FINITE AND BOUNDED AUSLANDER-REITEN COMPONENTS IN THE DERIVED CATEGORY

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ABSTRACT. We analyze Auslander-Reiten components for the bounded derived category of a finite-dimensional algebra. We classify derived categories whose Auslander-Reiten quiver has either a finite stable component or a stable component with finite Dynkin tree class or a bounded stable component. Their Auslander-Reiten quivers are determined. We also determine components that contain shift periodic complexes.

1. INTRODUCTION

In this paper we analyze the Auslander-Reiten triangles in the bounded derived category of a finite-dimensional indecomposable algebra $A$, denoted by $D^b(A)$. The bounded derived category of a finite-dimensional algebra is a triangulated category and Auslander-Reiten triangles are triangles with analogous properties to Auslander-Reiten sequences for finite-dimensional algebras. The conditions for the existence of such triangles in $D^b(A)$ have been determined by Happel in [H2].

Analogously to the classical Auslander-Reiten theory, which applies to Artin algebras, we can define Auslander-Reiten components of the bounded derived category. These are locally finite graphs, where the vertices correspond to indecomposable complexes in $D^b(A)$. We want to know how and if certain results on finite and dimension-bounded Auslander-Reiten components of finite-dimensional algebras extend to the bounded derived category.

In the second section we give a brief introduction to derived categories and we introduce Auslander-Reiten triangles as defined by Happel in [H1]. In the third section we deduce some properties of Auslander-Reiten triangles that will be used in the other sections.
In the fourth section we classify the bounded derived categories that have either a stable finite Auslander-Reiten component, a stable Auslander-Reiten component with finite Dynkin tree class or a stable bounded Auslander-Reiten component. In all these cases the Auslander-Reiten quiver is described completely. Also components with shift periodic modules are determined.

In the classical Auslander-Reiten theory finite components occur if and only if the algebra has finite representation type. We show that finite stable Auslander-Reiten components occur for $D^b(A)$ if and only if $A$ is simple. In this case each component of the Auslander-Reiten quiver is isomorphic to $A_1$.

In the classical Auslander-Reiten theory we call an Auslander-Reiten component bounded if the dimension of the modules appearing in this component is bounded. Motivated by this definition, we introduce bounded Auslander-Reiten components for the bounded derived category and show that the following are equivalent

1) The bounded derived category of $A$ has finite representation type, i.e. has only finitely many isomorphism classes of indecomposable objects up to shift;
2) There is a stable component with finite Dynkin tree class;
3) There is a bounded stable component.

In this case, the Auslander-Reiten quiver consists either only of one component $\mathbb{Z}[T]$ with $T \neq A_1$ a finite Dynkin diagram, or of infinitely many components $A_1$.

In the first case $A$ is derived equivalent to $kT$, which is a hereditary algebra of finite representation type. In the second case $A$ is simple. Finally we introduce shift periodic complexes in analogy to periodic modules. Possible tree classes for derived categories with shift periodic complexes are deduced. We show that their Auslander-Reiten quiver is either $\mathbb{Z}[T]$ with $T$ a finite Dynkin diagram or the component containing the shift periodic complex has tree class $A_\infty$.

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2. Preliminaries and Notation

Let $A$ denote a finite-dimensional indecomposable algebra over a field $k$ and $A\text{-mod}$ the category of finite-dimensional left $A$-modules. We denote by $\mathcal{P}$ the full subcategory in $A\text{-mod}$ of projective modules and $\mathcal{I}$ the full subcategory of injective $A$-modules. Let $C \in \{A\text{-mod}, \mathcal{P}, \mathcal{I}\}$.

Then $\text{Comp}^{*,?}(C)$ are the complexes that are bounded above if $* = -$, bounded below if $* = +$ and bounded if $? = b$. The homology is bounded if $? = b$. We denote by $D^b(A)$ the bounded derived category and by $K^{*,?}(C)$ the homotopy category.

Contractible complexes and homotopic to zero maps are related as stated in the following well-known result.

**Lemma 2.1.** Let $f : X \to Y$ be a map of complexes.

1. The map $f$ is homotopic to zero if and only if $f$ factors through a contractible complex.
2. The projection $p : \text{cone}(f) \to X[1]$ is a retraction if and only if $f$ is homotopic to zero.

The homotopy category and the derived categories are triangulated categories by [Wei, 10.2.4, 10.4.3] where the shift functor $[1]$ is the automorphism. The distinguished triangles are given up to isomorphism of triangles by

$$
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
\downarrow & & \downarrow \overset{0 \oplus \text{id}_Y}{\longrightarrow} \\
\text{cone}(f) & \overset{\text{id}_{X[1]} \oplus 0}{\longrightarrow} & X[1]
\end{array}
$$

for any morphism $f$.

It is difficult to calculate the morphisms in the derived category of $A$-modules. The following theorem provides an easier way to represent them.

**Theorem 2.2.** [Wei, 10.4.8] We have the following equivalences of triangulated categories

$$K^{-,b}(\mathcal{P}) \cong K^{+,b}(\mathcal{I}) \cong D^b(A).$$

We identify an $A$-module $X$ with the complex that has entry $X$ in degree 0 and entry 0 in all other degrees. By abuse of notation we call this complex $X$. A complex with non-zero entry in only one degree is also called a stalk complex. Note that $A\text{-mod}$ is equivalent to a full subcategory of $D^b(A)$ using this embedding.

Let $N$ be a left $A$-module and $\cdots \overset{d_2}{\longrightarrow} P_1 \overset{d_1}{\longrightarrow} P_0 \to N$ its minimal projective resolution. Let $N \to I_0 \overset{d_0}{\longrightarrow} I_1 \overset{d_1}{\longrightarrow} \cdots$ be its minimal injective resolution.
Then we denote throughout this paper by $pN$ the complex with $$(pN)^i := P_{-i}$$ and $d^i := d_{P_{-i}}$ for $i \leq 0$ and $(pN)^i := 0$ for $i > 0$. Similarly we define $iN$ to be the complex with $(iN)^n := I_n$ and $d^n := d^n_I$ for $n \geq 0$ and $(iN)^n := 0$ for $n < 0$.

**Remark 2.3.** Note that all indecomposable contractible complexes in $\text{Comp}^{-b}(\mathcal{P})$ have up to shift the form

$$
\cdots \to 0 \to P \xrightarrow{id} P \to 0 \to \cdots
$$

for an indecomposable projective module $P$ of $A$. We denote such a complex where $P$ occurs in degree 0 and 1 by $\bar{P}$.

Contractible complexes are projective in $\text{Comp}^{-b}(\mathcal{P})$ as the following lemma shows.

**Lemma 2.4.** A contractible complex in $\text{Comp}^{-b}(\mathcal{P})$ is projective.

**Proof.** Let $P$ be an indecomposable projective module and $\bar{P}$ the associated indecomposable contractible complex. By the previous remark, all contractible complexes are direct sums of shifts of such indecomposable complexes. Let $f : C \to D$ be a surjective map of complexes in $\text{Comp}^{-b}(\mathcal{P})$. This means that $f^i$ is surjective for all $i \in \mathbb{Z}$. Suppose there is a map $g : \bar{P} \to D$ of complexes. Then we construct a map $h : \bar{P} \to C$ as follows.

