The phase transition in the multi-type binomial random graph $G(n, P)$

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Abstract

We determine the asymptotic size of the largest component in the multi-type binomial random graph $G(n, P)$ near criticality using a refined branching process approach. In $G(n, P)$ every vertex has one of two types, the vector $n$ describes the number of vertices of each type and any edge $\{u, v\}$ is present independently with a probability that is given by an entry of the probability matrix $P$ according to the types of $u$ and $v$.

Among other results, we show that in the weakly supercritical regime of the phase transition, i.e. where the ‘distance’ to the critical point is given by some $\varepsilon = \varepsilon(n) \to 0$, with probability $1-o(1)$, the largest component in $G(n, P)$ is of size asymptotically $2\varepsilon\|n\|_1$ and all other components are at most of size $o(\varepsilon\|n\|_1)$.

1 Introduction

The study of random graphs was initiated by Paul Erdős and Alfréd Rényi in the late 1950s with the pair of papers [13, 14], and since then the theory of random graphs has been extensively developed by a vast range of authors. Besides being mathematically interesting objects by themselves random graphs have wide-ranging applications, for example, in neurobiology, statistical physics and modeling complex networks like the web [11, 22, 27, 30].

In the model $G(n, m)$ studied by Erdős and Rényi a graph is chosen with uniform probability from all graphs with $n$ vertices and exactly $m$ edges. Equivalently, $G(n, m)$ can also be seen as the $m$-th stage of a process that begins with an empty graph on $n$ vertices at time $0$ and adds new edges, one at a
time, uniformly at random among all edges not already present. This model is essentially equivalent to another random graph model, the binomial random graph \( G(n, p) \), introduced by Gilbert at about the same time, in which every possible edge occurs independently with probability \( p \).

One of the main subjects of the study of random graphs is to determine at what stage of the random graph process \( G(n, m) \) a particular property of the graph is likely to appear. Many properties were studied in [14] but perhaps the most extraordinary result was the discovery of a rapid change in the size of the largest component at time \( n^{2/3} \). More precisely, Erdős and Rényi showed that there are three stages that can be described as follows. Let \( c \) be a fixed constant and consider the process at time \( cn^{2/3} \): If \( c < 1 \), with high probability (whp for short, meaning with probability tending to 1 as \( n \) tends to infinity) all connected components of the random graph \( G(n, \frac{c}{2}n) \) have size \( O(\ln n) \), while if \( c = 1 \), whp the largest component is of size \( \Theta(n^{2/3}) \), and if \( c > 1 \), then whp there is a component of size \( \Theta(n^{2/3}) \), called the ‘giant component’, and all other components are of size \( O(\ln n) \). This rapid structural change is known as the phase transition of the random graph \( G(n, m) \).

In 1984 Bollobás investigated this phenomenon further and described in detail the behavior of \( G(n, m) \) when \( m \) is not much larger or smaller than \( n/2 \), i.e. at time \( \frac{n}{2} \) with \( c = 1 \pm \varepsilon \), where \( \varepsilon = \varepsilon(n) \rightarrow 0 \). His initial results were then further improved by Łuczak. They showed, in particular, that if \( c = 1 + \varepsilon \), where \( \varepsilon \rightarrow 0 \) and \( \varepsilon^3n \rightarrow \infty \) as \( n \rightarrow \infty \), then whp there is a unique largest component in \( G(n, \frac{(1+\varepsilon)n}{2}) \) that contains asymptotically \( 2\varepsilon n \) vertices and all other components are of size \( o(\varepsilon n) \). However, if \( c = 1 - \varepsilon \), where \( \varepsilon \rightarrow 0 \) and \( \varepsilon^3n \rightarrow \infty \) as \( n \rightarrow \infty \), then whp the largest component is of size \( o(n^{2/3}) \). For a comprehensive account of the results see [1, 3, 19].

In the meantime many of these results have been reproved and strengthened using various modern techniques, that provide appropriate means for studying more general random graph models and other discrete random structures such as random hypergraphs [2, 20]. These methods include tools like martingales [21], partial differential equations [29] and modifications of search algorithms [5, 21]. Influenced by real-world applications various others random graph models [4, 10, 24, 25] have been introduced trying to capture properties that these networks are empirically shown to posses.

The classical models \( G(n, p) \) and \( G(n, m) \) are homogeneous in the sense that all the vertices in the graph are treated in precisely the same way – an assumption that is not usually satisfied in real-world scenarios [30]. Removing this condition Bollobás, Janson and Riordan developed a very general theory of inhomogeneous random graphs that unifies many of the specific inhomogeneous random graph models introduced in recent years [4, 12]. In [5] several properties are analyzed including the degree distribution, the numbers of paths and cycles, the phase transition for the giant component and the typical distance between pairs of vertices in the giant component.

More recently, van der Hofstad has studied the critical behavior of the largest component in inhomogeneous random graphs in the so-called rank-1
case. In that paper, van der Hofstad mainly works with the Poisson random graph [28], where the degree of any vertex is close in distribution to a Poisson random variable, and shows that the critical behavior depends sensitively on the asymptotic properties of the degree sequence. The author also extends these results to the model studied by Chung and Lu in [9, 10] and the one studied by Britton, Deijfen and Martin-Löf in [8].

In this paper, we study a multi-type random graph model in which each vertex has one of \( k \) types and an edge between a pair of vertices of types \( i \) and \( j \) is present with probability \( p_{i,j} \) independently of all other pairs. The model we considered is similar to a finite version of inhomogeneous random graphs by Bollobás, Janson and Riordan [5]. In this paper, we determine the size of the largest component in the so-called *weakly* supercritical regime (Theorem 2.1), roughly speaking when the distance to the critical point of phase transition decreases to zero as the number of vertices tends to infinity. This case was not studied in [5].

In order to derive the main results, we apply a simple breath-first search approach to construct a rooted spanning tree of a component and couple it with a multi-type branching process with binomial offspring distributions, which is viewed as a random rooted tree. In addition, the width and the dual of that random rooted tree play important roles in the second moment analysis.

The results of this paper are indeed not surprising and the techniques used in the paper may look familiar. The main contribution of this paper is that it shows how a simple branching process approach combined with the concepts of tree width and dual trees can be nicely applied to multi-type random graphs all the way through the supercritical regime.

## 2 Model and main results

In this section we will first define the random graph model \( G(n, P) \) that we will investigate. After introducing some additional notation we state the main results in Section 2.3 and afterwards provide an outline of the proof and the methods involved. Then we discuss some related work on the general inhomogeneous random graph studied by Bollobás, Janson and Riordan [5] thus putting our results in context.

### 2.1 The multi-type binomial random graph model \( G(n, P) \)

Let \( k \in \mathbb{N} \) be fixed. Every vertex is associated with a *type* \( i \in \{1, \ldots, k\} \) and we denote by \( V_i \) the set of all vertices of type \( i \in \{1, \ldots, k\} \). Given an arbitrary vector \( n = (n_1, \ldots, n_k) \in \mathbb{N}^k \) and a symmetric matrix of probabilities \( P = (p_{i,j})_{i,j=1,\ldots,k} \in [0,1]^{k \times k} \) we consider the random graph \( G(n, P) \) on \( n_i \) vertices of type \( i \), for \( 1 \leq i \leq k \), with the following edge set: For each pair \( \{u, v\} \), where \( u \) is of type \( i \) and \( v \) is of type \( j \), we include the edge \( \{u, v\} \) independently of any other pair with probability \( p_{i,j} \) and exclude it with probability \( 1 - p_{i,j} \). For every pair \( (i, j) \in \{1, \ldots, k\}^2 \) we denote by \( \eta_{i,j} = p_{i,j} n_j \) the expected number
of neighbors of type \( j \) for a vertex of type \( i \) and furthermore by \( n = \sum_{i=1}^{k} n_i \) the total number of vertices in this graph. Without loss of generality we assume that \( n_1 \geq n_2 \geq \ldots \geq n_k \).

**Remark 1.** Observe that for \( k = 1 \) we obtain the classical binomial random graph \( G(n, p) \) where \( n = n_1 \) and \( p = p_{1,1} \).

We analyze the random graph model \( G(n, p) \) by investigating a closely linked \( k \)-type binomial branching process described as follows: Suppose there are \( k \) distinguished types of individuals. Fix a time \( t \in \mathbb{N}_0 \) and let \( I_t \) be a set of individuals (i.e. the population) at time \( t \), which we also call the \( t \)-th generation of individuals. Then, for each individual \( v \in I_t \) with type \( j' \in \{1, \ldots, k\} \), we associate a random vector \( \mathbf{X}_v = (X_{v,1}^1, \ldots, X_{v,k}^k) \), where for each \( j \in \{1, \ldots, k\} \) the random variable \( X_{v,j}^j \) is independent and binomially distributed with parameters \( n_j \) and \( p_{j',j} \) and thus with mean \( \eta_{j',j} \). Then the population \( I_{t+1} \) at time \( t + 1 \) will be a set containing exactly \( \sum_{v \in I_t} X_{v,j}^j \) new individuals of type \( j \), for each \( j \in \{1, \ldots, k\} \). In other words, the random variable \( X_{v,j}^j \) represents the number of children of type \( j \) that are born from the individual \( v \) before it dies. A \( k \)-type binomial branching process starting with an initial population \( I_0 \) is a sequence of random vectors \( (Z_t(1), \ldots, Z_t(k))_{t \in \mathbb{N}_0} \) generated by iterating the construction described above, where \( Z_t(j) \) is the random variable describing the number of individuals of type \( j \) in the \( t \)-th generation for each \( j \in \{1, \ldots, k\} \) and \( t \in \mathbb{N}_0 \). For \( i \in \{1, \ldots, k\} \) we denote by \( T_{n,i}^i \) a \( k \)-type binomial branching process starting with a single vertex of type \( i \). We may also use \( T_{n,p}^i \) to denote the rooted, possibly infinite, tree created by an instance of the branching process. The context will always clarify the notation. Furthermore, if the a statement is independent of the starting type we simply write \( T_{n,p} \) and we will refer to the matrix \( (\eta_{i,j})_{i,j \in \{1, \ldots, k\}} \) as the *offspring expectation matrix* of the process \( T_{n,p} \). Throughout the paper this will be a positive matrix, i.e. all the entries are positive.

