Application a cubic spline to calculate derivatives in the presence of a boundary layer

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Abstract. The problem of calculating the derivatives of a function with large gradients given at the grid nodes is considered. The decomposition of the function into regular and singular components is assumed. Such decomposition takes place in the presence of an exponential boundary layer. The case of the Bakhvalov mesh is considered. The error in calculating the derivatives based on the differentiation of the interpolating cubic spline is estimated. Error estimates are obtained taking into account uniformity in a small parameter.

1. Introduction

Cubic splines are widely used for smooth interpolation of functions [1]. It is of interest to use splines to approximate a function and its derivatives in the presence of a boundary layer. In [2] it is proved that in the presence of a boundary layer, the error of the cubic spline can grow unlimitedly with decreasing value of a small parameter. In [2] it was proved that in the case of the Shishkin mesh [3] it is possible to achieve that the interpolation error becomes uniform in a small parameter. In [4], [5] it was shown that, in the presence of an exponential boundary layer, the derivatives of the cubic spline approximate the derivatives of the function given at the nodes of Shishkin mesh with an error uniform in a small parameter. In [6] proposed to calculate derivatives of function with large gradients using differentiation of the spline, which is exact for the singular component of the function specified in the nodes of uniform grid.

In this paper, we study the application of a cubic spline on a Bakhvalov mesh [7] for approximating the derivatives of a function in the presence of an exponential boundary layer. Now we introduce the notations. Let \( \Omega \) be the grid of the interval \([0,1]\). If necessary, we consider the partition \( \Omega \) extended to the left of the point 0 with the step \( h_1 = x_1 - x_0 \) and to the right of the point 1 with the step \( h_N = x_N - x_{N-1} \). We set \( h = 1/N \). By \( C \) and \( C_j \) we mean positive constants that are independent of the parameter \( \varepsilon \) and \( N \). In this case, the same symbol \( C_j \) can denote different constants. We write \( f = O(g) \) if \( |f| \leq C|g| \); \( f = O^*(g) \) is true if \( f = O(g) \) and \( g = O(f) \). \( C[a,b] \), \( L_2[a,b] \) are spaces of continuous and
quadratically summable functions on \([a, b]\) with norms \(\| \cdot \|_{C[a,b]}\) and \(\| \cdot \|_{L_2[a,b]}\) accordingly, \((\cdot, \cdot)\) is the scalar product in \(L_2[0,1]\).

2. Preliminary information
Suppose that the following decomposition is true for the function \(u(x)\):

\[
    u(x) = q(x) + \Phi(x), \quad x \in [0,1],
\]

where

\[
    |q^{(j)}(x)| \leq C_1, \quad |\Phi^{(j)}(x)| \leq \frac{C_1}{\varepsilon} e^{-\alpha x/\varepsilon}, \quad 0 \leq j \leq 4,
\]

and the functions \(q(x)\) and \(\Phi(x)\) do not explicitly given, \(\alpha > 0, \varepsilon \in (0,1]\). Decomposition (1) is valid for the solution of a singularly perturbed boundary value problem \([3], [8]\).

We study the problem of cubic spline interpolation of a function (1).

Now we set the grid \(\Omega\) based on the work of N.S. Bakhvalov \([7]\). Let \(\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon}\}\).

In the case \(\sigma < 1/2\), the nodes of the grid \(\Omega\) are set in a form

\[
    x_n = \begin{cases} 
        -\frac{\varepsilon}{\alpha} \ln \left[1 - 2(1 - \varepsilon)n/N\right], & 0 \leq n \leq N/2, \\
        \sigma + (2n/N - 1)(1 - \sigma), & N/2 \leq n \leq N. 
    \end{cases}
\]

In the cases \(\sigma = 1/2\) and \(\varepsilon \geq e^{-1}\) we set the grid \(\Omega\) uniform with steps \(h_n = 1/N\). From (4) it follows

\[
    h_n = \frac{4\varepsilon}{\alpha} \ln \left[1 + \frac{2(1 - \varepsilon)N}{1 - 2(1 - \varepsilon)n/N}\right], \quad n = 1, 2, \ldots, N/2.
\]

For \(n > N/2\) \(h_n = 2(1 - \sigma)/N\).

