Context-independent quasi hidden variable (qHV) modelling of all joint von Neumann measurements for an arbitrary Hilbert space

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We prove the existence for each Hilbert space of the two new quasi hidden variable (qHV) models, noncontextual and context-independent, reproducing all the von Neumann joint probabilities via nonnegative values of real-valued measures and the quantum average of the product of mutually commuting bounded quantum observables – via the qHV (classical-like) average of the product of the corresponding random variables. The proved existence of a context-independent qHV model negates the general opinion that, in terms of random variables satisfying the functional condition required in quantum foundations, the Hilbert space description of all the joint von Neumann measurements for \( \dim \mathcal{H} \geq 3 \) can be reproduced only contextually. The existence of a noncontextual qHV model, in particular, implies that every \( N \)-partite quantum state admits a local quasi hidden variable (LqHV) model introduced in [Loubenets, J. Math. Phys. 53, 022201 (2012)]. The new results of the present paper point also to the generality of the quasi-classical probability model proposed in [Loubenets, J. Phys. A: Math. Theor. 45, 185306 (2012)].
The relation between the quantum probability model and the classical probability model has been a point of intensive discussions ever since the seminal publications of von Neumann\(^1\), Kolmogorov\(^2\), and Einstein, Podolsky and Rosen (EPR)\(^3\). In quantum theory, the interpretation of quantum measurements in classical probability terms, that is, via random variables and probability measures on a measurable space \((\Omega, \mathcal{F}_\Omega)\), is generally referred to as a hidden variable (HV) model, but a setting of a HV model depends essentially on its aim. Moreover, in the literature, the HV models are usually divided into noncontextual and contextual.
In a noncontextual HV model, each quantum observable $X$ on a Hilbert space $\mathcal{H}$ is represented on $(\Omega, \mathcal{F}_\Omega)$ by only one random variable $f_X$ with values in the spectrum $\text{sp}X$ of this observable $X$.

In a contextual HV model, not only there are quantum observables $X_\gamma$, $\gamma \in \Upsilon$, each modelled by a variety of random variables on $(\Omega, \mathcal{F}_\Omega)$, but also – which of these random variables represents an observable $X_\gamma$ under a joint von Neumann measurement depends specifically on a context of this measurement, i.e. on other compatible quantum observables measured jointly with $X_\gamma$.

In foundations of quantum theory, where a HV model aims to reproduce in classical probability terms the statistical properties of all quantum observables on a Hilbert space $\mathcal{H}$, the intention to mimic all the properties of quantum averages and quantum correlations is realized via some additional functional assumptions on a correspondence between random variables on $(\Omega, \mathcal{F}_\Omega)$ and quantum observables on $\mathcal{H}$. Under these functional assumptions, the Hilbert space description of all joint von Neumann measurements for $\dim \mathcal{H} \geq 3$ can be reproduced only contextually (see section II for details).

In quantum information theory, a HV model aims to reproduce only the probabilistic description of a quantum correlation scenario upon a state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, so that a noncontextual HV model is introduced directly via the noncontextual representation of all scenario joint probabilities in classical probability terms. This HV representation mimics by itself all the needed properties of quantum averages and quantum correlations, so that a setting of a HV model for a correlation scenario does not contain any additional assumptions on a correspondence between random variables and modelled quantum measurements. As a result, for $\dim(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N) \geq 3$, there exist quantum correlation scenarios admitting noncontextual HV models. Ever since the arguments of Bell, a noncontextual HV model for a quantum correlation scenario is generally referred to as local and a LHV model, for short (see section II for details).

From the point of view of quantum information applications, the existence of LHV models for some quantum correlation scenarios questioned whether there exists a probability model which could reproduce the probabilistic description of every quantum correlation scenario via random variables, each depending only on a setting of the corresponding measurement at the corresponding site, i.e. via "local" random variables.

As we proved in the Ref. 10, the answer to this question is positive – the probabilistic
description of every quantum correlation scenario does admit a new local probability model – a local quasi hidden variable (LqHV) model, where locality and the measure theory structure \((\Omega, \mathcal{F}_\Omega, \nu)\) inherent to a LHV model are preserved but positivity of a simulation measure \(\nu\) is dropped.

In a (deterministic) LqHV model, specified by a measure space \((\Omega, \mathcal{F}_\Omega, \nu)\), all the joint probabilities and all the quantum product averages of a correlation scenario are reproduced via nonnegative values of a normalized real-valued measure \(\nu\) and ”local” random variables on \((\Omega, \mathcal{F}_\Omega)\).

Moreover, we showed in the Ref.\(^{12}\) that the probabilistic description of every nonsignaling correlation scenario, not necessarily quantum, admits a LqHV model.

Based on our results in the Refs. 10, 12, we also introduced\(^\text{12}\) the notion of a quasi-classical probability model - a new general probability model, which is specified in terms of a measure space \((\Omega, \mathcal{F}_\Omega, \nu)\) with a normalized real-valued measure \(\nu\) and where a joint measurement with outcomes \((\lambda_1, \ldots, \lambda_n) \in \Lambda_{\theta_1} \times \cdots \times \Lambda_{\theta_n}\) and marginal measurements, represented by random variables \(f_{\theta_i}: \Omega \rightarrow \Lambda_{\theta_i}, i = 1, \ldots, n\), is possible if and only if

\[
\nu(f_{\theta_1}^{-1} \ldots f_{\theta_n}^{-1}(B)) \geq 0, \quad \forall B \subseteq \Lambda_{\theta_1} \times \cdots \times \Lambda_{\theta_n},
\]

(1)

where \(f_{\theta_1, \ldots, \theta_n} = (f_{\theta_1}, \ldots, f_{\theta_n})\) and \(f_{\theta_1, \ldots, \theta_n}^{-1}(B) = \{\omega \in \Omega \mid (f_{\theta_1}(\omega), \ldots, f_{\theta_n}(\omega)) \in B\}\). This new probability model reduces to the Kolmogorov probability model\(^\text{2,13}\) iff a real-valued measure \(\nu\) is positive.

If a quasi-classical probability model is applied for the description of quantum measurements, then, according to our terminology in the Ref. 10, 12, we refer to it as a qHV model.

In the present paper, we analyze further possibilities of the qHV approach\(^\text{10,12}\) and prove that, for each Hilbert space, the Hilbert space description of all joint von Neumann measurements can be reproduced via either of the two new qHV models, noncontextual or context-independent.

In the latter model of a completely new type, a quantum observable \(X\) can be modelled by a variety of random variables, but, in contrast to a contextual model, each of these random variables equivalently represents \(X\) under all joint von Neumann measurements, independently of their measurement contexts.

The proved existence of a noncontextual model for all joint von Neumann measurements,
in particular, implies that every $N$-partite quantum state admits a $LqHV$ model.

The proved existence of a context-independent $qHV$ model negates the general opinion that, via random variables satisfying the functional condition required in quantum foundations, the Hilbert space description of all the joint von Neumann measurements for $\dim \mathcal{H} \geq 3$ can be reproduced only contextually.

The paper is organized as follows. In section II, we shortly review the settings of the HV models available in the literature.

In section III, we recall the von Neumann formalism for the description of ideal (projective) quantum measurements and the notion of the spectral measure of a quantum observable.

In section IV, we generalize (lemma 3) some items of the Kolmogorov extension theorem to the case of consistent operator-valued measures and, based on this generalization, we prove (theorem 1) that all symmetrized products of spectral measures admit the representation via the uniquely defined self-adjoint operator-valued measure and the specific random variables on some specially constructed measurable space $(\Lambda, \mathcal{F}_\Lambda)$.

In section V, we apply the new mathematical results of section IV to qHV modelling (theorems 2, 3, corollaries 2, 3, propositions 2, 3) of the Hilbert space description of all joint von Neumann measurements for an arbitrary Hilbert space.

In section VI, we discuss the main new results of the present paper.

II. PRELIMINARIES

In this section, we shortly review settings of the HV models available in quantum foundations and quantum information.

In foundations of quantum theory, the first "no-go" theorem on non-existence of a noncontextual HV model reproducing the statistical properties of all quantum observables on $\mathcal{H}$ was introduced by von Neumann in 1932. However, analyzing this problem in 1964 - 1966, Bell explicitly constructed the noncontextual HV model for all quantum observables of the qubit (dim $\mathcal{H} = 2$) and argued that, though the proof of the von Neumann "no-go" theorem is mathematically correct, the setting of this theorem contains the linearity assumption which is inconsistent with the quantum formalism and is not, in particular, fulfilled in the specific HV model presented by him in the Ref. 9.
In 1967, Kochen and Specker introduced a new setting for a noncontextual HV model where a mapping $X \xrightarrow{\Phi} f_X$, $f_X(\Omega) = spX$, from the set of all quantum observables on $\mathcal{H}$ into the set of all random variables on a measurable space $(\Omega, \mathcal{F}_\Omega)$ was supplied by the physically motivated functional condition

$$\Phi(\varphi \circ X) = \varphi \circ \Phi(X) = \varphi \circ f_X$$

for all quantum observables $X$ on $\mathcal{H}$ and all Borel real-valued functions $\varphi : \mathbb{R} \to \mathbb{R}$. Since quantum observables $X_1, ..., X_n$ mutually commute iff there exist a quantum observable $Z$ and Borel functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that all $X_i = \varphi_i(Z)$, $i = 1, ..., n$, condition (2) does imply linearity of a mapping $\Phi$:

$$\Phi(X_1 + \cdots + X_n) = \Phi(X_1) + \cdots + \Phi(X_n) = f_{X_1} + \cdots + f_{X_n}$$

and also its multiplicativity

$$(\Phi(X_1 \cdots X_n))(\omega) = f_{X_1}(\omega) \cdots f_{X_n}(\omega), \quad \forall \omega \in \Omega,$$

but – only for mutually commuting quantum observables.

Kochen and Specker proved that, for a Hilbert space of a dimension $\dim \mathcal{H} \geq 3$, a noncontextual HV model supplied by the functional conditions (3), (4) cannot exist.

For the case where, in a setting of a HV model, a quantum observable $X$ on $\mathcal{H}$ can be represented by a variety $\{f^{(\theta)}_X, \theta \in \Theta_X\}$ of random variables on $(\Omega, \mathcal{F}_\Omega)$, it was proved (proposition 1.4.2 in the Ref. 6) that, for each Hilbert space $\mathcal{H}$, there exist a measurable space $(\Omega, \mathcal{F}_\Omega)$ and a mapping $\Psi$ from a set of random variables on $(\Omega, \mathcal{F}_\Omega)$ onto the set of all quantum observables on $\mathcal{H}$ such that (i) the HV representation $\langle X \rangle_{\rho} = \langle f^{(\theta)}_X \rangle_{HV}$ of the quantum average of each observable $X$ in a state $\rho$ and (ii) the functional condition

$$\Psi(\varphi \circ f^{(\theta)}_X) = \varphi \circ \Psi(f^{(\theta)}_X) = \varphi \circ X, \quad \forall X, \quad \forall \varphi : \mathbb{R} \to \mathbb{R},$$

similar by its sense to condition (2) are fulfilled for each of random variables $\{f^{(\theta)}_X, \theta \in \Theta_X\}$ representing a quantum observable $X$. If $\dim \mathcal{H} \geq 3$, then this mapping $\Psi$ cannot be injective due to the Kochen-Specker result and, therefore, the HV model specified by the above
setting cannot be noncontextual. Furthermore, for \( \dim \mathcal{H} \geq 3 \), this HV model implies the *contextual* description of a joint von Neumann measurement of mutually commuting quantum observables \( X_1, \ldots, X_n \) – in the sense that, under a joint von Neumann measurement, an observable \( X_i \) is, in general, represented by a random variable specific for a context of this joint measurement. Thus, for a Hilbert space of a dimension \( \dim \mathcal{H} \geq 3 \), the HV model specified by proposition 1.4.2 in the Ref. 6 is *contextual*.

