A GENERAL BACKWARDS CALCULUS OF VARIATIONS VIA DUALITY

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Abstract. We prove Euler-Lagrange and natural boundary necessary optimality conditions for problems of the calculus of variations which are given by a composition of nabla integrals on an arbitrary time scale. As an application, we get optimality conditions for the product and the quotient of nabla variational functionals.

1. Introduction

A time scale \( T \) is any nonempty closed subset of \( \mathbb{R} \). The theory of dynamic Euler-Lagrange equations is a recent field attracting considerable attention — see [5] [6] [7] [8] [13] [18] [21] [22] and references therein. For applications of the calculus of variations on time scales to economics see [1] [3] [17]. In particular, one obtains the well-known continuous [23], discrete [19], and quantum calculus of variations [4] by choosing \( \mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}, \) and \( q_N^\mathbb{N} := \{q^k | k \in \mathbb{N}_0 \}, \) \( q > 1, \) respectively.

This paper is dedicated to the study of general (non-classical) problems of calculus of variations on an arbitrary time scale \( T \). As a particular case, when \( T = \mathbb{R} \) one gets the generalized calculus of variations [12] with functionals of the form

\[
H \left( \int_a^b f(t, x(t), x'(t)) dt \right)
\]

where \( f \) has \( n \) components and \( H \) has \( n \) independent variables. Problems of calculus of variations of this form appear in practical applications of economics but cannot be solved using the classical theory (see [12] and the references therein).

In the literature of the calculus of variations on time scales, the problems are formulated and the results are proved in terms of the delta or the nabla calculus. Here we use a different approach. We make use of the duality technique recently introduced by Caputo [11], obtaining the results for the nabla variational problems directly from the results on the delta calculus of variations. Such duality theory has shown recently to be very useful in control theory [24].

In contrast with [20], we adopt here a backward perspective, which has proved useful, and sometimes more natural and preferable, with respect to applications in economics [11] [24] [3]. The advantage of the backward approach here promoted becomes evident when one considers that the time scales analysis has important

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implications for numerical analysts, who often prefer backward differences rather than forward differences to handle their computations. This is due to practical implementation reasons and better stability properties of implicit discretizations \cite{14, 24}.

The paper is organized as follows. In Section 2 some preliminaries on the recent duality theory on time scales are presented. Our results are then given in Section 3 we formulate the general (non-classical) problem of calculus of variations \cite{31} on an arbitrary time scale; we obtain a general formula for the Euler-Lagrange equations and natural boundary conditions (Theorem 3.2); and interesting corollaries are presented for the product (Corollary 3.1) and the quotient (Corollary 3.3). Finally, in Section 4 we illustrate the results of the paper with two examples.

2. Preliminaries

In this section we review some facts from \cite{11} which we need for the proof of our main result (Theorem 3.2). We begin by briefly recalling the basic definitions, notations, and facts concerning the delta and nabla differential calculus on time scales, which can be found in the monographs \cite{9, 10, 16}.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For each time scale $\mathbb{T}$ the following operators are used: the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$; the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\rho(t) := \sup \{ s \in \mathbb{T} : s < t \}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$; the forward graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$; and the backward graininess function $\nu : \mathbb{T} \rightarrow [0, \infty)$, defined by $\nu(t) = t - \rho(t)$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. We say that $t$ is isolated if $\rho(t) < t < \sigma(t)$, that $t$ is dense if $\rho(t) = t = \sigma(t)$. If $\sup \mathbb{T}$ is finite and left-scattered, we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{ \sup \mathbb{T} \}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood $U$ of $t$ such that $|f(\sigma(s)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$ for all $s \in U$. We call $f^\Delta(t)$ the derivative of $f$ at $t$ and $f$ is said delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. For $f : \mathbb{T} \rightarrow X$, where $X$ is an arbitrary set, we define $f^\sigma := f \circ \sigma$. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. We denote the set of all rd-continuous functions by $C^\mathrm{rd}_{\mathbb{T}} = C^\mathrm{rd}_{\mathbb{T}}(\mathbb{T}) = C^\mathrm{rd}_{\mathbb{T}}(\mathbb{T}; \mathbb{R})$. The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are delta differentiable and whose delta derivative is rd-continuous is denoted by $C^\mathrm{rd}_{\mathbb{T}} = C^\mathrm{rd}_{\mathbb{T}}(\mathbb{T}) = C^\mathrm{rd}_{\mathbb{T}}(\mathbb{T}; \mathbb{R})$.

