Analyticity and propagation of plurisubharmonic singularities

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Abstract

A variant of Siu’s analyticity theorem is proved for relative types of plurisubharmonic functions. Some results on propagation of plurisubharmonic singularities and maximality of pluricomplex Green functions with analytic singularities are derived.

1 Introduction

Given a complex manifold $X$, let $\text{PSH}(X)$ denote the class of all plurisubharmonic functions on $X$ and $\text{PSH}^-(X)$ its subclass of all non-positive functions.

We will say that $u \in \text{PSH}(X)$ has singularity at a point $\zeta \in X$ if $u(\zeta) = -\infty$. A basic characteristic of the singularity is its Lelong number

$$
\nu(u, \zeta) = \liminf_{x \to \zeta} \frac{u(x)}{\log |\varsigma(x)|} = dd^c u \wedge (dd^c \log |\varsigma(x)|)^{n-1}(\{\zeta\});
$$

here $d = \partial + \bar{\partial}$, $dd^c = (\partial - \bar{\partial})/2\pi i$, $n = \dim X$, and $\varsigma$ are local coordinates on a neighbourhood of $\zeta$ with $\varsigma(\zeta) = 0$.

A classical result due to Siu states that the function $x \mapsto \nu(u, x)$ is upper semicontinuous in the analytic Zariski topology; this means that the set

$$
S_\zeta(u, X) = \{\zeta \in X : \nu(u, \zeta) \geq c\}
$$

is an analytic variety of $X$ for any $u \in \text{PSH}(X)$ and $c > 0$. As a consequence, for an arbitrary analytic variety $Z$, the value $\nu(u, \zeta)$ is generically constant on $Z$, equal to $\inf\{\nu(u, \zeta) : \zeta \in Z\}$; it can be greater only on a proper analytic subset of $Z$.

Siu’s theorem was extended to directional Lelong numbers $\nu(u, \zeta, a)$, $a \in \mathbb{R}^n_+$, by Kiselman [K87], and to generalized (weighted) Lelong numbers $\nu(u, \varphi)$ with respect to exponentially H"older continuous plurisubharmonic weights $\varphi$ by Demailly [D87]. The analyticity theorems with respect to the standard and directional Lelong numbers give important information on asymptotic behaviour of plurisubharmonic functions near the singularity points: for example, $u(x) \leq c \log |\varsigma(x)| + O(1)$ as $x \to \zeta \in S_\zeta(u, X)$. Relations between the weighted Lelong numbers $\nu(u, \varphi)$ and the asymptotic behaviour of $u$ are not that direct.

In [R06], a notion of relative type $\sigma(u, \varphi)$ of $u$ with respect to a maximal plurisubharmonic weight $\varphi$ was introduced (see Section 2) and an analyticity theorem for the sets $\{\zeta : \sigma(u, \varphi) \geq c\} = \{\zeta : u(x) \leq c\varphi(x, \zeta) + O(1), \ x \to \zeta\}$ was proved, where $\varphi(x, \zeta) = \varphi(x, \zeta) \in \text{PSH}(X \times X)$ is such that $\varphi^{-1}_{\zeta}(-\infty) = \zeta$, $(dd^c\varphi)^n = 0$ on $\{x \neq \zeta\}$, and $e^\varphi$ is H"older continuous with respect to $\zeta$. The extra condition (comparing to Demailly’s result) on $(dd^c\varphi)^n$ is quite essential. Take, for example, the function $\varphi(x, \zeta) = \max\{|x_1 - \zeta_1| + \log(|x_1 - \zeta_1| + |x_2|), \log|x_2 - \zeta_2|\}$ in $\mathbb{C}^2 \times \mathbb{C}^2$; one has $\log|x_1| \leq \varphi(x, \zeta) + O(1)$ precisely when $\zeta \in \{(0, \zeta_2) : \zeta_2 \neq 0\}$ that is not an analytic variety. The reason here is that the values of the weighted Lelong numbers $\nu(u, \varphi_\zeta)$ and relative types $\sigma(u, \varphi_\zeta)$ depend on the singularity of $\varphi$ in opposite ways: while any jump of the singularity of $\varphi$ at a particular point $\zeta$ just increases the value of $\nu(u, \varphi_\zeta)$, it diminishes the type $\sigma(u, \varphi_\zeta)$. 


