ON THE ISOMETRIC DEFORMATION OF SURFACES VIA THE BÄCKLUND TRANSFORMATION

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Abstract. In trying to generalize Bianchi’s Bäcklund transformation of quadrics to Bäcklund transformations of isometric deformations of other (classes of) surfaces, we investigate basic features of the isometric deformation of surfaces via the Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed).

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1. Introduction

The classical problem of finding the isometric deformations of surfaces (see Eisenhart [7]) was stated in 1859 by the French Academy of Sciences as

To find all surfaces applicable to a given one.

Probably the most successful researcher of this problem is Bianchi, who in 1906 in [1] solved the problem for quadrics by introducing the Bäcklund transformation of surfaces isometric to quadrics and the isometric correspondence provided by the Ivory affine transformation. By 1909 Bianchi had a fairly complete treatment [2] and in 1917 he proved in [3] the rigidity of the Bäcklund transformation of isometric deformations of quadrics in the case of auxiliary surface plane: the only Bäcklund transformation with defining surface and auxiliary surface plane appears as the singular Bäcklund transformation of isometric deformations of quadrics.

In [6] we have managed to improve Bianchi’s result [3] to arbitrary auxiliary surface: the only Bäcklund transformation with defining surface is Bianchi’s Bäcklund transformation of isometric deformations of quadrics.

The question remained to consider isometric deformation of surfaces via the Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed) and without the assumption of defining surface; it is the purpose of this note to address this question.

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We shall consider the complexification

$$(\mathbb{C}^3, \langle \cdot, \cdot \rangle), \langle x, y \rangle := x^T y, \ |x|^2 := x^T x, \ x, y \in \mathbb{C}^3$$

of the real 3-dimensional Euclidean space; in this setting surfaces are 2-dimensional objects of $\mathbb{C}^3$ depending on two real or complex parameters.

Isotropic (null) vectors are those vectors of length 0; since most vectors are not isotropic we call a vector simply vector and we shall emphasize isotropic for isotropic vectors. The same denomination will apply in other settings: for example we call quadric a non-degenerate quadric (a quadric, a general nature as a consequence of an infinitesimal law via discretization (the converse is clear); the classical geometers were led to consider the Bäcklund transformation (a projectively equivalent to the complex unit sphere).

Consider Lie’s viewpoint: one can replace a surface $x \subset \mathbb{C}^3$ with a 2-dimensional distribution of contact elements (pairs of points and planes passing through those points; the classical geometers call them facets): the collection of its tangent planes (with the points of tangency highlighted); thus a contact element is the infinitesimal version of a surface (the integral element $(x, dx)[p]$ of the surface). Conversely, a 2-dimensional distribution of contact elements is not always the collection of the tangent planes of a surface (with the points of tangency highlighted), but the condition that a 2-dimensional distribution of contact elements is integrable (that is it is the collection of the tangent planes of a leaf (sub-manifold)) does not distinguish between the cases when this sub-manifold is a surface, curve or point, thus allowing the collapsing of the leaf.

A 3-dimensional distribution of contact elements is integrable if it is the collection of the tangent planes of an 1-dimensional family of leaves.

Tworollable (applicable or isometric) surfaces can be rolled (applied) one onto the other such that at any instant they meet tangentially and with same differential at the tangency point.

**Definition 1.1.** The rolling of two isometric surfaces $x_0, x \subset \mathbb{C}^3$ (that is $|dx_0|^2 = |dx|^2$) is the surface, curve or point $(R, t) \subset O_3(\mathbb{C}) \times \mathbb{C}^3$ such that $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$.

The rolling introduces the flat connection form (it encodes the difference of the second fundamental forms of $x_0$, $x$ and it being flat encodes the difference of the Gauß-Codazzi-Mainardi-Peterson equations of $x_0$, $x$).

**Definition 1.2.** Consider an integrable 3-dimensional distribution of contact elements $\mathcal{F} = (p, m)$ centered at $p = p(u, v, w)$, with normal fields $m = m(u, v, w)$ and distributed along the surface $x_0 = x_0(u, v)$. If we roll $x_0$ on an isometric surface $x$ (that is $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$), then the rolled distribution of contact elements is $(Rp + t, Rm)$ and is distributed along $x$; if it remains integrable for any rolling, then the distribution is called integrable rolling distribution of contact elements.

Since by infinitesimal rolling in an arbitrary tangential infinitesimal direction $\delta$ an initial contact element $\mathcal{F}$ which is common tangent plane to two isometric surfaces is replaced with an infinitesimally close contact element $\mathcal{F}'$ having in common with $\mathcal{F}$ the direction $\delta$, in the actual rolling problem we have contact elements centered on each other (the symmetric tangency configuration) and contact elements centered on another one $\mathcal{F}$ reflect in $\mathcal{F}$; note that we assumed a finite law of a general nature as a consequence of an infinitesimal law via discretization (the converse is clear); see Bianchi [3].

Thus for a theory of isometric deformation of surfaces with the assumption above we are led to consider, via rolling, certain 4-dimensional distributions of contact elements centered on the tangent planes of the considered surface $x_0$ and passing through the origin of the tangent planes (each point of each tangent plane is the center of finitely many contact elements) and their rolling counterparts on the isometric surface $x$.

After a thorough study of infinitesimal laws (in particular infinitesimal isometric deformations) and their iterations the classical geometers were led to consider the Bäcklund transformation (a finite law of a general nature) as a consequence of an infinitesimal law via discretization.
The Bäcklund transformation in the isometric deformation problem naturally appears by splitting the 4-dimensional distribution of contact elements above into an 1-dimensional family of 3-dimensional integrable rolling distributions of contact elements, thus introducing a spectral parameter \( z \) (the Bäcklund transformation is denoted \( B_z \)); each 3-dimensional distribution of contact elements is integrable (with leaves \( x^1 \)) regardless of the shape of the seed surface \( x^0 \).

The question appears if for the 4-dimensional distribution of contact elements above being integrable for any surface \( x \) isometric to \( x_0 \) and with a 2-dimensional family of leaves (in general surfaces), then it splits into an 1-dimensional family of 3-dimensional integrable rolling distributions of contact elements.

By imposing the initial collapsing ansatz of leaves of the 1-dimensional family of 3-dimensional integrable rolling distributions of contact elements to be a 2-dimensional family of curves (this ansatz individuates the defining surface \( x_0 \)) the 2-dimensional family of curves naturally splits into an 1-dimensional family of auxiliary surfaces \( x_z \) (1-dimensional families of curves) such that the contact elements centered on these auxiliary surfaces form a 3-dimensional integrable rolling distribution of contact elements whose integrability is independent of the shape of the seed surface \( x^0 \) isometric to the surface \( x_0^0 \subset x_0 \). For \( x_0 \) quadric the auxiliary surfaces \( x_z \) are ((isotropic) singular) confocal quadrics doubly ruled by collapsed leaves and with \( z \) being the spectral parameter of the confocal family (this includes in a general sense the (isotropic) planes of the pencil of (isotropic) planes through the axis of revolution for \( x_0 \) quadric of revolution (excluding (pseudo-)spheres) (in this case the actual isotropic singular quadric of the confocal family is the two isotropic planes of this pencil) or (isotropic) planes of the pencil of (isotropic) planes through an isotropic line (which is the actual isotropic singular quadric of its confocal family) for \( x_0 \) Darboux quadric with tangency of order 3 with the circle at infinity \( C(\infty) \)).

This collapsing ansatz allows us to simplify the denomination Bäcklund transformation of surfaces isometric to quadrics to Bäcklund transformation of quadrics; Bianchi’s Bäcklund transformation of quadrics is just the metric-projective generalization of Lie’s point of view on the Bäcklund transformation of the pseudo-sphere (one replaces ‘pseudo-sphere’ with ‘quadric’ and ‘circle’ with ‘conic’).

Note however that according to our note [6], there are no other instances of Bäcklund transformations with defining surface except Bianchi’s Bäcklund transformation of quadrics.

Bianchi considered the most general form of a Bäcklund transformation as the focal surfaces (one transform of the other) of a Weingarten congruence (congruence upon whose two focal surfaces the asymptotic directions correspond; equivalently the second fundamental forms are proportional). Note that although the correspondence provided by the Weingarten congruence does not give the applicability (isometric) correspondence, the Bäcklund transformation is the tool best suited to attack the isometric deformation problem via transformation, since it provides correspondence of the characteristics of the isometric deformation problem (according to Darboux these are the asymptotic directions), it is directly linked to the infinitesimal isometric deformation problem (Darboux proved that infinitesimal isometric deformations generate Weingarten congruences and Guichard proved the converse: there is an infinitesimal isometric deformation of a focal surface of a Weingarten congruence in the direction normal to the other focal surface; see Darboux [II, § 883-§ 924]) and it admits a version of the Bianchi Permutability Theorem for its second iteration.

1.1. Preliminaries and results from [6].

Let \( (u, v) \in D \) with \( D \) domain of \( \mathbb{R}^2 \) or \( \mathbb{C}^2 \) and \( x : D \to \mathbb{C}^3 \) be a surface.

For \( \omega_1, \omega_2 \in \mathbb{C}^3 \) valued 1-forms on \( D \) and \( a, b \in \mathbb{C}^3 \) we have

\[
a^T \omega_1 \wedge b^T \omega_2 = ((a \times b) \times \omega_1 + b^T \omega_1 a)^T \wedge \omega_2 = (a \times b)^T (\omega_1 \times \omega_2) + b^T \omega_1 \wedge a^T \omega_2; \]

in particular \( a^T \omega \wedge b^T \omega = \frac{1}{2} (a \times b)^T (\omega \times \omega) \).

