Field Theory of Quantum Antiferromagnets:
From The Triangular To The Kagome Lattice

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ABSTRACT

We analyse a family of models, that interpolates between the Triangular lattice antiferromagnet (TLAF) and the Kagome lattice antiferromagnet (KLAF). We identify the field theories governing the low energy, long wavelength physics of these models. Near the TLAF the low energy field theory is a nonlinear sigma model of a $SO(3)$ group valued field. The $SO(3)$ symmetry of the spin system is enhanced to a $SO(3)_R \times SO(2)_L$ symmetry in the field theory. Near the KLAF other modes become important and the field takes values in $SO(3) \times S_2$. We analyse this field theory and show that it admits a novel phase in which the $SO(3)_R$ spin symmetry is unbroken and the $SO(2)_L$ symmetry is broken. We propose this as a possible mechanism by which a gapless excitation can exist in the KLAF without breaking the spin rotation symmetry.
1 Introduction.

The possibility of novel groundstates has been motivating the study of two dimensional frustrated quantum antiferromagnets for quite some time now. In classical unfrustrated antiferromagnets, the ground state is the well known Neel state. The SO(3) spin symmetry of the system is broken down to SO(2). The low energy excitations are the two branches of gapless spin waves which are the Goldstone modes. A commonly observed effect of frustration is that the ground state becomes a spiral state. The spin arrangement remains periodic and is characterized by a spiral vector, $\mathbf{q}$. The SO(3) symmetry is now completely broken and there are three gapless Goldstone modes. If the parameters of the system are such that the effects of quantum fluctuations are not very large then this basic picture remains true in the quantum system with changed values of physical quantities like staggered magnetization, spin wave velocities etc. However strong quantum fluctuations can destroy the long range spin order and the system could go to a paramagnetic phase. In frustrated systems several alternate novel effects of quantum fluctuations have been proposed. One possibility [1] is the spin liquid groundstate that has the full symmetry of the hamiltonian. Closely related states are the chiral spin liquids or flux phases[21]. More recently, magnetic states characterized by long range order of higher tensor operators have been proposed [2].

Numerical and analytical studies [3, 4] indicate that the triangular lattice antiferromagnet (TLAF) has a Neel ordered spiral ground state with a spiral angle of $2\pi/3$. This is the so called $\sqrt{3} \times \sqrt{3}$ state. However, work on the spin
1/2 Kagome lattice antiferromagnet (KLAF) indicates the absence of any kind of long range Neel order \([5, 6, 7]\). The KLAF is therefore a potential candidate for novel groundstates.

The KLAF is experimentally realized in the magnetoplumbite type compound \(\text{Sr Cr}_8 \text{ Ga}_4 \text{ O}_{19}\) \([8]\). This is a layered compound containing planes of \(\text{Cr}^{3+}\) ions that form a \(S = \frac{3}{2}\) KLAF. About 80\% of the KLAF sites are occupied by the chromium ions. The inter Cr spacing is 2.9 \(\text{Å}\). Susceptibility measurements show a Curie-Weiss behaviour at high temperature with a Curie-Weiss temperature \(\theta_{CW} \sim 400 K\). There is a spin glass like cusp at \(T_g \sim 5 K\). The specific heat shows a \(T^2\) behaviour below \(T_g\) \([9]\). Neutron scattering however shows no Bragg peaks down to 1.5K \([10]\). There exists short range \(\sqrt{3} \times \sqrt{3}\) order with a correlation length of about 7\(\text{Å}\) at 1.5K. This has led to the speculation that the groundstate is characterized by the long range order of some order parameter that is invisible to the neutrons. Recent \(\mu\)sr studies on the compound lends support to a spin liquid type of ground state \([11]\).

Recently another system where the KLAF is experimentally realized has been reported \([12]\). This is the deuteronium jarosite, \((\text{D}_3\text{O})\text{Fe}_3(\text{SO}_4)(\text{OD})_6\). The \(\text{Fe}^{3+}\) atoms in this compound form layers of \(S = \frac{5}{2}\) KLAF. About 97\% of the KLAF sites are occupied by the iron ions. The inter Fe spacing is 3.67 \(\text{Å}\). The Curie-Weiss temperature is \(\sim 1500 K\). There is a spin glass type cusp at 13.8 \(K\). The specific heat goes as \(T^2\) below this temperature. Neutron scattering sees no long range order. There is short range order corresponding to the \(\sqrt{3} \times \sqrt{3}\) spin structure with a correlation length of about 19 \(\text{Å}\) at 1.9 \(K\).
The $T^2$ behaviour of the low temperature specific heat, absence of long range spin order and the presence of short range $\sqrt{3} \times \sqrt{3}$ order are common properties of both these systems indicating that these are universal properties of a KLAF. The specific heat behaviour indicates the presence of a gapless boson in the low temperature phase. However the neutron scattering shows absence of long range spiral order. Further, as mentioned above, numerical work indicates that all the symmetries of the hamiltonian are intact. What is the mechanism in these systems that produces a gapless boson while keeping the symmetries of the hamiltonian intact? In this paper, we address this puzzle and propose a possible solution for it. We work within the framework of the large $S$ semiclassical expansion. The fairly high value of the spin in the experimental systems indicates that these properties should be seen in this approximation.

The classical KLAF has infinitely many (apart from symmetry operations) degenerate groundstates including many with non-coplanar spin configurations. It exhibits the order from disorder phenomenon, i.e., the spin wave modes around the planar groundstates are softer, hence the fluctuations partially lift the ground-state degeneracy. However, there still remain infinitely many distinct planar ground state spin configurations [13, 14].

This property of the KLAF makes it difficult to study analytically. In this paper we consider instead, a one parameter family of models that interpolate between the TLAF and the KLAF. Such a model has also been considered by Zeng and Elser in [5], where they do a spin wave analysis of the model. We will refer to these models as the deformed triangular lattice antiferromagnet (DTLAF). The
model is defined by the Hamiltonian,

\[ H = J \left( \sum_{<i,j> \in K_B} \vec{S}_i \cdot \vec{S}_j + \chi \sum_{<i,j> \notin K_B} \vec{S}_i \cdot \vec{S}_j \right) \]  

(1)

Here \(<i,j>\) label the nearest neighbour sites on a triangular lattice. \(K_B\) denotes the set of nearest neighbour bonds that belong to the kagome lattice (which is a subset of the triangular lattice). When \(\chi = 1\), the model is the TLAF, whereas if \(\chi = 0\), it is the KLAF. It is also interesting to note that the structure of the Cr atoms in SCGO is made up of a two layers. The atoms in one plane lie on a Kagome lattice, while those on the upper layer lie on a triangular lattice whose lattice points lie over the centres of the hexagons in the kagome structure \[8\]. Therefore the DTLAF could be of direct relevance to SCGO.

An important property of the DTLAF which we will show in the next section is that the ground state is unique (upto symmetry operations) for all nonzero values of \(\chi\). For \(0 < \chi \leq 2\), the ground state is the \(\sqrt{3} \times \sqrt{3}\) state. Our strategy is then to study the model at \(\chi \neq 0\) and analyse the quantum groundstate as a function of \(\chi\). As mentioned earlier, short range \(\sqrt{3} \times \sqrt{3}\) order has been experimentally observed both in \(\text{SrCr}_8\text{Ga}_4\text{O}_{19}\) and in \((\text{D}_3\text{O})\text{Fe}_3(\text{SO}_4)(\text{OD})_6\). Theoretically also in a large N formalism, the fluctuations pick out the \(\sqrt{3} \times \sqrt{3}\) state \[15\]. This indicates that it should be meaningful to look upon the KLAF as the \(\chi \to 0\) limit of the DTLAF.

