A SCHWARZ LEMMA FOR A DOMAIN RELATED TO MU-SYNTHESIS

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Abstract. We prove a Schwarz lemma for a domain $E$ in $\mathbb{C}^3$ that arises in connection with a problem in $H^\infty$ control theory. We describe a class of automorphisms of $E$ and determine the distinguished boundary of $E$. We obtain a type of Schwarz-Pick lemma for a $2 \times 2 \mu$-synthesis problem.

1. Introduction

In this paper we study the complex geometry of a domain $E \subset \mathbb{C}^3$ which is relevant to some problems of analytic interpolation that arise in control engineering. Our main result is a Schwarz lemma for $E$, but we also identify a natural class of automorphisms of $E$ and determine the distinguished boundary of $E$.

Definition 1.1. The tetrablock is the domain

$$E = \{ x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 z w \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1 \}.$$

The closure of $E$ is denoted by $\overline{E}$.

$E$ is a polynomially convex, non-convex domain, is starlike about 0 and intersects $\mathbb{R}^3$ in a regular tetrahedron (Theorems 2.3, 2.4 and 2.5). To a first approximation one can think of $E$ as the set of linear fractional maps $(x_3 z - x_1)/(x_2 z - 1)$ that map the closed unit disc $\Delta$ into the open unit disc $D$, but this viewpoint, though useful, must be interpreted with care since it does not capture the case that $x_1 x_2 = x_3$: see Theorem 2.1 for a precise statement.

Here is our main result. To cut down on subscripts we write the typical point of $E$ as $(a, b, p)$.

Theorem 1.1. Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$ and let $x = (a, b, p) \in E$. The following conditions are equivalent:

1. there exists an analytic function $\varphi : \mathbb{D} \to \overline{E}$ such that $\varphi(0) = (0, 0, 0)$ and $\varphi(\lambda_0) = x$;

1' there exists an analytic function $\varphi : \mathbb{D} \to E$ such that $\varphi(0) = (0, 0, 0)$ and $\varphi(\lambda_0) = x$;

2. $\max \left\{ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right\} \leq |\lambda_0|$;
either \(|b| \leq |a|\) and
\[
\frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \leq |\lambda_0|
\]

or \(|a| \leq |b|\) and
\[
\frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \leq |\lambda_0|;
\]

there exists a \(2 \times 2\) function \(F\) in the Schur class such that
\[
F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F(\lambda_0) = A = [a_{ij}]
\]
where \(x = (a_{11}, a_{22}, \det A)\).

Recall that the Schur class (of type \(m \times n\)) is the set of analytic functions \(F\) on \(D\) with values in the space \(\mathbb{C}^{m \times n}\) of complex \(m \times n\) matrices such that \(||F(\lambda)|| \leq 1\) for all \(\lambda \in D\); here and elsewhere \(||.||\) denotes the usual operator norm (the largest singular value) of a matrix.

The starting point of our research was a certain special case of the \(\mu\)-synthesis problem, which arises in the \(H^\infty\) approach to the problem of robust control \([13, 14]\). Perhaps the most appealing still unsolved instance of \(\mu\)-synthesis is the spectral Nevanlinna-Pick problem (to construct an analytic square-matrix valued function on the disc subject to interpolation conditions and a bound on the spectral radius, \([8, 2]\)). In earlier work \([2, 4]\) some progress was made on the 2 \(\times 2\) spectral Nevanlinna-Pick problem via the analysis of a domain in \(\mathbb{C}^2\) known as the symmetrised bidisc. It transpired that this domain and its higher dimensional analogues have a rich geometry and function theory, of interest independently of their connections with engineering (for example \([5, 12, 15, 19, 10]\) among others). In an analogous way the study of the “next” special case of \(\mu\)-synthesis for \(2 \times 2\) matrix functions led us to analyse the tetrablock. In Section 9 we explain the connection between \(E\) and \(\mu\)-synthesis and give an application of our Schwarz lemma. Note the interesting fact that \(\mu\)-synthesis problems can be ill conditioned (Remark 9.2(iv)). We also prove a Schwarz-Pick lemma for the 2 \(\times 2\) \(\mu\)-synthesis problem.

In Section 2 we give a variety of characterizations of the open and closed tetrablocks and present some basic geometric properties of \(E\). In Section 3 we prove Theorem 1.1 and deduce a formula for the Carathéodory and Kobayashi distances of a general point of \(E\) from the origin. In Section 4 we show that there is no uniqueness statement for the extremal case and in Section 5 we describe all solutions of a Schwarz-type 2-point interpolation problem for \(E\). In Section 6 we identify a rich class of automorphisms of \(E\). In Section 7 we calculate the distinguished boundary of \(E\). In Section 8 we pose the question as to whether \(E\) is an analytic retract of a certain convex domain and prove a partial negative result.

We write \(T\) for the unit circle in \(\mathbb{C}\) and \(d\) for the pseudohyperbolic distance on \(D\). As usual, \(H^\infty\) denotes the Banach space of bounded analytic functions on \(D\) with supremum norm. An automorphism of a domain \(\Omega\) is a biholomorphic self-map of \(\Omega\); the automorphism group of \(\Omega\) will be denoted by \(\text{Aut} \, \Omega\). We denote by \(S_{m \times n}\) the class (slightly smaller than the Schur class) of analytic functions \(F : D \to \mathbb{C}^{m \times n}\) such that \(||F(\lambda)|| < 1\) for all \(\lambda \in D\).
For $Z \in \mathbb{C}^{m \times n}$ such that $\|Z\| < 1$ we denote by $\mathcal{M}_Z$ the matricial Möbius transformation defined for contractive $X \in \mathbb{C}^{m \times n}$ by

$$\mathcal{M}_Z(X) = (1 - ZZ^*)^{-\frac{1}{2}}(X - Z)(1 - Z^*X)^{-1}(1 - Z^*Z)^{\frac{1}{2}}.$$  

Recall that $\mathcal{M}_Z^{-1} = \mathcal{M}_{-Z}$ as self-mappings of the closed unit ball of $\mathbb{C}^{m \times n}$. We shall denote the $(i,j)$ entry of a matrix $A$ by $[A]_{ij}$.

This paper is based on the first-named author’s Ph.D. thesis [1].

2. Characterization of the tetrablock

The following rational functions of 4 variables play a central role in the study of $E$.

**Definition 2.1.** For $z \in \mathbb{C}$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ we define

$$\Psi(z, x) = \frac{x_3z - x_1}{x_2z - 1},$$

$$\Upsilon(z, x) = \Psi(z, x_2, x_1, x_3) = \frac{x_4z - x_2}{x_1z - 1},$$

$$D(x) = \sup_{z \in \mathbb{D}} |\Psi(z, x)| = ||\Psi(., x)||_{H^\infty}.$$  

We interpret $\Psi(., x)$ to be the constant function equal to $x_1$ in the event that $x_1x_2 = x_3$; thus $\Psi(z, x)$ is defined when $x_2 \neq 1$ or $x_1x_2 = x_3$. The quantity $D(x)$ is finite (and $\Psi(., x) \in H^\infty$) if and only if either $x_2 \in \mathbb{D}$ or $x_1x_2 = x_3$. Indeed, for $x_2 \in \mathbb{D}$, the linear fractional function $\Psi(., x)$ maps $\mathbb{D}$ to the open disc with centre and radius

$$D(x) = \frac{x_1 - \bar{x}_2x_3}{1 - |x_2|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_2|^2}$$

respectively. Hence

$$D(x) = \begin{cases} 
\frac{|x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3|}{1 - |x_2|^2} & \text{if } |x_2| < 1 \\
\frac{|x_1|}{|x_2|^2} & \text{if } x_1x_2 = x_3 \\
\infty & \text{otherwise.}
\end{cases}$$

Similarly, if $x_1 \in \mathbb{D}$, $\Upsilon(., x)$ maps $\mathbb{D}$ to the open disc with centre and radius

$$D(x) = \frac{x_2 - \bar{x}_1x_3}{1 - |x_1|^2}, \quad \frac{|x_1x_2 - x_3|}{1 - |x_1|^2}$$

respectively.

**Theorem 2.1.** For $x \in \mathbb{C}^3$ the following are equivalent.

1. $x \in \mathbb{E}$;
2. $||\Psi(., x)||_{H^\infty} < 1$ and if $x_1x_2 = x_3$ then, in addition, $|x_2| < 1$;
3. $||\Upsilon(., x)||_{H^\infty} < 1$ and if $x_1x_2 = x_3$ then, in addition, $|x_1| < 1$;
4. $|x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| < 1 - |x_2|^2$;
5. $|x_2 - \bar{x}_1x_3| + |x_1x_2 - x_3| < 1 - |x_1|^2$;
6. $|x_1|^2 + |x_2|^2 + |x_3|^2 + 2|x_2 - \bar{x}_1x_3| < 1$ and if $x_1x_2 = x_3$ then, in addition, $|x_2| < 1$;
7. $|x_1|^2 + |x_2|^2 + |x_3|^2 + 2|x_1 - \bar{x}_2x_3| < 1$ and if $x_1x_2 = x_3$ then, in addition, $|x_1| < 1$;
(5) \(|x_1|^2 + |x_2|^2 - |x_3|^2 + 2|x_1 x_2 - x_3| < 1\) and if \(x_1 x_2 = x_3\) then, in addition, 
\(|x_1| + |x_2| < 2;\)
(6) \(|x_1 - \bar{x}_2 x_3| + |x_2 - \bar{x}_1 x_3| < 1 - |x_3|^2;\)
(7) there exists a 2 \(\times\) 2 matrix \(A = [a_{ij}]\) such that \(|A| < 1\) and \(x = (a_{11}, a_{22}, \text{det} A);\)
(8) there exists a symmetric 2 \(\times\) 2 matrix \(A = [a_{ij}]\) such that \(|A| < 1\) and \(x = (a_{11}, a_{22}, \text{det} A);\)
(9) \(|x_3| < 1\) and there exist \(\beta_1, \beta_2 \in \mathbb{C}\) such that \(|\beta_1| + |\beta_2| < 1\) and 
\[x_1 = \beta_1 + \bar{\beta}_2 x_3, \quad x_2 = \beta_2 + \bar{\beta}_1 x_3.\]

**Proof.** Consider the case that \(x_1 x_2 = x_3\): conditions (1) to (8) (apart from (6)) easily reduce to the pair of statements \(|x_1| < 1, |x_2| < 1\). Hence we may suppose that \(x_1 x_2 \neq x_3\) for proof of the equivalence of statements (1) to (5), (7) and (8). It is clear that \(E\) is symmetric in its first two variables: \((x_1, x_2, x_3) \in E\) if and only if \((x_2, x_1, x_3) \in E\). Hence, if we show that (1) \(\iff\) (2) then it will follow also that (1) \(\iff\) (2') since \(T(x, x_1, x_2)\). We shall prove

\[
\begin{align*}
(1) & \iff (2) \iff (3) \iff (1) \iff (5) \iff (7) \iff (1) \iff (9) \\
& \iff (4) \iff (8) \iff (6)
\end{align*}
\]

and the equivalences \((n')\) follow by symmetry.

(1) \(\iff\) (2) Condition (1) is equivalent to

\[z(x_1 - x_3 w) \neq 1 - x_2 w\]

for all \(z, w \in \Delta\),
that is, \(|x_2| < 1\) and \(1 \notin \Psi(\Delta, x)\) for all \(z \in \Delta\). Hence (1) holds if and only if \(\Psi(\Delta, x)\) does not meet the complement of \(\mathbb{D}\), which is so if and only if (2) holds.

(2) \(\iff\) (3) By equation (1.6),

\[||\Psi(., x)||_{H^\infty} = D(x) = \frac{|x_1 - \bar{x}_2 x_3| + |x_1 x_2 - x_3|}{1 - |x_2|^2},\]

from which the equivalence is immediate.

(2) \(\iff\) (4) By the maximum principle, (2) holds if and only if \(|x_2| < 1\) and

\[|x_3 z - x_1|^2 < |x_2 z - 1|^2\]

for all \(z \in \mathbb{T}\).

On expanding and re-arranging we find that (2) \(\iff\) (4).

(1) \(\iff\) (5) The left hand side of (4') is unchanged if \(x_2, x_3\) are replaced by \(\bar{x}_3, \bar{x}_2\) respectively. Hence \((x_1, x_2, x_3) \in E\) if and only if \((x_1, x_3, \bar{x}_2) \in E\), which, by the equivalence (1) \(\iff\) (4), is so if and only if (5) holds.

The following is a routine calculation.

**Lemma 2.1.** If

\[A = \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix}\]

where \(bc = x_1 x_2 - x_3\) then \(\text{det} A = x_3\),

\[1 - A^* A = \begin{bmatrix} 1 - |x_1|^2 - |c|^2 & -b \bar{x}_1 - c x_2 \\ -b x_1 - c \bar{x}_2 & 1 - |x_2|^2 - |b|^2 \end{bmatrix}\]

and

\[\text{det}(1 - A^* A) = 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - |b|^2 - |c|^2.\]
(5)⇒(8)⇒(7)⇒(5) Suppose (5) holds. Choose (either) \( w \) such that \( w^2 = x_1x_2 - x_3 \) and let \( A = \begin{bmatrix} x_1 & w \\ w & x_2 \end{bmatrix} \). Since (5)⇔(1)⇔(4)⇔(4'), the diagonal entries of \( 1 - A^*A \) are positive (see equation (2.6)), and by equation (2.7)
\[
\det(1 - A^*A) = 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| > 0.
\]
Hence \( \|A\| < 1 \) and so (5)⇒(8).

Trivially (8)⇒(7). Suppose (7) holds. Since
\[
|a_{12}|^2 + |a_{21}|^2 ≥ 2|a_{12}a_{21}| = 2|x_1x_2 - x_3|,
\]
we have
\[
1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| ≥ 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - |a_{12}|^2 - |a_{21}|^2 = \det(1 - A^*A) > 0.
\]
Thus (7)⇒(5).

For the remaining implications we do not assume \( x_1x_2 ≠ x_3 \).

(1)⇒(6)⇒(9)⇒(1) Suppose (1). Then (4) and (4') hold, and on adding these two inequalities we obtain (6).

Now suppose (6). Certainly \( |x_3| < 1 \). Let
\[
(2.8) \quad β_1 = \frac{x_1 - \bar{x}_2x_3}{1 - |x_3|^2}, \quad β_2 = \frac{x_2 - \bar{x}_1x_3}{1 - |x_3|^2}.
\]
Inequality (6) tells us that \( |β_1| + |β_2| < 1 \) and it is immediate that
\[
β_1 + \bar{β}_2x_3 = x_1, \quad β_2 + \bar{β}_1x_3 = x_2.
\]
Hence (9) holds.

Suppose (9). Then \( |x_2| ≤ |β_1| + |β_2| < 1 \) and
\[
|x_1|^2 - |x_2|^2 = (|β_1|^2 - |β_2|^2)(1 - |x_3|^2) ≤ (|β_1| - |β_2|)(1 - |x_3|^2).
\]
Moreover \( x_2 - \bar{x}_1x_3 = β_2(1 - |x_3|^2) \), and so
\[
|x_1|^2 - |x_2|^2 + 2|x_2 - \bar{x}_1x_3| ≤ (|β_1| - |β_2| + 2|β_2|)(1 - |x_3|^2) < 1 - |x_3|^2.
\]
Thus (9)⇒(4)⇒(1).

There are analogous characterizations of \( \mathbb{E} \).

