SMOOTHNESS OF THE MODULI SPACE OF COMPLEXES OF COHERENT SHEAVES ON AN ABELIAN OR A PROJECTIVE K3 SURFACE

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Abstract. For an abelian or a projective K3 surface $X$ over an algebraically closed field $k$, consider the moduli space $\mathrm{Splcp}x_{X/k}^{\mathbb{A}}$ of the objects $E$ in $D^b(\mathrm{Coh}(X))$ satisfying $\mathrm{Ext}^1(E, E) = 0$ and $\mathrm{Hom}(E, E) \cong k$. Then we can prove that $\mathrm{Splcp}x_{X/k}^{\mathbb{A}}$ is smooth and has a symplectic structure.

1. Introduction

It was proved by Mukai in [6] that the moduli space of simple sheaves on an abelian or a projective K3 surface is smooth and has a symplectic structure. We will generalize this result to the moduli space of objects in the derived category of coherent sheaves, which is introduced in [3]. By [3], Theorem 4.4, the moduli space of (semi)stable objects with respect to a strict ample sequence in a derived category of objects in the derived category of coherent sheaves, which is introduced in [3]. By [4], Theorem 4.4, K3 surface is smooth and has a symplectic structure. We will generalize this result to the moduli space of (semi)stable objects with respect to a strict ample sequence in a derived category of coherent sheaves on an abelian or a projective K3 surface.

In this paper the author corrects the mistake. In the proof of the main results, we will use the trace map that also played a key role in [6]. More precisely, we will calculate the image by the trace map of the obstruction class for the deformation of vector bundles. By virtue of this consideration (Lemma 2.3) in section 2, the calculation of the trace becomes clear and the main result can be deduced from it.

The content of this paper was originally written as an appendix of [4]. However there was a mistake in the proof of the smoothness of $\mathrm{Splcp}x_{X/k}^{\mathbb{A}}$. In this paper the author corrects the mistake.

2. Obstruction Classes for the Deformation of Vector Bundles

First we recall the obstruction theory of the deformation of objects in the derived category of bounded complexes of coherent sheaves.

Let $S$ be a noetherian scheme and $X$ be a projective scheme flat over $S$. We fix an $S$-ample line bundle $\mathcal{O}_X(1)$ on $X$. Let $A$ be an artinian local ring over $S$ with residue field $k = A/m$ and $I$ be an ideal of $A$ such that $mI = 0$. Take a bounded complex $E^\bullet$ of $A/I$-flat coherent sheaves on $X_{A/I}$. Then there are integers $l$, $l'$ such that $E^i = 0$ for $i < l'$ and $i > l$. We can take a complex $V^\bullet = (V^i, d^i)$ of the form $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ and a quasi-isomorphism $V^\bullet \to E^\bullet$, where $V_i$ are free $A$ modules of finite rank, $V_i = 0$ for $i > l$ and $1 \ll m_i \ll m_{l-1} \ll \cdots \ll m_{l+1} \ll m_l \ll \cdots$. Take lifts

$$d^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$$

of the homomorphisms

$$d^i : V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_{A/I}}(-m_{i+1}).$$

Then we obtain homomorphisms

$$\delta^i := \delta d + d \delta : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to I \otimes_A V_{i+2} \otimes \mathcal{O}_{X_A}(-m_{i+2}).$$

We put

$$\omega(E^\bullet) := \{[\delta^i]\} \in H^2(\mathrm{Hom}(V^\bullet, V^\bullet \otimes I)) \cong \mathrm{Ext}^2(E^\bullet \otimes k, E^\bullet \otimes k) \otimes_k I.$$ 

Proposition 2.1. $\omega(E^\bullet) = 0$ if and only if $E^\bullet$ can be lifted to an object of $D^b(\mathrm{Coh}(X_A))$ of finite Tor dimension over $A$.

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Lemma 2.3.  

First note that the element \( \omega \) of \( \Theta \) is isomorphic to \( \mathit{Hom}(F, F) \otimes I \). Taking lifts \( \tilde{\omega} \) of \( \omega \) on \( X \), we put \( \theta_{\alpha\beta\gamma} := \varphi^{-1}_{\alpha\beta} \circ \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} \circ \text{id}_{F} : F_{|\alpha\beta\gamma} \rightarrow I \otimes F_{|\alpha\beta\gamma} \), where \( \alpha\beta\gamma := U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \). Then the cohomology class \( o(F) = [\{ \theta_{\alpha\beta\gamma} \}] \in \hat{H}^{2}(\End(F) \otimes I) \cong \text{Ext}^{2}(F, F \otimes I) \) can be defined. As is stated in [11, III, Proposition 7.1], we have the following proposition.

**Proposition 2.2.** \( o(F) = 0 \) if and only if \( F \) can be lifted to a locally free sheaf on \( X_{A} \).

A vector bundle \( F \) on \( X_{A/I} \) can be considered as the object of \( D^{b}(\text{Coh}(X_{A/I})) \) whose 0-th component is \( F \) and the other components are zero. We will show that \( \omega(F) \) and \( o(F) \) are the same element in \( \text{Ext}^{2}(F, F \otimes I) \).

We take a resolution of \( F \) by locally free sheaves:
\[
\cdots \rightarrow V^{2} \xrightarrow{d^{2}} V^{1} \xrightarrow{d^{1}} V^{0} \xrightarrow{\pi} F \rightarrow 0,
\]
where each \( V^{i} \) is isomorphic to \( U_{\alpha} \otimes \mathcal{O}_{X_{A/I}}(-m) \) for a free \( A \)-module \( V_{i} \) of finite rank and \( 1 \ll m_{1} \ll \cdots \ll m_{i} \ll m_{i+1} \ll \cdots \). Then we have a quasi-isomorphism \( \mathcal{H}(F, F) \otimes I \rightarrow \mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I \).

