Sub-Riemannian geometry on some step-two Carnot groups

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Abstract. This paper is a continuation of the previous work of the first author. We characterize a class of step-two groups introduced in [88], saying GM-groups, via some basic sub-Riemannian geometric properties, including the squared Carnot-Carathéodory distance, the cut locus, the classical cut locus, the optimal synthesis, etc. Also, the shortest abnormal set can be exhibited easily in such situation. Some examples of such groups are step-two groups of corank 2, of Kolmogorov type, or those associated to quadratic CR manifolds. As a byproduct, the main goal in [19] is achieved from the setting of step-two groups of corank 2 to all possible step-two groups, via a completely different method. A partial answer to the open questions [20, (29)-(30)] is provided in this paper as well. Moreover, we provide a entirely different proof, based yet on [88], for the Gaveau-Brockett optimal control problem on the free step-two Carnot group with three generators. As a byproduct, we provide a new and independent proof for the main results obtained in [103], namely, the exact expression of $d(g)^2$ for $g$ belonging to the classical cut locus of the identity element $o$, as well as the determination of all shortest geodesics joining $o$ to such $g$.

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1 Introduction

In the past several decades, step-two groups and their sub-Laplacians, as special Lie groups of polynomial volume growth or perfect sub-Riemannian manifolds, have attracted wide attention from experts in various fields, such as complex analysis, control theory, geometric measure theory, harmonic analysis, heat kernel, Lie group theory, probability analysis, PDE, sub-Riemannian geometry, etc. We only mention some relevant works here, [4–18,20–28,30,32,34–48,50–53,55–63,65–72,74–76,80–87,90,91,96,97,103,105,114,118,123,124,131,132]. The list is far from exhaustive and in fact is rather limited. More related papers can be found in the references therein as well as their subsequent researches.

In this present paper, we will restrict our attention to the sub-Riemannian geometry on step-two Carnot groups. Many relevant works can be found in the literature as cited before. However, some most fundamental problems are far from being solved, or even
poorly known, in this very fine framework. Recently, in [88] (cf. also [89]), the first
author used Loewner’s theorem to study two basic problems of sub-Riemannian geometry
on 2-step groups: one is to obtain the exact formula for the sub-Riemannian distance, that
is the Gaveau-Brockett optimal control problem; another is to characterize all (shortest)
normal geodesics from the identity element \( o \) to any given \( g \neq o \). In particular, there exists
an enormous class of 2-step groups, saying GM-groups (see Subsection 2.3 below for the
definition), which have some consummate sub-Riemannian geometric properties. More
precisely, the squared Carnot-Carathéodory distance \( d(g)^2 := d(o, g)^2 \) and the cut locus
of \( o \), \( \text{Cut}_o \) (namely the set of points where \( d^2 \) is not smooth) can be characterized easily
in such situation. For example, all Heisenberg groups even generalized Heisenberg-type
groups (so step-two groups of corank 1), and star graphs are GM-groups. We emphasize
that in general, the expression of \( d(g)^2 \) is extremely complicated. It is impossible to
provide an explicit expression, via a relatively simple inverse function, as in the most
special situation of generalized Heisenberg-type groups (cf. [63], [27] and [88]). We refer
the reader to [88] for more details. The work is a continuation of [88], one of our main goals
is to provide various equivalent characterizations of GM-groups via basic sub-Riemannian
geometric properties.

Moreover, the sub-Riemannian geometry in the setting of 2-step groups is not well
understood. Roughly speaking, the main reason for this is that the well-understood
examples are merely the Heisenberg group (cf. [63]) and generalized Heisenberg groups (cf.
[27]). Other known cases, such as generalized Heisenberg-type groups as well as the direct
product of a generalized Heisenberg group with a Euclidean space (in particular, step-two
groups of corank 1), are essentially the same. Hence, it is very meaningful and exigent to
supply some examples possessing richer sub-Riemannian geometric properties. Here, we
will provide more examples of GM-groups, such as groups of corank 2, of Kolmogorov type,
or those associated to quadratic CR manifolds. In particular, the aforementioned groups
may have complicated shortest abnormal set of \( o \), \( \text{Abn}_o^* \), that is, the set of the endpoints
of abnormal shortest geodesics starting from \( o \). The existence of non-trivial abnormal
shortest geodesics is closely related to the regularity of the Carnot-Carathéodory distance.
And its appearance makes an obstacle for us to deal with some topics, such as the heat
cornel asymptotics and geometric inequalities, etc. See for example [2, 20, 29, 30, 88, 105]
and the references therein for more details. Recall that (cf. [118, 119]) a sub-Riemannian
manifold is called \textit{ideal} if it is complete and has no non-trivial abnormal shortest geodesics.
In our setting, a step-two group \( G \) is ideal if and only if it is of Métivier type (see Subsection
2.2.1 for the definition).

Optimal syntheses (namely the collection of all arclength parametrized geodesics with
their cut times) are generally very difficult to obtain. In the setting of step-two groups, as
far as we know, a correct result about them can be found only on nonisotropic Heisenberg
groups. See [2, § 13] and Remark 1 below for more details. However, we can now give the
optimal synthesis from the identity element \( o \) on GM-groups. As a result, the classical
cut locus of \( o \), \( \text{Cut}_o^{\text{CL}} \), that is the set of points where geodesics starting at \( o \) cease to be
shortest, can be characterized on such groups as well.

We say that \( d^2 \) is \textit{semiconcave} (resp. \textit{semiconvex}) \textit{in a neighborhood of} \( g_0 \) if there exist
\[ C > 0 \text{ and } \delta > 0 \text{ such that} \]
\[
d(g_1 + g')^2 + d(g_1 - g')^2 - 2d(g_1)^2 \leq C |g'|^2 (\text{resp. } \geq -C |g'|^2), \tag{1.1} \]
\[ \text{for all } g_1 \pm g' \in B(g_0, \delta) = \{g \in \mathbb{R}^q \times \mathbb{R}^m; |g - g_0| < \delta \}. \]

Here we stress that \(| \cdot |\) denotes the usual Euclidean norm and \(g_1 \pm g'\) the usual operation in the Euclidean space. We also remark that this definition is independent of the choice of local coordinates around \(g_0\) (here we use the canonical one) since \(d^2\) is locally Lipschitz w.r.t. the usual Euclidean distance (see for example \([118, 119]\)). Set in the sequel\(^1\)

\[ \text{SC}^-_o := \{g; d^2 \text{ fails to be semiconcave in any neighborhood of } g\}, \tag{1.2} \]
\[ \text{SC}^+_o := \{g; d^2 \text{ fails to be semiconvex in any neighborhood of } g\}. \tag{1.3} \]

Recall that \(\text{SC}^-_o = \text{Abn}^*_o\) in the setting of Métivier groups, all free Carnot groups of step 2, as well as some other sub-Riemannian structures. See \([35, 54, 102, 20, \S\ 4.1 \text{ and } \S\ 4.2]\) and references therein for more details. And an open problem is raised in \([20, (29)]\), which asks whether it holds \(\text{SC}^-_o = \text{Abn}^*_o\) in the more general sub-Riemannian setting (where our \(\text{Abn}^*_o\) is noted by \(\text{Abn}(o)\)). In the framework of GM-groups, \(\text{Abn}^*_o\) can be described easily; as a byproduct, we give a positive answer to this open problem. Also, other related results and step-two groups can be found in Subsection 2.5.

In addition, the most challenging problem should be to study the sub-Riemannian geometry in the setting of free step-two groups with \(k\) generators \(N_{k,2} \cong \mathbb{R}^k \times \mathbb{R}^{k(k-1)/2}\) \((k \geq 3)\). Indeed, for any step-two group \(\mathbb{G}\) with \(k\) generators, that is the first layer in the stratification of Lie algebra has dimension \(k\), there exists some relation between \(\mathbb{G}\) and \(N_{k,2}\) by Rothschild-Stein lifting theorem (see \([124]\) or \([32]\)). Observe that \(N_{2,2}\) is exactly the Heisenberg group, which is well-known (cf. \([63]\) or \([27]\)). Recall that (cf. \([63]\) and \([34]\)) the original Gaveau-Brockett optimal control problem is to determine the sub-Riemannian distance on \(N_{k,2}\) with \(k \geq 3\). This is a long-standing open problem. Recently, it is completely solved on \(N_{3,2}\) in \([88, \S\ 11]\). Also remark that the main idea and method in \([88]\) can be adapted to general step-two groups and other situations.

In the setting of \(N_{3,2}\), the classical cut locus of \(o\), \(\text{Cut}^\text{CL}_o\), has been determined in \([113]\) and \([103]\) by completely different techniques; furthermore, the expression of \(d(g)^2\) with \(g \in \text{Cut}^\text{CL}_o\) has been obtained in \([103]\). Strictly speaking, we have used the above known results in the proof of \([88]\). Also notice that \(\text{Abn}^*_o\) and \(\text{Cut}^\text{CL}_o\) on \(N_{3,2}\) are relatively very simple. However, it is still an open problem to characterize the classical cut locus of \(o\) on \(N_{k,2}\) with \(k \geq 4\), see \([123]\) for more details. Motivated by this problem, we ask naturally if we can determine first \(d(g)^2\) for any \(g\) then \(\text{Cut}^\text{CL}_o\) on \(N_{3,2}\). This is exactly another main purpose of this work.

In the framework of step-two groups, first we recall that all shortest geodesics are normal (cf. \([3]\) or \([119, \S\ 2.4]\)). Next, up to a subset of measure zero, all normal geodesics

\(^1\)We would like to thank L. Rizzi for informing us of the addendum of \([20]\) that the definition of the failure of semiconcavity/semiconvexity for the open questions should be the one stated here in our situation (which is consistent with the classical definition of local semiconcavity/semiconvexity), rather than the one given in \([20]\). For more details, we refer to the addendum on L. Rizzi’s homepage.
from \( o \) to any given \( g \neq o \) have been characterized by [88, Theorem 2.4]. Moreover, it follows from [88, Theorem 2.5] that the squared distance has been determined in a symmetric, scaling invariant subset with non-empty interior. In particular, for the special case of \( N_{3,2} \), we can simplify the Gaveau-Brockett problem via an orthogonal-invariant property, and some useful results can be found in [88, §11]. Based on these known results, we can describe the squared distance on \( N_{3,2} \) first on some dense open subset, then on whole space via a limiting argument. Then, from the regularity of the squared sub-Riemannian distance, we can further determine the cut locus \( \text{Cut}_o \). Finally, all shortest geodesics joining \( o \) to any given point in \( \text{Cut}^{\text{CL}}_o \) are obtained by approximating them with that joining \( o \) to some points in \( (\text{Cut}_o)^c \), which are relatively easy to describe. As a consequence, we supply an independent and new proof for the main results obtained in [103].

Some applications will be given in a future work.

This paper is organized as follows. In Section 2, we collect some preliminary materials and give our main results, which will be proven in Section 3. In Section 4, we provide a sufficient condition for a step-two group to be GM-group by using semi-algebraic theory. As a consequence, we find that all step-two groups of corank 2 are GM-groups. Furthermore, we also prove in this section that there exist Métivier groups of corank 3 and of sufficiently large dimension which are not of GM-type. In Section 5, we consider the sub-Riemannian geometry in the setting of step-two K-type groups. Step-two groups associated to quadratic CR manifolds will be studied in Section 6. Finally, we give in Section 7 a completely different proof, based on [87], for the Gaveau-Brockett optimal control problem on \( N_{3,2} \). As an application, we provide a new and independent proof for the main results obtained in [103].

2 Preliminaries and main results

2.1 Step-two Carnot groups

Recall that a connected and simply connected Lie group \( G \) is a step-two Carnot group if its left-invariant Lie algebra \( g \) admits a stratification

\[
g = g_1 \oplus g_2, \quad [g_1, g_1] = g_2, \quad [g_1, g_2] = \{0\},
\]

where \([\cdot, \cdot]\) denotes the Lie bracket on \( g \). We identify \( G \) and \( g \) via the exponential map. As a result, \( G \) can be considered as \( \mathbb{R}^q \times \mathbb{R}^m \), \( q, m \in \mathbb{N}^r = \{1, 2, 3, \ldots\} \) (in this paper we use \( \mathbb{N} \) to denote the set of natural numbers \( \{0, 1, 2, \ldots\} \)), with the group law

\[
(x, t) \cdot (x', t') = \left(x + x', t + t' + \frac{1}{2} \langle UX, x' \rangle\right), \quad g := (x, t) \in \mathbb{R}^q \times \mathbb{R}^m,
\]

where

\[
\langle UX, x' \rangle := (\langle U^{(1)} x, x' \rangle, \ldots, \langle U^{(m)} x, x' \rangle) \in \mathbb{R}^m.
\]
Here $\mathbb{U} = \{U^{(1)}, \ldots, U^{(m)}\}$ is an $m$-tuple of linearly independent $q \times q$ skew-symmetric matrices with real entries and $\langle \cdot, \cdot \rangle$ (or $\cdot$ in the sequel when there is no ambiguity) denotes the usual inner product on $\mathbb{R}^q$. Furthermore, in this article, we will not distinguish row vectors from column vectors and we may write a column vector $t$ with scalar coordinates $t_1, \ldots, t_m$, simply as $(t_1, \ldots, t_m)$ unless otherwise stated in the context. Note that $m \leq \frac{q(q-1)}{2}$. We call such a group a step-two group of type $(q, m, \mathbb{U})$, which is denoted by $G(q, m, \mathbb{U})$ or $G$ for simplicity. One can refer to [49] or [32] for more details.

Let $U^{(j)} = (U^{(j)}_{l,k})_{1 \leq l,k \leq q}$ ($1 \leq j \leq m$). The canonical basis of $\mathfrak{g}_1$ is defined by the left-invariant vector fields on $G$:

$$X_l(g) := \frac{\partial}{\partial x_l} + \frac{1}{2} \sum_{j=1}^{m} \left( \sum_{k=1}^{q} U^{(j)}_{l,k} x_k \right) \frac{\partial}{\partial t_j}, \quad 1 \leq l \leq q.$$ 

And the canonical sub-Laplacian is $\Delta = \sum_{l=1}^{q} X_l^2$.

### 2.2 Left-invariant sub-Riemannian geometry on $G$: some elementary properties

Let us first recall some basic facts about the sub-Riemannian geometry in the framework of 2-step groups. In our setting, we will sometimes use equivalent definitions for some concepts in order to avoid recalling too many notations. We refer the reader to [2, 28–30, 105, 119, 129] and references therein for further details. Also notice that partial but not all results below remain valid in some more general setting.

The group $G = G(q, m, \mathbb{U})$ is endowed with the sub-Riemannian structure, namely a scalar product on $\mathfrak{g}_1$, with respect to which $\{X_l\}_{1 \leq l \leq q}$ are orthonormal (and the norm induced by this scalar product is denoted by $\| \cdot \|$). In the sequel, $m$ is called the corank of $G(q, m, \mathbb{U})$.

A horizontal curve $\gamma : [0, 1] \to G$ is an absolutely continuous path such that

$$\dot{\gamma}(s) = \sum_{j=1}^{q} u_j(s) X_j(\gamma(s)) \quad \text{for a.e. } s \in [0, 1],$$

and we define its length as follows

$$\ell(\gamma) := \int_{0}^{1} \| \dot{\gamma}(s) \| \, ds = \int_{0}^{1} \sqrt{\sum_{j=1}^{q} |u_j(s)|^2} \, ds.$$ 

The Carnot-Carathéodory (or sub-Riemannian) distance between $g, g' \in G$ is then

$$d(g, g') := \inf \{ \ell(\gamma); \, \gamma(0) = g, \, \gamma(1) = g', \, \gamma \text{ horizontal} \}.$$ 

A geodesic is a horizontal curve $\gamma$ satisfying: $\| \dot{\gamma}(s) \|$ is constant and for any $s_0 \in [0, 1]$ there exists a neighborhood $I$ of $s_0$ in $[0, 1]$ such that $\ell(\gamma|_I)$ is equal to the distance
between its endpoints. And a shortest geodesic is a geodesic \( \gamma \) which realizes the distance between its extremities, that is, \( \ell(\gamma) = d(\gamma(0), \gamma(1)) \).

By slightly abusing of notation in the sequel, \( 0 \) denotes the number 0 or the origin in the Euclidean space. Let \( o = (0,0) \) denote the identity element of \( G \). It is well-known that \( d \) is a left-invariant distance on \( G \). Hence we set in the following \( d(g) := d(g,o) \).

Recall that \( d^2 \) is locally Lipschitz on \( G \) with respect to the usual Euclidean distance.

The dilation on \( G \) is defined by
\[
\delta_r(x,t) := (r x, r^2 t), \quad \forall r > 0, \ (x,t) \in G.
\]

And the following scaling property is well-known:
\[
d(r x, r^2 t) = r d(x,t), \quad \forall r > 0, \ (x,t) \in G.
\]

### 2.2.1 Sub-Riemannian Hamiltonian and normal geodesics starting from \( o \)

In the setting of step-two Carnot groups, it is well-known that all shortest geodesics are projections of normal Pontryagin extremals, that is integral curves of the sub-Riemannian Hamiltonian in \( T^*G \). See for example [3, § 20.5] or [119, Theorem 2.22].

More precisely, the sub-Riemannian Hamiltonian in \( T^*G \cong (\mathbb{R}^q \times \mathbb{R}^m) \times (\mathbb{R}^q \times \mathbb{R}^m) \) is defined by
\[
H = H(x,t,\xi,\tau) := \frac{1}{2} \sum_{j=1}^{q} \zeta_j^2, \quad \zeta_j := \xi_j + \frac{1}{2} \sum_{k=1}^{m} \left( \sum_{l=1}^{q} U^{(k)}_{j,l} x_l \right) \tau_k, \quad 1 \leq j \leq q.
\]

And a normal Pontryagin extremal,
\[
(\gamma(s) := (x(s),t(s)),\xi(s),\tau(s)) : [0, 1] \rightarrow T^*G, \quad \text{with } \gamma(0) = o,
\]
is a solution of
\[
\dot{x}_k = \frac{\partial H}{\partial \xi_k}, \quad \dot{\xi}_j = \frac{\partial H}{\partial \tau_j}, \quad \dot{\xi}_k = - \frac{\partial H}{\partial x_k}, \quad \dot{\tau}_j = - \frac{\partial H}{\partial t_j}, \quad 1 \leq k \leq q, 1 \leq j \leq m.
\]

The covector \( (\xi(0), \tau(0)) \) (resp. \( (\xi(1), \tau(1)) \)) is called the initial (resp. final) covector of \( (\gamma(s), \xi(s), \tau(s)) \). Its projection
\[
\gamma(s) := \gamma(\xi(0), \tau(0); s) = \gamma(\xi(0), \tau(0))(s) = (x(s), t(s)) : [0, 1] \rightarrow G
\]
is said to be the normal geodesic starting from \( o \) with initial covector \( (\xi(0), \tau(0)) \).

Note that \( H \) is independent of \( t \). Hence we have
\[
\tau(s) \equiv \tau(0) := 2 \theta \in \mathbb{R}^m.
\]

Set in the following
\[
\tilde{U}(\theta) := \sum_{j=1}^{m} \theta_j U^{(j)} \text{ and } U(\theta) := i \tilde{U}(\theta), \quad \text{for } \theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m.
\]
Recall that a step-two group $G$ is a Métrivier group (or of Métrivier type) if $U(\theta)$ is invertible for any $\theta \neq 0$ (cf. [98]).

Let $\zeta(s) := \xi(s) + \bar{U}(\theta)x(s)$. Remark that $\xi(0) = \zeta(0)$. A simple calculation implies that

$$\zeta(s) = e^{2s\bar{U}(\theta)} \zeta(0), \quad x(s) = \int_0^s \zeta(r) \, dr, \quad t(s) = \frac{1}{2} \int_0^s \langle \bar{U}x(r), \zeta(r) \rangle \, dr. \quad (2.6)$$

In particular, we have

$$x(1) = \int_0^1 \zeta(r) \, dr = \sin U(\theta) e^{\bar{U}(\theta)} \zeta(0), \quad (2.7)$$

and $\gamma(\xi(0), \tau(0); s) = (x(s), t(s))$ is extendable and real analytic on $[0, +\infty)$. It is easy to check the following homogeneity property:

$$\gamma(\alpha \zeta_0, 2 \alpha \theta_0; s) = \gamma(\zeta_0, 2 \theta_0; \alpha s), \quad \forall \alpha > 0, \ s \geq 0, \ (\zeta_0, 2 \theta_0) \in \mathbb{R}^q \times \mathbb{R}^m.$$ 

From now on, the domain of the normal geodesic $\gamma(s) = \gamma(\zeta_0, 2 \theta_0; s)$ is $[0, +\infty)$ and that of $\gamma(s) = \gamma(\zeta_0, 2 \theta_0)(s)$ is $[0, 1]$ by default. Also remark that $\gamma = o$ if $\zeta_0 = 0$, which is trivial. And all normal geodesics are by convention starting from $o$ in this work.

Let $(x, t)$ denote the endpoint of $\gamma(\zeta, \tau)$, then by (2.6), that of $\gamma(-\zeta, \tau)$ is $(-x, t)$. Moreover, both $\gamma(\zeta, \tau)$ and $\gamma(-\zeta, \tau)$ have length $|\zeta|$, where $|\cdot|$ denotes the usual Euclidean norm. Combining this with the fact that the Carnot-Carathéodory distance is a left-invariant distance on $G$, we have the following simple but useful observation:

**Lemma 1.** In the setting of step-two groups, it holds that

$$d(x, t) = d(-x, t) = d(x, -t) = d(-x, -t), \quad \forall (x, t) \in G. \quad (2.8)$$

The following basic property is well-known:

**Lemma 2.** Let $0 \leq s_1 < s_2$. Assume that $\gamma(\zeta_0, 2 \theta_0; s) = \gamma(\zeta', 2 \theta'; s)$ for all $s_1 \leq s \leq s_2$. Then we have $\gamma(\zeta_0, 2 \theta_0; \cdot) \equiv \gamma(\zeta', 2 \theta'; \cdot)$. Moreover, it holds that $\zeta_0 = \zeta'$.

More information about such geodesics can be found in Proposition 1 below.

### 2.2.2 Sub-Riemannian exponential map, cut point and optimal synthesis

The sub-Riemannian exponential map based at $o$ is the smooth map defined by

$$\exp : \mathbb{R}^q \times \mathbb{R}^m \longrightarrow G$$

$$\zeta_0, 2 \theta_0 \longmapsto \gamma(\zeta_0, 2 \theta_0; 1).$$

In our setting, it is surjective and has the following property:

$$\gamma(\zeta_0, 2 \theta_0; s) = \exp \{ s (\zeta_0, 2 \theta_0) \}, \quad \forall s \geq 0, \ (\zeta_0, 2 \theta_0) \in \mathbb{R}^q \times \mathbb{R}^m.$$ 

See for example [2, § 8.6]. Furthermore, we have the following simple observations:
Lemma 3. Suppose that $\exp(w, \tau) = (x, t)$. Then we have
\[
\exp(r w, \tau) = (r x, r^2 t), \quad \forall r \neq 0, \tag{2.9}
\]
\[
\exp(-e^{\tilde{U}(\tau)} w, -\tau) = (-x, -t). \tag{2.10}
\]

Indeed, using (2.6), (2.9) is trivial, and an elementary computation implies (2.10).

Now assume that $\gamma(s) = \exp\{s (\zeta_0, 2 \theta_0)\}$ is parametrized by arclength (or arclength parametrized), namely $|\zeta_0| = 1$. Let $g_0 = \gamma(s_0)$. We say that $g_0$ is conjugate to $o$ along $\gamma$ if $s_0 (\zeta_0, 2 \theta_0)$ is a critical point of $\exp$. The cut time along $\gamma$ is defined as
\[
h_{\text{cut}} := h_{\text{cut}}(\gamma) = \sup\{s > 0; \gamma|[0, s] \text{ is a shortest geodesic}\}. \tag{2.11}
\]
When $h_{\text{cut}} < +\infty$, $\gamma(h_{\text{cut}})$ is said to be the cut point of $o$ along $\gamma$. And we say $\gamma$ has no cut point if $h_{\text{cut}} = +\infty$. The optimal synthesis from $o$ is the collection of all arclength parametrized geodesics with their cut times.

2.2.3 Shortest abnormal set and cut locus

A normal geodesic is said to be abnormal (i.e. singular) if it has two (so infinitely many) different normal lifts (see [120, Remark 8] and [119, Remark 2.4]). However, we stress that our definition of abnormal geodesic is not complete in general. In particular, on some sub-Riemannian manifolds, excluding our step-two groups, there are shortest geodesics which are not projections of normal Pontryagin extremals. For the original definition of abnormal geodesic as well as counter-examples, we refer the reader to [2,92,104,105,119] and the references therein for more details.

In this work, the (normal-) abnormal set of $o$, $\text{Abn}_o$ is defined by
\[
\text{Abn}_o := \{g; \text{there exists an abnormal (which is also normal) geodesic joining $o$ to $g$}\}.
\]

And we define the shortest abnormal set of $o$ as follows:
\[
\text{Abn}^*_{\text{nor}} := \{g; \text{there exists an abnormal shortest geodesic joining $o$ to $g$}\},
\]
which is a subset of $\text{Abn}_o$. The main difference between the two sets is that: in the definition of $\text{Abn}_o$, we do not care about minimality of geodesics, while this is needed in that of $\text{Abn}^*_{\text{nor}}$. Notice that $o \in \text{Abn}^*_{\text{nor}}$. Also remark that our $\text{Abn}_o$ is exactly $\text{Abn}^*_{\text{nor}}(e)$ in [82, § 2.7].

The following characterization of abnormal geodesics, which can be also considered as an improvement of Lemma 2, can be easily verified by (2.6) (see also [102, § 3.1.1] for an explanation from the original definition of abnormal (-normal) geodesics).

Proposition 1. Let $\theta \neq \theta'$. Then $\gamma(w, 2 \theta; \cdot) \equiv \gamma(w, 2 \theta'; \cdot)$ if and only if for $\sigma = \theta - \theta' \in \mathbb{R}^m \setminus \{0\}$, we have
\[
U(\sigma) U^k(\theta) w = 0, \quad \forall k \in \mathbb{N}, \tag{2.12}
\]
or equivalently,
\[
U(\sigma) e^{s \tilde{U}(\theta)} w = 0, \quad \forall s \in \mathbb{R}. \tag{2.13}
\]
Lemma 4. \( \gamma(w, 2\theta; \cdot) \) is abnormal if and only if there exists some \( \sigma \neq 0 \) such that (2.12) (or equivalently (2.13)) satisfies.

As a consequence, we get the following known fact:

**Corollary 1.** If \( \gamma(w, 2\theta) \) is abnormal, then so does \( \gamma(a w, 2b\theta) \) for any \( a, b \in \mathbb{R} \). In particular, for any \( 0 < a < 1 \), the restriction of \( \gamma(w, 2\theta) \) in \([0, a] \), \( \gamma(w, 2\theta)|_{[0,a]} = \gamma(a w, 2a\theta) \) is also abnormal.

A normal geodesic is called strictly normal if it is not abnormal. Let \( \gamma = \gamma(w, 2\theta) \) (resp. \( \gamma(w, 2\theta; \cdot) \)) and \( 0 < s_1 < s_2 < 1 \) (resp. \( 0 < s_1 < s_2 < +\infty \)). We consider the restriction of \( \gamma \) in \([s_1, s_2] \), \( \gamma|_{[s_1, s_2]} \) as well as

\[
\gamma^{s_1, s_2}(s) := \gamma(s_1)^{-1} \gamma(s_1 + s(s_2 - s_1)), \quad s \in [0, 1].
\]

By (2.6), a simple calculation shows that

\[
\gamma^{s_1, s_2} = \gamma((s_2 - s_1) e^2 \sigma \cdot \theta) w, 2(s_2 - s_1) \theta,
\]

which is a normal geodesic starting from \( o \). If \( \gamma^{s_1, s_2} \) is abnormal, then it follows from Proposition 1 that there exists a \( \sigma \in \mathbb{R}^m \setminus \{0\} \) such that

\[
(s_2 - s_1)^{k+1} U(\sigma) U(\theta) e^{2 s_1 \theta} w = 0, \quad \forall k \in \mathbb{N},
\]

which implies that \( \gamma = \gamma(w, 2\theta) \) itself (so \( \gamma(w, 2\theta; \cdot) \)) is also abnormal by (2.13). We say a normal geodesic \( \gamma(w, 2\theta) \) (resp. \( \gamma(w, 2\theta; \cdot) \)) does not contain abnormal segments if \( \gamma^{s_1, s_2} \) is not abnormal for any \( 0 \leq s_1 < s_2 \leq 1 \) (resp. \( 0 \leq s_1 < s_2 < +\infty \)). In conclusion, we get the following:

**Lemma 4.** In the framework of step-two groups, any strictly normal geodesic does not contain abnormal segments.

It is worthwhile to point out that the above property is no longer valid in general. See [99] for more details.

In this paper, the cut locus of \( o \), \( \text{Cut}_o \), is defined as

\[
\text{Cut}_o := S^c, \quad \text{with } S := \{g; d^2 \text{ is } C^\infty \text{ in a neighborhood of } g\}. \tag{2.14}
\]

Recall that (see for example [2, § 11.1])

\[
S = \{g; \text{there exists a unique shortest geodesic } \gamma \text{ from } o \text{ to } g, \text{ which is}
\]

not abnormal, and \( g \) is not conjugate to \( o \) along \( \gamma \}, \tag{2.15}

and it is open and dense in \( \mathbb{G} \). Hence \( \text{Cut}_o \) is closed. Furthermore, it has measure zero (cf. [118, Proposition 15]). Remark also that \( o \in \text{Abn}^*_o \subseteq \text{Cut}_o \).