Since both \( \times \) and \( \wedge \) are skew-symmetric, we have \( 2 \omega_1 \times \omega_2 = \omega_1 \times \omega_2 + \omega_2 \times \omega_1 = 2 \omega_2 \times \omega_1 \).

Consider the scalar product \( \langle \cdot, \cdot \rangle \) on \( M_3(\mathbb{C}) \) : \( \langle X, Y \rangle := \frac{1}{4} \text{tr}(X^T Y) \). We have the isometry
\[ \alpha : \mathbb{C}^3 \mapsto \mathbf{o}_3(\mathbb{C}), \quad \alpha \left( \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \right) = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix}, \quad x^T y = \langle \alpha(x), \alpha(y) \rangle = \frac{1}{2} \text{tr}(\alpha(x)^T \alpha(y)), \]

\[ \alpha(x \times y) = [\alpha(x), \alpha(y)] = \alpha(\alpha(x)y) = y \alpha^T - x \alpha^T, \quad \alpha(Rx) = R \alpha(x) R^{-1}, \quad x, y \in \mathbb{C}^3, \quad R \in \mathbf{o}_3(\mathbb{C}). \]

Let \( x \in \mathbb{C}^3 \) be a surface non-rigidly applicable (isometric) to a surface \( x_0 \subset \mathbb{C}^3 \):

\[ (x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0), \]

where \((R, t)\) is a sub-manifold in \( \mathbf{o}_3(\mathbb{C}) \times \mathbb{C}^3 \) (in general surface, but it is a curve if \( x_0, x \) are ruled and the rulings correspond under isometry or a point if \( x_0, x \) differ by a rigid motion). The sub-manifold \( R \) gives the rolling of \( x_0 \) on \( x \), that is if we rigidly roll \( x_0 \) on \( x \) such that points corresponding under the isometry will have the same differentials, \( R \) will dictate the rotation of \( x_0 \); the translation \( t \) will satisfy \( dt = -dRx_0 \).

For \((u, v)\) parametrization on \( x_0, x \) and outside the locus of isotropic (degenerate) induced metric of \( x_0 \), \( x \) we have \( N_0 := \frac{\partial x_0 \times \partial x_0}{|\partial x_0 \times \partial x_0|} \), \( N := \frac{\partial x \times \partial x}{|\partial x \times \partial x|} \) respectively positively oriented unit normal fields of \( x_0, x \) and \( R \) is determined by \( R = [\partial_u x \ \partial_v x \ N][\partial_u x_0 \ \partial_v x_0 \ \text{det}(R) N_0]^{-1} \); we take \( R \) with \( \text{det}(R) = 1 \); thus the rotation of the rolling with the other face of \( x_0 \) (or on the other face of \( x \)) is \( R' := R(I_3 - 2N_0N_0^T) = (I_3 - 2NN^T)R, \quad \text{det}(R') = -1 \).

With \( \omega := \alpha^{-1}(R^{-1}dR) = N_0 \times R^{-1}dRN_0 \) we have

\[ d\omega + \frac{1}{2} \omega \times \omega = 0, \quad \omega \times \wedge dx_0 = 0, \quad (\omega)^2 = 0. \]

Here we shall recall some results from [6].

Consider a surface \( x_0 = x_0(u, v) \subset \mathbb{C}^3 \) with unit normal field \( N_0 = N_0(u, v) \).

Consider a 3-dimensional distribution of contact elements with the symmetry of the tangency configuration, that is the contact elements are centered at \( x_0 + V, V = V(u, v, w), N_0^T V = 0, \quad du \wedge dv \wedge dw \neq 0 \) and have non-isotropic normal fields \( m = V \times N_0 + mN_0, \quad m = m(u, v, w) \subset \mathbb{C} \).

With \( \tilde{d} := \partial_u \cdot du + \partial_v \cdot dv + \partial_w \cdot dw = d + \partial_w \cdot dw \) if the distribution of contact elements is integrable and the rolled distribution remains integrable if we roll \( x_0 \) on an isometric surface \( x, (x, dx) = (R, t)(x_0, dx_0) \) (that is we replace \( x_0, V, m \) with \( x, RV, \tilde{d} \)), then along the leaves we have

\[ 0 = (Rm)^T \tilde{d}(RV + x) = m^T(\omega \times V + d(V + x_0) + \partial_w V dw), \]

or, assuming \( N_0^T(\partial_w V \times V) \neq 0 \),

\[ dw = \frac{N_0^T [V \times d(V + x_0)]}{N_0^T(\partial_w V \times V)} + m^T \frac{N_0^T(\omega \times N_0 + dN_0)}{N_0^T(\partial_w V \times V)}. \]

By applying the compatibility condition \( \tilde{d} \) to [1.3] and using the equation itself we get the integrability condition

\[ V \times \partial_w V \neq 0, \quad \frac{2[\partial_w V \times d(V + x_0)]^T \wedge (V \times dx_0)}{([\partial_w V \times V]^T) dx_0 \wedge dx_0} + (m^2 + |V|^2) K = 0, \]

\[ dm = -\partial_w m \frac{N_0^T [V \times d(V + x_0)]}{N_0^T(\partial_w V \times V)} + m \frac{N_0^T(\partial_w V \times dV)}{N_0^T(\partial_w V \times V)}, \quad du \wedge dv \wedge dw \neq 0. \]

Assume \( m \neq 0 \); excluding the case \( x_0 \) developable we can take \( m^2 \) from the first equation of [1.5] and replace it into the second one; applying the compatibility condition \( d \) to this equation and using the equation itself we get a relation, which coupled to differentiating the first equation of [1.5] with respect to \( w \) gives
Consider the general case \( m^T \partial_u V(m \times V) \times N_0 \neq 0 \) and all leaves of all rolled distributions of contact elements are isometric to the same surface \( y = y(u_1, v_1) = y(u_1(u, v, w), v_1(u, v, w)) \); if for a particular isometric deformation of \( x_0 \) one knows the isometric correspondence of all leaves to the surface \( y \), then one finds the isometric correspondence to \( y \) of all rolled leaves (including the case of leaves with degenerate metric) by composing the rolling of the particular leaves on \( y \) with the inverse of the rolling of \( x_0 \) on \( x \). For \( u_1, v_1 = ct \) from the three independent variables \( u, v, w \) only one remains independent, thus giving the submersion from the integrable rolling distribution

\[
(1.6) \quad \frac{2(\partial_u V \times dV)^T \wedge (\partial_u V \times dx_0)}{(\partial_u V \times V)^T(dx_0 \wedge dx_0)} + (m \partial_u m + V^T \partial_u V)K = 0, \quad du \wedge dv \wedge dw \neq 0
\]

and \([15] \) becomes

\[
\begin{align*}
\partial_w & N_0^T[\partial_u V \times d(V + x_0)] \wedge N_0^T(V \times dx_0) - \frac{N_0^T(\partial_u V \times dV)}{N_0^T(\partial_u V \times V)} \wedge N_0^T(\partial_u V \times dx_0) = 0, \\
\frac{1}{2} d(2\frac{[\partial_u V \times d(V + x_0)]^T \wedge (V \times dx_0)}{K(\partial_u V \times V)^T(dx_0 \wedge dx_0)} + |V|^2) = -\frac{2[\partial_u V \times d(V + x_0)]^T \wedge (\partial_u V \times dx_0)}{K(\partial_u V \times V)^T(dx_0 \wedge dx_0)} + V^T \partial_u V) \\
& \left( \frac{[\partial_u V \times d(V + x_0)]}{N_0^T(\partial_u V \times V)} \right) \left( \frac{1}{2}[\partial_u V \times d(V + x_0)]^T \wedge (V \times dx_0) + |V|^2 \right) N_0^T(\partial_u V \times dV) \right) = 0, \\
\end{align*}
\]

(1.7)

applying \( \partial_u \) to the second equation of \([15] \) and using the first one we get another second order equation in \( V \).

From \([16] \) we get the Weingarten congruence property of \( x \) and any leaf of the 3-dimensional integrable rolling distribution of contact elements with the symmetry of the tangency configuration (and thus Bäcklund transformation according to Bianchi’s definition).

We consider now the question whether the leaves of a 3-dimensional integrable rolling distribution of contact elements are isometric deformations of surfaces; we exclude the case \( m \) isotropic because all leaves are isotropic developables with degenerate 2-dimensional metric.

Consider a surface \( x_0 = x_0(u, v) \subset \mathbb{C}^3 \) with unit normal field \( N_0 = N_0(u, v) \).

Consider a 3-dimensional distribution of contact elements distributed along \( x_0 \), that is the contact elements are centered at \( x_0 + V \), \( V = V(u, v, w) \), du \wedge dv \wedge dw \neq 0 \) and have normal fields \( m = m(u, v, w) \subset \mathbb{C}^3 \).

With \( \tilde{d} \cdot := \partial_u \cdot du + \partial_v \cdot dv + \partial_w \cdot dw = d \cdot + \partial_w \cdot dw \) if the distribution of contact elements is integrable and the rolled distribution remains integrable if we roll \( x_0 \) on an isometric surface \( x, (x, dx) = (R, t)(x_0, dx_0) \) (that is we replace \( x_0, V, m \) with \( x, RV, Rm \)), then along the leaves we have

\[
(1.8) \quad 0 = (Rm)^T \tilde{d}(RV + x) = m^T(\omega \times V + d(V + x_0) + \partial_w V dw).
\]

Since we shall not need the integrability condition of this general integrable rolling distribution of contact elements, we shall not derive it.

