On Minimal Critical Exponent of Balanced Sequences

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joint work with Daniela Opočenská, Edita Pelantová
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Rational powers

**Definition**

Let $e \in \mathbb{Q}$. A word $z$ is an $e$-th power of a word $u$ if $z$ is a prefix of $u^\omega = uuuuu \ldots$ and $e = \frac{|z|}{|u|}$. We write $z = u^e$.

**Example**

- $abbabb = (abb)^2$
- $abbcabbcc = (abbc)^3$
- $abbabbab = (abb)^{8/3}$
- $starosta = (staro)^{8/5}$
Critical exponent

**Definition**

Let $u$ be a sequence. The *critical exponent* of $u$ $E(u) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } u\}$.

**Example**

The Thue–Morse sequence $u_{TM} = abbabaabbaababbabaab\ldots$

$u_{TM} = \varphi(u_{TM})$, where $\varphi : a \to ab, \ b \to ba$

$u_{TM}$ does not contain overlaps: $xwxwx$, where $w$ is a factor and $x$ is a letter. Hence $E(u_{TM}) = 2$. 
Dejean’s theorem (conjecture), 1972 – 2011: (proven by Dejean, Pansiot, Moulin Ollagnier, Mohammad-Noori, Carpi, Currie, Rampersad, Rao)
the least critical exponent of sequences over an alphabet of size $d$:
- $2$ for $d = 2$;
- $7/4$ for $d = 3$;
- $7/5$ for $d = 4$;
- $\frac{d}{d-1}$ for $d \geq 5$. 
Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme, 2019:**
  the least critical exponent of balanced sequences over an alphabet of size $d$ equals $\frac{d-2}{d-3}$ for $d \geq 5$
  - proven for $5 \leq d \leq 8$

- **Dolce, D., Pelantová, 2021:**
  - proven for $9 \leq d \leq 10$
  - disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$
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$$\frac{d}{d-1} < \frac{d-1}{d-2} < \frac{d-2}{d-3}$$
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Program

1 Preliminaries

2 History of our results
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2. History of our results
Definitions CoW

- *bispecial factor* of $u$
- *Parikh vector* $\vec{V}(u)$ of a factor $u$ of $u$

**Example**

Let $u_F = \text{abaababaababaa...}$

$u_F = \varphi(u_F)$, where $\varphi: a \rightarrow ab, \ b \rightarrow a$

*aba is a bispecial factor since aaba, baba and abab, abaa are factors of $u_F$*

$\vec{V}(aba) = (\frac{1}{2})$
Definitions CoW

- *return word* to a factor $u$ of $u$
- *derived sequence* $d_u(u)$ to a factor $u$ of $u$

**Example**

$u_F = \text{abaababaabaababaa}\ldots$

$r = \text{aba} \text{ and } s = \text{ab} \text{ are return words to the factor } u = \text{aba}$

$d_{u_F}(u) = \text{abaababaabaabababa}\ldots = \text{rsrrsr}\ldots$
(Asymptotic) critical exponent

- **Critical exponent of u**
  \[ E(u) = \sup\{ e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } u \} \]

- **Asymptotic critical exponent of u**
  \[ E^*(u) = \lim_{n \to \infty} \sup\{ e \in \mathbb{Q} : u^e \text{ is a factor of } u \text{ and } |u| \geq n \} \]

Evidently, \( E^*(u) \leq E(u) \).

**Proposition (D., Medková, Pelantová, 2020)**

Let \( u \) be a uniformly recurrent aperiodic sequence. Let \( w_n \) be the \( n \)-th bispecial of \( u \) and \( v_n \) a shortest return word to \( w_n \). Then

\[ E(u) = 1 + \sup\{ \frac{|w_n|}{|v_n|} : n \in \mathbb{N} \} \quad \text{and} \quad E^*(u) = 1 + \limsup_{n \to \infty} \frac{|w_n|}{|v_n|}. \]
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**Example**

\[
|w_n| = F_{n+2} + F_{n+1} - 2 \quad \text{and} \quad |v_n| = F_{n+1} \quad \text{with} \quad F_0 = 0, F_1 = 1
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\[
E(u_F) = 2 + \tau = 2 + \frac{1+\sqrt{5}}{2} = E^*(u_F) \quad \text{– minimal for Sturmian}
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Balanced sequences