The analysis of a spin system near a transition requires consideration of large amplitude fluctuations. Further since the correlation length near the transition is large, the lattice model can be approximated by a continuum field theory.
the physics is expected to be well described by a field theory of the soft modes of the system. This expectation has been well verified experimentally for the unfrustrated square lattice antiferromagnet where the physics is described well by the nonlinear sigma model \[16\]. The order parameter for this model is a unit vector field. The soft modes are the two Goldstone modes.

Field theories for frustrated systems, in particular for the TLAF, have been derived \[17\] and analysed using momentum space renormalization group techniques \[18, 19, 20\]. The order parameter here is a SO(3) group element. Physically a rotation group element can be looked upon as describing the orientation of a rigid body. In the spin system this orientation is specified by the sublattice magnetization and the chiral order parameter. The internal symmetry group of these field theories is \(SO(3) \times SO(2)\) which is larger than the \(SO(3)\) symmetry of the spin system. The extra \(SO(2)\) symmetry corresponds to rotations in the body fixed frame of the rigid bodies. The renormalization group analysis of these models\[18, 19, 20\] shows no novel phases. The system is either in the Neel ordered phase or in the usual paramagnetic phase with exponentially decaying correlation functions and gapped spin one magnon excitations. Thus the DTLAF also can be expected not to show any novel behaviour near \(\chi = 1\). However, near \(\chi = 0\) we expect some modes other than the Goldstone modes to soften, reflecting the infinite degeneracy that sets in at \(\chi = 0\). The field theory that includes these modes would be appropriate to study the physics near \(\chi = 0\).

In this paper we motivate a field theory to describe the Kagome end of the model and study its phase structure. We start by finding the classical ground-
states for different values of $\chi$ in section 2. We do the spin wave calculation in section 3, systematically parametrise the hard and soft fluctuations about the classical ground state in the region $0 < \chi < 1$ and identify the modes that soften when $\chi \to 0$. The field theory describing the system near $\chi = 1$ is derived in section 4. In section 5, we motivate the form of the field theory near $\chi = 0$ that includes large amplitude fluctuations of the modes that soften in this region. In section 6, we integrate out the Goldstone modes and obtain the effective theory of the new modes. The phases of this effective theory are analysed in section 7. We summarise our results in section 8.

2 Classical ground states.

In this section, we will analyse the ground state of the classical model. We will show that there are three different types of ground states corresponding to three ranges of the parameter $\chi$. The energy of the classical model can be written as,

$$E = J \sum_{<i,j> \in K_B} \vec{S}_i \cdot \vec{S}_j + J \chi \sum_{<i,j> \not\in K_B} \vec{S}_i \cdot \vec{S}_j$$

Here, $\vec{S}_i$ are vectors satisfying the constraint $\vec{S}_i \cdot \vec{S}_i = S^2$. $K_B$ denotes the set of bonds that belong to the Kagome lattice.

We begin with the parameter range $0 < \chi < 2$. The energy in equation (2) can be rewritten as,

$$\frac{E}{J} = \frac{1}{2} \left(1 - \frac{\chi}{2}\right) \sum_{\Delta \epsilon K_\Delta} \left(\sum_i \vec{S}_i\right)^2 + \frac{\chi}{4} \sum_{\Delta \not\in K_\Delta} \left(\sum_i \vec{S}_i\right)^2 - \frac{3S^2N}{2} \left(\frac{1 + \chi}{2}\right)$$

(3)
where the sum is over all the triangles that belong to the Kagome lattice. \( N \) is the total number of sites. In the range of \( \chi \) under consideration, the coefficients of the first two terms in equation (3) are positive. Thus the energy is minimized by spin configurations that satisfy the condition that the net magnetization of every triangle is zero. It is well known that the unique (upto symmetry operations) solution of this constraint is the spiral state with spiral angle equal to \( 2\pi/3 \).

Thus, this so called \( \sqrt{3} \times \sqrt{3} \) state is the unique, stable groundstate of the model when \( 0 < \chi < 2 \). At \( \chi = 0 \), of course, there are infinitely many other solutions to the constraint and the ground state is highly degenerate.

The ground state energy in this range of \( \chi \) is given by,

\[
E_{G.S} = -\frac{3JS^2}{4} \left( \frac{1 + \chi}{2} \right)
\]

Next we look at the range \( \chi \geq 2 \). We rewrite the energy as,

\[
\frac{E}{J} = \sum_{\Delta \in K} \frac{1}{2} \left( \vec{S}_{1K} + \vec{S}_{2K} + \frac{\chi}{2} \vec{S}_{NK} \right)^2 - \frac{3S^2N}{8} \left( 1 + \frac{\chi^2}{8} \right)
\]

Here the sum is over all the triangles that do not belong to the Kagome lattice. \( \vec{S}_{1K} \) and \( \vec{S}_{2K} \) are the spins at the two Kagome sites and \( \vec{S}_{NK} \) is the spin at the non-Kagome site in the centre of every hexagon.

In the range \( 2 \leq \chi \leq 4 \), the quantity \( (\vec{S}_{1K} + \vec{S}_{2K} + \frac{\chi}{2} \vec{S}_{NK}) \) can be made to be equal to zero on every triangle by a non-coplanar spin configuration described below. Consider any non-Kagome site and let \( \vec{S}_a \) and \( \vec{S}_b \) be the spins of the \( \sqrt{3} \times \sqrt{3} \) spiral state on the sites that surround it. Choose,
\[ \vec{S}_{1K} = \cos \theta \vec{S}_a + S \sin \theta \hat{z} \]
\[ \vec{S}_{2K} = \cos \theta \vec{S}_b + S \sin \theta \hat{z} \] (6)

If \( \theta \) satisfies the equation

\[ \sin^2 \theta = \frac{1}{3} \left( \frac{\chi^2}{4} - 1 \right) \] (7)

Then we have \(| \vec{S}_{1K} + \vec{S}_{2K} | = \frac{\chi}{2} S \). So if we choose

\[ \vec{S}_{NK} = -\frac{2}{\chi} (\vec{S}_{1K} + \vec{S}_{2K}) \]
\[ = \frac{2}{\chi} \left( \cos \theta \vec{S}_c - 2 S \sin \theta \hat{z} \right) \] (8)

Then the condition \( \vec{S}_{1K} + \vec{S}_{2K} + \frac{\chi}{2} \vec{S}_{NK} = 0 \) is satisfied in every triangle under consideration. Equation (7) always has a solution in the parameter range \( 2 \leq \chi \leq 4 \). Thus the non-coplanar configuration described in equations (6) and (8) is the stable ground state in the range of \( \chi \). The ground state energy in this range is given by,

\[ E_{G.S} = -\frac{3 S^2 J N}{2} \left( 1 + \frac{\chi^2}{8} \right) \] (9)

At \( \chi = 4 \), we have \( \theta = \pi/2 \). All the spins are then collinear. The spins on the Kagome lattice point up and the others point down. Examining the energy as written in equation (8), it is clear that this state (\( \theta = \pi/2 \)) will minimize the energy in the range \( \chi \geq 4 \). The ground state energy in this range being,
\[ E_{G,S} = -\frac{3S^2 JN}{2} (\chi - 1) \] (10)

