A GEOMETRIC INEQUALITY ON HYPERSURFACE IN HYPERBOLIC SPACE

HAIZHONG LI, YONG WEI, AND CHANGWEI XIONG

Abstract. In this paper, we use the inverse curvature flow to prove a sharp geometric inequality on star-shaped and two-convex hypersurface in hyperbolic space.

1. Introduction

The classical Alexandrov-Fenchel inequalities for closed convex hypersurface $\Sigma \subset \mathbb{R}^n$ state that

$$\int_{\Sigma} \sigma_m(\kappa)d\mu \geq C_{n,m}(\int_{\Sigma} \sigma_{m-1}(\kappa)d\mu)^{\frac{n-m-1}{n-m}}, \quad 1 \leq m \leq n-1 \quad (1)$$

where $\sigma_m(\kappa)$ is the $m$-th elementary symmetric polynomial of the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of $\Sigma$ and $C_{n,m} = \frac{\sigma_m(1, \cdots, 1)}{\sigma_{m-1}(1, \cdots, 1)}$ is a constant. When $m = 0$, (1) is interpreted as the classical isoperimetric inequality

$$|\Sigma|^{\frac{n-1}{n}} \geq \bar{C}_n Vol(\Omega)^{\frac{1}{n}}, \quad (2)$$

which holds on all bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\Sigma = \partial \Omega$. Here $|\Sigma|$ is the area $\Sigma$ and $\bar{C}_n$ is a constant depending only on dimension $n$.

Inequality (1) was generalized to star-shaped and $m$-convex hypersurface $\Sigma \subset \mathbb{R}^n$ by Guan and Li [8] using the inverse curvature flow recently, where $m$-convex means that the principal curvature of $\Sigma$ lies in the Garding’s cone

$$\Gamma_m = \{ \kappa \in \mathbb{R}^{n-1} | \sigma_i(\kappa) > 0, i = 1, \cdots, m \}.$$

Recently, Huisken [11] showed that in the case $m = 1$, the assumption star-shaped can be replaced by outward-minimizing.

In this paper, we consider the hyperbolic space $\mathbb{H}^n = \mathbb{R}^+ \times S^{n-1}$ endowed with the metric

$$\tilde{g} = dr^2 + \sinh^2 r g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard round metric on the unit sphere $S^{n-1}$. It’s a natural question to establish some analogue inequalities of (1) for closed hypersurface in $\mathbb{H}^n$. In the case of $m = 1$, $\sigma_1 = \sigma_1(\kappa)$ is just the mean curvature $H$ of $\Sigma$. Gallego and Solanes [6] have obtained a generalization of (1) to convex hypersurface in hyperbolic space using integral geometric methods,

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however, their result does not seem to be sharp. Denoting $\lambda(r) = \sinh r$, then $\lambda'(r) = \cosh r$. Recently, Brendle, Hung and Wang [3] proved the following inequality for star-shaped and mean convex (i.e., $H > 0$) hypersurface $\Sigma \subset \mathbb{H}^n$:

$$
\int_{\Sigma} (\lambda'H - (n-1)\langle \nabla \lambda', \nu \rangle) d\mu \geq (n-1)\omega_{n-1}^{-1} |\Sigma|^\frac{n-2}{n-1}
$$

(3)

where $|\Sigma|$ is the area of $\Sigma$ and $\omega_{n-1}$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. de Lima and Girao [4] also proved the following related inequality independently.

$$
\int_{\Sigma} \lambda'H d\mu \geq (n-1)\omega_{n-1} (-\frac{|\Sigma|}{\omega_{n-1}})^{\frac{n}{n-1}} + (\frac{|\Sigma|}{\omega_{n-1}})^{\frac{n}{n-1}}),
$$

(4)

Both inequalities (3) and (4) are sharp in the sense that equality holds if and only if $\Sigma$ is a geodesic sphere centered at the origin. Here, we say a closed hypersurface $\Sigma \subset \mathbb{H}^n$ is star-shaped if the unit outward normal $\nu$ satisfies $\langle \nu, \partial_r \rangle > 0$ everywhere on $\Sigma$, which is also equivalent to that $\Sigma$ can be parametrized by a graph

$$
\Sigma = \{ (r(\theta), \theta) | \theta \in S^{n-1} \}
$$

for some smooth function $r$ on $S^{n-1}$. We note that inequalities (3) and (4) have some applications in general relativity, see [3, 4, 14].

