A SHORT NOTE ON GOPPA CODES OVER ELEMENTARY ABELIAN 
$p$-EXTENSIONS OF $\mathbb{F}_{p^s}(x)$

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Abstract. In this note, we investigate Goppa codes which are constructed by means of Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$, where $p$ is a prime number and $s$ is a positive integer. We give a simple criterion for self-duality of these codes and list the second generalized Hamming weight of these codes.

1. Introduction

Let $\mathbb{F}_{p^s}$ be the finite field with $p^s$ elements of characteristic $p$ (where $s$ is a positive integer). A linear code is a $\mathbb{F}_{p^s}$-subspace of $\mathbb{F}_{p^s}^n$, the $n$-dimensional standard vector space over $\mathbb{F}_{p^s}$. Such codes are used for transmission of information. It was observed by Goppa in 1975 that we can use algebraic function field over $\mathbb{F}_{p^s}$ to construct a class of linear codes. In Goppa’s construction, we choose a divisor $G$ and $n$ rational places of the algebraic function field to form a linear code of length $n$. In this note, we study Goppa codes over Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$.

The properties of Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$ have been studied in [2], [1], etc. In [11], T. Johnsen, S. Manshadi and N. Monzavi determined parameters of Goppa codes $C_L(D, G)$ over plane projective curves $X$ with affine equation $A(y) = B(x)$, where $A(T) \in \mathbb{F}_{p^s}[T]$ is a separable, additive polynomial of degree $q = p^k$, for some $k$ and the degree $m$ of $B(T) \in \mathbb{F}_{p^s}[T]$ is not divisible by $p$. They studied codes over $X$ with the assumption that $\deg D \geq 4g - 2$. In [3], Garcia also studied Goppa codes over plane projective curves $X$ with affine equation $A(y) = B(x)$ but with a central hypothesis that there exists a subgroup $H$ of $\mathbb{F}_{p^s}\{0\}$ such that if $A(\beta) = B(\alpha)$ with $\alpha \in H \cup \{0\}$, then $\beta \in \mathbb{F}_{p^s}$. In this note, we investigate codes which are constructed by means of Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$ without the above assumptions. We determine a simple condition for self-duality of these codes and list their second generalized Hamming weights.

This note is organised as follows. In section 2, we recall some results about Goppa’s construction of linear codes, Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$ and generalized Hamming weight of linear codes. In section 3, we study the properties of one-point codes over this function field. In section 4, we determine a simple condition for self-duality of these codes. In section 5, we conclude the note listing the second generalized Hamming weights of these codes.

2010 Mathematics Subject Classification. 94B27, 14G15, 14H05.

Key words and phrases. Elementary Abelian $p$-Extension of $\mathbb{F}_{p^s}(x)$, Goppa codes, Generalized Hamming weight.
2. Preliminaries

2.1. Goppa code. Goppa’s construction is described as follows:

Let \( F/\mathbb{F}_{p^s} \) be an algebraic function field of genus \( g \). Let \( P_1, \cdots, P_n \) be pairwise distinct places of \( F/\mathbb{F}_{p^s} \) of degree 1. Let \( D := P_1 + \cdots + P_n \) and \( G \) be a divisor of \( F/\mathbb{F}_{p^s} \) such that \( \text{supp}(G) \cap \text{supp}(D) = \emptyset \). The Goppa code \( C \) associated with \( D \) and \( G \) is defined as

\[
C := \{(x(P_1), \cdots, x(P_n)) : x \in L(G)\} \subseteq \mathbb{F}_{p^r}^n.
\]

Then, \( C \) is an \([n, k, d]\) code with parameters \( k = \dim(L(G)) - \dim(L(G - D)) \) and \( d \geq n - \deg(G) \).

We can define another code with the divisors \( G \) and \( D \) by using local components of Weil differentials. We define the code \( C_{\Omega}(D, G) \subseteq \mathbb{F}_{p^r}^n \) by

\[
C_{\Omega}(D, G) := \{\omega_{P_i}(1), \cdots, \omega_{P_n}(1) : \omega \in \Omega_F(G - D)\}.
\]

Then, \( C_{\Omega}(D, G) \) is an \([n, k', d']\) code with parameters \( k' = i(G - D) - i(G) \) and \( d' \geq \deg(G) - (2g - 2) \).

\( C_{\Omega}(D, G) \) is the dual code of \( C \) with respect to Euclidean scalar product on \( \mathbb{F}_{p^r}^n \) i.e. \( C_{\Omega}(D, G) = C^\perp \). Let \( \eta \) be a Weil differential of \( F \) such that \( \nu_{P_i}(\eta) = -1 \) and \( \eta_{P_i}(1) = 1 \) for \( i = 1, \cdots, n \). Then, \( C_{\Omega}(D, G) = C_{\Omega}(D, G) = C_{\Omega}(D, D - G + (\eta)) \).

