Existence and smoothness of the Navier-Stokes equations and semigroups of linear operators

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Abstract

Based on Leray’s formulation of the Navier-Stokes equations and the conditions of the exact linear representation of the nonlinear problem found in this paper, a compact explicit expression for the exact operator solution of the Navier-Stokes equations is given. It is shown that the introduced linear operator for Leray’s equations is the generator of one-parameter contraction semigroup. This semigroup yields the existence of a unique and smooth classical solution of the associated Cauchy problem of Navier-Stokes equations in space $\mathbb{R}^3$ under smooth initial conditions.

1 The Navier-Stokes Equations

Incompressible flows of homogeneous fluids in space $\mathbb{R}^3$ are solutions of the system of equations [1]-[5]

$$\frac{\partial v}{\partial t} = \nu \Delta v - \sum_{j=1}^{3} v_j \frac{\partial v}{\partial x_j} - \nabla p + f,$$

$$\text{div} \ v \equiv \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0, \quad (x,t) \in \mathbb{R}^3 \times [0, \infty),$$

$$v \mid_{t=0} = v_0, \quad x \in \mathbb{R}^3,$$

where $v(x,t) \equiv (v_1, v_2, v_3)$ is the fluid velocity, $p(x,t)$ is the scalar pressure, the gradient operator is $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and the Laplace operator $\Delta \equiv$
$\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$, a kinematic constant viscosity $\nu \geq 0$, initial conditions $v \mid_{t=0} = v_0(x) \equiv (v_{01}, v_{02}, v_{03})$, $f = f(x, t)$ is given, externally applied force. Further we will consider Navier-Stokes equations in the absence of external forces, i.e. with $f = 0$.

It is well known [3], [4] that we can find an equation $p = p(v)$ to eliminate the pressure from (1). If we take the divergence at each side of the momentum equation (1), which yields, using (2) that

$$\nabla \cdot \nabla p = -\sum_{i,j=1}^{3} \frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j}. \tag{4}$$

The equation (4) with respect to the function $p$ is the well known Poisson equation, whose solution in $\mathbb{R}^3$ [3] has the form (for brevity, we will sometimes denote this solution as $p_v(x, t) \equiv p_v$)

$$p(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d\xi \sum_{i,j=1}^{3} \frac{\partial v_j(t, \xi)}{\partial \xi_i} \frac{\partial v_i(t, \xi)}{\partial \xi_j} \frac{1}{|x - \xi|}, \quad (\xi \in \mathbb{R}^3). \tag{5}$$

Substituting the solution (5) into (1), we obtain (Leray's Formulation of the Navier-Stokes Equation [3]) non-linear system of integro-differential equations containing only unknown functions $v_i$.

$$\frac{\partial v}{\partial t} = \nu \nabla v - \sum_{j=1}^{3} v_j \frac{\partial v}{\partial x_j} - \nabla p_v, \quad (x \in \mathbb{R}^3, t \geq 0). \tag{6}$$

It is easy to see [3] that the systems of the Navier-Stokes equations (1)-(3) are equivalent to the system of the equations (5),(6), provided that functions $v_0$ in initial conditions are smooth and

$$\sum_{i=1}^{3} \frac{\partial v_0}{\partial x_i} = 0, \quad (x \in \mathbb{R}^3). \tag{7}$$

Although the Leray's formulation of the Navier-Stokes equation has been known for a long time, it has not been widely used to analyze its solutions. The main theses of this can be traced to the following [3].

The equation (6) is quadratically nonlinear and contains a nonlocal, quadratically nonlinear operator. These facts make the Navier-Stokes equation hard to study analytically.

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Although (6) constitutes a closed system for \( v \), this formulation turns out to be not very useful for further analysis except by rather crude methods based on the energy principle. The main reason for this is that this formulation hides all the properties of vorticity stretching and interaction between rotation and deformation.