In this paper, we consider the case $m = 2$. We prove the following sharp inequality for star-shaped and two-convex hypersurface $\Sigma \subset \mathbb{H}^n$, where two-convex means that the principal curvature lies in the Garding’s cone $\Gamma_2$ everywhere on $\Sigma$.

**Theorem 1.** If $\Sigma \subset \mathbb{H}^n$ is a star-shaped and two-convex hypersurface, then

$$
\int_{\Sigma} \sigma_2 d\mu \geq \frac{(n-1)(n-2)}{2} \left( \frac{2}{\omega_{n-1}} |\Sigma|^\frac{n-1}{n-2} + |\Sigma| \right),
$$

(5)

where $\omega_{n-1}$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and $|\Sigma|$ is the area of $\Sigma$. The equality holds if and only if $\Sigma$ is a geodesic sphere.

Note that there exists at least one elliptic point on a closed, connected hypersurface $\Sigma$ in hyperbolic space $\mathbb{H}^n$. Proposition 3.2 in [1] shows that if $\sigma_2$ is positive, then $\sigma_1$ is automatically positive. So our assumption two-convex can also be replaced by $\sigma_2 > 0$ on $\Sigma$.

The proof of Theorem 1 follows a similar argument as in [3, 4, 8]. We evolve $\Sigma$ by a special case of the inverse curvature flow in [7], and consider the following quantity defined by

$$
Q(t) = |\Sigma|^{-\frac{n-3}{n-1}} \left( \int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma| \right).
$$

We show that $Q(t)$ is monotonically decreasing under the flow. Then we use the convergence result of the flow proved by Gerhardt to estimate a lower
bound of the limit of $Q(t)$:
\[
\liminf_{t \to \infty} Q(t) \geq \frac{(n-1)(n-2)}{2} \frac{\omega_{n-1}^2}{\omega_{n-1}^2}.
\]
In order to estimate this lim inf, we also use a sharp version Sobolev inequality on $S^{n-1}$ due to Beckner [2] as in [3]. Finally Theorem 1 follows easily from the monotonicity and the lower bound of $\liminf_{t \to \infty} Q(t)$.

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2. Preliminaries

Let $\Sigma \subset \mathbb{H}^n$ be a closed hypersurface with unit outward normal $\nu$. The second fundamental form $h$ of $\Sigma$ is defined by
\[
h(X, Y) = \langle \nabla_X \nu, Y \rangle
\]
for any two tangent fields $X, Y$. The principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ are the eigenvalues of $h$ with respect to the induced metric $g$ on $\Sigma$. For $1 \leq m \leq n-1$, the $m$-th elementary symmetric polynomial of $\kappa$ is defined as
\[
\sigma_m(\kappa) = \sum_{i_1 < i_2 < \cdots < i_m} \kappa_{i_1} \cdots \kappa_{i_m},
\]
which can also be viewed as function of the second fundamental form $h^j_i = g^{jk}h_{ki}$. In the sequel, we will simply write $\sigma_m$ for $\sigma_m(\kappa)$. We first collect the following basic facts on $\sigma_m$ (see, e.g, [9, 12, 13]):

