Continuous-spin mixed-symmetry fields in AdS(5)

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Abstract
Free mixed-symmetry continuous-spin fields propagating in AdS(5) space and flat R(4,1) space are studied. In the framework of a light-cone gauge formulation of relativistic dynamics, we build simple actions for such fields. The realization of relativistic symmetries on the space of light-cone gauge mixed-symmetry continuous-spin fields is also found. Interrelations between constant parameters entering the light-cone gauge actions and eigenvalues of the Casimir operators of space-time symmetry algebras are obtained. Using these interrelations and requiring that the field dynamics in AdS(5) be irreducible and classically unitary, we derive restrictions on the constant parameters and eigenvalues of the second-order Casimir operator of the $so(4,2)$ algebra.

Keywords: continuous-spin fields, mixed-symmetry fields, light-cone gauge approach

1. Introduction

In view of the aesthetic features of continuous-spin field theory, interest in this theory has periodically been renewed (see [1–16]). A review of recent developments in this topic may be found in [13]. One of the interesting features of a continuous-spin field is that this field is decomposed into an infinite chain of coupled scalar, vector, and totally symmetric tensor fields consisting of every spin just once. It is such a chain of scalar, vector and totally symmetric fields that enters the theory of higher-spin gauge field in AdS space [17]. We think, therefore, that certain interesting interrelations between continuous-spin field theory and higher-spin gauge theory might exist. Also, as noted in the literature, some regimes in string theory are related to continuous-spin field theory (see, e.g. [10]). Regarding string theory, we note that one of the interesting examples of the string model is realized as the type IIB superstring in the $AdS_5 \times S^5$ background [18]. It has been demonstrated in [19, 20] that it is the use of a
light-cone gauge formulation that considerably simplifies the action of the superstring in the AdS$_5 \times S^5$ background. Taking this into account, it seems then worthwhile to apply a light-cone gauge formulation in the study of a continuous-spin field in AdS$_5$ space. This is what we do in this paper.

In this paper, using a light-cone gauge formulation, we study a free mixed-symmetry continuous-spin bosonic field in AdS$_5$ space. The light-cone gauge formulation of relativistic dynamics in AdS space was developed first in [21], while, in [22], we applied this formulation to the study of finite-component mixed-symmetry massless fields in AdS$_5$. Later on, the approach in [21] was reformulated into a more convenient form in [23], while, in [24], we used such renewed light-cone gauge formulation in the study of a finite-component mixed-symmetry massive field in AdS$_5$ space. It is the light-cone gauge formulation in [23] that we are going to use in the study of a mixed-symmetry continuous-spin field in AdS$_5$ space in this paper.

We recall that, in manifestly Lorentz covariant formulations, mixed-symmetry fields propagating in five-dimensional space-time are described by tensor fields whose $so(4, 1)$ space-time tensor indices have the structure of the Young tableaux with two rows. A remarkable feature of the light-cone gauge formulation is that, in the framework of this formulation, mixed-symmetry fields propagating in five-dimensional space-time are described by complex-valued totally symmetric tensor fields of the $so(3)$ algebra. It is the use of the complex-valued totally symmetric tensor fields of the $so(3)$ algebra that allows us to obtain the simple light-cone gauge Lagrangian formulation for the mixed-symmetry continuous-spin field in AdS$_5$ space. Besides this, our approach allows us in a straightforward way to find interesting restrictions on the eigenvalues of Casimir operators of the $so(4, 2)$ algebra, which is algebra of relativistic symmetries of the continuous-spin field in AdS$_5$. We believe that, in future studies, these restrictions will be helpful in solving the problem of group theoretical interpretation of continuous-spin fields in AdS space. As a by-product, we also obtain the light-cone gauge Lagrangian formulation for a mixed-symmetry continuous-spin field in $R^{4,1}$ space.

This paper is organized as follows.

In section 2, we discuss a mixed-symmetry continuous-spin field in flat $R^{4,1}$ space. Namely, for such a field, we find a light-cone gauge action and obtain a light-cone gauge realization of the Poincaré algebra symmetries on the mixed-symmetry continuous-spin field. Also we find interrelations between constant parameters entering the light-cone gauge formulation of the continuous-spin field and three independent Casimir operators of the Poincaré algebra $iso(4, 1)$.

In section 3, we start with a brief review of the general light-cone gauge formalism developed in [23]. After this, we apply this formalism to studying the mixed-symmetry continuous-spin field in AdS$_5$ space. Also we present our new result for the light-cone gauge realization of the third-order and fourth-order Casimir operators of the $so(4, 2)$ algebra and find interrelations between three constant parameters entering the light-cone gauge Lagrangian formulation of the continuous-spin field in AdS$_5$ and three independent Casimir operators of the $so(4, 2)$ algebra. We demonstrate how, for a large radius of AdS space, light-cone gauge formulations in AdS space and flat space are related to each other.

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1 A discussion of other approaches which could be helpful in the study of various aspects of mixed-symmetry continuous-spin fields may found in [25–30].

2 Recently, in [16], two-rows tensor (tensor-spinor) fields have been used for the frame-like formulation of mixed-symmetry continuous-spin bosonic (fermionic) fields in AdS$_{d+1}$.

3 The light-cone gauge approach also turns out to be efficient in the study of interacting fields (see, e.g. [31, 32] and references therein). Recent developments in the light-cone approach may be found in [33–36].
In section 4, we study restrictions imposed on the constant parameters and the second-order Casimir operators of the \( so(4,2) \) algebra which are obtained by requiring that the field dynamics in AdS\(_5\) space be classically unitary and irreducible. We find interesting representations for eigenvalues of the Casimir operators for the mixed-symmetry continuous-spin field which are similar to the expressions for eigenvalues of the Casimir operators for positive-energy lowest weight unitary representations of the \( so(4,2) \) algebra.

In appendix A, we briefly review the Casimir operators of the \( so(4,2) \) algebra. In appendix B, we present useful relations for various spin operators.

2. Continuous-spin mixed-symmetry field in \( R^{4,1} \) space

2.1. Notation and conventions

Relativistic symmetries of field dynamics in \( R^{4,1} \) space are described by the Poincaré algebra \( iso(4,1) \). We use the following commutation relations for generators of the Poincaré algebra \( iso(4,1) \):

\[
[P^a, J^{bc}] = \eta^{ab} P^c - \eta^{ac} P^b,
\]
\[
[J^{ab}, J^{ce}] = \eta^{bc} J^{ae} + 3 \text{ terms},
\]
where \( \eta^{ab} \) is a mostly positive flat metric tensor. In this section, vector indices of the \( so(4,1) \) Lorentz algebra take values \( a, b = 0, 1, 2, 3 \). The generators \( P^a, J^{ab} \) are assumed to be anti-Hermitian.

The light-cone frame coordinates and the vector indices of the \( so(3) \) algebra are given by

\[
x^\pm = \frac{1}{\sqrt{2}} (x^4 \pm x^0), \quad x^I, \quad I, J, K = 1, 2, 3,
\]

where the coordinate \( x^+ \) is treated as an evolution parameter. The \( so(4,1) \) Lorentz algebra vector \( X^a \) is decomposed as \( X^+, X^-, X^I \) and a scalar product of Lorentz algebra vectors \( X^a \) and \( Y^b \) is represented as

\[
\eta_{ab} X^a Y^b = X^+ Y^- + X^- Y^+ + X^I Y^I.
\]

Relation (2.3) implies that, in the light-cone frame, nonvanishing elements of the flat metric \( \eta_{ab} \) are given by \( \eta_{+-} = 1, \eta_{-+} = 1, \eta_{IJ} = \delta_{IJ} \). Thus, for the covariant and contravariant components of vectors, we get the relations \( X^+ = X^- = X^I, \quad X^I = X_I \). In the light-cone frame, commutators of the Poincaré algebra generators are obtained from the ones in (2.1) by using the nonvanishing elements of the \( \eta^{ab} \) given by \( \eta^{+-} = 1, \eta^{-+} = 1, \eta^{IJ} = \delta^{IJ} \).

2.2. Field content

To discuss light-cone gauge description of a mixed-symmetry continuous-spin field in \( R^{4,1} \), we use the following set of complex-valued fields of the \( so(3) \) algebra,

\[
\phi^{I_1 \ldots I_n}(x), \quad n = h_2, h_2 + 1, \ldots, \infty,
\]

where \( h_2 \in \mathbb{N} \) is an integer which labels the mixed-symmetry continuous-spin field\(^4\). In (2.4), a field with \( n = 1 \) is a vector field of the \( so(3) \) algebra, while a field with \( n \geq 2 \) is a totally symmetric rank-\( n \) traceless tensor field of the \( so(3) \) algebra. Note that, in view of \( h_2 \in \mathbb{N} \), fields \( \phi^{I_1 \ldots I_n} \) with \( n = 0, 1, \ldots, h_2 - 1 \) do not enter the field content of the mixed-symmetry

\(^4\) Throughout this paper, \( \mathbb{N} \) stands for \( 1, 2, \ldots, \infty \), while \( \mathbb{N}_0 \) stands for \( 0, 1, 2, \ldots, \infty \).
continuous-spin field (2.4). Also, note that field \( \phi_{1\ldots n} \) with \( n = 0 \) stands for a scalar field of the \( so(3) \) algebra.

To streamline the presentation, we introduce creation operators \( \alpha^I, \nu \) and the respective annihilation operators \( \bar{\alpha}^I, \bar{\nu} \) which we refer to as oscillators in this paper. The oscillators, the Hermitian conjugation rule, and the vacuum \(|0\rangle\) are defined by the relations

\[
[\bar{\alpha}^I, \alpha^J] = \delta^{IJ}, \quad [\bar{\nu}, \nu] = 1, \quad \alpha^I \dagger = \bar{\alpha}^I, \quad \nu \dagger = \bar{\nu},
\]

(2.5)

Using the oscillators \( \alpha^I, \nu \), we collect fields (2.4) into a ket-vector defined by

\[
|\phi_{1\ldots n} \rangle = \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \alpha_{1} \ldots \alpha_{n} \phi_{1\ldots n}(x) |0\rangle.
\]

(2.7)

Ket-vector (2.7) satisfies the algebraic constraints

\[
(N_{\alpha} - N_{\nu})|\phi_{1\ldots n} \rangle = 0, \quad N_{\alpha} \equiv \alpha^I \bar{\alpha}^I, \quad N_{\nu} \equiv \nu \bar{\nu},
\]

(2.8)

\[
\bar{\alpha}^2|\phi_{1\ldots n} \rangle = 0, \quad \alpha^2 \equiv \alpha^I \bar{\alpha}^I.
\]

(2.9)

From constraint (2.8), we learn that the expansion of the \(|\phi_{1\ldots n} \rangle \) into the oscillators \( \alpha^I \) and \( \nu \) involves only those terms of the expansion whose powers in the \( \alpha^I \) are equal to powers in the \( \nu \). Constraint (2.9) tells us that fields \( \phi_{1\ldots n} \) (2.4) are traceless tensor fields of the \( so(3) \) algebra.

2.3. Light-cone gauge action and its relativistic symmetries

In terms of ket-vector \(|\phi_{1\ldots n} \rangle \) (2.7), light-cone gauge action of mixed-symmetry continuous-spin field takes the form

\[
S = \int dx^+ dx^- d^3x \ L, \quad L = \langle \phi | (\Box - m^2) |\phi \rangle,
\]

(2.10)

\[
\Box = 2\partial^+ \partial^- + \partial^I \partial^I, \quad \partial^+ = \partial/\partial x^-, \quad \partial^- = \partial/\partial x^+, \quad \partial^I = \partial/\partial x^I,
\]

(2.11)

where a bra-vector \( \langle \phi \rangle \) in (2.10) is obtained from ket-vector \(|\phi_{1\ldots n} \rangle \) (2.7) by using the rule \( \langle \phi \rangle = |\phi \rangle \dagger \).

