Borel–Padé vs Borel–Weniger method: a QED and a QCD example

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Abstract

Recently, Weniger (delta sequence) method has been proposed by the authors of Ref. \cite{1} for resummation of truncated perturbation series in quantum field theories. Those authors presented numerical evidence suggesting that this method works better than Padé approximants when we resum a function with singularities in the Borel plane but not on the positive axis. We present here numerical evidence suggesting that in such cases the combined method of Borel–Padé works better than its analog Borel–Weniger, and that it may work better or comparably well in some of the cases when there are singularities on the positive axis in the Borel plane.

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I. INTRODUCTION

In this letter we want to present some numerical results which allow us to compare the efficiency of the Borel–Padé method with that of the Borel–Weniger method for resummation of truncated perturbation series (TPS) in some physically significant scenarios. The scenarios we are referring to are those when the function, which we want to find through a resummation, is known to have certain singularity structure in the Borel plane. If there are singularities on the positive axis of the Borel plane, then we implicitly assume that in such cases we either know the correct prescription for integration in the Laplace–Borel integral, or we simply adhere to a certain adopted prescription.

(1) We will first illustrate the efficiency of the two methods on the QED example of the Euler–Heisenberg Lagrangian density, i.e., the one–loop fermion–induced effective action density in a strong uniform electromagnetic field \cite{2}–\cite{8}. In this case, the solution is known, and its real part can be written in the following form:

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\[ \text{Re}\delta \tilde{\mathcal{L}}(\tilde{a}; p) = \text{Re} \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) \left( -\frac{1}{w} \right) \left[ \frac{p \cos(w)}{\sin(w+ie')} \coth(pw) + \frac{1}{3}(1-p^2)-\frac{1}{w^2} \right], \tag{1} \]

where we use notations
\[ \tilde{a} \equiv \frac{ga}{m^2}, \quad \tilde{b} \equiv \frac{gb}{m^2}, \quad p \equiv \frac{\tilde{b}}{\tilde{a}} \equiv \frac{\tilde{b}}{\tilde{a}}, \quad \delta \tilde{\mathcal{L}} \equiv \delta \mathcal{L}/\left( \frac{m^4 \tilde{a}^2}{8\pi^2} \right). \tag{2} \]

Here, \( \delta \mathcal{L} \) is the actual Lagrangian density induced by the one–loop fluctuations of the fermions in the field; \( g \) is the field–to–fermion coupling parameter (in QED it is the positron charge \( e_0 \)); \( m \) is the mass of the fermion (electron); \( a \) and \( b \) are Lorentz–invariant expressions characterizing the electric and the magnetic fields \( \vec{E} \) and \( \vec{B} \), respectively

\[ \left( \frac{a}{b} \right) = \left[ \pm \vec{E}^2 \mp \vec{B}^2 + \sqrt{\left( \vec{E}^2 - \vec{B}^2 \right)^2 + 4 \left( \vec{E} \cdot \vec{B} \right)^2} \right]^{1/2}/\sqrt{2}. \tag{3} \]

Expression (1) can be obtained, for example, directly by integrating out the fermionic degrees of freedom in the path integral expression of the full effective action, then employing the proper–time integral representation for the difference of logarithms, evaluating the traces in the integrand, and subsequently performing Wick rotation by \(-\pi/4\) in the plane of the proper–time \( s \): \( a gs \leftrightarrow -iw+\epsilon' \). We refer to [3] for more details on the latter point. The perturbative expansion of the full solution (1), in powers of \( \tilde{a} \), is

\[ \delta \tilde{\mathcal{L}}_{\text{pert.}}(\tilde{a}; p) = \left[ c_1(p)1! \tilde{a}^2 + c_3(p)3! \tilde{a}^4 + c_5(p)5! \tilde{a}^6 + \cdots \right], \tag{4} \]

with coefficients
\[ c_1(p) = \frac{1}{45} \left[ (1-p^2)^2 + 7p^2 \right], \quad c_3(p) = \frac{1}{945} \left[ 2(1-p^2)^3 + 13p^2(1-p^2) \right], \text{ etc.} \tag{5} \]