The leaves for a particular isometric deformation \( x_0 \) are isometric to the same surface or to an 1-dimensional family of surfaces.

Since we already have a correspondence between the contact elements of leaves given by rolling, it is natural to require that the isometric correspondence is independent of the shape of \( x \).

However, for \( (m \times V) \times N_0 \neq 0 \) the distribution of contact elements into leaves changes with the shape of \( x \); thus all leaves of all rolled distributions of contact elements must be isometric to the same surface (and we have a submersion from the 3-dimensional integrable rolling distribution of contact elements to the distribution of tangent planes of the fixed surface), or \( (m \times V) \times N_0 = 0 \) and we have only a 2-dimensional integrable rolling distribution of contact elements with the leaf having metric independent of the shape of \( x \).

Consider the general case \( m^T \partial_u V(m \times V) \times N_0 \neq 0 \) and all leaves of all rolled distributions of contact elements are isometric to the same surface \( y = y(u_1, v_1) = y(u_1(u, v, w), v_1(u, v, w)) \); if for a particular isometric deformation of \( x_0 \) one knows the isometric correspondence of all leaves to the surface \( y \), then one finds the isometric correspondence to \( y \) of all rolled leaves (including the case of leaves with degenerate metric) by composing the rolling of the particular leaves on \( y \) with the inverse of the rolling of \( x_0 \) on \( x \). For \( u_1, v_1 = ct \) from the three independent variables \( u, v, w \) only one remains independent, thus giving the submersion from the integrable rolling distribution
of contact elements to the distribution of tangent planes of \( y \) (equivalently we count each contact element \((y,dy)\) with simple \( \infty \) multiplicity).

The function \( w = w(u,v,c) \) is given by the integration of (1.8); thus a-priori \( w \) depends also on \( \omega \) and we have

\[
|\{J_3 - \frac{\partial_w V m^T}{m^T \partial_w V}\}[\omega \times V + d(V + x_0)]|^2 = |dy - \frac{m^T[\omega \times V + d(V + x_0)]}{m^T \partial_w V}\partial_w|^2, \\
\forall \omega \text{ satisfying } (1.3).
\]

(1.9)

The leaves are isometric to different regions of \( y \) because the constant \( c \) in \( w = w(u,v,c) \) changes for \( \omega \) fixed and for different \( \omega \) \( w \) changes; however in order to be determined by (1.8), \( w \) is not allowed to be linked to \( \omega \) by any other relation, either functional (as a-priori (1.9) is) or differential; thus in (1.9) \( \omega \) cancels independently of \( w \) and outside \( w \) we can replace \( \omega \) with any other \( \hat{\omega} \) satisfying (1.3).

After some manipulation of these premises we get the tangency configuration \( V^TN_0 = 0 \) and

(1.10) \[ \tilde{dy} = R_1(J_3 - \frac{N_0 m^T}{m^T N_0}) d(V + x_0), \quad R_1 \subset O_3(\mathbb{C}), \quad du \land dv \land dw \neq 0. \]

Note that since along the leaves we have \( m^T \tilde{d}(V + x_0) = 0 \), \( R_1 \) is the rotation of the rolling of the leaves on \( y \); if we replace \( x_0 \) with an isometric surface \( x = Rx_0 + t \), then \( R_1 \) is replaced with \( R_1R^{-1} \).

Imposing the compatibility condition \( R_1^{-1}\tilde{d} \) on (1.10) we get

(1.11) \[ 0 = [R_1^{-1}\tilde{d}R_1(J_3 - \frac{N_0 m^T}{m^T N_0}) - \frac{\partial N_0 m^T}{m^T N_0}] \land \tilde{d}(V + x_0); \]

with \( R_1^{-1}\tilde{d}R_1 =: \Omega_j du + \Omega_2 dv + \Omega_3 dw \) this constitutes a linear system of 9 equations in the 9 entries of \( \Omega_j \), \( j = 1,2,3 \) with the rank of the matrix of the system being 6, so the rank of the augmented matrix of the system must be also 6 and the solution \( R_1^{-1}\tilde{d}R_1 \) must also satisfy the compatibility condition

(1.12) \[ \tilde{d}(R_1^{-1}\tilde{d}R_1) + \frac{1}{2}[R_1^{-1}\tilde{d}R_1, \land R_1^{-1}\tilde{d}R_1] = 0; \]

these are the necessary and sufficient conditions on the 3-dimensional tangential integrable rolling distribution of contact elements in order to obtain isometric correspondence of leaves of a general nature.

From (1.10) \( R_1 \) is determined modulo the rotation of the rolling of \( y \) on an isometric surface, so from the three constants introduced in the solution of (1.11) (which are actually functions of three variables \( u, v, w \)) satisfying (1.12) we must get dependence on two functions of a variable related to the isometric deformation problem.

By applying, if necessary, a change of variables \( w = w(\hat{w}, u, v) \) we have \( N_0^T[d(V + x_0) \times \land d(V + x_0)] \neq 0 \) and the above considered linear system is consistent for

(1.13) \[ 2N_0^T[\partial_w V \times d(V + x_0)] \frac{d(\frac{w}{m^T N_0} \land d(V + x_0))}{\frac{N_0}{N_0}[d(V + x_0) \times \land d(V + x_0)]} = 0, \quad du \land dv \land dw \neq 0; \]

if we further assume the symmetric tangency configuration \( m^TV = 0 \), then by using (1.5) (1.13) is satisfied.
1.2. The main program. Our purpose is to generalize Bianchi’s Bäcklund transformation of quadrics to Bäcklund transformation of isometric deformations of other (classes of) surfaces.

The question remained to consider isometric deformations of surfaces via the Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed) and without the assumption of defining surface.

Similarly to the quadrics case, we should get

- (Existence of Bäcklund transformation) necessary and sufficient conditions on the metric of the seed surface in order to get Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed). Once these conditions are satisfied, a certain differential system subjacent to such Bäcklund transformations introduces a constant (denoted \( z \) in \( B_z \)) and a second constant is introduced by finding the leaves of the integrable rolling distribution of contact elements.

If (classes of) surfaces other than quadrics and Bäcklund transformation with defining surface are found to satisfy these requirements, then we continue the program with

- (Inversion of the Bäcklund transformation) the metric of the leaves satisfies the necessary and sufficient conditions in order to get Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed); thus the seed can be considered leaf and the leaf seed.

- (Bianchi Permutability Theorem) If \( x^1 = B_{z_1}(x^0) \), \( x^2 = B_{z_2}(x^0) \), then one can find only by algebraic computations a surface \( B_{z_1}(x^2) = x^3 = B_{z_2}(x^1) \); thus \( B_{z_2} \circ B_{z_2} = B_{z_1} \circ B_{z_1} \) and once all B transforms of the seed \( x^0 \) are found, the B transformation can be iterated using only algebraic computations.

\[
\begin{array}{c}
  x^2 \\
  B_{z_1} \\
  # \\
  B_{z_2} \\
  x^0 \\
  B_{z_1} \\
  x^1
\end{array}
\]

- (Existence of 3-Möbius moving configurations) If \( x^1 = B_{z_1}(x^0) \), \( x^2 = B_{z_2}(x^0) \), \( x^4 = B_{z_3}(x^0) \) and by use of the Bianchi Permutability Theorem one finds \( B_{z_1}(x^2) = x^6 = B_{z_3}(x^4) \), \( B_{z_2}(x^4) = x^5 = B_{z_2}(x^1) \), \( B_{z_2}(x^1) = x^3 = B_{z_1}(x^2) \), \( B_{z_2}(x^3) = x^7 = B_{z_2}(x^5) \), \( B_{z_1}(x^6) = x'''' = B_{z_3}(x^3) \), \( B_{z_2}(x^5) = x''' = B_{z_1}(x^6) \), then \( x''' = x''' = x''' = x''' = x''' \); thus once all B transforms of the seed \( x^0 \) are found, the B transformation can be further iterated using only algebraic computations.

Out of this program we were able to partially fulfill the requirements of the first item: we found necessary and sufficient conditions on \( V \), \( \mathbf{m} \) (and their derivatives) and the metric of the seed surface \( x_0 \) which using (1.5) and adjoined to (1.7) constitutes a differential system on \( V \) (here compatibility conditions may impose conditions on the metric of the seed surface).

