NON-LEFT-COMPLETE DERIVED CATEGORIES

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Abstract. We give some examples of abelian categories \( \mathcal{A} \) for which the derived category \( D(\mathcal{A}) \) is not left-complete. Perhaps the most natural of these is where \( \mathcal{A} \) is the category of representations of the additive group \( \mathbb{G}_a \) over a field \( k \) of characteristic \( p > 0 \).

Contents

0. Assumed background 1
1. The counterexample 1
2. The proof 4
References 6

0. Assumed background

In this article we assume the reader is familiar with derived categories and with \( t \)-structures on them. See Verdier [5] for the theory of derived categories, and Beilinson, Bernstein and Deligne [1, Chapter 1] for an introduction to \( t \)-structures.

1. The counterexample

Suppose \( \mathcal{A} \) is an abelian category and \( D(\mathcal{A}) \) is its derived category. For any object \( x \in D(\mathcal{A}) \), we write \( x^{\geq n} \) for the truncation of \( x \) with respect to the standard \( t \)-structure. We have canonical maps \( x^{\geq n} \to x^{\geq n+1} \), and a (non-canonical) map

\[
\varphi_x : x \longrightarrow \text{Holim} \ x^{\geq n}.
\]

The category \( D(\mathcal{A}) \) is said to be left-complete if, for every object \( x \in D(\mathcal{A}) \), any map \( \varphi_x \) as above is an isomorphism. Even though the map \( \varphi_x \) is not canonical, it can be shown that, for given \( x \), if one \( \varphi_x \) is an isomorphism then they all are.

The reader can find much more about left-complete categories in Lurie [3, Section 7] or [1, Subsection 1.2.1, more precisely starting from Proposition 1.2.1.17]. See also Drinfeld and Gaitsgory [2].

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In this note we will see how to produce many \( \mathcal{A} \) for which \( \mathbf{D}(\mathcal{A}) \) is not left-complete. Our counterexamples will be of a very special form, which allows us to easily compute the homotopy inverse limit \( \text{Holim } x^{\geq n} \). Let us now sketch what we will do.

We will suppose that the abelian category \( \mathcal{A} \) satisfies the axiom [AB4], that is coproducts are exact; this makes it easy to compute coproducts in the derived category \( \mathbf{D}(\mathcal{A}) \), just form the coproduct as complexes. Suppose \( A \) is an object in our [AB4] abelian category \( \mathcal{A} \), and let

\[
x = \prod_{i=0}^{\infty} A[i].
\]

It is clear that, for \( n > 0 \), we have

\[
x^{\geq -n} = \prod_{i=0}^{n} A[i] = \prod_{i=0}^{n} A[i],
\]

where the last equality is because finite coproducts agree with finite products. Now the homotopy inverse limit of the products is a genuine inverse limit, and we have

\[
\text{Holim } x^{\geq n} = \prod_{i=0}^{\infty} A[i].
\]

Thus our problem becomes to decide whether the map

\[
\prod_{i=0}^{\infty} A[i] \xrightarrow{\varphi} \prod_{i=0}^{\infty} A[i]
\]

is an isomorphism. Note that in this case the map is canonical; our homotopy inverse limit happens to be a genuine inverse limit, removing the arbitrariness. The left hand side is easy to work with; its cohomology is \( A \) in each degree \( n \leq 0 \). What we will show is how to produce examples where the right hand side has lots more cohomology. More precisely, we have

\[
\prod_{i=0}^{\infty} A[i] = A[0] \oplus \left( \prod_{i=1}^{\infty} A[i] \right)
\]

and the expectation would be for the second term to have a vanishing \( H^0 \); what we will show is how to produce non-zero classes in

\[
H^0 \left( \prod_{i=1}^{\infty} A[i] \right).
\]

It is time to disclose what will be our choice for the category \( \mathcal{A} \) and for the object \( A \in \mathcal{A} \).

**Construction 1.1.** Let \( k \) be a field, let \( R_1 \) be a finitely generated \( k \) algebra, and let \( \mathfrak{m} \) be a \( k \)-point of \( \text{Spec}(R_1) \). In other words, \( \mathfrak{m} \subset R \) is a maximal ideal with \( R_1/\mathfrak{m} \cong k \). We make a string of definitions:

(i) \( R_n = \otimes_{i=1}^{n} R_1 \), where the tensor is over the field \( k \).
(ii) The inclusion $R_n \to R_{n+1}$ is the inclusion of the tensor product of the first $n$ terms.

(iii) $R = \text{colim} R_n$.

(iv) The map $\Phi_i : R_1 \to R$ is the inclusion of the $i$th factor.

(v) The category $A$ will be the category of all those $R$–modules, on which $\Phi_i(m)$ acts trivially for all but finitely many $i$.

The object $A \in A$ will be the colimit over $n$ of the $R_n$–modules $k = \otimes_{i=1}^n [R_1/m]$.

