MAHLER MEASURES AND \(L\)-VALUES OF ELLIPTIC CURVES OVER REAL QUADRATIC FIELDS

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Abstract. A famous formula of Rodriguez Villegas shows that the Mahler measures \(m(k)\) of 
\(P_k(x, y) = x + 1/x + y + 1/y + k\) can be written as a Kronecker-Eisenstein series. We prove 
that the degree of \(k\) in Villegas’ formula can be bounded by the class numbers of CM points. 
This fact allows us to systematically derive 28 new identities linking \(m(k)\) to \(L\)-values of cusp 
forms. Guided by Beilinson’s conjecture, we also prove 5 formulas that express \(L\)-values of CM 
elliptic curves over real quadratic fields to some \(2 \times 2\) determinants of \(m(k)\). This extends a 
recent work of Guo (the second author of this paper), Ji, Liu, and Qin, in which they dealt 
with the cases when \(k = 4 \pm 4\sqrt{2}\).

1. Introduction

The \((\text{logarithmic})\) Mahler measure of a Laurent polynomial \(P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\) is defined 
by 
\[
m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \cdots d\theta_n.
\]
There are many connections between the Mahler measures of certain polynomial families associated 
to elliptic curves and special values of their \(L\)-functions. For example, by using Beilinson’s 
regulator map, Deninger \[8\] conjectured that 
\[
m \left( x + \frac{1}{x} + y + \frac{1}{y} + 1 \right) = L'(E_1, 0) = \frac{15}{4\pi^2}L(E_1, 2),
\]
where \(E_1\) is the elliptic curve of conductor 15 given by the projective closure of 
\[x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0.\]

Equation (1.1) is the first conjectural formula that connects Mahler measure and \(L\)-function 
of elliptic curve. Note that \(E_1\) does not have CM. Hence, it is essentially difficult to prove (1.1). 
Almost two decades after it was proposed, this conjecture was finally proved by Rogers and 
Zudilin in \[19\].

In this paper, we study the Mahler measures of the polynomial family
\[P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k,\]
where \(k \in \mathbb{C} \setminus \{0, \pm 4\}\). One can easily check that the rational transformation 
\[
x = \frac{kX - 2Y}{2X(X - 1)}, \quad y = \frac{kX + 2Y}{2X(X - 1)}
\]
converts the zero locus of \(P_k(x, y) = 0\) to an elliptic curve \(E_k\) with Weierstrass equation 
\[
Y^2 = X^3 + \left(\frac{k^2}{4} - 2\right)X^2 + X.
\]
Note that $E_k$ is defined over $\mathbb{Q}$ if $k^2 \in \mathbb{Q}$. The inverse of transformation (1.2) is
\begin{equation}
X = -\frac{1}{xy}, \quad Y = \frac{(y - x)(1 + xy)}{2x^2y^2}.
\end{equation}

For simplicity, we write $m(k) = m(P_k)$ in the rest of this paper. Motivated by Deninger’s conjecture, Boyd [2] did a lot of numerical calculations and conjectured that for many $k \in \mathbb{Z} \setminus \{0, \pm 4\}$, $m(k)$ is a nonzero rational multiple of $L'(E_k, 0)$, i.e.
\begin{equation}
m(k) = r_k L'(E_k, 0)
\end{equation}
for some $r_k \in \mathbb{Q}^*$. Boyd listed his conjectural identities in a table of [2], he numerically verified those identities with at least 25 decimal places of precision. The famous modularity theorem tells us that if $E/\mathbb{Q}$ is an elliptic curve of conductor $N$, then there exists a newform $f \in S_2(\Gamma_0(N))$ such that
\begin{equation}
L(E, s) = L(f, s).
\end{equation}
According to the functional equation satisfied by $L(E, s)$, we have
\begin{equation}
L'(E, 0) = \pm \frac{N}{4\pi^2} L(E, 2).
\end{equation}
Thus we can rewrite (1.5) as
\begin{equation}
m(k) = r_k L'(f_N, 0) = r_k \frac{N}{4\pi^2} L(f_N, 2),
\end{equation}
where $f_N \in S_2(\Gamma_0(N))$ is the newform associated to $E_k$.

Recall that the classical Dedekind eta-function is the infinite product
\begin{equation}
\eta(\tau) = e^{\frac{2\pi i}{24}} \prod_{n=1}^\infty (1 - q^n),
\end{equation}
where $\tau \in \mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$ and $q = e^{2\pi i \tau}$. Let $\chi_{-d} = \left(\frac{-d}{n}\right)$ be the real odd Dirichlet character of conductor $d$ given by the Kronecker-Jacobi symbol $(\cdot)$. The modular lambda-function $\lambda(\tau)$ is defined by
\begin{equation}
\lambda(\tau) = 16 \frac{\eta(\tau/2)^8 \eta(2\tau)^{16}}{\eta(\tau)^{24}}.
\end{equation}

In [24], Villegas proved the following result that express $m(k)$ as a Kronecker-Eisenstein series:

**Theorem 1.1.** If $\tau$ lies in the region $\mathcal{F'} \subset \mathcal{H}$ formed by the geodesic triangle of vertices $i\infty, 0, 1/2$ and its reflection along the imaginary axis, then we have
\begin{equation}
m(k) = \text{Re} \left( \frac{16 \text{Im}(\tau)}{\pi^2} \sum_{m,n \in \mathbb{Z}}' \frac{\chi_{-4}(n)}{(4m\tau + n)^2(4m\tau + n)} \right),
\end{equation}
where $k = \frac{4}{\sqrt{n(\text{Im}(\tau))}}$ and $\sum_{m,n \in \mathbb{Z}}'$ means $(0, 0)$ is excluded from the summation.

Then, by taking $\tau$ to be some appropriate CM points (quadratic algebraic numbers in $\mathcal{H}$) in (1.8), he proved that
\begin{equation}
m(4\sqrt{2}) = L'(E_4\sqrt{2}, 0), \quad m(2\sqrt{2}) = L'(E_{2\sqrt{2}}, 0), \quad m(4i) = 2L'(E_4i, 0).
\end{equation}

Although much research has been done, only a few $k$ with $E_k$ defined over $\mathbb{Q}$ are rigorously proved to satisfy (1.5). Samart made a comprehensive table of these $k$ in [20], they are
\begin{equation}
k = 1, 3i, 5, 16, i, 3, 2, 3\sqrt{2}, 8, i\sqrt{2}, 4i, 2\sqrt{2}, 2i, 12, \sqrt{2}, 4\sqrt{2}.
\end{equation}
He also listed the conductors of each $E_k$, the rational numbers $r_k$, and their references. See [4, 14, 15, 16, 19, 21, 26] for more about this subject.
One may wonder what will happen if $E_k$ is defined over a number field $K$ with $r = [K : \mathbb{Q}] > 1$. In fact, by Theorem 1.1, if there is a CM point $\tau \in \mathcal{F}$ such that $k = \frac{4}{\sqrt{\lambda(2\tau)}}$, then we can still write $m(k)$ as a lattice sum, and thus it can also be written as $L$-value of a modular form. However, in order to relate $m(k)$ to the $L$-value of $E_k$, we need to turn to a $r \times r$ determinant with $m(k)$ as its entries according to Beilinson’s conjecture.

For example, in the real quadratic case $k = 4 \pm 4\sqrt{2}$, by taking $\tau = \frac{4\sqrt{2}}{2}$ and $i\sqrt{2}$ into (1.8), Guo, Ji, Liu, and Qin proved in [11] that

\[(1.10) \quad m(4 + 4\sqrt{2}) = \frac{16\sqrt{2}}{\pi^2} L(f, 2), \quad m(4 - 4\sqrt{2}) = \frac{16\sqrt{2}}{\pi^2} L(g, 2), \]

where $f(\tau) = \frac{n(8\tau)(16\tau)^5}{\eta(16\tau)^2}$ and $g = 2 \frac{n(8\tau)^2\eta(32\tau)^2}{\eta(16\tau)}$ are weight 2 cusp forms for $\Gamma_1(64)$. Then, by constructing the associated Beilinson regulator, they proved that

\[(1.11) \quad \left| \det \begin{pmatrix} m(4 + 4\sqrt{2}) & m(4 - 4\sqrt{2}) \\ m(4 - 4\sqrt{2}) & -m(4 + 4\sqrt{2}) \end{pmatrix} \right| = \frac{512}{\pi^4} L(E_{4\pm4\sqrt{2}}, 2). \]

The present paper is intended to extend the work of [11]. Since $\lambda(2\tau)$ is a modular function for $\Gamma_0(4)$ with rational Fourier coefficients. It is known that $\lambda(2\tau)$ are algebraic numbers when $\tau$ takes CM points. Moreover, the degree of $\lambda(2\tau)$ have something to do with the class number of $D_\tau$, the discriminant of $\tau$. We will explore the connection between them and this will help us find some CM points that make $\lambda(2\tau)$ look good. Then we will use these points to prove identities similar to (1.10). We summarize the results in the following theorem.

**Theorem 1.2.** We have identities of the form

\[ m(k) = \frac{c_k}{\pi^2} L(f_k, 2) \]

for the following 28 $k$:

\[
\begin{align*}
12 \pm 8\sqrt{2}, & \quad 8i\sqrt{3\sqrt{2} + 4}, \quad 8\sqrt{3\sqrt{2} - 4}, \quad 2i\sqrt{2\sqrt{2} - 2}, \quad 2\sqrt{2\sqrt{2} + 2} \\
4i\sqrt{2\sqrt{2} + 2}, & \quad 4\sqrt{2\sqrt{2} - 2}, \quad (8 \pm 4\sqrt{2})i, \quad \sqrt{2} \pm \sqrt{6}, \quad 4\sqrt{2} \pm 4\sqrt{6}, \quad 2\sqrt{3} \pm 2i \\
(32\sqrt{2} \pm 12\sqrt{7})i, & \quad \frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2}, \quad 24\sqrt{2} \pm 8\sqrt{14}, \quad 6 \pm 2i\sqrt{7}, \quad \frac{3}{2} \pm \frac{i\sqrt{7}}{2}, \quad \frac{3\sqrt{7}}{2} \pm \frac{i}{2}.
\end{align*}
\]

Where $f_k$ are normalized weight 2 cusp forms for $\Gamma_0(N_k)$ of the form

\[ r \sum_{m,n \in \mathbb{Z}} \chi_\lambda(n)(lm + sn)q^{am^2 + bmn + cn^2}, \]

$c_k$ are quadratic numbers such that $c_k^2 \in \mathbb{Q}$. To save space, we list the explicit expressions of $f_k$, the levels $N_k$ and $c_k$ in Table 2.

When $E_k$ is defined over a number field $K$, the approach of Guo et al. used to construct the Beilinson regulator requires that the rational transformation (1.2) is also defined over $K$. One can check that it is the case for $k = 12 \pm 8\sqrt{2}$. So we can follow their approach and prove the following formula.

