Local Weyl modules and cyclicity of tensor products for Yangians of $G_2$

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Abstract

Let $\mathfrak{g}$ be the exceptional complex simple Lie algebra of type $G_2$. We provide a concrete cyclicity condition for the tensor product of fundamental representations of the Yangian $Y(g)$. Using this condition, we show that every local Weyl module is isomorphic to an ordered tensor product of fundamental representations of $Y(g)$.

Key words: Yangian; $G_2$; Local Weyl modules; Cyclicity condition
1 Introduction

There is a rich structure theory behind the finite-dimensional representations of
Yangians $Y(\mathfrak{g})$ and quantum affine algebras $U_q(\hat{\mathfrak{g}})$ as the category of their finite-
dimensional representations is not semi-simple, where $\mathfrak{g}$ is a complex simple Lie
algebra of rank $l$. Finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ are pa-
parameterized by $l$-tuples of polynomials $P = (P_1(u), \ldots, P_l(u))$, where $P_i(0) = 1$, see
[6]. In [9], V. Chari and A. Pressley showed that in the class of all highest weight rep-
resentations associated to $P$, there is a unique (up to isomorphism) finite-dimensional
highest weight representation $W(P)$ such that any other representation in this class
is a quotient of $W(P)$. It has been established that $W(P)$ is isomorphic to an ordered
tensor product of fundamental representations of $U_q(\hat{\mathfrak{g}})$, and a proof of this fact can
be found in [3]. In [2], V. Chari provided a method to find a concrete cyclicity con-
dition for an ordered tensor product of Kirillov-Reshetikhin modules using braided
group action on the imaginary root vectors.

The finite-dimensional representation theory of $Y(\mathfrak{g})$ is an analogue of the one
of $U_q(\hat{\mathfrak{g}})$. Let $\pi$ be a $l$-tuple of polynomials. One can define the local Weyl module
of the Yangian $Y(\mathfrak{g})$ similarly. We gave the definition of local Weyl module $W(\pi)$
by generators and defining relations in [12], and proved that $W(\pi)$ is isomorphic to an
ordered tensor product of fundamental representations when $\mathfrak{g}$ is classical. The
main challenge in [12] was to find an explicit cyclicity condition for an ordered tensor
product, and the methodology will be introduced in Section 2 for more detail.

This work is a continuation of the paper [12]. We introduce a new algorithm to
compute certain associated polynomials when $\mathfrak{g}$ is of type $G_2$, which enables us to
provide an explicit cyclicity condition for an ordered tensor product of fundamental
representations, see Theorem 1. Using this condition, we show that the local Weyl
module $W(\pi)$ is isomorphic to an ordered tensor product of fundamental repr esenta-
tions in Theorem 2.

2 Preliminary

In this section, we give a brief review of the previous paper [12] and one question
left open in loc. cit.

Definition 2.1. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with rank $l$ and let $A = (a_{ij})_{i,j \in I}$,
where $I = \{1,2,\ldots,l\}$, be its Cartan matrix. Let $D = \text{diag}(d_1,\ldots,d_l)$, $d_i \in \mathbb{N}$, such
that $d_1, d_2, \ldots, d_l$ are co-prime and $DA$ is symmetric. The Yangian $Y(\mathfrak{g})$ is defined to
be the associative algebra with generators $x_{i,r}^\pm, h_{i,r}, i \in I, r \in \mathbb{Z}_{\geq 0}$, and the following
defining relations:

\[ [h_{i,r}, h_{j,s}] = 0, \quad [h_{i,0}, x^\pm_{j,s}] = \pm d_{ij} x^\pm_{j,s}, \quad [x^+_{i,r}, x^-_{j,s}] = \delta_{ij} h_{i,r+s}, \]

\[
[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = \pm \frac{1}{2} d_{ij} (h_{i,r} x^\pm_{j,s} + x^\pm_{j,s} h_{i,r}),
\]

\[
[x^+_{i,r+1}, x^-_{j,s}] - [x^+_{i,r}, x^-_{j,s+1}] = \pm \frac{1}{2} d_{ij} (x^+_{i,r} x^-_{j,s} + x^-_{j,s} x^+_{i,r}),
\]

\[
\sum_{\pi} [x^+_{i,r_{\pi(1)}}, [x^+_{i,r_{\pi(2)}}, \ldots, [x^+_{i,r_{\pi(m)}}, x^-_{j,s}] \ldots ]] = 0, \quad i \neq j,
\]

for all sequences of non-negative integers \( r_1, \ldots, r_m \), where \( m = 1 - a_{ij} \) and the sum is over all permutations \( \pi \) of \( \{1, \ldots, m\} \).

