On the Coupling of Relativistic Particle to Gravity and Wheeler-DeWitt Quantization

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Abstract

A system consisting of a point particle coupled to gravity is investigated. The set of constraints is derived. It was found that a suitable superposition of those constraints is the generator of the infinitesimal transformations of the time coordinate $t \equiv x^0$ and serves as the Hamiltonian which gives the correct equations of motion. Besides that, the system satisfies the mass shell constraint, $p^\mu p_\mu - m^2 = 0$, which is the generator of the worldsheet reparametrizations, where the momenta $p_\mu$, $\mu = 0, 1, 2, 3$, generate infinitesimal changes of the particle’s position $X^\mu$ in spacetime. Consequently, the Hamiltonian contains $p_0$, which upon quantization becomes the operator $-i\partial/\partial T$, occurring on the r.h.s. of the Wheeler-DeWitt equation. Here the role of time has the particle coordinate $X^0 \equiv T$, which is a distinct concept than the spacetime coordinate $x^0 \equiv t$. It is also shown how the ordering ambiguities can be avoided if a quadratic form of the momenta is cast into the form that instead of the metric contains the basis vectors.

1 Introduction

When quantizing gravity, one faces a tough problem, because time disappears from the equations. If gravity is coupled to matter, then the changes of matter configurations are supposed to have the role of time in quantum gravity (see, e.g, a review by Anderson [1]). Typically, matter is described by scalar, spinor or electromagnetic fields [2]. A different approach was explored by Rovelli [3,4] who considered as a model a single particle coupled to general relativity. In addition to particle’s coordinates $X^\mu (\tau)$, he also considered a clock variable, attached to the particle. In Ref. [5] a model without the clock variable was investigated and it was found that the particle’s coordinates $X^0$ (as well as $X^i$, $i = 1, 2, 3$) survive quantization and has the role of time in quantum gravity in the presence of the particle. The model was also extended to a system of particles [5] and further elaborated [6]. Recently that model was reconsidered by Struyve [7,8]. He put the action into such a form that the matter and gravity part had the same time parameter $\tau$. This required to insert an extra $\dot{X}^0$ into the gravity part of the action, and thus change the canonical momenta and the constraint. Instead of the usual mass shell constraint $p^\mu p_\mu - m^2 = 0$, he obtained a new, more complicated constraint that contained the Ricci scalar $R$. With this new
constraint, it turned out that upon quantization the time parameter $\tau$ disappeared from the equations. But Struyve also observed that by a suitable canonical transformation at the classical level and a unitary transformation at the quantum level one can arrive at the equations obtained in Ref. [5, 6].

In the present work we intend to clarify this important subject. Firstly, we observe that both the particle and gravity part of the action can be cast into the form in which they both have the same evolution parameter, namely, $t \equiv x^0$, while retaining the particle worldline parameter $\tau$ and the mass shell constraint $p^\mu p_\mu - m^2 = 0$. Rewriting the total action by employing the ADM (1+3) split and varying it with respect to the lapse and shift (considered as Lagrange multipliers), one obtains the constraints. As a Hamiltonian we take a superposition of those constraints and find that it leads to the correct equations of motion (the geodesic equation and the Einstein equations) by employing the ordinary Poisson brackets. By this we varify the correctness of the Hamiltonian so constructed.

To further explore the meaning of the quantities such as the particle’s momenta $p_\mu$, and the Hamiltonian, we perform the total variation of the action that includes a change $\delta x^\mu$ and $\delta \tau$ of the boundary. So we obtain the generator $H$ of the transformations $\delta t$, the generators $p_\mu$ of the transformations $\delta X^\mu$ (which are changes of particle’s position in spacetime), and the generator of the transformation $\delta \tau$ (which is proportional to the mass shell constraint). Such fundamental analysis, at each step covariant, convinces us that all the momenta $p_\mu$, $\mu = 0, 1, 2, 3$, take place within the formalism. At the classical level, the presence of the particle enables the identification (definition) of spacetime points. At the quantum level those particle variables $X^\mu$, including $X^0$, remain in the equations; the particle’s coordinate $X^0$ has the role of time. Because of the presence of the mass shell constraint, the Hamilton obtained by Rovelli, namely

$$H = \int d^3x \, N^\mu H^\mu_{\text{ADM}} - N^i p_i - N \sqrt{m^2 + p^2},$$

(1)

can be written as $H = \int d^3x \, N^\mu H^\mu_{\text{ADM}} - p_0$. Upon quantization, $p_0 \to \hat{p}_0 = -i \frac{\partial}{\partial X^0}$. We see that the presence of the particle “saves” the concept of spacetime, so that time, namely, $X^0$, is present both in the classical and quantum equations. Otherwise it would be difficult to retain in the quantum theory the concept of local Lorentz transformations of Eq. (1), and understand how different local inertial observers compare the observed values of $p_i$ without bringing $p_0$ into the description.

In Sec. 2 we first point out that the Einstein equations imply the relation

$$\frac{1}{8\pi G} \int d^3x \, \sqrt{-g} G_0^0 = -p_0.$$

Then we show that the analogous equation comes out in the ADM formalism. In Sec. 3 we discuss quantization of that model. At the end we also touch the problem of the ordering ambiguities and point out how they could be avoided.

