Renormalization-Group Approach to Spin-Wave Theory of Quantum Heisenberg Ferromagnet

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Abstract

The renormalization-group method is used to analyze the low-temperature behaviour of a two-dimensional, spin-$s$ quantum Heisenberg ferromagnet. A set of recursion equations is derived in an one-loop approximation. The low-temperature asymptotics of the correlation length and the uniform susceptibility are obtained. For small spins ($s = 1/2, 1$) the results are essentially different from those in the spin-wave theory.

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The revival of the interest in the theory of quantum magnetism has led to creation of new approaches of investigation. An effective continuum field theory, which is quantum-mechanical generalization of the classical nonlinear \( \sigma \)-model was derived from lattice, large spin Heisenberg model of antiferromagnets \([1]\). The two dimensional antiferromagnet on a square lattice was treated also by means of the Schwinger boson representation of the spin algebra. This representation allows an appropriate mean-field theory of the low-temperature behaviour of the system \([2]\). At the same time, a modified spin-wave theory of Heisenberg (anti)ferromagnets was formulated \([3,4]\). The usual spin-wave theory was supplemented with the constraint that the magnetization at each site is zero. This ensures that the sublattice rotational symmetry is not broken.

The results obtained in the papers are in quantitative agreement, but, as was recognized by the authors, it is difficult to continue them to smaller values of the spin.

The aim of the present investigation is to obtain a better understanding of a small spin quantum ferromagnet. The renormalization-group method is used to analyze the low-temperature behaviour of a two dimensional, spin-\( s \) quantum Heisenberg ferromagnet. A set of recursion equations is derived in an one-loop approximation. The spin-recursion relation shows that the effective spin increases (the spin rescaling factor is two) and hence in the limit of infinitely many recursions, the spin-wave approximation can be used. The low-temperature asymptotics of the correlation length and the uniform susceptibility are obtained solving the recursion relations. For small spins \((s = 1/2, 1)\) the results are essentially different from those in the spin-wave theory.

The spin-\( s \) quantum Heisenberg ferromagnet is defined by the hamiltonian

\[
\hat{h} = -J \sum_{<i,j>} \hat{S}_i \cdot \hat{S}_j
\]

where \( \hat{S}_i \) are spin operators on site \( i(j) \) of a two dimensional square lattice with number of sites \( N \) and lattice spacing \( a \). By \( <i,j> \) I denote the sum over the nearest neighbours.

To preserve the sublattice rotational symmetry in spin-wave theory of the disordered phase, Takahashy \([3]\) imposed a condition of zero sublattice magnetization \( \langle \hat{S}_i^3 \rangle = 0 \). Using
Holstein-Primakoff representation for the spin operators, one can rewrite it as a condition that the total number, on average, of spin-waves per site is $s$. To enforce the constraint, one introduces a new term in the hamiltonian $\hat{h} \rightarrow \hat{h} - \mu \sum \hat{S}_i^3$, with the chemical potential $\mu$ to be determined from the condition. The system is disordered at any temperature $T \neq 0$ and the chemical potential is positive $\mu(T) > 0$. When temperature goes to zero $\mu(T)$ reaches 0 and the spin-wave bosons Bose-condense at zero wave-vector. Due to Bose condensation at $T = 0$, the long-range ferromagnetic correlation is present.

The low-temperature features are determined by the behaviour of the system near the origin of the Brillouin zone. Following the renormalization group method one devides the Brillouin zone into two parts. Then the half of the zone which contains small wave-vectors is of interest to us, and the theory should be reformulated using the reduced Brillouin zone. In the site representation, this is equivalent to introduce two sublattices, $A$ and $B$. I identify the sublattices and refer to the new lattice, with number of sites $N_1 = \frac{1}{2}N$ and lattice spacing $a_1 = \sqrt{2}a$, as $L_1$. The hamiltonian (1) can be rewritten in the form

$$\hat{h} = -J \sum \nu \sum_{i \in L_1} \left( \hat{S}_i^A \cdot \hat{S}_{i+a \nu}^B + \hat{S}_i^B \cdot \hat{S}_{i+a \nu}^A \right) - \mu \sum_{i \in L_1} \left( \hat{S}_i^{A3} + \hat{S}_i^{B3} \right)$$

where the sum is over the two directions $a_x = (a, 0)$ and $a_y = (0, a)$ and over the sites of the new lattice $L_1$. The spin operators $\hat{S}_i^A$ and $\hat{S}_i^B$ can be realized using canonical bose creation and annihilation operators $\hat{a}_i^{A+}, \hat{a}_i^A$ and $\hat{a}_i^{B+}, \hat{a}_i^B$ (Holstein-Promakoff representation) Then, to leading order of $s^{-1}$ the hamiltonian (2) takes the form

