Tightness and Convergence of Trimmed Lévy Processes to Normality at Small Times

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Abstract

Let \((r,s)X_t\) be the Lévy process \(X_t\) with the \(r\) largest positive jumps and \(s\) smallest negative jumps up till time \(t\) deleted and let \(\tilde{(r)}X_t\) be \(X_t\) with the \(r\) largest jumps in modulus up till time \(t\) deleted. Let \(a_t \in \mathbb{R}\) and \(b_t > 0\) be non-stochastic functions in \(t\). We show that the tightness of \(((r,s)X_t - a_t)/b_t\) or \(\tilde{(r)}X_t - a_t)/b_t\) at \(0\) implies the tightness of all normed ordered jumps, hence the tightness of the untrimmed process \((X_t - a_t)/b_t\) at \(0\). We use this to deduce that the trimmed process \(((r,s)X_t - a_t)/b_t\) or \(\tilde{(r)}X_t - a_t)/b_t\) converges to \(N(0,1)\) or to a degenerate distribution if and only if \((X_t - a_t)/b_t\) converges to \(N(0,1)\) or to the same degenerate distribution, as \(t \downarrow 0\).

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1 Introduction and Main Results

Lévy processes can be seen as continuous analogues of random walks. Historically motivated by robust statistics, the concept of trimming has been thoroughly explored in the random walks literature to assess the effect of outliers. Here we construct an analogous process by removing a finite number of largest jumps from a Lévy process. For large time behaviour, i.e. as \(t \to \infty\), the trimmed Lévy process exhibits a similar structure to the trimmed sums of independent and identically distributed random variables. In this paper, however, we are concerned with small time convergence properties. Note that as \(t \to \infty\), an increasing number of jumps with bigger magnitude come into consideration for being removed, but as \(t \downarrow 0\), jumps of bigger size are gradually excluded from being removed in the trimming procedure. This makes trimming at small times a nontrivial effort with no exact random walk analogy. This also promises a fresh perspective in seeking out potential applications. Local structure of a process, i.e. small time behaviour, has recently gathered more interest as a subject of practical investigation. For example, Aït-Sahalia and Jacod \cite{Aït-Sahalia} have estimated an activity index at small times for high frequency financial data. Zheng

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et al. [18] have observed that self-propelled Janus particles exhibit asymmetric normal behaviour at small times. Hence small time results can be expected to become increasingly relevant in practical situations as measuring equipment improves in the modern digital age of “big” data. In this paper, we will show that “light” trimming, i.e. trimming off a finite number of large jumps, does not affect the asymptotic normality or degeneracy behaviour of a normed and centered Lévy process as $t \downarrow 0$.

As a continuation of the classical precedent in random walks, we can borrow from the rich repertoire of ideas in the literature. It has been shown that the convergence of the normed, centered random walk to a finite, non-degenerate random variable implies the convergence of the lightly trimmed sum (see for example Darling [4], Hall [8] and Mori [14]). However, the converse is known to be a much harder problem. Maller [11] first gave a partial converse in the case of the domain of attraction of normality for trimmed random walks under the assumption of a continuous and symmetric distribution for the increments of the random walk. Then Mori [14] completed the proof for the general case without extra assumptions only for asymptotic normality and admitted the difficulties in proving a similar result for a non-normal limit. In 1993, Kesten [10] then proved the most general case by showing that the convergence in distribution of normed and centered lightly trimmed and untrimmed random walks are equivalent as $n \to \infty$. The asymptotic normality property has also been investigated for other types of trimming in the random walks literature, see for example Griffin and Pruitt [6], Griffin and Mason [7].

The idea of removing jumps from a Lévy process is not at all new either. Rosiński [16] made use of “thinning” to generate one Lévy process from another by removing a finite number of jumps stochastically. By comparison, the “trimmed” processes introduced by Buchmann, Fan and Maller [2] have a more deterministic flavour, i.e. jumps are removed according to their sizes. In Buchmann et al. [2], representation formulae for the distribution of the trimmed process joint with its order statistics and quadratic variation are derived for positive and modulus trimming. The resulting trimmed processes no longer have independent stationary increments, hence are not Lévy processes. But their distributions can be written as mixtures of a truncated infinitely divisible distribution with a gamma random variable. This permits techniques for Lévy processes to be carried over to the trimmed processes. In Section 2, as preparatory material for the proofs of the main results, we revisit and extend the results in [2] to asymmetrical trimming.

Our main results are stated as Theorems 1.1 and 1.2 below. They state that light trimming has no effect, in a weak convergence sense, on the tightness or asymptotic normality at 0 of a normed and centered Lévy process. Our setup is as follows. Let $(X_t)_{t \geq 0}$ be a real valued Lévy process with canonical triplet $(\gamma, \sigma^2, \Pi)$, thus having characteristic function $E e^{i \theta X_t} = e^{t \Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with characteristic exponent

$$
\Psi(\theta) := i \theta \gamma - \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} \left( e^{i \theta x} - 1 - i \theta x 1_{\{|x| \leq 1\}} \right) \Pi(dx),
$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$. Here $\Pi$ is a Borel measure on $\mathbb{R}_+ := \mathbb{R} \setminus \{0\}$ with $\int_{\mathbb{R}_+} 1 \wedge x^2 \Pi(dx) < \infty$ and $\Pi((-x,x)^c) < \infty$ for all $x > 0$.

Denote the jump process of $X$ by $(\Delta X_t)_{t \geq 0}$, where $\Delta X_t = X_t - X_{t-}$, $t > 0$, with $\Delta X_0 \equiv 0$. In particular, denote the positive jumps by $\Delta X_t^+ = \Delta X_t \vee 0$ and the
magnitudes of the negative jumps by \( \Delta X_t^- = (-\Delta X_t) \vee 0 \). Note that \((\Delta X_t^+))_{t \geq 0} \) and \((\Delta X_t^-))_{t \geq 0} \) are non-negative independent processes. For any integers \( r, s > 0 \), let \( \Delta X_t^{(r)} \) be the \( r^{th} \) largest positive jump, and let \( \Delta X_t^{(s)} \) be the \( s^{th} \) largest jump in \( \{ \Delta X_s^-, 0 < s \leq t \} \), i.e. the jump with magnitude of the \( s^{th} \) smallest negative jump.

We further write \( \Delta X_t^{(r)} \) to denote the \( r^{th} \) largest jump in modulus up to time \( t \). For a formal definition of the ordered jumps, allowing tied values, we refer to Buchmann et al. \[2\] Section 2.1. The trimmed versions of \( X \) are defined as

\[
(r,s)X_t := X_t - \sum_{i=1}^{r} \Delta X_t^{(i)} + \sum_{j=1}^{s} \Delta X_t^{(j)}, \quad \text{and} \quad (r)\tilde{X}_t := X_t - \sum_{i=1}^{r} \tilde{\Delta} X_t^{(i)},
\]

which are termed asymmetrical trimming and modulus trimming respectively. Set

\[
(0,0)X_t = (0)\tilde{X}_t = (0)X_t = (0^-)X_t = X_t.
\]

By letting \( r = 0 \) or \( s = 0 \) in asymmetrical trimming, we obtain the one-sided trimmed processes,

\[
(r)X_t := X_t - \sum_{i=1}^{r} \Delta X_t^{(i)}, \quad \text{and} \quad (s^-)X_t := X_t + \sum_{i=1}^{s} \Delta X_t^{(i)}.
\]

These ideas are analogous to what has been called light trimming in the random walks literature, i.e. trimming off a bounded number of terms.

The positive, negative and two-sided tails of the Lévy measure \( \Pi \) are

\[
\tilde{\Pi}^+(x) := \Pi\{x, \infty\}, \quad \tilde{\Pi}^-(x) := \Pi\{\infty, -x\}, \quad \text{and} \quad \tilde{\Pi}(x) := \tilde{\Pi}^+(x) + \tilde{\Pi}^-(x), \quad x > 0.
\]

The restriction of \( \Pi \) on \( (0, \infty) \) is \( \Pi^+ \). Let \( \Pi^-(\cdot) = \Pi(\cdot) \) and \( \Pi^{\pm}(\cdot) = \Pi^+ + \Pi^- \).

For each \( x > 0 \), denote the truncated mean and second moment functions by

\[
\nu(x) = \gamma - \int_{x < |y| \leq 1} y \Pi(dy), \quad \text{and} \quad V(x) = \sigma^2 + \int_{|y| \leq x} y^2 \Pi(dy).
\]

Throughout the paper, we assume \( \Pi(0+) = \infty \) when dealing with modulus trimming and \( \Pi^+(0+) = \infty \) or \( \Pi^- (0+) = \infty \) (or both when appropriate) when dealing with one-sided trimming. In particular, these mean \( V(x) > 0 \) for all \( x > 0 \), and they ensure there are infinitely many jumps \( \Delta X_t, \Delta X_t^\pm \), a.s., in any bounded interval of time.

Analytical conditions for a Lévy process to be in the domain of attraction of a normal law as \( t \downarrow 0 \) or \( t \rightarrow \infty \) were studied in Doney and Maller \[5\]. \( X_t \) is in the domain of attraction of the normal law at 0, i.e. there exist some centering and norming functions \( a_t \in \mathbb{R} \) and \( b_t > 0 \) such that

\[
\frac{X_t - a_t}{b_t} \rightarrow N(0,1) \quad \text{as} \quad t \downarrow 0, \quad \text{if and only if} \quad \frac{x^2\Pi(x)}{V(x)} \rightarrow 0 \quad \text{as} \quad x \downarrow 0. \]

When \((6)\) holds, the norming function \( b_t \) is regularly varying with index 1/2 at 0 and the truncated second moment function \( V(x) \) is slowly varying at 0. For the
definition and properties of regular variation, we refer to [3]. The centering function \(a_t\) can differ for small and large time convergence; in particular, at small times, the centering function \(a_t\) can be chosen to be 0, i.e. \(X_t\) in the domain of attraction of the normal law \((X_t \in D(N))\) is equivalent to \(X_t\) in the centered domain of attraction of the normal law \((X_t \in D_0(N))\) (see for example Maller and Mason [13]).