2 Exact linear representation of a nonlinear problem

We define for a multi-index \((\alpha_1, ..., \alpha_1) \in \mathbb{N}^n\)

\[
D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}, \text{ where } D_k = \frac{\partial}{\partial x_k}, \quad \alpha := \sum_{i=1}^{n} \alpha_i.
\]

Let us take an arbitrary function \( u(x), x \in \mathbb{R}^n \) which is infinitely many times differentiable and decay sufficiently rapidly, i.e., \( u \in C^\infty(\mathbb{R}^n) \), and

\[
\lim_{|x| \to \infty} |x|^p D^\alpha u(x) = 0 \quad \text{for some } p \in \mathbb{N} \text{ and all } \alpha \in \mathbb{N}^n.
\]

Then, using this function, we define the linear space of analytic on all arguments functions \( G(u) := G(u, u_1, u_2, ...) \in C^\infty \), where \( u_i := D_k^i u(x) \) \((i, k \in \mathbb{N})\) and which decay sufficiently rapidly with \( u \) and all \( u_i \) provided \( |x| \to \infty \).

Since we can also consider these functions \( G(u(x)) \) as functions of \( x \in \mathbb{R}^n \) denoting \( G(u(x)) = \tilde{G}(x) \). In this case, it is obvious that the function \( \tilde{G}(x) \in C^\infty(\mathbb{R}^n) \) and decay sufficiently rapidly when \( |x| \to \infty \).

When endowed with the usual inner product

\[
< \tilde{F}(x), \tilde{G}(x) >= \int_{\mathbb{R}^n} \tilde{F}(x) \tilde{G}(x) \, dx
\]

the considered space becomes as a complete normed subspace \( L_2^2(\mathbb{R}^n) \) of the Hilbert space \( L^2(\mathbb{R}^n) \).

Let a differential operator \( A \) with domain \( D(A) \) on a Banach space \( L_2^2(\mathbb{R}^n) \) is of type

\[
A = \int_{\mathbb{R}^n} d\zeta F(u(\zeta)) \frac{\delta}{\delta u(\zeta)}, \quad \zeta \in \mathbb{R}^n,
\] (8)
where \( u(x) \) is a function of class described above, a \emph{predetermined} \( F(u(x)) := F(u, u_1, u_2, \ldots) \in L^2_u(\mathbb{R}^n) \) with \( u_i := D^i u(x) \) and we can consider \( F(u(x)) \) as the linear multiplication operator, \( \frac{\delta}{\delta u(\zeta)} \) is the functional derivative. We have specially transferred the symbol \( d\zeta \) at the beginning of the formula to emphasize that the right operator is executed first, and then the integration is performed.

Here by functional derivative we mean the linear mapping with the following property

\[
\frac{\delta u(x)}{\delta u(\zeta)} = \text{Dirac}(x - \zeta),
\]

the chain rule is also valid in this context.

Entered operator is a linear operation, i.e., operator \( A \) obeys the following properties:

\[
AcG(u(x)) = cAG(u(x)), \quad \text{for any } G \in L^2_u(\mathbb{R}^n) \text{ and any scalar } c \in \mathbb{R},
\]

\[
A[G(u(x)) + H(u(x))] = AG(u(x)) + AH(u(x)), \quad \text{for any } G, H \in L^2_u(\mathbb{R}^n).
\]

It is obvious that \( A^j u \in L^2_u(\mathbb{R}^n) \) for any \( j \in \mathbb{N} \) and operator \( A \) is \emph{continuous} and therefore it is \emph{bounded} and \emph{closable} on the space \( L^2_u(\mathbb{R}^n) \).

Let an operator \( A \) with some \( F(u(x)) \) is the \emph{generator of strongly continuous semigroup} \( (T(t))_{t \geq 0} \) then [6], p.50

\[
\frac{d}{dt}T(t)u(x) = T(t)Au(x) = AT(t)u(x) \quad \text{for all } t \geq 0. \tag{9}
\]

Note that if we denote

\[
v(x, t) := T(t)u(x), \tag{10}
\]

then from (9) \emph{three equivalent} equations follows.

First

\[
\frac{d}{dt}v(x, t) = Av(x, t), \tag{11}
\]

the \emph{linear} one with the \emph{functional derivative} and expression (10) is its \emph{general} solution.