**Theorem 1.3.** We have

\[(1.12) \quad \left| \det \begin{pmatrix} m(12 + 8\sqrt{2}) & m(12 - 8\sqrt{2}) \\ m(12 - 8\sqrt{2}) & m(12 + 8\sqrt{2}) \end{pmatrix} \right| = \frac{1024}{\pi^4} L(E_{12\pm8\sqrt{2}}, 2). \]
For the cases when \( k = \sqrt{2} \pm \sqrt{6}, \ 4\sqrt{2} \pm 4\sqrt{6}, \ \frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2}, \ 24\sqrt{2} \pm 8\sqrt{14} \). Although \( E_k \) will still be defined over real quadratic fields. The rational transformations (1.2) are no longer defined over the same field. However, by changing the Weierstrass model of \( P_k(x,y) = 0 \) to

\[
E'_k : Y^2 = X^3 + \left(\frac{k^2}{2} - 4\right)X^2 + 4X,
\]

we can still prove \( 2 \times 2 \) formulas for these \( k \):

**Theorem 1.4.** Let \( E'_k \) be the curve defined above, then we have

\[
\begin{align*}
(1.13) \quad & \det \left( \begin{array}{cc} m(\sqrt{2} + \sqrt{6}) & m(\sqrt{2} - \sqrt{6}) \\ 3m(\sqrt{2} - \sqrt{6}) & -m(\sqrt{2} + \sqrt{6}) \end{array} \right) = \frac{144}{\pi^4} L(E'_{\sqrt{2}+\sqrt{6}}, 2), \\
(1.14) \quad & \det \left( \begin{array}{cc} m(4\sqrt{2} + 4\sqrt{6}) & m(4\sqrt{2} - 4\sqrt{6}) \\ m(4\sqrt{2} - 4\sqrt{6}) & -3m(4\sqrt{2} + 4\sqrt{6}) \end{array} \right) = \frac{2304}{\pi^4} L(E'_{4\sqrt{2}+\sqrt{6}}, 2), \\
(1.15) \quad & \det \left( \begin{array}{cc} m\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2}\right) & m\left(\frac{3\sqrt{2}}{2} - \frac{\sqrt{14}}{2}\right) \\ 7m\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{14}}{2}\right) & -m\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2}\right) \end{array} \right) = \frac{196}{\pi^4} L(E'_{\frac{3\sqrt{2}}{2}+\frac{\sqrt{14}}{2}}, 2), \\
(1.16) \quad & \det \left( \begin{array}{cc} m(24\sqrt{2} + 8\sqrt{14}) & m(24\sqrt{2} - 8\sqrt{14}) \\ m(24\sqrt{2} - 8\sqrt{14}) & -7m(24\sqrt{2} + 8\sqrt{14}) \end{array} \right) = \frac{12544}{\pi^4} L(E'_{24\sqrt{2}+8\sqrt{14}}, 2).
\end{align*}
\]

This paper is organized as follows.

In Section 2, we study the modular function \( \lambda(2\tau) \) for \( \Gamma_0(4) \). It is proved that when \( \tau_0 \) is a CM point with discriminant \( D_{\tau_0} \), then \( \lambda(2\tau_0) \) will be an algebraic number with degree no more than \( h(D_{\tau_0})h(D_{4\tau_0}) \), where \( h(D) \) is the class number of primitive binary quadratic forms with discriminant \( D \).

In Section 3, we will use all negative discriminants \( D \) with \( h(D) \leq 2 \) and an algorithm to obtain a list (see Table 1) of CM points in \( \mathcal{F}' \) with \( h(D_{\tau_0})h(D_{4\tau_0}) \leq 4 \) and distinct \( \lambda(2\tau_0) \) values. Our idea is partly inspired by the work [13] of Huber, Schultz, and Ye, in which they searched CM points to derive many Ramanujan-Sato series for \( \frac{1}{\pi} \) of level 20.

In Section 4, we use the CM points in Table 1 to prove Theorem 1.2. Instead of providing the details for all cases, we only prove the case when \( k = 12 \pm 8\sqrt{2} \) as an illustration and list other results directly in Table 2. Once we know the CM points, the other cases can be proved in the same way. In addition to identities in Theorem 1.2, we also find that Table 2 reproduces all proven cases of \( m(k) \) when \( E_k \) have CM.

In Section 5, we briefly introduce Beilinson’s conjecture for curves over number fields. Our references are [9, 17].

In Section 6, we prove Theorem 1.3. Velu’s formula will be used to construct the isogeny between \( E_{12+8\sqrt{2}} \) and \( E_{12-8\sqrt{2}} \). This will help us to determine the shape of Beilinson regulator which is a \( 2 \times 2 \) determinant of \( m(12 \pm 8\sqrt{2}) \). When it comes to the \( L \)-functions of \( E_{12\pm8\sqrt{2}} \), the LMFDB database [27] helps a lot.

In Section 7, we prove Theorem 1.4. When dealing with these cases, we need to turn to the Weierstrass model \( E'_k \).

The computations were mainly performed in Mathematica. We also used SageMath and PARI/GP for elliptic curve-related calculations.

2. The degree of \( \lambda(2\tau) \)

Since according to (1.7), \( \lambda(2\tau) \) has the eta-quotient expression

\[
\lambda(2\tau) = 16\frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{16}} = 16q - 128q^2 + 704q^3 - 3072q^4 + \cdots,
\]
we can deduce (for example, from [6, Proposition 5.9.2]) that \( \lambda(2\tau) \) is a modular function for \( \Gamma_0(4) \).

By the theory of complex multiplication, we know that every modular function for \( \Gamma_0(m) \) is

\[
j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots
\]

is the modular invariant. To make it more clear, let

\[
C(m) = [\text{SL}(2, \mathbb{Z}) : \Gamma_0(m)] = m \prod_{p|m} \left(1 + \frac{1}{p}\right),
\]

and \( \gamma_i, \quad i = 1, \cdots, C(m) \)  be the right coset representatives of \( \Gamma_0(m) \) in \( \text{SL}(2, \mathbb{Z}) \). For each \( m \), there exists a polynomial (usually called the modular equation for \( \Gamma_0(m) \))

\[
\Phi_m(X, Y) \in \mathbb{Z}[X, Y]
\]
of degree \( C(m) \) that satisfies

(2.1) \[
\Phi_m(X, j(\tau)) = \prod_{i=1}^{C(m)} (X - j(m\gamma_i\tau)).
\]

**Theorem 2.1** ([7, Proposition 12.7]). Let \( f(\tau) \) be a modular function for \( \Gamma_0(m) \) whose \( q \)-expansion has rational coefficients. Then

1. \( f(\tau) \in \mathbb{Q}(j(\tau), j(m\tau)) \).
2. Assume in addition that \( f(\tau) \) is holomorphic on \( \mathcal{H} \), and let \( \tau_0 \in \mathcal{H} \). If

\[
\frac{\partial \Phi_m}{\partial X}(j(m\tau_0), j(\tau_0)) \neq 0,
\]

then \( f(\tau_0) \in \mathbb{Q}(j(\tau_0), j(m\tau_0)) \).

Now, back to the case \( m = 4 \) that we are interested in, we have \([\text{SL}(2, \mathbb{Z}) : \Gamma_0(4)] = 6 \). Let

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

be the generators of \( \text{SL}(2, \mathbb{Z}) \). One can verify that

(2.2) \[
\gamma_1 = I_2, \quad \gamma_2 = S, \quad \gamma_3 = ST, \quad \gamma_4 = ST^{-1}, \quad \gamma_5 = ST^{-2}, \quad \gamma_6 = ST^{-2}S
\]
is a set of right coset representatives of \( \Gamma_0(4) \) in \( \text{SL}(2, \mathbb{Z}) \).

For a negative integer \( D \) with \( D \equiv 0 \) or \( 1 \) mod 4, let \( h(D) \) be the class number of primitive binary quadratic forms of discriminant \( D \). By a CM point, we mean a point \( \tau_0 \in \mathcal{H} \) that satisfies

\[
a\tau_0^2 + b\tau_0 + c = 0,
\]

where \( a, b, c \) are integers with \( a > 0 \) and \( \gcd(a, b, c) = 1 \). Denote \( D_{\tau_0} := b^2 - 4ac \) be the discriminant of \( \tau_0 \). It is convenient to write \( \tau_0 \) as \((a, b, c)\).

**Theorem 2.2.** Let \( \tau_0 \in \mathcal{H} \) be a CM point. If

\[
\prod_{i \neq 1} (j(4\tau_0) - j(4\gamma_i\tau_0)) \neq 0,
\]

where \( \gamma_i \) are the coset representatives in (2.2), then \( \lambda(2\tau_0) \) is an algebraic number with degree no more than \( h(D_{\tau_0})h(D_{4\tau_0}) \).

**Proof.** It is clear that \( \lambda(2\tau) = 16 \frac{\eta(\tau)^8\eta(4\tau)^{16}}{\eta(2\tau)^{16}} \) is holomorphic on \( \mathcal{H} \) since

\[
\eta(\tau) = e^{2\pi i \tau} \prod_{n=1}^{\infty} (1 - q^n)
\]
has no zeros on $H$. Take the derivative of both sides of (2.1) with respect to $X$ and evaluate at $\tau = \tau_0$, we have

$$\frac{\partial \Phi_1}{\partial X}(j(4\tau_0), j(\tau_0)) = \prod_{i \neq 1} (j(4\tau_0) - j(4\gamma_i\tau_0)).$$

Thus, when the above equation is not zero, we know from Theorem 2.1 that

$$\lambda(2\tau_0) \in \mathbb{Q}(j(\tau_0), j(4\tau_0)).$$

Since $j(\tau_0)$ is an algebraic integer of degree $h(D_{\tau_0})$ by the theory of complex multiplication, we have

$$[\mathbb{Q}(\lambda(2\tau_0)) : \mathbb{Q}] \leq [\mathbb{Q}(j(\tau_0), j(4\tau_0)) : \mathbb{Q}]
= [\mathbb{Q}(j(\tau_0), j(4\tau_0)) : \mathbb{Q}(j(4\tau_0))][\mathbb{Q}(j(4\tau_0)) : \mathbb{Q}]
\leq [\mathbb{Q}(j(\tau_0)) : \mathbb{Q}][\mathbb{Q}(j(4\tau_0)) : \mathbb{Q}]
= h(D_{\tau_0})h(D_{4\tau_0}).$$

3. CM Points with $h(D_{\tau_0})h(D_{4\tau_0}) \leq 4$

Let $a, b, c \in \mathbb{Z}$ with $a > 0$, $\gcd(a, b, c) = 1$. If the quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is positive definite, then the discriminant $D = b^2 - 4ac$ of $f$ satisfies

$$D < 0, \quad D \equiv 0 \text{ or } 1 \mod 4.$$

We can factorize $f$ as the form

$$f(x, y) = a(x - \tau_0y)(x - \bar{\tau}_0y)$$

for a unique CM point $\tau_0 \in H$. It can be shown that $f$ is a reduced form (i.e. $|b| \leq a \leq c$, and $b \geq 0$ if either $|b| = a$ or $a = c$) if and only if $\tau_0$ lies in the fundamental domain

$$\mathcal{F} = \{\tau \in H \mid |\Re(\tau)| \leq 1/2, |\tau| \geq 1, \text{ and } \Re(\tau) \geq 0 \text{ if } |\Re(\tau)| = 1/2 \text{ or } |\tau| = 1\}$$

of $\text{SL}(2, \mathbb{Z})$. It is well known that there are exactly $h(D)$ CM points of discriminant $D$ in $\mathcal{F}$, each corresponds to a unique CM point with discriminant $D$.