The classical cut locus of \( o \), \( \text{Cut}^\text{CL}_o \) is defined as the set of points where geodesics starting at \( o \) cease to be shortest, that is

\[
\text{Cut}^\text{CL}_o := \{g; g \text{ is the cut point of } o \text{ along some arclength parametrized normal geodesic}\}.
\]

Now, we can give an affirmative answer to the open question [20, first part of (30)] in our framework, which follows from (2.15), Lemma 4 and [2, Theorem 8.72].
Theorem 1. In the setting of step-two Carnot groups, it holds that $\text{Cut}_o = \text{Cut}_o^\text{CL} \cup \text{Abn}_o^*$. 

2.3 Notations and results from [88]

Let us begin by recalling the initial reference set and the reference function, introduced in [88], which are defined respectively by

$$\Omega_* := \left\{ \tau \in \mathbb{R}^m; \max_{|x|=1} |U(\tau)^2 x, x| < \pi^2 \right\} = \{ \tau \in \mathbb{R}^m; \|U(\tau)\| < \pi \},$$  \hfill (2.16)

$$\phi(g; \tau) = \langle U(\tau) \cot U(\tau) x, x \rangle + 4 t \cdot \tau, \; \tau \in \Omega_*, \; g = (x, t) \in G. \hfill (2.17)$$

Notice that the function $\phi(g; \cdot)$ is well-defined provided the spectrum of $U(\tau)$ does not contain any $k \pi$ ($k \in \mathbb{Z} \setminus \{0\}$). Also, we will use its usual extension on $\overline{\Omega}_*$ (which is denoted by $\phi(g; \cdot)$ as well). And we have

Proposition 2 ([88], Proposition 2.1 and Remark 2.1). For any $g$, $\phi(g; \cdot)$ is smooth and concave in $\Omega_*$. Moreover, for every $g$, there exists an $\theta_g \in \Omega_*$ such that

$$\phi(g; \theta_g) = \sup_{\tau \in \Omega_*} \phi(g; \tau).$$

Let $\nabla_\theta = \left( \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_m} \right)$ denote the usual gradient on $\mathbb{R}^m$. Recall that (cf. [88, § 2])

$$\tilde{\mathbb{M}} := \left\{ \left( x, -\frac{1}{4} \nabla_\theta \langle U(\theta) \cot U(\theta) x, x \rangle \right); \; x \in \mathbb{R}^q, \; \theta \in \Omega_* \right\},$$  \hfill (2.18)

which is the union of disjoint and nonempty subsets

$$\mathbb{M} := \{ g; \; \exists \theta \in \Omega_* \text{ s.t. } \theta \text{ is a nondegenerate critical point of } \phi(g; \cdot) \text{ in } \Omega_* \}$$

$$= \{ g; \; \exists \theta \in \Omega_* \text{ s.t. the set of global maximizers of } \phi(g; \cdot) \text{ in } \Omega_* \text{ is } \{ \theta \} \},$$  \hfill (2.19)

and

$$\tilde{\mathbb{M}}_2 := \{ g; \; \text{the set of global maximizers of } \phi(g; \cdot) \text{ in } \Omega_* \text{ has at least two points} \}. \hfill (2.20)$$

Also recall that $\mathbb{M}$ is an open set, $\mathbb{M} \subseteq S$, $o \in \tilde{\mathbb{M}}_2 \subseteq \text{Abn}_o^* \subseteq \text{Cut}_o$ and (cf. [88, § 2])

$$d(g)^2 = \max_{\tau \in \Omega_*} \phi(g; \tau) \; \text{ for } g \in \tilde{\mathbb{M}}, \; \text{ and } \; d(g)^2 = \sup_{\tau \in \Omega_*} \phi(g; \tau) \; \text{ for } g \in \overline{\mathbb{M}}. \hfill (2.21)$$

And we have the following

Theorem 2 ([88], Theorems 2.4 and 2.5). Assume that $\zeta_0 \in \mathbb{R}^q \setminus \{0\}$ and $\theta_0 \in \Omega_*$. Then $\exp\{(\zeta_0, 2 \theta_0)\} = g_0 := (x_0, t_0)$ if and only if

$$x_0 = \left( \frac{U(\theta_0)}{\sin U(\theta_0)} e^{-U(\theta_0)} \right)^{-1} \zeta_0, \; t_0 = -\frac{1}{4} \nabla_\theta \langle U(\theta_0) \cot U(\theta_0) x_0, x_0 \rangle.$$
Furthermore, in such case, we have

\[ d(g_0)^2 = |\zeta_0|^2 = \frac{U(\theta_0)}{\sin U(\theta_0)} x_0^2 = \phi(g_0; \theta_0), \]

and the unique shortest geodesic from \( o \) to \( g_0 \) is \( \exp\{s (\zeta_0, 2 \theta_0)\} \) \((0 \leq s \leq 1)\), which is strictly normal if and only if \( g_0 \in \mathcal{M} \).

As a consequence, we yield immediately

**Corollary 2.** Let \( \gamma(s) := \exp\{s (\zeta_0, \tau_0)\} \) be an arclength parametrized geodesic, that is \(|\zeta_0| = 1\). Then its cut time \( h_{\text{cut}} = +\infty \) if \( \tau_0 = 0 \), and in such case \( \gamma \) is a ray in the first layer. In addition, we have \( h_{\text{cut}} \geq 2\pi/\|U(\tau_0)\| \) when \( \tau_0 \neq 0 \).

**Remark 1.** It follows from [88, Proposition 5.1 and/or Corollary 2.2] that if \( \mathbb{G} \) is not of Métévier type, then there exist \( 0 \neq \theta_0 \in \mathbb{R}^m \) and \( 0 \neq x_0 \in \ker \tilde{U}(\theta_0) \) such that \(|x_0| = 1\) and \( \exp\{s (x_0, \theta_0)\} = \exp\{s (x_0, 0)\} \) for all \( s > 0 \). Hence the cut time of \( \exp\{s (x_0, \theta_0)\} \) is equal to \( +\infty \) and the statement of [19, Theorems 6 and 7] is misleadingly phrased. However, a correct statement and their generalization can be found in Theorem 4, Corollaries 8 or 9 below. Also notice that our method to determine the cut time is completely different from theirs.

Another easy but very useful consequence is the following:

**Corollary 3.** It holds that \( \text{Cut}_{o}^{CL} \subseteq \tilde{\mathcal{M}}^{c} \).

Combining [88, Proposition 5.1 (b)] with Theorem 2 as well as Proposition 1, we have the following characterization of \( \tilde{\mathcal{M}}_2 \), \( \text{Abn}^{*}_o \) and \( \text{Abn}_o \):

**Proposition 3.** It holds that:

\[
\tilde{\mathcal{M}}_2 = \left\{ \gamma(s) = \gamma(\zeta, \tau; s); \gamma \text{ is abnormal, } |\zeta| = 1 \text{ and } 0 \leq s < \frac{2\pi}{\|U(\tau)\|} \right\}, \\
\text{Abn}^{*}_o = \left\{ \gamma(s) = \gamma(\zeta, \tau; s); \gamma \text{ is abnormal, } |\zeta| = 1 \text{ and } 0 \leq s \leq h_{\text{cut}}(\gamma) \right\}, \\
\text{Abn}_o = \left\{ \gamma(s) = \gamma(\zeta, \tau; s); \gamma \text{ is abnormal, } |\zeta| = 1, s \geq 0 \right\}.
\]

**Remark 2.** Obviously, the arclength parametrized geodesic \( \gamma(\zeta_0, \tau_0; s) \) is abnormal if and only if there exists a \( s_0 > 0 \) such that \( \|U(s_0 \tau_0)\| < 2\pi \) and \( \exp\{s_0 (\zeta_0, \tau_0)\} \in \tilde{\mathcal{M}}_2 \).

In order to describe \( \text{Abn}^{*}_o \) and \( \text{Abn}_o \), it suffices to determine \( \tilde{\mathcal{M}}_2 \), which is much less difficult. Moreover, we have

**Corollary 4.** In the setting of step-two groups, if \( \tilde{\mathcal{M}}_2 \subseteq \{(x, 0); x \in \mathbb{R}^q\} \), then \( \text{Abn}_o = \text{Abn}^{*}_o = \tilde{\mathcal{M}}_2 \).
Remark 3. (a) Recall that \( \mathbb{R}^q \times \{0\} \subseteq \tilde{M} \). If \((x_0, 0) \in \text{Abn}_o\) for some \(x_0 \in \mathbb{R}^q\), then we have also \((x_0, 0) \in \tilde{M}_2 \subseteq \text{Abn}_o\).

(b) In general, \( \text{Abn}_o = \text{Abn}_o^* \) does not imply \( \tilde{M}_2 \subseteq \mathbb{R}^q \times \{0\} \). In fact, let \( H^3 = \mathbb{R}^2 \times \mathbb{R} \) denote the Heisenberg group of real dimension 3 and consider \( G = H^3 \times H^3 \cong \mathbb{R}^4 \times \mathbb{R}^2 \). We have
\[
\text{Abn}_o = \text{Abn}_o^* = (\{0\} \times H^3) \cup (H^3 \times \{0\}),
\]
but
\[
\{0\} \cup (\{0\} \times M_{3H^3}) \cup (M_{3H^3} \times \{0\}) = \tilde{M}_2 \not\subseteq \mathbb{R}^4 \times \{0\},
\]
where \( M_{3H^3} = \{(x, t); x \in \mathbb{R}^2 \setminus \{0\}\} \) and \( o_{3H^3} \) denote the corresponding set \( M \) and identity element in the setting of \( H^3 \) respectively.

(c) The example in (b) also provides a group on which \( \tilde{M}_2 \not\subseteq \text{Abn}_o^* \) since \((0, 0, 1, o_{3H^3}) \in \tilde{M}_2 \not\subseteq H^3 \times H^3 = G\). Furthermore, another example of \( \tilde{M}_2 \not\subseteq \text{Abn}_o^* \) can be found in the proof of Proposition 7 (see Subsection 6.2 below), and that of \( \text{Abn}_o^* \not\subseteq \text{Abn}_o \) in Subsection 6.3 below.

(d) Obviously, a step-two group is of M´etivier type if and only if \( \text{Abn}_o^* = \{o\} \).

Recall that the nonempty open subset \( M \subset G \) is the set of points, \( g \), where the reference function \( \phi(g; \cdot) \) has a nondegenerate critical point in the initial reference set \( \Omega_\ast \). Observe that \( M \) is symmetric and scaling invariant; namely, if \( g \in M \), then we have \( g^{-1} = -g \in M \) and \( \delta_r(g) \in M \) for all \( r > 0 \). A step-two group \( G \) is said to be a GM-group (or of type GM) if it satisfies
\[
\tilde{M} = G. \quad \text{(GM)}
\]
Notice that if both \( G_1 \) and \( G_2 \) satisfy \( \text{(GM)} \), then so does the direct product \( G_1 \times G_2 \). See Appendix B for more details. Also remark that GM groups form a wild set. Indeed, for any given \( G(q, m, U) \), we can construct an uncountable number of GM-groups \( G(q + 2n, m, \tilde{U}) \). See [88, § 8.1] for more details.

Recall that the global reference set is a compact set in \( \mathbb{R}^m \) defined by (cf. [88, § 2.6])
\[
\mathcal{R} := \mathcal{R}, \quad \text{with} \quad \mathcal{R} := \left\{ \theta = \frac{1}{4} \nabla d(g)^2; g = (x, t) \notin \text{Cut}_o \right\} \quad \text{open.} \quad (2.22)
\]
It follows from [102, § 3] or [88, Proposition 5.2] that \( \mathcal{R} \cap \Omega_\ast \) is dense in \( \Omega_\ast \). Then
\[
\mathcal{R} \supseteq \Omega_\ast = \left\{ \theta; \|U(\theta)\| \leq \pi \right\}. \quad (2.23)
\]
Set
\[
\mathcal{V} := \{ \vartheta \in \mathbb{R}^m; \det(k\pi - U(\vartheta)) \neq 0, \forall k \in \mathbb{N}^* \}
\]
and
\[
\mathcal{W} := \exp(\mathbb{R}^q \times (2\mathcal{V}^c)), \quad (2.24)
\]
which is the set of the endpoints of “bad” normal geodesics, where “bad” normal geodesic
(resp. “good” normal geodesic) means \( \gamma = \gamma(w,2\theta) \) with \( \theta \in \mathcal{V} \) (resp. \( \theta \in \mathcal{V} \)). It is clearly
that \( \mathcal{W} \) is of measure zero.

Finally some notations of special functions related to \(-s \cot s\) are also recalled:

\[
f(s) := 1 - s \cot s, \quad \mu(s) := f'(s) = \frac{2s - \sin(2s)}{2 \sin^2 s}, \quad \psi(s) := \frac{f(s)}{s^2}.
\]  (2.25)

\section{2.4 Main results}

Our first result is the following:

\subsection{2.4.1 Properties of the global reference set \( \mathfrak{R} \)}

The following theorem should be useful to determine all shortest geodesics in the setting
of step-two groups.

\begin{t1}
\textbf{Theorem 3.} Let \( o \neq g \in \mathbb{G} \), and \( \gamma_g(s) \) (\( 0 \leq s \leq 1 \)) be a shortest geodesic joining \( o \) to \( g \). Then there exist \( \zeta \in \mathbb{R}^q \) with \( |\zeta| = d(g) \) and \( \theta \in \mathfrak{R} \) such that \( \gamma_g(s) = \exp\{s(\zeta,2\theta)\} \) for all \( 0 \leq s \leq 1 \).
\end{t1}

The following property is a direct consequence of (2.15) together with Lemma 4 and [2, Theorem 8.72]:

\begin{bln}
\textbf{Lemma 5.} Suppose that \( g_0 = \exp\{z_0,\tau_0\} \in \mathcal{S} \) and \( \exp\{s(\zeta_0,\tau_0)\} \) (\( 0 \leq s \leq 1 \)) is the unique shortest geodesic between \( o \) and \( g_0 \). Then \( \exp\{s(\zeta_0,\tau_0)\} \in \mathcal{S} \) and \( 2^{-1}s\tau_0 \in \mathfrak{R} \) for all \( 0 < s \leq 1 \).
\end{bln}

\begin{nRnM}
\textbf{Remark 4.} Recall that \( \widetilde{\mathcal{M}} = \mathcal{M} \cup \widetilde{\mathcal{M}}_2 \) and \( \widetilde{\mathcal{M}}_2 \subseteq \mathrm{Cut}_o = \mathcal{S}_c \). By the fact that \( \exp\{z,2\tau\} \in \widetilde{\mathcal{M}} \) whenever \( \tau \in \Omega_* \), a direct consequence of Lemma 5 is that \( \mathcal{M} \neq \emptyset \). See also [97, Proposition 6 and \$3\] for another explanation.

It follows from Lemma 5 that \( \mathfrak{R} \) is star-shaped w.r.t. the origin \( 0 \), that is, if \( \tau \in \mathfrak{R} \), then we have \( s\tau \in \mathfrak{R} \) for all \( s \in [0,1] \); in particular, it is path connected. Furthermore, it is the smallest compact set which satisfies the property in Theorem 3. Also notice that Lemma 5 provides a theoretical basis for the method proposed in [88, \$11\] to determine the squared sub-Riemannian distance for general non-GM groups.

\begin{RKn2}
\textbf{Remark 5.} Theorem 3 could be considered as a somewhat converse statement of [120, Proposition 4\] in our setting. In fact, from the proof of Theorem 3, we know that there exist \( \{g_j\}_{j=1}^{\infty} \subseteq \mathcal{S} \) with \( \gamma(\zeta^{(j)},2\theta^{(j)}) \) the shortest geodesic joining \( o \) to \( g_j \) such that \( g_j \to g \) and \( (\zeta^{(j)},2\theta^{(j)}) \to (\zeta,2\theta) \) as \( j \to +\infty \). As a result, every shortest geodesic can be induced by some limiting sub-differential in our situation.

Combining this with another basic property of \( \mathfrak{R} \), namely [88, Corollary 2.4], this is why it is called the global reference set.
2.4.2 Other sub-Riemannian geometric properties on step-two groups

Let us begin with an upper bound about the cut time of an arclength parametrized geodesic:

**Corollary 5.** Let

\[ C_R := \max_{\tau \in \mathbb{R}} \| U(\tau) \|, \quad C_\tau := \sup \{ s > 0; \ r \tau \in \mathcal{R}, \ \forall 0 \leq r \leq s \} \ (\tau \in \mathbb{R}^m). \]

For any arclength parametrized geodesic \( \gamma(s) = \exp\{s(\zeta, \tau)\} = \gamma(\zeta, \tau; s) \), its cut time satisfies

\[
\begin{align*}
      h_{cut} &\leq \sup \left\{ 2 C_\sigma; \sigma \in \mathbb{R}^m, \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot) \right\} \\
               &\leq \sup \left\{ \frac{2 C_R}{\| U(\sigma) \|}; \sigma \in \mathbb{R}^m, \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot) \right\}
\end{align*}
\]

(2.26) with the understanding \( \frac{2 C_R}{0} = +\infty \). In particular, assume moreover that \( \gamma(s) = \exp\{s(\zeta, \tau)\} \) is not abnormal, then \( h_{cut} \leq 2 C_\tau \leq \frac{2 C_R}{\| U(\tau) \|} \).

**Remark 6.** Fix \( (\zeta, \tau) \) and let \( \Pi_{(\zeta, \tau)} := \{ \sigma - \tau \in \mathbb{R}^m; \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot) \} \). Proposition 1 implies that \( \Pi_{(\zeta, \tau)} \) is a linear subspace of \( \mathbb{R}^m \). It is clear that the continuous function \( \sigma \mapsto \| U(\sigma) \| \) defined on \( \tau + \Pi_{(\zeta, \tau)} \) attains its minimum. Hence, the last “sup” in (2.26) can be replaced by “max”.

Moreover, we have the following:

**Lemma 6.** In the framework of step-two Carnot groups, \( \text{Abn}^*_o \) is a closed set.

On a step-two group \( \mathbb{G} \), notice that \( o \not\in \text{Cut}_o^{\text{CL}} \). Now assume that \( \text{Cut}_o^{\text{CL}} \cap \text{Abn}^*_o = \emptyset \) and \( \text{Abn}^*_o \neq \{o\} \). Let \( \gamma(s) = \exp\{s(\zeta, \tau)\} = \gamma(\zeta, \tau; s) \) be an arclength parametrized abnormal geodesic. Then it follows from Lemma 6 that its cut time is \( +\infty \). Using Corollary 5 and Remark 6, we obtain \( \gamma(\zeta, \tau; \cdot) = \gamma(\zeta, 0; \cdot) \), which implies \( \{\gamma(s); \ s \geq 0\} \subseteq \mathbb{R}^q \times \{0\} \) by (2.6). In conclusion, combining this with Corollaries 4 and 3, we get the following:

**Corollary 6.** In the setting of step-two Carnot groups, \( \text{Cut}_o^{\text{CL}} \cap \text{Abn}^*_o = \emptyset \) if and only if \( \tilde{\mathcal{M}}_2 \subseteq \mathbb{R}^q \times \{0\} \).

To finish this subsection, we provide the following:

**Proposition 4.** In the context of step-two Carnot groups, it holds that \( \overline{\mathcal{M}} = \mathcal{M} \).

\(^2\)For a general sub-Riemannian manifold \( M \), we can define the shortest abnormal set of \( y \in M \), \( \text{Abn}^*_y \), as the set of the endpoints of abnormal (not necessarily normal) shortest geodesics starting from \( y \) and \( \text{Abn}^*_y \) is a closed set as well. This is a result of the characterization of abnormal Pontryagin extremals via Lagrange multipliers rule and the compactness of minimal controls. We would like to thank L. Rizzi for informing us of this general result and providing a sketched proof. For the sake of completeness, we give a proof in the setting of step-two groups without using the notion of the endpoint map in Section 3.
2.4.3 Characterizations of GM-groups

GM-groups have very fine properties of sub-Riemannian geometry. More precisely,

**Theorem 4.** The following properties are equivalent:

(i) \( \mathbb{G} \) is of type GM;
(ii) \( \tilde{\mathbb{M}} \) is dense in \( \mathbb{G} \);
(iii) \( d(g)^2 = \sup_{\theta \in \Omega} \phi(g; \theta) \) for all \( g \in \mathbb{G} \);
(iv) The global reference set \( \mathcal{R} \) is equal to \( \{ \theta; \|U(\theta)\| \leq \pi \} \);
(v) For any arclength parametrized, strictly normal geodesic \( \gamma(s) = \exp(s(\zeta, \tau)) \), its cut time is equal to \( h_{\text{cut}}(\tau) := 2\pi/\|U(\tau)\| \), with the understanding \( h_{\text{cut}}(0) = +\infty \);
(vi) \( \text{Cut}_o^c \cap \partial \mathbb{M} = \emptyset \).

**Remark 7.** (1) The condition (ii) should be the easiest to check among all those.
(2) In Section 4, we can find that every step-two group of corank 1 or 2 is of type GM. As a result, from the above Property (v), we obtain the cut time of any arclength parametrized, strictly normal geodesic on \( \mathbb{G}(q, m, U) \) with \( m = 1 \) or \( 2 \), which coincides with that in [19, Theorems 6 and 7]. However, considering Remark 1, their results for the cut time of abnormal geodesics need more explanations. See (ii) of Corollary 9 and Remark 9 below for more details. Furthermore, we emphasize that Property (v) is an equivalent characterization of GM-groups and thus we have found all possible step-two groups satisfying this fine property.

Recall that \( \mathbb{M} \) is an open set. As a consequence, we obtain the following improvement of [88, Theorem 2.7]:

**Corollary 7.** A step-two group \( \mathbb{G} \) is of type GM if and only if \( \text{Cut}_o = \partial \mathbb{M} \).

**Corollary 8.** Assume that \( \mathbb{G} \) is of MÉtivier type. Then \( \mathbb{G} \) is a GM-group if and only if for any arclength parametrized geodesic \( \gamma(s) = \exp(s(\zeta, \tau)) \), its cut time is \( 2\pi/\|U(\tau)\| \).

If \( \mathbb{G} \) is of GM-type, then our Theorem 3 can be improved. Indeed, the parameter \( \theta \) can be further chosen as a maximum point of the reference function \( \phi(g; \cdot) \) on \( \Omega^*_s \). In other words, we have

**Theorem 5.** Assume that \( \mathbb{G} \) is of GM-type and \( o \neq g \in \mathbb{G} \). For any shortest geodesic \( \gamma_g(s) \) \( (0 \leq s \leq 1) \) joining \( o \) to \( g \), there exist \( \zeta \in \mathbb{R}^q \) and \( \theta \in \Omega^*_s \) such that

\[
\phi(g; \theta) = d(g)^2 = \sup_{\tau \in \Omega^*_s} \phi(g; \tau) = |\zeta|^2, \quad \gamma_g(s) = \exp(s(\zeta, 2\theta)), \quad \forall 0 \leq s \leq 1.
\]

Moreover, we have that \( \theta \in \partial \Omega^*_s \) if \( g \in \tilde{\mathbb{M}}^c \).

**Remark 8.** Suppose further that \( \Omega^*_s \) is strictly convex, namely, for any \( \tau \neq \tau' \in \Omega^*_s \) and \( 0 < s < 1 \), we have \( s\tau + (1 - s)\tau' \in \Omega^*_s \). For example, all MÉtivier groups satisfy this property, see [88, Lemma 9.1]. Then, for any \( g_0 \in \tilde{\mathbb{M}}^c \), the concave function \( \phi(g_0; \cdot) \) has a unique maximum point on \( \partial \Omega^*_s \). This simple observation is very useful to determine all shortest geodesic(s) from \( o \) to \( g_0 \in \tilde{\mathbb{M}}^c \).
Note that in (v) of Theorem 4, we only consider the cut time of any strictly normal geodesic. However, we can characterize GM-groups via the optimal synthesis from $o$, as well as the classical cut locus of $o$. More precisely, we have the following:

**Corollary 9.** The following properties are equivalent:

(i) $G$ is of type GM;

(ii) For any arclength parametrized geodesic $\gamma(s) = \exp\{s(\zeta, \tau)\} = \gamma(\zeta, \tau; s)$, its cut time is given by

$$h_{\text{cut}} = \max\left\{\frac{2\pi}{\|U(\sigma)\|}; \sigma \in \mathbb{R}^m, \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot)\right\},$$

with the understanding $\frac{2\pi}{0} = +\infty$;

(iii) $\text{Cut}_{o}^{\text{CL}} = \tilde{\mathbb{M}}^c$.

**Remark 9.** It follows from (ii) of Corollary 9 that the statement in [19, Theorems 6 and 7] is correct if we choose the covector suitably. To be more precise, let $G$ be a step-two group of corank 1 or 2, so it is a GM-group from Corollary 13 in Section 4 below. Let $\gamma(s) = \exp\{s(\zeta, \tau)\} = \gamma(\zeta, \tau; s)$ be an arclength parametrized geodesic on $G$. Recall that $\Pi(\zeta, \tau) := \{\sigma - \tau \in \mathbb{R}^m; \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot)\}$ is a linear subspace. Assume further that the continuous function $\sigma \mapsto \|U(\sigma)\|$ defined on $\tau + \Pi(\zeta, \tau)$ attains its minimum at $\tau$. Then the cut time of $\gamma$ is given by $2\pi/\|U(\tau)\|$.

### 2.5 On the lack of semi-concavity of $d^2$ on $\tilde{\mathbb{M}}_2$

Recall that $o \in \tilde{\mathbb{M}}_2 \subset G$ is the set of points, $g$, where the reference function $\phi(g; \cdot)$ has a degenerate (so infinite) critical point in the initial reference set $\Omega_\ast$. The following theorem is a kind of generalization of the first result in [102, Theorem 1.1], which will be proven by a completely different method.

**Theorem 6.** Let $g_0 = (x_0, t_0) \in \tilde{\mathbb{M}}_2$. Then there exist a unit vector $\nu_0 \in \mathbb{R}^m$ and a constant $c_0 > 0$ such that

$$d(x_0, t_0 + h\nu_0)^2 + d(x_0, t_0 - h\nu_0)^2 - 2d(x_0, t_0)^2 \geq c_0 h, \quad \forall h > 0. \quad (2.27)$$

In particular, together with (a) in Remark 3 and (2.8), it gives immediately

**Corollary 10.** If $g_0 = (x_0, 0) \in \text{Abn}_o^\ast$, there exist a unit vector $\nu_\ast \in \mathbb{R}^m$ and a constant $c_\ast > 0$ such that

$$d(x_0, h\nu_\ast)^2 - d(x_0, 0)^2 \geq c_\ast|h|, \quad \forall h \in \mathbb{R}. \quad (2.28)$$

A direct consequence of (2.27) is the following significantly weaker estimate:

$$\limsup_{g' \to o} \frac{d(g_0 + g')^2 + d(g_0 - g')^2 - 2d(g_0)^2}{|g'|^2} = +\infty, \quad (2.29)$$
which implies, from (1.1), the lack of semi-concavity of $d^2$ for any $g_0 \in \widetilde{M}_2$.

It follows from Lemma 6 that $\overline{M}_2 \subseteq \text{Abn}_o^*$. A very interesting phenomenon is that (2.29) can be no longer valid for $g_0 \in \overline{M}_2 \setminus \widetilde{M}_2$ even in the setting of GM-groups. A concrete example will be provided in Subsection 6.2 below.

However, recall that our $\text{SC}^-_o$ is defined by (1.2) instead of as the set of points where (2.29) satisfies. Obviously, $\text{SC}^-_o$ is closed. When the underlying group is of type GM, combining (ii) of Corollary 9 with Theorem 6 obtained above, we can characterize $\text{Abn}_o^*$ via $\widetilde{M}_2$; as a byproduct, we answer the open problem [20, (29)] affirmatively:

\textbf{Theorem 7.} In the setting of GM-groups, it holds that $\text{SC}^-_o = \text{Abn}_o^* = \overline{M}_2$.

Furthermore, combining Theorem 7, Lemmas 4 and 6 with [20, Corollary 30] (with little modification in its proof), we answer the open question [20, second part of (30)], when the underlying group is GM-group:

\textbf{Corollary 11.} In the framework of GM-groups, we have $\text{Cut}_o = \text{SC}^+_o \cup \text{SC}^-_o$.

Similarly, by Corollaries 4 and 6, we have the following result that provides an affirmative, also partial, answer to the open questions [20, (29)-(30)]:

\textbf{Corollary 12.} Let $G$ be a step-two group such that $\widetilde{M}_2 \subseteq \{(x,0); \; x \in \mathbb{R}^q\}$. Then it holds that $\text{SC}^-_o = \text{Abn}_o^* (= \widetilde{M}_2)$ and $\text{Cut}_o = \text{SC}^+_o \cup \text{SC}^-_o$.

Recall that $\widetilde{M}_2 = \{o\}$ if and only if $G$ is of Métivier type. In the sequel, a step-two group $G$ is said to be a $\text{SA-group}$ (or of type $SA$) if it satisfies

\[\{o\} \neq \widetilde{M}_2 \subseteq \{(x,0); \; x \in \mathbb{R}^q\}. \quad (SA)\]

Notice that star graphs and $N_{3,2}$, namely the free step-two Carnot group with three generators, are SA-groups. See [88] or §5 and §7 below for more details. Also remark that star graphs are both GM-groups and SA-groups, $N_{3,2}$ is the simplest example of SA-group that is not of type GM. Of course, we can provide an uncountable number of SA but not GM groups.

Furthermore, a trivial method to construct SA-groups can be found in Proposition 11 of Appendix B. See Appendix C for another method which is much more meaningful. In particular, SA-groups of corank 1 are the direct product of a Euclidean space $\mathbb{R}^k$ with a generalized Heisenberg group. However, for any $m \geq 2$, SA-groups with corank $m$ form a very complicated set.