Second

\[
\frac{d}{dt}v(x, t) = T(t)Au(x),
\]
but because for considered operator

\[ Au(x) = F(u(x)), \]

then as much as (see [6], p. 58) \( T(t) \) is a semigroup of algebra homomorphisms on \( D(A) \), i.e.,

\[ T(t)(h \cdot g) = T(t)h \cdot T(t)g \quad \text{for } h, g \in D(A) \text{ and } t \geq 0 \]

and, since we considering analytic functions \( F \), we have

\[ T(t)Au(x) = F(T(t)u(x)) \]

and come to

\[ \frac{d}{dt}v(x, t) = F(v(x, t), v_{x_1}(x, t), ..., D^\alpha v(x, t), ...), \tag{12} \]

i.e., to nonlinear PDE. The expression \( (10) \) is the general solution to \( (12) \) too!

This result has parallels with the well known connection between nonlinear ODEs and linear PDEs. There is a third equivalent equation

\[ Av(x, t) = F(v(x, t), v_{x_1}(x, t), ..., D^\alpha v(x, t), ...), \]

the nonlinear one with the functional derivative and with the same general solution.

From a comparison of expressions \( (8) \) and \( (12) \), it is easy to obtain a one-to-one relationship between the linear operator \( A \) and the nonlinear right-hand side of equation \( (12) \).

Note that the strongly continuous semigroup \( (T(t))_{t \geq 0} \) generated by the operator \( A \) of the type considered here with some \( F(u(x)) \) can be given by \[ \{6\}, \{7\} \]

\[ T(t) = \exp\{tA\} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0, \]

where the power series converges for every \( t \geq 0 \) \([6], \text{ pp. 52, 81} \).

Practically without changing the above reasoning, we can generalize the results obtained to more complex equations, when

\[ F(v(t)) = F(v(t, x), v_{x_1}(t, x), ..., D^\alpha v(t, x), ..., I_1 v(t, x), I_2 v(t, x) ...), \]
where $I_i$ are nonlinear integral operators.

Moreover, it is possible to consider in a completely similar way systems of nonlinear equations

$$\frac{d}{dt}v_i(x, t) = F_i(v_1(x, t), ...v_n(x, t)), \quad i = 1..n,$$

(13)

where all $F_i$ are functions of the type described above and if we introduce the following linear operator

$$A = \int_{\mathbb{R}^n} d\zeta \sum_{i=1}^{n} F_i(u_1(\zeta), ..., u_n(\zeta)) \frac{\delta}{\delta u_i(\zeta)}, \quad \zeta \in \mathbb{R}^n,$$

(14)

where rapidly decreasing $u_i \in C^\infty(\mathbb{R}^n)$, and $F_i(u_1(x), ...u_n(x))$ are in the space $L_2^2(\mathbb{R}^n)$ built on vector function $u(x) \equiv (u_1(x), ..., u_n(x))$.

Again assuming that operator $A$ is the generator of strongly continuous semigroup $(T_{sys}(t))_{t \geq 0}$. In this case the general solutions to the system (13) is

$$v_i(x, t) = T_{sys}(t)u_i(x),$$

(15)

where as before

$$T_{sys}(t) = \exp\{tA\} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0.$$

Thus we have obtained an important result that, under the above conditions, it is possible to reduce exactly a nonlinear problem to a linear one, which allows us to use for solving nonlinear problems the accumulated baggage for linear problems.$^1$

**Comment.** So far as $T(t)|_{t=0} = 1$ (identity operator) then from (10) or (15) it follows that function $u(x)$ is directly related to the initial conditions. However, we need to note one circumstance in previous reasoning. We use in definition of the operators $A$ the functional derivatives with respect to the variable $u$ and therefore cannot identify it with everyone fixed initial condition. That is to obtain the particular solution we have to substitute $u(x) = v(x, 0)$ into the general solution

$$v(x, t) = \{T(t)u(x)\} \big|_{u(x)=v(x,0)}.$$

(16)

$^1$Here we consider the case when the operator $A$ does not depend on $t$. It is possible to generalize the results obtained above for more general functions $F(t, x, u(x))$ (see the idea in [8]).
3 Solution of the Navier-Stokes problem