7
2. The Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed)

From (1.7) with \( m = V \times N_0 + mN_0 \), \( R_1^{-1}\hat{d}R_1 =: \alpha(\hat{\omega}_u du + \hat{\omega}_v dv + \hat{\omega}_w dw) = \alpha(\hat{\omega} + \hat{\omega}_v dw) \) and using (1.4) and (1.5) (in particular \(- \frac{1}{m} \omega^T (\partial_a V \times V) \times \omega^T N_0^T [\partial_a V \times d(V + x_0)] - \frac{1}{m} \omega^T N_0^T (dN_0 \times \wedge dN_0) = \frac{1}{2m} N_0^T [d(V + x_0) \times \wedge d(V + x_0)] + m N_0^T (dN_0 \times \wedge dN_0) \)) we get

\[
N_0^T [\hat{\omega} \times \wedge d(V + x_0)] - dN_0^T \wedge dV + \frac{1}{2m} N_0^T [d(V + x_0) \times \wedge d(V + x_0)] + \frac{m}{2} N_0^T (dN_0 \times \wedge dN_0) = 0, \\
-\hat{\omega}_T N_0 \wedge d(V + x_0) \times N_0 + \hat{\omega}_N \times N_0 \wedge \frac{N_0^T[V \times d(V + x_0)]}{m} - \frac{1}{2} N_0^T (dN_0 \times \wedge dN_0) N_0 \times V \\
+ dN_0 \wedge \frac{N_0^T[V \times d(V + x_0)]}{m} = 0, \\
N_0^T [\hat{\omega}_w \times d(V + x_0)] - N_0^T (\hat{\omega} \times \partial_w V) + dN_0^T \partial_w V + \frac{N_0^T[\partial_w V \times d(V + x_0)]}{m} = 0, \\
-\hat{\omega}_w N_0 d(V + x_0) \times N_0 + \hat{\omega}_w \times N_0 \wedge \frac{N_0^T[V \times d(V + x_0)]}{m} - \hat{\omega}^T N_0 N_0 \times \partial_w V \\
+ \hat{\omega}_N + dN_0 N_0^T (\partial_w V \times V) = 0.
\]

With \( \mathcal{M} := -\frac{1}{2m} + \frac{dN_0^T N_0 d(V + x_0) \times \wedge d(V + x_0)}{N_0^T [d(V + x_0) \times \wedge d(V + x_0)]} \), \( \mathcal{U} := \partial_a (V + x_0) \), \( \mathcal{V} := \partial_v (V + x_0) \) we have from the first equation of (2.1)

\[
\hat{\omega}_u = (c_1 + \mathcal{M})(N_0 \times \mathcal{U}) \times N_0 + c_2 (N_0 \times \mathcal{V}) \times N_0 + c_3 N_0, \\
\hat{\omega}_v = c_4 (N_0 \times \mathcal{U}) \times N_0 - (c_1 - \mathcal{M})(N_0 \times \mathcal{V}) \times N_0 + c_5 N_0;
\]

and multiplying the second equation of (2.1) on the left with \( \mathcal{U}^T \), \( \mathcal{V}^T \) we get

\[
c_3 = c_2 \frac{N_0^T(V \times \mathcal{V})}{m} + (c_1 - \mathcal{M}) \frac{N_0^T(V \times \mathcal{U})}{m} \\
+ \mathcal{U}^T - \frac{N_0^T (\partial_a N_0 \times \partial_v N_0) N_0 \times V + \partial_a N_0 N_0^T(V \times \mathcal{U})}{m} \frac{\partial_a N_0 N_0^T(V \times \mathcal{U})}{m}.
\]

\[
c_5 = -(c_1 + \mathcal{M}) \frac{N_0^T(V \times \mathcal{V})}{m} + c_4 \frac{N_0^T(V \times \mathcal{U})}{m} \\
+ \mathcal{V}^T - \frac{N_0^T (\partial_a N_0 \times \partial_v N_0) N_0 \times V + \partial_a N_0 N_0^T(V \times \mathcal{V})}{m} \frac{\partial_a N_0 N_0^T(V \times \mathcal{U})}{m}.
\]

With \( \hat{\omega}_w =: c_6 (N_0 \times \mathcal{U}) \times N_0 + c_7 (N_0 \times \mathcal{V}) \times N_0 + c_8 N_0 \) from the third equation of (2.1) we get

\[
c_6 = -c_4 N_0^T(\partial_w V \times \mathcal{U}) - (c_1 - \mathcal{M}) N_0^T(\partial_a V \times V) + \partial_v N_0 N_0^T \partial_w V + \frac{N_0^T(\partial_a V \times \mathcal{V})}{m}.
\]

\[
c_7 = (c_1 + \mathcal{M}) N_0^T(\partial_w V \times \mathcal{U}) + c_2 N_0^T(\partial_v V \times \mathcal{V}) + \partial_v N_0^T \partial_w V + \frac{N_0^T(\partial_a V \times \mathcal{U})}{m}.
\]

Multiplying the last equation of (2.1) on the left with \( \mathcal{U}^T \), \( \mathcal{V}^T \) we get four scalar equations; from one of them we get

\[
c_8 = -c_1 - \mathcal{M} \frac{N_0^T(\partial_w V \times \mathcal{V})}{m} + c_7 \frac{N_0^T(V \times \mathcal{V})}{m} - c_5 \frac{N_0^T(V \times \mathcal{U})}{m} - \frac{\partial_a N_0^T \partial_a V \times \mathcal{U}}{N_0^T(\partial_a V \times \mathcal{U})}.
\]

and the remaining three are identically satisfied.

Thus
\[\tilde{\omega}_u = c_1((N_0 \times \mathcal{U}) \times N_0 + \frac{N_0^T(V \times \mathcal{U})}{m} N_0) + c_2((N_0 \times \mathcal{V}) \times N_0 + \frac{N_0^T(V \times \mathcal{V})}{m} N_0) + \mathcal{M}((N_0 \times \mathcal{U}) \times N_0 - \frac{N_0^T(V \times \mathcal{U})}{m} N_0)\]

\[\tilde{\omega}_v = c_4((N_0 \times \mathcal{U}) \times N_0 + \frac{N_0^T(V \times \mathcal{U})}{m} N_0) - c_1((N_0 \times \mathcal{V}) \times N_0 + \frac{N_0^T(V \times \mathcal{V})}{m} N_0) + \mathcal{M}((N_0 \times \mathcal{V}) \times N_0 - \frac{N_0^T(V \times \mathcal{V})}{m} N_0)\]

\[\tilde{\omega}_w = c_1 \frac{N_0^T(\partial_w V \times \mathcal{V})}{N_0^T(\mathcal{U} \times \mathcal{V})}((N_0 \times \mathcal{U}) \times N_0 + \frac{N_0^T(V \times \mathcal{U})}{m} N_0) + \mathcal{M}N_0^T(\partial_w V \times \mathcal{V})}\]

These can be written for short as

\[\tilde{\omega}_u = c_1\tilde{\omega}_{uc1} + c_2\tilde{\omega}_{uc2} + \tilde{\omega}_{u1},\]

\[\tilde{\omega}_v = -c_1\tilde{\omega}_{vc2} + c_3\tilde{\omega}_{vc1} + \tilde{\omega}_{v1},\]

\[\tilde{\omega}_w = c_1(\tilde{\omega}_{w1} + \tilde{\omega}_{w1}^2) + c_2\tilde{\omega}_{w2} - c_4\tilde{\omega}_{w1} + \tilde{\omega}_{w1},\]

where \(\tilde{\mathcal{U}} := \frac{N_0^T(\partial_w V \times \mathcal{U})}{N_0^T(V \times \mathcal{U})}, \tilde{\mathcal{V}} := \frac{N_0^T(\partial_w V \times \mathcal{V})}{N_0^T(V \times \mathcal{V})}\).

Note \(\tilde{\omega}_{w1} = -(\mathcal{M}\tilde{\mathcal{V}} + \frac{\partial_x N_0^T(\partial_w V \times \mathcal{U})}{m} N_0^T(V \times \mathcal{V}) + \frac{\partial_x N_0^T(\partial_w V \times \mathcal{V})}{m} N_0^T(V \times \mathcal{U}) + \frac{\partial_u N_0^T(\partial_w V \times \mathcal{U})}{m} N_0^T(\mathcal{U} \times \mathcal{V}) N_0).

Now (1.2) becomes

\[\partial_u \tilde{\omega}_u - \partial_w \tilde{\omega}_w + \tilde{\omega}_w \times \tilde{\omega}_u = 0,\]

\[\partial_u \tilde{\omega}_v - \partial_w \tilde{\omega}_w + \tilde{\omega}_w \times \tilde{\omega}_u = 0,\]

(2.3)

or

\[(\partial_u c_4 - \partial_v c_1)\tilde{\omega}_{uc1} - (\partial_u c_1 + \partial_v c_2)\tilde{\omega}_{vc2} + \partial_u \tilde{\omega}_{w1} - \partial_v \tilde{\omega}_{w1} + \tilde{\omega}_{w1} \times \tilde{\omega}_{w1} + c_1(-\partial_u \tilde{\omega}_{w1} - \partial_v \tilde{\omega}_{w1} + \tilde{\omega}_{w1} \times \tilde{\omega}_{w1}) + c_2(\partial_u \tilde{\omega}_{w1} + \tilde{\omega}_{w1} \times \tilde{\omega}_{w1}) - (c_1^2 + c_2^2)\tilde{\omega}_{uc1} \times \tilde{\omega}_{vc2} = 0,\]
\[ \partial_\nu c_1 (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) - \partial_\nu c_1 \bar{w}_{uc1} + (\partial_\nu c_2 \bar{V} - \partial_\nu c_2) \bar{w}_{uc2} - \partial_\nu c_1 \bar{U} \bar{w}_{uc1} + \partial_\nu \bar{w}_{u1} - \partial_\nu \bar{w}_{u1} + \bar{w}_{u1} \times \bar{w}_{u1} + c_1 [\partial_\nu (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) - \partial_\nu \bar{w}_{uc1} + \bar{w}_{uc1} \times \bar{w}_{u1} + \bar{w}_{u1} \times (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2})] \\
+ c_2 [\partial_\nu (\bar{V} \bar{w}_{uc2}) - \partial_\nu \bar{w}_{uc2} + \bar{w}_{uc2} \times \bar{w}_{u1} + \bar{V} \bar{w}_{u1} \times \bar{w}_{uc2}] \\
- c_4 [\partial_\nu (\bar{U} \bar{w}_{uc1}) + \bar{U} \bar{w}_{u1} \times \bar{w}_{uc1}] + (c_1^2 + c_2 c_4) \bar{U} \bar{w}_{uc1} \times \bar{w}_{uc2} = 0, \\
- \partial_\nu \bar{w}_{uc1} - \bar{w}_{u1} - \bar{w}_{u1} \times \bar{w}_{u1} + c_1 [-\partial_\nu \bar{w}_{uc2} - \partial_\nu (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2})] \\
+ (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) \times \bar{w}_{u1} - \bar{w}_{u1} \times \bar{w}_{uc1} + c_2 [-\partial_\nu (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) + \bar{V} \bar{w}_{u1} \times \bar{w}_{uc1}] \\
+ c_4 [\partial_\nu \bar{w}_{uc1} + \partial_\nu (\bar{U} \bar{w}_{uc1}) - \bar{U} \bar{w}_{uc1} \times \bar{w}_{u1} + \bar{w}_{u1} \times \bar{w}_{uc1}] - (c_1^2 + c_2 c_4) \bar{V} \bar{w}_{uc1} \times \bar{w}_{uc2} = 0, \]

