Abstract: In this paper, we study an optimal control problem of linear backward stochastic differential equation (BSDE) with quadratic cost functional under partial information. This problem is solved completely and explicitly by using a stochastic maximum principle and a decoupling technique. By using the maximum principle, a stochastic Hamiltonian system, which is a forward-backward stochastic differential equation (FBSDE) with filtering, is obtained. By decoupling the stochastic Hamiltonian system, three Riccati equations, a BSDE with filtering, and a stochastic differential equation (SDE) with filtering are derived. We then get an optimal control with a feedback representation. An explicit formula for the corresponding optimal cost is also established. As illustrative examples, we consider two special scalar-valued control problems and give some numerical simulations.

Keywords: Linear quadratic optimal control; backward stochastic differential equation; filtering; Ricatti equation; feedback representation.

Mathematics Subject Classification: 93E20, 60H10

1 Introduction

A BSDE is an Itô SDE for which a random terminal rather than an initial condition on state has been specified. Bismut [1] first introduced a linear BSDE, which is an adjoint equation of stochastic optimal control problem. Pardoux and Peng [2] extended the linear BSDE to a general

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case. Since then, there has been considerable attention on related topics and their applications among researchers in mathematical finance and stochastic optimal control. See for example, El Karoui et al. [3], Ma and Yong [4], Kohlmann and Zhou [5].

Since BSDE stems from stochastic control theory, it is very natural and appealing to investigate optimal control problem of BSDE. Moreover, controlled BSDE is expected to have wide and important applications in various fields, especially in mathematical finance. In financial investment, a European contingent claim $\xi$, which is a random variable, can be thought as a contract to be guaranteed at maturity $T$. Peng [6] and Dokuchaev and Zhou [7] derived local and global stochastic maximum principles of optimality for BSDEs, respectively. Linear quadratic (LQ) optimal control problems of BSDEs have also been investigated. Lim and Zhou [8] discussed an LQ control problem of BSDE with a general setting and gave a feedback representation of the optimal control. Li et al. [9] extended the results in [8] to the case with mean-field term. Huang et al. [10] and Du et al. [11] considered LQ backward mean-field games. Du and Wu [12] concerned a stackelberg game for mean field linear BSDE with quadratic cost functionals.

In this paper, we investigate an LQ control problem of BSDE with partial information, which will be referred as a stochastic backward LQ control problem. We are devoted to deriving the optimal control with a feedback representation and establishing an explicit formula for the corresponding optimal cost. Note that the mentioned papers above are concentrated on the complete information case. The motivation of studying stochastic control problems with partial information arises naturally from the area of financial economics. In a portfolio and consumption problem, let $\{F_t\}_{t \geq 0}$ denote the flow of information generated by all market noises. In reality, the information available to an agent maybe less than the one produced by the market noises, that is, $\mathcal{G}_t \subseteq F_t$, where $\{\mathcal{G}_t\}_{t \geq 0}$ is the information available to the agent. There are considerable literatures on related topics [13, 14, 15, 16, 17, 18]. In particular, Huang et al. [15] derived a necessary condition for optimality of BSDE with partial information and applied their results to two classes of LQ problems. Wang et al. [16] and Wang et al. [17] concerned LQ problems with partially observable information driven by FBSDE and mean field FBSDE, respectively. LQ non-zero sum stochastic differential game of BSDE is considered in Wang et al. [18]. They obtained feedback Nash equilibrium points by FBSDE and Riccati equation under asymmetric information.

Our work distinguishes itself from existing literatures in the following aspects. (i) Both the generator of dynamic system and the cost functional contain diffusion terms $Z_1$ and $Z_2$. Moreover, our results are obtained under some usual conditions (see Assumptions $A1$ and $A2$ in section 2). In the literatures on this topic, diffusion terms $Z_1$ and $Z_2$ are usually assumed not to be contained into the generator (see [15], [17]), or there are some additional conditions to ensure the solvability of Riccati equation (see [16], [18]). (ii) Sufficient and necessary conditions of optimality are established, which provide an expression for optimal control via the solution of stochastic Hamiltonian system. (iii) Explicit representations for optimal control in terms
of three Riccati equations, a BSDE with filtering, and an SDE with filtering are obtained, as well as the associated optimal cost. The derivation of associated Riccati equations is extremely different from [8] and [9], since the stochastic Hamiltonian system is an FBSDE with filtering. Moreover, the uniqueness and existence of solution to BSDE (3.5) is first obtained, which is important in deriving explicit representations for optimal control and associated optimal cost.

(iv) Last but not least, we consider two special scalar-valued control problems of BSDEs with partial information. In the case of \( H = N_1 = 0 \), we obtain explicit solutions of the stochastic Hamiltonian system, as well as related Riccati equations. In the case of \( C_2 = 0 \), we give some numerical simulations to illustrate our theoretical results.

The rest of this paper is organized as follows. In Section 2, we formulate the stochastic backward LQ control problem and give some preliminary results. Section 3 aims to decouple the associated stochastic Hamiltonian system and derive some Riccati equations. In Section 4, we give explicit representations of optimal control and the associated optimal cost. Section 5 is devoted to solving two special scalar-valued control problems and giving some numerical simulations. Finally, we conclude this paper.

\section{Preliminaries}

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space and let \(T > 0\) be a fixed time horizon. Let \((W_{1t}, W_{2t}) : 0 \leq t \leq T\) be a \(\mathbb{R}^2\)-valued standard Wiener process, defined on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). \(\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}\) is a natural filtration of \((W_1, W_2)\) augmented by all \(\mathbb{P}\)-null sets. Let \(\mathcal{F}^\beta_t = \sigma\{\beta_s, 0 \leq s \leq t\}\) be the filtration generated by a stochastic process \(\beta\). Let \(\mathbb{R}^{n \times m}\) be the set of all \(n \times m\) matrices and \(\mathbb{S}^n\) be the set of all \(n \times n\) symmetric matrices. For a matrix \(M \in \mathbb{R}^n\), let \(M^\top\) be its transpose. The inner product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{R}^{n \times m}\) is defined by \(\langle M, N \rangle \mapsto \text{tr}(M^\top N)\) with an induced norm \(|M| = \sqrt{\text{tr}(M^\top M)}\). In particular, we denote by \(\mathbb{S}^n_+ (\hat{\mathbb{S}}^n_+)\) the set of all \(n \times n\) (uniformly) positive definite matrices. For any Euclidean space \(M\), we adopt the following notations:

\[
\mathcal{L}^2 F_T(\Omega; M) = \left\{ \zeta : \Omega \rightarrow M | \zeta \text{ is an } \mathcal{F}_T\text{-measurable random variable, } \mathbb{E}[|\zeta|^2] < \infty \right\};
\]

\[
\mathcal{L}^\infty(0, T; M) = \left\{ v : [0, T] \rightarrow M | \text{v is a bounded function} \right\};
\]

\[
\mathcal{L}^2_2(0, T; M) = \left\{ v : [0, T] \times \Omega \rightarrow M | v \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted stochastic process, } \mathbb{E} \left[ \int_0^T |v_t|^2 dt \right] < \infty \right\};
\]

\[
\mathcal{S}^2_2(0, T; M) = \left\{ v : [0, T] \times \Omega \rightarrow M | v \text{ is an } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted stochastic process and has continuous paths, } \mathbb{E} \left[ \sup_{t \in [0, T]} |v_t|^2 \right] < \infty \right\}.
\]

Consider a controlled linear BSDE

\[
\begin{align*}
\left\{ \begin{array}{l}
dY_t = (A_t Y_t + B_t v_t + C_{1t} Z_{1t} + C_{2t} Z_{2t}) dt + Z_{1t} dW_{1t} + Z_{2t} dW_{2t}, \quad t \in [0, T], \\
Y_T = \zeta,
\end{array} \right.
\end{align*}
\]
where $\zeta \in L^2_{\mathcal{F}^T}(\Omega; \mathbb{R}^n)$ and $v$, valued in $\mathbb{R}^m$, is a control process. Introduce an admissible control set

$$V_{ad}[0,T] = \left\{ v : [0,T] \times \Omega \to \mathbb{R}^m | v \text{ is } \mathcal{F}^W_t \text{ adapted, } E \left[ \int_0^T |v_t|^2 dt \right] < \infty \right\}.$$ 

Any $v \in V_{ad}[0,T]$ is called an admissible control.

