“Triviality” Made Easy: 
the real $(\lambda\Phi^4)_4$ story

M. Consoli

Istituto Nazionale di Fisica Nucleare, Sezione di Catania
Corso Italia 57, 95129 Catania, Italy

and

P. M. Stevenson

T. W. Bonner Laboratory, Physics Department
Rice University, Houston, TX 77251, USA

Abstract:

The real meaning of “triviality” of $(\lambda\Phi^4)_4$ theory is outlined. Assuming “triviality” leads to an effective potential that is just the classical potential plus the zero-point energy of the free-field fluctuations. This $V_{\text{eff}}$ gives spontaneous symmetry breaking. Its proper renormalization has the consequence that all scattering amplitudes vanish, self-consistently validating the original assumption. Nevertheless, the theory is physically distinguishable from a free field theory; it has a symmetry-restoring phase transition at a finite critical temperature.
1. The strong evidence that $\lambda\Phi^4$ theory is “trivial” in 4 dimensions \cite{1,2} seemingly conflicts with the textbook description of the Standard Model, in which $W$ and $Z$ masses arise from spontaneous symmetry breaking (SSB) in the $\lambda\Phi^4$ scalar sector. Current thinking holds that the theory can only be saved by a finite ultraviolet cutoff, thereby abandoning the one grand principle underlying the Standard Model — renormalizability. In our view, “triviality” is true, but its meaning and its consequences have not been properly understood.

Our earlier papers \cite{3} discuss the arguments in detail, but here our exposition is as terse as possible so that the overall picture can be seen whole. The key point is this: The effective potential of a “trivial” theory is not necessarily a trivial quadratic function. The effective potential is the classical potential plus quantum effects, and in a “trivial” theory the only quantum effect is the zero-point energy of the free-field vacuum fluctuations.

2. Consider the Euclidean action of classically-scale-invariant $\lambda\Phi^4$ theory:

\begin{equation}
S[\Phi] = \int d^4x \left( \frac{1}{2} \partial_\mu \Phi_B \partial_\mu \Phi_B + \frac{\lambda_B}{4!} \Phi_B^4 \right),
\end{equation}

and substitute

\begin{equation}
\Phi_B(x) = \phi_B + h(x),
\end{equation}

where $\phi_B$ is a constant. (To make the decomposition unambiguous we impose $\int d^4x \ h(x) = 0$ using a Lagrange multiplier $\eta$.) Upon expanding one obtains $S[\Phi] = S_0 + S_1 + S_2 + S_{\text{int}}$ where

\begin{align*}
S_0 &= \frac{\lambda_B}{4!} \phi_B^4 \int d^4x, \\
S_1 &= \left( \frac{\lambda_B}{6} \phi_B^3 - \eta \right) \int d^4x \ h(x), \\
S_2 &= \int d^4x \left( \frac{1}{2} \partial_\mu h \partial_\mu h + \frac{1}{2} \left( \frac{\lambda_B}{2} \phi_B^2 \right) h(x)^2 \right), \\
S_{\text{int}} &= \int d^4x \frac{\lambda_B}{4!} \left( 4\phi_B h(x)^3 + h(x)^4 \right).
\end{align*}

Consider the approximation in which we ignore $S_{\text{int}}$. It is then straightforward to compute the effective action by the standard functional methods. Briefly, the linear term $S_1$ effectively plays no role; the $S_0$ term simply reproduces itself in the effective action; and the $S_2$ term reproduces itself together with a zero-point energy contribution from the functional determinant. Thus, the (Euclidean) effective action is:

\begin{equation}
\Gamma = - \int d^4x \left[ \frac{1}{2} \partial_\mu h \partial_\mu h + \frac{1}{2} \left( \frac{\lambda_B}{2} \phi_B^2 \right) h(x)^2 + \left. V_{\text{eff}}(\phi_B) \right] ,
\end{equation}

1
where
\[ V_{\text{eff}}(\phi_B) = \frac{\lambda_B}{4!} \phi_B^4 + \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \ln(p^2 + \frac{1}{2} \lambda_B \phi_B^2). \] (8)

This effective action describes a free \( h(x) \) field with a \( \phi_B \)-dependent mass-squared, \( \frac{1}{2} \lambda_B \phi_B^2 \). The effective potential for \( \phi_B \) is just the classical potential plus the zero-point energy of the \( h(x) \) field.