To finish this section, we point out the following facts:

(1) The second result of [102, Theorem 1.1] says that on the free step-two Carnot group with $k$ ($k \geq 4$) generators, $N_{k,2}$, for any $g_0 = (x_0, t_0) \in \text{Abn}_o^*$, there exist a unit vector $\nu_s \in \mathbb{R}^{k(k-1)/2}$ and a constant $c_s > 0$ such that we have the following counterpart of (2.28):

\[d(x_0, t_0 + h \nu_s) - d(x_0, t_0) \geq c_s |h|, \quad -c_s \leq h \leq c_s.\]

Hence, (2.29) is valid on $N_{k,2}$ for any $g_0 \in \text{Abn}_o^*$. 

17
(2) \( N_{k,2} (k \geq 4) \) neither is a GM-group nor satisfies \( \overline{M}_2 \subseteq \{(x,0); \, x \in \mathbb{R}^k\} \).

3  Proof of main results

3.1 Proof of Theorem 3

Proof. We first assume that the shortest geodesic joining \( o \) to \( g \) is unique. By [88, Corollary 2.4], there exist \( w \in \mathbb{R}^q \) with \( |w| = d(g) \) and \( \theta \in \mathfrak{X} \) such that \( \gamma_{(w,2\theta)} \) is a shortest geodesic joining \( o \) to \( g \). By uniqueness we must have \( \gamma_g(s) = \gamma_{(w,2\theta)}(s) = \exp(s \cdot (w, 2\theta)) \) for all \( 0 \leq s \leq 1 \), which ends the proof in this case.

In general, from the characterization of the shortest geodesic in Subsection 2.2.1, there exist \( w_*, \theta_* \in \mathbb{R}^m \) (not necessarily belonging to \( \mathfrak{X} \)) such that \( \gamma_g = \gamma_{(w_*,2\theta_*)} \). We now prove that for each \( s_* \in (0, 1) \), the restriction of \( \gamma_g \) on the interval \( [0, s_*] \), \( \gamma_g|_{[0,s_*]} = \gamma_{(s_* w_*, 2s_* \theta_*)} := (\gamma_g)^{0,s_*} \) is the unique shortest geodesic joining \( o \) to \( \gamma_g(s_*) = \exp(s_* w_*, 2s_* \theta_*) \). Notice that \( (\gamma_g)^{0,s_*} \) is a shortest geodesic joining \( o \) to \( \gamma_g(s_*) \) since \( \gamma_g \) itself is shortest. To prove uniqueness, we argue by contradiction. Assume that there is another shortest geodesic \( \gamma_{s_*} \neq (\gamma_g)^{0,s_*} \) joining \( o \) to \( \gamma_g(s_*) \) with constant speed \( s_* \cdot |w_*| \). Then we construct a horizontal curve with constant speed \( |w_*| = d(g) \) defined by

\[
\tilde{\gamma}_{s_*}(s) := \begin{cases} 
\gamma_{s_*} \left( \frac{s}{s_*} \right), & 0 \leq s \leq s_*, \\
\gamma_g(s), & s_* \leq s \leq 1.
\end{cases}
\]

Obviously, \( \tilde{\gamma}_{s_*} \) is a shortest geodesic joining \( o \) to \( g \) as well. Again from the characterization of the shortest geodesic in Subsection 2.2.1, there exist \( w_{**} \in \mathbb{R}^q \) with \( |w_{**}| = d(g) \) and \( \theta_{**} \in \mathbb{R}^m \) such that \( \tilde{\gamma}_{s_*} = \gamma_{(w_{**},2\theta_{**})} \).

By the fact that \( \gamma_g(s) = \tilde{\gamma}_{s_*}(s) \) when \( s \in [s_*, 1] \), it follows from Lemma 2 that \( \gamma_g \) and \( \tilde{\gamma}_{s_*} \) coincide on the whole interval \( [0, 1] \). In particular, \( \gamma_g(s) = \tilde{\gamma}_{s_*}(s) \) for all \( 0 \leq s \leq s_* \), that is \( (\gamma_g)^{0,s_*} = \gamma_{s_*} \), which contradicts with our assumption.

For each \( s_* \in (0, 1) \), what we have proven at the beginning shows that there exist \( w(s_*) \in \mathbb{R}^q \) with \( |w(s_*)| = s_* d(g) \) and \( \theta(s_*) \in \mathfrak{X} \) such that

\[
\exp(s \cdot (w(s_*), 2\theta(s_*))) = (\gamma_g)^{0,s_*}(s) = \exp(s \cdot (s_* w_*, 2s_* \theta_*)), \quad \forall \, 0 \leq s \leq 1. \tag{3.1}
\]

From compactness of \( \mathfrak{X} \), we extract a sequence \( \{s_j\}_{j=1}^{+\infty} \subseteq (0, 1) \) such that \( s_j \to 1 \), \( w(s_j) \to w \) with \( |w| = d(g) \) and \( \theta(s_j) \to \theta \in \mathfrak{X} \) as \( j \to +\infty \). With \( s_* \) replaced by \( s_j \) in (3.1) and letting \( j \to +\infty \), we obtain that \( \exp(s \cdot (w_*, 2\theta_*)) = \gamma_g(s) = \exp(s \cdot (w, 2\theta)) \) for all \( 0 \leq s \leq 1 \), which ends the proof of the theorem.

3.2 Proof of Corollary 5

Proof. For convenience, we set

\[
\tilde{h} = \tilde{h}(\zeta, \tau) := \sup \{ 2C_\sigma; \, \sigma \in \mathbb{R}^m, \gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot) \}.
\]
For any $s* \in (0, h_{\text{cut}})$, we know that $\gamma(s*,\zeta,s*)$ is a shortest geodesic. It follows from Theorem 3 that there exist $\zeta(s*) \in \mathbb{R}^q$ and $\theta(s*) \in \mathfrak{R}$ such that $\gamma(s*,\zeta,s*) = \gamma(\zeta(s*),2\theta(s*))$. Then Lemma 2 implies that $s* \zeta = \zeta(s*)$ and

$$\gamma(\zeta, \sigma(s*); \cdot) = \gamma(\zeta, \tau; \cdot),$$

with $\sigma(s*) := \frac{2\theta(s*)}{s*}$.

Recalling that $\mathfrak{R}$ is star-shaped w.r.t. 0, by the fact that $\theta(s*) = \frac{s* \sigma(s*)}{2} \in \mathfrak{R}$, we have

$$\frac{s*}{2} \leq C_\sigma(s*) \leq \frac{\tilde{h}}{2},$$

which implies $s* \leq \tilde{h}$. Since $s* \in (0, h_{\text{cut}})$ is arbitrary, we obtain $h_{\text{cut}} \leq \tilde{h}$. To prove the second inequality, it suffices to observe that for each $\sigma \in \mathbb{R}^m$, we have $C_\sigma \leq \frac{C_m}{\|U(\sigma)\|}$ and this finishes the proof of Corollary 5.

### 3.3 Proof of Lemma 6

**Proof.** For any $\{g_j\}_{j=1}^{+\infty} \subseteq \text{Abn}_n^*$ such that $g_j \to g \in \mathcal{G}$ as $j \to +\infty$, our aim is to prove $g \in \text{Abn}_n^*$ as well. For each $j \in \mathbb{N}^*$, let $\gamma_j(s) (0 \leq s \leq 1)$ be an abnormal shortest geodesic joining $o$ to $g_j$. By Theorem 3, there exist $\zeta(j) \in \mathbb{R}^q$ with $|\zeta(j)| = d(g_j)$ and $\theta(j) \in \mathfrak{R}$ such that $\gamma_j = \gamma(\zeta(j),2\theta(j))$. Since $\gamma(\zeta(j),2\theta(j))$ is abnormal, from Proposition 1 there exists a $\sigma(j) \in S^{m-1}$ such that

$$U(\sigma(j))U(\theta(j))^k \zeta(j) = 0, \quad \forall k \in \mathbb{N}. \quad (3.2) \text{ Abn}$$

Notice that $|\zeta(j)| = d(g_j) \to d(g)$ as $j \to +\infty$. From compactness, up to subsequences, we may assume that $\zeta(j) \to \zeta_0$ with $|\zeta_0| = d(g)$, $\theta(j) \to \theta_0 \in \mathfrak{R}$ and $\sigma(j) \to \sigma_0 \in S^{m-1}$ as $j \to +\infty$. Observe that

$$\gamma(\zeta_0,2\theta_0)(1) = \exp(\zeta_0,2\theta_0) = \lim_{j \to +\infty} \exp(\zeta(j),2\theta(j)) = \lim_{j \to +\infty} g_j = g.$$

By the fact that $|\zeta_0| = d(g)$, we obtain that $\gamma(\zeta_0,2\theta_0)$ is a shortest geodesic joining $o$ to $g$. It remains to prove that $\gamma(\zeta_0,2\theta_0)$ is also abnormal. In fact, letting $j \to +\infty$ in (3.2), we get

$$U(\sigma_0)U(\theta_0)^k \zeta_0 = 0, \quad \forall k \in \mathbb{N},$$

which implies $\gamma(\zeta_0,2\theta_0)$ is abnormal by Proposition 1.

This ends the proof of Lemma 6. \qed

### 3.4 Proof of Proposition 4

**Proof.** Set

$$\Xi_1 := \{(x, \theta) \in \mathbb{R}^q \times \Omega_\ast; \det(-\text{Hess}_\theta(U(\theta) \cot U(\theta) x, x)) > 0\},$$

$$\Xi_2 := \{(x, \theta) \in \mathbb{R}^q \times \Omega_\ast; \det(-\text{Hess}_\theta(U(\theta) \cot U(\theta) x, x)) = 0\},$$

where $\Omega_\ast$ is a certain domain in $\mathbb{R}^m$.

19
and the map

\[ \kappa : \mathbb{R}^q \times \Omega_* \rightarrow \mathbb{M} \]

\[ (x, \theta) \mapsto \left( x, -\frac{1}{4} \nabla_{\theta} (U(\theta) \cot U(\theta) x, x) \right). \]

It follows from Proposition 2 that \( \Xi_1 \cup \Xi_2 = \mathbb{R}^q \times \Omega_* \). Recall that (cf. (2.18)-(2.20)) \( \kappa(\Xi_1) = \mathbb{M} \) and \( \kappa(\Xi_2) = \mathbb{M}_2 \).

Observe that the function \((x, \theta) \mapsto \det(-\text{Hess}_\theta (U(\theta) \cot U(\theta) x, x))\) is real analytic in \( \mathbb{R}^q \times \Omega_* \). We claim that \( \Xi_2 \) has an empty interior. Otherwise \( \Xi_2 \) should be \( \mathbb{R}^q \times \Omega_* \) (cf. [73, § 3.3 (b)])], which means \( \Xi_1 = \emptyset \) and thus \( \mathbb{M} = \emptyset \). This leads to a contradiction since \( \mathbb{M} \neq \emptyset \) from Remark 4. In conclusion, \( \Xi_1 \) is dense in \( \mathbb{R}^q \times \Omega_* \).

Now, we shall show that \( \mathbb{M}_2 \subseteq \mathbb{M} \). Fix \((x, t) \in \mathbb{M}_2 \). There exists a \( \theta \in \Omega_* \) such that \((x, \theta) \in \Xi_2 \) and \( \kappa(x, \theta) = (x, t) \). Since \( \Xi_1 \) is dense in \( \mathbb{R}^q \times \Omega_* \), there are \( \{(x^{(j)}, \theta^{(j)})\}_{j=1}^\infty \subseteq \Xi_1 \) such that \((x^{(j)}, \theta^{(j)}) \rightarrow (x, \theta)\) as \( j \rightarrow +\infty \). Hence, \( \mathbb{M} \ni \kappa(x^{(j)}, \theta^{(j)}) \rightarrow \kappa(x, \theta) = (x, t) \) as \( j \rightarrow +\infty \). As a result, we obtain that \( \mathbb{M}_2 \subseteq \mathbb{M} \) and thus \( \overline{\mathbb{M}} = \mathbb{M} \), which ends the proof of the proposition.

### 3.5 Proof of Theorem 4

**Proof.** (i) \( \Rightarrow \) (ii): This is evident.

(ii) \( \Rightarrow \) (iii): Just use (2.21).

(iii) \( \Rightarrow \) (iv): For any given \( g = (x, t) \in \mathcal{S} \), under our assumption, it follows from Proposition 2 that there exists a \( \theta_0 \in \Omega_* \) such that

\[ d(x, t)^2 = \phi((x, t); \theta_0) = \langle U(\theta_0) \cot U(\theta_0) x, x \rangle + 4 t \cdot \theta_0. \] \((3.3)\) \( t_{\min} \)

Since \( \mathcal{S} \) is open, there exists a \( r_0 > 0 \) such that

\[ \{x\} \times B(t, r_0) = \{x\} \times \{\tau; |\tau - t| < r_0\} \subseteq \mathcal{S}. \]

Then it follows from (iii) that we have for \( s \in B(t, r_0) \),

\[ d(x, s)^2 \geq \phi((x, s); \theta_0) = \langle U(\theta_0) \cot U(\theta_0) x, x \rangle + 4 s \cdot \theta_0. \] \((3.4)\) \( s_{\min} \)

So, the function \( s \mapsto d(x, s)^2 - 4 s \cdot \theta_0 \) has a local minimum at the point \( s = t \). As a result, we have \( \frac{1}{4} \nabla_t d(g)^2 = \theta_0 \in \Omega_* \) and consequently \( \mathcal{R} \subseteq \Omega_* \). The inverse inclusion is given by (2.23) and we obtain (iv).

(iv) \( \Rightarrow \) (v): Just combine Corollary 2 with Corollary 5.

(v) \( \Rightarrow \) (vi): We argue by contradiction. Assume that there exists a \( g \in \text{Cut}_o^c \cap \partial \mathbb{M} \).

Since \( \mathbb{M}_2 \subseteq \text{Abn}_o^c \subseteq \text{Cut}_o \), we have \( g \in \overline{\mathbb{M}} \setminus \mathbb{M} \). Then from [88, (1) of Remark 2.6] there exist \( \zeta \in \mathbb{R}^q \) with \( |\zeta| = d(g) \) and \( \theta \in \partial \Omega_* \) such that \( \gamma_{(\zeta, 2 \theta)} \) is a shortest geodesic joining \( o \) to \( g \). Since \( g \in \text{Cut}_o^c = \mathcal{S} \), it follows from (2.15) that \( \gamma_{(\zeta, 2 \theta)} \) is strictly normal. As a result, (v) implies \( g \in \text{Cut}_o^c \subseteq \text{Cut}_o \) and we obtain a contradiction.
(vi) ⇒ (i): We argue by contradiction. Assume that \( \overline{M} \subseteq \Omega \). Since \( \text{Cut}^c_o \) is dense in \( \Omega \) by [2, Theorem 11.8], we can pick a \( g \in (\Omega \setminus \overline{M}) \cap \text{Cut}^c_o \). From the characterization of the smooth points (2.15), there exists a unique shortest geodesic \( \gamma = \gamma_{(w,2\theta)} \) joining \( o \) to \( g \), which is not abnormal. We first claim that \( \theta \notin \Omega_* \), otherwise \( \theta \) should be a critical point of \( \phi(g; \cdot) \) in \( \Omega_* \) by Theorem 2 and thus \( g \in \overline{M} = \overline{M} \cup \overline{M}_2 \). Since \( g \notin \overline{M} \), then \( g \in \overline{M}_2 \subseteq \text{Cut}^c_o \), which gives a contradiction and proves this assertion.

We further claim that \( \theta \notin \partial \Omega_* \). Otherwise, Lemma 5 and Theorem 2 should imply that for any \( s_* \in (0, 1) \), we have that \( \exp(s_*(w,2\theta)) \in \overline{M} \). Thus \( g = \exp(w,2\theta) \in \overline{M} \), contradicting our assumption that \( g \in (\Omega \setminus \overline{M}) \cap \text{Cut}^c_o \).

As a result, there exists a \( s_0 \in (0, 1) \) such that \( s_0 \theta \in \partial \Omega_* \). Set

\[
g_0 = \exp(s_0(w,2\theta)) \in \text{Cut}^c_o = \mathcal{S},
\]

where the “\( \in \)” is given by Lemma 5. Similarly, we get that \( \exp(s_*(w,2\theta)) \in \overline{M} \) for all \( s_* \in (0, s_0) \) and \( g_0 \in \overline{M} \).

Now we are in a position to show that \( g_0 \notin \overline{M} \). We argue by contradiction. Assume that \( g_0 \in \overline{M} \), then there exists \( (w_*,2\theta_*) \) such that \( \theta_* \in \Omega_* \) and \( \gamma_{(w_*,2\theta_*)} \) is the unique shortest geodesic joining \( o \) and \( g_0 \). So \( \gamma_{(w_*,2\theta_*)} \) coincides with the restriction of \( \gamma \) in \([0, s_0]\), namely \( \gamma^{0,s_0} = \gamma_{(s_0w,2s_0\theta)} \). Hence, \( \gamma^{0,s_0} \) admits two different normal lifts. Consequently \( \gamma^{0,s_0} \) is also abnormal by definition, which contradicts with, by Lemma 4, the fact that \( \gamma \) is strictly normal.

After all, we have that \( g_0 \in \partial \overline{M} \cap \text{Cut}^c_o \), which leads to a contradiction. Therefore we finish the proof.

### 3.6 Proof of Theorem 5

**Proof.** By the definition of \( \overline{M} \) (see (2.18)), the second claim in Theorem 5 is a direct consequence of the first one, that needs to be proven. Indeed, from Corollary 7 we have \( \mathcal{S} = \overline{M} \). By Remark 5, there exist \( \{g_j = (\zeta(j),2\theta(j))\}_{j=1}^{+\infty} \subseteq \mathcal{S} = \overline{M} \) with \( \gamma_{(\zeta(j),2\theta(j))} \) \( (\{\theta(j)\}_{j=1}^{+\infty} \subseteq \Omega_* \) the unique shortest geodesic joining \( o \) to \( g_j \) such that:

\[
g_j \rightarrow g, (\zeta(j),2\theta(j)) \rightarrow (\zeta,2\theta) \text{ as } j \rightarrow +\infty, \quad \gamma_{(\zeta,2\theta)} = \gamma_g.
\]

It remains to prove that \( \phi(g;\theta) = d(g)^2 \) when \( \theta \in \partial \Omega_* \).

Notice that \( U(\tau)^2 \) is semi-positive definite for every \( 0 \neq \tau \in \mathbb{R}^m \). Let \( 0 \leq \lambda_1(\tau)^2 \leq \ldots \leq \lambda_q(\tau)^2 \) \( (\lambda_l(\tau) \geq 0, 1 \leq l \leq q) \) denote its eigenvalues and \( \{P_l(\tau)\}_{l=1}^q \) the corresponding set of pairwise orthogonal projections (that is, \( (P_k(\tau))(\mathbb{R}^q) \perp (P_l(\tau))(\mathbb{R}^q) \) for \( k \neq l \)). Then we have

\[
U(\tau)^2 = \sum_{l=1}^q \lambda_l(\tau)^2 P_l(\tau). \tag{3.5} \]

It follows from [77, Chapter two] that for every \( 1 \leq l \leq q \), \( \lambda_l(\tau) \) is a continuous function of \( \tau \neq 0 \) and homogeneous of degree 1, namely \( \lambda_l(s\tau) = s\lambda_l(\tau) \) for \( s > 0 \). However, \( P_l(\tau) \)
is not necessarily continuous, but it can be chosen to be symmetric and homogeneous of
degree 0, namely

\[ P_l(r \tau) = P_l(\tau) \quad \forall r \neq 0, \ 1 \leq l \leq q. \quad (3.6) \]

For the \( \theta \in \partial \Omega_+ \) obtained before, there exists an \( L \in \{1, \ldots, q\} \) such that

\[ \lambda_{L-1}(\theta)^2 < \pi^2 \quad \text{when} \ L > 1, \quad \text{and} \quad \lambda_L(\theta)^2 = \cdots = \lambda_q(\theta)^2 = \pi^2. \]

From the continuity of \( \{\lambda_l(\tau)\}_{l=1}^q \), there exist \( \delta \in \left(0, \frac{\pi^2}{4}\right) \) and \( r_0 \in \left(0, \frac{|\theta|}{2}\right) \) such that for \( \tau \in B(\theta, r_0) = \{\tau; |\tau - \theta| < r_0\} \), we have

\[ \lambda_{L-1}(\tau)^2 \leq \pi^2 - 4\delta \quad \text{when} \ L > 1, \quad \text{and} \quad \pi^2 - \delta \leq \lambda_L(\tau)^2 \leq \cdots \leq \lambda_q(\tau)^2 \leq \pi^2 + \delta. \]

For \( \tau \in B(\theta, r_0) \), let us set

\[ V(\tau) := \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_1} z (z - U(\tau)^2)^{-1}dz = \sum_{l=1}^{L-1} \lambda_l(\tau)^2 P_l(\tau) & \text{when} \ L > 1, \\ 0 & \text{when} \ L = 1 \end{cases}, \quad (3.7) \]

and the projection on (near \( \pi^2 \))-eigenspaces of \( U(\tau)^2 \)

\[ Q(\tau) := \frac{1}{2\pi i} \int_{\Gamma_2} (z - U(\tau)^2)^{-1}dz = \sum_{l=L}^{q} P_l(\tau), \quad (3.8) \]

where the contours \( \Gamma_1, \Gamma_2 \subseteq \mathbb{C} \) are defined by

\[ \Gamma_1 := \{z; \ \text{dist}(z, [0, \pi^2 - 4\delta]) = \delta\} \quad \text{and} \quad \Gamma_2 := \{z; \ \text{dist}(z, [\pi^2 - \delta, \pi^2 + \delta]) = \delta\} \]

respectively, with the counterclockwise orientation. From the integral representation, it is easy to see that the operator functions \( V(\tau) \) and \( Q(\tau) \) are continuous in \( B(\theta, r_0) \). It deduces from Theorem 2 that

\[
\sum_{l=1}^{q} \left( \frac{\lambda_l(\theta^{(j)})}{\sin \lambda_l(\theta^{(j)})} \right)^2 |P_l(\theta^{(j)}) x^{(j)}|^2 = \left| \frac{U(\theta^{(j)})}{\sin U(\theta^{(j)})} x^{(j)} \right|^2 = |\zeta^{(j)}|^2 \to |\zeta|^2, \quad \text{as} \ j \to +\infty.
\]

As a result, since

\[
|Q(\theta) x|^2 = \lim_{j \to +\infty} |Q(\theta^{(j)}) x^{(j)}|^2 = \lim_{j \to +\infty} \sum_{l=L}^{q} |P_l(\theta^{(j)}) x^{(j)}|^2
\]

\[
= \lim_{j \to +\infty} \sum_{l=L}^{q} \left( \frac{\lambda_l(\theta^{(j)})}{\sin \lambda_l(\theta^{(j)})} \right)^2 \left( \frac{\lambda_l(\theta^{(j)})}{\sin \lambda_l(\theta^{(j)})} \right)^2 |P_l(\theta^{(j)}) x^{(j)}|^2,
\]

and \( \sin s \sim (\pi - s) \) for \( s \) near \( \pi \), we get immediately

\[ |Q(\theta) x|^2 = 0. \quad (3.9) \]
Similarly, by the fact that  
\[ \left\langle \sqrt{V(\theta^j)} \cot \sqrt{V(\theta^j)} x^{(j)}, x^{(j)} \right\rangle = \langle U(\theta^j) \cot U(\theta^j) x^{(j)}, x^{(j)} \rangle + |Q(\theta^j) x^{(j)}|^2 - \sum_{l=1}^{q} (\lambda_l(\theta^j) \cot \lambda_l(\theta^j)) |P_l(\theta^j) x^{(j)}|^2, \]

we yield that  
\[ \left\langle \sqrt{V(\theta)} \cot \sqrt{V(\theta)} x, x \right\rangle = \lim_{j \to +\infty} \left\langle \sqrt{V(\theta^j)} \cot \sqrt{V(\theta^j)} x^{(j)}, x^{(j)} \right\rangle = \lim_{j \to +\infty} \langle U(\theta^j) \cot U(\theta^j) x^{(j)}, x^{(j)} \rangle. \tag{3.10} \]

Hence, it follows from Theorem 2 that  
\[ d(g)^2 = \lim_{j \to +\infty} d(g_j)^2 = \lim_{j \to +\infty} \phi(g_j; \theta^j) = \lim_{j \to +\infty} \left( \langle U(\theta^j) \cot U(\theta^j) x^{(j)}, x^{(j)} \rangle + 4 t(\cdot) \cdot \theta(\cdot) \right) = \left\langle \sqrt{V(\theta)} \cot \sqrt{V(\theta)} x, x \right\rangle + 4 t \cdot \theta. \]

Combining this with \( \theta \in \partial \Omega^* \) and the fact that the orthogonal projection of \( x \) on \( \pi^2 \)-eigenspace of \( U(\theta)^2 \) is zero (cf. (3.9)), it follows from [88, Remark 2.1] that  
\[ d(g)^2 = \langle U(\theta) \cot U(\theta) x, x \rangle + 4 t \cdot \theta = \phi((x,t); \theta). \tag{3.11} \]

This ends the proof of Theorem 5.

\[ \square \]

3.7 Proof of Corollary 9

Proof. (i) \( \Rightarrow \) (ii): Just combine Corollary 2 with Corollary 5 (cf. also Remark 6).

(ii) \( \Rightarrow \) (i): It is trivial because Theorem 4 (v) satisfies under our assumption.

(iii) \( \Rightarrow \) (i): It follows from Theorem 1 and [118, Proposition 15] that \( \tilde{\mathbb{M}}^c = \text{Cut}_{o}^{\text{CL}} \subseteq \text{Cut}_{\partial} \) is a set of measure zero, which means \( \mathbb{M} \) is dense in \( \mathbb{G} \). This is exactly (ii) of Theorem 4 and we obtain that \( \mathbb{G} \) is of type GM.

(i) + (ii) \( \Rightarrow \) (iii): From Corollary 3 it suffices to prove \( \tilde{\mathbb{M}}^c \subseteq \text{Cut}_{o}^{\text{CL}} \). Fix \( g \in \tilde{\mathbb{M}}^c \). It is clear that \( g \neq o \). Theorem 5 guarantees that there exist \( \zeta \in \mathbb{R}^q \setminus \{0\} \) and \( \theta \in \partial \Omega^* \) such that \( \gamma_* = \gamma(\zeta,2\theta) \) is a shortest geodesic joining \( o \) to \( g \). Consider the arclength parametrized geodesic \( \tilde{\gamma} := \gamma(\tilde{\zeta}, \tau; \cdot) \), where \( \tau := \frac{2\theta}{|\zeta|} \). Here and in the sequel, we adopt the convention  
\[ \hat{u} := \begin{cases} u, & \text{if } u \in \mathbb{R}^\ell \setminus \{0\}; \\ 0, & \text{if } u = 0. \end{cases} \tag{3.12} \]

To prove that \( g \in \text{Cut}_{o}^{\text{CL}} \), it remains to show that the cut time of \( \tilde{\gamma} \), \( h_{\text{cut}}(\tilde{\gamma}) \), equals \( |\zeta| \). First, notice that \( \tilde{\gamma}|_{[0, |\zeta|]} = \gamma(\zeta,2\theta) = \gamma_* \) is a shortest geodesic. Hence we get \( h_{\text{cut}}(\tilde{\gamma}) \geq |\zeta| \).
On the other hand, for any $\sigma \in \mathbb{R}^m$ such that $\gamma(\zeta, \sigma; \cdot) = \gamma(\hat{\zeta}, \tau; \cdot)$, we have $\gamma_\ast = \gamma(\zeta, \sigma; \cdot)$. Since $g \in \tilde{M}_2$, it follows from Theorem 2 that

$$\left\| U \left( \frac{|\sigma|}{2} \right) \right\| \geq \pi,$$

or equivalently $\frac{2\pi}{\|U(\sigma)\|} \leq |\sigma|$. Then from (iii), we obtain that $h_{\text{cut}}(\gamma) \leq |\sigma|$.

Therefore, we finish the proof of Corollary 9.

\section*{3.8 Proof of Theorem 6}

\textit{Proof.} Let $g_0 = (x_0, t_0) \in \tilde{M}_2$. Then (2.21) and (2.20) imply that there exists a $\theta_0 \in \Omega_\ast$ such that $d(g_0)^2 = \phi(g_0; \theta_0)$. Since $\Omega_\ast$ is open, there exists a $r_0 > 0$ such that $B(\theta_0, 2r_0) = \{ \tau \in \mathbb{R}^m; |\tau - \theta_0| < 2r_0 \} \subseteq \Omega_\ast$. From [88, Proposition 5.1 (c)] there exists a unit vector $\nu_0$ such that

$$t_0 \cdot \nu_0 = 0, \quad \phi(g_0; \theta_0 + s\nu_0) = \phi(g_0; \theta_0), \quad \forall s \in \mathbb{R} \text{ with } \theta_0 + s\nu_0 \in \Omega_\ast.$$

Consequently, for any $h > 0$, using [88, Theorem 2.1], we have that

$$d(x_0, t_0 + h\nu_0)^2 = \phi((x_0, t_0 + h\nu_0); \theta_0 + r_0\nu_0) = \phi(g_0; \theta_0) + 4h r_0 + 4h \nu_0 \cdot \theta_0,$$

$$d(x_0, t_0 - h\nu_0)^2 = \phi((x_0, t_0 - h\nu_0); \theta_0 - r_0\nu_0) = \phi(g_0; \theta_0) + 4h r_0 - 4h \nu_0 \cdot \theta_0.$$

As a result, we obtain

$$d(x_0, t_0 + h\nu_0)^2 + d(x_0, t_0 - h\nu_0)^2 - 2d(g_0)^2 \geq 8r_0 h, \quad \forall h > 0,$$

which finishes the proof of this theorem.

