How long does it take to form the Andreev quasiparticles?

R. Taranko and T. Domaniński

Institute of Physics, M. Curie Skłodowska University, 20-031 Lublin, Poland

(Dated: May 25, 2017)

We study transient effects in a setup, where the quantum dot (QD) is abruptly sandwiched between the metallic and superconducting leads. Focusing on the proximity-induced electron pairing, manifested by the in-gap bound states, we determine characteristic time-scale needed for these quasiparticles to develop. In particular, we derive analytic expressions for (i) charge occupancy of the QD, (ii) amplitude of the induced electron pairing, and (iii) the transient currents under equilibrium and nonequilibrium conditions. We also investigate the correlation effects within the Hartree-Fock-Bogoliubov approximation, revealing a competition between the Coulomb interactions and electron pairing.

I. INTRODUCTION

When a quantum impurity is attached to some superconducting bulk material it absorbs the Cooper pairs and develops the bound quasiparticle states in the sub-gap region $|\omega| \leq \Delta$ of its spectrum ($\Delta$ is the energy gap of superconducting reservoir) [1]. These Andreev (or Yu-Shiba-Rusinov) states have been observed in various STM studies for impurities deposited on superconducting substrates [2] and in the tunneling experiments via heterostructures comprising the quantum dots arranged in Josephson [3], Andreev [4] and/or more complex multi-terminal configurations [5]. Since such tunneling measurements can be nowadays done with state-of-art precision, one may also probe the time-resolved properties. We address this issue here, indicating some feasible methods that could determine the characteristic time-scales of these in-gap quasiparticles.

Time-resolved techniques become more and more popular, because they provide an insight into the many-body effects both, in macroscopic and nanoscopic systems. For instance, the pump-and-probe experiments [6] and the time-resolved ARPES [7] helped to determine the lifetime of the Bogoliubov quasiparticles in the high temperature superconductors. In nanoscopic systems, the transient effects have been investigated so far mainly for the quantum dots hybridized with the conducting (metallic) leads. For instance it has been shown that the Abrikosov-Suhl peak (appearing at the Fermi energy) develops on a period sensitive to the energies of in-gap quasiparticles.

The paper is organized as follows. In Sec. II we introduce the microscopic model and discuss the method accounting for the time-dependent phenomena. Sec. III presents a few analytical results obtained for the correlated quantum dot, such as: (i) charge occupancy, (ii) complex order parameter, and (iii) charge current for the unbiased and biased heterojunction. Next, in Sec. IV we discuss the correlation effects. In Sec. V we summarize our results and present some quantitative evaluations.

II. MICROSCOPIC MODEL

For description of the N-QD-S heterostructure we use the single impurity Anderson Hamiltonian

$$\hat{H} = \sum_{\sigma} \varepsilon_{\sigma} \hat{d}_{\sigma}^{\dagger} \hat{d}_{\sigma} + U \hat{n}_{\uparrow} \hat{n}_{\downarrow} + \sum_{\beta} \left( \hat{H}_{\beta} + \hat{V}_{\beta-QD} \right)$$  (1)

where $\beta$ refers to the normal ($N$) and superconducting ($S$) electrodes, respectively. As usually $\hat{d}_{\sigma}$ ($\hat{d}_{\sigma}^{\dagger}$) is the annihilation (creation) operator for the quantum dot (QD) electron with spin $\sigma$ and energy $\varepsilon_{\sigma}$. Potential of the Coulomb repulsion between the opposite spin electrons is denoted by $U$. We treat the external metallic lead as free fermion gas $\hat{H}_N = \sum_{k,\sigma} \varepsilon_{k,\sigma} \hat{c}_{k,\sigma}^{\dagger} \hat{c}_{k,\sigma}$ and describe the isotropic superconductor by the BCS model $\hat{H}_S = \sum_{q,\sigma} \varepsilon_{q,\sigma} \hat{c}_{q,\sigma}^{\dagger} \hat{c}_{q,\sigma} - \sum_{q} \Delta \left( \hat{c}_{q,\uparrow}^{\dagger} \hat{c}_{-q,\downarrow}^{\dagger} + \hat{c}_{-q,\downarrow} \hat{c}_{q,\uparrow} \right)$, where
\( \varepsilon_{k(q)} \) is the energy measured from the chemical potential \( \mu_{N(S)} \) and \( \Delta \) denotes the superconducting energy gap. Hybridization between the QD electrons and the metallic lead is given by \( \tilde{V}_{N-QD} = \sum_{k,\sigma} (V_k \hat{d}_{\sigma} \hat{c}_{k\sigma} + \text{h.c.}) \) and \( \tilde{V}_{S-QD} \) can be expressed by interchanging \( k \leftrightarrow q \).

