Quantum corrections to the dynamics of the Bose-Einstein condensate in a double-well potential

Yan Xu\textsuperscript{1,2}, Wei Fan\textsuperscript{1,*} and Bing Chen\textsuperscript{1}

\textsuperscript{1}College of Science, Shandong University of Science and Technology, 579 Quanwangang Road, Economic and Technical Development Zone, Qingdao, 266510, China

\textsuperscript{2}Center for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117542, Republic of Singapore

\*ashitakatosan@gmail.com

The dynamics of the Bose-Einstein condensate (BEC) in a double-well potential is often investigated under the mean-field theory (MFT). This works successfully for large particle numbers with dynamical stability. But for dynamical instabilities, quantum corrections to the MFT becomes important [Phys.Rev.A 64, 013605(2001)]. Recently the adiabatic dynamics of the double-well BEC is investigated under the MFT in terms of a dark variable [Phys.Rev.A 81, 043621(2010)], which generalizes the adiabatic passage techniques in quantum optics to the nonlinear matter-wave case. We give a fully quantized version of it using second-quantization and introduce new correction terms from higher order interactions beyond the on-site interaction, which are interactions between the tunneling particle and the particle in the well and interactions between the tunneling particles. If only the on-site interaction is considered, this reduces to the usual two-mode BEC.

Keywords: Bose-Einstein condensate; double-well potential

1. Introduction

The dynamics of the Bose-Einstein condensate (BEC) is often investigated under the mean-field theory (MFT), which provides a classical field equation for the nonlinear matter wave and is the classical limit of the quantum theory in large number limit. MFT works successfully in predicting experimental results as the particle numbers of BEC is very large, so the classical limit captures the essence. In the vicinity of a mean-field dynamical instability, however, the quantum correction to the MFT becomes important. They provide accurate predictions for the dynamics by combining the mean-field with the fluctuations. As most investigations are done under the MFT, it is interesting to look at their behavior under quantum corrections.

On the other hand, the analogy to quantum optics is noticed in investigating BEC, because of its macroscopic coherence that allows us to view it as a large atom. A recent example is the discovery of a dark variable for the double-well BEC by Ottaviani et al [2], which is a nonlinear matter-wave extension of Vitanov and...
Shore’s work of a two-state atom, where a dark variable is found by looking into the analogy between the optical two-state Bloch equation and the three-state stimulated Raman adiabatic passage (STRAP) equations. They compose three variables analogous to the Bloch spin from the ground states of the two wells and the dark variable is found by comparing the equation of the three variables with the STRAP equation. The adiabatic splitting, transport and self-trapping of the double-well BEC are investigated using this dark variable under the MFT. The STRAP can be utilized to transport BEC in multiple wells.

In the paper, we give a fully quantized version of the work of Ottaviani et al. We introduce further quantum corrections by including higher order interactions beyond the on-site interaction. If these quantum corrections are turned off, this model would reduce to the two-mode BEC widely used in quantum entanglement.

2. The on-site Interaction

In MFT, the two-mode approximation is applied to the GP equation and the dynamics of the double-well BEC is

\[ \mathcal{H} \left( \begin{array}{c} c_L \\ c_R \end{array} \right) = \frac{i}{\hbar} \frac{d}{dt} \left( \begin{array}{c} c_L \\ c_R \end{array} \right), \]  

where the Hamiltonian is given by

\[ \mathcal{H} = \left( \begin{array}{cc} \epsilon_L + U_L |c_L|^2 & \Omega \\ \Omega & \epsilon_R + U_R |c_R|^2 \end{array} \right). \]  

Introducing the field operator \( \hat{\Psi}(\vec{r}, t) \), the Hamiltonian can be written as

