FUJIKI RELATION ON SYMPLECTIC VARIETIES

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Abstract. We generalize Fujiki relation of Beauville-Bogomolov quadratic form on a projective symplectic variety. As an application, we study a fibre space structure of a projective symplectic variety.

1. Introduction

We start with the definition of a symplectic variety.

Definition 1.1. A compact Kähler variety $Z$ is said to be a symplectic variety if $Z$ satisfies the following conditions: For a smooth locus $U$ on $Z$, there exists a nondegenerate holomorphic 2-form $\omega$ on $U$. This form is extended to a regular form $\tilde{\omega}$ on $\tilde{Z}$, where $\nu: \tilde{Z} \to Z$ is a resolution of singularities of $Z$.

In [3, Theorem 8 (2)], Namikawa induced a quadratic form $q_Z$ on $H^2(Z, \mathbb{C})$, which is a natural extension of Beauville-Bogomolov quadratic form defined in [1, Théorème 5] on singular varieties. In [3, Theorem 4.7], Fujiki proved that Beauville-Bogomolov quadratic form has special relation with cup products. We prove that $q_Z$ has same properties.

Theorem 1.2. Let $Z$ be a $2n$-dimensional projective symplectic variety and $\nu: \tilde{Z} \to Z$ a resolution of singularities of $Z$. Assume that

1. The codimension of the singular locus of $Z$ is greater than four.
2. $Z$ has only $\mathbb{Q}$-factorial singularities.
3. $\dim H^1(Z, \mathcal{O}_Z) = 0$ and $\dim H^2(Z, \mathcal{O}_Z) = 1$.

According to [3, Theorem 8 (2)], we define the quadratic form $q_Z$ on $H^2(Z, \mathbb{C})$ by

$$q_Z(\alpha) := n \int_{Z} (\tilde{\omega})^{-1} (n-1) \alpha^2 + (1-2n) \left( \int_{Z} \tilde{\omega}^{-1-n} \tilde{\alpha} \int_{Z} \tilde{\omega}^{-n} \tilde{\omega}^{-1} \right)$$

where $\tilde{\alpha} := \nu^* \alpha$ and $\alpha \in H^2(Z, \mathbb{C})$. Then $q_Z(\alpha)$ satisfies the following equation:

$$C_Z q_Z(\alpha)^n = \alpha^{2n}.$$

Note that a constant $C_Z$ depends on only $Z$. Moreover the index of $q_Z$ is $(3, h^2(Z, \mathbb{C}) - 3)$.

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Remark 1. By [7, Theorem], when $Z$ is symplectic varieties with only terminal singularities, the condition of codimension of singular locus is always satisfied.

Remark 2. Namikawa obtained the index of $q_Z$ in [8, Corollary 8] by different method.

Theorem 1.3. Let $Z$ be a projective symplectic variety which satisfies the conditions of Theorem 1.2 and $D$ a Cartier divisor on $Z$. Then Riemann-Roch formula of $D$ is expressed as follows:

$$\chi(D) = \sum_{k=0}^{n} a_k q_Z(D)^k,$$

where $a_k$ $(0 \leq k \leq n)$ are constants which depend on only $Z$.

As an application of Theorem 1.2, we obtain the following result:

Corollary 1.4. Let $Z$ be a $2n$-dimensional projective symplectic variety which satisfies the conditions of Theorem 1.2. Assume that there exists a surjective morphism $f : Z \to B$ over a projective normal variety $B$ and a general fibre $F$ of $f$ has positive dimension. Then $F$ and $B$ has the following properties:

1. $\dim B = n$ and Picard number of $B$ is one.
2. $F$ is an $n$-dimensional abelian variety.
3. For the singular locus $Z_{\text{sing}}$ of $Z$, $f(Z_{\text{sing}})$ forms a proper closed subset of $B$ and the restriction $\omega$ to $F$ is identically zero.
4. For every effective divisor $D$ of $Z$, $\dim f(D) \geq \dim B - 1$.