**Lemma 2.** Denote $(T_{m-1})^j_i = \frac{\partial \sigma_m}{\partial h^j_i}$ and $(h^2)^j_i = g^{ij}g^{pk}h_{ki}h_{ip}$. We have
\[
\sum_{i,j} (T_{m-1})^j_i h^j_i = m\sigma_m,
\]
(6)
\[
\sum_{i,j} (T_{m-1})^j_i \delta^j_i = (n-m)\sigma_{m-1}
\]
(7)
\[
\sum_{i,j} (T_{m-1})^j_i (h^2)^j_i = \sigma_1 \sigma_m - (m+1)\sigma_{m+1}
\]
(8)

Moreover, if $\kappa \in \Gamma^+_m$, we have the following Newton-MacLaurin inequalities
\[
\frac{\sigma_{m-1}\sigma_{m+1}}{\sigma_m^2} \leq \frac{m(n-m-1)}{(m+1)(n-m)}
\]
(9)
\[
\frac{\sigma_1\sigma_{m-1}}{\sigma_m} \geq \frac{m(n-1)}{n-m},
\]
(10)
and the equalities hold in (9), (10) at a given point if and only if $\Sigma$ is umbilical there.
We now evolve $\Sigma \subset \mathbb{H}^n$ by the following evolution equation
\[ \partial_t X = F \nu, \]
where $\nu$ is the unit outward normal to $\Sigma_t = X(t, \cdot)$ and $F$ is the speed function which may depend on the position vector, principal curvatures and time. Let $g_{ij}$ be the induced metric and $d\mu_t$ be its area element on $\Sigma_t$. We have the following evolution equations.

**Proposition 3.** Under the flow (11), we have:
\[
\begin{align*}
\partial_t g_{ij} &= 2Fh_{ij} \\
\partial_t \nu &= -\nabla F, \\
\partial_t h_{ij} &= -\nabla^i \nabla_j F - F(h^2)_i^j + F\delta_i^j, \\
\partial_t d\mu &= F\sigma_1 d\mu, \\
\partial_t \sigma_m &= -\nabla^i((T_{m-1})^i_j \nabla_j F) - F(\sigma_1 \sigma_m - (m + 1)\sigma_{m+1}) \\
&\quad + (n - m)F\sigma_{m-1},
\end{align*}
\]
where in the last equality we used (7),(8) and the divergence free property of $(T_{m-1})^i_j$ (see [13]). □

**Proposition 4.** Under the flow (11), we have
\[
\frac{d}{dt} \int_{\Sigma} \sigma_m d\mu = (m + 1) \int_{\Sigma} F\sigma_{m+1} d\mu + (n - m) \int_{\Sigma} F\sigma_{m-1} d\mu.
\]

**Proof.** This proposition follows directly from (12), (13) and the divergence theorem. □

In [7] Gerhardt studied general inverse curvature flow of star-shaped hypersurface in hyperbolic space. For our purpose, we will use a special case of their result for the following flow
\[ \partial_t X = \frac{n - 2}{2(n - 1)} \frac{\sigma_1}{\sigma_2} \nu. \]

**Theorem 5** (Gerhardt [7]). If the initial hypersurface is star-shaped and strictly two-convex, then the solution for the flow (14) exists for all time $t > 0$ and the flow hypersurfaces converge to infinity while maintaining star-shapedness and strictly two-convex. Moreover, the hypersurfaces become
strictly convex exponentially fast and more and more totally umbilical in the sense of
\[ |h^i_j - \delta^i_j| \leq Ce^{-t^{n-1}}, \quad t > 0, \]
i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one.

3. Monotonicity

We define the quantity
\[ Q(t) = |\Sigma_t|^{-\frac{n-3}{n-1}} \left( \int_{\Sigma_t} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right), \]
where \(|\Sigma_t|\) is the area of \(\Sigma_t\). In this section, we show that \(Q(t)\) is monotone decreasing along the flow (14).

**Proposition 6.** Under the flow (14), the quantity \(Q(t)\) is monotone decreasing. Moreover, \(\frac{d}{dt}Q(t) = 0\) at some time \(t\) if and only if the surface \(\Sigma_t\) is totally umbilical.