Since our study refers to the subgap quasiparticle states, we assume the constant couplings \( \Gamma_{N(S)} = 2\pi \sum_{k(q)} |V_{k(q)}|^2 \delta(\omega - \varepsilon_{k(q)}) \). For the deep subgap regime \(|\omega| \ll \Delta \) (so-called, superconducting atomic limit) the coupling \( \Gamma_{S}/2 \) can be regarded as a qualitative measure of the induced pairing potential, whereas \( \Gamma_{N} \) controls the inverse life-time of the in-gap quasiparticles. As we shall see, both these couplings play important (but different) role in the transient phenomena.

### A. Sudden switching

We assume that all three constituents of the N-QD-S heterostructure are disconnected from each other until \( t \leq 0 \). Both external (N, S) reservoirs are suddenly coupled to the quantum dot

\[
V_{k(q)}(t) = \begin{cases} 
0 & \text{for } t \leq 0 \\ 
V_{k(q)} & \text{for } t > 0,
\end{cases}
\]

inducing the transient effects. The time-dependence of arbitrary physical observable \( \hat{O} \) is governed by the Heisenberg equation of motion \( i\hbar \frac{d}{dt} \hat{O} = \left[ \hat{O}, \hat{H} \right] \).

In what follows, we are going to determine expectation values of the QD occupation \( \langle \hat{n}_\sigma(t) \rangle \), the induced order parameter \( \langle \hat{d}_\sigma(t)\hat{d}_\sigma(t) \rangle \), and the transient charge currents flowing between the QD and external electrodes. These quantities are subject to the specific initial conditions at \( t = 0 \), which turn out to be meaningful.

### B. Laplace transforms

The strategy of our analysis is as follows. First, we formulate the differential equations of motion for the QD annihilation \( \hat{d}_\sigma(t) \) and creation \( \hat{d}^\dagger_\sigma(t) \) operators (and similar ones for the mobile electrons). In the next step we solve them analytically (for the uncorrelated quantum dot) introducing the Laplace transformations

\[
d_\sigma(s) = \int_0^\infty e^{-st}d_\sigma(t)dt \equiv \mathcal{L}\{d_\sigma(t)\}(s). \quad (3)
\]

Finally, we determine the time-dependent quantities for the statistically averaged observables using the inverse Laplace transforms. For example, the QD occupancy \( n_\sigma(t) \equiv \langle \hat{n}_\sigma(t) \rangle \) can be expressed by

\[
n_\sigma(t) = \mathcal{L}^{-1}\left\{ \mathcal{L}^{-1}\left\{ \hat{d}^\dagger_\sigma(s) \right\}(t) \mathcal{L}^{-1}\left\{ \hat{d}_\sigma(s) \right\}(t) \right\}, \quad (4)
\]

where \( \mathcal{L}^{-1}\{\hat{d}^\dagger_\sigma(s)\}(t) \) denotes the inverse Laplace transform of \( \hat{d}^\dagger_\sigma(s) \). We present the explicit expressions for \( \hat{d}_\sigma(t) \) and \( \hat{c}_\sigma(t) \) in the Appendix A.

In what follows we use the wide-band limit approximation (\( \Gamma_\beta = \text{const} \)) and assume \( e = \hbar = k_B = 1 \), so that energies, currents and time are expressed in units of \( \Gamma_S \), \( e\Gamma_S/\hbar \), \( h/\Gamma_S \), respectively. We also set the chemical potential \( \mu_S = 0 \) as the convenient reference energy.

### III. UNCORRELATED QD CASE

Let us address the transient effects appearing in a sub-gap spectrum of the uncorrelated \((U = 0)\) quantum dot, when the analytical expressions can be derived.