or for short

\[ (\partial_\nu c_4 - \partial_\nu c_3) \bar{w}_{uc1} - (\partial_\nu c_1 + \partial_\nu c_2) \bar{w}_{uc2} + C_1^0 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1 = 0, \]

\[ \partial_\nu c_1 (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) - \partial_\nu c_1 \bar{w}_{uc1} + (\partial_\nu c_2 \bar{V} - \partial_\nu c_2) \bar{w}_{uc2} - \partial_\nu c_1 \bar{U} \bar{w}_{uc1} \\
+ c_2^0 + c_1 C_1^2 + c_2 C_2^2 + c_4 C_4^2 + (c_1^2 + c_2 c_4) C_{124}^2 = 0, \]

\[ -\partial_\nu c_1 \bar{w}_{uc2} - \partial_\nu c_1 (\bar{V} \bar{w}_{uc1} + \bar{U} \bar{w}_{uc2}) - \partial_\nu c_2 \bar{V} \bar{w}_{uc2} - (\partial_\nu c_1 \bar{U} + \partial_\nu c_4) \bar{w}_{uc1} \\
+ c_2^0 + c_1 C_1^3 + c_2 C_2^3 + c_4 C_4^3 + (c_1^2 + c_2 c_4) C_{124}^3 = 0. \]

Multiplying each of these 3 vector equations on the left respectively with \((\bar{U} \times N_0)\), \((\bar{V} \times N_0)\), \(N_0^T \) we get 9 scalar equations linear in the 7 variables \(\partial_\nu c_1, \partial_\nu c_1, \partial_\nu c_2, \partial_\nu c_2, \partial_\nu c_3, \partial_\nu c_4)\) with the rank of the matrix of the system being 5; dividing the rows 1, 2, 4, 5, 7, 8 with \(N_0^T (\bar{U} \times \bar{V})\) the linear combinations of rows which are 0 are as follows:

\[ L_3 + \frac{N_0^T (\bar{V} \times \bar{V})}{m} L_1 - \frac{N_0^T (\bar{V} \times \bar{U})}{m} L_2 = 0, \]

\[ L_6 + \frac{N_0^T (\bar{V} \times \bar{U})}{m} L_4 - \frac{N_0^T (\bar{V} \times \bar{U})\bar{L}_5}{m} = 0, \]

\[ (2.4) \]

\[ L_9 + \frac{N_0^T (\bar{V} \times \bar{V})}{m} L_7 - \frac{N_0^T (\bar{V} \times \bar{U})}{m} L_8 = 0, \]

\[ L_7 + L_5 - \bar{V} L_1 + \bar{U} L_2 = 0. \]

The condition that the augmented matrix of the linear system satisfies the same linear combinations of rows being 0 imposes 4 equations in \(c_1, c_2, c_4\); one solves for some of \(c_1, c_2, c_4\) from these equations and then one imposes compatibility conditions on the derivatives of \(c_1, c_2, c_4\) and on the commuting of the mixed derivatives of these 3 functions of the variables \(u, v, w\) (using \(\partial_\nu c_1, \partial_\nu c_1, \partial_\nu c_2, \partial_\nu c_2, \partial_\nu c_3, \partial_\nu c_4)\) of the 7 variables above given as solutions of the linear system above); this may impose further functional relationships between \(c_1, c_2, c_4\). At the end of this process we must get precisely dependence on two functions of a variable, so all necessary conditions on \(V\) (and \(m\), but using \((1.5)\) this will become dependent on \(V\) and on the seed surface, adjoined to \((1.7)\) become the necessary and sufficient conditions to get Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed).

Since we must get dependence on two functions of a variable, from the 4 functional relationships imposed on \(c_1, c_2, c_4\) by the consistency requirement of the linear system above only at most two of them are functionally independent; if we get 2 functionally independent functional relationships, then one can solve for two of \(c_1, c_2, c_4\) as functions of the remaining third one and compatibility conditions on derivatives and mixed derivatives do not allow a space of two functions of a variable dependence of solutions; thus from the 4 functional relationships imposed on \(c_1, c_2, c_4\) by the consistency requirement of the linear system above only at most one of them is functionally independent; the conditions that arise from this premise will be conditions on \(V, m\) and the seed surface.

The coefficient of the quadratic term \(c_1^2 + c_2 c_4\) from the first equation of \((2.4)\) is \(-\frac{m_0^2}{m} N_0^T (\bar{U} \times \bar{V}) \neq 0\) and thus the equation is an independent (non-vacuous) functional relationship between \(c_1, c_2, c_4\).

The coefficient of the quadratic term \(c_1^2 + c_2 c_4\) from the second equation of \((2.4)\) is \(\frac{m_0^2}{m} N_0^T (\partial_\nu V \times \bar{U})\); thus the second equation is \(-\bar{U}\) the first equation.
The coefficient of the quadratic term \( \partial_1^2 + c_2c_4 \) from the third equation of (2.3) is \(-\frac{m^2}{m^2} N_0^T (\partial_w V \times V)\); thus the third equation is \( \tilde{V} \) the first equation.

The fourth equation of (2.3) does not contain quadratic terms, so it is identically satisfied. We thus get the relations:

\[
m^T C_j^2 = -\tilde{U} m^T C_j^1, \quad m^T C_j^3 = \tilde{V} m^T C_j^1,
\]

(2.5) \[-(\partial_w V \times N_0)^T C_j^1 + (U \times N_0)^T C_j^3 + (V \times N_0)^T C_j^2 = 0, \quad j = 0, 1, 2, 4,
\]

which after removing two obvious relations become:

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} + \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{U} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + (\tilde{V} \tilde{\omega}_{uc} + \tilde{U} \tilde{\omega}_{uc}) \times \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

\[
m^T [\tilde{V} \partial_u \tilde{\omega}_{uc} - \tilde{U} \partial_u \tilde{\omega}_{uc} - \partial_u \tilde{\omega}_{uc} + \tilde{\omega}_{uc} \times \tilde{\omega}_{uc} + \tilde{V} \tilde{\omega}_{uc} \times \tilde{\omega}_{uc}] = 0,
\]

The first and fifth relations are equivalent; so are the second and the fourth. We shall call these relations R1; they depend on \( V, m \) (and their derivatives) and the first and second fundamental forms of the seed surface \( x_0 \) (and their derivatives).

From the 9 scalar equations obtained by multiplying on the left respectively with \((U \times N_0)^T\), \((V \times N_0)^T\), \(N_0^T\) the 3 vector equations of (2.3), among which we have the linear combinations of rows (2.4), only the next 5 remain linearly independent:
\[ \partial_u c_1 = -\partial_v c_2 - \left[ C_0^1 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1 \right]^T \frac{(\mathbf{u} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})}; \]
\[ \partial_v c_1 = \partial_v c_4 + \left[ C_0^1 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1 \right]^T \frac{(\mathbf{v} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})}; \]
\[ \partial_w c_2 = (-\partial_v c_2 - [C_0^1 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1]^T \frac{(\mathbf{u} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})} - \partial_v c_2 \mathbf{V} - [C_0^2 + c_1 C_1^2 + c_2 C_2^2 + c_4 C_4^2 + (c_1^2 + c_2 c_4) C_{124}^2]^T \frac{(\mathbf{v} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})} + \partial_v c_2 \mathbf{V} - [C_0^2 + c_1 C_1^2 + c_2 C_2^2 + c_4 C_4^2 + (c_1^2 + c_2 c_4) C_{124}^2]^T \frac{(\mathbf{v} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})}; \]
\[ \partial_w c_1 = -\partial_v c_2 + \left[ C_0^1 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1 \right]^T \frac{(\mathbf{u} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})} - \partial_v c_4 \mathbf{U} + [C_0^2 + c_1 C_1^2 + c_2 C_2^2 + c_4 C_4^2 + (c_1^2 + c_2 c_4) C_{124}^2]^T \frac{(\mathbf{v} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})}; \]
\[ \partial_w c_4 = \left( \partial_v c_4 + [C_0^1 + c_1 C_1^1 + c_2 C_2^1 + c_4 C_4^1 + (c_1^2 + c_2 c_4) C_{124}^1]^T \frac{(\mathbf{u} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})} - \partial_v c_4 \mathbf{U} - [C_0^2 + c_1 C_1^2 + c_2 C_2^2 + c_4 C_4^2 + (c_1^2 + c_2 c_4) C_{124}^2]^T \frac{(\mathbf{v} \times N_0)}{N_0^3 (\mathbf{u} \times \mathbf{v})}; \]
\[ \frac{[(C'_4 + c_2C_{124}) \times (C'_0 + c_1C'_1 + c_2C'_2 + c_2C_{124})] \times m_T}{m_T(C'_4 + c_2C_{124})} \left( \mathbf{U} \times \mathbf{N}_0 \right) + \frac{m_T(C'_1 + 2c_1C_{124})}{m_T(C'_4 + c_2C_{124})} \]
\[
-\left(\partial_u(m^T C^0_0) + c_1 \partial_u(m^T C^1_0) + c_2 \partial_u(m^T C^2_0) + c_3^2 \partial_u(m^T C^1_{124})\right)
+ \frac{\partial_u(m^T C^1_0) + c_2 \partial_u(m^T C^1_{124})}{m^T(C^1_0 + c_2 C^1_{124})} \frac{m^T(C^1_0 + c_2 C^1_{124})}{m^T(C^1_0 + c_2 C^1_{124})} + (\partial_v c_2)
+ \frac{\partial_u C^2_0}{m^T(C^1_0 + c_2 C^1_{124})} + \frac{\partial_u C^2_0}{m^T(C^1_0 + c_2 C^1_{124})} + \frac{\partial_u C^2_0}{m^T(C^1_0 + c_2 C^1_{124})} \frac{m^T(C^1_0 + c_2 C^1_{124})}{m^T(C^1_0 + c_2 C^1_{124})} + (\partial_v c_2)
\]

