THE ACTIVE GRAVITATIONAL MASS OF
A HEAT CONDUCTING SPHERE OUT OF
HYDROSTATIC EQUILIBRIUM

L. Herrera* and A. Di Prisco*
Escuela de Física
Facultad de Ciencias
Universidad Central de Venezuela
Caracas, Venezuela.

Abstract

We obtain an expression for the active gravitational mass of a relativistic heat conducting fluid, just after its departure from hydrostatic equilibrium, on a time scale of the order of relaxation time.

It is shown that an increase of a characteristic parameter leads to larger (smaller) values of active gravitational mass of collapsing (expanding) spheres, enhancing thereby the instability of the system.

1 Introduction

In a recent series of works [1, 2, 3] the behaviour of dissipative systems at the very moment when they depart from hydrostatic equilibrium, has been studied.

It appears that a parameter formed by a specific combination of thermal relaxation time, temperature, proper energy density and pressure, may critically affect the evolution of the object.

*Postal address: Apartado 80793, Caracas 1080A, Venezuela; E-mail address: lherrera@telcel.net.ve
More specifically, it has been shown that in the equation of motion of any fluid element, the inertial mass term is multiplied by a factor, vanishing for a given value of that parameter (critical point) and changing of sign beyond that value.

Although the above mentioned parameter is constrained by causality requirements, it appears that in some cases these requirements do not prevent the system from reaching the critical point. Furthermore it might not be reasonable to apply, close to the critical point, restrictions obtained from a linear perturbative scheme (as is the case for causality conditions).

In order to delve more deeply into the physical nature of the critical point, we shall obtain here an expression for the active gravitational mass (Tolman mass) which explicitly contains the parameter mentioned above.

It will be seen that this expression yields larger (smaller) values for the active gravitational mass of the inner core of a collapsing (expanding) sphere, as we approach the critical point, this tendency persists beyond the critical point as the system moves away from it.

This result provides some hints about the way in which the evolution of the system is affected by the aforesaid parameter.

The paper is organized as follows.

In the next section the field equations, the conventions and other useful formulae are introduced. In section 3 we briefly present the equation for the heat conduction. The departure from hydrostatic equilibrium is considered in section 4. In section 5 we derive an expression for the Tolman mass and evaluate it at the very moment the system departs from hydrostatic equilibrium. Finally a discussion of this expression is presented in the last section.

2 Field Equations and Conventions.

We consider spherically symmetric distributions of collapsing fluid, which for sake of completeness we assume to be anisotropic, undergoing dissipation in the form of heat flow, bounded by a spherical surface Σ.

The line element is given in Schwarzschild-like coordinates by

\[ ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (1)
where \( \nu(t, r) \) and \( \lambda(t, r) \) are functions of their arguments. We number the coordinates: \( x^0 = t; \ x^1 = r; \ x^2 = \theta; \ x^3 = \phi. \)

The metric (1) has to satisfy Einstein field equations

\[
G^\nu_\mu = -8\pi T^\nu_\mu
\]  

which in our case read (3):

\[
-8\pi T^0_0 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)
\]

\[
-8\pi T^1_1 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \nu'\right)
\]

\[
-8\pi T^2_2 = -8\pi T^3_3 = -\frac{e^{-\nu}}{4}\left(2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \nu')\right) + \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right)
\]

\[
-8\pi T^0_1 = -\frac{\dot{\lambda}}{r}
\]

where dots and primes stand for partial differentiation with respect to \( t \) and \( r \) respectively.

In order to give physical significance to the \( T^\mu_\nu \) components we apply the Bondi approach [5].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates \((\tau, x, y, z)\)

\[
d\tau = e^{\nu/2}dt \quad dx = e^{\lambda/2}dr \quad dy = r\,d\theta \quad dz = r\sin\theta d\phi
\]