2.2. (\text{I}, p.200) Elementary Abelian \( p \)-Extensions of rational function field. Let \( K \) be a field of characteristic \( p > 0 \). Consider a function field \( F = K(x, y) \) with

\[
y^q + \mu y = f(x) \in K[x],
\]

where \( q = p^k > 1 \) is a power of \( p \) and \( 0 \neq \mu \in K \). Assume that \( m := \deg f > 0 \) is coprime to \( p \). Also assume that all the roots of the equation \( T^q + \mu T = 0 \) are in \( K \). Then the following holds:

1. \([F : K(x)] = q\), and \( K \) is the full constant field of \( F \).
2. \( F/K(x) \) is Galois. The set \( A := \{\gamma \in K : \gamma^q + \mu \gamma = 0\} \) is a subgroup of order \( q \) of the additive group of \( K \).
3. The pole \( P_\infty \in \mathbb{P}_{K(x)} \) of \( x \) in \( K(x) \) has a unique extension \( Q_\infty \in \mathbb{P}_F \), and \( e(Q_\infty|P_\infty) = q \). Hence \( Q_\infty \) is a place of \( F/K \) of degree one.
4. \( P_\infty \) is the only place of \( K(x) \) which ramifies in \( F/K(x) \).
5. The genus of \( F/K \) is \( g = (q - 1)(m - 1)/2 \).
6. The divisor of the differential \( dx \) is

\[
(dx) = (2g - 2)Q_\infty = ((q - 1)(m - 1) - 2)Q_\infty.
\]

7. The pole divisor of \( x \) is \((x)_\infty = qQ_\infty \) and the pole divisor of \( y \) is \((y)_\infty = mQ_\infty \).
8. Let \( r \geq 0 \). Then, the elements \( x^iy^j \) with

\[
0 \leq i, 0 \leq j \leq q - 1, qi + mj \leq r
\]

form a basis of the space \( L(rQ_\infty) \) over \( K \).
(9) For all \( \alpha \in K \), one of the following cases holds:

Case (1). The equation \( T^q + \mu T = f(\alpha) \) has \( q \) distinct roots in \( K \). In this case, for each \( \beta \) with \( \beta^q + \mu \beta = f(\alpha) \) there exists a unique place \( P_{\alpha,\beta} \in \mathbb{P}_F \) such that \( P_{\alpha,\beta} | P_\alpha \) and \( y(P_{\alpha,\beta}) = \beta \). Hence, \( P_\alpha \) has \( q \) distinct extensions in \( F/K(x) \), each of degree one.

Case (2). The equation \( T^q + \mu T = f(\alpha) \) has no root in \( K \). In this case, all extensions of \( P_\alpha \) in \( F \) have degree \( > 1 \).

**Remark 2.1.** The Hermitian function field over \( \mathbb{F}_{q^2} \) is defined by

\[
H = \mathbb{F}_{q^2}(x, y) \text{ with } y^q + y = x^{q+1}.
\]

This is a special case of Elementary Abelian \( p \)-Extension with \( K = \mathbb{F}_{q^2}, \mu = 1 \) and \( f(x) = x^{q+1} \).

**Remark 2.2.** \( \min\{q, m\} \geq 2 \).

### 2.3. Generalized Hamming weight of Linear codes

The support of a \([n, k]\) linear code \( C \) over \( \mathbb{F}_{p^r} \) is defined by

\[
supp(C) := \{i : x_i \neq 0 \text{ for some } x = (x_1, \ldots, x_n) \in C\}.
\]

For \( 1 \leq l \leq k \), the \( l \)th generalized Hamming weight of \( C \) is defined by

\[
d_l(C) := \min \{ | supp(D) | : D \text{ is a linear subcode of } C \text{ with } dim(D) = l \}
\]

In particular, the first generalized Hamming weight of \( C \) is the usual minimum distance. The weight hierarchy of the code \( C \) is the set of generalized Hamming weights \( \{d_1(C), \ldots, d_k(C)\} \). These notions of generalized Hamming weights for linear codes were introduced by Wei in his paper [12].

Few properties of generalized Hamming weight of \( C \) have been listed in the following theorems.

**Theorem 2.3.** [12] (Monotonicity) For an \([n, k]\) linear code \( C \) with \( k > 0 \), we have

\[
1 \leq d_1(C) < d_2(C) < \cdots < d_k(C) \leq n.
\]

Let \( H \) be a parity check matrix of \( C \), and let \( H_i, 1 \leq i \leq n \), be its column vectors. For \( I \subseteq \{1, \ldots, n\} \), let \( \langle H_i : i \in I \rangle \) denote the space generated by those vectors. Then

**Theorem 2.4.** [12] \( d_l(C) = \min \{ | I | : | I | - \text{rank}(\langle H_i : i \in I \rangle) \geq l \} \)

For Goppa code \( C_L(D, G) \), the \( l \)th generalized Hamming weight is given by the following theorem.