Here we present a more general analyticity result (Theorem 1) for the relative types. Its main feature is that we allow the singularity sets $\varphi^{-1}(\infty)$ consisting of several points, which makes it possible to apply the result to weights generated by finite holomorphic mappings. Another benefit is that the analyticity concerns a parameter space (as in [D87 Théorème 4.14]), which can thus give additional information on the asymptotic behaviour even at a fixed point (see, for example, Corollary 1). We derive some results on propagation of plurisubharmonic singularities (Corollary 2 and Theorem 2), which in turn imply certain global maximality properties of pluricomplex Green functions with non-isolated analytic singularities (Corollary 3).

2 Preliminaries

Throughout the note, the following notions will be used.

A function $u \in \text{PSH}(X)$ is said to be maximal on an open set $U \subseteq X$ if for any $v \in \text{PSH}(X)$ the condition $v \leq u$ on $X \setminus U$ implies $v \leq u$ on the whole $X$. A locally bounded $u$ is maximal on $U$ if and only if $(dd^c u)^n = 0$ there, $n = \dim X$.

Given a Stein manifold $X$, let us have a finite set $Z = \{\zeta_1, \ldots, \zeta_k\} \subset X$ and functions $\varphi_1, \ldots, \varphi_k$ such that $\varphi_j$ is plurisubharmonic near $\zeta_j$, locally bounded and maximal on a punctured neighbourhood of $\zeta_j$, and $\varphi_j(\zeta_j) = -\infty$. The function

$$G_{A,\{\varphi_j\}}(z) = \sup \{u(z) : u \in \text{PSH}^-(X), u \leq \varphi_j \text{ near } \zeta_j, 1 \leq j \leq k\}$$

is the Green–Zahariuta function of $X$ with the singularity $\varphi = \{\varphi_j\}$. The notion was introduced, for the continuous weights $\varphi_j$, in [Z84], see also [Z91]; the general case was treated in [R06]. The function $G_{A,\varphi}$ is plurisubharmonic in $X$, maximal on $X \setminus Z$ and satisfies $G_{A,\varphi}(x) = \varphi_j(x) + O(1)$ as $x \to \zeta_j$.

Let $\varphi \in \text{PSH}(X)$ be locally bounded on $X \setminus Z$ and such that its restriction to a neighbourhood of each point $\zeta_j$ is a maximal weight equivalent to $\varphi_j$ in the sense $\lim \varphi_j(x)/\varphi(x) = 1$; for example, one can take $\varphi = G_{A,\{\varphi_j\}}$. The relative type $\sigma(u, \varphi)$ of $u$ with respect to $\varphi$ was introduced in [R06] as

$$\sigma(u, \varphi) = \lim_{\varphi(x) \to -\infty} \inf \frac{u(x)}{\varphi(x)}.$$

In other words,

$$\sigma(u, \varphi) = \lim_{r \to -\infty} r^{-1} \Lambda(u, \varphi, r),$$

where $\Lambda(u, \varphi, r) := \sup\{u(x) : \varphi(x) < r\}$.

3 Analyticity theorem

Let now $X$ be a Stein manifold of dimension $n$ and $Y$ be a complex manifold of dimension $m$. Let $R : Y \to (-\infty, \infty]$ be a lower semicontinuous function on $Y$. We consider a continuous plurisubharmonic function $\varphi : X \times Y \to [-\infty, \infty)$ such that:

(i) $\varphi(x, y) < R(y)$ on $X \times Y$;

(ii) the set $\mathcal{Z}(y) = \{x : \varphi(x, y) = -\infty\}$ is finite for every $y \in Y$;

(iii) for any $y_0 \in Y$ and $r < R(y_0)$ there exists a neighbourhood $U$ of $y_0$ such that the set $\{(x, y) : \varphi(x, y) < r, y \in U\} \subseteq X \times Y$.
(iv) \((dd^c \varphi)^n = 0\) on \(\{ \varphi(x, y) > -\infty \}\);

(v) \(e^{\varphi(x,y)}\) is locally Hölder continuous in \(y\): every point \((x_0, y_0) \in X \times Y\) has a neighbourhood \(\omega\) such that

\[
|e^{\varphi(x_0, y_1)} - e^{\varphi(x_0, y_2)}| \leq M|\varsigma(y_1) - \varsigma(y_2)|^\beta, \quad (x, y_j) \in \omega, \tag{1}
\]

for some \(M, \beta > 0\) and suitable coordinates \(\varsigma\) on \(Y\).