**Assumption A1:** The coefficients of dynamic system satisfy

$$A, C_1, C_2 \in \mathcal{L}^\infty(0,T; \mathbb{R}^{n \times n}), \ B \in \mathcal{L}^\infty(0,T; \mathbb{R}^{n \times m}).$$

Under Assumption A1, dynamic system (2.1) admits a unique solution pair $(Y, Z_1, Z_2) \in \mathcal{S}^2_F(0,T; \mathbb{R}^n) \times \mathcal{L}^2_F(0,T; \mathbb{R}^n \times \mathbb{R}^n)$, which is called the corresponding state process, for any $v \in V_{ad}[0,T]$ (see Pardoux and Peng [2], Yong and Zhou [19]). We introduce a quadratic cost functional

$$J(v) = \frac{1}{2} E \left[ Y_0^\top G Y_0 + \int_0^T \left( Y_t^\top H Y_t + v_t^\top R v_t + Z_{1t}^\top N_{1t} Z_{1t} + Z_{2t}^\top N_{2t} Z_{2t} \right) dt \right]. \quad (2.2)$$

**Assumption A2:** The weighting matrices in cost functional satisfy

$$H, N_1, N_2 \in \mathcal{L}^\infty(0,T; \mathbb{S}_+^n), \ R \in \mathcal{L}^\infty(0,T; \tilde{S}_+^m), \ G \in \mathbb{S}_+^n.$$

Our stochastic backward LQ control problem can be stated as follows.

**Problem BLQ.** Find a $v^* \in V_{ad}[0,T]$ such that

$$J(v^*) = \inf_{v \in V_{ad}[0,T]} J(v). \quad (2.3)$$

Any $v^* \in V_{ad}[0,T]$ satisfying (2.3) is called an optimal control, and the state process $(Y^*, Z_1^*, Z_2^*)$ is called an optimal state process. Under Assumptions A1 and A2, Problem BLQ is uniquely solvable for any terminal state $\zeta \in L^2_{\mathcal{F}^T}(\Omega; \mathbb{R}^n)$ (see Li et al. [9]). We suppressed the time argument in the sequel of this paper wherever necessary, for the sake of notation simplicity. The following theorem is a necessary condition of optimality, which is easy to be obtained from Theorem 3.1 in Huang et al. [15].

**Theorem 2.1.** Under Assumptions A1 - A2, if $v^*$ is an optimal control of Problem BLQ and $(Y^*, Z_1^*, Z_2^*)$ is the corresponding optimal state process, then

$$\begin{cases} 
  dX^* = - \left( A^\top X^* + H Y^* \right) dt - \left( C_1^\top X^* + N_1 Z_1^* \right) dW_1 - \left( C_2^\top X^* + N_2 Z_2^* \right) dW_2, \\
  X^*_0 = - G Y^*_0
\end{cases} \quad (2.4)$$

admits a unique solution such that

$$E \left[ R_t v_t^* + B_t^\top X_t^* | \mathcal{F}^W_t \right] = 0, \ t \in [0,T], \ a.s..$$
According to the above analysis, we end up with a stochastic Hamiltonian system

\[
\begin{aligned}
    dY &= (AY + Bv + C_1Z_1 + C_2Z_2)\,dt + Z_1dW_1 + Z_2dW_2, \\
    dX &= -\left(A^\top X + HY\right)\,dt - \left(C_1^\top X + N_1Z_1\right)\,dW_1 - \left(C_2^\top X + N_2Z_2\right)\,dW_2, \\
    Y &= \zeta, \quad X_0 = -GY_0, \\
    \mathbb{E}[R_tv_t + B_t^\top X_t|\mathcal{F}_t^{W_1}] &= 0.
\end{aligned}
\]  

(2.5)

This is a coupled FBSDE with filtering. Note that the coupling comes from the last equation in (2.5), which is also called a stationarity condition. We point out that in our setting, the stationarity condition involves a conditional expectation, which makes the decoupling of this stochastic Hamiltonian system different and difficult. We now prove the sufficiency of the above result.

**Theorem 2.2.** Let Assumption A1 – A2 hold. If \((X^*, Y^*, Z_1^*, Z_2^*, v^*)\) is an adapted solution to stochastic Hamiltonian system (2.5), then \(v^*\) is an optimal control.

**Proof.** For any \(v \in \mathcal{V}_d(0, T]\), let \((Y, Z_1, Z_2)\) be the corresponding state process. Let \((\tilde{Y}, \tilde{Z}_1, \tilde{Z}_2)\) satisfies

\[
\begin{aligned}
    d\tilde{Y} &= \left[A^\top Y + B(v - v^*) + C_1\tilde{Z}_1 + C_2\tilde{Z}_2\right]\,dt + \tilde{Z}_1dW_1 + \tilde{Z}_2dW_2, \\
    \tilde{Y}_T &= 0.
\end{aligned}
\]

According to the existence and uniqueness of solution to BSDE, we have \(\tilde{Y} = Y - Y^*, \tilde{Z}_1 = Z_1 - Z_1^*, \tilde{Z}_2 = Z_2 - Z_2^*\). With the notation, we derive

\[
J(v) - J(v^*) = \mathbb{E}\left[Y^{*\top}G\tilde{Y}_0 + \int_0^T \left(Y^{*\top}HY + v^{*\top}R(v - v^*) + Z_1^{*\top}N_1\tilde{Z}_1 + Z_2^{*\top}N_2\tilde{Z}_2\right)\,dt\right] + \tilde{J},
\]

where

\[
\tilde{J} = \frac{1}{2}\mathbb{E}\left[\tilde{Y}_0^{\top}GY_0 + \int_0^T \left(\tilde{Y}^{*\top}HY + (v - v^*)^{\top}R(v - v^*) + \tilde{Z}_1^{\top}N_1\tilde{Z}_1 + \tilde{Z}_2^{\top}N_2\tilde{Z}_2\right)\,dt\right].
\]

It is easy to see that \(\tilde{J} \geq 0\) under Assumption A2. Further,

\[
\mathbb{E}\left[Y_0^{*\top}GY_0\right] = \mathbb{E}\left[\int_0^T \left(\langle A\tilde{Y} + B(v - v^*) + C_1\tilde{Z}_1 + C_2\tilde{Z}_2, X^*\rangle - \langle \tilde{Y}, A^\top X^* + HY^*\rangleight.ight.
\]

\[
- \langle \tilde{Z}_1, C_1^\top X^* + N_1Z_1^*\rangle - \langle \tilde{Z}_2, C_2^\top X^* + N_2Z_2^*\rangle\,dt\right]\]

\[
= \mathbb{E}\left[\int_0^T \left(\langle v - v^*, B^\top X^*\rangle - \langle \tilde{Y}, HY^*\rangle - \langle \tilde{Z}_1, N_1Z_1^*\rangle - \langle \tilde{Z}_2, N_2Z_2^*\rangle\right)\,dt\right].
\]

Thus, we have

\[
J(v) - J(v^*) = \mathbb{E}\left[\int_0^T \langle v - v^*, Rv^* + B^\top X^*\rangle\,dt\right] + \tilde{J} \geq 0.
\]

Then, \(v^*\) is an optimal control. \(\square\)
3 Decoupling stochastic Hamiltonian system \((2.5)\)

In this section, we use the decoupling method for general FBSDE introduced in [4] to solve stochastic Hamiltonian system \((2.5)\), which is an FBSDE with filtering. Different from the results in [8], we obtain three Riccati equations, an BSDE with filtering and an SDE with filtering. For simplicity of notation, we denote 
\[
\hat{\beta}_t = E[\beta_t|\mathcal{F}_t].
\]
To be precise, we assume that 
\[
Y = \Upsilon \hat{X} + \varphi,
\]
where \(\Upsilon\) is a differential and deterministic matrix-valued function with a terminal condition \(\Upsilon_T = 0\), and \(\varphi\) is a stochastic process satisfying the BSDE
\[
\begin{cases}
\frac{d\varphi}{dt} = \lambda dt + \eta_1 dW_1 + \eta_2 dW_2, \\
\varphi_T = \zeta,
\end{cases}
\]
for \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted processes \(\lambda, \eta_1\) and \(\eta_2\). According to Theorem 2.1 in Wang et al. [20] (see also Theorem 5.7 in Xiong [21] and Theorem 8.1 in Liptser and Shiryayev [22]), we have
\[
\begin{cases}
\frac{d\hat{X}}{dt} = - (A^T \hat{X} + H\hat{Y}) dt - (C_1^T \hat{X} + N_1 \hat{Z}_1) dW_1, \\
\hat{X}_0 = -G\hat{Y}_0.
\end{cases}
\]
Applying Itô formula to \((3.1)\), we get
\[
0 = dY - \dot{\Upsilon} \hat{X} dt - \Upsilon d\hat{X} - d\varphi
= (AY + Bv + C_1 Z_1 + C_2 Z_2) dt + Z_1 dW_1 + Z_2 dW_2 - \dot{\Upsilon} \hat{X} dt + \Upsilon \left( A^T \hat{X} + H\hat{Y} \right) dt
+ \Upsilon (C_1^T \hat{X} + N_1 \hat{Z}_1) dW_1 - \lambda dt - \eta_1 dW_1 - \eta_2 dW_2.
\]
This implies
\[
\begin{cases}
AY - BR^{-1}B^T \hat{X} + C_1 Z_1 + C_2 Z_2 - \dot{\Upsilon} \hat{X} + \Upsilon \left( A^T \hat{X} + H\hat{Y} \right) - \lambda = 0, \\
Z_1 + \Upsilon (C_1^T \hat{X} + N_1 \hat{Z}_1) - \eta_1 = 0, \\
Z_2 - \eta_2 = 0.
\end{cases}
\]
Assuming that \(I + \Upsilon N_1\) is invertible, we have
\[
\begin{cases}
Z_1 = \eta_1 - \tilde{\eta}_1 + (I + \Upsilon N_1)^{-1}(\tilde{\eta}_1 - \Upsilon C_1^T \hat{X}), \\
Z_2 = \eta_2.
\end{cases}
\]
Substituting \((3.1)\) and \((3.3)\) into the first equation in \((3.2)\), we obtain
\[
A(\Upsilon \hat{X} + \varphi) - BR^{-1}B^T \hat{X} + C_1(\eta_1 - \tilde{\eta}_1) + C_1(I + \Upsilon N_1)^{-1}(\tilde{\eta}_1 - \Upsilon C_1^T \hat{X})
+ C_2\eta_2 - \dot{\Upsilon} \hat{X} + \Upsilon A^T \hat{X} + \Upsilon H(\Upsilon \hat{X} + \varphi) - \lambda = 0.
\]
Then Υ satisfies a Riccati equation