Moreover, it is generally argued that this *contextuality* is always the case whenever, in a HV setting, a mapping \( \Psi \) from a set of random variables on \( (\Omega, \mathcal{F}_\Omega) \) onto the set of all quantum observables on \( \mathcal{H} \) is non-injective. As we prove below in section V, this opinion is misleading.

In *quantum information*, for some quantum correlation scenarios, their noncontextual HV description is possible, while for others – is impossible. Since contextual HV modelling is possible for all \( N \)-partite joint measurements on a \( N \)-partite quantum state, Bell presumed\(^{8,9}\) that, for an \( N \)-partite case, the only physical reason for non-existence of a noncontextual HV model is *quantum nonlocality*\(^{17}\) – in contrast to *locality* of parties’ measurements argued by Einstein, Podolsky and Rosen (EPR) in the Ref. 3.

Ever since this Bell’s conjecture, a HV model for a quantum correlation scenario is called *local* and is referred to as a LHV model if, in this model, each random variable modelling a party’s measurement depends only on a setting of this measurement at the corresponding site.

If an \( N \)-partite quantum state admits\(^{18}\) a single LHV model for all possible quantum correlation scenarios on this state, then this \( N \)-partite state is also referred to as *local*.

An arbitrary \( N \)-partite quantum correlation scenario does not need to admit a LHV model. Also, an arbitrary \( N \)-partite quantum state does not need to be local. However, not only every separable quantum state is local – Werner\(^{18}\) presented the example of a nonseparable (entangled) bipartite quantum state, which is local under all bipartite joint von Neumann measurements on this state. Moreover, there also exist\(^{7,19}\) nonseparable \( N \)-partite quantum states, that behave themselves as *local* under every \( N \)-partite quantum correlation scenario with some specific number \( S_n \leq S_n^{(0)} \) of quantum measurements at each \( n \)-th site.

Furthermore, as we proved in the Ref. 10, every \( N \)-partite quantum correlation scenario admits a *local* quasi hidden variable (LqHV) model, described in Introduction.
In view of the above analysis of different settings of the HV models available now in the literature and the new quasi hidden variable (qHV) approach, developed in the Refs. 10, 12 and outlined in Introduction, we put the following questions important for both – quantum foundations and quantum information.

1. Does every $N$-partite quantum state admit a LqHV model?

2. Can the Hilbert space description of all the von Neumann joint probabilities and all the quantum product averages be reproduced via a noncontextual qHV model, where a mapping $X \xrightarrow{\Phi} f_X$, $f_X(\Omega) = \text{sp}X$, satisfies in average both – the von Neumann linearity assumption and the Kochen-Specker assumptions (2), (4), in other words, satisfies for each quantum state $\rho$ the average relations

$$\langle \varphi \circ X \rangle_\rho = \langle \varphi \circ f_X \rangle_{\text{qHV}},$$

$$\langle X_1 + \cdots + X_n \rangle_\rho = \langle f_{X_1} + \cdots + f_{X_n} \rangle_{\text{qHV}},$$

$$\langle X_1 \cdots X_n \rangle_\rho = \langle f_{X_1} \cdots f_{X_n} \rangle_{\text{qHV}},$$

where relation (3) holds for all quantum observables $X$ and all bounded Borel functions $\varphi : \mathbb{R} \to \mathbb{R}$; relation (7) – for all bounded quantum observables $X_1, \ldots, X_n$ on $\mathcal{H}$ and relation (8) is fulfilled only for mutually commuting bounded quantum observables $X_1, \ldots, X_n$?

3. Does there exist a qHV model correctly reproducing all the von Neumann joint probabilities and all the quantum averages and where (i) a quantum observable $X$ can be represented by a variety $\{ f^{(\theta)}_X, \theta \in \Theta_X \}$ of random variables on $(\Omega, \mathcal{F}_\Omega)$, but each of these random variables equivalently represents a quantum observable $X$ under all joint von Neumann measurements, independently of their measurement contexts; (ii) the functional condition (5), required in quantum foundations, is fulfilled; (iii) the average relations (6) - (8) are fulfilled with arbitrary representatives $f^{(\theta_1)}_{X_1}, \ldots, f^{(\theta_n)}_{X_n}$ on $(\Omega, \mathcal{F}_\Omega)$ of quantum observables $X_1, \ldots, X_n$ on $\mathcal{H}$?

In what follows, we answer positively to all of these questions.

III. VON NEUMANN MEASUREMENTS

In the frame of the von Neumann formalism, states and observables of a quantum system are described, correspondingly, by density operators $\rho$ and self-adjoint linear operators $X$ on a complex separable Hilbert space $\mathcal{H}$, possibly infinite dimensional.
Let $L^{(s)}_H$ be the real vector space of all self-adjoint bounded linear operators on $H$. Equipped with the operator norm, this vector space is Banach. Denote by $\mathcal{X}_H \supset L^{(s)}_H$ the set of all quantum observables on $H$, bounded and unbounded, and by $\text{sp} X \subseteq \mathbb{R}$ the spectrum of a quantum observable $X$.

The probability that, under an ideal (errorless) measurement of a quantum observable $X \in \mathcal{X}_H$ in a state $\rho$, an observed value belongs to a Borel subset $B$ of $\mathbb{R}$ is given\textsuperscript{1,20,21} by

$$\text{tr}[\rho P_X(B)], \ B \in \mathcal{B}_\mathbb{R},$$

where $\mathcal{B}_\mathbb{R}$ is the Borel $\sigma$-algebra\textsuperscript{22} on $\mathbb{R}$ and $P_X$ is the spectral measure of an observable $X \in \mathcal{X}_H$, that is, the normalized projection-valued measure $P_X$ on $\mathcal{B}_\mathbb{R}$, uniquely corresponding to an observable $X$ due to the spectral theorem\textsuperscript{1,20}

$$X = \int_\mathbb{R} x P_X(dx).$$

The values $P_X(B), \ B \in \mathcal{B}_\mathbb{R}, \ P_X(\mathbb{R}) = \mathbb{1}_H$, of this measure are projections on $H$, satisfying the relations

$$P_X(B_1)P_X(B_2) = P_X(B_2)P_X(B_1) = P_X(B_1 \cap B_2), \ B_1, B_2 \in \mathcal{B}_\mathbb{R},$$

$$P_X(B) = 0, \ B \in \mathcal{B}_\mathbb{R}\setminus \text{sp} X.$$

For each $X \in \mathcal{X}_H$, its spectrum $\text{sp} X \in \mathcal{B}_\mathbb{R}$. Due to the second relation in (11), we further consider the spectral measure $P_X$, $X \in \mathcal{X}_H$, only on the trace $\sigma$-algebra

$$\mathcal{B}_{\text{sp} X} := \mathcal{B}_\mathbb{R} \cap \text{sp} X.$$ 

The measure $P_X$ is $\sigma$-additive in the strong operator topology\textsuperscript{21,23} in $L^{(s)}_H$, that is:

$$\lim_{n \to \infty} \left\| P_X(\bigcup_{i=1}^\infty B_i) \psi - \sum_{i=1}^n P_X(B_i) \psi \right\|_H = 0$$

for all $\psi \in H$ and all countable collections $\{B_i\}$ of mutually disjoint sets in $\mathcal{B}_{\text{sp} X}$.

\textbf{Remark 1} In this article, we follow the terminology of the Ref. 24. Namely, let $\mathfrak{B}$ be a Banach space and $\mathcal{F}_\Lambda$ be an algebra of subsets of a set $\Lambda$. We refer to an additive set function $m : \mathcal{F}_\Lambda \to \mathfrak{B}$ as a $\mathfrak{B}$-valued (finitely additive) measure on $\mathcal{F}_\Lambda$. If a measure $m$ on $\mathcal{F}_\Lambda$ is $\sigma$-additive in some topology in $\mathfrak{B}$, then we specify this in addition.
An ideal measurement \(^{(9)}\) of a quantum observable \(X\) in a state \(\rho\) is generally referred to as the von Neumann measurement.

The joint von Neumann measurement of several quantum observables \(X_1, \ldots, X_n \in \mathcal{X}_\mathcal{H}\) is possible\(^{21}\) if and only if all values of their spectral measures mutually commute:

\[
[P_{X_1}(B_{i_1}), P_{X_2}(B_{i_2})] = 0, \quad B_i \in \mathcal{B}_{\text{sp}X_i}, \quad i = 1, \ldots, n.
\] (14)

and is described in this case by the projection-valued product measure\(^{21}\)

\[
P_{X_1,\ldots,X_n}(B) := \int_{(x_1,\ldots,x_n) \in B} P_{X_1}(dx_1) \cdots P_{X_n}(dx_n), \quad B \in \mathcal{B}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n},
\] (15)

which is normalized \(P_{X_1,\ldots,X_n}(\text{sp}X_1 \times \cdots \times \text{sp}X_n) = \mathbb{I}_\mathcal{H}\) and defined on the \(\sigma\)-algebra

\[
\mathcal{B}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n} := \mathcal{B}_{\mathbb{R}^n} \cap (\text{sp}X_1 \times \cdots \times \text{sp}X_n)
\] (16)

of Borel subsets of \(\text{sp}X_1 \times \cdots \times \text{sp}X_n\).

For bounded quantum observables \(X_1, \ldots, X_n \in \mathcal{L}^{(s)}_\mathcal{H}\), condition (14) is equivalent to mutual commutativity \([X_{i_1}, X_{i_2}] = 0, \ i = 1, \ldots, n, \ \text{of these observables}. \) Therefore, for short, we further refer to arbitrary quantum observables \(X_1, \ldots, X_n\), bounded or unbounded, as mutually commuting if their spectral measures satisfy condition (14). The measure \(P_{X_1,\ldots,X_n}\) is also referred\(^{23}\) to as the joint spectral measure of mutually commuting quantum observables \(X_1, \ldots, X_n\).

The expression

\[
\text{tr}[\rho P_{X_1,\ldots,X_n}(B_1 \times \cdots \times B_n)] = \text{tr}[(\rho P_{X_1}(B_1) \cdot \cdots \cdot P_{X_n}(B_n))]
\] (17)

defines the probability that the observed values of mutually commuting quantum observables \(X_1, \ldots, X_n\) are in sets \(B_1 \in \mathcal{B}_{\text{sp}X_1}, \ldots, B_n \in \mathcal{B}_{\text{sp}X_n}\), respectively.

**IV. SYMMETRIZED PRODUCTS OF SPECTRAL MEASURES**

In this section, for our further consideration in section V, we introduce a new operator-valued measure induced by the symmetrized product of spectral measures of quantum observables and prove the extension theorem for the consistent family of these operator-valued measures.
For an \( n \)-tuple \((X_1, \ldots, X_n)\) of arbitrary mutually non-equal quantum observables \(X_1, \ldots, X_n \in \mathcal{X}_H\), let \( \mathcal{F}_{spX_1 \times \cdots \times spX_n} \) be the product algebra on the set \( \text{sp}X_1 \times \cdots \times \text{sp}X_n \subseteq \mathbb{R}^n \), that is, the algebra generated by all rectangles \( B_1 \times \cdots \times B_n \subseteq \text{sp}X_1 \times \cdots \times \text{sp}X_n \) with measurable sides \( B_i \in \mathcal{B}_{spX_i} \).

