GLOBAL ATTRACTORS FOR A MIXTURE PROBLEM IN ONE DIMENSIONAL SOLIDS WITH NONLINEAR DAMPING AND SOURCES TERMS

M. L. SANTOS
Institute of Exact and Natural Sciences, Doctoral Program in Mathematics
Federal University of Pará, Augusto corréa Street,
Number 01, 66075-110, Belém PA, Brazil

MIRELSON M. FREITAS
Federal University of Pará, Raimundo Santana Street s/n
Salinópolis PA, 68721-000, Brazil

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ABSTRACT. This paper is concerned with long-time dynamics of binary mixture problem of solids, focusing on the interplay between nonlinear damping and source terms. By employing nonlinear semigroups and the theory of monotone operators, we obtain several results on the existence of local and global weak solutions, and uniqueness of weak solutions. Moreover, we prove that such unique solutions depend continuously on the initial data. We also establish the existence of a global attractor, and we study the fractal dimension and exponential attractors.

1. Introduction. This discussion is devoted to a special case of a theory of binary mixture of solids with nonlinear damping and sources terms. It is important to note that the theory of mixtures of solids has been widely investigated in the last decades, see the references [3, 6, 7, 8] for a detailed presentation. Qualitative properties of solutions to the problem defining this kind of material have been the scope of many investigations. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature [1, 2, 14, 17].

The version of the theory of binary mixture of solids considered below bears the influence of nonlinear amplitude-modulated forcing terms that could either act as energy ”sinks” with a restoring effect (e.g., as a nonlinear refinement on Hooke’s law) or in the more interesting case as ”sources” that contribute to the build-up of energy and potentially lead to a blow-up of solutions. In addition, to counterbalance the effects of potentially destabilizing strong sources, the system incorporates internal viscous (frictional) damping. Besides the Hadamard well-posedness of this problem,
the influence of the source-damping interaction on the behavior of solutions is of
the main interest in this work.

From the above, the problem we want to study can be stated as follows:

Given \( L > 0 \) and \( T > 0 \), we consider a rod composed by a mixture of two
interacting continuous with reference configuration \([0, L]\). Denoting by
\( u(x, t), w(x, t) : [0, L] \times [0, T] \to \mathbb{R} \)
the displacement of each constituent, the considered particles are suppose to occupy
the same position at time \( t = 0 \), so that \( x = y \). For \( i = 1, 2 \), we also introduce the
mass densities \( \rho_i > 0 \), the partial stresses \( T_i \) and the external forces \( F_i \)
associated to \( u \) and \( w \) respectively. Indicating with \( P_i \) the internal body forces, the motion
equations are given by the differential system
\[
\begin{align*}
\rho_1 u_{tt} &= T_1x - P_1 + F_1, \\
\rho_2 w_{tt} &= T_2x + P_2 + F_2.
\end{align*}
\]  
(1.1)

We assume the constitutive equations of the partial stresses to be
\[
\begin{align*}
T_1 &= a_{11}u_x + a_{12}w_x, \\
T_2 &= a_{12}u_x + a_{22}w_x
\end{align*}
\]  
(1.2)

where
\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{bmatrix}
\]
is a positive definite (real) symmetric matrix, while the internal body forces are
supposed to have the form
\[
\begin{align*}
P_1 &= -f_1(u, w) \quad \text{and} \quad P_2 = f_2(u, w).
\end{align*}
\]  
(1.3)

Finally, taking
\[
\begin{align*}
F_1 &= -g_1(u_t) \quad \text{and} \quad F_2 = -g_2(w_t)
\end{align*}
\]
and substituting (1.2) and (1.3) into system (1.1), we end up with
\[
\begin{align*}
\rho_1 u_{tt} - a_{11}u_{xx} - a_{12}w_{xx} + g_1(u_t) &= f_1(u, w), \quad \text{in} \ (0, L) \times (0, T), \\
\rho_2 w_{tt} - a_{12}u_{xx} - a_{22}w_{xx} + g_2(w_t) &= f_2(u, w), \quad \text{in} \ (0, L) \times (0, T).
\end{align*}
\]  
(1.4)

The two equations in (1.4) are supplemented with the boundary and initial conditions
\[
\begin{align*}
u(0, t) &= u(L, t) = w(0, t) = w(L, t) = 0, \quad t > 0, \\
u(0) &= u_0 \in H^1_0(0, L), \quad u_t(0) = u_1 \in L^2(0, L), \\
w(0) &= w_0 \in H^1_0(0, L), \quad w_t(0) = w_1 \in L^2(0, L).
\end{align*}
\]  
(1.5)

Note that in light of the assumptions on the matrix \( A \) we have the relation
\[
a_{11} > 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}^2 > 0.
\]  
(1.6)

The main goal here is to study long-time behavior of binary mixture of solids with
nonlinear damping and sources terms, focusing on the interplay between nonlinear
damping and source terms. The sources may represent restoring forces, but may
also be focusing thus potentially amplifying the total energy. We will use nonlinear
semigroups and the theory of monotone operators to obtain several results on the
existence of local and global weak solutions, and uniqueness of weak solutions.
Moreover, we will prove that such unique solutions depend continuously on the
initial data. We will also prove the existence of a global attractor and we will study
the fractal dimension and exponential attractors. This is done by showing that the solution semigroup is gradient and quasi-stable in the sense of [9].

The paper is organized as follows: In the Section 2 we present the notations needed, we list the standing assumptions on the nonlinear terms and we summarize the main results. In the Section 3 we prove the existence of global solution. In Section 4 we study the global attractors. Finally, in Section 5 we study the fractal dimensional and exponential attractors.

2. Problem setting, assumptions and main results. In this section we will present the notations, assumptions and main results.

2.1. Notation. In throughout this paper, for scalar functions we will use the following notation

\[ \| u \|_{1,2} = \| u_x \|_2, \quad \| u \|_p = \| u \|_{L^p(0,L)}, \quad (u, v) = (u, v)_{L^2(0,L)}, \]

and for vector-value functions \( z = (u, w), \hat{z} = (\hat{u}, \hat{w}) \) we will use similar notation

\[ \| z \|_{1,2} = (\| u \|_{1,2}^2 + \| w \|_{1,2}^2)^{1/2}, \quad \| z \|_p = (\| u \|_{p}^p + \| w \|_{p}^p)^{1/p}, \quad (z, \hat{z}) = (u, \hat{u}) + (w, \hat{w}). \]

We recall the Poincaré's inequality

\[ \lambda_0 \| \varphi \|_2^2 \leq \| \varphi \|_{1,2}^2 \quad \text{for all} \quad \varphi \in H^1_0(0, L), \]

where \( \lambda_0 = \pi^2/L^2. \)

Let us consider the Hilbert space

\[ V = (H^1_0(0, L))^2, \quad H = (L^2(0, L))^2 \quad \text{and} \quad \mathcal{H} = V \times H, \quad (2.1) \]

with inner products given by

\[ (z, \hat{z})_V = a_{11}(u_x, \hat{u}_x) + a_{12}(u_x, \hat{w}_x) + a_{21}(w_x, \hat{u}_x) + a_{22}(w_x, \hat{w}_x), \]

\[ (z_1, \hat{z}_1)_H = \rho_1(u_1, \hat{u}_1) + \rho_2(w_1, \hat{w}_1), \quad (2.2) \]

and

\[ (U, \hat{U})_\mathcal{H} = (z, \hat{z})_V + (z_1, \hat{z}_1)_H, \]

respectively, where \( z = (u, w), \hat{z} = (\hat{u}, \hat{w}), z_1 = (u_1, w_1), \hat{z}_1 = (\hat{u}_1, \hat{w}_1), \) and \( U = (z, z_1), \hat{U} = (\hat{z}, \hat{z}_1). \) We denote by \( \| \cdot \|_V, \| \cdot \|_H \) and \( \| \cdot \|_\mathcal{H} \) the associated norms, that is,

\[ \| z \|_V^2 = a_{11} \| u_x \|_2^2 + 2a_{12}(u_x, w_x) + a_{22} \| w_x \|_2^2, \]

\[ \| z_1 \|_H^2 = \rho_1 \| u_1 \|_2^2 + \rho_2 \| w_1 \|_2^2, \]

and

\[ \| U \|_\mathcal{H}^2 = \| z \|_V^2 + \| z_1 \|_H^2. \quad (2.3) \]

In particular, there exists a positive constant \( K_0 \) such that

\[ \| z \|_V^2 \geq K_0 \| z \|_{1,2}^2. \quad (2.4) \]

Taking \( K_1 = \min\{K_0, \rho_1, \rho_2\} \) we obtain the equivalence of the norms

\[ \| U \|_\mathcal{H}^2 \geq K_1 \left( \| z \|_{1,2}^2 + \| z_1 \|_2^2 \right). \quad (2.5) \]

Using the preliminary notations, we can write the problem (1.4)-(1.5) as an Cauchy problem:

\[
\begin{aligned}
U_t + AU &= F(U), \\
U(0) &= U_0 \in \mathcal{H},
\end{aligned}
\quad (2.6)
\]
where
\[
\mathbf{AU} = \begin{pmatrix}
-\frac{a_{11}}{\rho_1} u_{xx} - \frac{a_{12}}{\rho_1} w_{xx} + \frac{1}{\rho_1} g_1(u_1) \\
-\frac{a_{21}}{\rho_2} u_{xx} - \frac{a_{22}}{\rho_2} w_{xx} + \frac{1}{\rho_2} g_2(w_1)
\end{pmatrix}^{tr}
\]
with the domain
\[
D(A) = (H^2(0, L) \cap H^1_0(0, L))^2 \times (H^1_0(0, L))^2.
\]
The function \( F : \mathcal{H} \to \mathcal{H} \) is defined by
\[
F(U) = \begin{pmatrix}
0 \\
0 \\
\frac{1}{\rho_1} f_1(z) \\
\frac{1}{\rho_2} f_2(z)
\end{pmatrix}^{tr}.
\]

Inspired by [18, 16, 11, 12] we use the following assumption.