\[ \begin{cases} 
\dot{\Upsilon} - \Upsilon A^\top - A\Upsilon - \Upsilon H\Upsilon + BR^{-1}B^\top + C_1(I + \Upsilon N_1)^{-1}YC_1^\top = 0, \\
\Upsilon_T = 0, 
\end{cases} \tag{3.4} \]

and ϕ satisfies a BSDE

\[ \begin{cases} 
\dot{\varphi} = [A\varphi + \Upsilon H\hat{\varphi} + C_1(\eta_1 - \hat{\eta}_1) + C_1(I + \Upsilon N_1)^{-1}\hat{\eta}_1 + C_2\eta_2]dt \\
+ \eta_1 dW_1 + \eta_2 dW_2, \quad \varphi_T = \zeta. \tag{3.5} 
\end{cases} \]

Riccati equation (3.4) admits a unique solution Υ ∈ L^∞(0, T; ℋ_1^2) under Assumptions A1 – A2 (see [9, 8]). Note that (3.5) is a BSDE with filtering, for which the solvability has not been given in literatures before. We will specified this problem in Section 4. In order to give the optimal control with a feedback representation, we conjecture that

\[ X = -\Gamma_1(Y - \hat{Y}) - \Gamma_2\hat{Y} - \psi, \tag{3.6} \]

where Γ_1 and Γ_2 are differential and deterministic matrix-valued functions with initial conditions Γ_10 = G and Γ_20 = G, respectively; ψ is a stochastic process satisfying an SDE

\[ \begin{cases} 
\dot{\psi} = \alpha_0 dt + \alpha_1 dW_1 + \alpha_2 dW_2, \\
\psi_0 = 0, 
\end{cases} \]

where \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are \{\mathcal{F}_t\}_{t \geq 0}-adapted processes. Note that

\[ \begin{cases} 
\dot{\hat{Y}} = \left( A\hat{Y} - BR^{-1}B^\top \hat{X} + C_1\hat{Z}_1 + C_2\hat{Z}_2 \right) dt + \hat{Z}_1 dW_1, \\
\hat{Y}_T = \hat{\zeta}, 
\end{cases} \]

where \( \hat{\zeta} = \mathbb{E}[\zeta | \mathcal{F}^W_T] \). Hence,

\[ \begin{cases} 
\dot{Y}(Y - \hat{Y}) = \left[ A(Y - \hat{Y}) + C_1(Z_1 - \hat{Z}_1) + C_2(Z_2 - \hat{Z}_2) \right] dt + (Z_1 - \hat{Z}_1)dW_1 + Z_2dW_2, \\
Y_T - \hat{Y}_T = \kappa - \hat{\kappa}. 
\end{cases} \]

Applying Itô formula to (3.6), we obtain

\[ 0 = dX + \dot{\Gamma}_1(Y - \hat{Y})dt + \Gamma_1 d(Y - \hat{Y}) + \dot{\Gamma}_2\hat{Y}dt + \Gamma_2 d\hat{Y} + d\psi \]
\[ = -\left( A^\top X + HY \right) dt - \left( C_1^\top X + N_1 Z_1 \right) dW_1 - \left( C_2^\top X + N_2 Z_2 \right) dW_2 \]
\[ + \dot{\Gamma}_1(Y - \hat{Y})dt + \Gamma_1 \left[ A(Y - \hat{Y}) + C_1(Z_1 - \hat{Z}_1) + C_2(Z_2 - \hat{Z}_2) \right] dt + \Gamma_1 (Z_1 - \hat{Z}_1)dW_1 \]
\[ + \Gamma_1 Z_2 dW_2 + \dot{\Gamma}_2 \hat{Y} dt + \Gamma_2 \left( A\hat{Y} - BR^{-1}B^\top \hat{X} + C_1\hat{Z}_1 + C_2\hat{Z}_2 \right) dt + \Gamma_2 \hat{Z}_1 dW_1 \]
\[ + \alpha_0 dt + \alpha_1 dW_1 + \alpha_2 dW_2. \]

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It yields
\[
\begin{cases}
- \left( A^\top X + HY \right) + \hat{\Gamma}_1(Y - \hat{Y}) + \Gamma_1 \left[ A(Y - \hat{Y}) + C_1(Z_1 - \hat{Z}_1) + C_2(Z_2 - \hat{Z}_2) \right] \\
\quad + \hat{\Gamma}_2 \hat{Y} + \Gamma_2 \left( A\hat{Y} - BR^{-1}B^\top \hat{X} + C_1\hat{Z}_1 + C_2\hat{Z}_2 \right) + \alpha_0 = 0, \\
- (C_1^\top X + N_1Z_1) + \Gamma_1(Z_1 - \hat{Z}_1) + \Gamma_2\hat{Z}_1 + \alpha_1 = 0, \\
- (C_2^\top X + N_2Z_2) + \Gamma_1Z_2 + \alpha_2 = 0.
\end{cases}
\]
Assuming that \( I + \Gamma_2Y \) is invertible, we arrive at
\[
\begin{align*}
\alpha_1 &= (N_1 - \Gamma_1)(\eta_1 - \hat{\eta}_1) + (N_1 - \Gamma_2)(I + \Upsilon N_1)^{-1} \left[ \hat{\eta}_1 + \Upsilon C_1^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}) \right] \\
&\quad - C_1^\top \Gamma_1(\varphi - \hat{\varphi}) - C_1^\top (\psi - \hat{\psi}) - C_1^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}), \\
\alpha_2 &= (N_2 - \Gamma_1)\eta_2 - C_2^\top \Gamma_1(\varphi - \hat{\varphi}) - C_2^\top (\psi - \hat{\psi}) - C_2^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}).
\end{align*}
\]
Further, it follows from (3.3) and (3.6) that
\[
\begin{align}
A^\top \Gamma_1(Y - \hat{Y}) + A^\top \Gamma_2\hat{Y} + A^\top \psi - HY + \hat{\Gamma}_1(Y - \hat{Y}) + \Gamma_1 A(Y - \hat{Y}) \\
+ \Gamma_1 C_1(\eta_1 - \hat{\eta}_1) + \Gamma_1 C_2(\eta_2 - \hat{\eta}_2) + \hat{\Gamma}_2 \hat{Y} + \Gamma_2 A \hat{Y} + \Gamma_2 BR^{-1}B^\top (\Gamma_2 \hat{Y} + \hat{\psi}) \\
+ \Gamma_2 C_1(I + \Upsilon N_1)^{-1} \left[ \hat{\eta}_1 + \Upsilon C_1^\top (\Gamma_2 \hat{Y} + \hat{\psi}) \right] + \Gamma_2 C_2 \hat{\eta}_2 + \alpha_0 = 0.
\end{align}
\]
Introduce
\[
\begin{align}
\begin{cases}
\hat{\Gamma}_1 + \Gamma_1 A + A^\top \Gamma_1 - H = 0, \\
\Gamma_{10} = G,
\end{cases}
\text{ (3.7)}
\end{align}
\]
and
\[
\begin{align}
\begin{cases}
d\psi = - \left[ A^\top \psi + \Gamma_2 BR^{-1}B^\top \hat{\psi} + \Gamma_2 C_1(I + \Upsilon N_1)^{-1} \left( \hat{\eta}_1 + \Upsilon C_1^\top \hat{\psi} \right) \\
\quad + \Gamma_2 C_2 \hat{\eta}_2 + \Gamma_1 C_1(\eta_1 - \hat{\eta}_1) + \Gamma_1 C_2(\eta_2 - \hat{\eta}_2) \right] dt \\
\quad + \left\{ (N_1 - \Gamma_1)(\eta_1 - \hat{\eta}_1) - C_1^\top \Gamma_1(\varphi - \hat{\varphi}) - C_1^\top (\psi - \hat{\psi}) \\
\quad + (N_1 - \Gamma_2)(I + \Upsilon N_1)^{-1} \left[ \hat{\eta}_1 + \Upsilon C_1^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}) \right] \\
\quad - C_1^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}) \right\} dW_1 \\
\quad + \left\{ (N_2 - \Gamma_1)\eta_2 - C_2^\top \Gamma_1(\varphi - \hat{\varphi}) + (\psi - \hat{\psi}) \right\} - C_2^\top (I + \Gamma_2Y)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi}) \right\} dW_2,
\end{cases}
\text{ (3.9)}
\end{align}
\]
There is a unique solution \( \Gamma_1 \in \mathcal{L}^\infty(0, T; S^n_{\alpha}) \) to Riccati equation (3.7), since Assumptions A1 and A2 hold (see Yong and Zhou [19]). Corollary 4.6 in Lim and Zhou [8] implies that (3.8) admits a unique solution \( \Gamma_2 \in \mathcal{L}^\infty(0, T; S^n_{\alpha}) \). Once \( Y, \Gamma_1, \Gamma_2 \) and the solution \( (\varphi, \eta_1, \eta_2) \) of (3.5) are known, the solvability of (3.9) will be obtained immediately.
4 Explicit representations of optimal control and optimal cost

