SS-Injective Modules and Rings

Adel Salim Tayyah
Department of Mathematics, College of Computer Science and Information Technology, Al-Qadisiyah University, Al-Qadisiyah, Iraq
Email: adils9888@gmail.com

Akeel Ramadan Mehdi
Department of Mathematics, College of Education, Al-Qadisiyah University, P. O. Box 88, Al-Qadisiyah, Iraq
Email: akeel_math@yahoo.com

March 19, 2018

Abstract

We introduce and investigate ss-injectivity as a generalization of both soc-injectivity and small injectivity. A module \(M\) is said to be ss-\(N\)-injective (where \(N\) is a module) if every \(R\)-homomorphism from a semisimple small submodule of \(N\) into \(M\) extends to \(N\). A module \(M\) is said to be ss-injective (resp. strongly ss-injective), if \(M\) is ss-\(R\)-injective (resp. ss-\(N\)-injective for every right \(R\)-module \(N\)). Some characterizations and properties of (strongly) ss-injective modules and rings are given. Some results of Amin, Yuosif and Zeyada on soc-injectivity are extended to ss-injectivity. Also, we provide some new characterizations of universally mininjective rings, quasi-Frobenius rings, Artinian rings and semisimple rings.

Key words and phrases: Small injective rings (modules); soc-injective rings (modules); SS-Injective rings (modules); Perfect rings; quasi-Frobenius rings.

2010 Mathematics Subject Classification: Primary: 16D50, 16D60, 16D80 ; Secondary: 16P20, 16P40, 16L60 .

* The results of this paper will be part of a MSc thesis of the first author, under the supervision of the second author at the University of Al-Qadisiyah.

1 Introduction

Throughout this paper, \(R\) is an associative ring with identity, and all modules are unitary \(R\)-modules. For a right \(R\)-module \(M\), we write \(\text{soc}(M)\), \(J(M)\), \(Z(M)\), \(Z_2(M)\), \(E(M)\) and \(\text{End}(M)\) for the socle, the Jacobson radical, the singular submodule, the second singular submodule, the injective hull and the endomorphism ring of \(M\), respectively. Also, we use \(S_r, S_\ell, Z_r, Z_\ell, Z_2\) and \(J\) to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, the
right second singular ideal, and the Jacobson radical of \( R \), respectively. For a submodule \( N \) of \( M \), we write \( N \subseteq \text{ess} M, N \ll M, N \subseteq \oplus M \), and \( N \subseteq \text{max} M \) to indicate that \( N \) is an essential submodule, a small submodule, a direct summand, and a maximal submodule of \( M \), respectively. If \( X \) is a subset of a right \( R \)-module \( M \), the right (resp. left) annihilator of \( X \) in \( R \) is denoted by \( r_R(X) \) (resp. \( l_R(X) \)). If \( M = R \), we write \( r_R(X) = r(X) \) and \( l_R(X) = l(X) \).

Let \( M \) and \( N \) be right \( R \)-modules, \( M \) is called soc-\( N \)-injective if every \( R \)-homomorphism from the soc\((N)\) into \( M \) extends to \( N \). A right \( R \)-module \( M \) is called soc-injective, if \( M \) is soc-\( R \)-injective. A right \( R \)-module \( M \) is called strongly soc-injective, if \( M \) is soc-\( N \)-injective for all right \( R \)-module \( N \) \[2\].

Recall that a right \( R \)-module \( M \) is called mininjective \[14\] (resp. small injective \[19\], principally small injective \[20\]) if every \( R \)-homomorphism from any simple (resp. small, principally small) right ideal to \( M \) extend to \( R \). A ring is called right mininjective (resp. small injective, principally small injective) ring, if it is right mininjective (resp. small injective, principally small injective) as right \( R \)-module. A ring \( R \) is called right Kasch if every simple right \( R \)-module embeds in \( R \) (see for example \[15\]). Recall that a ring \( R \) is called semilocal if \( R/J \) is a semisimple \[11\]. Also, a ring \( R \) is said to be right perfect if every right \( R \)-module has a projective cover. Recall that a ring \( R \) is said to be quasi-Frobenius (or \( QF \)) ring if it is right (or left) artinian and right (or left) self-injective; or equivalently, every injective right \( R \)-module is projective.

In this paper, we introduce and investigate the notions of ss-injective and strongly ss-injective modules and rings. Examples are given to show that the (strong) ss-injectivity is distinct from that of mininjectivity, principally small injectivity, small injectivity, simple J-injectivity, and (strong) soc-injectivity. Some characterizations and properties of (strongly) ss-injective modules and rings are given.

W. K. Nicholson and M. F. Youssif in \[14\] introduced the notion of universally mininjective ring, a ring \( R \) is called right universally mininjective if \( S_r \cap J = 0 \). In Section 2, we show that \( R \) is a right universally mininjective ring if and only if every simple right \( R \)-module is ss-injective. We also prove that if \( M \) is a projective right \( R \)-module, then every quotient of an ss-\( M \)-injective right \( R \)-module is ss-\( M \)-injective if and only if every sum of two ss-\( M \)-injective submodules of a right \( R \)-module is ss-\( M \)-injective if and only if \( \text{Soc}(M) \cap J(M) \) is projective. Also, some results are given in terms of ss-injectivity modules. For example, every simple singular right \( R \)-module is ss-injective implies that \( S_r \) projective and \( r(a) \subseteq \oplus R \) for all \( a \in S_r \cap J \), and if \( M \) is a finitely generated right \( R \)-module, then \( \text{Soc}(M) \cap J(M) \) is finitely generated if and only if every direct sum of ss-\( M \)-injective right \( R \)-modules is ss-\( M \)-injective if and only if every direct sum of \( N \) copies of ss-\( M \)-injective right \( R \)-module is ss-\( M \)-injective.

In Section 3, we show that a right \( R \)-module \( M \) is strongly ss-injective if and only if every small submodule \( A \) of a right \( R \)-module \( N \), every \( R \)-homomorphism \( \alpha : A \rightarrow M \) with \( \alpha(A) \) semisimple extends to \( N \). In particular, \( R \) is semiprimitive if every simple right \( R \)-module is strongly ss-injective, but not conversely. We also prove that if \( R \) is a right perfect ring, then a right \( R \)-module \( M \) is strongly soc-injective if and only if \( M \) is strongly ss-injective. A results \((2, \text{Theorem} 3.6 \text{and Proposition} 3.7)\) are extended. We prove that a ring \( R \) is right artinian if and only if every direct sum of strongly ss-injective right \( R \)-modules is injective, and \( R \) is QF ring if and only if every strongly ss-injective right \( R \)-module is projective.

In Section 4, we extend the results \((2, \text{Proposition} 4.6 \text{and Theorem} 4.12)\) from a soc-injective ring to an ss-injective ring (see Proposition \[4.14\] and Corollary \[4.15\]).

In Section 5, we show that a ring \( R \) is QF if and only if \( R \) is strongly ss-injective and right noetherian with essential right socle if and only if \( R \) is strongly ss-injective, \( l(J^2) \) is countable generated left ideal, \( S_r \subseteq \text{ess} R \), and the chain \( r(x_1) \subseteq r(x_2x_1) \subseteq \ldots \subseteq r(x_nx_{n-1} \ldots x_1) \subseteq \ldots \) terminates for every infinite sequence \( x_1, x_2, \ldots \) in \( R \) (see Theorem \[5.10\] and Theorem \[5.12\]). Finally, we prove that a ring \( R \) is QF if and only if \( R \) is strongly left and right ss-injective, left
Kasch, and $J$ is left $t$-nilpotent (see Theorem 5.15), extending a result of I. Amin, M. Yousif and N. Zeyada [2, Proposition 5.8] on strongly soc-injective rings.

General background materials can be found in [3], [9] and [10].

## 2 SS-Injective Modules

**Definition 2.1.** Let $N$ be a right $R$-module. A right $R$-module $M$ is said to be ss-$N$-injective, if for any semisimple small submodule $K$ of $N$, any right $R$-homomorphism $f : K \rightarrow M$ extends to $N$. A module $M$ is said to be ss-quasi-injective if $M$ is ss-$M$-injective. $M$ is said to be ss-injective if $M$ is ss-$R$-injective. A ring $R$ is said to be right ss-injective if the right $R$-module $R_R$ is ss-injective.

**Definition 2.2.** A right $R$-module $M$ is said to be strongly ss-injective if $M$ is ss-$N$-injective, for all right $R$-module $N$. A ring $R$ is said to be strongly right ss-injective if the right $R$-module $R_R$ is strongly ss-injective.

**Example 2.3.** (1) Every soc-injective module is ss-injective, but not conversely (see Example 5.8).
(2) Every small injective module is ss-injective, but not conversely (see Example 5.6).
(3) Every $\mathbb{Z}$-module is ss-injective. In fact, if $M$ is a $\mathbb{Z}$-module, then $M$ is small injective (by [19, Theorem 2.8] and hence it is ss-injective.
(4) The two classes of principally small injective rings and ss-injective rings are different (see [15, Example 5.2], Example 4.4 and Example 5.6).
(5) Every strongly soc-injective module is strongly ss-injective, but not conversely (see Example 5.8).
(6) Every strongly ss-injective module is ss-injective, but not conversely (see Example 5.7).

**Theorem 2.4.** The following statements hold:
(1) Let $N$ be a right $R$-module and let $\{M_i : i \in I\}$ be a family of right $R$-modules. Then the direct product $\prod_{i \in I} M_i$ is ss-$N$-injective if and only if each $M_i$ is ss-$N$-injective, for all $i \in I$.
(2) Let $M, N$ and $K$ be right $R$-modules with $K \subseteq N$. If $M$ is ss-$N$-injective, then $M$ is ss-$K$-injective.
(3) Let $M, N$ and $K$ be right $R$-modules with $M \cong N$. If $M$ is ss-$K$-injective, then $N$ is ss-$K$-injective.
(4) Let $M, N$ and $K$ be right $R$-modules with $K \cong N$. If $M$ is ss-$K$-injective, then $M$ is ss-$N$-injective.
(5) Let $M, N$ and $K$ be right $R$-modules with $N$ is a direct summand of $M$. If $M$ is ss-$K$-injective, then $N$ is ss-$K$-injective.

**Proof.** Clear.

**Corollary 2.5.** (1) If $N$ is a right $R$-module, then a finite direct sum of ss-$N$-injective modules is again ss-$N$-injective. Moreover, a finite direct sum of ss-injective (resp. strongly ss-injective) modules is again ss-injective (resp. strongly ss-injective).
(2) A direct summand of an ss-quasi-injective (resp. ss-injective, strongly ss-injective) module is again ss-quasi-injective (resp. ss-injective, strongly ss-injective).

**Proof.** (1) By taking the index $I$ to be a finite set and applying Theorem 2.4(1).
(2) This follows from Theorem 2.4(5).

**Lemma 2.6.** Every ss-injective right $R$-module is right mininjective.
Proof. Let $I$ be a simple right ideal of $R$. By [6, Lemma 3.8] we have that either $I$ is nilpotent or a direct summand of $R$. If $I$ is a nilpotent, then $I \subseteq J$ by [6, Corollary 6.2.8] and hence $I$ is a semisimple small right ideal of $R$. Thus every ss-injective right $R$-module is right mininjective.

It easy to prove the following proposition.

**Proposition 2.7.** Let $N$ be a right $R$-module. If $J(N)$ is a small submodule of $N$, then a right $R$-module $M$ is ss-$N$-injective if and only if any $R$-homomorphism $f : \text{soc}(N) \cap J(N) \rightarrow M$ extends to $N$.

**Proposition 2.8.** Let $N$ be a right $R$-module and \( \{ A_i : i = 1, 2, \ldots , n \} \) be a family of finitely generated right $R$-modules. Then $N$ is ss-$\bigoplus_{i=1}^n A_i$-injective if and only if any $R$-module $M$ is ss-$\bigoplus_{i=1}^n A_i$-injective, for all $i = 1, 2, \ldots , n$.

**Proof.** ($\Rightarrow$) This follows from Theorem 2.4(2),(4).

($\Leftarrow$) By [5, Proposition (I.4.1) and Proposition (I.1.2)] we have that $\text{soc}(\bigoplus_{i=1}^n A_i) \cap J(\bigoplus_{i=1}^n A_i) = (\text{soc} \cap J)(\bigoplus_{i=1}^n A_i) = \bigoplus_{i=1}^n (\text{soc}(A_i) \cap J(A_i))$. For $j = 1, 2, \ldots , n$, consider the following diagram:

\[
K_j = \text{soc}(A_j) \cap J(A_j) \xrightarrow{i_j} A_j \xrightarrow{i_{K_j}} N_i
\]

where $i_1$, $i_2$ are inclusion maps and $i_{K_j}$, $i_{A_j}$ are injection maps. By hypothesis, there exists an $R$-homomorphism $h_j : A_j \rightarrow N$ such that $h_j \circ i_2 = f \circ i_{K_j}$, also there exists exactly one homomorphism $h : \bigoplus_{i=1}^n A_i \rightarrow N$ satisfying $h_j = h \circ i_{A_j}$ by [9, Theorem 4.1.6(2)]. Thus $\circ i_{K_j} = h_j \circ i_2 = h \circ i_{A_j} \circ i_2 = h \circ i_1 \circ i_{K_j}$ for all $j = 1, 2, \ldots , n$. Let $(a_1, a_2, \ldots , a_n) \in \bigoplus_{i=1}^n (\text{soc}(A_i) \cap J(A_i))$, thus $a_j \in \text{soc}(A_j) \cap J(A_j)$, for all $i = 1, 2, \ldots , n$ and, $f(a_1, a_2, \ldots , a_n) = f(i_{K_1}(a_1)) + f(i_{K_2}(a_2)) + \ldots + f(i_{K_n}(a_n)) = (h \circ i_1)(a_1, a_2, \ldots , a_n)$. Thus $f = h \circ i_1$ and the proof is complete.

