The actually robust surface signature of a Hopf insulator: Bulk-to-boundary flow of Berry curvature beyond the anomaly inflow paradigm

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(Dated: October 25, 2019)

The Hopf insulator has been touted as the first-known 3D magnetic bulk-insulator with topological surface states, Here, we demonstrate that all surfaces of the Hopf insulator can be made insulating while preserving the bulk gap. Despite the absence of spectral flow, we propose that the sum of all surface states carry a nontrivial first Chern number, which is in one-to-one correspondence with the bulk homotopy invariant. This bulk-boundary correspondence stems from a novel mechanism for the bulk-to-boundary flow of Berry curvature, which is not a type of Callan-Harvey anomaly inflow. The topologically nontrivial surface states arise as non-chiral, Schrödinger-type modes on the domain wall of a generalized Weyl equation – describing a pair of opposite-chirality Weyl fermions acting as a dipolar source of Berry curvature. If the lattice regularization of the dipolar Weyl equation preserves discrete rotational symmetry, the lattice model manifests a generalized Thouless pump that translates charge by one lattice period over half an adiabatic cycle, but reverses the charge flow over the next half – its effect is to transfer angular momentum between surface modes localized to opposite surfaces of a finite slab, realizing a type of symmetry ‘teleportation’.

The Hopf insulator \[1\] is the paradigm for a topological insulator that is neither a stable topological insulator in the tenfold way, \[2, 3\] nor a fragile topological insulator \[4, 5\] in topological quantum chemistry. \[6, 7\] Having a \(k\)-dependent matrix Hamiltonian that is only a sum of Pauli matrices, the Hopf insulator is conceptually the simplest topological insulator. As originally touted by Moore, Ran and Wen, \[1\] the Hopf insulator is the first-known 3D magnetic bulk-insulator with topological surface states, as illustrated in Fig. 1(a). Contrary to some expectations, \[8–10\] we will demonstrate that all surfaces can be made insulating while preserving the bulk gap [cf. Fig. 1(b)]. What then is ‘topological’ about such surface states which can be spectrally disconnected from the bulk states?

We will show that the surface Chern number -- defined as the first Chern number of all surface-localized bands -- is in one-to-one correspondence with the integer-valued, bulk homotopy invariant \(\chi\). \(\chi\) classifies maps from the three-dimensional Brillouin zone (BZ) to the Bloch sphere of pseudospin-half wave functions, \[10, 11\] and is given by a gauge-invariant BZ-integral of the Abelian Chern-Simons three-form \[11, 12\]

\[
\chi = -\frac{1}{4\pi^2} \int_{BZ} F \cdot A d^3k, \tag{1}
\]

with \(F = \nabla \times A\) and \(A(k) = \langle u | i \nabla_k u \rangle\) the Berry connection of the energy-nondegenerate band. \[13\]

In the geometric theory of polarization, \[14, 15\] the charge polarization is proportional to \(\int_{BZ} A\): analogously, one may view \(\int_{BZ} A \cdot F\) as proportional to a real-space polarization of Berry curvature. \[16, 18\] Just as the charge polarization is revealed by the charge that accumulates at a surface termination, \[19\] it is known from the theory of magnetoelectric polarizability \[16, 20, 21\] that a Berry polarization of \(\chi\) contributes \((-\chi/2) e^2/h\) to the geometric component \((\sigma_{yx})\) of the surface anomalous Hall conductance. \[22, 23, 25\] Crucially, the bulk conduction and valence bands individually and equally contribute \((-\chi/2) e^2/h\), owing to the fundamental antisymmetry of all Pauli-matrix Hamiltonians: \(\sigma_y H(k) \sigma_y = -H(k)\). If all bands in the Hilbert space are accounted for, the net contribution to \(\sigma_{yx}\) should vanish, \[25\] hence there should exist surface band(s) which contribute \(\chi e^2/h\) to \(\sigma_{yx}\). This heuristic deduction is confirmed numerically in Fig. 1(c) and will be demystified in this Letter.

It will be shown that such topologically nontrivial surface bands originate from a bulk-to-boundary flow of Berry curvature – at the critical point between the trivial and Hopf insulator. Such flow does not occur via the Callan-Harvey anomaly inflow mechanism, \[20, 27\] which is understood through chiral zero modes of the Dirac equation with a mass coupling to a domain wall. \[28, 29\]
Instead, the surface states of the Hopf insulator arise as non-chiral, Schrödinger-type modes on the domain wall of a generalized Weyl equation – describing a pair of opposite-chirality Weyl fermions acting as a dipolar source of Berry curvature. The domain-wall mode with positive Schrödinger mass is characterized by an abnormal angular momentum ‘stolen’ from the filled Dirac sea.