Now we would like to give explicit formulas of optimal control and associated optimal cost in terms of Riccati equations (3.4), (3.7), (3.8), BSDE (3.5) and SDE (3.9). We first prove that (3.5) admits a unique solution. Consider a BSDE

\[
\begin{align*}
& dP = g(t, P, Q_1, Q_2, \hat{P}, \hat{Q}_1, \hat{Q}_2)dt + Q_1dW_1 + Q_2dW_2, \\
& P_T = \zeta,
\end{align*}
\]

where \( \hat{P}_t = \mathbb{E}[P_t|\mathcal{F}^W_t], \hat{Q}_{1t} = \mathbb{E}[Q_{1t}|\mathcal{F}^W_t], \hat{Q}_{2t} = \mathbb{E}[Q_{2t}|\mathcal{F}^W_t] \).

We assume that

**Assumption A3:** There exists a constant \( L \), such that, \( \mathbb{P} \)-a.s., for all \( t \in [0, T] \), \( p, q_1, q_2, \bar{p}, \bar{q}_1, \bar{q}_2, p', q'_1, q'_2, \bar{p}', \bar{q}'_1, \bar{q}'_2 \in \mathbb{R}^n \),

\[
\left| g(t, p, q_1, q_2, \bar{p}, \bar{q}_1, \bar{q}_2) - g(t, p', q'_1, q'_2, \bar{p}', \bar{q}'_1, \bar{q}'_2) \right| \\
\leq L(|p - p'| + |q_1 - q'_1| + |q_2 - q'_2| + |\bar{p} - \bar{p}'| + |\bar{q}_1 - \bar{q}'_1| + |\bar{q}_2 - \bar{q}'_2|).
\]

**Assumption A4:** \( g(\cdot, 0, 0, 0, 0, 0, 0) \in \mathcal{L}^2_F(0, T; \mathbb{R}^n) \).

**Lemma 4.1.** Let Assumptions A3 and A4 hold. For any \( \zeta \in \mathcal{L}^2_F(\Omega; \mathbb{R}^n) \), BSDE (4.1) admits a unique solution \((P, Q_1, Q_2) \in \mathcal{L}^2_F(0, T; \mathbb{R}^n) \times \mathcal{L}^2_F(0, T; \mathbb{R}^n) \times \mathcal{L}^2_F(0, T; \mathbb{R}^n) \).

**Proof.** We first introduce a norm on \( \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n}) \), which is equivalent to the canonical norm

\[
||u||_\delta = \left( \mathbb{E} \left[ \int_0^T |u_t|^2 e^{\delta t} dt \right] \right)^{\frac{1}{2}}, \delta > 0.
\]

The parameter \( \delta \) will be specified later. For any \((p, q_1, q_2) \in \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n})\), the following BSDE

\[
\begin{align*}
& dP = g(t, P, Q_1, Q_2, \hat{P}, \hat{Q}_1, \hat{Q}_2)dt + Q_1dW_1 + Q_2dW_2, \\
& P_T = \zeta
\end{align*}
\]

admits a unique solution \((P, Q_1, Q_2) \in \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n})\). We then introduce a mapping \((P, Q_1, Q_2) = I(p, q_1, q_2): \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n}) \to \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n})\) by

\[
\begin{align*}
& dP = g(t, P, Q_1, Q_2, \hat{P}, \hat{Q}_1, \hat{Q}_2)dt + Q_1dW_1 + Q_2dW_2, \\
& P_T = \zeta.
\end{align*}
\]

For any \((p, q_1, q_2), (p', q'_1, q'_2) \in \mathcal{L}^2_F(0, T; \mathbb{R}^{n+n+n})\), we denote \((P, Q_1, Q_2) = I(p, q_1, q_2), (P', Q'_1, Q'_2) = I(p', q'_1, q'_2), (\bar{p}, \bar{q}_1, \bar{q}_2) = (p - p', q_1 - q'_1, q_2 - q'_2)\) and \((\hat{P}, \hat{Q}_1, \hat{Q}_2) = (P - \hat{P}_t, \hat{Q}_{1t}, \hat{Q}_{2t})\).
$P', Z_1 - Z'_1, Q_2 - Q'_2$). Applying Itô formula to $|\widetilde{P}_t|^2 e^{\delta t}$ and taking conditional expectations, we get

$$
|\widetilde{P}_t|^2 + \mathbb{E} \left[ \int_t^T \delta e^{\delta(s-t)} |\widetilde{P}_s|^2 ds \right] + \mathbb{E} \left[ \int_t^T e^{\delta(s-t)} (|\widetilde{Q}_{1s}|^2 + |\widetilde{Q}_{2s}|^2) ds \right]
$$

$$
= 2 \mathbb{E} \left[ \int_t^T e^{\delta(s-t)} (P_s g(s, P', Q'_1, Q'_2, \widetilde{q}_1, \widetilde{q}_2) - g(s, P, Q_1, Q_2, \widetilde{p}, \widetilde{q}_1, \widetilde{q}_2)) ds \right]
$$

$$
\leq 2 \mathbb{E} \left[ \int_t^T e^{\delta(s-t)} \left( |\widetilde{P}_s| + |\widetilde{Q}_{1s}| + |\widetilde{Q}_{2s}| + |\widetilde{p}_s| + |\widetilde{q}_1| + |\widetilde{q}_2| \right) ds \right]
$$

$$
\leq \mathbb{E} \left[ \int_t^T e^{\delta(s-t)} \left( (2L + 4L^2 + \frac{\delta}{2}) |\widetilde{P}_s|^2 + \frac{1}{2} |\widetilde{Q}_{1s}|^2 + \frac{1}{2} |\widetilde{Q}_{2s}|^2 \right) ds \right]
$$

$$
+ \mathbb{E} \left[ \int_t^T e^{\delta(s-t)} \frac{6L^2}{\delta} \left( |\widetilde{p}_s|^2 + |\widetilde{q}_1|^2 + |\widetilde{q}_2|^2 \right) ds \right].
$$

Thus we have

$$
\left( \frac{\delta}{2} - 2L - 4L^2 \right) \mathbb{E} \left[ \int_0^T e^{\delta s} |\widetilde{P}_s|^2 ds \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{\delta s} \left( |\widetilde{Q}_{1s}|^2 + |\widetilde{Q}_{2s}|^2 \right) ds \right]
$$

$$
\leq \frac{6L^2}{\delta} \mathbb{E} \left[ \int_0^T e^{\delta s} \left( |\widetilde{p}_s|^2 + |\widetilde{q}_1|^2 + |\widetilde{q}_2|^2 \right) ds \right].
$$

Taking $\delta = 24L^2 + 4L + 1$, we arrive at

$$
\mathbb{E} \left[ \int_0^T e^{\delta s} \left( |\widetilde{P}_s|^2 + |\widetilde{Q}_{1s}|^2 + |\widetilde{Q}_{2s}|^2 \right) ds \right]
$$

$$
\leq \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{\delta s} \left( |\widetilde{p}_s|^2 + |\widetilde{q}_1|^2 + |\widetilde{q}_2|^2 \right) ds \right],
$$

which implies $||P, Q_1, Q_2||_{\delta} \leq \frac{1}{\sqrt{2}} ||\widetilde{P}, \widetilde{Q}_1, \widetilde{Q}_2||_{\delta}$. That is, $I$ is a contraction on $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n+n+n})$, endowed with the norm $||\cdot||_{\delta}$. According to the contraction mapping theorem, we know that there is a unique fixed point $(P, Q_1, Q_2) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n+n+n})$, such that $I(P, Q_1, Q_2) = (P, Q_1, Q_2)$, which is exactly the solution of (4.1). We now proceed to prove that $P \in S^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$. Using
Jensen inequality, Hölder’s inequality and Burkholder-Davis-Gundy’s inequality yields

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |P_t|^2 \right] \leq 4 \mathbb{E} \left[ |\zeta|^2 \right] + 4 \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T g(s, P, Q_1, Q_2, \tilde{Q}, \tilde{Q}_1, \tilde{Q}_2) ds \right|^2 \right] \\
+ 4 \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T Q_{1s} dW_{1s} \right|^2 \right] + 4 \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T Q_{2s} dW_{2s} \right|^2 \right] \\
\leq 4 \mathbb{E} \left[ |\zeta|^2 \right] + 4T \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_t^T |g(s, P, Q_1, Q_2, \tilde{P}, \tilde{Q}_1, \tilde{Q}_2)|^2 ds \right) \right] \\
+ 16 \mathbb{E} \left[ \int_0^T |Q_{1t}|^2 dt \right] + 16 \mathbb{E} \left[ \int_0^T |Q_{2t}|^2 dt \right] < \infty.
\]

Therefore, we obtain \( P \in \mathcal{S}^2_T(0, T; \mathbb{R}^n) \).