**Theorem 2.2.** For \( x ∈ \mathbb{C}^3 \) the following conditions are equivalent.

(0) \( 1 - x_1z - x_2w + x_3zw ≠ 0 \) for all \( z, w ∈ \mathbb{D} \);
(1) \( x ∈ \mathbb{E} \);
(2) \( \|Ψ(., x)\|_{H^∞} ≤ 1 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_2| ≤ 1 \);
(2') \( \|Υ(., x)\|_{H^∞} ≤ 1 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_1| ≤ 1 \);
(3) \( |x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| ≤ 1 - |x_2|^2 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_1| ≤ 1 \);
(3') \( |x_2 - \bar{x}_1x_3| + |x_1x_2 - x_3| ≤ 1 - |x_1|^2 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_2| ≤ 1 \);
(4) \( |x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_2 - \bar{x}_1x_3| ≤ 1 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_2| ≤ 1 \);
(4') \( |x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_1 - \bar{x}_2x_3| ≤ 1 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_1| ≤ 1 \);
(5) \( |x_1|^2 + |x_2|^2 - |x_3|^2 + 2|x_1x_2 - x_3| ≤ 1 \) and if \( x_1x_2 = x_3 \) then, in addition, \( |x_1| + |x_2| ≤ 2 \).
(6) \(|x_1 - \bar{x}_2x_3| + |x_2 - \bar{x}_1x_3| \leq 1 - |x_3|^2\) and if \(|x_3| = 1\) then, in addition, \(|x_1| \leq 1\);

(7) there exists a 2 \(\times\) 2 matrix \(A = [a_{ij}]\) such that \(||A|| \leq 1\) and \(x = (a_{11}, a_{22}, \det A)\);

(8) there exists a symmetric 2 \(\times\) 2 matrix \(A = [a_{ij}]\) such that \(||A|| \leq 1\) and \(x = (a_{11}, a_{22}, \det A)\);

(9) \(|x_3| \leq 1\) and there exist \(\beta_1, \beta_2 \in \mathbb{C}\) such that \(|\beta_1| + |\beta_2| \leq 1\) and

\[x_1 = \beta_1 + \bar{\beta}_2x_3, \quad x_2 = \beta_2 + \bar{\beta}_1x_3.\]

**Proof.** (0) \(\Rightarrow\) (1) Suppose (0) and consider any \(\zeta, \eta \in \Delta\). Since \((r\zeta, r\eta) \in \mathbb{D}^2\) for any \(r \in (0, 1)\) we have \(1 - x_1r\zeta - x_2r\eta + x_3r^2\zeta\eta \neq 0\). Hence \((rx_1, rx_2, r^2x_3) \in \mathbb{E}\) for \(r \in (0, 1)\), and so \(x \in \overline{\mathbb{E}}\).

(1) \(\Rightarrow\) (0) Suppose \(x \in \overline{\mathbb{E}}\) but \(1 - x_1z - x_2w + x_3zw = 0\) for some \(z, w \in \mathbb{D}\). Then \(z\Psi(w, x) = 1\) and so \(|\Psi(w, x)| > 1\). However, \(|\Psi(w, x)| < 1\) for all \(x \in \mathbb{E}\), and since \(x\) is a limit point of such \(x\) we have \(|\Psi(w, x)| \leq 1\), a contradiction.

Consider the case that \(x_1x_2 = x_3\). Condition (0) reduces to \(1 - x_1z \neq 0, 1 - x_2w \neq 0\) for all \(z, w \in \mathbb{D}\), that is, to

\[(2.9) \quad |x_1| \leq 1, \quad |x_2| \leq 1.\]

An analysis of cases shows that conditions (2) to (5) and (7) and (8) also reduce to this pair of inequalities. In particular, condition (5) becomes

\[|1 - |x_1|| - |x_2|| \geq 0 \quad \text{and} \quad (1 - |x_1|) + (1 - |x_2|) \geq 0,\]

which is equivalent to the relations \((2.8)\). Thus (0) to (5), (7) and (8) are all equivalent in the case that \(x_1x_2 = x_3\).

In the case that \(x_1x_2 \neq x_3\) the proof of equivalence of (0)-(5), (7) and (8) is much as for Theorem 2.1. It remains to prove (1) \(\Rightarrow\) (6) \(\Rightarrow\) (9) \(\Rightarrow\) (1), whether or not \(x_1x_2 = x_3\).

(1) \(\Rightarrow\) (6) We have \(|x_1| \leq 1\), for example from (2'), and on adding the inequalities in (4) and (4') we deduce (6).

(6) \(\Rightarrow\) (9) Suppose (6). Clearly \(|x_3| \leq 1\). If \(|x_3| < 1\) then the proof that (6) \(\Rightarrow\) (9) in Theorem 2.1 still applies. If \(|x_3| = 1\) then \(x_1 = \bar{x}_2x_3, |x_2| = |x_1| \leq 1\) and we find that (9) holds with \(\beta_1 = tx_1, \beta_2 = (1 - t)x_2\) for any \(t \in [0, 1]\).

(9) \(\Rightarrow\) (1) is proved just as in Theorem 2.1 \(\Box\)

**Remark 2.1.** (i) Further criteria for membership of \(E\) and \(\overline{E}\), in terms of the structured singular value, are given in Theorem 9.1 below.

(ii) Note the strange symmetry of \(E\) and \(\overline{E}\):

\[\begin{array}{c}
(x_1, x_2, x_3) \mapsto (x_1, \bar{x}_3, \bar{x}_2)
\end{array}\]

which we used in the proof and which follows from criterion (4').

(iii) In relation to conditions (8) we observe that, for any \(x \in \mathbb{C}^3\), there is either a unique symmetric 2 \(\times\) 2 matrix \(A\) such that \(x = (a_{11}, a_{22}, \det A)\) (when \(x_1x_2 = x_3\)), or precisely two such \(A\), corresponding to the square roots of \(x_1x_2 - x_3\). In the latter case the two \(A\)s are unitarily equivalent, by conjugation by diag(1, -1). Hence we can replace “There exists a ...” by “For every ...” in (8) if we wish.

(iv) Condition (9) of Theorem 2.1 furnishes a foliation of \(E\) by a family of geodesic discs. Indeed, for \(\beta_1, \beta_2\) such that \(|\beta_1| + |\beta_2| < 1\), the map

\[\varphi_{\beta_1, \beta_2} : \mathbb{D} \to E : \lambda \mapsto (\beta_1 + \bar{\beta}_2\lambda, \beta_2 + \bar{\beta}_1\lambda, \lambda)\]
satisfies $\Psi(\omega,.) \circ \varphi_{\beta_1,\beta_2} \in \text{Aut } \mathbb{D}$ for any $\omega \in T$, since $\Psi(\omega,.)$ is analytic from $E$ to $\mathbb{D}$ and

$$\Psi(\omega, \varphi_{\beta_1,\beta_2}(\lambda)) = c \frac{\alpha - \lambda}{1 - \bar{\omega}\lambda}$$

where

$$c = \omega \frac{1 - \bar{\omega}\beta_2}{1 - \bar{\omega}\beta_2} \in T, \quad \alpha = \frac{\bar{\omega}\beta_1}{1 - \bar{\omega}\beta_2} \in \mathbb{D}.$$ 

Since $\varphi_{\beta_1,\beta_2}$ has a left inverse modulo $\text{Aut } \mathbb{D}$ it is a complex geodesic of $E$. Since $\beta_1, \beta_2$ are determined, for any $x \in E$, by equations \(2.8\), each point of $E$ lies on a unique disc $\varphi_{\beta_1,\beta_2}(\mathbb{D})$. However, points of $\partial E$ of the form $(x_1, x_2, x_3)$ with $|x_3| = 1$ lie on infinitely many discs $\varphi_{\beta_1,\beta_2}(\Delta)$ (these are the points of the distinguished boundary of $E$; see Theorem \(2.4\) below).

(v) Here is a geometric interpretation of the parameters $\beta_1, \beta_2$ in conditions \(9\). For $x = (x_1, x_2, x_3) \in \overline{E}$ let $\bar{x} = (x_1, x_3, x_2)$. As we have observed, $\bar{x} \in E$. In view of the formulae \(2.4\) and \(2.8\) we find that the disc $\Psi(\mathbb{D}, \bar{x})$ has centre $\beta_1$ and radius $|\beta_2|$.

Note that any point $(x_1, x_2, x_3)$ of $\overline{E}$ satisfies $x_1x_2 = x_3$ if and only if any matrix representing it as in \(7\) of Theorem \(2.2\) is either upper or lower triangular. This motivates the following definition:

**Definition 2.2.** We say that a point $(x_1, x_2, x_3)$ of $\overline{E}$ is triangular if $x_1x_2 = x_3$.

The characterization theorems show the close relation between $E$ and two standard Cartan domains: the open unit balls $R_I(2, 2), R_{II}(2)$ of the spaces of $2 \times 2$ matrices and symmetric $2 \times 2$ matrices respectively. Denote by $\pi$ the mapping

$$\pi : \mathbb{C}^2 \times 2 \rightarrow \mathbb{C}^3 : A = [a_{ij}] \mapsto (a_{11}, a_{22}, \det A);$$

two of the assertions of Theorem \(2.1\) are that $E$ is the image under $\pi$ of both the Cartan domains $R_I(2, 2)$ and $R_{II}(2)$.

Condition \(2\) of Theorem \(2.1\) shows that we can almost identify $E$ with the space of M"obius transformations that map $\Delta$ to $\mathbb{D}$ via the correspondence $x \mapsto \Psi(., x)$. For non-triangular $x$ (and equivalently non-constant $\Psi(., x)$) this correspondence is bijective, but if $x$ is triangular then $\Psi(., x)$ is the constant function equal to $x_1$, and so the whole disc $\{(x_1, \lambda, x_1\lambda) : \lambda \in \mathbb{D}\} \subset E$ maps to the same constant function $\Psi(., x)$. It is nevertheless often useful to think of $E$ and $\overline{E}$ as sets of M"obius transformations. In particular this viewpoint reveals a natural binary operation on $E$, corresponding to the composition of M"obius transformations. We make this precise in Section \(3\).

We conclude this section with some basic geometric properties of $E$. Firstly, both $E$ and $\overline{E}$ are non-convex: if $x = (1, i, i)$ and $y = (-i, 1, -i)$ then $x, y \in \overline{E}$ but $\frac{1}{2}(x + y) \notin \overline{E}$. However, $\overline{E}$ is contractible by virtue of the following result.

**Theorem 2.3.** $E$ and $\overline{E}$ are starlike about $(0, 0, 0)$ but are not circled.

**Proof.** A straightforward verification shows that, for any $x \in \mathbb{C}^3, z \in \mathbb{C}$ and $r > 0$,

$$|1 - rzx|^2 - |rz| + 1 = r^2 \{1 - ||z||^2 - |r|^2\} + (1 - r)(1 + r - 2r|z|^2).$$

Consider $x \in \overline{E}, z \in \Delta$ and $0 \leq r < 1$. By Theorem \(2.2\) condition \(2\), we have $||\Psi(., x)||_{\infty} \leq 1$ and hence $|1 - rz|^2 - |rz| + 1 \geq 0$. It is also true that $1 - r > 0$ and $1 + r - 2r|z|^2 > 0$. It follows from the identity \(2.11\) that

$$|1 - rzx|^2 - |rz| + 1 > 0,$$
or equivalently \( \Psi(z, rx) \in \mathbb{D} \). Thus \( \Psi(., rx) \) maps \( \Delta \) into \( \mathbb{D} \), and so \( rx \in \mathbb{E} \). Thus \( \mathbb{E} \) and \( \mathbb{E}^\circ \) are starlike about \((0,0,0)\). The point \( x = (1,1,1) \) is in \( \mathbb{E} \) but \( ix \notin \mathbb{E}^\circ \), so that neither \( \mathbb{E}^\circ \) nor \( \mathbb{E} \) is circled. \( \square \)

Although \( \mathbb{E} \) is not convex, \( \mathbb{E} \cap \mathbb{R}^3 \) is.

**Theorem 2.4.** \( \mathbb{E} \cap \mathbb{R}^3 \) is the open tetrahedron with vertices \((1,1,1), (1,-1,-1), (-1,1,-1) \) and \((-1,-1,1)\).

**Proof.** Let \( x \in \mathbb{R}^3, |x_2| < 1 \) and suppose \( x \) non-triangular. The centre of the disc \( \Psi(\Delta, x) \) is real, to wit \( \frac{x_1-x_2x_3}{1-x_2^2} \), and hence the point \( \zeta \) of maximum modulus in \( \Psi(\Delta, x) \) is also real. It follows that \( \Psi(., x)^{-1}(\zeta) \in \mathbb{R} \), and so \( \Psi(., x) \) attains its maximum modulus over \( \Delta \) at either 1 or \(-1\). Hence \( x \in \mathbb{E} \) if and only if

\[
-1 < \Psi(-1, x) < 1 \quad \text{and} \quad -1 < \Psi(1, x) < 1,
\]

that is,

\[
(2.12) \quad -x_1 + x_2 - x_3 + 1 > 0, \quad -x_1 - x_2 + x_3 + 1 > 0
\]

\[
\quad x_1 + x_2 + x_3 + 1 > 0, \quad x_1 - x_2 - x_3 + 1 > 0.
\]

The four half-spaces defined by these inequalities intersect in the open tetrahedron with the four vertices in the statement of the theorem, and so \( x \in \mathbb{E} \) if and only if \( x \) lies in the tetrahedron.

If \( |x_2| \geq 1 \) then \( x \) belongs neither to \( \mathbb{E} \) nor to the tetrahedron. If \( x \) is triangular the inequalities (2.12) reduce to

\[
(1-x_1)(1+x_2) > 0, \quad (1-x_1)(1-x_2) > 0
\]

\[
(1+x_1)(1+x_2) > 0, \quad (1+x_1)(1-x_2) > 0,
\]

which is equivalent to \( |x_1| < 1, |x_2| < 1 \). Thus in all cases, for \( x \in \mathbb{R}^3 \) we have \( x \in \mathbb{E} \) if and only if \( x \) lies in the tetrahedron. \( \square \)

**Theorem 2.5.** \( \mathbb{E}^\circ \) is polynomially convex.

**Proof.** Let \( x \in \mathbb{C}^3 \setminus \mathbb{E}^\circ \). We must find a polynomial \( f \) such that \( |f| \leq 1 \) on \( \mathbb{E}^\circ \) and \( |f(x)| > 1 \). If \( x \) is triangular it suffices to take \( f(x) = x_1 \) or \( f(x) = x_2 \), and if any \( |x_j| > 1 \) we may take \( f(x) = x_j \), so we assume that \( x \) is non-triangular and \( x \in \Delta^3 \). By Theorem 2.2, condition (2), there exists \( z \in \mathbb{D} \) such that \( |\Psi(z, x)| > 1 \), while \( |\Psi(z, .)| \leq 1 \) on \( \mathbb{E} \). Let \( f_N \) be the polynomial given by \( f_N(x) = (x_1 - x_3 z)(1 + x_2 z + x_2^2 z^2 + \cdots + x_2^N z^N) \); then, for any \( y \in \Delta^3 \),

\[
|f_N(y) - \Psi(z, y)| \leq \frac{2|z|^{N+1}}{1-|z|}.
\]

Let \( 0 < \varepsilon < \frac{1}{3}(|\Psi(z, x)| - 1) \) and choose \( N \) so large that \( |f_N - \Psi(z, .)| < \varepsilon \) on \( \Delta^3 \). Then \( |f_N| < 1 + \varepsilon \) on \( \mathbb{E}^\circ \) and \( |f_N(x)| \geq 1 + 2\varepsilon \). Hence we can take \( f = (1+\varepsilon)^{-1}f_N \).

It follows that \( \mathbb{E} \) is a domain of holomorphy (for example [17, Theorem 3.4.2]). However, Theorem 2.4 shows that \( \mathbb{E} \) does not have a \( C^1 \) boundary, and consequently much of the theory of pseudoconvex domains does not apply to \( \mathbb{E} \).
3. A Schwarz lemma for the tetrablock

Criterion (7) of Theorem 2.2 tells us that \( x \in \overline{E} \) if and only if \( x = \pi(A) \) for some contractive \( 2 \times 2 \) matrix \( A \). It follows that any \( 2 \times 2 \) function \( F \) in the Schur class determines an analytic function \( \pi \circ F : \mathbb{D} \to \overline{E} \). The interpolation problem for \( \overline{E} \) can therefore be addressed with the aid of the rich classical interpolation theory of the Schur class: to prove Theorem 1.1 we shall use a refinement of the Schur-Nevanlinna reduction process for which the following result will be useful.

**Lemma 3.1.** Let \( Z \in \mathbb{C}^{2 \times 2} \) be such that \( \|Z\| < 1 \) and let \( 0 \leq \rho < 1 \). Let

\[
M(\rho) = \begin{bmatrix} [(1 - \rho^2 Z^* Z)(1 - Z^* Z)^{-1}]_{11} & [(1 - \rho^2)(1 - ZZ^*)^{-1} Z]_{21} \\ [(1 - \rho^2)Z^*(1 - ZZ^*)^{-1}]_{12} & [ZZ^* - \rho^2](1 - ZZ^*)^{-1} \end{bmatrix}.
\]

(1) There exists \( X \in \mathbb{C}^{2 \times 2} \) such that \( \|X\| \leq \rho \) and \( [M_{-Z}(X)]_{22} = 0 \) if and only if \( \det M(\rho) \leq 0 \).

(2) For any \( 2 \times 2 \) matrix \( X \), \( [M_{-Z}(X)]_{22} = 0 \) if and only if there exists \( \alpha \in \mathbb{C}^2 \setminus \{0\} \) such that

\[
X^* u(\alpha) = v(\alpha)
\]

where

\[
(3.2) \quad u(\alpha) = (1 - ZZ^*)^{-\frac{1}{2}} (\alpha_1 Ze_1 + \alpha_2 e_2), \quad v(\alpha) = -(1 - Z^* Z)^{-\frac{1}{2}} (\alpha_1 e_1 + \alpha_2 Z^* e_2)
\]

and \( e_1, e_2 \) is the standard basis of \( \mathbb{C}^2 \).

