CUTOFF PHENOMENON OF THE GLAUBER DYNAMICS FOR THE ISING MODEL ON COMPLETE MULTIPARTITE GRAPHS IN THE HIGH TEMPERATURE REGIME

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Abstract. In this paper, the Glauber dynamics for the Ising model on the complete multipartite graph $K_{n p_1, \ldots, n p_m}$ is investigated where $0 < p_i < 1$ is the proportion of the vertices in the $i$th component. We show that the dynamics exhibits the cutoff phenomena at $t_n := \frac{1}{2}(1 - \frac{1}{\beta_{cr}}) n \ln n$ with window size $O(n)$ in the high temperature regime $\beta < \beta_{cr}$ where $\beta_{cr}$ is a constant only depending on $p_1, \ldots, p_m$. Exponentially slow mixing is shown in the low temperature regime $\beta > \beta_{cr}$.

1. Introduction and preliminaries

Informally, the cutoff phenomenon is an abrupt transition of a Markov chain to its equilibrium when the system under consideration is sufficiently large (see Section 1.3 for a rigorous definition). To the author’s knowledge, the first rapid mixing result appeared in [4] on the symmetric group while considering random transpositions. Shortly afterward, Aldous and Diaconis [2] showed that the top-in-at-random card-shuffle precisely exhibits a cutoff phenomenon, initiating the whole industry of the cutoff phenomenon.

As pointed out in [12], only a few examples of cutoff were known regarding the Glauber dynamics of the Ising model (see Section 1.2 for formal definitions), such as that of [5, 10] on complete graphs and of [12, 13, 14] on lattices. Recent researches have mainly focused on lattices. A breakthrough paper by Lubetzky and Sly [12] showed cutoff with a continuous-time window $O(\ln \ln n)$ for this longstanding problem. An improvement on the window size to optimal $O(1)$ was made by the same authors in [14] with the information percolation framework. By the same technique, the authors illustrated the existence of cutoff in high enough temperatures for the Ising model of any sequence of graphs with a bounded degree in [15]. Mean-field Potts model on complete graphs was comprehensively explored in [3], again verifying the cutoff phenomenon in high temperatures. For the bipartite Potts model, Hernández, Kovchegov, and Otto [7] proved the cutoff phenomena in the high temperatures using their aggregate path coupling method.

The purpose of this paper is to investigate the Glauber dynamics for the Ising model on complete multipartite graphs. (Exact definitions are given in the rest of the introduction.) Indeed, we identify the critical temperature and establish cutoff in the high temperature regime. On the other hand, exponentially slow mixing is established in the low temperature regime. The significance of our setting is that...
complete multipartite graphs have an intermediate geometry between the complete graphs which have no geometry at all (e.g. [10]), and lattices which have a strong geometry (e.g. [12]). Thus, our result serves as a midway example between those two extreme cases. The method of proof hinges on generalizations of the tools in [10], notably the two-coordinate chain thereof.

Due to the nature of complete multipartite graphs, our model can be considered as a block spin Ising model with no interaction inside each block. Such mean-field block models naturally occur in statistical physics when modelling metamagnets (see [8]) and in studies on social interactions (see, e.g., [6]). A recent paper by Knöpfel et al. [9] contains an excellent introduction to this line of work.

When it comes to cutoff phenomenon on finite graphs, it is easy to convert the discrete-time results to that of the continuous-time and vice versa. Hence, we only consider discrete-time chains.

1.1. Notations. Boldface letters are used to denote vectors or matrices. Inequalities between vectors and matrices are defined element-wise. The dependence of any quantities on the number of vertices $n$ is understood throughout the paper. Some important quantities not depending on $n$ will be explicitly mentioned. We will write $e_j$ to be the $j$th vector in the standard basis of $\mathbb{R}^m$. The lower case $t$ will always denote time. Let $\circ$ denote the Hadamard product between matrices. More precisely, $B \circ C = (B_{ij}C_{ij})$ whenever $B = (B_{ij})$ and $C = (C_{ij})$ are matrices with the same dimensions.

1.2. Ising model and Glauber dynamics. Let $G = (V,E)$ be a finite graph with the vertex set $V$ and the edge set $E$. Elements of $\Omega := \{\pm 1\}^V$ are called configurations. In the absence of external fields, the Ising model on $G$ is a distribution $\mu$ called the Gibbs distribution on $\Omega$ given by

$$\mu(\sigma) := \frac{e^{-\beta H(\sigma)}}{Z(\beta)}$$

where $\sigma \in \Omega$, $\beta \geq 0$, $H(\sigma) = -\sum_{ij \in E} h_{ij} \sigma(i)\sigma(j)$, and $Z(\beta)$ is a normalizing factor. Assuming an isotropic interaction strength between the vertices, we set $h_{ij} = 1/|V|$. The physical interpretation of $H(\sigma)$ is the energy of the whole spin system with the configuration $\sigma$. We call each $\sigma(v)$ the spin at site $v$.

The Glauber dynamics for the Ising model is a reversible Markov chain with respect to the Gibbs distribution satisfying the following rule. At each time, choose a site uniformly at random in $V$ and update the spin at the chosen site according to $\mu$ conditioned on the set of configurations having the same spins at all the sites except the chosen one. The Glauber dynamics for the Gibbs distribution $\mu$ is irreducible, aperiodic, and reversible with $\mu$ as its unique stationary distribution. For the Ising model, it is easy to see that the probability of updating to $\pm 1$ at the chosen site $v$ is $r_{\pm}(S)$ where

$$r_{\pm}(x) := \frac{e^{\pm \beta x}}{e^{\beta x} + e^{-\beta x}} = \frac{1 \pm \tanh(\beta x)}{2}; \quad x \in \mathbb{R} \quad (1)$$

and $S = \sum_{v,v' \in E} \sigma(v')/|V|$ is the mean-field at $v$. 
1.3. Markov chain mixing and cutoff phenomenon. The total variation distance between two probability measures \( \nu_1 \) and \( \nu_2 \) on \( \Omega \) is defined by
\[
\| \nu_1 - \nu_2 \|_{TV} := \sup_{A \subseteq \Omega} | \nu_1(A) - \nu_2(A) | = \frac{1}{2} \sum_{x \in \Omega} | \nu_1(x) - \nu_2(x) |.
\]
The total variation distance is half of the \( L^1 \)-distance between the probability measures.

Let \( (\sigma_t) \) be the Markov chain of the Glauber dynamics for the Ising model. Define the worst-case total variation distance of the chains to the stationary distribution \( \mu \) at time \( t \) by
\[
d(t) := \max_{\sigma \in \Omega} \| \mathbb{P}_\sigma(\sigma_t \in \cdot) - \mu \|_{TV}
\]
where here and thereafter \( \mathbb{P}_\sigma \) denotes the probability given \( \sigma_0 = \sigma \). The mixing time is defined by
\[
t_{\text{mix}}(\varepsilon) := \min \{ t : d(t) \leq \varepsilon \}; \quad \varepsilon \in (0, 1).
\]
We say a sequence of Markov chains with corresponding mixing times \( t_{\text{mix}}^{(n)}(\varepsilon) \) exhibit a cutoff phenomenon if for every \( 0 < \varepsilon < 1/2 \),
\[
\lim_{n \to \infty} \frac{t_{\text{mix}}^{(n)}(\varepsilon)}{t_{\text{mix}}^{(n)}(1 - \varepsilon)} = 1.
\]
Furthermore, we say that the cutoff occurs at \( t_{\text{mix}}^{(n)}(\varepsilon) \), with window size \( O(w_n) \) if \( w_n = o(t_{\text{mix}}^{(n)}(\varepsilon)) \) and
\[
\lim_{\gamma \to \infty} \lim_{n \to \infty} \inf d_n(t_{\text{mix}}^{(n)} - \gamma w_n) = 1, \quad \lim_{\gamma \to \infty} \lim_{n \to \infty} d_n(t_{\text{mix}}^{(n)} + \gamma w_n) = 0.
\]

1.4. Magnetization chain on complete multipartite graphs. Now, we are in a place to consider a complete \( m \)-partite graph, a graph whose vertices are partitioned into \( m \) different independent sets, and every pair of vertices from different independent sets is connected by an edge. Each edge represents an interaction between the vertices. Denote this graph by \( K_{n_{p_1}, n_{p_2}, \ldots, n_{p_m}} \) which has \( n \) vertices and \( m \) partitions where \( \sum_{i=1}^m p_i = 1 \) and \( p_i > 0 \) for \( i = 1, 2, \ldots, m \). We fix the parameters \( m \) and \( p_i \)’s hereafter. Without loss of generality, we assume \( p_1 \leq p_2 \leq \cdots \leq p_m \). We may also assume that \( n_{p_i} \in \mathbb{N} \) for every \( i \) so that \( K_{n_{p_1}, n_{p_2}, \ldots, n_{p_m}} \) is well defined whenever such considerations are required. Let \( V = \bigcup_{i=1}^m J_i \) be the set of all vertices where \( J_i \) denotes the set of the \( i \)th partition of the vertices. Note \( n_{p_i} = |J_i| \).

We define \( \Omega_i := \{ \pm 1 \}^{J_i} \) for \( i = 1, \ldots, m \) so that \( \Omega = \prod_{i=1}^m \Omega_i \) is our configuration space. Each configuration \( \sigma \in \Omega \) has a unique representation \( (\sigma^{(1)}, \ldots, \sigma^{(m)}) \) \( \in \prod_{i=1}^m \Omega_i \) and both representations are understood throughout this paper. For each \( \sigma \in \Omega \), define the magnetization on \( J_i \) by \( S_i^{(i)}(\sigma) := \sum_{v \in J_i} \sigma(v)/n \), \( i = 1, \ldots, m \). For the Markov chain \( (\sigma_t)_{t \geq 0} = (\sigma^{(1)}_t, \ldots, \sigma^{(m)}_t)_{t \geq 0} \) starting at \( \sigma = (\sigma^{(1)}, \ldots, \sigma^{(m)}) \in \prod_{i=1}^m \Omega_i \), we define the corresponding magnetization on \( J_i \) by
\[
S_i^{(i)} := \frac{1}{n} \sum_{v \in V_i} \sigma_i^{(i)}(v) \text{ for } i \in \{1, \ldots, m\}, \ t \geq 0.
\]
We sometimes use the vector notation \( S_t := (S_{t}^{(1)}, \ldots, S_{t}^{(m)}) \) for \( t \geq 0 \). We call the process \( (S_t)_{t \geq 0} \) a magnetization chain. Proposition 2.1 shows that \( (S_t)_{t \geq 0} \) is in fact a Markov chain. Note that it is a projection of the whole Markov chain \( (\sigma_t)_{t \geq 0} \), so
mixing of the whole chain \((\sigma_t)_{t \geq 0}\) implies the mixing of the chain \((S_t)_{t \geq 0}\). Our aim is to show the converse in a certain sense.

1.5. Main results. Given the above definitions and notations, our main result establishes the cutoff phenomenon on complete multipartite graphs.

**Theorem 1.1** (Main result). For \(m \in \mathbb{N}\) and \(p_i > 0\) such that \(\sum_{i=1}^{m} p_i = 1\), the Glauber dynamics for the Ising model on the complete multipartite graph \(K_{np_1, \ldots, np_m}\) exhibits a cutoff at \(\frac{1}{2(1 - \beta/\beta_{cr})} \ln n\) with window size \(O(n)\) in the high temperature regime \(\beta < \beta_{cr}\) where \(\beta_{cr} = \beta_{cr}(p_1, \ldots, p_m)\) is a constant defined in equation (3).

**Theorem 1.2.** In the low temperature regime \(\beta > \beta_{cr}\), the dynamics is exponentially slow mixing, i.e., \(t_{mix} \geq C_1 \exp(C_2n)\) for some constants \(C_1, C_2 > 0\) not depending on \(n\).

A few remarks are in order. Our main result is obtained as a consequence of Theorem 5.1 and Theorem 5.4. In the low temperature regime \(\beta > \beta_{cr}\), the mixing time is exponentially slow, therefore identifying the critical temperature \(\beta_{cr}\). In the \(m = 1\) case, there are no spin interactions so the chain is equivalent to the lazy random walk on an \(n\)-dimensional hypercube, which has a cutoff at \((n \ln n)/2\) with window size \(O(n)\) (see [1] or [11, Chapter 18]). This result can be seen as a consequence of our main result since \(m = 1\) implies \(\beta_{cr} = \infty\) (see equation (3)).

1.6. Organization of the article. As mentioned earlier, our proof is based on the ideas of Levin, Luczak, and Peres [10]. We assume high temperatures until Section 6. We first observe that the magnetization chain is a Markov chain in its own right (Proposition 2.1). A suitable scaling of the magnetization chain leads to a contraction property (Proposition 2.8). This in turn gives a uniform variance bound of magnetizations in time (Sections 2 and 3). In Section 4, we construct a coupling of the magnetization chain so that it couples in \(\frac{1}{2(1 - \beta/\beta_{cr})} \ln n + O(n)\) steps with high probability. After the magnetization coupling phase, by considering the "2m-coordinate chain" inspired by [10], we can construct a post magnetization coupling to reach the full-mixing in another \(O(n)\) steps. This proves the upper bound (Theorem 5.1). We construct a suitable distinguishing-statistic of the magnetization chain [see 11, Chapter 7.3] to obtain the lower bound (Theorem 5.4). These upper and lower bound results establish the cutoff in the high temperature regime. Exponentially slow mixing in the low temperature regime is shown in Section 6.