In order to introduce the definition of nabla derivative, one defines a new set $\mathbb{T}_\kappa$ which is derived from $\mathbb{T}$ as follows: if $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{ m \}$; otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. In order to simplify expressions, and similarly as done with composition with $\sigma$, we define $f^\nabla(t) := f(\rho(t))$. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ if there is a number $f^\nabla(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ such that $|f(\rho(s)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon |\rho(t) - s|$ for all $s \in U$. We call $f^\nabla(t)$ the nabla derivative of $f$ at $t$. Moreover, we say that $f$ is nabla differentiable on $\mathbb{T}$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \rightarrow \mathbb{R}$. We say that function $f$ is ld-continuous if it is continuous at left-dense points and its right-sided limits exist (finite) at all right-dense points. The set of all ld-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by
and the calculus of variations. Typically, when one considers minimizing a function
This theory has been extended to many settings, including optimal control theory
in optimization theory that one learns in the earliest of courses beginning
present context, the word “duality” has a different meaning. The term duality does
not refer here to the classical concept, well known to researchers in optimization,
but to the one recently introduced in [11] (see also [24]).

Given a time scale \( \mathbb{T} \) we define the dual time scale \( \mathbb{T}^* := \{ s \in \mathbb{R} | -s \in \mathbb{T} \} \).
If \( \rho \) and \( \sigma \) denote the backward jump operator and the forward jump operator
associated to \( \mathbb{T} \), then we denote by \( \hat{\rho} \) and \( \hat{\sigma} \) the jump operators associated with \( \mathbb{T}^* \).
If \( \nu \) and \( \mu \) denote respectively the backward graininess function and the forward
graininess function associated to \( \mathbb{T} \), then we denote by \( \hat{\nu} \) and \( \hat{\mu} \) the graininess
functions associated to \( \mathbb{T}^* \).

Given a function \( f : \mathbb{T} \rightarrow \mathbb{R} \) we define the dual function \( f^* : \mathbb{T}^* \rightarrow \mathbb{R} \) by
\[ f^*(s) := f(-s) \text{ for all } s \in \mathbb{T}^*. \]
For a given quintuple \( (\mathbb{T}, \rho, \sigma, \nu, \mu) \) its dual will be
\( (\mathbb{T}^*, \hat{\rho}, \hat{\sigma}, \hat{\nu}, \hat{\mu}) \) where \( \hat{\rho}, \hat{\sigma}, \hat{\nu}, \hat{\mu} \) are given as follows:
\( \hat{\rho}(s) = -\sigma(-s), \hat{\sigma}(s) = -\rho(-s), \hat{\nu}(s) = \mu^*(s), \) and \( \hat{\mu}(s) = \nu^*(s). \)

Let \( f : \mathbb{T} \rightarrow \mathbb{R} \). The following holds:

(i) If \( f \) is delta (resp. nabla) differentiable at \( t_0 \in \mathbb{T}^* \) (resp. \( t_0 \in \mathbb{T}_n \)), then
\[ f^* \text{ is nabla (resp. delta) differentiable at } t_0 \in (\mathbb{T}^*)^* \text{ (resp. } t_0 \in (\mathbb{T}^*)_n\)\]
and \( f^*(t_0) = -(f^*)^*(t_0) \) (resp. \( f^*(t_0) = -(f^*)_\nabla(t_0) \)), where \( \Delta \) and \( \nabla \)
denote the delta and nabla derivatives for the time scale \( \mathbb{T} \); and \( \hat{\Delta} \) and \( \hat{\nabla} \)
denote the delta and nabla derivatives for the time scale \( \mathbb{T}^*. \)

(ii) \( f \) belongs to \( C_{\text{id}}^1 \) (resp. \( C_{\text{rd}}^1 \)) if and only if \( f^* \) belongs to \( C_{\text{id}}^1 \) (resp. \( C_{\text{rd}}^1 \)).

Let \( a, b \in \mathbb{T} \) with \( a \leq b \). We define the closed interval \( [a, b] \) in \( \mathbb{T} \) by \( [a, b] := \{ t \in \mathbb{T} : a \leq t \leq b \} \). Along this work we always assume that \( [a, b] \) denote an interval in a
given time scale \( \mathbb{T} \).

