THE WITTEN TOP CHERN CLASS VIA K-THEORY

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Abstract. The Witten top Chern class is the crucial cohomology class needed to state a conjecture by Witten relating the Gelfand–Dikii hierarchies to higher spin curves. In [PV01], Polishchuk and Vaintrob provide an algebraic construction of such a class. We present a more straightforward construction via K-theory. In this way we short-circuit the passage through bivariant intersection theory and the use of MacPherson’s graph construction. Furthermore, we show that the Witten top Chern class admits a natural lifting to the K-theory ring.

1. Introduction

1.1. Witten’s conjecture. In [Wit93], Witten conjectures that the intersection numbers of certain cohomology classes on the moduli stacks of stable r-spin curves encode a solution of the Gelfand–Dikii (also known as the higher KdV) hierarchy. (The conjecture is a generalization of the Kontsevich–Witten Theorem [Wit91, Kon92].) Witten’s formulation of the conjecture lacks both a rigorous definition of the moduli stack $\mathcal{M}_{g,n}(r,k)$ of stable r-spin curves and a construction of the crucial cohomology class on $\mathcal{M}_{g,n}(r,k)$ which is usually referred to as the Witten top Chern class.

1.2. Definition of the moduli stack. The moduli stack of stable r-spin curves was first defined by Jarvis [Jar00] and is the compactification of the stack $\mathcal{M}_{g,n}(r,k)$ labeled by the integer and nonnegative indexes $r, g, n,$ and $k = (k_1, k_2, \ldots, k_n)$ satisfying $r \geq 2, \, 2g - 2 + n > 0,$ and $2g - 2 - \sum_i k_i \in r\mathbb{Z}.$

The stack $\mathcal{M}_{g,n}(r,k)$ classifies smooth stable r-spin curves: $n$-pointed smooth stable curves $(C; s_1, \ldots, s_n)$ of genus $g$ equipped with a line bundle $L$ on $C$ and an isomorphism $f: L^\otimes r \cong \omega_C(-\sum_i k_i[s_i]).$ The compactified stack $\mathcal{M}_{g,n}(r,k)$ classifies stable curves $C$ equipped with a torsion free sheaf of rank one $L$ and a nonzero homomorphism $f$ as above. (Using stack-theoretic curves, this solution is rephrased in [AJ03] and can be modified as shown in [Chi]; however, the problem of constructing the Witten top Chern class has equivalent solutions with all these compactifications, Remark 4.0.10.)

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1.3. Witten’s definition of his class on an open substack. In [Wit93], the Witten top Chern class is defined on the open substack $\mathcal{V}$ of $\mathcal{M}_{g,n}(r,k)$ satisfying

$$H^0(C_x, L_x) = 0 \quad \text{for all} \quad x: \text{Spec} \mathbb{C} \to \mathcal{V},$$

where $((C_x; s_{1,x}, \ldots, s_{n,x}), L_x, f_x)$ is the $r$-spin curve represented by $x$. Over $\mathcal{V}$ there exists a vector bundle whose fibre on the point $x$ is the space $H^1(C_x, L_x)$. This is the higher direct image $R^1\pi_*\mathcal{L}$, where $\pi: \mathcal{C} \to \mathcal{V}$ is the universal curve and $\mathcal{L}$ on $\mathcal{V}$ is the universal line bundle. Then we take

$$c_{\text{top}}(R^1\pi_*\mathcal{L}).$$

If $g = 0$, the condition (1.3.1) is satisfied over the whole moduli stack. In general, $R^1\pi_*\mathcal{L}$ is not a vector bundle, so the usual definition of top Chern class does not apply. Witten sketches a generalization based on the index theory of elliptic operators. It is not clear, however, how to extend this approach to singular curves and to the whole stack $\mathcal{M}_{g,n}(r,k)$.

1.4. Polishchuk and Vaintrob’s construction. In [PV01], Polishchuk and Vaintrob provide an algebraic construction of the Witten top Chern class on the whole stack $\mathcal{M}_{g,n}(r,k)$. They work with the universal torsion free sheaf of rank one $\mathcal{L}$ on the universal family $\mathcal{C} \to \mathcal{M}_{g,n}(r,k)$, and they consider the pushforward $R\pi_*\mathcal{L}$ in the derived category, which is used to construct a $\mathbb{Z}/2\mathbb{Z}$-graded spinor bundle $S$. By extending MacPherson’s graph construction to 2-periodic complexes, Polishchuk and Vaintrob define the localized Chern character of $S$ in bivariant intersection theory. Passing to the Chow ring, they obtain a Chow cohomology class $c_{PV} \in A^{-\chi}(X)\mathbb{Q}$. Furthermore, such a class is compatible with Witten’s earlier definition and, by [Pol04], satisfies all the axioms of the cohomological field theory defined in [JKV01]—this is a preliminary condition to Witten’s conjecture [Wit93].

1.5. Our construction of the Witten top Chern class. We start from the pushforward $R\pi_*\mathcal{L}$ in the derived category, but follow a different path, which is more straightforward and closer to Witten’s original idea (1.3.2) (our approach is also alluded to in [Pol04] §2, p.2, Remark). We obtain the class $c_{PV}$ by working only with classes in $K$-theory, short-circuiting the passage through $\mathbb{Z}/2\mathbb{Z}$-graded spinor bundles, bivariant intersection theory, and the use of MacPherson’s graph construction.

We define the $K$-theory Euler class, a generalization of the total lambda class evaluated at $-1$, which by definition is

$$\lambda_{-1}: \text{Vect}(X) \to K_0(X)$$

$$V \mapsto \sum_i (-1)^i [A^i V].$$
The Witten top Chern class via $K$-theory

The $K$-theory Euler class $Ke(F_\bullet, a)$ is defined for a pair $(F_\bullet, a)$ where $F_\bullet$ is a bounded complex in degrees 1 and 0 of locally free coherent sheaves and $a: O \to \text{Sym}^{r-1} F_0 \otimes F_1$ is a closed and nondegenerate form (see Definition 3.1.1 and Definition 3.1.5).

Once $Ke(F_\bullet, a)$ is defined, then, the construction follows naturally. We apply it to a double complex $F_\bullet$ in degrees 1 and 0 representing $(R\pi_* \mathcal{L})^\vee$ in the derived category of $\mathcal{M}_{g,n}(r, \mathbb{k})$. Indeed, $F_\bullet$ is equipped with a closed and nondegenerate form $a$ defined using $L_0 \to \omega$ (Section 4).

For a vector bundle $V$, the total lambda class $\lambda_{-1}$ is related to the top Chern class in the Chow ring by

\[(1.5.1) \quad c_{\text{top}}(V) = \text{ch}(\lambda_{-1}(V^\vee)) \cdot \text{td}(V).\]

In Definition 3.5.1 and Definition 4.0.8 we define a Chow cohomology class $c_W$ by

\[(1.5.2) \quad c_W = \text{ch}(Ke(F_\bullet, a)) \frac{\text{td}(F_1^\vee)}{\text{td}(F_0^\vee)}.\]

In Theorem 5.4.1 we prove the identity $c_W = c_{PV}$.

It is also worth mentioning that our construction produces a $K$-class $K_W$ that lifts $c_W$ to the $K$-theory ring, Definition 4.0.8. We hope that such a lifting can not only clarify the definition of the Witten top Chern class, but also improve our understanding of the conjecture.

1.6. Structure of the paper. In Section 2 we introduce our notation. In Section 3 we define the $K$-theory Euler class. We provide our construction for the Witten top Chern class in Section 4 and prove the identity with the Polishchuk–Vaintrob class in Section 5. Finally, in Section 6 we give some explicit examples.

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2. Notation

2.1. Schemes. All schemes are of finite type over $\mathbb{C}$. 

2.2. Fibres. Let \( p: E \to X \) be a complex vector bundle on a scheme; for \( x \in X \) we denote by \( E_x \) the fibre \( p^{-1}(x) \) over \( x \in X \). For a morphism of vector bundles \( f: E \to F \), we denote by \( f_x: E_x \to F_x \) the morphism induced on the fibre over \( x \in X \). For coherent locally free sheaves \( K \) on \( X \) we denote by \( K_x \) the fibre of the corresponding vector bundle.

2.3. \( K \)-theory. We denote by \( \text{Vect}(X) \) the category of finite complex vector bundles on \( X \). We denote by \( K_0(X) \) and \( K'_0(X) \) the Grothendieck groups generated by coherent locally free sheaves and coherent sheaves on \( X \).

2.4. Symmetric and exterior product homomorphisms. Let \( E \) be a coherent locally free sheaf on a scheme \( X \). For any nonnegative integers \( h \) and \( k \), the homomorphisms
\[
(2.4.1) \quad \sigma_n: \text{Sym}^h E \otimes \text{Sym}^n E \to \text{Sym}^{h+n} E \quad \text{and} \quad \lambda_n: \Lambda^h E \otimes \Lambda^n E \to \Lambda^{h+n} E
\]
are the natural symmetric product and exterior product.

2.5. Symmetric and exterior powers of a complex. In this paper we work with complexes in degree 0 and 1; in this case, the symmetric and the exterior power in the derived category can be realized by bounded cohomological and homological Koszul complexes. Following [Gre89, §1], we write the complexes explicitly.

First, for any coherent locally free sheaf \( E \) on \( X \) and for any nonnegative integer \( h \), denote by
\[
s: \text{Sym}^h E \to \text{Sym}^{h-1} E \otimes E \quad \text{and} \quad l: \Lambda^h E \to \Lambda^{h-1} E \otimes E,
\]
the dual homomorphisms of the products \( \sigma_1 \) and \( \lambda_1 \) applied to \( E^\vee \). This allows us to define, for any \( d: V^0 \to V^1 \), the homomorphisms
\[
(2.5.1) \quad \text{Sym}^h V^0 \otimes \Lambda^k V^1 \to \text{Sym}^{h-1} V^0 \otimes V^0 \otimes \Lambda^k V^1
\]
\[
\to \text{Sym}^{h-1} V^0 \otimes V^1 \otimes \Lambda^k V^1 \to \text{Sym}^{h-1} V^0 \otimes \Lambda^{k+1} V^1,
\]
where the first homomorphism is \( s \) tensored with the identity on \( \Lambda^k V^1 \), the second homomorphism is the identity on \( \text{Sym}^{h-1} V^0 \) and \( \Lambda^k V^1 \) tensored with \( d \), and the third homomorphism is the identity on \( \text{Sym}^{h-1} V^0 \) tensored with the exterior product \( \lambda_1 \). In this way, for any complex of coherent locally free sheaves \( V^\bullet: 0 \to V^0 \to V^1 \to 0 \) we get the symmetric power complex \( \text{Sym}^N(V^\bullet) \)
\[
(2.5.2) \quad 0 \to \text{Sym}^N V^0 \to \text{Sym}^{N-1} V^0 \otimes V^1 \to \text{Sym}^{N-2} V^0 \otimes \Lambda^2 V^1 \to \cdots
\]
\[
\cdots \to \text{Sym}^2 V^0 \otimes \Lambda^{N-2} V^1 \to V^0 \otimes \Lambda^{N-1} V^1 \to \Lambda^N V^1 \to 0.
\]
Similarly, for any complex $W_\bullet : 0 \to W_1 \to W_0 \to 0$ of coherent locally free sheaves on $X$. We can define

$$\text{(2.5.3)} \quad \text{Sym}^h W_0 \otimes \Lambda^k W_1 \to \text{Sym}^h W_0 \otimes W_1 \otimes \Lambda^{k-1} W_1$$

$$\to \text{Sym}^h W_0 \otimes W_0 \otimes \Lambda^{k-1} W_1 \to \text{Sym}^{h+1} W_0 \otimes \Lambda^{k-1} W_1$$

as the composite $(\text{id} \otimes l) \circ (\text{id} \otimes d \otimes \text{id}) \circ (\sigma_1 \otimes \text{id})$. The exterior power complex $\Lambda^N(W_\bullet)$ is the complex

$$\text{(2.5.4)} \quad 0 \to \Lambda^N W_1 \to W_0 \otimes \Lambda^{N-1} W_1 \to \text{Sym}^2 W_0 \otimes \Lambda^{N-2} W_1 \to \cdots$$

$$\cdots \to \text{Sym}^{N-2} W_0 \otimes \Lambda^2 W_1 \to \text{Sym}^{N-1} W_0 \otimes W_1 \to \text{Sym}^N W_0 \to 0.$$

