Superluminality and UV Completion

G.M. Shore

Department of Physics
University of Wales, Swansea
Swansea SA2 8PP, U.K.
E-mail: g.m.shore@swansea.ac.uk

ABSTRACT: The idea that the existence of a consistent UV completion satisfying the fundamental axioms of local quantum field theory or string theory may impose positivity constraints on the couplings of the leading irrelevant operators in a low-energy effective field theory is critically discussed. Violation of these constraints implies superluminal propagation, in the sense that the low-frequency limit of the phase velocity \( v_{\text{ph}}(0) \) exceeds \( c \). It is explained why causality is related not to \( v_{\text{ph}}(0) \) but to the high-frequency limit \( v_{\text{ph}}(\infty) \) and how these are related by the Kramers-Kronig dispersion relation, depending on the sign of the imaginary part of the refractive index \( \text{Im} n(\omega) \) which is normally assumed positive. Superluminal propagation and its relation to UV completion is investigated in detail in three theories: QED in a background electromagnetic field, where the full dispersion relation for \( n(\omega) \) is evaluated numerically for the first time and the role of the null energy condition \( T_{\mu\nu}k^\mu k^\nu \geq 0 \) is highlighted; QED in a background gravitational field, where examples of superluminal low-frequency phase velocities arise in violation of the positivity constraints; and light propagation in coupled laser-atom \( \Lambda \) systems exhibiting Raman gain lines with \( \text{Im} n(\omega) < 0 \). The possibility that a negative \( \text{Im} n(\omega) \) must occur in quantum field theories involving gravity to avoid causality violation, and the implications for the relation of IR effective field theories to their UV completion, are carefully analysed.

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1. Introduction

Effective field theories have always played an important role in particle physics as low-energy phenomenological models describing the IR dynamics of renormalisable quantum field theories [1]. Key examples include the chiral Lagrangians describing the light pseudoscalar mesons arising from chiral symmetry breaking in QCD or those related to electroweak symmetry breaking in the absence of a light Higgs boson. More recently, the necessity of renormalisability as a criterion in constructing unified field theories has been reconsidered and it has become familiar to think of the standard model and its supersymmetric generalisations as IR effective theories of some more fundamental QFT or string theory at the Planck scale.

The question then arises whether all effective field theories admit a consistent UV completion, i.e. whether they can arise as the IR limit of a well-defined renormalisable QFT or string theory. If not, we may be able to use the criterion of the existence of a UV completion as a constraint on the parameters of the IR effective field theory, perhaps with interesting phenomenological implications.

At first sight, the answer certainly appears to be no. It is well-known that the parameters of chiral Lagrangians derived from known renormalisable theories such as QCD
satisfy constraints arising from analyticity and unitarity. However, in general the situation may not be so straightforward, particularly when we consider theories involving gravity. The purpose of this paper is to expose some of the subtleties that arise in establishing the relation between constraints on IR effective field theories and their UV completion, especially those that arise from considerations of causality and the absence of superluminal propagation.

The idea that the existence of a well-defined UV completion may place important constraints on the couplings of the leading irrelevant operators in low-energy effective actions has been revisited recently in ref. [2]. This paper highlights a number of examples where these couplings must satisfy certain positivity constraints to ensure that the effective theory does not exhibit unphysical behaviour such as superluminal propagation of massless particles or violations of analyticity or unitarity bounds on scattering amplitudes. This has been developed in ref. [3], where positivity constraints are derived on the couplings of the operators controlling WW scattering in an effective theory of electroweak symmetry breaking. Turning the argument around, these papers claim that an observed violation of these positivity constraints would signify a breakdown of some of the fundamental axioms of QFT and string theory, such as Lorentz invariance, unitarity, analyticity or causality.

In this paper, we examine these arguments critically and show that the assumed relation between the IR effective field theory and its UV completion may be significantly more subtle. We focus on the issue of superluminal propagation in the effective theory. This makes contact with our extensive body of work (see e.g. [4, 5, 6, 7, 8, 9, 10]) on superluminality, causality and dispersion in QED in curved spacetime, a subject we have given the name of ‘quantum gravitational optics’ [9]. As will become clear, our discussion is equally applicable to the issue of analyticity bounds, since each ultimately rests on the behaviour of the imaginary (absorptive) part of the refractive index or forward scattering amplitude.

We begin by discussing the general form of the effective action for low-energy electrodynamics and derive positivity constraints on the couplings of the leading irrelevant operators based on the requirement that light propagation is not superluminal. However, the ‘speed of light’ obtained from the IR effective theory is simply the zero-frequency limit of the phase velocity, \(v_{\text{ph}}(0)\). In section 3, we present a careful analysis of dispersion and light propagation and show that in fact the ‘speed of light’ relevant for causality is \(v_{\text{ph}}(\infty)\), i.e. the high-frequency limit of the phase velocity. Determining this requires a knowledge of the UV completion of the quantum field theory.

It therefore appears that causality and superluminal propagation place no restriction on the low-energy theory. However, on the basis of the usual axioms of quantum field theory (local Lorentz invariance, analyticity, microcausality) we can prove the Kramers-Kronig dispersion relation for the refractive index:

\[
n(\infty) = n(0) - \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega} \text{Im} \, n(\omega)
\]

It is usually assumed that \(\text{Im} \, n(\omega) > 0\), typical of an absorptive medium. (Similar relations hold for scattering amplitudes. Here, the imaginary part of the forward scattering amplitude, \(\text{Im} \, \mathcal{M}(s,0)\), is related via the optical theorem to the total cross section, ensur-
ing positivity.) If this is true, then eq.(1.1) implies that $v_{\text{ph}}(\infty) > v_{\text{ph}}(0)$. It follows that if $v_{\text{ph}}(0)$ is found to be superluminal in the low-energy theory then $v_{\text{ph}}(\infty)$ is necessarily also superluminal, in contradiction with causality. Violation of the positivity constraints on the couplings in the IR effective action would indeed then mean that the theory had no consistent UV completion. The central point of this paper, however, is to question the assumption that $\text{Im } n(\omega)$ is necessarily positive, especially in theories involving gravity.

First, we consider an example that is under complete control and where the conventional expectations are realised, viz. QED in a background electromagnetic field. The background field modifies the propagation of light due to vacuum polarisation and induces irrelevant operators in the low-energy effective action. Recent technical developments mean that we now have an expression for the vacuum polarisation valid for all momenta for general constant background fields [11] and in section 4 we use this to derive the full frequency dependence of the refractive index $n(\omega)$. This is the first complete numerical evaluation of the QED dispersion relation. This shows the expected features – a single absorption line with $\text{Im } n(\omega) > 0$ and the corresponding phase velocity satisfying $v_{\text{ph}}(\infty) > v_{\text{ph}}(0)$. The low-energy couplings, which determine the Euler-Heisenberg effective action, are consistent with the positivity constraints following from imposing $v_{\text{ph}}(0) < 1$.

An important feature of our analysis is the demonstration of the precise role of the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $k^\mu$ is the photon momentum. (See also, e.g., refs. [5, 12, 13].) The null projection of the energy-momentum tensor occurs universally in the dispersion relation for light propagation in background fields – the speed of light is determined by both the couplings and the sign of $T_{\mu\nu}k^\mu k^\nu$. This is greatly clarified by our use of the Newman-Penrose, or null-tetrad, formalism, which is by far the clearest language in which to analyse the propagation of light in general backgrounds.

We then consider the analogous case of QED in a background gravitational field, i.e. in curved spacetime. Here, current techniques only allow a determination of the refractive index and phase velocity in the low-frequency region. Moreover, the null-energy projection $T_{\mu\nu}k^\mu k^\nu$ (the Newman-Penrose scalar $\Phi_{00}$) is not the only quantity determining the speed of light. There is also a polarisation-dependent contribution given by the Weyl tensor projection $C_{\mu\lambda
u\rho}k^\lambda k^\rho a^\mu a^\nu$ (essentially the NP scalar $\Psi_0$), which gives rise to the phenomenon of gravitational birefringence. The remarkable feature of QED in curved spacetime is that, even for Weyl-flat spacetimes, the positivity constraints on the low-energy couplings which would prohibit superluminal propagation are violated [4]. We can readily find examples exhibiting a superluminal low-frequency phase velocity, $v_{\text{ph}}(0) > 1$. Furthermore, for Ricci-flat spacetimes it is possible to prove a simple sum rule [5] which shows that one polarisation is always superluminal, whatever the sign of the couplings. We illustrate these phenomena with examples of propagation in FRW and black-hole spacetimes in section 5.

This raises the central issue of this paper. If the KK dispersion relation holds in gravitational backgrounds and $\text{Im } n(\omega) > 0$, then necessarily $v_{\text{ph}}(\infty) > v_{\text{ph}}(0) > 1$ for certain spacetimes, in apparent violation of causality. (See ref.[8] and sections 3 and 5 for a more nuanced discussion). The most natural escape seems to be that for theories involving grav-

1Note that in general we set $c = 1$ throughout this paper.
ity we must have the possibility that $\text{Im} \, n(\omega) < 0$. But if so, this invalidates the conclusion that IR effective field theories with couplings violating the positivity constraints \textit{necessarily} have no consistent UV completion. Once we admit the possibility that a fundamental QFT satisfying the standard axioms (and surely QED in curved spacetime is such a theory) can have a negative absorptive part, $\text{Im} \, n(\omega) < 0$ or $\text{Im} \, \mathcal{M}(s,0) < 0$, then we must accept that the relation between the IR effective theory and its UV completion can be rather more subtle.

In order to understand better what a negative $\text{Im} \, n(\omega)$ would mean physically, in section 6 we introduce a fascinating topic in atomic physics – the interaction of lasers with so-called ‘Λ-systems’ exhibiting electromagnetically-induced transparency (EIT) and ‘slow light’. We discuss a variant in which the Λ-system is engineered to produce Raman \textit{gain} lines [14] rather than absorption lines in $\text{Im} \, n(\omega)$ and derive in detail an explicit example where $v_{ph}(\infty) < v_{ph}(0)$. This occurs as a result of the intensity gain of a light pulse propagating through the coupled laser-atom medium. Our conjecture is that gravity may be able to act in the same way.

The issue of whether the existence of a consistent UV completion does indeed impose positivity constraints on the couplings of an IR effective field theory therefore remains open. In particular, theories involving gravity present a serious challenge to the accepted wisdom on the relation of the IR and UV behaviour of QFT. A summary of our final conclusions is presented in section 7.
2. Superluminality constraints in low-energy quantum electrodynamics

The low-energy effective action for QED, valid for momenta below the scale of the electron mass $m$, is given by the well-known Euler-Heisenberg Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{m^4} \left( -\frac{1}{36} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{90} F_{\mu\nu} F^{\mu\lambda} F^{\nu\rho} \right)$$  \hspace{1cm} (2.1)

This is more conveniently written in terms of the two Lorentz scalars quadratic in the field strengths, $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and $\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$, as follows:

$$\mathcal{L} = -\mathcal{F} + \frac{2}{45} \frac{\alpha^2}{m^4} \left( c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 \right)$$  \hspace{1cm} (2.2)

where $c_1 = 4$ and $c_2 = 7$.

The higher-order terms omitted in eqs.

(2.1) or (2.2) are of two types: higher orders in derivatives, i.e. terms of $O(D^2 F^4 / m^6)$ which give contributions to the photon dispersion relation suppressed by $O(k^2 / m^2)^2$, and higher orders in the field strengths $\mathcal{F}, \mathcal{G}$ which are associated with higher powers of $\alpha$.

In this section, we consider photon propagation in a background electromagnetic field using this effective Lagrangian and discuss the constraints imposed on the coefficients $c_1, c_2$ of the leading (dim 8) irrelevant operators in the low-energy expansion (2.2) by the requirement that propagation is causal, i.e. subluminal.

The simplest way to study the causal nature of photon propagation is to use geometric optics (see, e.g. refs.

[8, 9] for reviews of our formalism). Since we shall be considering QED in a background gravitational field later in this paper, the following brief summary of geometric optics is sufficiently general to include the case of propagation in an arbitrary spacetime metric $g_{\mu\nu}$. The electromagnetic field in the (modified) Maxwell equations derived from eq.

(2.2) is written in terms of a slowly-varying amplitude (and polarisation) and a rapidly-varying phase as follows:

$$(A_\mu + i \epsilon B_\mu + \ldots) e^{i \theta / \epsilon}$$  \hspace{1cm} (2.3)

where $\epsilon$ is a parameter introduced to keep track of the relative orders of terms as the Maxwell and Bianchi equations are solved order-by-order in $\epsilon$. The wave vector is identified as the gradient of the phase, $k_\mu = \partial_\mu \vartheta$. We also write $A_\mu = A a_\mu$, where $A$ represents the amplitude itself while $a_\mu$ specifies the polarisation.

Solving the usual Maxwell equation $D_\mu F^{\mu\nu} = 0$, we find at $O(1/\epsilon)$,

$$k^2 = 0$$  \hspace{1cm} (2.4)

while at $O(1)$,

$$k^\mu D_\mu a^\nu = 0$$  \hspace{1cm} (2.5)

and

$$k^\mu D_\mu (\ln A) = -\frac{1}{2} D_\mu k^\mu$$  \hspace{1cm} (2.6)

\[2\text{Here, we are using ‘}k^2\text{‘ to denote two powers of momentum, not necessarily the contracted form }k^\mu k_\mu\text{, which can of course be zero on-shell even for large values of }|k|\text{. Similarly for ‘}D^2\text{‘.}\]
Eq. (2.4) shows immediately that \( k^\mu \) is a null vector. From its definition as a gradient, we also see

\[
k^\mu D_\mu k^\nu = k^\mu D^\nu k_\mu = \frac{1}{2} D^\nu k^2 = 0
\]

(2.7)

Light rays, or equivalently photon trajectories, are the integral curves of \( k^\mu \), i.e. the curves \( x^\mu(s) \) where \( dx^\mu/ds = k^\mu \). These curves therefore satisfy

\[
0 = k^\mu D_\mu k^\nu = \frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\mu\lambda} \frac{dx^\mu}{ds} \frac{dx^\lambda}{ds}
\]

(2.8)

This is the geodesic equation. We conclude that for the usual Maxwell theory in general relativity, light follows null geodesics. Eqs. (2.7), (2.5) show that both the wave vector and the polarisation are parallel transported along these null geodesic rays, while eq. (2.6), whose r.h.s. is just (minus) the optical scalar \( \theta \), shows how the amplitude changes as the beam of rays focuses or diverges.

As a consequence of only keeping terms in the effective action with no explicit derivatives acting on the field strengths, the modified light-cone condition derived from the low-energy effective action (2.2) remains homogeneous and quadratic in \( k^\mu \). It can therefore be written in the form:

\[
\mathcal{G}^{\mu\nu} k_\mu k_\nu = 0
\]

(2.9)

where \( \mathcal{G}^{\mu\nu} \) is a function of the background electromagnetic (or gravitational) field. In the gravitational case, \( \mathcal{G}^{\mu\nu} \) also involves the photon polarisation.

Now notice that in the discussion of the free Maxwell theory, we did not need to distinguish between the photon momentum \( p^\mu \), i.e. the tangent vector to the light rays, and the wave vector \( k_\mu \) since they were simply related by raising the index using the spacetime metric, \( p^\mu = g^{\mu\nu} k_\nu \). In the modified theory, however, there is an important distinction. The wave vector, defined as the gradient of the phase, is a covariant vector or 1-form, whereas the photon momentum/tangent vector to the rays is a true contravariant vector. The relation is non-trivial. In fact, given \( k_\mu \), we should define the corresponding ‘momentum’ as

\[
p^\mu = \mathcal{G}^{\mu\nu} k_\nu
\]

(2.10)

and the light rays as curves \( x^\mu(s) \) where \( dx^\mu/ds = p^\mu \). This definition of momentum satisfies

\[
G_{\mu\nu} p^\mu p^\nu = \mathcal{G}^{\mu\nu} k_\mu k_\nu = 0
\]

(2.11)

where \( G \equiv \mathcal{G}^{-1} \) defines a new effective metric which determines the light cones mapped out by the geometric optics light rays. (Indices are always raised or lowered using the true metric \( g_{\mu\nu} \).) The ray velocity \( v_{\text{ray}} \) corresponding to the momentum \( p^\mu \), which is the velocity with which the equal-phase surfaces advance, is given by (defining components in an orthonormal frame)

\[
v_{\text{ray}} = \frac{|p|}{p^0} = \frac{d|x|}{dt}
\]

(2.12)

along the ray. This is in general different from the phase velocity

\[
v_{\text{ph}} = \frac{\omega}{|k|}
\]

(2.13)
where the frequency $\omega = k^0$.

