On empty pentagons and hexagons in planar point sets

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Abstract

We give improved lower bounds on the minimum number of $k$-holes (empty convex $k$-gons) in a set of $n$ points in general position in the plane, for $k = 5, 6$.

Keywords: Empty polygon, planar point set, empty hexagon, empty pentagon

1 Introduction

We say that a set $P$ of points in the plane is in general position if it contains no three points on a line.

Let $P$ be a set of $n$ points in general position in the plane. A $k$-hole of $P$ (sometimes also called empty convex $k$-gon or convex $k$-hole) is a set of vertices of a convex $k$-gon with vertices in $P$ containing no other points of $P$.

Let $X_k(n)$ be the minimum number of $k$-holes in a set of $n$ points in general position in the plane. Horton [7] proved that $X_k(n) = 0$ for any $k \geq 7$ and for any positive integer $n$. The following bounds on $X_k(n), k = 3, 4, 5, 6$, are known (the letter $H$ denotes the number of vertices of the convex hull of the point set):

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*Work on this paper was supported by project 1M0545 of The Ministry of Education of the Czech Republic.

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The upper bounds were shown in [2], improving previous bounds of [9, 11, 14]. The lower bounds for \( k = 3, 4, 5 \) can be found in an updated version of the conference paper [6], also improving lower bounds from several papers. The lower bound on \( X_6(n) \) follows from a result of V. A. Koshelev [8].

In this paper we give the following improved lower bounds:

**Theorem 1**

\[
X_5(n) \geq \frac{n}{2} - O(1),
\]

\[
X_6(n) \geq \frac{n}{229} - 4.
\]

After finishing our research, we have learned that a group of researchers including Oswin Aichholzer, Ruy Fabila-Monroy, Clemens Huemer, and Birgit Vogtenhuber has very recently obtained a better bound \( X_5(n) \geq 3n/4 - o(n) \). Their result is not written yet. Their method does not seem to achieve our bound on \( X_6(n) \) but it also gives slight improvements on the lower bounds on \( X_3(n) \) and \( X_4(n) \) mentioned above.

## 2 Proofs

To prove the first inequality in Theorem 1, it suffices to prove that if \( P \) is a set of \( n > 20 \) points in general position in the plane then \( P \) contains a subset \( P' \) of eight points such that \( P' \) and \( P - P' \) can be separated by a line and at least four 5-holes of \( P \) intersect \( P' \). Indeed, if this is true then we can repeatedly remove eight points of \( P' \). Each removal decreases the number of points by 8 and the number of 5-holes by at least 4. Doing this as long as at least 21 points remain, we obtain the first inequality in Theorem 1.
Let $P$ be a set of $n > 20$ points in general position in the plane. For two points $x, y$ of $P$, we denote by $L(xy)$ the open halfplane to the left of the line $xy$ (oriented from $x$ to $y$). The complementary open halfplane is denoted by $R(xy)$. If $L(xy)$ contains exactly $k$ points of $P$, then we say that the oriented segment $xy$ is a $k$-edge of $P$.

Take a vertex $a$ of the convex hull of $P$. Order the other points radially around $a$ starting from the point on the convex hull clockwise from $a$. Let $a'$ be the 12-th point in this order. Then $aa'$ is an 11-edge. Since $X_5(10) > 0$ [5], $L(aa')$ contains a 5-hole, $D$, of $P$. In the rest of the proof, $D$ is fixed but $aa'$ may later denote other 11-edges.

