Localization transition on the Random Regular Graph as an unstable tricritical point in a log-normal Rosenzweig-Porter random matrix ensemble.

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Gaussian Rosenzweig-Porter (GRP) random matrix ensemble is the only one in which the robust multifractal phase and ergodic transition have a status of a mathematical theorem. Yet, this phase in GRP model is oversimplified: the spectrum of fractal dimensions is degenerate and the mini-band in the local spectrum is not multifractal. In this paper we suggest an extension of the GRP model by adopting a logarithmically-normal (LN) distribution of off-diagonal matrix elements. A family of such LN-RP models is parametrized by a symmetry parameter $p$ and it interpolates between the GRP at $p \to 0$ and Levy ensembles at $p \to \infty$. A special point $p = 1$ is shown to be the simplest approximation to the Anderson localization model on a random regular graph. We study in detail the phase diagram of LN-RP model and show that $p = 1$ is a tricritical point where the multifractal phase first collapses. This collapse is shown to be unstable with respect to the truncation of the log-normal distribution. We suggest a new criteria of stability of the non-ergodic phases and prove that the Anderson transition in LN-RP model is discontinuous at all $p > 0$.

I. INTRODUCTION

The development of Quantum Computing algorithms and the problem of Many Body Localization (MBL) [1] in interacting systems (e.g. in spin chains) ignited an interest to the single-particle (Anderson) localization on random graphs as a proxy for MBL. The analogy comes from the mapping in which bit strings of spins-$\frac{1}{2}$ (describing many-body configurations) correspond to sites on a graph and interaction provides transitions between these bit strings represented by a link between sites. The bit strings directly accessible from a given one are uniquely determined by the interaction Hamiltonian, so does the structure and topology of the corresponding graph.

Since the seminal work [2] there is a mounting evidence that a representative class of interacting systems can be modeled by a graph with a local tree structure but without a boundary. The simplest graph of such kind is a random regular graph (RRG), Fig. 1, which was suggested in Refs. [3, 4] as a toy model for many-body localization.

In particular, it was conjectured in these works [3, 4] that a special Non-Ergodic Extended (NEE) phase is realized on RRG which random eigenfunctions are multifractal. In view of the above analogy with the system of interacting qubits this statement, if correct, is of extreme importance. Recently such NEE states were suggested [5, 6] as mediators for implementing efficient population transfer in the Grover’s Quantum Search Algorithms with potential application to Machine Learning.

While the existence of NEE phase on RRG is still under debate [4, 7–13] it is rigorously proven to exist in a much simpler Rosenzweig-Porter (RP) random matrix model [14–20]. On the face of it, the RP model seems to have little to do with RRG. However, in this paper we show that the modification of RP model, the logarithmically normal RP model (LN-RP), offers the simplest approximation to RRG which accounts for a key difference between an RRG and a finite Cayley tree.

This LN-RP model is important on its own, as it interpolates between the Gaussian RP model and the Lévy random matrix models (see, e.g., [21, 22] and references therein) with power-law distribution of the off-diagonal matrix elements. We introduce a symmetry parameter $p$ that controls this interpolation with $p = 0$ corresponding to the Gaussian RP model and $p = \infty$ corresponding to the Lévy RP model.

FIG. 1. (Color online) Random regular graph with the branching number $K + 1 = 3$ and $N = 8$. 
A particular case of the Anderson model on RRG corresponds to \( p = 1 \) due to the hidden \( \beta \)-symmetry (see Eqs. (6.5)-(6.8) in Ref. [23], Eqs. (D.2), (D.17) in Ref. [4] and Appendix C) on the local Cayley tree. We show that \( p = 1 \) is the tricritical point such that for \( p < 1 \) the LN-RP model supports the NEE phase, while at \( p \geq 1 \) a direct transition from the localized to the Ergodic Extended (EE) phase occurs. However, this EE phase is unstable, as it results from large off-diagonal matrix elements from the tail of the logarithmically-normal distribution. Any truncation of this tail (as well as breaking the \( \beta \)-symmetry) is shown to lead to the reappearance of the NEE phase separating localized and EE phases.

The analytical theory of the Ergodic (ET) and Localization transitions (AT) developed in this paper is verified by extensive numerics based on the Kullback-Leibler divergence [24, 25] of certain correlation functions of wave function coefficients [26]. We also present the theory of this new measure of eigenfunction statistics.

The rest of the paper is organized as follows. In Sec. II we provide the motivation of the model with long-tailed distribution of off-diagonal elements and its relevance to the Anderson problem on RRG. In Sec. III we describe LN-RP model and introduce the symmetry parameter \( p \). In Sec. IV we present the phase diagram of LN-RP model based on the Anderson localization and Mott ergodicity criteria. Section V shows the numerical data confirming the analytical predictions of Sec. IV by making use of the two types of Kullback-Leibler divergence. The critical values of both ergodic and localization transitions with the corresponding critical exponents are extracted from the exact diagonalization with finite-size scaling analysis. In Sec. VI we show that for large symmetry parameter \( p \geq 1 \) the system is unstable with respect to the emergence of the multifractal phase by pushing the ergodic transition to smaller disorder values. In Sec. VII we develop a new theory of stability of non-ergodic states with respect to their hybridization. The analytical theory of the wave function support set fractal dimension is presented in Sec. VIII. Conclusions and discussion of the results are given in Sec. IX. The Appendices A – F give the details of the derivation of the above results and in-depth discussion of the new methods and ideas.

II. DEFINITIONS AND EMERGENCE OF LONG-RANGE MODELS ON TIGHT-BINDING HIERARCHICAL GRAPHS

Let us first sketch the mapping of the Anderson localization model with nearest neighbor hopping on RRG to the log-normal Rosenzweig-Porter random matrix model with infinite-range hopping. The reader interested in the properties of the LN-RP model itself can go directly to the next section.

The random \( K \)-regular graph (RRG) of \( N \) sites is a graph in which any site is connected to \( K + 1 \) other sites in a random way (see Fig. 1). The hopping amplitude

![Distribution of distances between two points on RRG](image)

**FIG. 2.** (Color online) Distribution of distances \( r \) between two points on RRG of \( N = 2^{13} = 8192 \) sites and branching number \( K = 2 \). The diameter of the graph \( d = 16 \) is only by 4 larger than the most probable distance \( r_\ast = 12 \), both being approximately equal to \( \ln N / \ln K \) in the limit \( N \rightarrow \infty \).

\[ V = 1 \] between nearest neighbor sites is fixed for all links. The Anderson localization model on RRG adds a random potential \( \varepsilon_n \) fluctuating independently at any site \( n \) with the site-independent distribution \( F(\varepsilon) \) characterized by the variance \( \langle \varepsilon_n^2 \rangle \sim W^2 \).

RRG is known to have locally a Cayley-tree structure with the branching number \( K \). However, in contrast to a finite Cayley tree which has a strict hierarchical structure and grows from a special point (the root) onwards, any point of RRG can be considered as a root of a local Cayley tree. This is because RRG has loops (of which overwhelming majority are long loops with the length of the order of the diameter of the graph) which connect one local tree with the other thus making all points of the graph statistically equivalent.

The local tree structure and the predominance of long loops on RRG lead to the exponential growth of the distribution \( p_N(r) \sim K^r \) of distances between points on this graph. This growth lasts nearly up to the maximal distance on RRG (the diameter \( d \)), followed by an abrupt drop to zero, Fig. 2 (see also Fig. 12 in [4]). The most probable distance, \( r = r_\ast \), differs from the graph diameter \( d \) only by few extra links and for large graphs of \( N \) points they are approximately equal \( r_\ast \approx d \approx \ln N / \ln K \). Moreover, due to the exponential growth of \( p_N(r) \) with \( r \), in the thermodynamic limit \( N \rightarrow \infty \) a finite fraction of pairs of sites on RRG is at the most probable distance, \( p_{N \rightarrow \infty}(r_\ast) \rightarrow f > 0 \). This “condensation of large distances” is the crucial point for the analogy between RRG and RP models.

Indeed, let’s consider the set of equally spaced sites on RRG with the most abundant distance \( r_\ast \approx d = 4 \) (see Fig. 2). An effective random matrix model involving only such (“marked”) sites which constitute a finite fraction (\( \approx 25\% \) at \( K = 2 \)) of all sites on RRG can be written using the Anderson impurity model (see Ap-
pendix A for more details). Those marked sites interact with each other through the remaining tree sites similar to the indirect interaction between Anderson impurities in a metal. This indirect interaction is long-range, as well as the RKKY interaction [27] of magnetic impurities mediated by electrons in a metal. Since all the marked sites are at a distance $r_s \simeq d$ from each other, the effective hopping matrix elements between them $H_{\text{eff}, n \neq m}$ can be all taken independent and identically distributed (i.i.d.). Thus we arrive at an infinite-range random matrix model of the Rosenzweig-Porter type. The diagonal matrix elements $H_{\text{eff}, nn} = \varepsilon_n$ have the same statistics $F(\varepsilon)$ as the on-site energies on RRG.

In the same way as RKKY interaction depends on the details of the Fermi surface in a metal, in our case the effective hopping matrix elements $H_{\text{eff}, n \neq m}$ between marked sites encode the hierarchical structure of the tree sites in their distribution function. In Appendices B – C we show that in contrast to the Gaussian Rosenzweig-Porter (RP) random matrix theory (RMT) [14, 15], the distributions of diagonal $F(\varepsilon)$ and off-diagonal $P(U)$ matrix elements for a RP model associated with RRG are drastically different. Unlike $F(\varepsilon)$ which is compact, $\langle \varepsilon^2 \rangle \propto \langle \varepsilon^2 \rangle^r$, the distribution of $P(U)$ of $H_{\text{eff}, n \neq m}$ has a fat tail which makes any moment $\langle |U|^q \rangle \sim N^{-\gamma_q}$ of $U$ characterized by its own exponent $\gamma_q$, the averages with large enough $q$ being divergent.