For given non-stochastic functions \(a_t \in \mathbb{R}\) and \(b_t > 0\), abbreviate the various centered and normed processes as

\[
S_t := \frac{X_t - a_t}{b_t}, \quad (r,s)S_t := \frac{(r,s)X_t - a_t}{b_t} \quad \text{and} \quad (r)\tilde{S}_t := \frac{(r)\tilde{X}_t - a_t}{b_t}.
\]

Also denote the one-sided versions (refer to (3)) as

\[
(r)S_t := \frac{(r)X_t - a_t}{b_t} \quad \text{and} \quad (s,-)S_t := \frac{(s,-)X_t - a_t}{b_t}.
\]

We will pursue a compactness argument by first proving that \((S_t)\) is relatively compact as \(t \downarrow 0\) if \((r,s)S_t\) or \((r)\tilde{S}_t\) is. This will imply that each sequence of \((S_t)\) has a convergent further subsequence. Then we will establish that each convergent subsequence has to converge to the same limit when \((r,s)S_t\) or \((r)\tilde{S}_t\) has a normal or degenerate limit as \(t \downarrow 0\).

The idea of the proof is inspired by Mori [14] in the random walks literature, but we will apply it to the continuous setting in the small time framework where some notable differences occur, especially in regard to the treatment of tied values in the large jumps of \(X\). Before proving the asymptotic normality result, we will establish equivalent conditions for the sequence of normed and centered Lévy process to be relatively compact. Since we are dealing with \(X_t\) on the real line, we can instead prove that, if \((r,s)S_t\) or \((r)\tilde{S}_t\) is tight at 0, then \(S_t\) is tight at 0, i.e.

\[
\lim_{x \to \infty} \limsup_{t \downarrow 0} P(|S_t| > x) = 0.
\]

Henceforth we will state theorems for both asymmetrical and modulus trimmed processes but only give detailed proofs for one type of trimming. All statements are also true for one-sided trimmed processes, as special cases of the asymmetrical trimmed process by taking \(r = 0\) or \(s = 0\). Mori [14] dealt only with modulus trimming while Kesten [10] dealt only with modulus and one-sided trimming.

Theorem 1.1 gives a thorough characterisation of tightness of the trimmed process, the ordered jumps and the untrimmed process.

**Theorem 1.1** (a) Fix \(r = 0,1,2,\ldots\) and \(s = 0,1,2,\ldots\). Suppose that \((r,s)S_t\) is tight as \(t \downarrow 0\) for some \(a_t \in \mathbb{R}\) and \(b_t > 0\). Then the following hold.

(i) \((\Delta X_t^{(j)}/b_t)\) is tight at 0 for all \(j \in \mathbb{N}\) and \(\lim_{x \to \infty} \limsup_{t \downarrow 0} t\Pi^+(xb_t) = 0\).

(ii) \((\Delta X_t^{(k,-)}/b_t)\) is tight at 0 for all \(k \in \mathbb{N}\) and \(\lim_{x \to \infty} \limsup_{t \downarrow 0} t\Pi^-(xb_t) = 0\).

(iii) \((\tilde{j})S_t\) is tight at 0 for all \(j = 1,2,\ldots\).

(iv) \((\tilde{k}^-)S_t\) is tight at 0 for all \(k = 1,2,\ldots\).
(v) \((S_t)\) is tight at 0.

(b) Suppose \((\{\bar{S}_t\})\) is tight at 0 for some \(a_t \in \mathbb{R}\) and \(b_t > 0\). Then \((S_t)\) is tight at 0 and \((\Delta X^{(j)}_t / b_t)\) is tight at 0 for some (hence all) \(j \in \mathbb{N}\) and \(\lim_{x \to \infty} \limsup_{t \downarrow 0} t \Pi(x b_t) = 0\).

With the help of Theorem 1.1 we can prove Theorem 1.2, the main result of the paper, showing that light trimming has no effect on asymptotic normality or degeneracy at 0.

**Theorem 1.2**

Suppose \(\Pi(0+) = \infty\). There exist non-stochastic functions \(a_t\) and \(b_t > 0\) such that, as \(t \downarrow 0\), for any \(r, s \in \mathbb{N}\),

\[
\frac{X_t - a_t}{b_t} \overset{D}{\to} N(0, 1) \quad \text{or a degenerate distribution},
\]

(8)

if and only if

\[
\frac{(r,s)X_t - a_t}{b_t} \overset{D}{\to} N(0, 1) \quad \text{or a degenerate distribution},
\]

(9)

or equivalently,

\[
\frac{\bar{X}_t - a_t}{b_t} \overset{D}{\to} N(0, 1) \quad \text{or a degenerate distribution}.
\]

(10)

**Outline of the Proof**

To show tightness, we make use of a key inequality (Prop. 3.3) in Section 3 that gives an upper bound to the distribution of the trimmed process. Before that, in Section 2 we investigate the limit of a truncated Lévy process as \(t \downarrow 0\), allowing a Poisson number of possible tied values, which can be applied in the distributional representation also developed in Section 2. Then in Section 4, by an important estimate on the tail probability of a Lévy process in Sato ([17]), we show by contradiction that each convergent subsequence has the same normal or degenerate limit at 0. Some auxiliary results concerning the quadratic variation process of \(X\), and the domain of partial attraction of the normal are in Section 5.

2 The Truncated Process

Note that by the Lévy-Itô decomposition, we can write

\[
X_t = \gamma t + \sigma Z_t + X_t^J,
\]

(11)

where \((Z_t)\) is a standard Brownian motion and the jump process is

\[
X_t^J = a.s. \lim_{\varepsilon \downarrow 0} \left( \sum_{0 < s \leq t} \Delta X_s 1_{\{|\Delta X_s| > \varepsilon\}} - t \int_{0 < |x| \leq 1} x \Pi(dx) \right).
\]
We will need notation and properties of inverse functions of $\Pi$ and $\Pi^\pm$. Define the right-continuous inverse of a nonincreasing monotone function $f : (0, \infty) \mapsto [0, \infty)$ as

$$f^+(x) = \inf\{y > 0 : f(y) \leq x\}, \quad x > 0. \quad (12)$$

The following setup and results are taken from Buchmann et al. [2]. We introduce three families of processes, indexed by $v > 0$, truncating jumps from sample paths of $X^f_t$. Let $v, t > 0$. When $\Pi(0+) = \infty$, we set

$$X^\pm_{t,v} := X^f_t - \sum_{0 < s \leq t} \Delta X_s 1_{\{\Delta X_s \geq \Pi^\pm_{-(v)}(v)\}}, \quad X^{-v}_{t,v} := X^f_t - \sum_{0 < s \leq t} \Delta X_s 1_{\{\Delta X_s \leq -\Pi^{-v}_{-(v)}(v)\}}, \quad (13)$$

and for the modulus case, we truncate from the original process $X_t$ instead of its jump process, i.e.

$$\tilde{X}^v_t := X_t - \sum_{0 < s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq \Pi^{-v}(v)\}}. \quad (14)$$

Under the assumption $\Pi(0+) = \infty$, $(X^\pm_{t,v})_{t \geq 0}$ and $(\tilde{X}^v_t)_{t \geq 0}$ are well defined Lévy processes with canonical triplets, respectively,

$$\left(\pm 1_{\{\Pi^\pm_{+(v)}(v) \leq 1\}} \int_{\Pi^\pm_{+(v)}(v) \leq x \leq 1} x \Pi^\pm(dx), \quad 0, \Pi^\pm(dx) 1_{0 < x \leq \Pi^\pm_{+(v)}(v)} \right), \quad (15)$$

and

$$\left(\gamma - 1_{\{\Pi^{-v}(v) \leq 1\}} \int_{\Pi^{-v}(v) \leq |x| \leq 1} x \Pi(dx), \quad \sigma^2, \Pi(dx) 1_{\{|x| \leq \Pi^{-v}(v)\}} \right). \quad (16)$$

Theorem 2.1 of [2] uses a pathwise construction method to derive a representation for the distribution of the positively trimmed process $(r)X_t$ or of the modulus trimmed process $(r)\tilde{X}_t$ with its corresponding ordered jumps, i.e. $\Delta X^r_t$ or $\Delta \tilde{X}^r_t$. We need to extend these expressions to the asymmetrically trimmed process $(r,s)X_t$ joint with both positive and negative ordered jumps $\Delta X^r_t$ and $\Delta X^{(s),-}_t$. Let $X^J_t = X^+_t - X^-_t$ where $X^\pm_t$ are the compensated sums of positive and negative jumps respectively. We can trim these to get $(r,s)X_t = \gamma t + \sigma Z_t + (r)X^+_t - (s)X^-_t$, where $(r)X^+_t$ and $(s)X^-_t$ are defined analogously as in [3]. These processes are non-negative and independent of each other. Therefore we can treat the positive and negative jump processes independently using our previous result on one-sided trimming in [2].

For each $r, s \in \mathbb{N}$, let $\Gamma_r$ and $\tilde{\Gamma}_s$ be standard Gamma random variables with parameters $r$ and $s$, independent of $(X_t)_{t \geq 0}$ as well as each other. Let $(Y^\pm_t)_{t \geq 0}$ be Poisson processes with unit mean, independent from $X, \Gamma$ and $\tilde{\Gamma}$. On the assumption that $\Pi^\pm(0+) = \infty$, we have, by Theorem 2.1 in [2], for each $t > 0$,

$$\left((r,s)X_t, \Delta X^r_t, \Delta X^{(s),-}_t\right) \overset{D}{=} \left(\tilde{X}^u_{t,v} + G^+_t - G^-_t, \Pi^+\left(v\right)\Pi^-(u)\right)_{v = \Gamma_r/t, u = \tilde{\Gamma}_s/t}, \quad (17)$$
where for \( w > 0 \),
\[
G_{t}^{\pm,w} = \Pi^{\pm,\psi}(w)Y_{\rho_{\pm}(w)} \quad \text{and} \quad \rho_{\pm}(w) = \Pi^{\pm,\psi}(w) - w
\]
and for each \( u > 0, v > 0 \),
\[
X_{t}^{u,v} := \gamma t + \sigma Z_{t} + X_{t}^{u,v} - X_{t}^{v,u}
\]
is infinitely divisible with characteristic triplet
\[
\left( \gamma_{u,v}, \sigma^{2}, \Pi(d\mathbf{x})1_{\{\Pi^{+,\psi}(u) < x < \Pi^{+,\psi}(v)\}} \right).
\]

Here
\[
\gamma_{u,v} = \gamma - 1_{\{\Pi^{+,\psi}(v) \leq 1\}} \int_{\Pi^{+,\psi}(v) \leq x \leq 1} x \Pi(dx) + 1_{\{\Pi^{-,\psi}(u) \leq 1\}} \int_{\Pi^{-,\psi}(u) \leq x \leq 1} x \Pi(dx).
\]

The processes \( G_{t}^{\pm,w} \) and \( \tilde{G}_{t}^{v} \) (below) are Poisson processes resulting from possible tied values in the ordered jumps. For completeness, we quote next the representation of the modulus trimmed process from [2] before proceeding to the proofs.