\section*{3.9 Proof of Theorem 7}

\textit{Proof.} First, for any arclength parametrized abnormal geodesic $\gamma(s) = \exp\{s(\zeta, \tau)\} = \gamma(\zeta, \tau; s)$ with cut time $h_{\text{cut}}$ and $s_\ast < h_{\text{cut}}$, we claim that $\gamma(s_\ast) = \gamma(\zeta, \tau; s_\ast) \in \tilde{M}_2$. In fact, from (ii) of Corollary 9, there exists a $\sigma \in \mathbb{R}^m$ such that $s_\ast < \frac{2\pi}{\|U(\sigma)\|}$ and $\gamma(\zeta, \sigma; \cdot) = \gamma(\zeta, \tau; \cdot)$. Then it follows from the first equation of Proposition 3 that

$$\gamma(s_\ast) = \gamma(\zeta, \tau; s_\ast) = \gamma(\zeta, \sigma; s_\ast) \in \tilde{M}_2.$$

Thus, the second equation in Proposition 3 implies that $\text{Abn}_o^\ast \subseteq \tilde{M}_2$.

Next, from definition the set $\text{SC}_o^\ast$ is closed. Moreover, Theorem 6 implies that $\tilde{M}_2 \subseteq \text{SC}_o^\ast$. In conclusion, we have that

$$\text{Abn}_o^\ast \subseteq \tilde{M}_2 \subseteq \text{SC}_o^\ast.$$

Finally, we recall that $\text{Abn}_o^\ast$ is closed (cf. Lemma 6), so the inclusion $\text{SC}_o^\ast \subseteq \text{Abn}_o^\ast$ can be deduced from [35, Theorem 1], which ends the proof of Theorem 7.
4 Step-two groups of Corank 2 are GM-groups

The purpose of this section is twofold. On one hand, we provide a sufficient condition for $\overline{M} = G$ by means of semi-algebraic theory. As a byproduct, we show that all $G(q, 2, U)$ are of type GM. On the other hand, we prove that there exists a Métivier group $G(4N, 3, U_N)$, which is not of type GM, for any $N \in \mathbb{N}^\star$.

Let us begin by recalling (cf. [31, Chapter 2]):

4.1 Semi-algebraic sets, mappings and dimension

A set $A \subseteq \mathbb{R}^q$ is semi-algebraic if it is the result of a finite number of unions and intersections of sets of the form $\{f = 0\}, \{g > 0\}$, where $f, g$ are polynomials on $\mathbb{R}^q$. If $A$ is a semi-algebraic set, then its complement, boundary and any Cartesian projection of $A$ are semi-algebraic sets. If $A$ and $B$ are semi-algebraic sets, then so does $A \times B$. Furthermore, any semi-algebraic set $A \subseteq \mathbb{R}^q$ is the disjoint union of a finite number of semi-algebraic sets $M_i$ in $\mathbb{R}^q$ where each $M_i$ is a smooth submanifold in $\mathbb{R}^q$ and diffeomorphic to $(0, 1)^{\dim M_i}$.

If $A \subseteq \mathbb{R}^q$ and $B \subseteq \mathbb{R}^r$ are two semi-algebraic sets. A mapping $h : A \to B$ is semi-algebraic if its graph is a semi-algebraic set in $\mathbb{R}^{q+r}$. If $S \subseteq A$ is a semi-algebraic set and $h : A \to B$ is a semi-algebraic mapping, then $h(S)$ is a semi-algebraic set in $\mathbb{R}^r$.

Let $A \subseteq \mathbb{R}^q$ be a semi-algebraic set. Its dimension, $\dim A$, can be defined in some algebraic way. The basic properties that we will use later are: (1) If $A$ is the finite union of semi-algebraic sets $A_1, \ldots, A_p$, then $\dim A = \max \{\dim A_i\}$. (2) If $A$ and $B$ are semi-algebraic sets, then $\dim(A \times B) = \dim A + \dim B$. (3) If $h : A \to B$ is a semi-algebraic mapping, then $\dim h(A) \leq \dim A$. (4) Moreover, if $A$ is a semi-algebraic set as well as a smooth submanifold in $\mathbb{R}^q$, its dimension as a semi-algebraic set coincides with its dimension as a smooth manifold. As a result, if $A \subseteq \mathbb{R}^q$ is a semi-algebraic set with $\dim A < q$, then it has measure 0 in $\mathbb{R}^q$ by the usual Morse-Sard-Federer Theorem (cf. [79, p. 72]).

4.2 A sufficient condition, from an algebraic point of view, for $\overline{M} = G$

Recall that $\Omega_*$ is defined by (2.16) and $U(\theta) (\theta \in \mathbb{R}^m)$ by (2.5). For $\theta \neq 0$, let $M(\theta)$ denote the multiplicity of the maximal eigenvalue of $U(\theta)^2$, and

$$M := \min_{\theta \neq 0} M(\theta). \quad (4.1)$$

We have the following:

Theorem 8. If $M \geq m$, then $\overline{M} = G$.

Proof. First observe that the open set $\Omega_*$ is a semi-algebraic set in $\mathbb{R}^m$ by the fact that

$$\Omega_*^c = \pi_2 \left( \{(x, \tau); (U(\tau)^2 x, x) \geq \pi^2\} \cap \{(x, \tau); |x|^2 = 1\} \right),$$

$$\overline{M} := \min_{\theta \neq 0} M(\theta). \quad (4.1)$$
where \( \pi_2 \) denotes the projection from \( \mathbb{R}^q \times \mathbb{R}^m \) to the second entry \( \mathbb{R}^m \). Then, \( \partial \Omega \) is a semi-algebraic set and \( \dim(\partial \Omega) = \dim(\overline{\Omega} \setminus \Omega) \leq m - 1 \) by [31, Propositon 2.8.13].

Set
\[
\Sigma := \{(U(\theta)^2 - \pi^2)y; \ \theta \in \partial \Omega, \ y \in \mathbb{R}^q\} \subseteq \mathbb{R}^q,
\]
that is, the set of points \( x \) such that there exists a \( \theta \in \partial \Omega \), satisfying that the orthogonal projection of \( x \) on \( \pi^2 \)-eigenspace of \( U(\theta)^2 \) is zero. From [88, Proposition 2.2], it remains to prove that \( \Sigma \) has measure 0.

Now, consider the map defined by
\[
\Psi : \mathbb{R}^m \times \mathbb{R}^q \longrightarrow \mathbb{R}^q \quad (\theta, y) \longmapsto \Psi(\theta, y) := (U(\theta)^2 - \pi^2)y.
\]
Notice that it is a semi-algebraic mapping. Then \( \Sigma = \Psi(\partial \Omega \times \mathbb{R}^q) \) is a semi-algebraic set. It suffices to prove that \( \dim \Sigma \leq q - 1 \).

For \( r \in \mathbb{N} \) satisfying \( r \leq q \), set
\[
\Pi_{q, r} := \{L; \text{there exist } 1 \leq j_1 < \ldots < j_r \leq q \text{ such that } L = \text{span}\{e_{j_1}, \ldots, e_{j_r}\}\},
\]
where \( \{e_1, \ldots, e_q\} \) denotes the standard orthonormal basis of \( \mathbb{R}^q \) and we adopt the convention that \( \text{span}\{\emptyset\} = \{0\} \). Then \( \Pi_{q, r} \) is a finite set of \( C_r^q \) elements. Remark that for a \( q \times q \) real matrix \( S \) with \( \text{rank}(S) \leq r \), there exists an \( L \in \Pi_{q, r} \) such that \( S(\mathbb{R}^q) = S(L) \).

Under our assumption, we have \( \text{rank}(U(\theta)^2 - \pi^2) \leq q - m \) for any \( \theta \in \partial \Omega \). Hence, we get
\[
\Sigma = \Psi(\partial \Omega \times \mathbb{R}^q) = \bigcup_{L \in \Pi_{q, q - m}} \Psi(\partial \Omega \times L).
\]

It is clear that for each \( L \in \Pi_{q, q - m} \), we have
\[
\dim(\partial \Omega \times L) = \dim(\partial \Omega) + \dim L \leq (m - 1) + (q - m) = q - 1,
\]
and as a result,
\[
\dim \Sigma = \max_{L \in \Pi_{q, q - m}} \dim(\partial \Omega \times L) \leq \max_{L \in \Pi_{q, q - m}} \dim(\partial \Omega \times L) \leq q - 1,
\]
which ends the proof of this theorem.

By the fact that \( \widetilde{U}(\theta) \) is skew-symmetric for any \( \theta \), we have \( M \geq 2 \). Then

**Corollary 13.** If \( G = G(q, 1, U) \) or \( G(q, 2, U) \), namely \( G \) is a step-two group of Corank 1 or 2, then it is a GM-group.

**Remark 10.** (1) Theorem 8 is sharp in the sense that \( G \) may not be of type GM if \( M < m \). The simplest example is the free Carnot group of step two and 3 generators studied in [88]. Notice that in such case, we have \( M = 2 < 3 = m \). Other interesting examples can be found in Subsection 4.3 below.

(2) Remark also that \( M \geq m \) is in general not necessary for \( \overline{M} = G \). See for example the star graphs studied in [88].

(3) We do not know whether there exists a simple algebraic characterization for GM-groups similar to that of Métivier groups.
4.3 Not all Métivier groups are of type GM

In the sequel, we will illustrate that when \( m = 3 \), the condition that \( q \) (instead of \( M \)) is sufficiently larger than \( m \) cannot guarantee \( M = G \), even in the case that \( G \) is a Métivier group.

**Proposition 5.** For any \( N \in \mathbb{N} \), there exists a Métivier group \( G = G(4N + 4, 3, U_N) \) which is not a GM-group.

**Proof.** Let \( \mathbb{H}(4n, 3) = \mathbb{G}(4n, 3, U_{\mathbb{H}(4n, 3)}) \) \( (n \in \mathbb{N}^*) \) denote the \((4n+3)\)-dimensional H-type group, that is \( U_{\mathbb{H}(4n, 3)} \) satisfies the following condition (cf. (2.5) for the related definition):

\[
U_{\mathbb{H}(4n, 3)}(\lambda) U_{\mathbb{H}(4n, 3)}(\lambda') U_{\mathbb{H}(4n, 3)}(\lambda) = 2 \lambda \cdot \lambda' \mathbb{I}_{4n}, \quad \forall \lambda, \lambda' \in \mathbb{R}^3.
\]

We remark that the \((4n+3)\)-dimensional quaternionic Heisenberg group provides a good example for it.

For a row vector \( \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 \), set

\[
\mathfrak{X}(\tau) := i \begin{pmatrix}
0 & 2^{-1} \tau_1 & 2^{-1} \tau_2 & 2^{-1} \tau_3 \\
-2^{-1} \tau_1 & 0 & -\tau_3 & \tau_2 \\
-2^{-1} \tau_2 & \tau_3 & 0 & -\tau_1 \\
-2^{-1} \tau_3 & -\tau_2 & \tau_1 & 0
\end{pmatrix},
\]

\( U_0(\tau) := \mathfrak{X}(\tau) \) and for \( N \in \mathbb{N}^* \),

\[
U_N(\tau) := \begin{pmatrix}
U_{\mathbb{H}(4n, 3)}(\frac{\tau}{3}) & 0 \\
0 & (4N)_{4 \times 4} & \mathfrak{X}(\tau)
\end{pmatrix}.
\]

Observe that

\[
U_N(\tau)^2 = \begin{pmatrix}
\frac{1}{4} |\tau|^2 \mathbb{I}_{4N+1} & \frac{1}{4} |\tau|^2 \mathbb{I}_3 - \frac{3}{4} \tau^T \tau \\
\frac{1}{4} |\tau|^2 \mathbb{I}_3 - \frac{3}{4} \tau^T \tau & \mathbb{I}_{4N+4}
\end{pmatrix},
\]

(4.2) whose eigenvalues are \( |\tau|^2 \) with the multiplicity 2 and \( \frac{|\tau|^2}{4} \) with the multiplicity \( 4N + 2 \). Hence, we get a Métivier group, saying that \( G = G(4N + 4, 3, U_N) \).

We only consider the case \( N \in \mathbb{N}^* \) here and the proof is similar when \( N = 0 \).

From now on, we fix \( N \in \mathbb{N}^* \), and write

\[
x = (\bar{x}^*, \bar{x}_1, \bar{x}_2) \in \mathbb{R}^{4N} \times \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^{4N+4}.
\]

In our situation, by (4.2), a direct calculation shows that the initial reference set, defined by (2.16), is given by

\[
\Omega_* = \{ \tau; |\tau| < \pi \},
\]

and the reference function (cf. (2.17)) is

\[
\phi((x,t); \tau) = 4 t \cdot \tau + \left( \frac{|\tau|}{2} \cot \frac{|\tau|}{2} \right) \left( |\bar{x}^*|^2 + |\bar{x}_1|^2 + |\bar{x}_2| + |\bar{x}_* \cdot \hat{\tau}|^2 \right)
\]

\[
+ (|\tau| \cot |\tau|) |\bar{x}_* - (\bar{x}_* \cdot \hat{\tau}) \hat{\tau}|^2.
\]
Here we have used the convention (3.12).

Observe that
\[ \phi((\bar{x}^*, \bar{x}_1, \bar{x}_e), t; \tau) = \phi((\bar{x}^*, \bar{x}_1, \bar{x}_e), -t; -\tau), \]  
\[ \phi((\bar{x}^*, \bar{x}_1, O \bar{x}_e), O t; O \tau) = \phi((\bar{x}^*, \bar{x}_1, \bar{x}_e), t; \tau), \quad \forall O \in O_3, \]  
(4.3) (symM1)  
(4.4) (symM2)

where \( O_3 \) denotes the \( 3 \times 3 \) orthogonal group. Without loss of generality, we may assume in the sequel that
\[ \bar{x}_e = |\bar{x}_e|e_1 = |\bar{x}_e|(1, 0, 0), \quad t = (t_1, t_2, 0) \text{ with } t_1, t_2 \geq 0. \]
\[ \text{(4.5) (nIc)} \]

Now, by recalling that (see (2.25))
\[ \psi(s) := \frac{1 - s \cot s}{s^2}, \quad \mu(s) := -(s \cot s)' = \frac{2s - \sin(2s)}{2s^3}, \]
we can write
\[ \phi((x, t); \tau) = 4t \cdot \tau + |\bar{x}_e|^2 + \left(\frac{|\tau|}{2} \cot \frac{|\tau|}{2}\right) (|\bar{x}^*|^2 + \bar{x}_1^2) 
- \psi\left(\frac{|\tau|}{2}\right) \frac{\tau_1^2}{4} |\bar{x}_e|^2 - \psi(|\tau|)(\tau_2^2 + \tau_3^2)|\bar{x}_e|^2. \]

Suppose that \( \theta \in \Omega_* \) is a critical point of \( \phi((x, t); \cdot) \) for some \((x, t)\) satisfying (4.5). Then we have
\[ 4t = \mu\left(\frac{|\theta|}{2}\right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2}{2|\theta|} \theta + \psi'\left(\frac{|\theta|}{2}\right) \frac{\theta_1^2}{4} |\bar{x}^*|^2 (\theta_1 \theta) + \psi\left(\frac{|\theta|}{2}\right) \frac{|\bar{x}_e|^2}{2} (\theta_1 e_1) 
+ \psi'(|\theta|) (\theta_2^2 + \theta_3^2) \frac{|\bar{x}_e|^2}{|\theta|} \theta + 2 \psi(|\theta|)|\bar{x}_e|^2 (\theta_2 \theta_2 + \theta_3 \theta_3). \]

We further assume that \( |\bar{x}^*|^2 + \bar{x}_1^2 \neq 0 \) and \( \bar{x}_e \neq 0 \). Using [88, Lemmas 3.1 and 3.2], the fact that \( t_3 = 0 \) implies \( \theta_3 = 0 \). Thus
\[ 4 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \mu\left(\frac{|\theta|}{2}\right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2}{2|\theta|} (\theta_1 \theta_2) + \psi'\left(\frac{|\theta|}{2}\right) \frac{\theta_1^2}{4} |\bar{x}^*|^2 (\theta_1 \theta_2) + \psi\left(\frac{|\theta|}{2}\right) \frac{|\bar{x}_e|^2}{2} (\theta_1 e_1) 
+ \psi'(|\theta|) \theta_2^2 \frac{|\bar{x}_e|^2}{|\theta|} (\theta_1 \theta_2) + 2 \psi(|\theta|)|\bar{x}_e|^2 (0) \left(\theta_2 \right) \]
\[ := \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_e); (\theta_1, \theta_2)). \]

Next, following the proof of [88, Proposition 10.3], we can establish the following lemma. For completeness, we include its proof in “Appendix A”.

Lemma 7. Suppose that \( |\bar{x}^*|^2 + \bar{x}_1^2 \neq 0 \) and \( \bar{x}_e \neq 0 \). Let
\[ B_{\mathbb{R}^2}(0, \pi) := \left\{ (v_1, v_2) \in \mathbb{R}^2; \sqrt{v_1^2 + v_2^2} < \pi \right\}, \]
\[ \mathbb{R}^2_\tau(\bar{x}^*, \bar{x}_1, \bar{x}_e) := \left\{ (u_1, u_2) \in \mathbb{R}^2; |u_1| < \frac{\pi}{4} \left( \frac{u_2^2}{|\bar{x}_1|^2} + |\bar{x}^*|^2 + \bar{x}_1^2 + |\bar{x}_e|^2 \right) \right\}. \]

Then \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_e); \cdot) \) is a \( C^\infty \)-diffeomorphism from \( B_{\mathbb{R}^2}(0, \pi) \) onto \( \mathbb{R}^2_\tau(\bar{x}^*, \bar{x}_1, \bar{x}_e) \).
Combining this with (4.3) and (4.4), we get that \( \mathbb{G} \setminus \mathbb{M} \) contains the subset

\[
\left\{(x, t); \; \tilde{x}_* \neq 0, \; |\tilde{x}|^2 + \tilde{x}_1^2 \neq 0, \; \frac{t \cdot \tilde{x}_*}{\pi |\tilde{x}_*|} > \frac{1}{\tilde{x}_*^2} \left| t - \frac{t \cdot \tilde{x}_*}{|\tilde{x}_*|} \right|^2 \frac{|\tilde{x}|^2 + \tilde{x}_1^2 + |\tilde{x}_*|^2}{16} \right\},
\]

which implies immediately \( \mathbb{M} \subseteq \mathbb{M} \subsetneq \mathbb{G} \).

\[\square\]

5 Sub-Riemannian geometry on step-two K-type groups

Let \( p_0, p_1 \in \{2, 3, 4, \ldots \} \) with \( p_0 \geq p_1 \). Consider a \( p_1 \times p_0 \) full-rank real matrix

\[
B = \begin{pmatrix}
\tilde{b}_1^T \\
\vdots \\
\tilde{b}_{p_1}^T
\end{pmatrix}
\]

with column vectors \( \tilde{b}_j \in \mathbb{R}^{p_0} \setminus \{0\}, \; 1 \leq j \leq p_1 \).

A step-two Kolmogorov type group (or K-type group, in short) of type \( B, \mathbb{G}^{(K)}_B \), is defined by \( \mathbb{G}(1 + p_0, p_1, U^{(K)}_B) \) with (see [32, § 4])

\[U^{(K),(j)}_B = \begin{pmatrix} 0 & \tilde{b}_j^T \\ -\tilde{b}_j & 0 \end{pmatrix}_{p_0 \times p_0}, \; 1 \leq j \leq p_1.
\]

Notice that we have for \( \tau \in \mathbb{R}^{p_1} \)

\[U^{(K)}_B(\tau) = i \begin{pmatrix} 0 & \tau^T B \\ -B^T \tau & 0 \end{pmatrix}_{p_0 \times p_0}, \; \quad U^{(K)}_B(\tau)^2 = \begin{pmatrix} |B^T \tau|^2 & 0_{1 \times p_0} \\ 0_{p_0 \times 1} & B^T \tau \tau^T B \end{pmatrix}.
\]

Hence the initial reference set, defined by (2.16), is

\[\Omega_* : = \Omega^{(K)}_B = \{ \tau \in \mathbb{R}^{p_1}; |B^T \tau| < \pi \}.
\]

In the rest of this section, we write

\[x = (x_1, x_*) \in \mathbb{R} \times \mathbb{R}^{p_0} = \mathbb{R}^{1+p_0}.
\]

A simple calculation shows that the reference function in this setting is given by

\[
\phi((x, t); \tau) : = \phi^{(K)}_B((x, t); \tau) = |x|^2 + 4 t \cdot \tau - \left[ |x_1^2 f(|B^T \tau|)| + |\tau \cdot B x_*|^2 \psi(|B^T \tau|) \right].
\]

Remark that the case \( B = I_n \) corresponds to the star graph \( K_{1,n} \), on which the squared sub-Riemannian distance and the cut locus have been characterized by [88, Theorem 10.1]. We will use this known result to deduce the counterpart for general \( \mathbb{G}^{(K)}_B \).
Notice that $BB^T$ is positive definite, then we can define an isomorphism:

$$T_B : \mathbb{R}^{p_1} \rightarrow \mathbb{R}^{p_1}, \quad T_B(\tau) := (BB^T)^{\frac{1}{2}} \tau. \quad (5.2)$$

Let $T_B^{-1}$ denote its inverse. Set

$$T := T(t) = T_B^{-1}(t), \quad t \in \mathbb{R}^{p_1}; \quad X_* := X_*(x_*) = T_B^{-1}(Bx_*), \quad x_* \in \mathbb{R}^{p_0}, \quad (5.3)$$

and

$$X_{**} := X_{**}(x_*) = x_* - B^T(T_B^{-1}(X_*)) = x_* - B^T(BB^T)^{-1}Bx_*, \quad x_* \in \mathbb{R}^{p_0}. \quad (5.4)$$

Observe that $B^T T_B^{-1}$ is an isometry on $\mathbb{R}^{p_1}$, then we have

$$\tau \cdot Bx_* = (B^T T_B^{-1}\tau, B^T T_B^{-1}Bx_*) = (BB^T T_B^{-1}\tau) \cdot (T_B^{-1}Bx_*) = (T_B\tau) \cdot X_*, \quad (5.5)$$

and similarly $t \cdot \tau = T \cdot (T_B\tau)$. Moreover, by the fact that $B^T (BB^T)^{-1}B$ is a projection on $\mathbb{R}^{p_0}$ and $|T_B(\tau)| = |B^T\tau|$ for any $\tau \in \mathbb{R}^{p_1}$, we find that

$$\phi_B^{(K)}((x_1, x_*), t; \tau) = \phi_{p_1}^{(K)}((x_1, X_*), T; T_B(\tau)) + |X_{**}|^2.$$

Thus, the following result is a direct consequence of [88, Theorem 10.1]:

**Theorem 9.** (1) We have

$$d(g)^2 = \sup_{\tau \in \Omega_B^{(K)}} \phi_B^{(K)}(g; \tau), \quad \forall g \in \Omega_B^{(K)}.$$

(2) The cut locus of $o$, Cut$_o$, is exactly

$$M^c = \left\{((0, x_*), t); |X_* \cdot T| \leq \frac{|X_*|^2}{\sqrt{\pi}} \sqrt{|T - (\hat{X}_* \cdot T) \hat{X}_*|}\right\},$$

where we have used the convention (3.12). And for $((0, x_*), t) \in M^c$, it holds that

$$d((0, x_*), t)^2 = |x_*|^2 + 4\pi \left|T - (\hat{X}_* \cdot T) \hat{X}_*\right|.$$

(3) If $(x, t) = ((x_1, x_*), t) \in M$, then there exists a unique $\theta = \theta(x, t) \in \Omega_B^{(K)}$ such that

$$t = \frac{1}{4} \nabla_{\theta} \left[x_1^2 f(|B^T\theta|) + |\theta \cdot Bx_*|^2 \psi(|B^T\theta|)\right].$$

Furthermore, we have

$$d(x, t)^2 = \phi_B^{(K)}((x, t); \theta) = \left|\frac{U_B^{(K)}(\theta)}{\sin U_B^{(K)}(\theta)} x\right|^2$$

$$= \left(\frac{|B^T\theta|}{\sin |B^T\theta|}\right)^2 x_1^2 + |x_*|^2 + \left(\frac{|B^T\theta|}{\sin |B^T\theta|} - 1\right)|\theta \cdot Bx_*|^2 \frac{|B^T\theta|^2}{|B^T\theta|^2}. \quad (5.5)$$
Remark 11. 1. Using [88, Corollary 10.1] and (iii) of Corollary 9, we find that
\[ \tilde{M}_2 = \{ ((0, x_*), 0); x_* \in \mathbb{R}^{p_0} \} = \text{Abn}_* = \text{Abn}_o, \quad \text{Cut}_o^{CL} = M^c \setminus \tilde{M}_2. \]

2. \( G^{(K)}_o \) is of type GM, then other properties in Theorem 4, Corollaries 7 and 9 are also valid.

To end this section, we describe all

5.1 Shortest geodesic(s) joining \( o \) to any given \( g \neq o \), as well as “bad” normal geodesics

Let us begin with the

5.1.1 Concrete expression of \( \gamma(w, 2\theta; s) \) on K-type groups

Let \( x = (x_1, x_*) \), \( \tilde{x} = (\tilde{x}_1, \tilde{x}_*) \in \mathbb{R} \times \mathbb{R}^{p_0} = \mathbb{R}^{1+p_0} \) and \( \tau \in \mathbb{R}^{p_1} \). A simple calculation gives that
\[ \langle \mathbb{U}_B^{(K)}(x, \tilde{x}) \rangle \cdot \tau = \langle \mathbb{U}_B^{(K)}(\tau) x, \tilde{x} \rangle = (\tilde{x}_1 B x_* - x_1 B \tilde{x}_*) \cdot \tau, \quad \forall \tau \in \mathbb{R}^{p_1}, \] which implies
\[ \langle \mathbb{U}_B^{(K)}(x, \tilde{x}) \rangle = x_1 B x_* - x_1 B \tilde{x}_* := x \star \tilde{x}. \]

Next we consider \( \exp\{\tilde{U}_B^{(K)}(\tau)\} \zeta(0) \) with \( \zeta(0) \in \mathbb{R}^{1+p_0} \) and \( \tau \in \mathbb{R}^{p_1} \). It can be treated via the Spectral Theorem or directly by the fact that
\[ \exp\{\tilde{U}_B^{(K)}(\tau)\} = \exp\{-iU_B^{(K)}(\tau)\} = \cos U_B^{(K)}(\tau) + \frac{\sin U_B^{(K)}(\tau)}{U_B^{(K)}(\tau)}, \] which is, by (5.1), equal to
\[ \left( \begin{array}{c} \cos (|B^T\tau|) \\ \mathbb{I}_{p_0} + \frac{\cos (|B^T\tau|) - 1}{|B^T\tau|^2} B^T\tau T B \end{array} \right) + \frac{\sin (|B^T\tau|)}{|B^T\tau|} \left( \begin{array}{cc} 0 & \tau^T B \\ -B^T \tau & \mathbb{O}_{p_0 \times p_0} \end{array} \right). \]

Now, we describe \( \gamma(w, 2\theta; s) := (x(s), t(s)) \) for given \( w := (w_1, w_*) \in \mathbb{R}^{1+p_0} \) and \( \theta \in \mathbb{R}^{p_1} \). Let us introduce the column vectors
\[ v_1 := \left( \begin{array}{c} \theta \cdot B w_* \\ -w_1 B^T \theta \end{array} \right), \quad v_2 := \left( \begin{array}{c} w_1 \\ \frac{\theta \cdot B w_*}{|B^T\theta|^2} B^T \theta \end{array} \right), \quad v_3 := \left( \begin{array}{c} 0 \\ w_* - \frac{\theta \cdot B w_*}{|B^T\theta|^2} B^T \theta \end{array} \right), \]
\[ w_1 := v_1 \ast v_2 = -\left( w_1^2 + \frac{(\theta \cdot B w_*)^2}{|B^T\theta|^2} \right) B B^T \theta, \]
\[ w_2 := v_1 \ast v_3 = -(\theta \cdot B w_*) B w_* + \frac{(\theta \cdot B w_*)^2}{|B^T\theta|^2} - B B^T \theta, \]
\[ w_3 := v_2 \ast v_3 = -w_1 B \left( w_* - \frac{\theta \cdot B w_*}{|B^T\theta|^2} B^T \theta \right). \]
It follows from (2.6) that
\[
\zeta(s) = \frac{\sin(2s |B^T\theta|)}{|B^T\theta|}v_1 + \cos(2s |B^T\theta|)v_2 + v_3, \tag{5.13}
\]
and
\[
x(s) = \frac{1 - \cos(2s |B^T\theta|)}{2|B^T\theta|^2}v_1 + \sin(2s |B^T\theta|)\frac{1}{2|B^T\theta|}v_2 + s v_3. \tag{5.14}
\]

The calculation of \(t(s)\) is cumbersome. However, (5.7)-(5.12), (2.6) together with (5.13) and (5.14) imply that
\[
\dot{t}(s) = \frac{1}{2} x(s) \ast \zeta(s)
= \frac{\cos(2s |B^T\theta|) - 1}{4|B^T\theta|^2}w_1 + \frac{1}{2} \left( \frac{1 - \cos(2s |B^T\theta|)}{2|B^T\theta|^2} - s \frac{\sin(2s |B^T\theta|)}{|B^T\theta|} \right)w_2
+ \frac{1}{2} \left( \frac{\sin(2s |B^T\theta|)}{2|B^T\theta|} - s \cos(2s |B^T\theta|) \right)w_3. \tag{5.15}
\]

