Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

Aritra Kundu¹,², Abhishek Dhar¹ and Sanjib Sabhapandit²,³

¹ International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India
² Raman Research Institute, Bangalore 560080, India
E-mail: aritrak@icts.res.in, abhishek.dhar@icts.res.in and sanjib.sabhapandit@gmail.com

Received 3 September 2019
Accepted for publication 14 November 2019
Published 19 February 2020

Abstract. We study a gas of point particles with hard-core repulsion in one dimension where the particles move freely in-between elastic collisions. We prepare the system with a uniform density on the infinite line. The velocities $\{v_i; i \in \mathbb{Z}\}$ of the particles are chosen independently from a thermal distribution. Using a mapping to the non-interacting gas, we analytically compute the equilibrium spatio-temporal correlations $\langle v_m^i(t) v_n^j(0) \rangle$ for arbitrary integers $m, n$. The analytical results are verified with microscopic simulations of the Hamiltonian dynamics. The correlation functions have ballistic scaling, as expected in an integrable model.

Keywords: classical integrability, correlation functions, exact results, transport properties

³ Author to whom any correspondence should be addressed.
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

1. Introduction

The study of equilibrium spatio-temporal correlations in one-dimensional systems of interacting particles with Hamiltonian dynamics is useful in understanding their transport properties. Non-integrable systems have a few conserved quantities and spatio-temporal correlation functions in one dimension show universal behavior that can be understood within the framework of fluctuating hydrodynamics of the conserved quantities [1–3]. In contrast, for integrable systems, which are characterized by a macroscopic number of conservation laws, the spatio-temporal correlations are non-universal and very limited results are known. Some progress has recently been made in understanding transport in integrable systems through the approach of generalized hydrodynamics for these macroscopic number of conserved quantities, which was introduced in [4, 5], along developed consequently in [6, 7] and references therein. Recently, this approach has been used to extract universal behaviour of correlation functions in integrable systems of conserved as well as non-conserved densities [7–9].

One of the simplest examples of an interacting particle system is the hard particle gas (HPG), where particles undergo elastic binary collisions, while in between the collisions, they move freely with constant velocity. The elastic collisions conserve energy and momentum. In general, when the masses of the particles are not all equal, the system is expected to be non-integrable. A particular case, where the masses of successive particles are taken to alternate between two values, has been studied both in the equilibrium and non-equilibrium setups, and is known to exhibit anomalous transport with the thermal conductivity diverging with system size [10–15]. In contrast, in the equal
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

mass HPG, the particles simply exchange their velocities during the elastic collisions, which makes the system integrable. Evidently, the conserved quantities in this system are \( \sum_i v_i^n \) for integer \( n \). It was shown by Jepsen [16] that one can effectively map the equal mass HPG to a gas of non-interacting particles and many exact results, such as tagged particle equilibrium velocity correlations, could be obtained. Some extensions to the case of hard rods were obtained in [17] and dynamical pair correlations were studied in [18] where closed expressions were found for particular choices of initial conditions. In recent work [19–21], considerable simplification of this approach was used to obtain exact results on properties of tagged particle displacements and velocity correlations. More recently [22], the correlations of energy, momentum and stretch was computed numerically in Toda chain and the hard particle case was studied as a limiting case of Toda potential. The Toda system has been treated using the framework of generalized hydrodynamics to understand dynamical properties in the system [23, 24].

In this paper, following the methods in [19], we compute dynamical correlations for all conserved quantities which, as noted above, are simply given by various powers of the velocity. Specifically, we considered a set of \( 2N+1 \) particles of unit masses that are initially distributed uniformly inside a one-dimensional box of length \( 2L \). The initial velocity of each particle is chosen independently from the Maxwell–Boltzmann distribution at temperature \( T \). The ordered particles have positions and velocities given by \( \{x_i, v_i\} \) for \( i = 1, 2 \ldots, 2N+1 \). We are interested in computing general spatio-temporal correlation functions, defined as

\[
\langle v_i^m(t); v_j^n(0) \rangle = \frac{\langle v_i^m(t)v_j^n(0) \rangle - \langle v_i^m(t) \rangle \langle v_j^n(0) \rangle}{\bar{v}^{m+n}}, \text{ with } \bar{v} = \sqrt{T},
\]

where \( m, n \) are positive integers and \( i, j \) are taken to be particles in the bulk, and the average is over initial configurations chosen from the equilibrium Gibb’s distribution at temperature \( T \) and uniform particle density. In the thermodynamic limit of \( N \to \infty, L \to \infty \) while keeping the density \( \rho = N/L \) constant, the correlation function depends only on the relative position of the two particles, \( r = i - j \).

