Notes on Weyl–Clifford algebras

Alexander Yu. Vlasov

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Abstract
Here is discussed generalization of Clifford algebras, \( l^n \)-dimensional Weyl–Clifford algebras \( T(n, l) \) with \( n \) generators \( t_k \) satisfying equation

\[
\left( \sum_{k=1}^{n} a_k t_k \right)^l = \sum_{k=1}^{n} (a_k)^l
\]

It is originated from two basic and well known constructions: representation of Clifford algebras via tensor products of Pauli matrices together with extension for \( l > 2 \) using Weyl commutation relations. Presentation of such general topics here may not pretend to entire originality or completeness and it is rather a preliminary excursus into this very broad and interesting area of research.

1 Introduction
Clifford algebras let us write “square root” of a quadratic form \(-Q(x, y)\) \([1]\). If \(-Q(x, y)\) is Euclidean distance \(Q(x, x) = -\sum_{k=1}^{n} x_k^2\) it corresponds to simple expressions \([1, 2]\) for generators \(e_k\) of real Clifford algebra \(\mathbb{C}l(n)\):

\[
\left( \sum_{k=1}^{n} x_k e_k \right)^2 = \sum_{k=1}^{n} x_k^2 \mathbb{1}
\]

(where \(\mathbb{1}\) is unit of the algebra, often omitted further for simplicity) and so

\[
\{e_k, e_j\} = e_k e_j + e_j e_k = 2 \delta_{jk}.
\]

It was real case and it is also useful to consider \(2^n\)-dimensional universal complex Clifford algebra \(\mathbb{C}l(n, \mathbb{C})\) \([3]\).

In the paper is discussed natural question about polynomial analogue of this construction, \(i.e.\) “\(l\)-th root” of polynomial \(P(x) = \sum_{k=1}^{n} x_k^l\) described by noncommutative Lamé equation \([4]\)

\[
\left( \sum_{k=1}^{n} x_k t_k \right)^l = \sum_{k=1}^{n} x_k^l, \quad x_k \in \mathbb{C}
\]

\[^{*}\text{E-mail: Alexander.Vlasov@Pbox.spbu.ru}\]

\[^{1}\text{See footnote 2 on page 3 for short historical reference.}\]
with $t_k$ are $n$ generators of a complex algebra $\mathfrak{T}(n,l)$: $\mathfrak{T}(n,2) \cong \mathfrak{Cl}(n,\mathbb{C})$.

In can be shown, that Eq. (1.3) follows from a polynomial analogue of Eq. (1.2), i.e.

$$t_j t_k = \zeta t_k t_j \quad (j < k), \quad (t_k)^l = \mathbf{I}.$$  \hspace{1cm} (1.4)

where $\zeta$ is primitive $l$-th root of unit

$$\zeta = e^{2\pi i/l} \hspace{1cm} (1.5)$$

and a proof is considered in Sec. 2. It is not discussed here, if Eq. (1.4) is necessary condition for Eq. (1.3), but instead of $\zeta$ defined by Eq. (1.4) it is possible to use $\zeta' = \zeta^m$ if $m$ and $l$ are relatively prime.

In Sec. 3 are described matrix representation of algebras $\mathfrak{T}(n,l)$ based on straightforward generalization of a Clifford algebra construction. In Sec. 4 the algebras and a limit $l \to \infty$ are discussed as particular case of Weyl representation of Heisenberg commutation relations. Due to such representations and properties of $\mathfrak{T}(n,l)$ here is used term $\text{Weyl–Clifford algebras}$. It should be mentioned also, that formally $\mathfrak{T}(n,l)$ is also particular example of general object, known as an algebra of quantum affine space \cite{5}, but it is not discussed here, because this paper is not devoted to immense theory of quantum groups \cite{10} having alternative prerequisites.

2 Noncommutative Lamé equation

Let us consider a proof of Eq. (1.3) for an algebra defined by Eqs. (1.4, 1.5). It is convenient to consider even more general case, when in Eq. (1.4) is not specified condition $t_k^l = \mathbf{I}$ and write instead of Eq. (1.3)

$$\left(\sum_{k=1}^{n} c_k t_k\right)^l = \sum_{k=1}^{n} c_k^l t_k^l.$$  \hspace{1cm} (2.1)

For simplicity of proof here is suggested, that all $t_k$ in Eq. (2.1) are invertible.

Let us prove first a lemma, that if $\mathbf{I}, \mathbf{r}$ are two elements of an associative algebra satisfying properties:

$$\mathbf{I} \mathbf{r} = \zeta \mathbf{r} \mathbf{I} \quad (\zeta = e^{2\pi i/l}), \quad \exists \mathbf{I}^{-1} : \mathbf{I}^{-1} \mathbf{I} = \mathbf{I}^{-1} = \mathbf{I},$$  \hspace{1cm} (2.2)

then for any coefficients $a, b \in \mathbb{C}$

$$(a \mathbf{I} + b \mathbf{r})^l = a^l \mathbf{I}^l + b^l \mathbf{r}^l.$$  \hspace{1cm} (2.3)

Proof: For invertible $\mathbf{I}$ it is possible to write

$$(a \mathbf{I} + b \mathbf{r})^l = (a + b \mathbf{r} \mathbf{I}^{-1}) \mathbf{I} (a + b \mathbf{r} \mathbf{I}^{-1}) \mathbf{I} \cdots (a + b \mathbf{r} \mathbf{I}^{-1}) \mathbf{I} =$$

$$= (a + b \mathbf{r} \mathbf{I}^{-1}) (a + \zeta b \mathbf{r} \mathbf{I}^{-1}) \cdots (a + \zeta^{l-1} b \mathbf{r} \mathbf{I}^{-1}) \mathbf{I}^l,$$  \hspace{1cm} (2.4)