Let \( \mathcal{H}^{\bullet}(F, F) \otimes I \rightarrow \mathcal{C}^{\bullet}(\mathcal{H}^{\bullet}(F, F) \otimes I) \) be the Čech resolution of \( \mathcal{H}^{\bullet}(F, F) \otimes I \) with respect to the covering \( \{ U_{\alpha} \} \) and
\[
\mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I \rightarrow \mathcal{C}^{\bullet}(\mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I)
\]
be that of \( \mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I \). Then we obtain a composition of isomorphisms
\[
f : \hat{H}^{2}(\mathcal{H}^{\bullet}(V^{\bullet}, F)) \xrightarrow{\sim} \hat{H}^{2}(\mathcal{C}^{\bullet}(\mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I)) \xrightarrow{\sim} \hat{H}^{2}(\End(F) \otimes I),
\]
where \( \mathcal{C}^{\bullet}(\mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I) = \Gamma(X, \mathcal{C}^{\bullet}(\mathcal{H}^{\bullet}(V^{\bullet}, F) \otimes I)) \).

**Lemma 2.3.** Under the above assumption and notation, we have \( f(\omega(F)) = o(F) \).

**Proof.** First note that the element \( \omega(F) \) is defined by
\[
\omega(F) = \{ (\pi \otimes \text{id}_{I}) \circ (\tilde{d}^{1} \circ \tilde{d}^{2}) \} \in \hat{H}^{2}(\mathcal{C}^{\bullet}(V^{\bullet}, F) \otimes I),
\]
where \( \tilde{d}^{i} : V_{i} \otimes \mathcal{O}_{X_{A}}(-m_{i}) \rightarrow V_{i+1} \otimes \mathcal{O}_{X_{A}}(-m_{i+1}) \) is a lift of \( d^{i} \). Replacing \( \{ U_{\alpha} \} \) by its refinement, we may assume that \( \ker d^{2}|_{U_{\alpha}} \), \( \im d^{2}|_{U_{\alpha}} \), \( \im d^{1}|_{U_{\alpha}}, V_{2} \otimes \mathcal{O}_{X_{A/I}}(-m_{2})|_{U_{\alpha}}, V_{1} \otimes \mathcal{O}_{X_{A/I}}(-m_{1})|_{U_{\alpha}} \) and \( F|_{U_{\alpha}} \) are all free sheaves. Then the exact sequences
\[
0 \rightarrow \ker d^{2}|_{U_{\alpha}} \xrightarrow{i_{0}} V_{2} \otimes \mathcal{O}_{X_{A/I}}(-m_{2})|_{U_{\alpha}} \xrightarrow{\pi_{1}} \im d^{2}|_{U_{\alpha}} \rightarrow 0,
\]
\[
0 \rightarrow \im d^{2}|_{U_{\alpha}} \xrightarrow{i_{1}} V_{1} \otimes \mathcal{O}_{X_{A/I}}(-m_{1})|_{U_{\alpha}} \xrightarrow{\pi_{2}} \im d^{1}|_{U_{\alpha}} \rightarrow 0,
\]
\[
0 \rightarrow \im d^{1}|_{U_{\alpha}} \xrightarrow{i_{0}} V_{0} \otimes \mathcal{O}_{X_{A/I}}(-m_{0})|_{U_{\alpha}} \xrightarrow{\pi_{0}} F|_{U_{\alpha}} \rightarrow 0
\]
split and we can take free \( \mathcal{O}_{U_{\alpha}} \)-modules \( F_{\alpha}, I_{1}^{\alpha}, I_{2}^{\alpha} \) such that \( F_{\alpha} \otimes A/I \cong F|_{U_{\alpha}} \) and \( I_{i}^{\alpha} \otimes A/I \cong \im d^{i}|_{U_{\alpha}} \) for \( i = 1, 2 \). Taking lifts \( i_{0}^{\alpha}, i_{1}^{\alpha}, \pi_{1}^{\alpha}, \pi_{2}^{\alpha}, \pi_{0}^{\alpha} \) of \( i_{0}, i_{1}, \pi_{1}^{\alpha}, \pi_{2}^{\alpha}, \pi_{0}^{\alpha} \), we obtain splitting exact sequences
\[
0 \rightarrow \ker p_{2}^{\alpha} \rightarrow V_{2} \otimes \mathcal{O}_{X_{A}}(-m_{2})|_{U_{\alpha}} \xrightarrow{\pi_{1}^{\alpha}} I_{2}^{\alpha} \rightarrow 0,
\]
\[
0 \rightarrow I_{2}^{\alpha} \xrightarrow{i_{1}^{\alpha}} V_{1} \otimes \mathcal{O}_{X_{A}}(-m_{1})|_{U_{\alpha}} \xrightarrow{\pi_{0}^{\alpha}} I_{1}^{\alpha} \rightarrow 0,
\]
\[
0 \rightarrow I_{1}^{\alpha} \xrightarrow{i_{0}^{\alpha}} V_{0} \otimes \mathcal{O}_{X_{A}}(-m_{0})|_{U_{\alpha}} \xrightarrow{\pi_{0}^{\alpha}} F_{\alpha} \rightarrow 0.
\]
Let
\[ \tilde{s}_2^\alpha : I_2^\alpha \rightarrow V_2 \otimes O_{X_A}(-m_2)|_{u_\alpha}, \]
\[ \tilde{r}_1^\alpha : V_1 \otimes O_{X_A}(-m_1)|_{u_\alpha} \rightarrow I_2^\alpha, \]
\[ \tilde{s}_1^\alpha : I_1^\alpha \rightarrow V_1 \otimes O_{X_A}(-m_1)|_{u_\alpha}, \]
\[ \tilde{r}_0^\alpha : V_0 \otimes O_{X_A}(-m_0)|_{u_\alpha} \rightarrow I_1^\alpha, \quad \nu_\alpha : F_\alpha \rightarrow V_0 \otimes O_{X_A}(-m_0)|_{u_\alpha} \]
be splittings. Put
\[ d_2^\alpha : V_2 \otimes O_{X_A}(-m_2)|_{u_\alpha} \rightarrow I_2^\alpha, \]