$$\hat{h} = sJ \sum \nu \sum_{i \in L_1} \left( \hat{a}_i^{A+} \hat{a}_i^A + \hat{a}_i^{A+} \hat{a}_{i+a \nu}^A + \hat{a}_i^B \hat{a}_i^{B+} + \hat{a}_i^B \hat{a}_{i+a \nu}^B - \hat{a}_i^{B+} \hat{a}_i^A - \hat{a}_i^{B+} \hat{a}_{i+a \nu}^A \right) - \mu \sum_{i \in L_1} \left( \hat{a}_i^{A+} \hat{a}_i^A + \hat{a}_i^{B+} \hat{a}_i^B + 2\mu - 2s \right) - 2Js^2N$$

The hamiltonian (3) is diagonalized by the canonical transformation

$$\hat{a}_i^A = \frac{1}{\sqrt{2}} \left( \hat{\alpha}_i + \hat{\beta}_i \right), \quad \hat{a}_i^{A+} = \frac{1}{\sqrt{2}} \left( \hat{\alpha}_i^{+} + \hat{\beta}_i^{+} \right), \quad \hat{a}_i^B = \frac{1}{\sqrt{2}} \left( \hat{\alpha}_i - \hat{\beta}_i \right), \quad \hat{a}_i^{B+} = \frac{1}{\sqrt{2}} \left( \hat{\alpha}_i^{+} - \hat{\beta}_i^{+} \right).$$

In momentum space the wave-vectors run the half of the Brillouin zone. The dispersion $\varepsilon_{\beta}^\beta$ of the $\hat{\beta}_i^{+}, \hat{\beta}_i$ modes is positive even at $T = 0$ ($\mu = 0$) and I refer to them as "fast modes". On the other hand, at zero temperature, $\varepsilon_{\alpha}^{\alpha}_{k=0} = 0$, and I refer to $\hat{\alpha}_i^{+}$ and $\hat{\alpha}_i$ as "slow modes".
To proceed further it is more convenient to represent the partition function in terms of path integral. In Matsubara representation

\[ Z = \int d\mu(a^{A+}, a^{A})d\mu(a^{B+}, a^{B}) \exp \left[ -\int d\tau \left( \sum_{i \in L} \left( a_{i}^{A+}a_{i}^{A} + a_{i}^{B+}a_{i}^{B} \right) + h(\tau) \right) \right] \]  

(5)

In formula (5) \( \beta \) is the inverse temperature, \( a_{i}^{A+}(\tau), a_{i}^{A}(\tau) \) and \( a_{i}^{B+}(\tau), a_{i}^{B}(\tau) \) are complex functions on imaginary time \( \tau \), subject to conditions \( a_{i}^{A+}(\tau)a_{i}^{A}(\tau) \leq 2s \) and \( a_{i}^{B+}(\tau)a_{i}^{B}(\tau) \leq 2s \), and \( d\mu(a^{R+}, a^{R}) \) is the usual path integral measure for bosons. The fields obey periodic boundary conditions. The Hamiltonian \( h(\tau) \) is obtained from Eqs (2) replacing the creation and annihilation operators by the complex fields. In particular, the spin operators are replaced by the vectors \( \vec{S}_{i}^{A}(\tau) \) and \( \vec{S}_{i}^{B}(\tau) \) that satisfy \( \vec{S}_{i}^{A}(\tau)^{2} = s^{2} \) and \( \vec{S}_{i}^{B}(\tau)^{2} = s^{2} \). The transformation (4) should be thought of as a change of variables in the path integral (5) with Jacobian equal to one. After this transformation, the spin vectors \( \vec{S}_{i}^{A} \) and \( \vec{S}_{i}^{B} \) depend on the complex fields \( \alpha_{i}^{+}(\tau), \alpha_{i}(\tau) \) and \( \beta_{i}^{+}(\tau), \beta_{i}(\tau) \).

I shall define the spin-vectors \( \vec{S}_{(1)i} \) by the equations

\[ \vec{S}_{i}^{A}\big|_{\beta_{i}^{+}=0, \beta_{i}=0} = \vec{S}_{i}^{B}\big|_{\beta_{i}^{+}=0, \beta_{i}=0} = \frac{1}{2} \vec{S}_{(1)i}(\alpha^{+}, \alpha) \]  

(6)

They depend on the slow modes and satisfy \( \vec{S}_{(1)i} = (2s)^{2} \).