This shows that photon propagation for low-energy QED in a background electromagnetic or gravitational field (i.e. curved spacetime) can be characterised as a bimetric theory – the physical light cones are determined by the effective metric $G_{\mu\nu}$ while the geometric null cones are fixed by the spacetime metric $g_{\mu\nu}$.

This also clarifies a potentially confusing point. In the perturbative situation we are considering here, where the effective Lagrangian $\mathcal{L}(\mathcal{F}, \mathcal{G}) = -\mathcal{F} + O(\alpha)$ and the effective metric $G_{\mu\nu} = g_{\mu\nu} + O(\alpha)$, subluminal propagation with $v_{\text{ph}} < 1$ corresponds to $k^2 < 0$, i.e. the wave-vector is spacelike. However, the analysis above makes it clear that in this case the photon momentum is nevertheless timelike, i.e. $p^2 > 0$ if $v_{\text{ph}} < 1$ (and $v_{\text{ray}} < 1$). For further discussion of this point and an illustration in the case of QED in a background FRW spacetime, see ref.[8].

Now consider a general low-energy effective action $\mathcal{L}(\mathcal{F}, \mathcal{G})$. The modified Maxwell equation is

$$\mathcal{L}_\mathcal{F}\partial_\mu F^{\mu\nu} + \mathcal{L}_\mathcal{G}\partial_\mu \tilde{F}^{\mu\nu} + \left(\mathcal{L}_\mathcal{F} F^{\mu\nu} F_{\lambda\rho} + \mathcal{L}_\mathcal{G} \tilde{F}^{\mu\nu} \tilde{F}_{\lambda\rho} + \mathcal{L}_\mathcal{F\mathcal{G}}(F^{\mu\nu} \tilde{F}_{\lambda\rho} + \tilde{F}^{\mu\nu} F_{\lambda\rho})\right)\partial_\mu F^{\lambda\rho} = 0$$

while the Bianchi identity remains

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

(2.14)

We now immediately restrict to the case of interest here, viz. $\mathcal{L}(\mathcal{F}, \mathcal{G}) = -\mathcal{F} + O(\alpha)$. It would be trivial to extend the results of this section to this more general case, which includes the important example of Born-Infeld theory. Applying geometric optics, we then find

$$k^2 a^\nu - k. a k^\nu - \mathcal{L}_\mathcal{F} k_\mu F^{\mu\nu} F_{\lambda\rho} k_\lambda a^\rho - \mathcal{L}_\mathcal{G} k_\mu \tilde{F}^{\mu\nu} \tilde{F}_{\lambda\rho} k_\lambda a^\rho - \mathcal{L}_\mathcal{F\mathcal{G}}(k_\mu F^{\mu\nu} \tilde{F}_{\lambda\rho} k_\lambda a^\rho + k_\mu \tilde{F}^{\mu\nu} F_{\lambda\rho} k_\lambda a^\rho) = 0$$

(2.16)

where the field strengths now refer to the background electromagnetic field.

The natural language to discuss photon propagation in background fields is the Newman-Penrose (NP), or null tetrad, formalism (see appendix A). At each point in spacetime, we establish a tetrad of null vectors $\ell^\mu, n^\nu, m^\mu, \bar{m}^\mu$ such that $\ell.n = 1, m.\bar{m} = -1$ with all others zero. We choose $\ell^\mu$ to lie along the (unperturbed) direction of propagation, i.e. $k^\mu = \sqrt{2}\omega \ell^\mu + O(\alpha)$, where $\omega$ is normalised to be the frequency. The polarisation vector is then

$$a^\mu = \alpha m^\mu + \beta \bar{m}^\mu$$

(2.17)

where $m^\mu(\bar{m}^\mu)$ corresponds to left (right) circular polarisation, i.e. photon helicity $+1(-1)$.

Substituting for the polarisation $a^\mu$ in eq.(2.16) and contracting successively with $\bar{m}^\nu$ and $m^\nu$ then yields the equations:

$$k^2 \alpha + 2\omega^2 \left(\mathcal{L}_\mathcal{F} F_{\mu\nu} F_{\lambda\rho} + \mathcal{L}_\mathcal{G} \tilde{F}_{\mu\nu} \tilde{F}_{\lambda\rho} + \mathcal{L}_\mathcal{F\mathcal{G}}(F_{\mu\nu} \tilde{F}_{\lambda\rho} + \tilde{F}^{\mu\nu} F_{\lambda\rho})\right)\ell^\mu \ell^\nu \left(\alpha m^\rho + \beta \bar{m}^\rho\right) = 0$$

(2.18)

3The essential simplification we make is that we can drop the factor $\mathcal{L}_\mathcal{F}$ in front of $\partial_\mu F^{\mu\nu}$ since the latter is already $O(\alpha)$ by virtue of the equations of motion, so we should approximate $\mathcal{L}_\mathcal{F} \simeq 1$ when we work consistently to $O(\alpha)$. 
and
\[ k^2 \beta + 2 \omega^2 \left( \mathcal{L}_F F_{\mu \nu} F_{\lambda \rho} + \mathcal{L}_G \tilde{F}_{\mu \nu} \tilde{F}_{\lambda \rho} + \mathcal{L}_{FG} (F_{\mu \nu} \tilde{F}_{\lambda \rho} + \tilde{F}_{\mu \nu} F_{\lambda \rho}) \right) \ell^\mu \ell^\nu (\alpha m^\rho + \beta \tilde{m}^\rho) = 0 \]
(2.19)

To simplify further, we now insert the explicit form for the background fields written in NP form. Here, the six independent components of the field strength \( F^{\mu \nu} \), i.e. the E and B fields, are represented by three complex scalars \( \phi_0, \phi_1, \phi_2 \) such that (see appendix A)
\[
F^{\mu \nu} = - (\phi_1 + \phi_1^*) [\ell^\mu n^\nu] + (\phi_1 - \phi_1^*) [m^\mu \tilde{m}^\nu] + \phi_2 [\ell^\mu m^\nu] + \phi_2^* [\ell^\mu \tilde{m}^\nu] - \phi_0^*[n^\mu m^\nu] - \phi_0 [n^\mu \tilde{m}^\nu]
\]
(2.20)

where the notation is \([ab] \equiv ab - ba\). Then, after some calculation, we find that eqs.(2.18), (2.19) can be written in matrix form as
\[
\begin{pmatrix}
 k^2 + 2 \omega^2 (\mathcal{L}_F F + \mathcal{L}_G) \phi_0 \phi_0^* & 2 \omega^2 (\mathcal{L}_F F - \mathcal{L}_G - 2i \mathcal{L}_{FG}) \phi_0 \phi_0^* \\
 2 \omega^2 (\mathcal{L}_F F - \mathcal{L}_G + 2i \mathcal{L}_{FG}) \phi_0 \phi_0 & k^2 + 2 \omega^2 (\mathcal{L}_F F + \mathcal{L}_G) \phi_0 \phi_0^*
\end{pmatrix}
\begin{pmatrix}
 \alpha \\
 \beta
\end{pmatrix}
= 0
\]
(2.21)

The coefficients have a particularly natural interpretation in terms of the variables
\[
\chi = \frac{1}{2} (F + iG) \quad \bar{\chi} = \frac{1}{2} (F - iG)
\]
(2.22)

with eq.(2.21) simplifying to
\[
\begin{pmatrix}
 k^2 + 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0^* & 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0^* \phi_0^* \\
 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0 & k^2 + 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0^*
\end{pmatrix}
\begin{pmatrix}
 \alpha \\
 \beta
\end{pmatrix}
= 0
\]
(2.23)

The dependence on the background field is especially illuminating. The energy-momentum tensor is
\[
T_{\mu \nu} = - F_{\mu \lambda} F_{\nu}^\lambda + \frac{1}{4} g_{\mu \nu} F^2
\]
(2.24)

and in the NP basis, this implies
\[
T_{\ell \ell} = T_{\mu \nu} \ell^\mu \ell^\nu = 2 \phi_0 \phi_0^*
\]
(2.25)

This is of course the component of the energy-momentum tensor arising in the 'null energy condition',
\[
T_{\mu \nu} \ell^\mu \ell^\nu > 0
\]
(2.26)

As we now see, it plays a vital role in the modified dispersion relations.

The dispersion relation follows directly from eq.(2.23) as the requirement that the determinant of the matrix vanishes, or in other words, the allowed values of \( k^2 \) are the eigenvalues of the matrix
\[
\begin{pmatrix}
 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0^* & 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0^* \phi_0^* \\
 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0 & 2 \omega^2 \mathcal{L}_{X\bar{X}} \phi_0 \phi_0^*
\end{pmatrix}
\]
(2.27)
This gives
\[ k^2 = -2\omega^2 \phi^* \phi \left( \mathcal{L}_{XX} \pm \sqrt{\mathcal{L}_{X\bar{X}} \mathcal{L}_{X\bar{X}}} \right) \] (2.28)
that is,
\[ k^2 = -\frac{1}{2} T_{\mu\nu} k^\mu k^\nu \left( \mathcal{L}_{FF} + \mathcal{L}_{GG} \pm \sqrt{(\mathcal{L}_{FF} - \mathcal{L}_{GG})^2 + 4\mathcal{L}_{FG}^2} \right) \] (2.29)

We see immediately what will later be established as a very general result, viz. that the modification in the light cone due to the background field is proportional to the projection of the energy-momentum tensor onto the null cone, i.e. the component \( T_{\mu\nu} k^\mu k^\nu \) which appears in the null energy condition and whose sign is therefore fixed. This dependence turns out to be completely general and is the origin of the universal features of light propagation observed, for example, in ref.[15].

Notice also that as a direct consequence of its derivation from the low-energy effective action, eq.(2.29) is homogeneous in \( k^\mu \) so there is no dispersion in this approximation. The result is of the general form considered above in eq.(2.9) and gives a phase velocity
\[ v_{ph}(0) = 1 - \frac{1}{2} T_{\mu\nu} \ell^\mu \ell^\nu \left( \mathcal{L}_{FF} + \mathcal{L}_{GG} \pm \sqrt{(\mathcal{L}_{FF} - \mathcal{L}_{GG})^2 + 4\mathcal{L}_{FG}^2} \right) \] (2.30)

The corresponding polarisation vectors are the solutions for \( \alpha, \beta \) in eq.(2.21), i.e. the eigenvectors of (2.27). This gives
\[ a^\mu_+ = \frac{1}{\sqrt{2}} (e^{i\theta} m^\mu + e^{-i\theta} \bar{m}^\mu) \]
\[ a^\mu_- = -\frac{i}{\sqrt{2}} (e^{i\theta} m^\mu - e^{-i\theta} \bar{m}^\mu) \] (2.31)
where the phase is defined from \( \sqrt{\mathcal{L}_{XX}} = re^{i\theta} \). These combinations \( a^\mu_\pm \) of the left and right circular polarisations represent orthogonal linear polarisations, with the direction fixed by the angle \( \theta \). The situation is therefore very similar to the case of the polarisation eigenstates for propagation in a gravitational background considered, e.g. in ref.[8].

Now, as explained above, in order to have subluminal propagation and \( v_{ph} < 1 \), we require \( k^2 < 0 \). Given the null energy condition \( T_{\mu\nu} k^\mu k^\nu > 0 \), it follows from eqs.(2.28),(2.29) that this is ensured if
\[ \det \begin{pmatrix} \mathcal{L}_{XX} & \mathcal{L}_{X\bar{X}} \\ \mathcal{L}_{\bar{X}X} & \mathcal{L}_{\bar{X}\bar{X}} \end{pmatrix} > 0, \quad \mathcal{L}_{X\bar{X}} > 0 \] (2.32)

\[ Q^2 = E^2 + B^2 - 2S \cdot \hat{k} - (E \cdot \hat{k})^2 - (B \cdot \hat{k})^2 \]
and \( S \) is the Poynting vector. From the identities in the appendix, it is easy to check in our conventions that \( Q^2 \) is indeed just \( 2 T_{\mu\nu} \ell^\mu \ell^\nu \). (See also ref.[17].)
or equivalently,

\[
\det \begin{pmatrix} L_{FF} & L_{FG} \\ L_{FG} & L_{GG} \end{pmatrix} > 0, \quad L_{FF} + L_{GG} > 0
\] (2.33)

These are the constraints placed on the coefficients of the leading irrelevant operators in a general low-energy effective action by the requirement of no superluminal propagation, at least (see section 3) the requirement that \( v_{\text{ph}}(0) < 1 \).

If we specialise to the Euler-Heisenberg action (2.2), we can see how this constraint is realised in low-energy effective QCD. We find

\[
L_{FF} = \frac{4}{45} \alpha^2 m_4 c_1, \quad L_{GG} = \frac{4}{45} \alpha^2 m_4 c_2, \quad L_{FG} = 0
\] (2.34)

and the positivity constraint reduces to simply \( c_1 > 0, c_2 > 0 \). Of course this is satisfied by the QED values \( c_1 = 4, c_2 = 7 \).

The new light cone condition for the Euler-Heisenberg action is simply

\[
k^2 + \frac{4}{45} \alpha^2 T_{\mu\nu} k^\mu k^\nu \left[ 4_+, 7_- \right] = 0
\] (2.35)

corresponding to the linear polarisations \( a_+^\mu, a_-^\mu \) respectively. This reproduces the results found in our earlier paper [5] where the dependence on the null energy component \( T_{\mu\nu} k^\mu k^\nu \) was first recognised. It was observed there that the positivity of \( c_1 \) and \( c_2 \) also ensured the positivity of the trace of the energy-momentum tensor, but subsequent work [18] has shown that this apparent link between causality and the conformal anomaly is merely a low-order artifact. It also reproduces the classic result for photon propagation in a purely magnetic background field [19, 20]. In this case, it is straightforward to check that the phase angle entering the formula for the polarisation eigenstates is just \( \tan \theta = B_1 / B_2 \) (where we take the direction of propagation to be the 3-axis), so that \( a_-^\mu \) lies parallel to \( \mathbf{B} \) while \( a_+^\mu \) is orthogonal.

To summarise, working in low-energy electrodynamics, we have seen how the light cone condition is modified by the component of the energy-momentum tensor which is fixed by the null energy condition. The eigenstates of definite phase velocity are linear polarisations with direction fixed by the background field. Ensuring \( v_{\text{ph}} < 1 \) imposes positivity constraints on the coefficients of the leading irrelevant operators in the effective action. The modified light cone condition is homogeneous and quadratic in \( k \) and there is no dispersion.

However, the absence of dispersion is merely an artifact of the low-energy approximation. In general, there is non-trivial dispersion and the light cone condition (2.29) only fixes the low-frequency limit \( v_{\text{ph}}(0) \) of the phase velocity. In the following section, we discuss more carefully what causality actually requires and find that, far from imposing any constraint on the low-frequency behaviour of the phase velocity, it imposes a constraint on the high-frequency limit, \( v_{\text{ph}}(\infty) < 1 \). In turn, this is controlled not by the low-energy effective action but by the UV limit of the quantum field theory.
3. Speeds of light, causality and dispersion relations

In order to understand whether causality does indeed impose constraints on the parameters of a low-energy effective action, we first need to understand more precisely exactly what we mean by the ‘speed of light’.

By definition, a low-energy effective action is an expansion in powers of derivatives acting on the fields. For example, the Euler-Heisenberg effective action (2.1) contains only terms of $O(F^4/m^4)$ leading, as we have seen, to a homogeneous dispersion relation for photon propagation with each term of $O(k^2)$. However, this neglects higher derivative terms beginning with $O(D^2F^4/m^6)$, which modify the dispersion relation with terms suppressed by $O(k^2/m^2)$. This is why the ‘speed of light’ derived in section 2 from the low-energy effective action is just the low-frequency limit of the phase velocity, i.e. $v_{\text{ph}}(\omega)|_{\omega \to 0}$.

In fact, there are many ‘speeds of light’, of which the phase velocity is just one. A comprehensive account is given in the classic text by Brillouin [21], which considers in detail the propagation of a sharp-fronted wave pulse in a medium with a single absorption band (see below). The essential ideas are explained in our review [8]. As well as the phase velocity $v_{\text{ph}}(\omega) = \frac{\omega}{k}$, we need to consider the group velocity $v_{\text{gd}}(\omega) = \frac{d\omega}{dk}$ and the wavefront velocity, i.e. the velocity at which the boundary between the regions of zero and non-zero disturbance of a wave pulse propagates. This definition of wavefront is identified with the characteristics of the partial differential equation governing the wave propagation and it is the wavefront velocity $v_{\text{wf}}$ is the speed of light which is relevant for causality.

Other definitions of the ‘speed of light’, each of which is useful in a particular context, can be found in ref.[8], e.g. signal velocity, energy-transfer velocity, etc. In addition, as we have seen in section 2, the ray velocity also plays a conceptually important role.

