We present closed form expressions for the ranks of all cohomology groups of holomorphic line bundles on several Calabi-Yau threefolds realised as complete intersections in products of projective spaces. The formulae have been obtained by systematising and extrapolating concrete calculations and they have been checked computationally. Although the intermediate calculations often involve laborious computations of ranks of Leray maps in the Koszul spectral sequence, the final results for cohomology follow a simple pattern. The space of line bundles can be divided into several different regions, and in each such region the ranks of all cohomology groups can be expressed as polynomials in the line bundle integers of degree at most three. The number of regions increases and case distinctions become more complicated for manifolds with a larger Picard number. We also find explicit cohomology formulae for several non-simply connected Calabi-Yau threefolds realised as quotients by freely acting discrete symmetries. More cases may be systematically handled by machine learning algorithms.

1. Introduction

It is difficult to underestimate the importance of cohomology computations in mathematics and theoretical physics. Despite this, and except in simple cases, cohomology computations are hard to carry out explicitly. One situation where closed form expressions are known to exist is the case of line bundles on projective spaces. The result, known as Bott’s formula, is strikingly simple:

\[ h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \frac{1}{n!} (1+k)(2+k) \cdots (n+k), \]

if \( k \geq 0 \), and 0 otherwise.

\[ h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0, \text{ if } 0 < i < n. \]

An algorithm generalising Bott’s formula to the case of toric line bundles has recently been proposed in Refs. [1–3]. This algorithm was developed in the context of string compactifications, where massless modes of the heterotic or type II string on compact Calabi-Yau manifolds are determined by vector bundle valued cohomology. However, passing from projective spaces or toric varieties to Calabi-Yau manifolds involves an additional layer of complication. Smooth Calabi-Yau manifolds can be realised as hypersurfaces or complete intersections in products of projective spaces or toric varieties. In the presence of such an embedding space, knowledge about vector bundle cohomology on the embedding space can be transferred to the bundle restricted to the Calabi-Yau sub-manifold, using the Koszul complex and its associated spectral sequence.

In general, cohomology computations with spectral sequences require explicit information about the ranks of the Leray maps. For complete intersection manifolds in products of projective spaces,\(^4,5\) this information can be obtained using a computational algorithm that relies on the Bott-Borel-Weil theorem.\(^6,7\) This algorithm has been implemented in Mathematica\(^8\) and applied to various problems related to string compactifications.\(^9–25\) The experience gained by computing a large number of examples has led to the observation, made in Refs. [14,15], that the ranks of cohomology groups of holomorphic line bundles on the tetraquadric Calabi-Yau manifold follow a certain pattern that can be expressed by a concrete formula.

The purpose of this note is to extend the above observation to several other complete intersection Calabi-Yau threefolds. We find formulae for line bundle cohomology for a number of other manifolds and this suggests that similar formulae may exist for large classes of manifolds. It is likely that they can be obtained systematically by making use of computer-aided learning techniques.

There is at least one class of Calabi-Yau threefolds for which the appearance of closed form expressions for line bundle cohomology should not be surprising: smooth complete intersections in \( \mathbb{P}^n \) with Picard number equal to one. Let \( X \subset \mathbb{P}^n \) be such a
manifold. All line bundles on $X$ can be obtained as restrictions of line bundles $\mathcal{L} = \mathcal{O}_X(n)$ on $\mathbb{P}^n$ and we denote these by $L = \mathcal{O}_X(k)$. Their first Chern class can be written as $c_1(\mathcal{O}_X(k)) = kj$, where $J$ is the restriction of $X$ to the Kähler form on $\mathbb{P}^n$. If $k > 0$, Kodaira’s vanishing theorem implies that $h^i(X, L) = 0$, for all $q > 0$. Hence $H^0(X, L)$ is the only non-trivial cohomology group and its rank equals the index of $L$. For negative line bundles the picture is reflected, due to Serre duality, $H^0(X, L) = H^q(X, \mathcal{L}^*)$, which implies that $h^1(X, L) = -\text{ind}(L)$ and that all other cohomologies are trivial. Using the Atiyah-Singer index theorem, the index of $L$ can be expressed as
\[
\text{ind}(L) = \frac{1}{6} d(X) k^3 + \frac{1}{12} d(X) c_3(TX) k.
\]
where $d(X)$ is the triple intersection number and $c_3(TX) = c_3(TX) \wedge J$ is the second Chern class of $X$. Together with the information that $H^i(X, \mathcal{O}_X) \cong H^q(X, \mathcal{O}_X) \cong C$ and that $H^i(X, \mathcal{O}_X)$ and $H^q(X, \mathcal{O}_X)$ are trivial, this fixes the cohomology ranks for all holomorphic line bundles on complete intersection Calabi-Yau threefolds in $\mathbb{P}^n$. There are five such manifolds, and the corresponding line bundle cohomology formulae were given in Ref. [7]:
\[
\begin{align*}
\hat{h}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k)) &= \max \left( \delta_1, 0 \right) \\
\hat{h}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k)) &= \max \left( \delta_2, \frac{1}{2} k^1 + \frac{9}{2} k, 0 \right) \\
\hat{h}^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(k)) &= \max \left( \delta_4, \frac{4}{3} k^1 + \frac{14}{3} k, 0 \right) \\
\hat{h}^0(\mathbb{P}^3, \mathcal{O}(k)) &= \max \left( \delta_6, 2 k^1 + 5 k, 0 \right) \\
\hat{h}^0(\mathbb{P}^3 \times \mathbb{P}^1, \mathcal{O}(k)) &= \max \left( \delta_8, \frac{8}{3} k^1 + \frac{16}{3} k, 0 \right)
\end{align*}
\]
In addition, we have $h^1(X, \mathcal{O}(k)) = h^1(X, \mathcal{O}(k)) = 0$ for all these manifolds and $h^0(X, \mathcal{O}(k)) = h^0(X, -\mathcal{O}(k))$ is obtained from the above results via Serre duality. We have used the notation commonly used in the physics literature by which, say, $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ denotes a Calabi-Yau threefold embedded in $\mathbb{P}^7$ and realised as the complete intersection of four hypersurfaces of degree 2.