Let

\[
\mathcal{P}_{(X_1, \ldots, X_n)} : \mathcal{F}_{spX_1 \times \cdots \times spX_n} \to \mathcal{L}^{(s)}(\mathcal{H}), \quad \mathcal{P}_{(X_1, \ldots, X_n)}(\text{sp}X_1 \times \cdots \times \text{sp}X_n) = \mathbb{I}_H,
\]

be the normalized (finitely additive) product measure on \( \mathcal{F}_{spX_1 \times \cdots \times spX_n} \), defined uniquely via its representation

\[
\mathcal{P}_{(X_1, \ldots, X_n)}(B_1 \times \cdots \times B_n) = \frac{1}{n!} \{ P_{X_1}(B_1) \cdot \ldots \cdot P_{X_n}(B_n) \}_{\text{sym}}
\]

on all rectangles \( B_1 \times \cdots \times B_n \) with \( B_i \in \mathcal{B}_{spX_i} \). The values of the measure \( \mathcal{P}_{(X_1, \ldots, X_n)} \) are self-adjoint bounded linear operators on \( \mathcal{H} \). Here, notation \( \{ Z_1 \cdot \ldots \cdot Z_n \}_{\text{sym}} \) means the sum constituting the symmetrization of the operator product \( Z_1 \cdot \ldots \cdot Z_n \), where \( Z_i \in \mathcal{L}^{(s)}(\mathcal{H}) \), with respect to all permutations of its factors.

For a collection \( \{X_1, \ldots, X_n\} \) of mutually commuting quantum observables, the measure \( \mathcal{P}_{(X_1, \ldots, X_n)} \) is projection-valued and constitutes the restriction to the product algebra \( \mathcal{F}_{spX_1 \times \cdots \times spX_n} \) of the joint spectral measure \( P_{X_1, \ldots, X_n} \) defined by relation (15) on the \( \sigma \)-algebra \( \mathcal{B}_{spX_1 \times \cdots \times spX_n} \).

If each observable \( X_i \) in a collection \( \{X_1, \ldots, X_n\} \) is bounded, i.e. \( X_i \in \mathcal{L}^{(s)}(\mathcal{H}) \), and, moreover, has only a discrete spectrum \( \text{sp}X_i = \{ x^{(k)}_i \in \mathbb{R}, k = 1, \ldots, K_{X_i} < \infty \} \), where each \( x^{(k)}_i \) is an eigenvalue of \( X_i \), then the product algebra \( \mathcal{F}_{spX_1 \times \cdots \times spX_n} \) is finite and coincides with the \( \sigma \)-algebra \( \mathcal{B}_{spX_1 \times \cdots \times spX_n} \) while the product measure \( \mathcal{P}_{(X_1, \ldots, X_n)} \) takes the form

\[
\mathcal{P}_{(X_1, \ldots, X_n)}(F) := \frac{1}{n!} \sum_{(x_1, \ldots, x_n) \in F} \{ P_{X_1}(\{x_1\}) \cdot \ldots \cdot P_{X_n}(\{x_n\}) \}_{\text{sym}}
\]

for all \( F \in \mathcal{F}_{spX_1 \times \cdots \times spX_n} \).

Consider the family

\[
\{ \mathcal{P}_{(X_1, \ldots, X_n)} \mid \{X_1, \ldots, X_n\} \subseteq \mathcal{X}_H, \ n \in \mathbb{N} \}
\]

of all normalized \( \mathcal{L}^{(s)}(\mathcal{H}) \)-valued measures, each specified by a tuple \( (X_1, \ldots, X_n) \) of mutually non-equal quantum observables on \( \mathcal{H} \). These measures satisfy the following consistency relations proved in appendix A.
Lemma 1 For every collection \( \{X_1, \ldots, X_n\} \subset \mathcal{X}_H \), \( n \in \mathbb{N} \), of quantum observables on \( \mathcal{H} \), the relation
\[
P_{(X_1, \ldots, X_n)}(B_1 \times \cdots \times B_n) = P_{(X_{i_1}, \ldots, X_{i_n})}(B_{i_1} \times \cdots \times B_{i_n}),
\]
holds for all permutations \( (1, \ldots, n) \) and the relation
\[
P_{(X_1, \ldots, X_n)}(\{ (x_1, \ldots, x_n) \in \text{sp}X_1 \times \cdots \times \text{sp}X_n \mid (x_{i_1}, \ldots, x_{i_k}) \in F \}) = P_{(X_{i_1}, \ldots, X_{i_k})}(F), \quad F \in \mathcal{F}_{\text{sp}X_{i_1} \times \cdots \times \text{sp}X_{i_k}},
\]
is fulfilled for each \( \{X_{i_1}, \ldots, X_{i_k}\} \subseteq \{X_1, \ldots, X_n\} \).

Note that, for the operator-valued measures \( \{P_{(X_1, \ldots, X_n)}\} \), relations (22), (23) are quite similar by their form to the Kolmogorov consistency conditions\(^2\)\(^\text{,13}\) for a family
\[
\{\mu_{(t_1, \ldots, t_n)} : \varepsilon_{\mathbb{R}^n} \rightarrow [0, 1] \mid \{t_1, \ldots, t_n\} \subset T, \ n \in \mathbb{N}\}
\]
of probability measures \( \mu_{(t_1, \ldots, t_n)} \), each specified by tuples \((t_1, \ldots, t_n)\) of mutually non-equal elements in an index set \( T \).

In view of this similarity and for our further consideration, in appendix B, we generalize to the case of consistent operator-valued measures some items of the Kolmogorov extension theorem\(^2\)\(^\text{,13}\) for consistent probability measures (24).

A. The extension theorem

Denote by \( \Lambda := \prod_{X \in \mathcal{X}_H} \text{sp}X \) the Cartesian product of the spectrums of all the quantum observables on \( \mathcal{H} \). By its definition\(^2\), \( \Lambda \) is the set of all real-valued functions
\[
\lambda : \mathcal{X}_H \rightarrow \bigcup_{X \in \mathcal{X}_H} \text{sp}X
\]
with values \( \lambda(X) \equiv \lambda_X \in \text{sp}X \).

Let the random variable \( \pi_{(X_1, \ldots, X_n)} : \Lambda \rightarrow \text{sp}X_1 \times \cdots \times \text{sp}X_n \) be the canonical projection on \( \Lambda \):
\[
\pi_{(X_1, \ldots, X_n)}(\lambda) := (\pi_{X_1}(\lambda), \ldots, \pi_{X_n}(\lambda)),
\]
\[
\pi_X(\lambda) := \lambda_X \in \text{sp}X.
\]
The set
\[ \mathcal{A}_\Lambda = \left\{ \pi_{(X_1, \ldots, X_n)}^{-1}(F) \subseteq \Lambda \mid F \in \mathcal{F}_{spX_1 \times \cdots \times spX_n}, \ \{X_1, \ldots, X_n\} \subseteq \mathcal{X}_H, \ n \in \mathbb{N} \right\} \] (27)
of all cylindrical subsets of \( \Lambda \) of the form
\[ \pi_{(X_1, \ldots, X_n)}^{-1}(F) := \{ \lambda \in \Lambda \mid (\pi_{X_1}(\lambda), \ldots, \pi_{X_n}(\lambda)) \in F \}, \] (28)
constitutes an algebra on \( \Lambda \) (proposition III.11.18 in the Ref. 22).

Due to lemma 3 proved in appendix B and generalizing some items of the Kolmogorov extension theorem to the case of consistent operator-valued measures, we have the following statement.

**Theorem 1 (The extension theorem)** Let \( \mathcal{H} \) be an arbitrary complex separable Hilbert space. For family \( \{ \mathcal{P}_{(X_1, \ldots, X_n)} : \mathcal{F}_{spX_1 \times \cdots \times spX_n} \to \mathcal{L}^{(s)}_{\mathcal{H}} \} \), there exists a unique normalized finitely additive \( \mathcal{L}^{(s)}_{\mathcal{H}} \)-valued measure
\[ \mathcal{M} : \mathcal{A}_\Lambda \to \mathcal{L}^{(s)}_{\mathcal{H}}, \quad \mathcal{M}(\Lambda) = I_{\mathcal{H}}, \] (29)
on \( (\Lambda, \mathcal{A}_\Lambda) \) such that
\[ \mathcal{P}_{(X_1, \ldots, X_n)}(F) = \mathcal{M} \left( \pi_{(X_1, \ldots, X_n)}^{-1}(F) \right), \quad F \in \mathcal{F}_{spX_1 \times \cdots \times spX_n}, \] (30)
in particular,
\[ \frac{1}{n!} \{ P_{X_1}(B_1) \cdot \cdots \cdot P_{X_n}(B_n) \}_{sym} = \mathcal{M}(\pi_{X_1}^{-1}(B_1) \cap \cdots \cap \pi_{X_n}^{-1}(B_n)), \] (31)
for all collections \( \{X_1, \ldots, X_n\} \subset \mathcal{X}_H, \ n \in \mathbb{N}, \) of quantum observables on \( \mathcal{H} \).

**Proof.** The family \( \{ \mathcal{P}_{(X_1, \ldots, X_n)} \} \) represents a particular example of a general family if, in the latter, we replace
\[ T \to \mathcal{X}_H, \quad \Lambda_t \to spX, \quad \tilde{\Lambda} \to \Lambda \] (32)
\[ \mathcal{F}_t \to \mathcal{B}_{spX}, \quad \mathcal{F}_{\Lambda_t \times \cdots \times \Lambda_t} \to \mathcal{F}_{spX_1 \times \cdots \times spX_n}. \]
Moreover, by lemma 1, the family \( \{ \mathcal{P}_{(X_1, \ldots, X_n)} \} \) satisfies the consistency conditions \( (\mathcal{B}2), \mathcal{B}2 \). Therefore, representation (30) follows explicitly from relation \( (\mathcal{B}8) \) in lemma 3. This proves the statement. 

Theorem 1 allows us to express all real-valued measures
\[ \text{tr}[\rho \mathcal{P}_{(X_1, \ldots, X_n)}(\cdot)], \quad \{X_1, \ldots, X_n\} \subset \mathcal{X}_H, \ n \in \mathbb{N}, \] (33)
for a quantum state $\rho$ on $\mathcal{H}$ via a single real-valued measure on the algebra $\mathcal{A}_\Lambda$.