For each \( v > 0 \), recall the modulus truncated process \( \overline{X}_{t}^{v} \) in (14) with canonical triplet
\[
\left( \overline{\gamma}_{v}, \sigma^{2}, \Pi(dx)1_{\{|x| < \Pi^{-}(v)\}} \right),
\]
where \( \overline{\gamma}_{v} = \gamma - 1_{\{\Pi^{-}(v) \leq 1\}} \int_{\Pi^{-}(v) \leq |x| \leq 1} x \Pi(dx) \) as defined in (16). Then, for each \( t > 0 \) and \( r \in \mathbb{N} \), we have
\[
\left( (r) \overline{X}_{t}, |\Delta \overline{X}_{t}^{(r)}| \right) \overset{D}{=} \left( \overline{X}_{t}^{v}, \Pi^{-}(v) \right) \bigg|_{v=\Gamma_{r}/t},
\]
where \( \overline{G}_{t}^{v} = \Pi^{-}(v)(Y_{\nu^{+}(v)}^{+} - Y_{\nu^{-}(v)}^{-}) \) and
\[
\kappa_{\pm}(v) = (\Pi(\Pi^{-}(v)) - v)1_{\{\Pi^{\pm,\psi}(v) \neq 0\}}.
\]

From the above analysis, we can write down the characteristic functions of the trimmed processes. For each \( \theta \in \mathbb{R} \) and \( v > 0 \), define
\[
\tilde{\Phi}(\theta, v) := i\theta \overline{\gamma}_{v} - \frac{1}{2} \sigma^{2} \theta^{2} + \int_{|x| < \Pi^{-}(v)} \left\{ e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1} \right\} \Pi(dx) + (e^{i\theta \Pi^{+}(v)} - 1) + \kappa_{+}(v)(e^{-i\theta \Pi^{-}(v)} - 1).
\]

Note that this is the characteristic exponent of \( \overline{X}_{t}^{v} + \sigma G_{t}^{v} \). Similarly for \( r, s \)-asymmetrical trimming, define, for each \( u, v > 0 \) and \( \theta \in \mathbb{R} \),
\[
\Phi(\theta, u, v) := i\theta \gamma_{u,v} - \frac{1}{2} \sigma^{2} \theta^{2} + \int_{(\Pi^{-,\psi}(u), \Pi^{+,\psi}(v))} \left\{ e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1} \right\} \Pi(dx) + \rho_{+}(v)(e^{i\theta \Pi^{+}(u)} - 1) + \rho_{-}(u)(e^{-i\theta \Pi^{-}(u)} - 1),
\]

\[ 7 \]
which is the characteristic exponent of $X_1^{u,v} + G_1^{+u,v} - G_1^{-u,v}$.

Then the characteristic functions of the trimmed processes are

$$E\left(e^{i\theta X_t}\right) = \int_{(0,\infty)} \exp(t\Phi(\theta,v))P(\Gamma_r \in tdv) \quad (25)$$

and

$$E\left(e^{i\theta X_t}\right) = \int_{0}^{\infty} \int_{0}^{\infty} \exp(t\Phi(\theta,u,v))P(\Gamma_s \in tdv)P(\Gamma_r \in tdv).$$

2.1 Normed and Centered Truncation

Suppose for some non-stochastic functions $a_t \in \mathbb{R}$ and $b_t > 0$ and a sequence $t_n \downarrow 0$, a Lévy process $X_t$ has a limit in distribution, i.e.

$$\frac{X_{t_n} - a_{t_n}}{b_{t_n}} \overset{D}{\rightarrow} Y, \quad \text{as} \quad t \downarrow 0,$$

(26)

for some a.s. finite nondegenerate random variable $Y$. By Maller and Mason (see Lemma 4.1), $Y$ has to be infinitely divisible, say with triplet $(\beta, \tau^2, \Lambda)$. We would like to investigate the convergence of the truncated processes with the same centering and norming, i.e. the asymmetrical truncated process $X_t^{u,t,v/t}$ and the modulus truncated process $\tilde{X}_t^{v,t}$ in [13] and [14] for appropriate $u, v > 0$ through the sequence $t_n$. However, in order to relate to the trimmed process, we need to consider not just the truncated processes but also the Poisson number of ties at each truncation level. With this restriction, we only get convergence through a subsequence in general. Nonetheless, this suffices for our purposes.

For each $t > 0$, $u, v > 0$, and $a_t \in \mathbb{R}$, $b_t > 0$ non-stochastic functions, abbreviate the normed, centred, truncated processes including the Poisson number of ties by

$$Z_t^{u,v} := \frac{X_t^{u,t,v/t} + G_t^{+u,v/t} - G_t^{-u,v/t} - a_t}{b_t} \quad \text{and} \quad \tilde{Z}_t^v := \frac{\tilde{X}_t^{v,t} + \tilde{G}_t^{v,t} - a_t}{b_t}.$$  

(27)

If $(X_{t_n} - a_{t_n})/b_{t_n}$ converges as in (26), we would like to show that $Z_t^{u,v}$ and $\tilde{Z}_t^v$ also have infinitely divisible limits at least through a subsequence of $t_n$. Let $\Lambda$ and $\overline{\Lambda}$ denote the tails of the Lévy measure $\Lambda$ of $Y$.

Lemma 2.1 Suppose $\Pi(0+) = \infty$ and for some non-stochastic functions $a_t$ and $b_t > 0$, and sequence $t_n \downarrow 0$

$$\frac{X_{t_n} - a_{t_n}}{b_{t_n}} \overset{D}{\rightarrow} Y, \quad \text{as} \quad n \rightarrow \infty$$

(28)

for some a.s. finite infinitely divisible distribution $Y$ with characteristic triplet $(\beta, \tau^2, \Lambda)$. Suppose further that $\Lambda \neq 0$ so there exists $l > 0$ such that $m := \overline{\Lambda}(l) > 0$. Then

(i) For each continuity point $v$ of $\overline{\Lambda}$ such that $v \in (0, m)$, $(\tilde{X}_{t_n}^{v,t_n} - a_{t_n})/b_{t_n}$ converges in distribution to an infinitely divisible random variable $Y^v$ as $n \rightarrow \infty$,
where $\tilde{Y}^v$ is the value at time 1 of a Lévy process with canonical triplet $(\tilde{\beta}_v, \tilde{\tau}_v^2, \tilde{\Lambda}_v)$ given by

$$
\tilde{\beta}_v = \beta - \mathbf{1}_{\{X^{-}(v) \leq 1\}} \int_{X^{-}(v) \leq |y| \leq 1} y \Lambda(dy), \quad \tilde{\tau}_v^2 = \tau^2, \quad \tilde{\Lambda}_v(dx) = \Lambda(dx) \mathbf{1}_{\{|x| < X^{-}(v)\}}.
$$

(29)

Similarly, for each continuity point $u > 0$ of $\bar{X}^{-,v}(-)$ and each continuity point $v > 0$ of $\bar{X}^{+,v}(-)$, such that $u, v \in (0, m)$, we have

$$
\frac{X_{t_{nk}}^{u/v, u/v} - a_{t_{nk}}}{b_{t_{nk}}} \xrightarrow{D} Y_{t_{nk}}^{u,v} \quad \text{as} \quad n \to \infty
$$

where $Y_{t_{nk}}^{u,v}$ has canonical triplet $(\beta_{u,v}, \tau_{u,v}^2, \Lambda_{u,v})$ given by

$$
\beta_{u,v} = \beta - \mathbf{1}_{\{X^{+,v}(v) \leq 1\}} \int_{X^{+,v}(v) \leq |y| \leq 1} y \Lambda(dy) + \mathbf{1}_{\{\bar{X}^{+,v}(u) \leq 1\}} \int_{\bar{X}^{+,v}(u) \leq |y| \leq 1} y \Lambda(dy),
$$

$$
\tau_{u,v}^2 = \tau^2, \quad \text{and} \quad \Lambda_{u,v}(dx) = \Lambda(dx) \mathbf{1}_{\{-\bar{X}^{+,v}(u) < x < \bar{X}^{+,v}(u)\}}.
$$

(30)

(ii) For each $u, v \in (0, m)$ that are continuity points of $\bar{X}^{-,v}$ and $\bar{X}^{+,v}$ respectively, there exists a subsequence $\{t_{nk} \downarrow 0\}$ and some infinitely divisible random variables $Y_{t_{nk}}^{u,v}$ and $\bar{Y}^v$ which may depend on the choice of subsequence such that

$$
Z_{t_{nk}}^{u,v} \xrightarrow{D} Y_{t_{nk}}^{u,v} \quad \text{and} \quad \bar{Z}_{t_{nk}}^{v} \xrightarrow{D} \bar{Y}^v \quad \text{as} \quad k \to \infty.
$$

(31)

In both (i) and (ii), the supports of the Lévy measures of the limit distributions of $Y_{t_{nk}}^{u,v}$ and $\bar{Y}^v$ include the sets $(-\bar{X}^{+,v}(u), \bar{X}^{+,v}(v))_*$ and $(-\bar{X}^{-}(v), \bar{X}^{-}(v))_*$ respectively.