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1We omit here the clock variable that is included in the Rovelli’s equation.
2 Gravity coupled to particle

The action for particle coupled to the gravitational field is

\[ I[X^\mu, g_{\mu\nu}] = m \int d\tau \left( g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \right)^{1/2} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \]  
(2)

The variation of this action with respect to the metric \( g_{\mu\nu} \) gives

\[ -\frac{1}{8\pi G} G^{\mu\nu} = \int d\tau \delta^4(x - X(\tau)) \frac{m \dot{X}^\mu \dot{X}^\nu}{(g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta)^{1/2} \sqrt{-g}} = T^{\mu\nu}, \]  
(3)

which are the Einstein equations in the presence of the stress-energy tensor \( T^{\mu\nu} \) of the particle.

From Eq. (3) we obtain

\[ -\frac{1}{8\pi G} \int G^{\mu\nu} \sqrt{-g} d\Sigma_\nu = \int T^{\mu\nu} \sqrt{-g} d\Sigma_\nu = p^\mu, \]  
(4)

where \( p^\mu \) is the particle’s momentum. Writing the hypersurface element as \( d\Sigma_\nu = n_\nu d\Sigma \) and taking coordinates such that \( n_\nu = (1, 0, 0, 0) \) and \( d\Sigma = d^3x \), we have

\[ -\frac{1}{8\pi G} \int G^{\mu0} \sqrt{-g} d^3x = \int T^{\mu0} \sqrt{-g} d^3x = \frac{m \dot{X}^\mu}{\sqrt{\dot{X}^\alpha \dot{X}_\alpha}}. \]  
(5)

Here we have used \( \int d\tau f(\tau)\delta(x^0 - X^0(\tau)) = \frac{f(\tau_c)}{X^0} \bigg|_{\tau_c} \), where \( \tau_c \) is the solution of the equation \( x^0 = X^0(\tau) \).

Because of the Bianchi identity, \( G^{\mu\nu;\nu} = 0 \) (implying \( T^{\mu\nu;\nu} = 0 \)), not all equations (3) are independent. The equations \( \frac{1}{8\pi G} G^{0\nu} + T^{0\nu} = 0 \) are constraints on initial data, and so are the equations \( \frac{1}{8\pi G} G^{0\nu} + T^{0\nu} = 0 \). We thus have four constraints

\[ \phi_\nu = \frac{1}{8\pi G} G^{0\nu} + T^{0\nu} = 0. \]  
(6)

Similarly, not all components of the metric \( g_{\mu\nu} \) are independent. The components \( g_{0\nu} \) can be chosen to be an artifact of a choice of coordinates and to have the role of Lagrange multipliers. The same holds for \( g^{0\nu} \). The variation of the action (2) with respect to \( g^{0\nu} \) (having the role of Lagrange multipliers) gives the constraints (6).

A linear superposition of the constraints (6) is the Hamiltonian:

\[ H = \int \alpha^\nu \phi_\nu \sqrt{-g} d^3x = \int \left( \frac{1}{8\pi G} G^{0\nu} + T^{0\nu} \right) g^{0\nu} \sqrt{-g} d^3x = 0, \]  
(7)

\(^2\)Such action makes sense if \( X^\mu(\tau) \) are not meant to be the coordinates of an exactly point particle, but coordinates of the center of mass of an extended object. Here we thus describe not a point particle, but an extended particle (object) coupled to gravity, and include into the description only a restricted set of the object’s variables, namely its center of mass coordinates.
where \( \alpha^\nu = g^{0\nu} \) are arbitrary functions of \( x^\mu \). We thus have

\[
\int \frac{1}{8\pi G} G_{0\nu} g^{0\nu} \sqrt{-g} \, d^3 x = -p_0, \tag{8}
\]

where

\[
p_0 = \int T_0^\nu \sqrt{-g} \, d\Sigma_\nu = \int T_0^0 \sqrt{-g} \, d^3 x = \frac{\partial L}{\partial \dot{X}^0} = -\frac{m g_{0\mu} \dot{X}^\mu}{\sqrt{g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta}}, \tag{9}
\]

The phase space form of the action \([2]\) is

\[
I[X^\mu, p_\mu, \pi^{ij}, q_{ij}, a, n, n^i] = \int d\tau \left[ \frac{\dot{X}^\mu}{\dot{X}^0} - \frac{\alpha}{2} \left( g_{\mu\nu} p_\mu p_\nu - m^2 \right) \right] \delta^4(x - X(\tau)) \, d^4 x
\]

\[
+ \int d^4 x \left( \pi^{ij} \dot{q}_{ij} - N \mathcal{H}^{ADM} + N^i \mathcal{H}^{ADM}_i \right), \tag{10}
\]

where \( \pi^{ij}, q_{ij}, i, j = 1, 2, 3, \) are the ADM phase space variables \([9, 10]\), and \( \mathcal{H}^{ADM}, \mathcal{H}^{ADM}_i \) the ADM expressions for the gravitation part of the constraints. Here \( \dot{X}^\mu \equiv dx^\mu/d\tau \) and \( \dot{q}_{ij} \equiv dq_{ij}/dt \). Later we will also have \( \dot{X}^i \equiv dx^i/d\tau \).

Because of the inserted \( \delta \)-function, the matter and the gravitational part of the action have both the same time parameter \( x^0 \equiv t \).

