REGULAR VARIATION AND FREE REGULAR INFINITELY DIVISIBLE LAWS

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Abstract. In this article the relation between the tail behaviours of a free regular infinitely divisible probability measure and its Lévy measure is studied. An important example of such a measure is the compound free Poisson distribution, which often occurs as a limiting spectral distribution of certain sequences of random matrices. We also describe a connection between an analogous classical result of Embrechts et al. [1979] and our result using the Bercovici-Pata bijection.

1. Introduction

The limiting spectral distribution (LSD) of product of two or more random matrices is important in the field of random matrix theory. It arises naturally, for example, in study of multivariate F-matrix (the product of mutually independent sample covariance matrix and the inverse of another sample covariance matrix). The limiting spectral distributions of F-matrices were studied in Wachter [1980], Bai et al. [1987]. In addition, products of random matrices arise in study of high dimensional time-series, for example, see Pan [2010], Pan and Gao [2012]. For a history of the product of random matrices the reader is referred to Bai et al. [2007].

The existence of a non-random LSD of the product of a sample covariance matrix and a non-negative definite Hermitian matrix, which are mutually independent, was given explicitly in terms of the Stieltjes transform in Silverstein [1995]. A stronger result in this direction is obtained using the moment method and truncation arguments in Bai et al. [2007], by replacing the non-negative definite assumption by a Lindeberg type one, on the entries of the Hermitian matrices. When one considers Wishart matrices, a more explicit description of the LSD can be given in terms of free probability; see Merlevède and Peligrad [2016], Chakrabarty et al. [2018].

It is well known in random matrix theory that the Marchenko-Pastur law (also called the free Poisson distribution) turns out to be the limiting spectral distribution of a sequence of Wishart random matrices \(W_N\). Suppose for each \(N \geq 1\), \(Y_N\) is an \(N \times N\) independent random Hermitian matrix with LSD \(\rho\). It can be shown that the expected empirical distribution of \(W_NY_N\) converges to \(m \boxtimes \rho\) as \(N \to \infty\) where \(\boxtimes\) denotes the free multiplicative convolution. It is not difficult to see that \(\rho\) is compactly supported if and only if so is \(m \boxtimes \rho\). Therefore it is natural to ask whether there is any relation between the tail behaviour of \(m \boxtimes \rho\) and \(\rho\)? In this paper, an affirmative answer is given to that question when \(\rho\) has a power law tail decay. Thus, based on the LSD of \(Y_N\), one can describe the tail behaviour of that of \(W_NY_N\). In general, it is very hard to write down an explicit formula for the limit distribution.

It is noteworthy that the probability measures of the form \(m \boxtimes \rho\) are free regular probability measures (see Arizmendi et al. [2013]) which form a special subclass of free infinitely divisible measures.
distributions (also called the \( \mathcal{H} \)-infinitely divisible distributions, see Bercovici and Voiculescu [1993]). The free cumulant transform of a free regular probability measure can be described through a Lévy-Khintchine representation. Interestingly, it turns out that \( \rho \) is the Lévy measure of \( m \boxplus \rho \). Therefore it is natural to wonder whether there is any relation between the tail behaviours of a free regular probability measure and its Lévy measure.

In classical probability theory, a classically infinitely divisible probability measure \( \mu \) also enjoys a Lévy-Khintchine representation in terms of its Lévy measure \( \nu \). In Embrechts et al. [1979], it was shown that for a positively supported classically infinitely divisible probability measure (a subordinator) \( \mu \), the tails of \( \mu \) and its Lévy measure \( \nu \) are asymptotically equivalent if and only if any one of \( \mu \) or \( \nu \) is subexponential. In analogy to the classical case, it is natural to pose whether free subexponentiality characterizes the tail equivalence of a free infinitely divisible probability measure and its free Lévy measure. But unfortunately the result can not be extended to the bigger class of free infinitely divisible probability measures. Although according to Arizmendi et al. [2013], the correct analogue of the positively supported classically infinitely divisible probability measures are the free regular probability measures, in this paper, we provide a partial answer in Theorem 2.2 by showing the tail equivalence of a free regular probability measure and its free Lévy measure in presence of regular variation. Note that regularly varying measures are the most important subclass of both free and classical subexponential distributions (Hazra and Maulik [2013]). As an application of this result, the exact tail behaviour of the free multiplicative convolution of Marchenko-Pastur law with another regularly varying measure is derived in Corollary 2.2.2. Besides, the connection of these results with the classical case is not a mere coincidence. From the famous result of Bercovici and Pata (Bercovici et al. [1999]), it is known that classical and free infinitely divisible laws are in a one-to-one correspondence. It is shown in Corollary 2.2.3 that in the regularly varying set-up, the classical infinitely divisible law and its image under the Bercovici-Pata bijection are tail equivalent.

In Section 2 the basic notations and transforms used in free probability are introduced. Subsequently, the main results of this article are stated. Section 3 collects the proofs. The proofs depend heavily on relations between the transforms and regular variation. To keep the article self contained main result of Hazra and Maulik [2013] is quoted in the Appendix.

2. Preliminaries and Main results

2.1. Notations and basic definitions: A real valued measurable function \( f \) defined on non-negative real line is called regularly varying (at infinity) with index \( \alpha \) if for every \( t > 0 \), \( f(tx)/f(x) \to t^\alpha \) as \( x \to \infty \). The function \( f \) is said to be a slowly varying function (at infinity) if \( \alpha = 0 \). Throughout this paper, regular variation of a function will always be considered at infinity. A distribution function \( F \) on \([0, \infty)\) has regularly varying tail of index \( -\alpha \) if \( \overline{F}(x) = 1 - F(x) \) is regularly varying of index \( -\alpha \). Since \( \overline{F}(x) \to 0 \) as \( x \to \infty \), it necessarily holds that \( \alpha \geq 0 \). For further details about regular variation see Resnick [1987].