**Assumption 2.1** (For existence of global attractors).
- \( g_1, g_2 \in C^1(\mathbb{R}) \) are monotone increasing functions with \( g_1(0) = g_2(0) = 0 \). In addition, there exist positive constants \( \alpha \) and \( \beta \) such that for all \( |s| \geq 1 \),
  \[
  \alpha_j |s|^2 \leq g_j(s)s \leq \beta_j |s|^2, \quad j = 1, 2.
  \]
- There exists a function \( F \in C^2(\mathbb{R}^2) \) such that
  \[
  \nabla F = (f_1, f_2).
  \]
- There exists a constant \( C > 0 \) such that
  \[
  |\nabla f_j(z)| \leq C \left(|u|^{p-1} + |w|^{p-1} + 1\right), \quad j = 1, 2, \quad \text{with } p \geq 1.
  \]
- There exist \( \alpha_0, \beta_0 > 0 \) such that
  \[
  F(z) \leq \alpha_0 \left(|u|^2 + |w|^2\right) + \beta_0,
  \]
  where
  \[
  0 \leq \alpha_0 < \frac{K_0 \lambda_0}{2}.
  \]

Moreover, we assume that
\[
\nabla F(z) \cdot z \leq F(z) + \alpha_0 \left(|u|^2 + |w|^2\right) + \beta_0.
\]

**Remark 2.2.** Observe that the assumption on the damping, implies that for any \( \delta > 0 \) there exists a \( C_\delta > 0 \) such that
\[
C_\delta(g_j(u) - g_j(w))(u - w) + \delta \geq |u - w|^2, \quad \text{for all } u, w \in \mathbb{R},
\]
\( \text{cf. [10, Proposition B.1.2]} \).

In order to describe the results we introduce the definition of weak solution to the problem (1.4)-(1.5).

**Definition 2.3.** A vector-valued function \( z = (u, w) \) is called a weak solution to (1.4)-(1.5) if
- \( z \in C([0, \infty); V) \), \( (z(0), z_t(0)) = U_0 \in \mathcal{H} \);
- \( z_t \in C([0, \infty); H) \cap L^2_{loc}(\mathbb{R}^+; H) \);
The weak solution $z = (u, w)$ satisfies the following identity in the sense of distributions for all $\varphi = (\phi, \psi) \in (H^1_0(0, L))^2$:

$$\frac{d}{dt}(z_t, \varphi)_H + (z, \varphi)_V + (\mathscr{G}(z_t), \varphi) = (\mathscr{F}(z), \varphi),$$

where

$$\mathscr{G}(z_t) = (g_1(u_t), g_2(w_t)), \quad \mathscr{F}(z) = (f_1(z), f_2(z)). \quad (2.15)$$

The weak solution $z = (u, w)$ is called strong if

$$(z, z_t) \in C([0, \infty); D(A)) \cap W^{1,\infty}_{loc}(\mathbb{R}^+, \mathcal{H}).$$

Also, we define the total energy by

$$E(t) = E(t) - \int_0^t F(z(t)) \, dx,$$

where $E(t)$ is the linear energy defined by

$$E(t) = \frac{1}{2} \left( \|z(t)\|_V^2 + \|z_t(t)\|_H^2 \right). \quad (2.17)$$

### 2.2. Main results.

In this part, we present the main results of this paper.

**Theorem 2.4** (Well-posedness). *Suppose that Assumption 2.1 hold. Then

- For any initial data $U_0 \in \mathcal{H}$, there exists a weak solution $U \in C([0, \infty); \mathcal{H})$ to (1.4)-(1.5) satisfying the following energy identity for all $s \in [0, t)$:

$$E(t) + \int_s^t (\mathscr{G}(z_t), z_t) \, d\tau = E(s). \quad (2.18)$$

- The weak solution $U \in C([0, \infty); \mathcal{H})$ depend continuously on initial data $U_0$ in the phase space $\mathcal{H}$. In particular, such solutions are unique.

- If $U_0 \in D(A)$ then the weak solution is strong.

**Theorem 2.5** (Existence of global attractors). *Suppose that Assumption 2.1 hold. Then the dynamical system $(\mathcal{H}, S_t)$ generated by the equation (1.4) is dissipative and asymptotically smooth, and hence, it has a compact global attractor $\mathbb{A}$. Moreover, the global attractor $\mathbb{A}$ is characterized by

$$\mathbb{A} = \mathcal{M}^u(\mathcal{N}), \quad (2.19)$$

where $\mathcal{N}$ is the set of stationary point of $S_t$ and $\mathcal{M}^u(\mathcal{N})$ is the unstable manifold of $\mathcal{N}$.

To obtain the next result, we will need to replace the assumption (2.8) by

$$\alpha_j \leq g_j(s) \leq \beta_j \quad \text{for all } s \in \mathbb{R}. \quad (2.20)$$

**Remark 2.6.** Observe that assumption (2.20) implies the monotonicity property

$$(g_j(u) - g_j(w))(u - w) \geq m_j|u - w|^2 \quad \text{for all } u, w \in \mathbb{R}, \ j = 1, 2, \quad (2.21)$$

where $m_j > 0$, $j = 1, 2$.

**Theorem 2.7** (Fractal dimension, regularity and exponential attractors). *Assume the assumptions of Theorem 2.5 with (2.8) replaced by (2.20), then

- The global attractor $\mathbb{A}$ has finite fractal dimension.

- The complete trajectories $(z(\cdot), z_t(\cdot))$ in $\mathbb{A}$ satisfy

$$\|z\|_{H^2_0(0, L)}^2 + \|z_t\|_{L^2}^2 \leq C,$$

for some constant $C > 0$.
3. Existence global. In this section, we obtain the result on the existence of global solution stated in Theorem 2.4.

We start by proving two lemmas which will be used in sequence.

**Lemma 3.1.** Suppose that (2.10) hold. Then the function $F : \mathcal{H} \to \mathcal{H}$ defined in (2.7) is locally Lipschitz continuous.

**Proof.** Firstly, we recall that $V = (H_0^1(0, L))^2$, $H = (L^2(0, L))^2$, $\mathcal{H} = V \times H$. Let $U, \hat{U} \in \mathcal{H}$ such that $\|U\|_{\mathcal{H}}, \|\hat{U}\|_{\mathcal{H}} \leq R$, where $R > 0$. By our notation we have $U = (z, z_1), \hat{U} = (\hat{z}, \hat{z}_1)$, where $z = (u, w), \hat{z} = (\hat{u}, \hat{w})$, $z_1 = (u_1, w_1), \hat{z}_1 = (\hat{u}_1, \hat{w}_1)$.