In the case of the pure magnetic field (p.m.f.), the corresponding expressions are simpler

\[ \delta \tilde{\mathcal{L}}_{\text{pert.}}(\tilde{b}; a=0) = \int_0^\infty dw \exp \left( -\frac{w}{b} \right) \left( -\frac{1}{w} \right) \left[ \frac{\coth(w)}{w} - \frac{1}{3} - \frac{1}{w^2} \right], \tag{6} \]

\[ \delta \tilde{\mathcal{L}}_{\text{pert.}}(\tilde{b}; a=0) = \left[ \tilde{c}_1 1! \tilde{b}^2 + \tilde{c}_3 3! \tilde{b}^4 + \cdots \right], \quad \tilde{c}_1 = \frac{1}{45}, \quad \tilde{c}_3 = -\frac{2}{945}, \ldots \tag{7} \]

We can now use (1), (3), and (7), as a laboratory for resummation methods, since the full (resummed) solutions (1) and (6) are known. Since (1) and (6) are Laplace–Borel integrals, it is natural to use these examples for testing combined resummation techniques which involve Borel transformation. Borel transform \( B_L \) of series (4) is

\[ B_L(w; p) = c_1(p)w + c_3(p)w^3 + c_5(p)w^5 + \cdots, \tag{8} \]

and analogously for (7). In Ref. [9], we used Borel–Padé technique for resummation, i.e., we applied various Padé approximants \([N/M]_B(w; p)\) to (8) and then employed the Laplace–Borel integral to obtain the resummed value

\[ ^1 [N/M]_B(w; p), \text{ being ratio of polynomials in } w \text{ of powers } N \text{ and } M, \text{ respectively } [1], \text{ is based solely on the truncated perturbation series (TPS) of } (8) \text{ involving only terms with } c_n; \ n \leq N+M. \]
BP^{[N/M]} \left[ \delta \hat{L}_{\text{pert.}} \right] \left( \tilde{a}; p \right) = \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) [N/M]_B(w; p). \quad (9)

The integration over poles in (9) was carried out according to the Cauchy principal value prescription, since the full solution (1) requires it.

Recently, the authors of [1] proposed the use of Weniger (delta sequence) transformations as an alternative to the use of Padé approximants, for direct resummation of truncated perturbation series. For a truncated perturbation series (TPS) of the form

\[ F_{[n+1]}(z) = \sum_{j=0}^{n+1} \gamma_j z^j \quad (10) \]

it is defined as [14]

\[
\delta_n^{(0)}(\zeta; \gamma_0, \ldots, \gamma_{n+1}) = \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} (\zeta+j)_{n-1} z^{n-j} F_{[j]}(z)}{\gamma_{j+1}} \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j} (\zeta+j)_{n-1} z^{n-j}}{\gamma_{j+1}} , \quad (11)
\]

where \((\zeta+j)_{n-1} \equiv \Gamma(\zeta+j+n-1)/\Gamma(\zeta+j)\) are the Pochhammer symbols and \(\zeta=1\) is usually taken. The approximant (11) is a ratio of two polynomials in \(z\) of power \(n\) each, and when expanded back in powers of \(z\) it reproduces all the terms of \(F_{[n+1]}\).

The authors [1] applied (11) directly to the TPS’s of \(\delta \hat{L}_{\text{pert.}}(\tilde{a}; p)/\tilde{a}^2\) of (4), and when re-expanding the approximant in powers of \(\tilde{a}\) they were able to predict the next coefficient in the series with a better precision than the one provided by the corresponding diagonal (or almost diagonal) Padé approximant. Further, in the case of the pure magnetic field they showed that the method (11), when applied directly to the TPS’s in \(\tilde{b}\) of the induced Lagrangian density [1] gave better results of resummation than the corresponding Padé approximants.