In the range \( \chi > 2 \), the system has non-zero magnetization. The average magnetization per site is given by,

\[
\begin{align*}
\vec{M} &= S \sin \theta \left( \frac{3}{4} - \frac{1}{\chi} \right) \hat{z} & \quad 2 \leq \chi \leq 4 \\
&= \frac{S}{2} \hat{z} & \quad \chi \geq 4
\end{align*}
\] (11)

To summarize, at \( \chi = 0 \) the model is exactly the Kagome lattice model the ground state is infinitely degenerate. As soon as we tune on \( \chi \), this infinite degeneracy is lifted and we have the \( \sqrt{3} \times \sqrt{3} \) spiral state as the unique (upto symmetry operations) ground state. This state remains the ground state until \( \chi = 2 \). The spins then start lifting off the plane. The spins on the Kagome sites having a \( \hat{z} \) component which is anti parallel to the \( \hat{z} \) component of the spins on the non-Kagome sites. This state is thus a combination of a spiral and ferrimagnetic state. At \( \chi = 4 \) all the spins are collinear and the transition to the ferrimagnetic state is complete. The ferrimagnetic state persists for all the values of \( \chi \geq 4 \).

This completes our analysis of the classical ground states. For the rest of the paper we will be focussing our attention on the region \( 0 < \chi \leq 2 \) and will be analyzing the fluctuations about the \( \sqrt{3} \times \sqrt{3} \) ground state.
3 Spinwave theory

In this section we do the spin wave analysis of our model hamiltonian in the region $0 < \chi < 2$. The calculation has been done earlier in reference \[5\]. Our aim here is to compute the gaps as a function of $\chi$ and explicitly identify the modes which soften as $\chi \to 0$. The hamiltonian is,

$$H = \sum_{<i,j>} J_{ij} \vec{S}_i \cdot \vec{S}_j$$  \hspace{3cm} (12)

where, $J_{ij} = \chi$ for i or j belonging to the Kagome lattice and $J_{ij} = 1$ when i and j both lie in the kagome lattice.

The unit cell, as shown in fig 1, is a set of 12 points. This is commensurate with the periodicity of the DTLAF and the $\sqrt{3} \times \sqrt{3}$ structure of the classical groundstate. Adapting our notation to what is suggested by the unit cell structure, we rewrite the hamiltonian, as,

$$H = \sum_{Ii\alpha,Jj\beta} \frac{1}{2} J_{Ii\alpha,Jj\beta} Tr[\vec{S}_{Ii\alpha} \vec{S}_{Jj\beta}]$$  \hspace{3cm} (13)

Where we have used the notation, $S_{Ii\alpha} = \frac{1}{2} \vec{S}_{Ii\alpha} \tau^a$, $\tau^a$ being the Pauli spin matrices. The index I labels the unit cell. The set $(i, \alpha)$ label the spins in each unit cell. $\alpha = 0, 1, 2$ is the sublattice index. $i = 0, ..., 3$ label the four different spins of each sublattice in the unit cell. The convention we are using to label the twelve spins in each unit cell is shown in fig 1. We then write the spins as,

$$S_{Ii\alpha} = \vec{s}\{n_\alpha - \frac{i}{\sqrt{\vec{s}}} [w_{Ii\alpha}, n_\alpha] - \frac{1}{2\vec{s}} [w_{Ii\alpha}, [w_{Ii\alpha}, n_\alpha]] \}$$  \hspace{3cm} (14)
This is the usual Holstein Primakoff transformation and \( n_\alpha = \frac{1}{2} \hat{n}_\alpha \cdot \vec{r} \), \( \hat{n}_\alpha \) being the classical groundstate spin configuration. \( \vec{s} = \sqrt{S(S+1)} \), so that the magnitudes of the spins are normalised to \( \vec{s} \cdot \vec{s} = S(S+1) \). Here, \( w_{Ii\alpha} = \frac{1}{2} \vec{w}_{Ii\alpha} \cdot \vec{r} \), with \( \vec{w}_{Ii\alpha} \) being perpendicular to the ground state spin orientation, \( \vec{n}_\alpha \). Hence \( w_{Ii\alpha} = \vec{P}_{Ii\alpha} \epsilon^1_\alpha + \vec{Q}_{Ii\alpha} \epsilon^2_\alpha \), with \( \epsilon^1_\alpha, \epsilon^2_\alpha, \hat{n}_\alpha \) forming an orthogonal set of basis vectors for each \( \alpha \) and \( [\vec{Q}_{Ii\alpha}, \vec{P}_{Jj\beta}] = i\delta_{IJ} \delta_{ij} \delta_{\alpha\beta} \). The hamiltonian expanded to the quadratic order in the fluctuations, \( w_{Ii\alpha} \), is given by,

\[
H = \frac{s^2}{2} \sum_{r=0,3} J^r_{Ii\alpha,Jj\beta} \text{Tr}[\frac{1}{2s} n_\alpha n_\beta (w_{Ii\alpha}^2 + w_{Jj\beta}^2) - \frac{1}{s} w_{Ii\alpha} n_\alpha w_{Jj\beta} n_\beta] \quad (15)
\]

we define the fourier transform as follows,

\[
w_{Kia} = \sum_{I} w_{Ii\alpha} \exp(-i\vec{K} \cdot \vec{I}) \quad (16)
\]

Where \( I \) is the unit cell index and \( \vec{K} \) takes values in the Brilloiun Zone. Now we can write \( H \) in the form,

\[
H = J \vec{s} \sum_{K,i\alpha,j\beta} [\vec{P}_{Kia} M_{ia,j\beta}^{-1} \vec{P}_{Kia} + \vec{Q}_{Kia} K_{ia,j\beta} \vec{Q}_{Kia}] \quad (17)
\]

For arbitrary \( \chi \) the matrices \( M^{-1} \) and \( K \) do not commute, hence it is not possible to directly diagonalise \( H \). We can define the matrix \( M^{-1} K = \Omega^2 \), and the left and right eigenvectors of this matrix, \( \Psi^r_{La} \) and \( \Psi^r_{Ra} \), are the normal modes of \( H \) and the eigenvalues \( \omega^2_{n,r} \) are the corresponding energy gaps. The old variables \( \vec{P} \) and \( \vec{Q} \) are written in terms of the new canonical variables \( P \) and \( Q \) as follows,

\[
\vec{P}_{Kia} = 2\Psi^r_{Ra,ia} \sqrt{\frac{c_{nr}}{\omega_{nr}}} P_{Kn} \quad (18)
\]

\[
\vec{Q}_{Kia} = 2\Psi^r_{La,ia} \sqrt{\frac{c_{nr}}{\omega_{nr}}} Q_{Kn} \quad (19)
\]

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The explicit form of $M^{-1}$ and $K$ and our derivation of the normal modes of $H$ are given in the Appendix-A. In terms of the new canonical variables $P_{nr}$ and $Q_{nr}$, the Hamiltonian is,

$$H = \frac{1}{2} \sum_{nr} \omega_{nr} (P_{Knr}P_{-Knr} + Q_{Knr}Q_{-Knr})$$

Explicit expressions for the gaps and the left and right eigenvectors of the matrix $\Omega^2$ are given in Appendix-B. The modes (0,0), (0,1), (0,2) are the soft modes, gapless for all $\chi$, which we shall address as the S-S modes. The modes (1,0), (1,1), (3,1), (1,2), (3,2), which are hard for non-zero $\chi$ but become gapless for $\chi = 0$, will be referred to as the H-S modes. The modes (2,0), (3,0), (2,1), (2,2), which remain hard for all $\chi$ will be referred to as the H-H modes. Among the H-S modes, the modes labelled (1,0), (1,1) and (1,2), become gapless at $\chi = 0$ simply because 3 points from each unit cell decouple from their neighbours at $\chi = 0$. This can be seen by looking at the expressions for the corresponding eigenvectors at $\chi = 0$, as given in Appendix B. Whereas the modes (3,1) and (3,2) are the ones which truly soften and become gapless at $\chi = 0$. A look at the contribution from these different modes to the reduction of the staggered magnetization, gives an idea about how these modes affect the physics close to the Kagome end.