We propose the surface Chern number \( C_s \) as a topological invariant for any three-spatial-dimensional crystalline Hamiltonian that is a two-by-two matrix at each \( k \in BZ \), with a spectral gap (at each \( k \)) that separates a low-energy and high-energy bulk band. Both bulk bands are assumed to have trivial first Chern class, meaning the first Chern number vanishes on any 2D cut of the BZ.\[^{30}\] For such class of Hamiltonians which include the Hopf insulator, we consider a surface termination whose reduced Brillouin zone (rBZ) is a 2D cut of BZ. Throughout this letter, ‘bulk’ objects or properties refer to the crystalline interior which locally has the symmetry of a 3D space group, up to exponentially-weak corrections due to a surface termination.

The triviality of the first Chern class implies any surface-localized mode is continuously deformable to a flat-band dispersion\[^{31, 32}\] and is therefore detachable from bulk bands. To ‘detach’ means to energetically separate a surface band from bulk bands while preserving all crystallographic spacetime symmetries of the surface, such that the projector to the surface-band wave function is continuous throughout \( rBZ \), as illustrated in Fig. 1(a-b). (In contrast, the surface states of a 3D Chern insulator\[^{33}\] are un-detachable.\[^{24}\]) The conceptually simplest formulation of the surface Chern number is as a \( rBZ \)-integral of the Berry curvature \( \mathcal{F} \) of the surface-band wave function:

\[
C_s := \frac{1}{2\pi} \int_{rBZ} Tr[\mathcal{F}(k) \cdot \hat{n}] \, d^2k,
\]

with \( \hat{n} \) the outward-normal vector for a chosen surface, and \( Tr \) a trace over all surface bands, independent of electron filling. The second half of the Letter will present a more general formulation of \( C_s \) – via eigenstates of the projected position operator – that is well-defined without need of detachment.

Let us demonstrate that \( C_s \) is both uniquely defined and invariant under continuous deformation of the surface-terminated Hamiltonian, so long as both bulk gap and bulk translational symmetry are maintained. While a surface-localized deformation can add to the number of in-gap surface bands, all additions must individually be topologically trivial\[^{18}\] and cannot modify \( C_s \). Moreover, a bulk deformation alone cannot change the Chern number of detached surface bands. What remains is to prove the absence of a continuous deformation between two Hamiltonians with the same bulk but with distinct surface Chern numbers: \( C_s > C_s' \). Let us assume to the contrary that such a deformation exists and is parametrized by \( \lambda \in [0, 1] \). It is simplest to view \( C_s, C_s' \) as applying to the top surface of a finite slab, with periodic boundary conditions in two surface-parallel directions. As \( \lambda \) is varied, we hypothesize that \( (C_s - C_s') \) quanta of Berry curvature can leave the top surface and enter the bulk.\[^{17, 18}\] Since the bulk Hamiltonians at \( \lambda=0 \) and \( \lambda=1 \) are indistinguishable, this Berry curvature must finally end up at the bottom surface of the slab. In the bulk perspective, there must then exist an adiabatic cycle for a bulk Hamiltonian \( H(k_x, k_y, k_z, \lambda) \) that is periodic in \( \lambda \), such that \( (C_s - C_s') \) quanta of Berry curvature is translated by a surface-normal lattice period over one cycle. If this translation occurs through the low-energy (resp. high-energy) band of \( H(k, \lambda) \), then \( (C_s - C_s') \) is identifiable as the second Chern number \( C_2 \) of the low-energy (resp. high-energy) band; \( C_2 \) is defined\[^{20}\] as the \( (k, \lambda) \)-integral of the second Chern character. The wrongness of our hypothesis is now self-evident, because \( C_2 \) necessarily vanishes for any band spanned by a single, analytic Bloch function of \( k \).\[^{24}\]

