COMBINATORIAL RICCI FLOW ON CUSPED 3-MANIFOLDS

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Abstract. Combinatorial Ricci flow on a cusped 3-manifold is an analogue of Chow-Luo’s combinatorial Ricci flow on surfaces and Luo’s combinatorial Ricci flow on compact 3-manifolds with boundary for finding complete hyperbolic metrics on cusped 3-manifolds. Dual to Casson and Rivin’s program of maximizing the volume of angle structures, combinatorial Ricci flow finds the complete hyperbolic metric on a cusped 3-manifold by minimizing the co-volume of decorated hyperbolic polyhedral metrics. The combinatorial Ricci flow may develop singularities. We overcome this difficulty by extending the flow through the potential singularities using Luo-Yang’s extension. It is shown that the existence of a complete hyperbolic metric on a cusped 3-manifold is equivalent to the convergence of the extended combinatorial Ricci flow, which gives a new characterization of existence of a complete hyperbolic metric on a cusped 3-manifold dual to Casson and Rivin’s program. The extended combinatorial Ricci flow also provides an effective algorithm for finding complete hyperbolic metrics on cusped 3-manifolds.

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1. Introduction

If $M$ is the interior of a compact oriented 3-manifold with boundary consisting of tori, $M$ is called a cusped 3-manifold, the complete hyperbolic metric on which is proved to be unique up to isometry [34, 38]. By subdividing $M$ into ideal tetrahedra and giving them shapes with hyperbolic ideal tetrahedra, Thurston [44] wrote down a system of gluing equations (now named after him) for the complex parameters of the hyperbolic ideal tetrahedra, the solution of which corresponds to a complete hyperbolic metric on the ideally triangulated manifold $M$.

However, it is difficult to solve Thurston’s gluing equation directly in practice. Motivated by [8], Casson and Rivin proposed to solve Thurston’s gluing equation by maximizing the volume function on the space of angle structures, which is a convex polytope of the dihedral angles of the tetrahedra in the triangulation. The readers can also refer to [4, 7, 24, 28, 30, 31, 41, 43] and others for topics related to Casson and Rivin’s program. The approach of combinatorial Ricci flow on cusped 3-manifolds introduced by Yang [51]
is dual to Casson and Rivin’s program, in which one works on decorated hyperbolic polyhedral metrics to ensure that there is no shearing around edges and finds the complete hyperbolic metrics on cusped 3-manifolds by adjusting the cone angles through adjusting the edge lengths according to the combinatorial Ricci curvature along the edges.

Suppose $M$ is an ideally triangulated cusped 3-manifold with a triangulation $\mathcal{T}$ and $E$ is the edge set in $\mathcal{T}$. A decorated hyperbolic polyhedral metric on $(M, \mathcal{T})$ is defined to be the hyperbolic conic metric obtained by isometrically gluing decorated hyperbolic ideal tetrahedra along the faces such that the decorations are preserved. The decorated hyperbolic polyhedral metrics on $(M, \mathcal{T})$ are parameterized by the edge lengths. The combinatorial Ricci curvature $K_i$ along an edge $i \in E$ is defined to be $2\pi$ less the cone angle along the edge $i \in E$.

Motivated by Chow-Luo’s combinatorial Ricci flow on surfaces [6] and Luo’s combinatorial Ricci flow on compact 3-manifolds with boundary [26], Yang [51] introduced the following combinatorial Ricci flow on cusped 3-manifolds

$$\frac{dl_i}{dt} = K_i,$$

where $l : E \to \mathbb{R}$ is the edge length function for the decorated hyperbolic polyhedral metric. The combinatorial Ricci flow (1.1) may develop singularities, which correspond to flat hyperbolic ideal tetrahedra [51]. Under an additional condition that the combinatorial Ricci flow (1.1) does not develop singularities along the flow in finite and infinite time, Yang [51] proved the convergence of the combinatorial Ricci flow (1.1). This condition is too strong to be satisfied in practice. In this paper, we shall remove this condition by extending the combinatorial Ricci flow (1.1) through the potential singularities of the flow. This gives rise to a new characterization of existence of a complete hyperbolic metric on a cusped 3-manifold using the extended combinatorial Ricci flow, which is dual to Casson-Rivin’s program. The extended combinatorial Ricci flow also provides an effective algorithm for finding the complete hyperbolic metric on a cusped 3-manifold.

**Theorem 1.1.** Suppose $(M, \mathcal{T})$ is an ideally triangulated cusped 3-manifold which supports a complete hyperbolic metric. Then a decorated hyperbolic polyhedral metric $l^*$ corresponds to a complete hyperbolic metric on $(M, \mathcal{T})$ if and only if the extended combinatorial Ricci flow converges exponentially fast to $l^*$ for any initial decorated hyperbolic polyhedral metric on $(M, \mathcal{T})$.

The decorated hyperbolic metric $l^*$ in Theorem 1.1 has zero combinatorial Ricci curvature and was proved to be unique up to change of decorations by Luo-Yang [32] using nonnegative angle structures and Fenchel dual of the volume function. We will provide a short and direct proof of Luo-Yang’s rigidity without involving angle structures and Fenchel dual. A result paralleling to Theorem 1.1 has been obtained by the author [49] for compact 3-manifolds with boundary, which confirms a conjecture of Luo [26] on the
global convergence of combinatorial Ricci flow on compact 3-manifolds with boundary. There have been many important work on combinatorial curvature flows on surfaces. See [6, 10, 16, 18, 19, 25, 52] and others.

The paper is organized as follows. In Section 2 we present preliminary results on decorated hyperbolic ideal tetrahedra and decorated hyperbolic polyhedral metrics on cusped 3-manifolds. In Section 3 we study the basic properties of the combinatorial Ricci flow (1.1) for decorated hyperbolic polyhedral metrics on cusped 3-manifolds. In Section 4 we extend the definition of dihedral angles in a decorated hyperbolic ideal tetrahedron and the definition of combinatorial Ricci curvature on cusped 3-manifolds. In Section 5 we extend the combinatorial Ricci flow (1.1) through the potential singularities and prove a generalization of Theorem 1.1.

2. Preliminary on decorated hyperbolic ideal tetrahedra and cusped 3-manifolds

In this section, we give some preliminaries on ideal triangulations of cusped 3-manifolds, decorated hyperbolic ideal tetrahedra and decorated hyperbolic polyhedral metrics on cusped 3-manifolds. For more details, please refer to [1, 7, 30, 32, 40, 42].

2.1. Triangulations of cusped 3-manifolds. Suppose $(M^*, T^*)$ is a closed 3-manifold with a triangulation $T^* = (V, E, F, T)$, where $V, E, F, T$ are the sets of vertices, edges, faces and tetrahedra respectively. If we remove the vertices from $T^*$, then we get an ideal triangulation $T$ of the manifold $M = M^* \setminus V$, which is composed of ideal tetrahedra. If for each vertex $p \in V$, the link of $p$ in $M^*$ is a torus, then $(M, T)$ is called an ideally triangulated cusped 3-manifolds. We still call $E, F, T$ as the sets of vertices, edges and tetrahedra in $T$ respectively. Each vertex $p \in V$ corresponds to a cusp of the ideally triangulated cusped 3-manifold $(M, T)$. For an ideally triangulated cusped 3-manifold $(M, T)$, the number of edges is equal to the number of tetrahedra in the triangulation. For simplicity, we will denote $N$ as the number of the edges and $s$ as the number of cusps in $T$ in the following.

2.2. Decorated hyperbolic ideal tetrahedron. A hyperbolic ideal tetrahedron is a hyperbolic tetrahedra in $\mathbb{H}^3$ with vertices at infinity. The vertices at infinity are different and not part of the hyperbolic ideal tetrahedron. A hyperbolic ideal tetrahedron could be taken as the convex hull of four distinct points $v_1, v_2, v_3, v_4$ in $\mathbb{H}^3$ such that $v_1, v_2, v_3, v_4$ are not in a round circle.