[More precisely, the exact effective potential is the ‘convex envelope’ of this \( V_{\text{eff}} \); Ritschel’s version of our calculation shows explicitly how this comes about [5]. \( V_{\text{eff}} \) is the usual “one-loop effective potential”. However, \( \Gamma \) is not the one-loop effective action.]

3. After subtracting a constant and performing the mass renormalization so that the second derivative of the effective potential vanishes at the origin, one has [4]:
\[ V_{\text{eff}} = \frac{\lambda_B}{4!} \phi_B^4 + \frac{\lambda_B^2}{256\pi^2} \left( \ln \frac{1}{2} \frac{\lambda_B \phi_B^2}{\Lambda^2} - \frac{1}{2} \right), \] (9)

where \( \Lambda \) is an ultraviolet cutoff. This function is just a sum of \( \phi_B^4 \ln \phi_B^2 \) and \( \phi_B^4 \) terms. It has a pair of minima at \( \phi_B = \pm v_B \) and may be re-written in the form:
\[ V_{\text{eff}} = \frac{\lambda_B^2}{256\pi^2} \left( \ln \frac{\phi_B^2}{v_B^2} - \frac{1}{2} \right). \] (10)

Comparing the equivalent forms (9) and (10) gives \( v_B \) in terms of \( \Lambda \). Hence, the mass-squared of the \( h(x) \) fluctuation field, \( \frac{1}{2} \lambda_B \phi_B^2 \), when evaluated in the SSB vacuum, is
\[ m_h^2 = \frac{1}{2} \lambda_B v_B^2 = \Lambda^2 \exp \left( -\frac{32\pi^2}{3\lambda_B} \right). \] (11)

Demanding that this particle mass be finite requires an infinitesimal \( \lambda_B \):
\[ \lambda_B = \frac{32\pi^2}{3} \ln(\Lambda^2/m_h^2) \rightarrow 0+. \] (12)

[This implies a negative \( \beta \) function: \( \Lambda \partial \lambda_B / \partial \Lambda = -b_0 \lambda_B^2 \), with \( b_0 = 3/16\pi^2 \).]

It follows that \( v_B \) goes to \( \infty \), but the depth of the SSB vacuum, \( \lambda_B^2 v_B^2 / 512\pi^2 = m_h^4 / 128\pi^2 \), remains finite. Thus, \( V_{\text{eff}}(\phi_B) \) becomes infinitely flat. However, the effective potential can be made manifestly finite by re-scaling the constant background field \( \phi_B \). One defines \( \phi_R \) as \( Z_\phi^{-1/2} \phi_B \), with \( Z_\phi \propto 1/\lambda_B \rightarrow \infty \), so that the combination \( \xi \equiv \frac{1}{2} \lambda_B Z_\phi \) remains finite. The physical mass is then finitely proportional to \( v_R = Z_\phi^{-1/2} v_B \); i.e.,
\[ m_h^2 = \xi v^2. \] The requirement that the second derivative of \( V_{\text{eff}} \) with respect to \( \phi_R \) at \( \phi_R = v_R \) should be \( m_h^2 \) fixes \( \xi \) to be \( 8\pi^2 \). Thus, one obtains:

\[ V_{\text{eff}} = \frac{\pi^2}{2} \phi_R^4 \left( \ln \frac{\phi_R^2}{v_R^2} - \frac{1}{2} \right), \quad (13) \]

and

\[ m_h^2 = 8\pi^2 v_R^2. \quad (14) \]

Although the constant field \( \phi \) requires an infinite re-scaling, the fluctuation field \( h(x) \) is not renormalized: in the effective action \( (7) \) the kinetic term for \( h(x) \) is already properly normalized. The different re-scaling of the zero-momentum mode \( \phi \) and the finite-momentum modes \( h(x) \) is the only truly radical feature of our analysis. We return to this issue in Sect. 5.

