SUBPRODUCT SYSTEMS WITH QUANTUM GROUP SYMMETRY

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Abstract. We introduce a class of subproduct systems of finite dimensional Hilbert spaces whose fibers are defined by the Jones–Wenzl projections in Temperley–Lieb algebras. The quantum symmetries of a subclass of these systems are the free orthogonal quantum groups. For this subclass, we show that the corresponding Toeplitz algebras are nuclear C∗-algebras that are KK-equivalent to C and obtain a complete list of generators and relations for them. We also show that their gauge-invariant subalgebras coincide with the algebras of functions on the end compactifications of the duals of the free orthogonal quantum groups. Along the way we prove a few general results on equivariant subproduct systems, in particular, on the behavior of the Toeplitz and Cuntz–Pimsner algebras under monoidal equivalence of quantum symmetry groups.

Introduction

The notion of a subproduct system, introduced by Shalit and Solel [SS09] and Bhat and Mukherjee [BM10], lies at the intersection of two lines of research. One is dilation theory for semigroups of completely positive maps, the other is noncommutative function theory for row contractions. Recall that a row contraction is an m-tuple of operators (S1,...,Sm) such that \( \sum_{i=1}^{m} S_i S_i^* \leq 1 \). By a result of Popescu [Pop91], a universal, in a suitable sense, model for row contractions is provided by the creation operators T1,...,Tm on the full Fock space \( F(\mathbb{C}^m) \). If we are interested in the row contractions satisfying in addition algebraic relations \( P_j(S_1,...,S_m) = 0 \) for some homogeneous polynomials \( P_j \) in m noncommuting variables, then such a model is often obtained by the compressions \( S_i = e_{H_i} T_i e_{H_i} \) of the operators \( T_i \) to the subspace \( H_n = \bigoplus_{n=0}^{\infty} H_n \subset F(\mathbb{C}^m) \), where \( H_n = I_n^+ \subset (\mathbb{C}^m)^\otimes n \) and \( I_n \) is the degree n component of the ideal I \( \subset \mathbb{C}[X_1,...,X_m] \) generated by \( P_j \) [Arv98, AP00]. The collection \( H = (H_n)_{n=0}^{\infty} \) has the property \( H_{k+l} \subset H_k \otimes H_l \), which in the terminology of [SS09] means that it is a standard subproduct system of finite dimensional Hilbert spaces.

Our main object of interest in this paper is the structure of the Toeplitz algebra \( T_H = C^*(S_1,...,S_m) \subset B(F_H) \) and its quotient \( O_H = T_H/K(F_H) \), called the Cuntz–Pimsner algebra of \( H \) by Viselter [Vis12]. Although the definition of these algebras is similar to such much more studied constructions as Cuntz–Krieger algebras, graph algebras and Pimsner algebras, a number of basic properties of the latter algebras do not have obvious analogues for \( T_H \) and \( O_H \) [Vis12], and a more or less complete understanding of \( T_H \) and \( O_H \) has been achieved only in a few cases. For example, Kakariadis and Shalit [KST19] made a comprehensive analysis of the case where the ideal I is generated by monomials. In the present paper we consider principal ideals \( I = \langle P \rangle \) generated by one quadratic polynomial. Furthermore, we require that the corresponding rank one projection \( e = [CP] \in B(\mathbb{C}^m \otimes \mathbb{C}^m) \) defines representations of Temperley–Lieb algebras \( TL_n(\lambda^{-1}) \) on \( (\mathbb{C}^m)^\otimes n \) for some \( \lambda \geq 4 \); recall that \( TL_n(\lambda^{-1}) \) is generated by projections \( e_j, 1 \leq j \leq n-1 \), satisfying the relations

\[ e_i e_j = e_j e_i \quad \text{if} \quad |i - j| \geq 2, \quad e_i e_{i\pm 1} e_i = \frac{1}{\lambda} e_i. \]
Concretely, we consider the polynomials $P = \sum_{i,j=1}^{m} a_{ij} X_i X_j$ ($m \geq 2$) such that $AA^*$ is a scalar multiple of a unitary matrix, where $A = (a_{ij})_{i,j}$. We call such polynomials Temperley–Lieb. The simplest example of such a polynomial is $X_1 X_2 - X_2 X_1$, which defines Arveson’s symmetric subproduct system $SSP_2$ \cite{Arv98}. In this case $\lambda = 4$. Since the Temperley–Lieb algebras for $\lambda \geq 4$ are all isomorphic, the entire collection that we consider can be thought of as a deformation/quantum analogue of $SSP_2$, and indeed, as we will see, the corresponding C$^*$-algebras $T_P$ and $O_P$ share a lot of properties. To be precise, our main results are proved for the subclass of polynomials such that $AA^* = \pm 1$. They have well-studied quantum symmetries - the free orthogonal quantum groups \cite{VDW96}. The general case is related to braided free orthogonal quantum groups and will be analyzed in detail in a separate publication.

Particular cases of the subproduct systems associated with Temperley–Lieb polynomials have already appeared in the literature. In addition to $SSP_2$, the case of polynomials $X_1 X_2 - q X_2 X_1$ ($q \in \mathbb{C}^*$) has been studied in \cite{AP00, BB02, SS09}, although without a detailed analysis of the associated C$^*$-algebras. Our main class of examples, corresponding to the case $AA^* = \pm 1$, has been studied by Andersson \cite{And15}, with an emphasis on the gauge-invariant part of $T_P$. It should be said that his paper contains interesting ideas, but suffers from numerous imprecise and outright wrong constructions and arguments. Recently, the case of polynomials $\sum_{i=1}^{m} (-1)^i X_i X_{n-i+1}$ has been studied by Arici and Kaad \cite{AK20}, and some of our results strengthen and generalize the results in their paper.

The contents of the paper is as follows. In Section 1 after a brief reminder on subproduct systems and associated C$^*$-algebras, we discuss the class of Temperley–Lieb polynomials. Using known formulas for the Jones–Wenzl projections in the Temperley–Lieb algebras we find some nontrivial relations in $T_P$.

In Section 2 we make a digression into the general theory of quantum group equivariant subproduct systems. By this we mean that each space $H_n$ is equipped with a unitary representation of a compact quantum group $G$ and the embedding maps $H_{k+l} \to H_k \otimes H_l$ are equivariant. A technical question that we study is under which conditions the action of $G$ on the Cuntz–Pimsner algebra $O_H$ is reduced, meaning that the averaging over $G$ defines a faithful cp map $O_H \to O_H^{G}$. Another question is what happens when we consider a monoidally equivalent compact quantum group $\tilde{G}$ and transform $H = (H_n)_{n=0}^\infty$ into a $\tilde{G}$-equivariant subproduct system $\tilde{H}$. We show that the Toeplitz algebras $T_H$ and $T_{\tilde{H}}$ correspond to each other under the equivalence of categories of $G$- and $\tilde{G}$-C$^*$-algebras defined in \cite{DRVV10}. The same is true for the reduced forms of the Cuntz–Pimsner algebras.

We want to make it clear that none of the results of this section is strictly speaking needed for the subsequent sections, but they provide a conceptual framework for the results in those sections.

In Section 3 we return to the Temperley–Lieb polynomials and from this point onward concentrate on the polynomials $P = \sum_{i,j=1}^{m} a_{ij} X_i X_j$ such that $AA^* = \pm 1$. As we already mentioned, every such polynomial has a large quantum symmetry group $O_P^+$, where by large we mean that every $O_P^+$-module $H_n$ is irreducible. We show that by starting from Arveson’s system $SSP_2$, corresponding to $P = X_1 X_2 - X_2 X_1$, then moving to the polynomials $q^{-1/2} X_1 X_2 \pm q^{1/2} X_2 X_1$ ($q > 0$) and then considering the general case, the results of Section 2 quickly lead to the conclusion that the Cuntz–Pimsner algebra $O_P$ is $O_P^+$-equivariantly isomorphic to the linking algebra $B(SU_{q^2}(2), O_P^+)$ (for suitable $q \in (0, 1]$ and $\tau = \pm 1$), which defines an equivalence between the representation categories of $SU_{q^2}(2)$ and $O_P^+$. We then find an explicit such isomorphism. It is worth stressing that the reason we know that $O_P$ cannot be smaller is that the action of $O_P^+$ on $B(SU_{q^2}(2), O_P^+)$ is reduced and ergodic, and therefore it does not admit any proper $O_P^+$-equivariant quotients. This is exactly the same argument as was used by Arveson for $SSP_2$ to conclude that the Cuntz–Pimsner algebra in his case is $C(S^3)$ \cite{Arv98}. This property can be viewed as a replacement of the gauge-invariant uniqueness theorem, which fails for general
subproduct systems [Vis12]; see the recent paper by Dor-on [Do21] for a related discussion.
Once $O_P$ has been computed, it is not difficult to show that the relations in $T_P$ that we found
in Section 2 are complete.

In Section 3 we consider the gauge-invariant part $T_P(0)$ of $T_P$. Since $H_n$, $n \geq 0$, exhaust the
irreducible $O_P^+$-modules up to isomorphism, $T_P(0)$ can be considered as an algebra of functions
on a compactification of the dual discrete quantum group $FO_P$ of $O_P^+$. A compactification $\tilde{FO}_P$
of $FO_P$ has been constructed for the polynomials $P = q^{-1/2}X_1X_2 \pm q^{1/2}X_2X_1$ by Tuset and
the second author [NT04] using an analogy with the algebra of equivariant pseudo-differential
operators of order zero on $SU(2)$, and for general $P$ by Vaes and Vergnioux [VV07] as a quantum
analogue of the end compactification of a free group. We show that, excluding possibly the
cases $O_P^+ \cong SU_{\pm 1}(2)$, we have $T_P(0) = C(\tilde{FO}_P)$. As a consequence, we now have an explicit
isomorphism $C(\partial \tilde{FO}_P) \cong \mathbb{T}B(SU_{\text{reg}}(2), O_P^+)$. The existence of such an isomorphism has been
known, but in an indirect way, via an identification of both sides with the Martin boundary
of $FO_P$ [NT04, VV07, VV08, DRV10].

In Section 4 we study $K$-theoretic properties of $T_P$. If we look again at $SSP_2$, then it follows already
from an index theorem of Vegunopalkrishna [Ven72] that the embedding map
$\mathbb{C} \to T_{X_1X_2-X_2X_1}$ is a $KK$-equivalence. At this point it is natural to expect that the same
is true for general $T_P$. For the polynomials $P = \sum_{i=1}^{m}(-1)^{i}X_1X_{m-i+1}$ this has indeed been
shown by Arici and Kaad [AK20]. They constructed an explicit inverse $T_P \to \mathbb{C}$ in the $KK$
category, which is a highly nontrivial task for $m \geq 3$ (and for $m = 2$ as well, if we want in
addition $SU(2)$-equivariance). It is plausible that with some modification their construction
and homotopy arguments work for general $P$. We take, however, a different route and use
the Baum–Connes conjecture for $FO_P$, as formulated by Meyer and Nest [MN06] and proved
by Voigt [Voi11], to reduce the problem to comparing the $K$-theory of crossed products. We
thus show that the embedding $\mathbb{C} \to T_P$ is a $KK^{\mathbb{Q}}$-equivalence without explicitly providing
an inverse.