Performing the integration over \( \tau \), we obtain

\[
I = \int dt \, d^3 x \left[ \frac{\delta^3(x - X(t))}{X^0} \left( p_\mu \dot{X}^\mu - \frac{\alpha}{2} \left( g_{\mu\nu} p_\mu p_\nu - m^2 \right) \right) \right]_{\tau_c} + \pi^{ij} \dot{q}_{ij} - N \mathcal{H}^{ADM} + N^i \mathcal{H}^{ADM}_i, \tag{11}
\]

where \( \tau_c \) is the solution of the equation \( x^0 - X^0(\tau) = 0 \). Expressing the metric according to the ADM split \([9,10]\),

\[
g_{\mu\nu} = \begin{pmatrix} N^2 - N^i N_i & -N_i \\ -N_j & -q_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1/N^2 & -N^i/N^2 \\ -N^j/N^2 & N^i N^j/N^2 - q^{ij} \end{pmatrix}, \tag{12}
\]

where \( N = \sqrt{1/g^{00}} \) and \( N_i = -g_{0i}, i = 1, 2, 3, \) are the laps and shift functions, we have

\[
I = \int dt \, d^3 x \left[ \delta^3(x - X(t)) \left( p_0 + p_i \dot{X}^i - \frac{\alpha}{2X^0} \left( \frac{1}{N^2} (p_0 - N^i p_i)^2 - q^{ij} p_i p_j - m^2 \right) \right) \right]_{\tau_c}
\]

\[
+ \pi^{ij} \dot{q}_{ij} - N \mathcal{H}^{ADM} + N^i \mathcal{H}^{ADM}_i, \tag{13}
\]

Here we identify \( \alpha/\dot{X}^0 \) with a new Lagrange multiplier according to \( \alpha/\dot{X}^0 = \lambda \), because \( \dot{X}^0 \) is arbitrary and has no longer a formal role of a velocity as it had in the original action \([10]\).

The variation of this action with respect to the 3-metric \( q_{ij} \) gives the \((ij)\)-components of the Einstein equation in the ADM split. The variation with respect to other variables
\[
\delta p_0 : 1 = \frac{\alpha}{X^0} \frac{1}{N^2}(p_0 - N^i p_i) = \frac{\alpha}{X^0} p^0 \Rightarrow p^0 = \frac{\alpha}{X^0},
\]
(14)
\[
\delta p_i : \dot{X}^i = \frac{\alpha}{X^0} \frac{N^i}{N^2}(p_0 - N^j p_j) - q^{ij} p_j = p^i \frac{\alpha}{X^0} \Rightarrow p^i = \frac{\dot{X}^i}{\alpha} = \dot{X}^i p^0,
\]
(15)
\[
\delta \alpha : \frac{1}{N^2}(p_0 - N^i p_i)^2 - q^{ij} p_i p_j - m^2 = 0,
\]
(16)
\[
\Rightarrow p_0 - N^i p_i = \pm N \sqrt{q^{ij} p_i p_j + m^2}
\]
(17)
\[
\delta N : H^{ADM} = \frac{1}{N}(p_0 - N^i p_i) \delta^3(x - X) = \sqrt{q^{ij} p_i p_j + m^2} \delta^3(x - X)
\]
(18)
\[
\delta N^i : \frac{\alpha}{X^0} \frac{1}{N^2}(p_0 - N^j p_j) \delta^3(x - X) = \frac{\alpha}{X^0} p^0 \delta^3(x - X) = p_i \delta^3(x - X)
\]
(19)

Here we simplified the notation so that now \(\dot{X}^0 = \dot{X}^0|_{\tau_c}\).

The canonical momenta \(p_\mu = \partial L^{(1)}_m / (\partial \dot{X}^\mu)\), calculated from the action (10), whose matter part contains the parameter \(\tau\), are the same as the canonical momenta \(p_i = \partial L^{(1)}_m / \partial \dot{X}^i\) and the quantity \(p_0\) obtained from the action (13) in which \(\tau\) was integrated out and the time parameter was \(t\). This can be seen from the relations (14)–(17).

Equations (16),(18) and (19) imply the following constraints [5, 6]:

\[
\chi = \frac{1}{N^2}(p_0 - N^i p_i)^2 - q^{ij} p_i p_j - m^2 = 0,
\]
(20)
\[
\phi = H^{ADM} - \frac{1}{N}(p_0 - N^i p_i) \delta^3(x - X) = 0,
\]
(21)
\[
\phi_i = H^{ADM}_i - p_i \delta^3(x - X) = 0.
\]
(22)

The Hamiltonian is a superposition of those constraints:

\[
H = \int d^3x \left(\lambda \chi \delta^3(x - X) + N^i \phi + N^i \phi_i\right) = 0.
\]
(23)

Using (20)–(22), we obtain

\[
H = \int d^3x \left[\lambda \left(\frac{1}{N^2}(p_0 - N^i p_i)^2 - q^{ij} p_i p_j - m^2\right) \delta^3(x - X) + N^i H^{ADM}_i - p_0 \delta^3(x - X)\right] = 0.
\]
(24)

The terms with \(N^i p_i\) have canceled out in the latter expression. The same Hamiltonian we also obtain from the action (13) according to the expression

\[
H = \int d^3x \left(p_i \dot{X}^i \delta^3(x - X) + \pi^i \ddot{q}_{ij} - L\right).
\]
(25)
From Eq. (24), after using (20) we obtain

\[ \int d^3x \left( N \mathcal{H}_{ADM} + N^i \mathcal{H}_i^{ADM} \right) = p_0. \]  

(26)

This corresponds to Eq. (25), it is its ADM split analog.