The function \(\varphi_y(x) = \varphi(x, y)\) is a maximal plurisubharmonic weight with poles at \(\mathcal{Z}(y)\); we will write this as \(\varphi_y(x) \in MW_{\mathcal{Z}(y)}\). In particular, given \(u \in PSH(X)\), the function

\[
r \mapsto \Lambda(u, \varphi_y, r) := \sup\{u(x) : \varphi_y(x) < r\}
\]

is convex and there exists the limit

\[
\sigma(u, \varphi_y) = \lim_{r \to -\infty} r^{-1} \Lambda(u, \varphi_y, r) = \liminf_{x \to \mathcal{Z}(y)} \frac{u(x)}{\varphi(x, y)},
\]

the relative type of \(u\) with respect to the weight \(\varphi_y\). We have thus

\[
u(x) \leq \sigma(u, \varphi_y)\varphi(x, y) + O(1), \quad x \to \mathcal{Z}(y). \tag{2}
\]

Denote

\[
S_c(u, \varphi, Y) = \{y \in Y : u(x) \leq c\varphi(x, y) + O(1) \text{ as } x \to \mathcal{Z}(y)\}.
\]

Equivalently, \(S_c(u, \varphi, Y) = \{y \in Y : \sigma(u, \varphi_y) \geq c\}\).

**Theorem 1** Let a continuous function \(\varphi \in PSH(X \times Y)\) satisfy the above conditions (i)–(v). Then for every \(u \in PSH(X)\) and \(c > 0\), the set \(S_c(u, \varphi, Y)\) is an analytic variety.

**Proof.** We will follow the lines of the proof of [R06, Theorem 7.1], which in turn is an adaptation of Kiselman’s and Demailly’s proofs of the corresponding variants of Siu’s theorem. Note that although the proof is quite short, it is based on such deep results as Demailly’s theorem on plurisubharmonicity of the function \(\Lambda(u, \varphi_y, r)\) and the Bombieri–Hörmander theorem.

By [DSS, Theorem 6.11], the function \(\Lambda(u, \varphi_y, \text{Re} \xi)\) is plurisubharmonic on the set \(\{(y, \xi) \in Y \times \mathbb{C} : \text{Re} \xi < R(y)\}\). Fix a pseudoconvex domain \(D \subseteq Y\) and denote \(R_0 = \inf \{R(y) : y \in D\} > -\infty\). Given \(a > 0\), the function

\[
(u, \xi) \mapsto \Lambda(u, \varphi_y, \text{Re} \xi) - a \text{Re} \xi
\]

is thus plurisubharmonic in \(D \times \{\text{Re} \xi < R_0\}\) and independent of \(\text{Im} \xi\), so by Kiselman’s minimum principle [K78], the function

\[
U_a(y) = \inf \{\Lambda(u, \varphi_y, r) - a(r - R_0) : r < R_0\}
\]

is plurisubharmonic in \(D\).

Let \(y \in D\). If \(a > \sigma(u, \varphi_y)\), then \(\Lambda(u, \varphi_y, r) > a(r - R_0)\) for all \(r \leq r_0 < R_0\). If \(r_0 < r < R_0\), then \(\Lambda(u, \varphi_y, r) - a(r - R_0) > \Lambda(u, \varphi_y, r_0)\). Therefore \(U_a(y) > -\infty\).

Now let \(a < \sigma(u, \varphi_y)\). In view of property (iii) and estimate (2), the exponential Hölder continuity implies the bound

\[
\Lambda(u, \varphi_z, r) \leq \Lambda(u, \varphi_y, \log(e^r + M|\varsigma(z)|^\beta)) \leq \sigma(u, \varphi_y)\log(e^r + M|\varsigma(z)|^\beta) + C
\]
in a neighbourhood $U_y$ of $y$ with the coordinates $z$ chosen so that $z(y) = 0$. Denote $r_z = \beta \log |z(z)|$, then

$$U_a(z) \leq N(u, \varphi_z, r_z) - ar_z \leq (\sigma(u, \varphi_y) - a)\beta \log |z(z)| + C_1, \quad z \in U_y. \quad (3)$$

Given $a, b > 0$, let $Z_{a,b}$ be the set of points $y \in D$ such that the function $\exp(-b^{-1}U_a)$ is not integrable near $y$. As follows from the Hörmander–Bombieri–Skoda theorem [H Theorem 4.4.4], all the sets $Z_{a,b}$ are analytic.