The staggered magnetization $M_I$ is given by,

$$M_I = \frac{\bar{s}}{12} \sum_{i,\alpha} U_{i\alpha}^\dagger \frac{\tau^1}{2} U_{i\alpha}$$

where, $U_{i\alpha} = \exp(\frac{\tau^i}{\sqrt{s}} w_{Ii\alpha})$

Expanding $M_I$ up to terms quadratic in $w_{Ii\alpha}$ we have,

$$M_I = \frac{\bar{s}}{12} \sum_{i\alpha} \frac{1}{2} \left[ \tau^1 - \frac{2i}{\sqrt{s}} w_{Ii\alpha} \tau^1 - \frac{2}{s} w_{Ii\alpha}^2 \tau^1 \right]$$

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the average staggered magnetization is,

\[
< M_I > = M_{cl}^I (1 - \Delta M_I)
\]

(23)

where \(\Delta M_I\) is given by,

\[
\Delta M_I = \frac{1}{24} \sum_{a,i,\alpha} [< w_{Ii\alpha}^a w_{Ii\alpha}^a > -1]
\]

(24)

The contributions to \(\Delta M_I\) coming from the hard and soft modes, plotted as a function of \(\chi\) are shown in fig.2. The contribution of the hard modes is seen to dominate close to \(\chi = 0\). This is because two of the H-S modes start softening in this region and give a large contribution to \(\Delta M\). This indicates that while deriving the low energy effective field theory of this model we should allow for large fluctuations of the modes (3,1) and (3,2) near \(\chi = 0\). Hence, near the KLAFF end, the theory should be described by 5 parameters which include three corresponding to the S-S modes and two corresponding to the H-S modes. But before we look at this theory, we take a brief look at the physics close to the TLAF.

4 The Field Theory near \(\chi = 1\)

As mentioned in the introduction, we expect the lattice spin system to be well described by a field theory near phase transitions where the physics is dominated by the low energy, long wavelength modes. This field theory has previously been derived in reference [17] for the TLAF. Near \(\chi = 1\), the low energy modes are the three Goldstone modes, the S-S modes. So we must take into consideration the
large amplitude fluctuations of these modes. To do this, we write the spins as,

\[ S_{I\alpha} = \tilde{s} U_{I\alpha}^\dagger n_{\alpha} U_{I\alpha} \]  

(25)

We separate the hard and the soft modes by rewriting the \( U_{I\alpha} \) as,

\[ U_{I\alpha} = \exp -\frac{i w_{I\alpha}}{\sqrt{s}} W_I \]  

(26)

Here \( w_{I\alpha} \) contains only the H-S and the H-H modes. In the derivation of the field theory, \( w_{I\alpha} \) are assumed to be small. There is no assumption about \( W_I \) and they can take any value. The \( W_I \) correspond to rigid rotations of all the spins in the unit cell. These are therefore exactly the Goldstone modes. Therefore, if \( w_{I\alpha} \) is assumed to be small, we have a parameterization of the spins such which allows for large fluctuations of the soft (S-S) modes and small fluctuations of the hard (H-S and H-H) modes.

The effective action in the long wavelength, low energy approximation is obtained by keeping only the terms quadratic in the hard fluctuations, and then integrating them out. This leaves us with the effective field theory of the soft modes. The details of this method of deriving the field theory will be described elsewhere. The final expression for the action that we get is,

\[ S = \int d^3 x \frac{1}{2} \sum_{\mu,a} \rho_{\mu}^0 L_{\mu}^a L_{\mu}^a \]  

(27)

where, \( L_{\mu}^a = \frac{1}{2} Tr [ r^a \partial_\mu W(x) W(x)^\dagger] \), \( \rho_0^0 = \frac{1}{4} \frac{4}{\sqrt{3}} \frac{(3-\chi)}{\chi(2-\chi)} \), \( \rho_0^{1,2} = \frac{1}{9} \frac{4}{\sqrt{3}} \frac{(3+7\chi)}{\chi(3+\chi)} \), \( \rho_i^0 = JS^2 \sqrt{3} (1 + \chi) \) and \( \rho_i^{1,2} = JS^2 \sqrt{3} \frac{\chi(5-\chi)}{(3+\chi)} \), for \( i = 1, 2 \).

Our expressions for the parameters, evaluated at \( \chi = 1 \) coincides with the values given in reference\[17]. Before ending this section we describe the symmetries of this model. The original spin hamiltonian is invariant under the \( SO(3) \)
spin rotations. This corresponds to the spins $S_{I\alpha}$ transforming as follows,

$$S_{I\alpha}^a \rightarrow (\Omega_R)_b^a S_{I\alpha}^b$$  \hspace{1cm} (28)

Where $\Omega_R$ is a $SO(3)$ matrix. In terms of the matrices $S_{I\alpha}$,

$$S_{I\alpha} \rightarrow X^\dagger S_{I\alpha} X$$  \hspace{1cm} (29)

where $X$ is the SU(2) representative of the matrix $\Omega_R$. From equations (25, 26), we see that this corresponds to the transformation,

$$W(x) \rightarrow W(x)X$$  \hspace{1cm} (30)

$L_\mu$ and hence the action in equation (27) are invariant under this transformation. We refer to this symmetry as the $SO(3)_R$ symmetry.

In addition the action is also invariant under the transformation,

$$W(x) \rightarrow YW(x)$$  \hspace{1cm} (31)

where $Y \in SO(2)$ and consists of matrices of the form $\exp i\theta \tau^3$. We refer to this symmetry as the $SO(2)_L$ symmetry. It acts on the spins as follows,

$$S_{I\alpha}^a \rightarrow (\Omega_L)_\beta^a S_{I\beta}^a$$  \hspace{1cm} (32)

where $\Omega_L$ is the $SO(2)$ matrix corresponding to $Y$. This transformation is not a symmetry of the lattice model. It only becomes so in the continuum field theory. The rudiment of this symmetry is observed in the lattice spin wave hamiltonian as a discrete $Z_3$ symmetry. This comes from the way the translation symmetry of the original spin hamiltonian is realised and is discussed in more detail in reference [19]. The full internal symmetry group of the model is therefore $SO(3)_R \times SO(2)_L$. 

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5 The field theory near $\chi = 0$

We now turn to the low energy physics near the $\chi = 0$. To include the large amplitude fluctuations of the H-S modes, we look for a parametrization of the spins as in equation (26), in which both the S-S and the H-S modes are allowed to have large fluctuations. We also want the parameterization to be in terms of quantities defined over the whole unit cell (i.e. independent of the indices $i$ and $\alpha$) just as the S-S modes were represented by $W_I$.

