An Optimal Plank Theorem

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Abstract

It is shown that for any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a real Hilbert space $H$, there exists a unit vector $v$ in $H$ such that

$$|\langle v_k, v \rangle| \geq \sin(\pi/2n)$$

for all $k$. This a sharp version of the plank theorem for real Hilbert spaces.

Introduction

A plank in a vector space $X$ is the region bounded by two parallel hyperplanes. The classical plank problem, conjectured by Tarski, states that if an $n$-dimensional convex body is covered by a collection of planks, then the sum of the widths of the planks should be at least the minimal with of the convex body they cover. Tarski proved it for the particular cases of the unit disc and

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*The author was supported by the Mexican National Council of Science and Technology (CONACYT) grant no. CVU579817.
the 3-dimensional sphere. Bang [1] solved the problem for arbitrary convex bodies. Bang [1] also asked whether the widths of the planks could be measured with respect to the convex body that it is covered. Ball [2] answered affirmatively this affine version of the plank problem for the most interesting case: when the convex body in question is symmetric. Ball’s plank theorem can be seen as a generalization of the Hahn-Banach theorem, a sharp quantitative version of the uniform boundedness principle, or a geometric pigeon-hole principle.

A plank in a normed space $X$ is a region of the form
\[ \{ x \in X : |\phi(x) - m| \leq w \} \]
where $\phi$ is a linear functional on $X^*$ of norm 1, $m$ a real number, and $w$ is a positive number. The number $w$ is called the half-width of the plank. Ball’s affine plank theorem states the following.

**Theorem 1 (The Plank Theorem [2]).** For any sequence $(\phi_k)_{k=1}^{\infty}$ of norm one functionals on a real Banach space $X$, $(m_k)_{k=1}^{\infty}$ a sequence of real numbers and non-negative numbers $(t_k)_{k=1}^{\infty}$ satisfying
\[ \sum_{k=1}^{\infty} t_k < 1, \]
there exists a unit vector $x$ in $X$ for which
\[ |\phi_j(x) - m_j| > t_j \]
for every $j$.

The Plank theorem is obviously sharp in the sense that the unit ball of $X$ can be covered by $n$ non-overlapping parallel planks whose half-widths add up to 1.

In the present discussion, we are interested in the affine problem in the case that the planks covering the convex body are symmetric about the origin:
so we are only interested in planks of the form

\[ \{ x \in X : |\phi(x)| \leq w \} \]

where \( \phi \) is a linear functional on \( X^* \) of norm 1 and \( w \) is a positive number. In this case, Ball’s plank theorem states the following.

**Theorem 2 (The Plank Theorem).** For any sequence \( (\phi_k)_{k=1}^{\infty} \) of norm one functionals on a (real) Banach space \( X \) and non-negative numbers \( (t_k)_{k=1}^{\infty} \) satisfying

\[ \sum_{k=1}^{\infty} t_k < 1, \]

there exists a unit vector \( x \) in \( X \) for which

\[ |\phi_j(x)| > t_j \]

for every \( j \).

For an arbitrary Banach space, the condition that the sequence of positive sequence of numbers \( (t_k)_{k=1}^{\infty} \) add up to at most 1 is sharp. This can be seen by taking the space \( X \) to be \( \ell_1 \) and the collection \( \phi_i \) to be the standard basis vectors in \( \ell_\infty \). For other spaces, such as Hilbert spaces, one might expect to be able to improve upon this condition. Ball [3] proved that for complex Hilbert spaces it is possible to beat any sequence for which \( \sum_k t_k^2 = 1 \).

**Theorem 3 (Complex Plank Theorem [3]).** For any sequence \( v_1, v_2, \ldots, v_n \) of unit vectors in a complex Hilbert space and positive real numbers \( t_1, t_2, \ldots, t_n \) satisfying

\[ \sum_{k=1}^{n} t_k^2 = 1 \]

there exists a unit vector \( z \in \mathbb{R}^n \) such that

\[ |\langle v_k, z \rangle| \geq t_k \]

for all \( k \).
On the other hand, for real Hilbert spaces, this is clearly not possible. Consider $2n$ vectors $v_1, v_2, \ldots, v_{2n}$ in $\mathbb{R}^2$ equally spaced around the circle: ($n$ vectors and their negatives). For any unit vector $v$ in $\mathbb{R}^2$ there is an $i$ such that

$$|\langle v_i, v \rangle| \leq \sin(\pi/2n).$$

The purpose of this paper is to show that this simple example gives the sharp version of the plank theorem for real Hilbert spaces: an asymptotic improvement by a factor of $\pi/2$. The main theorem of the paper is thus the following.