**Theorem 2.5.** [3] Let \( C = C_L(D, G) \) be a code of dimension \( k \) and \( a := \text{dim}(L(G-D)) \geq 0 \). Then for every \( l, 1 \leq l \leq k \),

\[
d_l(C) = \min \{ \text{deg}(D') : 0 \leq D' \leq D, \text{dim}(L(G-D+D')) \geq l + a \}
\]

\[
= \min \{ n - \text{deg}(D') : 0 \leq D' \leq D, \text{dim}(L(G-D')) \geq l + a \}.
\]
3. Goppa code over Elementary Abelian $p$-extensions of $\mathbb{F}_{p^s}(x)$

Let $Q$ be a rational place in $F$. A positive integer $l$ is called pole number at $Q$ if there exists $z \in F$ such that $(z)_Q = lQ$. Let $p_1 < p_2 < \cdots$ be the sequence of pole numbers at $Q$ (that is, $p_r$ is the $r$th pole number at $Q$); thus $\dim(\mathcal{L}(p_r Q)) = r$, so $p_1 = 0$.

Few properties of Elementary Abelian $p$-Extensions of $\mathbb{F}_{p^s}(x)$, observed from [7] and [10], are listed in the following theorem.

**Theorem 3.1. (Properties of function field)**

- The Weierstrass semigroup $H$ of $Q_{\infty}$ is generated by $m$ and $q$ i.e. $H = \langle q, m \rangle$.
- $H$ is symmetric numerical semigroup.
- $(2g - 1)$ is the largest gap number of $Q_{\infty}$.
- $(2g - 2)Q_{\infty}$ is a canonical divisor.
- $H^* = H^*(D, Q_{\infty}) := \{l \in \mathbb{N}_0 : C_L(D, lQ_{\infty}) \neq C_L(D, (l - 1)Q_{\infty})\}$. If $l_1, \cdots, l_g$ denotes the gap numbers of $Q_{\infty}$, then \( \{l_1, \cdots, l_g, n + 2g - 1 - l_g, \cdots, n + 2g - 1 - l_1\} = \{0, 1, \cdots, n + 2g - 1\} \setminus H^* \).
- The sequence \( (C_L(D, m_i Q_{\infty}))_{i=0, \cdots, n} \) satisfy the isometry dual condition where $H^* = \{m_1, \cdots, m_n\}$.
- The polynomial $T^n + \mu T - \prod_{i=1}^m (x - \alpha_i) \in \mathbb{F}_{p^s}(x)[T]$, where $\alpha_i \in K$, is absolutely irreducible.
- $\{1, y, \cdots, y^{g-1}\}$ is an integral basis of $F/\mathbb{F}_{p^s}(x)$ for all $P \in \mathbb{F}_{p^s}(x) \setminus P_\infty$.

In [7], Stichtenoth has investigated one-point Goppa codes over Hermitian function field. Using similar idea, we define codes over Elementary Abelian $p$-Extension of $\mathbb{F}_{p^s}(x)$ and determine its parameters.

Let $K := \mathbb{F}_{p^s}$, $s$ large enough. Let $m > 0$ be an integer coprime to $p$. Choose $m$ distinct elements $\alpha_1, \cdots, \alpha_m \in K$. Let $f(x) := \prod_{i=1}^m (x - \alpha_i)$. Denote by $P_{\alpha_i}$ the zero of $(x - \alpha_i)$ in $K(x)$. Let $\beta_1, \cdots, \beta_q$ denotes the zeroes of $T^n + \mu T$ in $K$ (choose $s$ large enough so that all the zeroes are in $K$). Then, for $1 \leq i \leq m$, $1 \leq j \leq q$, $P_{\alpha_i, \beta_j}$ are the places of $F/K(x)$ of degree one (such places exist by section 2).

**Definition 3.2.** For $r \in \mathbb{Z}$ we define

\[
C_r := C_L(D, rQ_{\infty}),
\]

where

\[
D := \sum_{i=1}^m \sum_{j=1}^q P_{\alpha_i, \beta_j}.
\]

Then, $C_r$ is a code of length $n := qm$ over the field $K$. For $r < 0$, $\mathcal{L}(rQ_{\infty}) = \{0\}$, therefore $C_r = \{(0, \cdots, 0)\}$. For $r > n + (2g - 2) = 2qm - q - m - 1$, $\dim(C_r) = n$, therefore $C_r = \mathbb{F}_{p^s}^q$. It remains to study codes $C_r$ with $0 \leq r \leq 2qm - q - m - 1$. 
3.1. Parameters of \( C_r \). Let \( J \) be the set of pole numbers of \( Q_\infty \). For \( b \geq 0 \), let
\[
J(b) := \{ u \in J | u \leq b \}.
\]
Then, \( |J(b)| = \dim(\mathcal{L}(bQ_\infty)) \). From section 2, we have:
\[
J(b) = \{ u \leq b | u = iq + jm \text{ with } i \geq 0 \text{ and } 0 \leq j \leq q - 1 \}.
\]
Hence
\[
|J(b)| = \{|(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0; \ j \leq q - 1 \text{ and } iq + jm \leq b|\}.
\]

**Theorem 3.3.** (\cite{7}, II.2.2 and II.2.3) \( C_\mathcal{L}(D, G) \) is an \([n, k, d]\) code with parameters \( k = \dim(\mathcal{L}(G)) - \dim(\mathcal{L}(G-D)) \) and \( d \geq n - \deg(G) \). If \( \deg(G) < n \), then \( k = \dim(\mathcal{L}(G)) \).

