RECOVERING BROWNIAN AND JUMP PARTS FROM HIGH-FREQUENCY OBSERVATIONS OF A LÉVY PROCESS

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Abstract. We introduce two general non-parametric methods for recovering paths of the Brownian and jump components from high-frequency observations of a Lévy process. The first procedure relies on reordering of independently sampled normal increments and thus avoids tuning parameters. The functionality of this method is a consequence of the small time predominance of the Brownian component, the presence of exchangeable structures, and fast convergence of normal empirical quantile functions. The second procedure amounts to filtering the increments and compensating with the final value. It requires a carefully chosen threshold, in which case both methods yield the same rate of convergence. This rate depends on the small-jump activity and is given in terms of the Blumenthal-Getoor index. Finally, we discuss possible extensions, including the multidimensional case, and provide numerical illustrations.

1. Introduction

Consider a Lévy process $X$ on $[0,1]$ and the decomposition

\[ X_t = Y_t + \sigma W_t, \quad t \in [0,1], \]

where $\sigma \geq 0$ and $W$ is a standard Brownian motion independent of the Lévy process $Y$ with no Brownian component. In this work we assume that $\sigma > 0$ and provide two methods to recover $W$, and thus also $Y$, from high-frequency observations $(X_{i/n})_{i=0,\ldots,n}$ (as $n \to \infty$) of a given sample path of $X$.

More precisely, we recover the path of the bridge $(W_t - W_1)_{t \in [0,1]}$ and the path of the drifted process $(Y_t + \sigma W_1)_{t \in [0,1]}$. It is not possible to recover $W_1$ because it is impossible to separate a linear drift from the Brownian path consistently (as the respective laws are equivalent). Note, however, that if $Y$ has bounded variation on compacts, there is a clear definition of the linear drift. Using the second method, this drift can be naturally separated from $Y$ (but not from $W$). Importantly, the proposed procedures do not require knowledge on the law of $X$, except for the parameter $\sigma$ (to some extent), which can be estimated efficiently from the given high-frequency observations [1, 14]. Furthermore, the first method, which is the main focus of this paper, completely avoids tuning parameters.

Apart from their intrinsic interest, the discussed procedures may be useful in a variety of applied areas. Oftentimes $\sigma W$ is interpreted as noise, see e.g. [4, 8, 21, 23, 24], and thus our methods recovers the signal $Y$ up to an unknown linear drift. This separation can then be used to answer various further questions regarding the observed trajectory and its decomposition. For example, what is the maximal fluctuation of the signal around its linear drift $\sup_{t \in [0,1]} |Y_t - tY_1|$ and how does it compare with the same quantity for the noise component?

Various statistical procedures may benefit from pre-separation of the Brownian part. According to [22, §5] ‘coexistence of the Gaussian part and the jump part makes the parametric estimation problem much more difficult and cumbersome’ and the common strategy then is to use thresholding. As was exemplified in [28] through simulations, a naive choice of the threshold may severely deteriorate estimation performance (see also Proposition 5 below). Thus our first procedure can be employed to avoid the difficult practical problem of threshold selection. Furthermore, it can be used as an alternative to [19, 20] to detect the presence of jumps. It must be noted, however, that essentially simpler problems than path decomposition may suffer from suboptimal rates coming from the latter part. Nevertheless, absence of tuning parameters may still seem attractive in applications.

Finally, our first method provides a coupling between the Lévy process $X$ and a Brownian motion by means of approximating $W$ which can be of independent interest, see [3] for an application where upper
bounds on the Wasserstein distance between the laws of a Lévy process and a Brownian motion are needed. This coupling is easy to simulate, making it suitable for (multilevel) Monte Carlo methods. We note here that, for any fixed $n$, the approximation of $W$ produced by the second method need not be a (discretely observed) Brownian motion.

1.1. Method I: reordering of normal increments. Our main procedure, which may be surprising at first, has a simple construction:

a) simulate an independent standard Brownian motion $W'$ on the grid $(i/n)_{i=1,\ldots,n}$,
b) reorder the increments of $W'$ according to the ordering of the increments of $X$.

We will show that the resultant skeleton $W^{(n)}$ (see (6) below for definition) satisfies:

\begin{equation}
(W_t - W_t^{(n)})_{t \in [0,1]} \overset{p}{\to} ((W_1 - W_1')t)_{t \in [0,1]}
\end{equation}

in supremum norm. In words, we recover the Brownian evolution up to some linear drift. In fact, we have a much stronger result in Theorem 1 establishing the speed of convergence, see also Figure 2 for a numerical illustration. The joint recovery is now straightforward:

\begin{equation}
\left( W_t^{(n)} - W_t^{(n)}(r), X_t - \sigma(W_t^{(n)} - W_t^{(n)}(r)) \right) \overset{p}{\to} \left( W_t - W_1 t, Y_t + \sigma W_1 t \right)
\end{equation}

in supremum norm, which is assumed throughout unless mentioned otherwise. Note, however, that only $X_{\{n\}/n}$ are available but we may replace $X$ above by its discretised version while relaxing to the convergence in Skorokhod $J_1$-topology [16, A2]. Alternatively, we may look at the difference of both sides and discretise all the processes involved.

Figure 1 illustrates the algorithm in the case $\sigma = 1$ and $Y$ being a variance gamma process. In addition, we remove the random drift in the approximation $X - W^{(n)}$ of $Y$ by matching the endpoints. In general, this is not possible in practice, and is done here only to assess the signal recovery.

Let us provide some intuition. On the small time scale an overwhelming number of the increments of $X$ are close to those of $\sigma W$. By self-similarity the scaled increments of $W$ are i.i.d. standard normal random variables, whose empirical quantile function exhibits fast convergence due to light tails of the normal distribution. At an intuitive level this explains that ordering the increments of $W'$ according to the increments of $W$ or $X$ may produce a well-coupled process. Nonetheless, the result may still look surprising even for the purely Brownian case. In some sense, a path of the Brownian bridge is determined by the ordering of its infinitesimal increments. Interestingly, in the case of no Brownian component ($\sigma = 0$), despite the increments following the same order, the limiting result of this procedure is a standard Brownian motion independent of the original process $X$, see Proposition 3 below.

1.2. Method II: threshold filter. A much more intuitive path decomposition method is based on a threshold filter. More precisely, we consider

\begin{equation}
\hat{W}_t^{(n)} = \sigma^{-1} \sum_{i \leq t n, |\Delta_t^a X| \leq a_n} \Delta_t^a X, \quad t \in [0,1],
\end{equation}

which is the (scaled) skeleton that results from removing the increments of $X$ whose size is larger than $a_n > 0$. Throughout this work we adopt the notation $\Delta_t^a X = X_{i/n} - X_{(i-1)/n}$ for $i = 1,\ldots,n$, which is standard in discretisation of processes [13].

A wide range of work has been done on the subject of estimation via threshold filters, see [13, Ch. 9 & 13] and [28, 5, 25]. Most of these works, however, explore the recovery of a generalised path-variation or related quantities, and it seems that path decomposition of a Lévy process has been overlooked in part. For instance, the conditions of [13, Thm 9.1.1] are not satisfied by the identity function $f : x \mapsto x$ when $Y$ has infinite variation on compacts. Moreover, even in the covered cases, the speed of convergence seems not to be known, see [13, Thm 13.1.1].

It is possible that this case has been overlooked since $\hat{W}^{(n)}$ may explode when $Y$ has infinite variation on compacts. It is thus very important in that case to subtract the term $\hat{W}_1^{(n)} t$ as it plays a crucial compensating role. We will show that with the right choice of threshold $a_n$, we have

\begin{equation}
(\hat{W}_t^{(n)} - \hat{W}_1^{(n)} t)_{t \in [0,1]} \overset{p}{\to} (W_t - W_1 t)_{t \in [0,1]}.
\end{equation}
see Theorem 4 below, which also provides the rate of convergence. Once again, we avoid writing this convergence in the form of (2), since then both sides may explode. However, if \( Y \) has bounded variation on compact intervals and possesses a linear drift \( \gamma_0 \) then

\[
\left( \tilde{W}_t^{(n)} \right)_{t \in [0,1]} \overset{p}{\rightarrow} \left( W_t + \gamma_0 \sigma^{-1} t \right)_{t \in [0,1]},
\]

see Proposition 6. For this method it may have been cleaner to avoid scaling by \( \sigma^{-1} \), but we prefer to be consistent: \( W^{(n)} \) and \( \tilde{W}^{(n)} \) both approximate the path of a standard Brownian motion. Note, however, that for any \( n \), the approximation \( W^{(n)} \) is a discretely observed Brownian motion while \( \tilde{W}^{(n)} \) need not be. We stress that for finite \( n \) this method depends on the tuning parameter \( \alpha_n \), which may present a serious hurdle in applications. In this sense, our main Method I is more robust.

