Models for Pairing Phenomena

Xiang-Xiang Sun and Shan-Gui Zhou *

Abstract  Pairing effects manifests themselves in many aspects in nuclear systems ranging from finite nuclei to nuclear matter and compact stars. Although with some specific features for nuclear systems, the mechanism of pairing between nucleons in these systems resembles that of electrons in superconductors. The Bardeen-Cooper-Schrieffer (BCS) theory, the first successful and microscopic theory for superconductivity, and the Bogoliubov transformation, the generalization of the BCS theory, have been widely used to describe pairing correlations in nuclear systems. To deal with the problem of particle number non-conservation in the BCS method and generalized Bogoliubov transformation, particle number projection techniques as well as several approaches which keep the particle number conserved, have been proposed. In the study of exotic nuclei, which are quantum open systems, the continuum contributions have to be taken into account. In this chapter, a thorough but brief discussion of pairing effects in nuclear systems will be introduced. Then nuclear models dealing with pairing correlations in nuclear structure properties will be presented to different extent of details. Although formulas are given, the emphasis is mainly put on the basic ideas concerning these models.

Outline

• Effects of nucleon pairing

Xiang-Xiang Sun
School of Nuclear Science and Technology, University of Chinese Academy of Sciences, Beijing 100049, China; CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China; e-mail: sunxiangxiang@ucas.ac.cn

Shan-Gui Zhou
CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China; School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China; e-mail: sgzhou@itp.ac.cn

* corresponding author
Since the early stage of nuclear physics, it has been known that an atomic nucleus is more stable if it consists of even number(s) of protons and/or neutrons, as is clearly seen both from the fact that there are much more even-even stable isotopes than odd-mass and odd-odd ones and from the odd-even effects in nuclear masses or binding energies. Such observations indicate that protons (neutrons) like to be coupled into pairs in atomic nuclei. Indeed, pairing effects have been observed in many essential nuclear phenomena [1–6]:

The odd-even effect in binding energy

The binding energy of an odd-even nucleus is found to be smaller than the arithmetic mean of binding energies of its two neighboring even-even nuclei. One can verify this fact by calculating, e.g., the neutron gaps for even-even nuclei

\[
\Delta(N, Z) = -\frac{1}{2}[E_B(Z, N-1) + E_B(Z, N+1) - 2E_B(Z, N)],
\]

with the latest nuclear masses given in AME2020 [7–9] and can find that they are almost larger than 1 MeV systematically. \(N\) and \(Z\) are the number of neutrons and charge number, respectively. Similar conclusions can also be drawn for the proton gaps. Note that, besides pairing effects, the odd-even mass staggering is also related to other mechanisms [10–12], such as the shell structure, single particle properties, and deformations.

Energy spectra

The ground states of all even-even nuclei have the spin-parity \(I^\pi = 0^+\), which can be understood as that two nucleons occupying time-reversal states couple with each other via the pairing interaction. For an even-even nucleus, the excitation energy of the first non-collective excited state can be well estimated as the energy corresponding to a broken pair [3]. The situation in an odd-even nucleus is different: The ground state of an odd-even nucleus is almost completely determined by the last unpaired nucleon and it can have many excited states in the same energy interval.

The moment of inertia

The moment of inertia of deformed nuclei can be extracted from the rotational spectra. The systematics of excitation spectra show that the moment of inertia from the pure single particle scheme deviates by a factor of two from the experimental values [3]. But when the pairing is included, theoretical calculations are in good agreement with experimental data. This is due to the
Models for Pairing Phenomena

The level density  

In the case of a few nucleons occupying a single $j$-shell, the energy spectra can be constructed and are all energetically degenerate corresponding to the various possibilities of angular momentum coupling. The number of states per energy unit can easily be estimated and it is found that the energy spectrum is directly related to the pairing strength.

Deformations  

In the well accepted mean-field picture, if proton or neutron orbitals below a major shell are all occupied, the nucleus generally has a spherical shape in its ground state. However, for open shell nuclei, single particle levels can be partially filled due to pairing correlations. The quantum correlations originating from the mix of different single particle orbitals drive these nuclei to be deformed. This is reflected by the fact that there is a shape transition from spherical shapes for closed shell nuclei to well deformed shapes for nuclei with half-filled shells.

The nuclear fission  

The spontaneous fission is a decay mode in which a nucleus splits into two or more lighter nuclei. This process requires the system to tunnel under the fission barrier. The character of the system, whether it is superfluid or not, affects much the fission dynamics. The occurrence of a systematic difference between even- and odd-mass nuclei in the fission of, e.g., $^{238}$U isotopes, is associated with pairing effects.

Halos  

Most of halo nuclei are close to or located at drip lines. For nuclei close to drip lines, the valence nucleons are weakly bound and pairing correlations provide the possibility to scatter valence nucleons back and forth in the continuum. Once the valence nucleons occupy orbitals with low orbital angular momenta, say, $s$- or $p$-wave, a halo can be formed. Therefore pairing correlations play a crucial role in the formation of nuclear halos. The occupation of $s$- or $p$-wave orbitals by weakly bound nucleon(s) leads to that halo nuclei have a pure neutron (proton) matter with a very low density surrounding a dense core, where pair condensate or strong di-neutron correlations may occur. Thus, halo nuclei may be an ideal prototype to search for evidence of induced pairing through surface vibrational coupling. The pygmy dipole resonances as a result of the oscillation between the core and the low density neutron matter are also connected with pairing correlations. Furthermore, halo configurations are related to other aspects of nuclear structures, including deformations and the shell evolution, and the interplay among them results in more fascinating and complicated phenomena.

Two-nucleon decays  

Nuclei are unbound with respect to nucleon(s) emission beyond the drip lines. This feature implies some new radioactivities of which mostly discussed are one- or two-nucleon decay. A handful of ground state two-proton emitters have been observed due to the Coulomb interaction which leads to a Coulomb barrier hindering the escape of protons from the parent nucleus. There may be also two-neutron radioactivity and the only candidate identified so far is $^{26}$O. The information of two-nucleon decay is particularly important for revealing the angular and energy correlations between
emitted nucleons [31], which are highly correlated to the pairing at the initial stage of the process. 

The backbending In a well deformed superfluid nucleus, the rotational spectrum follows the $I(I+1)$ law and the moment of inertia is almost a constant below a certain critical value of rotational frequency. In some deformed nuclei, the observed spectra indicate that there is a very steep increase of moment of inertia at high angular momenta. This is because when a deformed nucleus starts to rotate, the Coriolis force acts in opposite directions on the two nucleons of each time-reversal pair. As a consequence, the rotation tends to align the spins of the nucleons by successive breaking of pairs with high angular momentum.

The neutron-proton pairing For nuclei with $N \approx Z$, the protons and neutrons near the Fermi levels occupy identical orbitals, which allows for the appearance of pairs consisting of a neutron ($n$) and a proton ($p$). The binding energies show the characteristic $T(T+1)$ isorotational dependence on isospin $T$, which means the presence of an isovector pair condensate that rotates in isospace. Such rotation implies that the existence of the $np$ condensate is on an equal footing with the $nn$ and $pp$ condensates [32].

Pairing rotations The energies of $I^\pi = 0^+$ states relative to that of a reference nucleus are quadratic as a function of the number of additional pairs. A typical example is the energies of $0^+$ states of Sn isotopes with the reference nucleus $^{116}$Sn [33,34]. This parabolic behavior is similar to the rotational band in deformed nuclei. The pairing rotation in atomic nuclei is one kind of Nambu-Goldstone modes as a sequence of the spontaneous symmetry breaking in the $U(1)$ gauge space, namely the condensation of Cooper pairs [5,33,35,36].

In one word, pairing manifests itself in many aspects of nuclear physics. Besides the above-mentioned nuclear structure features, pairing is also related to nuclear astrophysics, such as the thermal evolution of neutron stars and glitches in pulsar stars [4]. In nuclear reactions, a typical example is that the enhanced two-nucleon transfer cross section is understood as arising from the collective pairing states [13]. This chapter mainly focuses on pairing effects in the study of nuclear structure.

2. The pairing mechanism

2.1 The pairing forces

Atomic nuclei are quantum many-body systems composed of protons and neutrons. Apart from relatively weak electric forces, the interaction between two protons is very similar to that between two neutrons. The isospin degree of freedom is used to distinguish proton ($\tau = 1/2$, $\tau_z = -1/2$) and neutron ($\tau = 1/2$, $\tau_z = +1/2$) and a nuclear state can be labeled with the isospin quantum number $T$, with the third component $T_z = (N - Z)/2$ for a nucleus with $N$ neutrons and $Z$ protons. When the relative orbital angular momentum $L$ is even, two nucleons can couple to $T = 0$
(isoscalar) and $T = 1$ (isovector) pairs with the spin $S$ of 1 or 0, respectively. Thus two neutrons can form a pair with $T = 1$, $T_z = 1$, and $S = 0$ and two protons with $T = 1$, $T_z = -1$, and $S = 0$. A neutron and a proton can couple to $S = 0$ and isospin $T = 1$ with $T_z = 0$ (isovector) or $T = 0$ (isoscalar) and $S = 1$ in order to ensure the antisymmetry of the total wave function of the two nucleons. The possible types of nucleon-nucleon pairs are listed in Table 1.

Table 1 Possible types of nucleon-nucleon pairs with isospin ($T$ and $T_z$) and total spin ($S$).

| type | $T$ | $T_z$ | $S$ |
|------|-----|-------|-----|
| pp   | 1   | -1    | 0   |
| pn   | 1   | 0     | 0   |
| nn   | 1   | 1     | 0   |
| pn   | 0   | 0     | 1   |

Pairing correlations and the phenomena associated with superfluidity in nuclear physics directly rely on the underlying interaction, i.e., the nucleon-nucleon ($NN$) interaction. Nowadays, the bare $NN$ interaction, the interaction of two nucleons in the free space, can be constructed by using some methods to reproduce nucleon-nucleon scattering phase shifts in different partial waves labeled by $^{2S+1}L_J$, where $S$, $L$, and $J$ represent the total spin, orbital, and total angular momenta, respectively. The bare $NN$ interaction is characterized by a strong repulsive core at short distances. The pairing gap is mainly determined by the attractive part of $NN$ interactions. In the $^1S_0$ channel, the $NN$ interaction is attractive when the inter nucleon distance is larger than $\sim 0.6$ fm. In pure neutron matter, neutrons can form Cooper pairs in the weak-coupling limit with this attractive interaction. One can also expect that the system undergoes a Bose-Einstein condensation into a single quantum state in the strong coupling regime \[4\]. The pairing interactions in other channels are also very important, especially for the study of dynamical and thermal evolution of neutron stars. For finite nuclei, although various approaches starting from realistic nuclear forces have been developed and produced encouraging results, there are still some issues to be explored further, such as the understanding of pairing correlations from bare $NN$ interactions, the influence of the medium polarization on pairing in finite nuclei, and impacts from Coulomb and three-nucleon forces \[37\]. Thus it is more straightforward to understand the pairing force in a phenomenological way.