Recall that a discriminant $D < 0$ is said to be fundamental if there is an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-N})$ with $N \geq 1$ and square-free, such that $D = d_K$, where

$$d_K = \begin{cases} N, & \text{if } N \equiv 1 \mod 4, \\ 4N, & \text{otherwise} \end{cases}$$

is the field discriminant of $K$. It is a famous result that Gauss conjectured and K. Heegner and H. M. Stark proved in [12] and [22] that the only negative fundamental discriminants $d_K$ with $h(d_K) = 1$ are

$$(3.1) \quad d_K = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$  

Later, Stark also determined all fundamental discriminants of class number 2 in [23], they are

$$(3.2) \quad d_K = -15, -20, -24, -35, -40, -51, -52, -88, -91, -115, -123, -148,$$

$$- 187, -232, -235, -267, -403, -427.$$  

Recall that an order $\mathcal{O}$ in a quadratic field $K$ is a $\mathbb{Z}$-submodule of the form

$$\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K,$$
where $\mathcal{O}_K$ is the ring of integers of $K$ and $f = [\mathcal{O}_K : \mathcal{O}]$ is the conductor of $\mathcal{O}$. In particular, $\mathcal{O}_K$ is the maximal order of conductor 1. The discriminant of $\mathcal{O}$ is defined by $f^2d_K$, and we have [7, Corollary 7.28]

$$h(f^2d_K) = \frac{h(d_K)f}{[\mathcal{O}_K : \mathcal{O}]} \prod_{p|f} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right),$$

By using (3.3), together with (3.1) and (3.2), we can determine all the discriminants $D$ such that $h(D) \leq 2$. To achieve this, consider the unit groups of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-N})$

$$\mathcal{O}_K = \begin{cases} \{\pm 1, \pm i\}, & \text{if } N = -1, \\ \{\pm 1, \frac{1}{2}(\pm 1 \pm i\sqrt{3})\}, & \text{if } N = -3, \\ \{\pm 1\}, & \text{otherwise.} \end{cases}$$

First, suppose that $\mathcal{O}_K^* = \{\pm 1\}$. If $f > 6$, then

$$f \prod_{p|f} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right) > 2,$$

and (3.3) tells us that the only possible $(d_K, f)$ such that

$$\begin{cases} d_K \neq -3 \text{ or } -4, \\ f > 1, \\ h(f^2d_K) \leq 2, \end{cases}$$

are

$$(-7, 2), (-8, 2), (-15, 2), (-8, 3), (-11, 3), (-7, 4).$$

Similarly, when $d_K = -3$ or $-4$, the only possible $(d_K, f)$ such that $f > 1$ and $h(f^2d_K) \leq 2$ are

$$(-3, 2), (-4, 2), (-3, 3), (-4, 3), (-3, 4), (-4, 4), (-3, 5), (-4, 5), (-3, 7).$$

This completes our goal.

**Theorem 3.1.** Let $D \equiv 0$ or $1 \pmod{4}$ be negative. Then

1. $h(D) = 1$ if and only if $D$ is one of the following numbers:

   $$-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163.$$

2. $h(D) = 2$ if and only if $D$ is one of the following numbers:

   $$-15, -20, -24, -32, -35, -36, -40, -48, -51, -52, -60, -64, -72, -75, -88, -91,$$

   $$-99, -100, -112, -115, -123, -147, -148, -187, -232, -235, -267, -403, -427.$$

Let $S, T$ be the generators of $\text{SL}(2, \mathbb{Z})$ as in Section 2. Note that

$$\mathcal{F} \cup S\mathcal{F} \cup ST\mathcal{F} \cup ST^{-1}\mathcal{F} \cup ST^2\mathcal{F} \cup ST^{-2}\mathcal{F} \cup ST^2S\mathcal{F} \cup ST^{-2}S\mathcal{F}$$

covers the region $\mathcal{F}'$ in Theorem 1.1; see Figure 1. We can now apply Algorithm 3.2 to get a list of CM points $\tau_0 \in \mathcal{F}'$ such that $h(D_{\tau_0})h(D_{4\tau_0}) \leq 4$ and have distinct values of $\lambda(2\tau_0)$ (by numerical calculation).

**Algorithm 3.2.**

**Input:** The discriminants in Theorem 3.1.

**Output:** A list (Table 1) of CM points in $\mathcal{F}'$ with $h(D_{\tau_0})h(D_{4\tau_0}) \leq 4$.

(1) For each $D$ in Theorem 3.1, determine all CM points of discriminant $D$ in the fundamental domain $\mathcal{F}$ of $\text{SL}(2, \mathbb{Z})$.

(2) Determine a full list of CM points that lie in $\mathcal{F}'$ with $h(D_{\tau_0}) \leq 2$ by using $S, ST, ST^{-1}, ST^2, ST^{-2}, ST^2S, ST^{-2}S$ (they come from the covering (3.4)) to translate the CM points obtained by (1).
(3) For each point $\tau_0$ obtained by (2). If $h(D_{\tau_0})h(D_{4\tau_0}) \leq 4$, then calculate $\lambda(2\tau_0)$ numerically and take a CM point for each different value to make Table 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The region $\mathcal{F}'$ in Theorem 1.1 and its covering (3.4).}
\end{figure}
### Table 1. The outputs of Algorithm 3.2

| $\tau_0$ | $h(D_{2\tau_0})$ | $h(D_{\tau_0})h(D_{4\tau_0})$ | $\lambda(2\tau_0)$ (take 5 valid digits) |
|----------|-----------------|-----------------|--------------------------------------|
| $(2, -2, 1)$   | 1               | 1               | $-1.0000$                             |
| $(4, 0, 1)$     | 1               | 1               | $0.50000$                             |
| $(8, -4, 1)$    | 1               | 1               | $2.00000$                             |
| $(16, 16, 5)$   | 1               | 2               | $-32.970$                             |
| $(16, 0, 1)$    | 1               | 2               | $0.97056$                             |
| $(1, 0, 1)$     | 1               | 2               | $0.029437$                            |
| $(5, -4, 1)$    | 1               | 2               | $33.970$                              |
| $(4, -4, 5)$    | 1               | 4               | $-0.030330$                           |
| $(20, -4, 1)$   | 1               | 4               | $1.030330$                            |
| $(8, 8, 3)$     | 1               | 2               | $-4.8284$                             |
| $(8, 0, 1)$     | 1               | 2               | $0.82842$                             |
| $(2, 0, 1)$     | 1               | 2               | $0.17157$                             |
| $(6, 4, 1)$     | 1               | 2               | $5.8284$                              |
| $(4, 4, 3)$     | 1               | 4               | $-0.20710$                            |
| $(12, 4, 1)$    | 1               | 4               | $1.20710$                             |
| $(3, 3, 1)$     | 1               | 2               | $-13.928$                             |
| $(1, 1, 1)$     | 1               | 2               | $-0.071796$                           |
| $(16, 4, 1)$    | 1               | 2               | $1.07179$                             |
| $(16, 12, 3)$   | 1               | 2               | $14.928$                              |
| $(4, 0, 3)$     | 1               | 4               | $0.066987$                            |
| $(12, 0, 1)$    | 1               | 4               | $0.93301$                             |
| $(4, 2, 1)$     | 1               | 1               | $0.50000 - 0.86602i$                  |
| $(4, -2, 1)$    | 1               | 1               | $0.50000 + 0.86602i$                  |
| $(7, 7, 2)$     | 1               | 2               | $-253.99$                             |
| $(1, 1, 2)$     | 1               | 2               | $-0.0039370$                          |
| $(32, 4, 1)$    | 1               | 2               | $1.0039$                              |
| $(32, 28, 7)$   | 1               | 2               | $254.99$                              |
| $(4, 0, 7)$     | 1               | 4               | $0.0039216$                           |
| $(28, 0, 1)$    | 1               | 4               | $0.99607$                             |
| $(2, 1, 1)$     | 1               | 1               | $0.031250 - 0.24803i$                 |
| $(2, -1, 1)$    | 1               | 1               | $0.031250 + 0.24803i$                 |
| $(4, 3, 1)$     | 1               | 1               | $0.50000 - 3.9686i$                   |
| $(4, -3, 1)$    | 1               | 1               | $0.50000 + 3.9686i$                   |
| $(8, 2, 1)$     | 1               | 1               | $0.96875 - 0.24803i$                  |
| $(8, -2, 1)$    | 1               | 1               | $0.96875 + 0.24803i$                  |
| $(4, -1, 1)$    | 2               | 4               | $0.50000 + 0.30096i$                  |
| $(4, 1, 1)$     | 2               | 4               | $0.50000 - 0.30096i$                  |
| $(8, 7, 2)$     | 2               | 4               | $0.50000 - 27.411i$                   |
| $(8, -7, 2)$    | 2               | 4               | $0.50000 + 27.411i$                   |
| $(2, 1, 2)$     | 2               | 4               | $0.00066519 - 0.036468i$               |
| $(2, -1, 2)$    | 2               | 4               | $0.00066519 + 0.036468i$               |
| $(6, 3, 1)$     | 2               | 4               | $1.4680 - 0.88368i$                   |
| $(6, -3, 1)$    | 2               | 4               | $1.4680 + 0.88368i$                   |
| $(8, 6, 3)$     | 2               | 4               | $-0.46808 - 0.88368i$                 |
| $(8, -6, 3)$    | 2               | 4               | $-0.46808 + 0.88368i$                 |
| $(16, 2, 1)$    | 2               | 4               | $0.99933 - 0.036468i$                 |
| $(16, -2, 1)$   | 2               | 4               | $0.99933 + 0.036468i$                 |
4. Mahler Measures and L-values of Modular Forms

To illustrate how the data in Table 1 allows us to write Mahler measures as L-values of modular forms, we calculate two examples in detail, they are \( m(12 \pm 8\sqrt{2}) \).

