On Adjacent Vertex-distinguishing Equitable-total Chromatic Number of $P_m \vee F_m$

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Abstract. Suppose the simple graph $G(V, E)$ is at least 2nd-order connected. We study the adjacent vertex-distinguishing equitable-total coloring of the join graph $P_m \vee F_m$, which belongs to the graph $G(V, E)$. By constructing the total coloring of $P_m \vee F_m$, we obtain the adjacent vertex-distinguishing equitable-total chromatic number of it, which proves to satisfy the adjacent vertex-distinguishing equitable-total coloring conjecture.

Keywords: Join graph, Adjacent Vertex-distinguishing Equitable-total Coloring, Adjacent Vertex-distinguishing Equitable-total Chromatic Number

1. Introduction
Suppose a simple graph is $G(V, E)$ with vertices $V$ and edges $E$. The simple graphs discussed in this article are finite, undirected and connected. $\Delta(G)$ is the maximum degree of graph $G(V, E)$, for a real number $x$ we denote by $\lfloor x \rfloor$ the greatest integer which at least $x$, and by $\lceil x \rceil$ the least integer which at most $x$.

Graph coloring is an important research problem. Many coloring conceptions or methods are introduced for the sake of computer and information science, equalization distribution and so on. Among them, the concept about the adjacent vertex-distinguishing total coloring (AVDTC briefly) is proposed by Zhang Zhongfu etc. in[1] (2004). To prove the conjecture about the adjacent vertex-distinguishing total chromatic number in [1] be true, some results about AVDTC of some special graphs or constructional graphs were discussed [3-5]. In 2005, the adjacent vertex-distinguishing equitable-total coloring (AVDETC briefly) is proposed by Zhongfu Zhang etc.[2] based on the concept of AVDETC of $G(V, E)$, but few papers about the AVDETC of $G(V, E)$ appear up to now for it more difficulty than AVDTC of $G(V, E)$.

The AVDETC of $G(V, E)$ is a normal total coloring of graph $G(V, E)$ to meet that, for any two different vertices which coloring sets are not the same, and the difference of any two colors to color the total of vertices and edges no more than 1. The minimum $k$ such that $G(V, E)$ has a $k$ - AVDETC we call the adjacent vertex-distinguishing equitable-total chromatic number of $G(V, E)$, that is...
\( \chi_{aet}(G) = \min \{ k | k - \text{AVDETC of } G \} \). In this article, we will explore the AVDETC of join graph \( P_n \lor F_m \), and the \( \chi_{aet}(P_n \lor F_m) \) obtained proves to satisfy the proposed coloring conjecture in [2].

**Conjecture** [1,2] For \( G(V, E) \) with at least order 2, we have
\[
\chi_{aet}(G) = \chi_{at}(G) \leq \Delta(G) + 3. \tag{1}
\]

For other terms or symbols of graphs not mentioned in the article, readers are invited to participate in the literature [6].

2. Main Results

At the beginning, we give some results about the total chromatic number of AVDTC and AVDETC of \( G(V, E) \) in the following.

**Lemma 1.1**[1,2] For every graph \( G(V, E) \) with at least order 2, if any of the vertices which degrees are maximum are not adjacent, then
\[
\chi_{aet}(G) = \chi_{at}(G) \geq \Delta(G) + 1. \tag{2}
\]

**Lemma 1.2**[1,2] If there are two vertices in the graph \( G(V, E) \) with at least order 2, which degrees are the maximum and adjacent, then
\[
\chi_{aet}(G) = \chi_{at}(G) \geq \Delta(G) + 2. \tag{3}
\]

The main work immediately followed is that, for join graph \( P_m \lor F_m \), we give a proposition, and use construction method to prove it on the basis of the above-mentioned lemmas, combined with the structural characteristics of join graph \( P_m \lor F_m \).