In fact, in characteristic 0, $\text{Sym}^N$ and $\Lambda^N$ are well defined functors up to homotopy [Del73]. We denote by $\text{Sym}^N(f)$ and $\Lambda^N(f)$ the homomorphism of complexes associated to a homomorphism $f$ of complexes in degrees 1 and 0.

2.5.5. **Proposition.** Let $d : W_1 \to W_0$ be a homomorphism of complex vector spaces, and let $H_1$ and $H_0$ be its the kernel and its cokernel. The cohomology of $\Lambda^N(W_\bullet)$ is given by $\text{Sym}^h H_0 \otimes \Lambda^{N-h} H_1$ for $0 \leq h \leq N$.

**Proof.** Let $V$ be the image of $d$. The complex $W_\bullet$ is the sum of $H_1 \xrightarrow{0} H_0$ and $V \xrightarrow{\text{id}} V$. Therefore, we get

$$\Lambda^N(W_\bullet) = \sum_{i=0}^{N} \Lambda^{N-i}(H_\bullet) \otimes \Lambda^i(V \xrightarrow{\text{id}} V).$$

Note that $\Lambda^i(V \xrightarrow{\text{id}} V)$ is exact for $i > 0$. The claim follows. \qed

3. **The $K$-theory Euler class**

The total lambda class evaluated at $-1$ is defined over vector bundles up to isomorphism:

$$\lambda_{-1} : \text{Vect}(X) \to K_0(X)$$

$$V \mapsto \sum_i (-1)^i [\Lambda^i V].$$

We now extend its definition. We choose an integer

$$m \geq 1.$$

We work with complexes $F_\bullet = (0 \to F_1 \to F_0 \to 0)$ of coherent locally free sheaves $F_i$ on the scheme $X$. Denote by $H_{1,x}$ and $H_{0,x}$ the kernel and the cokernel of $(F_\bullet)_x$. 
3.1. **Closed and nondegenerate forms.** Consider a homomorphism

\[ a : \mathcal{O}_X \to \text{Sym}^m F_0 \otimes F_1. \]

3.1.1. **Definition.** The form \( a \) is **closed** if \( d \circ a = 0 \) where \( d \) is the natural map

\[ d : \text{Sym}^m F_0 \otimes F_1 \to \text{Sym}^{m+1} F_0. \]

3.1.2. **Remark.** Consider the complex \( \Lambda^{m+1}(F_\bullet) \)

\[
\begin{align*}
0 \to & \Lambda^m F_1 \to F_0 \otimes \Lambda^m F_1 \to \text{Sym}^2 F_0 \otimes \Lambda^{m-1} F_1 \to \cdots \\
& \cdots \to \text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1 \to \text{Sym}^m F_0 \otimes F_1 \to \text{Sym}^{m+1} F_0 \to 0,
\end{align*}
\]

introduced in (2.5.4). Note that the form \( a \) is closed if and only if \( d(a(1)) \) vanishes.

By Proposition 2.5.5, the cohomology of the complex \( \Lambda^{m+1}(F_\bullet)_x \) at \((\text{Sym}^m F_0 \otimes F_1)_x\) is \( \text{Sym}^m H_{0,x} \otimes H_{1,x} \). For any \( x \in X \), if \( a \) is closed, \( a(1) \) induces an element in \( \text{Sym}^m H_{0,x} \otimes H_{1,x} \), which can be regarded as a linear system

\[
S_x : H^1_{1,x} \to \text{Sym}^m H_{0,x} = H^0(Y, \mathcal{O}_Y(m))
\]

with \( Y = \mathbb{P}H^1_{0,x} \).

3.1.5. **Definition.** A closed form \( a \) is **nondegenerate** if for any \( x \in X \) the image of \( S_x \) is a base point free linear system on \( \mathbb{P}H^1_{0,x} \).

3.1.6. **Remark.** Note that this condition implies \( \text{rk}(H_{0,x}) \leq \text{rk}(H_{1,x}) \) and, therefore, \( \text{rk}(F_0) \leq \text{rk}(F_1) \).

3.1.7. **Remark.** Note that a closed form \( a \) is nondegenerate if and only if for any \( x \in X \) the following condition is satisfied: an element \( v \in H^1_{0,x} \) is zero if

\[
\forall w \in H^1_{1,x}, \quad \langle S_x(w), v^m \rangle = 0.
\]

3.2. **The double complex \( L^{\bullet \bullet} \).** From now on we always assume that \( a \) is a closed and nondegenerate form. We write

\[
L^{h,k} = \text{Sym}^h F_0 \otimes \Lambda^k F_1.
\]

We define homomorphisms

\[
\tilde{d} : L^{h,k} \to L^{h+1,k-1},
\]

\[
\tilde{a} : L^{h,k} \to L^{h+m,k+1}
\]

of bidegrees \((1, -1)\) and \((m, 1)\). The homomorphism \( \tilde{d} \) is defined as the differential of the complex \( \Lambda^N(F_\bullet) \) at (2.5.1) with \( N = h + k \). Consider the natural homomorphisms described in (2.4.1)

\[
\sigma_m : \text{Sym}^h F_0 \otimes \text{Sym}^m F_0 \to \text{Sym}^{h+m} F_0, \quad \lambda_1 : F_1 \otimes \Lambda^k F_1 \to \Lambda^{k+1} F_1.
\]
The homomorphism $\tilde{a}$ is the composite of $\text{id} \otimes a \otimes \text{id}$ and $\sigma_m \otimes \lambda_l$

(3.2.2) $\text{Sym}^h F_0 \otimes \Lambda^k F_1 \xrightarrow{\text{id} \otimes a \otimes \text{id}} \text{Sym}^h F_0 \otimes \text{Sym}^m F_0 \otimes F_1 \otimes \Lambda^k F_1$

$\sigma_m \otimes \lambda_l \to \text{Sym}^{h+m} F_0 \otimes \Lambda^{h+1} F_1$.

3.2.3. Lemma. The bigraded sheaf $L^\bullet \bullet$ is a double complex with differentials $d$ and $\tilde{a}$.

Proof. We show $\tilde{d} \circ d = a \circ a = d \circ a + a \circ d = 0$. The homomorphism $\tilde{d}$ is the differential of the exterior power of $F_\bullet$; therefore $\tilde{d} \circ d = 0$. Furthermore, a local description of $a$ at $x \in X$ shows $a \circ a = 0$. We choose an open $U \ni x$ such that

$$a(1) = \sum_{l=0}^N P_l \otimes \sigma_l \in \text{Sym}^m F_0(U) \otimes F_1(U)$$

with $P_l \in \text{Sym}^m F_0(U)$ and $\sigma_l \in F_1(U)$. Then, $\tilde{a}_x : L_{x}^{h,k} \to L_{x}^{h+m,k+1}$ can be written, for $c_i \in (F_0)_x$ and $b_j \in (F_1)_x$, as the homomorphism sending $\prod_{1 \leq i \leq h} c_i \otimes \bigwedge_{1 \leq j \leq k} b_j \in L_{x}^{h,k}$ to

(3.2.4) $\sum_{l=0}^N \prod_{1 \leq i \leq h} c_i \cdot P_i(x) \otimes \sigma_l(x) \wedge \bigwedge_{1 \leq j \leq k} b_j$.

Using $\sigma_l \wedge \sigma_{l'} + \sigma_{l'} \wedge \sigma_l = 0$ we see that $\tilde{a}$ is a differential.

Finally, we show $d \circ a + a \circ d = 0$. We point out that $\tilde{d}_x : L_{x}^{h,k} \to L_{x}^{h+1,k-1}$ sends $\prod_{1 \leq i \leq h} c_i \otimes \bigwedge_{0 \leq j \leq k} b_j$ to

(3.2.5) $\sum_{j_0=1}^k (-1)^{j_0} \left( \prod_{i} c_i \cdot d_x(b_{j_0}) \otimes \bigwedge_{j \neq j_0} b_j \right)$.

with $d_x = (d_{F_\bullet})_x$, $c_i \in (F_0)_x$, and $b_j \in (F_1)_x$. Indeed, as illustrated in Section 2.2, the homomorphism $d_x$ is the composite

(3.2.6) $\text{Sym}^h (F_{0,x}) \otimes \Lambda^k (F_{1,x}) \to \text{Sym}^h (F_{0,x}) \otimes F_{1,x} \otimes \Lambda^{k-1} (F_{1,x}) \to$

$\to \text{Sym}^h (F_{0,x}) \otimes F_{0,x} \otimes \Lambda^{k-1} (F_{1,x}) \to \text{Sym}^{h+1} (F_{0,x}) \otimes \Lambda^{k-1} (F_{1,x})$,

where the first arrow denotes the identity homomorphism tensored by $l_x : \Lambda^{k} (F_{1,x}) \to F_{1,x} \otimes \Lambda^{k-1} (F_{1,x})$, which we write as

$$\bigwedge_{0 \leq j \leq k} b_j \mapsto \sum_{j_0=1}^k (-1)^{j_0} \left( b_{j_0} \otimes \bigwedge_{j \neq j_0} b_j \right).$$

Then the definition given in (3.2.4) implies (3.2.5). As a consequence of (3.2.5) and (3.2.4), the fact that $\tilde{d}$ and $\tilde{a}$ anticommute is just another way to say that $a$ is closed. $\square$

3.3. The complexes $K^\bullet \bullet$. For each $i = 0, \ldots, m$, set

$$K^{p,q}_i = \text{Sym}^{p+mq-i} F_0 \otimes \Lambda^{-p+q} F_1$$

with differentials of bidegrees $(1,0)$ and $(0,1)$

$$\tilde{d} : K^{p,q}_i \to K^{p+1,q}_i \quad \text{and} \quad \tilde{a} : K^{p,q}_i \to K^{p,q+1}_i.$$
This diagram illustrates $K_0^{\bullet \bullet}$.

\[
\begin{array}{c}
\cdots \rightarrow \text{Sym}^{3m+1} F_0 \otimes \Lambda^2 F_1 \rightarrow \cdots \\
\tilde{d} \quad \tilde{d} \quad \tilde{d} \\
\text{Sym}^{2m-1} F_0 \otimes \Lambda^3 F_1 \rightarrow \text{Sym}^{2m} F_0 \otimes \Lambda^2 F_1 \rightarrow \text{Sym}^{2m+1} F_0 \otimes F_1 \\
\text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1 \rightarrow \text{Sym}^m F_0 \otimes F_1 \rightarrow \text{Sym}^{m+1} F_0 \\
0 \rightarrow \mathcal{O}_X \rightarrow 0 \rightarrow 0
\end{array}
\]

Fig. 1 illustrates that $K_i^{\bullet \bullet}$ is nonzero only if $0 \leq -p + q \leq \text{rk}(F_1)$ and $p + mq \geq 0$.