Before working out these two examples, recall that the modular Weber functions (see, for example [7, §12]) are

\[
f(\tau) = q^{\frac{1}{24}} \frac{\eta((\tau + 1)/2)}{\eta(\tau)}, \quad f_1(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}, \quad f_2(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)}.
\]

They satisfy the following identities:

\[
f(\tau)f_1(\tau)f_2(\tau) = \sqrt{2}, \quad f_1(2\tau)f_2(\tau) = \sqrt{2},
\]

(4.1)

\[
j(\tau) = \frac{(f(\tau)^{24} - 16)^3}{f(\tau)^{24}} = \frac{(f_1(\tau)^{24} + 16)^3}{f_1(\tau)^{24}} = \frac{(f_2(\tau)^{24} + 16)^3}{f_2(\tau)^{24}},
\]

(4.2)

\[
\lambda(2\tau) = \frac{f_1(2\tau)^8f_2(2\tau)^{16}}{16}.
\]

Proof of the cases \( k = 12 \pm 8\sqrt{2} \) in Theorem 1.3. Take \( \tau_0 = (1, 0, 1) = i \). Since \( D(2\tau_0) = -16 \) and \( h(-16) = 1 \), we know that \( j(2\tau_0) \) is a rational integer. It is not difficult to find that

\[
j(2\tau_0) = 287496
\]

by numerical calculation. Thus, by (4.1) and some numerical calculation for \( f_1 \) and \( f_2 \), we have

\[
f_1(2\tau_0)^{24} = 512, \quad f_2(2\tau_0)^{24} = -280 + 192\sqrt{2}.
\]

Then (4.2) tells us that

\[
\lambda(2\tau_0) = 17 - 12\sqrt{2},
\]

and the corresponding

\[
k = \frac{4}{\sqrt{\lambda(2\tau_0)}} = 12 + 8\sqrt{2}.
\]

Now, all the ingredients are in place to use Theorem 1.1. Take \( \tau = \tau_0 \) in (1.8), we have

\[
m(12 + 8\sqrt{2}) = \text{Re} \left( \frac{16 \text{ Im}(\tau_0)}{\pi^2} \sum_{m,n \in \mathbb{Z}}' \frac{\chi_4(n)}{(4m\tau_0 + n)^2(4m\bar{\tau}_0 + n)} \right)
\]

\[
= \text{Re} \left( \frac{16}{\pi^2} \sum_{m,n \in \mathbb{Z}}' \frac{\chi_4(n)(4m\bar{\tau}_0 + n)}{4m\tau_0 + n^4} \right)
\]

\[
= \frac{16}{\pi^2} \sum_{m,n \in \mathbb{Z}}' \frac{\chi_4(n)n}{(16m^2 + n^2)^2}
\]

\[
= \frac{32}{\pi^2} \left( \frac{1}{2} \sum_{m,n \in \mathbb{Z}}' \frac{\chi_4(n)n}{(16m^2 + n^2)^2} \right).
\]

One can deduce from [6, Corollary 14.3.16] that the theta function

\[
f_{12+8\sqrt{2}} = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \chi_4(n)q^{16m^2+n^2} = q - 3q^9 + 2q^{17} - q^{25} + 10q^{41} - 7q^{49} + \cdots
\]

is a normalized cusp form in \( S_2(\Gamma_0(64)) \). This implies that

\[
m(12 + 8\sqrt{2}) = \frac{32}{\pi^2} L(f_{12+8\sqrt{2}}, 2).
\]
Next, take $\tau_1 = (5, -4, 1) = \frac{2 + i}{5}$. We can calculate that

\[
j(2\tau_1) = 287496, \quad f_1(2\tau_1)^{24} = 512, \quad f_2(2\tau_1)^{24} = -280 - 192\sqrt{2},
\]

\[
\lambda(2\tau_1) = 17 + 12\sqrt{2}, \quad k = \frac{4}{\sqrt{\lambda(2\tau_1)}} = 12 - 8\sqrt{2}.
\]

By taking $\tau = \tau_1$ in (1.8), we have

\[
m\left(12 - 8\sqrt{2}\right) = \frac{16}{5\pi^2} \sum_{m,n\in\mathbb{Z}}' \frac{\chi_4(n)\left(\frac{8m}{5} + n\right)}{(\frac{8m}{5} + n)^2 + (\frac{4n}{5})^2}^2
\]

\[
= \frac{64}{\pi^2} \left(\frac{1}{4} \sum_{m,n\in\mathbb{Z}}' \frac{\chi_4(n)(8m + 5n)}{16m^2 + 16mn + 5n^2}\right).
\]

Once again, according to [6, Corollary 14.3.16], the theta function

\[
f_{12-8\sqrt{2}} = \frac{1}{4} \sum_{m,n\in\mathbb{Z}} \chi_4(n)(8m + 5n)q^{16m^2+16mn+5n^2}
\]

\[
= q^5 - 3q^{13} + 5q^{29} + q^{37} - 3q^{45} - 7q^{53} + 5q^{61} + 2q^{85} + \cdots
\]

is a normalized cusp form in $S_2(\Gamma_0(64))$. This implies that

\[
m\left(12 - 8\sqrt{2}\right) = \frac{64}{\pi^2} L(f_{12-8\sqrt{2}}, 2).
\]

\[\square\]

For the other CM points in Table 1 with $h(D_{2m}) = 1$, one can also repeat the above calculation to obtain identities of the form

\[
m(k) = \frac{ck}{\pi^2} L(f_k, 2),
\]

where $f_k$ are normalized cusp forms in $S_2(\Gamma_0(N_k))$. To avoid lengthy calculations, we list the results directly in Table 2.

**Remark 4.1.**

1. As mentioned in the Introduction, the first three cases in table 2 are the results (1.9) proved by Villegas. The cases $k = 4 \pm 4\sqrt{2}$ have also been proven in [11]. And the cases $k = \sqrt[4]{8}(\sqrt{2} - 1)i$ and $\sqrt[4]{8}(\sqrt{2} + 1)$ are in fact proved by Samart in [21].

2. We do not guarantee that we have found all $k = \frac{4}{\sqrt{\lambda(2\tau_0)}}$ with degree $\leq 4$ that come from CM points. Since the degree of $\lambda(2\tau_0)$ may be strictly less than $h(D_{\tau_0})h(D_{4\tau_0})$.  


curves over number field. The content of 
This section is devoted to give a relatively detailed statement of Beilins on’s conjecture for 
K
We start with some 
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m + n)q^{m^2 + 4mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(-2m + n)q^{m^2 - 4mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(8m + 5n)q^{6m^2 + 16mn + 5n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(2m + 5n)q^{4m^2 + 4mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(-2m + n)q^{7m^2 - 4mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(3m - 2n)q^{2m^2 - 3mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - 2n)q^{2m^2 - mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(7m - 4n)q^{7m^2 - 7mn + 2n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - n)q^{3m^2 - 2mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(2m - n)q^{3m^2 - 2mn + 7n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(2m - n)q^{3m^2 - mn + 2n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(7m - 4n)q^{7m^2 - 7mn + 2n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - 2n)q^{2m^2 - mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - n)q^{3m^2 - 2mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(2m - n)q^{3m^2 - mn + 2n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - n)q^{3m^2 - 2mn + n^2} \]
\[ \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m - n)q^{3m^2 - 2mn + n^2} \]

5. BEILINSON’S CONJECTURE FOR CURVES OVER NUMBER FIELDS

This section is devoted to give a relatively detailed statement of Beilinson’s conjecture for curves over number field. The content of [9] is widely quoted.

We start with some K-theory settings. Let F be a field, the group K_{2}(F) can be defined as

\[ F^* \otimes_{\mathbb{Z}} F^*/\langle a \otimes (1 - a), a \in F, a \neq 0, 1 \rangle. \]
The class of $a \otimes b$ is usually denoted by $\{a, b\}$. One can see form definition that $K_2(F)$ is an abelian group (we use “+” for the operator) given by generators $\{a, b\}$ for $a, b \in F^*$ and relations

\[
\begin{align*}
\{a_1a_2, b\} &= \{a_1, b\} + \{a_2, b\}, \\
\{a, b_1b_2\} &= \{a, b_1\} + \{a, b_2\}, \\
\{a, 1 - a\} &= 0, \text{ for } a \in F, a \neq 0, 1.
\end{align*}
\]

Let $K$ be a number field with the ring of integers $\mathcal{O}_K$ and $C/K$ be a (non-singular, projective, geometrically irreducible) curve of genus $g$ with function field $F = K(C)$. Define the tame $K_2^T$ of $C$ as

\[
K_2^T(C) = \ker \left( K_2(F) \xrightarrow{T} \bigoplus_{x \in C(\mathbb{Q})} \mathbb{Q}^* \right),
\]

where the $x$-component of $T$ is given by the tame symbol

\[
(5.1) \quad T_x(\{a, b\}) = (-1)^{\text{ord}_x(a)\text{ord}_x(b)} a^{\text{ord}_x(b)} \mathbb{Q}^{\text{ord}_x(a)}(x).
\]

For each $\{a, b\} \in K_2^T(C)$, there is an almost everywhere defined 1-form

\[
\eta(a, b) = \log |a| d \arg b - \log |b| d \arg a
\]
on the Riemann surface $C(\mathbb{C})$, where $d \arg a$ is defined by $\Re(da/a)$. One can check that

\[
\langle \cdot , \cdot \rangle : H_1(C(\mathbb{C}); \mathbb{Z}) \times K_2^T(C)/\text{torsion} \to \mathbb{R},
\]

\[
(\gamma, \{a, b\}) \mapsto \frac{1}{2\pi} \int_\gamma \eta(a, b).
\]
is a well-defined pairing. (Note that when calculating the above integral, we choose a representative of $\gamma$ that avoids the set of zeros and poles of $a$ and $b$.) Since $\eta(a, b)$ changes sign under complex conjugation, it can be shown that $\langle \cdot , \cdot \rangle$ is in fact equal to zero on $H_1(C(\mathbb{C}); \mathbb{Z})^+$, the subgroup of $H_1(C(\mathbb{C}); \mathbb{Z})$ consisting of loops that invariant under complex conjugation. We only need to consider the above pairing for $\gamma \in H_1(C(\mathbb{C}); \mathbb{Z})^-$, the subgroup of $H_1(C(\mathbb{C}); \mathbb{Z})$ on which complex conjugation acts by multiplying $-1$. Thus, we obtain the regulator pairing

\[
(5.2) \quad \langle \cdot , \cdot \rangle : H_1(C(\mathbb{C}); \mathbb{Z})^- \times K_2^T(C)/\text{torsion} \to \mathbb{R}.
\]

From basic topology we know that $H_1(C(\mathbb{C}); \mathbb{Z})$ has rank $g$. Beilinson originally conjectured that $K_2^T(C)/\text{torsion}$ also has rank $g$ and the pairing (5.2) is non-degenerate. However, Bloch and Grayson discovered in [1] that the rank of $K_2^T(C)/\text{torsion}$ can sometimes $> g$. As a modification, we turn to a subgroup $K_2(C; \mathbb{Z})$ of $K_2^T(C)/\text{torsion}$ defined by

\[
K_2(C; \mathbb{Z}) = \ker \left( K_2(F) \xrightarrow{T} \bigoplus_{D \in \mathcal{D}} F(D)^* \right) \subset K_2^T(C)/\text{torsion},
\]

where $\mathcal{D}$ runs through all irreducible curves on a regular proper model $C/\mathcal{O}_K$ of $C/K$, and $F(D)$ is the residue field at $D$. The $D$-component of $T$ is given by the tame symbol similar to (5.1):

\[
T_D(\{a, b\}) = (-1)^{v_D(a)v_D(b)} a^{v_D(b)} \mathbb{Q}^{v_D(a)}(D),
\]

where $v_D$ is is the valuation corresponding to $D$. It is expected that $K_2(C; \mathbb{Z})$ has rank $g$.