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1. Temperley–Lieb subproduct systems

Recall, following [SS09, BM10], that a subproduct system $\mathcal{H}$ of finite dimensional Hilbert
spaces (over the additive monoid $\mathbb{Z}_+$) is a sequence of Hilbert spaces $(H_n)_{n=0}^\infty$
with isometries $w_{k,l}: H_{k+l} \to H_k \otimes H_l$ such that
\[
\dim H_0 = 1, \quad \dim H_1 = m < \infty, \quad (w_{k,l} \otimes 1)w_{k+l,n} = (1 \otimes w_{l,n})w_{k,l+n}.
\]
By [SS09, Lemma 6.1], we can assume that $H_0 = \mathbb{C}$, $H_{k+l} \subset H_k \otimes H_l$ and the isometries $w_{k,l}$
are simply the embedding maps. The subproduct systems satisfying this stronger property are
called standard. For such subproduct systems we denote by $f_n$ the projection $H_n^{\otimes n} \to H_n$.

Remark 1.1. We could also start with formally weaker axioms for subproduct systems: it is
enough to require that the identities $(w_{k,l} \otimes 1)w_{k+l,n} = (1 \otimes w_{l,n})w_{k,l+n}$ hold only up to phase
factors. Indeed, the same arguments as in [SS09, Lemma 6.1] show that we can still construct
a standard subproduct system out of such a datum, and this standard system remains the same
if we multiply $w_{k,l}$ by phase factors. On a more conceptual level, the possibility of getting rid
of phase factors in an essentially unique way follows from triviality of the cohomology groups
$H^2(\mathbb{Z}_+; \mathbb{T})$, $H^3(\mathbb{Z}_+; \mathbb{T})$ (see [CE56, Proposition X.4.1]).

Given a subproduct system $\mathcal{H} = (H_n)_{n=0}^\infty$, the associated Fock space is defined by
\[
\mathcal{F}_\mathcal{H} = \bigoplus_{n=0}^\infty H_n.
\]
For every $\xi \in H_1$, we define an operator

$$S_\xi : \mathcal{F}_H \to \mathcal{F}_H \text{ by } S_\xi \zeta = w_{\xi, n}(\xi \otimes \zeta) \text{ for } \zeta \in H_n.$$  

We will usually fix an orthonormal basis $(\xi_i)_{i=1}^m$ in $H_1$ and write $S_i$ for $S_{\xi_i}$. The Toeplitz algebra $\mathcal{T}_H$ of $H$ is defined as the unital C$^*$-algebra generated by $S_1, \ldots, S_m$.

If $\mathcal{H}$ is standard and $H = H_1$, it is convenient to identify $\mathcal{F}_H$ with a subspace of the full Fock space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$  

Consider the operators $T_i : \mathcal{F}(H) \to \mathcal{F}(H)$, $T_i \zeta = \xi_i \otimes \zeta$, and the projection

$$e_H : \mathcal{F}(H) \to \mathcal{F}_H.$$  

We obviously have $S_i = e_H T_i |_{\mathcal{F}_H}$. Then also $S_i^* = e_H T_i^* |_{\mathcal{F}_H}$, but since $H_{n+1} \subset H_1 \otimes H_n$, we actually have the stronger property

$$S_i^* = T_i^* |_{\mathcal{F}_H}.$$  

Since $1 - \sum_{i=1}^m T_i T_i^*$ is the projection onto $H^{\otimes 0} = \mathbb{C}$, we then get

$$e_0 = 1 - \sum_{i=1}^m S_i S_i^*,$$  

where $e_0$ is the projection $\mathcal{F}_H \to H_0$. As the vacuum vector $\Omega = 1 \in H_0$ is cyclic for $\mathcal{T}_H$, it follows that $\mathcal{K}(\mathcal{F}_H) \subset \mathcal{T}_H$. The Cuntz–Pimsner algebra of $H$ [Virs12] is defined by

$$\mathcal{O}_H = \mathcal{T}_H / \mathcal{K}(\mathcal{F}_H).$$

Once we fix an orthonormal basis $(\xi_i)_{i=1}^m$ in $H$, it is convenient to identify the tensor algebra $T(H)$ with the algebra $\mathbb{C}(X_1, \ldots, X_m)$ of polynomials in $m$ noncommuting variables. When we do this, we omit the symbol $\otimes$ for the product in $T(H)$, so we write $X_\xi$ instead of $\xi \otimes \xi$.

By [SS09 Proposition 7.2], there is a one-to-one correspondence between the standard subproduct systems with $H_1 = H$ and the homogeneous ideals $I$ in $\mathbb{C}(X_1, \ldots, X_m)$ such that the degree one homogeneous component $I_1$ of $I$ is zero. Namely, given such an ideal $I$, we define

$$H_n = I_n^{\perp} \subset H^{\otimes n}.$$  

In this work we mainly consider ideals $I = (P)$ generated by one homogeneous polynomial of degree 2. We denote by $\mathcal{H}_P$ the corresponding standard subproduct system, but then normally use the subscript $P$ instead of $\mathcal{H}_P$, so we write $\mathcal{F}_P$, $\mathcal{T}_P$, etc.

We concentrate on polynomials having a particular symmetry.

**Definition 1.2.** Given a Hilbert space $H$ of finite dimension $m \geq 2$, we say that a nonzero vector $\xi \in H \otimes H$ is **Temperley–Lieb**, if the corresponding projection $e = [\xi \xi] \in B(H \otimes H)$ satisfies

$$(e \otimes 1)(1 \otimes e)(e \otimes 1) = \frac{1}{\lambda} e \otimes 1 \text{ in } B(H \otimes H) \otimes H$$  

for some $\lambda > 0$.

As the following lemma shows, we then also have

$$(1 \otimes e)(e \otimes 1)(1 \otimes e) = \frac{1}{\lambda} 1 \otimes e.$$  

**Lemma 1.3.** Assume $e_1$ and $e_2$ are projections in a C$^*$-algebra $A$ with a faithful tracial state $\tau$ such that

$$\tau(e_1) = \tau(e_2) \text{ and } e_1 e_2 e_1 = \frac{1}{\lambda} e_1 \text{ for some } \lambda > 0.$$  

Then we also have $e_2 e_1 e_2 = \frac{1}{\lambda} e_2$.  

Proof. Consider the element \( p = e_2 - \lambda e_2 e_1 e_2 \). Then \( p^2 = p \) and \( \tau(p) = 0 \). Hence \( p = 0 \). \( \square \)

In order to describe explicitly the Temperley–Lieb tensors, it is convenient to view the space \( H \otimes H \) as the space of anti-linear operators \( H \to H \), with \( \zeta \otimes \eta \) corresponding to the operator \((\eta, \cdot)\zeta\). Equivalently, the tensor \( \xi_A \in H \otimes H \) corresponding to an anti-linear operator \( A: H \to H \) is given by

\[
\xi_A = \sum_i \xi_i \otimes A\xi_i,
\]

where \( (\xi_i)_i \) is any orthonormal basis in \( H \).

**Lemma 1.4.** Given \( A \neq 0 \), the tensor \( \xi_A \in H \otimes H \) is Temperley–Lieb if and only if \( A^2 \) is unitary up to a scalar factor, so that \( (A^2)^* A^2 = \alpha \) for some \( \alpha > 0 \), and then \( \lambda \) satisfying \( 1 \leq \lambda = \alpha^{-1} (\text{Tr} A^* A)^{-1} \).

**Proof.** Consider the polar decomposition \( A = U |A| \). A priori \( U \) is only an anti-linear partial isometry. We extend it to an anti-unitary \( \tilde{U} \). Let \( (\xi_i)_i \) be an orthonormal basis consisting of eigenvectors of \( |A| \), \( |A| \xi_i = \lambda_i \xi_i \). Put \( \zeta_i = \tilde{U} \xi_i \). Let \( (e_{ij})_{i,j} \) and \( (f_{ij})_{i,j} \) be the matrix units in \( B(H) \) corresponding to \((\xi_i)_i\) and \((\zeta_i)_i\), resp. As \( \xi_A = \sum_i \lambda_i \xi_i \otimes \zeta_i \), for the projection \( e = [C\xi_A] \) we have

\[
e = \beta \sum_{i,j} \lambda_i \lambda_j e_{ij} \otimes f_{ij}, \quad \text{where } \beta = (\text{Tr} A^* A)^{-1}.
\]

Since \( f_{i_1,j_1} e_{i_2,j_2} f_{i_3,j_3} = (\xi_{i_2}, \zeta_{j_2}) (\xi_{i_3}, \zeta_{j_3}) f_{i_1,j_3} \), we have

\[
(e \otimes 1)(1 \otimes e)(e \otimes 1) = \beta^3 \sum_{i_1,i_2,i_3,j_1,j_2,j_3} \delta_{i_1,i_2,i_3} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} (\xi_{i_2}, \zeta_{j_1}) (\xi_{i_3}, \zeta_{j_2}) e_{i_1,j_3} \otimes f_{i_1,j_3} \otimes f_{i_2,j_2}.
\]

It follows that \( 1 \) holds if and only if the following identities are satisfied:

\[
\beta^3 \lambda_{i_1} \lambda_{j_3} \sum_i \lambda_i^2 \lambda_{i_2} \lambda_{j_2} (\xi_{i_2}, \zeta_{i}) (\xi_{i_3}, \zeta_{j_2}) = \delta_{i_2,j_2} \beta^{-1} \lambda_{i_1} \lambda_{j_3}.
\]

As \( \lambda_{i_1} \lambda_{j_3} \neq 0 \) at least for some indices, it follows that if we introduce the unitary matrix \( V = ((\zeta_j, \xi_i))_{i,j} \) and the diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \), then we must have

\[
\lambda \beta^2 AV \Lambda^2 V^* \Lambda = I.
\]

This is equivalent to invertibility of \( \Lambda \) together with the identity \( \lambda \beta^2 V \Lambda^2 = \Lambda^{-2} V \). In terms of \( |A| \) and \( U \) this means that \( \lambda \beta^2 U |A|^2 = |A|^{-2} U \), or equivalently, \( \lambda \beta^2 AA^* = (A^* A)^{-1} \), which proves the lemma. \( \square \)

Using this lemma it is not difficult to parameterize the Temperley–Lieb tensors up to scalar factors and unitary transformations on \( H \). Namely, since \((U \otimes U) \xi_A = \xi_{U. AU^*}\) for any unitary \( U \) on \( H \), we want to classify up to unitary conjugacy antilinear operators \( A: H \to H \) such that \( A^2 \) is unitary. If \( A = U |A| \) is the polar decomposition of such an \( A \), then \( U |A| = |A|^{-1} U \). It follows that the spectrum \( \sigma(|A|) \) of \( |A| \) is closed under the transformation \( \beta \mapsto \beta^{-1} \), and if \( H_\beta \subset H \) is the spectral subspace for \( |A| \) corresponding to \( \beta \), then \( U H_\beta = H_{\beta^{-1}} \). This implies that the collection of pairs \((\beta, Z_\beta)\), where \( \beta \in S := \sigma(|A|) \cap (0, 1] \) and \( Z_\beta \) is the eigenvalues of \( U^2 \) on \( H_\beta \) counting with multiplicity, is an invariant of the unitary conjugacy class of \( A \). We claim that this is a complete invariant and the only restrictions on the sets \( S \) and \( Z_\beta \) are that \( Z_1 \) is closed under complex conjugation and

\[
\sum_{\beta \in S \cap (0, 1)} 2 |Z_\beta| + |Z_1| = m.
\]