**Proposition 1** Let $\mathcal{H}$ be a complex separable Hilbert space and $\{P_{(X_1, ..., X_n)}\}$ be family (21) of $L^1(\mathcal{H})$-valued measures (18). To every state $\rho$ on $\mathcal{H}$, there corresponds $\rho \mapsto \mu_\rho$ a unique normalized finitely additive real-valued measure

$$\mu_\rho : \mathcal{A}_\Lambda \rightarrow \mathbb{R}, \quad \mu_\rho(\Lambda) = 1,$$

on $(\Lambda, \mathcal{A}_\Lambda)$ such that

$$\text{tr}[\rho P_{(X_1, ..., X_n)}(F)] = \mu_\rho \left(\pi^{-1}_{(X_1, ..., X_n)}(F)\right), \quad F \in \mathcal{F}_{\text{sp} X_1 \times \cdots \times \text{sp} X_n},$$

in particular,

$$\frac{1}{n!}\text{tr}[\rho \{P_{X_1}(B_1) \cdot \cdots \cdot P_{X_n}(B_n)\}_{\text{sym}}] = \mu_\rho \left(\pi^{-1}_{X_1}(B_1) \cap \cdots \cap \pi^{-1}_{X_n}(B_n)\right), \quad B_i \in \mathcal{B}_{\text{sp} X_i}, \quad i = 1, ..., n,$$

for all collections $\{X_1, ..., X_n\} \subset \mathcal{X}_\mathcal{H}$, $n \in \mathbb{N}$, of quantum observables on $\mathcal{H}$. If $\rho_j \mapsto \mu_{\rho_j}$, $j = 1, ..., m$, then

$$\sum \alpha_j \rho_j \mapsto \sum \alpha_j \mu_{\rho_j}, \quad \alpha_j > 0, \quad \sum \alpha_j = 1.$$

**Proof.** For a state $\rho$ on $\mathcal{H}$, representation (30) implies

$$\text{tr}[\rho P_{(X_1, ..., X_n)}(F)] = \text{tr}[\rho \mathbb{M} \left(\pi^{-1}_{(X_1, ..., X_n)}(F)\right)]$$

for all sets $F \in \mathcal{F}_{\text{sp} X_1 \times \cdots \times \text{sp} X_n}$ and all finite collections $\{X_1, ..., X_n\} \subset \mathcal{X}_\mathcal{H}$. Introduce on the algebra $\mathcal{A}_\Lambda$ the set function

$$\mu_\rho(A) := \text{tr}[\rho \mathbb{M}(A)], \quad A \in \mathcal{A}_\Lambda,$$

defined uniquely $\rho \mapsto \mu_\rho$ to each state $\rho$. Since $\mathbb{M}$ is a normalized (finitely additive) measure on the algebra $\mathcal{A}_\Lambda$, also, $\mu_\rho$ is a normalized (finitely additive) measure on $\mathcal{A}_\Lambda$. Moreover, since $\mathbb{M}$ is a unique measure on $\mathcal{A}_\Lambda$, satisfying representation (30), the measure $\mu_\rho$, uniquely defined to each state $\rho$ by relation (39), is also a unique normalized real-valued measure on $\mathcal{A}_\Lambda$, satisfying representation (38), hence, (35) and (36).

Further, if $\rho_j \mapsto \mu_{\rho_j}$, then, due to definition (39) of this mapping, the measure $\sum \alpha_j \mu_{\rho_j}$, where $\alpha_j > 0$ and $\sum \alpha_j = 1$, is a unique real-valued measure, corresponding to the state $\sum \alpha_j \rho_j$ due to (39) and satisfying (35), (36). This completes the proof.

For bounded quantum observables, proposition 1 implies the following representation.
Corollary 1 Let $\mathcal{H}$ be a complex separable Hilbert space. The representation
\[
\frac{1}{n!} \text{tr}[\rho \{X_1 \cdot \ldots \cdot X_n\}_{\text{sym}}] = \int \pi_{X_1}(\lambda) \cdot \ldots \cdot \pi_{X_n}(\lambda) \, \mu_{\rho}(d\lambda)
\]
holds for all states $\rho$ and all finite collections $\{X_1, \ldots, X_n\}$ of bounded quantum observables on $\mathcal{H}$.

Proof. For bounded quantum observables $X_1, \ldots, X_n$, the operator $\rho \{X_1 \cdot \ldots \cdot X_n\}_{\text{sym}}$ is trace class, so that the quantum average $\text{tr}[\rho \{X_1 \cdot \ldots \cdot X_n\}_{\text{sym}}] < \infty$ exists for all states $\rho$. Combining (36) and (10), we derive (40). 

V. QUASI HIDDEN VARIABLE (QHV) MODELLING

As we discussed in section III, the joint von Neumann measurement of quantum observables $X_1, \ldots, X_n$ on $\mathcal{H}$ is possible if and only if these observables mutually commute and is described in this case by the joint spectral measure $P_{X_1, \ldots, X_n}$ defined by [15]. If a joint von Neumann measurement of mutually commuting quantum observables $X_1, \ldots, X_n$ is performed on a quantum system in a state $\rho$ on $\mathcal{H}$, then the expression
\[
\text{tr}[\rho P_{X_1, \ldots, X_n}(B)], \quad B \in \mathcal{B}_{\text{sp}X_1 \times \ldots \times \text{sp}X_n},
\]
gives the probability that these observables take values $x_1, \ldots, x_n$ such that $(x_1, \ldots, x_n) \in B$. In particular,
\[
\text{tr}[\rho \{P_{X_1}(B_1) \cdot \ldots \cdot P_{X_n}(B_n)\}]
\]
is the probability that the observed values of $X_1, \ldots, X_n$ are in sets $B_1 \in \mathcal{B}_{\text{sp}X_1}, \ldots, B_n \in \mathcal{B}_{\text{sp}X_n}$, respectively.

For a collection $\{X_1, \ldots, X_n\}$ of mutually commuting quantum observables, the measure $\mathcal{P}_{(X_1, \ldots, X_n)}$ discussed in theorem 1 coincides with the restriction of the joint spectral measure $P_{X_1, \ldots, X_n}$ to the algebra $\mathcal{F}_{\text{sp}X_1 \times \ldots \times \text{sp}X_n}$.

In the following sections, this allows us to analyze (theorem 2, 3) a possibility of modelling of the Hilbert space description [12] of all joint von Neumann measurements in measure theory terms.
A. A noncontextual qHV model

In this section, we formulate and prove the statements (theorem 1, proposition 3) which immediately give the positive answers to, correspondingly, questions (2) and (1), formulated in section II.

For our below consideration, we recall that if quantum observables $X_1, ..., X_n$ mutually commute, then, for each Borel function $\psi : \mathbb{R}^n \to \mathbb{R}$, the notation $\psi(X_1, ..., X_n)$ means the quantum observable

$$\psi(X_1, ..., X_n) := \int_{\mathbb{R}^n} \psi(x_1, ..., x_n) P_{X_1}(dx_1) \cdot \ldots \cdot P_{X_n}(dx_n).$$

(43)

If a real-valued function $\psi$ is bounded, then the quantum observable $\psi(X_1, ..., X_n)$ is also bounded.

Theorem 2 Let $\mathcal{H}$ be an arbitrary complex separable Hilbert space. There exist:

(i) a set $\Omega$ and an algebra $\mathcal{F}_\Omega$ of subsets of $\Omega$;

(ii) a one-to-one mapping $\Phi : \mathcal{X}_H \to \mathcal{F}_\Omega$ from the set $\mathcal{X}_H$ of all quantum observables on $\mathcal{H}$ into the set $\mathcal{F}_\Omega$ of all random variables on $(\Omega, \mathcal{F}_\Omega)$, with values $f_X := \Phi(X)$ satisfying the spectral correspondence rule $f_X(\Omega) = \text{sp}X$;

such that, to each quantum state $\rho$ on $\mathcal{H}$, there corresponds $(\rho \mapsto \nu_\rho$) a unique normalized real-valued measure $\nu_\rho$ on $(\Omega, \mathcal{F}_\Omega)$, satisfying the relation

$$\text{tr}[\rho \{P_{X_1}(B_1) \cdot \ldots \cdot P_{X_n}(B_n)\}] = \nu_\rho \left( f_{X_1}^{-1}(B_1) \cap \cdots \cap f_{X_n}^{-1}(B_n) \right)$$

(44)

in particular,

$$\text{tr}[\rho\{P_{X_1}(B_1) \cdot \ldots \cdot P_{X_n}(B_n)\}] = \nu_\rho \left( f_{X_1}^{-1}(B_1) \cap \cdots \cap f_{X_n}^{-1}(B_n) \right)$$

(45)

for all collections $\{X_1, ..., X_n\}$, $n \in \mathbb{N}$, of mutually commuting quantum observables on $\mathcal{H}$. If $\rho_j \mapsto \nu_{\rho_j}$, $j = 1, ..., m < \infty$, then $\sum \alpha_j \rho_j \mapsto \sum \alpha_j \nu_{\rho_j}$, for all $\alpha_j > 0$, $\sum \alpha_j = 1$. 

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Proof. In order to prove the existence point of theorem 2, let us take the measurable space \((\Lambda, A_\Lambda)\) and the random variables \(\pi_X(\lambda) = \lambda X \in \text{sp}X, X \in \mathfrak{X}_\mathcal{H}\), on this space, which are specified in section IV.A. Since \(\pi_{X_1} \neq \pi_{X_2} \iff X_1 \neq X_2\) and \(\pi_X(\Lambda) = \text{sp}X\), the set \(\{\pi_X \mid X \in \mathfrak{X}_\mathcal{H}\}\) of random variables is put into the one-to-one correspondence to the set \(\mathfrak{X}_\mathcal{H}\) of all quantum observables on \(\mathcal{H}\) and, for each random variable \(\pi_X\), the spectral correspondence rule \(\pi_X(\Lambda) = \text{sp}X\) is fulfilled.

Furthermore, by proposition 1, to each quantum state \(\rho\) on \(\mathcal{H}\), there corresponds a unique normalized real-valued measures \(\mu_\rho\), defined on the algebra \(A_\Lambda\) and satisfying representations (35), (36). For a collection \(\{X_1, ..., X_n\}\) of mutually commuting quantum observables, the measure \(P_{(X_1, ..., X_n)}\) in the left-hand sides of (35), (36) reduces to the joint spectral measure \(P_{X_1, ..., X_n}\). Therefore, with random variables \(\pi_{X_1}, ..., \pi_{X_n}\) and the measures \(\{\mu_\rho, \forall \rho\}\) in the right-hand side, representations (44), (45) hold for all states and all finite collections \(\{X_1, ..., X_n\}\) of mutually commuting quantum observables. This proves the existence point of theorem 2. Also, by proposition 1, if \(\rho\) \(\mapsto\) \(\mu_\rho\), \(j = 1, ..., m,\) then \(\sum \alpha_j \rho \mapsto \sum \alpha_j \mu_\rho\).

This completes the proof. □

Representations (44), (45) imply.

Corollary 2 In the setting of theorem 2, for all states \(\rho\) and all finite collections \(\{X_1, ..., X_n\}\) of mutually commuting quantum observables, the representation

\[
\text{tr}[\rho \psi(X_1, ..., X_n)] = \int_{\Omega} \psi(f_{X_1}(\omega), ..., f_{X_n}(\omega)) \nu_\rho(\text{d}\omega) \tag{46}
\]

holds for all bounded Borel functions \(\psi: \mathbb{R}^n \to \mathbb{R}\) and the representation

\[
\text{tr}[\rho(X_1 \cdot \cdot \cdot \cdot X_n)] = \int_{\Omega} f_{X_1}(\omega) \cdot \cdot \cdot f_{X_n}(\omega) \nu_\rho(\text{d}\omega) \tag{47}
\]

is fulfilled whenever mutually commuting quantum observables \(X_1, ..., X_n\) are bounded.

Proof. For a bounded Borel function \(\psi: \mathbb{R}^n \to \mathbb{R}\), the quantum observable (43) is bounded, so that the operator \(\rho \psi(X_1 \cdot \cdot \cdot \cdot X_n)\) is trace class for all states \(\rho\). This and relation (45) imply

\[
\text{tr}[\rho \psi(X_1, ..., X_n)] = \int_{\mathbb{R}^n} \psi(x_1, ..., x_n) \text{tr}[\rho(P_{X_1}(dx_1) \cdot \cdot \cdot P_{X_n}(dx_n))] \tag{48}
\]

\[
= \int_{\Omega} \psi(f_{X_1}(\omega), ..., f_{X_n}(\omega)) \nu_\rho(\text{d}\omega). \tag{49}
\]
The proof of representation (47) is quite similar to our proof of (40). Namely, for bounded quantum observables $X_1, \ldots, X_n$, the operator $\rho(X_1 \cdots X_n)$ is trace class. This and relations (15), (10) imply representation (47). This proves the statement.