Remark 3. Comparing the above corollary with [9, Theorem 2], there exist two difference: One is ampleness of $-K_B$ and another one is $\mathbb{Q}$-factoriality of $B$. In section 3, we construct an example such that $-K_B$ is not ample.

We prove Theorem 1.2 and 1.3 in Section 2. A proof of Corollary 1.4 and an example are given in Section 3.

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2. Proof of Theorem 1.2

(2.1) We start with the investigation of the rank of $q_Z$.

Lemma 2.1. The rank of $q_Z$ is greater than three.

Proof. Let $\nu : \tilde{Z} \to Z$ be a resolution of singularities of $Z$ and $\tilde{\omega}$ a extension of $\omega$ on $\tilde{Z}$. By the definition of $q_Z$, $q_Z(\tilde{\omega} + \tilde{\omega}) > 0$ and $q_Z(\tilde{\omega} - i\tilde{\omega}) > 0$. Let $\alpha$ be an ample divisor on $Z$. If we choose a suitable exceptional divisor $E$, $\nu^*\alpha - E$
becomes an ample divisor. From [1, Remark (1) (p24)], $\tilde{\omega}^{n-1}|_E = 0$. Hence
\[ q_Z(\alpha) = n \int_Z (\tilde{\omega})^{n-1}(\nu^*\alpha - E)^2. \]
Thus $q_Z(\alpha) > 0$ by Hodge-Riemann bilinear relation.

(2.2) Let $D$ be the Kuranishi space of $Z$, $Z$ the Kuranishi family and $Z_s$ the fibre at $s \in D$. By [3, Theorem 8], for every point of $s \in D$, there exists an isomorphism $\phi_s : H^2(Z, \mathbb{C}) \to H^2(Z_s, \mathbb{C})$. The period map $p : D \to \mathbb{P}(H^2(Z, \mathbb{C}))$ defined by $p(s) := \phi_s^{-1}(\omega_s)$, where $\omega_s$ is a symplectic form of $Z_s$.

**Lemma 2.2.** Let $\nu_s : \tilde{Z}_s \to Z_s$ be a resolution of $Z_s$. We define a quadric form on $H^2(Z_s, \mathbb{C})$ by
\[ q_{Z_s}(\alpha) := n \int_{\tilde{Z}_s} (\tilde{\omega}_s)^{n-1}\tilde{\alpha}^2 + (1 - 2n) \left( \int_{\tilde{Z}_s} \tilde{\omega}_s^{n-1} \tilde{\alpha} \int_{\tilde{Z}_s} \tilde{\omega}_s^{n-1} \tilde{\omega}_s \tilde{\alpha} \right), \]
where $\tilde{\alpha} := \nu^*_s\alpha$ and $\alpha \in H^2(Z_s, \mathbb{C})$. Then $q_{Z_s}$ is not depend the choice of $\tilde{Z}_s$ and $q_Z(\alpha) = q_Z(\phi_s^{-1}(\omega_s + \tilde{\omega}_s))q_{Z_s}(\phi_s(\alpha))$.

**Proof.** We prove that $q_{Z_s}$ is independent of the choice of resolution by similar argument in the proof of [3, Theorem 8 (2)] and obtain the rank of $q_{Z_s}$ is greater than three by similar argument of Lemma 2.1. The period map $p_s : D \to \mathbb{P}(H^2(Z_s, \mathbb{C}))$ is defined by $p_s(t) := \phi_s(p(t))$.

