A novel approach to
the computation of one-loop three- and four-point functions.
III - The infrared divergent case

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Abstract

This article is the third and last of a series presenting an alternative method to compute the one-loop scalar integrals. It extends the results of first two articles to the infrared divergent case. This novel method enjoys a couple of interesting features as compared with the methods found in the literature. It directly proceeds in terms of the quantities driving algebraic reduction methods. It yields a simple decision tree based on the vanishing of internal masses and one-pinched kinematic matrices which avoids a profusion of cases. Lastly, it extends to kinematics more general than the one of physical e.g. collider processes relevant at one loop. This last feature may be useful when considering the application of this method beyond one loop using generalised one-loop integrals as building blocks.

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1 Introduction

This article is the third of a triptych. The first one [1] presents a method exploiting a Stokes-type identity to compute one-loop three- and four-point scalar integrals for the real mass case. The second article [2] extends the results of the first paper to the case of general complex masses. The present article widens the results of [1] and [2] to the case where some internal masses are vanishing leading to infrared divergences. For the motivation of this work, we refer the reader to [1].

The scalar Feynman integrals for one-loop three- and four-point functions are all known and have been compiled in a useful article [3]. This article relies mainly on the results of other publications, especially the important work of Beenakker and Denner [4]. Let us mention also ref. [5] which provides a complete set of results for soft and/or collinear divergent four-point functions using different kind of IR regulators. The purpose of the present article is to extend these results for more general kinematics beyond those relevant for collider processes at the one-loop order. Note that despite the fact that some internal masses may vanish, the others can be real or complex and we treat both cases in this article. The soft and collinear divergences are dealt with using dimensional regularisation, $n = 4 - 2\varepsilon$, and doing an $\varepsilon$ expansion. The method presented enjoys a couple of interesting features as compared with the methods presented in [3] and references therein. Firstly, it directly proceeds in terms of the algebraic quantities and the derivation of the one-loop integral representations is more systematic and applies to the three- and four-point functions. Secondly, it unifies the different cases presented in [3], sixteen for the four-point case, with a rather simple decision tree relying on the vanishing of internal masses and the determinants of one-pinched kinematical matrices.

This novel method has one disadvantage already mentioned in [1] and [2], namely, an inherent increase of the number of dilogarithms involved. This increase does not show up for the three-point functions but, depending on the case, can be more serious for the four-point case. This point will not be discussed in this article because the method used here is the same, up to slight modifications, as the one used in [1] and [2]. So, the recipes proposed in the latter articles can be applied to counteract this increase of the number of dilogarithms.

The outline of this article follows closely the one of our preceding articles [1] and [2]. We start by considering the three-point function $I_3^n$ in a space-time dimension shifted by a small amount from 4 to $n$. The kinematics leading to infrared divergences is discussed. It is considered as a warm-up for sec. 3. We successively present two variants of the method. The simplest variant, labelled “direct way”, is presented in subsec. 2.1. It is well suited for the three-point function, but cannot be extended to the case of the four-point function. Then, in subsec. 2.2, practical implementation of the results of the preceding subsection is discussed and some explicit examples are computed and compared to [3]. In subsec. 2.3 we present an alternative coined “indirect way” easily applicable to the four-point case which is the subject of sec. 3. We first explain, in subsec. 3.1 how to extend the calculation of $I_4^1$ developed in [1]
to the case where the infrared divergences are regulated in $n$-dimension. The net result is that the four-point scalar integral can be decomposed on sectors labelled by three indices and a three dimensional integral over the first octant of $\mathbb{R}^3$ is associated to each sector. Then, two cases are distinguished depending on the sectors. In the first case, presented in 3.2, the determinant of the one-pinched kinematical matrix $S$ vanishes but not the internal mass associated to this sector: this case is met when a soft divergence appears. In the second case, presented in subsec. 3.3, both the determinant of the one-pinched $S$ matrix and the internal mass associated to this sector vanish: this case is met when a collinear or a soft and collinear divergence shows up. In sec. 3.4, the infrared divergent part of the scalar four-point integral is shown to be proportional to a three-point scalar integral as it should be. Some explicit examples are given and compared to the results found in the literature. We then conclude. Various appendices gather a number of utilities removed from the main text to facilitate its reading. Accordingly, in appendix A, we complete appendix D of [1] and appendix A of [2] by giving a missing case where the power of the integration variable is not an integer as required by dimensional regularisation. Appendix B shows how to compute an integral appearing in the three-point case in closed form. Appendix C collects a bunch of integrals required to compute the three- and four-point functions having soft and/or collinear divergences in the case of general complex masses. Then, appendix D goes through the examples given in subsec. 2.2 and explains in detail how the results obtained in the latter subsection can be found again from those derived in the “indirect way” case. Appendix E provides the way to compute the last integration in closed form in terms of dilogarithms for the case of infrared divergent integrals. It complements the appendix E of [1] and appendix B of [2]. Lastly, appendix F proves a tricky point used in sec. 3: for the real mass case, the sign of the vanishing imaginary part of the denominator in the last integral can be safely changed.

## 2 Three-point function with infrared divergences

![Figure 1: The triangle picturing the one-loop three-point function.](image)

When some internal masses vanish, divergences of collinear or soft origin appear and the approach shall be revisited. We regularise these divergences using dimensional regularisation, shifting the dimension of the space time by a small positive amount from 4 to $n = 4 - 2\varepsilon$
with $\varepsilon < 0$. After performing the loop momentum integral, instead of eq. (2.3) of ref. [1] we get:

\[
I_3^n = -\Gamma(1 + \varepsilon) \int \prod_{i=1}^{3} dz_i \delta(1 - \sum_{i=1}^{3} z_i) \left(-\frac{1}{2} z^T \cdot S \cdot z - i \lambda\right)^{1-\varepsilon} \tag{2.1}
\]

To appropriately shift the power of the denominator in eq. (2.1) so as to apply the Stokes identity (2.9) of ref. [1] as we did in the massive case, we use the following modified integral representation instead of identity (2.11) of the previous reference (cf. appendix B of [1]):

\[
\frac{1}{D^{1+\varepsilon}} = \frac{\nu}{B(2 - 1/\nu, 1/\nu)} \int_{0}^{+\infty} \frac{d\xi}{(D + \xi^\nu)^2}
\]

with $\nu = 1/(1 - \varepsilon)$. Instead of eq. (2.7) of ref. [1] we now get:

\[
I_3^n = -2^{1+\varepsilon} \frac{\Gamma(1 + \varepsilon)}{1 - \varepsilon} \frac{1}{B(1 + \varepsilon, 1 - \varepsilon)} \int_{0}^{+\infty} d\xi \int_{\Sigma_{bc}} dx_b dx_c \frac{dx_{b} dx_{c}}{(D(\alpha)(x_{b}, x_{c}) + \xi^\nu - i \lambda)^2} \tag{2.2}
\]

We otherwise proceed as in subsec. 2.1 of ref. [1]. The counterpart of eq. (2.24) of the same reference now reads:

\[
I_3^n = 2^\varepsilon \frac{\Gamma(1 + \varepsilon)}{1 - \varepsilon} \frac{1}{B(1 + \varepsilon, 1 - \varepsilon)} \times \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G)} \int_{0}^{+\infty} d\xi \int_{0}^{1} dx \frac{dx_{b} dx_{c}}{D(i)(x) + \xi^\nu - i \lambda} \tag{2.3}
\]

where $j \in S_3 \setminus \{i\}$ ($S_3 = \{1, 2, 3\}$). More precisely, we assume that $j$ is chosen to be $1 + (i$ modulo $3)$. Similarly to what we did for the three-point function in the massive case, one can also consider both a “direct way” and an “indirect way” in the IR case. We first focus on the “direct way” which provides a more straightforward and synthetic discussion of the various cases at hand. We then illustrate how these cases are involved in a few examples. The “indirect way” instead leads to a cumbersome split-up discussion. Notwithstanding the latter has its own interest. The calculation of the four-point one-loop integral relying on the approach described in this article proceeds along the “indirect way” as we found no extension of the “direct way” approach in this case. In refs. [6, 7] it was shown on general grounds using the decomposition²

\[
\det(S) I_4^n(S) = \sum_{i=1}^{4} \overline{b}_i I_3^n(S^{(i)}) - \det(G) (1 - 2 \varepsilon) I_4^{n+2}(S) \tag{2.4}
\]

that the infrared structure of any IR divergent four-point one-loop integral is carried by IR divergent three-point one-loop functions resulting from appropriate iterated pinchings.

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¹As in refs. [1, 2], we assume that the elements of the kinematic matrix $S$ have been made dimensionless by an appropriate rescaling.

²This decomposition has been discovered before and used for different purposes, see [8–10].
Therefore the comparison of the IR structures in both sides of eq. (2.4) proceeds most conveniently via a term by term comparison using the three-point one-loop functions decomposed according to the “indirect way” as well. In anticipation, we hereby give the key ingredients to perform this comparison, as well as the general recombination of these “indirect way” ingredients into the more compact expression obtained from the “direct way”, thereby checking their equivalence. The extensive collection of expressions computed in closed form which enable to perform detailed case-by-case comparisons is gathered in appendix D to lighten the presentation.

2.1 Direct way

Soft and/or collinear divergences are caused by some vanishing masses which make \( \det (S) \) vanish so that \( \Delta_2 = 0 \), whereas the other internal masses may or may not vanish as well - and may even be complex. We will keep the \(-i\lambda\) prescription having in mind that it is ineffective in the case of complex masses.

Starting from eq. (2.3) and performing the \( \xi \) integration using eq. (A.4), we end up with:

\[
P^n_3 = \frac{2\varepsilon}{\varepsilon} \Gamma(1+\varepsilon) \sum_{i\in S_3} \frac{\bar{b}_i}{\det(G)} \int_0^1 dx \ (D^{(i)}(x) - i\lambda)^{-1-\varepsilon}
\] (2.5)

In the general case \( D^{(i)}(x) \) depends on two internal masses squared \( m_j^2 \) and \( m_k^2 \) such that \( m_j^2 = D^{(i)}(0)/2 = \tilde{D}_{ik}/2 \) and \( m_k^2 = D^{(i)}(1)/2 = \tilde{D}_{ij}/2 \), cf. sec. 2 of [1]. We introduced the label \( k \) which is the only element of the complement of \( \{i,j\} \) in \( S_3 \). With our assumption on \( j \), this implies that \( k \equiv 1 + ((i+1) \mod 3) \). Let us focus on the function \( W \) given by:

\[
W \left( \det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik} \right) = \frac{2\varepsilon}{\varepsilon} \Gamma(1+\varepsilon) \int_0^1 dx \ (D^{(i)}(x) - i\lambda)^{-1-\varepsilon}
\] (2.6)

We remind (cf. eqs. (2.18), (2.19) and (2.20) of ref. [1]):

\[
D^{(i)}(x) = G^{(i)} x^2 - 2V^{(i)} x - C^{(i)}
\] (2.7)

with

\[
G^{(i)} = -S_{kk} + 2S_{kj} - S_{jj} = \det (G^{(i)})
\]

\[
V^{(i)} = S_{kj} - S_{jj} = \frac{1}{2} \left[ \det (G^{(i)}) - \tilde{D}_{ij} + \tilde{D}_{ik} \right]
\] (2.8)

\[
C^{(i)} = S_{jj} = -\tilde{D}_{ik}
\]

The Gram matrix \( G^{(i)} \) is built from the one-pinched \( S \) matrix \( S^{(i)} \), it is a real matrix which depends only on a squared external momentum in the three-point case. Notice that, in this case, \( G^{(i)} \) is a 1 \times 1 matrix and \( V^{(i)} \) a one-dimensional vector, this explains
the notations\footnote{Let us remind that $S_{jk} = S_{jk}^{(i)}$ for $j, k \neq i$.} used in eqs. (2.7) and (2.8). The knowledge of $\det (G^{(i)}), \tilde{D}_{ij}$ and $\tilde{D}_{ik}$ fully determines the polynomial $D^{(i)(j)}(x)$. These two internal masses may or may not vanish, hence three cases to be considered.

a) Neither $m^2_j$ nor $m^2_k$ vanishes

We perform a Taylor expansion\footnote{Here and below, only the terms in the $\varepsilon$-expansion providing the divergent and finite terms in the limit $\varepsilon \to 0$ are kept.} of $W\left(\det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik}\right)$ in $\varepsilon$:

$$W\left(\det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik}\right) = \frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left[ \int_0^1 \frac{dx}{D^{(i)(j)}(x) - i\lambda} - \varepsilon \int_0^1 \frac{dx \ln \left(\frac{D^{(i)(j)}(x) - i\lambda}{D^{(i)(j)}(x) - i\lambda}\right)}{D^{(i)(j)}(x) - i\lambda} \right]$$

(2.9)

Let us note $x_1$ and $x_2$ the two roots of $D^{(i)(j)}(x) - i\lambda$, given by, cf. eqs. (2.7), (2.8):

$$x_1 = \frac{\det (G^{(i)}) - \tilde{D}_{ij} + \tilde{D}_{ik} \pm \sqrt{\mathcal{K}\left(\det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik}\right) + i\lambda S_G}}{2 \det (G^{(i)})}$$

(2.10)

where $\mathcal{K}$ is the Källén function:

$$\mathcal{K}(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

(2.11)

and $S_G = \text{sign}(\det (G^{(i)}))$. Then, we introduce $J(x_1, x_2)$ and $K(x_1, x_2)$ defined by:

$$K(x_1, x_2) = \int_0^1 \frac{dx}{(x - x_1)(x - x_2)}$$

(2.12)

$$J(x_1, x_2) = \int_0^1 \frac{dx \ln \left(\frac{(x - x_1)(x - x_2)}{(x - x_1)(x - x_2)}\right)}{(x - x_1)(x - x_2)}$$

(2.13)

As it will become clear in the forthcoming paragraph on the origin of infrared singularities, only the case with $m^2_j$ and $m^2_k$ both real matters in practice, which makes the explicit calculation of $J(x_1, x_2)$ somewhat simpler\footnote{With real masses, $x_1$ and $x_2$ in eq. (2.10) have imaginary parts of opposite signs. This namely simplifies splittings and recombinations of logarithms of ratios in the explicit calculation of the function $J(x_1, x_2)$ computed in appendix B.}. The latter is provided in appendix B. The $x$ integration in the function $K(x_1, x_2)$ straightforwardly gives:

$$K(x_1, x_2) = \frac{1}{x_1 - x_2} \left[ \ln \left(\frac{x_1 - 1}{x_1}\right) - \ln \left(\frac{x_2 - 1}{x_2}\right) \right]$$

(2.14)
Thus $W \left( \det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik} \right)$ reads:

$$ W \left( \det (G^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik} \right) = \frac{1}{\varepsilon} \frac{\Gamma(1 + \varepsilon)}{\det (G^{(i)})} \left\{ \left[ 1 - \varepsilon \ln \left( \frac{\det (G^{(i)})}{2} - i \lambda \right) \right] K(x_1, x_2) - \varepsilon J(x_1, x_2) \right\} \quad (2.15) $$

**b) One and only one of $m_j^2$ and $m_k^2$ vanishes**

Let us assume that the vanishing internal mass is $m_j^2$. $D^{(i)(j)}(x)$ becomes:

$$ D^{(i)(j)}(x) = x \left( G^{(i)(j)}(x) - 2 V^{(i)(j)} \right) \quad (2.16) $$

From eqs. (2.6) and (2.8), $W \left( \det (G^{(i)}), \tilde{D}_{ij}, 0 \right)$ is thus of the form

$$ W \left( \det (G^{(i)}), \tilde{D}_{ij}, 0 \right) = \frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \int_0^1 dx x^{-1-\varepsilon} (a x + z)^{-1-\varepsilon} \quad (2.17) $$

where $a = \det (G^{(i)})$ is real and $z = - \det (G^{(i)}) + \tilde{D}_{ij} - i \lambda$ is complex. As $z$ and $a x + z$ have imaginary parts of the same sign $(a x + z)^{-1-\varepsilon}$ can be split as follows:

$$ (a x + z)^{-1-\varepsilon} = z^{-1-\varepsilon} \left( 1 + \frac{a}{z} x \right)^{-1-\varepsilon} $$

The r.h.s. of eq. (2.17) involves the Gauss hypergeometric function $2F1$:

$$ W \left( \det (G^{(i)}), \tilde{D}_{ij}, 0 \right) = - \frac{2^\varepsilon}{\varepsilon^2} \Gamma(1 + \varepsilon) z^{-1-\varepsilon} 2F1 \left( 1 + \varepsilon, -\varepsilon; 1 - \varepsilon; -\frac{a}{z} \right) \quad (2.18) $$

We use the identity [11]

$$ 2F1(a, b; c; w) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} 2F1(a, b; a + b - c + 1; 1 - w) $$

$$ + (1 - w)^{c - a - b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} 2F1(c - a, c - b; c - a - b + 1; 1 - w) $$

and the Pfaff identity

$$ 2F1(a, b; c; w) = (1 - w)^{-b} 2F1 \left( c - a, b; \frac{w}{w - 1} \right) $$

to rewrite:

$$ 2F1 \left( 1 + \varepsilon, -\varepsilon; 1 - \varepsilon; -\frac{a}{z} \right) $$

$$ = 2 \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left( -\frac{a}{z} \right)^\varepsilon \left( \frac{a + z}{z} \right)^{-\varepsilon} \left( -\frac{a}{z} \right)^{2\varepsilon} 2F1 \left( -2\varepsilon, -\varepsilon; 1 - \varepsilon; \frac{a + z}{a} \right) \quad (2.19) $$
Performing a Taylor expansion in $\varepsilon$ we get:

\[ _2F_1(-2\varepsilon, -\varepsilon; 1 - \varepsilon; \tau) = 1 + 2\varepsilon^2 \text{Li}_2(\tau) \]

and splitting $\ln((a + z)/z) = \ln(a + z) - \ln(z)$, we rewrite $W\left(\det (G^{(i)}), \tilde{D}_{ij}, 0\right)$ as:

\[
W\left(\det (G^{(i)}), \tilde{D}_{ij}, 0\right) = \frac{2\varepsilon}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{1}{\det (G^{(i)}) - \tilde{D}_{ij}} \left\{ \left( \frac{\det (G^{(i)})}{\det (G^{(i)}) - \tilde{D}_{ij} + i\lambda} \right)^{2\varepsilon} \left( \tilde{D}_{ij} - i\lambda \right)^{-\varepsilon} \left[ 1 + 2\varepsilon^2 \text{Li}_2 \left( \frac{\tilde{D}_{ij} - i\lambda}{\det (G^{(i)})} \right) \right] \right. \\
- 2 \left. \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left( \frac{\det (G^{(i)})}{\det (G^{(i)}) - \tilde{D}_{ij} + i\lambda} \right)^{\varepsilon} \left( \tilde{D}_{ij} - \det (G^{(i)}) - i\lambda \right)^{-\varepsilon} \right\} \tag{2.20}
\]

Making explicit $z = -\det (G^{(i)}) + \tilde{D}_{ij} - i\lambda$, $a + z = \tilde{D}_{ij} - i\lambda$ we get:

\[
W\left(\det (G^{(i)}), \tilde{D}_{ij}, 0\right) = -\frac{2\varepsilon}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{1}{\det (G^{(i)}) - \tilde{D}_{ij}} \left\{ \left( \frac{\det (G^{(i)})}{\det (G^{(i)}) - \tilde{D}_{ij} + i\lambda} \right)^{2\varepsilon} \left( \tilde{D}_{ij} - i\lambda \right)^{-\varepsilon} \left[ 1 + 2\varepsilon^2 \text{Li}_2 \left( \frac{\tilde{D}_{ij} - i\lambda}{\det (G^{(i)})} \right) \right] \right. \\
- 2 \left. \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left( \frac{\det (G^{(i)})}{\det (G^{(i)}) - \tilde{D}_{ij} + i\lambda} \right)^{\varepsilon} \left( \tilde{D}_{ij} - \det (G^{(i)}) - i\lambda \right)^{-\varepsilon} \right\} \tag{2.21}
\]

This formula is manifestly well-behaved as $\tilde{D}_{ij} \to 0$ ($m_k^2 \to 0$) yet it is not handy to expand around $\varepsilon = 0$. A more practical alternative may be obtained as follows. Firstly, we use the identities relating $\text{Li}_2(1 - w)$, $\text{Li}_2(w)$ and $\text{Li}_2(1/w)$ to change the argument of the $\text{Li}_2$ function, and the following relations:

\[
\ln \left( \frac{a}{z} \right) = \ln \left( -\frac{a}{z} \right) - i \pi S \left( az \right) \tag{2.22}
\]

\[
\ln \left( \frac{a + z}{a} \right) = \ln \left( -\frac{a + z}{a} \right) + i \pi S \left( az \right) \tag{2.23}
\]

\[
\ln \left( -\frac{a + z}{a} \right) = \ln \left( \frac{a + z}{z} \right) - \ln \left( -\frac{a}{z} \right) \tag{2.24}
\]

with

\[
S(z) = \text{sign} \left( \text{Im}(z) \right) \tag{2.25}
\]

so that the $\text{Li}_2$ function can be rewritten as:

\[
\text{Li}_2 \left( \frac{a + z}{a} \right) = \text{Li}_2 \left( -\frac{a}{z} \right) - \frac{\pi^2}{6} - \frac{1}{2} \ln^2 \left( -\frac{a}{z} \right) + \ln \left( -\frac{a}{z} \right) \ln \left( -\frac{a}{z} \right) \tag{2.26}
\]
Secondly, we Taylor expand around $\varepsilon = 0$ the $(-a/z)$ terms in eq. (2.20). We thus get:

$$ W \left( \det(\tilde{G}^{(i)}), \tilde{D}_{ij}, 0 \right) = -\frac{2^\varepsilon}{\varepsilon} \frac{\Gamma(1 + \varepsilon)}{z} \left[ \frac{2}{\varepsilon} (z)^{-\varepsilon} - \frac{1}{\varepsilon} (a + z)^{-\varepsilon} - 2 \varepsilon \text{Li}_2 \left( -\frac{a}{z} \right) \right] $$  \hspace{1cm} (2.27)

i.e. making explicit $z$ and $a$ in terms of $\det(\tilde{G}^{(i)})$ and $\tilde{D}_{ij}$:

$$ W \left( \det(\tilde{G}^{(i)}), \tilde{D}_{ij}, 0 \right) = \frac{1}{\varepsilon} \frac{\Gamma(1 + \varepsilon)}{\det(\tilde{G}^{(i)}) - \tilde{D}_{ij}} \times \left\{ \frac{2}{\varepsilon} \left[ \frac{1}{2} \left( \tilde{D}_{ij} - \det(\tilde{G}^{(i)}) \right) - i \lambda \right]^{-\varepsilon} - \frac{1}{\varepsilon} \left( \frac{\tilde{D}_{ij}}{2} - i \lambda \right)^{-\varepsilon} \\ - 2 \varepsilon \text{Li}_2 \left( \frac{\det(\tilde{G}^{(i)})}{\det(\tilde{G}^{(i)}) - \tilde{D}_{ij} + i \lambda} \right) \right\} $$  \hspace{1cm} (2.28)

which is both well behaved when $\tilde{D}_{ij} \to 0 \,(m_k^2 \to 0)$ and more compact.

c) Both $m_j^2$ and $m_k^2$ vanish

The function $D^{(i)(j)}(x)$ becomes:

$$ D^{(i)(j)}(x) = -G^{(i)(j)} \, x \,(1 - x) $$  \hspace{1cm} (2.29)

and we immediately get:

$$ W \left( \det(\tilde{G}^{(i)}), 0, 0 \right) = -\frac{1}{\varepsilon^2} \frac{\Gamma(1 + \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \left( \frac{-\det(\tilde{G}^{(i)})}{2} - i \lambda \right)^{-1-\varepsilon} $$  \hspace{1cm} (2.30)

In the limit $\tilde{D}_{ij} \to 0 \,(m_k^2 \to 0)$, eq. (2.28) or eq. (2.21) smoothly becomes eq. (2.30) as expected.

### 2.2 Practical implementation of the preceding cases and explicit examples

The various cases reviewed above may or may not be involved in a specific computation because some coefficients weighing the $W \left( \det(\tilde{G}^{(i)}), \tilde{D}_{ij}, \tilde{D}_{ik} \right)$ may vanish. In particular, as seen on eq. (2.5), when $\Delta_2 = 0$, the three-point function in dimension $4 - 2 \varepsilon$ is the sum of three two-point functions in dimension $2 - 2 \varepsilon$ These two-point functions correspond to the three distinct pinchings of the internal propagators of the three-point function. At first sight, one should worry that some of these two-point functions in low dimensions may badly diverge due to a threshold singularity which is however not present in the three-point function! For example, one of the pinchings of a three-point function having IR/collinear singularities would lead to a two-point function with the external legs on the mass shell.
of one of the propagators whereas the other propagator is massless. This would lead to a polynomial $D^{(i)(j)}(x) \propto x^2$ or $(1-x)^2$. Fortunately the corresponding $b$ coefficients weighting such pathological terms identically vanish, and the discussion which follows, illustrated with examples, elucidates why it happens so. Let us note $p_i^2 = s_i$ with $i = 1, 2, 3$.

1. A soft divergence occurs when the kinematic matrix $S$ has a vanishing line (and corresponding column). This happens whenever a massless propagator connects two vertices in which enter external momenta on the mass shells of the two other propagators. As the external momenta are real this case can occur only when the non vanishing internal masses are real. Let us assume, cf. fig. 1, that the internal mass squared $m_1^2$ vanishes whereas the external four-momenta $p_1$ and $p_2$ satisfy the mass shell conditions $s_1 = m_3^2$, $s_2 = m_2^2$. The $S$ matrix has the following texture:

$$S^{soft} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2m_2^2 & s_3 - m_2^2 - m_3^2 \\ 0 & s_3 - m_2^2 - m_3^2 & -2m_3^2 \end{pmatrix}$$

If one singles out row and column 1 in $S^{soft}$, the two-component vector $V^{(1)}$ is readily seen to vanish and so do the coefficients $b_2$ and $b_3$ which are proportional to the two components of $(G^{(1)})^{-1} V^{(1)}$; thus only $b_1$ differs from zero, cf. eq. (2.17) of ref. [1].