The idea of a pairing interaction was already proposed in the early developments of the traditional shell model \[38\], in which single particles move under a central potential with a strong spin-orbit interaction. One of the most important effects of the pairing interaction is that it can couple two identical nucleons to spin zero. This can be understood in the case of two nucleons interacting with a short-range attractive effective interaction in a single $j$-shell. The simplest example is the $\delta$-interaction with the strength $V_0$, $V(r_{12}) = -4\pi V_0 \delta(r_1 - r_2)$. \[2\]
Two identical nucleons in a shell model orbital with angular momentum $j$ coupling to a total angular momentum $J$ have a wave function $|jjJM\rangle$ and the interaction energy is

$$E_J = \langle jjJM|V(r_{12})|jjJM\rangle = -\frac{2j+1}{2}V_0 I(j) \left| \langle J\frac{1}{2}|\frac{1}{2}\rangle \right|^2,$$  

(3)

where

$$I(j) = \int R_j^2(r_{12})r_{12}^2dr_{12},$$  

(4)

is an integral depending on the radial wave function $R_j(r_{12})$ of the level $j$, and $\langle J\frac{1}{2}|\frac{1}{2}\rangle$ is the Clebsch-Gordan coefficient. The energy of the $J = 0$ state can be simplified as

$$E_0 = -\frac{2j+1}{2}V_0 I(j),$$  

(5)

and the energies of other states can be obtained,

$$E_2 \sim \frac{1}{4}E_0, \quad E_4 \sim \frac{9}{64}E_0, \quad E_6 \sim \frac{25}{256}E_0, \quad \cdots.$$  

(6)

This result shows that the interaction energy of the $J = 0$ state is much smaller than other states and thus the $J = 0$ pair is energetically favored.

Besides this $\delta$ interaction, many phenomenological pairing forces have been proposed, such as a constant pairing force \cite{3}, zero-range pairing force with or without density-dependence \cite{39,40}, Gogny force \cite{41}, and separable pairing force of finite range \cite{42}. They have been widely applied in modern nuclear density functional theories and can describe nuclear superfluidity successfully \cite{17,43–45}. Moreover, to simultaneously describe the density dependence of the neutron pairing gap for both symmetric and neutron matter, an isospin dependence in the effective pairing interaction has also been developed \cite{46,47} and applied to study the properties of finite nuclei \cite{48–50}.

As mentioned above, the long-range attractive part of the bare $NN$ force can lead to the nuclear superfluidity. Thus many efforts have been made to incorporate realistic forces, such as low-momentum $NN$ interactions and Argonne $v_{14}$ and $v_{18}$, to investigate nuclear structure by using mean-field methods \cite{51–54}. It has been shown that energy gaps for semi-magic nuclei can be reproduced in such kinds of calculations. It should be noted that the implementation of realistic forces is technically much more complicated than phenomenological pairing forces due to the complexities of bare $NN$ forces. The microscopic understanding of pairing correlations starting from the underlying forces is still an open question up to now.
2.2 Pairing models

Various approaches have been developed to understand the pairing phenomena in nuclear systems. The seniority model \[55,56\] solves the problem of \( N \) particles occupying the single \( j \)-shell and can be used to understand why the ground states of even-even nuclei have spin zero and the spin of an odd-mass nucleus is the same as the spin of the last unpaired nucleon. In 1950s the superconductivity in metals was understood by that two electrons of opposite momenta attract each other to form a bound state with zero momentum, which is known as the Cooper pair and it behaves like a boson \[57,58\]. This is the basic idea of the BCS theory of superconductivity \[58\]. The characteristic of a metallic superconductor is primarily the large energy gap in the spectrum, corresponding to the energy required to break a Cooper pair. Following the suggestions of Bohr, Mottelson, and Pines \[59\], the BCS theory was used to study atomic nuclei by Belyaev \[60\]. The BCS wave function can be generalized by using the Bogoliubov-Valatin transformation \[61,62\]. Both the BCS approximation and generalized Bogoliubov transformation have been widely applied in various mean-field models nowadays to incorporate pairing correlations. Since the BCS-type wave function does not keep the conservation of particle number, to deal with this problem many methods have been proposed, such as the Lipkin-Nogami method \[63,64\], exact solutions \[65,66\] and particle number conserved methods \[67\] for the pairing Hamiltonian, and the particle number projection technique in mean-field models \[68,70\]. In the following sections, the basic formulas of these methods and several relevant topics are introduced.

3. The seniority model

The ground state of an even-even nucleus has the spin-parity \( I^\pi = 0^+ \) and the spin of an odd-even nucleus is the same as that of last unpaired nucleon. This fact can be understood by using the seniority model, in which \( N \) nucleons interact through a constant pairing force in a single \( j \)-shell with the degeneracy of \( 2j + 1 \) \[55\]. Setting single particle energies of the orbitals in the \( j \)-shell to be zero, then the pairing Hamiltonian is

\[
H = -G \sum_{m, m'>0} a^\dagger_m a_{-m} a_{m'} a_{-m'} = -G \hat{S}_+ \hat{S}_- ,
\]

(7)

where \( m \) is the projection of \( j \) on the \( z \)-axis and the pair creation and annihilation operators are defined as

\[
\hat{S}_+ \equiv \sum_{m} s_+^{(m)} = \sum_{m=1}^{\Omega} a^\dagger_m a_{-m}, \quad \hat{S}_- = (\hat{S}_+)^\dagger ,
\]

(8)
with the number of paired states \( \Omega = (2j + 1)/2 \). The quasi-spin operator proposed in Ref. [71] is one of the methods to solve this problem and will be introduced here.

For each substate \( m \), let
\[
\begin{align*}
\hat{s}^+_m &= a^\dagger_m a^\dagger_{-m}, \\
\hat{s}^-_m &= a_{-m} a_{m}, \\
\hat{s}^0_m &= \frac{1}{2} \left( a^\dagger_m a_m + a^\dagger_{-m} a_{-m} - 1 \right).
\end{align*}
\]
One can get the commutation relations as:
\[
\begin{align*}
\left[ \hat{s}^+_m, \hat{s}^-_m \right] &= 2\hat{s}^0_m, \\
\left[ \hat{s}^+_m, \hat{s}^0_m \right] &= \hat{s}^-_m, \\
\left[ \hat{s}^0_m, \hat{s}^-_m \right] &= -\hat{s}^+_m.
\end{align*}
\]
It is found that such commutation properties are the same as those of angular momentum operators. \( \hat{s}^0_m \) has two eigenvalues of \(-1/2\) and \(1/2\) corresponding to whether the pair \((m, -m)\) is full or empty. Thus the operator \( \hat{s}^0_m \) is similar to the spin operator and is called “quasi-spin”. The total spin vector is
\[
\hat{S} = \sum_{m > 0} \hat{s}^0_m,
\]
and its third component is
\[
\hat{S}_0 = \sum_{m > 0} \hat{s}^0_m = \frac{1}{2} \sum_{m > 0} \left( a^\dagger_m a_m + a^\dagger_{-m} a_{-m} - 1 \right) = \frac{1}{2} \left( \hat{N} - \Omega \right),
\]
with the particle number operator \( \hat{N} \).

The pairing Hamiltonian can be rewritten as
\[
H = -G \left( \hat{S}^2 - \hat{S}_0^2 + \hat{S}_0 \right),
\]
and the pairing energy reads
\[
E(N, S) = -G \left[ S(S+1) - \frac{1}{4}(N-\Omega)^2 + \frac{1}{2}(N-\Omega) \right].
\]
When the number of particles is even, \( S = \Omega/2, \Omega/2-1, \cdots, \) and \( |N-\Omega|/2 \).

When the particle number is odd, \( S = (\Omega-1)/2, (\Omega-3)/2, \cdots, |N-\Omega|/2 \). If one defines \( S = (\Omega - \sigma)/2 \) with \( \sigma \) being the seniority quantum number, the energy is
\[
E(N, \sigma) = -\frac{G}{4} \left[ \sigma(\sigma+1) - 2\sigma(\Omega + 1) + 2N(\Omega + 1) - N^2 \right].
\]
The *seniority* quantum number represents the number of unpaired particles in \( j \)-shell. The corresponding pairing gaps are \( G(2\Omega + 1)/2 \) for even particle number and \( G\Omega \) for odd particle number. This simple model shows that

- The ground state for an even system has the minimal seniority \( \sigma = 0 \) and \( \sigma = 1 \) for an odd system;
- For even \( N \), the first excited state is given by \( \sigma = 2 \) and the excitation energy is \( G\Omega \), which is independent on \( N \);
- When the particle number is much smaller than the degeneracy of single \( j \)-shell \( (N \ll \Omega) \), the ground state energy increases with \( G\Omega \) multiplied by the number of pairs. This is related to the pair vibrational spectrum and a typical example is neutron pair vibration based on \(^{208}\text{Pb}\) \([72]\).

The nucleus \(^{210}\text{Po}\) can be taken an example. Its low-lying excited states with \( I^\pi = 2^+, 4^+, \) and \( 6^+ \) are nearly degenerate and the excitation energy of the \( 2^+ \) state is 1.18 MeV. This can be well explained by the seniority scheme as two protons occupying the \( 1h_9/2 \) orbital, which gives the excitation energy of the first excited state about 1 MeV with the pairing gap estimated by the empirical formula \((13.43 \pm 1.38)A^{-0.48\pm0.03} \) MeV \([73]\).

In addition, the seniority model is useful for nuclei with the number of protons (neutrons) close to magic numbers for which the configurations can be well described by single \( j \) components. There are many states which are the mixture of several \( j \) configurations even in semi-magic nuclei \([74,75]\). To study these nuclei, the seniority model has been generalized to the case of multi \( j \)-shell \([76,78]\). Recently, the generalized seniority model has been successfully applied to study various properties of Sn isotopes \([79,81]\).

### 4. The BCS model

Although the seniority model is successful for the description of the energy gaps related with the nuclear superfluidity, it is limited to the study of nuclei close to magic ones. The BCS model was firstly proposed to study superfluidity in metals \([58]\) and later applied to nuclear systems. This method can be combined easily with the mean-field models as shown in Refs. \([82,83]\) and has been widely applied to study nuclei close to the \( \beta \)-stability line. In this section, the BCS theory will be presented.