(3) In particular, if \( \det M(\rho) \leq 0 \) then an \( X \) such that \( \|X\| \leq \rho \) and \( [M_{-Z}(X)]_{22} = 0 \) is given by

\[
X = \begin{cases} 
\frac{u(\alpha)v(\alpha)^*}{\|u(\alpha)\|^2} & \text{if } [Z]_{22} \neq 0 \\
0 & \text{if } [Z]_{22} = 0
\end{cases}
\]

for any \( \alpha \in \mathbb{C}^2 \setminus \{0\} \) such that \( \langle M(\rho)\alpha, \alpha \rangle \leq 0 \).

**Proof.** We may write

\[
M_{-Z}(X) = (AX + B)(CX + D)^{-1}
\]

where

\[
A = (1 - ZZ^*)^{-\frac{1}{2}}, \quad B = (1 - ZZ^*)^{-\frac{1}{2}} Z, \\
C = (1 - Z^* Z)^{-\frac{1}{2}} Z^*, \quad D = (1 - Z^* Z)^{-\frac{1}{2}}.
\]

With this notation equations (3.2) become

\[
u(\alpha) = \alpha_1 C^* e_1 + \alpha_2 A^* e_2, \quad v(\alpha) = -\alpha_1 D^* e_1 - \alpha_2 B^* e_2.
\]
For any matrix $X$,

$$[M_{-Z}(X)]_{22} =$$

$$\iff \langle (AX + B)(CX + D)^{-1}e_2, e_2 \rangle = 0$$

$$\iff \text{for some non-zero } \xi \in \mathbb{C}^2 \langle (AX + B)\xi, e_2 \rangle = 0 \text{ and } (CX + D)\xi = e_2$$

$$\iff \text{for some non-zero } \xi \in \mathbb{C}^2 \xi \perp (X^*A^* + B^*)e_2 \text{ and } \xi \perp (X^*C^* + D^*)e_1$$

$$\iff \text{there exists } \alpha \in \mathbb{C}^2 \setminus \{0\} \text{ such that}$$

$$\alpha_1(X^*C^* + D^*)e_1 + \alpha_2(X^*A^* + B^*)e_2 = 0$$

$$\iff \text{there exists } \alpha \in \mathbb{C}^2 \setminus \{0\} \text{ such that}$$

$$X^*(\alpha_1C^*e_1 + \alpha_2A^*e_2) = -\alpha_1D^*e_1 - \alpha_2B^*e_2$$

(3.3) $$\iff \text{there exists } \alpha \in \mathbb{C}^2 \setminus \{0\} \text{ such that}$$

$$X^*u(\alpha) = v(\alpha).$$

Hence statement (2) holds. For any $\alpha$ there exists an $X$ such that $X^*u(\alpha) = v(\alpha)$ and $\|X\| \leq \rho$ if and only if $\|v(\alpha)\| \leq \rho\|u(\alpha)\|$. Now

$$\|v(\alpha)\|^2 - \rho^2\|u(\alpha)\|^2 = \langle (DD^* - \rho^2CC^*)e_1, e_1 \rangle \alpha_1\bar{\alpha}_1 + \langle (BD^* - \rho^2AC^*)e_1, e_1 \rangle \alpha_1\bar{\alpha}_2 +$$

$$\langle (DB^* - \rho^2CA^*)e_2, e_2 \rangle \alpha_2\bar{\alpha}_1 + \langle (BB^* - \rho^2AA^*)e_2, e_2 \rangle \alpha_2\bar{\alpha}_2$$

(3.4)$$= \langle M(\rho), \alpha \rangle.$$

Hence there exists an $X$ such that $\|X\| \leq \rho$ and $[M_{-Z}(X)]_{22} = 0$ if and only if $M(\rho)$ is not positive definite, that is, if and only if $\det M(\rho) \leq 0$, since it is easily seen that the $(1,1)$ entry of $M(\rho)$ is positive. Statement (1) follows.

When $\det M(\rho) \leq 0$ we may find $\alpha \neq 0$ such that $\langle M(\rho), \alpha, \alpha \rangle \leq 0$ and define $u(\alpha), v(\alpha)$ by equations (3.2). Then $\|v(\alpha)\| \leq \rho\|u(\alpha)\|$. If $u(\alpha) = 0$ then also $v(\alpha) = 0$ and equation (3.3) holds with $X = 0$ and we have

$$0 = [M(0)]_{22} = [Z]_{22}.$$

If $[Z]_{22} \neq 0$ then $u(\alpha) \neq 0$ and an $X$ satisfying the relations (3.6) and $\|X\| \leq \rho$ is $u(\alpha)v(\alpha)^*\|u(\alpha)\|^{-2}$. \hfill \Box

We denote by $B$ the Blaschke factor

(3.5) $$B(\lambda) = \frac{\lambda_0 - \lambda}{1 - \lambda_0\lambda}.$$

**Lemma 3.2.** Let $\lambda_0 \in \mathbb{D} \setminus \{0\}$, let $Z \in \mathbb{C}^{2 \times 2}$ satisfy $\|Z\| < 1$ and let $M(\cdot)$ be given by equation (3.1).

1. There exists a function $G$ such that

(3.6) $$G \in S_{2 \times 2}, \quad [G(0)]_{22} = 0 \quad \text{and} \quad G(\lambda_0) = Z$$

if and only if $\det M(\lambda_0) \leq 0$.

2. A function $G \in S_{2 \times 2}$ satisfies the conditions (3.6) if and only if there exists $\alpha \in \mathbb{C}^2 \setminus \{0\}$ such that $\langle M(\lambda_0), \alpha, \alpha \rangle \leq 0$ and a Schur function $Q$ such that $Q(0)^*\lambda_0(\alpha) = v(\alpha)$ and $G = M_{-Z} \circ (BQ)$, where $u(\alpha), v(\alpha)$ are given by equations (3.2).

3. In particular, if $[Z]_{22} \neq 0$ and $\alpha \in \mathbb{C}^2 \setminus \{0\}$ satisfies $\langle M(\lambda_0), \alpha, \alpha \rangle \leq 0$ then the function

(3.7) $$G(\lambda) = M_{-Z} \left( \frac{B(\lambda)u(\alpha)v(\alpha)^*}{\lambda_0\|u(\alpha)\|^2} \right)$$

satisfies the conditions (3.6).
Proof. (2) If $G$ satisfies the conditions (3.6), then $M_Z \circ G \in S_{2 \times 2}$ vanishes at $\lambda_0$ and hence is of the form $BQ$ for some Schur function $Q$. Then $G = M_Z \circ (BQ)$ and moreover

$$[M_Z(\lambda_0Q(0))]_{22} = [M_Z \circ (BQ)(0)]_{22} = [G(0)]_{22} = 0.$$ 

Since $||\lambda_0Q(0)|| \leq |\lambda_0|$ Lemma 3.1 tells us that there exists $\alpha \neq 0$ such that $\langle M(\lambda_0)|\alpha, \alpha \rangle \leq 0$ and $(\lambda_0Q(0))^*u(\alpha) = v(\alpha)$. Thus necessity holds in statement (2). The argument is reversible, and so (2) holds.

(3) is also an easy consequence of (2), obtained by taking $Q$ to be the constant function whose value is the unique rank 1 matrix satisfying $Q^*\bar{\lambda}_0u(\alpha) = v(\alpha)$ (as in Lemma 3.1(3)).

$k.n.$

Remark 3.1. If $[Z]_{22} = 0$ then the constant function $G(\lambda) = Z$ has the desired properties.

Lemma 3.3. Let $\varphi : D \to \overline{E}$ be analytic. If $\varphi$ maps some point of $D$ into $E$ then $\varphi(D) \subset E$.

Proof. Suppose that $\varphi(\lambda_0) \in E$ for some $\lambda_0 \in D$. Since $\varphi_2 : D \to \Delta$ is analytic and $|\varphi_2(\lambda_0)| < 1$ it follows from the Schwarz-Pick Lemma that $\varphi_2(D) \subset \Delta$. Fix $z \in \Delta$. The function $\lambda \mapsto \Psi(z, \varphi(\lambda))$ is well defined and analytic on $D$ and maps $\lambda_0$ into $\Delta$; hence it maps all of $D$ into $\Delta$. Now fix $\lambda \in D$: the map $\Psi(z, \varphi(\lambda))$ maps $\Delta$ to $D$, and hence, by Theorem 2.1, $\varphi(\lambda) \in \overline{E}$.

We now prove the Schwarz Lemma for $E$, the main result of the paper.

Proof of Theorem 1.1. It is clear from Lemma 3.3 that (1)$\iff$(1').

(1)$\Rightarrow$(2) Let $\varphi : D \to \overline{E}$ be as in (1). For any $\omega \in T$, $\Psi(\omega, \varphi(z))$ is an analytic self-map of $D$ and

$$\Psi(\omega, \varphi(0)) = \Psi(\omega, 0, 0, 0) = 0.$$ 

By Schwarz’ Lemma

$$|\Psi(\omega, x)| = |\Psi(\omega, \varphi(\lambda_0))| \leq |\lambda_0|.$$ 

On taking the supremum over $\omega \in T$ we find that $D(x) \leq |\lambda_0|$, that is,

$$\frac{|a - bp| + |ab - p|}{1 - |b|^2} \leq |\lambda_0|.$$ 

By the same reasoning with $a, b$ interchanged we have

$$\frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \leq |\lambda_0|,$$

and so (2) holds.

(2)$\Rightarrow$(3) is trivial.

(4)$\Rightarrow$(1) If $F = [F_{ij}]$ is as in (4) then the function

$$\varphi = (F_{11}, F_{22}, \det F)$$

is analytic in $D$, satisfies

$$\varphi(0) = (0, 0, 0), \quad \varphi(\lambda_0) = x.$$
Lemma 3.4. Let $a, b, p \in \mathbb{D}$ and $D(x) = |a|$. Consider the case that $ab = p$. Here $D(x) = |a|$, and so

$$|b| \leq |a| \leq |\lambda_0|.$$  

By Schwarz’ Lemma there are analytic self-maps $f, g$ of $\mathbb{D}$ such that $f(0) = g(0) = 0$, $f(\lambda_0) = a$ and $g(\lambda_0) = b$. The function $F = \text{diag}(f, g)$ then has the required properties, and so (3) $\Rightarrow$ (4) when $ab = p$.

Now consider the case that $ab \neq p$. We shall construct $F \in S_{2 \times 2}$ such that

$$F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \quad F(\lambda_0) = \begin{bmatrix} a \\ \lambda_0 w \\ w \\ b \end{bmatrix},$$

where $w$ is a square root of $(ab - p)/\lambda_0$. It suffices to find $G \in S_{2 \times 2}$ such that conditions (3.6) above hold for

$$Z = \begin{bmatrix} a/\lambda_0 & w \\ w & b \end{bmatrix},$$

for then the function $F(\lambda) = G(\lambda)\text{diag}(\lambda, 1)$ has the required properties. To obtain such a $G$ we shall invoke Lemma 3.2, which we can do provided that $||Z|| < 1$.

Let

$$a' = a/\lambda_0, \quad b' = b/\lambda_0, \quad p' = p/\lambda_0.$$  

Since $D(x) \leq |\lambda_0|$ we have

$$D(a', b, p') \leq 1 \text{ and } D(b', a, p') \leq 1,$$

so that $(a', b, p'), (a, b', p') \in \overline{E}$. Hence

$$|b| \leq |a| \leq |\lambda_0|, \quad |p| \leq |\lambda_0| < 1.$$  

It follows that $a \neq \bar{b}p$ and

$$|ab - p| < |a - \bar{b}p| + |ab - p| \leq |\lambda_0|(1 - |b|^2).$$

Moreover, since $(a, b', p') \in \overline{E}$, condition (5) of Theorem 2.2 shows that

$$1 - |a|^2 - |b'|^2 + |p'|^2 \geq \frac{2}{|\lambda_0|}|ab - p|.$$  

That is,

$$2 \leq Y_1 \overset{\text{def}}{=} \frac{|\lambda_0|}{|ab - p|} \left(1 - |a|^2 - \frac{|b|^2}{|\lambda_0|^2} + \frac{|p|^2}{|\lambda_0|^2}\right),$$

with strict inequality if and only if $(a, b', p') \in \mathbb{E}$, that is, if and only if $D(b, a, p) < |\lambda_0|$. Likewise

$$2 \leq Y_2 \overset{\text{def}}{=} \frac{|\lambda_0|}{|ab - p|} \left(1 - \frac{|a|^2}{|\lambda_0|^2} - |b|^2 + \frac{|p|^2}{|\lambda_0|^2}\right),$$

with strict inequality if and only if $D(a, b, p) < |\lambda_0|$.  

Lemma 3.4. Let $x = (a, b, p) \in \mathbb{E}, \lambda_0 \in \mathbb{D} \setminus \{0\}$ and $w^2 = (ab - p)/\lambda_0$. Suppose that $|b| \leq |a|$, $D(x) \leq |\lambda_0|$ and $ab \neq p$. Let $Z$ be defined by equation (3.5). Then $||Z|| \leq 1$, with equality if and only if $D(x) = |\lambda_0|$.  

Proof. We have

\[ 1 - Z^* Z = \begin{bmatrix} 1 - \frac{a}{\lambda_0}^2 - |w|^2 & -\frac{aw}{\lambda_0} - bw \\ -\frac{aw}{\lambda_0} - bw & 1 - |b|^2 - |w|^2 \end{bmatrix}, \]

\[ \det(1 - Z^* Z) = 1 - \left| \frac{a}{\lambda_0} \right|^2 - |b|^2 + \left| \frac{p}{\lambda_0} \right|^2 - 2\left| ab - p \right|/|\lambda_0| \]

(see equation \[(\text{3.10})\] above). Since \((a', b, p'), (a, b', p') \in \bar{\mathbb{E}}\), conditions \((3')\) and \((3)\) respectively of Theorem \(2.2\) show that the diagonal entries of \(1 - Z^* Z\) are non-negative, while condition \((5)\) of the same theorem shows that \(\det(1 - Z^* Z) \geq 0\). Hence \(1 - Z^* Z \geq 0\). By the corresponding conditions in Theorem \(2.1\) the diagonal entries and determinant are all strictly positive if and only if \((a', b, p'), (a, b', p') \in \mathbb{E}\), which is so if and only if \(D(x) < |\lambda_0|\). \(\square\)

To apply Lemma \(3.2\) we need to know the sign of \(\det M(|\lambda_0|)\). A routine (if laborious) calculation gives the following.

Lemma 3.5. Under the assumptions of Lemma \(3.4\) if \(D(x) < |\lambda_0|\) and \(M(|\lambda_0|)\) is defined by equation \(\text{(3.1)}\) then

\[ (3.12) \quad M(|\lambda_0|) \det(1 - Z^* Z) = \]

\[ \begin{bmatrix} 1 - |a|^2 - |b|^2 + |p|^2 - |ab - p| \left( \left| \lambda_0 \right| + \frac{1}{|\lambda_0|} \right) \\
(1 - |\lambda_0|^2) \left( \bar{w} + \frac{\bar{w}p}{\lambda_0} \right) & -|\lambda_0|^2 + |a|^2 + |b|^2 - \left| \frac{p}{\lambda_0} \right|^2 + |ab - p| \left( \left| \lambda_0 \right| + \frac{1}{|\lambda_0|} \right) \end{bmatrix} \]

and

\[ (3.13) \quad \det \left( M(|\lambda_0|) \det(1 - Z^* Z) \right) = -(y - y_1)(y - y_2) \]

where

\[ y = 2|ab - p|, \]

\[ y_1 = \left| \lambda_0 \right| \left( 1 - |a|^2 - \frac{b}{\lambda_0} \right)^2 + \left| \frac{p}{\lambda_0} \right|^2 = |ab - p|Y_1, \]

\[ y_2 = \left| \lambda_0 \right| \left( 1 - \frac{a}{\lambda_0} \right)^2 - |b|^2 + \left| \frac{p}{\lambda_0} \right|^2 = |ab - p|Y_2. \]

We resume the proof that \((3) \Rightarrow (4)\) when \(ab \neq p\) and \(|b| \leq |a|\). Suppose first that \(D(x) < |\lambda_0|\). By Lemmas \(3.4\) and \(3.5\) we have \(\|Z\| < 1\) and

\[ \det(M(|\lambda_0|) \det(1 - Z^* Z)) = -|ab - p|^2(2 - Y_1)(2 - Y_2) < 0, \]

since \(Y_1, Y_2 > 2\) by inequalities \(3.10\) and \(3.11\). Since \(\det(1 - Z^* Z) > 0\) it follows that \(\det M(|\lambda_0|) < 0\). By Lemma \(3.2\) there exists \(G \in S_{2 \times 2}\) such that \([G(0)]_{22} = 0\) and \(G(\lambda_0) = Z\). Let

\[ F(\lambda) = G(\lambda) \text{diag}(\lambda, 1), \quad \lambda \in \mathbb{D}. \]

Then \(F \in S_{2 \times 2}\),

\[ (3.14) \quad F(0) = \begin{bmatrix} 0 & \ast \\
0 & 0 \end{bmatrix} \text{ and } F(\lambda_0) = \begin{bmatrix} a & w \\
\lambda_0 w & b \end{bmatrix}. \]
where \( w^2 = (ab - p)/\lambda_0 \). Since \( \det F(\lambda_0) = p \) we have (3)\( \Rightarrow \) (4) in the case that \( |b| \leq |a| \) and \( D(x) < |\lambda_0| \). Similarly it holds if \( |a| \leq |b| \) and \( D(x) < |\lambda_0| \).