The above argument relied on identifying \( C_2 \) as the number of Berry-curvature quanta pumped by one lattice vector in an adiabatic cycle. Indeed, if \( H(k, \lambda) \) – with trivial first Chern class and \( C_2\neq0 \) – were diagonalized with open boundary conditions in the \( z \) direction, then the boundary theory describes \((3+1)D\) Weyl fermions with the net chiral charge\[^{35}\] of \( C_2 \), in the three-dimensional momentum-like space of \( (k_x, k_y, \lambda) \).\[^{18}\] (These Weyl fermions may be derived in a continuum model – as chiral zero modes of the \((4+1)D\) Dirac equation with a domain wall.\[^{20, 26}\]) Viewing the boundary theory as \((2+1)D\) fermions tuned by \( \lambda \), \( C_2 \) quanta of Berry curvature must be transferred at critical closings of the boundary gap, over one \( \lambda \)-cycle. Since the complete Hamiltonian – inclusive of boundary and bulk Hilbert spaces – is periodic in \( \lambda \), there must be a compensating pump of curvature through the bulk to exactly cancel the transfer of curvature at the boundary; this is a type of Callan-Harvey anomaly inflow.\[^{20}\] This proves that \( C_2\neq0 \) implies a Berry-curvature pump in the bulk; the converse statement – that a pump implies \( C_2\neq0 \) – has been proven by formulating the Chern-Simons magnetoelectric polarizability in terms of the ‘Berry-curvature dipole’.\[^{17, 18}\]

Our argument for the invariance of \( C_s \) shows that the Callan-Harvey inflow mechanism is not the origin of the Berry curvature on the Hopf-insulator surface. (In contrast, this inflow mechanism is responsible for the nontrivial surface anomalous Hall conductance of the ‘layered Haldane model’.\[^{18}\]) Understanding the Hopf insulator not only reveals a new mechanism for Berry curvature to flow to the surface, but also leads us to the advertised bulk-boundary correspondence \( \chi=C_2 \).

In the continuum perspective, the curvature may be understood from the perspective of nonchiral domain-wall modes of a dipolar Weyl equation [Eq. (3) below].
We propose that the dipolar Weyl equation describes the critical transition between a trivial and a Hopf insulator, in analogy with how the Dirac equation describes the critical point for an insulator with nontrivial Chern class.\[20\] We define a Weyl dipole as a dipolar source of Berry curvature; it is a two-fold-degenerate band touching with quadratic dispersion in all three momentum directions, in contrast with the linearly-dispersing Weyl points which act as monopole sources.\[36\] The Hamiltonian of a Weyl dipole is given by:

\[
H(k) = -B \cdot \sigma; \quad B(k) = z^3 \sigma; \quad \sigma := (\sigma_x, \sigma_y, \sigma_z)
\]

\[
z(k) = (z_1, z_2)^T = (k_x + ik_y, k_z + i\phi)^T,
\]

where \(\phi \in \mathbb{R}\) is an additional tuning parameter. Independent of \(\phi\), \(H\) has the rotational symmetry:

\[
U_\theta H(k) U_\theta^{-1} = H(k', k_y', k_z), \quad U_\theta = e^{i\theta \sigma_z / 2}
\]

with \(k_x' = k_x \cos \theta - k_y \sin \theta\), and \(U_\theta\) a spinor matrix representation with spin operator \(\sigma_z / 2\). The Weyl dipole moment lies parallel to the rotation-invariant \(k_z\)-axis, as illustrated in Fig. 2(a) by a unit positive (resp. negative) Berry flux out of the southern (resp. northern) hemisphere. For \(\phi \neq 0\), \(z \rightarrow B\) exhibits the standard Hopf map from \(S^3 \rightarrow S^2\), and the continuum analog of the Hopf invariant equals

\[
-\frac{1}{4\pi^2} \int \mathbf{F} \cdot \mathbf{A} d^3k = \frac{1}{2} \text{sign}[\phi],
\]

for both the low-energy wave function \((\sqrt{z^3} z)\) and its orthogonal complement \((\sigma_y z / \sqrt{z^3})\). \(\phi = 0\) is thus the critical point for a unit change in \(\chi\) for both low- and high-energy bands.

To manifest the nontrivial boundary mode that develops at criticality, we solve Eq. (3) with \(k_z \rightarrow -i\partial_z\) and \(\phi(z)\) having a domain-wall profile. It is analytically convenient to choose \(\phi(z) = \phi' z\) that is linear in the interval \(|z| < \Lambda\) (with \(\Lambda \gg |\phi'|^{-1/2}\)), with \(\phi\) tapering off to a constant outside this interval, as depicted in Fig. 2(b). Then a Gaussian-localized, spin-polarized mode \(\psi_\eta\) exists with a Schrödinger-type dispersion \(\varepsilon_\eta\):

\[
\psi_\eta = \kappa_\eta e^{-|\phi'|^2 / 2}, \quad \varepsilon_\eta = -\eta(k_x^2 + k_y^2 - |\phi'|),
\]

with \(\sigma_z \kappa_\pm = \pm \kappa_\pm\). The mass and spin depend on \(\text{sign}[\phi'] = -\eta\); we refer to modes with positive (resp. negative) mass in the \(xy\)-plane as electron-like (resp. hole-like).