**Claim 2.3.** Let $Q_s := \{ q_{Z_s}(\alpha) = 0 | \alpha \in H^2(Z_s, \mathbb{C}) \}$. Then $p_s(D) \subset Q_s$.

**Proof.** Let $\alpha \in H^2(Z_s, \mathbb{C})$. For the smooth locus of $U_s$ of $Z_s$, there exists Hodge decomposition of $H^2(U_s, \mathbb{C})$ and an isomorphism $H^2(U_s, \mathbb{C}) \cong H^2(Z, \mathbb{C})$ by [3, Theorem 8]. Hence we write $\alpha$ as $\alpha = a\omega_s + w + b\tilde{\omega}_s$. By direct calculation
\[ \int_{\tilde{Z}_s} \tilde{\alpha}^{n+1}\tilde{\omega}_s^{n-1} = (n + 1)q_{Z_s}(\alpha) \left( \int_{\tilde{Z}_s} \tilde{\alpha}\tilde{\omega}_s^{n-1}\tilde{\omega}_s^{n-1} \right)^{-1}. \]
Let $E$ be an irreducible component of the exceptional locus of $\tilde{Z}_s \to Z_s$. Assume $\alpha := \phi_s\phi_t^{-1}(\omega_t)$. Then $\tilde{\alpha}^{n+1}|_{U_t} = 0$ because $\alpha|_{U_t}$ is a holomorphic two form on the smooth locus $U_t$ of $Z_t$ and $U_t$ is deformation equivalent to $U_s$ by [3, Theorem 8]. By [3, Remark (1) (p24)], $\tilde{\omega}_s^{n-1}|_E = 0$. Hence the left hand side equals zero. If $t$ is very near to $s$, we obtain $\int_{\tilde{Z}_s} \tilde{\alpha}\tilde{\omega}_s^{n-1}\tilde{\omega}_s^{n} \neq 0$. Thus we obtain $q_{Z_s}(\alpha) = 0$.

We continue the proof of Lemma 2.2. By the above Claim, $\phi_s$ maps an open set of $Q_0$ to an open set of $Q_s$ isomorphically. Since both quadric surfaces are irreducible, we obtain that $\phi_s(Q_0) = Q_s$. Since $q_{Z_s}(\omega_s + \tilde{\omega}_s) = 1$, we are done.

(2.3) The following Lemma is the key of the proof of Theorem 1.2 and 1.3 which is based on arguments in [3, Lemma 1.9] and [4, Theorem 5.6].
Lemma 2.4. Let \( \nu : \tilde{Z} \to Z \) be a resolution of \( Z \) and \( D^0 \) the open set of \( D \) such that the morphism \( \tilde{Z} \to D \) is smooth over \( D^0 \). We fix one point \( s \in D^0 \). Let \( \alpha \in H^2(Z_s, \mathbb{C}) \) and \( \tau \) a \((k,k)\)-form on \( \tilde{Z}_s \). If \( \tau \) is a \((k,k)\)-form on every fibre \( \tilde{Z}_t \) near \( s \),

\[
\tau \cdot \nu^* \alpha^{2(n-k)} = C_k q_{Z_s}(\alpha)^{n-k},
\]

where \( C_k \) is a constant depending on only \( Z_s \).

Proof. Let \( S_s \) be the hypersurface in \( H^2(Z_s, \mathbb{C}) \) defined by

\[
\{ \tau \cdot \nu^* \alpha^{2(n-k)} = 0 | \alpha \in H^2(Z_s, \mathbb{C}) \}.
\]

If we prove that \( Q_s = S_s \), we obtain equation (1) because both hand sides of (1) have same degree and same zero locus. We choose an open set \( V \) of \( Q_s \) which is contained in \( p_s(D^0) \). For every point \( \alpha \in V \), there exists the point \( t \in D^0 \) such that \( \alpha \) defines a symplectic form on the smooth part of \( Z_t \). Thus \( \tau \cdot (\nu^* \alpha)^{n-k+1} = 0 \) in \( H^{2n+2}(\tilde{Z}_t, \mathbb{C}) \) and hence \( \tau \cdot (\nu^* \alpha)^{n-k+1} = 0 \) in \( H^{2n+2}(\tilde{Z}_s, \mathbb{C}) \). By analytic continuation, we obtain \( \tau \cdot (\nu^* \alpha)^{n-k+1} = 0 \) for every point of \( Q_s \). Therefore we obtain \( Q_s \subset S_s \). We prove the opposite inclusion. Assume the contrary. Then there exists \( \beta \in H^2(Z_s, \mathbb{C}) \) such that \( \tau \cdot \nu^* \beta^{2(n-k)} = 0 \) and \( \beta \notin Q_s \). We choose a general element \( \gamma \) of \( H^2(Z_s, \mathbb{C}) \) such that \( \tau \cdot \gamma^{2(n-k)} \neq 0 \). Then the line which pass \( \beta \) and \( \gamma \) intersect \( Q_s \) with two points. Let \( \delta_0 \) and \( \delta_1 \) be these points. We write

\[
\beta = \lambda_{11} \delta_0 + \lambda_{12} \delta_1, \quad \gamma = \lambda_{21} \delta_0 + \lambda_{22} \delta_1.
\]