2. Contraction of the magnetization chain in high temperatures

We describe the monotone coupling. Let \(I\) and \(U\) be independent uniform random variables over \(V\) and \([0, 1]\), respectively. We consider the collection of Markov chains with starting configurations \(\sigma \in \Omega\). Simultaneously define the next configurations at time \(t = 1\) by

\[
\sigma_1(i) = \begin{cases} 
\sigma(i) & \text{if } I \neq i \\
1_{U < r_+} + (\sum_{j \notin k} S(i)(\sigma)) - 1_{U \geq r_+} (\sum_{j \notin k} S(i)(\sigma)) & \text{if } I = i \in J_k
\end{cases}
\]

where \(r_+\) is defined in equation (1). Repeat this procedure independently for each time. It is clear that each Markov chain \((\sigma_t)_{t \geq 0}\) above is a version of the Glauber dynamics on the complete multipartite graph with starting state \(\sigma\)'s, defined on a common probability space. The above coupling is called a monotone coupling in
Lemma 2.2 (Contraction in mean for monotone coupling)

Let \( \sigma \) and \( \bar{\sigma} \) be two configurations. Similarly, we can define \( \text{dist} \) as the Hamming distance by \( \text{dist}(\sigma, \bar{\sigma}) = \frac{1}{2} \sum_{k \in \Omega} |\sigma(k) - \bar{\sigma}(k)| \).

Remark. By symmetry, \((\mathcal{S}_t(1), \ldots, \mathcal{S}_t(m))\) starting from \( \sigma \) and \((\mathcal{S}_t(1), \ldots, \mathcal{S}_t(m))\) starting from \(-\sigma\) have the same distributions. This can also be seen by the physical fact that the map \( \sigma \mapsto -\sigma \) just corresponds to flipping the reference axis to which we are measuring the spins of each site. This does not change the dynamics of the spin system.

Definition (Hamming distance). For two configurations \( \sigma \) and \( \sigma' \), denote the Hamming distance by \( \text{dist}(\sigma, \sigma') := \frac{1}{2} \sum_{k \in \Omega} |\sigma(k) - \sigma'(k)| \).

Remark. This is a metric on \( \Omega \), which is equal to the number of sites with different spins for two configurations. Similarly, we can define \( \text{dist}_i \) on \( \Omega_i \), respectively, but \( \text{dist}_i \) merely satisfy the triangle inequality.

Lemma 2.2 (Contraction in mean for monotone coupling). For a monotone coupling \((\sigma_t, \sigma'_t)_{t \geq 0}\) starting at \((\sigma, \sigma') = ((\sigma^{(1)}, \ldots, \sigma^{(2)}), (\sigma'^{(1)}, \ldots, \sigma'^{(2)}))\), we have

\[
\begin{pmatrix}
\text{Edist}_1(\sigma_t^{(1)}, \sigma'_t^{(1)}) \\
\vdots \\
\text{Edist}_m(\sigma_t^{(m)}, \sigma'_t^{(m)})
\end{pmatrix} \leq A^t
\begin{pmatrix}
\text{dist}_1(\sigma^{(1)}, \sigma'_0) \\
\vdots \\
\text{dist}_m(\sigma^{(m)}, \sigma'_0)
\end{pmatrix}
\]

where

\[
A = A_n := \begin{pmatrix}
a & b_1 & b_1 & \ldots & b_1 \\
b_2 & a & b_2 & \ldots & b_2 \\
b_3 & b_3 & a & \ldots & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_m & \ldots & \ldots & \ldots & a
\end{pmatrix}
\]

with \( a := 1 - 1/n \), \( b_k := p_k \beta/n \).
Proof: Assume $d(\sigma, \sigma') = 1$ with $-1 = \sigma(v) = -\sigma'(v)$ for some vertex $v$. Note $\sigma \leq \sigma'$. Since we are considering a monotone coupling, it holds that for each $i = 1, \ldots, m$,
\[
\text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)}) = \mathbb{I}_{v \in J_i} (1 - \mathbb{I}_{I = v}) + \mathbb{I}_{v \notin J_i} (\mathbb{I}_I \mathbb{I}_J A_i)
\]
where
\[
B_i = \left\{ r_+ \left( \sum_{l \neq i} S_l(\sigma) \right) \leq U < r_+ \left( \sum_{l \neq i} S_l(\sigma') \right) \right\}.
\]
Note that
\[
P(B_i) = \frac{1}{2} \left( \tanh \left( \beta \sum_{l \neq i} S_l(\sigma') \right) - \tanh \left( \beta \sum_{l \neq i} S_l(\sigma) \right) \right)
\]
\[
= \frac{1}{2} \left( \tanh \left( \beta \left( \sum_{l \neq i} S_l(\sigma) + \frac{2}{n} \right) \right) - \tanh \left( \beta \sum_{l \neq i} S_l(\sigma) \right) \right) \mathbb{I}_{v \notin J_i}
\]
\[
\leq \tanh \frac{\beta}{n} \mathbb{I}_{v \notin J_i}.
\]
Since $I$ and $U$ are independent, for $i = 1, \ldots, m$,
\[
\mathbb{E}\text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)}) \leq \mathbb{1}_{v \in J_i} (1 - \frac{1}{n}) + \mathbb{1}_{v \notin J_i} p_i \tanh \frac{\beta}{n}.
\]
Suppose $\text{dist}(\sigma, \sigma') = k > 1$. There exists $\sigma^0 := \sigma, \sigma^1, \ldots, \sigma^k := \sigma'$ such that $\text{dist}(\sigma^i, \sigma^{i+1}) = 1$. By the triangular inequality for $\text{dist}_i$ and the fact $\tanh(\beta/n) \leq \beta/n$,
\[
\mathbb{E}\text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)}) \leq (1 - \frac{1}{n})\text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)}) + \frac{p_i \beta}{n} \sum_{l \neq i} \text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)}).
\]
Furthermore, by the Markov property,
\[
\mathbb{E}[\text{dist}_i(\sigma_t^{(i)}, \sigma_t^{(i)}) | \sigma_t, \sigma'_t] \leq (1 - \frac{1}{n})\text{dist}_i(\sigma_t^{(i)}, \sigma_t^{(i)}) + \frac{p_i \beta}{n} \sum_{l \neq i} \text{dist}_i(\sigma_t^{(i)}, \sigma_t^{(i)}).
\]
By taking expectation and putting $x_{i,t} := \mathbb{E}\text{dist}_i(\sigma_1^{(i)}, \sigma_1^{(i)})$, we have
\[
\begin{pmatrix}
    \mathbb{E}x_{1,t} \\
    \vdots \\
    \mathbb{E}x_{m,t}
\end{pmatrix}
\leq A
\begin{pmatrix}
    \mathbb{E}x_{1,t-1} \\
    \vdots \\
    \mathbb{E}x_{m,t-1}
\end{pmatrix}.
\]
Iterating gives
\[
\begin{pmatrix}
    \mathbb{E}x_{1,t} \\
    \vdots \\
    \mathbb{E}x_{m,t}
\end{pmatrix}
\leq A^t
\begin{pmatrix}
    \mathbb{E}\text{dist}_1(\sigma_1^{(1)}, \sigma_1^{(1)}) \\
    \vdots \\
    \mathbb{E}\text{dist}_m(\sigma_1^{(m)}, \sigma_1^{(m)})
\end{pmatrix}.
\]

From now on, $A$ (which depends on the number of vertices $n$) always denotes the matrix defined in Lemma 2.2. Note that $A$ is a positive matrix, so by the Perron-Frobenius theorem, there exists the largest eigenvalue $g = g_n > 0$ with the left eigenvector $a^T := (a_1, \ldots, a_m) > 0$ normalized in $l^1$ norm. Note that $g$ has algebraic multiplicity 1 (see [16, Section 8.2] for a proof), so $a^T$ is unique.
We fix the following notations
\[ v := n(1 - g) \quad \text{and} \quad (2) \]
\[ \beta_{cr} := \frac{1}{(m - 1) \sum_{i=1}^{m} a_i p_i} \quad (3) \]
where \( g \) and \((a_1, \ldots, a_m)\) are defined in the previous paragraph. Another characterization of \( \beta_{cr} \) is given in Lemma 6.1. Insuk Seo commented\(^1\) that it can also be characterized as the threshold value of \( \beta \) that makes \( K \) positive definite where \( K \) is defined through the equation \( A = I - \frac{1}{n} K \), \( I \) being the \( m \)-by-\( m \) identity matrix. Proposition 2.3 connects the quantities \( v \) and \( \beta_{cr} \).

**Proposition 2.3.** The left eigenvector \( a^T \) only depends on \( p_1, \ldots, p_m \). Moreover, \( v \) only depends on \( p_1, \ldots, p_m \), and \( \beta \) through the following equation:
\[ v = 1 - \beta(m - 1) \sum_{i=1}^{m} a_i p_i. \]

Therefore, \( \beta_{cr} \) only depends on \( p_1, \ldots, p_m \), and we have \( v = 1 - \beta/\beta_{cr} \).

**Proof.** Since \( g \) satisfies
\[ 0 = (n/\beta)^m \det(A - gI) = \det(nA/\beta - ngI/\beta) \]
\[ = \det \left( \begin{array}{cccc}
\frac{n-1}{\beta} & p_1 & p_1 & \cdots & p_1 \\
p_2 & \frac{n-1}{\beta} & p_2 & \cdots & p_2 \\
p_3 & p_3 & \frac{n-1}{\beta} & \cdots & p_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_m & \cdots & \cdots & \cdots & \frac{n-1}{\beta}
\end{array} \right), \]
it holds that \((v - 1)/\beta \) is a root of a polynomial with coefficients only depending on \( p_1, \ldots, p_m \). Since \( a \) is in the kernel of the transpose of the above matrix, it only depends on \( p_1, \ldots, p_m \).

Finally, \( g = \|A^T a\|_1 = 1 - 1/n + \frac{\beta}{n}(m - 1) \sum_k a_k p_i \) implies \( v = 1 - \beta(m - 1) \sum_i a_i p_i \).

\[ \square \]

We collect further properties of the matrix \( A \) and its left eigenvector \( a^T \) in the next two lemmas.

**Lemma 2.4.** We have
\[ a_1 \geq \cdots \geq a_m \quad \text{and} \quad \sum_{i=1}^{m} a_i p_i \leq \frac{1}{m}. \]
The equality in the latter holds if and only if \( p_1 = \cdots = p_m \).

**Proof.** Recall that we assumed \( p_1 \leq \cdots \leq p_m \).

We claim that \( a_1 \geq \cdots \geq a_m \). To that end, fix \( i < j \). From \( a^T A = ga^T \), we have \((1 - \frac{1}{n}) a_i + \frac{\beta}{n} \sum_{k \neq i} a_k p_k - g a_i = 0 = (1 - \frac{1}{n}) a_j + \frac{\beta}{n} \sum_{k \neq j} a_k p_k - g a_j \). Then \((1 - \frac{1}{n} - g - \frac{\beta a_j}{n}) a_i = (1 - \frac{1}{n} - g - \frac{\beta a_j}{n}) a_j \), i.e., \((\beta p_i + 1 - v) a_i = (\beta p_j + 1 - v) a_j \). Thus, \( p_i \leq p_j \) implies \( a_i \geq a_j \), proving the claim.

\(^1\)personal communication
By Chebyshev’s sum inequality, since \(a_i \geq a_j\) and \(p_i \leq p_j\) whenever \(i < j\),
\[
\sum_{i=1}^{m} a_i p_i \leq \frac{1}{m} \left( \sum_{i=1}^{m} a_i \right) \left( \sum_{i=1}^{m} p_i \right) = \frac{1}{m}.
\]
The equality holds if and only if \(a_1 = \cdots = a_m\) or \(p_1 = \cdots = p_m\). The proof is now complete by noticing the fact that \((\beta p_i + 1 - v) a_i = (\beta p_j + 1 - v) a_j\) and \(a_1 = \cdots = a_m = 1/m\) imply \(p_1 = \cdots = p_m\). □

**Remark.** As a consequence, we obtain a lower bound \(\beta_{cr} \geq m/(m - 1)\).

**Lemma 2.5.** For \(0 \leq s \in S\) and \(p := (p_1, \ldots, p_m)^T\), we have
\[
\|A^i s\|_1 \leq g^i \left( \sum_{i=1}^{m} \frac{(s(i))^2}{p_i} \right)^{1/2}, \quad e_j^T A^i s \leq \sqrt{p_j g^i} \left( \sum_{i=1}^{m} \frac{(s(i))^2}{p_i} \right)^{1/2}.
\]
In particular, it holds that
\[
\|A^i p\|_1 \leq g^i, \quad e_j^T A^i p \leq \sqrt{p_j g^i}.
\]

**Proof.** We want to find a symmetric matrix \(C\) which is similar to \(A\). To that end, suppose that there exists an invertible diagonal matrix \(D = \text{diag}(d_1, \ldots, d_m)\) and a symmetric matrix \(C\) such that \(C = D^{-1} A D\). Then \(D A^T D^{-1} = C^T = C = D^{-1} A D\), so \(D^2 A^T = A D^2\), which leads to \(d_i^2 p_j = p_i d_j^2\) for \(i, j \in \{1, 2, \ldots, m\}\). With the above in mind, let \(D := \text{diag}(\sqrt{p_1}, \ldots, \sqrt{p_m})\) and \(C := (c_{ij})\) where \(c_{ii} = 1 - 1/n\) and \(c_{ij} = \sqrt{p_i p_j} \beta/n\) for \(i \neq j\). Note that \(C\) is real-symmetric and \(C = D^{-1} A D\). Then, by the spectral theorem for real symmetric matrices, \(\|C\|_2 = g\). Note that \(C\) and \(A\) have the same real eigenvalues since they are similar.