By (2.10) and the Mean Value Theorem, we have for $j = 1, 2$,

$$
\|f_j(z) - f_j(\hat{z})\|_2^2 \\
= \int_0^L |f_j(z) - f_j(\hat{z})|^2 \, dx \\
\leq C \int_0^L \left( |u|^{2(p-1)} + |\dot{u}|^{2(p-1)} + |w|^{2(p-1)} + |\dot{w}|^{2(p-1)} + 1 \right) |z - \hat{z}|^2 \, dx.
$$

(3.1)

All terms on the right-hand side of (3.1) are estimated analogously. Using the Hölder’s inequality, Sobolev embedding $H_0^1(0, L) \hookrightarrow L^2(0, L)$ for all $1 \leq q \leq \infty$, assumptions $p \geq 1$ and $\|u\|_{1,2} \leq C(R)$, we obtain

$$
\int_0^L |u|^{2(p-1)} |z - \hat{z}|^2 \, dx \leq C \left( \int_0^L |u|^{2p} \, dx \right)^{\frac{p-1}{p}} \left( \int_0^L |z - \hat{z}|^{2p} \, dx \right)^{\frac{1}{p}} \\
\leq C \|u\|_{1,2}^{2(p-1)} \|z - \hat{z}\|_{1,2}^2 \\
\leq C(R)^{2(p-1)} \|z - \hat{z}\|_{1,2}^2.
$$

(3.2)

Therefore, by (3.2) and (2.5) we see that

$$
\|f_j(z) - f_j(\hat{z})\|_2 \leq C(R) \|z - \hat{z}\|_{1,2} \leq C(R) \|U - \hat{U}\|_{\mathcal{H}}.
$$

Using the above estimate and definition of $F$ it is easy to see that there exists a constant $L(R) > 0$ such that

$$
\|F(U) - F(\hat{U})\|_{\mathcal{H}} \leq L(R) \|U - \hat{U}\|_{\mathcal{H}}.
$$

The proof is complete. \(\square\)

**Lemma 3.2.** Suppose that $z = (u, w)$ is a strong solution to (1.4)-(1.5). Then

- There exists a positive constant $C_0$ such that
  
  $$
  C_0 E(t) \leq E(t) + L\beta_0 \quad \text{for all } t \geq 0. 
  $$

(3.3)

- The following identity holds
  
  $$
  \frac{d}{dt} E(t) = -\langle \mathcal{G}(z_t), z_t \rangle \quad \text{for all } t \geq 0.
  $$

(3.4)
Proof. Using Poincaré’s inequality, (2.11) and (2.4) we obtain
\[ \int_0^T F(z) \, dx \leq \alpha_0 \|z\|_{2}^2 + L\beta_0 \leq \frac{\alpha_0}{\lambda_0} \|z\|_{L^2}^2 + L\beta_0 \leq \frac{2\alpha_0}{K\lambda_0} E(t) + L\beta_0. \]
Consequently,
\[ E(t) = E(t) - \int_0^L F(z) \, dx \geq \left( 1 - \frac{2\alpha_0}{K\lambda_0} \right) E(t) - L\beta_0 = C_0 E(t) - L\beta_0, \]
where \( C_0 = 1 - 2\alpha_0/K\lambda_0 \) is positive by assumption (2.12). Now, Multiplying (1.4) by \( z_t = (u_t, w_t) \), and integrating by parts we obtain (3.4). The proof is complete. \( \square \)

3.1. Proof of Theorem 2.4: The argument follows the lines of the proof of [18, Proposition 2.1] also see [12, Theorem 1] and [9, Theorem 2.1].

We need to verify that the operator \( A \) is maximal monotone. We recall that \( V = (H_0^1(0, L))^2 \), \( H = (L^2(0, L))^2 \), \( \mathcal{H} = V \times H \) and our vector notation \( U = (z, z_1) \), \( \hat{U} = (\hat{z}, \hat{z}_1) z = (u, w) \), \( z_1 = (u_1, w_1) \), \( \hat{z} = (\hat{u}, \hat{w}) \), \( \hat{z}_1 = (\hat{u}_1, \hat{w}_1) \). Inspired by [18] we write the operator \( A \) as
\[ A(U) = (-z_1, B(z) + G(z_1)), \quad (3.5) \]
where \( B : V \to V' \) and \( G : V \to V' \) are given by
\[ B(z) = \begin{pmatrix} -\frac{a_{11}}{\rho_1} u_{xx} - \frac{a_{12}}{\rho_2} w_{xx} \\ -\frac{a_{21}}{\rho_1} u_{xx} - \frac{a_{22}}{\rho_2} w_{xx} \end{pmatrix}^{tr}, \quad G(z_1) = \begin{pmatrix} \frac{1}{\rho_1} g_1(u_1) \\ \frac{1}{\rho_2} g_2(w_1) \end{pmatrix}^{tr}. \]

Step 1: \( A \) is monotone. By straightforward computation, for all \( U, \hat{U} \in \mathcal{D}(A) \) we get
\[ \langle A(U) - A\hat{U}, U - \hat{U} \rangle_{\mathcal{H}} = -\langle z_1 - \hat{z}_1, z - \hat{z} \rangle_V + \langle z - \hat{z}, z_1 - \hat{z}_1 \rangle_V 
+ \langle G(z_1) - G(\hat{z}_1), z_1 - \hat{z}_1 \rangle_H 
= \langle G(z_1) - G(\hat{z}_1), z_1 - \hat{z}_1 \rangle_H \geq 0. \]
where we have used the monotonicity of \( g_1 \) and \( g_2 \). Therefore \( A \) is monotone.

Step 1: \( A \) is maximal monotone. We need to prove that the range of \( A + I \) is all of \( \mathcal{H} \).
Let \( \hat{V} = (v, v_1) \in \mathcal{H} \). We will prove that there exists \( U = (z, z_1) \in \mathcal{D}(A) \) such that \( (A + I)U = \hat{V} \), that is,
\[ \begin{cases} -z_1 + z = v, \\
B(z) + G(z_1) + z_1 = v_1. \end{cases} \quad (3.7) \]
Observe that (3.7) is equivalent to
\[ B(z_1) + G(z_1) + z_1 = v_1 - B(v). \quad (3.8) \]
Since \( v \in V \), then the right hand side of (3.8) belongs to \( V' \). Thus, we define the operator \( S : V \to V' \) by
\[ S(z_1) = B(z_1) + G(z_1) + z_1. \]
So we need to prove that \( S \) is surjective. In view of [5, Corollary 1.2], we only need to prove that \( S \) is maximal monotone and coercive.

We split \( S \) as two operators \( C, G : V \to V' \):
\[ S(z_1) = C(z_1) + G(z_1), \]

\[ C(z_1) = B(z_1) + G(z_1), \quad G(z_1) = -z_1 + z. \]
where
\[ C(z_1) = B(z_1) + z_1. \]

**Step 3:** \( C \) is maximal monotone and coercive. In view of [5, Theorem 1.3], we only need to prove that \( C \) is monotone and hemicontinuous. Let \( z = (u, w) \in V \) and \( \hat{z} = (\hat{u}, \hat{w}) \in V \). We have
\[
\langle C(z) - C(\hat{z}), z - \hat{z} \rangle = \langle B(z - \hat{z}), z - \hat{z} \rangle + \langle z - \hat{z}, z - \hat{z} \rangle
= \|z - \hat{z}\|_V^2 + \|z - \hat{z}\|_H^2.
\]
(3.9)

This proves that \( C \) is monotone.

Next, we will show that \( C \) is hemicontinuous, that is, \( \lim_{\lambda \to 0} C(z + \lambda z_1) = C(z) \) for all \( z, z_1 \in V \). Let \( \hat{z} \in V \), then
\[
|\langle C(z + \lambda z_1), \hat{z} \rangle - \langle C(z), \hat{z} \rangle| = \|\langle B(z + \lambda z_1), \hat{z} \rangle - \langle B(z), \hat{z} \rangle + \langle z + \lambda z_1, \hat{z} \rangle - \langle z, \hat{z} \rangle\|
= |\lambda|\| (z_1, \hat{z})_V + (z_1, \hat{z})_H | \xrightarrow{\lambda \to 0} 0,
\]
(3.10)

which implies that \( C \) is hemicontinuous.

**Step 3:** \( G \) is maximal monotone and coercive. Since \( g_1 \) and \( g_2 \) are monotones it is easy to see that \( G \) is monotone. We now prove the hemicontinuity. Let \( z = (u, w), z_1 = (u, w_1) \in V \) and \( \hat{z} = (\hat{u}, \hat{w}) \in V \). Firstly, we observe that
\[
\langle G(z + \lambda z_1), \hat{z} \rangle = \int_0^L g_1(u + \lambda u_1)\hat{u} \, dx + \int_0^L g_2(w + \lambda w_1)\hat{w} \, dx.
\]
(3.11)

Since \( g_1 \) is continuous we have \( g_1(u + \lambda u_1)\hat{u} \to g_1(u)\hat{u} \) pointwise as \( \lambda \to 0 \). Moreover, by the assumption on the damping \( |g_1(s)| \leq \beta_1 (|s| + 1) \) for all \( s \). Therefore, if \( |\lambda| \leq 1 \), we have that
\[
|g_1(u + \lambda u_1)\hat{u}| \leq \beta_1 (|u + \lambda u_1| + 1)|\hat{u}| \leq \beta_1 (\|u\|_\infty + \|u_1\|_\infty + 1)\|u\|_\infty.
\]

By Lebesgue’s dominated convergence theorem we obtain
\[
\lim_{\lambda \to 0} \int_0^L g_1(u + \lambda u_1)\hat{u} \, dx = \int_0^L g_1(u)\hat{u} \, dx.
\]
(3.12)

Similarly,
\[
\lim_{\lambda \to 0} \int_0^L g_2(w + \lambda w_1)\hat{w} \, dx = \int_0^L g_2(w)\hat{w} \, dx.
\]
(3.13)

From (3.14) it follows that
\[
\lim_{\lambda \to 0} \langle G(z + \lambda z_1), \hat{z} \rangle = \langle G(z), \hat{z} \rangle.
\]
(3.14)

This proves that \( G \) is hemicontinuous.