It is convenient to represent the spin-vectors in the form

\[ \vec{S}_{i}^{A} = \frac{1}{2} \sqrt{1 - \frac{\vec{\varphi}_{i}^{A}\varphi_{i}^{A}}{s^{2}}} \vec{S}_{(1)i} + \vec{\varphi}_{i}^{A}\vec{e}_{i} + \vec{\bar{e}}_{i}\varphi_{i}^{A} ; \quad \vec{S}_{i}^{B} = \frac{1}{2} \sqrt{1 - \frac{\vec{\varphi}_{i}^{B}\varphi_{i}^{B}}{s^{2}}} \vec{S}_{(1)i} + \vec{\varphi}_{i}^{B}\vec{e}_{i} + \vec{\bar{e}}_{i}\varphi_{i}^{B} \]  

(7)

where the real vectors \( \vec{e}_{j}^{1} = \vec{e}_{j} + \vec{\bar{e}}_{j} \) and \( \vec{e}_{j}^{2} = \frac{1}{2} (\vec{e}_{j} - \vec{\bar{e}}_{j}) \) depend on the fields \( \alpha_{j}^{+}(\tau) \) and \( \alpha_{j}(\tau) \). They are orthogonal to the vectors \( \vec{S}_{(1)j} \) and to each other and \( \vec{e}_{j} \cdot \vec{e}_{j} = \frac{1}{2} \).

The representation (7) is not unique. There is an arbitrariness in the definition of the vectors \( \vec{e}_{j}, \vec{e}_{j} \) and the coefficients \( \vec{\varphi}_{j}^{A}, \varphi_{j}^{A}, \vec{\varphi}_{j}^{B}, \varphi_{j}^{B} \). It is clear from Eqs (7) that a local \( U(1) \) transformation of the vectors and the coefficients does not change the spin vectors \( \vec{S}_{i}^{A} \) and \( \vec{S}_{i}^{B} \). I shall use the following representation for the vectors \( \vec{e}_{j} \) and \( \vec{\bar{e}}_{j} \)

\[ \begin{align*}
\vec{e}_{i}^{+} &= -\frac{1}{4s} \alpha_{i}^{+} \alpha_{i} \quad \vec{e}_{i}^{-} = \frac{1}{4s} (4s - \alpha_{i}^{+} \alpha_{i}) \quad \vec{e}_{i}^{3} = -\frac{1}{4s} (4s - \alpha_{i}^{+} \alpha_{i}) \frac{1}{2} \alpha_{i} \\
\vec{\bar{e}}_{i}^{+} &= \frac{1}{4s} (4s - \alpha_{i}^{+} \alpha_{i}) \quad \vec{\bar{e}}_{i}^{-} = -\frac{1}{4s} \alpha_{i}^{+} \alpha_{i}^{+} \quad \vec{\bar{e}}_{i}^{3} = -\frac{1}{4s} \alpha_{i}^{+} (4s - \alpha_{i}^{+} \alpha_{i}) \frac{1}{2}
\end{align*} \]  

(8)
The next step of the renormalization group approach is the elimination of the fast modes. One can integrate over the fields $\beta_i^+(\tau)$ and $\beta_i(\tau)$ only perturbatively. For this reason I expand $\varphi_i^A$ and $\varphi_i^B$ in powers of $\beta_i^+(\tau)$ and $\beta_i(\tau)$ fields. Calculating the coefficients in the expansion to leading order of $s^{-1}$, one obtains

$$
\varphi_i^A \simeq \sqrt{s} \beta_i + \frac{\alpha_i}{4\sqrt{s}} \beta_i^+ \beta_i \quad \varphi_i^B \simeq -\sqrt{s} \beta_i + \frac{\alpha_i}{4\sqrt{s}} \beta_i^+ \beta_i
$$

(9)

where the terms proportional to $\beta_i \beta_i$ are dropped because they do not contribute to the effective action. Substituting (9) and (7) in the hamiltonian (2) and selecting the quadratic terms in $\beta_i^+$ and $\beta_i$ one gets

$$
h = -\frac{J}{2} \sum \sum [\tilde{S}_{(1)}]_i \cdot [\tilde{S}_{(1)}]_j + sJ \sum \sum [\beta_i^+ \beta_i + \beta_j^+ \beta_j + \beta_i^+ \beta_j + \beta_j^+ \beta_i]
$$

$$-\frac{J}{8s} \sum \sum \left[ (\tilde{\alpha}_i^+ + \tilde{\alpha}_j^+ + \tilde{\alpha}_i) \beta_i^+ \beta_i + \tilde{\alpha}_j^+ \beta_j + \tilde{\alpha}_j \beta_j^+ \beta_j \right]
$$

$$-sJ \sum \sum \left[ (1 - 2\tilde{e}_i \tilde{e}_j) \beta_i^+ \beta_j + (1 - 2\tilde{e}_i \tilde{e}_j) \beta_j^+ \beta_i \right]
$$

(10)

where $j = i + a$, and the terms proportional to $\beta_i \beta_j$ and $\beta_i^+ \beta_j^+$ are dropped again.