#### A. Time-dependent charge

We can determine the time-dependent QD occupancy \( n_\sigma(t) \) driven by the abrupt coupling to both external leads 3, using Eq. 4 and the Laplace transforms presented in Appendix A. In the superconducting atomic limit, \(|\Delta| \to \infty \), it takes the following form

\[
n_\sigma(t) = \mathcal{L}^{-1}\left\{ \mathcal{L}^{-1}\left\{ \hat{d}^\dagger_\sigma(s) \right\}(t) \mathcal{L}^{-1}\left\{ \hat{d}_\sigma(s) \right\}(t) \right\} n_\sigma(0) + \mathcal{L}^{-1}\left\{ \mathcal{L}^{-1}\left\{ \hat{d}^\dagger_\sigma(s) \right\}(t) \mathcal{L}^{-1}\left\{ \hat{c}_\sigma(s) \right\}(t) \right\} [1 - n_\sigma(0)], \quad (5)
\]

where \( f_N(\omega) = [1 + \exp((\omega - \mu_N)/T)]^{-1} \) is the normal electrode Fermi-Dirac distribution function,

\[
s_{1,2} = \frac{1}{2} \left[ i(\varepsilon_\uparrow - \varepsilon_\downarrow) - \Gamma_N \pm i\sqrt{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2} \right], \quad (6)
\]

\[
s_{3,4} = \frac{1}{2} \left[ i(\varepsilon_\uparrow - \varepsilon_\downarrow) - \Gamma_N \pm i\sqrt{\Gamma_N^2 + \Gamma_S^2} \right], \quad (7)
\]

Finally, we determine the time-dependent quantities for the statistically averaged observables using the inverse Laplace transforms. For example, the QD occupancy \( n_\sigma(t) \equiv \langle \hat{n}_\sigma(t) \rangle \) can be expressed by

\[
n_\sigma(t) = \mathcal{L}^{-1}\left\{ \mathcal{L}^{-1}\left\{ \hat{d}^\dagger_\sigma(s) \right\}(t) \mathcal{L}^{-1}\left\{ \hat{d}_\sigma(s) \right\}(t) \right\}, \quad (4)
\]

where \( \mathcal{L}^{-1}\{\hat{d}^\dagger_\sigma(s)\}(t) \) denotes the inverse Laplace transform of \( \hat{d}^\dagger_\sigma(s) \). We present the explicit expressions for \( \hat{d}_\sigma(t) \) and \( \hat{c}_\sigma(t) \) in the Appendix A.

In what follows we use the wide-band limit approximation (\( \Gamma_\beta = \text{const} \)) and assume \( e = \hbar = k_B = 1 \), so that energies, currents and time are expressed in units of \( \Gamma_S \), \( e\Gamma_S/\hbar \), \( h/\Gamma_S \), respectively. We also set the chemical potential \( \mu_S = 0 \) as the convenient reference energy.
we obtain 

and for $n_{-\sigma}(t)$ the auxiliary parameters $(s_1, s_2, s_3, s_4)$ should be 
replaced by $(s_3, s_4, s_1, s_2)$. 

Expressions appearing in Eq. (8) are rather lengthy, but in special 
cases they considerably simplify. One of such possibilities occurs for the QD coupled 
only to the superconducting lead ($\Gamma_N = 0$). QD occupancy is then 
characterized by non-vanishing quantum oscillations

$$n_{\sigma}(t) = \frac{\Gamma_S^2}{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2} \sin^2 \left( \frac{\sqrt{\delta}}{2} t \right) [1 - n_{-\sigma}(0)] + \cos^2 \left( \frac{\sqrt{\delta}}{2} t \right) + \frac{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2}{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2} \sin^2 \left( \frac{\sqrt{\delta}}{2} t \right) n_{\sigma}(0), \quad (8)$$

where $\delta = (\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2$. For $\varepsilon_\uparrow + \varepsilon_\downarrow = 0$ this equation 
(8) simplifies to

$$n_{\sigma}(t) = \cos^2 \left( \frac{\Gamma_{st}}{2} \right) n_{\sigma}(0) + \sin^2 \left( \frac{\Gamma_{st}}{2} \right) [1 - n_{-\sigma}(0)] \quad (9)$$

implying the oscillation period $T = 2\pi/\Gamma_S$ (unless $n_{\sigma}(0) = 1, n_{-\sigma}(0) = 0$ when QD occupancy is preserved).

When the QD is coupled to both normal and superconducting leads ($\Gamma_S \neq 0, \Gamma_N \neq 0$) the first two terms in Eq. (8) give the same result as that given in Eq. (8) for $\Gamma_N = 0$, but with the additional factor $\exp(-\Gamma_N t)$. These terms disappear at $t \to \infty$, and the asymptotic value of QD occupancy is expressed only by the last terms in Eq. (8) which depend on the normal lead electron spectrum. Fig. 1 presents $n_{\sigma}(t)$ obtained in absence of external voltage for several values of $\Gamma_N$, assuming $\varepsilon_\uparrow + \varepsilon_\downarrow = 0$ and $n_\uparrow(0) = 0 = n_\downarrow(0)$. Such oscillating behavior, with a period $2\pi/\Gamma_S$, is weighted by the factor $\sin^2 \left( \frac{\sqrt{\delta}}{2} t \right)$ and the envelope function $\exp(-\Gamma_N t)$. For $\varepsilon_\uparrow + \varepsilon_\downarrow \neq 0$ the quantum oscillations have a period $T = 2\pi/\sqrt{(\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2}$ in agreement with the predictions by J. Gramich et al [24]. Amplitude of these oscillations $\exp(-\Gamma_N t) \Gamma_S^2 / (\varepsilon_\uparrow + \varepsilon_\downarrow)^2 + \Gamma_S^2$ indicates the crucial role of metallic lead for the relaxation processes.