The key part of the proof is to find an 11-edge $bb'$ such that $b$ is a vertex of $D$ and the other four vertices of $D$ lie in $L(bb')$. To do it, we clockwise rotate a line $l$ starting from $l = aa'$ as follows. Initially we start to rotate $l$ at the midpoint of the segment $aa'$. During the rotation, the center of rotation may change at any moment but the rotated line $l$ cannot go over any point of $P$. We rotate as long as it is possible, until we reach a position $l = bb'$, where $b, b' \in P$, the point $b$ was originally to the left of $l$ and $b'$ was originally to the right of $l$. Thus, $b \in L(aa') \cup \{a'\}$ and $b' \in R(aa') \cup \{a\}$. There are no points of $P$ in the open wedges $R(aa') \cap L(bb')$ and $L(aa') \cap R(bb')$. The edge $bb'$ is an 11-edge of $P$. We distinguish three cases:

Case 1: The segments $aa'$ and $bb'$ internally cross, thus $a, a', b, b'$ are pairwise different.

Case 2: $b' = a$.

Case 3: $b = a'$.

Since $D$ lies in $L(aa')$, it also lies in $L(bb') \cup \{b\}$. The point $b$ may be a vertex of $D$ in Cases 1 and 2. All other vertices of $D$ lie in $L(bb')$. If $b$ is not a vertex of $D$, then we rename the points $b$ and $b'$ by $a$ and $a'$, respectively, and rotate a line $l$ in the same way as above from the position $l = aa'$. We reach some new position $l = bb'$. Repeat this process until the point $b$ coincides with one of the vertices of $D$. (Note that the line $l$ cannot rotate outside of $D$ forever, because $n > 20$.) Then we are in Case 1 or in Case 2, and the other four vertices of $D$ lie in $L(aa') \cap L(bb')$. In Case 1 or 2, we consider the 12-point set $Q := (P \cap L(bb')) \cup \{b\}$. Since $X_5(12) \geq 3$ [3], the set $Q$ contains at least three 5-holes of $P$. Together with $D$, these are at least four 5-holes of $P$ with vertices in the 13-point set $Q \cup \{b\} = P \cap \text{closure}(L(bb'))$. None of these 5-holes contains both $b$ and $b'$. Therefore, we can take $P'$ as the set of eight points of $L(bb')$ with largest distances to the line $bb'$. This finishes the proof of the first inequality in Theorem [4].
We remark without proof that a slightly better bound \((1/2 + c)n - \text{const}\) with \(c > 0\) can be obtained by using the fact that any sufficiently large set \(P\) contains linearly many disjoint 6-holes.

The above proof can be generalized to give the more general theorem below. The theorem below together with \(X_6(463) > 0\) (proved by V. A. Koshelev [8]) gives the second inequality in Theorem 1.

**Theorem 2** Suppose that \(X_k(s - 1) \geq 1\) and \(X_k(s) \geq t\) for some positive integers \(k, s, t\). Then \(X_k(n) \geq \frac{t+1}{s-k+1}(n - (2s - 2))\) for \(n \geq 2s - 2\).

**Proof.** If \(P\) is a set of \(n > 2s - 2\) points then \(P\) contains an \((s-1)\)-edge \(aa'\). Let \(D\) be a \(k\)-hole of \(P\) contained in \(L(aa')\). Analogously as in the previous proof, we find two \((s-1)\)-edges \(aa'\) and \(bb'\) such that \(b\) is a vertex of \(D\) and \(D\) lies in \(L(aa')\) and also in \(L(bb') \cup \{b\}\). In Case 1 or 2, we consider the \(s\)-point set \(Q := (P \cap L(bb')) \cup \{b\}\). Since \(X_k(s) \geq t\), the set \(Q\) contains at least \(t\) \(k\)-holes of \(P\). Together with \(D\), these are at least \(t+1\) \(k\)-holes of \(P\) with vertices in the \(s+1\)-point set \(Q \cup \{b\} = P \cap \text{closure}(L(bb'))\). None of these \(k\)-holes contains both \(b\) and \(b'\). Therefore, if we take \(P'\) as the set of \(s-k+1\) points of \(L(bb')\) with largest distances to the line \(bb'\) then removing the \(s-k+1\) points of \(P'\) from \(P\) decreases the number of \(k\)-holes by at least \(t+1\). Theorem 2 follows.

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