Turning to the Leray’s formulation of the Navier-Stokes equations (1), we will consider the vector function $u(x) \equiv (u_1(x), u_2(x), u_3(x))$, where arbitrary functions $u_i(x), x \in \mathbb{R}^3$ which, as we discussed in the previous section, are infinitely many times differentiable and decay sufficiently rapidly. In accordance with the trick described above let us introduce the linear differential operator

$$A = \int_{\mathbb{R}^3} d\xi \sum_{i=1}^{3} \{\nu \Delta \xi u_i(\xi) - \sum_{j=1}^{3} u_j(\xi) \frac{\partial u_i(\xi)}{\partial \xi_j} - \frac{1}{4\pi} \frac{\partial}{\partial \xi_i} \int_{\mathbb{R}^3} d\xi \sum_{i,j=1}^{3} \frac{\partial u_j(\xi)}{\partial \xi_i} \frac{\partial u_i(\xi)}{\partial \xi_j} \frac{1}{|\xi - \xi'|}\} \delta \frac{\partial }{\partial u_i(\xi)}.$$ \hspace{1cm} (17)

In short notation

$$A = \int_{\mathbb{R}^3} d\xi \sum_{i=1}^{3} \{\nu \Delta \xi u_i(\xi) - \sum_{j=1}^{3} u_j(\xi) \frac{\partial u_i(\xi)}{\partial \xi_j} - \nabla \xi p_u(\xi)\} \delta \frac{\partial }{\partial u_i(\xi)}.$$ \hspace{1cm} (18)

where

$$p_u(\xi) \equiv \frac{1}{4\pi} \left[ \int_{\mathbb{R}^3} d\xi \sum_{i,j=1}^{3} \frac{\partial u_j(\xi)}{\partial \xi_i} \frac{\partial u_i(\xi)}{\partial \xi_j} \frac{1}{|\xi - \xi'|} \right].$$

We restrict our attention to Navier-Stokes equations and correspondingly to operator $A$ on all of $\mathbb{R}^3$, thus avoiding the delicate discussion of boundary conditions.

As it follows from the results of the previous section we need to find out whether the operator is a generator of a semigroup. Let us describe some properties of the operator $A$ introduced above. First of all

$$A u_i = \nu \Delta u_i - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{4\pi} \frac{\partial}{\partial x_i} \int_{\mathbb{R}^3} d\xi \sum_{i,j=1}^{3} \frac{\partial u_j(\xi)}{\partial \xi_i} \frac{\partial u_i(\xi)}{\partial \xi_j} \frac{1}{|x - \xi|}$$ \hspace{1cm} (19)

and it is obvious that $A : L^2_u(\mathbb{R}^3) \mapsto L^2_u(\mathbb{R}^3)$ and $A$ is continuous and therefore it is bounded and closable on the space $L^2_u(\mathbb{R}^n)$ built on vector function $u(x) \equiv$
We can rewrite the expressions \( (19) \) in vector form for brevity
\[
A u = \nu \Delta u - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nabla p_u,
\]
where
\[
p_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d\xi \sum_{i,j=1}^{3} \frac{\partial u_j(\xi)}{\partial \xi_i} \frac{\partial u_i(\xi)}{\partial \xi_j} \frac{1}{|x-\xi|}.
\]

Further, by analogy with the proof of the energy principle, we have in view of (7) and that \( u \) is rapidly decreasing and hence \( p_u \) rapidly decreasing at infinity too one can obtain the well known result that inner product
\[
< A u, u > = -\nu \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \left( \frac{\partial c_i}{\partial x_j} \right)^2 dx \leq 0.
\]

Hence, the linear operator \( A \) on the Hilbert space \( L^2_0(\mathbb{R}^3) \) is dissipative [9]-[13].

By definition [14], the adjoint operator \( A^* \) to the bounded operator \( A \) on the Hilbert space satisfies the identity
\[
< Au, h > = < u, A^* h >, \quad \text{for any } u, h \in L^2(\mathbb{R}^3).
\]

In the case of the real vector Hilbert space we have for considered operator \( A \), taking into account (21), that
\[
< u, A^* u > = < A^* u, u > = < A u, u > \leq 0,
\]
so adjoint operator \( A^* \) is dissipative too.

Now we use one of the results of semigroup theory ([6], p. 84, 3.17 Corollary).