**Definition**

$u$ over $A$ balanced if $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$ for all $a \in A$

**Theorem (Graham 1973, Hubert 2000)**

$v$ recurrent aperiodic is balanced iff $v$ obtained from a Sturmian sequence $u$ over \{a, b\} by replacing

- a with a constant gap sequence $y$ over $A$,
- b with a constant gap sequence $y'$ over $B$,

where $A$ and $B$ disjoint. We write $v = \text{colour}(u, y, y')$. 

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**Example**

\( v = \text{colour}(u_F, y, y') \), where \( y = (0102)^\omega \) and \( y' = (34)^\omega \)

\( u_F = \text{abaababaabaabab} \ldots \)

\( v = 031042301402304 \ldots \pi(423) = \text{bab} \)
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2. History of our results
Motivation

- **Rampersad**, May 2020, One World Numeration Seminar: *Ostrowski numeration and repetitions in words*
  - question by Cassaigne: “What about the asymptotic version?”

- **D., Medková, Pelantová**, 2020: *Complementary symmetric Rote sequences: the critical exponent and the recurrence function*
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  *Complementary symmetric Rote sequences: the critical exponent and the recurrence function*
Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
- **complementary symmetric sequence**: language closed under exchange of 0 and 1

$$S(v) = S(v_0 v_1 v_2 v_3 v_4 \ldots) = (v_0 + v_1 \mod 2)(v_1 + v_2 \mod 2)(v_2 + v_3 \mod 2)\ldots$$

$$S(v_F) = S(00111001110001\ldots) = 0100101001001\ldots$$
Complementary symmetric Rote sequences

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$$S(v) = S(v_0v_1v_2v_3...v_n) = (v_0 + v_1 \mod 2)(v_1 + v_2 \mod 2)(v_2 + v_3 \mod 2)...$$

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**Theorem (Rote 1994)**

Let $u$ and $v$ be two binary sequences such that $u = S(v)$. Then $v$ is a CS Rote sequence iff $u$ is a Sturmian sequence.
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Let $u$ and $v$ be two binary sequences such that $u = S(v)$. Then $v$ is a CS Rote sequence iff $u$ is a Sturmian sequence.

- $E^*(v) = E^*(\hat{v})$, where $v$ is a CS Rote sequence associated with $u$ and $\hat{v} = \text{colour}(u, y, y')$ by $y = 0^\omega$ and $y' = (12)^\omega$.
- The minimal critical exponent of ternary balanced sequences is the same as the minimal critical exponent of CS Rote sequences, and it equals $2 + \frac{1}{\sqrt{2}}$. 
Complementary symmetric Rote sequences

- **Rote sequence**: binary sequence with complexity $2n$
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  $S(v) = S(v_0v_1v_2v_3v_4...)= (v_0+v_1 \mod 2)(v_1+v_2 \mod 2)(v_2+v_3 \mod 2)...
  
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*Let $u$ and $v$ be two binary sequences such that $u = S(v)$. Then $v$ is a CS Rote sequence iff $u$ is a Sturmian sequence.*

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- The minimal critical exponent of ternary balanced sequences is the same as the minimal critical exponent of CS Rote sequences, and it equals $2 + \frac{1}{\sqrt{2}}$. 
Computation of asymptotic critical exponent

Recall $E^*(v) = 1 + \limsup_{n \to \infty} \frac{|w_n|}{|v_n|}$

**Proposition (Dolce, D., Pelantová, 2020)**

Let $v = \text{colour}(u, y, y')$. For a sufficiently long bispecial $w$ in $v$ its projection $u = \pi(w)$ is a bispecial in $u$. The shortest return word to $w$ is of length $\min\{k|r| + \ell|s|\}$, where

1. $k\vec{V}(r) + \ell\vec{V}(s) = \begin{pmatrix} 0 \mod \text{Per}(y) \\ 0 \mod \text{Per}(y') \end{pmatrix}$;

2. $(\frac{\ell}{k})$ is the Parikh vector of a factor in $d_u(u)$.