**Figure 1.** the double complex $K_i^{\bullet \bullet}$.

Note that the direct sum of the double complexes $K_i^{\bullet \bullet}$ is equal to $L^{\bullet \bullet}$.

\[
\bigoplus_{i=0}^{m} K_i^{\bullet \bullet} = L^{\bullet \bullet}
\]

For any $i = 0, \ldots, m$, we denote by $(K_i^{\bullet}, D = \tilde{d} + \tilde{a})$ the total complex.
3.3.1. **Theorem.** There exists an integer $n_0$ such that, for $n \geq n_0$, the cohomology group $H^n(K^\bullet_\tau)$ vanishes and the sheaf $I = \text{im}(K^{n-1}_i \to K^n_i)$ is locally free.

**Proof.** We write $K^\bullet$ and $K^\bullet$ omitting $i$. We define

\begin{equation}
(3.3.2) \quad n_0 = (4m + 2 \text{rk}(\text{Sym}^m F_0))/(m + 1)
\end{equation}

and show the exactness of $K^{n-1} \to K^n \to K^{n+1}$ for $n \geq n_0$.

Since all the sheaves of $K^\bullet$ are coherent and locally free, it suffices to check the exactness after base change of $K^\bullet$ to $x$ for all closed points $x \in X$. The spectral sequence of complex vector spaces $E^{p,q}_2 \cong H^p_d(H^q_d(K^\bullet_\tau))$ abuts to the cohomology of $(K^\bullet_\tau, D_x)$. We show $E^{p,q}_2 = 0$ for $p + q \geq n_0$.

Write $F_{1,x} \to F_{0,x}$ as the sum of $H_{1,x} \xrightarrow{0} H_{0,x}$ and $V \xrightarrow{\text{id}} V$. In this way, each term $K^\bullet_{p,q}$ can be written as the sum of the complex vector spaces $\text{Sym}^r H_{0,x} \otimes \Lambda^s H_{1,x} \otimes \text{Sym}^{p+mq-i-r}V \otimes \Lambda^{-p+q-i}V$ for $r \in \{0, \ldots, p + mq - i\}$ and $s \in \{0, \ldots, -p + q\}$. Denote by $G^{p,q}$ the summand corresponding to $r = 0$ and $s = 0$ and denote by $Q^{p,q}$ the sum of the remaining terms in $K^\bullet_{p,q}$. In this way, each element $\alpha \in K^\bullet_{p,q}$ is given by $\alpha' \in G^{p,q}$ and $\alpha'' \in Q^{p,q}$. The morphism $a_x : C_x \to \text{Sym}^m F_{0,x} \otimes F_{1,x} = K^{0,1}$ splits as $(a'_x \in G^{0,1}, a''_x \in Q^{0,1})$.

The differentials $\overrightarrow{d}_x$ and $\overrightarrow{a}_x$ can be written as

\begin{align}
(3.3.3) & \quad G^{p,q} \oplus Q^{p,q} \to G^{p+1,q} \oplus Q^{p+1,q} & \overrightarrow{d}_x &= \begin{pmatrix} 0 & 0 \\ 0 & d'' \end{pmatrix}, \\
(3.3.4) & \quad G^{p,q} \oplus Q^{p,q} \to G^{p,q+1} \oplus Q^{p,q+1} & \overrightarrow{a}_x &= \begin{pmatrix} 0 \pi' \\ \pi'' & 0 \end{pmatrix},
\end{align}

where, by construction, $d''$ is exact, and we have $d'' \circ \overrightarrow{a} + \overrightarrow{a} \circ d'' = 0$ and $d'' \circ \overrightarrow{a} = 0$, as a consequence of $\overrightarrow{d}_x \circ \overrightarrow{a}_x + \overrightarrow{a}_x \circ \overrightarrow{d}_x = 0$, (Lemma 3.3.3).

Now, note that $H^d_{K^\bullet_\tau}(K^\bullet_\tau) = G^{p,q}$, by the exactness of $d''$. Furthermore, the vertical cohomology $H^d_{K^\bullet_\tau}(H^\bullet_d(K^\bullet_\tau))$ is the cohomology of the complex $G^\bullet$ where the differential $\alpha'$ is induced by $a'_x : C_x \to G^{0,1} = \text{Sym}^m H_{0,x} \otimes H_{1,x}$ as follows

\begin{equation}
(3.3.5) \quad \text{Sym}^h H_{0,x} \otimes \Lambda^k H_{1,x} \xrightarrow{\text{id} \otimes a'_x \otimes \text{id}} \text{Sym}^h H_{0,x} \otimes G^{0,1} \otimes \Lambda^k H_{1,x} = \text{Sym}^h H_{0,x} \otimes \text{Sym}^m H_{0,x} \otimes H_{1,x} \otimes \Lambda^k H_{1,x}
\end{equation}

Indeed, by $3.3.4$, we have $\overrightarrow{a}_x(\alpha', \alpha'') = (\overrightarrow{a'}(\alpha'), \overrightarrow{a}''(\alpha') + \overrightarrow{a}''(\alpha''))$. By the exactness of $d''$, the passage to cohomology with respect to $\overrightarrow{d}_x$ induces the homomorphism $\overrightarrow{\pi}$ of 3.3.5 on the cohomology groups $H^d_{K^\bullet_\tau}(K^\bullet_\tau)$. So,

\[ H^{d}_{K^\bullet_\tau}(K^\bullet_\tau) \to H^{d}_{K^\bullet_\tau}(K^\bullet_\tau) \to H^{d}_{K^\bullet_\tau}(K^\bullet_\tau), \]
is the cohomological Koszul complex associated to \((H_{1,x})' \rightarrow \text{Sym}^m H_{0,x}:

\begin{equation}
\text{Sym}^{h-m} H_{0,x} \otimes \Lambda^{k-1} H_{1,x} \rightarrow \text{Sym}^h H_{0,x} \otimes \Lambda^k H_{1,x}
\rightarrow \text{Sym}^{h+m} H_{0,x} \otimes \Lambda^{k+1} H_{1,x},
\end{equation}

for \(h = p + mq - i\) and \(k = -p + q\). After tensoring by \(\det(H_{1,x})'\), we obtain the homological Koszul complex

\begin{equation}
\text{Sym}^{w-m} H_{0,x} \otimes \Lambda^{v+1} H_{1,x} \rightarrow \text{Sym}^w H_{0,x} \otimes \Lambda^v H_{1,x}
\rightarrow \text{Sym}^{w+m} H_{0,x} \otimes \Lambda^{v-1} H_{1,x},
\end{equation}

with \(v = \text{rk}(H_{1,x}) - q + p\) and \(w = p + mq - i\).

We illustrate that the homomorphism of (3.3.8) can be regarded as the homological Koszul complex associated to the linear system

\begin{equation}
S_z: H_{1,x}^\vee \rightarrow \text{Sym}^m H_{0,x} = H^0(Y, \mathcal{O}_Y(m)),
\end{equation}

where \(Y = \mathbb{P} H_{0,x}'\). Indeed, denote by \(J_z\) the image of \(H_{1,x}^\vee\) via \(S_z\); recall that \(J_z\) is a base point free linear system because \(a\) is nondegenerate. Write \(H_{1,x}^\vee = N_z \oplus J_z\), where \(N_z\) is the kernel of \(S_z\). The composite homomorphism

\begin{equation}
\text{Sym}^{w-m} H_{0,x} \otimes \Lambda^{v-t+1} J_z \rightarrow \text{Sym}^w H_{0,x} \otimes \Lambda^{v-t} J_z
\rightarrow \text{Sym}^{w+m} H_{0,x} \otimes \Lambda^{v-t-1} J_z
\end{equation}

tensored by \(\Lambda^t N_z\).

In [Gre88], Green treats the case of the homological Koszul complexes induced by a base point free linear system. By [Gre88 Thm. 2], the sequence (3.3.7) is exact when \(J_z\) is base point free and the condition

\[ w \geq (v-t) + m + \dim(J_z) = v-t + m + \text{rk}(\text{Sym}^m H_{0,x}) - \text{rk}(J_z) \]

is satisfied. The middle term of (3.3.9) is nonzero only if \(v-t - \text{rk}(J_z) \leq 0\). Therefore, (3.3.9) is exact for any \(t\) if \(w \geq m + \text{rk}(\text{Sym}^m H_{0,x})\).

Recall that \(K^{p,q} \neq 0\) only if \(q-p \geq 0\); therefore, we restrict to the indexes \((p, q)\) satisfying \(q-p \geq 0\). In this way, we have \((m-1)(q-p) \geq 0\). Furthermore, we have \((m+1)(p+q) \geq (m+1)n_0\), which follows from \(p+q \geq n_0\). Summing up, we get \(2p+2mq \geq (m+1)n_0\) and, by (3.3.2), \(2p+2mq \geq 4m+2\text{rk}(\text{Sym}^m F_0)\), which implies

\[ p + mq - m \geq m + \text{rk}(\text{Sym}^m F_0). \]

We obtain

\[ w = p + mq - i \geq p + mq - m \geq m + \text{rk}(\text{Sym}^m F_0) \geq m + \text{rk}(\text{Sym}^m H_{0,x}); \]

therefore, (3.3.9) is exact and \(E^p_{2,q} = 0\) vanishes.