We now discuss the Poincaré algebra \( iso(4,1) \) symmetries of light-cone gauge action (2.10). As is known, the choice of the light-cone gauge spoils the manifest \( so(4,1) \) Lorentz algebra symmetries. Therefore, in order to show that the Poincaré algebra symmetries are still present, we should find an explicit realization of the Poincaré algebra symmetries on ket-vector \(|\phi_{1\ldots n} \rangle \) (2.7).

The representation for the generators of the Poincaré algebra in terms of differential operators acting on ket-vector \(|\phi_{1\ldots n} \rangle \) (2.7) is given by

\[
P^I = \partial^I, \quad P^+ = \partial^+,
\]

(2.12)

\[
J^{+-} = x^+ P^- - x^- P^+, \quad J^{+I} = x^+ \partial^I - x^I \partial^+, \quad J^{I+} = x^I \partial^+ - x^+ \partial^I, \quad M_{\mu},
\]

(2.13)

\[
P^- = -\frac{\partial^I \partial^I - m^2}{2\partial^+}, \quad J^{-I} = x^- \partial^I - x^I P^- + M^{-I},
\]

(2.14)
where operators $M^{-I}$, $M^{IJ}$ are defined as
\[ M^{-I} = M^{IJ} \frac{\partial}{\partial v^J} + \frac{1}{\partial v^I}, \]
\[ M^{IJ} = \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I, \] (2.16)

In (2.14) and (2.16), a quantity $M^{IJ}$ stands for a spin operator of the so(3) algebra. In (2.17), we present the well known realization of the $M^{IJ}$ on ket-vector $|\phi\rangle$ (2.7). Operator $M^{IJ}$ (2.16) does not depend on space-time coordinates and their derivatives. This operator acts only on spin indices of ket-vector $|\phi\rangle$ (2.7). The operator $M^I$ transforms as a vector of the so(3) algebra,
\[ [M^I, M^J] = \delta^{IJ} M^K - \delta^{IK} M^J, \] (2.18)
and satisfies the following commutation relations
\[ [M^I, M^J] = m^2 M^{IJ}. \] (2.19)

It is the equation (2.19) that are the basic equations of the light-cone gauge formulation of relativistic dynamics in the flat space. In the framework of the light-cone gauge formulation, the most difficult problem is to find a solution to the basic equation (2.19).

We now discuss our solution for the operator $M^I$ corresponding to the mixed-symmetry continuous-spin field in $R^{4,1}$. The solution for the operator $M^I$ we found is given by
\[ M^I = l S^I + g \alpha^I + A^I \bar{g}, \] (2.20)
\[ S^I \equiv \epsilon^{IJK} \alpha^J \bar{\alpha}^K, \] (2.21)
\[ A^I \equiv \alpha^I - \alpha^2 \frac{1}{2N_\alpha + 3} \bar{\alpha}^I, \] (2.22)
\[ N_\alpha \equiv \alpha^I \bar{\alpha}^I, \quad N_\nu \equiv \nu \bar{\nu}, \quad \alpha^2 \equiv \alpha^I \bar{\alpha}^I, \] (2.23)
\[ l \equiv \frac{i h_{2K}}{N_\nu (N_\nu + 1)}, \] (2.24)
\[ g \equiv g_\nu \bar{\nu}, \quad \bar{g} \equiv g \nu \bar{v}, \quad g_\nu \equiv \frac{((N_\nu + 1)^2 - h_2^2)}{(N_\nu + 1)^2 (2N_\nu + 3) F_\nu} \] \[ \frac{1}{2}, \] (2.25)
\[ F_\nu \equiv \kappa^2 - (N_\nu + 1)^2 m^2, \] (2.26)
where $\epsilon^{IJK}$ (2.21) stands for the Levi–Civita symbol of rank three with $\epsilon^{123} = 1$. In (2.24) and (2.26), a quantity $\kappa$ stands for a dimensionful constant parameter.

The following remarks are in order.

(i) From (2.24)–(2.26), we see that the mixed-symmetry continuous-spin field in $R^{4,1}$ space is labeled by three parameters: one integer $h_2 \in \mathbb{N}$, and two dimensionful parameters $m$ and $\kappa$.

(ii) In this paper, field $|\phi\rangle$ (2.7) with $m = 0$ is referred to as a massless continuous-spin field, while field $|\phi\rangle$ (2.7) with $m \neq 0$ is referred to as a massive continuous-spin field$^5$.

$^5$ For the massless continuous-spin field propagating in $R^{4,1}$, discussion about the operator $M^I$ can also be found in section 2 in [3].
(iii) If $\kappa h_2 = 0$, then, from (2.24), we see that operator $M^I$ (2.20) becomes real-valued. Therefore complex-valued fields (2.4) can be restricted to being real-valued. This case corresponds to the totally symmetric continuous-spin field.

(iv) If $\kappa h_2 \neq 0$, then considering the $lS^I$ term (2.24) and requiring the operator $M^I$ (2.20) to be Hermitian, we find that $\kappa h_2$ should be real-valued. This implies that the $\kappa$ should be real-valued. For definiteness, we assume that the $\kappa$ is strictly positive. Thus we have the classification

$$\kappa > 0, \quad h_2 \in \mathbb{N}, \text{ for a mixed-symmetry field;}$$

$$\kappa h_2 = 0, \quad h_2 \in \mathbb{N}_0, \text{ for a totally symmetric field.}$$

(v) Using (2.12)–(2.15), we verify that the light-cone gauge action (2.10) is invariant under the transformations of the Poincaré algebra $iso(4, 1)$ given by

$$\delta G|\phi\rangle = G|\phi\rangle,$$

where $G$ appearing on the r.h.s (2.29) stands for the differential operators given in (2.12)–(2.15).

As is known, the Poincaré algebra $iso(4, 1)$ has three independent Casimir operators. Now our aim is to express eigenvalues of the Casimir operators in terms of the three parameters $m$, $\kappa$, $h_2$.

### 2.4. Casimir operators of the Poincaré algebra $iso(4, 1)$

The three independent Casimir operators, which we denote as $C_2$, $C_{c3}$, $C_4$, can be expressed in terms of the generators of the Poincaré algebra (2.1) as

$$C_2 = P^a P_a,$$  \hspace{1cm} (2.30)

$$C_{c3} = -\frac{i}{8} \epsilon^{a_1 \ldots a_5} J^{a_1 a_2} P^{a_3} P^{a_4},$$  \hspace{1cm} (2.31)

$$C_4 = J^{abc} J^{def} P^c P^d P^e - \frac{1}{2} P^a P^b J^{a b},$$  \hspace{1cm} (2.32)

where $\epsilon^{a_1 \ldots a_5}$ (2.31) stands for the Levi–Civita symbol of rank five with $\epsilon^{01234} = 1$. Note that operator $C_4$ (2.32) admits the following representation:

$$C_4 = \frac{1}{8} C_{c2} C_{c2}, \quad C_{c2} \equiv \epsilon^{abcde} P^a J^b J^c J^d J^e.$$  \hspace{1cm} (2.33)

Plugging the generators of the Poincaré algebra (2.12)–(2.15) into (2.30)–(2.32), we find that the operator $C_2$ (2.30) is diagonalized,

$$C_2 = m^2,$$  \hspace{1cm} (2.34)

while the operators $C_{c3}$, $C_4$ (2.31) and (2.32) take the form

$$C_{c3} = \frac{i}{2} \epsilon^{ijk} M^i M^j M^k,$$

$$C_4 = M^I M^I - \frac{1}{2} m^2 M^I M^I.$$  \hspace{1cm} (2.35)
Finally, plugging the operators $M^I$, $M^J$ (2.17) and (2.20) into (2.35) and (2.36), we find that the operators $C_{\epsilon 3}$, $C_4$ are also diagonalized,

$$C_{\epsilon 3} = \kappa h_2,$$

(2.37)

$$C_4 = \kappa^2 + m^2 (h_2^2 - 1).$$

(2.38)

From (2.34), (2.37) and (2.38), we see how the eigenvalues of the three Casimir operators $C_2$, $C_{\epsilon 3}$, $C_4$ are expressed in terms of the three parameters $m$, $\kappa$, $h_2$. Note that eigenvalues of the $C_2$ for the ket-vector $|\phi\rangle$ and the bra-vector $\langle \phi| \equiv |\phi\rangle^\dagger$ are equal. The same holds true for eigenvalues of the $C_4$. Contrary to this, the eigenvalue of $C_{\epsilon 3}$ (2.35) for the ket-vector $|\phi\rangle$ is equal to $\kappa h_2$, while the eigenvalue of the $C_{\epsilon 3}$ for the bra-vector $\langle \phi|$ is equal to $-\kappa h_2^6$.

### 2.5. Irreducible classically unitary mixed-symmetry continuous-spin field

A detailed definition of classical unitarity and irreducibility of field dynamics may be found below in section 4. Briefly speaking, for the mixed-symmetry continuous-spin field in the flat space, classical unitarity amounts to the two conditions: (a) the operator $M^I$ (2.20) should be Hermitian; (b) the $F_\nu$ (2.26) should be non-negative, $F_\nu \geq 0$, for all $N_\nu = h_2, h_2 + 1, \ldots, \infty$. The irreducibility amounts to the condition $F_\nu \neq 0$, for all $N_\nu = h_2, h_2 + 1, \ldots, \infty$. If, for some value of $N_\nu = s$, the $F_\nu$ is equal to zero, then the field dynamics is referred to as reducible field dynamics.

Let us first discuss the irreducible classically unitary mixed-symmetry continuous-spin field. As we noted earlier, the Hermicity of operator $M^I$ (2.20) implies that $\kappa$ should be real-valued. Note also that, for the mixed-symmetry field, $\kappa \neq 0$ (2.27). Taking this into account, we see that requiring $F_\nu > 0$ for all $N_\nu = h_2, h_2 + 1, \ldots, \infty$, we find the inequality $m^2 \leq 0$. The cases $m^2 = 0$ and $m^2 \neq 0$ we refer to as massless and massive continuous-spin fields, respectively. Thus, we see that the massive continuous-spin field has tachyonic mass. In [8], we conjectured that the massive continuous-spin field is associated with the tachyonic UIR of the Poincaré algebra. Discussion on the tachyonic UIR of the Poincaré algebra may be found in [2].

### 2.6. Reducible mixed-symmetry continuous-spin field

Now let us discuss the reducible mixed-symmetry continuous-spin field. Requiring $F_\nu|_{N_\nu = h_1} = 0$, we find the relation

$$\kappa^2 = (h_1 + 1)^2 m^2.$$

(2.39)

Plugging $\kappa^2$ (2.39) into (2.26), we get

$$F_\nu = (h_1 - N_\nu)(h_1 + 2 + N_\nu)m^2.$$

(2.40)

Using $F_\nu$ (2.40), we can verify that Lagrangian (2.10) and Poincaré algebra transformations (2.29) describe the reducible mixed-symmetry continuous-spin field. Namely, decomposing $|\phi\rangle$ (2.7) as

$$|\phi\rangle = |\phi^{h_2, h_1}\rangle + |\phi^{h_1 + 1, \infty}\rangle,$$

(2.41)

First, in the framework of the light-cone approach, Casimir operator $C_{\epsilon 3}$ (2.35) and its eigenvalue in (2.37) were obtained in [3].
\[ |\phi^{MN}\rangle \equiv \sum_{n=0}^{N} \frac{\epsilon^n}{n! \sqrt{n!}} \phi^h \cdots \phi^h(x) |0\rangle, \]  
(2.42)

we can verify that Lagrangian (2.10) is factorized as
\[ \mathcal{L} = \mathcal{L}^{h_2, h_1} + \mathcal{L}^{h_1+1, \infty}, \]  
(2.43)

\[ \mathcal{L}^{MN} \equiv \langle \phi^{MN} | (\Box - m^2) | \phi^{MN} \rangle. \]  
(2.44)

This is to say that \( \mathcal{L}^{h_2, h_1} \) and \( \mathcal{L}^{h_1+1, \infty} \) (2.43) are invariant under the Poincaré algebra transformations (2.29). Using (2.40) and considering \( m^2 > 0 \), we see that \( F_\nu > 0 \) when \( N_\nu = h_2, h_2 + 1, \ldots, h_1 - 1 \) and \( F_\nu < 0 \) when \( N_\nu = h_1 + 1, h_1 + 2, \ldots, \infty \). This implies that, for \( m^2 > 0 \), the \( |\phi^{h_2, h_1}\rangle \) (2.41) describes a classically unitary massive finite-component field, while the \( |\phi^{h_1+1, \infty}\rangle \) (2.41) describes a classically non-unitary infinite-component field. Note also that, for \( m^2 < 0 \), the \( \kappa \) becomes imaginary in view of (2.39). This implies that, for \( m^2 < 0 \), both the \( |\phi^{h_2, h_1}\rangle \) and \( |\phi^{h_1+1, \infty}\rangle \) (2.41) describe classically non-unitary mixed-symmetry massive fields.