**Theorem 4.** For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in $\mathbb{R}^n$, there exists a unit vector $v \in \mathbb{R}^n$ such that

$$|\langle v_i, v \rangle| \geq \sin(\pi/2n)$$

for all $i \in \{1, 2, \ldots, n\}$.

The basic strategy in the proof of theorem 4 is the strategy followed by Ball in the proof the complex plank problem, but there is a fundamental difference. The main ingredient of the proof of Theorem 3 has no analogue in the real case. In [3], Ball studies the behaviour of a complex polynomial locally around 1 and, with the aid of the maximum modulus principle, manages to jump away from 1 to a point in the unit disk where this polynomial has large absolute value. In contrast, the proof of Theorem 4 relies on the extremal properties of trigonometric polynomials to produce this jump.

For rest of the discussion, we shall work with the following rescaled version of Theorem 4 which will suit our purposes better. We also assume that $n \geq 2$ so as to eliminate from the discussion the trivial case when $n = 1$.

**Theorem 4'.** For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a real Hilbert space $H$, there exists a vector $v \in H$ of norm $\sqrt{n}$ for which

$$|\langle v_k, v \rangle| \geq \sqrt{n} \sin(\pi/2n)$$

for all $k$. 

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Acknowledgements

The author thanks Professor Keith M. Ball for his guidance throughout the development of this research work and his numerous remarks and suggestions to improve the presentation of the results.

1 Inverse eigenvectors

In this section we introduce the notion of inverse eigenvectors. An inverse eigenvector of a matrix $M$ is a vector $x$ satisfying the equation $Mx = x^{-1}$ where $x^{-1}$ is the inverse of $x$ componentwise. Inverse eigenvectors arose naturally in the solution of the complex plank problem. In his paper [3], Ball transforms the complex plank problem to a problem concerning the location of inverse eigenvectors of a complex Gram matrix. Seven years later, Leung, Li and Rakesh [4] reformulated the problem of finding the polarization constant of $\mathbb{R}^n$ in terms of inverse eigenvalues and described the structure of the inverse eigenvalues for real positive symmetric matrices. The term inverse eigenvector for a vector $x$ satisfying $Mx = x^{-1}$ turns up for the first time in [5], where Ambrus used the methods in [3] to reformulate the strong polarization problem as a geometric question concerning the location of inverse eigenvectors and managed to solve the strong polarization problem for the planar case. In order to motivate the definition of inverse eigenvector, let us go back to our question.

Our problem consists of finding a vector $v$ of norm $\sqrt{n}$ which has large inner product with all the vectors $v_1, v_2, \ldots, v_n$. An obvious candidate for this vector $v$ would be one for which $\min_k |\langle v_k, v \rangle|$ is maximal. However, there seems to be no simple way to either manipulate or obtain useful information from this maximal condition. Instead we choose a unit vector $v$ for which the product $\prod_i |\langle v_i, v \rangle|$ is maximal, hoping that each of the factors will be large enough to get the desired inequality. For the product, we can use simple analytic tools to study the points for which it is locally extremal. Luckily, the
structure of these local optimisers can be described concisely as the following proposition shows.

**Proposition 5.** Let \(v_1, v_2, \ldots, v_n\) be a sequence for unit vectors in a real Hilbert space \(H\). Suppose that \(v\) is vector of norm \(\sqrt{n}\) chosen so as to maximize

\[
\prod_{k=1}^{n} |\langle v_k, v \rangle|.
\]

Then,

\[
v = \sum_{k=1}^{n} \frac{1}{\langle v_k, v \rangle} v_k
\]

(1)

**Proof.** Since \(v\) is a stationary point, by the method of Lagrange multipliers, the gradients of the objective function and the constraint should be scalar multiples of one another. Hence, there exists a real number \(\lambda\) such that

\[
v = \lambda \sum_{k=1}^{n} \frac{v_k}{\langle v_k, v \rangle} \prod_{k=1}^{n} |\langle v_k, v \rangle|.
\]

(2)

This gives equation (1) up to a constant and taking inner product with \(v\) shows that the constant must be 1.