At \((k_x, k_y) = 0\), the spin of the electron-like, domain-wall mode is opposite to that of the electron-like, bulk mode; the latter mode diagonalizes Eq. (3) with homogeneous \(\phi \neq 0\). This abnormal, domain-wall spin has been ‘stolen’ from the bulk, hole-like band at criticality. The unit difference in spin implies contrasting spin textures over the \((k_x, k_y)\)-plane: constant for the domain-wall mode, but skyrmionic for the \(k_z = 0\) bulk mode, as illustrated in Fig. 2(c). (In comparison, the meronic spin texture\[37\] of a \((2+1)D\) massive Dirac fermion is ‘half’ of a skyrmion.) From the contrasting spin texture, we deduce that the integrated Berry curvature vanishes for the domain-wall mode (for both \(\eta = \pm 1\), but equals \(+2\pi\) for the electron-like bulk mode, and \(-2\pi\) for the hole-like bulk mode.

We will see that the differential curvature between bulk and domain-wall modes is retained upon regularizing Eq. (3) on a lattice. The Hopf insulator is a lattice regularization satisfying that all bulk modes have trivial first Chern number; this would imply that the regularized domain-wall mode has a nontrivial first Chern number (=\(\eta\)) which depends on the orientation of the domain wall, like how our proposed surface Chern number depends on the surface orientation \(\hat{n}\) [cf. Eq. (2)]. Since \(\phi = 0\) describes a Weyl dipole and not a monopole, a regularization can be found without any fermion doubling,\[38\] meaning that the energy gap of the lattice model closes at a single isolated wavevector (the BZ center) when \(\phi = 0\). A lattice regularization satisfying the above-stated properties is given by the Moore-Ran-Wen tight-binding model,\[4\] though they did not emphasize the dipole picture that is presented here. Their lattice Hamiltonian has the form of Eq. (3) with \(z\) generalized to a \(2D\)-periodic function with a Wilson coupling:

\[
z_1 = \sin k_x + i \sin k_y, \quad z_2 = \sin k_x + i (\phi + \sum_{j=3}^3 \cos k_j - 3).
\]

The tight-binding Hilbert space may be viewed as comprising spin-half particles (\(\sigma_z = \pm 1\)) in each unit cell of a tetragonal lattice.

Let us prove \(\chi = C_s\) for the Moore-Ran-Wen model. Despite the specificity of the model, the correspondence would extend to the entire homotopy class of translation-invariant, two-by-two Hamiltonians. This is because \(\chi\) and \(C_s\) are individually invariant under any continuous, bulk-gap-preserving deformations that preserve the bulk-translational symmetry; this fact is well-known for \(\chi \in \mathbb{H}\) and has been proven above for \(C_s\). Hamiltonians with nonzero \(\chi\) may have any non-translational symmetry that does not invert the pseudoscalar: \(\chi \rightarrow -\chi\). Particularly, the Moore-Ran-Wen regularization retains an order-four rotational symmetry [cf. Eq. (4)], which not only allows us to utilize our rotation-symmetric domain-wall analysis, but also manifests several features unique
FIG. 3. Dispersion of the projected-\(z\) operator over a high-symmetry line in \(rBZ\), for a 50-cell-wide 001 slab. \(z\) is projected to the low-energy band of the Moore-Ran-Wen model in (a-c), and to the high-energy band in (d). \(\phi = -0.2\) in (a), and \(= 1\) in (b-d). (a-b) illustrate the bulk dispersion, while (c-d) depict the top and bottom surfaces respectively. The expected spin \(\langle \sigma_z / 2 \rangle\) is depicted according to a color scheme on the right.

to the crystalline Hopf insulator.