For each embedding $\sigma : K \to \mathbb{C}$, by applying $\sigma$ to coefficients of the equation defining $C$, we get a curve $C^\sigma$. Let $X^\sigma$ be the connected Riemann surface of genus $g$ associated to $C^\sigma(\mathbb{C})$ and let $X$ be the disjoint union of all $X^\sigma$. In fact, $X$ is exactly the Riemann surface corresponding to $C \times_{Q} \mathbb{C}$. Through the action on $\mathbb{C}$ in $C \times_{Q} \mathbb{C}$, the complex conjugation acts on $X$, then
on $H_1(X; \mathbb{Z})$. Similarly, one can define $H_1(X; \mathbb{Z})^-$ to be the subgroup of $H_1(X; \mathbb{Z})$ on which complex conjugation acts by multiplying $-1$. We have

$$H_1(X; \mathbb{Z})^- = \bigoplus_{\sigma} H_1(X^\sigma; \mathbb{Z})^-$$

and thus the rank of $H_1(X; \mathbb{Z})^-$ is $r = g[K : \mathbb{Q}]$.

To state Beilinson’s conjecture for curves over number fields, we extend the definition of the regulator pairing $\langle \cdot, \cdot \rangle$ to all $H_1(X; \mathbb{Z})^-$ by putting

$$\langle \gamma, \{a, b\} \rangle = \frac{1}{2\pi} \int_{\gamma} \eta(a^\sigma, b^\sigma)$$

for $\gamma \in H_1(X^\sigma; \mathbb{Z})^-$, where $a^\sigma$ is the function on $X^\sigma$ obtained by applying $\sigma$ to the coefficients of $a \in K(C)$.

**Conjecture 5.1 (Beilinson).** Let $C$ be a non-singular, projective, geometrically irreducible curve of genus $g$ defined over $K$ and let $X$ be defined as above, then

1. $K_2(C; \mathbb{Z})$ is a free abelian group of rank $r = g[K : \mathbb{Q}]$ and the pairing $\langle \cdot, \cdot \rangle : H_1(X; \mathbb{Z})^- \times K_2(C; \mathbb{Z}) \to \mathbb{R}$ is non-degenerate;
2. The absolute value $R$ (this is called the Beilinson regulator) of the determinant of the pairing in (1) with respect to $\mathbb{Z}$-bases of $H_1(X; \mathbb{Z})$ and $K_2(C; \mathbb{Z})$ is a nonzero rational multiple of $\pi^{-2r}L(C, 2)$, where $L(C, s)$ is the $L$-function of $C$.

The first part of Conjecture 5.1 is widely open. In fact, it is not yet known whether $K_2(C; \mathbb{Z})$ is finite generated, let alone to find its $\mathbb{Z}$-bases. However, if we can construct $r$ elements in $K_2(C; \mathbb{Z})$ with the associated $R \neq 0$, then these $r$ elements are linearly independent over $\mathbb{Z}$. In this case, we may be able to verify the second part of Conjecture 5.1.

### 6. Proofs of Theorem 1.3

When $k = 12 \pm 8\sqrt{2}$, under the rational transformation (1.3), we have Weierstrass equations

$$E_{12+8\sqrt{2}} : Y^2 = X^3 + (66 + 48\sqrt{2})X^2 + X,$$
$$E_{12-8\sqrt{2}} : Y^2 = X^3 + (66 - 48\sqrt{2})X^2 + X.$$  

They are defined over $K = \mathbb{Q}(\sqrt{2})$. Let $\sigma \in \text{Gal}(K/\mathbb{Q})$ be the nontrivial element that sends $\sqrt{2}$ to $-\sqrt{2}$. For simplicity, we write $E$ for $E_{12+8\sqrt{2}}$, then $E^\sigma$ is for $E_{12-8\sqrt{2}}$ according to notations in Section 5.

An elliptic curve over a (Galois) number field is said to be a $\mathbb{Q}$-curve if it is isogenic to its every Galois conjugations. Since $E$ has CM and every elliptic curve with CM is a $\mathbb{Q}$-curve (see [3, Section 2]), there must exist an isogeny from $E$ to $E^\sigma$. By using PARI/GP, we compute the lattices that associated to $E$ and $E^\sigma$, and discover that there seems to have an isogeny $\phi : E \to E^\sigma$ of degree 4 with kernel

$$G = \left\{ O, (0, 0), \left( -1, 4\sqrt{4 + 3\sqrt{2}} \right), \left( -1, -4\sqrt{4 + 3\sqrt{2}} \right) \right\}.$$

To obtain this isogeny, we first use Vélu’s formula (see [25, Theorem 12.16] or [10, Theorem 25.1.6]) to construct an elliptic curve

$$\tilde{E} : Y^2 = X^3 + (66 + 48\sqrt{2})X^2 + (1276 + 960\sqrt{2})X + 137464 + 96960\sqrt{2}$$
and isogeny \( \psi : E \to \tilde{E} \) with kernel \( \ker \psi = G \). This isogeny is given by \((X,Y) \mapsto (\psi_1(X), Y\psi_2(X))\), where
\[
\psi_1(X) = \frac{X^4 + 2X^3 - 2(127 + 96\sqrt{2})X^2 + 2X + 1}{X(X + 1)^2},
\]
\[
\psi_2(X) = \frac{(X - 1)(X^4 + 4X^3 + 2(131 + 96\sqrt{2})X^2 + 4X + 1)}{X^2(X + 1)^3}.
\]

Moreover, we have (by [10, Theorem 25.1.6])
\[
(6.1) \quad \psi^*(\omega_{\tilde{E}}) = \omega_E,
\]
where \( \omega_{\tilde{E}} \) and \( \omega_E \) are invariant differentials of \( \tilde{E} \) and \( E \) defined by \( \frac{dx}{y} \). Next, set \( u = \frac{3}{2} - \sqrt{2}, r = -\frac{49}{2} + 18\sqrt{2} \), one can easily verify that
\[
(6.2) \quad \varphi : \tilde{E} \to E^\sigma, \quad (X,Y) \mapsto (u^2X + r, u^3Y)
\]
is an isomorphism over \( K \). It follows that \( \phi = \varphi \circ \psi : E \to E^\sigma \) is our desire isogeny with kernel \( G \). We can explicitly write down its formula as \( \phi : (X,Y) \mapsto (\phi_1(X), Y\phi_2(X)) \), where
\[
\phi_1(X) = \frac{(X - 1)^2((17 - 12\sqrt{2})X^2 - 6(5 - 4\sqrt{2})X + 17 - 12\sqrt{2})}{4X(X + 1)^2},
\]
\[
\phi_2(X) = \frac{(99 - 70\sqrt{2})(X - 1)(X^4 + 4X^3 + 2(131 + 96\sqrt{2})X^2 + 4X + 1)}{8X^2(X + 1)^3}.
\]

By applying \( \sigma \) to coefficients of \( \phi \), we obtain an isogeny \( \phi^* : E^\sigma \to E \). One can check that \( \phi^* \circ \phi = [4] \).

Let \(|x| = 1\), we assume \( x = e^{i\theta} \). Then \( x + \frac{1}{x} + k = 2\cos \theta + k \in \mathbb{R} \) if \( k \in \mathbb{R} \). When \( |2\cos \theta + k| > 2 \), the equation
\[
(6.3) \quad y + \frac{1}{y} = -k - x - \frac{1}{x}
\]
have two real roots \( y_1(x), y_2(x) \). They are not equal since the discriminant \( \Delta = (x + \frac{1}{x} + k)^2 - 4 > 0 \). Note that \( y_1(x)y_2(x) = 1 \), without loss of generality we can always take \( y_1(x) \) to be the one that has absolute value greater than \( 1 \). However, when \( |2\cos \theta + k| \leq 2 \), the roots \( y_1, y_2 \) of (6.3) are conjugate on the unit circle \( S^1 = \{x = e^{i\theta} | \theta \in [-\pi, \pi] \} \). Notice that in this case, \( y_1 = y_2 \) only when \( 2\cos \theta + k = \pm 2 \).

If \( k = 12 + 8\sqrt{2} > 4 \), then for every \( \theta \in [-\pi, \pi] \) we have \( |2\cos \theta + k| > 2 \). Let
\[
y_1^E(x) = \frac{-x - \frac{1}{x} - 12 - 8\sqrt{2} - \sqrt{(x + \frac{1}{x} + 12 + 8\sqrt{2})^2 - 4}}{2},
\]
\[
y_2^E(x) = \frac{-x - \frac{1}{x} - 12 - 8\sqrt{2} + \sqrt{(x + \frac{1}{x} + 12 + 8\sqrt{2})^2 - 4}}{2}
\]
be the two roots of (6.3). As mentioned in the previous paragraph, \( y_1^E \) is taken to be the root that has absolute value greater than \( 1 \). Thus the circle \( S^1 \) can be lifted to a loop \( \gamma_E = \{(x, y_1^E(x)) | |x| = 1 \} \) on \( E(\mathbb{C}) \). We take the orientation of \( \gamma_E \) induced by \( \theta \) increasing. Since complex conjugation reverse the orientation, \( \gamma_E \) is an element of \( H_1(E(\mathbb{C}); \mathbb{Z})^- \). This path is called the Deninger path, one can see [5, §7.3] for more details.

However, if \( k = 12 - 8\sqrt{2} = 0.6862\cdots \), as \( x = e^{i\theta} \) varies on \( S^1 \), the discriminant \( \Delta = 0 \) when \( x = x_0 = -5 + 4\sqrt{2} + 2i\sqrt{10\sqrt{2} - 14} \) or \( x = \bar{x}_0 \). The parameter \( \theta_0 \in [-\pi, \pi] \) corresponding to
\[ x_0 = \theta_0 = \arctan \frac{2\sqrt{2+10\sqrt{2}}}{7} \]. Thus (6.3) has real roots

\[
y_1^{E^\sigma}(x) = \frac{-x^2 - (12 - 8\sqrt{2})x - 1 - \sqrt{(x^2 + (12 - 8\sqrt{2})x + 1)^2 - 4x^2}}{2x},
\]

\[
y_2^{E^\sigma}(x) = \frac{-x^2 - (12 - 8\sqrt{2})x - 1 + \sqrt{(x^2 + (12 - 8\sqrt{2})x + 1)^2 - 4x^2}}{2x}
\]

for \( \theta \in [-\theta_0, \theta_0] \). We illustrate in Figure 2 the paths of \( y_1^{E^\sigma} \) and \( y_2^{E^\sigma} \) when \( \theta \) varies from \(-\pi\) to \(\pi\). (Take the principal branch of the square root.) In particular, we indicate with thick lines the paths when \( \theta \in [-\theta_0, \theta_0] \).

Let \( \gamma_1^{E^\sigma} = \{(x, y_1^{E^\sigma}(x)) \mid \theta \in [-\theta_0, \theta_0]\} \), \( \gamma_2^{E^\sigma} = \{(x, y_2^{E^\sigma}(x)) \mid \theta \in [-\theta_0, \theta_0]\} \).

Take \( \gamma_{E^\sigma} = \gamma_1^{E^\sigma} \cup \gamma_2^{E^\sigma} \) with the orientation induced on \( \gamma_1^{E^\sigma} \) by \( \theta \) increasing and on \( \gamma_2^{E^\sigma} \) by \( \theta \) decreasing. One can check that \( \gamma_{E^\sigma} \) is an element of \( H_1(E^\sigma(C); \mathbb{Z})^- \).