We may encode the information contained in the above formulae in the following diagrams:

It is interesting to note that the same structure for the ranks of line bundle cohomology groups is present for threefolds with non-trivial canonical bundle, such as $\mathbb{P}^1$. The difference comes from the fact that in this case Serre duality operates between cohomology groups $H^i(\mathbb{P}^3, \mathcal{O}_P(k))$ and $H^i(\mathbb{P}^3, \mathcal{O}_P(k - 4))$.

We will shortly turn to the case of manifolds with $h^{1,1}(X) > 1$. Anticipating the results, we remark that the main features of the above formulae are retained in all the cases studied below. The ranks of all cohomology groups can be given by simple expressions, although the intermediate calculations involving the Koszul resolution and the associated spectral sequence, kernels and co-kernels of Leray maps and so on are quite non-trivial. More concretely, we find that the ranks of all cohomology groups can be expressed as polynomials of degree at most three in the line bundle integers and the form of these polynomials changes in different regions of the $k$-space.

2. Manifolds with $h^{1,1} > 1$

Before discussing Calabi-Yau threefolds, it is interesting to take a look at manifolds with Picard number greater than 1 for which line bundle cohomology formulae are known to exist. Products of projective spaces provide the simplest examples of such manifolds, and their line bundle valued cohomology can be obtained from Bott’s formula combined with Kinnell’s formula
\[
\hat{H}^i(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(k_1, k_2)) = \bigoplus_{j_1 + j_2 = k} \hat{H}^j(\mathbb{P}^n, \mathcal{O}(k_1)) \otimes \hat{H}^j(\mathbb{P}^m, \mathcal{O}(k_2)).
\]

For concreteness, consider the line bundle $\mathcal{L} = \mathcal{O}_P(k_1) \otimes \mathcal{O}_P(k_2)$, with cohomology ranks given by the following formulae and illustrated in Figure 1. Note that in this case there exist two lines, namely $k_1 = -1$ and $k_2 = -1$ along which all the cohomology ranks vanish and these lines separate the $k$-space into different regions in which the rank of one cohomology group does not vanish.

\[
\begin{align*}
\hat{h}^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k_1, k_2)) &= \begin{cases} 
(1 + k_1)(1 + k_2), & k_1, k_2 \geq 0 \\
0, & \text{otherwise}
\end{cases} \\
\hat{h}^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k_1, k_2)) &= \begin{cases} 
(1 + k_1)(1 + k_2), & k_1 \geq 0, k_2 \leq -2 \\
(1 + k_1)(-1 - k_2), & k_1 \leq -2, k_2 \geq 0 \\
0, & \text{otherwise}
\end{cases} \\
\hat{h}^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k_1, k_2)) &= \begin{cases} 
(1 + k_1)(1 + k_2), & k_1, k_2 \leq -2 \\
(1 + k_1)(1 + k_2), & k_1, k_2 \geq 0 \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

1 Such line bundles with entirely vanishing cohomology were recently studied for the case of toric varieties in [26].
Figure 1. Regions in $k$-space where the cohomology ranks take different polynomial forms of degree at most 2. In the blue regions the ranks are given by the index of the bundle, while in the red regions they vanish. Top left: $h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L})$, top right: $h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L})$, bottom: $h^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L})$.

2.1. Complete Intersections in Products of Projective Spaces

Let $X \subset \mathcal{A}$ be a smooth complete intersection Calabi-Yau three-fold in the ambient space $\mathcal{A} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, defined as the common zero locus of several multi-homogeneous polynomials. It is convenient to record the multi-degrees of the defining polynomials as a matrix, known as the configuration matrix, of the form

$$
\begin{bmatrix}
q_1^1 & \cdots & q_R^1 \\
\vdots & \ddots & \vdots \\
q_1^{m} & \cdots & q_R^{m}
\end{bmatrix}
$$

There are $R$ defining polynomials, one for every column of the configuration matrix, and the integer vector $q_a = (q_1^a, \ldots, q_R^a)$, containing the entries of the $a^{th}$ column of the above matrix, denotes the multi-degree of the $a^{th}$ polynomial with respect to the homogeneous coordinates of the projective ambient space factors. The two non-trivial Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$ are attached as a superscript. Such a complete intersection $X$ has vanishing first Chern class if and only if the sum of the degrees in each row of the configuration matrix equals the dimension of the corresponding projective space plus one.

For each projective factor we have an associated Kähler form and its restriction to $X$, which we denote by $J_r$, where $r = 1, \ldots, m$. The restrictions of the line bundles $\mathcal{L} = \mathcal{O}_X(k_1, \ldots, k_m) = \mathcal{O}_{\mathbb{P}^{n_1}}(k_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^{n_m}}(k_m)$ to $X$ are denoted by $L = \mathcal{O}_X(k_1, \ldots, k_m)$, with first Chern classes $c_1(\mathcal{O}_X(k_1, \ldots, k_m)) = \sum_k k_j J_r$. In the cases discussed below, the second cohomology of $X$ is spanned by (the classes of) the forms $J_r$ so that all line bundles on $X$ are of the form $\mathcal{O}_X(k_1, \ldots, k_m)$ and are classified by $m$-dimensional integer vectors $(k_1, \ldots, k_m)$. We also refer to the integers $k_r$ as “line bundle integers”. With this notation the defining polynomials of $X$ are sections of the bundle $\mathcal{U} = \mathcal{O}_X(q_1) \oplus \cdots \oplus \mathcal{O}_X(q_R)$ and $R = \text{rank}(\mathcal{N})$.