### 4.1 The BCS approximation

In fermion systems like atomic nuclei, the Kramers degeneracy ensures the existence of pairs of degenerate and mutually time-reversal conjugate states \((k, \bar{k})\). Nucleons occupying such states could be coupled strongly by a short-range force. Under the
independent particle approximation, the ground state properties are described by
filling the single particle levels from the bottom up to the Fermi level. As will be
shown below, in this case the occupation probability of each single particle level is
either zero or one. Due to the pairing interaction, pairs of nucleons can be scattered
from the levels below the Fermi level to those above. One then has to deal with
the occupation probabilities ranging from zero to one. Mathematically, this could
be solved by introducing the concept of quasiparticles. For stable nuclei, the BCS
approximation has turned out to be very useful.

The model Hamiltonian with a single particle term and a residual two-body in-
teraction is
\[
H = \sum_{n_1 n_2} e_{n_1 n_2} a_{n_1}^\dagger a_{n_2} + \frac{1}{4} \sum_{n_1 n_2 n_3 n_4} \bar{v}_{n_1 n_2 n_3 n_4} a_{n_1}^\dagger a_{n_2}^\dagger a_{n_4} a_{n_3},
\]
(16)
where \(\bar{v}_{n_1 n_2 n_3 n_4} = \langle n_1 n_2 | v | n_3 n_4 \rangle - \langle n_1 n_2 | v | n_4 n_3 \rangle\). This problem can be solved by
using the BCS approximation. The wave function for an even-even nucleus can be
approximated by the BCS wave function as
\[
|\text{BCS}\rangle = \prod_{k > 0} (u_k + v_k a_k^\dagger a_{\bar{k}}^\dagger) |0\rangle,
\]
(17)
where \(u_k\) and \(v_k\) represent variational parameters. The product runs over half of
the configuration space \((k > 0)\). \(|0\rangle\) represents the vacuum state of single particles. \(v_k^2\)
and \(u_k^2\) represent the probability that a certain pair of states \((k, \bar{k})\) is occupied and
empty, respectively. The norm of the state requires
\[
u_k^2 + u_k^2 = 1.
\]
(18)
It should be mentioned that generally \(u_k\) and \(v_k\) are complex and it is reasonable to
choose real \(u_k\) and \(v_k\) to satisfy the variation principle [3]. For \(k > 0\), that means \(\bar{k} < 0\), one has
\[
u_k^2 = u_k, \quad v_k^2 = -v_k.
\]
(19)
u_k and \(v_k\) can be determined in such a way that the total energy of the system is
minimal. As is seen in Eq. (17), the BCS wave function does not conserve the par-
ticle number. Under the condition that the expectation value of the particle number
operator has the desired value \(N\)
\[
\langle \text{BCS} | \hat{N} | \text{BCS} \rangle = 2 \sum_{k > 0} v_k^2 = N,
\]
(20)
the variational Hamiltonian reads
\[
H' = H - \lambda \langle \hat{N} \rangle.
\]
(21)
\(\lambda\) is the chemical potential or Fermi energy, which represents the increase of the
energy with respect to a change in the particle number,
The particle-number uncertainty reads

$$(\Delta N)^2 := \langle \text{BCS}|N^2|\text{BCS}\rangle - N^2 = 4 \sum_{k>0} u_k^2 v_k^2,$$

(23)

The expectation value of $H'$ with respect to the BCS state is

$$\langle \text{BCS}|H'|\text{BCS}\rangle = \sum_k \left[ (\varepsilon_{kk} - \lambda) v_k^2 + \frac{1}{2} \sum_{k'} \bar{v}_{kk'k} v_k^2 v_{k'} \right] + \sum_{kk'} \bar{v}_{kk'} u_k u_{k'} v_{k'} v_{k'}. \quad (24)$$

Under the condition of $v_k^2 + u_k^2 = 1$, the BCS wave function can be determined after obtaining $v_k$. The variation condition

$$\delta \langle \text{BCS}|H'|\text{BCS}\rangle = 0,$$

(25)

yields

$$\left[ \frac{\partial}{\partial v_k} \frac{v_k}{u_k} \frac{\partial}{\partial u_k} \right] \langle \text{BCS}|H'|\text{BCS}\rangle = 0. \quad (26)$$

After differentiating, one can get the BCS equation

$$2\tilde{\varepsilon}_k u_k v_k + \Delta_k (v_k^2 - u_k^2) = 0, \quad k > 0, \quad (27)$$

with

$$\tilde{\varepsilon}_k = \varepsilon_{kk} + \frac{1}{2} \sum_{k'} (\bar{v}_{kk'k} + \bar{v}_{kk'k'}) v_{k'}^2 - \lambda, \quad (28)$$

and

$$\Delta_k = - \sum_{k' > 0} \bar{v}_{kk'k} u_k u_{k'}. \quad (29)$$

Fig. 1 Occupation number as a function of single particle energy for pairing gap $\Delta = 0$ (left panel) and $\Delta \neq 0$ (right panel).
There is a trivial solution for the BCS equation [cf. Eq. (27)], i.e., the pairing gap \( \Delta = 0 \), \( v^2_k = 1 \) while \( u^2_k = 0 \) for all single particle states below the Fermi level \( (\tilde{\varepsilon}_k \leq \lambda) \). The non-trivial solution of the BCS equation is

\[
v^2_k = \frac{1}{2} \left[ 1 - \frac{\tilde{\varepsilon}_k}{\sqrt{\tilde{\varepsilon}_k^2 + \Delta^2_k}} \right], \quad u^2_k = \frac{1}{2} \left[ 1 + \frac{\tilde{\varepsilon}_k}{\sqrt{\tilde{\varepsilon}_k^2 + \Delta^2_k}} \right].
\]

Then the gap equation can be obtained by inserting Eq. (30) into Eq. (29)

\[
\Delta_k = -\frac{1}{2} \sum_{k' > 0} \bar{v}_{k'k} \bar{v}_{k'k} \frac{\Delta_{k'}}{\sqrt{\tilde{\varepsilon}_{k'} + \Delta_{k'}}}.
\]

In general, these equations are nonlinear and can be solved iteratively. One can get the occupation probability of each single particle level determined by the mean-field potential, shown in Fig. 1. For the trivial solution, \( v^2_k \) is a step function of the single particle energy. In the case of finite pairing gap, the single particle levels around the Fermi energy \( \lambda \) are partially occupied and particles can be scattered from below to above the Fermi energy due to pairing correlations.

The above-mentioned variation procedure usually leads to the ground state of the system, which can be compared with the results given by the seniority model to address the limitation of the BCS method when applying to nuclear systems. In the case of \( N \) particles occupying single \( j \)-shell, for the ground state with the seniority zero, one can find

\[
v_k = \sqrt{\frac{N}{2\Omega}}, \quad u_k = \sqrt{1 - \frac{N}{2\Omega}}.
\]

The pairing energy with the BCS model is

\[
E^{(N)}_{\text{BCS}} = -\frac{1}{2} GN\Omega \left( 1 - \frac{N}{2\Omega} + \frac{N}{2\Omega^2} \right),
\]

and this is close to the result obtained from Eq. (15) when \( \Omega \gg N \). The uncertainty in the particle number is given by

\[
\frac{\Delta N}{N} = \sqrt{\frac{2}{N} - \frac{N}{2\Omega}}.
\]

Thus the BCS approximation is suitable to study the ground state with a large particle number. For those systems with small particle number, the uncertainty from the BCS method is relatively large.

A way to suppress the error due to particle number fluctuation is the Lipkin-Nogami approximation [63,64], in which the variation Hamiltonian reads

\[
H'' = H - \lambda_1 \langle \hat{N} \rangle - \lambda_2 \langle \hat{N}^2 \rangle,
\]
where $\lambda_1$ represents the Lagrange multiplier used to constrain the average particle number and the parameter $\lambda_2$ is determined by

$$\lambda_2 = \frac{\langle \hat{H} \Delta \hat{N}_2^2 \rangle}{\langle \hat{N}^2 \Delta \hat{N}_2^2 \rangle}.$$  

(36)

$\hat{N}_2$ is the term of the particle number operator which projects onto two-quasiparticle states and $\Delta \hat{N}_2^2 = \hat{N}_2^2 - \langle \hat{N}_2^2 \rangle$ represents its variance. In this case, the Fermi energy is given by $\lambda = \lambda_1 + 4 \lambda_2 (N + 1)$. The Lipkin-Nogami method is an approximation to the particle number projected BCS theory and has also been applied in modern density functional calculations [84–87] to consider the energy corrections caused by the particle number fluctuation.

The wave function given in Eq. (17) corresponds to the case where all particles are paired. But for odd-mass or odd-odd nuclei, the valence particle is unpaired. Therefore, Eq. (17) cannot be directly used. One needs to block the valence orbital $k_b$ with $v_{k_b}^2 = 1$ and its time-reversal state is empty. The corresponding wave function reads [3, 88]

$$a_{k_b}^{\dagger} \prod_{k > 0, k \neq k_b} (u_k + u_k a_{k_b}^{\dagger} a_{k_b}^{\dagger}) |0\rangle.$$  

(37)

Applying this wave function to solve the variation problem, one can find that the occupation probabilities for other states are the same as Eq. (30). The unpaired particle does not contribute to the pairing gap and the total particle number is $1 + 2 \sum_{k \neq k_b} v_{k_b}^2$. The changes on pairing gaps and occupation probabilities caused by blocking unpaired particle are called blocking effects. It should be mentioned that this blocking procedure breaks the time-reversal symmetry and leads to the appearance of currents, which can be avoided by using the equal filling approximation [89, 90]. This method will be introduced in Sec. 5.3.

### 4.2 The BCS approximation with resonant states

Nowadays, with the worldwide development of radioactivity-ion-beam facilities, more and more exotic nuclear phenomena have been observed [15, 25, 91, 92]. For a suitable description of exotic nuclei, the contribution from the continuum should be included [31, 93, 94]. However, the conventional BCS method is only valid for bound states and not justified for exotic nuclei because it cannot include properly the contribution of continuum states [95, 96]. For the extremely neutron-rich (or proton-rich) nuclei near drip lines, the contribution from the continuum plays an important role as schematically shown in Fig. 2. The conventional BCS approach involves unphysical states and the density contributed from continuum is non-local. For these nuclei, one must either investigate the detailed properties of continuum states and include the coupling between the bound state and the continuum by extending the BCS method to resonant BCS (rBCS) theory or treat pairing correlations by using
the generalized Bogoliubov transformation. In the following, the way to include the contribution of resonant states within the rBCS method will be given.