Fig. 2 shows the QD occupancies for several initial conditions obtained for $\mu_N = \mu_S$, assuming $\varepsilon_\sigma = 0$. For $n_\uparrow(0) = n_\downarrow(0)$ the quantum oscillations are damped (see Fig. 1 and this effect is caused by the third and 
fourth terms on the right h.s. of Eq. (8) originating from the coupling $\Gamma_N$ to the normal lead. On the other hand, for the initial condition $n_{\sigma}(0) = 1 = n_{-\sigma}(0) = 0$, the transient effects look differently. This stems from the fact that electron pairing is inefficient, because it can affect 
only the empty or doubly occupied configurations and such exponential decrease (or increase) of the QD occupancy is due to the coupling with the normal electrode. Let us also remark, that for $\Gamma_S = 0$ Eq. (8) simplifies to the standard formula obtained by the non-equilibrium Green’s function method [25].

B. Development of the proximity effect

In this section we calculate the time-dependent order parameter $\langle \hat{d}_{\sigma}^\dagger \hat{d}_{\sigma} \rangle$ for $|\Delta| \to \infty$ limit. Using the expressions for QD operators given in Appendix A, we obtain 

$$\chi(t) = \langle \hat{d}_{\uparrow}^\dagger \hat{d}_{\downarrow} \rangle = \left[ \frac{\Gamma_S}{2} \right]^{-1} \left\{ \frac{s + i\varepsilon_\downarrow + \Gamma_N/2}{(s-s_1)(s-s_2)} \right\} (t) \chi_{\uparrow}(t) = \left\{ \frac{1}{(s-s_3)(s-s_4)} \right\} (t) n_\uparrow(0)$$
The imaginary part Im of the order parameter. On the other hand, the real part (shown in Fig. 4) is intimately related with the transient current flowing between the proximitized QD and the superconducting lead in analogy to the Josephson junction comprising two superconducting pieces, differing in phase of the order parameter. We notice that a period of the dumped quantum oscillations depends on the excitation energy between the subgap Andreev quasiparticles \[ \int e^{-\Gamma_N t} \frac{\Gamma_S}{4\pi} \text{Im} \left[ \int d\omega \left[ 1 - f_N(\varepsilon) \right] \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i\omega)} \right) \left( t - \frac{1}{(s - s_3)(s - s_4)(s - i\omega)} \right) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_3)(s - s_4)(s - i\omega)} \right) (t) \right) \right] \] simplifies to

\[ \chi(t) = [1 - n_{\downarrow}(0) - n_{\uparrow}(0)] e^{-\Gamma_N t} \left( \varepsilon_{\uparrow} + \varepsilon_{\downarrow} \right) \left[ 1 - \cos \left( \sqrt{\delta t} \right) - i\sqrt{\delta t} \sin \left( \sqrt{\delta t} \right) \right] / 2 \delta + \frac{\Gamma_S \Gamma_N}{4\pi} \text{Im} \left[ \int d\omega \left[ 1 - f_N(\varepsilon) \right] \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)(s - i\omega)} \right) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_3)(s - s_4)(s - s_5)} \right) (t) \right] \] obtained for the same model parameters as in Fig. 4, which originates from both electrodes.

Assuming the initial QD occupancy \[ n_{\sigma}(0) = 0 \] we show in Fig. 3 the imaginary part \[ \text{Im} \chi(t) = -e^{-\Gamma_N t} \Gamma_S \left[ (\varepsilon_{\uparrow} + \varepsilon_{\downarrow})^2 + \Gamma_S^2 \right]^{1/2} \sin \left( \sqrt{\varepsilon_{\uparrow} + \varepsilon_{\downarrow}} \right) / 2 \Gamma_S \] obtained for the same model parameters as in Fig. 4. We notice, that a period \( T \) of the dumped quantum oscillations depends on the excitation energy between the subgap Andreev quasiparticles via \[ T = 2\pi / \sqrt{(\varepsilon_{\uparrow} + \varepsilon_{\downarrow})^2 + \Gamma_S^2} \]. For \( \Gamma_N = 0 \) these oscillations are intimately related with the transient current \[ j_{\sigma}(t) \] flowing between the proximitized QD and the superconducting lead in analogy to the Josephson junction comprising two superconducting pieces, differing in phase of the order parameter. On the other hand, the real part (shown in Fig. 4) evolves monotonously to some asymptotic value except of one particular case \( \Gamma_N = 0 \) when for \( \varepsilon_{\uparrow} + \varepsilon_{\downarrow} = 0 \) the real part of \( \chi(t) \) vanishes.