Our main results include the following explicit asymptotic \( (t \to \infty) \) form for the velocity correlations, in terms of the scaling variable \( l = r/(\rho \bar{v} t) \):

\[
\rho \bar{v} t \langle v_r^m(t); v_0^n(0) \rangle = \left(l^m - \delta_m\right)\left(l^n - \delta_n\right)f(l),
\]

where \( f(\omega) = e^{-\omega^2/2}/\sqrt{2\pi} \) and \( \delta_n = \int_{-\infty}^{\infty} d\omega \omega^n f(\omega) \). From these, in particular, we extract the correlations for stretch, momenta and energy as

\[
\rho \bar{v} t \langle s_r(t)s_0(0) \rangle = \frac{1}{\rho^2} f(l)
\]

\[
\rho \bar{v} t \langle v_r(t)v_0(0) \rangle = \bar{v}^2 l^2 f(l)
\]

\[
\rho \bar{v} t \langle e_r(t); e_0(0) \rangle = \frac{\bar{v}^4}{4} \left(l^4 - 2l^2 + 1 \right) f(l).
\]

The paper is organized as follows: in section 2, we define the model precisely and explain the idea of computing the correlations exactly using the mapping to the non-interacting system. Then in section 3 we present results for the correlations which are

https://doi.org/10.1088/1742-5468/ab5d0c
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

4https://doi.org/10.1088/1742-5468/ab5d0c

J. Stat. Mech. (2020) 023205

obtained in the asymptotic limit of large time. We also verify the results from simulations of the microscopic system. Finally we conclude in section 4.

2. Main steps of the calculation

We consider a gas of \( 2N + 1 \) ordered particles with positions \( \{x_i\} \), with \( i = 1, \ldots, 2N + 1 \), initially distributed uniformly in the interval \([-L, L]\). The velocities \( \{v_i\} \) are chosen from the Gibbs distribution \( P(\{v_i\}) = \prod_i [\exp(-v_i^2/2T)/\sqrt{2\pi T}] \). Since we are interested only in the thermodynamic limit and in bulk properties, we do not need to include a confining wall. The particles move freely with constant velocity between elastic collisions with the neighbors. Under a collision the particles simply exchange their velocities. This allows us to make a mapping from interacting system of particles (figure 1(a)), where the particles collide with each other, to a system of non-interacting particles (figure 1(b)), where the particles evolve independently and pass through each other, and we exchange labels whenever particles cross. In fact, this non-interacting picture allows one to evolve the system directly to any final time and the corresponding trajectory in the case where the particles are undergoing collisions can be obtained by relabeling the tags of the particles. The probability of obtaining the trajectories in the non-interacting picture is same as that of the interacting system.

This mapping to the non-interacting gas was used by Jepsen [16] to obtain an exact solution for velocity-velocity autocorrelation functions in the hard-particle gas. A simpler approach was recently proposed in [19–21] to obtain two particle distribution, tagged particle statistics and also a particular case of velocity correlations with \( m = n = 1 \) in equation (1).

In the non-interacting picture, a particle at \( x \) with velocity \( v \) travels to \( y \) at time \( t \) such that \( y = x + vt = x + \sigma_tv/\bar{v} \) where we have defined \( \sigma_t = \bar{v}t \). For a particle with velocity chosen from the Maxwell distribution, the single particle propagator giving the probability of finding the particle at \( y \) at time \( t \) is then given by

\[
G(y, t|x, 0) = \int_{-\infty}^{\infty} dv \delta(y - x - \sigma_tv/\bar{v}) \frac{e^{-v^2/2\bar{v}^2}}{\sqrt{2\pi\bar{v}^2}} = \frac{1}{\sigma_t} f\left(\frac{y - x}{\sigma_t}\right),
\]

where \( f \) is the Maxwell velocity distribution function.

\[ \text{(6)} \]
where the function \( f(\omega) = e^{-\omega^2/2}/\sqrt{2\pi} \). Note that most parts of the derivations given below go through for arbitrary scaling function \( f(\omega) \) with finite moments.

Our strategy is to compute the correlations of velocity via the two-time joint distribution function \( P(x, j, 0; y, k, t) \) defined as the probability of the \( j \)th (ordered) particle being at \( x \) at time \( t = 0 \) and the \( k \)th (ordered) particle being at \( y \) at time \( t \). Following [19], this joint PDF can be readily computed using the single particle propagator and through the mapping to non-interacting particles. We now give the details.

### 2.1. Computation of the joint probability distribution of two particles

In terms of the non-interacting gas picture, the joint probability density, \( P(x, j, 0; y, k, t) \), of the \( j \)th (ordered) particle being at \( x \) at time \( t = 0 \) and the \( k \)th (ordered) particle being at \( y \) at time \( t \), has two independent contributions \( P_1(x, j, 0; y, k, t) \) and \( P_2(x, j, \tilde{x}, 0; y, k, \tilde{y}, t) \) that are defined below (see figure 2).