**Theorem 3.4.** Suppose that \( 0 \leq r \leq 2qm - q - m - 1 \). Then the following holds:

(1) \( \dim(C_r) = \dim(\mathcal{L}(rQ_\infty)) - \dim(\mathcal{L}((r - qm)Q_\infty)) \). For \( 0 \leq r < qm \) (i.e. \( \deg(G) < n \)),
\[
\dim(C_r) = \dim(\mathcal{L}(rQ_\infty)) = \dim(\mathcal{L}(Q_\infty)) = |J(r)|.
\]

For \( qm - q - m - 1 < r < qm \) (i.e. \( 2g - 2 < \deg(G) < n \)), Riemann-Roch theorem yields
\[
\dim(C_r) = \dim(\mathcal{L}(rQ_\infty)) = \deg(rQ_\infty) + 1 - g = r + 1 - (q - 1)(m - 1)/2.
\]

(2) The inequality \( d \geq qm - r \) directly follows from Theorem 3.3. If \( r = qb \), where \( 0 \leq b < m \), choose \( b \) distinct elements from the set \( \{\alpha_1, \ldots, \alpha_m\} \) (where as before \( \alpha_i \) are such that \( f(x) = \prod_{i=1}^{m}(x - \alpha_i) \)). Let us call these elements \( \gamma_1, \ldots, \gamma_b \). Then the element
\[
z_1 := \prod_{j=1}^{b}(x - \gamma_j) \in \mathcal{L}(rQ_\infty)
\]
has exactly \( qb = r \) distinct zeros in \( D \). The weight of the corresponding codeword in \( C_r \) is \( qm - r \). Hence, \( d = qm - r \).

Similarly, if \( r = cm \), where \( 0 \leq c < q \), choose \( c \) distinct elements from the set \( \{\beta_1, \ldots, \beta_q\} \). Let us call these elements \( \tau_1, \ldots, \tau_c \). Then the element
\[
z_2 := \prod_{j=1}^{c}(y - \tau_j) \in \mathcal{L}(rQ_\infty)
\]
has exactly \( cm = r \) distinct zeros in \( D \). The weight of the corresponding codeword in \( C_r \) is \( qm - r \). Hence, \( d = qm - r \).

If \( r = qb \) and \( C_r \) is MDS code then, \( d = n - k + 1 \) implies \( g = 0 \) which is not possible. Similarly for \( r = cm \).

\[ \square \]

Using the idea from \([8]\), we have the following result for minimum distance of \( C_r \).

**Theorem 3.5.** Assume \( m > q \). For \( qm \leq r \leq 2qm - q - m - 1 \) we have \( 0 \leq r^\perp := 2qm - q - m - 1 - r \leq qm - q - m - 1 \). Let \( t^\perp \leq r^\perp \) be the largest integer such that \( t^\perp \) is a pole number at \( Q_\infty \) i.e. \( t^\perp = aq + bm \) where \( 0 \leq a \leq m - 2 \) and \( 0 \leq b \leq q - 1 \). Then, the minimum distance of \( C_r \) satisfies

\[ d(C_r) = a + 2. \]

**Proof.** Let \( H \) be a parity check matrix of \( C_r \). From section 2, we have \( \{1, x, y, \cdots, x^a, x^{a-1}y, \cdots, y^b\} \) is a basis for \( \mathcal{L}(t^\perp Q_\infty) \). Choose \( \beta \in K \) such that \( \beta^a + \mu \beta = 0 \). Let \( H_1 \) be a submatrix of \( H \) with columns corresponding to \( P_{\alpha_1, \beta}, \cdots, P_{\alpha_a+2, \beta} \). We write \( H_1 \) in the following form using row reduction.

\[
H_1 = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{a+2} \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{a+2}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1^a & \alpha_2^a & \alpha_3^a & \cdots & \alpha_{a+2}^a \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

Here, \( \text{rank}(H_1) = a + 1 \) and \( H_1 \) has \( a + 2 \) columns, so the columns of \( H_1 \) are linearly dependent. Therefore, \( d(C_r) \leq a + 2 \).