For not very large branching number $K$ the distribution of $P(U)$ associated with RRG in its delocalized phase is logarithmically normal. The log-normal distribution of effective hopping reflects the hierarchical structure of the tree sites and follows from the representation of $H_{\text{eff}, n \neq m}$ as a product of one-point Green’s functions on a tree along the path between the sites $n$ and $m$. Due to the random graph structure the logarithm $\ln H_{\text{eff}, n \neq m}$ is represented by the sum of large number $r_s$ of nearly independent elements and, thus, its distribution can be approximated by Gaussian (see Appendices A and C for details of derivation and limits of applicability). This leads to the log-normal distribution of the hopping term itself.

We will see that the Rosenzweig-Porter RMT with the compact distribution of diagonal elements and log-normal distribution of off-diagonal elements (LN-RP) is very rich with numerous potential applications which justify its detailed consideration independently of the analogy with the Anderson model on RRG.

III. LOG-NORMAL ROSENZWEIG-PORTER

As shown in Appendices B – C, the distribution function $P(U)$ of off-diagonal elements of the RP ensemble associated with RRG is of the “multifractal” form of the large deviation ansatz:

$$P(U) \sim U^{-1} \exp \left[ -\ln N \Phi \left( \frac{\ln U}{\ln N} \right) \right], \quad (1)$$

where $\Phi(x)$ is a certain function and the large parameter $\ln N$ is proportional to the diameter of RRG. This form is very special, as $\ln N$ appears both in front of the function $\Phi(x)$ and in its argument. It emerges in many different physical problems ranging from distribution of amplitudes of multifractal wave functions to statistics of work in driven systems out-of-equilibrium [28].

The simplest choice of $\Phi(x)$ is a linear function which gives rise to a power-law distribution $P(U)$. However, in many relevant cases where the Central Limit Theorem applies to $\ln U$, the distribution $P(U)$ is log-normal which corresponds to a parabolic function $\Phi(x)$:

$$P(U) = \frac{A}{U} \exp \left[ -\frac{\ln^2(U/\gamma)}{2p \ln(U^{p})} \right], \quad U^{p} \sim N^{-\gamma/2}. \quad (2)$$

This distribution is controlled by two parameters: the parameter $\gamma > 0$ that governs the typical value of the hopping matrix element in LN-RP model and the symmetry parameter $p$.

The reason we refer to this parameter as the symmetry parameter is related to the basic symmetry on the Cayley tree (see Appendix C) which gives rise to the duality relation:

$$P(U^{-1}) = U^{4} P(U), \quad \Leftrightarrow p = 1. \quad (3)$$

When applied to Eq. (2) this relation requires $p = 1$. However, it is useful to keep this parameter free to inter-polate between the LN-RP with long-tail cut in $P(U)$ (the case $p \rightarrow 0$ which is equivalent to the Gaussian RP [14, 15]) and the case $p \rightarrow \infty$ when $P(U)$ approaches the Lévy power-law distribution.

Another model parameter $\gamma$ is related to the Lyapunov exponent $\lambda$ on the disordered Cayley tree. It is defined [4] via the exponential decay of the typical absolute value of the Green’s function $|G_r|_{\text{typ}}$ with the distance $r$:

$$\lambda = - \lim_{r \rightarrow \infty} r^{-1} \ln |G_r|_{\text{typ}}. \quad (4)$$

By substituting $r = r_s \simeq d = \ln N/\ln K$ and $|G_r|_{\text{typ}} \sim N^{-\gamma/2}$ in Eq. (4) one immediately obtains:

$$\gamma = \frac{2\lambda}{\ln K}. \quad (5)$$

As shown in Appendix C, the log-normal distribution of $P(U)$ is asymptotically exact for RRG at small disorder. It is also quantitatively accurate in the entire delocalized phase for not very large branching number $K$.

It is important to note that for the distribution Eq. (2) the scaling of the typical value of $U$ with $N$ differs from that of the mean value. The latter exists only for $p < 2$ and is given by $\langle U \rangle \sim N^{-\gamma_{av}/2}$, with

$$\gamma_{av} = \gamma (1 - p/2), \quad (0 < p < 2). \quad (6)$$
IV. PHASE DIAGRAM OF LN-RP. COLLAPSE OF THE MULTIFRACTAL PHASE AT THE TRICRITICAL POINT AT $p = 1$.

It was first shown in Ref. [15] that the Rosenzweig-Porter RMT with a Gaussian $P(U)$ has three phases: ergodic, $\gamma < \gamma_{ET}$, multifractal, $\gamma_{ET} < \gamma < \gamma_{AT}$, and localized, $\gamma > \gamma_{AT}$ and two transitions between them at $\gamma_{ET} = 1$ and $\gamma_{AT} = 2$. The same transition points are expected for the LN-RP in the limit $p \to 0$. In this section we consider simple “rule of thumb” criteria formulated in Refs. [29, 30] which show how the phase diagram of LN-RP is modified as the symmetry parameter $p$ increases. The physical meaning of this criterion is that the number of sites in resonance with the given site is the sum $P$ of which the subscript $m$ refers to the on-site energies. The probability that $P$ of two sites $n$ and $m$ are in resonance is:

$$S_1 = \sum_{m=1}^{N} (|H_{n,m}|)_W < \infty \quad (7)$$

converges in the limit $N \to \infty$ then the states are Anderson localized. Here $(..)_W$ stands for the disorder averaging.

The physical meaning of this criterion is that the number is sites in resonance with a given site $n$ is finite. Indeed, consider for simplicity the box-shaped distribution $F(\varepsilon)$ of on-site energies. The probability that two sites $n$ and $m$ are in resonance is:

$$P_{n,m} = W^{-2} \int_{-W/2}^{W/2} d\varepsilon_n \int_{-W/2}^{W/2} d\varepsilon_m \int_{|\varepsilon_n - \varepsilon_m|}^{\infty} P(H_{nm}) d(H_{nm}). \quad (8)$$

Then simple integration over $(\varepsilon_n + \varepsilon_m)/2$ and integration by parts over $\varepsilon_n - \varepsilon_m$ gives:

$$P_{n,m} = \int_{0}^{W} dU P(U) \left( 2U \frac{W}{W^2} - \frac{U^2}{W^2} \right) + \int_{W}^{\infty} P(U) dU, \quad (9)$$

where $U = |H_{nm}|$.

One can easily see that at $U_{\text{typ}} \sim N^{-\gamma/2} \ll O(1)$ the last integral in Eq. (9) is always small and the second term in the first integral is at most of the same order as the first term. Thus with the accuracy up to a constant of order 1 we obtain:

$$P_{n,m} \sim \langle |H_{nm}| \rangle_W W, \quad (10)$$

where the subscript $W$ in $(..)_W$ implies that the distribution $P(U)$ should be truncated at $U_{\text{max}} = W$. The number of sites in resonance with the given site is the sum $\sum_{m} P_{n,m}$ which coincides with Eq. (7) up to a prefactor of order $O(1)$.

Importantly, the above derivation (9) gives an elaboration to Eq. (7). Indeed, in the case of the long-tailed distribution $P(U)$, one should cut it off at $U_{\text{max}} = W \sim O(1)$ in order to obtain a correct sufficient criterion of Anderson localization. Note that such a cutoff of $P(U)$ is automatically embedded into the RRG/LN-RP correspondence (see Appendix B).

The second criterion suggested in Refs. [29, 30] and nicknamed in [30] the Mott’s criterion is a sufficient criterion of ergodicity. It states that if the sum

$$S_2 = \sum_{m=1}^{N} (|H_{nm}|^2)_W \to \infty \quad (11)$$

diverges in the limit $N \to \infty$ then the system is in the (fully) ergodic phase [30].

Note that similar to Eq. (7), the averaging in Eq. (11) should be done with the distribution truncated at $U_{\text{max}} \sim O(1)$. The reason for that is that rare large matrix elements $|H_{nm}| \gg O(1)$ split the resonance pair of levels so much that they are pushed at the Lifshitz tail of the spectrum and do not affect statistics of states in the body of spectrum that we are studying [31].

The physical meaning of Eq. (11) is that the Breit-Wigner width $\Gamma$ that quantifies the escape rate of a particle created at a given site $n$, is much larger than the spread of energy levels $W \sim O(1)$ due to disorder. In other words, the fulfillment of the Mott’s criterion implies that the width $\Gamma$ is of the same order as the total spectral bandwidth and thus there are no mini-bands (which width is $\Gamma$) in the local spectrum. As the presence of such mini-bands is suggested [32–34] as a “smoking gun” evidence of the non-ergodic extended (e.g. multifractal) phase, the fulfillment of the Mott’s criterion (11) immediately implies that the system is in the ergodic extended phase.

The above non-ergodic extended (e.g., multifractal) phase realizes provided that in the limit $N \to \infty$:

$$S_1 \to \infty, \quad S_2 \to 0. \quad (12)$$

The case of a finite $S_2$ in the limit $N \to \infty$ is more delicate and may imply merely weak ergodicity [35].

The fact that it is the second moment of $|H_{nm}| = U$ which enters Eq. (11) is related to the Fermi Golden Rule determining the Breit-Wigner width:

$$\Gamma = 2\pi \sum_{m=1}^{N} \langle \rho_m |H_{nm}|^2 \rangle_W \approx 2\pi \rho \sum_{m=1}^{N} (|H_{nm}|^2)_W, \quad (13)$$

where $\rho_m$ and $\rho \sim W^{-1}$ are the density of final states and the density of on-site energies, respectively. The perturbative Eq. (13) is valid as long as $\Gamma \lesssim W$ and one can neglect contribution of off-diagonal matrix elements $H_{nm}$ to the density of states. Then the total spectral bandwidth is limited by $W$. In the opposite limit $\Gamma \gg W$ the total spectral bandwidth $\rho^{-1} \approx \Gamma$ is dominated by the off-diagonal matrix elements and should be determined self-consistently

$$\rho^2 \sum_{m=1}^{N} (|H_{nm}|^2)_W \sim 1. \quad (14)$$
The resulting phase diagram for the log-normal Rosenzweig-Porter ensemble, Eq. (2), which is the main result of this paper, is presented in Fig. 3.

It is remarkable that the point \( p = 1 \) which corresponds to the Anderson model on RRG, is the tricritical point on this phase diagram. At this point in the pure log-normal RP ensemble and in LN-RP ensemble with \( P(U) \) truncated at \( U > U_{\text{max}} \sim O(1) \), the multifractal phase vanishes. However, as we demonstrate in Sec. VI, the collapse of the multifractal phase at \( p \geq 1 \) and thus the tricritical point, is unstable with respect to any truncation of \( P(U) \) with \( U_{\text{max}} \ll O(1) \).