Denote by \(H\) the Gram matrix associated to the sequence of unit vectors \((v_k)_{k=1}^{n}\), that is, \(H_{ij} = \langle v_i, v_j \rangle\), and let \(w\) be the vector in \(\mathbb{R}^n\) given by

\[
w_k = \frac{1}{\langle v_k, v \rangle}
\]

(3)

for all \(k\). Then \(w\) satisfies

\[
(Hw)_j = \sum_{i=1}^{n} h_{ji} w_i = \langle v_j, \sum_{i=1}^{n} w_i v_i \rangle = \langle v_j, v \rangle = \frac{1}{w_j}.
\]

Therefore, \(w\) satisfies the following equation \(Hw = w^{-1}\) where \(w^{-1}\) is defined as the inverse of the vector \(w\) componentwise, i.e.

\[
w^{-1} = \left(\frac{1}{w_1}, \ldots, \frac{1}{w_n}\right).
\]

This observation leads us naturally to the following definition.
DEFINITION 6. Let $M$ be a $n \times n$ matrix. We say that $w$ is an inverse eigenvector of $M$ if

$$Mw = w^{-1}$$

(4)

In their paper [4], Leung, Li and Rakesh describe the structure of inverse eigenvectors for real Gram matrices. This is summarized in the following proposition.

PROPOSITION 7 ([4]). Let $H$ be a $n \times n$ real Gram matrix then:

a) there is at most one inverse eigenvector in each quadrant of $\mathbb{R}^n$,

b) there is an inverse eigenvector in a quadrant $Q$ of $\mathbb{R}^n$ if and only if

$$Q \cap \text{Ker}(H) = \{0\},$$

c) there is an inverse eigenvector in $Q$ of $\mathbb{R}^n$ if and only if $\prod_i |w_i|$ has a maximum in $Q$. Moreover, the maximizer in $Q$ is the unique eigenvector.

Proposition 7 shows that there are at most $2^n$ inverse eigenvectors for a given real Gram matrix $H$, in contrast with the complex case, where the equation $H \bar{z} = z^{-1}$ has infinitely many solutions.

On the other hand, given an inverse eigenvector $w$ of $H$, one can set

$$v = \sum_{k=1}^{n} w_k v_k.$$ 

It is clear that $v$ would satisfy equations (1) and (3). Theorem 4′ is thus a consequence of the following.

THEOREM 8. Let $H$ be a real Gram matrix. Then, there exist an inverse eigenvector $w$ of $H$ for which

$$\|w\|_{\infty} \leq n^{-\frac{1}{2}} \csc(\pi/2n).$$
To find a suitable eigenvector \( w \) notice that \( w \) defined as in equation (3) is a local extremal point for the function

\[
\prod_{k=1}^{n} |\langle v_k, v \rangle|,
\]

which in terms of \( w \) is given by

\[
\prod_{k=1}^{n} \frac{1}{|w_k|}
\]

subject to the constraint

\[
\|v\|^2 = \left\| \sum w_k v_k \right\|^2 = w^\top H w = n
\]

In the light of Proposition 5 this would suggest that we try to find a vector \( w \) to minimize \( \prod |w_k| \) subject to the constrain \( w^\top H w = n \). Unfortunately, this minimum is always 0. To deal with this problem, we choose \( u \) so as to maximize \( \prod |u_k| \) subject to the constrain \( w^\top H^{-1} w = n \), in the hope that the maximum would be converted into a minimum of the original problem via the natural bijection between the inverse eigenvectors of \( H \) and \( H^{-1} \). That is, if \( u \) is an inverse eigenvector of \( H^{-1} \), then \( w = u^{-1} \) is an inverse eigenvector of \( H \). We have the following lemma which is a slight variation of Lemma 7 in [3].

