Emptiness formation in polytropic quantum liquids

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Large deviations
- emptiness formation probability

Imagine a Fermi gas on a line

What is the probability to have region of size $R$ empty of particles in the ground state?

$$P_{\text{EFP}}(R) = \int_{|x_i| > R} d^N x |\Psi_{\text{GS}}(x_1, x_2, \ldots, x_N)|^2$$

Abanov 2003
Path integral

\[ P_{\text{EFP}}(R) = \lim_{\beta \to \infty} \frac{1}{Z} \text{Tr} \left( e^{-\beta H} |2R\rangle \langle 2R| e^{-\beta H} \right) \]

Hydrodynamic description

\[ R \gg \xi, \rho_0^{-1} \]

\[ P_{\text{EFP}}(R) = \frac{1}{Z} \int \mathcal{D}[\rho, j] e^{-S[\rho, j]} \]

\[ S[\rho, j] = \frac{\rho_0 R^2}{\xi} \int dt dx \left[ \frac{j^2}{2\rho} + V(\rho) \right] \]

Eq. of state: pressure-density relation at \( T = 0 \)

\[ P(\rho) = \rho \partial_\rho V(\rho) - V(\rho) \]
Optimal fluctuation – instanton of hydrodynamical fields

\[ P_{\text{EFP}}(R) \sim e^{-S_{\text{opt}}} \]

Abanov 2003

\[ S_{\text{opt}} = \frac{\pi \rho_0^3}{2} \times R \times R/v_0 \gg 1 \]

\[ \sim \text{ area in space-time, \# of missing particles} \]

Boundary is an astroid

\[ x^{2/3} + \tau^{2/3} = R^{2/3} \]

Courtesy of chabad.org
Previous results

- Random Matrices
- Spin chains
- Weakly interacting Bose gas

In all cases

\[ S_{opt} \sim \frac{\rho_0 R^2}{\xi} \]

for \( R \gg \xi, \rho_0^{-1} \)

This talk — Polytropic Quantum Liquid

\[ V(\rho) \sim \rho^\gamma \]

\[ S_{opt} = f(\gamma) \frac{\rho_0 R^2}{\xi} \]

known values

\[ f(3) = \frac{\pi}{2} \] free fermions, RMT

\[ f(2) = 1.70(1) \] weakly interacting bosons

Yeh & Kamenev 2202
Main Result

\[ f(\gamma) = \frac{\pi \cdot 2^{\frac{\gamma-5}{\gamma-1}} \left[ \Gamma \left( \frac{\gamma+1}{\gamma-1} \right) \right]^2}{\Gamma \left( \frac{3\gamma-1}{2\gamma-2} \right) \left[ \Gamma \left( \frac{\gamma+1}{2\gamma-2} \right) \right]^3} \]
Looking for optimal solution

Hydrodynamical equations of motion in imaginary time

continuity

\[ \partial_\tau \rho + \partial_x (\rho v) = 0 \]

Euler

\[ \partial_\tau v + v \partial_x v = \rho^{\gamma-2} \partial_x \rho \]

+ boundary conditions imposed in distant past and future

Riemann invariant

\[ \lambda(x, \tau) = v + i \frac{2}{\gamma - 1} \rho^{\frac{\gamma-1}{2}} \]

Complex velocity

\[ w(x, \tau) = v + i \rho^{\frac{\gamma-1}{2}} \]

\[ \partial_\tau \lambda + w(\lambda, \bar{\lambda}) \partial_x \lambda = 0 \]

\[ \partial_\tau \bar{\lambda} + \bar{w}(\lambda, \bar{\lambda}) \partial_x \bar{\lambda} = 0 \]

Free fermions \( \gamma = 3 \quad w = \lambda \quad > \quad \) Complex Burgers \( \partial_\tau \lambda + \lambda \partial_x \lambda = 0 \)
Hodograph Transform

\[ \lambda(x, \tau), \bar{\lambda}(x, \tau) \longrightarrow x(\lambda, \bar{\lambda}), \tau(\lambda, \bar{\lambda}) \]

\[ \partial_{\bar{\lambda}} x - w(\lambda, \bar{\lambda}) \partial_{\bar{\lambda}} \tau = 0 \]
\[ \partial_{\lambda} x - \bar{w}(\lambda, \bar{\lambda}) \partial_{\lambda} \tau = 0. \]

free fermions \( w = \lambda \) solve the equations with

characteristics \( x - \lambda \tau = \partial_{\lambda} \mathcal{V} \)

\[ \mathcal{V}(\lambda, \bar{\lambda}) = F_0(\lambda) + G_0(\bar{\lambda}) \]
\[ F_0(\lambda) = \overline{G_0(\bar{\lambda})} = \sqrt{\lambda^2 + 1} \]

found from an electrostatic (RH) problem corresponding to the emptiness boundary condition at \( \tau = 0 \)
Ballistic Ansatz for general $\gamma$

$$x - w\tau = \partial_\lambda \mathcal{V}$$
$$x - \bar{w}\tau = \partial_{\bar{\lambda}} \mathcal{V}$$