Similar to the decorated ideal triangles in [30], one can also introduce the decoration to parameterize ideal tetrahedron by edge length. Suppose $\tau$ is a hyperbolic ideal tetrahedron generalized by $v_1, v_2, v_3, v_4 \in \partial \mathbb{H}^3$. $H_1, H_2, H_3, H_4$ are four horospheres attached to the ideal vertices $v_1, v_2, v_3, v_4$ respectively. Then $\sigma = (\tau, (H_1, H_2, H_3, H_4))$ is called a decorated hyperbolic
ideal tetrahedron and \((H_1, H_2, H_3, H_4)\) is called a decoration. For the edge 
\(e = v_iv_j\) of the hyperbolic ideal tetrahedron \(\tau\), if \(H_i \cap H_j = \emptyset\), we define 
the edge length \(l_{ij}\) of \(v_iv_j\) to be the hyperbolic distant of \(H_i \cap e\) and \(H_j \cap e\). If 
\(H_i \cap H_j \neq \emptyset\), then \(-l_{ij}\) is the hyperbolic distant of \(H_i \cap e\) and \(H_j \cap e\). Using 
Penner’s cosine law, the length of the arc between \(H_i \cap v_iv_j\) and \(H_i \cap v_kv_l\) in 
\(H_i \cap \triangle ijk\) is given by \(e(l_{ij} - l_{ij} - l_{kl})/2\). Note that \(H_i \cap \tau\) is a Euclidean triangle. 
Using Penner’s cosine law for decorated hyperbolic ideal triangles, we have 
the following characterization of nondegenerate decorated hyperbolic ideal 
tetrahedra in terms of the edge lengths.

**Lemma 2.1** ([2] Lemma 5.2.3, [30] Lemma 2.5, [32] Lemma 2.1). Suppose 
\(\tau = \{1234\}\) is an ideal tetrahedron. Then \((l_{12}, \ldots, l_{34}) \in \mathbb{R}^6\) corresponds to 
the edge lengths of a nondegenerate decorated hyperbolic ideal tetrahedron if 
and only if

\[
e^{(l_{ij}+l_{ik})/2} + e^{(l_{ik}+l_{jk})/2} > e^{(l_{ik}+l_{jk})/2} \quad (2.1)
\]

for \(\{i, j, k, h\} = \{1, 2, 3, 4\}\). Furthermore, when \((l_{12}, \ldots, l_{34}) \in \mathbb{R}^6\) is the 
edge length of a nondegenerate hyperbolic ideal tetrahedron, the dihedral 
angle \(\alpha_{ij}\) along the edge \(v_iv_j\) equals the inner angle opposite to the edge 
with length \(e^{(l_{ij}+l_{lk})/2}\) in the triangle formed by three edges with lengths 
\(e^{(l_{ij}+l_{lk})/2}, e^{(l_{ik}+l_{jk})/2}\) and \(e^{(l_{ik}+l_{jk})/2}\).

Lemma 2.1 implies that the dihedral angles along the edges in a decorated 
hyperbolic ideal tetrahedron is independent of the choice of decorations and 
the dihedral angles along opposite edges are equal. As a direct consequence 
of Lemma 2.1, the admissible space of edge lengths for a decorated hyperbolic 
ideal tetrahedron \(\sigma\) is given by

\[
\mathcal{L}_\sigma = \{(l_{12}, \ldots, l_{34}) \in \mathbb{R}^6 | e^{(l_{ij}+l_{ik})/2} + e^{(l_{ik}+l_{jk})/2} > e^{(l_{ik}+l_{jk})/2}, \{i, j, k, h\} = \{1, 2, 3, 4\}\}.
\]

The admissible space \(\mathcal{L}_\sigma\) is nonconvex and invariant under the change of 
decorations. Note that \(\mathbb{R}^6 \setminus \mathcal{L}_\sigma = V_{12}^{34} \cup V_{13}^{24} \cup V_{14}^{23}\), where

\[
V_{ij}^{kh} = \{(l_{12}, \ldots, l_{34}) \in \mathbb{R}^6 | e^{(l_{ik}+l_{jk})/2} + e^{(l_{ih}+l_{jk})/2} \leq e^{(l_{ij}+l_{kh})/2}\}.
\]

It is directly to see that \(V_{ij}^{kh} = V_{kh}^{ij}\) and

\[
V_{ij}^{kh} = \{(l_{12}, \ldots, l_{34}) \in \mathbb{R}^6 | l_{ij} \geq -l_{kh} + 2 \ln\left(e^{(l_{ih}+l_{jk})/2} + e^{(l_{ih}+l_{jk})/2}\right)\}
\]
is the closed region above an analytical function defined on \(\mathbb{R}^5\). Furthermore, 
\(V_{12}^{34}, V_{13}^{24}, V_{14}^{23}\) are mutually disjoint, which follows from the fact that three 
positive constants \(a, b, c\) can not satisfy \(a \geq b + c\) and \(b \geq a + c\) simultaneously. 
This implies the admissible space \(\mathcal{L}_\sigma = \mathbb{R}^6 \setminus (V_{12}^{34} \cup V_{13}^{24} \cup V_{14}^{23})\) is homotopy 
equivalent to \(\mathbb{R}^6\) and then simply connected. In summary, we have the 
following corollary of Lemma 2.1.
Corollary 2.1. The admissible space $L_\sigma$ of edge lengths for a decorated hyperbolic ideal tetrahedron being nondegenerate is simply connected. Furthermore, the boundary of $L_\sigma$ consists of three disjoint analytical hypersurface of $\mathbb{R}^6$.

The idea to prove simply connectivity of admissible space by homotopy equivalence was first introduced by the author in [47] and then further applied in [22, 23, 48] to prove simply connectivity of admissible spaces in different cases. It seems that this is a basic technique for handling similar problems.

For a decorated hyperbolic ideal tetrahedron $\sigma$, one has the following Schlafli formula [3, 27, 33, 41, 45],

$$-2d(vol_\sigma) = \sum_{i<j} l_{ij} d\alpha_{ij}, \quad (2.2)$$

where $vol_\sigma$ is the volume of the hyperbolic ideal tetrahedron. The volume $vol_\sigma$ could be explicitly computed using the formula $vol_\sigma = \frac{1}{2} \sum_{i<j} \Lambda(\alpha_{ij})$, where $\Lambda(x) = -\int_0^x \ln |2 \sin t| dt$ is the Lobachevsky function. One can refer to [33, 39] for this formula.

Dual to the volume, the co-volume of a decorated hyperbolic ideal tetrahedron $\sigma$ is defined to be

$$cov_\sigma = 2vol_\sigma + \sum_{i<j} \alpha_{ij} l_{ij}.$$

By Schlafli formula (2.2), we have

$$d(cov_\sigma) = 2d(vol_\sigma) + \sum_{i<j} d(\alpha_{ij} l_{ij}) = \sum_{i<j} \alpha_{ij} dl_{ij},$$

which implies $\frac{dcov_\sigma}{dl_{ij}} = \alpha_{ij}$. Then we have

$$\frac{\partial \alpha_{ij}}{\partial l_{kh}} = \frac{\partial^2 cov_\sigma}{\partial l_{kh} \partial l_{ij}} = \frac{\partial^2 cov_\sigma}{\partial l_{ij} \partial l_{kh}} = \frac{\partial \alpha_{kh}}{\partial l_{ij}},$$

which implies the Jacobian matrix $J_\sigma$ of the dihedral angles with respect to the edge lengths in a decorated hyperbolic ideal tetrahedron is symmetric. By Corollary 2.1, the following integral

$$F_\sigma(l) = \int_0^l \sum_{i<j} \alpha_{ij} dl_{ij} \quad (2.3)$$

is a well-defined smooth function on $L_\sigma$. As $dF_\sigma = \sum_{i<j} \alpha_{ij} dl_{ij} = d(cov_\sigma)$, $F_\sigma$ differs from the co-volume function $cov_\sigma$ by a constant. Luo-Yang [32] found the deep relationship between the volume function $vol_\sigma$ and the co-volume function $cov_\sigma$ via Fenchel dual. The readers can refer to [32] for more details on this.