where Eq. (2.3) produced from Eq. (2.4) by sequential transition of all terms $\mathbf{I}$ at right side of expression using relation $\mathbf{I} (\mathbf{r} \mathbf{I}^{-1}) = \zeta (\mathbf{r} \mathbf{I}^{-1}) \mathbf{I}$ following from Eq. (2.2).
Let us note, that if $x$ is a complex number, it is possible to write
\[ (x - 1)(x - \zeta)(x - \zeta^2) \cdots (x - \zeta^{l-1}) = x^l - 1, \]
(2.6) because $\zeta^k (k = 0, \ldots, l - 1)$ are $l$ different roots of $x^l - 1 = 0$. The analogue of Eq. (2.6) for homogeneous polynomials with two variables $x, y \in \mathbb{C}$ is (cf. with formal substitution $x \to x/y$):
\[ (x - y)(x - \zeta y)(x - \zeta^2 y) \cdots (x - \zeta^{l-1} y) = x^l - y^l, \]
(2.7) or after change $y \to -y$:
\[ (x + y)(x + \zeta y)(x + \zeta^2 y) \cdots (x + \zeta^{l-1} y) = x^l + (-1)^{l-1} y^l, \]
(2.8) but because Eq. (2.8) is pure algebraic identity, it is possible to write instead of $x, y \in \mathbb{C}$ any commuting elements $a, b$ of an algebra
\[ ab = ba \implies \prod_{k=0}^{l-1} (a + \zeta^k b) = a^l + (-1)^{l-1} b^l. \]
(2.9) Using Eq. (2.9) with $a = aI$ and $b = b\lambda t^{-1}$, it is possible to rewrite Eq. (2.8)
\[ (aI + b\lambda t)^l = \left( \prod_{k=0}^{l-1} (a + \zeta^k \lambda t^{-1}) \right) t^l = a^l t^l + (-1)^{l-1} b^l (\lambda t^{-1})^l t^l = a^l t^l + b^l \lambda t^{-l}, \]
(2.10) where $(\lambda t^{-1})^l = \zeta^{-(1+2+\ldots+l-1)} \lambda^{l-1} t^{-l} = (-1)^{l-1} \lambda^{l-1} t^{-l}$. □

*Note:* For more rigour way to produce Eq. (2.9) from Eq. (2.8), it is possible to consider polynomials $r_{kl}(\lambda)$ defined by relation
\[ (x - 1)(x - \lambda)(x - \lambda^2) \cdots (x - \lambda^{l-1}) \equiv \sum_{k=0}^{l} r_{kl}(\lambda) x^k, \]
then for $l$-th root of unit, $\zeta$: $r_{kl}(\zeta) = \delta_{k0} - \delta_{kl}$ due to Eq. (2.9), but product in Eq. (2.9) is represented as series $\sum_{k=0}^{l} r_{kl}(\zeta) a^k (-b)^{l-k}$ and we have necessary result.

The similar idea may be used for proof of the Eq. (2.3) without additional condition about existence of $\Gamma^{-1}$. It could be enough to show
\[ \Gamma t = \lambda \Gamma t \implies (aI + b\lambda t)^l = \sum_{k=0}^{l} (-1)^{l-k} r_{kl}(\lambda) t^k \lambda^{l-k} \]
(2.11)
Such approach makes possible to prove Eq. (2.1) without additional condition about invertibility of $t_k$, but it is not discussed in present paper.

The “$\lambda$-deformed binomial coefficients” in Eq. (2.11) sometime are denoted as
\[ (-1)^{l-k} r_{kl}(\lambda) \equiv \begin{bmatrix} l \end{bmatrix}_\lambda \]
and may be explicitly written as
\[ \begin{bmatrix} l \end{bmatrix}_\lambda = \frac{[l]!}{[l-k]!} \lambda^{k-1} = \sum_{j=0}^{k-1} \lambda^j, \quad [k]_\lambda = \prod_{j=1}^{k} [j]_\lambda. \]

It should be mentioned also, that because the proof is based on Eq. (2.6) with $l$ different roots of unit, the same condition is satisfied for any substitution $\zeta \rightarrow \zeta^j$ if number $j$ is coprime for $l$, i.e. does not have common divisors with $l$.

If $l$ is prime, $j$ may be any natural number $0 < j < l$. Only for Clifford algebras, i.e. $l = 2$ the construction does not produce any new nontrivial solution.

Using formula Eq. (2.3) with different elements $r, l$ satisfying Eq. (2.2), it is simple to prove Eq. (2.1) for any natural number $n$. First, let us consider two elements $t_1, t_2$ ($n = 2$):
\[ t_1 t_2 = \zeta t_2 t_1. \quad (2.12) \]

The Eq. (2.1) for $t_1, t_2$ follows from Eq. (2.3) with $l = t_1$ and $r = t_2$:
\[ (a_1 t_1 + a_2 t_2)^l = a_1^l t_1^l + a_2^l t_2^l. \quad (2.13) \]

Now Eq. (2.1) is proved for $n = 2$. For other $n > 2$ it is possible to use induction: let Eq. (2.1) be true for some $n \geq 2$ and prove it for $n + 1$. It is enough to use Eq. (2.3) for
\[ l = t_{n+1}, \quad r = \sum_{k=1}^{n} a_k t_k, \quad (2.14) \]

These elements satisfy Eq. (2.2); $l$ is invertible (but $r$ maybe not, for example, if $t_k^l = 1$ and $\sum_{k=1}^{n+1} a_k^l = 0$, then $r^l = 0$ and $\sum r^{-1}$) and $t_{n+1} r = \zeta t_{n+1}$ because $t_{n+1}^l t_k = \zeta t_k t_{n+1}$ for all terms $t_k$ ($k < n + 1$) in $r$. So we have
\[ \left( \sum_{k=1}^{n+1} a_k t_k \right)^l = \left( a_{n+1} t_{n+1} + \sum_{k=1}^{n} a_k t_k \right)^l = a_{n+1}^l t_{n+1}^l + \sum_{k=1}^{n} a_k^l t_k^l = \]
\[ = a_{n+1}^l t_{n+1}^l + \sum_{k=1}^{n} a_k^l t_k^l = \sum_{k=1}^{n+1} a_k^l t_k^l \]
and Eq. (2.1) is proved for all $n > 0$ by induction. □

It also proves Eq. (1.3), because $t_k^l = 1$ for generators of $\mathfrak{T}(n,l)$, they are all invertible $t_k^{-1} = t_k^{l-1}$ and algebra $\mathfrak{T}(n,l)$ is associative by definition.
3 Representations of Weyl–Clifford Algebras

Representations of Weyl–Clifford algebras defined by Eqs. (1.4, 1.5) and satisfying Eq. (1.3) may be originated from two basic constructions: universal Clifford algebras $\mathfrak{Cl}(n, \mathbb{C}) \cong T(n, 2)$ and Weyl pair representation of $T(2, l)$. Construction of $T(2n, l)$ from Weyl representation of Heisenberg relation with $n$ coordinates and momenta used below has analogy with construction of Clifford algebra $\mathfrak{Cl}(2n, \mathbb{C})$ represented as tensor product of complex $2 \times 2$ matrices (Pauli matrices) [2]. Note: Description of representation $T(2n, l)$ here is close to [7].