Observe that for \(x, y \in \mathbb{R}^m\), \(\|xy^T\|_2 = \|x\|_2 \|y\|_2\). This can be easily checked by the equalities
\[
\|xy^T\|_2 = \sup_{\|z\|_2 = 1} \|xy^T z\|_2 = \sup_{\|z\|_2 = 1} \|y^T z\|_2 = \|y\|_2 \|x\|_2.
\]

Let \(1 := (1, \ldots, 1)^T\). The case \(s = 0\) is trivial, so assume \(s > 0\). Since \(s 1^T\) has rank 1, \(C D^{-1} s 1^T D\) has rank 1. Also, its elements are positive, so it has a positive eigenvalue by the Perron-Frobenius theorem. Thus, \(\text{Tr}(C D^{-1} s 1^T D)\) is equal to its spectral radius, from which the following inequality follows:
\[
\|A^i s\|_1 = 1^T A^i s = 1^T D \cdot C D^{-1} s = \text{Tr}(C D^{-1} s 1^T D) \leq \|C D^{-1} s 1^T D\|_2 \leq \|C\|_2 \|D^{-1} s 1^T D\|_2 = g \|D^{-1} s\|_2 \|D\|_2 = g^{1/2} \left( \sum_{i=1}^{m} \frac{(s(i))^2}{p_i} \right)^{1/2}.
\]

Similarly,
\[
e_j^T A^i s \leq \|C\|_2 \|D^{-1} s e_j^T D\|_2 = g^{1/2} \|D^{-1} s\|_2 \|De_j\|_2 = \sqrt{p_j} g^{1/2} \left( \sum_{i=1}^{m} \frac{(s(i))^2}{p_i} \right)^{1/2}.
\]

**Remark.** Another relatively simple proof of \(\|A^i s\|_1 \leq g^{1/2} \left( \sum_{i=1}^{m} \frac{(s(i))^2}{p_i} \right)^{1/2}\) can be given as follows. By the Cauchy-Schwartz inequality, we have \(\sum_{i=1}^{m} s_i^2/p_i \geq \sum_{i=1}^{m} s_i\). Then \(\|A^i s\|_1 \leq \|D^{-1} A^i D D^{-1} s\|_2 \leq \|C D^{-1} s\|_2 \leq g^{1/2} \|D^{-1} s\|_2\).
From now on, for brevity, we use the notation \( p := (p_1, \ldots, p_m)^T \).

**Lemma 2.6.** For a monotone coupling \((\sigma_t, \sigma'_t)_{t \geq 0}\) starting at \((\sigma, \sigma')\), we have
\[
\mathbb{E} \sum_{i=1}^m a_i \text{dist}_t(\sigma_t, \sigma'_t) \leq g t^4 \sum_{i=1}^m a_i \text{dist}_t(\sigma, \sigma').
\]

Moreover, for \(i = 1, \ldots, m\),
\[
\mathbb{E} \text{dist}_i(\sigma_t^{(i)}, \sigma'_t^{(i)}) \leq n \sqrt{p_t} g t^4.
\]

**Proof.** From Lemma 2.2,
\[
\mathbb{E} \sum_{i=1}^m a_i \text{dist}_i(\sigma_t, \sigma'_t) = a^T \left( \begin{array}{c} \mathbb{E} \text{dist}_1(\sigma_t^{(1)}, \sigma'_t^{(1)}) \\ \vdots \\ \mathbb{E} \text{dist}_m(\sigma_t^{(m)}, \sigma'_t^{(m)}) \end{array} \right) \leq a^T A t^4 \left( \begin{array}{c} \text{dist}_1(\sigma^{(1)}, \sigma'^{(1)}) \\ \vdots \\ \text{dist}_m(\sigma^{(m)}, \sigma'^{(m)}) \end{array} \right) \leq g t^4 a^T \left( \begin{array}{c} \text{dist}_1(\sigma^{(1)}, \sigma'^{(1)}) \\ \vdots \\ \text{dist}_m(\sigma^{(m)}, \sigma'^{(m)}) \end{array} \right) \leq g t^4 \sum_{i=1}^m a_i \text{dist}_i(\sigma, \sigma')
\]

Notice that \(\text{dist}_k(\sigma_t^{(k)}, \sigma'_t^{(k)}) \leq n p_k\) for each \(k\), so Lemma 2.5 implies
\[
\mathbb{E} \text{dist}_i(\sigma_t^{(i)}, \sigma'_t^{(i)}) \leq n e_i^T A t^4 p \leq n \sqrt{p_t} g t^4.
\]

We would like to translate Lemma 2.6 to the case of magnetization chains, which is done in Proposition 2.8.

**Lemma 2.7.** For starting magnetizations \(s = (s_t^{(1)}, \ldots, s_t^{(m)}) \geq (s'^{(1)}, \ldots, s'^{(m)}) = s'\), the magnetization chains satisfy
\[
0 \leq \left( \begin{array}{c} \mathbb{E}_s s_t^{(1)} - \mathbb{E}_{s'} s_t'^{(1)} \\ \vdots \\ \mathbb{E}_s s_t^{(m)} - \mathbb{E}_{s'} s_t'^{(m)} \end{array} \right) \leq A t^4 \left( \begin{array}{c} s^{(1)} - s'^{(1)} \\ \vdots \\ s^{(m)} - s'^{(m)} \end{array} \right).
\]

**Remark.** We say such pairs of starting magnetizations are **monotone pairs**.

**Proof.** Let \((\sigma_t, \sigma'_t)\) be a monotone coupling starting from \((\sigma, \sigma')\) where \(\sigma \geq \sigma'\) and \(S_t^{(i)}(\sigma) = s_t^{i}, S_t^{(i)}(\sigma') = s_t'^{i}\) for \(i = 1, \ldots, m\). Such a monotone coupling exists because of the given condition \(s_t \geq s_t'\) for each \(i\). Since \(\sigma_t \geq \sigma'_t\), we have \(s_t - s_t' = \mathbb{E} \text{dist}_i(\sigma_t, \sigma'_t)\) for each \(i\). By monotonicity, \(\sigma_t^{(i)} \geq \sigma'_t^{(i)}\) for each \(i\). Thus, \(S_t^{(i)} - S_t'^{(i)} = |S_t^{(i)} - S_t'^{(i)}| = \frac{1}{n} \mathbb{E} \text{dist}_i(\sigma_t^{(i)}, \sigma'_t^{(i)}) \geq 0\) for each \(i\). Then, by Lemma 2.2,
\[
0 \leq \left( \begin{array}{c} \mathbb{E}_s s_t^{(1)} - \mathbb{E}_{s'} s_t'^{(1)} \\ \vdots \\ \mathbb{E}_s s_t^{(m)} - \mathbb{E}_{s'} s_t'^{(m)} \end{array} \right) = \left( \begin{array}{c} \mathbb{E}_{\sigma, \sigma'} |S_t^{(1)} - S_t'^{(1)}| \\ \vdots \\ \mathbb{E}_{\sigma, \sigma'} |S_t^{(m)} - S_t'^{(m)}| \end{array} \right) \leq A t^4 \left( \begin{array}{c} s^{(1)} - s'^{(1)} \\ \vdots \\ s^{(m)} - s'^{(m)} \end{array} \right).
\]

Now, we can complete the proof since we have \(\mathbb{E}_s S_t^{(i)} - \mathbb{E}_{s'} S_t'^{(i)} = \mathbb{E}_s S_t^{(i)} - \mathbb{E}_{s'} S_t^{(i)}\) for each \(i\) by Proposition 2.1.

Recall that \(\circ\) denotes a Hadamard product. \(\square\)
Moreover, not depending on the coupling, we have
\[ \mathbb{E}_{\sigma,\sigma'} [a \circ S_t - a \circ S'_t] \leq g' \| a \circ s - a \circ s' \|_1. \]

Proposition 2.8. For a monotone coupling \((\sigma_t, \sigma'_t)_{t \geq 0}\) starting at \((\sigma, \sigma')\) with magnetizations \((s, s')\), we have
\[ \mathbb{E}_{\sigma,\sigma'} [a \circ S_t - a \circ S'_t] \leq g' \| a \circ s - a \circ s' \|_1. \]

Moreover, not depending on the coupling, we have
\[ \| \mathbb{E}_s a \circ S_t - \mathbb{E}_{s'} a \circ S'_t \|_1 \leq g' \| a \circ s - a \circ s' \|_1. \]

Proof. For any magnetizations \(s = s(0)\) and \(s' = s(m)\), there exists \(s(1), \ldots, s(m-1) \in \mathcal{S} \subseteq \mathbb{R}^m\) such that \(s(i-1) - s(i) = e_i(s^{(i)} - s^{(i)})\) for \(i = 1, \ldots, m\). In particular, \(s(i-1)\) and \(s(i)\) are a monotone pair for each \(i\). Then we can consider a monotone coupling \((\sigma(i), t), \ldots, (\sigma(m), t)_{t \geq 0}\) with starting states \((\sigma(0), \ldots, \sigma(m))\) such that \(\sigma_t = \sigma(0), t\)
\[ \sigma'_t = \sigma(m), t \text{ for } t \geq 0, \text{ and the magnetization of the starting configuration } \sigma(i) \text{ is } s(i) \text{ for } i = 0, \ldots, m. \]

Let \(S(j)_t\) be the magnetization chain corresponding to \(\sigma(j), t\) for \(j = 0, \ldots, m\). By telescoping, Lemma 2.7 gives
\[ \mathbb{E}_{\sigma,\sigma'} [a \circ S_t - a \circ S'_t] \leq \sum_{j=1}^{m} \mathbb{E}_{\sigma(j),\sigma(j)} [a \circ S(j)_{t-1}, t - a \circ S(j)_t, t] \]
\[ \leq \sum_{j=1}^{m} a^T A e_j |s(j) - s'(j)| = g' \sum_{j=1}^{m} a_j |s(j) - s'(j)| = g' \| a \circ s - a \circ s' \|_1. \]

Then, the triangle inequality and Proposition 2.1 imply
\[ \| \mathbb{E}_s a \circ S_t - \mathbb{E}_{s'} a \circ S'_t \|_1 \leq g' \| a \circ s - a \circ s' \|_1. \]

3. Variance bound of the magnetization in high temperatures

The next lemma is a generalization of Lemma 2.6 in [10] to Markov chains with a finite state space in \(\mathbb{R}^m\). Observe that for square-integrable \(\mathbb{R}^m\)-valued i.i.d. random vectors \(X, Y\), we have \(\text{Var}X = \frac{4}{2} \mathbb{E}\|X - Y\|_2^2\).

Lemma 3.1. Let \((Z_t)_{t \geq 0}\) be a Markov chain in a finite state space \(\tilde{S} \subseteq \mathbb{R}^m\). Suppose that there exists \(0 < r < 1\) such that for any \(\theta, \theta' \in \tilde{S}\),
\[ \| \mathbb{E}_\theta Z_t - \mathbb{E}_{\theta'} Z'_t \|_1 \leq r^t \| \theta - \theta' \|_1. \]

Then, for the \(l^2\) norm variance,
\[ \sup_{\theta \in \tilde{S}} \text{Var}_\theta Z_t \leq m \sup_{\theta \in \tilde{S}} \text{Var}_\theta Z_1 \min\{t, (1 - r^2)^{-1}\}. \]

Proof. Put \(v_t := \sup_{\theta \in \tilde{S}} \text{Var}_\theta Z_t\). Let \((Z_t)\) and \((Z'_t)\) be independent copies of the chain starting from \(\theta \in \tilde{S}\). The idea is to condition on the first step. Note that \(\|x\|_2 \leq \|x\|_1 \leq \sqrt{m} \|x\|_2\) for \(x \in \mathbb{R}^m\). Then by the observation right before the statement of this lemma,
\[ \frac{1}{2} \mathbb{E}_\theta \| Z_1 - Z'_1 \|_2^2 \leq m \frac{1}{2} \mathbb{E}_\theta \| Z_1 - Z'_1 \|_2^2 \leq mv_1. \]

By the assumption and Markov property, we have
\[ \| \mathbb{E}_\theta [Z_t | Z_1] - \mathbb{E}_\theta [Z'_t | Z'_1] \|_1 \leq \| \mathbb{E}_Z [Z_{t-1}] - \mathbb{E}_Z [Z'_{t-1}] \|_1 \leq r^{t-1} \| Z_1 - Z'_1 \|_1. \]
Thus, for $\theta \in \hat{S}$,
\[
\Var_{\theta}(E_{\theta}(Z_{t}|Z_{1})) = \frac{1}{2} \mathbb{E}_{\theta} \|E_{\theta}(Z_{t},Z_{t-1}) - E_{\theta}(Z_{t}',Z_{t-1}')\|^2 \leq \frac{1}{2} \mathbb{E}_{\theta} \|E_{\theta}(Z_{t},Z_{t-1}) - E_{\theta}(Z_{t}',Z_{t-1}')\|^2 \\
\leq \frac{1}{2} \mathbb{E}_{\theta}\left[r^{2(t-1)} \|Z_{1} - Z_{1}'\|^2\right] \leq m\nu_{1} r^{2(t-1)}.
\]
By the Markov property, for every $\theta \in \hat{S}$, $\Var_{\theta}(Z_{t}|Z_{1}) \leq \nu_{t-1}$, so
\[
\sup_{\theta \in \hat{S}} \mathbb{E}_{\theta}\left[\Var_{\theta}(Z_{t}|Z_{1})\right] \leq \nu_{t-1}.
\]
The total variance formula holds since we are using the $l^2$ norm. Thus, taking supremum over $\theta \in \hat{S}$ in the total variance formula $\Var_{\theta} Z_{t} = \mathbb{E}_{\theta}\left[\Var_{\theta}(Z_{t}|Z_{1})\right] + \Var_{\theta}(E_{\theta}(Z_{t}|Z_{1}))$, we have $\nu_{t} \leq \nu_{t-1} + m\nu_{1} r^{2(t-1)}$. Upon iterating,
\[
\nu_{t} \leq m\nu_{1} \sum_{i=1}^{t} r^{2(t-i)} \leq m\nu_{1} \min \left\{ t, (1 - r^2)^{-1} \right\}.
\]

The following proposition is an important result bounding the variance of magnetization chains uniformly in time.