**Program implemented by Daniela Opočenská:**
Input: slope $\alpha$ quadratic irrational, $\text{Per}(y)$, $\text{Per}(y')$
Output: $E^*(v)$, where $v = \text{colour}(u, y, y')$
Completion of table

| $d$ | $\alpha$ | $y$          | $y'$         | $E(v)$         | $E^*(v)$         |
|-----|----------|--------------|--------------|----------------|-----------------|
| 3   | $[0, 2]$ | $(01)^\omega$| $2^\omega$   | $2 + \frac{1}{\sqrt{2}}$ | $2 + \frac{1}{\sqrt{2}}$ |
| 4   | $[0, 2, 1]$ | $(01)^\omega$ | $(23)^\omega$ | $1 + \frac{1 + \sqrt{5}}{4}$ | $1 + \frac{1 + \sqrt{5}}{4}$ |
| 5   | $[0, 2]$  | $(0102)^\omega$ | $(34)^\omega$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| 6   | $[0, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2]$ | $0^\omega$ | $(123415321435)^\omega$ | $\frac{4}{3}$ | $\frac{4}{3}$ |
| 7   | $[0, 1, 1, 3, 1, 2, 1]$ | $(01)^\omega$ | $(234526432546)^\omega$ | $\frac{5}{4}$ | $\frac{5}{4}$ |
| 8   | $[0, 1, 3, 1, 2]$ | $(01)^\omega$ | $(234526732546237526432576)^\omega$ | $\frac{6}{5} = 1.2$ | $\frac{12 + 3\sqrt{2}}{14} \approx 1.16$ |
| 9   | $[0, 1, 2, 3, 2]$ | $(01)^\omega$ | $(234567284365274863254768)^\omega$ | $\frac{7}{6}$ | $1 + \frac{2\sqrt{2} - 1}{14} \approx 1.13$ |
| 10  | $[0, 1, 4, 2, 3]$ | $(01)^\omega$ | $(234567284963254768294365274869)^\omega$ | $\frac{8}{7}$ | $1 + \frac{\sqrt{13}}{26} \approx 1.139$ |

**Table:** Baranwal, Rampersad, Shallit, Vandomme: $d$-ary balanced sequences with the least critical exponent.
Computation of critical exponent

Recall \( E(v) = 1 + \sup \{ \frac{|w_n|}{|v_n|} : n \in \mathbb{N} \} \)

Our result: \( E(v) = \max \left\{ E^*(v), 1 + \frac{|w_i|}{|v_i|} \right\} \) for finitely many \( i \)
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Proposition (Dolce, D., Pelantová, 2020)

Let $v = \text{colour}(u, y, y')$. Let $w$ be a bispecial factor of $v$ with projection $u = \pi(w)$ in $u$. The shortest return word to $w$ is of length $\min \{ k|r| + \ell|s| \}$, where

1. $k\vec{V}(r) + \ell\vec{V}(s) = (0 \mod n, 0 \mod n')$, where $n \in \text{gap}(y, |u_a|)$ and $n' \in \text{gap}(y', |u_b|)$;
2. $(\ell_k)$ is the Parikh vector of a factor in $d_u(u)$.

Example

For $y = (0102)\omega$, we have $\text{gap}(y, 1) = \{2, 4\}$ and $\text{gap}(y, m) = \{4\}$ for $m \geq 2$. 
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Recall \( E(\mathbf{v}) = 1 + \sup \{|w_n| : n \in \mathbb{N}\} \)

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**Proposition (Dolce, D., Pelantová, 2020)**

\( \mathbf{v} = \text{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}') \). Let \( w \) be a bispecial factor of \( \mathbf{v} \) with projection \( \mathbf{u} = \pi(w) \) in \( \mathbf{u} \). The shortest return word to \( w \) is of length \( \min \{|k|_r + |\ell|_s\} \), where

1. \( k \vec{V}(r) + \ell \vec{V}(s) = (0 \mod n', 0 \mod n') \), where \( n \in \text{gap}(\mathbf{y}, |u|_a) \) and \( n' \in \text{gap}(\mathbf{y}', |u|_b) \);
2. \( (\frac{\ell}{k}) \) is the Parikh vector of a factor in \( d_u(u) \).