3.1 Clifford Algebras

Let $\sigma_1, \sigma_2, \sigma_3 = i\sigma_1\sigma_2$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.1)$$

These matrices satisfy equations Eq. (1.2) for three generators of Clifford algebra

$$\sigma_k^2 = 1, \quad \sigma_k\sigma_j = -\sigma_j\sigma_k, \quad (3.2)$$

but if to consider universal Clifford algebras without extra relations between generators like $\sigma_3 = i\sigma_1\sigma_2$, then any two Pauli matrices, say $\sigma_1, \sigma_2$, may be used as generators of $\mathfrak{Cl}(2, \mathbb{C})$ represented as algebra $C(2 \times 2)$ of all complex $2 \times 2$ matrices.

Due to Eq. (1.2) may be maximum $2^n$ different products for $n$ generators $e_k$ and the Clifford algebras with maximal dimension, $\mathfrak{Cl}(n, \mathbb{C})$ are called universal, because of homomorphism to any other (associative) Clifford algebra with $n$ generators [2]

$$\mathfrak{Cl}(2n, \mathbb{C}) \cong \mathbb{C}(2^n \times 2^n), \quad \mathfrak{Cl}(2n + 1, \mathbb{C}) \cong \mathbb{C}(2^n \times 2^n) \oplus \mathbb{C}(2^n \times 2^n). \quad (3.3)$$

As generators of $\mathfrak{Cl}(2n, \mathbb{C})$ may be used $2n$ elements:

$$e_{2k-1} = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1, \quad (3.4)$$

$$e_{2k} = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1, \quad (3.5)$$

where $k = 1, \ldots, n$.

Representation of $\mathfrak{Cl}(2n + 1, \mathbb{C})$ also may be based on the same construction, it is enough to consider it as subalgebra of $\mathbb{C}(2^{n+1} \times 2^{n+1}) \cong \mathbb{C}(2^n \times 2^n) \otimes \mathbb{C}(2 \times 2)$ with last generator defined as

$$e_{2n+1} = \sigma_3 \otimes \cdots \otimes \sigma_3. \quad (3.6)$$

Because $\sigma_3$ together with $1$ produce algebra $D(2, \mathbb{C})$ of all diagonal $2 \times 2$ complex matrices, $\mathfrak{Cl}(2n + 1, \mathbb{C}) \cong \mathfrak{Cl}(2n, \mathbb{C}) \otimes D(2, \mathbb{C}) \cong \mathfrak{Cl}(2n, \mathbb{C}) \oplus \mathfrak{Cl}(2n, \mathbb{C})$. 

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It should be mentioned also, that the algebra, of course, may be defined as subalgebra of $\mathfrak{gl}(2n+2, \mathbb{C})$ without last generator $e_{2n+2}$ described by Eq. (3.3) for $k = n+1$. The construction Eq. (3.6) of $e_{2n+1}$ (with $\sigma_3$ in last term) instead of Eq. (3.4) (for generator $e_{2n+1}$ of $\mathfrak{gl}(2n+2, \mathbb{C})$ with $\sigma_3$ in last term) is convenient only because $\sigma_3$ is diagonal for this particular representation.

### 3.2 Weyl pair

Weyl relations [3] are similar with Eq. (1.4) for $n = 2$

$$UV = \zeta VU, \quad U^\dagger = U^{-1}, \quad V^\dagger = V^{-1}. \quad (3.7)$$

If $U$ and $V$ are linear operator on $l$-dimensional space, then $\zeta^l = 1$ follows from Eq. (3.7), because det($UV$) = det($\zeta VU$) = $\zeta^l$ det($UV$) and det($UV$) $\neq 0$. Elements $U^\dagger$ and $V^\dagger$ commute with all other elements of group generated by $U$, $V$ and for irreducible representation must be proportional to unit due to Schur lemma [3]. It is possible to choose $U^\dagger = V^\dagger = 1$ using unessential complex multiplier and so we have precisely generators of $\mathfrak{sl}(2, l)$.

A natural example of matrix representation is Weyl pair [3], i.e. two $l \times l$ unitary matrices:

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & 0 & \zeta^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta^{l-1} \end{pmatrix}. \quad (3.8)$$

It is clear, that if det($UV$) $= 0$, the inference used above to prove $\zeta^l = 1$ does not work, and it is really possible to suggest solution for arbitrary $\lambda \in \mathbb{C}$. Let us consider matrices

$$S^{(a)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^{l-1} \end{pmatrix}. \quad (3.9)$$

In Eq. (3.7) was used $U = S^{(1)}$ and $V = V^\zeta$ only with $\zeta^l = 1$, but for $S \equiv S^{(0)}$ and $V^\lambda$ it is possible to write for any $\lambda \in \mathbb{C}$

$$SV^\lambda = \lambda V^\lambda S, \quad \det(S) = 0, \quad |\lambda| \neq 1 \Rightarrow (V^\lambda)^\dagger \neq (V^\lambda)^{-1}. \quad (3.10)$$

The Eq. (3.10) has nontrivial $l \times l$ matrix representation for any $l > 1$.

Let us consider instead of matrices $U, V$ Eq. (3.8) two other matrices $U'$, $V'$ defined as

$$U' = M^{-1}UM, \quad V' = M^{-1}VM. \quad (3.11)$$
where $M$ is arbitrary unitary matrix. It is clear, Weyl relations Eq. (3.7) also
ture for the matrices $U'$, $V'$.