A distribution \( F \) on \([0, \infty)\) is called (classical) subexponential if \( \overline{F}^{(n)}(x) \sim \overline{F}(x) \) as \( x \to \infty \), for all \( n \geq 0 \). Here \( \overline{F}^{(n)} \) denotes the \( n \)-th (classical) convolution of \( F \). Both regular variation and subexponentiality of a probability measure \( \mu \) is defined thorough its distribution function \( \mu \).

The real line and the complex plane will be denoted by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. The notations \( \mathbb{C}^+ \) and \( \mathbb{C}^- \) are used for the upper and the lower halves of the complex plane, respectively, namely, \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \Re z > 0 \} \) and \( \mathbb{C}^- = -\mathbb{C}^+ \), while \( \mathbb{R}_+ := [0, \infty) \). For a complex number \( z \), \( \mathbb{R}z \) and \( \Im z \) denote its real and imaginary parts, respectively. Given positive numbers \( \eta, \delta \)
and $M$, let us define the following cone:
\[
\Gamma_{\eta} = \{ z \in \mathbb{C}^+ : |\Re z| < \eta \Im z \} \quad \text{and} \quad \Gamma_{\eta,M} = \{ z \in \Gamma_{\eta} : |z| > M \}.
\]
Then we shall say that $f(z) \to l$ as $z$ goes to $\infty$ non-tangentially, abbreviated by “n.t.”, if for any $\epsilon > 0$ and $\eta > 0$, there exists $M \equiv M(\eta,\epsilon) > 0$, such that $|f(z) - l| < \epsilon$, whenever $z \in \Gamma_{\eta,M}$. This is same as saying that the convergence in $\mathbb{C}^+$ is uniform in each cone $\Gamma_{\eta}$.

The boundedness can be defined analogously.

We use the notations $f(z) \approx g(z)$, $f(z) = o(g(z))$ and $f(z) = O(g(z))$ as $z \to \infty$ n.t.” to mean, respectively, that $f(z)/g(z)$ converges to a non-zero finite limit”, “$f(z)/g(z) \to 0$” and “$f(z)/g(z)$ stays bounded as $z \to \infty$ n.t.” If the limit is 1 in the first case, we write $f(z) \sim g(z)$ as $z \to \infty$ n.t. For $f(z) = o(g(z))$ as $z \to \infty$ n.t., we shall also use the notations $f(z) \ll g(z)$ and $g(z) \gg f(z)$ as $z \to \infty$ n.t.

Following Bercovici and Voiculescu [1993], we recall that a non-commutative probability space $(\mathbf{A}, \phi)$ is said to be a $\mathcal{W}^*$-probability space if $\mathbf{A}$ is a non-commutative von Neumann algebra and $\phi$ is a normal faithful trace. A family of unital von Neumann subalgebras $(\mathbf{A}_i)_{i \in I} \subset \mathbf{A}$ in a $\mathcal{W}^*$-probability space is called free if $\phi(a_1a_2\cdots a_n) = 0$ whenever $\phi(a_j) = 0, a_j \in \mathbf{A}_j$, and $i_1 \neq i_2 \neq \cdots \neq i_n$. A self-adjoint operator $X$ is affiliated with $\mathbf{A}$ if $f(X) \in \mathbf{A}$ for any bounded Borel function $f$ on $\mathbb{R}$. In this case it is said that $X$ is a (non-commutative) random variable. For a self-adjoint operator $X$ affiliated with $\mathbf{A}$, the distribution of $X$ is the unique measure $\mu_X$ in $\mathcal{M}$ satisfying
\[
\phi(f(X)) = \int f(x) \, d\mu_X(dx)
\]
for every Borel bounded function $f$ on $\mathbb{R}$. The self-adjoint operators $\{X_i\}_{1 \leq i \leq p}$, affiliated with a von Neumann algebra $\mathbf{A}$, are called free if and only if the algebras generated by $\{f(X_i) : f \text{ is bounded measurable}\}_{1 \leq i \leq p}$ are free.

$\mathcal{M}$ and $\mathcal{M}_+$ are the set of probability measures supported on $\mathbb{R}$ and $\mathbb{R}_+$ respectively. By $\mathcal{M}_p$ we mean the set of probability measures on $[0,\infty)$ whose $p$-th moment is finite and do not have the $(p+1)$-th moment. The set $\mathcal{M}_{p,\alpha}$ will contain all probability measures in $\mathcal{M}_p$ with regularly varying tail index $-\alpha$ such that $p \leq \alpha \leq p+1$.

For a probability measure $\mu \in \mathcal{M}$, its Cauchy transform is defined as
\[
G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t} \, d\mu(t), \quad z \in \mathbb{C}^+.
\]
Note that $G_\mu$ maps $\mathbb{C}^+$ to $\mathbb{C}^-$. Set $F_\mu = 1/G_\mu$, which maps $\mathbb{C}^+$ to $\mathbb{C}^+$.