2. The main results: rates of convergence and limit laws

Denote the Lévy triplet (see [27, §2, Def. 8.2]) of \( X \) by \( (\gamma, \sigma^2, \Pi(dx)) \) and write \( \Pi(x) = \Pi(\mathbb{R} \setminus (-x,x)) \) for any \( x > 0 \). Throughout this work we assume that \( \sigma > 0 \) unless stated otherwise, and use the notation in (1). The quality of decomposition of the path of \( X \) crucially depends on the activity of small jumps. Therefore,
we define two indices $0 \leq \beta_* \leq \beta^* \leq 2$ capturing some main characteristics:

\[ \beta^* = \inf \left\{ p \geq 0 : \int_{(-1,1)} |x|^p \Pi(dx) < \infty \right\}, \]

\[ \beta_* = \inf \left\{ p \geq 0 : \liminf_{x \to 0} x^p \Pi(x) = 0 \right\}. \]

The index $\beta^*$ is known as the Blumenthal-Getoor index, whereas $\beta_*$ reminds Pruitt’s index [26], which must lie between $\beta_*$ and $\beta^*$. Importantly, $\beta_* = \beta^*$ under some weak regularity assumptions, such as $\Pi(x)$ being regularly varying at 0 with some index $-\alpha$, in which case

\[ \beta^* = \beta_* = \alpha, \]

a simple consequence of the standard theory of regular variation [2, §1].

2.1. Method I. As mentioned in §1.1 above, we consider a standard Brownian motion $W'$ independent of $X$ and $W$. Let $\pi$ be the (random) permutation of the indices $1, \ldots, n$ such that the ordering of $\Delta^n_{\pi(s)} W'$ coincides with that of $\Delta^n_s X$. In other words, if $s$ is a permutation such that $\Delta^n_s X$ is an increasing sequence then $\Delta^n_{\pi(s)} W'$ is also an increasing sequence. Such permutation $\pi$ is a.s. unique since there are a.s. no ties in either sequence. Finally, we take the corresponding partial sum process

\[ W^{(n)}_t = \sum_{i \leq nt} \Delta^n_{\pi(i)} W', \quad t \in [0,1], \]

which is a Brownian random walk. This can be seen by noting that the composition of a fixed permutation and a random uniform permutation is also uniformly distributed, and so the increments $\Delta^n_i W^{(n)}$ are i.i.d. indeed. We may also keep the bridges in-between discretisation, which would then yield a standard Brownian motion. Note that the joint process $(W^{(n)}_{i/n}, W_{i/n})$, $i = 1, \ldots, n$, has exchangeable increments but is not a random walk. Next we state the main result.

**Theorem 1.** For any $p \in (\beta^*, 2] \cup \{2\}$ it holds that

\[ n^{(2-p)/4} \sup_{t \in [0,1]} \left| W_t - W_t^{(n)} - (W_1 - W_1^{(n)}) t \right| \xrightarrow{p} 0. \]

Moreover, this convergence fails for any $p \in [0, \beta_*) \cup \{0\}$.

It is noted that the final statement of Theorem 1 implies that with some positive probability the quantity on the left hand side of (7) becomes arbitrarily large for some large $n$. Thus we establish the exact convergence rate in the logarithmic sense when $\beta_* = \beta^*$ and, in particular, this rate is $n^{-{(2-\alpha)/4}}$ in the regularly varying case (5). Note as well that Theorem 1 also implies the convergence of the bivariate approximation in (3) with exactly the same rate. The convergence in (3) requires access to the parameter $\sigma$. However, this parameter may be estimated first without deteriorating the resulting convergence speed in (3). For instance, according to [12] (see also [14]), one can construct an estimator of $\sigma$ based on threshold filters (analogous to our Method II) with error decay $O_p(n^{-1/2})$ if $\beta^* < 1$ and, otherwise, $O_p(n^{-(2-p)/2})$ for any $p \in (\beta^*, 2]$.

2.2. Method I. Extensions.

2.2.1. Dynamics under a dominating probability measure. It suffices to let $W$ be a Brownian motion independent of the pure-jump Lévy process $Y$ under some probability measure $Q$ dominating $\mathbb{P}$, that is, $Q \gg \mathbb{P}$. Indeed, in that case the limit in Theorem 1 holds under $Q$ and thus, under $\mathbb{P}$. For example, by Girsanov’s theorem, we may consider $W_t = B_t + \int_0^t f(s)ds$, $t \in [0,1]$, where $f$ is an adapted process satisfying $\int_0^1 f(t)^2 dt < \infty$ and $B$ is a standard Brownian motion under $\mathbb{P}$. In fact, the process $Y$ may also be a rather general pure-jump semimartingale under $\mathbb{P}$, possibly dependent on $W$, see [18, Thm 2.3] and [15, Thm III.3.24, p. 172], but also Proposition 2 below.
2.2.2. Extensions under exchangeability. Further generalisations are possible. Namely, the convergence in (7) is guaranteed for any process \( Y \) (possibly dependent on the Brownian motion \( W \)) such that the increments of the bivariate process \((W,Y)\) are exchangeable and \( Y \) satisfies:

\[
 n^{2-p/2} \mathbb{E} \left( (\Delta_t^X)^2 \wedge \frac{\log n}{n} \right) \to 0.
\]

It would be interesting to understand if exchangeability can be replaced by another structural assumption. One way is to ensure that (14) below is sufficient for the corresponding partial sums to vanish. In this regard we point out that a martingale assumption [13, Eq. (2.2.35)] seems to be of no immediate use because of inherent reorderings.

2.2.3. Multidimensional case. Our method readily applies in a multivariate setting, where \( Y \) is an \( \mathbb{R}^d \)-valued pure-jump Lévy process, \( W \) is an independent \( d \)-dimensional Brownian motion with standard but possibly correlated components and \( \sigma \) is a diagonal scaling matrix. Indeed, by applying the decomposition procedure to every component of \( X \) we may recover the entire path of the \( d \)-dimensional bridge \((W_t - tW)_{t \in [0,1]}\). One may use a single one-dimensional standard Brownian motion \( W' \) for every component of \( X \), which may seem counter-intuitive as the dependence structure across coordinates is captured by the reorderings of increments. In a degenerate case when the rank of the correlation matrix (suppose it is known) is smaller than \( d \), we may use appropriate directions to reduce the number of required one-dimensional reconstructions to the given rank. Finally, we point out that the generalisations discussed above still apply in this context, see §2.2.1 in particular.

2.3. Method I. Further results. We have a more precise result in the case when \( Y \) is a general piecewise constant process, including the compound Poisson process case. Note that one may always add a linear drift to \( Y \) since this does not affect \( W^{(n)} \). The following result is stated for the discrete skeleton, since it may fail otherwise as the maximal deviation of \( W \) from its discrete skeleton is of the same order:

\[
(8) \quad \sup_{t \in [0,1]} |W_t - W_{\lfloor tn \rfloor/n}| = \Theta_p \left( \frac{\log n}{n} \right),
\]

meaning that the function on the right-hand side is, up to multiplicative constants, both an upper and lower asymptotic bound for the left-hand side. Throughout, the space \( D[0,1] \) of right-continuous functions with left-hand limits on \([0,1]\) is endowed with the standard Skorohod \( J_1 \)-topology [16, Ch. 16 and A2].

**Proposition 2.** Let \((Y_t)_{t \in [0,1]}\) be a piecewise constant process (not necessarily Lévy) independent of \( W \) with jumps \( J_1, \ldots, J_N \) at times \( T_1, \ldots, T_N \in (0,1) \). Then the limit

\[
\sqrt{\frac{n}{2 \log n}} \left( W_{\lfloor tn \rfloor/n} - W_t^{(n)} - (W_1 - W_1^{(n)})t \right) \overset{p}{\to} \sum_{i=1}^{N} \text{sign}(J_i)(t - 1_{\{T_i \leq t\}})
\]

holds in \( D[0,1] \).

Interestingly, only the signs of the jump sizes \( J_i \) appear in the limit. In the purely Brownian case \((Y = 0)\) the effective convergence rate is \( \sqrt{\log \log n/n} \), see Appendix A.

The final result covers the case \( \sigma = 0 \). That is, we apply our procedure for a Lévy process without Gaussian part: \( X = Y \). Interestingly, this ‘coupling’ yields, in the limit, a Brownian motion independent of \( X \). We believe that the following result is true even in the highest activity case \( \beta^* = 2 \), but its proof seems to require a more careful analysis.