Denote by \( V_m(a) \), \( m \geq 1 \) and \( a \in \mathbb{C} \), the finite-dimensional irreducible representation of \( Y(\mathfrak{sl}_2) \) associated to the Drinfeld polynomial \((u-a)(u-(a+1)) \ldots (u-(a+m-1))\).

**Proposition 2.2 (Proposition 3.5, [8]).** The module \( V_m(a) \) has a basis \( \{w_0, w_1, \ldots, w_m\} \) on which the action of \( Y(\mathfrak{sl}_2) \) is given by

\[
x^+_k w_s = (s + a)^k (s + 1) w_{s+1}, \quad x^-_k w_s = (s + a - 1)^k (m - s + 1) w_{s-1},
\]

\[
h_k w_s = ((s + a - 1)^k s (m - s + 1) - (s + a)^k (s + 1) (m - s)) w_s.
\]

Suppose that a fixed reduced expression of the longest element of the Weyl group of \( \mathfrak{g} \) is \( w_0 = s_{r_1} s_{r_2} \ldots s_{r_p} \), where \( s_{r_j} \), for \( 1 \leq j \leq p \), are simple reflections. Denote the \( i \)-th fundamental weight by \( \omega_i \) and the \( i \)-th fundamental representation of \( Y(\mathfrak{g}) \) by \( V_a(\omega_i) \). Let \( v^+ \) and \( v^- \) be the highest and lowest weight vectors in the fundamental representation \( V_a(\omega_i) \), respectively. Suppose \( s_{r_{j+1}} s_{r_{j+2}} \ldots s_{r_p} (\omega_i) = m_j \omega_{r_j} + \sum_{n \neq r_j} c_n \omega_n \).

Then

\[
v^- = (x^-_{r_1,0})^{m_1} (x^-_{r_2,0})^{m_2} \ldots (x^-_{r_p,0})^{m_p} v^+.
\]

Define \( s_{r_{j+1}} s_{r_{j+2}} \ldots s_{r_p} \) by \( \sigma_j \) and \((x^-_{r_{j+1},0})^{m_{j+1}} (x^-_{r_{j+2},0})^{m_{j+2}} \ldots (x^-_{r_p,0})^{m_p} v^+ \) by \( v_{\sigma_j}(\omega_i) \). Let \( Y_i \) be the subalgebra generated by \( \{x^\pm_{i,r}, h_{i,r} | r \in \mathbb{Z}_{\geq 0}\} \) for \( i \in I \), which is isomorphic to \( Y(\mathfrak{sl}_2) \). Denote by \( Y_{r_j}(v_{\sigma_j}(\omega_i)) \) the \( Y_{r_j} \)-module generated by the extremal vector \( v_{\sigma_j}(\omega_i) \). We remark that it has been established in Section 5 in [12] that \( Y_{r_j}(v_{\sigma_j}(\omega_m)) \) is a highest weight representation and the degree of its associated polynomial is \( m_j \).

In [12], we showed:

**Theorem 5.2** [12] An ordered tensor product \( L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k}) \) is a highest weight representation if for all \( 1 \leq j \leq p \) and \( 1 \leq m < n \leq k \), when \( b_m = r_j \), the difference of the number \( \frac{a_m}{d_{r_j}} \) and any root of the associated polynomial of \( Y_{r_j}(v_{\sigma_j}(\omega_{m})) \) does not equal 1.
We proved the above theorem by adopting the ideas in [2]. To find an explicit cyclicity condition, it is enough to compute the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{bm})})$ for $1 \leq j \leq p$ and $1 \leq m \leq l$. When $g$ is a classical simple Lie algebra, by computing the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_1)})$ using some of the defining relations of $Y(g)$, a concrete cyclicity condition of $L$ was obtained in Theorem 5.18 in [12].

For the following reasons, it becomes complicated if one tries to describe a concrete cyclicity condition for the tensor product when $g$ is an exceptional simple Lie algebra. In this case, $m_j$ may be greater than or equal to 3. The computations of the eigenvalue of $h_{r_j,k}$ on the weight vector $v_{\sigma_j(\omega)}$, $3 \leq k \leq m_j$, are tedious if one uses defining relations of $Y(g)$. In addition, the path from the highest weight vector $v^+$ to the lowest one $v^-$ is more subtle than in the case when $g$ is a classical simple Lie algebra. For instance, when $g$ is of type $F_4$, in $V_s(\omega_2)$, a path is:

$$v^- = x_{2,0}^{-}x_{1,0}^{-}(x_{3,0}^{-})^2(x_{2,0}^{-})^2x_{1,0}^{-}(x_{3,0}^{-})^2x_{2,0}^{-}(x_{4,0}^{-})^4(x_{3,0}^{-})^4(x_{2,0}^{-})^3(x_{1,0}^{-})^3(x_{3,0}^{-})^2$$

$$x_{2,0}^{-}x_{3,0}^{-}(x_{4,0}^{-})^4(x_{3,0}^{-})^2x_{2,0}^{-}(x_{1,0}^{-})^2x_{2,0}^{-}v^+.$$