The equations of motion obtained from the Hamiltonian (24) are:

\[ \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial X^i}, \quad \dot{X}^i = \{X^i, H\} = \frac{\partial H}{\partial X^i}, \quad (27) \]

\[ \dot{\pi}^{ij} = \{\pi^{ij}, H\} = -\frac{\delta H}{\delta q^{ij}}, \quad \dot{q}^{ij} = \{q^{ij}, H\} = \frac{\delta H}{\delta \pi^{ij}}, \quad (28) \]

where the usual Poisson brackets relations have been used. The quantity \( p_0 \) is given by Eq. (17), which comes from the constraint (22).

Besides Eq. (27), there is also the equation of motion for \( p_0 \), namely,

\[ \frac{\partial H}{\partial p_0} = -\frac{\partial L}{\partial p_0} = 0. \]  

(29)

Namely, the same equations (27), (28) also follow directly from the phase space action (13) according to the Euler-Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{X}^i} - \frac{\partial L}{\partial X^i} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{p}_i} - \frac{\partial L}{\partial p_i} = 0, \quad -\frac{\partial L}{\partial p_0} = 0, \quad (30) \]

\[ \frac{d}{dt} \frac{\delta L}{\delta \dot{q}^{ij}} - \frac{\delta L}{\delta q^{ij}} = 0, \quad \frac{d}{dt} \frac{\delta L}{\delta \dot{\pi}^{ij}} - \frac{\delta L}{\delta \pi^{ij}} = 0. \quad (31) \]

Equations (27) together with (29) are equivalent to the geodesic equation.

The same constraints (20)–(22) also follow directly from the action (10) which contains the “time” parameter \( \tau \) and the velocities \( \dot{X}^\mu \). Then alls quantities \( p_\mu = (p_0, p_i) \) have the role of canonical momenta derivable according to \( p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} \) from such \( \tau \)-depended Lagrangian. The Hamiltonian defined in terms of a superposition of those contraints is again given by Eq. (23) in which the parameter \( \lambda \) is replaced by another parameter, namely \( \alpha \). The equations of motion for \( X^\mu, p_\mu \) are

\[ \dot{p}_\mu = \{p_\mu, H\}, \quad \dot{X}^\mu = \{X^\mu, H\}. \]  

(32)

Explicitly this gives

\[ \dot{p}_\mu = -\frac{\partial H}{\partial X^\mu} = -\frac{\alpha}{2} \partial_\mu g^{\alpha\beta} p_\alpha p_\beta = \frac{\alpha}{2} \partial_\mu g_{\alpha\beta} p^\alpha p^\beta, \]  

(33)

\[ \dot{X}^\mu = \frac{\partial H}{\partial p_\mu} = \alpha g^\mu. \]  

(34)

From the latter equations, after using \( \alpha = \sqrt{\dot{X}^\mu \dot{X}_\mu/m} \), we obtain

\[ \frac{1}{\sqrt{X^2}} \frac{d}{d\tau} \left( \frac{\dot{X}_\mu}{\sqrt{X^2}} \right) - \frac{1}{2} \partial_\mu g_{\alpha\beta} \frac{\dot{X}^\alpha \dot{X}^\beta}{X^2} = 0, \]  

(35)
or equivalently,
\[
\frac{1}{\sqrt{X^2}} \frac{d}{d\tau} \left( \begin{array}{c} \dot{X}^\mu \\
\sqrt{X^2} \end{array} \right) + \Gamma^\mu_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta = 0,
\]
which is the equation of geodesic.

We see that regardless of whether we start (i) from the original phase space action (10), or (ii) from the action (13), we obtain the same Hamiltonian (24). In both cases the Hamiltonian is a superposition of the constraints (21)–20), obtained by varying the action with respect to the Lagrange multipliers \(\alpha, N\) and \(N^i\). In the second case, the Hamiltonian can also be obtained by using the expression (25).

In general, the total variation of an action \(I = \int d^4 x \mathcal{L}(\phi^a, \partial_\mu \phi^a)\) that includes a change \(\delta x^\mu\) of the boundary, is (see, e.g. [11])

\[
\bar{\delta} I = \int_R d^4 x \, \delta \mathcal{L} + \int_{R-R'} d^4 x \, \mathcal{L} = \int_R d^4 x \, \delta \mathcal{L} + \int_B d\Sigma_\mu \mathcal{L} \delta x^\mu.
\]

Assuming that the equations of motion are satisfied, we have

\[
\bar{\delta} I = \int_B d\Sigma_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \mathcal{L} \delta x^\mu \right) = \int d^4 x \, \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \mathcal{L} \delta x^\mu \right).
\]

Here \(\delta \phi^a = \delta \phi^a(x) = \delta \phi^a + \partial_\mu \phi^a \delta x^\mu\), we obtain

\[
\bar{\delta} I = \int_B d\Sigma_\mu \left[ \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \left( \mathcal{L} \delta x^\mu - \frac{\partial \mathcal{L}}{\partial \phi^a} \partial_\nu \phi^a \right) \delta x^\nu \right].
\]