**Proposition 3.** If \( \sigma = 0 \) and \( \beta^* < 2 \), then the distributional convergence

\[
(X_t, W_t^{(n)}) \overset{d}{\to} (X_t, B_t)
\]

holds in \( D[0,1] \), where \( B \) is a standard Brownian motion independent of \( X \).
2.4. Method I. Numerical illustration. We conclude the discussion of Method I with numerical illustrations of Theorem 1 and Proposition 2. For this example we assume \(\sigma = 1\), \(W\) is a 3-dimensional Bessel process (a Brownian motion under an equivalent probability measure) and \(Y\) is an independent strictly \(\alpha\)-stable process. We consider various values of \(\alpha\) and, as the other parameters are less relevant, we fix the skewness parameter at \(\beta = 0.5\) and take unit scale. We work with 5 approximation levels \(n = 10^3, 10^4, \ldots, 10^7\) and for each scenario we compute the maximal difference between the discretised bridge on a uniform grid of \(N = 10^7\) points and its approximation at level \(n\):

\[
\sup_{i \leq N} \left| W_{i/N} - \frac{i}{N} W_1 - (W_{i/N}^{(n)} - \frac{i}{N} W_1^{(n)}) \right|.
\]

We point out that we do not resample \(W'\), i.e. we use the same path for each of the resolution levels \(n\). We replicate the procedure 100 times to estimate the expected value of the quantity in (9) and its standard deviation, see Table 1. For the sake of comparison, we also take \(n = 1\), where \(W_1^{(n)} = W_1\) and (9) quantifies the discrepancy between two independent bridges, resulting in 1.202(0.3333).

| \(n\)     | \(\alpha = 0.2\)  | \(\alpha = 0.6\)  | \(\alpha = 1\)   | \(\alpha = 1.4\) | \(\alpha = 1.8\) | \(\alpha = 1.99\) |
|----------|--------------------|--------------------|-------------------|-------------------|------------------|-------------------|
| \(10^3\) | .1590(0.430)       | .1704(0.438)       | .1912(0.598)      | .2330(0.553)      | .3115(0.825)     | .3399(0.786)      |
| \(10^4\) | .0626(.0175)       | .0741(.0199)       | .0980(.0241)      | .1468(.0415)      | .2623(.0715)     | .3061(.0793)      |
| \(10^5\) | .0243(.0086)       | .0326(.0083)       | .0536(.0137)      | .1002(.0306)      | .2275(.0649)     | .2958(.0793)      |
| \(10^6\) | .0090(.0033)       | .0145(.0036)       | .0290(.0079)      | .0704(.0221)      | .2007(.0551)     | .2905(.0811)      |
| \(10^7\) | .0031(.0017)       | .0056(.0018)       | .0153(.0043)      | .0501(.0160)      | .1773(.0511)     | .2886(.0821)      |

Table 1. Means (and standard deviations) of the errors given by (9) when \(Y\) is an \(\alpha\)-stable process.

Figure 2 provides the log-log plot together with lines corresponding to the theoretical rates given by Theorem 1. That is, the lines pass through the given value at \(n = 10^5\) and their slopes are given by \(-(2 - \alpha)/4\).

Next assume \(\sigma\) and \(W\) are as in the first paragraph of §2.4 and \(Y\) is a Poisson process with intensity 3. The Figure 3 below exemplifies the limit established in Proposition 2 above. Note how the signs of the jumps in the limit are opposite to those of \(Y\).

2.5. Method II. Finally, we turn our attention to the filtering method described in §1.2. Here, the main approximant is the process \(\tilde{W}_t^{(n)}\) defined in (4).

**Theorem 4.** For any \(p \in (\beta^*, 2]\) it holds that

\[
n(2 - p)/4 \sup_{t \in [0,1]} \left| W_t - \tilde{W}_t^{(n)} - (W_1 - \tilde{W}_1^{(n)})t \right|_p^p \to 0,
\]

assuming that the sequence \(a_n\) satisfies

\[
\lim \inf_{n \to \infty} \frac{na_n^2}{\sigma^2 \log n} \geq 2 - \beta^* \quad \text{and} \quad n^{1/2 - \epsilon} a_n \to 0 \quad \text{for all } \epsilon > 0.
\]
The limit (10) also holds for \( p = 2 \) and any \( \beta^* \) if \( a_n \to 0 \) and \( \lim \inf_{n \to \infty} na_n^2 / \log n > 0. \)

In words, the threshold \( a_n \) should (roughly) be of order \( n^{-1/2} \), but not smaller than \( c \sqrt{\log n / n} \) for a certain constant \( c > 0 \). The choice \( a_n = n^{-1/2} \log n \), for example, satisfies (11) in all cases. Both upper and lower bounds on \( a_n \) can be relaxed, but then the rate would deteriorate. The allowed relaxations of the bounds and the corresponding rates are stated in Lemma 14 of §5. In some sense, the rate is much less sensitive to increasing \( a_n \) than to decreasing it.

Importantly, the same (logarithmic) rate can be obtained using filtering method with an appropriate \( a_n \). We expect, however, that the choice of such threshold for a finite \( n \) may have a serious impact on the quality of decomposition, see also [28]. Our main Method I has no such issues. Let us also supplement Theorem 4 with the negative results showing that the stated threshold bounds and the convergence rates are optimal.

**Proposition 5.** The following statements are true

(a) If \( p \in [0, \beta^*) \) then the limit (10) fails.

(b) If \( \lim \inf_{n \to \infty} na_n^2 / (\sigma^2 \log n) < 2 - \beta^* \) then the limit (10) fails for some \( p \in (\beta^*, 2] \).

(c) If \( \Pi \neq 0 \), \( a_n \to 0 \) and \( n^{1/2-\epsilon} a_n \to 0 \) for some \( \epsilon > 0 \) then the limit (10) fails for some \( p \in (\beta^*, 2] \).

As mentioned above, compensation by \( \hat{W}_1^{(n)} t \) is not required in the case when \( Y \) has bounded variation on compact intervals (implying \( \beta^* \in [0, 1] \)). We have the following additional result:

**Proposition 6.** If \( p \in (\beta^*, 1] \) and \( a_n \) satisfies (11), it holds that

(12) \[ n^{(1-p)/2} \sup_{t \in [0,1]} \left| W_t + \gamma_0 \sigma^{-1} t - \hat{W}_1^{(n)} t \right| \rightarrow 0, \]

where \( \gamma_0 \) is the linear drift of \( X \). The limit (12) is also true for \( p = 1 \) if \( Y \) has bounded variation on compact intervals, \( a_n \to 0 \) and \( \lim \inf_{n \to \infty} na_n^2 / \log n > \sigma^2 \).

Observe that in the case considered by Proposition 6 we may recover \( \sigma W_t + \gamma_0 t \) and the pure jump part \( Y_t - \gamma_0 t \), whereas separating the linear drift from the Brownian motion is impossible. The convergence rate, however, is worse than in Theorem 4.

### 2.6. Methods I and II. Numerical comparison

To compare both methods, we test them on the process \( X = \sigma W + Y \) where \( \sigma = 1, W \) is a 3-dimensional Bessel process and \( Y \) is a strictly \( \alpha \)-stable process with \( \alpha = 1.2 \) and skewness parameter \(-0.5\). In the application of Method II we used the threshold \( a_n = n^{-1/2} \log n \), which satisfies (11) independently of \( \sigma \) and \( \alpha \).

Figure 4 displays the errors \( W_t - W_1^{(n)} - (W_t - W_1^{(n)}) t \) and \( W_t - \hat{W}_1^{(n)} - (W_t - \hat{W}_1^{(n)}) t \) under both methods. Table 2 reports the mean and standard deviation of the maximal absolute value (on the skeleton) of the two error processes after repeating the procedure described in the previous paragraph for 1000 independent
paths of $X$. The comparison of the test statistics in Table 2 suggests that Method I outperforms Method II by a constant factor in this example.

Consider the process of interest

$$\left(n^{(2-p)/4}(W_t - W_t^{(n)} - (W_1 - W_1^{(n)})t)\right)_{t\in[0,1]},$$

with $p \in (0, 2]$ and let $\xi_{ni}$ be its $i$th increment, that is, we apply $\Delta^n_i$. According to (8) we may restrict our attention to the partial sum process $(\sum_{i \leq t n} \xi_{ni})_{t\in[0,1]}$. Observe that

$$\xi_{ni} = n^{(2-p)/4} \left( \Delta^n_i W - \Delta^n_{\pi(i)} W' - \frac{1}{n} \sum_{j \leq n} (\Delta^n_j W - \Delta^n_j W') \right)$$

$$= n^{-p/4} \left( Z_i - Z'_{\pi(i)} - \frac{1}{n} \sum_{j \leq n} (Z_j - Z'_{j}) \right),$$

where $Z_1, \ldots, Z_n$ and $Z'_1, \ldots, Z'_n$ are i.i.d. standard normal variables. Recall that $\pi$ is the permutation such that $Z'_{\pi(i)}$ has the same ordering as

$$Z_i + \frac{1}{\sigma} \sqrt{n} \Delta^n_i Y.$$ 

Additionally, we define the permutation $\nu$ so that $Z_{\nu(i)}$ is ordered according to (13) and thus the orderings of $Z_{\nu(i)}$ and $Z'_{\pi(i)}$ coincide. In the decomposition

$$\xi_{ni} = \tilde{\xi}_{ni} + \hat{\xi}_{ni} = n^{-p/4} \left( Z_{\nu(i)} - Z'_{\pi(i)} - \frac{1}{n} \sum_{j \leq n} (Z_j - Z'_{j}) \right) + n^{-p/4} (Z_i - Z_{\nu(i)}),$$

the second term does not depend on $W'$, whereas the first term essentially corresponds to comparing certain order statistics (the order is random and dependent on $X$).
The strategy is to split the analysis of the partial sum process of $\xi_{ni}$ into that of the partial sum processes of $\xi_{ni}$ and $\xi_{ni}$. Importantly, $(\xi_{ni})_{i=1,\ldots,n}$ and $(\xi_{ni})_{i=1,\ldots,n}$ are both exchangeable. In fact, this is true for any process $Y$ as long as $(\Delta^n W, \Delta^n X)$ is exchangeable. Thus the general theory in [17, Thm 3.13] for exchangeable increment processes is applicable. In this respect, since $\sum_{i\leq n} \hat{\xi}_i = \sum_{i\leq n} \hat{\xi}_{ni} = 0$, we note that the convergence in probability of the partial sum processes of $\hat{\xi}_{ni}$ and $\xi_{ni}$ to 0 is equivalent to, respectively, the limits

$$\sum_{i\leq n} \hat{\xi}_{ni}^2 \xrightarrow{p} 0 \quad \text{and} \quad \sum_{i\leq n} \xi_{ni}^2 \xrightarrow{p} 0.$$  

Lemma 7 below establishes the first limit for any $p > 0$. The second convergence depends on the choice of $p$: it holds for large enough $p$ and fails for sufficiently small $p$.