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a rd-continuous (resp. ld-continuous). Then,
\[ \int_a^b f(t) \Delta t = \int_{-b}^{-a} f^*(s) \hat{\nabla} s \text{ (resp. } \int_a^b f(t) \nabla t = \int_{-b}^{-a} f^*(s) \hat{\Delta} s\).

Lemma 2.1 ([11]). For a given Lagrangian \( L : [a, b]_\nu \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) the following
identity holds:
\[ \int_a^b L \left( t, x^\rho(t), x^\nabla(t) \right) \nabla t = \int_{-b}^{-a} L^* \left( s, (x^*)^\sigma(s), (x^*)^\nabla s \right) \hat{\Delta} s \]
for all functions \( x \in C_{\text{id}}^1 ([a, b]) \), where the dual Lagrangian \( L^* : [-b, -a]_\nu \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \)
is defined by \( L^*(s, x, v) = L(-s, x, -v) \) for all \( (s, x, v) \in [-b, -a]_\nu \times \mathbb{R} \times \mathbb{R} \).

A note on the use of the term duality is in order. Duality theory is a standard
concept in optimization theory that one learns in the earliest of courses beginning
with its applications in linear programming and then in nonlinear programming.
This theory has been extended to many settings, including optimal control theory
and the calculus of variations. Typically, when one considers minimizing a function
\( f(\cdot) \) over a set \( C \), the dual problem is one of maximizing a related function. In
the present context, the word “duality” has a different meaning. The term duality does
not refer here to the classical concept, well known to researchers in optimization,
but to the one recently introduced in [11] (see also [24]).
3. Main Results

Throughout we consider $A, B \in \mathbb{T}$ with $A < B$. Now let $[a, b]$ with $a, b \in \mathbb{T}$, $b < B$ and $a > A$, be a subinterval of $[A, B]$. The general (non-classical) problem of the calculus of variations on time scales under our consideration consists of extremizing (i.e., minimizing or maximizing)

$$\mathcal{L}[x] = H \left( \int_a^b f_1(t, x^\rho(t), x^{\nabla}(t)) \nabla t, \ldots, \int_a^b f_n(t, x^\rho(t), x^{\nabla}(t)) \nabla t \right)$$

(iii) for all $x \in C^1_{ld}$. Using parentheses around the end-point conditions means that these conditions may or may not be present. We assume that:

(i) the function $H : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives with respect to its arguments and we denote them by $H_i'$, $i = 1, \ldots, n$;

(ii) functions $(t, y, v) \to f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, $i = 1, \ldots, n$, have partial continuous derivatives with respect to $y, v$ for all $t \in [a, b]$ and we denote them by $f_{iy}$, $f_{iv}$;

(iii) $f_i$, $i = 1, \ldots, n$, and their partial derivatives are ld-continuous in $t$ for all $x \in C^1_{ld}$.

A function $x \in C^1_{ld}$ is said to be an admissible function provided that it satisfies the end-points conditions (if any is given). The following norm in $C^1_{ld}$ is considered:

$$||x||_1 = \sup_{t \in [a, b]} |x^\rho(t)| + \sup_{t \in [a, b]} |x^{\nabla}(t)|.$$

Definition 3.1. An admissible function $\tilde{x}$ is said to be a weak local minimizer (resp. weak local maximizer) for (3.1) if there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \leq \mathcal{L}[x]$ (resp. $\mathcal{L}[\tilde{x}] \geq \mathcal{L}[x]$) for all admissible $x$ with $||x - \tilde{x}||_1 < \delta$.

The next theorem gives necessary optimality conditions for problem (3.1).

Theorem 3.2. If $\tilde{x}$ is a weak local solution of problem (3.1), then the Euler-Lagrange equation

$$\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( \int_a^b f_{iy}(t, \tilde{x}^\rho(t), \tilde{x}^{\nabla}(t)) \nabla t \right) = 0$$

holds for all $t \in [a, b]$; where $\mathcal{F}_i[\tilde{x}] = \int_a^b f_i(t, \tilde{x}^\rho(t), \tilde{x}^{\nabla}(t)) \nabla t$, $i = 1, \ldots, n$. Moreover, if $x(a)$ is not specified, then