*Note:* Formally in definition of $\mathfrak{F}(2, l)$ below could be possible to use
arbitrary nonsingular matrices $U'$, $V'$, $\det(U'V') \neq 0$, but such matrices again
might be expressed using $U$, $V$ via Eq. (3.11) with nonsingular matrix $M$.

It is also true, that any matrices satisfying Weyl relations Eq. (3.7) may be
expressed using Weyl pair Eq. (3.3) via Eq. (3.14) for some matrix $M$ up to
unessential complex multiplier, i.e. all Weyl pairs are (unitary) equivalent.

An outline of proof follows. Let us consider matrices $U'$, $V'$ satisfying
Eq. (3.7) and let $e$ is eigenvector of $V'$ with eigenvalue $\mu$, then

$$V'e = \mu e \Rightarrow \zeta V'U' e = U'V'e = U'\mu e \Rightarrow V'(U'e) = (\mu/\zeta)(U'e). \tag{3.12}$$

So $(U'e)$ is other eigenvector of $V'$ with eigenvalue $(\mu/\zeta)$ and sequential application
of $U'$ Eq. (3.12) generates all $l$ different eigenvectors of $V'$ with eigenvalues
$\mu \zeta^k$, $k = 0, \ldots, l - 1$. It is possible to choose $e^{(k)} = U'^k e$ as basis of vector
space. In the basis $U'$ and $V'$ are represented as $U$, $V$ Eq. (3.8), because $U'$
performs left cyclic shift of elements of basis $U': e^{(k)} \mapsto e^{(k-l \mod l)}$ and $V'$ is
diagonal by definition $V'e^{(k)} = \mu \zeta^k e^{(k)}$. The Eq. (3.11) is simply formula of
transformation to the new basis $e^{(k)}$. If matrices $U'$, $V'$ are unitary, then matrix
of transformation $M$ also should be unitary and $|\mu| = 1$. □

As an interesting example, let us find transformation $F$ for pair

$$U' = V^{-1} = F^{-1}UF, \quad V' = U = F^{-1}VF. \tag{3.13}$$

Eigenvectors of $V'$, i.e. $U$ may be simply found. Let us start with $f = (1, 1, \ldots, 1)/\sqrt{l}$ and write

$$f^{(k)} = U'^k f \Rightarrow (f^{(k)})_j = \zeta^{-(j-1)(k-1)}/\sqrt{l} \Rightarrow F_{kj} = \zeta^{-(j-1)(k-1)}/\sqrt{l}. \tag{3.14}$$

The unitary matrix $F$ defined by Eq. (3.14) is called discrete (or quantum)
Fourier transform.

It should be mentioned also, that if $l$ is not prime, then Eq. (3.7) together
with discussed representation for $\zeta = \exp(2\pi i/l)$, has nonequivalent reducible
representations for any factor $m$, $l = mk$ and $\zeta' = \zeta^m = \exp(2\pi i/k)$ with
matrices

$$U' = U^m, \quad V' = V, \quad (\mathfrak{F}; U' = M^{-1}UM). \tag{3.15}$$

The representation is reducible, because it is equivalent with direct sum of
$m$ representations Eq. (3.8) with dimensions $k$ ($l = mk$), i.e. with $\bigoplus_{j=1}^m U$, $\bigoplus_{j=1}^m \zeta^{j-1} V$.

Let us consider case $l = 2$. Here is $U = \sigma_1$ and $V = \sigma_3$. There are three
Pauli matrices and for general case $l > 2$ it is also possible to define together
with $U$ and $V$ third matrix and use it for definition of $\mathfrak{F}(n, l)$ similar with
constructions of generators of Clifford algebras described above.
But here is necessary to mention some difference between case $l = 2$ and the general case $l > 2$. Let us consider Eq. (1.4) for three generators:

$$t_1 t_2 = \zeta t_2 t_1, \quad t_2 t_3 = \zeta t_3 t_2, \quad t_1 t_3 = \zeta t_3 t_1. \quad (3.16)$$

Here is clear, that $t_1, t_2, t_3$ in Eq. (3.16) are not equivalent (ordered triple), unlike the case with changed order in last equation (cyclic triple)

$$t_1 t_2 = \zeta t_2 t_1, \quad t_2 t_3 = \zeta t_3 t_2, \quad t_3 t_1 = \zeta t_1 t_3,$$

but for $l = 2$ both set of equations are the same because $\zeta = \zeta^{-1} = -1$ for $l = 2$ and all three Pauli matrices have equal status. Another difference of case $l > 2$ is because due to inequality $t_k^{-1} \neq t_k$, there are few different ways to construct an analogue of “Pauli triple” Eq. (3.1). Many different variants of triples with $U, V, U^{-1} = U^\dagger, V^{-1} = V^\dagger$ and products are represented on a diagram Fig. 1.

![Diagram](image.png)

Figure 1: The diagram. Arrow from $A$ to $B$ means $AB = \zeta BA.$

On the diagram it is possible to see different ordered triples with property Eq. (3.16) together with cyclic ones. All ordered triples are appropriate for construction of $\mathfrak{F}(n, l)$ used below. In the paper are discussed only few different combinations. For example it is useful sometime to use initial Weyl pair $U, V$ Eq. (3.8) as two first generators

$$\tau^w_1 = U, \quad \tau^w_2 = V, \quad \tau^w_3 = \nu U^\dagger V. \quad (3.17)$$

where complex coefficient

$$\nu = \zeta^{(l+1)/2} = e^{\pi i (l+1)/l} \quad (3.18)$$
is used to satisfy condition \((\tau_3^w)^l = I\).

Other choice

\[
\tau_1 = U, \quad \tau_2 = \bar{v}UV, \quad \tau_3 = V
\]  

(3.19)
is convenient for construction of representation in Sec. 3.3 due to direct analogy with Pauli matrices used for construction of Clifford algebras above in Sec. 3.1, say for \(l = 2\): \(\tau_i = \sigma_i\) and also \(\tau_3\) is always diagonal like \(\sigma_3\).