It seems to us that step-by-step computations are needed to compute the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{bm})})$. The amount of time required to compute these associated polynomials would be tremendous using exactly the same approach as in [12].

3 A concrete cyclicity condition and local Weyl modules

From now on, let $g$ denote the simple Lie algebra of type $G_2$, unless the contrary is stated. Let $\alpha_1$ and $\alpha_2$ be the simple long and short roots, respectively (as labelled in [1]) and let $\omega_1$ and $\omega_2$ be the fundamental weights. The Cartan matrix of $g$ is

$$\left(\begin{array}{cc} 2 & -1 \\ -3 & 2 \end{array}\right).$$

Let $D = \left(\begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array}\right)$, so $DA$ is symmetric. The Weyl group $\mathcal{W}$ of $g$ is generated by $s_1$ and $s_2$ such that $s_1(\alpha_1) = -\alpha_1$, $s_2(\alpha_1) = \alpha_1 + 3\alpha_2$, $s_1(\alpha_2) = \alpha_1 + \alpha_2$, and $s_2(\alpha_2) = -\alpha_2$. A reduced expression of the longest element in Weyl group is $w_0 = s_1s_2s_1s_2s_1s_2$. Using the method indicated in Section 2 of this paper, we found a path form $v^+$ to $v^-$. 

Lemma 3.1. Let $a \in \mathbb{C}$.

1. In $V_a(\omega_1)$, $v^- = x_{1,0}^{-}(x_{2,0}^{-})^3(x_{1,0}^{-})^2x_{2,0}^{-}v^+$.

2. In $V_a(\omega_2)$, $v^- = x_{2,0}^{-}x_{1,0}^{-}(x_{2,0}^{-})^2x_{1,0}^{-}x_{2,0}^{-}v^+.$
Note that $x_{1,r}^\pm, h_{1,r}$ do not satisfy the defining relations of $Y(\mathfrak{sl}_2)$. Therefore, we need to re-scale the generators. Let $\tilde{x}_{1,r}^\pm = \frac{\sqrt{3}}{2 + r} x_{1,r}^\pm$ and $\tilde{h}_{1,r} = \frac{1}{2 + r} h_{1,r}$, then $\tilde{x}_{1,r}^\pm, \tilde{h}_{1,s}$ satisfy the defining relations of $Y(\mathfrak{sl}_2)$. Now we are in the position to compute the associated polynomial of $Y_{\eta}(v_{\sigma_j(\omega_m)})$, where $b_m \in I = \{1, 2\}$.

**Proposition 3.2.** Let $v^+ \in V_a(\omega_1)$.

| Item | $Y(\mathfrak{sl}_2)$-module | Associated polynomial |
|------|-------------------------------|----------------------|
| 1    | $Y_1(v^+)$                    | $u - \frac{a}{3}$    |
| 2    | $Y_2(x_{1,0}v^+)$             | $(u - (a + \frac{s}{3}))(u - (a + \frac{s}{2}))(u - (a - \frac{1}{2}))$ |
| 3    | $Y_1((x_{2,0}^-)^3 x_{1,0}^+ v^+)$ | $(u - \frac{a + 2}{3})(u - \frac{a + 1}{3})$ |
| 4    | $Y_2((x_{1,0}^+)^2 (x_{2,0}^-)^2 x_{1,0}^+ v^+)$ | $(u - (a + \frac{s}{3}))(u - (a + \frac{s}{2}))(u - (a - \frac{s}{2}))$ |
| 5    | $Y_1((x_{2,0}^-)^3 (x_{1,0}^-)^2 (x_{1,0}^+)^2 v^+)$ | $u - \frac{a + s}{3}$ |

**Proof.** We omit the proof of items 1, 2 and 5 since these proofs can be checked using the same approach as in paper [12]. Lemma 3.6 is devoted to proving the third item. The fourth item is proved in Lemma 3.8. □