In particular, for \( \mu_N = 0 \), Eq. (10) simplifies to

\[ - \frac{\Gamma_N}{2\pi} \int d\omega f_N(\omega) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i\omega)} \right) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_3)(s - s_4)(s - i\omega)} \right) (t) \] \[ - \frac{\Gamma_N}{2\pi} \int d\omega \left[ 1 - f_N(\omega) \right] \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i\omega)} \right) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_3)(s - s_4)(s - i\omega)} \right) (t) \] \[ \text{Re} \chi(t) = \left\langle \hat{d}_{\uparrow}(t) \hat{d}_{\downarrow}(t) \right\rangle \] obtained for \( \Gamma_S = 1 \) and other parameters like in Fig. 4.

C. Transient currents for unbiased junction

In this section we calculate the currents \( j_{N\sigma}(t) \) and \( j_{S\sigma}(t) \) flowing from QD to the normal and superconducting leads, respectively. For instance \( j_{N\sigma}(t) = \left\langle \frac{dN_{\sigma}(t)}{dt} \right\rangle \), where \( \dot{N}_{\sigma}(t) \) counts the total number of electrons in \( N \) electrode, simplifies to the standard formula \[ j_{N\sigma}(t) = 2 \int d\omega f_N(\omega) e^{-i\omega t} \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)(s - s_5)} \right) \chi^{-1} \left( \frac{s + i\varepsilon + \Gamma_N/2}{(s - s_3)(s - s_4)(s - s_5)} \right) (\omega) \] \[ \text{Re} \chi(t) = \left\langle \hat{d}_{\uparrow}(t) \hat{d}_{\downarrow}(t) \right\rangle \] obtained for \( \Gamma_S = 1 \) and other parameters like in Fig. 4.

Using

\[ \hat{\sigma}_{\sigma}(t) = \hat{\sigma}_{\sigma}(0) e^{-i \int_0^t \epsilon_{\sigma}(t')dt'} - i \int_0^t dt' V_{\text{K}} e^{-i \int_0^{t'} \epsilon_{\sigma}(t'')dt''} \hat{\sigma}_{\sigma}(t'') \] \[ \text{Re} \chi(t) = \left\langle \hat{d}_{\uparrow}(t) \hat{d}_{\downarrow}(t) \right\rangle \] and assuming the static energies \( \epsilon_{\sigma}(t) = \epsilon_{\sigma} \) we obtain

\[ j_{N\sigma}(t) = -\Gamma_N n_{\sigma}(t) + \frac{\Gamma_N}{\pi} \text{Re} \left( \int_{-\infty}^{\infty} d\omega f_N(\omega) e^{-i\omega t} \right) \]
we present transient behavior of the current \( j_{N-\sigma}(t) \) the parameters \((s_1, s_2, s_3, s_4)\) should be replaced by \((s_3, s_4, s_1, s_2)\), respectively. In particular, for

\[
\Gamma_N \neq 0 \quad \text{we arrive at}
\]

where \( \omega_{\pm} = \frac{\Gamma_{\delta}}{2} \pm \omega \).

In figure 5 we present transient behavior of the current \( j_{N\sigma}(t) \) induced by an abrupt coupling of the QD to external electrodes \([2]\) in absence of any bias voltage. We have done the calculations for zero temperature \((T = 0)\). In analogy to the time-dependent occupancy \( \langle \Delta \sigma \rangle \) presented in Appendix B we obtain in the limit \(|\Delta| \to \infty \) the following expression

For \( j_{S-\sigma}(t) \) the auxiliary parameters \((s_1, s_2, s_3, s_4)\) should be replaced by \((s_3, s_4, s_1, s_2)\).