(i) In the non-interacting gas picture, \( P_1(x, j, 0; y, k, t) \) is the probability that \( j \)th particle, which is at \( x \) at time \( t = 0 \), becomes the \( k \)th particle at \( y \) at time \( t \) (shown in figure 2(a)). This is obtained by picking one of the non-interacting particles at random at time \( t = 0 \), with probability \((2N+1)/(2L)\), evolving with the propagator from \((x, 0)\) to \((y, t)\) and multiplying by the probability that this is the \( j \)th particle at time \( t = 0 \) and the \( k \)th particle at time \( t \). We thus get

\[
P_1(x, j, 0; y, k, t) = \frac{2N + 1}{2L} G(y, t|x, 0) F_{1N}(x, j, 0; y, k, t),
\]

where \( F_{1N}(x, j, 0; y, k, t) \) is the probability that there are \( j - 1 \) particles to the left of \( x \) at \( t = 0 \) and \( k - 1 \) particles to the left of \( y \) at time \( t \). The explicit form of \( F_{1N} \) can be obtained by combinatorial arguments [19] and is given in the appendix.

(ii) Similarly, for the non-interacting gas, \( P_2(x, j, \tilde{x}, 0; y, k, \tilde{y}, t) \) is the probability that the \( j \)th particle at position \( x \) at \( t = 0 \) becomes another particle at \( \tilde{y} \) at time \( t \), and some other particle at \( \tilde{x} \) at time \( t = 0 \) becomes the \( k \)th particle at position \( y \) at time \( t \).
time \( t \) (shown in figure 2(b)). In this case, two particles are chosen randomly at \( t = 0 \), with probability \((2N + 1)(2N)/(2L)^2\), and are evolved with the propagator for the transition \((x, \tilde{x}, t = 0) \rightarrow (\tilde{y}, y, t)\). Finally we need to multiply this with the probability, \( P_{2N}(x, j, \tilde{x}, 0; y, k, \tilde{y}, t) \), that there are \( j - 1 \) particles to the left of \( x \) at \( t = 0 \) and \( k - 1 \) particles to the left of \( y \) at time \( t \), given that there is a particle at \( \tilde{x} \) at \( t = 0 \) and a particle at \( \tilde{y} \) at time \( t \). Combining these, we get

\[
P_2(x, j, \tilde{x}, 0; y, k, \tilde{y}, t) = \frac{(2N + 1)(2N)}{(2L)^2} G(y, t|\tilde{x}, 0)G(\tilde{y}, t|x, 0)P_{2N}(x, j, \tilde{x}, 0; y, k, \tilde{y}, t). \tag{8}
\]

The explicit form of \( F_{2N} \) can be obtained by combinatorial arguments [19] and is given in the appendix.

### 2.2. Relating the joint probability distribution to correlations

The contribution to velocity correlations in the interacting system comes from two separate processes corresponding to the two joint probability distributions from the previous section. Hence we obtain

\[
\mathcal{F}^{(m,n)}(k, j, t) = \langle v_k^m(t)v_j^n(0)\rangle/\bar{v}^{(m+n)} = \langle v_k^m(t)v_j^n(0)\rangle_{\text{I}}/\bar{v}^{m+n} + \langle v_k^m(t)v_j^n(0)\rangle_{\text{II}}/\bar{v}^{m+n},
\]

where \([\text{I}]\) and \([\text{II}]\) can be computed as follows.

(i) The first contribution comes when the \( j \)th particle at \( x \) at time \( t = 0 \) becomes the \( k \)th particle at \( y \) at time \( t \). In the non-interacting picture, these two particles are correlated and have the same velocity, i.e. \( v_k(t)/\bar{v} = v_j(0)/\bar{v} = (y - x)/\sigma_t \). Multiplying this with the appropriate probability \( P_1(x, j, 0; y, k, t) \) and integrating over all possible initial and final positions, we get the first contribution to velocity correlations

\[
[\text{I}] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left( \frac{y - x}{\sigma_t} \right)^{m+n} P_1(x, j, 0; y, k, t),
\]

\[
= \rho \sigma_t \int_{-\infty}^{\infty} \frac{dx}{\sigma_t} \int_{-\infty}^{\infty} \frac{dy}{\sigma_t} \left( \frac{y - x}{\sigma_t} \right)^{m+n} f \left( \frac{y - x}{\sigma_t} \right) F_{1N}(x, j, y, k), \tag{9}
\]

where we have replaced \((2N + 1)/(2L)\) with the density \( \rho \) in the thermodynamic limit.