For a decorated hyperbolic ideal tetrahedron $\sigma = (\tau, \{H_1, H_2, H_3, H_4\})$, using the relationship between edge lengths and inner angles in a Euclidean
triangle \( \tau \cap H_i \), one has the following result on the Jacobian matrix \( J_\sigma \) of the dihedral angles with respect to the edge lengths.

**Lemma 2.2 (\cite{32, 51}).** For a decorated hyperbolic ideal tetrahedron \( \sigma \), the Jacobian matrix of the six dihedral angles with respect to the six edge lengths is given by

\[
J_\sigma = \left( \frac{\partial \alpha}{\partial l} \right) = \frac{1}{2} \begin{pmatrix} M & M \\ M & M \end{pmatrix},
\]

where \( M \) is a \( 3 \times 3 \) matrix given by

\[
M = \begin{pmatrix} \cot \alpha_{13} + \cot \alpha_{14} & -\cot \alpha_{14} & -\cot \alpha_{13} \\ -\cot \alpha_{14} & \cot \alpha_{12} + \cot \alpha_{14} & -\cot \alpha_{12} \\ -\cot \alpha_{13} & -\cot \alpha_{12} & \cot \alpha_{12} + \cot \alpha_{13} \end{pmatrix}.
\]

Here the labels for the six columns and rows in \( J_\sigma \) are \( 12, 13, 14, 34, 24, 23 \) so that \( i \)-th column and \( (i+3) \)-th column, \( i \)-th row and \( (i+3) \)-th row correspond to opposite edges. Furthermore, \( J_\sigma \) is positive semi-definite with null space spanned by the following 4 linearly independent vectors

\[
(1, 1, 1, 0, 0, 0)^T, (1, 0, 0, 0, 1, 1)^T, (0, 1, 0, 1, 0, 1)^T, (0, 0, 1, 1, 1, 0)^T.
\]

Lemma 2.2 implies \( F_\sigma \) is locally convex on \( L_\sigma \) and strictly locally convex on \( L_\sigma \cap \text{Ker}(J_\sigma)^\perp \).

**2.3. Decorated hyperbolic polyhedral metrics on triangulated cusped 3-manifolds.** Suppose \((M, T)\) is an ideally triangulated cusped 3-manifold. For simplicity, we denote the tetrahedra in \((M, T)\) as \( \sigma_1, \cdots, \sigma_N \) in the following.

**Definition 2.1.** A decorated hyperbolic polyhedral metric on \((M, T)\) is a map \( l : E \to \mathbb{R} \) such that for any ideal tetrahedron \( \tau = \{ijkh\} \in T \), the six real numbers \( l_{ij}, l_{ik}, l_{ih}, l_{jk}, l_{jh}, l_{kh} \) form the edge lengths of a nondegenerate decorated hyperbolic ideal tetrahedron.

We denote the admissible space of decorated hyperbolic polyhedral metrics on \((M, T)\) as \( \mathcal{L}(T) \) in the following. For simplicity of notations, we will label the edges with one index when we are handling problems for ideally triangulated cusped 3-manifolds and label the edges with two indices when we are handling problems for a single ideal tetrahedron in the following.

For the following applications, we introduce the following two notations.

**Definition 2.2 (\cite{7, 51}).** The cusp relation matrix \( C \) is an \( s \times N \) matrix with the \((i, j)\)-entry \( c_{ij} \) equal to the number of ends of edge \( j \) on the cusp \( i \).

The cusp relation matrix \( C \) has the following properties.

**Proposition 2.1 (\cite{51, 32, 35, 51}).** Suppose \( C \) is the cusp relation matrix for an ideally triangulated cusped 3-manifold \((M, T)\). Then

\(1\): \( \text{Rank}(C) = s \), which implies \( |V| \leq |E| = |T| \) for an ideally triangulated cusped 3-manifold \((M, T)\).
(2): Two decorated hyperbolic polyhedral metrics \( l_A, l_B \) correspond to the same hyperbolic polyhedral metric if and only if \( l_A \) and \( l_B \) differ by a change of decorations, which is equivalent to \( l_A - l_B \in \text{Im}(C^T) \).

**Definition 2.3** ([51]). The incident matrix \( G \) is the \( N \times 6N \) matrix, so that for \( n = 1, \cdots, 6 \) and \( i, j = 1, \cdots, N \), the \((i, 6j + n - 6)\)-entry of \( G \) is 1 if and only if the \( n \)-th edge in the tetrahedron \( \sigma_j \) is from the \( i \)-th edge in \( E = E(T) \).

Set \( J = \text{diag}\{J_1, \cdots, J_N\} \), where \( J_i \) is the Jacobian matrix for the tetrahedron \( \sigma_i \). Then \( J \) is symmetric and positive semi-definite by Lemma 2.2. By the chain rules, the Jacobian matrix \( \Lambda \) of the combinatorial Ricci curvature \( K \) with respect to the edge length \( l \in \mathcal{L}(T) \) is given by

\[
\Lambda := \left( \frac{\partial K}{\partial l} \right) = -GG^T.
\] (2.4)

The Jacobian matrix \( \Lambda \) has the following useful property (see for instance [5] Page 1354, [51] Proposition 4.13).

**Theorem 2.1.** The Jacobian matrix \( \Lambda \) of the combinatorial Ricci curvature \( K \) with respect to the edge length \( l \in \mathcal{L}(T) \) is symmetric and negative semi-definite. Furthermore,

\[ \text{Ker}(\Lambda) = \text{Im}(C^T) \]

and \( \Lambda \) is strictly negative definite on \( \text{Ker}(\Lambda)^\perp = \text{Ker}(C) \).

Theorem 2.1 implies the decorated hyperbolic polyhedral metric is locally determined by combinatorial Ricci curvature up to change of decorations. The global rigidity of decorated hyperbolic polyhedral metric has been proved by Luo-Yang [32].

Using the co-volume function \( \text{cov}_{\sigma} \) for a single decorated hyperbolic ideal tetrahedron \( \sigma \), one can define the following co-volume function \( \text{cov} \) for a decorated hyperbolic polyhedral metric \( l \in \mathcal{L}(T) \)

\[ \text{cov} = \sum_{\sigma \in T} \text{cov}_{\sigma}, \]

which is a locally convex function defined on the admissible space \( \mathcal{L}(T) \) with \( \frac{\partial \text{cov}}{\partial l_i} = 2\pi - K_i \) and strictly locally convex on \( \mathcal{L}(T) \cap \text{Ker}(C) \).

### 3. Basic properties of combinatorial Ricci flow

Following Chow-Luo’s combinatorial Ricci flow for circle packing metrics on surfaces [6] and Luo’s combinatorial Ricci flow for hyper-ideal polyhedral metrics on compact 3-manifolds with boundary [26], Yang [51] introduced the following combinatorial Ricci flow for decorated hyperbolic polyhedral metrics on cusped 3-manifolds.
Definition 3.1. [51] Suppose \((M, T)\) is an ideally triangulated cusped 3-manifold. The combinatorial Ricci flow for decorated hyperbolic polyhedral metrics on \((M, T)\) is defined to be
\[
\frac{dl_i}{dt} = K_i \tag{3.1}
\]
with \(l(0) = l_0 \in \mathcal{L}(T)\), where \(l_i\) is the length of edge \(i \in E\) and \(K_i\) is the combinatorial Ricci curvature along the edge \(i\).