In view of expressions (41), (42) for von Neumann joint probabilities, theorem 2 proves that, for an arbitrary Hilbert space $\mathcal{H}$, the Hilbert space description of all joint von Neumann measurements can be reproduced in terms of a single measurable space $(\Omega, \mathcal{F}_\Omega)$ via the set $\{\nu_\rho, \forall \rho\}$ of normalized real-valued measures, where each $\nu_\rho$ is defined uniquely to a state $\rho$ on $\mathcal{H}$, and the set random variables $\{f_X : \Omega \to \text{sp}X, f_X(\Omega) = \text{sp}X, X \in \mathcal{X}_\mathcal{H}\}$ of random variables, where each $f_X$ is defined uniquely to a quantum observable $X$ on $\mathcal{H}$ – that is, via a noncontextual quasi hidden variable (qHV) model.

In this new model, (i) each quantum observable on $\mathcal{H}$ is modelled on $(\Omega, \mathcal{F}_\Omega)$ by only one random variable representing this quantum observable under all joint von Neumann measurements, regardless of their contexts; (ii) all the von Neumann joint probabilities are reproduced due to the noncontextual representations (44), (45) via nonnegative values of real-valued measures and (iii) the quantum averages are reproduced due to the noncontextual representations (46) (47) via the classical-like averages of the corresponding expressions for the random variables.

The specific example of a noncontextual qHV model for all joint von Neumann measurements is given in the proof of theorem 2.

For all joint von Neumann measurements on a quantum state $\rho$, theorem 2 and corollary 2 imply the following probability model.

**Proposition 2** Let $\rho$ be a state on an arbitrary complex separable Hilbert space. There exist a measure space $(\Omega, \mathcal{F}_\Omega, \nu_\rho)$ with a normalized real-valued measure $\nu_\rho$ and a set

$$\{f_X : \Omega \to \text{sp}X \mid f_X(\Omega) = \text{sp}X, \quad X \in \mathcal{X}_\mathcal{H}\}$$

of random variables one-to-one corresponding to the set $\mathcal{X}_\mathcal{H}$ of all quantum observables on $\mathcal{H}$ such that all the von Neumann joint probabilities (41), (42) admit the noncontextual qHV representation

$$\text{tr}[\rho P_{X_1, \ldots, X_n}(F)] = \nu_\rho \left(f_{X_1, \ldots, X_n}^{-1}(F)\right), \quad F \in \mathcal{F}_{\text{sp}X_1 \times \cdots \times \text{sp}X_n},$$

$$f_{(X_1, \ldots, X_n)} : = (f_{X_1}, \ldots, f_{X_n}), \quad n \in \mathbb{N},$$
in particular,
\[
\text{tr}[\rho\{P_{X_1}(B_1) \cdots P_{X_n}(B_n)\}] = \nu_\rho \left( f_{X_1}^{-1}(B_1) \cap \cdots \cap f_{X_n}^{-1}(B_n) \right)
\]  
\[
\equiv \int_\Omega \chi_{f_{X_1}^{-1}(B_1)}(\omega) \cdots \chi_{f_{X_n}^{-1}(B_n)}(\omega) \nu_\rho(\text{d}\omega),
\]
\[
B_i \in B_{sp(X_i)}, \quad i = 1, \ldots, n \in \mathbb{N},
\]
implicating the noncontextual qHV representation
\[
\text{tr}[\rho(X_1 \cdots X_n)] = \int_\Omega f_{X_1}(\omega) \cdots f_{X_n}(\omega) \nu_\rho(\text{d}\omega)
\]  
for the quantum product average whenever mutually commuting quantum observables $X_1, \ldots, X_n$, $n \in \mathbb{N}$, are bounded.

Recall that, in probability theory, a measurement situation (an experiment) is generally described\textsuperscript{12} via the Kolmogorov probability model based on the notion of a probability space\textsuperscript{2}, that is, a measure space $(\Omega, \mathcal{F}, \tau)$ where a measure $\tau$ is a probability one.

In view of this, proposition 2 points to the generality of the quasi-classical probability (qHV) model introduced in the Ref. 12 and incorporating the Kolmogorov probability model as a particular case. This new probability model is also outlined in Introduction.

Representation (51) is, in particular, fulfilled for all $N$-partite joint von Neumann measurements on an $N$-partite quantum state. This implies the following statement important for quantum information applications.

**Proposition 3** Every $N$-partite quantum state $\rho$ admits a local qHV (LqHV) model, that is, for each state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, all the $N$-partite joint von Neumann probabilities
\[
\text{tr}[\rho\{P_{X_1}(B_1) \otimes \cdots \otimes P_{X_N}(B_N)\}]
\]  
admit the LqHV representation\textsuperscript{7,10}
\[
\text{tr}[\rho\{P_{X_1}(B_1) \otimes \cdots \otimes P_{X_N}(B_N)\}] = \int_\Omega P_{X_1}(B_1; \omega) \cdots P_{X_N}(B_N; \omega) \nu_\rho(\text{d}\omega)
\]  
in terms of a single measure space $(\Omega, \mathcal{F}, \nu_\rho)$, with a normalized real-valued measure $\nu_\rho$, and conditional probability distributions $P_{X_n}(\cdot; \omega)$, $n = 1, \ldots, N$, each depending only on the corresponding quantum observable $X_n$ on a Hilbert space $\mathcal{H}_n$ at $n$-th site.
Proof. For a state \( \rho \) and quantum observables

\[
X_1 \otimes \mathbb{I}_{\mathcal{H}_2} \otimes \cdots \otimes \mathbb{I}_{\mathcal{H}_N},
\]

\[
\ldots, \nonumber
\]

\[
\mathbb{I}_{\mathcal{H}_1} \otimes \cdots \otimes \mathbb{I}_{\mathcal{H}_{N-1}} \otimes X_{\mathcal{H}_N},
\]
onumber

on \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \), representation (51) has the LqHV form

\[
\text{tr}[\rho \{ P_{X_1}(B_1) \otimes \cdots \otimes P_{X_N}(B_N) \}] = \int_{\Omega} \chi_{f_{X_1}^{-1}(B_1)}(\omega) \cdot \ldots \cdot \chi_{f_{X_N}^{-1}(B_N)}(\omega) \nu_\rho(d\omega),
\]

(56)

\[
B_1 \in \mathcal{B}_{spX_1}, \ldots, B_N \in \mathcal{B}_{spX_N}.
\]

This proves the statement. □

In theorem 2, a mapping \( \Phi \) does not need to satisfy for each Borel function \( \varphi : \mathbb{R} \to \mathbb{R} \) and each quantum observable \( X \) the functional relation \( \Phi(\varphi \circ X) = \varphi \circ \Phi(X) \) required by Kochen and Specker in the Ref. 5. Moreover, in view of the Kochen-Specker result, for a mapping \( \Phi \) in theorem 2, relation (2) cannot be fulfilled whenever \( \dim \mathcal{H} \geq 3 \).

Correspondingly, in a noncontextual qHV model specified for each Hilbert space \( \mathcal{H} \) by theorem 2, the Kochen-Specker assumption (7) does not need to hold for \( \dim \mathcal{H} = 2 \) and cannot be fulfilled if \( \dim \mathcal{H} \geq 3 \). Nevertheless, for each Hilbert space \( \mathcal{H} \), this new qHV model reproduces all the von Neumann joint probabilities and all the quantum product averages via "noncontextual" random variables. Also, in this noncontextual model, all the physically motivated average relations (6) - (8), argued in question (2) of section II, follow explicitly from the noncontextual representations (46), (47).

B. A context-independent qHV model

It is generally argued that if a mapping \( \Psi \) from a set of random variables on \((\Omega, \mathcal{F}_\Omega)\) onto the set of all quantum observable on \( \mathcal{H} \) is non-injective, then the corresponding model reproducing in terms of random variables the statistical properties of all quantum observables on \( \mathcal{H} \) needs to be contextual in the sense that, under a joint von Neumann measurement of mutually commuting quantum observables \( X_1, \ldots, X_n \), each observable \( X_i \) is modelled by a random variable specific for a context of this joint measurement.

In what follows, we prove (theorem 3) that, for all joint von Neumann measurements, the existence of a noncontextual qHV model, specified by theorem 2, implies the existence of a
new qHV model with the setting described, in general, in question (3) formulated in section II. In this new qHV model, the functional condition (5), required in quantum foundations, is fulfilled for each Hilbert space, nevertheless, this model is not contextual.

Consider first the following property proved in appendix C.

**Lemma 2** Let \( \Phi : X_H \to \mathfrak{F}(\Omega, \mathcal{F}_\Omega) \), \( f_X := \Phi(X) \) and \( \{\nu_\rho, \forall \rho\} \) be, correspondingly, the mapping and the measures specified in theorem 2. Then

\[
\nu_\rho(f^{-1}_{\varphi(X)}(B) \cap f^{-1}_{Y_1}(B_1) \cap \cdots \cap f^{-1}_{Y_m}(B_m)) = \nu_\rho((\varphi \circ f_X)^{-1}(B) \cap f^{-1}_{Y_1}(B_1) \cap \cdots \cap f^{-1}_{Y_m}(B_m)),
\]

\( B \in \mathcal{B}_{sp(X)}, \ B_i \in \mathcal{B}_{sp(Y_i)}, \ i = 1, \ldots, m \in \mathbb{N}, \)

for all states \( \rho \), all Borel functions \( \varphi : \mathbb{R} \to \mathbb{R} \) and all finite collections \( \{X, Y_1, \ldots, Y_m\} \) of mutually commuting quantum observables on \( \mathcal{H} \).

From relations (41), (45) and property (57) it follows that if \( f_X, f_{\varphi(X)} \in \Phi(X_H) \) are random variables specified in theorem 2, then the random variables \( f_{\varphi(X)} \) and \( \varphi \circ f_X \) on \( (\Omega, \mathcal{F}_\Omega) \) equivalently represent the quantum observable \( \varphi(X) \) under all joint von Neumann measurements.

However, as it is discussed at the end of section IV.A, for a Borel function \( \varphi : \mathbb{R} \to \mathbb{R} \) and a quantum observable \( X \), the random variable \( \varphi \circ f_X \) does not need to coincide with the random variable \( f_{\varphi(X)} \).

The following theorem (proved in appendix D) answers positively to question (3) put in section II.

**Theorem 3** Let \( \mathcal{H} \) be an arbitrary complex separable Hilbert space. There exist:

(i) a measurable space \( (\Omega, \mathcal{F}_\Omega) \);

(ii) a mapping \( \Psi : \mathfrak{F}_{X_H} \to X_H \) from a set \( \mathfrak{F}_{X_H} \) of random variables \( g \) on \( (\Omega, \mathcal{F}_\Omega) \) onto the set \( X_H \) of all quantum observables on \( \mathcal{H} \), with the spectral correspondence rule \( g(\Omega) = \text{sp}X \)

\( \quad \text{for each } g \in \Psi^{-1}(\{X\}) \) and the functional condition

\[
\varphi \circ g \in \Psi^{-1}(\{\varphi \circ X\}), \ \forall g \in \Psi^{-1}(\{X\}),
\]

(58)

for all Borel functions \( \varphi : \mathbb{R} \to \mathbb{R} \);

such that, to each quantum state \( \rho \) on \( \mathcal{H} \), there corresponds \( (\rho \mapsto \nu_\rho) \) a unique normalized
real-valued measure $\nu_\rho$ on $(\Omega, \mathcal{F}_\Omega)$ satisfying the context-independent relation

$$\text{tr}[\rho \{P_{X_1}(B_1) \cdots P_{X_n}(B_n)\}] = \nu_\rho(\{g_1^{-1}(B_1) \cap \cdots \cap g_n^{-1}(B_n)\}),$$

$$\forall g_i \in \Psi^{-1}(\{X_i\}), \ B_i \in \mathcal{B}_{spX_i}, \ i = 1, \ldots, n,$$

for all collections $\{X_1, \ldots, X_n\} \subset \mathcal{X}_\mathcal{H}$, $n \in \mathbb{N}$, of mutually commuting quantum observables on $\mathcal{H}$. If $\rho_j \overset{\mathcal{M}}{\to} \nu_\rho^j$, $j = 1, \ldots, m < \infty$, then $\sum \alpha_j \rho_j \overset{\mathcal{M}}{\to} \sum \alpha_j \nu_\rho^j$, for all $\alpha_j > 0$, $\sum \alpha_j = 1$.