\[ d_1^\alpha : V_1 \otimes O_{X_A}(-m_1)|_{u_\alpha} \rightarrow I_1^\alpha, \]
\[ \tau_\alpha : V_0 \otimes O_{X_A}(-m_0)|_{u_\alpha} \rightarrow I_1^\alpha, \]
\[ \sigma_\alpha : V_1 \otimes O_{X_A}(-m_1)|_{u_\alpha} \rightarrow I_2^\alpha, \]
\[ \tau_\alpha : V_0 \otimes O_{X_A}(-m_0)|_{u_\alpha} \rightarrow I_2^\alpha, \]
\[ \nu_\alpha : F_\alpha \rightarrow V_0 \otimes O_{X_A}(-m_0)|_{u_\alpha} \]
We consider the following diagram:
\[
\begin{array}{ccc}
\text{Hom}(V^0, F \otimes I) & \rightarrow & \text{Hom}(V^1, F \otimes I) \\
\downarrow & & \downarrow \\
C^0(\text{Hom}(V^0, F \otimes I)) & \rightarrow & C^0(\text{Hom}(V^1, F \otimes I)) \\
\downarrow & & \downarrow \\
C^1(\text{Hom}(V^0, F \otimes I)) & \rightarrow & C^1(\text{Hom}(V^1, F \otimes I)) \\
\downarrow & & \downarrow \\
C^2(\text{Hom}(V^0, F \otimes I)) & \rightarrow & C^2(\text{Hom}(V^1, F \otimes I)) \\
\end{array}
\]
where we put \( V^i := V_i \otimes O_{X_A}(-m_i) \) for \( i = 0, 1, 2 \). The image of \( \omega(F) \) in \( H^2(C^\bullet(\text{Hom}^\bullet(V^\bullet, F) \otimes I)) \) can be represented by
\[ \left\{ (\pi \otimes \text{id}_I) \circ d^1 \circ d^2|_{u_\alpha} \right\} \in C^0(\text{Hom}(V^2, F) \otimes I), \]
which defines the same element in \( H^2(C^\bullet(\text{Hom}^\bullet(V^\bullet, F) \otimes I)) \) as
\[ \left\{ (\pi \otimes \text{id}_I) \circ d^1 \circ d^2 \circ (\sigma_\alpha - \sigma_\beta) \right\} \in C^1(\text{Hom}(V^1, F) \otimes I). \]
On the other hand, the image of the element
\[ \left\{ (\pi \otimes \text{id}_I) \circ \left( d_1^\alpha - d^1 \circ (1 - d^2 \circ \sigma_\alpha) \right) \right\} \in C^0(\text{Hom}(V^1, F) \otimes I) \]
by the homomorphism \( C^0(\text{Hom}(V^1, F) \otimes I) \rightarrow C^0(\text{Hom}(V^2, F) \otimes I) \) is
\[ \left\{ (\pi \otimes \text{id}_I) \circ \left( d_1^\alpha - d^1 \circ (1 - d^2 \circ \sigma_\alpha) \right) \right\} \circ d^2_\alpha \]
\[ = \left\{ (\pi \otimes \text{id}_I)(d_1^\alpha \circ d^2_\alpha - d^1 \circ d^2_\alpha + d^1 \circ d^2 \circ \sigma_\alpha \circ d^2_\alpha) \right\} \]
\[ = \left\{ (\pi \otimes \text{id}_I)\left( -d^1 \circ d^2 \circ \sigma_\alpha \circ d^2_\alpha \right) \right\} \]
\[ = \left\{ (\pi \otimes \text{id}_I)\left( d_1^\alpha \circ d^2_\alpha + d^1 \circ d^2 \circ s^2_\alpha \circ \tilde{s}^2_\alpha \circ \tilde{p}^2_\alpha \right) \right\} \]
\[ = \left\{ (\pi \otimes \text{id}_I)\left( -d^1 \circ d^2 \circ s^2_\alpha \circ \tilde{p}^2_\alpha \right) \right\} \]
\[ = \left\{ (\pi \otimes \text{id}_I) \circ d^1 \circ (d^2_\alpha - d_{\nu_\alpha}) \circ s^2_\alpha \circ \tilde{p}^2_\alpha \right\} \]
\[ = 0. \]

Since
\[ \left\{ (\pi \otimes \text{id}_I) \circ d^1 \circ d^2 \circ (\sigma_\alpha - \sigma_\beta) \right\} + d\left\{ (\pi \otimes \text{id}_I) \circ \left( d_1^\alpha - d^1 \circ (1 - d^2 \circ \sigma_\alpha) \right) \right\} \]
\[ = \left\{ (\pi \otimes \text{id}_I) \circ d^1 \circ d^2 \circ (\sigma_\alpha - \sigma_\beta) \right\} + \left\{ (\pi \otimes \text{id}_I) \circ \left( d_1^\alpha - d^1 \circ (1 - d^2 \circ \sigma_\alpha) \right) \right\} |_{u_\alpha \cup U_\beta} \]
\[ - \left\{ (\pi \otimes \text{id}_I) \circ \left( d_1^\alpha - d^1 \circ (1 - d^2 \circ \sigma_\alpha) \right) \right\} |_{u_\alpha \cap U_\beta} \]
\[ = - \left\{ (\pi \otimes \text{id}_I) \circ (d_1^\alpha - d_3^\beta) \right\} , \]
we can see that \( \left\{ (\pi \otimes \text{id}_I) \circ \tilde{d}^1 \circ \tilde{d}^2 (\sigma_\alpha - \sigma_\beta) \right\} \) and \( \left\{ (\pi \otimes \text{id}_I) \circ (d^1_\alpha - d^2_\beta) \right\} \) define the same element in \( H^2(C^* (\text{Hom}^*(V^* , F) \otimes I)) \). We can see that the element \( -\left\{ (\pi \otimes \text{id}_I) \circ (d^1_\alpha - d^2_\beta) \right\} \) defines the same element as