**Lemma 9.** Suppose that \( H \) is a real Gram matrix and \( w \) is a vector for which

\[
\prod_{k=1}^{n} |w_k|
\]

is locally extremal subject to the condition

\[
w^\top H w = n. \tag{5}
\]

Then, \( w \) is an inverse eigenvector for \( H \).
Lemma 9 yields a vector \( u \) for which \( \prod |u_k| \) is maximal subject to the constrain \( u^\top H^{-1} u = n \). Set \( w = u^{-1} \). Thus, \( w \) is an inverse eigenvector of \( H \). Moreover, since \( u \) has been selected as to maximize \( \prod |u_k| \) we have that if \( c \) is a vector such that \( \prod |c_k| = 1 \), then

\[
\prod |c_k u_k| = \prod |u_k|
\]

and therefore

\[
\sum_{jk} c_j u_j H_{jk}^{-1} c_k u_k \geq n
\]

The problem is to show \( \|w\|_\infty \leq n^{-\frac{1}{2}} \csc(\pi/2n) \).

If one were to prove the classic plank theorem for Hilbert spaces using inverse eigenvectors, one would only have to show the following weaker version of Theorem 8.

**Theorem 10.** Let \( H \) be a real Gram matrix. Then, there exist an inverse eigenvector \( w \) of \( H \) for which

\[
\|w\|_\infty \leq \sqrt{n}.
\]

This is an analogue of Bang’s lemma in [1] (also see [2] for a proof of Bang’s lemma in the form described here). To see this, we rewrite Theorem 10 as follows.

**Theorem 11.** Let \( H \) be a real Gram matrix. Then, there exist a vector \( w \) such that \( |w_i| \leq \sqrt{n} \) for all \( i \) and

\[
w_j \sum_k H_{jk} w_k = 1
\]

and recall that Bang’s lemma states slightly more than the following.

**Lemma 12.** Let \( H \) be a real Gram matrix. Then, there exist a vector of signs \( \varepsilon \in \{-1, 1\}^n \) such that

\[
\varepsilon_j \sum_k H_{jk} \varepsilon_k \geq \frac{1}{n}
\]
We will give a simple proof of Theorem 10 using inverse eigenvectors. Actually, we will prove something stronger.

**Theorem 13.** Let $H$ be a real $n \times n$ Gram matrix with $n \geq 2$. Then, there exists an inverse eigenvector $w$ of $H$ for which

$$
\|w\|_H = \sup_{\|x\|=1} (wx) \top H(wx) \leq \sqrt{n - 1}
$$

where $wx$ is just the coordinate-wise product of the vectors $w$ and $x$.

In terms of a sequence of vectors in a Hilbert space this says that if $v_1, \ldots, v_n$ is a sequence of unit vectors in a real Hilbert space $H$, then there exists a vector $w$ in $\mathbb{R}^n$ such that

$$
\left\| \sum x_k w_k v_k \right\|_H \leq \sqrt{n - 1} \|x\|_2
$$

for all vector $x$ in $\mathbb{R}^n$ and

$$
\langle v_k, \sum w_k v_k \rangle = \frac{1}{w_k} \geq \frac{1}{\sqrt{n - 1}}.
$$

This resembles a plank-type theorem of Nazarov [6], (also stated in [3]) that states the following.

**Theorem.** Let $f_i$ be unit functions in $L_1$ which satisfies

$$
\left\| \sum a_j f_j \right\| \leq M \|a\|_{\ell^2}
$$

for some $M$ and all $a \in \ell^2$. Let $t_j$ be a sequence of positive numbers with $\sum t_j^2 = 1$. Then there is a function $g \in L_\infty$ with norm at most $15M^2$ and

$$
| \langle f_j, g \rangle | \geq t_j
$$

for every $j$. 
2 The final transformation

In this section, we make a final transformation of the statement of Theorem 4 and then prove it. Define a matrix $M$ by

$$M_{ij} = w_j H_{jk} w_k$$

for all $j, k$. $M$ is a positive matrix and its inverse is given by

$$M^{-1}_{ik} = u_j H_{jk}^{-1} u_k.$$  

Observe that

$$m_{kk} = |w_k|^2.$$

If we denote by $1$ the vector whose entries are all equal to 1, then

$$(M1)_j = \sum_k w_j H_{jk} w_k$$

$$= w_j \sum_k H_{jk} w_k$$

$$= w_j (Hw)_j = 1$$

where the last identity is guaranteed by the fact that $w$ is an inverse eigenvector of $H$. Hence, to prove Theorem 8 it suffices to show the following.