**Proof:** Assume $\Pi(0+) = \infty$. We will prove the case with modulus truncation and to ease notation we will write $t$ for $t_{nk}$. We thus assume $(X_t - a_t)/b_t$ converges as $t \downarrow 0$ but make no assumption regarding the limit distribution other than that it is a.s. finite. By Kallenberg’s conditions (Theorem 15.14, Kallenberg [9]), we have the following limits for each continuity point $x > 0$ of $\bar{X}(-)$:

$$
\lim_{t \downarrow 0} t \Pi^\pm(xb_t) = \bar{X}^\pm(x), \quad \lim_{t \downarrow 0} \frac{tV(xb_t)}{b_t^2} = \tau^2 + \int_{|y| \leq x} y^2 \Lambda(dx), \quad \lim_{t \downarrow 0} \frac{t\nu(b_t) - a_t}{b_t} = \beta.
$$

(32)

By properties of inverse monotone functions (Proposition 0.1 in Resnick p.5 [15]), the first relation in (32) implies that $\Pi^\pm(v/t)/b_t \to \bar{X}^\pm(v)$ for each continuity point $v > 0$ of $\bar{X}(-)$. By (33) and (35), we have

$$
E \left( \exp(i\theta \bar{Z}^v_t) \right) = \exp \left\{ i\theta \left( \frac{t \bar{\tau}^2(v/t) - a_t}{b_t} - t \int_{b_t \leq |x| \leq 1} \frac{x}{b_t} \Pi(dx) \right) \right. 
$$

$$
- \frac{1}{2} \frac{t^2 \theta^2}{b_t^2} + t \int_{|x| < \bar{X}^{-}(v/t)} \left( e^{i\theta x/b_t} - 1 - i\theta x/b_t \mathbf{1}_{|x| \leq b_t} \right) \Pi(dx) 
$$

$$
+ t\kappa^+(v/t) \left( e^{i\theta \bar{X}^{+}(v/t)/b_t} - 1 \right) + t\kappa^-(v/t) \left( e^{-i\theta \bar{X}^{-}(v/t)/b_t} - 1 \right) \right\}.
$$

(33)
By (20), the resulting centering, i.e. the first line on the RHS of (33), equals

\[
\left( \frac{t\gamma - at}{b_t} - 1_{\{\Pi^+(v/t)\leq 1\}} t \int_{\Pi^+(v/t) \leq |x| \leq 1} \frac{x}{b_t} \Pi(dx) - t \int_{|x| < 1, |x| \leq \Pi^+(v/t)} \frac{x}{b_t} \Pi(dx) \right)
\]

\[
= \left( \frac{t\gamma - at}{b_t} - 1_{\{\Pi^+(v/t)\leq b_t\}} t \int_{\Pi^+(v/t) \leq |x| \leq b_t} \frac{x}{b_t} \Pi(dx) - t \int_{|x| < 1, |x| \leq \Pi^+(v/t)} \frac{x}{b_t} \Pi(dx) \right)
\]

\[
= \frac{t\nu(b_t) - at}{b_t} - 1_{\{\Pi^+(v/t)/b_t\leq 1\}} \int_{\Pi^+(v/t)/b_t \leq |x| \leq 1} x \Pi(b_t dx)
\]

\[
\to_{t \downarrow 0} - 1_{\{\Lambda^+(v)\leq 1\}} \int_{\Lambda^+(v) \leq |x| \leq 1} x \Lambda(dx) := \tilde{\beta}_v.
\]

In the last line of (34), note that \( \Lambda^+(v) > 0 \) for \( v \in (0, m) \) which is a continuity point of \( \Lambda \), hence making use of (32) and dominated convergence, we arrive at the limit \( \tilde{\beta}_v \). We break up the second line in (33) into two parts. Recall that we have assumed \( \Lambda \neq 0 \) and thus for each \( v \in (0, m) \) a continuity point of \( \Lambda \), where \( m = \Lambda(l) > 0 \) for some \( l > 0 \), we have \( \Pi^+(v/t)/b_t \to \Lambda^-(v) \geq \Lambda^-(m) \geq l > 0 \). So \( \varepsilon b_t < \Pi^+(v/t) \) for all \( 0 < \varepsilon < \min(l, 1) \), \( v \in (0, m) \) and all small \( t \).

Now first consider the integral on \( \{|x| \leq \varepsilon b_t\} \). We have

\[
- \frac{t\sigma^2 \theta^2}{2b_t^2} + t \int_{|x| \leq \varepsilon b_t} \left( e^{i\theta x/b_t} - 1 - i\theta x/b_t \right) \Pi(dx)
\]

\[
= - \frac{t\sigma^2 \theta^2}{2b_t^2} + t \int_{|x| \leq \varepsilon b_t} \left( \frac{(i\theta x)^2}{2b_t^2} + O\left( \frac{|x|^3}{b_t^3} \right) \right) \Pi(dx)
\]

\[
= - \frac{t\theta^2}{2b_t^2} \left( \sigma^2 + \int_{|x| \leq \varepsilon b_t} x^2 \Pi(dx) \right) + t \int_{|x| \leq \varepsilon b_t} O\left( \frac{|x|^3}{b_t^3} \right) \Pi(dx)
\]

\[
= - \frac{t\theta^2 V(\varepsilon b_t)}{2b_t^2} + O\left( \frac{\varepsilon V(\varepsilon b_t)}{b_t^2} \right).
\]

By (32), we have that

\[
\lim_{\varepsilon \downarrow 0} \lim_{t \downarrow 0} \frac{t V(\varepsilon b_t)}{b_t^2} = \tau^2.
\]

The second term in (35) is \( O(\varepsilon) \) as \( t \downarrow 0 \) hence arbitrarily small. So the expression in (35) tends to \( -\theta^2 \tau^2/2 \) as \( t \downarrow 0 \) then \( \varepsilon \downarrow 0 \).

Next consider the component of the integral in the second line of (33) on \( \varepsilon b_t < |x| < \Pi^+(v/t) \):

\[
t \int_{\varepsilon b_t < |x| < \Pi^+(v/t)} \left( e^{i\theta x/b_t} - 1 - i\theta x/b_t 1_{|x| \leq b_t} \right) \Pi(dx)
\]

\[
= t \int_{|x| < \Pi^-(v/t)/b_t} \left( e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1} \right) \Pi(b_t dx)
\]

\[
\to \int_{|x| < \Lambda^-(v)} \left( e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1} \right) \Lambda(dx) \text{ as } t \downarrow 0 \text{ and then } \varepsilon \to 0.
\]

(36)
Therefore the overall limit as $t \downarrow 0$ for the second line in (33) is

\[- \frac{1}{2} \theta^2 \tau^2 + \int_{|x|<\Lambda^+(v)} \left( e^{i\theta x} - 1 - i\theta x 1_{|x| \leq 1} \right) \Lambda(dx). \tag{37}\]

From here we can see that the support of the limit Lévy measure is $(-\Lambda^+(v), \Lambda^+(v))$, without considering the ties component. The ties component will enlarge the support by including one or both boundary points. This proves Part (i), for the convergence of $(\tilde{X}_{t_n}^{v/t_n} - a_{t_n})/b_{t_n}$.

For Part (ii), the Poisson number of ties are added in to $\tilde{Z}_v^n$ in (27). This corresponds to the last line of (33). As before, we fix $v$ to be a continuity point of $\Lambda^+$ and $v \in (0, m)$. By (22), the ties disappear if $\tilde{\Pi}^+(v/t)$ is not an atom of $\Pi^{|}\cdot|$. Let $\{t_n\} \downarrow 0$ be the given sequence. If there exists a subsequence $\{t_{n_k}\} \downarrow 0$ such that $\tilde{\Pi}^+(v/t_{n_k})$ is a continuity point of $\Pi$ for all $\{t_{n_k}\}$, then the ties components converge to 0 as $k \to \infty$, and Part (ii) of the Lemma is true for this subsequence.

Suppose this is not the case. Henceforth without loss of generality, we assume further that $\Pi^{|}\{\tilde{\Pi}^+(v/t_n)\} \neq 0$ for all $n \in \mathbb{N}$. Observe from (27) that

\[\tilde{Z}_v^{t_n} = \frac{X_{t_n}^{v/t_n}}{b_{t_n}} - \frac{a_{t_n}}{b_{t_n}} + \frac{G_{t_n}^{v/t_n}}{b_{t_n}}. \tag{38}\]

Since we have shown in Part (i) that the first term in (38) converges to an infinitely divisible random variable with characteristic triplet $(\beta_v, \tau_v^2, \Lambda_v)$, we only need to show that $G_{t_n}^{v/t_n}/b_{t_n}$ has a limit through a subsequence. Recall from (21)-(22),

\[\frac{G_{t_n}^{v/t_n}}{b_{t_n}} = \frac{1}{b_{t_n}} \left( Y_{t_n\kappa^+(v/t_n)} - Y_{t_n\kappa^-(v/t_n)} \right)\]

where $Y^\pm$ are Poisson processes with unit mean, independent from $X_{t_n}^{v/t_n}$. By (22),

\[tk^\pm(v/t) = t \left( \frac{\tilde{\Pi}^+(v/t) - v}{v} \right) \Pi^{|}\{\tilde{\Pi}^+(v/t)\} = \int_v^{\Pi^+(v/t)} g^\pm(\tilde{\Pi}^+(v/t))du \tag{39}\]

where $g^\pm = d\Pi^\pm/d\Pi^{|}$ are the Radon-Nikodym derivatives of $\Pi^\pm$ with respect to $\Pi^{|}$. Since $\tilde{\Pi}^+(v/t)$ is an atom of $\Pi^{|}$, then we have

\[g^\pm(\tilde{\Pi}^+(v/t)) = \frac{\Pi^{|}\{\tilde{\Pi}^+(v/t)\}}{\Pi^{|}\{\tilde{\Pi}^+(v/t)\}}. \]

For each $w > 0$, $t > 0$, define

\[\lambda_{t}^\pm(w) = \int_0^w g^\pm(\tilde{\Pi}^+(u/t))du. \tag{40}\]