**Theorem 2.1** For a join graph \( P_m \lor F_m \), where \( P_m \) is a path with \( m(m \geq 2) \) vertices and \( F_m \) is a fan with \( m + 1(m \geq 2) \) vertices, then
\[
\chi_{aet}(P_m \lor F_m) = \begin{cases} 7, m = 2; \\ 8, m = 3; \\ 2m + 1, m \geq 4. \end{cases} \tag{4}
\]

**Proof.** For a path \( P_m(V_1, E_1) \) and a fan \( F_m(V_2, E_2) \). Let \( V_1 = \{u_1, u_2, \cdots, u_m\}, V_2 = \{v_0, v_1, v_2, \cdots, v_m\} \), now the following three cases will be considered.

**Case 1** If \( m = 2 \), the join graph \( P_m \lor F_m \) just is the complete graph \( K_5 \), so we have \( \chi_{aet}(P_2 \lor F_2) = 7[2] \).

**Case 2** If \( m = 3 \), \( P_m \lor F_m \) is complete graph \( K_7 \), so \( \chi_{aet}(P_3 \lor F_3) = 8[2] \).

**Case 3** If \( m > 3 \), there is only one vertex \( v_0 \) which degree is maximum \( (\Delta(P_m \lor F_m) = d(v_0) = 2m) \) in \( P_m \lor F_m \), so \( \chi_{aet}(P_m \lor F_m) \geq 2m + 1 \) according to Lemma 1.1. Therefore we only need to prove that there exists an \( (2m + 1) \)-AVDETC. We construct the map \( \phi \) from \( P_m \lor F_m \) to \( \{1, 2, \cdots, 2m + 1\} \) in the light of the following two cases, and in the interest of simplicity we have taken \( \alpha \) to be \( \left\lceil \frac{m}{2} \right\rceil \) and consecutive integers \( 1, 2, \cdots, m \) to be \( \{m\} \) respectively in the rear of the proof.

**Case 3.1** If \( m \) is an odd, let
\[
\begin{align*}
\phi(v_0) & = 2m + 1, \phi(v_s) = s, s = \{m\}; \\
\phi(u_t) & = m + t, t = \{m\}; \phi(v_0 u_t) = s + 1, s = \{m\}; \\
\phi(v_0 u_t) & = s + t + \alpha, s = \alpha + 1, \cdots, t = (3\alpha - s - 1); \\
\phi(v_s u_t) & = s + t - 3\alpha + 2, s = \alpha + 1, \cdots, t = 3\alpha - s - \cdots, m; \\
\phi(u_t u_{s+1}) & = m + \alpha + s + 3, s = \{m - \alpha - 2\}, \\
\phi(u_t u_{s+1}) & = s - m + \alpha + 2, s = m - \alpha - 1, \cdots, m - 1; \\
\phi(v_s v_{s+1}) & = m + \alpha + s + 2, s = \{\alpha - 1\}; \\
\phi(v_s v_{s+1}) & = 1, s = \alpha, \alpha + 2, \cdots, m - 1 \ (\text{if } \alpha \text{ is an even}) \text{ or } m - 2 \ (\text{if } \alpha \text{ is an odd}),
\end{align*}
\]
\[ \phi(v_sv_{s+1}) = \frac{s+3\alpha+1}{2}, \quad s = \alpha + 1, \alpha + 3, \ldots, m - 1 \text{ (if } \alpha \text{ is an odd) or } m - 2 \text{ (if } \alpha \text{ is an even).} \]

Under the coloring, \( \phi \) has been an \((2m + 1)\)-AVDTC of \( P_m \lor I_m \), in order to make the \( \phi \) to be an \((2m + 1)\)-AVDETC of \( P_m \lor I_m \), we redefine the color of the edges \( v_sv_{m-[\frac{s}{2}]} = 1 \) \((s \equiv 0 \text{ (mod 2)} \) and \( s < \alpha \)) such that \( \phi(v_sv_{m-[\frac{s}{2}]}) = 1 \), for the new coloring \( \phi \), we will show that \( \phi \) is an \((2m + 1)\)-AVDETC of \( P_m \lor I_m \) with respect to the following two cases.