**Lemma 14.** Suppose that $M$ is a symmetric positive matrix satisfying

- $M1 = 1$, and
- whenever $c$ is a vector such that $\prod |c_k| = 1$, then

$$c^\top M^{-1} c \geq n.$$  

Then $m_{kk} \leq n^{-1} \csc^2(\pi/2n)$ for all $k$.

In the same way Lemma 13 reduces to the following.
Lemma 15. Suppose that \( n \geq 2 \) and \( M \) is an \( n \times n \) symmetric positive matrix satisfying

- \( M\mathbf{1} = \mathbf{1} \), and
- whenever \( c \) is a vector such that \( \prod |c_k| = 1 \), then
  \[ c^\top M^{-1}c \geq n. \]

Then \( \|M\|_2 \leq n - 1 \).

We will first give the proof for Lemma 15 and make some useful remarks that lead us to the proof of Lemma 14.

Proof of lemma 15. First notice that if we let \( c = Mb \) then the second condition of the lemma can be restated as follows: \( \prod |(Mb)_k| = 1 \) implies

\[ b^\top Mb \geq n. \]

Or equivalently, for any \( b \) with

\[ b^\top Mb = n, \]

\( \prod |(Mb)_k| \leq 1 \). The proof consists of looking at 2-dimensional slices of the ellipsoid defined by

\[ \mathcal{E} = \{ x : x^\top Mx = n \}. \]

So we will “cut” \( \mathcal{E} \) with subspaces of dimension 2 of \( \mathbb{R}^n \) which contain the vector \( \mathbf{1} \). Thus, given a vector \( v \in \mathcal{E} \) orthogonal to \( \mathbf{1} \), we let \( H_v \) be the 2 dimensional subspace span by \( \mathbf{1} \) and \( v \),

\[ H_v = \text{span}\{\mathbf{1}, v\}. \]  \hspace{1cm} (6)

We denote by \( \mathcal{E}_v \) the ellipse we get by intersecting \( \mathcal{E} \) and \( H_v \),

\[ \mathcal{E}_v = \mathcal{E} \cap H_v. \]  \hspace{1cm} (7)
Notice that we can parameterize the ellipse $\mathcal{E}_v$ as follows: given an angle $\theta \in [0, 2\pi]$ we define

$$v_\theta = \cos \theta \mathbf{1} + \sin \theta \mathbf{v}.$$  \hfill (8)

Any vector in $\mathcal{E}_v$ is of the form (8) for some $\theta \in [0, 2\pi]$ and that every vector $v_\theta \in \mathcal{E}_v$ for every $\theta \in [0, 2\pi]$. Hence,

$$\mathcal{E}_v = \{v_\theta : \theta \in [0, 2\pi]\}.$$  

Define the trigonometric polynomial $T_v$ by

$$T_v(\theta) = \prod_{k=1}^{n} (Mv_\theta)$$

$$= \prod_{k=1}^{n} (\cos \theta + (Mv)_k \sin \theta)$$  \hfill (9)

$$= \prod_{k=1}^{n} (\cos \theta + (Mv)_k \sin \theta)$$  \hfill (10)

Notice that $T_v(0) = 1$. We now compute the first and second derivatives of $T_v$ at 0. For any $\theta$ such that $T_v(\theta)$ is not 0 we have

$$\frac{T'_v(\theta)}{T_v(\theta)} = -\sum_{i=1}^{n} \frac{\sin \theta - (Mv)_k \cos \theta}{\cos \theta + (Mv)_k \sin \theta}$$  \hfill (11)

Evaluating equation (11) at 0, we see that $T'_v(0) = 0$. Taking derivatives on both sides of equation (11) yields

$$\frac{T''_v(\theta)T_v(\theta) - (T'_v(\theta))^2}{T_v(\theta)^2} = -\sum_{i=1}^{n} \frac{1 + (Mv)^2_k}{(\cos \theta + (Mv)_k \sin \theta)^2}$$  \hfill (12)

