EXACT MULTIPLICITY OF STATIONARY LIMITING PROBLEMS OF A CELL POLARIZATION MODEL

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Abstract. We show existence, nonexistence, and exact multiplicity for stationary limiting problems of a cell polarization model proposed by Y. Mori, A. Jilkine and L. Edelstein-Keshet. It is a nonlinear boundary value problem with total mass constraint. We obtain exact multiplicity results by investigating a global bifurcation sheet which we constructed by using complete elliptic integrals in a previous paper.

1. Introduction. We are interested in wave-pinning in a reaction-diffusion model for cell polarization proposed by Y. Mori, A. Jilkine and L. Edelstein-Keshet [8] and [9].

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The model

\[
(\text{TP}) \begin{cases}
\varepsilon W_t = \varepsilon^2 W_{xx} + W(W-1)(V+1-W) & \text{in } (0,1) \times (0,\infty), \\
\varepsilon V_t = D V_{xx} - W(W-1)(V+1-W) & \text{in } (0,1) \times (0,\infty), \\
W_x(0,t) = W_x(1,t) = 0, & V_x(0,t) = V_x(1,t) = 0 \text{ in } (0,\infty), \\
W(x,0) = W_0(x), & V(x,0) = V_0(x) \text{ in } (0,1)
\end{cases}
\]

is proposed in [9], where \( W = W(x,t) \) denotes the density of an active protein, \( V = V(x,t) \) denotes the density of an inactive protein, \( \varepsilon, D \) are diffusion coefficients, \( W_0(x) \) denotes an initial density of the active protein, and \( V_0(x) \) denotes an initial density of the inactive protein. The mass conservation

\[
\int_0^1 (W(x,t) + V(x,t)) dx = \int_0^1 (W_0(x) + V_0(x)) dx = m
\]

holds, where \( m \) is the total mass determined by the mass of the initial densities \( W_0(x) \) and \( V_0(x) \).

The stationary problem of (TP) is

\[
(\text{SP}) \begin{cases}
\varepsilon^2 W_{xx} + W(W-1)(V+1-W) = 0 & \text{in } (0,1), \\
D V_{xx} - W(W-1)(V+1-W) = 0 & \text{in } (0,1), \\
W(x) > 0, & V(x) > 0 \text{ in } (0,1), \\
W_x(0) = W_x(1) = 0, & V_x(0) = V_x(1) = 0, \\
\int_0^1 (W(x) + V(x)) dx = m,
\end{cases}
\]

where \( W = W(x) \), \( V = V(x) \), and \( m \) is a given initial total mass determined by initial densities. Letting \( D \to \infty \) in (SP), we obtain a limiting equation. In addition, we concentrate on monotone increasing solutions for simplicity, since we can obtain other solutions by reflecting this kind of solutions. Thus we obtain the following stationary limiting problem

\[
(\text{SLP}) \begin{cases}
\varepsilon^2 W_{xx} + W(W-1)(\tilde{V} + 1-W) = 0 & \text{in } (0,1), \\
W_x(0) = W_x(1) = 0, \\
W_x(x) > 0 \text{ in } (0,1), & \tilde{V} > 0, \\
\int_0^1 W(x) dx + \tilde{V} = m,
\end{cases}
\]

where \( m, \varepsilon \) are given positive constants, \( W = W(x) \) is an unknown function, and \( \tilde{V} \) is an unknown nonnegative constant. Here, we note that we may omit a condition \( W(0) > 0 \) since we can derive it from other conditions.

Interesting bifurcation diagrams are obtained in [9] by numerical computations. Kuto and Tsujikawa [5] obtained several mathematical results for (SLP) with suitable change of variables. (see, also [3] and [4]) We have obtained the exact expressions of all the solutions of (SLP) by using the Jacobi elliptic functions and complete elliptic integrals in Mori, Kuto, Nagayama, Tsujikawa and Yotsutani [7]. The method to obtain all the exact solutions is essentially based on the method which started in Lou, Ni and Yotsutani [6]. It is developed by Kosugi, Morita and Yotsutani [2] to investigate the Cahn-Hilliard equation treated in Carr, Gurtin and Semrod [1].
Now, let us introduce an auxiliary problem to investigate (SLP). Let $\tilde{V} > 0$ be given, let us consider the problem

$$
\begin{align*}
\text{(AP; } \tilde{V}) \quad &\left\{ \begin{array}{l}
\varepsilon^2 W_{xx} + W(W - 1)(\tilde{V} + 1 - W) = 0 \quad \text{in } (0,1), \\
W_x(0) = W_x(1) = 0, \\
W_x(x) > 0 \quad \text{in } (0,1).
\end{array} \right.
\end{align*}
$$

(1.5)

The following fact is fundamental (see, e.g. Smoller and Wasserman [11], Smoller [10], and Theorem 2.1 in [7]).

There exists a solution of (AP; $\tilde{V}$), if and only if $(\tilde{V}, \varepsilon^2) \in \mathcal{G}$, where

$$
\mathcal{G} := \left\{ (\tilde{V}, \varepsilon^2) : 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\}.
$$

(1.8)

Moreover, the solution is unique, is represented by elliptic integrals, and has properties

$$
0 < W(x; \tilde{V}, \varepsilon^2) < \tilde{V} + 1,
$$

(1.9)

$$
W(x; \tilde{V}, \varepsilon^2) = \tilde{V} + 1 - \tilde{V} \cdot W \left(1 - x; \frac{1}{\tilde{V}^2}, \frac{\varepsilon^2}{\tilde{V}^2}\right).
$$

(1.10)

Let us define the global bifurcation sheet $S$ by

$$
S := \left\{ (\tilde{V}, \varepsilon^2, m(\tilde{V}, \varepsilon^2)) : (\tilde{V}, \varepsilon^2) \in \mathcal{G} \right\},
$$

(1.11)

where

$$
m(\tilde{V}, \varepsilon^2) := \int_0^1 W(x; \tilde{V}, \varepsilon^2) \, dx + \tilde{V}.
$$

(1.12)

We note that

$$
m(\tilde{V}, \varepsilon^2) = 2\tilde{V} + 2 - \tilde{V} \cdot m \left(\frac{1}{\tilde{V}^2}, \frac{\varepsilon^2}{\tilde{V}^2}\right) \quad \text{for any } \tilde{V} > 0, \varepsilon > 0
$$

(1.13)

by (1.10), and which implies

$$
m(1, \varepsilon^2) = 2 \quad \text{for any } \varepsilon^2 \in \left(0, \frac{1}{\pi^2}\right).
$$

(1.14)

We see from Theorem 2.2 in [7] that $m(\tilde{V}, \varepsilon^2)$ is represented by complete elliptic integrals. For given $m > 0$, level curve with the height $m$ of the global bifurcation sheet $S$ corresponds to the bifurcation diagram in the plane $(\tilde{V}, \varepsilon^2)$ for (SLP) with given $m$. Thus, for each $m$, we can obtain the bifurcation diagram by

$$
\left\{ (\tilde{V}, \varepsilon^2) \in \mathcal{G} : m(\tilde{V}, \varepsilon^2) = m \right\}.
$$

(1.15)

In Figure 1.1, we show the global bifurcation sheet and bifurcation diagrams of (SLP) which are obtained numerically in [7]. The function $m(\tilde{V}, \varepsilon^2)$ has the following properties.

**Proposition 1.1.** Let $\tilde{V} \in (0, \infty)$ be fixed. The following holds:

(i) $m(\tilde{V}, \varepsilon^2) \to \tilde{V} + 1$ as $\varepsilon^2 \to \tilde{V}/\pi^2$.

(ii) For $\tilde{V} \in (0,1)$, $m(\tilde{V}, \varepsilon^2) \to 2\tilde{V} + 1$ as $\varepsilon^2 \to 0$.

(iii) For $\tilde{V} \in (1,\infty)$, $m(\tilde{V}, \varepsilon^2) \to \tilde{V} + 1$ as $\varepsilon^2 \to 0$.

(1.16) (1.17) (1.18)

The following theorems are the main results of the paper.
Theorem 1.1. Let \( m(\tilde{V}, \varepsilon^2) \) be the function defined by (1.12), and \( \tilde{V} > 0 \) be fixed. It holds that

\[
\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} < 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \quad \text{with} \quad \tilde{V} \in (0, 1). \tag{1.19}
\]

\[
\frac{\partial m(1, \varepsilon^2)}{\partial \varepsilon} \equiv 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{1}{\pi^2}\right). \tag{1.20}
\]

\[
\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} > 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \quad \text{with} \quad \tilde{V} \in (1, \infty). \tag{1.21}
\]

As applications of Proposition 1.1 and Theorem 1.1, we can obtain exact multiplicity results for (SLP).

Theorem 1.2. Let \( 0 < m \leq 1 \) be given. There exists no solution of (SLP).

Theorem 1.3. Let \( 1 < m < 2 \) be given. The following holds:

(i) There exists no solution of (SLP) for \( \tilde{V} \in (0, (m - 1)/2] \cup [m - 1, 1] \cup [m, \infty) \).

(ii) There exists the unique \( \varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi) \) such that \( W(x; \tilde{V}, \varepsilon^2(\tilde{V})) \) is a solution of (SLP) for \( \tilde{V} \in ((m - 1)/2, m - 1) \).

(iii) There exists the unique \( \varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi) \) such that \( W(x; \tilde{V}, \varepsilon^2(\tilde{V})) \) is a solution of (SLP) for \( \tilde{V} \in (1, m) \).

Theorem 1.4. Let \( m = 2 \) be given. The following holds:

(i) There exists no solution of (SLP) for \( \tilde{V} \in (0, 1/2] \cup [2, \infty) \).

(ii) There exists the unique \( \varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi) \) such that \( W(x; \tilde{V}, \varepsilon^2(\tilde{V})) \) is a solution of (SLP) for \( \tilde{V} \in (1/2, 1) \).

(iii) For \( \tilde{V} = 1 \), there exists no solution of (SLP) for \( \varepsilon \in [1/\pi, \infty) \), and there exists the unique solution \( W(x; 1, \varepsilon^2) \) of (SLP) for \( \varepsilon \in (0, 1/\pi) \).

(iv) There exists the unique \( \varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi) \) such that \( W(x; \tilde{V}, \varepsilon^2(\tilde{V})) \) is a solution of (SLP) for \( \tilde{V} \in (1, 2) \).
Figure 1.2. Profiles of $W(x; \tilde{V}, \varepsilon^2)$ for $m = 2$.

We show several profiles of $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ corresponding to $(\tilde{V}, \varepsilon^2(\tilde{V}))$ assured by Theorem 1.4 in Figure 1.2.

**Theorem 1.5.** Let $2 < m < 3$ be given. The following holds:

(i) There exists no solution of (SLP) for $\tilde{V} \in (0, (m - 1)/2] \cup [1, m - 1] \cup ]m, \infty)$.  
(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}/\pi})$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in ((m - 1)/2, 1)$.  
(iii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}/\pi})$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (m - 1, m)$.