Let $H$ be the subalgebra of $Y(\mathfrak{g})$ generated by all $h_{i,k}$ and $h_i(u) = 1 + h_{i,0} u^{-1} + h_{i,1} u^{-2} + \ldots$, where $i \in I = \{1, 2\}$ and $k \in \mathbb{Z}_{\geq 0}$. We use alternate generators for $H$ as given in [10]. Let

$$H_i(u) = \sum_{k=0}^{\infty} H_{i,k} u^{-k-1} := \ln (h_i(u)).$$

An explicit computation shows that

$$H_i(u) = h_{i,0} u^{-1} + (h_{i,1} - \frac{1}{2} (h_{i,0})^2) u^{-2} + (h_{i,2} - h_{i,0} h_{i,1} + \frac{1}{3} (h_{i,0})^3) u^{-3} + \ldots \ (3.1)$$

**Lemma 3.3** (Corollary 1.5, [10]). Let $\mathfrak{g}$ be a complex simple Lie algebra.

$$[H_{i,k}, x_{j,l}^\pm] = \pm d_{i j} a_i x_{j,l+1}^\pm \pm \sum_{0 \leq s \leq k-2 \atop k+s \text{ even}} 2^{s-k} (d_{i a_j})^{k+1-s} \binom{k+1}{s} \binom{k}{k+s} x_{j,l+s}^\pm.$$

Let $V$ be a finite-dimensional highest weight representation of $Y(\mathfrak{sl}_2)$ whose associated polynomial is $\pi$. Let $v^+$ and $v^-$ be highest and lowest weight vectors of $V$, respectively.

**Lemma 3.4.** $h(u) v^- = \frac{\pi(u-1)}{\pi(u)} v^-.

**Proof.** It is enough to consider the case when $V$ is irreducible. It was established in Proposition 3.1 [3] that $h(u) v^- = \frac{p(u)}{p(u+1)} v^-$, where $p(u)$ is the associated polynomial of the right dual $V^\vee$. It was showed in Proposition 2.4 in [3] that $p(u) = \pi(u-1)$. Therefore this lemma is proved. □
To show Lemma 3.6, we need the following corollary. We remark that in the proof of the corollary, we use some algorithm which did not show up in [12].

**Corollary 3.5.** In the representation $Y_2(x_{1,0}^+)$,

1. $$(x_{2,1}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,1} x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^-) x_{1,0}^+ = (a + \frac{3}{2}) + (a - \frac{3}{2})^2 + (a - \frac{1}{2})^2 (x_{2,0}^-)^3 x_{1,0}^+.$$  
2. $$(x_{2,2}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,2} x_{2,0}^- + (x_{2,0}^-)^2 x_{2,2}^-) x_{1,0}^+ = (a + \frac{3}{2})^2 + (a - \frac{3}{2})^2 + (a - \frac{1}{2})^2 (x_{2,0}^-)^3 x_{1,0}^+.$$  

**Proof.** It follows from Proposition 3.2 that the associated polynomial of the highest weight representation $Y_2(x_{1,0}^+)$ is $(u - (a + \frac{3}{2})) (u - (a + \frac{3}{2})) (u - (a - \frac{1}{2}))$. Therefore, $h_2(u)(x_{1,0}^+) = \frac{u-(a+\frac{3}{2})}{u-(a+\frac{3}{2})} x_{1,0}^+$ and $h_2(u)((x_{2,0}^-)^3 x_{1,0}^+) = \frac{u-(a+\frac{3}{2})}{u-(a+\frac{3}{2})} (x_{2,0}^-)^3 x_{1,0}^+$ by Lemma 3.4. Thus

$$H_2(u)(x_{1,0}^+) = \left( \ln(1 - (a - \frac{3}{2})u^{-1}) - \ln(1 - (a + \frac{3}{2})u^{-1}) \right) x_{1,0}^+$$

and

$$H_2(u)((x_{2,0}^-)^3 x_{1,0}^+) = \left( \ln(1 - (a + \frac{5}{2})u^{-1}) - \ln(1 - (a - \frac{1}{2})u^{-1}) \right) (x_{2,0}^-)^3 x_{1,0}^+.$$  

In particular, we have both $H_{2,1}(x_{2,0}^-)^2 x_{1,0}^+ = \frac{1}{2} ((a - \frac{1}{2})^2 - (a - \frac{5}{2})^2) (x_{2,0}^-)^3 x_{1,0}^+$ and $H_{2,2}(x_{2,0}^-)^3 x_{1,0}^+ = \frac{1}{3} ((a - \frac{1}{2})^3 - (a + \frac{5}{2})^3) (x_{2,0}^-)^3 x_{1,0}^+$. By Lemma 3.3, $[H_{2,1}, x_{2,0}^-] = -2 x_{2,1}^-$, and $[H_{2,2}, x_{2,0}^-] = -2 x_{2,2}^- - \frac{2}{3} x_{2,0}^-$. We are going to show the first item in this corollary.