(ii) The second contribution is when the \( j \)th particle at \( x \) with velocity \( v_j(0)/\bar{v} = (y - x)/\sigma_t \) at \( t = 0 \) becomes another particle at \( \tilde{y} \) time \( t \), and some other particle at \( \tilde{x} \) with velocity \( v_k(t)/\bar{v} = (y - \tilde{x})/\sigma_t \) at time \( t = 0 \) becomes the \( k \)th particle at \( y \) at time \( t \). Multiplying this with the appropriate probability \( P_2(x, j, \tilde{x}, 0; y, k, \tilde{y}, t) \) and integrating over all position variables, we get
3. Asymptotic results in the thermodynamic limit

3.1. Computation of velocity correlations

Next we use the explicit forms of $F_{1N}, F_{2N}$ given in the Appendix. One can perform the integrals over $\bar{x}, \bar{y}$ in the second term $[\Pi]$ in equation (10). We also make a change of integration variables from $x, y$ to the variables $z = (x - y)/\sigma$, and $\bar{z} = (x + y)/\sigma$, in equations (9) and (10). After some algebra, we finally obtain the following expression for the velocity correlations, for $j$ in the bulk:

$$C^{(m,n)}(r, t) = \mathcal{F}^{(m,n)}(j, j + r, t) = [I] + [\Pi]$$

$$= \rho \sigma, \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \frac{dz}{2} \frac{d\phi}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{2\pi} D(z, \theta, \phi)$$

$$\times e^{-2N(1 - \cos \phi)} e^{i\rho \sigma \tau (-z \sin \phi + z \sin \theta) e^{i(\phi - \theta) + e^{-2\rho \sigma \tau} Q(z)(1 - \cos \theta)},$$

where $Q(z) = z \int_{0}^{\pi} dw \ w f(\omega) + \int_{0}^{\infty} \omega f(\omega)$,

$$D(z, \theta, \phi) = (\rho \sigma)^{-1} z^{m+n} f(z) + \Delta_1^n(z) + \Delta_2^n(z) e^{-i\phi}$$

$$+ \Delta_1^m(z) + \Delta_2^m(z) e^{i\phi} + \Delta_1^m(z) + \Delta_2^m(z) e^{-i\phi},$$

and the $\Delta_{1,2}^p$ are related to moments of the propagator

$$\Delta_1^p(z) = \int_{z}^{\infty} d\omega \ \omega^p f(\omega), \Delta_2^p(z) = \int_{-\infty}^{z} d\omega \ \omega^p f(\omega).$$

Since in equation (11) the $\phi$ term comes with a factor of $N$, a saddle point analysis reveals that the major contribution comes from $\phi = 0$. We do a Taylor expansion around $\phi = 0$, up to second order, first perform the resulting Gaussian integral in $\phi$ and then perform the resulting Gaussian integral in the variable $\bar{z}$. This leads to following simpler form

$$C^{(m,n)}(r, t) = \rho \sigma, \int_{-\infty}^{\infty} dz \int_{-\pi}^{\pi} D(z, \theta, 0) e^{-2\rho \sigma \tau} e^{-i\phi} e^{-2\rho \sigma \tau} Q(z)(1 - \cos \theta) e^{-i\phi r}.$$
We define a new scaling variable \( l = r/(\rho \sigma_t) \) and rewrite the above equation as

\[
C^{(m,n)}(r = \rho \sigma_t l, t) = \rho \sigma_t \int_{-\infty}^{\infty} dz \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} D(z, \theta, 0)e^{-\rho \sigma_t I(z, \theta)},
\]

where, \( I(z, \theta) = [2Q(z)(1 - \cos \theta) - iz \sin \theta + i\theta l] \). Now we are interested in the large time (hence \( \rho \sigma_t >> 1 \)) behaviour and so we again use saddle point methods. Our strategy is to find \((z^*, \theta^*)\), where the function \( I(z, \theta) \) is minimum, and expand both the functions \( I \) and \( D \) in equation (15) around this minimum. The minimum is obtained by solving the equations

\[
\begin{align*}
2Q(z^*) \sin \theta^* - iz^* \cos \theta^* + il &= 0, \\
2Q'(z^*)(1 - \cos \theta^*) - i \sin \theta^* &= 0
\end{align*}
\]

whose solution is \( z^* = l, \theta^* = 0 \). We define new scaling variables \((w, u)\) through \( z = l + w\sqrt{\epsilon} \) and \( \theta = u\sqrt{\epsilon} \), where \( \epsilon = 1/\rho \sigma_t \), which gives

\[
I(l + w\sqrt{\epsilon}, u\sqrt{\epsilon}) = \left( u^2 Q(l) - iuw \right) + C_1 \epsilon^{1/2} + C_2 \epsilon + \mathcal{O}(\epsilon^{3/2}),
\]

with \( C_1 = \left( u^2 w Q'(l) + \frac{1}{6} iwu^3 \right) \), \( C_2 = -\frac{1}{12} u^2 (-6w^2 Q''(l) + u^2 Q(l) - 2iuv) \), and