### 2.7 Totally symmetric continuous-spin field

From (2.28), we see that the totally symmetric continuous-spin field is realized by considering the following two cases:

\[ h_2 = 0, \quad \kappa \text{ - arbitrary}, \]  
(2.45)

\[ h_2 \neq 0, \quad \kappa = 0. \]  
(2.46)

Case \( h_2 = 0, \kappa\)-arbitrary. Setting \( h_2 = 0 \) in (2.7), we get ket-vector \(|\phi\rangle\) entering Lagrangian for a totally symmetric field (2.10). Also, setting \( h_2 = 0 \) in (2.20)–(2.26), we see that the operator \( M^I \) is simplified as
\[ M^I = g^I \bar{\alpha} + A^I \bar{g}, \]  
(2.47)

\[ g = g_\nu \bar{g}, \quad \bar{g} = v_\nu g_\nu, \quad g_\nu = \left( \frac{F_\nu}{(N_\nu + 1)(2N_\nu + 3)} \right)^{1/2}, \]  
(2.48)

\[ F_\nu = \kappa^2 - (N_\nu + 1)^2 m^2, \]  
(2.49)

where the operators \( A^I, N_\nu \) take the same form as in (2.22) and (2.23).

The following remarks are in order:

(i) As the \( IS^I \)-term (2.20) does not appear in (2.47), the Hermitian operator \( M^I \) (2.47) turns out to be real-valued. For this reason the complex-valued fields (2.4) can be restricted to being real-valued.

(ii) For the mixed-symmetry field, the Hermicity of the operator \( M^I \) (2.47), in view of the \( IS^I \)-term (2.47), implies that the \( \kappa \) should be real-valued. For the totally symmetric field, the \( IS^I \)-term (2.20) does not appear in (2.47). Therefore, for the totally symmetric field, the Hermicity of the operator \( M^I \) (2.47) does not imply that only real-valued \( \kappa \) is admitted.
(iii) As the $lS^l$-term (2.20) does not appear in (2.47), all that is required for the classical unitarity and irreducibility of the totally symmetric field is to respect the condition $F_\nu > 0$ for all $N_\nu = 0, 1, \ldots, \infty$. From (2.49), we see that the above-mentioned condition is satisfied provided

$$\kappa^2 > m^2, \quad m^2 \leq 0. \quad (2.50)$$

From (2.50), we see that, for massless field, $m = 0$, the parameter $\kappa$ should be real-valued, while, for a massive field, $m^2 < 0$, the parameter $\kappa$ can be real-valued or purely imaginary.

(iv) To get the reducible totally symmetric field we consider equation $F_\nu |_{\nu_i = s} = 0$. The solution of this equation is given by $\kappa^2 = (s + 1)^2 m^2$. Using such a solution in (2.49), we get

$$F_\nu = (s - N_\nu)(s + 2 + N_\nu)m^2. \quad (2.51)$$

Now decomposing $|\phi\rangle$ (2.7) as

$$|\phi\rangle = |\phi^{0, s}\rangle + |\phi^{s+1, \infty}\rangle, \quad (2.52)$$

where $|\phi^{0, s}\rangle, |\phi^{s+1, \infty}\rangle$ are defined as in (2.42), we can verify that Lagrangian (2.10) is factorized as

$$\mathcal{L} = \mathcal{L}^{0, s} + \mathcal{L}^{s+1, \infty}, \quad (2.53)$$

where $\mathcal{L}^{0, s}, \mathcal{L}^{s+1, \infty}$ are defined as in (2.44). This is to say that $\mathcal{L}^{0, s}$ and $\mathcal{L}^{s+1, \infty}$ (2.53) are invariant under the Poincaré algebra transformations (2.29). Using (2.51) and considering $m^2 > 0$, we verify that $F_\nu > 0$ when $N_\nu = 0, 1, \ldots, s - 1$ and $F_\nu < 0$ when $N_\nu = s + 1, s + 2, \ldots, \infty$. This implies that, for $m^2 > 0$, the $|\phi^{0, s}\rangle$ in (2.52) describes a classically unitary massive spin-$s$ field, while the $|\phi^{s+1, \infty}\rangle$ in (2.52) describes a classically non-unitary infinite-component field. On the contrary, for $m^2 < 0$, we have $F_\nu < 0$ when $N_\nu = 0, 1, \ldots, s - 1$ and $F_\nu > 0$ when $N_\nu = s + 1, s + 2, \ldots, \infty$. Therefore, for $m^2 < 0$, the $|\phi^{0, s}\rangle$ in (2.52) describes a classically non-unitary massive spin-$s$ field, while the $|\phi^{s+1, \infty}\rangle$ in (2.52) describes a classically unitary infinite-component field.

Case $h_2 \neq 0, \kappa = 0$. The totally symmetric field given in (2.46) turns out to be equivalent to the infinite component field $|\phi^{s+1, \infty}\rangle$ (2.52). To see this we set $\kappa = 0$ in (2.20)–(2.26) and, on the one hand, we obtain the operator $\mathcal{M}$ as in (2.47) and (2.48) with the following expression for $g_v$

$$g_v = \left( \frac{h_2^2 - (N_\nu + 1)^2}{(N_\nu + 1)(2N_\nu + 3) m^2} \right)^{1/2}. \quad (2.54)$$

On the other hand, the infinite component $|\phi^{s+1, \infty}\rangle$ (2.52) is described by $g_v$ given in (2.48) with $F_v$ as in (2.51),

$$g_v = \left( \frac{(s + 1)^2 - (N_\nu + 1)^2}{(N_\nu + 1)(2N_\nu + 3) m^2} \right)^{1/2}. \quad (2.55)$$

Making the identification $h_2 = s + 1$, we see that expressions for $g_v$ in (2.54) and (2.55) coincide.
3. Continuous-spin mixed-symmetry field in AdS$_5$ space

3.1. Notation and conventions

Relativistic symmetries of field dynamics in AdS$_5$ space are described by the $so(4,2)$ algebra. We use the following commutators for generators of the $so(4,2)$ algebra

$$
[D,P^a] = -P^a, \quad [P^a,J^{bc}] = \eta^{ab}K^c - \eta^{ac}K^b,
$$

$$
[D,K^a] = K^a, \quad [P^a,K^b] = \eta^{ab}D - J^{ab},
$$

$$
[J^{ab},J^{ce}] = \eta^{bc}J^{ae} + 3 \text{ terms},
$$

(3.1)

where $\eta^{ab}$ stands for the mostly positive flat metric tensor. The vector indices of the $so(3,1)$ Lorentz algebra take values $a,b = 0,1,2,3$. The generators $P^a$, $K^a$, $D$, $J^{ab}$ are assumed to be anti-Hermitian.

We use the Poincaré parametrization of AdS$_5$ space,

$$
dx^2 = R^2 \left( -dx^0dx^0 + dx^idx^i + dx^3dx^3 + dzdz \right),
$$

(3.2)

where $R$ is the radius of the AdS$_5$ space. The vector indices of the $so(2)$ algebra take the values $i,j,k = 1,2$. The light-cone frame coordinates $x^\pm$, $x'$ and their derivatives $\partial^\pm$, $\partial'$ are defined as

$$
x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad x' = x^i, z,
$$

(3.4)

$$
\partial^+ = \partial/\partial x^-, \quad \partial^- = \partial/\partial x^+, \quad \partial' = \partial / \partial x^i, \quad \partial = \partial / \partial z,
$$

(3.5)

where $x^+$ is taken to be the light-cone time. In the light-cone frame, the $so(3,1)$ Lorentz algebra vector $X^a$ is decomposed as $X^+, X^-, X'$. A scalar product of the $so(3,1)$ Lorentz algebra vectors $X^a$ and $Y^b$ is represented then as

$$
\eta_{ab}X^a Y^b = X'^+ Y'^- + X'^- Y'^+ + X'^i Y'^i.
$$

(3.6)

From (3.6), we see that, in the light-cone frame, nonvanishing elements of the flat metric $\eta_{ab}$ are given by $\eta_{++} = 1$, $\eta_{--} = 1$, $\eta_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta symbol. Therefore, in the light-cone frame, commutators for generators of the $so(4,2)$ algebra are obtained from the ones in (3.1) by using values of the inverse flat metric given by $\eta_{++} = 1$, $\eta_{--} = 1$, $\eta_{ij} = \delta_{ij}$.

In accordance with the decomposition for the coordinates $x'$ (3.4), the $so(3)$ algebra vector $X'$ is decomposed as $X'$, $X'$. A scalar product of the $so(3)$ algebra vectors $X' = X'$, $X'$ and $Y' = Y'$, $Y'$ is represented then as

$$
X'^i Y'^j = X'^i Y'^j + X'^i Y'^j.
$$

(3.7)

In what follows, the notation $\delta^{ij}$ stands for the Kronecker delta symbol. This symbol is decomposed as $\delta^{ij} = \delta^{ij}, \delta^{zz}$, where $\delta^{zz} = 1$.

3.2. Field content

To discuss the light-cone gauge description of a mixed-symmetry continuous-spin field propagating in AdS$_5$ space we use the following set of complex-valued fields of the $so(3)$ algebra...
where \( h_2 \in \mathbb{N} \) is an integer which labels the mixed-symmetry continuous-spin field. In (3.8), a field with \( n = 1 \) is a vector field of the \( so(3) \) algebra, while a field with \( n \geq 2 \) is a totally symmetric rank-\( n \) traceless tensor field of the \( so(3) \) algebra. Note that, in view of \( h_2 \in \mathbb{N} \), fields \( \phi^{I_1\ldots I_n} \) with \( n = 0, 1, \ldots, h_2 - 1 \) do not enter the field content of the mixed-symmetry continuous-spin field (3.8). Also note that field \( \phi^{I_1\ldots I_n} \) with \( n = 0 \) stands for a scalar field of the \( so(3) \) algebra.

Using the oscillators \( \alpha^I, \nu \) (2.5) and (2.6), we collect fields (3.8) into a ket-vector defined by

\[
|\phi\rangle = \sum_{n=h_2}^{\infty} \frac{\nu^n}{n!^2} \alpha^{I_1} \ldots \alpha^{I_n} \phi^{I_1\ldots I_n}(x, z)|0\rangle.
\]  

(Ket-vector (3.9) satisfies the algebraic constraints

\[
(N_\alpha - N_\nu)|\phi\rangle = 0, \quad N_\alpha \equiv \alpha^I \bar{\alpha}^I, \quad N_\nu \equiv \nu \bar{\nu},
\]  

(3.10)

\[
\bar{\alpha}^2|\phi\rangle = 0, \quad \bar{\nu}^2 \equiv \nu \bar{\nu}.
\]  

(3.11)

Constraint (3.11) amounts to the requirement that fields \( \phi^{I_1\ldots I_n} \) (3.8) are traceless tensor fields of the \( so(3) \) algebra.