Finally, since \(H^n((K^*_x)_{\geq 0})\) vanishes for every \(x \in X\) and for \(n \geq n_0\), we have, for all \(n \geq n_0\),

\[ \ker(D_n)_x = \text{im}(D_{n-1})_x, \]
where \((D_n)_x\) is the homomorphism induced by the total differential \(D_n: K^n \to K^{n+1}\) on the fibre over \(x\). In fact, the rank of \(\text{ker}(D_n)_x\) is equal to the rank of \(\text{im}(D_{n-1})_x\) and is constant, because it is upper and lower semi-continuous. Therefore, the sheaf \(I = \text{im}(K^{n-1} \to K^n)\) is locally free. □

3.4. The \(K\)-class. By Theorem 3.3.1 the alternate sum of the cohomology sheaves of \(K^\bullet_i\) is a well defined class in the \(K\)-theory group \(K_0(X)\). The fact that \(I = \text{im}(K^{n-1} \to K^n)\) is locally free for \(n \geq n_0\) allows us to define the \(K\)-class
\[
\text{cl}(K^\bullet_i) = \sum_{n<n_0} (-1)^n [K^n_i] + (-1)^{n_0} [I] \in K_0(X).
\]
which lifts the natural class \(\sum_{n} (-1)^n [H^n(K^\bullet_i)] \in K_0(X)\).

3.4.1. Definition. Let \(F^\bullet\) be a complex in degrees 1 and 0 with a closed and nondegenerate form \(a\). Then, the \(K\)-theory Euler class of \((F^\bullet, a)\) is
\[
\text{Ke}(F^\bullet, a) = \sum_{0 \leq i \leq m} \text{cl}(K^\bullet_i).
\]

3.4.2. Proposition. Let \(V\) be a vector bundle of finite rank on \(X\). Consider the complex \(V \to 0\) in degrees 1 and 0. A closed and nondegenerate form \(a\) is necessarily the zero homomorphism \(O \to 0\). The \(K\)-theory Euler class extends the total lambda class evaluated at \(-1\) in the sense that
\[
(3.4.3) \quad \text{Ke}(V \to 0, a = 0) = \lambda_{-1}(V).
\]

Proof. By Definition 3.4.1 we have
\[
\text{Ke}(V \to 0, 0) = \sum_{0 \leq i \leq m} \text{cl}(K^\bullet_i) = \sum_{0 \leq i \leq m} \sum_{p, q \geq 0} \sum_{p+mq = i} (-1)^{q-p} [\Lambda^{q-p}V] = \sum_{0 \leq i \leq m} \sum_{q \geq 1} (-1)^{i} [\Lambda^{i}V].
\]

3.5. The cohomology class. We define the cohomological realization of \(\text{Ke}(F^\bullet, a)\) by analogy with 1.5.1.

3.5.1. Definition. Let \(E^0 \to E^1\) be a complex of coherent and locally free sheaves in degrees 0 and 1, and \(b: \text{Sym}^m E^0 \otimes E^1 \to \mathcal{O}_X\) a homomorphism such that \(b^\vee\) is a closed and nondegenerate form for \((E^\bullet)^\vee\). Then,
\[
c_{\text{top}}(E^\bullet, b) = \text{ch}(\text{Ke}((E^\bullet)^\vee, b^\vee)) \cdot \text{td}(E^1) / \text{td}(E^0)
\]
is the top Chern class of \((E^\bullet, b)\) in the rational Chow ring \(A^*(X)_\mathbb{Q}\).

3.5.2. Remark. Clearly, by Proposition 3.4.2 and 1.5.1, we have
\[
c_{\text{top}}(0 \to V, b = 0) = c_{\text{top}}(V).
\]
3.6. Dependence on $F_\bullet$ and $a$. We investigate how $Ke$ depends on the complex $F_\bullet$ and on the form $a$.

3.6.1. Theorem. The class $Ke(F_\bullet, a)$ depends on $F_\bullet$ up to quasiisomorphism and on the form $a$ up to homotopy.

Proof. We prove the claim in two lemmata.

3.6.2. Lemma. Let $a$ and $a'$ be two closed and nondegenerate forms $O_X \to \text{Sym}^m F_0 \otimes F_1$ defining homotopic homomorphisms of complexes:

\[
\cdots \to \text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1 \to \text{Sym}^m F_0 \otimes F_1 \to \text{Sym}^{m+1} F_0 \to 0
\]

Then, for any $i = 0, \ldots, m$, the total complex $(K^i \cdot, D = \tilde{d} + \tilde{a})$ and the total complex $(K^i \cdot, D' = \tilde{d} + \tilde{a}')$ are isomorphic. In particular, in K-theory, we have the identity $Ke(F_\bullet, a) = Ke(F_\bullet, a')$.

Proof. Let $h$ be the homotopy $O_X \to \text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1$ making the diagram

\[
\begin{array}{ccc}
\text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1 & \xrightarrow{d} & \text{Sym}^m F_0 \otimes F_1 \\
\downarrow{h} & & \downarrow{a-a'} \\
O_X & & 0
\end{array}
\]

commutative:

\[
d \circ h = a - a'.
\]

In fact, $h$ allows us to define a map $\tilde{h}: K^{p,q}_i \to K^{p-1,q+1}_i$ as the natural composite homomorphism

\[
\text{Sym}^{p+mq-1} F_0 \otimes O_X \otimes \Lambda^{q-p} F_1
\]

\[
\to \text{Sym}^{p+mq-1} F_0 \otimes \text{Sym}^{m-1} F_0 \otimes \Lambda^2 F_1 \otimes \Lambda^{q-p} F_1
\]

\[
\to \text{Sym}^{p-1+mq+1-i} F_0 \otimes \Lambda^{q-p+2} F_1.
\]

More explicitly, if we write $h$ locally as $\sum H_i \otimes \sigma^+_i \wedge \sigma^-_i$, the homomorphism $\tilde{h}$ sends $\prod_{1 \leq i \leq h} c_i \otimes \bigwedge_{1 \leq j \leq k} b_j$ to

\[
\sum \prod_{1 \leq i \leq h} c_i \cdot H_i \otimes \sigma^+_i \wedge \sigma^-_i \wedge \bigwedge_{1 \leq j \leq k} b_j.
\]

(3.6.4)

Now, by means of $\tilde{h}$ we define an isomorphism between the two double complexes. First, we need to show two relations: (3.6.5) and (3.6.7).

By (3.6.3), we have immediately

\[
\tilde{a} - \tilde{a}' = \tilde{d} \circ \tilde{h} - \tilde{h} \circ \tilde{d}.
\]

(3.6.5)
Furthermore, we have
\[(3.6.6) \quad \tilde{d} \circ \tilde{h} \circ \tilde{h} = -\tilde{h} \circ \tilde{h} \circ \tilde{d} + 2\tilde{h} \circ \tilde{d} \circ \tilde{h}\]
as a straightforward consequence of \(\tilde{d}(\tilde{h}(h)) = 2\tilde{h}(\tilde{d}(h))\), which we can verify by means of the local presentation of \(\tilde{h}\) given above. Indeed, apply \(\tilde{h}\) to
\[
\tilde{d}(h) = \sum_i H_i \cdot d(\sigma_i^+) \otimes \sigma_i^- - \sum_i H_i \cdot \sigma_i^- \otimes d(\sigma_i^+)
\]
and compare with the image via \(\tilde{d}\) of
\[
\tilde{h}(h) = 2 \sum_{l,q} H_l \cdot H_q \otimes \sigma_l^+ \wedge \sigma_l^- \wedge \sigma_q^+ \wedge \sigma_q^-.
\]

Using (3.6.6), we prove by induction the following equation for \(n \geq 0\)
\[(3.6.7) \quad n(\tilde{d} \circ \tilde{h}^{n+1}) = -\tilde{h}^{n+1} \circ \tilde{d} + (n+1)(\tilde{h} \circ \tilde{d} \circ \tilde{h}^n),
\]
where \(\tilde{h}^n\) is the composition of \(\tilde{h}\) iterated \(n\) times. First, applying the equation for \(n - 1\), we see that the right hand side equals
\[
(-\tilde{h}^{n+1} \circ \tilde{d} + \tilde{h} \circ \tilde{d} \circ \tilde{h}^n) + n(\tilde{h} \circ \tilde{d} \circ \tilde{h}^{n-1}) \circ \tilde{h} = (-\tilde{h}^{n+1} \circ \tilde{d} + \tilde{h} \circ \tilde{d} \circ \tilde{h}^n) + (\tilde{h}^n \circ \tilde{d} + (n-1)(\tilde{d} \circ \tilde{h}^n)) \circ \tilde{h}.
\]
By (3.6.6), for all integers \(0 \leq i \leq n+1\), we have \(-\tilde{h}^i \circ \tilde{d} \circ \tilde{h}^{n+1-i} + \tilde{h}^{i+1} \circ \tilde{d} \circ \tilde{h}^{n-i} = -\tilde{h}^{i+1} \circ \tilde{d} \circ \tilde{h}^{n-i} + \tilde{h}^{i+2} \circ \tilde{d} \circ \tilde{h}^{n-i-1}\), which implies (3.6.7) by iterated application.

Now we introduce a homomorphism of total complexes:
\[
e^{-\tilde{h}} := \sum_{j \geq 0} (-\tilde{h})^j / j!.
\]
Since \(\tilde{h}\) respects the total grading, it is an endomorphism of \(K^n\). It is well defined (in characteristic 0) because the summands above vanish as soon as \(j > (\text{rk} F_1)/2\) (this immediately follows from (3.6.4)). For all \(n\), \(e^{-\tilde{h}}\) is in fact an automorphism of \(K^n\), because \(e^\tilde{h}\) is also well defined and is the inverse of \(e^{-\tilde{h}}\). Finally, putting together (3.6.5) and (3.6.7), we prove
\[
e^{-\tilde{h}} \circ D = D' \circ e^{-\tilde{h}}.
\]
Indeed, by (3.6.5), we have \(D' = \tilde{d} + \tilde{a} + \tilde{d} \circ \tilde{h} - \tilde{h} \circ \tilde{d}\), and it is enough to show
\[
e^{-\tilde{h}} \circ \tilde{d} = \tilde{d} \circ e^{-\tilde{h}} + \tilde{d} \circ \tilde{h} \circ e^{-\tilde{h}} + \tilde{h} \circ \tilde{d} \circ e^{-\tilde{h}},
\]
because \(\tilde{h}\) and \(\tilde{a}\) clearly commute. Indeed, in the equation above, the terms of degree \(n\) in \(\tilde{h}\) yield precisely the equation (3.6.7). □
3.6.8. Lemma. Let \((F_\bullet,a)\) and \((\Phi_\bullet,\alpha)\) be two pairs formed by a complex and a closed and nondegenerate form. Let \(f : \Phi_\bullet \to F_\bullet\) be a quasiisomorphism with
\[(3.6.9) \quad a = \alpha \circ \Lambda^{m+1}(f)\]
Let \(\Gamma_i^{*\cdot}\) and \(K_i^{*\cdot}\) be the double complexes associated to \((\Phi_i,\alpha)\) and \((F_\bullet,a)\). Then, the total complexes \(K_i^\bullet\) and \(\Gamma_i^\bullet\) are quasiisomorphic. In particular, we have \(\text{Ke}(F_\bullet,a) = \text{Ke}(\Phi_\bullet,\alpha)\).