The free cumulant transform $(C_\mu)$ and the Voiculescu transform $(\phi_\mu)$ of a probability measure $\mu$ is defined as
\[
C_\mu(z) = z\phi_\mu\left(\frac{1}{z}\right) = z F^{-1}_\mu\left(\frac{1}{z}\right) - 1,
\]
for $z$ in a domain $D_\mu \subset \mathbb{C}_-$ such that $1/z \in \Gamma_{\eta,M}$ where $F^{-1}_\mu$ is defined; The free additive convolution of two probability measures $\mu_1, \mu_2$ on $\mathbb{R}$ is defined as the probability measure $\mu_1 * \mu_2$ on $\mathbb{R}$ such that $\phi_{\mu_1 * \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$ or equivalently $C_{\mu_1 * \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z)$ for $z \in D_{\mu_1} \cap D_{\mu_2}$. It turns out that $\mu_1 * \mu_2$ is the distribution of the sum $X_1 + X_2$ of two free random variables $X_1$ and $X_2$ having distributions $\mu_1$ and $\mu_2$ respectively. On the other hand, the free multiplicative operation $\boxtimes$ on $\mathcal{M}$ is defined as follows (see Bercovici and Voiculescu [1993]). Let $\mu_1, \mu_2$ be probability measures on $\mathbb{R}$, with $\mu_1 \in \mathcal{M}_+$ and let $X_1, X_2$ be free random variables such that $\mu_{X_1} = \mu_1$. Since $\mu_1$ is supported on $\mathbb{R}_+$, $X_1$ is a positive self-adjoint operator and $\mu_1^{1/2}$ is uniquely determined by $\mu_1$. Hence the distribution $\mu_1^{1/2}X_2X_1^{1/2}$
of the self-adjoint operator $X_{1}^{1/2}X_{2}X_{1}^{1/2}$ is determined by $\mu_{1}$ and $\mu_{2}$. This measure is called 
the free multiplicative convolution of $\mu_{1}$ and $\mu_{2}$ and it is denoted by $\mu_{1}\boxtimes\mu_{2}$. This operation 
on $\mathcal{M}$ is associative and commutative.

We recall (Theorem 1.3 and Theorem 1.5 of Benaych-Georges [2006]) the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms for probability measures $\mu$ with finite $p$ moments and summarize the following expressions from Hazra and Maulik [2013]:

$$r_{G_\mu}(z) = z^{p+1} \left( G_\mu(z) - \sum_{j=1}^{p+1} m_{j-1}(\mu) z^{-j} \right)$$

(2.1)

and

$$r_{\phi_\mu}(z) = z^{p-1} \left( \phi_\mu(z) - \sum_{j=0}^{p-1} \kappa_{j+1}(\mu) z^{-j} \right),$$

(2.2)

where $\{m_j(\mu) : j \leq p\}$ and $\{\kappa_j(\mu) : j \leq p\}$ denotes the moment and free cumulant sequences of the probability measure $\mu$, respectively.

2.2. Classical infinite divisibility and known results. A probability measure $\mu$ is called 
classically infinitely divisible, if for every $n \in \mathbb{N}$, there exists a probability measure $\mu_n$ such 
that $\mu = \mu_n*\mu_n*\ldots*\mu_n (n \text{ times})$, where $*$ is the classical convolution of probability measures. 
A detailed description about classical infinite divisibility can be found in Sato [2013]. It is well 
known that a probability measure $\mu$ on $\mathbb{R}$ is classically infinitely divisible if and only if its classical 
cumulant transform $C_{\mu}^\ast(w) := \log \int_{\mathbb{R}} e^{ixw}d\mu(x)$ has the following Lévy-Khintchine 
representation (see Sato [2013] or Barndorff-Nielsen et al. [2006])

$$C_{\mu}^\ast(w) = i\eta w - \frac{1}{2}aw^2 + \int_{\mathbb{R}} \left( e^{iwt} - 1 - iwt 1_{[-1,1]}(t) \right) d\nu(t), \quad w \in \mathbb{R},$$

(2.3)

where $\eta \in \mathbb{R}$, $a \geq 0$ and $\nu$ is a Lévy measure on $\mathbb{R}$, that is, $\int_{\mathbb{R}} \min(1,t^2) d\nu(t) < \infty$ 
and $\nu(\{0\}) = 0$. If this representation exists, the triplet $(\eta, a, \nu)$ is called the classical 
characteristic triplet of $\mu$ and the triplet is unique.

Another form of $C_{\mu}^\ast(w)$ is given by

$$C_{\mu}^\ast(w) = i\gamma w + \int_{\mathbb{R}} \left( e^{iwt} - 1 - \frac{iwt}{1+t^2} \right) \frac{1+t^2}{t^2} d\sigma(t), \quad w \in \mathbb{R},$$

where $\gamma$ is a real constant and $\sigma$ is a finite measure on $\mathbb{R}$.

One has the following relationships between the two representations (see equations (2.3) 
below definition 2.1 in Barndorff-Nielsen et al. [2006]):

$$a = \sigma(\{0\}),$$

$$d\rho(t) = \frac{1+t^2}{t^2} 1_{\mathbb{R}\setminus\{0\}} d\sigma(t),$$

(2.4)

$$\eta = \gamma + \int_{\mathbb{R}} t \left( 1_{[-1,1]}(t) - \frac{1}{1+t^2} \right) d\rho(t).$$

(2.5)

In general, when one does not have the Brownian component, it is easier to consider the Laplace transform of the measure, for example, in the case of compound Poisson. In this 
situation let us recall the classical result which studies the tail equivalence of Lévy measure
and the infinitely divisible distribution. In Embrechts et al. [1979] it was shown subexponentiality is a property which makes an infinitely divisible measure and its Lévy measure tail equivalent.

**Theorem 2.1** (Embrechts et al., 1979). Let \( \mu \) be a classical infinitely divisible probability measure on \([0, \infty)\). Suppose \( \mu \) has the Lévy-Khintchine representation of the form, 

\[
f(s) = \int_0^\infty e^{-st} d\mu(t) = \exp \left\{ -as - \int_0^\infty (1 - e^{-st}) d\nu(t) \right\}
\]

where \( \nu \) is a Lévy measure satisfying \( \int_0^\infty \min\{1, t^2\} d\nu(t) < \infty \). Then the following statements are equivalent:

(a) \( \mu \) is subexponential,

(b) \( \nu \) is subexponential,

(c) and \( \mu(x, \infty) \sim \nu(x, \infty) \) as \( x \to \infty \).