Theorem 3 implies the following statement proved in appendix D.

**Corollary 3** In the setting of theorem 3, for all states $\rho$ and all finite collections $\{X_1, \ldots, X_n\} \subset \mathcal{X}_\mathcal{H}$ of mutually commuting quantum observables on $\mathcal{H}$, the context-independent representation

$$\text{tr}[\rho \psi(X_1, \ldots, X_n)] = \int_{\Omega} \psi(g_1(\omega), \ldots, g_n(\omega)) \nu_\rho(\omega) \, d\omega,$$

$$\forall g_i \in \Psi^{-1}(\{X_i\}), \ i = 1, \ldots, n,$$

holds for all bounded Borel functions $\psi : \mathbb{R}^n \to \mathbb{R}$ and the context-independent representation

$$\text{tr}[\rho(X_1 \cdot \ldots \cdot X_n)] = \int_{\Omega} g_1(\omega) \cdot \ldots \cdot g_n(\omega) \nu_\rho(\omega) \, d\omega,$$

$$\forall g_i \in \Psi^{-1}(\{X_i\}), \ i = 1, \ldots, n,$$

is fulfilled whenever mutually commuting quantum observables $X_1, \ldots, X_n$ are bounded.

**Remark 2** Representations (59), (60), (61) are context-independent in the sense that, independently of a context of a joint measurement, in the right-hand sides of these representations, there can stand an arbitrary random variable $g_i \in \Psi^{-1}(\{X_i\})$, representing on $(\Omega, \mathcal{F}_\Omega)$ a quantum observable $X_i$.

From theorem 3 it follows that, for each Hilbert space $\mathcal{H}$, the Hilbert space description of all the joint von Neumann measurements (equivalently, of all the quantum observables on $\mathcal{H}$) can be reproduced via a qHV model, where a quantum observable $X$ on $\mathcal{H}$ can be represented on $(\Omega, \mathcal{F}_\Omega)$ by a variety of random variables, but each of these random variables equivalently models $X$ under all joint von Neumann measurements, independently of their measurement contexts – in other words, via a context-independent qHV model.
The specific example of a context-equivalent qHV model is given in the proof of theorem 3.

In a context-independent model, the functional condition (5), required in quantum foundations (see Introduction), constitutes our condition (58) and is, therefore, fulfilled for each Hilbert space.

In view of the Kochen-Specker result, a mapping $\Psi$ in theorem 3 cannot be injective whenever $\dim \mathcal{H} \geq 3$. For $\dim \mathcal{H} = 2$, this mapping does not need to be injective.

Correspondingly, a context-independent qHV model cannot be noncontextual whenever $\dim \mathcal{H} \geq 3$ and does not need to be noncontextual if $\dim \mathcal{H} = 2$. Nevertheless, in this new model, in contrast to a contextual one, all the von Neumann joint probabilities and all the quantum averages are reproduced via the context-independent representations.

Thus, in contrast to the wide-spread opinion, a model reproducing the statistical properties of all quantum observables via random variables does not need to be contextual whenever, in this model, a mapping $\Psi$ from a set of random variables onto the set of all quantum observables is non-injective.

VI. CONCLUSIONS

In the present paper, we have introduced the two new quasi hidden variable (qHV) models correctly reproducing in measure theory terms the Hilbert space description of all joint von Neumann measurements for an arbitrary Hilbert space. These new models answer positively to either of three questions formulated in section II and are important for both – quantum foundations and quantum applications.

For this aim, we had first to generalize (lemma 3) some items of the Kolmogorov extension theorem to the case of consistent operator-valued measures. This generalization allowed us to express (theorem 1) all symmetrized finite products of the spectral measures of quantum observables via a unique self-adjoint operator-valued measure and the specific random variables defined on some specially constructed measurable space.

Based on these new mathematical result, we further analyzed modelling of all joint von Neumann probabilities in qHV terms. We proved that, for each Hilbert space, the Hilbert space description of all joint von Neumann measurements can be reproduced via either of the two new quasi hidden variable (qHV) models, noncontextual or context-independent.
In both of these qHV models, all the von Neumann joint probabilities are represented (theorems 2, 3) via nonnegative values of real-valued measures and all the quantum averages – via the qHV average (corollaries 2, 3) of the corresponding expressions for random variables.

In a noncontextual qHV model (theorem 2, corollary 2), each quantum observable $X$ on a Hilbert space $\mathcal{H}$ is modelled on a measurable space $(\Omega, \mathcal{F}_\Omega)$ by only one random variable $f_X$, representing this quantum observable under all joint von Neumann measurements, regardless of their contexts. In this model, the Kochen-Specker\textsuperscript{5} functional assumptions (2), (3), (4) do not need to hold and, moreover, cannot be fulfilled whenever $\dim \mathcal{H} \geq 3$.

Nevertheless, this new model correctly reproduces the Hilbert space description and the properties of all the von Neumann joint probabilities and all the quantum averages via the noncontextual representations (44) - (47). Moreover, in this model, the von Neumann linearity assumption and the Kochen-Specker assumptions (2), (4) are fulfilled in average in the sense of relation (7) and relations (6), (8), respectively.

The specific example of a noncontextual qHV model for all joint von Neumann measurements is given in the proof of theorem 2.

The proved existence of a noncontextual qHV model, in particular, implies (proposition 3) that every $N$-partite quantum state admits a local qHV (LqHV) model introduced in the Refs. 10.

Also, note that it is specifically the qHV approach that allowed us to construct\textsuperscript{10} the quantum analogs of Bell-type inequalities\textsuperscript{27} and to find\textsuperscript{10} the new exact upper bounds on violation of a Bell-type inequality by a $N$-partite quantum state – the problem which has been intensively discussed in the literature since the publication\textsuperscript{28} of Tsirelson and which is now important for a variety of quantum information processing tasks.

In a context-independent qHV model (theorem 3, corollary 3), a quantum observable $X$ on a Hilbert space $\mathcal{H}$ can be represented on a measurable space $(\Omega, \mathcal{F}_\Omega)$ by a variety of random variables, but each of these random variables equivalently represents $X$ under all joint von Neumann measurements, independently of their measurement contexts. In this model, the functional condition (5) (equivalently, (58)), generally required in quantum foundations, is fulfilled, so that, for $\dim \mathcal{H} \geq 3$, this model cannot be noncontextual. For $\dim \mathcal{H} = 2$, a context-independent qHV model does not need to be noncontextual.

Nevertheless, in contrast to a contextual model, this new model reproduces the Hilbert...
space description of all the von Neumann joint probabilities and all the quantum averages via the context-independent representations (59) - (61). Also, in this model, due to representation (61), the von Neumann linearity assumption and the Kochen-Specker assumption (4) are fulfilled in average in the sense of relations (7) and (8), respectively, and with arbitrary representatives on \((\Omega, \mathcal{F}_\Omega)\) of quantum observables \(X_1, \ldots, X_n\) on \(\mathcal{H}\).

The specific example of a context-independent qHV model is presented in the proof of theorem 3.

The proved existence of a context-independent qHV model negates the general opinion that, in terms of random variables satisfying the functional condition (5), the Hilbert space description of all the joint von Neumann measurements for \(\dim \mathcal{H} \geq 3\) can be reproduced only contextually.

The results of the present paper also underline the generality of the quasi-classical probability model proposed in the Ref. 12.

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**Appendix A: proof of lemma 1**

Relation (22) follows explicitly from the symmetrized form of the right-hand side in (19). In order to prove (23), let us first take a collection \(\{X_1, \ldots, X_n\}\) of bounded quantum observables with discrete spectrums. In this case, the measure \(\mathcal{P}_{(X_1, \ldots, X_n)}\) is given by representation (20) and taking into the account that \(P_{X_i}(\text{sp}X_i) = \mathbb{I}_\mathcal{H}\), we have:

\[
\mathcal{P}_{(X_1, \ldots, X_n)}\left(\{(x_1, \ldots, x_n) \in \text{sp}X_1 \times \cdots \times \text{sp}X_n \mid (x_{i_1}, \ldots, x_{i_k}) \in F\}\right)
= \frac{1}{n!} \sum_{(x_1, \ldots, x_n) \in F} \left\{ P_{X_1}(\{x_1\}) \cdots P_{X_n}(\{x_n\}) \right\}_{\text{sym}}
= \frac{1}{k!} \sum_{(x_{i_1}, \ldots, x_{i_k}) \in F} \left\{ P_{X_{i_1}}(\{x_{i_1}\}) \cdots P_{X_{i_k}}(\{x_{i_m}\}) \right\}_{\text{sym}}
= \mathcal{P}_{(X_{i_1}, \ldots, X_{i_k})}(F) .
\]

For an arbitrary collection \(\{X_1, \ldots, X_n\} \subset \mathcal{X}_\mathcal{H}\) of quantum observables on \(\mathcal{H}\), let \(\mathcal{E}\) be the set of all rectangles \(E := B_{i_1} \times \cdots \times B_{i_k}\) with measurable sides \(B_i \in \mathcal{B}_{\text{sp}X_i}\). Since the product
algebra \( F_{\text{sp} X_1 \times \cdots \times \text{sp} X_k} \) consists of all finite unions of mutually disjoint rectangles from \( \mathcal{E} \), every set \( F \in F_{\text{sp} X_1 \times \cdots \times \text{sp} X_k} \) admits a finite decomposition

\[
F = \bigcup_{m=1, \ldots, M} E_m, \quad E_{m_1} \cap E_{m_2} = \emptyset, \quad E_m \in \mathcal{E}, \quad M < \infty. \tag{A2}
\]

Taking into the account that \( \mathcal{P}(X_1, \ldots, X_n) \) and \( \mathcal{P}(X_1, \ldots, X_k) \) are finitely additive measures and also, relations (A2), (19) and \( \mathcal{P}_{X_i}(\text{sp} X_i) = I_H \), we have:

\[
\mathcal{P}(X_1, \ldots, X_n) \left( \big\{(x_1, \ldots, x_n) \in \text{sp} X_1 \times \cdots \times \text{sp} X_n \mid (x_{i_1}, \ldots, x_{i_k}) \in F \big\} \right) \tag{A3}
\]
\[
= \sum_{m=1, \ldots, M} \mathcal{P}(X_1, \ldots, X_n) \left( \big\{(x_1, \ldots, x_n) \in \text{sp} X_1 \times \cdots \times \text{sp} X_n \mid (x_{i_1}, \ldots, x_{i_k}) \in E_m \big\} \right)
\]
\[
= \sum_{m=1, \ldots, M} \mathcal{P}(X_1, \ldots, X_k) (E_m)
\]
\[
= \mathcal{P}(X_1, \ldots, X_k) (F).
\]

This proves lemma 1.