Case 3.1.1 If \( \alpha \) is an even, then we have

\[
C(v_0) = \{(2m + 1)\}, \quad \text{and} \quad C(v_1) = \{1, 2, m + \alpha + 2, t + 2 \mid t = \langle m \rangle \}.
\]

If \( 1 < s \leq \alpha - 1 \) and \( s \equiv 0 \text{ (mod 2)} \),

\[
C(v_s) = \{1, s + 1, m + \alpha + s, m + \alpha + 1 + s\} \cup \{s + t + 1 \mid t = \langle m \rangle\} \cup \{m + 1 + \left[\frac{s}{2}\right]\},
\]

If \( 1 < s \leq \alpha - 1 \) and \( s \equiv 1 \text{ (mod 2)} \),

\[
C(v_s) = \{1, s + 1, m + \alpha + s, m + \alpha + 1 + s\} \cup \{s + t + 1 \mid t = \langle m \rangle\}.
\]

If \( s = \alpha + k(s \leq m - 1) \) and \( \text{and } s \equiv 1 \text{ (mod 2)} \),

\[
C(v_s) = \left\{1, \frac{s+3\alpha}{2}, s + 1, s + t + \alpha \mid t = \langle 3\alpha - s - 1 \rangle \right\}
\]

\[
\cup \left\{s + t - 3\alpha + 2 \mid t = 3\alpha - s, 3\alpha - s + 1, \ldots, s + m - 3\alpha \right\},
\]

If \( \alpha = \alpha + k(s \leq m - 1) \) and \( k \equiv 0 \text{ (mod 2)} \),

\[
C(v_s) = \left\{1, \frac{s+3\alpha}{2}, s, s + 1, s + t + \alpha \mid t = \langle 3 - s - 1 \rangle \right\}
\]

\[
\cup \left\{s + t - 3\alpha + 2 \mid t = 3\alpha - s, 3\alpha - s + 1, \ldots, s + m - 3\alpha \right\},
\]

If \( 1 < c \leq \alpha - 1 \) and \( c \equiv 0 \text{ (mod 2)} \),

\[
C(v_s) = \left\{1, m, m + 1, m + \alpha + t, m + \alpha + 1 \right\} \cup \{s + t + 1 \mid t = \langle \alpha - 1 \rangle\}
\]

\[
\cup \left\{m + 1 + \alpha + s + k \mid s = \langle \alpha - 1 \rangle \right\},
\]

If \( 1 < c \leq \alpha - 1 \) and \( c \equiv 1 \text{ (mod 2)} \),

\[
C(v_s) = \left\{1, m, m + 1, m + \alpha + t, m + \alpha + 1 \right\} \cup \{s + t + 1 \mid t = \langle \alpha - 1 \rangle\}
\]

\[
\cup \left\{m + 1 + \alpha + s + k \mid s = \langle \alpha - 1 \rangle \right\},
\]

Case 3.1.2 If \( \alpha \) is an odd, then we obtain

\[
C(v_0) = \{1, 2, \ldots, 2m + 1\}, \quad C(v_1) = \{1, 2, m + \alpha + 2, t \mid t = \langle m \rangle\}.
\]

If \( 1 < s \leq \alpha - 1 \) and \( s \equiv 0 \text{ (mod 2)} \),

\[
C(v_s) = \{1, s, s + 1, m + \alpha + s, m + \alpha + 1 + s, s + t + 1 \mid t = \langle m \rangle\} \cup \{m + 1 + \left[\frac{s}{2}\right]\},
\]