To illustrate this point, let us consider the case where $m_1 = 0$, $m_2 = m_3 = m$, $s_1 = s_2 = m^2$ and $s_3$ arbitrary. In this case, since $b_1$ is the only non vanishing coefficient, the polynomial $D^{(1)(2)}(x)$ involved in $I^3$ corresponds to the one appearing in the two-point function obtained by pinching the internal line with four-momentum $q_1$ (cf. fig. 1). This polynomial involves two masses (equal here) and this example corresponds to case a of the preceding section. In this simple case, the two roots of the polynomial $D^{(1)(2)}(x)$ is given by:

$$x_{1,2} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4(m^2 - i\lambda)}{s_3}}$$

with the property:

$$1 - x_1 = x_2$$

Injecting this property into eqs. (2.14) and (B.8), we get for the functions $J(x_1, x_2)$ and $K(x_1, x_2)$ in eq. (2.15):

$$J(x_1, x_2) = \frac{2}{x_1 - x_2} \left\{ \text{Li}_2 \left( \frac{x_2}{x_1} \right) - \text{Li}_2 \left( \frac{x_1}{x_2} \right) \right\} + \ln \left( -\frac{x_2}{x_1} \right) \ln \left( -(x_1 - x_2)^2 \right)$$

$$K^2(x_1, x_2) = \frac{2}{x_1 - x_2} \ln \left( -\frac{x_2}{x_1} \right)$$

We check numerically that we recover the result of [3].
2. A collinear divergence occurs when two internal masses vanish whereas the external four-momentum which enters into the vertex connecting the two adjacent massless propagators is lightlike (massless collinear splitting at this vertex). Note that the non vanishing internal mass can be real or complex. Let us assume that the labels of the two massless propagators are 1 and 2, with \( s_2 = 0 \): the \( S \) matrix has the following texture:

\[
S^{\text{coll}} = \begin{pmatrix}
  0 & 0 & s_1 - m_3^2 \\
  0 & 0 & s_3 - m_3^2 \\
 s_1 - m_3^2 & s_3 - m_3^2 & -2 m_3^2 \\
\end{pmatrix}
\] (2.35)

If one singles out row and column 3 in \( S^{\text{coll}} \), the Gram matrix \( G^{(3)} \) and the vector \( V^{(3)} \) read:

\[
G^{(3)} = \begin{pmatrix}
  2 s_1 & s_1 + s_3 \\
 s_1 + s_3 & 2 s_3 \\
\end{pmatrix}
\] (2.36)

\[
V^{(3)} = \begin{pmatrix}
  s_1 + m_3^2 \\
 s_3 + m_3^2 \\
\end{pmatrix}
\] (2.37)

A simple calculation yields:

\[
\sum_{i \in S_3 \setminus \{3\}} \left[ (G^{(3)})^{-1} \cdot V^{(3)} \right]_i = 1
\] (2.38)

so that \( \bar{b}_3 \) given by eq. (2.17) of ref. [1] vanishes, whereas \( \bar{b}_1 \) and \( \bar{b}_2 \) generically differ from zero.

Let us compute \( I^n_3 \) for this specific case. As \( \bar{b}_3 = 0 \), the relevant polynomials \( D^{(i)(j)}(x) \) are those of the two-point functions obtained in the two pinching configurations (cf. figure (1)) where either the internal line with four-momentum \( q_1 \) or the one with the four-momentum \( q_2 \) shall be pinched. As these two lines are associated to vanishing masses and the third propagator is associated to a non vanishing mass, the two polynomials both have one vanishing mass, this corresponds to case \( \mathbf{b} \) of the previous section. Starting with eq. (2.5) \( I^n_3 \) reads:

\[
I^n_3 = \frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left[ \frac{\bar{b}_1}{\det(G)} \int_0^1 dx \left( D^{(1)(2)}(x) - i \lambda \right)^{-1-\varepsilon} + \frac{\bar{b}_2}{\det(G)} \int_0^1 dx \left( D^{(2)(3)}(x) - i \lambda \right)^{-1-\varepsilon} \right]
\] (2.39)

It is better to change \( x \leftrightarrow 1 - x \) in the second integral in order to have a polynomial of the type (2.16) and write \( I^n_3 \) as:

\[
I^n_3 = \frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left[ \frac{\bar{b}_1}{\det(G)} \int_0^1 dx \left( D^{(1)(2)}(x) - i \lambda \right)^{-1-\varepsilon} + \frac{\bar{b}_2}{\det(G)} \int_0^1 dx \left( D^{(2)(1)}(x) - i \lambda \right)^{-1-\varepsilon} \right]
\]

\[
= \frac{\bar{b}_1}{\det(G)} W \left( \det (G^{(1)}), \tilde{D}_{12}, 0 \right) + \frac{\bar{b}_2}{\det(G)} W \left( \det (G^{(2)}), \tilde{D}_{21}, 0 \right)
\] (2.40)
Noting that \( b_1 = (s_3 - m_3^2)(s_1 - s_3) \) and \( b_2 = (s_1 - m_3^2)(s_3 - s_1) \), determining \( \det (G^{(1)}) \), \( \det (G^{(2)}) \) and \( \tilde{D}_{12} \) from the \( S \) matrix elements (cf. eq. (2.8)) and applying directly the result of eq. (2.28), we get:

\[
I_n^3 = \frac{\Gamma(1 + \varepsilon)}{s_1 - s_3} \left\{ -\frac{1}{\varepsilon^2} \left[ (-s_3 + m_3^2 - i\lambda)^{-\varepsilon} - (-s_1 + m_3^2 - i\lambda)^{-\varepsilon} \right] + \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i\lambda} \right) - \text{Li}_2 \left( \frac{s_1}{s_1 - m_3^2 + i\lambda} \right) \right\} \tag{2.41}
\]

Using the Landen identity (B.3) we recover the formula (4.8) of ref. [3] after some algebra.

3. Both a soft and a collinear divergence may occur at the same time, thereby proceeding from both cases 1. and 2. above.

Let us take the example of case 2 and specify \( s_1 = m_3^2 \). The texture of the \( S \) matrix becomes:

\[
S^{cs} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & s_3 - m_3^2 \\
0 & s_3 - m_3^2 & -2m_3^2
\end{pmatrix} \tag{2.42}
\]

Only \( b_1 \) does not vanish, so \( I_n^3 \) reads simply:

\[
I_n^3 = \frac{2\varepsilon}{\varepsilon} \frac{\Gamma(1 + \varepsilon)}{\det(G)} \frac{b_1}{\Gamma(1 - \varepsilon)} \int_0^1 dx (D^{(1)(2)}(x) - i\lambda)^{-1-\varepsilon} = \frac{\tilde{b}_1}{\det(G)} W \left( \det (G^{(1)}), \tilde{D}_{12}, 0 \right) \tag{2.43}
\]

Using eq. (2.28) and expressing \( \det (G^{(1)}) \) and \( \tilde{D}_{12} \) in terms of the \( S \) matrix elements (cf. eq. (2.8)), we get:

\[
I_n^3 = \frac{\Gamma(1 + \varepsilon)}{m_3^2 - s_3} \left\{ -\frac{1}{\varepsilon^2} \left[ (-s_3 + m_3^2 - i\lambda)^{-\varepsilon} + \frac{1}{2\varepsilon^2} (m_3^2 - i\lambda)^{-\varepsilon} \right] + \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i\lambda} \right) \right\} \tag{2.44}
\]

After some algebra, we recover\(^6\) formula (4.11) of ref. [3].

### 2.3 Indirect way

The present subsection provides the calculation according to the “indirect way”. The results presented are valid for both real and complex masses; unless explicitly specified the \( i\lambda \)

---

\(^6\)In contrast to the present eq. (2.44), eq. (4.11) of ref. [3] contains a factor \( \Gamma^2(1 - \varepsilon)/\Gamma(1 - 2\varepsilon) \) and an extra term \( +\pi^2/12 \) inside the brackets. Yet they cancel against each other when performing the \( \varepsilon \)-expansion at the appropriate order.

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prescription is kept having in mind that it is ineffective in the case of complex masses. Let us start from eq. (2.3). The $x$ integration is traded for a $\rho$ integration in a way very similar to the four-dimensional case (see subsubsec. 2.2.2 of ref. [1]) and we get:

$$I^n_3 = \sum_{i \in S_3} \frac{\bar{t}_i}{\det(G)} \sum_{j \in S_3 \setminus \{i\}} \frac{\bar{b}_{ij}^{(i)}}{\det(G^{(i)})} L^n_3 \left(0, \Delta_1^{(i)}, \tilde{D}_{ij}\right)$$

(2.45)

with:

$$L^n_3 \left(0, \Delta_1^{(i)}, \tilde{D}_{ij}\right) = \kappa_{IR} \int_0^{+\infty} d\xi \int_0^{+\infty} d\rho \left(\frac{\xi^\nu - i\lambda}{\xi^\nu - \Delta_1^{(i)} - i\lambda}(\xi^\nu + \rho^2 + \tilde{D}_{ij} - i\lambda)^{1/2}\right)$$

(2.46)

and:

$$\kappa_{IR} = 2^\varepsilon \Gamma(1 + \varepsilon) \frac{1}{\Gamma(1 - \varepsilon)} B(1 + \varepsilon, 1 - \varepsilon), \quad \nu = \frac{1}{1 - \varepsilon}$$

To handle the cases with soft and/or collinear divergences, the two relevant configurations are 1) $\tilde{D}_{ij} \neq 0$ and 2) $\tilde{D}_{ij} = 0$. Note that when both $\Delta_2$ and $\Delta_1^{(i)}$ vanish $L^n_3(0,0,\tilde{D}_{ij})$ is weighting a vanishing $b$, therefore it shall not be considered (cf. subsec. 2.2).

The $\rho$ integration can be done using appendix A as in the four dimensional case and as the outcome of this integration, we shall distinguish two cases depending on the sign of $\text{Im}(\Delta_1^{(i)})$.

1) $\tilde{D}_{ij} \neq 0$

1.a) $\text{Im}(\Delta_1^{(i)}) > 0$

This case covers in particular real masses. After the $\rho$ integration, $L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij})$ becomes:

$$L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij}) = \kappa_{IR} \int_0^{+\infty} d\xi \int_0^{+\infty} d\xi \left(\frac{1}{(\xi^\nu - i \lambda)(\xi^\nu - (1 - z^2) \Delta_1^{(i)} + z^2 \tilde{D}_{ij} - i \lambda)^{1/2}}\right)$$

(2.47)

The $\xi$ integration is performed first, using eq. (A.4) of the appendix A and $L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij})$ becomes:

$$L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij}) = -\frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \int_0^{1} dz \left(z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} - i \lambda\right)^{1-\varepsilon}$$

(2.48)
Expanding\(^7\) the r.h.s. of eq. (2.48) around \(\varepsilon = 0\) then gives:

\[
L^n_3 \left( 0, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = 2^\varepsilon \Gamma(1 + \varepsilon) \left[ -\frac{1}{\varepsilon} \int_0^1 \frac{dz}{z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} - i \lambda} + \int_0^1 \frac{dz}{z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} - i \lambda} \int_0^1 \frac{dz \ln(z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} - i \lambda)}{z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} - i \lambda} \right] (2.49)
\]

1.b) \(\text{Im}(\Delta_1^{(i)}) < 0\)

This case occurs only with complex masses, in a way such that the \(i \lambda\) prescriptions are overshadowed and ineffective: the latter are therefore dropped in this part. After \(\rho\) integration performed as in the four-dimensional case, we get (see eq. (A.7) of appendix A):

\[
L^n_3 \left( 0, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = -\kappa_{IR} \int_0^{+\infty} \frac{d\xi}{\xi^\nu - i \lambda} \left[ i \int_0^{+\infty} \frac{dz}{-\xi^\nu + \tilde{D}_{ij} z^2 + (1 + z^2) \Delta_1^{(i)}} + \int_1^{+\infty} \frac{dz}{\xi^\nu + \tilde{D}_{ij} z^2 - (1 - z^2) \Delta_1^{(i)}} \right] (2.50)
\]

Using identity (A.4), \(L^n_3 \left( 0, \Delta_1^{(i)}, \tilde{D}_{ij} \right)\) reads:

\[
L^n_3 \left( 0, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = \frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left[ -i \int_0^{+\infty} dz \left( z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} \right)^{-1-\varepsilon} + \int_1^{+\infty} dz \left( z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} \right)^{-1-\varepsilon} \right] (2.51)
\]

Expanding the terms in the square bracket in eq. (2.51) around \(\varepsilon = 0\) then gives:

\[
L^n_3 \left( 0, \Delta_1^{(i)}, \tilde{D}_{ij} \right) = 2^\varepsilon \Gamma(1 + \varepsilon) \left\{ \frac{1}{\varepsilon} \int_0^{+\infty} \frac{dz}{z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) + \Delta_1^{(i)}} + \int_1^{+\infty} \frac{dz}{z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)}} \right. \\
- \left. i \int_0^{+\infty} dz \ln(-z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)}) z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) + \Delta_1^{(i)} \right. \\
- \left. \int_1^{+\infty} dz \ln(z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)}) z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} \right\} (2.52)
\]

\(^7\)Let us remind that only the terms in the \(\varepsilon\)-expansion which provide the divergent and finite terms in the limit \(\varepsilon \to 0\) are kept.
For eqs. (2.49) and (2.52), the z integration can be then performed using appendix E.

Since the cuts of \((z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)})^{-1-\epsilon}\) and \(\ln(z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)})\) are the same, the discussion carried to extend the “indirect way” to the general complex mass case holds likewise here (cf. sec. 2 of ref. [2]). Eq. (2.51) can be rewritten as:

\[
L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij}) = -\frac{2^\epsilon}{\epsilon} \frac{\Gamma(1+\epsilon)}{2} \left\{ \int_0^1 + \int_{+\infty}^1 \right\} dz \left( z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} \right)^{-1-\epsilon} \tag{2.53}
\]

In eq. (2.53) let us add a vanishing contribution along the “contour at \(\infty\)” in the north-east quadrant \(\{\text{Re}(z) > 0, \text{Im}(z) > 0\}\) so as to concatenate the two contributions. The connected contour thus obtained can in turn be deformed into a finite contour \(\hat{(0, 1)}_{i,j}\) stretched from 0 to 1 as pictured on fig. 2 thereby unifying eqs. (2.48) and (2.53):

![Figure 2: Location of the relevant discontinuity cut \(C_{i,j}\) with respect to the two half straight lines \([0, +\infty]\) and \([1, +\infty]\) and deformation of the contour \((0, 1)\) partly wrapping the extremity of the cut.](image)

\[
L^n_3(0, \Delta_1^{(i)}, \tilde{D}_{ij}) = -\frac{2^\epsilon}{\epsilon} \frac{\Gamma(1+\epsilon)}{2} \int_{(0, 1)_{i,j}} dz \left( z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)} \right)^{-1-\epsilon} \tag{2.54}
\]

2) \(\tilde{D}_{ij} = 0\)

Here again we shall distinguish two cases depending on the sign of \(\text{Im}(\Delta_1^{(i)})\).
2.a) $\text{Im}(\Delta_1^{(i)}) > 0$
This case covers in particular real masses. The calculation initially amounts to setting $\tilde{D}_{ij} = 0$ in eq. (2.48), which becomes:

$$L_3^n(0, \Delta_1^{(i)}, 0) = -\frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left(-\Delta_1^{(i)} - i \lambda\right)^{-1-\varepsilon} \int_0^1 dz \left(1 - z^2\right)^{-1-\varepsilon}$$

(2.55)

The integration over $z$ is performed using eq. (C.6) and $L_3^n(0, \Delta_1^{(i)}, 0)$ becomes:

$$L_3^n(0, \Delta_1^{(i)}, 0) = \frac{1}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{\Gamma(1 - \varepsilon)^2}{\Gamma(1 - 2 \varepsilon)} \left(-2 \Delta_1^{(i)} - i \lambda\right)^{-1-\varepsilon}$$

(2.56)

2.b) $\text{Im}(\Delta_1^{(i)}) < 0$
Here again, this case occurs only with complex masses, with the $i \lambda$ prescriptions overshadowed and therefore dropped. The calculation initially amounts anew to setting $\tilde{D}_{ij} = 0$ in eq. (2.51) which becomes:

$$L_3^n(0, \Delta_1^{(i)}, 0) = \frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \left[-i \left(-\Delta_1^{(i)}\right)^{-1-\varepsilon} \int_0^{+\infty} dz \left(1 + z^2\right)^{-1-\varepsilon}
- \left(\Delta_1^{(i)}\right)^{-1-\varepsilon} \int_{1}^{+\infty} dz \left(z^2 - 1\right)^{-1-\varepsilon}\right]$$

(2.57)

The $z$ integrals are computed in appendix C (eqs. (C.4) and (C.5)). Hence for $L_3^n(0, \Delta_1^{(i)}, 0)$:

$$L_3^n(0, \Delta_1^{(i)}, 0) = -\frac{2^{-\varepsilon}}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{\cos(\pi \varepsilon)} \times \left[i \sin(\pi \varepsilon) \left(-\Delta_1^{(i)}\right)^{-1-\varepsilon} + \left(\Delta_1^{(i)}\right)^{-1-\varepsilon}\right]$$

(2.58)

Since $\text{Im}(\Delta_1^{(i)}) < 0$, $(\Delta_1^{(i)})^{-1-\varepsilon}$ may be rewritten as $-e^{i\pi \varepsilon} (-\Delta_1^{(i)})^{-1-\varepsilon}$, so that $L_3^n(0, \Delta_1^{(i)}, \tilde{D}_{ij})$ simplifies into:

$$L_3^n(0, \Delta_1^{(i)}, 0) = \frac{1}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \left(-2 \Delta_1^{(i)}\right)^{-1-\varepsilon}$$

(2.59)

which coincides with eq. (2.56).

Let us show now the equivalence between the “indirect way” and the “direct way” starting from the integral representation of $L_3^n(0, \Delta_1^{(i)}, \tilde{D}_{ij})$ and disregarding the fact that some $\tilde{D}_{ij}$ may or may not vanish. Thus, $I_3^n$ now reads:

$$I_3^n = -\frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \sum_{i \in S_3} \frac{\bar{b}_i}{\det(G)} \sum_{j \in S_3 \setminus \{i\}} \frac{\bar{b}_j^{(i)}}{\det(G^{(i)})} \times \int_{(0,1)_{i,j}} dz \left(z^2 (\tilde{D}_{ij} + \Delta_1^{(i)}) - \Delta_1^{(i)}\right)^{-1-\varepsilon}$$

(2.60)
Sticking to the general complex mass case, we perform the following change of variable: $s = b^{(i)}_j z$ in such a way that the two integrands corresponding to the sum over $j$ (at $i$ fixed) in eq. (2.60) are the same (use eqs. (2.38) and (2.40) of [1]). Specifying the two elements of $S_3 \setminus \{i\}$ to be $k \equiv 1 + ((i + 1) \mod 3)$ and $l \equiv 1 + (i \mod 3)$, the two integrals are concatenated into a single one integrated along the contour $I_{k,l}^{(i)} \equiv -b^{(i)}_k (0,1)_{i,k} \cup b^{(i)}_l (0,1)_{i,l}$ in the complex $s$-plane:

$$I_3^n = -\frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G) \det(G^{(i)})}$$

$$\times \int_{I_{k,l}^{(i)}} ds \left( \frac{s^2 + \det(S^{(i)})}{\det(G^{(i)})} \right)^{-1-\varepsilon}$$

$$= -\frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G) \det(G^{(i)})}$$

$$\times \int_{-b^{(i)}_k}^{b^{(i)}_l} ds \left( \frac{s^2 + \det(S^{(i)})}{\det(G^{(i)})} \right)^{-1-\varepsilon}$$

(2.61)

Figure 3: Example of a contour deformation involving a triangle with one distorted side, for which no cut crosses the straight base $[-b^{(i)}_k, b^{(i)}_l]$.

As in the case $\Delta_2 \neq 0$ treated in [2], the contour $I_{k,l}^{(i)}$ can be deformed into the straight line $[-b^{(i)}_k, b^{(i)}_l]$ as depicted on figure 3:

$$I_3^n = -\frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G) \det(G^{(i)})}$$

$$\times \int_{-b^{(i)}_k}^{b^{(i)}_l} ds \left( \frac{s^2 + \det(S^{(i)})}{\det(G^{(i)})} \right)^{-1-\varepsilon}$$

(2.62)

Performing the change of variable: $s = -\overline{b}^{(i)}_k - \det(G^{(i)}) u$ and following the procedure given by eqs. (2.47) to (2.49) of ref. [1] leads to:

$$I_3^n = \frac{2\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \sum_{i \in S_3} \frac{\overline{b}_i}{\det(G)} \int_0^1 du \left( D^{(i)\{i\}}(u) \right)^{-1-\varepsilon}$$

which is namely eq. (2.5).
It is instructive to recover the results of the “direct way” from the ones of the “indirect way” using for the latter the closed form formulae. These formulae can be obtained for the case $\tilde{D}_{ij} \neq 0$ by using results of appendix E and for $\tilde{D}_{ij} = 0$ using eq. (2.56). This exercise is performed with great details in appendix D.

3 Four-point function with Infrared divergences

Figure 4: The box picturing the one-loop four-point function.

In the case where infrared divergences appear, the latter can be regularised by dimensional regularisation shifting the space-time dimension $n = 4 - 2\varepsilon$ slightly above 4 ($\varepsilon < 0$). The Feynman parametrisation of $I_4^n$ reads:

$$I_4^n = \Gamma(2 + \varepsilon) \int_0^1 \prod_{i=1}^4 \delta(1 - \sum_{i=1}^4 z_i) \left( -\frac{1}{2} Z^T \cdot S \cdot Z - i\lambda \right)^{-2-\varepsilon}$$

(3.1)

where $Z$ is a column 4-vector whose components are the $z_i$. The power $-2 - \varepsilon$ in eq. (3.1) is not an integer. Notwithstanding, the tricks and techniques elaborated in section 3 of ref. [1] can be used with a slight adaptation. Let us sketch the different steps for this special case.

3.1 Computation of $I_4^n$

We make use of identity (2.28) of ref. [1] to shift the power of the denominator in the integrand, choosing $\mu = 5/2$ and $\nu = 2/(1 - 2\varepsilon)$ so that $I_4^n$ is recast as:

$$I_4^n = \frac{2^{3+\varepsilon}}{B(2 + \varepsilon, 1/2 - \varepsilon)} \frac{\Gamma(2 + \varepsilon)}{(1 - 2\varepsilon)} \int_0^{+\infty} d\xi \int_{\Sigma_{bcd}} dx_b dx_c dx_d \frac{dx_d}{(D^{(a)}(x_b, x_c, x_d) + \xi^\nu - i\lambda)^{5/2}}$$

(3.2)
Step 1 is very similar to the case $n = 4$ and we get:

$$I^n_4 = \frac{2^{3+\varepsilon}}{3 B(2 + \varepsilon, 1/2 - \varepsilon)} \frac{\Gamma(2 + \varepsilon)}{(1 - 2 \varepsilon)} \times \sum_{i=1}^{4} \frac{\tilde{b}_i}{\det(G)} \int_{0}^{+\infty} d\xi \int_{0}^{+\infty} d\rho \int_{0}^{+\infty} d\sigma \frac{1}{\Delta_3 - \xi^\nu + i \lambda}$$

$$\times \prod_{k,l \in \Sigma_{kl}} \left( \tilde{D}_{ijk} + \xi^\nu + \rho^2 + \sigma^2 - i \lambda \right)^{1/2}$$

with the same notational conventions as in eqs. (3.6) and (3.17) of sec. 3 of ref. [1]. Likewise, steps 2 and 3 are identical to section 3 of ref. [1] but the power of the variable $\xi$: $\nu$ instead of 2, and will not be repeated here. Regarding step 4, the integration shall be performed over the variables $\xi$, $\rho$ then $\sigma$ in the corresponding $L^n_4(\Delta_3, \Delta^{(i)}_2, \Delta^{(ij)}_1, \tilde{D}_{ijk})$ now given by:

$$L^n_4(\Delta_3, \Delta^{(i)}_2, \Delta^{(ij)}_1, \tilde{D}_{ijk}) = \kappa \int_{0}^{+\infty} d\xi \int_{0}^{+\infty} d\rho \int_{0}^{+\infty} d\sigma \frac{1}{\xi^\nu - \Delta_3 - i \lambda}$$

$$\times \frac{1}{\xi^\nu + \rho^2 - \Delta^{(i)}_2 - i \lambda} \frac{1}{\xi^\nu + \rho^2 + \sigma^2 - \Delta^{(ij)}_1 - i \lambda}$$

$$\times \frac{1}{(\tilde{D}_{ijk} + \xi^\nu + \rho^2 + \sigma^2 - i \lambda)^{1/2}}$$

with

$$\kappa = \frac{2^{4+\varepsilon}}{3 B(2 + \varepsilon, 1/2 - \varepsilon)} \frac{\Gamma(2 + \varepsilon)}{B(3/2, 1/2) B(1, 1/2) (1 - 2 \varepsilon)}$$

reminiscent of eq. (3.35) of sec. 3 of ref. [1]. Infrared divergences in the four-point function are not dominant Landau-type singularities but subleading ones. Such an infrared divergence corresponds to the vanishing determinant $\det(S^{(i)})$ of some reduced $S$ matrix (or some $\Delta^{(i)}_2$), associated with some three-point functions which are obtained from the four-point function considered by one pinching [6]. The various cases of vanishing kinematic matrices associated with three-point functions plagued with infrared soft or collinear singularities have been evoked in sec. 2. Let us note that the method developed in section 3 of ref. [1] is still valid because we never divide by $\Delta^{(i)}_2$ per se but by $\Delta^{(i)}_2 - \xi^\nu - \rho^2$. The divergences will show up when performing the integrations over the parameters of eq. (3.4). Note also that if we face a case when IR divergences arise whereas there are some non vanishing internal masses, only some of the contributions (let us call them sector) “$i, j, k$”, not all, are plagued with IR divergences; the other ones, to which corresponds $\Delta^{(i)}_2 \neq 0$, will be treated as in the $n = 4$ case. Besides, in any IR-divergent sector “$i$” for which $\Delta^{(i)}_2 = 0$, we do not have to sum over all three sub-sectors $j \in S_4 \setminus \{i\}$ because, as was seen in the IR-divergent three-point function case, some of the $\tilde{b}_i^{(j)}$ vanish.