For sufficiently negative \(\phi\), \(H(k) \approx \phi^2 \sigma_z\) implies that both low- and high-energy bands are spin-polarized and the Hopf invariant vanishes. It is instructive to represent both bands by basis functions that are extended as Bloch functions with wavevector \((k_x, k_y) \in rBZ\), but exponentially-localized in the \(z\)-direction as Wannier functions.\[39\]-[41] In units of the lattice period, the position\-al center \(\bar{Z}\) of a hybrid function is related\[13\] to the Berry phase as \(\bar{Z}(k_x, k_y) = \int_{rBZ} A_z(k) dk_z / 2 \pi\). Owing to the discrete translational symmetry along \(z\), the \(rBZ\) centers \(\bar{Z}\) are arranged as a Wannier-Stark ladder\[12\]-[14] with unit spacing between adjacent rungs. The low-energy band having only spin-down character implies the Berry connection can be trivialized, implying \(\bar{Z} = 0\) (modulo integer) for all \((k_x, k_y)\). For negative, finite \(\phi\), the low-energy band acquires some spin-up character except at the four rotation-invariant \((k_x, k_y)\)-points [denoted as \(\Gamma, X, Y, M\) in Fig. 3(a)], where distinct rotational representations cannot mix. Despite the nonzero dispersion, each hybrid function retains the topology of a ‘flat sheet’ which is pinned to integer values at the rotation-invariant \((k_x, k_y)\)-points.

As negative \(\phi\) is tuned through zero, the spins of the bulk eigenstates of \(H(0) = \phi^2 \sigma_z\) are not inverted; nevertheless, the Hopf invariant increases by unity [cf. Eq. (1)]. Analogously, the Berry phase of the low-energy mode in the continuum model is \(\int_{\text{top}} A_z(0, 0, k_z) dk_z = \pi \text{sign}[\phi]\), which implies a \(2\pi\)-quantum of Berry phase is transferred from the high- to low-energy band (at the \(rBZ\) center). Therefore, there must be a discontinuous change in the connectivity of the Wannier-Stark ladder, resulting in upward-protruding (resp. downward-protruding) sheets for the high-energy (resp. low-energy) band, as illustrated in Fig. 3(b-d). The triviality of the first Chern class ensures the periodicity of \(\bar{Z}\) over \(rBZ\), hence no net charge flows if \((k_x, k_y)\) is varied along \(\Gamma M \Gamma\). Despite the absence of a conventional Thouless pump over an adiabatic cycle,\[43\] observe that the hybrid center is translated by exactly one lattice period at the half-way mark before returning to its original position. Such a returning Thouless pump implies that the Wannier function – obtained by one-dimensional Fourier transform of the hybrid Bloch-Wannier functions – cannot be localized to a single lattice site. Such a topological obstruction is a necessary condition for a quantized, real-space polarization of Berry curvature.

The above-described connectivity of Wannier-Stark sheets holds for an infinite ladder, but is frustrated for a finite slab with rotation-invariant surface terminations – meaning that protrusions of the lowermost (resp. uppermost) sheet of the high-energy (resp. low-energy) band are incompatible with the confining surface potentials. These two frustrated sheets are continuously deformable to the surface bands of the finite-slab Hopf insulator.\[31\],[45\] For these surface bands, the spin at the \(rBZ\)-center is undetermined from a bulk analysis but determined by our domain-wall analysis, which asserts that the electron-like (resp. hole-like) mode localized to the bottom (resp. top) surface has spin down (resp. up); this is confirmed by a numerical calculation illustrated in Fig. 3(c-d). A colloquial interpretation is that the lowermost sheet of the high-energy band ‘acquires’ a spin-down parton from the low-energy band, which has been ‘left over’ at the bottom surface due to a returning Thouless pump in the upward direction. Effectively, there is a mutual exchange of rotational representations between surface bands localized to opposite surfaces of a finite slab, manifesting a type of symmetry ‘teleportation’.

The unit gain of angular momentum for the top-surface band introduces an irremovable twisting of the wave function that is quantified by the Chern number \(C_s = 1\).\[46\] Bearing in mind that the surface-normal vector \(\mathbf{n}\) flips between top and bottom surface [cf. Eq. (2)], we arrive at \(C_s = \chi = 1\) for both surfaces. It is straightforward to generalize this equality for any integer value – by substituting \(k_z \rightarrow N k_z\; (N \in \mathbb{Z})\) in the Moore-Ran-Wen model, such that \(\phi = 0\) marks a critical point with \(|N|\) Weyl dipoles. The nontrivial Berry curvature implies that the full subspace of surface states (on a single surface) cannot be realized by any strictly-2D, tight-binding lattice model. This is a non-chiral analog of how chiral fermions on topological-insulating surfaces\[20\] evade the fermion-doubling theorem.\[35\],[47\]