Consistency condition (Euler-Poisson eq)

$$\partial_\lambda \partial_{\bar{\lambda}} \mathcal{V} = \frac{n}{\lambda - \bar{\lambda}} \left( \partial_\lambda \mathcal{V} - \partial_{\bar{\lambda}} \mathcal{V} \right)$$

$$\gamma = \frac{2n + 3}{2n + 1}$$

For $n = 0$ - Laplace equation (free fermions)

For $n =$ integer $>0$ a closed form solution can be found (Kamchatnov’s book)

**Strategy**: solve for infinite sequence of $\gamma$ corresponding to integer $n$ and continue analytically for any value of $\gamma$
Solution for \( n = 0, 1, 2, \ldots \)

\[
\mathcal{V} = \frac{F_0(\lambda) + G_0(\bar{\lambda})}{(\lambda - \bar{\lambda})^n} + \sum_{m=1}^{n-1} a_m \frac{F_m(\lambda) + (-1)^m G_m(\bar{\lambda})}{(\lambda - \bar{\lambda})^{n+m}}
\]

\[
F_{m-1} = \partial_{\lambda} F_m \quad G_{m-1} = \partial_{\bar{\lambda}} G_m
\]

\[
a_m = -\frac{(n + m - 1)(n - m)}{m} a_{m-1} \quad a_0 = 1
\]

\[n = 0 \quad \mathcal{V} = F_0(\lambda) + G_0(\bar{\lambda})\]

\[n = 1 \quad \mathcal{V} = \frac{F_0(\lambda) + G_0(\bar{\lambda})}{\lambda - \bar{\lambda}}\]
Boundary conditions in \((\lambda, \bar{\lambda})\) plane

\[ x - w\tau = \partial_\lambda \mathcal{V} \]

1. Particles accumulate avoiding emptiness region

\[ \partial_\lambda \mathcal{V} \big|_{|\lambda| \to \infty} = \pm 1 \]

2. Density decays as

\[ \rho \to 1 + 1/x^2 \text{ for } x \to \infty \]

\[ \partial_\lambda \mathcal{V} \big|_{\lambda \to i(2n+1)} \sim \frac{1}{\sqrt{\lambda^2 + (2n + 1)^2}} \]

Branch points at \(x = \pm 1\) for complex functions

\[ \lambda(x, \tau = 0), \bar{\lambda}(x, \tau = 0) \]
Dynamic density profile

\[ F_{n-1} = \frac{\lambda}{n!} \left[ \lambda^2 + (2n + 1)^2 \right]^{\frac{2n-1}{2}} \]

\[ G_{n-1} = (-1)^n F_{n-1} \]

\[
\begin{align*}
 n = 0 & \quad \nu = F_0 + G_0 \\
 n = 1 & \quad \nu = \frac{F_0 + G_0}{\lambda - \bar{\lambda}} \\
 n = 2 & \quad \nu = \frac{F_0 + G_0}{(\lambda - \bar{\lambda})^2} - 2 \frac{F_1 + G_1}{(\lambda - \bar{\lambda})^3}
\end{align*}
\]

\[
\begin{align*}
 F_0 &= G_0(\lambda) = \sqrt{\lambda^2 + 1} \\
 F_0 &= -G_0(\lambda) = \lambda \sqrt{\lambda^2 + 9} \\
 F_1 &= G_1(\lambda) = \frac{\lambda}{2} (\lambda^2 + 25)^{3/2} \\
 F_0 &= G_0(\lambda) = \partial_\lambda F_1
\end{align*}
\]

Solving \( x - w \tau = \partial_\lambda \nu \) for \( \lambda, \bar{\lambda} \) and extracting \( \rho(x, \tau) \)
Inside Emptiness

\[ \rho = 0 \quad \Rightarrow \quad \lambda = \bar{\lambda} = w = \bar{w} = v(x, \tau) \]

Ballistic evolution \[ x - v\tau = X_n(v) \]

\[ X_0(v) = \frac{v}{\sqrt{v^2 + 1}} \]
\[ X_1(v) = \frac{3v}{\sqrt{v^2 + 9}} - \frac{v^3}{2(v^2 + 9)^{3/2}} \]
\[ X_2(v) = \frac{15v}{8\sqrt{v^2 + 25}} - \frac{5v^3}{4(v^2 + 25)^{3/2}} + \frac{3v^5}{8(v^2 + 25)^{5/2}} \]

\[ X_n(v) \sim 1 + \frac{1}{v^{2n+2}} \quad \text{as} \quad v \to \infty \]
Boundary of emptiness region (tangent method)

\[ x - v \tau = X(v) \] defines a surface \[ \nu(x, \tau) \]