\[
- \left\{ (\pi \otimes \text{id}_I) \circ ((d^1_\beta - d^1_\gamma) \circ \tau_\beta - (d^1_\alpha - d^1_\beta) \circ \tau_\alpha + (d^1_\alpha - d^1_\beta) \circ \tau_\alpha) \right\} \\
= \left\{ (\pi \otimes \text{id}_I) \circ (d^1_\beta - d^1_\gamma) \circ (\tau_\alpha - \tau_\beta) \right\} \in C^2(\text{Hom}(V^0, F) \otimes I)
\]

in \( H^2(C^* (\text{Hom}^*(V^* , F) \otimes I)) \). Thus \( \omega(F) \) is equal to the element given by

\[
\left\{ (\pi \otimes \text{id}_I) \circ (d^1_\beta - d^1_\gamma) \circ (\tau_\alpha - \tau_\beta) \right\} \in C^2(\text{Hom}(V^0, F) \otimes I)
\]

in \( H^2(C^* (\text{Hom}^*(V^* , F) \otimes I)) \). On the other hand, the element \( o(F) \) is given by

\[
\left\{ (\pi \otimes \nu_\alpha)^{-1} \circ \pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha - \text{id}_{F_\alpha} \right\}
\]

in \( H^2(\text{End}(F) \otimes I) \), whose image in \( H^2(C^* (\text{Hom}^*(V^* , F) \otimes I)) \) is represented by

\[
\left\{ (\pi \otimes \nu_\alpha)^{-1} \circ \pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha \circ \pi_\alpha - \pi_\alpha \right\}
\]

\[
= \left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ (\pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha \circ \pi_\alpha - \pi_\gamma \circ \nu_\alpha \circ \pi_\alpha) \right\}
\]

\[
= \left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ (\pi_\gamma \circ \nu_\beta \circ \pi_\beta \circ \nu_\alpha - 1) \circ \nu_\alpha \circ \pi_\alpha \right\}
\]

\[
= \left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ (-d^1_\beta \circ \tau_\beta) \circ (1 - d^1_\alpha \circ \tau_\alpha) \right\}
\]

\[
= \left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ (\pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) - \pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) \circ d^1_\alpha \circ \tau_\alpha) \right\}.
\]

Here we have

\[
\pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) \circ d^1_\alpha \circ \tau_\alpha
\]

\[
= \pi_\gamma \circ d^1_\beta \circ (\tilde{v}^1_\alpha \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 - \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0) \circ d^1_\alpha \circ \tau_\alpha
\]

\[
= \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ d^1_\alpha \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha
\]

\[
= \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha
\]

\[
= \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha - \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha
\]

\[
= \pi_\gamma \circ d^1_\beta \circ \tilde{s}^1_\alpha \circ \tilde{r}^\beta_0 \circ \tilde{v}^1_\alpha \circ \tilde{p}^1_\alpha \circ \tau_\alpha
\]

So the image of \( o(F) \) in \( H^2(C^* (\text{Hom}^*(V^* , F) \otimes I)) \) is

\[
\left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ d^1_\beta \circ (\tau_\alpha - \tau_\beta) \right\} = \left\{ (\pi_\gamma \circ \nu_\alpha)^{-1} \circ \pi_\gamma \circ (d^1_\beta - d^1_\alpha) \circ (\tau_\alpha - \tau_\beta) \right\}
\]

\[
= \left\{ (\pi \otimes \text{id}_I) \circ (d^1_\beta - d^1_\alpha) \circ (\tau_\alpha - \tau_\beta) \right\}
\]

Thus we have the equality \( f(\omega(F)) = o(F) \).

**Remark 2.4.** Several authors introduced obstruction classes for the deformation of vector bundles and coherent sheaves. For example, [2, Chap 2, Appendix] is a good reference. However, it is not so clear that these definitions are all equivalent.

3. Smoothness and symplectic structure

Let \( X \) be a projective scheme over a noetherian scheme \( S \), which is flat over \( S \). We define a functor \( \text{Splicpx}_{X/S} \) of the category of locally noetherian schemes to that of sets by putting

\[
\text{Splicpx}_{X/S}(T) := \left\{ E^* \left| \begin{array}{c}
E^* \text{ is a bounded complex of } T-\text{flat coherent } \\
O_X \text{-modules such that for any } t \in T, \\
E^*(t) \text{ satisfies the following condition } (*),
\end{array} \right. \right\} / \sim,
\]

where \( \sim \) denotes the equivalence relation such that...

where $T$ is a locally noetherian scheme over $S$ and $E^\bullet \sim F^\bullet$ if there is a line bundle $L$ on $T$ such that $E^\bullet \cong F^\bullet \otimes L$ in $D(X_T)$. Here $D(X_T)$ is the derived category of $\mathcal{O}_{X_T}$-modules and the condition $(\ast)$ is

\[
\text{(\ast)} \quad \text{Ext}^i(E^\bullet(t), E^\bullet(t)) \cong \begin{cases} 0 & \text{if } i = -1 \\ k(t) & \text{if } i = 0. \end{cases}
\]

Note that we denote $E^\bullet \otimes^L k(t)$ by $E^\bullet(t)$. Let $\text{Splcpx}_{X/S}^{\text{et}}$ be the étale sheafification of $\text{Splcpx}_{X/S}$.