**Appendix B: a generalization of the Kolmogorov extension theorem**

In this appendix, we generalize (lemma 3) to the case of consistent operator-valued measures some items of the Kolmogorov consistency theorem \(^{2,13}\) for a family of consistent probability measures \(^{23}\).

For an uncountable index set \( T \), consider a family \( \{(\Lambda_t, \mathcal{F}_{\Lambda_t}), t \in T\} \) of measurable spaces, where each \( \Lambda_t \) is a non-empty set and \( \mathcal{F}_{\Lambda_t} \) is an algebra of subsets of \( \Lambda_t \).

Let \( \mathcal{F}_{\Lambda_{t_1} \times \cdots \times \Lambda_{t_n}} \) be the product algebra on \( \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \), that is, the algebra generated by all rectangles \( F_1 \times \cdots \times F_n \subseteq \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \) with measurable sides \( F_k \in \mathcal{F}_{\Lambda_{t_k}} \).

Denote by \( \tilde{\Lambda} := \prod_{t \in T} \Lambda_t \) the Cartesian product \(^{23}\) of all sets \( \Lambda_t \), \( t \in T \), that is, \( \tilde{\Lambda} \) is the collection of all functions \( \lambda : T \to \bigcup_{t \in T} \Lambda_t \) with values \( \lambda(t) := \lambda_t \in \Lambda_t \).

The set of all cylindrical subsets of \( \tilde{\Lambda} \) of the form

\[
\mathcal{J}(t_1, \ldots, t_n)(F) := \{ \lambda \in \tilde{\Lambda} \mid (\lambda_{t_1}, \ldots, \lambda_{t_n}) \in F \}, \quad F \in \mathcal{F}_{\Lambda_{t_1} \times \cdots \times \Lambda_{t_n}}, \tag{B1}
\]

where \( \{t_1, \ldots, t_n\} \subseteq T, n \in \mathbb{N} \), constitutes \(^{13,22}\) an algebra on \( \tilde{\Lambda} \) that we further denote by \( \mathcal{A}_{\tilde{\Lambda}} \).

Since \( \mathcal{J}(t_1, \ldots, t_n)(F) \equiv \pi_{(t_1, \ldots, t_n)}^{-1}(F) \), where the function

\[
\pi_{(t_1, \ldots, t_n)} : \tilde{\Lambda} \to \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \tag{B1}
\]

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is the canonical projection

$$
\pi(t_1, \ldots, t_n)(\lambda) := (\pi_{t_1}(\lambda), \ldots, \pi_{t_n}(\lambda)), \quad \pi_t(\lambda) := \lambda_t,
$$

we have

$$
A_{\tilde{\Lambda}} = \left\{ \pi^{-1}_{(t_1, \ldots, t_n)}(F) \subseteq \tilde{\Lambda} \mid F \in \mathcal{F}_{\Lambda t_1 \times \cdots \times \Lambda t_n}, \quad \{t_1, \ldots, t_n\} \subset T, \ n \in \mathbb{N} \right\}.
$$

Introduce a family

$$\{\mathfrak{M}(t_1, \ldots, t_n) : \mathcal{F}_{\Lambda t_1 \times \cdots \times \Lambda t_n} \to \mathcal{L}_{\mathcal{H}}^{(s)} \mid \mathfrak{M}(t_1, \ldots, t_n)(\Lambda t_1 \times \cdots \times \Lambda t_n) = \mathbb{I}_H, \ \{t_1, \ldots, t_n\} \subset T, \ n \in \mathbb{N} \}$$

of normalized finitely additive $\mathcal{L}_{\mathcal{H}}^{(s)}$-valued measures $\mathfrak{M}(t_1, \ldots, t_n)$, each specified by a finite collection $\{t_1, \ldots, t_n\} \subset T$ of indices and having values that are self-adjoint bounded linear operators on $\mathcal{H}$.

Let, for each finite index collection $\{t_1, \ldots, t_n\} \subset T$, these measures satisfy the consistency condition

$$\mathfrak{M}(t_1, \ldots, t_n)(F_1 \times \cdots \times F_n) = \mathfrak{M}(t_1, \ldots, t_n)(F_1 \times \cdots \times F_n),$$

for all permutations $\left( t_{i_1}, \ldots, t_{i_n} \right)$ and the consistency condition

$$\mathfrak{M}(t_1, \ldots, t_n)\left( \{ (\lambda_1, \ldots, \lambda_n) \in \Lambda t_1 \times \cdots \times \Lambda t_n \mid (\lambda_{i_1}, \ldots, \lambda_{i_k}) \in F \} \right) = \mathfrak{M}(t_{i_1}, \ldots, t_{i_k})(F), \quad F \in \mathcal{F}_{\Lambda t_{i_1} \times \cdots \times \Lambda t_{i_k}},$$

for each $\{t_{i_1}, \ldots, t_{i_k}\} \subseteq \{t_1, \ldots, t_n\}$.

**Lemma 3** Let $\mathcal{H}$ be a complex separable Hilbert space. For a family $\{B_4\}$ of normalized finitely additive $\mathcal{L}_{\mathcal{H}}^{(s)}$-valued measures $\mathfrak{M}(t_1, \ldots, t_n)$ satisfying the consistency conditions $\{B_2\}$, there exists a unique normalized finitely additive $\mathcal{L}_{\mathcal{H}}^{(s)}$-valued measure

$$\mathfrak{M} : A_{\tilde{\Lambda}} \to \mathcal{L}_{\mathcal{H}}^{(s)}, \quad \mathfrak{M}(\tilde{\Lambda}) = \mathbb{I}_H,$$

on $(\tilde{\Lambda}, A_{\tilde{\Lambda}})$ such that

$$\mathfrak{M}\left( \pi^{-1}_{(t_1, \ldots, t_n)}(F) \right) = \mathfrak{M}(t_1, \ldots, t_n)(F)$$

for all sets $F \in \mathcal{F}_{\Lambda t_1 \times \cdots \times \Lambda t_n}$ and an arbitrary index collection $\{t_1, \ldots, t_n\} \subset T, n \in \mathbb{N}$. 27
Proof. Our proof of lemma 3 is quite similar to the proof of the corresponding items in the Kolmogorov extension theorem for consistent probability measures. Let $\mathcal{A}_\tilde{\Lambda}$ be algebra on $\tilde{\Lambda}$. Suppose that a set $A \in \mathcal{A}_\tilde{\Lambda}$ admits representation $A = \pi^{-1}_{(t_1, \ldots, t_n)}(F)$, where $\{t_1, \ldots, t_n\} \subset T$ and $F \in \mathcal{F}_{\Lambda_{t_1} \times \cdots \times \Lambda_{t_n}}$, and take

$$M(A) := M_{(t_1, \ldots, t_n)}(F).$$

(B9)

In order to show that relation (B9) defines correctly a set function $M$ on $\mathcal{A}_\tilde{\Lambda}$, we must prove that this relation implies a unique value of $M$ on a set $A$ even if this set $A$ admits two different representations, say:

$$A = \pi^{-1}_{(t_1, \ldots, t_k)}(F) \equiv \{\lambda \in \tilde{\Lambda} \mid (\lambda_{t_1}, \ldots, \lambda_{t_k}) \in F\},$$

(B10)

$$A = \pi^{-1}_{(t_j, \ldots, t_m)}(F') \equiv \{\lambda \in \tilde{\Lambda} \mid (\lambda_{t_j}, \ldots, \lambda_{t_m}) \in F'\}$$

for some sets $F \in \mathcal{F}_{\Lambda_{t_1} \times \cdots \times \Lambda_{t_k}}$ and $F' \in \mathcal{F}_{\Lambda_{t_j} \times \cdots \times \Lambda_{t_m}}$ and some index collections $\{t_{i_1}, \ldots, t_{i_k}\}$; $\{t_j, \ldots, t_m\} \subset T$.

Denote

$$\{t_{i_1}, \ldots, t_{i_k}\} \cup \{t_j, \ldots, t_m\} = \{t_1, \ldots, t_n\}. $$

(B11)

From representations (B3) it follows that sets $F$ and $F'$ are such that, for a point $(\lambda_1, \ldots, \lambda_n)$ in the set $\Lambda_{t_1} \times \cdots \times \Lambda_{t_n}$, the condition $(\lambda_{i_1}, \ldots, \lambda_{i_k}) \in F$ and the condition $(\lambda_{j_1}, \ldots, \lambda_{j_m}) \in F'$ are equivalent, that is:

$$\{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \ldots, \lambda_{i_k}) \in F \}$$

$$= \{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \ldots, \lambda_{j_m}) \in F' \}.$$  

(B12)

Due to relations (B9) - (B3) and the consistency conditions (B2), (B2), we have:

$$M(\pi^{-1}_{(t_{i_1}, \ldots, t_{i_k})}(F)) = M_{(t_{i_1}, \ldots, t_{i_k})}(F)$$

(B13)

$$= M_{(t_{i_1}, \ldots, t_{n})}(\{(\lambda_1, \ldots, \lambda_n) \in \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \mid (\lambda_{i_1}, \ldots, \lambda_{i_k}) \in F\})$$

$$= M_{(t_{i_1}, \ldots, t_{n})}(\{(\lambda_1, \ldots, \lambda_n) \in \Lambda_{t_1} \times \cdots \times \Lambda_{t_n} \mid (\lambda_{j_1}, \ldots, \lambda_{j_m}) \in F'\})$$

$$= M_{(t_{j_1}, \ldots, t_{m})}(F')$$

$$= M(\pi^{-1}_{(t_{j_1}, \ldots, t_{j_m})}(F')).$$

Thus, relation (B9) defines a unique set function $M : \mathcal{A}_\tilde{\Lambda} \to \mathcal{L}_H$ satisfying condition (B8). Since $\tilde{\Lambda} = \pi^{-1}_{(t_1, \ldots, t_n)}(\Lambda_{t_1} \times \cdots \times \Lambda_{t_n})$ and $M_{(t_1, \ldots, t_n)}(\Lambda_{t_1} \times \cdots \times \Lambda_{t_n}) = \Pi_H$, from (B9) it follows that this set function $M$ is normalized, that is, $M(\tilde{\Lambda}) = \Pi_H$. 