**Proposition 3.2.** Let $\beta < \beta_{cr}$. For an arbitrary starting configuration $s$ and $t \geq 0$, we have
\[
\sum_{i=1}^{m} \Var_{s}(S^{(i)}_{t}) = C/n
\]
where $C > 0$ only depends on $p_{1, \ldots, p_{m}}$, and $\beta$.

**Proof.** Observe that $\sum_{i=1}^{m} \Var_{s}(a_{i}S^{(i)}_{t}) = \Var_{s}(a \circ S_{t})$. Note that increments of $S_{t}$ are bounded by $2/n$ in absolute value. Then, from Lemma 2.4, we have
\[
\sum_{i=1}^{m} \Var_{s}a_{i}S^{(i)}_{t} \leq a_{m}^{2}(2/n)^{2}.
\]
By Lemma 2.4, Proposition 2.8, and Lemma 3.1, we have
\[
a_{m}^{2} \sum_{i=1}^{m} \Var_{s}(S^{(i)}_{t}) \leq \sum_{i=1}^{m} \Var_{s}(a_{i}S^{(i)}_{t}) \leq m \frac{4a_{m}^{2}}{2} \frac{1}{n^2} \frac{1}{1 - g^2} = \frac{4ma_{m}^{2}}{vn(1 + g)} \leq \frac{4ma_{m}^{2}}{vn}.
\]

Note that Proposition 2.3 assures $\nu > 0$. 

We also establish a bound for the expected magnetization on subsets of partitions. To that end, we need the following observation.

**Lemma 3.3.** For each $i \in V$, $E_{\mu}(\sigma(i)) = 0$ where $\mu$ is the Gibbs distribution. In particular, we have $E_{\mu}(S^{(i)}) = 0$.

**Proof.** Since $\mu(\sigma) = \mu(-\sigma)$ for each configuration $\sigma$ and $\sigma \mapsto -\sigma$ is a bijection from $\Omega$ into itself, we have $E_{\mu}(\sigma(i)) = \sum_{\sigma} \sigma(i) \mu(\sigma) = \sum_{\sigma, \sigma(i) = 1} \mu(\sigma) - \sum_{\sigma, \sigma(i) = -1} \mu(\sigma) = 0$. 

**Proposition 3.4** (Expected magnetization bound). Let $\beta < \beta_{cr}$ and $1 \leq i \leq m$. For any $B \subseteq J_i$, and a chain $(\sigma_{t})_{t \geq 0}$ starting at $\sigma \in \Omega$, define $M_{t}(B) := \frac{1}{2} \sum_{k \in B} \sigma_{t}(k)$. Then
\[
|E_{\sigma} M_{t}(B)| \leq |B| g^{t} / \sqrt{n}.
\]
Furthermore, for \( t \geq \frac{1}{2(1-\beta/\beta_c)} n \ln n \), we have
\[
\var_{\sigma}(M_t(B)) = O(n) , \quad \mathbb{E}_{\sigma}[M_t(B)] = O(\sqrt{n}).
\]

\textbf{Proof.} Let "\( + \)" denote the configuration such that all spins are 1 and "\( - \)" denote the configuration with all spins \(-1\). Let \((\sigma_t^+, \sigma_t^+, \sigma_t^-)\) be a monotone coupling with starting configuration \((+, \mu, -)\) where \( \mu \) is the stationary distribution. Let \( i \in \{1, \ldots, m\} \). By Lemma 2.6 and Lemma 3.3,
\[
\mathbb{E}_+[M_t(J_i^+)] \leq \mathbb{E}_{+, \mu}[M_t(J_i^+)] - M_t(J_i)^\mu + \mathbb{E}_{\mu}[M_t(J_i)^\mu] \leq n \sqrt{p_t} g^4.
\]

Then, by symmetry, for \( v \in J_i \), \( \mathbb{E}_+[M_t(v)] \leq n \sqrt{p_t} g^4 / |J_i| = g^4 / \sqrt{p_t} \). Thus, by summing over sites in \( B \), \( \mathbb{E}_+[M_t(B)^+] \leq |B| g^4 / \sqrt{p_t} \). However, for any configuration \( \sigma \), by monotonicity, \( \mathbb{E}_+[M_t(B)^+] \geq \mathbb{E}_\sigma[M_t(B)] \geq \mathbb{E}_[-M_t(B)^-] \). Considering the remark after Proposition 2.1, \( \mathbb{E}_[-M_t(B)^-] = -\mathbb{E}_+[M_t(B)^+] \). Thus, \( |\mathbb{E}_\sigma[M_t(B)]| \leq |\mathbb{E}_+[M_t(B)^+]| \leq |B| g^4 / \sqrt{p_t} \) for any \( \sigma \).

Now, by Proposition 3.2, \( O(1/n) = \operatorname{Var} S_t^{(i)} = \operatorname{Var}(M_t(J_i)2/n) \), so
\[
\var_{\mathbb{E}_+[M_t(J_i)]} = O(n).
\]

Thus, for \( t \geq \frac{1}{2(1-\beta/\beta_c)} n \ln n \),
\[
\mathbb{E}_+[M_t(J_i)^2] = \operatorname{Var} + (\mathbb{E}_+[M_t(J_i)])^2 = O(n)
\]

However, by symmetry, for any fixed \( v_1, v_2 \in J_i \),
\[
\mathbb{E}_+[M_t(J_i)^2] = np_i + \left(\frac{np_i}{2}\right) \mathbb{E}_+[\sigma_i^+(v_1) \sigma_i^+(v_2)]).
\]

Thus,
\[
|\mathbb{E}_+[\sigma_i^+(v_1) \sigma_i^+(v_2)]| = O(1/n).
\]

Likewise, for \( B \subseteq J_i \),
\[
\mathbb{E}_+[M_t(B)^2] = |B| + \left(\frac{|B|}{2}\right) \mathbb{E}_+[\sigma_i^+(v_1) \sigma_i^+(v_2)] \leq O(n).
\]

Similarly, \( \mathbb{E}_-[M_t(B)^2] \leq O(n) \), so from \( (M_t(B))^2 \leq (M_t(B)^+)^2 + (M_t(B)^-)^2 \),
\[
\mathbb{E}[M_t(B)^2] = O(n)
\]

whenever \( t \geq \frac{1}{2(1-\beta/\beta_c)} n \ln n \). Thus, for \( t \geq \frac{1}{2(1-\beta/\beta_c)} n \ln n \),
\[
\var_{\sigma}(M_t(B)) = O(n).
\]

Lastly, for \( t \geq \frac{1}{2(1-\beta/\beta_c)} n \ln n \), from Jensen’s inequality,
\[
\mathbb{E}\sigma[M_t(B)] \leq \sqrt{\mathbb{E}\sigma[M_t(B)]^2} = \sqrt{(\mathbb{E}\sigma[M_t(B)])^2 + \var\sigma(M_t(B))},
\]
\[
\leq |\mathbb{E}\sigma[M_t(B)]| + \sqrt{\var\sigma(M_t(B))} = O(\sqrt{n}).
\]

\( \Box \)
4. Couplings

Fix the notation
\[ t_n := \frac{1}{2(1 - \beta/\beta_{cr})^n} n \ln n. \]

**Definition** (Modified matching). Let \( \sigma \in \Omega \) and \( \sigma' \in \Omega \) have magnetizations \( s \in \mathcal{S} \) and \( s' \in \mathcal{S} \), respectively. Consider two copies of the graph, \( V = \bigcup_i J_i \) and \( V' = \bigcup_i J'_i \). Let \( i \in \{1, \ldots, m\} \). If \( s(i) \geq s'(i) \), then it is possible to match each site in \( J'_i \) with +1 spin to a site in \( J_i \) with +1 spin. Any leftover sites in \( J'_i \) are arbitrarily matched to the leftover sites in \( J_i \). We match the sites in a similar way whenever \( s(i) \leq s'(i) \). This defines a bijection \( f_{\sigma, \sigma'} : V \to V' \).

We call this bijection a *modified matching* of \( \sigma \) and \( \sigma' \).

**Definition** (Modified monotone update and coupling). Let \( f_{\sigma, \sigma'} : V \to V' \) be a modified matching of \( \sigma, \sigma' \in \Omega \). Let \( I \) and \( U \) be uniformly distributed over \( V = \bigcup_{i=1}^m J_i \) and \( [0, 1] \subseteq \mathbb{R} \), respectively, and be independent. Suppose \( I \in J_\eta \) for some \( \eta \in \{1, \ldots, m\} \) is the chosen site in \( V \). Consider the case \( \sum_{v \notin J_\eta} \sigma(v) \leq \sum_{v \notin J_\eta} \sigma'(v) \).

If
\[ U < \frac{1 + \tanh \left( \beta \sum_{v \notin J_\eta} \sigma(v) \right)}{2}, \]
then update the chosen site \( I \) of \( V \) by +1 and \( f_{\sigma, \sigma'}(I) \) of \( V' \) by +1. If
\[ U \geq \frac{1 + \tanh \left( \beta \sum_{v \notin J_\eta} \sigma'(v) \right)}{2}, \]
then update the chosen site \( I \) of \( V \) by -1 and \( f_{\sigma, \sigma'}(I) \) of \( V' \) by -1. Otherwise, if
\[ \frac{1 + \tanh \left( \beta \sum_{v \notin J_\eta} \sigma'(v) \right)}{2} \leq U < \frac{1 + \tanh \left( \beta \sum_{v \notin J_\eta} \sigma'(v) \right)}{2}, \]
then update the chosen site \( I \) of \( V \) by -1 and \( f_{\sigma, \sigma'}(I) \) of \( V' \) by +1. The other case \( \sum_{v \notin J_\eta} \sigma'(v) > \sum_{v \notin J_\eta} \sigma'(v) \) can similarly be updated.

Given the chosen site \( I \), we call the above procedure of deciding the updating spin in the two chains a *modified monotone update* with respect to the given modified matching.

Now, fix a modified matching \( f_{\sigma, \sigma'} \) of \( \sigma \) and \( \sigma' \). Let \( \sigma_t \) and \( \sigma'_t \) be chains starting at \( \sigma \) and \( \sigma' \), respectively. Repeating the above procedure independently for each step with respect to \( f_{\sigma, \sigma'} \) gives a coupling of the Glauber dynamics. We call this coupling a *modified monotone coupling* with respect to the given modified matching.

**Remark.** Lemma 2.2 and its consequences hold with a suitable distance function for a modified coupling with respect to a given modified matching.

We first construct a coupling such that the magnetizations agree after \( t_n + O(n) \) steps in the next two lemmas.

**Lemma 4.1** (Lemma 2.4, [10]). Let \( (W_t)_{t \geq 0} \) be a non-negative supermartingale with a stopping time \( \tau \) satisfying

(i) \( W_0 = k \)
(ii) \( W_{t+1} - W_t \leq B < \infty \)
(iii) \( \text{Var}(W_{t+1}|F_t) > \sigma^2 > 0 \) on the event \( \{\tau > t\} \). Then for \( u > \frac{4k^2}{\sigma^2} \),
\[
P_k(\tau > u) \leq \frac{4k}{\sigma\sqrt{u}}.
\]

**Lemma 4.2** (Magnetization coupling). Let \( \beta < \beta_{cr} \). For any configurations \( \sigma \) and \( \sigma' \), there exists a coupling \( (\sigma_t, \sigma'_t) \) with starting states \( (\sigma, \sigma') \) satisfying the following condition. If \( \tau_{mag} := \min\{t \geq 0 : S_t = S'_t\} \), then for large \( \gamma n \),
\[
P_{\sigma,\sigma'}(\tau_{mag} > t_n + \gamma n) \leq \frac{c}{\sqrt{\gamma}}
\]
where \( c > 0 \) is a constant not depending on \( \sigma, \sigma', \) or \( n \).

**Proof.** Let \( (\sigma_t, \sigma'_t) \) be a monotone coupling with starting states \( (\sigma, \sigma') \). Put \( Y_{i,t} := \frac{\beta}{2} a_i |S^{(i)}_1 - S^{(i)}_t| \) for \( i = 1, \ldots, m \) and \( Y_{tot,t} := \sum_{i=1}^m Y_{i,t} \). Define
\[
\tau := \min\{t \geq t_n : \max_{1 \leq i \leq m} Y_{i,t}/a_i \leq 1\}.
\]

By Proposition 2.8,
\[
\mathbb{E}_{\sigma,\sigma'}[Y_{tot,t_n}] \leq c\sqrt{n}
\]
for some \( c > 0 \).

We construct a coupling such that \( (Y_{tot,t})_{t_n \leq t < \tau} \) is a positive supermartingale with bounded increments and the conditional probability of not being lazy is bounded away from zero uniformly in time and \( n \).

To that end, consider a time \( t_n \leq t < \tau \). Define \( K_l := \bigcup_{i:Y_{i,t}/a_i \leq 1} J_i \), \( L_l := \bigcup_{i:Y_{i,t}/a_i > 1} J_i \), and \( L'_l := \bigcup_{i:Y_{i,t}/a_i > 1} J'_i \). Note that \( L_l \neq \emptyset \) since \( t < \tau \). Choose a site equiprobably over \( V = K_l \cup L_l \). Let \( f_l \) be the modified matching of \( \sigma_l \) and \( \sigma'_l \). If a site in \( K_l \) is chosen, then use the modified monotone update with respect to \( f_l \) to update \( (\sigma_l, \sigma'_l) \). If a site in \( L_l \) is chosen, then independently choose another site equiprobably over \( L'_l \) (which can be the same site) to update \( \sigma'_l \) independent of \( \sigma_l \).

It is easy to check that the above is a coupling of the Glauber dynamics.