**Theorem 1.6.** Let $m \geq 3$ be given. The following holds:

(i) There exists no solution of (SLP) for $\tilde{V} \in (0, m - 1] \cup [m, \infty)$,  
(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}/\pi})$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (m - 1, m)$.
We see from Theorem 1.4 and Figure 1.2 that the symmetric breaking occurs at secondary bifurcation point for the case \( m = 2 \) and there is an imperfect transcritical bifurcation phenomena in the neighborhood of \( m = 2 \). Moreover, this case is very delicate, since the shape of bifurcation curves drastically change.

We will discuss continuity, smoothness and limiting values of end points of \( \varepsilon^2(\tilde{V}) \) including the unique secondary bifurcation point in a forthcoming paper.

This paper is organized as follows. In Section 2 we give proofs of Theorems 1.2 - 1.6 by using Proposition 1.1 and Theorem 1.1, which we prove in subsequent sections. In Section 3 we prepare some fundamental facts to give proofs of Proposition 1.1 and Theorem 1.1. In Section 4 we give a proof of Proposition 1.1. In Section 5 we give a proof of Theorem 1.1 by using Propositions 5.1 - 5.6 which we prove in subsequent sections. Propositions 5.1 and 5.2 show the expression of \( \partial m(\tilde{V}, \varepsilon^2) / \partial \varepsilon \) by using parameters \( h \) and \( s \). Propositions 5.3 and 5.4 show properties of \( j(h, s) \), which decides sign of the \( \partial m(\tilde{V}, \varepsilon^2) / \partial \varepsilon \). Propositions 5.5 and 5.6 show that \( j(h, s) \) is concave in \( s \) for each fixed \( h \). In the proof of Proposition 5.6 we use Lemmas 9.1 - 9.6. In particular, Lemma 9.2 is main calculation. We prove Lemma 9.2 using by Lemmas A.1 - A.5, Lemmas B.1 and B.2 in Appendix A and B respectively.

In Section 6 we give a proofs of Propositions 5.1 and 5.2. In Section 7 we give a proofs of Propositions 5.3 and 5.4. In Section 8 we give a proof of Proposition 5.5. In Section 9 we give a proof of Proposition 5.6. In Appendix A we give proofs of several inequalities including ratios of complete elliptic integrals. In Appendix B we treat more complicated inequalities.

2. Proofs of Theorem 1.2 - 1.6. In this section we give proofs of Theorems 1.2 - 1.6 by using Proposition 1.1 and Theorem 1.1 which we prove later. It is easy to see that the following lemma holds.

**Lemma 2.1.** Let \( \tilde{V} > 0 \) and \( m > 0 \). The following equivalence holds:

(i) It holds that

\[
\begin{align*}
\tilde{V} + 1 < m < 2\tilde{V} + 1, \\
0 < \tilde{V} < 1,
\end{align*}
\]

\[
\Leftrightarrow \max \left( 0, \frac{m - 1}{2} \right) < \tilde{V} < \min (1, m - 1).
\]

(ii) It holds that

\[
\begin{align*}
\tilde{V} < m < \tilde{V} + 1, \\
\tilde{V} > 1,
\end{align*}
\]

\[
\Leftrightarrow \max (1, m - 1) < \tilde{V} < m.
\]

**Lemma 2.2.** Let \( D_m \) be defined by

\[
D_m := \left( \max \left( 0, \frac{m - 1}{2} \right), \min (1, m - 1) \right) \cup (\max (1, m - 1), m).
\]

Then it holds that

\[
D_m = \emptyset \quad \text{for} \quad 0 \leq m \leq 1,
\]

\[
D_m = \left( \frac{m - 1}{2}, m - 1 \right) \cup (1, m) \quad \text{for} \quad 1 < m < 2,
\]
\[ D_m = \left( \frac{1}{2}, 1 \right) \cup (1, 2) \quad \text{for} \quad m = 2, \]
\[ D_m = \left( \frac{m-1}{2}, 1 \right) \cup (m-1, m) \quad \text{for} \quad 2 < m < 3, \]
\[ D_m = (m-1, m) \quad \text{for} \quad m \geq 3. \]

We obtain conclusions of Theorems 1.2 - 1.6 by using Proposition 1.1, Theorem 1.1, Lemma 2.1, Lemma 2.2 and (1.14).

3. Preliminaries. In this section we prepare some fundamental facts to give proofs of Proposition 1.1 and Theorem 1.1.

3.1. Definition of the elliptic functions. Let \( sn(x, k) \) and \( cn(x, k) \) be Jacobi’s elliptic functions. The following properties hold:

\[ sn^{-1}(z, k) = \int_0^z \frac{1}{\sqrt{1-k^2 \xi^2} \sqrt{1-\xi^2}} \, d\xi \quad (-1 \leq z \leq 1), \]
\[ sn^2(x, k) + cn^2(x, k) = 1, \quad cn(0, k) = 1. \]

Let \( k \in [0, 1) \) and \(-1 < \nu < 1\). The complete elliptic integrals of the first, second and third kind are defined by

\[ K(k) := \int_0^1 \frac{1}{\sqrt{1-k^2 t^2} \sqrt{1-t^2}} \, dt, \quad E(k) := \int_0^1 \frac{\sqrt{1-k^2 t^2}}{\sqrt{1-t^2}} \, dt, \]
and
\[ \Pi(\nu, k) := \int_0^1 \frac{1}{(1+\nu t^2) \sqrt{1-k^2 t^2} \sqrt{1-t^2}} \, dt, \]
respectively. We see that \( K(k) \) is monotone increasing in \( k \),

\[ K(0) = \frac{\pi}{2}, \quad \lim_{k \to 1} K(k) = \infty \]
and \( E(k) \) is monotone decreasing in \( k \),

\[ E(0) = \frac{\pi}{2}, \quad \lim_{k \to 1} E(k) = 1. \]

The following formulas for the complete elliptic integrals are fundamental:

\[ \frac{d}{dk}K(k) = -\frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k}, \quad \frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}, \]
\[ \frac{\partial}{\partial k} \Pi(\nu, k) = \frac{kE(k)}{(k^2 + \nu)(1-k^2)} - \frac{k \Pi(\nu, k)}{k^2 + \nu}, \]
\[ \frac{\partial}{\partial \nu} \Pi(\nu, k) = \frac{(k^2 - \nu^2) \Pi(\nu, k)}{2(1+\nu)(k^2+\nu)} - \frac{K(k)}{2(1+\nu)} + \frac{E(k)}{2(1+\nu)(k^2+\nu)}, \]
and

\[ \frac{d}{dh}K(\sqrt{h}) = \frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{2h(1-h)}, \quad \frac{d}{dh}E(\sqrt{h}) = \frac{E(\sqrt{h}) - K(\sqrt{h})}{2h}, \]
\[ \frac{d}{dH}K(\sqrt{1-H^2}) = \frac{E(\sqrt{1-H^2}) - H^2 K(\sqrt{1-H^2})}{H(1-H^2)}, \]
\[ \frac{d}{dH}E(\sqrt{1-H^2}) = \frac{H(K(\sqrt{1-H^2}) - E(\sqrt{1-H^2}))}{1-H^2}. \]

It is easy to see that the following inequalities hold.
Lemma 3.1. It holds that
\[
\sqrt{1 - h} < \frac{E(\sqrt{h})}{K(\sqrt{h})} < 1 - \frac{h}{2} < 1 \quad (0 < h < 1),
\]
and
\[
H < \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} < 1 + \frac{H^2}{2} < 1 \quad (0 < H < 1).
\]

3.2. All exact solutions for \((AP; \tilde{V})\). In this section we show exact solution for \((AP; V)\) and an exact expression of \(m(\tilde{V}, \varepsilon^2)\) defined by (1.12).

Theorem A (Theorem 2.1 of [7]). Let \(\tilde{V} > 0\). There exists a solution of \((AP; \tilde{V})\), if and only if \((\tilde{V}, \varepsilon^2) \in \mathcal{G}\), where \(\mathcal{G}\) is defined by (1.8). Moreover, the solution is unique and it has properties (1.9) and (1.10).

The solution \(W(x; \tilde{V}, \varepsilon^2)\) is represented by

\[
W(x; \tilde{V}, \varepsilon^2) = \frac{\tilde{V} + 2}{3} + \frac{1}{\sqrt{3}} \sqrt{V^2 + \tilde{V} + 1} \cdot \beta \cdot (1 - hs) \text{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot \text{cn}^2(K(\sqrt{h})x, \sqrt{h}),
\]

where \((h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2))\) is the unique solution of the following system of transcendental equations

\[
\begin{align*}
E(h, s) &= \frac{\varepsilon}{\sqrt{V^2 + \tilde{V} + 1}}, \quad (3.4) \\
A(h, s) &= \frac{1}{3\sqrt{3}} (1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2), \quad (3.5) \\
0 < h < 1, & \quad 0 < s < 1, \quad (3.6)
\end{align*}
\]

where

\[
E(h, s) := \frac{\sqrt{2s(1-s)(1-sh)} / K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h^2 + 4)s^2 - 4(1 + h)s + 3}}, \quad (3.7)
\]

\[
A(h, s) := \frac{2hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h^2 + 4)s^2 - 4(1 + h)s + 3}}, \quad (3.8)
\]

Here, \(\text{sn}(\cdot, \cdot), \text{cn}(\cdot, \cdot)\) are Jacobi’s elliptic function. \(K(\cdot)\) is complete elliptic integral of the first kind.

Theorem B (Theorem 2.2 of [7]). Let \(W(x; \tilde{V}, \varepsilon^2)\) be the unique solution of \((AP; \tilde{V})\), and \(m(\tilde{V}, \varepsilon^2)\) is defined by (1.12). Then (1.13) and (1.14) hold.
Moreover, it holds that
\[
m(\bar{V}, \varepsilon^2) = \frac{4\bar{V} + 2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{V^2 + \bar{V} + 1} \cdot \mathcal{M}(h, s),
\]
(3.9)
\[
\mathcal{M}(h, s) := -\left( hs^2 - 2(1 + h)s + 3 \right) + 4(1 - s)(1 - sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})
\]
(3.10)
\[
\sqrt{3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(1 + h)s + 3},
\]
where \( h = h(\bar{V}, \varepsilon^2), s = s(\bar{V}, \varepsilon^2) \) are given in Theorem A. Here, \( K(\cdot) \) is the complete elliptic integral of the first kind, and \( \Pi(\cdot, \cdot) \) is the complete elliptic integral of the third kind.

3.3. Properties of fundamental functions. The following lemmas are used in the proofs of Theorems A and B. We also use them in this paper. We have the following lemma by Lemma 3.2 and the proof of Lemma 3.4 in [2].