**Remark 4.1.** Equation (3.5) is a linear BSDE with filtering, where the generator satisfies Assumptions A3–A4. Then it follows that (3.5) admits a unique solution \((\varphi, \eta_1, \eta_2) \in \mathcal{S}_F^2(0, T; \mathbb{R}^n) \times \mathcal{L}_F^2(0, T; \mathbb{R}^n) \times \mathcal{L}_F^2(0, T; \mathbb{R}^n)\).

We have the following theorem which specifies the solvability of stochastic Hamiltonian system (2.5) and gives some relations between the forward component and the backward components.

**Theorem 4.1.** Under Assumptions A1 – A2, stochastic Hamiltonian system (2.5) admits a unique solution \((X, Y, Z_1, Z_2, \nu)\). Moreover, we have the following relations

\[
\begin{cases}
Y = \Upsilon \hat{X} + \varphi, \\
Z_1 = \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1}(\hat{\eta}_1 - \Upsilon C_1^T \hat{X}), \\
Z_2 = \eta_2, \\
\nu = -R^{-1}B^T \hat{X}, \\
Y_0 = (I + \Upsilon_0 G)^{-1} \varphi_0,
\end{cases}
\]

where \(\Upsilon\) and \((\varphi, \eta_1, \eta_2)\) are solutions to (3.4) and (3.5), respectively.

**Proof.** Consider the following SDE with filtering

\[
\begin{cases}
d\hat{X} = - \left[ A^T \hat{X} + H(\Upsilon \hat{X} + \varphi) \right] dt \\
- \left[ C_1^T \hat{X} + N_1(\eta_1 - \hat{\eta}_1) + N_1(I + \Upsilon N_1)^{-1}(\tilde{\eta}_1 - \Upsilon C_1^T \hat{X}) \right] dW_1 \\
- \left( C_2^T \hat{X} + N_2 \eta_2 \right) dW_2,
\end{cases}
\]

\(\hat{X}_0 = - (I + G\Upsilon_0)^{-1} G \varphi_0\).
where $\Upsilon$ and $(\varphi, \eta_1, \eta_2)$ are solutions of (3.4) and (3.5), respectively. According to Theorem 2.1 in Wang et al. [20], we get

$$\begin{cases}
d\hat{X} = - \left( A^\top \hat{X} + H \Upsilon \hat{X} + H \varphi \right) dt \\
- \left[ (I + N_1 \Upsilon)^{-1} C_1^\top \hat{X} + N_1 (I + \Upsilon N_1)^{-1} \hat{\eta}_1 \right] dW_1,
\end{cases} \quad (4.3)$$

By using Itô formula, $\hat{Y}$ satisfies

$$d\hat{Y} = \left[ \Upsilon A^\top + A \Upsilon + \Upsilon H \Upsilon - BR^{-1} B^\top - C_1 (I + \Upsilon N_1)^{-1} \Upsilon C_1^\top \right] \hat{X} dt
- \Upsilon \left( A^\top \hat{X} + H \Upsilon \hat{X} + H \varphi \right) dt - \Upsilon \left[ (I + N_1 \Upsilon)^{-1} C_1^\top \hat{X} + N_1 (I + \Upsilon N_1)^{-1} \hat{\eta}_1 \right] dW_1
+ A \varphi + \Upsilon H \varphi + C_1 (\eta_1 - \hat{\eta}_1) + C_1 (I + \Upsilon N_1)^{-1} \hat{\eta}_1 + C_2 \eta_2 \right] dt + \eta_1 dW_1 + \eta_2 dW_2
= \left[ A \hat{Y} - BR^{-1} B^\top \hat{X} + \Upsilon \left( \eta_1 - \hat{\eta}_1 \right) + C_1 (I + \Upsilon N_1)^{-1} (\hat{\eta}_1 - \Upsilon C_1^\top \hat{X}) + C_2 \eta_2 \right] dt
+ \left[ \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1} (\hat{\eta}_1 - \Upsilon C_1^\top \hat{X}) \right] dW_1 + \eta_2 dW_2,$$

with an initial condition $Y_0 = (I + \Upsilon_0 G)^{-1} \varphi_0$. Defining

$$\begin{cases}
\hat{Z}_1 = \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1} (\hat{\eta}_1 - \Upsilon C_1^\top \hat{X}), \\
\hat{Z}_2 = \eta_2, \\
\hat{\nu} = -R^{-1} B^\top \hat{X},
\end{cases}$$

It is obvious that $(\hat{X}, \hat{Y}, \hat{Z}_1, \hat{Z}_2, \hat{\nu})$ is a solution to stochastic Hamiltonian system (2.5).

We now turn to prove the uniqueness. Suppose that equation (2.5) admits two solutions $(X, Y, Z_1, Z_2, \nu)$ and $(X', Y', Z'_1, Z'_2, \nu')$, respectively. Let $(\hat{X}, \hat{Y}, \hat{Z}_1, \hat{Z}_2, \hat{\nu}) = (X - X', Y - Y', Z_1 - Z'_1, Z_2 - Z'_2, \nu - \nu')$. Thus $(\hat{X}, \hat{Y}, \hat{Z}_1, \hat{Z}_2, \hat{\nu})$ satisfies

$$\begin{cases}
d\hat{Y} = \left( A \hat{Y} + B \hat{\nu} + C_1 \hat{Z}_1 + C_2 \hat{Z}_2 \right) dt + \hat{Z}_1 dW_1 + \hat{Z}_2 dW_2, \\
d\hat{X} = - \left( A^\top \hat{X} + H \hat{Y} \right) dt - \left( C_1^\top \hat{X} + N_1 \hat{Z}_1 \right) dW_1 - \left( C_2^\top \hat{X} + N_2 \hat{Z}_2 \right) dW_2, \\
\hat{Y}_0 = 0, \quad \hat{X}_0 = -G \hat{Y}_0, \\
E[R_t \hat{\nu}_t + B_t^\top \hat{X}_t | \mathcal{F}_t^{W_1}] = 0.
\end{cases}$$
Applying Itô formula to \( \tilde{Y}^\top \tilde{X} \), we obtain

\[
\mathbb{E} \left[ \tilde{Y}_0^\top G \tilde{Y}_0 \right] = -\mathbb{E} \left[ \int_0^T \tilde{Y}^\top \left( A^\top \tilde{X} + H \tilde{Y} \right) dt \right] - \mathbb{E} \left[ \int_0^T \tilde{Z}_1^\top \left( C_1^\top \tilde{X} + N_1 \tilde{Z}_1 \right) dt \right] - \mathbb{E} \left[ \int_0^T \tilde{Z}_2^\top \left( C_2^\top \tilde{X} + N_2 \tilde{Z}_2 \right) dt \right]
\]

We adopt the same procedure as in the proof of Theorem 2.2. Since \( G, H, R, N_1, N_2 \) satisfy Assumption A2, it follows that

\[
\mathbb{E} \left[ \int_0^T \tilde{X}^\top R \tilde{X} dt \right] = 0.
\]

Recalling \( R \) is uniformly positive, it yields

\[
B_1^\top \tilde{X}_t = 0, \quad \text{a.e. } t \in [0, T], \quad \mathbb{P} - \text{a.s.}
\]

With the equality, \((\tilde{Y}, \tilde{Z}_1, \tilde{Z}_2)\) satisfies

\[
\begin{align*}
\begin{cases}
  d\tilde{Y} = \left( A \tilde{Y} + C_1 \tilde{Z}_1 + C_2 \tilde{Z}_2 \right) dt + \tilde{Z}_1 dW_1 + \tilde{Z}_2 dW_2, \\
  \tilde{Y}_T = 0.
\end{cases}
\end{align*}
\]

(4.4)

It is easy to see that (4.4) admits a unique solution \((\tilde{Y}, \tilde{Z}_1, \tilde{Z}_2) \equiv 0\). Then

\[
\begin{align*}
\begin{cases}
  d\tilde{X} = -A^\top \tilde{X} dt - C_1^\top \tilde{X} dW_1 - C_2^\top \tilde{X} dW_2, \\
  \tilde{X}_0 = -G \tilde{Y}_0.
\end{cases}
\end{align*}
\]

Hence it follows from the uniqueness of solution that \( \tilde{X} \equiv 0 \). The proof is completed. \( \square \)

To summarize the above analysis, we establish the following main result.