If $y \not\in S_c(u, \varphi, D)$ and $\sigma(u, \varphi_y) < a < c$, then $U_a(y) > -\infty$ and so, by Skoda’s theorem [H Theorem 4.4.5], $y \not\in Z_{a,b}$ for all $b > 0$.

If $y \in S_c(u, \varphi, D)$, $a < c$, and $b < (c - a)\beta(2m)^{-1}$, then (3) implies $y \in Z_{a,b}$. Thus, $S_c(u, \varphi, D)$ coincides with the intersection of all the sets $Z_{a,b}$ with $a < c$ and $b < (c - a)\beta(2m)^{-1}$, and is therefore analytic. $\square$

4 Dependence on coordinates

By a classical result (again due to Siu), standard Lelong numbers are independent of the choice of coordinates. The following statement can be viewed as a bridge between Siu’s analyticity and invariance theorems.

**Corollary 1** Let $\varphi \in MW_0$ satisfy $|e^{\varphi(a)} - e^{\varphi(b)}| \leq M|a - b|^\beta$, $\beta > 0$, on a pseudoconvex neighbourhood $X$ of $0 \in \mathbb{C}^n$, and let $Y$ be a complex manifold in $GL_n(\mathbb{C})$. Then for every $u \in PSH(X)$, the sets

$$\{(\zeta, A) \in X \times Y : u(x) \leq \varphi(Ax - \zeta) + O(1) \text{ as } x \rightarrow A^{-1}\zeta\}$$

and

$$\{(\zeta, A) \in X \times Y : u(x) \leq \varphi(Ax - \zeta) + O(1) \text{ as } x \rightarrow \zeta\}$$

are analytic varieties in $X \times Y$. In particular, the set

$$S(u, \varphi, Y) = \{A \in Y : u(x) \leq \varphi(Ax) + O(1) \text{ as } x \rightarrow 0\}$$

is analytic in $Y$. The functional $u \mapsto \sigma(u, \varphi_A)$, where $\varphi_A(x) = \varphi(Ax)$, is independent of $A \in GL_n(\mathbb{C})$ if and only if $\varphi(x) = c \log |x| + O(1)$ for some constant $c > 0$.

**Proof.** The analyticity follows directly from Theorem [H]. To prove the last assertion, consider the Green–Zahariuta function $G_\varphi$ for the singularity $\varphi$ in the unit ball $\mathbb{B}$. Since $\varphi(x) = \varphi(Ax) + O(1)$ for any unitary $A$, we have $G_\varphi(x) = \chi(\log |x|)$, where $\chi$ is a convex increasing function on $(-\infty, 0)$. The equation $(dd^cG_\varphi)^n = 0$ outside $0$ implies $\chi'' = 0$, and the condition $G_\varphi = 0$ on $\partial \mathbb{B}$ gives then $\chi(t) = ct, c > 0$. $\square$

**Remark.** For the case $\varphi(x) = \max_k \log |x_k|^{a_k}$ and $Y = GL_n(\mathbb{C})$, similar analyticity theorems were proved in [D93] and [K91].

5 Analytic singularities

Let $F : X \times Y \rightarrow \mathbb{C}^n$ be a holomorphic mapping such that its zero set $|Z_F|$ is of codimension $n$ and moreover, $|Z_F| \cap \{(x, y_0) : x \in X\}$ is finite for any $y_0 \in Y$. Then the function $\varphi(x, y) = \log |F(x, y)|$ satisfies conditions (i)-(v) on $X' \times Y$ for any domain $X' \subset X$; condition (iv) follows from King’s formula $(dd^c \log |F|)^n = |Z_F|$. This observation can be used in finding analytic majorants for plurisubharmonic singularities.
Corollary 2 Let $f = (f', f'')$ be a finite equidimensional holomorphic mapping on a complex manifold $X$. If $u \in \text{PSH}(X)$ satisfies $u \leq \log |f'| + O(1)$ on an open set $\omega \subset X$ intersecting every irreducible component of the zero set of $f'$, then $u \leq \log |f'| + O(1)$ locally on $X$.