Even all appropriate triples on the diagram Fig. 1 are only small part of possible variants, because similarly with Eq. (3.11), it is possible to write most general choice

\[
\tau'_1 = M^{-1}UM, \quad \tau'_2 = M^{-1}VM, \quad \tau'_3 = \nu M^{-1}U^\dagger V M = \nu \tau_1^{-1} \tau_2',
\]  

(3.20)

where \(M\) is arbitrary unitary (or nonsingular) matrix, if we are looking for unitary (nonsingular) representations. All triples on diagram Fig. 1 may be expressed using Eq. (3.20).

The triples produce some example of \(\mathfrak{T}(3, l)\), but here, similarly with universal Clifford algebras, it is useful to consider case with maximal dimension for given set of generators. Due to Eq. (3.20) it must be no more than \(l^n\) linearly independent products of \(n\) generators \(\mathfrak{T}(n, l)\) and construction provided below in Sec. 3.3 has this maximal dimension \(l^n\) as complex algebra. Generators \(t_1 = U\) and \(t_2 = V\) may be appropriate for \(\mathfrak{T}(2, l)\) if to prove that \(l^2\) different products \(U^k V^j\) \(k, j = 0, \ldots, l - 1\) are basis for algebra of \(l \times l\) complex matrices and so \(\mathfrak{T}(l, 2) \cong \mathbb{C}(l \times l)\).

Let us consider usual basis \(E^{ab}\) of \(l \times l\) complex matrices: \((E^{ab})_{jk} = \delta_{aj} \delta_{bk}, \quad a, b, j, k = 1, \ldots, l\). All matrices of this basis are possible to express as linear combinations of \(U^k V^j, U^k, V^j\), because \(E_{11} = \sum_{k=1}^{l} V^k / l\) and \(E^{ab} = U^{l-a+1} E^{11} U^{-1} V^{b-1}\). So \(U^k V^j\) \(k, j = 0, \ldots, l - 1\) are basis of \(\mathbb{C}(l \times l)\).

Both \(U\) and \(V\) may be expressed with any pair of elements between triple \(\tau_1, \tau_2, \tau_3\) (or \(\tau'_1, \tau'_2, \tau'_3\)) and so any such pair may be also used as generators of \(\mathfrak{T}(2, l)\). Certainly, it is also true in general case with \(\tau'_k\) Eq. (3.19).

### 3.3 Representations of \(\mathfrak{T}(n, l)\)

Similarly with case \(l = 2\) with Clifford algebras discussed below, generators of \(\mathfrak{T}(2n, l)\) may be represented as

\[
t_{2(k-1)} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_1 \otimes I \otimes \cdots \otimes I, \quad k = 1, \ldots, n
\]

(3.21)

\[
t_{2k} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_2 \otimes I \otimes \cdots \otimes I, \quad k = 1, \ldots, n
\]

(3.22)

where \(k = 1, \ldots, n\).

To check that these \(2n\) generators \(t_j\) Eqs. (3.21, 3.22) satisfy Eqs. (1.4, 1.5) it is enough to consider three different cases:
1. $t_{2k-1}t_{2k} = \zeta t_{2k}t_{2k-1}, k \geq 1$

$$t_{2k-1} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_1 \otimes I \otimes \cdots \otimes I_{n-k}$$

$$t_{2k} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_2 \otimes I \otimes \cdots \otimes I_{n-k}$$

2. $t_{2k-1}t_{2k+j} = \zeta t_{2k+j}t_{2k-1}, k \geq 1, j \geq 1$

$$t_{2k-1} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_1 \otimes I \otimes \cdots \otimes I_{n-k}$$

$$t_{2k+j} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_2 \otimes I \otimes \cdots \otimes I_{n-k}$$

3. $t_{2k}t_{2k+j} = \zeta t_{2k+j}t_{2k}, k \geq 1, j \geq 1$

$$t_{2k} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_2 \otimes I \otimes \cdots \otimes I_{n-k}$$

$$t_{2k+j} = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_3 \otimes I \otimes \cdots \otimes I_{n-k}$$

where $\tau_i$ are arbitrary ($i = 1, 2, 3$). In each case only one pair of terms marked by ‘$\downarrow$’ in both tensor products, is not commutative and any such pair has proper order $(12, 13, 23)$, cf. Eq. (3.16). Other property, $t_k^k = I_1$ is also true, because $\tau_1^l = \tau_2^l = \tau_3^l = I_1$. So Eq. (3.4) is proved for all $t_j$.

It is also possible to check that different products of $2n$ elements $t_j$, Eqs. (3.21)–(3.22) generate full matrix algebra $\mathbb{C}(l^n \times l^n) \cong \mathbb{C}(l \times l)^{\otimes n}$. Let us denote

$$\tau_{i:k} \equiv I_{\otimes \cdots \otimes} I \otimes \tau_i \otimes I \otimes \cdots \otimes I_{n-k}, \quad (3.23)$$

It is possible to express Eq. (3.23) using $t_j$

$$\tau_{3:k}^j = \bar{\nu} t_{2k-1}^{j-1} t_{2k}, \quad \tau_{1:k} = t_{2k-1}^{j-1} \tau_{3;1}^j \cdots \tau_{3;k-1}^j, \quad \tau_{2;k} = t_{2k}^{j-1} \tau_{3;1}^j \cdots \tau_{3;k-1}^j, \quad (3.24)$$

where $\nu$ is complex coefficient defined in Eq. (3.18). It was shown earlier that $\tau_1^j, j, k = 0, \ldots, l - 1$ are basis of $\mathbb{C}(l \times l)$ and so $2n$ elements $\tau_{1:k}, \tau_{2;k}$ generate $\mathbb{C}(l^n \times l^n)$.