Then, denoting the Minkowski components of the energy tensor by a bar, we have

\[
\bar{T}^0_0 = T^0_0 \quad \bar{T}^1_1 = T^1_1 \quad \bar{T}^2_2 = T^2_2 \quad \bar{T}^3_3 = T^3_3 \quad \bar{T}_{01} = e^{-(\nu+\lambda)/2}T_{01}
\]
Next, we suppose that when viewed by an observer moving relative to these coordinates with proper velocity $\omega$ in the radial direction, the physical content of space consists of an anisotropic fluid of energy density $\rho$, radial pressure $P_r$, tangential pressure $P_\perp$, and radial heat flux $\dot{q}$. Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

$$
\begin{pmatrix}
\rho & -\dot{q} & 0 & 0 \\
-\dot{q} & P_r & 0 & 0 \\
0 & 0 & P_\perp & 0 \\
0 & 0 & 0 & P_\perp
\end{pmatrix}
$$

Then a Lorentz transformation readily shows that

$$
T_0^0 = \bar{T}_0^0 = \frac{\rho + P_r \omega^2}{1 - \omega^2} + \frac{2Q\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} (7)
$$

$$
T_1^1 = \bar{T}_1^1 = -\frac{P_r + \rho \omega^2}{1 - \omega^2} - \frac{2Q\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} (8)
$$

$$
T_2^2 = T_3^3 = \bar{T}_2^2 = \bar{T}_3^3 = -P_\perp (9)
$$

$$
T_{01} = e^{(\nu+\lambda)/2} \bar{T}_{01} = -\frac{(\rho + P_r)\omega e^{(\nu+\lambda)/2}}{1 - \omega^2} - \frac{Qe^{\nu/2}e^\lambda}{(1 - \omega^2)^{1/2}(1 + \omega^2)} (10)
$$

with

$$
Q \equiv \frac{\dot{q} e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} (11)
$$

Note that the coordinate velocity in the $(t, r, \theta, \phi)$ system, $dr/dt$, is related to $\omega$ by

$$
\omega = \frac{dr}{dt} e^{(\lambda - \nu)/2} (12)
$$

At the outside of the fluid distribution, the spacetime is that of Vaidya, given by

$$
ds^2 = \left(1 - \frac{2M(u)}{R}\right) du^2 + 2dud\mathcal{R} - \mathcal{R}^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) (13)
$$
where \( u \) is a time-like coordinate such that \( u = \text{constant} \) is (asymptotically) a null cone open to the future and \( \mathcal{R} \) is a null coordinate \( (g_{\mathcal{R}\mathcal{R}} = 0) \). It should be remarked, however, that strictly speaking, the radiation can be considered in radial free streaming only at radial infinity. The two coordinate systems \((t, r, \theta, \phi)\) and \((u, \mathcal{R}, \theta, \phi)\) are related at the boundary surface and outside it by

\[
u = t - r - 2M \ln \left( \frac{r}{2M} - 1 \right) \tag{14}
\]

\[
\mathcal{R} = r \tag{15}
\]

In order to match smoothly the two metrics above on the boundary surface \( r = r_\Sigma(t) \), we have to require the continuity of the first fundamental form across that surface. As result of this matching we obtain

\[
[P]\Sigma = \left[ Q e^{\lambda/2} \left( 1 - \omega^2 \right)^{1/2} \right]_\Sigma = [\hat{q}]\Sigma \tag{16}
\]

expressing the discontinuity of the radial pressure in the presence of heat flow, which is a well known result [6].

Next, it will be useful to calculate the radial component of the conservation law

\[
T^\mu_{\nu;\mu} = 0 \tag{17}
\]

After tedious but simple calculations we get

\[
(-8\pi T_1^1)' = \frac{16\pi}{r} \left( T_1^1 - T_2^2 \right) + 4\pi\nu' \left( T_1^1 - T_0^0 \right) + \frac{e^{-\nu}}{r} \left( \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\nu}\nu}{2} \right) \tag{18}
\]

which in the static case becomes

\[
P_r' = -\frac{\nu'}{2} (\rho + P_r) + \frac{2(P_\perp - P_r)}{r} \tag{19}
\]

representing the generalization of the Tolman-Oppenheimer-Volkof equation for anisotropic fluids [7].
3 Heat Conduction Equation.

In the study of star interiors it is usually assumed that the energy flux of radiation (and thermal conduction) is proportional to the gradient of temperature (Maxwell-Fourier law or Eckart-Landau in general relativity). However it is well known that the Maxwell-Fourier law for the radiation flux leads to a parabolic equation (diffusion equation) which predicts propagation of perturbation with infinite speed (see [8]–[10] and references therein). This simple fact is at the origin of the pathologies [11] found in the approaches of Eckart [12] and Landau [13] for relativistic dissipative processes.