**Corollary 2.9.** Let $M$ be a right $R$-module and $1 = e_1 + e_2 + \ldots + e_n$ in $R$ such that $e_i$ are orthogonal idempotent. Then $M$ is ss-injective if and only if $M$ is ss-$e_i R$-injective for every $i = 1, 2, \ldots , n$.

(2) For idempotents $e$ and $f$ of $R$. If $eR \cong fR$ and $M$ is ss-$eR$-injective, then $M$ is ss-$fR$-injective.

**Proof.** (1) From [3, Corollary 7.3], we have $R = \bigoplus_{i=1}^n e_i R$, thus it follows from Proposition 2.8 that $M$ is ss-injective if and only if $M$ is ss-$e_i R$-injective for all $1 \leq i \leq n$.

(2) This follows from Theorem 2.4(4).

**Proposition 2.10.** A right $R$-module $M$ is ss-injective if and only if $M$ is ss-$P$-injective, for every finitely generated projective right $R$-module $P$.

**Proof.** ($\Rightarrow$) Let $M$ be a ss-injective $R$-module, thus it follows from Proposition 2.8 that $M$ is ss-$R^n$-injective for any $n \in \mathbb{Z}^+$. Let $P$ be a finitely generated projective $R$-module, thus by [11, Corollary 5.5], we have that $P$ is a direct summand of a module isomorphic to $R^m$ for some $m \in \mathbb{Z}^+$. Since $M$ is ss-$R^m$-injective, thus $M$ is ss-$P$-injective by Theorem 2.4(2),(4).

($\Leftarrow$) By the fact that $R$ is projective.
**Proposition 2.11.** The following statements are equivalent for a right $R$-module $M$.

1. Every right $R$-module is ss-$M$-injective.
2. Every simple submodule of $M$ is ss-$M$-injective.
3. $\text{soc}(M) \cap J(M) = 0$.

**Proof.** (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3) Assume that $\text{soc}(M) \cap J(M) \neq 0$, thus $\text{soc}(M) \cap J(M) = \bigoplus_{i \in I} x_i R$ where $x_i R$ is a simple small submodule of $M$, for each $i \in I$. Therefore, $x_i R$ is ss-$M$-injective for each $i \in I$ by hypothesis. For any $i \in I$, the inclusion map from $x_i R$ to $M$ is split, so we have that $x_i R$ is a direct summand of $M$. Since $x_i R$ is small submodule of $M$, thus $x_i R = 0$ and hence $x_i = 0$ for all $i \in I$ and this a contradiction.

**Lemma 2.12.** Let $M$ be an ss-quasi-injective right $R$-module and $S = \text{End}(M_R)$, then the following statements hold:

1. $l_M r_R(m) = Sm$ for all $m \in \text{soc}(M) \cap J(M)$.
2. $r_R(m) \subseteq r_R(n)$, where $m \in \text{soc}(M) \cap J(M)$, $n \in M$ implies $Sn \subseteq Sm$.
3. $l_S(m R \cap r_M(\alpha)) = l_S(m) + S\alpha$, where $m \in \text{soc}(M) \cap J(M)$, $\alpha \in S$.
4. If $kR$ is a simple submodule of $M$, then $Sk$ is a simple left $S$-module, for all $k \in J(M)$. Moreover, $\text{soc}(M) \cap J(M) \subseteq \text{soc}(sM)$.
5. $\text{soc}(M) \cap J(M) \subseteq r_M(J(S))$.
6. $l_S(A \cap B) = l_S(A) + l_S(B)$, for every semisimple small right submodules $A$ and $B$ of $M$.

**Proof.** (1) Let $n \in l_M r_R(m)$, thus $r_R(m) \subseteq r_R(n)$. Now, let $\gamma : m R \longrightarrow M$ is given by $\gamma(m r) = nr$, thus $\gamma$ is a well define $R$-homomorphism. By hypothesis, there exists an endomorphism $\beta$ of $M$ such that $\beta_{m R} = \gamma$. Therefore, $n = \gamma(m) = \beta(m) \in Sm$, that is $l_M r_R(m) \subseteq Sm$. The inverse inclusion is clear.

(2) Let $n \in M$ and $m \in \text{soc}(M) \cap J(M)$. Since $r_R(m) \subseteq r_R(n)$, then $n \in l_M r_R(m)$. By (1), we have $n \in Sm$ as desired.

(3) If $f \in l_S(m) + S\alpha$, then $f = f_1 + f_2$ such that $f_1(m) = 0$ and $f_2 = g\alpha$, for some $g \in S$. For all $n \in m R \cap r_M(\alpha)$, we have $n = mr$ and $\alpha(n) = 0$ for some $r \in R$. Since $f_1(n) = f_1(m r) = f_1(m) = 0$ and $f_2(n) = g(\alpha(n)) = g(0) = 0$, thus $f \in l_S(m R \cap r_M(\alpha))$ and this implies that $l_S(m) + S\alpha \subseteq l_S(m R \cap r_M(\alpha))$. Now, we will prove that the other inclusion. Let $g \in l_S(m R \cap r_M(\alpha))$. If $r \in r_R(\alpha(m))$, then $\alpha(m r) = 0$, so $mr \in m R \cap r_M(\alpha)$ which yields $r_R(\alpha(m)) \subseteq r_R(g(m))$. Since $m \in \text{soc}(M) \cap J(M)$, thus $\alpha(m) \in \text{soc}(M) \cap J(M)$. By (2), we have that $g(m) = \gamma(\alpha(m))$ for some $\gamma \in S$. Therefore, $g - \gamma \alpha \in l_S(m)$ which leads to $g \in l_S(m) + S\alpha$. Thus $l_S(m R \cap r_M(\alpha)) = l_S(m) + S\alpha$.

(4) To prove $Sk$ is simple left $S$-module, we need only show that $Sk$ is cyclic for any nonzero element in it. If $0 \neq k \in Sk$, then $\alpha : k R \longrightarrow \alpha(k R)$ is an $R$-isomorphism. Since $\alpha \in S$, then $\alpha(k R) \ll M$. Since $M$ is ss-quasi-injective, thus $\alpha^{-1} : \alpha(k R) \longrightarrow K$ has an extension $\hat{\beta} \in S$ and hence $\beta(\alpha(k)) = \alpha^{-1}(\alpha(k)) = k$, so $k \in S \alpha k$ which leads to $Sk = S \alpha k$. Therefore $Sk$ is a simple left $S$-module and this leads to $\text{soc}(M) \cap J(M) \subseteq \text{soc}(sM)$.

(5) If $m R$ is simple and small submodule of $M$, then $m \neq 0$. We claim that $\alpha(m) = 0$ for all $\alpha \in J(S)$, thus $m R \subseteq r_M(J(S))$. Otherwise, $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. Thus $\alpha : m R \longrightarrow \alpha(m R)$ is an $R$-isomorphism. Now, we need prove that $r_R(\alpha(m)) = r_R(m)$. Let $r \in r_R(m)$, so $\alpha(m) r = \alpha(m r) = \alpha(0) = 0$ which leads to $r_R(m) \subseteq r_R(\alpha(m))$. The other inclusion, if $r \in r_R(\alpha(m))$, then $\alpha(m r) = 0$, that is $mr \in \ker(\alpha) = 0$, so $r \in r_R(m)$. Hence $r_R(\alpha(m)) = r_R(m)$. Since $m, \alpha(m) \in \text{soc}(M) \cap J(M)$, thus $S \alpha m = Sm$ (by (2)) and this implies that $m = \beta \alpha(m)$ for some $\beta \in S$, so $(1 - \beta \alpha)(m) = 0$. Since $\alpha \in J(S)$, then the element $\beta \alpha$ is quasi-regular by [3 Theorem 15.3]. Thus $1 - \beta \alpha$ is invertible and hence $m = 0$ which is a contradiction. This shows that $\text{soc}(M) \cap J(M) \subseteq r_M(J(S))$.  

5
(6) Let \( \alpha \in I_S(A \cap B) \) and consider \( f : A + B \rightarrow M \) is given by \( f(a + b) = \alpha(a) \), for all \( a \in A \) and \( b \in B \). Since \( M \) is ss-quasi-injective, thus there exists \( \beta \in S \) such that \( f(a + b) = \beta(a + b) \). Thus \( \beta(a + b) = \alpha(a) \), so \( (\alpha - \beta)(a) = \beta(b) \) which yields \( \alpha - \beta \in IS(A) \). Therefore, \( \alpha = \alpha - \beta + \beta \in IS(A) + IS(B) \) and this implies that \( IS(A \cap B) \subseteq IS(A) + IS(B) \). The other inclusion is trivial and the proof is complete.

\[ \square \]

**Remark 2.13.** Let \( M \) be a right \( R \)-module, then \( D(S) = \{ \alpha \in S = \text{End}(M) \mid r_M(\alpha) \cap mR \neq 0 \} \) for each \( 0 \neq m \in \text{soc}(M) \cap J(M) \) is a left ideal in \( S \).

**Proof.** This is obvious.

\[ \square \]

**Proposition 2.14.** Let \( M \) be an ss-quasi-injective right \( R \)-module. Then \( r_M(\alpha) \not\subseteq r_M(\alpha - \alpha\gamma\alpha) \), for all \( \alpha \notin D(S) \) and for some \( \gamma \in S \).

**Proof.** For all \( \alpha \notin D(S) \), By hypothesis, we can find \( 0 \neq m \in \text{soc}(M) \cap J(M) \) such that \( r_M(\alpha) \cap mR = 0 \). Clearly, \( r_M(\alpha) \cap mR = 0 \). Thus \( r \in r_M(\alpha(m)) \) and hence \( \alpha(m) \subseteq r_M(\alpha(m)) \subseteq r_M(\alpha(m)) \subseteq S\alpha m \) by Lemma 2.12(2). Therefore, \( m \in \ker(1 - \gamma\alpha) \) for some \( \gamma \in S \). Since \( m \neq 0 \), thus \( 1 - \gamma\alpha \) is not monomorphism and hence the inclusion holds. Now, let \( \alpha \in J(S) \) we have \( \beta\alpha \) is a quasi-regular element by [3]. Theorem 15.3] and hence \( 1 - \beta\alpha \) is isomorphism for all \( \beta \in S \), which completes the proof.

\[ \square \]

**Theorem 2.16.** (ss-Baer’s condition) The following statements are equivalent for a ring \( R \).

(1) \( M \) is an ss-injective right \( R \)-module.

(2) If \( S_r \cap J = A \oplus B \) and \( \alpha : A \rightarrow M \) is an \( R \)-homomorphism, then there exists \( m \in M \) such that \( \alpha(a) = ma \) for all \( a \in A \) and \( mB = 0 \).

(3) If \( S_r \cap J = A \oplus B \), and \( \alpha : A \rightarrow M \) is an \( R \)-homomorphism, then there exists \( m \in M \) such that \( \alpha(a) = ma \), for all \( a \in A \) and \( mB = 0 \).

**Proof.** (1)\( \Rightarrow \) (2) Define \( \gamma : S_r \cap J \rightarrow M \) by \( \gamma(a + b) = \alpha(a) \) for all \( a \in A, b \in B \). By hypothesis, there is a right \( R \)-homomorphism \( \beta : R \rightarrow M \) is an extension of \( \gamma \), so if \( m = \beta(1) \), then \( \alpha(a) = \gamma(a) = \beta(a) = \beta(1)a = ma \), for all \( a \in A \). Moreover, \( mb = \beta(b) = \gamma(b) = \alpha(0) = 0 \) for all \( b \), so \( mB = 0 \).

(2)\( \Rightarrow \) (1) Let \( \alpha : I \rightarrow M \) be any right \( R \)-homomorphism, where \( I \) is any semisimple small right ideal in \( R \). By (2), there exists \( m \in M \) such that \( \alpha(a) = ma \) for all \( a \in I \). Define \( \beta : R_R \rightarrow M \) by \( \beta(r) = mr \) for all \( r \in R \), thus \( \beta \) extends \( \alpha \).

(2)\( \Leftrightarrow \) (3) Clear.

A ring \( R \) is called right universally mininjective ring if it satisfies the condition \( S_r \cap J = 0 \) (see for example [14]). In the next results, we give new characterizations of universally mininjective ring in terms of ss-injectivity and soc-injectivity.

**Corollary 2.17.** The following are equivalent for a ring \( R \).

(1) \( R \) is right universally mininjective.

(2) \( R \) is right mininjective and every quotient of a soc-injective right \( R \)-module is soc-injective.

(3) \( R \) is right mininjective and every quotient of an injective right \( R \)-module is soc-injective.
(4) $R$ is right mininjective and every semisimple submodule of a projective right $R$-module is projective.
(5) Every right $R$-module is ss-injective.
(6) Every simple right ideal is ss-injective.