**Proof.** Under the flow (14), Proposition 4 and (12) imply that
\[
\frac{d}{dt} \int_{\Sigma} \sigma_2 d\mu = \frac{3(n-2)}{2(n-1)} \int_{\Sigma} \frac{\sigma_1 \sigma_3}{\sigma_2} d\mu + \frac{(n-2)^2}{2(n-1)} \int_{\Sigma} \frac{\sigma_1^2}{\sigma_2} d\mu \tag{15}
\]
and
\[
\frac{d}{dt} |\Sigma_t| = \frac{(n-2)}{2(n-1)} \int_{\Sigma} \frac{\sigma_1^2}{\sigma_2} d\mu. \tag{16}
\]
Combining (15), (16) and (9), we have
\[
\frac{d}{dt} \left( \int_{\Sigma} \sigma_2 d\mu - (n-2)|\Sigma_t| \right) \leq \frac{n-3}{n-1} \int_{\Sigma} \sigma_2 d\mu. \tag{17}
\]
By applying the Newton-MacLaurin inequality (10) in (16), we also have
\[
\frac{d}{dt} |\Sigma_t| \geq |\Sigma_t|. \tag{18}
\]
Then combining (17) and (18) gives that
\[
\frac{d}{dt} \left( \int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right) \leq \frac{n-3}{n-1} \left( \int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right). \tag{19}
\]

From Proposition 8 in the next section and (19), we know that the quantity
\[
\int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t|
\]
is nonnegative along the flow (14). Then inequalities (18) and (19) implies that
\[
\frac{d}{dt} Q(t) \leq 0.
\]
If the equality holds, the inequalities (9) and (10) assume equalities everywhere on \( \Sigma_t \). Then \( \Sigma_t \) is totally umbilical. \( \square \)

4. The Asymptotic Behavior of Monotone Quantity

In this section, we use the convergence result of the flow (14) proved in \([7]\) to estimate the lower bound of the limit of \( Q(t) \). First we need the following sharp Sobolev inequality on \( S^{n-1} \) (\([2]\)).

**Lemma 7.** For every positive function \( f \) on \( S^{n-1} \), we have
\[
\int_{S^{n-1}} f^{n-3} \, d\text{vol}_{S^{n-1}} + \frac{n-3}{n-1} \int_{S^{n-1}} f^{n-5} |\nabla f|^2 \, d\text{vol}_{S^{n-1}} \\
\geq \omega_{n-1}^{\frac{2}{n-1}} \left( \int_{S^{n-1}} f^{n-1} \, d\text{vol}_{S^{n-1}} \right)^{\frac{n-3}{n-1}}.
\]
Moreover, equality holds if and only if \( f \) is a constant.

**Proof.** From Theorem 4 in \([2]\), for any positive smooth function \( w \) on \( S^{n-1} \), we have the following inequality
\[
\frac{4}{(n-1)(n-3)} \int_{S^{n-1}} |\nabla w|^2 \, d\text{vol}_{S^{n-1}} + \int_{S^{n-1}} w^2 \, d\text{vol}_{S^{n-1}} \\
\geq \omega_{n-1}^{\frac{2}{n-1}} \left( \int_{S^{n-1}} w \frac{2(n-1)}{n-3} \, d\text{vol}_{S^{n-1}} \right)^{\frac{n-3}{n-1}}.
\]
Moreover equality holds if and only if \( w \) is constant. For any positive function \( f \) on \( S^{n-1} \), by letting \( w = f^\frac{n-1}{n-3} \), we have
\[
\int_{S^{n-1}} f^{n-3} \, d\text{vol}_{S^{n-1}} + \frac{n-3}{n-1} \int_{S^{n-1}} f^{n-5} |\nabla f|^2 \, d\text{vol}_{S^{n-1}} \\
\geq \omega_{n-1}^{\frac{2}{n-1}} \left( \int_{S^{n-1}} f^{n-1} \, d\text{vol}_{S^{n-1}} \right)^{\frac{n-3}{n-1}}
\]
and equality holds if and only if \( f \) is a constant. \( \square \)