Noticing that
\[
\int_0^s r \sin(2r |B^T\theta|) \, dr = -\frac{2s |B^T\theta| \cos(2s |B^T\theta|) + \sin(2s |B^T\theta|)}{4|B^T\theta|^2}, \tag{5.16}
\]
\[
\int_0^s r \cos(2r |B^T\theta|) \, dr = \frac{2s |B^T\theta| \sin(2s |B^T\theta|) - 1 + \cos(2s |B^T\theta|)}{4|B^T\theta|^2}, \tag{5.17}
\]
from (5.15), we get that
\[
t(s) = \frac{\sin(2s |B^T\theta|) - 2s |B^T\theta|}{8|B^T\theta|^3}w_1
+ \frac{s |B^T\theta| + s |B^T\theta| \cos(2s |B^T\theta|) - \sin(2s |B^T\theta|)}{4|B^T\theta|^3}w_2
+ \frac{-s |B^T\theta| \sin(2s |B^T\theta|) + 1 - \cos(2s |B^T\theta|)}{4|B^T\theta|^2}w_3. \tag{5.18}
\]

In particular, we can get some information about the set of the endpoints of nontrivial “bad” normal geodesics, as well as

\subsection{The nontrivial “bad” normal geodesics}

We suppose in this subsection that
\[w = (w_1, w_\ast) \neq 0, \quad \text{and} \quad |B^T\theta| = k\pi \quad \text{with} \quad k \in \mathbb{N}^*.
\]
Recall that the isomorphism \(T_B\) is defined by (5.2) and \(|T_B(\tau)| = |B^T\tau|\) for any \(\tau \in \mathbb{R}^n\).
Set
\[
W_\ast := T_B^{-1}(Bw_\ast), \quad \tilde{\eta} := \frac{T_B(\theta)}{|T_B(\theta)|} = \frac{T_B(\theta)}{|B^T\theta|} = k \pi.
\]
Observe that from (5.5), we have that
\[
\frac{\theta \cdot Bw_*}{|B^T\theta|} = \frac{T_B(\theta) \cdot W_*}{|B^T\theta|} = \hat{\eta} \cdot W_* := \tilde{w}_1. \tag{5.20}
\]

Let
\[
(x, t) = ((x_1, x_*), t) := \exp(w, 2 \theta) = \gamma_{(w, 2\theta)}(1), \quad (x(s), t(s)) := \gamma_{(w, 2\theta)}(s).
\]

Taking \( s = 1 \) and \( |B^T\theta| = k \pi \) in (5.14), we deduce that
\[
x = \left(0, w_* - \frac{\theta \cdot Bw_*}{|B^T\theta|^2} B^T\theta\right), \tag{5.21}
\]
namely \( x_1 = 0 \) and
\[
x_* = w_* - (\hat{\eta} \cdot W_*) \frac{B^T\theta}{|B^T\theta|}, \tag{5.22}
\]
where we have used (5.20). Similarly, (5.18) implies that
\[
t = -\frac{1}{4 |B^T\theta|^2} w_1 + \frac{1}{2 |B^T\theta|^2} w_2 \tag{5.23}
\]
\[
= -\frac{\theta \cdot Bw_*}{2 |B^T\theta|^2} Bw_* + \frac{|B^T\theta|^2 w_1^2 + 3 (\theta \cdot Bw_*)^2}{4 |B^T\theta|^4} BB^T\theta.
\]
In other words, by (5.20), we have
\[
t = -\hat{\eta} \cdot W_* \frac{Bw_*}{2 |B^T\theta|} + \frac{w_1^2 + 3 (\hat{\eta} \cdot W_*)^2}{4 |B^T\theta|^2} BB^T\theta. \tag{5.24}
\]

Recall that \( T_B := (BB^T)^{\frac{1}{2}} \), \( X_* := T_B^{-1}(Bx_*) \) and \( T := T_B^{-1}(t) \). Applying \( T_B^{-1}B \) to both sides of (5.22), it follows from (5.19) and (5.20) that
\[
X_* = W_* - (\hat{\eta} \cdot W_*) \hat{\eta} = W_* - \tilde{w}_1 \hat{\eta}. \tag{5.25}
\]

And similarly, applying \( T_B^{-1} \) to both sides of (5.24), we obtain
\[
T = -\frac{\tilde{w}_1}{2 k \pi} W_* + \frac{w_1^2 + 3 \tilde{w}_1^2}{4 k \pi \tilde{\eta}} \hat{\eta}. \tag{5.26}
\]

We split it into cases.

**Case 1.** \( t = 0 \) so \( T = 0 \). In such case, taking inner product with \( \hat{\eta} \) on both sides of (5.26) and using (5.20), we yield \( w_1 = \tilde{w}_1 = 0 \). So \( \theta \cdot Bw_* = 0 \) and \( w_* = x_* \) because of (5.21). Hence, the vectors defined by (5.9)-(5.12) in this situation are \( w_1 = w_2 = w_3 = 0 \), \( v_1 = v_2 = 0 \) and \( v_3 = x \) respectively. In conclusion, by (5.14) and (5.18), a simple calculation shows that the “bad” normal geodesic from \( o \) to \(((0, x_*), 0)\) is the straight segment \( \gamma_{((0, x_*), 0)}(s) \).

From now on, we further assume that:
Case 2. \( t \neq 0 \) so \( T \neq 0 \). In such case, we have \( w_1^2 + \tilde{w}_1^2 > 0 \). Taking inner product with \( \hat{\eta} \) on both sides of (5.25), by (5.20), we get \( X_* \cdot \hat{\eta} = 0 \). Inserting (5.25) into (5.26), we obtain that

\[
T = -\frac{\tilde{w}_1}{2k\pi} X_* + \frac{w_1^2 + \tilde{w}_1^2}{4k\pi} \hat{\eta}. \tag{5.27}
\]

And we will consider the cases \( X_* \neq 0 \) and \( X_* = 0 \).

(I) Assume that \( X_* \neq 0 \). In such case, \( X_* \) and the unit vector \( \hat{\eta} = \frac{T \theta}{k\pi} \) are orthogonal. It is easy to solve out \((w_1, \tilde{w}_1)\) as well as \( \theta \) from (5.27). Using (5.19), a direct calculation shows that

\[
\tilde{w}_1 = -2k\pi \frac{T \cdot \hat{X}_*}{|X_*|}, \quad \theta = k\pi T^{-1}_B \left( \frac{T - (T \cdot \hat{X}_*) \hat{X}_*}{|T - (T \cdot \hat{X}_*) \hat{X}_*|} \right),
\]

\[
w_1 = \pm \sqrt{4k\pi \left| T - (T \cdot \hat{X}_*) \hat{X}_* \right| - 4k^2 \pi^2 \left( \frac{T \cdot \hat{X}_*}{|X_*|} \right)^2} \tag{5.28}
\]

(II) Assume that \( X_* = 0 \), that is \( Bx_* = 0 \). In such case, by (5.19), (5.27) implies that

\[
\theta = k\pi T^{-1}_B \left( \hat{T} \right), \quad (w_1, \tilde{w}_1) = \sqrt{4k\pi |T|} (\cos \sigma, \sin \sigma) (\sigma \in \mathbb{R}). \tag{5.29}
\]

In conclusion, we have always

\[
X_* \cdot \hat{\eta} = 0. \tag{5.30}
\]

And \((x, t) = ((0, x_*), t) (t \neq 0)\) is the endpoint of some “bad” normal geodesic \( \gamma((w_1, w_*), 2\theta) \) satisfying \(|B^T \theta| = k\pi \) \((k \in \mathbb{N}^*)\) if and only if

\[
|X_* \cdot T| \leq \frac{|X_*|^2}{\sqrt{k\pi}} \sqrt{|T - (\hat{X}_* \cdot T) \hat{X}_*|}. \tag{5.31}
\]

In such case, taking \((w_1, \tilde{w}_1, \theta)\) as in (5.28) for \( X_* \neq 0 \), or in (5.29) for \( X_* = 0 \), it follows from (5.22) and (5.20) that

\[
w_* = x_* - 2k\pi \frac{T \cdot \hat{X}_*}{|X_*|} B^T \theta = x_* + \frac{\tilde{w}_1}{k\pi} B^T \theta. \tag{5.32}
\]

Now recall that (see (5.21)) \( x_* = w_* - \frac{\theta \cdot Bw_*}{|B^T \theta|^2} B^T \theta \). Substituting this as well as (5.20) and \(|B^T \theta| = k\pi \) in (5.9)-(5.12), we get that

\[
v_3 = \begin{pmatrix} 0 \\ x_* \end{pmatrix}, \quad v_2 = \begin{pmatrix} w_1 \\ \frac{\tilde{w}_1}{k\pi} B^T \theta \end{pmatrix}, \quad v_1 = k\pi \left( \frac{\tilde{w}_1}{k\pi} B^T \theta \right),
\]

\[
w_2 = -\left( \theta \cdot Bw_* \right) Bx_* = -k\pi \tilde{w}_1 Bx_*, \quad w_3 = -w_1 Bx_*,
\]

34
and by (5.23),
\[ w_1 = 2w_2 - 4|B^T\theta|^2 t = -2k\pi (\tilde{w}_1Bx_s + 2k\pi t). \]

Substituting them into (5.14) and (5.18), and replacing $|B^T\theta|$ by $k\pi$, we obtain finally the expression of $\gamma((w_1,w_*),2\theta)(s)$ as follows:

\[
\begin{align*}
x(s) &= s \begin{pmatrix} 0 \\ x_s \end{pmatrix} + \frac{\sin(2sk\pi)}{2k\pi} \left( \begin{pmatrix} w_1 \\ \tilde{w}_1 \end{pmatrix} B^T\theta \right) + \frac{1-\cos(2sk\pi)}{2k\pi} \left( \begin{pmatrix} \tilde{w}_1 \\ -w_1 \end{pmatrix} B^T\theta \right) \\
t(s) &= s \frac{\sin(2sk\pi)}{2k\pi} + \frac{1-\cos(2sk\pi)}{4k^2\pi^2} \tilde{w}_1Bx_s \\
&+ \frac{sk\pi \sin(2sk\pi)-1+\cos(2sk\pi)}{4k^2\pi^2} \tilde{w}_1Bx_s.
\end{align*}
\]

(5.33) \hspace{1cm} \text{bngE}

Moreover, by (5.22) and (5.20), we get that
\[
|w_*|^2 = \left| x_* + \tilde{w}_1 \frac{B^T\theta}{|B^T\theta|} \right|^2 = |x_*|^2 + \tilde{w}_1^2,
\]
since (5.5) says that
\[
x_* \cdot B^T\theta \frac{B^T\theta}{|B^T\theta|} = X_* \cdot \frac{T_B\theta}{|B^T\theta|} = X_* \cdot \tilde{\eta} = 0,
\]
where we have used (5.30) in the last equality. In conclusion,
\[
\ell^2(\gamma((w_1,w_*),2\theta)) = w_1^2 + |w_*|^2 = |x_*|^2 + \tilde{w}_1^2 \\
= |x_*|^2 + 4k\pi \left| T - (T \cdot \tilde{X}_s) \tilde{X}_s \right|.
\]

(5.34) \hspace{1cm} \text{Kg1}

5.1.3 Shortest geodesic(s), as well as normal geodesics from $o$ to any given $g \neq o$

Recall that the “good” normal geodesics from $o$ to any given $g \neq o$ are characterized by [88, Theorem 2.4] in the general setting of step-two groups. Also in our special setting of $K$-type groups, the “bad” normal geodesics from $o$ to any given $g \in \mathcal{W} \setminus \{o\}$ are characterized in the last subsection. Hence, we only study in the sequel the shortest geodesic(s) from $o$ to any given $g_0 \neq o$.

Consider the cases $g_0 \in \mathcal{M}$, $g_0 \in \text{Abn}_o^* \setminus \{o\}$ and $g_0 \in \text{Cut}_o \setminus \text{Abn}_o^* = \text{Cut}_o^{\text{CL}}$.

1. If $g_0 \in \mathcal{M}$, there exists a unique shortest geodesic steering $o$ to $g_0$, and its equation is well-known by Theorem 2 together with (5.14) and (5.18).

2. For $g_0 \in \text{Abn}_o^* \setminus \{o\}$, the unique shortest geodesic is a straight segment and it is abnormal.

3. From now on, assume that $g_0 \in \text{Cut}_o \setminus \text{Abn}_o^*$. Using Theorem 5, any shortest geodesic joining $o$ to $g_0$ is given by $\gamma(w,2\theta)$, where
\[
(w,2\theta) := ((w_1,w_*),2\theta) \in \mathbb{R}^{1+p_0} \times \mathbb{R}^{p_1} \text{ with } |w| = d(g_0), \]
\[ |B^T\theta| = \pi, \text{ and } \exp(w,2\theta) = g_0. \]

More precisely, by the results known in Subsection 5.1.2, we yield:
Proposition 6. Let $g_0 \in \text{Cut}_o \setminus \text{Abn}_o^* = \text{Cut}^{CL}_o$, namely $g_0 = ((0, x_*), t)$ with $t \neq 0$ and

$$|X_* \cdot T| \leq \frac{|X_*|^2}{\sqrt{\pi}} \sqrt{|T - (\hat{X}_* \cdot T) \hat{X}_*|},$$

where $T$ and $X_*$ are defined by (5.3). Then any shortest geodesic from $o$ to $g_0$,

$$\gamma_{((w_1, w_*), 2\theta)}(s) = (x(s), t(s)),$$

can be written as in (5.33) with $k = 1$, $w_*$ defined by (5.32), and $(w_1, \bar{w}_1, \theta)$ by (5.28) for $X_* \neq 0$, or by (5.29) for $X_* = 0$.

Remark 12. If we suppose further that $X_* \neq 0$ and $|X_* \cdot T| < \frac{|X_*|^2}{\sqrt{\pi}} \sqrt{|T - (\hat{X}_* \cdot T) \hat{X}_*|}$, (5.28) implies that there are exactly two distinct shortest geodesics joining $o$ to $g_0$.

To end this part, we determine on K-type groups

5.1.4 Optimal synthesis

In the special case of K-type groups, combining (ii) of Corollary 9 with Proposition 1 and the result in Case 1 of Subsection 5.1.2, we obtain the following:

Corollary 14. Let $\exp\{s (w, 2 \theta)\}$ be an arclength parametrized geodesic. Then its cut time $h_{\text{cut}}$ equals $+\infty$ when $w = (0, w_*)$ with $\theta \cdot B w_* = 0$, and $\pi/\|U(\theta)\|$ otherwise.

6 Sub-Riemannian geometry on step-two groups associated to quadratic CR manifolds

Let $m, n \in \mathbb{N}^*$ with $n \geq m$. Consider a full-rank $m \times n$ real matrix

$$A = (\tilde{a}_1, \ldots, \tilde{a}_n) \quad \text{with column vectors} \quad \tilde{a}_j := (a_{1,j}, \ldots, a_{m,j})^T \in \mathbb{R}^m.$$ 

A step-two group associated to quadratic CR manifolds of type $A$, $G^{(\text{CR})}_A$, is defined by $G(2n, m, U^{(\text{CR})}_A)$ with (cf. for example [114] for more details)

$$U^{(\text{CR}), (j)}_A = \left( \begin{array}{cc} 0 & a_{j,1} \\ -a_{j,1} & 0 \\ \vdots & \mathllap{\cdot} \\ 0 & a_{j,n} \\ -a_{j,n} & 0 \end{array} \right), \quad 1 \leq j \leq m.$$
For example, if \( m = 1 \) and \( A_1 = (1, \ldots, 1) \), \( G^{(\text{CR})}_{A_1} \) is the Heisenberg group of real dimension \( 2n + 1, \mathbb{H}^{2n+1} \). Moreover, for \( m = n \) and \( A = I_n, G^{(\text{CR})}_{I_n} \) is the direct product of \( n \) copies of Heisenberg group \( \mathbb{H}^3 \), namely \[ G^{(\text{CR})}_{I_n} = \mathbb{H}^3 \times \cdots \times \mathbb{H}^3. \]

Observe that we have for \( \tau \in \mathbb{R}^m \),
\[ U_{A}^{(\text{CR})}(\tau) = i \begin{pmatrix} 0 & \tilde{a}_1 \cdot \tau \\ -\tilde{a}_1 \cdot \tau & 0 \\ \vdots \\ 0 & \tilde{a}_n \cdot \tau \\ -\tilde{a}_n \cdot \tau & 0 \end{pmatrix}. \] (6.1) uc

Then the initial reference set in this situation is given by
\[ \Omega_* := \Omega^{(\text{CR})}_A = \bigcap_{j=1}^{n} \{ \tau \in \mathbb{R}^m; |\tilde{a}_j \cdot \tau| < \pi \}. \] (6.2) rs

We identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) in the usual way, so \( \mathbb{R}^2n \) with \( \mathbb{C}^n \). Write in the sequel
\[ z = (z_1, \ldots, z_n) \in \mathbb{C}^n. \]

A simple calculation shows that the reference function is
\[ \phi((z,t); \tau) := \phi^{(\text{CR})}_A((z,t); \tau) = \sum_{j=1}^{n} |z_j|^2 (\tilde{a}_j \cdot \tau) \cot(\tilde{a}_j \cdot \tau) + 4 t \cdot \tau. \] (6.3) n

Recall that \( f(s) = 1 - s \cot s \) and \( \mu(s) = f'(s) \) (see (2.25)). The gradient and the Hessian matrix of \( \phi((z,t); \cdot) \) at \( \tau \in \Omega_* \) are clearly
\[ \nabla_\tau \phi((z,t); \tau) = -\sum_{j=1}^{n} \mu(\tilde{a}_j \cdot \tau)|z_j|^2 \tilde{a}_j + 4 t, \] (6.4) H

\[ \text{Hess}_\tau \phi((z,t); \tau) = -A \Lambda(z; \tau) A^T = -A \sqrt{\Lambda(z; \tau)} (A \sqrt{\Lambda(z; \tau)})^T, \] (6.5) Hn

respectively, where
\[ \Lambda(z; \tau) = \begin{pmatrix} \mu'(\tilde{a}_1 \cdot \tau) |z_1|^2 \\ \vdots \\ \mu'(\tilde{a}_n \cdot \tau) |z_n|^2 \end{pmatrix} \geq 0, \quad \forall \tau \in \Omega_*, \]

since \( \mu'(s) > 0 \) for all \( -\pi < s < \pi \) (see for example Lemma 8 in Subsection 7.2 below).

Recall that \( M \) is defined in Subsection 2.3. We can characterize the squared sub-Riemannian distance as well as the cut locus in the following theorem.
Theorem 10. (1) We have
\[ d(g)^2 = \sup_{\tau \in \Omega^{(CR)}_A} \phi^{(CR)}_A(g; \tau), \quad \forall g \in \mathbb{G}^{(CR)}_A. \]

(2) The cut locus of \( o \), \( \text{Cut}_o \), is \( M^c \), where
\[ M = \left\{ (z, t); \text{span}\{\vec{a}_j; |z_j| \neq 0\} = \mathbb{R}^m \text{ and } \exists \theta \in \Omega_* \text{ s.t. } t = \frac{1}{4} \sum_{j=1}^n \mu(\vec{a}_j \cdot \theta) |z_j|^2 \vec{a}_j \right\}. \]

(3) If \((z, t) \in M\), then there exists a unique \( \theta = \theta(z, t) \in \Omega^{(CR)}_A \) such that
\[ t = \frac{1}{4} \sum_{j=1}^n \mu(\vec{a}_j \cdot \theta) |z_j|^2 \vec{a}_j. \]

Moreover, we have
\[ d(z, t)^2 = \phi((z, t); \theta) = \sum_{j=1}^n \left( \frac{\vec{a}_j \cdot \theta}{\sin(\vec{a}_j \cdot \theta)} \right)^2 |z_j|^2 \]
\[ = \sum_{j=1}^n (\vec{a}_j \cdot \theta) \cot(\vec{a}_j \cdot \theta) |z_j|^2 + 4 t \cdot \theta. \]

Proof. Let \( r(A) \) denote the rank of a real matrix \( A \). Using the basic property \( r(AA^T) = r(A) \), it deduces from (6.5) that
\[ r(\text{Hess}_r \phi((z, t); \tau)) = r\left( A \sqrt{A(z; \tau)} \right) = \dim \text{span}\{\vec{a}_j; |z_j| \neq 0\}, \]
since \( \mu'(s) > 0 \) for \( s \in (-\pi, \pi) \) (see Lemma 8 in Subsection 7.2 below). Combining this with (6.4), we get immediately the characterization of \( M \). And the third assertion of this theorem follows directly from Theorem 2.

Moreover, it follows from [88, Proposition 2.2] that
\[ M \supseteq \{(z, t); |z_j| \neq 0, \forall 1 \leq j \leq n\}, \]
which is dense in \( \mathbb{G}^{(CR)}_A \). Then \( \mathbb{G}^{(CR)}_A \) is of type GM. By Theorem 4 and Corollary 7, we deduce the first assertion, \( \text{Cut}_o = M^c = \partial M \), as well as other sub-Riemannian geometric properties.

Now we describe

6.1 Shortest geodesic(s) joining \( o \) to any \( g \neq o \)

Let us begin with the
6.1.1 Sub-Riemannian exponential map

In the setting of step-two groups associated to quadratic CR manifolds, the equation of the normal geodesic (cf. (2.6)) as well as the sub-Riemannian exponential map becomes very concise via their special group structure, namely (6.1). More precisely, if

\[ p = (p_1, \ldots, p_n) \in \mathbb{C}^n \cong \mathbb{R}^{2n} \]

and \( \theta \in \mathbb{R}^m \), let

\[
\exp \{ s (p, 2 \theta) \} := (z(p, 2 \theta; s), t(p, 2 \theta; s)) := (z(s), t(s)), \tag{6.9}
\]

and

\[
\exp(p, 2 \theta) := (z(p, 2 \theta), t(p, 2 \theta)) := (z, t). \tag{6.10}
\]

It follows from (2.6) that

\[
\zeta_j(s) = e^{-2 i s (\bar{a}_j \cdot \theta)} p_j, \quad z_j(s) = \frac{1 - e^{-2 i s (\bar{a}_j \cdot \theta)}}{2 i (\bar{a}_j \cdot \theta)} p_j, \quad 1 \leq j \leq n,
\]

and for \( 1 \leq k \leq m \),

\[
t_k(s) = 2^{-1} \sum_{j=1}^{n} a_{k,j} \int_{0}^{s} \left\langle \left( \begin{array}{c} 0 & 1 \\ -1 & 0 \end{array} \right) z_j(r), \zeta_j(r) \right\rangle \, dr
\]

\[
= -2^{-1} \sum_{j=1}^{n} a_{k,j} \int_{0}^{s} \Re(i z_j(r) \cdot \overline{\zeta_j(r)}) \, dr
\]

\[
= \sum_{j=1}^{n} \frac{2 s (\bar{a}_j \cdot \theta) - \sin(2 s (\bar{a}_j \cdot \theta))}{8 (\bar{a}_j \cdot \theta)^2} \| p_j \|^2 a_{k,j}.
\]

In conclusion, the normal geodesic with the initial covector \((p, 2 \theta)\), \( \gamma(p, 2 \theta; s) \), is given by

\[
\begin{cases}
  z_j(s) = \frac{1 - e^{-2 i s (\bar{a}_j \cdot \theta)}}{2 i (\bar{a}_j \cdot \theta)} p_j, & 1 \leq j \leq n, \\
  t(s) = \sum_{j=1}^{n} \frac{2 s (\bar{a}_j \cdot \theta) - \sin(2 s (\bar{a}_j \cdot \theta))}{8 (\bar{a}_j \cdot \theta)^2} \| p_j \|^2 \bar{a}_j.
\end{cases} \tag{6.11}
\]

In particular, we yield the expression of \( \exp \{ (p, 2 \theta) \} \),

\[
\begin{cases}
  z_j(p, 2 \theta) = \frac{1 - e^{-2 i (\bar{a}_j \cdot \theta)}}{2 i (\bar{a}_j \cdot \theta)} p_j, & 1 \leq j \leq n, \\
  t(p, 2 \theta) = \sum_{j=1}^{n} \frac{2 (\bar{a}_j \cdot \theta) - \sin(2 (\bar{a}_j \cdot \theta))}{8 (\bar{a}_j \cdot \theta)^2} \| p_j \|^2 \bar{a}_j.
\end{cases} \tag{6.12}
\]

6.1.2 Shortest geodesic(s) joining \( o \) to any given \( g \neq o \)

We are in a position to determine all shortest geodesics from \( o \) to any given \( o \neq g_0 := (z_0, t_0) \) with \( z_0 := (z_0^{(0)}, \ldots, z_0^{(0)}) \). First, notice that in our framework, the set defined by
(2.18) is
\[ \tilde{M} = \left\{ (z,t); \exists \theta \in \Omega_* \text{ s.t. } t = \frac{1}{4} \sum_{j=1}^{n} \mu(\tilde{a}_j \cdot \theta) |z_j|^2 \tilde{a}_j \right\}. \]

And we consider the following two cases \( g_0 \in \tilde{M} \setminus \{o\} \) and \( g_0 \in \tilde{M}^c \).

Case 1. \( g_0 \in \tilde{M} \setminus \{o\} \), namely \( \phi(g_0;\cdot) \) attains its maximum at some point in \( \Omega_* \), saying \( \theta_0 \). It follows from Theorem 2 that there exists a unique shortest geodesic joining \( o \) to \( g_0 \), that is, \( \exp\{s (p(g_0), 2 \theta_0)\} \) \( (0 \leq s \leq 1) \) with
\[ p(g_0) = \frac{U(\theta_0)}{\sin U(\theta_0)} e^{-\tilde{U}(\theta_0)} z_0, \quad \text{i.e.} \quad p_j(g_0) = \frac{\tilde{a}_j \cdot \theta_0}{\sin (\tilde{a}_j \cdot \theta_0)} e^{i(\tilde{a}_j, \theta_0)} z_j(0), \quad \forall 1 \leq j \leq n. \] (6.13)

Substituting this in (6.11), we yield its concise expression. Moreover, it is strictly normal if \( g_0 \in \tilde{M} \) and also abnormal for \( g_0 \in \tilde{M} \setminus (\tilde{M} \cup \{o\}) \).

Case 2. \( g_0 \in \tilde{M}^c \), that is \( \phi(g_0;\cdot) \) only attains its supremum in \( \Omega_* \) at some \( \theta \in \partial \Omega_* \). In order to characterize all shortest geodesics from \( o \) to \( g_0 \), we set
\[ E(g_0) := \left\{ \theta \in \partial \Omega_*; \ \phi(g_0;\theta) = \sup_{\tau \in \Omega_*} \phi(g_0;\tau) = d(g_0)^2 \right\}. \]

From Theorem 5, it remains to determine all \( (p(g_0), 2 \theta(g_0)) \) such that
\[ \theta(g_0) \in E(g_0), \quad |p(g_0)| = d(g_0) \quad \text{and} \quad \exp\{s (p(g_0), 2 \theta(g_0))\} = g_0. \] (6.14)

Up to rearrangements, we may assume that there exists an \( L \in \mathbb{N}, L < n \) such that
\[ |\tilde{a}_j \cdot \theta(g_0)| < \pi \quad \text{for} \quad 1 \leq j \leq L, \quad \text{and} \quad |\tilde{a}_j \cdot \theta(g_0)| = \pi \quad \text{if} \quad L + 1 \leq j \leq n. \] (6.15)

By (6.12), we obtain that
\[ p_j(g_0) = \frac{2i \tilde{a}_j \cdot \theta(g_0)}{1 - e^{-2i(\tilde{a}_j, \theta(g_0))}} z_j(0), \quad \forall 1 \leq j \leq L, \quad z_j(0) = 0 \quad \text{for} \quad L + 1 \leq j \leq n, \] (6.16)
and (6.14) holds if and only if \( (p_{L+1}(g_0), \ldots, p_n(g_0)) \) is a solution of the following equation:
\[ \sum_{j=L+1}^{n} \frac{1}{\tilde{a}_j \cdot \theta(g_0)} |p_j(g_0)|^2 \tilde{a}_j = 4t_0 - \sum_{j=1}^{L} \mu(\tilde{a}_j \cdot \theta(g_0)) |z_j(0)|^2 \tilde{a}_j. \] (6.17)

To prove this result, it suffices to show that \( |p(g_0)|^2 = d(g_0)^2 \) under our assumptions (6.15)-(6.17), and other claims are clear.

Indeed, from (3.11) and (6.16), we have
\[ d(g_0)^2 = \phi(g_0;\theta(g_0)) = \sum_{j=1}^{L} (\tilde{a}_j \cdot \theta(g_0)) \cot(\tilde{a}_j \cdot \theta(g_0)) |z_j(0)|^2 + 4t_0 \cdot \theta(g_0). \] (6.18)
When (6.17) holds, taking inner product with $\theta(g_0)$ on both sides of (6.17), we obtain that

$$
\sum_{j=L+1}^{n} |p_j(g_0)|^2 = 4t_0 \cdot \theta(g_0) - \sum_{j=1}^{L} (\bar{a}_j \cdot \theta(g_0)) \mu(\bar{a}_j \cdot \theta(g_0)) |z_j(0)|^2.
$$

Summing with $\sum_{j=1}^{L} |p_j(g_0)|^2$ on both sides of last equality, it follows from (6.16) that

$$
|p(g_0)|^2 = 4t_0 \cdot \theta(g_0) + \sum_{j=1}^{L} \left( \left( \frac{\bar{a}_j \cdot \theta(g_0)}{\sin(\bar{a}_j \cdot \theta(g_0))} \right)^2 - (\bar{a}_j \cdot \theta(g_0)) \mu(\bar{a}_j \cdot \theta(g_0)) \right)|z_j(0)|^2.
$$

By (6.18) and the elementary identity

$$
\left( \frac{s}{\sin s} \right)^2 - s \mu(s) = s \cot s,
$$

we get $|p(g_0)|^2 = d(g_0)^2$.