The combinatorial Ricci flow (3.1) for decorated hyperbolic polyhedral metric is invariant under the change of decoration [51]. In other words, if \(l(t)\) is a solution of the combinatorial Ricci flow (3.1) and \(\tilde{l}(t) = l(t) + CTx\) for some \(x \in \mathbb{R}^V\), then \(\frac{d\tilde{l}_i}{dt} = K_i(\tilde{l})\), which implies \(\tilde{l}(t)\) is also a solution of the combinatorial Ricci flow (3.1). This follows from the fact that the dihedral angles and combinatorial Ricci curvatures are invariant under the change of decorations. This also implies the combinatorial Ricci flow (3.1) is a well-defined deformation of the hyperbolic polyhedral metrics on \((M, T)\). As the combinatorial Ricci flow (3.1) is an ODE system, the short time existence of the solution of (3.1) follows from the standard ODE theory [21].

Lemma 3.1 ([51]). Suppose \((M, T)\) is an ideally triangulated cusped 3-manifold. Then \(\sum_{i \sim p} l_i\) is invariant along the combinatorial Ricci flow (3.1), where \(p \in V\) is a cusp of \((M, T)\) and \(i \in E\) is an edge of \((M, T)\).

Proof. This follows from the fact that
\[
\frac{d(\sum_{i \sim p} l_i)}{dt} = \sum_{i \sim p} K_i = 0
\]
for cusped 3-manifolds. Q.E.D.

As a direct consequence of Lemma 3.1, \(\sum_{i=1}^{N} l_i\) is invariant along the combinatorial Ricci flow (3.1).

Remark 3.1. By Lemma 3.1, we can assume \(\sum_{i \sim p} l_i(0) = 0, \forall p \in V\), which is possible by Proposition [21]. Then the solution of the combinatorial Ricci flow (3.1) stays in the plane \(\{l \in \mathbb{R}^N \mid \sum_{i \sim p} l_i = 0, \forall p \in V\}\) by Lemma 3.1, which is in fact \(\text{Ker}(C)\). For simplicity, we will assume \(\sum_{i \sim p} l_i(0) = 0\) in the following.

Along the combinatorial Ricci flow (3.1), the combinatorial Ricci curvature evolves according to the following equation
\[
\frac{dK_i}{dt} = \sum_j \frac{\partial K_i}{\partial l_j} \frac{dl_j}{dt} = (\Lambda K)_i.
\]

As \(\Lambda\) is negative semi-definite, we can define the combinatorial Laplacian \(\Delta\) for a decorated hyperbolic polyhedral metric \(l\) as \(\Delta = \Lambda\). Then the combinatorial Ricci curvature evolves according to a discrete heat equation
\[
\frac{dK}{dt} = \Delta K
\]
along the combinatorial Ricci flow (3.1). This is very similar to Hamilton’s Ricci flow on 3-manifolds [20] and matches Luo’s motivation to define a combinatorial Ricci flow for compact 3-manifolds with boundary [26].

**Remark 3.2.** By Lemma 2.2 and the formula (2.4), the combinatorial Laplacian $\Delta$ is intrinsic in the sense that it is independent of choice of decorations and depends only on the hyperbolic polyhedral metrics. This implies the combinatorial Laplacian $\Delta$ is well-defined. Using the combinatorial Laplacian $\Delta$, the author [50] introduced the following combinatorial Calabi flow

$$\frac{dl_i}{dt} = -\Delta K_i$$

for decorated hyperbolic polyhedral metrics on cusped 3-manifolds. The basic properties of the combinatorial Calabi flow (3.2), including the stability of the combinatorial Calabi flow and others, were established in [50]. For a single ideal tetrahedron, the author [50] further proved that for any reasonable prescribed dihedral angles, the combinatorial Calabi flow (3.2) exists for all time and converges exponentially fast to a decorated hyperbolic polyhedral metric with the prescribed dihedral angles.

The following property is a direct consequence of the smoothness of dihedral angles in a decorated hyperbolic ideal tetrahedron with respect to the edge lengths.

**Lemma 3.2** ([51]). If the solution of combinatorial Ricci flow (3.1) converges to some decorated hyperbolic polyhedral metric $l^* \in \mathcal{L}(T)$, then $K(l^*) = 0$.

Furthermore, we have the following stability for the combinatorial Ricci flow (3.1).

**Theorem 3.1.** Suppose $(M, T)$ is an ideally triangulated cusped 3-manifold and $l^* \in \mathcal{L}(T) \cap \text{Ker}(C)$ is a decorated hyperbolic polyhedral metric with $K(l^*) = 0$. Then there exists a positive constant $\delta$ such that if the initial value $l(0) = l_0 \in \mathcal{L}(T) \cap \text{Ker}(C)$ of the combinatorial Ricci flow (3.1) satisfies

$$||l_0 - l^*|| < \delta,$$

then the combinatorial Ricci flow (3.1) exists for all time and converges exponentially fast to the decorated hyperbolic polyhedral metric $l^*$.

**Proof.** Set $\Gamma(l) = K(l)$. Take the combinatorial Ricci flow (3.1) as an autonomous system $\frac{dl}{dt} = \Gamma(l)$. Then $l^*$ is a equilibrium point of the system by $K(l^*) = 0$ and $D_l\Gamma|_{l^*} = \left(\frac{\partial K}{\partial l}\right)|_{l^*} = \Lambda(l^*) \leq 0$ by Theorem 2.1. Further note that the solution of the combinatorial Ricci flow (3.1) stays in $\text{Ker}(C) = \text{Ker}(\Lambda)^\perp$ by Lemma 3.1 and $\Lambda$ is strictly negative definite on $\text{Ker}(C) = \text{Ker}(\Lambda)^\perp$ by Theorem 2.1 we have $l^*$ is a
local attractor of the combinatorial Ricci flow \(3.1\). Then the conclusion follows from Lyapunov Stability Theorem \([37]\), Chapter 5). Q.E.D.

For general initial values, the combinatorial Ricci flow may develop singularities, which corresponds to the boundary of the admissible space \(L(T)\) \([51]\). Under a very strong condition that the combinatorial Ricci flow does not develop singularities in finite and infinite time, Yang \([51]\) obtained the convergence of the combinatorial Ricci flow \(3.1\). However, the condition that the combinatorial Ricci flow \(3.1\) does not develop singularities in finite and infinite time is difficult to be satisfied in practice. We shall overcome this difficulty by extending the combinatorial Ricci flow \(3.1\) through the potential singularities of the flow.

4. Extension of the dihedral angles and the combinatorial Ricci curvature

4.1. Extension of dihedral angles. Recall the admissible space of edge lengths for a decorated hyperbolic ideal tetrahedron \(\sigma\) is given by

\[
L_\sigma = \mathbb{R}^6 \setminus (V_{12}^{34} \cup V_{13}^{24} \cup V_{14}^{23}),
\]

where each \(V_{ij}^{kl}, \{i, j, k, l\} = \{1, 2, 3, 4\}\), is the closed region above an analytical function defined on \(\mathbb{R}^5\) by Corollary 2.1.

**Lemma 4.1.** Suppose \(\sigma = (\tau, \{H_1, H_2, H_3, H_4\})\) is a decorated hyperbolic ideal tetrahedron and \(\vec{l} \in \partial V_{ij}^{kh}\). If \(l \in L_\sigma\) tends to \(\vec{l}\), then \(\alpha_{ij} = \alpha_{kh} \rightarrow \pi\) and \(\alpha_{ik}, \alpha_{ih}, \alpha_{jk}, \alpha_{jh} \rightarrow 0\). As a consequence, the dihedral angles of an ideal tetrahedron could be extended by constants to be a continuous function defined on \(\mathbb{R}^6\).