### 3.3. Light-cone gauge action

In terms of ket-vector \(|\phi\rangle \) (3.9), light-cone gauge action of the mixed-symmetry continuous-spin field takes the form

\[
S = \int d^4x \mathcal{L}, \quad d^4x = dx^+ dx^- d^2x,
\]  

(3.12)

\[
\mathcal{L} = \langle \phi| (\Box - \frac{1}{2} A)|\phi\rangle, \quad \Box = 2 \partial^+ \partial^- + \partial^I \partial^I + \partial^z \partial^z,
\]  

(3.13)

where the D’Alembertian operator (3.13) takes the same form as in the flat space. In (3.13), a bra-vector \( \langle \phi \) is obtained from ket-vector \( |\phi\rangle \) (3.9) by using the rule \( \langle \phi | = |\phi\rangle^\dagger \). An operator \( A \) appearing in (3.13) does not depend on the space-time coordinates and their derivatives. This operator acts only on spin indices of the ket-vector \( |\phi\rangle \) (3.9). For the reader’s convenience, we note that for a massive scalar field in AdS\(_d+1\), the operator \( A \) takes the form

\[
A = m^2 R^2 + \frac{d^2 - 1}{4},
\]  

(3.14)

where \( m \) stands for the mass parameter of the scalar field.

### 3.4. Realization of relativistic symmetries

As relativistic symmetries of fields in AdS\(_d\) space are described by the \( so(4, 2) \) algebra we now discuss the \( so(4, 2) \) algebra symmetries of light-cone gauge action (3.12). A choice of the light-cone gauge spoils the manifest \( so(3, 1) \) Lorentz algebra symmetries. Therefore, in order to demonstrate that the symmetries of \( so(4, 2) \) algebra are still present we should find an explicit realization of the \( so(4, 2) \) algebra symmetries on the ket-vector \(|\phi\rangle \) (3.9). Now, using
the general light-cone gauge approach in [23], we proceed to a discussion of the light-cone
gauge realization of the $so(4, 2)$ algebra symmetries on the ket-vector $|\phi\rangle$ (3.9).

The representation for the generators of the $so(4, 2)$ algebra in terms of differential operators acting on the ket-vector $|\phi\rangle$ (3.9) is given by

$$P^i = \partial^i, \quad P^+ = \partial^+, \quad (3.15)$$

$$J^{+-} = x^+ P^+ - x^- \partial^+, \quad J^{++} = x^+ \partial^+ - x^i \partial^i, \quad (3.16)$$

$$J^{ij} = x^i \partial^j - x^j \partial^i + M^{ij}, \quad (3.17)$$

$$D = x^+ P^- + x^- \partial^+ + x^i \partial^i + \frac{3}{2}, \quad (3.18)$$

$$K^+ = -\frac{1}{2} (2x^+ x^- + x^i x^j) \partial^+ + x^+ D, \quad (3.19)$$

$$K^i = -\frac{1}{2} (2x^+ x^- + x^i x^j) \partial^i + x^i D + M^{ij} x^j + M^- x^+, \quad (3.20)$$

$$P^- = -\frac{\partial^i \partial^j}{2 \partial^+} + \frac{1}{2z^+ \partial^+} A, \quad (3.21)$$

$$J^{-i} = x^- \partial^i - x^i \partial^+ + M^- i, \quad (3.22)$$

$$K^- = -\frac{1}{2} (2x^+ x^- + x^i x^j) P^- + x^- D + \frac{1}{\partial^+} x^i \partial^j M^{ij} - \frac{x^j}{2z^+ \partial^+} [M^{ij}, A] + \frac{1}{\partial^+} B, \quad (3.23)$$

where

$$M^{-i} \equiv M^{ij} \frac{\partial^j}{\partial^+} - \frac{1}{2z^+ \partial^+} [M^{ij}, A], \quad M^{-i} = -M^{i-}. \quad (3.24)$$

From (3.15)–(3.24), we see that the differential operators are expressed in terms of space-time coordinates $x^i$, $x^\pm$, the spatial derivatives $\partial^i$, $\partial^\pm$, and operators $A$, $B$, $M^{ij}$. The operators $A$, $B$, $M^{ij}$ are independent of the space-time coordinates and the space-time derivatives. These operators act only on spin indices of the ket-vector $|\phi\rangle$. We now turn to discussion of the realization of the operators $A$, $B$, $M^{ij}$ on the ket-vector $|\phi\rangle$ (3.9).

The operators $M^{ij}$ are spin operators of the $so(3)$ algebra, while the operator $M^i$ is the spin operator of the $so(2)$ algebra. Realization of the spin operator $M^{ij}$ on the space of ket-vector $|\phi\rangle$ (3.9) is well-known,

$$M^{ij} = \alpha^i \alpha^j - \alpha^j \alpha^i. \quad (3.25)$$

In [23], we found the following general representation for the operators $A$ and $B$:

$$A = C_2 + 2B^i + 2M^{ij} M^{ij} + \frac{1}{2} M^{ij} M^{ij} + \frac{15}{4}, \quad (3.26)$$

$$B = B^i + M^{ij} M^{ij}, \quad (3.27)$$

where $C_2$ in (3.26) stands for an eigenvalue of the second-order Casimir operator of the $so(4, 2)$ algebra, while $B^i$ in (3.26) and (3.27) stands for the $z$-component of a vector operator.
$B^l = B^l, B^r$. The operator $B^l$ acts only on spin indices of the ket-vector $|\phi\rangle$ (3.9) and transforms as a vector of the $so(3)$ algebra,

$$[B^l, M^{jk}] = \delta^{lj} B^k - \delta^{lk} B^j. \quad (3.28)$$

Besides this, the operator $B^l$ should satisfy the following defining equations:

$$[B^l, B^j] = (C_2 + M^2 + 4)M^{lj}, \quad (3.29)$$

$$M^2 \equiv M^{lj}M^{lj}, \quad (3.30)$$

where $C_2$ (3.29) is an eigenvalue of the second-order Casimir operator of the $so(4, 2)$ algebra. It is the equations (3.29) that are the basic equations of light-cone gauge formulation of relativistic dynamics in $AdS_5$ space. The basic equations (3.29) are the AdS counterparts of the ones in the flat space (2.19). We see that, in the flat space, the basic equations (2.19) are governed by the eigenvalue of the second-order Casimir operator of the Poincaré algebra, $C_2 = m^2$, (2.30) and (2.34), while, in $AdS_5$ space, the basic equations (3.29) are governed by the eigenvalue of the second-order Casimir operator of the $so(4, 2)$ algebra $C_2$ (for a brief review of the Casimir operators, see appendix A).

Finding a solution of the basic equation (3.29) is the most difficult point in the framework of the light-cone gauge formulation of relativistic dynamics in $AdS_5$ space. We find the following operator $B^l$ which satisfies equation (3.29)

$$B^l = lS^l + g\bar{\alpha}^l + A^l\bar{g}, \quad (3.31)$$

$$S^l \equiv \epsilon^{ijk}\bar{\alpha}^i\bar{\alpha}^k, \quad (3.32)$$

$$A^l \equiv \alpha^l - \alpha^2 \frac{1}{2N_{\alpha} + 3}\bar{\alpha}^l, \quad (3.33)$$

$$N_\alpha \equiv \alpha^l\bar{\alpha}^l, \quad N_\nu = \nu\bar{\nu}, \quad \alpha^2 = \alpha^l\alpha^l, \quad (3.34)$$

$$l \equiv \frac{i\hbar^2\kappa}{N_\nu (N_\nu + 1)} \quad (3.35)$$

$$g \equiv g_{\nu\bar{\nu}}, \quad \bar{g} = g_{\bar{\nu}\nu}, \quad g_{\nu\bar{\nu}} = \left[ \frac{(N_\nu + 1)^2 - \hbar_2^2}{(N_\nu + 1)^2}(2N_\nu + 3)F_{\nu}\right]^{1/2}, \quad (3.36)$$

$$F_{\nu} \equiv \kappa^2 - (C_2 - \hbar_2^2 + 5)(N_\nu + 1)^2 + (N_\nu + 1)^4, \quad (3.37)$$

where $\epsilon^{ijk}$ (3.32) stands for the Levi–Civita symbol of rank three with $\epsilon^{123} = 1$. In (3.35) and (3.37), a quantity $\kappa$ stands for a dimensionless constant parameter, while the $C_2$ is an eigenvalue of the second-order Casimir operator of the $so(4, 2)$ algebra. Note that $C_2$ is also dimensionless. Thus we see that the mixed-symmetry continuous-spin field in $AdS_5$ space is labeled by three parameters: one integer $\hbar_2 \in \mathbb{N}$, and two dimensionless parameters, $C_2$ and $\kappa$. Helpful formulas for the operators $S^l, A^l, M^{lj}$ may be found in appendix B.

The following remarks are in order.

(i) If $\kappa \hbar_2 = 0$, then, using (3.31) and (3.35), we see that the operator $B^l$ becomes real-valued. For such $\kappa$ and $\hbar_2$, the complex-valued fields (3.8) can be restricted to being real-valued and this case corresponds to totally symmetric fields.
(ii) If $\kappa h^2 \neq 0$, requiring the operator $B^I$ (3.31) to be Hermitian, we find that $\kappa h^2$ should be real-valued, i.e. $\kappa$ should be real-valued. To be definite let us assume that, for the ket-vector $|\phi\rangle$ (3.9), $\kappa$ is strictly positive. Thus we have the following classification:

\begin{align}
\kappa > 0, \quad h^2 \in \mathbb{N}, \text{ for a mixed-symmetry field;} \\
\kappa h^2 = 0, \quad h^2 \in \mathbb{N}_0, \text{ for a totally symmetric field.}
\end{align}

(iii) We recall that, in section 3, to study the mixed-symmetry continuous-spin field in $R^{4,1}$ space we used one integer $h^2 \in \mathbb{N}$ and two dimensionful parameters $m$ and $\kappa$, while, in this section, to study the mixed-symmetry continuous-spin field in AdS$_5$ space, we use one integer $h^2 \in \mathbb{N}$ and two dimensionless parameters, $C_2$ and $\kappa$. Obviously, for the case of AdS$_5$ space, the possibility of using of the dimensionless parameters $C_2$ and $\kappa$ is related to the radius of the AdS$_5$ space. Namely, as the radius of the AdS$_5$ space $R$ is a dimensionful parameter, all dimensionful parameters entering the game can be made dimensionless by multiplying them with a suitable power of the $R$.

(iv) Using the notation $n_{\text{AdS}}, C_{2\text{AdS}}$ and $n_{\text{flat}}, m_{\text{flat}}$ for the parameters entering the respective actions of the continuous-spin fields in AdS and flat spaces, we note that, for large $R$, these parameters are related as

\begin{align}
C_{2\text{AdS}}_{|R \to \infty} &\to R^2m_{\text{flat}}^2, \\
\kappa_{\text{AdS}}_{|R \to \infty} &\to R\kappa_{\text{flat}}.
\end{align}

Using (3.40), we note then that, for the large $R$, the operator $B^I$ (3.31) in AdS space and the operator $M^I$ (2.20) in flat space are related as

\begin{align}
B^I_{|R \to \infty} &\to RM^I.
\end{align}

Taking into account that the spin operator $M^I$ of the $so(3)$ algebra takes the same form in AdS space (3.25) and flat space (2.17), and using (3.40) and (3.41), we learn that, for the large radius $R$, the basic equation (3.29) in AdS space become the basic equation (2.19) in flat space.

(v) We verify that the light-cone gauge action (3.12) is invariant under the transformations of the $so(4,2)$ algebra given by

\begin{align}
\delta_G |\phi\rangle = G |\phi\rangle,
\end{align}

where $G$ appearing on r.h.s (3.42) stands for differential operators given in (3.15)–(3.23).