Let us consider a sector $i$ which diverges in the IR region. We have to distinguish two cases: 1) when $\Delta^{(i)}_2 = 0$ whereas $\tilde{D}_{ijk} \neq 0$ in which case there are only soft divergences, 2) when
both $\Delta_{2}^{\{i\}} = 0$ and $\tilde{D}_{ijk} = 0$ in which case there are collinear or both soft and collinear divergences. In this section, we will treat at once the real and complex mass case. Some of the cases correspond to real or complex masses, others to complex masses only. For those corresponding real or complex masses, we keep the imaginary part $-i\lambda$ explicitly having in mind that with complex masses this $-i\lambda$ is ineffective. Let us stress in passing that we also keep an infinitesimal prescription $-i\lambda$ in the pole term where we put $\Delta_{2}^{\{i\}} = 0$.

### 3.2 $\Delta_{2}^{\{i\}} = 0$ and $\tilde{D}_{ijk} \neq 0$

$$L_{4}^{a}(\Delta_{3}, 0, \Delta_{1}^{\{i,j\}}, \tilde{D}_{ijk}) = \kappa \int_{0}^{+\infty} d\xi \int_{0}^{+\infty} d\rho \int_{0}^{+\infty} d\sigma \frac{1}{(\xi' - \Delta_{3} - i\lambda)(\xi' + \rho^{2} - i\lambda)} \times \frac{1}{(\xi' + \rho^{2} + \sigma^{2} - \Delta_{1}^{\{i,j\}} - i\lambda)(\tilde{D}_{ijk} + \xi' + \rho^{2} + \sigma^{2} - i\lambda)^{1/2}}$$

(3.5)

In the complex mass case, $\text{Im}(\Delta_{3})$ and $\text{Im}(\Delta_{1}^{\{i,j\}})$ have arbitrary signs and $\text{Im}(\tilde{D}_{ijk})$ is negative, so we have to distinguish between different cases.

Let us define the function $M_{2}(\xi')$ as:

$$L_{4}^{a}(\Delta_{3}, 0, \Delta_{1}^{\{i,j\}}, \tilde{D}_{ijk}) = \kappa \int_{0}^{+\infty} d\xi \frac{1}{\xi' - \Delta_{3} - i\lambda} M_{2}(\xi')$$

(3.6)

The integrations over $\sigma$ and $\rho$ are identical with those appearing in the massive cases, we thus borrow the results derived in subsec. 3.4 of ref. [1] and in subsec. 3.1 of ref. [2] with $\Delta_{2}^{\{i\}} = 0$, explicitly $^{8}$:

- for $\text{Im}(\Delta_{1}^{\{i,j\}}) \geq 0$,

$$M_{2}(\xi') = \frac{1}{2} B(1/2, 1/2) \int_{0}^{1} \frac{du}{u^{2}(\tilde{D}_{ijk} + \Delta_{1}^{\{i,j\}}) - \Delta_{1}^{\{i,j\}}} \times \left[ \frac{1}{(\xi' - i\lambda)^{1/2}} - \frac{1}{(\xi' + u^{2}(\tilde{D}_{ijk} + \Delta_{1}^{\{i,j\}}) - \Delta_{1}^{\{i,j\}} - i\lambda)^{1/2}} \right]$$

(3.7)

$^{8}$The integration over $\sigma$ is of the “second kind” (cf. eqs. (A.6) and (A.7)) while the integration over $\rho$ is of the “first kind” (cf. eq. (A.1)). For both integrations, the power $\nu$ appearing in the eqs. (A.1), (A.6) and (A.7) is taken equal to 2.
- for $\text{Im}(\Delta_1^{(i,j)}) < 0$,

$$M_2(\xi') = -\frac{1}{2} B(1/2, 1/2) \left\{ i \int_0^{+\infty} \frac{du}{u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) + \Delta_1^{(i,j)}} \right.$$ 

$$\times \left[ \frac{1}{(\xi' - i \lambda)^{1/2}} - \frac{1}{(\xi' - u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) - \Delta_1^{(i,j)})^{1/2}} \right]$$ 

$$+ \int_1^{+\infty} \frac{du}{u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) - \Delta_1^{(i,j)}}$$ 

$$\times \left[ \frac{1}{(\xi' - i \lambda)^{1/2}} - \frac{1}{(\xi' + u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) - \Delta_1^{(i,j)})^{1/2}} \right] \right\} \ (3.8)$$

The integration over $\xi$ to obtain $L^n_4$ is of the “second kind” (cf. eqs. (A.6) and (A.7) with $\nu = 2/(1 - 2 \varepsilon)$). Let us go through the different cases with respect to the sign of the imaginary part of $\Delta_3$ and $\Delta_1^{(i,j)}$. Sticking with the notations of section 3 of ref. [1], we introduce:

$$P_{ijk} = \tilde{D}_{ijk} + \Delta_1^{(i,j)}$$

$$R_{ij} = -\Delta_1^{(i,j)}$$

$$T = -\Delta_3$$

The strategy for computing the different integrals is the same for all cases and very similar to section 3 of ref. [2]. We display the successive steps for the first case tackled only and we give the final result for the three others.

### 3.2.1 $\text{Im}(\Delta_3) > 0, \text{Im}(\Delta_1^{(i,j)}) > 0$

In this case, the three complex numbers $\Delta_3, \Delta_1^{(i,j)}$ and $\tilde{D}_{ijk}$ have an imaginary part of the same sign. We start with eq. (3.7) for $M_2(\xi')$ and use eq. (A.6) for the $\xi$ integration to get:

$$L^n_4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = F(\varepsilon) \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}}$$

$$\times \left[ \int_0^1 \frac{dz}{((1 - z^2) T - i \lambda)^{1+\varepsilon}} \right.$$ 

$$- \int_0^1 \frac{dz}{z^2 (u^2 P_{ijk} + R_{ij}) + (1 - z^2) T - i \lambda)^{1+\varepsilon}} \right] \ (3.9)$$

where

$$F(\varepsilon) = 2^{1+\varepsilon} \Gamma(1 + \varepsilon) \ (3.10)$$
To facilitate the reading, we will define some steps in a similar way as in the subsec. 3.4 of ref. [1]. Let us proceed along them.

1) We change $u = \sqrt{y/x}$ and $z = \sqrt{x}$ and exchange the $y$ and the $x$ integration so that:

$$L_4^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = -\frac{F(\varepsilon)}{4} \int_0^1 \frac{dy}{\sqrt{y}} \int_y^1 dx \frac{1}{y P_{ijk} + x R_{ij}}$$

$$\times \left[ \frac{1}{(y P_{ijk} + x (R_{ij} - T) + T - i \lambda)^{1+\varepsilon}} - \frac{1}{((1 - x) T - i \lambda)^{1+\varepsilon}} \right]$$

(3.11)

2) We set $y = u^2$ and perform a partial fraction decomposition on the variable $x$ to write $L_4^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk})$ in the following form:

$$L_4^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = -\frac{F(\varepsilon)}{2} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} (T - i \lambda)} \int_{u^2}^1 dx$$

$$\times \left\{ (T - R_{ij}) \left[ u^2 P_{ijk} + x (R_{ij} - T) + T - i \lambda \right]^{-1-\varepsilon}$$

$$- T \left[ [(1 - x) T - i \lambda]^{-1-\varepsilon} + \frac{R_{ij}}{u^2 P_{ijk} + x R_{ij}} \right]$$

$$\times \left[ u^2 P_{ijk} + x (R_{ij} - T) + T - i \lambda \right]^{-\varepsilon} - [(1 - x) T - i \lambda]^{-\varepsilon} \right\}$$

(3.12)

In eq. (3.12), the last line provides a contribution of order $\varepsilon$ only: for the computation of the one-loop four-point function it can thus be dropped.\(^9\)

3) Thus, ignoring these terms, the integration in $x$ is readily done providing:

$$L_4^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \frac{2^\varepsilon \Gamma(1+\varepsilon)}{T} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda \sigma_0}$$

$$\times \left\{ u^2 (P_{ijk} + R_{ij} - T) + T - i \lambda \right\}^{-\varepsilon}$$

$$- \left[ u^2 P_{ijk} + R_{ij} - i \lambda \right]^{-\varepsilon} - (T - i \lambda)^{-\varepsilon} (1 - u^2)^{-\varepsilon} \right\}$$

(3.13)

\(^9\)Similar truncations of $\varepsilon$ expansions will be performed everywhere throughout this section. In the perspective of computing generalised one-loop building blocks to be used in computations beyond one-loop, one might be led to keep further terms evanescent with $\varepsilon$ to the appropriate order, whenever such terms would hit $1/\varepsilon$ poles generated by the extra integrations, cf. introduction of [1].
where we have introduced $\sigma_0 = \text{sign}(R_{ij}/T)$ in the real mass case. In the complex mass case all "$-i\lambda$" and "$-i\sigma_0 \lambda$" contour prescriptions are ineffective and irrelevant, and all three terms in eq. (3.13) can be straightforwardly expanded in powers of $\varepsilon$. In contrast the real mass case requires a more cautious treatment which we elaborate below. We note that, when $u^2 \rightarrow -R_{ij}/P_{ijk}$,

$$u^2 (P_{ijk} + R_{ij} - T) + T \rightarrow (T - R_{ij}) \frac{\bar{D}_{ijk}}{P_{ijk}} \neq 0$$

$$1 - u^2 \rightarrow \frac{\bar{D}_{ijk}}{P_{ijk}} \neq 0$$

i.e. the pole at $u^2 = -R_{ij}/P_{ijk}$ is distinct from the branch points of the first and third functions in the numerator. Therefore we can readily perform an expansion around $\varepsilon = 0$ for the first and third term as:

$$\frac{1}{\varepsilon} \left[ u^2 (P_{ijk} + R_{ij} - T) + T - i \lambda \right]^{-\varepsilon} = \frac{1}{\varepsilon} - \ln \left[ u^2 (P_{ijk} + R_{ij} - T) + T - i \lambda \right] + \mathcal{O}(\varepsilon)$$

$$\frac{1}{\varepsilon} (T - i \lambda)^{-\varepsilon} \frac{1}{(1 - u^2)^{-\varepsilon}} = -\frac{1}{\varepsilon} - \ln(T - i \lambda) - \ln(1 - u^2) + \mathcal{O}(\varepsilon)$$

The $1/\varepsilon$ poles cancel between these two contributions leaving only logarithms. On the other hand the contribution coming from the second term requires some care since pole and branch point coincide. If this singularity lies outside the integration region the contour prescription for the pole is irrelevant and can be dropped. If the singularity lies inside $[0, 1] \sigma_0 = -$ however, a pinching occurs at the singular point in the limit $\lambda \rightarrow 0^+$. A too early expansion in powers of $\varepsilon$ before performing the integration over $u$ would lead to a divergence order by order in $\varepsilon$. On the other hand, we note that for $u^2 = -R_{ij}/P_{ijk}$ the numerator of the second term - i.e. the pole residue - is $(-i \lambda)^{-\varepsilon} \rightarrow 0$ with $\lambda \rightarrow 0^+$ for any fixed $\varepsilon < 0$. Therefore, up to terms vanishing $\propto \lambda^{-\varepsilon}$ we can make the replacement

$$\int_0^1 du \frac{u^2 P_{ijk} + R_{ij} - i \lambda}{u^2 P_{ijk} + R_{ij} + i \lambda} \rightarrow \int_0^1 du \frac{u^2 P_{ijk} + R_{ij} - i \lambda}{u^2 P_{ijk} + R_{ij} - i \lambda}$$

Details are provided in appendix F. Finally, we perform a partial expansion around $\varepsilon = 0$ and we get:

$$L^0_4(\Delta_3, 0, \Delta_1^{(ij)}, \bar{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon)$$

$$\times \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda} \left[ \ln \left( (T - i \lambda) (1 - u^2) \right) - \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T - i \lambda \right) \right] \right. - \frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda} + \left. \int_0^1 du \ln(u^2 P_{ijk} + R_{ij} - i \lambda) \right\}$$

(3.14)
Let us notice that the imaginary part of the argument of each logarithm in eq. (3.14) keeps a constant sign when \(u\) spans \([0, 1]\). This is obviously true for the real mass case and in the case of complex masses, this is easily verified keeping in mind the assumptions: \(\text{Im}(T) < 0\) and \(\text{Im}(R_{ij}) < 0\). For the sake of coherence with respect to ref. [2], the pole residue contributions will be added and subtracted for the two logarithms making up the first term inside the curly brackets of eq. (3.14). This latter equation recast in the form:

\[
L^n_4(\Delta_3, 0, \Delta_i^{(ij)}, \tilde{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon)
\times \left\{ -\frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda} + \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda} \ln(u^2 P_{ijk} + R_{ij} - i \lambda) \right\}
\times \left[ \ln \left( \left( T - i \lambda \right) \left( 1 - u^2 \right) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} - i \lambda \right) \right]
- \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij} - i \lambda} \ln \left( \frac{T - R_{ij} - i \lambda \sigma_1}{T - i \lambda \sigma_1} \right)
\right\}
\]

(3.15)

with \(\sigma_1 = \text{sign}\left(\frac{(P_{ijk} + R_{ij})}{P_{ijk}}\right)\) for the real mass case and:

\[
L^n_4(\Delta_3, 0, \Delta_i^{(ij)}, \tilde{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon)
\times \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( T \left( 1 - u^2 \right) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \right\}
- \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( u^2 \left( P_{ijk} + R_{ij} - T \right) + T - i \lambda \right) + \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} - i \lambda \right) \right]
- \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{T - R_{ij}}{T} \right) + \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T} \right) \right] \right\}
\]

(3.16)

for the complex mass case. The relevant integrals are given in appendices E of this paper, E of ref. [1] and B of ref. [2].

Eq. (3.16) matches eq. (3.19) of ref. [2] obtained in the corresponding general complex mass case 1.(a), considering the latter in the limit \(\text{Re}(\Delta_i^{(ij)}) = \text{Re}(Q_i + T) \to 0\) while keeping an
infinitesimal positive imaginary part $\text{Im}(\Delta^{(ij)}_2) = \lambda$. One then formally gets:

\[
\text{r.h.s. (3.19) of [2]} \rightarrow \frac{1}{T} \left\{ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( T (1 - u^2) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) - \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T \right) + \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \right] \\
- \ln (Q_i + T) \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} + \int_0^1 du \ln(u^2 P_{ijk} + R_{ij}) \right. \\
- \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{T - R_{ij}}{T} \right) + \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T} \right) \right] \right\}
\]

where the divergent term $\ln(Q_i + T)$ corresponds to the “dressed pole” $(2 e^{-\gamma}e^\varepsilon)/\varepsilon$.

**3.2.2 $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta^{(ij)}_1) < 0$**

Compared to the previous case, one uses eq. (3.8) for $M_2(\xi^\nu)$ and the $\xi$ integration is carried out with the help of eqs. (A.6) and (A.7) depending on the different terms. Then, performing the first step described in subsubsec. 3.2.1, we obtain:

\[
L^n_4(\Delta_3, 0, \Delta^{(ij)}_1, \widetilde{D}_{ijk}) \\
= - \frac{F(\varepsilon)}{4} \left\{ \int y \int_0^1 \frac{dy}{y} \int_0^1 \frac{dx}{y P_{ijk} - x R_{ij}} \left( (T(1 - x))^{-1 - \varepsilon} \right) \\
+ e^{i\pi\varepsilon} \int y \int_0^1 \frac{dy}{y} \int_0^1 \frac{dx}{y P_{ijk} - x R_{ij}} (-y P_{ijk} + x R_{ij} - T (1 + x))^{-1 - \varepsilon} \right. \\
+ i \int y \int_0^1 \frac{dy}{y} \int_0^1 \frac{dx}{y P_{ijk} - x R_{ij}} (-y P_{ijk} + x R_{ij} + T (1 - x))^{-1 - \varepsilon} \right. \\
+ \left. \int y \int_0^1 \frac{dy}{y} \int_0^1 \frac{dx}{y P_{ijk} + x R_{ij}} \left[ (T (1 - x))^{-1 - \varepsilon} \right] \right. \\
- \left. (y P_{ijk} + x R_{ij} + T (1 - x))^{-1 - \varepsilon} \right] \\
+ \left. \int y \int_0^1 \frac{dy}{y} \int_0^1 \frac{dx}{y P_{ijk} + x R_{ij}} \left[ (T (1 - x))^{-1 - \varepsilon} \right] \right. \\
- \left. (y P_{ijk} + x R_{ij} + T (1 - x))^{-1 - \varepsilon} \right] \right\} \quad (3.17)
\]
We set $y = u^2$, perform a partial fraction decomposition on the variable $x$ and expand around $\varepsilon = 0$. The $x$ integration is readily done and $L^u_4(\Delta_3, 0, \Delta^{(ij)}_1, \tilde{D}_{ijk})$ is written as:

$$
L^u_4(\Delta_3, 0, \Delta^{(ij)}_1, \tilde{D}_{ijk})
= -\frac{2\varepsilon}{T} \Gamma(1 + \varepsilon) \left\{ \frac{T^\varepsilon}{\varepsilon} \left[ i \int_0^{\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} + \int_1^{\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \right]
+ i \int_0^{\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} \left[ \ln \left( \frac{u^2 P_{ijk} - R_{ij}}{u^2 P_{ijk}} \right) - \ln \left( \frac{R_{ij} - T}{R_{ij}} \right) \right]
- \int_0^{\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{-u^2 P_{ijk}}{R_{ij}} \right) - \ln \left( \frac{P_{ijk} u^2 + T}{T - R_{ij}} \right) \right]
+ \int_1^{\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) + T}{u^2 P_{ijk} + T} \right)
+ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) + T}{u^2 P_{ijk} + T} \right) - \ln(1 - u^2) \right] \right\} (3.18)
$$

In this case, we have $\text{Im}(R_{ij}) > 0$, $\text{Im}(P_{ijk}) < 0$ and $\text{Im}(R_{ij} - T) > 0$. Furthermore,

- $u^2 P_{ijk} - R_{ij} = (1 + u^2) \Delta^{(ij)}_1 + u^2 \tilde{D}_{ijk}$
  thus $\text{Im}(u^2 P_{ijk} - R_{ij}) < 0$ when $u \in [0, \infty[$,

- $u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta^{(ij)}_1$
  thus $\text{Im}(u^2 P_{ijk} + R_{ij}) < 0$ when $u \in [1, \infty[$,

- $u^2 P_{ijk} + T = u^2 (\tilde{D}_{ijk} + \Delta^{(ij)}_1) - \Delta_3$
  thus $\text{Im}(u^2 P_{ijk} + T) < 0$ when $u \in [0, \infty[$,

- $u^2 (P_{ijk} + R_{ij} - T) + T = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta_3$
  thus $\text{Im}(u^2 (P_{ijk} + R_{ij} - T) + T) < 0$ when $u \in [0, 1]$. 

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Rearrangements and simplifications similar to those done in the massive case 2.(a) of ref. [2] can be performed, which lead to the following alternative expression:

\[
L^\prime_4(\Delta_3, 0, \Delta_{1^{(i,j)}}, \tilde{D}_{ij}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon) \times \left\{ -\frac{1}{\varepsilon} \int_{0}^{1} \frac{du}{u^2 P_{ijk} + R_{ij}} + \int_{0}^{1} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln\left(\frac{u^2 P_{ijk} + R_{ij}}{u^2 P_{ijk} + R_{ij}}\right) + \ln\left(\frac{u^2 (P_{ijk} + R_{ij} - T) + T}{P_{ijk}}\right) - \ln\left(\frac{T (P_{ijk} + R_{ij})}{P_{ijk}}\right) - \ln\left(\frac{T (P_{ijk} + R_{ij})}{P_{ijk}}\right) - \eta\left(\frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T}\right) - \int_{0}^{\sqrt{-R_{ij}/P_{ijk}}} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln\left(\frac{T - R_{ij}}{T}\right) \right\}
\] (3.19)

The contour \(\bigcirc\) can be deformed into a contour \((0, 1)^+\) stretched from 0 to 1 and which eventually wraps from above the cut of \(\ln(u^2 P_{ijk} + R_{ij})\) emerging from the branch point \(u_0 = \sqrt{-R_{ij}/P_{ijk}}\), whenever the latter lies in the “north-east” quadrant \(\{\text{Re}(u) > 0, \text{Im}(u) > 0\}\).

Eq. (3.19) can be compared with eq. (3.37) of [2] obtained in the corresponding general complex mass case 2.(a), considering the latter in the limit \(\text{Re}(\Delta_{2^{(i)}}) = \text{Re}(Q_i + T) \to 0\) while keeping an infinitesimal positive imaginary part \(\text{Im}(\Delta_{2^{(i)}}) = \lambda\). Whereas it appeared convenient to formulate eq. (3.37) of ref. [2] in terms of manifestly vanishing pole residues, this is no longer the case for eq. (3.19) since the pole has become also the branch point of \(\ln(u^2 P_{ijk} + (R_{ij} + Q_i + T))\) in the limit \((Q_i + T) \to 0\). For the purpose of the comparison the \(\eta\) functions introduced in eq. (3.37) of ref. [2] containing \(Q_i + T\) shall thus be made explicit in terms of constant logarithms, part of which then cancel against the constant logarithms which were subtracted so as to build the explicitly vanishing pole residues. One then formally...
get:

\[ \text{r.h.s. (3.37) of [2]} \rightarrow \frac{1}{T} \]

\[
\times \left\{ -\ln (Q_i + T) \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} + R_{ij}} + \int_{(0,1)^+} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{u^2 P_{ijk} + R_{ij}} \right) \\
+ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( T (1 - u^2) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) - \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) + T}{u^2 P_{ijk} + R_{ij}} + \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \right] \\
- \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T} \right) \right\}
\]

where the divergent term \( \ln(Q_i + T) \) corresponds to the “dressed pole” \((2 e^{-\gamma_E})^\varepsilon/\varepsilon\).

### 3.2.3 \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_1^{ij}) > 0 \)

In this case, we start with eq. (3.7) for \( M_2(\xi^\nu) \) and use eq. (A.7) for all the \( \xi \) integrations. We give the result for \( L^4_4(\Delta_3, 0, \Delta_1^{ij}, \bar{D}_{ijk}) \) after the intermediate step 1 of subsubsec. 3.2.1.

\[
L^4_4(\Delta_3, 0, \Delta_1^{ij}, \bar{D}_{ijk}) = \frac{F(\varepsilon)}{4} \left\{ i e^{-i \pi \varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_y^\infty \frac{dx}{y P_{ijk} + x R_{ij}} [\left( y P_{ijk} + x R_{ij} - T (1 + x) \right)^{-1-\varepsilon} \\
- (T (1 + x))^{-1-\varepsilon}] \\
+ \int_1^\infty \frac{dy}{\sqrt{y}} \int_y^\infty \frac{dx}{y P_{ijk} + x R_{ij}} [\left( y P_{ijk} + x R_{ij} + T (1 - x) \right)^{-1-\varepsilon} \\
- (T (1 - x))^{-1-\varepsilon}] \\
+ \int_0^1 \frac{dy}{\sqrt{y}} \int_1^\infty \frac{dx}{y P_{ijk} + x R_{ij}} [\left( y P_{ijk} + x R_{ij} + T (1 - x) \right)^{-1-\varepsilon} \\
- (T (1 - x))^{-1-\varepsilon}] \right\} (3.20)
\]
Then, performing the steps 2 and 3 of this subsubsec. yields:

\[
L^n_4(\Delta_3, 0, \Delta^{i,j}_1, \tilde{D}_{ijk})
= \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon)
\left\{-\frac{(-T)^{-\varepsilon}}{\varepsilon} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}}
\right.
+ i \int_{0}^{+\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) - T}{R_{ij} - T} \right) - \ln (u^2 + 1) \right]
\left.+ \int_{1}^{+\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) + T}{R_{ij} - T} \right) - \ln (u^2 - 1) \right]
+ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{R_{ij} - T} \right) \right\}
\right.
\right. (3.21)

In this case, we have \(\text{Im}(R_{ij} - T) < 0\). Furthermore,

- \(u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} - (1 - u^2) \Delta^{i,j}_1\)
  thus \(\text{Im}(u^2 P_{ijk} + R_{ij}) < 0\) when \(u \in [0, 1]\),

- \(u^2 (P_{ijk} + R_{ij} - T) - T = u^2 \tilde{D}_{ijk} + (1 + u^2) \Delta_3\)
  thus \(\text{Im}(u^2 (P_{ijk} + R_{ij} - T) - T) < 0\) when \(u \in [0, \infty[\).

- \(u^2 (P_{ijk} + R_{ij} - T) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3\)
  thus \(\text{Im}(u^2 (P_{ijk} + R_{ij} - T) + T) < 0\) when \(u \in [1, \infty[\).

Rearrangements and simplifications similar to those done in the massive case 1.(c) of [2] can be performed, which lead to the following alternative expression:

\[
L^n_4(\Delta_3, 0, \Delta^{i,j}_1, \tilde{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon)
\times \left\{-\frac{1}{\varepsilon} \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}}
\right. + \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{u^2 P_{ijk} + R_{ij}}{P_{ijk}} \right)
\left.+ \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( 1 - u^2 \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right)
\right.
\right.
\left.- \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T \right) + \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right)
\right.\right.
\left.- \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, T - R_{ij} \right) \right]\right. - \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{T - R_{ij}}{T} \right) \right\} (3.22)

The deformation of the contour \(\overline{0, 1}\) into a contour \((0, 1)^+\) stretched from 0 to 1 may eventually wrap from above the cut of \(\ln(u^2 (P_{ijk} + R_{ij} - T) + T)\) emerging from the branch point \(u_0 = \sqrt{-T/(P_{ijk} + R_{ij} - T)}\), whenever the latter lies in the “north-east” quadrant.
In a way similar to the previous case, eq. (3.22) matches eq. (3.33) of [2] obtained in the corresponding general complex mass case 1.(c), considering the latter in the limit \( \text{Re}(\Delta_2^{(i)}) = \text{Re}(Q_i + T) \to 0 \) while keeping an infinitesimal positive imaginary part \( \text{Im}(\Delta_2^{(i)}) = \lambda \).