We now combine the method (11) with the Borel transformation \(\int \to \int\), and compare the results of resummation obtained in this way with the results of the corresponding Borel–Padé approximants of Ref. [9]. Formula (11) is applied to the Borel transform \(\int\) divided by \(w\). We identify \(z \equiv w^2\) (we thank the authors of [15] for pointing out that this clarification was missing in the original version of the preprint). In the ensuing Borel–Weniger approximant, we integrate in the Laplace–Borel integral over the poles of the integrand with the Cauchy principal value prescription, just as in Borel–Padé approximant (4), in accordance with the full known solution (1).

The results of these calculations are presented in Figs. 1(a)–(d), as functions of the electric field strength parameter \(\tilde{a}\), for various values of \(p \equiv \tilde{b}/\tilde{a} = 0, 0.5, 1.5, 5.0\). In Fig. 2 we present the analogous results for the case of the pure magnetic field (p.m.f.), as function of

\[ P \frac{N}{M} \left[ \delta \hat{L}_{\text{pert.}} \right] \left( \tilde{a}; p \right) = \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) [N/M]_B(w; p) \quad (9) \]

2 Various QCD and QED applications of the Borel–Padé approach with the principal value prescription can be found in [11]–[13]. The novel method of Ref. [13] is, in addition, well suited for obtaining the imaginary part of \(\delta \hat{L}\).

3 The approximants (11) applied to the TPS’s of the series (7) divided by \(\tilde{b}^2\).
the magnetic field parameter $\tilde{b}$. N3 and $[3/4]$ denote the Borel–Weniger and the Borel–Padé resummations based on the truncated Borel transform (8) with the first four nonzero terms (i.e., three terms beyond the leading order); N5 and $[5/6]$ are based on the first six terms in (8). Comparison with the exact solutions, also present in the Figures, shows that Borel–Padé is better than the corresponding Borel–Weniger, except in the case of $p=5.0$ (electric field combined with a much stronger magnetic field). Fig. 2 suggests that Borel–Padé is better than Borel–Weniger for resummation of functions whose Borel transforms have singularities only outside the positive axis. Further, comparison of Fig. 2 with the results of Table I of Ref. [1] suggests that Borel–Padé and Borel–Weniger methods are much more efficient than Weniger method in resumming series with singularities in the Borel plane. Weniger method in the p.m.f. case is better than Padé method [1].

We can also do analogous calculations for the induced energy densities $\tilde{\delta U}$

$$\tilde{\delta U} = a \frac{\partial \text{Re} \delta \mathcal{L}}{\partial a} \bigg|_{\tilde{b}} - \text{Re} \delta \mathcal{L},$$

$$\tilde{\delta U}(\tilde{a}; p) = \text{Re} \int_0^{\infty} dw \exp \left( -\frac{w}{a} \right) \frac{(-1)}{w} \left[ -\frac{pw}{\sin^2(w+i\epsilon)} \coth(pw) + \frac{1}{3}(1+p^2) + \frac{1}{w^2} \right],$$

$$\tilde{\delta U}^{\text{pert.}} = \left[ d_1(p)! \tilde{a}^2 + d_3(p)3! \tilde{a}^4 + \cdots \right],$$

$$d_1(p) = \frac{1}{45} \left[ 3 + 5p^2 - p^4 \right], \quad d_3(p) = \frac{1}{945} \left[ 10 + 21p^2 - 7p^4 + 2p^6 \right], \quad \text{etc.}$$

where $\tilde{\delta U} \equiv 8\pi^2 \delta U/(m^4 \tilde{a}^2)$. In that case, the simple Borel transform has a double–pole structure on the positive real axis, and the Padé and Weniger approximants have trouble simulating such multiple poles adequately. Therefore, we employ a slightly modified Borel transform in the case of the induced energy densities