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} = \int d^3r \hat{\Psi}^\dagger \hat{H}_0 \hat{\Psi} + \frac{g}{2} \int d^3r \hat{\Psi}^\dagger \hat{\Psi} \hat{\Phi}, \]  

where \( \hat{\Phi}(\vec{r}, t) = \hat{a}_L(t) \phi_L(\vec{r}) + \hat{a}_R(t) \phi_R(\vec{r}) \), is the single particle Hamiltonian with \( V(\vec{r}, t) \) the double-well potential and \( g = 4\pi a_s/m \) is the nonlinear interaction parameter with \( a_s \) the s-wave scattering length. Only spatial degrees of freedom and two-body interactions are considered. Under the two mode approximation, where all modes are omitted except the condensate modes, the field operator can be expanded \( \hat{\Psi}(\vec{r}, t) = \hat{a}_L(t) \phi_L(\vec{r}) + \hat{a}_R(t) \phi_R(\vec{r}) \), where \( \hat{a}_L, \hat{a}_R \) and \( \phi_L, \phi_R \) are the annihilation operator and the ground state of the left and the right well respectively. The overlap between the two ground states is neglected because it is small compared with the on-site part, \( \phi_L^* \phi_R \ll |\phi_L|^2 \). Now the second quantized Hamiltonian becomes

\[ \mathcal{H} = \epsilon_L \hat{a}_L^\dagger \hat{a}_L + \Omega \hat{a}_L^\dagger \hat{a}_R + \frac{U_0}{2} \hat{a}_L^\dagger \hat{a}_L \hat{a}_R^\dagger \hat{a}_L \]

\[ + \epsilon_R \hat{a}_R^\dagger \hat{a}_R + \Omega \hat{a}_R^\dagger \hat{a}_L + \frac{U_0}{2} \hat{a}_R^\dagger \hat{a}_R \hat{a}_L^\dagger \hat{a}_R, \]
with the parameters

\[ \epsilon_{L,R} = \int d^3r \phi^*_L H_0 \phi_R, \]
\[ \Omega = \int d^3r \phi^*_L H_0 \phi_R = \int d^3r \phi^*_R H_0 \phi_L, \]
\[ U_0 = g \int d^3r |\phi_L|^4 = \int d^3r |\phi_R|^4. \]

with \( \epsilon_{L,R} \) the chemical potential of the two wells respectively and \( U_0 \) the nonlinear on-site interaction between particles in the same well. The tunneling rate \( \Omega \) between the two wells is negative, but it can also be defined to be positive by adding corresponding minus signs in the Hamiltonian.

It is convenient to rewrite the Hamiltonian by the Schwinger angular momentum operators

\[ \hat{J}_x = \left( \hat{a}^\dagger_R \hat{a}_L + \hat{a}^\dagger_L \hat{a}_R \right) / 2, \]
\[ \hat{J}_y = \left( \hat{a}^\dagger_R \hat{a}_L - \hat{a}^\dagger_L \hat{a}_R \right) / 2i, \]
\[ \hat{J}_z = \left( \hat{a}^\dagger_R \hat{a}_R - \hat{a}^\dagger_L \hat{a}_L \right) / 2, \]

where \( \hat{J}_x,y \) corresponds to the correlation between the two wells and \( \hat{J}_z \) is the particle number difference between the two wells. The Hamiltonian under these angular momentum operators becomes

\[ \mathcal{H} = \frac{E}{2} N + \epsilon \hat{J}_z + 2\Omega \hat{J}_x + U_0 \left( \hat{J}_z^2 + \frac{N^2}{4} - \frac{N}{2} \right) \]
\[ = U_0 \hat{J}_z^2 + \epsilon \hat{J}_z + 2\Omega \hat{J}_x, \]

with \( N = N_L + N_R \) the total particle number, \( E = \epsilon_L + \epsilon_R \) the sum of the two chemical potentials and \( \epsilon = \epsilon_R - \epsilon_L \) the difference of the two chemical potential. Terms containing \( N \) and \( E \) are assumed to be conserved and are neglected from the Hamiltonian. This is the often used Hamiltonian in quantum entanglement, which only the on-site interaction is included. The dynamics of the system manifests in the evolution of these angular momentum operators

\[ \frac{d}{dt} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix} = \begin{pmatrix} -iU_0 & -\left( \epsilon + 2U_0 \hat{J}_z \right) & 0 \\ \epsilon + 2U_0 \hat{J}_z & -iU_0 & -2\Omega \\ 0 & 2\Omega & 0 \end{pmatrix} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}. \]