**Theorem 4.2.** Let Assumptions A1 – A2 hold and let \( \zeta \in \mathcal{L}_F^2(\Omega; \mathbb{R}^n) \) be given. Let \( \tilde{Y}, \Gamma_1, \Gamma_2 \) be the solutions of Riccati equations (3.4), (3.7) and (3.8), respectively. Let \((\varphi, \eta_1, \eta_2)\) and \(\psi\) be the solutions of (3.5) and (3.9), respectively. Then the BSDE with filtering

\[
\begin{align*}
\begin{cases}
  dY = \left( AY + BR^{-1} B^\top \Gamma_2 \tilde{Y} + BR^{-1} B^\top \psi + C_1 Z_1 + C_2 Z_2 \right) dt + Z_1 dW_1 + Z_2 dW_2, \\
  Y_T = \zeta
\end{cases}
\end{align*}
\]

admits a unique solution \((Y, Z_1, Z_2)\). By defining

\[
\begin{align*}
\begin{cases}
  X = -\Gamma_1 (Y - \tilde{Y}) - \Gamma_2 \tilde{Y} - \psi, \\
  v = R^{-1} B^\top \Gamma_2 \tilde{Y} + R^{-1} B^\top \psi,
\end{cases}
\end{align*}
\]


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the 5-tuple \((X,Y,Z_1,Z_2,v)\) is an adapted solution to FBSDE (2.5) and \(v\) is an optimal control of Problem BLQ. The corresponding optimal cost is

\[
J(v) = \frac{1}{2} \mathbb{E} \left[ \langle \zeta, \Sigma_T \hat{\zeta} \rangle \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle H\varphi, \varphi \rangle - \langle \varphi, H\varphi \rangle \right) dt \right] \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle N_1(I + \Upsilon N_1)^{-1} - \Sigma \rangle \hat{\eta}_1, \hat{\eta}_1 \right) \right] \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle (N_1(\eta_1 - \hat{\eta}_1), \eta_1 - \hat{\eta}_1) + \langle N_2\eta_2, \eta_2 \rangle \right) dt \right] \\
- \mathbb{E} \left[ \int_0^T \left( \varphi, \Sigma \left( C_1(I + \Upsilon N_1)^{-1}\hat{\eta}_1 + C_2\hat{\eta}_2 \right) \right) dt \right]
\]

(4.5)

where \(\Sigma\) is the solution of

\[
\begin{align*}
\dot{\Sigma} + \Sigma(A + \Upsilon H) + (A + \Upsilon H)^\top \Sigma - H &= 0, \\
\Sigma_0 &= G(I + \Upsilon_0 G)^{-1}.
\end{align*}
\]

**Proof.** We need only to prove (4.5). Substituting (3.1), (3.3) into the cost functional, we derive

\[
J(v) = \frac{1}{2} \langle G(I + \Upsilon_0 G)^{-1} \varphi_0, (I + \Upsilon_0 G)^{-1} \varphi_0 \rangle + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle \Upsilon \hat{X} + \varphi, H(\Upsilon \hat{X} + \varphi) \rangle \right) dt \right] \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle R^{-1}B^\top \hat{X}, B^\top \hat{X} \rangle \right) dt \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle \eta_2, N_2\eta_2 \rangle \right) dt \right] \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1}(\hat{\eta}_1 - \Upsilon C_1^\top \hat{X}), N_1 \left[ \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1}(\hat{\eta}_1 - \Upsilon C_1^\top \hat{X}) \right] \right) \right] \\
= \frac{1}{2} \langle G(I + \Upsilon_0 G)^{-1} \varphi_0, (I + \Upsilon_0 G)^{-1} \varphi_0 \rangle \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle \hat{X}, \Upsilon H \varphi + BR^{-1}B^\top + C_1(I + \Upsilon N_1)^{-1}\Upsilon N_1 \Upsilon (I + N_1 \Upsilon)^{-1}C_1^\top \hat{X} \rangle \right) dt \right] \\
+ \mathbb{E} \left[ \int_0^T \langle \Upsilon H \varphi, \hat{X} \rangle \right] \\
- \mathbb{E} \left[ \int_0^T \langle \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1}\hat{\eta}_1, N_1 \left[ \eta_1 - \hat{\eta}_1 + (I + \Upsilon N_1)^{-1}\hat{\eta}_1 \right] \right] \\
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle \eta_2, N_2\eta_2 \rangle + \langle \varphi, H\varphi \rangle \right) dt \right].
\]
Applying Itô formula to $\langle \tilde{X}, \Upsilon \tilde{X} \rangle$, we have

$$
\langle (I + G\Upsilon_0)^{-1}G\varphi_0, \Upsilon_0(I + G\Upsilon_0)^{-1}G\varphi_0 \rangle
\quad
= -\mathbb{E} \left[ \int_0^T \left\langle \tilde{X}, \Upsilon [(A + \Upsilon H)^\top \tilde{X} + Q\tilde{\varphi}] \right\rangle dt \right] - \mathbb{E} \left[ \int_0^T \left\langle (A + \Upsilon H)^\top \tilde{X} + H\varphi, \Upsilon \tilde{X} \right\rangle dt \right]
\quad
+ \mathbb{E} \left[ \int_0^T \left\langle \tilde{X}, (\Upsilon A^T + A\Upsilon + \Upsilon HY - BR^{-1}B^T - C_1(I + \Upsilon N_1)^{-1}YC_1^\top) \tilde{X} \right\rangle dt \right]
\quad
+ \mathbb{E} \left[ \int_0^T \left\langle (I + N_1\Upsilon)^{-1}C_1^\top \tilde{X} + N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1, \Upsilon [I + N_1\Upsilon)^{-1}C_1^\top \tilde{X} + N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1] \right\rangle dt \right]
\quad
= -\mathbb{E} \left[ \int_0^T \left\langle \tilde{X}, [\Upsilon HY + BR^{-1}B^T + C_1(I + \Upsilon N_1)^{-1}\Upsilon N_1\Upsilon(I + N_1\Upsilon)^{-1}C_1^\top] \tilde{X} \right\rangle dt \right]
\quad
+ \mathbb{E} \left[ \int_0^T \left\langle N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1, \Upsilon N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1 \right\rangle dt \right]
\quad
+ 2\mathbb{E} \left[ \int_0^T \left\langle (I + N_1\Upsilon)^{-1}C_1^\top \tilde{X}, \Upsilon N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1 \right\rangle dt \right] - 2\mathbb{E} \left[ \int_0^T \left\langle \Upsilon H\varphi, \tilde{X} \right\rangle dt \right].
$$

With the equality, we derive

$$
J(v) = \frac{1}{2} \langle (I + G\Upsilon_0)^{-1}G\varphi_0, \varphi_0 \rangle + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left\langle H\varphi, \varphi \right\rangle dt \right]
\quad
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle N_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1, \tilde{\eta}_1 \rangle + \langle N_1(\eta_1 - \tilde{\eta}_1), \eta_1 - \tilde{\eta}_1 \rangle + \langle N_2\eta_2, \eta_2 \rangle \right) dt \right].
$$

Recalling that $\varphi$ satisfies (3.5) and applying Itô formula to $\langle \tilde{\varphi}, \Sigma \tilde{\varphi} \rangle$, we have

$$
\langle (I + G\Upsilon_0)^{-1}G\varphi_0, \varphi_0 \rangle = \mathbb{E} \left[ \langle \tilde{\zeta}, \Sigma T\tilde{\zeta} \rangle \right] - \mathbb{E} \left[ \int_0^T \left( \langle \tilde{\varphi}, H\tilde{\varphi} \rangle + \langle \tilde{\eta}_1, \Sigma \tilde{\eta}_1 \rangle \right) dt \right]
\quad
- 2\mathbb{E} \left[ \int_0^T \left\langle \tilde{\varphi}, \Sigma [C_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1 + C_2\tilde{\eta}_2] \right\rangle dt \right].
$$

We obtain

$$
J(v) = \frac{1}{2} \mathbb{E} \left[ \langle \tilde{\zeta}, \Sigma T\tilde{\zeta} \rangle \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle H\varphi, \varphi \rangle - \langle \tilde{\varphi}, H\tilde{\varphi} \rangle \right) dt \right]
\quad
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left[ N_1(I + \Upsilon N_1)^{-1} - \Sigma \right] \tilde{\eta}_1, \tilde{\eta}_1 \right] \right) dt \right]
\quad
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle N_1(\eta_1 - \tilde{\eta}_1), \eta_1 - \tilde{\eta}_1 \rangle + \langle N_2\eta_2, \eta_2 \rangle \right) dt \right]
\quad
- \mathbb{E} \left[ \int_0^T \left\langle \tilde{\varphi}, \Sigma [C_1(I + \Upsilon N_1)^{-1}\tilde{\eta}_1 + C_2\tilde{\eta}_2] \right\rangle dt \right].
$$

Then our claims follow.

\[\square\]

**Remark 4.2.** When we consider the complete information case, i.e., $W_2$ disappears in (2.1). Let $\zeta$ be an $\mathcal{F}_1^{W_1}$-measurable square integrable random variable. Let $v$ be an $\{\mathcal{F}_t^{W_1}\}_{t \geq 0}$-adapted
and square integrable stochastic process. Then Theorem 4.2 coincides with Theorem 3.2 in Lim and Zhou [8].