4. What about the interaction term \( S_{\text{int}} \) that we neglected? It generates a 3-point vertex \( \lambda_B \phi_B \) and a 4-point vertex \( \lambda_B \). Since our renormalization requires these to be of order \( \sqrt{\epsilon} \) and \( \epsilon \), respectively (where \( \epsilon \sim 1/\ln \Lambda \), or \( \epsilon = 4 - d \) in dimensional regularization), these interactions are of infinitesimal strength. This is true to all orders because any diagram with \( T \) three-point vertices, \( F \) four-point vertices, and \( L \) loops is, at most, of order \( (\sqrt{\epsilon})^T (1/\epsilon)^F (1/\epsilon)^L = \epsilon^{T/2 + F - L} \). It is a topological identity that \( T/2 + F - L = n/2 - 1 \), where \( n \) is the number of external legs. Hence, the full 3-point function vanishes like \( \epsilon^{1/2} \); the full 4-point function vanishes like \( \epsilon \), etc. Thus, we obtain “triviality” as a direct consequence of the way we were obliged to renormalize the effective potential. Our initial approximation of ignoring the interaction terms \( S_{\text{int}} \) is seen to be self-consistently justified because, physically, \( S_{\text{int}} \) produces no interactions. Thus, our starting point is not actually an approximation but rather an ansatz that produces a solution of the theory.

The subtlety, though, is that \( S_{\text{int}} \), while too weak to produce physical interactions, can seemingly give contributions to the propagator and to the effective potential. The above \( \epsilon \)-counting argument applied to the \( n = 2 \) case implies that there are finite contributions to the propagator from arbitrarily complicated diagrams. Similarly, in the \( n = 0 \) case there are \( \mathcal{O}(1/\epsilon) \) and finite contributions to the vacuum diagrams, and hence to \( V_{\text{eff}} \). However, our claim is that all of these contributions will be re-absorbed by the renormalization process; the unmeasurable quantities \( \lambda_B, Z_\phi, v_B \), etc., may change, but the physical results \( (13, 14) \) will not. This “exactness conjecture” is supported by three arguments: (i) Since the theory has no physical interactions it would be paradoxical for the effective potential to have a form other than that produced by the classical potential plus free-field fluctuations. How, physically, can there be non-trivial contributions to the effective potential due
to interactions when, physically, there are no interactions? (ii) In the Gaussian approximation, which accounts for all the “cactus” (“superdaisy”) diagrams generated by $S_{\text{int}}$, one finds exactly the same physical results \cite{13, 14}. Things are different at the bare level, but the physical results are nevertheless exactly the same \cite{6}. (iii) The effective potential computed on the lattice in the appropriate region of bare parameters agrees very nicely with the one-loop form \cite{6}. This is in spite of the fact that $\lambda_B \ln \Lambda$ is of order unity in this region, and so naïvely the two-loop contribution would be expected to be as large as the one-loop contribution.