Clearly, \( Y_{tot,t} \) has bounded increment with the above coupling. Let \( I \) be a random variable uniformly distributed over \( V \) which is independent of \( F_t \). Let \( E = \{I \in L_l, \sigma(I) = +1, \sigma_{l+1}(I) = +1 \} \) and \( F = \{I \in L_l, \sigma(I) = -1, \sigma_{l+1}(I) = +1 \} \). Since \( L_l \neq \emptyset \) implies \( |L_l|/n \geq p_1 \), we obtain that \( \mathbb{P}(Y_{tot,l+1} \neq Y_{tot,t}|F_t) \) is bounded below by

\[
\mathbb{P}(Y_{tot,l+1} \neq Y_{tot,t}, I \in L_l|F_t) \geq \mathbb{P}(E \cup F|F_t) \geq \frac{|L_l| + \sum_{i \in L_l} \sigma_{l+1}(i) \left(1 - \tanh(\beta(1 - p_1))\right)^2}{2n}
\]

\[
+ \frac{|L_l| - \sum_{i \in L_l} \sigma_{l+1}(i) \left(1 - \tanh(\beta(1 - p_1))\right)^2}{2n}
\]

\[
\geq p_1 \left(1 - \tanh(\beta(1 - p_1))\right)^2 > 0.
\]

Finally, we need to show the supermartingale property. Consider \( Y_{tot+1}/a_1 - Y_{tot}/a_1 \). Suppose \( J_1 \subseteq K_l \). Then by a direct calculation, on the event \( \{J_1 \subseteq K_l\} \), it holds
that \( \mathbb{E}(Y_{1,t+1}/a_1 - Y_{1,t}/a_1 | \mathcal{F}_t) \) is bounded above by

\[
\begin{align*}
&\leq \left( p_1 - \frac{|S_t^{(1)} - S_t^{(1)}|}{2} \right) \left( \frac{\tanh(\beta \sum_{j \neq 1} S_t^{(j)}) - \tanh(\beta \sum_{j \neq 1} S_t^{(j)'})}{2} 
- \frac{|S_t^{(1)} - S_t^{(1)}|}{2} \left( 1 - \frac{\tanh(\beta \sum_{j \neq 1} S_t^{(j)}) - \tanh(\beta \sum_{j \neq 1} S_t^{(j)'})}{2} \right) \right) \\
&\leq \frac{1}{2} \left( -|S_t^{(1)} - S_t^{(1)}| + p_1 \tanh(\beta \sum_{j \neq 1} S_t^{(j)}) \right).
\end{align*}
\]

Suppose \( J_1 \subseteq L_t \). Note that \( Y_{1,t} > 1 \) implies \( (S_{t+1}^{(1)} - S_{t+1}^{(1)})(S_t^{(1)} - S_t^{(1)}) \geq 0 \) and \( |S_t^{(1)} - S_t^{(1)}| > 0 \). Let \( \xi = (S_t^{(1)} - S_t^{(1)})/|S_t^{(1)} - S_t^{(1)}| \in \{ \pm 1 \} \). Then by equation (5) in Section 5.2, on the event \( \{ J_1 \subseteq L_t \} \), \( \mathbb{E}(Y_{1,t+1}/a_1 - Y_{1,t}/a_1 | \mathcal{F}_t) \) is equal to

\[
\begin{align*}
&= \xi \frac{n}{2} \left( \mathbb{E}(S_{t+1}^{(1)} - S_{t+1}^{(1)} | \sigma_t) - \mathbb{E}(S_t^{(1)} - S_t^{(1)} | \sigma_t') \right) \\
&= \xi \frac{n}{2} \left( -S_t^{(1)} + p_1 \tanh(\sum_{j \neq 1} S_t^{(j)}) \right) \\
&- \xi \frac{n}{2} \left( -S_t^{(1)} + p_1 \tanh(\sum_{j \neq 1} S_t^{(j)}) \right) \\
&= \xi \left( -(S_t^{(1)} - S_t^{(1)}) + p_1 \left( \tanh(\sum_{j \neq 1} S_t^{(j)}) - \tanh(\sum_{j \neq 1} S_t^{(j)}) \right) \right) \\
&\leq \frac{1}{2} \left( -|S_t^{(1)} - S_t^{(1)}| + p_1 \tanh(\sum_{j \neq 1} S_t^{(j)}) \right).
\end{align*}
\]

Since either \( J_1 \subseteq L_t \) or \( J_1 \subseteq K_t \) must hold, \( \mathbb{E}(Y_{1,t+1}/a_1 - Y_{1,t}/a_1 | \mathcal{F}_t) \) is equal to

\[
\begin{align*}
&= \mathbb{E}(Y_{1,t+1}/a_1 - Y_{1,t}/a_1 | \mathcal{F}_t) \\
&\leq \frac{1}{2} \left( -|S_t^{(1)} - S_t^{(1)}| + p_1 \tanh(\sum_{j \neq 1} S_t^{(j)}) \right) \\
&\leq \frac{1}{2} \left( -|S_t^{(1)} - S_t^{(1)}| + p_1 \tanh(\sum_{j \neq 1} S_t^{(j)}) \right).
\end{align*}
\]

Thus,

\[
\mathbb{E}(Y_{1,t+1}/a_1 | \mathcal{F}_t) \leq (1 - \frac{1}{n})Y_{1,t}/a_1 + \frac{\beta p_1}{n} \sum_{j \neq 1} Y_{j,t}/a_j.
\]

Putting in the matrix form with \( \mathbf{Y}_t := (Y_{1,t}/a_1, \ldots, Y_{m,t}/a_m)^T \), we have

\[
\mathbb{E}(Y_{tot,t+1}/\mathcal{F}_t) = a^T \mathbb{E}(\mathbf{Y}_{t+1} | \mathcal{F}_t) \leq a^T A \mathbf{Y}_t = ga^T \mathbf{Y}_t = gY_{tot,t}.
\]

Since \( \beta < \beta_c \) implies \( g < 1 \) by Proposition 2.3, the supermartingale property is established.

With the above coupling, by Lemma 4.1, for large \( \gamma n \),

\[
P_{\sigma,\sigma'}(\tau > t_n + \gamma n | \sigma_{t_n}, \sigma_{t_n}') \leq e^{n \left( \frac{\| (S_{t_n}^{(1)}, \ldots, S_{t_n}^{(m)}) - (S_{t_n}^{(1)}, \ldots, S_{t_n}^{(m)}) \|_1}{\gamma n} \right)}.
\]
for some $c' > 0$ not depending on $n$. Taking expectation,
$$\mathbb{P}_{\sigma,\sigma'}(\tau > t_n + \gamma n) \leq O(\gamma^{-1/2}).$$
Note $\sigma_\tau$ has at most $m$ more $+1$ spin sites than $\sigma'_\tau$, so $0 \leq Y_{\text{tot},\tau} \leq a_1 m$ by Lemma 2.4. At $\tau$, construct a modified matching of $\sigma_\tau$ and $\sigma'_\tau$, and use the modified monotone coupling with respect to this modified matching from then on. At $\tau_{\text{mag}}$, we construct another modified matching of the sites to do a new modified monotone coupling so that $(S^{i(1)}_t, \ldots, S^{i(m)}_t) = (S^{i(1)}_{t'}, \ldots, S^{i(m)}_{t'})$ forever after $\tau_{\text{mag}}$.

By Lemma 2.4, a modified version of Proposition 2.8, and the strong Markov property, we have
$$\mathbb{P}_{\sigma,\sigma'}(\tau_{\text{mag}} > \tau + \gamma' n|\sigma_\tau, \sigma'_\tau) \leq \mathbb{P}_{\sigma,\sigma'}(Y_{\text{tot},\tau+\gamma' n} \geq a_m |\sigma_\tau, \sigma'_\tau)$$
$$\leq \mathbb{E}_{\sigma,\sigma'}[Y_{\text{tot},\tau+\gamma' n}|\sigma_\tau, \sigma'_\tau]/a_m$$
$$\leq g^{\gamma' n}Y_{\text{tot},\tau}/a_m \leq g^{\gamma' n}a_1 m/a_m \leq e^{-\nu' \gamma'/2} a_1 m/a_m.$$
Thus,
$$\mathbb{P}_{\sigma,\sigma'}(\tau_{\text{mag}} > t_n + (\gamma + \gamma') n) \leq O(\gamma^{-1/2}) + e^{-\nu' \gamma'/2} a_1 m/a_m,$$
and putting $\gamma = \gamma'$ yields
$$\mathbb{P}_{\sigma,\sigma'}(\tau_{\text{mag}} > t_n + \gamma n) \leq O(\gamma^{-1/2}).$$

\[\square\]

**Definition** (Good configurations). Define the set of "good" configurations by
$$\tilde{\Omega} := \{\sigma \in \Omega : |S^{i(1)}(\sigma)| \leq p_i/2, \ i = 1, \ldots, m\}.$$ 
For $\sigma = (\sigma^{(1)}, \ldots, \sigma^{(m)}) \in \tilde{\Omega}$ and each $i$, define
$$u_\sigma^{(i)} := |\{v \in J_i : \sigma^{(i)}(v) = 1\}|, \ v_\sigma^{(i)} := |\{v \in J_i : \sigma^{(i)}(v) = -1\}|.$$
Define
$$\Lambda := \{(u_1, v_1, u_2, v_2, \ldots, u_m, v_m) \in \mathbb{N}^{2m} : |J_i|/4 \leq u_i \land v_i, \ i = 1, \ldots, m\}.$$ 

**Remark.** Note that $\sigma \in \tilde{\Omega} \iff (u_\sigma^{(1)}, v_\sigma^{(1)}, \ldots, u_\sigma^{(m)}, v_\sigma^{(m)}) \in \Lambda$. In other words, $\Lambda$ is another representation of good configurations $\tilde{\Omega}$. We omit the starting state and write $u_i$ instead of $u_\sigma^{(i)}$ for convenience.

**Lemma 4.3** (Lemma 3.3, [10]). For any subset $A \subseteq \Omega$ and stationary distribution $\pi$,
$$d_n(t_0 + t) = \max_{\sigma \in \Omega} \|\mathbb{P}_\sigma(\sigma_{t_0+t} \in \cdot) - \pi\|_{TV}$$
$$\leq \max_{\sigma \in A} \|\mathbb{P}_\sigma(\sigma_t \in \cdot) - \pi\|_{TV} + \max_{\sigma \in \Omega} \mathbb{P}_\sigma(\sigma_{t_0} \notin A).$$
Recall that we are assuming the high temperature regime. By Proposition 3.4, there exists $\delta > 0$ such that $\max_{\sigma \in \Omega, 1 \leq i \leq m} |E_\sigma S^{i(0)}_{\delta n}| \leq p_i/4$. Hence, by Proposition 3.2, for large $n$,
$$\mathbb{P}_\sigma(\sigma_{\delta n} \notin \tilde{\Omega}) \leq \sum_{i=1}^m \mathbb{P}_\sigma(|S^{i(0)}_{\delta n}| > p_i/2) \leq \sum_{i=1}^m \mathbb{P}_\sigma(|S^{i(0)}_{\delta n} - E_\sigma S^{i(0)}_{\delta n}| > p_i/4)$$
$$\leq \frac{16}{p_1} \sum_{i=1}^m \text{Var}_\sigma S^{i(0)}_{\delta n} = O(1/n).$$
Combining with Lemma 4.3,
\[ d_n(\delta n + t) \leq \max_{\sigma \in \tilde{\Omega}} \| \mathbb{P}_\sigma(\sigma_t \in \cdot) - \mu \|_{TV} + O(1/n). \] (4)

**Definition (2m-coordinate chain).** Let \( \tilde{\sigma} \in \Omega \) be a reference configuration. For \( \sigma \in \Omega \) and each \( i \), define
\[ U_i(\sigma) := |\{v \in J_i : \sigma^{(i)}(v) = \tilde{\sigma}^{(i)}(v) = 1\}|, \]
\[ V_i(\sigma) := |\{v \in J_i : \sigma^{(i)}(v) = \tilde{\sigma}^{(i)}(v) = -1\}|. \]

For a chain \((\sigma_t)\) with the starting configuration \(\sigma_0 \in \Omega\), define the 2m-coordinate chain with respect to \(\tilde{\sigma}\) by
\[ U_t := (U^{(1)}_t, V^{(1)}_t, \ldots, U^{(m)}_t, V^{(m)}_t) := (U_1(\sigma_t), V_1(\sigma_t), \ldots, U_m(\sigma_t), V_m(\sigma_t)). \]

It is easy to see that the 2m-coordinate chain is again a Markov chain in its state space \(\mathcal{U} \subseteq \mathbb{N}^{2m}\) and determines the magnetization chain \((S_t^{(1)}, \ldots, S_t^{(m)})\) through the relation \(S_t^{(i)} = 2(U^{(i)}_t - V^{(i)}_t)/n - (\tilde{u}_i - \tilde{v}_i)/n\) for \(i = 1, \ldots, m\).

Symmetry gives us the following lemma which is an adaptation of Lemma 3.4 in [10].

**Lemma 4.4.** Let \((\sigma_t)\) be a chain starting at \(\sigma \in \Omega\). Consider the corresponding 2m-coordinate chain starting at \(u \in \mathcal{U}\). Then
\[ \| \mathbb{P}_\sigma(\sigma_t \in \cdot) - \nu \|_{TV} = \| \mathbb{P}_u((U^{(1)}_t, V^{(1)}_t, \ldots, U^{(m)}_t, V^{(m)}_t) \in \cdot) - \nu \|_{TV} \]
where \(\nu\) is the stationary distribution of the 2m-coordinate chain.