Proof. Let $\varphi_N(x,y) = \log(|f'(x) - f'(y)| + |f''(x) - f''(y)|^N)$, $N \in \mathbb{Z}_+$, and let $X' \subset X$ be such that $\omega' = X' \cap \omega$ intersects all irreducible components of the set $Z' = \{ x \in X' : f'(x) = 0 \}$. Then, by Theorem 1, $S(u, \varphi, X')$ is an analytic variety. By the assumption, $S(u, \varphi, X') \cap \omega' \supset S(\log |f'|, \varphi, X') \cap \omega'$. Therefore, $S(u, \varphi, X')$ contains all irreducible components of $S(\log |f'|, \varphi, X')$ that pass through $\omega$. Observe now that $S(\log |f'|, \varphi, X') = Z'$, which implies $u \leq \varphi_N + C$ on $Z'$.

Given $a \in Z'$, we can assume $D = \{ x : \max\{|f'(x)|, |f''(x) - f''(a)|\} < 1 \} \subset X'$. Therefore, $u \leq g_N + C$, where $g_N(x) = \max\{\log |f'(x)|, \log |f''(x) - f''(a)|\}$ is the Green-Zahariuta function for the singularity $\varphi_N$ in $D$. Taking $N \to \infty$, we get $u \leq \log |f'| + C$ in $D$. □

A more accurate analysis allows us to weaken the assumptions on the mapping $f'$ in Corollary 2. To this end, it is convenient to use the notion of complex spaces.

For a closed complex subspace $A$ of $X$, let $\mathcal{I}_A = (\mathcal{I}_{A,x})_{x \in X}$ be the associated coherent sheaf of ideals in the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on $X$, and let $|A|$ be the variety in $X$ locally defined as the common set of zeros of holomorphic functions with germs in $\mathcal{I}_A$, i.e., $|A| = \{ x : \mathcal{I}_{A,x} \neq \mathcal{O}_{X,x} \}$.

Recall that an ideal $\mathcal{J} \subset \mathcal{I} \subset \mathcal{O}_{X,x}$ is called a reduction of $\mathcal{I}$ if its integral closure coincides with that of $\mathcal{I}$; the analytic spread of $\mathcal{I}$ equals the minimal number of generators of its reductions $|NRC|$.

We will say that a complex space $A$ is integrally generic at $x \in |A|$ if the analytic spread of $\mathcal{I}_{A,x}$ equals $\text{codim}_{\mathcal{O}_x}|A|$. This is equivalent to saying that there exist functions $h_k \in \mathcal{I}_{A,x}$, $k = 1, \ldots, \text{codim}_{\mathcal{O}_x}|A|$, such that $\log |h| = \log |f| + O(1)$, where $f = (f_1, \ldots, f_s)$ are generators of $\mathcal{I}_{A,x}$, see $[NRC]$. A space $A$ is integrally generic if it is so at each $x \in |A|$.

We will write $u \leq \log |\mathcal{I}_A|$ if a function $u$ satisfies $u \leq \log |f| + O(1)$ for local generators $f$ of $\mathcal{I}_A$.

Theorem 2 Let $A$ be an integrally generic complex space on $X$ and $\omega$ be an open set intersecting every irreducible component of $|A|$. If a function $u \in \text{PSH}(X)$ satisfies $u \leq \log |\mathcal{I}_A|$ on $\omega$, then it satisfies the relation everywhere in $X$.

Proof. Denote by $Z_l$, $l = 1, 2, \ldots$, the irreducible components of $|A|$. We will first prove near all points of the set $Z^*_l = Z_l \setminus \cup_{k \neq l} Z_k$.

Let $\text{codim} Z_l = p$. For an arbitrary point $z \in Z^*_l \cap \partial \omega$, there is a neighbourhood $U$ of $z$, a holomorphic mapping $h : U \to \mathbb{C}^p$, and a linear mapping $U \to \mathbb{C}^{n-p}$, such that $|A| \cap U = Z^*_l \cap U$, $\log |h| \leq \log |\mathcal{I}_A|$, and for every $y \in V$, the mapping $F_y : x \mapsto (h(x) - h(y), L(x) - L(y))$ is finite in $U$. By Corollary 2, we get then $u \leq \log |h| + O(1) \leq \log |\mathcal{I}_A|$ on $\omega \cup U$.

Now we can repeat the procedure with $\omega \cup U$ instead of $\omega$. Since the sets $Z^*_l$ are connected, it gives us the desired bounded near every point of $|A|^* = \cup Z^*_l$.