Let us consider the action (13), identify \(\phi^a = (X^i, q_{ij})\), and take coordinates in which the surface element is \(d\Sigma_\mu = (d\Sigma_0, 0, 0, 0)\), \(d\Sigma_0 = d^3 x\). Then Eq. (39) gives

\[
\bar{\delta} I = \int_{t_1}^{t_3} d^3 x \left[ \frac{\partial \mathcal{L}}{\partial X^i} \delta X^i + \frac{\partial \mathcal{L}}{\partial q_{ij}} \delta q_{ij} + \left( \mathcal{L} - \frac{\partial \mathcal{L}}{\partial X^i} \dot{X}^i - \frac{\partial \mathcal{L}}{\partial q_{ij}} \dot{q}_{ij} \right) \delta t \right].
\]

The quantities \(\partial \mathcal{L}/\partial \dot{X}^i = p_i\) and \(\partial \mathcal{L}/\partial \dot{q}_{ij} = \pi_{ij}\) are, respectively, the generators of the infinitesimal translations \(X^i \to X^i + \delta X^i\) and \(q_{ij} \to q_{ij} + \delta q_{ij}\). The expression in front of \(\delta t\) is just the negative of the Hamiltonian (25).

If we consider the original phase space action (10) and perform the change \(\tau \to \tau + \delta \tau\), then we obtain

\[
\bar{\delta}' I = \int d\tau \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{X}^\mu} \dot{X}^\mu + L \delta \tau \right) = \int d\tau \left( p_\mu \delta X^\mu + (L - p_\mu \dot{X}^\mu) \delta \tau \right).
\]

Here \(p_\mu\) are the generators of the infinitesimal transformations \(X^\mu \to X^\mu + \delta X^\mu\), where \(\delta X^\mu = X'^\mu(\tau') - X^\mu(\tau) = \delta X^\mu + \dot{X}^\mu \delta \tau\). The quantity in front of \(\delta \tau\) is the generator of infinitesimal transformations \(\tau \to \tau + \delta \tau\); it is equal to \(\frac{1}{2}(g^{\mu\nu}p_\mu p_\nu - m^2)\). This is
the Hamiltonian for the relativistic particle and it gives the correct equations of motion (namely, that of a geodesic).

We see that by considering the total variation of the action \((10)\) that includes a change of \(\tau\), we find not only that \(p_i\) are the generators of the infinitesimal “translations” of \(X^\mu\), \(i = 1, 2, 3\), but also that \(p_0\) is the generator of the translations of \(X^0\). As \(p_i\) do not vanish, also \(p_0\) does not vanish; \(p_\mu = (p_0, p_i)\) are the canonical momenta, conjugated to the particle’s position variables \(X^\mu(\tau) = (X^0(\tau), X^i(\tau))\). Those variables are distinct objects than the spacetime coordinates \(x^\mu = (x^0, x^i)\), \(x^0 \equiv t\). Because the particle is embedded in spacetime, in the action \((10)\) there occurs \(\delta^4(x - X(\tau))\), which says precisely that, namely that the particles is described by a worldline \(x^\mu = X^\mu(\tau)\).

Let us now follow the approach by Rovelli \([3,4]\) and see what do we obtain if instead of the phase space action \((10)\) we use the action \((2)\), express the metric according to Eq. \((12)\), and fix the parameter so that \(\tau = x^0 \equiv t\). The action \((2)\) then reads

\[
I = m \int dt \sqrt{N^2 - (\dot{X} + N)^2} + \int dt d^3x \left( \pi^{ij} \dot{q}_{ij} - L_G + \mathcal{H}_{ADM} + N^i \mathcal{H}_{ADM}^i \right),
\]

where \((\dot{X} + N)^2 = q_{ij}(\ddot{X}^i + N^i)(\ddot{X}^j + N^j)\). We will also write \(p^2 = q_{ij}p^i p^j\). The particle momentum is

\[
p = -\frac{m(\dot{X} + N)}{\sqrt{N^2 - (\dot{X} + N)^2}},
\]

from which we have

\[
N^2 - (\dot{X} + N)^2 = \frac{N^2 m^2}{m^2 + p^2}, \quad \text{and} \quad \dot{X} + N = \frac{p N}{\sqrt{m^2 + p^2}}.
\]

The Hamiltonian is given by

\[
H = p\dot{X} - L_m + \int \left( \pi^{ij} \dot{q}_{ij} - L_G \right) d^3x,
= -Np - N\sqrt{m^2 + p^2} + \int d^3x \left( N\mathcal{H}_{ADM} + N^i \mathcal{H}_{ADM}^i \right).
\]

If we vary the action \((12)\) with respect to \(N\) and \(N^i\), we obtain the following constraints \([3,4]\):

\[
\mathcal{H}_{ADM} = \sqrt{m^2 + p^2} \delta^3(x - X),
\]

\[
\mathcal{H}_{ADM}^i = p_i \delta^3(x - X).
\]

The Hamiltonian \((15)\) is a superposition of those constraints with the coefficients \(N, N^i\), and is therefore equal to zero (in the weak sense, i.e., on the constraint surface in the phase space).