**Lemma A.** Let \( \mathcal{E}(h, s) \) be defined by (3.7). The derivative of \( \mathcal{E}(h, s) \) with respect to \( s \) satisfies
\[
\frac{\partial}{\partial s} \mathcal{E}(h, s) \begin{cases} > 0, & s \in (0, \sigma(h)), \quad h \in [0, 1), \\ = 0, & s = \sigma(h), \quad h \in [0, 1), \\ < 0, & s \in (\sigma(h), 1), \quad h \in [0, 1), \end{cases}
\]
(3.11)
where \( \sigma(h) := 1/(1 + \sqrt{1 - h}) \). Moreover,
\[
\mathcal{E}(h, \sigma(h)) = \frac{1}{\sqrt{2(2 - h)K(\sqrt{h})}},
\]
(3.12)
\[
\frac{d}{dh} \mathcal{E}(h, \sigma(h)) < 0 \quad \text{for } h \in [0, 1),
\]
(3.13)
and
\[
\mathcal{E}(0, \sigma(0)) = \frac{1}{\pi}, \quad \mathcal{E}(h, \sigma(h)) \to 0 \quad \text{as } h \to 1.
\]
(3.14)
In addition,
\[
\mathcal{E}(0, s) = \frac{2\sqrt{2}s(1 - s)}{\pi \sqrt{4s^2 - 4s + 3}}.
\]
(3.15)

It is easy to see the following lemmas (see, Lemmas 6.2 - 6.5 in [7]).

**Lemma B.** Let
\[
r(v) := \frac{\sqrt{3}}{9} \cdot \frac{(1 - v)(2v + 1)(v + 2)}{\sqrt{v^2 + v + 1}}.
\]
(3.16)
Then \( r(v) \) is monotone decreasing in \((0, \infty)\) and
\[
r(0) = \frac{2\sqrt{3}}{9}, \quad r(1) = 0, \quad r(v) \to -\frac{2\sqrt{3}}{9} \quad \text{as } v \to \infty.
\]
(3.17)

**Lemma C.** Let \( \mathcal{A}(h, s) \) be defined by (3.8). Then
\[
\mathcal{A}(h, 0) = \frac{2\sqrt{3}}{9}, \quad \mathcal{A}(h, 1) = -\frac{2\sqrt{3}}{9} \quad \text{for all } h \in [0, 1),
\]
(3.18)
\[
\mathcal{A}(h, s) < 0 \quad \text{for all } (h, s) \in (0, 1) \times (0, 1).
\]
(3.19)
Lemma D. Let $\tilde{V} > 0$ be fixed. There exists a unique curve
\[ s(h; \tilde{V}) \in C([0, 1]) \cap C^\infty((0, 1)) \] (3.20)
such that
\[ \mathcal{A}(h, s(h; \tilde{V})) = \frac{1}{3\sqrt{3}} \cdot \frac{(1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2)}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} \cdot \sqrt{\tilde{V}}^2 - 4s(0; \tilde{V}) + 3 \]
(0 < s(h; \tilde{V}) < 1). (3.21)
Moreover,
\[ s(0; \tilde{V}) = \frac{1}{2} - \frac{1 - \tilde{V}}{\sqrt{2}\sqrt{(\tilde{V} + 2)(2\tilde{V} + 1)}} \] (3.22)
and
\[ s(h; 1) = \frac{1}{1 + \sqrt{1 - h}} \quad (0 < h < 1). \] (3.23)

Lemma E. Let $\mathcal{E}(h, s)$ be defined by (3.7), and $s(h; \tilde{V})$ defined in Lemma D, then for each fixed $\tilde{V} > 0$,
\[ \mathcal{E}(0, s(0; \tilde{V})) = \frac{\sqrt{3}\sqrt{\tilde{V}}}{\pi \sqrt{\tilde{V}^2 + \tilde{V} + 1}}, \] (3.24)
\[ \mathcal{E}(h, s(h; \tilde{V})) \to 0 \text{ as } h \to 1, \] (3.25)
and
\[ \frac{d\mathcal{E}(h, s(h; \tilde{V}))}{dh} < 0 \text{ in } (0, 1). \] (3.26)

4. Proof of Proposition 1.1. We will show (i). Let $\tilde{V} \in (0, 1)$ be fixed. We note that $(\tilde{V}, \varepsilon^2) \to (\tilde{V}, \tilde{V}/\pi^2)$ corresponds to
\[ (h, s(h; \tilde{V})) \to (0, s(0; \tilde{V})) = \left(0, \frac{1}{2} - \frac{1 - \tilde{V}}{\sqrt{2}\sqrt{(\tilde{V} + 2)(2\tilde{V} + 1)}}\right) \] (4.1)
by Lemma E and (3.22). Hence we get $m(\tilde{V}, \varepsilon^2) \to m(\tilde{V}, \tilde{V}/\pi^2)$ as $\varepsilon^2 \to \tilde{V}/\pi^2$ and
\[ m\left(\tilde{V}, \frac{\tilde{V}}{\pi^2}\right) = \frac{4\tilde{V} + 2}{3} + \frac{\sqrt{3}}{3} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \mathcal{A}(0, s(0; \tilde{V})) \]
\[ = \frac{4\tilde{V} + 2}{3} + \frac{\sqrt{3}}{3} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \frac{1 - 2s(0; \tilde{V})}{4s(0; \tilde{V})^2 - 4s(0; \tilde{V}) + 3} \]
\[ = \frac{4\tilde{V} + 2}{3} + \frac{\sqrt{3}}{3} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \frac{1 - \tilde{V}}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} \]
\[ = \tilde{V} + 1 \] (4.2)
by (3.10) and (3.22). Thus we obtain (1.16) in view of (1.13) and (1.14).
We will show (ii). Let $\tilde{V} \in (0, 1)$ be fixed. We see that $(\tilde{V}, \varepsilon) \to (\tilde{V}, 0)$ corresponds to
\[ (h, s(h; \tilde{V})) \to (1, s(1; \tilde{V})) \] (4.3)
by Lemma E and (3.21), where $s(1; \tilde{V})$ is the unique solution of
\[ \mathcal{A}(1, s) = r(\tilde{V}), \] (4.4)
and $0 < s(1; \tilde{V}) < 1$, where

$$A(1, s) = \frac{2(1-s)^2(s+1)}{(3s^2 - 2s + 3)^{3/2}}$$  \hspace{1cm} (4.5)$$

and $r(v)$ is defined by (3.16). In fact, it holds that

$$A(1, 0) = \frac{2\sqrt{3}}{9}, \quad A(1, 1) = 0, \quad 0 < r(\tilde{V}) < \frac{2\sqrt{3}}{9}$$  \hspace{1cm} (4.6)$$

and $A(1, s)$ is monotone decreasing in $s \in (0, 1)$.

By solving (4.4), we obtain

$$9\tilde{V}(\tilde{V} + 1)s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V} + 1) = 0,$$  \hspace{1cm} (4.7)$$

and

$$s(1; \tilde{V}) = \frac{7\tilde{V}^2 + 7\tilde{V} + 4 - 2(2\tilde{V} + 1)\sqrt{(2\tilde{V} + 4)(1 - \tilde{V})}}{9\tilde{V}(\tilde{V} + 1)},$$  \hspace{1cm} (4.8)$$

since

$$A(1, s)^2 - r(\tilde{V})^2$$

$$= 9\tilde{V}s^2 + (8\tilde{V}^2 + 2\tilde{V} + 8)s + 9\tilde{V}$$

$$= \frac{27(3s^2 - 2s + 3)^3(\tilde{V}^2 + \tilde{V} + 1)^3}{\left((9\tilde{V} + 9)s^2 - (8\tilde{V}^2 + 14\tilde{V} + 14)s + 9\tilde{V} + 9\right)\cdot (9(\tilde{V} + 1)\tilde{V}s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V} + 1))}$$

$$= \frac{9\tilde{V}s^2 + (8\tilde{V}^2 + 2\tilde{V} + 8)s + 9\tilde{V}}{27(3s^2 - 2s + 3)^3(\tilde{V}^2 + \tilde{V} + 1)^3}$$

$$\cdot \left((9\tilde{V} + 9)\left(s - \frac{4\tilde{V}^2 + 7\tilde{V} + 7}{9\tilde{V} + 1}\right)^2 + \frac{16\tilde{V} + 8)(1 - \tilde{V})(\tilde{V} + 2)^2}{9\tilde{V} + 9}\right)$$

$$\cdot (9(\tilde{V} + 1)\tilde{V}s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V} + 1)).$$  \hspace{1cm} (4.9)$$

We have

$$\lim_{h \to 1} M(h, s(h, \tilde{V})) = \frac{s(1; \tilde{V}) + 1}{\sqrt{3s(1; \tilde{V})^2 - 2s(1; \tilde{V}) + 3}}$$

$$= A(1, s(1; \tilde{V})) \cdot \frac{3s(1; \tilde{V})^2 - 2s(1; \tilde{V}) + 3}{2(1 - s(1; \tilde{V}))^2}$$

$$= \frac{1}{6\sqrt{3}} \cdot \frac{(1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2)(3s(1; \tilde{V})^2 - 2s(1; \tilde{V}) + 3)}{(\tilde{V}^2 + \tilde{V} + 1)^{3/2}(1 - s(1; \tilde{V}))^2},$$  \hspace{1cm} (4.10)$$

by $0 < s(1; \tilde{V}) < 1$ and

$$\lim_{h \to 1} \frac{(1 - hs(h; \tilde{V}))\Pi(-hs(h; \tilde{V}), \sqrt{h})}{K(\sqrt{h})} = 1.$$  \hspace{1cm} (4.11)$$
Hence
\[
\lim_{\varepsilon \to 0} m(\tilde{V}, \varepsilon^2) = \frac{4\tilde{V} + 2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1} \cdot \lim_{h \to 1} M(h, s(h; \tilde{V}))
\]
\[
= \frac{4\tilde{V} + 2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2 + \tilde{V} + 1}
\]
\[
\cdot \left( \frac{1}{6\sqrt{3}} \cdot \frac{(1 - \tilde{V})(2\tilde{V} + 1)(\tilde{V} + 2)(3s^2 - 2s + 3)}{(V^2 + V + 1)^{3/2}(1 - s)^2} \right) - (2\tilde{V} + 1) + (2\tilde{V} + 1)
\]
\[
= \frac{1}{18} \left( 2\tilde{V} + 1 \right) \frac{(9\tilde{V}(\tilde{V} + 1)s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V} + 1))}{(V^2 + V + 1)(1 - s)^2} + (2\tilde{V} + 1).
\]
\[
= 2\tilde{V} + 1
\]
(4.12)
by (4.10) and (4.7), where \( s = s(1; \tilde{V}) \). Thus, we get (1.17).

We obtain (iii) by (ii) and (1.13).

5. Proof of Theorem 1.1. We prepare several propositions to prove Theorem 1.1. We will give proofs of them in subsequent sections.

The following properties show the expression of \( \partial m(\tilde{V}, \varepsilon^2)/\partial \varepsilon \).