The rest points can be treated as in the proof of $[RS, \text{Lemma 4.2}]$. Namely, fix a point $z \in |A| \setminus |A|^*$, $\text{codim}_{|A|}|A| = p$. By Thie’s theorem, there exist local coordinates $x = (x', x'')$, $x' = (x_1, \ldots, x_p)$, $x'' = (x_{p+1}, \ldots, x_n)$, centered at $z$, and balls $B' \subset \mathbb{C}^p$, $B'' \subset \mathbb{C}^{n-p}$ such that $B' \times B'' \subset V$, $|A| \cap (B' \times B'')$ is contained in the cone $\{|x'| \leq \gamma |x''|\}$ with some constant $\gamma > 0$, and the projection of $|A| \cap (B' \times B'')$ onto $B''$ is a ramified covering with a finite number of sheets.
Let $h = (h_1, \ldots, h_p)$ satisfy $\log |h| \leq \log |I_A|$ on $V$. Take $r_1 = 2\gamma r_2$ with a sufficiently small $r_2 > 0$ so that $B_{r_1} \subset B'$ and $B_{r_2}'' \subset B''$, then for some $\delta > 0$ we have $|h| \geq \delta$ on $\partial B_{r_1} \times B_{r_2}''$.

Given a point $x_0'' \in B_{r_2}''$, denote by $R(x_0'')$ and $S(x_0'')$ the intersections of the set $B_{r_1}' \times \{x_0''\}$ with the varieties $|A|$ and $|A| \setminus |A|^*$, respectively. Since the projection is a ramified covering, $R(x_0'')$ is finite for any $x_0'' \in B_{r_2}''$, while $S(x_0'')$ is empty for almost all $x_0'' \in B_{r_2}''$ because $\dim S \leq n - p - 1$; we denote the set of all such generic $x_0''$ by $E$.

Given $x_0'' \in E$, the function $\nu(x') = \log |h(x', x_0'')|/\delta$ is nonnegative on $\partial B_{r_1}'$ and maximal on $B_{r_1}' \setminus R(x_0'')$, since the map $h(\cdot, x_0'') : B_{r_1}' \to \mathbb{C}^p$ has no zeros outside $R(x_0'')$. Since $u$ satisfies $u \leq \log |h| + O(1)$ locally near points of $|A|^*$, we have then $u(x', x_0'') \leq \nu(x') + C$ on the whole ball $B_{r_1}'$, where $C = \sup_V u$.

As $x_0'' \in E$ is arbitrary, this gives us $u \leq \log |h| - \log \delta + C$ on $B_{r_1}' \times E$. The continuity of the function $\log |h|$ extends this relation to the whole set $B_{r_1}' \times B_{r_2}''$, which completes the proof. \hfill $\Box$

6 Green functions

The result can be applied to investigation of maximality properties for Green functions with analytic singularities.

The Green function $G_A$ with singularities along a complex space $A$ is the upper envelope of the class of all functions $u \in PSH^{-}(X)$ such that $u \leq \log |I_A|$. This function is plurisubharmonic in $X$ and satisfies $G_A \leq \log |I_A|$, see [RS].

When $|A|$ is discrete, $G_A$ is maximal on $X \setminus |A|$. In the case $\dim |A| > 0$, the Green function has additional maximality properties. Namely, if $I_A$ has $p < n$ global generators, then $G_A$ is maximal on the whole $X$, and for an arbitrary complex space $A$, the function $G_A$ is locally maximal outside a discrete subset $J_A$ of $|A|$ consisting of all points $x \in |A|$ such that the analytic spread of $I_{A,x}$ equals $n$ [RS, Theorem 4.3]; in [R98], $J_A$ was called the complete indeterminacy locus. (A function $v$ is said to be locally maximal on an open set $\omega$ if every point of $\omega$ has a neighbourhood where $v$ is maximal.)

We do not know if the function $G_A$ is always maximal on $X \setminus J_A$; what we can prove is the following result.

Corollary 3 If $A$ is an arbitrary closed complex space on $X$, then the function $G_A$ is maximal outside an analytic subset $J$ of $|A|$, nowhere dense in each positive-dimensional component of $|A|$. If $\dim X = 2$, then $J$ coincides with the complete indeterminacy locus $J_A$. If $A$ is integrally generic, then $J = \emptyset$.

Proof. By [RS, Proposition 3.5], the set $|A|$ can be decomposed into the disjoint union of local (not necessarily closed) analytic varieties $J_k$, $1 \leq k \leq n$, such that $\text{codim } J_k \geq k$ and for each $a \in J_k$, the ideal $I_{A,a}$ has analytic spread at most $k$. In view of Theorem 2 this implies the claims. \hfill $\Box$

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