We now show how the wavefront velocity is related to the phase velocity. The following proof follows a paper by Leontovich [22], as summarised in ref.[8]. We show that for a very general set of PDEs the wavefront velocity associated with the characteristics is identical to the $k \to \infty$ limit of the phase velocity, i.e.

$$v_{\text{wf}} = \lim_{k \to \infty} \frac{\omega}{|k|} = \lim_{\omega \to \infty} v_{\text{ph}}(\omega) \quad (3.1)$$

The first step is to recognise that any second order PDE can be written as a system of first order PDEs by considering the first derivatives of the field as independent variables. Thus, if we consider a general second order wave equation for a field $u(t,x)$, in just one space dimension for simplicity, the system of PDEs we need to solve is

$$a_{ij} \frac{\partial \phi_i}{\partial t} + b_{ij} \frac{\partial \phi_i}{\partial x} + c_{ij} \phi_j = 0 \quad (3.2)$$

where $\phi_i = \{u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\}$.

Making the ‘geometric optics’ ansatz

$$\phi_i = \varphi_i \exp i(ut - kx) \quad (3.3)$$

where the frequency-dependent phase velocity is $v_{\text{ph}}(k) = \omega(k)/k$, and substituting into eq.(3.2) we find

$$\left(i\omega a_{ij} - ikb_{ij} + c_{ij}\right)\varphi_j = 0 \quad (3.4)$$
The condition for a solution,
\[
\det \left[ a_{ij} v_{\text{ph}}(k) - b_{ij} - \frac{i}{k} c_{ij} \right] = 0 \quad (3.5)
\]
then determines the phase velocity.

On the other hand, we need to find the characteristics of eq.(3.2), i.e. curves \( C \) on which Cauchy’s theorem breaks down and the evolution is not uniquely determined by the initial data on \( C \). The derivatives of the field may be discontinuous across the characteristics and these curves are associated with the wavefronts for the propagation of a sharp-fronted pulse. The corresponding light rays are the ‘bicharacteristics’.

We therefore consider a characteristic curve \( C \) in the \((t, x)\) plane separating regions where \( \phi_i = 0 \) (ahead of the wavefront) from \( \phi_i \neq 0 \) (behind the wavefront). At a fixed point \((t_0, x_0)\) on \( C \), the absolute derivative of \( \phi_i \) along the curve, parametrised as \( x(t) \), is just
\[
\frac{d\phi_i}{dt} = \frac{\partial \phi_i}{\partial t} \bigg|_0 + \frac{\partial \phi_i}{\partial x} \bigg|_0 \frac{dx}{dt} \quad (3.6)
\]
where \( dx/dt = v_{\text{wf}} \) gives the wavefront velocity. Using this to eliminate \( \frac{\partial \phi_i}{\partial t} \) from the PDE eq.(3.2) at \((t_0, x_0)\), we find
\[
\left( -a_{ij} \frac{dx}{dt} + b_{ij} \right) \frac{\partial \phi_j}{\partial x} \bigg|_0 + a_{ij} \frac{d\phi_j^{(0)}}{dt} + c_{ij} \phi_j^{(0)} = 0 \quad (3.7)
\]
Now since \( C \) is a wavefront, on one side of which \( \phi_i \) vanishes identically, the second two terms above must be zero. The condition for the remaining equation to have a solution is simply
\[
\det \left[ a_{ij} v_{\text{wf}} - b_{ij} \right] = 0 \quad (3.8)
\]
which determines the wavefront velocity \( v_{\text{wf}} \). The proof is now evident. Comparing eqs.(3.5) and (3.8), we clearly identify
\[
v_{\text{wf}} = v_{\text{ph}} |_{k \to \infty} \quad (3.9)
\]

The conclusion is that the wavefront velocity is in fact the high-frequency limit of the phase velocity. That is, the ‘speed of light’ which is relevant for causality is not the low-frequency phase velocity \( v_{\text{ph}}(0) \) but its high-frequency limit \( v_{\text{ph}}(\infty) \).

In flat spacetime, it is therefore natural, and indeed true, that causality requires \( v_{\text{ph}}(\infty) < 1 \). In curved spacetime, however, the requirement of causality is much more subtle. Indeed, as explained in ref.[8] (see also ref.[12, 23]), it is in principle possible to maintain a causal theory even with \( v_{\text{ph}}(\infty) > 1 \) provided that the background spacetime exhibits a modified version of ‘stable causality’. This possible loophole is most easily illustrated for the case considered in section 2 where the apparently superluminal wave equation is homogeneous and quadratic, although the physical idea remains relevant for general propagation equations. In this case, the dispersion relation may be written as \( G^{\mu\nu} k_\mu k_\nu = 0 \). As explained in section 2, the effective light cone is then determined by an ‘effective metric’ \( G_{\mu\nu} \), where \( G = G^{-1} \), which depends on the background fields and which
we think of as being perturbatively close to the spacetime metric $g_{\mu \nu}$. This generalises the standard general relativity assumption that light follows null geodesics, in which case the dispersion relation is just $g^{\mu \nu} k_\mu k_\nu = 0$.

The concept of stable causality is summarised in the following definition and theorem [24]:

- A spacetime manifold $(\mathcal{M}, g_{\mu \nu})$ is stably causal if the metric $g_{\mu \nu}$ has an open neighbourhood such that $\mathcal{M}$ has no closed timelike or null curves with respect to any metric belonging to that neighbourhood.

- Stable causality holds everywhere on $\mathcal{M}$ if and only if there is a globally defined function $f$ whose gradient $D_\mu f$ is everywhere non-zero and timelike with respect to $g_{\mu \nu}$.

According to this theorem, the absence of causality violation in the form of closed timelike or null curves is assured if we can find a globally defined function $f$ whose gradient is timelike with respect to the effective metric $G_{\mu \nu}$ for light propagation. $f$ then acts as a global time coordinate. That this can occur is demonstrated for the example of 'superluminal' light propagation in a Friedmann-Robertson-Walker spacetime in ref.[8].

Setting this curved-spacetime subtlety aside for the moment, our conclusion here is that the restriction imposed by causality on the propagation of light is not concerned with the low-frequency, IR limit $v_{\text{ph}}(0)$ determined by the low-energy effective action. Rather, causality imposes a bound on the high-frequency limit, $v_{\text{ph}}(\infty) < 1$. This is determined by the UV nature of the quantum field theory.

At first sight, therefore, it appears that causality imposes no constraint on the structure of low-energy effective theories. However, there is one further twist. The (complex) refractive index $n(\omega)$ satisfies a Kramers-Kronig (KK) dispersion relation\(^5\)

\[ n(\infty) = n(0) - \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega} \text{Im} \ n(\omega) \] (3.10)

In a conventional dispersion relation, the imaginary part of the refractive index, i.e. the absorption coefficient, is positive, $\text{Im} \ n(\omega) > 0$. This implies that $n(\infty) < n(0)$ and so, recalling $v_{\text{ph}}(\omega) = 1/\text{Re} \ n(\omega)$, that $v_{\text{ph}}(\infty) > v_{\text{ph}}(0)$. If so, the phase velocity deduced from the low-energy effective action would be a lower bound on the wavefront velocity $v_{\text{wf}} = v_{\text{ph}}(\infty)$ and so a superluminal $v_{\text{ph}}(0)$ would indeed represent a violation of causality.

Similar KK dispersion relations arise in quantum field theory applied to correlation functions and scattering amplitudes $\mathcal{M}(s,t)$. The axiomatic inputs into their derivation are the standard ones of local quantum field theory, notably micro-causality, viz. the vanishing of commutators of field operators evaluated at spacelike separated points. In conventional QFT, the optical theorem relates the imaginary part $\text{Im} \ \mathcal{M}(s,0)$ of a forward scattering amplitude via unitarity to the total cross section, so positivity is assured.

It therefore appears that positivity of $\text{Im} \ n(\omega)$, and consequently $v_{\text{ph}}(\infty) > v_{\text{ph}}(0)$, is guaranteed in any situation of interest. This would ensure the validity of causality bounds derived from the low-energy effective action. However, this conclusion may be too fast. For

\(^5\)In what follows, we always refer to this as the ‘KK dispersion relation’ to avoid any possible confusion with our use so far of ‘dispersion relation’ to describe the dependence of the frequency $\omega(k)$ on wave-number $k$ determined by the light-cone condition. We also drop the convention $c = 1$ in this section.
example, consider an optical medium with a single absorption band. Its refractive index may be modelled as [21]

\[ n^2(\omega) = 1 - \frac{M^2}{\omega^2 - \omega_0^2 + i\delta\omega} \]  

where \( M \) determines the absorption strength, \( \omega_0 \) the characteristic frequency of the medium and \( \delta \) the width of the absorption band. The real and imaginary parts of \( n(\omega) \) are sketched in Fig. 1.

\[ \text{Figure 1:} \text{ The left hand figure shows the real part of the refractive index } \text{Re } n(\omega) \text{ for the single absorption band model (3.11). The dashed line is } n(\omega) = 1. \text{ The right hand plot is the corresponding Im } n(\omega). \]

\[ \text{Figure 2:} \text{ The left hand figure shows the frequency dependence of the phase velocity } v_{\text{ph}}(\omega) \text{ (solid green line) and the group velocity } v_{\text{gp}}(\omega) \text{ (dashed blue line) for the single absorption band model. The constant wavefront velocity } v_{\text{wf}} = 1 \text{ is indicated by the long-dashed red line. The right hand figure shows the corresponding indices, } n_{\text{ph}} = \text{Re } n(\omega) = 1/v_{\text{ph}} \text{ and } n_{\text{gp}}(\omega) = 1/v_{\text{gp}}. \]

This is the standard situation. The imaginary part Im \( n(\omega) \) is positive, while Re \( n(\omega) \) initially rises (‘normal dispersion’) then falls rapidly through the absorption band (‘anomalous dispersion’) passing through Re \( n = 1 \) at the characteristic frequency \( \omega_0 \) before asymptotically approaching \( n(\infty) = 1 \) from below. In terms of the phase velocity, this gives \( v_{\text{ph}}(0) < 1 \) and \( v_{\text{ph}}(\infty) = v_{\text{wf}} = 1 \), with an interesting behaviour in the vicinity of the characteristic frequency \( \omega_0 \). The bound \( v_{\text{ph}}(\infty) > v_{\text{ph}}(0) \) clearly holds. In the following section,
we derive the explicit behaviour of \( v_{\text{ph}}(\omega) \) for QED in a background electromagnetic field and show how the above features are realised in a QFT context.

It will be clear, however, that all this would be reversed if we were in a situation where \( \text{Im} \, n(\omega) < 0 \). In fact, this can easily be realised in, for example, atomic physics systems such as we describe in section 6. It corresponds to the system exhibiting \textit{gain} rather than absorptive scattering. The big question is whether such a phenomenon can arise in QFT. In section 5, we show that for QED in a gravitational background we can indeed find a superluminal low-frequency phase velocity \( v_{\text{ph}}(0) > 1 \). This poses a clear dilemma in what is after all a very conventional QFT, albeit involving gravity: either we really find \( v_{\text{ph}}(\infty) > v_{\text{ph}}(0) > 1 \) in which case we would be forced to reassess fundamental issues regarding causality, or \( \text{Im} \, n(\omega) < 0 \), i.e. propagation and scattering in a gravitational field can exhibit the characteristics of gain as well as dispersion.

All this will be explored in detail in future sections. It is immediately clear, however, that we need to be extremely careful before drawing the conclusion that apparently causality-violating values of couplings in low-energy effective actions necessarily imply the absence of a well-defined UV completion within the usual axioms of QFT, possibly including gravity.

4. The QED refractive index

In this section, we find an explicit representation of the dispersion relation for photon propagation in QED in a constant background electromagnetic field. This shows how the frequency dependence of the refractive index described above is actually realised in a well-understood quantum field theory.

The field-theoretic calculations required to evaluate an effective action valid for all momenta in a general background field are still beyond current techniques, so here we restrict the discussion to constant background fields. Also, since we are concerned only with analysing photon propagation, it is sufficient to evaluate the vacuum polarisation tensor (the second functional derivative of the effective action) in the given background. The state of the art in such calculations are those of Schubert and Gies [11, 25, 26] using the worldline path integral approach to QFT. This generalises the earlier classic work on QED in background fields using the more conventional Schwinger proper time or heat kernel methods (see for example [27, 19, 20]). Here, we start from the result obtained in ref.[11] for the vacuum polarisation tensor in a constant background electromagnetic field and translate it into the NP formalism before considering in detail its consequences for photon propagation, including the realisation of the KK dispersion relation.

Our starting point is therefore the formula (eq.(4.11) of ref.[11]) for the one-loop QED vacuum polarisation:

\[
\Pi_{\mu\nu}(k) = \frac{\alpha}{4\pi} \int_0^\infty ds \, e^{-ism^2} \int_{-1}^1 dv \left[ \frac{z_+z_-}{\tanh z_+ \tanh z_-} \exp \left( -\frac{s}{2} \sum_{\alpha=+,-} C_\alpha k.Z^2_\alpha \right) \times \sum_{\alpha,\beta=+,-} \left( S_{\alpha\beta}(Z^2_\alpha k.Z^2_\beta k - (Z^2_\alpha k)_\mu(Z^2_\beta k)_\nu) + A_{\alpha\beta} (Z_\alpha k)_\mu(Z_\beta k)_\nu - (g_{\mu\nu} k^2 - k_\mu k_\nu)(v^2 - 1) \right) \right]
\]
where
\[ z_+ = iesa \quad z_- = -esb \quad Z_+ = \frac{aF - b\tilde{F}}{a^2 + b^2} \quad Z_- = -i\frac{bF + a\tilde{F}}{a^2 + b^2} \] (4.2)

with
\[ a = \sqrt{\sqrt{F^2 + G^2} + F} \quad b = \sqrt{\sqrt{F^2 + G^2} - F} \] (4.3)

The explicit expressions for the dynamical coefficients \( C_{\alpha}, S_{\alpha\beta}, A_{\alpha\beta} \) are given later.

The dispersion relation for photon propagation is
\[ k^2 a_\nu - k.a k_\nu - \Pi_{\mu\nu}a^\mu = 0 \] (4.4)

Clearly, given the vacuum polarisation in the form (4.1), it is far from transparent what its implications are for the physics of photon propagation. However, simplifications and physical meaning rapidly become apparent as we translate into the null tetrad, NP form. In fact, it may well be that the most efficient method for calculating vacuum polarisation in background fields would be to adopt the NP formalism from the outset. Indeed, this may offer the only realistic hope of reproducing similar vacuum polarisation calculations for background gravitational fields (see section 5).

4.1 General structure of the vacuum polarisation and dispersion relations

As before, we take \( k^\mu = \sqrt{2}\omega\ell^\mu + O(\alpha) \) and polarisation vectors \( a^\mu = a_\alpha m^\mu + \beta \bar{m}^\mu \) and look for the eigenvectors and eigenvalues of the dispersion relation. Contracting with \( m_\nu, \bar{m}_\nu \) in eq.(4.4), we have
\[ \begin{pmatrix} k^2 + \Pi_{m\bar{m}} & \Pi_{\bar{m}m} \\ \Pi_{mm} & k^2 + \Pi_{m\bar{m}} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \] (4.5)

where \( \Pi_{m\bar{m}} = \Pi_{\mu\nu}m^\mu \bar{m}^\nu \) etc.

The first step is to translate the various ‘kinematic’ terms in the vacuum polarisation tensor into NP form. The necessary intermediate formulae are collected in the appendix.