The Atiyah-Singer index theorem applied to such line bundles now leads to

$$
\text{ind}(L) = \sum_{i=0}^3 (-1)^i h^i(X, L) = \int_X \left( c_1(L) + \frac{1}{12} c_2(TX) \wedge c_1(L) \right)
= \frac{1}{6} d_{\mathbb{C}} k' k'' + \frac{1}{12} c_2 k'.
$$
where \( d_{i,a} = \int_{X} J_i \wedge J_a \) are the triple intersection numbers and \( c'_1 = \int_{X} c_1(TX) \wedge J_i \) are the components of the second Chern class of \( X \). Summation over repeated indices is understood. This still provides one easy-to-compute relation between the four cohomology ranks but, unlike in the Picard number one case, there is strong evidence the formulae are correct. The formulae presented below have been checked for all line bundles with integers in the range \(-10 \leq k \leq 10\), and in same cases for many more.

2.2. The Bicubic Manifold

Let \( X \) be a generic threefold in the ambient space \( \mathcal{A} = \mathbb{P}^2 \times \mathbb{P}^2 \) defined by the configuration matrix

\[
\begin{bmatrix}
3 & & 3,83 \\
3 & & 3
\end{bmatrix}
\]

and \( L = \mathcal{O}_X(k_1, k_2) \) a line bundle over \( X \). Due to the symmetry of this configuration we have \( h^0(\mathcal{O}_X(k_1, k_2)) = h^0(\mathcal{O}_X(k_2, k_1)) \), so without loss of generality we can assume that \( k_1 \leq k_2 \). The corresponding cohomology formulae are given below, together with two plots in Figure 2 showing the regions where the expressions take different forms.

\[
h^0(X, L) = \begin{cases} 
\frac{1}{2}(1 + k_1)(2 + k_2), & k_1 = 0, k_2 \geq 0 \\
\text{ind}(L), & k_1, k_2 > 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
h^1(X, L) = \begin{cases} 
\frac{1}{2}(-1 + k_1)(-2 + k_2), & k_1 = 0, k_2 > 0 \\
-\text{ind}(L), & k_1 < 0, k_2 > -k_1 \\
0, & \text{otherwise},
\end{cases}
\]

Here, the index is explicitly given by \( \text{ind}(L) = \frac{1}{2}(k_1 + k_2)(2 + k_1 k_2) \).
The expressions given above for the semi-axis $k_i = 0, k_j > 0$ and, implicitly by symmetry, for the semi-axis $k_i > 0, k_j = 0$ can be combined into the single formula

$$h^0(X, L) = \frac{1}{4}(1 + k_i)(2 + k_i)(1 + k_j)(2 + k_j).$$

$$h^1(X, L) = \frac{1}{4}(-1 + k_i)(-2 + k_i)(-1 + k_j)(-2 + k_j),$$

$$h^2(X, L) = h^3(X, L) = 0.$$ Computing the characteristic, one recovers the formula for the index, as expected.

The above results have been inferred from the results of explicit cohomology calculations for many values of $k_i$, $k_j$, using the computer code. However, the cubic manifold is sufficiently simple so that we can derive these formulae with relative ease by either combining vanishing theorems with the index or else from the sequence (2.2). For $k_i > 0$ and $k_j > 0$, Kodaira’s vanishing theorem ensures that $H^0(X, L)$ is the only non-trivial cohomology group, and its rank equals the index of $L$. Similarly, when $k_i < 0$ and $k_j < 0$, the only non-trivial cohomology is $H^1(X, L)$. Hence we only need to study the points lying in the second and fourth quadrant, as well as the points lying along the lines $k_i = 0$ and $k_j = 0$. For this, we can make use of the embedding of $X$ in $\mathbb{P}^2 \times \mathbb{P}^2$. In fact, due to the symmetry of the problem, it suffices to study only the points lying in the second quadrant and on its boundary.

For the bicubic the Koszul sequence specialises to

$$0 \to \mathcal{L} \otimes \mathcal{N}^* \to \mathcal{L} \to \mathcal{L}|_X \to 0,$$

where $\mathcal{N} = \mathcal{O}_X(3, 3)$ and $p$ is the defining polynomial. Passing to cohomology, we have the following long exact sequence:

$$H^q(\mathcal{L} \otimes \mathcal{N}^*) \to H^q(\mathcal{L}) \to H^q(X, \mathcal{L}|_X) \to \cdots$$

(2.6)

$$H^q(X, L) \simeq \text{Coker} \left( H^q(\mathcal{L} \otimes \mathcal{N}) \to H^q(\mathcal{L}) \right) \oplus \text{Ker} \left( H^{q+1}(\mathcal{L} \otimes \mathcal{N}^*) \to H^{q+1}(\mathcal{L}) \right).$$

Along the semi-line $k_1 = 0, k_2 \geq 0$, the only non-trivial bundle-valued cohomology groups on $\mathcal{A} = \mathbb{P}^2 \times \mathbb{P}^2$ that appear in the long exact sequence (2.6) are $H^q(\mathcal{L}, \mathcal{L})$ and $H^q(\mathcal{A}, \mathcal{L} \otimes \mathcal{N}^*)$. Hence $H^0(X, L) \simeq H^0(\mathcal{L}), \ H^1(X, L) \simeq H^1(\mathcal{A})$. The corresponding ranks can be obtained using Bott’s formula, and this leads to the corresponding results given in (2.4) and (2.5). Also, it follows that $H^2(X, L)$ and $H^3(X, L)$ are trivial. Due to the symmetry of the configuration, the semi-line $k_1 \leq 0, k_2 = 0$ is Serre dual to the semi-line $k_1 = 0, k_2 \geq 0$ and hence $H^0(X, L)$ and $H^1(X, L)$ are trivial in this case.