### 2.2 Notation

Given a graph \( G \) with components \( C_1, \ldots, C_r \) such that \( |C_1| \geq |C_2| \geq \ldots \geq |C_r| \), we denote by \( L_i(G) = C_i \) the \( i \)-th largest component and by \( L_i(G) = \{|C_i| \text{ size for any } i \in \{1, \ldots, r\} \} \) of \( G \) and set \( L_i(G) = \emptyset \) and \( L_i(G) = 0 \) if \( i > r \).

Throughout the paper we focus on the case \( k = 2 \) and we comment on the situation for arbitrary \( k > 2 \) at the end of the paper. Without loss of generality we will assume that \( n_1 \geq n_2 \). Furthermore, unless specified explicitly, all asymptotic statements are to be understood in terms of \( n_1 \) and \( n_2 \) being *large enough* yet fixed and we use the notation \( \min\{n_1, n_2\} = n_2 \to \infty \) for this. Moreover, we will use the following standard notation to describe asymptotic statements: For any real functions \( f = f(n_1, n_2) \) and \( g = g(n_1, n_2) \) we write: \( f = O(g) \) if \( \exists C > 0, n_0 \) such that \( |f(n_1, n_2)| \leq C|g(n_1, n_2)| \) for all \( n_1 \geq n_2 \geq n_0 \); \( f = o(g) \) if \( \forall C > 0, \exists n_0 \) such that \( |f(n_1, n_2)| \leq C|g(n_1, n_2)| \) for all \( n_1 \geq n_2 \geq n_0 \); \( f = \Omega(g) \) if \( g = O(f) \); \( f = \Theta(g) \) if \( f = O(g) \) and \( f = \Omega(g) \) and \( f \sim g \) if \( f - g = o(g) \).
2.3 Main results

We show that in \( G(n, P) \) there is a phase transition with respect to the size of the largest component and determine where it occurs. Primarily we show that shortly after the phase transition, i.e. in the weakly supercritical regime, there is a unique largest component of order asymptotically \( 2\varepsilon n \). In fact we provide a slightly stronger result.

**Theorem 2.1.** For \( n_1 \in \mathbb{N} \) and \( n_2 \in \mathbb{N} \) with \( n_1 \geq n_2 \), let \( n = n_1 + n_2 \) and let \( \varepsilon = \varepsilon(n_1, n_2) > 0 \) with \( \varepsilon = o(1) \). Furthermore, let

\[
P = (p_{i,j})_{i,j \in \{1,2\}} = (p_{i,j}(n_1, n_2, \varepsilon))_{i,j \in \{1,2\}} \in [0,1]^{2 \times 2}
\]

be a symmetric matrix of probabilities such that the following conditions are satisfied:

\[
\varepsilon^3 n_2 \eta_{2,1} \to \infty, \quad (1)
\]

\[
\eta_{i,1} + \eta_{i,2} = 1 + \varepsilon + o(\varepsilon), \quad \text{for any } i \in \{1,2\}, \quad (2)
\]

where \( \eta_{i,j} = p_{i,j} n_j \) for every pair \( (i, j) \in \{1,2\}^2 \).

Then, for \( i \in \{1,2\} \), we have whp

\[
|L_1(G(n, P)) \cap V_i| = (2 + o(1))\varepsilon n_i,
\]

and for every \( r \geq 2 \)

\[
|L_r(G(n, P)) \cap V_i| = o(\varepsilon n_i).
\]

Therefore, in particular, whp

\[
L_1(G(n, P)) = (2 + o(1))\varepsilon n,
\]

and for every \( r \geq 2 \)

\[
L_r(G(n, P)) = o(\varepsilon n).
\]

**Remark 2.** Observe that, up to the term \( \eta_{2,1} \), Condition (1) mirrors the condition \( \varepsilon^3 n \to \infty \) that is necessary and sufficient to have a unique largest component in \( G(n,(1+\varepsilon)/n) \) whp. In \( G(n,(1+\varepsilon)/n) \) the average degree is \( 1 + \varepsilon = \Theta(1) \) and therefore it does not influence the asymptotic statement of Condition (1).

In \( G(n, P) \) however, \( P \) can be such that we are close to criticality but still the average number \( \eta_{2,1} = \eta_{1,2} \frac{n_2}{n} \) of neighbors of the opposite type for a given vertex is \( o(1) \). Roughly speaking, it is reasonable that if \( \eta_{2,1} \) is ‘very small’ the random graph \( G(n, P) \) may have two largest components, one of each type, that coexist independently since the probability of adding any edge between them is negligible. In particular this happens when the probability \( p_{1,2} \) is equal to zero and therefore \( \eta_{1,2} = \eta_{2,1} = 0 \).

**Remark 3.** Assuming that \( P \) is a symmetric matrix simply reflects the fact that \( G(n, P) \) is an undirected random graph.
Note that the parameter $\varepsilon > 0$ describes the distance to the critical point for the emergence of the giant component in a sense that we will explain now. Roughly speaking, for some time, the breadth-first exploration process of a component in $G(n, P)$ looks like a 2-type binomial branching process. This can be described by an appropriate coupling of the two processes. If the branching process dies out its total population should be rather ‘small’, thus, by the coupling, the explored component is also ‘small’. It is well-known that for a 2-type binomial branching process the property of survival has a threshold and that the critical point is characterized by the largest eigenvalue

$$
\lambda = \frac{\eta_{1,1} + \eta_{2,2}}{2} + \frac{1}{2} \sqrt{(\eta_{1,1} + \eta_{2,2})^2 + 4 (\eta_{1,2}\eta_{2,1} - \eta_{1,1}\eta_{2,2})}
$$

of its offspring expectation matrix $(\eta_{i,j})_{i,j \in \{1,2\}}$. If $\lambda > 1$, the process has a positive probability of survival, while if $\lambda \leq 1$, it dies out with probability 1.

Next, let us compute $\lambda$ for $T_{n, P}$ with parameters as in Theorem 2.1. Condition (2) states that for any $\delta > 0$ there is an $n_0 = n_0(\delta)$ such that we have

$$
|\eta_{i,1} + \eta_{i,2} - (1 + \varepsilon)| \leq \frac{\delta}{2} \varepsilon,
$$

for $i \in \{1, 2\}$ and all $n_1 \geq n_2 \geq n_0$. This implies

$$
\eta_{1,2}\eta_{2,1} - \eta_{1,1}\eta_{2,2} \leq \left(1 + \varepsilon + \frac{\delta}{2} \varepsilon\right)^2 - (\eta_{1,1} + \eta_{2,2}) \left(1 + \varepsilon + \frac{\delta}{2} \varepsilon\right)
$$

and similarly

$$
\eta_{1,2}\eta_{2,1} - \eta_{1,1}\eta_{2,2} \geq \left(1 + \varepsilon - \frac{\delta}{2} \varepsilon\right)^2 - (\eta_{1,1} + \eta_{2,2}) \left(1 + \varepsilon - \frac{\delta}{2} \varepsilon\right)
$$

for all $n_1 \geq n_2 \geq n_0$. Therefore we can bound the argument of the square root in Equation (3) from above by

$$
(\eta_{1,1} + \eta_{2,2})^2 + 4 (\eta_{1,2}\eta_{2,1} - \eta_{1,1}\eta_{2,2}) \leq \left(2 \left(1 + \varepsilon + \frac{\delta}{2} \varepsilon\right) - (\eta_{1,1} + \eta_{2,2})\right)^2
$$

and from below by

$$
(\eta_{1,1} + \eta_{2,2})^2 + 4 (\eta_{1,2}\eta_{2,1} - \eta_{1,1}\eta_{2,2}) \geq \left(2 \left(1 + \varepsilon - \frac{\delta}{2} \varepsilon\right) - (\eta_{1,1} + \eta_{2,2})\right)^2.
$$

Thus, by Equation (3) and since $\delta$ was arbitrary, we obtain the following asymptotic estimate for the largest eigenvalue

$$
\lambda = 1 + \varepsilon + o(\varepsilon).
$$

In other words, $\varepsilon$ describes how close $\lambda$ is to 1.

Now let us get back to stating our results. We also analyze the corresponding weakly subcritical regime.
Theorem 2.2. For \( n_1 \in \mathbb{N} \) and \( n_2 \in \mathbb{N} \) with \( n_1 \geq n_2 \), let \( n = n_1 + n_2 \) and let \( \varepsilon = \varepsilon(n_1, n_2) > 0 \) with \( \varepsilon = o(1) \). Furthermore, let

\[
P = (p_{i,j})_{i,j \in \{1,2\}} = (p_{i,j}(n_1, n_2, \varepsilon))_{i,j \in \{1,2\}} \in [0,1]^{2 \times 2}
\]

be a symmetric matrix of probabilities such that the following conditions are satisfied:

1. \( \varepsilon^3 n_2 \to \infty \),
2. \( \eta_{i,1} + \eta_{i,2} = 1 - \varepsilon + o(\varepsilon) \), for any \( i \in \{1,2\} \),

where \( \eta_{i,j} = p_{i,j} n_j \) for every pair \( (i, j) \in \{1,2\}^2 \).