**Proof.** (1)⇔(2)⇔(3)⇔(4) By [14] Lemma 5.1 and [2] Corollary 2.9. (1)⇔(5)⇔(6) By Proposition 2.11. □

**Theorem 2.18.** If $M$ is a projective right $R$-module. Then the following statements are equivalent.
(1) Every quotient of an ss-M-injective right $R$-module is ss-M-injective.
(2) Every quotient of a soc-M-injective right $R$-module is ss-M-injective.
(3) Every quotient of an injective right $R$-module is ss-M-injective.
(4) Every sum of two ss-M-injective submodules of a right $R$-module is ss-M-injective.
(5) Every sum of two soc-M-injective submodules of a right $R$-module is ss-M-injective.
(6) Every sum of two injective submodules of a right $R$-module is ss-M-injective.
(7) Every semisimple small submodule of $M$ is projective.
(8) Every simple small submodule of $M$ is projective.
(9) $\text{soc}(M) \cap J(M)$ is projective.

**Proof.** (1)⇒(2)⇒(3), (4)⇒(5)⇒(6) and (9)⇒(7)⇒(8) are obvious.

(8)⇒(9) Since $\text{soc}(M) \cap J(M)$ is a direct sum of simple submodules of $M$ and since every simple in $J(M)$ is small in $M$, thus $\text{soc}(M) \cap J(M)$ is projective.

(3)⇒(7) Consider the following diagram:

$$
\begin{array}{ccc}
0 & \to & K & \xrightarrow{i} & M \\
& & f & \downarrow & \\
E & \xrightarrow{h} & N & \to & 0
\end{array}
$$

where $E$ and $N$ are right $R$-modules, $K$ is a semisimple small submodule of $M$, $h$ is a right $R$-epimorphism and $f$ is a right $R$-homomorphism. We can assume that $E$ is injective (see, e.g. [6] Proposition 5.2.10). Since $N$ is ss-$M$-injective, thus $f$ can be extended to an $R$-homomorphism $g : M \to N$. By projectivity of $M$, thus $g$ can be lifted to an $R$-homomorphism $\tilde{g} : M \to E$ such that $h \circ \tilde{g} = g$. Define $\tilde{f} : K \to E$ is the restriction of $\tilde{g}$ over $K$. Clearly, $h \circ \tilde{f} = f$ and this implies that $K$ is projective.

(7)⇒(1) Let $N$ and $L$ be right $R$-modules with $h : N \to L$ is an $R$-epimorphism and $N$ is ss-$M$-injective. Let $K$ be any semisimple small submodule of $M$ and let $f : K \to L$ be any left $R$-homomorphism. By hypothesis $K$ is projective, thus $f$ can be lifted to $R$-homomorphism $g : K \to N$ such that $h \circ g = f$. Since $N$ is ss-$M$-injective, thus there exists an $R$-homomorphism $\tilde{g} : M \to N$ such that $\tilde{g} \circ i = g$. Put $\tilde{f} = h \circ \tilde{g} : M \to L$. Thus $\tilde{f} \circ i = h \circ \tilde{g} \circ i = h \circ g = f$. Hence $L$ is an ss-$M$-injective right $R$-module.

(1)⇒(4) Let $N_1$ and $N_2$ be two ss-$M$-injective submodules of a right $R$-module $N$. Thus $N_1 + N_2$ is a homomorphic image of the direct sum $N_1 \oplus N_2$. Since $N_1 \oplus N_2$ is ss-$M$-injective, thus $N_1 + N_2$ is ss-$M$-injective by hypothesis.

(6)⇒(3) Let $E$ be an injective right $R$-module with submodule $N$. Let $Q = E \oplus E$. $K = \{(n,n) | n \in N\}$, $\tilde{Q} = Q/K$, $H_1 = \{y + K \in \tilde{Q} | y \in E \oplus 0\}$, $H_2 = \{y + K \in \tilde{Q} | y \in 0 \oplus E\}$. Then $\tilde{Q} = H_1 + H_2$. Since $(E \oplus 0) \cap K = 0$ and $(0 \oplus E) \cap K = 0$, thus $E \cong H_i, i = 1, 2$. Since $H_1 \cap H_2 = \{y + K \in \tilde{Q} | y \in N \oplus 0\} \equiv \{y + K \in \tilde{Q} | y \in 0 \oplus N\}$, then $H_1 \cap H_2 \cong N$ under $y \mapsto y + K$ for all $y \in N \oplus 0$. By hypothesis, $\tilde{Q}$ is ss-$M$-injective. Since $H_1$ is injective, thus $\tilde{Q} = H_2 \oplus A$ for some submodule $A$ of $\tilde{Q}$, so $A \cong (H_1 + H_2)/H_1 \cong H_2/H_1 \cap H_2 \cong E/N$. By Theorem 2.4.5, $E/N$ is ss-$M$-injective. □
Corollary 2.19. The following statements are equivalent.
(1) Every quotient of an ss-injective right $R$-module is ss-injective.
(2) Every quotient of a soc-injective right $R$-module is ss-injective.
(3) Every quotient of a small injective right $R$-module is ss-injective.
(4) Every quotient of an injective right $R$-module is ss-injective.
(5) Every sum of two ss-injective submodules of any right $R$-module is ss-injective.
(6) Every sum of two soc-injective submodules of any right $R$-module is ss-injective.
(7) Every sum of two small injective submodules of any right $R$-module is ss-injective.
(8) Every sum of two soc-injective submodules of any right $R$-module is ss-injective.
(9) Every semisimple small submodule of any projective right $R$-module is projective.
(10) Every semisimple small submodule of any finitely generated projective right $R$-module is projective.
(11) Every semisimple small submodule of $R_R$ is projective.
(12) Every simple small submodule of $R_R$ is projective.
(13) $S_r$ is projective.
(14) $S_r$ is projective.

Proof. The equivalence of (1), (2), (4), (5), (6), (8), (11), (12) and (13) is from Theorem 2.18.
(1)⇒(3)⇒(4), (5)⇒(7)⇒(8) and (9)⇒(10)⇒(13) are clear.
(14)⇒(9) By [2, Corollary 2.9].
(13)⇒(14) Let $S_r = (S_r \cap J) \oplus A$, where $A = \bigoplus_{i \in I} S_i$ and $S_i$ is a right simple and summand of $R_R$
for all $i \in I$. Thus $A$ is projective, but $S_r \cap J$ is projective, so it follows that $S_r$ is projective.  

Theorem 2.20. If every simple singular right $R$-module is ss-injective, then $r(a) \subseteq R_R$ for
every $a \in S_r \cap J$ and $S_r$ is projective.

Proof. Let $a \in S_r \cap J$ and let $A = RaR + r(a)$. Thus there exists a right ideal $B$ of $R$ such that
$A \oplus B \subseteq R_R$. Suppose that $A \oplus B \neq R_R$, thus we choose $I \subseteq R$ such that $A \oplus B \subseteq I$ and so
$I \subseteq R_R$. By hypothesis, $R/I$ is a right ss-injective. Consider the map $\alpha : aR \rightarrow R/I$ is given
by $\alpha(ar) = r + I$ which is a well-defined $R$-homomorphism. Thus there exists $c \in R$ such that
$1 + I = ca + I$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$ which leads to $1 \in I$, a contradiction.
Thus $A \oplus B = R$ and hence $RaR + (r(a) \oplus B) = R$. Since $RaR \ll L_R$, thus $r(a) \subseteq R_R$. Put
$r(a) = (1 - e)R$, for some $e^2 = e \in R$, so it follows that $ax = aex$ for all $x \in R$ and hence
$aR = aeR$. Let $\gamma : eR \rightarrow aeR$ be defined by $\gamma(er) = aer$ for all $r \in R$. Then $\gamma$ is a well-
definite $R$-epimorphism. Clearly, $\ker(\gamma) = eR \cap r(a)$. Hence $\gamma$ is an isomorphism and so $ar$ is
projective. Since $S_r \cap J$ is a direct sum of simple small right ideals, thus $S_r \cap J$ is projective and
it follows from Corollary 2.19 that $S_r$ is projective.  

Corollary 2.21. The following statements are equivalent for a ring $R$.
(1) $R$ is right mininjective and every simple singular right $R$-module is ss-injective.
(2) $R$ is right universally mininjective.

Proof. By Theorem 2.20 and [14, Lemma 5.1].  

Recall that a ring $R$ is called zero insertive, if $aRb = 0$ for each $a, b \in R$ with $ab = 0$ (see [19]). Note that if $R$ is zero insertive ring, then $RaR + r(a) \subseteq R_R$ for every $a \in R$ (see [19, Lemma 2.11]).

Proposition 2.22. Let $R$ be a zero insertive ring. If every simple singular right $R$-module is
ss-injective, then $R$ is right universally mininjective.
Proof. Let \( a \in S_r \cap J \). We claim that \( RaR + r(a) = R \), thus \( r(a) = R \) (since \( RaR \ll R \)), so \( a = 0 \) and this means that \( S_r \cap J = 0 \). Otherwise, if \( RaR + r(a) \subseteq R \), then there exists a maximal right ideal \( I \) of \( R \) such that \( RaR + r(a) \subseteq I \). Since \( I \subseteq R \) is ss-injective by hypothesis. Consider \( \alpha : aR \to R/I \) is given by \( \alpha(ar) = r + I \) for all \( r \in R \) which is a well-defined \( R \)-homomorphism. Thus \( 1 + I = ca + I \) for some \( c \in R \). Since \( ca \in RaR \subseteq I \), thus \( 1 \in I \) and this contradicts with a maximality of \( I \), so we must have \( RaR + r(a) = R \) and this completes the proof. \( \square \)

Theorem 2.23. If \( M \) is a finitely generated right \( R \)-module, then the following statements are equivalent.

1. \( \text{soc}(M) \cap J(M) \) is a Noetherian \( R \)-module.
2. \( \text{soc}(M) \cap J(M) \) is finitely generated.
3. Any direct sum of ss-\( M \)-injective right \( R \)-modules is ss-\( M \)-injective.
4. Any direct sum of soc-\( M \)-injective right \( R \)-modules is ss-\( M \)-injective.
5. Any direct sum of injective right \( R \)-modules is ss-\( M \)-injective.
6. \( K^{(S)} \) is ss-\( M \)-injective for every injective right \( R \)-module \( K \) and for any index set \( S \).
7. \( K^{(N)} \) is ss-\( M \)-injective for every injective right \( R \)-module \( K \).

Proof. (1)\( \Rightarrow \) (2) and (3)\( \Rightarrow \) (4)\( \Rightarrow \) (5)\( \Rightarrow \) (6)\( \Rightarrow \) (7) Clear.

(2)\( \Rightarrow \) (3) Let \( E = \bigoplus_{i \in I} M_i \) be a direct sum of ss-\( M \)-injective right \( R \)-modules and \( f : N \to E \) be a right \( R \)-homomorphism, where \( N \) is a semisimple small submodule of \( M \). Since \( \text{soc}(M) \cap J(M) \) is finitely generated, thus \( N \) is finitely generated and hence \( f(N) \subseteq \bigoplus_{j \in J} M_j \), for some finite subset \( I_1 \) of \( I \). Since a finite direct sums of ss-\( M \)-injective right \( R \)-modules is ss-\( M \)-injective, thus \( \bigoplus_{j \in J} M_j \) is ss-\( M \)-injective and hence \( f \) can be extended to an \( R \)-homomorphism \( g : M \to E \).

Thus \( E \) is ss-\( M \)-injective.

(7)\( \Rightarrow \) (1) Let \( N_1 \subseteq N_2 \subseteq \ldots \) be a chain of submodules of \( \text{soc}(M) \cap J(M) \). For each \( i \geq 1 \), let \( E_i = E(M/N_i) \), \( E = \bigoplus_{i = 1}^{\infty} E_i \) and \( M_i = \prod_{j = 1}^{\infty} E_j = E_i \oplus \bigoplus_{j \neq i} E_j \), then \( M_i \) is injective. By hypothesis, \( \bigoplus_{i = 1}^{\infty} M_i = \bigoplus_{i = 1}^{\infty} E_i \oplus \bigoplus_{j \neq i} E_j \) is ss-\( M \)-injective, so it follows from Theorem 2.4(5) that \( E \) itself is ss-\( M \)-injective. Define \( f : U = \bigcup_{i = 1}^{\infty} N_i \to E \) by \( f(m) = (m+N_i) \). It is clear that \( f \) is a well-defined \( R \)-homomorphism. Since \( M \) is finitely generated, thus \( \text{soc}(M) \cap J(M) \) is a semisimple small submodule of \( M \) and hence \( \bigcup_{i = 1}^{\infty} N_i \) is a semisimple small submodule of \( M \), so \( f \) can be extended to a right \( R \)-homomorphism \( g : M \to E \). Since \( M \) is finitely generated, we have \( g(M) \subseteq \bigoplus_{i = 1}^{n} E(M/N_i) \) for some \( n \) and hence \( f(\bigcup_{i = 1}^{\infty} N_i) \subseteq \bigoplus_{i = 1}^{n} E(M/N_i) \). Since \( \pi_i f(x) = \pi_i (x + N_j)_{\geq 1} = x + N_i \), for all \( x \in U \) and \( i \geq 1 \), where \( \pi_i : \bigoplus_{i = 1}^{n} E(M/N_i) \to E(M/N_i) \) be the projection map, thus \( \pi_i f(U) = U/N_i \) for all \( i \geq 1 \). Since \( f(U) \subseteq \bigoplus_{i = 1}^{n} E(M/N_i) \), thus \( U/N_i = \pi_i f(U) = 0 \), for all \( i \geq n + 1 \), so \( U = N_i \) for all \( i \geq n + 1 \) and hence the chain \( N_1 \subseteq N_2 \subseteq \ldots \) terminates at \( N_{n+1} \).