Finally, we have the situation $k_1 < 0, k_2 > 0$. In this case, the only non-trivial cohomologies on $\mathcal{A} = \mathbb{P}^2 \times \mathbb{P}^2$ are $H^2(\mathcal{L}, \mathcal{L})$ and $H^2(\mathcal{A}, \mathcal{L} \otimes \mathcal{N}^*)$. It follows that $H^0(X, L)$ and $H^1(X, L)$ are trivial and

$$H^2(X, L) \simeq \text{Ker} \left( H^2(\mathcal{A} \otimes \mathcal{N}^*) \to H^1(\mathcal{L}) \right),$$

$$H^3(X, L) \simeq \text{Coker} \left( H^2(\mathcal{A} \otimes \mathcal{N}^*) \to H^1(\mathcal{L}) \right).$$

The rank of the map $p$ that appears in (2.7) turns out to be always maximal which can be shown by methods of commutative algebra. Hence, for the line bundles for which $h^2(\mathcal{A} \otimes \mathcal{N}^*) \leq h^2(\mathcal{L})$, we conclude that $H^2(X, L)$ is trivial, while $h^2(X, L)$ must equal $\text{ind}(L)$. Similarly, when $h^2(\mathcal{A} \otimes \mathcal{N}^*) > h^1(\mathcal{A}, \mathcal{L})$, $H^2(X, L)$ is trivial and $h^2(X, L) = -\text{ind}(L)$. The boundary between these two phases is given by $\text{ind}(L) = 0$, which corresponds to the line $k_2 = -k_1$.

2.3. Another Hypersurface with Picard Number Two

Let $X$ be a generic member of the family of threefolds defined in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^3$ by the configuration matrix

$$\begin{pmatrix} \mathbb{P}^1 & 2 \\ \mathbb{P}^3 & 1 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix}^{2.86} \begin{pmatrix} 4 \end{pmatrix}$$

and $L = \mathcal{O}_X(k_1, k_2)$ a line bundle over $X$. From our explicit cohomology calculations for may values of $k_i, k_j$ we infer the following formulae:

$$h^0(X, L) = \begin{cases} k_1 + 1, & k_1 \geq 0, k_2 > 0 \\ \text{ind}(L), & k_2 < 0, k_2 = -4k_1 \\ 0, & \text{otherwise} \end{cases}$$

(2.9)

$$h^1(X, L) = \begin{cases} \frac{32}{3} k_1(k_1 - k_1^2) + \text{ind}(L), & k_1 < 0, k_2 > -4k_1 \\ \text{ind}(L), & k_1 < -1, -4k_1 > k_2 > 0 \\ \text{ind}(L), & k_1 \geq 0, k_2 = -4k_1 \\ 0, & \text{otherwise} \end{cases}$$

(2.10)

where $\text{ind}(L) = \frac{1}{4}(6k_1(1 + k_1^2) + k_2(11 + k_1^2))$. The different regions in $k$-space are shown in Figure 3.
and surprisingly, the answer seems to be “no”. Even at higher creases significantly with increasing co-dimension it is reasonable that the basic structure of the final result remains unchanged. The above formulae can, in principle, be shown to hold in a way similar to the previous case of the bicubic manifold, that is, by starting with the sequence (2.2). The novelty here (and also the feature which makes the proof more difficult) are the regions $k_1 \leq -1, k_2 = -4k_1$ and $k_1 \leq -1, k_2 > -4k_1$ which are cones in the $k$-space whose tips are away from the origin. In these regions, the cohomology groups are given by

$$H^0(X, L) \cong \text{Ker}(H^1(A, L \otimes N^*) \to H^1(A, L))$$
$$H^1(X, L) \cong \text{Coker}(H^1(A, L \otimes N^*) \to H^1(A, L))$$
$$H^2(X, L) \cong H^1 \cong 0$$

but the ranks of the maps involved in the expressions for $H^0(X, L)$ and $H^2(X, L)$ are non-maximal.

### 2.4. A Co-Dimension Two Manifold with Picard Number Two

It is a reasonable question to ask whether the appearance of exact cohomology formulae is general, at least within the class of complete intersection Calabi-Yau manifolds, or merely an accidental phenomenon particular to certain manifolds. Without aiming at any kind of general proof, we can probe this question by studying a number of additional examples with the aim of finding polynomial expressions of degree at most three for the ranks of all line bundle valued cohomology groups.

The lesson learnt from the previous two examples is that relatively simple cohomology formulae appear irrespective of the details of the Koszul sequence. In particular, we have shown that details of how cohomologies on the ambient space relate to those on the Calabi-Yau sub-manifold or specific properties of the maps involved do not matter. The basic structure of the final result remains unchanged.