Then we have whp

\[
|L_1(G(n, P)) \cap V_i| = (\rho_\varepsilon + o(1)) n_i,
\]

where \( \rho_\varepsilon \) is the unique positive solution for the equation

\[
1 - \rho_\varepsilon - \exp\left(- (1 + \varepsilon) \rho_\varepsilon\right) = 0.
\]

Note that analogously to Theorem 2.1 the parameter \( \varepsilon > 0 \) describes the distance to the critical point from below. This case is much easier to analyze. The largest eigenvalue \( \lambda \) of the offspring expectation matrix of \( G(n, P) \) with parameters as in Theorem 2.2 satisfies

\[
\lambda = 1 - \varepsilon + o(\varepsilon),
\]

by Condition (6).

Next, for sake of completeness, we will analyze the size of the largest component in the regimes where the distance to the critical value is a fixed constant, i.e. in particular independent of \( n_1 \) and \( n_2 \). We believe that there are various different ways of proving the following two theorems. For instance fitting our model to the setting of inhomogeneous random graphs \([5]\) would yield the results, however it is unclear, at least to us, whether this can be done in a simple way.

In the supercritical regime the largest component will already be a giant component, i.e. it is unique and of linear size.

Theorem 2.3. For \( n_1 \in \mathbb{N} \) and \( n_2 \in \mathbb{N} \) with \( n_1 \geq n_2 \), let \( n = n_1 + n_2 \) and let \( \varepsilon > 0 \) be a fixed constant. Furthermore, let

\[
P = (p_{i,j})_{i,j \in \{1,2\}} = (p_{i,j}(n_1, n_2, \varepsilon))_{i,j \in \{1,2\}} \in [0,1]^{2 \times 2}
\]

be a symmetric matrix of probabilities such that the following conditions are satisfied:

1. \( n_2 \eta_{2,1} \to \infty \),
2. \( \eta_{i,1} + \eta_{i,2} = 1 + \varepsilon + o(1) \), for any \( i \in \{1,2\} \),

where \( \eta_{i,j} = p_{i,j} n_j \) for every pair \( (i, j) \in \{1,2\}^2 \). Let \( \rho_\varepsilon \) be the unique positive solution for the equation

\[
1 - \rho_\varepsilon - \exp\left(- (1 + \varepsilon) \rho_\varepsilon\right) = 0.
\]

Then, for \( i \in \{1,2\} \), we have whp

\[
|L_1(G(n, P)) \cap V_i| = (\rho_\varepsilon + o(1)) n_i,
\]
and for every \( r \geq 2 \)
\[
|L_r(G(n, P)) \cap V_i| = o(n_i).
\]
Therefore, in particular, whp
\[
L_1(G(n, P)) = (\rho_\varepsilon + o(1))n,
\]
and for every \( r \geq 2 \)
\[
L_r(G(n, P)) = o(n).
\]

Similarly, we also get a stronger upper bound on the size of all components in the subcritical regime, i.e. with constant distance to the critical point.

**Theorem 2.4.** For \( n_1 \in \mathbb{N} \) and \( n_2 \in \mathbb{N} \) with \( n_1 \geq n_2 \), let \( n = n_1 + n_2 \) and let \( 1 > \varepsilon > 0 \) be a fixed constant. Furthermore, let
\[
P = (p_{i,j})_{i,j \in \{1,2\}} = (p_{i,j}(n_1, n_2, \varepsilon))_{i,j \in \{1,2\}} \in [0,1]^{2\times 2}
\]
be a symmetric matrix of probabilities such that the following conditions are satisfied:
\[
\eta_{i,1} + \eta_{i,2} = 1 - \varepsilon + o(1), \text{ for any } i \in \{1,2\},
\]
where \( \eta_{i,j} = p_{i,j}n_j \) for every pair \( (i,j) \in \{1,2\}^2 \).

Then we have whp
\[
L_1(G(n, P)) \leq 3\varepsilon^{-2} \ln n = O(\log n).
\]

The most difficult of these results is Theorem 2.1, to which we will dedicate most of this paper. In the next section we provide a sketch of its proof.

### 2.4 Sketch of proof of Theorem 2.1

For the proof and extend the method employed by Bollobás and Riordan [7] for the study of the weakly supercritical regime of \( G(n, p) \). To prove Theorem 2.1 we consider the set \( S \) of vertices in ‘large’ components. The first goal is to show that the size \( |S| \) of this set is concentrated around \((2 + o(1))\varepsilon n\). We do this by applying Chebyshev’s inequality: we calculate asymptotically matching upper and lower bounds for the expected size of \( S \) by coupling the breadth-first component exploration process from below and above with multi-type branching processes. Once this is done, using a more refined version of this idea, we show that the square of this expectation is an upper bound for the second moment of the size of \( S \), therefore the variance of the size of \( S \) is indeed ‘small’ compared to the square of the expectation and concentration follows by Chebyshev’s inequality. So now we know that whp the appropriate number of vertices lie in ‘large’ components, but there might be several distinct such components all of which may also be much smaller than claimed in Theorem 2.4. However, we can also construct the random graph via a two-round exposure: In the first round we reduce the probability of including each edge by a tiny bit and note that the above arguments will still hold in this setting. In the second round...
we once again look at each pair not yet connected by an edge and ‘sprinkle’ an edge in between with a tiny probability independently for each such pair. By choosing the magnitude of these probabilities appropriately we can ensure that the resulting random graph can still be coupled with $G(n, P)$ such that it is included in $G(n, P)$ as a subgraph. Analyzing the probability that ‘large’ components are connected by at least one edge and using the union bound we show whp that all large components must be connected in $G(n, P)$.

2.5 Related work

The general inhomogeneous random graph model $G^V(n, c\kappa_n)$ studied by Bollobás, Janson and Riordan [5] is closely related to the model $G(n, P)$ that we have introduced earlier in this paper. Therefore let us roughly describe $G^V(n, c\kappa_n)$: For any $n \in \mathbb{N}$ consider a random sequence $x_n = (x_1, \ldots, x_n)$ of points from a separable metric space $(\mathcal{S}, \mu)$ and let $\nu_n$ be the empirical distribution of $x_n$. Assume that $\nu_n$ converges in probability to $\mu$ as $n \to \infty$, then the triple $(\mathcal{S}, \mu, (x_n)_{n \geq 1})$ is called a vertex space. Furthermore let $\kappa_n$ be a symmetric non-negative measurable function on $(\mathcal{S} \times \mathcal{S}, \mu)$ and let $c > 0$ be a constant. Then the random graph $G^V(n, c\kappa_n)$ is a graph with vertex set $[n]$, where each pair of vertices $\{k, l\}$ is connected by an edge with probability $p_{k,l} := \min\{1, c\kappa_n(x_k, x_l)/n\}$ independently of all other pairs.

Under some extra conditions on $\kappa_n$, it is proved in [5] that with respect to the parameter $c$ there is a phase transition concerning the size of the largest component. In particular, the existence and uniqueness of the giant component in $G^V(n, c\kappa_n)$ in the supercritical regime are proved by using an appropriated multi-type branching process and an integral operator $T_\kappa$. The critical point of the phase transition is characterized by $c_0 := \|T_\kappa\|^{-1}$, i.e. if $c \leq c_0$, then the random graph contains only small components, if $c > c_0$, then there is a giant component of size asymptotically $\rho_c n$, where $\rho_c$ is independent of $n$ and grows linearly in $c - c_0 > 0$.

By contrast the focus in our paper lies on the weakly supercritical regime (Theorem 2.1), i.e. the distance $\varepsilon = \varepsilon(n)$ from the critical point of the phase transition may tend to zero as $\|n\|_1 \to \infty$. The analysis in this regime is in general quite sophisticated, compared with the supercritical regime, i.e. when $\varepsilon > 0$ is a constant independent of $n$.

3 Multi-type binomial branching processes in the supercritical regime

As already mentioned, we will use (multi-type) branching processes to study the component sizes of the random graph $G(n, P)$. Hence, in this section we state various properties of these processes.

We start this section by stating a simplified (2-type binomial) version of a key result concerning the survival probability of a general multi-type branching
Lemma 3.1 (e.g. [17]). Given a 2-type binomial branching process with parameters $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$, with $n_1 \geq n_2$, and

$$P = (p_{i,j})_{i,j \in \{1,2\}} = (p_{i,j}(n_1,n_2))_{i,j \in \{1,2\}} \in [0,1]^{2 \times 2},$$

such that there is a $m \in \mathbb{N}$ for which the entries of the matrix $P^m$ are all positive. Let $\lambda = \lambda(n_1,n_2)$ be the largest eigenvalue of its offspring expectation matrix $(\eta_{i,j})_{i,j \in \{1,2\}}$ and let $(\rho_1,\rho_2)$ be the pair of survival probabilities. Then

- if $\lambda \leq 1$, we have $\rho_1 = \rho_2 = 0$,
- if $\lambda > 1$, then $(\rho_1,\rho_2)$ is the unique positive simultaneous solution of the equations

$$F_i(\rho_1,\rho_2) = 0, \quad i \in \{1,2\}, \quad (11)$$

where

$$F_i(\rho_1,\rho_2) = 1 - \rho_i - \left(1 - \frac{\eta_{i,1}\rho_1}{n_1}\right)^{n_1} \left(1 - \frac{\eta_{i,2}\rho_2}{n_2}\right)^{n_2}. \quad (12)$$

Remark 4. By Condition (1) (resp. (8)) it is easy to check that the entries of $P^2$ are positive in all the cases that are considered in this paper.