**Proof.** For \(l \in L_\sigma\), by the cosine law for Euclidean triangle \(\tau \cap H_i\), we have

\[
\cos \alpha_{ij} = \cos \alpha_{kh} = \frac{y_{jk}^2 + y_{jh}^2 - y_{kh}^2}{2y_{jk}y_{jh}},
\]

where \(y_{jk} = e^{(l_{jk} + l_{ih})/2}, y_{jh} = e^{(l_{ij} + l_{ih})/2}, y_{kh} = e^{(l_{ij} + l_{ih})/2}\). For \(\vec{l} \in \partial V_{ij}^{kh}\), we have \(\vec{y}_{kh} = \vec{y}_{jk} + \vec{y}_{jh}\). This implies \(\cos \alpha_{ij} = \cos \alpha_{kh} \rightarrow -1\) as \(l \rightarrow \vec{l}\).

Therefore, \(\alpha_{ij} = \alpha_{kh} \rightarrow \pi\) as \(l \rightarrow \vec{l}\). \(\alpha_{ik}, \alpha_{ih}, \alpha_{jk}, \alpha_{jh} \rightarrow 0\) can be proved similarly.

Define \(\tilde{\alpha}_{ij}\) as follows

\[
\tilde{\alpha}_{ij} = \begin{cases} 
\pi, & l \in V_{ij}^{kh}; \\
\alpha, & l \in L_\sigma; \\
0, & l \in V_{ik}^{jh} \cup V_{ih}^{jk}.
\end{cases}
\]

Then \(\tilde{\alpha}_{ij}\) is a continuous function defined on \(\mathbb{R}^6\) extending \(\alpha_{ij}\). The other dihedral angles can be extended similarly. Q.E.D.

Note that for the extended dihedral angles, we still have \(\tilde{\alpha}_{ij} = \tilde{\alpha}_{kh}\) and the sum of six extended dihedral angles is \(2\pi\). The extension of dihedral angles
in Lemma 4.1 is essentially Luo-Yang’s extension in [32]. Before going on, we recall the following definition and theorem of Luo in [29].

**Definition 4.1.** A differential 1-form \( w = \sum_{i=1}^{n} a_i(x) dx^i \) in an open set \( U \subset \mathbb{R}^n \) is said to be continuous if each \( a_i(x) \) is continuous on \( U \). A continuous differential 1-form \( w \) is called closed if \( \int_{\partial\tau} w = 0 \) for each triangle \( \tau \subset U \).

**Theorem 4.1** ([29], Corollary 2.6). Suppose \( X \subset \mathbb{R}^n \) is an open convex set and \( A \subset X \) is an open subset of \( X \) bounded by a \( C^1 \) smooth codimension-1 submanifold in \( X \). If \( w = \sum_{i=1}^{n} a_i(x) dx_i \) is a continuous closed 1-form on \( A \) so that \( F(x) = \int_{a}^x w \) is locally convex on \( A \) and each \( a_i \) can be extended continuous to \( X \) by constant functions to a function \( \tilde{a}_i \) on \( X \), then \( \tilde{F}(x) = \int_{a}^x \sum_{i=1}^{n} \tilde{a}_i(x) dx_i \) is a \( C^1 \)-smooth convex function on \( X \) extending \( F \).

By Theorem 4.1 we have the following result.

**Lemma 4.2.** The function \( F_\sigma \) defined on \( L_\sigma \) in (2.3) could be extended to be a function

\[
\tilde{F}_\sigma(l) = \int_0^l \sum_{i<j} \tilde{\alpha}_{ij} dl_{ij},
\]

which is \( C^1 \) smooth and convex on \( \mathbb{R}^6 \) with \( \nabla \tilde{F}_\sigma = \tilde{\alpha} \).

\( \tilde{F}_\sigma \) is the extended co-volume function up to a constant obtained by Luo-Yang [32] using Fenchel dual. Here we give a direct extension of \( F_\sigma \).

**Remark 4.1.** The idea of extension was first introduced by Bobenko-Pinkall-Springborn [2] to extend a locally convex function on a nonconvex domain to a convex function and solved affirmatively a conjecture of Luo [25] on the global rigidity of piecewise linear metrics on surfaces. Luo [29] systematically studied the method of extension and proved the global rigidity of inversive distance circle packing metrics for nonnegative inversive distance. The method of extension has lots of applications, see [12, 13, 14, 15, 17, 18, 19, 32, 46, 47, 48] and others.

**4.2. Extension of the combinatorial Ricci curvature.** Using the extension of dihedral angles in a decorated hyperbolic ideal tetrahedron, we can extend the definition of combinatorial Ricci curvature for \( l \in \mathbb{R}^N \) as follows

\[
\tilde{K}_i(l) = 2\pi - \sum_{\sigma \in T} \tilde{\alpha}_i(l),
\]

where \( \tilde{\alpha}_i \) is the extension of the dihedral angle \( \alpha_i \) along the edge \( i \in E \) in a tetrahedron \( \sigma \). If \( l \in \mathbb{R}^N \) corresponds to a nondegenerate decorated hyperbolic polyhedral metric in \( L(T) \), then \( \tilde{K} = K \), which implies \( \tilde{K} \) is an extension of \( K \). Paralleling to the combinatorial Ricci curvature \( K \), the extended combinatorial Ricci curvature \( \tilde{K} \) has the following property on cusped 3-manifolds.
Lemma 4.3. Suppose \((M, T)\) is an ideally triangulated cusped 3-manifolds. Then the extended combinatorial Ricci curvature \(\widetilde{K}\) satisfies 
\[
\sum_{i \sim p} \widetilde{K}_i = 0, \quad \forall p \in V,
\]
which is equivalent to \(\widetilde{K} \in \text{Ker}(C)\).

Following Luo-Yang [32], we call \(l \in \mathbb{R}^N\) as a generalized decorated hyperbolic polyhedral metric in the following. Using the extension \(\widetilde{F}_\sigma\) of \(F_\sigma\), we can define the following Ricci energy function
\[
\widetilde{F}(l) = \sum_{\sigma \in T} \widetilde{F}_\sigma - 2\pi \sum_{i=1}^N l_i
\]
for generalized decorated hyperbolic polyhedral metrics \(l \in \mathbb{R}^N\). By Lemma 4.2, we have

Lemma 4.4. \(\widetilde{F}(l)\) is a \(C^1\) smooth convex function globally defined on \(\mathbb{R}^N\) with \(\nabla \widetilde{F} = -\widetilde{K}\).

Remark 4.2. One can also define \(\widetilde{F}(l)\) directly by extending \(-\int_0^l \sum_{i=1}^N K_i dl_i\) to \(-\int_0^l \sum_{i=1}^N \tilde{K}_i dl_i\), which is well-defined, \(C^1\) smooth and convex on \(\mathbb{R}^N\) by Theorem 4.1 and Theorem 2.1.

Using the Ricci energy function \(\widetilde{F}\), one can prove the following rigidity for generalized decorated hyperbolic polyhedral metric, which has been obtained by Luo-Yang [32]. However, our proof is direct, short and does not involve angle structures and Fenchel dual of the volume function. For completeness, we present the proof here.

Theorem 4.2 ([32], Theorem 3.2). Suppose a decorated hyperbolic polyhedral metric \(l_A \in \mathcal{L}(T)\) and a generalized decorated hyperbolic polyhedral metric \(l_B \in \mathbb{R}^N\) have the same generalized combinatorial Ricci curvature \(\widetilde{K}\). Then \(l_A\) and \(l_B\) differ by a change of decoration.