Now, since \( C \) and \( G \) are both maximal monotone and \( D(C) = D(G) = V \), by [5, Theorem 1.5], we conclude that \( S = C + G \) is maximal monotone. Moreover, since \( g_1, g_2 \) are monotones and \( C_0 = G_0 = 0 \), then from (3.9) it follows that, for all \( z \in V \),
\[
\langle S z, z \rangle = \langle C z - C 0, z - 0 \rangle + \langle G z - G 0, z - 0 \rangle
\geq \|z\|_V^2 + \|z\|_H^2
\geq |\langle z \rangle|_V^2.
\]

Therefore \( S \) is coercive. Consequently the maximal monotonicity of \( A \) is proved.

**Step 4:** Global solution: Since \( A \) is maximal monotone and \( F \) locally Lipschitz, cf. Lemma 3.1, then by [9, Theorem 7.2] for all \( U_0 \in D(A) \) there exists a \( t_{\text{max}} < \infty \) and a unique strong solution \( U \) for (2.6) defined on the interval \( [0, t_{\text{max}}) \). Moreover,
if $U_0 \in \mathcal{H}$ then (2.6) has a unique weak solution $U \in C([0,t_{\text{max}}];\mathcal{H})$ and such solutions satisfy $\limsup_{t \to t_{\text{max}}} \|U(t)\|_\mathcal{H} = \infty$, provided $t_{\text{max}} < \infty$. A standard approximation argument shows that the energy identity (2.18) hold. To prove the existence global, we need to prove that $t_{\text{max}} = \infty$. Let $U$ a strong solution on $[0,t_{\text{max}})$. By (3.3) it follows that

$$ E(t) \leq \frac{1}{C_0} (\mathcal{E}(t) + L\beta_0) \leq \frac{1}{C_0} (\mathcal{E}(0) + L\beta_0) \quad \text{for all } t \in [0,t_{\text{max}}). \quad \text{(3.15)} $$

Using density argument, we obtain that (3.15) is also satisfied for weak solutions. We conclude that $t_{\text{max}} = \infty$.

**Step 5: Continuous dependence:** The solution of (2.6), denoted by $U(t)$, can be expressed by the following variation of constants formula

$$ U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}F(U(s)) \, ds. $$

Since, by Lemma 3.1 $F$ is locally Lipschitz continuous, we conclude that there exists a constant $C_T > 0$ such that

$$ \|U^1(t) - U^2(t)\|_\mathcal{H} \leq C_T\|U_0^1 - U_0^2\|_\mathcal{H} \quad \text{for all } t \in [0,T]. \quad \text{(3.16)} $$

Thus, the proof of Theorem 2.4 is complete.

4. **Global attractors.** This section is dedicated to proof the existence of global attractor for the dynamical system $(\mathcal{H},S_t)$ generated by the problem (1.4)-(1.5).

**Lemma 4.1.** Suppose that Assumption 2.1 hold. Then the system $(\mathcal{H},S_t)$ generated by the problem (1.4)-(1.5) is gradient.

**Proof.** We can take the energy functional $\mathcal{E}(t)$ defined in (2.16) as a Lyapunov function $\Phi$. Indeed, let $U = (z_0, z_1) \in \mathcal{H}$, then by (3.4) we obtain

$$ \frac{d}{dt} \Phi(S_t U) = -\langle G(z_t), z_t \rangle \leq 0. \quad \text{(4.1)} $$

This shows that $t \mapsto \Phi(S_t U)$ is a non-increasing function.

Suppose that $\Phi(S_t U) = \Phi(U)$ for all $t \geq 0$. Then,

$$ \langle G(z_t), z_t \rangle = 0, \quad \text{for all } t \geq 0. \quad \text{(4.2)} $$

We use (2.8) and (4.2) to get

$$ \|z_t(t)\|_2^2 = 0, \quad \text{for all } t \geq 0. \quad \text{(4.3)} $$

Hence,

$$ z_t(t) = 0, \quad \text{for all } t \geq 0, \text{ a.e. in } (0,L), $$

which implies that

$$ z(t) = z_0, \quad \text{for all } t \geq 0. $$

Therefore, $S_t U = U = (z_0, 0)$ for all $t \geq 0$. The proof is complete.

In next, we will prove that the dynamical system $(\mathcal{H},S_t)$ has a bounded absorbing set. The proof is inspired by the argument of [16, Lemma 4.6], also see [9, Lemma 3.2].

**Lemma 4.2.** Suppose that Assumption 2.1 hold. Then the dynamical $(\mathcal{H},S_t)$ has a bounded absorbing set $B \subset \mathcal{H}$. 
Proof. Multiplying (1.4) by $z = (u, w)$ and integrating over $(0, L)$ we obtain
\[
\frac{d}{dt}(z_t, z)_H + \|z\|_V^2 + (\mathcal{G}(z_t), z) = (\mathcal{F}(z), z) + \|z_t\|_H^2. \tag{4.4}
\]
Integrating (4.4) from 0 to $T$ we get
\[
\int_0^T \|z\|_V^2 \, dt = \int_0^T (\mathcal{F}(z), z) \, dt - ((z_t(T), z(T))_H - (z_t(0), z(0))_H)
+ \int_0^T \|z_t\|_H^2 \, dt - \int_0^T (\mathcal{G}(z), z) \, dt. \tag{4.5}
\]
By (2.9), (2.13), Poincaré’s inequality, (2.4) and (2.12) we obtain
\[
\int_0^T (\mathcal{F}(z), z) \, dt \leq \int_0^T \int_0^L F(z) \, dx \, dt + \int_0^T \alpha_0 \|z\|_2^2 \, dt + LT\beta_0
\leq \int_0^T \int_0^L F(z) \, dx \, dt + \frac{\alpha_0}{\lambda_0} \|z\|_{1,2}^2 \, dt + LT\beta_0
\leq \int_0^T \int_0^L F(z) \, dx \, dt + \frac{1}{2} \int_0^T \|z\|_V^2 \, dt + LT\beta_0.
\]
Using the last estimate in (4.5) we see that
\[
\int_0^T \mathcal{E}(t) \, dt \leq -((z_t(T), z(T))_H - (z_t(0), z(0))_H)
+ \frac{3}{2} \int_0^T \|z_t\|_H^2 \, dt - \int_0^T (\mathcal{G}(z), z) \, dt + LT\beta_0. \tag{4.6}
\]
We need to estimate all terms on the right-hand side of (4.6).

**Estimate for the term:** $-((z_t(T), z(T))_H - (z_t(0), z(0))_H)$. Using Hölder’s inequality we have
\[
-((z_t(T), z(T))_H - (z_t(0), z(0))_H)
\leq C (\|z_t(T)\|_H^2 + \|z(T)\|_H^2 + \|z(0)\|_H^2 + \|z(0)\|_H^2)
\leq C(E(T) + E(0)). \tag{4.7}
\]
Using Lemma 3.2 in (4.7) we get
\[
-((z_t(T), z(T))_H - (z_t(0), z(0))_H) \leq C_1(E(T) + E(0)) + C_1. \tag{4.8}
\]

**Estimate for the term:**
\[
\frac{3}{2} \int_0^T \|z_t\|_H^2 \, dt.
\]
We introduce the sets:
\[
Z_1 := \{(x, t) \in (0, L) \times (0, T) : |u_t(x, t)| < 1\},
Z_2 := \{(x, t) \in (0, L) \times (0, T) : |u_t(x, t)| \geq 1\}. \tag{4.9}
\]
By (2.8) we have $g_1(s)s \geq \alpha_1|s|^2$ for $|s| \geq 1$, thus
\[
\int_0^T \|u_t\|_2^2 \, dt = \int_{Z_1} |u_t|^2 \, dx + \int_{Z_2} |u_t|^2 \, dxdt \\
\leq TL + \frac{1}{\alpha_1} \int_{Z_2} g_1(u_t)u_t \, dx \\
\leq TL + \frac{1}{\alpha_1} \int_0^T \int_{Z_2} g_1(u_t)u_t \, dxdt.
\] (4.10)

Similarly,
\[
\int_0^T \|w_t\|_2^2 \, dt \leq TL + \frac{1}{\alpha_2} \int_0^T \int_{Z_2} g_2(w_t)w_t \, dxdt.\] (4.11)