**Proof.** Since \(\mu(\sigma) = e^{\beta n \sum_{i \neq j} S^{(i)}(\sigma)S^{(j)}(\sigma)} / Z(\beta)\), given the 2m-coordinate \(u' \in \mathcal{U}\), the conditional \(\mu\)-probability of the configurations is equiprobable. In other words, \(\mu(\cdot | \Omega(u'))\) is uniform where \(\Omega(u')\) is the set of configurations having the 2m-coordinate \(u'\). Also, by symmetry,
\[ \mathbb{P}_\sigma(\sigma_t \in \cdot | U_t = u') \]
is uniform over \(\Omega(u')\). Thus,
\[ \mathbb{P}_\sigma(\sigma_t = \eta) - \mu(\eta) = \sum_{u' \in \mathcal{U}} \frac{\mathbbm{1}\{\eta \in \Omega(u')\}}{|\Omega(u')|} (\mathbb{P}_{u'}(U_t = u') - \mu(\Omega(u'))) (\mathbb{P}_{u'}(U_t = u') - \mu(\Omega(u'))) \]
Taking absolute values, applying the triangular inequality, summing over \(\eta\), and changing the order of summation shows
\[ \| \mathbb{P}_\sigma(\sigma_t \in \cdot) - \mu \|_{TV} \leq \| \mathbb{P}_u((U^{(1)}_t, V^{(1)}_t, \ldots, U^{(m)}_t, V^{(m)}_t) \in \cdot) - \nu \|_{TV}. \]
The reverse inequality holds since the 2m-coordinate chain is a function of the original chain \((\sigma_t)\).

**Remark.** This lemma lets us look at the 2m-coordinate chain instead of the original chain when considering the total variation distance.

Fix a good configuration \(\tilde{\sigma} \in \tilde{\Omega}\). Recall \(\tau_{mag}\) defined in Lemma 4.2. We use the following coupling after \(\tau_{mag}\), which is a generalization of Lemma 3.5 of [10].
Lemma 4.5 (Post magnetization coupling). Let \( \tilde{\sigma} \in \tilde{\Omega} \) be a good configuration. Suppose that two configurations \( \sigma_0, \sigma'_0 \) satisfy \( S^{(i)}(\sigma_0) = S^{(i)}(\sigma'_0) \) for \( i = 1, \ldots, m \).

With respect to the good configuration \( \tilde{\sigma} \), define

\[
\Theta := \left\{ \sigma \in \Sigma : \min \{ U_i(\sigma), \tilde{u} - U_i(\sigma), V_i(\sigma), \tilde{v} - V_i(\sigma) \} \geq \frac{|J|}{16} \right\},
\]

for each \( i \). Then there exists a coupling \( (\sigma_i, \sigma'_i) \) of the Glauber dynamics with starting states \( (\sigma_0, \sigma'_0) \) satisfying:

(i) \( S_t = S'_t \) for all \( t \geq 0 \)

(ii) If \( R^{(i)}_t := U^{(i)}_t - U^{(i)}_t \), then \( \mathbb{E}_{\sigma_0, \sigma'_0} (R^{(i)}_{t+1} - R^{(i)}_t | \sigma_1, \sigma'_1) = -\frac{R^{(i)}_t}{n} \), \( i = 1, \ldots, m \)

(iii) There exists \( c > 0 \) not depending on \( n \) such that on the event \( \{ \sigma_t, \sigma'_t \in \Theta \} \),

\[
\mathbb{P}_{\sigma_0, \sigma'_0} (R^{(i)}_{t+1} - R^{(i)}_t \neq 0 | \sigma_t, \sigma'_t) \geq c > 0 \text{ for all } i = 1, \ldots, m.
\]

Proof. We inductively define the coupling. The random spin \( S \) determined by the randomness \( I \) and \( U \) is

\[
S = \sum_{i=1}^m (\mathbb{1}_{I \in J_i, U \leq r_+ (\Sigma, \Theta)} - \mathbb{1}_{I \in J_i, U > r_+ (\Sigma, \Theta)}).
\]

Suppose that \( (\sigma_t, \sigma'_t) \) is given such that the statements hold for some \( t \geq 0 \). Let \( \sigma_{t+1} \) be determined \( I \) and \( U \). If \( I \in J_i \) for some \( i \), then choose \( I' \) randomly from \( \{ v \in J'_i : \sigma'_i(v) = \sigma_t(I) \} \). Update the primed chain by

\[
\sigma'_{t+1}(v) = \begin{cases} 
\sigma'_i(v) & \text{if } v \neq I' \\
S & \text{if } v = I'. 
\end{cases}
\]

By the induction hypothesis \( S_t = S'_t \), we have \( \{ v \in J'_i : \sigma'_i(v) = \sigma_t(I) \} \neq \emptyset \) and \( (\sigma'_i) \) satisfies the Glauber dynamics. Also, \( S_{t+1} = S'_{t+1} \) with this coupling.

For \( i = 1, \ldots, m \), put

\[
A_i(\sigma) := \{ v \in J_i : \sigma(v) = \tilde{\sigma}(v) = 1 \},
\]

\[
B_i(\sigma) := \{ v \in J_i : \sigma(v) = -1, \tilde{\sigma}(v) = 1 \},
\]

\[
C_i(\sigma) := \{ v \in J_i : \sigma(v) = 1, \tilde{\sigma}(v) = -1 \},
\]

so \(|A_i(\sigma)| = U_i(\sigma), |B_i(\sigma)| = \tilde{u} - U_i(\sigma), |C_i(\sigma)| = \tilde{v} - V_i(\sigma), \) and \(|D_i(\sigma)| = V_i(\sigma)\).

Now we calculate \( R^{(1)}_{t+1} - R^{(1)}_t \) with the above coupling. The following table shows the one-step dynamics of \( R^{(i)}_t \).

| \( I \) \ | \( I' \) \ | \( S \) \ | \( R^{(1)}_{t+1} - R^{(1)}_t \) |
|---|---|---|---|
| \( B_1(\sigma_1) \) \ | \( D_1(\sigma'_1) \) \ | 1 \ | -1 |
| \( C_1(\sigma_1) \) \ | \( A_1(\sigma'_1) \) \ | -1 \ | -1 |
| \( A_1(\sigma_1) \) \ | \( C_1(\sigma'_1) \) \ | -1 \ | 1 |
| \( D_1(\sigma_1) \) \ | \( B_1(\sigma'_1) \) \ | 1 \ | 1 |
| otherwise \ | otherwise \ | otherwise \ | 0 |
Likewise, thus by a direct calculation,

\[ \mathbb{P}_{\sigma_0, \sigma'_0}(R_{t+1}^{(1)} - R_t^{(1)} = -1| \sigma_t, \sigma'_t) = a(U_t^{(1)}, V_t^{(1)}, U_{2,t}, V_{2,t}) \]

\[ = \frac{\bar{u}_1 - U_t^{(1)}}{n} \frac{V_t^{(1)}}{\bar{u}_1 - U_t^{(1)} + V_t^{(1)}} r_+ (\sum_{j \neq 1} S_t^{(j)}) + \frac{\bar{v}_1 - V_t^{(1)}}{n} \frac{U_t^{(1)}}{\bar{u}_1 - U_t^{(1)} + V_t^{(1)}} r_- (\sum_{j \neq 1} S_t^{(j)}) \]

\[ = \frac{\bar{u}_1 - U_t^{(1)}}{n} \frac{V_t^{(1)} + R_t^{(1)}}{\bar{u}_1 - U_t^{(1)} + V_t^{(1)}} r_+ (\sum_{j \neq 1} S_t^{(j)}) + \frac{\bar{v}_1 - V_t^{(1)}}{n} \frac{U_t^{(1)} + R_t^{(1)}}{\bar{u}_1 - U_t^{(1)} + V_t^{(1)}} r_- (\sum_{j \neq 1} S_t^{(j)}) \].

Thus, by a direct calculation,

\[ \mathbb{E}_{\sigma_0, \sigma'_0}(R_{t+1}^{(1)} - R_t^{(1)}| \sigma_t, \sigma'_t) = b - a \]

\[ = -\frac{R_t^{(1)}}{n} \left( r_+ (\sum_{j \neq 1} S_t^{(j)}) + r_- (\sum_{j \neq 1} S_t^{(j)}) \right) = -\frac{R_t^{(1)}}{n}. \]

Moreover, on the event \( \{ \sigma_t, \sigma'_t \in \Theta \} \), \( (\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_m, \bar{v}_m) \in \bar{\Lambda} \) implies \( U_t^{(1)} \leq \bar{u}_1 - |J_1|/16 \leq 3|J_1|/4 - |J_1|/16 = 11|J_1|/16 \), and \( \bar{u}_1 - U_t^{(1)} \leq 3|J_1|/4 - |J_1|/16 = 11|J_1|/16 \). The same upper bound holds for \( \bar{v}_1 - V_t^{(1)} \) and \( V_t^{(1)} \). Thus, on the event \( \{ \sigma_t, \sigma'_t \in \Theta \} \),

\[ \mathbb{P}_{\sigma_0, \sigma'_0}(R_{t+1}^{(1)} - R_t^{(1)} \neq 0| \sigma_t, \sigma'_t) \geq b \geq \frac{p_1}{16} \frac{1}{16} r_+ (\sum_{j \neq 1} S_t^{(j)}) + \frac{p_1}{16} \frac{1}{16} r_- (\sum_{j \neq 1} S_t^{(j)}) \]

\[ = \frac{p_1}{352}. \]

Similarly, for \( i > 1 \), \( \mathbb{P}_{\sigma_0, \sigma'_0}(R_{t+1}^{(1)} - R_t^{(1)} \neq 0| \sigma_t, \sigma'_t) \geq p_i/352 \geq p_1/352 > 0 \), which concludes the induction.

5. Upper and Lower Bounds in the High Temperature Regime

5.1. Upper Bound.

**Theorem 5.1.** For \( \beta < \beta_{cr} \), we have

\[ \lim_{\gamma \to \infty} \limsup_{n \to \infty} d_n(t_n + \gamma n) = 0. \]
Thus, taking supremum over $A \subseteq \mathcal{U}$ and $u \in \tilde{\Lambda}$,
\[
\max_{u \in \tilde{\Lambda}} \|P_u(U_t \in \cdot) - \nu\|_{TV} \leq \max_{u \in \tilde{\Lambda}} \|P_u(U_t' \in \cdot) - \nu\|_{TV}.
\]
Also, from inequality (4) and Lemma 4.4,
\[
d_n(\delta n + t) \leq \max_{\sigma \in \Omega} \|P_{\sigma}(\sigma_t \in \cdot) - \mu\| + O(1/n)
\leq \max_{u \in \tilde{\Lambda}} \|P_u(U_t \in \cdot) - \nu\|_{TV} + O(1/n).
\]
For $2m$-coordinate chains $U_t$ and $U'_t$ with respect to a fixed $\tilde{\sigma} \in \tilde{\Omega}$ starting at $u \in \mathcal{U}$ and $u' \in \mathcal{U}$, respectively, put
\[
\tau_{tot,c} := \min\{t \geq 0 : U_t = U'_t\}.
\]
It is a standard fact [11, Section 5.2] that
\[
\|P_u(U_t \in \cdot) - P_{u'}(U'_t \in \cdot)\|_{TV} \leq P_{u,u'}(\tau_{tot,c} > t).
\]
Combining all the above results, it suffices to bound
\[
\max_{u \in \tilde{\Lambda}, u' \in \mathcal{U}} P_{u,u'}(\tau_{tot,c} > t).
\]
With the above considerations, fix a good starting configuration $\tilde{\sigma} \in \tilde{\Omega}$ with the associated $2m$-coordinates $\tilde{u} = (\tilde{u}_1, \tilde{v}_1, \ldots, \tilde{u}_m, \tilde{v}_m) \in \tilde{\Lambda}$ and an arbitrary starting configuration $\sigma' \in \Omega$. Put
\[
t_n(\gamma) := t_n + \gamma n, \quad H_M := \{\tau_{mag} \leq t_n(\gamma)\}.
\]
The first step is the magnetization coupling phase. By Lemma 4.2, there exists a coupling $(\sigma_t, \sigma'_t)$ for $t \leq t_n(\gamma)$ with starting configurations $(\tilde{\sigma}, \sigma')$ such that
\[
P_{\tilde{\sigma},\sigma'}(H_M^c) \leq O(1/\sqrt{\gamma}).
\]
The next step is the $2m$-coordinate chain coupling phase. For $i = 1, \ldots, m$, define
\[
\tau_{i,c} := \min\{t \geq 0 : (U_t^{(i)}, V_t^{(i)}) = (U_t'^{(i)}, V_t'^{(i)})\},
\]
\[
\Theta_i := \left\{\sigma \in \Omega : \min\{U_i(\sigma), \tilde{u}_i - U_i(\sigma), V_i(\sigma), \tilde{v}_i - V_i(\sigma)\} \geq \frac{|J_i|}{16}\right\},
\]
\[
H_i(t) := \{\sigma_t^{(i)}, \sigma'_t^{(i)} \in \Theta_i\}, \quad H_i := \bigcap_{t \geq t_n(\gamma), t_{\gamma} = 2\gamma} H_i(t), \quad H_{tot} := \bigcap_{i=1}^m H_i.
\]
We have defined the two coordinate chains with respect to $\tilde{\sigma}$. On the event $H_M$, for $t \geq t_n(\gamma)$, we use the coupling in Lemma 4.5, while on the event $H_M^c$, we let the chains run independently for $t \geq t_n(\gamma)$ since we do not care about this un-probable event.
Our first claim is that
\[ P_{\tilde{\sigma}, \sigma'}(H_i^c) \leq \gamma O(1/n), \quad i = 1, \ldots, m. \]

To that end, observe that
\[ \{ \sigma^{(i)}_t \notin \Theta_i \} \subseteq \{ U^{(i)}_t < |J_i|/16 \} \cup \{ \tilde{u}_i - U^{(i)}_t < |J_i|/16 \} \cup \{ V^{(i)}_t < |J_i|/16 \} \cup \{ \tilde{v}_i - V^{(i)}_t < |J_i|/16 \}. \]