In the last relation the coefficients of \(\partial_u c_2, \partial_v c_2\) are 0; thus this relation is a functional relationship (which is actually rational) between \(u\) and \(c_2\). Since \(c_1\) and \(c_2\) are functionally independent, this last relation being identically 0 gives relations depending on \(V, m\) (and their derivatives) and the first and second fundamental form (and their derivatives) of the seed surface \(x_0\). We shall call these relations (and those obtained by exchanging the rôle played by \(c_2\) and \(c_4\)) R2.

The remaining 4 equations can be written as:

\[
\begin{align*}
\partial_u c_1 &= -\partial_v c_2 + D_1, \\
\partial_v c_1 &= D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4, \\
\partial_u c_2 &= D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7, \\
\partial_v c_2 &= D_8 + \partial_u c_2 \hat{V} - \partial_v c_2 \hat{U},
\end{align*}
\]

(2.7)

where \(D_j = D_j(c_1, c_2, u, v, w), j = 1, \ldots, 8\) are rational functions of \(c_1\) and \(c_2\) and depend otherwise on \(V, m\) (and their derivatives) and on the first and second fundamental form of the seed surface \(x_0\) (and their derivatives).

Imposing the three compatibility conditions \(\partial_v(\partial_u c_1) = \partial_u(\partial_v c_1), \partial_u(\partial_u c_1) = \partial_u(\partial_u c_1), \partial_u(\partial_v c_1) = \partial_u(\partial_v c_1)\) and using the equations (2.7) themselves we get the relations

\[
\begin{align*}
-\partial^2_{c_2} c_2 + \partial_{c_2} D_1(D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) + \partial_{c_2} D_1 \partial_v c_2 + \partial_u D_1 \\
-(\partial_v c_1, D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4)(-\partial_u c_2 + D_1) - (\partial_v c_1, D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) \partial_u c_2 \\
- \partial_v D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4 - (\partial^2_{c_2} c_2 D_3 + \partial^2_{c_2} c_2 D_4) = 0, \\
-[\partial_v, D_5(D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) + \partial_v D_5 \partial_u c_2 + \partial_v D_5 + \partial^2_{c_2} c_2 \hat{V} + \partial_{c_2} \partial_v \hat{V} \\
- \partial^2_{c_2} \hat{U} - \partial_{c_2} \partial_v \hat{U} + \partial_{c_2} D_1(D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) \\
+ \partial_{c_2} D_1(D_5 + \partial_u c_2 \hat{V} - \partial_v c_2 \hat{U} + \partial_v D_1) \\
-(\partial_v c_1, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7)(-\partial_v c_2 + D_1) - (\partial_v, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) \partial_u c_2 \\
- (\partial_v, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) - (\partial^2_{c_2} c_2 D_6 + \partial^2_{c_2} c_2 D_7) = 0, \\
(\partial_v c_1, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7)(D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) + (\partial_v, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) \partial_u c_2 \\
+ \partial_v D_5 \partial_u c_2 + \partial_v D_5 + \partial^2_{c_2} c_2 \hat{V} + \partial_{c_2} \partial_v \hat{V} - \partial^2_{c_2} \hat{U} \\
- \partial_{c_2} \partial_v \hat{U} + D_4 D_5(D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) + \partial_{c_2} D_5 \partial_v c_2 + \partial_{c_2} D_5 \\
+ \partial^2_{c_2} c_2 \hat{V} + \partial_{c_2} \partial_v \hat{V} - \partial^2_{c_2} \hat{U} - \partial_{c_2} \partial_v \hat{U} \\
-(\partial_v, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7)(D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) \\
-(\partial_v, D_5 + \partial_u c_2 D_6 + \partial_v c_2 D_7) - (\partial^2_{c_2} c_2 D_6 + \partial^2_{c_2} c_2 D_7) = 0.
\]
These constitute a linear system in the variables $\partial_u^2 c_2, \partial_{uw}^2 c_2, \partial_v^2 c_2$ with the determinant of the matrix of the system being

$$
\begin{vmatrix}
-D_3 & -D_4 & -1 \\
-D_6 & -D_7 + \tilde{V} & -\tilde{U} \\
\tilde{V} & -D_6 + \tilde{V} - \tilde{U} & -D_7 - \tilde{U}
\end{vmatrix} = -\begin{vmatrix}
-D_6 + D_3 \tilde{U} & -D_7 + \tilde{V} + D_4 \tilde{U} \\
\tilde{V} + D_3 (D_7 + \tilde{U}) & -D_6 + \tilde{V} - \tilde{U} + D_4 (D_7 + \tilde{U})
\end{vmatrix}.
$$

Using $D_3 = -\frac{m^T C_1}{m^T (C_4^1 + C_2 C_{124})} + \frac{m^T C_1}{m^T (C_4^1 + C_2 C_{124})}$, $D_4 = m^T (C_4^1 + C_2 C_{124})$, $D_6 = \tilde{U} D_3$, $D_7 = -\tilde{V} - \tilde{U} D_4$ the coefficient of the highest order term $c_4^3$ in the numerator (after bringing to a common denominator $[m^T (C_4^1 + C_2 C_{124})]^4$) of the above determinant is $2 (m^T C_1) \tilde{U}^2 \neq 0$, so this determinant is not 0.

This allows us to solve for

$$
\begin{align*}
\partial_u^2 c_2 &= F_1 (\partial_u c_2, \partial_v c_2, c_1, c_2, u, v, w), \\
\partial_{uw}^2 c_2 &= F_2 (\partial_u c_2, \partial_v c_2, c_1, c_2, u, v, w), \\
\partial_v^2 c_2 &= F_3 (\partial_u c_2, \partial_v c_2, c_1, c_2, u, v, w),
\end{align*}
$$

where $F_j, j = 1, 2, 3$ depend quadratically on $\partial_u c_2, \partial_v c_2$, rationally on $c_1, c_2$ and depend otherwise on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives).

Imposing the two compatibility conditions $\partial_u (\partial_u^2 c_2) = \partial_u (\partial_{uw}^2 c_2)$, $\partial_u (\partial_v^2 c_2) = \partial_v (\partial_{uw}^2 c_2)$ and using the equations (2.7), (2.8) themselves we get the relations

$$
\begin{align*}
\partial_{u,c_2} F_1 F_2 + \partial_{u,c_2} F_1 F_3 + \partial_{c_1} F_1 (D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) + \partial_{c_2} F_1 \partial_u c_2 + \partial_u F_1 \\
- (\partial_{u,c_2} F_2 F_2 + \partial_{u,c_2} F_2 F_3 + \partial_{c_1} F_2 (-\partial_u c_2 + D_1) + \partial_{c_2} F_2 \partial_u c_2 + \partial_u F_2) = 0,
\end{align*}
$$

$$
\begin{align*}
\partial_{u,c_2} F_3 F_1 + \partial_{u,c_2} F_3 F_2 + \partial_{c_1} F_3 (-\partial_v c_2 + D_1) + \partial_{c_2} F_3 \partial_u c_2 + \partial_u F_3 \\
- (\partial_{u,c_2} F_2 F_2 + \partial_{u,c_2} F_2 F_3 + \partial_{c_1} F_2 (D_2 + \partial_u c_2 D_3 + \partial_v c_2 D_4) + \partial_{c_2} F_2 \partial_u c_2 + \partial_u F_2) = 0.
\end{align*}
$$

These can be written as

$$
(2.9) \quad G_1 (\partial_u c_2, \partial_v c_2, c_1, c_2, u, v, w) = 0, \quad G_2 (\partial_u c_2, \partial_v c_2, c_1, c_2, u, v, w) = 0,
$$

where $G_1, G_2$ depend cubically on $\partial_u c_2, \partial_v c_2$, rationally on $c_1, c_2$ and depend otherwise on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives).

If $G_1, G_2$ are functionally independent in $\partial_u c_2, \partial_v c_2$, then one can solve them for $\partial_u c_2 = H_1 (c_1, c_2, u, v, w)$, $\partial_v c_2 = H_2 (c_1, c_2, u, v, w)$: replacing this into (2.4) we get all derivatives of $c_1$ and $c_2$ as functions of $c_1$ and $c_2$ and depend otherwise on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives). Imposing the compatibility conditions of commuting of mixed derivatives of $c_1$ and $c_2$ and using the derivatives of $c_1$ and $c_2$ themselves as functions of $c_1$ and $c_2$ we get 6 compatibility conditions which must be identically satisfied in $c_1$ and $c_2$, but in this case the space of solutions depends on constants and not on two functions of a variable.