The main result is

**Theorem 1.2.** Assume that $k = R_1/m$ is not projective over the localization $(R_1)_m$ of the ring $R_1$ at the maximal ideal $m$. With the category $A$ and the object $A \in A$ as in Construction 1.1, there is a non-zero element in

$$H^0 \left( \prod_{i=1}^{\infty} A[i] \right).$$

**Remark 1.3.** The case where $R_1 = k[x]/(x^p)$ is of particular interest. If the field $k$ is of characteristic $p$ then the category $A$ happens to be the category of representations of the additive group $\mathbb{G}_a$, and we learn that its derived category is not left-complete.

**Remark 1.4.** We trivially have

$$\prod_{i=1}^{\infty} A[i] = \left( \prod_{i=1}^{n} A[i] \right) \oplus \left( \prod_{i=n+1}^{\infty} A[i] \right),$$

and hence

$$H^0 \left( \prod_{i=1}^{\infty} A[i] \right) = H^0 \left( \prod_{i=1}^{n} A[i] \right) \oplus H^0 \left( \prod_{i=n+1}^{\infty} A[i] \right).$$

On the other hand, with the finite product we have no problem computing

$$H^0 \left( \prod_{i=1}^{n} A[i] \right) = H^0 \left( \prod_{i=1}^{n} A[i] \right) = 0,$$

and Theorem 1.2 now allows us to deduce that

$$H^0 \left( \prod_{i=n+1}^{\infty} A[i] \right) \neq 0.$$

Translating we have that

$$H^n \left( \prod_{i=1}^{\infty} A[i] \right) \neq 0$$

for all $n \geq 0$. The complexes $A[i]$, $i > 0$ all belong to $D(A)^{<0}$, but the product $\prod_{i=1}^{\infty} A[i]$ is not bounded above.

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2. The proof

We begin with a little lemma.

**Lemma 2.1.** Let $k$ be a field, and let $R$ and $S$ be finitely generated $k$-algebras. Suppose further that we are given $k$-points of $\text{Spec}(R)$ and $\text{Spec}(S)$; that is $m \subset R$ and $n \subset S$ are maximal ideals, with

$$R/m \cong k \cong S/n.$$  

Let $E$ be an injective envelope of $k = R/m$ over the ring $R$, and $F$ an injective envelope of $k = S/n$ over the ring $S$. Then $E \otimes_k F$ is an injective envelope of $k$ over the ring $R \otimes_k S$.

**Proof.** We will first prove the case where $R$ and $S$ are polynomial rings.

Let $R' = k[x_1, x_2, \ldots, x_m]$ be a polynomial ring, and let $m$ be the maximal ideal generated by $\{x_1, x_2, \ldots, x_m\}$. Then we know the injective envelope $E'$ of $k = R'/m$ explicitly: it is the quotient of $S = k[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1}]$ by the $R'$-submodule generated by all monomials $x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$ with at least one of the $i_j > 0$. As a $k$-vector space $E' = k[x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}]$, and the $R'$-module structure is obvious when we declare $x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m} = 0$ if some $i_j > 0$. If $S' = k[y_1, y_2, \ldots, y_n]$ and $n \subset S'$ is the ideal generated by $\{y_1, y_2, \ldots, y_n\}$, then the fact that

$$E' \otimes_k F' = k[x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}] \otimes_k k[y_1^{-1}, y_2^{-1}, \ldots, y_n^{-1}]$$

is the injective hull of $k$ over $R' \otimes S'$ is by inspection.

Now for the general case: assume $R = R'/I$ and $S = S'/J$ where $R'$ and $S'$ are polynomial rings, and $I \subset R'$ and $J \subset S'$ are ideals contained in the $m$ and $n$ above. Then the injective hull $E$ of $k = R/m$ over the ring $R$ is the largest $R$-submodule of the $R'$-module $E'$, that is the $R'$-submodule $E \subset E'$ of all elements annihilated by the ideal $I$. The lemma therefore comes down to the fact that the submodule of $E' \otimes_k F'$ annihilated by the ideal $I \otimes_k S' + R' \otimes_k J$ is precisely $E \otimes_k F$.  

**Proof of Theorem 1.2** Let $\overline{R}$ be the localization of $R_1$ at the maximal ideal $m$. We are assuming that $k$ is not projective over $\overline{R}$, that is the projective dimension of $k$ is at least one. Choose and fix a minimal free resolution of $k = \overline{R}/m\overline{R}$ as an $\overline{R}$-module. Let us write this resolution as

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$ 

Then the modules $P_i$ are all finite and free over the ring $\overline{R}$, the differentials are all matrices over $\overline{R}$, and the minimality guarantees that the entries in these matrices all belong to the ideal $\overline{m} = m\overline{R} \subset \overline{R}$. Now let $E$ be the $\overline{R}$-injective envelope of the module $k$; applying the functor $\text{Hom}_{\overline{R}}(-, E)$ to the projective resolution above, we produce an injective resolution $I^*$ of $k$, which we write out as

$$0 \longrightarrow k \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$
We know that each $I^j = \text{Hom}(P_j, E)$ is a finite coproduct of copies of $E$, and that the differentials $I^j \to I^{j+1}$ are matrices whose entries belong to the ideal $m$. The fact that the projective dimension of $k$ is at least one tells us that $P_1 \neq 0$, and therefore $I^1 \neq 0$. Note that an injective envelope $E$ of $k$ over the localized ring $\overline{R} = (R_1)_m$ is also an injective envelope of $k$ over the ring $R_1$, hence we have produced an injective resolution of $k$ over $R_1$. Next we

(i) Choose a non-zero element $a$ in the image of the map $k \to I^0$.