Indeed, it suffices to explain how to choose an orthonormal basis in which \( A \) has a normal form. For every \( \beta \in S \cap (0, 1) \) we choose an orthonormal basis \((\xi_{\beta i})_i\) in \( H_\beta \) consisting of
eigenvectors of $U^2$, $U^2 \xi_{\beta i} = z_{\beta i} \xi_{\beta i}$. Then the vectors $\zeta_{\beta i} = U \xi_{\beta i}$ form an orthonormal basis in $H_{\beta^{-1}}$. Each two-dimensional space spanned by $\xi_{\beta i}$ and $\zeta_{\beta i}$ is invariant under $U$ and $|A|$, hence under $A$ and $A^\ast$, and the matrix of the restriction of $A$ to this space is

$$
\begin{pmatrix}
0 & z_{\beta i} \beta^{-1}
\end{pmatrix}.
$$

Next, assuming 1 is an eigenvector of $|A|$, consider the space $H_1$. It is invariant under $U$, so here we can use the well-known description of anti-unitary matrices due to Wigner. It will be convenient to slightly modify it though. If $H_{1z} \subset H_1$ is the spectral subspace for $U^2$ corresponding to $z \in \mathbb{T}$, then $U H_{1z} = H_{1z}$. It follows that, similarly to the previous paragraph, the direct sum of the spaces $H_{1z}$ for $z \neq 1$ decomposes into a direct sum of two-dimensional $U$-invariant subspaces such that on each of them $U$ has the form

$$
\begin{pmatrix}
0 & z
1 & 0
\end{pmatrix}, \quad 0 < \arg(z) \leq \pi.
$$

Finally, consider $H_{11}$. The real subspace of vectors fixed by $U$ is a real form of $H_{11}$. Choose an orthonormal basis $g_1, \ldots, g_k$ in this Euclidean subspace. If $k = 2l$ is even, we define an orthonormal basis $(\xi_j)_j$ in $H_{11}$ by

$$
\xi_j = \frac{1}{\sqrt{2}}(g_j + ig_{k-j+1}) \quad \text{and} \quad \xi_{k-j+1} = \frac{1}{\sqrt{2}}(g_j - ig_{k-j+1}), \quad 1 \leq j \leq l.
$$

If $k = 2l + 1$, then, in order to get a basis, in addition to the above vectors we need to take $\xi_{l+1} = g_{l+1}$. In both cases the matrix of $U|H_{11}$ in the basis $(\xi_j)_j$ is

$$
\begin{pmatrix}
0 & \cdots & 1
\cdots & \cdots & \cdots
1 & \cdots & 0
\end{pmatrix}.
$$

This finishes the proof of our claim. The discussion can be summarized as follows.

**Proposition 1.5.** Assume $H$ is a Hilbert space of dimension $m = 2l + r$ ($l \in \mathbb{N}$, $r \in \{0, 1\}$) and $A: H \to H$ is an anti-linear operator such that $A^2$ is unitary. Then there is an orthonormal basis $(\xi_i)_{i=1}^m$ in which $A$ has the form

$$
\begin{pmatrix}
0 & a_m
\vdots & \ddots
a_1 & 0
\end{pmatrix}
$$

for some $a_i \in \mathbb{C}^*$, $|a_i a_{m-i+1}| = 1$ for all $i$, and then $\lambda$ satisfying (L.3) for $\xi = \xi_A = \sum_i a_i \xi_i \otimes \xi_{m-i+1}$ is given by

$$
\lambda^{1/2} = \sum_{i=1}^m |a_i|^2 \geq m.
$$

The numbers $a_1, \ldots, a_m$ are uniquely determined by $A$ up to permutations of pairs of coefficients $(a_i, a_{m-i+1})$ ($1 \leq i \leq l$) if we in addition require $0 < a_i \leq 1$ ($1 \leq i \leq l$), $a_{l+1} = 1$ if $m$ is odd, and $0 \leq \arg(a_{m-i+1}) \leq \pi$ whenever $a_i = 1$ ($1 \leq i \leq l$).

Identifying $H$ with the space of homogeneous linear polynomials in $m$ variables, we can equivalently say that a homogeneous quadratic polynomial

$$
\sum_{i,j=1}^m a_{ij} X_i X_j
$$

is Temperley–Lieb if and only if the matrix $A = (a_{ij})_{i,j}$ is such that $A\tilde{A}$ is a nonzero scalar multiple of a unitary matrix, where $\tilde{A} = (\tilde{a}_{ij})_{i,j}$. By a rescaling and a unitary change of
variables, every such polynomial can be written as
\[ \sum_{i=1}^{m} a_i X_i X_{m-i+1}, \quad \text{with} \quad |a_i a_{m-i+1}| = 1 \quad \text{for all} \quad i, \]
and such a presentation is unique up to permutations of pairs \((a_i, a_{m-i+1})\) if we put restrictions on the coefficients \(a_i\) as in Proposition 1.5.

Given a Temperley–Lieb polynomial \(P = \sum_{i,j=1}^{m} a_{ij} X_i X_j\), consider the corresponding standard subproduct system \(H_P = (H_n)_{n=0}^{\infty}\). If \(e \in B(H \otimes H)\) is the projection onto \(\mathbb{C}P\), then
\[ H_n = f_n H^{\otimes n}, \quad f_n = 1 - \sum_{i=0}^{n-2} 1^{\otimes i} \otimes e \otimes 1^{\otimes (n-i-2)} \quad \text{for} \quad n \geq 2, \]
and \(f_0 = 1 \in \mathbb{C}, \ f_1 = 1 \in B(H)\). Since the projections \(1^{\otimes i} \otimes e \otimes 1^{\otimes (n-i-2)}\) satisfy the Temperley–Lieb relations, by computations of Jones \cite{Jon83} and Wenzl \cite{Wen87} (see \cite{NT13} Lemma 2.5.8 and its proof), we have
\[ f_{n+1} = 1 \otimes f_n - [2]_q \phi(n)(1 \otimes f_n)(e \otimes 1^{\otimes (n-1)})(1 \otimes f_n), \quad n \geq 0, \quad \text{(1.4)} \]
where \(q \in (0, 1]\) is such that \(\lambda^{1/2} = q + q^{-1}\), \([k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}\) (with the convention \([k]_1 = k\) and
\[ \phi(n) = \frac{[n]_q}{[n+1]_q} = \frac{q^n - q^{-n}}{q^{n+1} - q^{-n-1}}. \quad \text{(1.5)} \]

Note that \(\phi(n) \to q\) as \(n \to +\infty\).

**Lemma 1.6.** For all \(n \geq 0\), we have \(\dim H_n = \lfloor n+1 \rfloor_t\), where \(t \in (0, 1]\) is such that \(t + t^{-1} = m\).

**Proof.** It is known that the projection \([2]_q \phi(n)(1 \otimes f_n)(e \otimes 1^{\otimes (n-1)})(1 \otimes f_n)\) in \((1.4)\) is equivalent to \(e \otimes f_{n-1}\), see, e.g., again \cite{NT13} Lemma 2.5.8. Hence the dimensions of the spaces \(H_n = f_n H^{\otimes n}\) satisfy the recurrence relation
\[ \dim H_{n+1} = m \dim H_n - \dim H_{n-1}. \]
This gives the result. \(\square\)

We next want to describe certain relations in \(\mathcal{T}_P\). As we will see later, these relations are complete, at least for Temperley–Lieb polynomials of a particular type.

Let us first introduce some notation. Since \(\mathcal{K}(\mathcal{F}_P) \subset \mathcal{T}_P\), we can view the algebra \(c = C(\mathbb{Z}_+ \cup \{+\infty\})\) of converging sequences as a subalgebra of \(\mathcal{T}_P\), with the projection \(e_n \in c\) corresponding to the projection \(\mathcal{F}_P \to H_n\). Denote by \(\gamma\) the shift to the left on \(c\).

**Proposition 1.7.** Consider a polynomial \(P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}\) such that \(|a_i a_{m-i+1}| = 1\) for all \(i\) \((m \geq 2)\). Let \(q \in (0, 1]\) be such that
\[ \sum_{i=1}^{m} |a_i|^2 = q + q^{-1}. \]
Then we have the following relations in \(\mathcal{T}_P\):
\[ f S_i = S_i \gamma(f) \quad (f \in c, \ 1 \leq i \leq m), \quad \sum_{i=1}^{m} S_i S_i^* = 1 - e_0, \quad \sum_{i=1}^{m} a_i S_i S_{m-i+1} = 0, \]
\[ S_i^* S_j + a_i \bar{a}_j \phi S_{m-i+1} S_{m-j+1}^* = \delta_{ij} 1 \quad (1 \leq i, j \leq m), \]
where the element \(\phi = (\phi(n))_{n=0}^{\infty} \in c\) is defined by \((1.5)\).
Proof. The first relation simply reflects the fact that $S_tH_n \subset H_{n+1}$. The second relation is \([1,2]\), it holds in any subproduct system of finite dimensional Hilbert spaces. The third relation is an immediate consequence of the definition of $H_P$. So it is only the last relation that is not obvious.

Denote by $e_{ij}$ the matrix units in $B(H)$. The projection $e \in B(H \otimes H)$ onto $\mathbb{C}P$ is

$$e = \frac{1}{2^q} \sum_{i,j=1}^m a_i \bar{a}_j e_{ij} \otimes e_{m-i+1,m-j+1}.$$  

Therefore by \([1,2]\) we have

$$f_{n+1} = 1 \otimes f_n - \phi(n) \sum_{i,j} a_i \bar{a}_j e_{ij} \otimes f_n(e_{m-i+1,m-j+1} \otimes 1^{\otimes(n-1)}) f_n.$$  

From this, for every $\xi \in H_n$, we get

$$S_t^*S_t \xi = \delta_{ij} \xi - \phi(n) a_i \bar{a}_j e_{ij} \otimes f_n(e_{m-i+1,m-j+1} \otimes 1^{\otimes(n-1)}) f_n \xi.$$  

By observing that by \([1,1]\) we have

$$f_n(e_{kl} \otimes 1^{\otimes(n-1)}) f_n = f_n T_k T_l^* f_n = S_k S_l^*$$  

on $H_n$, we obtain the last relation in $T_P$. \hfill \Box

2. Equivariant subproduct systems

As a preparation for a more thorough study of $T_P$, in this section we consider subproduct systems that are equivariant with respect to a compact quantum group.

Let us fix the notation and recall some basic notions, see \([NT13]\) for more details. By a compact quantum group $G$ we mean a Hopf $\ast$-algebra $(\mathbb{C}[G], \Delta)$ that is generated by matrix coefficients of finite dimensional unitary right comodules. We have a one-to-one correspondence between such comodules and the finite dimensional unitary representations of $G$, that is, unitaries $U \in B(H_U \otimes \mathbb{C}[G])$ such that $(\iota \otimes \Delta)(U) = U_{12} U_{13}$. Namely, the right comodule structure on $H_U$ is given by

$$\delta_U : H_U \rightarrow H_U \otimes \mathbb{C}[G], \quad \delta_U(\xi) = U(\xi \otimes 1).$$  

The tensor product $U \otimes V$ of two unitary representations (denoted also by $U \oplus V$ or $U \times V$) is defined by $U_{13} V_{23}$. We denote by $\text{Irr}(G)$ the set of the isomorphism classes of irreducible unitary representations of $G$. For every $s \in \text{Irr}(G)$ we fix a representative $U_s \in B(H_s) \otimes \mathbb{C}[G]$.

Denote by $h$ the Haar state on $\mathbb{C}[G]$ and by $L^2(G)$ the corresponding GNS-space. We view $\mathbb{C}[G]$ as a subalgebra of $B(L^2(G))$. The (reduced) $C^\ast$-algebra $C(G)$ of continuous functions on $G$ is defined as the norm closure of $\mathbb{C}[G]$.

