A remark on the structure of torsors under an affine group scheme

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1 Introduction

Any affine group scheme $G$ over a field $k$ is the inverse limit over a directed poset $I$ of affine group schemes of finite type over $k$. Let $P$ be a torsor under $G$, i.e. a non-empty affine scheme over $k$ with a $G$-action such that the morphism $G \times_k P \to P \times_k P, (g, p) \mapsto (gp, p)$ is an isomorphism.

Theorem 1 Consider an algebraically closed field $k$ and an affine group scheme $G$ over $k$ written as above as an inverse limit $G = \varprojlim G_i$ over $I$ of algebraic groups $G_i$. Assume that one of the following conditions holds:

i) The poset $I$ is (at most) countable;

ii) The cardinality of $I$ is strictly less than the one of $k$.

Then any $G$-torsor $P$ over $k$ is trivial, i.e. $P(k) \neq \emptyset$.

The proof of the first part relies on the following proposition. The second part is an application of a Hilbert Nullstellensatz in infinitely many dimensions.

Proposition 2 Any torsor $P$ under an affine group scheme $G$ over a field $k$ is the directed inverse limit of affine schemes of finite type over $k$ with faithfully flat transition maps. Here the directed poset over which the limit is taken, can be chosen to have the same cardinality as $I$, a poset for $G$. 
The theorem implies the following result which motivated the present note.

**Corollary 3** Let $\mathcal{T}$ be a neutral Tannakian category over an algebraically closed field $k$. Assume that there is a set $J$ of objects which generate $\mathcal{T}$ as a tensor category satisfying one of the following two conditions:

i) $J$ is countable;

ii) The cardinality of $J$ is strictly less than the one of $k$.

Then any two fibre functors of $\mathcal{T}$ over $k$ are isomorphic.

### 2 Proofs

Consider an affine group scheme $G = \text{spec } A$ over a field $k$ given as an inverse limit $G = \lim_{\rightarrow} \tilde{G}_i$ over a directed poset $I$ of affine group schemes $\tilde{G}_i$ of finite type over $k$. Such a representation is always possible by [W2] 3.3 Corollary. Writing $\tilde{G}_i = \text{spec } \tilde{A}_i$, the projection $G \to \tilde{G}_i$ corresponds to a homomorphism of commutative Hopf algebras $\tilde{A}_i \to A$ over $k$. The image $A_i$ in $A$ of $\tilde{A}_i$ is a sub Hopf algebra of $A$ by [MM] Lemma 4.6 ii). The transition maps between the $\tilde{A}_i$'s correspond to inclusion maps in $A$ between the $A_i$'s and the natural Hopf-algebra map

$$\lim_{\rightarrow} \tilde{A}_i \hookrightarrow \lim_{\rightarrow} A_i \subset A$$

is an isomorphism. Hence we have

$$\lim_{\rightarrow} \tilde{A}_i \cong \lim_{\rightarrow} A_i = A.$$  

Setting $G_i = \text{spec } A_i$ we have $G = \lim_{\rightarrow} G_i$ with the same poset $I$ as before and the coordinate rings $A_i$ now being sub Hopf algebras of $A$. For $j \geq i$ in $I$ the inclusion $A_i \subset A_j$ makes $A_j$ a faithfully flat $A_i$-algebra and $G_j \to G_i$ a faithfully flat morphism by [W2] 14.1 Theorem.

Let $P = \text{spec } B$ be a $G$-torsor. The isomorphism $G \times_k P \cong P \times_k P$ corresponds to an isomorphism of $R$-algebras $A \otimes_k R \cong B \otimes_k R$ where we have set $R = B$. As any $k$-algebra, $R$ is faithfully flat. The image $Q_i$ in $B \otimes_k R$ of $A_i \otimes_k R$ is a finitely generated $R$-sub Hopf algebra of $B \otimes_k R$ and for $j \geq i$ the inclusion $Q_i \subset Q_j$ in $B \otimes_k R$ makes $Q_j$ a faithfully flat $Q_i$-algebra. Choose finitely many generators $q^{(\alpha)}$ of the $R$-algebra $Q_i$ and write them in the form $q^{(\alpha)} = \sum_v b^{(\alpha)}_v \otimes r^{(\alpha)}_v$ with...
$b^{(a)}_\omega \in B$ and $r^{(a)}_\omega \in R$. Consider the subalgebra $B_i \subset B$ over $k$ generated by the elements $b^{(a)}_\omega$. Since the composed maps

$$B_i \otimes_k R \hookrightarrow Q_i \otimes_k R \xrightarrow{\text{mult}} Q_i \subset B \otimes_k R$$