\(^3\)In the approach considered by Rovelli, also a term due to a clock variable on the particle’s world line was included.
But if we consider the form (2) of the action, before fixing $\tau$, then we obtain the momentum $p_0 = \partial L / \partial \dot{X}^0$, besides the momenta $p_i = \partial L / \partial \dot{X}^i$. Together all those momenta $p_\mu = (p_0, p_i)$ are constrained according to

$$g^{\mu\nu} p_\mu p_\nu - m^2 = \frac{1}{n^2} (p_0 - N^i p_i)^2 - q^{ij} p_i p_j - m^2 = 0. \quad (48)$$

Solving the latter equation for $p_0$, we obtain Eq. (17), i.e., $p_0 = N^i p_i + N \sqrt{q^{ij} p_i p_j + m^2}$. Using the latter relation in Eq. (45), we obtain

$$H_G \equiv \int d^3 \mathbf{x} \left( N \mathcal{H}^{ADM} + N^i \mathcal{H}_i^{ADM} \right) = p_0. \quad (49)$$

This means that the gravitational Hamiltonian is equal to the particle momentum $p_0$ which, as we have seen before, is the generator of the transformation $X^0 \to X^0 + \delta X^0$. Altogether, $p_\mu = (p_0, p_i)$ generate the transformations $X^\mu \to X^\mu + \delta X^\mu$, i.e., they shift the particle’s position in spacetime. Recall that Eq. (49) is consistent with Eq. (8) that we obtained directly from Einstein’s equations.

A different role has the $H$ of Eq. (45). It is the generator of the transformation $t \to t + \delta t$ which in the passive interpretation is just a change of a coordinate, a reparametrization. And because the action is invariant under reparametrizations of $x^\mu$, the corresponding generators, defined in Eq. (39), vanish, and so does the $H$ in Eq. (45).

According to the well known Einstein’s hole argument, spacetime points cannot be identified. A way to identify them is to fill spacetime with a reference fluid. If instead of a fluid we have particles, then spacetime points are identified on the worldlines of those particles. In the simplified model of a single particle, spacetime points are identified along the worldline of that particle. From Eq. (41) we read that $\delta x^\mu = (\delta x^0, \delta x^i)$ is a different sort of transformation than $\delta x^\mu = (\delta x^0, \delta x^i)$, $\delta x^0 \equiv \delta t$.

An alternative approach was considered by Struyve [7]. He started from the action (2) and cast it into such form that both terms, the particle’s and the gravitational, had the same “time” parameter $\tau$. For that purpose the integral $\int dt \, d^3 \mathbf{x} \sqrt{-g} R$ was transformed into $\int d\tau \, \dot{X}^0 \, d^3 \mathbf{x} \sqrt{-\tilde{g}} \tilde{R}$, where $\tilde{R}$ and $\tilde{g}$ were properly adjusted $R$ and $g$ that took into account the relation $x^0 = X^0(\tau)$. Because of the occurrence of $\dot{X}^0$ in the gravity part of the action, the null-component of the canonical momentum was not $p_0 = m \dot{X}_0 / \sqrt{\dot{X}^2}$, but $p_0 = m \dot{X}_0 / \sqrt{\dot{X}^2} + \int d^3 \mathbf{x} \sqrt{-\tilde{g}} \tilde{R}$. Because of the extra term in $p_0$, the constraint no longer had the simple form (48). Consequently, instead of Eq. (49) in which $p_0 \neq 0$, a different equation was obtained, namely, $H_G = p_0$, where $p_0$ turned out to be zero (i.e., it vanished weakly). It was then concluded that, because of the constraint $p_0 \approx 0$, a particle cannot give rise to time in the quantum version of the theory and that the notorious problem of time still existed. Struyve also observed that there exists a canonical transformation that relates the approach based on the action (2) to the approach based on his modified action,
and that upon quantization, those two approaches are related by the corresponding unitary transformations.

## 3 Quantization

We have arrived, following different paths, to the equation (49), i.e.,

$$H = H_G - p_0 = 0,$$

(50)

where $p_0$ is related to $p_i$ according to Eq. (17), which comes from the mass shell constraint (17), and where $H_G = \int d^3x \left( N \mathcal{H}_{ADM} + N^i \mathcal{H}^i_{ADM} \right)$ contains the canonical momenta conjugated to the 3-metric $q_{ij}$.

Upon quantization, the particle coordinates $X^\mu$ and momenta $p_\mu$ become operators satisfying

$$[\hat{X}^\mu, \hat{X}^\nu] = 0, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{X}^\mu, \hat{p}_\nu] = i\delta_\mu^\nu.$$

(51)

Similarly, the gravity variables become operators satisfying

$$[\hat{q}_{ij}, \hat{q}_{mn}] = 0, \quad [\hat{\pi}_{ij}, \hat{\pi}_{mn}] = 0, \quad [\hat{q}_{ij}, \hat{\pi}_{mn}] = i\delta_{ij}^{mn}.$$

(52)

In the Schrödinger representation in which $X^\mu$ and $q_{ij}$ are diagonal, Eqs. (51) and (52) are satisfied by $\hat{X}_\mu = X_\mu$, $\hat{q}_{ij} = q_{ij}$, $\hat{p}_\mu = -i\partial/\partial X_\mu$ and $\hat{\pi}_{ij} = -i\delta/\partial q_{ij}$.

The constraints (20)–(22) become operator equations acting on the state vector, which can be represented as a function of $X^\mu$ and a functional of $q_{ij}(x)$, namely $\Psi[X^\mu, q_{ij}(x)]$.