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In order to prove that the normalized set function $\mathcal{M} : \mathcal{A}_\Lambda \rightarrow \mathcal{L}_H$ is additive, let us consider in the algebra $\mathcal{A}_\Lambda$ two disjoint sets

$$A_1 = \pi_{(t_1, \ldots, t_k)}^{-1}(F_1), \quad A_2 = \pi_{(t_j, \ldots, t_{jm})}^{-1}(F_2), \quad A_1 \cap A_2 = \emptyset,$$

specified by some index collections $\{t_1, \ldots, t_k\}, \{t_j, \ldots, t_{jm}\} \subseteq \{t_1, \ldots, t_n\} \subset T$ and sets $F_1 \in \mathcal{F}_{\lambda_1 \times \cdots \times \lambda_{ik}}$ and $F_2 \in \mathcal{F}_{\lambda_j \times \cdots \times \lambda_{jm}}$. Since $A_1 \cap A_2 = \emptyset$, the sets $F_1, F_2$ in (B14) are such that, for a point $(\lambda_1, \ldots, \lambda_n)$ in $\Lambda_1 \times \cdots \times \Lambda_{tn}$, conditions $(\lambda_i, \ldots, \lambda_{ik}) \in F_1$ and $(\lambda_j, \ldots, \lambda_{jm}) \in F_2$ are mutually exclusive, that is:

$$\{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_1 \times \cdots \times \Lambda_{tn} \mid (\lambda_i, \ldots, \lambda_{ik}) \in F_1 \} \cap \{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_1 \times \cdots \times \Lambda_{tn} \mid (\lambda_j, \ldots, \lambda_{jm}) \in F_2 \} = \emptyset.$$

Taking into the account relations (B9), (B14), (B3), the consistency conditions (B2), (B2), and also that each $\mathcal{M}_{(t_1, \ldots, t_n)}$ is a finitely additive measure on $\mathcal{A}_\Lambda$, we have

$$\mathcal{M}(A_1 \cup A_2) = \mathcal{M}_{(t_1, \ldots, t_n)}(\{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_1 \times \cdots \times \Lambda_{tn} \mid (\lambda_i, \ldots, \lambda_{ik}) \in F_1 \text{ or } (\lambda_j, \ldots, \lambda_{jm}) \in F_2 \})$$

$$= \mathcal{M}_{(t_1, \ldots, t_n)}(\{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_1 \times \cdots \times \Lambda_{tn} \mid (\lambda_i, \ldots, \lambda_{ik}) \in F_1 \}) + \mathcal{M}_{(t_1, \ldots, t_n)}(\{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_1 \times \cdots \times \Lambda_{tn} \mid (\lambda_j, \ldots, \lambda_{jm}) \in F_2 \})$$

$$= \mathcal{M}_{(t_1, \ldots, t_k)}(F_1) + \mathcal{M}_{(t_j, \ldots, t_{jm})}(F_2)$$

$$= \mathcal{M}(A_1) + \mathcal{M}(A_2).$$

Therefore, the normalized set function $\mathcal{M}$ on $\mathcal{A}_\Lambda$ defined via relation (B9) is additive and constitutes a finitely additive measure on $\mathcal{A}_\Lambda$, see remark 1.

Thus, the set function $\mathcal{M} : \mathcal{A}_\Lambda \rightarrow \mathcal{L}_H$ defined by relation (B9) constitutes a unique normalized finitely additive $\mathcal{L}_H$-valued measure satisfying representation (B8). This completes the proof of lemma 3.

**Appendix C: proof of lemma 2**

For a quantum observable $X \in \mathcal{X}_H$, bounded or unbounded, and a Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, the quantum observable $\varphi(X) \equiv \varphi \circ X$ is defined as

$$\varphi(X) := \int_{\mathbb{R}} \varphi(x) P_X(dx)$$

(C1)
and has the spectrum \( \text{sp}\varphi(X) = \varphi(\text{sp}X) \). Its spectral measure \( P_{\varphi(X)} \) satisfies the relation

\[
P_{\varphi(X)}(B) = P_X(\varphi^{-1}(B)), \quad B \in \mathcal{B}_{\text{sp}\varphi(X)}.
\]

Due to \( \text{(C2)} \), if quantum observables \( X, Y_1, \ldots, Y_m \) mutually commute, then the same is true for quantum observables \( \varphi(X), Y_1, \ldots, Y_m \).

For arbitrary mutually commuting observables \( X, Y_1, \ldots, Y_m \) on \( \mathcal{H} \), relation \( \text{(45)} \) and property \( \text{(C2)} \) imply

\[
\nu(\varphi^{-1}(f^{-1}(B)) \cap f_Y^{-1}(B_1) \cap \cdots \cap f_Y^{-1}(B_m)) = \text{tr}[\rho \{P_X(\varphi^{-1}(B)) \cdot P_{Y_1}(B_1) \cdot \cdots \cdot P_{Y_m}(B_m)\}]
\]

This proves lemma 2.

**Appendix D: proof of theorem 3 and corollary 3**

In order to prove the existence point of theorem 3, let us take the specific noncontextual qHV model that we used for the proof of theorem 2. Namely, let \((\Lambda, \mathcal{F}_\Lambda)\) and \(\pi_X : \Lambda \to \text{sp}X, X \in \mathcal{X}_H\), be the measurable space and the random variables specified in theorem 1. Then the one-to-one mapping \(\tilde{\Phi} : \mathcal{X}_H \to \mathcal{F}(\Lambda, \mathcal{F}_\Lambda)\), defined by \(\tilde{\Phi}(X) = \pi_X, \forall X \in \mathcal{X}_H\), satisfies the setting of theorem 2.

Note that, for each Borel function \(\varphi : \mathbb{R} \to \mathbb{R}\) and each observable \(X\), the random variable \(\varphi \circ \pi_X : \Lambda \to \text{sp}\varphi(X)\) satisfies the spectral correspondence rule \((\varphi \circ \pi_X)(\Lambda) = \text{sp}\varphi(X)\) but does not coincide with the random variable \(\pi_{\varphi(X)}\) and does not belong to the image \(\tilde{\Phi}(\mathcal{X}_H)\). However, due to representation \(\text{(45)}\) and property \(\text{(57)}\), the random variables \(\varphi \circ \pi_X\) and \(\pi_{\varphi(X)}\) equivalently model the quantum observable \(\varphi(X)\) under all joint von Neumann measurements, *independently* of their measurements contexts.

Taking all this into the account, let \(\phi_X^{(\theta)}(Y_\theta) = X\), with a Borel function \(\phi_X^{(\theta)} : \mathbb{R} \to \mathbb{R}\), an observable \(Y_\theta \in \mathcal{X}_H\) and \(\theta \in \Theta_X\), be a possible functional representation of a quantum observable \(X \in \mathcal{X}_H\) via a quantum observable \(Y_\theta\). Here, \(\Theta_X\) is an index set of all such
representations of an observable X. Since there is always the trivial representation, let \( \theta_0 \), where \( \phi^{(\theta_0)} \equiv 1 \), \( Y_{\theta_0} = X \), the set \( \Theta_X \) is non-empty for each \( X \).

Let \([\pi_X]\) be the set of the following random variables on \((\Lambda, \mathcal{F}_\Lambda)\):

\[
[\pi_X] = \{ g^{(\theta)}_X | \theta \in \Theta_X \}, \quad \text{where} \quad g^{(\theta)}_X = \phi^{(\theta)}_X \circ \pi_{Y_{\theta}},
\]

(D1)

\[
\phi^{(\theta)}_X \circ Y_{\theta} = X, \quad \phi^{(\theta)}_X : \mathbb{R} \to \mathbb{R}, \quad Y_{\theta} \in \mathcal{X}_H.
\]

By lemma 2, all random variables in \([\pi_X]\) equivalently represent an observable \( X \) under all joint von Neumann measurements. Moreover, by our construction of the set \([\pi_X]\), each random variable \( g^{(\theta)}_X \in [\pi_X] \) satisfies the spectral correspondence rule \( g^{(\theta)}_X(\Lambda) = \text{sp} X \).

In order to prove that

\[ X_1 \neq X_2 \Rightarrow [\pi_{X_1}] \cap [\pi_{X_2}] = \emptyset, \]  

(D2)

let us suppose that, for some observables \( X_1 \neq X_2 \), the intersection \([\pi_{X_1}] \cap [\pi_{X_2}] \neq \emptyset \). Then \( \text{sp} X_1 = \text{sp} X_2 \) and a common random variable \( g \in [\pi_{X_1}] \cap [\pi_{X_2}] \) admits either of representations

\[
g = \phi_{X_1} \circ \pi_{Y_1}, \quad g = \phi_{X_2} \circ \pi_{Y_2}, \quad \text{where}
\]

(D3)

\[
\phi_{X_1}(Y_1) = X_1, \quad \phi_{X_2}(Y_2) = X_2, \quad Y_1, Y_2 \in \mathcal{X}_H,
\]

with

\[
\pi_{Y_1}^{-1}(\phi_{X_1}^{-1}(B)) = (\phi_{X_1} \circ \pi_{Y_1})^{-1}(B)
\]

(D4)

\[
= (\phi_{X_2} \circ \pi_{Y_2})^{-1}(B) = \pi_{Y_2}^{-1}(\phi_{X_2}^{-1}(B))
\]

for each \( B \in \mathcal{B}_{\text{sp} X_1} = \mathcal{B}_{\text{sp} X_2} \). Due to (C2), (H) and (D4), we have:

\[
P_{X_1}(B) = P_{\phi_{X_1}(Y_1)}(B) = P_{\phi^{-1}_{X_1}(B)}(B)
\]

(D5)

\[
= M(\pi_{Y_1}^{-1}(\phi_{X_1}^{-1}(B)))
\]

\[
= M((\pi_{Y_2}^{-1}(\phi_{X_2}^{-1}(B))))
\]

\[
= P_{Y_2}(\phi_{X_2}^{-1}(B)) = P_{\phi_{X_2}(Y_2)}(B) = P_{X_2}(B),
\]

\[ B \in \mathcal{B}_{\text{sp} X_1} = \mathcal{B}_{\text{sp} X_2}. \]

In view of the spectral theorem (10), relation \( P_{X_1}(B) = P_{X_2}(B), \forall B \), implies \( X_1 = X_2 \). Thus, if \([\pi_{X_1}] \cap [\pi_{X_2}] \neq \emptyset \), then \( X_1 = X_2 \). This proves (D2).
Let
\[ \mathcal{F}_X := \bigcup_{X \in \mathcal{X}_H} [\pi_X] \]  
be the union of all disjoint sets \([\pi_X], \forall X \in \mathcal{X}_H\), of random variables and \(\Psi: \mathcal{F}_X \rightarrow \mathcal{X}_H\) be the mapping on this set defined via the relation
\[ \Psi(g) = X, \quad g \in \mathcal{F}_X \cap [\pi_X], \quad X \in \mathcal{X}_H, \]  
and having, therefore, the preimage
\[ \Psi^{-1}(\{X\}) = [\pi_X], \quad X \in \mathcal{X}_H. \]  

For a Borel function \(\varphi: \mathbb{R} \rightarrow \mathbb{R}\) and an arbitrary \(g^{(\theta)}_X \in [\pi_X] = \Psi^{-1}(\{X\})\), consider the random variable \(\varphi \circ g^{(\theta)}_X\). Due to our above construction of the set \([\pi_X]\), a random variable \(g^{(\theta)}_X\) admits a representation
\[ g^{(\theta)}_X = \phi^{(\theta)}_X \circ \pi_{Y_{\theta}}, \]  
where a Borel function \(\phi^{(\theta)}_X: \mathbb{R} \rightarrow \mathbb{R}\) and a quantum observable \(Y_{\theta}\) are such that \(\phi^{(\theta)}_X \circ Y_{\theta} = X\). We have:
\[ \varphi \circ g^{(\theta)}_X = (\varphi \circ \phi^{(\theta)}_X) \circ \pi_{Y_{\theta}} \in [\pi_{(\varphi \circ \phi^{(\theta)}_X) \circ Y_{\theta}}] = [\pi_{\varphi \circ X}] = \Psi^{-1}(\{\varphi \circ X\}). \]  

Thus, the mapping \(\Psi\), defined by relation (D7), satisfies the functional condition (58).

Combining all this with representation (45), property (57), definitions (D1), (D7) we prove the existence point of theorem 3, in particular, the context-independence of representation (59).

The relation \(\sum \alpha_j \rho_j \mapsto \sum \alpha_j \nu_{\rho_j}\) in theorem 3 follows explicitly from relation (37) in proposition 1. This completes the proof of theorem 3.

In corollary 3, the context-independent representations (60), (61) follow from (59), (43), (47) and are proved quite similarly to our proof of (46), (47).

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26 The notation $\Psi^{-1}(\{X\}) := \{ g \in \mathfrak{F}_{\mathfrak{H}} \mid \Psi(g) = X \}$ means the preimage of an observable $X$ under the mapping $\Psi$.

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