**Proposition 8.** Under the flow (14), we have
\[
\liminf_{t \to \infty} Q(t) \geq \frac{(n-1)(n-2)}{2} \omega_{n-1}^\frac{2}{n-1}.
\] (20)

**Proof.** Recall that star-shaped hypersurfaces can be written as graphs of function \( r = r(t, \theta) \), \( \theta \in S^{n-1} \). Denote \( \lambda(r) = \sinh(r) \), then \( \lambda'(r) = \cosh(r) \).

We next define a function \( \varphi \) on \( S^{n-1} \) by \( \varphi(\theta) = \Phi(r(\theta)) \), where \( \Phi(r) \) is a positive function satisfying \( \Phi' = 1/\lambda \). Let \( \theta = \{\theta^j\}, j = 1, \ldots, n-1 \) be a coordinate system on \( S^{n-1} \) and \( \varphi_i, \varphi_{ij} \) be the covariant derivatives of \( \varphi \) with respect to the metric \( g_{S^{n-1}} \). Define
\[
v = \sqrt{1 + |\nabla \varphi|_{S^{n-1}}^2}.
\]

From \([7]\), we know that
\[
\lambda = O(e^{-\lambda r}), \quad |\nabla \varphi|_{S^{n-1}} = O(e^{-\lambda r}) \quad (21)
\]
Since $\lambda' = \sqrt{1 + \lambda^2}$, we have

$$\lambda' = \lambda(1 + \frac{1}{2} \lambda^{-2} + O(e^{-\frac{4t}{n-1}})) \quad (22)$$

From (21) we also have

$$\frac{1}{v} = 1 - \frac{1}{2} |\nabla \varphi|^2_{g_{S^{n-1}}} + O(e^{-\frac{4t}{n-1}}) \quad (23)$$

In terms of $\varphi$, we can express the metric and second fundamental form of $\Sigma$ as following (see, e.g, [3, 5])

$$g_{ij} = \lambda^2 (\sigma_{ij} + \varphi_i \varphi_j)$$

$$h_{ij} = \frac{\lambda'}{v \lambda} g_{ij} - \frac{\lambda}{v} \varphi_{ij},$$

where $\sigma_{ij} = g_{S^{n-1}}(\partial_{\theta_i}, \partial_{\theta_j})$. Denote $a_i = \sum_k \sigma_{ik} \varphi_{ki}$ and note that $\sum_i a_i = \Delta_{S^{n-1}} \varphi$. By (21), the principal curvatures of $\Sigma_t$ has the following form

$$\kappa_i = \frac{\lambda'}{v \lambda} - \frac{a_i}{v \lambda} + O(e^{-\frac{4t}{n-1}}), \quad i = 1, \cdots, n-1.$$ 

Then we have

$$\sigma_2 = \sum_{i < j} \kappa_i \kappa_j$$

$$= \frac{(n-1)(n-2)}{2} \left( \frac{\lambda'}{v \lambda} \right)^2 - (n-2) \frac{\lambda' \Delta_{S^{n-1}} \varphi}{v^2 \lambda^2} + O(e^{-\frac{4t}{n-1}}).$$

By using (22) and (23),

$$\sigma_2 = \frac{(n-1)(n-2)}{2} (1 + \lambda^{-2} - |\nabla \varphi|^2_{g_{S^{n-1}}})$$