**Remark 4.3.** In Huang et al. [23], an optimal control for Problem BLQ with feedback representation is given. We point out that their results rely on the condition that the solution of (3.5) satisfies $\eta_2 = 0$.

## 5 One-dimensional case

In this section, we consider two scalar-valued backward LQ problems with partial information and give more detailed analyses. In the case of $H = N_1 = 0$, we work out an explicit control problem and show the detailed procedure to obtain the feedback representation of optimal control using our theoretical results. In the case of $C_2 = 0$, we give some numerical simulations to illustrate our theoretical results, since we can not obtain explicit solutions of related stochastic Hamiltonian system and Riccati equation.

### 5.1 Special case: $H = N_1 = 0$

Under Assumptions $A_1$ and $A_2$, let all the coefficients of (2.1) and (2.2) are constants, and

$$\zeta = e^{(a_{12}b_1 - \frac{1}{2}c_1^2)T + 6W_1 + cW_2}.$$

In this case, (2.1) is given by

$$\begin{cases}
    dY_t = (AY_t + Bu_t + C_1Z_{1t} + C_2Z_{2t}) dt + Z_{1t}dW_{1t} + Z_{2t}dW_{2t}, & t \in [0, T], \\
    Y_T = \zeta.
\end{cases}$$

The cost functional takes the form of

$$J(v) = \frac{1}{2}E \left[ GY_T^2 + \int_0^T (Rv_t^2 + N_2Z_{2t}^2) dt \right].$$

Then Problem BLQ is stated as follows.

**Problem BLQA.** Find a $v^* \in \mathcal{V}[0, T]$ such that

$$J(v^*) = \inf_{v \in \mathcal{V}[0, T]} J(v),$$

where the admissible control set is given by

$$\mathcal{V}[0, T] = \left\{ v : [0, T] \times \Omega \to \mathbb{R} | v is \ \{F_t\}_{t \geq 0}-adapted, \ E \left[ \int_0^T v_t^2 dt \right] < \infty \right\}.$$
The corresponding stochastic Hamiltonian system reads

\[
\begin{align*}
&dY_t = (AY_t + Bv_t + C_1 Z_{1t} + C_2 Z_{2t}) dt + Z_{1d}W_{1t} + Z_{2d}W_{2t}, \\
&dX_t = -AX_t dt - C_1 X_t dW_{1t} - (C_2 X_t + N_2 Z_{2t}) dW_{2t}, \\
&Y_T = \zeta, \quad X_0 = -GY_0, \\
&E[Rv_t + BX_t|F^W_t] = 0.
\end{align*}
\] (5.1)

We introduce

\[
\begin{align*}
\dot{\Upsilon}_t - (2A - C_1^2)\Upsilon_t + \frac{B^2}{R} = 0, \\
\Upsilon_T = 0,
\end{align*}
\]

and

\[
\begin{align*}
&d\varphi_t = (A\varphi_t + C_1 \eta_{1t} + C_2 \eta_{2t}) dt + \eta_{1t} dW_{1t} + \eta_{2t} dW_{2t}, \\
&\varphi_T = \zeta.
\end{align*}
\]

It is easy to see that

\[
\Upsilon_t = \begin{cases}
\frac{B^2}{R(2A - C_1^2)} \left(1 - e^{(2A - C_1^2)(t-T)}\right), & 2A - C_1^2 \neq 0, \\
\frac{B^2}{R(2A - C_1^2)} (t-T), & 2A - C_1^2 = 0,
\end{cases}
\]

and

\[
\begin{align*}
\varphi_t &= \exp \left[(a - bC_1 - cC_2 - A)T \right. \\
&\quad + (A + bC_1 + cC_2 - \frac{1}{2}b^2 - \frac{1}{2}c^2)t + b W_{1t} + c W_{2t}\right], \\
\eta_{1t} &= b\varphi_t, \\
\eta_{2t} &= c\varphi_t.
\end{align*}
\] (5.2)

Taking \( t = 0 \) in (5.2), we have

\[
\varphi_0 = \exp \left[(a - bC_1 - cC_2 - A)T \right].
\]

Then it follows from (5.1) and (5.2) that

\[
\begin{align*}
&dX_t = -AX_t dt - C_1 X_t dW_{1t} - (C_2 X_t + N_2 \eta_{2t}) dW_{2t}, \\
&X_0 = -(I + G\Upsilon_0)^{-1}G\varphi_0,
\end{align*}
\]

which admits a unique solution

\[
X_t = \Psi_t \left[ X_0 - \int_0^t \Psi_s^{-1} C_2 N_2 \eta_{2s} ds - \int_0^t \Psi_s^{-1} N_2 \eta_{2s} dW_{2s} \right],
\] (5.3)

with

\[
\Psi_t = \exp \left[-(A + \frac{1}{2}C_1^2 + \frac{1}{2}C_2^2)t - C_1 W_{1t} - C_2 W_{2t}\right].
\]
Further,

\[ v_t = -R^{-1}B\hat{X}_t = -R^{-1}BX_0\exp\left[-\left(A + \frac{1}{2}C_2^2\right)t - C_1W_t\right]. \tag{5.4} \]

Theorem 2.2 implies that \( v \) given by (5.4) is an optimal control of Problem BLQA.

In the following, we aim to derive a feedback representation of \( v \). For this end, we introduce

\[
\begin{align*}
\dot{\Gamma}_{1t} + 2A\Gamma_{1t} &= 0, \\
\Gamma_{10} &= G, \\
\end{align*}
\tag{5.5}
\]

\[
\begin{align*}
\dot{\Gamma}_{2t} + 2A\Gamma_{2t} + \left(\frac{B^2}{R} + C_1^2\Upsilon_t\right)\Gamma_{2t}^2 &= 0, \\
\Gamma_{20} &= G, \\
\end{align*}
\tag{5.6}
\]

and

\[
\begin{align*}
d\psi_t = & -\left[A\psi_t + \left(\frac{B^2\Gamma_{2t}}{R} + C_1^2\Gamma_{2t}\Upsilon_t\right)\hat{\psi}_t + C_1\Gamma_{2t}\hat{\eta}_{1t}\right] \\
&+ C_1\Gamma_{1t}(\eta_{1t} - \hat{\eta}_{1t}) + C_2\Gamma_{1t}(\eta_{2t} - \hat{\eta}_{2t}) + C_2\Gamma_{1t}\hat{\eta}_{2t} \\
& - \left[\Gamma_{1t}(\eta_{1t} - \hat{\eta}_{1t}) + \Gamma_{2t}\hat{\eta}_{1t} + C_1(\Gamma_{2t}\hat{\varphi}_t + \hat{\psi}_t) \\
&+ C_1\Gamma_{1t}(\varphi_t - \hat{\varphi}_t) + C_1(\psi_t - \hat{\psi}_t)\right]dW_{1t} \\
&+ C_2(1 + \Gamma_{2t}\Upsilon_t)^{-1}(\Gamma_{2t}\hat{\varphi}_t + \hat{\psi}_t) + C_2(\psi_t - \hat{\psi}_t)\right]dW_{2t}, \\
\end{align*}
\]

\( \psi_0 = 0. \)

Solving (5.5) and (5.6), we get

\[ \Gamma_{1t} = Ge^{-2At}, \]

and

\[ \Gamma_{2t} = \frac{G\exp(-2At)}{1 + G\int_0^t \exp(-2As)(\frac{B^2}{R} + C_1^2\Upsilon_s)ds}, \]

respectively.

According to Theorem 2.1 in Wang et al. [20], we have

\[
\begin{align*}
d\hat{\psi}_t = & -\left(A_t\hat{\psi}_t + B_t\right)dt - \left(C_1\hat{\psi}_t + D_t\right)dW_{1t}, \\
\hat{\psi}_0 = 0, \\
\end{align*}
\]

where

\[
\begin{align*}
A_t &= A + \frac{B^2\Gamma_{2t}}{R} + C_1^2\Gamma_{2t}\Upsilon_t, \\
B_t &= C_1\Gamma_{2t}\hat{\eta}_{1t} + C_2\Gamma_{2t}\hat{\eta}_{2t}, \\
D_t &= \Gamma_{2t}\hat{\eta}_{1t} + C_1\Gamma_{2t}\hat{\varphi}_t. \\
\end{align*}
\]
Similarly, we derive
\[ \hat{\psi}_t = \Phi_t \left[ - \int_0^t \Phi_s^{-1} \left( B_s + C_1 D_s \right) ds - \int_0^t \Phi_s^{-1} D_s dW_{1s} \right], \]
where
\[ \Phi_t = \exp \left[ - \int_0^t (A_s + \frac{1}{2} C_1^2) ds - C_1 W_{1t} \right]. \]

Then Theorem 4.2 implies that (5.4) admits a feedback representation below
\[ v_t = R^{-1} B_2 \hat{Y}_t + R^{-1} B \hat{\psi}_t, \]
where \( Y_t \) satisfies
\[
\begin{cases}
    dY_t = (A Y_t + B^2 R^{-1} \Gamma_{2t} \hat{Y}_t + B^2 R^{-1} \hat{\psi}_t + C_1 Z_{1t} + C_2 Z_{2t}) dt + Z_{1t} dW_{1t} + Z_{2t} dW_{2t}, \\
    Y_T = \zeta.
\end{cases}
\]
(5.7)

The corresponding optimal cost is
\[ J(v) = \frac{G \varphi_0^2}{2 + 2GY_0} + \frac{1}{2} \mathbb{E} \left[ \int_0^T \langle N_2 \eta_{2t}, \eta_{2t} \rangle dt \right]. \]

**Remark 5.1.** It follows from Theorem 4.1 that the solution of (5.7) is given by
\[
\begin{cases}
    Y_t = \Upsilon_t \hat{X}_t + \varphi_t, \\
    Z_{1t} = \eta_{1t} - C_1 \Upsilon_t \hat{X}_t, \\
    Z_{2t} = \eta_{2t}.
\end{cases}
\]

Here, \( X \) and \((\varphi, \eta_1, \eta_2)\) are given by (5.3) and (5.2), respectively. Note that equation (5.7) is a BSDE with filtering, which is difficult to obtain the explicit solution in general.