For the required contractions, we find
\[ m.Z_\pm.\ell = \frac{a - ib}{a^2 + b^2} \phi_0 \quad \bar{m}.Z_\pm.\ell = \mp \frac{a + ib}{a^2 + b^2} \phi_0^* \]
\[ \ell.Z_\pm^2.\ell = \pm \frac{1}{a^2 + b^2} 2\phi_0\phi_0^* \quad m.Z_\pm^2.\ell = \mp \frac{1}{a^2 + b^2} 2\phi_0\phi_1^* \quad \bar{m}.Z_\pm^2.\ell = \mp \frac{1}{a^2 + b^2} 2\phi_1\phi_0^* \]
\[ m.Z_\pm^2.m = \bar{m}.Z_\pm^2.\bar{m} = \pm \frac{1}{a^2 + b^2} 2\phi_0\phi_1^* \quad m.Z_\pm^2.\bar{m} = \bar{m}Z_\pm^2.m = \frac{1}{2} \pm \frac{1}{a^2 + b^2} 2\phi_1\phi_1^* \] (4.6)

where we have also used the results:
\[ a^2 + b^2 = 2\sqrt{F^2 + G^2} = 4\sqrt{X\chi} \]
\[ a^2 - b^2 = 2F = 4 \text{Re} (\phi_0\phi_2 - \phi_1^2) \quad 2ab = 2G = 4 \text{Im} (\phi_0\phi_2 - \phi_1^2) \] (4.7)
This immediately begins to show the simplifications which arise by using the NP components of the electromagnetic field strengths. Several remarkable cancellations now occur. To illustrate with just two examples, we find:

\[
S_{++} [ (Z_+^2)_{m\ell} \ell Z_+^{2 \ell} \ell \ell - (Z_+^2 \ell)_{m}(Z_+^{2 \ell})_{m}] \\
= 4\phi_0^2 \frac{1}{(a^2 + b^2)^2} \left( \phi_0^* \phi_2 - (\phi_1^*)^2 \right) S_{++} \\
= \frac{1}{4X} \phi_0^* \phi_2 S_{++} \tag{4.8}
\]

while

\[
S_{++} [ (Z_+^2)_{m\ell} \ell Z_+^{2 \ell} \ell \ell - (Z_+^2 \ell)_{m}(Z_+^{2 \ell})_{m}] \\
= \left[ \frac{1}{2} + \frac{1}{a^2 + b^2} 2\phi_1 \phi_1^* \right] \frac{1}{a^2 + b^2} 2\phi_0 \phi_0^* \phi_0 \frac{1}{a^2 + b^2} 2\phi_0 \phi_1^* S_{++} \\
= \frac{1}{4X} \phi_0 \phi_0^* S_{++} \tag{4.9}
\]

Collecting terms and substituting back into eq.(4.1), we therefore find the following simplified form for the relevant components of the vacuum polarisation tensor in the NP basis:

\[
\Pi_{AB}(k) = \frac{\alpha}{4\pi} \int_0^\infty ds e^{-ism^2} \int_{-1}^{1} dv \frac{z_+ z_-}{\tanh z_+ \tanh z_-} \exp \left( -is\omega^2 \phi_0 \phi_0^* \frac{(C_+ - C_-)}{2\sqrt{X}} \right) \tilde{\Pi}_{AB} \\
\tag{4.10}
\]

where

\[
\tilde{\Pi}_{\bar{m}m} = \omega^2 \phi_0 \phi_0^* \frac{1}{2\sqrt{X}} \left( S_{++} - S_{+-} + S_{-+} - S_{--} + A_{++} + A_{+-} - A_{-+} - A_{--} \right) \\
\tilde{\Pi}_{m\bar{m}} = \omega^2 \phi_0 \phi_0^* \frac{1}{2\sqrt{X}} \left( S_{++} - S_{+-} + S_{-+} - S_{--} + A_{++} + A_{+-} - A_{-+} - A_{--} \right) \\
\tilde{\Pi}_{mm} = \omega^2 \phi_0 \phi_0 \frac{1}{2X} \left( S_{++} - S_{+-} - S_{-+} + S_{--} + A_{++} + A_{+-} + A_{-+} + A_{--} \right) \\
\tilde{\Pi}_{\bar{m}\bar{m}} = \omega^2 \phi_0^* \phi_0 \frac{1}{2X} \left( S_{++} - S_{+-} - S_{-+} + S_{--} + A_{++} + A_{+-} - A_{-+} + A_{--} \right) \tag{4.11}
\]

Notice that up to this point we have assumed nothing about the explicit form of the coefficients $C_\alpha, S_{\alpha\beta}$ and $A_{\alpha\beta}$. Self-consistency is guaranteed, however, by certain explicit properties, e.g. both $A_{\alpha\beta}$ and $S_{\alpha\beta}$ are symmetric on $\alpha, \beta$ while all $S_{\alpha\beta}$ are real, $A_{++}, A_{--}$ are real and $A_{+-}, A_{-+}$ are pure imaginary. Imposing these properties, we therefore find in the $m, \bar{m}$ sector:

\[
\left( \begin{array}{cc}
\tilde{\Pi}_{\bar{m}m} & \tilde{\Pi}_{m\bar{m}} \\
\tilde{\Pi}_{\bar{m}m} & \tilde{\Pi}_{\bar{m}m}
\end{array} \right) = \frac{1}{2\omega^2} \left( \begin{array}{cc}
A \frac{1}{\sqrt{X}} \phi_0 \phi_0^* & B^* \frac{1}{\sqrt{X}} \phi_0 \phi_0^* \\
B \frac{1}{\sqrt{X}} \phi_0 \phi_0 & A \frac{1}{\sqrt{X}} \phi_0 \phi_0^*
\end{array} \right) \tag{4.12}
\]

with

\[
A = S_{++} - S_{+-} + A_{++} - A_{--} \tag{4.13}
\]

and

\[
B = S_{++} - 2S_{+-} + S_{-+} + A_{++} + 2A_{+-} + A_{--} \tag{4.14}
\]
This matrix has eigenvalues

\[ \frac{1}{2} \omega^2 \phi_0 \phi_0^* \frac{1}{\sqrt{\chi \chi'}} (A \pm \sqrt{B B'}) \quad (4.15) \]

Finally, putting all this together, we find the dispersion relation for photon propagation in a constant background electromagnetic field, as determined by the vacuum polarisation tensor (4.1), is:

\[ k^2 + \frac{\alpha}{4\pi} \int_0^{\infty} ds e^{-ism^2} \int_{-1}^{1} dv \frac{z_+ z_-}{\tanh z_+ \tanh z_-} \exp \left( -is \frac{1}{8} T_{\mu \nu} k^\mu k^\nu \frac{1}{\sqrt{\chi \chi'}} (C_+ - C_-) \right) \]

\[ \times \left[ \frac{1}{8} T_{\mu \nu} k^\mu k^\nu \frac{1}{\sqrt{\chi \chi'}} (A \pm \sqrt{B B'}) \right] = 0 \quad (4.16) \]

corresponding (see section 2) to linear polarisation vectors

\[ a^\mu_+ = \frac{1}{\sqrt{2}} \left( e^{i\theta} m + e^{-i\theta} \bar{m} \right) \]

\[ a^\mu_- = -\frac{i}{\sqrt{2}} \left( e^{i\theta} m - e^{-i\theta} \bar{m} \right) \quad (4.17) \]

with the phase defined by \( \phi_0^* \sqrt{B / \chi} = re^{i\theta} \).

The most remarkable feature of this result is the appearance of the null energy combination \( T_{\mu \nu} k^\mu k^\nu = 4\omega^2 \phi_0 \phi_0^* \), both as an overall factor just as in the low-frequency limit and also in the exponent which, as we shall see, controls the high-frequency limit. This is an exceptionally clear demonstration of the direct relation, valid for all frequencies or momenta, between the null energy condition and the presence or absence of superluminal propagation of light.

### 4.2 Dynamics and the frequency dependence of the refractive index

Having established that the relation between photon propagation and the null energy condition remains valid for all frequencies, the next step is to investigate the detailed dynamics of the dispersion relation. This is encoded in the coefficients \( C_\alpha, S_{\alpha \beta} \) and \( A_{\alpha \beta} \), which are functions only of the two Lorentz invariant combinations of the background electromagnetic field strengths, \( \mathcal{F} \) and \( \mathcal{G} \).

The required formulae can be extracted from the results derived in ref.[11]. We find

\[ C_\alpha = -\frac{1}{z_\alpha} \left( \frac{\cosh z_\alpha v}{\sinh z_\alpha} - \coth z_\alpha \right) \]

\[ S_{\alpha \beta} = -\frac{\cosh z_\alpha v \cosh z_\beta v}{\cosh z_\alpha \cosh z_\beta} + \frac{\sinh z_\alpha v \sinh z_\beta v}{\sinh z_\alpha \sinh z_\beta} \]

\[ A_{\alpha \beta} = -\left( \frac{\cosh z_\alpha v}{\sinh z_\alpha} - \coth z_\alpha + \tanh z_\alpha \right) (\alpha \leftrightarrow \beta) + \frac{\sinh z_\alpha v \sinh z_\beta v}{\cosh z_\alpha \cosh z_\beta} \quad (4.18) \]
Expanding these in a weak field expansion, we find
\[
C_\alpha = \frac{1}{2} (1 - v^2) \left( 1 - \frac{1}{12} (1 - v^2) z_\alpha^2 + \ldots \right)
\]
\[
S_{\alpha\beta} = - (1 - v^2) \left( 1 - \frac{1}{2} (1 - \frac{1}{3} v^2) (z_\alpha^2 + z_\beta^2) + \ldots \right)
\]
\[
A_{\alpha\beta} = - \frac{1}{4} (1 - v^2)^2 z_\alpha z_\beta + \ldots
\]
(4.19)

Recalling \(z_+ = i e sa\) and \(z_- = -e sb\), and noting \(a + ib = 2\sqrt{\lambda}\), we verify that the combinations of the \(S_{\alpha\beta}\) and \(A_{\alpha\beta}\) occurring in eqs.(4.13) and (4.14) have precisely the correct \(X, \bar{X}\) dependence to cancel the inverse field strength factors. Obviously this must happen to ensure that the dispersion relation involves only positive powers of the background field strengths, but it is a very non-trivial consistency check. For the coefficients \(A\) and \(B\) we find explicitly
\[
A = - e^2 s^2 \sqrt{X \bar{X}} (1 - v^2)(3 - \frac{1}{3} v^2) + \ldots
\]
\[
B = e^2 s^2 X (1 - v^2)^2 + \ldots
\]
(4.20)
while
\[
C_+ - C_- = e^2 s^2 \frac{1}{6} (1 - v^2)^2 \sqrt{X \bar{X}}
\]
(4.21)

While we could continue this analysis for a general constant background field with arbitrary \(F\) and \(G\), since our main interest is in the frequency dependence of the dispersion relation we now simplify to the important special case of a pure magnetic field.

In this case, \(F = \frac{1}{2} B^2, G = 0\) so \(a = \sqrt{2F}, b = 0\) and \(z_+ = i es\sqrt{2F}, z_- = 0\). It is convenient to introduce the notation \(z = -iz_+ = esB\), with the hyperbolic functions of \(z_+\) becoming trigonometric functions of \(z\). Then, since \(B\) is real, the eigenvalues of the

\[\text{The dictionary between our notation and that of ref.}[11] \text{ is:}\]
\[
C_\alpha = - \frac{1}{z_\alpha} (A_{B12} - A_{B11})
\]
\[
S_{\alpha\beta} = S^0_{B12} S^0_{B12} - S^0_{F12} S^0_{F12}
\]
\[
A_{\alpha\beta} = - \left[ (A^0_{B12} - A^0_{B11}) (A^0_{B12} - A^0_{B22} + A^0_{F22}) \right] + A^0_{F12} A^0_{F12}
\]

where the worldline formalism coefficient functions are:
\[
S^0_{B12} = \frac{\sinh z_\alpha v}{\sinh z_\alpha}
\]
\[
S^0_{F12} = G_{F12} \frac{\cosh z_\alpha v}{\cosh z_\alpha}
\]
\[
A^0_{B12} = \frac{\cosh z_\alpha v}{\sinh z_\alpha} - \frac{1}{z_\alpha}
\]
\[
A^0_{B11} = A^0_{B22} = \coth z_\alpha - \frac{1}{z}
\]
\[
A^0_{F11} = A^0_{F22} = \tanh z_\alpha
\]
\[
G^2_{F12} = 1.
\]

---

While we could continue this analysis for a general constant background field with arbitrary \(F\) and \(G\), since our main interest is in the frequency dependence of the dispersion relation we now simplify to the important special case of a pure magnetic field.

In this case, \(F = \frac{1}{2} B^2, G = 0\) so \(a = \sqrt{2F}, b = 0\) and \(z_+ = i es\sqrt{2F}, z_- = 0\). It is convenient to introduce the notation \(z = -iz_+ = esB\), with the hyperbolic functions of \(z_+\) becoming trigonometric functions of \(z\). Then, since \(B\) is real, the eigenvalues of the

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\]
\[
A^0_{F11} = A^0_{F22} = \tanh z_\alpha
\]
\[
G^2_{F12} = 1.
\]
dispersion relation simply involve the combinations: 7
\[
\frac{1}{2}(A + B) = \frac{\cos z v}{\cos z} - \frac{v \sin z v}{\sin z} - \frac{2 \cos z v - \cos z}{\cos z \sin^2 z} = -\frac{1}{4}(1 - v^2)(1 + \frac{v^2}{3})z^2 + O(z^4)
\]
\[
\frac{1}{2}(A - B) = (1 - v^2) - \frac{\cos z v}{\cos z} + \frac{v \sin z v}{\sin z} = -\frac{1}{2}(1 - v^2)(1 - \frac{v^2}{3})z^2 + O(z^4) \quad (4.22)
\]
while the exponent involves
\[
C_+ - C_- = -\frac{1}{2}(1 - v^2) + \frac{\cos z v - \cos z}{z \sin z} = \frac{1}{24}(1 - v^2)^2 z^2 + O(z^4) \quad (4.23)
\]

Collecting everything, we find that the dispersion relation (4.16) for a pure magnetic background field becomes:
\[
k^2 + \frac{\alpha}{4\pi} \int_{0}^{\infty} \frac{ds}{s} e^{-ism^2} \int_{-1}^{1} dv \frac{z}{\tan z} \exp \left( -is^3 e^2 B^2 \omega^2 \frac{1}{2z^2} (C_+ - C_-) \right) \times \left[ s^2 e^2 B^2 \omega^2 \frac{1}{2z^2} (A \pm B) \right] = 0 \quad (4.24)
\]
This is our final analytic result for the dispersion relation. We highlight three features. First, notice how the null energy combination \( T_{\mu \nu} T^{\mu \nu} \) automatically projects out the relevant component of the background field, in this case the component orthogonal to the direction of propagation, i.e. \( B^2_{\perp} = B^2_1 + B^2_2 \).

Next, note that the angle \( \theta \) in eq.(4.17) defining the direction of linear polarisation for the eigenvalues of \( k^2 \) is given in this case by \( \tan \theta = -\frac{\text{Im} \phi_0}{\text{Re} \phi_0} = \frac{B_1}{B_2} \). This shows that the eigenvectors \( a_{\pm}^\mu \) correspond to polarisation perpendicular (+) and parallel (−) to the projection of the \( B \) field in the plane orthogonal to the direction of propagation.

Finally, as we now show explicitly, the presence of the factor \( \omega^2 \) in the exponent in eq.(4.24) (inherited from the null energy term \( T_{\mu \nu} k^\mu k^\nu \)) means that for high frequencies the exponent will oscillate rapidly and in fact drive the whole integral to zero. This is the mechanism which ensures that in the high frequency limit the QED refractive index becomes unity, ensuring \( v_{\text{ref}} = 1 \).

To see this in detail, consider eq.(4.24) in the weak field approximation. The appropriate small dimensionless parameter is \( e B_{\perp}/m^2 \). The dispersion relation reduces to
\[
k^2 - \frac{\alpha}{8\pi} \left( \frac{eB_{\perp}}{m^2} \right)^2 \omega^2 \int_{-1}^{1} dv \left( 1 - v^2 \right) \left\{ \frac{1}{2} \left( 1 + \frac{v^2}{3} \right) \right\} \left( 1 - \frac{v^2}{4} \right) \times \int_{0}^{\infty} dt \ t \ \exp \left( -\frac{1}{48} \left( 1 - v^2 \right)^2 \left( \frac{eB_{\perp}}{m^2} \right)^2 \frac{\omega^2 \ t^3}{m^2} \right) = 0 \quad (4.25)
\]

These expressions can be directly compared with the classic work on photon propagation in a magnetic field by Tsai and Erber [19, 20]. The equivalences are: \( \frac{1}{2}(A + B) \rightarrow -\frac{\text{Im} \phi_0}{\text{Re} \phi_0} N_{\perp} \) and \( \frac{1}{2}(A - B) \rightarrow -\frac{\text{Im} \phi_0}{\text{Re} \phi_0} N_{\parallel} \), eqs.(49) and (48) of ref.[19, 20] respectively. The exponent also reproduces the equivalent expression in ref.[19, 20].
for the $\{+\}$ eigenvalues respectively. The corresponding result for the weak-field QED refractive index is therefore

$$n(\omega) = 1 - \frac{\alpha}{16\pi} \left( \frac{eB_\perp}{m^2} \right)^2 \int_{-1}^{1} dv \left( 1 - v^2 \right) \left\{ \frac{1}{2} \left( 1 + \frac{v^2}{3} \right) \right\}$$

$$\times \int_{0}^{\infty} dt \ t \ \exp \left( -i \left[ t + \frac{1}{48} (1 - v^2)^2 \left( \frac{eB_\perp}{m^2} \right)^2 \frac{\omega^2}{m^2} \right] \right) \ (4.26)$$

The (inverse) phase velocity and the absorption coefficient are determined by the real and imaginary parts of $n(\omega)$. The important observation is that the term in the exponent proportional to $t^3$ is vital for the high frequency behaviour of $n(\omega)$. Although it is suppressed by the weak-field parameter $eB_\perp/m^2$, it cannot be dropped because it is accompanied by the arbitrarily large $\omega^2/m^2$ factor. This is why it is necessary to evaluate this exponential contribution to the vacuum polarisation, rather than restrict to a simple power series expansion in the background field. Indeed, this is one of the major challenges in extending this analysis to background gravitational fields.