Thus replacing $T_v(0) = 1$ and $T'_v(0) = 0$ in equation (12), we get

$$|T''_v(0)| = n + \|Mv\|^2$$

We are now in a position to apply the following well known inequality for trigonometric polynomials.
**Theorem** (Bernstein’s Inequality). Let $\mathcal{T}_n$ be the set of trigonometric polynomials of degree at most $n$. If $T \in \mathcal{T}_n$, then

$$\|T'\|_\infty \leq n \|T\|_\infty$$

(13)

where $\|T\|_\infty$ denotes the uniform norm of $T$ on $[0, 2\pi]$.

Applying Bernstein’s inequality twice, we get the following inequality for the second derivative of $T_v$,

$$\|T''_v\|_\infty \leq n^2 \|T_v\|_\infty.$$  

(14)

Since $v_\theta \in \mathcal{E}$,

$$|T_v(\theta)| = \prod_{k=1}^{n} |(Mv_\theta)| \leq 1$$

for all $\theta$ and thus

$$\|T_v\|_\infty \leq 1$$

for all $v \in \mathcal{E}$. Hence by inequality (14),

$$n + \|Mv\|^2 = |T''_v(0)| \leq \|T''_v\|_\infty \leq n^2$$

for all $v \in \mathcal{E}$ orthogonal to $1$. Therefore,

$$\|Mv\|^2 \leq n(n - 1)$$

(15)

for all $v \in \mathcal{E}$ orthogonal to $1$.

Let $v \in \mathcal{E}$ be an eigenvector orthogonal to $1$ associated to the possible largest eigenvalue $\lambda$. For this eigenvector $v$ we have that

$$\|Mv\|^2 = v^\top M^\top Mv = \lambda v^\top Mv = \lambda n$$

and hence by (15),

$$\lambda \leq n - 1.$$  

The norm $\|M\|_2$ is the maximum of $1$ and $\lambda$ which, in either case, is less than or equal to $n - 1$.  

\[\square\]
Remark 16. To get the classic result for Hilbert spaces, observe that
\[ m_{kk} = e_k^T M e_k \leq \|M\|_2 \leq n - 1 < n. \]

where \( e_k \) is the \( k \)-th canonical vector. This would give a proof for Lemma 10. However, one can try to make a better selection of the vector \( v \) so as to get a much better estimate of \( m_{kk} \) for all \( k \). In other words, we could select the 2 dimensional slice of \( \mathcal{E} \) more carefully so that we get a better bound for \( m_{kk} \) for all \( k \). The natural choice of 2 dimensional subspace to cut \( \mathcal{E} \) so as to get a better estimate for \( m_{kk} \) would be
\[ H = \{ x1 + ye_k : x, y \in \mathbb{R} \}. \]

However \( e_k \) is not orthogonal to \( 1 \) so we project it into the space orthogonal to \( 1 \). Doing so and normalizing so that the projection belongs to the ellipsoid \( \mathcal{E} \), we get that \( H \) is equal to \( H_{v_k} = \text{span}\{1, v_k\} \) where
\[ v_k = \frac{nc_k - 1}{\sqrt{nm_{kk} - 1}} \quad \text{(16)} \]

Then,
\[ \|M v_k\|^2 = \frac{n^2 \|Me_k\|^2 - n}{nm_{kk} - 1} \quad \text{(17)} \]

and
\[ \|Me_k\|^2 = \sum_{j=1}^{n} m_{kj}^2 = m_{kk}^2 + \sum_{j \neq k} m_{kj}^2. \quad \text{(18)} \]

and therefore by (15)
\[ \|M v_k\|^2 \leq n(n - 1). \quad \text{(19)} \]

Replacing equations (17) and (18) in (19) and rearranging, we obtain
\[ n^2 m_{kk}^2 + n^2 \sum_{j \neq k} m_{kj}^2 - n \leq n(n - 1)(nm_{kk} - 1). \]