**Proof.** We denote by \(\widetilde{a}\) and \(\widetilde{d}\) the differentials of \(K_i^\bullet\) as in Section 3.3. We denote by \(\tilde{\delta}\) and \(\tilde{\alpha}\) the analog differentials of bidegrees \((1,0)\) and \((0,1)\) of \(\Gamma_i^{*\cdot}\).

Define \(\widetilde{f}: \Gamma_i^{*\cdot} \to K_i^{*\cdot}\) by the equation
\[\widetilde{f} = \Lambda^{m+q-p-i}(f) : \text{Sym}^{p+m-q-i}\Phi_0 \otimes \Lambda^{-p+q}\Phi_1 \to \text{Sym}^{p+m-q-i}F_0 \otimes \Lambda^{-p+q}F_1.\]

Note that \(\widetilde{f}\) satisfies \(\widetilde{f} \circ \tilde{\delta} = \tilde{d} \circ \widetilde{f}\) and, by \[(3.6.9), \quad \tilde{f} \circ \tilde{\alpha} = \tilde{\alpha} \circ \widetilde{f};\] therefore, \(\tilde{f}\) is a homomorphism of double complexes which respects the filtrations \(\sum_{q \geq t} K_i^{p,q}\) and \(\sum_{q \geq t} \Gamma_i^{p,q}\).

Note that \(\tilde{f}\) is a morphism of filtered differential graded modules and induces homomorphisms of bigraded modules \(\phi_l : \text{Sym}^{l,q}(K_i^{*\cdot}) \to \text{Sym}^{l,q}(\Gamma_i^{*\cdot})\) for \(l \geq 0\). Note that \(\phi_l\) is an isomorphism, because \(\Lambda^N(f) : \Lambda^N(\Phi_\bullet) \to \Lambda^N(F_\bullet)\) is a quasiisomorphism. Then, by [McC01 §3.1, Thm. 3.2], \(\phi_l\) is an isomorphism for every \(l \geq 1\).

The theorem follows from Lemma 3.6.2 and Lemma 3.6.8. 

\[\square\]

3.7. Passage to the derived category. In the next section we consider a complex \(E^\bullet\) in degrees 0 and 1 of coherent locally free sheaves on \(X\) and a morphism in the derived category \(\mathcal{D}^b(X)\)
\[\tau : \text{Sym}^{m+1}(E^\bullet) \to O_X[-1]\]
inducing a base point free linear system \(S_x : H^1(E^\bullet_x) \to \text{Sym}^m H^0(E^\bullet_x)^\vee\) for any \(x \in X\). When \(X\) is quasi-projective, Definition 3.4.1 and Definition 3.5.1 descend to the derived category in the same way as in [PV01].

Indeed, by [PV01 Prop. 4.7], for \(h\) and \(k\) sufficiently large, there exists a complex \(C^\bullet = (C^0 \to O_X(-h)^{\oplus k})\) of coherent locally free sheaves on \(X\), quasiisomorphic to \(E^\bullet\), and satisfying
\[(3.7.1) \quad \text{Hom}_{K^b(X)}(\text{Sym}^{m+1}(C^\bullet), O_X[-1]) \cong \text{Hom}_{\mathcal{D}^b(X)}(\text{Sym}^{m+1}(C^\bullet), O_X[-1]),\]
where \(K^b(X)\) is the homotopic category of bounded complexes of coherent sheaves on \(X\). Therefore, the morphism \(\tau\) in the derived category is lifted up to homotopy by the homomorphism of complexes \(\gamma_\bullet : \text{Sym}^{m+1}(C^\bullet) \to O_X[-1]\).

Let \(c\) be the homomorphism \(\gamma_1 : \text{Sym}^m C^0 \otimes C^1 \to O_X\). Note that \(c^\vee\) is a closed and nondegenerate form for \((C^\bullet)^\vee\); therefore, we define
\[(3.7.2) \quad \text{Ke}((E^\bullet)^\vee, \tau^\vee) = \text{Ke}((C^\bullet)^\vee, c^\vee).\]
Furthermore, Definition 3.5.1 applies to $C^\bullet$ and $c$. We define
\[
\text{top}(E^\bullet, \tau) = \text{top}(C^\bullet, c).
\]
The definitions above do not depend on the choice of $C^\bullet$ and $c$. Indeed, by [PV01, Lem. 4.8], for any quasiisomorphisms $C^\bullet \to E^\bullet$ and $D^\bullet \to E^\bullet$ of complexes in degrees 0 and 1 and for any homomorphisms lifting $\tau$
\[
c: \text{Sym}^m C^0 \otimes C^1 \to \mathcal{O}_X, \quad d: \text{Sym}^m D^0 \otimes D^1 \to \mathcal{O}_X,
\]
there exists a complex $E^\bullet = (E^0 \to \mathcal{O}_X(-h)^{\otimes k})$, and two quasiisomorphisms $f: E^\bullet \to C^\bullet$, $g: E^\bullet \to D^\bullet$. By [PV01, Prop. 4.7], for a sufficiently large $h$, we can choose $E^\bullet$ in such a way that the isomorphism (3.7.1) holds.

The homomorphisms $c = c \circ \text{Sym}^{m+1}(f)$ and $d = d \circ \text{Sym}^{m+1}(g)$ lift $\tau$. Therefore, by Lemma 3.6.8, we have
\[
\text{Ke}((C^\bullet)^\vee, c^\vee) = \text{Ke}((E^\bullet)^\vee, \bar{c}^\vee) \quad \text{and} \quad \text{Ke}((D^\bullet)^\vee, d^\vee) = \text{Ke}((E^\bullet)^\vee, \bar{d}^\vee),
\]
whereas, by Lemma 3.6.2, we have
\[
\text{Ke}((E^\bullet)^\vee, \bar{c}^\vee) = \text{Ke}((E^\bullet)^\vee, \bar{d}^\vee).
\]

3.7.4. Remark. The $K$-class $\text{Ke}(F^\bullet, a)$ of Definition 3.4.1 only depends on $F^\bullet$ and on the homomorphism $\tau$ induced by $a$ in the derived category. The $K$-class can only be defined when there exists a suitable homomorphism $\tau$. It would be interesting to determine more accurately how it depends on it.

4. The Witten top Chern class

We choose nonnegative integers $r$, $g$, $n$, and $(k_1, \ldots, k_n) = k$ satisfying the conditions $r \geq 2$, $2g - 2 + n > 0$, and $2g - 2 - \sum k_i \in r\mathbb{Z}$. Let $\mathcal{M}_{g,n}(r,k)$ be the moduli stack of stable $r$-spin curves. By [Jar00], an object of $\mathcal{M}_{g,n}(r,k)$ is a set of data including:

1. a stable curve

2. a torsion free sheaf of rank one $L$ on $C$,

3. an injective homomorphism

\[
f: L^{\otimes r} \to \omega_{C/X} \left( - \sum_i k_i [s_i(X)] \right).
\]

The data $(\pi: C \to X; L; f)$ determine $\tau: \text{Sym}^r(R\pi_*L) \to \mathcal{O}_X[-1]$ in the derived category $D^b(X)$. Indeed, by [PV01, Sect. 5], the homomorphism $\text{Sym}^r(R\pi_*L) \to \mathcal{O}_X[-1]$ is the composite of the trace homomorphism
from Grothendieck duality $R\pi_*\omega_{C/X} \to \mathcal{O}_X[-1]$ and the homomorphism 
$\text{Sym}^r(E^\bullet) \to R\pi_*\omega_{C/X}$ induced by 
\begin{equation}
\gamma: L^\otimes r \xrightarrow{f} \omega_{C/X} \left(- \sum_i k_i[s_i(X)]\right) \to \omega_{C/X}.
\end{equation}

It is straightforward that the class $R\pi_*L$ can be represented by a complex $E^\bullet$ in degrees 0 and 1. On the other hand, for every $x \in X$, the homomorphism $\tau$ induces
\begin{equation}
\text{Sym}^r - 1 H^0(E_x^\bullet) \otimes H^1(E_x^\bullet) \to C,
\end{equation}
where $H^i(E_x^\bullet)$ is isomorphic to $H^i(C_x, L_x)$. The homomorphism $(4.0.6)$ coincides with the composite homomorphism
\begin{equation}
\tau_x: \text{Sym}^r - 1 H^0(E_x^\bullet) \to H^0(C_x, L_x^\otimes r^{-1}) \to H^0(C_x, \omega_{C_x} \otimes L_x^\vee) \xrightarrow{\sim} (H^1(E_x^\bullet))^\vee.
\end{equation}

Note that an element $v \in H^0(E_x^\bullet)$ is zero if for every $w \in H^1(E_x^\bullet)$ we have $\langle \tau_x(v^r - 1), w \rangle = 0$ (the injectivity of the middle homomorphism in $(4.0.7)$ is crucial and is guaranteed by $k_i \geq 0$). This means that $\tau$ induces a base point free linear system; therefore, choosing $m = r - 1$, we get a well defined $K$-class $K\varepsilon((E^\bullet)^\vee, \tau^\vee) \in K^0(X)$.

For any object of $\mathcal{S}_{g,n}(r, kkk)(X)$, we define the classes $K\varepsilon((E^\bullet)^\vee, \tau^\vee)$ and $c_{\text{top}}(E^\bullet, \tau)$ in $K^0(X)$ and $A^*(X)_Q$. The constructions are compatible with morphism in the category $\mathcal{S}_{g,n}(r, kkk)$ and naturally induce a $K$-theory class in $K^0(\mathcal{S}_{g,n}(r, kkk))$ and a Chow cohomology class in $A^*(\mathcal{S}_{g,n}(r, kkk))_Q$.

4.0.8. Definition. We define the Witten top Chern class in K-theory as
\begin{equation}
K_W = K\varepsilon((E^\bullet)^\vee, \tau^\vee)
\end{equation}
and the Witten top Chern class as
\begin{equation}
c_W = c_{\text{top}}(E^\bullet, \tau).
\end{equation}

4.0.10. Remark (Witten top Chern class using other compactifications). In this paper we work with Jarvis’s compactification [Jar00]. In fact, one can give a different compactification of the stack of smooth $r$-spin curves $\mathcal{S}_{g,n}(r, kkk)$ by means of (balanced) twisted curves in the sense of Abramovich and Vistoli. In this way, for instance, Abramovich and Jarvis rephrase the original construction of Jarvis, [AJ03]. A different approach exploits Olsson’s description of the category of twisted curves $\mathcal{S}_{g,n}(r, k)$: in [CM], we consider the stack $\mathcal{S}_{g}(r)$ of $r$-spin structures over twisted curves, which is étale over Olsson’s stack of twisted curves $\mathcal{M}_g$:
\begin{equation}
\mathcal{S}_{g}(r) \to \mathcal{M}_g
\end{equation}
(in sketching the procedure, we are omitting the markings for simplicity). We proceed by classifying all the compactifications of $\mathcal{M}_g$ contained in $\mathcal{M}_g$. Then, base change leads to new compactifications.
Although the new compactifications are not isomorphic to the preexisting ones, the rational Chow rings are isomorphic and the construction can be applied without modifications and yields the same class in rational cohomology. This happens because, the pushforward homomorphism $R(\pi_{\mathcal{C}})_*$ of the $r$-spin structure $\mathcal{L}$ on the twisted curve $\pi_{\mathcal{C}}: \mathcal{C} \to X$ in the derived category factors through the derived category of the coarse space $C$. Now, for stacks of Deligne–Mumford type, the pushforward homomorphism via $\mathcal{C} \to C$ is exact on coherent sheaves [AV02, Lem. 2.3.4]. Finally, this procedure yields the class $c_W$, because it carries $r$-spin structures on twisted curves to Jarvis’s $r$-spin structures on stable curves defined using relatively torsion-free sheaves (see [AJ03 §3, §4.3]).