In the low-frequency limit, the $t$ integral is purely real and reduces to $\int_{0}^{\infty} dt \ t \ e^{-it} = -1$. This leaves

$$n(0) = 1 + \frac{\alpha}{90\pi} \left( \frac{eB_\perp}{m^2} \right)^2 [4_+, 7_-] \ (4.27)$$

in agreement with the result (2.35) derived from the Euler-Heisenberg low-energy effective action.

In order to map out the full frequency dependence of the refractive index, we need the following special functions defined by Airy integrals: 8

$$\int_{0}^{\infty} dt \ t \ \cos \left[ x \left( t + \frac{t^3}{3} \right) \right] = \ -\frac{1}{x^2} \ G(x^{\frac{2}{3}}) \ (4.28)$$

and

$$\int_{0}^{\infty} dt \ t \ \sin \left[ x \left( t + \frac{t^3}{3} \right) \right] = \ -\frac{1}{\sqrt{3}} \ K_\frac{2}{3} \left( \frac{2}{3} x \right) \ (4.29)$$

We follow the notation of ref.[28]. $Gi(x)$, defined via the integral

$$Gi(x) = \frac{1}{\pi} \int_{0}^{\infty} dt \ \sin \left( xt + \frac{t^3}{3} \right)$$

can be expressed in terms of the Airy functions $Ai(x)$ and $Bi(x)$, or alternatively hypergeometrics, as follows:

$$Gi(x) = \frac{1}{3} Bi(x) + \int_{0}^{\infty} dt \ \left[ Ai(x)Bi(t) - Ai(t)Bi(x) \right]$$

$$= \frac{1}{3} Bi(x) - \frac{1}{2\pi} z^{\frac{5}{2}} \ _6F_4 \left( \begin{array}{c} 1, 2, 5, 7, 4, 1 \end{array} ; \begin{array}{c} 6, 6, 3, 1296 z^6 \end{array} \right) - \frac{1}{40\pi} z^{\frac{7}{2}} \ _6F_4 \left( \begin{array}{c} 1, 7, 4, 5, 11, 6 \end{array} ; \begin{array}{c} 6, 3, 3, 1296 z^6 \end{array} \right)$$

(Note the typographical error in ref.[28] where the factor ‘$z^5$’ is omitted.) The prefactor in our definition (4.28) of $G(x)$ compensates the asymptotic behaviour $Gi(x) \sim -\frac{1}{x^2} + \ldots$ as $x \rightarrow \infty$. In eqs.(4.30) and (4.31) the change of variable $u = (1 - v^2)$ has been made to simplify the final formulae.
Here, we have defined $G(x) = -\pi x^2 \frac{d}{dx} Gi(x)$, where the function $Gi(x)$ is related to the standard Airy functions.

After some further simplification, and introducing the rescaled frequency $\Omega = \frac{eB_\perp m^2}{\omega m}$, we find the following results for the real and imaginary parts of the refractive index:

\[
\text{Re } n(\Omega) = 1 + \frac{\alpha}{\pi} \left(\frac{eB_\perp}{m^2}\right)^2 \frac{1}{24} \int_0^1 du\ u\ (1-u)^{-\frac{1}{2}} \left\{ \frac{(1 - \frac{u}{4})}{(1 + \frac{u}{2})} \right\} G\left(\frac{1}{4} u \Omega\right)^{-\frac{3}{4}}
\]

and

\[
\text{Im } n(\Omega) = 1 + \frac{\alpha}{\pi} \left(\frac{eB_\perp}{m^2}\right)^2 \frac{2}{3\sqrt{3}} \int_0^1 du\ u^{-1}\ (1-u)^{-\frac{1}{2}} \left\{ \frac{(1 - \frac{u}{4})}{(1 + \frac{u}{2})} \right\} \Omega^{-2} K_{\frac{3}{2}}\left(\frac{3}{8} u \Omega\right)^{-1}
\]

\[\text{(4.30)}\]

\[\text{(4.31)}\]

Figure 3: This figure shows the real part of the QED refractive index $\text{Re } n(\Omega)$ for a weak background magnetic field. The horizontal axis $\ln \Omega$ measures frequency on a logarithmic scale, with $\Omega = \frac{eB_\perp m^2}{\omega m}$. The magnified vertical scale corresponds to setting $\frac{\alpha}{\pi} \left(\frac{eB_\perp}{m^2}\right)^2 = 1$ in eq.(4.30). The solid (dotted) line shows the refractive index for the polarisation $a_+^\parallel$ ($a_+^\perp$) orthogonal (parallel) to the magnetic field.

These expressions are evaluated numerically and the results plotted in Figs. 3,4,5. The refractive index for QED shows the same essential features as the single absorption band model described in section 3. The absorption coefficient $\text{Im } n(\Omega)$ due to pair creation is positive and shows the expected single peak. The real part $\text{Re } n(\Omega)$ initially rises from the values (4.27) then falls away gradually to just below 1 before approaching the asymptotic value 1 from below, with frequency dependence $\Omega^{-\frac{4}{3}}$. In terms of the phase velocity $v_{ph}(\omega) = 1/\text{Re } n(\omega)$, the low-frequency limit is subluminal, while asymptotically $v_{ph}(\omega)$ approaches $v_{wf} = 1$ from the superluminal side.

It is worth noting that the absorption band in QED is very broad. This means that there is an appreciable difference between the frequency $\Omega_{\max} \simeq 2.5$ defining the maximum of the absorption coefficient and the frequency $\Omega_0 \simeq 20$ where (with the definition following
the simple model (3.11)) the refractive index passes through 1. The fall-off of Re $n(\Omega)$ is slow and it has only a very shallow dip into the ‘superluminal’ regime. (These effects are partially masked by the use of the log scale for $\Omega$ in the figures.) The corresponding frequencies $\omega_{\text{max}}$ and $\omega_0$ are inversely proportional to the background field $B_\perp$.

We now comment briefly on the experimental implications of these results. The most striking phenomenon is ‘vacuum electromagnetic birefringence’. A general linear polarisation is a superposition of the eigenstates $a_{\uparrow}$ and $a_{\downarrow}$ which propagate with different phase velocities. After propagating through some distance, this velocity difference introduces a phase difference between the eigenstates, and the resulting wave then describes elliptic polarisation with the principal axis of the ellipse rotated relative to the initial direction of linear polarisation.\footnote{The PVLAS collaboration \cite{29, 30} has recently reported the observation of a rotation of the polarisation plane for linearly polarised light in a 5.5T magnetic field using their high-sensitivity optical ellipsometer. The origin of this observation is not yet understood. The vacuum electromagnetic birefringence described here is negligible in this system, prompting the speculation that the effect may be due to the existence of a new ultra-light axion-like particle with mass in the region of $10^{-3}$eV.}

Experiments to look for vacuum electromagnetic birefringence are currently being planned \cite{16, 17} using either an X-ray free-electron laser or a Thomson backscattering source to provide a high-frequency polarised pulse to scatter from a high-intensity laser background. Even for such high frequencies, however, we still only have $\frac{\omega}{m} \sim 10^{-2}$ and the attainable background fields from petawatt lasers with intensity around $10^{22}$ Wcm$^{-2}$ correspond to $eB_\perp/m^2 \sim 10^{-4}$. This means that currently envisageable experiments will only be sensitive to the low-frequency sector of the weak-field QED refractive index in Fig. 3, i.e. the region of normal dispersion (rising Re $n(\omega)$). The anomalous dispersion (falling Re $n(\omega)$) region correlated with the peak of the absorption coefficient Im $n(\omega)$ remains beyond the direct reach of current technology.
To conclude this section, we have shown in explicit detail how the KK dispersion relation (3.10) controls the relation between the IR and UV limits of the phase velocity in a well-understood quantum field theory. The absorption coefficient Im $n(\omega)$ due to pair creation is positive, in accordance with expectations from the optical theorem, ensuring that the low-frequency phase velocity $v_{\text{ph}}(0)$ is less than its UV limit $v_{\text{ph}}(\infty) = v_{\text{wf}}$. Causality requires $v_{\text{wf}} \leq 1$ and as expected we find that $v_{\text{wf}} = 1$ in QED, a result which arises technically through the appearance of a rapidly-oscillating exponential factor in the vacuum polarisation. It therefore follows that in QED there is indeed a causality bound $v_{\text{ph}}(0) < 1$. As we have seen, the IR limit of the phase velocity is determined by a combination of the null projection of the energy momentum tensor and the coefficients of the leading irrelevant operators in the low-energy (Euler-Heisenberg) effective action. It finally follows that, under the assumption that the null energy condition $T_{\mu\nu}k^\mu k^\nu \geq 0$ holds, causality does impose the positivity bound (2.33) on these parameters.

5. Superluminality and QED in gravitational fields

We now consider photon propagation in a background gravitational field, taking into account the quantum corrections induced by vacuum polarisation. Remarkably, in this case the low-frequency phase velocity $v_{\text{ph}}(0)$ can become superluminal.

The low-energy effective action for QED in a background gravitational field (the analogue of the Euler-Heisenberg action) was first calculated by Drummond and Hathrell in the original paper [4] in which they discovered the possibility of vacuum-polarisation induced superluminal propagation. Integrating out the electron field, the one-loop effective action is

$$\int dx \sqrt{-g} \mathcal{L} = \int dx \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \\
+ \left. \frac{\alpha}{720\pi} \frac{1}{m^2} \left( aR F_{\mu\nu} F^{\mu\nu} + b R_{\mu\nu\lambda\rho} F^{\mu\lambda} F^{\nu\rho} + c R_{\mu\nu\lambda\rho} F^{\mu\lambda} F^{\nu\rho} \right) \right] (5.1)$$

The coefficients of the leading irrelevant operators of type $\mathcal{R} FF$ are $a = -5$, $b = 26$ and $c = -2$.

To study photon propagation with this effective action, we again use the geometric optics methods described in section 2, making the additional assumption that the background field is slowly-varying on scales comparable to the wavelength of light. In contrast to the electromagnetic backgrounds, we cannot restrict to constant gravitational fields since for a spacetime of constant curvature\footnote{A spacetime of constant curvature is characterised by $R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}$, $R_{\mu\nu\lambda\rho} = \frac{1}{12} R (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda})$.} the only independent component is the Ricci scalar, $R = R_{\mu\nu\lambda\rho} k^\mu k^\nu k^\lambda k^\rho$. There is a further term, $dD_{\mu} F^{\mu\lambda} D_{\nu} F_{\nu\lambda}$ with $d = -24$, but this only affects the dispersion relation at $O(\alpha k^2) \sim O(\alpha^2)$ since its contribution to the Maxwell equation involves the equation of motion factor $D_{\mu} F^{\mu\lambda}$ which is itself already of $O(\alpha)$. The Ricci scalar term also does not contribute to the dispersion relation at $O(\alpha)$ for the same reason.
which does not affect the dispersion relation at $O(\alpha)$. In this case, omitting the terms involving derivatives of the curvatures, which are suppressed by $O(\lambda L)$ for wavelength $\lambda$ and curvature scale $L$, the modified Maxwell equation gives rise to the following relation (the analogue of eq.(2.16) for a background electromagnetic field):

$$k^2 a'' - k a k'' - \frac{\alpha}{720 \pi m^2} \left(2 b R \lambda \rho (k^\lambda k^\rho a'' - k^\lambda k^\rho a') - 8 c R \lambda \mu \rho k^\lambda k^\rho a'' a^\mu \right) = 0 \quad (5.2)$$

We now re-express this in terms of the Ricci and Weyl tensors and introduce the NP formalism. In an analogous way to the electromagnetic field strengths, the ten independent components of the Weyl tensor are represented by five complex scalars $\Psi_i \ (i = 0, \ldots, 4)$, where

$$\begin{align*}
\Psi_0 &= -C_{\mu \nu \lambda \rho} \ell^\mu m^\nu \ell^\lambda m^\rho \\
\Psi_1 &= -C_{\mu \nu \lambda \rho} \ell^\mu n^\nu \ell^\lambda m^\rho \\
\Psi_2 &= -C_{\mu \nu \lambda \rho} \ell^\mu m^\nu \bar{m}^\lambda n^\rho \\
\Psi_3 &= -C_{\mu \nu \lambda \rho} \ell^\mu n^\nu \bar{m}^\lambda n^\rho \\
\Psi_4 &= -C_{\mu \nu \lambda \rho} n^\mu \bar{m}^\nu \bar{m}^\lambda n^\rho
\end{align*} \quad (5.3)$$

The independent components of the Ricci tensor are described by four real and three complex scalars:

$$\begin{align*}
\Phi_{00} &= -\frac{1}{2} R_{\mu \nu} \ell^\mu n^\nu \\
\Phi_{11} &= -\frac{1}{4} R_{\mu \nu} (\ell^\mu n^\nu + m^\mu \bar{m}^\nu) \\
\Phi_{22} &= -\frac{1}{2} R_{\mu \nu} m^\mu n^\nu \\
\Lambda &= \frac{1}{24} \left[ 1 - \frac{1}{12} R_{\mu \nu} (\ell^\mu n^\nu - m^\mu \bar{m}^\nu) \right]
\end{align*} \quad (5.4)$$

As before, we take $k_\mu = \sqrt{2} \omega \ell^\mu$ and the polarisation vectors $a^\mu = \alpha m^\mu + \beta \bar{m}^\mu$. Then, eq.(5.2) becomes

$$\begin{pmatrix}
k^2 + \frac{\alpha}{90 \pi \omega^2 \frac{\omega^2}{m^2}} (b + 2c) \Phi_{00} & -\frac{\alpha}{90 \pi \omega^2 \frac{\omega^2}{m^2}} 2c \Psi^*_0 \\
-\frac{\alpha}{90 \pi \omega^2 \frac{\omega^2}{m^2}} 2c \Psi_0 & k^2 + \frac{\alpha}{90 \pi \omega^2 \frac{\omega^2}{m^2}} (b + 2c) \Phi_{00}
\end{pmatrix} \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = 0 \quad (5.5)$$

where we have used the important identity $C_{\mu \nu \lambda \rho} \ell^\mu m^\nu \ell^\lambda \bar{m}^\rho = 0$. The eigenvalues give the new light cone

$$k^2 + \frac{\alpha}{90 \pi \omega^2 \frac{\omega^2}{m^2}} \left[(b + 2c) \Phi_{00} \pm 2c \sqrt{\Psi_0 \Psi^*_0} \right] = 0 \quad (5.6)$$

The polarisation eigenstates are

$$\begin{align*}
a^\mu_+ &= \frac{1}{\sqrt{2}} \left( e^{i \theta} m^\mu + e^{-i \theta} \bar{m}^\mu \right) \\
a^\mu_- &= -\frac{i}{\sqrt{2}} \left( e^{i \theta} m^\mu - e^{-i \theta} \bar{m}^\mu \right)
\end{align*} \quad (5.7)$$
with the phase defined by $\Psi_0^* = |\Psi_0|e^{2i\theta}$.

The modified light cone is therefore

$$k^2 + \frac{\alpha}{360 \pi m^2} \left[ -(b + 2c) R_{\mu\nu} k^\mu k^\nu \pm 4c \left| C_{\mu\nu\lambda\rho} k^\mu m^\nu k^\lambda m^\rho \right| \right] = 0$$

(5.8)

corresponding to a phase velocity

$$v_{ph}(0) = 1 - \frac{\alpha}{360 \pi m^2} \left[ -(b + 2c) R_{\mu\nu} k^\mu k^\nu \pm 4c \left| C_{\mu\nu\lambda\rho} k^\mu m^\nu k^\lambda m^\rho \right| \right]$$

(5.9)

Notice immediately that using the Einstein equation, the Ricci tensor can be re-expressed in terms of the energy-momentum tensor by $R_{\mu\nu} = 8\pi T_{\mu\nu}$ (in $G = 1$ units) and the first term can be written in familiar form involving the null energy projection $T_{\mu\nu} k^\mu k^\nu$. In the gravitational case, however, the phase velocity also has a new, polarisation-dependent contribution involving the Weyl curvature. The relation between superluminality and the null energy condition is therefore more subtle for QED in a gravitational background field.