Let $C_2 = \frac{3 \max \{g_1, g_2\}}{2} \left(2L + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)$. Then, by (4.10) and (4.11) we obtain that
\[
\frac{3}{2} \int_0^T \|z_t\|_H^2 \, dt \leq C_2 T + C_2 \int_0^T (\mathcal{G}(z_t), z_t) \, dt.\] (4.12)

**Estimate for the term:**
\[-\int_0^T (\mathcal{G}(z_t), z) \, dt.
\]

By Hölder’s and Young’s inequalities we have
\[
-\int_0^T \int_0^L g_1(u_t)u \, dxdt \\
\leq \left( \int_0^T \|u\|_2^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_0^L g_1(u_t)^2 \, dxdt \right)^{\frac{1}{2}} \\
\leq \epsilon \int_0^T E(t) \, dt + C_1 \int_0^T \int_0^L g_1(u_t)^2 \, dxdt \\
= \epsilon \int_0^T E(t) \, dt + C_1 \int_{Z_1} g_1(u_t)^2 \, dxdt + C_1 \int_{Z_2} g_1(u_t)^2 \, dxdt.
\] (4.13)

Since $g_1$ is continuous and increasing we see that
\[
\int_{Z_1} g_1(u_t)^2 \, dxdt \leq \max \{g_1(-1)^2, g_1(1)^2\} LT.\] (4.14)

By (2.8), we have $|g_1(s)s| \leq \beta_1|s|^2$ for $|s| \geq 1$. Therefore
\[
\int_{Z_2} g_1(u_t)^2 \, dxdt \leq \beta_1 \int_0^T \int_0^L g_1(u_t)u_t \, dxdt.\] (4.15)

Substituting the estimates (4.14) and (4.15) in (4.13) we have
\[
-\int_0^T \int_0^L g_1(u_t)u \, dxdt \leq \epsilon \int_0^T E(t) \, dt + C_1 T + C_1 \int_0^T \int_0^L g_1(u_t)u_t \, dxdt.\] (4.16)

Similarly,
\[
-\int_0^T \int_0^L g_2(w_t)w \, dxdt \leq \epsilon \int_0^T E(t) \, dt + C_1 T + C_1 \int_0^T \int_0^L g_2(w_t)w_t \, dxdt.\] (4.17)

Combining (4.16) and (4.17) we obtain
\[
-\int_0^T (\mathcal{G}(z_t), z) \, dt \leq 2\epsilon \int_0^T E(t) \, dt + C_1 T + C_1 \int_0^T (\mathcal{G}(z_t), z_t) \, dt.
\]
We choose $\epsilon = C_0^4$, then by Lemma 3.2 we conclude that there exists a constant $C_3 > 0$ such that
\[
- \int_0^T \langle \mathcal{G}(z_t), z_t \rangle \, dt \leq \frac{1}{2} \int_0^T \mathcal{E}(t) \, dt + C_3T + C_3 \int_0^T \langle \mathcal{G}(z_t), z_t \rangle \, dt.
\]  
(4.18)
We apply the estimates (4.8), (4.12) and (4.18) in (4.6) to get
\[
\frac{1}{2} \int_0^T \mathcal{E}(t) \, dt \leq C_1 \mathcal{E}(T) + \mathcal{E}(0) + (C_2 + C_3) \int_0^T \langle \mathcal{G}(z_t), z_t \rangle \, dt + C_1 + (C_2 + C_3)T.
\]
Using the fact that $\mathcal{E}(T) \leq \mathcal{E}(t)$ and energy identity (2.18) we have
\[
\frac{T}{2} \mathcal{E}(T) \leq (C_1 - C_2 - C_3)\mathcal{E}(T) + (C_1 + C_2 + C_3)\mathcal{E}(0) + C_1 + (C_1 + C_2)T.
\]
Hence, taking $T > 0$ sufficiently large such that $T > 2C_2$, we obtain
\[
\mathcal{E}(T) \leq \gamma_T \mathcal{E}(0) + K_T.
\]
where
\[
\gamma_T = \frac{2(C_1 + C_2 + C_3)}{T - 2(C_1 - C_2 - C_3)}, \quad K_T = \frac{2C_1 + 2(C_1 + C_2)T}{T - 2(C_1 - C_2 - C_3)}.
\]
By iterating the estimate on intervals $[mT, (m+1)T]$, $m = 1, 2, \ldots$, we have
\[
\mathcal{E}(mT) \leq \gamma_T^m \mathcal{E}((m-1)T) + K_T \leq \gamma_T^m \mathcal{E}(0) + K_T \sum_{k=1}^{m-1} \gamma_T^k.
\]
Using the fact that $\gamma_T < 1$ and the argument presented in [16, p.p. 2485–2486] it follows that there exists constants $\gamma, \alpha > 0$ such that
\[
\mathcal{E}(t) \leq \gamma \mathcal{E}(0) e^{-\alpha t} + \frac{K_T}{1 - \gamma_T} \quad \text{for all } t \geq 0.
\]
Combining the last estimate and Lemma 3.2 we obtain
\[
\|S_t U_0\|_{\mathcal{H}}^2 \leq \frac{2\gamma}{C_0} \mathcal{E}(0) e^{-\alpha t} + \frac{2K_T + 2L\beta_0(1 - \gamma_T)}{C_0(1 - \gamma_T)} \quad \text{for all } t \geq 0.
\]
Therefore, the closed ball $B = B(0, R_0)$ in $\mathcal{H}$ of center zero and radius $R_0$, where
\[
R_0^2 = 1 + \frac{2K_T + 2L\beta_0(1 - \gamma_T)}{C_0(1 - \gamma_T)}
\]
is a bounded absorbing set. The proof is complete.

Remark 4.3. Since the set $B$ given in Lemma 4.2 is a bounded absorbing set for $(\mathcal{H}, S_t)$, then there exists a time $t_B > 0$ such that $S_t B \subset B$ for all $t \geq t_B$. Therefore, $B_0 = \cup_{t \geq t_B} S_t B$ is a bounded positively invariant absorbing set with $B_0 \subset B$. 

\[\square\]
4.1. Proof of Theorem 2.5 (Existence of global attractors). In this part, we conclude the proof of Theorem 2.5. We start by citing a criterion of asymptotic smoothness by [10, Theorem 7.1.11] which will be used in sequence.

**Lemma 4.4.** Let \((X,S_t)\) be a dynamical system on a complete metric space \(X\) endowed with a metric \(d\). Assume that for any bounded positively invariant set \(B\) in \(X\) and for any \(\epsilon > 0\) there exists \(T = T(\epsilon, B)\) such that

\[
d(S_{T}y_1, S_{T}y_2) \leq \epsilon + \Psi_{\epsilon,B,T}(y_1, y_2) \quad \text{for all } y_1, y_2 \in B.
\]

where \(\Psi_{\epsilon,B,T}(y_1, y_2)\) is a functional defined on \(B \times B\) such that

\[
\lim_{m \to \infty} \liminf_{n \to \infty} \Psi_{\epsilon,B,T}(y_n, y_m) = 0.
\]

for every sequence \(\{y_n\}\) from \(B\). Then \((X,S_t)\) is an asymptotically smooth dynamical system.

Next, we will show that the dynamical system \((\mathcal{H},S_t)\) is asymptotically smooth. The argument of the next lemma is similar to the proof of Lemma 4.2 in [16].

**Lemma 4.5.** Suppose that Assumption 2.1 hold. Let \(B\) a bounded invariant set in \(\mathcal{H}\) and \(S_t U^j = (z^j, z_t^j) = (u^j, w^j, u_t^j, w_t^j)\), \(j = 1, 2\) weak solutions of (1.4)-(1.5) with initial condition \(U^1, U^2 \in B\). Then, for any \(\epsilon > 0\), there exists positive constants \(C_B, C_\epsilon\) such that for \(T > 0\) sufficiently large the following estimate hold

\[
E_z(T) \leq \epsilon + 2 \left( C_B + \frac{C_\epsilon}{T} \right) \int_0^T \|z\|_{p+1} \, dt.
\]  

where \(E_z\) is the linear energy corresponding to \(z = z^1 - z^2\) defined by

\[
E_z(t) = \frac{1}{2} (\|z_t^1\|_{\mathcal{H}}^2 + \|z_t^2\|_{\mathcal{H}}^2) \quad \text{for all } t \in [0, T].
\]

**Proof.** Since \(z^1 = (u^1, w^1)\) and \(z^2 = (u^2, w^2)\) are weak solutions, then \(u = u^1 - u^2\) and \(w = w^1 - w^2\) satisfies

\[
\begin{cases}
\rho_1 u_t - a_{11} u_{xx} - a_{12} w_{xx} + g_1(u_t^1) - g_1(u_t^2) = f_1(z^1) - f_1(z^2), \\
\rho_2 u_t - a_{12} u_{xx} - a_{22} w_{xx} + g_2(u_t^1) - g_2(u_t^2) = f_2(z^1) - f_2(z^2).
\end{cases}
\]

Multiplying (4.20) by \(z = (u, w)\) and integrating over \((0, L)\), we obtain

\[
\frac{d}{dt}(z^1, z^2)_{\mathcal{H}} + 2E_z(t) = 2\|z_t^1\|_{\mathcal{H}}^2 - (\mathcal{G}(z^1) - \mathcal{G}(z^2), z) + (\mathcal{F}(z^1) - \mathcal{F}(z^2), z).
\]  

Integrating (4.21) from 0 to \(T\) we have

\[
\int_0^T E_z(t) \, dt = -((z_t(T), z(T))_{\mathcal{H}} - (z_t(0), z(0))_{\mathcal{H}}) + \int_0^T \|z_t\|_{\mathcal{H}}^2 \, dt - \frac{1}{2} \int_0^T (\mathcal{G}(z_t^1) - \mathcal{G}(z_t^2), z) \, dt
\]

\[
+ \frac{1}{2} \int_0^T (\mathcal{F}(z^1) - \mathcal{F}(z^2), z) \, dt.
\]

We will estimate all terms on the right-hand side of (4.22).