If \( 1 < s \leq \alpha - 1 \) and \( s \equiv 1 \text{ (mod 2)} \),

\[
C(v_s) = \{s, s + 1, m + \alpha + s, m + \alpha + 1 + s, s + t + 1 \mid t = \langle m \rangle\}.
\]
\[ C(v_a) = \{1, \alpha, \alpha + 1, 2m + 1, \alpha + t + 1 | t = \langle m \rangle \}; \]
\[ C(v_{a+1}) = \{1, \alpha + 1, \alpha + 2, 2a + t + 1 | t = \langle 2a - 2 \rangle \} \cup \{t - 2a + 3 | t = 2a - 1 \}; \]
If \( s = \alpha + k(s \leq m - 1) \) and \( k \equiv 0 \, (\text{mod} \, 2) \),
\[ C(v_s) = \left\{ 1, \frac{s+3a}{2}, s + 1, s + t + \alpha | t = \langle 3a - s - 1 \rangle \right\} \]
\[ \cup \{s + t - 3a + 2 | t = 3a - s, 3a - s + 1, \cdots, s + m - 3a \}; \]
If \( s = \alpha + k(s \leq m - 1) \) and \( k \equiv 1 \, (\text{mod} \, 2) \),
\[ C(v_s) = \left\{ 1, \frac{3a+s+1}{2}, s + 1, s + t + \alpha | t = \langle 3a - s - 1 \rangle \right\} \]
\[ \cup \{ \frac{m+3a}{2}, m + 1, m + t + \alpha | t = \langle 3a - m - 1 \rangle \} \]
\[ \cup \{s + t - 3a + 2 | t = 3a - m, 3a - m + 1, \cdots, 2m - 3a \}; \]
\[ C(v_m) = \{m + 1, m + 2, 3a + 3, 2 + s | s = \langle \alpha \rangle \} \cup \{m + 2 + s | s = \langle \alpha - 1 \rangle \}; \]
If \( 2 \leq t < \alpha - 2 \),
\[ C(u_t) = \{m + t, m + t + 1, 3a + t + 1, 3a + t + 2, s + t + 1 | s = \langle \alpha \rangle \} \]
\[ \cup \{m + s + t + 1 | s = \langle \alpha - 1 \rangle \}; \]
\[ C(u_{a-2}) = \{1, 2m + 1, m + \alpha - 2, m + \alpha - 1, m + \alpha - 1 | s = \langle \alpha \rangle \} \]
\[ \cup \{m + \alpha + s - 1 | s = \langle \alpha - 1 \rangle \}; \]
If \( \alpha - 1 \leq t \leq \alpha \),
\[ C(u_t) = \{t - \alpha + 2, t - \alpha + 3, m + t, m + t + 1, s + t + 1 | s = \langle \alpha \rangle \} \]
\[ \cup \{m + s + t + 1 | s = \langle \alpha - 1 \rangle \}; \]
If \( t = \alpha + k, k = \langle m - \frac{3}{2} (\alpha + 1) + 1 \rangle \)
\[ C(u_t) = \{m + t, m + t + 1, s + t + 1 | s = \langle \alpha \rangle \} \cup \{s + 1 | s = \langle k + 2 \rangle \} \]
\[ \cup \{m + 1 + \alpha + s + k | s = \langle \alpha - k - 1 \rangle \}; \]
If \( \alpha \) is an odd, and \( t = m - \frac{1}{2} (\alpha + 1) + k, k = \langle \alpha \rangle, m + t + 2, m + t + 3 \),
\[ C(u_t) = \{1, m + t, m + t + 1, s + 1 | s = \langle k + 2 \rangle \} \]
\[ \cup \{m + 1 + \alpha + s + k | s = \langle \alpha - k - 1 \rangle \}; \]
\[ C(u_m) = \{s | s = \langle \alpha + 1 \rangle, 2m \} \cup \{s + m + 1 | s = \langle \alpha \rangle \}. \]
We can verify that \( \phi \) satisfies Definition 1 considering all the coloring sets one by one. So \( \phi \) is an \((2m + 1)\)-AVDTC of \( P_m \cup V_m \); Furthermore, in the case of \( m \) an odd, for any color \( s \) in the set \( C \), we can find that
\[ |V_s \cup E_s| = \begin{cases} \alpha + 2, s = \langle 3a + 2, \rangle, 2m + 1; \\ \alpha + 3, s = \langle 3a + 3, \cdots, 2m \rangle. \end{cases} \]
Especially, when \( n = 5, |V_s \cup E_s| = 5, s = 1, 2, \cdots, 10; |V_s \cup E_s| = 4, s = 11 \). so the \( \phi \) is an \((2m + 1)\)-AVDETC of \( P_m \cup V_m \).