Equation “Eq. (9)” is the second quantized version of the dynamics given in Ref. [2] where it is described by the evolution equation of the Bloch spin vector [20],

\[ \frac{d}{dt} \begin{pmatrix} \mu \\ \nu \\ \omega \end{pmatrix} = \begin{pmatrix} 0 & -(\epsilon + U) & 0 \\ -(\epsilon + U \omega) & 0 & -2\omega \\ 0 & 2\omega & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \\ \omega \end{pmatrix}, \]

The elements of the Bloch spin vector \( \mu, \nu \) and \( \omega \) are composed from the two modal populations.

The two evolution equations would resemble each other with the angular momentum operator corresponding to the Bloch spin vector, if we normalize the angular momentum operator by \( N/2, 2\hat{J}_{x,y,z}/N \sim u, v, w \). The nonlinear interaction parameter becomes \( g = 4\pi Na_s/2m \) under this normalization, which becomes the same
\( g \) in “Eq. (10)”. Without the normalization, the adiabatic splitting, transport and self-trapping can be given by the motion of the angular momentum operator on the Bloch sphere with radius \( j = N/2 \), while in Ref. 2 it corresponds to the motion of the Bloch spine in the unit sphere. The adiabatic transport corresponds to the variance of \( J_z \) from \(-N/2\) to 0 then to \( N/2 \) and the self-trapping corresponds to its variance from \(-N/2\) to 0 then back to \( N/2 \).

Except the resemblance of their appearance, these two equations are quite different. The first one is the quantized version of the second one and it is the evolution equation for quantum operators rather than classical variables. We can approximate the operators with their expectation values, with the lowest-order approximation corresponding the MFT and the second-order approximation corresponding to the equation given by Ref. 1. Also two diagonal terms of \(-iU_0\) emerge in the quantized version, which puts a phase factor to \( J_z \) and \( J_y \). So the phases are also important in the dynamics. The phases and particle numbers can be investigated under the mean field approximation 21, where they exhibit oscillations in the phase space. In the angular momentum space, we can choose the eigenstates of \( \hat{J}_z \) as the basis set \( \{ |jm\rangle \} \). The state of the system at time \( t \) is given by \( |\Psi(t)\rangle = \exp(-iHt)|\Psi_0\rangle \), where \( |\Psi_0\rangle = \left| \frac{\pi}{2} \frac{\pi}{2} \right\rangle \) is the initial state with the left well population. The adiabatic transport of the system then means the evolution from \( |\Psi_0\rangle \) to \( |\Psi(t)\rangle = \left| \frac{\pi}{2} \frac{\pi}{2} \right\rangle \).

3. Higher Order Interactions

The above quantized version of Ref. 2 only includes the nonlinear effects up to the on-site interaction. It would be interesting to investigate the effects of the neglected interactions, which becomes useful when its behavior is investigated under a wide parameter regime. To do this, we add to the Hamiltonian “Eq. (4)” the overlapping part \( \phi_L^{\ast} \phi_R \) and introduce two new parameters \( U_t \) and \( U_{tt} \) to describe their effects, where \( U_t \) captures interactions between the tunneling particle and the on-site particle and \( U_{tt} \) captures interactions between the tunneling particles themselves. We term the tunneling particle as ‘tunnelier’ for simplicity. Now the full Hamiltonian is given by