We call a branching process that has a positive survival probability super-critical and otherwise we call it subcritical.

Remark 5. There is a very simple way to see that the survival probabilities must satisfy these equations: We consider the extinction probabilities before and after the first step of the process and apply the Binomial Theorem.

Therefore, as the conditions of Theorem 2.1 imply that the largest eigenvalue of the expectation matrix is strictly larger than 1, the associated branching process will have a positive survival probability that is given implicitly by Equations (11). Solving these equations explicitly seems hard, but we only need some information about the asymptotic behavior of the unique positive solution. We provide this information in the next section.

3.1 Asymptotic survival probability

Lemma 3.2. Under the conditions as in Theorem 2.1 the survival probabilities of $\mathcal{T}_{n,p}$ satisfy

$$\rho_1 \sim \rho_2 \sim 2\varepsilon.$$

Proof. The key idea is to find suitably close bounding functions for the $F_i$’s defined in Equation (12), for which the asymptotic values of the zeros can be computed easily, and then to observe that these coincide for the upper and lower bound.
We will couple component exploration processes in $G(n, \mathcal{P})$ with 2-type binomial branching processes. Given one particular component exploration process we will need a good upper bound on the probability that this process discovers a component that has at least $l_j$ vertices of type $j$, for at least one $j \in \{1, 2\}$ and carefully chosen parameters $l_1 = l_1(n_1, n_2, \varepsilon) \in \mathbb{N}$ and $l_2 = l_2(n_1, n_2, \varepsilon) \in \mathbb{N}$. However, by the coupling, this probability is bounded by the probability that

3.2 Dual processes

We will couple component exploration processes in $G(n, \mathcal{P})$ with 2-type binomial branching processes. Given one particular component exploration process we will need a good upper bound on the probability that this process discovers a component that has at least $l_j$ vertices of type $j$, for at least one $j \in \{1, 2\}$ and carefully chosen parameters $l_1 = l_1(n_1, n_2, \varepsilon) \in \mathbb{N}$ and $l_2 = l_2(n_1, n_2, \varepsilon) \in \mathbb{N}$. However, by the coupling, this probability is bounded by the probability that
the total number of offspring of type \( j \), \(|T_{i,P}^j \cap V_j|\), of the associated branching process \( T_{i,P}^j \), is at least \( l_j \). Since this probability is 1 if the process survives, this reduces to analyzing the conditional probability given the event \( D_i \) that the process dies out. We call the resulting 2-type binomial branching process the dual process and we can describe its offspring distributions as follows. We need to know, for a vertex \( v \) of type \( i \in \{1, 2\} \) and a potential child \( u \) of type \( j \in \{1, 2\} \), whether \( u \) is indeed a child of \( v \) in the dual process, i.e. conditioned on \( D_i \). Denote by \( e \) the event that \( u \) is a child of \( v \) in \( T_{i,P}^j \), and by \( D_u \) the event that the branching process started with \( u \) dies out. Note that conditioning on \( e \) will decrease the probability of \( D_i \). More precisely, calculating \( \mathbb{P}(D_i|e) \) and \( \mathbb{P}(D_i|\neg e) \) by conditioning on the number of children of \( v \) shows that

\[
\mathbb{P}(D_i|e) = \mathbb{P}(D_u) = 1 - \rho_j, \quad (13)
\]

Therefore we get

\[
\pi_{i,j} = \mathbb{P}(e|D_i) = \frac{\mathbb{P}(D_i|e) \mathbb{P}(e)}{\mathbb{P}(D_i|e) \mathbb{P}(e) + \mathbb{P}(D_i|\neg e) \mathbb{P}(\neg e)} = \frac{\mathbb{P}(D_u) \mathbb{P}(e)}{\mathbb{P}(D_u) \mathbb{P}(e) + \mathbb{P}(\neg e)} = \frac{p_{i,j}(1 - \rho_j)}{1 - \rho_j p_{i,j}} = p_{i,j}(1 - \rho_j)(1 + O(n_j^{-2}\rho_j)). \quad (14)
\]

The last equation holds since, by Condition (2), \( p_{i,j} = O(n_j^{-1}) \) for each pair \((i, j) \in \{1, 2\}^2\). We write \( \Pi = (\pi_{i,j})_{i,j \in \{1, 2\}} \), \( h_{i,j} = \pi_{i,j} n_j \) and denote the dual process of \( T_{i,P}^j \) by \( T_{i,\Pi}^j \). Intuitively it is obvious that the dual process of any supercritical process is subcritical. Nevertheless we give a short proof for the processes that we use.

**Lemma 3.3.** Let \( T_{i,P}^j \) be a 2-type binomial branching process satisfying the conditions of Theorem 2.1 Then the expectations of the dual process \( T_{i,\Pi}^j \) satisfy

\[
\lambda = 1 - \varepsilon + o(\varepsilon), \quad \text{for } i \in \{1, 2\},
\]

and therefore \( \lambda = 1 - \varepsilon + o(\varepsilon) \), where \( \lambda \) is the largest eigenvalue of the offspring expectation matrix \((h_{i,j})_{i,j \in \{1, 2\}}\).

**Proof.** By the previous calculation (15), Lemma 5.2 and Condition (2) we get

\[
h_{i,1} + h_{i,2} = (\eta_{i,1} + \eta_{i,2})(1 - 2\varepsilon) = 1 - \varepsilon + o(\varepsilon), \quad \text{for } i \in \{1, 2\}.
\]

The second statement follows by calculation analogous to Equation (7).
The benefit of using this subcritical process is that we can find a closed form for the finite expectation $E\left(|T_{n,\Pi}^i \cap V_j|\right)$ for $j \in \{1, 2\}$.

**Lemma 3.4.** Let $T_{n,p}$ be a 2-type binomial branching process satisfying the conditions of Theorem 2.1. Then the associated dual process $\mathcal{T}_{n,\Pi}$ satisfies

$$E\left(|T_{n,\Pi}^i \cap V_j|\right) = O\left(\varepsilon^{-1}\right), \text{ for } (i, j) \in \{1, 2\}^2. \quad (16)$$

Moreover, for any $l_1 = l_1(n_1, n_2, \varepsilon) \in \mathbb{N}$ and $l_2 = l_2(n_1, n_2, \varepsilon) \in \mathbb{N}$, this implies that

$$P\left(|T_{n,p}^i \cap V_1| \geq l_1 \lor |T_{n,p}^i \cap V_2| \geq l_2\right) \leq 2\varepsilon + O\left(\varepsilon^{-1}l_1^{-1} + \varepsilon^{-1}l_2^{-1}\right) + o(\varepsilon),$$

and in particular

$$P\left(|T_{n,p}^i \cap V_1| \geq l_1 \lor |T_{n,p}^i \cap V_2| \geq l_2\right) \leq (2 + o(1))\varepsilon, \quad (17)$$

if $\varepsilon^2 l_1 \to \infty$ and $\varepsilon^2 l_2 \to \infty$.

**Proof.** Fix a pair $(i, j) \in \{1, 2\}^2$ and consider the dual process $\mathcal{T}_{n,\Pi}^i$. For a vertex of type $j$ created in this process let $t \geq 1$ be the generation in which it is born and associate it with a string $\sigma \in \Sigma = \{1, 2\}^{t+1}$ that is the finite sequence of types of all its ancestors, i.e. its ‘family line’. Therefore, clearly, $\sigma(0) = i$ and $\sigma(t) = j$. Let us consider the set the injective function

$$f : \Sigma \to \left\{(1, 1), (1, 2), (2, 1), (2, 2)\right\}^t,$$

and denote its image by $\Sigma' = f(\Sigma)$. Thus $f$ can be seen as a bijection that maps a string $\sigma \in \Sigma$ starting with $i$ and ending in $j$ to a string $\sigma' \in \Sigma'$ starting with $(i, 1)$ or $(i, 2)$ and ending with $(1, j)$ or $(2, j)$ for any pair $(i, j) \in \{1, 2\}^2$. Therefore, applying the symbolic method (e.g. [13]) we can compute the formal ordinary generating function $F_{i,j}((1, 1), (1, 2), (2, 1), (2, 2))$ of the set of strings in $\Sigma'$ that correspond to ancestry lines in $\Sigma$ starting with $i$ and ending with $j$, $(i, j) \in \{1, 2\}^2$. Now define an evaluation function

$$\hat{g} : \left\{\{(1, 1), (1, 2), (2, 1), (2, 2)\} \to \mathbb{R}_{>0}, \right\}$$

$$\{i, j\} \mapsto h_{i,j},$$

that canonically extends to a function $g : \Sigma' \to \mathbb{R}_{>0}$ and note that the expected number of offspring with a fixed ancestry line $\sigma \in \Sigma$ is precisely $g(f(\sigma))$. Therefore we have

$$E\left(|T_{n,\Pi}^i \cap V_i|\right) = g(F_{i,i}((1, 1), (1, 2), (2, 1), (2, 2)))$$

$$= \frac{1 - h_{3-1,3-i}}{1 - h_{1,1} - h_{2,2} + h_{1,1} h_{2,2} - h_{1,2} h_{2,1}}, \text{ for } i \in \{1, 2\}, \quad (18)$$
and
\[ \mathbb{E} \left( \left| T_{n,P}^i \cap V_{3-i} \right| \right) = g \left( F_{i,3-i} \left( (1,1), (1,2), (2,1), (2,2) \right) \right) \]
\[ = \frac{h_{i,3-i}}{1 - h_{1,1} - h_{2,2} + h_{1,1}h_{2,2} - h_{1,2}h_{2,1}}, \quad \text{for } i \in \{1,2\}. \]  

(19)

Since the numerator is bounded for small enough \( \varepsilon \) this is essentially a geometric series and thus converges if the absolute value of \( h_{1,1} + h_{2,2} - h_{1,1}h_{2,2} + h_{1,2}h_{2,1} \) is smaller than 1. Due to Condition (2), and since all offspring expectations \( \eta_{i,j}, (i,j) \in \{1,2\}^2 \) are positive, we may parametrize
\[ \eta_{i,j} = c_{i,j} + \delta_{i,j}, \]  

(20)

where \( c_{i,j} \in [0, 1] \) is a fixed constant and \( \delta_{i,j} = \delta_{i,i}(n_1, n_2, \varepsilon) = o(1) \) for \( i \in \{1,2\} \). Thus Condition (2) implies that
\[ \eta_{i,3-i} = c_{i,3-i} + \varepsilon - \delta_{i,i} + o(\varepsilon), \]  

(21)

where \( c_{i,3-i} = 1 - c_{i,i} \in [0, 1] \) is also a fixed constant for \( i \in \{1,2\} \).