**Theorem 3.1.** $\text{Splcpx}_{X/S}^{\text{et}}$ is represented by an algebraic space over $S$.

(Proof is in [3], Theorem 0.2. This result was generalized by Lieblich in [5] for $X$ proper over $S$.)

**Theorem 3.2.** If $X$ is an abelian or a projective K3 surface over an algebraically closed field $k$, $\text{Splcpx}_{X/k}^{\text{et}}$ is smooth over $k$.

**Proof.** Take an artinian local ring $A$ over $k$ with residue field $k = A/m$ and an ideal $I$ of $A$ such that $mI = 0$. It is sufficient to show that $\text{Splcpx}_{X/k}(A) \to \text{Splcpx}_{X/k}(A/I)$ is surjective. Indeed we can take a scheme $U$ locally of finite type over $k$ and a morphism $p : U \to \text{Splcpx}_{X/k}$ such that the composite $U \to \text{Splcpx}_{X/k} \to \text{Splcpx}_{X/k}^{\text{et}}$ is étale and surjective. Take any artinian local ring $A$ over $k$ with residue field $k = A/m$ and an ideal $I$ of $A$ such that $mI = 0$. Take any member $x \in U(A/I)$. By the surjectivity of $\text{Splcpx}_{X/k}(A) \to \text{Splcpx}_{X/k}(A/I)$, we can take an element $y \in \text{Splcpx}_{X/k}(A)$ such that $y \otimes A/I = p(x)$.

Then $\iota \circ p : U \to \text{Splcpx}_{X/k}^{\text{et}}$ is étale, there is an element $z \in U(A)$ such that $z \otimes A/I = x$ and $(\iota \circ p)(z) = y$. Thus $U$ is smooth over $k$.

Let $E^\bullet$ be an $A/I$-valued point of $\text{Splcpx}_{X/k}^{\text{et}}$. Put $E^\bullet_0 := E^\bullet \otimes k$ and

\[l' := \min \{ i \mid H^i(E^\bullet_0 \otimes^L k(x)) \neq 0 \text{ for some } x \in X \}.\]

We may assume that $E^\bullet$ is of the form

\[
\cdots \to 0 \to 0 \to E^\prime \xrightarrow{d^\prime} V^\prime \xrightarrow{d^\prime} \cdots \xrightarrow{V^i} \cdots \xrightarrow{d^i} V^i \xrightarrow{d^i} 0 \to 0 \cdots,
\]

where $E^\prime$ is a vector bundle on $X_{A/I}$, $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ with $V_i$ a finite dimensional vector space over $k$, $\mathcal{O}_X(1)$ a fixed ample line bundle on $X$ and $1 \ll m_1 \ll m_2 \ll \cdots \ll m_{l+1}$. We can see that $d^\prime \otimes k(x)$ is not injective for some $x \in X$. Take a resolution

\[
\cdots \xrightarrow{\cdots} V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i) \xrightarrow{\cdots} \cdots \xrightarrow{\cdots} V_l \otimes \mathcal{O}_{X_{A/I}}(-m_l) \xrightarrow{\pi} E^\prime \xrightarrow{\pi} 0,
\]

where each $V_i$ is a vector space over $k$ of finite dimension and

\[m_{l+1} \ll m_l \ll \cdots \ll m_i \ll m_{i-1} \ll \cdots.\]

We put $V^i = V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i)$ for $i \leq l$ and $V^i = 0$ for $i > l$. Let $V^\bullet$ be the complex

\[
\cdots \to V^i \to V^{i+1} \to \cdots \to V^l \xrightarrow{d^\prime \otimes \pi} V^{l+1} \to \cdots \to V^l \to 0 \to \cdots.
\]

Then there is a canonical quasi-isomorphism

\[V^\bullet \to E^\bullet.\]

Put $V^\bullet_0 := V^\bullet \otimes k$. Let

\[
\text{tr}^\bullet : \mathcal{H}om^\bullet(E^\bullet_0, E^\bullet_0) \xrightarrow{\sim} \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(E^\bullet_0, E^\bullet_0), \mathcal{O}_X) \to \mathcal{O}_X
\]

be the dual of the canonical morphism

\[
\mathcal{O}_X \to \mathcal{H}om^\bullet(E^\bullet_0, E^\bullet_0); \quad 1 \mapsto \text{id}_{E^\bullet_0}.
\]

Note that $\text{tr}^p = 0$ on $\mathcal{H}om^p(E^\bullet_0, E^\bullet_0)$ for $p \neq 0$ and $\text{tr}^0(\{x^i\}) = \sum_i (-1)^i \text{tr}(x^i)$ for $x^i \in \mathcal{H}om(E^i_0, E^0_0)$. $\text{tr}^\bullet$ is also introduced in [2, Chapter 10]. There is a commutative diagram

\[
\begin{array}{ccc}
\text{Ext}_X^2(E^\bullet_0, E^\bullet_0) & \xrightarrow{H^2(\text{tr}^\bullet)} & H^2(X, \mathcal{O}_X) \\
\downarrow s_1 \cong & & \downarrow s_2 \cong \\
\text{Hom}_{D(X)}(E^\bullet_0, E^\bullet_\vee) & \xrightarrow{\cdot} & H^0(X, \mathcal{O}_X)\vee
\end{array}
\]
where $s_1, s_2$ are the isomorphisms determined by Grothendieck-Serre duality and the bottom row is the dual of $k = H^0(\mathcal{O}_X) \to \text{Hom}_{D(X)}(E_0^\bullet, E_0^\bullet)$, which is bijective since $E_0^\bullet$ is simple. Thus the homomorphism

$$\text{Ext}^2_X(E_0^\bullet, E_0^\bullet) \xrightarrow{H^2(\text{tr})} H^2(X, \mathcal{O}_X)$$

is an isomorphism.