Integrating over the fields $\beta_i^+(\tau)$ and $\beta_i(\tau)$ we lose the rotational invariance of the theory. This is well known disadvantage of the spin-wave approximation. To restore the rotational symmetry, and to obtain a rotationally invariant effective action one has to average it over the elements of the group of rotation $\mathbb{R}$. Under the nonlinear rotational transformations of the fields $\alpha_i^+$ and $\alpha_i$ the scalar product of the vectors $\tilde{e}_i$ and $\tilde{e}_j$ changes the phase $\tilde{e}_i \cdot \tilde{e}_j \rightarrow \exp[i(\delta_i - \delta_j)]\tilde{e}_i \cdot \tilde{e}_j$, where $\delta_i$ depends on the fields $\alpha_i^+$ and $\alpha_i$ and on the group elements. Then the module of the scalar product is invariant and the phase $\gamma_{ij}$ transforms as an $U(1)$ gauge field on a square lattice $\gamma^\prime_{ij} = \gamma_{ij} + \delta_i - \delta_j$. Hence, the phases enter the invariant action in the same way as $U(1)$ gauge fields enter a gauge invariant action. It is not difficult to check that these terms have higher dimension and one has to drop them. This means to replace $\tilde{e}_i \cdot \tilde{e}_j$ and $\tilde{e}_j \cdot \tilde{e}_i$ by $|\tilde{e}_i \cdot \tilde{e}_j|$ in the hamiltonian (10). Straightforward calculation using (8) leads to the result
\[ |\vec{e}_i \cdot \vec{e}_j| = \frac{1}{2} - \frac{1}{32s^2} \left( \vec{S}_{(1)i} - \vec{S}_{(1)j} \right)^2 \]  

(11)

I shall treat the other terms in the same way. Using the representation (8) and (6), one gets

\[ \vec{S}_{(1)j} \cdot (\vec{e}_i \alpha_i^+ + \vec{e}_i \alpha_i) + \vec{S}_{(1)i} \cdot (\vec{e}_j \alpha_j^+ + \vec{e}_j \alpha_j) = -\frac{1}{2\sqrt{s}} \left( \vec{S}_{(1)i} - \vec{S}_{(1)j} \right)^2 + \ldots \]  

(12)

where the dots stand for terms which become irrelevant after group averaging.

Integrating over the fast modes \( \beta_i^+ (\tau) \) and \( \beta_i (\tau) \) only the tadpole graphs contribute to the one-loop approximation. Calculating them at zero temperature and taking into account the Eqs (11) and (12) I obtain the effective hamiltonian

\[ h_{\text{eff}} = -J' \sum \sum_i \vec{S}_{(1)i} \cdot \vec{S}_{(1)j}, \]  

where

\[ J' = \frac{1}{2} J \left[ 1 - \frac{1}{2s} \left( \frac{1}{2} + \frac{1}{\pi^2} \right) \right] \]  

and \( j = i + a_\nu \). The sum is over the sites of the lattice \( L_1 \) and over the two directions of the previous lattice. I want to represent the sum as a sum over the nearest neighbours of the lattice \( L_1 \). For this reason I write the hamiltonian in momentum representation

\[ h_{\text{eff}} = -J' \sum \vec{S}_{(1)} \cdot \vec{S}_{(1)}, \]  

(13)

The effective hamiltonian \( h_{(1)} \) is a hamiltonian of a Heisenberg ferromagnet with spin \( s_1 = 2s \) (see eq (6)), defined on a square lattice with spacing \( a_1 = \sqrt{2}a \), and with exchange constant \( J_1 = \frac{1}{4} J \left[ 1 - \frac{1}{2s} \left( \frac{1}{2} + \frac{2}{\pi^2} \right) \right] \).