Set cubic spline \(S_3(x,u) \in S(\Omega, 3, 1)\) from conditions

\[
    S_3(x_n, u) = u(x_n), \quad 0 \leq n \leq N, \quad S'_3(0, u) = u'(0), \quad S'_3(1, u) = u'(1).
\]

**Theorem 1** There are such constants \(C_1\) and \(\beta > 0\), independent of \(\varepsilon, N\), such that the estimates hold

\[
    \| S_3(x,u) - u(x) \|_{C[x_{n-1},x_n]} \leq C_1 \begin{cases} 
        N^{-4}, & 0 < n < N/2, \\
        N^{-4} \ln \left[1 + \frac{1}{N}\right] + N^{-4}, & n = N/2, \\
        \frac{1}{N^4} e^{-\beta(n-1-N/2)} + \frac{1}{N^2}, & N/2 < n \leq N
    \end{cases}.
\]

**Proof.** The proof is similar to the theorem 3 in \([2]\), where the error of the cubic spline is estimated in the case of the Shishkin mesh. Here we give only the main idea of the proof.

It is sufficient to consider the case \(u(x) = \Phi(x)\), since the function \(q(x)\) has bounded derivatives. Let \(e(x) = S_3(x, \Phi) - \Phi(x)\). As it was shown in \([2]\), it is enough to estimate the function \(e''(x) = S''_3(x, \Phi) - \Phi''(x)\). According to \([9]\) \(S''_3(x, \Phi) = P\Phi''(x)\), where \(P\) is the projector on \(L_2[0,1]\) orthogonal to \(S(\Omega, 1, 1)\). Denote by \(gI(x) \in S(\Omega, 1, 1)\) the linear interpolant of \(\Phi''(x)\) at the nodes of the mesh. Let \(gI(x) \in S(\Omega, 1, 1)\), \(gI(x) = gI(x)\) for \(x \in [0, x_{N/2-2}]\) and \(gI(x) = 0\) for \(x \in [x_{N/2-1}, 1]\). Then we have

\[
    e''(x) = P(\Phi''(x) - gI(x)) + (gI(x) - \Phi''(x)).
\]
Since the estimate \((gI(x) - \Phi''(x))\) is trivially obtained from (5) and the error estimate of the linear interpolation formula, then it remains to estimate \(P(\Phi''(x) - gI(x))\) in (7). We estimate \(P(\Phi''(x) - gI(x))\) by analogy with [2]. Thus, we obtain the estimate of \(|e''(x)|, x \in [x_{n-1}, x_n]\) depending on \(n\). Then we obtain the estimate (6) using the formula

\[
e(x) = \int_{x_{n-1}}^{x_n} G(x, s)e''(s)ds,
\]

where

\[
G(x, s) = \frac{1}{h_n} \begin{cases} 
(x - x_{n-1})(x_n - s), & x_{n-1} \leq x \leq s, \\
(s - x_{n-1})(x_n - x), & s < x \leq x_n
\end{cases}
\]
is Green function. This briefly proves the theorem.

3. Estimation of the error in the calculation of derivatives

**Lemma 1** For \(\varepsilon < \frac{1}{2}\) the sequence of steps \(h_n\) for \(n \leq N/2\) monotonically increases and estimates hold

\[
h_n = \begin{cases} 
O^*(\frac{\varepsilon}{N/2-n}), & 1 \leq n \leq N/2 - 1, \\
O^*(\varepsilon \ln(1 + \frac{N}{N/2})), & n = N/2, \\
O^*(1/N), & N/2 + 1 \leq n \leq N.
\end{cases}
\]  

(8)

The proof follows from the specification of grid nodes in accordance with (4). We denote \(K = 2(1 - \varepsilon)\).