$$\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) \left( \int_a^{a'} f_{iuc}(t, \tilde{x}^\rho(t), \tilde{x}^{\nabla}(t)) \nabla t \right.$$

$$\left. - f_{iu}(a, \tilde{x}^\rho(a), \tilde{x}^{\nabla}(a)) \right) = 0;$$

if $x(b)$ is not specified, then

$$\sum_{i=1}^n H_i'(\mathcal{F}_1[\tilde{x}], \ldots, \mathcal{F}_n[\tilde{x}]) f_{iu}(b, \tilde{x}^\rho(b), \tilde{x}^{\nabla}(b)) = 0.$$
Proof. Since \( \tilde{x} \) is a weak local extremizer for (3.1), it follows by Lemma 2.1 that \( \tilde{x}^* \) is a weak local extremizer for the dual problem

\[
\mathcal{L}^*[x^*] = H \left( \int_{-b}^{-a} f_1^*(s, (x^*)^\sigma(s), (x^*)^\Lambda(s)) \Delta s, \right.
\]

\[
\ldots, \int_{-b}^{-a} f_n^*(s, (x^*)^\sigma(s), (x^*)^\Lambda(s)) \Delta s \bigg),
\]

\[
(x^*(-a) = x_a) \quad (x^*(-b) = x_b)
\]

over all \( x^* \in C^1_{rd}([-b, -a]). \) Applying [20 Theorem 3.2] we conclude that \( \tilde{x}^* \) satisfies the following conditions:

\[
\sum_{i=1}^{n} H'_i(\mathcal{F}^*_i[\tilde{x}^*], \ldots, \mathcal{F}^*_n[\tilde{x}^*]) \left( (f^*_i)(s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s)) \right.
\]

\[
\left. - (f'^*_i)(s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s)) \right) = 0
\]

for all \( s \in [-b, -a]^n \), where \( \mathcal{F}^*_i[\tilde{x}^*] = \int_{-b}^{-a} f_1^*(s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s)) \Delta s, i = 1, \ldots, n. \)

Moreover, if \( x^*(-b) \) is not specified, then

\[
(3.4) \quad \sum_{i=1}^{n} H'_i(\mathcal{F}^*_i[\tilde{x}^*], \ldots, \mathcal{F}^*_n[\tilde{x}^*]) f^*_i(-b, (\tilde{x}^*)^\sigma(-b), (\tilde{x}^*)^\Lambda(-b)) = 0;
\]

if \( x^*(-a) \) is not specified, then

\[
(3.5) \quad \sum_{i=1}^{n} H'_i(\mathcal{F}^*_i[\tilde{x}^*], \ldots, \mathcal{F}^*_n[\tilde{x}^*]) \left( f^*_i(\hat{\rho}(-a), (\tilde{x}^*)^\sigma(\hat{\rho}(-a)), (\tilde{x}^*)^\Lambda(\hat{\rho}(-a))) \right.
\]

\[
\left. + \int_{\hat{\rho}(-a)}^{-a} f'^*_i(s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s)) \Delta s \right) = 0.
\]

By Lemma 2.1

\[
\mathcal{F}^*_i[\tilde{x}^*] = \int_{-b}^{-a} f_1^*(s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s)) \Delta s
\]

\[
= \int_{-b}^{-a} f_i(t, \tilde{x}^\rho(t), \tilde{x}^\Lambda(t)) \Delta t
\]

\[
= \mathcal{F}_i[\tilde{x}],
\]

\( i = 1, \ldots, n, \) and \( t \in [a, b]. \) From the duality of the Euler-Lagrange equations [11 Theorem 6.10] it follows that

\[
(f^*_i)^\Lambda \left( s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s) \right) - f'^*_i \left( s, (\tilde{x}^*)^\sigma(s), (\tilde{x}^*)^\Lambda(s) \right)
\]

\[
= f^\Lambda_i \left( t, \tilde{x}^\rho(t), \tilde{x}^\Lambda(t) \right) - f^\Lambda_i \left( t, \tilde{x}^\rho(t), \tilde{x}^\Lambda(t) \right),
\]