### 3.1. Preparatory results.

**Lemma 7.** Let $Z_{(1)} < \cdots < Z_{(n)}$ and $Z'_{(1)} < \cdots < Z'_{(n)}$ be two independent ordered sequences of $n$ standard normal random variables. Then the limit

$$a_n \sum_{i\leq n} \left( Z(i) - Z(i)' - \frac{1}{n} \sum_{j\leq n} (Z(j) - Z(j)') \right)^2 \xrightarrow{p} 0$$

holds whenever $a_n \log \log n \to 0$.

**Proof.** Letting $\mu_n$ be the inner sum of the statement we note that

$$\sum_{i\leq n} \left( Z(i) - Z(i)' - \frac{1}{n} \mu_n \right)^2 = \sum_{i\leq n} \left( Z(i) - Z(i)' \right)^2 - \frac{1}{n} \mu_n^2.$$  

Since $\mu_n/\sqrt{n} \sim N(0,2)$ has constant distribution we have $\mu_n^2/(n \log n) \xrightarrow{p} 0$. Thus it is left to prove that

$$a_n \sum_{i\leq n} \left( Z(i) - Z(i)' \right)^2 = a_n n \| F_n^{-1} - G_n^{-1} \|^2 \xrightarrow{p} 0,$$

where $\|f\|^2 = \int_0^1 f^2(x) dx$, and $F_n^{-1}$ and $G_n^{-1}$ are the right-inverses of the empirical distributions of $Z$ and $Z'$, respectively.

Let $\Phi$ denote the standard normal distribution and let $\tilde{F}_n = \sqrt{n}(F_n^{-1} - \Phi^{-1})$ and $\tilde{G}_n = \sqrt{n}(G_n^{-1} - \Phi^{-1})$ be the respective normalised empirical quantile processes. By Minkowski inequality we have

$$n \| F_n^{-1} - G_n^{-1} \|^2 = \| \tilde{F}_n - \tilde{G}_n \|^2 \leq 2(\| \tilde{F}_n \|^2 + \| \tilde{G}_n \|^2).$$

From [7, Thm 4.6(ii)] it can be deduced that $\| \tilde{F}_n \|^2 / \log \log n \xrightarrow{p} 1$. The same is true of $\tilde{G}_n$, completing the proof. \qed

It can be shown that $\log \log n$ is the "right" scale, see Appendix A. The following permutation Lemma (with $q = 2$) is crucial to upper bound $\sum_{i\leq n} (Z_i - Z_{\nu(i)})^2$. In this lemma it is more convenient to swap $\nu$ for $\nu^{-1}$.

**Lemma 8.** Let $z_1 \leq \cdots \leq z_n$ be $n \geq 1$ ordered real numbers. For arbitrary $y_i \in \mathbb{R}$ consider a permutation $\nu$ such that $z_{\nu^{-1}} + y_{\nu^{-1}}$ is ordered. Then we have

$$\sum_{i\leq n} |z_i - z_{\nu(i)}|^q \leq 2^q \sum_{i\leq n} (|y_i|^q \wedge m^q), \quad q \geq 1,$$

with $m = z_n - z_1$.

**Proof.** Without loss of generality we assume that $\nu$ has exactly one cycle, since otherwise we just sum over the cycles and increase the respective $m$ if needed. Moreover, the result is trivial for a cycle of length 1.

It is a basic fact that

$$|z_i - z_j|^q \leq (|y_i| + |y_j|)^q \leq 2^q(|y_i|^q + |y_j|^q)$$

whenever $i < j$ and $\nu(i) > \nu(j)$ or $i > j$ and $\nu(i) < \nu(j)$, i.e., if the order is flipped. Furthermore, this bound is still true when $|y_i|$ is replaced by $|y_i| \wedge m$. 


We call the ordered sequence $\nu(i), \nu^2(i), \ldots$ the successors of $i$. For each $i$ satisfying $i < \nu(i)$, we define:

$$b(i)$$

is the first successor of $i$ such that $\nu(b(i)) < \nu(i) \leq b(i)$,

and note that $b(i)$ is well defined, see Example 3.1. From $(15)$ we have the bound:

$$|z_i - z_{\nu(i)}|^q \leq |z_i - z_{b(i)}|^q \leq 2^{q-1}(|y_i|^q \wedge m^q + |y_{b(i)}|^q \wedge m^q).$$

The case $i > \nu(i)$ is analogous but with inequalities reversed in the definition of $b(i)$. By summing up over all $i$ we get the upper bound for $\sum_{i \leq n} |z_i - z_{\nu(i)}|^q$. This bound needs to be reduced since the same $b$ may appear multiple times.

Suppose $i_1, \ldots, i_k$ with $k > 1$ are all the indices with

$$b^* = b(i_1) = \cdots = b(i_k).$$

Without loss of generality we assume that $b^* > \nu(b^*)$ and so $i_j < \nu(i_j)$ for all $j = 1, \ldots, k$. Moreover, let the numbering be such that the path from $i_1$ to $b^*$ passes through $i_2, \ldots, i_k$ in this order. Note that $i_2 < \nu(i_1)$ implies that $b(i_1)$ occurs before $i_2$, a contradiction. Thus we have

$$i_1 < \nu(i_1) \leq i_2 < \nu(i_2) \leq \cdots < i_k < \nu(i_k) \leq b^*.$$

Hence, it holds that

$$|z_{i_1} - z_{\nu(i_1)}|^q + \cdots + |z_{i_k} - z_{\nu(i_k)}|^q \leq |z_{i_1} - b^*|^q,$$

implying that only one term $2^{q-1}(|y_i|^q \wedge m^q + |y_{b(i)}|^q \wedge m^q)$ out of $k$ is necessary. The proof is now complete. \(\square\)

Note that the constant $2^q$ in front of the upper bound can not be reduced in general. For example, let $q = n = 2$ and $z_1 = 0, z_2 = 1, y_1 = 1/2 + \epsilon, y_2 = -y_1$ with some $\epsilon > 0$. Then $z_1 + y_1 > z_2 + y_2$ and the bound reads $2 \leq 2(1 + 2\epsilon)^2$.

**Example.** Consider the permutation: $1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 1$. The summary of indices is given below:

| $i$ | direction | $b(i)$ | $\# y_i$ in the bound |
|-----|-----------|--------|-------------------|
| 1   | $\rightarrow$ | 5      | 2                 |
| 2   | $\rightarrow$ | 4      | 1                 |
| 3   | $\rightarrow$ | 5      | 1                 |
| 4   | $\leftarrow$ | 3      | 2                 |
| 5   | $\leftarrow$ | 1      | 2                 |

Note that the pair $(3, 5)$ was not used in the construction of our bound.

Finally, we need some estimates for the Lévy processes $Y$.

**Lemma 9.** The following statements hold for any Lévy process $Y$ without Brownian component:

(a) For any positive decreasing sequence $a_n \downarrow 0$ satisfying $a_n \sqrt{n} \rightarrow \infty$, we have the limit $\mathbb{P}(|Y_{1/n}| > a_n) \rightarrow 0$ and, for sufficiently large $n$, the following bound holds:

$$n\mathbb{P}(|Y_{1/n}| > a_n) \geq \frac{1}{2} \Pi(2a_n).$$

(b) For any $p \in (\beta^*, 2] \cup \{2\}$, we have

$$n^{2-p/2} \mathbb{E}
\left(Y_{1/n}^2 \wedge \frac{\log n}{n}\right) \rightarrow 0.$$

(c) If $\beta^* < 2$ then

$$\sqrt{n \log n} \mathbb{E}(|Y_{1/n}| \wedge 1) \rightarrow 0.$$

The proof is based on some standard techniques and is deferred to §4 in order to keep the presentation focused.
3.2. **Proofs of the main results.** In the following we say that events \((A_n)_{n \in \mathbb{N}}\) have high probability (for all large \(n\)) if \(\mathbb{P}(A_n) \to 1\). Clearly, any finite collection of events with high probability jointly have high probability.