Let \( M_1 = \{x, y\} \) and \( M_2 = \{\phi^*(x), \phi^*(y)\} \), where the \( x, y \) in \( M_1 \) are functions in \( K(E) \) by (1.2) and the \( x, y \) in \( M_2 \) are functions in \( K(E^\sigma) \), they pull back by \( \phi \) to functions in \( K(E) \).

Since our polynomial family \( P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k \) is tempered, i.e. the set of roots of all the face polynomials of \( P_k \) consists of roots of unity only (see [24, 18] for more about this notion), we know from [24, III.9] that \( nM_1 \in K_2(E; \mathbb{Z}) \) for some \( n \in \mathbb{N} \). Therefore, one can use \( M_1 \) and \( M_2 \) to verify the second part of Beilinson’s conjecture. The Beilinson regulator for \( \gamma_{E^\sigma}, \gamma_{E^\sigma} \in H_1(X; \mathbb{Z})^- \) and \( M_1, M_2 \) now reads

\[
R = \left| \det \begin{pmatrix} \langle \gamma_{E^\sigma}, M_1 \rangle & \langle \gamma_{E^\sigma}, M_1 \rangle \\ \langle \gamma_{E^\sigma}, M_2 \rangle & \langle \gamma_{E^\sigma}, M_2 \rangle \end{pmatrix} \right|.
\]

**Proof of (1.12).** By definition, we calculate that

\[
\langle \gamma_{E^\sigma}, M_1 \rangle = \frac{1}{2\pi} \int_{\gamma_{E^\sigma}} \eta(x, y)
\]

\[
= \frac{1}{2\pi} \int_{\gamma_{E^\sigma}} \log |x| \Im \left( \frac{dy}{y} \right) - \log |y| \Im \left( \frac{dx}{x} \right)
\]

\[
= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |y_1^{E^\sigma}(e^{i\theta})| d\theta
\]

\[
= -m \left( 12 + 8\sqrt{2} \right),
\]

where the last equality follows by Jensen’s formula (see [2, §§1A] or [15, §§3.2]). Similarly, we have

\[
\langle \gamma_{E^\sigma}, M_1 \rangle = -2m \left( 12 - 8\sqrt{2} \right).
\]
Also, for the second row of (6.4), we have
\[
\langle \gamma_E, M_2 \rangle = \langle \gamma_E, \{ \phi^*(x), \phi^*(y) \} \rangle = \langle \phi_* \gamma_E, \{ x, y \} \rangle
\]
and
\[
\langle \gamma_{E^*}, M_2 \rangle = \langle \gamma_{E^*}, \{ \phi^*(x)^\sigma, \phi^*(y)^\sigma \} \rangle = \langle \gamma_{E^*}, \{ (\phi^\sigma)^*(x), (\phi^\sigma)^*(y) \} \rangle \quad \text{(here } x, y \in \mathbb{C}(\mathbb{C})) ,
\]
\[
= \langle (\phi^\sigma)_* \gamma_{E^*}, \{ x, y \} \rangle.
\]

In order to determine \(\phi_* \gamma_E\) and \((\phi^\sigma)_* \gamma_{E^*}\). One can calculate that

\[
\int_{\gamma_E} \omega_E = \int_{\gamma_E}^1 \frac{dX}{2Y} = \int_{\gamma_E}^1 \frac{ydx}{x(1-y^2)} \quad \text{(because } \frac{dX}{2Y} = \frac{ydx}{x(1-y^2)} \text{ by (1.4))}
\]

\[
= \int_{\gamma_E}^\pi y_1^E(e^{i\theta})de^{i\theta}
\]

\[
\approx 0.27152i
\]

and

\[
\int_{\gamma_{E^*}} \omega_{E^*} = \int_{\gamma_{E^*}}^1 \omega_{E^*} + \int_{\gamma_{E^*}}^\pi \omega_{E^*}
\]

\[
= \int_{-\theta_0}^{\theta_0} y_{E^*}^{\sigma}(e^{i\theta})de^{i\theta} + \int_{-\theta_0}^{\theta_0} y_{E^*}^{\sigma}(e^{i\theta})de^{i\theta}
\]

\[
\approx 3.1651i.
\]

Compare the above integrals of \(\omega_E\) and \(\omega_{E^*}\) with the periods of lattices associated to \(E\) and \(E^*\) calculated by PAR/\(\text{GP}\), we can deduce that \(\gamma_E\) and \(\gamma_{E^*}\) are in fact generators of \(H_1(E; \mathbb{Z})^-\) and \(H_1(E^*; \mathbb{Z})^-\). Since \(\phi\) and \(\phi^\sigma\) are defined over \(\mathbb{R}\), we know that \(\phi_*\) maps \(H_1(E(\mathbb{C}); \mathbb{Z})^-\) to \(H_1(E^*(\mathbb{C}); \mathbb{Z})^-\) and \((\phi^\sigma)_*\) maps \(H_1(E^*(\mathbb{C}); \mathbb{Z})^-\) to \(H_1(E(\mathbb{C}); \mathbb{Z})^-\). Thus, there must exist \(a, b \in \mathbb{Z}\) such that

\[
\phi_* \gamma_E = a \gamma_{E^*}, \quad (\phi^\sigma)_* \gamma_{E^*} = b \gamma_E.
\]

We have \(ab = 4\) because \(\phi^\sigma \circ \phi = [4]\). According to (6.1) and (6.2), we have

\[
\phi^*(\omega_{E^*}) = (\psi \circ \psi^*)^*(\omega_{E^*}) = \psi^*(\frac{1}{u} \omega_E) = \frac{1}{u} \omega_E.
\]

Therefore

\[
\int_{\phi_* \gamma_E} \omega_{E^*} = \int_{\gamma_E} \phi^*(\omega_{E^*}) = \frac{1}{u} \int_{\gamma_E} \omega_E \approx 3.1651i
\]

according to (6.5). Since \(a\) is an integer, by comparing (6.6) with (6.7), it follows that \(a = 1\) and then \(b = 4\).

Immediately, we have

\[
\langle \gamma_E, M_2 \rangle = \langle \gamma_{E^*}, M_1 \rangle = -2m \left(12 - 8\sqrt{2}\right),
\]

\[
\langle \gamma_{E^*}, M_2 \rangle = 4 \langle \gamma_E, M_1 \rangle = -4m \left(12 + 8\sqrt{2}\right),
\]
and

\[(6.8) \quad R = 4 \left( m \left( 12 + 8\sqrt{2} \right)^2 - m \left( 12 - 8\sqrt{2} \right)^2 \right). \]

Beilinson’s conjecture implies that $R$ should be some rational multiple of $\pi^{-4}L(E, 2)$.

In [3], Bruin and Ferraguti described an algorithm to express explicitly the $L$-functions of elliptic curves completely defined over quadratic number fields without CM as the product of two $L$-functions associated to a pair of conjugate newforms. Fortunately, in our CM case, we can seek help from the LMFDB database [27].

Since $E$ has $j$-invariant 41113158120 + 29071392966$\sqrt{2}$ and conductor norm 32. One can search in LMFDB and find that $E$ is isomorphic over $K$ to the elliptic curve with LMFDB label 2.2.8.1-32.1-a8. Also, its $L$-function with label 4-2e11-1.1-c1e2-0-0 can be written as

\[L(E, s) = L(f_{64}, s)L(f_{32}, s),\]

where

\[f_{64} = \frac{\eta(8\tau)^8}{\eta(4\tau)^2\eta(16\tau)^2} = q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + \cdots \in S_2(\Gamma_0(64)),\]

\[f_{32} = \eta(4\tau)^2\eta(8\tau)^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} + \cdots \in S_2(\Gamma_0(32)).\]

One can use Sturm bound to prove that

\[
\frac{1}{2} \sum_{m,n\in\mathbb{Z}} \chi_{-4}(n)mq^{16m^2+n^2} = \frac{f_{64} + f_{32}}{2},
\]

\[
\frac{1}{4} \sum_{m,n\in\mathbb{Z}} \chi_{-4}(n)(8m + 5n)q^{16m^2+16mn+5n^2} = \frac{f_{64} - f_{32}}{4}.
\]

According to (6.8) as well as to Table 2, we finally have

\[
R = 4 \left( \frac{32}{\pi^2} \cdot \frac{L(f_{64}, 2) + L(f_{32}, 2)}{2} \right)^2 - \left( \frac{64}{\pi^2} \cdot \frac{L(f_{64}, 2) - L(f_{32}, 2)}{4} \right)^2
\]

\[= \frac{4096}{\pi^4}L(f_{64}, 2)L(f_{32}, 2)
\]

\[= \frac{4096}{\pi^4}L(E, 2).
\]

Now, by a simplification, we can obtain (1.12). \hfill \Box

7. The Other Four Cases

For the remaining four cases $k = \sqrt{2} \pm \sqrt{6}$, $4\sqrt{2} \pm 4\sqrt{6}$, $\frac{3\sqrt{2} \pm \sqrt{17}}{2}$, and $24\sqrt{2} \pm 8\sqrt{14}$, there will be some trouble if we continue to use the rational transformation (1.2). Since for instance when $k = \sqrt{2} \pm \sqrt{6}$, $E_k$ has Weierstrass equation $Y^2 = X^3 + \sqrt{3}X^2 + X$ under (1.2). It is defined over $K = \mathbb{Q}(\sqrt{3})$. However, under this transformation, the functions $x = \frac{(\sqrt{2} \pm \sqrt{6})X - 2Y}{2X(X-1)}$ and $y = \frac{(\sqrt{2} \pm \sqrt{6})X + 2Y}{2X(X-1)}$ are not defined over $K$. It turns out that $\{x, y\}$ should not be an element of $K(T)(E_k)$ by definition.

To settle this issue, we turn to the rational transformation

\[(7.1) \quad x = \frac{kX - \sqrt{2}Y}{X(X-2)}, \quad y = \frac{kX + \sqrt{2}Y}{X(X-2)}.
\]

Under this transformation, we obtain elliptic curves

\[E_k : Y^2 = X^3 + \left( \frac{k^2}{2} - 4 \right)X^2 + 4X.
\]
And one can easily check that the functions $xy$ and $\frac{x}{y}$ are now defined over $K$ for our four interested pairs of $k$. So we will use

\[(7.2) \quad M_1 = \left\{ xy, \frac{x}{y} \right\} \quad \text{and} \quad M_2 = \left\{ \phi^*(xy), \phi^*\left(\frac{x}{y}\right) \right\}.
\]

to compute the Beilinson regulators in this section.