**Case 3.2** If \( m \) is an even, let
\[ \phi(v_0) = 2m + 1, \phi(v_0) = s, s = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + 1, s = \langle m \rangle; \]
\[ \phi(v_0v_0) = m + t, t = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + 1, s = \langle m \rangle; \]
\[ \phi(v_0v_0) = m + t + 1, t = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + t + 1, s = \langle \alpha \rangle, t = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + t + 1, s = \langle \alpha \rangle, t = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + t - 3a + 2, s = \alpha + 1, m, t = \langle 3a - s \rangle; \]
\[ \phi(v_0v_0) = s + t - 3a + 2, s = \alpha + 1, m, t = \langle 3a - s \rangle; \]
\[ \phi(v_0v_0) = s, t = \langle m \rangle; \]
\[ \phi(v_0v_0) = s + m + 2 + s, s = \langle \alpha - 1 \rangle; \]
\[ \phi(v_0v_0) = 1, s = \alpha, \alpha + 2, \cdots, m - 1 \) (if \( \alpha \) is an odd) or \( m - 2 \) (if \( \alpha \) is an even),
\[ \phi(v_0v_0) = \frac{3}{2} (\alpha + 1) + \frac{s}{2}, s = \alpha + 1, \alpha + 3, \cdots, m - 1 \) (if \( \alpha \) is an even) or \( m - 2 \) (if \( \alpha \) is an odd).
Although, $\phi$ has been an $(2m + 1)$-AVDTC of $P_m \lor F_m$, in order to make the $\phi$ to be an $(2m + 1)$-AVDETC of $P_m \lor F_m$, we redefine the color of the edges $v_3 u_{m-1}[\frac{s}{2}](s \equiv 0(\text{mod } 2)$ and $s < \alpha$) such that $\phi(v_3 u_{m-1}[\frac{s}{2}]) = 1$. The new coloring is also a normal total coloring and, for any arbitrary two adjacent vertices, we now illustrate that they have different sets of colors with this new coloring by considering two cases.

**Case 3.2.1** If $\alpha$ is an odd, then

- $C(v_0) = \{(2m + 1)\}$,
- $C(v_1) = \{1, 2, m + \alpha + 3, t | t = \langle m \rangle\}$;
- If $1 < s \leq \alpha - 1$ and $s \equiv 0(\text{mod } 2)$,
- $C(v_s) = \{s, s + 1, m + \alpha + 1 + s, m + \alpha + 2 + s, s + t + 1 | t = \langle m \rangle\}/(m + 1 + \frac{s}{2})$;
- If $1 \leq s \leq \alpha - 1$ and $s \equiv 1(\text{mod } 2)$,
- $C(v_s) = \{1, 2, m + \alpha + 1 + s, m + \alpha + 2 + s, s + t + 1 | t = \langle m \rangle\}$;
- $C(v_{\alpha+1}) = \{1, 2, \alpha + 1, \alpha + 2, 2\alpha + t + 1 | t = \langle m \rangle\}$;
- If $s = \alpha + k(s \leq m - 1)$ and $k \equiv 1(\text{mod } 2)$,
- $C(v_s) = \{1, \alpha + 1, \alpha + 2, 2\alpha + t + 1 | t = \langle m \rangle\}$.