**Example**

For \( \mathbf{y} = (0102)\omega \), we have \( \text{gap}(\mathbf{y}, 1) = \{2, 4\} \) and \( \text{gap}(\mathbf{y}, m) = \{4\} \) for \( m \geq 2 \).
Computation of critical exponent

Program implemented by Opočenská:
Input: slope $\alpha$ quadratic irrational, $y, y'$
Output: $E(v)$, where $v = \text{colour}(u, y, y')$

Description of algorithms for computation of (asymptotic) critical exponent published:
Dolce, D., Pelantová: *On balanced sequences and their critical exponent*, arXiv 2021
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| 9   | $[0, 1, 2, 3, 2]$ | $(01)^{\omega}$ | $(23456728436527457663254768)^{\omega}$ | $\frac{7}{6} \approx 1.167$ | $1 + \frac{2\sqrt{2}-1}{14} \approx 1.13$ |
| 10  | $[0, 1, 4, 2, 3]$ | $(01)^{\omega}$ | $(234567284963254768294365274869)^{\omega}$ | $\frac{8}{7} \approx 1.14$ | $1 + \frac{\sqrt{13}}{20} \approx 1.139$ |
| 11  | $[0, 1, 5, 1, 1, 1, 1, 2]$ | $(01)^{\omega}$ | $(234567892A436587294A638527496A832547698A)^{\omega}$ | $\frac{10}{9} \approx 1.11$ | $\frac{415+5\sqrt{105}}{424} \approx 1.0996$ |
| 12  | $[0, 1, 1, 3, 2]$ | $(012345)^{\omega}$ | $(6789AB)^{\omega}$ | $\frac{11}{10} = 1.1$ | $\frac{8-\sqrt{2}}{6} \approx 1.0976$ |

Table: Baranwal, Rampersad, Shallit, Vandomme: $d$-ary balanced sequences with the least critical exponent.
Towards a new conjecture

- **Dvořáková**, September 2021, WORDS 2021: *Critical exponent of balanced sequences*
  - conjecture $\frac{d-2}{d-3}$ refuted by examples over 11 and 12 letters
  - new conjecture: $\frac{d-1}{d-2}$ or $\frac{d}{d-1}$?
  - **Shur**: the lower bound $\frac{d-1}{d-2}$

- **D., Opočenská, Pelantová, Shur**, 2021: *On minimal critical exponent of balanced sequences*, arXiv 2021
  - new conjecture $\frac{d-1}{d-2}$ for $d \geq 11$
  - proven for even $d \geq 12$
Lower bounds

**Observation 7** If $4 \in \text{gap}(y, 1)$ and $a_1 = 1$ and $a_2 \geq 2$, then $E(v) \geq \frac{10}{9}$.

*Proof.* Use Proposition 5 with $u = a$ and $f = ababab^2ab$.

**Observation 8** If $6 \in \text{gap}(y', 2)$ and $a_1 = 1$, $a_2 = 2$ and $a_3 \geq 2$, then $E(v) \geq \frac{6}{5}$.

*Proof.* We use Proposition 5 with $u = b^2$ and $f = b^2abab^2aba$.

**Observation 9** If $7 \in \text{gap}(y', 2)$ and $a_1 \geq 2$, then $E(v) \geq \frac{6}{5}$.

*Proof.* We apply Proposition 5 with $u = b^2$ and the following $f$:

- If $a_1 \geq 4$, then $f = b^5ab^2$.
- If $a_1 = 3$, then $f = b^3ab^4a$.
- If $a_1 = 2$ and $a_2 \geq 2$, then $f = b^2ab^2ab^2ab$.
- If $a_1 = 2$ and $a_2 = 1$, then $f = b^2ab^3ab^2a$.