**Proposition 5.1.** Let \( \tilde{V} > 0 \) be fixed. Then, \( m(\tilde{V}, \varepsilon^2) \) satisfies the following equation

\[
\frac{\partial m(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} = -\frac{M_s \cdot A_h - M_h \cdot A_s}{A_s \cdot \frac{d\xi(h, s(h; \tilde{V}))}{dh}},
\]
(5.1)

where \( \xi(h, s) \) is defined by (3.7), \( s(h; \tilde{V}) \) is defined in Lemma D,

\[
A_s(h, s) = -32 (D^{K^2})^{-5/2} s (1 - s) (1 - hs)
\]
\[
\cdot \left( (h^2 - h + 1) h^2 s^4 - 2(h + 1) h^2 s^3 + 6h^2 s^2 - 2(h + 1) hs + h^2 - h + 1 \right),
\]
(5.2)

\[
A_h(h, s) = -16 (D^{K^2})^{-5/2} s^2 (1 - s)^2 (1 - hs) \left( s^3 h^2 + (-2s^3 + 3s^2 - 3s + 2) h - 1 \right).
\]
(5.3)

\[
M_s(h, s) = -2s^{-1} (D^{K^2})^{-3/2} K(\sqrt{h})^{-1}
\]
\[
\cdot \left( -3 (1 - hs^2) (hs^2 - 2hs + 1) (hs^2 - 2s + 1) \Pi(-hs, \sqrt{h}) + s \cdot D^{K^2} \cdot E(\sqrt{h}) \right.
\]
\[
+ (hs^2 - 2s + 1) (2h^2 s^3 - hs^3 - hs^2 - 4hs + 2s^2 - s + 3) K(\sqrt{h}) \big) \bigg),
\]
(5.4)

\[
M_h(h, s) = 2h^{-1} (1 - h)^{-1} (D^{K^2})^{-3/2} K(\sqrt{h})^{-2} (1 - s) \left( \left( - (1 - hs) \cdot D^{K^2} \cdot E(\sqrt{h}) \right. \right.
\]
\[
+ (1 - h) (h^2 s^4 + 2h^2 s^3 - 8hs^3 + 8hs^2 - 6hs + 4s^2 - 4s + 3) K(\sqrt{h}) \bigg) \Pi(-hs, \sqrt{h})
\]
\[
+ D^{K^2} \cdot E(\sqrt{h}) K(\sqrt{h})
\]
\[
- (1 - h) \left( h^2 s^4 - 4 h s^3 + 4 h s^2 - 4 h s + 4 s^2 - 4 s + 3 \right) K(\sqrt{h})^2, \quad (5.5)
\]
\[
D^{K^2}(h, s) := (3s^4 - 4s^3 + 4s^2) h^2 + (-4s^3 + 2s^2 - 4s) h + 4s^2 - 4s + 3. \quad (5.6)
\]

Moreover, it holds that
\[
D^{K^2}(h, s) > 0 \quad (0 < h < 1, \ 0 < s < 1). \quad (5.7)
\]

**Proposition 5.2.** Let \( \bar{V} > 0 \) be fixed. Then, the following equation holds:
\[
\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s = \frac{32s(1-s)^2(1-hs)^2}{h(1-h)(D^{K^2})^{-1} \cdot K(\sqrt{h})^2} \cdot \frac{\sqrt{s} \cdot \mathcal{N}^{III}}{(1-s)(1-sh)} \cdot \mathcal{J}, \quad (5.8)
\]

where
\[
\mathcal{N}^{III}(h, s) := \left( 2s^4h^4 - (2s^4 - 4s^3) h^3 \right.
\]
\[
+ (2s^4 - 4s^3 + 12s^2 - 4s + 2) h^2 - (4s + 2) h + 2 \Big) E(\sqrt{h})
\]
\[
- \left( s^4h^4 + (-3s^4 + 4s^3 - 6s^2) h^3 \right.
\]
\[
+ (2s^4 - 4s^3 + 6s^2 + 4s + 1) h^2 + (-4s - 3) h + 2 \Big) K(\sqrt{h}), \quad (5.9)
\]
\[
\mathcal{J}(h, s) := -\frac{\sqrt{1-s)(1-sh)}}{\sqrt{s}} \cdot \Pi(-hs, \sqrt{h})
\]
\[
+ \frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}} \cdot \mathcal{N}^{KE} K(\sqrt{h}) E(\sqrt{h}) + \mathcal{N}^{K^2} K(\sqrt{h})^2 \mathcal{N}^{III}, \quad (5.10)
\]
\[
\mathcal{N}^{KE}(h, s) := -h^3 s^3 - h^2 s^3 + 6h^2 s^2 - 3h^2 s - 3hs - 2h + 2, \quad (5.11)
\]
\[
\mathcal{N}^{K^2}(h, s) := (1 - h) \left( h^2 s^3 - 3h^2 s^2 + 3hs + h - 2 \right). \quad (5.12)
\]

Moreover, it holds that
\[
\mathcal{N}^{III}(h, s) > 0 \quad (0 < h < 1, \ 0 < s < 1). \quad (5.13)
\]

We note that \( (\bar{V}, \varepsilon^2) \) with \( 0 < \bar{V} < 1 \) and \( 0 < \varepsilon^2 < \bar{V}/\pi^2 \) corresponds to \( (h, s) \) with \( 0 < h < 1 \) and \( 0 < s < 1/(1 + \sqrt{1-h}) \) by (3.23).

Now we investigate \( \mathcal{J}(h, s) \).

**Proposition 5.3.** Let \( \mathcal{J}(h, s) \) be defined by (5.10) For each fixed \( h \in (0, 1) \), it holds that
\[
\lim_{s \to 0} \mathcal{J}(h, s) = 0, \quad (5.14)
\]
\[
\mathcal{J} \left( h, \frac{1}{1 + \sqrt{1-h}} \right) = 0 \quad (5.15)
\]

for each fixed \( h \in (0, 1) \).

**Proposition 5.4.** Let \( \mathcal{J}(h, s) \) be defined by (5.10). Then \( \mathcal{J}(h, s) \) satisfies the following equation
\[
\frac{\partial}{\partial s} \mathcal{J}(h, s) = \frac{F(h, s)K(\sqrt{h})^3}{\sqrt{s}(1-s)(1-sh) \cdot (\mathcal{N}^{III})^2}, \quad (5.16)
\]
where
\[
F(h, s) := h^4 C^{F_0}(h, U(h))s^8 - 2h^4 C^{F_1}(h, U(h))s^7 + 2h^4 C^{F_2}(h, U(h))s^6 \\
- 2h^3 C^{F_3}(h, U(h))s^5 + 2h^2 C^{F_4}(h, U(h))s^4 - 2h C^{F_5}(h, U(h))s^3 \\
+ 2h^2 C^{F_6}(h, U(h))s^2 - 2h C^{F_1}(h, U(h))s + C^{F_0}(h, U(h)),
\]
(5.17)
\[
C^{F_0}(h, u) := 2 \left( h^2 - h + 1 \right)^2 u^3 - 3 \left( 2 - h \right) \left( h^2 - h + 1 \right) u^2 \\
+ 3 \left( h^2 - 2h + 2 \right) \left( 1 - h \right)^2 u - \left( 2 - h \right) \left( 1 - h \right)^3,
\]
(5.18)
\[
C^{F_1}(h, u) := 4 \left( h + 1 \right) \left( h^2 - h + 1 \right) u^3 - 3 \left( 1 - h \right) \left( 3h^2 - 3h + 4 \right) u^2 \\
+ 6 \left( h^2 - 2h + 2 \right) \left( 1 - h \right)^2 u - \left( 4 - 3h \right) \left( 1 - h \right)^3,
\]
(5.19)
\[
C^{F_2}(h, u) := 4 \left( 4h^2 - h + 4 \right) u^3 - \left( 1 - h \right) \left( 23h^2 - 6h + 25 \right) u^2 \\
+ 2 \left( 3h^2 - 4h + 4 \right) \left( 1 - h \right)^2 u + \left( 3h + 1 \right) \left( 1 - h \right)^3,
\]
(5.20)
\[
C^{F_3}(h, u) := 4 \left( h + 1 \right) \left( 2h^2 + 5h + 1 \right) u - \left( 1 - h \right) \left( 16h^3 + 31h^2 + 41h - 4 \right) u^2 \\
+ 2 \left( h^2 + 6h - 10 \right) \left( 1 - h \right)^2 u + \left( 5h + 12 \right) \left( 1 - h \right)^3,
\]
(5.21)
\[
C^{F_4}(h, u) := 2 \left( h^4 + 2h^3 + 29h^2 + 2h + 1 \right) u^3 \\
- \left( 1 - h \right) \left( 8h^4 + 17h^3 + 87h^2 - 5h - 2 \right) u^2 \\
- \left( 8h^3 - 33h^2 + 30h + 10 \right) \left( 1 - h \right)^2 u - \left( 2h^2 - 21h - 6 \right) \left( 1 - h \right)^3,
\]
(5.22)
\[
U(h) := \frac{E(\sqrt{h})}{K(\sqrt{h})}.
\]
(5.23)

Let us consider properties of \( F(h, s) \).

**Proposition 5.5.** Let \( h \in (0, 1) \) be fixed. Then,
\[
F(h, 0) = C^{F_0} \left( h, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right) > 0,
\]
(5.24)
\[
F \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) = -\frac{8(1 - h)^4}{\left( 1 + \left( 1 - h \right)^{1/2} \right)^4} \cdot g_1 \left( \sqrt{1 - h}, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right) < 0,
\]
(5.25)
where \( F(h, s) \) is defined by (5.17),
\[
g_1(H, u) := \left( (1 + H^2)u - 2H^2 \right) \left( -\left( 1 + H^2 \right)u^2 + 2(1 + H^4)u - (1 + H^2)H^2 \right).
\]
(5.26)

**Proposition 5.6.** Let \( h \in (0, 1) \) be fixed. Then, the following equation
\[
F(h, s) = 0 \quad 0 < s < \frac{1}{1 + \sqrt{1 - h}}
\]
(5.27)
in \( s \) has the unique solution, where \( F(h, s) \) is defined by (5.17).

**Proof of Theorem 1.1.** We have
\[
\mathcal{J}(h, s) > 0 \quad 0 < h < 1, \quad 0 < s < \frac{1}{1 + \sqrt{1 - h}}
\]
(5.28)
by Propositions 5.3 - 5.6. Hence, we obtain (1.19) by Propositions 5.1 and 5.2, Lemmas C and E. □
We get (1.20) by (1.14). We obtain (1.21) by (1.19) and (1.13).

6. Proofs of Propositions 5.1 and 5.2. Let \( \mathcal{M}(h, s) \), \( \mathcal{E}(h, s) \), \( \mathcal{A}(h, s) \) be defined by (3.10), (3.7), (3.8) respectively, \( s(h; \hat{V}) \) is defined in Lemma D.