On the other hand, we choose any \( a + 1 \) distinct columns from \( H \). Let us call this matrix \( H_2 \). Since each column of \( H \) corresponds to a place \( P_{\alpha, \beta} \) of degree 1, we reorder columns of \( H_2 \) according to \( \alpha' \)'s as follows.

\[
P_{\alpha_1, \beta_1, 1}, \ P_{\alpha_2, \beta_2, 1}, \cdots, \ P_{\alpha_1, \beta_1, w_1} \\
P_{\alpha_1, \beta_2, 1}, \ P_{\alpha_2, \beta_2, 2}, \cdots, \ P_{\alpha_1, \beta_2, w_2} \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
P_{\alpha_1, \beta_{\gamma, 1}}, \ P_{\alpha_2, \beta_{\gamma, 2}}, \cdots, \ P_{\alpha_1, \beta_{\gamma, w_{\gamma}}}
\]

where \( \alpha_i \)'s are pairwise distinct and \( w_1 + w_2 + \cdots + w_{\gamma} = a + 1 \) with \( w_1 \geq w_2 \geq \cdots \geq w_{\gamma} \geq 1 \). For \( 0 \leq j_i \leq w_i - 1 \); \( 1 \leq i \leq \gamma \), \( x^{i-1}y^j \) belongs to basis of \( \mathcal{L}(t^\perp Q_\infty) \). We rewrite these
basis elements in the form

\[
\begin{align*}
1, & \quad y, \quad y^2, \quad \cdots, \quad y^{w_1-1} \\
x, & \quad xy, \quad xy^2, \quad \cdots, \quad xy^{w_2-1} \\
x^2, & \quad x^2y, \quad x^2y^2, \quad \cdots, \quad x^2y^{w_3-1} \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
x^{\gamma-1}, & \quad x^{\gamma-1}y, \quad x^{\gamma-1}y^2, \quad \cdots, \quad x^{\gamma-1}y^{w_{\gamma}-1}
\end{align*}
\]

Then, we extract an \((a+1) \times (a+1)\) submatrix \(H'\) of \(H_2\) such that each row corresponds to a function above in the given order. That is, \(H' = [H'_{i,j}],\ i, j = 1, 2, \cdots, \gamma\) where \(H'_{i,j}\) is a \((w_i \times w_j)\) matrix with \(H'_{i,j} = \alpha_j^{-1}B_{i,j}\) with

\[
B_{i,j} = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\beta_{j,1} & \beta_{j,2} & \beta_{j,3} & \cdots & \beta_{j,w_j} \\
\beta_{j,1}^2 & \beta_{j,2}^2 & \beta_{j,3}^2 & \cdots & \beta_{j,w_j}^2 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\beta_{j,1}^{w_j-1} & \beta_{j,2}^{w_j-1} & \beta_{j,3}^{w_j-1} & \cdots & \beta_{j,w_j}^{w_j-1}
\end{bmatrix}
\]

Then, from \([8]\), Lemma 2 and Lemma 3,

\[
det(H') = \left( \prod_{i=1}^{\gamma} det(B_{i,i}) \right) \left( \prod_{j=2}^{\gamma} \rho_j^{w_j} \right)
\]

where

\[
\rho_j = \prod_{i=1}^{j-1} (\alpha_j - \alpha_i), \ j = 2, 3, \cdots, \gamma.
\]

And any \(a + 1\) columns of \(H\) are linearly independent over \(K\). Hence, \(d(C_r) \geq a + 2. \ □\)

4. Condition for self-duality of codes

A linear code \(C\) is called self-dual if \(C = C^\perp\), where \(C^\perp\) is the dual of \(C\) with respect to Euclidean scalar product on \(\mathbb{F}_p^n\). Self-dual codes are an important class of linear codes. We give a simple criterion for self-duality of codes over Elementary Abelian \(p\)-Extensions of \(\mathbb{F}_p^n\) by using the following theorem from \([8]\).

**Definition 4.1.** We call two divisors \(G\) and \(H\) equivalent with respect to \(D\) if there exists \(u \in F\) such that \(H = G + (u)\) and \(u(P_i) = 1\), for all \(i = 1, \cdots, n\).

**Theorem 4.2** (\([8]\), 4.15). Suppose \(n > 2g + 2\). Let \(G\) and \(H\) be two divisors of the same degree \(m\) on a function field of genus \(g\). If \(C_L(D,G)\) is not equal to 0 nor to \(K^n\) and \(2g - 1 < m < n - 1\), then \(C_L(D,G) = C_L(D,H)\) if and only if \(G\) and \(H\) are equivalent with respect to \(D\).

Let \(f(x)\) and \(D\) as before. Clearly, \(n = qm > 2g + 2\). Let \(\eta := \frac{f'(x)dx}{f(x)}\) be a differential then we get, \(\nu_P(\eta) = -1\) and \(res_P(\eta) = 1\) for all \(P \in supp(D)\). Therefore, for any divisor \(G\) on \(F\) with \(supp(D) \cap supp(G) = \emptyset\), we have \(C_L(D,G) = C_L(D, D + \eta - G)\).
Theorem 4.3. The dual code of $C_r$ is given by

$$C_r^\perp = \bar{a} C_{2qm-q-m-1-r}$$

where $(\bar{a})^{-1} = ((f'(x))(P_{a_1,b_1}), \cdots, (f'(x))(P_{a_m,b_m})) \in (\mathbb{F}_q^*)^n$. Hence, if $qm-q-m+1 \leq r \leq qm-2$ then $C_r$ is quasi-self-dual if and only if $r = \frac{(2qm-q-m-1)}{2}$.