### 5.2 Special case: \( C_2 = 0 \)

In this case, (2.1) is written as
\[
\begin{cases}
    dY = (A Y + Bv + C_1 Z_1) dt + Z_1 dW_1 + Z_2 dW_2, t \in [0, T], \\
    Y_T = \zeta.
\end{cases}
\]

Cost functional (2.2) takes the form of
\[ J(v) = \frac{1}{2} \mathbb{E} \left[ GY_0^2 + \int_0^T \left( HY^2 + Rv^2 + N_1 Z_1^2 + N_2 Z_2^2 \right) dt \right]. \]

Then Problem BLQ is formulated as follows.

**Problem BLQB.** Find a \( v^* \in \mathcal{V}[0, T] \) such that
\[ J(v^*) = \inf_{v \in \mathcal{V}[0, T]} J(v). \]
The corresponding stochastic Hamiltonian system reads

\[
\begin{align*}
    dY &= (AY + Bv + C_1Z_1) \, dt + Z_1dW_1 + Z_2dW_2, \\
    dX &= -(AX + HY) \, dt - (C_1X + N_1Z_1) \, dW_1 - N_2Z_2dW_2, \\
    Y &= \zeta, \quad X_0 = -GY_0, \\
    \mathbb{E}[R_tv_t + B_tX_t|\mathcal{F}_t^{W_1}] &= 0.
\end{align*}
\]

According to Theorem 4.2, the optimal control is

\[
v_t = R_t^{-1}B_t\Gamma_2t\hat{Y}_t + R_t^{-1}B_t\hat{\psi}_t
\]

with

\[
\begin{align*}
    dY &= \left[AY + B(R^{-1}B\Gamma_2t\hat{Y} + R^{-1}B\hat{\psi}) + C_1Z_1\right] \, dt + Z_1dW_1 + Z_2dW_2, \\
    \hat{Y}_T &= \zeta.
\end{align*}
\]

The corresponding Riccati equations are

\[
\begin{align*}
    \dot{\Upsilon} - 2A\Upsilon - HT^2 + B^2R^{-1} + C_1^2\Upsilon(1 + \Upsilon N_1)^{-1} &= 0, \\
    \Upsilon_T &= 0, \\
    \dot{\Gamma}_1 + 2A\Gamma_1 - H &= 0, \\
    \Gamma_{10} &= G, \\
    \dot{\Gamma}_2 + 2A\Gamma_2 + [B^2R^{-1} + C_1^2\Upsilon(1 + \Upsilon N_1)^{-1}]\Gamma_2^2 - H &= 0, \\
    \Gamma_{20} &= G.
\end{align*}
\]

Equations (3.5) and (3.9) are reduced to

\[
\begin{align*}
    d\varphi &= \left[A\varphi + \Upsilon H\hat{\varphi} + C_1(\eta_1 - \hat{\eta}_1) + C_1(1 + \Upsilon N_1)^{-1}\hat{\eta}_1\right] \, dt + \eta_1dW_1 + \eta_2dW_2, \\
    \varphi_T &= \zeta,
\end{align*}
\]

and

\[
\begin{align*}
    d\psi &= - \left\{A\psi + \Gamma_2B^2R^{-1}\hat{\psi} + \Gamma_1C_1(\eta_1 - \hat{\eta}_1) + \Gamma_2C_1(1 + \Upsilon N_1)^{-1}\left[\hat{\eta}_1 + \Upsilon C_1\hat{\psi}\right]\right\} \, dt \\
    &\quad + \left[(N_1 - \Gamma_1)(\eta_1 - \hat{\eta}_1) + (N_1 - \Gamma_2)(I + \Upsilon N_1)^{-1}\hat{\eta}_1 - C_1(1 + \Upsilon N_1)^{-1}(\Gamma_2\hat{\varphi} + \hat{\psi})
    - C_1\Gamma_1(\varphi - \hat{\varphi}) - C_1(\psi - \hat{\psi})\right] \, dW_1 + (N_2 - \Gamma_1)\eta_2dW_2, \\
    \psi(0) &= 0,
\end{align*}
\]

respectively.

Note that it is hard to obtain a more explicit expression of \(v\) due to the complexity of (5.8) and (5.9). In the following, we hope to give numerical solutions for this case with certain particular coefficients. Let \(T = 1, A = 2, B = 3t + 2, C_1 = t - 2, G = 2, H = e^{-0.05t}, R = \)
Figure 1: The solutions of $\Upsilon, \Gamma_1, \Gamma_2$

Figure 2: Numerical simulations of $\hat{\varphi}, \hat{\eta}_1$ and $\hat{\psi}$
Figure 3: Numerical simulations of $\tilde{Y}$ and $\tilde{Z}_1$

Figure 4: Numerical simulation of $v$
2t + 1, N_1 = t(T - t), N_2 = 2 and \( \zeta = T + \sin(W_{1T}) + \cos(2W_{2T}) \). Applying Runge-Kutta method, we generate the dynamic simulations of \( \Upsilon, \Gamma_1 \) and \( \Gamma_2 \), shown in Figure 1.

It seems that there is no existing literature on numerical methods of equation (5.9), which is a BSDE with filtering. Using Theorem 2.1 in Wang et al. [20] again, we get

\[
\begin{align*}
\hat{\varphi}'' &= [(A + Y Q)\hat{\varphi} + C_1(1 + Y N_1)^{-1}\hat{\eta}_1] dt + \hat{\eta}_1 dW_1, \\
\hat{\varphi}_T &= \zeta, \\
\hat{\psi}' &= \left\{ [A + \Gamma_2 B^2 R^{-1} + \Gamma_2 C_2^2 Y (1 + Y N_1)^{-1}] \hat{\psi} + \Gamma_2 C_1 (1 + Y N_1)^{-1} \hat{\eta}_1 \right\} dt \\
&\quad + \left[(N_1 - \Gamma_2)(I + Y N_1)^{-1} \hat{\eta}_1 - C_1 (1 + Y N_1)^{-1} \hat{\psi}\right] dW_1, \\
\hat{\psi}_0 &= 0.
\end{align*}
\]

Applying the numerical method introduced in Ma et al. [21], we generate the dynamic simulations of \( \hat{\varphi} \) and \( \hat{\eta}_1 \), shown in Fig. 2. For more information about numerical methods for BSDEs, please refer to Peng and Xu [25], Zhao et al. [26] and the references therein. The simulation of \( \hat{\psi} \) is also shown in Figure 2.

From Theorem 4.1 and Theorem 4.2 we have \( \hat{Y} = Y \hat{X} + \hat{\varphi}, \hat{X} = -\Gamma_2 \hat{Y} - \hat{\psi} \) and \( \hat{Z}_1 = (I + Y N_1)^{-1} (\hat{\eta}_1 - Y C_1 \hat{X}) \). Then the dynamic simulations of \( \hat{Y} \) and \( \hat{Z}_1 \) are similarly generated, shown in Figure 3. Further, from Theorem 4.2 we also generate the dynamic simulation of \( \nu \), which is presented in Figure 4.

6 Conclusion

We investigate an LQ control problem of BSDE with partial information, where both the generator of dynamic system and the cost functional contain diffusion terms \( Z_1 \) and \( Z_2 \). This problem is solved completely and explicitly under some standard conditions. An feedback representation of optimal control and an explicit formula of corresponding optimal cost are given in terms of three Riccati equations, a BSDE with filtering and an SDE with filtering. Moreover, we work out two special scalar-valued control problems to illustrate our theoretical results.

Note that the coefficients in the generator of state equation and the weighting matrices in the cost functional are deterministic. If the coefficients are random, there will be an essential difficulty in solving the case. Since \( \mathbb{E}[A_t Y_t | \mathcal{F}^W_t] = A_t \mathbb{E}[Y_t | \mathcal{F}^W_t] \) is no longer true if \( A \) is an \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted stochastic process. We will investigate the stochastic case in future.

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