### 3.2.4 \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_1^{(i,j)}) < 0 \)

One uses eq. (3.8) for \( M_2(\xi') \) and the \( \xi \) integration is carried out with the help of eqs. (A.6) and (A.7) depending on the terms. Then, the step 1 (cf. subsubsec. 3.2.1) leads to:

\[
L_i^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \frac{F(\varepsilon)}{4} \left\{ i \left( -T \right)^{-1-\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^\infty \frac{dx}{y P_{ijk} - x R_{ij}} (x - 1)^{-1-\varepsilon}
- \varepsilon^{-i \pi \varepsilon} \left( -T \right)^{-1-\varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^\infty \frac{dx}{y P_{ijk} - x R_{ij}} (1 + x)^{-1-\varepsilon}
+ i \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y P_{ijk} - x R_{ij}} (-y P_{ijk} + x R_{ij} + T (1 - x))^{-1-\varepsilon}
- i \varepsilon^{-i \pi \varepsilon} \int_0^\infty \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y P_{ijk} + x R_{ij}} [(y P_{ijk} + x R_{ij} - T (1 + x))^{-1-\varepsilon}
- (-T (1 + x))^{-1-\varepsilon}]
- \int_1^\infty \frac{dy}{\sqrt{y}} \int_1^y \frac{dx}{y P_{ijk} + x R_{ij}} [(y P_{ijk} + x R_{ij} + T (1 - x))^{-1-\varepsilon}
- (T (1 - x))^{-1-\varepsilon}] \right\} \tag{3.23}
\]

At the end of the step 5, we get:

\[
L_i^n(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = -\frac{2^\varepsilon}{T} \Gamma(1 + \varepsilon) \left\{ -\frac{(T)^{-\varepsilon} \varepsilon}{\varepsilon} \left[ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} + \int_1^{+\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \right]
+ i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} \left[ \ln \left( \frac{R_{ij} - u^2 P_{ijk}}{R_{ij}} \right) - \ln \left( \frac{P_{ijk} u^2 - T}{u^2 P_{ijk}} \right) \right]
- i \int_0^{+\infty} \frac{du}{u^2 P_{ijk} - R_{ij}} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) - T}{u^2 P_{ijk} - T} \right) - \ln(u^2 + 1) \right]
- \int_1^{+\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( \frac{u^2 (P_{ijk} + R_{ij} - T) + T}{u^2 P_{ijk} + R_{ij}} \right) - \ln(u^2 - 1) \right]
- \int_1^{+\infty} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{-u^2 P_{ijk}}{R_{ij}} \right) \right\} \tag{3.24}
\]

In this case, we have \( \text{Im}(R_{ij}) > 0, \text{Im}(P_{ijk}) < 0 \). Furthermore,
\[ u^2 P_{ijk} - R_{ij} = u^2 \tilde{D}_{ijk} + (u^2 + 1) \Delta_1^{(i,j)} \]
thus \( \text{Im}(u^2 P_{ijk} - R_{ij}) < 0 \) when \( u \in [0, \infty[. \)

\[ u^2 P_{ijk} + R_{ij} = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_1^{(i,j)} \]
thus \( \text{Im}(u^2 P_{ijk} + R_{ij}) < 0 \) when \( u \in [1, \infty[. \)

\[ u^2 P_{ijk} - T = u^2 (\tilde{D}_{ijk} + \Delta_1^{(i,j)}) + \Delta_3 \]
thus \( \text{Im}(u^2 P_{ijk} - T) < 0 \) when \( u \in [0, \infty[. \)

\[ u^2 (P_{ijk} + R_{ij} - T) - T = u^2 \tilde{D}_{ijk} + (u^2 + 1) \Delta_3 \]
thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} - T) - T) < 0 \) when \( u \in [0, \infty[. \)

\[ u^2 (P_{ijk} + R_{ij} - T) + T = u^2 \tilde{D}_{ijk} + (u^2 - 1) \Delta_3 \]
thus \( \text{Im}(u^2 (P_{ijk} + R_{ij} - T) + T) < 0 \) when \( u \in [1, \infty[. \)

Rearrangements and simplifications similar to those done in the massive case 2.(c) of [2] can be performed, which lead to the following alternative expression:

\[
L_4^0(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon) \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}}
\times \left\{ -\frac{1}{\varepsilon} + \ln(u^2 P_{ijk} + R_{ij}) + \ln \left( T \left( 1 - u^2 \right) \right) - \ln \left( \frac{T(P_{ijk} + R_{ij})}{P_{ijk}} \right)
- \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T \right) + \ln \left( \frac{(P_{ijk} + R_{ij})(T - R_{ij})}{P_{ijk}} \right)
- \eta \left( \frac{T(P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T} \right) - \ln \left( \frac{T - R_{ij}}{T} \right) \right\}
(3.25)
\]

As in the previous cases, the contours \( \oint \) can be deformed into contours \( \hat{(0,1)} \) stretched from 0 to 1. They eventually wrap from above the cuts of \( \ln(u^2 P_{ijk} + R_{ij}) \) emerging from the branch point \( \sqrt{-R_{ij}/P_{ijk}} \) and of \( \ln(u^2 (P_{ijk} + R_{ij} - T) + T) \) emerging from the branch point \( \sqrt{-T/(P_{ijk} + R_{ij} - T)} \), respectively, whenever either of these branch points or both lie in the “north-east” quadrant.

In a way similar to the previous case, eq. (3.25) matches eq. (3.41) of [2] obtained in the corresponding general complex mass case 2.(c), considering the latter in the limit \( \text{Re}(\Delta_2^{(i)}) = \text{Re}(Q_i + T) \rightarrow 0 \) while keeping an infinitesimal positive imaginary part \( \text{Im}(\Delta_2^{(i)}) = \lambda. \)

### 3.3 \( \Delta_2^{(i)} = 0 \) and \( \tilde{D}_{ijk} = 0 \)

In this case, we have \( P_{ijk} = -R_{ij} \) and \( Q_i = -T \), so that eqs. (3.7) and (3.8) become:
- for \( \text{Im}(\Delta_{1}^{(i,j)}) > 0) \),

\[
M_{2}(\xi') = \frac{1}{2} B(1/2, 1/2) \int_{0}^{1} \frac{dz}{\Delta_{1}^{(i,j)}(z^{2} - 1)} \times \left[ \frac{1}{(\xi' - i \lambda)^{1/2}} - \frac{1}{(\xi' + \Delta_{1}^{(i,j)}(z^{2} - 1) - i \lambda)^{1/2}} \right]
\]

(3.26)

- for \( \text{Im}(\Delta_{1}^{(i,j)}) < 0) \),

\[
M_{2}(\xi') = -\frac{1}{2} B(1/2, 1/2) \left\{ i \int_{0}^{+\infty} \frac{dz}{\Delta_{1}^{(i,j)}(z^{2} + 1)} \times \left[ \frac{1}{(\xi' - i \lambda)^{1/2}} - \frac{1}{(\xi' - \Delta_{1}^{(i,j)}(1 + z^{2}))^{1/2}} \right] 
+ \int_{1}^{+\infty} \frac{dz}{\Delta_{1}^{(i,j)}(z^{2} - 1)} \times \left[ \frac{1}{(\xi' - i \lambda)^{1/2}} - \frac{1}{(\xi' + \Delta_{1}^{(i,j)}(z^{2} - 1))^{1/2}} \right] \right\}
\]

(3.27)

Here again, the \( \xi \) integration will be of the type (A.6) or (A.7) (cf. appendix (A)). Let us go through the different cases according the signs of \( \text{Im}(\Delta_{3}) \) and \( \text{Im}(\Delta_{1}^{(i,j)}) \).

### 3.3.1 \( \text{Im}(\Delta_{3}) > 0, \text{Im}(\Delta_{1}^{(i,j)}) > 0 \)

We proceed along the two first steps described in subsubsec. 3.2.1. We borrow the result obtained in eq. (3.11), set \( P_{ijk} = -R_{ij} \) and make the following change of variables \( x = y + (1 - y) v \). The quantity \( L_{4}^{\nu}(\Delta_{3}, 0, \Delta_{1}^{(i,j)}, 0) \) reads now:

\[
L_{4}^{\nu}(\Delta_{3}, 0, \Delta_{1}^{(i,j)}, 0) = -\frac{F(\varepsilon)}{4 R_{ij}} \int_{0}^{1} \frac{dv}{v} \left[ \frac{1}{[v R_{ij} + (1 - v) T - i \lambda]^{1+\varepsilon}} - \frac{1}{[(1 - v) T - i \lambda]^{1+\varepsilon}} \right] 
\]

\[
\times \int_{0}^{1} dy \frac{y^{-1/2} (1 - y)^{-1-\varepsilon}}{(1 - y)^{-1-\varepsilon}}
\]

(3.28)

The integrals over \( y \) and \( v \) are unnested and are computed easily using eqs. (C.3), (C.6) and (C.10). Notice that \( \text{Im}(v R_{ij} + (1 - v) T - i \lambda) \) never changes its sign when \( v \) spans \([0, 1]\), this is obviously true in the real mass case and due to the fact that \( \text{Im}(T) \) and \( \text{Im}(R_{ij}) \) have
a same sign imaginary part in the complex mass case. Inserting the explicit results for the
$y$ and $v$ integrals, we get:

$$L_n^*(\Delta_3, 0, \Delta_i^{ij}, 0) = \frac{1}{2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{R_i T} \times \left[ \frac{1}{\varepsilon^2} (2 R_i - i \lambda)^{-\varepsilon} + \text{Li}_2 \left( \frac{T - R_i}{T - i \lambda} \right) - \pi^2 \right]$$ (3.29)

Equation (3.29) displays explicitly the singularity of $L_n^*(\Delta_3, 0, \Delta_i^{ij}, 0)$ when $\varepsilon \to 0$. In eq.
(3.29) we use the property $\text{Li}_2(z) + \text{Li}_2(1 - z) = \pi^2/6 - \ln(z) \ln(1 - z)$ to obtain the following alternative form suitable for further comparisons:

$$L_n^*(\Delta_3, 0, \Delta_i^{ij}, 0) = \frac{1}{2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{R_i T} \times \left\{ \frac{1}{\varepsilon^2} (2 R_i - i \lambda)^{-\varepsilon} - \text{Li}_2 \left( \frac{R_i - i \lambda}{T - i \lambda} \right) - \ln(R_i - i \lambda) - \ln(T - i \lambda) \right\}$$ (3.30)

3.3.2 $\text{Im}(\Delta_3) > 0$, $\text{Im}(\Delta_i^{ij}) < 0$

The starting point is eq. (3.17) with $P_{ij} = -R_{ij}$. In the first line of eq. (3.17) we rescale
$y = x u^2$ so that the double integral factorises into a product of two unnested integrals over
$u$ and over $x$. The integrals of the second and third lines of eq. (3.17) yield no divergences
when $\varepsilon \to 0$, we thus take $\varepsilon = 0$ in them. We then make the following change of variables:
$x = y - (y - 1) v$ in the penultimate line, and $x = y - (1 - y) v$ in the last line. We obtain:

$$L_n^*(\Delta_3, 0, \Delta_i^{ij}, 0) \quad \text{by the change of variables}$$

$$\begin{align*}
F(\varepsilon) & = \frac{1}{2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{R_i T} \times \left[ \frac{1}{\varepsilon^2} (2 R_i - i \lambda)^{-\varepsilon} - \text{Li}_2 \left( \frac{R_i - i \lambda}{T - i \lambda} \right) - \ln(R_i - i \lambda) - \ln(T - i \lambda) \right] \\
& + \text{other terms}
\end{align*}$$ (3.31)

Let us compute the different terms. Using eq. (C.3), the first integral is readily given by:

$$\int_0^{+\infty} \frac{2 du}{1 + u^2} \int_0^1 \frac{dx}{\sqrt{x}} (1 - x)^{-1 - \varepsilon} = \pi B \left( \frac{1}{2}, -\varepsilon \right)$$ (3.32)
In the second and third lines of eq. (3.31) the \( x \) integration is easily performed after a partial fraction decomposition on the \( x \) variable. We get:

\[
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^{+\infty} \frac{dx}{(y + x) \left( (x + y) R_{ij} - T (1 + x) \right)} = \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y - 1} \left[ \ln \left( \frac{y R_{ij} - T}{R_{ij} - T} \right) - \ln(y) \right] \tag{3.33}
\]

\[
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_1^{+\infty} \frac{dx}{(y + x) \left( (x + y) R_{ij} + T (1 - x) \right)} = \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y + 1} \ln \left( \frac{R_{ij}}{R_{ij} - T} \right) \tag{3.34}
\]

We write the last two integrals of eq. (3.31) as:

\[
\int_1^{+\infty} \frac{dy}{\sqrt{y}} (y - 1)^{-1-\varepsilon} \int_1^{y-1} \frac{dv}{v} \left[ T^{-1-\varepsilon} (v - 1)^{-1-\varepsilon} - (-v R_{ij} - T (1 - v))^{-1-\varepsilon} \right] = \int_1^{+\infty} \frac{dy}{\sqrt{y}} (y - 1)^{-1-\varepsilon} \left[ E_1(y) - E_1(1^+) \right] + E_1(1^+) B \left( \frac{1}{2} + \varepsilon, -\varepsilon \right) \tag{3.35}
\]

\[
\int_0^{1} \frac{dy}{\sqrt{y}} (1 - y)^{-1-\varepsilon} \int_0^{y-1} \frac{dv}{v} \left[ T^{-1-\varepsilon} (1 + v)^{-1-\varepsilon} - (-v R_{ij} + T (1 + v))^{-1-\varepsilon} \right] = \int_0^{1} \frac{dy}{\sqrt{y}} (1 - y)^{-1-\varepsilon} \left[ E_2(y) - E_2(1^-) \right] + E_2(1^-) B \left( \frac{1}{2}, -\varepsilon \right) \tag{3.36}
\]

with:

\[
E_1(y) = \int_1^{y-1} \frac{dv}{v} \left[ T^{-1-\varepsilon} (v - 1)^{-1-\varepsilon} - (-v R_{ij} - T (1 - v))^{-1-\varepsilon} \right] \tag{3.37}
\]

\[
E_2(y) = \int_0^{y-1} \frac{dv}{v} \left[ T^{-1-\varepsilon} (1 + v)^{-1-\varepsilon} - (-v R_{ij} + T (1 + v))^{-1-\varepsilon} \right] \tag{3.38}
\]

When \( y \to 1^+ \), the upper bound of the integral defining the function \( E_1(y) \) goes to \( +\infty \) and likewise for \( E_2(y) \) when \( y \to 1^- \). The quantities \( E_1(y) - E_1(1^+) \) and \( E_2(y) - E_2(1^-) \) are given by integrals between \( y/(y - 1) \) and \( +\infty \) and between \( y/(1 - y) \) and \( +\infty \) respectively. As, in \( E_1(y) - E_1(1^+) \), \( y \) is greater than 1 and so is the lower bound \( y/(y - 1) \), the singular support \( v = 1 \) of the distribution \( (v - 1)^{-1-\varepsilon} \) lies outside the range of integration thus the first term of the r.h.s. of eq. (3.37) can be taken at \( \varepsilon = 0 \). In \( E_2(y) - E_2(1^-) \), the integrand is non singular either and the first term of the r.h.s. of eq. (3.38) can also be taken at \( \varepsilon = 0 \). These two terms give:

\[
\int_1^{+\infty} \frac{dy}{\sqrt{y}} (y - 1)^{-1-\varepsilon} \left[ E_1(y) - E_1(1^+) \right] = -\frac{1}{T} \int_1^{+\infty} \frac{dy}{\sqrt{y}} \frac{1}{y - 1} \ln \left( \frac{y R_{ij} - T}{R_{ij} - T} \right) \tag{3.39}
\]

\[
\int_0^{1} \frac{dy}{\sqrt{y}} (1 - y)^{-1-\varepsilon} \left[ E_2(y) - E_2(1^-) \right] = -\frac{1}{T} \int_0^{1} \frac{dy}{\sqrt{y}} \frac{1}{y - 1} \ln \left( \frac{y R_{ij} - T}{R_{ij} - T} \right) \tag{3.40}
\]
The combined r.h.s. of eqs. (3.39) and (3.40) cancel against the first term in eq. (3.33). The expressions of $E_1(1^+)$ and $E_2(1^-)$ are computed using eqs. (C.13) and (C.15):

$$E_1(1^+) = -K_4(T - R_{ij}, -T) = \frac{1}{T} \left\{ -\frac{1}{\varepsilon} (-R_{ij})^{-\varepsilon} + \ln(T) - \ln(T - R_{ij}) 
+ \varepsilon\left[ \text{Li}_2\left(\frac{R_{ij}}{T}\right) + \ln(-R_{ij}) \ln\left(\frac{T - R_{ij}}{T}\right) \right] \right\}$$

$$E_2(1^-) = -K_3(T - R_{ij}, T) = \frac{1}{T} \ln\left(\frac{T - R_{ij}}{T}\right) \left[ 1 - \varepsilon \ln(T) \right]$$

Putting everything together, we get for $L_4^a(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$:

$$L_4^a(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{F(\varepsilon)}{4R_{ij} T} \left\{ i \pi B\left(\frac{1}{2}, -\varepsilon\right) (1 - \varepsilon \ln(T)) 
- \int_0^{\infty} \frac{dy}{\sqrt{y}} \frac{1}{y - 1} \ln(y) + i \int_0^{\infty} \frac{dy}{\sqrt{y}} \frac{1}{y + 1} \ln\left(\frac{R_{ij}}{R_{ij} - T}\right) 
+ B\left(\frac{1}{2}, -\varepsilon\right) \ln\left(\frac{T - R_{ij}}{T}\right) (1 - \varepsilon \ln(T)) 
+ B\left(\frac{1}{2} + \varepsilon, -\varepsilon\right) \left[ -\frac{1}{\varepsilon} (-R_{ij})^{-\varepsilon} + \ln(T) - \ln(T - R_{ij}) 
+ \varepsilon\left[ \text{Li}_2\left(\frac{R_{ij}}{T}\right) + \ln(-R_{ij}) \ln\left(\frac{T - R_{ij}}{T}\right) \right] \right] \right\}$$

Extracting the Euler Beta functions from eqs. (C.2), (C.5) and (C.3), (C.6) and using the fact that $\ln(-R_{ij}) = \ln(R_{ij}) - i\pi$, eq. (3.43) can be cast in the following form:

$$L_4^a(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{1}{2} \Gamma(1 + \varepsilon) \Gamma(1 - \varepsilon) \frac{1}{R_{ij} T} \left\{ \frac{1}{\varepsilon^2} (2R_{ij})^{-\varepsilon} - \text{Li}_2\left(\frac{R_{ij}}{T}\right) - \ln(R_{ij}) - \ln(T) \ln\left(\frac{T - R_{ij}}{T}\right) \right\}$$

Eqs.(3.44) and (3.30) have the same analytic expression.

**3.3.3 Im($\Delta_3$) < 0, Im($\Delta_1^{\{i,j\}}$) > 0**

We start with eq. (3.20) with $P_{ijk} = -R_{ij}$. We then set $x = y + (1 + y) v$ in the first integral of eq. (3.20), $x = y + (y - 1) v$ in the second integral and $x = y + (1 - y) v$ in the third one.
and we get:

\[
L_4^n(\Delta_3, 0, \Delta_{i,j}^{(i,j)}, 0) = \frac{F(\varepsilon)}{4R_{ij}} \left\{ i e^{-i\pi\varepsilon} \int_0^{+\infty} \frac{dy}{\sqrt{y}} (1+y)^{-1-\varepsilon} \times \int_0^{+\infty} \frac{dv}{v} \left[ \frac{1}{[vR_{ij} - (1+v) T]^{1+\varepsilon}} - \frac{1}{[-(1+v) T]^{1+\varepsilon}} \right] \right. \\
+ \left. \int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \times \int_0^{+\infty} \frac{dv}{v} \left[ \frac{1}{[vR_{ij} - (1+v) T]^{1+\varepsilon}} - \frac{1}{[-(1+v) T]^{1+\varepsilon}} \right] \right. \\
+ \left. \int_1^{1} \frac{dy}{\sqrt{y}} (1-y)^{-1-\varepsilon} \times \int_1^{+\infty} \frac{dv}{v} \left[ \frac{1}{[vR_{ij} + (1-v) T]^{1+\varepsilon}} - \frac{1}{[(1-v) T]^{1+\varepsilon}} \right] \right\} 
\]

(3.45)

Here again the integrals over \( y \) and \( v \) are unnested. Since the signs of \( \text{Im}(R_{ij}) \) and \( \text{Im}(T) \) are mutually opposite, let us note that the imaginary parts of each of the terms raised to the power \( 1+\varepsilon \) in denominators in the \( v \) integrals remain constant over the corresponding ranges of integration over \( v \). The first two integrals on \( v \) of eq. (3.45) are given by eq. (C.13) with \( A = R_{ij} - T \) and \( B = -T \) while the last one is given by eq. (C.15) with \( A' = R_{ij} - T \) and \( B' = T \). As for the \( y \) integration, they can be read from eqs. (C.1) to (C.6). All ingredients combine into:

\[
L_4^n(\Delta_3, 0, \Delta_{i,j}^{(i,j)}, 0) = \frac{1}{2} \Gamma(1+\varepsilon) \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{1}{R_{ij} T} \times \left\{ \frac{1}{\varepsilon^2} (2R_{ij})^{-\varepsilon} - \text{Li}_2 \left( \frac{R_{ij}}{T} \right) - [\ln (R_{ij}) - \ln (-T) - i \pi] \ln \left( \frac{T - R_{ij}}{T} \right) \right\} 
\]

(3.46)

In the present case \( \text{Im}(-R_{ij}) \) and \( \text{Im}(T) > 0 \) so that \( \ln (-T) = \ln (T) - i \pi \) thus eq. (3.46) can be rewritten:

\[
L_4^n(\Delta_3, 0, \Delta_{i,j}^{(i,j)}, 0) = \frac{1}{2} \Gamma(1+\varepsilon) \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{1}{R_{ij} T} \times \left\{ \frac{1}{\varepsilon^2} (2R_{ij})^{-\varepsilon} - \text{Li}_2 \left( \frac{R_{ij}}{T} \right) - [\ln (R_{ij}) - \ln (T)] \ln \left( \frac{T - R_{ij}}{T} \right) \right\} 
\]

(3.47)

i.e. again the same analytic form as the previous two cases, cf. eqs. (3.30) and (3.44).

3.3.4  \( \text{Im}(\Delta_3) < 0, \text{Im}(\Delta_{i,j}^{(i,j)}) < 0 \)

The starting point is now eq. (3.23) with \( P_{ijk} = -R_{ij} \). In the first two integrals we rescale \( y = x u^2 \) to factorise the double integral into a product of two unnested integrals over \( u \)
and over $x$. Using results given in eqs. (C.1) and (C.2), these unnested integrals are readily performed yielding:

\[
L_4^\alpha(\Delta_3, 0, \Delta_1^{(i,j)}, 0) = \frac{F(\varepsilon)}{4 R_{ij}} \left\{ i \pi (-T)^{-1-\varepsilon} \left[ -i e^{-i\pi \varepsilon} B \left( \frac{1}{2}, \frac{1}{2} + \varepsilon \right) - B \left( \frac{1}{2} + \varepsilon, -\varepsilon \right) \right] 
- i \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y+x} (R_{ij} (x+y) + T (1-x))^{-1-\varepsilon} 
+ i e^{-i\pi \varepsilon} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y-x} \times \left[ (-R_{ij} (y-x) - T (1+x))^{-1-\varepsilon} - (-T)^{-1-\varepsilon} (1+x)^{-1-\varepsilon} \right] 
+ \int_1^{+\infty} \frac{dy}{\sqrt{y}} \int_1^y \frac{dx}{y-x} \times \left[ (-R_{ij} (y-x) + T (1-x))^{-1-\varepsilon} - (-T)^{-1-\varepsilon} (x-1)^{-1-\varepsilon} \right] \right\} 
\]

(3.48)

In the three remaining integrals in eq. (3.48), the first two ones remain finite when $\varepsilon \to 0$ and are thus computed in this limit, which yields:

\[
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{dx}{y+x} (R_{ij} (x+y) + T (1-x))^{-1} 
= \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{1}{1+y} \left[ \ln \left( \frac{y R_{ij} + T}{R_{ij}} \right) \right] - \ln(y) 
\]

(3.49)

\[
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^y \frac{dx}{y-x} [(-R_{ij} (y-x) - T (1+x))^{-1-\varepsilon} - (-T)^{-1} (1+x)^{-1}] 
= \frac{1}{T} \int_0^{+\infty} \frac{dy}{\sqrt{y}} \int_0^1 \frac{1}{1+y} \ln \left( \frac{y R_{ij} + T}{T} \right) 
\]

(3.50)

Making the change of variable $u = 1/y$ we readily see that

\[
\int_0^{+\infty} \frac{dy}{\sqrt{y}} \frac{\ln(y)}{1+y} = - \int_0^{+\infty} \frac{du}{\sqrt{u}} \frac{\ln(u)}{1+u} = 0
\]

the combination \{ $- i \times (3.49) + i \times (3.50)$ \} thus gives $i \pi \ln(R_{ij}/T)$. In the third integral, we make the change of variable $x = y - (y-1) v$, the third integral thus becomes

\[
\int_1^{+\infty} \frac{dy}{\sqrt{y}} (y-1)^{-1-\varepsilon} \int_0^1 \frac{dv}{v} \left[ (-v R_{ij} + T (v-1))^{-1-\varepsilon} - (-T)^{-1-\varepsilon} (1-v)^{-1-\varepsilon} \right]
\]

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The $v$ integration is performed using the result of $K_2(-R_{ij}, -T)$ (cf. eq. (C.10)). After some algebra, eq. (3.48) thus reads:

$$L_n^4(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{1}{2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{R_{ij} T} \left[ \frac{1}{\varepsilon^2} (2 R_{ij})^{-\varepsilon} + \text{Li}_2 \left( \frac{T - R_{ij}}{T} \right) - \frac{\pi^2}{6} \right]$$  \hspace{1cm} (3.51)

Eq. (3.51) is identical to eq. (3.29); since Im($R_{ij}$) and Im($T$) have the same sign as in subsubsec. 3.3.1, eq. (3.51) can be recast into a form identical to eq. (3.30):

$$L_n^4(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0) = \frac{1}{2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{1}{R_{ij} T} \times \left\{ \frac{1}{\varepsilon^2} (2 R_{ij})^{-\varepsilon} - \text{Li}_2 \left( \frac{R_{ij}}{T} \right) - [\ln(R_{ij}) - \ln(T)] \ln \left( \frac{T - R_{ij}}{T} \right) \right\}$$  \hspace{1cm} (3.52)

In summary in all four cases $L_n^4(\Delta_3, 0, \Delta_1^{\{i,j\}}, 0)$ takes the same analytical form. Compared with the multiplicity of forms met in the general complex mass case, and still with the diverse cases met in subsec. 3.2, this simplification come from the coalescence of the pole and branch points all at the value 1 which is the end-point singularity causing the appearance of the soft and collinear singularity in all four cases.