Furthermore, simple diagrammatic arguments can immediately establish part of the “exactness conjecture”. By $\epsilon$ counting it follows that finite contributions to the 2-point function come only from terms that gain a $1/\epsilon$ from every loop. Such terms cannot depend on the external momentum $p$, so the additional contributions only affect the mass renormalization. Similarly, to obtain a net $1/\epsilon$ contribution from a vacuum diagram, one must gain a $1/\epsilon$ from every loop. Such terms obey naive dimensional analysis and are proportional to $\phi^4$. The associated sub-leading finite contributions will involve $\phi^4 \ln \phi^2$. However, one cannot obtain any other functional dependence on $\phi$; terms with two or more powers of $\ln \phi$ will be suppressed by one or more powers of $\epsilon$. Thus the effective potential, at any order, is a sum of $\phi^4$ and $\phi^4 \ln \phi^2$ terms. It can therefore always be parametrized as $A \phi^4 (\ln(\phi^2/v^2) - 1/2)$. All that one cannot show by this simple argument is that, after renormalization, the coefficient $A$ must be $\pi^2$.

5. As we have seen, the interactions of the $h(x)$ field vanish because $\lambda_B \to 0$, but the effective potential is non-trivial because there one has $Z_\phi \to \infty$ to compensate for $\lambda_B \to 0$. Thus, it is crucial for our picture that the $Z_{\phi}^{1/2}$ re-scaling of the constant background field $\phi_B$ is quite distinct from the $Z_{h}^{1/2} = 1$ re-scaling of the fluctuation field $h(x)$. The decomposition $\Phi_B(x) = \phi_B + h_B(x)$, which separates the zero 4-momentum mode from the finite-momentum modes, is a Lorentz invariant decomposition for a scalar field. Hence, we can see no valid objection to treating the re-scaling of $\phi$ and $h(x)$ separately. The separation of the zero mode is particularly straightforward and natural in a finite-volume context \cite{5}. The situation is directly analogous to Bose-Einstein condensation where the lowest state must be given special treatment because it, and it alone, acquires a macroscopic occupation number.

$V_{\text{eff}}$ is the generator of the zero-momentum Green’s functions \cite{4}:

$$V_{\text{eff}}(\phi_R) = V_{\text{eff}}(v_R) - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{\phi_R}(0, 0, \ldots; v_R)(\phi_R - v_R)^n.$$  \hfill (15)
The $\Gamma^{(n)}_R$'s at zero momentum, being derivatives of the renormalized effective potential, are finite. However, at finite momentum, the $\Gamma^{(n)}_R$'s vanish for $n \geq 3$, corresponding to ‘triviality’. This just means that the $p^\mu \to 0$ limit is not smooth: The zero mode has non-trivial interactions, but the finite-momentum modes do not. The 2-point function is a special case: at finite momentum it is $\Gamma^{(2)}_R(p) = p^2 + m^2_R$, which is the (Euclidean) inverse propagator of a free field of mass $m^2_R$. It does have a smooth limit at $p^\mu = 0$, because we required

$$\frac{d^2 V_{\text{eff}}(\phi_R)}{d\phi_R^2} \bigg|_{\phi_R = v_R} = m^2_R.$$  (16)

Physically, the point is this: The $h(x)$ fluctuations (which in some sense are infinitesimal on the scale of $\phi_R$ if they were finite on the scale of $\phi_B$) are sensitive only to the quadratic dependence of $V_{\text{eff}}$ in the immediate neighbourhood of $v_R$. This quadratic dependence should mimic the potential for a free field of mass $m_h$ for self consistency.

6. Although this solution to $\lambda\Phi^4$ theory is “trivial” (meaning, it has no observable particle interactions), it is not entirely trivial — it is physically distinguishable from a free field theory. One can see this by considering finite temperatures. The free thermal fluctuations add to $V_{\text{eff}}$ a term

$$\frac{1}{\beta} \int \frac{d^3p}{(2\pi)^3} \ln[1 - \exp\{-\beta(p^2 + 8\pi^2\phi^2_R)^{1/2}\}],$$  (17)

where $\beta = 1/T$. This term leads to a first-order symmetry-restoring phase transition at a finite, not an infinite, temperature: $T_c = 2.77v_R$ (i.e., $T_c = 0.31m_h$). It is $v_R$, not $v_B$, that sets the scale because the depth of the SSB vacuum (invariant under $\phi$ re-scalings) was $\frac{1}{2}\pi^2v^4_R$. Thus, the non-trivial self-interactions of the zero mode, responsible for the non-trivial shape of $V_{\text{eff}}$, do reveal themselves in the finite-temperature behaviour of the theory.