Proof.

$$C_r^\perp = C_{C}(D, D + (\eta) - rQ_\infty)$$
$$= C_{C}(D, D + (f'(x)) - (f(x)) + (dx) - rQ_\infty)$$
$$= C_{C}(D, D + (f'(x)) - D + qmQ_\infty + (2g-2)Q_\infty - rQ_\infty)$$
$$= C_{\bar{C}}(D, (f'(x)) + (qm + (2g-2-r)Q_\infty)$$
$$= eC_{\bar{C}}(D, (2qm-q-m-1-r)Q_\infty)$$
$$= eC_{2qm-q-m-1-r}$$

Now it follows directly from Theorem 4.2 that $C_r$ is quasi-self-dual if and only if $r = \frac{(2qm-q-m-1)}{2}$. \hfill \Box

Corollary 4.4. For $0 \leq r \leq 2qm - q - m - 1$, define $s := 2qm - q - m - 1 - r$. The dimension of $C_r$ is given by

$$\text{dim}(C_r) = \begin{cases} |J(r)| & \text{for } 0 \leq r < qm, \\ qm - |J(s)| & \text{for } qm \leq r \leq 2qm - q - m - 1. \end{cases}$$

Let $G$ be a divisor of $F$ with $\text{deg}(G) = qm - \frac{q}{2} - \frac{m}{2} - \frac{1}{2}$. Clearly, $n > 2g+2$ and $2g-1 < \text{deg}(G) < n-1$. Let $H := D + (\eta) - G$. Then, $\text{deg}(G) = \text{deg}(H)$. From Theorem 4.2, it follows that

Theorem 4.5. $C_{\bar{C}}(D, G)$ is self-dual if and only if $2G$ is equivalent to $(f'(x)) + (2qm-q-m-1)Q_\infty$ with respect to $D$.

Proof. By Theorem 4.2,

$$C_{\bar{C}}(D, G) = C_{\bar{C}}(D, D + (\eta) - G)$$
$$\Leftrightarrow G = D + (\eta) - G + (u) \text{ for some } u \in F \text{ such that } u(P) = 1 \text{ for each } P \in \text{supp}(D)$$
$$\Leftrightarrow (u) + (\eta) = 2G - D$$
$$\Leftrightarrow (u) + (f'(x)) + (dx) - (f(x)) = 2G - D$$
$$\Leftrightarrow (u) + (f'(x)) + [(q - 1)(m - 1) - 2]Q_\infty - D + qmQ_\infty = 2G - D$$
$$\Leftrightarrow (f'(x)) = 2G - (2qm - q - m - 1)Q_\infty - (u).$$

\hfill \Box

Example 4.6. Let $p = 2$. Let $K = \mathbb{F}_4$. Let $\omega$ be a primitive element of $\mathbb{F}_4$. Consider $F = K(x, y)$ with

$$y^2 + y = x(x - 1)(x - \omega)$$
Therefore, all roots of $T^2 + T$ is in $K$. The genus of $F/K$ is $g = 1$. Let
\[ f(x) = x(x - 1)(x - \omega). \]

Let $P_0, P_1$ and $P_\omega$ denotes zero in $K(x)$ of $x, (x - 1)$ and $(x - \omega)$ respectively. Then, each of $P_0, P_1$ and $P_\omega$ has exactly two extensions in $F$. Similarly, the zero of $(x - \omega^2)$ denoted by $P_{\omega^2}$ has two extensions in $F$, say, $Q_1$ and $Q_2$. Let $D = (f(x))_0$ and let $G$ be a divisor in $F$ equivalent to $Q_1 + Q_2 + Q_\infty$ with respect to $D$. Then $C_L(D, G)$ is self-dual. Conversely, if $C_L(D, G)$ is self-dual code with $D$ as above then $2G$ is equivalent to $2(Q_1 + Q_2 + Q_\infty)$ with respect to $D$.

5. Generalized Hamming weight of code $C_r$

Using the idea of [5], we have the following lemma.

**Lemma 5.1.** Let $r \leq qm$ be a pole number at $Q_\infty$. Then, $r = iq + jm$ where $i \geq 0$ and $0 \leq j \leq q - 1$. If either $i = 0$ or $j = 0$ then $\exists$ a divisor $0 \leq D' \leq D$ such that $rQ_\infty \sim D'$.

**Proof.** For $i = 0$ and $j = 0$, $D' = 0$ works.

If $i = 0$ and $j \neq 0$ then, $r = jm$. With notation as in section 3, choose $j$ elements from $\beta_1, \ldots, \beta_q$. Denote these elements by $\tau_1, \ldots, \tau_j$. Define $g := \prod_{i=1}^j (y - \tau_i)$. Then, $(g) = D' - rQ_\infty$. Therefore, $D' \sim rQ_\infty$.