V. KULLBACK-LEIBLER (KL) MEASURE

The numerical verification of the phase diagram and determination of the critical exponents at the Anderson localization and ergodic transitions is done in this paper using the Kullback-Leibler divergence (KL) [24–26] of certain correlation functions of random eigenstates (for more detailed multifractal analysis of this model see [37]).

The Kullback-Leibler correlation functions \( KL_1 \) and \( KL_2 \) are defined as follows [26]. The first one is defined in terms of wave functions of two neighboring in energy states:

\[
KL_1 = \sum_i |\psi_\alpha(i)|^2 \ln \left( \frac{|\psi_\alpha(i)|^2}{|\tilde{\psi}_{\alpha+1}(i)|^2} \right). \tag{21}
\]

The second one is similar but the states \( \psi \) and \( \tilde{\psi} \) correspond to different (and totally uncorrelated) disorder realizations:

\[
KL_2 = \sum_i |\psi(i)|^2 \ln \left( \frac{|\psi(i)|^2}{|\tilde{\psi}(i)|^2} \right). \tag{22}
\]

The idea to define such two measures is the following. In the ergodic phase each of the states has an amplitude \( |\psi(i)|^2 \sim N^{-1} \) of the same order of magnitude. Then the logarithm of their ratio is of order \( O(1) \), and for the normalized states

\[
KL_1 \sim KL_2 \sim O(1). \tag{23}
\]

For fully-ergodic states in the Wigner-Dyson limit the eigenfunction coefficients are fully uncorrelated, even for the neighboring in energy states. Thus there is no difference between \( KL_1 \) and \( KL_2 \). Using the Porter-Thomas distribution one finds:

\[
KL_1 = KL_2 = 2. \tag{24}
\]

Deeply in the localized phase \( \ln |\psi_\alpha(i)|^2 \sim -|i - i_\alpha|/\xi \), where \( i_\alpha \) is the position of the localization center. Since the positions of localization centers \( i_\alpha \) are not correlated even for the states neighboring in the energy, \( KL_1 \) and \( KL_2 \) are proportional to \( L \sim \ln N \) and divergent in the thermodynamic limit:

\[
KL_1 \sim KL_2 \propto \ln N \to \infty. \tag{25}
\]
The properties of KL, Eqs. (24), (25) are fully confirmed by numerics presented in Fig. 4.

A qualitative difference between KL1 and KL2 is in the multifractal NEE phase. In this phase the neighboring in energy states $|\psi_\alpha(i)|^2$ and $|\psi_{\alpha+1}(i)|^2$ are most probably belonging to the same support set and hence they are strongly overlapping: $|\psi_\alpha(i)|^2 \approx |\psi_{\alpha+1}(i)|^2$. Furthermore, eigenfunctions on the same fractal support set can be represented as: $\psi_\alpha(i) = \Psi(i) \phi_\alpha(i)$, where $\Psi(i)$ is the multifractal envelope on the support set and $\phi_\alpha(i)$ is the fast oscillating function with the Porter-Thomas statistics [3]. Thus the ratio $|\psi_\alpha(i)|/|\psi_{\alpha+1}(i)|$ and hence KL1 has the same statistics as in the ergodic phase. In other words, KL1 is not sensitive to the ergodic transition but is very sensitive to the localization one, Fig. 4.

In contrast, the eigenfunctions $\psi(i)$ and $\tilde{\psi}(i)$ corresponding to different realizations of a random Hamiltonian in KL2, overlap very poorly. This is because the fractal support sets which contain a vanishing fraction of all the sites, do not typically overlap when taken at random. Therefore the ratio $\ln(|\psi(i)|/|\tilde{\psi}(i)|) \approx \ln N$ in KL2 is divergent in the thermodynamic limit in the multifractal phase, very much like in the localized one. This makes KL2 very sensitive to the ergodic transition, Fig. 4.

A more detailed theory of KL1 and KL2 in the multifractal phase is given in Appendix D. The main conclusion of the analysis done in Appendix D is that the curves for KL1($\gamma, N$) for different $N$ have an intersection point at the critical point $\gamma = \gamma_{AT}$ of the Anderson localization transition. At the same time, the intersection point for curves for KL2($\gamma, N$) coincides with the ergodic transition [26], provided that it is continuous and well separated from the Anderson localization transition. If the localization and ergodic transition merge together and the multifractal state exists only at the transition point (as in 3D Anderson model) then intersection of KL2 curves is smeared out and may disappear whatsoever. However, the intersection of KL1 curves remains sharp in this case too (see Fig. 4).

The intersection of finite-size curves for KL1 and KL2 helps to locate numerically the critical points $\gamma_{AT}$ and $\gamma_{ET}$. First, we checked that for the well studied 3d Anderson transition the intersection point of KL1 curves exactly corresponds to the known critical disorder $W \approx 16.56$, while KL2 curves show no intersection whatsoever (see Fig. 4). The results for $\gamma_{AT}$ and $\gamma_{ET}$ for LN-RP model are shown in the Table 6. They coincide (for $p > 1$ after the extrapolation) with the theoretical prediction Eq. (19), (20) with the deviation less than 6%.

The next step is to analyze the finite-size scaling (FSS) by a collapse of the data for KL1 and KL2 at different $N$ in the vicinity of the localization and ergodic transition, respectively. To this end we use the form of FSS derived in Appendix D:

\[ KL1 = \Phi_1(\ln N|\gamma - \gamma_{AT}|^{\nu_1}), \]  

\[ KL2 - KL2_{\nu}(N) = \Phi_2(\ln N|\gamma - \gamma_{ET}|^{\nu_2}). \]  

The input data for the collapse is KL1 and KL2 versus $\gamma$ and $W$ for 7 values of $N$ is shown in Fig. 4. The fitting parameters extracted from the best collapse are $\nu_1$ ($\nu_2$) and the critical points $\gamma_{AT}$ ($\gamma_{ET}$). The critical value of $KL2_{\nu}(N) = KL2(\gamma_{ET}, N)$ is determined by the best fitting for $\gamma_{ET}$. For the localization transition where the critical point $\gamma_{AT}$ is well defined by the intersection in KL1, one may look for the best collapse by fitting only $\nu_1$.

This procedure of the finite-size scaling has been tested for the 3D Anderson model with sizes $L = 8 - 32$. The
FIG. 5. (Color online) **The best collapse of the KL1 and KL2 data for LN-RP** with $p = 1$ and $p = 0.5$. The collapse for KL1 and KL2 is done in the vicinity of the localization (for KL1) and ergodic (for KL2) transitions by recursive procedure that finds $\gamma_c$ and $\nu$ by minimizing the mean square deviation of data from a smooth scaling function which is updated at any step of the procedure. (insets) The critical value of $\gamma_1$ and $\gamma_2$ as a function of $\ln N$. It stays almost a constant for KL1 and for KL2 at $p = 0.5$ when the ergodic transition is continuous and well separated from the localized one but it grows logarithmically in $N$ at $p = 1$ when the ergodic and localization transitions merge together. This growth is the reason of smearing of the intersection of KL2 curves in Fig. 4. The exponent $\nu$ significantly depends on $p$ and is consistent with $\nu_1 \approx \nu_2 = 0.5$ at $p = 1$ and $\nu_2 = 1$ at $p = 0.5$.

results for the scaling collapse of data are presented in Appendix E. Note that in this case there is no inter-

| $p$ | $\gamma_{ET}$ [exact] | $\gamma_{ET}$ [mean] |
|-----|----------------------|----------------------|
| 0.01 | 2.00 [2]             | 1.00 [1]             |
| 1/2  | 2.80 [3/3 = 2.67]    | 2.06 [2]             |
| 3/4  | 3.43 [16/16 = 3.20]  | 2.96 [3]             |
| 1    | 4.10 [4, 4.04, 4]    | 3.00 [4, 4.10, 4]    |
| 5/4  | 4.81 [5, 5.02, 5]    | 4.45 [5, 5.19, 5]    |
| 3/2  | 5.51 [5, 5.84, 6]    | 5.18 [5, 5.83, 6]    |

FIG. 6. **Comparison of analytical predictions (blue), Eq. (19), (20), and numerical data for the transition points $\gamma_{ET}$ and $\gamma_{ET}$ for LN-RP model.** Numerical data (black) is obtained by exact diagonalization of LN-RP random matrices with $N = 512 - 32768$ from the intersection points in KL1 and KL2, Fig. 4 and from finite-size scaling, Fig. 5. For $p > 1$ a linear in $1/\ln N$ extrapolation to $N \to \infty$ of the position of the intersection point for two consecutive $N$ is shown in red.

| $p$ | $\nu_1$          | $\nu_2$          |
|-----|------------------|------------------|
| 0.01 | 1.02 ± 0.03 [1.00]| 0.98 ± 0.05 [1.00]|
| 1/2  | 0.70 ± 0.07 [0.75]| 1.18 ± 0.15 [1.00]|
| 3/4  | 0.64 ± 0.07 [0.625]| 0.79 ± 0.12 [0.75]|
| 1    | 0.56 ± 0.07 [0.50]| 0.48 ± 0.08 [0.50]|
| 5/4  | 0.57 ± 0.08 [0.50]| 0.50 ± 0.10 [0.50]|
| 3/2  | 0.57 ± 0.10 [0.50]| 0.46 ± 0.10 [0.50]|

FIG. 7. **Critical exponents $\nu_1$ and $\nu_2$ in the finite-size scaling, Eq. (26),** obtained from the best collapse of KL1 and KL2 data, respectively, see Fig. 5. In the [...] are the conjectured values of $\nu_1$ and $\nu_2$.