In particular, if $((p_1(g_0), \ldots, p_{L+1}(g_0), \ldots, p_n(g_0)), 2\theta(g_0))$ satisfies the condition (6.14), then so does $((p_1(g_0), \ldots, w_{L+1}p_{L+1}(g_0), \ldots, w_n p_n(g_0)), 2\theta(g_0))$ for any complex numbers $(w_{L+1}, \ldots, w_n)$ satifying

$$
|w_{L+1}| = \ldots = |w_n| = 1.
$$

Thus, there exist infinitely many shortest geodesics from $o$ to $g_0 \in \widetilde{M}^c$ if

$$
4t_0 - \sum_{j=1}^{L} \mu(\bar{a}_j \cdot \theta(g_0)) |z_j(0)|^2 \bar{a}_j \neq 0.
$$

In such case, it follows from [123, Lemma 9] that $g_0$ belongs to the classical cut locus of $o$, $\text{Cut}_{o}^{\text{cl}}$. This provides another explanation for such $g_0 \in \widetilde{M}^c$ belonging to $\text{Cut}_{o}^{\text{cl}}$ besides (iii) of Corollary 9. Finally we remark that the meaning for the case $L = 0$ is clear in the above discussion.

Step-two groups associated to quadratic CR manifolds have very rich sub-Riemannian geometric properties. First, we provide an example of such groups on which (2.27) is no longer valid for some $g_0 \in \text{Abn}_{o}^{*} \setminus \widetilde{M}_2$.

6.2 (2.29) can be false at a point belonging to the shortest abnormal set on GM-groups

**Proposition 7.** There exist $\mathcal{G}_{A}^{\text{(CR)}}$, $g_0 \in \text{Abn}_{o}^{*} \setminus \widetilde{M}_2$ and $c_0 > 0$ such that

$$
d(g_0 + hg)^2 + d(g_0 - hg)^2 - 2d(g_0)^2 \leq 0,
$$

(6.19) semiconc

for all $g = (z, t) \in \mathbb{C}^n \times \mathbb{R}^m$ with $|g|^2 = |z|^2 + |t|^2 = 1$ and $0 < h < c_0$.  

41
Proof. Setting
\[ \tilde{a}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{a}_2 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \tilde{a}_3 := \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
and \( A := (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \), we obtain a step-two group associated to quadratic CR manifolds \( G_{A}^{(CR)} \). In our case, we have

\[ \Omega^* = \Omega_{A}^{(CR)} = \{ \tau; -\pi < \tau_1 \pm \tau_2 < \pi \}. \]

Consider \( g_0 := (0, e_1) \in \mathbb{C}^3 \times \mathbb{R}^2 \) with \( e_1 = (1, 0) \). Using the first result in Theorem 10 and (6.3), we have that \( g_0 \not\in \tilde{M} \) (see (2.18) for its definition) since the unique \( \theta_0 \in \Omega^* \) satisfying
\[
d(0, e_1)^2 = \sup_{\tau \in \Omega^*} \phi((0, e_1); \tau) = \sup_{\tau \in \Omega^*} 4(e_1 \cdot \tau) = 4\pi = 4(e_1 \cdot \theta_0), \]  

is \( \theta_0 = (\pi, 0) \in \partial \Omega^* \).

Next, we will show that \( g_0 \in \text{Abn}^*_o \). Let
\[
p = (p_1, p_2, p_3) := (2\sqrt{\pi}, 0, 0) \in \mathbb{C}^3.\]

Observe that, by (6.12), we have
\[
\exp(p, 2\theta_0) = (0, e_1).\]

Combining this with (6.20), we find that \( \gamma_{(p, 2\theta_0)} \) is a shortest geodesic from \( o \) to \( g_0 \). It remains to prove that it is also abnormal.

Indeed, set \( e_2 = (0, 1) \). From (6.1), \( p \) as well as \( U_A^{(CR)}(\theta_0)^k p \ (k \in \mathbb{N}^*) \) belongs to the kernel of \( U_A^{(CR)}(e_2) \), namely
\[
U_A^{(CR)}(e_2)U_A^{(CR)}(\theta_0)^k p = 0, \quad \forall k \in \mathbb{N}. \]

Hence, Proposition 1 implies that \( \gamma_{(p, 2\theta_0)} \) is also abnormal. So \( (0, e_1) \in \text{Abn}_o^* \).

Now, we are in a position to prove (6.19) under our assumptions. Set in the sequel
\[
\Omega_\phi := \left\{ \tau; \frac{3\pi}{4} < \tau_1 \pm \tau_2 < \pi \right\} \subseteq \Omega^*.\]

First, we suppose that \( g = (z, t) \) with \( z_j \neq 0 \) for all \( j = 1, 2, 3 \). Then (6.8) implies that \( g(\pm h) := g_0 \pm h g \in \mathbb{M} \) for any \( h \neq 0 \). Let us begin with \( g(h) \). By Theorem 2, there exists a unique \( (\zeta(h), \theta(h)) \in (\mathbb{C}^3 \setminus \{0\}) \times \Omega^* \) such that \( \exp\{(\zeta(h), 2\theta(h))\} = g(h) \).

Next, we claim that \( \theta(h) \in \Omega_\phi \) for any \( |g| = 1 \) and any \( 0 < h \leq c_0 \) with \( c_0 < 1/4 \) small enough. Otherwise, by compactness there exists a \( (\zeta^*, \theta^*) \in (\mathbb{C}^3 \setminus \{0\}) \times \Omega^* \setminus \Omega_\phi \) such that \( |\zeta^*| = d(g_0) \) and \( \exp(\zeta^*, 2\theta^*) = g_0 \), which implies \( \gamma_{(\zeta^*, 2\theta^*)} \) is also a shortest geodesic.
from $o$ to $g_0$. From Case 2 in Subsection 6.1.2, we have that $\gamma(\zeta, 2 \theta^*) = \gamma(0, 2 \theta_0)$ since $E(g_0) = \{\theta_0\}$. Furthermore, $p^* := (p_1^*, p_2^*, p_3^*) \in \mathbb{C}^3$, satisfying $|p^*|^2 = 4\pi$, is a solution of

$$\sum_{j=1}^{3} |p_j^*|^2 \tilde{a}_j = 4\pi e_1. \quad (6.23)$$

From Lemma 2, we get $p^* = \zeta^*$, and Proposition 1 implies that

$$U_{A}^{(\text{CR})}(\theta_0 - \theta^*) p^* = 0. \quad (6.24)$$

If $p_1^* \neq 0$, using (6.1), we obtain that $\tilde{a}_1 \cdot \theta^* = \tilde{a}_1 \cdot \theta_0 = \pi$, so $\theta^* = (\pi, 0)$ since $\theta^* \in \Omega_\zeta$, which contradicts with the fact $\theta^* \in \Omega_\Phi$. In the opposite case $p_1^* = 0$, it follows from (6.23) that $|p_2^*| = |p_3^*| = \sqrt{2}\pi$. Using (6.1), we get that $\tilde{a}_j \cdot (\theta_0 - \theta^*) = 0$ for $j = 2, 3$ since $p_2^*, p_3^* \neq 0$. Observe that $\tilde{a}_2$ and $\tilde{a}_3$ are linearly independent, and thus we have $\theta_0 = \theta^*$, which leads to a contradiction as well.

Consequently, for such $g(h)$, by (3) of Theorem 10, we get that

$$d(g(h))^2 = \phi(g(h); \theta(h)) = \sum_{j=1}^{3} (\tilde{a}_j \cdot \theta(h)) \cot(\tilde{a}_j \cdot \theta(h)) |z_j|^2 h^2 + 4 (e_1 + h t) \cdot \theta(h) \leq 4 (e_1 + h t) \cdot \theta(h),$$

where we have used in the inequality that $\theta(h) \in \Omega_\Phi$. Let $(a, b) := 4 (e_1 + h t)$. Observe that $a \geq 2 \geq |b|$ since $0 < h \leq c_0 < \frac{1}{4}$ and $|t| \leq 1$ from our assumption. So the function defined on $\Omega_\tau$, $\kappa(\tau_1, \tau_2) := a \tau_1 + b \tau_2$ attains its maximum at $\theta_0$. In conclusion,

$$d(g(h))^2 \leq 4 (e_1 + h t) \cdot \theta_0, \quad \forall |g| = 1, \ 0 < h < c_0 \ll 1. \quad (6.25)$$

And similarly, we have

$$d(g(-h))^2 \leq 4 (e_1 - h t) \cdot \theta_0, \quad \forall |g| = 1, \ 0 < h < c_0 \ll 1. \quad (6.26)$$

We use (6.20) together with (6.25) and (6.26), and obtain (6.19) under the additional condition that $(z, t)$ satisfying $z_j \neq 0$ for all $j = 1, 2, 3$.

Finally a limiting argument finishes the proof of this proposition. \hfill $\Box$

To finish this section, we provide an example of $G_{A}^{(\text{CR})}$ on which $\text{Abn}_o \neq \text{Abn}_o^*$.\hfill $\Box$

### 6.3 The shortest abnormal set is not always equal to the abnormal set

**Proposition 8.** There exists a step-two group associated to quadratic CR manifolds $G_{A}^{(\text{CR})}$ such that $\text{Abn}_o^* \subsetneq \text{Abn}_o$.\hfill $\Box$
Proof. Setting 
\[ \tilde{a}_1 = \left( \frac{2^{-1}}{0} \right), \quad \tilde{a}_2 = \left( \frac{1}{1} \right), \quad \tilde{a}_3 = \left( \frac{1}{-1} \right) \] 
and \( A = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \),
we obtain a step-two group associated to quadratic CR manifolds \( G^{(\text{CR})}_A \). And we will show that the point \( g_0 := (0, e_1) \in \mathbb{C}^3 \times \mathbb{R}^2 \) with \( e_1 = (1, 0) \) satisfies \( g_0 \in \text{Abn}_o \setminus \text{Abn}_o^* \).

In our situation, (6.2) implies that the initial reference set is given by 
\[ \Omega_* = \Omega^{(\text{CR})}_A = \{ \tau = (\tau_1, \tau_2); -\pi < \tau_1 \pm \tau_2 < \pi \}. \]
We argue as in the proof of Proposition 7, and get that 
\[ d(g_0)^2 = \sup_{\tau \in \Omega_*} \phi(g_0; \tau) = \sup_{\tau \in \Omega_*} (e_1 \cdot \tau) = 4 \pi = 4 (e_1 \cdot \theta_0), \]
where \( \theta_0 = (\pi, 0) \in \partial \Omega_* \) is the unique maximum point of \( \phi(g_0; \cdot) \) in \( \Omega_* \). So \( g_0 \in \mathbb{M}^c \). Via a simple calculation, Case 2 in Subsection 6.1.2 implies that any shortest geodesic joining \( o \) to \( g_0 \) can be written as \( \gamma_p(s) := \exp\{s (p, 2 \theta_0)\} \) \( (0 \leq s \leq 1) \) where \( p := (0, p_2, p_3) \in \mathbb{C}^3 \) with \( |p_2| = |p_3| = \sqrt{2\pi} \).

We claim that all \( \gamma_p \) are strictly normal, so \( g_0 \notin \text{Abn}_o^* \). We argue by contradiction: assume that there exists \( p^* = (0, p_2^*, p_3^*) \) with \( |p_2^*| = |p_3^*| = \sqrt{2\pi} \) such that \( \gamma_{p^*} \) is abnormal. It follows from Proposition 1 that there exists a \( \sigma \in \mathbb{R}^2 \setminus \{0\} \) such that 
\[ U^{(\text{CR})}_A(\sigma) U^{(\text{CR})}_A(\theta_0)^k p^* = 0, \quad \forall \ k \in \mathbb{N}. \] (6.27)
In particular, we yield \( U^{(\text{CR})}_A(\theta_0)^k(\sigma) = 0 \). Using (6.1), we obtain that \( \tilde{a}_j \cdot \sigma = 0 \) for \( j = 2, 3 \) since \( p_2^*, p_3^* \neq 0 \). Notice that \( \tilde{a}_2 \) and \( \tilde{a}_3 \) are linearly independent. Hence we have \( \sigma = 0 \), which leads to a contradiction.

On the other hand, we set 
\[ p_* = (2\sqrt{2\pi}, 0, 0) \in \mathbb{C}^3, \quad \theta_* = (2\pi, 0), \]
and consider the normal geodesic \( \gamma_*(s) := \exp(s (p_*, 2 \theta_*)) \) \( (0 \leq s \leq 1) \). It follows from (6.12) that \( \exp\{s (p_*, 2 \theta_*)\} = g_0 \), that is, \( \gamma_* \) is a normal geodesic joining \( o \) to \( g_0 \).

Furthermore, let \( e_2 = (0, 1) \). From (6.1), a simple computation shows that 
\[ U^{(\text{CR})}_A(e_2) U^{(\text{CR})}_A(\theta_*)^k p_* = 0, \quad \forall \ k \in \mathbb{N}. \] (6.28)
Combining this with Proposition 1, \( \gamma_* \) is also abnormal. As a result, we have \( g_0 \in \text{Abn}_o \), which ends the proof of this proposition. \( \square \)

7 Gaveau-Brockett optimal control problem on \( N_{3,2} \)

The purpose of this section is to provide a new and independent proof, based on [88], for the Gaveau-Brockett optimal control problem on the free Carnot group of step two and 3 generators \( N_{3,2} \). More precisely, we will give a different proof for Theorem 12 below. For this purpose, we start by
7.1 Preliminaries and known results obtained in [88]

Recall that \( N_{3,2} = \mathbb{R}^3 \times \mathbb{R}^3 \) with
\[
U(\tau) := i \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{pmatrix}, \quad \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3,
\]
that is, \( \langle Ux, x' \rangle = x \times x' \), where “\( \times \)” denotes the cross product on \( \mathbb{R}^3 \) and hence \( \tilde{U}(\tau) x = \tau \times x \).

In our situation, by considering \( \tau \) as a column vector,
\[
\frac{U(\tau)}{\sin U(\tau)} = \frac{|\tau|}{\sin |\tau|} I_3 - \left( \frac{|\tau|}{\sin |\tau|} - 1 \right) \frac{\tau \tau^T}{|\tau|^2}
\]
and the initial reference set and the reference function are given respectively by \( \Omega_* = \{\tau; |\tau| < \pi\} \) and
\[
\phi((x, t); \tau) = (|\tau| \cot |\tau|) |x|^2 + \frac{1 - |\tau| \cot |\tau|}{|\tau|^2} (\tau \cdot x)^2 + 4 t \cdot \tau. \tag{7.1}
\]
See [88, §11] for more details.

We will solve the Gaveau-Brockett optimal control problem on \( N_{3,2} \), namely to find the exact expression of \( d(x, t)^2 \). Using a limiting argument, the scaling property (see (2.2)), and an orthogonal invariance, namely (cf. [88, Lemma 11.1])
\[
d(x, t)^2 = d(Ox, Ot)^2, \quad \forall (x, t) \in N_{3,2}, \forall O \in O_3, \tag{7.2}
\]
where \( O_3 \) denotes the \( 3 \times 3 \) orthogonal group, it suffices to determine \( d(e_1, t_1 e_1 + t_2 e_2)^2 \) with \( t_2 > 0 \).

To begin with, we recall some notations and known results:
\[
\Omega_+ := \{(u_1, u_2) \in \mathbb{R}^2; u_2 > 0, u_1^2 + u_2^2 < \pi^2\}, \tag{7.3}
\]
\[
\mathbb{R}^2_+ := \left\{(u_1, u_2) \in \mathbb{R}^2; u_2 > \frac{2}{\sqrt{\pi}} \sqrt{|u_1|} \geq 0\right\}, \tag{7.4}
\]
\[
\mathbb{R}^2_{<,+} := \left\{(u_1, u_2); u_1 > 0, 0 < u_2 < \frac{2}{\sqrt{\pi}} \sqrt{u_1}\right\}. \tag{7.5}
\]
Set in the sequel
\[
\sqrt{2\pi} < \vartheta_1 < \frac{3}{2} \pi \text{ such that } \tan \vartheta_1 = \vartheta_1, \tag{7.6}
\]
\[
K_3(v_1, v_2) := 2 \psi(r) + \frac{\psi'(r)}{r} v_2^2 \text{ with } r := \sqrt{v_1^2 + v_2^2}, \tag{7.7}
\]
and
\[
\Omega_{-4} := \left\{(v_1, v_2); v_2 < 0, \pi < v_1 < r = \sqrt{v_1^2 + v_2^2} < \vartheta_1, K_3(v_1, v_2) < 0\right\}
\]
\[
= \{(v_1, v_2); v_2 < 0 < v_1, K_3(v_1, v_2) < 0, \pi \neq r < \vartheta_1\}. \tag{7.8}
\]
Indeed, to show the last equality in (7.8), by [88, Lemma 3.4], we have that $\pi < r < \vartheta_1$ and

$$0 > K_3(v_1, v_2) = 4 \left[ \sum_{j=1}^{+\infty} v_2^2 \left( (j \pi)^2 - r^2 \right)^{-2} + \sum_{j=2}^{+\infty} \left( (j \pi)^2 - r^2 \right)^{-1} - \frac{1}{r^2 - \pi^2} \right]$$

$$> 4 \frac{\pi^2 - v_1^2}{(r^2 - \pi^2)^2},$$

which implies $v_1 > \pi$ since $v_1 > 0$.

Moreover, for suitable $E \subseteq \mathbb{R}^2$, we define the smooth function $\Lambda$,

$$\Lambda(v_1, v_2) := v_2 \left[ \frac{\psi'(r)}{r} v_2 + 2 \psi(r) e_2 \right], \quad v = (v_1, v_2) \in E, \ r = |v|. \quad (7.9)$$

The following results can be found or deduced directly from [88, § 11]:

\textbf{Theorem 11} ([88]). \textit{It holds that:}

(i) $d(e_1, t_1 e_1 + t_2 e_2) = d(e_1, |t_1| e_1 + t_2 e_2)$.

(ii) $\Lambda$ is a $C^\infty$-diffeomorphism from $\Omega_+$ onto $\mathbb{R}^2_\varphi$.

(iii) For suitable $(\theta_1, \theta_2) \in \mathbb{R}^2$, set

$$\theta := (\theta_1, \theta_2, 0), \quad t_\theta := \frac{1}{4}(\Lambda(\theta_1, \theta_2), 0), \quad g_\theta := (e_1, t_\theta). \quad (7.10)$$

Then

$$d(g_\theta)^2 = \phi(g_\theta; \theta) = \frac{\theta_1^2}{|\theta|^2} + \left( \frac{\theta_2}{\sin |\theta|} \right)^2 = \left| \frac{U(\theta)}{\sin U(\theta)} e_1 \right|^2, \quad \forall (\theta_1, \theta_2) \in \Omega_+. \quad (7.11)$$

(iv) For any $\alpha \geq 0$, $\tilde{t}(\alpha) := 4^{-1}(\alpha^2, \frac{2}{\pi} \alpha) \in \partial \mathbb{R}^2_\varphi$ and $d(e_1, (\tilde{t}(\alpha), 0))^2 = 1 + \alpha^2$.

(v) $\text{Abn}_o^* = \text{Abn}_o = \{(x, 0); \ x \in \mathbb{R}^3\} = \tilde{M}_2$ and

$$\left\{(x, t); \ x \neq 0, \ t \neq 0, \ \left| t - \langle t, \frac{x}{|x|} \rangle \frac{x}{|x|} \right| > \frac{1}{\sqrt{\pi}} \sqrt{|x| |t \cdot x|} \right\} \subseteq S.$$

(vi) $\Lambda$ is a $C^\infty$-diffeomorphism from $\Omega_{-,4}$ onto $\mathbb{R}^2_{<,+}$.

(vii) We have $d(0, t)^2 = 4\pi |t|$ for all $t \in \mathbb{R}^3$.

In conclusion, via a limiting argument, it remains to determine $d(g_\theta)^2$ with $(\theta_1, \theta_2) \in \Omega_{-,4}$. Indeed, we have the following:

\textbf{Theorem 12} ([88], Theorem 11.3). \textit{(7.11) remains valid for any $(\theta_1, \theta_2) \in \Omega_{-,4}$.}
7.2 Properties of some functions related to \(-s \cot s\)

Recall that the functions \(f\), \(\mu\) and \(\psi\) are defined by (2.25). The following lemma can be found in [63, Lemme 3, p. 112] or [27, Lemma 1.33]:

**Lemma 8.** The function \(\mu\) is an odd function, and a monotonely increasing diffeomorphism between \((−\pi, \pi)\) and \(\mathbb{R}\).

In the sequel, let us define

\[
\varphi_0(s) := \left(\frac{s}{\sin s}\right)^2 - 1, \quad s \in \mathbb{R},
\]

(7.12) nvp0

and for \(s > 0\),

\[
h(s) := \psi'(s) s^3 \sin^2 s = s^2 + s \sin s \cos s - 2 \sin^2 s,
\]

(7.13) ndH

\[
\varphi_1(s) := \frac{s^2 - \sin^2 s}{s - \sin s \cos s} \left(\frac{\varphi_0(s)}{\mu(s)} = \frac{\varphi_0(s)}{s^2 \psi'(s) + 2 s \psi(s)}\right),
\]

(7.14) nvp1

\[
\varphi_2(s) := \frac{s (s^2 - \sin^2 s)}{s^2 + s \sin s \cos s - 2 \sin^2 s} \left(\frac{\varphi_0(s)}{\mu(s) - 2 s \psi(s)} = \frac{\varphi_0(s)}{s^2 \psi'(s)}\right),
\]

(7.15) nvp2

and

\[
\varphi_3(s) := \sqrt{\varphi_1(s) \varphi_2(s)}.
\]

(7.16) nvp3

For \(k \in \mathbb{N}^*\), let \(\vartheta_k\) denote the unique solution of \(s = \tan s\) on \((k\pi, (k + \frac{1}{2})\pi)\).

We will need the following lemma in order to prove Theorem 12:

**Lemma 9.** We have

1. \(h(r) > 0\) for all \(r > 0\). So, \(\psi'(r) > 0\) for \(0 < r \not\in \{k\pi; \; k \in \mathbb{N}^*\}\).
2. \(\varphi_1\) is strictly increasing on \((0, +\infty)\).
3. \(\varphi_2\) is strictly increasing on \(\cup_{k=1}^{+\infty}(k\pi, \vartheta_k)\).
4. \(\varphi_3\) is strictly increasing on \((\pi, +\infty)\).

**Proof.** Notice that (1) can be found in [103, Lemma 3.1], and (2) as well as the strict monotonicity of \(\varphi_3\) on \(\cup_{k=1}^{+\infty}(k\pi, \vartheta_k)\) can be found in the proof of [103, Lemma 3.4]. For the sake of clarity, we will provide a complete proof which is not complicated.

We begin with the proof of (1). Obviously, it suffices to prove the first claim. Indeed, when \(r \geq \frac{\pi}{2}\), we have

\[
h'(r) = r [2 + \cos(2r)] - \frac{3}{2} \sin(2r) \geq \frac{\pi - 3}{2} > 0.
\]

(7.17) fprime

In the opposite case \(r \in (0, \frac{\pi}{2})\), we have the elementary inequality \(\sin r > r \cos r\) and it is clear that

\[
h''(r) = 2 - 2 \cos(2r) - 2r \sin(2r) = 4 \sin r (\sin r - r \cos r) > 0,
\]
which implies that $h'(r) > \lim_{r \to 0^+} h'(r) = 0$ for $r \in (0, \frac{\pi}{2})$. Combining this with (7.17), we get that $h(r) > \lim_{r \to 0^+} h(r) = 0$ for $r > 0$, which ends the proof of the first assertion.

To prove (2), let us set

$$F(r) := \frac{r - \sin r \cos r}{r^2 - \sin^2 r} \quad \text{and} \quad G(r) := \frac{-\sin^2 r + r \sin r \cos r}{r (r^2 - \sin^2 r)}. \quad (7.18)$$

Remark that

$$F(r) = \frac{1}{\varphi_1(r)}, \quad F(r) + 2G(r) = \frac{1}{\varphi_2(r)}, \quad F(r) + G(r) = \frac{1}{r}. \quad (7.19)$$

A simple computation gets that

$$F'(r) = -2 \frac{(r \cos r - \sin r)^2}{(r^2 - \sin^2 r)^2} < 0, \quad \forall r \in (0, +\infty) \setminus \{\vartheta_k; \ k \in \mathbb{N}^*\}, \quad (7.20)$$

which implies the strict monotonicity of $\varphi_1$.

We return to the proof of (3). Using (7.19), we have that

$$\left(\frac{1}{\varphi_2(r)}\right)' = 2(F(r) + G(r))' - F'(r) = -\frac{2}{r^2} + 2 \frac{(r \cos r - \sin r)^2}{(r^2 - \sin^2 r)^2} = 2 \frac{(r^2 \cos r - r \sin r - r^2 + \sin^2 r)(r^2 \cos r - r \sin r + r^2 - \sin^2 r)}{r^2 (r^2 - \sin^2 r)^2}.$$ 

Observe that for $r \in (k\pi, \vartheta_k)$ with $k$ odd, we have that $r \cos r < \sin r < 0$, so

$$\left\{\begin{array}{l}
r^2 \cos r - r \sin r - r^2 + \sin^2 r = r (r \cos r - \sin r) - (r^2 - \sin^2 r) < 0 \\
r^2 \cos r - r \sin r + r^2 - \sin^2 r = r^2 (\cos r + 1) - \sin r (r + \sin r) > 0
\end{array}\right.$$ 

Similarly, if $r \in (k\pi, \vartheta_k)$ with $k$ even, we have that $0 < \sin r < r \cos r$ and

$$\left\{\begin{array}{l}
r^2 \cos r - r \sin r - r^2 + \sin^2 r = r^2 (\cos r - 1) + \sin r (-r + \sin r) < 0 \\
r^2 \cos r - r \sin r + r^2 - \sin^2 r = r (r \cos r - \sin r) + (r^2 - \sin^2 r) > 0
\end{array}\right.$$ 

Hence we have that $\left(\frac{1}{\varphi_2}\right)' < 0$ on $\cup_{k=1}^{+\infty} (k\pi, \vartheta_k)$. Finally, a direct computation gives

$$\varphi_2(k\pi) = k\pi \quad \text{and} \quad \varphi_2(\vartheta_k) = \vartheta_k \quad \forall k \geq 1,$$

which finishes the proof of the strict monotonicity of $\varphi_2$ on $\cup_{k=1}^{+\infty} (k\pi, \vartheta_k)$. We are in a position to prove the strict monotonicity of $\varphi_3$. By the fact that $\varphi_3 = \sqrt{\varphi_1 \varphi_2}$, it following from (2) and (3) that $\varphi_3$ is strictly increasing on $\cup_{k=1}^{+\infty} (k\pi, \vartheta_k)$. Then
it remains to prove that it is also strictly increasing on $\cup_{k=1}^{+\infty}(\vartheta_k, (k + 1)\pi)$. Indeed, by using (7.19) again, we have that
\[
\left(\frac{1}{\varphi_3^2(r)}\right)' = \left[(F(r) + 2G(r))F'(r)\right]' = \left[\left(\frac{2}{r} - F(r)\right)F(r)\right]' \\
= \left[\left(\frac{2}{r} - F(r)\right)'F(r) + (F(r) + 2G(r))F'(r)\right] \\
= -\frac{2}{r^2}F(r) + 2G(r)F'(r).
\]
From (7.18) and (7.20), the last term equals
\[
-2\frac{(r - \sin r \cos r)(r^2 - \sin^2 r)^2 + 2r (r \cos r - \sin r)^2 (\sin^2 r + r \sin r \cos r)}{r^2 (r^2 - \sin^2 r)^3}.
\]
Note that we have for $r > \vartheta_1 > 4$,
\[
(r - \sin r \cos r)(r^2 - \sin^2 r)^2 + 2r (r \cos r - \sin r)^2 (\sin^2 r + r \sin r \cos r) \\
\geq \left(r - \frac{1}{2}\right) (r^2 - 1)^2 - 2r (r + 1)^2 \left(\frac{r}{2} + 1\right) \\
= (r + 1)^2 \left[(r - \frac{1}{2}) (r - 1)^2 - r^2 - 2r\right] \\
\geq (r + 1)^2 \left[3 (r - 1)^2 - r^2 - 2r\right] > 2r (r - 4) + 3 > 0,
\]
which proves our lemma.

\section{7.3 Determination of $\mathcal{W}$ in our situation}

In order to prove Theorem 12, we will use [88, Corollary 2.1] in which we have assumed that $g$ does not belong to $\mathcal{W}$ (cf. (2.24)).