Since $P$ is projective, there is a map $q : P \to C^0$ such that $f^0q = g^0$. We then set $h^1 := d^1_C q$ and $h^0 := q$ and $h^i := 0$ for all other degrees. We visualize this in the diagram

$$
\begin{array}{cccccc}
\cdots & \to & 0 & \to & P & \xrightarrow{id} & P & \to & 0 & \to & \cdots \\
\cdots & \downarrow{f^{-1}} & \to & C^{-1} & \downarrow{g^0} & \to & C^0 & \xrightarrow{id} & P & \to & 0 & \to & \cdots \\
\cdots & \downarrow{D^{-1}} & \to & D^{-1} & \downarrow{f^0} & \to & D^0 & \xrightarrow{id} & P & \to & 0 & \to & \cdots \\
\end{array}
$$

Then $h$ is a map of complexes. Furthermore $f^1h^1 = f^1d^1_C q = d^1_D f^0 q = d^1_D g^0 = g^1$. This gives the proof. \hfill $\square$

Finally we define for a complex $X$, the complex $\sigma_{\leq n}(X)$ to be the complex with $\sigma_{\leq n}(X)^i := X^i$ for $i \leq n$ and $d_{\sigma_{\leq n}(X)}^i := d_X^i$ for $i < n$ and $\sigma_{\leq n}(X)^i := 0$ for $i > n$. We define $\sigma_{\geq n}(X)$ analogously.
Next we introduce Auslander-Reiten theory for triangulated categories. We state the existence conditions for Auslander-Reiten triangles in the bounded derived category of a finite-dimensional algebra and prove some properties that will be needed in the other sections.

For an introduction to triangulated categories we refer to [H1, I.1.1]. Let \( T \) be a triangulated category with translation functor \( T \).

**Definition 2.5.** [H1, I.4.1] ([Auslander-Reiten triangles]) A distinguished triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \) is called an Auslander-Reiten triangle if the following conditions are satisfied:

1. The objects \( X, Z \) are indecomposable
2. The map \( w \) is non-zero
3. If \( f : W \rightarrow Z \) is not a retraction, then there exists \( f' : W \rightarrow Y \) such that \( vf' = f \).

We introduce the following conditions.

1. If \( f : X \rightarrow W \) is not a section, then there exists \( f' : Y \rightarrow W \) such that \( f'u = f \).
2. The map \( u \) is not a section.
3. The map \( v \) is not a retraction.

By [H1, I.4.2] we have that the condition (1)+(2)+(3) is equivalent to the condition (1)+(2)+(3') and also to the condition (1)+(2)+(3'').

The condition (2) is equivalent to

1. The map \( u \) is not a section.
2. The map \( v \) is not a retraction.

We refer to \( w \) as the connecting homomorphism of an Auslander-Reiten triangle. We say that the Auslander-Reiten triangle \( X \rightarrow Y \rightarrow Z \rightarrow TX \) starts in \( X \), has middle term \( Y \) and ends in \( Z \). Note also that an Auslander-Reiten triangle is uniquely determined up to isomorphisms of triangles by the isomorphism class of the element it ends or starts with. The Auslander-Reiten translation \( \tau \) is defined as the functor on the set of all isomorphism classes of indecomposable objects that appear at the end of an Auslander-Reiten triangle to the set of indecomposable objects that appear at the start of an Auslander-Reiten sequence. Then \( \tau \) sends the isomorphism class of \( Z \) to the isomorphism class of \( X \).

The Auslander-Reiten translation of \( A \text{-mod} \) will be denoted by \( \tau_A \) in this paper to avoid confusion.

Analogously to the classical Auslander-Reiten theory we can define irreducible maps, minimal maps, left almost split maps and right almost split...
Lemma 2.6. Let $N, M \in T$ and let $f : N \to M$ be an irreducible map in $D^b(A)$.

1. Let $N \xrightarrow{g} Q \xrightarrow{E} \to TN$ be the Auslander-Reiten triangle, then there is a retraction $s : Q \to M$ such that $f = sg$.

2. Let $L \to B \xrightarrow{h} M \to TL$ be an Auslander-Reiten triangle, then there is a section $r : N \to B$ such that $f = hr$.

Let from now on $k$ be an algebraically closed field. If $T$ is a Krull-Schmidt category we define the Auslander-Reiten quiver to be the labelled graph $\Gamma(T)$ with vertices the isomorphism classes of indecomposable objects. For two indecomposable objects $X, Y$ there are $d_{X,Y}$ arrows from $X$ to $Y$ where $d_{X,Y} = \dim_k \text{Irr}(X,Y)$. (see [H1] I.4.7 and [K] 2.7)

We call a connected component of $\Gamma(T)$ an Auslander-Reiten component.

From now on let $T = D^b(A)$. By [K] 2.6 we have that $D^b(A)$ is a Krull-Schmidt category. Therefore Auslander-Reiten components of $D^b(A)$ are well-defined. Using [K] 2.6 combined with [K] B.2 shows that $\text{Comp}^{-b}(P)$ and $\text{Comp}^{+,-b}(I)$ are also Krull-Schmidt categories. They are full subcategories of the abelian category $\text{Comp}(A)$ which are closed under direct summands and extensions. We can therefore refer to $[A]$ for their Auslander-Reiten sequences.

Lemma 2.7. [H1] I.4.3, I.4.5] Let $X \xrightarrow{u} M \xrightarrow{v} Z \to TX$ be an Auslander-Reiten triangle and $M \cong M_1 \oplus M_2$, where $M_1$ is indecomposable. Let $i : M_1 \to M$ be an inclusion and $p : M \to M_1$ a projection. Then $vi : M_1 \to Z$ and $pu : X \to M$ are irreducible maps. Furthermore $u$ is minimal left almost split and $v$ is minimal right almost split.

Let $\nu_A$ denote the Nakayama functor of $A$ and let

$$\nu_A^{-1} := \text{Hom}_A(\text{Hom}_k(\cdot, k), A).$$

We denote by $\nu$ the left derived functor of $\nu_A$ on $D^b(A)$ and by $\nu^{-1}$ the right derived functor of $\nu_A^{-1}$. Then $\nu$ maps a complex $X \in \text{Comp}^b(P)$ to $\nu(X) \in \text{Comp}^b(I)$, where $\nu(X)^i := \nu_A(X^i)$ and $d^i_{\nu(X)} := \nu_A(d^i_X)$ for all $i \in \mathbb{Z}$.

The conditions for the existence of Auslander-Reiten triangles in a triangulated category have been determined in [VR] I.2.4]. It is shown that every
indecomposable element $X$ there is an Auslander-Reiten triangle that ends in $X$ and one that starts in $X$, if and only if the category has a Serre functor.

A specialization of this result is given in the next theorem for the case of $\text{D}^b(A)$.

**Theorem 2.8.** [12, 1.4] (1) Let $Z \in K^{-,b}(P)$ be indecomposable. Then there exists an Auslander-Reiten triangle ending in $Z$ if and only if $Z \in K^b(P)$. The triangle is of the form $\nu(Z)[-1] \rightarrow Y \rightarrow Z \rightarrow \nu(Z)$ for some $Y \in K^{-,b}(P)$.