The results are shown in the Tables 6, 7 while representative samples of the data collapse are shown in Fig. 5. On the basis of these numerical results we conjecture the dependence of the critical exponents $\nu_1$ and $\nu_2$ on the symmetry parameter which is shown in Fig. 8.
FIG. 9. (Color online) Phase diagram of LN-RP model with $U_{\text{typ}} \sim N^{-\gamma}$ truncated at $U_{\text{max}} \sim N^{-\gamma_{tr}}$ ($\gamma > \gamma_{tr} > 0$). (Left panel) Phase diagram in the plane $\gamma - \gamma_{tr}$ at a fixed value $p = 1$ of the symmetry parameter. (Right panel) Phase diagram in the plane $\gamma - p$ at fixed $\gamma_{tr} = 0.3$. At any $\gamma_{tr} > 0$ the multifractal NEE phase emerges at $p \geq 1$ and fills the gap between the ergodic and localized phases. At a small $\gamma_{tr}$ the Anderson localization transition (blue line) is almost unaffected by truncation, while the ergodic transition (orange line) is pushed to smaller values of $\gamma$. Thus the multifractal NEE phase substitutes the ergodic one as the truncation parameter $\gamma_{tr}$ increases demonstrating the fragility of the ergodic phase which existence is due to atypically large values of the transition matrix elements $U$.

VI. TRUNCATED LN-RP AND FRAGILITY OF ERGODIC PHASE.

The main result, Fig. 3, of Sec. IV confirmed numerically in Sec. V is the collapse of the multifractal phase at $p \geq 1$ and existence of the tricritical point in LN-RP which is associated, via the qualitative arguments of Sec. II, with the localization transition on RRG.

In this section we show that the ergodic phase that emerges at the localization transition in this tricritical point (and for all $p \geq 1$) is unstable with respect to a deformation of LN-RP model such that $P(U)$ is cut from above at:

$$U_{\text{max}} \sim N^{-\gamma_{tr}/2} \ll O(1), \quad (\gamma_{tr} > 0).$$

As the result of this truncation the multifractal phase re-appears by substituting a part of the ergodic phase in a non-truncated LN-RP model (see Fig. 9) [38]. To this end we use the expression that generalizes Eq. (16):

$$\int_0^{N^{-\gamma_{tr}/2}} dU U^p P(U) \sim$$

$$\begin{cases} N^{-\frac{pq}{2}} (1 - pq), & \gamma_{tr} < \gamma (1 - pq) \\ - \frac{1}{p^2} \frac{\gamma_{tr}^{2(1 - pq)} + \frac{1}{4} pq \gamma_{tr}}{N}, & \gamma_{tr} > \gamma (1 - pq) \end{cases}$$

and apply the same criteria Eq. (7), (11) to find the critical points of the localization and ergodic transitions. Then we obtain for the critical point $\gamma_{AT}$ of the Anderson localization transition:

$$\gamma_{AT} = 2p - (p - 1)\gamma_{tr} + \sqrt{(2p - (p - 1)\gamma_{tr})^2 - \gamma_{tr}^2},$$

if $\gamma_{tr} > \gamma_{AT} (1 - p)$. In the opposite case truncation does not affect $\gamma_{AT}$.

For the critical point $\gamma_{ET}$ of the ergodic transition in the same way we find for $\gamma_{tr} > \gamma_{ET} (1 - 2p)$:

$$\gamma_{ET} = 2p - (2p - 1)\gamma_{tr} + \sqrt{(2p - (2p - 1)\gamma_{tr})^2 - \gamma_{tr}^2},$$

The results of Eq. (29), (30) are plotted in Fig. 9.

One can see that at any positive non-zero $\gamma_{tr}$ the multifractal NEE phase emerges at $p \geq 1$ in between of the localized and ergodic ones. At small $\gamma_{tr}$ the line of localization transition is almost insensitive to truncation, while the line of ergodic transition is pushed to smaller values of $\gamma$ corresponding to larger typical transition matrix elements $U$ (smaller effective disorder). This proves the fact that the ergodic phase in LN-RP with $p \geq 1$ is very fragile and exists only due to atypically large transition matrix elements. It is substituted by the multifractal NEE phase as soon as such matrix elements are made improbable by truncation.

We believe that this scenario of the multifractal phase emergence at $p \geq 1$ is quite generic and happens for the wide class of perturbations of the LN-RP model [39]. In the case of RRG corresponding to the tricritical point, $p = 1$, of the non-truncated LN-RP model, the effect of the local Cayley tree structure in the exact mapping of the Anderson model on RRG onto LN-RP model is unexplored in detail and might, in principle, lead to an effective truncation of the above type. In any scenario the tricritical nature of $p = 1$ point in LN-RP makes this case (and the corresponding case of RRG) significantly more complicated and different from the conventional Anderson localization transition in finite dimensions. This is the reason, in our opinion, of the long-lasting debates on the existence of NEE phase in the Anderson model on RRG (see the debates in [4, 7–11] and references therein).
We assume that both states are multifractal with fractal support sets as it is shown in Fig. 10(a) and (b).

By the constraint, for brevity we conveniently rewrite Eq. (C9)

\[ \sigma(g, D_1) = 2D_1 - \frac{1}{p_D} \left( g - \gamma_D \right)^2. \]  

If \( \sigma(g, D_1) < 0 \) (region I in Fig. 16 of Appendix F), the sum, Eq. (31), is dominated by a single term with the largest \( G_{ij} \). For positive \( \sigma(g, D_1) > 0 \) (region II in Fig. 16 of Appendix F), many terms contribute to this sum and the distribution \( P(V = \| V_{\mu, \nu} \|) \) becomes Gaussian. In general, there are both contributions given by

\[ P(V) = \int_{g \in I} dg \ N^{\sigma(g, D_1)} \delta(V - N^{-D_1} - g) + \int_{g \in II} dg \ P(g) \left\{ \frac{1}{g - \sum_{ij} G_{ij} \psi_{\mu}(i) \psi_{\nu}(j)} \right\}. \]  

The condition of stability of the multifractal phase against hybridization is derived similar to the Anderson criteria of stability, Eq. (7), of the localized phase done in Sec. VII. The difference is that now we have to replace the matrix element between the resonant sites by the matrix element \( V \) between the resonant non-ergodic states and take into account that on each of \( M = N^{1-D_1} \) different support sets there are \( m = N^{D_1} \) wave functions which belong to the same mini-band and thus are already in resonance with each other. Therefore the total number of independent states-candidates for hybridization with a given state should be smaller than the total number of states \( Mm = N \) and larger than the number of support sets \( M \). This number is in fact equal to their geometric mean \( \sqrt{NM} = M \sqrt{m} = N^{1-D_1} \).

With this comment, the criterion of stability of the multifractal phase reads in the limit \( N \to \infty \) as

\[ N^{1-D_1} \int_0^W dV \ V \ P(V) < \infty. \]  

The contribution of the Gaussian part of \( P(V) \) to Eq. (35) is:

\[ N^{1-D_1} \sqrt{\langle V^2 \rangle} = N^{1-D_1} - \frac{1}{2} \gamma_{\text{eff}}(D_1) < \infty, \]  

where

\[ \langle V^2 \rangle \equiv N^{-\gamma_{\text{eff}}}. \]  

The contribution of the first (log-normal) term in Eq. (34) to the stability criterion is:

\[ N^{1-D_1} \int_{g \in I} dg \ N^{\sigma(g, D_1) - g - D_1} \equiv N^{1-D_1 - \Delta(D_1)} < \infty. \]  

Thus the multifractal phase is stable against hybridization if the following inequalities are both fulfilled

\[ \frac{1}{2} D_1 + \frac{1}{2} \gamma_{\text{eff}}(D_1) \geq 1, \]  

and

\[ \frac{1}{2} D_1 + \Delta(D_1) \geq 1. \]  

The function \( \gamma_{\text{eff}}(D_1) \) and \( \Delta(D_1) \) are computed in Appendix F.

A particular case \( D_1 = 0 \) of Eqs. (39), (40) describes the stability criterion of the localized phase. If the
In this section we re-consider the phase diagram of non-truncated LN-RP, $\gamma_{tr} \leq 0$, from the viewpoint of stability criteria given in the previous section by Eqs. (39), (40) and derive the expression for the fractal dimension $D_1(\gamma)$ of the support set of multifractal wave functions.

To this end in Fig. 11 we plot

$$\gamma_{\text{eff}}(\alpha) \gamma + \alpha = \begin{cases} 1 + 3\alpha - 2\sqrt{2\alpha p} , & 4\alpha < 2p, \frac{1}{2p}, \\ 1 \gamma_{\text{ET}}(p) + \alpha , & \text{otherwise}, \end{cases},$$

(41)

and

$$\frac{2\Delta(\alpha)}{\gamma} + \alpha = \begin{cases} 1 + 3\alpha - 2\sqrt{2\alpha p} , & p < 8\alpha < \frac{1}{p}, \\ 2/\gamma_{\text{AT}}(p) - \alpha , & 8\alpha < p, \frac{1}{p}, \\ 1 + 3\alpha + 2\sqrt{2\alpha p} , & 8\alpha > \frac{1}{p}, \end{cases},$$

(42)

calculated in Appendix F as functions of $\alpha = D_1/\gamma$. Here $\gamma_{\text{AT}}(p) \geq 2$ and $\gamma_{\text{ET}}(p) \geq 1$ are given by Eqs. (19) and (20), respectively.

According to the stability criteria Eqs. (39), (40) the functions Eqs. (41), (42) should be compared to $2/\gamma$, see Fig. 11. First, we note that the localized phase which formally corresponds to $D_1 = 0$, is stable if the lowest of the blue and orange curves in Fig. 11 is higher than $2/\gamma$ at $\alpha = 0$ and it is unstable otherwise. One can see that at $\alpha = 0$ for all values of $p$ the log-normal contribution (orange curve) is lower than the Gaussian one (blue curve). This means that the stability of the localized phase is always determined by the log-normal part of $P(V)$. Moreover, since at $\alpha = 0$ Eqs. (41), (42) reduce to $\alpha + \gamma_{\text{eff}}(\alpha)/\gamma = 1$ and $\alpha + 2\Delta(\alpha)/\gamma = 2/\gamma_{\text{AT}}$, respectively, the stability of the localized phase implies that $\gamma > \gamma_{\text{AT}}(p) \geq 2$ in agreement with (19).