$$H_{2,1}(x_{2,0}^-)^3 x_{1,0}^+ = [H_{2,1}, (x_{2,0}^-)^3] x_{1,0}^+ + (x_{2,0}^-)^3 [H_{2,1}, x_{1,0}^+]$$

$$= -2 (x_{2,1}^- (x_{2,0}^-)^2 + x_{2,0}^- x_{2,1} x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^-) x_{1,0}^+ + \frac{1}{2}((a - \frac{3}{2})^2 - (a - \frac{3}{2})^2) (x_{2,0}^-)^3 x_{1,0}^+.$$  

Therefore,

$$H_{2,1}(x_{2,0}^-)^2 + x_{2,0}^- x_{2,1} x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^- x_{1,0}^+ = \frac{1}{4}((a + \frac{5}{2})^2 - (a - \frac{1}{2})^2 + (a + \frac{3}{2})^2 - (a - \frac{3}{2})^2) (x_{2,0}^-)^3 x_{1,0}^+$$

$$= (3a + \frac{3}{2})(x_{2,0}^-)^3 x_{1,0}^+.$$  

The second item in this corollary can be obtained similarly, so we omit the proof. ☐
Lemma 3.6. The associated polynomial of the representation $Y_1\left((x^{-}_{2,0})^3x^{-}_{1,0}v^+\right)$ is given by $(u - \frac{a+2}{3})(u - \frac{a+1}{3})$.

Proof. The associated polynomial of the representation $Y_1\left((x^{-}_{2,0})^3x^{-}_{1,0}v^+\right)$ is of degree 2, say $(u - a_1)(u - a_2)$. The eigenvalues of $(x^{-}_{2,0})^3x^{-}_{1,0}v^+$ under $\tilde{h}_{1,1}$ and $\tilde{h}_{1,2}$ will tell us the values of $a_1$ and $a_2$. We first compute the eigenvalues of $(x^{-}_{2,0})^3x^{-}_{1,0}v^+$ under $H_{1,1}$ and $H_{1,2}$.

$$H_{1,1}(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= [H_{1,1}, x^{-}_{2,0}](x^{-}_{2,0})^2x^{-}_{1,0}v^+ + x^{-}_{2,0}[H_{1,1}, x^{-}_{2,0}]x^{-}_{2,0}x^{-}_{1,0}v^+$$
$$+ (x^{-}_{2,0})^2[H_{1,1}, x^{-}_{2,0}]x^{-}_{1,0}v^+ + (x^{-}_{2,0})^3H_{1,1}x^{-}_{1,0}v^+$$
$$= 3x^{-}_{2,1}(x^{-}_{2,0})^2x^{-}_{1,0}v^+ + 3x^{-}_{2,0}x^{-}_{1,0}x^{-}_{1,0}v^+ + 3(x^{-}_{2,0})^2x^{-}_{1,0}v^+$$
$$+ \frac{1}{2}((a - 3)^2 - a^2)(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= 3\left((a - \frac{1}{2})^2 + (a + \frac{1}{2}) + (a + \frac{3}{2}) - (a + \frac{3}{2})\right)(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= 6a(x^{-}_{2,0})^3x^{-}_{1,0}v^+,$$

where the third equality follows from the first item of Corollary 3.5.

$$H_{1,2}(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= [H_{1,2}, x^{-}_{2,0}](x^{-}_{2,0})^2x^{-}_{1,0}v^+ + x^{-}_{2,0}[H_{1,2}, x^{-}_{2,0}]x^{-}_{2,0}x^{-}_{1,0}v^+$$
$$+ (x^{-}_{2,0})^2[H_{1,2}, x^{-}_{2,0}]x^{-}_{1,0}v^+ + (x^{-}_{2,0})^3H_{1,2}x^{-}_{1,0}v^+$$
$$= (3x^{-}_{2,2} + \left(\frac{3}{2}\right)^2x^{-}_{2,0})(x^{-}_{2,0})^2x^{-}_{1,0}v^+ + x^{-}_{2,0}(3x^{-}_{2,2} + \left(\frac{3}{2}\right)^2x^{-}_{2,0})x^{-}_{2,0}x^{-}_{1,0}v^+$$
$$+ (x^{-}_{2,0})^2(3x^{-}_{2,2} + \left(\frac{3}{2}\right)^2x^{-}_{2,0})x^{-}_{1,0}v^+ - \frac{1}{3}((a - 3)^3 - a^3)(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= 3\left((a - \frac{1}{2})^2 + (a + \frac{1}{2})^2 + (a + \frac{3}{2})^2 + \frac{9}{4} - (a^2 + 3a + 3)\right)(x^{-}_{2,0})^3x^{-}_{1,0}v^+$$
$$= (6a^2 + 6)(x^{-}_{2,0})^3x^{-}_{1,0}v^+.$$