Repeating the procedure one obtains the recursion equation

\[ s_{n+1} = 2s_n, \quad a_{n+1} = \sqrt{2}a_n, \quad J_{n+1} = \frac{1}{4} J_n \left[ 1 - \frac{1}{2s_n} \left( \frac{1}{2} + \frac{2}{\pi^2} \right) \right] \]  

(13)

It is readily to solve them

\[ s_n = 2^n s, \quad a_n = 2^n a, \quad J_n = \frac{1}{4^n} J \prod_{i=1}^n \left[ 1 - \frac{1}{2i} \left( \frac{1}{2} + \frac{2}{\pi^2} \right) \right] = \frac{1}{4^n} J \rho_n (s) \]  

(14)

I shall consider the equation for the chemical potential. Making the transformation (4) and rewriting it in momentum space representation one obtains
\[
\frac{1}{N_1} \sum_q \left[ \langle \alpha_q^+ \alpha_q \rangle + \langle \beta_q^+ \beta_q \rangle \right] = 2s = s_1 \tag{15}
\]

where \( \vec{q} \) runs the Brillouin zone of \( L_1 \) lattice. The dispersion of \( \beta_q \) bosons is positive, hence, when the temperature goes to zero, \( \langle \beta_q^+ \beta_q \rangle \) goes to zero exponentially and can be dropped in comparison to \( \langle \alpha_q^+ \alpha_q \rangle \). As a result the equation for the chemical potential in low-temperature limit takes the form \( \frac{1}{N_1} \sum_q \langle \alpha_q^+ \alpha_q \rangle = s_1 \). In the limit of infinitely many recursions, one can calculate \( \langle \alpha_q^+ \alpha_q \rangle \) in the spin-wave approximation \([3]\). Then

\[
\mu(\beta \to \infty) = \frac{1}{\beta} e^{-4\pi s^2 J \rho(s) \beta} \tag{16}
\]

where

\[
\rho(s) = \prod_{l=1}^{\infty} \left[ 1 - \frac{1}{2l} s \left( \frac{1}{2} + \frac{2}{\pi^2} \right) \right] \tag{17}
\]

To obtain the correlation length one has to consider the asymptotic behaviour of the two-point static correlation function at long distance. In momentum space this translates to small wave-vectors. Each recursion separates smaller and smaller area around the origin of the Brillouin zone. In the limit of infinitely many recursions the correlation function can be calculated in the spin-wave approximation. Using the relation between the chemical potential and the correlation length in spin-wave theory \([3]\), I obtain the low-temperature asymptotic of the correlation length

\[
\xi(\beta \to \infty) = \lim_{n \to \infty} \frac{1}{2} \left( \frac{s_n J_n a_n^2}{\mu(\beta)} \right)^{\frac{1}{2}} = \frac{1}{2} a \sqrt{s J \rho(s) \beta e^{2\pi s^2 J \rho(s) \beta}} \tag{18}
\]

Let us add a new (Zeeman) term in the hamiltonian \( h \to h - 2H \sum_i \hat{S}_i^3 \), where \( H \) is an external constant magnetic field. Then the uniform susceptibility is proportional to \( d^2 F / dH^2 \big|_{H=0} \), where \( F \) is the free energy of the system. The recursion does not change the magnetic field. For the free energy one obtains \( F = \frac{1}{2} F_1 + \ldots \), where \( F_1 \) is the free energy of the system which is defined by the hamiltonian \( h_{(1)} \) with an additional Zeeman term. The dots stand for terms independent on the magnetic field. The multiplier \( \frac{1}{2} \) in front of \( F_1 \) is a consequence of the fact that the number of sites of the lattice \( L_1 \) is \( N_1 = \frac{1}{2} N \). Hence \( \chi(\beta \to \infty) = \frac{1}{2} \chi_1(\beta \to \infty) \) and to leading order of small temperature
\[ \chi(\beta \to \infty) = \lim_{n \to \infty} \frac{1}{2\pi} \chi_n(\beta \to \infty) = \frac{1}{3\pi s J \rho(s)} e^{4\pi s^2 J \rho(s) \beta} \] (19)

where \( \chi_n(\beta \to \infty) \) is calculated in the spin-wave approximation [3].

From Eq.(17) follows that \( \rho(s \to \infty) = 1 \). Substituting this result in formulas (16),(18) and (19), one gets the results from the spin-wave theory [3]. On the other hand \( \rho(s)_{|s=\frac{1}{2}} = 0.1327 \) \( \rho(s)_{|s=1} = 0.4462 \) and one can see that the results for small spins are essentially different from those in the spin-wave theory.

In conclusion I shall summarize what has been discussed here. Ideas and tools developed in the papers of K.G.Wilson [5], A.A.Migdal [6], L.P.Kadanoff [7], A.M.Polyakov [8], D.Nelson, R.A.Pelcovits, S.Chakravarty and B.I.Halperin [9,10] have been adapted to formulate a renormalization group approach in the spin-wave theory. The formalism allows the analysis of systems with small spin.

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