Here, the probability measure \( \nu \), supported on the interval \((1, \infty)\), is defined by 

\[
\nu(1, x] = \nu(1, \infty) / \nu(1, \infty).
\]

The remarkable feature of this result is that tail equivalence gives subexponentiality. In subsection 2.4 we will address the partial extension of this result in the free setting.

**2.3. Free infinite divisibility and free regular probability measures.** Free infinitely divisible probability measures are defined in analogy with classical infinitely divisible probability measures. A probability measure \( \mu \) is called free infinitely divisible, if for every \( n \in \mathbb{N} \), there exists a probability measure \( \mu_n \) such that

\[
\mu = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n (n \text{ times}) \quad \text{holds.}
\]

Also a probability measure \( \mu \) on \( \mathbb{R} \) is \( \boxplus \)-infinitely divisible i.e. free infinitely divisible if and only if there exists a finite measure \( \sigma \) on \( \mathbb{R} \) and a real constant \( \gamma \), such that

\[
\phi_\mu(z) = \gamma + \int_\mathbb{R} \frac{1 + zt}{t} d\sigma(t), \quad z \in \mathbb{C}_+.
\]

We now recall the following from Arizmendi et al. [2013]:

A probability measure \( \mu \) on \( \mathbb{R} \) is \( \boxplus \)-infinitely divisible if and only if the free cumulant transform has the representation:

\[
C_{\mu}(z) = \eta z + az^2 + \int_\mathbb{R} \left( \frac{1}{1 - zt} - \frac{1 - tz}{1 - tz} \mathbf{1}_{[-1,1]}(t) \right) d\nu(t), \quad z \in \mathbb{C}_-.
\]

We now recall the following from Arizmendi et al. [2013]:

\[
d\sigma(t) = a\delta_0(dt) + \frac{t^2}{1 + t^2} d\nu(t), \quad (2.8)
\]

\[
\gamma = \eta - \int_\mathbb{R} t \left( \mathbf{1}_{[-1,1]}(t) - \frac{1}{1 + t^2} \right) d\nu(t). \quad (2.8)
\]

The proof of our main theorem (Theorem 2.2) demands the finiteness of the Lévy measure and we are only aware of the fact that the Lévy measure appearing the Voiculescu transform is finite.

**Remark 2.1** (Bercovici et al. [1999]). The **Bercovici-Pata bijection** between the set of classical infinitely divisible probability measures \( \mathcal{I}(\mathbb{R}) \) and the set of free infinitely divisible
probability measures \( I (\mathbb{H}) \) is the mapping \( \Lambda : I (*) \to I (\mathbb{H}) \) that sends the measure \( \mu \) in \( I (*) \) with classical characteristic triplet \((\eta, a, \nu)\) to the measure \( \Lambda (\mu) \) in \( I (\mathbb{H}) \) with free characteristic triplet \((\eta, a, \nu)\).

For a free infinitely divisible probability measure \( \mu \) on \( \mathbb{R} \) where the Lévy measure (Definition 2.1 in Barndorff-Nielsen et al. [2006]) \( \nu \) satisfies \( \int_{\mathbb{R}^+} \min (1, t) \, d\nu (t) < \infty \) the Lévy-Khintchine representation (2.7) reduces to

\[
C^{\mathbb{H}}_{\mu} (z) = \eta' z + \int_{\mathbb{R}} \left( \frac{1}{1 - zt} - 1 \right) \, d\nu (t), \quad z \in \mathbb{C},
\]

(2.9)

where \( \eta' \in \mathbb{R} \). The measure \( \mu \) is called a free regular infinitely divisible distribution (or regular \( \mathbb{H} \)-infinitely divisible measure) if \( \eta' \geq 0 \) and \( \nu ((-\infty, 0]) = 0 \).

The most typical example is compound free Poisson distributions. If the drift term \( \eta \) is zero and the Lévy measure \( \nu \) is \( \lambda \rho \) for some constant \( \lambda > 0 \) and a probability measure \( \rho \) on \( \mathbb{R} \), then we call \( \mu \) a compound free Poisson distribution with rate \( \lambda \) and jump distribution \( \rho \). To clarify these parameters, we denote \( \mu = \pi (\lambda, \rho) \).

**Example 2.1** (Arizmendi et al., 2013, Remark 8).

1. The Marchenko-Pastur law \( m \) is a compound free Poisson with rate 1 and jump distribution \( \delta_1 \).
2. The compound free Poisson \( \pi (1, \rho) \) coincides with the free multiplication \( m \boxplus \rho \).

We shall use both the Voiculescu transform and the cumulant transform to state our theorems. The notations \( \mu_{\mathbb{H}, V}^{\gamma, \sigma} \) and \( \mu_{\mathbb{H}, C}^{\eta', 0, \nu} \) shall occur whenever we write the Voiculescu transform or the cumulant transform of a free regular probability measure \( \mu \) respectively. The indices \( V \) and \( C \) are used to distinguish between the occurrence in Voiculescu transform or in the cumulant transform. The occurrences of \( \gamma, \sigma, \eta' \) and \( \nu \) in the indices are clear from (2.10) and (2.11) while in \( \mu_{\mathbb{H}, C}^{\eta', 0, \nu} \), the index 0 is to indicate the non existence of the Gaussian part in the representation of the cumulant transform. Let \( \mu_{\mathbb{H}, V}^{\gamma, \sigma} = \mu_{\mathbb{H}, C}^{\eta', 0, \nu} \) be a free regular infinitely divisible probability measure. Then its Voiculescu and cumulant transforms have the representations:

\[
\phi_{\mu_{\mathbb{H}, V}^{\gamma, \sigma}} (z) = \gamma + \int_{-\infty}^{\infty} \frac{1 + tz}{z - t} \, d\sigma (t),
\]

(2.10)

\[
C_{\mu_{\mathbb{H}, C}^{\eta', 0, \nu}} (z) = \eta' z + \int_{\mathbb{R}} \left( \frac{1}{1 - zt} - 1 \right) \, d\nu (t)
\]

(2.11)

respectively following (2.6) and (2.9).