Let us begin with the

\subsection{7.3.1 Expression of $\gamma(w, 2\theta; s) = (x(s), t(s))$ on $N_{3,2}$}

We first recall the convention (3.12). As in the setting of K-type groups, an elementary computation gives that
\[
\cos(2sU(\theta))w = \cos(2s|\theta|)(w - (w \cdot \hat{\theta}) \hat{\theta}) + (w \cdot \hat{\theta}) \hat{\theta}, \\
\frac{\sin(2sU(\theta))}{U(\theta)}w = \frac{\sin(2s|\theta|)}{|\theta|}(w - (w \cdot \hat{\theta}) \hat{\theta}) + 2s(w \cdot \hat{\theta}) \hat{\theta}, \\
\bar{U}(\theta)\frac{\sin(2sU(\theta))}{U(\theta)}w = \sin(2s|\theta|)(\hat{\theta} \times w),
\]
where we have used the fact that $\mathcal{U}(\tau)x = \tau \times x$. Consequently, using (2.6), we obtain that

$$\zeta(s) = \dot{x}(s) = \cos(2sU(\theta))w + \mathcal{U}(\theta) \frac{\sin(2sU(\theta))}{U(\theta)}w$$

$$= \cos(2s|\theta|)(w - (w \cdot \hat{\theta})\hat{\theta}) + (w \cdot \hat{\theta})\hat{\theta} + \sin(2s|\theta|)(\hat{\theta} \times w),$$

$$x(s) = \frac{\sin(2s|\theta|)}{2|\theta|}(w - (w \cdot \hat{\theta})\hat{\theta}) + s(w \cdot \hat{\theta})\hat{\theta} + \frac{1 - \cos(2s|\theta|)}{2|\theta|}(\hat{\theta} \times w).$$ (7.21) \text{exptN}

Then using (2.6) again we have

$$\dot{t}(s) = \frac{1}{2} x(s) \times \zeta(s) = \frac{1}{2} \left( \frac{\sin(2s|\theta|)}{2|\theta|} - s \cos(2s|\theta|) \right) u_1 + \frac{1 - \cos(2s|\theta|)}{4|\theta|} u_2$$

$$+ \frac{1}{2} \left( s \sin(2s|\theta|) - \frac{1 - \cos(2s|\theta|)}{2|\theta|} \right) u_3,$$ (7.22) \text{exptN}

with

$$u_1 = (w - (w \cdot \hat{\theta})\hat{\theta}) \times [(w \cdot \hat{\theta})\hat{\theta}] = -(w \cdot \hat{\theta})(\hat{\theta} \times w),$$ (7.23)

$$u_2 = (w - (w \cdot \hat{\theta})\hat{\theta}) \times (\hat{\theta} \times w) = (|w|^2 - (w \cdot \hat{\theta})^2)\hat{\theta},$$ (7.24)

$$u_3 = (w \cdot \hat{\theta})\hat{\theta} \times (\hat{\theta} \times w) = -(w \cdot \hat{\theta})[w - (w \cdot \hat{\theta})\hat{\theta}],$$ (7.25)

where we have used the well-known vector triple product expansion:

$$(a \times (b \times c)) = (a \cdot c)b - (a \cdot b)c, \quad \forall a, b, c \in \mathbb{R}^3.$$ (7.26) \text{triex}

As a result, by using (5.16) and (5.17), we can write

$$t(s) = \frac{1 - \cos(2s|\theta|) - s|\theta|\sin(2s|\theta|)}{4|\theta|^2} u_1 + \frac{2s|\theta| - \sin(2s|\theta|)}{8|\theta|^2} u_2$$

$$+ \frac{\sin(2s|\theta|) - s|\theta| - s|\theta|\cos(2s|\theta|)}{4|\theta|^2} u_3.$$ (7.27) \text{exptpN}

Now, we provide the

\[ 7.3.2 \quad \text{Description of } \mathcal{W} \cap \{(e_1, \frac{1}{4}(u_1, u_2, 0)); \; u_1, u_2 \in \mathbb{R}\} \]

In this subsection, we suppose that $|\theta| = k\pi$ with $k \in \mathbb{N}^*$. Substituting this with $s = 1$ in (7.21) and (7.27), $(x(w, 2\theta), t(w, 2\theta)) := \exp(w, 2\theta)$ is given by

$$\begin{cases}
  x(w, 2\theta) = (w \cdot \hat{\theta})\hat{\theta} \\
  t(w, 2\theta) = \frac{1}{4|\theta|} u_2 - \frac{1}{2|\theta|} u_3 = \frac{w \cdot \hat{\theta}}{2|\theta|} w + \frac{|w|^2 - 3(w \cdot \hat{\theta})^2}{4|\theta|} \hat{\theta}.
\end{cases}$$ (7.28) \text{expmap}

Then we have the following lemma, which implies that the set $\mathcal{W}$ is negligible in the “subspace” as well.
Lemma 10. Let \( t = \frac{1}{4}(u, 0) = \frac{1}{3}(u_1, u_2, 0) \) such that \((e_1, t) \in \mathcal{W}\). Then we have \(16u_1^2 = k^2 \pi^2 u_2^2\) for some \(k \in \mathbb{N}^*\).

Proof. Let \((w, 2\theta) \in \mathbb{R}^3 \times \mathbb{R}^3\) with \(w := (w_1, w_2, w_3)\) such that \(\exp(w, 2\theta) = (e_1, t)\) and \(|\theta| = k\pi\) for some \(k \in \mathbb{N}^*\). It follows from (7.28) that

\[
e_1 = (w \cdot \hat{\theta}) \hat{\theta}, \quad t = \frac{w \cdot \hat{\theta}}{|\theta|} w + \frac{|w|^2 - (w \cdot \hat{\theta})^2}{4|\theta|} \hat{\theta}.
\]

The first equality implies that \(\theta = \pm (k\pi, 0, 0) := (\theta_1, 0, 0)\), \(w \cdot \hat{\theta} = \pm 1\). Furthermore, \(w \cdot \hat{\theta}\) and \(\theta_1\) have the same sign. Taking inner product on both sides of the second identity in (7.29) with \(\hat{\theta}\), we obtain

\[
t \cdot \hat{\theta} = \frac{|w|^2 - (w \cdot \hat{\theta})^2}{4|\theta|},
\]

and \(\theta_1 u_1 \geq 0\).

Multiplying both sides of (7.30) by \(-\hat{\theta}\), and summing with both sides of the second equation in (7.29) respectively, we have

\[
t - (t \cdot \hat{\theta}) \hat{\theta} = \frac{w \cdot \hat{\theta}}{2|\theta|} (w - (w \cdot \hat{\theta}) \hat{\theta}).
\]

In particular, we get that \(w_3 = 0\) and \(u_2 = 2 \frac{w \cdot \hat{\theta}}{|\theta|} w_2\).

By Pythagoras Theorem, we can write

\[
|t|^2 = |(t \cdot \hat{\theta}) \hat{\theta}|^2 + |t - (t \cdot \hat{\theta}) \hat{\theta}|^2 = (t \cdot \hat{\theta})^2 + \frac{(w \cdot \hat{\theta})^2}{4|\theta|^2} (|w|^2 - (w \cdot \hat{\theta})^2)
\]

\[
= (t \cdot \hat{\theta})^2 + \frac{1}{4|\theta|^2} (4t \cdot \theta)
\]

\[
= \frac{1}{|\theta|^2} ((t \cdot \theta)^2 + (t \cdot \theta)),
\]

where we have used (7.31) and Pythagoras Theorem in the second “=”, (7.30) and \(|w \cdot \hat{\theta}| = 1\) in the third “=”. Inserting \(t = \frac{1}{4}(u_1, u_2, 0)\) and \(\theta = \pm (k\pi, 0, 0)\) in the last equation gives the desired result. \(\square\)

Remark 13. Assume \(t = \frac{1}{4}(u_1, \frac{2}{\sqrt{k\pi}} \sqrt{u_1}, 0)\) with \(u_1 > 0\) and \(k \in \mathbb{N}^*\). It follows from the proof above that the "bad" normal geodesic joining \(o\) to \((e_1, t) \in \mathcal{W}\) is

\[
\gamma_{(w, 2\theta)}(s) \ (0 \leq s \leq 1) \ \text{with} \ w = (1, \sqrt{k\pi} u_1, 0) \ \text{and} \ \theta = (k\pi, 0, 0).
\]
More precisely, from (7.21) and (7.27), \( \gamma_{(w, \theta)}(s) := (x(s), t(s)) \) \((0 \leq s \leq 1)\) is given by:

\[
\begin{align*}
  x(s) & = \begin{pmatrix} s \\ 0 \end{pmatrix} + \frac{\sin(2 s k \pi)}{2k \pi} \begin{pmatrix} 0 \\ \sqrt{k \pi} u_{1} \end{pmatrix} + \frac{1 - \cos(2 s k \pi)}{2k \pi} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
  t(s) & = \frac{1}{4} \left[ -\frac{\sin(2 s k \pi) - s k \pi - s k \pi \cos(2 s k \pi)}{k^2 \pi^2} \begin{pmatrix} 0 \\ \sqrt{k \pi} u_{1} \end{pmatrix} + \frac{2 s k \pi - \sin(2 s k \pi)}{2k^2 \pi^2} \begin{pmatrix} k \pi u_{1} \\ 0 \end{pmatrix} \right].
\end{align*}
\]

We are in a position to provide the

\subsection*{7.4 Proof of Theorem 12}

Recall that \( \psi(s) = \frac{1 - s \cot s}{s^2} \) and \( \varphi_0(s) \) is defined by (7.12). Set in the following:

\[
\Phi(w) := \varphi_0(|w|) \frac{w^2}{|w|^2}, \quad w = (w_1, w_2, w_3) \in \mathbb{R}^3.
\]

Let \((\theta_1, \theta_2) \in \Omega_{\theta, \omega}^+\). Recall that (cf. (7.10))

\[
\theta := (\theta_1, \theta_2, 0), \quad (u_1, u_2) := \Lambda(\theta_1, \theta_2) \in \mathbb{R}^2_{\theta, \omega}+, \quad t_{\theta} := \frac{1}{4} (u_1, u_2, 0), \quad g_{\theta} := (e_1, t_{\theta}).
\]

By Lemma 10, via a limiting argument, we may suppose in the sequel that \( g_{\theta} \notin \mathcal{W} \). Then all the normal geodesics joining \( o \) to \( g_{\theta} \) are “good” ones, so [88, Corollary 2.1] gives that

\[
d(e_1, t_{\theta})^2 = \inf_{\tau \in \Upsilon_{\theta}} \left[ \frac{\tau_1^2}{|\tau|^2} + \left( \frac{\tau_2}{\sin |\tau|} \right)^2 \right] = \inf_{\tau \in \Upsilon_{\theta}} \{ \Phi(\tau) + 1 \}, \tag{7.32}
\]

where \( \Upsilon_{\theta} \) denotes the set of \( \tau = (\tau_1, \tau_2, 0) \) such that \( \Lambda(\tau_1, \tau_2) = (u_1, u_2) \) (see (7.9)), namely,

\[
\begin{align*}
(0 <) u_1 & = \frac{\psi(|\tau|)}{|\tau|} \tau_1 \tau_2^2 \\
(0 <) u_2 & = \tau_2 K_3(\tau_1, \tau_2) = \tau_2 \left( \frac{\psi(|\tau|)}{|\tau|} \tau_2^2 + 2 \psi(|\tau|) \right). \tag{7.33}
\end{align*}
\]

Here are some direct observations.
Observations: Under the above assumptions, for any $\tau \in \Upsilon_\theta$, we have
1. $|\tau| \notin \{k \pi; k \in \mathbb{N}^*\}$ since $(e_1, t_\theta) \notin \mathcal{W}$.
2. Moreover $\tau_2 \neq 0$ and $\tau_1 > 0$ since $u_1 > 0$ and $\psi' > 0$ from (1) of Lemma 9.
3. Furthermore $|\tau| > \pi$. Otherwise $|\tau| < \pi$, $u_2 > 0$ and the second equation in (7.33) imply that $\tau_2 > 0$. So $\tau \in \Omega_\tau$. Hence it follows from (ii) of Theorem 11 that $(u_1, u_2) \in \mathbb{R}_>^2$. This leads to a contradiction.
4. The following equalities hold, in particular for $\theta$,

$$
\Phi(\tau) = \varphi_2(|\tau|) \frac{|\tau|}{\tau_1} u_1 = \varphi_1(|\tau|) \left( \frac{\tau_1}{|\tau|} u_1 + \frac{\tau_2}{|\tau|} u_2 \right) = \varphi_3(|\tau|) \sqrt{u_1 \left( u_1 + \frac{\tau_2}{\tau_1} u_2 \right)}. 
$$

(7.34)

Indeed, the first “=” follows from the definition of $\varphi_2$ (see (7.15)) and the first equation in (7.33), the second (resp. third) one from (7.33) and (7.14) (resp. (7.16) and the first two equalities).

It remains to show that

$$
\Phi(\theta) < \Phi(\tau), \quad \forall \tau \in \Upsilon_\theta \setminus \{\theta\}. 
$$

(7.35)

And we split the proof into three cases.

Case 1: $\tau \in \Upsilon_\theta$ with $|\tau| < |\theta|$. In such case, we get $|\tau| \in (\pi, |\theta|)$. Moreover, we have $\tau_2 > 0$. Otherwise $\tau_2 < 0$, and combining with the fact that $u_2 > 0$ and the second equation in (7.33), we have $K_2(\tau_1, \tau_2) = \psi(|\tau|) \tau_2^2 + 2 \psi(|\tau|) < 0$. So $(\tau_1, \tau_2) \in \Omega_\tau$ and it follows from (vi) of Theorem 11 that $(\tau_1, \tau_2) = (\theta_1, \theta_2)$, which is a contradiction.

Next, remark that $s \psi(s) = \frac{1}{s} - \cot s (< 0)$ is strictly increasing on $(\pi, \theta_1)$, then we have $2 |\tau| \psi(|\tau|) < 2 |\theta| \psi(|\theta|)$. By (7.33), we can write

$$
u_2 \frac{|\tau|}{\tau_2} - u_1 \frac{|\tau|}{\tau_1} = 2 |\tau| \psi(|\tau|) < 2 |\theta| \psi(|\theta|) = u_2 \frac{|\theta|}{\theta_2} - u_1 \frac{|\theta|}{\theta_1}.
$$

By the fact that $u_1, u_2, \tau_1, \tau_2, \theta_1 > 0$ and $\theta_2 < 0$, the last inequality implies that

$$0 < \frac{\tau_1}{|\tau|} < \frac{\theta_1}{|\theta|} \quad \text{and so} \quad \frac{|\tau_2|}{|\tau|} = \sqrt{1 - \left( \frac{\tau_1}{|\tau|} \right)^2} > \sqrt{1 - \left( \frac{\theta_1}{|\theta|} \right)^2} = \frac{|\theta_2|}{|\theta|} > 0.
$$

Then we have

$$
\Phi(\theta) = \left[ \left( \frac{|\theta|}{\sin |\theta|} \right)^2 - 1 \right] \left( \frac{\theta_2}{|\theta|} \right)^2 < \left[ \left( \frac{|\tau|}{\sin |\tau|} \right)^2 - 1 \right] \left( \frac{\tau_2}{|\tau|} \right)^2 = \Phi(\tau),
$$

since the function $\left( \frac{s}{\sin s} \right)^2 > 1$ is strictly decreasing on $(\pi, \theta_1)$ (cf. [27, (1.45)]) , which ends the proof in this case.
**Case 2:** \( \tau \in \Upsilon_\theta \) with \( |\tau| \geq |\theta|, \tau \neq \theta \) and \( \tau_2 < 0 \). We argue as in the beginning of Case 1, we have that \( K_3(\tau_1, \tau_2) = \frac{\psi'(|\tau|)}{|\tau|} \tau_2^2 + 2\psi(|\tau|) < 0 \) and \( |\tau| \notin (\pi, \vartheta_1) \). Moreover, since \( \psi' \) is always positive (see (1) of Lemma 9) and \( \psi \) is negative only on \( \cup_{k=1}^{+\infty}(k\pi, \vartheta_k) \), then we get that \( |\tau| \in \cup_{k=2}^{+\infty}(k\pi, \vartheta_k) \).

We begin with the case where \( \frac{\tau_2}{|\tau|} > \frac{\theta_2}{|\theta|} \). Then we yield that

\[
0 < \frac{\tau_1}{|\tau|} < \frac{\theta_1}{|\theta|}, \text{ so } \frac{|\tau|}{\tau_1} > \frac{|\theta|}{\theta_1} > 0.
\]

Combining this with the first equality in (7.34), we get

\[
\Phi(\theta) = \varphi_2(|\theta|) u_1 \frac{|\theta|}{\theta_1} \left| \varphi_2(|\tau|) u_1 \frac{|\tau|}{\tau_1} = \Phi(\tau),
\]

where we have used, in the inequality, the fact that \( u_1 > 0 \) and \( \varphi_2 (\geq \pi) \) is strictly increasing on \( \cup_{k=1}^{+\infty}(k\pi, \vartheta_k) \) from Lemma 9.

We continue with the opposite case \( \frac{\tau_2}{|\tau|} < \frac{\theta_2}{|\theta|} \), which is equivalent to \( \frac{\theta_2}{|\theta|} < \frac{\tau_2}{|\tau|} \). Similarly, we have \( \frac{\tau_1}{|\tau|} > \frac{\theta_1}{|\theta|} > 0 \). Hence, via the second equality in (7.34),

\[
0 < \Phi(\theta) = \varphi_1(|\theta|) \left( u_1 \frac{\theta_1}{|\theta|} + u_2 \frac{\theta_2}{|\theta|} \right) < \varphi_1(|\tau|) \left( u_1 \frac{\tau_1}{|\tau|} + u_2 \frac{\tau_2}{|\tau|} \right) = \Phi(\tau),
\]

where we have used the fact that \( u_1, u_2 > 0 \) and \( \varphi_1 (\geq 0) \) is strictly increasing on \( (0, +\infty) \) from Lemma 9.

**Case 3:** \( \tau \in \Upsilon_\theta \) with \( |\tau| \geq |\theta| \) and \( \tau_2 > 0 \). By the third equality in (7.34), we get

\[
\Phi(\theta) = \varphi_3(|\theta|) \sqrt{u_1 \left( u_1 + u_2 \frac{\theta_2}{\theta_1} \right)} < \varphi_3(|\tau|) \sqrt{u_1 \left( u_1 + u_2 \frac{\tau_2}{\tau_1} \right)} = \Phi(\tau),
\]

where we have used, in the inequality, the fact that \( u_1, u_2 > 0, \frac{\theta_2}{\theta_1} < 0 \), \( \frac{\tau_2}{\tau_1} > 0 \), and \( \varphi_3 (\geq 0) \) is strictly increasing on \( (\pi, +\infty) \) from Lemma 9.

This finishes the proof of Theorem 12.

### 7.5 Some consequences

In this sub-section, we provide some applications of Theorems 11 and 12. More precisely, we determine the exact formulas of \( d(g)^2 \) on the whole space via a limiting argument, the cut locus \( \text{Cut}_o \) as well as all shortest geodesics from \( o \) to any given \( g \neq o \). For the sake of clarity, we will first reformulate Theorem 12.

Recall that \( \vartheta_1 \) is the unique solution of \( \tan s = s \) on \( (\pi, \frac{3}{2}\pi) \). Also for \( \pi < s < \vartheta_1 \),

\[
f(s) = 1 - s \cot s, \quad \mu(s) = f'(s), \quad \psi(s) = \frac{f(s)}{s^2}, \quad \varphi_0(s) = \left( \frac{s}{\sin s} \right)^2 - 1,
\]

\[
\varphi_1(s) = \frac{\varphi_0(s)}{\mu(s)}, \quad \varphi_2(s) = \frac{\varphi_0(s)}{s^2 \psi'(s)}, \quad \varphi_3(s) = \sqrt{\varphi_1(s) \varphi_2(s)}.
\]
Theorem 13. Let $u_1, u_2 > 0$ such that $u_2 < \frac{2}{\sqrt{\pi}} \sqrt{u_1}$. Suppose that

$$\tilde{\theta} := \tilde{\theta}(u_1, u_2) = (\theta_1, \theta_2) \quad \text{with} \quad \theta_2 < 0 < \theta_1 < |\tilde{\theta}| (\neq \pi) < \vartheta_1$$

is the unique solution of:

$$u_1 = \frac{\psi'(\tilde{\theta})}{|\tilde{\theta}|} \theta_1 \theta_2^2 \quad u_2 = \theta_2 \left( \frac{\psi'(\tilde{\theta})}{|\tilde{\theta}|} \theta_2^2 + 2 \psi(\tilde{\theta}) \right).$$

Then we have $\theta_1 > \pi$ and

$$d\left( e_1, \frac{1}{4}(u_1, u_2, 0) \right)^2 = \left( \frac{\theta_1}{|\tilde{\theta}|} \right)^2 + \left( \frac{\theta_2}{\sin|\tilde{\theta}|} \right)^2 = \varphi_1(|\tilde{\theta}|) \left( u_1 \frac{\theta_1}{|\tilde{\theta}|} + u_2 \frac{\theta_2}{|\tilde{\theta}|} \right) + 1$$

$$= \varphi_2(|\tilde{\theta}|) u_1 \frac{|\tilde{\theta}|}{\theta_1} + 1 = \varphi_3(|\tilde{\theta}|) \sqrt{u_1 \left( u_1 + u_2 \frac{\theta_2}{\theta_1} \right)} + 1. \quad (7.36)$$

Combining this with Theorem 11, it only remains to find the

7.5.1 Exact expression of $d(e_1, \frac{\beta}{4} e_1)^2$ with $\beta > 0$

We have the following result, which is exactly [103, Theorem 1.4] up to a scaling property (cf. (2.2)) and an orthogonal invariance (see (7.2)) combining with (i) of Theorem 11.

Corollary 15. Let $t(\beta) = \frac{1}{4}(\beta, 0, 0)$ with $\beta > 0$. Then it holds that

$$d(e_1, t(\beta))^2 = \varphi_3(r) \beta + 1,$$

where $r$ is the unique solution of the following equation in $(\pi, \vartheta_1)$:

$$-2 \psi(r) \sqrt{r^2 + 2 r \frac{\psi(r)}{\psi'(r)}} = \beta. \quad (7.37)$$

Proof. Let $0 < \epsilon < \frac{2}{\sqrt{\pi}} \sqrt{\beta}$ and $t(\beta, \epsilon) = \frac{1}{4}(\beta, \epsilon, 0)$. Suppose that $\tilde{\theta}_\epsilon := (\theta_1(\epsilon), \theta_2(\epsilon))$ is the unique solution of

$$\theta_2(\epsilon) < 0 < \pi < \theta_1(\epsilon) < |\tilde{\theta}_\epsilon| < \vartheta_1,$$

$$\beta = \frac{\psi'(|\tilde{\theta}_\epsilon|)}{|\tilde{\theta}_\epsilon|} \theta_1(\epsilon) \theta_2^2(\epsilon), \quad \epsilon = \theta_2(\epsilon) \left( \frac{\psi'(|\tilde{\theta}_\epsilon|)}{|\tilde{\theta}_\epsilon|} \theta_2^2(\epsilon) + 2 \psi(|\tilde{\theta}_\epsilon|) \right). \quad (7.38)$$

By the compactness of $B_{\mathbb{R}^2}(0, \vartheta_1)$, up to subsequences, we may take $\epsilon_j \to 0^+$ as $j \to +\infty$ such that the corresponding $\tilde{\theta}_{\epsilon_j} \to \tilde{\theta}_0 := (\theta_1^{(0)}, \theta_2^{(0)})$. Obviously $\pi \leq \theta_1^{(0)} \leq \vartheta_1$ and $\theta_2^{(0)} \leq 0$.  

55
We claim that $\theta_2(0) \neq 0$ so $\tilde{\theta}_0 \notin \{(\pi, 0), (\vartheta_1, 0)\}$ and $\pi < r := |\tilde{\theta}_0| < \vartheta_1$ by (7.38). Indeed, this is ensured by the choice of $\Omega_{-4}$ in [88, § 11]. More precisely, we argue by contradiction: suppose that $\theta_2(\epsilon) \rightarrow 0^-$. Then the first equation in (7.38) implies that

$$
\lim_{j \rightarrow +\infty} \frac{\psi'(|\tilde{\theta}_{\epsilon_j}|)}{|\tilde{\theta}_{\epsilon_j}|} = +\infty, \quad \text{so} \quad |\tilde{\theta}_{\epsilon_j}| \rightarrow \pi^+ \text{ and } \theta_1(\epsilon_j) \rightarrow \pi^+,
$$

since $\pi < \theta_1(\epsilon_j) < |\tilde{\theta}_{\epsilon_j}| < \vartheta_1$ and $\psi'(s) \rightarrow +\infty (\pi < s < \vartheta_1)$ only if $s \rightarrow \pi^+$. Moreover, a direct calculation shows that (see also [88, Lemma 3.4])

$$
\lim_{s \rightarrow \pi^+} (s - \pi) \psi(s) = -\frac{1}{\pi}, \quad \lim_{s \rightarrow \pi^+} (s - \pi)^2 \psi'(s) = \frac{1}{\pi}.
$$

Combining this with (7.38), we get that

$$
\lim_{j \rightarrow +\infty} \frac{1}{\pi} \left( \frac{\theta_2(\epsilon_j)}{|\tilde{\theta}_{\epsilon_j}| - \pi} \right)^2 = \beta, \quad 0 = -\frac{2}{\pi} \lim_{j \rightarrow +\infty} \frac{\theta_2(\epsilon_j)}{|\tilde{\theta}_{\epsilon_j}| - \pi} = 2 \sqrt{\beta} > 0.
$$

This leads to a contradiction.

In conclusion, by the continuity of $d^2$ and the last equality in (7.36), we obtain that $d(e_1, t(\beta))^2 = \varphi_3(r) \beta + 1$, where $\pi < r < \vartheta_1$ satisfies

$$
\beta = \frac{\psi'(r)}{r} \theta_1(0^+) \theta_2(0^+)^2, \quad \frac{\psi'(r)}{r} \theta_2(0^+)^2 + 2 \psi(r) = 0. \tag{7.39}
$$

That is

$$
\beta = -2 \psi(r) \sqrt{r^2 - (\theta_2(0^+))^2} = -2 \psi(r) \sqrt{r^2 + 2r \frac{\psi'(r)}{\psi(r)}}.
$$

Note that the RHS of the last equality is exactly $\frac{4}{P'(s)}$ with the function $P$ defined in [103, (3.3)]. Then from [103, Lemma 3.5], we know that $P$ is a strictly increasing diffeomorphism between $(\pi, \vartheta_1)$ and $(0, +\infty)$, which justifies the uniqueness of the solution $r$ in $(\pi, \vartheta_1)$.

### 7.5.2 The cut locus on $N_{3,2}$

We can characterize the cut locus of $o$ in $N_{3,2}$ from the exact formula for $d^2$ as well. Recall that $\mathcal{S}$ denotes the set of points $g$ such that $d^2$ is $C^\infty$ in a neighborhood of $g$. We have the following result:

**Proposition 9.** It holds that $\mathcal{S} \supseteq \{(x, t); x \text{ and } t \text{ are linearly independent}\}$ on $N_{3,2}$.

**Proof.** Using the scaling property (cf. (2.2)), the orthogonal invariance (see (7.2)) as well as (i) of Theorem 11, it suffices to show that $d^2$ is smooth at $(e_1, \frac{1}{4}(u_1, u_2, 0))$ with $u_2 > 0$ and $u_1 \geq 0$. By recalling the notations defined by (7.3)-(7.9), we divide it into cases.
Case (1): \((u_1, u_2) \in \mathbb{R}^2_{>,+}\). Consider the smooth map

\[
\Pi : O_3 \times (0, +\infty) \times \Omega_{-,4} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 = N_{3,2}
\]

\[
(O, r, (\theta_1, \theta_2)) \longmapsto (r O e_1, \frac{r^2}{4} O (\Lambda(\theta_1, \theta_2), 0)),
\]

where \(O_3\) denotes the \(3 \times 3\) orthogonal group. A direct computation shows that the differential of \(\Pi\) at the point \((I_3, 1, \Lambda^{-1}(u_1, u_2))\) is invertible since \(\Lambda\) is a \(C^\infty\)-diffeomorphism from \(\Omega_{-,4}\) onto \(\mathbb{R}^2_{<,+}\) by (vi) of Theorem 11. As a result, from the inverse function theorem, Theorem 13 and the fact that the function \(O, r \eta \in R^3\) onto \(\mathbb{R}^3\) is smooth on \((0, +\infty) \times \Omega_{-,4}\), we have \((e_1, \frac{1}{4}(u_1, u_2, 0)) \in S\).

Case (2): \((u_1, u_2) \in \mathbb{R}^2\). Similarly, we have \((e_1, \frac{1}{2}(u_1, u_2, 0)) \in S\).

Case (3): \((u_1, \frac{2}{\sqrt{\pi}} \sqrt{u_1})\) with \(u_1 > 0\). In such case, it can be proven by using [20, Theorem 26] with the function defined in [88, § 11.2, Step 2], via Lemma 4 and the concrete characterization of \(\text{Abn}_n^*\) obtained in (v) of Theorem 11. However, we will provide here a direct proof, which is of independent interest.

Set \(\Omega_{+,1} := \Omega_+ \cap \{(v_1, v_2); v_1 > 0\}\), \(\mathbb{R}^2_{>,+} := \mathbb{R}^2 > \cap \{(u_1, u_2); u_1 > 0\}\).

Using the polar coordinate in \(\mathbb{R}^2 \setminus \{(v_1, 0); v_1 \leq 0\}\) where \(v_1 = r \cos \eta\) and \(v_2 = r \sin \eta\) with \(-\pi < \eta < \pi\), we introduce another map \(\Theta : (r, \eta) \mapsto (r, \rho := \frac{\sin \eta}{\sin r})\) with suitable domain.