**Proof of Proposition 5.1.** Let \( \hat{V} > 0 \) be fixed. Then \( s = s(h; \hat{V}) \) by lemma D. By the equation of (1.12), we get

\[
\frac{\partial m(\hat{V}, \varepsilon^2)}{\partial \varepsilon} = -\frac{d\mathcal{M}(h, s(h; \hat{V}))}{dh} \frac{\partial \varepsilon}{\partial h}.
\]

Now, it hold that

\[
\frac{d\mathcal{E}(h, s(h; \hat{V}))}{dh} = \sqrt{\frac{3}{V^2 + V + 1}} \frac{\partial \varepsilon}{\partial h},
\]

\[
\frac{d\mathcal{M}(h, s(h; \hat{V}))}{dh} = \mathcal{M}_h + \mathcal{M}_s \cdot \frac{ds(h; \hat{V})}{dh}
\]

and

\[
\mathcal{A}_h + \mathcal{A}_s \cdot \frac{ds(h; \hat{V})}{dh} = 0.
\]

Therefore,

\[
\frac{\partial m(\hat{V}, \varepsilon^2)}{\partial \varepsilon} = \frac{\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s}{\mathcal{A}_s \cdot d\mathcal{E}(h, s(h; \hat{V}))}.
\]  \hspace{1cm} (6.1)

Thus, we get (5.1).

We have (5.7) by

\[
\mathcal{D}^K(h, s) = (3s^4 - 4s^3 + 4s^2) \left( h - \frac{2s^2 - s + 2}{s(3s^2 - 4s + 4)}\right)^2 + \frac{8(s^2 - s + 1)(s - 1)^2}{3s^2 - 4s + 4} > 0.
\]

Thus, we complete the proof. \(\square\)

**Proof of Proposition 5.2.** We have (5.2) - (5.5). Multiplying \( \mathcal{M}_s \) by \( \mathcal{A}_h \), we get

\[
\mathcal{M}_s \cdot \mathcal{A}_h
\]

\[
= 32s (1-s)^2 (1-hs) (h^2 s^3 - 2hs^3 + 3hs^2 - 3hs + 2h - 1)
\]

\[
\cdot \left( 3(hs^2 - 1)(hs^2 - 2hs + 1)(hs^2 - 2s + 1) \Pi(-hs, \sqrt{h})
\right.
\]

\[
+ s\mathcal{D}^K E(\sqrt{h})
\]

\[
+ (hs^2 - 2s + 1) (2h^2 s^3 - hs^3 - hs^2 - 4hs + 2s^2 - s + 3) K(\sqrt{h})
\]

\[
\cdot (\mathcal{D}^K)^{-4} K(\sqrt{h})^{-1}.
\]

Multiplying \( \mathcal{M}_h \) by \( \mathcal{A}_s \), we get

\[
\mathcal{M}_h \cdot \mathcal{A}_s
\]

\[
= -4 (1-s) (hs^2 - 2hs + 1)(hs^2 - 2s + 1)(1-hs^2)
\]

\[
\cdot \left( (1-hs) \mathcal{D}^K E(\sqrt{h})
\right.
\]
Let \( E \) and \( F \) be fixed. We have
\[
\begin{align*}
&- (1 - h) \left( h^2 s^4 + 2h^2 s^3 - 8hs^3 + 8hs^2 - 6hs + 4s^2 - 4s + 3 \right) K(\sqrt{h}) 
\Pi(-hs, \sqrt{h}) \\
&- D^{K^2}E(\sqrt{h})K(\sqrt{h}) \\
&(1 - h) \left( h^2 s^4 - 4hs^3 + 4hs^2 - 4hs + 4s^2 - 4s + 3 \right) K(\sqrt{h})^2 \\
&\cdot (h(1 - h))^{-1} \left( D^{K^2} \right)^{-3} K(\sqrt{h})^{-2}.
\end{align*}
\]
Hence we obtain
\[
M_s \cdot A_h - M_h \cdot A_s
= \frac{32s(1 - s)^2(1 - hs)^2}{h(1 - h)(D^{K^2})^3 \cdot K(\sqrt{h})^2}
\cdot \left( -N^{II} \cdot \Pi(-hs, \sqrt{h}) + N^{KE} \cdot K(\sqrt{h})E(\sqrt{h}) + N^{K^2}K(\sqrt{h})^2 \right)
\]
(6.2)
by direct calculation. Thus (5.8) is obvious from (6.2).

We show (5.13). We see from the proof of Lemma E of [7] that
\[
E(\sqrt{h}) - K(\sqrt{h})\sqrt{1 - h} > 0 \quad (0 < h < 1),
\]
\[
F_1(h, s) > 0 \quad (0 < h < 1, \ 0 < s < 1),
\]
and
\[
N^{II}(h, s) = F_1(h, s)E(\sqrt{h}) - F_2(h, s)K(\sqrt{h})
= K(\sqrt{h}) \left( F_1(h, s) \frac{E(\sqrt{h})}{K(\sqrt{h})} - F_2(h, s) \right)
> K(\sqrt{h})(F_1(h, s)\sqrt{1 - h} - F_2(h, s))
> 0 \quad (0 < h < 1, \ 0 < s < 1),
\]
where
\[
F_1(h, s) := 2s^2 h^4 - (2s^4 - 4s^3) h^3 \\
+ (2s^4 - 4s^3 + 12s^2 - 4s + 2) h^2 - (4s + 2) h + 2,
\]
\[
F_2(h, s) := s^4 h^4 + (-3s^4 + 4s^3 - 6s^2) h^3 \\
+ (2s^4 - 4s^3 + 6s^2 + 4s + 1) h^2 + (-4s - 3) h + 2.
\]
Thus we complete the proof.

7. Proofs of Propositions 5.3 and 5.4. We prepare two lemmas to prove Proposition 5.3.

**Lemma 7.1.** Let \( h \in (0, 1) \) be fixed. \( J(h, s) \) defined by (5.10) satisfies (5.14).

**Proof.** Let \( h \in (0, 1) \) be fixed. We have
\[
J(h, s) \cdot \sqrt{s}
= -\sqrt{(1 - s)(1 - sh)} \cdot \Pi(-hs, \sqrt{h}) \\
+ \sqrt{(1 - s)(1 - sh)} \cdot K(\sqrt{h}) \cdot \frac{N^{KE}(h, s)E(\sqrt{h}) + N^{K^2}(h, s)K(\sqrt{h})}{N^{II}(h, s)} = (*).
\]
We note that
\[
\mathcal{N}^E(h, 0) = 2 \left( h^2 - h + 1 \right) E(\sqrt{h}) - (1 - h) (2 - h) K(\sqrt{h}) \\
\geq \left( 2 \left( h^2 - h + 1 \right) \sqrt{1 - h} - (1 - h) (2 - h) \right) K(\sqrt{h}) \\
= \sqrt{1 - h} \left( 4 - 2h + 3 \sqrt{1 - h} \right) \left( \sqrt{1 - h} - 1 \right)^2 > 0 \quad (0 < h < 1).
\]
Hence we get
\[
(*)|_{s = 0} = \Pi(0, \sqrt{h}) + K(\sqrt{h}) \cdot \frac{\mathcal{N}^K E(h, 0) E(\sqrt{h}) + \mathcal{N}^K^2(h, 0) K(\sqrt{h})}{\mathcal{N}^E(h, 0)} \\
= -K(\sqrt{h}) + K(\sqrt{h}) \cdot \frac{2 \left( h^2 - h + 1 \right) E(\sqrt{h}) - (1 - h) (2 - h) K(\sqrt{h})}{2 \left( h^2 - h + 1 \right) E(\sqrt{h}) - (1 - h) (2 - h) K(\sqrt{h})} = 0.
\]
Thus we obtain
\[
(*) = c_1^2(\sqrt{h}) \cdot s + c_2^2(\sqrt{h}) \cdot s^2 + \cdots,
\]
near \( s = 0 \), where \( c_i^2(\sqrt{h}), (i = 1, 2, \cdots) \) are some constants. Therefore we have
\[
\mathcal{J}(h, s) = c_1^2(\sqrt{h}) \cdot \sqrt{s} + c_2^2(\sqrt{h}) \cdot s \sqrt{s} + \cdots,
\]
which implies (5.14).

\[\square\]

**Lemma 7.2.** Let \( h \in (0, 1) \) be fixed. \( \mathcal{J}(h, s) \) defined by (5.10) satisfies (5.15).

**Proof.** We have
\[
\mathcal{J}(h, s) = -Q(h, s) \cdot \frac{\partial m(\hat{V}, \varepsilon^2)}{\partial \varepsilon},
\]
where
\[
Q(h, s) := A_s \cdot \frac{dQ(h, s; \hat{V})}{dh} \cdot \frac{h (1 - h) (\mathcal{D}^K)^3 \cdot K(\sqrt{h})^2}{32s(1-s)^2(1-hs)^2} \cdot \frac{(1-s)(1-sh)}{\sqrt{s} \cdot \mathcal{N}^E}
\]
by (5.8) and (5.1). Thus
\[
\mathcal{J} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) = -Q \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) \cdot \frac{\partial m(1, \varepsilon^2)}{\partial \varepsilon} = 0
\]
by (1.14).

\[\square\]

**Proof of Proposition 5.3.** It is obvious from Lemmas 7.1 and 7.2.

**Proof of Proposition 5.4.** We have
\[
\frac{\partial}{\partial s} \left( -\frac{(1-s)(1-sh)}{\sqrt{s}} \Pi(-hs, \sqrt{h}) \right) = \frac{s E(\sqrt{h}) + (1-s)K(\sqrt{h})}{2s \sqrt{s(1-s)(1-sh)}}, \quad (7.1)
\]
\[
\frac{\partial}{\partial s} \left( \frac{(1-s)(1-sh)}{\sqrt{s}} \right) = \frac{hs^2 - 1}{2s \sqrt{s(1-s)(1-sh)}}, \quad (7.2)
\]
\[
\frac{\partial}{\partial s} \mathcal{N}^E = 4h \left( 2h^3 s^3 - 2h^2 s^3 - 3h^2 s^2 + 2hs^3 - 3hs^2 + 6hs - h - 1 \right) E(\sqrt{h})
\]
\[ + (1 - h) \left( h^2 s^3 - 2 hs^3 + 3 hs^2 - 3 hs + 1 \right) K(\sqrt{h}), \]  
\[ \frac{\partial}{\partial s} \left( N^K E(\sqrt{h}) + N^K E(\sqrt{h}) \right) \]
\[ = -3hK(\sqrt{h}) \left( (h^2 s^3 + s^3 h - 4sh + h + 1) E(\sqrt{h}) \right) \]
\[ - (1 - h) \left( s^2 h - 2sh + 1 \right) K(\sqrt{h}). \]

Hence, we obtain (5.16) by direct calculation. Thus, we complete proof. \( \square \)

8. **Proof of Proposition 5.5.** We begin with the following lemmas.