**Estimate for the term:** \(-((z_t(T), z(T))_{\mathcal{H}} - (z_t(0), z(0))_{\mathcal{H}})\). By Hölder’s inequality we deduce that

\[
-((z_t(T), z(T))_{\mathcal{H}} - (z_t(0), z(0))_{\mathcal{H}}) \leq C(E_z(T) + E_z(0)).
\]  

(4.23)
Estimate for the term:
\[ \int_0^T \| z_t \|^2_{L^2} \, dt. \]

For any \( \delta > 0 \), by (2.14) we obtain
\[
\int_0^T \| u_t \|^2_2 \, dt \leq LT \delta + C \delta \int_0^T \int_L^L (g_1(u_1^1) - g_1(u_2^1)) u_t \, dx \, dt. \tag{4.24}
\]

Similarly,
\[
\int_0^T \| w_t \|^2_2 \, dt \leq LT \delta + C \delta \int_0^T \int_L^L (g_2(u_1^1) - g_2(u_2^1)) w_t \, dx \, dt. \tag{4.25}
\]

From (4.24) and (4.25) it follows that
\[
\int_0^T \| z_t \|^2_{L^2} \, dt \leq CLT + C \delta \int_0^T (G(z_1^1) - G(z_2^1), z_t) \, dt. \tag{4.26}
\]

Estimate for the term:
\[ -\frac{1}{2} \int_0^T (G(z_1^1) - G(z_2^1), z) \, dt. \tag{4.27} \]

We introduce the sets:
\[
X_1 := \{(x, t) \in (0, L) \times (0, T) : |u_t(x, t)| < 1 \};
\]
\[
X_2 := \{(x, t) \in (0, L) \times (0, T) : |u_t(x, t)| \geq 1 \}.
\]

It follows from (2.8) that
\[
- \int_0^T \int_0^L (g_1(u_1^1) - g_1(u_2^1)) u \, dx \, dt \\
\leq \int_{X_1} \| g_1(u_1^1) - g_1(u_2^1) \| u \, dt + \int_{X_2} \beta_1 (|u_1^1| + |u_2^1|) |u| \, dx \, dt \\
\leq \int_0^T \int_0^L (|g_1(u_1^1)| + |g_1(u_2^1)|) |u| \, dx \, dt + \int_0^T \int_0^L \beta_1 (|u_1^1| + |u_2^1|) |u| \, dx \, dt \tag{4.27}
\]
\[
\leq C_B \int_0^T \| u \|_{p+1} \, dt \\
\leq C_B \int_0^T \| z \|_{p+1} \, dt.
\]

Similarly,
\[
- \int_0^T \int_0^L (g_2(u_1^1) - g_2(u_2^1)) w \, dx \, dt \leq C_B \int_0^T \| z \|_{p+1} \, dt. \tag{4.28}
\]

Combining (4.27)-(4.28) we obtain
\[
-\frac{1}{2} \int_0^T (G(z_1^1) - G(z_2^1), z) \, dt \leq C_B \int_0^T \| z \|_{p+1} \, dt. \tag{4.29}
\]

Estimate for the term:
\[ \frac{1}{2} \int_0^T (F(z_1^1) - F(z_2^1), z) \, dt. \tag{4.30} \]
Using (2.10), Hölder’s inequality with exponents $p_1 = \frac{2(p+1)}{p-1}$, $p_2 = p + 1$, $p_3 = 2$ we obtain that
\[
\int_0^T \int_0^L (f_1(z^1) - f_2(z^2)) u \, dx \, dt \\
\leq C \int_0^T \int_0^L \left( |u|^p - 1 + |u^2|^p - 1 + |w^1|^p - 1 + |w^2|^p - 1 + 1 \right) |z| |u| \, dx \, dt \\
\leq C_B \int_0^T \|z\|_{p+1} \|u\|_2 \, dt \\
\leq C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.31)

Similarly,
\[
\int_0^T \int_0^L (f_2(z^1) - f_2(z^2)) w \, dx \, dt \leq C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.32)

Therefore, from (4.31) and (4.32), it follows that
\[
\frac{1}{2} \int_0^T \left( |\mathcal{F}(z^1) - \mathcal{F}(z^2), z \right) \, dx \, dt \leq C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.33)

Using the estimates (4.23), (4.26), (4.29) and (4.33) in (4.22) we obtain
\[
\int_0^T E_z(t) \, dt \leq C(E_z(T) + E_z(0)) + CLT\delta \\
+ C_\delta \int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \, dt + C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.34)

Now we multiply (4.20) by $z_t = (u_t, w_t)$ and we integrate over $(0, L)$ to get
\[
\frac{d}{dt} E_z(t) + \langle \mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t \rangle = \langle \mathcal{F}(z^1) - \mathcal{F}(z^2), z_t \rangle.
\] (4.35)

Integrating the last identity from $s$ to $T$ we obtain
\[
E_z(T) + \int_s^T \langle \mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t \rangle \, dt = E_z(s) + \int_s^T \langle \mathcal{F}(z^1_t) - \mathcal{F}(z^2_t), z_t \rangle \, dt.
\] (4.36)

Making $s = 0$ in (4.36) we get
\[
\int_0^T \langle \mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t \rangle \, dt \leq E_z(0) + E_z(T) + \int_0^T \langle \mathcal{F}(z^1_t) - \mathcal{F}(z^2_t), z_t \rangle \, dt.
\] (4.37)

Applying the estimate (4.37) in (4.34) and using the fact that $E_z(0) \leq C_B$, we obtain that
\[
\int_0^T E_z(t) \, dt \leq C_\delta E_z(T) + C_{\delta, B} + CLT\delta \\
+ C_\delta \int_0^T \langle \mathcal{F}(z^1_t) - \mathcal{F}(z^2_t), z_t \rangle \, dt + C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.38)

The argument that lead to (4.33) can be used now to ensure that
\[
\int_0^T \langle \mathcal{F}(z^1_t) - \mathcal{F}(z^2_t), z_t \rangle \, dt \leq C_B \int_0^T \|z\|_{p+1} \, dt.
\] (4.39)
Substituting the estimate (4.39) into (4.38) we get

\[
\int_0^T E_z(t) \, dt \leq C_\delta E_z(T) + C_\delta B + CLT \delta + C_\delta B \int_0^T \|z\|_{p+1} \, dt. \tag{4.40}
\]

Integrating (4.36) from 0 to T with respect to s and using that \((\mathcal{G}(z_1^{-}(t)) - \mathcal{G}(z_1^{+}(t)), z_1) \geq 0\), we obtain

\[
TE_z(T) \leq \int_0^T E_z(t) \, dt + \int_0^T \int_s^T (\mathcal{F}(z_1^{-}) - \mathcal{F}(z_1^{+}), z_1) \, dt \, ds. \tag{4.41}
\]

Combining (4.39), (4.40) and (4.41) we have

\[
TE_z(T) \leq C_\delta E_z(T) + C_\delta B + CLT \delta + \left( TC_B + C_\delta B \right) \int_0^T \|z\|_{p+1} \, dt. \tag{4.42}
\]

Therefore, given \(\delta > 0\), choosing \(T\) sufficiently large such that \(T > 2C_\delta\), we see that

\[
E_z(T) \leq \epsilon + 2 \left( C_B + \frac{C_\epsilon}{T} \right) \int_0^T \|z\|_{p+1} \, dt. \tag{4.43}
\]

Now, given \(\epsilon > 0\), let \(\delta > 0\) sufficiently small and \(T\) sufficiently large such that

\[
\delta < \frac{\epsilon}{4CL} \quad \text{and} \quad \frac{2C_\delta B}{T} < \frac{\epsilon}{2}.
\]