\[
H = \epsilon_L \hat{a}_L^{\dagger} \hat{a}_L + \Omega \hat{a}_L^{\dagger} \hat{a}_R + \frac{U_0}{2} \hat{a}_L^{\dagger} \hat{a}_L \hat{a}_L^{\dagger} \hat{a}_L + \\
+ U_t \left( \hat{a}_L^{\dagger} \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L \right) + \frac{U_{tt}}{2} \left( \hat{a}_L^{\dagger} \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L + \hat{a}_L^{\dagger} \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L \right) + \\
+ \epsilon_R \hat{a}_R^{\dagger} \hat{a}_R + \Omega \hat{a}_R^{\dagger} \hat{a}_L + \frac{U_0}{2} \hat{a}_R^{\dagger} \hat{a}_R \hat{a}_R^{\dagger} \hat{a}_R + \\
+ U_t \left( \hat{a}_R^{\dagger} \hat{a}_R \hat{a}_L + \hat{a}_L^{\dagger} \hat{a}_R \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L \hat{a}_R \right) + \frac{U_{tt}}{2} \left( \hat{a}_R^{\dagger} \hat{a}_L \hat{a}_R + \hat{a}_R^{\dagger} \hat{a}_L \hat{a}_R + \hat{a}_L^{\dagger} \hat{a}_R \hat{a}_R \right),
\]

with the newly introduced parameters given by \( U_t = g \int d^3r \, |\psi_L|^2 \langle \psi_L^{\ast} \psi_R \rangle = g \int d^3r \, |\psi_R|^2 \langle \psi_R^{\ast} \psi_L \rangle \) and \( U_{tt} = g \int d^3r \, (\psi_L^{\ast} \psi_R)^2 = g \int d^3r \, (\psi_R^{\ast} \psi_L)^2 \).
and the on-site particle adds a term $U$ Hamiltonian by comparing it with “Eq. (8)”. The interaction between the tunnelier and the right tunnelier. We can see the modifications introduced to the left well), $\hat{U}$

carried by $\hat{\Omega}$, the interaction between the left and the right tunneliers adds a term $U_L(N-1)$, the interaction between the right tunneliers adds a term $U_R(N-1)$ and $U_{L,R}$ the interaction between the left tunnelier and the right tunnelier. We can see the modifications introduced to the Hamiltonian by comparing it with “Eq. (5)”. The interaction between the tunnelier and the on-site particle adds a term $U_t(N-1)$ to the original tunneling parameter $\Omega$, the interaction between the left and the right tunneliers adds a term $-2U_{lt}$ to the original nonlinear on-site interaction parameter $U_0$ and the interaction of the left tunneliers and that of the right tunneliers add another nonlinear terms $U_{lt} \left( J_x^2 - J_y^2 \right)$ to the whole Hamiltonian. The equation of motion of these operators is

$$\begin{pmatrix}
  \frac{d}{dt} \hat{J}_x \\
  \frac{d}{dt} \hat{J}_y \\
  \frac{d}{dt} \hat{J}_z
\end{pmatrix} =
\begin{pmatrix}
  -i(U_0 - U_{lt}) & \epsilon + 2(U_0 - U_{lt}) & 0 \\
  \epsilon + 2(U_0 - 3U_{lt}) & -i(U_0 - 3U_{lt}) & -2(\Omega + U_t(N-1)) \\
  0 & 2(\Omega + U_t(N-1) + 2U_{lt}) & -i2U_{lt}
\end{pmatrix}
\begin{pmatrix}
  \hat{J}_x \\
  \hat{J}_y \\
  \hat{J}_z
\end{pmatrix}. \tag{13}
$$

Compared with “Eq. (9)”, this equation is non-symmetric and there is even a phase term of $\hat{J}_z$. By the definition of $\hat{J}_z$, there is no phase term as the phases carried by $\hat{a}_{L,R}$ is canceled out in $\hat{a}_{L,R}^\dagger \hat{a}_{L,R}$. The origin of this phase term of $\hat{J}_z$ manifests in the commutation relationship of $\hat{J}_x$ and $\hat{J}_y$, which is introduced in deriving this equation of motion from the Hamiltonian “Eq. (12)”.