(vi) For the case of a finite-component mixed-symmetry massive field in AdS$_5$ space, detailed exposition of the procedure for solving the basic equation (3.29) may be found in appendix C in [24]. Adaptation of the procedure in appendix C in [24] to the case of a continuous-spin mixed-symmetry field in AdS$_5$ space is straightforward. Useful relations for various spin operators needed for the analysis of the basic equation (3.29) are presented in appendix B in this paper.

On the one hand, the $so(4,2)$ algebra has three independent Casimir operators. On the other hand, we found that the Lagrangian of the mixed-symmetry continuous-spin field in AdS$_5$ space depends on three parameters: one integer, $h^2 \in \mathbb{N}$, one dimensionless real-valued parameter, $\kappa$, and the eigenvalue of the second-order Casimir operator $C_2$. Our aim is to express eigenvalues of third-order and fourth-order Casimir operators of the $so(4,2)$ algebra in terms of the $h^2$, $\kappa$, and $C_2$. To this end we now present our new results for the light-cone gauge representation for third-order and fourth-order Casimir operators of the $so(4,2)$ algebra, which has not been discussed in [23, 24].
3.5. Casimir operators of the so(4, 2) algebra

We find that third-order and fourth-order Casimir operators of the so(4, 2) algebra, denoted as $C_{\epsilon, 3}$ and $C_4$ respectively, can be expressed in terms of eigenvalues of the second-order Casimir operator $C_2$ and the spin operators $B_I$, $M_{IJ}$ as follows

$$C_{\epsilon, 3} = \frac{i}{2} \epsilon^{IJK} B_I M^{JK},$$

(3.43)

$$C_4 = B_I B_I - \frac{1}{2} (C_2 + 2) M^2 - \frac{1}{4} (M^2)^2, \quad M^2 \equiv M^{IJ} M^{IJ}.$$  

(3.44)

The statement that operators $C_{\epsilon, 3}$, $C_4$ (3.43) and (3.44) are indeed Casimir operators of the so(4, 2) algebra can be directly verified using (3.28) and (3.29). See also helpful relations in appendix B. For the reader’s convenience, in appendix A, we briefly review a manifestly six-dimensional covariant representation for the Casimir operators of the so(4, 2) algebra.

Plugging the operators $M_{IJ}$, $B_I$ (3.25) and (3.31) into (3.43) and (3.44), we find that the operators $C_{\epsilon, 3}$, $C_4$ are diagonalized,

$$C_{\epsilon, 3} = \kappa \hbar^2,$$

(3.45)

$$C_4 = \kappa^2 + (C_2 - \hbar^2 + 4)(\hbar^2 - 1).$$

(3.46)

From (3.45) and (3.46), we see how the eigenvalues of the Casimir operators $C_{\epsilon, 3}$, $C_4$ are expressed in terms of $\kappa$, $\hbar^2$ and $C_2$. Note that eigenvalues of the $C_2$ for the ket-vector $|\phi\rangle$ and the bra-vector $\langle\phi|$ are the same. The same holds true for eigenvalues of the $C_4$. Contrary to this, the eigenvalue of the $C_{\epsilon, 3}$ (3.43) for the ket-vector $|\phi\rangle$ is equal to $\kappa \hbar^2$ (3.45), while the eigenvalue of the $C_{\epsilon, 3}$ (3.43) for the bra-vector $\langle\phi|$ is equal to $-\kappa \hbar^2$. By definition, the ket-vector $|\phi\rangle$ and the bra-vector $\langle\phi|$ have one and the same label, $h^2$. This implies that ket-vector $|\phi\rangle$ (3.9) is related to the representation of the so(4, 2) algebra labeled by $C_2$, $\kappa$, $\hbar^2$, while the bra-vector $\langle\phi|$ is related to the representation of the so(4, 2) algebra labeled by $C_2$, $-\kappa$, $\hbar^2$.

3.6. Totally symmetric continuous-spin field

From (3.39), we learn that a totally symmetric continuous-spin field is realized by considering the following two cases:

$$h_2 = 0, \quad \kappa - \text{arbitrary},$$

(3.47)

$$h_2 \neq 0, \quad \kappa = 0.$$  

(3.48)

Case $h_2 = 0$, $\kappa$-arbitrary. Setting $h_2 = 0$ in (3.9), we get ket-vector $|\phi\rangle$ entering Lagrangian for the totally symmetric field (3.13). Also, setting $h_2 = 0$ in (3.31)–(3.37), we see that the operator $B^I$ is simplified as

$$B^I = g\tilde{\alpha}^I + A^I \tilde{g},$$

(3.49)

$$g = g_{uv}, \quad \tilde{g} = v g_v, \quad g_v = \left( \frac{F_v}{(N_v + 1)(2N_v + 3)} \right)^{1/2}, \quad F_v \equiv \kappa^2 - (C_2 + 5)(N_v + 1)^2 + (N_v + 1)\hat{\kappa},$$

(3.50)

$$F_v \equiv \kappa^2 - (C_2 + 5)(N_v + 1)^2 + (N_v + 1)\hat{\kappa}. $$

(3.51)
where the operators $\Lambda^I$, $N_\nu$ are defined in (3.32) and (3.33).

The following remarks are in order:

(i) As the $\mathcal{LS}$ term (3.31) does not appear in (3.49), the Hermitian operator $B^I$ (3.49) turns out to be real-valued. Therefore the complex-valued fields (3.8) can be restricted to being real-valued.

(ii) We recall that, for the mixed-symmetry field, the Hermicity of the operator $B^I$ (3.31) implies, in view of the $\mathcal{LS}$ term (3.31), that the $\kappa$ should be real-valued. For the totally symmetric field, the $\mathcal{LS}$ term (3.31) does not appear in (3.49). Therefore, for the totally symmetric field, the Hermicity of the operator $B^I$ (3.49) does not imply that only real-valued $\kappa$ are admitted. Namely, from (3.49)–(3.51), we see that the Hermicity of the operator $B^I$ implies that $\kappa^2$ should be real-valued. In other words, for the totally symmetric field, the $\kappa$ can be real-valued or purely imaginary. For more discussion on this, see section 4.

Case $h_2 \neq 0$, $\kappa = 0$. In section 4, we will demonstrate that this case is realized as some infinite-component field entering a reducible continuous-spin field that has $h_2 = 0$, $\kappa \neq 0$.

4. (Ir)reducible classically unitary mixed-symmetry continuous-spin field in $\text{AdS}_5$ space

The Lagrangian of the mixed-symmetry continuous-spin field in $\text{AdS}_5$ (3.13) depends on the three parameters $C_2$, $\kappa$ and $h_2$. We now discuss restrictions imposed on these parameters for irreducible and reducible classically unitary dynamical systems. We start with our definition of classically unitary reducible and irreducible systems.

A light-cone gauge Lagrangian (3.13) is constructed out of complex-valued fields. In order for the light-cone gauge action be real-valued the parameter $\kappa$ (3.35) entering operator $B^I$ (3.31) should be real-valued, while the $F_\nu$ defined in (3.37) should be positive for all eigenvalues $N_\nu = h_2, h_2 + 1, \ldots, \infty$. Introducing the notation

$$F_\nu(n) = \kappa^2 - \mu(n + 1)^2 + (n + 1)^4, \quad F_\nu(n) \equiv F_\nu|_{N_\nu = n},$$

$$\mu \equiv C_2 - h_2^2 + 5,$$  (4.1)

we note that, depending on the behaviour of the $F_\nu(n)$, we use the following terminology

$$F_\nu(n) \geq 0 \text{ for all } n = h_2, h_2 + 1, \ldots, \infty, \text{ classically unitary system;}$$

$$F_\nu(n) \neq 0 \text{ for all } n = h_2, h_2 + 1, \ldots, \infty, \text{ irreducible system;}$$

$$F_\nu(n_r) = 0 \text{ for some } n_r \in h_2, h_2 + 1, \ldots, \infty, \text{ reducible system.}$$  (4.2)

This is to say that, if $F_\nu(n)$ (4.1) is positive for all $n$ (4.3), then we will refer to the fields (3.9) as a classically unitary system, while if $F_\nu(n)$ (4.1) has no roots (4.4), then we will refer to the fields (3.9) as an irreducible system. For the case (4.4), our Lagrangian (3.13) describes an infinite chain of coupling fields (3.9). Relation (4.5) tells us that, if $F_\nu(n)$ (4.1) has roots, then we will refer to the fields (3.9) as a reducible system. For the case of the reducible system, the Lagrangian (3.13) is factorized and describes finite and infinite decoupled chains of fields.

Taking into account the definitions presented in (4.3)–(4.5), we now define (ir)reducible classically unitary systems in the following way:
\( F_\nu(n) > 0 \) for all \( n = h_2, h_2 + 1, \ldots, \infty \), irreducible classically unitary system; \hspace{1cm} (4.6)

\( F_\nu(n_r) = 0 \) for some \( n_r \in h_2, h_2 + 1, \ldots, \infty \),

\( F_\nu(n) > 0 \) for all \( n = h_2, h_2 + 1, \ldots, \infty \) and \( n \neq n_r \), reducible classically unitary system. \hspace{1cm} (4.7)

For the mixed-symmetry field, the \( \kappa \) associated with the ket-vector \( |\phi\rangle \) is strictly positive, \( \kappa > 0 \) (3.38). Keeping this in mind, we now summarize our study of equations (4.6) and (4.7) as the following three statements.

**Statement 1.** We classify the solutions of equation (4.6) as type I, IIA, IIB, and III solutions. These solutions are as follows.

**Type I solutions:**
\[ \mu < 2\kappa; \quad \kappa > 0. \] \hspace{1cm} (4.8)

**Type IIA solutions:**
\[ \mu > 2\kappa, \quad \kappa > 0 \]
\[ \mu = p^2 + q^2, \quad \kappa = pq. \] \hspace{1cm} (4.9)

\[ 0 < p < h_2 + 1, \quad 0 < q < h_2 + 1; \quad p \neq q. \] \hspace{1cm} (4.10)

**Type IIB solutions:**
\[ \mu > 2\kappa, \quad \kappa > 0, \] \hspace{1cm} (4.11)
\[ \mu = p_k^2 + q_k^2, \quad \kappa = p_k q_k, \] \hspace{1cm} (4.12)
\[ p_k = h_2 + 1 + k + \epsilon_p, \quad q_k = h_2 + 1 + k + \epsilon_q, \quad k \in \mathbb{N}_0. \] \hspace{1cm} (4.13)
\[ 0 < \epsilon_p < 1, \quad 0 < \epsilon_q < 1, \quad \epsilon_p \neq \epsilon_q. \] \hspace{1cm} (4.14)

**Type III solutions:**
\[ \mu = 2\kappa, \quad \kappa > 0, \] \hspace{1cm} (4.15)
\[ \mu = 2p_k^2, \quad \kappa = p_k^2, \] \hspace{1cm} (4.16)
\[ p_k = k + \epsilon_p, \quad 0 < \epsilon_p < 1, \quad k \in \mathbb{N}_0. \] \hspace{1cm} (4.17)

We now comment on statement 1.

(i) For the type I solutions, using (4.2) and (4.8), we get the following restriction for the eigenvalue of the second-order Casimir operator \( C_2 \):
\[ C_2 < 2\kappa + h_2^2 - 5. \] \hspace{1cm} (4.19)

(ii) For the type II and III solutions, we can get various interesting representations for the eigenvalues of the Casimir operators. Namely, using (4.2), (4.10), (4.13) and (4.17), we see that \( C_2 \) and \( C_3, C_4 \) (3.45) and (3.46) can be represented as
\[ C_2 = p^2 + q^2 + h_2^2 - 5, \] \hspace{1cm} (4.20)
\[ C_{\epsilon 3} = pqh_3, \quad (4.21) \]

\[ C_4 = (p^2 - 1)(q^2 - 1) + h_2^2(p^2 + q^2 - 1), \quad (4.22) \]

where, for the type IIB solutions, values of \( p, q \) are given in (4.14). Note that, for the type III solutions, we should set \( p = q \) in (4.20)–(4.22), where values of \( p \) are given in (4.18).