(2) Let $X \in K^{+,b}(I)$ be indecomposable, then there exists an Auslander-Reiten triangle starting in $X$ if and only if $X \in K^b(I)$. The triangle is of the form $X \rightarrow Y \rightarrow \nu^{-1}(X)[1] \rightarrow X[1]$ for some $Y \in K^{-,b}(P)$.

From this result we deduce that the translation $\tau$ is given by $\nu[-1]$ and $\tau$ is natural equivalence from $K^b(P)$ to $K^b(I)$. So every Auslander-Reiten triangle is isomorphic to

$$
\nu(X)[-1] \rightarrow \text{cone}(w)[-1] \rightarrow X \xrightarrow{w} \nu(X)
$$

for $X \in K^b(P)$ and some map $w : X \rightarrow \nu(X)$.

Not all irreducible maps appear in Auslander-Reiten triangles.

**Lemma 2.9.** Let $f : B \rightarrow C$ be an irreducible map in $\text{D}^b(A)$ that does not appear in an Auslander-Reiten triangle. Then $B, C \notin K^b(P)$ and $B, C \notin K^b(I)$.

**Proof.** By 2.8 and 2.6 it is clear that $B \notin K^b(I)$ and $C \notin K^b(P)$. Let us assume that $B \in K^b(P)$ and let $n \in \mathbb{N}$ be minimal such that $B^n \neq 0$. Then $f$ factorizes through $\sigma^{\geq n-1}(C)$, where $C$ is represented as a complex in $\text{Comp}^{-,b}(P)$. Let $f = hg$ be this factorization, then $g$ is not a section, as $f$ is not a section and $h$ is not a retraction as $\sigma^{\geq n-1}(C) \neq C$. This is a contradiction to the fact that $f$ is irreducible. Therefore $B \notin K^b(P)$. Analogously, we can show that $C \notin K^b(I)$. \[\square\]

More results on these irreducible maps can be found in [HKR].

### 3. Auslander-Reiten triangles

In this section we deduce some properties of Auslander-Reiten triangles and introduce stable components. The following lemma determines the relation between irreducible maps, retractions and sections in $K^{-,b}(P)$ and $\text{Comp}^{-,b}(P)$. Note that by duality the same is true if we replace $K^{-,b}(P)$ by $K^{+,b}(I)$ and $\text{Comp}^{-,b}(P)$ by $\text{Comp}^{+,b}(I)$. 
Lemma 3.1. Let $B,C \in \text{Comp}^{-b}(\mathcal{P})$ be complexes that are not contractible. Let $f : B \to C$ be a map of complexes.

(1) Let $C, B$ be indecomposable. The map $f$ is irreducible in $\text{Comp}^{-b}(\mathcal{P})$ if and only if $f$ is irreducible in $K^{-b}(\mathcal{P})$.

(2) Let $C$ be indecomposable. The map $f$ is a retraction in $\text{Comp}^{-b}(\mathcal{P})$ if and only if $f$ is a retraction in $K^{-b}(\mathcal{P})$.

(3) Let $B$ be indecomposable. The map $f$ is a section in $\text{Comp}^{-b}(\mathcal{P})$ if and only if $f$ is a section in $K^{-b}(\mathcal{P})$.

Proof. We first give a proof of (2). Let $f : B \to C$ be a retraction in $\text{Comp}^{-b}(\mathcal{P})$. Then $f$ is clearly a retraction in $K^{-b}(\mathcal{P})$. Let $f$ be a retraction in $K^{-b}(\mathcal{P})$, then there is a map $g : C \to B$ such that $fg$ is homotopic to $id_C$. Therefore $fg - id_C$ factors through a contractible complex $P$ via $s : C \to P$ and $t : P \to C$. Then $(t,f)(\gamma_g) = id_C$. As $C$ does not have a contractible summand, we have that $fg$ is an isomorphism. The proof of (3) is analogous.

We prove (1). Let $f : B \to C$ be an irreducible map in $K^{-b}(\mathcal{P})$, then by (2) and (3) $f$ is also an irreducible map in $\text{Comp}^{-b}(\mathcal{P})$. Suppose now that $f$ is irreducible in $\text{Comp}^{-b}(\mathcal{P})$ and let $gh$ be homotopic to $f$ for some $g : D \to C$ and $h : B \to D$. Then $gh - f$ factors through a contractible complex $P$ via $s : B \to P$ and $t : P \to C$. So $f = (g,-t)(\gamma_g)$ factors through $D \oplus P$ in $\text{Comp}^{-b}(\mathcal{P})$. Therefore $(\gamma_g)$ is a section in $\text{Comp}^{-b}(\mathcal{P})$ or $(g,-t)$ is a retraction in $\text{Comp}^{-b}(\mathcal{P})$. This means that $h$ is a section in $K^{-b}(\mathcal{P})$ or $g$ is a retraction in $K^{-b}(\mathcal{P})$. Therefore $f$ is irreducible in $K^{-b}(\mathcal{P})$. □

We can therefore choose for an irreducible map in $K^{-b}(\mathcal{P})$ an irreducible map in $\text{Comp}^{-b}(\mathcal{P})$ that represents this map. For the rest of this paper all irreducible maps in $K^{-b}(\mathcal{P})$ or $K^{+b}(\mathcal{I})$ will be represented by irreducible maps in $\text{Comp}^{-b}(\mathcal{P})$ or $\text{Comp}^{+b}(\mathcal{I})$ respectively.

Lemma 3.2. Let

$$w := 0 \to E \to C \xrightarrow{\nu} P \to 0$$

be an Auslander-Reiten sequence in $\text{Comp}^{-b}(\mathcal{P})$. Then $P \in \text{Comp}^{b}(\mathcal{P})$. We identify $\nu(P)$ with an indecomposable complex in $\text{Comp}^{-b}(\mathcal{P})$. Then $w$ is isomorphic to

$$0 \to \nu(P)[-1] \to \text{cone}(w)[-1] \to P \to 0.$$ 

Proof. We consider $w \in \text{Hom}_{D^b(A)}(P,E[1])$ and the distinguished triangle $E \to C \to P \xrightarrow{w} E[1]$ corresponding to the Auslander-Reiten sequence $w$. 
As this sequence does not split, the map $w$ is not homotopic to zero by 2.1. Let $M \in \text{Comp}^{-b}(P)$ and let $f : M \to P$ be a map in $\text{Comp}^{-b}(P)$ representing a map in $K^{-b}(P)$ that is not a retraction. Then $f$ is not a retraction in $\text{Comp}^{-b}(P)$ by 3.1 and factors through $\sigma$ in $\text{Comp}^{-b}(P)$. By 2.5

$$E \to C \to P \xrightarrow{w} E[1]$$

is an Auslander-Reiten triangle. It follows from 2.8 that $P \in K^{-b}(P)$. As $P$ is indecomposable, we have $P \in \text{Comp}^{-b}(P)$ and $E \cong \nu(P)[-1]$. □

The next theorem determines the relation between Auslander-Reiten triangles in $D^b(A)$ and Auslander-Reiten sequences in $\text{Comp}^{-b}(P)$. The analogous statement for self-injective algebras was given by [W, 2.3, 2.2].