Now suppose that \( D(x) = |\lambda_0| \). Write \( \lambda = \lambda_0(1 + \varepsilon)^2 \) for \( \varepsilon > 0 \) so small that \( |\lambda| < 1 \). Note that

\[
\left( \frac{w}{1 + \varepsilon} \right)^2 = \frac{ab - p}{\lambda_0}.
\]

By the above reasoning there exists \( F_\varepsilon \in S_{2 \times 2} \) such that

\[
F_\varepsilon(0) = \begin{bmatrix} 0 & \ast \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F_\varepsilon(\lambda_0) = \begin{bmatrix} a/ (1 + \varepsilon) \lambda_0 w & \frac{w}{1 + \varepsilon} \\ \frac{2}{\lambda_0} \end{bmatrix}.
\]

By Montel’s theorem some subsequence of \( F_\varepsilon \) converges uniformly on compact subsets of \( \mathbb{D} \) as \( \varepsilon \to 0 \) to an analytic function \( F \). Clearly \( F \) is in the Schur class and satisfies equation (3.14). Hence (3) \( \Rightarrow \) (4).

**Corollary 1.** For any \( x = (a, b, p) \in \mathbb{E} \)

\[
C_\mathbb{E}(0, x) = \kappa_\mathbb{E}(0, x) = \delta_\mathbb{E}(0, x)
\]

\[
= \max \left\{ \tanh^{-1} \left[ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \right], \tanh^{-1} \left[ \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right] \right\}
\]

where \( C_\mathbb{E}, \kappa_\mathbb{E} \) and \( \delta_\mathbb{E} \) are the Carathéodory distance, Kobayashi distance and Lempert functions of \( \mathbb{E} \) respectively.

For definitions of \( C_\mathbb{E}, \kappa_\mathbb{E} \) and \( \delta_\mathbb{E} \) see for example [14, Chapter 1].

**Proof.** The equation

\[
\delta_\mathbb{E}(0, x) = \max \left\{ \tanh^{-1} \left[ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \right], \tanh^{-1} \left[ \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right] \right\}
\]

is simply a re-statement of the equivalence (1) \( \Leftrightarrow \) (2) of Theorem [11]. By definition

\[
C_\mathbb{E}(0, x) = \sup_{F \in \mathbb{E}} \tanh^{-1} |F(x)|
\]

over all analytic maps \( F : \mathbb{E} \to \mathbb{D} \) such that \( F(0) = 0 \). On taking \( F = \Psi(\omega, \cdot), \omega \in \mathbb{T} \), we find

\[
C_\mathbb{E}(0, x) \geq \sup_{\omega \in \mathbb{T}} \tanh^{-1} |\Psi(\omega, x)| = \tanh^{-1} \left[ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \right]
\]

and by symmetry

\[
C_\mathbb{E}(0, x) \geq \max \left( \tanh^{-1} \left[ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} \right], \tanh^{-1} \left[ \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right] \right) = \delta_\mathbb{E}(0, x).
\]

It is always true that

\[
C_\mathbb{E} \leq \kappa_\mathbb{E} \leq \delta_\mathbb{E};
\]

the Corollary follows.

**Corollary 2.** If \( (a, b, p) \in \mathbb{E} \) and \( |b| \leq |a| \) then

\[
\frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \leq \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}.
\]
A SCHWARZ LEMMA

Proof. This follows from the implication (3)⇒(2) of Theorem 1.1 with \( \lambda_0 = D(a, b, p) \).

The proof of Theorem 1.1 not only demonstrates the existence of an interpolating function \( \varphi : \mathbb{D} \to \mathbb{E} \) when \( D(x) \leq |\lambda_0| \) but also shows us how to construct a suitable \( \varphi \).

Algorithm

Let \( x = (a, b, p) \in \mathbb{E} \), let \( \lambda_0 \in \mathbb{D} \setminus \{0\} \) and suppose that \( |b| \leq |a| \) and

\[
\frac{|a - bp| + |ab - p|}{1 - |b|^2} < |\lambda_0|.
\]

An analytic function \( \varphi : \mathbb{D} \to \mathbb{E} \) such that \( \varphi(0) = (0, 0, 0) \) and \( \varphi(\lambda_0) = x \) can be found as follows.

1. If \( b = 0 \) then let \( \varphi(\lambda) = \lambda x/\lambda_0 \). Otherwise:
2. Choose \( w \) such that \( w^2 = (ab - p)/\lambda_0 \) and let
   \[
   Z = \begin{bmatrix}
   a/\lambda_0 & w \\
   w & b
   \end{bmatrix};
   \]
   then \( ||Z|| < 1 \).
3. Let \( M(.) \) be defined by equation (3.1) and choose \( \alpha \in \mathbb{C}^2 \setminus \{0\} \) such that \( \langle M(\lambda_0|\alpha, \alpha \rangle \leq 0 \) (such an \( \alpha \) exists by Lemmas 3.4 and 3.5).
4. Let vectors \( u, v \in \mathbb{C}^2 \) and the Blaschke factor \( B \) be given by equations (3.2) and (3.5); note that by Lemma 3.1, \( u \neq 0 \) since \( |Z|_2 < b \).
5. Let \( F = [F_{ij}] \) be defined by
   \[
   F(\lambda) = M_{-Z} \left( \frac{B(\lambda)uv^*}{||u||^2} \right) \text{diag}(\lambda, 1);
   \]
   then \( F \in S_{2 \times 2} \).
6. Let \( \varphi = (F_{11}, F_{22}, \det F) \). Then \( \varphi \) is analytic, maps \( \mathbb{D} \) to \( \mathbb{E} \) and satisfies \( \varphi(0) = (0, 0, 0), \varphi(\lambda_0) = x \).

Remark 3.2. (i) When \( b = 0 \) we have

\[
D(\lambda x/\lambda_0) = \left| \frac{\lambda}{\lambda_0} \right| D(x) \leq |\lambda|
\]

and so the simple recipe in Step 1 of the algorithm does indeed produce a mapping \( \varphi \) such that \( \varphi(\mathbb{D}) \subset \mathbb{E} \). In general, though, this recipe is insufficient. Consider for example the point \( x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \mathbb{E} \); here \( D(x) = \frac{2}{3} \). Take \( \lambda_0 \) slightly greater than \( \frac{2}{3} \). One can easily check that \( D(\lambda x/\lambda_0) > 1 \) for real \( r \) close to 1. Hence \( \lambda \mapsto \lambda x/\lambda_0 \) does not map \( \mathbb{D} \) into \( \mathbb{E} \).

(ii) In the case that \( D(x) = \lambda_0 \) a modification of our construction will produce an interpolating function \( \varphi \); one uses the singular value decomposition of \( Z \). The construction is similar to the one in the next section; the details are left to the reader.

4. NON-UNIQUENESS IN THE SCHWARZ LEMMA

In contrast to Schwarz’ original Lemma, there is no uniqueness statement in the case that the necessary and sufficient condition (2) of Theorem 1.1 holds with equality. Here is a numerical example.
Let \( x = (a, b, p) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2}) \). We have, since \(|b| \leq |a|\),
\[
\max \left\{ \left| a - \bar{b}p \right| + \left| ab - p \right|, \left| b - \bar{a}p \right| + \left| ab - p \right| \right\} = \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2} = \frac{3}{5}.
\]

Let \( \lambda_0 = -\frac{4}{5} \). We shall construct infinitely many analytic \( \varphi : \mathbb{D} \to \mathbb{E} \) such that \( \varphi(0) = (0, 0, 0) \) and \( \varphi(\lambda_0) = x \).

Let \( w^2 = \frac{5}{\lambda_0 - 4} = \frac{15}{32} \) and
\[
Z = \begin{bmatrix} a/\lambda_0 & w & b \\ w & 1/4 & \\ b & & 1 \end{bmatrix}.
\]

Then \(|Z| = 1\) and since \( Z \) is Hermitian we may diagonalise \( Z \) as follows:
\[
Z = \begin{bmatrix} -\frac{5}{8} & w & 0 \\ w & 1 & 0 \\ b & 0 & 1 \end{bmatrix} = U^* \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{3}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix} U,
\]
where \( U \) is the unitary matrix
\[
U = \begin{bmatrix} 8w & 4w & -3 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{39}} \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{65} \end{bmatrix}.
\]

If \( G \) is a Schur function such that \( G(\lambda_0) = Z \) then \( U^*GU \) is a Schur function whose value at \( \lambda_0 \) is \( \text{diag}(-1, \frac{3}{8}) \), from which it is clear that \( U^*GU = \text{diag}(-1, g) \) for some scalar function \( g \) in the Schur class satisfying \( g(\lambda_0) = \frac{3}{8} \). We then have
\[
|G(0)|_{22} = \frac{1}{13}(10g(0) - 3).
\]

It follows that the set of functions \( G \) in the Schur class such that \( G(\lambda_0) = Z \), \( |G(0)|_{22} = 0 \) consists precisely of the functions \( U \text{diag}(-1, g)U^* \) where \( g \) is a function in the Schur class such that \( g(0) = 3/10 \) and \( g(-4/5) = 5/8 \). There are infinitely many such \( g \), since the pseudohyperbolic distance \( d\left(\frac{3}{10}, \frac{5}{8}\right) = \frac{2}{5} < \frac{4}{5} = |\lambda_0| \).

For any such \( g \) we define \( \varphi : \mathbb{D} \to \mathbb{E} \) by \( \varphi = (F_{11}, F_{22}, \text{det} F) \) where \( F(\lambda) = U \text{diag}(-1, g)U^* \text{diag}(\lambda, 1) \). Note that \( \varphi_3(\lambda) = -\lambda g(\lambda) \), so that distinct \( g \) give rise to different mappings \( \varphi \), all analytic and satisfying \( \varphi(0) = (0, 0, 0), \varphi(\lambda_0) = x \).

5. All Interpolating Functions

The algorithm in Section 3 produces a single analytic function \( \varphi : \mathbb{D} \to \mathbb{E} \) satisfying a pair of interpolation conditions. Our method of proof in fact gives more: a description of all such functions. In an engineering context one could use the freedom in the solution to meet further performance specifications.

**Theorem 5.1.** Let \( x = (a, b, p) \in \mathbb{E} \) and \( \lambda_0 \in \mathbb{D} \setminus \{0\} \) and suppose that \( ab \neq p, |b| \leq |a| \) and \( D(x) < |\lambda_0| \). The set \( \mathcal{I} \) of analytic functions \( \varphi : \mathbb{D} \to \mathbb{E} \) such that \( \varphi(0) = (0, 0, 0) \) and \( \varphi(\lambda_0) = x \) can be described as follows.

Let \( w^2 = (ab - p)/\lambda_0 \) and let \( \xi_1, \xi_2 \) be the roots of the equation \( \xi + 1/\xi = Y_2 \) where \( Y_2 \) is defined by equation \( 3.17 \).

For any \( \sigma > 0 \) let
\[
Z(\sigma) = \begin{bmatrix} a/\lambda_0 & \sigma w \\ \sigma^{-1} w & b \end{bmatrix}
\]
and let \( M(\cdot) \) be defined by equation \( 3.24 \) with \( Z = Z(\sigma) \). For any \( \sigma \) such that
\[
\xi_1 < \sigma^2 < \xi_2
\]

(5.1)
we have \( ||Z(\sigma)|| < 1 \) and \( M(|\lambda_0|) \) is not positive definite. Furthermore, for any \( \alpha \in \mathbb{C}_2 \setminus \{0\} \) such that
\[ (5.2) \quad \langle M(|\lambda_0|)\alpha, \alpha \rangle \leq 0 \]
and any \( 2 \times 2 \) function \( Q \) in the Schur class such that
\[ (5.3) \quad Q(0)^* \lambda_0 u(\alpha) = v(\alpha), \]
where \( u(\alpha), v(\alpha) \) are given by equations (3.2), the function \( \pi \circ F \) belongs to \( \mathcal{I} \) where
\[ (5.4) \quad F(\lambda) = M_{-Z(\sigma)} \circ (BQ)(\lambda) \text{diag}(\lambda, 1). \]
Conversely, every function in \( \mathcal{I} \) is of the form \( \pi \circ F \) for some choice of \( \sigma, \alpha \) and \( Q \) satisfying the conditions (5.1), (5.2) and (5.3) respectively and for \( F \) given by equation (5.4).

**Proof.** A slight modification of the proof of Lemma 3.4 shows that \( ||Z(\sigma)|| < 1 \) if and only if
\[ 1 - |b|^2 - \sigma^2 \left| \frac{ab - p}{\lambda_0} \right| > 0 \]
and 1 - \( \left| \frac{a}{\lambda_0} \right|^2 - |b|^2 + \left| \frac{p}{\lambda_0} \right|^2 - \left| \frac{ab - p}{\lambda_0} \right| (\sigma^2 + \frac{1}{\sigma^2}) > 0, \]
that is, if and only if
\[ \sigma^2 < K \] and \( \sigma^2 + \frac{1}{\sigma^2} < Y_2 \]
where
\[ (5.5) \quad K \overset{\text{def}}{=} \frac{|\lambda_0|(1 - |b|^2)}{|ab - p|}. \]
By the inequalities (5.9) and (5.11),
\[ (5.6) \quad K > 1 \] and \( Y_2 > 2. \]
Moreover
\[ (5.7) \quad K + \frac{1}{K} - Y_2 = \frac{|a - \bar{b}p|^2}{|\lambda_0|(1 - |b|^2)|ab - p|} > 0. \]
Figure 1 is a plot of \( \xi + 1/\xi \) against \( \xi \) that incorporates the relations (5.6) and (5.7). It is clear from the plot that \( \xi < K \) and \( \xi + 1/\xi < Y_2 \) precisely when \( \xi_1 < \xi < \xi_2 \), or equivalently, when \( \xi + 1/\xi < Y_2 \), the inequality \( \xi < K \) then being automatically satisfied. It follows that \( ||Z(\sigma)|| < 1 \) if and only if \( \xi_1 < \sigma^2 < \xi_2 \).
We claim that, for the same range of values of \( \sigma \), \( \det M(\lambda_0) < 0 \). Indeed, a straightforward calculation gives

\[
\det(M(\lambda_0)) \det(1 - Z(\sigma)^*Z(\sigma)) = -(y - y_1)(y - y_2)
\]

where

\[
y = |ab - p|\sigma^2 + \frac{1}{\sigma^2}, \quad y_j = |ab - p|Y_j, \quad j = 1, 2.
\]

Since

\[
Y_1 - Y_2 = \frac{1 - |\lambda|^2}{|(ab - p)\lambda|}(|a|^2 - |b|^2) \geq 0,
\]

it is clear that \( \det M(\lambda_0) < 0 \) when \( y < y_2 \), or equivalently, when \( \sigma^2 + \frac{1}{\sigma^2} < Y_2 \), which is to say, when \( \xi_1 < \sigma^2 < \xi_2 \). Thus \( ||Z(\sigma)|| < 1 \) and \( M(\lambda_0) \) is not positive definite when condition (5.1) is satisfied.

Suppose that \( \sigma, \alpha \) and \( Q \) satisfy conditions (5.1), (5.2) and (5.3). By Lemma 8.2, the function \( G = M_{Z(\sigma)} \circ (BQ) \) belongs to \( S_{2 \times 2} \) and satisfies \( |G(0)| = 0 \) and \( G(\lambda_0) = Z(\sigma) \). Hence \( F \) given by equation (5.4) satisfies

\[
F \in S_{2 \times 2}, \quad F(0) = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F(\lambda_0) = \begin{bmatrix} a & \sigma w \\ \lambda_0 \sigma^{-1} w & b \end{bmatrix}.
\]

Thus the function \( \varphi = \pi \circ F \) is analytic from \( \mathbb{D} \) to \( \mathbb{E} \) and satisfies \( \varphi(0) = (0, 0, 0) \) and \( \varphi(\lambda_0) = (a, b, p) = x \). Thus \( \varphi \in \mathcal{I} \).