**Proof of Theorem 1.** Recall that it is left to consider the quantities in (14). Lemma 7 implies that \(\sum_{i \leq n} \tilde{Z}_{ni}^2 \xrightarrow{p} 0\) for any \(p > 0\) because the sum can be reordered so that both \(Z\) and \(Z'\) appear in increasing order. Thus is is left to (i) show \(\sum_{i \leq n} \tilde{Z}_{ni}^2 \xrightarrow{p} 0\) for \(p > \beta_s\) (and \(p = 2\) when \(\beta^* = 2\)), and (ii) to disprove this for \(p \in (0, \beta_s)\). For \(p = 0\) the convergence in (7) always fails as a consequence of presence of \(n\) independent scaled Brownian bridges between the grid points, see also (8).

Part (i). By standard extreme value theory \([9, (3.65)]\) we have

\[
M_n - 2\sqrt{2\log n} \xrightarrow{p} 0, \quad \text{where} \quad M_n = \max_{i \leq n} Z_i - \min_{i \leq n} Z_i.
\]

According to (13) and Lemma 8 there is the bound

\[
\sum_{i \leq n} \tilde{Z}_{ni}^2 = n^{-p/2} \sum_{i \leq n} (Z_i - Z_{\nu(i)})^2 \leq 4n^{-p/2} \sum_{i \leq n} \left( \frac{n}{\sigma^2} (\Delta_i^0 Y)^2 \wedge M_n^2 \right).
\]

With high probability \(M_n^2 < 9\log n\) for all large \(n\). Moreover, by Lemma 9(b),

\[
\mathbb{E} \left[ n^{-p/2} \sum_{i \leq n} \left( n(\Delta_i^0 Y)^2 \wedge \log n \right) \right] = n^{1-p/2} \mathbb{E} \left( nY_{1/n}^2 \wedge \log n \right) \to 0
\]

whenever \(p > \beta^*\) or \(p = 2 = \beta^*\). Hence we also have \(\sum_{i \leq n} \tilde{Z}_{ni}^2 \xrightarrow{p} 0\) for such a \(p\), proving the first claim.

Part (ii). Assume that \(p \in (0, \beta_s)\) and recall that \(M_n < 3\sqrt{\log n}\) with high probability for all large \(n\). Note that \(\beta_s > 0\) implies \(\Pi(\mathbb{R}) = \infty\), and so \(Y\) is not compound Poisson. Let \(I\) be the set of indices \(i\) such that

\[
(16) \quad \sqrt{n}|\Delta_i^0 Y|/\sigma > 6\sqrt{\log n}.
\]

The cardinality \(N = |I|\) is Binomial\((n, p_n)\) distributed, where \(p_n\) satisfies

\[
np_n = n\mathbb{P}\left( |Y_{1/n}| > a_n/\sigma \right) \geq \frac{1}{2} \Pi(ca_n), \quad \text{for} \quad a_n = \sqrt{\log n/n},
\]

some \(c > 0\) and all large \(n\), see Lemma 9(a). This implies that \(np_n \to \infty\) and so

\[
N = np_n(1 + o_p(1)).
\]

Moreover, Lemma 9(a) shows that \(p_n \to 0\) and so \(N/n \xrightarrow{p} 0\).

Let \(N'\) be the analogue of \(N\), but with 6 replaced by 3 in (16). From the definition of \(\nu\) (see also (13)) and the above bound on \(M_n\), we conclude that all \(Z_{\nu(i)}, i \in I,\) must be among the \(N'\) largest or among the \(N'\) smallest values of \(Z\) with high probability. As with \(N\), we see that \(N'/n \xrightarrow{p} 0\) and thus \(Z(n') \xrightarrow{p} -\infty\) and \(Z(n-N') \xrightarrow{p} +\infty\). The corresponding \(Z_i, i \in I,\) however, are chosen independently of \(Y\) so by the law of large numbers, \([N'/2]\) of their moduli \(|Z_i|\) must be bounded above by \(\Phi^{-1}(4/5)\) with high probability for all large \(n\). Finally, we get the following bound with high probability for all sufficiently large \(n:\)

\[
\sum_{i \leq n} (Z_i - Z_{\nu(i)})^2 \geq 3N \geq \Pi(ca_n).
\]

Choose \(q \in (p, \beta_s)\) and note that necessarily \(x^q\Pi(x) \to \infty\) as \(x \downarrow 0\). Thus for some \(c_1 > 0\) and all large \(n\) we have the bound

\[
n^{-p/2}\Pi(ca_n) \geq c_1 n^{-p/2} n^{q/2}(\log n)^{-q/2} \to \infty.
\]

This shows that

\[
\sum_{i \leq n} \tilde{Z}_{ni}^2 = n^{-p/2} \sum_{i \leq n} (Z_i - Z_{\nu(i)})^2 \xrightarrow{p} \infty,
\]

instead of convergence to 0. The proof is now complete. \(\square\)
Lemma 10. Let \((Z_1, \ldots, Z_n)\) be exchangeable and independent of \((Z'_1, \ldots, Z'_n)\). For any \(1 \leq i_1 < \cdots < i_k \leq n\) and \(y_1, \ldots, y_k \in \mathbb{R}\) define
\[
\tilde{Z}_i = Z_i + \sum_{u=1}^{k} y_u 1_{\{i=i_u\}}, \quad i = 1, \ldots, n.
\]
Assume there are no ties a.s. and let \(\nu\) and \(\pi\) be permutations such that the orderings of \((\tilde{Z}_i)\), \((Z_{\nu(i)})\) and \((Z'_{\pi(i)})\) coincide. Then the sequence \(((Z_i, Z_{\nu(i)}, Z'_{\pi(i)}))_{i \notin \{i_1, \ldots, i_k\}}\) of length \(n-k\) is exchangeable.

Proof. Any finite sequence of random variables is exchangeable if and only if it can be represented as arbitrary random variables that are independently and uniformly permuted, see e.g. [17, Prop 1.8]. Therefore, we may prove the result by conditioning in a way that only the order of the variables \((Z_i)\) is random and then removing such conditioning. Thus, we henceforth assume that the sequence \((Z'_i)\) is non-random and \((Z_i)\) is the result of uniformly permuting non-random numbers.

Let the permutation \(s\) be such that \(Z_{s(1)} < \cdots < Z_{s(n)}\). The permutation \(s^{-1}\) maps \(s(i_u)\) to \(i_u\) for \(u = 1, \ldots, k\) and is otherwise independently and uniformly distributed. The sequences \((Z_{\nu(i)})\) and \((Z'_{\pi(i)})\) are obtained by sorting \((Z_i)\) and \((Z'_i)\) in increasing order and then permuting according to \(s^{-1}\). We conclude that the law of the sequence \(((Z_i, Z_{\nu(i)}, Z'_{\pi(i)}))_{i \notin \{i_1, \ldots, i_k\}}\) is invariant under uniform permutations of \(\{1, \ldots, n\}\) \(\backslash\) \(\{i_1, \ldots, i_k\}\), completing the proof. \(\square\)

Proof of Proposition 2. As in the Proof of Theorem 1, we consider the increments \(\xi_{ni}\) but with the scaling \(\sqrt{n}/(2\log n)\). Let \(Z^{(j)}\) be the corresponding order statistics and recall that (see [6, Thm 1]), as \(n \to \infty\),
\[
Z^{(n)} - \frac{2\log n}{\sqrt{n}} \overset{p}{\to} 0 \quad \text{and} \quad \Delta_n := \max_{i<n}(Z(i+1) - Z^{(i)}) \overset{p}{\to} 0.
\]

First, we focus on the partial sum process corresponding to
\[
\hat{\xi}_{ni} = (2 \log n)^{-1/2}(Z_i - Z_{\nu(i)}).
\]
We will work on the event \(\{N_k = k_{+}\}\) for \(k_{+} + k_{-} \geq 1\), where \(N_k\) is the number of positive/negative jumps \(J\); the case of no jumps is trivial. Now the following is true for large \(n\) with probability arbitrarily close to 1. The indices \([T_j n]\) must be different (the set of such is denoted by \(I\)), every \(\sqrt{n}|J_j|/\sigma\) must be larger than \(Z^{(n)} - Z^{(1)}\), and the latter is smaller than \(3\sqrt{\log n}\). Hence for each \(i = \lceil nT_j \rceil \in I\) the quantity \(Z_i + \sqrt{n}\Delta_i Y/\sigma\) must be among \(k_{+}\) largest if the corresponding \(J_j > 0\) or \(k_{-}\) smallest if \(J_j < 0\). Thus
\[
(2 \log n)^{-1/2}Z_{\nu(i)} \overset{p}{\to} \pm 1
\]
according to the sign of the respective jump \(J_j\). However, the variables \(Z_i\) do not depend on the choice of indices \(i\), so
\[
\hat{\xi}_{ni} \overset{p}{\to} -\text{sign}(J_j), \quad i/n = \lceil nT_j \rceil \to T_j,
\]
where \(j\) is the corresponding jump index. It is thus left to show that the partial sum process of \(\hat{\xi}_{ni}\) with \(i \in I\) excluded converges in probability to \((k_{+} - k_{-})t\) in supremum norm. But the vector \(\hat{\xi}_{ni}, i \notin I\) is also exchangeable, see Lemma 10, and so according to [17, Thm 3.13] it is sufficient to show that
\[
\sum_{i \notin I} \hat{\xi}_{ni} \overset{p}{\to} k_{+} - k_{-} \quad \text{and} \quad \sum_{i \notin I} \hat{\xi}_{ni}^2 \overset{p}{\to} 0.
\]