**Theorem 3.3.** Let $P \in \text{Comp}^{-b}(P)$ be an indecomposable complex that is not contractible. We identify $\nu(P)$ with an indecomposable complex in $\text{Comp}^{-b}(P)$. Let $w : P \to \nu(P)$ be a map in $\text{Comp}^{-b}(P)$. Then

$$0 \to \nu(P)[-1] \to \text{cone}(w)[-1] \to P \to 0$$

is an Auslander-Reiten sequence in $\text{Comp}^{-b}(P)$ if and only if $w$ induces an Auslander-Reiten triangle in $D^b(A)$.

**Proof.** Let $w : P \to \nu(P)$ induce an Auslander-Reiten triangle in $D^b(A)$. We have an exact sequence

$$0 \to \nu(P)[-1] \to \text{cone}(w)[-1] \xrightarrow{\sigma} P \to 0.$$ 

Furthermore $\nu(P)[-1]$ and $P$ are indecomposable. Let $M$ be a complex in $\text{Comp}^{-b}(P)$ and let $f : M \to P$ be a non-split map in $\text{Comp}^{-b}(P)$. Then by 3.1 $f$ is not a retraction in $K^{-b}(P)$. Therefore there is a map $f_1 : M \to \text{cone}(w)[-1]$ such that $\sigma f_1 = f$ in $K^{-b}(P)$. Then $f - \sigma f_1$ factors through a contractible complex $P_2$. Let $f - \sigma f_1 = gh$ where $h : M \to P_2$ and $g : P_2 \to P$. As $P_2$ is projective in $\text{Comp}^{-b}(P)$ by 2.4 there is a map $s : P_2 \to \text{cone}(w)[-1]$ such that $g = \sigma s$. We set $f' = f_1 + sh$, then $\sigma f' = \sigma f_1 + \sigma sh = f$. Therefore $\sigma$ is right almost split and the exact sequence is therefore an Auslander-Reiten sequence. Assume now that the exact sequence

$$0 \to \nu(P)[-1] \to \text{cone}(w)[-1] \xrightarrow{\sigma} P \to 0$$

is an Auslander-Reiten sequence. Then the statement follows from the proof of 3.2. □
Note that if \( P \in \text{Comp}^b(\mathcal{P}) \) is contractible, then there is no Auslander-Reiten sequence in \( \text{Comp}^{-b}(\mathcal{P}) \) that ends in \( P \), as contractible complexes are projective objects in \( \text{Comp}^{-b}(\mathcal{P}) \) by [2,4].

We call an Auslander-Reiten component \( \Lambda \) stable, if all vertices appear at the start and at the end of an Auslander-Reiten triangle. By [2,8] this is equivalent to the fact that all vertices in the component are in \( K^b(I) \) and \( K^b(\mathcal{P}) \).

The Auslander-Reiten quiver of \( D^b(A) \) does not contain loops by [XZ, 2.2.1]. So we can apply Riedtmann’s Structure Theorem [Ri, p.206].

**Corollary 3.4.** Let \( \Lambda \) be a stable Auslander-Reiten component of \( D^b(A) \). Then \( \Lambda \cong \mathbb{Z}[T]/I \) where \( T \) is a tree and \( I \) is an admissible subgroup of \( \text{aut}(\mathbb{Z}[T]) \).

It is easy to see for which algebras stable components appear.

**Corollary 3.5.** All complexes that appear in Auslander-Reiten triangles are contained in stable components if and only if \( A \) is Gorenstein.

If \( A \) has finite global dimension, then the Auslander-Reiten quiver is a stable translation quiver. If \( A \) is self-injective and not semi-simple, then the stable Auslander-Reiten components are isomorphic to \( \mathbb{Z}[A_\infty] \) by [W, 3.7]. The non-stable components have been determined in [HKR, 5.7].

4. Finite and bounded Auslander-Reiten components

In this section, we determine the tree class of bounded and finite stable Auslander-Reiten components. We show that finite stable components can only appear if \( A \) is simple. Bounded stable components appear if and only if the representation type of \( D^b(A) \) is finite. This is also equivalent to the fact that the Auslander-Reiten quiver has a component with tree class finite Dynkin. We describe the Auslander-Reiten quiver concretely in these cases.

We start with the following easy lemma.

**Lemma 4.1.** The following are equivalent:

1. There is a stable Auslander-Reiten component of \( D^b(A) \) isomorphic to \( A_1 \);
2. The bounded derived category of \( A \) has an Auslander-Reiten triangle with middle term zero;
3. The algebra \( A \) is simple;
4. The Auslander-Reiten quiver of \( D^b(A) \) is the union of infinitely many components \( A_1 \).
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Proof. Obviously, part (2) follows from (1). If \( \nu(X)[-1] \to 0 \to X \xrightarrow{w} \nu(X) \) is an Auslander-Reiten triangle, then \( w \) is an isomorphism. Also for all indecomposable objects \( M \in D^b(A) \) with \( M \neq X \) we have \( \text{Hom}_{D^b(A)}(M, X) = 0 \) and \( \text{rad} \text{End}(X) = 0 \) by 2.5 (3). Let \( X \) be represented by an indecomposable complex in \( \text{Comp}^{-b}(\mathcal{P}) \). Without loss of generality we assume that \( X^0 \neq 0 \) and \( X^i = 0 \) for all \( i > 0 \). As \( X \) does not have contractible summands, the map \( d_{X}^{-1} \) is not surjective. Suppose \( X^{-1} \neq 0 \). Then there is a map \( h \) from the stalk complex of \( X^0 \) to \( X \) in \( K^{-b}(\mathcal{P}) \) given by \( h^0 = \text{id}_{X^0} \). This map is non-zero in \( K^{-b}(\mathcal{P}) \) as the map \( d_{X}^{-1} \) is not surjective. This gives a contradiction. Assume now that \( X^0 \) is not simple. Then there is a non-zero map from the stalk complex of the projective cover of \( \text{soc} X^0 \) to \( X \) in \( K^{-b}(\mathcal{P}) \) that is not an isomorphism. This gives also a contradiction. Therefore \( X^0 \) is simple and projective. Furthermore \( \nu_A(X^0) \cong X^0 \) as \( w \) is an isomorphism. Therefore \( X^0 \) is injective. By [13, 1.8.5] we have that \( A \) is simple and \( X^0 \) is the only simple module in \( A \) up to isomorphism. Therefore (3) follows from (2). Clearly (3) implies (4) and (4) implies (1).

This lemma shows that if \( A \) is not simple then the stable Auslander-Reiten components are the components on which \( \tau \) is an automorphism. In \( \text{Comp}^{-b}(\mathcal{P}) \) we have an analogous result.

Lemma 4.2. The following are equivalent:

1. The category \( \text{Comp}^{-b}(\mathcal{P}) \) has an Auslander-Reiten sequence with contractible middle term;
2. The algebra \( A \) is simple;
3. The Auslander-Reiten quiver of \( \text{Comp}^{-b}(\mathcal{P}) \) is isomorphic to \( A\mathbb{N}^\infty \).