Note that there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}\text{om}^\bullet(E^l[-l'], I \otimes E^\bullet) & \xrightarrow{\tau} & \mathcal{H}\text{om}(E^l, I \otimes E^l) \\
\downarrow & & \downarrow \langle -1 \rangle^l \text{tr} \\
\mathcal{H}\text{om}^\bullet(E^\bullet, I \otimes E^\bullet) & \xrightarrow{\text{tr}} & \mathcal{O}_X \otimes I.
\end{array}$$

From the above commutative diagram, we obtain a commutative diagram

$$\begin{array}{ccc}
\text{Ext}^2(E^l[-l'], I \otimes E^\bullet) & \xrightarrow{\tau} & \text{Ext}^2(E^l, I \otimes E^l) \\
\downarrow \langle \langle \rangle \rangle & & \downarrow \langle -1 \rangle^l \text{tr}^2 \\
\text{Ext}^2(E^\bullet, I \otimes E^\bullet) & \xrightarrow{H^2(\text{tr})} & H^2(\mathcal{O}_X) \otimes I.
\end{array}$$

Note that the morphism

$$\text{Hom}(E_0^\bullet, E_0^\bullet) \to \text{Hom}(E_0^\bullet, E_0^\bullet[-l'])$$

is not zero, since the image of $\text{id}$ by this morphism is the canonical morphism $\iota : E_0^\bullet \to E_0^\bullet[-l']$ which is not zero because

$$H^0(l \otimes k(x)) : \ker(d^l_{E_0^\bullet} \otimes k(x)) = H^0(E_0^\bullet \otimes k(x)) \to H^0(E_0^\bullet[-l'] \otimes k(x)) = E_0^\bullet \otimes k(x)$$

is not zero. By Grothendieck-Serre duality, we can see that

$$\iota^* : \text{Ext}^2(E_0^\bullet[-l'], E_0^\bullet) \to \text{Ext}^2(E_0^\bullet, E_0^\bullet)$$

is not zero. Since $\text{Ext}^2(E_0^\bullet, E_0^\bullet) \cong k$, $\iota^*$ is surjective. So the morphism

$$\sigma : \text{Ext}^2(E^l[-l'], I \otimes E^\bullet) \to \text{Ext}^2(E^\bullet, I \otimes E^\bullet)$$

is also surjective.

Take an obstruction class $\omega(E^\bullet) \in \text{Ext}^2(E^\bullet, I \otimes E^\bullet)$ for the lifting of $E^\bullet$ to an $A$-valued point of $\text{SpLCS}_{X/k}$. Then there is a member $\varphi = [\langle \varphi^i \rangle] \in \text{Ext}^2(E^l[-l'], I \otimes E^\bullet)$ such that $\sigma(\varphi) = \omega(E^\bullet)$. Here $\varphi^i : V^i \to I \otimes E^{i+2}$ ($i \leq l'$) and $\varphi^i = 0$ for $i > l'$. There is an element $\gamma = (\gamma^i) \in \text{Hom}^1(V^i, I \otimes E^\bullet)$ such that

$$\gamma^i+1 \circ d_{V^i}^+ \circ d_{E^i}^+ \circ \gamma^i = d^i+1 \circ d^i - \varphi^i \quad \text{for} \; i \geq l' - 1$$

$$\gamma^{i-1} \circ d_{V^i}^{-2} = \pi \circ d^{i-1} \circ d^{i-2} - \varphi^{i-2},$$

where $d^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$ is a lift of $d_{V^i}$. We can see that the image of $\varphi$ by the morphism $\tau : \text{Ext}^2(E^l[-l'], I \otimes E^\bullet) \to \text{Ext}^2(E^l, I \otimes E^l)$ is given by $[\pi \circ d^{i-1} \circ d^{i-2}]$, which is just the obstruction class $\omega(E^l)$. By Lemma 2.3, we have $\omega(E^l) = o(E^l)$. We can see that $\text{H}^2(\text{tr})(o(E^l)) = o(\text{det}(E^l))$. Since the Picard scheme $\text{Pic}_{X/k}$ is smooth over $k$, we have $o(\text{det}(E^l)) = 0$. So we have

$$\text{H}^2(\text{tr})(\omega(E^\bullet)) = \text{H}^2(\text{tr})(\sigma(\varphi))$$

$$= \langle -1 \rangle^{l'} \text{H}^2(\text{tr})(\tau(\varphi))$$

$$= \langle -1 \rangle^{l'} \text{H}^{2}(\text{tr})(\omega(E^l))$$

$$= \langle -1 \rangle^{l'} \text{H}^2(\text{tr})(o(E^l))$$

$$= \langle -1 \rangle^{l'} o(\text{det}(E^l)) = 0.$$ 

Since the morphism

$$\text{H}^2(\text{tr}) : \text{Ext}^2(E, I \otimes E) \to \text{H}^2(\mathcal{O}_X) \otimes I$$

is isomorphic, we have $\omega(E^\bullet) = 0$. Thus $\text{SpLCS}_{X/k}$ is smooth over $k$. 