On the other hand, we know that \( \sum_{j \neq k} m_{kj} = 1 - m_{kk} \) since \( M1 = 1 \) and so
\[ \frac{(m_{kk} - 1)^2}{n - 1} \leq \sum_{j \neq k} m_{kj}^2. \]
Hence,

\[ n^2 m_{kk}^2 + n^2 \frac{(m_{kk} - 1)^2}{n - 1} - n \leq n(n - 1)(nm_{kk} - 1). \]

Simplifying above inequality yields

\[ n^2 m_{kk}^2 - 2nm_{kk} + 1 \leq (n - 1)^2(nm_{kk} - 1). \]

Substituting \( t = nm_{kk} \) we get

\[ t^2 - 2t + 1 \leq (n - 1)^2(t - 1) \]

which is true if and only if

\[ 1 \leq t \leq 1 + (n - 1)^2 \]

which corresponds to

\[ \frac{1}{n} \leq m_{kk} \leq \frac{1 + (n - 1)^2}{n} \]  \hspace{1cm} (20)

For all \( k \).

The leftmost inequality of (20) is the minimal condition \( m_{kk} \) should satisfy since \( M \) is positive and \( M1 = 1 \). The right hand side gives an improvement over the classic plank theorem for Hilbert spaces. In other words, this grants that if \( v_1, \ldots, v_n \) is a set of unit vectors on a Hilbert space \( H \) there exists a unit vector \( v \in H \) such that

\[ |\langle v_i, v \rangle| \geq \frac{1}{\sqrt{1 + (n - 1)^2}} \]

However, this is far form being optimal. In fact, this is asymptotically equivalent to the classic result. We will give a slightly different argument for the optimal bound.

**Proof of lemma 14.** Notice that if we let \( c = Mb \) then the second condition of lemma \( 14 \) states that if \( \prod |(Mb)_j| = 1 \),

\[ b^\top Mb \geq n. \]
Let us assume, for a contradiction, that one of the diagonal entries is too large. Thus, assume that there exists \( k \) such that
\[
m_{kk} > \frac{1}{n \sin^2(\pi/2n)}
\]

Consider the following vector
\[
v_k^{(\alpha)} = -\sqrt{\alpha}v_k
\]
where \( v_k \) is defined as in (16) and
\[
\alpha = \frac{\cot^2(\pi/2n)}{nm_{kk} - 1}
\]

The first thing we should notice is that \( \alpha \in (0, 1) \). For each \( \theta \in [0, 2\pi] \) define
\[
v_{\theta}^{(\alpha)} = \cos \theta \textbf{1} + \sin \theta v_k^{(\alpha)}.
\]
(21)
It is easy to see that \( v_{\theta}^{(\alpha)} \) is just a parametrisation of a 2-dimensional ellipsoid inside \( E \). In fact, if \( \theta \neq 0 \) or \( \pi \),
\[
v_{\theta}^{(\alpha)^\top} M v_{\theta}^{(\alpha)} = n \cos^2 \theta + \alpha n \sin^2 \theta
\]
\[
= n(\cos^2 \theta + \alpha \sin^2 \theta) < n
\]
and \( v_{\theta}^{(\alpha)^\top} M v_{\theta}^{(\alpha)} = n \) if and only if \( \theta = 0 \) or \( \pi \).

Remark 17. \( \{v_{\theta} : \theta \in [0, 2\pi]\} \) is the ellipsoid that one gets by taking the slice with the 2-dimensional subspace \( H_{v_k} \) of the ellipsoid
\[
E^{(\alpha)} = \{ x : x^\top M^{(\alpha)} x = n \}
\]
where
\[
M^{(\alpha)} = \frac{1}{\alpha} M - \frac{1 - \alpha}{\alpha} \frac{1}{n} \otimes \textbf{1}
\]
In geometrical terms, we obtain the ellipsoid \( E^{(\alpha)} \) by shrinking the axes of the ellipsoid \( E \) by a factor of \( \alpha \) except for the axis \( \textbf{1} \) which remains untouched.
Define the trigonometric polynomial $T_{\nu_k}^{(\alpha)}$ by

$$T_{\nu_k}^{(\alpha)}(\theta) = \prod_{j=1}^{n} (M_{\nu_k}^{(\alpha)})_j$$

or equivalently

$$T_{\nu_k}^{(\alpha)}(\theta) = \prod_{j=1}^{n} \left( \cos \theta + (M_{\nu_k}^{(\alpha)})_j \sin \theta \right). \quad (22)$$