If \( c_{1,2} + c_{2,1} > 0 \), using Equations (15), (20) and (21), we get
\[ 1 - h_{3-i,3-i} = c_{3-i,i} + o(1), \]
and
\[ h_{i,3-i} = c_{i,3-i} + o(1), \]
as well as
\[ h_{1,1} + h_{2,2} - h_{1,1}h_{2,2} + h_{1,2}h_{2,1} = 1 - \varepsilon(c_{1,2} + c_{2,1}) + o(\varepsilon). \]

Hence Equation (10) follows from Equations (18) and (19).

Similarly, if \( c_{1,2} = c_{2,1} = 0 \), we obtain, by Equations (15), (20) and (21),
\[ 1 - h_{3-i,3-i} = 2\varepsilon - \delta_{3-i,3-i} + o(\varepsilon + \delta_{3-i,3-i}), \]
and
\[ h_{i,3-i} = \varepsilon - \delta_{i,i} + o(\varepsilon + \delta_{i,i}), \]
as well as
\[ h_{1,1} + h_{2,2} - h_{1,1}h_{2,2} + h_{1,2}h_{2,1} = 1 - \varepsilon(3\varepsilon - \delta_{1,1} - \delta_{2,2}) + o(\varepsilon^2 + \varepsilon\delta_{1,1} + \varepsilon\delta_{2,2}). \]

Thus Equation (10) also follows from Equations (18) and (19).

As mentioned before, by conditioning on \( D_i \) (the event that the primal process \( T_{n,P}^i \) dies out) and applying Markov’s inequality we get
\[ P \left( \left| T_{n,P}^i \cap V_1 \right| \geq l_1 \lor \left| T_{n,P}^i \cap V_2 \right| \geq l_2 \right) \]
\[ = P \left( \neg D_i \right) P \left( \left| T_{n,P}^i \cap V_1 \right| \geq l_1 \lor \left| T_{n,P}^i \cap V_2 \right| \geq l_2 \middle| \neg D_i \right) \]
\[ + P(D_i) P \left( \left| T_{n,P}^i \cap V_1 \right| \geq l_1 \lor \left| T_{n,P}^i \cap V_2 \right| \geq l_2 \middle| D_i \right) \]
\[ = \rho_i + P(D_i) P \left( \left| T_{n,P}^i \cap V_1 \right| \geq l_1 \lor \left| T_{n,P}^i \cap V_2 \right| \geq l_2 \middle| D_i \right) \]
\[ = \rho_i + P(D_i) \left( P \left( \left| T_{n,P}^i \cap V_1 \right| \geq l_1 \right) + P \left( \left| T_{n,P}^i \cap V_2 \right| \geq l_2 \right) \right) + o(\varepsilon) \]
\[ \leq 2\varepsilon + o \left( \varepsilon^{-1}l_1^{-1} + \varepsilon^{-1}l_2^{-1} \right) + o(\varepsilon), \]
completing the proof. \[\blacksquare\]
3.3 The width of a tree

The last tool that we need for the proof is the concept of the width of a rooted tree. The width $w(T)$ of a rooted tree $T$ is defined as the supremum of the sizes of all its generations.

**Lemma 3.5.** Let $T_{n_1,n_2}^i$ be a 2-type branching process satisfying the conditions of Theorem 2.1 and denote by $D_i$ the event that this process dies out. Then for any $M = M(n_1, n_2) \in \mathbb{N}$ such that $\varepsilon M \to \infty$ we have

$$P\left(\{w(T_{n_1,n_2}^i) \geq M\} \cap D_i\right) = o(\varepsilon).$$

**Proof.** Denote $W_M = \{w(T_{n_1,n_2}^i) \geq M\}$ and let us construct $T_{n_1,n_2}^i$ generation by generation and stop as soon as we see the first generation of size at least $M$ if there is one. Then we have $M_1$ vertices of type 1 and $M_2$ vertices of type 2 where $M_1 + M_2 = M$. Since each of the vertices of this generation starts an independent copy of $T_{n_1,n_2}^1$, respectively $T_{n_1,n_2}^2$ we get for the probability of dying out given that $W_M$ holds

$$P(D_i|W_M) = (1 - \rho_1)^{M_1}(1 - \rho_2)^{M_2} \leq e^{-(\rho_1 M_1 + \rho_2 M_2)} \sim e^{-2\varepsilon M} = o(1),$$

where the asymptotic statements hold due to Lemma 3.2 and since $\varepsilon M \to \infty$. Hence,

$$P(\neg D_i|W_M) = 1 - o(1),$$

and by the laws of conditional probability and Lemma 3.2 we have

$$P(W_M \cap D_i) = P(W_M \cap \neg D_i) \cdot \frac{P(D_i|W_M)}{P(\neg D_i|W_M)} \leq o(P(\neg D_i)) = o(\varepsilon),$$

proving Lemma 3.5.

4 Large components: proof of Theorem 2.1

We will now use the 2-type binomial branching process to analyze the size of the giant component in $G(n, P)$ in the supercritical regime. The main idea is to couple the component exploration process in $G(n, P)$ with instances of the 2-type binomial branching process. Given a vertex $v$ of type $i$ in $G(n, P)$ we denote its component by $C_v$. Furthermore let $T_v$ be the random spanning-tree rooted at $v$ constructed by exploring new neighbors in $C_v$ via a breadth-first search.

**Lemma 4.1.** Given any vector $n \in \mathbb{N}^2$, any symmetric matrix $P \in [0, 1]^{2 \times 2}$ and any vertex of type $i \in \{1, 2\}$, the following two statements hold.

(i) There is a coupling of the random rooted trees $T_v$ and $T_{n_1,n_2}^i$ such that $T_v \subset T_{n_1,n_2}^i$. In particular, $|C_v \cap V_j| \leq |T_{n_1,n_2}^i \cap V_j|$, for $j \in \{1, 2\}$.
Lemma 4.2. Let vector of parameters $L$ contain at least $l \in F$ or $\epsilon$.

Proof. For the first statement, we generate $T_v$ and $T_{\nu, P}$ simultaneously, restoring the set of potential neighbors in the breadth-first search by adding fictional vertices of the same type to the vertex set of $T_v$ and $T_{\nu, P}$ is at least $m_r$ for some $r \in \{1, 2\}$.

For the second statement we proceed as before with the slight change that in each step we choose for any type $j \in \{1, 2\}$ exactly $n_j - m_j$ from all possible new neighbors of this type and only add those independently with probability $p_{s,j}$, where $s$ is the type of the current vertex, and ignore all other vertices. Until we have encountered a total of at least $m_r$ vertices of type $r$ in $T_{\nu, P}$, for some $r \in \{1, 2\}$ there are always enough vertices of each type to choose from. Hence, for $j \in \{1, 2\}$, we have $(T_v \cap V_j) \subset (T_{\nu, P} \cap V_j) \subset (T_v \cap V_j)$ unless all of them contain at least $m_r$ vertices of type $r$.

Using Lemma 3.3 and Lemma 4.1 we can now establish the expectation of the number of vertices in ‘large’ components. For any type $i \in \{1, 2\}$, we denote by $S_i, L = S_i, L(G(n, P))$ the set of all vertices of type $i$ in components that contain at least $l_j$ vertices of type $j$, for some $j \in \{1, 2\}$ and a properly chosen vector of parameters $L = (l_1, l_2) \in \mathbb{N}^2$. Moreover, for $i \in \{1, 2\}$, we denote by $s_i, L = |S_i, L|$ the cardinality of this set.

Lemma 4.2. Let $L = (l_1, l_2) = (l_1(n_1, n_2, \varepsilon), l_2(n_1, n_2, \varepsilon)) \in \mathbb{N}^2$ such that $\varepsilon l_i \rightarrow \infty$, for $i \in \{1, 2\}$. Then

$$E(s_i, L) \leq (2 + o(1))\varepsilon n_i, \text{ for } i \in \{1, 2\}. $$

Proof. For $i \in \{1, 2\}$, by Lemma 3.3 and linearity of expectation, we have

$$E(s_i, L) = \sum_{u \text{ of type } i} P(|C_u \cap V_1| \geq l_1 \lor |C_u \cap V_2| \geq l_2)$$

$$\leq n_i P(|T_{\nu, P} \cap V_1| \geq l_1 \lor |T_{\nu, P} \cap V_2| \geq l_2) \sim 2\varepsilon n_i, $$

where the last step holds by Equation (17) in Lemma 3.3.