Notice \( \tilde{u}_i \geq |J_i|/4 \) implies
\[ \{ U^{(i)}_t < |J_i|/16 \} \subseteq \{ \tilde{u}_i - U^{(i)}_t > 3|J_i|/16 \}, \]
\[ \{ \tilde{u}_i - U^{(i)}_t < |J_i|/16 \} \subseteq \{ U^{(i)}_t > 3|J_i|/16 \}. \]

Similarly, \( \tilde{v}_i \geq |J_i|/4 \) implies
\[ \{ V^{(i)}_t < |J_i|/16 \} \subseteq \{ \tilde{v}_i - V^{(i)}_t > 3|J_i|/16 \}, \]
\[ \{ \tilde{v}_i - V^{(i)}_t < |J_i|/16 \} \subseteq \{ V^{(i)}_t > 3|J_i|/16 \}. \]

Put
\[ \tilde{A}_i := \{ k \in J_i : \bar{\sigma}(k) = 1 \}, \quad i = 1, \ldots, m. \]

Then, following the notation in Proposition 3.4, \( |M_t(\tilde{A}_i)| = |U_t^{(i)} - (\tilde{u}_i - U_t^{(i)})| \)
implies
\[ \{ U_t^{(i)} < |J_i|/16 \} \cup \{ \tilde{u}_i - U_t^{(i)} < |J_i|/16 \} \subseteq \{ |M_t(\tilde{A}_i)| \geq |J_i|/8 \}. \]

Similarly, \( |M_t(J_i \setminus \tilde{A}_i)| = |V_t^{(i)} - (\tilde{v}_i - V_t^{(i)})| \)
implies
\[ \{ V_t^{(i)} < |J_i|/16 \} \cup \{ \tilde{v}_i - V_t^{(i)} < |J_i|/16 \} \subseteq \{ |M_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/8 \}. \]

Combining all the above results, we obtain
\[ \{ \sigma^{(i)}_t \notin \Theta_i \} \subseteq \{ |M_t(\tilde{A}_i)| \geq |J_i|/8 \} \cup \{ |M_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/8 \}. \]

A parallel argument for the primed chain shows
\[ \{ \sigma^{*(i)}_t \notin \Theta_i \} \subseteq \{ |M'_t(\tilde{A}_i)| \geq |J_i|/8 \} \cup \{ |M'_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/8 \}. \]

In conclusion,
\[ H_i(t)^c = \{ \sigma^{(i)}_t \notin \Theta_i \} \cup \{ \sigma^{*(i)}_t \notin \Theta_i \} \]
\[ \subseteq \{ |M_t(\tilde{A}_i)| \geq |J_i|/8 \} \cup \{ |M_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/8 \} \]
\[ \cup \{ |M'_t(\tilde{A}_i)| \geq |J_i|/8 \} \cup \{ |M'_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/8 \}. \]

Define
\[ B := \bigcup_{t \in [t_n(\gamma), t_n(2\gamma)]} \{ |M_t(\tilde{A}_i)| \geq |J_i|/8 \}, \quad Y := \sum_{t \in [t_n(\gamma), t_n(2\gamma)]} \mathbb{1}_{\{ |M_t(\tilde{A}_i)| \geq |J_i|/16 \}}. \]

Since \( M_t(\tilde{A}_i) \) has increments in \( \{-1, 0, 1\} \), we have \( B \subseteq \{ Y \geq |J_i|/16 \} \). By Cheby-
shev’s inequality, \( P_{\tilde{\sigma}, \sigma'}(B) \leq c \mathbb{E}_{\tilde{\sigma}, \sigma'}(Y)/n \) for some constant \( c > 0 \). From Proposition 3.4, for \( t \geq t_n \), \( \mathbb{E}_{\tilde{\sigma}, \sigma'}(|M_t(\tilde{A}_i)|) \geq |J_i|/16 = O(1/n) \), so \( \mathbb{E}_{\tilde{\sigma}, \sigma'}(Y) = \gamma O(1) \). Thus, \( P_{\tilde{\sigma}, \sigma'}(B) = \gamma O(1/n) \). Similar results hold for \( \bigcup_{t \in [t_n(\gamma), t_n(2\gamma)]} \{ |M_t(J_i \setminus \tilde{A}_i)| \geq |J_i|/16 \} \).
Thus, on \(|J_i|/8\), \(\bigcup_{t \in [t_n(\gamma), t_n(2\gamma)]} \{|M'_i(\hat{A}_i)| \geq |J_i|/8\}\), and \(\bigcup_{t \in [t_n(\gamma), t_n(2\gamma)]} \{|M'_i(J_i \setminus \hat{A}_i)| \geq |J_i|/8\}\). In conclusion,

\[
\mathbb{P}_{\sigma, \sigma'}(H_i^c) = \mathbb{P}_{\sigma, \sigma'} \left( \bigcup_{t \in [t_n(\gamma), t_n(2\gamma)]} H_i(t)^c \right) \leq 4\gamma O(1/n),
\]

which proves our first claim.

From the first claim,

\[
\mathbb{P}_{\sigma, \sigma'}(H_i^c) \leq \sum_{i=1}^{m} \mathbb{P}_{\sigma, \sigma'}(H_i^c) = \gamma O(1/n).
\]

Now, condition on the event \(H_M\). Recalling the fact that Lemma 4.5 assures \(S_t = S'_t\) for \(t \geq t_n(\gamma)\) on the event \(H_M\), we can make \(R_{i}^{(i)}\) stay zero after \(\tau_{i,c}\), using the modified monotone update on \(J_i\) whenever a site in \(J_i\) is chosen to be updated. Thus, on \(H_M\),

\[
\tau_{t_{\text{tot},c}} = \max_{1 \leq i \leq m} \tau_{i,c}.
\]

Our second claim is that

\[
\mathbb{P}_{\sigma, \sigma'}(\tau_{i,c} > t_n(2\gamma), H_i, H_M) = O(1/\sqrt{\gamma}), \quad i = 1, \ldots, m.
\]

From Lemma 4.1 and Lemma 4.5, \(\mathbb{P}_{\sigma, \sigma'}(\tau_{i,c} > t_n(2\gamma), H_i, H_M | \sigma_{t_n(\gamma)}, \sigma'_{t_n(\gamma)}) \leq c|R_{t_n(\gamma)}^{(i)}|/\sqrt{n\gamma}\) for some \(c > 0\). Taking expectation yields,

\[
\mathbb{P}_{\sigma, \sigma'}(\tau_{i,c} > t_n(2\gamma), H_i, H_M) \leq \frac{c\mathbb{E}_{\sigma, \sigma'} |R_{t_n(\gamma)}^{(i)}|}{\sqrt{n\gamma}}.
\]

However, for any \(t > 0\), \(|R_{t}^{(i)}| = |U'_t - U_t| = |M'_i(\hat{A}_i) - M_i(\hat{A}_i)|\), so from Proposition 3.4, \(\mathbb{E}_{\sigma, \sigma'} |R_{t_n(\gamma)}^{(i)}| \leq \mathbb{E}_{\sigma'} |M'_{t_n(\gamma)}(\hat{A}_i)| + \mathbb{E}_{\sigma} |M_{t_n(\gamma)}(\hat{A}_i)| = O(\sqrt{n})\), which proves our second claim.

From the second claim,

\[
\mathbb{P}_{\sigma, \sigma'}(\tau_{t_{\text{tot},c}} > t_n(2\gamma), H_{\text{tot}}, H_M) \leq \sum_{i=1}^{m} \mathbb{P}_{\sigma, \sigma'}(\tau_{i,c} > t_n(2\gamma), H_{\text{tot}}, H_M)
\]

\[
\leq \sum_{i=1}^{m} \mathbb{P}_{\sigma, \sigma'}(\tau_{i,c} > t_n(2\gamma), H_i, H_M) = O(1/\sqrt{\gamma}).
\]

Combining all the above results,

\[
\mathbb{P}_{\sigma, \sigma'}(\tau_{t_{\text{tot},c}} > t_n(2\gamma))
\]

\[
\leq \mathbb{P}_{\sigma, \sigma'}(\tau_{t_{\text{tot},c}} > t_n(2\gamma), H_{\text{tot}}, H_M) + \mathbb{P}_{\sigma, \sigma'}(H_{\text{tot}}^c) + \mathbb{P}_{\sigma, \sigma'}(H_M^c)
\]

\[
= O(1/\sqrt{\gamma}) + \gamma O(1/n) + O(1/\sqrt{\gamma}).
\]

Finally,

\[
d_n(t_n + (2\gamma + \delta)n) \leq O(1/\sqrt{\gamma}) + \gamma O(1/n) + O(1/n),
\]

which gives us the result upon taking limits. \(\square\)
5.2. Lower Bound. We first analyze the drift of magnetization chains. Let $1 \leq i \leq m$ and $\mathcal{F}_t$ be the $\sigma$-algebra generated by $S_t^{(1)}, \ldots, S_t^{(m)}$. By a direct calculation,

$$
\mathbb{E}[S_{t+1}^{(i)} - S_t^{(i)} | \mathcal{F}_t] = \frac{2}{n} p_i \left(\frac{|J_i|}{2 |J_i|} r_+ (\sum_{j \neq i} S_t^{(j)}) - \frac{2}{n} p_i \frac{|J_i|}{2 |J_i|} r_- (\sum_{j \neq i} S_t^{(j)}) \right) 
= \frac{2 p_i - S_t^{(i)}}{2} r_+ (\sum_{j \neq i} S_t^{(j)}) - \frac{2 p_i + S_t^{(i)}}{2} r_- (\sum_{j \neq i} S_t^{(j)}) 
= \frac{1}{n} \left( -S_t^{(i)} + p_i \tanh(\beta \sum_{j \neq i} S_t^{(j)}) \right).
$$

The following simple lemma is the main tool to get the lower bound in Theorem 5.4.

**Lemma 5.2** (Proposition 7.9, [11]). Let $f : S \to \mathbb{R}$ be a measurable function and $\nu_1, \nu_2$ be two probability measures on $S$. Let $\sigma^2 := \max \{ \text{Var}_{\nu_1} f, \text{Var}_{\nu_2} f \}$. If $|\mathbb{E}_1 f - \mathbb{E}_2 f| \geq \sigma$, then

$$
\|\nu_1 - \nu_2\|_{TV} \geq \frac{1}{\sigma^2}.
$$

Positive starting configurations give us the following result.

**Lemma 5.3.** Let $s \geq 0$ be the starting magnetization. Then, for $t \geq 0$,

$$
\mathbb{E}_s \|S_t\| \leq g^t \left( \sum_{i=1}^m \frac{(s^{(i)})^2}{p_i} \right)^{1/2} + O(1/\sqrt{n}).
$$

*Proof.* Consider the case that $|J_i|$ is odd for each $i = 1, \ldots, m$. Let $\nu$ be the starting distribution such that $s^+_i = (\frac{1}{4}, \ldots, \frac{1}{4})$ with probability $\frac{1}{2}$ and $s^-_i = (-\frac{1}{4}, \ldots, -\frac{1}{4})$ with probability $\frac{1}{2}$.

By Lemma 2.7, since $s \geq s^+_i$ in this case,

$$
0 \leq \mathbb{E}_s \nu(S_t - S'_t) \leq \frac{1}{2} A^t (s - s^+_i) + \frac{1}{2} A^t (s - s^-_i) = A^t s.
$$

However, $\mathbb{E}_s S_t^{(i)} = 0$ for $i = 1, \ldots, m$ by the remark after Proposition 2.1. Thus, $0 \leq \mathbb{E}_s S_t \leq A^t s$, so by Lemma 2.5,

$$
0 \leq \sum_{i=1}^m \mathbb{E}_s S_t^{(i)} \leq \|A^t s\|_1 \leq g^t \left( \sum_{i=1}^m \frac{(s^{(i)})^2}{p_i} \right)^{1/2}.
$$

From Proposition 3.2 and Cauchy-Schwarz inequality, since $0 \leq \mathbb{E}_s S_t^{(i)}$ for $i = 1, \ldots, m$,

$$
\mathbb{E}_s \|S_t\| = \sum_{i=1}^m \mathbb{E}_s \|S_t^{(i)}\| \leq \sum_{i=1}^m \left( \mathbb{E}_s S_t^{(i)} + \sqrt{\text{Var}_s S_t^{(i)}} \right) = \sum_{i=1}^m \mathbb{E}_s S_t^{(i)} + \sum_{i=1}^m \sqrt{\text{Var}_s S_t^{(i)}} 
\leq g^t \left( \sum_{i=1}^m \frac{(s^{(i)})^2}{p_i} \right)^{1/2} + \left( m \sum_{i=1}^m \text{Var}_s S_t^{(i)} \right)^{1/2} = g^t \left( \sum_{i=1}^m \frac{(s^{(i)})^2}{p_i} \right)^{1/2} + O(1/\sqrt{n}).
$$

Other cases of $|J_i|$ can similarly be shown by considering $0$ instead of $\frac{1}{n}$ whenever the partition has even number of sites.  \(\Box\)
Finally, we prove the lower bound.

**Theorem 5.4.** For \( \beta < \beta_{cr} \), we have

\[
\lim_{\gamma \to \infty} \lim_{n \to \infty} d_n(t_n - \gamma n) = 1.
\]

**Proof.** Since the magnetization chain is a projection of the original chain, it suffices to provide a lower bound on the total variation norm of the magnetization chain. Using \( \tanh \) to provide a lower bound on the total variation norm of the magnetization chain.