Proof. (2) \( \Rightarrow \) (3), (1) Let \( A \) be simple and let \( S \) be the simple \( A \)-module. Then \( S \) is projective and injective. Then by [4, 1.1] the connecting homomorphism \( w \) can be taken to be the identity. Let \( \bar{S} \) be the associated contractible complex. The Auslander-Reiten sequence ending in \( \bar{S} \) is then of the form \( 0 \to S[-1] \to \bar{S} \to S \to 0 \), where the map \( \bar{S} \to S \) is given by

\[
\begin{array}{ccccccc}
\cdots & - & 0 & S & \xrightarrow{id} & S & 0 & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & - & 0 & \bar{S} & S & 0 & \cdots
\end{array}
\]

and the map from \( S[-1] \to \bar{S} \) by
The Auslander-Reiten quiver of $\text{Comp}^{-b}(\mathcal{P})$ is given by

\[
\cdots \rightarrow 0 \rightarrow 0 \rightarrow S \rightarrow 0 \rightarrow \cdots
\]

and is isomorphic to $A^\infty_\infty$. Clearly (3) implies (1).

(1) $\Rightarrow$ (2) Suppose that $\text{Comp}^{-b}(\mathcal{P})$ has an Auslander-Reiten sequence with contractible middle term. Then by 3.2 and 3.3 there is an Auslander-Reiten triangle in $K^{-b}(\mathcal{P})$ with trivial middle term. By the previous lemma, $A$ has to be simple. $\square$

**Lemma 4.3.** Let $\tau(C) \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{w} \tau(C)[1]$ be a distinguished triangle in $D^b(A)$ and $M$ an indecomposable element in $D^b(A)$. Then

\[
\text{Hom}(M, \tau(C)) \xrightarrow{f^*} \text{Hom}(M, B) \xrightarrow{g^*} \text{Hom}(M, C)
\]

and

\[
\text{Hom}(C, M) \xrightarrow{\bar{g}} \text{Hom}(B, M) \xrightarrow{\bar{f}} \text{Hom}(\tau(C), M)
\]

are exact. If the triangle is an Auslander-Reiten triangle then the following holds.

1. The map $f^*$ is injective if and only if $M[1] \not\cong C$;
2. The map $\bar{g}$ is injective if and only if $M[-1] \not\cong \tau(C)$;
3. The map $g^*$ is surjective if and only if $M \not\cong C$;
4. The map $\bar{f}$ is surjective if and only if $M \not\cong \tau(C)$.

**Proof.** By [H1, I.1.2(b)] the sequence $\text{Hom}(M, \tau(C)) \xrightarrow{f^*} \text{Hom}(M, B) \xrightarrow{g^*} \text{Hom}(M, C)$ is exact. We assume from this point on that the considered triangle is an Auslander-Reiten triangle. Suppose there is an $h : M \rightarrow \tau(C)$ such that $fh = 0$. Then by the axiom (TR3) of [H1 I.1.1] there is a map $j : M[1] \rightarrow C$ such that the following diagram of distinguished triangles commutes:

\[
\begin{array}{ccc}
M & \rightarrow & 0 & \rightarrow & M[1] & \rightarrow & M[1] \\
\downarrow_{h} & & \downarrow_{j} & & \downarrow_{\text{id}} & & \downarrow_{h[1]} \\
\tau(C) & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{w} & \tau(C)[1].
\end{array}
\]
If $M[1] \not\cong C$ then $w_j = 0$ by (2.5) (3’). This forces $h[1] = 0$ and therefore $h = 0$. So in this case $f^*$ is injective. If $M[1] \cong C$, then $f^*$ is not injective as $w[-1] : M \to \tau(C)$ is mapped to zero. This proves (1). If $M \cong C$ then (2”) implies that $g^*$ is not surjective. Surjectivity for $M \not\cong C$ uses (2.5) (3). This proves (3).

The remaining cases are proven similarly.

The following version of Lemma [13, 4.13.4] holds for our setup.

**Lemma 4.4.** Let $C$ and $B$ be indecomposable complexes such that there is a non-zero map from $C$ to $B$ in $D^b(A)$ that is not an isomorphism. Suppose there is no chain of irreducible maps in $D^b(A)$ from $C$ to $B$ of length less than $n$.

1. Assume $B$ lies in a component $\Lambda$ of the Auslander-Reiten quiver of $D^b(A)$ such that all elements of $\Lambda$ are in $K^b(P)$. Then there exists a chain of irreducible maps in $\Lambda$

   $$P^0 \xrightarrow{g_1} P^1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} P^n \xrightarrow{g_n} B$$

   and a map $h : C \to P^0$ with $g^n \cdots g^1 h \not= 0$ in $K^{-b}(P)$.

2. Assume $C$ lies in a component $\Lambda$ of the Auslander-Reiten quiver of $D^b(A)$ such that all elements of $\Lambda$ are in $K^b(I)$. Then there exists a chain of irreducible maps in $\Lambda$

   $$C \xrightarrow{g_1} P^1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} P^n \xrightarrow{g_n} P^n$$

   and a map $h : P^n \to B$ with $h g^n \cdots g^1 \not= 0$ in $K^{+b}(I)$.

**Proof.** We give a proof of (1) as the proof of (2) is analogous. The proof follows by induction on $n$. Assume first $n = 1$. Then there is no irreducible map from $C$ to $B$. Let $\nu(B)[-1] \xrightarrow{\ell} E \xrightarrow{l} B \to \nu(B)$ be the Auslander-Reiten triangle ending in $B$. By the assumption, there is a non zero map $t : C \to B$, that is not an isomorphism. Therefore we have some $\sigma : C \to E$ such that $t = l \sigma$. Let $E := \bigoplus_{i=1}^m E_i$ with $E_i$ indecomposable for all $1 \leq i \leq m$. Then $t = \sum_{i=1}^m l_i \sigma_i$ for some maps $l_i : E_i \to B$ and $\sigma_i : C \to E_i$. Clearly there exists an $1 \leq s \leq m$ such that $l_s \sigma_s \not= 0$ in $K^b(P)$ as $t \not= 0$ in $K^b(P)$. By (2.7) $g_1 := l_s$ is irreducible and $E_s$ lies in $\Lambda$. Furthermore by the induction assumption $\sigma_s$ is not an isomorphism as there is no irreducible map from $C$ to $B$. We can therefore use the same argument in the induction step on the map $\sigma_s$. □
We define certain length functions of complexes. Similar length functions have been defined for example in [W]. We will use the following notation.

**Definition 4.5.** We denote by $l_c$ the function that maps an element $B \in \text{Comp}^b(A)$ to the length of a composition series of $\bigoplus_{i \in \mathbb{Z}} B^i$. For $Y \in \text{Comp}^b(P)$ we denote by $l(Y)$ the number of projective indecomposable summands in $\bigoplus_{i \in \mathbb{Z}} Y^i$. For $X \in K^b(P)$ we denote by $l_p(X) := \min\{l(Y) | Y \in \text{Comp}^b(P) \text{ and } Y \cong X \text{ in } K^b(P)\}$.

We define analogously $l_i$ for elements of $K^b(I)$ counting the number of indecomposable injective modules.

We call a stable Auslander-Reiten component $\Lambda$ bounded if $l_p$ and $l_i$ take bounded values on $\Lambda$.

A complex $Y \in \text{Comp}^b(P)$ with $l_p(X) = l(Y)$ and $Y \cong X$ in $K^b(P)$ is called homotopically minimal in [Kr, B2]. Krause also shows that such a homotopically minimal element to a complex is uniquely determined up to isomorphism of complexes.

Note that $l_p$, $l_c$ and $l_i$ are defined for complexes of a stable Auslander-Reiten component of $D^b(A)$. Also a bounded Auslander-Reiten component is always stable, as all elements in the component are both in $K^b(P)$ and $K^b(I)$.