Proof. Define
\[
f(t) = \widetilde{F}(tl_B + (1-t)l_A)
\]
for \(t \in [0, 1]\). Then \(f(t)\) is a \(C^1\) smooth convex function on \([0, 1]\) by the \(C^1\) smoothness and convexity of \(\widetilde{F}\) in Lemma 4.4. By direct calculations, we have
\[
f'(t) = -\sum_{i=1}^N (l_{B,i} - l_{A,i}) \tilde{K}_i(tl_B + (1-t)l_A),
\]
which gives
\[
f'(0) = -\sum_{i=1}^N (l_{B,i} - l_{A,i}) \tilde{K}_i(l_A), \quad f'(1) = -\sum_{i=1}^N (l_{B,i} - l_{A,i}) \tilde{K}_i(l_B).
\]
By the condition that \(l_A \in \mathcal{L}(T)\) and \(l_B \in \mathbb{R}^N\) have the same generalized combinatorial Ricci curvature, we have \(f'(0) = f'(1)\). Note that \(f\) is a
$C^1$ smooth convex function, $f'(t)$ is an increasing function of $t \in [0, 1]$. Therefore, $f'(t)$ is a constant on $[0, 1]$ by $f'(0) = f'(1)$.

Note that $\tilde{K}$ is smooth around $l_A \in L(T)$, which implies that $f(t)$ is smooth around 0. Specially, $f(t)$ is a smooth function on $[0, \epsilon)$ for some $\epsilon \in (0, 1)$. Therefore, we have $f''(t) = 0$ for $t \in [0, \epsilon)$. By direct calculations, we have

$$f''(t) = -(l_B - l_A)^T \Lambda (l_B - l_A) = 0, \quad t \in [0, \epsilon).$$

Then $l_B - l_A \in \text{Ker}(\Lambda) = \text{Im}(C^T)$ by Theorem 2.1. Therefore, $l_B = l_A + C^T x$ for some $x \in \mathbb{R}^k$, which is equivalent to that $l_A$ and $l_B$ differ by a change of decoration. Q.E.D.

5. Extension of the combinatorial Ricci flow

5.1. Definition of the extended combinatorial Ricci flow. Using the extension of combinatorial Ricci curvature, we can define the extension of combinatorial Ricci flow for generalized decorated hyperbolic polyhedral metric $l \in \mathbb{R}^N$ as

$$\frac{dl_i}{dt} = \tilde{K}_i$$

(5.1)

with $l(0) = l_0 \in \mathbb{R}^N$.

5.2. Uniqueness of the solution of extended combinatorial Ricci flow. Although the term $\tilde{K}$ on the righthand side of the extended combinatorial Ricci flow (5.1) is $C^0$ and not $C^1$ on $\mathbb{R}^N$, we still have the uniqueness of the solution of extended combinatorial Ricci flow (5.1), which is a consequence of the convexity of $C^1$ smooth Ricci energy function $\tilde{F}(l)$ defined on $\mathbb{R}^N$.

**Theorem 5.1.** The solution of the extended combinatorial Ricci flow (5.1) is unique for any initial generalized decorated hyperbolic polyhedral metric $l(0) \in \mathbb{R}^N$.

**Proof.** The proof is paralleling to that of Theorem 5.1 in [49], we only sketch the proof here. The idea of the proof comes from Ge-Hua [11].

By Lemma 4.4, $\tilde{F}$ is a $C^1$ smooth convex function on $\mathbb{R}^N$. We can use the standard technique in PDE to mollify $\tilde{F}$ to be $\tilde{F}_\epsilon$, which is a $C^\infty$ convex function on $\mathbb{R}^N$ such that $\tilde{F}_\epsilon \to \tilde{F}$ in $C^1_{loc}(\mathbb{R}^N)$ as $\epsilon \to 0$. For the smooth convex function $\tilde{F}_\epsilon$, we have

$$\left(\nabla \tilde{F}_\epsilon(l_1) - \nabla \tilde{F}_\epsilon(l_2)\right) \cdot (l_1 - l_2) \geq 0,$$

(5.2)

which is a consequence of the monotonicity of the function

$$f(t) = \left(\nabla \tilde{F}_\epsilon(l_1) - \nabla \tilde{F}_\epsilon(l_1 + t(l_2 - l_1))\right) \cdot (l_1 - l_2).$$

Let $\epsilon \to 0$, (5.2) gives

$$\left(\nabla \tilde{F}(l_1) - \nabla \tilde{F}(l_2)\right) \cdot (l_1 - l_2) \geq 0,$$

which is equivalent to

$$\left(\tilde{K}(l_1) - \tilde{K}(l_2)\right) \cdot (l_1 - l_2) \leq 0.$$
If \( l_1(t) \) and \( l_2(t) \) are two solutions of the extended flow \((5.1)\) with the same initial value \( l(0) \in \mathbb{R}^N \), then for the function \( g(t) = ||l_1(t) - l_2(t)||^2 \), we have \( g(0) = 0 \), \( g(t) \geq 0 \) and
\[
\frac{dg}{dt} = 2(\bar{l}_1 - \bar{l}_2)(\bar{\kappa}_1 - \bar{\kappa}_2) \leq 0,
\]
which implies \( g(t) \equiv 0 \). Therefore, \( l_1(t) \equiv l_2(t) \). Q.E.D.

As a corollary of Theorem \ref{thm:uniqueness} we have the following result which shows that the solution of the extended combinatorial Ricci flow \((5.1)\) extends the solution of the combinatorial Ricci flow \((1.1)\) for any initial decorated hyperbolic polyhedral metric in \( \mathcal{L}(T) \).

**Corollary 5.1.** For any initial decorated hyperbolic polyhedral metric \( l(0) \in \mathcal{L}(T) \), denote the solutions of the combinatorial Ricci flow \((1.1)\) and the extended combinatorial Ricci flow \((5.1)\) as \( l(t) \) and \( \bar{l}(t) \) respectively. Then \( \bar{l}(t) = l(t) \) whenever \( l(t) \) exists in \( \mathcal{L}(T) \).

**Remark 5.1.** Although the solution of the extended combinatorial Ricci flow \((5.1)\) is unique, there may exist some other different extensions of the solution of the combinatorial Ricci flow \((1.1)\) on \( \mathbb{R}^N \). The uniqueness of extension of the solution of combinatorial Ricci flow \((1.1)\) depends on the choice of extension of the combinatorial Ricci curvature. Here we choose a natural extension of the combinatorial Ricci curvature, but there may exist some other different extension of the combinatorial Ricci curvature on \( \mathbb{R}^N \). The author thanks Tian Yang for pointing this out to the author.

### 5.3. Longtime existence of the extended combinatorial Ricci flow

We have the following longtime existence for the solution of the extended combinatorial Ricci flow \((5.1)\).

**Theorem 5.2.** The solution of the extended combinatorial Ricci flow \((5.1)\) exists for all time for any initial generalized decorated hyperbolic polyhedral metric \( l(0) \in \mathbb{R}^N \).

**Proof.** To prove the longtime existence of the solution of the extended combinatorial Ricci flow \((5.1)\), we just need to prove that the solution \( l(t) \) is bounded for any finite time \([0, T]\) with \( T < +\infty \).

As the extended dihedral angles are uniformly bounded by \( \pi \) and we are working on a cusped 3-manifold \( M \) with a fixed triangulation \( T \), the extended combinatorial Ricci curvature \( \bar{\kappa} \) is uniformly bounded by some constant \( C \) depending only on the triangulation \( T \), which implies that the solution \( l(t) \) of the extended combinatorial Ricci flow \((5.1)\) satisfies
\[
|l_i(t)| \leq |l_i(0)| + CT < +\infty
\]
for any \( t \in [0, T] \). Therefore, the solution of the extended combinatorial Ricci flow \((5.1)\) exists for all time for any initial generalized decorated hyperbolic polyhedral metric \( l(0) \in \mathbb{R}^N \). Q.E.D.
5.4. Convergence of the extended combinatorial Ricci flow.

**Lemma 5.1.** If the solution \( l(t) \) of the extended combinatorial Ricci flow (5.1) on an ideally triangulated cusped 3-manifold \((M, T)\) converges to some generalized decorated hyperbolic polyhedral metric \( \bar{l} \in \mathbb{R}^N \), then \( \bar{K}(\bar{l}) = 0 \).