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Let \( E = (a_1, \ldots, a_m) > 0 \) with \( \|a\|_1 = 1 \) being the left eigenvector of \( A \) with eigenvalue \( g \). Then \( E(a^T S_{t+1} | F_t) \geq a^T AS_t - a^T x \), i.e.,

\[
E\left( \sum_{i=1}^{m} a_i S_{t+1}^i | F_t \right) \geq g E\left( \sum_{i=1}^{m} a_i S_t^i \right) - \frac{\beta^2}{3n} \sum_{i=1}^{m} a_i p_i \left( \sum_{j \neq i} S_t^j \right)^2 .
\]

Observe that

\[
\sum_{i=1}^{m} a_i p_i \left( \sum_{j \neq i} S_t^j \right)^2 \leq \sum_{k=1}^{m} a_k p_k \left( \sum_{j=1}^{m} |S_t^j| \right)^2 = \left( \sum_{k=1}^{m} a_k p_k \right) \|S_t\|_1^2 .
\]

Thus, upon taking expectation in equation (6),

\[
E\left( \sum_{i=1}^{m} a_i S_{t+1}^i \right) \geq g E\left( \sum_{i=1}^{m} a_i S_t^i \right) - \frac{\beta^2}{3n} \left( \sum_{i=1}^{m} a_i p_i \right) E\|S_t\|_1^2 .
\]

We claim that,

\[
E\|S_t\|_1^2 \leq (E\|S_t\|_1)^2 + O(1/n).
\]

Since \( E\|S_t\|_1^2 = (E\|S_t\|_1)^2 + \text{Var}\|S_t\|_1 \), it suffices to show \( \text{Var}\|S_t\|_1 \leq O(1/n) \).

However, from Proposition 3.2,

\[
\text{Var}\|S_t\|_1 = \sum_{i=1}^{m} \text{Var}\|S_t^i\| + 2 \sum_{i>j} \text{Cov}(\|S_t^i\|, \|S_t^j\|)
\]

\[
\leq \sum_{i=1}^{m} \text{Var}S_t^i + 2 \sum_{i>j} \sqrt{\text{Var}S_t^i} \sqrt{\text{Var}S_t^j}
\]

\[
\leq \sum_{i=1}^{m} \text{Var}S_t^i + \sum_{i>j} (\text{Var}S_t^i + \text{Var}S_t^j) = m \sum_{i=1}^{m} \text{Var}S_t^i = O(1/n),
\]

which proves the claim.

Put \( Z_t := \sum_{i=1}^{m} a_i S_t^i / g_t \). Then, from the claim above,

\[
E Z_{t+1} - E Z_t \geq -\frac{\beta^2}{3n g_t} \left( (E\|S_t\|_1)^2 + O(1/n) \right).
\]
Assume that $s \geq 0$ is a non-negative starting magnetization. Recalling the definition $v := n(1 - g)$, from Lemma 5.3 and the fact $\sum_i (s^{(i)})^2 / p_i \leq 1$,

$$
\mathbb{E}_n Z_{t+1} - \mathbb{E}_n Z_t \geq - \frac{\beta^2 \sum_i a_i p_i}{3n g^{t+1}} \left( \left( g^t \sum_i (s^{(i)})^2 / p_i \right)^{1/2} + O(1/\sqrt{n}) \right)^2 + O(1/n)
$$

$$
\geq - \frac{\beta^2 \sum_i a_i p_i}{3(n - v)} \left( g^t \sum_i (s^{(i)})^2 / p_i + O(1/\sqrt{n}) + \frac{1}{g^t} O(1/n) \right).
$$

Iterating from 0 to $t - 1$,

$$
\mathbb{E}_n Z_t - Z_0 \geq - \frac{\beta^2 \sum_i a_i p_i}{3(n - v)} \left( 1 - g^t \sum_i \frac{(s^{(i)})^2}{p_i} + tO(1/\sqrt{n}) + \frac{n - v - 1}{g^t} O(1/n) \right)
$$

$$
= - \frac{\beta^2 \sum_i a_i p_i}{3v(1 - v/n)} \left( 1 - g^t \sum_i \frac{(s^{(i)})^2}{p_i} \right) - \frac{\beta^2 \sum_i a_i p_i}{3(n - v)} tO(1/\sqrt{n})
$$

$$
- \frac{\beta^2 \sum_i a_i p_i}{3v} \left( \frac{1}{g^t} - 1 \right) O(1/n).
$$

For brevity, let us prefer to use $v$ rather than use $\beta_{cr}$ in view of Proposition 2.3. Consider the step $t_* := t_n - \gamma n/v = \frac{1}{2v} n \ln n - \frac{2n}{v}$. Observe that $1 - 1/x \geq e^{-1/(x-1)}$ for $x > 1$ implies

$$
g^{t_*} \geq \frac{e^\gamma}{n^{n/(2(n-v))}}.
$$

Then

$$
\mathbb{E}_n Z_{t_*} - \sum_{i=1}^m a_i s_i \geq - \frac{\beta^2 \sum_i a_i p_i}{3v(1 - v/n)} \left( 1 - \frac{e^\gamma}{n^{n/(2(n-v))}} \right) \sum_{i=1}^m \frac{(s^{(i)})^2}{p_i}
$$

$$
- \frac{\beta^2 \sum_i a_i p_i}{3(n - v)} \left( \frac{1}{2v} n \ln n - \frac{\gamma n}{v} \right) O(1/\sqrt{n})
$$

$$
- \frac{\beta^2 \sum_i a_i p_i}{3v} \left( \frac{n^{n/(2(n-v))}}{e^\gamma} - 1 \right) O(1/n).
$$

The right-hand side of the above inequality converges to $- \frac{\beta^2 \sum_i a_i p_i \sum_i (s^{(i)})^2 / p_i}{3v}$ as $n \to \infty$ for every $\gamma > 0$.

We claim that if $n$ is large enough, then there exists $s > 0$ such that

$$
\sum_{i=1}^m a_i s_i - \frac{\beta^2 \sum_i a_i p_i \sum_i (s^{(i)})^2 / p_i}{3v} > 0.
$$

Consider $s = \zeta p$ where $0 < \zeta < 1$ is a constant to be determined. We want to find $\zeta$ such that

$$
\sum_{i=1}^m a_i p_i \zeta - \frac{\beta^2 \sum_i a_i p_i \sum_i (p_i \zeta)^2 / p_i}{3v} > 0,
$$

which is equivalent to

$$
3v > \beta^2 \zeta.
$$

From Proposition 2.3, $v > 0$, so $\frac{3v}{\beta^2} p > s > 0$ assures that the inequality in the claim holds, and such a positive magnetization $s \in \mathcal{S}$ exists since $n$ is large and $0 \leq \beta < \beta_{cr}$ (if $\beta = 0$, choose $s = p$).
By the last claim, for large $n$, there exists $s \in S$ and $\varepsilon > 0$ such that
\[ E_s \left( \sum_{i=1}^{m} a_i S^{(i)}_t \right) \geq 2\varepsilon g^t \geq \frac{\varepsilon e^\gamma}{n^{(2(n-v))}} \geq \varepsilon e^\gamma \sqrt{n}. \]

Proposition 3.2 and the Cauchy-Schwartz inequality imply $\text{Var}(\sum_{i=1}^{m} a_i S^{(i)}_t) = O(\frac{1}{n})$ as $n \to \infty$. Thus, by Lemma 3.3 and Lemma 5.2, for some $c > 0$,
\[ \lim_{\gamma \to \infty} \liminf_{n \to \infty} d_n(t_n - \frac{\gamma n}{v}) \geq \lim_{\gamma \to \infty} 1 - \frac{c}{\varepsilon e^\gamma} = 1. \]

\[ \Box \]

6. Exponentially slow mixing in the low temperature regime

Using a standard bottleneck ratio argument, we can show that the mixing time for the Glauber dynamics is exponential in the low temperature regime. The bottleneck ratio is defined as
\[ \Phi := \min_{A : \mu(A) \leq 1/2} \frac{\sum_{x \in A, y \notin A} \mu(x)P(x,y)}{\mu(A)} \]
where $P$ is the transition matrix of the Glauber dynamics. The bottleneck ratio gives a lower bound of the mixing time (see [11, Theorem 7.4]):
\[ t_{\text{mix}} \geq \frac{1}{4\Phi}. \]

We need another characterization of the critical temperature $\beta_{cr}$.

**Lemma 6.1.** We have that
\[ \beta_{cr} = \frac{\sum_i a_i^2 p_i}{(\sum_i a_i p_i)^2 - \sum_i a_i^2 p_i^2} \]

**Proof.** From $\mathbf{a}^T \mathbf{A} = g \mathbf{a}^T$, equation (3), and Proposition 2.3, we have
\[ \sum_i a_i p_i = \left( p_k + \frac{1}{\beta_{cr}} \right) a_k \]
for each $k = 1, \ldots, m$. Multiplying $a_k p_k$ to both sides and summing over $k$ yields the result. \[ \Box \]

**Proof of Theorem 1.2.** It suffices to show that $\Phi \leq c_1 \exp(-c_2 n)$ for some positive constants $c_1, c_2 > 0$. By symmetry of the Hamiltonian, we have that $\mu(A) \leq 1/2$ where $A := \{ \sigma : \sum_i S^{(i)}(\sigma) > 0 \}$. Since the only way to go from $A$ to $A^c$ is to go through $B := \{ \sigma : |\sum_i S^{(i)}(\sigma)| \leq 1/n \}$, it holds that
\[ \sum_{x \in A, y \notin A} \mu(x)P(x,y) \leq \mu(B). \]

Note that for any $\sigma \in \Omega$,
\[ \mu(\sigma) = \frac{\exp \left( \frac{\beta n}{2} \left( (\sum_i S^{(i)}(\sigma))^2 - \sum_i (S^{(i)}(\sigma))^2) \right) \right)}{Z(\beta)}. \]

By the Cauchy-Schwartz inequality,
\[ \mu(B) \leq \left( \frac{n}{\lceil n/2 \rceil} \right) \frac{\exp \left( \frac{\beta n}{2} (1 - \frac{1}{m}) (\frac{1}{n})^2 \right)}{Z(\beta)} \leq \left( \frac{n}{\lceil n/2 \rceil} \right) / Z(\beta) \]
where \( \lesssim \) denotes that the inequality holds for sufficiently large \( n \) up to a constant not depending on \( n \). Using Stirling’s formula,

\[
\Phi \lesssim \frac{\exp(n \ln 2)}{Z(\beta)\mu(A)}.
\]

Now, consider the configurations with exactly \( k_i np_i \) many "+" spins in \( J_i \) where \( 1/2 \leq k_i \leq 1 \) for each \( i = 1, \ldots, m \) and there exists at least one \( i \) such that \( 1/2 < k_i \). These configurations are members of \( A \) and there are at least \( \prod_{i=1}^{m} \binom{np_i}{k_i} \) many such configurations. Using Stirling’s formula again, we obtain

\[
Z(\beta)\mu(A) \gtrsim \left( \frac{1}{\prod_{i=1}^{m} (1-k_i)p_i(1-k_i)} \right)^n e^{\frac{\beta n}{2} \left( (\sum_i (2k_i-1)p_i)^2 - \sum_i (2k_i-1)^2 p_i^2 \right)}.
\]

Define a function \( f \) through the equation

\[
e^{n f(k_1, \ldots, k_m)} := \left( \frac{1}{\prod_{i=1}^{m} (1-k_i)p_i(1-k_i)} \right)^n e^{\frac{\beta n}{2} \left( (\sum_i (2k_i-1)p_i)^2 - \sum_i (2k_i-1)^2 p_i^2 \right)}.
\]

Put \( (k_1, \ldots, k_m) = (1/2, \ldots, 1/2) + \gamma (v_1, \ldots, v_m) \) where \( v_i \geq 0 \) for each \( i = 1, \ldots, m \), \( \gamma \in \mathbb{R} \), and \( \sum_i v_i^2 \neq 0 \). Fixing \( v_i \)'s, we can regard \( f \) as a one-variable function of \( \gamma \), say \( f = f(\gamma) \), and this is equivalent to fixing a direction in \( \mathbb{R}^m \). A little calculation shows that

\[
f(\gamma) = 2\beta \gamma^2 \left( \sum_i v_i p_i \right)^2 - \sum_i v_i^2 p_i^2 \right) - \sum_i p_i (1/2 - \gamma v_i) \ln(1/2 - \gamma v_i) + (1/2 + \gamma v_i) \ln(1/2 + \gamma v_i) \right) \]

\[
f'(\gamma) = 4\beta \gamma \left( \left( \sum_i v_i p_i \right)^2 \right. - \sum_i v_i^2 p_i^2 \right) \left. - \sum_i p_i v_i \right) \left( - \ln(1/2 - \gamma v_i) + \ln(1/2 + \gamma v_i) \right) \]

\[
f''(\gamma) = 4\beta \left( \left( \sum_i v_i p_i \right)^2 \right. - \sum_i v_i^2 p_i^2 \right) \left. - \sum_i p_i v_i^2 \right) \left( \frac{1}{1/2 - \gamma v_i} + \frac{1}{1/2 + \gamma v_i} \right)
\]

where ' denotes a differentiation in \( \gamma \). Note that \( f(0) = \ln 2 \) and \( f'(0) = 0 \). Thus, it suffices to show that there is a direction \( (v_1, \ldots, v_m) \) such that \( f''(0) > 0 \). Lemma 6.1 shows that the direction \( (v_1, \ldots, v_m) = (a_1, \ldots, a_m) \) satisfies \( f''(0) > 0 \) whenever \( \beta > \beta_{cr} \), which completes the proof. □

**Remark.** Combined with the non-exponential mixing time of \( O(n \ln n) \) whenever \( \beta < \beta_{cr} \), the above proof shows that \( \inf_{v \geq 0, v \neq 0} \frac{\sum_i v_i^2 p_i}{\sum_i v_i p_i} \) is achieved with the direction \( (v_1, \ldots, v_m) = a^T \).

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