The following theorem is essentially proved in [2],II-10. We give a proof again.
We denote this complex by $\alpha$. Note that there are canonical isomorphisms \( X/k \) for any algebraic space $U$ étale over $\text{Splcpx}_{X/k}^\alpha$, where $k[\epsilon]$ is the $k$-algebra generated by $\epsilon$ with $\epsilon^2 = 0$ and $U \to k[\epsilon]$ is the morphism induced by the ring homomorphism
\[
k[\epsilon] \to k; \quad \epsilon \mapsto 0.
\]

There is an étale covering $\coprod_i U_i \to \text{Splcpx}_{X/k}^\alpha$ such that $U_i \to \text{Splcpx}_{X/k}^\alpha$ factors through $\text{Splcpx}_{X/k}$, that is, there is a universal family $E_U$ on each $X_U$. Let $U$ be an affine scheme étale over $\coprod_i U_i$ and $E_U$ be the pull-back of the universal family. Take any element $v \in T_{\text{Splcpx}_{X/k}^\alpha}(U)$. Then $U[k[\epsilon]] \to U$ factors through $\text{Splcpx}_{X/k}^\alpha$ factors through $\coprod_i U_i$, since $\coprod_i U_i$ is étale over $\text{Splcpx}_{X/k}^\alpha$. Let $E_{i[U]} \in \text{Splcpx}_{X/k}(U[k[\epsilon]])$ be the pull-back of the universal family. We can take a complex $\tilde{V}^\bullet$ of the form $\tilde{V}^i = V_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X_U[k[\epsilon]]}(-m_i)$ and a quasi-isomorphism $\tilde{V}^\bullet \to E_{i[U]}^\bullet$, where $V_i$ is a locally free sheaf of finite rank on $U$, $V_i = 0$ for $i \geq 1$, $\mathcal{O}_X(1)$ is a fixed ample line bundle on $X$ and $\cdots \gg m_i \gg m_{i+1} \gg \cdots$. Let $V^\bullet$ be the pull-back of $\tilde{V}^\bullet$ by $X \times U \xrightarrow{\text{id}_X \times i_0} X \times U[k[\epsilon]]$. Then we obtain an element
\[
[(d_{i^*}^V - d_{i^*}^V \otimes 1)] \in H^1(\text{Hom}(\tilde{V}^\bullet, \epsilon k[\epsilon] \otimes \tilde{V}^\bullet)) \cong \text{Ext}^1(E_U^\bullet, E_U^\bullet),
\]
which is independent of the choice of the representative $\tilde{V}^\bullet$. We can see that the mapping $v \to [(d_{i^*}^V - d_{i^*}^V \otimes 1)]$ defines an isomorphism
\[
T_{\text{Splcpx}_{X/k}^\alpha}(U) \xrightarrow{\sim} H^0(U, \text{Ext}^1_{X_U/U}(E_U^\bullet, E_U^\bullet)).
\]

For an affine scheme $U$ étale over $\coprod_i U_i$, there is a canonical pairing:
\[
\alpha_U : \text{Ext}^1_{X_U/U}(E_U^\bullet, E_U^\bullet) \times \text{Ext}^1_{X_U/U}(E_U^\bullet, E_U^\bullet) \rightarrow \text{Ext}^2_{X_U/U}(E_U^\bullet, E_U^\bullet) \rightarrow g \circ h.
\]

Note that there are canonical isomorphisms
\[
\text{Ext}^2_{X_U/U}(E_U^\bullet, E_U^\bullet) \cong \text{Ext}^0_{X_U/U}(E_U^\bullet, E_U^\bullet)^\vee \cong \mathcal{O}_U.
\]

Then we can obtain a pairing
\[
\alpha : T_{\text{Splcpx}_{X/k}^\alpha} \times T_{\text{Splcpx}_{X/k}^\alpha} \rightarrow \mathcal{O}_{\text{Splcpx}_{X/k}^\alpha}
\]
by patching $\alpha_U$.

Now we will see that $\alpha$ is skew-symmetric. Take any $k$-valued point $p$ of $\text{Splcpx}_{X/k}^\alpha$. $p$ corresponds to a complex
\[
\cdots \rightarrow V_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_i) \xrightarrow{d_{i^*}} V_{i+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_{i+1}) \xrightarrow{d_{i^*}+\epsilon \omega} \cdots \rightarrow V_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_i) \rightarrow 0 \rightarrow \cdots
\]

We denote this complex by $V^\bullet$. Let us consider the restriction
\[
\alpha(p) : \text{Ext}^1(V^\bullet, V^\bullet) \times \text{Ext}^1(V^\bullet, V^\bullet) \rightarrow \text{Ext}^2(V^\bullet, V^\bullet) \cong k
\]
of the pairing $\alpha$. Take any element $v = \{[\omega^{\epsilon}]\} \in H^1(\text{Hom}(V^\bullet, V^\bullet)) \cong \text{Ext}^1(V^\bullet, V^\bullet)$ and let $\tilde{V}^\bullet$ be a member of $\text{Splcpx}_{X/k}(k[\epsilon])$ which corresponds to $v$. $\tilde{V}^\bullet$ can be given by the complex
\[
\cdots \rightarrow V_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_i) \otimes \mathcal{O}_{X_U}(-m_{i+1}) \otimes k[\epsilon] \rightarrow \cdots
\]
Consider the surjection $k[t]/(t^3) \to k[\epsilon]; \ t \mapsto \epsilon$ and the extension
\[
d_{i^*} + \epsilon \omega : V_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_i) \otimes \mathcal{O}_{X_U}(-m_{i+1}) \otimes k[t]/(t^3) \rightarrow V_{i+1} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-m_{i+1}) \otimes k[t]/(t^3)
\]
of the homomorphism $d_{v, *}^i + e^v : V_i \otimes \mathcal{O}_X(-m_i) \otimes k[\epsilon] \to V_{i+1} \otimes \mathcal{O}_X(-m_{i+1}) \otimes k[\epsilon]$. Then the obstruction class $\omega(V^\bullet)$ for the lifting of $V^\bullet$ to a member of $\text{Splcir}_{X/k}(k[t]/(t^j))$ with respect to the surjection $k[t]/(t^j) \to k[\epsilon]$; $t \mapsto \epsilon$ is given by $[\{(d_{v, *}^{i+1} + t v^{i+1}) \circ (d_{v, *}^i + t v^i)\}] \in (t^2) \otimes \text{Ext}^2(V^\bullet, V^\bullet)$. However, 

$$(d_{v, *}^{i+1} + t v^{i+1}) \circ (d_{v, *}^i + t v^i) = d_{v, *}^{i+1} \circ d_{v, *}^i + t(d_{v, *}^{i+1} \circ v^i + v^{i+1} \circ d_{v, *}^i) + t^2 v^{i+1} \circ v^i = t^2 v^{i+1} \circ v^i.$$ 

Then $\alpha(p)(v, v) = v \circ v = [(v^{i+1} \circ v^i)] = \omega(V^\bullet) = 0$ since $\text{Splcir}_{X/k}$ is smooth over $k$.