Then

\[
E_z(T) \leq \epsilon + 2 \left( C_B + \frac{C_\epsilon}{T} \right) \int_0^T \|z\|_{p+1} \, dt.
\]

Therefore (4.19) holds. The proof is complete. \(\Box\)

**Lemma 4.6.** Suppose that Assumption 2.1 hold. Then the dynamical system \((H, S_t)\) is asymptotically smooth.

**Proof.** In order to prove Lemma 4.6 we will use the Lemma 4.4. Let \(B\) bounded positively invariant set. By Lemma 4.5 for any \(\epsilon > 0\) there exists \(T = T(\epsilon, B) > 0\) such that

\[
E_z(T) \leq \epsilon + \Psi_{\epsilon, B, T}(U^1, U^2), \tag{4.44}
\]

where

\[
\Psi_{\epsilon, B, T}(U^1, U^2) = 2 \left( C_B + \frac{C_\epsilon}{T} \right) \int_0^T \|z^1(t) - z^2(t)\|_{p+1} \, dt. \tag{4.45}
\]

We recall that \(V = (H^1_0(0, L))^2, H = (L^2(0, L))^2\) and \(\mathcal{H} = V \times H\). Let \(U^n = (z^n, z^n_t) \in B\), by positive invariance, we have

\[
\|S_t U^n\|^2_{\mathcal{H}} = \|z^n(t)\|^2_V + \|z^n_t(t)\|^2_H \leq C_B \quad \text{for all } t \geq 0. \tag{4.46}
\]

Consequently,

\[
z^n \text{ is bounded in } L^\infty(0, T; V),
\]

\[
z^n_t \text{ is bounded in } L^\infty(0, T; H). \tag{4.47}
\]

By Simon’s compactness theorem [19] there exists a subsequence such that

\[
z^n \to z \quad \text{strongly in } C \left([0, T]; (L^{p+1}(0, L))^2\right). \tag{4.48}
\]

It follows from (4.48) that

\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_0^T \|z^m - z^n\|_{p+1} \, dt = 0. \tag{4.49}
\]
Finally, by (4.49) we conclude that
\[ \liminf_{m \to \infty} \liminf_{n \to \infty} \Psi_{\epsilon, B, T}(U^n, U^m) = 0. \]
Then the asymptotic smoothness follows from Lemma 4.4. The proof is complete. □

Since we have shown that \((\mathcal{H}, S_t)\) is a dissipative asymptotically smooth dynamical system, we can apply the result in [10, Theorem 7.2.3] to conclude that \((\mathcal{H}, S_t)\) has a compact global attractor \(A\). Characterization (2.19) follows from [10, Theorem 2.4.5] (also see [4, 13, 20, 15]). Thus the proof of Theorem 2.5 is complete.

5. Fractal dimension, regularity and exponential attractors. This section is devoted to prove the Theorem 2.7.

5.1. Quasistability. In this part, we will establish the quasi-stability of dynamical system \((\mathcal{H}, S_t)\) generated by the problem (1.4)-(1.5). The proof is inspired by the argument of [16, Lemma 5.3].

Lemma 5.1. Under the assumptions of Lemma 4.5, with (2.8) replaced by (2.20), given a bounded invariant set \(B\), there exist constants \(\alpha, \gamma > 0\) and \(C_B > 0\) such that
\[ E_z(t) \leq \gamma E_z(0)e^{-\alpha t} + C_B \sup_{s \in [0, t]} \|z(s)\|_{2p}^2 \text{ for all } t \geq 0. \] (5.1)

where \(S_t U^j = (z^j, z_t^j) = (u^j, w^j, u_t^j, w_t^j), j = 1, 2\) weak solutions of (1.4)-(1.5) with initial condition \(U^1, U^2 \in B\) and \(z = (u, w) = z^1 - z^2\).

Proof. We proceed as in the proof of Lemma 4.5, since (2.20) implies (2.8). We only need estimate the right-hand side of (4.22). Throughout the proof by \(C > 0\) we will always denote various constants independent on \(B, t\).

Estimate for the term:
\[ -\left((z_t(T), z(T))_H - (z_t(0), z(0))_H\right) \leq C(E_z(T) + E_z(0)). \] (5.2)

Estimate for the term:
\[ \frac{1}{2} \int_0^T \left(\mathcal{F}(z^1) - \mathcal{F}(z^2), z\right) dt \]
Using (2.10), Hölder’s inequality with exponents \(p_1 = \frac{2p}{p-1}, p_2 = 2p, p_3 = 2\) and embedding \(L^{2p}(0, L) \hookrightarrow L^2(0, L)\) we have
\[ \int_0^T \int_0^L (f_1(z^1) - f_1(z^2))u \, dx \, dt \leq C \int_0^T \int_0^L (|u|^p + |z^1|^p + |u_t|^p + |w^2|^p + 1)|z||u| \, dx \, dt \]
\[ \leq C_B \int_0^T \|z\|_{2p} \|u\|_2 \, dt \]
\[ \leq C_B \int_0^T \|z\|_{2p} \|u\|_{2p} \, dt \]
\[ \leq C_B \int_0^T \|z\|_{2p}^2 \, dt. \] (5.3)
Similarly,
\[ \int_0^T \int_0^L (f_2(z^1) - f_2(z^2))w \, dx \, dt \leq C_B \int_0^T \|z\|^2_{2p} \, dt. \] (5.4)

It follows from (5.3) and (5.4) that
\[ \frac{1}{2} \int_0^T \int_0^L (\mathcal{F}(z^1) - \mathcal{F}(z^2), z) \, dt \leq C_B \int_0^T \|z\|^2_{2p} \, dt. \] (5.5)

**Estimate for the term:**
\[ \int_0^T \|z_t\|^2_H \, dt. \]

By (2.21) it is easy to conclude that
\[ \int_0^T \|z_t\|^2_H \, dt \leq C_B \int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \, dt. \] (5.6)

Using that \(|g_j(u) - g_j(w)| \leq m_j |u - w|\) for all \(u, w \in \mathbb{R}\), cf. Remark 2.6, Hölder’s and Young’s inequalities and embedding \(L^{2p}(0, L) \hookrightarrow L^2(0, L)\) we obtain that
\[ \int_0^T \int_0^L (g_1(u^1_t) - g_1(u^2_t))u \, dx \, dt \leq \frac{1}{4} \int_0^T \rho_1 \|u_t\|^2_2 \, dt + C \int_0^T \|u\|^2_{2p} \, dt \]
\[ \leq \frac{1}{4} \int_0^T \rho_1 \|u_t\|^2_2 \, dt + C \int_0^T \|u\|^2_{2p} \, dt \] (5.7)

Similarly,
\[ \int_0^T \int_0^L (g_2(w^1_t) - g_2(w^2_t))w \, dx \, dt \leq \frac{1}{4} \int_0^T \rho_2 \|w_t\|^2_2 \, dt + C \int_0^T \|w\|^2_{2p} \, dt. \] (5.8)

By (5.7) and (5.8) we have that
\[ -\frac{1}{2} \int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z) \, dt \leq \frac{1}{4} \int_0^T \|z_t\|^2_H \, dt + C \int_0^T \|z\|^2_{2p} \, dt \]
\[ \leq \frac{1}{2} \int_0^T E_z(t) \, dt + C \int_0^T \|z\|^2_{2p} \, dt. \] (5.9)

Using the estimates (5.2), (5.5), (5.6) and (5.9) in (4.22), we get that
\[ \int_0^T E_z(t) \, dt \leq C(E_z(T) + E_z(0)) + C_B \int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \, dt + C_B \int_0^T \|z\|^2_{2p} \, dt. \] (5.10)

The energy identity (4.36) implies that
\[ \int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \, dt = E_z(0) - E_z(T) + \int_0^T (\mathcal{F}(z^1) - \mathcal{F}(z^2), z_t) \, dt. \] (5.11)
In next, we estimate the forcing term. For any $\epsilon > 0$, by Hölder’s and Young’s inequalities we have

$$
\int_0^L (f_1(z^1) - f_1(z^2))u_t \, dx
\leq C \int_0^L (1 + |u^1|^p + |u^2|^p + |u^1|^p + |u^2|^p) |z||u| \, dx
\leq C(1 + \|u^1\|_{2p}^p + \|u^2\|_{2p}^p + \|u^1\|_{2p}^p + \|u^2\|_{2p}^p) \|z\|_{2p}\|u_t\|_2
\leq \frac{\epsilon}{2} \|u_t\|_2^2 + C_{\epsilon,B} \|z\|_{2p}^2
\leq \frac{\epsilon}{2} \|u_t\|_2^2 + C_{\epsilon,B} \|z\|_{2p}^2.
$$