Since we only need to look at the sums, we may permute the indices arbitrarily. In this paragraph we assume that \(Z\) is an increasing sequence, and that the elements of \(I\) are given by \(i_1 < \cdots < i_k\). For \(i > i_k\) we have \(\nu(i) = i - k_{+}\), for \(i < i_1\) we have \(\nu(i) = i + k_{-}\) and between any two \(i_j\) and \(i_{j+1}\), the permutation \(\nu\) displaces every index a fixed amount bounded by \(k\). Furthermore, the indices \(i_j\) are chosen uniformly at random (and then sorted), implying \((Z_{i_k} - Z_{i_k})/\sqrt{2\log n} \overset{p}{\to} 0\). Thus
\[
\sum_{i < i_1} \hat{\xi}_{ni} \leq k \frac{Z_{i_1} - Z_{i_1}}{\sqrt{2\log n}} \overset{p}{\to} 0,
\]
\[
\sum_{i < i_1} \hat{\xi}_{ni} = \sum_{j \leq k_{-}} \frac{Z_j - Z_{i_1 + j - 1}}{\sqrt{2\log n}} \overset{p}{\to} -k_{-} \quad \text{and} \quad \sum_{i < i_1} \hat{\xi}_{ni} = \sum_{j \geq k_{+}} \frac{Z_{n-j+1} - Z_{i_k - j + 1}}{\sqrt{2\log n}} \overset{p}{\to} k_{+},
\]
which yield the first limit in (17). A simple induction on \( k \) shows that the bound \( \sum_{i \in I} |Z_i - Z_{\nu(i)}| \leq k(Z_n - Z_1) \) holds, establishing the second limit in (17):

\[
\sum_{i \in I} \xi_{ni}^2 \leq \frac{\Delta n}{2\log n} \sum_{i \in I} |Z_i - Z_{\nu(i)}| \leq k\Delta_n \frac{Z_n - Z_1}{2\log n} \overset{P}{\to} 0.
\]

It remains to show that the partial sums of \( \tilde{\xi}_{ni} \) vanish in probability. Observe that the sequence \( \tilde{\xi}_{ni} \) need not be exchangeable. Nevertheless, we may condition on the number of jumps and note that \( (\tilde{\xi}_{ni})_{i \in I} \) (of length \( n - k \)) is exchangeable. Indeed, we need only apply Lemma 10 after conditioning on the ordered values of \( Z \) and \( Z' \). Now \( \sum_{i \in I} \tilde{\xi}_{ni}^2 \leq \sum_{i \leq n} \tilde{\xi}_{ni}^2 \overset{P}{\to} 0 \) according to Lemma 7. Moreover,

\[
\sum_{i \notin I} \tilde{\xi}_{ni} = - \sum_{i \in I} \tilde{\xi}_{ni} \overset{P}{\to} 0,
\]

because for \( i \in I \), both \( Z_{\nu(i)} \) and \( Z'_{\tau(i)} \) become \( \pm \sqrt{2\log n + o_P(1)} \) (with the same sign) and hence \( \tilde{\xi}_{ni} \overset{P}{\to} 0 \). This yields

\[
\sum_{i \notin I} \tilde{\xi}_{ni} \overset{P}{\to} 0, \quad \sum_{i \in I} \tilde{\xi}_{ni} \overset{P}{\to} 0,
\]

completing the proof.

\[\Box\]

**Proof of Proposition 3.** Note that the bivariate increments \( \xi_{ni} = (\Delta_n X, \Delta_n W^{(n)}) \) are exchangeable. Moreover, the partial sums of the first coordinate corresponds to the process \( X \) observed on the grid \( 1/n, \ldots, 1 \), and those of the second coordinate correspond to some Brownian motion (dependent on \( X \)) observed on the same grid. Now we apply [17, Thm 3.13] to each coordinate separately, and then jointly. It is only required to show that the cross-variation vanishes:

\[
\sum_{i \leq n} (\Delta_n X)(\Delta_n W^{(n)}) \overset{P}{\to} 0.
\]

Recall that \( \max_{i \leq n} |\Delta_n W| = O_P(\sqrt{\log n / n}) \), and hence we are done in the case when \( X \) has bounded variation on compacts. In general, by [11, Thm 2.3], it is sufficient to show that

\[
\sqrt{n \log n} \mathbb{E}(|X_{1/n}| \wedge 1) \to 0,
\]

so Lemma 9(c) completes the proof.

\[\Box\]

## 4. Some Estimates for Lévy Processes

This section is devoted to some basic bounds for Lévy processes at small times. Here we prove the three statements in Lemma 9, and also lay foundations needed in the proofs underlying Method II in §5. Recall that \((\gamma, 0, \Pi)\) is the Lévy triply of \( Y \) having no Brownian part. For any \( x \in (0, 1] \) define the standard quantities:

\[
m(x) = \gamma - \int_{x \leq |y| < 1} y \Pi(dy), \quad v(x) = \int_{|y| < x} y^2 \Pi(dy).
\]

In the case when \( Y \) has bounded variation on compacts we can express the linear drift as \( \gamma_0 = m(0) \). We also let

\[
Y_t = m(x)t + J^x_{t, 1} + J^x_{t, 2}
\]

be the Lévy -Itô decomposition of \( Y \), where \( J^x_{t, 1} \) is the martingale corresponding to the compensated jumps of \( Y \) of magnitude less than \( x \) and \( J^x_{t, 2} \) is driftless compound Poisson process containing all jumps of \( Y \) of magnitude at least \( x \). In particular, \( \mathbb{E}[(J^x_{t, 1})^2] = v(x)t \). Finally, we consider the integrals

\[
I_q = \int_{(-1, 1)} |x|^q \Pi(dx), \quad q \geq 0.
\]

and recall the following useful lemma (see, e.g. [10, Lem. 9]).

**Lemma 11.** If \( I_q < \infty \) for some \( q \in [0, 2] \), then for any \( x \in (0, 1] \), we have

\[
\Pi(x) \leq \Pi(1) + I_q x^{-q}, \quad |m(x)| \leq |\gamma| + I_q x^{-q-1}, \quad v(x) \leq I_q x^{2-q}.
\]
Lemma 12. For any $p \in (0, 2]$, $K > 0$, $t > 0$ and $x \in (0, 1)$, we have
\[
\mathbb{E}(|Y_t|^p) \leq (m(x)t)^{2} + v(x)t + K\Pi(x)t,
\]
\[
\mathbb{P}(|Y_t| \geq K) \leq (m(x)t^2 + v(x)t)/K^2 + \Pi(x)t.
\]
Proof. Fix $t > 0$ and define the event $A = \bigcap_{s \leq t}(J_s^{x,2} = 0)$ of not observing any jump from $J_s^{x,2}$ on the time interval $[0, t]$. Clearly $1 - \mathbb{P}(A) = 1 - e^{-\Pi(x)t} \leq \Pi(x)t$. Consider the elementary inequality $|Y_t|^p \leq (m(x)t)^{2} + \mathbb{E}[(J_t^{x,1})^2]^{p/2} + K(1 - \mathbb{P}(A))$, because $\mathbb{E}J_t^{x,1} = 0$. The first inequality readily follows. Using Markov’s inequality we readily get
\[
\mathbb{P}(|Y_t| \geq K) = \mathbb{P}(|Y_t| \geq K) \leq \mathbb{E}(Y_t^2 \wedge K)/K^2
\]
and the second result follows from the first with $p = 2$. \hfill \Box

Lemma 13. For any $\epsilon > 0$ and $a_t \downarrow 0$ satisfying $a_t/\sqrt{t} \to \infty$ as $t \downarrow 0$ we have
\[
\lim_{t \downarrow 0} \frac{\mathbb{P}(|X_t| > a_t)}{\Pi(a_t(1 + \epsilon))} \geq 1.
\]
Proof. Take $x = x_t = a_t(1 + \epsilon)$ and consider the event that $J_s^{x,2}$ has exactly one jump in $[0, t]$, which yields the lower bound
\[
\mathbb{P}(|X_t| > a_t) \geq t(1 + o(1))\Pi(x_t)\mathbb{P}(|J_t^{x,1}| + |m(x_t)|t < a_t\epsilon),
\]
where $J_t^{x} = J_t^{x,1} + \sigma W$. Here we use $t\Pi(x_t) \to 0$ which follows from Lemma 11 with $q = 2$ and the assumption $ta_t^{-2} \to 0$. Furthermore, we have $|m(x_t)|t/a_t \to 0$ and $\mathbb{P}(|J_t^{x,1}| > a_t\epsilon/2) \to 0$ which follows from Markov’s inequality and the fact that $\mathbb{E}((J_t^{x,1})^2)/\sigma_t^2 = t(\sigma^2 + v(x_t))/\sigma_t^2 \to 0$. This completes the proof. \hfill \Box

Proof of Lemma 9. Part (a). The inequality follows from Lemma 13. The limit is a consequence of the second inequality in Lemma 12 with $t = 1/n$ and $K = x = a_n$, Lemma 11 with $q = 2$ and the fact that $a_n\sqrt{n} \to \infty$.