If the localized phase is unstable then different localized states hybridize and form a multifractal state with $D_1 > 0$. Those states are, however, unstable until their support set reaches the fractal dimension $D_1^{\text{min}} > 0$ where Eqs. (39), (40) are first both fulfilled.

As the parameter $\gamma$ decreases below the critical value $\gamma_{\text{AT}}$, the fractal dimension $D_1(\gamma)$ increases from $D_1^{\text{min}}$ being always determined by the intersection of the horizontal line $y = 2/\gamma > 2/\gamma_{\text{AT}}(p)$ (red line in Fig. 11) with the blue line. Thus the stable fractal dimension $D_1(\gamma)$ is always determined by the Gaussian part of $P(V)$ and according to the second line of Eq. (41) and Fig. 11 is equal to:

$$D_1(\gamma) = 2 - \frac{\gamma}{\gamma_{\text{ET}}(p)} , \quad p \leq 1 .$$

(43)

At $\gamma = \gamma_{\text{ET}}$ the fractal dimension $D_1(\gamma)$ reaches unity, and at this point a continuous ergodic transition happens. Thus the critical point of ergodic transition coincides with that determined by Eq. (20).

Note that while $D_1(\gamma)$ is linear in $\gamma$, as for the Gaussian RP model [15], other fractal dimensions $D_q (q \geq 1)$ are not necessarily equal to $D_1(\gamma)$ as it was the case in Ref. [15]. The calculation of $D_q$ with $q > 1$ goes beyond the scope of this paper and will be studied elsewhere [37].

Note that, unlike the ergodic transition, the Anderson transition is discontinuous: the stable fractal dimension $D_1(\gamma)$ is separated by a finite gap $D_1^{\text{min}} = D_1(\gamma_{\text{AT}})$ from

![Fig. 11. (Color online) The functions (41)(blue curve) and (42)(orange curve) entering inequalities Eqs. (39), (40) in different regions of $p$: (left) $p < 1/2$; (middle) $1/2 \leq p \leq 1$; (right) $p > 1$. Intervals of $\alpha = D_1/\gamma$ with different functional dependence are shown by dashed vertical lines. The Anderson localization transition corresponds to the lower of the blue and orange curves equal to $2/\gamma$ at $\alpha = 0$. This transition is always determined by the orange curve representing the log-normal part of the distribution $P(V)$. On the contrary, the stable fractal dimension $D_1(\gamma) = 2 - \gamma/\gamma_{\text{ET}}(p)$ for $\gamma \leq \gamma_{\text{AT}}$ is always determined by the blue curve representing the Gaussian part of the distribution $P(V)$. The Anderson transition in all cases but $p = 0$ is discontinuous, with the minimal stable fractal dimension of the support set being $D_1^{\text{min}} = D_1(\gamma_{\text{AT}}) = 2 - \gamma_{\text{AT}}/\gamma_{\text{ET}}(p) > 0$ (shown by a gray dotted arrow). The ergodic transition corresponds to $D_1(\gamma) = 1$ and it is continuous. For $p \geq 1$ there is no solution $D_1 < 1$ to the system of inequalities Eqs. (39), (40) in the region of parameters where the localized phase is unstable. In this case the multifractal phase is absent.](image-url)
the localized state $D_1 = 0$:

$$D_1^{\text{min}} = \begin{cases} 
2 - \frac{\gamma_{tr}(p)}{\gamma_{tr}(p)}, & 0 < p < 1 \\
1, & p \geq 1
\end{cases}$$

(44)

This gap is shown by the gray dotted arrow in Fig. 11. The right panel of Fig. 11 demonstrates that for $p \geq 1$ the minimal fractal dimension $D_1^{\text{min}} = 1$, so that the multifractal phase is no longer possible in LN-RP model with $\gamma_{tr} \leq 0$. However, as it is shown in Sec. VI, it appears if $\gamma_{tr} > 0$.

IX. CONCLUSION AND DISCUSSION

In this paper we introduce a log-normal Rosenzweig-Porter (LN-RP) random matrix ensemble characterized by a long-tailed distribution of off-diagonal matrix elements with the variance controlled by the symmetry parameter $p$. We calculate the phase diagram of LN-RP using the recently suggested Anderson localization and Mott ergodicity criteria for random matrices. An alternative approach based on the analysis of stability with respect to hybridization of multifractal wave functions developed in this paper gives results identical to those obtained from the above criteria and consistent with numerical calculations. It also helps to compute the dimension $D_1$ of the eigenfunction fractal support set and show that the Anderson localization transition is discontinuous with $D_1^{\text{min}} > 0$ at all $p > 0$.

This LN-RP model has many potential applications and we use it to develop an alternative approach to the localization problem on random regular graph. It is based on the partition of all sites on RRG into two groups: (i) the “marked sites” remote from each other at the most abundant distance of the order of the graph diameter and (ii) the “tree sites” at much smaller distance from each other. This partition is only meaningful for the graphs in which the marked sites take a finite fraction of all sites like in the graphs with a local tree structure. Then we study an effective random matrix model involving only the marked sites and show that it is a special case $p = 1$ of the log-normal Rosenzweig-Porter random matrix ensemble introduced in this paper. An important result of this paper is that in this $p = 1$ LN-RP model arising from the above partition, there is a direct transition from the localized to the ergodic phase similar to the one obtained in Refs. [7, 8]. However, the point $p = 1$ appears to be very special: it is a tricritical point of the LN-RP model which is unstable to deformations of this model. In particular, it is unstable to truncation of the far tail in the log-normal distribution considered in Sec. VI and sensitive to modification of the loop statistics in RRG [40] leading to the different (possibly non-convex) distribution of off-diagonal matrix elements in the corresponding LN-RP model.

We would like to emphasize that our mapping of the localization problem on RRG to LN-RP random matrix ensemble is an approximation which is justified only qualitatively. Note that the approach adopted in Ref. [7, 8] is not free of approximations too. While the local tree structure of RRG is treated exactly in the framework of the supersymmetric sigma-model, the final solution rests on the “self-consistency” condition (Eqs. (24) and (30) in Ref. [8]):

$$g_0(Q_0) = \int DQ_0 \left[ g_0(Q_0) \right]^K e^{-STr[-2g(Q_0-Q_0^g)^2+\eta\Lambda Q_0^g]}. \quad (45)$$

This condition is an approximation, as the corresponding equations do not take into account a detailed topology of the graph (statistics of loop lengths, etc.) but only (i) the local tree structure (encoded in the non-linear term $[g_0(Q_0^g)]^K$), and (ii) the statistical homogeneity of the graph as a whole (encoded in the fact that only the zero spatial mode component $Q_0$ of the supersymmetric $Q(i)$-field enters in Eq. (45)).

In a sense, this approximation is in many respects similar to our mapping onto LN-RP model. Indeed, the zero spatial mode component $Q_0$ is known to describe the Wigner-Dyson random matrix ensembles [41]. The Gaussian Rosenzweig-Porter model is a Wigner-Dyson random matrix ensemble with parametrically enhanced fluctuations of diagonal matrix elements. The breakdown of basis-rotation invariance by the special diagonal (like in the Rosenzweig-Porter ensemble) is described in Eq. (45) by the ‘gradient’ term $-2g(Q_0-Q_0^g)^2$ which becomes non-zero (in contrast to the Wigner-Dyson case) due to the presence of the “external” $Q_0$ supermatrix in the non-linear integral equation (45).

Therefore, one may conjecture that our mapping onto the $p = 1$ LN-RP model is equivalent to the self-consistent approximation of Ref. [8] and earlier works by Mirlin and Fyodorov (see Refs. [42, 43] and references therein). An additional support to this conjecture comes from the fact that at $p = 1$ the critical exponent $\nu_1 = \nu_2 \equiv \nu$ reaches its mean-field value $\nu = 1/2$ (see Table 7 and Fig. 8). Whether or not both approximations give correct predictions for the phases on RRG is, in our opinion, still an open issue.

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Appendix A: RRG-to-LN-RP correspondence.

As mentioned in Sec. II, the local tree structure and the predominance of long loops on RRG lead to ”condensation of large distances” when most of the pairs of sites are located at a certain distance of the order of the graph diameter $d \simeq \ln N / \ln K$. This leads to the set of equally spaced sites on RRG with the most abundant distance $r_s \approx d - 4$. Those marked sites interact with each other through the remaining tree sites similar to the indirect interaction between Anderson impurities in a metal. This indirect interaction is long-range and the effective hopping matrix elements $(H_{\text{eff}})_{nm}$ of such a model can be found using the Anderson impurity model.

Indeed, let us consider the marked sites $n$ and $m$ as Anderson impurities imbedded into the Cayley tree which sites are connected by a hopping $V$. Those impurities (which are not directly connected) are supposed to be connected with the neighboring tree sites by the same hopping $V$. Then the impurity Green’s function $G = (E - H_{\text{eff}})^{-1}$ can be expressed through an exact Green’s function $\hat{G}$ on a Cayley tree as follows:

$$G_{nm} = g_n V^2 G_{m'n'} g_m + g_n \delta_{nm}, \quad (A1)$$

where $n'$ and $m'$ are the sites on a tree neighboring to the marked sites (’impurities’) $m$ and $n$, respectively, and $g_{nm} = g_n \delta_{nm} = (E - \varepsilon_n)^{-1} \delta_{nm}$ is the bare impurity Green’s function. Thus, inverting Eq. (A1) and assuming $V^2 \| \hat{G} \| \ll 1$ we obtain:

$$G^{-1} = \hat{g}^{-1} (1 + V^2 \hat{G})^{-1} \approx \hat{g}^{-1} - V^2 \hat{G}. \quad (A2)$$

Substituting here $G^{-1} = (E - \hat{H}_{\text{eff}})$ and $\hat{g}^{-1} = E - \hat{\varepsilon}$ one obtains:

$$\hat{H}_{\text{eff}}(E) = \hat{\varepsilon} + V^2 \hat{G}(E). \quad (A3)$$

where $\hat{G}(E)$ is an exact Green’s function on a Cayley tree and $\hat{\varepsilon} = \text{diag} \{ \varepsilon_n \}$.