**Lemma 8.1.** Let \( F(h, u) \) and \( C^F(h, u) \) be defined by (5.17) and (5.18) respectively, then (5.24) holds.

**Proof.** We have
\[ F(h, 0) = C^F \left( h, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right), \]
\[ C^F(h, H) = H^3 \left( H^2 + H + 1 \right) \left( 2 H^2 + 3 H + 2 \right) (1 - H)^4 > 0, \]
\[ C^F_u(h, u) \]
\[ = 6 \left( h^2 - h + 1 \right)^2 u^2 - 6 (2 - h) (1 - h) \left( h^2 - h + 1 \right) u + 3 \left( h^2 - 2h + 2 \right) (1 - h)^2 \]
\[ = 6 \left( h^2 - h + 1 \right)^2 \left( u - \frac{1}{2} \cdot \frac{(2 - h)(1 - h)}{h^2 - h + 1} \right)^2 + \frac{3}{2} (1 - h)^2 h^2 > 0 \]
for \( 0 < h < 1, H < u < 1 \), where \( H = \sqrt{1 - h} \). Thus, we complete the proof by using \( \sqrt{1 - h} < E(\sqrt{h})/K(\sqrt{h}) < 1 \) due to Lemma 3.1. \( \square \)

**Lemma 8.2.** Let \( F(h, u) \) and \( g_1(H, u) \) be defined by (5.17) and (5.26) respectively, then (5.25) holds.

**Proof.** We have
\[ F \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) = F \left( h, \frac{1}{1 + H} \right) = -\frac{8H^8}{(1 + H)^4} \cdot g_1 \left( H, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right), \]  
where \( H = \sqrt{1 - h} \). Hence we may show that
\[ g_1(H, u) > 0 \quad (0 < H < 1, \quad H < u < 1) \]  
(8.2)
by Lemma 3.1.

It is easy to see that
\[ ((1 + H^2)u - 2H^2) \geq H(1 - H^2) > 0 \quad (0 < H < 1, \quad H < u < 1) \]

and
\[ - (1 + H^2)u^2 + 2(1 + H^4)u - (1 + H^2)H^2 \]
\[ \geq \min \left\{ 2H(1 - H)^2(H^2 + H + 1), (1 - H^2)^2 \right\} > 0 \quad (0 < H < 1, \quad H < u < 1). \]

Thus we have (8.2). Therefore we complete the proof. \( \square \)

Thus we obtain Proposition 5.5 by Lemmas 8.1 and 8.2.
9. Proof of Proposition 5.6. We prepare several lemmas to prove Proposition 5.6.

**Lemma 9.1.** Let $F(h, u)$ be defined by (5.17). It holds that

\[
\frac{\partial F}{\partial s}(h, s) = 8h^4C^{F_0}s^7 - 14h^4C^{F_1}s^6 + 12h^4C^{F_2}s^5 - 10h^3C^{F_3}s^4 \\
+ 8h^2C^{F_4}s^3 - 6h^2C^{F_5}s^2 + 4h^2C^{F_6}s - 2hC^{F_7},
\]

(9.1)

\[
\frac{\partial^2 F}{\partial s^2}(h, s) = 56h^4C^{F_0}s^6 - 84h^4C^{F_1}s^5 + 60h^4C^{F_2}s^4 - 40h^3C^{F_3}s^3 \\
+ 24h^2C^{F_4}s^2 - 12h^2C^{F_5}s + 4h^2C^{F_6},
\]

(9.2)

\[
\frac{\partial^3 F}{\partial s^3}(h, s) = 336h^4C^{F_0}s^5 - 420h^4C^{F_1}s^4 + 240h^4C^{F_2}s^3 - 120h^3C^{F_3}s^2 \\
+ 48h^2C^{F_4}s - 12h^2C^{F_5},
\]

(9.3)

\[
\frac{\partial^4 F}{\partial s^4}(h, s) = 1680h^4C^{F_0}s^4 - 1680h^4C^{F_1}s^3 + 720h^4C^{F_2}s^2 - 240h^3C^{F_3}s \\
+ 48h^2C^{F_4},
\]

(9.4)

\[
\frac{\partial^5 F}{\partial s^5}(h, s) = 6720h^4C^{F_0}s^3 - 5040h^4C^{F_1}s^2 + 1440h^4C^{F_2}s - 240h^3C^{F_3},
\]

(9.5)

\[
\frac{\partial^6 F}{\partial s^6}(h, s) = 20160h^4C^{F_0}s^2 - 10080h^4C^{F_1}s + 1440h^4C^{F_2},
\]

(9.6)

\[
\frac{\partial^7 F}{\partial s^7}(h, s) = 40320h^4C^{F_0}s - 10080h^4C^{F_1},
\]

(9.7)

\[
\frac{\partial^8 F}{\partial s^8}(h, s) = 40320h^4C^{F_0} > 0 \quad (0 < h < 1),
\]

(9.8)

where

\[
C^{F_0} = C^{F_0}(h, U(h)), \quad C^{F_1} = C^{F_1}(h, U(h)), \quad C^{F_2} = C^{F_2}(h, U(h)),
\]

\[
C^{F_3} = C^{F_3}(h, U(h)), \quad C^{F_4} = C^{F_4}(h, U(h)), \quad U(h) = E(\sqrt{h})/E(\sqrt{1-h})
\]

are defined by (5.18) - (5.23).

**Proof.** We obtain identities in (9.1) - (9.8) by direct calculations and the inequality in (9.8) by Lemma 8.1. \(\square\)

**Lemma 9.2.** Let $F(h, u)$ be defined by (5.17). It holds that

\[
\frac{\partial F}{\partial s} \left( \frac{h}{1 + \sqrt{1-h}} \right) = \frac{\partial F}{\partial s} \left( \frac{1 - H^2}{1 + H} \right) \\
= \frac{32H^7(1-H)}{(1+H)^3} \cdot g_1 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad (0 < h < 1),
\]

(9.9)

\[
\frac{\partial^2 F}{\partial s^2} \left( \frac{h}{1 + \sqrt{1-h}} \right) = \frac{\partial^2 F}{\partial s^2} \left( \frac{1 - H^2}{1 + H} \right) \\
= -\frac{16H^6(1-H)^2}{(1+H)^2} \cdot g_2 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) < 0 \quad \left( 1 - \frac{31^2}{100^2} \leq h < 1 \right),
\]

(9.10)
\[
\begin{align*}
\frac{\partial^3 F}{\partial s^3} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) &= \frac{\partial^3 F}{\partial s^3} \left( 1 - H^2, \frac{1}{1 + H} \right) \\
&= \frac{48H^5 (1 - H)^3}{1 + H} \cdot g_3 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) \begin{cases}
< 0 & (0 < h < h_0), \\
= 0 & (h = h_0), \\
> 0 & (h_0 < h < 1),
\end{cases} \tag{9.11}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^4 F}{\partial s^4} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) &= \frac{\partial^4 F}{\partial s^4} \left( 1 - H^2, \frac{1}{1 + H} \right) \\
&= 48H^2 (1 - H)^2 \cdot g_4 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) > 0 \quad (0 < h < 1 - \frac{1}{10^2}), \tag{9.12}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^5 F}{\partial s^5} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) &= \frac{\partial^5 F}{\partial s^5} \left( 1 - H^2, \frac{1}{1 + H} \right) \\
&= -480H^3 (1 + H) (1 - H)^3 \cdot g_5 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) < 0 \quad (0 < h < 1), \tag{9.13}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^6 F}{\partial s^6} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) &= \frac{\partial^6 F}{\partial s^6} \left( 1 - H^2, \frac{1}{1 + H} \right) \\
&= 1440H^2 (1 + H)^2 (1 - H)^4 \cdot g_6 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) > 0 \quad (0 < h < 1), \tag{9.14}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^7 F}{\partial s^7} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) &= \frac{\partial^7 F}{\partial s^7} \left( 1 - H^2, \frac{1}{1 + H} \right) \\
&= -10080H (1 + H)^3 (1 - H)^5 \cdot g_7 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) < 0 \quad (0 < h < 1), \tag{9.15}
\end{align*}
\]

where \( H := \sqrt{1 - h} \),

\[
\begin{align*}
g_2(H, u) &:= -2 \left( H^2 + 1 \right) (5H^2 + 3H + 5) u^3 \\
&\quad - \left( -12H^6 + 4H^5 - 37H^4 - 22H^3 - 37H^2 + 4H - 12 \right) u^2 \\
&\quad - 2H^2 (18H^4 + H^3 + 8H^2 + H + 18) u + H^4 (15H^2 + 2H + 15), \tag{9.16}
\end{align*}
\]

\[
\begin{align*}
g_3(H, u) &:= -2 \left( H^2 + 1 \right) (8H^2 + 9H + 8) u^3 \\
&\quad + (8H^6 - 12H^5 + 55H^4 + 66H^3 + 55H^2 - 12H + 8) u^2 \\
&\quad - 2H^2 (19H^4 + 3H^3 + 10H^2 + 3H + 19) u + H^4 (17H^2 + 6H + 17), \tag{9.17}
\end{align*}
\]

\[
\begin{align*}
g_4(H, u) &:= (72H^6 - 16H^5 - 104H^3 - 16H + 72) u^3 \\
&\quad - (8H^8 - 64H^7 + 32H^6 - 148H^5 - 258H^4 - 148H^3 + 32H^2 - 64H + 8) u^2 \\
&\quad + 2H^2 (49H^6 - 92H^5 + 51H^4 - 36H^3 + 51H^2 - 92H + 49) u \\
&\quad - H^4 (47H^4 - 76H^3 + 34H^2 - 76H + 47), \tag{9.18}
\end{align*}
\]
Let
\begin{align*}
g_5(H,u) & := - ( -28H^6 - 12H^5 + 12H^4 + 48H^3 + 12H^2 - 12H - 28 ) u^3 \\
& + H ( 8H^6 - 95H^5 + 36H^4 + 110H^3 + 36H^2 - 95H + 8 ) u^2 \\
& + 2H^2 ( 9H^6 - 22H^5 + 7H^4 - 8H^3 + 7H^2 - 22H + 9 ) u \\
& - H^4 ( 9H^4 - 20H^3 - 2H^2 - 20H + 9 ), \tag{9.19}
\end{align*}
\begin{align*}
g_6(H,u) & := ( 28H^6 + 28H^5 - 12H^4 - 12H^2 + 28H + 28 ) u^3 \\
& - H^2 ( 65H^4 - 17H^3 - 80H^2 - 17H + 65 ) u^2 \\
& + 2H^2 ( 3H^6 - 15H^5 + H^4 - 4H^3 - 2H^2 - 15H + 3 ) u \\
& - H^4 ( 3H^4 - 15H^3 - 8H^2 - 15H + 3 ), \tag{9.20}
\end{align*}
\begin{align*}
g_7(H,u) & := 4 ( 2H^2 + 3H + 2 ) ( H^4 - H^2 + 1 ) u^3 \\
& - 3H^2 ( 4H^4 + H^3 - 2H^2 + H + 4 ) u^2 \\
& - 6H^3 ( H^4 + 1 ) u + H^5 ( 3H^2 + 2H + 3 ). \tag{9.21}
\end{align*}
Here, \( h_0 \) is a real number with \( h \in (1 - 31^2/100^2, 1 - 1/10^2) \) which appears in Lemma B.2.