Note that $\tilde{\Pi}^-(u/t) = \tilde{\Pi}^+(v/t)$ for each $u \in (v, t\tilde{\Pi}^+(v/t)-)$. By (39), we can write

\[tk^\pm(v/t) = \lambda_{t}^+(t\tilde{\Pi}^-(v/t)-) - \lambda_{t}^-(v). \tag{41}\]
Observe that \( \lambda^\pm(t\overline{\Pi}^-(v/t)-) \) and \( \lambda^\pm(v) \) are nondecreasing in \( v \). Therefore by Helly’s selection theorem, there exists a subsequence \( \{t_{n_k} \downarrow 0\} \) of \( \{t_n\} \) and measurable functions \( h^\pm(\cdot) \) and \( l^\pm(\cdot) \) such that

\[
\lambda^\pm(t_{n_k}\overline{\Pi}^-(v/t_{n_k}^-)-) \to h^\pm(v) \quad \text{and} \quad \lambda^\pm_{t_{n_k}}(v) \to l^\pm(v) \quad \text{as} \quad k \to \infty. \tag{42}
\]

Therefore \( 0 < t_{n_k}\kappa^\pm(v/t_{n_k}) \to h^\pm(v)-l^\pm(v) =: \lambda^\pm(v) \). We claim that these quantities are finite for each \( v \in (0, m) \). To see this, note that for each \( v \in (0, m) \), we have \( \overline{\Lambda}^+(v) \geq l > 0 \). Hence there exists a \( \delta > 0 \) such that \( c_v := \overline{\Lambda}^-(v) - \delta > 0 \). Since we have \( \overline{\Pi}^-(v/t)/b_t \to \overline{\Lambda}^- \), then we have \( \overline{\Pi}^-(v/t) \geq b_tc_v \), for all sufficiently small \( t \). Hence

\[
t\overline{\Pi}^-(v/t)- \leq t\overline{\Pi}(b_tc_v) \to \overline{\Lambda}(c_v) < \infty. \tag{43}
\]

This shows that for each \( v \in (0, m) \), we have \( t\kappa^\pm(v/t) < \infty \) for all sufficiently small \( t > 0 \). To summarise, by (34), (37) and (42), we have \( E(\exp(i\theta\widetilde{T}_{t_{n_k}})) \) tends, as \( k \to \infty \), to

\[
\exp\left\{ i\theta\widetilde{\beta}_v - \frac{1}{2}\theta^2\tau^2 + \int_{|x|<\overline{\Lambda}^-(v)} \left( e^{i\theta x} - 1 - i\theta x1_{|x|\leq 1} \right) \Pi(dx) \right. \\
\left. \quad + \lambda^+(v) \left( e^{i\theta\overline{\Lambda}^+(v)} - 1 \right) + \lambda^-(v) \left( e^{-i\theta\overline{\Lambda}^-(v)} - 1 \right) \right\} := \widetilde{\psi}_v(\theta). \tag{44}
\]

Note that (44) is the characteristic function of the limit random variable, say \( \widetilde{Y}_v \), which is a convolution of an infinitely divisible random variable with canonical triplet \( (\widetilde{\beta}_v, \tau^2, \overline{\Lambda}_v) \) and two independent Poisson numbers at \( \pm\overline{\Lambda}^+(v) \) respectively.

This completes the proof of the modulus truncation. Asymmetrical truncation can be computed analogously. \( \square \)

3 Inequalities

In this section, we will derive inequalities that compare the trimmed processes with the ordered jumps. First let us write out the marginal distribution of the \((r+1)^{st}\) ordered jump from the representations in (17) and (21).

**Lemma 3.1** Let \( y > 0 \). The tail of marginal distribution of the \((r+1)^{st}\) largest jump in modulus is

\[
P(|\overline{\Delta}X_t^{(r+1)}| > y) = \int_0^{\overline{\Pi}(y)} P(\Gamma_{r+1} \in dv) = \int_0^{\overline{\Pi}(y)} P(\Gamma_r \in dv) - e^{-\overline{\Pi}(y)}(\overline{\Pi}(y))^r/r!. \tag{45}
\]

Denote the \((r+1)^{st}\) largest positive and negative jumps in magnitude by \( \Delta X_t^{(r+1),\pm} \) respectively. We have

\[
P(\Delta X_t^{(r+1),\pm} > y) = \int_0^{\overline{\Pi}^+(y)} P(\Gamma_{r+1} \in dv) = \int_0^{\overline{\Pi}^+(y)} P(\Gamma_r \in dv) - e^{-\overline{\Pi}^+(y)}(\overline{\Pi}^+(y))^r/r!. \tag{46}
\]
Hence,
\[ e^{-t\Pi(y)} \frac{(t\Pi(y))^{r+1}}{(r+1)!} \leq P(\tilde{\Delta}X_t^{(r+1)} > y) \leq \frac{(t\Pi(y))^{r+1}}{(r+1)!}, \tag{47} \]
and similarly,
\[ e^{-t\Pi^\pm(y)} \frac{(t\Pi^\pm(y))^{r+1}}{(r+1)!} \leq P(\Delta X_t^{(r+1),\pm} > y) \leq \frac{(t\Pi^\pm(y))^{r+1}}{(r+1)!}. \tag{48} \]

**Proof:** From the representation in (21), we have
\[ P(\tilde{\Delta}X_t^{(r+1)} > y) = P(\Gamma_{r+1} < t\Pi(y)). \tag{49} \]
This gives the first identity in (45). Then integrate by parts to get
\[ \int_0^{t\Pi(y)} \frac{1}{r!} x^r e^{-x} dx = \frac{1}{r!} \left( -(t\Pi(y))^r e^{-t\Pi(y)} + \int_0^{t\Pi(y)} r x^{r-1} e^{-x} dx \right). \tag{50} \]
Then we can read off the second identity in (45). \(46\) can be proved similarly. The inequality in (47) comes easily by observing that
\[ e^{-t\Pi(y)} \int_0^{t\Pi(y)} \frac{1}{r!} x^r dx \leq \int_0^{t\Pi(y)} e^{-x} \frac{x^r}{r!} dx \leq \int_0^{t\Pi(y)} \frac{x^r}{r!} dx. \tag{48} \]
\(48\) can be proved similarly. □

**Remark 3.2** From \(45\), we can see that the tail of the cdf of the ordered jumps satisfies
\[ P(\tilde{\Delta}X_t^{(r)} > y) = P(\Gamma_r < t\Pi(y)) \quad \text{and} \quad P(\tilde{\Delta}X_t^{(r)} \geq y) = P(\Gamma_r < t\Pi(y-)). \tag{51} \]
Therefore the discontinuity points of the distribution of the \(r\)th order statistics coincide with the atoms of the Lévy measure \(\Pi^{|\cdot|}\), which are at most countable.

Next we need to establish an approximation procedure to deal with the atoms of the Lévy measure. Recall from \(2\) that any Lévy measure \(\Pi\) with \(\Pi(0+) = \infty\) can be approximated by a sequence of absolutely continuous Lévy measures, \(\Pi_n\), with each \(\Pi_n(0+) = \infty\) and \(\Pi_n \to \Pi\) vaguely. Hence \(\Pi_n^- (v) \to \Pi^- (v)\) for each continuity point \(v > 0\) of \(\Pi^-\), also \(\Pi_n^+ (v) \to \Pi^+ (v)\) and \(\Pi_n^{+,-} (v) \to \Pi^{+,-} (v)\) for each continuity point \(v\) of \(\Pi^{+,-}\) and \(\Pi^{+,-}\) respectively. Let \(X_t\) and \(X_t(n)\) be Lévy processes with characteristic triplets \((\gamma, \sigma^2, \Pi)\) and \((\gamma, \sigma^2, \Pi_n)\) respectively.

We can approximate the distributions of the truncated processes and hence the trimmed processes as follows. Recall the truncated processes in \((14)\) and \((18)\). For each \(t > 0\) and continuity points \(u, v > 0\) of \(\Pi^-\), we have, as \(n \to \infty\),
\[ \tilde{X}_t^{u/v} (n) \overset{D}{\to} \tilde{X}_t^{u/v}, \quad \text{and} \quad X_t^{u/t,v/t} (n) \overset{D}{\to} X_t^{u/t,v/t}. \tag{52} \]
Similarly the modulus trimmed process satisfies
\[ |\Delta X_t^r(n) - |\Delta X_t^r| > x | \rightarrow \mathcal{L} |\Delta X_t^r| > x, \quad n \to \infty. \] (53)

Also for the order statistics, we have, as \( n \to \infty, \)
\[ |\Delta X_t^r(n)| \overset{D}{\rightarrow} |\Delta X_t^r|, \quad \text{and} \quad \Delta X_t^r(n) \overset{D}{\rightarrow} \Delta X_t^r, \quad \text{as} \quad n \to \infty. \] (54)

Therefore at continuity points \( x > 0, \) as \( n \to \infty, \)
\[ P\left( |\Delta X_t^r(n)| > x \right) \rightarrow P\left( |\Delta X_t^r| > x \right), \]
\[ P\left( \Delta X_t^r(n) > x \right) \rightarrow P\left( \Delta X_t^r > x \right) \quad \text{and} \quad P\left( \Delta X_t^{(s,-)}(n) > x \right) \rightarrow P\left( \Delta X_t^{(s,-)} > x \right). \] (55)

Recall the trimmed processes \( (r) \tilde{X}_t, \) \( (r) X_t \) and \( (s,-) X_t \) defined in [2] and [3]. We state our main inequality relating the trimmed process with the normed order statistics in the next lemma. A version of the following inequality appeared in Buchmann et al. [2] in which only the maximal trimmed process is considered.