(iii) Another interesting representation for the eigenvalues of the Casimir operators is obtained by using, in place of the \( p, q, \) and \( h_2 \), new parameters \( E_0, H_1, H_2 \) defined by the relations

\[ p = E_0 - 2, \quad (4.23) \]
\[ q = H_1 + 1, \quad (4.24) \]
\[ h_2 = H_2. \quad (4.25) \]

Plugging (4.23)–(4.25) into (4.20)–(4.22), we find the following expressions:

\[ C_2 = E_0(E_0 - 4) + H_1(H_1 + 2) + H_2^2, \]
\[ C_{\epsilon 3} = (E_0 - 2)(H_1 + 1)H_2, \quad (4.26) \]
\[ C_4 = (E_0 - 1)(E_0 - 3)H_1(H_1 + 2) + H_2^2\left( E_0(E_0 - 4) + H_1(H_1 + 2) + 4 \right). \]

We now explain our motivation for introducing the parameters \( E_0, H_1, H_2 \). To this end we use the notation \( D(E_0, h_1, h_2) \) for a positive-energy lowest weight representation of the \( so(4, 2) \) algebra, where \( E_0 \) is a lowest eigenvalue of the energy operator, while \( h_1, h_2 \) label the highest weight of the \( so(4) \) algebra representation. Eigenvalues of the Casimir operators for the \( D(E_0, h_1, h_2) \) are given in appendix A in (A.11). We see then that relations in (4.26) are similar to the ones in (A.11). It is the similarity of relations in (4.26) and (A.11) that motivates us to introduce the parameters \( E_0, H_1, H_2 \) in (4.23)–(4.25).

(iv) For the type II and III solutions, the classical unitarity restrictions imposed on the parameters \( p, q, h_2 \) can straightforwardly be expressed in terms of the \( E_0, H_1, H_2 \) defined in (4.23)–(4.25). Namely, for the type IIB solutions, we get the restrictions

\[ E_0 > H_1 + 2, \quad H_1 > H_2, \quad \text{for IIB solutions} \]

while, for the type IIA and III solutions, we get the restrictions

\[ 2 < E_0 < H_2 + 3, \quad -1 < H_1 < H_2, \quad E_0 \neq H_1 + 3, \quad \text{for IIA solutions}; \]
\[ E_0 = H_1 + 3, \quad -1 < H_1, \quad \text{for III solutions}. \]

Note that restrictions (4.28) are easily obtained by using the classical unitarity restrictions (4.11) and relations (4.23)–(4.25). To get (4.29), we use \( p > 0 \) (4.18), and make the identification \( p = q \) in (4.23) and (4.24). To get (4.27) we note that, for the type IIB solutions, the classical unitarity restrictions given in (4.14) and (4.15) amount to the restrictions

\[ -1 < p - q < 1, \quad p > h_2 + 1, \quad q > h_2 + 1, \quad p \neq q, \quad \text{for IIB solutions}. \]
Using (4.23)–(4.25), we see that restrictions in (4.30) amount to restrictions in (4.27).

(v) As we noted above, the parameters $E_0$, $h_1$, $h_2$ label the positive-energy lowest weight representation of the $so(4,2)$ algebra denoted by $D(E_0, h_1, h_2)$. We now compare unitarity restrictions imposed on the $E_0$, $h_1$, $h_2$ with the classical unitarity restrictions imposed on the $\mathcal{E}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$ in (4.27). To this end we recall that, if $h_1 > h_2$, then the $D(E_0, h_1, h_2)$ is realized as unitary representation of the $so(4,2)$ algebra provided the $E_0$, $h_1$, $h_2$ satisfy the following restrictions:

$$E_0 \geq h_1 + 2, \quad \text{for } h_1 > h_2.$$  \hspace{1cm} (4.31)

The restrictions in the first line in (4.27) are remarkably similar to the ones in (4.31). Note, however, that the label $h_1$ in (4.31) is an integer, while our label $\mathcal{H}_1$ is not an integer. Also note that, contrary to (4.31), for the type IIB solutions, we have the additional restrictions given in the second line in (4.27).

**Statement 2.** The solution to equation (4.7) with one root of $F_\nu$, denoted by $h_1$ is given by

$$\kappa^2 = (h_1 + 1)^2 \left(C_2 - h_1(h_1 + 2) - h_2^2 + 4\right),$$  \hspace{1cm} (4.32)

$$2h_1(h_1 + 1) + h_2^2 - 4 < C_2 < 2h_1(h_1 + 3) + h_2^2,$$  \hspace{1cm} (4.33)

$$F_\nu = \left((h_1 + 1)^2 - (N_\nu + 1)^2\right)\left(C_2 - h_1(h_1 + 2) - h_2^2 + 4 - (N_\nu + 1)^2\right).$$  \hspace{1cm} (4.34)

We now comment on statement 2.

(i) Lagrangian (3.13), with the $C_2$ and $F_\nu$ as in (4.33) and (4.34), describes a reducible classically unitary system. This is to say that decomposing ket-vector $|\phi\rangle$ (3.9) as

$$|\phi\rangle = |\phi^{h_1,h_1}\rangle + |\phi^{h_1,+1,\infty}\rangle,$$  \hspace{1cm} (4.35)

$$|\phi^{M,N}\rangle \equiv \sum_{n=H}^{N} \frac{\epsilon^n}{n!} \alpha^h \ldots \alpha^{h_1+1,\infty}(x,z)|0\rangle,$$  \hspace{1cm} (4.36)

we verify that Lagrangian (3.13) is factorized as

$$\mathcal{L} = \mathcal{L}^{h_1,h_1} + \mathcal{L}^{h_1,+1,\infty},$$  \hspace{1cm} (4.37)

$$\mathcal{L}^{M,N} \equiv \langle \phi^{M,N} | (\Box - \frac{1}{\epsilon^2} A) |\phi^{M,N}\rangle.$$  \hspace{1cm} (4.38)

Namely, $\mathcal{L}^{h_1,h_1}$ and $\mathcal{L}^{h_1,+1,\infty}$ (4.37) are invariant under the $so(4,2)$ algebra transformations (4.32).

(ii) The $|\phi^{h_1,h_1}\rangle$ (4.35) describes a massive finite-component field associated with positive-energy lowest weight representation of the $so(4,2)$ algebra which we denote as $D(E_0, h_1, h_2)$. The eigenvalue of the second-order Casimir operator $C_2$, the labels $E_0$, $h_1$, $h_2$ and a mass parameter $m^2$ for $|\phi^{h_1,h_1}\rangle$ are related as

$$C_2 = E_0(E_0 - 4) + h_1(h_1 + 2) + h_2^2,$$  \hspace{1cm} (4.39)

$$m^2 \equiv (E_0 - 2)^2 - h_2^2.$$  \hspace{1cm} (4.40)

Using (4.39), we can represent relations (4.32) and (4.33) as
\[ \kappa = (E_0 - 2)(h_1 + 1), \]  
\[ h_1 + 2 < E_0 < h_1 + 4. \]  

In turn, using the mass parameter \( (4.40) \), we can represent relation \( (4.42) \) as
\[ 0 < m^2 < 4(h_1 + 1). \]  

(iii) For a particular \( C_2 \) in \( (4.34) \), the root \( h_1 \) becomes doubly-degenerate. Namely, the appearance of the doubly-degenerate root \( h_1 \) in \( (4.34) \) implies the following relations for \( \kappa, C_2, \) and \( F_\nu \):
\[ \kappa = (h_1 + 1)^2, \]  
\[ C_2 = 2h_1(h_1 + 2) + h_2^2 - 3, \]  
\[ F_\nu = \left( (h_1 + 1)^2 - (N_\nu + 1)^2 \right)^2. \]  

Lagrangian \( (3.13) \), with \( F_\nu \) as in \( (4.46) \), describes a reducible classically unitary system. Namely, for \( F_\nu \) given in \( (4.46) \), the ket-vector \( |\phi\rangle \) and the Lagrangian are decomposed as in \( (4.35) \) and \( (4.37) \) respectively. The \( |\phi^{h_1,h_1}\rangle \) \( (4.35) \) describes a classically unitary finite-component massive field, while the \( |\phi^{h_1+1,\infty}\rangle \) \( (4.35) \) describes a classically unitary infinite-component spin field. The lowest eigenvalue of the energy operator and mass parameter \( m^2 \) of the \( |\phi^{h_1,h_1}\rangle \) are given by
\[ E_0 = h_1 + 3, \quad m^2 = 2h_1 + 1, \quad \text{for massive } |\phi^{h_1,h_1}\rangle. \]  

Note that \( E_0 \) \( (4.47) \) is obtained by plugging \( (4.45) \) into \( (4.39) \). In turn, plugging \( E_0 \) \( (4.47) \) into \( (4.40) \) leads to \( m^2 \) as given in \( (4.47) \).

**Statement 3.** The solution to equation \( (4.7) \) with two roots of \( F_\nu \) is given by
\[ \kappa = (h_1 + 1)(h_1 + 2), \]  
\[ C_2 = 2h_1(h_1 + 2) + h_2^2, \]  
\[ F_\nu = \left( (h_1 + 1)^2 - (N_\nu + 1)^2 \right) \left( (h_1 + 2)^2 - (N_\nu + 1)^2 \right). \]  

We now comment on statement 3.

(i) From \( (4.50) \), we see that the \( F_\nu \) has two roots. Namely, the \( F_\nu \) is equal to zero for \( N_\nu = h_1 \) and \( N_\nu = h_1 + 1 \).

(ii) Lagrangian \( (3.13) \), with \( F_\nu \) as in \( (4.50) \), describes a reducible classically unitary field. Namely, using the \( F_\nu \) given in \( (4.50) \) and the notation given in \( (4.36) \) and \( (4.38) \), we can decompose \( |\phi\rangle \) \( (3.9) \) and Lagrangian \( (3.13) \) as
\[ |\phi\rangle = |\phi^{h_1,h_1}\rangle + |\phi^{h_1+1,h_1+1}\rangle + |\phi^{h_1+2,\infty}\rangle, \]  
\[ \mathcal{L} = \mathcal{L}^{h_1,h_1} + \mathcal{L}^{h_1+1,h_1+1} + \mathcal{L}^{h_1+2,\infty}. \]  

(iii) In \( (4.51) \), the \( |\phi^{h_1,h_1}\rangle \) describes a classically unitary finite-component massive field associated with the representation \( D(E_0, h_1, h_2) \), the \( |\phi^{h_1+1,h_1+1}\rangle \) describes a classically unitary finite-component massless field associated with the representation \( D(E_0, h_1 + 1, h_2) \).
while the $|\phi^{h+2,\infty}\rangle$ describes a classically unitary infinite-component spin field. The lowest eigenvalues of the energy operator and mass parameters $m^2$ of the fields $|\phi^{h,1}\rangle$ and $|\phi^{h+1,1,1+1}\rangle$ are given by

$$E_0 = h + 4, \quad m^2 = 4(h + 1), \quad \text{for massive } |\phi^{h,1}\rangle,$$

$$E_0 = h + 3, \quad m^2 = 0, \quad \text{for massless } |\phi^{h+1,1,1+1}\rangle.$$  

The three statements presented above can easily be proved by noticing that $F_v(n)$ (4.1) has at most two roots. This is to say that we are going to analyse the following three cases:

1. $F_v(n)$ has no roots; (2) $F_v(n)$ has one root; (3) $F_v(n)$ has two roots. Let us analyse these three cases in turn.

(i) Solutions without roots of $F_v(n)$. Such solutions lead to the type I, II, and III solutions described above in statement 1. We consider them separately.