The small fluctuations of the classical ground state configuration due to the H-S modes can be written using equation (25) as,

$$S_{i\alpha} = \tilde{s}n_\alpha + i\sqrt{\tilde{s}}[n_\alpha, w_{i\alpha}]$$  (33)

where,

$$w_{i\alpha} = [P^+ \Psi^{3,1}_i + P^- \Psi^{3,2}_i] e^1_\alpha$$ with $P^+ = (P^-)^* = P_1 - iP_2$.

We find that if we parameterise the $U_{i\alpha}$ matrices in terms of a unit vector $\hat{m}_I$ as follows,

$$U_{i\alpha} = \exp i\frac{(i-1)\phi \tau^3}{2} \exp i\frac{\phi m_I}{2} \exp i\frac{(1-2i)\phi \tau^3}{4}$$  (34)

where, $\phi = 2\pi/3$, $m_I = \hat{m}_I \hat{\tau}$. Then the small fluctuations in equation (33) are exactly reproduced when $\hat{m}_I$ is taken to be a small deviation from the $z$ axis. Namely, $m_I = \tau^3 + \pi^1 \tau^1 + \pi^2 \tau^2$ and taking up to linear terms in $\pi^a$. Where $\pi_1 = \frac{1}{3}P_2$ and $\pi_2 = \frac{1}{3}P_1$. Equation (34) thus gives a parameterisation, in terms of a unit vector field, of the large amplitude fluctuations caused by the H-S modes.

The complete expression for $U_{i\alpha}$ including the effects of the H-S, the H-H and the S-S modes can be written as,

$$U_{i\alpha} = \exp \frac{i\omega_{i\alpha}}{\sqrt{s}} V_I W_I$$  (35)
where the $w_{Ii\alpha}$ is expanded in terms of the H-H modes alone, the $V_I$ is given by the R.H.S of equation (34) and the SU(2) matrix $W_I$ contains the S-S modes. The expression (34) shows that the H-S modes cause a deformation of the spin arrangements within the unit cell. The fluctuations corresponding to $W_I$ cause rigid rotations of the spins within a unit cell as before and are the Goldstone modes.

We now examine the transformation properties of the new fields under the symmetries of the theory. First we consider the $SO(3)_R$ spin rotation symmetry of the hamiltonian. The transformation of the spins under this symmetry is given in equation (29). This transformation of the spins is obtained if $W_I$ and $m_I$ transform as follows,

$$W_I \rightarrow W_I X$$  \hspace{1cm} (36)$$

and

$$m_I \rightarrow m_I$$  \hspace{1cm} (37)$$

$m$ is therefore a spin singlet. Next we consider the $SO(2)_L$ symmetry described in section 5. As mentioned there, this is not a symmetry of the spin system but is however a symmetry of the low energy, longwavelength field theory near $\chi = 1$. We assume that this symmetry persists near $\chi = 0$ also. The transformation of the spins in equation (32) is obtained if we have,

$$W \rightarrow Y W$$  \hspace{1cm} (38)$$

and

$$m_I \rightarrow Y m_I Y^\dagger$$  \hspace{1cm} (39)$$
Equations (36 -39) then specify the transformation properties of the fields under the $SO(3)_R \times SO(2)_L$ symmetry of the low energy theory.

We now motivate the form of the action that will effectively describe the the phases of the DTLAF for small $\chi$. We split up the action as,

$$S = S_W[W] + S_{int}[W, m] + S_m[m] \quad (40)$$

As stated above, we assume that the full symmetry of the model to be $SO(3)_R \times SO(2)_L$ in the continuum limit. Retaining terms quadratic in the derivatives, the most general form of the $S_m$ is,

$$S_m = \int d^3x \frac{1}{g_2} \partial_\mu m^a \partial_\mu m^a + V(m^3) \quad (41)$$

This action is trivially invariant under the $SO(3)_R$ symmetry since $\hat{m}$ is a singlet under this symmetry. We have taken the derivative terms to be fully $SO(3)_L$ symmetric. We could have introduced an XY anisotropy but it does not make any qualitative difference in the one-loop approximation we will be working with. $V(m^3)$ however is symmetric only under $SO(2)_L$. At the classical level, a model defined by $S_m$ has two phases. The disordered $SO(2)_L$ symmetric phase which occurs when $V(m^3)$ is minimised at $m^3 = \pm 1$ and the ordered $SO(2)_L$ broken phase when it is minimised at $m^3 \neq 1$. For definiteness, we take the potential to be,

$$V(m^3) = \frac{\lambda_0}{2}(m^3 - \eta_0)^2 \quad (42)$$

Thus for $\eta_0 > 1$, we have the symmetric phase (classically), there are two modes with equal gaps which are equal to $g_2\lambda_0(\eta_0 - 1)/2$. For $\eta_0 < 1$, the $SO(2)_L$ symmetry is broken. There is one gapless Goldstone mode and the other mode
has a gap equal to $g_2\lambda\sqrt{(1 - \eta_0^2)/2}$. The spin wave analysis in section 3 showed that the two H-S modes had equal gaps which went to zero as $\chi \to 0$. We therefore take the unrenormalised value of $\eta_0$ to be equal to 1.

The general form of $S_W$ that retains terms quadratic in the derivatives and consistent with the symmetries of the theory is given by equation (27). To motivate the form of the interaction term, we note that the deviation of $m^3$ from $\pm 1$ implies that the spin configuration is nonplanar. To see this, we define vectors $\hat{C}_{Ii}$ as,

$$C_{Ii} = -\frac{2i}{3\sqrt{3}} \sum_\alpha [S_{Ii\alpha}, S_{Ii\alpha+1}]$$

where as usual $C_{Ii} = \hat{C}_{Ii}.\hat{\tau}$. $\hat{C}_{Ii}$ is the normal to the plane on which the 3 spins labelled by a particular value of $i$ lie. Using equation (34) we have

$$C_{Ii} = e^{i(i-1)\phi_{\tau^3}^3} e^{i\phi_{mI}^3} \tau^3 e^{-i\phi_{mI}^3} e^{-i(i-1)\phi_{\tau^3}^3}$$

It is clear that when $\hat{m}$ deviates from $\hat{z}$, the vectors $\hat{C}_{Ii}$ are non coplanar. It is known that nonplanarity of the background spin configuration makes the gapless spin waves stiffer [13]. We therfore write down an interaction term of the form,

$$S_{int} = \int d^3x f(m^3)L^a_\mu L^a_\mu$$

Where $f(m^3)$ increases as $|m^3|$ decreases. For simplicity, we take $f(m^3) = -\alpha(m^3)^2$ with $\alpha > 0$.

6 Integrating out the W fields

We now investigate the phases of the field theory that has been proposed in the previous section. In particular we are interested in seeing if there is a phase
in which the $SO(2)_L$ symmetry is broken and the $SO(3)_R$ spin symmetry is unbroken. At values of $\chi$ where the system is effectively described by a field theory of form given in equation (27), it is known that this does not happen \cite{19,20}. However, as mentioned in the previous section, this does occur in the field theory given in equation (40) at the classical level if $\eta_0 < 1$. We have also argued that the unrenormalised value of $\eta_0$ is equal to 1. The potential $V(m^3)$ in equation (42) will get modified by the fluctuations of both the $W$ and the $\hat{m}$ fields. In this section we will integrate out the $W$ fields and compute the above mentioned change. We then investigate the effect of the $\hat{m}$ fluctuations by a renormalization group analysis of $S_m$ in the next section.