\( i = 1, \ldots, n. \) This establishes relation (3.2). Now assume that \( x(b) \) is not specified. By duality, condition (3.4) holds. Since

\[
f^*_i(-b, (\tilde{x}^*)^\sigma(-b), (\tilde{x}^*)^\Lambda(-b)) = -f^*_i(b, \tilde{x}^\rho(b), \tilde{x}^\Lambda(b),
\]
If \( x(a) \) is not specified, then by duality condition (3.5) holds. Observe that

\[
  f_{iv}^*(\rho(-a), (\hat{x}^*)^\sigma(\rho(-a)), (\hat{x}^*)^\Delta(\rho(-a))) = -f_{iv}(\sigma(a), \hat{x}^\rho(\sigma(a)), \hat{x}^\nabla(\sigma(a)),
\]

and

\[
  \int_{\rho(-a)}^{-a} f_{iv}^*(s, (\hat{x}^*)^\rho(s), (\hat{x}^*)^\Delta(s)) \Delta s = \int_{a}^{\sigma(a)} f_{iv}(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) \nabla t.
\]

From the above it follows (3.3). □

Choosing \( T = \mathbb{R} \) in Theorem 3.2 we immediately obtain the following result:

**Corollary 3.3.** (Th. 3.1 and Eq. (4.1) in [12]) If \( \hat{x} \) is a solution of the problem

\[
  \mathcal{L}[x] = H \left( \int_{a}^{b} f_{1}(t, x(t), x'(t)) dt, \ldots, \int_{a}^{b} f_{n}(t, x(t), x'(t)) dt \right) \rightarrow \text{extr}
\]

\[
  (x(a) = x_a) \quad (x(b) = x_b)
\]

then the Euler-Lagrange equation

\[
  \sum_{i=1}^{n} H_i^*(\mathcal{F}_1[\hat{x}], \ldots, \mathcal{F}_n[\hat{x}]) \left( f_{iv}(t, \hat{x}(t), \hat{x}'(t)) - \frac{d}{dx} f_{iv}(t, \hat{x}(t), \hat{x}'(t)) \right) = 0
\]

holds for all \( t \in [a, b] \), where \( \mathcal{F}_i[\hat{x}] = \int_{a}^{b} f_{i}(t, \hat{x}(t), \hat{x}'(t)) dt, i = 1, \ldots, n \). Moreover, if \( x(a) \) is not specified, then

\[
  \sum_{i=1}^{n} H_i^*(\mathcal{F}_1[\hat{x}], \ldots, \mathcal{F}_n[\hat{x}]) f_{iv}(a, \hat{x}(a), \hat{x}'(a)) = 0;
\]

if \( x(b) \) is not specified, then

\[
  \sum_{i=1}^{n} H_i^*(\mathcal{F}_1[\hat{x}], \ldots, \mathcal{F}_n[\hat{x}]) f_{iv}(b, \hat{x}(b), \hat{x}'(b)) = 0.
\]

**Corollary 3.4.** If \( \hat{x} \) is a solution of the problem

\[
  \mathcal{L}[x] = \left( \int_{a}^{b} f_{1}(t, x^\rho(t), x^\nabla(t)) dt \right) \left( \int_{a}^{b} f_{2}(t, x^\rho(t), x^\nabla(t)) dt \right) \rightarrow \text{extr}
\]

\[
  (x(a) = x_a) \quad (x(b) = x_b)
\]

then the Euler-Lagrange equation

\[
  \mathcal{F}_2[\hat{x}] \left( f_{1v}(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) - f_{1y}(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) \right)
\]

\[
  + \mathcal{F}_1[\hat{x}] \left( f_{2v}(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) - f_{2y}(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) \right) = 0
\]
transforms as 

\[
\mathcal{F}[\tilde{x}] \left( \int_a^{\sigma(a)} f_1(t, \tilde{x}'(t), t) \nabla t - f_{1v}(\sigma(a), \tilde{x}'(\sigma(a)), \nabla \tilde{x}(\sigma(a))) \right)
\]

Moreover, if \( x(a) \) is not specified, then

\[
\mathcal{F}[\tilde{x}] \left( \int_a^{\sigma(a)} f_1(t, \tilde{x}'(t), t) \nabla t - f_{1v}(\sigma(a), \tilde{x}'(\sigma(a)), \nabla \tilde{x}(\sigma(a))) \right) + \mathcal{F}[\tilde{x}] \left( \int_a^{\sigma(a)} f_2(t, \tilde{x}'(t), t) \nabla t - f_{2v}(\sigma(a), \tilde{x}'(\sigma(a)), \nabla \tilde{x}(\sigma(a))) \right) = 0;
\]

if \( x(b) \) is not specified, then

\[
\mathcal{F}[\tilde{x}] f_{1v}(b, \tilde{x}'(b), \nabla \tilde{x}(b)) + \mathcal{F}[\tilde{x}] f_{2v}(b, \tilde{x}'(b), \nabla \tilde{x}(b)) = 0.
\]