To overcome such difficulties, different relativistic theories with non-vanishing relaxation times have been proposed in the past [14]–[17]. The important point is that all these theories provide a heat transport equation which is not of Maxwell-Fourier type but of Cattaneo type [18], leading thereby to a hyperbolic equation for the propagation of thermal perturbation.

Accordingly we shall describe the heat transport by means of a relativistic Israel-Stewart equation [10]. Although a complete treatment of dissipative processes requires the inclusion of viscous stresses as well as the coupling between these and the heat flow, we shall assume here for simplicity vanishing viscosity. Thus we have

\[ \tau \frac{Dq^\alpha}{Ds} + q^\alpha = \kappa P^{\alpha\beta} (T_{,\beta} - Ta_{,\beta}) - \tau u^\alpha q_\beta a^{\beta} - \frac{1}{2} \kappa T^2 \left( \frac{\tau}{\kappa T^2} u^\beta \right) q^{,\beta} \] (20)

with

\[ u^\mu = \left( \frac{e^{-\nu/2}}{(1 - \omega^2)^{1/2}}, \frac{\omega e^{-\lambda/2}}{(1 - \omega^2)^{1/2}}, 0, 0 \right) \] (21)

\[ q^\mu = Q \left( \omega e^{(\lambda - \nu)/2}, 1, 0, 0 \right) \] (22)

where \( \kappa, \tau, T, q^\beta \) and \( a^{\beta} \) denote thermal conductivity, thermal relaxation time, temperature, the heat flow vector and the components of the four acceleration, respectively. Also, \( P^{\alpha\beta} \) is the projector onto the hypersurface orthogonal to the four velocity \( u^\alpha \).
4 Thermal Conduction and Departure from Hydrostatic Equilibrium.

Let us now consider a spherically symmetric fluid distribution which initially may be in either hydrostatic and thermal equilibrium (i.e. $\omega = Q = 0$), or slowly evolving and dissipating energy through a radial heat flow vector.

Before proceeding further with the treatment of our problem, let us clearly specify the meaning of “slowly evolving”. That means that our sphere changes on a time scale which is very large as compared to the typical time in which it reacts on a slight perturbation of hydrostatic equilibrium. This typical time is called hydrostatic time scale. Thus a slowly evolving system is always in hydrostatic equilibrium (very close to), and its evolution may be regarded as a sequence of static models linked by (6). This assumption is very sensible, since the hydrostatic time scale is usually very small.

In fact, it is of the order of 27 minutes for the sun, 4.5 seconds for a white dwarf and $10^{-4}$ seconds for a neutron star of one solar mass and 10 Km radius [19].

In terms of $\omega$ and metric functions, slow evolution means that the radial velocity $\omega$ measured by the Minkowski observer, as well as time derivatives are so small that their products and second order time derivatives may be neglected (an invariant characterization of slow evolution may be found in [20]).

Thus

$$\ddot{\nu} \approx \lambda \dot{\nu} \approx \lambda^2 \approx \dot{\nu}^2 \approx \omega^2 \approx \dot{\omega} = 0$$

(23)

As it follows from (18) and (11), $Q$ is of the order $O(\omega)$. Therefore in the slowly evolving regime, relaxation terms may be neglected and (20) becomes the usual Landau-Eckart transport equation [21].

Then, using (23) and (18) we obtain (19), which as mentioned before is the equation of hydrostatic equilibrium for an anisotropic fluid. This is in agreement with what was mentioned above, in the sense that a slowly evolving system is in hydrostatic equilibrium.

Let us now return to our problem. Before perturbation, the two possible initial states of our system are characterized by:

1. Static

$$\dot{\omega} = \dot{Q} = \omega = Q = 0$$

(24)
2. Slowly evolving

\[ \dot{\omega} = \dot{Q} = 0 \]  
\[ Q \approx O(\omega) \neq 0 \quad (small) \]

where the meaning of “small” is given by (23).

Let us now assume that our system is submitted to perturbations which force it to depart from hydrostatic equilibrium but keeping the spherical symmetry. We shall study the perturbed system on a time scale which is small as compared to the thermal adjustment time.