**Proof of (1.13).** Under (7.1), we have

\[E'_{\sqrt{3}+\sqrt{6}}: Y^2 = X^3 + 2\sqrt{3}X^2 + 4X,
\]

\[E'_{\sqrt{3}-\sqrt{6}}: Y^2 = X^3 - 2\sqrt{3}X^2 + 4X.
\]

They are defined over $K = \mathbb{Q}(\sqrt{3})$. Let $\sigma \in \text{Gal}(K/\mathbb{Q})$ be the nontrivial element. Then we can write $E$ for $E'_{\sqrt{3}+\sqrt{6}}$ and $E^\sigma$ for $E'_{\sqrt{3}-\sqrt{6}}$. There is an isogeny $\phi : E \to E^\sigma$ defined over $K$ of degree $3$, it is given by

\[\langle X, Y \rangle \mapsto \left( \frac{3X(X^2 + 4\sqrt{3}X + 12)}{(3X + 2\sqrt{3})^2}, \frac{3\sqrt{3}Y(X + 2\sqrt{3})(X^2 + 4)}{(3X + 2\sqrt{3})^3} \right).
\]

We have $\phi^* \circ \phi = [-3]$ and $\phi^*(\omega_{E^\sigma}) = \sqrt{3}\omega_E$.

Take

\[\gamma_E = \{ (x, y^E) | \theta \in [-\theta_1, \theta_1] \} \cup \{ (x, y^E_2) | \theta \in [-\theta_1, \theta_1] \} \in H_1(E; \mathbb{Z})^-
\]

and

\[\gamma_{E^\sigma} = \{ (x, y_1^{E^\sigma}) | \theta \in [\theta_2, 2\pi - \theta_2] \} \cup \{ (x, y_2^{E^\sigma}) | \theta \in [\theta_2, 2\pi - \theta_2] \} \in H_1(E^\sigma; \mathbb{Z})^-,
\]

where

\[\theta_1 = \pi - \arctan \sqrt{ \frac{(\sqrt{2} - 1) (\sqrt{3} - 1)}{2} }, \quad \theta_2 = \pi - \arctan \sqrt{ \frac{(\sqrt{2} + 1) (\sqrt{3} + 1)}{2} }.
\]

By calculating the integral of invariant differential, we know that $\gamma_E$ and $\gamma_{E^\sigma}$ are generators of $H_1(X; \mathbb{Z})^-$ and

\[(7.3) \quad \phi_*\gamma_E = \pm 3\gamma_{E^\sigma}, \quad (\phi^*)_*\gamma_{E^\sigma} = \mp \gamma_E.
\]

Next, take $M_1$ and $M_2$ as (7.2). In this setting, we have

\[\langle \gamma_E, M_1 \rangle = \frac{1}{2\pi} \int_{\gamma_E} \eta \left( xy, \frac{x}{y} \right) \]

\[\quad = \frac{1}{2\pi} \int_{\gamma_E} \log |xy| \Im \left( \frac{dy}{y} \right) - \log \left| \frac{x}{y} \right| \Im \left( \frac{d(xy)}{xy} \right) \]

\[\quad = \frac{1}{2\pi} \int_{\gamma_E} \log |y| \Im \left( -\frac{dy}{y} + \frac{dx}{x} \right) + \log |y| \Im \left( \frac{x dy + y dx}{xy} \right) \]

\[\quad = \frac{1}{\pi} \int_{\gamma_E} \log |y| \Im \left( \frac{dx}{x} \right) \]

\[\quad = \pm 4m \left( \sqrt{2} + \sqrt{6} \right).\]

Similarly, we have $\langle \gamma_{E^\sigma}, M_1 \rangle = \pm 4m \left( \sqrt{2} - \sqrt{6} \right)$. Also, according to (7.3) and the calculations in the previous section, we have

\[\langle \gamma_E, M_2 \rangle = \pm 12m \left( \sqrt{2} - \sqrt{6} \right), \quad \langle \gamma_{E^\sigma}, M_2 \rangle = \mp 4m \left( \sqrt{2} + \sqrt{6} \right).
\]

We search in [27] and find that $E$ is isomorphic over $K$ to the elliptic curve 2.2.12.1-16.1-a1, its $L$-function $4-48e2-1.1-c1e2-0-2$ can be written as

\[L(E, s) = L(f_{48}, s)L(g_{48}, s),\]
where \( f_{48} \) is the newform 48.2.c.a.47.1 and \( g_{48} \) is its dual form 48.2.c.a.47.2. One can use Sturm bound to prove that
\[
-\frac{1}{6} \sum_{m,n \in \mathbb{Z}} \chi_4(m-2n) q^{m^2 - mn + n^2} = \frac{3 + i\sqrt{3}}{6} f_{48} + \frac{3 - i\sqrt{3}}{6} g_{48},
\]
\[
-\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \chi_4(3m-2n) q^{3m^2 - 3mn + n^2} = \frac{1 - i\sqrt{3}}{2} f_{48} + \frac{1 + i\sqrt{3}}{2} g_{48}.
\]
Hence, we have
\[
R = \left| \det \left( \frac{\langle \gamma_E, M_1 \rangle}{\langle \gamma_E, M_2 \rangle}, \frac{\langle \gamma_{E^*}, M_1 \rangle}{\langle \gamma_{E^*}, M_2 \rangle} \right) \right| = 16m \left( \sqrt{2 + \sqrt{6}} \right)^2 + 48m \left( \sqrt{2 - \sqrt{6}} \right)^2
\]
\[
= 16 \left( \frac{6\sqrt{3}}{\pi^2} \right)^2 \left( \frac{3 + i\sqrt{3}}{6} L(f_{48},2) + \frac{3 - i\sqrt{3}}{6} L(g_{48},2) \right)^2 + 48 \left( \frac{2\sqrt{3}}{\pi^2} \right)^2 \left( \frac{1 - i\sqrt{3}}{2} L(f_{48},2) + \frac{1 + i\sqrt{3}}{2} L(g_{48},2) \right)^2
\]
\[
= \frac{2304}{\pi^4} L(f_{48},2) L(g_{48},2)
\]
\[
= \frac{2304}{\pi^4} L(E,2).
\]
A simplification immediately yields (1.13). \( \square \)

**Proof of (1.14).** Under (7.1), we have
\[
E'_{4\sqrt{2}+4\sqrt{6}}: Y^2 = X^3 + (60 + 32\sqrt{3})X^2 + 4X,
\]
\[
E'_{4\sqrt{2}-4\sqrt{6}}: Y^2 = X^3 + (60 - 32\sqrt{3})X^2 + 4X.
\]
They are defined over \( K = \mathbb{Q}(\sqrt{3}) \). Let \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) be the nontrivial element. Then we can write \( E \) for \( E'_{4\sqrt{2}+4\sqrt{6}} \) and \( E^* \) for \( E'_{4\sqrt{2}-4\sqrt{6}} \). There is an isogeny \( \phi: E \to E^* \) defined over \( K \) of degree 3, it is given by \( (X,Y) \mapsto (\phi_1(X), Y \phi_2(X)) \), where
\[
\phi_1(X) = \frac{3X \left( (7 - 4\sqrt{3})X^2 + (12 - 8\sqrt{3})X + 12 \right)}{(3X + 6 - 4\sqrt{3})^2},
\]
\[
\phi_2(X) = \frac{(2\sqrt{3} - 3)^3 (X - 4\sqrt{3}) (X^2 + 12X + 4)}{(3X + 6 - 4\sqrt{3})^3}.
\]
We have \( \phi^* \circ \phi = [-3] \) and \( \phi^*(\omega_{E^*}) = (3 + 2\sqrt{3})\omega_E \).

Take
\[
\gamma_E = \{(x, y^E) | \theta \in [-\pi, \pi]\} \in H_1(E; \mathbb{Z})^{-}
\]
and
\[
\gamma_{E^*} = \{(x, y^{E^*}) | \theta \in [-\pi, \pi]\} \in H_1(E^*; \mathbb{Z})^{-}.
\]
By calculating the integral of invariant differential, we know that \( \gamma_E \) and \( \gamma_{E^*} \) are generators of \( H_1(X; \mathbb{Z})^{-} \) and
\[
\phi_* \gamma_E = \pm \gamma_{E^*}, \quad (\phi^*)_* \gamma_{E^*} = \pm 3 \gamma_E.
\]
Hence, we have
\[
\langle \gamma_E, M_1 \rangle = \pm 2m \left( 4\sqrt{2} + 4\sqrt{6} \right), \quad \langle \gamma_{E^\sigma}, M_1 \rangle = \pm 2m \left( 4\sqrt{2} - 4\sqrt{6} \right),
\]
\[
\langle \gamma_E, M_2 \rangle = \mp 2m \left( 4\sqrt{2} - 4\sqrt{6} \right), \quad \langle \gamma_{E^\sigma}, M_2 \rangle = \pm 6m \left( 4\sqrt{2} + 4\sqrt{6} \right).
\]

We search in [27] and find that \( E \) is isomorphic over \( K \) to the elliptic curve 22.12.1-256.1-c8, its \( L \)-function \( 4-192e2-1.1-c1e2-0-1 \) can be written as
\[
L(E, s) = L(f_{192}, s)L(g_{192}, s),
\]
where \( f_{192} \) is the newform 192.2.c.a.191.1 and \( g_{192} \) is its dual form 192.2.c.a.191.2. One can use Sturm bound to prove that
\[
\frac{1}{2} \sum_{m, n \in \mathbb{Z}} \chi_4(n)q^{12m^2+n^2} = \frac{f_{192} + g_{192}}{2},
\]
\[
\frac{1}{2} \sum_{m, n \in \mathbb{Z}} \chi_4(n)q^{4m^2+3n^2} = \frac{g_{192} - f_{192}}{2i\sqrt{3}}.
\]

Hence, we have
\[
R = 12m \left( 4\sqrt{2} + 4\sqrt{6} \right)^2 + 4m \left( 4\sqrt{2} - 4\sqrt{6} \right)^2
\]
\[
= 12 \left( \frac{16\sqrt{3}}{\pi^2} \cdot L(f_{192}, 2) + L(g_{192}, 2) \right)^2 + 4 \left( \frac{48\sqrt{3}}{\pi^2} \cdot L(g_{192}, 2) - L(f_{192}, 2) \right)^2
\]
\[
= \frac{9216}{\pi^4} L(f_{192}, 2)L(g_{192}, 2)
\]
\[
= \frac{9216}{\pi^4} L(E, 2).
\]

A simplification immediately yields (1.14). \( \square \)

**Proof of (1.15).** Under (7.1), we have
\[
E'_{\sqrt{7} + \sqrt{\frac{3}{7}}} : Y^2 = X^3 + \frac{3\sqrt{7}}{2}X^2 + 4X,
\]
\[
E'_{\sqrt{7} - \sqrt{\frac{3}{7}}} : Y^2 = X^3 - \frac{3\sqrt{7}}{2}X^2 + 4X.
\]

They are defined over \( K = \mathbb{Q}(\sqrt{7}) \). Let \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) be the nontrivial element. Then we can write \( E \) for \( E'_{\sqrt{7} + \sqrt{\frac{3}{7}}} \) and \( E^\sigma \) for \( E'_{\sqrt{7} - \sqrt{\frac{3}{7}}} \). There is an isogeny \( \phi : E \to E^\sigma \) defined over \( K \) of degree 7, it is given by \((X, Y) \mapsto \left( \frac{7X\phi_2(X)}{\phi_1(X)^2}, \frac{7\sqrt{7}Y\phi_3(X)}{\phi_1(X)^3} \right) \), where
\[
\phi_1(X) = 7X^3 + 14\sqrt{7}X^2 + 56X + 8\sqrt{7},
\]
\[
\phi_2(X) = X^6 + 8\sqrt{7}X^5 + 168X^4 + 240\sqrt{7}X^3 + 1232X^2 + 448\sqrt{7}X + 448,
\]
\[
\phi_3(X) = X^9 + 6\sqrt{7}X^8 + 96X^7 + 120\sqrt{7}X^6 + 720X^5 + 544\sqrt{7}X^4 + 2752X^3
\]
\[+ 1536\sqrt{7}X^2 + 3584X + 512\sqrt{7}.
\]

We have \( \phi^\sigma \circ \phi = [-7] \) and \( \phi^\sigma(\omega_{E^\sigma}) = \sqrt{7}\omega_E \).