$$MB_U(w; p) = d_1(p) \frac{w^2}{2} + d_3(p) \frac{w^4}{4} + d_5(p) \frac{w^6}{6} + \cdots,$$

which has no multiple–pole structure – all the poles are simple. The (modified) Laplace–Borel integral in this case is

$$\tilde{\delta U}(\tilde{a}; p) = \frac{1}{\tilde{a}} \int_0^{\infty} dw \exp \left( -\frac{w}{\tilde{a}} \right) MB_U(w; p),$$

where again the Cauchy principal value has to be taken, once $MB_U(w; p)$ is replaced by its Padé or Weniger approximants. For details, we refer to Ref. [4] where Borel–Padé was employed also for the induced energy densities. Weniger formula (11) is now applied to the modified Borel transform (16) divided by $w^2$. The results are presented in Figs. 3(a)–(d), as functions of $\tilde{a}$ at fixed $p=0.0, 0.5, 1.5, 5.0$, respectively. We present the solutions of Borel–Weniger and Borel–Padé based on the first four (N3, [4/4]) and six (N5, [6/6]) nonzero terms of the modified Borel transform of the energy density. We see that for the induced

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4 In the case of the pure magnetic field, the energy density is the same as the Lagrangian density, except for the sign change.
energy density the situation is less clear. In the cases $p = 0$, 0.5 and 5.0 the Borel–Padé and Borel–Weniger resummations are apparently of comparable quality, while at $p = 1.5$ the Borel–Padé appears to work better.

We can see these trends also if we compare the perturbation coefficients predicted by these two methods with the exact ones. These results are written in Table I for the case of the Lagrangian density (predicted $c_9$ and $c_{13}$) and in Table II for the case of the energy density (predicted $d_9$ and $d_{13}$). Predictions of Borel–Padé and Borel–Weniger are of comparable quality in the cases of $p = 0, 0.5, 5.0$ for energy density and in the case of $p = 5.0$ for Lagrangian density. In other cases, predictions of Borel–Padé are better. In fact, in the

| approximant | $p = 0.0$ and p.m.f | $p = 0.5$ | $p = 1.5$ | $p = 5.0$ |
|-------------|---------------------|-----------|-----------|-----------|
| N3          | $c_9 = 2.1666 \cdot 10^{-6}$ | $c_9 = 3.524 \cdot 10^{-6}$ | $c_9 = 4.320 \cdot 10^{-4}$ | $c_9 = 596.91$ |
| $[3/4]$     | $c_9 = 2.1637 \cdot 10^{-6}$ | $c_9 = 3.648 \cdot 10^{-6}$ | $c_9 = 5.866 \cdot 10^{-4}$ | $c_9 = 595.28$ |
| exact       | $c_9 = 2.1644 \cdot 10^{-6}$ | $c_9 = 3.711 \cdot 10^{-6}$ | $c_9 = 6.166 \cdot 10^{-4}$ | $c_9 = 596.24$ |
| N5          | $c_{13} = 2.2212 \cdot 10^{-8}$ | $c_{13} = 3.725 \cdot 10^{-8}$ | $c_{13} = 2.460 \cdot 10^{-5}$ | $c_{13} = 3823.65$ |
| $[5/6]$     | $c_{13} = 2.2215 \cdot 10^{-8}$ | $c_{13} = 3.804 \cdot 10^{-8}$ | $c_{13} = 3.157 \cdot 10^{-5}$ | $c_{13} = 3824.42$ |
| exact       | $c_{13} = 2.2215 \cdot 10^{-8}$ | $c_{13} = 3.805 \cdot 10^{-8}$ | $c_{13} = 3.161 \cdot 10^{-5}$ | $c_{13} = 3824.45$ |

TABLE I. Coefficients $c_9$ and $c_{13}$ of the perturbation series for the induced Lagrangian density, as predicted by various Borel–Weniger and Borel–Padé approximants. We include exact values for comparison.