It follows from equation (3.11) and the above computations that

$$\tilde{h}_{1,1}(x^{-}_{2,0})^3x^{-}_{1,0}v^+ = (\frac{2a}{3} + 2)(x^{-}_{2,0})^3x^{-}_{1,0}v^+;$$
$$\tilde{h}_{1,2}(x^{-}_{2,0})^3x^{-}_{1,0}v^+ = (2\left(\frac{a}{3}\right)^2 + \frac{4a}{3} + \frac{14}{9})(x^{-}_{2,0})^3x^{-}_{1,0}v^+.$$

It follows from relation (5.1) in [12] that $a_1 + a_2 = \frac{2a}{3} + 1$ and $a_1^2 + a_2^2 + a_1 + a_2 = 2\left(\frac{a}{3}\right)^2 + \frac{4a}{3} + \frac{14}{9}$. Then $a_1 = \frac{a+1}{3}$ and $a_2 = \frac{a+2}{3}$, or vice-versa with $a_1$ and $a_2$ switched. Thus the associated polynomial of the representation $Y_1\left((x^{-}_{2,0})^3x^{-}_{1,0}v^+\right)$ is $(u - \frac{a+2}{3})(u - \frac{a+1}{3})$. \qed
Similar to the proof of Corollary 3.5 we can prove the following corollary. We remark that: $h_1(u)((x_{-2})^3x_{1,0}v^+) = \frac{u-a}{u-(a+2)}(x_{-2})^3x_{1,0}v^+$.

**Corollary 3.7.** In the representation $Y_1((x_{-2})^3x_{1,0}v^+)$,

1. $(x_{-1}^-x_{1,0}^-x_{1,1}^-)(x_{-2})^3x_{1,0}v^+ = ((a + 1) + (a + 2))(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+$;
2. $(x_{-1}^-x_{1,0}^-x_{1,2}^-)(x_{-2})^3x_{1,0}v^+ = ((a + 1)^2 + (a + 2)^2)(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+$;
3. $(x_{-1}^-x_{1,0}^-x_{1,3}^-)(x_{-2})^3x_{1,0}v^+ = ((a + 1)^3 + (a + 2)^3)(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+$.

**Lemma 3.8.** The associated polynomial of the module $Y_2((x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+)$ is given by $(u - (a + 3/2))(u - (a + 5/2))(u - (a + 7/2))$.

**Proof.** The associated polynomial of the representation $Y_2((x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+)$ is of degree 3, say $(u - a_1)(u - a_2)(u - a_3)$. The eigenvalues of $(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+$ under $h_{2,1}$, $h_{2,2}$ and $h_{2,3}$ will tell us the values of $a_1$, $a_2$ and $a_3$.

$$
H_{2,1}(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= [H_{2,1}, x_{-1}^-]x_{1,0}^-[x_{-2}]^3x_{1,0}v^+ + x_{-1}^-[H_{2,1}, x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ + (x_{-1}^-)^2H_{2,1}(x_{-2})^3x_{1,0}v^+ \\
= 3(x_{-1}^-x_{1,0}^-x_{1,1}^-)(x_{-2})^3x_{1,0}v^+ - \frac{1}{2}((a + \frac{5}{2})^2 - (a - \frac{1}{2})^2)(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= ((6a + 9) - (3a + 3))(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= (3a + 6)(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+,
$$

where the third equality follows from the first item of Corollary 3.7.