Recall from Sato [17] Theorem 27.4 that Lévy processes with infinite activity have continuous distributions. Thus the truncated processes \( X_t^{u,v} \) or \( \tilde{X}_t \) also have continuous distributions, and consequently the trimmed processes also have continuous distributions. To see that this is so, note that from (17) and for each \( x > 0, \)
\[ P^{(r,s)} X_t > x = \int_u^v P(X_t^{u,v} + G_t^{r,v} - G_t^{-u} > x)P(\Gamma_t \in tdu, \tilde{\Gamma}_s \in tdu). \] (56)

Here \( X_t^{u,v} \) is a Lévy process with infinite Lévy measure (albeit on finite support), therefore its distribution is continuous. Recall also that the convolution of any distribution with a continuous distribution is still continuous. Hence \( X_t^{u,v} + G_t^{r,v} - G_t^{-u} \) also has a continuous distribution (see e.g. Lemma 27.1 in Sato [17]). Hence for each fixed \( t > 0, \) the trimmed process has a continuous cdf. This fact is important in the proof of Proposition 3.3.

**Proposition 3.3** Assume \( \bar{\Pi}(0+) = \infty. \) For each \( t, x > 0, r, s \in \mathbb{N}, \) let \( a_t \in \mathbb{R} \) be any non-stochastic function. We have
\[ 4P(|(r,s) X_t - a_t| > x) \geq \max \left( P(\Delta X_t^{(r+1)} > 4x), P(\Delta X_t^{(s+1,-)} > 4x) \right). \] (57)

By letting \( r = 0 \) or \( s = 0, \) we get
\[ 4P(|(r) X_t - a_t| > x) \geq P(\Delta X_t^{(r+1)} > 4x) \] (58)
and
\[ 4P(|(s,-) X_t - a_t| > x) \geq P(\Delta X_t^{(s+1,-)} > 4x). \] (59)

Similarly the modulus trimmed process satisfies
\[ 4P(|(r) \tilde{X}_t - a_t| > x) \geq P(|\Delta \tilde{X}_t^{(r+1)}| > 4x). \] (60)
Proof: First we will prove (57). Assume $\Pi(0+) = \infty$. Suppose that $4x > 0$ is a continuity point of $\Pi$. By the representation in (17), write $Y_{t,v}^\pm := Y_{t,v}(v)$, we have

$$P\left(\left|X^u,v_t - a_t\right| > x\right) = \int_{u,v \in (0,\infty)} P\left(\left|X^u,v_t - a_t + \Pi^+,v(v)Y_{t,v}^+ - \Pi^-,v(u)Y_{t,v}^-\right| > x\right) P(\Gamma_r \in tdu, \tilde{\Gamma}_s \in tdu)$$

$$= \sum_{j,k=0}^{\infty} \int_{u,v \in (0,\infty)} P\left(\left|X^u,v_t - a_t + \Pi^+,v(v)k - \Pi^-,v(u)j\right| > x\right) P(Y_{t,v}^+ = k, Y_{t,v}^- = j) P(\Gamma_r \in tdu, \tilde{\Gamma}_s \in tdu)$$

$$\geq \frac{1}{2} \sum_{j,k=0}^{\infty} \int_{u,v \in (0,\infty)} P\left(|\tilde{X}^u,v_t| > 2x\right) P(Y_{t,v}^+ = k, Y_{t,v}^- = j) P(\Gamma_r \in tdu, \tilde{\Gamma}_s \in tdu).$$

(61)

Here $\tilde{X}^u,v_t$ is a symmetrised version of $X^u,v_t$, obtained by subtracting an independent copy. In the last inequality, we used the symmetrisation inequality, i.e., with $\tilde{X}$ a symmetrised copy of $X$,

$$P(|X_t| > x) \geq \frac{1}{2} P(|\tilde{X}_t| > 2x) \quad \text{for} \quad t > 0.$$ 

Hence $\tilde{X}^u,v_t$ has Lévy measure

$$\tilde{\Pi}(\cdot) = \Pi_{u,v}(\cdot) + \Pi_{u,v}(-\cdot),$$

where $\Pi_{u,v}$ is the Lévy measure of the truncated process $X^u,v_t$. Therefore we can express the tail function as $\tilde{\Pi}(x) = 2\Pi_{u,v}(x)$, with positive and negative parts

$$\begin{cases} 
\tilde{\Pi}^+(x) = 2(\Pi^+(x) - \Pi^+(\Pi^+,v(v)-), & 0 < x < \Pi^+,v(v), \\
\tilde{\Pi}^-(x) = 2(\Pi^-(x) - \Pi^-(\Pi^-,v(u)-), & 0 < x < \Pi^-,v(u). 
\end{cases}$$

(62)

Summing over $j$ and $k$, (61) becomes

$$\frac{1}{2} \int_{u,v \in (0,\infty)} P\left(|\tilde{X}^u,v_t| > 2x\right) P(\Gamma_r \in tdu, \tilde{\Gamma}_s \in tdu).$$

(63)

Recall the continuous approximation of the Lévy measure from (52) - (55), to get

$$|\tilde{X}^u,v_t(n)| \xrightarrow{D} |\tilde{X}^u,v_t| \quad \text{as} \quad n \to \infty.$$ 

By Sato [17], Theorem 27.4, the distribution of a Lévy process with infinite activity is continuous. Hence for each $x > 0$, we have the following approximation. By the dominated convergence theorem, (63) equals

$$\lim_{n \to \infty} \frac{1}{2} \int_{u,v \in (0,\infty)} P\left(|\tilde{X}^u,v_t(n)| > 2x\right) P(\Gamma_r \in tdu, \tilde{\Gamma}_s \in tdu).$$

(64)
Note that by Lévy’s maximal inequality, for each \( x > 0 \),
\[
2P(|X_t| > x) \geq P(\sup_{0 < s \leq t} |\Delta X_s| > 2x) \geq P(\sup_{0 < s \leq t} \Delta X_s > 2x) \lor P(\sup_{0 < s \leq t} \Delta X_s^- > 2x).
\]

On the set \( \{0 < 4x < \Pi_n^{1,\pm}(v)\} = \{v < \Pi_n^{1}(4x)\} \), apply this inequality to (61) and (64) to get
\[
4P(|^{(r,s)}X_t - a_t| > x) \\
\geq \lim_{n \to \infty} \int_0^{\Pi_n^{1}(4x)} \int_{u(0,\infty)} P(\sup_{0 \leq s \leq t} \Delta X_s^{u,v}(n) > 4x)P(\Gamma_r \in tdv, \Gamma_s \in tdu) \\
= \lim_{n \to \infty} \int_0^{\Pi_n^{1}(4x)} \int_{u(0,\infty)} \left[1 - e^{-2t(\Pi_n^{1}(4x) - \Pi_n^{1,\pm}(v/t) - 0)}\right] P(\Gamma_r \in dv, \Gamma_s \in du) \\
\geq \lim_{n \to \infty} \int_0^{\Pi_n^{1}(4x)} \left(1 - e^{-\Pi_n^{1}(4x) + v}\right) P(\Gamma_r \in dv). \quad (65)
\]
where the equality follows from (62) and the last inequality follows since \( \Pi_n \) is continuous and thus \( \Pi_n^{1}(\Pi_n^{1,\pm}(v/t) - v) = v \) for each \( v > 0 \).

We simplify the RHS in (65) as the limit of
\[
\int_0^{\Pi_n^{1}(4x)} P(\Gamma_r \in dv) - e^{-\Pi_n^{1}(4x)} \int_0^{\Pi_n^{1}(4x)} \frac{u^{r-1}}{(r-1)!}dv. \quad (66)
\]

On the assumption that \( 4x \) is a continuity point of \( \Pi \), we have \( \Pi_n^{1}(4x) \to \Pi^{1}(4x) \) as \( n \to \infty \). Hence we have the limit of (66), as \( n \to \infty \), to be
\[
\int_0^{\Pi^{1}(4x)} P(\Gamma_r \in dv) - e^{-\Pi^{1}(4x)} \frac{(\Pi^{1}(4x))^r}{r!} = P(\Delta X_t^{(r+1)} > 4x).
\]
The last equality is due to (16).

Similarly, if we consider the set \( \{0 < 4x < \Pi_n^{1,\pm}(v)\} \) and use \( P(\sup_{0 < s \leq t} \Delta X_s > 4x) \) instead of \( P(\sup_{0 < s \leq t} \Delta X_s > 4x) \) in (65), we will arrive at (67). (60) can be proved similarly. So far we have proved that the inequalities (57)–(60) hold for \( 4x \) being a continuity point of \( \Pi \) or \( \Pi^{\pm} \) respectively. To eliminate this assumption, let \( x > 0 \) be any arbitrary point. We can choose a sequence \( x_n \downarrow x \) such that the members of the sequence \( \{4x_n\} \) are all continuity points of \( \Pi \). Therefore for each \( x_n \) we have
\[
4P(|^{(r,s)}X_t - a_t| > x_n) \geq P(\Delta X_t^{(r+1)} > 4x_n). \quad (67)
\]
Recall the discussion below (56) that \(^rX_t\) has a continuous cdf. Thus, we can take \( n \to \infty \) in the LHS of (67) to get
\[
\lim_{n \to \infty} 4P(|^{(r,s)}X_t - a_t| > x_n) = 4P(|^{(r,s)}X_t - a_t| > x).
\]
From the distribution of the ordered jumps in Lemma 3.1, (45) -(46), and noting that $\Pi^\pm$ are right continuous functions, we have $\Pi^\pm(x_n) \to \Pi^\pm(x)$, so

$$\lim_{n \to \infty} P(\Delta X_t^{(r+1)} > 4x_n) = \lim_{n \to \infty} \int_0^{\Pi^+ (4x_n)} P(\Gamma_{r+1} \in dv) = \int_0^{\Pi^+ (4x)} P(\Gamma_{r+1} \in dv) = P(\Delta X_t^{(r+1)} > 4x).$$

(68)

This completes the proof of Proposition 3.3.