Notice that

$$(M_{\nu_k}^{(\alpha)})_k = -\sqrt{\alpha} (M_{\nu_k})_k = \cot \left( \frac{\pi}{2n} \right)$$

and so the $k$-th factor of $T_{\nu_k}^{(\alpha)}$ is equal to 0 if and only if

$$\cos \theta = \cot \left( \frac{\pi}{2n} \right) \sin \theta$$

which happens if and only if $\theta = \frac{\pi}{2n}$ or $\pi + \frac{\pi}{2n}$. Hence, $T_{\nu_k}^{(\alpha)}$ has a root at $\theta = \frac{\pi}{2n}$ and $\pi + \frac{\pi}{2n}$. Expanding the product we get

$$T_{\nu_k}^{(\alpha)}(\theta) = \cos^n \theta + \sum_j (M_{\nu_k}^{(\alpha)})_j \cos^{n-1} \theta \sin \theta + \sin^2 \theta \psi(\theta)$$

where $\psi$ is a trigonometric polynomial of degree at most $n - 2$. On the other hand,

$$\sum_j (M_{\nu_k}^{(\alpha)})_j = \mathbf{1}^\top M_{\nu_k}^{(\alpha)} = \mathbf{1}^\top \nu_k^{(\alpha)} = 0$$

and therefore,

$$T_{\nu_k}^{(\alpha)}(\theta) = \cos^n \theta + \sin^2 \theta \psi(\theta). \quad (23)$$

It is easy to see that $\cos n\theta$ is of the form (23); thus, taking the difference of $T_{\nu_k}^{(\alpha)}(\theta)$ and $\cos n\theta$ we get

$$Q(\theta) = T_{\nu_k}^{(\alpha)}(\theta) - \cos n\theta$$

$$= \sin^2 \theta \psi(\theta)$$

where $\psi$ is a trigonometric polynomial of degree at most $n - 2$. 18
Observe that $Q$ has roots at 0 and $\pi$, where $T_{v_k}^{(\alpha)}$ and $\cos n\theta$ are both 1, and at $\frac{\pi}{2n}$ and $\pi + \frac{\pi}{2n}$, where both functions are equal to 0.

For a contradiction, let us assume that

$$|T_{v_k}^{(\alpha)}(\theta)| < 1$$

for all $\theta \in \left[ \frac{\pi}{n}, \frac{(n-1)\pi}{n} \right] \cup \left[ \frac{(n+1)\pi}{n}, \frac{(2n-1)\pi}{n} \right]$. The extrema of $\cos n\theta$ on $[0, 2\pi)$ are located at $\theta_k = \frac{kn}{n}$ for $k \in \{0, \ldots, 2n-1\}$ so

$$\text{sgn } Q(\theta_k) = (-1)^{k+1}.$$

Thus, by the intermediate value theorem, for each $k \in \{1, \ldots, n-1\} \cup \{n+1, \ldots, 2n-2\}$ there is a $\varphi_k \in \left( \frac{kn}{n}, \frac{(k+1)n}{n} \right)$ such that $Q(\varphi_k) = 0$.

Hence, $Q$ has $2n$ distinct roots on the interval $[0, 2\pi)$. However, $Q(\theta) = \sin^2(\theta)\psi(\theta)$ so it could not have more than $2n - 2$ distinct roots.

Hence, there exist $\theta \in \left[ \frac{\pi}{n}, \frac{(n-1)\pi}{n} \right] \cup \left[ \frac{(n+1)\pi}{n}, \frac{(2n-1)\pi}{n} \right]$ such that

$$|T_{v_k}^{(\alpha)}(\theta)| = \prod_j |(M_{v_\theta}^{(\alpha)})_j| \geq 1$$

and $v_\theta^{(\alpha)\top} M_{v_\theta}^{(\alpha)} < n$ which is a contradiction to the second condition of Lemma 14. \(\Box\)

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