Thus \( \text{soc}(M) \cap J(M) \) is a Noetherian \( R \)-module. \( \square \)

Corollary 2.24. If \( N \) is a finitely generated right \( R \)-module, then the following statements are equivalent.

1. \( \text{soc}(N) \cap J(N) \) is finitely generated.
2. \( M^{(S)} \) is ss-\( N \)-injective for every soc-\( N \)-injective right \( R \)-module \( M \) and for any index set \( S \).
3. \( M^{(S)} \) is ss-\( N \)-injective for every ss-\( N \)-injective right \( R \)-module \( M \) and for any index set \( S \).
(4) $M^{(N)}$ is ss-N-injective for every soc-N-injective right $R$-module $M$.
(5) $M^{(N)}$ is ss-N-injective for every ss-N-injective right $R$-module $M$.

Proof. By Theorem \ref{2.23}.

**Corollary 2.25.** The following statements are equivalent.
(1) $S_r \cap J$ is finitely generated.
(2) Any direct sum of ss-injective right $R$-modules is ss-injective.
(3) Any direct sum of soc-injective right $R$-modules is ss-injective.
(4) Any direct sum of small injective right $R$-modules is ss-injective.
(5) Any direct sum of injective right $R$-modules is ss-injective.
(6) $M^{(S)}$ is ss-injective for every injective right $R$-module $M$ and for any index set $S$.
(7) $M^{(S)}$ is ss-injective for every soc-injective right $R$-module $M$ and for any index set $S$.
(8) $M^{(S)}$ is ss-injective for every small injective right $R$-module $M$ and for any index set $S$.
(9) $M^{(N)}$ is ss-injective for every ss-injective right $R$-module $M$ and for any index set $S$.
(10) $M^{(N)}$ is ss-injective for every soc-injective right $R$-module $M$.
(11) $M^{(N)}$ is ss-injective for every small injective right $R$-module $M$.
(12) $M^{(N)}$ is ss-injective for every ss-injective right $R$-module $M$.

Proof. By applying Theorem \ref{2.23} and Corollary \ref{2.24}.

**Remark 2.26.** Let $M$ be a right $R$-module. We denote that $r_u(N) = \{a \in S_r \cap J \mid Na = 0\}$ and $l_M(K) = \{m \in M \mid mK = 0\}$ where $N \subseteq M$ and $K \subseteq S_r \cap J$. Clearly, $r_u(N) \subseteq (S_r \cap J)_R$ and $l_M(K) \subseteq \mathfrak{s}M$, where $S = \text{End}(M_R)$ and we have the following:
(1) $N \subseteq l_Mr_u(N)$ for all $N \subseteq M$.
(2) $K \subseteq r_u l_M(K)$ for all $K \subseteq S_r \cap J$.
(3) $r_lI = r_u(N)$ for all $N \subseteq M$.
(4) $l_Mr_u l_M(K) = l_M(K)$ for all $K \subseteq S_r \cap J$.

Proof. This is clear.

**Lemma 2.27.** The following statements are equivalent for a right $R$-module $M$:
(1) $R$ satisfies the ACC for right ideals of form $r_u(N)$, where $N \subseteq M$.
(2) $R$ satisfies the DCC for $l_M(K)$, where $K \subseteq S_r \cap J$.
(3) For each semisimple small right ideal $I$ there exists a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$.

Proof. (1)$\iff$(2). Clear.

(2)$\implies$(3) Consider $\Omega = \{l_M(A) \mid A \text{ is finitely generated right ideal and } A \subseteq I \}$ which is non empty set because $M \in \Omega$. Now, let $K$ be a finitely generated right ideal of $R$ and contained in $I$. such that $l_M(K)$ is minimal in $\Omega$. Put $B = K + xR$, where $x \in I$. Thus $B$ is a finitely generated right ideal contained in $I$ and $l_M(B) \subseteq l_M(K)$. But since $l_M(K)$ is minimal in $\Omega$, thus $l_M(B) = l_M(K)$ which yields $l_M(K)(x) = 0$ for all $x \in I$. Therefore, $l_M(K)_I = 0$ and hence $l_M(K) \subseteq l_M(I)$. But $l_M(I) \subseteq l_M(K)$, so $l_M(I) = l_M(K)$.

(3)$\implies$(1) Suppose that $r_u(M_1) \subseteq r_u(M_2) \subseteq \ldots \subseteq r_u(M_n) \subseteq \ldots$, where $M_i \subseteq M$ for each $i$. Put $D_i = l_Mr_u(M_i)$ for each $i$, and $I = \bigcup_{i=1}^{\infty} r_u(M_i)$, then $I \subseteq S_r \cap J$. By hypothesis, there exists a finitely generated right ideal $K$ of $R$ and contained in $I$ such that $l_M(I) = l_M(K)$. Since $K$ is a finitely generated, thus there exists $t \in \mathbb{N}$ such that $K \subseteq r_u(M_n)$ for all $n \geq t$, that is $l_M(K) \supseteq l_Mr_u(M_n) = D_n$ for all $n \geq t$. Since $l_M(K) = l_M(I) = l_M(\bigcup_{i=1}^{\infty} r_u(M_i)) = \bigcap_{i=1}^{\infty} l_Mr_u(M_i) = \bigcap_{i=1}^{\infty} D_i \subseteq \bigcap_{n \geq t} D_n$, thus $l_M(K) = D_n$ for all $n \geq t$. Since $D_n = l_Mr_u(M_n)$, thus $r_u(M_n) = r_u l_Mr_u(M_n) = r_u(D_n) = r_u(l_M(K))$ for all $n \geq t$. Thus $r_u(M_n) = r_u(M_t)$ for all $n \geq t$. and hence (3) implies (1), which completes the proof.

\[ \square \]
The first part in following proposition is obtained directly by Corollary 2.25. but we will prove it by different way.

**Proposition 2.28.** Let $E$ be an ss-injective right $R$-module. Then $E^{(N)}$ is ss-injective if and only if $R$ satisfies the ACC for right ideals of form $r_a(N)$, where $N \subseteq E$.

**Proof.** ($\Rightarrow$) Suppose that $r_a(N_1) \subseteq r_a(N_2) \subseteq \ldots \subseteq r_a(N_m) \subseteq \ldots$ be a strictly chain, where $N_i \subseteq E$. Thus we get, $I_{E r_a}(N_1) \supseteq I_{E r_a}(N_2) \supseteq \ldots \supseteq I_{E r_a}(N_m) \supseteq \ldots$. For each $i \geq 1$, so we can find $t_i \in I_{E r_a}(N_i) \setminus I_{E r_a}(N_{i+1})$ and $a_{i+1} \in r_a(N_{i+1})$ such that $t_i a_{i+1} \neq 0$. Let $L = \bigcup_{i=1}^{m} r_a(N_i)$, then for all $\ell \in L$ there exists $m_\ell \geq 1$ such that $\ell \in r_a(N_i)$ for all $i \geq m_\ell$ and this implies that $t_i \ell = 0$ for all $i \geq m_\ell$. Put $i = (t_i)_i$, we have $i_\ell \in E^{(N)}$ for every $\ell \in L$. Consider $\alpha_i : L \rightarrow E^{(N)}$ is given by $\alpha_i(\ell) = i_\ell$, then $\alpha_i$ is a well-defined $R$-homomorphism. Since $L$ is a semisimple small right ideal, thus $\alpha_i$ extends to $\gamma : R \rightarrow E^{(N)}$ (by hypothesis) and hence $\alpha_i(\ell) = i_\ell = \gamma(\ell) = \gamma(1) \ell$. Thus there exists $k \geq 1$ such that $\gamma(1) \ell = 0$ for all $i \geq k$ and all $\ell \in L$ (since $\gamma(1) \in E^{(N)}$), but this contradicts with $t_i a_{k+1} \neq 0$.

($\Leftarrow$) Let $\alpha : I \rightarrow E^{(N)}$ be an $R$-homomorphism, where $I$ is a semisimple small right ideal, thus it follows from Lemma 2.27 that there is a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$. Since $E^{(N)}$ is ss-injective, thus $\alpha = a$. for some $a \in E^{N}$. Write $K = \bigoplus_{i=1}^{m} r_i R$, so we have $\alpha(r_i) = a r_i \in E^{(N)}$, $i = 1, 2, \ldots, m$. Thus there exists $a \in E^{(N)}$ such that $a r_i = a_\alpha r_i$ for all $n \in \mathbb{N}$, $i = 1, 2, \ldots, m$, where $a_\alpha$ is the $n$th-coordinate of $a$. Since $K$ is generated by $\{r_1, r_2, \ldots, r_m\}$, thus $a r = a_\alpha r$ for all $r \in K$. Therefore, $a_n - a_\alpha \in l_M(K) = l_M(I)$ for all $n \in \mathbb{N}$ which leads to $a_n r = a_\alpha r$ for all $r \in I$ and $n \in \mathbb{N}$, so $a r = a_\alpha r$ for all $r \in I$. Thus there exists $a \in E^{(N)}$ such that $\alpha(r) = a_\alpha r$ for all $r \in I$ and this means that $E^{(N)}$ is ss-injective.

**Theorem 2.29.** The following statements are equivalent for a ring $R$:

(1) $S_r \cap J$ is finitely generated.

(2) $\bigoplus_{i=1}^{m} E(M_i)$ is ss-injective right $R$-module for every simple right $R$-modules $M_i$, $i \geq 1$.

**Proof.** (1)$\Rightarrow$(2) By Corollary 2.25.

(2)$\Rightarrow$(1) Let $I_1 \subseteq I_2 \subseteq \ldots$ be a properly ascending chain of semisimple small right ideals of $R$. Clearly, $I = \bigcup_{i=1}^{m} I_i \subseteq S_r \cap J$. For every $i \geq 1$, there exists $a_i \in I_i$, $a_i \notin I_i$ and consider $N_i/I_i \subseteq \bigoplus_{i=1}^{m} (a_i R + I_i)/I_i$, so $K_i = (a_i R + I_i)/N_i$ is a simple right $R$-module. Define $\alpha_i : (a_i R + I_i)/I_i \rightarrow (a_i R + I_i)/N_i$ by $\alpha_i(x + I_i) = x + N_i$ which is right $R$-epimorphism. Let $E(K_i)$ be the injective hull of $K_i$ and $i_i : K_i \rightarrow E(K_i)$ be the inclusion map. By injectivity of $E(K_i)$, there exists $\beta_i : I_i \rightarrow E(K_i)$ such that $\beta_i = i_i \alpha_i$. Since $a_i \notin N_i$, then $\beta_i(a_i + I_i) = i_i(\alpha_i(a_i + I_i)) = a_i + N_i \neq 0$ for each $i \geq 1$. If $b \in I$, then there exists $n_b \geq 1$ such that $b \in I_i$ for all $i \geq n_b$ and hence $\beta_i(b + I_i) = 0$ for all $i \geq n_b$. Thus we can define $\gamma : I \rightarrow \bigoplus_{i=1}^{m} E(K_i)$ by $\gamma(b) = (\beta_i(b + I_i))_i$.

Then there exists $\gamma : R \rightarrow \bigoplus_{i=1}^{m} E(K_i)$ such that $\gamma(1) = \gamma$ (by hypothesis). Put $\gamma(1) = (c_i)_i$, thus there exists $n \geq 1$ with $c_i = 0$ for all $i \geq n$. Since $\beta_i(b + I_i)_i = \gamma(b) = \gamma(b) = \gamma(1)b = (c_i b)_i$ for all $b \in I$, thus $\beta_i(b + I_i) = c_i b$ for all $i \geq 1$, so it follows that $\beta_i(b + I_i) = 0$ for all $i \geq n$ and all $b \in I$ and this contradicts with $\beta_n(a_n + I_n) \neq 0$. Hence (2) implies (1).

### 3 Strongly SS-Injective Modules

**Proposition 3.1.** The following statements are equivalent:

(1) $M$ is a strongly ss-injective right $R$-module.
(2) Every $R$-homomorphism $\alpha : A \to M$ extends to $N$, for all right $R$-module $N$, where $A \ll N$ and $\alpha(A)$ is a semisimple submodule in $M$.

Proof. (2)⇒(1) Clear.

(1)⇒(2) Let $A$ be a small submodule of $N$, and $\alpha : A \to M$ be an $R$-homomorphism with $\alpha(A)$ is a semisimple submodule of $M$. If $B = \ker(\alpha)$, then $\alpha$ induces an embedding $\bar{\alpha} : A/B \to M$ defined by $\bar{\alpha}(a + B) = \alpha(a)$, for all $a \in A$. Clearly, $\bar{\alpha}$ is well defined because if $a_1 + B = a_2 + B$ we have $a_1 - a_2 \in B$, so $\alpha(a_1) = \alpha(a_2)$, that is $\bar{\alpha}(a_1 + B) = \bar{\alpha}(a_2 + B)$. Since $M$ is strongly ss-injective and $A/B$ is semisimple and small in $N/B$, thus $\bar{\alpha}$ extends to an $R$-homomorphism $\gamma : N/B \to M$. If $\pi : N \to N/B$ is the canonical map, then the $R$-homomorphism $\tilde{\beta} = \gamma \circ \pi : N \to M$ is an extension of $\alpha$ such that if $a \in A$, then $\tilde{\beta}(a) = \gamma(\pi(a)) = \gamma((a + B)) = \bar{\alpha}(a + B) = \alpha(a)$ as desired.