\[
D(l + w\sqrt{\epsilon}, u\sqrt{\epsilon}) = \delta_m \delta_n + D_1 \epsilon^{1/2} + D_2 \epsilon + \mathcal{O}(\epsilon^{3/2}),
\]

with \( D_1 = iu \left[ \Delta_m^m(l) \Delta_n^n(l) - \Delta_m^m(l) \Delta_n^n(l) \right] \),

\[
D_2 = f(l)l^{m+n} - \frac{1}{2} u^2 \left[ \Delta_m^m(l) \Delta_n^n(l) + \Delta_m^m(l) \Delta_n^n(l) \right] + iuw f(l) \left( l^n \delta_m + l^m \delta_n \right),
\]

and \( \delta_p = \Delta_p^p(z) + \Delta_p^p(z) = \int_{-\infty}^{\infty} \omega^p f(\omega)d\omega \).

For \( f(\omega) = e^{-\omega^2/2\sqrt{\pi}} \), we have \( \delta_m = [1 + (-1)^m](m - 1)!!/2 \), where !! denotes the double factorial. Consequently, we get the following series expansion in powers of \( \sqrt{\epsilon} \):

\[
D(l + w\sqrt{\epsilon}, u\sqrt{\epsilon})e^{-C_1 \epsilon^{1/2} - C_2 \epsilon + \ldots} = \sum_{k=0}^{\infty} a_k(u, w)(\sqrt{\epsilon})^k,
\]

where \( a_k(u, w) \) are polynomials in \( u \) and \( w \). Using equations (18), (19), (22) in equation (16), we then get

\[
C^{(m,n)}(r = \rho \sigma_t l, t) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dw \frac{d}{2\pi} e^{-u^2 Q(l) - iuw} \sum_{k=0}^{\infty} a_k(u, w)(\sqrt{\epsilon})^k.
\]

Finally, we perform Gaussian integrations over the variables \( u \) and \( w \) to get

\[
C^{(m,n)}(r = \rho \sigma_t l, t) = \delta_m \delta_n + (l^m - \delta_m)(l^n - \delta_n) f(l) / \rho \sigma_t + \mathcal{O}(\epsilon^{3/2}),
\]

where \( \delta_n \) is defined in (21). The scaled correlation function after subtracting off the mean is then finally given, to leading order, by

\[\text{https://doi.org/10.1088/1742-5468/ab5d0c}\]
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

\[ \rho \sigma_t \langle v^m_{r \to \rho \sigma_t}(t); v^n_0(0) \rangle = (l^m - \delta_m)(l^n - \delta_n) f(l). \]  

(25)

3.2. Computation of other correlations

Here we compute some other correlations which are of interest from the point of hydrodynamics. We define the stretch variable \( s_t = q_{t+1} - q_t \). To compute the stretch correlations, we note that

\[ \langle s_r(t)s_0(0) \rangle = \langle [q_{r+1}(t) - q_r(t)](q_r(0) - q_0(0)) \rangle \]

\[ = -[\langle q_{r+1}(t)q_0(0) \rangle - 2\langle q_r(t)q_0(0) \rangle + \langle q_{r-1}(t)q_0(0) \rangle] \]

\[ \approx -\partial^2 \langle \langle q_r(t)q_0(0) \rangle \rangle, \quad (26) \]

where we have used the translation symmetry of the system. Now taking two time derivatives gives

\[ \partial^2 \langle \langle s_r(t)s_0(0) \rangle \rangle = -\partial^2 \langle \langle v_r(t)v_0(0) \rangle \rangle, \quad (27) \]

where we used the results \( \langle v_r(t)q_0(0) \rangle = \langle v_r(0)q_0(-t) \rangle \) and \( \langle v_r(0)v_0(-t) \rangle = \langle v_r(t)v_0(0) \rangle \), following from time-translation invariance. The stretch correlation can be written in terms of velocity correlations as follows:

\[ \langle s_r(t)s_0(0) \rangle = \langle s_r(0) \rangle \langle s_0(0) \rangle - \int_0^t dt' \int_0^{t'} dt'' \partial^2 \partial^2 \langle v_r(t'')v_0(0) \rangle, \quad (28) \]

where we used the fact that \( \langle v_r(t)s_0(0) \rangle = 0 \). This finally gives

\[ \langle s_r(t); s_0(0) \rangle = \langle s_r(t)s_0(0) \rangle - \langle s_r(0) \rangle \langle s_0(0) \rangle = \frac{1}{\rho^2 \sigma_t} e^{-\frac{1}{2}(\frac{\bar{v}}{\rho \sigma_t})^2}. \quad (29) \]

The energy correlations are easily expressed in terms of velocity correlations.