The two previous examples had co-dimension one. Since the complexity of the calculation based on the sequence (2.2) increases significantly with increasing co-dimension it is reasonable to ask whether the same might hold for the final result. Perhaps surprisingly, the answer seems to be “no”. Even at higher co-dimension, the final formula for the cohomology dimensions remains a cubic in the line bundle integers $k_1$, for each region in $k$-space. The purpose of this section is to illustrate this hypothesis with a co-dimension two complete intersection Calabi-Yau threefold.

Thus, let $X$ be a generic threefold of the family in the ambient space $A = \mathbb{P}^2 \times \mathbb{P}^3$ defined by the configuration matrix

$$\begin{bmatrix}
\mathbb{P}^2 & 2 & 1 \\
\mathbb{P}^3 & 2 & 2
\end{bmatrix}^{2,62}$$

and $L = O_x(k_1, k_2)$ a line bundle over $X$. It turns out that all explicit cohomology calculations for specific values of $k_1, k_2$ are consistent with the following formulae:

$$h^0(X, L) = \begin{cases}
\frac{1}{2}(1 + k_3)(2 + k_3), & k_1 \geq 0, k_2 = 0 \\
\frac{1}{2}(1 - k_3)(2 - k_3), & k_1 < 0, k_2 = -6k_1 \\
8k_1(2 - 3k_2^2) + \text{ind}(L), & k_1 < 0, k_2 > -6k_1 \\
0 & \text{otherwise}
\end{cases}$$

$$h^1(X, L) = \begin{cases}
\frac{1}{2}(1 - k_3)(2 - k_3), & k_1 > 0, k_2 = 0 \\
\text{Max}(-\text{ind}(L), 0), & k_1 > 0, k_2 < 0, 3k_1 > -4k_2 \\
\text{Max}(-\text{ind}(L), 0), & k_1 < 0, k_2 > 0, 3k_1 > -4k_2 \\
-\text{ind}(L) + \frac{1}{2}(1 - k_3)(2 - k_3), & k_1 < 0, k_2 = -6k_1 \\
8k_1(2 - 3k_2^2), & k_1 < 0, k_2 > -6k_1 \\
0 & \text{otherwise}
\end{cases}$$

where $\text{ind}(L) = \frac{1}{2}(6k_2^2k_3 + 9k_1(1 + k_2^2) + k_2(11 + k_2^2))$. The regions associated to the case distinctions in the above formulae are
shown in Figure 4. Evidently, the structure of these results is quite similar to what we have seen for co-dimension one.

### 2.5. A Hypersurface with Picard Number Three

The previous examples suggest that the general structure of the cohomology formulae is, as one would expect, insensitive to the realisation of $X$ as an embedding in a product of projective spaces and in particular to the co-dimension of $X$. What is more important is the rank of Pic($X$). As we will see, while the structure of the formulae remains unchanged for larger Picard numbers, the number of case distinctions increases. We illustrate this with two examples at Picard numbers three and four.

Let $X$ be a generic threefold in the family defined in the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ by the configuration matrix

$$
\begin{pmatrix}
\mathbb{P}^1 & 2 \\
\mathbb{P}^1 & 2 \\
\mathbb{P}^2 & 3
\end{pmatrix}^{1,75}
$$

and $L = \mathcal{O}_X(k_1, k_2, k_3)$ a line bundle over $X$. Due to the symmetry between the two $\mathbb{P}^1$ factors, we can assume, without loss of generality, that $k_1 \leq k_2$.

The index of $L = \mathcal{O}_X(k_1, k_2, k_3)$ is given by

$$\text{ind}(L) = (3k_1k_2 + 2k_1k_3 + 2k_2k_3 + 2k_1 + 2k_2 + 3k_3)$$

and appears in various places below. If $k_3 = 0$, the following cohomology formulae hold:

$$h^0(X, L) = \begin{cases} 
(1 + k_1)(1 + k_2), & k_1 \geq 0, k_2 \geq 0 \\
0, & \text{otherwise}
\end{cases} \quad (2.13)$$

$$h^1(X, L) = \begin{cases} 
(1 - k_1)(1 + k_2), & k_1 < 0, k_2 \geq 0 \\
0, & \text{otherwise}
\end{cases} \quad (2.14)$$

If $k_1 < 0, h^0(L) = 0$ and

$$h^1(X, L) = \begin{cases} 
\text{Max}(\text{ind}(L), 0), & k_1 > 0, k_2 > 0 \\
0, & \text{otherwise.}
\end{cases} \quad (2.15)$$

If $k_1 > 0$, and assuming again that $k_1 \leq k_2$ we have:

$$h^0(X, L) = \begin{cases} 
\frac{1}{2}(1 + k_1)(2 + k_1), & k_1 = 0, k_2 = 0 \\
\text{ind}(L), & k_1 \geq 0, k_2 > 0 \\
(1 - k_1)(1 + 2k_1 + k_2), & k_1 < 0, k_2 \geq -2k_1 - 1, \\
9k_1(1 - k_1^2) + \text{ind}(L), & k_1 < 0, k_2 \geq -2k_1 - 1, \\
0, & \text{otherwise}
\end{cases} \quad (2.16)$$

$$h^1(X, L) = \begin{cases} 
\frac{1}{2}(-1 + k_1)(-2 + k_1), & k_1 = 0, k_2 = 0 \\
\text{Max}(\text{ind}(L), 0), & k_1 < 0, k_2 < -3k_1 \\
0, & \text{or} k_1 < 0, \\
9k_1(1 - k_1^2) - \text{ind}(L), & k_1 < -1, \\
0, & \text{or} k_1 \geq -2k_1 - 1, \\
\text{Max}(\text{ind}(L), 9k_1(1 - k_1^2)), & k_1 < -1, k_2 \geq 0, \\
0, & \text{otherwise}
\end{cases} \quad (2.17)$$

While complicated, these formulae follow the same pattern as encountered in the earlier examples for manifolds with Picard numbers one and two: the $k$-space can be divided into several regions, and in each such region the ranks of bundle-valued coho-
mology groups can be expressed as polynomials in the \( k \)-integers of degree at most three. Note, however, that these regions are not cones in general. For instance, in the present example, when \( k_1 > 0, k_2 > 0 \) and \( k_i < 0 \) we have \( h^1(X, L) = \text{Max}(0, \text{ind}(L)) \). This means that the region where \( h^1(X, L) = -\text{ind}(L) \) is bounded by the cubic surface \( \text{ind}(L) = 0 \).