We remark that \( \lambda_{sr} \neq 0 \) because \( \beta, \gamma \) and \( \delta_s \) are mutually distinct. Since \( \tau \cdot (\nu^* \delta_0)^{n-k+1} = \tau \cdot (\nu^* \delta_1)^{n-k+1} = 0 \),

\[
\tau \cdot (\nu^* \beta)^{2(n-k)} = 2^{(n-k)} C_{n-k} \lambda_{11}^{n-k} \lambda_{12}^{n-k} \lambda_{21}^{n-k} \lambda_{22}^{n-k} = 0
\]

\[
\tau \cdot (\nu^* \gamma)^{2(n-k)} = 2^{(n-k)} C_{n-k} \lambda_{21}^{n-k} \lambda_{22}^{n-k} \lambda_{11}^{n-k} \lambda_{12}^{n-k} \neq 0.
\]

That derives a contradiction. \( \square \)

(2.4) Proof of Theorem 1.2. From Lemma 2.4, we obtain

\[
C q_{Z_s}(\phi_s(\alpha))^n = \phi_s(\alpha)^{2n}.
\]

Since \( \phi_s(\alpha)^{2n} = \alpha^{2n} \), we obtain that \( C q_{Z}(\alpha)^n = \alpha^{2n} \) by Lemma 2.2. We investigate the index of \( q_Z \). From [8, Proposition 9], there exists Hodge decomposition of \( H^2(Z, \mathbb{C}) \). Let \( A \) be an ample divisor on \( Z \) and \( H \) an element of \( H^{1,1}(Z, \mathbb{C})_R \) such that \( q_Z(H, A) = 0 \). We consider the following equation:

\[
C q_Z(\lambda H + A)^n = (\lambda H + A)^{2n}.
\]

If we compare \( \lambda \) term of both hand sides, we obtain \( H.A^{2n-1} = 0 \) from the assumption \( q_Z(A, H) = 0 \). By Hodge-Riemann bilinear relation,

\[
H^2.A^{2n-2} . A^{2n} \leq (H.A^{2n-1})^2 = 0.
\]
Hence $H^2A^{2n-2} < 0$ if $H \not\equiv 0$. Comparing $\lambda^2$ term of the both hand side of the equation (2), we obtain $qZ(H) < 0$ if $H \not\equiv 0$. Combining Lemma 2.1, the index of $qZ$ is $(3, h^2(Z, \mathbb{C}) - 3)$. 

(2.5) Proof of Theorem 1.3. From the proof of (1) of [6, Theorem 8],

$$\phi_s : H^2(Z, \mathbb{C}) \cong H^2(Z, \mathbb{C}) \to H^2(Z_s, \mathbb{C})$$

is isomorphism. Hence $\chi_Z(D) = \chi_{Z_s}(\phi_s(D))$. By Lemma 2.2, it is enough to prove that there exists constants $C_k$ ($0 \leq k \leq n$) and they satisfy

$$\chi_{Z_s}(D) = \sum_{i=0}^{n} C_k q_{Z_s}(D)^k,$$

due to a Cartier divisor $D$ of $Z_s$. Let us consider $\chi_{Z_s}(\nu^*D)$. By [6, Theorem 6], $Z_s$ has only rational singularities. Thus $\chi_{Z_s}(D) = \chi_{Z_s}(\nu^*D)$. By Serre duality

$$\chi_{Z_s}(D) = \chi_{Z_s}(-D).$$

Hence each term of Riemann-Roch formula of $\chi_{Z_s}(\nu^*D)$ is expressed

$$(\text{Polynomial of Chern classes of } Z_s) \cdot (\nu^*D)^{2k}.$$ 

Since polynomials of Chern classes remain $(k, k)$-form under small deformation, we obtain each term of Riemann-Roch formula is expressed as $C_k q_{Z_s}(D)^k$ by Lemma 2.4. 