$$
H_{2,2}(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= [H_{2,2}, x_{1,0}^-][x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ + x_{-1}^-[H_{2,2}, x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ \\
+ (x_{-1}^-)^2H_{2,2}(x_{-2})^3x_{1,0}v^+ \\
= (3x_{-1}^- + \frac{9}{4}x_{1,0}^-)[x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ + x_{-1}^-[3x_{-1}^- + \frac{9}{4}x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ \\
- \frac{1}{3}((a + \frac{5}{2})^3 - (a - \frac{1}{2})^3)(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= (3a^2 + 12a + \frac{57}{4})(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+.
$$

$$
H_{2,3}(x_{-1}^-)^2(x_{-2})^3x_{1,0}v^+ \\
= [H_{2,3}, x_{1,0}^-][x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ + x_{-1}^-[H_{2,3}, x_{1,0}^-][x_{-2}]^3x_{1,0}v^+ \\
+ (x_{-1}^-)^2H_{2,3}(x_{-2})^3x_{1,0}v^+.
$$
\[
(3x_{1,0}^3 + \frac{27}{4} x_{1,0} x_{2,0})^3 x_{1,0} v^+ + x_{1,0}^3 (3x_{1,0}^3 + \frac{27}{4} x_{1,0} x_{2,0})^3 x_{1,0} v^+
- \frac{1}{4} ((a + \frac{5}{2})^3 - (a - \frac{1}{2})^3) (x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+
\]
\[
= (3a^3 + 18a^2 + \frac{171}{4} a + \frac{75}{2}) (x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+.
\]

It follows from equation (3.1) and the above computations that
\[
h_{2,1}(x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+ = 3(a + \frac{7}{2}) (x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+;
\]
\[
h_{2,2}(x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+ = 3(a + \frac{7}{2}) (x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+;
\]
\[
h_{2,3}(x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+ = 3(a + \frac{7}{2}) (x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+.
\]

Note that
\[
\frac{u - (a_1 - 1)}{u - a_1} \cdot \frac{u - (a_2 - 1)}{u - a_2} \cdot \frac{u - (a_3 - 1)}{u - a_3}
= 1 + 3u^{-1} + (a_1 + a_2 + a_3 + 3)u^{-2} + (a_1^2 + a_2^2 + a_3^2 + 2a_1 + 2a_2 + 2a_3 + 1)u^{-3}
+ (a_1^3 + a_2^3 + a_3^3 + 2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_1a_2 + a_1a_3 + a_2a_3 + a_1 + a_2 + a_3)u^{-4}
+ \ldots
\]

Computations show that \(a_1 = a + \frac{3}{2}, a_2 = a + \frac{5}{2}\) and \(a_3 = a + \frac{7}{2}\), or vice-versa with \(a_1, a_2\) and \(a_3\) switched. Thus the associated polynomial of the representation \(Y_2((x_{1,0})^2 (x_{2,0})^3 x_{1,0} v^+)\) is given by \((u - (a + \frac{3}{2})) (u - (a + \frac{5}{2})) (u - (a + \frac{7}{2}))\). \(\square\)

We will omit the proof of the following proposition since it can be proved using some of the defining relations of \(Y(g)\).

**Proposition 3.9.** Let \(v^+ \in V_a(\omega_2)\).

| Item | \(Y(\mathfrak{sl}_2)\)-module | Associated polynomial |
|------|-----------------|------------------|
| 1    | \(Y_2(v^+)\)    | \(u - a\)        |
| 2    | \(Y_1(x_{2,0} v^+)\) | \(u - (\frac{4}{3} + \frac{1}{2})\) |
| 3    | \(Y_2(x_{1,0} x_{2,0} v^+)\) | \((u - (a + 3)) (u - (a + 2))\) |
| 4    | \(Y_1((x_{2,0})^2 x_{1,0} x_{2,0} v^+)\) | \(u - (\frac{7}{3} + \frac{5}{6})\) |
| 5    | \(Y_2(x_{1,0} (x_{2,0})^2 x_{1,0} x_{2,0} v^+)\) | \(u - (a + 5)\) |

We summarize all results in Propositions 1 and 2 into the coming corollary. Denote by \(T(b_m, r_j)\) the set of all possible roots of the associated polynomial of \(Y_{r_j}(v_{r_j(\omega_m)})\).

**Corollary 3.10.** \(T(1, 1) = \{\frac{a}{3}, \frac{a+1}{3}, \frac{a+2}{3}, \frac{a+3}{3}\}\), \(T(1, 2) = \{a - \frac{1}{2}, a + \frac{1}{2}, a + \frac{3}{2}, a + \frac{5}{2}, a + \frac{7}{2}\}\), \(T(2, 1) = \{\frac{a}{2} + \frac{1}{2}, \frac{a}{2} + \frac{3}{2}, \frac{a}{2} + \frac{5}{2}\}\) and \(T(2, 2) = \{a, a + 2, a + 3, a + 5\}\).
By Theorem 5.2 in [12] and Proposition 3.8 in [7], we have the following theorem.