First, using (4.1), we note that, if $\mu < 2\kappa, \kappa > 0$, then $F_v(n) > 0$ for all $n$ (4.6). This gives type I solutions in (4.8).

Second, we consider the case $\mu > 2\kappa, \kappa > 0$. Such $\mu$ and $\kappa$ can be presented in terms of two positive nonequal numbers $p, q$ as

$$\mu = p^2 + q^2, \quad \kappa = pq, \quad p \neq q, \quad p > 0, \quad q > 0.$$  

Plugging $\mu$ and $\kappa$ (4.55) into (4.1), we represent $F_v(n)$ as

$$F_v(n) = \left((n+1)^2 - p^2\right)\left((n+1)^2 - q^2\right).$$  

Using (4.56), it is easy to see that equation (4.6) lead to the type IIA solutions (4.9)–(4.11) and the type IIB solutions (4.12)–(4.15).

Third, we consider the case $\mu = 2\kappa, \kappa > 0$. Then we can represent $\mu$, $\kappa$, and $F_v(n)$ as

$$\mu = 2p^2, \quad \kappa = p^2, \quad p > 0, \quad F_v(n) = ((n+1)^2 - p^2)^2.$$  

Using $F_v(n)$ (4.57), it is easy to see that equation (4.6) lead to the type III solutions in (4.16)–(4.18).

At the end of our discussion of statement 1 we explain our two motivations for introducing the parameters $p$ and $q$ (4.55). First, the use of such parameters allows us to obtain a factorized representation for $F_v$ given in (4.56), which turns out to be very convenient for the analysis of the requirement of the classical unitarity. Second, in terms of $p$ and $q$, the eigenvalues of the Casimir operators $C_2, C_3, C_4$ take a simple and convenient form, given in (4.20)–(4.22).

(ii) Solution with one root of $F_v(n)$. Solutions of equation (4.7) with one root of $F_v(n)$ are described in statement 2.

First, we outline the derivation of relations in (4.32)–(4.34). Denoting one root of $F_v$ as $h_1$, we see that equation (4.7) can be represented as

$$F_v(h_1) = 0, \quad F_v(n) > 0 \quad \text{for} \quad n = h_2, h_2 + 1, \ldots, h_1 - 1, h_1 + 1, h_1 + 2, \ldots, \infty.$$  

Using (4.1), we note then that the equation $F_v(h_1) = 0$ leads to the following value of $\kappa^2$:

$$\kappa^2 = (h_1 + 1)^2 \left(C_2 - h_1(h_1 + 2) - h_2^2 + 4\right).$$  


Plugging $\kappa^2$ (4.59) into (3.37), we get $F_v$ given in (4.34), while plugging $\kappa^2$ (4.59) into (4.1), we cast the $F_v(n)$ into the following factorized form:

$$F_v(n) = \left((h_1 + 1)^2 - (n + 1)^2\right)\left(C_2 - h_1(h_1 + 2) - h_2^2 - (n + 1)^2 + 4\right).$$  \hspace{1cm} (4.60)

Now, using $F_v(n)$ (4.60) and considering inequalities $F_v(n) > 0$ in (4.7), we find the restrictions on the $C_2$.

$$2h_1(h_1 + 1) + h_2^2 - 4 < C_2 < 2h_1(h_1 + 3) + h_2^2.$$  \hspace{1cm} (4.61)

For the reader’s convenience, we note that the left inequality in (4.61) is obtained by requiring $F_v(n) > 0$ for $n = h_2, h_2 + 1, \ldots, h_1 - 1$, while the right inequality in (4.61) is obtained by requiring $F_v(n) > 0$ for $n = h_1 + 1, h_1 + 2, \ldots, \infty$. Note that using the left inequality (4.61) in (4.59), we find $\kappa^2 > 0$, as it should be for the real-valued $\kappa$.

Second, we outline the derivation of relations in (4.44)–(4.46). From (4.60), we see that, if $C_2$ takes the value

$$C_2 = 2h_1(h_1 + 2) + h_2^2 - 3,$$  \hspace{1cm} (4.62)

then the root $h_1$ in (4.60) becomes double-degenerate. Plugging $C_2$ (4.62) into (4.59), we get $\kappa$ given in (4.44), while plugging $C_2$ (4.62) into (4.34), we get $F_v$ given in (4.46).

(iii) **Solution with two roots of** $F_v(n)$. Solutions of equation (4.7) with two roots of $F_v(n)$ are described in statement 3. Denoting two roots of $F_v$ as $h_1$ and $H_1$,

$$F_v(h_1) = 0, \quad F_v(H_1) = 0, \quad h_1 < H_1,$$  \hspace{1cm} (4.63)

we see that equation (4.63) lead to the following relations:

$$\kappa = (h_1 + 1)(H_1 + 1),$$  \hspace{1cm} (4.64)

$$C_2 = H_1(H_1 + 2) + h_1(h_1 + 2) + h_2^2 - 3.$$  \hspace{1cm} (4.65)

Plugging $\kappa$, $C_2$ (4.64) and (4.65) into (4.1), we find the following factorized representation for $F_v(n)$:

$$F_v(n) = \left((h_1 + 1)^2 - (n + 1)^2\right)\left((H_1 + 1)^2 - (n + 1)^2\right).$$  \hspace{1cm} (4.66)

Lagrangian (3.13), with $\kappa$ and $C_2$ as in (4.64) and (4.65), describes a reducible mixed-symmetry continuous-spin field. Namely, using $\kappa$ and $C_2$ given in (4.64) and (4.65) and the notation given in (4.36) and (4.38), we can decompose ket-vector $|\phi\rangle$ (3.9) and Lagrangian (3.13) as

$$|\phi\rangle = |\phi^{h_2, h_1}\rangle + |\phi^{h_1+1, H_1}\rangle + |\phi^{H_1+1, \infty}\rangle,$$  \hspace{1cm} (4.67)

$$\mathcal{L} = \mathcal{L}^{h_2, h_1} + \mathcal{L}^{h_1+1, H_1} + \mathcal{L}^{H_1+1, \infty}.$$  \hspace{1cm} (4.68)

Let us consider the cases $h_1 + 1 = H_1$, and $h_1 + 1 < H_1$ separately.

Case $h_1 + 1 = H_1$. It is this case that respects the classical unitarity. This case is described in statement 3. Namely, setting $h_1 + 1 = H_1$ in (4.66), we see that $|\phi^{h_2, h_1}\rangle$ describes a classically unitary massive finite-component field, the $|\phi^{h_1+1, h_1+1}\rangle$ describes a classically unitary massless finite-component field, while the $|\phi^{h_1+1, \infty}\rangle$ describes a classically unitary infinite-component spin field. Setting $h_1 + 1 = H_1$ in (4.64) and (4.65), we find the $\kappa$ and $C_2$ given in (4.48) and (4.49). In turn, using $C_2$ (4.49), we find then the $E_0$
and $m^2$ of the fields $|\phi^{h_2,h_1}\rangle$ and $|\phi^{h_1+1,h_1+1}\rangle$ given in (4.53) and (4.54). Plugging $\kappa$ (4.48) and $C_2$ (4.49) into (3.37), we get $F_\nu$ (4.50).

Case $h_1 + 1 < H_1$. This case does not respect the classical unitarity and therefore this case is not discussed in statement 3. This is to say that, for this case, we get classically non-unitary fields in the decomposition (4.67). Namely, from $F_\nu(n)$ (4.66), we see that the $|\phi^{h_2,h_1}\rangle$ describes a classically unitary massive finite-component field, the $|\phi^{h_1+1,H_1}\rangle$ describes a classically non-unitary partial-massless finite-component field, while the $|\phi^{H_1+1,\infty}\rangle$ describes a classically unitary infinite-component spin field. The mass parameters of the fields $|\phi^{h_2,h_1}\rangle$ and $|\phi^{h_1+1,H_1}\rangle$ are given by

$$m^2 = (H_1 + 1)^2 - h_1^2,$$

for massive $|\phi^{h_2,h_1}\rangle$, \hspace{1cm} (4.69)

$$m^2 = (h_1 + 1)^2 - H_1^2,$$

for partial-massless $|\phi^{h_1+1,H_1}\rangle$. \hspace{1cm} (4.70)

Mass parameter $m^2$ (4.70) can be represented in the form

$$m^2 = -k(2H_1 - k), \quad k \equiv H_1 - h_1 - 1,$$

for partial-massless $|\phi^{h_1+1,H_1}\rangle$. \hspace{1cm} (4.71)

From (4.71), we learn that $|\phi^{h_1+1,H_1}\rangle$ describes a depth-$k$ partial-massless mixed-symmetry field. The lowest eigenvalues of the energy operator are given by

$$E_0 = H_1 + 3,$$

for massive $|\phi^{h_2,h_1}\rangle$, \hspace{1cm} (4.72)

$$E_0 = h_1 + 3,$$

for partial-massless $|\phi^{h_1+1,H_1}\rangle$. \hspace{1cm} (4.73)

The $|\phi^{h_2,h_1}\rangle$ (4.67) is associated with representation $D(E_0,h_1,h_2)$ which is unitary for $E_0$ (4.72) and $h_1 + 1 < H_1$. The $|\phi^{h_1+1,H_1}\rangle$ (4.67) is associated with representation $D(E_0,h_1,h_2)$ which is non-unitary for $E_0$ (4.73) and $h_1 + 1 < H_1$. Note that $E_0$ (4.73) can be represented as $E_0 = H_1 + 2 - k$, where $k$ is given in (4.71).

4.1. **Irreducible totally symmetric continuous-spin field with $h_2 = 0$, $\kappa$-arbitrary**

Totally symmetric fields are defined by relations (3.47) and (3.48). At the end of section 3, we noted that, for the case (3.47), the $\kappa$ is real-valued or purely imaginary. We consider the cases $\kappa^2 > 0$ and $\kappa^2 < 0$ in turn.

Case $h_2 = 0$, $\kappa^2 > 0$. For definiteness, we assume that the $\kappa$ is strictly positive, $\kappa > 0$. We note then that, for totally symmetric field, solutions to equation (4.6) are obtained from statement 1 by setting $h_2 = 0$.

Case $h_2 = 0$, $\kappa^2 < 0$. For this case, it is easy to see that the solution to equation (4.6), with $F_\nu$ as in (3.51), is given by

$$C_2 < \kappa^2 - 4,$$

for $\kappa^2 \leq 0$. \hspace{1cm} (4.74)

4.2. **Reducible totally symmetric field with $h_2 = 0$, $\kappa$-arbitrary**

To study solutions of equation (4.7), with $F_\nu$ as in (3.51), we consider the equation $F_\nu(s) = 0$. The solution of this equation is given by

$^7$Recent interesting discussion of mixed-symmetry partial-massless (A)dS fields may be found in [37, 38].
\[ \kappa^2 = (s + 1)^2(C_2 + 5) - (s + 1)^4. \]  
(4.75)

Plugging \( \kappa^2 \) (4.75) into (3.51), we get

\[ F_v = (N_v + 1)^2 - (s + 1)^2 \left( (N_v + 1)^2 + (s + 1)^2 - C_2 - 5 \right). \]  
(4.76)

On the one hand, using the notation \( F_v(n) \equiv F_v|_{N_v=n} \), we get

\[ F_v(n) = (n + 1)^2 - (s + 1)^2 \left( (n + 1)^2 + (s + 1)^2 - C_2 - 5 \right). \]  
(4.77)

On the other hand, setting \( h_1 = s, h_2 = 0 \) in (4.60), we also obtain \( F_v(n) \) as in (4.77). Taking this into account we note then that, for a totally symmetric field, solutions to equation (4.7) are obtained from statements 2 and 3 by setting \( h_1 = s, h_2 = 0 \).