| approximant | $p = 0.0$ | $p = 0.5$ | $p = 1.5$ | $p = 5.0$ |
|-------------|-----------|-----------|-----------|-----------|
| N3          | $d_9 = 2.3752 \cdot 10^{-5}$ | $d_9 = 3.8124 \cdot 10^{-5}$ | $d_9 = 3.312 \cdot 10^{-4}$ | $d_9 = -452.06$ |
| $[4/4]$     | $d_9 = 2.3658 \cdot 10^{-5}$ | $d_9 = 3.7974 \cdot 10^{-5}$ | $d_9 = 4.529 \cdot 10^{-6}$ | $c_9 = -458.32$ |
| exact       | $d_9 = 2.3808 \cdot 10^{-5}$ | $d_9 = 3.8085 \cdot 10^{-5}$ | $d_9 = 2.503 \cdot 10^{-5}$ | $c_9 = -464.01$ |
| N5          | $d_{13} = 3.3319 \cdot 10^{-7}$ | $d_{13} = 5.4289 \cdot 10^{-7}$ | $c_{13} = 8.162 \cdot 10^{-5}$ | $c_{13} = -2977.3$ |
| $[6/6]$     | $d_{13} = 3.3309 \cdot 10^{-7}$ | $d_{13} = 5.4291 \cdot 10^{-7}$ | $c_{13} = -1.571 \cdot 10^{-6}$ | $c_{13} = -2991.7$ |
| exact       | $d_{13} = 3.3322 \cdot 10^{-7}$ | $d_{13} = 5.4301 \cdot 10^{-7}$ | $c_{13} = -2.537 \cdot 10^{-6}$ | $c_{13} = -2976.7$ |

TABLE II. Coefficients $d_9$ and $d_{13}$ of the perturbation series for the induced energy density, as predicted by various Borel–Weniger and Borel–Padé approximants. For comparison, exact values are included as well.

The case $p = 5.0$ of the energy density, the modified Borel–Weniger is slightly, but discernibly, better than the modified Borel–Padé. Comparing predictions of Table I (for $p = 0.0$) with those of Tables II and III of Ref. 1 suggests strongly that the discussed Borel–Padé and Borel–Weniger methods are better than Weniger method in predicting the coefficients $c_n$. Weniger method is better than Padé method in predicting $c_n$’s.

(2) The second example to compare the efficiency of the Borel–Padé and Borel–Weniger methods will be taken from QCD, and it will have to do with the “fixing” of a pole of a Borel transform rather than with a resummation. We look at the Bjorken polarized sum rule.
(BjPSR), which involves the isotriplet combination of the first moments over $x_{\text{Bj}}$ of proton and neutron polarized structure functions

$$\int_0^1 dx_{\text{Bj}} \left[ g_1^{(p)}(x_{\text{Bj}}; Q_{\text{ph}}^2) - g_1^{(n)}(x_{\text{Bj}}; Q_{\text{ph}}^2) \right] = \frac{1}{6} g_A \left[ 1 - S(Q_{\text{ph}}^2) \right]. \tag{18}$$

Here, $p^2 = -Q_{\text{ph}}^2 < 0$ is $\gamma^*$ momentum transfer. At $Q_{\text{ph}}^2 = 3\text{GeV}^2$ where three quarks are assumed active ($n_f = 3$), and if taking $\overline{\text{MS}}$ scheme and renormalization scale ($\text{RScI}$) $Q_0^2 = Q_{\text{ph}}^2$, we have the following TPS of the BjPSR observable $S(Q_{\text{ph}}^2)$ available \([10] - [17]\):

$$S_{[2]}(Q_{\text{ph}}^2; Q_0^2 = Q_{\text{ph}}^2; c_2^{\text{MS}}, c_3^{\text{MS}}) = a_0(1 + 3.583 a_0 + 20.215 a_0^2), \tag{19}$$

with:

$$a_0 = a(\ln Q_0^2; c_2^{\text{MS}}, c_3^{\text{MS}}, \ldots), \quad n_f = 3, \quad c_2^{\text{MS}} = 4.471, \quad c_3^{\text{MS}} = 20.99. \tag{20}$$

Here we denoted by $a$ the strong coupling parameter $\alpha_s/\pi$.