In absence of the external voltage \( \mu_N = \mu_S = 0 \) the formula \([10]\) simplifies because the last two terms cancel each other. Under such conditions

\[
\varepsilon_\sigma = 0 \quad \text{we obtain}
\]

When the energy levels of QD are initially empty/full we obtain \( j_{\sigma}(t) = \pm \frac{\Gamma_3}{2\sqrt{\delta}} \sin \sqrt{\delta t} \) e\(^{-\Gamma_N t}\). Contrary to this behavior, for different initial occupancies, \( n_\sigma(0) = 0, \quad n_{-\sigma}(0) = 1 \), the transient current \( j_{\sigma}(t) \) vanishes. We assign the latter feature to inefficiency of the proximity effect that could operate only by mixing the empty with the doubly occupied QD configurations.
We have checked, that the charge is properly conserved in our system
\[ j_{S\sigma}(t) + j_{N\sigma}(t) + \frac{d}{dt} n_{\sigma}(t) = 0. \] (18)

Furthermore, we have also found the following exact relationship
\[ j_{S\sigma}(t) = -\Gamma_S \text{Im} \left( \hat{d}_+^\dagger \hat{d}_+ \right). \] The current \( j_{S\sigma}(t) \) can be hence inferred from what is shown in Fig. 4.

D. Transient currents of biased system

In absence of external voltage (\( \mu_N = \mu_S = 0 \)) the time-dependent QD occupancy, \( n_{\sigma}(t) \), and the charge currents, \( j_{N(S)\sigma}(t) \), provide indirect information about appearance of the quasiparticle states inside the subgap energy regime \( (-\Delta, \Delta) \). For practical reasons, however, much more convenient way to probe the time-scale of the Andreev/Shiba quasiparticles would be possible by studying transient properties of the biased system \( \mu_N \neq \mu_S \). Following the steps discussed in section [11] we shall consider here the time-dependent conductance \( G_\sigma(\mu, t) \) as a function of the biased voltage \( \mu \equiv \mu_N - \mu_S \). Using Eq. (13) one obtains the conductance (in \( \frac{e^2}{h} \) units) given by

\[
G_\sigma(\mu, t) = \Gamma_N \text{Re} \left[ e^{-i\mu t} \mathcal{L}^{-1} \left\{ \frac{s + i\varepsilon - \sigma + \Gamma_N/2}{(s - s_1)(s - s_2)(s - i\mu)} \right\}(t) \right] - \frac{\Gamma_N^2}{2} \mathcal{L}^{-1} \left\{ \frac{s + i\varepsilon - \sigma + \Gamma_N/2}{(s - s_3)(s - s_4)(s - i\mu)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i\mu)} \right\}(t) + \left( \frac{\Gamma_N^2 \Gamma_S^2}{8} \right) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_1)(s - s_2)(s - i\mu)} \right\}(t) \mathcal{L}^{-1} \left\{ \frac{1}{(s - s_3)(s - s_4)(s - i\mu)} \right\}(t).
\]

In particular, for \( \varepsilon = 0 \) the differential conductance (19) simplifies in the steady limit \( t \to \infty \) to the known result

\[
G_\sigma(\mu, \infty) = \frac{\Gamma_N^2 \Gamma_S^2}{4 \left( \mathcal{L}^+ \right)^2 + (\mathcal{L}^- - \mu)^2} \left[ \left( \mathcal{L}^+ \right)^2 + (\mathcal{L}^- + \mu)^2 \right].
\]

The local extrema of this expression for \( \Gamma_S > \Gamma_N \) are at \( \mu = \pm \frac{1}{2} \sqrt{\Gamma_S^2 - \Gamma_N^2} \) and tell us about the energy of the Andreev states, whereas the coupling \( \Gamma_N \) characterizes the inverse life-time of these quasiparticles. Such information is encoded in Eq. (19) along with evolution of the in-gap states driven by the sudden switching at \( t = 0 \) all the way to the steady limit asymptotic \( t \to \infty \).

Fig. 5 presents this differential conductance obtained numerically for \( \Gamma_N/\Gamma_S = 0.2 \) and 0.7. We can notice, that \( G_\sigma(\mu, t) \) approaches the steady-limit value with the Lorentzian-type quasiparticle peaks centered at \( \pm \frac{1}{2} \sqrt{\Gamma_S^2 - \Gamma_N^2} \). Such structure emerges gradually at a characteristic time \( \tau \), that depends on the ratio \( \Gamma_N/\Gamma_S \). The steady-limit values are approached with an envelope function \( 1 - \exp(-t/\tau) \), where \( \tau \simeq 2/\Gamma_N \). At an early stage of this process we also observe the dumped quantum oscillations with a period \( T \simeq 2\pi/\Gamma_S \).