Remark 3.5. In the particular case \( \mathbb{T} = \mathbb{R} \), Corollary 3.4 gives a result of [12]: the Euler-Lagrange equation associated with the product functional

\[
\mathcal{L}[x] = \left( \int_a^b f_1(t, x(t), x'(t)) dt \right) \left( \int_a^b f_2(t, x(t), x'(t)) dt \right)
\]

is

\[
\mathcal{F}[\tilde{x}] \left( f_{1v}(t, x(t), x'(t)) - \frac{d}{dt} f_{1v}(t, x(t), x'(t)) \right)
\]

\[
+ \mathcal{F}[\tilde{x}] \left( f_{2v}(t, x(t), x'(t)) - \frac{d}{dt} f_{2v}(t, x(t), x'(t)) \right) = 0
\]

and the natural condition at \( t = a \), when \( x(a) \) is free, becomes

\[
\mathcal{F}[\tilde{x}] f_{1v}(a, x(a), x'(a)) + \mathcal{F}[\tilde{x}] f_{2v}(a, x(a), x'(a)) = 0.
\]

Corollary 3.6. If \( \tilde{x} \) is a solution of the problem

\[
\mathcal{L}[x] = \left( \int_a^b f_1(t, x(t), x'(t)) dt \right) \left( \int_a^b f_2(t, x(t), x'(t)) dt \right)
\]

\[
\begin{align*}
\rightarrow \text{ext} \\
(x(a) = x_a) \quad (x(b) = x_b)
\end{align*}
\]

then the Euler-Lagrange equation

\[
\begin{align*}
f_{1v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) - f_{1v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) \\
- Q \left( f_{2v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) - f_{2v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) \right) = 0
\end{align*}
\]

holds for all \( t \in [a, b, \sigma] \), where \( Q = \frac{\mathcal{F}[\tilde{x}]}{\mathcal{F}_{1v}[\tilde{x}]} \). Moreover, if \( x(a) \) is not specified, then

\[
\int_a^{\sigma(a)} f_{1v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) dt - f_{1v}(\sigma(a), \tilde{x}'(\sigma(a)), \nabla \tilde{x}(\sigma(a)))
\]

\[
- Q \left( \int_a^{\sigma(a)} f_{2v}(t, \tilde{x}'(t), \nabla \tilde{x}(t)) dt - f_{2v}(\sigma(a), \tilde{x}'(\sigma(a)), \nabla \tilde{x}(\sigma(a))) \right) = 0;
\]

if \( x(b) \) is not specified, then

\[
f_{1v}(b, \tilde{x}'(b), \nabla \tilde{x}(b)) - Q f_{2v}(b, \tilde{x}'(b), \nabla \tilde{x}(b)) = 0.
\]
Remark 3.7. In the particular situation $T = \mathbb{R}$, Corollary 3.6 gives the following result of [12]: the Euler-Lagrange equation associated with the quotient functional

$$L[x] = \frac{\int_a^b f_1(t, x(t), x'(t))dt}{\int_a^b f_2(t, x(t), x'(t))dt}$$

is

$$f_1y(t, x(t), x'(t)) - Qf_2y(t, x(t), x'(t)) - \frac{d}{dt}[(f_1(t, x(t), x'(t)) - Qf_2(t, x(t), x'(t)))] = 0$$

and the natural condition at $t = a$, when $x(a)$ is free, becomes

$$f_1v(a, x(a), x'(a)) - Qf_2v(a, x(a), x'(a)) = 0.$$

4. Examples

Example 4.1. Consider the problem

$$L[x] = \left(\int_{-1}^0 (x^\nabla(t))^2\nabla t\right)\left(\int_{-1}^0 tx^\nabla(t)\nabla t\right) \rightarrow \min$$

$$x(-1) = 1, \quad x(0) = 0.$$ 

If $\tilde{x}$ is a local minimum of (4.1), then the Euler-Lagrange equation must hold, i.e.,

$$2\tilde{x}\nabla \nabla(t)Q_2 + Q_1 = 0,$$

where

$$Q_1 = \mathcal{F}_1[\tilde{x}] = \int_{-1}^0 (\tilde{x}\nabla(t))^2\nabla t, \quad Q_2 = \mathcal{F}_2[\tilde{x}] = \int_{-1}^0 t\tilde{x}\nabla(t)\nabla t.$$