Resonant states are important for determining the pairing properties of the ground state of nuclei far from the $\beta$-stability line [17]. Considering the contribution of continuum with the BCS approximation can make this method suitable for the study of exotic nuclei. This idea was first realized in the Hartree-Fock framework [97]. The main point is that the continuum can be discretized and regarded as a set of discrete states with the corresponding level density, more details can be found in Refs. [97, 98]. In the rBCS method, pairing gaps for bound states are

$$\Delta_i = \sum_j V_{i,j}^2 u_j v_j + \sum_\nu V_{\nu,i} \int g_\nu(\epsilon) u_\nu(\epsilon) v_\nu(\epsilon) d\epsilon,$$  \hspace{0.5cm} (38)

and averaged pairing gaps for resonances are

$$\Delta_\nu = \sum_j V_{\nu,j}^2 u_j v_j + \sum_{\nu'} V_{\nu',\nu} \int g_{\nu'}(\epsilon') u_{\nu'}(\epsilon') v_{\nu'}(\epsilon') d\epsilon', \hspace{0.5cm} (39)$$

where $V$ is the interaction matrix element and $g_\nu(\epsilon)$ is the total level density, which takes into account the variation of the localization of scattering states in the energy region of a resonance (i.e., the width effect) and goes to a delta function in the limit of a very small width. Both bound and resonant states contribute to the total particle number, which reads

$$N = \sum_i v_i^2 + \sum_\nu \int g_\nu(\epsilon) v_\nu^2(\epsilon) d\epsilon.$$  \hspace{0.5cm} (40)

The implementation of the rBCS method in mean-field models can be achieved straightforwardly by substituting the gap equations in the conventional BCS method.

![Figure 2](image-url)
5. The generalized Bogoliubov transformation

Many properties of nuclei can be described in terms of independent particle moving in a mean-field potential [3]. In modern nuclear density functional theories, this mean-field potential can be obtained by using the Hartree or Hartree-Fock (HF) method. By combining mean-field models with the BCS method, the pairing interaction is treated as a residual interaction and the occupation probabilities are determined by solving the BCS equations after obtaining single particle levels. In other words, the BCS method does not treat the mean-field and pairing correlations on the same footing. As a generalization of the HF+BCS method, the Hartree-Fock-Bogoliubov (HFB) theory can treat the mean-field and pairing correlations self-consistently. It has been shown that by formulating the generalized Bogoliubov transformation in the coordinate-space representation, the continuum effects can be properly taken into account [25, 39, 96, 99]. In this section, the main formulas of the HFB theory will be shown and some related topics will be given. For odd-mass or odd-odd nuclei, the blocking effects must be considered. Therefore the blocking method in the HFB theory is also presented.
5.1 The Hartree-Fock-Bogoliubov theory

The HFB wave function $|\Phi\rangle$ is represented as the vacuum with respect to quasiparticles

$$\beta_k|\Phi\rangle = 0, \quad k = 1, \ldots, M,$$

(41)

where $\beta_k$ is the quasiparticle annihilation operator [3] and $M$ is the number of quasiparticle states. The HFB wave function can be constructed as

$$|\Phi\rangle = \prod_k \beta_k |\rangle,$$

(42)

with the bare vacuum $|\rangle$. The most general linear transformation from the particle operators \{a^\dagger_l, a_l\} to the quasiparticle operators \{\beta^\dagger_k, \beta_k\} has the form

$$\begin{pmatrix} \beta^\dagger_k \\ \beta_k \end{pmatrix} = \begin{pmatrix} U^\dagger & V^\dagger \\ V & U \end{pmatrix} \begin{pmatrix} a^\dagger_l \\ a_l \end{pmatrix} = \mathcal{W} \begin{pmatrix} a^\dagger_l \\ a_l \end{pmatrix}.$$  

(43)

Since the quasiparticle operators \{\beta^\dagger_k, \beta_k\} should obey the same fermion commutation relations as the particles, the matrix $\mathcal{W}$ is unitary, i.e., $\mathcal{W}^\dagger \mathcal{W} = 1$. The corresponding density matrices are defined as

$$\rho_{ll'} = \langle \Phi | a^\dagger_{l'} a_l | \Phi \rangle = \sum_k V_{l'k} V^*_k,$$

$$\kappa_{ll'} = \langle \Phi | a_{l'} a_l | \Phi \rangle = \sum_k U_{l'k} V^*_k.$$

(44)

$\rho$ and $\kappa$ are called the normal and abnormal densities (or density matrix and pairing tensor), respectively.

A generalized density matrix can be constructed from $\rho$ and $\kappa$

$$\mathcal{R} = \begin{pmatrix} \langle \Phi | a^\dagger_{l'} a_l | \Phi \rangle & \langle \Phi | a^\dagger_{l'} a_l | \Phi \rangle \\ \langle \Phi | a_{l'} a_l | \Phi \rangle & \langle \Phi | a_{l'} a_l | \Phi \rangle \end{pmatrix} = \begin{pmatrix} \rho & \kappa \\ -\kappa^* & 1 - \rho^* \end{pmatrix}.$$  

(45)

As for the eigenvalue of $\mathcal{R}$, one has

$$\mathcal{W}^\dagger \mathcal{R} \mathcal{W} = \begin{pmatrix} \langle \Phi | \beta^\dagger_{l'} \beta_l | \Phi \rangle & \langle \Phi | \beta_{l'} \beta_l | \Phi \rangle \\ \langle \Phi | \beta^\dagger_{l'} \beta_l | \Phi \rangle & \langle \Phi | \beta_{l'} \beta_l | \Phi \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$  

(46)

and $\mathcal{R}^2 = \mathcal{R}$, which means that the eigenvalues of $\mathcal{R}$ are 0 and 1 with corresponding eigenvectors \begin{pmatrix} U \\ V \end{pmatrix} and \begin{pmatrix} V^* \\ U^* \end{pmatrix}.

The expectation value of $H'$ [cf. Eq. (21)] with respect to quasiparticle vacuum reads
\[ E[R] = \langle \Phi | H' | \Phi \rangle = \sum_{l_2l_2} (\epsilon_{l_2l_2} - \lambda \delta_{l_2l_2}) \rho_{l_2l_2} + \frac{1}{2} \sum_{l_1l_2l_3l_4} \bar{v}_{l_1l_2l_3l_4} \rho_{l_3l_4} \rho_{l_1l_2} + \frac{1}{4} \sum_{l_1l_2l_3l_4} \bar{v}_{l_1l_2l_3l_4} \kappa_{l_1l_2} \kappa_{l_3l_4}, \]  

which is a functional of the general density matrix \( R \). The variation of the energy functional is

\[ \delta E = E[R + \delta R] - E[R] = \sum_{kk'} \mathcal{H}_{kk'} \delta R_{kk'}, \]  

where the Hamiltonian matrix \( \mathcal{H} \) is defined as

\[ \mathcal{H}_{kk'} = \frac{\partial E[R]}{\partial R_{kk'}}. \]  

Since

\[ \frac{\partial E}{\partial \rho_{kk'}} = \epsilon_{kk'} - \lambda \delta_{kk'} + \sum_{pp'} \bar{v}_{kk'pp'} \rho_{pp'} \equiv \epsilon_{kk'} - \lambda \delta_{kk'} + \Gamma_{kk'}, \]

\[ -\frac{\partial E}{\partial \kappa_{kk'}} = \frac{1}{2} \sum_{pp'} \bar{v}_{kk'pp'} \kappa_{pp'} \equiv \Delta_{kk'}, \]  

\( \mathcal{H} \) can be expressed explicitly as

\[ \mathcal{H} = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^* \end{pmatrix}, \]  

where \( h = \epsilon - \lambda + \Gamma \) is the single particle energy and \( \Delta \) is the pairing potential.

One gets the HFB equation

\[ \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^* \end{pmatrix} \begin{pmatrix} U_k \\ V_k \end{pmatrix} = E_k \begin{pmatrix} U_k \\ V_k \end{pmatrix}, \]  

where \( E_k \) is the quasiparticle energy and \( \begin{pmatrix} U_k, V_k \end{pmatrix}^T \) are quasiparticle wave functions.

It should be noted that for the study of ground state of an even-even nucleus, the generalized Bogoliubov transformation can be reduced to the BCS-transformation by transforming the quasiparticle basis to the canonical basis [3].

The HFB theory is the basis of modern nuclear density functional theory, which provides an amazingly successful description of the complicated many-body system in nuclei all over the chart of nuclides [17, 43, 44, 83, 100, 101]. The HFB theory provides a unified description of particle-hole (ph) and particle-particle (pp) correlations [3] on a mean-field level by using two average potentials: The self-consistent Hartree–Fock field \( \Gamma \) which is attributed to long range \( ph \)-correlations, and a pairing potential \( \Delta \) which corresponds to the \( pp \)-correlations. In nuclear density functional theory, the effective nucleon-nucleon interactions are constructed from basic sym-
metries of the nuclear force and the involved parameters are determined by fitting to characteristic experimental data of finite nuclei and nuclear matter. These effective interactions are usually adopted in $ph$-channel in non-relativistic density functional or covariant density functional theories while another phenomenological interactions are mostly used for $pp$-channel. It should be noted that in HFB calculation with the Gogny force, the interactions used in $ph$-channel is a finite range central potential in the $pp$ channel [45][102][103].

5.2 Selected topics

As mentioned before, the generalized Bogoliubov transformation has been widely used in the study of nuclear structure because it can provide a self-consistent treatment of mean-field and pairing correlations, which is particularly important for describing the properties of exotic nuclei. Thus in this part, several selected topics related to exotic nuclei will be introduced, including the continuum effects, the influence of pairing on nuclear size, and various pairing forces adopted in density functional theory calculations.