**Proof.** As \( l(t) \) is a solution of (5.1), which exists for all time by Theorem 5.2, there exists \( \xi_n \in (n, n+1) \) such that
\[
 l(n+1) - l(n) = l'(\xi_n) = \bar{K}(l(\xi_n)) \to 0, \quad n \to \infty,
\]
by the convergence of the solution \( l(t) \). As \( l(t) \) converges to \( \bar{l} \), we have \( l(\xi_n) \to \bar{l} \). By the continuity of \( \bar{K} \), we have \( \bar{K}(\bar{l}) = 0 \). Q.E.D.

The following result is a generalization of Theorem 1.1 which gives a new characterization of the existence of a complete hyperbolic metric on an ideally triangulated cusped 3-manifold \((M, T)\) dual to Casson and Rivin’s program.

**Theorem 5.3.** Suppose there exists a complete hyperbolic metric on the ideally triangulated cusped 3-manifold \((M, T)\). Then \( l^* \in L(T) \) is a decorated hyperbolic polyhedral metric with zero combinatorial Ricci curvature if and only if the solution of extended combinatorial Ricci flow (5.1) with initial value \( l(0) \in \mathbb{R}^N \) exists for all time and converges exponentially fast to \( l^* \).

**Proof.** Suppose the solution of extended combinatorial Ricci flow (5.1) exists for all time and converges exponentially fast to \( l^* \), then \( \bar{K}(l^*) = 0 \) by Lemma 5.1. As there exists a nondegenerate decorated hyperbolic polyhedral metric with zero combinatorial Ricci curvature on the ideal triangulated cusped 3-manifold \((M, T)\) by assumption, we have \( l^* \) is the nondegenerate decorated hyperbolic polyhedral metric with zero combinatorial Ricci curvature by Theorem 4.2. Therefore, \( l^* \in L(T) \) and \( K(l^*) = 0 \).

Suppose \( l^* \in L(T) \) is a decorated hyperbolic polyhedral metric with zero combinatorial Ricci curvature. Along the extended combinatorial Ricci flow (5.1), we still have
\[
 \frac{d(\sum_{i \sim p} l_i)}{dt} = \sum_{i \sim p} \bar{K}_i = 0
\]
by Lemma 4.3 which implies \( \sum_{i \sim p} l_i \) is invariant along the extended combinatorial Ricci flow (5.1) for any cusp \( p \in V \). Without loss of generality, we assume that the initial decorated hyperbolic polyhedral metric \( l(0) \) and \( l^* \) are in \( \text{Ker}(C) \), where
\[
 \text{Ker}(C) = \{ l \in \mathbb{R}^N \mid \sum_{i \sim p} l_i = 0, \forall p \in V \}.
\]
Then the solution \( l(t) \) of the extended combinatorial Ricci flow (5.1) stays in \( \text{Ker}(C) \).

If \( l^* \in L(T) \) is a decorated hyperbolic polyhedral metric with \( K(l^*) = 0 \), then we have \( \nabla \bar{F}(l^*) = -\bar{K}(l^*) = 0 \). As \( \bar{F}(l) \) is a \( C^1 \) smooth convex
function on $\mathbb{R}^N$, we have $l^*$ is a minimal point of $\tilde{F}$ on $\mathbb{R}^N$. Further note that $Hess\tilde{F}(l^*) = -\Lambda(l^*)$ is positive semi-definite with $Ker(Hess\tilde{F}(l^*))^\perp = Ker(\Lambda)^\perp = Ker(C)$ by Theorem 2.1. $\tilde{F}$ is a convex function on $Ker(C)$ and strictly locally convex around $l^*$ in $Ker(C) \cap L(T)$. Then we have $\lim_{t \in Ker(C), t \to \infty} F(l) = +\infty$, which implies $\tilde{F}|_{Ker(C)}$ is proper. This follows from the following result of convex function, the proof of which could be found in [17] (Lemma 4.6).

**Lemma 5.2.** Suppose $f(x)$ is a $C^1$ smooth convex function on $\mathbb{R}^n$ with $\nabla f(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, $f(x)$ is $C^2$ smooth and strictly convex in a neighborhood of $x_0$, then $\lim_{x \to \infty} f(x) = +\infty$.

For any initial value $l(0) \in \mathbb{R}^N$, we have

$$\frac{d\tilde{F}(l(t))}{dt} = -\sum_i \tilde{K}_i^2 \leq 0,$$  \hspace{1cm} (5.3)

which implies $\tilde{F}(l(t))$ is bounded along the extended combinatorial Ricci flow (5.1). Combining with the properness of convex function $\tilde{F}|_{Ker(C)}$, the solution $l(t)$ of the extended combinatorial Ricci flow (5.1) lies in a compact subset $\Omega \subset Ker(C)$.

By the fact that $\tilde{F}(l(t))$ is bounded and (5.3), the limit $\lim_{t \to +\infty} \tilde{F}(l(t))$ exists. This implies

$$\tilde{F}(l(n + 1)) - \tilde{F}(l(n)) = -\sum_i \tilde{K}_i^2(l(\xi_n)) \to 0, n \to \infty,$$

for some $\xi_n \in (n, n + 1)$. By the compactness of $\Omega$, we have a subsequence of $l(\xi_n)$ converges to some $\xi^* \in \Omega \subset Ker(C)$. The continuity of $\tilde{K}$ then implies $\tilde{K}(\xi^*) = 0$. Therefore, we have

$$\tilde{K}(l^*) = \tilde{K}(\xi^*) = 0$$

with $l^* \in L(T) \cap KerC$ and $\xi^* \in KerC$. By the rigidity of the extended combinatorial Ricci curvature $\tilde{K}$ in Theorem 4.2, we have $l^* = \xi^*$. Therefore, $l(\xi_n) \to l^*$.

At $l^*$, we have

$$\left. \frac{\partial \tilde{K}}{\partial l} \right|_{l=l^*} = \left. \frac{\partial K}{\partial l} \right|_{l=l^*} = \Lambda(l^*) \leq 0.$$

Furthermore, the solution $l(t)$ stays in $Ker(C) = Ker(\Lambda)^\perp$ along the extended combinatorial Ricci flow (5.1) and $\Lambda(l^*)$ is strictly negative definite on $Ker(C)$ by Theorem 2.1. This implies $l^*$ is a local attractor of the extended combinatorial Ricci flow (5.1). As $l(\xi_n) \to l^*$, it follows from Lyapunov Stability Theorem ([37] Chapter 5) that the solution $l(t)$ of the extended combinatorial Ricci flow (5.1) converges exponentially fast to $l^*$. Q.E.D.
Remark 5.2. By the proof of Theorem 5.3, \( \sum_{i \sim p} l_i \) is invariant along the extended combinatorial Ricci flow (5.1) for any cusp \( p \in V \), which implies \( \sum_{i=1}^N l_i \) is a constant along the extended combinatorial Ricci flow (5.1). Therefore, the Ricci energy \( \tilde{\mathcal{F}} \) differs from the co-volume \( \text{cov} \) by a constant along the extended combinatorial Ricci flow (5.1). Note that the extended combinatorial Ricci flow (5.1) finds the complete hyperbolic metric on the cusped 3-manifold \((M, T)\) by minimizing the Ricci energy \( \tilde{\mathcal{F}} \), this implies that the extended combinatorial Ricci flow (5.1) find the complete hyperbolic metric on \((M, T)\) also by minimizing the co-volume function \( \text{cov} \) along the flow. This is dual to Casson-Rivin’s program to find complete hyperbolic metrics on \((M, T)\) by maximizing the volume on angle structures.