So $\mathfrak{T}(2n, l) \cong \mathbb{C}(l^n \times l^n)$. Let us prove, that $\mathfrak{T}(2n + 1, l) = \bigoplus^l \mathbb{C}(l^n \times l^n)$. It also has analogue with Clifford algebra. $\mathfrak{T}(2n + 1, l)$ is considered as subalgebra of $\mathbb{C}(l^{n+1} \times l^{n+1}) \cong \mathbb{C}(l^n \times l^n) \otimes \mathbb{C}(l \times l)$ with last generator

$$t_{2n+1} = \tau_3 \otimes \cdots \otimes \tau_3, \quad (3.25)$$

Here again $\tau_3 = V$ generates algebra $D(l, \mathbb{C})$ of all diagonal $l \times l$ complex matrices and so $\mathfrak{T}(2n + 1, l) \cong \mathbb{C}(l^n \times l^n) \otimes D(l, \mathbb{C}) \cong \bigoplus^l \mathbb{C}(l^n \times l^n) \cong \bigoplus^l \mathfrak{T}(2n, l)$. It is also possible to consider $\mathfrak{T}(2n + 1, l)$ as subalgebra of $\mathfrak{T}(2n + 2, l)$ — similarly with Clifford algebra in such a case instead of Eq. (3.25) is used generator of $\mathfrak{T}(2n + 2, l)$ next to the last, i.e. $t_{2n+1}$ described by Eq. (3.21) for $k = n + 1$.
with $\tau_1$ in last term instead of $\tau_3$. Algebra generated by such elements is not algebra of all diagonal matrices $D(l, \mathbb{C})$, but isomorphic with it.

It is proved that all Weyl–Clifford algebras $\mathfrak{T}(n, l)$ have maximal dimensions $l^n$ and may be represented as

$$\mathfrak{T}(2n, l) \cong \mathbb{C}(l^n \times l^n), \quad \mathfrak{T}(2n + 1, l) \cong \mathfrak{T}(2n, l) \oplus \cdots \oplus \mathfrak{T}(2n, l).$$

(3.26)

Of course, in such matrix representation instead of $\tau_i$ in Eq. (3.19) in generators Eqs. (3.21, 3.22) etc. could be used any other triple $\tau'_i$ in Eq. (3.20) (see for example Eqs. (4.29, 4.30) below), but it is more general to consider

$$t'_j = M^{-1} t_j M,$$

(3.27)

where for $\mathfrak{T}(2n, l)$ and $\mathfrak{T}(2n - 1, l)$, represented as subalgebra of $\mathfrak{T}(2n, l)$, $M \in GL(2^n, \mathbb{C})$ (or $M \in U(2^n)$ for unitary representations).

4 Weyl relations and $\mathfrak{T}(n, \infty)$

In this section is discussed, how $\mathfrak{T}(n, l)$ is related with Weyl representation of Heisenberg commutation relations [3] with non-canonical commutator form.

4.1 Weyl–Heisenberg relations

It was shown in Sec. 3.2, that for finite-dimensional case Weyl pair Eq. (3.7) is example of $\mathfrak{T}(2, l)$ generators Eq. (1.4) and might be represented by matrices Eq. (3.8), but Weyl constructions for infinite-dimensional space $l = \infty$ and arbitrary finite $n$ let us also introduce $\mathfrak{T}(n, \infty)$.

For $l = \infty$ Weyl relations based on $n$-parametric group:

$$W(t) \equiv W(t_1, t_2, \ldots, t_n) = e^{i(t_1 c_1 + t_2 c_2 + \cdots + t_n c_n)},$$

(4.1)

where $c_k$ some operators on infinite-dimensional Hilbert space with property:

$$[c_j, c_k] \equiv c_j c_k - c_k c_j = i h_{kj} \mathbb{1},$$

(4.2)

where $h_{kj}$ is antisymmetric commutator form $h(\cdot, \cdot)$. For two real vectors of parameters $t, t'$, and elements of group $W$ expressed by Eq. (4.1) with Eq. (4.2), it is possible to write general form of Weyl relations [3]

$$W(t)W(t') = e^{ih(t,t')}W(t')W(t).$$

(4.3)

For even $n = 2m$ commutator form $h$ may be represented in canonical way

$$h_c = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
-1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
0 & 0 & -1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}.$$
In such a case for $2m$ operators $c_k$ is used notation
\[ p_k = c_{2k-1}, \quad q_k = c_{2k}, \] (4.5)
called canonical coordinates and momenta, commutation relations Eq. (4.2) are rewriting for canonical form $h_c$ in such notations as
\[ [q_k, p_j] = i\delta_{kj} I, \quad [q_k, q_j] = [p_k, p_j] = 0, \] (4.6)
and coincide with Heisenberg commutation relations.

In such a case instead of one group $W$ Eq. (4.1) can be used two groups
\[ U(a) = e^{i(a_1 p_1 + a_2 p_2 + \cdots + a_m p_m)}, \]
\[ V(b) = e^{i(b_1 q_1 + b_2 q_2 + \cdots + b_m q_m)}, \] (4.7)
with properties
\[ U(a)U(a') = U(a + a'), \quad V(b)V(b') = U(b + b'), \] (4.9)
(i.e. $U$ and $V$ are $m$-parametric abelian groups of transformations) and rewrite Eq. (4.10) as
\[ U_k(a) = e^{ia p_k}, \quad V_k(b) = e^{ib q_k} \] (4.11)
and rewrite Eq. (4.10) as
\[ U_k(a)V_k(b) = e^{ia b} V_k(b)U_k(a), \quad U_k(a)V_j(b) = V_j(b)U_k(a), \quad k \neq j. \] (4.12)

Formally Eq. (4.12) follows from Eq. (4.6) if to use Campbell–Hausdorff series \[ e^{a + b} = e^a e^b e^{-[a, b]/2}, \] (4.13)
In simplest case of $n = 2$ there is pair of operators
\[ U(a)V(b) = e^{ia b} V(b)U(a), \quad U(a) = e^{iap}, \quad V(b) = e^{ibq}, \quad a, b \in \mathbb{R}. \] (4.14)

The continuous case can be considered \[ \text{as limit } l \to \infty \text{ of operators Eq. (3.8), because due to Eq. (3.7) it was possible to write in “discrete” case an analogue of Eq. (4.14)} \]
\[ U^a V^b = e^{2\pi i a b / l} V^b U^a, \quad a, b \in \mathbb{Z}. \] (4.15)

\[ \text{Such formal calculations with the series are not necessary correct for unbounded operators and in more rigor consideration Eq. (4.12) does not follow from Eq. (4.4) for arbitrary } p, q. \]
In the limit \( l \to \infty \) instead of vector space \( \mathbb{C}^l \) we have space of complex-valued functions \( \psi(q), \ q \in \mathbb{R} \) and operators

\[
U(a): \psi(q) \mapsto \psi(q + a), \quad V(b): \psi(q) \mapsto e^{ibq} \psi(q)
\]

(4.16)

with infinitesimal generators in Eq. (4.14) are represented as

\[
\mathbf{p}: \psi(q) \mapsto \frac{1}{i} \frac{d\psi}{dq}, \quad \mathbf{q}: \psi(q) \mapsto q \psi(q).
\]

(4.17)

It is Schrödinger representation of Heisenberg commutation relations.