Our Chern-number analysis may be verified by detecting surface bands and numerically calculating \(C_s\) through Eq. (2). It is possible to formulate \(C_s\) without detecting surface bands, but requiring the milder condition that a spectral gap exists throughout \(rBZ\) and separates two orthogonal subspaces \(P\) and \(Q\). Such a gap always exists for certain surface terminations of the considered class of Hamiltonians, because the surface bands are continuously deformable to hybrid Bloch-Wannier bands which disperse like ‘flat sheets’.\[43\],[48\] This ‘sheet’ topology applies to all surfaces of the Hopf insulator, implying that a finite-size, Hopf-insulator crystal can in principle be truly insulating – in both bulk and surface.) Let us consider a semi-infinite geometry that is periodic in...
$x$ and $y$, and with the position operator $z$ taking only negative values. We then diagonalize the projected position operator $P_z P_z$ and define $C^s_\mathcal{P} [n]$ as the first Chern number of the $n$ Bloch-Wannier eigenbands whose eigenvalues lie closest to $z = 0$. Viewed as a sequence in $\mathcal{n}$, $C^s_\mathcal{P} [n]$ has a unique accumulation point ($= C^s_\mathcal{S}$) because all bulk eigenbands have trivial first Chern class. Analogously defining $C^s_\mathcal{Q}$ as the accumulation point for the orthogonal subspace $\mathcal{Q}$, the surface Chern number is then $C_s := C^s_\mathcal{P} + C^s_\mathcal{Q}$. This formulation of $C_s$ is manifestly invariant if a Chern-insulating layer is adsorbed onto the surface. Practically, one may calculate $C^s_\mathcal{P} + C^s_\mathcal{Q}$ to exponential accuracy on a wide slab.

What does $\chi = C_s$ imply for the magnetoelectric response of the Hopf insulator? It is known that the geometric contribution to the frozen-lattice, orbital magnetoelectric polarizability equals $(\theta/2\pi \mod 1)e^2/h$, with the ‘axion angle’ $\theta$ equal to an integrated Chern-Simons three-form of the Berry connection.\[10, 20\] For the considered class of Hamiltonians, $\theta$ is gauge-invariant and simplifies to $\pi \chi$, thus the Hopf insulator with odd $\chi$ exemplifies the simplest axion insulator\[10, 20, 21, 50-52\] with only a Pauli-matrix Hamiltonian. The possibilities for surface anomalous Hall conductance on different facets of a finite-size, Hopf-insulator crystal are rich, and depend on the electrochemical potential as well as the energy dispersion of surface bands. Generically surface bands are partially filled, such that each facet behaves like a 2D anomalous Hall metal.\[52\] The differing crystallographic symmetries on distinct facets allow for non-identical fillings, e.g., it is possible for the rotation-invariant 001-surface bands of the Moore-Ran-Wen model to be completely filled, with the rotation-invariant side-surface bands completely depleted – this leads to $[C_s]$ number of chiral modes\[22, 34, 36\] that run around the edges of the 001 facet. Such a scenario is realizable if the electrochemical potential on distinct surfaces are tunable, perhaps by differential doping or gating.

Our last comment is aimed toward widening the search for solid-state realizations of the Hopf insulator, which would complement existing ultracold-atomic platforms.\[57, 58\] The first solid-state proposal of the Hopf insulator, anomalies and characteristic classes, and Zhu Penghao for informative discussions on the non-geometric magnetoelectric polarizability. We would also like to thank Barry Bradlyn, Ken Shiozaki, Judith Höller, Nicholas Read, G. M. Graf, I. Souza, T. Pahomi, D. Gresh for useful discussions. AN and AAS acknowledge the support of the SNF Professorship grant along with SNSF NCCR MARVEL and QSIT programs. AA was supported initially by the Yale Postdoctoral Prize Fellowship, and subsequently by the Gordon and Betty Moore Foundation EPIQS Initiative through Grant No. GBMF4305 at the University of Illinois.

**Acknowledgments** We thank Joel Moore, Eduardo Fradkin and Michael Stone for their expert advice on the Hopf insulator, anomalies and characteristic classes, and Zhu Penghao for informative discussions on the non-geometric magnetoelectric polarizability. We would also like to thank Barry Bradlyn, Ken Shiozaki, Judith Höller, Nicholas Read, G. M. Graf, I. Souza, T. Pahomi, D. Gresh for useful discussions. AN and AAS acknowledge the support of the SNF Professorship grant along with SNSF NCCR MARVEL and QSIT programs. AA was supported initially by the Yale Postdoctoral Prize Fellowship, and subsequently by the Gordon and Betty Moore Foundation EPIQS Initiative through Grant No. GBMF4305 at the University of Illinois.

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