\[ \langle e_r(t); e_0(0) \rangle = \langle e_r(t)e_0(0) \rangle - \langle e_r(t) \rangle \langle e_0(0) \rangle = \frac{1}{4} \langle [v^2_r(t) - \langle v^2_r(t) \rangle] [v^2_0(0) - \langle v^2_0(0) \rangle] \rangle. \]

Using the known form of the velocity correlations from (25), we then get

\[ \langle e_r(t); e_0(0) \rangle = \bar{v}^4 \left( \frac{\rho}{\rho \sigma_t} \right)^4 - 2 \left( \frac{\rho}{\rho \sigma_t} \right)^2 + 1 \right] f \left( \frac{\rho}{\rho \sigma_t} \right). \quad (30) \]

Following arguments similar to those in [19], we now show the asymptotic exact results obtained above can be understood from a heuristic argument. Since the initial velocities are chosen independently for each particle, the contribution to the correlation function \( \langle v^m_r(t); v^n_0(0) \rangle \) is non-zero only when the velocity of the \( r \)-th particle at time \( t \) is the same as that of the zero-th particle at time 0. The initial velocity distribution of each particle is chosen from a Maxwell distribution \( e^{-\frac{v^2}{2\bar{v}^2}} / \sqrt{2\pi \bar{v}} \), with \( \bar{v}^2 = k_B T \). The velocity correlation function is thus approximately given by

\[ \langle v^m_r(t); v^n_0(0) \rangle \approx \int_{-\infty}^{\infty} dv \left[ \frac{v^n - \langle v^n \rangle}{\bar{v}^m} \right] e^{-\frac{v^2}{2\bar{v}^2}} / \sqrt{2\pi \bar{v}} \frac{1}{\rho t} \frac{\delta (v - \rho t)}{v} \frac{1}{v} f \left( \frac{v}{\bar{v}} \right). \quad (31) \]

\[ = \frac{1}{\rho \sigma_t} (l^m - \delta_m)(l^n - \delta_n) f(l), \quad (32) \]
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

which reproduces (25).

In the next section we verify this result from simulations with the microscopic dynamics of the hard particle gas.

3.3. Numerical verification with Hamiltonian evolution of the hard-particle gas

We now present results from numerical simulations of the hard particle gas. In our simulations we considered a gas of hard point particles with density \( \rho = 1 \), moving on a ring. We choose the initial condition from an equilibrium distribution at temperature \( T \), i.e. we distribute the particles uniformly in space with unit density and velocity of each particle is independently chosen from the distribution \( e^{-v^2/(2T)}/\sqrt{2\pi T} \). The mapping to the independent particle picture means that the time-evolution of this system can be done very efficiently. Basically, starting from any given initial condition, we evolve the non-interacting gas up to time \( t \). In order to get the actual positions, we can get the correct tag of the interacting particle by simply sorting their final positions

Figure 3. Scaled velocity correlation functions at different times, for various choices of \( m, n \) as obtained from simulations of the HPG. The black curves are from the analytical result as given by equation (24). The curves are for \( t \ll L \). The parameters in the simulation are \( N = 1001 \), \( L = 1000 \) and temperature \( T = 1 \).
and taking into account the effect of periodic boundaries. We then compute the correlations \( \langle v^m(t); v^n_0(0) \rangle \) by taking averages over initial conditions. In our simulations we took 1000 particles and took averages over \( 10^8 \) initial conditions. In figure 3 we plot the scaled correlation functions \( t \langle v^m(t); v^n_0(0) \rangle \) (corresponding to ballistic scaling), obtained from the microscopic simulations and compared with the corresponding theoretical predictions from equation (25), for several choices of \( m, n \). We find very good agreement between simulations and theory, even at relatively early times. At very short times there is a small deviation from the theory and this is expected, since the theory makes predictions for long time behaviour, well after the transient dynamics. On the other hand, finite size effects would show up at times \( t \gtrsim O(L) \). The peak of the correlation functions at time \( t \), for odd values of \( m, n \), is given by \( r_{\text{peak}} \approx \bar{v} t \sqrt{m+n} \). In contrast to this, the sound velocity for the HPG is given by \( \sqrt{3\bar{v}} \). Using this, we estimate that finite size effects would show up in correlation functions of order \( m+n \), at times approximately \( t^* \sim L/(\bar{v}\sqrt{m+n}) \).

4. Conclusion

Using the mapping of the HPG to a non-interacting gas, we have computed exact long-time spatio-temporal correlation functions of arbitrary powers of velocities. This gives us correlations of all conserved quantities in the HPG. We have shown how other correlations such as the ‘stretch’ variable can be obtained. We have verified our analytic results through direct simulations of the HPG using a very efficient numerical scheme, which again relies on the mapping to non-interacting particles. While we obtained the correlations to leading order in \( 1/(\rho \bar{v} t) \), our approach can be readily extended to systematically compute the corrections, which can sometimes be important. Indeed, our approach here has been used in earlier work to compute tagged particle correlations, where the higher order corrections are relevant [19–21]. Another interesting direction would be to consider the derivation of Euler equations for the hard-point gas (see for example [25]) by extending the present approach to include inhomogenous distributions. The hard particle gas is in the same class as the harmonic chain, as an example of a ‘non-interacting’ integrable model [26] but the dynamics is non-Gaussian and so the computation of correlations is somewhat more non-trivial than for the harmonic chain. We expect that our results will provide a bench-mark to test predictions from generalized hydrodynamics of integrable systems.