**Theorem 3.11.** Let $L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k})$ be an ordered tensor product of fundamental representations of $Y(g)$, and define $S(1,1) = \{3,4,5,6\}$, $S(1,2) = \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}$, $S(2,1) = \{\frac{9}{2}, \frac{13}{2}\}$ and $S(2,2) = \{1,3,4,6\}$.

1. If $a_j - a_i \notin S(b_i,b_j)$ for $1 \leq i < j \leq k$, then $L$ is a highest weight representation of $Y(g)$.

2. If $a_j - a_i \notin S(b_i,b_j)$ for $1 \leq i \neq j \leq k$, then $L$ is an irreducible representation of $Y(g)$.

**Remark 3.12.**

(i). In Section 6.2 [2], V. Chari gave the set $S(i_1,i_2)$, $i_1 \leq i_2$ of values of $a_1^{-1}a_2$ for which the tensor product $V(i_1,a_1) \otimes V(i_2,a_2)$ may fail to be irreducible as a module over the quantum loop algebra. Note that, in the following discussion, we interchange the node labels on the Dynkin diagram of $G_2$ as used in that paper. V. Chari found that

- $S(1,1) = \{q^6,q^8,q^{10},q^{12}\}$
- $S(1,2) = \{q^3,q^7\}$
- $S(2,1) = \{q^7,q^{11}\}$
- $S(2,2) = \{q^2,q^6,q^8,q^{12}\}$

(ii). In this paragraph, let $g$ be any finite-dimensional complex simple Lie algebra of rank $l$. The fundamental representations of $Y(g)$ can be treated as special cases of the Kirillov-Reshetikhin modules, which are the finite-dimensional irreducible representations associated to an $l$-tuple of polynomials $\pi = (\pi_1(u), \pi_2(u), \ldots, \pi_l(u))$ such that $\pi_i(u) = (1 - au^{-1})(1 - (a+1)u^{-1}) \ldots (1 - (a+m-1)u^{-1})$ and $\pi_j(u) = 1$, for all $j \neq i$. The methods used in this paper could shed some light on obtaining a concrete cyclicity condition for the tensor product of Kirillov-Reshetikhin modules of $Y(g)$.

We close this section by providing the structure of $W(\pi)$. To obtain an upper bound on the dimension of $W(\pi)$, we use the dimension of the local Weyl module $W(\lambda)$ of the current algebra $g[t]$, which is given in [11].

**Proposition 3.13** (Corollary 9.5, [11]). Let $\lambda = m_1\omega_1 + m_2\omega_2$. Then

$$\dim (W(\lambda)) = \left( \dim (W(\omega_1)) \right)^{m_1} \left( \dim (W(\omega_2)) \right)^{m_2}.$$ 

It follows from Theorem 3.11 and Proposition 2.15 in [5] that
Proposition 3.14. Let $\pi = (\pi_1(u), \pi_2(u))$ be a pair of monic polynomials in $u$, and let $\pi_i(u) = \prod_{j=1}^{m_i} (u - a_{i,j})$. Let $S = \{a_{1,1}, \ldots, a_{1,m_1}, a_{2,1}, \ldots, a_{2,m_2}\}$ be a multi-set of roots. Let $a_1 = a_{i,j}$ be one of the numbers in $S$ with maximal real part and let $b_1 = i$. Similarly, let $a_r = a_{s,t}(r \geq 2)$ be one of the numbers in $S \setminus \{a_1, \ldots, a_{r-1}\}$ with maximal real part and let $b_r = s$. Let $L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k})$, where $k = m_1 + m_2$. Then $L$ is a highest weight representation and its associated polynomials are $\pi_1(u)$ and $\pi_2(u)$.

Theorem 3.15. The local Weyl module $W(\pi)$ of $Y(\mathfrak{g})$ associated to $\pi$ is isomorphic to $L$ as in Proposition 3.14.

Proof. Let $\lambda = m_1 \omega_1 + m_2 \omega_2$. On the one hand, $\dim(W(\pi)) \leq \dim(W(\lambda))$ by Theorem 3.8 [12]; on the other hand, $\dim(W(\pi)) \geq \dim(L)$ by the maximality of the local Weyl modules of Yangians. Note that as $G_2$-modules, $W(\omega_i) \cong KR(\omega_i) \cong V_a(\omega_i)$ and the latter isomorphism follows easily from the main theorem of Section 2.3 in [4] and Theorem 6.3 in [5]. In particular, $\dim(W(\omega_i)) = \dim(V_a(\omega_i))$ for any $a \in \mathbb{C}$. Therefore, $\dim(W(\lambda)) = \dim(L)$. This implies that $\dim(W(\pi)) = \dim(L)$, and therefore $W(\pi) \cong L$.

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