Next we will see that $\alpha$ is nondegenerate. The canonical isomorphism

$$\mathcal{R} \text{Hom}(E_{u, *}, E_{v, *}) \sim \mathcal{R} \text{Hom}(E_{u, *}, E_{v, *})^\vee$$

induces the composite isomorphism by Grothendieck-Serre duality

$$\text{Ext}^1(E_{u, *}, E_{v, *}) \sim \text{Ext}^1(\mathcal{R} \text{Hom}(E_{u, *}, E_{v, *}), \mathcal{O}_U) \sim \text{Hom}(\text{Ext}^1(E_{u, *}, E_{v, *}), \mathcal{O}_U),$$

which is just the homomorphism induced by $\alpha$. Thus $\alpha$ is nondegenerate.

Finally we will show that $\alpha$ is $d$-closed. For an affine scheme $U$ étale over $\coprod_i U_i$, take $u, v, w \in T_{\text{Splcir}_{X/k}}(U)$. Let $E^\bullet \in \text{Splcir}_{X/k}(U)$ be the pullback of the universal family. We may assume that there exists a complex $V^\bullet$ of the form $V_i = V_i \otimes \mathcal{O}_X(-m_i)$ such that $V^\bullet$ is quasi-isomorphic to $E^\bullet$ and that $V_i$ are vector spaces of finite dimension over $k$ and $m_i$ are integers. Take $u \in T_U(U)$. $u$ can be regarded as a derivation $\mathcal{O}_U \to \mathcal{O}_U$ over $\mathcal{O}_U$, which is canonically extended to a derivation

$$D_u : V_{i'}^\vee \otimes \mathcal{O}_U(m_i - m_{i'}) \otimes \mathcal{O}_U \to V_i^\vee \otimes \mathcal{O}_U(m_i - m_{i'}) \otimes \mathcal{O}_U$$

for $i \leq j$. We have $d_{v, *}^{i+1} \circ D_u(d_{v, *}^i) + D_u(d_{v, *}^{i+1}) \circ d_{v, *}^i = 0$ for any $i$. So we have $[\{D_u(d_{v, *}^i)\}] \in \text{Ext}^1(V^\bullet, V^\bullet)$, which corresponds to $u$ by the isomorphism $T_U(U) \sim \text{Ext}^1(V^\bullet, V^\bullet)$. Note that for $u, v \in T_U(U)$ we have

$$\alpha(u, v) = \{D_u(d_{v, *}^{i+1}) \circ D_v(d_{v, *}^i)\} \in \text{Ext}^2(V^\bullet, V^\bullet) \cong \mathcal{H}^1(U, \mathcal{O}_U).$$

For $u, v, w \in T_U(U)$, we have

$$d_\alpha(u, v, w) = \{D_u(\alpha(u, v)) + D_v(\alpha(w, u)) + D_w(\alpha(u, v)) + \alpha(w, [u, v]) + \alpha([u, w], v) + \alpha([u, v], w)\}$$

$$= \{D_u(D_v(d_{v, *}^{i+1}) \circ d_{v, *}^i) + D_v(D_w(d_{w, *}^{i+1}) \circ d_{w, *}^i) + D_w(D_u(d_{u, *}^{i+1}) \circ d_{u, *}^i) + D_{u,v}(d_{v, *}^{i+1}) \circ (D_u D_v \circ d_{v, *}^i) + D_{v,w}(d_{w, *}^{i+1}) \circ (D_v D_w \circ d_{w, *}^i) + D_{w,u}(d_{w, *}^{i+1}) \circ (D_w D_u \circ d_{u, *}^i)\}$$

$$= \{D_{u,v}(d_{v, *}^{i+1}) \circ D_w(d_{w, *}^i) + D_{v,w}(d_{w, *}^{i+1}) \circ D_v(d_{v, *}^i) + D_{w,u}(d_{w, *}^{i+1}) \circ D_u(d_{u, *}^i) + D_{u,v}(d_{v, *}^{i+1}) \circ D_w(d_{w, *}^i) + D_{v,w}(d_{w, *}^{i+1}) \circ D_v(d_{v, *}^i) + D_{w,u}(d_{w, *}^{i+1}) \circ D_u(d_{u, *}^i)\}$$

Here note that

$$D_u D_v D_w(d_{v, *}^{i+1} \circ d_{v, *}^i) = D_u(D_v(D_w(d_{w, *}^{i+1}) \circ d_{v, *}^i) + D_u(D_v(D_w(d_{w, *}^{i+1}) \circ d_{v, *}^i)))$$

$$= D_u(D_v(D_w(d_{w, *}^{i+1}) \circ d_{v, *}^i) + D_w(d_{v, *}^{i+1} \circ d_{v, *}^i))$$

$$= D_u(D_v D_w(d_{w, *}^{i+1}) \circ d_{v, *}^i) + D_w(d_{v, *}^{i+1} \circ d_{v, *}^i) \circ D_v(d_{v, *}^i).$$

So $\alpha$ is a closed 2-form. \qed
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