Take
\[
\gamma_E = \{(x, y_1^E) \mid \theta \in [-\theta_1, \theta_1]\} \cup \{(x, y_2^E) \mid \theta \in [-\theta_1, \theta_1]\} \in H_1(E; \mathbb{Z})^{-}
\]
and
\[
\gamma_{E^\sigma} = \{(x, y_1^{E^\sigma}) \mid \theta \in [-\theta_2, \theta_2]\} \cup \{(x, y_2^{E^\sigma}) \mid \theta \in [-\theta_2, \theta_2]\} \in H_1(E^\sigma; \mathbb{Z})^{-},
\]
where
\[
\theta_1 = \pi - \arctan \frac{\sqrt{552\sqrt{2} - 433 - 4\sqrt{7}(2993 - 1428\sqrt{2})}}{47},
\]
\[
\theta_2 = \arctan \frac{\sqrt{552\sqrt{2} - 433 + 4\sqrt{7}(2993 - 1428\sqrt{2})}}{47}.
\]

By calculating the integral of invariant differential, we know that \(\gamma_E\) and \(\gamma_{E^*}\) are generators of \(H_1(X; \mathbb{Z})\) and
\[
\phi_\ast \gamma_E = \pm 7\gamma_{E^*}, \quad (\phi^\ast) \ast \gamma_{E^*} = \mp \gamma_E.
\]

Next, take \(M_1\) and \(M_2\) as (7.2). We have
\[
\langle \gamma_E, M_1 \rangle = \pm 4m \left( \frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2} \right), \quad \langle \gamma_E, M_1 \rangle = \pm 4m \left( \frac{3\sqrt{2}}{2} - \frac{\sqrt{14}}{2} \right),
\]
\[
\langle \gamma_E, M_2 \rangle = \pm 28m \left( \frac{3\sqrt{2}}{2} - \frac{\sqrt{14}}{2} \right), \quad \langle \gamma_{E^*}, M_2 \rangle = \mp 4m \left( \frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2} \right).
\]

We search in [27] and find that \(E\) is isomorphic over \(K\) to the elliptic curve 22.2.28.1-1.1-a2, its \(L\)-function \(4-28e2-1.1-c1e2-0-1\) can be written as
\[
L(E, s) = L(f_{28}, s)L(g_{28}, s),
\]
where \(f_{28}\) is the newform 28.2.d.a.27.1 and \(g_{28}\) is its dual form 28.2.d.a.27.2. One can use Sturm bound to prove that
\[
-\frac{1}{14} \sum_{m,n \in \mathbb{Z}} \chi_4(n)(m-2n)q^m = \alpha_1 f_{28}(2\tau) + \alpha_2 g_{28}(2\tau) + \beta_1 f_{28}(4\tau) + \beta_2 g_{28}(4\tau),
\]
\[
-\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \chi_4(n)(7m-4n)q^{7m} = \alpha_1 f_{28}(2\tau) + \alpha_2 g_{28}(2\tau) + \beta_1 f_{28}(4\tau) + \beta_2 g_{28}(4\tau),
\]
where \(\alpha_1 = \frac{7-3\sqrt{7}}{14}, \beta_1 = \frac{7+3\sqrt{7}}{14}, \alpha_2 = \frac{1+i\sqrt{7}}{2}, \beta_2 = -3-i\sqrt{7}\). Hence, we have
\[
R = 16m \left( \frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2} \right)^2 + 112m \left( \frac{3\sqrt{2}}{2} - \frac{\sqrt{14}}{2} \right)^2
\]
\[
= 16 \left( \frac{14\sqrt{7}}{\pi^2} \left( \alpha_1 L(f_{28}(2\tau), 2) + \alpha_2 L(g_{28}(2\tau), 2) + \beta_1 L(f_{28}(4\tau), 2) + \beta_2 L(g_{28}(4\tau), 2) \right) \right)^2
\]
\[
+ 112 \left( \frac{2\sqrt{7}}{\pi^2} \left( \alpha_1 L(f_{28}(2\tau), 2) + \alpha_2 L(g_{28}(2\tau), 2) + \beta_1 L(f_{28}(4\tau), 2) + \beta_2 L(g_{28}(4\tau), 2) \right) \right)^2
\]
\[
= \frac{21952}{\pi^4} \left( \frac{\alpha_1}{2^2} + \frac{\beta_1}{4^2} \right) L(f_{28}, 2) + \left( \frac{\alpha_1}{2^2} + \frac{\beta_1}{4^2} \right) L(g_{28}, 2)^2
\]
\[
+ \frac{3136}{\pi^4} \left( \frac{\alpha_2}{2^2} + \frac{\beta_2}{4^2} \right) L(f_{28}, 2) + \left( \frac{\alpha_2}{2^2} + \frac{\beta_2}{4^2} \right) L(g_{28}, 2)^2
\]
\[
= \frac{3136}{\pi^4} L(f_{28}, 2)L(g_{28}, 2)
\]
\[
= \frac{3136}{\pi^4} L(E, 2).
\]

A simplification immediately yields (1.15). \(\square\)
Proof of (1.16). Under (7.1), we have
\[
E'_{24\sqrt{7}+8\sqrt{14}} : Y^2 = X^3 + (1020 + 384\sqrt{7})X^2 + 4X,
\]
\[
E'_{24\sqrt{7}-8\sqrt{14}} : Y^2 = X^3 + (1020 - 384\sqrt{7})X^2 + 4X.
\]
They are defined over \( K = \mathbb{Q}(\sqrt{7}) \). Let \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) be the nontrivial element. Then we can write \( E \) for \( E'_{24\sqrt{7}+8\sqrt{14}} \) and \( E'' \) for \( E'_{24\sqrt{7}-8\sqrt{14}} \). There is an isogeny \( \phi : E \rightarrow E'' \) defined over \( K \) of degree 7, it is given by \((X, Y) \mapsto \left( \frac{7X\phi(X)}{\phi_1(X)^2}, \frac{-(21-8\sqrt{7})Y\phi(X)}{\phi_1(X)^4} \right) \), where
\[
\phi_1(X) = 7X^3 - 14\left(15 + 8\sqrt{7}\right)X^2 + 308X + 8\left(21 - 8\sqrt{7}\right),
\]
\[
\phi_2(X) = \left(127 - 48\sqrt{7}\right)X^6 + 44\left(21 - 8\sqrt{7}\right)X^5 + 12\left(371 - 32\sqrt{7}\right)X^4
\]
\[\quad - 96\left(189 + 104\sqrt{7}\right)X^3 + 560\left(139 + 48\sqrt{7}\right)X^2 - 448\left(15 + 8\sqrt{7}\right)X + 448,
\]
\[
\phi_3(X) = X^9 - 6\left(15 + 8\sqrt{7}\right)X^8 - 48\left(6289 + 2380\sqrt{7}\right)X^7 - 96\left(215258 + 80356\sqrt{7}\right)X^6
\]
\[\quad + 288\left(2959 + 1112\sqrt{7}\right)X^5 + 64\left(13721325 + 5186128\sqrt{7}\right)X^4
\]
\[\quad + 256\left(2104321 + 795348\sqrt{7}\right)X^3 + 1536\left(1029 + 388\sqrt{7}\right)X^2
\]
\[\quad - 256\left(6727 + 2544\sqrt{7}\right)X - 512\left(21 + 8\sqrt{7}\right).
\]
We have \( \phi'' \circ \phi = [-7] \) and \( \phi''(\omega_{E''}) = (21 + 8\sqrt{7})\omega_E \).
Take
\[
\gamma_E = \{(x, y^E_1) \mid \theta \in [-\pi, \pi]\} \in H_1(E; \mathbb{Z})^-
\]
and
\[
\gamma_{E''} = \{(x, y^{E''}_1) \mid \theta \in [-\pi, \pi]\} \in H_1(E''; \mathbb{Z})^-.
\]
By calculating the integral of invariant differential, we know that \( \gamma_E \) and \( \gamma_{E''} \) are generators of \( H_1(X; \mathbb{Z})^- \) and
\[
\phi'' \gamma_E = \pm \gamma_{E''}, \quad (\phi'')^{-1} \gamma_{E''} = \mp \gamma_E.
\]
Next, take \( M_1 \) and \( M_2 \) as (7.2). We have
\[
\langle \gamma_E, M_1 \rangle = \pm 2m \left(24\sqrt{2} + 8\sqrt{14}\right), \quad \langle \gamma_{E''}, M_1 \rangle = \pm 2m \left(24\sqrt{2} - 8\sqrt{14}\right),
\]
\[
\langle \gamma_E, M_2 \rangle = \pm 2m \left(24\sqrt{2} - 8\sqrt{14}\right), \quad \langle \gamma_{E''}, M_2 \rangle = \mp 14m \left(24\sqrt{2} + 8\sqrt{14}\right).
\]
We search in [27] and find that \( E \) is isomorphic over \( K \) to the elliptic curve \( 2.28.1-256.1-j8 \), its \( L \)-function \( 4-448e2-1.1-c1e2-0-0 \) can be written as
\[
L(E, s) = L(f_{448}, s)L(g_{448}, s),
\]
where \( f_{448} \) is the newform \( 448.2.f.b.447.1 \) and \( g_{448} \) is its dual form \( 448.2.f.b.447.2 \). One can use Sturm bound to prove that
\[
\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \chi_4(n)mq^{28m^2+n^2} = \frac{f_{448} + g_{448}}{2},
\]
\[
\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \chi_4(n)mq^{4m^2+7n^2} = \frac{g_{448} - f_{448}}{2i\sqrt{7}}.
\]
Hence, we have
\[
R = 28m \left( 24\sqrt{2} + 8\sqrt{14} \right)^2 + 4m \left( 24\sqrt{2} - 8\sqrt{14} \right)^2
\]
\[
= 28 \left( \frac{16\sqrt{7}}{\pi^2} \cdot \frac{L(f_{448}, 2) + L(g_{448}, 2)}{2} \right)^2 + 4 \left( \frac{112\sqrt{7}}{\pi^2} \cdot \frac{L(g_{448}, 2) - L(f_{448}, 2)}{2i\sqrt{7}} \right)^2
\]
\[
= \frac{50176}{\pi^4} L(f_{448}, 2)L(g_{448}, 2)
\]
\[
= \frac{50176}{\pi^4} L(E, 2).
\]
A simplification immediately yields (1.16). □

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