5. Compatibility with the Polishchuk–Vaintrob class

5.1. Notation. The key ingredient of the construction is a 2-periodic complex.

5.1.1. Definition. A 2-periodic complex of sheaves on a scheme $Y$ is a $\mathbb{Z}/2\mathbb{Z}$-graded sheaf $W = W^+ \oplus W^-$ on $Y$ equipped with $d^+: W^+ \to W^-$ and $d^-: W^- \to W^+$ with $d^- \circ d^+ = d^+ \circ d^- = 0$. We say that $W$ is a complex of coherent (respectively quasicoherent) sheaves if $W^+$ and $W^-$ are coherent (respectively quasicoherent). We say that $W$ is exact if
\[ \cdots \xrightarrow{d^-} W^+ \xrightarrow{d^+} W^- \xrightarrow{d^-} W^+ \xrightarrow{d^+} W^- \xrightarrow{d^-} \cdots \]
is exact.

We denote by $E = \text{Spec}(\text{Sym}^* E^\vee)$ the total space of a coherent locally free sheaf $E$ on a scheme $X$.

5.2. The construction of Polishchuk and Vaintrob. We recall the construction from [PV01]. We choose an integer $m \geq 1$. Consider a complex $E^\bullet$ in degrees 0 and 1 of coherent locally free sheaves on $X$ and a homomorphism $b: \text{Sym}^m E^0 \otimes E^1 \to \mathcal{O}_X$ such that $b^\vee$ is a closed and nondegenerate form for $(E^\bullet)^\vee$.

Let $E^0$ be the total space of $E^0$, $p$ the projection $E^0 \to X$, and $i: X \to E^0$ the zero section. The pullback of $E^0 \to E^1$ via $p$ corresponds to a section $\delta: \mathcal{O}_{E^0} \to p^* E^1$. Compose the natural homomorphism $E^0 \to \text{Sym}^m E^0$ with the homomorphism $\text{Sym}^m E^0 \to (E^1)^\vee$ induced by $b$. The pullback via $p$ corresponds to a section $\alpha: \mathcal{O}_{E^0} \to p^*(E^1)^\vee$.

Denote by $S^h$ the sheaf $\Lambda^h p^*(E^1)^\vee$ on $E^0$; then, $\delta$ and $\alpha$ allow us to define homomorphisms $\tilde{\delta}: S^{h+1} \to S^h$ and $\tilde{\alpha}: S^h \to S^{h+1}$. The constructions of $\tilde{\delta}$ and of $\tilde{\alpha}$ are analogous to the ones of $\tilde{d}$ and $\tilde{a}$. For example, to define $\tilde{\delta}$, we take the natural map $\Lambda^{h+1} p^*(E^1)^\vee \to \Lambda^h p^*(E^1)^\vee \otimes p^*(E^1)^\vee$, the obvious
pairing $p^*(E^1)^\vee \otimes p^*(E^1) \to \mathcal{O}_{E^0}$, and the composite homomorphism

$$\bar{\delta}: \Lambda^{h+1} p^*(E^1)^\vee \to \Lambda^h p^*(E^1)^\vee \otimes p^*(E^1)^\vee \otimes \mathcal{O}_{E^0},$$

$$\text{id} \otimes \text{id} \otimes h^0 p^*(E^1)^\vee \otimes p^*(E^1)^\vee \otimes p^* E^1 \to \Lambda^h p^*(E^1)^\vee.$$

Define $d^+: S^+ \to S^-$ and $d^-: S^- \to S^+$ by summing for $h$ even and $h$ odd the homomorphisms $\bar{\delta}: S^h \to S^{h-1}$ and $\bar{\alpha}: S^h \to S^{h+1}$. It is shown in [PV01] that the homomorphisms $d^+ \circ d^-$ and $d^- \circ d^+$ vanish. We denote by $S = S^+ \oplus S^-$ the 2-periodic complex of coherent locally free sheaves on $E^0$ with differential $d = d^+ \oplus d^-$. (In fact $S$ is the spinor bundle associated to the orthogonal bundle $p^* E^1 \oplus p^*(E^1)^\vee$ and the differential is naturally induced by the isotropic section $\delta \otimes \alpha$.)

The complex

$$(5.2.1) \quad \ldots \xrightarrow{d^-} S^+ \xrightarrow{d^+} S^- \xrightarrow{d^-} S^+ \xrightarrow{d^+} S^- \xrightarrow{d^-} \ldots$$

is exact on $E^0 \setminus i(X)$, [PV01] [§3.1] (exactness is a consequence of a nondegeneracy condition [PV01] §4.1, (4.2)] on $a$, which is equivalent to Definition 3.1.5. In [BFM75], a localized Chern character $ch_X$ is defined in the rational bivariant Chow group $A(X \to Y)_{\mathbb{Q}}$ for a finite complex of coherent locally free sheaves on a scheme $Y$ exact outside a closed subscheme $X$. In [PV01], Section 2.2, this construction is adapted to 2-periodic complexes and applied to $S$ to produce a character $ch_X(S) \in A(X \to E^0)_{\mathbb{Q}}$, which allows us to define

$$(5.2.2) \quad c_{PV}(E^\bullet, b) = \text{td}(E^1) \cdot \text{ch}^g_X(S) \cdot [p] \in A^*(X)_{\mathbb{Q}}.$$  

By the same argument as in Section 5.4, this definition only depends on $E^\bullet$ and on the homomorphism in the derived category $\text{Sym}^{m+1}(E^\bullet) \to \mathcal{O}_X[-1]$ induced by $b$. Therefore, for any object $(\pi: C \to X; L; f)$ of the stack $\mathcal{F}_{g,n}(r, k)$, we choose $m = r - 1$, and we apply $c_{PV}$ to a complex $E^\bullet$ representing $R\pi_* L$ and a homomorphism $b$ representing $\tau: \text{Sym}^r(R\pi_* L) \to \mathcal{O}_X[-1]$. We obtain the Polishchuk–Vaintrob class

$$(5.2.3) \quad c_{PV} \in A^*(\mathcal{F}_{g,n}(r, k))_{\mathbb{Q}}.$$

We want to show $c_{PV} = c_W$ in $A^*(\mathcal{F}_{g,n}(r, k))_{\mathbb{Q}}$. This amounts to showing $c_{PV}(E^\bullet, b) = c_{top}(E^\bullet, b)$. First, we show that Polishchuk and Vaintrob’s localized Chern character descends to a homomorphism from a $K$-theory group to the rational bivariant Chow group.

5.3. Preliminaries on 2-periodic complexes. We need to introduce some notation.

5.3.1. Definition. Let $Y$ be a scheme and $X$ a closed subscheme. Write $\mathfrak{A}$ and $\mathfrak{F}$ for the categories of coherent sheaves and coherent locally free sheaves on $Y$. Denote by $\text{Ch}_{\mathbb{Z}/2}\mathfrak{A}$ and $\text{Ch}_{\mathbb{Z}/2}\mathfrak{F}$ the category of 2-periodic complexes $W = W^+ \oplus W^-$ of coherent sheaves and coherent locally free sheaves on $Y$. 

Write \( \text{Ch}_{Z/2}^X \mathfrak{A} \) and \( \text{Ch}_{Z/2}^X \mathfrak{F} \) for the full subcategories of objects which are exact on \( Y \setminus X \).

We consider the Grothendieck groups \( K_0(\text{Ch}_{Z/2}^X \mathfrak{A})_Y \), \( K_0(\text{Ch}_{Z/2}^X \mathfrak{F})_Y \), \( K_0(\text{Ch}_{Z/2}^X \mathfrak{A})_Y \), and \( K_0(\text{Ch}_{Z/2}^X \mathfrak{F})_Y \) generated by the objects above modulo the following relations:

\begin{enumerate}[(i)]
  \item \([W_1] = [W_2]\) if there exists a quasiisomorphism \( W_1 \to W_2 \);
  \item \([W] = [W_1] + [W_2]\) if there is a sequence \( 0 \to W_1 \to W \to W_2 \to 0 \) which is exact in all degrees of \( Z/2\mathbb{Z} \).
\end{enumerate}

5.3.2. Remark. Let \( Y \) be a smooth scheme and \( X \) a closed subscheme; then, Polishchuk and Vaintrob’s localized Chern character descends to an homomorphism on

\[ \text{ch}^Y_X : K_0(\text{Ch}_{Z/2}^X \mathfrak{F})_Y \to A(X \to Y)_\mathbb{Q}. \]

In order to define the homomorphism, we need to prove that \( \text{ch}^Y_X \) is defined on the objects of \( \text{Ch}_{Z/2}^X \mathfrak{F} \), and is compatible with (i) quasiisomorphisms and (ii) exact sequences.

In [PV01 §2.2], a technical condition is required for the existence of \( \text{ch}^Y_X(W) \): on \( Y \setminus X \), the sheaves \( \text{im} d^+ \) and \( \text{im} d^- \) have to be subbundles of \( W^+ \) and \( W^+ \). This condition is satisfied, because \( Y \) is smooth. Indeed, for any object \( W \) in \( \text{Ch}_{Z/2}^X \mathfrak{F} \), we can write resolutions on \( Y \setminus X \) of \( \text{im} d^+ \) (and of \( \text{im} d^- \)) of arbitrary length:

\[ 0 \to \text{im} d^+ \to W^+_{Y \setminus X} \to W^+_{Y \setminus X} \to \cdots \to W^+_{Y \setminus X} \to \text{im} d^+ \to 0. \]

Take a sequence as above of length \( l > \dim(Y) + 2 \). By [BS58 §4, Lem. 9], for \( Y \) smooth and \( W^+, W^- \in \mathfrak{F} \) we have \( \text{im} d^+ \in \mathfrak{F} \).

Finally \( \text{ch}^Y_X \) is compatible with the relations of \( K_0(\text{Ch}_{Z/2}^X \mathfrak{F})_Y \) because we have the identity \( \text{ch}^Y_X(W) = \text{ch}^Y_X(W_1) + \text{ch}^Y_X(W_2) \) for any exact sequence \( 0 \to W_1 \to W \to W_2 \to 0 \) by [PV01 Prop. 2.3,(iv)]. Note that this is sufficient by the same argument of [Ful84 Exa. 18.1.4]: for any quasiisomorphism \( f : W_1 \to W_2 \), we have the exact sequence \( 0 \to W_1 \to \text{Cone}(f) \to W_2 \to 0 \), with \( \text{ch}^Y_X(\text{Cone}(f)) = 0 \) by the exactness of \( \text{Cone}(f) \) on \( Y \).