Since the coefficients of $\partial_u^2 c_2^3, \partial_v c_3^3$ in $G_1$ are not 0, $G_1$ is a non-vacuous functional relationship between $\partial_u c_2, \partial_v c_2$, so $G_2$ must be a multiple (namely $-\frac{\tilde{V}}{\tilde{U}}$) of it.

We thus have $G_2 + \frac{\tilde{V}}{\tilde{U}} G_1 = 0$; after bringing to a common denominator the numerator will be quadratic in $\partial_u c_2$, $\partial_v c_2$ and polynomial in $c_1, c_2$ which must be identically 0. We shall call these relations (and those obtained by exchanging the rôles played by $c_2$ and $c_4$) R3.
We can solve $G_1 = 0$ for $\partial_u c_2 = H(\partial_v c_2, c_1, c_2, u, v, w)$ (there are 3 choices of $H$); replacing this into (2.7) we get

$$\partial_u c_1 = -\partial_v c_2 + D_1,$$
$$\partial_v c_1 = D_2 + H(\partial_v c_2, c_1, c_2, u, v, w)D_3 + \partial_v c_2 D_4,$$
$$\partial_w c_1 = D_5 + H(\partial_v c_2, c_1, c_2, u, v, w)D_6 + \partial_v c_2 D_7,$$
$$\partial_u c_2 = H(\partial_v c_2, c_1, c_2, u, v, w),$$

(2.10) $$\partial_w c_2 = D_8 + H(\partial_v c_2, c_1, c_2, u, v, w)\tilde{V} - \partial_v c_2 \tilde{U},$$

and (2.8) becomes

$$\partial_{\partial_v c_2} H F_2(H, \partial_v c_2, c_1, c_2, u, v, w) + \partial_v c_1 H(-\partial_u c_2 + D_1) + \partial_v c_2 H H + \partial_u H = F_1(H, \partial_v c_2, c_1, c_2, u, v, w),$$

$$\partial_{\partial_v c_2} H F_3(H, \partial_v c_2, c_1, c_2, u, v, w) + \partial_v c_1 H(D_2 + HD_3 + \partial_v c_2 D_4) + \partial_v c_2 H \partial_v c_2 + \partial_v D_8$$

$$= F_2(H, \partial_v c_2, c_1, c_2, u, v, w),$$

$$\partial^2_{\partial_v c_2} c_2 = F_3(H, \partial_v c_2, c_1, c_2, u, v, w).$$

The first two relations are valid for functionally independent $c_1, c_2, \partial_v c_2$, so impose conditions on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives).

Imposing the compatibility condition $\partial_u(\partial_v c_2) = \partial_u(\partial_w c_2)$ on the last two equations of (2.10) and using the equations of (2.10) themselves and $\partial^2_{\partial_v c_2} c_2 = F_3(H, \partial_v c_2, c_1, c_2, u, v, w)$ we get

$$\partial_{\partial_v c_2} H \partial_v c_1 D_8(D_2 + HD_3 + \partial_v c_2 D_4) + \partial_v c_2 D_8 \partial_v c_2 + \partial_v D_8$$

$$+ \partial_v c_1 H(D_2 + HD_3 + \partial_v c_2 D_4) + \partial_v c_2 H \partial_v c_2 + \partial_v H)\tilde{V} + \partial_v \tilde{V} F_3(H, \partial_v c_2, c_1, c_2, u, v, w)\tilde{U} - \partial_v c_2 \partial_v \tilde{U}$$

$$+ \partial_v c_1 H(D_5 + HD_6 + \partial_v c_2 D_7) + \partial_v c_2 H(D_8 + H\tilde{V} - \partial_v c_2 \tilde{U}) + \partial_v H - \partial_v D_8(-\partial_v c_2 + D_1) + \partial_v c_2 D_8 H + \partial_v D_8$$

$$+ (\partial_{\partial_v c_2} H F_2(H, \partial_v c_2, c_1, c_2, u, v, w) + \partial_v c_1 H(-\partial_v c_2 + D_1) + \partial_v c_2 H H + \partial_u H)\tilde{V} + \partial_v H \tilde{V}$$

$$- F_2(H, \partial_v c_2, c_1, c_2, u, v, w)\tilde{U} - \partial_v c_2 \partial_v \tilde{U}) = 0.$$

This is valid for functionally independent $c_1, c_2, \partial_v c_2$, so impose conditions on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives).

Thus (2.10) together with $\partial^2_{\partial_v c_2} c_2 = F_3(H, \partial_v c_2, c_1, c_2, u, v, w)$ is in involution and we know the solution must depend on two functions of a variable.

The conditions on $V, m$ (and their derivatives) and on the first and second fundamental form of the seed surface $x_0$ (and their derivatives) obtained at this last step will be called the relations R4. Note that here we don’t need to exchange the rôle played by $c_2$ and $c_4$, since there will be 9 cases to discuss instead of 3.

For the relations R1,R2,R3,R4 we can replace the seed $x_0$ with any other isometric surface; this will remove the dependence on the second fundamental form of the seed surface $x_0$ (and its derivatives). Then we can replace $m$ and its derivatives from (1.5), thus giving a differential system on $V$ depending only on the first fundamental form of the seed surface $x_0$, which adjoined to (1.7) constitutes a differential system on $V$ (here compatibility conditions may impose conditions on the metric of the seed surface) which completely describes the Bäcklund transformation with isometric correspondence of leaves of a general nature (independent of the shape of the seed surface).
2.1. The relations R2. After bringing to a common denominator $[m^T(C_1^4 + c_2C_{124}^4)]^3N_0^T(U \times V)$ the last relation of (2.6) becomes a quartic polynomial $P_1$ in $c_1, c_2$ being identically 0. We shall consider the coefficients of decreasing powers of $c_1, c_2$, establishing recurrence relations.

From the coefficient of $c_1^4$ we get $(U \times N_0)^T C_2 = -U(U \times N_0)^T C_1^4$, which is obvious and thus removed; from the coefficient of $c_2^4$ we get $(V \times N_0)^T C_2^4 = \tilde{V}(V \times N_0)^T C_2^4$, which is again obvious and thus removed.

From the coefficient of $c_1^2c_2$ we get $(V \times N_0)^T C_2^4 = -\tilde{U}(U \times N_0)^T (\tilde{C}_1^4 + \tilde{U}C_1^1)$; from the coefficient of $c_1^3c_2$ we get $(V \times N_0)^T C_2^4 = (V \times N_0)^T (\tilde{C}_1^4 - 2\tilde{U}C_1^1 - 2\tilde{C}_1^2) - (U \times N_0)^T (U \tilde{C}_1^1 + C_1^4)$.

Using the relations thus far obtained, the coefficient of $c_1^4$ is 0.

From the coefficient of $c_1^2c_2$ we get $(U \times N_0)^T C_2^4 = \frac{1}{2m^T C_{124}^4 c_1^4} [2(m^T C_{124}^1)^2(U \times N_0)^T (U \tilde{C}_1^1 + C_1^4) - 2\tilde{U}m^T C_{124}^1 m^T C_1^4(U \times N_0)^T C_1^4 - m^T C_1^1 m^T C_{124}^2(U \times N_0)^T (U \tilde{C}_1^1 + C_1^4) + 2\tilde{V}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - 2\tilde{U}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - 2m^T C_1^4(U \times N_0)^T (V C_1^4 - 2U C_1^1 - 2C_1^2)]$.

Using the relations thus far obtained, the coefficient of $c_1^2c_2$ is 0.

From the coefficient of $c_1^3c_2$ we get $(V \times N_0)^T C_2^4 = \frac{1}{2m^T C_{124}^4 c_1^4} [-2(m^T C_{124}^1)^2(V \times N_0)^T (U \tilde{C}_1^1 + C_1^4) + 2\tilde{U}m^T C_{124}^1 m^T C_1^4(U \times N_0)^T C_1^4 + m^T C_1^1 m^T C_{124}^2(V \times N_0)^T (U \tilde{C}_1^1 + C_1^4) + 2\tilde{V}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - \tilde{U}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] + m^T C_1^1 m^T C_{124}^2(U \times N_0)^T C_1^4$.

Using the relations thus far obtained, the coefficients of $c_1c_2$ and $c_1$ are 0.

From the coefficient of 1 we get $(V \times N_0)^T C_1^4 = \frac{1}{2m^T C_{124}^4 c_1^4} [2m^T C_{124}^1 m^T C_1^4(V \times N_0)^T (U \tilde{C}_1^1 + C_1^4) + m^T C_{124}^1 m^T C_1^4(U \times N_0)^T C_1^4 - m^T C_1^1 m^T C_{124}^2(V \times N_0)^T (U \tilde{C}_1^1 + C_1^4) - \tilde{V}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - \tilde{U}N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] - N_0^T(U \times \tilde{V})[m^T C_{124}^1 \partial_u(m^T C_1^4) - \partial_u(m^T C_{124}^1)m^T C_1^4] + m^T C_1^1 m^T C_{124}^2(U \times N_0)^T C_1^4]$.

Since these satisfy $2m^T C_1^4(\text{coef of } c_2^4) = m^T C_{124}^1(\text{coef of } c_2^4)$, we only need to further consider the coefficients of $c_2^2, c_2, c_1$.

By exchanging the rôle played by $c_2$ and $c_1$ and after bringing to a common denominator $[m^T C_1^4 + c_4C_{124}^4]3N_0^T(U \times V)$ the relation corresponding to the last relation of (2.6) by exchanging the rôle played by $c_2$ and $c_1$ becomes a quartic polynomial $P_2$ in $a_1, a_2$. Again we shall consider the coefficients of decreasing powers of $a_1, a_2$, establishing recurrence relations.