The surface has folds when \[ -\tau = \partial_v X(v) \]

Emptiness boundary \[ x(\tau) \] is Legendre Transform of \[ X(v) \]

Higher singularities – *cusps* – appear when two folds coalesce

“A transparent torus is rarely seen. Let us consider a different transparent body – a bottle (preferably milk). In Fig. 5 two cusp points are visible. By moving the bottle a little we may satisfy ourselves that the cusp singularity is stable. So we have convincing experimental confirmation of Whitney’s theorem.”

*Vladimir Arnold, “Catastrophe Theory”*
Universal behaviour near cusps

We have two types of cusps

**Soft** \( v \to 0 \) \( (x, \tau) \to (0, \pm \tau_c) \)

\[
x = (\tau - \tau_c)v - b_n v^3 \Rightarrow \tau - \tau_c \propto |x|^{2/3}
\]

**Hard** \( v \to \infty \) \( (x, \tau) \to (\pm 1, 0) \)

\[
\tau = \frac{1}{v}(x - 1) + \frac{c_n}{v^{2n+3}} \Rightarrow 1 - x \propto |\tau|^{2n+2 \over 2n+3}
\]

For free fermions: symmetry between soft and hard cusps

\[
x \to \tau
\]

\[
v \to 1/v
\]
Universal density profiles near emptiness boundary

\[ \tau \neq 0 \quad \rho \propto (x - x(\tau))^{1/(\gamma - 1)}, \quad x > x(\tau), \]

– same as exponent predicted by Thomas-Fermi at \( \tau = 0 \)

\[ \tau = 0 \quad \rho \propto (x - x(0))^{-2/(\gamma + 1)}, \quad x > 1 \]

NB: Don’t trust the polytropic law down to zero density: square root density profile near the boundary
Calculation of Emptiness Probability – Instanton action

From density asymptotics

\[ \rho(x, 0) = 1 + \frac{\alpha}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right) \]

and correlation length

\[ \frac{1}{\xi} = \rho_0^{1/(2n+1)} \]

\[ \partial_{\rho_0} S_{\text{opt}} = \frac{\pi R^2}{\xi} \alpha \]

\[ \alpha = \frac{1}{2(2n+1)} \left[ \frac{(2n+1)!!}{2^n n!} \right]^2 \]

is extracted from the divergence

\[ x \sim \frac{(2n-1)!!}{n!} \frac{\lambda^{n+1}}{(\lambda - \bar{\lambda})^n \sqrt{\lambda^2 + (2n+1)^2}} \]

as \( \lambda \to i(2n+1) \)
Result for Polytropic Emptiness

\[ S_{\text{opt}} = \frac{\rho_0 R^2}{\xi} f(n) \]

\[ f(n) = \frac{\pi \Gamma^2 (2n + 2)}{2^{4n+1} \Gamma(n + 2) \Gamma^3(n + 1)} \]

The result can be analytically continued to any real \( n \)

| \( n \) | \( n = 0 \) (\( \gamma = 3 \)) | \( n = 1 \) (\( \gamma = 5/3 \)) | \( n = 2 \) (\( \gamma = 7/5 \)) |
|---|---|---|---|
| \( f(n) \) | 1.56 \( \pm \) 0.02 | 1.76 \( \pm \) 0.02 | 1.85 \( \pm \) 0.02 |
| Eq. (52) | \( \pi/2 \approx 1.571 \) | \( 9\pi/16 \approx 1.767 \) | \( 75\pi/128 \approx 1.841 \) |

\[ f(\gamma) = \frac{\pi \ 2^{\gamma-5}}{\Gamma \left( \frac{3\gamma-1}{2\gamma-2} \right) \Gamma \left( \frac{\gamma+1}{2\gamma-2} \right)} \left[ \Gamma \left( \frac{\gamma+1}{\gamma-1} \right) \right]^2 \]

\[ \Gamma \left( \frac{\gamma+1}{\gamma-1} \right) \]
Conclusions

- First calculation of EFP in polytropic quantum liquid as a function of polytropic index.
- Example of interacting system, beyond free fermionic models
- Universal features, including shape of Emptiness Boundary singularities

Outlook

- Calculation of subleading terms (logarithmic for both fermions and weakly interacting bosons)
- Statistical models with polytropic coarse grained e.o.s?
- Real time dynamics from $\tau \rightarrow it$? Loschmidt echo, return probability
- Other physical models: magnetic impurities in SC

Cheers!