For Weyl-flat spacetimes, the situation is similar to the electromagnetic case. Assuming the null-energy condition holds, the sign of $R_{\mu\nu} k^\mu k^\nu$ is fixed and the question of whether $v_{ph}(0)$ is superluminal is determined by the coefficients of the leading irrelevant operators in the low-energy effective action. Specifically, a subluminal $v_{ph}(0) < 1$ requires a bound $(b + 2c) < 0$. Remarkably, this is violated by QED in a background curved spacetime where, as we have seen, $b = 26$ and $c = -2$. Both polarisations have the same phase velocity and there is no birefringence.

For Ricci-flat spacetimes, i.e. vacuum solutions of Einstein’s equation where $T_{\mu\nu} = 0$, eqs.(5.9) shows that for some directions and polarisations, photons must propagate with a superluminal $v_{ph}(0) > 1$ independent of the values of the low-energy couplings. If one polarisation is subluminal, the other is necessarily superluminal, satisfying the polarisation sum rule $\sum_{\pm} \delta v_{ph}(0) = 0$. The theory exhibits vacuum gravitational birefringence.

We illustrate this with two examples. First, consider a Friedmann-Robertson-Walker (FRW) spacetime [4, 8]. This has vanishing Weyl tensor, is spatially isotropic, and has Ricci curvature corresponding to the energy-momentum tensor $T_{\mu\nu} = (\rho + P)e_{\mu}^\ell e_{\nu}^\ell - P g_{\mu\nu}$, where $\rho$ is the energy density, $P$ is the pressure and $e_{\mu}^\ell$ specifies the time direction in a comoving orthonormal frame. The speed of light is the same in all directions and is polarisation independent. Eqns.(5.8), (5.9) give

$$k^2 = \frac{22}{45} \frac{\alpha}{m^2} T_{\mu\nu} k^\mu k^\nu$$

(5.10)

and

$$v_{ph}(0) = 1 + \frac{11}{45} \frac{\alpha}{m^2} (\rho + P)$$

(5.11)

This confirms the surprising result that the gravitational effect of a positive null energy projection $T_{\mu\nu} k^\mu k^\nu$ is to increase the phase velocity, resulting in superluminal propagation in Weyl-flat spacetimes.

\[12\] Note that $\frac{1}{2}(b + 2c) = 11$, revealing the universal factor 11 first identified by Latorre et al. in ref.[15] as characterising ‘energy density’ effects on the speed of light. Note that the polarisation sum $\sum_{\pm} v_{ph}(0)$ for electromagnetic fields involves the coefficients $4 + 7 = 11$. This universality arises because of the common dependence on the one-electron-loop vacuum polarisation.
As an example of a Ricci-flat background with non-vanishing Weyl tensor, consider Schwarzschild spacetime. In the NP formalism, a standard choice of null tetrad $\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu$ is made which reflects the properties of the null geodesics. Specifically, in terms of a conventional orthonormal tetrad $(e_t^\mu, e_r^\mu, e_\theta^\mu, e_\phi^\mu)$, we assign

$$
\ell^\mu = \frac{1}{\Delta}(r^2, \Delta, 0, 0) \\
n^\mu = \frac{1}{2r^2}(r^2, -\Delta, 0, 0) \\
m^\mu = \frac{1}{\sqrt{2r}}(0, 0, 1, i \csc \theta)
$$

(5.12)

where $\Delta = r^2 - 2Mr$. We restrict ourselves to planar motion, so without loss of generality set $\theta = \pi/2$ from now on.

To consider modifications to the principal null geodesics (radial directed) we choose $k^\mu = \sqrt{2} \omega \ell^\mu$. The derivation of the new light cones follows precisely eqs.(5.5) to (5.7) and we find

$$
k^2 \pm \frac{\alpha}{45\pi} \frac{\omega^2}{m^2} c |\Psi_0| = 0
$$

(5.13)

Now, it is a crucial feature of Schwarzschild spacetime (and indeed a large class of familiar black hole spacetimes such as Kerr) that it is of Petrov type D and the only non-vanishing NP Weyl scalar is $\Psi_2 = -M/r^3$. This is a consequence of the Goldberg-Sachs theorem and reflects the shear-free character of the principal null geodesics. It follows immediately that for radial motion, the light cone $k^2 = 0$ remains unchanged and $v_{ph}(0) = 1$.

Next, consider the modified null geodesics in the $r, \phi$ plane. Expressing the tangent vector $L^\mu = \frac{dx^\mu}{ds}$ in terms of the NP basis gives

$$
L^\mu = \left( \frac{r^2}{\Delta}, F, 0, \frac{D}{r^2} \right) = \frac{1}{2}(1 + F)\ell^\mu + \frac{r^2}{\Delta}(1 - F)n^\mu - \frac{i}{\sqrt{2}} \frac{D}{r}(m^\mu - \bar{m}^\mu)
$$

(5.14)

The Schwarzschild metric is

$$
ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{\Delta}{r^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
$$

The $t, \phi$ geodesic equations in the $\theta = \frac{\pi}{2}$ plane are

$$
\frac{d^2 t}{ds^2} + \left( \frac{\Delta'}{\Delta} - \frac{2}{r} \right) \frac{dt}{ds} \frac{dr}{ds} = 0 \\
\frac{d^2 \phi}{ds^2} + 2 \frac{d\phi}{ds} \frac{dr}{ds} = 0
$$

and integrate to give

$$
\frac{dt}{ds} = \frac{r^2}{\Delta} \\
\frac{d\phi}{ds} = \frac{1}{r^2} D
$$

Substituting into the metric, we then find that for a null interval

$$
\frac{dr}{ds} = \left( 1 - \frac{D^2 \Delta}{r^4} \right)^{\frac{1}{2}}
$$

The stated result for $L^\mu = \frac{dx^\mu}{ds}$ follows immediately. The radial geodesic is the special case where the impact parameter vanishes, $D = 0$. 

\[ \text{---} \]
where $D$ is the impact parameter and we have abbreviated $F = \left( 1 - \frac{D^2 \Delta}{r^4} \right)^{\frac{1}{2}}$. A convenient choice for the orthogonal circular polarisation vectors is

$$ M^\mu = \frac{1}{\sqrt{2r}} (0, \frac{D \Delta}{r^2}, i, -\alpha) = \frac{1}{\sqrt{2r}} \frac{D}{\Delta} \left( \frac{\Delta}{2r^2} \ell^\mu - n^\mu \right) + i \frac{1}{2} (1 + F) m^\mu + \frac{i}{2} (1 - F) \bar{m}^\mu \quad (5.15) $$

where $L.M = 0, \ M.\bar{M} = -1$. Now set $k^\mu = \sqrt{2} \omega L^\mu$ and take the polarisation eigenstates as

$$ a^\mu_+ = \alpha M^\mu + \beta \bar{M}^\mu, \quad a^\mu_- = -\frac{i}{\sqrt{2}} (M^\mu - \bar{M}^\mu) \quad (5.21) $$

where $\alpha = \Psi_2$ is real, the corresponding polarisation eigenstates are simply

$$ a^\mu_+ \text{ is the linear polarisation transverse to the direction of } k^\mu \text{ in the plane of motion, while } a^\mu_- \text{ is the orthogonal linear polarisation along } e_\theta. \text{ This provides an important example of gravitational birefringence with a superluminal } v_{ph}(0).$$

This brings us to the key question. If it is possible to have a superluminal phase velocity in the low-frequency limit, how can this be reconciled with causality, given the KK dispersion relation?

First, we review briefly what is known directly about dispersion for photon propagation in background gravitational fields. Adapting the pioneering background field calculations
of Barvinsky et al. [32, 33], we constructed the QED effective action at $O(RFF)$ to all orders in derivatives [7, 6]. This action is\(^{14}\)

\[
\int d^{4}x \sqrt{-g}\, \mathcal{L} = \int d^{4}x \sqrt{-g}\, \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right.
\]

\[
+ \frac{1}{m^{2}} \left( \bar{a}_{0} R F_{\mu\nu} F^{\mu\nu} + \bar{b}_{0} R_{\mu\rho\lambda\nu} F^{\mu\lambda} F^{\nu\rho} + \bar{c}_{0} R_{\mu\lambda\rho\nu} F^{\mu\lambda} F^{\nu\rho} \right)
\]

\[
+ \frac{1}{m^{4}} \left( \bar{a}_{1} R D_{\mu} F^{\mu\lambda} D_{\nu} F^{\nu\lambda} + \bar{b}_{1} R_{\mu\lambda\rho\nu} D_{\lambda} F^{\rho\lambda} D_{\mu} F^{\nu\rho} + \bar{c}_{1} R_{\mu\lambda\rho\nu} D_{\sigma} F^{\sigma\rho} D^{\lambda} F^{\mu\nu} \right)
\]

\[
+ \frac{1}{m^{6}} \left( \bar{b}_{2} R_{(1)} D_{\mu} F_{(1)} D_{\nu} F_{(1)} + \bar{b}_{3} R_{(2)} D_{\mu} F_{(2)} D_{\nu} F_{(2)} + \bar{c}_{3} R_{(3)} D_{\mu} F_{(3)} D_{\nu} F_{(3)} \right) \right]
\]

(5.22)

In this formula, the $\bar{a}_{n}$, $\bar{b}_{n}$, $\bar{c}_{n}$ are known form factor functions of three operators, i.e.

\[
\bar{a}_{n} \equiv a_{n} \left( \frac{D_{(1)}^{2}}{m^{2}}, \frac{D_{(2)}^{2}}{m^{2}}, \frac{D_{(3)}^{2}}{m^{2}} \right)
\]

(5.23)

where the first entry ($D_{(1)}^{2}$) acts on the first following term (the curvature), etc. It reduces to the Drummond-Hathrell action (5.1) in the low-energy limit.

This action contains all the information required to extend the dispersion relation from the zero-frequency limit into the low-frequency region. Extracting this from the action involves a number of subtleties described in full in ref.[6]. The result is an expression resembling eq.(5.8), viz.

\[
k^{2} + \frac{\alpha}{360 \pi m^{2}} \left[ F \left( \frac{k.D}{m^{2}} \right) R_{\mu\nu} k^{\mu} k^{\nu} \pm G \left( \frac{k.D}{m^{2}} \right) |C_{\mu\nu\lambda\rho} k^{\mu} m^{\nu} k^{\lambda} m^{\rho}| \right] = 0
\]

(5.24)

where the constant coefficients are replaced by functions of the operator $k.D$, which describes the variations of the curvature along the unperturbed null geodesics. The precise form of the functions $F$ and $G$ in terms of the form factors in eq.(5.22) is given in ref.[6].

However, it seems unlikely that this will be sufficient to describe the high-frequency behaviour of the dispersion relation. An equivalent approach applied to electrodynamics, in which we keep all orders in derivatives but restrict to lowest order in the field strengths, would miss the crucial exponent factor in the vacuum polarisation. This arises naturally in both the heat kernel and worldline path integral approaches and it would be extraordinary if the gravitational case was very different. This leads us to conjecture the following form

\[^{14}\text{This expression is slightly simplified from the form quoted in refs.[7, 6]. Here, we have used the identity}
\]

\[
\int d^{4}x \sqrt{-g}\, R_{\mu\nu} D_{\mu} D_{\nu} F_{\lambda\rho} F_{\lambda\rho} = - \int d^{4}x \sqrt{-g}\, R_{\mu\nu} D_{\mu} F_{\lambda\nu} D_{\lambda} F_{\nu\rho} + \frac{1}{4} \int d^{4}x \sqrt{-g}\, D^{2} R F_{\lambda\rho} F_{\lambda\rho}
\]

to relate two terms considered independent in refs.[7, 6]. The form factors are adapted accordingly. We have also omitted a term $D_{\mu} F^{\mu\lambda} D_{\lambda} F_{\nu\lambda}$ since, as explained above, it does not contribute to the dispersion relation at $O(\alpha)$.\]
for the dispersion relation for QED in a gravitational background, analogous to eqs.(4.16) and (4.24) for the electromagnetic case:

\[ k^2 + \frac{\alpha}{\pi} \int_0^\infty ds \ N(s, R) \ \exp \left[ -is \left( 1 + s^2 \Omega^2 P(s, R) \right) \right] = 0 \] (5.25)

Here, \( \Omega \sim \frac{R}{m^2} \frac{\omega}{m} \), where \( R \) denotes some generic curvature component, and \( N(s, R) \) and \( P(s, R) \) can be expanded in powers of curvatures and derivatives, with appropriate powers of \( s \). Neglecting the exponent factor is only valid for small \( \Omega \), i.e. for wavelengths satisfying \( \lambda \gg \frac{\lambda_c^2}{L} \), where \( \lambda_c \) is the electron Compton wavelength and \( L \) is a typical curvature scale. This is too near the IR to see the interesting structure in the refractive index.

Assuming this picture is correct, we will not be able to complete a direct evaluation of the full frequency dependence of the refractive index for QED in gravitational fields until new QFT techniques are developed which allow a calculation of the exponent contribution to the vacuum polarisation. The gravitational case has two main extra difficulties over pure electromagnetism – one is simply the plethora of indices associated with higher powers of curvature, but more importantly we cannot restrict to constant background fields since, as explained above, constant curvature spacetimes do not have the required spacetime anisotropy needed to modify the dispersion relation. The most promising approach would seem to involve formulating the heat kernel or worldline path integral methods using the NP formalism from the outset. For example, this would allow us to study black hole backgrounds in terms of only one non-vanishing NP curvature scalar \( \Psi_2 \). Even so, the analysis looks far from straightforward.

Now return to the KK dispersion relation (3.10). Recall that this predicts, under the usual assumption \( \text{Im} \ n(\omega) > 0 \), that the high-frequency limit \( v_{wf} = v_{ph}(\infty) \) of the phase velocity is greater than \( v_{ph}(0) \). However, for QED in gravitational fields, we have seen that it is possible to have a superluminal low-frequency limit \( v_{ph}(0) > 1 \). The question is, how can this be reconciled with causality?

We now consider the possible resolutions in turn:

1. It is possible that the KK dispersion relation does not hold in curved spacetime. However, the assumptions in the derivation of eq.(3.10) and similar relations for scattering amplitudes are the just the fundamental axioms of QFT, especially local Lorentz invariance and micro-causality, i.e. the vanishing of commutators of local operators for spacelike separations. These should also apply in curved spacetime, certainly for globally hyperbolic spacetimes (which admit a foliation into spacelike hypersurfaces), and we have examples of superluminal \( v_{ph}(0) \) in such spacetimes. Invalidity of the KK dispersion relation itself therefore looks unlikely.

2. It may be that \( v_{ph}(\infty) \) is indeed perturbatively greater than 1 and the physical light cone lies outside the geometric one, but that nevertheless the spacetime satisfies the criteria for stable causality with respect to the effective metric characterising the physical light cone. This is the situation discussed in section 3. However, this seems unlikely. If our conjecture (5.25) about the form of the vacuum polarisation is correct, it seems extremely probable that in the high-frequency limit the presence of the factor \( \Omega^2 \) will result in the exponent
term oscillating rapidly and, just as in the electromagnetic case, driving the entire integral to zero. The QFT evidence appears to point to $v_{ph}(\infty) = 1$.

3. The potential paradox would be resolved if it is possible to have $\text{Im } n(\omega) < 0$ in gravity. At first sight, this appears to be ruled out by the usual QFT identification of the imaginary part of a forward scattering amplitude with the total cross-section via the optical theorem. Indeed, the imaginary part of the vacuum polarisation should be related, just as in the case of a background electromagnetic field, to $e^+e^-$ pair creation, which is certainly absorptive. Another important physical process, photon splitting in a background field [34, 35] is also associated with $\text{Im } n(\omega)$ positive.

It is hard to see a way out. One line of thinking, as we explore further in the next section, is that a negative $\text{Im } n(\omega)$ corresponds to gain, i.e. an amplification of the amplitude and intensity of the light wave, rather than absorption. This may give us some hope that gravity is special, due to its ability to focus (see also ref.[36]). Recalling the geometric optics equation (2.6) for the variation of the amplitude $A$ along a geodesic, we readily see that including the $O(\alpha)$ contribution from the vacuum polarisation in the low-energy effective action (5.1) gives schematically

$$k^\mu D_\mu (\ln A) = -\frac{1}{\sqrt{2}} \omega \theta + \alpha k^\mu D_\mu \mathcal{R}$$

The optical scalar $\theta = \ell^\mu ; _\mu$ is the expansion coefficient (we have set $k^\mu = \sqrt{2}\omega \ell^\mu$ here), so for focusing geodesics ($\theta < 0$) the amplitude is increased. This is a purely classical effect. However, there is an additional $O(\alpha)$ quantum contribution depending on the change in the curvature along a geodesic. This arises from the vacuum polarisation and will in general show a frequency dependence. The expansion scalar $\theta$ and associated Raychaudhuri equation equally show $O(\alpha)$ corrections [37]. It seems natural that the $O(\alpha)$ corrections to the geodesics induced by the vacuum polarisation should be able to enhance focusing as well as divergence, corresponding to amplification as well as attenuation. This would suggest that a negative $\text{Im } n(\omega)$ may after all be a viable possibility in gravity.