**Remark 4.6.** Let $\Lambda$ be a stable component. Then by [3.5] every Auslander-Reiten triangle in $\Lambda$ is isomorphic to

$$\tau(P) \to \text{cone}(w)[-1] \to P \xrightarrow{w} \tau(P)[1]$$

where

$$0 \to \tau(P) \to \text{cone}(w)[-1] \to P \to 0$$

is a short exact sequence in $\text{Comp}^b(P)$ and $w : P \to \tau(P)[1]$ is a representative in $\text{Comp}^b(P)$ of the connecting morphism. Then $\text{cone}(w)[-1] \in \text{Comp}^b(P)$. Therefore $l_p$ satisfies $l_p(\text{cone}(w)[-1]) \leq l_p(\tau(P)) + l_p(P)$ with equality if and only $\text{cone}(w)[-1]$ does not have a contractible summand in $\text{Comp}^b(P)$.

With exactly the same proof as in [B, 4.14.1] we have

**Lemma 4.7.** Let $P_0, \ldots, P_{2^n-1} \in \text{Comp}^b(A)$ be indecomposable and assume that $l_c(P_i) \leq n$ for all $i$. If the maps $f_i : P_{i-1} \to P_i$ are not isomorphisms for $1 \leq i \leq 2^n - 1$, then $f_{2^n-1} \cdots f_2 f_1 = 0$. 
Suppose $f : C \to B$ is a map in $\text{Comp}^{-b}(\mathcal{P})$ that represents an irreducible map in $K^{-b}(\mathcal{P})$, where $C$ and $B$ are indecomposable complexes. Then by 3.1 $f$ is irreducible in $\text{Comp}^{-b}(\mathcal{P})$ and therefore $f$ is not an isomorphism.

We therefore have the following result.

**Corollary 4.8.** Let $P_0, \ldots, P_{2^n-1} \in \text{Comp}^b(P)$ be indecomposable such that $l_c(P_i) \leq n$ for all $i$. If the $f_i : P_{i-1} \to P_i$ are irreducible maps in $K^{-b}(\mathcal{P})$ for $1 \leq i \leq 2^n - 1$, then $f_{2^n-1} \cdots f_2 f_1 = 0$.

We can now determine some properties of bounded components.

**Theorem 4.9 (bounded components).** Let $\Lambda$ be a stable bounded Auslander-Reiten component of $D^b(A)$. We assume that $A$ is not simple. Then $\Lambda$ is the only component of the Auslander-Reiten quiver of $D^b(A)$. Furthermore $A$ has finite global dimension and is of finite representation type.

**Proof.** By assumption, there is an $n \in \mathbb{N}$ such that $l_p(M) \leq n$ and $l_i(M) \leq n$ for all complexes $M \in \Lambda$. Let $R, S \in K^{-b}(\mathcal{P})$ be indecomposable complexes such that there are non zero maps $g : R \to N$ and $f : M \to S$ in $K^{-b}(\mathcal{P})$ for some $N, M \in \Lambda$. Let $u := n \dim A$. Suppose there is no chain of irreducible maps from $R$ to $N$. Then by 4.4 part (1) there exists a chain of irreducible maps of length $2^u$ in $\Lambda$ that is not zero. This is a contradiction to 4.8 as $l_c$ takes values at most $u$ on $\Lambda$. Therefore there is a chain of irreducible maps from $R$ to $N$. So $R \in \Lambda$. Analogously $S$ lies in the component $\Lambda$. For any complex $M$ the connecting homomorphism $w$ is a non zero map from $M$ to $\tau(M)[1]$. We also have that $\tau(M) \in \Lambda$ as the middle term of an Auslander-Reiten triangle is not trivial by 4.3. Therefore $\tau(M), \tau(M)[1] \in \Lambda$. Thus the $[1]$ shift acts on the component.

Let $A = \bigoplus_{i=1}^n P_i$ be a decomposition of $A$ into indecomposable projective summands $P_i$. Let $C$ be an element of $\Lambda$. As $[1]$ acts on $\Lambda$ we can assume without loss of generality that there exists a non zero map $f$ in $D^b(A)$ from $P_i$ to $C$ for some $1 \leq i \leq n$. Therefore $P_i \in \Lambda$ and as $A$ is indecomposable we have $P_j \in \Lambda$ for all $1 \leq j \leq n$. For all indecomposable elements $X$ in $K^{-b}(\mathcal{P})$ there is an $s_x \in \mathbb{Z}$ such that there is a non-zero map $P_i \to X[s_x]$. Therefore $X[s_x] \in \Lambda$ using the first part of the proof. As the $[1]$ shift acts on the component, every indecomposable complex of $D^b(A)$ belongs to $\Lambda$. Therefore $\Lambda$ is the Auslander-Reiten quiver. The stalk complex of an indecomposable $A$-module $U$ is in $\Lambda$. A stalk complex in $D^b(A)$ can be identified with a complex in $K^b(\mathcal{P})$ if and only if the stalk has a finite projective resolution. Therefore $A$ has finite global dimension as all
indecomposable complexes in $K^{-b}(P)$ are in $K^b(P)$. As the dimension of the indecomposable $A$-modules are bounded, we know by [ARS 1.5] that $A$ has finite representation type.

We can now determine finite components.

**Theorem 4.10** (finite components). Let $\Lambda$ be a finite Auslander-Reiten component of $D^b(A)$ such that all elements in $\Lambda$ belong to $K^b(P)$. Then $A$ is simple and $\Lambda$ is isomorphic to $A_1$.

*Proof.* Suppose that $A$ is not simple. As $\Lambda$ is a finite component and all vertices of $\Lambda$ are in $K^b(P)$, the translation $\tau$ is an automorphism on $\Lambda$. Therefore the component $\Lambda$ is stable and bounded. By 4.9 the shift $[1]$ acts on $\Lambda$ which is a contradiction, as $\Lambda$ contains only finitely many vertices. Therefore $A$ is simple and $\Lambda$ is isomorphic to $A_1$ by 4.1. □

If $A$ has finite global dimension then $D^b(A) \cong K^b(P)$, and 4.10 gives the next corollary.

**Corollary 4.11.** Let $A$ be a finite-dimensional indecomposable algebra of finite global dimension. Suppose that the Auslander-Reiten quiver of $D^b(A)$ has a finite component $\Lambda$, then $A$ is simple.

In the case of $\text{Comp}^{-b}(P)$ there are no finite components.

**Corollary 4.12.** There is no finite Auslander-Reiten component $\Lambda$ of the Auslander-Reiten quiver of $\text{Comp}^{-b}(P)$, such that all elements in $\Lambda$ are in $\text{Comp}^b(P)$.

*Proof.* Suppose $\Lambda$ is a finite component such that all elements in $\Lambda$ are in $\text{Comp}^b(P)$. By 3.3 we have that the corresponding Auslander-Reiten component in $D^b(A)$ is finite. Therefore $A$ is simple by 4.10. But by 4.2 we have $\Lambda \cong A_1^\infty$ which is a contradiction to the finiteness of $\Lambda$. □

In analogy to the theory of algebras, we say that $D^b(A)$ has finite representation type if and only if $D^b(A)$ has finitely many indecomposable complexes up to shift. We call an indecomposable complex $X$ in a stable Auslander-Reiten component shift periodic, if there are $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $\tau^n(X) = X[m]$. If we have shift periodic modules in a component, we can construct subadditive functions.