Similarly for $j = 0$ and $i \neq 0$. \hfill \Box

**Definition 5.2.** A positive integer $r \leq qm$ is said to have property (*) if $r$ is a pole number at $Q_\infty$, $r = iq + jm$ for $i \geq 0$, $0 \leq j \leq q - 1$ and either $i = 0$ or $j = 0$.

**Theorem 5.3.** As before, let $C_r = C_L(D, rQ_\infty)$. If for $1 \leq l \leq k$, $r - p_l$ or $qm - r + p_l$ has the property (*) then, $d_l(C_r) \leq qm - r + p_l$.

**Proof.** If $r - p_l$ has the property (*), then according to Lemma 5.1, there exists a divisor $0 \leq D' \leq D$ such that $(r - p_l)Q_\infty \sim D'$. Thus, $\dim(\mathcal{L}(rQ_\infty - D')) = \dim(\mathcal{L}(p_lQ_\infty)) = l$. Hence, from Theorem 2.5, $d_l(C_r) \leq qm - r + p_l$.

Now, if $qm - r + p_l$ has the property (*) then according to Lemma 5.1 there exists a divisor $0 \leq D'' \leq D$ such that $(qm - r + p_l)Q_\infty \sim D''$. Also, $qmQ_\infty \sim D$. Thus, $D - rQ_\infty + p_lQ_\infty \sim D''$. Therefore, $D' := D - D'' \sim (r - p_l)Q_\infty$. Hence, $d_l(C_r) \leq qm - r + p_l$. \hfill \Box

An immediate corollary of the above theorem is the following.

**Corollary 5.4.** Assume that $m > q$. If for $1 \leq r < qm$, $r$ or $qm - r$ has the property (*), then the minimum distance of $C_r$ is $d = qm - r$.

**Remark 5.5.** For $C_r := C_L(D, rQ_\infty)$, we have $d_k(C_r) = n = qm$ where $k = \dim(C_r)$.
Theorem 5.6. Let $C = C_L(D, G)$ be a code of dimension $k$ and $\dim(\mathcal{L}(G - D)) = a > 0$. Then, for $1 \leq l \leq k$, we have $d_l(C) \leq \deg(D')$ for every effective divisor $D' \leq D$ such that $\dim(\mathcal{L}(D')) > l$.

Using the above theorem, we get the following result.

Theorem 5.7. Assume that $m > q$. If $qm \leq r \leq 2qm - q - m - 1$, then $d_2(C_r) \leq \min\{2q, m\}$.

Proof. Since $p_3 = \min\{2q, m\}$ therefore, from Lemma 5.1, $\exists \ 0 \leq D' \leq D$ such that $p_3 Q_\infty \sim D'$. Then, $\dim(\mathcal{L}(D')) = \dim(\mathcal{L}(p_3 Q_\infty)) = 3$. Hence, $d_2(C_r) \leq \deg D' = p_3 = \min\{2q, m\}$.

Theorem 5.8. Assume $m > 2q$. Suppose $qm \leq r \leq 2qm - q - m - 1$. Then $0 \leq r' := 2qm - q - m - 1 - r \leq qm - q - m - 1$. Let $t' \leq r'$ be the largest integer such that $t'$ is a pole number of $Q_\infty$ i.e. $t' = aq + bm$ where $0 \leq a \leq m - 3$ and $0 \leq b \leq q - 1$. Then,

$$d_2(C_r) = a + 3.$$

Proof. Let $H$ be a parity check matrix for $C_r$ over $K$. Choose $\beta \in K$ such that $\beta^q + \mu \beta = 0$. $\{1, x, y, \cdots, x^a, x^{a-1}y, \cdots, y^b\}$ is a basis for $\mathcal{L}(t' Q_\infty)$. Let $H_1$ be a submatrix of $H$ with columns corresponding to $P_{\alpha_1, \beta}, \cdots, P_{\alpha_{a+3}, \beta}$ (possible since $a + 3 \leq m$). By using row reduction, we make $H_1$ as follows.

$$H_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{a+3} \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{a+3}^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_1^a & \alpha_2^a & \alpha_3^a & \cdots & \alpha_{a+3}^a \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Here, $\text{rank}(H_1) = a + 1$ and $H_1$ has $a + 3$ columns. So, by Theorem 2.4, $d_2(C_r) \leq a + 3$.

On the other hand from Theorem 2.3, $d_2(C_r) \geq d_1(C_r) + 1 = (a + 2) + 1 = a + 3$. □

5.1. Feng-Rao Distances on Numerical Semigroups.

The Feng-Rao distances on numerical semigroups is defined in [9]. We will explain it briefly in this subsection.