IV. CORRELATION EFFECTS

Two-body interactions \( U\hat{n}_\uparrow \hat{n}_\downarrow \) (with the repulsive Coulomb potential \( U > 0 \)) can be expected to compete with the proximity-induced electron pairing. In the steady limit, this issue has been addressed by numerous methods [27]. The effective pairing (which is spectroscopically manifested by the in-gap states) depends predominantly on the ratio \( U/\Gamma_S \) and is also sensitive to QD level \( \varepsilon_\sigma \). Experimental realizations of the correlated quantum dot in N-QD-S geometry [28–31] revealed that the Coulomb potential \( U \) safely exceeds (at least one order of magnitude) the superconducting energy gap \( \Delta \). Under such circumstances the correlation effects show up in the subgap regime \( |\omega| < \Delta \) merely by a quantum phase transition (or crossover) from the BCS-type (spinless) state \( \psi(0) + v \chi \uparrow \downarrow \) to the singly occupied (spinful) configuration \( |\sigma\rangle \). Such changeover occurs upon increasing the ratio \( U/\Gamma_S \). The many-body Kondo effect may eventually appear only in the latter case, above some critical \( U_{cr} \sim \Gamma_S [31, 32] \).

The aforementioned quantum phase transition can be qualitatively captured already within the lowest order (Hartree-Fock-Bogoliubov) decoupling scheme

\[
\hat{d}_+^\dagger \hat{d}_+^\dagger \hat{d}_+ \hat{d}_+ \simeq n_\uparrow(t) \hat{d}_+^\dagger \hat{d}_+ + n_\downarrow(t) \hat{d}_-^\dagger \hat{d}_- - n_\uparrow(t)n_\downarrow(t) + \chi^*(t) \hat{d}_+^\dagger \hat{d}_+ + \chi(t) \hat{d}_-^\dagger \hat{d}_- - |\chi(t)|^2.
\]

Using this approximation (20) one can incorporate the
Hartree-Fock terms into the renormalized energy level \( \tilde{\varepsilon}_\sigma \equiv \varepsilon_\sigma + U n_{-\sigma}(t) \), whereas the anomalous (pair source and drain) terms rescale the effective pairing potential \( \tilde{\Gamma}_S/2 \equiv \Gamma_S/2 - U \chi(t) \). These corrections \([20]\) can yield a crossing of the Andreev quasiparticle energies at some critical ratio \( U/\Gamma_S \), dependent also on \( \varepsilon_\sigma \). In Josephson (S-QD-S’) junctions the same effect causes reversal of the d.c. tunneling current, so called, \( 0 - \pi \) transition \([33, 44]\). For the N-QD-S heterostructure its influence is noticeable, but in less spectacular way.

Unfortunately, the analytical method we have used in Sec. III is useless in the case when the renormalized QD energy levels and effective pairing potential are time-dependent. So our strategy to include the Coulomb correlation effects is as follows. Firstly we have checked that the time-dependence of the QD occupation, \( n_{\sigma}(t) \), the induced pairing \( \langle c_1^\dagger(t)c_1^\dagger(t) \rangle \) and the current flowing in the system of the proximitized QD coupled only to the normal lead are exactly the same as those calculated for the QD coupled with the normal and superconducting leads if the replacement \( \Delta_S = \frac{\Gamma_S}{2} \) is made (here \( \Delta_S \) is the induced pairing of the proximitized QD). Next, we consider the Coulomb correlations within the system of the proximitized QD with \( \Delta_S = \frac{\Gamma_S}{2} \) coupled to the normal lead. Applying the Hartree-Fock-Bogoliubov approximation we next find the solution for the proximitized QD described by the effective energy levels \( \varepsilon_\sigma \equiv \varepsilon_\sigma + U \langle n_{-\sigma}(t) \rangle \) and \( \Delta = \frac{\Gamma_S}{2} + U \langle c_1^\dagger(t)c_1(t) \rangle \). The quantities, \( n_{\sigma}(t) \), \( \langle c_1^\dagger(t)c_1^\dagger(t) \rangle \) and \( j_{N\sigma}(t) \) can be found solving the closed system of the equation of motion for the functions \( n_{\sigma}(t) \), \( \langle c_1^\dagger(t)c_1^\dagger(t) \rangle \), \( \langle c_1^\dagger(t)c_\sigma(0) \rangle \) and \( \langle c_\sigma(t)c_{k-\sigma}(0) \rangle \). We have solved numerically such differential equations.