Let $\Lambda$ and $C$ be two Auslander-Reiten components of $D^b(A)$. Then we denote by $\text{Hom}_{D^b(A)}(C, \Lambda[i])$ the union of $\text{Hom}_{D^b(A)}(M, N[i])$ for all objects $N \in \Lambda$ and $M \in C$. 
Theorem 4.13. Let $C$ be a stable component of the Auslander-Reiten quiver of $D^b(A)$. Suppose there is a complex $X \in C$ that is shift periodic. Let $T$ be the tree class of $C$, then $T$ is a finite Dynkin diagram or $A_{\infty}$.

(a) If $T$ is a finite Dynkin diagram, then the Auslander-Reiten quiver is equal to $C$ and $D^b(A)$ has finite representation type.

(b) Suppose $Q$ is a stable component of the Auslander-Reiten quiver of $D^b(A)$, that is not a shift of $C$. If the set $\text{Hom}_{D^b(A)}(C, Q[i])$ or $\text{Hom}_{D^b(A)}(Q[i], C)$ is non zero for some $i \in \mathbb{Z}$, then the tree class of $Q$ is either Euclidean or infinite Dynkin.

Proof. Let $n \in \mathbb{N}$, $m \in \mathbb{Z}$ be such that $\tau^n(X) = X[m]$. We consider the following subadditive function for all $M \in C$

$$d(M) := \sum_{i=1}^{n} \sum_{j \in \mathbb{Z}} \dim k \text{Hom}_{D^b(A)}(\tau^i(X), M[j]).$$

This function takes finite values as for two complexes $L, N \in K^b(P)$ the set $\text{Hom}_{K^b(P)}(L, N[s])$ is non zero only for finitely many values of $s \in \mathbb{Z}$.

Let $\tau(D) \xrightarrow{f} B \xrightarrow{g} D \rightarrow \tau(D)[-1]$ be an Auslander-Reiten triangle. By (4.3) we have $d(\tau(D)) + d(D) \leq d(B)$. As $C$ contains the element $X$, the function $d$ is a subadditive function that is not additive by (4.3) (3). Therefore $T$ is a finite Dynkin diagram or $A_{\infty}$ by [HPR, p.289]. The shift $[m]$ induces an automorphism of finite order on $T$ as $[m]$ commutes with $\tau$. Then we have for all complexes $M \in C$ that $M[l] = \tau^t(M)$ for some $t \in \mathbb{N}$ and $l \in \mathbb{Z}$. It follows that on each $\tau$-orbit of $C$, the length function $l_p$ is bounded. If $T$ is finite there are finitely many $\tau$-orbits and therefore $C$ is bounded. Then $C$ is the only component of the Auslander-Reiten quiver by (4.9). Furthermore $D^b(A)$ has only finitely many complexes up to shifts.

Without loss of generality, let $\text{Hom}_{D^b(A)}(X, L) \neq 0$ for some $L \in Q$. We define $d(M)$ as above for all $M \in Q$. Then $d$ is an additive function by (4.3) on $Q$ and therefore the tree class of $Q$ is Euclidean or infinite Dynkin. □

Note that if we consider the Auslander-Reiten quiver of $A$ then by [ARS, VII 2.1, VI 1.4] the Auslander-Reiten quiver has a finite component if and only if there is a component whose modules have bounded dimension. In this case $A$ has finite representation type and the Auslander-Reiten quiver consists of only one finite component. The stable Auslander-Reiten quiver has then tree class a finite Dynkin diagram.
The analogous theorem for the Auslander-Reiten quivers of $D^b(A)$ that have bounded components is given next.

**Theorem 4.14.** The following are equivalent:

1. The Auslander-Reiten quiver of $D^b(A)$ has a bounded component.
2. The representation type of $D^b(A)$ is finite.
3. The Auslander-Reiten quiver of $D^b(A)$ has a stable component whose tree class is finite Dynkin.

**Proof.** (1) $\implies$ (2)

Let $C$ be a bounded component. Let $M \in C$ be represented by an indecomposable complex in $\text{Comp}^b(P)$. There is an $n_0 \in \mathbb{Z}$ such that $M^{n_0} \neq 0$ and $M^i = 0$ for all $i > n_0$. Take an indecomposable summand $P$ of $M^{n_0}$. If $P[-n_0] \neq M$ there is a map $\psi : P[-n_0] \to M$ induced by the embedding of $P$ into $M^{n_0}$. This map is non-zero in $D^b(A)$, as $M$ does not have contractible direct summands. By the proof of 4.9 we have $P[-n_0] \in C$.

If we represent all elements in $C$ by indecomposable complexes in $\text{Comp}^b(P)$ then there is an $n \in \mathbb{N}$ such that $l_c$ takes values $\leq n$ for all elements in $C$. By 4.8 every chain of irreducible maps of length $> 2^n$ is therefore zero.

As $P[-n_0] \in C$ we know by 4.4 that there is a chain of irreducible maps of length at most $2^n$ that connects $P[-n_0]$ and $M$. There are only finitely many elements in the component $C$ that are connected to $P[-n_0]$ by a chain of irreducible maps of length $\leq 2^n$. Therefore there are only finitely many indecomposable complexes $L$ in $D^b(A)$ such that $L^{n_0}$ contains $P$ as a summand and $L^i = 0$ for $i > n_0$.

Therefore $D^b(A)$ has finite representation type.

(2) $\implies$ (3)

Let $D^b(A)$ have finite representation type. Let $N$ be an indecomposable $A$-module. Then the complexes $\sigma \geq n(pN)$ are indecomposable for all $-n \in \mathbb{N}$. If $N$ has infinite projective dimension, then they are pairwise non-isomorphic in $K^{-b}(P)$ up to shift. The same holds for a module with infinite injective dimension if we consider $\sigma \leq n(iN)$ for $n \in \mathbb{N}$. Therefore $A$ has finite global dimension and all Auslander-Reiten components are bounded and stable. If there is a finite component then by 4.10 the Auslander-Reiten component consists of copies of $A_1$. Otherwise the Auslander-Reiten quiver has only one component by 4.9. This component then contains a shift periodic module. Therefore the tree class is finite Dynkin or $A_\infty$ by 4.13. As $[1]$ acts as the identity on $A_\infty$ such a component can only occur if the representation type
of $D^b(A)$ is not finite. Therefore the Auslander-Reiten quiver consists of one component $\mathbb{Z}[T]$ where $T$ is a finite Dynkin diagram.

(3) $\implies$ (1)

Suppose now that $D^b(A)$ has a stable Auslander-Reiten component whose tree class $T$ is a finite Dynkin diagram. Then $T$ is either $A_n$ for $n \geq 2$, $D_n$ for $n \geq 3$, $E_6$, $E_7$ or $E_8$. We index the vertices in $\mathbb{Z}[T]$ by pairs $(t, i)$ where $i \in \mathbb{Z}$ denotes the $i$-th copy of $T$ and $t$ denotes the vertex of $T$. By 4.6 the function $l_p$ is subadditive on stable Auslander-Reiten components.