Conversely, suppose that \( \varphi \in \mathcal{I} \). The radial limit function of \( \varphi \), which we shall again denote by \( \varphi \), maps \( \mathbb{T} \) almost everywhere to \( \mathbb{E} \). By a theorem of F. Riesz, or directly from inner-outer factorization, there exist \( f, g \in H^\infty \) such that \( fg = \varphi_1 \varphi_2 - \varphi_3 \) and \( |f| = |g| \) a.e. on \( \mathbb{T} \). Since \( fg(0) = 0 \) we can assume that \( g(0) = 0 \). Let

\[
(5.8) \quad F = \begin{bmatrix} \varphi_1 & f \\ g & \varphi_2 \end{bmatrix}.
\]

We have \( \pi \circ F = \varphi \). By Lemma 2.1, \( 1 - F^*F \) has diagonal entries \( 1 - |\varphi_1|^2 - |\varphi_1 \varphi_2 - \varphi_3|^2 \) and \( 1 - |\varphi_2|^2 - |\varphi_1 \varphi_2 - \varphi_3|^2 \) and determinant \( 1 - |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 - 2|\varphi_1 \varphi_2 - \varphi_3|^2 \) a.e. on \( \mathbb{T} \), and since these three functions are non-negative by Theorem 2.2 it follows that \( F \) is in the Schur class. From the facts that \( ab \neq p \), \( \pi \circ F(\lambda_0) = (a, b, p) \) and

\[
F(0) = \begin{bmatrix} 0 & f(0) \\ 0 & 0 \end{bmatrix}
\]

one sees that \( F \) is non-constant and hence \( |f(0)| < 1 \). Thus \( ||F(0)|| < 1 \) and so \( F \in S_{2 \times 2} \). We shall show that \( F \) can be written in the form (5.4) for some choice of \( \sigma, \alpha \) and \( Q \).

Note that \( (fg)(\lambda_0) = ab - p \neq 0 \), so that \( f(\lambda_0), g(\lambda_0) \) are nonzero. Let \( \sigma = f(\lambda_0)/w \); then \( g(\lambda_0) = \lambda_0 \sigma^{-1} w \). We can suppose that \( \sigma > 0 \) (if necessary replace \( F \) by \( U^* Fu \) for some constant diagonal unitary \( U \)). Thus

\[
F(\lambda_0) = \begin{bmatrix} a & \sigma w \\ \lambda_0 \sigma^{-1} w & b \end{bmatrix}.
\]
Since the first column of $F(0)$ is zero we may write $F(\lambda) = G(\lambda)\text{diag}(\lambda, 1)$ for some $G \in S_{2 \times 2}$. We have
\[ G(0) = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \quad G(\lambda_0) = \begin{bmatrix} a/\lambda_0 & \sigma w \\ \sigma^{-1} w & b \end{bmatrix} = Z(\sigma). \]
Since $||G(\lambda_0)|| < 1$ it follows that condition (5.1) holds and thence that $M(\lambda_0)$ is not positive definite. By Lemma 3.2(2) there exist $\alpha, Q$ such that conditions (5.2), (5.3) hold and $F$ is given by equation (5.4).

**Remark 5.1.** The $2 \times 2$ functions $Q$ in the Schur class that satisfy condition (5.3) can easily be parametrised by standard Nevanlinna-Pick theory (see for example [6, Theorem 18.5.2 and Example 18.5.2]).

### 6. Automorphisms of the tetrablock

In this section we shall use “composition” on $\mathbb{E}$ to describe a large group of automorphisms of $\mathbb{E}$.

Let $x, y \in \mathbb{E}$. A simple calculation shows that
\[ \Psi(., x) \circ \Psi(., y) = \Psi(., x \circ y) \]
where
\[ x \circ y = \frac{1}{1 - x_2 y_1} \begin{pmatrix} x_1 - x_3 y_1, y_2 - x_2 y_3, x_1 y_2 - x_3 y_3 \end{pmatrix} \]
(6.1)
\[ = \begin{pmatrix} \Psi(y_1, x), \Psi(x_2, y), \frac{x_1 y_2 - x_3 y_1}{1 - x_2 y_1} \end{pmatrix}. \]

Note that $1 - x_2 y_1 \neq 0$ since $|x_2| < 1, |y_1| < 1$, and hence $x \circ y$ is defined. We shall define $x \circ y$ by equation (6.1) for any $x, y \in \mathbb{C}^3$ such that $x_2 y_1 \neq 1$. For $x, y \in \mathbb{E}$, $x \circ y$ can fail to be defined, but if it is defined then $\Psi(., x \circ y)$ is a self map of $\Delta$ and so $x \circ y \in \mathbb{E}$ (in the triangular case we do have $|\Psi(y_1, x)| \leq 1, |\Psi(x_2, y)| \leq 1$ by virtue of Theorem 2.2 conditions (2) and (2')).

We can think of $\circ$ as a disguised form of matrix multiplication. For $x \in \mathbb{E}$ let
\[ M_x \overset{\text{def}}{=} \begin{bmatrix} x_3 & -x_1 \\ x_2 & -1 \end{bmatrix}. \]

In the customary association of Möbius transformations with $2 \times 2$ matrices $\Psi(., x)$ corresponds to the non-zero multiples of $M_x$ and so $\Psi(., x \circ y)$ corresponds to
\[ M_{x \circ y} = \lambda \begin{bmatrix} x_3 & -x_1 \\ x_2 & -1 \end{bmatrix} \begin{bmatrix} y_3 & -y_1 \\ y_2 & -1 \end{bmatrix} = \lambda \begin{bmatrix} x_3 y_3 - x_1 y_2 & -x_3 y_1 + x_1 \\ x_2 y_3 - y_2 & -x_2 y_1 + 1 \end{bmatrix} \]
where $\lambda$ is chosen to make the $(1, 1)$ entry of the product equal to $-1$.

It follows from the associativity of matrix multiplication that $\circ$ is associative: for $u, v, w \in \mathbb{E}$, $(u \circ v) \circ w = u \circ (v \circ w)$ provided both sides are defined, for both sides have representing matrices proportional to $M_u M_v M_w$.

We define left and right actions of $\text{Aut} \mathbb{D}$ on both $\mathbb{E}$ and $\mathbb{E}^\sharp$. Let us write
\[ \mathbb{E}^\sharp \overset{\text{def}}{=} \{ x \in \mathbb{E} : |x_1| < 1, |x_2| < 1 \}, \]
so that $\mathbb{E} \subset \mathbb{E}^\sharp \subset \mathbb{E}$. It is clear from equations (6.1) that $x \circ y$ is defined if $x, y \in \mathbb{E}$ and one of $x, y$ lies in $\mathbb{E}^\sharp$, and moreover that $\mathbb{E}^\sharp$ is closed under the operation $\circ$. Thus $(\mathbb{E}^\sharp, \circ)$ is a semigroup, with identity $(0, 0, -1)$. It contains $\text{Aut} \mathbb{D}$ in a sense we now explain.
Consider any \( v \in \text{Aut} \mathbb{D} \). We can write
\[
(6.3) \quad v(z) = \frac{\omega z - \alpha}{\alpha z - 1} = \Psi(z, \omega \alpha, \alpha, \omega)
\]
for some \( \alpha \in \mathbb{D} \) and \( \omega \in \mathbb{T} \). Let
\[
(6.4) \quad \tau(v) = (\omega \alpha, \alpha, \omega),
\]
so that \( v = \Psi(., \tau(v)) \). Clearly \( \tau(v) \) is non-triangular, and since \( ||v||_\infty = 1 \) it follows from Theorem 2.2 that \( \tau(v) \in \mathbb{E} \). The first two components of \( \tau(v) \) have modulus less than one, so that \( \tau(v) \in \mathbb{E}^2 \).

**Lemma 6.1.** \( \tau : \text{Aut} \mathbb{D} \to \mathbb{E}^2 \) is a unital monomorphism of semigroups.

**Proof.** For \( v, \chi \in \text{Aut} \mathbb{D} \) we have
\[
\Psi(., \tau(v \circ \chi)) = v \circ \chi = \Psi(., \tau(v)) \circ \Psi(., \tau(\chi)) = \Psi(., \tau(v \circ \tau(\chi))).
\]
Hence \( \tau(v \circ \chi) = \tau(v) \circ \tau(\chi) \). If \( \iota \) is the identity automorphism on \( \mathbb{D} \), then by equation (6.1) we have \( \tau(\iota) = (0,0,-1) \), which is the identity element of \( \mathbb{E}^2 \). It is clear that \( \tau \) is injective, and so \( \tau \) is a unital monomorphism. \( \square \)

Henceforth we write \( v \cdot x \) for \( \tau(v) \circ x \) and \( x \cdot v \) for \( x \circ \tau(v) \).

**Lemma 6.2.** For any automorphism \( v \) of \( \mathbb{D} \) and any \( x \in \mathbb{E} \) we have \( v \cdot x \in \mathbb{E} \) and \( x \cdot v \in \mathbb{E} \). Moreover, if \( x \) is in \( \mathbb{E} \) then so are \( v \cdot x \) and \( x \cdot v \).

**Proof.** Since \( x \in \mathbb{E} \) and \( \tau(v) \in \mathbb{E}^2 \) it follows that \( \tau(v) \circ x \) exists and belongs to \( \mathbb{E} \). Likewise \( x \circ \tau(v) \in \mathbb{E} \). If further \( x \in \mathbb{E} \) then \( \Psi(., x) \) maps \( \Delta \) into \( \mathbb{D} \), and since
\[
(6.5) \quad \Psi(., v \cdot x) = \Psi(., \tau(v) \circ x) = v \circ \Psi(., x),
\]
it follows that \( \Psi(., v \cdot x) \) also maps \( \Delta \) into \( \mathbb{D} \). Now \( v \cdot x \) is triangular if and only if \( v \circ \Psi(., x) \) is constant, which is so if and only if \( x \) is triangular. Hence, by Theorem 2.2, if \( x \) is non-triangular, \( v \cdot x \) lies in \( \mathbb{E} \). In the case of triangular \( x \) the same conclusions hold: here \( v \cdot x \) is triangular, and in view of Theorem 2.2 Condition (2), we need also to check that the second component of \( v \cdot x \) lies in \( \mathbb{D} \).

By equations (6.1), this component is \( \Upsilon(\alpha, x) \), which equals \( x_2 \) and does lie in \( \mathbb{D} \). Likewise if \( x \in \mathbb{E} \) then \( x \cdot v \) lies in \( \mathbb{E} \). \( \square \)

Accordingly there are maps
\[
m_1 : \mathbb{E} \times \text{Aut} \mathbb{D} \to \mathbb{E} : (x,v) \mapsto x \cdot v, \quad m_2 : (\text{Aut} \mathbb{D}) \times \mathbb{E} \to \mathbb{E} : (v,x) \mapsto v \cdot x,
\]
which restrict to maps \( \mathbb{E} \times \text{Aut} \mathbb{D} \to \mathbb{E} \) and \( \text{Aut} \mathbb{D} \times \mathbb{E} \to \mathbb{E} \).

**Theorem 6.1.** The maps \( m_1 \) and \( m_2 \) define right and left group actions of \( \text{Aut} \mathbb{D} \) on \( \mathbb{E} \) (and by restriction on \( \mathbb{E} \)) which commute with each other. Moreover the actions on \( \mathbb{E} \) and \( \mathbb{E} \) are by maps that are holomorphic in a neighbourhood of \( \mathbb{E} \).

**Proof.** It follows from equations (6.1) that \( v \cdot x = (0,0,-1) \circ x = x \), and similarly \( x \cdot v = x \) for any \( x \in \mathbb{E} \). From the homomorphic property of \( \tau \) and the associativity of \( \circ \) we have
\[
v \cdot (\chi \cdot x) = \tau(v) \circ (\tau(\chi) \circ x) = (\tau(v) \circ \tau(\chi)) \circ x = \tau(v \circ \chi) \circ x = (v \circ \chi) \cdot x.
\]
Thus \( m_2 \) is a left action of \( \text{Aut} \mathbb{D} \) on \( \mathbb{E} \) and \( \mathbb{E} \). Similarly \( m_1 \) is a right action.

We must show that the left and right actions commute, that is, that \( v \cdot (x \cdot \chi) = (v \cdot x) \cdot \chi \) for \( v, \chi, x \) as above. This also follows from the associativity of the operation \( \circ \).
Finally, the actions of $\text{Aut} \mathbb{D}$ on $\mathbb{E}$ are given by rational functions: if $\nu$ is given by equation (6.3) then
\[
\nu \cdot x = \tau(\nu) \circ x = (\omega \alpha, \bar{\alpha}, \omega) \circ x
= \frac{1}{1 - \bar{\alpha}x_1}(\omega(\alpha - x_1), x_2 - \bar{\alpha}x_3, \omega(\alpha x_2 - x_3)).
\]
For fixed $\nu \in \text{Aut} \mathbb{D}$ this is clearly an analytic function of $x$ in the set $\{x \in \mathbb{C}^3 : |x_1| < 1/|\alpha|\}$, which is a neighbourhood of $\bar{\mathbb{E}}$.

It follows from Theorem 6.1 that, for $\nu, \chi \in \text{Aut} \mathbb{D}$, there are commuting elements $L_\nu, R_\chi \in \text{Aut} \mathbb{E}$ given by
\[
L_\nu = m_2(\nu, .), R_\chi = m_1(., \nu).
\]
Another automorphism of $\mathbb{E}$ is the “flip” $F$:
\[
F(x_1, x_2, x_3) = (x_2, x_1, x_3), \quad x \in \mathbb{E}^3.
\]
One can verify from equations (6.1) that
\[
F(x \circ y) = F(y) \circ F(x), \quad x, y \in \mathbb{E}^3.
\]
Moreover
\[
F(\tau(\nu)) = F(\omega \alpha, \bar{\alpha}, \omega) = (\alpha, \omega \alpha, \omega) = \tau(\nu_*)
\]
where $\nu_* \in \text{Aut} \mathbb{D}$,
\[
\nu_*(z) = \frac{\omega(\bar{\omega}z - 1)}{\omega \alpha z - 1}.
\]

**Theorem 6.2.** The set
\[
G = \{L_\nu R_\chi F^\nu : \nu, \chi \in \text{Aut} \mathbb{D}, \nu = 0 \text{ or } 1\}
\]
constitutes a group of automorphisms of $\mathbb{E}$.

**Proof.** It is clear that $G \subset \text{Aut} \mathbb{E}$. We need relations between the generators. For $x \in \mathbb{E}, \nu \in \text{Aut} \mathbb{D}$,
\[
FL_\nu(x) = F(\nu \cdot x) = F(\tau(\nu) \circ x) = F(x) \circ F(\tau(\nu)) = F(x) \circ \tau(\nu_*) = F(x) \cdot \nu_* = R_\nu F(x).
\]
Similarly $FR_\nu = L_\nu F$. It follows easily that $G$ is closed under the group operation and inversion, hence is a subgroup of $\text{Aut} \mathbb{E}$.

We propose the following natural

**Conjecture 1.** $G = \text{Aut} \mathbb{E}$: every automorphism of $\mathbb{E}$ is of the form $L_\nu R_\chi F^\nu$ for some $\nu, \chi \in \text{Aut} \mathbb{D}$ and $\nu = 0$ or 1.

**Remark 6.1.** The orbit of $(0, 0, 0)$ under $G$ is the set $\mathcal{T}$ of triangular points in $\mathbb{E}$. Observe that $L_\nu, R_\chi$ and $F$ all leave $\mathcal{T}$ invariant (if $\Psi(., x)$ is constant then so is $\nu \circ \Psi(., x)$), and so $\mathcal{T}$ is invariant under $G$. Moreover $G$ acts transitively on $\mathcal{T}$, as the following lemma shows.