Acknowledgments

AD and SS would like to acknowledge the support from the Indo–French Centre for the promotion of advanced research (IFCPAR) under Project No. 5604-2.
Appendix

A.1. Computation of $F_{1N}$ and $F_{2N}$

The expressions for $F_{1N}(x,y,j,k,t)$ and $F_{2N}(x,y,\tilde{x},\tilde{y},j,k,t)$ were explicitly computed in [19], using combinatorial arguments. Here we give the explicit expressions:

$$F_{1N}(x,y,j,k,t) = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [H(x,y,\theta,\phi,t)]^{2N} e^{-i\phi(2N+2-k+j)} e^{-i\theta(k-j)},$$

$$F_{2N}(x,y,\tilde{x},\tilde{y},j,k,t) = \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [H(x,y,\theta,\phi,t)]^{2N-1} e^{-i\phi(2N+2-k+j)} e^{-i\theta(k-j)} e^{-i\phi\chi_1} e^{-i\theta\chi_2},$$

where $\chi_1, \chi_2$ are defined as follows

(i) $\tilde{x} < x$ and $\tilde{y} < y$, then $\chi_1 = 1$ and $\chi_2 = 0$,

(ii) $\tilde{x} < x$ and $\tilde{y} > y$, then $\chi_1 = 0$ and $\chi_2 = 1$,

(iii) $\tilde{x} > x$ and $\tilde{y} < y$, then $\chi_1 = 0$ and $\chi_2 = -1$,

(iv) $\tilde{x} > x$ and $\tilde{y} > y$, then $\chi_1 = -1$ and $\chi_2 = 0$.

The function $H(x,y,\theta,\phi,t)$ is defined as

$$H(x,y,\theta,\phi,t) = p_{++}(x,y,t) e^{i\phi} + p_{--}(x,y,t) e^{-i\phi} + p_{-+}(x,y,t) e^{i\theta} + p_{+-}(x,y,t) e^{-i\theta},$$

$$= 1 - (1 - \cos \phi)(p_{++} + p_{--}) + i \sin \phi(p_{++} - p_{--}),$$

$$- (1 - \cos \theta)(p_{++} + p_{--}) + i \sin \theta(p_{++} - p_{--}),$$

where $p_{+-}(x,y,t)$ is defined as the probability that a single non-interacting particle is to the left of $x$ at time $t = 0$ and to the right of $y$ at time $t$, and $p_{++}, p_{--}$ and $p_{+-}$ are defined similarly. Their explicit forms are

$$p_{++}(x,y,t) = \frac{1}{2L} \int_{-L}^{x} dx' \int_{y}^{L} dy' G(y',t|x',0),$$

$$p_{--}(x,y,t) = \frac{1}{2L} \int_{-y}^{L} dy' \int_{-L}^{x} dx' G(y',t|x',0),$$

$$p_{-+}(x,y,t) = \frac{1}{2L} \int_{x}^{L} dx' \int_{y}^{L} dy' G(y',t|x',0),$$

$$p_{+-}(x,y,t) = \frac{1}{2L} \int_{-L}^{x} dx' \int_{-y}^{L} dy' G(y',t|x',0).$$

https://doi.org/10.1088/1742-5468/ab5d0c
We note that \( p_{-+} + p_{+-} + p_{--} + p_{++} = 1 \). We can explicitly find the expressions for \( p_{\pm\pm} \) using the exact propagator equation (6). In the large \( N, L \) limit, expanding in \( 1/L \), we get

\[
\begin{align*}
  p_{-+} &= \frac{\sigma_t}{2L} \left( -\frac{z}{2} + Q(z) \right) + \mathcal{O}(1/L^2), \\
  p_{+-} &= \frac{\sigma_t}{2L} \left( \frac{z}{2} + Q(z) \right) + \mathcal{O}(1/L^2), \\
  p_{++} &= \frac{1}{2} + \frac{\sigma_t}{2L} \left( \frac{z}{2} - Q(z) \right) + \mathcal{O}(1/L^2), \\
  p_{--} &= \frac{1}{2} + \frac{\sigma_t}{2L} \left( \frac{z}{2} - Q(z) \right) + \mathcal{O}(1/L^2),
\end{align*}
\]

where \( z = (x - y)/\sigma_t \) and \( \bar{z} = (x + y)/\sigma_t \) and

\[
Q(z) = z \int_0^\infty dw f(w) + \int_z^\infty dw w f(w).
\]