3. Fibre space structure

(3.1) We prove Corollary 1.4 in three steps:

**Step 1:** $\dim B = n$.

**Step 2:** For a general fibre $F$ of $f$, the restriction of a symplectic form on the smooth locus of $F$ is identically zero.

**Step 3:** $F$ is an Abelian variety and $f(Z_{\text{sing}})$ forms a proper closed subset of $B$.

**Step 4:** $\rho(B) = 1$

**Step 5:** For every effective divisor $D$ on $Z$, $\dim f(D) \geq n - 1$.

We fix some notations. Let $A$ be an ample divisor on $Z$ and $H$ an ample divisor on $B$.

(3.2) Step 1. From Theorem 1.2,

$$CZqZ(A + \lambda f^*H)^n = (A + \lambda f^*H)^{2n}.$$ 

Since $CZqZ(f^*H) = (f^*H)^{2n} = 0$, we obtain

$$CZ(qZ(A) + 2\lambda qZ(A, f^*H))^n = (A + \lambda f^*H)^{2n}.$$
If we compare both hand sides the above equation, we obtain
\[ A^k (f^*H)^{2n-k} = 0, \quad (k < n) \]
\[ 2nA^{2n-1} f^*H = C_Z q_Z(A, f^*H) \cdot q_Z(A)^{n-1}. \]

By Lemma 2.1, \( q_Z(A) > 0 \). Hence \( q_Z(A, f^*H) > 0 \) and
\[ A^k (f^*H)^{2n-k} > 0, \quad (k \geq n). \]

Hence \( \dim B = n \).

(3.3) **Step 2.** Let \( F \) be a general fibre of \( f \) and \( U_F \) the smooth part of \( F \). In order to prove \( \omega|_{U_F} \equiv 0 \), we prove
\[ \int_F \omega \wedge \bar{\omega} A^{n-2} = 0. \]

We have
\[ \int_F \omega \wedge \bar{\omega} A^{n-2} = c(\omega \bar{\omega} A^{n-2}(f^*H)^n), \]
where \( c \) is a nonzero constant. Hence we will show \( \omega \bar{\omega} A^{n-2}(f^*H)^n = 0 \). By Theorem 1.2,
\[ C_Z q_Z(\omega + \bar{\omega} + sA + tf^*H, \omega + \bar{\omega} + sA + tf^*H)^n = (\omega + \bar{\omega} + sA + tf^*H)^{2n}. \]

Calculating the left hand side, we obtain
\[ C_Z(q_Z(\omega + \bar{\omega}) + s^2 q_Z(A) + 2sq_Z(\omega + \bar{\omega}, A) + 2tq_Z(\omega + \bar{\omega}, f^*H) + 2stq_Z(A, f^*H))^n. \]

From the definition of \( q_Z \) in Theorem 1.2,
\[ q_Z(\omega + \bar{\omega}, A) = q_Z(\omega + \bar{\omega}, f^*H) = 0. \]

Thus we conclude that \( \omega \bar{\omega} A^{n-2}(f^*H)^n = 0 \) by comparing the \( s^{n-2}t^n \) term of both hands sides.

(3.4) **Step 3.** Let \( U \) be the smooth locus of \( Z \). If we choose a point \( x \) of \( B \) generally, \( f^o : U \to B \) is smooth at \( x \) and \( U_F = F \cap U \). We consider the following diagram:
\[
\begin{array}{c}
0 \to f^* T_{B,x} \to T_U \to T_{U_F} \to 0 \\
\downarrow \\
0 \to \Omega^1_{U_F} \to \Omega^1_U \to f^* \Omega^1_{B_F} \to 0.
\end{array}
\]

From the above diagram and Step 2, \( h^0(T_{U_F}) = n \). Then \( F \) is an Abelian variety by the following Lemma.