**Lemma 6.3.** If $x$ is a triangular point of $\mathbb{E}$ then $\nu \cdot x \cdot \chi = (0, 0, 0)$ where $\nu, \chi \in \text{Aut} \mathbb{D}$ are given by
\[
(6.6) \quad \nu(z) = \frac{z - x_1}{x_1 z - 1} \quad \text{and} \quad \chi(z) = \frac{z + x_2}{x_2 z + 1}.
\]
Furthermore, if we have for some non-zero \( \zeta \) since

\[
M_x \text{ has rank one and}
\]

\[
M_{v,x}\chi = \lambda M(v)M(x) = \lambda \begin{bmatrix}
1 & -x_1 \\
x_1 & -1
\end{bmatrix}
\begin{bmatrix}
x_2 & -1 \\
x_1 & 1
\end{bmatrix}
= \lambda \begin{bmatrix}
0 & 0 \\
(1 - |x_1|^2)(1 - |x_2|^2)
\end{bmatrix}
\]

for some non-zero \( \lambda \). It follows that \( v \cdot x \cdot \chi = (0, 0, 0) \). \( \square \)

By combining this lemma with the Schwarz Lemma, Theorem 1.1, we can obtain an explicit necessary and sufficient condition for the existence of an analytic map \( \varphi : \mathbb{D} \rightarrow \mathbb{E} \) mapping any given pair of points in \( \mathbb{D} \) to a given pair of points \( x, y \in \mathbb{E} \) of which one is triangular.

**Corollary 3.** Let \( x, y \in \mathbb{E} \), let \( \lambda_1, \lambda_2 \) be distinct points of \( \mathbb{D} \) and suppose that \( x_1x_2 = x_3 \). There exists an analytic map \( \varphi : \mathbb{D} \rightarrow \mathbb{E} \) such that \( \varphi(\lambda_1) = x \) and \( \varphi(\lambda_2) = y \) if and only if

\[
\max \left\{ \frac{(1 - |x_1|^2)|y_3 - y_1y_2| + |y_1 - \bar{y}_2y_3 - x_2(1 + |y_1|^2 - |y_2|^2 - |y_3|^2) + x_1^2(\bar{y}_1 - y_2\bar{y}_3)|}{|1 - x_1y_1|^2 - |y_2 - x_1y_2|^2}, \frac{(1 - |x_2|^2)|y_3 - y_1y_2| + |y_2 - \bar{y}_1y_3 - x_1(1 - |y_1|^2 + |y_2|^2 - |y_3|^2) + x_2^2(\bar{y}_2 - y_1\bar{y}_3)|}{|1 - x_2y_2|^2 - |y_1 - x_2y_2|^2} \right\} \leq |d(\lambda_1, \lambda_2)|.
\]

**Proof.** Let \( v, \chi \) be given by equations (6.6), so that \( v \cdot x \cdot \chi = (0, 0, 0) \), and let \( y' = v \cdot y \cdot \chi \). Some automorphism of \( \mathbb{D} \) maps \( \lambda_1, \lambda_2 \) to \( 0, d(\lambda_1, \lambda_2) \), and so, by Theorem 1.1, the required map \( \varphi \) exists if and only if

\[
\max \left\{ \frac{|y_1' - \bar{y}_2y_3'| + |x_1' - x_2y_2'|}{1 - |y_2'|^2}, \frac{|y_2' - \bar{y}_1y_3'| + |x_1' - x_2y_2'|}{1 - |y_1'|^2} \right\} \leq |d(\lambda_1, \lambda_2)|.
\]

We have

\[
M_{y'} = \zeta \begin{bmatrix}
1 & -x_1 \\
x_1 & -1
\end{bmatrix} M_y \begin{bmatrix}
1 & \bar{x}_2 \\
x_2 & 1
\end{bmatrix}
\]

for some non-zero \( \zeta \). Hence \( y_1'y_2' - y_3' = \det M_{y'} = -\zeta^2(1 - |x_1|^2)(1 - |x_2|^2)(y_1y_2 - y_3) \).

Furthermore, if \( J = \text{diag}(-1, 1) \),

\[
M_y J M_{y'}^* = \begin{bmatrix}
|y_1|^2 - |y_3|^2 & y_1 - \bar{y}_2y_3 \\
y_1 - \bar{y}_2y_3 & 1 - |y_2|^2
\end{bmatrix}.
\]

Since

\[
\begin{bmatrix}
1 & \bar{x}_2 \\
x_2 & 1
\end{bmatrix} J \begin{bmatrix}
1 & \bar{x}_2 \\
x_2 & 1
\end{bmatrix} = (1 - |x_2|^2)J,
\]

we have

\[
M_y J M_{y'}^* = |\zeta|^2(1 - |x_2|^2) \begin{bmatrix}
y_3 - x_1y_2 & x_1 - y_1 \\
x_1 - y_1 & 1 - \bar{x}_1y_1
\end{bmatrix} J \begin{bmatrix}
y_3 - x_1y_2 & x_1 - y_1 \\
x_1 - y_1 & 1 - \bar{x}_1y_1
\end{bmatrix}^*.
\]

The entries in the second column of this identity give us

\[
y_1' - \bar{y}_2y_3' = |\zeta|^2(1 - |x_2|^2)\{(x_1 - y_1)(1 - \bar{x}_1y_1) - (y_3 - x_1y_2)(x_1\bar{y}_3 - \bar{y}_2)\},
\]

\[
1 - |y_2'|^2 = |\zeta|^2(1 - |x_2|^2)\{|1 - x_1\bar{y}_1|^2 - |x_1\bar{y}_3 - \bar{y}_2|^2\}.
\]
On substituting these formulae and their symmetric analogues into the criterion \([6.7]\) we obtain the statement in the lemma. \(\square\)

Here is a less concrete but more assimilable version of this result.

**Corollary 4.** If \(x, y \in E\) and at least one of \(x, y\) is a triangular point then

\[ C_E(x, y) = K_E(x, y) = \delta_E(x, y). \]

The result is immediate from Corollary \([2]\) the invariance of \(C_E, K_E\) and \(\delta_E\) under automorphisms and Remark \([6.1]\).

### 7. The distinguished boundary of the tetrablock

Let \(\Omega\) be a domain in \(\mathbb{C}^n\) with closure \(\bar{\Omega}\) and let \(A(\Omega)\) be the algebra of continuous scalar functions on \(\bar{\Omega}\) that are holomorphic on \(\Omega\). A **boundary** for \(\Omega\) is a subset \(C\) of \(\bar{\Omega}\) such that every function in \(A(\Omega)\) attains its maximum modulus on \(C\). It follows from the theory of uniform algebras \([11, Corollary 2.2.10]\) that (at least when \(\bar{\Omega}\) is polynomially convex, as in the case of \(E\)) there is a smallest closed boundary of \(\Omega\), contained in all the closed boundaries of \(\Omega\) and called the **distinguished boundary** of \(\Omega\) (or the Shilov boundary of \(A(\bar{\Omega})\)). In this section we shall determine the distinguished boundary of \(E\); we denote it by \(bE\).

Clearly, if there is a function \(g \in A(E)\) and a point \(p \in E\) such that \(g(p) = 1\) and \(|g(x)| < 1\) for all \(x \in E \setminus \{p\}\), then \(p\) must belong to \(bE\). Such a point \(p\) is called a **peak point** of \(E\) and the function \(g\) a **peaking function** for \(p\).

An analytic disc in \(E\) is a non-constant analytic function \(f : \mathbb{D} \to E\). It follows easily from the maximum modulus principle that no element of the image \(f(\mathbb{D})\) can be a peak point.

**Theorem 7.1.** For \(x \in C^3\) the following are equivalent.

1. \(x_1 = \bar{x}_2 x_3, |x_3| = 1\) and \(|x_2| \leq 1\);
2. either \(x_1 x_2 \neq x_3\) and \(\Psi(x)\) is an automorphism of \(\mathbb{D}\) or \(x_1 x_2 = x_3\) and \(|x_1| = |x_2| = |x_3| = 1\);
3. \(x\) is a peak point of \(E\);
4. there exists a \(2 \times 2\) unitary matrix \(U\) such that \(x = \pi(U)\);
5. there exists a symmetric \(2 \times 2\) unitary matrix \(U\) such that \(x = \pi(U)\);
6. \(x \in bE\), the distinguished boundary of \(E\);.
7. \(x \in \bar{E}\) and \(|x_3| = 1\).

**Proof.** We first prove the equivalence of conditions (1) to (5); the proof is most easily presented as two completely separate cases. We first consider the simpler case \(x_1 x_2 = x_3\). We show that each of the conditions is equivalent to the applicable part of (2): \(x_1 x_2 = x_3\) and \(|x_1| = |x_2| = |x_3| = 1\).

(1)\(\Leftrightarrow\)(2) If (1) holds for a triangular point \((x_1, x_2, x_3)\) then \(|x_1 x_2| = |x_3| = 1\), and since \(|x_1| \leq 1, |x_2| \leq 1\) we must have \(|x_1| = |x_2| = |x_3| = 1\), and hence (2) holds. The converse is elementary.

(2)\(\Leftrightarrow\)(3) Let \(x\) satisfy (2). Define \(g : \bar{E} \to \mathbb{C}\) by

\[ g(y_1, y_2, y_3) = (\bar{x}_1 y_1 + \bar{x}_2 y_2 + \bar{x}_3 y_3 + 1)/4. \]

For \(y\) in \(E\) we have \(|y_i| \leq 1, i = 1, 2, 3\), and so \(|g(y)| \leq 1\). Further, if \(|g(y)| = 1\), then each \(\bar{x}_i y_i\) must be 1 and so \(y = x\). This shows that \(g\) is a peaking function for \(x\) relative to \(E\). Hence \(x\) is a peak point of \(E\).
(3)⇒(2) Consider a triangular point $x$ that is a peak point of $\overline{E}$. Suppose that $|x_3| < 1$. Note that $|x_1 x_2| < 1$ and so either $|x_1| < 1$ or $|x_2| < 1$. We assume that $|x_1| < 1$. Consider the function given by $g(z) = (z, x_2, x_3)$. It follows from condition (3) of Theorem 2.2 that $g(\mathbb{D}) \subset \overline{E}$. Thus $g$ is an analytic disc in $\overline{E}$ which contains the point $x$. This contradicts the hypothesis that $x$ is a peak point, and so we have $|x_3| = 1$. Since $|x_1 x_2| = 1$ it is also true that $|x_1| = |x_2| = 1$.

(2)⇒(5)⇒(4)⇒(2) If $x$ satisfies (2) then it is clear that the diagonal matrix $U = \text{diag}(x_1, x_2)$ satisfies condition (5). Trivially (5)⇒(4). Any unitary which is triangular is diagonal and hence (4) implies (2).

Thus conditions (1) to (5) are equivalent in the triangular case.

Now consider the non-triangular case, $x_1 x_2 \neq x_3$. Note that if $x \in \overline{E}$ then $|x_1| < 1$ and $|x_2| < 1$, for otherwise conditions (3) and (3′) of Theorem 2.2 show that $x \in \overline{E}$ is triangular.

(1)⇒(2) If $x$ satisfies (1) then $\bar{x}_1 x_3 = x_2 \bar{x}_3 x_3 = x_2$ and so

$$\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1} = \frac{x_3 \bar{z} - x_3 \bar{x}_2}{x_2 \bar{z} - 1} = \frac{x_3 \bar{z} - \bar{x}_2}{x_2 \bar{z} - 1}$$

and $\Psi(., x)$ is an automorphism of $\mathbb{D}$. Conversely, if $\Psi(., x)$ is an automorphism of $\mathbb{D}$ then $x \in \overline{E}$ and the image $\Psi(\mathbb{D}, x)$ has centre 0 and radius 1. As we noted in equation (2.3) the centre is $(x_1 - \bar{x}_2 x_3)/(1 - |x_2|^2)$ and the radius is $|x_1 x_2 - x_3|/(1 - |x_2|^2)$. Thus $x_1 = x_2 x_3, x_2 = \bar{x}_1 x_3$ and $|x_2 \bar{x}_2 x_3 - x_3|/(1 - |x_2|^2) = |x_3| = 1$. Hence (2) implies (1).

(4)⇒(1)⇒(5)⇒(4) Suppose (4): there exists a unitary matrix

$$U = \begin{bmatrix} x_1 & b \\ c & x_2 \end{bmatrix},$$

such that $\det U = x_1 x_2 - bc = x_3$. It is immediate that $|x_3| = |\det U| = 1$ and $|x_1| \leq ||U|| = 1, |x_2| \leq 1$. Since the columns of $U$ are orthonormal $x_1 \bar{b} + c \bar{x}_2 = 0$ and so

$$0 = b(x_1 \bar{b} + c \bar{x}_2) = x_1 |b|^2 + bc\bar{x}_2 = x_1(1 - |x_2|^2) + (x_1 x_2 - x_3)\bar{x}_2 = x_1 - \bar{x}_2 x_3.$$ 

Thus (4)⇒(1). Suppose (1) holds. Let $\zeta \in \mathbb{T}$ be a square root of $-x_3$. Then $\bar{x}_1 \zeta + \zeta x_2 = 0$ and so

$$U = \begin{bmatrix} x_1 & \zeta \sqrt{1 - |x_2|^2} \\ \zeta \sqrt{1 - |x_2|^2} & x_2 \end{bmatrix}$$

is a symmetric unitary matrix satisfying the conditions of (5). Trivially (5)⇒(4).

(2)⇒(3) Suppose $x$ satisfies (2) (and is non-triangular). We will exhibit a peaking function for $x$. Let $v$ be the inverse of the automorphism $\Psi(., x)$ of $\mathbb{D}$. Since

$$\Psi(., v \cdot x) = \Psi(., \tau(v) \circ x) = v \circ \Psi(., x) = \text{id}_E = \Psi(., 0, 0, -1),$$

it follows that $v \cdot x = (0, 0, -1)$.

There is a natural right action of $\text{Aut} \mathbb{D}$ on $A(E)$: for $\chi \in \text{Aut} \mathbb{D}, g \in A(E)$,

$$g \cdot \chi(x) = g(\chi \cdot x) = g(\tau(\chi) \circ x).$$

If $g$ is a peaking function for a point $y \in \overline{E}$ then $g \cdot \chi^{-1}$ is a peaking function for $\chi \cdot y$. Thus it suffices to find a peaking function for the point $(0, 0, -1)$. Consider the function $g(y) = ((y_3 - y_1 y_2) - 1)/2$ on $\overline{E}$. It follows from condition (3) of Theorem 2.2 that $|y_3 - y_1 y_2| \leq 1$ and hence $|g(y)| \leq 1$ on $\overline{E}$. Certainly $|g(0, 0, -1)| = 1$, and if $|g(y)| = 1$, then we must have $y_3 - y_1 y_2 = -1$ and, again by condition (3)
of Theorem 2.2, $|y_1|^2 = |y_2|^2 = 0$. Thus $y = (0, 0, -1)$, and hence $g$ peaks at the point $(0, 0, -1)$. Consequently $x$ is a peak point and (2)⇒(3).

(3)⇒(2) Suppose the non-triangular point $x$ is a peak point of $\mathbb{E}$ but $\Psi(x, x)$ is not an automorphism of $\mathbb{D}$; we shall show that $x$ lies on an analytic disc in $\mathbb{E}$ and obtain a contradiction. The conclusion is trivial if $x \in \partial \mathbb{E}$, the topological boundary of $\mathbb{E}$. By condition (2) of Theorems 2.2 and 2.4, $\|\psi\|_{H^\infty} = 1$ and so the closed disc $\Psi(\Delta, x)$ is a proper subset of $\Delta$ that touches $T$ at a unique point, $\zeta$ say, so that $\Psi(\eta, x) = \zeta$ for some $\eta \in T$. Let us make use of the Cayley transform

$$C_\eta(z) = \frac{\eta + z}{\eta - z},$$

which maps $\Delta$ to the closed right half plane $\mathbb{C}_+$ and maps $\eta$ to $\infty$. The (non-constant) M"obius transformation $C_\zeta \circ \Psi(., x) \circ C^{-1}_\eta$ maps $\mathbb{C}_+$ to a proper subset of itself and fixes $\infty$; it follows that

$$C_\zeta \circ \Psi(., x) \circ C^{-1}_\eta(z) = az + b$$

for some $a > 0$ and $b$ such that $\text{Re } b > 0$. Let $F(z, w) = az + b + w\text{Re } b$. For each $w \in \mathbb{D}$, $F(., w)$ is non-constant and maps $\mathbb{C}_+$ to a proper subset of itself. Thus $C^{-1}_\zeta \circ F(., w) \circ C_\eta$ is a non-constant M"obius transformation that maps $\Delta$ to itself, hence can be written $\Psi(., f(w))$ for some $f(w) \in \mathbb{E}$. The map $f$ is rational and bounded on $\mathbb{D}$ and so is an analytic disc in $\mathbb{E}$, and $f(0) = x$. This is a contradiction and so (3)⇒(2).

We have proved the equivalence of conditions (1) to (5) in both the triangular and non-triangular cases. Next we show that (3)⇒(6). By [11, Theorem 2.3.5], for any uniform algebra whose character space is metrizable, the set of peak points is a boundary. Thus the set $P$ of peak points of $\mathbb{E}$ is a boundary for $\mathbb{E}$; it is clearly contained in every boundary of $\mathbb{E}$, so that (3)⇒(6). By the equivalence of (1) and (3), $P$ is closed in $\mathbb{E}$. Hence $P$ is the smallest closed boundary of $\mathbb{E}$, that is $P = b\mathbb{E}$ and so (6)⇒(3).