It is known from \([18] - [19]\) that the Borel transform $B_S(z)$ of $S$ has the lowest positive pole at $z_{\text{pole}} = 1/\beta_0 = 4/9$ (leading infrared renormalon) and that this pole has a much stronger residuum than the highest negative pole at $z_{\text{pole}} = -1/\beta_0$ (leading ultraviolet renormalon). The question we raise here is: How well can Padé and Weniger approximants to the Borel transform $B_S(z)$ determine the next coefficient $r_3$ of the term $r_3 a_0^3$ in the TPS \([14]\), via the requirement that $z_{\text{pole}} = 4/9$? For that, we have to know well the actual $r_3$. That term can be determined reasonably well on the basis of two approximants discussed in \([21]\) — the effective charge approximant (ECH) $A_3^{(\text{ECH})}(c_3)$ with $c_3 \approx 20$, and another, also RScI– and scheme–independent approximant $A_{3/2}^{(\text{ECH})}(c_3)$ with $c_3 \approx 15.5$. These two approximants give the correct location of the leading infrared renormalon pole, and when we expand them back in powers of $a_0$ we obtain $r_3 \approx 129.4$ and $r_3 \approx 130.8$, respectively. Therefore, we can estimate with high confidence the actual $r_3$: $r_3 = 130 \pm 1$.

It is important to consider the RScl– and scheme–invariant Borel transform when we want to apply Padé or Weniger approximants to it, so that the predicted values of $r_3$ will be independent of the RScl– and scheme in which we work at the intermediate stage. Such a Borel transform has been used in \([21]\), and we use its variant $\tilde{B}_S(z)$ as specified in \([21]\) [cf. Eqs. (18)–(20) there]. Such a Borel transform reduces (up to a $z$–dependent nonsingular factor) to the usual Borel transform in the approximation of the one–loop evolution. The resulting power expansion of $\tilde{B}_S(z)$ up to $\sim z^3$ will depend on the coefficient $r_3$

$$\tilde{B}_S(z) = 1 + \frac{32}{81} (\gamma - 1) y + 0.02078 y^2 + \frac{8}{729} (-21.88 + \frac{1}{6} r_3) y^3 + O(y^4), \tag{21}$$

where $\gamma = 0.577...$ is Euler constant, and $y \equiv 2\beta_0 z$. If we apply \([21]/[1\,2]\) and \([1\,2]/\text{Padé}\) approximants to the TPS \([21]\) and demand $z_{\text{pole}} = 1/\beta_0$ ($y_{\text{pole}} = 2$), we obtain predictions $r_3 = 137.0$ and $r_3 = 128.0$, respectively. The prediction of \([1\,2]\) is significantly better, and this could possibly be explained with the more involved denominator structure of \([1\,2]\) in comparison to \([2\,1]\). When applying to \([21]\) Weniger formula \([14]\) ($\delta_2^{(0)}$ with $\zeta = 1$), we obtain $r_3 = 135.3$. This is further away from the actual value of $130 \pm 1$, than the prediction of \([1\,2]\). In both \([1\,2]\) and $\delta_2^{(0)}$, the denominators are polynomials of quadratic degree in $z$.

To summarize this QCD example: We applied Padé and Weniger approximants to a (TPS of a) Borel transform of the Bjorken polarized sum rule and demanded that the leading infrared renormalon pole be reproduced correctly. Weniger approximant $\delta_2^{(0)}$ then...
apparently gives a somewhat worse prediction for the next coefficient than the corresponding Padé approximant $[1/2]$.