In Ref. [2] the nonlinear parameter $U_0$ generates a rich adiabatic dynamics of the double-well BEC. Here the quantum corrections introduced by $U_{lt}$ and $U_t$ makes the dynamics even more complex. When $U_{lt}$ and $U_t$ are small compared with $U_0$, the influence may not be very explicit; but when they are large enough, the formulations above would lose sense as the two-mode assumption no longer holds true. Numerical simulations are needed to investigate their influence to the dynamics under various parameter regimes.
4. Conclusion

We have obtained quantum corrections of higher order interactions for the quantized dynamics of the double-well BEC and given a fully quantized expression of it under the two-mode approximation. Two new parameters are obtained to express the influence of the interactions between the tunneling particle and the on-site particle and between the tunneling particles. So the dynamics of the double-well BEC becomes more rich under this quantized version. This allows numerical simulation for a wide range of parameter regime and new phenomenon may be obtained.

Acknowledgements

Project supported by the National Natural Science Foundation of China (Grant No. 11105086), the Natural Science Foundation of Shandong Province (Grant No. BS2011DX029, ZR2009AM026), the basic scientific research business expenses of the central university, and the basic scientific research project of Qingdao(Grant No. 11-2-4-4-(6)-jch).

References

1. J. R. Anglin and A. Vardi, Phys. Rev. A 64, 013605 (2001).
2. C. Ottaviani, V. Ahufinger, R. Corbalán, and J. Mompart, Phys. Rev. A 81, 043621 (2010).
3. N. V. Vitanov and B. W. Shore, Phys. Rev. A 73, 053402 (2006).
4. K. Bergmann, H. Theuer, and B. W. Shore, Rev. Mod. Phys. 70, 1003 (1998).
5. A. D. Greentree, J. H. Cole, A. R. Hamilton, and L. C. L. Hollenberg, Phys. Rev. B 70, 235317 (2004).
6. J. H. Cole, A. D. Greentree, L. C. L. Hollenberg, and S. Das Sarma, Phys. Rev. B 77, 235418 (2008).
7. V. Nesterenko, A. Novikov, F. F. de Souza Cruz, and E. Lapolli, Laser Physics 19, 616 (2009).
8. E. M. Graefe, H. J. Korsch, and D. Witthaut, Phys. Rev. A 73, 013617 (2006).
9. M. Rab, J. H. Cole, N. G. Parker, A. D. Greentree, L. C. L. Hollenberg, and A. M. Martin, Phys. Rev. A 77, 061602 (2008).
10. M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
11. A. Srensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature 409, 63 (2001).
12. J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature 455, 1216 (2008).
13. A. Micheli, D. Jaksch, J. I. Cirac, and P. Zoller, Phys. Rev. A 67, 013607 (2003).
14. A. P. Hines, R. H. McKenzie, and G. J. Milburn, Phys. Rev. A 67, 013609 (2003).
15. H. T. Ng and P. T. Leung, Phys. Rev. A 71, 013601 (2005).
16. P. Facchi, G. Florio, U. Marzolino, G. Parisi, and S. Pascazio, Journal of Physics A: Mathematical and Theoretical 42, 055304 (2009).
17. J. Vidal, G. Palacios, and C. Aslangul, Phys. Rev. A 70, 062304 (2004).
18. H. Wichterich, J. Vidal, and S. Bose, Phys. Rev. A 81, 032311 (2010).
19. P. Ribeiro, J. Vidal, and R. Mosseri, Phys. Rev. E 78, 021106 (2008).
20. C. Lee, W. Hai, L. Shi, and K. Gao, Phys. Rev. A 69, 033611 (2004).
21. S. Kohler and F. Sols, Phys. Rev. Lett. 89, 060403 (2002).