(1)⇒(7) If (1) holds then, by condition (3) of Theorem 2.2, $x \in \mathbb{E}$ and hence (7) holds. If (7) holds then, by condition (6) of Theorem 2.2, $x_1 = \bar{x}_2x_3$, while by condition (3) of the same theorem $|x_2| \leq 1$. Thus (7)⇒(1). □

Corollary 5. $b\mathbb{E}$ is homeomorphic to $\Delta \times T$.

For the map $\Delta \times T \rightarrow b\mathbb{E} : (x_2, x_3) \mapsto (\bar{x}_2x_3, x_2, x_3)$ is a homeomorphism.

Corollary 6. $b\mathbb{E}$ is the closure of $\text{Aut } \mathbb{D}$ in $\mathbb{E}$.

Proof. By the definition (6.3), the monomorphism $\tau$ identifies an automorphism of $\mathbb{D}$ with a point $(\omega\alpha, \bar{\alpha}, \omega)$ with $\omega \in T, \alpha \in \mathbb{D}$. Thus $\tau$ identifies $\text{Aut } \mathbb{D}$ with the set $\{x : x_1 = \bar{x}_2x_3, |x_2| < 1, |x_3| = 1\}$, which is clearly a dense subset of $b\mathbb{E}$.

8. The analytic retraction problem

We have seen in Corollary 4 that $C_\xi, K_\xi$ and $\delta_\xi$ all agree at any pair of points of which one is triangular. Is it true that $C_\xi = K_\xi = \delta_\xi$? Note that the analogous equality holds for any convex domain, by a theorem of Lempert [16], and so in particular for the convex domain $R_I(2, 2)$, the unit ball of the space of $2 \times 2$ complex matrices. We have also seen that $\mathbb{E}$ is closely related to $R_I(2, 2)$, by the analytic
surjection $\pi : R_I \to E$. We ask: is $E$ an analytic retract of $R_I(2,2)$? In other words, do there exist analytic maps $h : E \to R_I$ and $f : R_I \to E$ such that $f \circ h = \text{id}_E$? If the answer is yes then it follows that $C_E = K_E = \delta_E$, since the following observation is a consequence of the fact that analytic maps are contractive for $C$ and $\delta$.

**Lemma 8.1.** Let $E, B$ be domains and let $E$ be an analytic retract of $B$.

(1) $C_E = C_B|E$ and $\delta_E = \delta_B|E$.

(2) If $C_B = \delta_B$ then $C_E = \delta_E$.

We could therefore resolve the question (does $C_E = \delta_E$?) if we could find an analytic $h : E \to R_I$ such that $\pi \circ h = \text{id}_E$. In fact there is no such $h$, and we conjecture that $E$ is not an analytic retract of $R_I(2,2)$.

**Theorem 8.1.** The map $\pi : R_I(2,2) \to E$ has no analytic right inverse.

**Proof.** Suppose $h = [h_{ij}] : E \to R_I$ satisfies $\pi \circ h = \text{id}_E$. Then $h_{11}(x) = x_1$, $h_{22}(x) = x_2$ and $h_{12}h_{21}(x) = x_1x_2 - x_3$. Let us write $P(x) = x_1x_2 - x_3$. Since $P$ is an irreducible polynomial and $h_{12}h_{21} = P$, it follows that $P$ divides one of $h_{12}, h_{21}$ — say $h_{21} = Pg$ and hence $h_{12} = 1/g$ where $g, 1/g$ are analytic scalar functions on $E$ and $Pg, 1/g$ are bounded. By equation (2.7), for any $x \in E$,

$$0 < \det(1-h(x)^*h(x)) = 1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - |x_1x_2 - x_3|^2 |g(x)|^2 - \frac{1}{|g(x)|^2}$$

(8.1) 

Consider any non-triangular point $y \in \partial E$. By conditions (5) of Theorems 2.1 and 2.2,

$$1 - |y_1|^2 - |y_2|^2 + |y_3|^2 - 2|y_1y_2 - y_3| = 0$$

and therefore

$$1 - |x_1|^2 - |x_2|^2 + |x_3|^2 - 2|x_1x_2 - x_3| \to 0 \text{ as } x \to y, \ x \in E.$$

It follows from inequalities (8.1) that

$$|x_1x_2 - x_3|^2 |g(x)|^2 + \frac{1}{|g(x)|^2} - 2|x_1x_2 - x_3| \to 0 \text{ as } x \to y.$$

By a refinement of the inequality of the means,

$$|x_1x_2 - x_3|^2 |g(x)|^2 - \frac{1}{|g(x)|^2} \to 0 \text{ as } x \to y.$$

Since $1/P(x)$ tends to the finite limit $1/P(y)$ as $x \to y$, we have

$$|g(x)|^2 - \frac{1}{|P(x)g(x)|^2} \to 0 \text{ as } x \to y,$$

and since $Pg$ is bounded on $E$,

$$|P(x)^2 |g(x)|^4| \to 1 \text{ as } x \to y.$$

Fix non-zero $\beta_1, \beta_2$ such that $|\beta_1| + |\beta_2| < 1$ and let $\beta = (\beta_1, \beta_2)$. We shall restrict the relation (8.2) to the disc $\varphi_\beta(D) \subset E$ and obtain a contradiction.

We claim that $\varphi_\beta(D)$ contains a unique triangular point of $E$. Indeed, $\varphi_\beta(\lambda)$ is triangular if and only if

$$(\beta_1 + \bar{\beta}_2 \lambda)(\bar{\beta}_2 + \beta_1 \lambda) = \lambda,$$
or
\[(8.3)\]
\[
\lambda^2 - \frac{1 - |\beta_1|^2 - |\beta_2|^2}{\beta_1 \beta_2} \lambda + \frac{\beta_1 \beta_2}{\beta_1 \beta_2} = 0.
\]
If the roots of this equation are \(\lambda_1, \lambda_2\) then \(|\lambda_1 \lambda_2| = 1\) and
\[
|\lambda_1 + \lambda_2| - 2 = \frac{1 - |\beta_1|^2 - |\beta_2|^2}{|\beta_1 \beta_2|} - 2 = \frac{1 - (|\beta_1| + |\beta_2|)^2}{|\beta_1 \beta_2|} > 0,
\]
so that exactly one of \(\lambda_1, \lambda_2\) belongs to \(\mathbb{D}\) — say \(|\lambda_1| < 1, |\lambda_2| > 1\). Note also that \(\varphi_\beta(\mathbb{T}) \subset \partial \mathbb{E}\) contains no triangular points.

Write down the explicit inner-outer factorisation of \(P \circ \varphi_\beta\):
\[
P \circ \varphi_\beta(\lambda) = \bar{\beta}_1 \beta_2 (\lambda - \lambda_1)(\lambda - \lambda_2) = v_\beta(\lambda)q_\beta(\lambda)
\]
where
\[
v_\beta(\lambda) = \frac{\lambda - \lambda_1}{\lambda_1 \lambda - 1}, \quad q_\beta(\lambda) = \bar{\beta}_1 \beta_2 (\bar{\lambda}_1 \lambda - 1)(\lambda - \lambda_2).
\]
Observe that \(q_\beta\) is bounded away from zero on \(\mathbb{D}\). Let
\[
\psi_\beta = q_\beta(g \circ \varphi_\beta)^2.
\]
Since \(g \circ \varphi_\beta\) is analytic on \(\mathbb{D}\) and \(v_\beta q_\beta g \circ \varphi_\beta \in H^\infty\), it follows that \(g \circ \varphi_\beta \in H^\infty\). Hence both \(\psi_\beta\) and \(1/\psi_\beta \in H^\infty\). Moreover, by relation (8.2),
\[
|\psi_\beta(\lambda)|^2 \to 1 \text{ as } \lambda \to \omega \in \mathbb{T}.
\]
Thus the radial limits of \(\psi_\beta\) have modulus 1 everywhere on \(\mathbb{T}\), so that \(\psi_\beta\) is inner. Since \(1/\psi_\beta \in H^\infty\), \(\psi_\beta\) is constant, and hence
\[
1 = |\psi_\beta(0)| = |q_\beta(0)g \circ \varphi_\beta(0)^2| = |\bar{\beta}_1 \beta_2 \lambda_2 g(\beta_1, \beta_2, 0)^2|
\]
and therefore
\[
|\beta_1 \beta_2 g(\beta_1, \beta_2, 0)^2| = \frac{1}{|\lambda_2|} = |\lambda_1|.
\]
Thus \(|\lambda_1|\) is the modulus of an analytic function of \(\beta\) on the domain \(\{(\beta_1, \beta_2) : \beta_1 \neq 0, \beta_2 \neq 0, |\beta_1| + |\beta_2| < 1\}\), and hence \(\log |\lambda_1|\) is a pluriharmonic function on this domain.

Let \(u(z)\) be the unique root in \(\mathbb{D}\) of the quadratic equation \((8.3)\) with \(\beta_1 = \beta_2 = z\). On the planar domain \(0 < |z| < \frac{1}{2}\), \(u\) does not vanish and \(\log |u(\cdot)|\) is a harmonic function. We have
\[
z^2 u(z)^2 - (1 - 2z\bar{z})u(z) + z^2 = 0.
\]
Implicit differentiation of this relation (together with the implicit function theorem) yields
\[
\frac{\partial u}{\partial z} = \frac{-2u(z\bar{u} + z)}{2z^2 u - 1 + 2z\bar{z}} = u \frac{\partial u}{\partial z}
\]
(one can check that the denominator of the middle term never vanishes when \(|z| < \frac{1}{2}\)),
\[
\frac{\partial^2 u}{\partial \bar{z} \partial z} = \frac{2(u - 4z\bar{z}u - 4z^2)}{(2z^2 u - 1 + 2z\bar{z})^2}.
\]
Thus
\[
\frac{\partial^2 \log |u(z)|}{\partial \bar{z} \partial z} = \Re \left\{ \frac{1}{u} \frac{\partial^2 u}{\partial \bar{z} \partial z} - \frac{1}{u^2} \frac{\partial u}{\partial \bar{z}} \frac{\partial u}{\partial z} \right\}
\]
\[
= -2 \Re \left\{ \frac{1}{(2z^2 u - 1 + 2z\bar{z})^3} \right\}.
\]
The right hand side is non-zero whenever \( z, u \) are both real – for example, when \( z = \frac{1}{3}, u = \frac{2-\sqrt{5}}{2} \). This is a contradiction, and so the postulated analytic \( h : E \to R_I \) does not exist. \( \square \)

**Remark 8.1.** *A fortiori* \( \pi : R_{II}(2) \to E \) has no right inverse either.

### 9. Relation to the \( \mu \)-synthesis Problem

In the theory of robust control the *structured singular value* of an \( m \times n \) matrix \( A \), denoted by \( \mu(A) \), is a cost function that generalizes the usual operator norm of \( A \) and encodes structural information about the perturbations of \( A \) that are being studied. In this context a “structure” is identified with a linear subspace of \( \mathbb{C}^{n \times m} \).

Let \( E \) be such a subspace, and write

\[
\mu_E(A) = (\inf \{ ||X|| : X \in E, 1 - AX \text{ is singular} \})^{-1},
\]

where we adopt the natural interpretation that \( \mu_E(A) = 0 \) in the event that \( 1 - AX \) is non-singular for all \( X \in E \). If \( E = \mathbb{C}^{n \times m} \) then \( \mu_E = ||.|| \), while if \( m = n \) and \( E \) is the space of scalar multiples of the identity matrix then \( \mu_E \) is the spectral radius. For a given \( E \subset \mathbb{C}^{n \times m} \) the \( \mu \)-synthesis problem is to construct, if possible, an analytic \( m \times n \)-matrix-valued function on \( \mathbb{D} \) or the right half plane subject to a finite number of interpolation conditions such that

\[
\mu_E(F(\lambda)) \leq 1 \text{ for all } \lambda \text{ in the domain of } F.
\]

In the case that \( \mu_E = ||.|| \) the \( \mu \)-synthesis problem is the classical Nevanlinna-Pick problem, for which there is a detailed theory (e.g. [6]). More generally, engineers have had some success in computing numerical solutions of \( \mu \)-synthesis problems [18], but there is a dearth of convergence results and existence theorems. At present there is not even a sufficient theory to enable the numerical methods to be tested satisfactorily. There is a clear need for a better understanding of the solvability or otherwise of \( \mu \)-synthesis problems. Bercovici, Foiaş and Tannenbaum [7, 8, 9] obtained some solvability criteria with the aid of variants of the commutant lifting theorem; however, the criteria they obtained are not easy to check. A solution for the special case of the spectral Nevanlinna-Pick problem (that is, with \( \mu_E \) being the spectral radius) for \( 2 \times 2 \) matrix functions and 2 interpolation points follows from the theory [2, 3] of the symmetrised bidisc. For more than two interpolation points even this very special case of \( \mu \)-synthesis is not yet well understood.

In the engineering literature (for example [14]) the space \( E \) of matrices is usually taken to be given by a block diagonal structure. If we confine ourselves to \( 2 \times 2 \) matrices the next natural case for study is that of the space of diagonal matrices:

\[
E = \text{Diag} = \{ \text{diag}(z, w) : z, w \in \mathbb{C} \}.
\]

This paper arises out of a study of the \( \mu \)-synthesis problem for \( 2 \times 2 \) matrices where \( \mu = \mu_{\text{Diag}} \). There is a simple connection between \( E \) and the set of matrices for which \( \mu < 1 \).

**Theorem 9.1.** An element \( x \) of \( \mathbb{C}^3 \) belongs to \( E \) if and only if there exists \( A \in \mathbb{C}^{2 \times 2} \) such that \( \mu_{\text{Diag}}(A) < 1 \) and \( x = \pi(A) \). Similarly, \( x \in \bar{E} \) if and only if there exists \( A \in \mathbb{C}^{2 \times 2} \) such that \( \mu_{\text{Diag}}(A) \leq 1 \) and \( x = \pi(A) \).
Henceforth we shall write $\mu$ for $\mu_{\text{Diag}}$.

**Proof.** For $r > 0$ and $A = [a_{ij}] \in \mathbb{C}^{2 \times 2}$ observe that $\mu(A) \leq 1/r$ if and only if $\|X\| \geq r$ whenever $X \in \text{Diag}$ and det$(1 - AX) = 0$. If $X = \text{diag}(z, w)$ then

$$
\det(1 - AX) = (1 - a_{11}z)(1 - a_{22}w) - a_{12}a_{21}zw
$$

(9.1)

Thus $\mu(A) \leq 1/r$ if and only if the zero variety of the polynomial (9.1) in $z, w$ does not meet the open bidisc $r\mathbb{D} \times r\mathbb{D}$.

Suppose that $\mu(A) < 1$ and $x = (a_{11}, a_{22}, \det A)$. For some $r > 1$ we have $\mu(A) \leq 1/r$, and so the zero variety of the polynomial (9.1) is disjoint from $(r\mathbb{D})^2$, hence *a fortiori* from $\Delta^2$. Thus $x \in \mathbb{E}$.

Conversely, if $x \in \mathbb{E}$, then the zero variety of (9.1) is disjoint from $(r\mathbb{D})^2$ for some $r > 1$, and hence the matrix

$$
A = \begin{bmatrix}
x_1 & x_1 x_2 - x_3 \\
1 & x_2
\end{bmatrix}
$$

satisfies $\mu(A) < 1$ and $x = \pi(A)$.

The proof of the second statement is similar. \(\square\)

We have found the bounded 3-dimensional domain $\mathbb{E}$ more amenable to study than the unbounded 4-dimensional domain

$$
\Sigma \overset{\text{def}}{=} \{A \in \mathbb{C}^{2 \times 2} : \mu(A) < 1\}.
$$

Every analytic function $F : \mathbb{D} \to \Sigma$ induces an analytic function $\pi \circ F : \mathbb{D} \to \mathbb{E}$, where $\pi$ is the map defined in equation (2.10) having the property that $A \in \Sigma$ if and only if $\pi(A) \in \mathbb{E}$. Conversely, every analytic $\varphi : \mathbb{D} \to \mathbb{E}$ lifts to a map $F : \mathbb{D} \to \Sigma$ such that $\pi \circ F = \varphi$, for example,

$$
F = \begin{bmatrix}
\varphi_1 & \varphi_1(\varphi_2 - \varphi_3) \\
1 & \varphi_2
\end{bmatrix}.
$$

More is true: the interpolation problems for $\Sigma$ and $\mathbb{E}$ are equivalent in the following sense.