Overall, however, the situation remains puzzling. Obviously what is needed to replace speculation by hard physics is a direct calculation of the momentum dependence of the vacuum polarisation for QED in a curved spacetime. This challenging technical problem remains open.

6. Laser-atom interactions in $\Lambda$-systems: EIT and Raman gain lines

In this section, we develop some intuition on how it may be possible to have $v_{ph}(\infty) < v_{ph}(0)$ and $\text{Im } n(\omega) < 0$ by studying light propagation in an atomic physics context. Specifically, we consider so-called ‘$\Lambda$-systems’, illustrated in Fig. 6.

Here, we have three atomic energy levels together with two lasers with frequencies $\omega_A$ and $\omega_B$ tuned nearly, but not exactly, to the frequency differences $\omega_{31} = \omega_3 - \omega_1$ and $\omega_{32} = \omega_3 - \omega_2$ of the levels. One of these lasers will be the ‘coupling’ (or ‘pump’) field while the other will be a ‘probe’ beam whose propagation through the coupled laser-atom system we wish to study. Experimentally, the atomic levels are typically realised by suitable hyperfine levels in ultracold gases of alkali metal atoms, Na, Rb, Cs.
The principal application of this set-up is to electromagnetically-induced transparency (EIT). For a review, see [38]. This allows the well-publicised phenomenon of ‘slow light’. In this case, laser B is a strong, coupling field and A is the probe. This gives rise to a double absorption line with an intermediate region where Im \( n(\omega) = 0 \) (transparency) and the refractive index is linear and rapidly rising (normal dispersion), giving rise to an extremely small group velocity for the probe light. This is the physics behind the slow light experiments. These have achieved group velocities as low as a few ms\(^{-1}\) in an ultracold gas of Na atoms [39]. Subsequent experiments [40, 41] have even demonstrated the possibility of stopping and reconstructing a light pulse in a Λ-system in either ultracold Na or warm Rb vapour cells.

For our purposes, we are more concerned with the opposite case where A is the coupling (or Raman pump) field while B is the probe laser. Here, we find a single Raman gain line, i.e. Im \( n(\omega) < 0 \). This demonstrates how it is possible to achieve the desired condition \( v_{ph}(\infty) < v_{ph}(0) \). Although not directly relevant to our discussion here, this also allows the remarkable phenomenon of a negative group velocity. To achieve this, we introduce two coupling fields so that we have a gain doublet with an intermediate region characterised by virtual transparency with a strong linear anomalous dispersion. If the slope of Re \( n(\omega) \) is sufficiently large and negative, this gives a negative group velocity – a light pulse traversing a cell with this coupled laser-atom system is advanced sufficiently that its peak leaves the cell before entering it. Although certainly counter-intuitive, this astonishing effect is not at all incompatible with causality and has been demonstrated experimentally by Wang et al. [14, 42, 43] in Λ-systems in warm atomic Cs gas.

We now present a unified analysis of Λ-systems which allows us to describe both the EIT and Raman gain scenarios in a common framework. EIT is obtained from our general formulae in one limit, while the complementary limit describes Raman gain lines.

It is most convenient to use a density matrix formulism. Writing the wave function for the 3-state Λ-system shown in Fig. 6 as

\[
|\psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle + c_3(t)e^{-i\omega_3 t}|3\rangle
\]  

we define density matrix elements as \( \rho_{ij} = c_i c_j^* \). The diagonal elements \( \rho_{11}, \rho_{22}, \rho_{33} \) are referred to as the ‘populations’ while the off-diagonal elements \( \rho_{31}, \rho_{32}, \rho_{21} \) are the ‘coherences’. It is also useful to introduce the notation \( \hat{\sigma}_{ij} = |i\rangle\langle j| \).

The laser-atom interaction is described by the potential \( V = e\mathbf{r}\cdot\mathbf{E}\cos\omega t \), where \( \omega \) is...
the laser frequency, with matrix elements
\[ V_{ij} = e\mu_{ij}|E| \cos \omega t = \Omega_{ij} \cos \omega t \] (6.2)
which defines the Rabi frequencies \( \Omega_{ij} \) in terms of the dipole matrix elements \( \mu_{ij} \). The polarisation of this laser-atom system is given in terms of the density matrix elements by
\[ P = N(\mu_{13}\rho_{31}e^{-i\omega_{31}t} + \mu_{23}\rho_{32}e^{-i\omega_{32}t} + \text{c.c.}) \] (6.3)
where \( N \) is the atomic number density, since the diagonal elements do not contribute by the dipole selection rule. Abbreviating \( \Omega_{31} = \Omega_A \) and \( \Omega_{32} = \Omega_B \) for convenience, and anticipating the solution of the Schrödinger equation for the time dependence of the density matrix to set
\[ \rho_{31} = \Omega_A e^{i\Delta_1 t} \hat{\rho}_{31} \quad \rho_{32} = \Omega_B e^{i\Delta_2 t} \hat{\rho}_{32} \] (6.4)
we have
\[ P = N(\mu_{13}\Omega_A e^{-i\omega_A t} \hat{\rho}_{31} + \mu_{23}\Omega_B e^{-i\omega_B t} \hat{\rho}_{32} + \text{c.c.}) \] (6.5)
Recalling that in general \( P = \chi \epsilon_0 |E| \cos \omega t \), we find the dielectric susceptibilities for a probe laser A in the coupled laser B–atom system, and vice-versa, are
\[ \chi_A = \frac{2N}{\epsilon_0} |\mu_{13}|^2 \hat{\rho}_{31} \quad \chi_B = \frac{2N}{\epsilon_0} |\mu_{23}|^2 \hat{\rho}_{32} \] (6.6)
This is related to the refractive index by \( n(\omega) = 1 + \chi/2 \), so we eventually find the following essential relations between the refractive indices for lasers A and B and the reduced density matrix elements \( \hat{\rho}_{ij} \):
\[ n_A(\omega_A) = 1 + \frac{N}{\epsilon_0} |\mu_{13}|^2 \hat{\rho}_{31} \quad n_B(\omega_B) = 1 + \frac{N}{\epsilon_0} |\mu_{23}|^2 \hat{\rho}_{32} \] (6.7)

The next step is to calculate the density matrix elements. They satisfy the Schrödinger equation
\[ \frac{d\rho}{dt} = [H_I, \rho] \] (6.8)
where the laser-atom interaction (6.2) is (implementing the standard ‘rotating-wave approximation’ [44])
\[ H_I = -\frac{1}{2}(\sigma_A e^{i\Delta_1 t}\hat{\sigma}_{31} + \sigma_B e^{i\Delta_2 t}\hat{\sigma}_{32} + \text{c.c.}) \] (6.9)
The important equations are those for the coherences. With the initial condition \( \rho_{11} = 1, \rho_{22} = \rho_{33} = 0 \) corresponding to a population initially in state \( |1\rangle \), we find
\[ \frac{d\rho_{31}}{dt} = \frac{i}{2} \Omega_A e^{i\Delta_1 t} + \frac{i}{2} \Omega_B e^{i\Delta_2 t} \rho_{21} - \frac{1}{2} \gamma_{31} \rho_{31} \]
\[ \frac{d\rho_{32}}{dt} = \frac{i}{2} \Omega_B e^{i\Delta_1 t} \rho_{21}^* - \frac{1}{2} \gamma_{32} \rho_{32} \]
\[ \frac{d\rho_{21}}{dt} = -\frac{i}{2} \Omega_A e^{i\Delta_1 t} \rho_{32}^* + \frac{i}{2} \Omega_B e^{-i\Delta_2 t} \rho_{31} - \frac{1}{2} \gamma_{21} \rho_{21} \] (6.10)
where we have included the total widths $\gamma_{ij}$ describing the spontaneous radiative decay of state $|3\rangle$ and dephasing [38]. It is straightforward to see that the time-dependence implied by these equations is consistent with eq.(6.4). For convenience, we also set $\rho_{21} = \exp(i\delta t)\hat{\rho}_{21}$, where $\delta = \Delta_1 - \Delta_2$. Then, with notation $\tilde{\Delta}_1 = \Delta_1 - i\gamma_{31}$ etc., we obtain the following set of algebraic equations for the reduced matrix elements $\hat{\rho}_{ij}$:

$$2\tilde{\Delta}_1 \hat{\rho}_{31} = 1 + \frac{\Omega_B}{\Omega_A} \hat{\rho}_{21}$$

$$2\tilde{\Delta}_2 \hat{\rho}_{32} = \frac{\Omega_A}{\Omega_B} \hat{\rho}_{21}^*$$

$$2\delta \hat{\rho}_{21} = -\Omega_A \Omega_B^* \hat{\rho}_{32}^* + \Omega_B^* \Omega_A \hat{\rho}_{31}$$  \hspace{1cm} (6.11)

Solving these by elimination of $\hat{\rho}_{21}$, we find the master formulae relevant for a general $\Lambda$-system:

$$\hat{\rho}_{31} = \frac{1}{2\Delta_1} \left( 1 + \frac{||\Omega_A|^2}{4\Delta_2^2}\right) \left[ 1 + \frac{||\Omega_A|^2}{4\Delta_2^2} - \frac{||\Omega_B|^2}{4\Delta_1^2}\right]^{-1}$$

$$\hat{\rho}_{32} = \frac{||\Omega_A|^2}{8\Delta_1^2\Delta_2^*} \left[ 1 + \frac{||\Omega_A|^2}{4\Delta_2^2} - \frac{||\Omega_B|^2}{4\Delta_1^2}\right]^{-1}$$  \hspace{1cm} (6.12)

At this point, we specialise to the two cases of interest – EIT and slow light, and Raman gain lines.

### 6.1 EIT and slow light

The standard EIT scenario is where $B$ is the coupling laser while $A$ is a weak probe. In this case, we are interested in the refractive index $n_A(\omega_A)$ for propagation of the probe laser light through the coupled laser B–atom system. This is given by the reduced density matrix element $\hat{\rho}_{31}$. Since the probe laser is weak, we assume $|\Omega_A|^2/\Delta_2 \ll 1$ and drop the corresponding term in eq.(6.12). We also assume that $\gamma_{21} \simeq 0$. This gives

$$\hat{\rho}_{31} = \frac{1}{2} \frac{1}{\Delta_1 - |\Omega_B|^2 \delta - i\frac{\gamma_{31}}{2}}$$  \hspace{1cm} (6.13)

In terms of the refractive index, this is:

$$\text{Re } n_A(\omega_A) = 1 + \frac{N}{2\epsilon_0} |\mu_{13}|^2 \frac{\left( \Delta_1 \delta - \frac{|\Omega_B|^2}{4}\right)\delta}{\left( \Delta_1 \delta - \frac{|\Omega_B|^2}{4}\right)^2 + \frac{1}{4} \gamma_{31}^2 \delta^2}$$

$$\text{Im } n_A(\omega_A) = \frac{N}{4\epsilon_0} |\mu_{13}|^2 \frac{\gamma_{31} \delta^2}{\left( \Delta_1 \delta - \frac{|\Omega_B|^2}{4}\right)^2 + \frac{1}{4} \gamma_{31}^2 \delta^2}$$  \hspace{1cm} (6.14)

The dependence on the probe frequency $\omega_A = \omega_{31} - \Delta_1$, for fixed $\omega_B$, is sketched in Fig. 7. The main feature in $\text{Im } n_A(\omega_A)$ is a double absorption line with an intermediate region around $\omega_A \simeq \omega_B + \omega_{21}$ ($\delta = 0$) where $\text{Im } n_A(\omega_A) \simeq 0$. At $\delta = 0$, the imaginary part vanishes so there is no absorption. This is electromagnetically-induced transparency.
However, although there is near complete transparency in this frequency region, there is significant dispersion – the refractive index $\Re n_A(\omega_A)$ rises steeply (normal dispersion) and nearly linearly. Recalling that the group velocity is given by

$$v_{gp}(\omega) = (n + \omega \frac{dn}{d\omega})^{-1}$$

(6.15)

a large positive slope $dn/d\omega$ corresponds to a small group velocity. Linearity implies negligible pulse distortion. This is the phenomenon of ‘slow light’, which has allowed group velocities as low as 17 ms$^{-1}$ to be achieved in ultracold Na atom cells [39].

### 6.2 Raman gain lines and gain-assisted anomalous dispersion

To describe the Raman gain scenario, we reverse the roles of the lasers so that now A corresponds to the coupling field and B is the probe. This time, therefore, we are interested in the refractive index $n_B(\omega_B)$ given by the reduced coherence $\hat{\rho}_{32}$ in eq.(6.12). In the relevant experiment [14, 42], we are concerned with the two-photon Raman transition $|1\rangle \rightarrow |2\rangle$ via the intermediate state $\sim |3\rangle$, so laser A is interpreted as the Raman pump and B the Raman probe field.

The relevant approximation here is that (while $|\Omega_A| \gg |\Omega_B|$) both fields are weak in the sense $|\Omega_A|^2/\Delta_2 \delta \ll 1$ and $|\Omega_B|^2/\Delta_1 \delta \ll 1$, while for such Raman experiments we can neglect the width of the state $|3\rangle$ so that $\gamma_{31} \simeq \gamma_{32} \simeq 0$. We also assume the common detuning $\bar{\Delta} = (\Delta_1 + \Delta_2)/2$ is much greater than the differential detuning $\delta = \Delta_1 - \Delta_2$.

With these simplifications, eq.(6.12) reduces to

$$\hat{\rho}_{32} = \frac{|\Omega_A|^2}{8\Delta^2} \frac{1}{\delta + \frac{1}{2}\gamma_{21}}$$

(6.16)

The corresponding refractive index is

$$\Re n_B(\omega_B) = 1 + \frac{N}{8\epsilon_0} |\mu_{23}|^2 \frac{|\Omega_A|^2}{\Delta^2} \frac{\delta}{\delta^2 + \frac{1}{4}\gamma_{21}^2}$$

$$\Im n_B(\omega_B) = -\frac{N}{16\epsilon_0} |\mu_{23}|^2 \frac{|\Omega_A|^2}{\Delta^2} \frac{\gamma_{21}}{\delta^2 + \frac{1}{4}\gamma_{21}^2}$$

(6.17)
The refractive index is sketched as a function of $\omega_B = \omega_{32} - \Delta_2$, for fixed $\omega_A$, in Fig. 8.

![Graph of refractive index](image.png)

**Figure 8**: The real (left) and imaginary (right) parts of the refractive index for the single Raman gain line given by eq.(6.17). This illustrates how it is possible to have $v_{ph}(\infty) < v_{ph}(0)$ in a system with $\text{Im} n(\omega) < 0$.

Here, the physics is quite different. This time we have $\text{Im} n_B(\omega_B) < 0$, indicating *gain* rather than absorption. The single Raman gain line is centred on $\omega_B \simeq \omega_A - \omega_{21}$ ($\delta = 0$). For low frequencies, the refractive index $\text{Re} n_B(\omega_B \simeq 0)$ is less than 1, corresponding to a superluminal phase velocity $v_{ph}(\omega_B \simeq 0) > 1$. For large $\omega_B$, the phase velocity tends to 1 as usual, $v_{ph}(\infty) = 1$. This behaviour of the refractive index and phase velocity is therefore compatible with the KK dispersion relation with $\text{Im} n_B(\omega_B) < 0$.

The Raman gain line therefore exhibits precisely the behaviour we speculate is necessary to resolve the potential paradox associated with the superluminal phase velocity in QED in a gravitational field. The essential feature is that the coupled laser-atom $\Lambda$-system in this case exhibits gain rather than absorption, i.e. $\text{Im} n(\omega) < 0$ rather than the more familiar $\text{Im} n(\omega) > 0$. Unlike the situation with absorption, where photons are scattered out of the probe beam, in the case of gain the intensity of the probe beam is increased as it passes through the laser-atom medium. This amplification of the electric field of the probe laser is due to the additional photons in the probe beam coming from the second stage of the Raman transition. It is maximised when the differential detuning $\delta \simeq 0$.

The crucial point for our considerations is that it is possible to have $v_{ph}(\infty) < v_{ph}(0)$ provided $\text{Im} n(\omega) < 0$, which means that the light must gain in intensity due to its passage through the medium. In the next section, we discuss whether this physical mechanism could be realised in the very different scenario of light propagation in gravitational fields.