Then, immediately after perturbation (“immediately” understood in the sense above), we have for the first initial condition (static)

\[ \omega = Q = 0 \]  
\[ \dot{\omega} \approx \dot{Q} \neq 0 \quad (small) \]

whereas for the second initial condition (slowly evolving)

\[ Q \approx O(\omega) \neq 0 \quad (small) \]
\[ \dot{Q} \approx \dot{\omega} \neq 0 \quad (small) \]

As it was shown in [1], it follows from (18) and (7)–(10) that after perturbation, we have for both initial conditions (see [1] for details)

\[ -e^{(\nu-\lambda)/2} R = (\rho + P_r) \dot{\omega} + \dot{Q} e^{\lambda/2} \]

where \( R \) denotes the left-hand side of the TOV equation, i.e.

\[ R \equiv \frac{dP_r}{dr} + \frac{4\pi r P_r^2}{1 - 2m/r} + \frac{P_r m}{r^2 (1 - 2m/r)} + \frac{4\pi r \rho P_r}{1 - 2m/r} + \]
\[ + \frac{\rho m}{r^2 (1 - 2m/r)} - \frac{2 (P_\perp - P_r)}{r} \]
\[ = P_r' + \frac{\nu'}{2} (\rho + P_r) - \frac{2}{r} (P_\perp - P_r) \]
The physical meaning of $R$ is clearly inferred from (32). It represents the total force (gravitational + pressure gradient + anisotropic term) acting on a given fluid element. Obviously, $R > 0/R < 0$ means that the total force is directed inward/outward of the sphere.

Let us now turn back to thermal conduction equation (20). Evaluating it immediately after perturbation, we obtain for both initial configurations (static and slowly evolving) (see [1] for details)

$$\tau \dot{Q} e^{\lambda/2} = -\kappa T \dot{\omega}$$

Finally, combining (31) and (33), one obtains

$$\dot{\omega} = -\frac{e^{(\nu-\lambda)/2} R}{(\rho + P_r)} \times \frac{1}{1 - \left(\frac{\kappa T}{\tau (\rho + P_r)}\right)}$$

or, defining the parameter $\alpha$ by

$$\alpha \equiv \frac{\kappa T}{\tau (\rho + P_r)}$$

$$- e^{(\nu-\lambda)/2} R = (\rho + P_r) \dot{\omega} (1 - \alpha)$$

This last expression has the obvious “Newtonian” form

Force = mass $\times$ acceleration

since, as it is well known, $(\rho + P_r)$ represents the inertial mass density and by “acceleration” we mean the time derivative of $\omega$ and not $(a_\mu a^\mu)^{1/2}$. If $\alpha < 1$, then an outward/inward acceleration ($\dot{\omega} > 0/\dot{\omega} < 0$) is associated with an outwardly/inwardly ($R < 0/R > 0$) directed total force (as one expects!). However, if $\alpha = 1$, we obtain that $\dot{\omega} \neq 0$ even though $R = 0$. Still worse, if $\alpha > 1$, then an outward/inward acceleration is associated with an inwardly/outwardly directed total force!.

As mentioned before, the critical point may be restricted by causality conditions, particularly in the pure bulk or shear viscosity case [3], however this is not so in the general case [4]. Independently of this fact, it is clear from (36), that the “effective” inertial mass term decreases as $\alpha$ increases. In the next section we shall obtain an expression for the active gravitational mass explicitly containing $\alpha$. 

9
5 The Tolman mass

The Tolman mass for a spherically symmetric distribution of matter is given by (eq.(24) in [4]):

\[ m_T = 4\pi \int_0^r r^2 e^{(\nu+\lambda)/2} \left( T_0^0 - T_1^1 - 2T_2^2 \right) dr \]

\[ + \frac{1}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \left( g^{\alpha\beta} \sqrt{-g} \right) / \partial t} \right) g^{\alpha\beta} dr \]  

(37)

where \( L \) denotes the usual gravitational lagrangian density (eq.(10) in [4]). Although Tolman’s formula was introduced as a measure of the total energy of the system, with no commitment to its localization, we shall define the mass within a sphere of radius \( r \), inside \( \Sigma \), as

\[ m_T = 4\pi \int_0^r r^2 e^{(\nu+\lambda)/2} \left( T_0^0 - T_1^1 - 2T_2^2 \right) dr \]

\[ + \frac{1}{2} \int_0^r r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \left( g^{\alpha\beta} \sqrt{-g} \right) / \partial t} \right) g^{\alpha\beta} dr \]  

(38)

This (heuristic) extension of the global concept of energy to a local level [22] is suggested by the conspicuous role played by \( m_T \) as the “effective gravitational mass”, which will be exhibited below.