**Theorem 9.2.** Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $A_k = [a_{ij}^k] \in \Sigma$, $1 \leq k \leq n$. The following conditions are equivalent.

1. There exists an analytic function $F : \mathbb{D} \to \Sigma$ such that $F(\lambda_k) = A_k$, $1 \leq k \leq n$;
2. there exists an analytic function $\varphi : \mathbb{D} \to \mathbb{E}$ such that $\varphi(\lambda_k) = \pi(A_k)$ and, if $A_k$ is a diagonal matrix, then

$$
\varphi'_1(\lambda_k) = a_{22}^k \varphi'_1(\lambda_k) + a_{11}^k \varphi'_2(\lambda_k), 1 \leq k \leq n.
$$

**Proof.** (1) $\Rightarrow$ (2) is easy, for we may take $\varphi = \pi \circ F$. Then $\varphi_3 = \varphi_1 \varphi_2 - F_{12} F_{21}$ and $\varphi_1(\lambda_k) = a_{11}^k$, $\varphi_2(\lambda_k) = a_{22}^k$. If $a_{12}^k = a_{21}^k = 0$ then $F_{12}(\lambda_k) = F_{21}(\lambda_k) = 0$ and

$$
\varphi'_3(\lambda_k) = \varphi'_1(\varphi_2(\lambda_k) + \varphi_1 \varphi'_2(\lambda_k) - F_{12}(\lambda_k) F_{21}(\lambda_k) - F_{12}(\lambda_k) F_{21}(\lambda_k)
$$

$$
= a_{22}^k \varphi'_1(\lambda_k) + a_{11}^k \varphi'_2(\lambda_k).
$$

(2)$\Rightarrow$(1) Let $\varphi$ be as in (2), so that

$$
(\varphi_1 \varphi_2 - \varphi_3)(\lambda_k) = a_{11}^k a_{22}^k - \det A_k = a_{12}^k a_{21}^k
$$

and, if $a_{12}^k = a_{21}^k = 0$, then $(\varphi_1 \varphi_2 - \varphi_3)'(\lambda_k) = 0$. Choose an analytic function $g$ in $\mathbb{D}$ such that
Theorem 9.3. Let

\( g(\lambda_k) = a_{21}^k, 1 \leq k \leq n; \)

(ii) \( g \) has simple zeros at those \( \lambda_k \) such that \( a_{21}^k = 0 \) and no other zeros;

(iii) if \( a_{21}^k = 0 \) and \( a_{12}^k \neq 0 \) then \( g'(\lambda_k) = (\varphi_1 \varphi_2 - \varphi_3)'(\lambda_k)/a_{12}^k. \)

Let \( f = (\varphi_1 \varphi_2 - \varphi_3)/g \) and let

\[
F = \begin{bmatrix} \varphi_1 & f \\ g & \varphi_2 \end{bmatrix}.
\]

\( F \) is analytic in \( \mathbb{D} \) and \( \pi \circ F = \varphi \), so that \( F(\mathbb{D}) \subseteq \Sigma \). Note that, if \( A_k \) is diagonal, then \( \varphi_1 \varphi_2 - \varphi_3 \) has a multiple zero at \( \lambda_k \) and \( g \) has a simple zero, so that \( f(\lambda_k) = 0 = a_{12}^k \). If \( a_{21}^k = 0 \) and \( a_{12} \neq 0 \) then L'Hopital’s rule gives \( f(\lambda_k) = a_{12}^k \). Hence \( F(\lambda_k) = A_k, 1 \leq k \leq n. \)

Remark 9.1. (i) The theorem remains true if we replace \( \Sigma \) and \( \mathbb{E} \) by their closures.

(ii) Generically the target matrices are non-diagonal, in which case the interpolation problem for \( \varphi : \mathbb{D} \to \mathbb{E} \) does not involve conditions on \( \varphi' \).

(iii) The problem of finding \( F \) satisfying (1) in Theorem 9.2 is called the structured Nevanlinna-Pick problem in [7].

On putting together Theorems 1.1 and 9.2, we obtain a Schwarz lemma for \( \tilde{\Sigma} \).

Theorem 9.3. Let \( \lambda_0 \in \mathbb{D} \setminus \{0\}, \zeta \in \mathbb{C} \) and \( A_1, A_2 \in \mathbb{C}^{2 \times 2} \), where \( A_2 \) is not diagonal,

\[
A_1 = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ \zeta & 0 \end{bmatrix} \text{ and } \pi(A_2) = (a, b, p) \in \mathbb{E}.
\]

There exists an analytic \( 2 \times 2 \) matrix function \( F \) such that \( F(0) = A_1, F(\lambda_0) = A_2 \) and \( \mu(F(\lambda)) \leq 1 \) for all \( \lambda \in \mathbb{D} \) if and only if

\[
\max \left\{ \frac{|a - \bar{b}p| + |ab - p|}{1 - |b|^2}, \frac{|b - \bar{a}p| + |ab - p|}{1 - |a|^2} \right\} \leq |\lambda_0| \quad \text{if } \zeta \neq 0
\]

\[
\left( \frac{a}{\lambda_0}, \frac{b}{\lambda_0}, \frac{p}{\lambda_0^2} \right) \in \mathbb{E} \quad \text{if } \zeta = 0.
\]

Proof. If \( \zeta \neq 0 \) then \( A_1 \) is not diagonal, and so by Theorem 9.2 and Remark 9.1(i), there is a function \( F \) with the required properties if and only if there exists an analytic function \( \varphi : \mathbb{D} \to \mathbb{E} \) such that \( \varphi(0) = (0, 0, 0) \) and \( \varphi(\lambda_0) = (a, b, p) \). By Theorem 1.1, this is so if and only if the first inequality in conditions (9.2) holds.

If \( \zeta = 0 \) then \( A_1 = 0 \) and the desired \( F \) exists if and only if there is a function \( G \) in the \( 2 \times 2 \) Schur class such that \( G(\lambda_0) = A_2/\lambda_0 \), which is so if and only if \( \|A_2/\lambda_0\| \leq 1 \), and so, by condition (7) of Theorem 2.2, if and only if \( (a/\lambda_0, b/\lambda_0, p/\lambda_0^2) \in \mathbb{E} \). \( \square \)

Remark 9.2. (i) This result is a solvability criterion for an extremely special type of \( 2 \)-point \( \mu \)-synthesis problem. It falls far short of what control engineers would like to know, but it does reveal some of the analytic subtleties of \( \mu \)-synthesis and may be a starting point for the solution of more general problems.

(ii) In control problems the interpolation conditions are typically “tangential”, that is, of the forms \( F(\lambda_j)x_j = y_j \) and \( x_j^*F(\lambda_j) = y_j^* \) for suitable vectors \( x_j, y_j \), rather than \( F(\lambda_j) = A_j \) as studied here, but a solution of the general problem must of course include our type of constraint.
state their most relevant result, specialised to the situation we are studying here. For example, by condition (5), it is equivalent to

$$|\lambda_0|^4 - (|a|^2 + |b|^2 + 2|ab - p|)|\lambda_0|^2 + |p|^2 \geq 0$$

and if \(ab = p\) then \(|a| + |b| \leq 2|\lambda_0|\).

(iv) Observe that a 2-point \(\mu\)-synthesis problem can be ill-conditioned. For example, if \((a, b, p) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\), then there exists an analytic function \(F_\zeta\) in \(\mathbb{D}\) such that

$$\mu(F_\zeta(\lambda)) \leq 1 \text{ for all } \lambda \in \mathbb{D} \text{ and } F_\zeta(0) = A_1, \quad F_\zeta(\lambda_0) = A_2$$

if and only if

$$|\lambda_0| \geq \begin{cases} \frac{2}{\sqrt{2}} & \text{if } \zeta \neq 0 \\ \frac{\sqrt{2}}{3} & \text{if } \zeta = 0. \end{cases}$$

It follows that if \(\frac{2}{3} < |\lambda_0| < \frac{\sqrt{2}}{3}\) the \(F_\zeta\) cannot be locally bounded as \(\zeta \to 0\). For such \(\lambda_0\), if \(\zeta\) is close to zero then the solutions of the interpolation problem are very sensitive to small changes in \(\zeta\). Any numerical method for the computation of solutions is likely to be unreliable for such data.

(v) The proof shows how to construct a solution of a 2-point problem of the type in Theorem 9.3, at least in the case that the inequality in conditions 9.2 holds strictly. For then, if \(\zeta = 0\), we may define \(\varphi = \pi \circ F\) where \(F(\lambda) = \lambda A_2 / \lambda_0\), while if \(\zeta \neq 0\) then we may take \(\varphi = \pi \circ F\) where \(F\) is constructed according to the algorithm in Section 3.

Similarly, by putting together Theorem 9.2 and Corollary 3 we obtain a partial Schwarz-Pick lemma for \(\Sigma\).

**Theorem 9.4.** Let \(\lambda_1, \lambda_2\) be distinct points in \(\mathbb{D}\), let \(A, B\) be non-diagonal \(2 \times 2\) matrices such that \(\mu(A) \leq 1, \mu(B) \leq 1\) and \(A\) is triangular. There exists an analytic \(2 \times 2\) matrix function \(F\) on \(\mathbb{D}\) such that \(F(\lambda_1) = A, F(\lambda_2) = B\) and \(\mu(F(\lambda)) \leq 1\) for all \(\lambda \in \mathbb{D}\) if and only if

$$\max \left\{ \frac{1}{1 - |x_1|^2} |y_3 - y_1y_2| + |y_1 - y_2y_3| - x_1(1 + |y_1|^2 - |y_2|^2 - |y_3|^2) + x_1^2(y_1 - y_2y_3), \right. \frac{1}{1 - |x_2|^2} |y_3 - y_1y_2| + |y_2 - y_1y_3| - x_2(1 - |y_1|^2 + |y_2|^2 - |y_3|^2) + x_2^2(y_2 - y_1y_3), \right\} \leq \ |d(\lambda_1, \lambda_2)|$$

where \(\pi(A) = x, \pi(B) = y\).

One can derive a somewhat more complicated criterion in the case of diagonal \(A\); however, diagonal target matrices can be most simply analysed by a form of the Schur reduction procedure.

Bercovici, Foiaş and Tannenbaum [7] use operator-theoretic methods to study a much more general \(\mu\)-synthesis problem than the special cases in Theorems 9.3 and 9.4 but they obtain a less detailed result. For the purpose of comparison we shall state their most relevant result, specialised to the situation we are studying here (2 \(\times\) 2-matrix functions, \(\mu = \mu_{\text{Diag}}\).

Suppose we are given distinct points \(\lambda_1, \ldots, \lambda_n \in \mathbb{D}\) and \(2 \times 2\) matrices \(A_1, \ldots, A_n\). For any analytic function \(F : \mathbb{D} \to \mathbb{C}^{2 \times 2}\) let

$$\mu_{\infty}(F) = \sup_{z \in \mathbb{D}} \mu(F(z)).$$
We seek to minimise \( \mu^\infty(F) \) over all analytic interpolating functions \( F \); the formula is in terms of operators. Let \( k_\lambda \) be the Szegő kernel:
\[
k_\lambda(z) = \frac{1}{1 - \lambda z}, \quad z \in \mathbb{D},
\]
let \( H^2 \) denote the Hardy space on the disc and let
\[
\mathcal{M} = \text{span} \{ k_{\lambda_1} \otimes \xi_1, \ldots, k_{\lambda_n} \otimes \xi_n : \xi_1, \ldots, \xi_n \in \mathbb{C}^2 \},
\]
which is a \( 2n \)-dimensional subspace of the Hilbert space \( H^2 \otimes \mathbb{C}^2 \). Corresponding to \( 2 \times 2 \) matrices \( F_1, \ldots, F_n \) we define a linear operator \( A(F_1, \ldots, F_n) \) on \( \mathcal{M} \) by
\[
A(F_1, \ldots, F_n) k_\lambda \otimes \xi = k_\lambda \otimes F^*_\lambda \xi.
\]

Theorem 5 of [7] states the following. The infimum of \( \mu^\infty(F) \) over all bounded rational analytic \( 2 \times 2 \) functions \( F \) on \( \mathbb{D} \) such that
\[
F(\lambda_j) = A_j, \quad 1 \leq j \leq n,
\]
is equal to
\[
\inf \{ ||A(D_1 A_1 D_1^{-1}, \ldots, D_n A_n D_n^{-1})|| : D_1, \ldots, D_n \in \text{Diag} \cap GL_2(\mathbb{C}) \}.
\]

This result gives the infimum over the infinite-dimensional set of \( F \)s in terms of an infimum over an \( n \)-dimensional set of \( D \)s; existing packages for the numerical solution of \( \mu \)-synthesis problems work by attempting to solve this \( n \)-dimensional (non-convex, unbounded) optimization problem. Note, however, that there is no assertion as to whether the infima are attained.

Both this paper and [7] seek to reduce \( \mu \)-synthesis problems to classical Nevanlinna-Pick problems, in one case via the geometry of \( \mathbb{E} \), in the other by diagonal scaling. We believe the two approaches complement each other, and that there is scope for further progress on \( \mu \)-synthesis problems through a study of \( \mathbb{E} \) and possibly higher-dimensional analogues.

Added in proof: Some of the questions raised in this paper are answered in the eprint “The automorphism group of the tetrablock”, arXiv:0708.0689. The automorphisms described in Section 6 do indeed comprise all automorphisms of \( \mathbb{E} \), so that Conjecture 1 is true. It is also shown that \( \mathbb{E} \) is not an analytic retract of \( R_I(2, 2) \) or \( R_{II}(2) \).

References
[1] Abouhajar, A. A., Function theory related to \( H^\infty \) control, PhD Thesis, Newcastle University, 2007.
[2] Agler, J. and Young, N. J., The two-point spectral Nevanlinna-Pick problem, Integral Equations Operator Theory, 37 (2000) 375–385.
[3] Agler, J. and Young, N. J., A Schwarz lemma for the symmetrised bidisc, Bull. London Math. Soc., 33 (2001), 175-186.
[4] Agler, J. and Young, N. J., The two-by-two spectral Nevanlinna-Pick problem, Trans. Amer. Math. Soc., 356 (2004), no. 2, 573-585.
[5] Agler, J. and Young, N. J., The hyperbolic geometry of the symmetrized bidisc, J. Geom. Anal., 14 (2004), no. 3 375-403.
[6] Ball, J. A., Gohberg, I. and Rodman, L., Interpolation of rational matrix functions, OT45, Birkhäuser Verlag, Basel, 1990.
[7] Bercovici, H., Foiaš, C. and Tannenbaum, A., Structured interpolation theory, Operator Theory: Advances and Applications, 47 (1990), 195-220.
[8] Bercovici, H., Foiaş, C. and Tannenbaum, A., Spectral variants of the Nevanlinna-Pick interpolation problem, in Signal processing, scattering and operator theory, and numerical methods (Amsterdam, 1989), 23–45, Progr. Systems Control Theory, 5, Birkhäuser, Boston, 1990.

[9] Bercovici, H., Foiaş, C. and Tannenbaum, A., A spectral commutant lifting theorem, Trans. Amer. Math. Soc., 325 (1991), no. 2, 741-763.

[10] Bharali, G., Nonisotropically balanced domains, Lempert function estimates, and the spectral Nevanlinna-Pick problem, arXiv:math/0601107v3.

[11] Browder, A., Introduction to function algebras, W. A. Benjamin Inc., New York, 1969.

[12] Costara, C., The symmetrised bidisc and Lempert’s theorem, Bull. London Math. Soc., 36 (2004), 656-662.

[13] Doyle, J. C., Analysis of feedback systems with structured uncertainties, IEE Proceedings D, Control Theory and Applications 129 (1982), no. 6, 242-250.

[14] Doyle, J. C. and Packard, A., The complex structured singular value, Automatica J. IFAC, 29 (1993), no. 1, 71-109.

[15] Edigarian, A. and Zwonek, W., Geometry of the symmetrized polydisc, Arch. Math. (Basel), 84 (2005), no. 4, 364–374.

[16] Jarnicki, M. and Pflug, P., Invariant distances and metrics in complex analysis – revisited, Dissertationes Math. (Rozprawy Mat.) Volume 430, Polish Academy of Sciences, Warsaw, 2005.

[17] Krantz, S. G., Function theory of several complex variables, John Wiley and Sons, New York, 1982.

[18] Matlab µ-Analysis and Synthesis Toolbox, The Math Works Inc., Natick, Massachusetts (http://www.mathworks.com/products/muanalysis/).

[19] Pflug, P. and Zwonek, W., Description of all complex geodesics in the symmetrized bidisc, Bull. London Math. Soc., 37 (2005), no. 4, 575-584.

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