(ii) Choose a non-zero element $b \in I^1$, with $mb = 0$.

If we view $k$ as a module over the ring $R_n = \otimes_{i=1}^n R_1$, then the tensor product $J_n^* = \otimes_{i=1}^n I^*$ is certainly a resolution of $k$ as an $R_n$ module, and Lemma 2.1 guarantees further that

(iii) Each $J_n^i$ is injective as a module over $R_n$.

(iv) Let the inclusion $J_n^* \to J_{n+1}^*$ be the map taking $x \in J_n^*$ to

$$x \otimes a \in J_n^* \otimes I^0 \subset J_n^* \otimes I^* = J_{n+1}^*,$$

where $a \in I^0$ is as in (i) above. We define $J^*$ to be

$$J^* = \text{colim} J_n^*;$$

then $J^*$ is an injective resolution of $k$ in the category $A$.

To prove the theorem we need to find a non-zero element in $H^0(\prod_{i>0} k[i])$, and our next observation is that the product in the derived category $\prod_{i>0} k[i]$ is obtained as the ordinary product of injective resolutions. The complex $J^*[i]$ is an injective resolution of $k[i]$, and hence the derived product $\prod_{i>0} J^*[i]$ is just the usual product $\prod_{i>0} J^*[i]$. Now for every $i \geq 1$ let

$$S_i = \{i^2 + 1, \ldots, i^2 + i\},$$

and observe that the sets $S_i$ are disjoint. In the injective $R_{i^2+i}$–module

$$J_{i^2+i}^j = \prod_{\sum \ell_m = i} I^{\ell_1} \otimes I^{\ell_2} \otimes \cdots \otimes I^{\ell_{i^2+i}},$$

or more specifically in the summand

$$(I^0)^{\otimes i^2} \otimes (I^1)^{\otimes i},$$

we take the term

$$\lambda_i = a^{\otimes i^2} \otimes b^{\otimes i},$$

where $a \in I^0$ and $b \in I^1$ are as in (i) and (ii) above. The embedding $J_{i^2+i}^* \to J^*$ of (iv) gives us an element which we will denote $\lambda_i \in J^i$. The elements $\lambda_i$ have the properties

(v) Each $\lambda_i$ is a cycle; the differential $J^i \to J^{i+1}$ kills $\lambda_i$.

(vi) $\Phi_j(m)\lambda_i = 0$ for all $i$ and $j$. 

We are assuming $i > 0$, so each $\lambda_i$ must be a boundary because $H^i(J^*) = 0$. But if $\mu_i \in J^{i-1}$ maps to $\lambda_i$, then there must exist an integer $j \in S_i$ so that $\Phi_j(\mathfrak{m})$ does not kill $\mu_i$. Now form the element

$$\prod_{i=1}^{\infty} \lambda_i \in \prod_{i=1}^{\infty} J^i,$$

where the product is in the category of all $R$–modules.

**Caution 2.2.** The reader is reminded that the category $\mathcal{A}$ is a subcategory of the category of $R$–modules. Both categories have infinite products; the products in the category of $R$–modules are just the usual cartesian products, while the products in $\mathcal{A}$ are subtler. To form the product in $\mathcal{A}$ of a bunch of objects in $\mathcal{A}$, one first forms the usual cartesian product, and then consider inside it the largest object belonging to $\mathcal{A}$, that is the collection of all elements satisfying part (v) of Construction 1.1.

The element $\prod_{i=1}^{\infty} \lambda_i$ is a degree 0 cycle in the complex $\prod_{i \geq 1} J^i[\mathfrak{m}]$, and it is annihilated by $\Phi_j(\mathfrak{m})$ for all $j$. By Caution 2.2 we have that $\prod_{i=1}^{\infty} \lambda_i$ belongs to $\prod_{i=1}^{\infty} J^i$ even when the product is understood in $\mathcal{A}$. However, it is not a boundary in $\mathcal{A}$. If we try to express $\prod_{i=1}^{\infty} \lambda_i$ as the boundary of

$$\prod_{i=1}^{\infty} \mu_i \in \prod_{i=1}^{\infty} J^{i-1},$$

then we discover that each $\mu_i$ fails to be annihilated by some $\Phi_j(\mathfrak{m})$ with $j \in S_i$. As the $S_i$ are disjoint, this produces infinitely many $\Phi_j(\mathfrak{m})$ not annihilating $\prod_{i=1}^{\infty} \mu_i$, meaning it does not belong to $\mathcal{A}$.

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