We assume that $l_p$ is not additive for only finitely many Auslander-Reiten triangles. Then we can choose an $l \in \mathbb{Z}$ such that $l_p$ is additive for all Auslander-Reiten triangles with vertices $(t, j)$ where $j > l$ and $t$ any vertex of $T$. We denote $x_{t,i} := l_p((t, i))$. Let the values $x_{t,j}$ be given for a fix $j > l$ and all vertices $t$ of $T$.

If $T = A_n$ then we can calculate the values of $x_{t,j+1}$ from the left to the right as follows:

\[ x_{1,j} \quad x_{2,j} \quad \cdots \quad x_{n-1,j} \quad x_{n,j} \]
\[ x_{2,j} - x_{1,j} \quad x_{3,j} - x_{1,j} \quad \cdots \quad x_{n,j} - x_{1,j} \]

Clearly this gives a contradiction as $l_p$ cannot be additive on the Auslander-Reiten triangles ending in $(n, j+1)$. Therefore $l_p$ is not additive for infinitely many Auslander-Reiten triangles.

If $T = D_n$ we have the following

\[ x_{1,j} \quad x_{2,j} \quad \cdots \quad x_{n-2,j} \quad x_{n-1,j} \]
\[ x_{2,j} - x_{1,j} \quad x_{3,j} - x_{1,j} \quad \cdots \quad x_{n-1,j} - x_{1,j} \]
\[ x_{n,j} - x_{1,j}. \]

Then the values $x_{n,i}$ are strictly decreasing for strictly increasing $i > j$. This is a contradiction as they have to be positive integers for all $i \in \mathbb{Z}$.

For $E_6$, $E_7$ and $E_8$ we consider the following diagram.
In the case of $E_6$ we have $x_{6,j+1} = x_{4,j} - x_{1,j}$. If we consider $E_7$ we have $x_{6,j+1} = x_{7,j} + x_{4,j} - x_{1,j}$ and $x_{7,j+1} = x_{4,j} - x_{1,j}$. Finally for $E_8$ we have $x_{6,j+1} = x_{7,j} + x_{4,j} - x_{1,j}$, $x_{7,j+1} = x_{8,j} - x_{1,j}$ and $x_{8,j+1} = x_{4,j} - x_{1,j}$.

In the case $E_6$ we have by the same argument that $x_{6,j+4} = -x_{1,j}$, in the case $E_7$ we have that $x_{3,j+20} = -x_{3,j} + x_{4,j}$ and for $E_8$ we have $x_{1,j+15} = -x_{1,j}$. Those are negative values as $-x_{3,j} + x_{4,j} = -x_{4,j} - 1$ and we obtain a contradiction to the assumption that $l_p$ is additive on all but finitely manyAuslander-Reiten triangles in the component.

As $l_p$ is not additive for infinitely many Auslander-Reiten triangles in $C$, there have to be infinitely many complexes that are homotopic to zero in the Auslander-Reiten component of $Comp^b(P)$ that is associated to $C$ by $4.6$.

As there are only finitely many indecomposable complexes homotopic to zero in $Comp^b(P)$ up to shift by $2.3$ we deduce that a shift $[m]$ induces an automorphism on $\mathbb{Z}[T]$ for some $m \in \mathbb{N}$. Therefore $\mathbb{Z}[T]$ is a bounded component. \hfill \Box

Note that we only require one component to be bounded or to have tree class a finite Dynkin diagram in order to deduce that the representation type of $D_b(A)$ is finite.

We can describe the Auslander-Reiten quiver and derived category more precisely in the case of the previous theorem. Note also that not all bounded components need to be finite as it is the case for the Auslander-Reiten quiver of an algebra.

**Theorem 4.15.** Let one of the conditions of $4.14$ be true. Then $A$ is either simple and the Auslander-Reiten quiver of $D_b(A)$ consists of countably many copies of $A_1$ or the Auslander-Reiten quiver consists of one component $\mathbb{Z}[D]$
where $D$ is a finite Dynkin diagram and $D \neq A_1$. In the second case $A$ is derived equivalent to $kD$.

Proof. Suppose the bounded component is finite, then the first case holds by 4.3. If the bounded component is not finite then by 4.9 the Auslander-Reiten quiver consists of only one component which needs to be $\mathbb{Z}[D]$ for $D$ a finite Dynkin diagram and $D \neq A_1$ by 4.14. As $D^b(A)$ is of finite representation type, it is discrete in the sense of [Y, 1.1]. By [BGS, Theorem A,B] the algebra $A$ is derived equivalent to $kQ$ where $Q$ is a finite Dynkin diagram. By [H1, I.5.5] we have $\bar{Q} = \bar{D}$. Then $D^b(kQ) \cong D^b(kD)$ by [H1, I.5.6], which proves the theorem. □

Now we know that there is only one Auslander-Reiten component if $D \neq A_1$. Let $\Gamma_0$ be the set of isomorphism classes of indecomposable objects in $D^b(A)$. Then we call $D^b(A)$ locally finite if $\sum_{M \in \Gamma_0} \dim_k \text{Hom}_{D^b(A)}(M,N)$ is finite for all $N \in \Gamma_0$. So using [XZ, 3.1.6] we have that $D^b(A)$ is locally finite if and only if $A$ is derived equivalent to an hereditary algebra of finite representation type.

References

[A] M. Auslander, *Functors and morphisms determined by objects* Lecture Notes in Pure Appl. Math. Vol. 37, Dekker, New York, 1978.

[ARS] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997.

[ASS] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras*, London Mathematical Society Student Texts, 65. Cambridge Univ. Press, Cambridge, 2006.

[B] D. Benson, *Representations and cohomology 1*, Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, 1998.

[BGS] G. Bobiński, C. Geiß and A. Skowroński, *Classification of discrete derived categories*, Cent. Eur. J. Math. 2 (2004), no. 1, 19-49.

[H1] D. Happel, *Triangulated Categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[H2] D. Happel, *Auslander-Reiten Triangles in Derived categories of Finite-Dimensional Algebras*, Proc. Amer. Math. Soc. 112 (1991), no. 3, 641-648.

[HKR] D. Happel, B. Keller and I. Reiten *Bounded derived categories and repetitive algebras*, J. Algebra 319 (2008), no. 4, 1611-1635.
[HPR] D. Happel, U. Preiser and C. M. Ringel, *Vinberg’s characterization of Dynkin diagrams using subadditive functions with application to $D_T^{-}$-periodic modules*. Representation Theory II, Lecture Notes in Math. 832 (1980), Springer, Berlin, 1980, 280-294.

[K] B. Keller, *Derived categories and tilting*, lecture notes, to be found at [http://people.math.jussieu.fr/keller/publ/index.html](http://people.math.jussieu.fr/keller/publ/index.html).

[Kr] H. Krause, *The stable derived category of a Noetherian scheme* Compos. Math. 141 (2005), no. 5, 1128-1162.

[Ri] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), no. 2, 199-224.

[Wei] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

[W] W. Wheeler, *The triangulated structure of the stable derived category*, J. Algebra 165 (1994), no. 1, 23-40.

[VR] M. Van den Bergh and I. Reiten, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. 15 (2002), no. 2, 295-366.

[V] D. Vossieck, *The algebras with discrete derived category* J. Algebra 243 (2001), no. 1, 168-176.

[XZ] J. Xiao, B. Zhu, *Locally finite triangulated categories*, J. Algebra 290 (2005), no. 2, 473-490.

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