Lemma 4.3. Let $L = (l_1, l_2) = (l_1(n_1, n_2, \varepsilon), l_2(n_1, n_2, \varepsilon)) \in \mathbb{N}^2$ such that $l_i = o(\varepsilon n_i)$, for $i \in \{1, 2\}$. Then

$$E(s_i, L) \geq (2 + o(1))\varepsilon n_i, \text{ for } i \in \{1, 2\}. $$
Proof. We apply Lemma 4.1(ii) with \( m = L \), since \( l_i = o(\varepsilon n_i) \), for \( i \in \{1, 2\} \), and note that the parameters of the coupling branching process satisfy Conditions (1) and (2). Hence, for \( i \in \{1, 2\} \), this yields by linearity of expectation

\[
E(s_{i,L}) = \sum_{v \text{ of type } i} P(|C_v \cap V_1| \geq l_1 \lor |C_v \cap V_2| \geq l_2) \\
\geq n_i P(|T^i_{n-m,P} \cap V_1| \geq l_1 \lor |T^i_{n-m,P} \cap V_2| \geq l_2) \\
\geq n_i P(T^i_{n-m,P} \text{ survives}) \sim 2\varepsilon n_i,
\]

and the last step holds due to Lemma 3.2.

In the next lemma we will show that \( s_{i,L}(G(n,P)) \), i.e. the number of vertices of type \( i \) in large components, is concentrated around its expectation for \( i \in \{1, 2\} \).

Lemma 4.4. Let \( L = (l_1,l_2) = (l_1(n_1,n_2,\varepsilon),l_2(n_1,n_2,\varepsilon)) \in \mathbb{N}^2 \) such that \( \varepsilon^2 l_i \to \infty \) and \( l_i = o(\varepsilon n_i) \), for \( i \in \{1, 2\} \). Then whp

\[
s_{i,L}(G(n,P)) = (2 + o(1))\varepsilon n_i, \text{ for } i \in \{1, 2\}.
\]

Proof. Lemmas 4.2 and 4.3 show that \( E(s_{i,L}) \sim 2\varepsilon n_i \), hence it is sufficient to show the upper bound

\[
E(s_{i,L}^2) \leq (4 + o(1))\varepsilon^2 n_i^2 \sim E(s_{i,L})^2 \text{ for } i \in \{1, 2\}.
\]

The reason for this is the classical second moment method (e.g. [1, 19]): Equation (22) implies that for the random variable \( s_{i,L} \) the variance is of smaller order than the square of the expectation, i.e.

\[
V(s_{i,L}) = E(s_{i,L}^2) - E(s_{i,L})^2 \leq o\left(E(s_{i,L})^2\right), \text{ for } i \in \{1, 2\},
\]

which provides concentration by using Chebyshev’s inequality.

Without loss of generality fix a type \( i \in \{1, 2\} \) for the rest of the proof. Furthermore, fix a vertex \( v \) of type \( i \) in \( G(n,P) \). Once again we explore the component \( C_v \) of that vertex in a breadth-first search generating a tree \( T_v' \subset C_v \). However, we will stop the exploration immediately, even midway through revealing the neighbors of one particular vertex, if one of the following two events occurs:

(i) we have already reached a total of \( l_j \) vertices of type \( j \) for some \( j \in \{1, 2\} \);

(ii) there are \( \varepsilon l_2 \) vertices that have been reached (i.e. that are a child of some earlier vertex) but not yet fully explored (flipped a coin for each possible neighbor).

Note that in Condition (ii) we do not distinguish the types of vertices. Any vertex that has been reached but not fully explored is called boundary vertex.
Observe that this process will create at most $\varepsilon l_2 + 1 \leq 2\varepsilon l_2$ boundary vertices. Furthermore, denote by $A$ the event that the process stops due to (i) or (ii), rather than because it has revealed the whole component $C_v$. Note that

$$\{|C_v \cap V_1| \geq l_1 \lor |C_v \cap V_2| \geq l_2\} \implies A,$$  

(23)

a fact that we will use later on. Now we estimate the probability that $A$ holds: By the coupling in Lemma 4.1(i) we may assume that $T_u \subset T_v \subset T_{n,P}$ and, since we proceed in a breadth-first way, at every point of time all the boundary vertices are contained in at most two consecutive generations. Hence if $A$ holds, either $|T_{n,P} \cap V_j| \geq l_j$, for some $j \in \{1, 2\}$, or the total number of offspring of the process $T_{n,P}$ is finite and $w(T_{n,P}) \geq \frac{\varepsilon l_2}{2}$. As calculated in Lemma 3.3 the probability that the first case occurs is asymptotically at most $2\varepsilon$, while for the second case we calculated in Lemma 3.5 that the probability of having large width but still dying out is $o(\varepsilon)$, hence

$$P(A) \leq (2 + o(1))\varepsilon.$$  

(24)

We use this to relate the second moment to the expectation on the conditional probability space, where we condition on $A$ holding. We replace $s_{i,L}$, $i \in \{1, 2\}$, by a sum of indicator random variables as usual and by Implication (23) we get

$$E[s_{i,L}^2] = \sum_{v \text{ of type } i} E[1_{\{|C_v \cap V_1| \geq l_1 \lor |C_v \cap V_2| \geq l_2\}} s_{i,L}]$$

$$\leq n_i E[1_A s_{i,L}]$$

$$= n_i P(A)E[s_{i,L} | A]$$

$$\leq (2 + o(1))\varepsilon n_i E[s_{i,L} | A].$$  

(25)

For the rest of this proof we will compute an asymptotic upper bound for the conditional expectation $E[s_{i,L} | A]$. Now for any vertex $u \notin T_u'$ of type $i$ we reveal its component as before in a breadth-first manner but ignore any vertices that are in $T_u'$, i.e. we explore in $G' = G(n, P) \setminus V(T_u')$ until we have revealed the whole component in this subgraph. Moreover, we couple the generated tree $T_u'$ with $T_{n,P}$ such that $T_{n,P}' \subset T_{n,P}$. We denote by $D_i$ the event that this instance of $T_{n,P}'$ dies out and note, in particular, that $D_i$ is independent of the event $A$, hence

$$P(\neg D_i | A) = P(\neg D_i) = (2 + o(1))\varepsilon$$  

(26)

by Lemma 4.2. Let us observe that $|T_{n,P}'| \leq |C_u|$ and furthermore that equality holds unless $G(n, P)$ contains an edge connecting a boundary vertex to a vertex of $T_{n,P}'$. Therefore, for any given $r \in \mathbb{N}$, we have

$$P(|C_u| \neq |T_{n,P}'| | D_i \land A \land \{|T_{n,P}'| = r\}) \leq 2\varepsilon l_2 r \max \{p_{j,j'} | j, j' \in \{1, 2\}\},$$

by the Union Bound, as there are at most $2\varepsilon l_2$ boundary vertices. Note that by Condition (2) we have

$$\max \{p_{j,j'} | j, j' \in \{1, 2\}\} \leq (1 + \varepsilon)n_2^{-1} \leq 2n_2^{-1}.$$
Hence, by the law of total probability,
\[ P(|C_u| \neq |T''_u| \mid D_i, \mathcal{A}) \leq 4\varepsilon l_2 n_2^{-1} E[|T''_u| \mid D_i]. \]

In order to simplify notation we will write
\[ X_{u,L} = \{|C_u \cap V_1| \geq l_1 \vee |C_u \cap V_2| \geq l_2 \} \]
for the event that the component of \( u \) is large. Hence it follows that
\[
P(X_{u,L} \mid D_i, \mathcal{A}) \leq P(\neg D_i) + P(D_i) P(X_{u,L} \mid D_i, \mathcal{A})
\]
\[
\leq P(\neg D_i) + P(X_{u,L} \cap \{|C_u| = |T''_u|\} \mid D_i, \mathcal{A})
\]
\[
+ P(X_{u,L} \cap \{|C_u| \neq |T''_u|\} \mid D_i, \mathcal{A})
\]
\[
\leq P(\neg D_i) + P(|T''_u \cap V_1| \geq l_1 \vee |T''_u \cap V_2| \geq l_2 \mid D_i)
\]
\[
+ l_1^{-1} E[|T''_u| \mid D_i]
\]
\[
\leq P(\neg D_i) + l_1^{-1} E[|T''_{n,1} \cap V_1|] + l_2^{-1} E[|T''_{n,2} \cap V_2|]
\]
\[
+ 4\varepsilon l_2 n_2^{-1} E[|T''_u| \mid D_i],
\]
where the last step holds due to Markov’s inequality. Furthermore, we know that these expectations are all of order \( O(\varepsilon^{-1}) \) by Equation (16) in Lemma 3.4. Additionally, by our assumptions on \( L \), the coefficients \( l_1^{-1} \), \( l_2^{-1} \) and \( 4\varepsilon l_2 n_2^{-1} \) are all of order \( o(\varepsilon) \) and therefore using Equation (28) we get
\[ P(X_{u,L} \mid \mathcal{A}) \leq (2 + o(1))\varepsilon. \]

Thus, since there are at least \( n_i - l_i \) vertices of type \( i \) for which we can apply this bound, we get
\[ E[s_i, L \mid \mathcal{A}] \leq l_i + (n_i - l_i) P(X_{u,L} \mid \mathcal{A}) \leq l_i + (2 + o(1))\varepsilon n_i = (2 + o(1))\varepsilon n_i. \]

Inserting this into inequality (25) and then applying Chebyshev’s inequality finishes the proof of Lemma 4.4.