Appendix B: Full and cavity Green’s functions on a Cayley tree

Note that $\hat{G}(E)$ with components $G_E(i,j) \equiv G(i,j)$ in Eq. (A3) is the full two-point Green’s function on a Cayley tree. It is convenient to express it in terms of the product $\Pi(i,j)$ of the cavity Green’s functions $G_{p\rightarrow p_\sim}$, where $p_\sim$ is an immediate descendant of $p$ along the path from $i$ to $j$ (see Fig. 12):

$$\Pi(i,j) = \prod_{i < p \leq j} G_{p \rightarrow p_\sim}. \quad (B1)$$

Here $G_{i \rightarrow k}$ is the Green’s function on a sub-tree such that the link from the site $i$ to its immediate descendant $k$ is cut. The advantage of such an object is that it can be found by a recursion relation:

$$G_{i \rightarrow k}^{-1} = E - \varepsilon_i - \sum_{j(i)}^K G_{j \rightarrow i}, \quad (B2)$$

where $\varepsilon_i$ is the random on-site energy and $j(i) = 1, ..., K$ are immediate predecessors of $i$. Thus knowing the Green’s functions on the outer sites of a tree (’leaves’) one can find $G_{i \rightarrow k}$ at any other site by iteration.

The general expression for the two-point Green’s function on a tree reads as follows [44]:

$$G(i,j) = \Pi(i,l) G(l,l) \Pi(j,l). \quad (B3)$$

In terms of the cavity Green’s functions Eq.(B3) takes the form:

$$G(i,j) = G_{i \rightarrow l_\sim} ... G_{l_\sim \rightarrow l} G(l,l) G_{l \rightarrow j} ... G_{j \rightarrow j}. \quad (B4)$$

where $l_\sim$ and $l_+ \sim$ are the neighbors of $l$ along the paths from $i$ to $l$ and from $l$ to $j$, respectively (see Fig.12), and

$$G(l,l) = \frac{1}{E - \xi_l - \sum_{n(l)=1}^K G_{n(l) \rightarrow l} - G_{k(l) \rightarrow l}}. \quad (B5)$$

In Eq. (B4) we denote the immediate predecessors of $l$ by $n(l) = 1, ..., K$, while $k(l)$ is its descendant. Note that two of $n(l)$ are necessarily $l_\sim$ and $l_+ \sim$ (see Fig.12).

In a particular case when $i$ and $j > i$ lie on the same

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Google Quantum Research Award “Ergodicity breaking in Quantum Many-Body Systems” (V. E. K.).
Eq. (B4) is a product of the cavity Green’s functions along the path from \( i \) to \( j \), with the single exception that at the common descendant \( l \) of \( i \) and \( j \) the cavity Green’s function is replaced by the full one. This is the minimal modification of the product which makes the Green’s function \( G(i,j) \) (as well as \( G(i,l) \)) singular at the eigen-energy \( E = E_n \).

However, this modification is crucial not only for the correct spectral properties. It drastically restricts fluctuations of \( |G(i,j)| \). Indeed, from the recursion relation Eq. (B2) it follows that if at some step \( G_{j \rightarrow i} \) is anomalously large than at the next step \( G_{i \rightarrow k} \) will be anomalously small, so that the product \( \Pi(\cdot) \) it follows that if at some step \( G_{j \rightarrow i} \) is anomalously large than at the next step \( G_{i \rightarrow k} \) will be anomalously small, so that the product \( \Pi(\cdot) \) is of order one. At the same time anomalously small \( G_{j \rightarrow i} \) does not result in an anomalously large \( G_{i \rightarrow k} \). For this to happen all other terms in r.h.s. of Eq. (B2) should be anomalously small which is much less probable. This means that a product \( \Pi(i,j) \) along a long path is typically small, as for large number of terms in the product the probability to have a small \( G_{j \rightarrow i} \) increases. However, there are rare events when \( \Pi(i,j) \) is anomalously large. This happens only when the last term in the product \( G_{p \rightarrow p} \) with \( p_- = i \) (which cannot be compensated) is anomalously large. These rare events are responsible for the symmetric distribution of \( y = |G(i,j)|^{-1} \) discussed in Appendix C.

Now let us consider Eq. (B4). One can easily see that the anomalously large end-terms \( G_{i \rightarrow 1} \) and \( G_{1 \rightarrow i} \) in \( \Pi(i,1) \) and \( \Pi(j,1) \) cancel out by the corresponding terms in the denominator of Eq. (B5), if we neglect very improbable events when they both are large. With this restriction one may write:

\[
r^{-1} \ln |G(i,j)| = \begin{cases} r^{-1} \ln |\Pi(i,j)|, & \text{if } \ln |\Pi(i,j)| < 0, \\ \approx 0, & \text{otherwise} \end{cases},
\]

where \( r = |i-j| \gg 1 \).

Note that the above analysis applies also to one-dimensional Anderson model which formally corresponds to \( K = 1 \). Then Eq. (B6) is consistent with the exact result [45] that \( T = |G(i,j)|^2 \leq 1 \), where \( T \) is the transmission coefficient through a chain of length \( L \).

This intrinsic cutoff at large values \( |i-j| = d \) of \( |G(i,j)| \) affects the Anderson localization (7) and the Mott ergodicity (11) principles for the corresponding LNRP random matrix model and leads to the phase diagram with the tricritical point at \( p = 1 \).

Appendix C: 'Multifractal' distribution of \( |\Pi(i,j)| \)

Now we consider generic properties of the distribution of the product \( \Pi_r \equiv |\Pi(i,j)| \) of cavity Green’s functions \( G_{p \rightarrow p} \) on a Cayley tree at large distances \( r \equiv |i-j| \).

As it is shown in Ref. [4], in the limit of a long path \( r \gg 1 \) the distribution function \( F(y) \) of \( y = \Pi_r^{-1} \) has a special symmetry:

\[
F(y) = F(1/y). \tag{C1}
\]

Correspondingly, the distribution function \( P(\Pi_r) \) obeys the symmetry:

\[
P(1/\Pi_r) = \Pi_r^4 P(\Pi_r). \tag{C2}
\]

In order to proceed further we make use of the expression for \( F(y) \) in terms of its moments \( I_n = \int F(y)y^{-2m}\,dy \):

\[
F(y) = \frac{2}{y} \int_{B} \frac{dm}{2\pi} y^{2m} (I_m)^{r}, \tag{C3}
\]

where the integration is performed over the Bromwich contour parallel to the imaginary axis \((z \in [-\infty, +\infty])\) on the positive side of the real one \((c > 0)\). Eq. (C3) is nothing but a Mellin transform which allows to restore the distribution function, given that the (analytically continued) moments \( I_m \) are known.

The moments \( I_m \) at \( m \in [0,1] \) obey the following symmetry which reflects the symmetry Eq. (C1) [4]:

\[
I_m = I_{1-m}, \tag{C4}
\]

with a minimum at \( m = 1/2 \) and:

\[
I_0 = I_1 = 1, \quad \partial_m I_m|_{m=1} = -\partial_m I_m|_{m=0}. \tag{C5}
\]

This symmetry is another representation of the basic \( \beta \)-symmetry on a Cayley tree established in the seminal work [23].

Computing the Mellin transform in the saddle-point approximation one finds with the exponential accuracy:

\[
\ln(y F(y)) = r (\ln I_m - m \partial_m \ln I_m)|_{m=m_*}, \tag{C6}
\]

where \( m_* \) is found from the stationarity condition:

\[
\frac{1}{2} (\partial_m \ln I_m)|_{m=m_*} = - \frac{\ln y}{r}. \tag{C7}
\]

Eq. (C7) implies that \( m_* \) is a function of the argument \( \ln(y)/r \). Then it follows from Eq. (C6) that:

\[
F(y) \sim y^{-1} \exp \left[ -r \Theta \left( \frac{\ln y}{r} \right) \right], \quad (r \gg 1), \tag{C8}
\]

where \( \Theta(x) \) some function of \( \ln(y)/r \).

The form Eq. (C8) is very special. A large parameter \( r \) appears both in front of \( \Theta(x) \) and in its argument in a reciprocal way. This form is known as the large deviation, or multifractal ansatz. It appears in many different problems of statistical mechanics (see e.g. Ref. [28] and references therein) and is a non-trivial generalization of Central Limit Theorem when the logarithm of the fluctuating quantity is a sum of many terms with special correlations between them.
The simplest choice of the function $\mathcal{G}(x)$ is a linear function which corresponds to a power-law distribution. A parabolic function $\mathcal{G}(x)$ appears when $\ln y$ is the sum of nearly uncorrelated terms which leads to the logarithmically-normal distribution $P(\Pi_r)$:

$$P(\Pi_r) = \frac{A(r)}{\Pi_r} \exp \left[ -\frac{\ln^2(\Pi_r/\Pi_{typ})}{2p \ln(\Pi_{typ})} \right], \quad A(r) = \frac{1}{\sqrt{2\pi p \lambda r}}. \quad (C9)$$

where $\Pi_{typ} \sim e^{-\lambda r}$, with the Lyapunov exponent $\lambda$, is the typical value of $\Pi_r$, and $p$ is the symmetry parameter. The symmetry Eq. (C2) corresponds to $p = 1$. Any $p \neq 1$ modifies the power of $\Pi_r$ in r.h.s. of the symmetry relation, Eq. (C2).

Note also that the product $\Pi_{r_1}\Pi_{r_2}$ has the log-normal distribution Eq. (C9) with $r = r_1 + r_2$, if both $\Pi_{r_1}$ and $\Pi_{r_2}$ are distributed log-normally as in Eq. (C9) with $r = r_1$ and $r_2$, respectively. This implies that the distribution of the product $\Pi(i,j)$ depends only on the distance between the points $i$ and $j$, no matter whether $i$ and $j$ are on the same path passing through the root of a tree or they are on two different branches (as in Fig. 12). The property $P(\Pi_{r_1}\Pi_{r_2}) = P(\Pi_{r_1+r_2})$ is required of any sensible distribution of the product of local quantities on a Cayley tree.

The distribution of the off-diagonal matrix elements $U = |G(i,j)|_{i,j\neq s,d}$ of the corresponding RP RMT can be found from $P(\Pi_r)$ by setting $r \approx d = \ln N/\ln K$, $U = \Pi_r$, and employing Eq. (B6) which imposes a cutoff at $U_{\text{max}} \sim N^0$.