### 3.4 Consistency checks and explicit examples

In refs. [6,7] the infrared structure of any IR divergent $N$-point one-loop integral was shown to be carried by IR divergent three-point one-loop functions resulting from appropriate iterated pinchings. In the following this feature is explicitly verified for the formulae obtained in this article for the four-point functions, when compared with those for the three-point functions, when the latter are formulated most conveniently according to the so-called “indirect way” for this purpose.

In the case of purely soft divergence i.e. whenever some $\Delta_2^{\{ij\}} = 0$ whereas $\tilde{D}_{ijk} \neq 0$ the various cases cf. eqs. (3.14), (3.19), (3.22) and (3.25) can be encompassed in one single formula. For this purpose let us introduce the following notation, where for any complex $Q$ we denote $Q_R \equiv \text{Re}(Q)$ and $Q_I \equiv \text{Im}(Q)$:

$$\int_{(\bar{0},1)} du F(u) = \begin{cases} \int_0^{-S_A} du F(u) + \int_{+\infty}^1 du F(u) & \text{if } 0 < -B_I/A_I < 1 \text{ and } A_I [A_R B_I - A_I B_R] > 0 \\ \int_0^1 du F(u) \end{cases}$$  \hspace{1cm} (3.53)

with $S_A = \text{sign}(A_I)$, whether

$$F(u) = \frac{\ln(A u^2 + B) - \ln(A u_0^2 + B)}{u^2 - u_0^2} \quad \text{with} \quad u_0^2 \neq -\frac{B}{A}$$

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or

\[ F(u) = (Au^2 + B)^{-1-\varepsilon} \]

The condition “\(0 < -B_1/A_1 < 1\) and \(A_1[A_R B_I - A_I B_R] > 0\)” is the condition for the discontinuity cuts of \(\ln(Au^2 + B)\) to cross the real axis between 0 and 1 (cf. appendix D of [2]). This enables us to rewrite \(L_n^4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk})\) in a generic way in the various cases:

\[
L_n^4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \frac{2\varepsilon}{T} \Gamma(1 + \varepsilon) \\
\times \left\{ -\frac{1}{\varepsilon} \int_{(0,1)} du \left( u^2 P_{ijk} + R_{ij} \right)^{-1-\varepsilon} - U \left( \Delta_3, \Delta_1^{(ij)}, \tilde{D}_{ijk} \right) \\
+ \int_{(0,1)} du \frac{u^2 P_{ijk} + R_{ij}}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( T (1 - u^2) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \\
- \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T \right) \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \right\} \quad (3.54)
\]

with:

\[
U(\Delta_3, \Delta_1^{(ij)}, \tilde{D}_{ijk}) \\
= \begin{cases} \\
0 & \text{if } \text{Im}(\Delta_3) > 0 \text{ Im}(\Delta_1^{(ij)}) > 0 \\
\int_{T^+} \frac{du}{u^2 P_{ijk} + R_{ij}} \ln \left( \frac{T-R_{ij}}{T} \right) & \text{if } \text{Im}(\Delta_3) > 0 \text{ Im}(\Delta_1^{(ij)}) < 0 \\
\int_{T^+} \frac{du}{u^2 P_{ijk} + R_{ij}} \eta \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \ln \left( \frac{T-R_{ij}}{T} \right) & \text{if } \text{Im}(\Delta_3) < 0 \text{ Im}(\Delta_1^{(ij)}) > 0 \\
\int_{T^+} \frac{du}{u^2 P_{ijk} + R_{ij}} + \eta \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \ln \left( \frac{T-R_{ij}}{T} \right) & \text{if } \text{Im}(\Delta_3) < 0 \text{ Im}(\Delta_1^{(ij)}) < 0 \end{cases} \quad (3.55)
\]

where the contour \(\Gamma^+\) is the closed contour encircling the “north-east” quadrant clockwise (cf. subsec. 3.1 of [2]). Depending on the location of the cuts of \((u^2 P_{ijk} + R_{ij})^{-1-\varepsilon}\), the first term of eq. (3.54) is the same as those which appear in eqs. (2.48) or (2.51), \(L_n^4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk})\) can thus be written as:

\[
L_n^4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) = \frac{1}{T} L_n^3(0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) + \tilde{L}_n^4(\Delta_3, 0, \Delta_1^{(i,j)}, \tilde{D}_{ijk}) \quad (3.56)
\]
with
\[
\tilde{L}_4^n(\Delta_3, 0, \Delta^{(i,j)}, \tilde{D}_{ijk}) = \frac{1}{T} \left\{ \int_{(0,1)} \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \ln \left( T \left( 1 - u^2 \right) \right) - \ln \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}} \right) \right] \\
- \ln \left( u^2 (P_{ijk} + R_{ij} - T) + T \right) + \ln \left( \frac{(P_{ijk} + R_{ij}) (T - R_{ij})}{P_{ijk}} \right) \\
- \int_0^1 \frac{du}{u^2 P_{ijk} + R_{ij}} \left[ \eta \left( \frac{T (P_{ijk} + R_{ij})}{P_{ijk}}, \frac{T - R_{ij}}{T} \right) + \ln \left( \frac{T - R_{ij}}{T} \right) \right] \\
- U \left( \Delta_3, \Delta^{(i,j)}, \tilde{D}_{ijk} \right) \right\}
\]
(3.57)
and \( L^n_4(0, \Delta^{(i,j)}, \tilde{D}_{ijk}) \) is given by eq. (2.48) or eq. (2.53) depending on the sign of the imaginary part of \( \Delta^{(i,j)} \).

For the cases where \( \Delta^{(i)} = 0 \) and \( \tilde{D}_{ijk} = 0 \), a unique formula for \( L^n_4(\Delta_3, 0, \Delta^{(i,j)}, 0) \) was found above whatever the signs of \( \text{Im}(\Delta_3) \) and \( \text{Im}(\Delta^{(i,j)}) \). The decomposition of the form (3.56) stills holds, the IR divergent part of \( L^n_4(\Delta_3, 0, \Delta^{(i,j)}, 0) \) is the same as in the three-point case (cf. eqs. (2.56) and (2.59)), whereas now:
\[
\tilde{L}_4^n(\Delta_3, 0, \Delta^{(i,j)}, 0) = -\frac{1}{2 R_{ij} T} \left[ \text{Li}_2 \left( \frac{R_{ij}}{T} \right) + [\ln(R_{ij}) - \ln(T)] \ln \left( \frac{T - R_{ij}}{T} \right) \right]
\]
(3.58)
Coming back to \( I^n_4 \) and using the results of subsec. 2.3, we have:
\[
I^n_4 = \sum_{i \in S_4} \frac{\tilde{b}_i}{\det (G)} \sum_{j \in S_4 \backslash \{i\}} \frac{\bar{b}_j^{(i)}}{\det (G^{(i)})} \frac{W \left( \det (G^{(i,j)}), \tilde{D}_{ijk}, \tilde{D}_{ijl} \right)}{T} \\
+ \sum_{i \in S_4} \frac{\tilde{b}_i}{\det (G)} \sum_{j \in S_4 \backslash \{i\}} \frac{\bar{b}_j^{(i)}}{\det (G^{(i)})} \sum_{k \in S_4 \backslash \{i,j\}} \frac{b_k^{(i,j)}}{\det (G^{(i,j)})} 
\tilde{L}_4^n(\Delta_3, 0, \Delta^{(i,j)}, \tilde{D}_{ijk})
\]
(3.59)
where \( l \in S_4 \backslash \{i, j, k\} \). The first term in the r.h.s. of eq. (3.59) is nothing but the combination of the three-point functions such as in refs. [6, 7] decomposed according to the so called “direct way” made explicit in subsec. 2.1. Note that when \( \tilde{D}_{ijk} = 0 \), the quantity \( \tilde{L}_4^n(\Delta_3, 0, \Delta^{(i,j)}, 0) \) which does actually not depend on \( k \) factors out from the sum over \( k \) which then yields trivially - 1. The functions \( W(\det (G^{(i,j)}), \tilde{D}_{ijk}, \tilde{D}_{ijl}) \) has been defined by eq. (2.6). Its arguments have one extra subscript, tracing back the extra pinching which was involved compared with the three-point case. We recap here the different results concerning this function.
\[
W \left( \det (G^{(i,j)}), \tilde{D}_{ijk}, \tilde{D}_{ijl} \right) = \frac{2^\varepsilon}{\varepsilon} \Gamma(1 + \varepsilon) \int_0^1 dx \left( (D^{(i,j)}(k))(x) - i \lambda \right)^{-1-\varepsilon}
\]
(3.60)
where
\[ D^{(i,j)(k)}(x) = G^{(i,j)(k)} x^2 - 2 V^{(i,j)(k)} x - C^{(i,j)(k)} \] (3.61)

with:
\[ G^{(i,j)(k)} = -S_{il} + 2 S_{kl} - S_{kk} = \det (G^{(i,j)}) \]
\[ V^{(i,j)(k)} = S_{kl} - S_{kk} = \frac{1}{2} \left[ \det (G^{(i,j)}) - \tilde{D}_{ijk} + \tilde{D}_{ijl} \right] \] (3.62)
\[ C^{(i,j)(k)} = S_{kk} = -\tilde{D}_{ijk} \]

Note that \( W(\det (G^{(i,j)}), \tilde{D}_{ijk}, \tilde{D}_{ijl}) \) is symmetric under the exchange of \( i \) and \( j \).

* If \( \tilde{D}_{ijk} \) and \( \tilde{D}_{ijl} \) both differ from zero, \( W(\det (G^{(i,j)}), \tilde{D}_{ijk}, \tilde{D}_{ijl}) \) is given by eq. (2.15).
* If only \( \tilde{D}_{ijl} \) vanishes, \( W(\det (G^{(i,j)}), \tilde{D}_{ijk}, 0) \) is read from eq. (2.28).
* If \( \tilde{D}_{ijk} \) and \( \tilde{D}_{ijl} \) both vanish, \( W(\det (G^{(i,j)}), 0, 0) \) is given by eq. (2.30).

For practical purpose let us stress that we do not have to compute a dedicated formula for each IR four-point case as in ref. [3] where 16 cases were distinguished. Indeed, for each contribution labelled by the index \( i \), we merely distinguish two cases: either \( \Delta^{(i)}_2 \neq 0 \) for which we use the generic formula suited to the massive case, or \( \Delta^{(i)}_2 = 0 \) for which we use the appropriate formula suited to the IR case at hand. The massive case is split depending on the vanishing of \( \text{Im}(\Delta_3) \) (namely if one internal mass squared has an imaginary part different from zero). The IR case is also divided in two cases: \( \tilde{D}_{ijk} = 0 \) and \( \tilde{D}_{ijl} \neq 0 \). The latter is furthermore separated according to the fact that one or several internal masses squared have a non vanishing imaginary part. All these cases are depicted on the decision tree presented in fig. 5, for each case the appropriate formula is given. Notice that when \( \text{Im}(\Delta_3) \neq 0 \), it may appear that \( \Delta^{(i)}_2 \) and/or \( \Delta^{(i,j)}_1 \) are real, in this case they must be understood as having a “\( +i\lambda \)” prescription. The expense paid by the present method is a possible proliferation of dilogarithms, a counteraction against which would require some extra work. This point will be commented in some more details in the examples studied below.

1) Two opposite external masses

In this case, all internal masses are zero and two opposite external legs (say 1 and 3) have non lightlike four momenta (our convention is depicted in fig. 4). The texture of the \( S \) matrix is:

\[ S = \begin{pmatrix}
0 & 0 & s_{23} & s_1 \\
0 & 0 & s_3 & s_{12} \\
s_{23} & s_3 & 0 & 0 \\
s_1 & s_{12} & 0 & 0
\end{pmatrix} \] (3.63)
with \( s_i = p_i^2 \) and \( s_{ij} = (p_i + p_j)^2 \) (all the momenta are taken ingoing). All the contributions \( i \) are such that \( \Delta_{2}^{(i)} = 0 \) and \( \tilde{D}_{ijk} = 0 \). In this case, \( R_{ij} = -\Delta_{1}^{(i,j)} \) is symmetric under the
exchange of \(i\) and \(j\). The four-point function is given by, cf. eq. (3.29):

\[
I_n^4 = \frac{1}{2 \det(S)} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \sum_{i \in S_4} \sum_{j > i} \frac{1}{\Delta^{(i,j)}_1} \left( \frac{\bar{b}_i^{(i)} \bar{b}_j^{(j)}}{\det(G^{(i)})} + \frac{\bar{b}_j^{(j)} \bar{b}_i^{(i)}}{\det(G^{(j)})} \right) \\
\times \left[ \frac{1}{\varepsilon^2} \left( -2 \Delta^{(i,j)}_1 \right)^{-\varepsilon} + \text{Li}_2 \left( \frac{T - R_{ij}}{T - i \lambda} \right) - \frac{\pi^2}{6} \right] 
\]

Due to the hollow texture of the reduced \(S\) matrices, a bunch of \(b^{(i)}_j\) coefficients vanish. Eq. (3.64) thus simplifies into:

\[
I_n^4 = \frac{1}{2 \det(S)} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \sum_{i=1}^2 \sum_{j=3}^4 \frac{1}{\Delta^{(i,j)}_1} \left( \frac{\bar{b}_i^{(i)} \bar{b}_j^{(j)}}{\det(G^{(i)})} + \frac{\bar{b}_j^{(j)} \bar{b}_i^{(i)}}{\det(G^{(j)})} \right) \\
\times \left[ \frac{1}{\varepsilon^2} \left( -2 \Delta^{(i,j)}_1 \right)^{-\varepsilon} + \text{Li}_2 \left( 1 - \frac{R_{ij} - i \lambda}{T - i \lambda} \right) - \frac{\pi^2}{6} \right] 
\]

Expressing the \(b_i\) and \(b^{(i)}_j\) coefficient as well as the various determinants as functions of the \(S\) matrix elements, we get:

\[
I_n^4 = \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2 \varepsilon)} \frac{2}{d} \\
\left\{ (-s_1 - i \lambda)^{-\varepsilon} + (-s_3 - i \lambda)^{-\varepsilon} - (-s_{12} - i \lambda)^{-\varepsilon} - (-s_{23} - i \lambda)^{-\varepsilon} \\
- \text{Li}_2 \left( 1 - \frac{s_{12} + i \lambda}{d/\Sigma + i \lambda} \right) - \text{Li}_2 \left( 1 - \frac{s_{23} + i \lambda}{d/\Sigma + i \lambda} \right) \\
+ \text{Li}_2 \left( 1 - \frac{s_3 + i \lambda}{d/\Sigma + i \lambda} \right) + \text{Li}_2 \left( 1 - \frac{s_1 + i \lambda}{d/\Sigma + i \lambda} \right) \right\} 
\]

with:

\[
d = s_1 s_3 - s_{12} s_{23} \\
\Sigma = s_1 + s_3 - s_{12} - s_{23} 
\]

Eq. (3.68) involves four dilogarithms as in the refs. [12, 13]. The arguments of the dilogarithms in eq. (3.66) vs. in ref. [12] are seemingly different, namely ref. [12] involves \(\text{Li}_2(1 - (w + i \lambda)/(d/\Sigma))\) whereas \(\text{Li}_2(1 - (w + i \lambda)/(d/\Sigma + i \lambda))\) appears in eq. (3.66), where \(w\) stands for \(s_1, s_3, s_{12}\) or \(s_{23}\). However the \(i \lambda\) prescriptions matter only when the real parts of the arguments of the dilogarithms are greater than 1 i.e. whenever \(w \Sigma/d < 0\), in which case the signs of \((d/\Sigma) - w\) and of \(\Sigma/d\), which respectively control the signs of the \(i \lambda\) prescriptions in either case, are the same: the two results in eq. (3.66) and in ref. [12] are actually identical.
This is to be compared with the formula given in ref. [3]. The latter was taken from ref. [13] which involves five dilogarithms instead of four. The authors of ref. [13] used the so-called Mantel identity which entails nine dilogarithms to prove that the four-dilogarithm and five-dilogarithm results are actually equivalent. The Mantel identity happens to be a corollary of the Hill identity, the former is derived by applying the latter three times to some suitable combinations of variables\textsuperscript{10}. The continuation of the Hill identity to any arbitrary two complex variables however requires additional combinations of $\eta$ functions handling the mismatch between the various discontinuities of the dilogarithms involved, and these $\eta$ functions are often skipped in the literature\textsuperscript{11}. An even busier modification is then required for the Mantel identity. This drove of $\eta$ functions makes the analytical check of the equivalence between the four-dilogarithm and five-dilogarithm expressions extremely awkward in general, and to our understanding this drove of $\eta$ functions was not accounted in ref. [13]. We did perform numerical tests accounting for these $\eta$ functions, which verified the equivalence for the configurations probed.

2) A simple case with one internal masses

With the same notations of the preceding example, the texture of the $S$ matrix is:

\[
S = \begin{pmatrix}
0 & 0 & s_{23} - m_3^2 & 0 \\
0 & 0 & 0 & s_{12} \\
s_{23} - m_3^2 & 0 & -2m_3^2 & s_4 - m_3^2 \\
0 & s_{12} & s_4 - m_3^2 & 0
\end{pmatrix}
\] (3.69)

In this case, the sector $i = 1$ has no soft or collinear divergence and is computed using the massive formula. The other sectors correspond to the cases: $\Delta_{2}^{(i)} = 0$, $\bar{D}_{ijk} \neq 0$ and $\Delta_{2}^{(i)} = 0$, $\bar{D}_{ijk} = 0$. The four-point amplitude can be cast in a divergent part and a finite one. The divergent part is given by:

\[
(I_{4}^{n})_{\text{div}} = \sum_{i \in S_{4} \setminus \{1\}} \tilde{b}_{i} \frac{\det(G)}{\det(G_{\{i\}})} \sum_{j \in S_{4} \setminus \{i\}} \tilde{b}_{j}^{(i)} \frac{W \left( \det(G_{\{i,j\}}, \bar{D}_{ijk}, \bar{D}_{ijl} \right)}{T} \) (3.70)

As several $\tilde{b}_{j}^{(i)}$ vanish due to the hollow texture of reduced $S$ matrices, we actually have to\textsuperscript{10}See [14], chap. 1, p. 2-3 and chap. 2, p. 17-18.\textsuperscript{11}See however [15].
compute\textsuperscript{12}:

\[
(I^2_4)_{\text{div}} = \frac{\tilde{b}_2}{\det(S)} \left[ \frac{\tilde{b}_1^{(2)}}{\det(G^{(2)})} W \left( \det(G^{(1,2)}), \tilde{D}_{124}, 0 \right) + \frac{\tilde{b}_1^{(2)}}{\det(G^{(2)})} W \left( \det(G^{(2,4)}), \tilde{D}_{124}, 0 \right) \right]
+ \frac{\tilde{b}_3}{\det(S)} \frac{\tilde{b}_1^{(3)}}{\det(G^{(3)})} W \left( \det(G^{(1,3)}), 0, 0 \right)
+ \frac{\tilde{b}_4}{\det(S)} \frac{\tilde{b}_2^{(4)}}{\det(G^{(4)})} W \left( \det(G^{(2,4)}), \tilde{D}_{124}, 0 \right)
\]

where \( W \left( \det(G^{(1,2)}), \tilde{D}_{124}, 0 \right) \), \( W \left( \det(G^{(2,4)}), \tilde{D}_{124}, 0 \right) \) are given by eq. (2.28) and \( W \left( \det(G^{(1,3)}), 0, 0 \right) \) by eq. (2.30). We get:

\[
(I^2_4)_{\text{div}} = \frac{1}{\varepsilon} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon) \Gamma(1 - 2 \varepsilon)}{\Gamma(1 + \varepsilon)} \frac{1}{s_{12} (s_{23} - m_3^2)}
\left[ \frac{2}{\varepsilon} (-s_{23} + m_3^2 - i \lambda)^{-\varepsilon} + \frac{1}{\varepsilon} (-s_{12} - i \lambda)^{-\varepsilon} \right.
- \frac{1}{\varepsilon} (-s_4 + m_3^2 - i \lambda)^{-\varepsilon} - \frac{1}{2 \varepsilon} (m_3^2 - i \lambda)^{-\varepsilon}
+ \varepsilon \text{Li}_2 \left( \frac{s_4}{s_4 - m_3^2 + i \lambda} \right)
- 2 \varepsilon \text{Li}_2 \left( \frac{s_{23}}{s_{23} - m_3^2 + i \lambda} \right) + \varepsilon \frac{\pi^2}{12} \right]
\]

Expanding eq. (3.72) in \( \varepsilon \), we recover the results of ref. [3] (eq. (4.27)) taken from ref. [16] for the terms proportional to \( 1/\varepsilon^2 \) and \( 1/\varepsilon \). Concerning the finite part, we obtain a host of terms for which it is cumbersome to verify analytically that they do reduce to the finite part of eq. (4.27) of ref. [3]. We verified numerically that they do indeed.

\section{Summary and outlook}

In this article we presented an extension of a novel approach developed in companion articles [1] and [2] to the case of vanishing internal masses involving soft and/or collinear divergences. For this latter case, the method remains very similar to the massive cases: the three- and four-point functions are split into “sectors” whose coefficients are expressed in terms of algebraic kinematical invariants involved in reduction algorithms. Each “sector” may diverge or not when the IR regulator is sent to zero yielding to a simple decision tree to compute the relevant integrals. This avoids the computation of the numerous different integrals over Feynman parameters as it is usually done in the literature. This extension also applies to general kinematics beyond the one relevant for one-loop collider processes, offering a

\textsuperscript{12} We will use the following properties: \( \det(G^{(i,j)}) \) is symmetric under the permutation \( i \leftrightarrow j \) and \( \tilde{D}_{ijk} \) is symmetric under any permutation of the set \( \{ i, j, k \} \)
potential application to the calculation of two-loop processes using one-loop (generalised) $N$-point functions as building blocks as discussed in the introduction of [1].

One drawback of the present method is the proliferation of dilogarithms in the expression of the four-point function computed in closed form. This requires some extra work to be better apprehended, in order to counteract it. But as the method used hereby is the same as in the real mass case, up to slight modifications, any solution found for the latter case can be applied in the infrared divergent case. This issue will be addressed in a future article.

The last goal is to provide the generalised one-loop building blocks entering as integrands in the computation of two-loop three- and four-point functions by means of an extra numerical double integration. In this respect, let us mention that the expansion around $\varepsilon = 0$ of the results given in this article has been truncated in order to keep only the divergent and the constant terms. This is sufficient for any one-loop computation but may be not enough for two-loop applications of the method. This article contains already a lot of results, so the expansion around $\varepsilon = 0$ at the necessary orders is postponed to a future work.

In memoriam

This work has been initiated by Prof. Shimizu after a visit to LAPTh. He explained us his idea about the numerical computation of scalar two-loop three- and four-point functions, he shared his notes partly in English, partly in Japanese with us and he encouraged us to push this project forward. J.Ph. G. would like to thank Shimizu-sensei for giving him a taste of the Japanese culture and for his kindness.

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A Two basic integrals

In what follows $A$ and $B$ are assumed dimensionless and complex valued, the signs of their real parts is unknown, and the signs of their imaginary parts may or may not be the same either.
A.1 First kind

The computation of the three- and four-point functions in a space-time of arbitrary dimensions, involves the following extension of the case treated in appendix D of ref. [1]:

\[ K(\nu) = \int_0^\infty \frac{d\xi}{(\xi^\nu + A)(\xi^\nu + B)} \]  

(A.1)

After partial fraction decomposition the r.h.s. of eq. (A.1) becomes:

\[ K(\nu) = \frac{1}{B - A} \int_0^\infty d\xi \left[ \frac{1}{\xi^\nu + A} - \frac{1}{\xi^\nu + B} \right] \]  

(A.2)

Let us assume that \( \nu > 1 \) such that the r.h.s. of eq. (A.1) can be split into a difference of two convergent integrals at infinity, which can be separately computed using appendix B of ref. [1] with \( \mu = 1 \). \( K(\nu) \) thus reads:

\[ K(\nu) = \frac{1}{B - A} \frac{1}{\nu} B \left( 1 - \frac{1}{\nu} \frac{1}{\nu} \right) \left[ A^{\frac{1}{\nu} - 1} - B^{\frac{1}{\nu} - 1} \right] \]  

(A.3)

regardless of the signs of \( \text{Im}(A) \) vs. \( \text{Im}(B) \). In the case of the three-point function, \( \nu = 1/(1 - \varepsilon) \) which is slightly less than 1 for \( \varepsilon < 0 \), the r.h.s. of eq. (A.3) can be analytically continued in \( \nu \) as long as \( \nu \neq -1/n \) or \( \nu \neq 1/(n + 1) \) with \( n \) an arbitrary positive integer. So the result of eq. (A.3) can be used in the case where the dimension of the space-time is shifted by a small positive amount from \( n = 4 \) to \( n = 4 - 2\varepsilon \). We also note that the limit \( \nu \to 1 \) of \( K(\nu) \) leads to eq. (D.4) of ref. [1].

A practical case met in sec. 2 is \( A = -i\lambda \) and \( B \) remaining an arbitrary complex number with an imaginary part different from 0. In the limit \( \lambda \to 0^+ \), eq. (A.3) becomes:

\[ \int_0^{+\infty} \frac{d\xi}{(\xi^\nu - i\lambda)(\xi^\nu + B)} = -\frac{1}{\varepsilon} \frac{1}{\Gamma(1 + \varepsilon)\Gamma(1 - \varepsilon)} B^{-1-\varepsilon} \]  

(A.4)

This is a well-known fact that the two limits \( \lambda \to 0^+ \) and \( \varepsilon \to 0 \) do not commute.