5.2.1 The generalized Bogoliubov transformation and continuum spectra

The advantage of the HFB theory is that, based on the quasiparticle transformation, it unifies the self-consistent description of single particle orbitals and the BCS pairing theory into a single variation theory. By solving the HFB equation, one can get the quasiparticle energy spectrum, which contains discrete bound states, resonances, and non-resonant continuum states [99]. For a self-bound system, i.e., $\lambda < 0$, the matter density calculated from quasiparticle wave functions $V_k$ with their quasiparticle energies being positive is always local [96]. Therefore the contribution of continuum effects can be self-consistently taken into account in the HFB theory. The solution of the HFB equations is not so straightforward because the continuum states must satisfy the scattering boundary conditions. It has been shown that using the Green’s function method in coordinate space or the Berggren basis, the ground state from the HFB theory can be obtained and the bound, resonant, and scattering quasiparticles are well defined [104][107]. Alternatively, solving the HFB equation in coordinate space with the box boundary condition, the quasiparticle spectrum consists of bound and discretized continuum states and the corresponding ground state can be approximately obtained. It has been shown that by choosing an appropriate box size, the ground state can be accurately calculated [40][108][110]. Besides, several basis expansion methods have also been proven to be valid for weakly bound nuclei [111][115]. With the box boundary condition in HFB calculations, the resonant states can be well located by using the the stabilization method [109].
A schematic picture of the HFB quasiparticle spectrum is shown in Fig. 4. The bound states locate in the region \( 0 < E_k < -\lambda \). When solving the HFB equation with the box boundary condition, the continuum is discretized and can be replaced by a set of discrete non-resonant states with \(-\lambda < E_k < E_{\text{cut}}\), where \( E_{\text{cut}} \) is the cut-off energy in the quasiparticle space. With the increase of the box size, the level density in continuum becomes more dense. It should be noted that when studying exotic nuclei by solving the wave function in a box, a proper description should be independent on \( E_{\text{cut}} \) and the box size \( R_{\text{Box}} \). In the HFB theory in coordinate representation, the contribution from the non-resonance continuum state plays an important role and can be included by calculating the densities with the lower component of quasiparticle wave function, which is always local \([39, 96]\). By using the canonical transformation \([3]\), one can get the single particle levels in the canonical basis. It is found that some bound single particle states might correspond to quasiparticle states with \( E_k > -\lambda \) such that the couplings between single particle states below the Fermi energy and those in the continuum can be properly treated. Additionally, the treatment and influence of the quasiparticle state with energy larger than \( E_{\text{cut}} \).
Pairing correlations can significantly affect properties of the nuclei close to drip lines due to the presence of the vast continuum space available for pair scattering [39]. A typical example is the effect of pairing on nuclear halos, in which the continuum induced by pairing correlations changes the asymptotic behavior of particle density and the occupation probabilities of single particle states near the Fermi energy in even-even weakly bound system thus influencing its spatial extension [18, 116–122]. In addition, the pairing coupling to positive-energy single particle states also influences the nuclear binding [39]. Particularly, the strong coupling to the continuum lowers the Fermi energy thus influences the range of bound nuclei and impacts the limit of the nuclear landscape [123–125].

5.2.2 Phenomenological pairing force: finite range vs. zero range

Usually, there are two-kind of pairing interactions commonly used in modern nuclear density functional calculations: zero-range forces with or without a density dependence [39, 40] and finite range forces. The latter includes Gogny [41, 126, 127] and separable pairing forces [42]. The zero-range force is relatively simple for numerical calculations, but it allows a coupling to the very highly excited states. Therefore an energy cutoff has to be introduced and the interaction strength has to be properly renormalized with respect to, e.g., pairing gaps [86]. The Gogny force has a better treatment for the coupling to the highly excited states, but it involves more sophisticated numerical techniques. The separable pairing force is numerically simpler than the Gogny force and has also been widely used nowadays.

In Ref. [128], the proper form of the pairing interaction is discussed in the framework of the RCHB theory. The even-even Ni isotopes ranging from the proton drip line to the neutron drip line are taken as examples. The pairing correlations are described by using a density-dependent force with zero range and the finite range Gogny force and the results from these two pairing forces were compared. Through the comparisons of the two-neutron separation energies $S_{2n}$, the neutron, proton, and matter rms radii, good agreements have been found between the calculations with both interactions and the empirical values. In Fig. 5 for example, the two-neutron separation energies of Ni isotopes are shown as a function of the neutron number $N$, including the experimental data (solid points) and results from the RCHB with $\delta$-force (open circles), RCHB with Gogny force (triangles), and HFB with SkP interaction [39] (stars) and SIII interaction [129] (pluses). The RCHB results with $\delta$-force and Gogny force are almost identical. There is a strong kink at $N = 28$ and a weaker one at $N = 50$. The neutron drip line position is predicted at $^{100}$Ni in both calculations. The empirical data are known only up to $N = 50$. Comparing with the available empirical data, the general trend and gradual decline of $S_{2n}$ have been well reproduced. It is shown clearly in Ref. [128] that after proper renormalization (e.g., fixing the corresponding pairing energies), observables including two-neutron separation energies and rms radii remain the same, even when the interaction strength
is changed within a reasonable region. A further study [42] has shown that the same average gaps can be given by the Gogny force D1S and $\delta$ force if the size of the strength is adjusted properly but the individual matrix elements of the forces and the matrix elements of the pairing field are very different from each other. Additionally, the separable approximation is very similar to the full Gogny force. In conclusion, for the study of ground state properties by using mean-field approaches, the results from the above-mentioned pairing forces are almost the same.

5.2.3 Pairing correlations and nuclear size

The radius of a nucleus is determined by the spatial distribution of the matter density. The asymptotic behavior of neutron ground state density from HF calculations can be approximated as [116]

$$\rho(r) \propto \exp\left(-\mu r/r^2\right),$$

(53)

where $\mu = \sqrt{-2m\epsilon_k/\hbar}$ and $\epsilon_k$ is the single particle energy of the least bound orbital with $l = 0$. The corresponding mean square radius deduced from the asymptotic solution is

$$\langle r^2 \rangle_{HF} \propto \frac{\hbar^2}{2m|\epsilon_k|},$$

(54)

which diverges in the limit $\epsilon_k \to 0$. This mechanism, i.e., valence nucleon(s) occupying weakly-bound and low-$l$ orbital, has been used in early interpretations of the nuclear halo phenomenon [130,132].
When pairing correlations are included, the mean square radius deduced from the asymptotic HFB density is

$$\langle r^2 \rangle_{\text{HFB}} \propto \frac{\hbar^2}{2m(E_k - \lambda)}$$  (55)

with the lowest discrete quasiparticle energy $E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta_k^2}$. When the pairing gap $\Delta_k$ is finite, the radius does not diverge in the limit of small separation energy $\epsilon_k \simeq \lambda \to 0$.

The asymptotic HF and HFB densities characterized by $l = 0$ orbitals were compared in Ref. [116] and it has been emphasized that pairing correlations reduce the nuclear size and an extreme halo with infinite radius cannot be formed in superfluid nuclear systems. In this sense, pairing correlations act against the formation of an infinite matter radius. This is the so-called “pairing anti-halo effect” [116].

In fact, the limiting condition $\epsilon_k \to 0$ and the radius deduced from this $l = 0$ orbital alone correspond to an extremely ideal situation, which is difficult to be found in real nuclei. In Ref. [120], the influences of pairing correlations on the nuclear size and on the formation of a nuclear halo were studied in details by using the self-consistent RCHB theory [40]. It has been shown that pairing correlations not only influence the radius of the orbital, but also affect the occupation probabilities of the orbitals close to the Fermi level. As a result, the nuclear radius is dominated by their competition.

### 5.3 Blocking effects

To describe odd-mass or odd-odd nuclei, the blocking effect has to be taken into account. In this part, the method describing the blocking effect in the HFB approach is introduced. To treat pairing correlations with the generalized Bogoliubov transformation, the quasiparticle concept is adopted and the ground state of an even-even nucleus $|\Phi\rangle$ [cf. Eq. (42)] is represented as a vacuum with respect to quasiparticles [3]. For odd-mass and odd-odd nuclei, in practice, the ground state can be constructed as one quasiparticle state

$$|\Phi_1\rangle = \beta_1^\dagger |\Phi_0\rangle = \beta_1^\dagger \prod_k \beta_k |0\rangle,$$  (56)

where $\beta_1^\dagger$ corresponds to the quasiparticle level to be blocked. This one quasiparticle state $|\Phi_1\rangle$ can be regarded as the vacuum with respect to the set of quasiparticle operators $(\beta_1', \ldots, \beta_M')$ with

$$\beta_1' = \beta_1^\dagger, \quad \beta_2' = \beta_2, \quad \ldots, \quad \beta_M' = \beta_M,$$  (57)
and the exchange of the operators $\beta_1^\dagger \leftrightarrow \beta_1$ forms a new set of quasiparticle operators $(\beta_1', \ldots, \beta_M', \beta_1^\dagger, \ldots, \beta_M^\dagger)$, which corresponds to the exchange of the columns $(U_{i1}, V_{i1}) \leftrightarrow (V_{i1}^*, U_{i1}^*)$ in the matrix $W \ [\text{cf. Eq. (43)}]$. Therefore, the blocking effect in the odd system can be realized by exchanging the creator $\beta_1^\dagger$ with the corresponding annihilator $\beta_1$ in the quasiparticle space.

Next the procedure will be presented of implementing the blocking in axially deformed nuclei [133]. For a fully paired and axially symmetric deformed system with the time reversal symmetry, the projection of the total angular momentum on the symmetry axis $\Omega$ is a good quantum number and each single particle state has a degeneracy of two. The HFB equation can be reduced to half dimension $M/2$ and decomposed into degenerate blocks with quantum numbers $+\Omega$ or $-\Omega$. The dimension of the corresponding density and abnormal density matrices is $M$.

For an odd system with the $k_b$-th level blocked in the $+\Omega$ subspace, the time reversal symmetry is violated and currents appear in the system. These currents are axially symmetric, i.e., $\Omega$ remains a good quantum number, but the quasiparticle energies are no longer degenerate for the two subspaces, because the subspace with $+\Omega$ contains the odd particle and the subspace with $-\Omega$ contains an empty level. Therefore, in principle, one has to diagonalize the HFB equation in the whole quasiparticle space composed of $+\Omega$ and $-\Omega$ subspaces. With the equal filling approximation [89, 90], which has been shown to be valid in dealing with blocking effects [134], one can average over the two configurations of a particle in the $+\Omega$ space and in the $-\Omega$ space. The corresponding currents in two subspaces cancel each other and can be neglected. In this way one can diagonalize the HFB equation in the $+\Omega$ subspace or the $-\Omega$ subspace and the resulted fields are time reversal symmetric. In practice the density matrix $\rho$ and the abnormal density $\kappa$ in two subspaces are

$$\rho = (V^*V^T)_{M\times M} + \frac{1}{2} \left( U_{k_b} U_{k_b}^T - V_{k_b} V_{k_b}^T \right), \tag{58}$$

$$\kappa = (V^*U^T)_{M\times M} - \frac{1}{2} \left( U_{k_b} V_{k_b}^* + V_{k_b}^* U_{k_b}^T \right), \tag{59}$$

where $V_{k_b}$ and $U_{k_b}$ are column vectors in the matrices $V$ and $U$ corresponding to the blocked level.

6. Issues with particle number

Both the BCS method and the HFB theory have been widely applied to describe nuclear superfluidity, but the BCS-type and HFB wave functions do not keep the particle number conserved, which is connected with the Nambu-Goldstone mode of a broken $U(1)$ phase symmetry [33]. In addition, the spontaneous breaking of $U(1)$ symmetry leads to a sharp phase transition, i.e., the pairing energy turns from zero to finite at a critical pairing strength [3]. Many efforts have been made to remedy the
problem of particle-number violation in the mean-field model, configuration space, and *ab initio* calculations (see Ref. [135] and references therein). The simplest way to treat the uncertainty in the particle number is the Lipkin-Nogami method mentioned before, but which only considers the corrections on the energy. Many exact solutions of the pairing Hamiltonian have also been proposed [136]. In mean-field models, the restoration of particle number of BCS-type or HFB wave function can be achieved by using the particle number projection (PNP) method. In this section, PNP in mean-field models and a particle number conservation method in the many-body configuration space are introduced.