Fig. 7 presents the influence of the Coulomb potential \( U \) on the complex order parameter \( \chi(t) \) (unbiased system is considered). The imaginary part (that is strictly related to the transient current) shows the damped quantum oscillations. Both the period and amplitude of such oscillations are substantially suppressed by the Coulomb potential. We interpret this fact as a signature of the competition between electron pairing and electrostatic repulsion. The real part of \( \chi(t) \) is characterized by the quantum oscillations too, however it asymptotically approaches the non-zero (stationary limit) solution. Fig. 8 shows this behavior for \( \varepsilon_\sigma = 0 \) and \( \Gamma_N/\Gamma_S = 0.2 \), which indicates a competing relationship between the induced electron pairing and the on-dot repulsion.

In Fig. 6 we show influence of the Coulomb potential \( U \) on the QD occupancy for \( \uparrow \) electrons. Besides the quantum oscillations, similar to the ones observed for the complex order parameter \( \chi(t) \), we notice partial reduction of the QD charge with increasing \( U \). Apparently this is caused by the Hartree term, which lifts the renormalized level \( \varepsilon_\sigma(t) \).
\( \Gamma_N \) controls the rate at which the stationary limit behavior is achieved, whereas period of the damped quantum oscillations is substantially reduced by Coulomb repulsion \( U \).

Acknowledgments

We thank V. Janiš for instructive discussions and A. Baumgartner for useful remarks concerning observability of the transient effects in two- and three-terminal heterostructures. This work is supported by the National Science Centre (Poland) through the grant DEC-2014/13/B/ST3/04451 (TD).

Appendix A: Laplace transforms

In this Appendix we present explicit formulas for the Laplace transforms of the quantum dot \( \hat{d}_\sigma(t) \) and the mobile electrons \( \hat{c}_{q\sigma}(t) \) operators. For arbitrary value of the energy gap \( \Delta \) we obtain the following transforms

\[
\hat{d}_\uparrow(s) = \frac{M_\uparrow^-(s)\hat{A}(s) - iK(s)\hat{B}(s)}{M_\uparrow^+(s)M_\uparrow^-(s) + |K(s)|^2} \tag{A1}
\]

\[
\hat{d}_\downarrow(s) = \frac{M_\downarrow^-(s)\hat{B}^\dagger(s) + iK(s)\hat{A}^\dagger(s)}{M_\downarrow^+(s)M_\downarrow^-(s) + |K(s)|^2} \tag{A2}
\]

where

\[
M_\sigma^{(+)}(s) = s \pm i\varepsilon_\sigma + \sum_k \frac{V^2_k}{s \pm i\varepsilon_k} + \sum_q \frac{V^2_q}{s^2 + \varepsilon^2_q + \Delta^2},
\]

\[
K(s) = \sum_q \frac{-V^2_q \Delta}{s^2 + \varepsilon^2_q + |\Delta|^2},
\]

\[
\hat{A}(s) = -i \sum_k \frac{V_k \hat{c}_k(0)}{s + i\varepsilon_k} - \sum_q \frac{V_q}{s^2 + \varepsilon^2_q + |\Delta|^2} \times \left( -\Delta \hat{c}_{-q\uparrow}(0) + i(s - i\varepsilon_q)\hat{c}_{q\uparrow}(0) \right) + \hat{d}_\uparrow(0),
\]

\[
\hat{B}(s) = i \sum_k \frac{V_k \hat{c}_k^\dagger(0)}{s - i\varepsilon_k} + \sum_q \frac{V_q}{s^2 + \varepsilon^2_q + |\Delta|^2} \times \left( -\Delta^\ast \hat{c}_{q\downarrow}(0) + i(s + i\varepsilon_q)\hat{c}_{-q\downarrow}(0) \right) + \hat{d}_\downarrow(0).
\]

The Laplace transform for \( \hat{c}_{q\sigma}(s) \) is given by

\[
\hat{c}_{q\sigma}(s) = \frac{1}{s^2 + \varepsilon^2_q + |\Delta|^2} \left( -iV_q(s - i\varepsilon_q)\hat{d}_\sigma(s) - \alpha\Delta V_q\hat{d}_{q\uparrow}(s) + i\alpha\Delta V_q\hat{d}_{q\downarrow}(s) \right),
\]

where \( \alpha = +(-) \) for \( \sigma \uparrow \downarrow \). The expressions for \( \hat{d}_\uparrow(s) \) and \( \hat{d}_\downarrow(s) \) can be obtained taking the hermitian conjugate of \( \hat{d}_\uparrow(s) \) and \( \hat{d}_\downarrow(s) \), respectively.