A.2 Second kind

The most general case for the integral

\[ J(\nu) = \int_0^{+\infty} \frac{d\xi}{(\xi^\nu + A)\sqrt{\xi^\nu + B}} \]  

(A.5)

has been treated in appendix A of ref. [2], let us just recap the results. Two cases can be distinguished according to the signs of the imaginary parts of \( A \) and \( B \).
1) Im(A) and Im(B) of the same sign

\[ J(\nu) = \frac{1}{\nu} B \left( \frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu} \right) \int_0^1 dz \left( (1 - z^2) A + z^2 B \right)^{-3/2 + 1/\nu} \]  \hspace{1cm} (A.6)

2) Im(A) and Im(B) of opposite signs

\[ J(\nu) = -\frac{1}{\nu} B \left( \frac{3}{2} - \frac{1}{\nu}, \frac{1}{\nu} \right) \times \left[ e^{-iS_B \pi/\nu} \int_0^{+\infty} dz \left( B z^2 - (1 + z^2) A \right)^{-3/2 + 1/\nu} \right. \\
\left. + \int_1^{+\infty} dz \left( B z^2 + (1 - z^2) A \right)^{-3/2 + 1/\nu} \right] \]  \hspace{1cm} (A.7)

with \( S_B = \text{sign}(\text{Im}(B)) \).

Let us remind that the two cases 1) vs. 2) can be reunified by seeing eq. (A.7) as an analytic continuation in \( A \) of eq. (A.6) which possibly requires a deformation of the contour \([0, 1]\) originally drawn along the real axis in eq. (A.6), cf. appendix A of [2] for more details.

**B  \hspace{1cm} The function \( J(x_1, x_2) \)**

This appendix computes the function

\[ J(x_1, x_2) = \int_0^1 dx \frac{\ln((x - x_1) (x - x_2))}{(x - x_1) (x - x_2)} \]

Using partial fraction decomposition, \( J(x_1, x_2) \) can be written as:

\[ J(x_1, x_2) = \frac{1}{x_1 - x_2} \left[ \int_0^1 dx \frac{\ln(x - x_1)}{x - x_1} - \int_0^1 dx \frac{\ln(x - x_2)}{x - x_2} \right. \\
\left. + \int_0^1 dx \frac{\ln(x - x_2) - \ln(x_1 - x_2)}{x - x_1} - \int_0^1 dx \frac{\ln(x - x_1) - \ln(x_2 - x_1)}{x - x_2} \right. \\
\left. + \ln(x_1 - x_2) \int_0^1 dx \frac{dx}{x - x_1} - \ln(x_2 - x_1) \int_0^1 dx \frac{dx}{x - x_2} \right] \]  \hspace{1cm} (B.1)
With the help of appendix E of [1] (cf. also appendix B of ref. [17]), we immediately get:

\[
\int_0^1 dx \frac{\ln(x - x_1) - \ln(x_2 - x_1)}{x - x_2} = R'(x_1, x_2)
\]

\[
= \text{Li}_2 \left( \frac{x_2}{x_2 - x_1} \right) - \text{Li}_2 \left( \frac{x_2 - 1}{x_2 - x_1} \right) + \eta \left( -x_1, \frac{1}{x_2 - x_1} \right) \ln \left( \frac{x_2}{x_2 - x_1} \right) - \eta \left( 1 - x_1, \frac{1}{x_2 - x_1} \right) \ln \left( \frac{x_2 - 1}{x_2 - x_1} \right)
\]

(B.2)

where \(R'(x_1, x_2)\) is given by eq. (E.15) of [1]. Since \(x_1\) and \(x_2\) have imaginary parts of opposite signs all the \(\eta\) functions vanish in eq. (B.2). Then using the Landen identity:

\[
\text{Li}_2(z) + \text{Li}_2 \left( \frac{z}{z - 1} \right) = -\frac{1}{2} \ln^2(1 - z)
\]

we can rewrite eq. (B.2) as:

\[
\int_0^1 dx \frac{\ln(x - x_1) - \ln(x_2 - x_1)}{x - x_2}
\]

\[
= \text{Li}_2 \left( \frac{x_2 - 1}{x_2 - x_1} \right) - \text{Li}_2 \left( \frac{x_2}{x_1} \right) + \frac{1}{2} \left[ \ln^2(x_1 - 1) - \ln^2(x_1) - 2 \ln(x_1 - x_2) \ln \left( \frac{x_1 - 1}{x_1} \right) \right]
\]

(B.4)

Substituting into eq. (B.1) and easily computing the remaining \(x\) integrals, we get:

\[
J(x_1, x_2)
\]

\[
= \frac{1}{x_1 - x_2} \left\{ \frac{1}{2} \left[ \ln^2(1 - x_1) - \ln^2(-x_1) \right] - \frac{1}{2} \left[ \ln^2(1 - x_2) - \ln^2(-x_2) \right] + \text{Li}_2 \left( \frac{x_1 - 1}{x_2 - 1} \right) - \text{Li}_2 \left( \frac{x_1}{x_2} \right) - \text{Li}_2 \left( \frac{x_2 - 1}{x_1 - 1} \right) + \text{Li}_2 \left( \frac{x_2}{x_1} \right) + \frac{1}{2} \left[ \ln^2(x_2 - 1) - \ln^2(x_2) - 2 \ln(x_2 - x_1) \ln \left( \frac{x_2 - 1}{x_2} \right) \right] - \frac{1}{2} \left[ \ln^2(x_1 - 1) - \ln^2(x_1) - 2 \ln(x_1 - x_2) \ln \left( \frac{x_1 - 1}{x_1} \right) \right] + \ln(x_1 - x_2) \ln \left( \frac{x_1 - 1}{x_1} \right) - \ln(x_2 - x_1) \ln \left( \frac{x_2 - 1}{x_2} \right) \right\}
\]

(B.5)

\footnote{The subtlety discussed in appendix E of [1] does not show up here because \(x_1\) and \(x_2\) have imaginary parts of opposite signs.}
The following identity:

\[
\ln^2(1 - x_1) - \ln^2(-x_1) - \ln^2(x_1 - 1) + \ln^2(x_1) = -2 i \pi S(x_1) \ln \left( \frac{x_1 - 1}{x_1} \right)
\]

where \( S(x_1) \) is the sign of the imaginary part of \( x_1 \), allows to simplify eq. (B.5) into:

\[
J(x_1, x_2) = \frac{1}{x_1 - x_2} \left\{ \text{Li}_2 \left( \frac{x_1 - 1}{x_2 - 1} \right) - \text{Li}_2 \left( \frac{x_1}{x_2} \right) - \text{Li}_2 \left( \frac{x_2 - 1}{x_1 - 1} \right) + \text{Li}_2 \left( \frac{x_2}{x_1} \right) \\
+ [2 \ln(x_1 - x_2) - i \pi S(x_1)] \ln \left( \frac{x_1 - 1}{x_1} \right) \\
- [2 \ln(x_2 - x_1) - i \pi S(x_2)] \ln \left( \frac{x_2 - 1}{x_2} \right) \right\}
\]

(B.6)

Then, we use the identity relating \( \text{Li}_2(z) \) and \( \text{Li}_2(1/z) \) \[11\], and also the following relation:

\[
2 \ln (x_1 - x_2) - i \pi S(x_1) = 2 \ln (x_2 - x_1) - i \pi S(x_2) = \ln \left( -(x_1 - x_2)^2 \right)
\]

(B.7)

whose validity relies on the fact that \( x_1 \) and \( x_2 \) have imaginary parts of opposites signs. This yields:

\[
J(x_1, x_2) = \frac{1}{x_1 - x_2} \left\{ 2 \text{Li}_2 \left( \frac{x_1 - 1}{x_2 - 1} \right) - 2 \text{Li}_2 \left( \frac{x_1}{x_2} \right) + \frac{1}{2} \ln^2 \left( \frac{x_1 - 1}{x_2 - 1} \right) - \frac{1}{2} \ln^2 \left( \frac{x_1}{x_2} \right) \\
+ \ln \left( -(x_1 - x_2)^2 \right) \left[ \ln \left( \frac{x_1 - 1}{x_1} \right) - \ln \left( \frac{x_2 - 1}{x_2} \right) \right] \right\}
\]

(B.8)

C Herbarium of utilitarian integrals

This appendix collects a bunch of integrals appearing in the three- and four-point cases to make the reading easier.

The first series appears under the following forms and can be computed in terms of the Euler Beta function:

\[
\int_0^1 dz \ (1 + z^2)^{-1-\varepsilon} = \frac{1}{2} \int_0^1 dy \ (1 + y)^{-1-\varepsilon} = \frac{1}{2} B \left( \frac{1}{2}, \frac{1}{2} + \varepsilon \right)
\]

(C.1)

\[
\int_1^\infty dz \ (z^2 - 1)^{-1-\varepsilon} = \frac{1}{2} \int_1^\infty dy \ (y - 1)^{-1-\varepsilon} = \frac{1}{2} B \left( \frac{1}{2} + \varepsilon, -\varepsilon \right)
\]

(C.2)

\[
\int_0^1 dz \ (1 - z^2)^{-1-\varepsilon} = \frac{1}{2} \int_0^1 dy \ (1 - y)^{-1-\varepsilon} = \frac{1}{2} B \left( \frac{1}{2}, -\varepsilon \right)
\]

(C.3)
Using the duplication formula for the Gamma functions [11], the $z$ integrals computed in closed form read:

\[
\int_0^{+\infty} dz \left( 1 + z^2 \right)^{-1-\epsilon} = \frac{\tan(\pi \epsilon)}{2 \epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} 2^{-2\epsilon}
\]

(C.4)

\[
\int_1^{+\infty} dz \left( z^2 - 1 \right)^{-1-\epsilon} = -\frac{1}{2 \epsilon} \frac{1}{\cos(\pi \epsilon)} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} 2^{-2\epsilon}
\]

(C.5)

\[
\int_0^1 dz \left( 1 - z^2 \right)^{-1-\epsilon} = -\frac{1}{2 \epsilon} 2^{-2\epsilon} \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}
\]

(C.6)

For the four-point functions in the real mass case or in the complex mass case when sign(Im($R_{ij}$)) = sign(Im($T$)), the following integral needs to be evaluated:

\[
K_2(R_{ij}, T) = \int_0^1 \frac{dv}{v} \left[ \frac{1}{v R_{ij} + (1 - v) T - i \lambda} \right]^{1+\epsilon} - \frac{1}{[(1 - v) T - i \lambda]^{1+\epsilon}}
\]

(C.7)

Note that in both cases, Im($v R_{ij} + (1 - v) T - i \lambda$) has a constant sign when $v$ spans $[0, 1]$.

After a partial fraction decomposition w.r.t. the variable $v$, $K_2(R_{ij}, T)$ can be written as:

\[
K_2(R_{ij}, T) = \frac{1}{T} \int_0^1 \frac{dv}{v} \left\{ -(R_{ij} - T) \left[ R_{ij} v + T (1 - v) - i \lambda \right]^{1-\epsilon} - (T - i \lambda)^{-\epsilon} (1 - v)^{-1-\epsilon}
\]

\[
+ \frac{1}{v} \left[ (R_{ij} v + T (1 - v) - i \lambda)^{-\epsilon} - (T - i \lambda)^{-\epsilon} \right]
\]

\[
+ \frac{(T - i \lambda)^{-\epsilon}}{v} \left[ 1 - (1 - v)^{-\epsilon} \right] \right\}
\]

(C.8)

The first two terms of eq. (C.8) which yield a $1/\epsilon$ pole are integrated in closed form, whereas the last three terms of eq. (C.8) which are not divergent can be expanded around $\epsilon = 0$ up to order $\epsilon$:

\[
K_2(R_{ij}, T) = \frac{1}{T} \left[ \frac{1}{\epsilon} (R_{ij} - i \lambda)^{-\epsilon}
\]

\[
- \epsilon \int_0^1 \frac{dv}{v} \ln \left( R_{ij} v + T (1 - v) - i \lambda \right) - \ln \left( T - i \lambda \right)
\]

\[
+ \epsilon (T - i \lambda)^{-\epsilon} \int_0^1 \frac{dv}{v} \ln(1 - v) \right]
\]

(C.9)

Since sign(Im($R_{ij} v + T (1 - v) - i \lambda$)) = sign(Im($T - i \lambda$)) when $v \in [0, 1]$ the logarithms in the first integral can be combined together. The last integration is performed explicitly and we end with:

\[
K_2(R_{ij}, T) = \frac{1}{T} \left[ \frac{1}{\epsilon} (R_{ij} - i \lambda)^{-\epsilon} + \epsilon \Li_2 \left( \frac{T - R_{ij}}{T - i \lambda} \right) - \frac{\epsilon^2}{6} \right]
\]

(C.10)
With respect to the preceding case, two new integrals show up when \( \text{sign(Im}(T)) \neq \text{sign(Im}(R_{ij}))):\)

\[
K_3(A, B) = \int_0^{+\infty} \frac{dv}{v} [(Av + B)^{-1-\varepsilon} - (B(1 + v))^{-1-\varepsilon}] 
\]

\[
K_4(A', B') = \int_1^{+\infty} \frac{dv}{v} [(A'v + B')^{-1-\varepsilon} - (B'(1 - v))^{-1-\varepsilon}] 
\]

where \( \text{sign(Im}(A v + B)) \) (resp. \( \text{sign(Im}(A' v + B'))) \) keeps a constant sign when \( v \) spans \([0, +\infty[\) (resp. \([1, +\infty[\)). To compute \( K_3(A, B) \), we expand the r.h.s. of eq. (C.11) around \( \varepsilon = 0 \) and we get:

\[
K_3(A, B) = \frac{1}{B} \ln \left( \frac{B}{A} \right) [1 - \varepsilon \ln(B)] 
\]

To compute \( K_4(A', B') \), we expand \( (A'v + B')^{-1-\varepsilon} \) in eq. (C.12) around \( \varepsilon = 0 \), keeping in mind that \( \text{sign(Im}(A' + B')) = \text{sign(Im}(A')) \), and we get:

\[
K_4(A', B') = \frac{1}{B'} \left\{ -\frac{1}{\varepsilon} (-B')^\varepsilon + (\ln(A' + B') - \ln(A')) \\
+ \varepsilon \left[ \text{Li}_2 \left( -\frac{B'}{A'} \right) - \frac{\pi^2}{6} - \frac{1}{2} (\ln^2(A' + B') - \ln^2(A')) \right] \right\} 
\]

With the additional assumption that \( \text{sign(Im}(A')) = -\text{sign(Im}(B')) \) and after some algebra \( K_4(A', B') \) can be recast in the alternative more useful form:

\[
K_4(A', B') = \frac{1}{B'} \left\{ -\frac{1}{\varepsilon} (A' + B')^\varepsilon + (\ln(-B') - \ln(A')) \\
+ \varepsilon \left[ \text{Li}_2 \left( \frac{A' + B'}{B'} \right) + \ln (A' + B') \ln \left( -\frac{A'}{B'} \right) \right] \right\} 
\]

Notice that this additional assumption is always fulfilled in the cases met.

D Detailed comparisons with the “direct way” (cf. sec. 2)

Our present goal is to check that the “indirect way” leads to the same results for infrared divergent three-point functions as the “direct way”. By performing explicitly the sum over the \( j \) index in eq. (2.45), we recover the results of section (2). This part is not necessary for the understanding of the method proposed in this article and can be skipped in a first reading.
Real mass case

Let us start by the real mass case. We successively revisit the examples examined in subsection (2.2).

1. Occurrence of a soft divergence

Let us recap the texture of the $S$ matrix (cf. eq. (2.31)):

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2m_2^2 & s_3 - m_2^2 - m_3^2 \\ 0 & s_3 - m_2^2 - m_3^2 & -2m_3^2 \end{pmatrix}$$

thus $\det(S) = 0$. Let us single out row and column 1. We readily see that the vector $V^{(1)}$ vanishes and so do the coefficients $\tilde{b}_2$ and $\tilde{b}_3$; thus $\tilde{b}_1 = \det(G)$ since the coefficients $\tilde{b}_i$ fulfil $\sum_{i \in S_3} \tilde{b}_i = \det(G)$. The three-point function is thus given by (cf. eq. (2.45)):

$$I_3^n = \frac{\tilde{b}_2^{(1)}}{\det(G^{(1)})} L_3^n (0, \Delta_1^{(1)}, \tilde{D}_{12}) + \frac{\tilde{b}_3^{(1)}}{\det(G^{(1)})} L_3^n (0, \Delta_1^{(1)}, \tilde{D}_{13})$$

The only relevant reduced $S$ matrix is:

$$S^{(1)} = \begin{pmatrix} -2m_2^2 & s_3 - m_2^2 - m_3^2 \\ s_3 - m_2^2 - m_3^2 & -2m_3^2 \end{pmatrix}$$

whose determinant $\det(S^{(1)}) = -K(s_3, m_2^2, m_3^2)$ involves the Källén function given by eq. (2.11). The associated Gram “matrix” degenerates into a single scalar:

$$G^{(1)(2)} = \det(G^{(1)}) = (2s_3)$$

One easily reads the $\tilde{b}_j^{(1)}$ coefficients from the reduced Gram matrix $G^{(1)(2)}$ and the vector $V^{(1)(2)}$ (cf. the group of eqs. (2.31) of ref. [1]):

$$\tilde{b}_2^{(1)} = m_2^2 - m_3^2 - s_3, \quad \tilde{b}_3^{(1)} = m_3^2 - m_2^2 - s_3$$

Since $\tilde{D}_{12} = 2m_2^2$ and $\tilde{D}_{13} = 2m_3^2$, the quantities $L_3^n (0, \Delta_1^{(1)}, \tilde{D}_{12})$ and $L_3^n (0, \Delta_1^{(1)}, \tilde{D}_{13})$ are given by eq. (2.49). The $\tilde{D}_{12} + \Delta_1^{(1)}$, $\tilde{D}_{13} + \Delta_1^{(1)}$ and $\Delta_1^{(1)}$ terms are given by eqs. (2.38) and (2.40) of [1]:

$$\tilde{D}_{12} + \Delta_1^{(1)} = \frac{(s_3 + m_2^2 - m_3^2)^2}{2s_3}$$

$$\tilde{D}_{13} + \Delta_1^{(1)} = \frac{(s_3 + m_2^2 - m_3^2)^2}{2s_3}$$

$$\Delta_1^{(1)} = \frac{K(s_3, m_2^2, m_3^2)}{2s_3}$$
The roots of the denominator of eq. (2.49) are such that:

\[(\bar{z}^{12})^2 = \frac{\Delta_1^{(1)} + i\lambda}{\bar{D}_{12} + \Delta_1^{(1)}} = \frac{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}{(s_3 + m_2^2 - m_3^2)^2}\]

\[(\bar{z}^{13})^2 = \frac{\Delta_1^{(1)} + i\lambda}{\bar{D}_{13} + \Delta_1^{(1)}} = \frac{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}{(s_3 + m_2^2 - m_3^2)^2}\]  (D.9)  (D.10)

with \(\sigma_s = \text{sign}(s_3)\). We specify:

\[\bar{z}^{12} = \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s} \quad , \quad \bar{z}^{13} = \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}\]  (D.11)

Introducing two new quantities: \(\bar{x}_1\) and \(\bar{x}_2\) which are the two roots of the equation \(D^{(1)(2)}(x) = 0\) appearing in the “direct way”:

\[\bar{x}_1 = \frac{s_3 + m_2^2 - m_3^2 + \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}}{2s_3}\]  (D.12)

\[\bar{x}_2 = \frac{s_3 + m_2^2 - m_3^2 - \sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}}{2s_3}\]  (D.13)

Notice that the quantities \(1 - \bar{x}_1\) and \(1 - \bar{x}_2\) are the roots of the equation \(D^{(1)(3)}(x) = 0\). The quantities \(\bar{z}^{12}\), one of the roots of the equation \((\bar{D}_{12} + \Delta_1^{(1)}) \bar{z}^2 - \Delta_1^{(1)} - i\lambda = 0\), and \(\bar{z}^{13}\), a root of the equation \((\bar{D}_{13} + \Delta_1^{(1)}) \bar{z}^2 - \Delta_1^{(1)} - i\lambda = 0\), can be related to the roots \(\bar{x}_1\) and \(\bar{x}_2\) by the following relations:

\[\bar{z}^{12} = \frac{\bar{x}_1 - \bar{x}_2}{2 - \bar{x}_1 - \bar{x}_2} \quad , \quad \bar{z}^{13} = \frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\]  (D.14)

Putting things together, the \(I^n_3\) can be written:

\[I^n_3 = -\frac{2^\varepsilon \Gamma(1 + \varepsilon)}{2\sqrt{\mathcal{K}(s_3, m_2^2, m_3^2) + i\lambda\sigma_s}} \left\{-\frac{1}{\varepsilon} \left[\ln \left(\frac{\bar{z}^{12} - 1}{\bar{z}^{12} + 1}\right) + \ln \left(\frac{\bar{z}^{13} - 1}{\bar{z}^{13} + 1}\right)\right] + \bar{H}_{0,1} \left(\bar{D}_{12} + \Delta_1^{(1)}, -\Delta_1^{(1)} - i\lambda\right) + \bar{H}_{0,1} \left(\bar{D}_{13} + \Delta_1^{(1)}, -\Delta_1^{(1)} - i\lambda\right)\right\}\]  (D.15)

with:

\[\bar{H}_{0,1} (A, B) = 2A \sqrt{-\frac{B}{A}} H_{0,1} (A, B)\]  (D.16)

which is effectively the content between the curly brackets of eq. (E.8) in appendix E. Expressing all the arguments of logarithms and dilogarithms in eq. (E.8) in terms of \(\bar{x}_1\) and
\(\bar{x}_2\), we get:

\[
I_3^n = - \frac{2\varepsilon \Gamma(1 + \varepsilon)}{(\bar{x}_1 - \bar{x}_2) \det (G^{(1)})} \times \left\{ -\frac{1}{\varepsilon} \left[ \ln \left( \frac{1 - \bar{x}_1}{1 - \bar{x}_2} \right) + \ln \left( \frac{-\bar{x}_2}{\bar{x}_1} \right) \right] + \ln \left( \frac{-\bar{x}_2}{1 - \bar{x}_2} \right) \left[ \ln \left( \frac{(2 - \bar{x}_1 - \bar{x}_2)^2}{4} \det (G^{(1)}) + i \lambda \sigma_4 \right) \right. \\
\left. + \frac{1}{2} \left[ \ln \left( \frac{4(1 - \bar{x}_1)(1 - \bar{x}_2)}{2 - \bar{x}_1 - \bar{x}_2)^2} \right) + \ln \left( -\frac{4(\bar{x}_1 - \bar{x}_2)^2}{(2 - \bar{x}_1 - \bar{x}_2)^2} \right) \right] \right. \\
\left. + \ln \left( \frac{-\bar{x}_2}{\bar{x}_1} \right) \left[ \ln \left( \frac{(\bar{x}_1 + \bar{x}_2)^2}{4} \det (G^{(1)}) + i \lambda \sigma_4 \right) \right. \\
\left. + \frac{1}{2} \left( \ln \left( \frac{4\bar{x}_1 \bar{x}_2}{(\bar{x}_1 + \bar{x}_2)^2} \right) + \ln \left( -\frac{4(\bar{x}_1 - \bar{x}_2)^2}{(\bar{x}_1 + \bar{x}_2)^2} \right) \right) \right] \\
+ \text{Li}_2 \left( \frac{1 - \bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) - \text{Li}_2 \left( \frac{\bar{x}_1 - 1}{\bar{x}_1 - \bar{x}_2} \right) + \text{Li}_2 \left( \frac{\bar{x}_1}{\bar{x}_1 - \bar{x}_2} \right) - \text{Li}_2 \left( -\frac{\bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) \right\} \\
(D.17)
\]

As \(\text{Im}(\bar{x}_1)\) and \(\text{Im}(\bar{x}_2)\) have opposite signs, eq. (D.17) can be rearranged as:

\[
I_3^n = - \frac{2\varepsilon \Gamma(1 + \varepsilon)}{(\bar{x}_1 - \bar{x}_2) \det (G^{(1)})} \times \left\{ -\frac{1}{\varepsilon} \left[ \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right] \\
+ \left[ \ln \left( \det (G^{(1)}) + i \lambda \sigma_4 \right) + \frac{1}{2} \ln \left( -\bar{x}_1 + \bar{x}_2 \right)^2 \right] \left[ \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right] \\
+ \frac{1}{2} \ln \left( \frac{-1 - \bar{x}_1}{1 - \bar{x}_2} \right) \ln \left( (1 - \bar{x}_1)(1 - \bar{x}_2) \right) + \frac{1}{2} \ln \left( \frac{-\bar{x}_2}{\bar{x}_1} \right) \ln(\bar{x}_1 \bar{x}_2) \\
- \frac{1}{2} \ln^2 \left( \frac{\bar{x}_1 - 1}{\bar{x}_1 - \bar{x}_2} \right) + \frac{1}{2} \ln^2 \left( \frac{1 - \bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) - \frac{1}{2} \ln^2 \left( -\frac{-\bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) + \frac{1}{2} \ln^2 \left( \frac{\bar{x}_1}{\bar{x}_1 - \bar{x}_2} \right) \\
+ 2 \text{Li}_2 \left( \frac{\bar{x}_1 - 1}{\bar{x}_2 - 1} \right) + \frac{1}{2} \ln^2 \left( \frac{1 - \bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) - 2 \text{Li}_2 \left( \frac{\bar{x}_1}{\bar{x}_2} \right) - \frac{1}{2} \ln^2 \left( -\frac{\bar{x}_2}{\bar{x}_1} \right) \right\} \\
(D.18)
\]

Using the relation between \(\ln(z)\) and \(\ln(-z)\), after some algebra the quantity

\[
E = \frac{1}{2} \ln \left( \frac{-1 - \bar{x}_1}{1 - \bar{x}_2} \right) \ln \left( (1 - \bar{x}_1)(1 - \bar{x}_2) \right) + \frac{1}{2} \ln \left( \frac{-\bar{x}_2}{\bar{x}_1} \right) \ln(\bar{x}_1 \bar{x}_2) \\
- \frac{1}{2} \ln^2 \left( \frac{\bar{x}_1 - 1}{\bar{x}_1 - \bar{x}_2} \right) + \frac{1}{2} \ln^2 \left( \frac{1 - \bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) - \frac{1}{2} \ln^2 \left( -\frac{-\bar{x}_2}{\bar{x}_1 - \bar{x}_2} \right) + \frac{1}{2} \ln^2 \left( \frac{\bar{x}_1}{\bar{x}_1 - \bar{x}_2} \right)
\]

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can be rewritten:

\[
E = \frac{1}{2} \left[ 2 \ln(\bar{x}_1 - \bar{x}_2) - i \pi S(\bar{x}_1) \right] \left[ \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right]
\]

\[
= \frac{1}{2} \ln \left( - (\bar{x}_1 - \bar{x}_2)^2 \right) \left[ \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right]
\]

with \( S(\bar{x}_1) = \text{sign}(\text{Im}(\bar{x}_1)) \). Substituting eq. (\ref{eq:D.19}) into eq. (\ref{eq:D.18}), we end up with:

\[
I_3^3 = \frac{-2^\varepsilon \Gamma(1 + \varepsilon)}{(\bar{x}_1 - \bar{x}_2) \det (G^{(1)})} \times \left\{ \frac{1}{\varepsilon} \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right\}
\]

\[
+ \left[ \ln \left( \det (G^{(1)}) + i \lambda \sigma_s \right) + \ln \left( - (\bar{x}_1 - \bar{x}_2)^2 \right) \right] \left[ \ln \left( \frac{\bar{x}_1 - 1}{\bar{x}_1} \right) - \ln \left( \frac{\bar{x}_2 - 1}{\bar{x}_2} \right) \right]
\]

\[
+ 2 \text{Li}_2 \left( \frac{\bar{x}_1 - 1}{\bar{x}_2 - 1} \right) + \frac{1}{2} \ln^2 \left( \frac{1 - \bar{x}_2}{\bar{x}_1 - 1} \right) - 2 \text{Li}_2 \left( \frac{\bar{x}_1}{\bar{x}_2} \right) - \frac{1}{2} \ln^2 \left( \frac{\bar{x}_2}{\bar{x}_1} \right)
\]

\[
= \frac{-2^\varepsilon \Gamma(1 + \varepsilon)}{\varepsilon \det (G^{(1)})} \left\{ [1 - \varepsilon \ln (\det (G^{(1)}) + i \lambda \sigma_s)] \det (G^{(1)}) \right\}
\]

where \( K(\bar{x}_1, \bar{x}_2) \) is given by eq. (\ref{eq:2.14}) and \( J(\bar{x}_1, \bar{x}_2) \) by eq. (\ref{eq:B.8}). Last we note that the prescription “+i \lambda \sigma_s” in eq.(\ref{eq:D.20}) can be replaced by “-i \lambda” as it matters only when \( \det (G^{(1)}) < 0 \). Thus eq. (\ref{eq:D.20}) is nothing but eq. (\ref{eq:2.15}): the indirect and direct ways lead to the same result indeed.