Consider the $\ast$-algebra $\mathcal{U}(G) = \mathbb{C}[G]^\ast$ dual to the coalgebra $(\mathbb{C}[G], \Delta)$, with the involution $\omega^*(x) = \overline{\omega(S(x)^{\ast})}$. Every finite dimensional unitary representation $U$ of $G$ defines a $\ast$-representation

$$\pi_U : \mathcal{U}(G) \rightarrow B(H_U), \quad \pi_U(\omega) = (\iota \otimes \omega)(U).$$  

The representations $\pi_s$ defined by $U_s$, $s \in \text{Irr}(G)$, give rise to an isomorphism

$$\mathcal{U}(G) \cong \prod_{s \in \text{Irr}(G)} B(H_s),$$  

and in what follows we are not going to distinguish between these two algebras. Consider the subalgebra

$$c_v(\hat{G}) = \bigoplus_{s \in \text{Irr}(G)} B(H_s) \subset \mathcal{U}(G).$$
It coincides with the subalgebra of $C(G)^* \subset \mathcal{U}(G)$ spanned by the linear functionals $h(\cdot x)$ for $x \in \mathbb{C}[G]$.

Assume now that $A$ is a C*-algebra and $\alpha : A \to A \otimes C(G)$ is a *-homomorphism such that $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$. We say that $\alpha$ is a (right) action of $G$ on $A$, or that $A$ is a $G$-C*-algebra, if either of the following equivalent conditions is satisfied:

1. the linear space $\alpha(A)(1 \otimes C(G))$ is a dense subspace of $A \otimes C(G)$ (the Podleś condition);
2. there is a dense *-subalgebra $\mathcal{A} \subset A$ on which $\alpha$ defines a coaction of the Hopf algebra $(\mathbb{C}[G], \Delta)$, that is, $\alpha(A) \subset \mathcal{A} \otimes_{\text{alg}} \mathbb{C}[G]$ and $(\iota \otimes \varepsilon)\alpha(a) = a$ for all $a \in \mathcal{A}$,

where $\otimes_{\text{alg}}$ denotes the purely algebraic tensor product. Then the largest subalgebra as in (ii) is given by

$$\mathcal{A} = \text{span}\{(\iota \otimes h)(\alpha(a)(1 \otimes x)) : a \in A, \ x \in \mathbb{C}[G]\},$$

denoted by $\mathcal{A} = c_c(\hat{G}) \triangleright A$.

An action is called reduced if $\alpha$ is injective, or equivalently, by faithfulness of $h$ on $C(G)$, if the conditional expectation

$$E = (\iota \otimes h)\alpha : A \to A^G = \{a \in A : \alpha(a) = a \otimes 1\}$$

is faithful. For reduced actions the subalgebra of regular elements can also be described as

$$\mathcal{A} = \{a \in A : \alpha(a) \in A \otimes_{\text{alg}} \mathbb{C}[G]\}, \quad (2.1)$$

since if $\alpha(a) \in A \otimes_{\text{alg}} \mathbb{C}[G]$, then $\alpha(a - (\iota \otimes \varepsilon)\alpha(a)) = 0$.

Given any action $\alpha : A \to A \otimes C(G)$, we get a reduced action on $A_r = A / \ker \alpha$, since

$$\ker \alpha = \{a \in A : E(a^*a) = 0\}.$$

We say that $A_r$ is the reduced form of the $G$-C*-algebra $A$. Note that $A$ and $A_r$ have the same subalgebra $\mathcal{A}$ of regular elements.

If $G$ is coamenable, that is, the counit $\varepsilon$ on $\mathbb{C}[G] \subset C(G)$ is bounded, then $(\iota \otimes \varepsilon)\alpha = \iota$ on $A$, so all actions of $G$ are reduced.

**Lemma 2.1.** Assume $\alpha : A \to A \otimes C(G)$ is a reduced action and $X \subset A$ is a finite dimensional $G$-invariant subspace, so that $\alpha(X) \subset X \otimes C(G)$. Then all elements of $X$ are regular.

*Proof.* By assumption, $X$ is a $c_c(\hat{G})$-submodule of $A$. As $X$ is finite dimensional, the action of $B(H_s)$ must be zero for all but a finite number of $s \in \text{Irr}(G)$. In other words, there is a finite subset $F \subset \text{Irr}(G)$ such that for all $\phi \in A^*$ the space $(\phi \otimes \iota)\alpha(X) \subset C(G) \subset L^2(G)$ is orthogonal to the matrix coefficients of $U_s$ for all $s \notin F$. It follows that this space is contained in the space $\mathbb{C}[G]_F$ spanned by the matrix coefficients of representations $U_s, s \in F$. But then $\alpha(X) \subset A \otimes \mathbb{C}[G]_F$, so $X$ consists of regular elements by (2.1). \qed

Everything above of course also applies to the left actions $\alpha : A \to C(G) \otimes A$.

Assume now we are given two compact quantum groups $G$ and $\hat{G}$. They are said to be monoidally equivalent if their categories of finite dimensional unitary representations are equivalent as C*-tensor categories. By [Ulb88, BDRV06], every such equivalence, up to a natural isomorphism, is defined by a bi-Galois object/linking algebra. This is a unital C*-algebra $B = B(G, \hat{G})$ equipped with two commuting reduced actions

$$\delta : B \to C(G) \otimes B, \quad \hat{\delta} : B \to B \otimes C(\hat{G})$$
that are ergodic and free; the precise meaning of the last property will not be important to us. Then an equivalence $F: \text{Rep} \ G \to \text{Rep} \  \tilde{G}$ is defined by mapping a finite dimensional unitary right comodule $(H_U, \theta_U)$ for $(\mathbb{C}[G], \Delta)$ into the cotensor product

$$H_U \boxtimes_G B = \{ X \in H_U \otimes B : (\delta_U \otimes \iota)(X) = (\iota \otimes \delta)(X) \},$$

with the right $(\mathbb{C}[\tilde{G}], \Delta)$-comodule structure given by $\iota \otimes \tilde{\delta}$ and the scalar product

$$(X, Y) = Y^* X \in \tilde{G} B = \mathbb{C}1,$$

where we interpret $\zeta^* \xi$ for $\xi, \zeta \in H_U$ as $(\xi, \zeta)$. The tensor structure on $F$ is defined by

$$(H_U \boxtimes_G B) \otimes (H_V \boxtimes_G B) \to H_U \otimes H_V \boxtimes_G B, \quad X \otimes Y \mapsto X_{13} Y_{23}.$$

In a similar way one can define an equivalence between the categories of reduced $G$-$C^*$-algebras and $\tilde{G}$-$C^*$-algebras, but there is a small technical issue that has to be taken care of [DRVV10]. Let $B = B(G, \tilde{G}) \subset B$ be the subalgebra of regular elements with respect to the $G$-action, equivalently, with respect to the $\tilde{G}$-action. Given a reduced action $\alpha: A \to A \otimes C(G)$, consider the subalgebra $A \subset A$ of regular elements and define

$$A \boxtimes_G B = \{ X \in A \otimes_{\text{alg}} B : (\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X) \} \subset A \otimes B.$$

The right action of $\tilde{G}$ on $A \boxtimes_G B$ is again given by $\iota \otimes \tilde{\delta}$.

In some cases it is possible to use exactly the same definition of $A \boxtimes_G B$ as for finite dimensional comodules.

**Lemma 2.2.** Assume $\alpha: A \to A \otimes C(G)$ is a reduced action of a compact quantum group $G$ on a $C^*$-algebra $A$ and there is a net of completely bounded $G$-equivariant maps $\theta_i: A \to A$ of finite rank such that $\theta_i(a) \to a$ for all $a \in A$ and $\sup_i \| \theta_i \|_{cb} < \infty$. Then, for any compact quantum group $\tilde{G}$ monoidally equivalent to $G$, we have

$$A \boxtimes_G B(\psi, \tilde{G}) = \{ X \in A \otimes B(\psi, \tilde{G}) : (\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X) \}.$$

**Proof.** If $(\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X)$, then the elements $X_i = (\theta_i \otimes \iota)(X)$ have the same property and converge to $X$. Therefore it suffices to show that $X_i \in A \otimes_{\text{alg}} B(G, \tilde{G})$. Lemma 2.1 implies that the finite dimensional space $\theta_i(A)$ is contained in $A$, hence $X_i \in A \otimes_{\text{alg}} B(G, \tilde{G})$ and then, using again that $(\alpha \otimes \iota)(X_i) = (\iota \otimes \delta)(X_i)$, we must have $X_i \in A \otimes_{\text{alg}} B(G, \tilde{G})$. \qed

Consider a reduced action $\alpha: A \to A \otimes C(G)$ and a unitary, not necessarily finite dimensional, representation $U \in M(\mathcal{K}(H) \otimes C(G))$ of $G$. Assume we are given a covariant representation $\pi: A \to B(H)$, meaning that

$$(\pi \otimes \iota)\alpha(a) = U(\pi(a) \otimes 1)U^*.$$

Given a monoidally equivalent compact quantum group $\tilde{G}$, we can extend the functor

$$F = \cdot \boxtimes_G B(G, \tilde{G}): \text{Rep} \ G \to \text{Rep} \  \tilde{G}$$

to all unitary representations, since they decompose into irreducible ones. Therefore we get a Hilbert space $H \boxtimes_G B(G, \tilde{G})$. By taking any state $\psi$ on $B(G, \tilde{G})$ and considering the associated GNS-representation $\pi_\psi: B(G, \tilde{G}) \to B(H_\psi)$, we can view $H \boxtimes_G B(G, \tilde{G})$ as a subspace of $H \otimes H_\psi$. It follows that $\pi \otimes \pi_\psi$ defines by restriction a representation $\pi \boxtimes \iota$ of $A \boxtimes_G B(G, \tilde{G})$ on $H \boxtimes_G B(G, \tilde{G})$.

**Lemma 2.3.** If the representation $\pi$ is faithful, then $\pi \boxtimes \iota$ is faithful as well.

**Proof.** Since the representation $\pi \boxtimes \iota$ is covariant and the $\tilde{G}$-action on $A \boxtimes_G B(G, \tilde{G})$ is reduced, it suffices to check that the representation is faithful on

$$(A \boxtimes_G B(G, \tilde{G})) \tilde{G} = A^G \otimes 1.$$
Decompose $H$ into isotypical components, $H \cong \bigoplus_{s \in \text{Irr}(G)} L_s \otimes H_s$, where $L_s$ are some Hilbert spaces. Then the representation $\pi$ restricted to $A^G$ has the form $\pi(a) = (\theta_s(a) \otimes 1)_{s \in \text{Irr}(G)}$ for some representations $\theta_s: A^G \to B(L_s)$. But then

$$H \Box_G B(G, \tilde{G}) \cong \bigoplus_{s \in \text{Irr}(G)} L_s \otimes (H_s \Box_G B(G, \tilde{G})),$$

and the representation $\pi \Box \iota$ on $A^G \otimes 1$ is simply

$$(\pi \Box \iota)(a \otimes 1) = (\theta_s(a) \otimes 1 \otimes 1)_{s \in \text{Irr}(G)},$$

which is obviously faithful. \hfill $\square$

As we already mentioned, for the coamenable compact quantum groups all actions are reduced. As soon as $G$ is noncoamenable, there are nonreduced actions, for example, the action by right translations on the universal completion $C_u(G)$ of $\mathbb{C}[G]$. An interesting question is whether a quotient of a reduced action is reduced. We do not know an answer, but here is a partial result.

**Lemma 2.4.** Assume $\alpha: A \to A \otimes C(G)$ is a reduced action of a compact quantum group $G$ on a $C^*$-algebra $A$, $I \subset A$ is a closed $G$-invariant ideal. Assume also that the following conditions are satisfied:

(i) $G$ is monoidally equivalent to a coamenable compact quantum group;

(ii) there is a net of completely bounded $G$-equivariant maps $\theta_i: I \to I$ of finite rank such that $\theta_i(a) \to a$ for all $a \in I$ and $\sup_i ||\theta_i||_{cb} < \infty$.