On the other hand, even though Tolman’s definition is not without its problems [22, 23], we shall see that \( m_T \), as defined by (38), is a good measure of the active gravitational mass, at least for the system under consideration. After some simple but tedious calculations, it can be shown that (38) may be written as (see [24] for details).

\[ m_T = e^{(\nu+\lambda)/2} \left[ m(r, t) - 4\pi r^3 T_1^1 \right] \]  

(39)

where the mass function \( m(r, t) \) is defined by [23, 26]

\[ m(r, t) = \frac{1}{2} r R_{232}^3 \]  

(40)

and the Riemann component for metric (4) is given by

\[ R_{232}^3 = 1 - e^{-\lambda} \]  

(41)
Using field equations, (40) may be written in the most familiar form

\[ m(r, t) = 4\pi \int_0^r r^2 T_0^0 \, dr \quad (42) \]

or alternativately [24]

\[ m(r, t) = \frac{4\pi}{3} r^3 \left( T_0^0 + T_1^1 - T_2^2 \right) + \frac{r}{2} C_{232}^3 \quad (43) \]

where \( C_{232}^3 \) denotes the corresponding component of the Weyl tensor.

It is worth noticing that this is, formally, the same expression for \( m_T \) in terms of \( m \) and \( T_1^1 \), that appears in the static (or quasi-static) case (eq.(25) in [20]).

Replacing \( T_1^1 \) by (4), and \( m \) by (40) and (41), one may also obtain

\[ m_T = e^{(\nu - \lambda)/2} \frac{r^2}{2} \quad (44) \]

This last equation brings out the physical meaning of \( m_T \) as the active gravitational mass. Indeed, it can be easily shown [27] that the gravitational acceleration \( (a) \) of a test particle, instantaneously at rest in a static gravitational field, as measured with standard rods and coordinate clock is given by

\[ a = -\frac{e^{(\nu - \lambda)/2} \nu'}{2} = -\frac{m_T}{r^2} \quad (45) \]

A similar conclusion may be obtained by inspection of eq.(13) (valid only in the static or quasi-static case) [28]. In fact, the first term on the right side of this equation (the “gravitational force” term) is a product of the “passive” gravitational mass density \( (\rho + P_r) \) and a term proportional to \( m_T/r^2 \).

We shall now consider another expression for \( m_T \), which appears to be more suitable for the treatment of the problem under consideration. This latter expression will be evaluated immediately after the system departs from equilibrium. Therefore the physical meaning of \( m_T \) as the active gravitational mass obtained for the static (and quasi-static) case, may be safely extrapolated to the non-static case within the time scale mentioned above.

The required expression for the Tolman mass will be obtained as follows (see [24] for details). Taking the \( r \)-derivative of (44) and using (43) and (39) we obtain the following differential equation for \( m_T \)
\[ rm_T' - 3m_T = e^{\frac{(\nu+\lambda)}{2}} \left[ 4\pi r^3 (T_1^1 - T_2^2) - 3W_s \right] \]
\[ + \frac{e^{\frac{\lambda-\nu}{2}}}{4} \left( \dot{\lambda} + \frac{\dot{\lambda}^2}{2} - \dot{\lambda} \dot{\nu} \right) \]  
(46)

where \( W_s \) is given by
\[ W_s = \frac{r^3 e^{-\lambda}}{6} \left( \frac{e^\lambda}{r^2} - \frac{1}{4} - \frac{\nu' \lambda'}{4} - \frac{\nu''}{2} - \frac{\lambda'}{2} \right) \]  
(47)