□

4 Proof of Theorems

Proof: [Proof of Theorem 1.1 (a):] Take $r, s \in \mathbb{N}$. Let $\left(\left(r, s\right)S_t\right)$ be tight. From Proposition 3.3, (57), we have for each $x > 0$ and $t > 0$,

$$4P \left(\left|\left(\left(r, s\right)X_t - a_t\right) \right| > xb_t\right) \geq \max \left(P \left(\Delta X_t^{(r+1)} > 4xb_t\right) , P \left(\Delta X_t^{(s+1),-} > 4xb_t\right)\right).$$

(69)

Take $\limsup_{t \downarrow 0}$ and then $\lim_{x \to \infty}$ to obtain

$$0 = \lim_{x \to \infty} \limsup_{t \downarrow 0} 4P \left(\left|r, s\right|S_t \right| > x\right) \geq \lim_{x \to \infty} \limsup_{t \downarrow 0} \left(P \left(\Delta X_t^{(r+1)}/b_t > 4x\right) , P \left(\Delta X_t^{(s+1),-}/b_t > 4x\right)\right).$$

(70)

This implies that $\left(\Delta X_t^{(r+1)}/b_t\right)$ and $\left(\Delta X_t^{(s+1),-}/b_t\right)$ are tight families as $t \downarrow 0$. Hence there exists $x_0 > 0$ such that

$$\lim_{t \downarrow 0} P(\Delta X_t^{(r+1)}/b_t > x) \leq 1/2 \quad \text{for all } x > x_0.$$ (71)

For such a $x > x_0$, suppose there exists a sequence $\{t_k\} \downarrow 0$ such that $t_k \Pi^+(b_{t_k} x) \to \infty$ as $k \to \infty$. Then by (46)

$$P(\Delta X_{t_k}^{(r+1)} > b_{t_k} x) = \int_0^{t_k \Pi^+(b_{t_k} x)} P(\Gamma_{r+1} \in dv) \to 1 \quad \text{as } k \to \infty,$$

(72)

which contradicts (71). Therefore we have shown that

$$\limsup_{t \downarrow 0} t \Pi^+ (b_t x) < \infty$$

for each $x > x_0$. By (48), we have for each $x > 0$,

$$P \left(\Delta X_t^{(r+1)} > b_t x\right) \geq e^{-\Pi^+(b_t x)} \frac{(\Pi^+(b_t x))^{r+1}}{r+1!}.$$ (73)

So we must have that $\lim_{x \to \infty} \limsup_{t \downarrow 0} t \Pi^+ (b_t x) = 0$. By the same reasoning we also have $\lim_{x \to \infty} \limsup_{t \downarrow 0} t \Pi^- (b_t x) = 0$. 17
Conversely, assume \( \lim_{x \to \infty} \limsup_{t \downarrow 0} t \Pi_t^+(b_t x) = 0 \). By (48), for each \( r \in \mathbb{N} \), \( x > 0 \),
\[
\lim_{x \to \infty} \limsup_{t \downarrow 0} P(\Delta X_t^{(r)} > xb_t) \leq \lim_{x \to \infty} \limsup_{t \downarrow 0} \frac{(t \Pi_t^+(b_t x))^r}{r!} = 0
\]
This proves statements (ii) and (iii). Recall the fact that the sum of tight families is again a tight family. Then since \((s, -)S_t = (r, s)S_t + \sum_{i=1}^r \Delta X_t^{(i)} / b_t \), we have \((s, -)S_t\) is also tight at 0. Similarly, since \((r)S_t = (r, s)S_t - \sum_{i=1}^s \Delta X_t^{(i)} / b_t \), we also have \((r)S_t\) is tight at 0. Note that \( S_t = (s, -)S_t - \sum_{i=1}^s \Delta X_t^{(i)} / b_t \), we have \( S_t \) is tight at 0. This completes the proof of Part (a) and Part (b) is proved similarly. □

Before proving the main theorem, we write down a useful lemma to eliminate the easy direction.

**Lemma 4.1** If there exists a subsequence \( t_k \downarrow 0 \) such that \((X_{t_k} - a_{t_k}) / b_{t_k} \) \( \xrightarrow{P} 0 \) as \( k \to \infty \) or \((X_{t_k} - a_{t_k}) / b_{t_k} \) \( \xrightarrow{D} N(0, 1) \) as \( k \to \infty \), then \( \Delta X_{t_k}^{(i)}/b_{t_k} \) \( \xrightarrow{P} 0 \) and \( \Delta X_{t_k}^{(i)+}/b_{t_k} \) \( \xrightarrow{P} 0 \) for \( i = 1, 2, 3, \ldots \) as \( k \to \infty \).

**Proof:** Either convergence implies, by (32), that \( t_k \Pi(b_{t_k} x) \to 0 \) for all \( x > 0 \), and this implies
\[
P(\bigl|\Delta X_{t_k}^{(1)}/b_{t_k} > \varepsilon\bigr|) = 1 - e^{-t_k \Pi(b_{t_k} \varepsilon)} \to 0
\]
for any \( \varepsilon > 0 \). Hence \( \Delta X_{t_k}^{(1)}/b_{t_k} \) \( \xrightarrow{P} 0 \). Thus \( \Delta X_{t_k}^{(i)+}/b_{t_k} \) \( \xrightarrow{P} 0 \) for \( i = 1, 2, \ldots \) as \( k \to \infty \). □

**Proof:** [Proof of Theorem 1.2] Necessity follows from Lemma 4.1. We shall prove the sufficiency. Assume (9). If \( \sigma^2 > 0 \) we have the truncated second moment function \( V(x) \geq \sigma^2 > 0 \), thus
\[
\frac{x^2 \Pi(x)}{V(x)} \to 0.
\]
By (6), this implies \( X_t \) is in the domain of attraction of a normal distribution at 0, in which case (5) holds with \( N(0, \sigma^2) \) on the RHS. But then \( \sigma^2 = 1 \) since the limit distribution is \( N(0, 1) \). So we can suppose \( \sigma^2 = 0 \) in what follows.

Frist we will deal with the degenerate limit. Suppose, without loss of generality, that the limit distribution is degenerate at 0. Then the LHS of (57), with \( x \) replaced by \( xb_t \), goes to 0 as \( t \downarrow 0 \), so we must have that
\[
\Delta X_{t}^{(r+1)}/4b_t \xrightarrow{P} 0 \quad \text{and} \quad \Delta X_{t}^{(s+1),-}/4b_t \xrightarrow{P} 0.
\]
By (48), this implies, for each \( x > 0 \),
\[
0 = \lim_{t \downarrow 0} P(\Delta X_t^{(r+1),+} > 4xb_t) \geq \lim_{t \downarrow 0} e^{-t \Pi^+(4xb_t)} \frac{(t \Pi^+(4xb_t))^{r+1}}{(r+1)!}.
\]
By a similar argument as in the proof of Theorem 1.1 the degeneracy of \((r, s)S_t\) implies \( \limsup_{t \downarrow 0} t \Pi_t^x(xb_t) < \infty \) for \( x > 0 \). Therefore as \( t \downarrow 0 \), we in fact have
$\lim_{t \to 0} t\Pi^\pm (4xzbt) = 0$ for all $x > 0$. As in Lemma 4.1, we have $\Delta X_t^{(i), \pm} / b_t \to 0$, $i = 1, 2, \ldots$. Therefore the original normed and centered process also converges, that is

$$S_t = (r,s)S_t + \sum_{i=1}^r \frac{\Delta X_t^{(i)}}{b_t} - \sum_{j=1}^s \frac{\Delta X_t^{(j)},-}{b_t} \overset{P}{\to} 0.$$ 

This completes the proof for the case with a degenerate limit.

Now we can concentrate on the non-trivial case where the limit distribution of $(r,s)S_t$ is $N(0,1)$. If implies $(r,s)S_t$ is tight at 0. From Theorem 1.1 then, $S_t$ is tight at 0, which is equivalent to saying that $S_t$ is relatively compact. Therefore every sequence has a convergent subsequence. In fact, $S_t$ is stochastically compact, i.e. no subsequence could have a degenerate limit in distribution. If this were not so, there would be a subsequence, say $(t_k)$, through which $(r,s)S_t$ converged to a degenerate distribution. Then by Lemma 4.1 $\Delta X_t k$ would tend to 0 in probability, and so the trimmed process $(r,s)X_t k - a_k)/b_k$ would converge to the same degenerate distribution. But this contradicts the assumption that $(r,s)S_t \to N(0,1)$ as $t \to 0$.

Therefore, for each sequence $(t_k)$, there exists a further subsequence (also denoted $(t_k)$) such that $(X_t k - a_k)/b_k \overset{D}{\to} Z$ as $k \to \infty$ for some a.s. finite non-degenerate infinitely divisible random variable $Z$ with canonical triplet $(\alpha, \tau^2, \Pi_z)$, say. By (32) this implies for any $x > 0$ which is a continuity point of $\Pi_z$,

$$\lim_{k \to \infty} t_k \Pi(b_kx) = \Pi_z(x) \quad \text{and} \quad \lim_{k \to \infty} \frac{t_k V(b_kx)}{b_k^2} = \tau^2 + \int_{|y| \leq x} y^2 \Pi_z(dy). \quad (74)$$

We will show that $\Pi_z(\cdot) \equiv 0$. Suppose not. Then there exists $l > 0$ such that $m = \Pi_z(l) > 0$. By the representation in (17), for any $x > 0$ and $t > 0$,

$$P((r,s)S_t_k > x) = \int_{u,v \in (0,\infty)} P(Z_{t_k}^{u,v} > x)P(\Gamma_r \in du, \tilde{\Gamma}_s \in du) \geq \int_{u,v \in (0,m)} P(Z_{t_k}^{u,v} > x)P(\Gamma_r \in du, \tilde{\Gamma}_s \in du), \quad (75)$$

where

$$Z_t^{u,v} := \frac{X_t^{u/t, v/t} + G^{+,u/t} - G^{-,u/t} - a_t}{b_t}.$$ 

Recall that, by Lemma 2.1 along a further subsequence of $(t_k)$ (still denoted $(t_k)$), we have $Z_{t_k}^{u,v} \overset{D}{\to} Y^{u,v}$ for each $u, v \in (0,m)$ as $k \to \infty$ where $Y^{u,v}$ is an infinitely divisible distribution with support including the set $(-\Pi^{+,+}_z(u), \Pi^{+,+}_z(v))$. 