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References

[1] X. Bao, F. Bonahon, Hyperideal polyhedra in hyperbolic 3-space, Bull. Soc. Math. France 130 (2002) 457-491.
[2] A. Bobenko, U. Pinkall, B. Springborn, Discrete conformal maps and ideal hyperbolic polyhedra, Geom. Topol. 19 (2015), no. 4, 2155-2215.
[3] F. Bonahon, A Schläfi-type formula for convex cores of hyperbolic 3-manifolds, J. Differential Geom. 50 (1998), 25-58.
[4] K. Chan, Constructing hyperbolic 3-manifolds, Undergraduate thesis with Craig Hodgson, University of Melbourne, 2002.
[5] Y.-E. Choi, Positively oriented ideal triangulations on hyperbolic three-manifolds, Topology 43 (2004) 1345-1371.
[6] B. Chow, F. Luo, Combinatorial Ricci flows on surfaces, J. Differential Geometry, 63 (2003), 97-129.
[7] Futer, David; Guéritaud, François From angled triangulations to hyperbolic structures. Interactions between hyperbolic geometry, quantum topology and number theory, 159-182, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.
[8] Y. C. de Verdière, Un principe variationnel pour les empilements de cercles, Invent. Math. 104(3) (1991) 655-669.
[9] H. Ge, Combinatorial Calabi flows on surfaces, Trans. Amer. Math. Soc. 370 (2018), no. 2, 1377-1391.
[10] H. Ge, B. Hua, On combinatorial Calabi flow with hyperbolic circle patterns, Adv. Math. 333 (2018), 523-538.
[11] H. Ge, B. Hua, 3-dimensional combinatorial Yamabe flow in hyperbolic background geometry, Trans. Amer. Math. Soc. 373 (2020), no. 7, 5111-5140.
[12] H. Ge, W. Jiang, On the deformation of discrete conformal factors on surfaces, Calc. Var. Partial Differential Equations 55 (2016), no. 6, Art. 136, 14 pp.
[13] H. Ge, W. Jiang, On the deformation of inversive distance circle packings, I, Trans. Amer. Math. Soc. 372 (2019), no. 9, 6231-6261.
[14] H. Ge, W. Jiang, On the deformation of inversive distance circle packings, II, J. Funct. Anal. 272 (2017), no. 9, 3573-3595.
[15] H. Ge, W. Jiang, On the deformation of inversive distance circle packings, III, J. Funct. Anal. 272 (2017), no. 9, 3596-3609.
[16] H. Ge, X. Xu, 2-dimensional combinatorial Calabi flow in hyperbolic background geometry, Differential Geom. Appl. 47 (2016) 86-98.
[17] H. Ge, X. Xu, On a combinatorial curvature for surfaces with inversive distance circle packing metrics, J. Funct. Anal. 275 (2018), no. 3, 523-558.
[18] X. D. Gu, R. Guo, F. Luo, J. Sun, T. Wu, A discrete uniformization theorem for polyhedral surfaces II, J. Differential Geom. 109 (2018), no. 3, 431-466.
[19] X. D. Gu, F. Luo, J. Sun, T. Wu, A discrete uniformization theorem for polyhedral surfaces, J. Differential Geom. 109 (2018), no. 2, 223-256.
[20] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982), no. 2, 255-306.
[21] P. Hartman, Ordinary differential equations. John Wiley & Sons, Inc., New York-London-Sydney 1964 xiv+612 pp.
[22] X. He, X. Xu, Thurston's sphere packings on 3-dimensional manifolds, I, arXiv:1904.11122v3 [math.GT].
[23] X. He, X. Xu, Thurston's sphere packings on 3-dimensional manifolds, II, In preparation.
[24] M. Lackenby, Word hyperbolic Dehn surgery, Invent. Math. 140 (2000) 243-282.
[25] F. Luo, Combinatorial Yamabe flow on surfaces, Commun. Contemp. Math. 6 (2004), no. 5, 765-780.
[26] F. Luo, A combinatorial curvature flow for compact 3-manifolds with boundary, Electron. Res. Announc. Amer. Math. Soc. 11 (2005), 12-20.
[27] F. Luo, 3-dimensional Schl"afli formula and its generalization, Commun. Contemp. Math. 10 (2008), 835-842.
[28] F. Luo, Triangulated 3-manifolds: from Haken’s normal surfaces to Thurston’s algebraic equation, Interactions between hyperbolic geometry, quantum topology and number theory, 183-204, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.
[29] F. Luo, Rigidity of polyhedral surfaces, III, Geom. Topol. 15 (2011), 2299-2319.
[30] F. Luo, A note on complete hyperbolic structures on ideal triangulated 3-manifolds, Topology and geometry in dimension three, Contemporary Mathematics 560 (American Mathematical Society, Providence, RI, 2011) 19-26.
[31] F. Luo, Volume Optimization, Normal Surfaces and Thurston’s Equation on Triangulated 3-Manifolds, Journal of Differential Geometry, 93:2 (2013), pp. 299-326.
[32] F. Luo, T. Yang, Volume and rigidity of hyperbolic polyhedral 3-manifolds. J. Topol. 11 (2018), no. 1, 1-29.
[33] J. Milnor, The Schl"afli differential equality, John Milnor Collected papers, Vol. 1, Geometry (Publish or Perish, Inc., Houston, TX, 1994).
[34] George D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973, Annals of Mathematics Studies, No. 78.
[35] Walter D. Neumann, *Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic 3-manifolds*, Topology ’90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 243-271.
[36] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*. Comm. Math. Phys. 113 (1987), no. 2, 299-339.
[37] L.S. Pontryagin, *Ordinary Differential Equations*, Addison-Wesley Publishing Company Inc., Reading, 1962.
[38] Gopal Prasad, *Strong rigidity of Q-rank 1 lattices*, Invent. Math. 21 (1973), 255-286.
[39] John G. Ratcliffe, *Foundations of hyperbolic manifolds*. Second edition. Graduate Texts in Mathematics, 149. Springer, New York, 2006. xii+779 pp.
[40] Igor Rivin, *On geometry of convex ideal polyhedra in hyperbolic 3-space*, Topology 32 (1993), no. 1, 87-92.
[41] Igor Rivin, *Euclidean structures on simplicial surfaces and hyperbolic volume*. Ann. of Math. (2) 139 (1994), no. 3, 553-580.
[42] Igor Rivin, *A characterization of ideal polyhedra in hyperbolic 3-space*, Ann. of Math. (2) 143 (1996), no. 1, 51-70.
[43] Igor Rivin, *Combinatorial optimization in geometry*. Adv. in Appl. Math. 31 (2003), no. 1, 242-271.
[44] W. Thurston, *Geometry and topology of 3-manifolds*. Princeton lecture notes 1976, [http://www.msri.org/publications/books/gt3m](http://www.msri.org/publications/books/gt3m).
[45] E.B. Vinberg, *Geometry. II*, Encyclopaedia of Mathematical Sciences, 29, Springer-Verlag, New York, 1988.
[46] X. Xu, *Rigidity of inversive distance circle packings revisited*. Adv. Math. 332 (2018), 476-509.
[47] X. Xu, *On the global rigidity of sphere packings on 3-dimensional manifolds*, J. Differential Geom. 115 (2020), no. 1, 175-193.
[48] X. Xu, *A new proof of Bowers-Stephenson conjecture*, arXiv:1904.11127 [math.GT], accepted by Math. Res. Lett. 2020.
[49] X. Xu, *Combinatorial Ricci flow on compact 3-manifolds with boundary*, arXiv:2009.02496.
[50] X. Xu, *Combinatorial Calabi flow on cusped 3-manifolds*, In preparation.
[51] T. Yang, *A combinatorial curvature flow for ideal triangulations*, Doctoral dissertation, University of Melbourne, 2019.
[52] X. Zhu, X. Xu, *Combinatorial Calabi flow with surgery on surfaces*. Calc. Var. Partial Differential Equations 58 (2019), no. 6, Paper No. 195, 20 pp.

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