**Proof.** We get identities in (9.9) - (9.15) by direct calculations. We have inequality in (9.9) by Lemma 8.2, We obtain (9.10), (9.11), (9.12), (9.13), (9.14), (9.15) by Lemma A.1, Lemma B.2, Lemma A.2, Lemma A.3, Lemma A.4, Lemma A.5 in Appendix A and B, respectively.

Table 9.1 shows the sign of \( F(0, h), \cdots, \partial^8 F(0, h)/\partial s^8 \) and \( F(1/(1 + \sqrt{1-h}), h), \cdots, \partial^8 F(1/(1 + \sqrt{1-h}), h)/\partial s^8 \), which we will prove in subsequent lemmas 9.3-9.6.

Here * means +, − or 0.

**Lemma 9.3.** Let \( F(h, u) \) be defined by (5.17). It holds that
\begin{align*}
\frac{\partial F}{\partial s} (h,0) & = \frac{1}{5040h^3} \cdot \frac{\partial^7 F}{\partial s^7} (h,0) < 0 \quad (0 < h < 1), \tag{9.22}
\frac{\partial^2 F}{\partial s^2} (h,0) & = \frac{1}{360h^3} \cdot \frac{\partial^6 F}{\partial s^6} (h,0) > 0 \quad (0 < h < 1), \tag{9.23}
\frac{\partial^3 F}{\partial s^3} (h,0) & = \frac{1}{20h^2} \cdot \frac{\partial^5 F}{\partial s^5} (h,0) < 0 \quad (0 < h < 1), \tag{9.24}
\frac{\partial^5 F}{\partial s^5} (h,s) & < 0 \quad (0 \leq h < 1, \quad 0 \leq s \leq \frac{1}{1 + \sqrt{1-h}}), \tag{9.25}
\frac{\partial^6 F}{\partial s^6} (h,s) & > 0 \quad (0 \leq h < 1, \quad 0 \leq s \leq \frac{1}{1 + \sqrt{1-h}}), \tag{9.26}
\frac{\partial^7 F}{\partial s^7} (h,s) & < 0 \quad (0 \leq h < 1, \quad 0 \leq s \leq \frac{1}{1 + \sqrt{1-h}}). \tag{9.27}
\end{align*}

**Proof.** We obtain (9.27) by (9.8) and (9.15). Hence we get (9.26) by (9.27) and (9.14). Therefore, we obtain (9.25) by (9.26) and (9.13). Thus we obtain equalities
in (9.22), (9.23), (9.24) by Lemma 9.1, and inequalities by (9.27), (9.26), (9.25) with $s = 0$.

**Lemma 9.4.** Let $F(h, u)$ be defined by (5.17). It holds that
\[
\frac{\partial^2 F}{\partial s^2} \left( h_0, \frac{1}{1 + \sqrt{1 - h}} \right) < 0, \tag{9.28}
\]
\[
\frac{\partial^4 F}{\partial s^4} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) > 0 \quad (0 < h \leq h_0). \tag{9.29}
\]

**Proof.** We obtain (9.28) by (9.10) and $h_0 \in (1 - 31^2/100^2, 1)$. We get (9.29) by (9.12) and $h_0 \in (0, 1 - 1/10^2)$. Thus we complete the proof. \qed

**Lemma 9.5.** Let $h \in (0, h_0]$ be fixed. Then, (5.27) in $s$ has the unique solution.

**Proof.** Let us fixed $h \in (0, h_0]$. We get
\[
\frac{\partial^4 F}{\partial s^4} (h, s) > 0 \quad (0 \leq s \leq \frac{1}{1 + \sqrt{1 - h}}) \tag{9.30}
\]
by (9.29) and (9.25). Hence we obtain

$$\frac{\partial^3 F}{\partial s^3} (h, s) \leq 0 \quad \left( 0 \leq s \leq \frac{1}{1 + \sqrt{1 - h}} \right)$$  \hspace{1cm} (9.31)

by (9.30) and (9.11) with $0 < h \leq h_0$.

On the other hand, we have

$$\frac{\partial F}{\partial s} (h, 0) < 0, \quad \frac{\partial F}{\partial s} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) > 0$$  \hspace{1cm} (9.32)

by (9.22) and (9.9), respectively. Therefore there exists the unique $s_1(h) \in (0, 1/(1 + \sqrt{1 - h}))$ such that

\begin{align*}
\frac{\partial F}{\partial s} (h, s) &< 0 \quad (0 \leq s < s_1(h)), \\
\frac{\partial F}{\partial s} (h, s_1(h)) &> 0, \\
\frac{\partial F}{\partial s} (h, s) &< 0 \quad \left( s_1(h) < s \leq \frac{1}{1 + \sqrt{1 - h}} \right)
\end{align*}  \hspace{1cm} (9.33, 9.34, 9.35)

by (9.31)

Now, we have

$$F(h, 0) > 0, \quad F \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) < 0$$  \hspace{1cm} (9.36)

due to (5.24) and (5.25). Consequently, there exists the unique $s_0(h) \in (0, 1/(1 + \sqrt{1 - h}))$ such that

\begin{align*}
F (h, s) &> 0 \quad (0 \leq s < s_0(h)), \\
F (h, s_0(h)) &= 0, \\
F (h, s) &< 0 \quad \left( s_0(h) < s \leq \frac{1}{1 + \sqrt{1 - h}} \right)
\end{align*}  \hspace{1cm} (9.37, 9.38, 9.39)

by (9.33), (9.34), (9.35). Thus we complete the proof. \hfill \Box

**Lemma 9.6.** Let $h \in (h_0, 1)$ be fixed. Then, (5.27) in $s$ has the unique solution.

**Proof.** Let us fixed $h \in (h_0, 1)$. We have

$$\frac{\partial^3 F}{\partial s^3} (h, 0) < 0, \quad \frac{\partial^3 F}{\partial s^3} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) > 0$$  \hspace{1cm} (9.40)

by (9.30) and (9.11) with $h_0 < h < 1$, respectively. Hence there exists the unique $s_3(h) \in (0, 1/(1 + \sqrt{1 - h}))$ such that

\begin{align*}
\frac{\partial^3 F}{\partial s^3} (h, s) &< 0 \quad (0 \leq s < s_3(h)), \\
\frac{\partial^3 F}{\partial s^3} (h, s_3(h)) &= 0, \\
\frac{\partial^3 F}{\partial s^3} (h, s) &> 0 \quad \left( s_3(h) < s \leq \frac{1}{1 + \sqrt{1 - h}} \right)
\end{align*}  \hspace{1cm} (9.41, 9.42, 9.43)

by (9.25).
On the other hand, we get
\[ \frac{\partial^2 F}{\partial s^2} (h, 0) > 0, \quad \frac{\partial^2 F}{\partial s^2} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) < 0 \] (9.44)
by (9.23) and (9.10), respectively. Thus there exists the unique \( \hat{s}_2(h) \in (0, 1/(1 + \sqrt{1 - h})) \) such that
\[ \frac{\partial^2 F}{\partial s^2} (h, s) > 0 \quad (0 \leq s < \hat{s}_2(h)), \] (9.45)
\[ \frac{\partial^2 F}{\partial s^2} (h, \hat{s}_2(h)) = 0, \] (9.46)
\[ \frac{\partial^2 F}{\partial s^2} (h, s) < 0 \quad \left( \hat{s}_2(h) < s \leq \frac{1}{1 + \sqrt{1 - h}} \right) \] (9.47)
by (9.41), (9.42) and (9.43).

Moreover, we obtain
\[ \frac{\partial F}{\partial s} (h, 0) < 0, \quad \frac{\partial F}{\partial s} \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) > 0 \] (9.48)
by (9.22) and (9.9), respectively. Therefore there exists the unique \( \hat{s}_1(h) \in (0, 1/(1 + \sqrt{1 - h})) \) such that
\[ \frac{\partial F}{\partial s} (h, s) < 0 \quad (0 \leq s < \hat{s}_1(h)), \] (9.49)
\[ \frac{\partial F}{\partial s} (h, \hat{s}_1(h)) = 0, \] (9.50)
\[ \frac{\partial F}{\partial s} (h, s) < 0 \quad \left( \hat{s}_1(h) < s \leq \frac{1}{1 + \sqrt{1 - h}} \right) \] (9.51)
by (9.45), (9.46) and (9.47).

Now, we have
\[ F(h, 0) > 0, \quad F \left( h, \frac{1}{1 + \sqrt{1 - h}} \right) < 0 \] (9.52)
due to (5.24) and (5.25). Consequently, there exists the unique \( \hat{s}_0(h) \) such that
\[ F(h, s) > 0 \quad (0 \leq s < \hat{s}_0), \] (9.53)
\[ F(h, \hat{s}_0) = 0, \] (9.54)
\[ F(h, s) < 0 \quad \left( \hat{s}_0 < s \leq \frac{1}{1 + \sqrt{1 - h}} \right) \] (9.55)
by (9.49), (9.50) and (9.51). Thus we complete the proof. \( \square \)

**Proof of Proposition 5.6** We obtain conclusions by Lemma 9.5 and Lemma 9.6. \( \square \)

**Appendix A. Inequalities including complete elliptic integrals I.** We prepare several lemmas to prove Proposition 5.6.

**Lemma A.1.** Let \( g_2(H, u) \) be defined by (9.16). Then
\[ g_2 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad \left( 0 < H \leq \frac{31}{100} \right) \] (A.1)
Proof. We may show that
\[ g_2(H, u) > 0 \quad \left( 0 < H \leq \frac{31}{100}, \ H < u < \frac{1 + H^2}{2} \right) \quad (A.2) \]
by Lemma 3.1.

We have
\[ g_2(H, H) = 2H^2(6H^4 - 13H^3 - 10H^2 - 13H + 6)(1 - H)^2 > 0, \]
\[ g_{2,u}(H, u) = -6 \left( H^2 + 1 \right) \left( 5H^2 + 3H + 5 \right) u^2 \]
\[ + \left( 24H^6 - 8H^5 + 74H^4 + 44H^3 + 74H^2 - 8H + 24 \right) u \]
\[ - 2H^2 \left( 18H^4 + H^3 + 8H^2 + H + 18 \right), \]
\[ g_{2,u}(H, H) = 2H \left( 12H^6 - 37H^5 + 27H^4 - 16H^3 + 27H^2 - 37H + 12 \right) > 0, \]
\[ g_{2,u} \left( H, \frac{1 + H^2}{2} \right) \]
\[ = \frac{1}{2} \left( 9H^6 - 35H^5 + 27H^4 - 14H^3 + 27H^2 - 35H + 9 \right) (H + 1)^2 > 0 \]
for \( 0 < H \leq \frac{31}{100} \) by virtue of Sturm’s theorem. Hence we obtain (A.2). Thus we complete the proof.

Lemma A.2. Let \( g_4(H, u) \) be defined by (9.18). Then
\[ g_4 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) > 0 \quad \left( \frac{1}{10} \leq H < 1 \right). \quad (A.3) \]

Proof. We may show that
\[ g_4(H, u) > 0 \quad \left( \frac{1}{10} \leq H < 1, \ H < u < \frac{1 + H^2}{2} \right) \quad (A.4) \]
by Lemma 3.1.