**Lemma 3.1.** Let \( F \) be a normal variety such that \( K_F \sim O_F \) and \( h^0(T_{U_F}) = \dim F \), where \( U_F \) is the smooth locus of \( F \). Then \( F \) is an Abelian variety.
Proof. Since $F$ is normal, $\dim(F \setminus U_F) \leq \dim F - 2$. Hence $h^0(\Theta_F) = \dim F$ by analytic continuation. Let $\tilde{F}$ be a resolution of $F$. Then $h^0(T_{\tilde{F}}) = h^0(\Theta_F) = \dim F$. Since $K_F \sim O_F$, we have an injection $T_{\tilde{F}} \to O_F^{\dim F-1}$. Hence $h^{\dim F-1}(O_{\tilde{F}}) = h^0(\Omega_{\tilde{F}}^{\dim F-1}) \geq \dim F$. By Serre duality $h^{\dim F-1}(O_{\tilde{F}}) = h^1(K_{\tilde{F}})$.

From Grauert-Riemenschneider vanishing theorem, $h^1(K_{\tilde{F}}) = h^1(\pi_* K_{\tilde{F}}) = h^1(O_F)$. Hence $h^1(F, O_F) \geq \dim F$. By [3, Theorem 13], $F$ is an abelian variety. \hfill \Box

Since $F$ is a complete intersection in $Z$, $Z$ is smooth in a neighbourhood of $F$ and $f(Z_{\text{sing}})$ forms a proper closed subset of $B$.

(3.5) Step 4. Let $D$ be a Cartier divisor on $B$. We prove that $f^*(D + \lambda H) \equiv 0$ for a suitable number $\lambda$. From Theorem [12], $q_Z$ is nondegenerate. Hence it is enough to prove that

$$q_Z(f^*(D + \lambda H)) = q_Z(f^*(D + \lambda H), A) = 0.$$  

From Theorem [12], we obtain $C_Zq_Z(f^*(D + \lambda H))^n = (f^*(D + \lambda H))^{2n} = 0$ for every $\lambda$. Hence if we choose $\lambda$ suitably, we obtain $q_Z(f^*(D + \lambda H), A) = 0$. \hfill \Box

(3.6) Step 5. Let $D$ be an effective divisor on $Z$. We derive a contradiction assuming that $\dim f(D) < \dim B - 1$. Under this assumption, we obtain that $D, f^*H^{n-1}$ is numerically trivial. The following equation

$$C_Zq_Z(D + tf^*H)^n = (D + tf^*H)$$

tells us that $q_Z(D, f^*H) = 0$ because $D^n f^*H^n = 0$. We consider the following equation:

$$(sD + tf^*H + A)^{2n} = C_Zq_Z(sD + tf^*H + A)^n.$$  

Comparing $st^{n-1}$-term of the both hand sides, we obtain that $q_Z(D, A)q_Z(A, f^*H) = 0$. For an effective divisor $E$ on $Z$, we consider the following equation:

$$C_Zq_Z(tE + A)^n = (tE + A)^{2n}.$$  

Comparing $t$-term of the both hand sides, we obtain that $q_Z(D, A) > 0$ and $q_Z(f^*H, A) > 0$. That derives a contradiction. \hfill \Box

(3.7) Under the conditions of Corollary [14], there exists an example such that $-K_B$ is not ample.

Example. Let $E_i$ be an elliptic curve. We consider the abelian 6-fold $\tilde{Z} := E_1 \times \cdots \times E_6$. Let $G$ be the finite subgroup of $GL(6, \mathbb{C})$ generated by

$$\begin{pmatrix}
\zeta_6 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_6^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_3^2
\end{pmatrix}$$

and

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
where $\zeta_n$ means a $n$-th root of unity. Then the quotient $Z := \tilde{Z}/G$ satisfies the conditions Theorem 1.2 and it admits a fibration $Z \to (E_1 \times E_3 \times E_5)/G$. By direct calculation, $K_{E^3/G} \equiv 0$.

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