Then the action of $G$ on $A$ defines reduced actions of $G$ on $I$ and $A/I$.

For the proof we need the observation from [BDRV06] that if $\tilde{G}$ is coamenable, then $B(G, \tilde{G})$ is a nuclear $C^*$-algebra. This follows from a general result in [DLRZ02] stating that for coamenable $\tilde{G}$ a $G$-equivariant $C^*$-algebra is nuclear if and only if the fixed point algebra is nuclear.

**Proof of Lemma 2.4.** If $A \subset A$ is the subalgebra of regular elements, then $\theta_i(I) \subset A$ by Lemma 2.1. As $\theta_i(a) \to a$ for all $a \in I$, it follows that $I = I \cap A$ is dense in $I$. Since $\alpha|_I$ is a coaction of $(\mathbb{C}[G], \Delta)$ on $I$, we conclude that $\alpha|_I$ is an action of $G$ on $I$. This action is obviously reduced.

Now, consider a coamenable compact quantum group $\tilde{G}$ monoidally equivalent to $G$. Let us fix an equivalence of the representation categories of $\tilde{G}$ and $\tilde{G}$ and consider the corresponding bi-Galois objects $B(G, \tilde{G})$ and $B(\tilde{G}, G)$. We then have bi-equivariant isomorphisms

$$B(G, \tilde{G}) \Box_{\tilde{G}} B(\tilde{G}, G) \cong C(G), \quad B(\tilde{G}, G) \Box_G B(G, \tilde{G}) \cong C(\tilde{G}),$$

since the bi-Galois object corresponding to the identity functor on $\text{Rep} G$ is $C(G)$. Then, modulo these isomorphisms, we have a natural isomorphism $A \cong A \Box_{\tilde{G}} B(G, \tilde{G}) \Box_{\tilde{G}} B(\tilde{G}, G)$ for any reduced action $\alpha: A \to A \otimes C(G)$, defined by

$$A \to A \Box_{\tilde{G}} C(G), \quad a \mapsto \alpha(a).$$

Put

$$\tilde{I} = I \Box_{\tilde{G}} B(G, \tilde{G}), \quad \tilde{A} = A \Box_{\tilde{G}} B(G, \tilde{G}), \quad \tilde{C} = \tilde{A}/\tilde{I}.$$

Denote by $\tilde{\alpha}$ the action of $\tilde{G}$ on $\tilde{A}$. The action of $\tilde{G}$ on $\tilde{C}$ is reduced by coamenability. As $B(\tilde{G}, G)$ is a nuclear $C^*$-algebra, we have a short exact sequence

$$0 \to \tilde{I} \otimes B(\tilde{G}, G) \to \tilde{A} \otimes B(\tilde{G}, G) \to \tilde{C} \otimes B(\tilde{G}, G) \to 0.$$

By identifying $A$ with $\tilde{A} \Box_{\tilde{G}} B(\tilde{G}, G)$ as explained above and denoting $\tilde{C} \Box_{\tilde{G}} B(\tilde{G}, G)$ by $C$, we then get an exact sequence

$$0 \to (\tilde{I} \otimes B(\tilde{G}, G)) \cap (\tilde{A} \Box_{\tilde{G}} B(\tilde{G}, G)) \to A \to C \to 0.$$
We claim that $(\tilde{I} \boxtimes B(\tilde{G}, G)) \cap (\tilde{A} \boxtimes B(\tilde{G}, G)) = I$. It is clear that
\[
\tilde{I} \boxtimes B(\tilde{G}, G) \subset (\tilde{I} \boxtimes B(\tilde{G}, G)) \cap (\tilde{A} \boxtimes B(\tilde{G}, G)) \\
\subset \{ x \in \tilde{I} \otimes B(\tilde{G}, G) : (\tilde{\alpha} \otimes \iota)(x) = (\iota \otimes \tilde{\delta})(x) \},
\]
where $\tilde{\alpha}$ and $\tilde{\delta}$ denote the actions of $\tilde{G}$ on $\tilde{A}$ and $B(\tilde{G}, G)$. The last set coincides with $\tilde{I} \boxtimes B(\tilde{G}, G) = I$ by Lemma 2.2 applied to $\tilde{G}$ and the cb maps $\tilde{\theta}_i = \theta_i \otimes 1_{\tilde{I} \boxtimes B(\tilde{G}, G)}$ on $\tilde{I}$. This proves our claim.

Therefore we have a $G$-equivariant short exact sequence $0 \to I \to A \to C \to 0$ where all actions are reduced, which means that the action of $G$ on $A/I$ is reduced. \qed

Remark 2.5. What we tacitly used in the above argument is that if $A \to C$ is a surjective homomorphism of $G$-$C^*$-algebras, then we get a surjective map $A \to C$ between the subalgebras of regular elements, and then, for any compact quantum group $\hat{G}$ monoidally equivalent to $G$, the map $A \boxtimes B(G, \hat{G}) \to C \boxtimes B(\hat{G}, G)$ is surjective as well. This implies that if $0 \to I \to A \to C \to 0$ is a short exact sequence of reduced $G$-$C^*$-algebras, then the sequence of $\hat{G}$-$C^*$-algebras
\[
0 \to I \boxtimes B(G, \hat{G}) \to A \boxtimes B(G, \hat{G}) \to C \boxtimes B(G, \hat{G}) \to 0
\]
might not be exact in the middle, but at least the reduced form of $(A \boxtimes B(G, \hat{G}))/ (I \boxtimes B(G, \hat{G}))$ is $C \boxtimes B(G, \hat{G})$. \hfill \Diamond

Let us finally turn to equivariant subproduct systems. Assume $G$ is a compact quantum group and $\mathcal{H} = (H_n)_{n=0}^\infty$ is a subproduct system of finite dimensional $G$-modules. By this we mean that we are given unitary representations $U_n$ of $G$ on $H_n$ and the structure maps $H_{k+l} \to H_k \otimes H_l$ are $G$-intertwiners.

In this case we have a unitary representation
\[
U_\mathcal{H} = \bigoplus_{n=0}^\infty U_n
\]
of $G$ on $F_\mathcal{H}$. It defines reduced right actions of $G$ on $\mathcal{K}(\mathcal{H})$ and $T_\mathcal{H}$ by
\[
\alpha(T) = U_\mathcal{H}(T \otimes 1) U_\mathcal{H}^*.
\]
To see that we indeed get an action on the Toeplitz algebra, fix an orthonormal basis in $H_1$ and let $u_{ij}$ be the matrix coefficients of $U_1$ in this basis. Then
\[
\alpha(S_i) = \sum_j S_i \otimes u_{ij}.
\]
This implies that $\alpha$ defines a coaction of $(\mathbb{C}[G], \Delta)$ on the unital $*$-algebra generated by the operators $S_i$, hence it defines an action of $G$ on $T_\mathcal{H}$.

We then get an action of $G$ on $O_\mathcal{H}$. It is not clear to us whether this action is always reduced, but by Lemma 2.4 we at least have the following.

Proposition 2.6. If $\mathcal{H} = (H_n)_{n=0}^\infty$ is a subproduct system of finite dimensional $G$-modules and $G$ is monoidally equivalent to a coamenable compact quantum group, then the action of $G$ on $O_\mathcal{H}$ is reduced.

Assume now that $\hat{G}$ is a compact quantum group monoidally equivalent to $G$ and fix a bi-Galois object $B = B(G, \hat{G})$. Given a subproduct system $\mathcal{H} = (H_n)_{n=0}^\infty$ of finite dimensional $G$-modules, we get a subproduct system $\bar{\mathcal{H}} = (H_n \boxtimes_G B)_{n=0}^\infty$ of finite dimensional $\hat{G}$-modules. Note that even if $\mathcal{H}$ is standard, $\bar{\mathcal{H}}$ is usually not.

By definition we have a canonical unitary isomorphism $F_{\mathcal{H}} \cong F_{\bar{\mathcal{H}}} \boxtimes_G B$. It follows that we get a representation of $T_{\mathcal{H}} \boxtimes_G B$ on $F_{\mathcal{H}}$. This representation is faithful by Lemma 2.3.
Corollary 2.8. \( z \) matrices diag(\( O \) quantum group \( U \)) \( \tau = 1 \), in which case it is now on consider only the polynomials generically rather small: if the numbers \( |u| \) are all different, it consists of the unitary diagonal matrices diag(\( z_1, \ldots, z_m \)) such that \( z_i z_{m-i+1} = 1 \).

Remark 3.1. The quantum group \( O^+ \) is a genuine group only when \( m = 2 \), \( |a_1| = |a_2| = 1 \) and \( \tau = 1 \), in which case it is \( SU(2) \). The classical part of \( O^+ \), that is, the stabilizer of \( P \) in \( U(m) \), is generically rather small: if the numbers \( |a_i| \) are all different, it consists of the unitary diagonal matrices diag(\( z_1, \ldots, z_m \)) such that \( z_i z_{m-i+1} = 1 \).

3. Toeplitz and Cuntz–Pimsner algebras of Temperley–Lieb subproduct systems with large symmetry

We now return to the subproduct systems defined by Temperley–Lieb polynomials and from now on consider only the polynomials \( P = \sum_{i=1}^m a_i X_i X_{m-i+1} \) such that
\[
a_i a_{m-i+1} = -\tau \in \{-1, 1\} \quad \text{for all} \quad 1 \leq i \leq m.
\]
A distinguishing property of these polynomials is that they have large quantum symmetries, the free orthogonal quantum groups \([VDW96]\). Namely, for every \( P \) as above consider the free orthogonal quantum group \( O^+_P = O^+_{F_P} \) defined by the matrix
\[
F_P = \begin{pmatrix}
0 & a_m \\
. & . \\
1 & 0
\end{pmatrix}
\]
(3.1)
The algebra \( C(O^+_P) \) of regular functions on \( O^+_P \) is a universal unital \( * \)-algebra with generators \( u_{ij} \), \( 1 \leq i, j \leq m \), and relations
\[
u_{ij}^* = a_i^{-1} a_j u_{m-j+1, m-i+1}, \quad \text{the matrix} \quad (u_{ij})_{i,j} \quad \text{is unitary}.
\]
The comultiplication is given by \( \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \).

Consider the standard subproduct system \( \mathcal{H}_P = (H_n)_n \) defined by \( P \). By definition, the quantum group \( O^+_P \) has a unitary representation \( U_1 = (u_{ij})_{i,j} \) on \( H_1 \) and the representation \( U_1 \otimes U_1 \) leaves \( P \in H_1 \otimes H_1 \) invariant. Therefore we get unitary representations \( U_n \) of \( O^+_P \) on \( H_n = f_n H_n^{\otimes n} \). The spin \( n \) representations \( U_n \), \( n \geq 0 \), exhaust all irreducible representations of \( O^+_P \) up to equivalence, see \([Ban96]\) or \([NT13]\) Section 2.5.

Remark 3.1. The quantum group \( O^+_P \) is a genuine group only when \( m = 2 \), \( |a_1| = |a_2| = 1 \) and \( \tau = 1 \), in which case it is \( SU(2) \). The classical part of \( O^+_P \), that is, the stabilizer of \( P \) in \( U(m) \), is generically rather small: if the numbers \( |a_i| \) are all different, it consists of the unitary diagonal matrices diag(\( z_1, \ldots, z_m \)) such that \( z_i z_{m-i+1} = 1 \).
Let us try to understand the Cuntz–Pimsner algebra $O_P$. In the simplest case $P = X_1X_2 - X_2X_1$ the subproduct system $\mathcal{H}_P$ is Arveson’s symmetric subproduct system SSP$_2$ and $O_P$ is isomorphic to $C(S^3) \cong C(SU(2))$ by [Arv98] Theorem 5.7.