Corollary 3.2. (1) Let $M$ be a semisimple right $R$-module. If $M$ is a strongly ss-injective, then $M$ is small injective.

(2) If every simple right $R$-module is strongly ss-injective, then $R$ is semiprimitive.

Proof. (1) By Proposition 3.1

(2) By (1) and applying [19, Theorem 2.8].

Remark 3.3. The converse of Corollary 3.2 is not true (see Example 3.8).

Theorem 3.4. If $M$ is a strongly ss-injective (or just ss-$E(M)$-injective) right $R$-module, then for every semisimple small submodule $A$ of $M$, there is an injective $R$-module $E_A$ such that $M = E_A \oplus T_A$ where $T_A$ is a submodule of $M$ with $T_A \cap A = 0$. Moreover, if $A \neq 0$, then $E_A$ can be taken $A \leq^{ess} E_A$.

Proof. Let $A$ be a semisimple small submodule of $M$. If $A = 0$, we are done by taking $E_A = 0$ and $T_A = M$. Suppose that $A \neq 0$ and let $i_1, i_2$ and $i_3$ be inclusion maps and $D_A = E(A)$ be the injective hull of $A$ in $E(M)$. Since $M$ is strongly ss-injective, thus $M$ is ss-$E(M)$-injective. Since $A$ is a semisimple small submodule of $M$, it follows from [9, Lemma 5.1.3(a)] that $A$ is a semisimple small submodule in $E(M)$ and hence there exists an $R$-homomorphism $\alpha : E(M) \to M$ such that $\alpha i_3 i_1 = i_3$. Put $\beta = \alpha i_2$, thus $\beta : D_A \to M$ is an extension of $i_3$. Since $A \leq^{ess} D_A$, then $\beta$ is a monomorphism. Put $E_A = \beta(D_A)$. Since $E_A$ is an injective submodule of $M$, thus $M = E_A \oplus T_A$ for some submodule $T_A$ of $M$. Since $\beta(A) = A$, thus $A \leq \beta(D_A) = E_A$ and this means that $T_A \cap A = 0$. Moreover, define $\tilde{\beta} = \beta : D_A \to E_A$, thus $\tilde{\beta}$ is an isomorphism. Since $A \leq^{ess} D_A$, thus $\tilde{\beta}(A) \leq^{ess} E_A$. But $\tilde{\beta}(A) = \beta(A) = A$, so $A \leq^{ess} E_A$.

Corollary 3.5. If $M$ is a right $R$-module has a semisimple small submodule $A$ such that $A \leq^{ess} M$, then the following conditions are equivalent.

(1) $M$ is injective.

(2) $M$ is strongly ss-injective.

(3) $M$ is ss-$E(M)$-injective.

Proof. (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(1) By Theorem 3.4 we can write $M = E_A \oplus T_A$ where $E_A$ injective and $T_A \cap A = 0$. Since $A \leq^{ess} M$, thus $T_A = 0$ and hence $M = E_A$. Therefore, $M$ is an injective $R$-module.

Example 3.6. $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective. In particular, $\mathbb{Z}_4$ is not ss-$\mathbb{Z}_{2^\infty}$-injective.

Proof. Assume that $\mathbb{Z}_4$ is strongly ss-injective $\mathbb{Z}$-module. Let $A = \ll 2 = \{0, 2\}$. It is clear that $A$ is a semisimple small and essential submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module. Thus by Corollary 3.5 we have that $\mathbb{Z}_4$ is injective $\mathbb{Z}$-module and this is a contradiction. Thus $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective. Since $E(\mathbb{Z}_{2^\infty}) = \mathbb{Z}_{2^\infty}$ as $\mathbb{Z}$-module, thus $\mathbb{Z}_4$ is not ss-$\mathbb{Z}_{2^\infty}$-injective, by Corollary 3.5.
**Corollary 3.7.** Let $M$ be a right $R$-module such that $soc(M) \cap J(M)$ is small submodule in $M$ (in particular, if $M$ is finitely generated). If $M$ is strongly ss-injective, then $M = E \oplus T$, where $E$ is injective and $T \cap soc(M) \cap J(M) = 0$. Moreover, if $soc(M) \cap J(M) \neq 0$, then we can take $soc(M) \cap J(M) \leq \text{ess } E.

\text{Proof. By taking } A = soc(M) \cap J(M) \text{ and applying Theorem 3.4.} \ \square

The following example shows that the converse of Theorem 3.4 and Corollary 3.7 is not true.

**Example 3.8.** Let $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module. Since $J(M) = 0$ and $soc(M) = M$, thus $soc(M) \cap J(M) = 0$. So, we can write $M = 0 \oplus M$ with $M \cap (soc(M) \cap J(M)) = 0$. Let $N = \mathbb{Z}_8$ as $\mathbb{Z}$-module. Since $J(N) =\langle \bar{2} \rangle$ and $soc(N) =\langle \bar{4} \rangle$. Define $\gamma : soc(N) \cap J(N) \rightarrow M$ by $\gamma(\bar{4}) = \bar{3}$, thus $\gamma$ is a $\mathbb{Z}$-homomorphism. Assume that $M$ is strongly ss-injective, thus $M$ is ss-$N$-injective, so there exists $\mathbb{Z}$-homomorphism $\beta : N \rightarrow M$ such that $\beta \circ i = \gamma$, where $i$ is the inclusion map from $soc(N) \cap J(N)$ to $N$. Since $\beta(J(N)) \subseteq J(M)$, thus $\bar{3} = \gamma(\bar{4}) = \beta(\bar{4}) \in \beta(J(N)) \subseteq J(M) = 0$ and this contradiction, so $M$ is not strongly ss-injective $\mathbb{Z}$-module.

**Corollary 3.9.** The following statements are equivalent:

1. $soc(M) \cap J(M) = 0$, for all right $R$-module $M$.
2. Every right $R$-module is strongly ss-injective.
3. Every simple right $R$-module is strongly ss-injective.

\text{Proof. By Proposition 2.11.} \ \square

Recall that a ring $R$ is called a right $V$-ring ($GV$-ring, $SL$-ring, respectively) if every simple (simple singular, singular, respectively) right $R$-module is injective. A right $R$-module $M$ is called strongly $s$-injective if every $R$-homomorphism from $K$ to $M$ extends to $N$ for every right $R$-module $N$, where $K \subseteq Z(N)$ (see [22]). A submodule $K$ of a right $R$-module $M$ is called $t$-essential in $M$ (written $K \subseteq_{t\text{-ess}} M$) if for every submodule $L$ of $M$, $K \cap L \subseteq Z_2(M)$ implies that $L \subseteq Z_2(M)$. $M$ is said to be $t$-semisimple if for every submodule $A$ of $M$ there exists a direct summand $B$ of $M$ such that $B \subseteq_{t\text{-ess}} A$ (see [4]). In the next results, we will give some relations between ss-injectivity and other injectivities and we provide many new equivalences of $V$-rings, $GV$-rings, $SL$ rings and $QF$ rings.

**Lemma 3.10.** Let $M/N$ be a semisimple right $R$-module and $C$ any right $R$-module. Then every homomorphism from a right submodule (resp. a right semisimple submodule) $A$ of $M$ to $C$ can be extended to a homomorphism from $M$ to $C$ if and only if every homomorphism from a right submodule (resp. a right semisimple submodule) $B$ of $N$ to $C$ can be extended to a homomorphism from $M$ to $C$.

\text{Proof.} ($\Rightarrow$) is obtained directly.

($\Leftarrow$) Let $f$ be a right $R$-homomorphism from a right submodule $A$ of $M$ to $C$. Since $M/N$ is semisimple, thus there exists a right submodule $L$ of $M$ such that $A + L = M$ and $A \cap L \subseteq N$ (see [11] Proposition 2.1)). Thus there exists a right $R$-homomorphism $g : M \rightarrow C$ such that $g(x) = f(x)$ for all $x \in A \cap L$. Define $h : M \rightarrow C$ such that for any $x = a + \ell$, $a \in A$, $\ell \in L$, 

$h(x) = f(a) + g(\ell)$. Thus $h$ is a well define $R$-homomorphism, because if $a_1 + \ell_1 = a_2 + \ell_2$, $a_i \in A$, $\ell_i \in L$, $i = 1, 2$, then $a_1 - a_2 = \ell_2 - \ell_1 \in A \cap L$, that is $f(a_1 - a_2) = g(\ell_2 - \ell_1)$ which leads to $h(a_1 + \ell_1) = h(a_2 + \ell_2)$. Thus $h$ is a well define $R$-homomorphism and extension of $f$. \ \square
Corollary 3.11. For right $R$-modules $M$ and $N$, then the following hold:
(1) If $M$ is finitely generated and $M/J(M)$ is semisimple right $R$-module, then $N$ is right soc-$M$-injective if and only if $N$ is right ss-$M$-injective.
(2) If $M/soc(M)$ is a semisimple right $R$-module, then $N$ is soc-$M$-injective if and only if $N$ is $M$-injective.
(3) If $R/S_r$ is semisimple right $R$-module, then $N$ is soc-injective if and only if $N$ is injective.
(4) If $R/S_r$ is semisimple right $R$-module, then $N$ is ss-injective if and only if $N$ is small injective.

Proof. (1). $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Since $N$ is a right ss-$M$-injective, thus every homomorphism from a semisimple small submodule of $M$ to $N$ extends to $M$. Since $M$ is finitely generated, thus $J(M) \ll M$ and hence every homomorphism from any semisimple submodule of $J(M)$ to $N$ extends to $M$. Since $M/J(M)$ is semisimple, thus every homomorphism from any semisimple submodule of $M$ to $N$ extends to $M$ by Lemma 3.10. Therefore $N$ is a soc-$M$-injective right $R$-module.

(2). $(\Rightarrow)$ Since $N$ is soc-$M$-injective. Thus every homomorphism from any submodule of $soc(M)$ to $N$ extends to $M$. Since $M/soc(M)$ is semisimple, thus Lemma 3.10 implies that every homomorphism from any submodule of $M$ to $N$ extends to $M$. Hence $N$ is $M$-injective.

$(\Leftarrow)$ Clear.

(3) By (2).

(4) Since $R/S_r$ is semisimple right $R$-module, thus $J(R/S_r) = 0$. By [9, Theorem 9.1.4(b)], we have $J \subseteq S_r$ and hence $J = J \cap S_r$. Thus $N$ is ss-injective if and only if $N$ is small injective.

Corollary 3.12. Let $R$ be a semilocal ring, then $S_r \cap J$ is finitely generated if and only if $S_r$ is finitely generated.

Proof. Suppose that $S_r \cap J$ is finitely generated. By Corollary 2.25 every direct sum of soc-injective right $R$-modules is ss-injective. Thus it follows from Corollary 3.11(1) and [2 Corollary 2.11] that $S_r$ is finitely generated. The converse is clear.

Theorem 3.13. If $R$ is a right perfect ring, then a right $R$-module $M$ is strongly soc-injective if and only if $M$ is strongly ss-injective.

Proof. $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Let $R$ be a right perfect ring and $M$ be a strongly ss-injective right $R$-module. By [11 Theorem 3.8], $R$ is a semilocal ring and hence by [11 Theorem 3.5], we have every right $R$-module $N$ is semilocal and hence $N/J(N)$ is semisimple right $R$-module. Since $R$ is a right perfect ring, thus the Jacobson radical of every right $R$-module is small by [7 Theorem 4.3 and 4.4, p. 69]. Thus $N/J(N)$ is semisimple and $J(N) \ll N$, for any $N \in Mod-R$. Since $M$ is strongly ss-injective, thus every homomorphism from a semisimple small submodule of $N$ to $M$ extends to $N$, for every $N \in Mod-R$, and this implies that every homomorphism from any semisimple submodule of $J(N)$ to $M$ extends to $N$, for every $N \in Mod-R$. Since $N/J(N)$ is semisimple right $R$-module, for every $N \in Mod-R$. Thus Lemma 3.10 implies that every homomorphism from any semisimple submodule of $N$ to $M$ extends to $N$, for every $N \in Mod-R$ and hence $M$ is strongly soc-injective.

Corollary 3.14. A ring $R$ is QF ring if and only if every strongly ss-injective right $R$-module is projective.

Proof. $(\Rightarrow)$ If $R$ is QF ring, then $R$ is a right perfect ring, so by Theorem 3.13 and [2 Proposition 3.7] we have every strongly ss-injective right $R$-module is projective.

$(\Leftarrow)$ By hypothesis we have every injective right $R$-module is projective and hence $R$ is QF ring (see for instance [6 Proposition 12.5.13]).
Theorem 3.15. The following statements are equivalent for a ring $R$.

1. Every direct sum of strongly ss-injective right $R$-modules is injective.
2. Every direct sum of strongly soc-injective right $R$-modules is injective.
3. $R$ is right artinian.

**Proof.** 
(1)⇒(2) Clear.

(2)⇒(3) Since every direct sum of strongly soc-injective right $R$-modules is injective, thus $R$ is right noetherian and right semiartinian by [2, Theorem 3.3 and Theorem 3.6], so it follows from [18 Proposition 5.2, p.189] that $R$ is right artinian.

(3)⇒(1) By hypothesis, $R$ is right perfect and right noetherian. It follows from Theorem 3.13 and [2, Theorem 3.3] that every direct sum of strongly ss-injective right $R$-modules is strongly soc-injective. Since $R$ is right semiartinian, so [2, Theorem 3.6] implies that every direct sum of strongly ss-injective right $R$-modules is injective.