In analogy to the classical case, \( \sigma \) in (2.6) (or in (2.10)) will again be called the free Lévy measure of the free regular infinitely divisible probability measure \( \mu \). Theorem 4.1 and Theorem 4.2 of Appendix shall be used to study the tail equivalence of a free regular probability measure and its free Lévy measure.

2.4. **Main results.** Now we are ready to state the main results of the paper while keeping in mind all the notations defined above. The following theorem gives us the tail equivalence between a free regular probability measure and its free Lévy measure occurring in the Voiculescu transform.

**Theorem 2.2.** Suppose that \( \mu_{\mathbb{H}, V}^{\gamma, \sigma} \) is free regular infinitely divisible measure. Assume that either \( \mu_{\mathbb{H}, V}^{\gamma, \sigma} \) or \( \sigma \) is concentrated on \([0, \infty)\). Then the following statements are equivalent:


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Let Corollary 2.2.2.

distributions. Therefore as a corollary of the the above corollary we get the following:

and therefore there is no ambiguity in talking about its tail behaviour.

Remark 2.2. Note that in Corollary 2.2.1, the measure \( \nu \) may not be a finite measure. Although \( \nu \) being a Lévy measure of a free regular probability measure we have \( \nu(1, \infty) < \infty \) and therefore there is no ambiguity in talking about its tail behaviour.

The most important free regular probability measures are the compound free Poisson distributions. Therefore as a corollary of the the above corollary we get the following:

Corollary 2.2.1. Suppose \( \mu_{\eta,\nu}^{(0, t)} \) is a free regular infinitely divisible measure. Then the following are equivalent:

1. \( \mu_{\eta,\nu}^{(0, t)} \) has regularly varying tail of index \(-\alpha\).
2. \( \nu \) has regularly varying tail of index \(-\alpha\).

If either of the above holds, then \( \mu_{\eta,\nu}^{(0, t)} (x, \infty) \sim \nu (x, \infty) \) as \( x \to \infty \).

Corollary 2.2.2. Let \( \rho \) be a positively supported probability measure. then for the compound free Poisson distribution \( \mu = \pi (1, \rho) \) which coincides with the free multiplication \( m \boxtimes \rho \), the following are equivalent.

1. The tail of \( \mu \) is regularly varying with index \(-\alpha\).
2. The tail of \( \rho \) is regularly varying with index \(-\alpha\).

If any of the above holds, then \( \mu (y, \infty) \sim \rho (y, \infty) \) as \( y \to \infty \).

Now we relate our result for the free regular probability measures (Corollary 2.2.1) and the famous classical result (stated in Theorem 2.1) via the notion of Bercovici-Pata bijection (definition 2.1).

Corollary 2.2.3. Suppose \( \alpha \geq 0 \), \( \eta' > 0 \) and \( \nu \in \mathcal{M}_+ \) satisfies \( \int_{\mathbb{R}_+} \min (1, t) d\nu (t) < \infty \).

Then the classical infinitely divisible probability measure \( \mu_{\eta',\nu}^{(0, t)} \) has regularly varying tail of index \(-\alpha\) if and only if the free regular infinitely divisible probability measure \( \mu_{\eta',\nu}^{(0, t)} \), the image of \( \mu_{\eta',\nu}^{(0, t)} \) under Bercovici-Pata bijection, has regularly varying tail of index \(-\alpha\). In either case,

\[ \mu_{\eta',\nu}^{(0, t)} (x, \infty) \sim \mu_{\eta',\nu}^{(0, t)} (x, \infty) \text{ as } x \to \infty. \]

Now we describe two situations where the above results can be applied. The first one is for random matrices while the other one is for the free stable laws.

Example 2.2. As mentioned in the introduction, \( m \boxtimes \rho \) often occurs as a limiting spectral distribution. For example consider for all \( N \geq 1 \), \( W_N = \sum_{k=1}^{M_N} X_N X_N^* \) where \( X_N \) is a complex Gaussian random matrix with i.i.d. entries and the sequence \( \{M_N\}_{N \geq 1} \) is such that \( \lim_{N \to \infty} N/M_N = \lambda \in (0, \infty) \). Also take \( Y_N \), for all \( N \geq 1 \) to be random complex Hermitian matrices independent of the entries of \( X_N \). Suppose there exists a non random probability measure \( \rho \) on \( \mathbb{R} \) such that empirical spectral distribution of \( Y_N \) converges to \( \rho \) weakly in probability. In this setup when \( \lambda = 1 \), Theorem 2.3 of Chakraborty et al. [2018] tells us that
the expected empirical spectral distribution of \( W_N Y_N \) converges to \( m \boxplus \rho \) weakly as \( N \to \infty \). Therefore if we take \( \rho \) to be regularly varying with tail index \(-\alpha\), \( \alpha \geq 0 \), we are able to conclude that the tail of the limiting spectral distribution of \( W_N Y_N \) is same as that of \( \rho \) using Corollary 2.2.2.

**Example 2.3.** Following Bercovici et al. [1999] we define two probability measures \( \mu \) and \( \nu \) to be equivalent (denote as \( \mu \sim \nu \)) if \( \mu(S) = \nu(aS + b) \) for every Borel set \( S \subseteq \mathbb{R} \), for some \( a \in \mathbb{R}_+ \) and \( b \in \mathbb{R} \). A measure \( \mu \) (excluding point mass measures) is said to be \( \boxplus\)-stable if for every \( \nu_1, \nu_2 \in \mathcal{M} \) such that \( \nu_1 \sim \nu_2 \), it follows that \( \nu_1 \boxplus \nu_2 \sim \mu \). Associated with every \( \boxplus\)-stable measure \( \mu \) there is a number \( \alpha \in (0, 1) \) such that the measure \( \mu \boxplus \mu \) is a translate of the measure \( \mathcal{D}_{1/2} \mu \) where \( \mathcal{D}_a \mu(S) = \mu(aS) \). The number \( \alpha \) is called the stability index of \( \mu \). The probability measure \( \mu^{(2)} \) will be the image of \( \mu \) under the map \( t \to t^2 \) on \( \mathbb{R} \).