Now we can finally prove Theorem 2.1.

**Proof of Theorem 2.1.** Let us first introduce some further notation. Let
\[ \omega = \varepsilon^3 n_2 \eta_{2,1} \]
and note that \( \omega \to \infty \) by Condition (1) of Theorem 2.1. We set
\[ l_j = \frac{\varepsilon n_j}{\ln \omega} = o(\varepsilon n_j), \quad \text{for } j \in \{1, 2\}, \]
and conclude.
and note that we could replace $\ln \omega$ by any function $\hat{\omega}$ such that $\hat{\omega} \to \infty$ but growing very slowly compared to $\omega$. Moreover, observe that by Condition (2) we have

$$\varepsilon^2 l_j = \frac{\varepsilon^3 n_i}{\ln \omega} = \frac{\omega \eta_{1,1}^{-1}}{\ln \omega} \to \infty, \text{ for } j \in \{1, 2\}. \tag{29}$$

Essentially, we know so far that the random graph $G(n, P)$ satisfying the conditions of Theorem 2.1, contains the ‘right’ number of vertices in large components. It only remains to show that all these components are connected if we ‘sprinkle’ some more edges. Formally this can be done as follows. We set

$$P^b = \frac{\varepsilon}{\ln \omega} P,$$

$$P^a = \left(1 - \frac{\varepsilon}{\ln \omega}\right) P,$$

and denote by

$$P^c = \left(p_{j,j'}^c\right)_{j,j' \in \{1, 2\}}$$

the probability matrix with entries

$$p_{j,j'}^c := p_{j,j'}^a + p_{j,j'}^b - p_{j,j'}^a p_{j,j'}^b,$$

for $(j, j') \in \{1, 2\}^2$.

Note that, for large enough $n_1$ and $n_2$, we have $p_{j,j'}^c \leq p_{j,j'}$ for each pair $(j, j') \in \{1, 2\}^2$ and thus by coupling

$$G(n, P^c) \subset G(n, P).$$

Furthermore, we construct $G(n, P^a)$ and $G(n, P^b)$ independently and retrieve $G(n, P^c)$ by coupling it with their union

$$G(n, P^c) = G(n, P^a) \cup G(n, P^b) \supset G(n, P^a),$$

hence

$$G(n, P^a) \subset G(n, P^c) \subset G(n, P).$$

Observe that $P^a = (1 - o(\varepsilon)) P$ and thus due to Equations (28) and (29) we can apply Lemma 4.4 with $L = (l_1, l_2)$ to $G(n, P^a)$. Thus, whp for $i \in \{1, 2\}$, there are $(2 + o(1))\varepsilon n_i$ vertices of type $i$ in large components in $G(n, P^a)$, i.e. components containing at least $l_j$ vertices of type $j$ for some $j \in \{1, 2\}$. Therefore let us call this event $\mathcal{E}$.

Assume $\mathcal{E}$ holds. Thus there are asymptotically $(2 + o(1))\varepsilon n_i$ vertices of type $i$ for $i \in \{1, 2\}$. Let $\mathcal{U}$ denote the set of all large components in $G(n, P^a)$. Then for any component $C \in \mathcal{U}$ and any $i \in \{1, 2\}$ we say which type $i$ is a witness for $C$ being large if $|C \cap V_i| > \frac{1}{2} l_i$. Observe that having a witness is a necessary condition for any component to be large, hence each large component $C \in \mathcal{U}$ has at least one witness, yet it is not a sufficient condition. For $i \in \{1, 2\}$ we define the set $\mathcal{U}^i \subset \mathcal{U}$ of large components for that type $i$ is a witness and write $\mathcal{U}^i = \{U_1^i, \ldots, U_{r_i}^i\}$. Intuitively, it should not be the case that one of these sets is empty. We prove this by a counting argument.
Claim 4.5. $U^1$ and $U^2$ are not empty, i.e. $r_1 > 0$ and $r_2 > 0$.

Proof. Without loss of generality assume towards contradiction that $r_1 = 0$, and thus clearly $r_2 > 0$. Observe that this implies that

$$|U_j^2 \cap V_1| \leq \frac{1}{2} l_1 = \frac{\varepsilon n_1}{2 \ln \omega},$$

and

$$|U_j^2 \cap V_2| \geq l_2 = \frac{\varepsilon n_2}{\ln \omega},$$

for $j \in \{1, \ldots, r_2\}$. Counting vertices of both types separately, we therefore get for type 1

$$(2 + o(1)) \varepsilon n_1 = \sum_{j=1}^{r_2} |U_j^2 \cap V_1| \leq \frac{r_2 \varepsilon n_1}{2 \ln \omega}$$

and for type 2

$$(2 + o(1)) \varepsilon n_2 = \sum_{j=1}^{r_2} |U_j^2 \cap V_2| \geq \frac{r_2 \varepsilon n_2}{\ln \omega}.$$ 

This shows

$$(4 + o(1)) \ln \omega \leq r_2 \leq (2 + o(1)) \ln \omega,$$

a contradiction, hence Claim 4.5 holds.

Observe that $p^b_{1,2} = \varepsilon (\ln \omega)^{-1} p_{1,2}$ and, by the definition of witnesses, obviously

$$|U_j^i| \geq |U_j^i \cap V_i| > \frac{1}{2} l_i = \frac{\varepsilon n_i}{2 \ln \omega},$$

for $i \in \{1, 2\}$ and $j \in \{1, \ldots, r_i\}$. Hence, if we estimate the number of vertices of type $i$ by only summing over the components in $U^i$ we get

$$\frac{r_i \varepsilon n_i}{2 \ln \omega} < \sum_{j=1}^{r_i} |U_j^i \cap V_i| \leq (2 + o(1)) \varepsilon n_i,$$

and consequently

$$r_i \leq (4 + o(1)) \ln \omega,$$

for $i \in \{1, 2\}$. Therefore the probability that in $G(n, P^b)$ there is no edge between two fixed components $U_{j_1}^1 \in U^1$ and $U_{j_2}^2 \in U^2$, with $j_1 \in \{1, \ldots, r_1\}$ and $j_2 \in \{1, \ldots, r_2\}$, is at most

$$(1 - p^b_{1,2}) |U_{j_1}^1| |U_{j_2}^2| \leq \exp \left( - \frac{\varepsilon p_{1,2}}{\ln \omega} \cdot \frac{\varepsilon n_1}{2 \ln \omega} \cdot \frac{\varepsilon n_2}{2 \ln \omega} \right) = \exp \left( - \frac{\omega}{4 \ln^3 \omega} \right),$$

by Equation (27), since $\eta_{2,1} = p_{2,1} n_1$. Taking the union bound for $r_1 + r_2 - 1$ of these events shows that the probability that in $G(n, P^c)$ all components that were large in $G(n, P^a)$ are connected is at least

$$1 - (r_1 + r_2 - 1) \exp \left( - \frac{\omega}{4 \ln^3 \omega} \right) \geq 1 - (4 + o(1)) \ln \omega \exp \left( - \frac{\omega}{4 \ln^3 \omega} \right) = 1 - o(1).$$

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Thus, \( whp \) the largest component \( L_1 (G(n, P^c)) \subset G(n, P^c) \) satisfies
\[
\frac{|L_1 (G(n, P^c)) \cap V_i|}{n_i} = (2 + o(1)) \varepsilon,
\]
for \( i \in \{1, 2\} \). Additionally, applying Lemma 4.4 to \( G(n, P) \) with the same choice of \( L \) as before shows that \( whp \) the total number of vertices of type \( i \in \{1, 2\} \) in large components only increases by a factor \( 1 + o(1) \) if we go from \( G(n, P^a) \) to \( G(n, P) \). Therefore, \( whp \) the component \( L_1 (G(n, P^c)) \) will be contained in the component \( L_1 = L_1 (G(n, P)) \) and also, for \( i \in \{1, 2\} \),
\[
\frac{|L_1 (G(n, P^c)) \cap V_i|}{n_i} = (1 + o(1)) \frac{|L_1 (G(n, P^c)) \cap V_i|}{n_i} = (2 + o(1)) \varepsilon.
\]

Moreover, \( whp \) the second-largest component \( L_2 (G(n, P)) \) can contain at most all the vertices that are contained in a large component in \( G(n, P) \) but were not already contained in a large component in \( G(n, P^a) \), hence
\[
\frac{|L_2 (G(n, P)) \cap V_i|}{n_i} = o(\varepsilon),
\]
for \( i \in \{1, 2\} \). Therefore, clearly we have \( whp \) that \( L_1 \) is the unique largest component of \( G(n, P) \) and has size \( L_1 \sim 2\varepsilon n \), while all other components are of size \( o(\varepsilon n) \).

5 Weakly subcritical regime: proof of Theorem 2.2

Most of the work for this regime has already been done in Section 3.2 since the dual process of a weakly supercritical branching process is weakly subcritical. Therefore we will keep the proof short.