Next, take $q \in (0, 1]$, $\tau = \pm 1$ and consider the polynomial $P = q^{-1/2}X_1X_2 - \tau q^{1/2}X_2X_1$. From Proposition 1.7 we see that the following relations are satisfied in $O_P$:

$$S_1S_1^* + S_2S_2^* = 1, \quad q^{-1/2}S_1S_2 - \tau q^{1/2}S_2S_1 = 0,$$

$$S_1S_1^* + S_2S_2^* = 1, \quad S_2^*S_2 + q^2S_1S_1^* = 1, \quad S_1S_2 - \tau qS_2S_1^* = 0.$$

These are the relations in $C[SU_\tau(2)] = C[O_P^+]$ for the generators $u_{21}$ and $u_{22}$, which are usually denoted by $\gamma$ and $\alpha^*$, resp. In view of (2.2) and coamenability of $SU_\tau(2)$ we conclude that there is a well-defined $SU_\tau(2)$-equivariant surjective $*$-homomorphism $C(SU_\tau(2)) \to O_P$, $\gamma \mapsto S_1$, $\alpha \mapsto S_2$. This homomorphism must be injective, since the action of $SU_\tau(2)$ on $C(SU_\tau(2))$ defined by $\Delta$ is reduced and ergodic. Thus, $O_P \cong C(SU_\tau(2))$, which is a quantum analogue of Arveson’s result for $q = \tau = 1$.

Consider now the general case $P = \sum_{i=1}^m a_iX_iX_{m-i+1}$. Let $q \in (0, 1]$ be such that

$$\sum_{i=1}^m |a_i|^2 = q + q^{-1}. \quad (3.2)$$

Then there is a unique up to a natural isomorphism monoidal equivalence between $O_P^+$ and $SU_\tau(2)$, see [BDRV06] or again [NT13] Section 2.5. Such an equivalence $F$ maps the spin $n$ representations of $O_P^+$ into spin $n/2$ representations of $SU_\tau(2)$. In both representation categories there is a unique up to a scalar factor morphism $H_{k+1} \to H_k \otimes H_l$. By Remark 1.1 it follows that $F$ maps $\mathcal{H}_P$ into an $SU_\tau(2)$-equivariant subproduct system isomorphic to $\mathcal{H}_{q^{-1/2}X_1X_2-\tau q^{1/2}X_2X_1}$. By Proposition 2.6 and Corollary 2.8 we conclude that $O_P \cong C(SU_\tau(2)) \boxtimes_{SU_\tau(2)} B(SU_\tau(2), O_P^+) \cong B(SU_\tau(2), O_P^+)$. Once we know that such an isomorphism is to be expected, it is not difficult to construct it from scratch using known generators and relations of $B(SU_\tau(2), O_P^+)$. Namely, consider the matrices $F_P$, given by (3.1), and

$$F_{q,\tau} = \begin{pmatrix} 0 & -\tau q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}. \quad \text{Then by [BDRV06] Theorem 5.5 and Remark 5.7, } B(SU_\tau(2), O_P^+) \text{ is a universal unital C*-algebra with generators } y_{ij}, 1 \leq i \leq 2, 1 \leq j \leq m, \text{ and relations } \quad Y = F_{q,\tau}^*YF_{q,\tau}^{-1}, \quad Y \text{ is unitary,} \quad \delta : B(SU_\tau(2), O_P^+) \to B(SU_\tau(2), O_P^+) \otimes C(O_P^+) \text{ is given by } \delta(y_{ij}) = \sum_{k=1}^m y_{ik} \otimes u_{kj} \text{ and the left action of } SU_\tau(2) \text{ is defined in a similar way.}$$

Consider the elements $y_j = y_{2j}$. The relation $Y = F_{q,\tau}^*YF_{q,\tau}^{-1}$ means that

$$y_{ij} = q^{1/2}a_jy_{m-j+1},$$

and then a simple computation shows that unitarity of $Y$ is equivalent to the relations

$$\sum_{i=1}^m y_iy_i^* = 1, \quad \sum_{i=1}^m a_iy_iy_{m-i+1} = 0,$$

$$y_i^*y_j + qa_i\bar{a}_jy_{m-i+1}y_{m-j+1} = \delta_{ij}1 \quad (1 \leq i, j \leq m).$$
These are exactly the relations in $\mathcal{O}_P$ that we get from Proposition 1.7. We thus arrive at the following result.

**Proposition 3.2.** Assume $P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all $i$ ($m \geq 2$), and let $q \in (0, 1]$ be given by (3.2). Then we have an $O_P^+$-equivariant isomorphism $B(SU_{\tau q}(2), O_P^+) \cong \mathcal{O}_P$, $y_i \mapsto S_i$ ($1 \leq i \leq m$).

**Proof.** By the above discussion, the homomorphism $B(SU_{\tau q}(2), O_P^+) \rightarrow \mathcal{O}_P$, $y_i \mapsto S_i$, is well-defined, surjective and $O_P^+$-equivariant. It must be injective, since the action of $O_P^+$ on the linking algebra $B(SU_{\tau q}(2), O_P^+)$ is reduced and ergodic. □

Since $B(SU_{\tau q}(2), O_P^+)$ is nuclear by coamenability of $SU_{\tau q}(2)$, we get the following corollary.

**Corollary 3.3.** The $C^*$-algebras $\tilde{T}_P$ and $\mathcal{O}_P$ are nuclear.

We are now ready to describe relations in $\tilde{T}_P$. Recall from Section 4 that we can view the $C^*$-algebra $c = C(\mathbb{Z}_+ \cup \{+\infty\})$ of converging sequences as a subalgebra of $\tilde{T}_P$ and we denote by $\gamma$ the shift to the left on $c$.

**Theorem 3.4.** Assume $P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all $i$ ($m \geq 2$), and let $q \in (0, 1]$ be given by (3.2). Then $\tilde{T}_P$ is a universal $C^*$-algebra generated by $\tilde{S}_1, \ldots, \tilde{S}_m$ and $c$ satisfying the relations

$$fS_i = S_i\gamma(f) \quad (f \in c, \ 1 \leq i \leq m),$$

$$\sum_{i=1}^{m} S_iS_i^* = 1 - e_0, \quad \sum_{i=1}^{m} a_i S_i S_{m-i+1} = 0,$$

$$S_i^* S_j + a_i \bar{a}_j \phi S_{m-i+1} S_{m-j+1} = \delta_{ij} 1 \quad (1 \leq i, j \leq m),$$

where the element $\phi \in c$ is defined by $\phi(n) = \frac{[n]_q}{[n+1]_q}$.

**Proof.** By Proposition 1.7 we already know that the above relations are satisfied in $\tilde{T}_P$. Consider a universal $C^*$-algebra $\tilde{T}_P$ generated $\tilde{S}_1, \ldots, \tilde{S}_m$ and $c$ satisfying these relations. Let $\pi: \tilde{T}_P \rightarrow T_P$ be the quotient map.

Every element of the $*$-algebra generated by $\tilde{S}_1, \ldots, \tilde{S}_m$ and $c$ can be written as a linear combination of elements of the form $f\tilde{S}_{i_1} \ldots \tilde{S}_{i_k}\gamma^{l} \ldots \tilde{S}_{j}\gamma^{l'}$, where $f \in c$ and $k, l \geq 0$. As $e_0\tilde{S}_i = 0$, it follows $e_0\tilde{f}\tilde{T}_P e_0 = \mathbb{C} e_0$, that is, $e_0$ is a minimal projection in $\tilde{T}_P$. Hence the closed ideal $I = \langle e_0 \rangle$ generated by $e_0$ is isomorphic to the algebra of compact operators on some Hilbert space.

Next, since

$$\sum_{i=1}^{m} \tilde{S}_i e_n \tilde{S}_i^* = e_{n+1}, \quad \sum_{i=1}^{m} \tilde{S}_i \tilde{S}_i^* = e_{n+1}(1 - e_0) = e_{n+1},$$

by induction on $n$ we conclude that $e_n \in I$ for all $n \geq 0$, that is, $e_0 \subset I$. It follows that the images of $\tilde{S}_i$ in $\tilde{T}_P/I$ satisfy the defining relations of $B(SU_{\tau q}(2), O_P^+) \cong \mathcal{O}_P$. On the other hand, by construction, $\mathcal{O}_P$ is a quotient of $\tilde{T}_P/I$. It follows that $\pi: \tilde{T}_P \rightarrow \tilde{T}_P$ gives rise to an isomorphism $\tilde{T}_P/I \cong \mathcal{O}_P$.

Therefore we have a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow I \longrightarrow \tilde{T}_P \longrightarrow \mathcal{O}_P \longrightarrow 0 \\
\pi|_I \downarrow \quad \pi \downarrow \quad \text{id} \downarrow \\
0 \longrightarrow K(F_P) \longrightarrow T_P \longrightarrow \mathcal{O}_P \longrightarrow 0
\end{array}
$$

with exact rows and surjective $\pi$. As $I$ is a simple $C^*$-algebra, the map $\pi|_I$ must be an isomorphism, hence $\pi: \tilde{T}_P \rightarrow T_P$ is an isomorphism as well. □
Remark 3.5. Since $\mathcal{T}_P$ is generated by $S_1, \ldots, S_m$, the relations in $\mathcal{T}_P$ can be written without using the subalgebra $c$. Namely, instead of the first two relations we could require the elements $p_n$ defined by

$$p_n = \sum_{i_1, \ldots, i_n=1}^m S_{i_1} \cdots S_{i_n} S_{i_1}^* \cdots S_{i_n}^*, \quad n \geq 1,$$

to be projections, and then define the element $\phi$ for the last relation by

$$\phi = qp_1 + \sum_{n=1}^\infty (\phi(n) - q)(p_n - p_{n+1}),$$

which makes sense, as $p_1 \geq p_2 \geq \ldots$ and therefore the projections $p_n - p_{n+1}$ are mutually orthogonal.

Indeed, then the projections $p_n$ together with the unit generate a copy of $c$, with $e_n = p_n - p_{n+1}$ for $n \geq 0$, where $p_0 = 1$. As $p_{n+1} = \sum_i S_i p_n S_i^*$, we have

$$(1 - p_{n+1})S_i p_n S_i^* (1 - p_{n+1}) \leq \sum_j (1 - p_{n+1}) S_j p_n S_j^* (1 - p_{n+1}) = (1 - p_{n+1}) p_{n+1} (1 - p_{n+1}) = 0,$$

so that $(1 - p_{n+1}) S_i p_n = 0$. A similar computation gives $p_{n+1} S_i (1 - p_n) = 0$. It follows that $p_{n+1} S_i = S_i p_n$ for all $n \geq 0$. This is equivalent to our first relation $f S_i = S_i \gamma(f)$.

4. GAUGE ACTION AND COMPACTIFICATIONS OF THE DUAL DISCRETE QUANTUM GROUPS

We continue to consider the Temperley–Lieb polynomials $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ such that $a_{i} \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$.

As the $O^+_P$-modules $H_n$ exhaust all irreducible representations of $O^+_P$ up to equivalence, the algebra of bounded functions on the dual discrete quantum group $\mathbb{F}O_P = \hat{O}^+_P$ is by definition

$$\ell^\infty(\mathbb{F}O_P) = \ell^\infty - \bigoplus_{n=0}^\infty B(H_n) \subset B(\mathcal{F}P).$$

For $x \in \ell^\infty(\mathbb{F}O_P)$, we denote by $x_n$ its component in $B(H_n)$.