We give a proper example where Corollary 2.2.2 follows directly. From the appendix of Bercovici et al. [1999] we get that the Voiculescu transform of a \( \boxplus \)-stable probability measure with stability index \( \alpha \in (0, 1) \) is of the form \( \phi(z) = -e^{\alpha \rho x} z^{-\alpha+1} \) where \( \rho \) is called the asymmetry coefficient. Now using Theorem 4.1 of Appendix we can conclude that the \( \boxplus\)-stable probability measures in \( \mathcal{M}_0 \) with stability index \( \alpha \in (0, 1) \) are exactly regularly varying probability measures with tail index \(-\alpha\).

Let \( \mu_\alpha \) be a regularly varying symmetric free \( \alpha\)-stable law with \( 0 < \alpha < 2 \). Then \( \mu_\alpha^{(2)} = \rho_{\alpha/2} \boxplus m \), where \( \rho_{\alpha/2} \) is a free positive \( \alpha/2 \) stable law.

The above statement can be verified by the following arguments. First from Corollary 21 of Pérez-Abreu and Sakuma [2012] observe that the positive \( \alpha/2 \)-stable law \( \mu_\alpha^{(2)} \) enjoys the relation \( \mu_\alpha^{(2)} = (\rho_\beta \boxplus \rho_\beta) \boxplus m \), where \( \rho_\beta \) is a free positive \( 2\alpha/(2+\alpha) \) stable law. Now applying Proposition 13 of Arizmendi E. and Pérez-Abreu [2009], it follows that \( \rho_\beta \boxplus \rho_\beta = \rho_{\alpha/2} \). Hence \( \mu_\alpha^{(2)} = \rho_{\alpha/2} \boxplus m \) where \( \mu_\alpha^{(2)} \) and \( \rho_{\alpha/2} \) are both are regularly varying of tail indices \(-\alpha/2\) as both are in \( \mathcal{M}_0 \). The Corollary 2.2.2 can be seen as generalizing this behaviour for much more general class of probability measures.

3. Proofs

We shall use the notation \( \mathcal{F} \) for the probability measure \( \tau \). Also fix the notations \( m_=-1(\tau) = \gamma \) and \( m_0(\tau) = \sigma(\mathbb{R}_+) \). Recall the remainder terms of the Cauchy and Voiculescu transforms as defined in (2.1) and (2.2) respectively.

**Lemma 3.1.** Let \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) be a regular \( \boxplus\)-infinitely divisible probability measure.

(a) Voiculescu transform of \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) and Cauchy transform of \( \mathcal{F} \) are related by

\[
\phi_{\nu_{\mathcal{F}}^{\gamma, \sigma}}(z) = m_-(\sigma) - m_0(\sigma) z + (1 + z^2) m_0(\sigma) G_\mathcal{F}(z). \tag{3.1}
\]

(b) If either \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) or \( \sigma \) has its support concentrated on \([0, \infty)\), then so does the other. Further, in this case, \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) and \( \sigma \) have same number of moments.

(c) If both \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) and \( \sigma \) have \( p \) moments, then the \( p \) cumulants of \( \nu_{\mathcal{F}}^{\gamma, \sigma} \) and \( p \) moments of \( \sigma \) satisfy the relation

\[
r_\phi_{\nu_{\mathcal{F}}^{\gamma, \sigma}}(z) = m_{p-2}(\sigma) z^{-2} + (1 + z^{-2}) m_0(\sigma) r_{G_\mathcal{F}}(z). \tag{3.3}
\]

and the remainder terms of \( \phi_{\nu_{\mathcal{F}}^{\gamma, \sigma}} \) and \( G_\mathcal{F} \) satisfy

\[
r_{\phi_{\nu_{\mathcal{F}}^{\gamma, \sigma}}}(z) = m_{p-1}(\sigma) z^{-1} + m_p(\sigma) z^{-p} + (1 + z^{-p}) m_0(\sigma) r_{G_\mathcal{F}}(z). \tag{3.3}
\]
Proof. (a) Using (2.10), we have
\[ \phi_{\mu^{\gamma,\sigma}_{\Pi,V}}(z) = \gamma + m_0(\sigma) \int_0^\infty \frac{1 + tz}{z - t} d\sigma(t) \]
\[ = \gamma + m_0(\sigma) G_{\sigma}(z) + m_0(\sigma) z \int_0^\infty \frac{t}{z - t} d\sigma(t) \]
\[ = m_{-1}(\sigma) + m_0(\sigma) G_{\sigma}(z) - m_0(\sigma) z + m_0(\sigma) z^2 G_{\sigma}(z). \]

(b) When \( \mu^{\gamma,\sigma}_{\Pi,V} \) is concentrated on the positive axis, Lemma 4.1 of Appendix shows that \( \sigma \) is also concentrated on \([0, \infty)\). Conversely, if \( \sigma \) is concentrated on \((0, \infty)\), then \( \mu^{\gamma,\sigma}_{\Pi,V} \) is also concentrated on \([0, \infty)\) by Lemma 4.2(a). Since \( \sigma \) is minorized to \([0, \infty)\), \( \mu^{\gamma,\sigma}_{\Pi,V} \) and \( \sigma \) will have same number of moments by Lemma 4.2(b, c).