Equation (46) can be formally integrated to obtain
\[ m_T = \left( m_T \right)_\Sigma \left( \frac{r}{r_{\Sigma}} \right)^3 \]
\[ - r^3 \int_r^{r_{\Sigma}} e^{\frac{(\nu+\lambda)}{2}} \left[ \frac{8\pi}{r} (T_1^1 - T_2^2) + \frac{1}{r^4} \int_0^r 4\pi \tilde{r}^3 (T_0^0)' d\tilde{r} \right] dr \]
\[- r^3 \int_r^{r_{\Sigma}} e^{\frac{(\lambda-\nu)}{2}} \left( \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda} \dot{\nu}}{2} \right) dr \]  
(48)

In the static (or quasi-static) case (\( \ddot{\lambda} = \dot{\lambda} = \dot{\nu} = 0 \)) the expression above is identical to eq.(32) in [20].

We shall now proceed to evaluate (48) immediately after perturbation. Using (7)–(10) and (27)–(30), we see that up to first order we get immediately after perturbation (for both initial conditions)
\[ T_0^0 = \rho \quad T_1^1 = -P_r \quad T_2^2 = -P_r \]
\[ \ddot{\lambda} = \dot{\nu} = 0 \]
\[ \ddot{\lambda} = -8\pi r \rho e^{\frac{(\nu+\lambda)}{2}} \left[ (\rho + P_r) \dot{\omega} + \dot{g} e^{\lambda/2} \right] \]  
(49)

Replacing (49) into (48) and using (33) and (36), we obtain finally
\[ m_T = \left( m_T \right)_\Sigma \left( \frac{r}{r_{\Sigma}} \right)^3 \]
\[ + 4\pi r^3 \int_r^{r_{\Sigma}} e^{\frac{(\nu+\lambda)}{2}} \left[ \frac{2}{r} (P_r - P_{\perp}) - \frac{1}{r^4} \int_0^r \tilde{r}^3 \rho' d\tilde{r} \right] dr \]
\[ + 4\pi r^3 \int_r^{r_{\Sigma}} e^\lambda (\rho + P_r) \dot{\omega} (1 - \alpha) dr \]  
(50)
where the general expression for \((m_T)_\Sigma\) can be obtained from (39), (8), (11) and (16)

\[
(m_T)_\Sigma = m_\Sigma + \frac{4\pi r_\Sigma^3 \dot{q}_\Sigma (1 + 2\omega_\Sigma)}{1 - \omega_\Sigma^2} + 4\pi r_\Sigma^3 \left( \frac{\rho_\Sigma \omega_\Sigma^2}{1 - \omega_\Sigma^2} \right)_\Sigma
\]  

which, after perturbation reduces to

\[
(m_T)_\Sigma = m_\Sigma + 4\pi r_\Sigma^3 \dot{q}_\Sigma
\]  

6 Discussion

Let us now consider a sphere of radius \(r\) within \(\Sigma\). Immediately after perturbation the Tolman mass of this internal core is given by (50). The relevance of the two terms in the first integral has already been discussed [24] and therefore they shall not be considered here.

Instead, we shall focus on the last term in (50). If the system starts to collapse (\(\dot{\omega} < 0\)) this last term tends to decrease the value of the Tolman mass, leading thereby to a weaker collapse. Inversely, if the system starts to expand (\(\dot{\omega} > 0\)), the last term in (50) contributes positively to the Tolman mass of the core, leading to a weaker expansion. Thus, in both cases this last term tends to stabilize the system. This is so long as \(\alpha < 1\).

If \(\alpha > 1\) the inverse picture follows. In this case for initially collapsing (expanding) configurations the last term in (50) becomes positive (negative) leading to stronger collapse (expansion).

In general the system becomes more and more unstable as \(\alpha\) grows.

It should be noticed that in the comments above we have assumed \(\alpha\) to be constant throughout the fluid distribution. This of course is a rather crude approximation as it is evident from (35). Therefore, a wide variety of scenarios may be considered from different radial dependence of that parameter.

Finally, observe that in the dissipationless case (\(\alpha = 0\)), an inflationary equation of state (\(\rho = -P_r\)) is equivalent to the critical point (\(\alpha = 1\)) in the heat conducting situation. In both cases the stabilizer term in (50) vanishes.