Consider the unitary representation $z \mapsto V_z$ of $\mathbb{T}$ on $\mathcal{F}P$, where $V_z$ is the unitary that acts on $H_n$ by the scalar $z^n$. Then we get an action $\text{Ad} V$ of $\mathbb{T}$ on $B(\mathcal{F}P)$, and $\ell^\infty(\mathbb{F}O_P)$ coincides with the fixed point subalgebra of $B(\mathcal{F}P)$ with respect to this action. The automorphisms $\text{Ad} V_z$ leave $\mathcal{T}_P$ globally invariant and define the gauge action $\sigma$ of $\mathbb{T}$ on $\mathcal{T}_P$, given by

$$\sigma_z(S_i) = z S_i.$$

Note that on $O_P \cong B(SU_{\tau q}(2), O^+_P)$ the gauge action coincides with the action of the maximal torus $\mathbb{T} \subset SU_{\tau q}(2)$.

Consider the gauge-invariant subalgebra

$$\mathcal{T}^{(0)}_P := \mathcal{T}^\sigma_P = \mathcal{T}_P \cap \ell^\infty(\mathbb{F}O_P)$$

of $\mathcal{T}_P$. Since it is unital and contains

$$c_0(\mathbb{F}O_P) := c_0 - \bigoplus_{n=0}^\infty B(H_n) = K(\mathcal{F}P)^\sigma,$$

it can be thought of as an algebra of continuous functions on a compactification of $\mathbb{F}O_P$.

Assume $\sum_i |a_i|^2 > 2$. Following [VV07], consider the inductive system of $O^+_P$-equivariant ucp maps

$$\psi_{n,n+k} : B(H_n) \to B(H_{n+k}), \quad T \mapsto f_{n+k}(T \otimes 1)f_{n+k},$$

where
where we use that $H_{n+k} \subset H_n \otimes H_k$, and define

\[
C(\mathcal{F}O_P) = \{ x \in \ell^\infty(\mathcal{F}O_P) \mid \psi_{n+k}(x_n) = x_{n+k} \text{ for all } n \text{ large enough and } k \geq 0 \},
\]

\[
= \{ x \in \ell^\infty(\mathcal{F}O_P) \mid \lim_{n \to +\infty} \sup_{k \geq 0} \| \psi_{n+k}(x_n) - x_{n+k} \| = 0 \}.
\]

Clearly, $c_0(\mathcal{F}O_P) \subset C(\mathcal{F}O_P)$. It is shown in [VV07] that $C(\mathcal{F}O_P)$ is a C*-algebra. This construction is a quantum analogue of the end compactification of a free group. Our goal in this section is to prove the following.

**Theorem 4.1.** Assume $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1,1\}$ for all $i$, and $\sum_{i=1}^m |a_i|^2 > 2$. Then $T_P^{(0)} = C(\mathcal{F}O_P)$.

In the proof we are not going to use that $C(\mathcal{F}O_P)$ is an algebra, so as a byproduct we will reprove that this is indeed the case.

Let $q \in (0,1)$ be given by (3.2). The proof of the theorem relies on the following key known estimate.

**Lemma 4.2.** There is a constant $C > 0$ depending only on $q$ such that

\[
\| f_{n+1} - (1 \otimes f_n)(f_n \otimes 1) \| \leq C q^n \quad \text{for all } n \geq 0.
\]

A stronger result is proved in [VV07] Lemma 8.4 using a generalization of Wenzl’s recursive formula for $f_n$ established in [FK97]. Let us show that the particular case we need is a consequence of basic properties of the representation theory of $SU_q(2)$.

**Proof of Lemma 4.2.** It suffices to prove the estimate in the category $\text{Rep} SU_q(2)$ equivalent to $\mathcal{F}O_P$. Furthermore, since $\text{Rep} SU_q(2)$ can be obtained from $\text{Rep} SU_q(2)$ by introducing new associativity morphisms such that $(H_1 \otimes H_1) \otimes H_1 \to H_1 \otimes (H_1 \otimes H_1)$ is the multiplication by $-1$, it is enough to consider $\text{Rep} SU_q(2)$. In other words, we may assume that $P = q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1$.

As $H_{n+1} \subset (H_1 \otimes H_n) \cap (H_n \otimes H_1)$, we just need to show that the restriction of $1 \otimes f_n$ to $(H_n \otimes H_1) \otimes H_{n+1}$ has norm $\leq C q^n$. Since the $SU_q(2)$-module $(H_n \otimes H_1) \otimes H_{n+1} \cong H_{n-1}$ is irreducible, this norm is equal to $\| (1 \otimes f_n) \xi \|$ for any unit vector $\xi$ in this module. As the vector $\xi$ we take a highest weight vector. A formula for this vector is not difficult to find, up to a scalar factor it is

\[
[n]_q^{1/2} X_1^n X_2 - q^{(n+1)/2} \zeta_n X_1,
\]

where $\zeta_n$ is a unit vector of weight $n/2 - 1$ in $H_n$, see the first paragraph of [NT13] Section 2.5]. Therefore we see that there is a unit vector $\xi_n \in (H_n \otimes H_1) \otimes H_{n+1}$ such that

\[
\| \xi_n - X_1^n X_2 \| \leq a q^n
\]

for a constant $a$ depending only on $q$. But then

\[
\| (1 \otimes f_n) \xi_n \| \leq a q^n + \| X_1 f_n(X_1^{n-1} X_2) \| \leq a q^n + a q^{n-1} + \| f_n \xi_{n-1} \| = a q^n + a q^{n-1},
\]

which proves the lemma. \qed

Similarly to the operators $S_i$ defined using the multiplication on the left, we can use the multiplication on the right and define

\[
R_i : \mathcal{F}P \to \mathcal{F}P, \quad R_i \xi = f_{n+1}(\xi X_i) \quad \text{for } \xi \in H_n.
\]

**Lemma 4.3.** For all $1 \leq i, j \leq m$, we have $[S_i, R_j] = 0$. There is a constant $C$ depending only on $q$ such that

\[
\|[S_i, R_j]|_{H_n} \| \leq C q^n \quad \text{for all } n \geq 0.
\]
Proof. For $\xi \in H_n$, we have $S_i R_j \xi = f_{n+2}(X_i \xi X_j) = R_j S_i \xi$, which proves the first claim. To prove the second claim, take a unit vector $\xi \in H_n$ for some $n \geq 1$. Then

$$S_i^* R_j \xi = S_i^* f_{n+1}(\xi X_j) = T_i^* f_{n+1}(\xi X_j),$$

where we used (1.1). By Lemma 4.2 the last expression is $Cq^n$-close to

$$T_i^*(1 \otimes f_n)(\xi X_j) = f_n(T_i^*(\xi)X_j) = R_j S_i^* \xi,$$

which gives the required estimate. \hfill $\square$

**Corollary 4.4.** For every $S \in T_P$ and every $R$ in the $C^*$-algebra $C^*(R_1, \ldots, R_m)$, we have $[S, R] \in K(T_P)$.

**Remark 4.5.** The last corollary is true for $q = 1$ as well, since in this case the proof of Lemma 4.2 gives $\|f_{n+1} - (1 \otimes f_n)(f_n \otimes 1)\| \leq Cn^{-1/2}$. \hfill $\diamond$

Similarly to (1.2), we have $\sum_{i=1}^m R_i R_i^* = 1 - e_0$. Define a contractive cp map

$$\Theta : B(F_P) \to B(F_P), \quad \Theta(T) = \sum_{i=1}^m R_i TR_i^*. $$

Since $K(F_P) \subset T_P$, by Corollary 4.3 this map leaves $T_P$ globally invariant. It also leaves $\ell^\infty(\mathcal{F}O_P)$ globally invariant and we have

$$\Theta^k(x)_{n+k} = \psi_{n,n+k}(x_n) \quad \text{for all} \quad x \in \ell^\infty(\mathcal{F}O_P), \quad n, k \geq 0. \tag{4.1}$$

Denote by $A_P$ the unital $*$-subalgebra of $T_P$ generated by the elements $S_i$, $1 \leq i \leq m$. Let $A_P^{(0)}$ be the gauge-invariant part of $A_P$.

**Lemma 4.6.** For every $x \in A_P^{(0)}$, there exists a constant $C > 0$ such that

$$\|x_{n+k} - \Theta^k(x)_{n+k}\| \leq Cq^n \quad \text{for all} \quad n, k \geq 0. $$

**Proof.** By Lemma 4.5 we can find a constant $\tilde{C}$ depending on $x$ such that

$$\|x_n - \Theta(x)_{n}\| \leq \tilde{C}q^n \quad \text{for all} \quad n \geq 0.$$

As $\Theta$ is a contraction mapping $B(H_{n+k-l})$ into $B(H_{n+k})$, it follows that

$$\|\Theta^l(x)_{n+k} - \Theta^{l+1}(x)_{n+k}\| \leq \tilde{C}q^{n+k-l} \quad \text{for all} \quad n \geq 0, \quad k \geq l \geq 0.$$

Summing up over $l = 0, \ldots, k - 1$ we get the required estimate, with $C = \tilde{C}q(1 - q)^{-1}$. \hfill $\square$

**Proof of Theorem 4.1.** The inclusion $A_P^{(0)} \subset C(\mathcal{F}O_P)$ follows from (4.1) and Lemma 4.6. Hence $T_P^{(0)} \subset C(\mathcal{F}O_P)$.

In order to prove the opposite inclusion, take any $x \in \ell^\infty(\mathcal{F}O_P)$ such that

$$x_{n_0+k} = \psi_{n_0,n_0+k}(x_{n_0})$$

for some $n_0$ and all $k \geq 0$. Since the maps $\psi_{n_0,n_0+k}$ are $O_P^{(0)}$-equivariant, we may assume that $x_{n_0}$ lies in a spin $l$ spectral component of $B(H_{n_0})$. Note that as $H_{n_0} \otimes H_{n_0} \cong H_0 \oplus H_2 \oplus \cdots \oplus H_{2n_0}$, $l$ must be an integer $\leq n_0$.

As $O_P^{(0)} \cong B(SU_q(2), O_P^+)$, the multiplicity of the spin $l$ component of $O_P^{(0)}$ is 1. Fix a nonzero, hence injective, $O_P^+$-equivariant map $H_l \to O_P^+$ and lift it to an $O_P^+$-equivariant map $\pi : H_l \to A_P^{(0)}$. By Lemma 4.6 applied to $\pi(\xi)$ for the elements $\xi$ of a basis of $H_l$, we can find $C > 0$ such that

$$\|\pi(\xi)_{n+k} - \Theta^k(\pi(\xi))_{n+k}\| \leq Cq^n \|\xi\| \quad \text{for all} \quad \xi \in H_l \quad \text{and} \quad n, k \geq 0. \tag{4.2}$$

Next, we claim that there exist $\alpha > 0$ and $n_1 \geq 0$ such that $\|\pi(\xi)_{n}\| \geq \alpha \|\xi\|$ for all $\xi \in H_l$ and $n \geq n_1$. If this is not true, then by compactness of the unit sphere of $H_l$ we can find a unit
vector $\xi \in H_t$ such that $\lim \inf_n \|\pi(\xi)_n\| = 0$. But then by (4.2) we must have $\lim_n \|\pi(\xi)_n\| = 0$, that is, the image of $\pi(\xi)$ in $O_P^{(0)}$ is zero, which is a contradiction. Thus, our claim is proved.