Theorem 3.16. If $R$ is a right $t$-semisimple, then a right $R$-module $M$ is injective if and only if $M$ is strongly $s$-injective.

**Proof.** 
($⇒$) Obvious.

($⇐$) Since $M$ is strongly $s$-injective, thus $Z_2(M)$ is injective by [22 Proposition 3, p.27]. Thus every homomorphism $f : K \to M$, where $K \subseteq Z_2^t$ extends to $R$ by [22 Lemma 1, p.26]. Since $R$ is a right $t$-semisimple, thus $R/Z_2^t$ is a right semisimple (see [4, Theorem 2.3]). So by applying Lemma 3.10, we conclude that $M$ is injective.

Corollary 3.17. The following statements are equivalent for a ring $R$.

1. $R$ is right SI and right $t$-semisimple.
2. $R$ is semisimple.

**Proof.** 
(1)⇒(2). Since $R$ is a right SI ring, thus every right $R$-module is strongly $s$-injective by [22 Theorem 1, p.29]. By Theorem 3.16 we have every right $R$-module is injective and hence $R$ is semisimple ring.

(2)⇒(1). Clear.

Corollary 3.18. If $R$ is a right $t$-semisimple ring, then $R$ is right $V$-ring if and only if $R$ is right $GV$-ring.

**Proof.** ($⇒$). Clear.

($⇐$). By [22 Proposition 5, p.28] and Theorem 3.16

Corollary 3.19. If $R$ is a right $t$-semisimple ring, then $R/S_r$ is noetherian right $R$-module if and only if $R$ is right noetherian.

**Proof.** If $R/S_r$ is a noetherian right $R$-module, thus every direct sum of injective right $R$-modules is strongly $s$-injective by [22 Proposition 6]. Since $R$ is right $t$-semisimple, so it follows from Theorem 3.16 that every direct sum of injective right $R$-modules is injective and hence $R$ is right noetherian. The converse is clear.

4 SS-Injective Rings

We recall that the dual of a right $R$-module $M$ is $M^d = \text{Hom}_R(M, R_R)$ and clearly that $M^d$ is a left $R$-module.
Proposition 4.1. The following statements are equivalent for a ring $R$.

1. $R$ is a right ss-injective ring.
2. If $K$ is a semisimple right $R$-module, $P$ and $Q$ are finitely generated projective right $R$-modules, $\beta : K \to P$ is an $R$-monomorphism with $\beta(K) \triangleleft P$ and $f : K \to Q$ is an $R$-homomorphism, then $f$ can be extended to an $R$-homomorphism $h : P \to Q$.
3. If $M$ is a right semisimple $R$-module and $f$ is a nonzero monomorphism from $M$ to $R$ such that $f(M) \triangleleft R$, then $M^d = Rf$.

Proof. (2)$\Rightarrow$(1) Clear.

(1)$\Rightarrow$(2) Since $Q$ is finitely generated, there is an $R$-epimorphism $\alpha_i : R^n \to Q$ for some $n \in \mathbb{Z}^+$. Since $Q$ is projective, there is an $R$-homomorphism $\alpha_2 : Q \to R^n$ such that $\alpha_1 \alpha_2 = \alpha_Q$. Define $\tilde{\beta} : K \to \beta(K)$ by $\tilde{\beta}(a) = \beta(a)$ for all $a \in K$. Since $R$ is a right ss-injective ring (by hypothesis), it follows from Proposition 2.8 and Corollary 2.5(1) that $R^n$ is a right ss-$P$-injective $R$-module. So there exists an $R$-homomorphism $h : P \to R^n$ such that $hi = \alpha_2 f \tilde{\beta}^{-1}$. Put $g = \alpha_i h : P \to Q$. Thus $gi = (\alpha_i h) i = \alpha_1 (\alpha_2 f \tilde{\beta}^{-1}) = f \tilde{\beta}^{-1}$ and hence $(g\beta)(a) = g(i(\tilde{\beta}(a))) = (f \tilde{\beta}^{-1})(\tilde{\beta}(a)) = f(a)$ for all $a \in K$. Therefore, there is an $R$-homomorphism $g : P \to Q$ such that $g\beta = f$.

(1)$\Rightarrow$(3) Let $g \in M^d$, we have $gf^{-1} : f(M) \to R$. Since $f(M)$ is a semisimple small right ideal of $R$ and $R$ is a right ss-injective ring (by hypothesis), thus $gf^{-1} = a$. for some $a \in R$. Therefore, $g = af$ and hence $M^d = Rf$.

(3)$\Rightarrow$(1) Let $f : K \to R$ be a right $R$-homomorphism, where $K$ is a semisimple small right ideal of $R$ and let $i : K \to R$ be the inclusion map, thus by (2) we have $K^d = Ri$ and hence $f = ci$ in $K^d$ for some $c \in R$. Thus there is $c \in R$ such that $f(a) = ca$ for all $a \in K$ and this implies that $R$ is a right ss-injective ring.

Example 4.2. (1) Every universally mininjective ring is ss-injective, but not conversely (see Example 5.7).

(2) The two classes of universally mininjective rings and soc-injective rings are different (see Example 5.7 and Example 5.8).

Corollary 4.3. Let $R$ be a right ss-injective ring. Then:

1. $R$ is a right mininjective ring.
2. $lr(a) = Ra$, for all $a \in S_r \cap J$.
3. $r(a) \subseteq r(b)$, $a \in St \cap J$, $b \in R$ implies $Rb \subseteq Ra$.
4. $l(bR \cap r(a)) = l(b) + Ra$, for all $a \in S_r \cap J$, $b \in R$.
5. $l(K_1 \cap K_2) = l(K_1) + l(K_2)$, for all semisimple small right ideals $K_1$ and $K_2$ of $R$.

Proof. (1) By Lemma 2.6.

(2), (3), (4) and (5) are obtained by Lemma 2.12.

The following is an example of a right mininjective ring which is not right ss-injective.

Example 4.4. (The Björk Example 15, Example 2.5). Let $F$ be a field and let $a \mapsto \bar{a}$ be an isomorphism $F \to \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let $R$ denote the left vector space on basis $\{1, t\}$, and make $R$ into an $F$-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. By 15 Example 2.5 we have $R$ is a right mininjective local ring. It is mentioned in [2] Example 4.15, that $R$ is not right soc-injective. Since $R$ is a local ring, thus by Corollary 3.11(1), $R$ is not right ss-injective ring.

Theorem 4.5. Let $R$ be a right ss-injective ring. Then:

1. $S_r \cap J \subseteq Z_r$.
2. If the ascending chain $r(a_1) \subseteq r(a_2 a_1) \subseteq \ldots$ terminates for any sequence $a_1, a_2, \ldots$ in $Z_r \cap S_r$, then $S_r \cap J$ is right $t$-nilpotent and $S_r \cap J = Z_r \cap S_r$.
Proof. (1) Let $a \in S_r \cap J$ and $bR \cap r(a) = 0$ for any $b \in R$. By Corollary 4.3(4), $l(b) + Ra = l(bR \cap r(a)) = l(0) = R$, so $l(b) = R$ because $a \in J$, implies that $b = 0$. Thus $r(a) \subseteq^{ess} R_R$ and hence $S_r \cap J \subseteq Z_R$.

(2) For any sequence $x_1, x_2, \ldots$ in $Z_r \cap S_r$, we have $r(x_1) \subseteq r(x_2 x_1) \subseteq \ldots$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r(x_m \ldots x_1) = r(x_{m+1} x_m \ldots x_1)$. If $x_m \ldots x_2 x_1 \neq 0$, then $x_m \ldots x_2 x_1 \cap r(x_{m+1}) \neq 0$ and hence $0 \neq x_m \ldots x_2 x_1 r \in r(x_{m+1})$ for some $r \in R$. Thus $x_{m+1} x_m \ldots x_2 x_1 r = 0$ and this implies that $x_m \ldots x_2 x_1 r = 0$, a contradiction. Thus $Z_r \cap S_r$ is right t-nilpotent, so $Z_r \cap S_r \subseteq J$. Therefore, $S_r \cap J = Z_r \cap S_r$ by (1). □

Proposition 4.6. Let $R$ be a right ss-injective ring. Then:

(1) If $Ra$ is a simple left ideal of $R$, then $soc(aR) \cap J(aR)$ is zero or simple.

(2) $rl(S_r \cap J) = S_r \cap J$ if and only if $rl(K) = K$ for all semisimple right ideals $K$ of $R$.

Proof. (1) Suppose that $soc(aR) \cap J(aR)$ is a nonzero. Let $x_1 R$ and $x_2 R$ be any simple small right ideals of $R$ with $x_i \in aR$, $i = 1, 2$. If $x_1 R \cap x_2 R = 0$, then by Corollary 4.3(5) $l(x_1) + l(x_2) = R$. Since $x_i \in aR$, thus $x_i = ar_i$ for some $r_i \in R$, $i = 1, 2$, that is $l(a) \subseteq l(ar_i) = l(x_i)$, $i = 1, 2$. Since $Ra$ is a simple, then $l(a) \subseteq^{max} R$, that is $l(x_1) = l(x_2) = l(a)$. Therefore, $l(a) = R$ and hence $a = 0$ and this contradicts the minimality of $Ra$. Thus $soc(aR) \cap J(aR)$ is simple.

(2) Suppose that $rl(S_r \cap J) = S_r \cap J$ and let $K$ be a semisimple small right ideal of $R$, trivially we have $K \subseteq rl(K)$. If $K \cap rR = 0$ for some $x \in rl(K)$, then by Corollary 4.3(5) $l(K \cap rR) = l(K) + l(xR) = R$, since $x \in rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$. If $y \in l(K)$, then $yx = 0$, that is $y(xr) = 0$ for all $r \in R$ and hence $l(K) \subseteq l(xR)$. Thus $l(xR) = R$, so $x = 0$ and this means that $K \subseteq^{ess} rl(K)$. Since $K \subseteq^{ess} rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$, it follows that $K = rl(K)$. The converse is trivial. □

Lemma 4.7. The following statements are equivalent.

(1) $rl(K) = K$, for all semisimple small right ideals $K$ of $R$.

(2) $r(l(K) \cap Ra) = K + r(a)$, for all semisimple small right ideals $K$ of $R$ and all $a \in R$.

Proof. (1)⇒(2). Clearly, $K + r(a) \subseteq r(l(K) \cap Ra)$ by [3 Proposition 2.16]. Now, let $x \in r(l(K) \cap Ra)$ and $y \in l(aK)$. Then $yaK = 0$ and $y \in l(ax)$, Thus $l(aK) \subseteq l(ax)$, and so $ax \in rl(ax) \subseteq rl(aK) = aK$, since $aK$ is a semisimple right ideal of $R$. Hence $ax = ak$ for some $k \in K$, and so $(x - k) \in r(a)$. This leads to $x \in K + r(a)$, that is $r(l(K) \cap Ra) = K + r(a)$.

(2)⇒(1). By taking $a = 1$. □

Recall that a right ideal $I$ of $R$ is said to be lie over a summand of $R_R$, if there exists a direct decomposition $R_R = A_R \oplus B_R$ with $A \subseteq I$ and $B \cap I \ll R_R$ (see [13]) which leads to $I = A \oplus (B \cap I)$.

Lemma 4.8. Let $K$ be an $m$-generated semisimple right ideal lies over summand of $R_R$. If $R$ is right ss-injective, then every homomorphism from $K$ to $R_R$ can be extended to an endomorphism of $R_R$.

Proof. Let $\alpha : K \longrightarrow R$ be a right $R$-homomorphism. By hypothesis, $K = eR \oplus B$, for some $e^2 = e \in R$, where $B$ is an $m$-generated semisimple small right ideal of $R$. Now, we need prove that $K = eR \oplus (1 - e)B$. Clearly, $eR + (1 - e)B$ is a direct sum. Let $x \in K$, then $x = a + b$ for some $a \in eR$, $b \in B$, so we can write $x = a + eb + (1 - e)b$ and this implies that $x \in eR \oplus (1 - e)B$. Conversely, let $x \in eR \oplus (1 - e)B$. Thus $x = a + (1 - e)b$, for some $a \in eR$, $b \in B$. We obtain $x = a + (1 - e)b = (a - eb) + b \in eR \oplus B$. It is obvious that $(1 - e)B$ is an $m$-generated semisimple small right ideal. Since $R$ is a right ss-injective, there exists $\gamma \in End(R_R)$ such that $\gamma((1 - e)B) = \alpha(1 - e)B$. Define $\beta : R_R \longrightarrow R_R$ by $\beta(x) = \alpha(ex) + \gamma((1 - e)x)$, for all $x \in R$ which is a well defined $R$-homomorphism. If $x \in K$, then $x = a + b$ where $a \in eR$ and $b \in (1 - e)B$, so $\beta(x) = \alpha(ex) + \gamma((1 - e)x) = \alpha(a) + \gamma(b) = \alpha(a) + \alpha(b) = \alpha(x)$ which yields $\beta$ is an extension of $\alpha$. □
Corollary 4.9. Let $R$ be a semiregular ring (or just every finitely generated semisimple right ideal lies over a summand of $R_R$). If $R$ is a right ss-injective ring, then every $R$-homomorphism from a finitely generated semisimple right ideal to $R$ extends to $R$.

Proof. By [13, Theorem 2.9] and Lemma 4.8.

Corollary 4.10. Let $S_r$ be a finitely generated and lie over a summand of $R_R$, then $R$ is a right ss-injective ring if and only if $R$ is right soc-injective.