### 6.1 Exact solutions for pairing Hamiltonian

The exact numerical solution of the pairing model has been proposed in the 1960s by Richardson and Sherman [65, 66]. This method has been extended to a family of exactly-solvable models, called the Richardson–Gaudin (RG) models and widely applied in various areas of quantum many-body systems, such as mesoscopic systems, condensed matter, quantum optics, cold atomic gases, and atomic nuclei [136–138]. In this method, one does not need to diagonalize the pairing Hamiltonian, but instead to solve a set of non-linear equations, called Richardson’s equation, for parameters in the pairing wave functions. More details of this method can be found in Refs. [5, 136, 137].

In this part, a shell-model-like approach (SLAP) for the pairing, dubbed the particle number conservation (PNC) method developed in the 1980s [67] will be introduced. In the PNC method, the pairing Hamiltonian is directly diagonalized in the many-body configuration space and it has been shown to be more accurate than the BCS calculation as compared with the exact solution [139]. Furthermore, the blocking effects are taken into account automatically and both odd-mass and even-even nuclei can be treated on the same footing. It has been demonstrated that the number of configurations with significant contributions to the low-lying excited states of a nucleus is quite limited [140]. Consequently the concept of many-body configuration truncation is introduced instead of the single particle state truncation used in the BCS or generalized Bogoliubov method. Extensive studies and discussions on the validity of the truncated many-body configuration spaces as well as the application of the PNC method can be found in Refs. [141–145]. In particular the presence of the low-lying seniority $\sigma = 0$ solutions, which are usually poorly described by using the standard BCS approximation or HFB theory, has been found to play a role in the interpretation of the spectra of rotating nuclei.

In the SLAP for pairing, the model Hamiltonian reads

$$H = H_{\text{p.p.}} + H_{\text{pair}},$$

where $H_{\text{p.p.}} = \sum_{\nu} \varepsilon_{\nu} a_{\nu}^{\dagger} a_{\nu}$ and the pairing Hamiltonian $H_{\text{pair}} = -G \sum_{\mu, \nu \neq 0} \delta_{\mu, \nu} a_{\mu}^{\dagger} a_{\nu}^{\dagger} a_{\nu} a_{\mu}$ with $G$ the average strength, $\varepsilon_{\nu}$ the single particle energy, and $\nu$ the notation of the
each level. In the case of axially deformed nuclei, \( \nu \equiv (\Omega, \pi) \). \( \nu \) represents the time-reversal state of \( \nu \).

For an even-even nucleus with the total particle number \( N = 2n \), the multi-particle configurations (MPC) used to diagonalize the Hamiltonian are constructed as the following:

(a) The fully paired configurations with the seniority \( \sigma = 0 \):
\[
|p_1 \tilde{p}_1 \cdots p_n \tilde{p}_n \rangle = a_{p_1}^\dagger a_{\tilde{p}_1}^\dagger \cdots a_{p_n}^\dagger a_{\tilde{p}_n}^\dagger |0\rangle,
\]
(b) The configurations with two unpaired particles, i.e., seniority \( \sigma = 2 \):
\[
|\mu \nu p_1 \tilde{p}_1 \cdots p_{n-1} \tilde{p}_{n-1} \rangle = a_{\mu}^\dagger a_{\nu}^\dagger a_{p_1}^\dagger a_{\tilde{p}_1}^\dagger \cdots a_{p_{n-1}}^\dagger a_{\tilde{p}_{n-1}}^\dagger |0\rangle,
\]
where \( \mu \) and \( \nu \) denote two unpaired levels. The MPCs with larger \( \sigma \) can also be constructed in this way [67].

In realistic calculations, the MPC space has to be truncated. Only configurations with excitation energies smaller than \( E_c \) are used to diagonalize the Hamiltonian (60), where \( E_c \) is the cutoff energy. The corresponding nuclear wave function can be expanded as
\[
\psi^\beta = \sum_{p_1,\cdots,p_n} V_{p_1,\cdots,p_n}^\beta |p_1 \tilde{p}_1 \cdots p_n \tilde{p}_n \rangle + \sum_{\mu,\nu} \sum_{p_1,\cdots,p_{n-1}} V_{\mu \nu p_1,\cdots,p_{n-1}}^\beta |\mu \nu p_1 \tilde{p}_1 \cdots p_{n-1} \tilde{p}_{n-1} \rangle + \cdots,
\]
where \( \beta = 0 \) (ground state), 1, 2, 3, \cdots (excited states). The occupation probability of the \( i \)th-level for state \( \beta \) is
\[
n_i^\beta = \sum_{p_1,\cdots,p_{n-1}} |V_{p_1,\cdots,p_{n-1},i}^\beta|^2 + \sum_{\mu,\nu} \sum_{p_1,\cdots,p_{n-2}} |V_{\mu \nu p_1,\cdots,p_{n-2},i}^\beta|^2 + \cdots, \quad i = 1, 2, 3, \cdots.
\]

This SLAP for pairing has been implemented in the relativistic mean field model [146, 148] and Skyrme Hartree-Fock model [149, 150] and it turns out that this hybrid model is valid for both ground state properties and low-lying spectra. In addition, it has also been applied in several cranking models to study the rotational properties of ground state bands and low-lying high-\( K \) multi-quasiparticle bands [147, 149, 151, 152].

6.2 Particle number projection

In the HFB theory, the wave functions are the vacua of the corresponding quasiparticle operators, which do not represent states with good particle number. The nonconservation of particle number can be restored by projecting an HFB state \( |\Psi\rangle \) onto a state with good particle number [68, 70, 153, 154].
\[ |\Phi^N\rangle = \rho^N |\Psi\rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{e^{\phi N}} e^{i\phi N} |\Psi\rangle, \] (65)

with the particle number projection operator
\[ \rho^N = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi (N-N)} d\phi. \] (66)

Then the projected energy is given by
\[ E^N = \langle \Phi^N | \hat{H} | \Phi^N \rangle = \int_0^{2\pi} d\phi E[\phi] \mathcal{N}^N(\phi), \] (67)

with
\[ E[\phi] = \frac{\langle \Phi_0 | \hat{H} | \Phi_\phi \rangle}{\langle \Phi_0 | \Phi_\phi \rangle}, \quad \mathcal{N}^N(\phi) = \frac{e^{-iN\phi}}{2\pi} \frac{\langle \Phi_0 | \Phi_\phi \rangle}{\langle \Phi^N | \Phi^N \rangle}, \] (68)

where \(|\Phi_\phi\rangle = e^{i\phi N} |\Psi\rangle\). It has been shown in Refs. [68, 69] that the energy kernel \(E[\phi]\) is similar to Eq. (47) with modified expressions for the pairing field and the HF potential and can be calculated by the generalized Wick’s theorem
\[ E[\phi] = \frac{\langle \Phi_0 | \hat{H} | \Phi_\phi \rangle}{\langle \Phi_0 | \Phi_\phi \rangle} = \sum_\mu \Gamma_{\mu\mu}^0 + \frac{1}{2} \sum_{\mu\nu} \bar{v}_{\mu\nu\mu\nu} \rho_{\mu\nu}^0 + \frac{1}{4} \sum_{\mu\nu} \bar{v}_{\mu\nu\mu\nu} \kappa_{\mu\mu}^0 \kappa_{\nu\nu}^0, \] (69)

where \(\bar{v}_{\mu\nu}\) and \(\bar{v}_{\nu\nu}\) denote the effective vertices in the \(ph\) and \(pp\) channels. The normal and anomalous transition density matrices are
\[ \rho_{\mu\nu}^0 = \frac{\langle \Phi_0 | d^\dagger_{\nu} d_{\mu} | \Phi_\phi \rangle}{\langle \Phi_0 | \Phi_\phi \rangle}, \]
\[ \kappa_{\mu\nu}^0 = \frac{\langle \Phi_0 | d^\dagger_{\nu} d_{\mu} | \Phi_\phi \rangle}{\langle \Phi_0 | \Phi_\phi \rangle}. \] (70)

The projected HFB equation read
\[ \left( \begin{array}{c} e^N + \Gamma^N + \lambda^N \\ -\Delta^N \end{array} \right) \left( \begin{array}{c} U^N \\ V^N \end{array} \right) = E \left( \begin{array}{c} U^N \\ V^N \end{array} \right), \] (71)

where \(\Gamma^N = \frac{\partial E^N}{\partial \rho}\) and \(\Delta^N = -\frac{\partial E^N}{\partial \kappa}\). All the quantities keep the good quantum number \(N\). Detailed formulas can be found in Ref. [154].

The implementation of PNP in the BCS model is relatively simple, which was applied using different methods shown in Refs. [3,155,158]. The PNP technique has been applied in mean-field models based on the HFB or HF+BCS equations. One can make the projection before or after variation and the latter is technically much
more easier and has been widely used in beyond-mean-field calculations nowadays. The mean field calculations with and without the PNP method have shown that the uncertainties caused by the particle number fluctuation might be important for determining the positions of drip lines and the stability of rare isotopes [159][162]. More details and related topics on PNP can be found in a recent review [135].

**Further Reading**

Pairing effects are directly related to many aspects of nuclear physics. In this chapter, only the basic picture and methodology of the pairing effects in atomic nuclei are introduced, especially focusing on the nuclear structure study. Various additional topics related to pairing are thoroughly covered in the following textbooks:

- P. Ring and P. Schuck, The nuclear many-body problem, Springer-Verlag, 1980.
- W. Greiner and J.A. Maruhn, Nuclear models, Springer-Verlag, 1996.
- D.M. Brink and R.A. Broglia, Nuclear Superfluidity: Pairing in Finite Systems, Cambridge University Press, 2005.
- D.J. Rowe and J.L. Wood, Fundamentals of nuclear models, World Scientific, 2010.
- R.A. Broglia and V. Zelevinsky (eds.), Fifty Years of Nuclear BCS: Pairing in Finite Systems, World Scientific, Singapore, 2013.