Similarly,

$$
\int_0^L (f_2(z^1) - f_2(z^2))w_t \, dx \leq \frac{\epsilon}{2} \|w_t\|_2^2 + C_{\epsilon,B} \|z\|_{2p}^2.
$$

By (5.12) and (5.13) we obtain

$$
(\mathcal{F}(z^1) - \mathcal{F}(z^2), z_t) \leq \epsilon E_z(t) + C_{\epsilon,B} \|z\|_{2p}^2.
$$

Consequently,

$$
\int_0^T (\mathcal{F}(z^1) - \mathcal{F}(z^2), z_t) \, dt \leq \epsilon \int_0^T E_z(t) \, dt + C_{\epsilon,B} \int_0^T \|z\|_{2p}^2 \, dt.
$$

Using the estimate (5.15) in (5.11) it follows that

$$
\int_0^T (\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \, dt \leq E_z(0) - E_z(T) + \epsilon \int_0^T E_z(t) \, dt + C_{\epsilon,B} \int_0^T \|z\|_{2p}^2 \, dt.
$$

By (5.16) and (5.10) with $\epsilon > 0$ small enough, we have

$$
\int_0^T E_z(t) \, dt \leq (C - C_B)E_z(T) + (C + C_B)E_z(0) + C_B \int_0^L \|z\|_{2p}^2 \, dt.
$$

Integrating (4.36) from 0 to $T$ with respect to $s$ and using that $(\mathcal{G}(z^1_t) - \mathcal{G}(z^2_t), z_t) \geq 0$, we obtain

$$
TE_z(T) \leq \int_0^T E_z(t) \, dt + \int_0^T \int_s^T (\mathcal{F}(z^1) - \mathcal{F}(z^2), z_t) \, dt \, ds.
$$

Using the estimate (5.15) in (5.18) with $\epsilon > 0$ small enough, we obtain

$$
TE_z(T) \leq 2 \int_0^T E_z(t) \, dt + C_{B,T} \int_0^L \|z\|_{2p}^2 \, dt.
$$

Combining (5.17) and (5.19) we get

$$
TE_z(T) \leq 2(C - C_B)E_z(T) + 2(C + C_B)E_z(0) + C_{B,T} \int_0^L \|z\|_{2p}^2 \, dt.
$$

We choose $T > 4C$ to get

$$
E_z(T) \leq \gamma T E_z(0) + C_{B,T} \sup_{s \in [0,T]} \|z(s)\|_{2p}^2,
$$

where

$$
\gamma_T = \frac{2(C + C_B)}{T - 2(C - C_B)} < 1.
$$
By iterating the estimate on intervals $[mT, (m + 1)T]$, $m = 1, 2, \ldots$, we have
\[ E_z(mT) \leq \gamma_T E_z((m - 1)T) + C_{B,T} \sup_{s \in [0,t]} \|z(s)\|_{2p}^2 \]
\[ \leq \gamma_T^m E_z(0) + C_{B,T} \sum_{k=1}^{m-1} \|z(s)\|_{2p}^2. \]
Using the fact that $\gamma_T < 1$ and the argument presented in [10, p.p. 745–747] it follows that there exists constants $\gamma, \alpha, C_B > 0$ such that
\[ E_z(t) \leq \gamma E_z(0)e^{-\alpha t} + C_B \sup_{s \in [0,t]} \|z(s)\|_{2p}^2 \quad \text{for all } t \geq 0. \]
The proof is complete. \hfill \Box

5.2. Proof of Theorem 2.7 (Fractal dimension). Firstly, we observe that (3.16) implies
\[ \|S_t U^1 - S_t U^2\|_{\mathcal{H}}^2 \leq a(t)\|U^1 - U^2\|_{\mathcal{H}}^2, \quad \text{for all } U^1, U^2 \in B, t \geq 0, \]
where $a(t) = C_T$. We recall that $V = (B^1_0(0, L))^2$, $H = (L^2(0, L))^2$ and $\mathcal{H} = V \times H$. Let $B \subset \mathcal{H}$ a bounded positively invariant set, cf. Remark 4.3. By Lemma 5.1 we have
\[ \|S_t U^1 - S_t U^2\|_{\mathcal{H}}^2 \leq b(t)\|U^1 - U^2\|_{\mathcal{H}}^2 + c(t) \sup_{s \in [0,t]} \left[ \mu_X(z^1(s) - z^2(s)) \right]^2 \quad \forall U^1, U^2 \in B, \]
where $b(t) = \gamma e^{-\alpha t}$, $c(t) = C_B$ and $\mu_V(\cdot)$ is the compact seminorm in $V$ defined by
\[ \mu_V(z) = \|z\|_{2p}, \quad z \in V. \]
Therefore, we can apply the result in [10, Theorem 7.9.6] to conclude that the global attractor $\mathcal{A}$ has finite fractal dimension. The proof is complete.

5.3. Proof of Theorem 2.7 (Regularity). Since we have shown that $(\mathcal{H}, S_t)$ is quasi-stable on the global attractor $\mathcal{A}$, from [10, Theorem 7.9.8] it follows that any complete trajectory $(z(\cdot), z_t(\cdot))$ in $\mathcal{A}$ enjoys the following regularity properties
\[ z_t \in L^\infty(\mathbb{R}; V) \cap C(\mathbb{R}; H), \quad z_{tt} \in L^\infty(\mathbb{R}; V). \]
Using (1.4) and the fact that nonlinear terms are continuous we conclude that $w_{xx}, w_{xxx} \in L^\infty(\mathbb{R}; L^2(0, L))$, consequently $z \in L^\infty(\mathbb{R}; (H^2(0, L) \cap H^3_0(0, L))^2)$ and
\[ \|z\|_{(H^2(0, L) \cap H^3_0(0, L))^2}^2 + \|z_t\|_{1,2}^2 \leq C. \]
The proof is complete.

5.4. Proof of Theorem 2.7 (Exponential attractors). Let $B_0$ the positively invariant absorbing set given by Remark 4.3. Then, for any $U \in B_0$ there exists $C_{B_0} > 0$ such that
\[ \|S_t U\|_{\mathcal{H}} \leq C_{B_0}, \quad \text{for all } t \in [0,T]. \]
Using this in (1.4) and the fact that nonlinear terms are locally Lipschitz continuous on the phase space $\mathcal{H}$ we can deduce that
\[ \left\| \frac{d}{dt} S_t U \right\|_{\mathcal{H}_{-1}} \leq C_{B_0}, \quad \text{for all } t \in [0,T]. \]
Hence,
\[ \| S_{t_1} U - S_{t_2} U \|_{\mathcal{H}^{-1}} \leq \int_{t_1}^{t_2} \left\| \frac{d}{dt} S_t U \right\|_{\mathcal{H}^{-1}} \, dt \leq C_{B_0} |t_1 - t_2|, \quad \text{for all } t_1, t_2 \in [0, T]. \] (5.22)

This proves that \( t \mapsto S_t U \) is Hölder continuous in \( \mathcal{H}^{-1} \) for any \( U \in B_0 \) with exponent \( \gamma = 1 \). Applying the abstract result [10, Theorem 7.9.9] we obtain the existence of a generalized exponential attractor \( \mathcal{A}_{\exp} \) whose fractal dimension is finite in \( \mathcal{H}^{-1} \).

Now, we will prove the existence of exponential attractors in \( \mathcal{H}^{-\eta} \) for any \( \eta \in (0, 1] \). If \( \eta \in (0, 1) \), it follows from interpolation theorem that
\[ \| U \|_{\mathcal{H}^{-\eta}} \leq C \| U \|_{\mathcal{H}^{1-\eta}}^{1-\eta} \| U \|_{\mathcal{H}^{-1}}^\eta \leq C_{B_0} \| U \|_{\mathcal{H}^{-1}}^\eta. \]

In particular,
\[ \| S_{t_1} U - S_{t_2} U \|_{\mathcal{H}^{-\eta}} \leq C_{B_0} \| S_{t_1} U - S_{t_2} U \|_{\mathcal{H}^{-1}}^\eta. \]

Using the last inequality in (5.22) we deduce that
\[ \| S_{t_1} U - S_{t_2} U \|_{\mathcal{H}^{-\eta}} \leq C_{B_0} |t_1 - t_2|^\eta, \quad \text{for all } t_1, t_2 \in [0, T]. \]

Therefore, \( t \mapsto S_t U \) is Hölder continuous in \( \mathcal{H}^{-\eta} \) for any \( U \in B_0 \) with exponent \( \gamma = \eta \). Using again the abstract result in [10, Theorem 7.9.9] we conclude existence of a generalized exponential attractor, with finite fractal dimension in \( \mathcal{H}^{-\eta} \). The proof is complete.

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E-mail address: ls@ufpa.br
E-mail address: mirelson@ufpa.br