(c) If both \( \mu^{\gamma,\sigma}_{\Pi,V} \) and \( \sigma \) have \( p \) moments finite, considering Laurent series expansion of \( G_{\sigma} \) in (3.1) and the fact that \( m_j(\sigma) = m_0(\sigma) m_j(\sigma) \) for \( 0 \leq j \leq p \), we have
\[ \phi_{\mu^{\gamma,\sigma}_{\Pi,V}}(z) = m_{-1}(\sigma) - m_0(\sigma) z + (1 + z^2) \sum_{j=1}^{p+1} m_{j-1}(\sigma) z^{-j} \]
\[ + (1 + z^2) z^{-(p+1)} m_0(\sigma) r_{G_{\sigma}}(z) \]
\[ = \sum_{j=1}^{p} (m_{j-2}(\sigma) + m_j(\sigma)) z^{-(j-1)} + z^{-(p-1)} (m_{p-1}(\sigma) z^{-1} \]
\[ + m_p(\sigma) z^{-2} + (1 + z^{-2}) m_0(\sigma) r_{G_{\sigma}}(z)). \]

Since \( r_{G_{\sigma}}(z) = o(1) \) as \( z \to \infty \) n.t., we have
\[ m_{p-1}(\sigma) z^{-1} + m_p(\sigma) z^{-2} + (1 + z^{-2}) m_0(\sigma) r_{G_{\sigma}}(z) = o(1) \]
as \( z \to \infty \) n.t. Thus, by uniqueness of Laurent series expansion (which is equivalent to the uniqueness of Taylor series expansion given in Lemma A.1 of Benaych-Georges, 2006), we obtain (3.2) as well as (3.3).

It can be shown, using the expansions of Voiculescu and Cauchy transforms, that, if \( \mu^{\gamma,\sigma}_{\Pi,V} \) is a compactly supported probability measure, then \( \sigma \) is also compactly supported and their cumulants and moments are related exactly by the formula stated in (3.2). Further note that \( m_{p-2}(\sigma) + m_{p}(\sigma) \) is also the classical cumulant of a classical infinitely divisible distribution.

It is obvious that Lemma 3.1(b) shows the assumptions on the supports of \( \mu^{\gamma,\sigma}_{\Pi,V} \) or \( \sigma \) in Theorem 2.2 are actually equivalent.

Proof of Theorem 2.2. First assume that \( \mu^{\gamma,\sigma}_{\Pi,V} \) has regularly varying tail. Then, for some nonnegative integer \( p \) and \( \alpha \in [p, p+1] \), the measure \( \mu^{\gamma,\sigma}_{\Pi,V} \in \mathcal{M}_{p,\alpha} \). Also, by Lemma 3.1(b), we have \( \sigma \in \mathcal{M}_{p,\alpha} \) as well. Furthermore evaluating (3.3) at \( z = iy \) and equating the real and the imaginary parts respectively, we have
\[ (1 - y^{-2}) m_0(\sigma) \Re r_{G_{\sigma}}(iy) - m_p(\sigma) y^{-2} = \Re \phi_{\mu^{\gamma,\sigma}_{\Pi,V}}(iy) \tag{3.4} \]
and
\[ (1 - y^{-2}) m_0(\sigma) \Im r_{G_{\sigma}}(iy) - m_{p-1}(\sigma) y^{-1} = \Im \phi_{\mu^{\gamma,\sigma}_{\Pi,V}}(iy). \tag{3.5} \]
Now if $\alpha \in [p, p + 1)$, using Theorem 4.1, we have from (3.5), as $y \to \infty$,

$$(1 - y^{-2}) m_0(\sigma) \Re r_{\sigma} (iy) - m_{p-1}(\sigma) y^{-1} = \Re r_{\phi,\gamma,\sigma} (iy) - \frac{\pi(p+1-\alpha)}{\cos \frac{\pi(\alpha-p)}{2}} y^p \mu_{\gamma,\sigma} (y, \infty),$$

which is regularly varying of index $-(\alpha - p)$ with $\alpha - p < 1$. Thus, as $y \to \infty$,

$$m_0(\sigma) \Re r_{\sigma} (iy) \sim (1 - y^{-2}) m_0(\sigma) \Re r_{\sigma} (iy) \sim -\frac{\pi(p+1-\alpha)}{\cos \frac{\pi(\alpha-p)}{2}} y^p \mu_{\gamma,\sigma} (y, \infty)$$

and is also regularly varying of index $-(\alpha - p)$ and again by Theorem 4.1, $\sigma$ and hence $\sigma$ has regularly varying tail of index $-\alpha$ and

$$\Re r_{\sigma} (iy) \sim -\frac{\pi(p+1-\alpha)}{\cos \frac{\pi(\alpha-p)}{2}} y^p \sigma (y, \infty) \text{ as } y \to \infty.$$  

Putting two asymptotic equivalences together, we get $\mu_{\gamma,\sigma} (y, \infty) \sim m_0(\sigma) \sigma (y, \infty) = \sigma (y, \infty)$ as same argument works for the case $\alpha = p+1$ with the help of Theorem 4.2 and equation (3.4).