More aspects of pairing effects, underlying mechanism, and more detailed derivations of specific implementations of pairing models, are discussed in the following review articles:

- R.A. Broglia, J. Terasaki, and N. Giovanardi, The Anderson–Goldstone–Nambu mode in finite and in infinite systems, Phys. Rep. 335, 1 (2000)
- W. von Oertzen and A. Vitturi, Pairing correlations of nucleons and multinucleon transfer between heavy nuclei, Rep. Prog. Phys. 64, 1247 (2001)
- D.J. Dean and M. Hjorth-Jensen, Pairing in nuclear systems: From neutron stars to finite nuclei, Rev. Mod. Phys. 75, 607 (2003)
- J. Dukelsky, S. Pittel, and G. Sierra, Colloquium: Exactly solvable Richardson-Gaudin models for many-body quantum systems, Rev. Mod. Phys. 76, 643 (2004)
- J. Meng, H. Toki, S.-G. Zhou, S.Q. Zhang, W.H. Long, and L.S. Geng, Relativistic continuum Hartree-Bogoliubov theory for ground-state properties of exotic nuclei, Prog. Part. Nucl. Phys. 57, 470 (2006)
- G.C. Strinati, P. Pieri, G. Röpke, P. Schuck, and M. Urban, The BCS–BEC crossover: From ultra-cold Fermi gases to nuclear systems, Phys. Rep. 738, 1 (2008)
- S. Frauendorf and A. Macchiavelli, Overview of neutron–proton pairing, Prog. Part. Nucl. Phys. 78, 24 (2014)
- N.Q. Hung, N.D. Dang, and L.G. Moretto, Pairing in excited nuclei: A review, Rep. Prog. Phys. 82, 056301 (2019)
Acknowledgements

The authors would like to thank Wen-Hui Long, Peng-Wei Zhao, Kai-Yuan Zhang, Shuang-Quan Zhang, and Zhen-Hua Zhang for their comments and suggestions. The authors have been partly supported by the National Key R&D Program of China (Grant No. 2018YFA0404402), the National Natural Science Foundation of China (Grants No. 11525524, No. 12070131001, No. 12047503, No. 11961141004, No. 11975237, No. 11575189, and No. 11790325), the Key Research Program of Frontier Sciences of Chinese Academy of Sciences (Grant No. QYZDB-SSWSYS013), the Strategic Priority Research Program of Chinese Academy of Sciences (Grants No. XDB34010000 and No. XDPB15), the Inter-Governmental S&T Cooperation Project between China and Croatia, and the IAEA Coordinated Research Project “F41033”.

References

1. A. Bohr, B.R. Mottelson, *Nuclear Structure, Volume 1: Single Particle Motion* (World Scientific, Singapore, 1998)
2. A. Bohr, B.R. Mottelson, *Nuclear Structure, Volume 2: Nuclear Deformations* (World Scientific, Singapore, 1998)
3. P. Ring, P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag Berlin Heidelberg, 1980)
4. D.J. Dean, M. Hjorth-Jensen, Rev. Mod. Phys. 75, 607 (2003)
5. D.M. Brink, R.A. Broglia, *Nuclear Superfluidity: Pairing in Finite Systems* (Cambridge University Press, 2005)
6. R.A. Broglia, V. Zelevinsky (eds.), *Fifty Years of Nuclear BCS: Pairing in Finite Systems* (World Scientific, Singapore, 2013)
7. F. Kondev, M. Wang, W. Huang, S. Naimi, G. Audi, Chin. Phys. C 45(3), 030001 (2021)
8. W. Huang, M. Wang, F. Kondev, G. Audi, S. Naimi, Chin. Phys. C 45(3), 030002 (2021)
9. M. Wang, W. Huang, F. Kondev, G. Audi, S. Naimi, Chin. Phys. C 45(3), 030003 (2021)
10. W. Satula, J. Dobaczewski, W. Nazarewicz, Phys. Rev. Lett. 81, 3599 (1998)
11. F.R. Xu, R. Wyss, P.M. Walker, Phys. Rev. C 60, 051301(R) (1999)
12. J. Dobaczewski, P. Magierski, W. Nazarewicz, W. Satula, Z. Szymański, Phys. Rev. C 63(2), 024308 (2001)
13. W. von Oertzen, A. Vitturi, Rep. Prog. Phys. 64(10), 1247 (2001)
14. G.F. Bertsch, in *Fifty Years of Nuclear BCS: Pairing in Finite Systems*, ed. by R.A. Broglia, V. Zelevinsky (World Scientific, Singapore, 2012), pp. 26–39
15. I. Tanihata, H. Savajols, R. Kanungo, Prog. Part. Nucl. Phys. 68, 215 (2013)
16. J. Meng, P. Ring, Phys. Rev. Lett. 77, 3963 (1996)
17. J. Meng, H. Toki, S.G. Zhou, S.Q. Zhang, W.H. Long, L.S. Geng, Prog. Part. Nucl. Phys. 57(2), 470 (2006)
18. J. Meng, S.G. Zhou, J. Phys. G: Nucl. Part. Phys. 42(9), 093101 (2015)
19. K. Hagino, H. Sagawa, J. Carbonell, P. Schuck, Phys. Rev. Lett. 99, 022506 (2007)
20. H. Sagawa, K. Hagino, Eur. Phys. J. A 51(8), 102 (2015)
21. R. Kanungo, I. Tanihata, in *Fifty Years of Nuclear BCS: Pairing in Finite Systems*, ed. by R.A. Broglia, V. Zelevinsky (World Scientific, Singapore, 2012), pp. 379–392
22. T. Nakamura, A.M. Vinodkumar, T. Sugimoto, N. Aoi, H. Baba, D. Bazin, N. Fukuda, T. Gomi, H. Hasegawa, N. Imai, M. Ishihara, T. Kobayashi, Y. Kondo, T. Kubo, M. Miura, T. Motobayashi, H. Otsu, A. Saito, H. Sakurai, S. Shimoura, K. Watanabe, Y.X. Watanabe, T. Yakushiji, Y. Yanagisawa, K. Yoneda, Phys. Rev. Lett. 96, 252502 (2006)
23. R. Kanungo, A. Sanetullaev, J. Tanaka, S. Ishimoto, G. Hagen, T. Myo, T. Suzuki, C. Andreoiu, P. Bender, A.A. Chen, B. Davids, J. Fallis, J.P. Fortin, N. Galinski, A.T. Gallant, P.E. Garrett, G. Hackman, B. Hadinia, G. Jansen, M. Keefe, R. Krücken, J. Lighthall, E. McNeice, D. Miller, T. Otsuka, J. Purcell, J.S. Randhawa, T. Roger, A. Rojas, H. Savajols, A. Shotter, I. Tanihata, I.J. Thompson, C. Unsworth, P. Voss, Z. Wang, Phys. Rev. Lett. 114, 192502 (2015)

24. T. Nakamura, N. Kobayashi, Y. Kondo, Y. Satou, J.A. Tostevin, Y. Utsuno, N. Aoi, H. Baba, N. Fukuda, J. Gibelin, N. Inabe, M. Ishihara, D. Kameda, T. Kubo, T. Motobayashi, T. Ohnishi, N.A. Orr, H. Otsu, T. Otsuka, H. Sakurai, T. Sumikama, H. Takeda, E. Takeshita, M. Takechi, S. Takeuchi, Y. Togano, K. Yoneda, Phys. Rev. Lett. 112, 142501 (2014)