Proof of Theorem 2.2  Let the conditions be as in Theorem 2.2. Then, analogously to the proof of Lemma 3.4 we calculate the expected total size of \( T_{n,P}^i \), for \( i \in \{1, 2\} \), and get
\[
E (|T_{n,P}^i|) = \frac{1 + \eta_{i,2-i} - \eta_{i,1,3-i}}{1 - (\eta_{1,1} + \eta_{2,2} - \eta_{1,1,2,2} + \eta_{1,2,2,1})} \sim \varepsilon^{-1},
\]
where the asymptotic statement holds since we sum over all vertices. Now let \( L = \delta n^{2/3} \), for any fixed constant \( \delta > 0 \) and write \( S_L \) for the set of vertices in components of size at least \( L \) and \( s_i = |S_L| \). Then with the coupling as in Lemma 4.1(i) we get, by applying Markov’s inequality twice and linearity of
proof of Lemma 3.2. We use the same bounding functions for the

\[ F \]

Essentially the proof in Section 4 works also when the distance from the critical point is a constant, except for some of the arguments used for calculating the survival probabilities.

\[ \text{Proof of Theorem 2.3.} \]

by Condition (5). Hence, since \( \delta > 0 \) was arbitrary, \( \text{whp} \) all components are of size \( o(n^{2/3}) \).

Remark 6. This result can be slightly strengthened: Let \( \omega = \varepsilon^3n \to \infty \) and replace \( L \) by \( \hat{L} = \delta n^{2/3} \omega^{-1/6+c} \) for any \( 0 < c < 1/6 \).

6 SuperCritical regime: proof of Theorem 2.3

Essentially the proof in Section 4 works also when the distance from the critical point is a constant, except for some of the arguments used for calculating the survival probabilities.

Proof of Theorem 2.3. Let the conditions be as in Theorem 2.3. We will only show the computation for the survival probabilities, which is very similar to the proof of Lemma 3.2. We use the same bounding functions for the \( F_i \)'s defined in Equation (12) given by Lemma 3.1.

So if \( \rho_1 \geq \rho_2 \), then \( F_1(\rho_1, \rho_2) \leq F_1(\rho_1, \rho_1) \) and \( F_2(\rho_1, \rho_2) \geq F_2(\rho_2, \rho_2) \) and analogously if \( \rho_1 < \rho_2 \), we have \( F_2(\rho_1, \rho_2) < F_2(\rho_2, \rho_2) \) and \( F_1(\rho_1, \rho_2) > F_1(\rho_1, \rho_1) \). Thus without loss of generality assume \( \rho_1 \geq \rho_2 \) and let us now consider the bounding functions \( F_i(\rho_i, \rho_i) \) for \( i \in \{1, 2\} \).

\[
F_i(\rho_i, \rho_i) = 1 - \rho_i - \left(1 - \frac{\eta_{i,1}\rho_i}{n_1}\right)^{n_1} \left(1 - \frac{\eta_{i,2}\rho_i}{n_2}\right)^{n_2} = 1 - \rho_i - \exp\left(-\left(\eta_{i,1} + \eta_{i,2}\right)\rho_i - O\left(\frac{\eta_{i,1}^2\rho_i^2}{n_1} + \frac{\eta_{i,2}^2\rho_i^2}{n_2}\right)\right),
\]

by the Taylor-expansion of the natural logarithm around 1. Since \( \eta_{i,1} \leq 1 + 2\varepsilon \) and \( \eta_{i,2} \leq 1 + 2\varepsilon \), by the conditions of Theorem 2.1 and the fact that \( \rho_i \leq 1 \), since it is a probability, we have

\[
F_i(\rho_i, \rho_i) = 1 - \rho_i - \exp\left(-\frac{\eta_{i,1} + \eta_{i,2} + O\left(n_2^{-1}\right)}{\rho_i}\right) = 1 - \rho_i - \exp\left(-(1 + \varepsilon_i)\rho_i\right) = f(\varepsilon_i, \rho_i),
\]

where \( \varepsilon_i = \eta_{i,1} + \eta_{i,2} - 1 + O\left(n_2^{-1}\right) \sim \varepsilon_i \), for \( i \in \{1, 2\} \), by Conditions 1 and 2. It is well-known that \( f(\varepsilon, \rho) = 0 \) has exactly one solution for any fixed \( \varepsilon > 0 \), so let us consider this solution \( (\varepsilon, \rho) \).
Let $D = \mathbb{R}_{>0} \times (0, 1)$ and note that the function $f : D \to \mathbb{R}$ is continuous and its partial derivative with respect to the first variable does not vanish on $D$, therefore we can apply the classical implicit function theorem in $\mathbb{R}^2$. Hence, for the solution $(\varepsilon, \rho)$ to $f = 0$ there is an open set $U$ with $\varepsilon \in U$ and an open set $V$ with $\rho \in V$ such that

$$\{(u, g(u)) \mid u \in U\} = \{(u, v) \in U \times V \mid f(u, v) = 0\},$$

where $g$ is a continuous function on $U$. Let $i \in \{1, 2\}$, as $|\varepsilon_i - \varepsilon| = o(1)$, hence $\varepsilon_i \in U$ for large enough $n_1$ and $n_2$, this implies that $f(\varepsilon_i, g(\varepsilon_i)) = 0$, for $i \in \{1, 2\}$. Therefore by uniqueness of the solution for a given $\varepsilon_i$ we have $g(\varepsilon_i) = \rho_2$ and since $g$ is continuous we get

$$\rho_1 \sim \rho_2 \sim \rho_\varepsilon, \quad (30)$$

where $\rho_\varepsilon = g(\varepsilon)$. Note that $\rho_\varepsilon$ is a constant independent of $n_1$ and $n_2$.

The rest of the proof follows the lines of the proof of Theorem 2.1 in Sections 3.2, 3.3 and 4, by replacing $\rho_1 \sim \rho_2 \sim 2\varepsilon$ with Statement (30).

7 Subcritical regime: proof of Theorem 2.4

In the subcritical regime, where the distance to the critical point is a constant, one can obtain a strong upper bound on the size of all components by a standard application of large deviation inequalities.

Proof of Theorem 2.4 Let the conditions be as in Theorem 2.4. We fix a vertex $v$ of type $i$ and explore its component $C_v$ in $G(n, P)$, denote the resulting spanning tree by $T_v$ and couple this process with a 2-type branching process $T_{n, p}^i$ as in Lemma 4.1(i) such that $T_v \subseteq T_{n, p}^i$. Let $S_L$ be the event that $G(n, P)$ contains a component of size at least $L$ for some appropriately chosen parameter $L = L(n_1, n_2, \varepsilon)$. We want to show that

$$\mathbb{P}(S_L) = o(1).$$

Let us denote the possibly infinite sequence of vertices born in $T_{n, p}^i$, with respect to the breadth-first exploration, by $\sigma = (v_1, v_2, v_3, \ldots)$, with $v_1 = v$.

For any vertex $u \in V_1 \cup V_2$ let $X_u$ be the random variable that counts the number of children of $u$ and has a distribution $Bin(n_1, p_{j,1}) + Bin(n_2, p_{j,2})$, where $j \in \{1, 2\}$ is the type of $u$. Then consider the random variables

$$X_{v, L} := \min\{L, |\sigma|\} \quad X_{v, r} \leq \sum_{r=1}^{L} X_{v, r} =: X_{v, L}^*, \quad (31)$$

where $\{v_{|\sigma|+1}, \ldots, v_L\}$ is an arbitrary sequence of distinct additional vertices. Notice that $X_{v, L}^*$ is a sum of independent Bernoulli random variables whose expectation satisfies

$$|\mathbb{E}(X_{v, L}^*) - L(1 - \varepsilon)| \leq \gamma, \quad (32)$$

$$\text{where } \gamma = \varepsilon(1 - \gamma) + \frac{1}{2}\varepsilon^2.$$
for some $\gamma = \gamma(n) = o(L)$, by Condition 10. Now observe that $|T_{n,p}| \geq L$ implies $X_{v,L} \geq L - 1$. Therefore we get by application of the Union Bound
\[
P(S_L) \leq nP(|C_v| \geq L) \leq nP(|T_{n,p}| \geq L) \leq nP(X_{v,L} \geq L - 1).
\]
Furthermore, by the definition of $X^*_{v,L}$ in Statement 31 and application of a Chernoff Bound (e.g. [19], page 29) we get
\[
P(S_L) \leq nP(X^*_{v,L} \geq L - 1) \leq n \exp \left( \frac{(\varepsilon L - 1 - \gamma)^2}{2(\varepsilon L - 1 - \gamma) + 1/3(\varepsilon L - 1 - \gamma)} \right)
\]
for any $L > 3\varepsilon^{-2}\ln n$, completing the proof.

8 Discussion

In the previous sections we showed that the emergence of the giant component in the multi-type random graph $G(n, P)$ is very similar to the behavior of the binomial random graph $G(n, p)$, at least when each row of the expectation matrix is scaled similarly. With this scaling it should not be too difficult to transfer the proofs here to the general $k$-type case, for $k \in \mathbb{N}$, although the sprinkling might impose more complex additional restrictions.

On the other hand, it would be interesting to generalize this method to more general cases without Condition 2.

**Question 1.** Let $n_1, n_2 \in \mathbb{N}$, $\varepsilon = \varepsilon(n_1, n_2) > 0$, with $\varepsilon = o(1)$ and $P = P(n_1, n_2, \varepsilon) \in [0, 1]^{2 \times 2}$ such that

$\varepsilon^3 n_2 n_{2,1} \to \infty$.

Moreover, let $\lambda = 1 + \varepsilon + o(\varepsilon)$ be the largest eigenvalue of the offspring expectation matrix $(\eta_{i,j})_{i,j=1,2}$ of a $2$-type binomial branching process with parameters $n$ and $P$. Is it true that its survival probabilities $\rho_1$ and $\rho_2$ satisfy

$\rho_1 \sim \rho_2 \sim 2\varepsilon$?

If so, does it also imply whp

$L_1(G(n, P)) \sim 2\varepsilon n$?

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