Part (b). From Lemma 12 with $x_n^2 = n^{-1} \log n$ we have the bound
\[
n^{-p/2}\mathbb{E}\left(Y_{1/n}^2, \frac{\log n}{n}\right) \leq n^{-p/2}m(x_n)^2 + n^{1-p/2}v(x_n) + n^{-p/2}\log(n)\Pi(x_n).
\]
Assume that $\beta^* < 2$, pick $q < p$ such that $I_q < \infty$, and apply Lemma 11. The first term vanishes because $-p/2 + (q - 1)^* \leq 0$. The second term vanishes because $1 - p/2 - (2 - q)/2 < 0$. The third term vanishes since $-p/2 + q/2 < 0$. Finally, if $\beta^* = 2$ then taking $q = p = 2$, proceeding as in the previous case and using the facts that $x^2\Pi(x) \to 0$ and $v(x) \to 0$ as $x \to 0$, gives the result.

Part (c). Applying Lemma 12 with $x = n^{-1/4}$ to $Y$ gives
\[
n\log(n)[\mathbb{E}((Y_{1/n}^2) \wedge 1)]^2 \leq 2n^{-1} \log(n)m(n^{-1/4})^2 + 2n^{-1}(v(n^{-1/4}) - \sigma^2)
\]
\[+ 2n^{-1}\log(n)\Pi(n^{-1/4})^2.
\]
Take $q$ satisfying $\beta^* \vee 1 < q < 2$ and apply Lemma 11 to show that this quantity indeed tends to 0. \hfill \Box

5. Proofs for Method II

Without loss of generality we assume throughout this section that $\sigma = 1$, which is based on a simple rescaling argument. The main ingredient in the proof of Theorem 4 is the following result.

Lemma 14. Assume that $\int_{|x| > 1} x^2 \Pi(dx) < \infty$. Then for $p \in (\beta^*, 2]$ we have
\[
n^{-p/2}\mathbb{E}\left(Y_{1/n}^2, 1_{\{x_{1/n} \leq a_n\}}\right) \to 0,
\]
\[
n^{-p/2}\mathbb{E}\left(W_{1/n}^2, 1_{\{x_{1/n} > a_n\}}\right) \to 0,
\]
assuming that the positive sequence $a_n$ satisfies
\[
\lim_{n \to \infty} \frac{na_n^2}{\log n} > 2 - p \quad \text{and} \quad a_n^{1/2 - \delta} \to 0 \quad \text{for some} \quad \delta < \frac{p - \beta^*}{2(2 - \beta^*)}.
\]
This result is also true for $p = 2$ and any $\beta^*$, in which case the upper bound on $a_n$ is replaced by $a_n \to 0$.

**Proof.** The set inclusion $\{|X_1/n| \leq a_n\} \subset \{|Y_1/n| \leq 2a_n\} \cup \{|W_1/n| > a_n\}$ implies

$$E(Y^2_{1/n}1_{|X_1/n| \leq a_n}) \leq E(Y^2_{1/n}1_{|Y_1/n| \leq 2a_n}) + E(Y^2_{1/n}1_{|W_1/n| > a_n}).$$

Since $E(Y^2_{1/n}) \sim \int_{\mathbb{R}} x^2 \Pi(dx)/n$, it follows by Mill’s ratio that

$$(18) \quad E(Y^2_{1/n}1_{|W_1/n| > a_n}) = 2E(Y^2_{1/n})\Phi(-\sqrt{n}a_n) \sim \frac{2e^{-na_n^2/2}}{\sqrt{2\pi na_n}} \int_{\mathbb{R}} x^2 \Pi(dx).$$

The assumed lower bound on $a_n$ implies that $a_n^2 n/2 > (1 - p/2) \log n$ for all large enough $n$, showing that the term in (18) is $o(n^{p/2-2})$; the case $p = 2$ needs special attention but is otherwise straightforward. To bound the other expectation, we use Lemmas 12 and 11 to obtain

$$E(Y^2_{1/n}1_{|Y_1/n| \leq 2a_n}) \leq m(a_n)^2/n^2 + v(a_n)/n + 4a_n^2\Pi(a_n)/n$$

(19) for any $q \in (\beta^*, p)$ whenever $a_n \leq 1$ (i.e., for all large enough $n$). Note that $na_n^2 \to \infty$ and it is enough to establish an upper bound on $na_n^{2q}/n$. But we have assumed that this term is $n^{1+2(p-q) - (1/2 + \delta)} o(1)$ for certain $\delta$. Equating the power to $-2 + p/2$ we find that $c_n = 1 + (2 - q)/[2(2 - q)] \uparrow (p - \beta^*)/(2 - \beta^*)$ as $q \downarrow \beta^*$, and hence we have the bound $o(n^{-2 + p/2})$. If $p = 2$ then we may take $q = 2$ to see that the result is $o(n^{-1})$ as claimed.

With regard to the second statement, we fix any sequence $b_n \downarrow 0$ with values in $(0, 1)$ and denote $c_n = a_n(1 - b_n)$. As before, we have the inequality

$$E(W^2_{1/n}1_{|X_1/n| > a_n}) = E(W^2_{1/n}1_{|Y_1/n| > a_n b_n}) + E(W^2_{1/n}1_{|W_1/n| > c_n}).$$

The second term on the right may be written as

$$(20) \quad 2n^{-1}E(W^2_{1/n}1_{|W_1/n| > c_n}) = 2n^{-1} \int_{c_n^{1/\sqrt{n}}}^{\infty} x^2 e^{-x^2/2} \sqrt{2\pi} dx \sim \frac{2c_n e^{-n c_n^2/2}}{\sqrt{2\pi n}}.$$ 

This term is only made larger by taking $c_n$ smaller, and so we may assume that $c_n^2 n/2 = (1 - p/2 + \epsilon) \log n$ for some $\epsilon > 0$ and all large $n$. Thus the term in (20) is $o(n^{p/2-2})$. On the other hand, according to Lemma 12, the first term satisfies

$$E(W^2_{1/n}1_{|Y_1/n| > a_n b_n}) = \mathbb{P}(|Y_1/n| > a_n b_n)/n$$

$$\leq \frac{m(a_n)^2/n^2 + v(a_n)/n^2}{a_n^{-2} b_n^{-2} + \Pi(a_n)/n^2},$$

when $a_n < 1$. Since $na_n^2 \to \infty$, the argument used in (19) completes the proof upon taking $b_n$ such that $na_n^2 b_n^2 \geq 1$. \qed

5.1. **Convergence results.**

**Proof of Theorem 4.** Without loss of generality we assume that the jumps of $X$ are bounded, since they are below a threshold $K \to \infty$ with probability tending to 1. As in the proof of Theorem 1, it suffices to prove the result when the supremum is taken over the grid $(i/n)_{i=1,\ldots,n}$. On that grid, $W_{i/n} - W$ is a random walk and so $(W_{i/n} - \hat{W}_{i/n}) - (W_{i-1/n} - \hat{W}_{i-1/n})_{i=1,\ldots,n}$ has exchangeable increments, while being 0 at $i = n$. Thus, the first assertion is equivalent to:

$$n^{(2-p)/2} \sum_{i \leq n} \left( \frac{V_i^{(n)}}{n} - \frac{1}{n} \sum_{j \leq n} V_j^{(n)} \right)^2 \to 0, \quad \text{where} \quad V_i^{(n)} = \begin{cases} \Delta_i W, & |\Delta_i X| > a_n, \\ -\Delta_i Y, & |\Delta_i X| \leq a_n. \end{cases}$$

This will be shown by proving convergence in $L^1$. As before, the expectation of the sum above may be rewritten as

$$(21) \quad (n-1) \left( \mathbb{E} \left[ \left( V_i^{(n)} \right)^2 \right] - \left( \mathbb{E} V_i^{(n)} \right)^2 \right).$$