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REFERENCES

[1] U. D. Jentschura, J. Becher, E. J. Weniger, and G. Soff, [hep-ph/9911265].
[2] W. Heisenberg and H. Euler, Z. Physik 98, 714 (1936);
[3] V. Weisskopf, Kgl. Danske Videnskab. Selskabs. Mat.-fys. Medd. 14, No. 6 (1936).
[4] J. Schwinger, Phys. Rev. 82, 664 (1951).
[5] W. Greiner, B. Müller, and J. Rafelski, Quantum electrodynamics of strong fields, 594 pp. (Berlin, Heidelberg, Springer 1985);
[6] W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics, 244 pp. (Springer 1985).
[7] J. Schwinger, Particles, Sources and Fields, Vol. II, 306 pp., Chapters 4–8 (Addison–Wesley, 1989).
[8] C. Itzykson and J.-B. Zuber, Quantum Field Theory, 705 pp., Chapter 4–3 (McGraw-Hill, New York, 1980).
[9] G. Cvetič and Ji-Young Yu, [hep-ph/9911370].
[10] George A. Baker, Jr. and Peter Graves–Morris, Padé Approximants, 2nd edition, 746 pp. (Encyclopedia of Mathematics and Its Applications, Vol. 59), edited by Gian-Carlo Rota (Cambridge University Press, 1996).
[11] P. A. Rączka, Phys. Rev. D 43, R9 (1991).
[12] M. Pindor, [hep-ph/9903151].
[13] U. D. Jentschura, [hep-th/0001135], to appear in Phys. Rev. D.
[14] E. J. Weniger, Ann. Phys. (N.Y.) 246, 133 (1996), especially Appendix A.
[15] U. D. Jentschura, E. J. Weniger, and G. Soff, [hep-ph/0005198].
[16] S. G. Gorishny and S. A. Larin, Phys. Lett. B 172, 109 (1986); E. B. Zijlstra and W. Van Neerven, Phys. Lett. B 297, 377 (1992).
[17] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 259, 345 (1991).
[18] D. J. Broadhurst and A. L. Kataev, Phys. Lett. B 315, 179 (1993).
[19] X. Ji, Nucl. Phys. B 448, 51 (1995); C. N. Lovett-Turner and C. J. Maxwell, Nucl. Phys. B 452, 188 (1995).
[20] G. Cvetič, [hep-ph/0003123], to appear in Phys. Lett. B.
[21] G. Grunberg, Phys. Lett. B 304, 183 (1993).
FIG. 1. Borel–Padé approximants ([3/4], [5/6]) and the corresponding Borel–Weniger approximants (N3, N5) to the induced dispersive Lagrangian density (1), as functions of \( \tilde{a} = \tilde{g}a/m^2 \), for various values of \( p = \tilde{b}/\tilde{a} \): (a) \( p = 0.0 \); (b) \( p = 0.5 \); (c) \( p = 1.5 \); (d) \( p = 5.0 \). The numerically exact curves are included for comparison.
FIG. 2. Borel–Padé approximants ([3/4], [5/6]) and Borel–Weniger approximants (N3, N5) to the induced dispersive Lagrangian density (6), as functions of $\tilde{b}$, for the pure magnetic field case ($\tilde{a} = 0$). The numerically exact curve is included for comparison.
FIG. 3. Modified Borel–Padé ([4/4], [6/6]) and the corresponding modified Borel–Weniger (N3, N5) approximants to the induced energy densities \( \tilde{a} = \tilde{b}/\tilde{a} \), as functions of \( \tilde{a} \), at fixed values of \( \tilde{p} = \tilde{b}/\tilde{a} \): (a) \( \tilde{p} = 0.0 \); (b) \( \tilde{p} = 0.5 \); (c) \( \tilde{p} = 1.5 \); (d) \( \tilde{p} = 5.0 \).