Notice that we have \(r > 0\) and \(\eta \in \left(0, \frac{\pi}{2}\right)\) (resp. \(\left(-\frac{\pi}{2}, 0\right)\)) on \(\Omega_{+,1}\) (resp. \(\Omega_{-,4}\)). Moreover, it is not hard to show that \(\Theta\) is injective on \(\Omega_{+,1}\) (resp. \(\Omega_{-,4}\)) and its Jacobian determinant is \(\frac{\sin \eta}{\sin r}\). It follows from the global inverse function Theorem that \(\Theta\) is a \(C^\infty\)-diffeomorphism from \(\Omega_{+,1}\) (resp. \(\Omega_{-,4}\)) onto \(\Omega_+ := \Theta(\Omega_{+,1})\) (resp. \(\Omega_- := \Theta(\Omega_{-,4})\)). From (7.3), (7.8), (7.7) and (7.13), a direct calculation yields that

\[
\tilde{\Omega}_+ = \{(r, \rho); 0 < \rho \sin r < 1, \ 0 < r < \pi\},
\]

\[
\tilde{\Omega}_- = \{(r, \rho); \pi < r < \vartheta_1, \ \rho > 0, \ r^2 h(r) + 2 r^2 \psi(r) < 0\}.
\]

By (ii) and (vi) of Theorem 11, \(\Xi := \Lambda \circ \Theta^{-1}: (r, \rho) \mapsto (u_1, u_2)\) is a \(C^\infty\)-diffeomorphism from \(\tilde{\Omega}_+\) (resp. \(\tilde{\Omega}_-\)) onto \(\mathbb{R}^2_{>,+}\) (resp. \(\mathbb{R}^2_{<,+}\)). Using (7.9), (7.13) and the definition of \(\psi\) (cf. (2.25)), it can be written explicitly:

\[
\begin{align*}
\begin{cases}
 u_1 &= \left(r + \sin r \cos r - 2 \frac{\sin^2 r}{r}\right) \sqrt{1 - \rho^2 \sin^2 r} \rho^2 \\
 u_2 &= \sin r \left(r + \sin r \cos r - 2 \frac{\sin^2 r}{r}\right) \rho^3 + 2 \left(\frac{\sin r}{r} - \cos r\right) \rho
\end{cases}
\end{align*}
\]

(7.40)
Figure 1: Plot of $\tilde{\Omega} = \tilde{\Omega}_+ \cup \tilde{\Omega}_- \cup (\{\pi\} \times (0, +\infty))$

A key observation is that $\Xi$ is also meaningful on $(\pi, \rho)$ for $\rho > 0$ and $\Xi(\pi, \rho) = (\pi \rho^2, 2\rho)$ is a bijection from $\{\pi\} \times (0, +\infty)$ to $\left\{(u_1, u_2) : u_2 = \frac{2}{\sqrt{\pi}} \sqrt{u_1} > 0\right\}$. Set

$$\tilde{\Omega} := \tilde{\Omega}_+ \cup \tilde{\Omega}_- \cup (\{\pi\} \times (0, +\infty)).$$

See the plot in Figure 1.

Moreover, notice that $\Xi$ is $C^\infty$ on $\tilde{\Omega}$. A direct computation shows that the Jacobian determinant of $\Xi$ at $(\pi, \rho)$ $(\rho > 0)$ equals

$$J(\Xi)(\pi, \rho) = 2 \left(\pi^2 \rho^4 + 4 \rho^2\right) > 0.$$

As a consequence, $\Xi$ is a $C^\infty$-diffeomorphism from $\tilde{\Omega}$ onto $(0, +\infty) \times (0, +\infty)$.

In conclusion, by (iii), (iv) of Theorem 11 as well as Theorem 12, we have

$$d(e_1, \frac{1}{4}(u_1, u_2, 0))^2 = (r^2 - \sin^2 r) \rho^2 + 1, \quad \forall u_1, u_2 > 0,$$

where $(r, \rho) = \Xi^{-1}(u_1, u_2)$. Finally, by using the smooth map

$$\tilde{\Pi} : O_3 \times (0, +\infty) \times \tilde{\Omega} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 = N_{3,2}$$

$$(O, R, (r, \rho)) \longmapsto (RO e_1, \frac{R^2}{4} O (\Xi(r, \rho), 0)),$$

we can get that $(e_1, \frac{1}{4}(u_1, \frac{2}{\sqrt{\pi}} \sqrt{u_1}, 0)) \in S$.

This completes the proof of this proposition.
Indeed, the relation "\( \supseteq \)" in Proposition 9 can be improved to "\( = \)". In other words, we have the following:

**Corollary 16.** On \( N_{3,2} \), we have \( \text{Cut}_o = \{ (x, t); x \text{ and } t \text{ are linearly dependent} \} \).

**Proof.** It follows from Proposition 9 that

\[
\text{Cut}_o \subseteq \{ (x, t); x \text{ and } t \text{ are linearly dependent} \}. \tag{7.42}
\]

Moreover, we have that (see (v) of Theorem 11):

\[
\text{Abn}^*_o = \{ (x, 0); x \in \mathbb{R}^3 \} = \tilde{M}_2. \tag{7.43}
\]

By Theorem 1, it remains to show that the classical cut locus of \( o \) is

\[
\text{Cut}^\text{CL}_o = \{ (x, t); t \neq 0 \text{ and } x = \lambda t \text{ for some } \lambda \in \mathbb{R} \} := E, \tag{7.44}
\]

which has been already proven in [113] and [103] by completely different technique. Indeed, once we get (7.42) and (7.43), via [123, Lemma 9], (7.44) is a direct consequence of the simple fact that there exist two distinct shortest geodesics from \( o \) to any \( g \in E \). See Subsection 7.5.3 below for more details.

7.5.3 Description of shortest geodesic(s) from \( o \) to any given \( g \neq o \)

Let us begin by the following observation:

**Lemma 11.** Let \( \text{SO}_3 \) denote the \( 3 \times 3 \) special orthogonal group, and \( (x, t) = \exp(w, \tau) \). Then we have

\[
\exp(O w, O \tau) = (O x, O t), \quad \forall O \in \text{SO}_3, \tag{7.45}
\]

\[
\exp(O e^{\tilde{U}(\tau)} w, O \tau) = (O x, O t), \quad \forall O \in O_3 \setminus \text{SO}_3. \tag{7.46}
\]

**Proof.** Using the well-known basic property of the cross product:

\[
O (\tau \times \eta) = (O \tau) \times (O \eta), \quad \forall O \in \text{SO}_3, \tag{7.47}
\]

(7.45) can be checked directly by (7.21) and (7.27), or explained by [101, § 2.1].

To obtain (7.46), we use the fact that \( \tilde{U}(\tau) \eta = \tau \times \eta \) as well as (7.47), and get that

\[
\tilde{U}(O \tau) \eta = (O \tau) \times \eta = O (\tau \times (O^T \eta)) = O \tilde{U}(\tau) O^T \eta, \quad \forall O \in \text{SO}_3, \tag{7.48}
\]

which implies that

\[
\tilde{U}(O \tau) = O \tilde{U}(\tau) O^T, \quad \forall O \in \text{SO}_3. \tag{7.49}
\]

Then for any \( O \in O_3 \setminus \text{SO}_3 \), we have \( -O \in \text{SO}_3 \), and (7.45) implies that

\[
\exp(-O w, -O \tau) = (-O x, -O t).
\]

Consequently, applying (2.10) to the last equation and using (7.49), we obtain (7.46).
Combining (7.45) and (7.46) with (2.9), it suffices to determine all shortest geodesics from $o$ to $g$, where: (1) $g = (e_1, \frac{1}{4}(u_1, u_2, 0))$ with $u_1 \geq 0$ and $u_2 > 0$; (2) $g = (e_1, 0)$; (3) $g = (0, e_1)$ or $g = (e_1, \frac{1}{4} \beta e_1)$ with $\beta > 0$.

Case 1. $g = (e_1, \frac{1}{4}(u_1, u_2, 0))$ with $u_1 \geq 0$ and $u_2 > 0$. In such case, we have $g \in S$. So there exists a unique shortest geodesic from $o$ to $g$, which is strictly normal. If $(u_1, u_2) \in \mathbb{R}_+^2 \cup \mathbb{R}_+^2$, it follows from [88, Theorem 2.4 and Theorem 2.5] that the shortest geodesic is given by $\gamma_{(w,2}\theta)$ with $w = \frac{U(\theta)}{\sin U(\theta)} e^{-\tilde{U}(\theta)} e_1$ and $\theta = (\Lambda^{-1}(u_1, u_2), 0)$. For $u_2 = \frac{2}{\sqrt{\pi}} \sqrt{u_1} > 0$, then it is given by $\gamma_{(w,2}\theta)$ with $w = (1, \sqrt{\pi} u_1, 0)$ and $\theta = (\pi, 0, 0)$, see Remark 13 for more details.

Case 2. $g = (e_1, 0) \in \text{Abn}_o^* \setminus \{o\}$. The unique shortest geodesic joining $o$ to $g$ is a straight segment and it is abnormal.

Case 3. $g = (0, e_1)$ or $g = (e_1, \frac{1}{4} \beta e_1)$ with $\beta > 0$. In such case, $g \in E$. By (7.45), a trivial observation is that there exist at least two distinct shortest geodesics from $o$ to $g$. Indeed, a complete description can be found in [103, §3]. However, we will provide a completely different method to get it, which can be considered as a direct consequence of our main results, namely Theorems 11 and 13. More precisely, we will use Remark 5 to determine the parameter $(w, \theta)$ of any shortest geodesic from $o$ to $g$, $\gamma_{(w,2}\theta)$.

Case 3 (a). Let us begin with the case $g_{(\beta)} = (e_1, \frac{1}{4}(\beta, 0, 0))$ where $\beta > 0$. By Remark 5, there exist $\{g_n := ((x^{(n)}, t^{(n)}))_{n=1}^{\infty} \subset \text{Cut}_o^c$, with the corresponding unique shortest geodesic $\gamma_{(w^{(n)},2}\theta^{(n)})$, such that $(g_n, w^{(n)}, \theta^{(n)}) \to (g_{(\beta)}, w, \theta)$ as $n \to +\infty$. Without loss of generality, we may assume that $|x^{(n)}| \neq 0$ and $x^{(n)} \cdot t^{(n)} > 0$ for all $n \geq 1$. For each $n \geq 1$, we pick an orthogonal matrix $O^{(n)} \in O_3$ such that

$$O^{(n)} x^{(n)} = |x^{(n)}| e_1, \quad O^{(n)} t^{(n)} = \frac{|x^{(n)}|^2}{4} (u_1^{(n)}, u_2^{(n)}, 0) \text{ with } u_1^{(n)}, u_2^{(n)} > 0. \quad (7.50)$$

Combining this with the fact that $(x^{(n)}, t^{(n)}) \to (e_1, \frac{1}{4} \beta e_1)$ as $n \to +\infty$, we get that $(u_1^{(n)}, u_2^{(n)}) \to (\beta, 0)$ as $n \to +\infty$. By arguing as in the proof of (7.39) and using (7.45), (7.46) as well as (2.9), we get that

$$O^{(n)} \theta^{(n)} \to (\Theta_1, \Theta_2, 0) \text{ with } \Theta_1 = \sqrt{r^2 + 2r \frac{\psi(r)}{\psi'(r)}}, \quad \Theta_2 = -\sqrt{-2r \frac{\psi(r)}{\psi'(r)}},$$

where $r$ is the unique solution of (7.37) in $(\pi, \vartheta_1)$. Since $O_3$ is compact, up to subsequences, we may further assume that $O^{(n)} \to O'$ as $n \to +\infty$. Moreover, it follows from the first equation in (7.50) that the orthogonal matrix $O'$ satisfies $O' e_1 = e_1$. Since $O' \theta = (\Theta_1, \Theta_2, 0)$, we yield that

$$\theta = (\Theta_1, |\Theta_2| \cos \sigma, |\Theta_2| \sin \sigma) \text{ for some } \sigma \in \mathbb{R},$$

$$w = \lim_{n \to +\infty} w^{(n)} = \lim_{n \to +\infty} \frac{U(\theta^{(n)})}{\sin U(\theta^{(n)})} e^{-\tilde{U}(\theta^{(n)})} x^{(n)} = \frac{U(\theta)}{\sin U(\theta)} e^{-\tilde{U}(\theta)} e_1,$$

where we have used (2.7) in the second “=“ of the last formula.
In other words, we have proven that every shortest geodesic from $o$ to $g_\beta$ can be expressed as $\gamma_{(w,2\theta)}$, where the parameter $(w, \theta)$ have the form

$$\theta(\sigma) := (\Theta_1, |\Theta_2| \cos \sigma, |\Theta_2| \sin \sigma), \quad w(\sigma) := \frac{U(\theta(\sigma))}{\sin U(\theta(\sigma))} e^{-\tilde{U}(\theta(\sigma))} e_1. \quad (7.51)$$

Furthermore, we will prove that the converse is also valid, namely every such parameter provides a shortest geodesic steering $o$ to $g_\beta$.

Let us fix such a shortest geodesic $\gamma_{(w(\sigma_0),2\theta(\sigma_0))}$, with $\sigma_0 \in \mathbb{R}$. By (7.45), for any $\alpha \in \mathbb{R}$, $\gamma_{(O(\alpha)w(\sigma_0),2O(\alpha)\theta(\sigma_0))}$ is also a shortest geodesic from $o$ to $g_\beta$, where

$$O(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} \in \text{SO}_3.$$

It remains to show that

$$O(\alpha) w(\sigma_0), O(\alpha) \theta(\sigma_0)) = (w(\sigma_0 + \alpha), \theta(\sigma_0 + \alpha)), \quad \forall \alpha \in \mathbb{R}, \quad (7.52)$$

since $(w(\sigma_0 + \alpha), \theta(\sigma_0 + \alpha))$ runs over all possible $(w(\sigma), \theta(\sigma))$ as the parameter $\alpha$ runs over $\mathbb{R}$.

In fact, it follows from (7.49) that

$$O(\alpha) w(\sigma_0) = O(\alpha) \frac{U(\theta(\sigma_0))}{\sin U(\theta(\sigma_0))} e^{-\tilde{U}(\theta(\sigma_0))} O(\alpha)^T O(\alpha) e_1$$

$$= \frac{U(O(\alpha) \theta(\sigma_0))}{\sin U(O(\alpha) \theta(\sigma_0))} e^{-\tilde{U}(O(\alpha) \theta(\sigma_0))} e_1. \quad (7.53)$$

To finish the proof of (7.52), it suffices to notice that we have obviously

$$O(\alpha) \theta(\sigma_0) = \theta(\sigma_0 + \alpha).$$

Via some elementary but tedious calculations, our result can be identified with that of [103, Theorem 3.2] with $(y, y^\perp, \theta) = \left(\sqrt{\frac{\beta}{2}} e_2, \sqrt{\frac{\beta}{2}} e_3, r\right)$ therein, where we identify $\wedge^2 \mathbb{R}^3$ with $\mathbb{R}^3$ via the map $T$ defined by

$$T : \wedge^2 \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$a \wedge b \longmapsto a \times b.$$

**Case 3 (b).** In the opposite case where $g = (0, e_1)$, similarly, it follows from Remark 5 and (7.28) that all shortest geodesics joining $o$ to $g$ are given by $\gamma_{(w,2\theta)}$, where

$$\theta = (\pi, 0, 0), \quad w = 2\sqrt{\pi} (0, -\cos \sigma, -\sin \sigma) \text{ with } \sigma \in \mathbb{R}.$$

Via some elementary but tedious calculations, our result can be identified with that of [103, Theorem 3.2] with $(y, y^\perp, \theta) = (e_2, e_3, \pi)$ therein (via the map $T$ as well).
Appendix A: Proof of Lemma 7

The proof essentially follows that of [88, Proposition 10.3] and we include it for the sake of completeness.

Set \( v := (v_1, v_2) \in \mathbb{R}^2 \). Then from the definition of \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_*) ; \cdot) \), for \( |\bar{x}^*|^2 + \bar{x}_1^2 \neq 0 \) and \( \bar{x}_* \neq 0 \), we have

\[
(u_1, u_2) := \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_*) ; (v_1, v_2)) = \nabla_v \left[ \left( \frac{v_1^2}{4} \psi \left( \frac{|v_1|}{2} \right) v_2^2 \psi(|v_2|) \right) |\bar{x}_*|^2 + f \left( \frac{|v_1|}{2} \right) (|\bar{x}^*|^2 + \bar{x}_1^2) \right].
\]  

(8.1)

More precisely,

\[
\begin{align*}
    u_1 &= v_1 \left[ \mu \left( \frac{|v_1|}{2} \right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2}{2 |v_1|} + \psi' \left( \frac{|v_1|}{2} \right) \frac{v_1^2}{4} |\bar{x}_*|^2 + \psi \left( \frac{|v_1|}{2} \right) \frac{|\bar{x}^*|^2}{2} + \psi'(|v_1|) v_2^2 \frac{|\bar{x}_*|^2}{|v_1|} \right] \\
    u_2 &= v_2 \left[ \mu \left( \frac{|v_1|}{2} \right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2}{2 |v_1|} + \psi' \left( \frac{|v_1|}{2} \right) \frac{v_1^2}{4} |\bar{x}_*|^2 + \psi'(|v_1|) v_2^2 \frac{|\bar{x}_*|^2}{|v_1|} + 2 \psi(|v_1|) |\bar{x}_*|^2 \right].
\end{align*}
\]

(8.2)

It follows from [88, Lemma 3.3] that

\[
0 \leq \frac{\psi'(|v_1|)}{|v_1|} |v_1| v_2^2 \leq \frac{\pi}{4} v_2^2 \left( 2 \psi(|v_1|) + \frac{\psi'(|v_1|)}{|v_1|} v_2^2 \right)^2, \quad \forall |v_1| < \pi,
\]

which implies that

\[
|u_1| \leq \mu \left( \frac{|v_1|}{2} \right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2}{2} + \psi' \left( \frac{|v_1|}{2} \right) \frac{|v_1|^2}{8} + \psi \left( \frac{|v_1|}{2} \right) \frac{|v_1|}{2} \frac{|\bar{x}_*|^2}{2} + \frac{\pi}{4} v_2^2 \left( 2 \psi(|v_1|) + \frac{\psi'(|v_1|)}{|v_1|} v_2^2 \right)^2 |\bar{x}_*|^2.
\]

Using the identity \( s^2 \psi'(s) + 2s \psi(s) = \mu(s) \), the second equality in (8.2) implies that

\[
|u_1| \leq \mu \left( \frac{|v_1|}{2} \right) \frac{|\bar{x}^*|^2 + \bar{x}_1^2 + |\bar{x}_*|^2}{2} + \frac{\pi}{4} \frac{u_2^2}{|\bar{x}_*|^2} < \frac{\pi}{4} \left( \frac{u_2^2}{|\bar{x}_*|^2} + |\bar{x}^*|^2 + \bar{x}_1^2 + |\bar{x}_*|^2 \right),
\]

where we have used in "<" the fact that \( \mu(\pi/2) = \pi/2 \) and \( \mu \) is strictly increasing on \((-\pi, \pi)\) (see Lemma 8), so \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_*); \cdot) \) is from \( B_{\mathbb{R}^2}(0, \pi) \) to \( \mathbb{R}^2(\bar{x}^*, \bar{x}_1, \bar{x}_*) \).

Next, it follows from (8.1) that the Jacobian of \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_*); \cdot) \) is given by

\[
\text{Hess}_v \left[ \left( \frac{v_1^2}{4} \psi \left( \frac{|v_1|}{2} \right) v_2^2 \psi(|v_2|) \right) |\bar{x}_*|^2 + f \left( \frac{|v_1|}{2} \right) (|\bar{x}^*|^2 + \bar{x}_1^2) \right] > 0,
\]

since \( |\bar{x}^*|^2 + \bar{x}_1^2 \neq 0, \bar{x}_* \neq 0 \), \( \text{Hess}_v f(|v|) > 0 \) and \( \text{Hess}_v (v_2^2 \psi(|v_2|)) \geq 0 \) (so symmetrically \( \text{Hess}_v (v_2^2 \psi(|v_1|)) \geq 0 \)) for \( |v| < \pi \), which can be found in the proof of [88, Proposition 10.1].

Finally, using Hadamard's theorem, it remains to show that \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_*); \cdot) \) is proper.

To this end, we consider the behaviour of \( \{v_j^{(j)} \}_{j=1}^{+\infty} \subseteq B_{\mathbb{R}^2}(0, \pi) \) satisfying \( v_j^{(j)} = (v_1^{(j)}, v_2^{(j)}) \rightarrow \)
\[ \partial B_{R^2}(0, \pi) \] and we split it into cases.

1. If \( |v(j)| \rightarrow \pi^+ \) and \( |v(j)| \geq \epsilon > 0 \), then \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_s); v(j)) \rightarrow \infty \).

2. If \( |v(j)| \rightarrow \pi^- \) and \( |v(j)| \rightarrow 0^+ \), set \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_s); v(j)) = (u_1(j), u_2(j)) \).

In the case \( |u_1(j)| \rightarrow +\infty \), it is easy to see \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_s); v(j)) \rightarrow \infty \).

In the opposite case, assume \( |u_1(j)| \rightarrow a (\geq 0) \). Then from the first equation of (8.2), using the fact that \( s^2 \psi'(s) + 2s \psi(s) = \mu(s) \) as well as \( \mu(\pi/2) = \pi/2 \), we have that

\[
a = \frac{\pi}{4} \left( |\bar{x}|^2 + \bar{x}_1^2 + |\bar{x}_s|^2 \right) + \lim_{j \rightarrow +\infty} \frac{1}{\pi} \left( \frac{v_2(j)}{\pi - |v(j)|} \right)^2 |\bar{x}_s|^2,
\]

where we have used the fact that (see also [88, Lemma 3.4])

\[
\lim_{s \rightarrow \pi^-} (\pi - s)^2 \psi'(s) = \frac{1}{\pi}.
\]

Since

\[
\lim_{s \rightarrow \pi^-} (\pi - s) \psi(s) = \frac{1}{\pi},
\]

the second equation of (8.2) gives that

\[
\lim_{j \rightarrow +\infty} |u_2(j)| = \lim_{j \rightarrow +\infty} \frac{2}{\pi} \frac{|v_2(j)|}{\pi - |v(j)|} |\bar{x}_s|^2 = \frac{2}{\pi} \frac{|\bar{x}_s|}{\sqrt{\pi}} \sqrt{a - \frac{\pi}{4} \left( |\bar{x}^*|^2 + \bar{x}_1^2 + |\bar{x}_s|^2 \right)}.
\]

In conclusion, we have \( \Upsilon((\bar{x}^*, \bar{x}_1, \bar{x}_s); v(j)) \rightarrow \partial B_{R^2}(\bar{x}^*, \bar{x}_1, \bar{x}_s) \) when \( v(j) \rightarrow \partial B_{R^2}(0, \pi) \) in \( B_{R^2}(0, \pi) \), which means it is proper.

This finishes the proof of this lemma.

9 Appendix B: Properties of the direct product of two 2-step groups

Axb Assume that \( \mathbb{G}_j = \mathbb{G}(q_j, m_j, U_j) \) \((j = 1, 2)\), where

\[
U_j = \{ U_j^{(1)}, \ldots, U_j^{(m_j)} \}.
\]

Consider the direct product \( \mathbb{G}_1 \times \mathbb{G}_2 := \mathbb{G}(q_1 + q_2, m_1 + m_2, U) \) with \( U \) defined by

\[
\left\{ \left( \begin{array}{c} \mathbb{G}_1^{(1)} \times \mathbb{G}_2^{(1)} \\ \mathbb{G}_1^{(m_1)} \times \mathbb{G}_2^{(m_1)} \\ \mathbb{G}_1^{(2)} \times \mathbb{G}_2^{(2)} \\ \mathbb{G}_1^{(m_2)} \times \mathbb{G}_2^{(m_2)} \end{array} \right), \ldots, \left( \begin{array}{c} \mathbb{G}_1^{(1)} \times \mathbb{G}_2^{(1)} \\ \mathbb{G}_1^{(m_1)} \times \mathbb{G}_2^{(m_1)} \\ \mathbb{G}_1^{(2)} \times \mathbb{G}_2^{(2)} \\ \mathbb{G}_1^{(m_2)} \times \mathbb{G}_2^{(m_2)} \end{array} \right) \right\},
\]

where \( \mathbb{G}_k^{(1)} \times \mathbb{G}_k^{(i)} \) denotes the \( k_1 \times k_2 \) null matrix. Let \( g_j := (x_j, t_j) \in \mathbb{G}_j, j = 1, 2 \). We identify \((g_1, g_2)\) with \((\langle x_1, x_2 \rangle, \langle t_1, t_2 \rangle)\). Then, it is clear that

\[
\phi(G_1 \times G_2) = \phi(G_1) \times \phi(G_2),
\]

\[
\mathbb{M}^{(G_1 \times G_2)} = \mathbb{M}^{(G_1)} \times \mathbb{M}^{(G_2)}, \quad d_{G_1 \times G_2}((g_1, g_2))^2 = d_{G_1}(g_1)^2 + d_{G_2}(g_2)^2,
\]
and the meaning of the notations herein is obvious. It is clear that \((\gamma_{G_1}(s), \gamma_{G_2}(s))\) is a normal geodesic if and only if both \(\gamma_{G_1}\) and \(\gamma_{G_2}\) are normal geodesics. Moreover, the normal geodesic \((\gamma_{G_1}(s), \gamma_{G_2}(s))\) is abnormal if and only if \(\gamma_{G_1}\) or \(\gamma_{G_2}\) is abnormal. Also, it is shortest if and only if both \(\gamma_{G_1}\) and \(\gamma_{G_2}\) are shortest. Hence, we have the following well-known properties: on \(G_1 \times G_2\), it holds that:

\[
\text{Cut}_o^{(G_1 \times G_2)} = \left( \text{Cut}_o^{(G_1)} \times G_2 \right) \cup \left( G_1 \times \text{Cut}_o^{(G_2)} \right),
\]

(9.1)

\[
\text{Cut}_o^{\text{CL}} (G_1 \times G_2) = \left( \text{Cut}_o^{\text{CL}} (G_1) \times G_2 \right) \cup \left( G_1 \times \text{Cut}_o^{\text{CL}} (G_2) \right),
\]

(9.2)

\[
\text{Abn}_o^* (G_1 \times G_2) = \left( \text{Abn}_o^* (G_1) \times G_2 \right) \cup \left( G_1 \times \text{Abn}_o^* (G_2) \right),
\]

(9.3)

and the meaning of the notations herein is obvious.

By the fact that \(M(G_1 \times G_2) = M(G_1) \times M(G_2)\), we have the following:

**Proposition 10.** \(G_1 \times G_2\) is of type GM if and only if both \(G_1\) and \(G_2\) are GM-groups.

Similarly, on the direct product of the Euclidean space \(\mathbb{R}^k\) with a step-two group \(G\), we have

\[
\text{Cut}_o^{(\mathbb{R}^k \times G)} = \mathbb{R}^k \times \text{Cut}_o^{(G)}, \quad \text{Cut}_o^{\text{CL}} (\mathbb{R}^k \times G) = \mathbb{R}^k \times \text{Cut}_o^{\text{CL}} (G),
\]

(9.4)

\[
\text{Abn}_o^* (\mathbb{R}^k \times G) = \mathbb{R}^k \times \text{Abn}_o^* (G),
\]

(9.5)

and the meaning of the notations herein is obvious. In particular,

**NSA Proposition 11.** Let \(G\) be a step-two group. Consider the direct product of the Euclidean space \(\mathbb{R}^k\) with \(G\). Then:

1. \(\mathbb{R}^k \times G\) is of type GM if and only if \(G\) is a GM-group;

2. \(\mathbb{R}^k \times G\) is of type SA if and only if \(G\) is a Métivier or SA-group.

### 10 Appendix C: Construction of SA-groups

**Axco** In [88, § 8.1], there is a simple method to construct an uncountable number of GM-groups (resp. GM-groups of Métivier type) from any given step-two group (resp. Métivier group). We will use the same method to produce SA-groups. More precisely, assume that \(m \geq 2\) and \(G_j = G(q_j, m, U_j)\) \((j = 1, 2)\), where

\[
U_j = \{U_j^{(1)}, \ldots, U_j^{(m)}\}.
\]

We consider \(G := G(q_1 + q_2, m, U)\) with \(U\) defined by

\[
\left\{ \begin{pmatrix} U_1^{(1)} \\ U_2^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} U_1^{(m)} \\ U_2^{(m)} \end{pmatrix} \right\}.
\]
Let \( g = (x_1, x_2, t) \in \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \times \mathbb{R}^m \). Obviously, we have:

\[
\Omega^*_G = \Omega^*_{G_1} \cap \Omega^*_{G_2},
\]

and the meaning of the notations herein is clear. Moreover, \( g \in \tilde{M}_2(G) \) if and only if there are \( \theta \in \Omega^*_{G_1} \cap \Omega^*_{G_2} \) and \( t', t'' \in \mathbb{R}^m \) such that:

\[
t = t' + t'', \quad t' = -\frac{1}{4} \nabla_{\theta} \langle U^{(G_1)}(\theta) \cot U^{(G_1)}(\theta) x_1, x_1 \rangle, \quad t'' = -\frac{1}{4} \nabla_{\theta} \langle U^{(G_2)}(\theta) \cot U^{(G_2)}(\theta) x_2, x_2 \rangle,
\]

and the sum of two positive semidefinite matrices

\[
\left( -\text{Hess}_{\theta} \langle U^{(G_1)}(\theta) \cot U^{(G_1)}(\theta) x_1, x_1 \rangle \right) + \left( -\text{Hess}_{\theta} \langle U^{(G_2)}(\theta) \cot U^{(G_2)}(\theta) x_2, x_2 \rangle \right)
\]

is singular. In such case, we have \((x_1, t') \in \tilde{M}_2(G_1)\) and \((x_2, t'') \in \tilde{M}_2(G_2)\). Thus, we get immediately the following:

**Proposition 12.** With the above notations, \( G \) is a SA-group when \( G_1 \) is of type SA and \( G_2 \) is of type Métivier or SA.

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