Case with \( n = 2m > 2 \) is similar, for finite \( l \) it is possible to consider pairs of operators

\[
U_k = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes U \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]

(4.18)

\[
V_k = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes V \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1},
\]

(4.19)

where \( k = 1, \ldots, m \) and \( U, V \) are \( l \times l \) unitary matrices Eq. (3.8). Matrices \( U_k \) and \( V_k \) are generators of \( \mathbb{C}(l^m \times l^m) \) similar with \( t_k \) in Eqs. (4.12, 4.13).

For continuous case \( l \to \infty \) instead of vector space \( \mathbb{C}^l \) we have space of complex-valued functions \( \psi(q) \equiv \psi(q_1, \ldots, q_m), \ q \in \mathbb{R}^m \) and operators

\[
U_k(a): \psi(\ldots, q_k, \ldots) \mapsto \psi(\ldots, q_k + a, \ldots), \quad V_k(b): \psi(q) \mapsto e^{ibq_k} \psi(q)
\]

(4.20)

with infinitesimal generators in Eq. (4.11) are written in the Schrödinger representation as

\[
\mathbf{p}_k: \psi(q) \mapsto \frac{1}{i} \frac{\partial \psi}{\partial q_k}, \quad \mathbf{q}_k: \psi(q) \mapsto q_k \psi(q).
\]

(4.21)

4.2 Weyl – Clifford relations

Let us return to general Weyl construction of group \( W(t) \) Eq. (4.1) for operators \( c_k \) with arbitrary commutator form \( h_{kj} \) Eq. (4.2). It is possible rewrite general Weyl relation Eq. (4.3) with arbitrary form as

\[
W_k(t_k)W_j(t_j) = e^{ih_{kj}t_kt_j}W_j(t_j)W_k(t_k), \quad W_k(t) \equiv e^{itc_k}.
\]

(4.22)

It is similar with Eqs. (1.4, 1.5) for equal \( t_k = \hat{t} \), where

\[
\hat{t} = \sqrt{2\pi/l}.
\]

(4.23)
and special commutator form $h = h^+_-$, where

$$
h^+_- = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & \ldots \\
-1 & 0 & 1 & 1 & 1 & \ldots \\
-1 & -1 & 0 & 1 & 1 & \ldots \\
-1 & -1 & -1 & 0 & 1 & \ldots \\
-1 & -1 & -1 & -1 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \tag{4.24}
$$

because for such a choice we can rewrite Eq. (4.22) as

$$W_k(\hat{t})W_j(\hat{t}) = \zeta W_j(\hat{t})W_k(\hat{t}), \quad k < j. \tag{4.25}$$

Let us consider linear transformation to new set of generators in Weyl group $W$ Eq. (4.1) with some matrix $G$

$$\epsilon'_k = \sum_{j=1}^{n} G_{kj} \epsilon_j. \tag{4.26}$$

For such transformation commutator form Eq. (4.2) may be rewritten using matrix notation:

$$h' = GhG^T, \tag{4.27}$$

where $G^T$ is transposed matrix.

Matrices $L$ of appropriate transformations Eq. (4.27); $h^+_+ = Lh^+_-$ from $h^+_+$ Eq. (4.4) to $h^+_+$ Eq. (4.24), can be found using comparison of generators $U_k, V_k$ Eqs. (4.18, 4.19) and $\xi_{2k-1}, \xi_{2k}$ Eqs. (3.21, 3.22). For example

$$L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}, \tag{4.28}
$$

but here is convenient to use triple Eq. (3.17) instead of Eq. (3.19) together with other representation of $\mathbb{F}(n,l)$

$$\begin{align*}
\xi_{2k-1}^w &= \alpha_k (U^+V) \otimes \cdots \otimes (U^+V) \otimes U \otimes I \otimes \cdots \otimes I, \tag{4.29} \\
\xi_{2k}^w &= \alpha_k (U^+V) \otimes \cdots \otimes (U^+V) \otimes V \otimes I \otimes \cdots \otimes I. \tag{4.30}
\end{align*}$$

Cf. with transformation of matrix of quadratic form, i.e. $q' = G^T q G$. 

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where \( k \geq 1 \) and \( \alpha_k = \zeta^{- (k-1)(l-1)/2} \). Using Eqs. (4.18, 4.19) and Eq. (3.17), it is possible to write

\[
\begin{align*}
\tilde{t}_{2k-1}^w &= \alpha_k U_k \prod_{j=1}^{k-1} (U_j^* V_j), \quad \tilde{t}_{2k}^w = \alpha_k V_k \prod_{j=1}^{k-1} (U_j^* V_j).
\end{align*}
\]

(4.31)

In exponential form it can be written

\[
\begin{align*}
\tilde{t}_{2k-1}^w &= \alpha_k e^{i \left( p_k + \sum_{j=1}^{k-1} (-p_j + q_j) \right)}, \quad \tilde{t}_{2k}^w = \alpha_k e^{i \left( q_k + \sum_{j=1}^{k-1} (-p_j + q_j) \right)},
\end{align*}
\]

(4.32)

i.e. can be described by transformation Eq. (4.26) with matrix

\[
L' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 1 & 1 & 0 \\
-1 & 1 & -1 & 1 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]

(4.33)

The matrix \( L \) above Eq. (4.28) was calculated using similar expansion, but with \( \tau_i \), Eq. (3.19) instead of \( \tau_i^w \), Eq. (5.17) used for \( L' \) Eq. (4.33).