If $\Delta V(m^3)$ is the change in the bare potential due to the $W$ fluctuations, then we have,

$$e^{-\int_x \Delta V(m^3)} = \int_W e^{-(S_W[W]+S_{int}[W;m^3])}$$

(46)

$S_W$, as stated earlier, is of the form given in equation (27). It is known \cite{19}, that the two renormalised spin wave velocities tend to become equal. So we make the simplifying assumption of space-time isotropy and work with $S_W$ of the form,

$$S_W = \int d^3x \frac{1}{g_1} \sum_{a=1}^2 L^a_\mu L^a_\mu + \frac{1}{g_3} L^3_\mu L^3_\mu$$

(47)

We first consider the weak coupling regime when $g_1, g_3 << 1$. In this regime the $W$ fields are ordered and the $SO(3)_R$ symmetry is broken. The $W$ integration can be done semiclassically and we get,

$$\Delta V(m^3) = -(g_1 + g_2)\alpha (m^3)^2$$

(48)

Thus in the weak coupling regime, where the $W$ field is ordered, we have $\eta_0 \rightarrow$
\[ \eta_0/(1 - \frac{(g_1 + g_3 \alpha)}{\Lambda}) \]. Therefore, in the ordered phase, the \( W \) field fluctuations increase the value of \( \eta_0 \).

Next we consider the strong coupling regime, \( g_1, g_3 >> 1 \), where the \( W \) fields are disordered. In this regime, the \( SO(3)_R \) symmetry is unbroken. We first rewrite the theory in terms of a set of three orthogonal vectors defined as below

\[ \phi^a_r = \frac{1}{2\gamma_a} tr(\tau_a W^\dagger \tau_r W) \] (49)

Where \( \tau_a \) are the Pauli matrices, the indices \( a, r = 1, 2, 3 \) and \( \frac{1}{\gamma_1} = \frac{1}{\gamma_2} = \frac{1}{g_1} - \alpha (m^3)_2 \), \( \frac{1}{\gamma_3} = \frac{1}{g_3} - \alpha (m^3)_2 \). From the definition, \( \phi^a_r \) satisfy the orthogonality conditions,

\[ \sum_{r=1,3} \phi^a_r \phi^b_r = \frac{1}{\gamma_a} \delta^{ab} \] (50)

The action in equation(46) can be rewritten in terms of these fields as,

\[ S_W + S_{int} = \int d^3x \sum_a \sum_{r=1,3} [\partial_\mu \phi^a_r \partial_\mu \phi^a_r + i\Lambda^{ab}(\phi^a_r \phi^b_r - \frac{1}{\gamma_a} \delta^{ab})] \] (51)

\( \Lambda^{ab} \) are Lagrange multiplier fields that impose the constraint in equation (50).

The action is quadratic in the \( \phi \) fields and they can be integrated out. We are then left with the integral over the \( \Lambda \) fields with an effective action given by,

\[ S_{eff} = \frac{1}{2} Tr ln(-\partial_\mu^2)\delta^{ab} + i\Lambda^{ab}) - i \int d^3x \frac{1}{\gamma_a} \Lambda^{aa} \] (52)

We now do the integration over the \( \Lambda \) fields in the saddle point approximation. This is a well known technique that is exact in the large \( N \) limit where the index \( r \) runs from 1 to \( N \) and the coupling constants are suitably rescaled. The saddle point equations are,

\[ \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2 + i\Lambda} \right)^{ab} = \frac{2}{\gamma_a} \delta^{ab} \] (53)
In the strong coupling regime, the solution is $i\Lambda^{ab} = M_a^2\delta^{ab}$, where $M_a^2$ are non-zero. The $W$ fields are thus disordered with correlation lengths $\xi_a = M_a^{-1}$. In the saddle point approximation then, $\Delta V(m^3)$ is given by,

$$\Delta V(m^3) = \frac{1}{2} Tr \ln(-\partial_{\mu}^2 + M_a^2)\delta^{ab} - \int d^3x \frac{1}{\gamma_a} M_a^2$$

(54)

Here $M_a$ are the solutions of the saddle point equations. Thus both $\gamma_a$ and $M_a$ in equation (54) are functions of $m^3$. To see the form of the dependence of $\Delta V(m^3)$ on $m^3$ in equation (54), we differentiate it with respect to $(m^3)^2$. Using the saddle point equation (53), we obtain,

$$\frac{\partial \Delta V}{\partial (m^3)^2} = \sum_a M_a^2\alpha$$

(55)

Thus $\Delta V$ is a monotonically increasing function of $(m^3)^2$ and is minimised at $m^3 = 0$. Therefore, in the strong coupling regime, the $W$ fluctuations decrease the value of $\eta_0$.

The important conclusion that we draw from the above results is that in the weak coupling regime, where the $W$ fields are ordered and the $SO(3)_R$ symmetry is broken, the $W$ field fluctuations increase $\eta_0$ and therefore tend to restore the $SO(2)_L$ symmetry. On the other hand in the strong coupling regime when the $W$ fields are disordered, and the $SO(3)_R$ symmetry is unbroken, the fluctuations decrease the value of $\eta_0$ and hence tend to break the $SO(2)_L$ symmetry.

7 The $\hat{m}$ field fluctuations

In this section, we investigate the effects of the $\hat{m}$ field fluctuations by a renormalization group analysis of $S_m$. The theory has three coupling constants, $g_2, \lambda$
and $\eta$. The one loop renormalization group equations that govern their flow can be computed using standard techniques. They turn out to be,

$$\frac{\partial g_2}{\partial l} = -g_2 + g_2^2 \quad (56)$$
$$\frac{\partial \lambda}{\partial l} = 3\lambda(1 - g_2) \quad (57)$$
$$\frac{\partial \eta}{\partial l} = 2g_2(1 + \eta) \quad (58)$$

These equations can be explicitly solved to get,

$$g_2 = \frac{g_{20}\exp(-l)}{1 - g_{20}(1 - \exp(-l))} \quad (59)$$
$$\lambda = \lambda_0 (1 - g_{20}(1 - \exp(-l)))^3 \exp(3l) \quad (60)$$
$$1 + \eta = (1 + \eta_0)(1 - g_{20}(1 - \exp(-l)))^{-2} \quad (61)$$

When $g_{20} < 0$, $g_2$ flows to 0 and $\lambda$ flows to $\infty$. Therefore, in this range of $g_{20}$, the $SO(2)_L$ symmetry will be broken if $\eta(\infty) < 1$ and will be intact otherwise. The phase boundary is then given by the equation,

$$(1 + \eta_0) = 2(1 + g_{20})^2 \quad (62)$$

Thus the $\hat{m}$ field fluctuations do not succeed in restoring the $SO(2)_L$ symmetry everywhere. There is a region of the couplings $g_{20}$ and $\eta_0$ shown in figure (3) for which the $SO(2)_L$ symmetry remains broken.

We can use the vectors $\hat{C}_{Ii}$ defined in equation (43) to define an order parameter for this transition in terms of the spins. We define

$$\Psi = 1 - \hat{C}_{Ii}\hat{C}_{Ii+1} \quad (63)$$
\(\Psi_I\) can be expressed in terms of \(W_I\) and \(\hat{m}_I\). It is independent of \(W_I\) (since it is a spin singlet) and is equal to

\[
\Psi_I = \frac{9}{8} \sin^2(\theta)(3\cos^2(\theta) + 1)
\]  

(64)

So \(\Psi\) is 0 in the \(SO(2)_L\) unbroken phase and is \(\neq 0\) in the broken phase.