The following lemma was pointed out to me by Charles Walter.

5.3.4. Lemma. We have an isomorphism \( K_0(\text{Ch}_{Z/2}^X \mathfrak{A})_Y \to K_0(Y) \) defined by \([W] \mapsto [H^+(W)] - [H^-(W)]\). Furthermore, for \( Y \) smooth and quasiprojective, we also have \( K_0(\text{Ch}_{Z/2}^X \mathfrak{A})_Y \cong K_0(\text{Ch}_{Z/2}^X \mathfrak{F})_Y \).

Proof. The homomorphism \( K_0(Y) \to K_0(\text{Ch}_{Z/2}^X \mathfrak{A})_Y \) is induced by the functor sending a sheaf \( F \) to the 2-periodic complex \( F \oplus 0 = (\cdots \to F \to 0 \to F \to 0 \to \cdots) \). Composing \([W] \mapsto [H^+(W)] - [H^-(W)]\) after \([F] \mapsto [F] \oplus [0]\) we get the identity in \( K_0(\mathfrak{A}) = K_0(Y) \). Conversely, reversing the order of the composition, we obtain the homomorphism \([W] \mapsto [H^+(W) \oplus 0] - [H^-(W) \oplus 0]\).
This functor descends to the identity in $K_0(\text{Ch}_{\mathbb{Z}/2} \mathfrak{A})$, because the sequence in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$

$$0 \rightarrow \ker d^+ \oplus \ker d^- \rightarrow W \rightarrow \im d^+ \oplus \im d^- \rightarrow 0,$$

is exact ($\ker d^+ \oplus \ker d^-$ and $\im d^+ \oplus \im d^-$ are $\mathbb{Z}/2\mathbb{Z}$-graded sheaves equipped with the zero differential).

If $X$ is smooth, the $K$-theory groups $K_0(\text{Ch}_{\mathbb{Z}/2} \mathfrak{A})$ and $K_0(\text{Ch}_{\mathbb{Z}/2} \mathfrak{F})$ are isomorphic, because the localization of $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$ and $\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$ induces equivalent triangulated categories. This follows from the natural inclusion in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$ and the fact that for any object $W \in \text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$ there exists an exact sequence $0 \rightarrow L_n \rightarrow L_{n-1} \cdots \rightarrow L_0 \rightarrow W \rightarrow 0$, where each $L_i$ is an object of $\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$.

1. If $0 \rightarrow F \rightarrow L_p \rightarrow L_{p-1} \cdots \rightarrow L_0 \rightarrow W \rightarrow 0$ is exact with $F, W$ in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$ and $L_i$ in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$, then, for $p \geq \dim(X)-1$, $F$ belongs to $\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$.

2. Any $W$ in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$ is a quotient of an object $L$ in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}$.

Note that [1] follows immediately from [BS58, Lem. 9] and [2] can be shown using the natural functor

$$\text{Hom}_{\mathfrak{A}}(P, W^+) \rightarrow \text{Hom}_{\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}} \left( \left( P \oplus P, \begin{pmatrix} 0 & \text{id} \\ 0 & 0 \end{pmatrix} \right), \left( W^+ \oplus W^-, \begin{pmatrix} 0 & d^- \\ d^+ & 0 \end{pmatrix} \right) \right),$$

given by $\phi \mapsto \phi \oplus (d_+ \circ \phi)$ and the functor $\phi \mapsto (d_- \circ \phi) \oplus \phi$ defined on $\text{Hom}_{\mathfrak{A}}(P, W^-)$. To see this, take an object $W^+ \oplus W^-$ in $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$, and, using [BS58, Lem. 10], take two surjections $(P^+ \rightarrow W^+) \in \text{Hom}_{\mathfrak{A}}(P, W^+)$ and $(P^- \rightarrow W^-) \in \text{Hom}_{\mathfrak{A}}(P, W^-)$. Then, applying the functors above, we obtain two morphisms to $W^+ \oplus W^-$. By construction, the sum is surjective.

For the rest of the section we consider a coherent locally free sheaf $V$ on a smooth scheme $X$, and we write $\mathcal{V}$ for the total space of $V$, $p: \mathcal{V} \rightarrow X$ for the projection, and $i: X \rightarrow \mathcal{V}$ for the zero section.

Note that, for any coherent 2-periodic complex $W$ on $X$ exact outside $i(X)$, the sheaves $p_*H^+(W)$ and $p_*H^-(W)$ are coherent, because $H^+(W)$ and $H^-(W)$ are supported on $i(X)$. The pushforward $p_*$ induces an exact functor from $\text{Ch}_{\mathbb{Z}/2} \mathfrak{A}$ to the category of 2-periodic complexes of coherent sheaves on $X$. Therefore, we can define a homomorphism

$$\varphi: K_0(\text{Ch}_{\mathbb{Z}/2} \mathfrak{F}) \rightarrow K_0(X)$$

mapping as $[W] \mapsto [p_*H^+(W)] - [p_*H^-(W)]$.

5.3.7. Lemma. For any vector bundle $\mathcal{V}$ on a smooth scheme $X$, the homomorphism $\varphi$ is a bijection.
Proof. Note that \( K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V \) is the kernel of the homomorphism
\[
K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V \to K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_{\mathbb{V} \setminus \{\text{id}(X)\}}.
\]
On the other hand the kernel of the homomorphism \( K'_0(\mathcal{V}) \to K'_0(\mathcal{V} \setminus \{\text{id}(X)\}) \) is identified (via \( p_* \)) with \( K'_0(X) \). Finally, note that \( \varphi \) is the restriction to \( K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V \) of the isomorphism \( K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V \to K'_0(\mathcal{V}) \) from Lemma \ref{lem:isomorphism}.

5.3.8. **Lemma.** Let \( V \) be a coherent locally free sheaf on \( X \). The diagram
\begin{equation}
\begin{array}{ccc}
K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V & \xrightarrow{\text{ch}_V^X} & A(X \to \mathbb{V})_Q \\
\varphi \downarrow & & \downarrow [p_* \cdot \text{td}(V)] \\
K'_0(X) & \xrightarrow{\sim} & K_0(X) \xrightarrow{\text{ch}} A^*(X)_Q
\end{array}
\end{equation}
is commutative.

**Proof.** The Koszul complex induced by the tautological section \( \mathcal{V} \to p^* \mathcal{V} \)
\[
0 \to \Lambda^{rk(V)}(p^* \mathcal{V})^\vee \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} \Lambda^2(p^* \mathcal{V})^\vee \xrightarrow{d_1} \cdots \xrightarrow{d_2} \Lambda^1(p^* \mathcal{V})^\vee \xrightarrow{d_1} \mathcal{O}_V \to 0,
\]
is exact off \( i(X) \); this happens in general for a vector bundle with a nowhere vanishing section, see [Ful84] B.3.4 and A.5]. Denote by \( \Lambda(p^* \mathcal{V})^\vee \) the 2-periodic complex given by summing the exterior powers \( \Lambda^i(p^* \mathcal{V})^\vee \) and the homomorphisms \( d_i \) for \( i \) even and \( i \) odd. For any coherent locally free sheaf \( A \) on \( X \), the 2-periodic complex \( p^* A \otimes \Lambda(p^* \mathcal{V})^\vee \) is exact off \( X \). We define \( \kappa: K'_0(X) \to K_0(\text{Ch}^X_{\mathbb{Z}/2} \mathcal{F})_V \) as the homomorphism \([A] \mapsto [p^* A \otimes \Lambda(p^* \mathcal{V})^\vee] \).

Note that \( \kappa \) commutes with \( \varphi \) and \( K'_0(X) \to K_0(X) \). Indeed, consider the complexes \( A \otimes \text{Sym}^N(V \xrightarrow{\text{id}} V) \), and denote by
\[
T_N = (\cdots \to T^+_N \to T^-_N \to T^+_N \to T^-_N \to \cdots)
\]
the 2-periodic complex given by summing over even and odd degrees. Then, note that, by the projection formula, we can decompose the complex \( p_*(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee) \) into the sum of the complexes \( T_N \). The complex \( T_N \) is exact if \( N > 0 \), so we have
\[
\varphi(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee) = [H^+(p_*(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee))] - [H^-(p_*(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee))] = [A].
\]
Therefore \( \kappa \) is an isomorphism.

By [PV01] Prop. 2.3, (iv)], we have
\[
\text{ch}_V^X(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee) \cdot [p] \cdot \text{td}(V) = \text{ch}(A)
\]

(\text{indeed, the formula above can be rewritten in terms of the standard localized Chern character from [Ful84] and follows from [BFM75] Prop. 3.4, Ch. 1]).

The diagram commutes, because we have
\[
\text{ch}_X^V(\kappa[A]) \cdot [p] \cdot \text{td}(V) = \text{ch}_X^V(p^* A \otimes \Lambda(p^* \mathcal{V})^\vee) \cdot [p] \cdot \text{td}(V) = \text{ch}(A).
\]
5.4. The identity. We show the identity between $c_{PV}$ and $c_W$. For any object $(\pi: C \to X; L; f)$ of the stack $\mathcal{F}_{g,n}(r, k)$, we have a complex $E^\bullet$ and a homomorphism $b: \text{Sym}^m E^0 \otimes E^1 \to \mathcal{O}_X$, whose dual is closed and nondegenerate for $(E^\bullet)^\vee$. We recall that $c_{PV}(F, a) = \text{td}(E^1) \cdot \text{ch}_{X}(S) \cdot [p]$, where $S$ is the 2-periodic complex at (5.2.1).

5.4.1. Theorem. We have

\begin{align*}
\varphi[S] &= \text{Ke}((E^\bullet)^\vee, b^\vee) \quad \text{in } K_0^{(X)} \\
\therefore c_{PV}(E^\bullet, b) &= c_{\text{top}}(E^\bullet, b) \quad \text{in } A^*(X)_{\mathbb{Q}}.
\end{align*}

Therefore $c_{PV} = c_W$ in $A(\mathcal{F}_{g,n}(r, k))_{\mathbb{Q}}$.