In the heat conducting case \(\alpha \neq 0\), an inflationary equation of state leads to an effective inertial mass density equal to \(-\kappa T/\tau\), as follows from (31) and (33). In this case, according to (33), (48) and (49) the last integral in (50) should be replaced by
This last integral contributes negatively to Tolman mass in the case $\dot{\omega} > 0$, yielding stronger expansions.

In other words, in what concerns eq. (50), an equation of state of the above-mentioned form ($\rho = -P_r$) is equivalent (in the dissipative case) to a situation with $\alpha > 1$.

Of course one might ask if a real physical system may reach (or even go beyond) the critical point. The answer to this question seems to be affirmative as suggested by the example provided in [3]. Indeed it is shown in that reference that a mixture of matter and neutrinos with typical values of temperature and energy density, corresponding to the moment of birth of a neutron star in a supernova explosion may lead to values of $\alpha$ equal to or even greater than 1.

However, it is not our purpose here to discuss about the plausibility to reach the critical point but rather to bring out the physical meaning of $\alpha$ and the critical point.

Finally it is worth noticing that evaluating the mass function from (39) and using (50), we obtain similar conclusions about the relation between $\alpha$ and the mass function as those obtained for the Tolman mass. However, unlike the Tolman expression, the mass function can not be interpreted (for a part of the configuration) as the active gravitational mass and therefore the stability/instability criteria look less convincing when using $m(r, t)$ instead of $m_T(r, t)$.

References

[1] Herrera L., Di Prisco A., Hernández-Pastora J. L., Martín J. and Martínez J., 1997 Class. Quant. Grav., 14, 2239.

[2] Herrera L. and Martínez J., 1997 Class. Quant. Grav., 14, 2697.

[3] Herrera L. and Martínez J., 1998 Class. Quant. Grav., 15, 407.

[4] Tolman R., 1930, Phys. Rev., 35, 875.

[5] Bondi H., 1964, Proc. R. Soc. London, A281, 39.
[6] Santos N. O., 1985, Mon. Not. R. Astron. Soc., 216, 403.

[7] Bowers R. and Liang E., 1974, Astrophys. J., 188, 657.

[8] Joseph D. and Preziosi L., 1989, Rev. Mod. Phys., 61, 41.

[9] Jou D., Casas-Vázquez J. and Lebon G., 1988, Rep. Prog. Phys., 51, 1105.

[10] Maartens R., Preprint astro-ph 9609119

[11] Hiscock W. and Lindblom L., 1983, Ann. Phys. NY, 151, 466.

[12] Eckart C., 1940, Phys. Rev., 58, 919.

[13] Landau L. and Lifshitz E., 1959, Fluid Mechanics (Pergamon Press, London).

[14] Israel W., 1976, Ann. Phys., NY, 100, 310.

[15] Israel W. and Stewart J., 1976, Phys. Lett., A58, 2131; 1979, Ann. Phys., NY, 118, 341.

[16] Pavón D., Jou D. and Casas-Vázquez J., 1982, Ann. Inst. H Poincaré, A36, 79.

[17] Carter B., 1976, Journées Relativistes, ed. Cahen M., Deveber R. and Geheniahau J., (ULB).

[18] Cattaneo C., 1948, Atti. Semin. Mat. Fis. Univ. Modena, 3, 3.

[19] Kippenhahn R. and Weigert A., 1990, Stellar Structure and Evolution, (Springer Verlag, Berlin).

[20] Herrera L. and Santos N.O., 1995, Gen. Rel. Gravit. 27, 1071.

[21] Herrera L. and Di Prisco A., 1997, Phys. Rev., 55, 2044.

[22] Cooperstock F. I., Sarracino R. S. and Bayin S. S., 1981, J. Phys. A 14, 181.

[23] Devitt J. and Florides P. S., 1989, Gen. Rel. Grav. 21, 585.
[24] Herrera L., Di Prisco A., Hernández-Pastora J. L. and Santos N. O., 1998, Phys. Let. A., 237, 113

[25] Misner C. and Sharp D., 1964, Phys. Rev. 136, B571.

[26] Cahill M. and McVittie G., 1970, J. Math. Phys. 11, 1382.

[27] Grøn Ø., 1985, Phys. Rev. D, 31, 2129.

[28] Lightman A., Press W., Price R. and Teukolsky S., 1975, Problem Book in Relativity and Gravitation (Princeton University Press, Princeton).