If $Q_2 = 0$, then also $Q_1 = 0$. This contradicts the fact that on any time scale a global minimizer for the problem

$$\mathcal{F}_1[x] = \int_{-1}^0 (x^\nabla(t))^2\nabla t \rightarrow \min$$

$$x(-1) = 1, \quad x(0) = 0$$

is $\bar{x}(t) = -t$ and $\mathcal{F}_1[\bar{x}] = 1$. Hence, $Q_2 \neq 0$ and equation (4.2) implies that the extremals for problem (4.1) are those satisfying the delta differential equation

$$(4.3) \quad \tilde{x}\nabla \nabla(t) = -\frac{Q_1}{2Q_2}$$

subject to boundary conditions $x(-1) = 1$ and $x(0) = 0$. A solution of (4.3) depends on the time scale. Let us solve, for example, this equation on $T = \mathbb{R}$ and on $T = \{-1, -\frac{1}{2}, 0\}$. On $T = \mathbb{R}$ we obtain

$$(4.4) \quad x(t) = -\frac{Q_1}{4Q_2}t^2 - \frac{4Q_2 + Q_1}{4Q_2}t.$$ 

Substituting (4.4) into functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ gives

$$(4.5) \quad \begin{cases} \frac{48Q_2^2 + Q_1^2}{48Q_2^2} = Q_1 \\ \frac{12Q_2 - Q_1}{24Q_2} = Q_2. \end{cases}$$
Solving the system of equations (4.5) we obtain
\[
\begin{align*}
Q_1 &= 0 \\
Q_2 &= 0,
\end{align*}
\]
Therefore,
\[
\tilde{x}(t) = -t^2 - 2t
\]
is an extremal for problem (4.1) on \( T = \mathbb{R} \).

The solution of (4.3) on \( T = \{-1, -\frac{1}{2}, 0\} \) is
\[
x(t) = \begin{cases} 
1 & \text{if } t = -1 \\
\frac{1}{2} + \frac{Q_1}{16Q_2} & \text{if } t = -\frac{1}{2} \\
0 & \text{if } t = 0.
\end{cases}
\]

(4.6)

Constants \( Q_1 \) and \( Q_2 \) are determined by substituting (4.6) into functionals \( F_1 \) and \( F_2 \). The resulting system of equations is
\[
\begin{align*}
1 + \frac{Q_2^2}{16Q_2^2} &= Q_1 \\
\frac{1}{2} - \frac{Q_1}{32Q_2} &= Q_2.
\end{align*}
\]
Since system of equations (4.7) has no real solutions, we conclude that there exists no extremizer for problem (4.1) on \( T = \{-1, -\frac{1}{2}, 0\} \) among the set of functions that we consider to be admissible.

**Example 4.2.** Consider now the problem
\[
(4.8)
\]
If \( \tilde{x} \) is a local minimizer for (4.8), then the Euler-Lagrange equation must hold, i.e.,
\[
0 = [2\tilde{x}\nabla (t) - Q(1 + 2\tilde{x}\nabla (t))]\nabla t, \quad t \in [-2, 0],
\]
where
\[
Q = \frac{\int_{-2}^{0}(\tilde{x}\nabla (t))^2\nabla t}{\int_{-2}^{0}(\tilde{x}\nabla (t) + (\tilde{x}\nabla (t))^2)^2\nabla t}.
\]
Therefore,
\[
0 = 2\tilde{x}\nabla\nabla (t) - Q2\tilde{x}\nabla\nabla (t), \quad t \in [-2, 0].
\]
As \( x(-2) = 4 \) and \( x(0) = 0 \), we have \( Q \neq 1 \). Thus \( \tilde{x}\nabla\nabla (t) = 0 \). The solution of the delta differential equation \( x\nabla\nabla (t) = 0 \), \( x(-2) = 4 \), \( x(0) = 0 \), does not depend on the time scale: \( \tilde{x}(t) = -2t \) is an extremal for problem (4.8).

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