2. Occurrence of a collinear divergence

We recap the texture of the \( S \) matrix in this case (cf. eq. (\ref{eq:2.35})):}

\[
S = \begin{pmatrix}
0 & 0 & s_1 - m_3^2 \\
0 & 0 & s_3 - m_3^2 \\
s_1 - m_3^2 & s_3 - m_3^2 & -2m_3^2
\end{pmatrix}
\]

\[
\text{(D.21)}
\]

Obviously, we have \( \det(S) = 0 \). As explained in section 2, (eqs. (\ref{eq:2.36}) and (\ref{eq:2.37})), the coefficient \( \tilde{b}_3 \) vanishes whereas \( \tilde{b}_2 \) and \( \tilde{b}_1 \) are different from zero. So the three-point function is given by:

\[
I_3^n = \frac{\tilde{b}_1}{\det (G)} \left[ \frac{\tilde{b}_2^{(1)}}{\det (G^{(1)})} L_3^n \left( 0, \Delta_1^{(1)}, \bar{D}_{12} \right) + \frac{\tilde{b}_3^{(1)}}{\det (G^{(1)})} L_3^n \left( 0, \Delta_1^{(1)}, \bar{D}_{13} \right) \right]
\]

\[
+ \frac{\tilde{b}_2}{\det (G)} \left[ \frac{\tilde{b}_3^{(2)}}{\det (G^{(2)})} L_3^n \left( 0, \Delta_1^{(2)}, \bar{D}_{21} \right) + \frac{\tilde{b}_3^{(2)}}{\det (G^{(2)})} L_3^n \left( 0, \Delta_1^{(2)}, \bar{D}_{23} \right) \right]
\]

\[
\text{(D.22)}
\]

As \( \bar{D}_{12} = \bar{D}_{21} = 2m_3^2 \) and \( \bar{D}_{13} = \bar{D}_{23} = 0 \), \( L_3^n(0, \Delta_1^{(1)}, \bar{D}_{12}) \) and \( L_3^n(0, \Delta_1^{(2)}, \bar{D}_{12}) \) are given by eq. (\ref{eq:2.49}) whereas \( L_3^n(0, \Delta_1^{(1)}, \bar{D}_{13}) \) and \( L_3^n(0, \Delta_1^{(2)}, \bar{D}_{23}) \) are given by (\ref{eq:2.56}).
Let us first focus on the first term of the r.h.s. of eq. (D.22). The relevant reduced $S$ matrix is:

$$S^{(1)} = \begin{pmatrix} 0 & s_3 - m_3^2 \\ s_3 - m_3^2 & -2m_3^2 \end{pmatrix}$$ (D.23)

whose determinant is: $\det(S^{(1)}) = -(s_3 - m_3^2)^2$. The $1 \times 1$ associated Gram matrix is: $G^{(1)(2)} = (2s_3)$ and the reduced $b$ coefficients are given by:

$$\frac{b_2^{(1)}}{\det(G^{(1)})} = -\frac{s_3 + m_3^2}{2s_3}, \quad \frac{b_3^{(1)}}{\det(G^{(1)})} = -\frac{s_3 - m_3^2}{2s_3}$$ (D.24)

whereas:

$$\Delta_1^{(1)} = \frac{(s_3 - m_3^2)^2}{2s_3}, \quad \tilde{D}_{12} + \Delta_1^{(1)} = \frac{(s_3 + m_3^2)^2}{2s_3}$$ (D.25)

The square of the root of the polynomial $(\tilde{D}_{12} + \Delta_1^{(1)}) z^2 - \Delta_1^{(1)} - i\lambda$ is:

$$\bar{z}^2 = \frac{(s_3 - m_3^2)^2}{(s_3 + m_3^2)^2} + i\lambda \sigma_s$$ (D.26)

with $\sigma_s = \text{sign}(s_3)$. Whereas

$$\sqrt{\bar{z}^2} = \left| \frac{s_3 - m_3^2}{s_3 + m_3^2} \right| + i\lambda \sigma_s$$

a more handy choice for further manipulations is instead:

$$\bar{z} = \frac{s_3 - m_3^2}{s_3 + m_3^2} + i\lambda \sigma_s \sigma_r$$ (D.27)

where $\sigma_r = \text{sign}((s_3 - m_3^2)/(s_3 + m_3^2))$. Let us note:

$$\Sigma^n_3(s_3) = \frac{\tilde{b}_2^{(1)}}{\det(G^{(1)})} L_3^n \left(0, \Delta_1^{(1)}, \tilde{D}_{12}\right) + \frac{\tilde{b}_3^{(1)}}{\det(G^{(1)})} L_3^n \left(0, \Delta_1^{(1)}, \tilde{D}_{13}\right)$$ (D.28)

we get:

$$\Sigma^n_3(s_3) = \frac{2^\varepsilon \Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left[ -\frac{1}{\varepsilon} \ln \left( -\frac{m_3^2}{s_3} + i\lambda \sigma_s \sigma_r \right) + H_{0,1} \left( \tilde{D}_{12} + \Delta_1^{(1)}, -\Delta_1^{(1)} - i\lambda \right) - \frac{1}{\varepsilon^2} \frac{\Gamma(1 - \varepsilon)^2}{\Gamma(1 - 2\varepsilon)} \left( -\frac{2(s_3 - m_3^2)^2}{s_3} - i\lambda \right)^{-\varepsilon} \right]$$ (D.29)
Using the definition of $H_{0,1}(X,Y)$ (cf. eq. (D.16)) and eq. (E.8), we have:

\[
\begin{align*}
\tilde{H}_{0,1}(D_{12} + \Delta^{(1)}_1, -\Delta^{(1)}_1 - i \lambda) \\
&= \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) - \text{Li}_2 \left( - \frac{m_3^2}{s_3 - m_3^2} + i \lambda \sigma_s \sigma_r \right) \\
&\quad + \ln \left( - \frac{m_3^2}{s_3} + i \lambda \sigma_s \sigma_r \right) \left[ \ln \left( \frac{(s_3 + m_3^2)^2}{2 s_3} + i \lambda \sigma_s \right) + \frac{1}{2} \ln \left( \frac{4 s_3 m_3^2}{(s_3 + m_3^2)^2} - i \lambda \sigma_s \right) \\
&\quad + \frac{1}{2} \ln \left( - \frac{4 (s_3 - m_3^2)^2}{(s_3 + m_3^2)^2} - i \lambda \sigma_s \right) \right] \tag{D.30}
\end{align*}
\]

which can be rewritten as:

\[
\begin{align*}
\tilde{H}_{0,1}(D_{12} + \Delta^{(1)}_1, -\Delta^{(1)}_1 - i \lambda) \\
&= \ln \left( - \frac{m_3^2}{s_3} + i \lambda \sigma_s \sigma_r \right) \left[ - \ln (2 s_3 - i \lambda \sigma_s) + \frac{1}{2} \ln \left( 4 s_3 m_3^2 - i \lambda \sigma_s \right) \\
&\quad + \frac{1}{2} \ln \left( - 4 (s_3 - m_3^2)^2 - i \lambda \sigma_s \right) \right] \\
&\quad + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) - \frac{\pi^2}{6} \\
&\quad + \ln \left( - \frac{m_3^2}{s_3 - m_3^2} + i \lambda \sigma_s \sigma_r \right) \ln \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) \tag{D.31}
\end{align*}
\]

Putting eq. (D.31) into eq. (D.29), the ln(2) drop out and we end with:

\[
\begin{align*}
\Sigma^n_3(s_3) &= \frac{\Gamma(1 + \epsilon)}{2 (m_3^2 - s_3)} \times \left\{ - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[ \ln \left( \frac{(s_3 - m_3^2)^2}{s_3} - i \lambda \right) - \ln \left( - \frac{m_3^2}{s_3} + i \lambda \sigma_s \sigma_r \right) \right] \\
&\quad - \frac{1}{2} \ln^2 \left( - \frac{(s_3 - m_3^2)^2}{s_3} - i \lambda \right) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) \\
&\quad + \ln \left( - \frac{m_3^2}{s_3} + i \lambda \sigma_s \sigma_r \right) \left[ - \ln (s_3 - i \lambda \sigma_s) \\
&\quad + \frac{1}{2} \ln \left( s_3 m_3^2 - i \lambda \sigma_s \right) + \frac{1}{2} \ln \left( - (s_3 - m_3^2)^2 - i \lambda \sigma_s \right) \right] \\
&\quad + \ln \left( - \frac{m_3^2}{s_3 - m_3^2} + i \lambda \sigma_s \sigma_r \right) \ln \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) \right\} \tag{D.32}
\end{align*}
\]

$\Sigma^n_3(s_3)$ can be shown to be equal to the following quantity:

\[
\begin{align*}
\Upsilon^n_3(s_3) &= \frac{\Gamma(1 + \epsilon)}{2 (m_3^2 - s_3)} \left\{ - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left[ 2 \ln (- s_3 + m_3^2 - i \lambda) - \ln (m_3^2 - i \lambda) \right] \\
&\quad - \ln^2 (- s_3 + m_3^2 - i \lambda) + \frac{1}{2} \ln^2 (m_3^2 - i \lambda) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i \lambda} \right) \right\} \tag{D.33}
\end{align*}
\]

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To show that, one has to distinguish the following cases: 1) \( s_3 < -m_3^2 \), 2) \(-m_3^2 < s_3 < 0\), 3) \(0 < s_3 < m_3^2\) and 4) \(m_3^2 < s_3\). For each of them, some tedious algebra performed on the r.h.s. of eqs. (D.32) shows that the two results are equal. Let us discuss in detail only the case \( s_3 > m_3^2 \), for which \( \sigma_s = \sigma_r = + \). The argument of the dilogarithm in eq. (D.32) has a real part greater than 1, hence:

\[
\text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \sigma_s \sigma_r \right) = \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} - i \lambda \right) = \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i \lambda} \right)
\]

The logarithms in eq. (D.32) can be modified in such a way that their arguments are ratios of positive quantities, \( \Sigma_3^n(s_3) \) thus reads:

\[
\Sigma_3^n(s_3) = \frac{\Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ \ln \left( \frac{(s_3 - m_3^2)^2}{s_3} \right) - \ln \left( \frac{m_3^2}{s_3} \right) - 2 i \pi \right] 
- \frac{1}{2} \left( \ln \left( \frac{(s_3 - m_3^2)^2}{s_3} \right) - i \pi \right)^2 + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i \lambda} \right) 
+ \left( \ln \left( \frac{m_3^2}{s_3} \right) + i \pi \right) \right. 
\times \left[ -\ln(s_3) + \frac{1}{2} \left( \ln \left( s_3 m_3^2 \right) + \ln \left( (s_3 - m_3^2)^2 \right) - i \pi \right) \right] 
\left. + \left( \ln \left( \frac{m_3^2}{s_3 - m_3^2} \right) + i \pi \right) \ln \left( \frac{s_3}{s_3 - m_3^2} \right) \right\} \tag{D.34}
\]

Splitting logarithms of ratios, we get:

\[
\Sigma_3^n(s_3) = \frac{\Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ 2 \left( \ln \left( s_3 - m_3^2 \right) - i \pi \right) - \ln \left( m_3^2 \right) \right] 
- \left( \ln^2 \left( s_3 - m_3^2 \right) - 2 i \pi \ln \left( s_3 - m_3^2 \right) - \pi^2 \right) 
+ \frac{1}{2} \ln^2 \left( m_3^2 \right) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i \lambda} \right) \right\} \tag{D.35}
\]

In eq. (D.35), for the case at hand, we recognise eq. (D.33). Similar handling can be performed in the other three cases so as to reach the same conclusion. We thus conclude:

\[
\Sigma_3^n(s_3) = \Upsilon_3(s_3) 
= \frac{\Gamma(1 + \varepsilon)}{(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} \left( -s_3 + m_3^2 - i \lambda \right)^{-\varepsilon} + \frac{1}{2 \varepsilon^2} \left( m_3^2 - i \lambda \right)^{-\varepsilon} 
+ \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2 + i \lambda} \right) \right\} \tag{D.36}
\]
So long for the first term of eq. (D.22). The second term can be obtained from the first one by replacing $s_3$ by $s_1$. The coefficients $\bar{b}_1$ and $\bar{b}_2$ as well as $\det(G)$ are easily extracted from the $S$ matrix cf. eq. (D.21):

$$\bar{b}_1 = (s_3 - m_3^2) (s_1 - s_3), \quad \bar{b}_2 = (s_1 - m_3^2) (s_3 - s_1)$$

$$\det(G) = -(s_1 - s_3)^2$$

(D.37)

Thus we finally get:

$$I^n_3 = \frac{\bar{b}_1}{\det(G)} \Sigma^n_3(s_3) + \frac{\bar{b}_2}{\det(G)} \Sigma^n_3(s_1)$$

$$= \frac{\Gamma(1 + \varepsilon)}{(s_1 - s_3)} \left\{ -\frac{1}{\varepsilon^2} \left[ (-s_3 + m_3^2 - i \lambda)^{-\varepsilon} - (-s_1 + m_3^2 - i \lambda)^{-\varepsilon} \right] \right.$$  

$$+ \log \left( \frac{s_3}{s_1} \right) - \log \left( \frac{s_1}{s_3} \right) \right\}$$

which coincides with eq. (2.41): the direct and indirect ways lead to the same result.

3. Concomitant occurrence of a soft and a collinear divergences

Here again, we start to recap the texture of the $S$ matrix:

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_3 \\ 0 & s_3 & 0 \end{pmatrix}$$

(D.38)

Obviously, $\det(S) = 0$. As in example “1. Occurrence of a soft divergence”, the coefficients $\bar{b}_2$ and $\bar{b}_3$ vanish whereas $\bar{b}_1 = \det(G)$, and, as $\bar{D}_{12} = \bar{D}_{13} = 0$, $L(0, \Delta^{(1)}_1, 0)$ is given by eq. (2.56), thus:

$$I^n_3 = \frac{\bar{b}^{(1)}_2}{\det(G^{(1)})} L^n_3 (0, \Delta^{(1)}_1, 0) + \frac{\bar{b}^{(1)}_3}{\det(G^{(1)})} L^n_3 (0, \Delta^{(1)}_1, 0)$$

(D.39)

The relevant reduced $S$ matrix is:

$$S^{(1)} = \begin{pmatrix} 0 & s_3 \\ s_3 & 0 \end{pmatrix}$$

(D.40)

whose determinant is $\det(S^{(1)}) = -s_3^2$. The $1 \times 1$ associated Gram “matrix” is $G^{(1)(2)} = (2 s_3)$ and the reduced $\bar{b}$ coefficients are given by:

$$\frac{\bar{b}^{(1)}_2}{\det(G^{(1)})} = \frac{\bar{b}^{(1)}_3}{\det(G^{(1)})} = -\frac{1}{2}$$

(D.41)

14Contrary to the example 3 in subsec. 2.2, we choose to set $m_3^2 = 0$ because, for the purpose of this appendix, a non vanishing mass does not bring anything new with respect to the previous case.
whereas:

\[ \Delta \! \! \! ^{(1)} = \frac{s_3}{2} \]  \hspace{1cm} (D.42)

We thus obtain:

\[ I_n^3 = -L_n^3 \left(0, \Delta \! \! \! ^{(1)}, 0\right) \]

\[ = -\frac{1}{\varepsilon^2} \Gamma(1 + \varepsilon) \frac{\Gamma^2(1 - \varepsilon)}{\Gamma(1 - 2\varepsilon)} \left(-s_3 - i\lambda\right)^{-1 - \varepsilon} \]

\[ = W \left(\det (G^{(1)}), 0, 0\right) \]

(cf. eqs. (2.5), (2.6) and (2.30)), so the direct and indirect ways lead to the same results.

**Complex mass case**

We now treat the complex mass case. As discussed in section (2), the only relevant case is the collinear case where the non vanishing internal mass, say \(m_R^2\) and \(m_I^2\), is complex\(^{15}\): \(m_R^2 = m_R^2 + i m_I^2\), and \(m_R^2 > 0, m_I^2 < 0\)\(^{16}\). The \(S\) matrix is given by eq. (D.21) and \(I_{3}^{n}\) by eq. (D.22). As in the real mass case, let us focus on the first line of eq. (D.22), the second line can be obtained by changing \(s_3\) in \(s_1\). To compute \(L(0, \Delta \! \! \! ^{(1)}), \bar{D}_{12}\) and \(L(0, \Delta \! \! \! ^{(1)}, \bar{D}_{13})\), we have to determine which formulae to use, depending on the sign of \(\text{Im}(\Delta \! \! \! ^{(1)})\). The quantity \(\Delta \! \! \! ^{(1)}\) is given by eq. (D.25) which reads:

\[ \Delta \! \! \! ^{(1)} = \frac{1}{2s_3} \left((s_3 - m_R^2)^2 - m_I^4 - 2i m_I^2 (s_3 - m_R^2)\right) \]  \hspace{1cm} (D.43)

When \((s_3 - m_R^2)/s_3 > 0\) i.e. either \(s_3 > m_R^2\) or \(s_3 < 0\), \(L(0, \Delta \! \! \! ^{(1)}), \bar{D}_{12}\) is given by eq. (2.49) as in the real mass case, and when \((s_3 - m_R^2)/s_3 < 0\) i.e. \(0 < s_3 < m_R^2\), \(L(0, \Delta \! \! \! ^{(1)}), \bar{D}_{12}\) is given by eq. (2.52). As discussed previously, there is no such dichotomy for \(L(0, \Delta \! \! \! ^{(1)}, \bar{D}_{13})\) which is given by eq. (2.56).

Let us first consider \(s_3 > m_R^2\) or \(s_3 < 0\). We define \(\bar{z} = (s_3 - m_3^2)/(s_3 + m_3^2)\) reminiscent of eq. (D.27) and \(\Sigma_3^3(s_3)\) is given by eq. (D.32), in which the infinitesimal imaginary parts \(\propto \lambda\)

\(^{15}\)We follow the convention of appendix C of ref. [2]

\(^{16}\)One could be tempted, in this subsec. to recover the real mass case results by setting \(m_I^2 = -\lambda\). Doing that could lead to wrong formulæ because, when deriving the complex mass case, we have already assumed that \(|m_I^2| \gg \lambda\) and so dropped some \(i\lambda\) terms.
are dropped out except for arguments of logarithms which do not depend on $m_3^2$, namely:

$$\Sigma^a_3(s_3) = \frac{\Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ \ln \left( -\frac{(s_3 - m_3^2)^2}{s_3} \right) - \ln \left( -\frac{m_3^2}{s_3} \right) \right] - \frac{1}{2} \ln^2 \left( -\frac{(s_3 - m_3^2)^2}{s_3} \right) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} \right) 
+ \ln \left( -\frac{m_3^2}{s_3} \right) 
\times \left[ -\ln (s_3 - i \lambda \sigma_3) + \frac{1}{2} \left[ \ln (s_3 m_3^2) + \ln (- (s_3 - m_3^2)^2) \right] \right] 
+ \ln \left( -\frac{m_3^2}{s_3 - m_3^2} \right) \ln \left( \frac{s_3}{s_3 - m_3^2} \right) \right\} \quad (D.44)$$

Logarithms of products and ratios in eq. (D.44) are further split. For this purpose we use eq. (E.1) as well as:

$$\ln \left( (s_3 - m_3^2)^2 \right) = 2 \ln(s_3 - m_3^2) - 2 i \pi \theta (-s_3 + m_R^2)$$

and, for any real $a$ and complex $b$:

$$\ln \left( \frac{b}{a} \right) = -\ln (a + i \lambda S(b)) + \ln(b)$$
$$\ln(a b) = \ln (a - i \lambda S(b)) + \ln(b)$$

We also use the notation $S(z) = \text{sign} (\text{Im}(z))$. Let us note that $S \left( (s_3 - m_3^2)^2 \right) = \text{sign}(s_3 - m_R^2) \equiv \sigma_p$ and that $\ln(s_3 - i \lambda \sigma_3)$ is equivalent to $\ln(s_3 + i \lambda)$. Then, to compactify eq. (D.44), we take advantage of the following relations:

$$\ln \left( -s_3 \pm i \lambda \right) = \ln \left( |s_3| \right) \pm i \pi \theta(s_3)$$
$$\ln \left( s_3 \pm i \lambda \right) = \ln \left( |s_3| \right) \pm i \pi \theta(-s_3)$$
$$\theta(\pm s_3) = \frac{1 \pm \sigma_3}{2}$$
$$\theta(-s_3 + m_R^2) = \frac{1 - \sigma_p}{2} \quad (D.45)$$
Eq. (D.44) can be rewritten:

$$\Sigma^n_3(s_3) = \frac{\Gamma(1+\varepsilon)}{2(m^2_3-s_3)} \left\{ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ 2 \left( \ln \left( s_3 - m^2_3 \right) - i \pi \right) - \ln \left( m^2_3 \right) \right] \right. $$

$$- \frac{1}{2} \left[ 2 \ln \left( s_3 - m^2_3 \right) - i \pi \left( 1 - \frac{\sigma_p}{2} + \frac{\sigma_p \sigma_s}{2} \right) \right]^2 $$

$$+ 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m^2_3} \right) + \left( \ln (m^2_3) + i \pi \frac{1 + \sigma_s}{2} \right) $$

$$\times \left[ \frac{1}{2} \ln \left( m^2_3 \right) + \ln \left( s_3 - m^2_3 \right) - i \pi (3 - \sigma_s) \right] $$

$$+ \left( \ln (m^2_3) - \ln \left( s_3 - m^2_3 \right) + i \pi \right) \left( i \pi \frac{1 - \sigma_s}{2} - \ln \left( s_3 - m^2_3 \right) \right) $$

$$+ \frac{i}{2} \ln \left( s_3 \right) \left( 1 - \sigma_s \right) \left( 1 + \sigma_p \right) \right\} \tag{D.46}$$

Let us treat the case where $s_3 > m^2_R$. We have $\sigma_p = +$ and $\sigma_s = +$. By expanding eq. (D.46), we get:

$$\Sigma^n_3(s_3) = \frac{\Gamma(1+\varepsilon)}{2(m^2_3-s_3)} \left\{ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ 2 \left( \ln \left( s_3 - m^2_3 \right) - i \pi \right) - \ln \left( m^2_3 \right) \right] \right. $$

$$- \left( \ln^2 \left( s_3 - m^2_3 \right) - 2 i \pi \ln \left( s_3 - m^2 \right) - \pi^2 \right) $$

$$+ \frac{1}{2} \ln \left( m^2_3 \right) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m^2_3} \right) \right\} \tag{D.47}$$

As $\ln(s_3 - m^2_3) - i \pi = \ln(-s_3 + m^2_3)$, we recover $\Upsilon^n_3(s_3)$ given by eq. (D.33). The same exercise can be easily done for the case $s_3 < 0$ leading to the same conclusion.