Here an important comment is at place. Eq. (C9) with $p = 1$ and $r = L$ applies also to a one-dimensional Anderson model at weak disorder and localization length $\xi \ll L$ much smaller than the system size $L$. Indeed, denoting $\ln \Pi_{typ} = L/(2g) \equiv x/2$ one reduces Eq. (C9) to the distribution function of $\sqrt{T}$ of Ref. [45], where $T \ll 1$ is the transmission coefficient. The comparison with the exact result of Ref. [45] could give a more precise limit of applicability of the log-normal distribution than Eq. (B6).

It shows that the log-normal distribution of $T = |G(i,j)|^2$ valid for $\ln(1/G) \gtrsim x$, is modified for $1 \lesssim \ln(1/G) \ll x$ (where $G = |G(i,j)|$ for brevity) by an additional pre-exponential factor $x^{-1} \ln(1/G)$ which is not essential unless in the vicinity of $G = 1$. Thus in 1D case the essential cut-off of the log-normal distribution, indeed, happens at $G = 1$.

One can expect that on the Cayley tree the additional factor which replaces $x^{-1} \ln(1/G)$ is a function of $\ell^{-1} = \ln(1/G)/\ln N$. Such a factor is of order $O(1)$ for any $G \sim N^{-\gamma_{tr}}, \gamma_{tr} \sim 1$ and thus cannot lead to efficient truncation of the log-normal distribution.

One can show that the distribution Eq. (C9) with $p = 1$ is asymptotically exact on an infinite Cayley tree in the limit of small disorder. In particular, for a granular Cayley tree described by the non-linear sigma model (NLσM)

![FIG. 13. (Color online) Plots of $\ln I_m$ from Eq. (C12) for $W = 20$ and the parabolic dependence $-3.05m(1-m)$.](image)

The moments $I_m$ are given by [7]:

$$I_m = \sqrt{\frac{2}{\pi g}} \left[ K_{m+1/2}(g) g \sinh g \right. \quad (C10)$$

$$+ K_{m-1/2}(g) \left( g \cosh g - m \sinh g \right) ,$$

where $g$ is the dimensionless conductance (the coefficient in front of $(\nabla Q)^2$ in the NLσM). In the limit of large inter-grain conductance $g \gg 1$ one obtains:

$$\ln(I_m) \approx -(2g)^{-1} \ln(1-m), \quad (C11)$$

which according to Eq. (C3) implies the log-normal distribution of $g$ and $\Pi_r$. The asymptotic expression Eq. (C11) reproduces Eq. (C10) very accurately down to $g \sim 0.1$.

The same is true for an ordinary Cayley tree with a single orbital per site. In this case the two-brick approximation (Eq. (90) in Ref. [4]) gives for $I_m$:

$$I_m = \left[ \frac{(2m-1) \ln \left( \frac{W}{T} \right)}{(2m-1) \sinh \left( \ln \left( \frac{W}{T} \right) \right)} \right]. \quad (C12)$$

One can show that $\ln I_m$ from Eq. (C12) is approaching Eq. (C11) with $(2g)^{-1} \rightarrow (W - 2)^2/6$ for $W \rightarrow 2$ and remains an almost perfect parabola in a broad interval $2 < W \lesssim 10$ (see Fig. 13). We conclude therefore that the log-normal distribution of $G_r$ is quantitatively accurate in the whole range of disorder strengths up to the Anderson transition point $W_c \sim K \ln K$ if the branching number $K_{\leq 12} \sim O(1)$.

However, for large $\ln(W/2) \gg 1$

$$\ln I_m \approx (|2m-1| - 1) \ln(W/2) \quad (C13)$$

is linear in $m$ everywhere except for a small interval of the width $\sim 1/\ln(W/2)$ in the vicinity of the minimum at $m = 1/2$ in which $\ln I_m$ can be approximated by a parabola (see Fig. 14). In this case the saddle point Eq. (C7) does not have a solution for $|\ln y| > r \ln(W/2)$ (see Fig. 14), and the distribution $F(y)$ is truncated. For $|\ln y| < r \ln(W/2)$ we obtain:

$$F(y) \sim C \exp \left[-\frac{\ln^2 y}{\Sigma}\right], \quad (|\ln y| < r \ln(W/2)), \quad (C14)$$
where $C = e^{-r \ln(W/2)}$ and $\Sigma = (2/3) r \ln^2(W/2)$.

Then from the results of Appendix B it follows that $P(U)$ is truncated from below at $U_{\min} \approx N^{-\ln(W/2)}$ and from above at $U_{\max} \approx 1$. Between these limits it can be approximated by:

$$P(U) \sim \frac{A}{U} \exp \left( -\frac{\ln^2(U)}{(2/3) \ln^2(W/2)} \right) \left( \frac{U_{\text{typ}}}{U} \right),$$

(C15)

where $U_{\text{typ}} \approx N^{-\ln(W/2)}$ and $d(N) = \ln N / \ln K$.

One can see that the probability to find $U$ larger than the typical one is considerably smaller than the one resulting from the forward scattering approximation (FSA):

$$P_{\text{FSA}} \sim \frac{A'}{U} \exp \left( -\frac{\ln^2(U/U_{\text{typ}})}{d(N) \ln^2(W/2)} \right).$$

(C16)

Furthermore, because of the resonances (neglected in the FSA but captured by Eq. (C12)) the transition matrix elements $U$ are never described by the FSA, no matter how large is $\ln(W/2)$.

**Appendix D: Kullback-Leibler measures in the multifractal phase**

In this section we give a more detailed quantitative description of $KL_2$. In order to do this we employ the ansatz for the wavefunction moments:

$$M_q = \left\langle \sum_i |\psi(i)|^{2q} \right\rangle = N^{-D_q(q-1)} f_q(L/\xi_q),$$

(D1)

where $D_q$ is the fractal dimension and $f_q(x)$ is the crossover scaling function:

$$f_q(L/\xi_q \to \infty) \rightarrow \begin{cases} \text{const}, & \text{multifractal phase} \\ \text{const.} \ N(q-1)/D_q - 1, & \text{ergodic phase} \\ \text{const.} \ N(q-1)D_q & \text{localized phase} \end{cases}$$

(D2)

Note that for the graphs with the local tree structure the length scale $L \propto \ln N$, so that the scaling function is in general a function of two arguments $\ln N/\xi_q$ and $N/\xi_q$, representing the length- and volume scaling [12, 13]. On the finite-dimensional lattices $N \propto L^d$, and the volume scaling can be represented as the length scaling in the modified scaling function. In this case a single argument $L/\xi_q$ is sufficient.

When $L \propto \ln N$ the volume scaling is the leading one for $L \gg \xi_q$, and it is this scaling that provides the asymptotic behavior Eq. (D2). The length scaling is important in the crossover region $L \lesssim \xi_q$. Below for brevity we will use the short-hand notation $L/\xi_q$ in all the cases.

There are two trivial cases: $M_0 = N$ and $M_1 = 1$ (which follows from the normalization of wave function). As a consequence we have $D_0 = 1$ and

$$f_0(x) = f_1(x) \equiv 1.$$  

Next using the statistical independence of $\psi$ and $\tilde{\psi}$ in Eq. (22) and normalization of wave functions we represent

$$KL_2 = \left\langle \sum_i |\psi(i)|^2 \ln |\psi(i)|^2 \right\rangle - N^{-1} \left\langle \sum_i \ln |\psi(i)|^2 \right\rangle.$$  

(D4)

Now we express both terms in Eq. (D4) in terms of $M_q$ using the identity:

$$\ln |\psi_0(i)|^2 = \lim_{\epsilon \to 0} \epsilon^{-1} |\psi_0(i)|^2 - 1.$$  

(D5)

The first term is equal to:

$$\left\langle \sum_i \lim_{\epsilon \to \infty} |\psi(i)|^{2(1+\epsilon)} - |\psi(i)|^2 \right\rangle = \lim_{\epsilon \to \infty} \left[ \frac{1}{\epsilon} (M_{1+\epsilon} - 1) \right].$$  

(D6)

The second term can be expressed as:

$$-\frac{1}{N} \left\langle \sum_i \lim_{\epsilon \to \infty} \frac{|\psi(i)|^{2\epsilon} - 1}{\epsilon} \right\rangle = -\lim_{\epsilon \to 0} \left[ \frac{1}{\epsilon} (N^{-1} M_{1-\epsilon} - 1) \right].$$  

(D7)

Now expanding $M_{1+\epsilon}$ and $M_{1-\epsilon}$ in the vicinity of $q = 0, 1$ and defining

$$f_{1+\epsilon}(x) = 1 + \epsilon \varphi_1(x) + O(\epsilon^2);$$  

$$f_{1-\epsilon}(x) = 1 - \epsilon \varphi_0(x) + O(\epsilon^2),$$  

(D8)

(D9)

we obtain:

$$KL_2 = KL_2_c(N) + \varphi_0(L/\xi_0) + \varphi_1(L/\xi_1),$$  

(D10)

where $KL_2_c$ is logarithmically divergent:

$$KL_2_c = \ln N (1 - \partial D_c |_{\epsilon=0} - D_1) + \text{const.}$$  

(D11)

Here we used the identity for $\alpha_0$ describing the typical value of the wave function amplitude $|\psi_{\text{typ}}^2| = N^{-\alpha_0}$:

$$\alpha_0 = \frac{d\tau_c}{d\epsilon} |_{\epsilon=0} = \partial \epsilon [D_c(\epsilon - 1)] |_{\epsilon=0}.$$  

(D12)
Note that, generally speaking, the characteristic lengths $\xi_0 \sim \vert \gamma - \gamma_c \vert^{-\nu(0)}$ and $\xi_1 \sim \vert \gamma - \gamma_c \vert^{-\nu(1)}$ may have different critical exponents $\nu(0)$ and $\nu(1)$. If this is the case, the smallest one will dominate the finite-size corrections near the critical point:

$$KL2 - KL2_c(N) = \Phi_2(L, |\gamma - \gamma_c|^{\nu_2}), \quad \nu_2 = \min\{\nu(0), \nu(1)\},$$

(Eq. (D13)) is used in this paper for the numerical characterization of the phases. Deeply in the multifractal phase and at $L \gg 1$ the scaling function $\Phi_2(x)$ according to Eq. (D2) is a constant. Then $KL2_c(N)$ and $KL2$ are both logarithmically divergent, as $\alpha_0 > 1$ and $D_1 < 1$ in Eq. (D11) in the multifractal phase.