Take $k \to \infty$ on both sides of (75) and apply Fatou’s lemma to get

$$\lim_{k \to \infty} P((r,s)S_t_k > x) \geq \int_{u,v \in (0,m)} \liminf_{k \to \infty} P(Z_{t_k}^{u,v} > x)P(\Gamma_r \in du, \tilde{\Gamma}_s \in du) \geq \int_{u,v \in (0,m)} P(Y^{u,v} > x)P(\Gamma_r \in du, \tilde{\Gamma}_s \in du). \quad (76)$$

Let $U_t$ be any Lévy process with Lévy measure $\Pi_U$. Define the support of $\Pi_U$ by $S_{\Pi_U}$ and let $c = \inf\{a > 0 : S_{\Pi_U} \subset \{x : |x| \leq a\}\}$. By Sato [17] (Theorem 26.1,
p.168), for any \( \alpha > 1/c \) and any \( t > 0 \), we have an estimate for the tail probability of \( U_t \) as

\[
e^{\alpha r \log r} P(|U_t| > r) \to \infty \quad \text{as} \quad r \to \infty.
\]

Note that \( \Pi_{z}^{-}(m) = \Pi_{z}^{+}(l) = l \), thus \( l \) is in the support of the Lévy measure of \( Y_{u,v} \) for all \( u, v \in (0, m) \) and \( 1/l = 1/\Pi_{z}^{+}(m) > 1/\Pi_{z}^{+,c}(u) \wedge 1/\Pi_{z}^{-,c}(v) \). We can apply the above result to \( Y_{u,v} \) to get

\[
limit_{x \to \infty} e^{x \log x/l} P(|Y_{u,v}| > x) = \infty.
\]

(77)

It follows from Egorov’s theorem that there exists a subset \( E \) of the interval \( (0, m) \) with positive Lebesgue measure such that (77) holds uniformly on \( E \). Multiply \( e^{x \log x/l} \) on both sides of (76). Then the modified RHS of (76) tends to infinity as \( x \to \infty \), while the modified LHS of (76) converges to zero as a result of the estimate

\[
e^{x \log x/l} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \leq e^{x \log x/l} O(e^{-x^2/2}) \to 0 \quad \text{as} \quad x \to \infty.
\]

(78)

This contradiction proves that \( \Pi_{z}(\cdot) \equiv 0 \) and therefore \( Z \) is Gaussian. This means that \( Z \) is \( N(0, \tau^2) \) for some \( \tau^2 > 0 \) (else \( Z \) would be degenerate, which case we eliminated earlier). Here we use \( ' \) to indicate that \( \tau' \) depends on the subsequence. We have shown that for each sequence, there exists a subsequence \( t' \) such that \( S_{t'} \to N(0, \tau^2) \). By the assumption in (9), we have through this subsequence that \( (r,s)X_{t'} - a_{t'} / b_{t'} \to N(0,1) \). This forces \( \tau^2 = 1 \). Since this is true for all sub-sequences, we have completed the proof for the case when the limit distribution is normal.

This completes the proof for \((r,s)X_t\). The version for \((r)\tilde{X}_t\) goes through with virtually the same argument. \( \square \)

5 Related Results

Define the quadratic variation process of \( X_t \) as \( V_t := \sigma^2 t + \sum_{s \leq t}(\Delta X_s)^2 \) and let the trimmed versions of \( V_t \) be

\[
(r,s)V_t := V_t - \sum_{i=1}^{r}(\Delta X_t^{(i)})^2 - \sum_{j=1}^{s}(\Delta X_t^{(j),-})^2 \quad \text{and} \quad (r)\tilde{V}_t := V_t - \sum_{i=1}^{r}(\tilde{\Delta} X_t^{(i)})^2
\]

respectively corresponding to asymmetrical and modulus trimming. We can easily deduce from Theorem 1.2 the relationship between the trimmed quadratic variation processes and the untrimmed version.

**Corollary 5.1** Under the assumptions of Theorem 1.2, for any \( r, s \in \mathbb{N}, b_t > 0 \) and \( \tau^2 > 0 \), as \( t \downarrow 0 \),

\[
\frac{(r,s)V_t}{b_t^2} \overset{P}{\to} \tau^2 \quad \text{or} \quad \frac{(r)\tilde{V}_t}{b_t^2} \overset{P}{\to} \tau^2 \quad \text{if and only if} \quad \frac{V_t}{b_t^2} \overset{P}{\to} \tau^2.
\]

(79)
Furthermore, we have (79) being equivalent to the existence of \( a_t \in \mathbb{R}, \ b_t > 0 \) such that
\[
\frac{X_t - a_t}{b_t} \xrightarrow{D} N(0, \tau^2), \quad \text{as} \quad t \downarrow 0.
\] (80)

The \( b_t \) in (79) and (80) can be taken the same.

**Proof:** [Proof of Corollary 5.1] The quadratic variation process of \( X_t \) with triplet \((\gamma, \sigma^2, \Pi)\) is a Lévy subordinator with drift \( \sigma^2 \) and Lévy measure \( \Pi_q \) where \( \Pi_q(x) = \Pi(\sqrt{x}) \) for each \( x > 0 \). Apply Theorem 1.2 to \( V_t \) with centering function 0 and norming function \( b_t^2 \) to get necessity. Sufficiency is a consequence of Lemma 4.1. This completes the proof of (79). The second statement comes from applying the Kallenberg convergence criterion (32) for subordinators. Note that (80) holds if and only if for each \( x > 0 \), as \( t \downarrow 0 \),
\[
t \Pi(xb_t^2) \rightarrow 0 \quad \text{and} \quad tV(xb_t^2) b_t^2 \rightarrow \tau^2.
\] (81)

Also that \( V_t/b_t^2 \xrightarrow{P} \tau^2 \) holds if and only if for each \( x > 0 \), as \( t \downarrow 0 \),
\[
t \Pi(xb_t^2) \rightarrow 0 \quad \text{and} \quad tV(xb_t^2) b_t^2 \rightarrow \tau^2.
\] (82)

Observe that \( t \Pi(xb_t^2) = t \Pi(\sqrt{x}b_t) \) and
\[
t \int_{0 \leq |y| \leq \sqrt{x}b_t} y \Pi_q(dy) = t \int_{0 \leq |y| \leq \sqrt{\tau}b_t} y^2 \Pi(dy) = \frac{tV(\sqrt{x}b_t)}{b_t^2}.
\]

Hence the two conditions in (81) and (82) are equivalent. This completes the proof.

The next theorem gives a subsequential version of Theorem 1.2. We say that \( X_t \) is in the domain of partial attraction of normal distribution if there exist sequences \( t_k \downarrow 0, \ a_k \in \mathbb{R} \) and \( b_k > 0 \) such that
\[
\frac{X_{t_k} - a_k}{b_k} \rightarrow N(0, 1).
\] (83)

A necessary and sufficient condition for (83) is that
\[
\liminf_{t \downarrow 0} \frac{x^2 \Pi(x)}{V(x)} = 0.
\]

**Theorem 5.2** Assume \( \Pi(0+) = \infty \). (83) holds if and only if, for any \( r, s \in \mathbb{N} \), there exist sequences \( t_k' \downarrow 0, \ a_k' \) and \( b_k' > 0 \) such that
\[
\frac{(r,s)X_{t_k'} - a_k'}{b_k'} \rightarrow N(0, 1),
\] (84)
or, equivalently,
\[
\frac{(r)\Xi_{t_k'} - a_k'}{b_k'} \rightarrow N(0, 1).
\] (85)
Proof: That (83) implies (84) or (85) is obvious by Lemma 4.1. In this case we can choose the same sequences, i.e., \((t_k', a_k', b_k') = (t_k, a_k, b_k')\). For the converse, write \((r,s) S_{t_k'} = \left( (r,s) X_{t_k'} - a_k' \right) / b_k'\). The convergence of \((r,s) S_{t_k'} \to N(0,1)\) as \(k \to \infty\) implies the tightness of \((r,s) S_{t_k'}\) as \(k \to \infty\). By the same argument as before, we can deduce that \((r,s) S_{t_k'}\) is stochastically compact as \(k \to \infty\). Therefore there exists a subsequence \(\{t_n\} \downarrow 0\) of \(\{t_k\}\) such that \(S_{t_n} \to Z\), where \(Z\) is an infinitely divisible distribution. By the same reasoning as (74) – (78), we see that \(Z\) has to be Gaussian. Necessarily, \(Z\) is standard normal. (85) implies (83) can be proved similarly.

Next, we will give two easy corollaries with degenerate limit distributions.

Corollary 5.3 (Weak Derivative at 0) Suppose \(\Pi(0+) = \infty\). As \(t \downarrow 0\), we have
\[
\frac{X_t}{t} \to d_X \quad \text{if and only if} \quad \frac{(r,s) X_t}{t} \to d_X \quad \text{or} \quad \frac{(r) \tilde{X}_t}{t} \to d_X,
\] or equivalently as \(x \to 0\),
\[
\sigma^2 = 0, \quad x \Pi(x) \to 0, \quad \text{and} \quad \nu(x) \to d_X.
\] (86)

Corollary 5.4 (Relative Stability) Suppose \(\Pi(0+) = \infty\). As \(t \downarrow 0\), there exists a norming function \(b_t \downarrow 0\) such that
\[
\frac{X_t}{b_t} \to 1 \quad \text{if and only if} \quad \frac{(r,s) X_t}{b_t} \to 1 \quad \text{or} \quad \frac{(r) \tilde{X}_t}{b_t} \to 1,
\] or equivalently as \(x \to 0\),
\[
\sigma^2 = 0, \quad \text{and} \quad \frac{\nu(x)}{x \Pi(x)} \to \infty.
\] (88)

Furthermore, \(b_t\) is regularly varying with index 1.

Note that if \(X\) is a subordinator, (86) holds and \(d_X\) is the drift coefficient.

Proof: [Proof of Corollaries 5.3 and 5.4:] This is a simple consequence of Theorem 1.2 with degenerate limit. That (86) is equivalent to (87) is proved in Theorem 2.1 of Doney and Maller [5]. The equivalence of (88) and (89) is proved in Theorem 2.2 of Doney and Maller [5].

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