Recall that a ring $R$ is called right minannihilator if every simple right ideal $K$ of $R$ is an annihilator; equivalently, if $rl(K) = K$ (see [14]).

Lemma 4.11. A ring $R$ is a right minannihilator if and only if $rl(K) = K$ for any simple small right ideal $K$ of $R$.

Lemma 4.12. A ring $R$ is a left minannihilator if and only if $lr(K) = K$ for any simple small left ideal $K$ of $R$.

Corollary 4.13. Let $R$ be a right ss-injective ring, then the following hold:
(1) If $rl(S_r \cap J) = S_r \cap J$, then $R$ is right minannihilator.
(2) If $S_l \subseteq S_r$, then:
   i) $S_l = S_r$.
   ii) $R$ is a left minannihilator ring.

Proof. (1) Let $aR$ be a simple small right ideal of $R$, thus $rl(a) = aR$ by Proposition 4.6(2). Therefore, $R$ is a right minannihilator ring.

(2) i) Since $R$ is a right ss-injective ring, thus it is right mininjective and it follows from [14, Proposition 1.14 (4)] that $S_l = S_r$.

   ii) If $Ra$ is a simple small left ideal of $R$, then $lr(a) = Ra$ by Corollary 4.3(2) and hence $R$ is a left minannihilator ring.

Proposition 4.14. The following statements are equivalent for a right ss-injective ring $R$.
(1) $S_l \subseteq S_r$.
(2) $S_l = S_r$.
(3) $R$ is a left mininjective ring.

Proof. (1)⇒(2) By Corollary 4.13(2) (i).

(2)⇒(3) By Corollary 4.13(2) and [15, Corollary 2.34], we need only show that $R$ is right mininjective ring. Let $aR$ be a simple small right ideal, then $Ra$ is a simple small left ideal by [14, Theorem 1.14]. Let $0 \neq x \in rl(aR)$, then $l(a) \subseteq l(x)$. Since $l(a) \subseteq^{\max} R$, thus $l(a) = l(x)$ and hence $Rx$ is simple left ideal, that is $x \in S_r$. Now, if $Rx = Re$ for some $e = e^2 \in R$, then $e = rx$ for some $0 \neq r \in R$. Since $(e - 1)e = 0$, then $(e - 1)rx = 0$, that is $(e - 1)ra = 0$ and this implies that $ra \in eR$. Thus $raR \subseteq eR$, but $eR$ is semisimple right ideal, so $raR \subseteq R$ and hence $ra = 0$. Therefore, $rx = 0$, that is $e = 0$, a contradiction. Thus $x \in J$ and hence $x \in S_r \cap J$. Therefore, $aR \subseteq rl(aR) \subseteq S_r \cap J$. Now, let $aR \cap yR = 0$ for some $y \in rl(aR)$, thus $l(aR) + l(yR) = l(aR \cap yR) = R$. Since $y \in rl(aR)$, thus $l(aR) \subseteq l(yR)$ and hence $l(yR) = R$, that is $y = 0$. Therefore, $aR \subseteq^{ess} rl(aR)$, so $aR = rl(aR)$ as desired.

(3)⇒(1) Follows from [15, Corollary 2.34].
Recall that a ring \( R \) is said to be right minfull if it is semiperfect, right mininjective and \( \text{soc}(eR) \neq 0 \) for each local idempotent \( e \in R \) (see [15]). A ring \( R \) is called right min-PF, if it is a semiperfect, right mininjective, \( S_\ell \subseteq \text{ess } R_R \), \( \text{lr}(K) = K \) for every simple left ideal \( K \subseteq R e \) for some local idempotent \( e \in R \) (see [15]).

**Corollary 4.15.** Let \( R \) be a right ss-injective ring, semiperfect with \( S_\ell \subseteq \text{ess } R_R \). Then \( R \) is right minfull ring and the following statements hold:

1. Every simple right ideal of \( R \) is essential in a summand.
2. \( \text{soc}(eR) \) is simple and essential in \( eR \) for every local idempotent \( e \in R \). Moreover, \( R \) is right finitely cogenerated.
3. For every semisimple right ideal \( I \) of \( R \), there exists \( e \) finitely cogenerated.
4. \( S_\ell \subseteq \text{ess } \ell \leq \text{ess } R_R \).
5. If \( I \) is a semisimple right ideal of \( R \) and \( aR \) is a simple right ideal of \( R \) with \( I \cap aR = 0 \), then \( \text{lr}(I \oplus aR) = \text{lr}(I) \oplus \text{lr}(aR) \).
6. \( \text{lr}(\bigoplus_i a_i R) = \bigoplus_i \text{lr}(a_i R) \), where \( \bigoplus_i a_i R \) is a direct sum of simple right ideals.
7. The following statements are equivalent.
   a. \( S_\ell = \text{rl}(S_\ell) \).
   b. \( K = \text{rl}(K) \) for every semisimple right ideals \( K \) of \( R \).
   c. \( kR = \text{lr}(kR) \) for every simple right ideals \( kR \) of \( R \).
   d. \( S_\ell = \text{ess } S_\ell \).
   e. \( \text{soc}(Re) \) is simple for all local idempotent \( e \in R \).
   f. \( \text{soc}(Re) = S_\ell e \) for every local idempotent \( e \in R \).
   g. \( R \) is left mininjective.
   h. \( L = \text{lr}(L) \) for every semisimple left ideals \( L \) of \( R \).
   i. \( R \) is left minfull ring.
   j. \( S_\ell \cap J = \text{rl}(S_\ell \cap J) \).
   k. \( K = \text{rl}(K) \) for every semisimple small right ideals \( K \) of \( R \).
   l. \( L = \text{lr}(L) \) for every semisimple small left ideals \( L \) of \( R \).
8. If \( R \) satisfies any condition of (7), then \( \text{rl}(S_\ell \cap J) \subseteq \text{ess } R_R \).

**Proof.** (1), (2), (3), (4), (5) and (6) are obtained by Corollary 3.11 and [24, Theorem 4.12].

(7) The equivalence of (a), (b), (c), (d), (e), (f), (g), (h) and (i) follows from Corollary 3.11 and [24, Theorem 4.12].

(b)\( \iff \) (j) Clear.

(j)\( \iff \) (k) By Proposition 4.12.

(k)\( \iff \) (e) By Corollary 4.13.

(i)\( \iff \) (l) Clear.

(l)\( \iff \) (d) Let \( Ra \) be a simple left ideal of \( R \). By hypothesis, \( \text{lr}(A) = A \) for any simple small left ideal \( A \) of \( R \). By Lemma 4.12 \( \text{lr}(A) = A \) for any simple left ideal \( A \) of \( R \) and hence \( \text{lr}(Ra) = Ra \). Thus \( R \) is a right min-PF ring and it follows from [14, Theorem 3.14] that \( S_\ell = S_\ell \).

(8) Let \( K \) be a right ideal of \( R \) such that \( \text{rl}(S_\ell \cap J) \cap K = 0 \). Then \( \text{lr}(S_\ell \cap J) = 0 \) and we have \( K \subseteq \text{lr}(S_\ell \cap J) = S_\ell \cap J = S_\ell \cap J \). Now, \( \text{rl}(S_\ell \cap J) \cap K = 0 \). Since \( R \) is left Kasch, then \( S_\ell \cap J \cap l(K) = \text{rl}(S_\ell \cap J) \cap K = 0 \). Thus \( l(K) = R \) and hence \( K = 0 \), so \( \text{rl}(S_\ell \cap J) \subseteq \text{ess } R_R \). \( \blacksquare \)

Recall that a right \( R \)-module \( M \) is called almost-injective if \( M = E \oplus K \), where \( E \) is injective and \( K \) has zero radical (see [23]). After reflect on [23, Theorem 2.12] we found it is not true always and the reason is due to the homomorphism \( h : (L + J)/J \rightarrow K \) in the part (3)\( \Rightarrow \) (1) of the proof of Theorem 2.12 in [23] is not well define, in particular see the following example.
Example 4.16. In particular from the proof of the part (3)⇒(1) in [23 Theorem 2.12], we consider \( R = \mathbb{Z}/8 \) and \( M = K = \langle \bar{4} \rangle = \{0, 4\} \). Thus \( M = E \oplus K, \) where \( E = 0 \) is a trivial injective \( R \)-module and \( J(K) = 0 \). Let \( f : L \to K \) is the identity map, where \( L = K \). So, the map \( h : (L + J)/J \to K \) which is given by \( h(\ell + J) = f(\ell) \) is not well define, because \( J = \bar{4} + J \) but \( h(J) = f(\bar{0}) = 0 \neq \bar{4} = f(\bar{4}) = h(\bar{4} + J) \).

The following example shows that there is a contradiction in [23 Theorem 2.12].

Example 4.17. Assume that \( R \) is a right artinian ring but not semisimple (this claim is found because for example \( \mathbb{Z}/8 \) satisfies this property). Now, let \( M \) be a simple right \( R \)-module, then \( M \) is almost-injective. Clearly, \( R \) is semilocal (see [9 Theorem 9.2.2]), thus \( M \) is injective by [23 Theorem 2.12]. Therefore, \( R \) is V-ring and hence \( R \) is a right semisimple ring but this contradiction. In other word, Since \( \mathbb{Z}/8 \) is semilocal ring and \( \langle \bar{4} \rangle = \{0, 4\} \) is almost injective as \( \mathbb{Z}/8 \)-module, then \( \langle \bar{4} \rangle \) is injective by [23 Theorem 2.12]. Thus \( \langle \bar{4} \rangle \subseteq \oplus \mathbb{Z}/8 \) and this contradiction.

Theorem 4.18. The following statements are equivalent for a ring \( R \).

1. \( R \) is semiprimitive and every almost-injective right \( R \)-module is quasi-continuous.
2. \( R \) is right ss-injective and right min annihilator ring, \( J \) is right artinian, and every almost-injective right \( R \)-module is quasi-continuous.
3. \( R \) is a semisimple ring.

Proof. (1)⇒(2) and (3)⇒(1) are clear.

(2)⇒(3) Let \( M \) be a right \( R \)-module with zero radical. If \( N \) is an arbitrary nonzero submodule of \( M \), then \( N \oplus M \) is quasi-continuous and by [12 Corollary 2.14], \( N \) is \( M \)-injective. Thus \( N \leq \oplus M \) and hence \( M \) is semisimple. In particular \( R/J \) is semisimple \( R \)-module and hence \( R/J \) is artinian by [9 Theorem 9.2.2(b)], so \( R \) is semilocal ring. Since \( J \) is a right artinian, then \( R \) is right artinian. So it follows from Corollary [4.15(7)] that \( R \) is right and left mininjective. Thus [14 Corollary 4.8] implies that \( R \) is \( QF \) ring. By hypothesis, \( R \oplus (R/J) \) is quasi-continuous (since \( R \) is self-injective), so again by [12 Corollary 2.14] we have that \( R/J \) is injective. Since \( R \) is \( QF \) ring, then \( R/J \) is projective (see [9 Theorem 13.6.1]). Thus the canonical map \( \pi : R \to R/J \) is splits and hence \( J \leq \oplus R \), that is \( J = 0 \). Therefore \( R \) is semisimple.

5 STRONGLY SS-INJECTIVE RINGS

Proposition 5.1. A ring \( R \) is strongly right ss-injective if and only if every finitely generated projective right \( R \)-module is strongly ss-injective.

Proof. Since a finite direct sum of strongly ss-injective modules is strongly ss-injective, so every finitely generated free right \( R \)-module is strongly ss-injective. But a direct summand of strongly ss-injective is strongly ss-injective. Therefore, every finitely generated projective is strongly ss-injective. The converse is clear.
Proof. Let \( x \in l(A \cap B) \) and define \( \alpha : A + B \rightarrow R_R \) by \( \alpha(a + b) = xa \) for all \( a \in A \) and \( b \in B \). Clearly, \( \alpha \) is well define, because if \( a_1 + b_1 = a_2 + b_2 \), then \( a_1 - a_2 = b_2 - b_1 \), that is \( x(a_1 - a_2) = 0 \), so \( \alpha(a_1 + b_1) = \alpha(a_2 + b_2) \). The map \( \alpha \) induces an \( R \)-homomorphism \( \tilde{\alpha} : (A + B)/B \rightarrow R_R \) which is given by \( \tilde{\alpha}(a + B) = xa \) for all \( a \in A \). Since \( (A + B)/B \subseteq soc(R/B) \cap J(R/B) \) and \( R \) is a strongly right ss-injective, \( \tilde{\alpha} \) can be extended to an \( R \)-homomorphism \( \gamma : R/B \rightarrow R_R \). If \( \gamma(1 + B) = y \), for some \( y \in R \), then \( y(a + b) = xa \), for all \( a \in A \) and \( b \in B \). In particular, \( ya = xa \), for all \( a \in A \) and \( yb = 0 \), for all \( b \in B \). Hence \( x = (x - y) + y \in l(A) + l(B) \). Therefore, \( l(A \cap B) \subseteq l(A) + l(B) \). Since the converse is always holds, thus the proof is complete.

Recall that a ring \( R \) is said to be right simple \( J \)-injective if for any small right ideal \( I \) and any \( R \)-homomorphism \( \alpha : I \rightarrow R_R \) with simple image, \( \alpha = c \cdot R \), for some \( c \in R \) (see [21]).