Proof. Define $L^+ \to L^-$ and $L^- \to L^+$ by summing, for $k$ even and $k$ odd, all the homomorphisms $\tilde{d}: L^{h,k} \to L^{h+1,k-1}$ and $\tilde{a}: L^{h,k} \to L^{h+m,k+1}$. Now, $L = L^+ \oplus L^-$ is a 2-periodic complex of quasicoherent sheaves by Lemma 5.2.8. It is easy to see that the 2-periodic complex $p_*S$ is equal to $L$ (with $p_*\delta = \tilde{d}$ and $p_*\tilde{\varphi} = \tilde{a}$). The functor $p_*$ is exact on complexes which are exact off $i(X)$; therefore, since the complex $S$ in (5.2.1) is exact off $i(X)$ by [PV01 S3.1], we have $p_*(H^+(S)) = H^+(p_*S) = H^+(L)$ and $p_*(H^-(S)) = H^+(p_*S) = H^-(L)$. This implies (5.4.2). Finally, Lemma 5.3.8 implies (5.4.3): $c_{PV}(E^\bullet, b) = \text{td}(E^1) \cdot \text{ch}_{X}(S) \cdot [p] = \text{td}(E^1) \cdot \text{td}(E^0)^{-1} \cdot \text{ch}(\varphi[S]) = \frac{\text{td}(E^1)}{\text{td}(E^0)} \text{ch}(\text{Ke}((E^\bullet)^\vee, b^\bullet)) = c_{\text{top}}(E^\bullet, b)$. \hfill \Box

5.4.4. Remark. The identity of Theorem 5.4.1 guarantees that the class is concentrated in degree $-\chi(C, L) = n_1 - n_0$. Furthermore, the axioms of cohomological field theory stated by Jarvis, Kimura, and Vaintrob in [JKV01] are proven by Polishchuk and Vaintrob in [PV01] and [Pol04]. The proofs are given on the level of $K$-theory of complexes on a vector bundle that are strictly exact off the zero section. Note that the equivalence of triangulated categories shown in Lemma 5.3.4 implies that working in the $K$-theory of the base scheme $X$ is equivalent to working in the $K$-theory of complexes on a vector bundle $V \to X$ that are strictly exact off the zero section. This indicates that the arguments used by Polishchuk and Vaintrob can be restated in $K_0(X)$, without passing through $V$. In the cases that we have checked, however, this passage to $X$ does not appear to simplify significantly the proofs of the cohomological field theory axioms.
6. Computations

6.1. Compatibility with Witten’s definition. Assume that for every point \( x \in X \) we have \( H^0(C_x, \mathcal{L}_x) = 0 \). Then, \( R^0\pi_* \mathcal{L} = 0 \) and \( R^1\pi_* \mathcal{L} \) is a bundle. By Definition 4.0.8 we have \( c_W = c_{\text{top}}(R^\bullet \pi_* \mathcal{L}, 0) \). By Definition 3.5.1 we have

\[
c_{\text{top}}(R^\bullet \pi_* \mathcal{L}, 0) = \text{ch}(\text{Ke}((R^\bullet \pi_* \mathcal{L})^\vee, 0)) \cdot \text{td}(R^1\pi_* \mathcal{L}).
\]

So, by Proposition 3.4.2, we have

\[
\text{Ke}((R^\bullet \pi_* \mathcal{L})^\vee, 0) = \lambda_{-1}((R^1\pi_* \mathcal{L})^\vee).
\]

Finally, by (1.5.1), we have

\[
c_W = c_{\text{top}}(R^\bullet \pi_* \mathcal{L}, 0) = \lambda_{-1}((R^1\pi_* \mathcal{L})^\vee) \cdot \text{td}(R^1\pi_* \mathcal{L}) = c_{\text{top}}(R^1\pi_* \mathcal{L}),
\]

which agrees with Witten’s definition (1.3.2).

6.2. The case when \( R^0\pi_* \mathcal{L} \) and \( R^1\pi_* \mathcal{L} \) are vector bundles. Let \((\pi: C \to X; L, f)\) be an object of \( \mathcal{G}_{g,n}((r, k)) \) over which \( R^i\pi_* \mathcal{L} \) is a vector bundle of rank \( h^i = h^i(C, L) \). Denote by \( a \) the form induced by Serre duality

\[
b: \text{Sym}^{r-1}(R^0\pi_* \mathcal{L}) \otimes (R^1\pi_* \mathcal{L}) \to \mathcal{O}_X.
\]

By the same argument as in Section 4, the form \( b^\vee \) is closed and nondegenerate. By Theorem 3.3.1 we have, for \( t_0 = r - 1 + (h^1 - 1 + r - 1) \),

\[
\text{Ke}(R^p\pi_* \mathcal{L}^\vee, b^\vee) = \sum_{h \leq (r-1)k + t_0} (-1)^k [\text{Sym}^h(R^0\pi_* \mathcal{L})^\vee] [\Lambda^k(R^1\pi_* \mathcal{L})^\vee].
\]

Therefore, via Definition 3.5.1 we can compute explicitly the pullback of \( c_W \) under the morphism \( X \to \mathcal{G}_{g,n}(r, k) \).

6.3. The case \( r = 2 \), theta characteristics. For \( r = 2 \) and \( k = 0 \), the class \( c_W \) is a cycle of codimension 0, Remark 5.4.4. So the class \( c_W \) is determined by a number. We claim that this number is 1 on the connected component compactifying even theta characteristics \( (h^0(C, L) \text{ even}) \) and \(-1\) on the component compactifying the odd ones \( (h^0(C, L) \text{ odd}) \). Indeed, since we only need to determine the multiplicity of \( c_W \) we can consider \( L \to C \to X \) where \( X \) is a point, \( C \) is a smooth curve, and \( L \) is a line bundle satisfying \( L^\otimes 2 \cong \omega_C \). The complex \( H^\bullet(C, L) \) with zero differential represents \( R\pi_* \mathcal{L} \).

The nondegenerate form is the perfect pairing \( H^0(C, L) \otimes H^1(C, L) \to \mathbb{C} \). We
write $H^\vee_i = (H^i(C, L))^\vee$ and $h^i = h^i(C, L)$ and describe $L^{••}$ as follows.

$$
\begin{array}{ccccccc}
0 & 0 & \cdots & \\
\Lambda^h H_1^\vee & H_0^\vee \otimes \Lambda^h H_1^\vee & \cdots & \\
\Lambda^{h-1} H_1^\vee & H_0^\vee \otimes H_1^\vee & \cdots & \\
\cdots & \cdots & \cdots & \\
\mathbb{C} & H_0^\vee & \cdots & \text{Sym}^r H_0^\vee & \cdots \\
\end{array}
$$

Note that the differential $\tilde{a}$ is zero and the differential $\bar{a}$ is always exact, except on $\Lambda^6 H_1^\vee$. The Witten top Chern class in $K$-theory is equal to $(-1)^{h^i}[\Lambda^h H_1^\vee] = (-1)^{h^i}[\mathbb{C}]$. The cohomology class is $c_W = (-1)^{h_i}$.

### 6.4. Genus one

We consider the case $g = 1$, $n = 1$, and $k = (0)$, the moduli stack compactifying elliptic $r$-spin curves. Indeed, again the class $c_W$ is a cycle of codimension 0. Remark [5.2]. We can consider $L \to C \xrightarrow{\pi} X$ where $X$ is a point, $C$ is a smooth curve of genus 1, and $L$ is a line bundle satisfying $L^{\otimes r} \cong \omega_C \cong \mathcal{O}_C$. The complex $H^\bullet(C, L)$ with zero differential represents $R\pi_\ast L$. Note that, if $L$ is not trivial, we have $H^i(C, L) = 0$ and, therefore, $c_W = 1_X$.

We assume that $L$ is trivial; therefore, we have $h^1(C, L) = 1$. We write $H^1 = H^1(C, L)$ and $H_0^\vee = (H^0(C, L))^\vee$. The closed and nondegenerate form is given by the isomorphism $\text{Sym}^{r-1} H^0 = (H^0)^{\otimes r-1} \cong H^0(L^{\otimes r-1}) \cong H_1^\vee$. We describe $L^{••}$ as follows.

$$
\begin{array}{ccccccc}
0 & 0 & \cdots & \\
H_1^\vee & H_0^\vee \otimes H_1^\vee & \cdots & (H_0^\vee)^{\otimes r-2} \otimes H_1^\vee & (H_0^\vee)^{\otimes r} \otimes H_1^\vee & \\
\mathbb{C} & H_0^\vee & \cdots & (H_0^\vee)^{\otimes r-2} & (H_0^\vee)^{\otimes r} & \\
\end{array}
$$

The differential $\tilde{a}$ is zero, and the differential

$$
\bar{a}: (H_0^\vee)^{\otimes i} \to (H_0^\vee)^{\otimes i+r-1} \otimes H_1^\vee
$$

is always exact, except on $H_0^\vee$, $H_0^\vee \otimes H_1^\vee$, . . . , and $(H_0^\vee)^{\otimes r-2} \otimes H_1^\vee$. Therefore, the Witten top Chern class in the $K$-theory of a closed point is equal to $-\sum_{i=1}^{r-2} (H_0^\vee)^{\otimes i} \otimes H_1^\vee = (1)[\mathbb{C}^{\otimes r-1}]$. Finally, via Chern character, we get $c_W = -(r-1)1_X$. This is consistent with Witten’s predictions and can be deduced also from the cohomological field theory axioms [JKV01].

### References

[AJ03] D. Abramovich and Tyler J. Jarvis, *Moduli of twisted spin curves*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 685–699.

[AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic).
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[BFM75] Paul Baum, William Fulton, and Robert MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 101–145.

[BS58] Armand Borel and Jean-Pierre Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136.

[Chi] A. Chiodo, *Stable twisted curves and their $r$-spin structures*, Preprint: math.AG/0603687.

[Chi03] ________, *Higher spin curves and Witten’s top Chern class*, Ph.D. thesis, University of Cambridge, 2003.

[Del73] Pierre Deligne, *Cohomologie à supports propres*, Théorie des topos et cohomologie étale des schémas. Tome 3, exposé XVII, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 305.

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.

[Gre88] Mark L. Green, *A new proof of the explicit Noether-Lefschetz theorem*, J. Differential Geom. **27** (1988), no. 1, 155–159.

[Gre89], *Koszul cohomology and geometry*, Lectures on Riemann surfaces (Trieste, 1987), World Sci. Publishing, Teaneck, NJ, 1989, pp. 177–200.

[Jar00] Tyler J. Jarvis, *Geometry of the moduli of higher spin curves*, Internat. J. Math. **11** (2000), no. 5, 637–663.

[JKV01] Tyler J. Jarvis, Takashi Kimura, and Arkady Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. **126** (2001), no. 2, 157–212.

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[McC01] John McCleary, *A user’s guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[Ols] Martin Olsson, *On (log) twisted curves*, Preprint: http://www.math.ias.edu/~molsson/logcurves.pdf.

[Pol04] A. Polishchuk, *Witten’s top Chern class on the moduli space of higher spin curves*, Hertling, Claus (ed.) et al., Frobenius manifolds, Quantum cohomology and singularities. Proceedings of the workshop, Bonn, Germany, July 8-19, 2002, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 253–264.

[PV01] A. Polishchuk and A. Vaintrob, *Algebraic construction of Witten’s top Chern class*, Advances in algebraic geometry motivated by physics (Lowell, MA, 2000) (Providence, RI), Contemp. Math., vol. 276, Amer. Math. Soc., 2001, pp. 229–249.

[Wit91] Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

[Wit93] ________, *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 255–269.

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