8 Summary

To summarize, we have studied the DTLAF which interpolates between the TLAF and the KLAF. The classical ground state, in the region, \(0 < \chi \leq 2\), is the \(\sqrt{3}\times \sqrt{3}\) Neel ordered state.

We have computed the spin wave spectrum of the DTLAF in the above mentioned regime. There are 3 gapless Goldstone modes which we have called the S-S modes. 5 have gaps which go to zero as \(\chi \to 0\), the H-S modes. The remaining 4 have a gap throughout the region and we have called them the H-H modes. The S-S and the H-S modes are important for the field theory that would describe the low energy long wavelength physics of the system in the small \(\chi\) region. There are \(3+5=8\) such modes. In the \(\chi \to 0\) limit, the system decouples into the KLAF and a bunch of decoupled individual spins (3 per unit cell) sitting of the triangular lattice sites that do not belong to the Kagome lattice. If we are interested only in the spins of the KLAF, then we have only 5 of these 8 modes are left. We then allowed for large fluctuations of these modes and found that they can be thought of as fluctuations of an order parameter that takes values in \(SO(3) \times S_2\). Namely a \(SO(3)\) matrix \(W\) and a unit vector \(\hat{m}\).

Based on this, we have written down an effective action in terms of the fluc-
tuations of $W$ and $\hat{m}$. We assume that the symmetry of the theory is enhanced to $SO(3)_R \times SO(2)_L$ in the continuum limit as it happens in the $\chi = 1$ end. We also allow for a simple interaction between these fields that is consistent with symmetry requirements and other known facts about the system. We have then integrated out the $W$ fields in the weak and strong coupling regimes, and have analysed the resulting effective theory of the $\hat{m}$ fields by a one loop renormalization group calculation.

We find that in the region where $g_1$ is small and the $W$ field is ordered, the $SO(2)_L$ symmetry remains unbroken and the gap of the $\hat{m}$ field is increased due to quantum fluctuations. In the regime $g_1 > 1$ where the spins are quantum disordered and the $SO(3)_R$ spin symmetry is unbroken, the $W$ field fluctuations drive the $\hat{m}$ system to a phase where the $SO(2)_L$ symmetry between is broken and there exists one gapless Goldstone mode in the spectrum.

This is our proposal for the mechanism that produces a gapless excitation while keeping the symmetries of the hamiltonian intact. While we have shown the existence of this phase in the continuum field theory, we cannot say if the spin system is actually realised in this phase. To answer this question within the framework we are working in, we have to derive the values of the coupling constants in the field theory from the spin system as we have done in the $\chi = 1$ end. This work is in progress.
A The Matrices $M^{-1}$ and $K$

$$M^{-1}_{i\alpha,j\beta} = \frac{1}{2}[A_{i,j} \otimes I_{\alpha,\beta} + 2(B^0 + B^1 + B^2)_{i\alpha,j\beta}]$$

$$K_{i\alpha,j\beta} = \frac{1}{2}[A_{i,j} \otimes I_{\alpha,\beta} - (B^0 + B^1 + B^2)_{i\alpha,j\beta}]$$

where,

$$A_{i,j} = \begin{bmatrix}
\chi + 2 & 0 & 0 & 0 \\
0 & \chi + 2 & 0 & 0 \\
0 & 0 & \chi + 2 & 0 \\
0 & 0 & 0 & 3\chi
\end{bmatrix}$$

the matrices $B_{0,1,2}$ are given by,

$$B_0 = \frac{1}{2} \begin{bmatrix}
0 & \tilde{B} & \tilde{B}^T \\
\tilde{B}^T & 0 & \tilde{B} \\
\tilde{B} & \tilde{B}^T & 0
\end{bmatrix}$$

and

$$B_1 + B_2 = \frac{1}{2} \begin{bmatrix}
0 & \tilde{B}_1 & \tilde{B}_2^\dagger \\
\tilde{B}_1^\dagger & 0 & \tilde{B}_3 \\
\tilde{B}_2 & \tilde{B}_3^\dagger & 0
\end{bmatrix}$$
where,
\[
\tilde{B}_{i,j} = \begin{bmatrix}
1 & 1 & 0 & \chi \\
0 & 1 & 1 & \chi \\
1 & 0 & 1 & \chi \\
\chi & \chi & \chi & 0
\end{bmatrix}
\]

and
\[
\tilde{B}_1(K_1, K_2, K_3) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \exp(-iK_2) - 1 & 0 & 0 \\
0 & 0 & \exp(iK_2) - 1 & \chi(\exp(-iK_3) - 1) \\
0 & \chi(\exp(iK_1) - 1) & 0 & 0
\end{bmatrix}
\]

\[
\tilde{B}_2(K_1, K_2, K_3) = \tilde{B}_1(K_2, K_3, K_1)
\]

\[
\tilde{B}_3(K_1, K_2, K_3) = \tilde{B}_1(K_3, K_1, K_2)
\]

The matrices \(M^{-1}\) and \(K\) do not commute for arbitrary \(\chi\), so in order to diagonalise the hamiltonian we define the normal modes as the left and right eigenvectors of the matrix \(\Omega^2 = KM^{-1}\) as follows,

\[
\Omega^2 \Psi_{nr}^R = \omega_{nr}^2 \Psi_{nr}^R \tag{65}
\]

\[
\Omega^2 \Psi_{nr}^L = \omega_{nr}^2 \Psi_{nr}^L \tag{66}
\]

where the indices \((n,r)\) are similar to the indices \((i,\alpha)\) The \(\Psi_{L,R}\) have the properties,

\[
(\Psi_{nr}^R, \Psi_{n'r'}^R) = \delta_{n,n'}\delta_{r,r'} \tag{67}
\]

\[
\sum_{n,r} \Psi_{nr}^L \Psi_{n'r}^{R*} = 1 \tag{68}
\]
Also,

\begin{align}
M^{-1}\Psi_{nr}^r &= c_{nr}\Psi_{L}^r \\
K\Psi_{L}^r &= c'_{nr}\Psi_{R}^r \\
\omega_{nr}^2 &= c_{nr}c'_{nr}
\end{align}

The \( w_{K_{i\alpha}} \) can be expressed in terms of these normal modes as follows,

\begin{align}
\tilde{P}_{K_{i\alpha}} &= \sum_{nr} 2\sqrt{c'_{nr}\omega_{nr}} \Psi_{Ri\alpha}^r P_{nr} \\
\tilde{Q}_{K_{i\alpha}} &= \sum_{nr} 2\sqrt{c_{nr}\omega_{nr}} \Psi_{Li\alpha}^r Q_{nr}
\end{align}

This is a canonical transformation since

\[ [Q_{nr}, P_{n'r'}] = i\delta_{n,n'}\delta_{r,r'} \]

Written in terms of \( P_{n,r} \) and \( Q_{n,r} \) the Hamiltonian is,

\[ H = \frac{1}{2} \sum_{nr} \omega_{nr}(P_{Kn}P_{Kn} + Q_{Kn}Q_{Kn}) \]

B The Eigenvalues and Eigenvectors of \( \Delta\Omega^2 \)

As we saw in Appendix A, the matrices \( M^{-1} \) and \( K \) occur naturally as direct products of certain \( 4 \times 4 \) and \( 3 \times 3 \) matrices and so the eigen vectors also have this form and this is written as follows.