In order to quantize Eq. (50), we take the gauge $N = 1$, $N^i = 0$, and so we obtain

$$\int d^3x \left( \frac{-1}{\kappa} G_{ijkl} \pi^{ij} \pi^{kl} + \kappa \sqrt{q} R^{(3)} \right) = p_0,$$

(53)

where $\kappa = 1/(16\pi G)$ and $G_{ijkl} \pi^{ij} \pi^{kl}$ Wheeler-DeWitt metric. The latter equation can be written in the following compact form:

$$\frac{1}{\kappa} G^{ab}(x)b(x') \pi_{a}(x) \pi_{b}(x') + V[q^a(x)] = -p_0,$$

(54)

where $\pi_a(x) \equiv \pi^{ij}(x)$, $G^{ab}(x)b(x') = G_{ijkl}(x)\delta^3(x - x')$, $q^a(x) \equiv q_{ij}(x)$, and $V[q^a(x)] = -\kappa \sqrt{q} R^{(3)}$. Upon quantization, $\pi_a(x)$ become the operators $\hat{\pi}_a(x) = -i\delta/\partial q^a(x) \equiv -i\partial_a(x)$, and $p_0$ becomes $\hat{p}_0 = -i\partial/\partial T$, where $T \equiv X^0$.

The notorious ordering ambiguity can be avoided if we proceed as follows. First, let me illustrate the procedure for the constraint (20). It can be written in the following form:

$$\gamma^\mu p_\mu \gamma^\nu p_\nu - m^2 = 0.$$

(55)
Here \( \gamma^\mu \) are the generators of the Clifford algebra, satisfying
\[
\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu}.
\] (56)

They have the role of the basis vectors \([13,14]\) (see also \([15\text{–}17]\)) in a curved spacetime \(M\).

The quantum version of Eq. (55) is
\[
(\gamma^\mu \hat{p}_\mu \gamma^\nu \hat{p}_\nu - m^2) \Psi = 0,
\] (57)
where \(\Psi\) is a scalar wave function. Using \(\hat{p}_\mu = -i\partial_\mu\) and \(\partial_\mu \gamma^\nu = \Gamma^{\nu}_{\mu\rho} \gamma^\rho\), where \(\Gamma^{\nu}_{\mu\rho}\) is the connection in \(M\), equation (57) becomes
\[
(-\Gamma^{\nu}_{\mu\rho} \gamma^\mu \gamma^\rho \partial_\nu - \gamma^\mu \gamma^\rho \partial_\mu \partial_\nu - m^2) \Psi = 0,
\] (58)
i.e.,
\[
(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \Gamma^{\nu}_{\mu\rho} g^{\mu\rho} \partial_\nu + m^2) \Psi = (D_\mu D^\mu + m^2) \Psi = 0.
\] (59)
Here \(D_\mu\) is the covariant derivative of the tensor calculus; acting on a vector components, it gives \(Da_\mu = \partial_\mu a_\nu + \Gamma^{\nu}_{\mu\rho} a^\rho\).

There is no ordering ambiguity in Eq. (57), because \(\gamma^\mu \hat{p}_\mu \gamma^\nu \hat{p}_\nu\) is the product of two vector momentum operators \([12]\). \(\hat{p}_\mu = \gamma^\mu \hat{p}_\mu\) and is invariant under general coordinate transformations. Using a different product, for instance, \(\hat{p}_\mu \gamma^\mu \gamma^\nu \hat{p}_\nu\) would make no sense, because such an object is not invariant and not a product of two vector operators. In Ref. \([18]\) it was shown what happens if \(\Psi\) in Eq. (57) is a spinor field, expanded in term of the spinor basis \(\xi_\alpha\), according to \(\Psi = \psi_\alpha \xi_\alpha\). Then, instead of (59) one obtains
\[
(D_\mu D^\mu + m^2) \psi_\delta + \frac{1}{2} [\gamma^\mu, \gamma^\nu]_\alpha R^\alpha_{\mu\nu\beta} \psi_\beta = 0,
\] (60)
where \(D_\mu\) contains also the spin connection, determined by the relation \(\partial_\mu \xi_\alpha = \Gamma^{\beta}_{\mu\alpha} \xi_\beta\). Analogously we can find the corresponding equation for a vector field.

In a similar way also the ordering ambiguity in Eq. (54) can be avoided by introducing the superspace analog of \(\gamma^\mu\) and rewrite Eq. (54) in the form
\[
\frac{1}{\kappa} G^{a(x)} \pi_a(x) G^{b(x')} \pi_b(x') + V[\pi] = -p_0,
\] (61)
where \(G^{a(x)}\) are the generators of the Clifford algebra in the superspace \(S\), satisfying
\[
G^{a(x)} \cdot G^{b(x')} = \frac{1}{2} \left(G^{a(x)} G^{b(x')} + G^{b(x')} G^{a(x)}\right) = G^{a(x)b(x')}.
\] (62)