$$- (n-2) \lambda^{-1} \Delta_{S^{n-1}} \varphi + O(e^{-\frac{4t}{n-1}}).$$

On the other hand,

$$\sqrt{\det g} = (\lambda^{n-3} + O(e^{\frac{(n-3)t}{n-1}})) \sqrt{\det g_{S^{n-1}}}. $$
So we have
\[
\int_{\Sigma_t} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \\
= \int_{S^{n-1}} \lambda^{n-1} (\sigma_2 - \frac{(n-1)(n-2)}{2}) dvol_{S^{n-1}} + O(e^{\frac{(n-5)t}{n-1}}) \\
= \frac{(n-1)(n-2)}{2} \int_{S^{n-1}} (\lambda^{n-3} - \lambda^{n-1} |\nabla \varphi|_{S^{n-1}}^2) dvol_{S^{n-1}} \\
- (n-2) \int_{S^{n-1}} \lambda^{n-2} \Delta_{S^{n-1}} \varphi dvol_{S^{n-1}} + O(e^{\frac{(n-5)t}{n-1}}) \\
= \frac{(n-1)(n-2)}{2} \int_{S^{n-1}} (\lambda^{n-3} - \lambda^{n-1} |\nabla \varphi|_{S^{n-1}}^2) dvol_{S^{n-1}} \\
+ (n-2)^2 \int_{S^{n-1}} \lambda^{n-3} \nabla \lambda \nabla \varphi dvol_{S^{n-1}} + O(e^{\frac{(n-5)t}{n-1}}). \\
\]

Since \( \nabla \lambda = \lambda \nabla \varphi \), by using (22), we deduce that
\[
\int_{\Sigma_t} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \\
= \frac{(n-1)(n-2)}{2} \int_{S^{n-1}} (\lambda^{n-3} - \frac{n-3}{n-1} \lambda^{n-5} |\nabla \lambda|^2) dvol_{S^{n-1}} + O(e^{\frac{(n-5)t}{n-1}}). \tag{24}
\]

Moreover,
\[
|\Sigma_t|^{\frac{n-3}{n-1}} = \left( \int_{S^{n-1}} \lambda^{n-1} dvol_{S^{n-1}} \right)^{\frac{n-3}{n-1}} + O(e^{\frac{(n-5)t}{n-1}}). \tag{25}
\]

Using Lemma 7 we can complete the proof of Proposition 8 by combining (24) and (25).

We now complete the proof of Theorem 1

**Proof of Theorem 1** Since \( Q(t) \) is monotone decreasing, we have
\[
Q(0) \geq \liminf_{t \to \infty} Q(t) \geq \frac{(n-1)(n-2)}{2} \omega_{n-1}^{\frac{2}{n-1}}. 
\]
This gives that the initial hypersurface \( \Sigma \) satisfies
\[
\left( \int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma| \right) \geq \frac{(n-1)(n-2)}{2} \omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}},
\]
which is equivalent to the inequality (5) in Theorem 1. Now we assume that equality holds in (5), which implies that \( Q(t) \) is a constant. Then Proposition 6 implies \( \Sigma_t \) is umbilical and therefore a geodesic sphere. It is also easy to see that if \( \Sigma \) is a geodesic sphere of radius \( r \), then the area of \( \Sigma \)
is $|\Sigma| = \omega_{n-1} \sinh^{n-1} r$ and the integral of $\sigma_2$ is

$$\int_{\Sigma} \sigma_2 d\mu = \frac{(n-1)(n-2)}{2} \omega_{n-1} \coth r \sinh^{n-1} r$$

$$= \frac{(n-1)(n-2)}{2} \omega_{n-1} (\sinh^{n-1} r + \sinh^{n-3} r)$$

$$= \frac{(n-1)(n-2)}{2} \left( |\Sigma| + \omega_{n-1} \frac{2}{n-1} |\Sigma| \frac{n-3}{n-1} \right).$$

Hence the equality holds in (5) on a geodesic sphere. This completes the proof of Theorem 1.

□

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Department of mathematical sciences, and Mathematical Sciences Center, Tsinghua University, 100084, Beijing, P. R. China E-mail address: hli@math.tsinghua.edu.cn

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China E-mail address: wei-y09@mails.tsinghua.edu.cn

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China E-mail address: xiongcw10@mails.tsinghua.edu.cn