\( \Psi_{L,Ri,\alpha}^n = \Phi_{L,Ri}^n X_{\alpha}^r \) where,

\[
X^0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad X^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix} \quad X^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha^2 \\ \alpha \end{pmatrix}
\]

where, \( \alpha^3 = 1 \)
The $\Phi^p_{L,R,i}$ are given by,

$$
\Phi^0_{00} = \frac{1}{3-\chi} \begin{pmatrix} \chi \\ \chi \\ 6-5\chi \end{pmatrix} \quad \Phi^0_{L} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
$$

$$
\Phi^{10}_{00} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \Phi^{10}_{L} = \frac{1}{\sqrt{3(3-\chi)}} \begin{pmatrix} 5\chi - 6 \\ 5\chi - 6 \\ 5\chi - 6 \end{pmatrix}
$$

$$
\Phi^{20}_{00} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha^2 \\ 0 \end{pmatrix} \quad \Phi^{20}_{L} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha^2 \\ 0 \end{pmatrix}
$$

$$
\Phi^{30}_{00} = \Phi^{20*}_{R} \quad \Phi^{30}_{L} = \Phi^{20*}_{L}
$$

$$
\Phi^{01}_{00} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Phi^{01}_{L} = \frac{1}{(7\chi+3)} \begin{pmatrix} 5\chi \\ 5\chi \\ (6-\chi) \end{pmatrix}
$$

$$
\Phi^{11}_{00} = \frac{1}{\sqrt{3(3+7\chi)}} \begin{pmatrix} \chi - 6 \\ \chi - 6 \\ 15\chi \end{pmatrix} \quad \Phi^{11}_{L} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
$$

$$
\Phi^{21}_{00} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha^2 \\ \alpha \end{pmatrix} \quad \Phi^{21}_{L} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \alpha^2 \\ \alpha \end{pmatrix}
$$

$$
\Phi^{31}_{00} = \Phi^{21*}_{R} \quad \Phi^{31}_{L} = \Phi^{21*}_{L} \\
\Phi^{i2}_{00} = \Phi^{i1*}_{R} \quad \Phi^{i2}_{L} = \Phi^{i1*}_{L}
$$

and
\[ c_{00} = \frac{9\chi(2-\chi)}{3-\chi}, \quad c'_{00} = 0, \quad \omega^2_{00} = 0 \]

\[ c_{10} = \frac{3-\chi}{4}, \quad c'_{10} = 2\chi, \quad \omega^2_{10} = \frac{\chi(3-\chi)}{2} \]

\[ c_{20} = \frac{3+\chi}{2}, \quad c'_{20} = \frac{3+2\chi}{4}, \quad \omega^2_{20} = \frac{(3+\chi)(2\chi+3)}{8} \]

\[ c_{30} = c_{20}, \quad c'_{30} = c'_{20}, \quad \omega^2_{30} = \omega^2_{20} \]

\[ c_{01} = 0, \quad c'_{01} = \frac{9\chi(4+\chi)}{2(\chi+3)}, \quad \omega^2_{01} = 0 \]

\[ c_{11} = 2\chi, \quad c'_{11} = \frac{(3+7\chi)}{8}, \quad \omega^2_{11} = \frac{\chi(3+7\chi)}{4} \]

\[ c_{21} = \frac{\chi+3}{2}, \quad c'_{21} = \frac{\chi+3}{2}, \quad \omega^2_{21} = \frac{(\chi+3)(2\chi+3)}{8} \]

\[ c_{31} = \frac{\chi}{2}, \quad c'_{31} = \frac{(\chi+3)}{2}, \quad \omega^2_{31} = \frac{\chi(\chi+3)}{4} \]

\[ c_{i2} = c_{i1}, \quad c'_{i2} = c'_{i1}, \quad \omega^2_{i2} = \omega^2_{i1}, \quad i = 0, \ldots, 3 \]
References

[1] P. Fazekas and P.W. Anderson, Philos. Mag. 30, 423 (1974).

[2] P. Chandra and P. Coleman, Phys.Rev.Lett. 66, 100, (1991).

[3] D.A. Huse and V. Elser, Phys.Rev.Lett, 60, 2531, (1988).

[4] Rajiv.R.P. Singh and D.A. Huse, Phys.Rev.Lett, 68, 1766 (1992).

[5] C. Zeng and V. Elser, Phys.Rev. B 42, 8436, (1990).

[6] J.T. Chalker and J.F.G. Eastmond, Phys.Rev. B 46, 14201 (1992).

[7] P.W. Leung and V. Elser, Phys.Rev. B 47, 5459 (1993)

[8] X. Obradors, A. Labarta, A. Isalgue, J. Tejada, J. Rodriguez and M. Perret, Solid.State.communications, 65, 189, (1988).

[9] A.P. Ramirez, G.P. Espinosa and A.S. Cooper, Phys.Rev.Lett., 64, 2070, (1990); A.P. Ramirez, G.P. Espinosa and A.S. Cooper, Phys.Rev. B 45, 2505 (1992).

[10] C. Broholm, G. Aeppli, G.P. Espinosa and A.S. Cooper Phys.Rev.Lett. 65, 3173, (1990).

[11] Y.J. Uemura, A. Keren, K. Kojima, L.P. Le, G. M. Luke, W.D. Wu, Y. Ajiro, T. Asaro, Y. Kuriyama, M. Mekata, H. Kikuchi and K. Kakurai. Phys. Rev. Lett. 73, 3306 (1994).
[12] A.S. Wills, A. Harrison, S.A.M. Mentink, T.E. Mason and Z. Tun, "Magnetic Correlations in deuteronium jarosite, a model $S = \frac{5}{2}$ Kagome antiferromagnet", cond-mat 9607106.

[13] J.T. Chalker, P.C.W. Holdsworth and E.F. Shender, Phys.Rev.Lett. 68, 855 (1992); J.N. Reimers and A.J. Berlinsky Phys.Rev. B 48, 9539 (1993).

[14] A. Chubukov, Phys.Rev.Lett. 69, 832, (1992).

[15] S. Sachdev Phys. Rev. B 45, 12377, (1992).

[16] S. Chakravarty, B.I. Halperin and D.R. Nelson, Phys.Rev.Lett. 60, 1057, (1988); Phys.Rev.B 39, 2344, (1989).

[17] T. Dombre and N. Read, Phys.Rev.B 39, 6797, (1989).

[18] D.H. Friedan, Annals of Physics, 163, 318, (1985).

[19] P. Azaria, B. Delamotte and D. Mouhanna, Phys.Rev.Lett. 68, 1762, (1992).

[20] P. Azaria, B. Delamotte, F. Deldue and T. Jolicouer, Nuclear Physics B 485, (1993).

[21] Fradkin, Eduardo. Field Theories of Condensed Matter Systems. -Redwood City: Addison-Wesley, 1991.
figure captions

fig.1: A section of the triangular lattice showing the unit cell consisting of 12 points and the labelling of the points within a unit cell. The non-Kagome points labelled $(3, \alpha)$ are marked with dark spots and the bonds connected to them are the non-kagome bonds.

fig.2: Reduction in the staggered magnetization, $\Delta M$, as a function of $\chi$ and the contributions to this from the S-S modes ($\Delta SM$) and from the H-H and H-S modes ($\Delta HM$). Near $\chi = 0$ $\Delta HM$ is seen to dominate over $\Delta SM$.

fig.3: Phase diagram showing the boundary dividing the $SO(2)_L$ broken phase from the unbroken phase.