\(^4\)Here the symbol \(\partial_\mu\) denotes a generic derivative that can act on any Clifford algebra-valued field. For instance, if acting on a scalar field it behaves as the partial derivative and if acting on basis vectors \(\gamma^\nu\) it determines the connection. More details and a justification why the same symbol \(\partial_\mu\) can be used in both cases is provided in Ref. \([17]\).
The quantum version of Eq. (61) is
\[
\left( \frac{1}{\kappa} C^a(x) \pi_a(x) G^{b(x')} \pi_b(x') + V[q] \right) \Psi = i \frac{\partial \Psi}{\partial T}, \quad (63)
\]
where \( \Psi = \Psi[T, X^i, q^a(x)] \). In the latter equation \( \pi_a(x) = -i \partial_b(x) \) acts on \( G^{b(x')} \). Analogously as in the finite dimensional case, the derivative \( \partial_a(x) \) acting on \( G^{b(x')} \) gives the connection according to
\[
\partial_a(x) G^{b(x')} = \Gamma^{(a)}_{b(c)} G^{c(x')}.
\]
Equation (63) then becomes
\[
\left( \frac{1}{\kappa} C^a(x) G^{b(x)} \partial_a(x) \pi_b(x') + \Gamma^{b(x')}_{a(c)} C^{a(c(x'))} \partial_b(x') + V[q] \right) \Psi = \left( D_a(x) D^a(x) + V[q] \right) = i \frac{\partial \Psi}{\partial T}. \quad (65)
\]
The connection is given by
\[
\Gamma^{c(x'')}_{a(b)} = \frac{1}{2} C^{c(x'')}_{a(b)} \left( G_a(x) d(x''), b(x') + G_d(x''), b(x''), a(x) - G_a(x) b(x'), d(x'') \right), \quad (66)
\]
where the comma denotes the functional derivative. Using the established techniques for the superspace calculations (see, e.g., [19]), the connection (66) can be calculated for the Wheeler-DeWitt metric by using
\[
\partial_{c(x'')} G^{a(x) b(x')} = \frac{\delta}{\delta q_{mn}} G_{ij kl} \delta(x - x') \text{, and } \frac{\delta}{\delta q_{mn}} g_{ij}(x') = \delta^{(m)}_{(i} \delta^{(n)}_{j)}. \quad (67)
\]
Because the ordering issues regarding the Wheeler-DeWitt equation are not the main topics of this paper, they will be discussed in more detail elsewhere.

4 Conclusion

We have considered the relativistic particle coupled to gravity and analysed the constraints satisfied by such system. The constraints follow directly from the action (2) by varying it with respect to the non dynamical components of the metric \( g^\mu \), namely, \( g^0\mu \), which gives the \((0\mu)\)-components of Einstein’s equations: \( \phi_\mu = \frac{1}{8\pi G} G_{0\mu} + T_{0\mu} = 0 \). The Hamiltonian is a linear superposition of those constraints, \( H = \int \sqrt{-g} d^3 x \phi_\mu \sqrt{-g} d^3 x = H_g + H_m = 0 \), where \( H_m = \int T_{0\mu} g^{0\mu} \sqrt{-g} d^3 x = p_0 \) is the particle momentum. If we perform the ADM split of the action (2) or its phase space form (10), then the non dynamical variables are the lapse and shift functions, \( N \) and \( N^i \). The constraints \( \phi_i \) come from varying the action with respect to \( N \) and \( N^i \). The time variable in the matter and the gravity part of the action are the same, namely \( t \equiv x^0 \). In addition, the matter part \( I_m \) contains the worldline parameter \( \tau \) and the term \( \delta^4(x - X(\tau)) \) which tells that the worldline is embedded in spacetime and thus satisfies the parametric equation \( x^\mu = X^\mu(\tau) \). Because
\( I_m \) is invariant under reparametrizations of \( \tau \), the particle momenta \( p_\mu = \partial L / \partial \dot{X}^\mu \) satisfy the mass shell constraint, \( \chi = p_\mu p^\mu - m^2 = 0 \). The Hamiltonian, which is a superposition of the constraints \( \chi, \phi \) and \( \phi_i \) gives the correct equations of motion for all the dynamical variables by using the ordinary Poisson brackets. The matter part of the Hamiltonian is \( p_0 \), as it should be, because—as we have seen—it also comes directly from the Einstein equations.

Upon quantization, \( p_0 \) becomes the operator \( \hat{p}_0 = i \partial / \partial T \), where \( T \equiv X^0 \) donotes the time coordinate of the particle. Altogether, \( X_\mu, \mu = 0, 1, 2, 3 \), denote position of the particle in spacetime. The Hamilton constraint \( H = H_g + H_m = 0 \), i.e., \( H_g = -p_0 \), becomes the Schrödinger-like equation \( H_g \Psi = i \partial \psi / \partial T \), where \( \Psi = \Psi[T, X^i, q_{ij}] \). The time and hence spacetime in this approach does not disappear upon quantization.

Spacetime in the quantized theory disappears if no matter is present. According to Einstein’s hole argument, spacetime points cannot be identified. They can be identified in the presence of a reference fluid. If there is no reference fluid and instead there are particles, then spacetime points can be identified on the worldlines of the particles. In this paper we have considered a simplified model with only one particle present, and found that its time coordinate \( T \) remains in the quantized theory. In the usual approaches in which matter is given by some fields, such as a scalar or spinor field, spacetime points cannot be directly identified as in the case of particles and one faces the notorious problem of time, namely, how those fields could give rise to time. There is a lot of discussion of how to resolve it, but so far no generally accepted resolution has been found. In the model in which gravity is coupled to particles, the problem of time does not exist.

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