But now the result follows from

$$n^{2-p/2} \mathbb{E} \left( V_i^{(n)} \right)^2 = n^{2-p/2} \mathbb{E} (W_{i/n}^2 1_{\{|X_{i/n}| > a_n\}}) + n^{2-p/2} \mathbb{E} (Y_{i/n}^2 1_{\{|X_{i/n}| \leq a_n\}}) \to 0,$$

where in the last step we apply Lemma 14. \qed
Proof of Proposition 6. In view of Theorem 4 it is sufficient to show the stated convergence for $t = 1$, which is equivalent to
\[ n^{(1-p)/2} \left( \sum_{i \leq n} V_i^{(n)} + \gamma_0 \right) \xrightarrow{P} 0. \]
Arguments similar to those in Lemma 14 give $n^{3/2-p/2} \mathbb{E}\left(||W_{1/n}|1_{\{|X_{1/n}| > a_n\}}\right) \to 0$. Hence we only need to show that
\[ n^{(1-p)/2} \left( \gamma_0 - \sum_{i \leq n} \Delta_i^n Y 1_{\{||\Delta_i^n X|| \leq a_n\}} \right) \xrightarrow{P} 0, \]
and for this the following is sufficient:
\[ n^{(1-p)/2} \mathbb{E}\left| n - nY_{1/n} 1_{\{|X_{1/n}| \leq a_n\}} \right| \to 0. \]
Write $Y_{1/n} = \gamma_0/n + J_{1/n}$, where $J_t$ is the uncompensated sum of the jumps of $Y$ on $(0, t]$. We note that $n^{(1-p)/2} \mathbb{P}(|X_{1/n}| > a_n) \to 0$ so it remains to prove
\[ n^{3/2-p/2} \mathbb{E}(|J_{1/n}| 1_{\{|X_{1/n}| \leq a_n\}}) \to 0. \]
Again, by the arguments in Lemma 14 we may replace the indicator by $1_{\{|J_{1/n}| \leq 2a_n\}}$. It is left to note that the simple structure of $J$ allows for an improved bound as compared to Lemma 12:
\[ \mathbb{E}(|J_{1/n}| \wedge a_n) \leq n^{-1} \int_{(-a_n,a_n)} |y| \Pi(dy) + a_n \Pi(a_n)/n, \]
where the first part corresponds to the sum of the absolute jumps of $J$ of size smaller than $a_n$ and the second part to $a_n$ times the probability of observing at least one jump whose size is at least $a_n$ in absolute value. Both terms are $o(a_n^{-\eta}/n)$ for $q > \beta^*$ and hence $o(n^{-3/2+p/2})$, proving the main claim. For $p = 1$ we additionally observe that the display above is $o(n^{-1})$ because necessarily $x \Pi(x) \to 0$ as $Y$ is of bounded variation.

5.2. Cases when convergence fails. Here we prove the negative results yielding Proposition 5. Consider the $n/2$ epoch, assuming $n$ is even in the following, and rewrite the difference of interest as the sum of independent symmetric terms:
\[ n^{(2-p)/4} (W_{1/2} - W_{1/2}^{(n)} - (W_{1} - W_{1}^{(n)})/2) = \frac{1}{2} \sum_{i \leq n/2} n^{(2-p)/4} (V_i^{(n)} - V_{i/2}^{(n)}). \]
Now the standard result [16, Ex. 4.18] states that converges to $0$ in probability is equivalent to convergence to $0$ of the sum of the expected (truncated) squares:
\[ n \mathbb{E}\left( n^{1-p/2} (V_1^{(n)} - V_2^{(n)})^2 \wedge 1 \right) \to 0. \]
Hence we only need to disprove the latter.

First, we consider the case when the threshold $a_n$ is too small:

Lemma 15. If $\lim \inf n a_n^2 / \log n < 2 - \beta^*$ then (22) fails for some $p \in (\beta^*, 2]$.

Proof. We may assume that $a_n > n^{-1/2}$ for all large enough $n$, because otherwise $V_{1}^{(n)} = W_{1/n}$ with probability bounded away from 0, and the contradiction can be easily derived. To get an appropriate lower bound on the expectation in (22), we consider the event
\[ \{\Delta_0^n W > a_n, \Delta_0^n Y \geq 0, |\Delta_2^n W| < a_n/2, |\Delta_1^n Y| \leq a_n/2\}. \]
We may assume that $\mathbb{P}(Y_{1/n} \geq 0) \geq 1/2$ since we may flip the signs otherwise. On this event we have $V_1 - V_2 \geq a_n/2$. Using the independence and the fact that $\mathbb{P}(|\Delta_2^n W| < a_n/2, |\Delta_1^n Y| \leq a_n/2)$ is bounded away from 0 (as $a_n > n^{-1/2}$ for large $n$) we get a bound
\[ n \mathbb{E}\left( n^{1-p/2} (V_1^{(n)} - V_2^{(n)})^2 \wedge 1 \right) \geq c n (a_n^2 n^{1-p/2} \wedge 1) \mathbb{P}(W_{1/n} > a_n) \geq c n^{1-p/2} \mathbb{P}(W_{1/n} > a_n) \]
for some $c > 0$. Note that $n a_n^2 / (2 \log n) < 1 - \beta^*/2 - \epsilon$ for some $\epsilon > 0$ along some subsequence. Along that subsequence we have $\mathbb{P}(W_1 > a_n \sqrt{n}) \geq n^{-1+\beta^*/2+\epsilon/2}$ for large $n$. Thus the contradiction is obtained for any $p < \beta^* + \epsilon/2$. \qed
Finally, we consider the case when $a_n$ is too large:

**Lemma 16.** Assume that $a_n \to 0$ and $\Pi \neq 0$. If \( \limsup_{n \to \infty} a_n n^{1/2-\epsilon} > 0 \) for some $\epsilon > 0$ then (22) fails for some $p \in (\beta_*, 2]$.

**Proof.** First suppose $\beta_* > 0$. Without real loss of generality we may assume that $a_n > n^{-1/2+\epsilon}$ for some $\epsilon \leq \beta_*/4$. In this case we focus on the event
\[
|\Delta^n_0 W| \leq a_n/2, a_n/4 < |\Delta^n_0 Y| \leq a_n/2, |\Delta^n_0 W| \leq a_n/2, |\Delta^n_0 Y| \leq a_n/8,
\]
yielding a lower bound
\[
n\mathbb{E}\left(n^{1-p/2}(V_{1}^{(n)} - V_{2}^{(n)})^2 \wedge 1\right) \geq c n^{2-p/2} a_n^{2} p (|Y_{1/n}| > a_n/4)
\]
for large enough $n$. According to Lemma 13 this is further lower bounded by
\[
e' n^{1-p/2} a_n^{2} \Pi (a_n/2) \geq c' n^{1-p/2} a_n^{2-\nu} \geq c'' n^{1-p/2} n^{(2-\nu)(-1/2+\epsilon)},
\]
for some $e', c'' > 0$ and $\nu < \beta_*$. The power is non-negative when $p \leq \nu + 2\epsilon(2 - \nu)$, but the right side can be made larger than $\beta_*$ by taking $\nu$ close to $\beta_*$; recall that our assumptions imply $\beta_* < 2$. In the case $\beta_* = 0$ we choose $p < 4\epsilon$ and lower bound the expression in (23) by some positive number as before, since $\Pi(\mathbb{R}) > 0$.

**Proof of Proposition 5.** Lemma 15 and Lemma 16 establish the second and third claims, respectively. The proof of first claim is analogous to that in Lemma 16. Note that the assumptions imply both $a_n \to 0$ and $\beta_* > 0$, so $\Pi \neq 0$. \hfill $\Box$

**Appendix A. Purely Brownian case**

An interesting problem is to identify the exact rate of convergence of the skeletons in Proposition 2 for the purely Brownian case, that is, when $Y = 0$. Here we show that this rate is $\sqrt{\log \log n/n}$. We cannot, however, establish the limit law, nor its existence.

As in §3, consider the variables
\[
\xi_{ni} = \frac{1}{\sqrt{\log \log n}} \left( Z_{n(i)} - Z'_{n(i)} - \frac{1}{n} \sum_{j \leq n} (Z_{j} - Z'_{j}) \right), \quad i = 1, \ldots, n.
\]
For any $n \in \mathbb{N}$, let $S_{i}^{(n)} = \sum_{i \leq \leq n} \xi_{ni}$ be its cumulative sum process. We clearly have $S_{0}^{(n)} = S_{1}^{(n)} = 0$ and the jumps of $S^{(n)}$ are exchangeable.

Following the proof of Lemma 7, more specifically, the bounds in terms of the functions $\tilde{F}_n$ and $\tilde{G}_n$, we easily deduce that for any $a > 4$, the quadratic variation of $S^{(n)}$ satisfies
\[
\mathbb{P} \left( \sum_{i \leq n} \xi_{ni}^2 > a \right) \to 0.
\]
The stated tightness in turn implies, according to [17, Lem. 3.9], that the processes $S^{(n)}$ are tight (and nonvanishing) in the Skorohod space $\mathcal{D}[0, 1]$. This establishes the claimed rate of convergence of skeletons.

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