**Proposition 1.** Let \( (A, D(A)) \) be a densely defined operator on a Banach space \( X \). If both \( A \) and its adjoint \( A^* \) are dissipative, then the closure \( \overline{A} \) of \( A \) generates a contraction semigroup on \( X \).

A contraction semigroup is a special case of strongly continuous semigroup \( (T(t))_{t \geq 0} \) [6], [7]. In each case, the semigroup is given by
\[
T(t) = \exp\{tA\} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0.
\]
where the power series converges for every \( t \geq 0 \).

So the general solution to the Lerays system (6) is as follows

\[
v(x, t) = \exp\{tA\}u(x) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} u(x), \quad t \geq 0,
\]

(22)

where \( u(x) \) is arbitrary vector function.

And its particular solutions

\[
v(x, t) = \{\exp\{tA\}u(x)\} |_{u=v_0} = \{\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} u(x)\} |_{u(x)=v_0}, \quad t \geq 0,
\]

(23)

provided that \( v_0 := v_0(x) \) is divergence-free vector field and the scalar pressure \( p \) restored through known solution \( v \) by (5) leads us to the solution of the Navier-Stokes equations (1)-(3).

Because the operator \( A \) is the generator of a contraction semigroup, it follows that the semigroup yields solutions of the associated abstract Cauchy problem ([6], p.145). Furthermore, if \( u(x) \in D(A) \) the map \( v(x, t) := T(t)u(x) \) is the unique classical solution of the X-valued initial value problem (or abstract Cauchy problem).

Returning to the Navier-Stokes equations, we come to the conclusion that the operator solution of the system of equations in form (23) provided that initial condition \( v_0 = v(x, 0) \) is divergence-free is the unique classical solution to (6).

That is, we have proved

**Proposition 2.** Let initial conditions \( v(x, 0) \) be any smooth rapidly decreasing, divergence-free vector field. Take \( f(x, t) \) to be identically zero. Then there exist unique smooth rapidly decreasing functions \( p(x, t), v_i(x, t) \) on \( \mathbb{R}^3 \times [0, \infty) \) that satisfy (1)-(3).

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**References**

[1] O. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows (2nd edition), Gordon and Breach, New York, 1969.
[2] C. Fefferman, Existence and smoothness of the Navier-Stokes equation, http://claymath.org/MilleniumPrizeProblems/Navier-StokesEquations, Clay Mathematics Institute, Cambridge, MA, 2000.

[3] Andrew J. Majda, Andrea L. Bertozzi, Vorticity and incompressible flow, Cambridge University Press, 2001.

[4] C. Foias, O. Manley, R. Rosa, R. Temam - Navier-Stokes equations and turbulence (Encyclopedia of mathematics and its applications 83) Cambridge University Press, 2001

[5] Lemarié-Rieusset Pierre Gilles, The Navier-Stokes Problem in the 21st Century, Chapman and Hall, 2020.

[6] Klaus-Jochen Engel, Rainer Nagel, One-parameter semigroups for linear evolution equations, Springer, 2000.

[7] M. Renardy, R. Rogers, An introduction to partial differential equations. Texts in Applied Mathematics 13 (Second ed.), Springer-Verlag, 2004.

[8] Yu. N. Kosovtsov, Formal exact operator solutions to nonlinear differential equations, Preprint, http://arxiv.org/abs/math-ph/0910.3923v1, 2009.

[9] E. Hille, R. S. Phillips: Functional Analysis and Semi-Groups. American Mathematical Society, 1975.

[10] S. G. Krein, Linear differential equations in Banach space, American Mathematical Society, Translations of Mathematical Monographs, Vol. 29, 1971.

[11] Klaus-Jochen Engel, Rainer Nagel, A Short Course on Operator Semigroups, Springer, 2010.

[12] G. Lumer, R. S. Phillips, Dissipative operators in a Banach space. Pacific J. Math. 11: 679698, 1961.

[13] R. S. Phillips, Dissipative operators and hyperbolic systems of partial differential equations, Trans. Amer. Math. Soc, 90, 193-254, 1959.

[14] N. Dunford, J. T. Schwartz Linear Operators, Part I General Theory, Wiley Classics Library, 1958.