**Observation 10** If $8 \in \text{gap}(y', 2)$ and $a_1 \geq 2$, then $E(v) \geq \frac{7}{6}$.

*Proof.* We apply Proposition 5 with $u = b^2$ and the following $f$:

- If $a_1 \geq 5$, then $f = b^5ab^3$.
- If $a_1 \in \{3, 4\}$, then $f = b^3ab^4ab$.
- If $a_1 = 2$ and $a_2 \geq 3$, then $f = b^2ab^2ab^2ab^2a$.
- If $a_1 = 2$ and $a_2 \in \{1, 2\}$, then $f = b^2ab^3ab^2ab$. 
### Completion of table – continued

| $d$  | $\alpha$          | $y$      | $y'$                        | $E(v)$   | $E^*(v)$ |
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| 3    | [0, 2]            | (01)$^\omega$ | $2^\omega$                  | $2 + \frac{1}{\sqrt{2}}$ | $2 + \frac{1}{\sqrt{2}}$ |
| 4    | [0, 2, 1]         | (01)$^\omega$ | (23)$^\omega$                | $1 + \frac{1+\sqrt{5}}{4}$ | $1 + \frac{1+\sqrt{5}}{4}$ |
| 5    | [0, 2]            | (0102)$^\omega$ | (34)$^\omega$                | $\frac{3}{2}$ | $\frac{3}{2}$ |
| 6    | [0, 1, 2, 1, 1, 1, 1, 2] | 0$^\omega$ | (123415321435)$^\omega$     | $\frac{3}{3}$ | $\frac{4}{3}$ |
| 7    | [0, 1, 1, 3, 1, 2, 1] | (01)$^\omega$ | (234526432546)$^\omega$     | $\frac{5}{4}$ | $\frac{5}{4}$ |
| 8    | [0, 1, 3, 1, 2]   | (01)$^\omega$ | (234526732546237526432576)$^\omega$ | $\frac{6}{5} = 1.2$ | $\frac{12+3\sqrt{2}}{14} \approx 1.16$ |
| 9    | [0, 1, 2, 3, 2]   | (01)$^\omega$ | (234567284365274863254768)$^\omega$ | $\frac{7}{6} \approx 1.167$ | $1 + \frac{2\sqrt{2}-1}{14} \approx 1.13$ |
| 10   | [0, 1, 4, 2, 3]   | (01)$^\omega$ | (234567284963254768294365274869)$^\omega$ | $\frac{8}{7} \approx 1.14$ | $1 + \frac{\sqrt{13}}{26} \approx 1.139$ |
| 11   | [0, 1, 5, 1, 1, 1, 2] | (01)$^\omega$ | (234567892A436587294A638527496A832547698A)$^\omega$ | $\frac{10}{9} \approx 1.11$ | $\frac{415+5\sqrt{105}}{424} \approx 1.0996$ |
| 12   | [0, 1, 1, 3, 2]   | (012345)$^\omega$ | (6789AB)$^\omega$ | $\frac{11}{10} = 1.1$ | $\frac{8-\sqrt{2}}{6} \approx 1.0976$ |
| $d \geq 14$ even | [0, 1, 1, $\lfloor d/4 \rfloor$, 1] | (12...d/2)$^\omega$ | ($1'2'...d/2'$)$^\omega$ | $\frac{d-1}{d-2}$ | $1 + \frac{2}{\tau^{d-2}}$, where $\tau^{N+1} < d/2 < \tau^{N+2}$ |

**Table:** Baranwal, Rampersad, Shallit, Vandomme: $d$-ary balanced sequences with the least critical exponent.
Open problems

- Proof of conjecture $\frac{d-1}{d-2}$ for odd $d \geq 13$
  - using our computer program done for $13 \leq d \leq 33$

- Minimal asymptotic critical exponent of $d$-ary balanced sequences
  - Is there an analogy of Dejean’s conjecture for $E^*$?
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Thank you for attention