2.6. The Tetraquadric Manifold

Let \( X \) be the family of threefolds defined in the ambient space \( \mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) by the configuration matrix

\[
\begin{bmatrix}
\mathbb{P}^1 & 2 & 2 & 2
\end{bmatrix}
\]

\[\text{(2.18)}\]

and \( L = \mathcal{O}_X(k_1, k_2, k_3, k_4) \) a line bundle over \( X \), with index given by

\[
\text{ind}(L) = 2(k_1 + k_2 + k_3 + k_4) + 2(k_1k_2k_3 + k_1k_3k_4 + k_1k_2k_4 + k_2k_3k_4).
\]

For the tetraquadric manifold exact line bundle cohomology formulae have already appeared in Refs. [14,15]. Although equivalent, the formulae presented below are simpler and are expressed in terms of polynomials of degree at most 3 in the line bundle integers.

For the purpose of clarity, we assume, without loss of generality, that \( k_1 \leq k_2 \leq k_3 \leq k_4 \) and, as before, we only present the formulae for \( h^0(X, L) \) and \( h^1(X, L) \), the other two cohomology groups being obtained by Serre duality.

For \( k_i < 0 \), Kodaira’s vanishing theorem implies that \( h^0(X, L) = h^1(X, L) = 0 \). Similarly, for \( k_i = 0 \) we have

\[
h^0(X, L) = \begin{cases}
1 & k_1 = k_2 = k_3 = 0 \\
0 & \text{otherwise}
\end{cases}
\]  

\[\text{(2.19)}\]

\[
h^1(X, L) = \begin{cases}
(1 + k_1)(1 + k_4) & k_1 = 0, k_2 = 0 \\
k_1 \geq 0, k_2 > 0 & \text{otherwise}
\end{cases}
\]  

\[\text{(2.20)}\]

These formulae have been checked to hold for all line bundles with integers in the range \(-30 \leq k_i \leq 30\).

From now on we assume that \( k_1, k_4 > 0 \). Then:

\[
h^0(X, L) = \begin{cases}
\text{Max}(0, \text{ind}(L)) & \text{otherwise}
\end{cases}
\]

\[
h^1(X, L) = \begin{cases}
(1 + k_1)(1 + k_4) & k_1 = 0, k_2 = 0 \\
k_1 \geq 0, k_2 > 0 & \text{otherwise}
\end{cases}
\]

\[\text{(2.21)}\]

\[
h^1(X, L) = \begin{cases}
(1 + k_1)(1 + k_4) & k_1 < 0, k_2 = 0 \\
k_1 < 0, k_2 > 0 & \text{otherwise}
\end{cases}
\]

\[\text{(2.22)}\]

3. Non-Simply Connected Calabi-Yau Threefolds

We can enlarge the class of manifolds for which exact cohomology formulae can be studied by looking at non-simply connected Calabi-Yau threefolds realised as free quotients of complete intersections by discrete symmetries. More concretely, let \( G \) be a finite group and \( G \times X \rightarrow X \) a free holomorphic action of \( G \) on the Calabi-Yau threefold \( X \). Then the quotient \( X/G \) is a smooth Calabi-Yau threefold with fundamental group isomorphic to \( G \). Smooth quotients of complete intersection Calabi-Yau threefolds have been systematically studied in [27–31] (see also the reviews [12,13]).

If \( L \rightarrow X \) is a line bundle equivariant with respect to the action of \( G \) on \( X \), then \( L \) is the pull-back of a line bundle on the quotient \( X/G \). In the examples discussed below, all line bundles will be equivariant.
3.1. A $\mathbb{Z}_5$-Quotient of the Quintic Threefold

The quintic family $\mathbb{P}^4[5]^{1,101}$ contains manifolds that admit a freely-acting $\mathbb{Z}_5$-symmetry. Let $z_0, z_1, z_2, z_3, z_4$ be homogeneous coordinates on $\mathbb{P}^4$ and consider the $\mathbb{Z}_5$-action generated by

$$z_i \to \zeta^iz_i,$$

where $\zeta$ is a non-trivial fifth root of unity. There are 26 monomials invariant under this action. Let $X$ be a quintic manifold defined as the zero locus of a generic linear combination of these invariant monomials. Then $X$ admits a smooth quotient with Hodge numbers $(h^1, h^2, h^3, h^4) = (1, 21)$.

All line bundles $L = O_\mathbb{P}(k)$ are equivariant with respect to the above action and we denote by $\tilde{L}$ the line bundle on $X/\mathbb{Z}_5$ whose pullback is $L$. Then the following cohomology formulae hold:

$$h^i(X/\mathbb{Z}_5, \tilde{L}) = \begin{cases} \text{ind}(\tilde{L}), & k > 0 \\ 1, & k = 0 \\ 0, & k < 0. \end{cases}$$

(3.1)

Here, the index is given by $\text{ind}(\tilde{L}) = \frac{1}{5}\text{ind}(L) = \frac{1}{6}k^3 + \frac{10}{12}k$.