Before leaving this section, however, we briefly mention an intriguing experimental application of the Raman gain scenario. This involves a system with *two* Raman pump fields with slightly different frequencies. This produces a *gain doublet*, with refractive index as sketched in Fig. 9.

The region between the two gain lines has $\text{Im} n(\omega) \simeq 0$, so there is virtually no gain, or absorption, and once again the system is transparent to the probe light. The refractive index in this region is steeply falling, and can be made approximately linear as shown – we therefore have a frequency range exhibiting transparency with linear anomalous dispersion.
Figure 9: The real (left) and imaginary (right) parts of the refractive index for the Raman gain doublet, exhibiting the phenomenon of gain-assisted anomalous dispersion and negative group velocity.

From eq.(6.15), we see that the group velocity in this region may be made extremely large and even, if the slope \( dn/d\omega \) is sufficiently large, we may find \( v_{gp} \) negative.

Remarkably, the phenomenon of a negative group velocity with negligible pulse distortion has been realised experimentally [14, 42, 43]. The set-up for these experiments uses a \( \Lambda \)-system where the energy levels shown in Fig. 6 are hyperfine levels of Cs atoms with \(|1\rangle = 6S_{1/2}|F = 4, m = -4\rangle\), \(|2\rangle = 6S_{1/2}|F = 4, m = -2\rangle\), and \(|3\rangle = 6P_{3/2}|F = 4, m = -3\rangle\) with the probe laser close to resonance with the Cs \( D_2 \) line. A laser pulse was passed through the Cs vapour cell with a group velocity of \( -0.003c \), so that it exits the cell before entering it. For more details of this intriguing experiment, see refs.[14, 42, 43].

7. Superluminality and UV completion

In this paper, we have analysed the propagation of light and the dispersion relation for the refractive index in three distinct quantum systems. For QED in a background electromagnetic field, we have used a recent evaluation of the one-loop vacuum polarisation to plot both the real and imaginary parts of \( n(\omega) \) over the full frequency range. Expressing the vacuum polarisation in the Newman-Penrose formalism allowed an elegant identification of the role of the null energy projection \( T_{\mu\nu}k^\mu k^\nu \), clarifying its relation to the possible occurrence of superluminal propagation. For QED in a gravitational field, only the low-frequency range of the refractive index is currently well understood. We reviewed and extended some of our earlier work in this field, demonstrating the occurrence of a superluminal low-frequency phase velocity, \( v_{ph}(0) > 1 \), for certain spacetimes. This was illustrated with examples of both Ricci-flat and Weyl-flat spacetimes, the former exhibiting gravitational birefringence as well as superluminality. Finally, we gave a unified treatment of propagation in coupled laser-atom \( \Lambda \)-systems exhibiting either electromagnetically-induced transparency with ‘slow light’ or Raman gain lines, where gain-assisted anomalous dispersion permitted the remarkable and experimentally observed phenomenon of a negative group velocity.

The purpose of drawing these various examples together here is to shed light on the proposal that the fundamental axioms of local quantum field theory or string theory imply...
constraints on the couplings of the corresponding IR effective field theory. That is, does the requirement that an effective field theory should have a consistent UV completion necessarily imply positivity constraints on the couplings of the leading irrelevant operators, with implications for the phenomenology of the low-energy theory?

In the case of QED in flat spacetime, this proposal works beautifully. As we explained, causality itself only requires the \( \text{UV limit} \) of the phase velocity to be subluminal, \( v_{\text{ph}}(\infty) < 1 \). But provided the imaginary part of the refractive index is positive, which we confirmed by an explicit evaluation showing \( \text{Im} \ n(\omega) \) has a single absorption line in a background electromagnetic field, the KK dispersion relation ensures \( v_{\text{ph}}(0) < v_{\text{ph}}(\infty) \). So causality does indeed require \( v_{\text{ph}}(0) < 1 \), which implies positivity constraints on the leading irrelevant operators of the low-energy effective theory for QED, viz. the Euler-Heisenberg effective action. Of course, explicit perturbative calculations confirm that these are satisfied.

QED in curved spacetime, on the other hand, presents a serious problem. Here we find examples where the low-energy effective theory has a superluminal phase velocity, \( v_{\text{ph}}(0) > 1 \). At first sight, this appears to be incompatible with causality. We identified three ways in which causality could be maintained: first, that an extension of the notion of ‘stable causality’ may permit \( v_{\text{ph}}(\infty) > 1 \) without causal violations; second, that the KK dispersion relation is not valid in its standard form for QFT in curved spacetime; and third, that at least in gravitational theories, \( \text{Im} \ n(\omega) \) (and by extension \( \text{Im} \ M(s,0) \)) may be negative. To explore the plausibility of this last option, we showed how a negative \( \text{Im} \ n(\omega) \) arises in Raman A-systems, where it is associated with gain rather than absorption. We have not found a similar example of \( \text{Im} \ n(\omega) < 0 \) in a non-gravitational quantum field theory and speculate that its occurrence for QED in curved spacetime may be related to the ability of gravitational interactions to focus geodesics and amplify light waves.

We do not yet know with certainty which of these is the physically realised option. However, each of them would invalidate the use of \( v_{\text{ph}}(0) < 1 \) (or related analyticity constraints on scattering amplitudes \( M(s,t) \)) as a criterion constraining the couplings of the IR effective field theory, since we must allow the possibility that the UV completion may involve gravity. This would appear to undermine the use of the positivity constraints as restrictions on low-energy phenomenology, though with better understanding they may still provide important information on the nature of the UV theory. In any case, understanding how to accommodate a superluminal low-frequency phase velocity with causality in quantum field theory is now an urgent problem whose resolution will surely shed light on the quantum theory of gravity.
A. Appendix: Electrodynamics in Newman-Penrose formalism

We collect here some formulae involving the Newman-Penrose formalism which are used in the main text. A useful review is contained in ref. [31]. The essential feature is the introduction of a null tetrad with vierbeins $e_A^\mu$ where the index ‘$A$’ refers to four null vectors $\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu$. These are chosen so that their only non-vanishing scalar products are $\ell.n = 1$ and $m.\bar{m} = -1$.\(^{15}\) In this basis, the metric is therefore

$$\eta_{AB} = e_A^\mu e_B^\nu g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (A.1)$$

It follows that

$$g_{\mu\nu} = (\ell_\mu n_\nu) - (m_\mu \bar{m}_\nu) \quad (A.2)$$

with notation $(ab) = ab + ba$. We also use $[ab] = ab - ba$. It follows that the trace in the null basis is $\text{tr}M_{AB} = \eta^{BA}M_{AB} = M_{(\ell n)} - M_{(m\bar{m})}$.

To describe the electromagnetic field strengths, we introduce three complex scalars $\phi_0, \phi_1, \phi_2$ defined as

$$\begin{align*} 
\phi_0 &= F_{\mu\nu} \ell^\mu n^\nu \\
\phi_1 &= \frac{1}{2} F_{\mu\nu} (\ell^\mu n^\nu + \bar{m}^\mu m^\nu) \\
\phi_2 &= F_{\mu\nu} \bar{m}^\mu m^\nu
\end{align*} \quad (A.3)$$

Inverting this, we recover the formula for $F_{\mu\nu}$ quoted in eq. (2.20), viz.

$$F_{\mu\nu} = - (\phi_1 + \phi_1^*) [\ell^\mu n^\nu] + (\phi_1 - \phi_1^*) [m^\mu \bar{m}^\nu] + \phi_2 [\ell^\mu m^\nu] + \phi_2^* [\bar{m}^\mu \bar{m}^\nu] - \phi_0^* [m^\mu m^\nu] - \phi_0 [\ell^\mu \bar{m}^\nu] \quad (A.4)$$

In terms of the vierbein basis, this is

$$F_{AB} = \begin{pmatrix} 0 & (\phi_1 + \phi_1^*) & \phi_0 & \phi_0^* \\
-(\phi_1 + \phi_1^*) & 0 & -\phi_2^* & -\phi_2 \\
-\phi_0 & \phi_2^* & 0 & -(\phi_1 - \phi_1^*) \\
-\phi_0^* & \phi_2 & (\phi_1 - \phi_1^*) & 0 \end{pmatrix} \quad (A.5)$$

It is useful to write explicit expressions for the $\phi_i$ in terms of the usual field strengths $E$ and $B$ in flat spacetime. Defining

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad (A.6)$$

\(^{15}\)In flat spacetime, a standard representation of these vectors would be $\ell^\mu = \frac{1}{\sqrt{2}} (1, 0, 0, 1)$, $n^\mu = -\frac{1}{\sqrt{2}} (1, 0, 0, -1)$, $m^\mu = \frac{1}{\sqrt{2}} (0, 1, i, 0)$, $\bar{m}^\mu = \frac{1}{\sqrt{2}} (0, 1, -i, 0)$. This is the origin of the factor $\sqrt{2}$ when we express the wave-vector as $k^\mu = \sqrt{2}\omega \ell^\mu$, identifying $\omega$ as the frequency (see section 2). We use a metric $g_{\mu\nu}$ with signature $(+,-,-,-)$. 


we find

\[ E_1 = \text{Re} (\phi_0 - \phi_2) \quad B_1 = - \text{Im} (\phi_0 - \phi_2) \]
\[ E_2 = \text{Im} (\phi_0 + \phi_2) \quad B_2 = \text{Re} (\phi_0 + \phi_2) \]
\[ E_3 = -2\text{Re} \phi_1 \quad B_3 = 2\text{Im} \phi_1 \]  \hspace{1cm} (A.7)

and the inverse relations

\[ \phi_0 = \frac{1}{2} ((E_1 + B_2) + i(E_2 - B_1)) \]
\[ \phi_1 = -\frac{1}{2} (E_3 - iB_3) \]
\[ \phi_2 = \frac{1}{2} ((-E_1 - B_2) + i(E_2 + B_1)) \]  \hspace{1cm} (A.9)

The corresponding results for the dual field strength tensor \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \) are \( \tilde{F}^{\mu\nu} = -i(\phi_1 - \phi_1^*)[\ell^\mu n^\nu] + i(\phi_1 + \phi_1^*)[m^\mu \bar{m}^\nu] + i\phi_2[\ell^\mu m^\nu] - i\phi_2^*[\ell^\mu \bar{m}^\nu] + i\phi_0^*[n^\mu m^\nu] - i\phi_0[n^\mu \bar{m}^\nu] \) that is,

\[
\tilde{F}_{AB} = \begin{pmatrix}
0 & i(\phi_1 - \phi_1^*) & i\phi_0 & -i\phi_0^*
\-i(\phi_1 - \phi_1^*) & 0 & i\phi_2 & -i\phi_2^*
\-i\phi_0 & -i\phi_2 & 0 & -(\phi_1 + \phi_1^*)
i\phi_0^* & i\phi_2^* & i(\phi_1 + \phi_1^*) & 0
\end{pmatrix} \]  \hspace{1cm} (A.11)

We also need expressions for the NP components of the squares of the field strengths \( (F^2)_{\mu\nu}, (F\tilde{F})_{\mu\nu} \) and \( (\tilde{F}\tilde{F})_{\mu\nu} \). In matrix notation, we find

\[
(F^2)_{AB} = \begin{pmatrix}
2\phi_0\phi_0^* & (F^2)_{en} & 2\phi_0\phi_1^* & 2\phi_1\phi_0^*
\(F^2)_{en} & 2\phi_2\phi_2^* & 2\phi_1\phi_2^* & 2\phi_2\phi_1^*
2\phi_0\phi_1^* & 2\phi_1\phi_2^* & 2\phi_0\phi_2^* & (F^2)_{m\bar{m}}
2\phi_1\phi_0^* & 2\phi_2\phi_1^* & (F^2)_{m\bar{m}} & 2\phi_2\phi_0^*
\end{pmatrix} \]  \hspace{1cm} (A.12)

where \( (F^2)_{en} = (F^2)_{n\ell} = (\phi_1 + \phi_1^*)^2 - \phi_0\phi_2 - \phi_0^*\phi_2^* \) and \( (F^2)_{m\bar{m}} = (F^2)_{m\bar{m}} = -(\phi_1 - \phi_1^*)^2 + \phi_0\phi_2 + \phi_0^*\phi_2^* \). Also,

\[
(F\tilde{F})_{AB} = i \begin{pmatrix}
0 & (F\tilde{F})_{en} & 0 & 0
\(F\tilde{F})_{en} & 0 & 0 & 0
0 & 0 & 0 & (F\tilde{F})_{m\bar{m}}
0 & 0 & (F\tilde{F})_{m\bar{m}} & 0
\end{pmatrix} \]  \hspace{1cm} (A.13)

where \( (F\tilde{F})_{en} = (F\tilde{F})_{n\ell} = (F\tilde{F})_{m\bar{m}} = (F\tilde{F})_{m\bar{m}} = (\phi_1 + \phi_1^*)(\phi_1 - \phi_1^*) - \phi_0\phi_2 + \phi_0^*\phi_2^* \).

Finally,

\[
(\tilde{F}\tilde{F})_{AB} = \begin{pmatrix}
2\phi_0\phi_0^* & (\tilde{F}\tilde{F})_{en} & 2\phi_0\phi_1^* & 2\phi_1\phi_0^*
(\tilde{F}\tilde{F})_{en} & 2\phi_2\phi_2^* & 2\phi_1\phi_2^* & 2\phi_2\phi_1^*
2\phi_0\phi_1^* & 2\phi_1\phi_2^* & 2\phi_0\phi_2^* & (\tilde{F}\tilde{F})_{m\bar{m}}
2\phi_1\phi_0^* & 2\phi_2\phi_1^* & (\tilde{F}\tilde{F})_{m\bar{m}} & 2\phi_2\phi_0^*
\end{pmatrix} \]  \hspace{1cm} (A.14)
where here \((\tilde{F} \tilde{F})_{tn} = (\tilde{F} \tilde{F})_{nt} = -(\phi_1 - \phi_1^*)^2 + \phi_0 \phi_2 + \phi_0^* \phi_2^*\) and \((\tilde{F} \tilde{F})_{mm} = (\tilde{F} \tilde{F})_{mm} = (\phi_1 + \phi_1^*)^2 - \phi_0 \phi_2 - \phi_0^* \phi_2^*\).

The Lorentz invariants are

\[
\mathcal{F} = 4 F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (E^2 - B^2) = 2 \text{Re} (\phi_0 \phi_2 - \phi_1^2)
\]

\[
\mathcal{G} = 4 F_{\mu\nu} \tilde{F}^{\mu\nu} = E B = 2 \text{Im} (\phi_0 \phi_2 - \phi_1^2)
\] (A.15)

and so \(\mathcal{X} = \frac{1}{2} (\mathcal{F} + i\mathcal{G}) = \phi_0 \phi_2 - \phi_1^2\). We also note the useful identity

\[
(\tilde{F} \tilde{F})_{\mu\nu} - (FF)_{\mu\nu} = 2 F_{\mu\nu}
\] (A.16)

The energy-momentum tensor is defined in pure Maxwell electrodynamics as

\[
T_{\mu\nu} = -F_{\mu\lambda} F^{\nu\lambda} + \frac{1}{4} g_{\mu\nu} F^2
\] (A.17)

In terms of the NP field strength scalars, this can be written as

\[
T_{\mu\nu} = 2 \left( \phi_1 \phi_1^*(\ell_\mu n_\nu) + \phi_1 \phi_1^*(m_\mu \bar{m}_\nu) - \phi_2 \phi_1^*(\ell_\mu m_\nu) - \phi_2^* \phi_1(\ell_\mu \bar{m}_\nu) - \phi_0^* \phi_1(n_\mu m_\nu) 
- \phi_0 \phi_1^*(n_\mu \bar{m}_\nu) + \phi_2 \phi_2^* \ell_\mu \ell_\nu + \phi_0 \phi_0^* n_\mu \bar{n}_\nu + \phi_2 \phi_0^* m_\mu m_\nu + \phi_2^* \phi_0 \bar{m}_\mu \bar{m}_\nu \right)
\] (A.18)

or rather more concisely,

\[
T_{AB} = 2 \begin{pmatrix}
\phi_0 \phi_0^* & \phi_1 \phi_1^* & \phi_0 \phi_1^* & \phi_1 \phi_0^* \\
\phi_1 \phi_1^* & \phi_2 \phi_2^* & \phi_1 \phi_2^* & \phi_2 \phi_1^* \\
\phi_0 \phi_1^* & \phi_1 \phi_2^* & \phi_0 \phi_2^* & \phi_1 \phi_1^* \\
\phi_1 \phi_0^* & \phi_2 \phi_1^* & \phi_1 \phi_1^* & \phi_2 \phi_0^*
\end{pmatrix}
\] (A.19)

This factorisation property is the origin of many of the simplifications which follow from the NP formalism for electrodynamics.
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