It turns out that we don't get more relations than those obtained by the polynomial $P_1$ being identically 0: the coefficient of $c_1^4$ in $P_2$ is the coefficient of $c_1^4$ in $P_1$; the coefficient of $c_2^4$ in $P_2$ is the coefficient of $c_2^4$ in $P_1$; the coefficient of $c_1c_2$ in $P_2$ is the coefficient of $c_1c_2$ in $P_1$.
of $c_1c_4^3$ in $P_2$ is $-\text{the coefficient of } c_1c_2^3$ in $P_1$; the coefficient of $c_1^2c_2$ in $P_2$ is the coefficient of $c_1^2c_2^3$ in $P_1$; the coefficient of $c_1^2$ in $P_2$ is 0; the coefficient of $c_1c_2^3$ in $P_2$ is the coefficient of $c_1c_2^3$ in $P_1$. For the coefficient of $c_1c_4$ in $P_2$, by replacing $(U \times N_0)T C_2^3$ and $(V \times N_0)T C_3^3$ with their values it becomes $-\text{the coefficient of } c_2^3$ in $P_1$. The coefficient of $c_2$ in $P_2$ is 0; the coefficient of $c_1^2$ in $P_2$ is $-\frac{m^T C_2^3}{m^T C_2^3}$ the times the coefficient of $c_2^3$ in $P_1$; the coefficient of $c_1c_4$ in $P_2$ is $-\frac{m^T C_1^2}{m^T C_1^2}$ times the coefficient of $c_1c_4^3$ in $P_1$; the coefficient of $c_1$ in $P_2$ is $-\frac{m^T C_1^2m^T C_1^4}{(m^T C_1^2)^2}$ times the coefficient of $c_1^3$ in $P_1$; the coefficient of $c_4$ in $P_2$ is $-\frac{m^T C_2^3m^T C_1^4}{(m^T C_1^2)^2}$ times the coefficient of $c_2^3$ in $P_1$; the coefficient of $c_3$ in $P_2$ is $-\frac{m^T C_1^2}{m^T C_1^2}$ times the coefficient of $c_2^3$ in $P_1$; the coefficient of $c_2^3$ in $P_1$; the coefficient of $1$ in $P_2$ is $-\frac{m^T C_2^3m^T C_1^4}{(m^T C_1^2)^2}$ times the coefficient of $c_2^3$ in $P_1$.

2.2. The relations R1 and R2 revisited. The relations that depend only on the second fundamental form of $x_0$ (and not its derivatives) are of the form $\partial_u N_0^2 A + \partial_v N_0^2 B + C = 0$, where A, B, C do not depend on the second fundamental form of $x_0$. From each such relation we get four ones: with $u^1 := u$, $u^2 := v$, $g_{jk} = \partial_{x^j} x^k \partial_{x^k} x_0$, we have $g^{12} \partial_{u^1} x^k A = 0$, $g^{23} \partial_{u^2} x^k B = 0$, $g^{23} \partial_{u^2} x^k A + g^{12} \partial_{u^1} x^k B = 0$, $C = 0$; if $A = 0$ (or $B = 0$), then $B = 0$ (or $A = 0$) and we obtain only three relations.

For the fifth equation of R1 we have $A = -N_0^T (\partial_u V \times V) N_0^T (U \times V) N_0^T (U \times V) [N_0^T (\partial_u V \times V)] (N_0 \times U) N_0 \times V - N_0^T (\partial_u V \times V) [N_0^T (U \times V)] (N_0 \times U) N_0 \times V + N_0^T (\partial_u V \times V) [N_0^T (U \times V)] (N_0 \times U) N_0 \times V - [N_0^T (U \times V)] (N_0 \times U) N_0 \times V = 0$, where $g^{12} \partial_{u^1} x^k A = 0$, $g^{23} \partial_{u^2} x^k B = 0$, $g^{23} \partial_{u^2} x^k A + g^{12} \partial_{u^1} x^k B = 0$, $C = 0$; if $A = 0$ (or $B = 0$), then $B = 0$ (or $A = 0$) and we obtain only three relations.

From the second equation of R1 we have $A = N_0^T (\partial_u V \times V) N_0^T (U \times V) [-N_0^T (\partial_u V \times V)] (N_0 \times U) N_0 \times V - V^T V(N_0 \times U) N_0 \times V + V^T V(N_0 \times U) N_0 \times V + N_0^T (\partial_v V \times V) [N_0^T (U \times V)] (N_0 \times U) N_0 \times V + [N_0^T (U \times V)] (N_0 \times U) N_0 \times V = 0$, where $g^{12} \partial_{u^1} x^k A = 0$, $g^{23} \partial_{u^2} x^k B = 0$, $g^{23} \partial_{u^2} x^k A + g^{12} \partial_{u^1} x^k B = 0$, $C = 0$; if $A = 0$ (or $B = 0$), then $B = 0$ (or $A = 0$) and we obtain only three relations.

From the fourth equation of R1 we have $A = N_0^T (\partial_u V \times V) N_0^T (U \times V) [-N_0^T (\partial_u V \times V)] (N_0 \times U) N_0 \times V - V^T V(N_0 \times U) N_0 \times V + V^T V(N_0 \times U) N_0 \times V + N_0^T (\partial_v V \times V) [N_0^T (U \times V)] (N_0 \times U) N_0 \times V + [N_0^T (U \times V)] (N_0 \times U) N_0 \times V = 0$, where $g^{12} \partial_{u^1} x^k A = 0$, $g^{23} \partial_{u^2} x^k B = 0$, $g^{23} \partial_{u^2} x^k A + g^{12} \partial_{u^1} x^k B = 0$, $C = 0$; if $A = 0$ (or $B = 0$), then $B = 0$ (or $A = 0$) and we obtain only three relations.

From the seventh equation of R1 we have $A = N_0^T (\partial_u V \times V) N_0^T (U \times V) [-N_0^T (\partial_u V \times V)] (N_0 \times U) N_0 \times V - V^T V(N_0 \times U) N_0 \times V + V^T V(N_0 \times U) N_0 \times V + N_0^T (\partial_v V \times V) [N_0^T (U \times V)] (N_0 \times U) N_0 \times V + [N_0^T (U \times V)] (N_0 \times U) N_0 \times V = 0$, where $g^{12} \partial_{u^1} x^k A = 0$, $g^{23} \partial_{u^2} x^k B = 0$, $g^{23} \partial_{u^2} x^k A + g^{12} \partial_{u^1} x^k B = 0$, $C = 0$; if $A = 0$ (or $B = 0$), then $B = 0$ (or $A = 0$) and we obtain only three relations.
From the eighth equation of R1 we have $A = N^0_0(\partial_u V × V)^2 N^0_0(\partial_u V × V)[N^0_0(\partial_u V × V)] V × N_0 = [V × N_0][N_0 × U × N_0 + (V × N_0)]^2 (U × N_0)(N_0 × V) × N_0 = 0$, $B = (N^0_0(\partial_u V × V))^2 N^0_0(\partial_u V × V)(V × N_0) = 0$.

From the ninth equation of R1 we have $A = (V × N_0)^2 (\partial_u V × V) N_0^2(\partial_u V × V)(V × N_0)(V × N_0) + 4 N_0^2(\partial_u V × V)(V × N_0)(V × N_0) + 4 V^2 U(\partial_u V × V)(V × N_0)(V × N_0) × N_0 - 4 U^2 (\partial_u V × V)(V × N_0)(V × N_0) × N_0 - 4 V^2 (\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$, $C = (\partial_u V × V)(V × N_0)(V × N_0) + 4 V^2 U(\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$.

From the first equation of R2 we have $A = (V × N_0)^2 (\partial_u V × V) N_0^2(\partial_u V × V)(V × N_0)(V × N_0) + (1/2)[U × N_0]^2 (\partial_u V × V) N_0^2(\partial_u V × V)(V × N_0)(V × N_0) = 0$, $B = (1/2)(\partial_u V × V) N_0^2(\partial_u V × V)(V × N_0)(V × N_0) + 2 V^2 U(\partial_u V × V)(V × N_0)(V × N_0) = 0$, $C = (\partial_u V × V)(V × N_0)(V × N_0) + 2 U^2 (\partial_u V × V)(V × N_0)(V × N_0) = 0$.

From the second equation of R2 we have $A = N^0_0(\partial_u V × V)^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + N^0_0(\partial_u V × V)^2 (V × N_0)(N_0 × V) × N_0 = 0$, $B = (N^0_0(\partial_u V × V))^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + 2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$, $C = (\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$.

From the third equation of R2 we have $A = N^0_0(\partial_u V × V)^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + N^0_0(\partial_u V × V)^2 (V × N_0)(N_0 × V) × N_0 = 0$, $B = (N^0_0(\partial_u V × V))^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + 2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$, $C = (\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$.

From the fourth equation of R2 we have $A = (\partial_u V × V)^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + N^0_0(\partial_u V × V)^2 (V × N_0)(N_0 × V) × N_0 = 0$, $B = (\partial_u V × V)^2 N^0_0(\partial_u V × V)(V × N_0)(V × N_0) + 2 (\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$, $C = (\partial_u V × V)(V × N_0)(V × N_0) × N_0 = 0$.

The remaining equations of R1 (the third, sixth and seventh) depend a-priori also linearly on the first derivatives of the second fundamental form of $x_0$: using the Codazzi-Mainardi-Peterson equations of $x_0$ the (linear) dependence on the first derivatives of the second fundamental form of the third and sixth equations of R1 disappears.

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