is the inclusion $B_i \otimes_k R \subset B \otimes_k R$ it follows that $B_i \otimes_k R \subset Q_i$ and by construction of $B_i$ also $B_i \otimes_k R = Q_i$ in $B \otimes_k R$. The inclusion maps $Q_i \subset Q_j$ do not necessarily descend to the $B_i$’s. To remedy this, let $\Omega$ be the set of finite subsets $\omega \subset I$ having a maximal element. Denote by $i(\omega) \in \omega$ the (uniquely determined) maximal element of $\omega$. Setting $\omega_1 \leq \omega_2$ if $\omega_1 \subset \omega_2$ we get a partial ordering of $\Omega$. It is directed since $I$ is directed: For $\omega_1, \omega_2 \in \Omega$ set $\omega_3 = \omega_1 \cup \omega_2 \cup \{k\}$ where $k \in I$ is such that $k \geq i(\omega_1), k \geq i(\omega_2)$. Then $\omega_3 \in \Omega$ and $\omega_1 \leq \omega_3, \omega_2 \leq \omega_3$. For every $\omega \in \Omega$ let $B_\omega$ be the $k$-algebra in $B$ generated by the $k$-algebras $B_i$ for $i \in \omega$. For $i \in \omega$ we have $i \leq i(\omega)$ and hence $B_i \otimes_k R = Q_i \subset Q_{i(\omega)} = B_{i(\omega)} \otimes_k R$. It follows that $B_\omega \otimes_k R = Q_{i(\omega)}$. In particular $B_\omega$ is non-zero. For $\omega_1 \leq \omega_2$ the inclusion $B_{\omega_1} \subset B_{\omega_2}$ is faithfully flat since the inclusion of $R$-algebras $B_{\omega_1} \otimes_k R = Q_{i(\omega_1)} \subset Q_{i(\omega_2)} = B_{\omega_2} \otimes_k R$ is faithfully flat. Tensoring the inclusion

$$\varprojlim \omega B_\omega \subset B$$

with $R$, we obtain an isomorphism

$$\left( \varprojlim \omega B_\omega \right) \otimes_k R = \varprojlim \omega (B_\omega \otimes_k R) = \varprojlim \omega Q_{i(\omega)} = B \otimes_k R.$$ 

Hence we have $B = \varprojlim \omega B_\omega$. This implies proposition 2.

With notations as before we set $P_\omega = \text{spec} B_\omega$. Since $B_\omega \neq 0$ we have $P_\omega \neq \emptyset$. For $\omega_1 \leq \omega_2$ the morphism $P_{\omega_2} \rightarrow P_{\omega_1}$, is faithfully flat and in particular surjective. Since $P_{\omega_1}$ and $P_{\omega_2}$ are of finite type over $k$, the morphism induces a surjection between the sets of closed points $|P_{\omega_2}| \rightarrow |P_{\omega_1}|$. If $k$ is algebraically closed the closed points are in natural bijection with the $k$-rational points, and hence the map $P_{\omega_2}(k) \rightarrow P_{\omega_1}(k)$ is surjective. Note that $P_\omega(k) \neq \emptyset$ by the Nullstellensatz since $P_\omega \neq \emptyset$ is of finite type over $k$. If $I$ is countable, then $\Omega$ is countable as well and

$$P(k) = \varprojlim \omega P_\omega(k)$$

is an inverse limit of non-empty sets where the transition functions are surjective. Inductively we can choose a cofinal sequence $\omega_1 \leq \omega_2 \leq \ldots$ in $\Omega$ and it
follows that

\[ P(k) = \lim_{\nu} P_{\omega}(k) \]

is non-empty. This proves the assertion of theorem 1 under assumption i).

Note that for uncountable directed posets \( I \) an inverse limit of non-empty sets \( X_i \) with surjective transition maps \( \pi_{ji} \) can well be empty, see e.g. [W1]. On the other hand according to [HM] Proposition 2.7 the inverse limit \( \lim_{\nu} X_i \) will be non-empty (and the projections \( \pi_{ji} \) are closed surjective continuous maps between quasicompact non-empty \( T_1 \)-spaces \( X_i \)). For the Zariski topologies on the \( P_\omega(k) \) all conditions are satisfied except that the transition maps need not map closed sets to closed sets. In fact Hochschild and Moore who are dealing with groups \( X_i \) use a suitable coset topology for their arguments. It does not seem to be applicable here. I do not know if torsors for general affine group schemes over an algebraically closed field are trivial.

We now prove the assertion of theorem 1 under condition ii). For finite \( I \) the group \( G \) is algebraic and the assertion well known. We may therefore assume that \( I \) is infinite. With notations as before, we have \( A = \lim_{\nu} A_i \) and \( B = \lim_{\nu} B_\omega \) where the cardinalities of the index sets \( I \) and \( \Omega \) are the same. The algebra \( B_\omega \) being finitely generated over \( k \) it follows that \( B \) is a quotient of the polynomial ring \( S = k[T_i, i \in I] \) in infinitely many variables \( T_i \) indexed by \( I \). Since \( B \neq 0 \) because of \( P \neq \emptyset \), there is an ideal \( s \) in \( S \) with \( B \cong S/s \) and \( s \neq S \).

We now refer to [L] Theorem from which it follows that for \( \text{card } I < \text{card } k \) the ideal \( s \) has an “algebraic zero”. This implies that there is a maximal ideal \( m \) in \( S \) with \( s \subset m \) and \( S/m = k \). The composition

\[ B \cong S/s \longrightarrow S/m = k \]

provides a \( k \)-rational point of \( P \) as claimed.

Generalizations of the Hilbert Nullstellensatz to polynomial rings in infinitely many variables go back to Krull [K] § 3, Satz 4 who treats the case of countable \( I \) and remarks that the case of arbitrary \( I \) could be done along the same lines.

Corollary 3 is a consequence of theorem 1 because for two fibre functors \( \eta_1 \) and \( \eta_2 \) of \( \mathcal{T} \) the affine \( k \)-scheme \( P = \text{Isom}^{\otimes}(\eta_1, \eta_2) \) is a torsor under the affine group scheme \( G = \text{Aut}^{\otimes}(\eta_1) \). Arguing as in the proof of [DM] Proposition 2.8 one sees that \( G \) satisfies the condition of theorem 1 by our assumption on \( \mathcal{T} \). Hence \( P(k) \neq \emptyset \) and thus there exists an isomorphism between \( \eta_1 \) and \( \eta_2 \) over \( k \).
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