3.2. A $\mathbb{Z}_3$-Quotient of the Bicubic Threefold

The family of bicubic manifolds contains a sub-family which admits free quotients by a following $\mathbb{Z}_3$-action. Introducing homogeneous coordinates $x_0, x_1, x_2$ and $y_0, y_1, y_2$ for the two ambient space $\mathbb{P}^2$ factors, the free $\mathbb{Z}_3$-action is generated by

$$x_i \to \omega^i x_i, \quad y_i \to \omega^i y_i,$$

with $\omega$ a non-trivial cube root of unity. There are 34 monomials invariant under the above action and we consider a cubic manifold $X$ defined by a generic linear combination of the invariant monomials. By quotienting, a Calabi-Yau threefold with Hodge numbers $(2, 29)$ and fundamental group $\mathbb{Z}_3$ is obtained.

As in the previous example, all line bundles $L = O_\mathbb{P}(k_1, k_2)$ are equivariant with respect to the above $\mathbb{Z}_3$-action. We denote by $\tilde{L}$ the bundle on $X/\mathbb{Z}_3$ whose pullback is $L$. Then the following cohomology formulae hold:

$$h^i(X/\mathbb{Z}_3, \tilde{L}) = \begin{cases} \frac{1}{6}(-1 + k_2)(-2 + k_2), & k_1 = 0, k_2 > 0, \\ \frac{1}{6}k_2^2, & k_1 = 1 \mod 3 \text{ or} \\ \frac{1}{6}k_2^2 + \frac{k_2}{2} + 1, & k_1 = 0, k_2 > 0, \\ \frac{1}{6}k_2^2 + \frac{k_2}{2}, & k_1 = 0 \mod 3 \\ \frac{1}{2}(k_1 + k_2)(2 + k_1 k_2), & k_1, k_2 > 0 \\ 0, & \text{otherwise}. \end{cases}$$

(3.2)

4. Conclusions

The evidence gathered from the examples presented in this note suggests that the existence of relatively simple formulae for line bundle cohomology on Calabi-Yau three-folds is a generic phenomenon. We have studied several complete intersection Calabi-Yau threefolds and some of their quotients by freely acting discrete symmetries, with Picard numbers ranging from one to four and we have found a common pattern. The space of line bundle integers can be divided into several different regions, and in each such region the ranks of all cohomology groups can be expressed as a polynomial in the line bundle integers of degree at most three.

The formulae presented here were found by computing cohomology dimensions for a large number of line bundles and by looking for patterns in this data. The computations were carried out using a Mathematica implementation of a computational algorithm that relies on the Bott-Borel-Weil theorem and spectral sequences techniques applied to the Koszul sequence (2.2). Although conceptually straightforward, these calculations are often laborious and computationally intensive. Despite the intricacy of the intermediate computations and the presence of Leray maps with non-maximal ranks, the final cohomology results fall into the simple pattern described above. It is relatively straightforward to fix the cubic polynomials which describe the cohomology dimensions by matching to sufficiently many line bundle cohomologies. The more difficult part of extracting the correct formulae from the data is to establishing the regions of validity for these polynomials.

For each Calabi-Yau manifold that we have analysed, the cohomology dimensions have been computed for line bundles with integers in the range $10 \leq k_1 \leq 10$. For each manifold, their number is significantly larger than what is required in order to fix all the coefficients in the general Ansatz for the cohomology formula, yet all cohomology results are correctly described by this formula. While this is of course not a proof it provides a non-trivial check of our results.

For relatively simple cases the cohomology formulae can be proved by chasing through the long exact cohomology sequences associated to the Koszul sequence (2.2) and, where required, computing ranks of maps using methods of commutative algebra. We have carried this out explicitly for the bi-cubic in $\mathbb{P}^2 \times \mathbb{P}^2$. For more complicated examples this approach, while possible in principle, becomes extremely cumbersome and it would not be practical to carry this out even for a modest number of manifolds.
There are two other potential ways of deriving or extracting cohomology formulae in a systematic way. For our examples, the structure of the formulae turns out to be independent of the ambient space. This suggests that there may be an alternative, more intrinsic method to compute these cohomologies which bypasses the embedding into the ambient space and works on the Calabi-Yau manifold only. We do not currently know how such a method would work - or if it even exists - but it would certainly be interesting to pursue this further.

From a practical point of view, our results suggest a very concrete problem in machine learning. Such techniques have recently been applied to problems in geometry and string theory for the pioneering papers see Refs. [34–39]. We can use the line bundle cohomology data on a given manifold, as computed by the methods described in Ref. [8], and train a neural network. However, unlike for most applications of machine learning, the goal would not merely be to have the neural network predict further cohomology results for individual line bundles, but rather to extract concrete formulae from the trained neural network. Such an approach would facilitate extracting cohomology formulae in a systematic way and for a large number of manifolds. Work in this direction is currently underway.

After this paper appeared, Ref. [40] was submitted to the arXiv. This work discusses related problems in the context of Calabi-Yau hypersurfaces in toric four-folds, extracting information about line bundle cohomology using machine learning techniques. The basic structure of their results is similar to the one presented here.

Acknowledgements

We are grateful to James Gray, Damian Rössler and Shing-Tung Yau for insightful discussions and to Magdalena Larfors and Robin Schneider for pointing out a mistake in one of our formulæ.[41] A. L. is partially supported by the EPSRC network grant EP/N007158/1.

Conflict of Interest

The authors declare no conflict of interest.