25. S.G. Zhou, PoS INPC2016, 373 (2017)

26. X.X. Sun, J. Zhao, S.G. Zhou, Phys. Lett. B 785, 530 (2018)

27. X.X. Sun, S.G. Zhou, Sci. Bull. 66(20), 2072 (2021)

28. B. Blank, M. Płoszajczak, Rep. Prog. Phys. 71(4), 046301 (2008)

29. M. Pfützner, M. Karny, L.V. Grigorenko, K. Riisager, Rev. Mod. Phys. 84, 567 (2012)

30. S.G. Mayer, J.H.D. Jensen, Elementary Theory of Nuclear Shell Structure (New York: Wiley, 1955)

31. J. Margueron, H. Sagawa, K. Hagino, Phys. Rev. C 81, 044313 (2010)

32. J. Margueron, H. Sagawa, K. Hagino, Phys. Rev. C 77, 054309 (2008)

33. M. Yamagami, Y.R. Shimizu, T. Nakatsukasa, Phys. Rev. C 80, 064301 (2009)

34. W.J. Chen, C.A. Bertulani, F.R. Xu, Y.N. Zhang, Phys. Rev. C 91, 047303 (2015)

35. F. Barranco, R.A. Broglia, G. Colò, G. Gori, E. Vigezzi, P.F. Bortignon, Eur. Phys. J. A 21(1), 57 (2004)

36. J. Dechargé, D. Gogny, Phys. Rev. C 21, 1568 (1980)

37. Y. Tian, Z.Y. Ma, P. Ring, Phys. Lett. B 676(1), 44 (2009)

38. T. Duguet, in Fifty Years of Nuclear BCS: Pairing in Finite Systems (World Scientific, Singapore, 2012), pp. 229–242

39. T. Duguet, Prog. Part. Nucl. Phys. 78, 24 (2014)
58. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. 108, 1175 (1957)
59. A. Bohr, B.R. Mottelson, D. Pines, Phys. Rev. 110, 936 (1958)
60. S.T. Belyaev, Mat. Fys. Medd. Dan. Vid. Selsk. 31(11), 1 (1959)
61. N.N. Bogoljubov, Il Nuovo Cimento 7(6), 794 (1958)
62. J.G. Valatin, Phys. Rev. 122, 1012 (1961)
63. H.J. Lipkin, Ann. Phys. 9(2), 272 (1960)
64. Y. Nogami, Phys. Rev. 134, B313 (1964)
65. J.A. Sheikh, P. Ring, Nucl. Phys. A 665(3), 71 (2000)
66. M. Anguiano, J. Egido, L. Robledo, Nucl. Phys. A 696(3), 467 (2001)
67. J. Zeng, T. Cheng, Nucl. Phys. A 108(1), 1 (1957)
68. A. Bohr, B.R. Mottelson, D. Pines, Phys. Rev. 110, 936 (1958)
69. S.T. Belyaev, Mat. Fys. Medd. Dan. Vid. Selsk. 31(11), 1 (1959)
70. N.N. Bogoljubov, Il Nuovo Cimento 7(6), 794 (1958)
71. J.G. Valatin, Phys. Rev. 122, 1012 (1961)
72. H.J. Lipkin, Ann. Phys. 9(2), 272 (1960)
73. Y. Nogami, Phys. Rev. 134, B313 (1964)
74. J.A. Sheikh, P. Ring, Nucl. Phys. A 665(3), 71 (2000)
75. M. Anguiano, J. Egido, L. Robledo, Nucl. Phys. A 696(3), 467 (2001)
76. J. Zeng, T. Cheng, Nucl. Phys. A 108(1), 1 (1957)
92. T. Yamaguchi, H. Koura, Y. Litvinov, M. Wang, Prog. Part. Nucl. Phys. 120, 103882 (2021)
93. J. Dobaczewski, N. Michel, W. Nazarewicz, M. Płoszajczak, J. Rotureau, Prog. Part. Nucl. Phys. 59(1), 432 (2007)
94. C.W. Johnson, K.D. Launey, N. Auerbach, S. Bacca, B.R. Barrett, C.R. Brune, M.A. Caprio, P. Descouvemont, W.H. Dickhoff, C. Elster, P.J. Fasano, K. Fossez, H. Hergert, M. Hjorth-Jensen, L. Hlophe, B. Hu, R.M.I. Betan, A. Idini, S. König, K. Kravvaris, D. Lee, J. Lei, A. Mercenne, R.N. Perez, W. Nazarewicz, F.M. Nunes, M. Płoszajczak, J. Rotureau, G. Ru-pak, A.M. Shirokov, I. Thompson, J.P. Vary, A. Volya, F. Xu, R.G.T. Zegers, V. Zelevinsky, X. Zhang, J. Phys. G: Nucl. Part. Phys. 47(12), 123001 (2020)
95. A. Bulgac, arXiv:nucl-th/9907088 (1999)
96. J. Dobaczewski, H. Flocard, J. Treiner, Nucl. Phys. A 422(1), 103 (1984)
97. N. Sandulescu, N. Van Giai, R.J. Liotta, Phys. Rev. C 61, 061301(R) (2000)
98. N. Sandulescu, L.S. Geng, H. Toki, G.C. Hillhouse, Phys. Rev. C 68, 054323 (2003)
99. J. Dobaczewski, W. Nazarewicz, in Fifty Years of Nuclear BCS: Pairing in Finite Systems, ed. by R.A. Broglia, V. Zelevinsky (World Scientific, Singapore, 2012), pp. 40–60
100. J.W. Negele, Rev. Mod. Phys. 54, 913 (1982)
101. R.O. Jones, Rev. Mod. Phys. 87, 897 (2015)
102. S. Pérus, M. Martini, Eur. Phys. J. A 50(5), 88 (2014)
103. J.F. Berger, J.P. Blaizot, D. Bouche, P. Chaix, J.P. Delaroche, M. Dupuis, M. Girod, J. Gogny, B. Grasmatocicos, D. Iracane, J. Lachkar, F. Mariotte, N. Pillet, N. Van Giai, Eur. Phys. J. A 53(10), 214 (2017)
104. S.T. Belyaev, A.V. Smirnov, S.V. Tolokonnikov, S.A. Fayans, Sov. J. Nucl. Phys. 45, 783 (1987)
105. N. Michel, K. Matsuyanagi, M. Stoitsof, Phys. Rev. C 78, 044319 (2008)
106. Y. Zhang, M. Matsuo, J. Meng, Phys. Rev. C 83, 054301 (2011)
107. T.F. Sun, Z.X. Liu, L. Qian, B. Wang, W. Zhang, Phys. Rev. C 99, 054316 (2019)
108. M. Gross, N. Sandulescu, N. Van Giai, R.T. Liotta, Phys. Rev. C 64, 064321 (2001)
109. J.C. Pei, A.T. Kruppa, W. Nazarewicz, Phys. Rev. C 84, 024311 (2011)
110. S. Typel, Front. Phys. 6, 73 (2018)
111. W. Pannert, P. Ring, J. Boguta, Phys. Rev. Lett. 59, 2420 (1987)
112. C.E. Price, G.E. Walker, Phys. Rev. C 36, 354 (1987)
113. M. Stoitsof, P. Ring, D. Vretenar, G.A. Lalazissis, Phys. Rev. C 58, 2086 (1998)
114. S.G. Zhou, J. Meng, P. Ring, Phys. Rev. C 68, 034323 (2003)
115. H. Nakada, K. Takayama, Phys. Rev. C 98, 011301(R) (2018)
116. K. Bennaccer, J. Dobaczewski, M. Płoszajczak, Phys. Lett. B 496(3–4), 154 (2000)
117. S.G. Zhou, J. Meng, P. Ring, E.G. Zhao, Phys. Rev. C 82, 011301(R) (2010)
118. K. Hagiino, H. Sagawa, Phys. Rev. C 84, 011303(R) (2011)
119. J.C. Pei, Y.N. Zhang, F.R. Xu, Phys. Rev. C 87, 051302(R) (2013)
120. Y. Chen, P. Ring, J. Meng, Phys. Rev. C 89, 014312 (2014)
121. J.C. Pei, G.I. Fann, R.J. Harrison, W. Nazarewicz, Y. Shi, S. Thornton, Phys. Rev. C 90, 024317 (2014)
122. X.X. Sun, J. Zhao, S.G. Zhou, Nucl. Phys. A 1003, 122011 (2020)
123. S. Goriely, N. Chamel, J.M. Pearson, Phys. Rev. Lett. 102, 152503 (2009)
124. X.W. Xia, Y. Lim, P.W. Zhao, H.Z. Liang, X.Y. Qu, Y. Chen, H. Liu, L.F. Zhang, S.Q. Zhang, Y. Kim, J. Meng, At. Data Nucl. Data Tables 121-122, 1 (2018)
125. K. Zhang, M.K. Cheoun, Y.B. Choi, P.S. Chong, J. Dong, Z. Dong, X. Du, L. Geng, E. Ha, X.T. He, C. Heo, M.C. Ho, E.J. In, S. Kim, Y. Kim, C.H. Lee, J. Lee, H. Li, Z. Li, T. Luo, J. Meng, M.H. Mun, Z. Niu, C. Pan, P. Papakonstantinou, X. Shang, C. Shen, G. Shen, W. Sun, X.X. Sun, C.K. Tam, Thaivayongnou, C. Wang, X. Wang, S.H. Wong, J. Wu, X. Wu, X. Xie, Y. Yan, R.W.Y. Yeung, T.C. Yiu, S. Zhang, W. Zhang, X. Zhang, Q. Zhao, S.G. Zhou, At. Data Nucl. Data Tables 144, 101488 (2022)
126. J. Berger, M. Girod, D. Gogny, Nucl. Phys. A 428, 23 (1984)
127. T. Gonzalez-Llarena, J.L. Egido, G.A. Lalazissis, P. Ring, Phys. Lett. B 379(1), 13 (1996)
128. J. Meng, Phys. Rev. C 57, 1229 (1998)
129. M. Beiner, H. Flocard, N. Van Giai, P. Quentin, Nucl. Phys. A 238(1), 29 (1975)
130. G.F. Bertsch, B.A. Brown, H. Sagawa, Phys. Rev. C 39, 1154 (1989)
131. H. Sagawa, Phys. Lett. B 286(1), 7 (1992)
132. Z.Y. Zhu, W.Q. Shen, Y.H. Cai, Y.G. Ma, Phys. Lett. B 328(1-2), 1 (1994)
133. L.L. Li, J. Meng, P. Ring, E.G. Zhao, S.G. Zhou, Chin. Phys. Lett. 29(4), 042101 (2012)
134. G.F. Bertsch, C.A. Bertulani, W. Nazarewicz, N. Schunck, M.V. Stoitsov, Phys. Rev. C 79, 034306 (2009)
135. J.A. Sheikh, J. Dobaczewski, P. Ring, L.M. Robledo, C. Yannouleas, J. Phys. G: Nucl. Part. Phys. 48, 123001 (2021)
136. J. Dukelsky, S. Pittel, G. Sierra, Rev. Mod. Phys. 76, 643 (2004)
137. J. Dukelsky, S. Pittel, in Fifty Years of Nuclear BCS: Pairing in Finite Systems, ed. by R.A. Broglia, V. Zelevinsky (WORLD SCIENTIFIC, Singapore, 2012), pp. 200–211
138. C. Qi, T. Chen, Phys. Rev. C 92, 051304(R) (2015)
139. H. Molique, J. Dudek, Phys. Rev. C 56, 1795 (1997)
140. C.S. Wu, J.Y. Zeng, Phys. Rev. C 39, 666 (1989)
141. C.S. Wu, J.Y. Zeng, Phys. Rev. Lett. 66, 1022 (1991)
142. C.S. Wu, J.Y. Zeng, Phys. Rev. C 44, 2566 (1991)
143. J.Y. Zeng, Y.A. Lei, T.H. Jin, Z.J. Zhao, Phys. Rev. C 50, 746 (1994)
144. J.Y. Zeng, S.X. Liu, Y.A. Lei, L. Yu, Phys. Rev. C 63, 024305 (2001)
145. J.Y. Zeng, S.X. Liu, L.X. Gong, H.B. Zhu, Phys. Rev. C 65, 044307 (2002)
146. J. Meng, J.Y. Guo, L. Liu, S.Q. Zhang, Front. Phys. China 1(1), 38 (2006)
147. Z. Shi, Z.H. Zhang, Q.B. Chen, S.Q. Zhang, J. Meng, Phys. Rev. C 97, 034317 (2018)
148. B.W. Xiong, Phys. Rev. C 101, 054305 (2020)
149. W.Y. Liang, C.F. Jiao, Q. Wu, X.M. Fu, F.R. Xu, Phys. Rev. C 92, 064325 (2015)
150. A.C. Dai, F.R. Xu, W.Y. Liang, Chin. Phys. C 43(8), 084101 (2019)
151. X.M. Fu, F.R. Xu, J.C. Pei, C.F. Jiao, Y. Shi, Z.H. Zhang, Y.A. Lei, Phys. Rev. C 87, 044319 (2013)
152. Z.H. Zhang, M. Huang, A.V. Afanasjev, Phys. Rev. C 101, 054303 (2020)
153. M. Bender, T. Duguet, D. Lacroix, Phys. Rev. C 79, 044319 (2009)
154. J.A. Sheikh, P. Ring, E. Lopes, R. Rossignoli, Phys. Rev. C 66, 044318 (2002)
155. B. Bayman, Nucl. Phys. 15, 33 (1960)
156. K. Dietrich, H.J. Mang, J.H. Pradal, Phys. Rev. 135, B22 (1964)
157. V.N. Fomenko, J. Phys. A: Gen. Phys. 3(1), 8 (1970)
158. D. Janssen, P. Schuck, Z. Physik A 301(3), 255 (1981)
159. N. Schunck, J.L. Egido, Phys. Rev. C 78, 064305 (2008)
160. N. Schunck, L.M. Robledo, Rep. Prog. Phys. 79(11), 116301 (2016)
161. R. An, G.F. Shen, S.S. Zhang, L.S. Geng, Chin. Phys. C 44(7), 074101 (2020)
162. M. Verriere, N. Schunck, D. Regnier, Phys. Rev. C 103, 054602 (2021)