**Corollary 5.3.** Every strongly right ss-injective ring is right simple \( J \)-injective.

**Proof.** By Proposition 5.1.

**Remark 5.4.** The converse of Corollary 5.3 is not true (see Example 5.7).

**Proposition 5.5.** Let \( R \) be a right Kasch and strongly right ss-injective ring. Then:

1. \( rl(K) = K \), for every small right ideal \( K \). Moreover, \( R \) is right minannihilator.
2. If \( R \) is left Kasch, then \( r \subseteq ess R_R \).

**Proof.** (1) By Corollary 5.3 and [21] Lemma 2.4.

(2) Let \( K \) be a right ideal of \( R \) and \( r(J) \cap K = 0 \). Then \( Kr(J) = 0 \) and we obtain \( K \subseteq lrl(J) = J \), because \( R \) is left Kasch. By (1), we have \( r(J + l(K)) = r(J) \cap K = 0 \) and this means that \( J + l(K) = R \) (since \( R \) is left Kasch). Thus \( K = 0 \) and hence \( r(J) \subseteq ess R_R \).

The following examples show that the classes of rings: strongly ss-injective rings, soc-injective rings and of small injective rings are different.

**Example 5.6.** Let \( R = \mathbb{Z}_{(p)} = \{ \frac{m}{n} \mid p \) does not divide \( n \} \), the localization ring of \( \mathbb{Z} \) at the prime \( p \). Then \( R \) is a commutative local ring and it has zero socle but not principally small injective (see [20] Example 4)). Since \( S_r = 0 \), thus \( R \) is strongly soc-injective ring and hence \( R \) is strongly ss-injective ring.

**Example 5.7.** Let \( R = \left\{ \begin{pmatrix} n & x \\ 0 & n \end{pmatrix} \mid n \in \mathbb{Z}, x \in \mathbb{Z}_2 \right\} \). Thus \( R \) is a commutative ring. \( J = S_r = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\} \) and \( R \) is small injective (see [19] Example(i)). Let \( A = J \) and \( B = \left\{ \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z} \right\} \), then \( l(A) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\} \) and \( l(B) = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in \mathbb{Z}_2 \right\} \). Thus \( l(A) + l(B) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\} \).

Since \( A \cap B = 0 \), thus \( l(A \cap B) = R \) and this implies that \( l(A) + l(B) \neq l(A \cap B) \). Therefore \( R \) is not strongly ss-injective and not strongly soc-injective by Proposition 5.2.

**Example 5.8.** Let \( F = \mathbb{Z}_2 \) be the field of two elements, \( F_i = F \) for \( i = 1, 2, 3, ..., Q = \prod_{i=1}^{\infty} F_i \), \( S = \bigoplus F_i \). If \( R \) is the subring of \( Q \) generated by 1 and \( S \), then \( R \) is a Von Neumann regular ring (see [22] Example (1), p.28)). Since \( R \) is commutative, thus every simple \( R \)-module is injective by [10] Corollary 3.73]. Thus \( R \) is \( V \)-ring and hence \( J(N) = 0 \) for every right \( R \)-module \( N \). It follows from Corollary 3.9 that every \( R \)-module is strongly ss-injective. In particular, \( R \) is strongly ss-injective ring. But \( R \) is not soc-injective (see [22] Example (1)).
Example 5.9. Let \( R = \mathbb{Z}_2[x_1, x_2, \ldots] \) where \( \mathbb{Z}_2 \) is the field of two elements, \( x_i^2 = 0 \) for all \( i \neq j \) and \( x_i^2 = x_j^2 \neq 0 \) for all \( i \) and \( j \). If \( m = x_i^2 \), then \( R \) is a commutative, semiprimary, local, soc-injective ring with \( J = \text{span}\{m, x_1, x_2, \ldots\} \), and \( R \) has simple essential socle \( J^2 = \mathbb{Z}_2m \) (see [2] Example 5.7). It follows from [2] Example 5.7 that the \( R \)-homomorphism \( \gamma: J \rightarrow R \) which is given by \( \gamma(a) = a^2 \) for all \( a \in J \) with simple image can be not extended to \( R \), then \( R \) is not simple \( J \)-injective and not small injective, so it follows from Corollary 5.3 that \( R \) is not strongly ss-injective.

Recall that \( R \) is said to be right minsymmetric ring if \( aR \) is simple right ideal then \( Ra \) is simple left ideal (see [14]). Every right mininjective ring is right minsymmetric by [14, Theorem 1.14].

Theorem 5.10. A ring \( R \) is \( QF \) if and only if \( R \) is a strongly right ss-injective and right noetherian ring with \( S_r \subseteq \text{ess} R_r \).

Proof. \((\Rightarrow)\) This is clear.

\((\Leftarrow)\) By Corollary 4.3, \( R \) is right minsymmetric. It follows from [19, Lemma 2.2] that \( R \) is right perfect. Thus \( R \) is strongly right soc-injective, by Theorem 3.13. Since \( S_r \subseteq \text{ess} R_r \), so it follows from [2, Corollary 3.2] that \( R \) is self-injective and hence \( R \) is \( QF \). \(\square\)

Corollary 5.11. For a ring \( R \) the following statements are true.

\( (1) \) \( R \) is semisimple if and only if \( S_r \subseteq \text{ess} R_r \) and every semisimple right \( R \)-module is strongly soc-injective.

\( (2) \) \( R \) is \( QF \) if and only if \( R \) is strongly right ss-injective, semiperfect with essential right socle and \( R/S_r \) is noetherian as right \( R \)-module.

Proof. \((1)\) Suppose that \( S_r \subseteq \text{ess} R_r \) and every semisimple right \( R \)-module is strongly soc-injective, then \( R \) is a right noetherian right \( V \)-ring by [2, Proposition 3.12], so it follows from Corollary 3.9 that \( R \) is strongly right ss-injective. Thus \( R \) is \( QF \) by Theorem 5.10. But \( J = 0 \), so \( R \) is semisimple. The converse is clear.

\((2)\) By [14, Theorem 2.9], \( J = Z_r \). Since \( R/Z \) is a homomorphic image of \( R/Z \) and \( R \) is a semilocal ring, thus \( R \) is a right \( t \)-semisimple. By Corollary 3.19, \( R \) is right noetherian, so it follows from Theorem 5.10 that \( R \) is \( QF \). The converse is clear. \(\square\)

Theorem 5.12. A ring \( R \) is \( QF \) if and only if \( R \) is a strongly right ss-injective, \( 1/(J^2) \) is a countable generated left ideal, \( S_r \subseteq \text{ess} R_r \) and the chain \( r(x_1) \subseteq r(x_2x_1) \subseteq \ldots \subseteq r(x_nx_{n-1}\ldots x_1) \subseteq \ldots \) terminates for every infinite sequence \( x_1, x_2, \ldots \) in \( R \).

Proof. \((\Rightarrow)\) Clear.

\((\Leftarrow)\) By [19, Lemma 2.2], \( R \) is right perfect. Since \( S_r \subseteq \text{ess} R_r \), thus \( R \) is right Kasch (by [14, Theorem 3.7]). Since \( R \) is strongly right ss-injective, thus \( R \) is right simple \( J \)-injective, by Corollary 5.3. Now, by Proposition 5.5(1) we have \( r(S_r) \cap J = S_r \cap J \), so it follows from Corollary 4.15(7) that \( S_r = S_r \). By [15, Lemma 3.36], \( S_r^2 = l(J^2) \). The result now follows from [21, Theorem 2.18]. \(\square\)

Remark 5.13. The condition \( S_r \subseteq \text{ess} R_r \) in Theorem 5.10 and Theorem 5.12 can be not deleted, for example, \( \mathbb{Z} \) is strongly ss-injective noetherian ring but not \( QF \).

The following two results are extension of Proposition 5.8 in [2].

Corollary 5.14. The following statements are equivalent.

\( (1) \) \( R \) is a \( QF \) ring.

\( (2) \) \( R \) is a left perfect, strongly left and right ss-injective ring.
Proof. By Corollary [5.3] and [21] Corollary 2.12.

**Theorem 5.15.** The following statements are equivalent:

1. R is a QF ring.
2. R is a strongly left and right ss-injective, right Kasch and J is left t-nilpotent.
3. R is a strongly left and right ss-injective, left Kasch and J is left t-nilpotent.

Proof. (1)⇒(2) and (1)⇒(3) are clear.

(3)⇒(1) Suppose that xR is simple right ideal. Thus either rl(x) = xR ⊆ R or x ∈ J. If x ∈ J, then rl(x) = xR (since R is right minannihilator), so it follows from Theorem [5.4] that rl(x) ⊆ ess E ⊆ R. Therefore, rl(x) is essential in a direct summand of R for every simple right ideal xR. Let K be a maximal left ideal of R. Since R is left Kasch, thus r(K) ≠ 0 by [10] Corollary 8.28. Choose 0 ≠ y ∈ r(K), so K ⊆ l(y) and we conclude that K = l(y). Since Ry ≅ R/l(y), thus Ry is simple left ideal. But R is left mininjective ring, so yR is right simple ideal by [14] Theorem 1.14 and this implies that r(K) ⊆ ess eR for some e² = e ∈ R (since r(K) = r(l(y))). Thus R is semiperfect by [15] Lemma 4.1 and hence R is left perfect (since J is left t-nilpotent), so it follows from Corollary [5.14] that R is QF.

(2)⇒(1) It is similar to the proof of (3)⇒(1). 

**Theorem 5.16.** The ring R is QF if and only if R is strongly left and right ss-injective, left and right Kasch, and the chain l(a₁) ⊆ l(a₁a₂) ⊆ l(a₁a₂a₃) ⊆ ... terminates for every a₁, a₂, ..., ∈ Zₖ.

Proof. (⇒) Clear.

(⇐) By Proposition [5.5], l(J) is essential in R. Thus J ⊆ Zₖ. Let a₁, a₂, ..., ∈ J , we have l(a₁) ⊆ l(a₁a₂) ⊆ l(a₁a₂a₃) ⊆ ... Thus there exists k ∈ N such that l(a₁...aₖ) = l(a₁...a₄a₅a₆...aₖ+1) (by hypothesis). Suppose that a₁...aₖ ≠ 0, so R(a₁...aₖ) ∩ l(aₖ+1) ≠ 0 (since l(aₖ+1) is essential in R). Thus ra₁...aₖ ≠ 0 and ra₁...a₄a₅a₆...aₖ+1 = 0 for some r ∈ R, a contradiction. Therefore, a₁...aₖ = 0 and hence J is left t-nilpotent, so it follows from Theorem [5.15] that R is QF.

**Corollary 5.17.** The ring R is QF if and only if R is strongly left and right ss-injective with essential right socle, and the chain r(a₁) ⊆ r(a₂a₁) ⊆ r(a₃a₂a₁) ⊆ ... terminates for every infinite sequence a₁, a₂, ..., in R.

Proof. By [19] Lemma 2.2 and Corollary [5.14]

**References**

[1] W. A. Adkins and S. H. Weintraub, Algebra: an approach via module theory, Springer-Verlag, New York, 1992.

[2] I. Amin, M. Yousif and N. Zeyada, Soc-injective rings and modules, Commun. Algebra, 33(2005), 4229-4250.

[3] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1974.

[4] Sh. Asgari, A. Haghany and Y. Tolooei, T-Semisimple Modules and T-Semisimple Rings, Commun. Algebra, 41(2013), 1882-1902.

[5] L. Bican, T. Kepka and P. Němec, Rings, Modules, and Preradicals, Marcel Dekker, Inc., New York, 1982.

[6] P. E. Bland, Rings and Their Modules, Walter de Gruyter and Co., Berlin, 2011.
[7] J. L. Chen, N. Q. Ding and H. Marubayashi, Advances in Ring Theory, Proceedings of the 4th China-Japan-Korea, International Conference, 2005.

[8] P. A. Grillet, Abstract Algebra, GTM 242, Springer, 2007.

[9] F. Kasch, Modules and Rings, Academic Press, New York, 1982.

[10] T. Y. Lam, Lectures on Modules and Rings, GTM 189, Springer-Verlag, New York, 1999.

[11] C. Lomp, On semilocal modules and rings, Commun. Algebra, 27(1999), 1921-1935.

[12] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, Cambridge University Press, Cambridge, 1990.

[13] W. K. Nicholson, Semiregular modules and rings, Canadian J. Math., 28(1976), 1105-1120.

[14] W. K. Nicholson and M. F. Yousif, Mininjective Rings, J. Algebra, 187(1997), 548-578.

[15] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius Rings, Cambridge Tracts in Math.(158), Cambridge University Press, Cambridge, 2003.

[16] D. S. Passman, A course in ring theory, AMS Chelsea Publishing, 2004.

[17] L. Shen and J. Chen, New characterizations of quasi-Frobenius rings, Commun. Algebra, 34(2006), 2157-2165.

[18] B. Stenström, Rings of Quotients, Springer-Verlage, New York, 1975.

[19] L. V. Thuyet and T. C. Quynh, On small injective rings and modules, J. Algebra and Its Applications, 8(2009), 379-387.

[20] Y. Xiang, Principally Small Injective Rings, Kyungpook Math. J., 51(2011), 177-185.

[21] M. F. Yousif and Y. Q. Zhou, FP-Injective, Simple-Injective and Quasi-Frobenius Rings, Commun. Algebra, 32(2004), 2273-2285.

[22] N. Zeyada, S-Injective Modules and Rings, Advances in Pure Math., 4(2014), 25-33.

[23] N. Zeyada, S. Hussein and A. Amin, Rad-injective and Almost-injective Modules and Rings, Algebra Colloquium, 18(2011), 411-418.