To get the converse statement, we shall start with $\sigma$ to be regularly varying with index $-\alpha$. Thus $\sigma \in \mathcal{M}_{p,0}$ for some integer $p \geq 0$. Lemma 3.1(b) gives $\mu_{\gamma,\sigma} \in \mathcal{M}_p$ also, and we get the equations (3.4) and (3.5). Arguing exactly the same way like above we shall be able to conclude that $\mu_{\gamma,\sigma}$ is regularly varying with tail index $-\alpha$. \hfill \Box

**Proof of Corollary 2.2.1.** First we observe the following with the notations $\sigma$, $\nu$ and $a$ be as in (2.8). Suppose $a = 0$. Then from (2.8), taking integral from $x$ to infinity on both sides we get,

$$\nu(x, \infty) = \sigma(x, \infty) + \int_x^{\infty} \frac{1}{t^2} d\sigma(t) \leq (1 + \frac{1}{x^2}) \sigma(x, \infty), \text{ since } x < t.$$  

Therefore,

$$\sigma(x, \infty) \leq \nu(x, \infty) \leq (1 + \frac{1}{x^2}) \sigma(x, \infty).$$

Taking limit as $x \to \infty$ we get

$$\sigma(x, \infty) \sim \nu(x, \infty) \text{ as } x \to \infty. \quad \text{(3.6)}$$

Now Corollary 2.2.1 is immediate from Theorem 2.2 and the equation (3.6) as

$$\mu_{\gamma,\sigma}^C (x, \infty) = \mu_{\gamma,\sigma} (x, \infty) \overset{\text{Theorem 2.2}}{\sim} \sigma(x, \infty) \overset{(3.6)}{\sim} \nu(x, \infty).$$  

\hfill \Box

**Proof of Corollary 2.2.2.** This result follows directly from Corollary 2.2.1 while noticing from the example 2.1 that $\rho$ is the Lévy measure of the compound free Poisson distribution $\mu = m \Rightarrow \rho$.

\hfill \Box

**Proof of Corollary 2.2.3.** Suppose the classical infinitely divisible probability measure $\mu_{\gamma,0,\nu}^C$ has regularly varying tail, then by Theorem 2.1 we have the measure $\nu$ in the Laplace transform has the same regularly varying tail. Now both measures in the Laplace transform and the Fourier transform is same $\nu$ by [Sato, 2013, Remark 21.6] since the measure $\nu$ satisfies the conditions $a = 0$ in (2.3), $\int_{-\infty}^{0} dv(t) = 0$, $\int_{0}^{1} t d\nu(t) < \infty$ and $\eta > 0$. Then the relation
If any of the above statements holds, we also have, as $y \to \infty$ and $z \to \infty$ n.t.,
\[ \Re r_\phi (iy) \gg y^{-1} \] as $y \to \infty$ and $z \to \infty$ n.t.,
\[ \Re r_\phi (iy) \sim \Re r_G (iy) \sim \frac{\pi(p+1-\alpha)}{\cos\frac{\alpha}{2}} y^p \mu (y, \infty) \gg \frac{1}{y} \] and $\Im r_\phi (iy) \gg \Re r_G (iy) \gg \frac{1}{y}$.

If $\alpha > p$ and any of the statements (i)-(iii) holds, we further have, as $y \to \infty$,
\[ \Re r_\phi (iy) \sim \Re r_G (iy) \sim -\frac{\pi(p+2-\alpha)}{\sin\frac{\alpha}{2}} y^p \mu (y, \infty) . \]

If $\alpha = p = 0$ and any of the statements (i)-(iii) holds, we further have, as $y \to \infty$,
\[ \Re r_\phi (iy) \sim \Re r_G (iy) \sim -\mu (y, \infty) . \]

Next we consider the case $\alpha = p + 1$.

**Theorem 4.2.** Let $\mu$ be a probability measure in the class $\mathcal{M}_p$ and $\alpha \in (p, p + 1)$. The following statements are equivalent:

(i) $y \mapsto \mu (y, \infty)$ is regularly varying of index $-(p + 1)$.
(ii) $y \mapsto \Re r_G (iy)$ is regularly varying of index $-1$.
(iii) $y \mapsto \Re r_\phi (iy)$ is regularly varying of index $-1$, $y^{-1} \ll \Re r_\phi (iy) \ll y^{-(1-\beta/2)}$ as $y \to \infty$ and $z^{-1} \ll r_\phi (z) \ll z^{-\beta}$ as $z \to \infty$ n.t.

If any of the above statements holds, we also have, as $z \to \infty$ n.t., $z^{-1} \ll r_G (z) \sim r_\phi (z) \ll z^{-\beta}$ as $y \to \infty$,
\[ y^{-(1+\beta/2)} \ll \Re r_\phi (iy) \sim \Re r_G (iy) \sim \frac{\pi}{2} y^p \mu (y, \infty) \ll y^{-(1-\beta/2)} \]
and
\[ y^{-1} \ll \Im r_\phi (iy) \sim \Im r_G (iy) \ll y^{-(1-\beta/2)} . \]

The following two lemmas have been used in the course of the proof of Lemma 3.1.

**Lemma 4.1** (Arizmendi et al., 2013, Theorem 9). Let $\mu \in \mathcal{M}$, the following statements are equivalent.

(a) $\mu$ is free regular.
(b) \( \mu \) is \( \boxplus \)-infinitely divisible, \( \sigma (-\infty, 0) = 0 \) and \( \phi_{\mu}(-0) \geq 0 \), where \( \sigma \) is the measure appearing in (2.6).

Before stating the following lemma, recall that a subset of \( \mathbb{R} \) is said to be minorized (respectively, majorized), if it is contained in an interval of type \((a, \infty)\) (respectively, of type \((-\infty, a)\)), for some real number \( a \).

**Lemma 4.2** (Benaych-Georges, 2006, Section 2). Let \( \mu^{\gamma, \sigma}_{\boxplus V} \) be a \( \boxplus \)-infinitely divisible probability measure with free Lévy measure \( \sigma \).

(a) If the support of \( \sigma \) is contained in \((0, \infty)\), then \( \mu^{\gamma, \sigma}_{\boxplus V} \) is concentrated on \([0, \infty)\).

(b) If \( p \) is an integer and \( \sigma \) has \( p \) moments, then so does \( \mu^{\gamma, \sigma}_{\boxplus V} \).

(c) Suppose moreover, that \( p \) is even or that the support of \( \sigma \) is minorized or majorized. In this case, if \( \mu^{\gamma, \sigma}_{\boxplus V} \) admits a moment of order \( p \), then the same holds for \( \sigma \).

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