In the case where $0 < s_3 < m^2_R$, $L(0, \Delta^{(1)}_1, \widetilde{D}_{12})$ is given by eq. (2.52) i.e.:

$$L(0, \Delta^{(1)}_1, \widetilde{D}_{12}) = \frac{2^\varepsilon \Gamma(1+\varepsilon)}{2(\widetilde{D}_{12} + \Delta^{(1)}_1)} \left[ \frac{1}{\varepsilon} \left( i \pi S(i \tilde{z}) + \ln \left( \frac{1 + \tilde{z}}{1 - \tilde{z}} \right) \right) \right] $$

$$- \tilde{H}_{0,\infty}(-\widetilde{D}_{12} - \Delta^{(1)}_1, -\Delta^{(1)}_1) $$

$$- \tilde{H}_{1,\infty}(\widetilde{D}_{12} + \Delta^{(1)}_1, -\Delta^{(1)}_1) \tag{D.48}$$

where $\tilde{z} = \sqrt{\Delta^{(1)}_1/(\widetilde{D}_{12} + \Delta^{(1)}_1)}$. We have chosen for the root of the equation $(\widetilde{D}_{12} + \Delta^{(1)}_1) \tilde{z}^2 + \Delta^{(1)}_1 = 0$, $\tilde{z} = i \tilde{z}$. The functions $\tilde{H}_{1,\infty}(x, y)$ is given by the expression in curly brackets of eq. (E.15) and $\tilde{H}_{0,\infty}(x, y)$ by the expression in square brackets in eq. (E.19) of appendix E.
Putting all things together, i.e.: 
\[ H_{0,\infty}(-\tilde{D}_{12} - \Delta_1^{(1)}, -\Delta_1^{(1)}) = i \pi S(i \tilde{z}) \left[ 2 \ln (2 i \tilde{z}) + \ln \left( -\tilde{D}_{12} - \Delta_1^{(1)} \right) \right. \]
\[ + \eta \left( -\tilde{D}_{12} - \Delta_1^{(1)}, \tilde{z}^2 \right) \]  
\[ + \pi^2 \]  
\[ \left. \right] \quad (D.49) \]

\[ H_{1,\infty}(-\tilde{D}_{12} + \Delta_1^{(1)}, -\Delta_1^{(1)}) = \ln \left( \frac{1 + \tilde{z}}{1 - \tilde{z}} \right) \left[ \ln \left( -\tilde{D}_{12} + \Delta_1^{(1)} \right) + \frac{1}{2} \ln (1 - \tilde{z}^2) + \ln (2 \tilde{z}) \right. \]
\[ + \eta \left( -\tilde{D}_{12} + \Delta_1^{(1)}, \frac{-\tilde{D}_{12}}{\tilde{D}_{12} + \Delta_1^{(1)}} \right) \]  
\[ + \frac{\pi^2}{2} \] 
\[ - i \pi S(\tilde{z}) \ln \left( \frac{\tilde{z} + 1}{2 \tilde{z}} \right) - \text{Li}_2 \left( \frac{\tilde{z} + 1}{2 \tilde{z}} \right) + \text{Li}_2 \left( \frac{\tilde{z} - 1}{2 \tilde{z}} \right) \]  
\[ \quad (D.50) \]

The quantities \( \tilde{D}_{12} + \Delta_1^{(1)} \) and \( \Delta_1^{(1)} \) can be expressed in terms of \( s_3 \) and \( m_R^2 \) with the group of eqs. (D.25). The expression obtained contains the ratio \( (s_3 - m_R^2)/(s_3 + m_R^2) \) given by:
\[ \frac{s_3 - m_R^2}{s_3 + m_R^2} = \frac{s_3 - m_R^4 - m_R^4}{(s_3 + m_R^2)^2 + m_R^4} \]  
\[ \quad (D.51) \]

which has a negative real part and a positive imaginary part for \( 0 < s_3 < m_R^2 \). This implies that:
\[ S(i \tilde{z}) = -1, \quad S(\tilde{z}) = +1 \]  
\[ \quad (D.52) \]

In addition, since \( \text{Im}((s_3 - m_R^2)^2) = -2 m_R^4 (s_3 - m_R^2) < 0 \) and \( \text{Im}((s_3 + m_R^2)^2) = 2 m_R^2 (s_3 + m_R^2) < 0 \), one can show that:
\[ \eta \left( \frac{(s_3 + m_R^2)^2}{2 s_3}, \frac{4 m_R^2 s_3}{(s_3 + m_R^2)^2} \right) = \eta \left( \frac{(s_3 + m_R^2)^2}{2 s_3}, \frac{(s_3 - m_R^2)^2}{(s_3 + m_R^2)^2} \right) = 0 \]

Putting all things together, \( L(0, \Delta_1^{(1)}, \tilde{D}_{12}) \) becomes:
\[ L(0, \Delta_1^{(1)}, \tilde{D}_{12}) = \frac{2^\varepsilon \Gamma(1 + \varepsilon) s_3}{(s_3 + m_R^2)(s_3 - m_R^2)} \left\{ \frac{1}{\varepsilon} \right\} \ln \left( \frac{s_3}{m_R^2} \right) - i \pi \ln \left( \frac{s_3}{s_3 - m_R^2} \right) \]
\[ + i \pi \left[ 2 \ln \left( \frac{2 i (s_3 - m_R^2)}{s_3 + m_R^2} \right) + \ln \left( \frac{(s_2 - m_R^2)^2}{2 s_3} \right) \right] \]
\[ + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_R^2} \right) + \ln \left( \frac{m_R^2}{s_3 - m_R^2} \right) \ln \left( \frac{s_3}{s_3 - m_R^2} \right) \]
\[ - \ln \left( \frac{s_3}{m_R^2} \right) \left[ \ln \left( \frac{(s_3 + m_R^2)^2}{2 s_3} \right) + \frac{1}{2} \ln \left( \frac{4 m_R^2 s_3}{(s_3 + m_R^2)^2} \right) \right. \]
\[ \left. + \ln \left( \frac{2 s_3 - m_R^2}{s_3 + m_R^2} \right) \right\} \]  
\[ \quad (D.53) \]
Substituting eq. (D.53) into eq. (D.28) with the explicit values for $\vec{b}_2^{(1)}$, $\vec{b}_3^{(1)}$ and det $(G^{(1)})$ and using eq. (2.56) for $L(0, \Delta_1^{(1)}, \vec{D}_{13})$, we get:

$$\Sigma_3^n(s_3) = \frac{\Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ \ln \left( \frac{s_3}{m_3^2} \right) + \ln \left( -\frac{(s_3 - m_3^2)^2}{s_3} \right) - i\pi \right] 
- \frac{1}{2} \ln^2 \left( -\frac{(s_3 - m_3^2)^2}{s_3} \right) - \frac{3\pi^2}{2} + i\pi \ln \left( \frac{s_3}{s_3 - m_3^2} \right) 
+ i\pi \left[ 2 \ln \left( \frac{s_3 - m_3^2}{s_3 + m_3^2} \right) + \ln \left( \frac{(s_3 - m_3^2)^2}{s_3} \right) - \ln \left( \frac{s_3}{m_3^2} \right) \right] 
\times \left[ \ln \left( \frac{(s_3 + m_3^2)^2}{s_3} \right) + \frac{1}{2} \ln \left( \frac{m_3^2}{s_3 + m_3^2} \right) + \ln \left( \frac{s_3 - m_3^2}{s_3 + m_3^2} \right) \right) 
+ 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} \right) + \ln \left( -\frac{m_3^2}{s_3 - m_3^2} \right) \ln \left( \frac{s_3}{m_3^2} \right) \right\}$$

(D.54)

Keeping in mind that $0 < s_3 < m_R^2$, we split the logarithms and expand the terms to end with:

$$\Sigma_3^n(s_3) = \frac{\Gamma(1 + \varepsilon)}{2(m_3^2 - s_3)} \left\{ -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ 2 \ln (s_3 - m_3^2) - 2i\pi - \ln (m_3^2) \right] 
- \ln^2 (s_3 - m_3^2) - \pi^2 - 2i\pi \ln (s_3 - m_3^2) \right\} + \frac{1}{2} \ln^2 (m_3^2) + 2 \text{Li}_2 \left( \frac{s_3}{s_3 - m_3^2} \right)$$

(D.55)

and using $\ln(s_3 - m_3^2) = \ln(-s_3 + m_3^2) + i\pi$, we again recover eq. (D.33). We note that the same formula holds both for $\text{Im}(\Delta_1^{(1)}) > 0$ i.e. either $s_3 < 0$ or $s_3 > m_R^2$, and for $\text{Im}(\Delta_1^{(1)}) < 0$ i.e. $0 < s_3 < m_R^2$. This is because in the last case, the integration contour $\int_0^{+\infty} + \int_{+\infty}^{1}$ can actually be deformed into $\int_0^{1}$; i.e. eq. (2.52) can be deformed into eq. (2.49) by means of the Cauchy theorem. Indeed, when $0 < s_3 < m_R^2$, $\text{Im}(z^2 (\vec{D}_{12} + \Delta_1^{(1)}) - \Delta_1^{(1)})$ never vanishes as $z$ spans the real interval $[0, 1]$ hence the cut of $\ln(z^2 (\vec{D}_{12} + \Delta_1^{(1)}) - \Delta_1^{(1)})$ in the half plane $\{ \text{Re}(z) > 0 \}$ entirely lies inside the “south-east” quadrant $\{ \text{Re}(z) > 0, \text{Im}(z) < 0 \}$.

$\Sigma_3^n(s_1)$ is read from eq. (D.55) by replacing $s_3$ by $s_1$ and the coefficients $\vec{b}_1$ and $\vec{b}_2$ as well as det $(G)$ are still given by eq. (D.37). So, in the complex mass case the same result is obtained as in the real mass case for $I_3^n$ and this leads to the conclusion that for the case of complex masses, the “direct way” and the “indirect way” also coincide.

### E Basic integrals in terms of dilogarithms and logarithms

In presence of vanishing internal masses, specific integrals of the following type

$$H = \int_b^a du \ln(Au^2 + B)$$

$$A u^2 + B$$
for the contour $[0, 1]$ as well as the two other contours $[0, +\infty]$ and $[1, +\infty]$ are involved.

We hereby compute all the above types of integrals successively. The presentation is ordered according to the integration contours $(a, b)$ considered. We last provide an extra load of back-up integrals. This appendix often makes use of the identity

$$\ln(z) = \ln(-z) + i \pi S(z) \quad (E.1)$$

where $S(z)$ is given by eq. (2.25).

### E.1 H-type integrals for the IR case

#### E.1.1 First kind

$$H_{0,1}(A, B) = \int_0^1 du \frac{\ln(A u^2 + B)}{A u^2 + B}$$

The cases of real masses ($A$ real) and of complex masses ($\text{Im}(A) \neq 0$) are treated all at once considering $A$ and $B$ both complex yet such that $\text{sign}(\text{Im}(A u^2 + B))$ is kept constant when $u$ spans the range $[0, 1]$, as is always the case for all our needs (cf. sections 2 and 3). We write:

$$H_{0,1}(A, B) = \frac{1}{A} \int_0^1 du \frac{C_A + \ln(u^2 - \bar{u}^2)}{u^2 - \bar{u}^2} \quad (E.2)$$

where

$$C_A = \ln(A - i \lambda S(-\bar{u}^2)) \quad \text{if} \ \text{Im}(A) = 0$$

$$C_A = \ln(A) + \eta(A, -\bar{u}^2) \quad \text{otherwise} \quad (E.3)$$

and $\bar{u}^2 = -B/A$. The $\eta$ function is given by eq. (E.6) of ref. [1]. The term $\ln(u^2 - \bar{u}^2)$ can be split without $\eta$ function since $\bar{u}$ and $-\bar{u}$ have imaginary parts of opposite signs. Performing a partial fraction decomposition, we get:

$$H_{0,1}(A, B) = \frac{1}{2 A \bar{u}} \left[ C_A \int_0^1 du \left( \frac{1}{u - \bar{u}} - \frac{1}{u + \bar{u}} \right) 
+ \int_0^1 du \frac{\ln(u - \bar{u})}{u - \bar{u}} - \int_0^1 du \frac{\ln(u - \bar{u})}{u + \bar{u}} 
+ \int_0^1 du \frac{\ln(u + \bar{u})}{u - \bar{u}} - \int_0^1 du \frac{\ln(u + \bar{u})}{u + \bar{u}} \right] \quad (E.5)$$
We can rearrange the terms of the r.h.s. of eq. (E.5) in the following way:

$$H_{0,1}(A, B) = \frac{1}{2A\bar{u}} \left\{ C_A \left[ \ln \left( \frac{\bar{u} - 1}{\bar{u}} \right) - \ln \left( \frac{\bar{u} + 1}{\bar{u}} \right) \right] + \frac{1}{2} \left[ \ln^2(1 - \bar{u}) - \ln^2(-\bar{u}) - \ln^2(1 + \bar{u}) + \ln^2(\bar{u}) \right] + \int_0^1 du \ln \frac{u + \bar{u}}{u - \bar{u}} - \int_0^1 du \ln \frac{u - \bar{u}}{u + \bar{u}} \right\}$$

(E.6)

Using eq. (E.1) we can write eq. (E.6) in the following way:

$$H_{0,1}(A, B) = \frac{1}{2A\bar{u}} \left\{ \ln \left( \frac{\bar{u} - 1}{\bar{u} + 1} \right) \left[ C_A + \frac{1}{2} [\ln(\bar{u} - 1) + \ln(\bar{u} + 1)] + i\pi S(-\bar{u}) + \ln(2\bar{u}) \right] + R'(-\bar{u}, \bar{u}) \right\}$$

(E.7)

where the function $R'$ has been defined in eq. (E.11) of [1]. Using eq. (E.15) of [1] with $y = -\bar{u}$ and $z = \bar{u}$ and rearranging the term in square brackets, we get:

$$H_{0,1}(A, B) = \frac{1}{2A\bar{u}} \left\{ \ln \left( \frac{\bar{u} - 1}{\bar{u} + 1} \right) \left[ C_A + \frac{1}{2} \left[ \ln \left( 1 - \bar{u}^2 \right) + \ln \left( -4\bar{u}^2 \right) \right] \right] + \text{Li}_2 \left( \frac{\bar{u} + 1}{2\bar{u}} \right) - \text{Li}_2 \left( \frac{\bar{u} - 1}{2\bar{u}} \right) \right\}$$

(E.8)

E.1.2 Second kind

With complex masses we need also to compute:

$$H_{1,\infty}(A, B) = \int_1^\infty du \frac{\ln(Au^2 + B)}{Au^2 + B}$$

(E.9)

where $A$ and $B$ are complex yet such that sign(Im($Au^2 + B$)) is kept constant while $u$ spans $[1, +\infty]$. The quantity $H_{1,\infty}(A, B)$ can be written as:

$$H_{1,\infty}(A, B) = \frac{1}{A} \int_1^\infty du \frac{C'_A + \ln(u^2 - \bar{u}^2)}{u^2 - \bar{u}^2}$$

(E.10)

where $\bar{u}^2 = -B/A$ and $C'_A$ is given by:

$$C'_A = \ln(A) + \eta(A, 1 - \bar{u}^2)$$

(E.11)

\[\text{The subtlety discussed in [1] does not appear in this case.}\]
We perform a partial fraction decomposition and, writing $H_{1,\infty}(A, B)$ as a sum of terms which are individually divergent when $u \to \infty$, we face a situation similar to the one met in subsec. B.2 of [2]. We proceed likewise, introducing a large $u$-cut-off $\Lambda$ and write $H_{1,\infty}(A, B)$ as:

$$H_{1,\infty}(A, B) = \frac{1}{2A\bar{u}} \lim_{\Lambda \to +\infty} \mathcal{H}_{1,\infty}^\Lambda(A, B)$$

where

$$\mathcal{H}_{1,\infty}^\Lambda(A, B) = \left\{ C'_A \int_1^\Lambda du \left[ \frac{1}{u - \bar{u}} - \frac{1}{u + \bar{u}} \right] 
+ \int_1^\Lambda du \frac{\ln(u - \bar{u})}{u - \bar{u}} - \int_1^\Lambda du \frac{\ln(u + \bar{u})}{u + \bar{u}} 
+ \int_1^\Lambda du \frac{\ln(u + \bar{u}) - \ln(2\bar{u})}{u - \bar{u}} + \ln(2\bar{u}) \int_1^\Lambda \frac{du}{u - \bar{u}} 
- \int_1^\Lambda du \frac{\ln(u - \bar{u}) - \ln(-2\bar{u})}{u + \bar{u}} - \ln(-2\bar{u}) \int_1^\Lambda \frac{du}{u + \bar{u}} \right\}$$

(E.12)

We express $\mathcal{H}_{1,\infty}^\Lambda(A, B)$ in term of the function $R^\Lambda(y, z)$ defined by eq. (B.9) of [2]:

$$\mathcal{H}_{1,\infty}^\Lambda(A, B) = \left\{ [C'_A + \ln(2\bar{u})] \ln \left( \frac{\Lambda - \bar{u}}{1 - \bar{u}} \right) - [C'_A + \ln(-2\bar{u})] \ln \left( \frac{\Lambda + \bar{u}}{1 + \bar{u}} \right) 
+ \frac{1}{2} \left[ \ln^2(\Lambda - \bar{u}) - \ln^2(1 - \bar{u}) \right] - \frac{1}{2} \left[ \ln^2(\Lambda + \bar{u}) - \ln^2(1 + \bar{u}) \right] 
+ R^\Lambda(-\bar{u}, \bar{u}) - R^\Lambda(\bar{u}, -\bar{u}) \right\}$$

(E.13)

Using eq. (B.14) of [2] for the $R^\Lambda$ terms we take the limit $\Lambda \to \infty$. The terms proportional to $\ln^2(\Lambda)$ and those proportional to $\ln(\Lambda)$ drop out and we get:

$$H_{1,\infty}(A, B) = \frac{1}{2A\bar{u}} \left\{ [C'_A + \ln(-2\bar{u})] \ln \left( \frac{1 + \bar{u}}{1 - \bar{u}} \right) - [C'_A + \ln(2\bar{u})] \ln \left( 1 - \bar{u} \right) 
+ \frac{1}{2} \left[ \ln^2(1 + \bar{u}) - \ln^2(1 - \bar{u}) + \ln^2(2\bar{u}) - \ln^2(-2\bar{u}) \right] 
- \text{Li}_2 \left( \frac{\bar{u} + 1}{2\bar{u}} \right) + \text{Li}_2 \left( \frac{\bar{u} - 1}{2\bar{u}} \right) \right\}$$

(E.14)

Noting that:

$$\ln \left( \frac{1 + \bar{u}}{1 - \bar{u}} \right) = \ln(1 + \bar{u}) - \ln(1 - \bar{u})$$

eq. (E.14) becomes after some algebra:

$$H_{1,\infty}(A, B) = \frac{1}{2A\bar{u}} \left\{ \ln \left( \frac{1 + \bar{u}}{1 - \bar{u}} \right) \left[ C'_A + \frac{1}{2} \ln(1 - \bar{u}^2) + \ln(2\bar{u}) \right] + \frac{\pi^2}{2} 
- i\pi \text{S}(\bar{u}) \ln \left( \frac{\bar{u} + 1}{2\bar{u}} \right) - \text{Li}_2 \left( \frac{\bar{u} + 1}{2\bar{u}} \right) + \text{Li}_2 \left( \frac{\bar{u} - 1}{2\bar{u}} \right) \right\}$$

(E.15)
We remark that the same combination of dilogarithms - up to a sign - appears in eq. (E.15) and in $H_{0,1}(A, B)$ so that we can rewrite $H_{1,\infty}$ as:

$$
H_{1,\infty}(A, B) = -H_{0,1}(A, B) + \frac{1}{2A\bar{u}} \left\{ i \pi S(\bar{u}) \left[ 2 \ln(2\bar{u}) + C_A \right] + \pi^2 
+ \left[ \eta(A, -\bar{u}^2) - \eta(A, 1 - \bar{u}^2) \right] \ln \left( \frac{1 + \bar{u}}{1 - \bar{u}} \right) \right\}
$$

(E.16)

E.1.3 Third kind

With complex masses a third kind of integrals has also to be considered:

$$
H_{0,\infty}(A, B) = \int_{0}^{\infty} du \frac{\ln(Au^2 + B)}{Au^2 + B} \tag{E.17}
$$

where $A$ and $B$ are complex yet such that sign($\text{Im}(Au^2 + B)$) is kept constant while $u$ spans $[0, +\infty]$. The quantity $H_{0,\infty}(A, B)$ can be split as:

$$
H_{0,\infty}(A, B) = H_{0,1}(A, B) + H_{1,\infty}(A, B) \tag{E.18}
$$

From eq. (E.16) and reminding that the assumption on the sign of $\text{Im}(Au^2 + B)$ implies that $\eta(A, -\bar{u}^2) = \eta(A, 1 - \bar{u}^2)$, we immediately get:

$$
H_{0,\infty}(A, B) = \frac{1}{2A\bar{u}} \left[ i \pi S(\bar{u}) \left[ 2 \ln(2\bar{u}) + C_A \right] + \pi^2 \right] \tag{E.19}
$$

As happened for $K_{0,\infty}^C(A, B)$ (cf. appendix B of [2]), $H_{0,\infty}(A, B)$ contains only logarithmic terms.

E.2 An extra load of back-up integrals

We also need the following load of simpler integrals:

$$
W_1(u_0^2) = \int_{0}^{1} du \frac{\ln(1 - u^2)}{u^2 - u_0^2} \tag{E.20}
$$

$$
W_2(u_0^2) = \int_{0}^{1} du \frac{\ln(u)}{u^2 - u_0^2} \tag{E.21}
$$

$$
W_3(u_0^2) = \int_{1}^{\infty} du \frac{\ln(u^2 - 1)}{u^2 - u_0^2} \tag{E.22}
$$

$$
W_4(u_0^2) = \int_{0}^{\infty} du \frac{\ln(u)}{u^2 - u_0^2} \tag{E.23}
$$

$$
W_5(u_0^2) = \int_{0}^{\infty} du \frac{\ln(u^2 + 1)}{u^2 + u_0^2} \tag{E.24}
$$
For all these integrals, $u_0^2$ is assumed to be a complex number, this is indeed the case because these integrals appear in the computation of the four-point function in the IR case where $u_0^2$ is either a complex number with an imaginary part $\propto \lambda$ or a genuine complex number.

One might compute these integrals using specific values for $A$ and $B$ in $K^{BC}(A, B)$, $K^{CCR}(A, B)$ and $K^{C)(A, B)}$ given in appendices E of ref. [1] and B of ref. [2], however these integrals are simple enough to be computed directly (we verified that the results can be retrieved using the K-type integrals after some transformation on the dilogarithms). We give here the result of these integrals without any details:

\begin{align}
W_1(u_0^2) &= \frac{1}{2u_0} \left[ \text{Li}_2 \left( \frac{2}{1+u_0} \right) - \text{Li}_2 \left( \frac{2}{1-u_0} \right) - 2 \ln(2) \ln \left( \frac{u_0 + 1}{u_0 - 1} \right) \right] \\
W_2(u_0^2) &= \frac{1}{2u_0} \left[ \text{Li}_2 \left( \frac{1}{u_0} \right) - \text{Li}_2 \left( \frac{-1}{u_0} \right) \right] \\
W_3(u_0^2) &= -W_1(u_0^2) + \frac{1}{2u_0} i S(u_0) \pi \ln \left( 1 - u_0^2 \right) \\
W_4(u_0^2) &= \frac{1}{4u_0} i S(u_0) \pi \ln \left( -u_0^2 \right) \\
W_5(u_0^2) &= \frac{\pi}{u_0} \ln(1 + u_0)
\end{align}

with $u_0 = \sqrt{u_0^2}$.

**F Change of contour prescription for the pole in the IR four-point integral**

The appendix legitimates the replacement

\begin{align}
\int_0^1 du \frac{u^2 P_{ijk} + R_{ij} - i \lambda} {u^2 P_{ijk} + R_{ij} + i \lambda} \to \int_0^1 du \frac{u^2 P_{ijk} + R_{ij} - i \lambda} {u^2 P_{ijk} + R_{ij} + i \lambda} 
\end{align}

when $\lambda \to 0^+$ for fixed $0 < -\varepsilon \ll 1$, whenever $P_{ijk}$ and $R_{ij}$ are both real with $0 < -R_{ij}/P_{ijk} < 1$. Intuitively this replacement is based on the fact that for any function $f(u)$ analytic along $[0, 1]$ and $0 < u_0 < 1$,

\begin{align}
\int_0^1 du \frac{f(u)} {u - u_0 - i \lambda} - \int_0^1 du \frac{f(u)} {u - u_0 + i \lambda} \to 2i \pi f(u_0) \text{ when } \lambda \to 0^+
\end{align}

which vanishes if $u_0$ is a zero of $f(u)$. However the situation is made trickier when $f(u) = (u - u_0)^{-\varepsilon}$ thus has a branch point at $u = u_0$ and a cut running along part of the interval of integration. To make the above argument apply one could think of shifting the branch point and cut by a contour prescription $-ia\lambda$ with $a > 1$ so as to pass whether above or below the pole while remaining on the same side of the cut. We would then get a residue value
\( \propto \lambda^{-\varepsilon} \) vanishing in the limit \( \lambda \to 0^+ \) keeping \( \varepsilon < 0 \) fixed. The “hierarchised lambdalogy” underpinning this disentanglement of pole from branch point may look awkward to the rigorous reader, let us thus back up this hand waving argument more rigorously as follows.

We first perform a partial fraction decomposition of the pole term:

\[
\frac{1}{u^2 P_{ijk} + R_{ij} + is\lambda} = \frac{1}{P_{ijk} 2 \sqrt{-R_{ij}/P_{ijk}}} \left[ \frac{1}{u - (u_0 - i s\lambda')} - \frac{1}{u + (u_0 - i s\lambda')} \right]
\]

where \( s = \pm \), \( u_0 = \sqrt{-R_{ij}/P_{ijk}} \) is assumed in \( ]0,1[ \) and \( \lambda' = \lambda/(2u_0 P_{ijk}) \). We focus on the pole at \( +u_0 \) in decomposition (F.2): since the pole at \( -u_0 \) involved in the second term lies outside the integration region the contour prescription for this pole is irrelevant and so is the corresponding pole term in the discussion. We then study the legitimacy of the replacement

\[
\int_{-\infty}^{+\infty} dv \left[ v^4 + (2u_0)^2v^2 + (2u_0a)^2 \right]^{-\varepsilon/2}
\]

is convergent when \( -\varepsilon > 0 \) is small enough. It provides an “integrable hat” for the application of Lebesgue’s theorem of dominated convergence. When \( \lambda \to 0^+ \) keeping \( -\varepsilon > 0 \) fixed and small enough, the integral in eq. (F.4) has the limit

\[
\mathcal{L} = (2u_0)^{-\varepsilon} \int_{-\infty}^{+\infty} \frac{dv}{v^2 + 1} (\sigma v - ia)^{-\varepsilon}
\]

which is finite regardless of \( a > 0 \) and \( \varepsilon \) small enough (its actual value, readily computable using the residue theorem, is irrelevant for the conclusion). We thus see that \( \delta \sim \mathcal{O}(\lambda^{-\varepsilon}) \) as anticipated. q.e.d.

\(^{18}\) We keep track of this multiplicative change to control various normalizations in the reasoning so as to check the independence of the conclusion w.r.t. any assumption of “hierarchised lambdalogy”.

\[
\frac{1}{u^2 P_{ijk} + R_{ij} + is\lambda} \rightarrow \frac{1}{P_{ijk} 2 \sqrt{-R_{ij}/P_{ijk}}} \left[ \frac{1}{u - (u_0 - i s\lambda')} - \frac{1}{u + (u_0 - i s\lambda')} \right]
\]
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