We finish our discussion of the reducible totally symmetric field with \( h_2 = 0 \) and arbitrary \( \kappa \) by the following comment. Let us decompose \( |\phi\rangle \) (3.9) as

\[ |\phi\rangle = |\phi^{0,\infty}\rangle + |\phi^{s+1,\infty}\rangle, \]  
(4.78)

where \( |\phi^{0,\infty}\rangle, |\phi^{s+1,\infty}\rangle \) are defined as in (4.36). If \( F_v \) takes the form given in (4.76), then Lagrangian (3.13) is factorized as

\[ \mathcal{L} = \mathcal{L}^{0,\infty} + \mathcal{L}^{s+1,\infty}, \]  
(4.79)

where \( \mathcal{L}^{0,\infty}, \mathcal{L}^{s+1,\infty} \) are defined as in (4.38), while the operator \( g_v \) (3.50) takes the form

\[ g_v = \left( \frac{(N_v + 1)^2 - s^2}{(N_v + 1)(2N_v + 3)} \left( (N_v + 1)^2 + (s + 1)^2 - C_2 - 5 \right) \right)^{1/2}. \]  
(4.80)

4.3. Totally symmetric field with \( h_2 \neq 0, \kappa = 0 \)

Now, as promised at the end of section 3, we are going to demonstrate that a totally symmetric field (3.48) is equivalent to the infinite component field \( |\phi^{s+1,\infty}\rangle \) (4.78). To prove the equivalence we should match the field contents and the operators \( B^l \). First, we match the field contents. To this end, we make the identification \( h_2 = s + 1 \), and note that \( |\phi\rangle \) (3.9) coincides with \( |\phi^{s+1,\infty}\rangle \) (4.78). Second, we match the operators \( B^l \). To this end we set \( \kappa = 0 \) in (3.31)–(3.37) and obtain the operator \( B^l \) as in (3.49) and (3.50) with the following expression for \( g_v \)

\[ g_v = \left( \frac{(N_v + 1)^2 - h_2^2}{(N_v + 1)(2N_v + 3)} \left( (N_v + 1)^2 + h_2^2 - C_2 - 5 \right) \right)^{1/2}. \]  
(4.81)

On the one hand, operator \( B^l \) (3.49) and (3.50), with \( g_v \) as in (4.81), describes the field \( |\phi\rangle \) with \( h_2 \neq 0, \kappa = 0 \). On the other hand, the infinite component \( |\phi^{s+1,\infty}\rangle \) (4.78) is described by operator \( B^l \) (3.49) and (3.50) with \( g_v \) given in (4.80). Identifying that \( h_2 \equiv s + 1 \), we see that the expressions for \( g_v \) given in (4.80) and (4.81) coincide. Thus the operators \( B^l \) are also matched.

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Appendix A. Casimir operators of the so(4, 2) algebra

To discuss Casimir operators of the so(4, 2) algebra it is convenient to use a manifestly six-dimensional covariant approach. In this approach, generators of the so(4, 2) algebra denoted by $J^{AB}$ satisfy the commutation relations

$$[J^{AB}, J^{CE}] = \eta^{RC} J^{AE} + 3 \text{ terms}, \quad \eta^{AB} = (- - + + +),$$

where vector indices of the so(4, 2) algebra take values $A, B, C, E = 0', 0, 1, 2, 3, 4$. In terms of the generators $J^{AB}$, the Casimir operators of the so(4, 2) algebra can be presented as

$$C_2 = \frac{1}{2} J^{A_1 A_2} J^{A_2 A_1},$$

(A.2)

$$C_3 = -\frac{i}{48} \epsilon^{A_1 ... A_6} J^{A_1 A_2} J^{A_3 A_4} J^{A_5 A_6},$$

(A.3)

$$C_4 = \frac{1}{8} (J^{A_1 A_2} J^{A_3 A_1})^2 + \frac{3}{2} J^{A_1 A_2} J^{A_3 A_4} J^{A_4 A_5} - \frac{1}{4} J^{A_1 A_2} J^{A_3 A_1} J^{A_5 A_4} J^{A_4 A_3},$$

(A.4)

where $\epsilon^{A_1 ... A_6}$ stands for the Levi–Civita symbol of rank six with $\epsilon^{0'1234} = 1$. Our choice of the particular form of the fourth-order Casimir operator $C_4$ (A.4) is motivated by the following relation for the $C_4$ (A.4):

$$C_4 = \frac{1}{128} X^{AB} X^{AB},$$

(A.5)

$$X^{AB} \equiv \epsilon^{A B C D E F} J^{C D} J^{E F}.$$

(A.6)

In [21], we demonstrated that relation (A.2) allows us to find the representation for the operator $A$ given in (3.26). Now, using the light-cone gauge realization of generators of the so(4, 2) algebra in terms of the differential operators given in (3.15)–(3.23), we can verify that the Casimir operators defined in (A.3) and (A.4) take the form given in (3.43) and (3.44). To this end we relate the six-dimensional notation for the generators in (A.1) to the four-dimensional notation for the generators in (3.1). Namely, let us decompose six-dimensional coordinates $x^A$ as

$$x^A = x^{\oplus}, x^{\ominus}, x^a, \quad a = 0, 1, 2, 3, \quad x^{\ominus} = \frac{1}{\sqrt{2}} (x^4 + x^{0'}), \quad x^{\oplus} = \frac{1}{\sqrt{2}} (x^4 - x^{0'}).$$

(A.7)

It is easy then to see that, in the frame of the coordinates $x^{\oplus, \ominus}, x^a$, the generators $J^{AB}$ and the flat metric tensor $\eta^{AB}$ (A.1) are decomposed as

$$J^{\oplus a} = J^{\oplus a}, \quad J^{\ominus a} = J^{\ominus a}, \quad J^{ab},$$

(A.8)

$$\eta^{\oplus a} = \eta^{\ominus a}, \quad \eta^{\oplus a}, \quad \eta^{ab}, \quad \eta^{\oplus a} = 1, \quad \eta^{\ominus a} = 1.$$

(A.9)

Generators $F^{ab}$ in (A.8) are identified with the $J^{ab}$ appearing in (3.1), while the remaining generators in (3.1) and (A.8) are identified in the following way:

$$P^a = J^{\oplus a}, \quad K^a = J^{\ominus a}, \quad D = J^{\ominus a}.$$

(A.10)

Making use of (A.10) and the light-cone gauge realization for the generators in (3.15)–(3.23), we verified that expressions for $C_3, C_4$ in (A.3) and (A.4) lead to expressions for $C_3, C_4$ in (3.43) and (3.44).
For the $D(E_0, h_1, h_2)$, which is the positive-energy lowest weight representation of the $so(4, 2)$ algebra, eigenvalues of the Casimir operators (A.2)–(A.4) are given by

$$C_2 = E_0(E_0 - 4) + h_1(h_1 + 2) + h_2^2,$$
$$C_{r3} = (E_0 - 2)(h_1 + 1)h_2,$$
$$C_4 = (E_0 - 1)(E_0 - 3)h_1(h_1 + 2) + h_2^2(E_0(E_0 - 4) + h_1(h_1 + 2) + 4).$$

(A.11)

Appendix B. Useful relations for various spin operators

Creation operators $\alpha^I$ and the respective annihilation operators $\bar{\alpha}^I$ are referred to as oscillators. The oscillators, Hermitian conjugation rule, and the vacuum $|0\rangle$ are defined by the relations

$$[\bar{\alpha}^I, \alpha^J] = \delta^{IJ}, \quad \alpha^I = \alpha^I, \quad \alpha^I|0\rangle = |0\rangle,$$

(B.1)

where $\delta^{IJ}$ stands for the Kronecker delta symbol. Vector indices of the $so(3)$ algebra take values $I, J, K = 1, 2, 3$. We use the following notation for various operators constructed out of the oscillators

$$S^I \equiv \epsilon^{IJK} \alpha^J \bar{\alpha}^K,$$
$$M^{IJ} \equiv \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I,$$
$$A^I \equiv \alpha^I - \alpha^2 \frac{1}{2N_\alpha + 3} \bar{\alpha}^I,$$
$$N_\alpha \equiv \alpha^I \bar{\alpha}^I,$$
$$\bar{\alpha}^2 \equiv \bar{\alpha}^I \alpha^I,$$

(B.2)\quad(B.3)\quad(B.4)\quad(B.5)

where $\epsilon^{IJK}$ stands for the Levi–Civita symbol of rank three with $\epsilon^{123} = 1$. The operators $M^{IJ}$ and $S^I$ are related as

$$M^{IJ} = \epsilon^{IJK} S^K, \quad S^I = \frac{1}{2} \epsilon^{IJK} M^{JK}.$$

(B.6)

For $M^{IJ}$ and $S^I$, we note the following commutation relations and Hermitian conjugation rules

$$[M^{IJ}, M^{KL}] = \delta^{IK} M^{JL} + 3 \text{ terms}, \quad M^{IJ\dagger} = -M^{IJ},$$
$$[S^I, S^J] = -M^{IJ}, \quad S^I\dagger = -S^I.$$  

(B.7)\quad(B.8)

The following relations turn out to be helpful when studying the defining equations in (2.19) and (3.29).

$$A^I A^J - A^J A^I = 0,$$
$$\bar{\alpha}^I S^J - \bar{\alpha}^J S^I = -\epsilon^{IJK} (N_\alpha + 2) \alpha^K + \epsilon^{IJK} \alpha^K \bar{\alpha}^2,$$
$$S^I \bar{\alpha}^J - S^J \bar{\alpha}^I = \epsilon^{IJK} N_\alpha \bar{\alpha}^K - \epsilon^{IJK} \alpha^K \bar{\alpha}^2,$$
$$A^I S^J - A^J S^I = \epsilon^{IJK} N_\alpha \bar{\alpha}^K - \epsilon^{IJK} \alpha^K \bar{\alpha}^2 \frac{1}{2N_\alpha + 5} \bar{\alpha}^2.$$  

(B.9)\quad(B.10)\quad(B.11)\quad(B.12)
\[ S^I A^J - S^J A^I = -\epsilon^{IJK} A^K (N_\alpha + 2) + \epsilon^{IJK} \alpha^2 \frac{1}{2N_\alpha + 5} \hat{\alpha}^2. \quad (B.13) \]

\[ A^I \bar{\alpha}^J - (I \leftrightarrow J) = M^{IJ}. \quad (B.14) \]

\[ \bar{\alpha}^I A^J - (I \leftrightarrow J) = -\frac{2N_\alpha + 3}{2N_\alpha + 1} M^{IJ}. \quad (B.15) \]

\[ M^{IJ} A^I = A^I (N_\alpha + 2) - \alpha^2 \frac{1}{2N_\alpha + 5} \hat{\alpha}^2, \quad (B.16) \]

\[ M^{IJ} \bar{\alpha}^I = -N_\alpha \alpha^2 + \alpha^2 \frac{1}{2N_\alpha + 5} \hat{\alpha}^2, \quad (B.17) \]

\[ M^{IJ} S^I = S^I, \quad (B.18) \]

\[ M^{IJ} M^{KL} M^{KJ} - (I \leftrightarrow J) = -M^{IKL} M^{IJK}. \quad (B.19) \]

For the computation of eigenvalues of the Casimir operators (2.35), (2.36), (3.43) and (3.44), we use the following relations:

\[ A^I \bar{\alpha}^I = N_\alpha - \alpha^2 \frac{1}{2N_\alpha + 3} \hat{\alpha}^2, \quad (B.20) \]

\[ \bar{\alpha}^I A^I = \frac{(2N_\alpha + 3)(N_\alpha + 1)}{2N_\alpha + 1} - \alpha^2 \frac{1}{2N_\alpha + 5} \hat{\alpha}^2. \quad (B.21) \]

\[ S^I S^I = -N_\alpha (N_\alpha + 1) + \alpha^2 \hat{\alpha}^2, \quad (B.22) \]

\[ M^{IJ} M^{IJ} = 2S^I S^I. \quad (B.23) \]

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