Keywords

Calabi-Yau manifolds, cohomology, string theory

Received: October 11, 2019
Published online: November 10, 2019

[1] R. Blumenhagen, B. Jurke, T. Rahn, H. Roschy, J. Math. Phys. 2010, 51, 103525, 1003.5217.
[2] T. Rahn, H. Roschy, J. Math. Phys. 2010, 51, 103520, 1006.2392.
[3] S.-Y. Jow, Cohomology of Toric Line Bundles via Simplicial Alexander Duality, 1006.0780.
[4] P. Candelas, A. Dale, C. Lutken, R. Schimmrigk, Nucl. Phys. 1988, 8298, 493.
[5] P. Candelas, C. A. Lutken, R. Schimmrigk, Nucl. Phys. 1988, B306, 113.
[6] T. Hübisch, Calabi-Yau Manifolds: A Bestiary for Physicists, World Scientific. 1991.
[7] L. B. Anderson, Heterotic and M-theory Compactifications for String Phenomenology, 0808.3621.
[8] L. B. Anderson, J. Gray, Y.-H. He, S.-J. Lee, A. Lukas, CICY package, based on methods described in arXiv:0911.1569, arXiv:0911.0865, arXiv:0805.2875, hep-th/0703249, hep-th/0702210.
[9] L. B. Anderson, Y.-H. He, A. Lukas, JHEP 2007, 0707, 049, hep-th/0702210.
[10] L. B. Anderson, Y.-H. He, A. Lukas, JHEP 2008, 0807, 104, 0805.2875.
[11] L. B. Anderson, J. Gray, Y.-H. He, A. Lukas, JHEP 2010, 1002, 054, 0911.1569.
[12] L. B. Anderson, J. Gray, A. Lukas, E. Palti, Phys. Rev. 2011, D84, 106005, 1106.4804.
[13] L. B. Anderson, J. Gray, A. Lukas, E. Palti, JHEP 2012, 1206, 113, 1202.1757.
[14] A. Constantin, Heterotic String Models on Smooth Calabi-Yau Three-folds (DPhil Thesis), Oxford U. 2013, 1808.09993.
[15] E. I. Buchbinder, A. Constantin, A. Lukas, JHEP 2014, 1403, 025, 1311.1941.
[16] L. B. Anderson, A. Constantin, J. Gray, A. Lukas, E. Palti, JHEP 2014, 1004, 047, 1307.4787.
[17] Y.-H. He, S.-J. Lee, A. Lukas, C. Sun, Heterotic Model Building: 16 Special Manifolds, 1309.0223.
[18] E. I. Buchbinder, A. Constantin, A. Lukas, JHEP 2014, 1406, 100, 1404.2767.
[19] E. I. Buchbinder, A. Constantin, A. Lukas, Phys. Lett. 2015, B748, 251, 1409.2412.
[20] E. I. Buchbinder, A. Constantin, A. Lukas, Phys. Rev. 2015, D91, 046010, 1412.8696.
[21] L. B. Anderson, A. Constantin, S.-J. Lee, A. Lukas, Phys. Rev. 2015, D91, 046008, 1411.0034.
[22] A. Constantin, A. Lukas, C. Mishra, JHEP 2016, 03, 173, 1509.02729.
[23] E. I. Buchbinder, A. Constantin, J. Gray, A. Lukas, Phys. Rev. 2016, D94, 046005, 1606.04032.
[24] A. P. Braun, C. R. Brodie, A. Lukas, Heterotic Line Bundle Models on Elliptically Fibered Calabi-Yau Three-folds, 1706.07688.
[25] S. Blesneag, E. I. Buchbinder, A. Constantin, A. Lukas, E. Palti, JHEP 2018, 04, 139, 1801.09645.
[26] K. Altmann, J. Buczyński, L. Kastner, A.-L. Winz, Immaculate Line Bundles on Toric Varieties, 1808.09312.
[27] P. Candelas, A. Constantin, C. Mishra, Adv. High Energy Phys. 2018, 005, 1003.3235.
[28] V. Braun, JHEP 2011, 1104, 005, 1003.3235.
[29] P. Candelas, A. Constantin, Fortsch. Phys. 2012, 60, 345, 1010.1878.
[30] P. Candelas, A. Constantin, A. Mishra, Fortsch. Phys. 2016, 64, 463, 1511.01703.
[31] A. Constantin, J. Gray, A. Lukas, JHEP 2017, 01, 001, 1607.01830.
[32] R. Davies, Adv. High Energy Phys. 2011, 2011, 901989, 1103.3156.
[33] P. Candelas, A. Constantin, C. Mishra, Fortsch. Phys. 2018, 66, 1800029, 1602.06303.
[34] Y.-H. He, Deep-Learning the Landscape, 1706.02714.
[35] F. Ruehle, JHEP 2017, 08, 038, 1706.07024.
[36] D. Kreff, R.-K. Seong, Phys. Rev. 2017, D96, 066014, 1706.03346.
[37] J. Carifio, J. Halverson, D. Krioukov, B. D. Nelson, JHEP 2017, 09, 157, 1707.00655.
[38] K. Bull, Y.-H. He, V. Jejjala, C. Mishra, Machine Learning CICY Three-folds, 1806.03121.
[39] M. Demirtas, C. Long, L. McAllister, M. Stillman, The Kreuzer-Skarke Atlas, 1808.01282.
[40] D. Klaewer, L. Schlechter, Machine Learning Line Bundle Cohomologies of Hypersurfaces in Toric Varieties, 1809.02547.
[41] M. Larfors, R. Schneider, Line bundle cohomologies on CICYs with Picard number two, 1906.00392.