We have
\[ g_4(H, H) = 2H^2(-4H^4 + 101H^3 + 140H^2 + 101H - 4)(1 - H)^4 > 0, \]
\[ g_{4,u}(H, H) = 2H(-8H^6 + 205H^5 - 27H^4 - 60H^3 - 27H^2 + 205H - 8) \]
\[ \cdot (1 - H)^2 > 0, \]
\[ g_{4,uu}(H, u) = 2 \left( 216H^6 - 48H^5 - 312H^3 - 48H + 216 \right) u \]
\[ - 16H^8 + 128H^7 - 658H^6 + 296H^5 + 516H^4 \]
\[ + 296H^3 - 658H^2 + 128H - 16, \]
\[ g_{4,uu}(H, H) = -16H^8 + 560H^7 - 754H^6 + 296H^5 - 108H^4 \]
\[ + 296H^3 - 754H^2 + 560H - 16 > 0, \]
\[ g_{4,uu} \left( H, \frac{1 + H^2}{2} \right) \]
\[ = 2(100H^6 - 160H^5 - H^4 + 130H^3 - H^2 - 16H + 100)(H + 1)^2 > 0 \]
for \( \frac{1}{10} \leq H < 1 \) by virtue of Sturm’s theorem. Hence we obtain (A.4). Thus we complete the proof.
Lemma A.3. Let \( g_5(H, u) \) be defined by (9.19). Then
\[
g_5 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad (0 < H < 1).
\]  
(A.5)

Proof. We may show that
\[
g_5(H, u) > 0 \quad (0 < H < 1, \ H < u < 1)
\]  
(A.6)
by Lemma 3.1.

We have
\[
\begin{align*}
g_5(H, H) &= 2H^3(27H^2 + 40H + 27)(1 - H)^4 > 0, \\
g_{5, u}(H, H) &= 2H^2(59H^4 + 19H^3 + 4H^2 + 19H + 59)(1 - H)^2 > 0, \\
g_{5, uu}(H, u) &= 2 \left( 84H^6 + 36H^5 - 36H^4 - 144H^3 - 36H^2 + 36H + 84 \right) u \\
&\quad + 2H \left( 8H^6 - 95H^5 + 36H^4 + 110H^3 + 36H^2 - 95H + 8 \right), \\
g_{5, uu}(H, H) &= 2H(92H^6 - 59H^5 - 34H^3 - 59H + 92) > 0, \\
g_{5, u}(H, 1) &= 2(H + 1)(8H^6 - 19H^5 + 91H^4 - 17H^3 - 91H^2 - 40H + 84) > 0
\end{align*}
\]
for \( 0 < H < 1 \) by virtue of Sturm’s theorem. Hence we obtain (A.6). Thus we complete the proof.

Lemma A.4. Let \( g_6(H, u) \) be defined by (9.20). Then,
\[
g_6 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad (0 < H < 1).
\]  
(A.7)

Proof. We may show that
\[
g_6(H, u) > 0 \quad (0 < H < 1, \ H < u < 1)
\]  
(A.8)
by Lemma 3.1.

We have
\[
\begin{align*}
g_6(1 - H^2, u) &= 2H^3(17H^4 - H^3 - 8H^2 - H + 17)(1 - H)^2 > 0, \\
g_{6, u}(H, H) &= 2H^2(45H^6 - 38H^5 - 2H^3 - 38H + 45) > 0, \\
g_{6, uu}(H, u) &= 2 \left( 84H^6 + 84H^5 - 36H^4 - 156H^3 - 36H^2 + 84H + 84 \right) u \\
&\quad - 2H^2 \left( 65H^4 - 17H^3 - 80H^2 - 17H + 65 \right), \\
g_{6, uu}(H, H) &= 2H(84H^6 + 19H^5 - 19H^4 - 76H^3 - 19H^2 + 19H + 84) > 0, \\
g_{6, u}(H, 1) &= 2(H + 1)(19H^5 + 82H^4 - 38H^3 - 101H^2 + 84) > 0
\end{align*}
\]
for \( 0 < H < 1 \) by virtue of Sturm’s theorem. Hence we obtain (A.8). Thus we complete the proof.

Lemma A.5. Let \( g_7(H, u) \) be defined by (9.21). Then
\[
g_7 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad (0 < H < 1).
\]  
(A.9)

Proof. We may show that
\[
g_7(H, u) > 0 \quad (0 < H < 1, \ H < u < 1)
\]  
(A.10)
by Lemma 3.1.
We have
\[ g_7(H, u) = 2H^3(4H^4 + 5H^3 + 6H^2 + 5H + 4)(1 - H)^2 > 0, \]
\[ g_{7, u}(H, H) = 6H^2(4H^6 + H^5 - H^4 - 4H^3 - H^2 + H + 4) > 0, \]
\[ g_{7, uu}(H, u) = 24(2H^2 + 3H + 2)(H^3 - H^2 + 1)u \]
\[ -6H^2(4H^4 + H^3 - 2H^2 + H + 4), \]
\[ g_{7, uu}(H, H) = 6H(8H^6 + 8H^5 - H^4 - 10H^3 - H^2 + 8H + 8) > 0, \]
\[ g_{7, uu}(H, 1) = 6H(1 + 1)(4H^5 + 7H^4 - 5H^3 - 8H^2 + 4H + 8) > 0 \]
for \( 0 < H < 1 \) by virtue of Sturm’s theorem. Hence we obtain (A.10). Thus we complete the proof. \( \square \)

Appendix B. Inequalities including complete elliptic integrals II.

**Lemma B.1.** It holds that
\[
\frac{d}{dH} \left( \frac{K(\sqrt{1 - H^2})^3}{(1 - H^2)^2} \cdot g_3 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) \right) < 0 \quad (0 < H < 1). \tag{B.1}
\]

**Proof.** We have
\[
\frac{d}{dH} \left( \frac{K(\sqrt{1 - H^2})^3}{(1 - H^2)^2} \cdot g_3 \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right) \right) = -\frac{K(\sqrt{1 - H^2})^3}{H(1 - H)^3(1 + H)^3} \cdot f \left( H, \frac{E(\sqrt{1 - H^2})}{K(\sqrt{1 - H^2})} \right), \tag{B.2}
\]
where
\[
f(H, u) := c_3(H) \cdot u^3 + c_2(H) \cdot u^2 + c_1(H) \cdot u + H^4(21H^4 - 6H^3 - 133H^2 - 24H - 30), \tag{B.3}
\]
\[
c_1(H) := -3H^2(18H^6 - 8H^5 - 63H^4 + 24H^3 - 45H^2 - 14H - 20), \tag{B.4}
\]
\[
c_2(H) := 3H(8H^7 - 8H^6 - 7H^5 + 34H^4 - 73H^3 - 44H^2 - 54H + 4), \tag{B.5}
\]
\[
c_3(H) := -(40H^6 - 48H^5 + 87H^4 + 120H^3 + 135H^2 + 6H + 8). \tag{B.6}
\]

We may show that
\[
f(H, u) > 0 \quad (0 < H < 1, \ H < u < 1) \tag{B.7}
\]
by Lemma 3.1.

We obtain
\[
f(H, H) = 10H^3(12H^5 - 35H^4 - 94H^3 + 33H^2 + 8H + 40)(1 - H)^2 > 0,
\]
and \( c_3(H) > 0 \) for \( H \in (0, 1) \) by Sturm’s theorem. Moreover, we get
\[
f_{u}(H, u) = 3c_3(H) \cdot u^2 + 2c_2(H) \cdot u + c_1(H) > 0 \quad (0 < H < 1),
\]
since
\[
c_2(H)^2 - 3c_3(H) \cdot c_1(H)
\]
\[
= -9H^2(H + 1)^3(-64H^{11} + 320H^{10} - 1008H^8 - 103H^7
\]
\[
-183H^6 + 3211H^5 - 1837H^4 + 1004H^3 - 548H^2 + 232H + 144) < 0,
\]
for \( 0 < H < 1 \) by virtue of Sturm’s theorem.
Therefore, we get (B.7). Thus we complete the proof. □

**Lemma B.2.** There exists the unique $h_0 \in [1 - (31/100)^2, 1 - (1/10)^2]$ such that

\[
g_3 \left( \sqrt{1 - h}, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right) < 0 \quad (0 < h < h_0), \tag{B.8}
\]

\[
g_3 \left( \sqrt{1 - h}, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right) = 0, \tag{B.9}
\]

\[
g_3 \left( \sqrt{1 - h}, \frac{E(\sqrt{h})}{K(\sqrt{h})} \right) > 0 \quad (h_0 < h < 1). \tag{B.10}
\]

**Proof.** We have

\[
g_3 \left( \frac{1}{10}, E \left( \sqrt{1 - \left( \frac{1}{10} \right)^2} \right) \right) / K \left( \sqrt{1 - \left( \frac{1}{10} \right)^2} \right) > 0,
\]

since it holds that

\[
\frac{1}{10} < E \left( \sqrt{1 - \left( \frac{1}{10} \right)^2} \right) / K \left( \sqrt{1 - \left( \frac{1}{10} \right)^2} \right) < \frac{121}{400} = \frac{1}{4} \left( 1 + \frac{1}{10} \right)^2
\]

by Lemma 3.1, and

\[
g_3 \left( \frac{1}{10}, u \right)
\]

\[
= -\frac{45349}{2500} u^3 + \frac{1855347}{250000} u^2 - \frac{194049}{500000} u + \frac{1777}{1000000} > 0 \quad \left( \frac{1}{10} < u < \frac{121}{400} \right)
\]

by Sturm’s theorem.

We have

\[
g_3 \left( \frac{31}{100}, E \left( \sqrt{1 - \left( \frac{31}{100} \right)^2} \right) \right) / K \left( \sqrt{1 - \left( \frac{31}{100} \right)^2} \right) < 0,
\]

since it holds that

\[
\frac{31}{100} < E \left( \sqrt{1 - \left( \frac{31}{100} \right)^2} \right) / K \left( \sqrt{1 - \left( \frac{31}{100} \right)^2} \right) < 1
\]

by Lemma 3.1, and

\[
g_3 \left( \frac{31}{100}, u \right)
\]

\[
= -\frac{316740017}{12500000} u^3 + \frac{1501548449781}{125000000000} u^2 - \frac{2033076415239}{500000000000} u + \frac{189263623177}{10000000000000} < 0 \quad \left( \frac{31}{100} < u < 1 \right)
\]

by Sturm’s theorem.
Thus, there exists the unique $H_0 \in (1/10, 31/100)$ such that
\[
g_3 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) > 0 \quad (0 < h < H_0),
g_3 \left( H_0, \frac{E(\sqrt{1-H_0^2})}{K(\sqrt{1-H_0^2})} \right) = 0,
g_3 \left( H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} \right) < 0 \quad (H_0 < h < 1).
\]
Consequently we complete the proof by putting $h := 1 - H^2$ and $h_0 := 1 - H_0^2$. □

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