Clearly, $\phi$ is an $(2m + 1)$-AVDTC of $P_m \lor F_m$, and for each $s \in C$, we have that

$$|V_s \cup E_s| = \{\alpha + 3, s = 1, 2, \ldots, m - 1, 3\alpha + 3, \ldots, 2m\}.$$

so the $\phi$ is an $(2m + 1)$-AVDETC of $P_m \lor F_m$.
$C(v_a) = \{1, 2m + 1, \alpha, \alpha + 1, \alpha + t + 1 \mid t = \langle m \rangle \}$;
$C(v_{a+1}) = \{1, \alpha + 1, \alpha + 2, 2\alpha + t + 1 \mid t = \langle 2\alpha \rangle \} \cup \{ t - 2\alpha + 3 \mid t = 2\alpha - 1 \}$;
If $s = \alpha + k(s \leq m - 1)$ and $k \equiv 1 \pmod{2}$,
$C(v_s) = \{1, \frac{s + 3\alpha + 1}{2} + 1, s, s + 1, s + t + \alpha + 1 \mid t = \langle 3\alpha - s \rangle \}$
$\cup \{ s + t - 3\alpha + 2 \mid t = 3\alpha - s, 3\alpha - s + 1, m + s - 3\alpha \}$.
If $s = \alpha + k(s \leq m - 1)$ and $k \equiv 0 \pmod{2}$,
$C(v_s) = \{1, \frac{s + 3\alpha + 1}{2} + 1, s, s + 1, s + t + \alpha \mid t = \langle 3\alpha - s \rangle \}$
$\cup \{ s + t - 3\alpha + 2 \mid t = 3\alpha - s, 3\alpha - s + 1, m + s - 3\alpha \}$.

If $c = \langle I + 1 \rangle$ and $4 \equiv 1 \pmod{2}$,
$C(u_c) = \{1, m + 1, m + 2, 2 + s \mid s = \langle \alpha \rangle \} \cup \{ m + 2 + s \mid s = \langle \alpha \rangle \};$
If $2 \leq t \leq \alpha$,
$C(u_t) = \{ m + t, m + t + 1, t - 1, t, s + t + 1 \mid s = \langle \alpha \rangle \} \cup \{ m + s + t + 1 \mid s = \langle \alpha \rangle \};$
If $t = \alpha + k, k = \langle m - \frac{3}{2} \alpha - 1 \rangle,$
$C(u_t) = \{ m + t, m + t + 1, t - 1, t, s + t + 1 \mid s = \langle \alpha \rangle \} \cup \{ s + 1 \mid s = \langle k \rangle \}$
$\cup \{ m + 1 + \alpha + s + k \mid s = \langle \alpha - k \rangle \};$
If $t = m + k - \frac{1}{2} \alpha - 1, k = \langle \frac{1}{2} \alpha \rangle$,
$C(u_t) = \{ m + t, m + t + 1, t - 1, t, s + 1 \mid s = \langle \alpha \rangle \} \cup \{ s + t + 1 \mid s = \langle \alpha \rangle \}$
$\cup \{ m + 1 + s + t \mid s = \langle \frac{1}{2} \alpha + k + 1 \rangle \} / (m - k + 2 + \frac{1}{2} \alpha) ;$
$C(u_m) = \{ s \mid s = \langle \alpha + 1 \rangle, 2m \} \cup \{ s + m + 1 \}.$

With this case, it is obviously to find that $\phi$ is an $(2m + 1)$-AVDTC of $P_m \vee F_m$, and for any $s \in C$, we can obtain that
If $s = \langle m + \frac{1}{2} \alpha \rangle, m + \frac{1}{2} \alpha + 2, \ldots, 3\alpha + 2, 2m + 1, |V_s \cup E_s| = \alpha + 2$;
If $s = m + \frac{1}{2} \alpha + 1, 3\alpha + 3, \ldots, 2m, |V_s \cup E_s| = \alpha + 3$.
So the $\phi$ is an $(2m + 1)$-AVDETC of $P_m \vee F_m$.

The proof is completed.

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