Now, fix $\varepsilon > 0$ and choose $n \geq \max\{n_0, n_1\}$ such that $Cq^n \|x\| < \varepsilon \alpha$. For the unique $\xi \in H_t$ such that $\pi(\xi)_n = x_n$ we have $\|\xi\| \leq \alpha^{-1} \|x\|$. For all $k \geq 0$ we have

$$x_{n+k} = \Theta^k(x)_{n+k} = \Theta^k(\pi(\xi))_{n+k}.$$

Hence, applying again (4.2), we get

$$\|x_{n+k} - \pi(\xi)_{n+k}\| \leq Cq^n \|\xi\| < \varepsilon.$$

Therefore, modulo the compacts, $x$ is $\varepsilon$-close to $A_P^{(0)}$. Hence $x \in T_P^{(0)}$. \hfill $\square$

5. K-theory

We continue to consider the same class of polynomials $P = \sum_{i=1}^m a_i x_i x_{m-i+1}$ as in the previous two sections.

Since $O_P \cong B(SU_q, O_P^+)$, by [DCY15 Example 7.5] we have

$$K_0(O_P) \cong \mathbb{Z}/(m-2)\mathbb{Z}, \quad K_1(O_P) = \begin{cases} \mathbb{Z}, & \text{if } m = 2, \\ 0, & \text{if } m \geq 3. \end{cases}$$

This already gives a lot of information about the $K$-theory of $T_P$ (for example, $K_1(T_P) = 0$ if $m \geq 3$), but is not quite enough to fully determine it. Our goal is to prove the following.

**Theorem 5.1.** Assume $P = \sum_{i=1}^m a_i x_i x_{m-i+1}$ is a polynomial such that $a_i a_{m-i+1} = -\tau \in \{-1, 1\}$ for all $i (m \geq 2)$. Then the embedding map $\mathbb{C} \to T_P$ is a $KK^{O_P^+}$-equivalence.

For the polynomials $P = \sum_i (-1)^i x_i x_{m-i+1}$ this upgrades the $KK^{SU_q(2)}$-equivalence of Arici–Kaad [AK20] to the equivariant $KK$-category of the whole quantum symmetry group.

Our proof relies on the strong Baum–Connes conjecture for the dual of $SU_q(2)$ established by Voigt [Voi11]. More precisely, we need the following standard consequence of this result.

**Proposition 5.2.** Assume $q \in [-1, 1] \setminus \{0\}$, and $A$ and $B$ are separable $SU_q(2)$-$C^*$-algebras. Then a class $x \in KK^{SU_q(2)}(A, B)$ is a $KK^{SU_q(2)}$-equivalence if and only if it defines a $KK$-equivalence between $A \rtimes SU_q(2)$ and $B \rtimes SU_q(2)$.

**Proof.** A class $x$ is a $KK^{SU_q(2)}$-equivalence if and only if it induces an isomorphism

$$KK^{SU_q(2)}(C, A) \cong KK^{SU_q(2)}(C, B)$$

for every separable $SU_q(2)$-$C^*$-algebra $C$. The fact that $SU_q(2)$ is torsion-free and satisfies the strong Baum–Connes conjecture implies that the localizing subcategory of the triangulated category $KK^{SU_q(2)}$ generated by the separable $C^*$-algebras with trivial $SU_q(2)$-action coincides with $KK^{SU_q(2)}$ [Voi11]. Since the functors $KK^{SU_q(2)}(\cdot, A)$ and $KK^{SU_q(2)}(\cdot, B)$ are cohomological and transform the $c_0$-direct sums into direct products [MN08], it follows that in order to check (5.1) it suffices to consider $C$ with trivial $SU_q(2)$-action. For such $C$ we have the Green–Julg isomorphisms

$$KK^{SU_q(2)}(C, A) \cong KK(C, A \rtimes SU_q(2)) \quad \text{and} \quad KK^{SU_q(2)}(C, B) \cong KK(C, B \rtimes SU_q(2)),$$

see [Ver02 Theorem 5.7]. Therefore $x$ is a $KK^{SU_q(2)}$-equivalence if and only if it induces an isomorphism $KK(C, A \rtimes SU_q(2)) \cong KK(C, B \rtimes SU_q(2))$ for every separable $C^*$-algebra $C$, that is, if and only if it defines a $KK$-equivalence between $A \rtimes SU_q(2)$ and $B \rtimes SU_q(2)$. \hfill $\square$

Combining this with the Universal Coefficient Theorem [RS87], we get the following.
Corollary 5.3. Assume $A$ and $B$ are separable $SU_q(2)$-$C^*$-algebras such that $A \rtimes SU_q(2)$ and $B \rtimes SU_q(2)$ satisfy the UCT. Then a class $x \in KK^{SU_q(2)}(A, B)$ is a $KK^{SU_q(2)}$-equivalence if and only if it induces an isomorphism $K_*(A \rtimes SU_q(2)) \cong K_*(B \rtimes SU_q(2))$.

Let us now fix our conventions and notation for the crossed products. Given a compact quantum group $G$, consider its right regular representation $V \in \mathcal{M}(K(L^2(G)) \otimes C(G))$, defined by

$$V(a \xi_h \otimes \xi) = \Delta(a)(\xi_h \otimes \xi), \quad a \in C(G), \quad \xi \in L^2(G),$$

where $\xi_h = 1 \in C(G) \subset L^2(G)$. Then $V(a \otimes 1)V^* = \Delta(a)$. The integrated form of $V$ is the representation

$$\rho = \pi_V: C^*(G) = c_0(\hat{G}) = c_0 - \bigoplus_{s \in \text{Irr}(G)} B(H_s) \to B(L^2(G)),$$

$$\rho(\omega) = (a \otimes \omega)(V) \quad \text{for} \quad \omega \in c_c(\hat{G}) \subset C^*(G).$$

Given an action $\alpha: A \to A \otimes C(G)$, the crossed product is defined by

$$A \rtimes G = \alpha(A)(1 \otimes \rho(C^*(G))) \subset M(A \otimes K(L^2(G))).$$

Proof of Theorem 5.1. Let $q \in (0, 1]$ be given by (8.2). By [Voi11, Theorem 8.5], the functor $A \mapsto A \rtimes SU_q(2)$ extends to an equivalence of the equivariant $KK$-categories. Therefore by Proposition 2.7 it suffices to prove the theorem for the polynomials

$$P = q^{-1/2}X_1X_2 - \tau q^{1/2}X_2X_1, \quad q \in (0, 1], \quad \tau = \pm 1.$$

Consider the short exact sequence $0 \to \mathcal{K}(F_P) \to T_P \to O_P \to 0$. Passing to crossed products we get a short exact sequence

$$0 \to \mathcal{K}(F_P) \rtimes SU_q(2) \to T_P \rtimes SU_q(2) \to O_P \rtimes SU_q(2) \to 0. \quad (5.2)$$

Since $O_P \cong C(SU_q(2))$ by Proposition 3.2, by the Takesaki–Takai duality we have $O_P \rtimes SU_q(2) \cong K(L^2(SU_q(2)))$. Since the action of $SU_q(2)$ on $\mathcal{K}(F_P)$ is implemented by the unitary representation $U_P = \bigoplus_{n=0}^\infty U_n$, we also have an isomorphism

$$\mathcal{K}(F_P) \rtimes SU_q(2) \cong \mathcal{K}(F_P) \otimes \rho(C^*(SU_q(2))), \quad X \mapsto U_P^* XU_P.$$

As $C^*(SU_q(2)) = c_0 - \bigoplus_{n=0}^\infty B(H_n)$, it follows that we can write (5.2) as

$$0 \to c_0 - \bigoplus_{n=0}^\infty \mathcal{K}(F_P \otimes H_n) \to T_P \rtimes SU_q(2) \to \mathcal{K}(L^2(SU_q(2))) \to 0. \quad (5.3)$$

In this picture the canonical homomorphisms of $C^*(SU_q(2))$ into $M(K(F_P) \rtimes SU_q(2))$ and $O_P \rtimes SU_q(2)$ are $(\pi_{U_P \otimes U_n})_{n=0}^\infty$ since $(U_P^*)_1 V_2 = (U_P^*)_2 V_1 = (U_P^*)_3 V_3$ and $\rho$, resp.

From (5.3) it is clear that $T_P \rtimes SU_q(2)$ is a type I $C^*$-algebra, hence it satisfies the UCT, and $K_1(T_P \rtimes SU_q(2)) = 0$. Since $C^*(SU_q(2))$ is also of type I with trivial $K_1$-group, by Corollary 5.3 we just need to show that the canonical embedding $\pi: C^*(SU_q(2)) \to T_P \rtimes SU_q(2)$, $x \mapsto 1 \otimes \rho(x)$, induces an isomorphism of the $K_0$-groups.

We identify $K_0(C^*(SU_q(2)))$ with the representation ring $R(SU_q(2)) = \bigoplus_{n=0}^\infty \mathbb{Z}[U_n]$; in other words, $[U_n] \in K_0(C^*(SU_q(2)))$ denotes the class of a rank one projection in $B(H_n)$. From (5.3) we see that $K_0(T_P \rtimes SU_q(2))$ is a free abelian group with generators $[p_n], \ n \in \mathbb{Z}_+ \cup \{\infty\}$, where $p_n$ is a rank one projection in $K(F_P \otimes H_n)$ for $n \in \mathbb{Z}_+$ and $p_\infty$ is a projection in $T_P \rtimes SU_q(2)$ such that its image in $K(L^2(SU_q(2)))$ is a rank one projection. As $p_\infty$, we can take the image of $1 \in B(H_0)$ under $\pi: C^*(SU_q(2)) \to T_P \rtimes SU_q(2)$, so that

$$\pi_*([U_0]) = [p_\infty].$$

Now, let us fix $n \geq 1$ and compute $\pi_*([U_n])$. We have

$$\pi_*([U_n]) = c[p_\infty] + \sum_{k=0}^\infty c_k[p_k] \quad (5.4)$$
for some $c, c_k \in \mathbb{Z}$, with only finitely many nonzero coefficients. The homomorphism $T_P \times SU_{\tau q}(2) \to K(L^2(SU_{\tau q}(2)))$ kills all the projections $p_n$, $n \geq 0$. On the other hand, its composition with $\pi$ is the right regular representation $\rho$. As the multiplicity of $U_n$ in $V$ is $\dim H_n = n+1$, we conclude that $c = n+1$.

For $k \geq 0$, consider the representation of $T_P \times SU_{\tau q}(2)$ on $F_P \otimes H_k$. Since the multiplicity of every isotopical component of $U_P \otimes U_k$ is finite, this is a representation by compact operators. Thus, we get a homomorphism $K_0(T_P \times SU_{\tau q}(2)) \to K_0(F_P \otimes H_k)$. Applying it to (5.4) we obtain

$$(\pi_{U_P \otimes U_k})_*([U_n]) = [n+1](\pi_{U_P \otimes U_k})_*([U_0]) + c_k[p_k] \quad \text{in} \quad K_0(F_P \otimes H_k).$$

Therefore, if we denote by $m_{lk}$ the multiplicity of $U_l$ in $U_P \otimes U_k$, then

$$c_k = m_{nk} - (n+1)m_{0k}.$$

Since $U_P = \bigoplus_{i=0}^{\infty} U_i$ and $U_l \otimes U_k \cong U_{|l-k|} \oplus U_{|l-k|+2} \oplus \cdots \oplus U_{l+k}$, we have

$$m_{lk} = \frac{1}{2}(|l+k| - |l-k|) + 1, \quad \text{hence} \quad c_k = \frac{1}{2}(k - n - |k - n|).$$

To summarize, for all $n \geq 0$ we have

$$\pi_*([U_n]) = (n+1)[p_\infty] + \sum_{k=0}^{n-1} (k - n)[p_k].$$

This shows that $\pi_*: R(SU_{\tau q}(2)) \to K_0(T_P \times SU_{\tau q}(2))$ is indeed an isomorphism. $\square$

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