The scaling function $\Phi_2(x)$ is also a constant deeply in the ergodic phase but in this case $\alpha_0 = D_1 = 1$ and the logarithmic divergence of $KL2_c$ is gone. As the result $KL2 = 2$ is independent of $N$ deeply in the ergodic phase.

At the continuous ergodic transition $\alpha_0 = D_1 = 1$, and the critical value $KL2_c(N)$ of $KL2$ is independent of $N$. This results in crossing at $\gamma = \gamma_{ET}$ of all the curves for $KL2$ at different values of $N$ which helps to identify the ergodic transition [26].

However, if the ergodic transition coincides with the Anderson localization transition and is discontinuous, (i.e. $\alpha_0$ and $D_1$ are not equal to 1 at the transition), the critical value $KL2_c(N)$ is no longer $N$-independent. In this case the crossing is smeared out and can disappear whatsoever. Nonetheless, by subtracting $KL2_c$ from $KL2$ one can still identify the transition from the condition of the best collapse by choosing an optimal $\gamma_c$ in Eq. (D13). However, it is safer to use $KL1$ in this case.

The derivation of finite size scaling (FSS) for $KL1$ proceeds in the same way by plugging the identity Eq. (D5) into:

$$KL1 = \left\langle \sum_i |\psi_\alpha(i)|^2 \ln |\psi_\alpha(i)|^2 \right\rangle - \left\langle \sum_i |\psi_\alpha(i)|^2 \ln |\psi_{\alpha+1}(i)|^2 \right\rangle,$$  \hspace{1cm} (D14)

and employing the ansatz:

$$\left\langle \sum_i |\psi_E(i)|^{2q_1} |\psi_{E+\omega}(i)|^{2q_2} \right\rangle \sim N^{1+\beta} N_\omega^{\alpha} \times F_{q_1,q_2}(L/\xi_{q_1}, L/\xi_{q_2}),$$  \hspace{1cm} (D15)

where $N_\omega = 1/\langle \rho \omega \rangle$ and $\rho$ is the mean DoS.

Applying for large $\omega \sim \rho^{-1}$ ($N_\omega \simeq 1$) the “decoupling rule”:

$$\left\langle \sum_i |\psi_E(i)|^{2q_1} |\psi_{E+\omega}(i)|^{2q_2} \right\rangle \sim \sum_i \left\langle \psi_E(i)^{2q_1} \right\rangle \left\langle \psi_{E+\omega}(i)^{2q_2} \right\rangle,$$  \hspace{1cm} (D16)

and for small $\omega \sim \delta$ ($N_\omega \simeq N$) the “fusion rule”:

$$\left\langle \sum_i |\psi_E(i)|^{2q_1} |\psi_{E+\omega}(i)|^{2q_2} \right\rangle \sim \left\langle \sum_i |\psi_E(i)|^{2q_1+2q_2} \right\rangle,$$  \hspace{1cm} (D17)

one easily finds:

$$\beta = -2 + D_{q_1}(1 - q_1) + D_{q_2}(1 - q_2),$$  \hspace{1cm} (D18)

$$\alpha + \beta = -1 + D_{q_1+q_2}(1 - q_1 - q_2).$$
Due to the “fusion rule” for $\psi_\alpha$ and $\psi_{\alpha+1}$ we obtain from Eq. (D1):

$$\langle \sum_i |\psi_\alpha(i)|^{2q_1} |\psi_{\alpha+1}(i)|^{2q_2} \rangle \sim F_{q_1,q_2}(L/\xi_{q_1+q_2}) \times N^{-D_{q_1+q_2}(q_1+q_2-1)}. \tag{D19}$$

Substituting Eq. (D19) in Eqs. (D5), (D14) we observe cancelation of the leading logarithmic in $N$ terms in $KL1$ deeply in the multifractal phase:

$$KL1_c = \text{const.} \tag{D20}$$

We obtain:

$$KL1 = \Phi_1(L|\gamma - \gamma_c|^{\nu_1}). \tag{D21}$$

where $\nu_1 = \nu^{(1)} \geq \nu_2$ and the crossover scaling function $\Phi_1(x)$ is:

$$\Phi_1(x) = \partial_x f_{1+}(x) - \partial_x f_{1,c}(x)|_{x=0}. \tag{D22}$$

As it is seen from Eq. (D21), $KL1$ is independent of $N$ at the Anderson transition point $\gamma = \gamma_{AT}$. Thus all curves for $KL1$ at different values of $N$ intersect at $\gamma = \gamma_{AT}$. This gives us a powerful instrument to identify the Anderson localization transition point.

**Appendix E: Finite-size scaling collapse for $KL1$ and $KL2$ for 3D Anderson model**

In Fig. 15 we present the result for the data collapse for $KL1$ and $KL2$ in the vicinity of the localization transition in the 3d Anderson model. This result demonstrates that our iteration procedure is convergent and gives a good approximation for the critical point from the collapse of $KL2$ data which do not show any intersection of $KL2$ vs. $W$ curves at the critical point. From this collapse we found the critical exponents:

$$\nu_1 = 1.43 \pm 0.15, \quad \nu_2 = 1.13 \pm 0.2. \tag{E1}$$

Note that from the results of Appendix D it follows that quite generally at the same critical point:

$$\nu_2 \leq \nu_1, \tag{E2}$$

since $\nu_2$ is given by the minimal of the two values $\nu^{(0)}$ and $\nu^{(1)}$ (see Eq. (D13)) corresponding to the wave function moments Eq. (D1) with $q = 0$ and $q = 1$, respectively. At the same time, $\nu_1 = \nu^{(1)}$. Our result Eq. (E1) satisfies the inequality Eq. (E2). On the theory side it follows from nowhere that there is only one single critical exponent $\nu$ of any FSS in a situation where there is a continuous multitude of multifractal dimensions. In our opinion, it is more natural to assume that the exponent $\nu$ is specific to the quantity which FSS is studied, as it is shown in the Appendix D for $KL1$ and $KL2$.

However, our samples are too small and our FSS analysis is too simplistic (e.g. it does not take into account irrelevant scaling exponents) to claim that $\nu_1$ and $\nu_2$ are really different.

Note that for different ergodic and Anderson localization transitions the inequality (E2) is not valid in general and thus, cannot be applied to the LN-RP at $p < 1$, while for $p \geq 1$ it is saturated, see Table 7 and Fig. 8.

**Appendix F: Analysis of stability**

In this section we calculate the contributions to $P(V)$ from the log-normal and Gaussian parts to Eq. (34).

One can easily compute the variance of the Gaussian part of $P(V)$ leaving in it only the bi-diagonal terms with $i = i'$ and $j = j'$:

$$\langle V^2 \rangle = \int_{P_{II}} dg N^{-\frac{1}{N^2}(g - \gamma)^2 - 2g} \tag{F1}$$

$$\sim \max_{g \in II} \left\{ N^{-\frac{1}{N^2}(g - \gamma)^2 - 2g} \right\} = N^{-\gamma_{eff}}. \tag{F2}$$

The maximum in Eq. (F1) at $g$ belonging to region II in Fig. 16 can be reached (i) inside the region II at $g = g_1$, (ii) at the border of this region at $g = g_2$, and (iii) at the cut-off of $P(g)$ at $g = 0$ (see Fig. 16 and Fig. 17(left)).

The expression for $\gamma_{eff}(D_1)$ takes the form:

$$\gamma_{eff}(D_1) = \begin{cases} \gamma(1-p), & \frac{p}{2} < D_1 < 1, \ p < \frac{1}{2} \\ 2D_1 + \gamma - 2\sqrt{2D_1\gamma p}, & D_1 < \min \left( \frac{p^2}{2}, \frac{\gamma^2}{8p} \right). \end{cases} \tag{F3}$$

Next we compute the function

$$\Delta(D_1) = -\max_{g \in I} \{ \sigma(g, D_1) - g - D_1 \}. \tag{F4}$$

in Eq. (38).

The details of the calculation which is similar to calculation of $\gamma_{eff}(D_1)$ in Eq. (F1) are illustrated in Fig. 17(right). The resulting expression for $\Delta(D_1)$ is:

![FIG. 16. (Color online) Regions of $g$ contributing to the log-normal (I) and Gaussian (II) parts of the distribution function $P(U_{\mu,\nu})$.](image)
\[ \Delta(D_i) = \begin{cases} \frac{\gamma}{2} \left( 1 - \frac{g^*}{p} \right) - D_i, & 0 < D_i < \frac{2p}{\gamma}, \ p < 1 \\ \frac{\gamma}{2} + \frac{\gamma}{2} - \sqrt{2D_1 \gamma p}, & \frac{2p}{\gamma} < D_i < \frac{2p}{\gamma^2}, \ p < 1 \\ \frac{\gamma}{2p} - D_i, & 0 < D_i < \frac{2p}{\gamma^2}, \ p \geq 1 \end{cases} \] (F4)

FIG. 17. (Color online) (Left panel) Different possible positions \( g_1^*, g_2^* \) or \( g^* = 0 \) that maximize Eq. (F1) in region II depending on \( p, \gamma \) and \( D_1 \). The configuration of maximum realized in each sector of parameters is shown by an ikon in the corresponding sector. (Right panel) Different possible positions \( g_1^*, g_2^* \) or \( g^* = 0 \) that maximize Eq. (F3) in region I. The configuration of maximum realized in each sector of parameters is shown by an ikon in the corresponding sector. For \( D_1 > \gamma/8p \) the maximum in Eq. (F3) is reached at the edge of the right segment of region I, \( g = g_2^* \) (not to be confused with the edge of the left segment \( g = g_2^* \), see Fig. 16). It leads to a higher branch of the orange curve \( \Delta(\alpha) + \alpha/2 \) in Fig. 11 (not shown in Fig. 11) which is separated by a gap from the blue curve in Fig. 11 and thus is irrelevant for our analysis.
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