Let $A$ be a numerical subgroup. If for a set $G \subseteq A$, every $x \in A$ can be written as a linear combination

$$x = \sum_{g \in G} \lambda_g g,$$

where finitely many $\lambda_g \in \mathbb{N} \cup \{0\}$ are non-zero, then we say that $A$ is generated by $G$. It is well known that every numerical semigroup is finitely generated. An element $x \in A$ is said to be irreducible if $x = a + b$ for $a, b \in A$ implies $a.b = 0$. Every generator set contains the
set of irreducible elements and the set of irreducibles actually generates $A$. The number of irreducible elements is called the embedding dimension of $A$. We enumerate the elements of $A$ in increasing order

$$A = \{ \rho_1 = 0 < \rho_2 < \cdots \}.$$ 

For $a, b \in \mathbb{Z}$ given, we say that $a$ divides $b$, and write

$$a \le_A b, \text{ if } b - a \in A.$$ 

The binary relation is an order relation.

The set $D(y)$ denotes the set of divisors of $y$ in $A$, and for given $M = \{ m_1, \ldots, m_r \} \subseteq A$, we write $D(M) = D(m_1, \ldots, m_r) = \bigcup_{i=1}^{r} D(m_i)$.

**Definition 5.9.** Let $A$ be a numerical subgroup, that is, a submonoid of $\mathbb{N}$ such that $| (\mathbb{N} \setminus A) | < \infty$ and $0 \in A$. We call $g := | (\mathbb{N} \setminus A) |$ the genus of $A$. The unique element $c \in A$ such that $c - 1 \notin A$ and $c + l \in A$ for all $l \in \mathbb{N}$ is called the conductor of $A$. The (classical) Feng-Rao distance of $A$ is defined by the function

$$\delta_{FR}: \quad A \to \mathbb{N} \quad x \mapsto \delta_{FR}(x) := \min \{| D(m_1) | : m_1 \ge x, \ m_1 \in A \}$$

There are some well-known results about the function $\delta_{FR}$ for an arbitrary semigroup $A$. One of the important result is that $\delta_{FR}(x) \ge x + 1 - 2g$ for all $x \in A$ with $x \ge c$. The proof of the following theorem can be found in [9].

**Theorem 5.10.** Let $A = \{ 0 = \rho_1 < \rho_2 < \cdots < \rho_n < \cdots \}$ be an embedding dimension two numerical subgroup. Then

$$d_r(C_t) \ge \delta_{FR}(l + 1) + \rho_r$$

for $r = 1, \cdots, k_t$, where $C_t$ is a code in an array of codes as in [4] and $k_t$ is the dimension of $C_t$.

Using the above theorem, we determine the second generalized Hamming weight of code $C_r$ (as defined in section 3).

**Theorem 5.11.** Assume $m > q$. For $r < qm$, $d_2(C_r) \ge qm - r + q$.

**Theorem 5.12.** Assume $m > q$. For $r < qm$, if $r - q$ or $qm - r + q$ satisfies the property (*) then,

$$d_2(C_r) = qm - r + q.$$ 

**Proof.** Applying Theorem 5.3, we get $d_2(C_r) \le qm - r + q$.

On the other hand, as $\rho_2 = q$ and as the dual code of $C_r$ form an array of codes (for details see [4]), from Theorem 4.2 and the Theorem 5.10, we have

$$d_2(C_r) = d_2(C_{2qm-qm-1-r}) \ge \delta_{FR}(2qm - q - m - r) + q.$$
Since \( r < qm \), we have
\[
d_2(C_r) \geq \delta_{FR}(2qm - q - m - r) + q \\
\geq 2qm - q - m - r + 1 - (qm - q - m + 1) + q \\
= qm - r + q.
\]

Hence proved. \( \square \)

The following result is stated in Munuera [5].

**Theorem 5.13.** Let \( C_L(D,G) \) be a code of dimension \( k \) and abundance \( \dim(C_L(D,G - D)) =: a \geq 0 \). If there is a rational point \( Q \) not in \( D \) and \( C_L(D,G - p_rQ) \neq \{0\} \), where \( p_r \) is \( r \)th pole number at \( Q \), then for every \( l \), \( 1 \leq l \leq k \),
\[
d_l(C_L(D,G)) \leq d_l(C_L(D,G - p_lQ)).
\]

Using the above theorem and Theorem 5.11, we get the following result.

**Theorem 5.14.** Assume \( m > q \). For \( r < qm \). If \( r - q \) and \( qm - r + q \) doesn’t satisfy the property (*), then
\[
qm - r + q \leq d_2(C_r) \leq qm - \overline{(r - q)}.
\]

where \( \overline{(r - q)} \) is the largest pole number less than or equal \((r - q)\) that satisfies the property (*).

6. Concluding remarks

In this note, we have defined one-point Goppa codes over Elementary Abelian \( p \)-Extension of \( \mathbb{F}_{p^s}(x) \) and determined its dimension and exact minimum distance in few cases. We have also given a simple criterion for one-point Goppa codes to be quasi-self-dual and Goppa codes with divisor \( G \) (not necessarily one-point) to be self-dual. We have listed exact second generalized Hamming weight of these codes in few cases.

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