7. We have discussed only the $N = 1$ theory, but everything can be generalized to the $O(N)$-symmetric case. There will be $N - 1$ massless, non-interacting Goldstone fields. Their zero-point energy is only an infinite constant, so the shape of the effective potential should be identical to the $N = 1$ case. This is our second “exactness conjecture”. It is supported by lattice evidence and by a non-Gaussian variational calculation.

We considered only the classically-scale-invariant (CSI) theory here, but everything can be generalized to include a bare $\frac{1}{2}m^2_R\Phi^2_B$ term in the Lagrangian. However, not only is the CSI theory simpler, it is the most attractive possibility. The only mass scale in the Standard Model would be $v_R$, arising from dimensional transmutation. One would
have a definite prediction for the Higgs mass; $m_{h}^{2} = 8\pi^{2}v_{R}^{2}$, which implies $m_{h} = 2.2$ TeV. (There are relatively small corrections due to the gauge and Yukawa couplings. These couplings would also induce weak interactions of the Higgs.)

It is usually believed that a Higgs above 800 GeV is either impossible [2] or must have a huge width and be associated with strongly interacting longitudinal gauge bosons. These beliefs stem from the false notion that $m_{h}^{2}$ is proportional to “$\lambda v_{R}^{2}$”. “Triviality” means that the “renormalized coupling” $\lambda_{R}$ vanishes; it does not mean that $m_{h}$ must also be zero in the continuum limit [10].

Acknowledgements

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-92ER40717.
References

[1] For a review of the rigorous results, see R. Fernández, J. Fröhlich, and A. D. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory* (Springer-Verlag, Berlin, 1992).

[2] For a review of lattice calculations, see *Lattice Higgs Workshop*, ed. Berg et al, World Scientific (1988).

[3] M. Consoli and P. M. Stevenson, *Resolution of the $\lambda\Phi^4$ puzzle and a 2 TeV Higgs boson*, Rice University preprint, DE-FG05-92ER40717-5, July 1993, submitted to Physical Review D. ([hep-ph 9303255](https://arxiv.org/abs/hep-ph/9303255)); M. Consoli and P. M. Stevenson, *The non-trivial effective potential of the ‘trivial’ $\lambda\Phi^4$ theory: a lattice test*, Rice University preprint DE-FG05-92ER40717-9, Oct. 1993, to be published in Z. Phys. C. ([hep-ph 9310338](https://arxiv.org/abs/hep-ph/9310338)).

[4] S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888.

[5] U. Ritschel, Phys. Lett. B 318 (1993) 617.

[6] V. Branchina, M. Consoli and N. M. Stivala, Z. Phys. C 57 (1993) 251.

[7] A. Agodi, G. Andronico, and M. Consoli *The real test of “triviality” on the lattice*, University of Catania preprint (Feb. 94) ([hep-th 9402071](https://arxiv.org/abs/hep-th/9402071)).

[8] A. Agodi, G. Andronico, and M. Consoli *Lattice computation of the effective potential in O(2)-invariant $\lambda\Phi^4$ theory*, University of Catania preprint (Apr. 94) ([hep-lat 9404010](https://arxiv.org/abs/hep-lat/9404010)).

[9] U. Ritschel, *Non-Gaussian corrections to Higgs mass in Autonomous $\lambda\phi^4_{3+1}$*, Essen preprint, to be published in Z. Phys. C.

[10] K. Huang, E. Manousakis, and J. Polonyi, Phys. Rev. D 35 (1987) 3187; K. Huang, Int. J. Mod. Phys. A 4 (1989) 1037; in Proceedings of the DPF Meeting, Storrs, CT, 1988.