Now, substituting these asymptotic expressions of \( p_{\pm\pm} \) in equation (A.3) for large \( N \), keeping only the dominant terms, one finds

\[
[H(x, y, \theta, t)]^{2N} = e^{-2N(1-\cos\phi)} e^{i\rho \sigma_t \bar{z} \sin\phi} e^{-2\rho \sigma_t Q(z)(1-\cos\theta)} e^{i\rho \sigma_t z \sin\theta}.
\]

References

[1] Van Beijeren H 2012 Exact results for anomalous transport in one-dimensional hamiltonian systems Phys. Rev. Lett. 108 180601
[2] Spohn H 2014 Nonlinear fluctuating hydrodynamics for anharmonic chains J. Stat. Phys. 154 1191–227
[3] Narayan O and Ramaswamy S 2002 Anomalous heat conduction in one-dimensional momentum-conserving systems Phys. Rev. Lett. 89 200601
[4] Castro-Alvaredo O A, Doyon B and Yoshimura T 2016 Emergent hydrodynamics in integrable quantum systems out of equilibrium Phys. Rev. X 6 041065
[5] Bertini B, Collura M, De Nardis J and Fagotti M 2016 Transport in out-of-equilibrium xxz chains: exact profiles of charges and currents Phys. Rev. Lett. 117 207201
[6] Doyon B, Yoshimura T and Caux J S 2016 Soliton gases and generalized hydrodynamics Phys. Rev. Lett. 120 045301
[7] Bastianello A, Doyon B, Watts G and Yoshimura T 2018 Generalized hydrodynamics of classical integrable field theory: the sinh-gordon model SciPost Phys. 4 045
[8] Doyon B and Spohn H 2017 Drude weight for the Lieb–Liniger Bose gas SciPost Phys. 3 039
[9] Doyon B 2018 Exact large-scale correlations in integrable systems out of equilibrium SciPost Phys. 5 54
[10] Casati G 1986 Energy transport and the fourier heat law in classical systems Found. Phys. 16 51–61
[11] Garritdo P I, Hurtado P I and Nadrowski B 2001 Simple one-dimensional model of heat conduction which obeys foquier’s law Phys. Rev. Lett. 86 5486
[12] Grassberger P, Nadler W and Yang L 2002 Heat conduction and entropy production in a one-dimensional hard-particle gas Phys. Rev. Lett. 89 180601
[13] Dhar A 2001 Heat conduction in a one-dimensional gas of elastically colliding particles of unequal masses Phys. Rev. Lett. 86 3554–7
[14] Hurtado P I and Garritdo P I 2016 A violation of universality in anomalous fourier’s law Sci. Rep. 6 38823
[15] Chen S, Wang J, Casati G and Benenti G 2014 Nonintegrability and the fourier heat conduction law Phys. Rev. E 90 032134
[16] Jepsen D W 1965 Dynamics of a simple many body system of hard rods J. Math. Phys. 6 405

https://doi.org/10.1088/1742-5468/ab5d0c
Dynamical correlations of conserved quantities in the one-dimensional equal mass hard particle gas

[17] Lebowitz J and Percus J 1967 Kinetic equations and density expansions: exactly solvable one-dimensional system Phys. Rev. 155 122
[18] Lebowitz J L, Percus J K and Sykes J 1968 Time evolution of the total distribution function of a one-dimensional system of hard rods Phys. Rev. 171 224–35
[19] Sabhapandit S and Dhar A 2015 Exact probability distribution for the two-tag displacement in single-file motion J. Stat. Mech. P07024
[20] Hegde C, Sabhapandit S and Dhar A 2014 Universal large deviations for the tagged particle in single-file motion Phys. Rev. Lett. 113 120601
[21] Roy A, Narayan O, Dhar A and Sabhapandit S 2013 Tagged particle diffusion in one-dimensional gas with Hamiltonian dynamics J. Stat. Phys. 150 851–66
[22] Kundu A and Dhar A 2016 Equilibrium dynamical correlations in the Toda chain and other integrable models Phys. Rev. E 94 062130
[23] Doyon B 2019 Generalized hydrodynamics of the classical toda system J. Math. Phys. 60 073302
[24] Spohn H 2019 Generalized gibbs ensembles of the classical toda chain J. Stat. Phys. 196 1–19
[25] Boldrighini C, Dobrushin R and Sukhov Y M 1983 One-dimensional hard rod caricature of hydrodynamics J. Stat. Phys. 31 577–616
[26] Spohn H 2018 Interacting and noninteracting integrable systems J. Math. Phys. 59 091402

https://doi.org/10.1088/1742-5468/ab5d0c