It is possible also to check directly, that matrices Eq. (4.28) and Eq. (4.33) describe transformations from \( h^+ \) to \( h_c \), i.e. \( h^+ = L h_c L^T = L h_c L'^T \). Such test was useful, because formally some part of consideration above was based on arithmetic of ring \( \mathbb{Z}_l \), not \( \mathbb{C} \). For example, using Clifford algebras and Jordan-Wigner construction of generators Eqs. (3.4, 3.5) with Pauli matrices, it could be possible also find a wrong form of Eq. (4.33) with 1 instead of \( -1 \) and same error can appear for any \( l \) if to use \( U^{l-1}, V^{l-1} \) instead of \( U^l, V^l \) in some expressions above.

It was shown above, that algebra \( \mathcal{C}(n,l) \) for \( l \to \infty \) may be described by usual Weyl construction with special commutator form, but it is clear, that there is a problem with condition \( t_k^w = 1 \). The condition is more appropriate in finite-dimensional case (see explanation after Eq. (3.7) and (3)). For infinite-dimensional case it is useful to consider weakened definition of Weyl–Clifford algebra \( \mathcal{C}(n,l) \) without this condition. An example could be based on group \( W \) Eq. (4.28) already discussed earlier as a stimulus to use commutation form \( h^+ \) in construction of Weyl–Clifford algebras, but here is simpler again to “split” \( W \) into two groups \( U, V \).

Let us use \( U_k(a) \) and \( V_k(b) \) with fixed \( a, b \) instead of \( U_k, V_k \) in definitions of generators \( t_k^w \) in Eq. (4.31).

\[
\begin{align*}
\tilde{t}_{2k-1} = U_k(a) \prod_{j=1}^{k-1} (U_j^*(a)V_j(b)), \quad \tilde{t}_{2k} = V_k(b) \prod_{j=1}^{k-1} (U_j^*(a)V_j(b)),
\end{align*}
\]

(4.34)

\[
\tilde{t}_j \tilde{t}_k = \zeta \tilde{t}_k \tilde{t}_j \quad (j < k), \quad \zeta = e^{iab}.
\]

(4.35)
Instead of Eq. (1.3) it is possible to write Eq. (2.1) for given \( l \) using Eq. (4.35) with fixed \( a \neq 0 \) and \( b = 2\pi/(la) \): 
\[
\left( \sum_{k=1}^{n} c_k \tilde{t}_k \right)^l = \sum_{k=1}^{n} c_k^l \tilde{t}_k^l.
\]
The proof of Eq. (2.1) may be found in Sec. 2 (\( \tilde{t}_k \) are invertible). The only difference with Eq. (1.3) here are terms \( \tilde{t}_k \neq 1 \) I. These elements are central in algebra \( \tilde{\mathfrak{S}}(n,l) \) generated by all possible products of \( \tilde{t}_k \), \( \tilde{t}_k \tilde{t}_j = \tilde{t}_j \tilde{t}_k \) (because \( \zeta = 1 \)) \( \forall k,j \). In such a case we have the algebra \( \tilde{\mathfrak{S}}(n,l) \) represented as tensor product of \( \mathfrak{S}(n,l) \) on some abelian subalgebra \( \mathfrak{C}(n,l) \) (dim \( \mathfrak{C}(n,l) \leq \infty \)) generated by all possible products of \( n \) elements \( \tilde{t}_k \).

It is possible to find more general expression instead of Eq. (4.34) if to use \( U_k(a_k) \) and \( V_k(b_k) \) with pairs \( a_k \neq 0, b_k = \lambda/a_k \) for some fixed \( \lambda \)
\[
\tilde{t}_{2k-1} = U_k(a_k)\Pi_k, \quad \tilde{t}_{2k} = V_k(\lambda/a_k)\Pi_k, \quad \Pi_k \equiv \prod_{j=1}^{k-1} \left( U_j(a_j)V_j(\lambda/a_j) \right), \quad (4.36)
\]
\[
\tilde{t}_j \tilde{t}_k = \zeta \tilde{t}_k \tilde{t}_j \quad (j < k), \quad \zeta = e^{i\lambda} \quad (4.37)
\]
and it is also satisfies Eq. (2.1) with given power \( l \) for \( \lambda = 2\pi/l \) (and also for \( \lambda = 2\pi m/l \), i.e. for any rational multiple of 2\( \pi \) it is possible to write Eq. (2.1) for some \( l \)).

It maybe looks strange, why it was only one appropriate value of parameter \( \tilde{\lambda} \) Eq. (4.23) in initial expression Eq. (4.25) with group \( \mathfrak{W} \) and \( n \)-parameter family Eq. (4.36) for construction Eq. (4.34) with two groups \( U, V \). Really two constructions are equal and have even “bigger” set of solutions, than it is represented in Eq. (4.36).

Let us consider this complete set. Linear transformation Eq. (4.26) is called symplectic, if it saves canonical form \( h_c \)
\[
h_c = Sh_cS^T \quad (4.38)
\]
Family used above in Eq. (4.36) was based on particular symplectic transformation diagonal in canonical basis
\[
D = \begin{pmatrix}
a_1 & 0 & 0 & 0 & \cdots \\
0 & 1/a_1 & 0 & 0 & \cdots \\
0 & 0 & a_2 & 0 & \cdots \\
0 & 0 & 0 & 1/a_2 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad (4.39)
\]
but it is not all possible symplectic transformations, it is only most simple case. For nonstandard canonical form \( h_c^+ \) also exists group of linear transformations \( N \) with property
\[
h_c^+ = Nh_c^+N^T \quad (4.40)
\]

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It was found earlier $h_+ = L h_c L^T$ ($h_c = L^{-1} h_+ L^{-1T}$) for matrix $L \text{ Eq. (4.28)}$
and it is possible also to associate $N$ with any symplectic transformation $S$

$$N S = L S L^{-1}.$$ \hspace{1cm} (4.41)

It is simply to check Eq. (4.40) for $N_S$

$$N_S h_+ N_S^T = L S L^{-1} h_+ L^{-1T} S^T L^T = L S h_c S^T L^T = L h_c L^T = h_+$$

and so group $N$ Eq. (4.40) is isomorphic with symplectic group.

Initial expression Eq. (4.25) looks less general, than Eq. (4.34) rather due to technical problems.

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