**Theorem 2** For some constant \(C\) the following estimates hold for \(x \in [x_{n-1}, x_n]\)

\[
\varepsilon|S_3'(x, u) - u'(x)| \leq \frac{C}{N^3}, \quad n < \frac{N}{2},
\]  

(9)

\[
\varepsilon|S_3'(x, u) - u'(x)| \leq \frac{C}{N^4} \ln^3 \left(1 + \frac{K}{N\varepsilon}\right) + \frac{C\varepsilon}{N^3}, \quad n = \frac{N}{2},
\]  

(10)

\[
\varepsilon|S_3'(x, u) - u'(x)| \leq \frac{C}{N^4}e^{-\beta(n-N/2)} + \frac{C\varepsilon}{N^3}, \quad n > \frac{N}{2}, \quad \beta > 0,
\]  

(11)

\[
\varepsilon^2|S_3''(x, u) - u''(x)| \leq \frac{C}{N^2}, \quad n < \frac{N}{2},
\]  

(12)

\[
\varepsilon^2|S_3''(x, u) - u''(x)| \leq \frac{C}{N^4} \ln^2 \left(1 + \frac{K}{N\varepsilon}\right) + \frac{C\varepsilon^2}{N^2}, \quad n = \frac{N}{2},
\]  

(13)

\[
\varepsilon^2|S_3''(x, u) - u''(x)| \leq \frac{C}{N^4}e^{-\beta(n-N/2)} + \frac{C\varepsilon^2}{N^2}, \quad n > \frac{N}{2}, \quad \beta > 0,
\]  

(14)

where \(\beta\) does not depend on \(\varepsilon\).

**Proof.** Consider two cases: \(\varepsilon \leq 1/N\) and \(\varepsilon \geq 1/N\).

The case \(\varepsilon \leq 1/N\). We define a third-degree polynomial \(P_n(x)\) based on the expansion of the function \(\Phi(x)\) in a Taylor series near the node \(x_{n+3}\):

\[
\Phi(x) = P_n(x) + \frac{1}{3!} \int_{x_{n+3}}^{x} (x - s)^3\Phi^{(4)}(s)ds.
\]  

(15)

We make estimates on each grid interval \([x_{n-1}, x_n]\) in depending on \(n\).
The case $n < N/2$. According to (2), (8), (15)

$$|\Phi(x) - P_n(x)| \leq C \frac{h^n}{\varepsilon^n} e^{-x_{n-1}/\varepsilon} \leq \frac{C}{(\frac{N}{2} - n)^4} e^{-x_{n-1}/\varepsilon}.$$  (16)

Given (4), from (16) we get

$$|\Phi(x) - P_n(x)| \leq \frac{C(N/2 - n + n\varepsilon + 2(1 - \varepsilon))^4}{4^4(N/2 - n)^4} \leq \frac{C_1}{N^4}, \quad n < \frac{N}{2}. \quad (17)$$

Now we evaluate $\varepsilon|\Phi'(x) - P_n'(x)|$ on the interval $[x_{n-1}, x_n]$ for $n < N/2$. Differentiating (15), we obtain

$$\Phi'(x) = P_n'(x) + \frac{1}{2} \int_{x_{n+1}}^x (x - s)^2 \Phi^{(4)}(s)ds. \quad (18)$$

Further, by analogy with the derivation of the estimate (17), we obtain

$$\varepsilon|\Phi'(x) - P_n'(x)| \leq \frac{C}{N^3}, \quad x \in [x_{n-1}, x_n], \quad n < \frac{N}{2}. \quad (19)$$

Similarly, one can get the estimate

$$\varepsilon^2|\Phi''(x) - P_n''(x)| \leq \frac{C}{N^2}, \quad x \in [x_{n-1}, x_n], \quad n < \frac{N}{2}. \quad (20)$$

Now we estimate $|S_n'(x, \Phi) - P_n'(x)|$ on the interval $[x_{n-1}, x_n]$, $n < N/2$.

Let $Q(x)$ be the polynomial of the third degree in the interval $[x_{n-1}, x_n]$. Then according to [2]

$$h_n \|Q'(x)\|_{C[x_{n-1}, x_n]} \leq C \|Q(x)\|_{C[x_{n-1}, x_n]}, \quad (20)$$

where $C$ is independent of $Q(x)$.

Note that $S_n'(x, \Phi) - P_n'(x)$ is a polynomial of the third degree, therefore, in accordance with (20)

$$h_n \|S_n'(x, \Phi) - P_n'(x)\|_{C[x_{n-1}, x_n]} \leq C \|S_n(x, \Phi) - P_n(x)\|_{C[x_{n-1}, x_n]}. \quad (21)$$

Given the estimates (17) and (6) for $n < N/2$, we get

$$\|S_n(x, \Phi) - P_n(x)\|_{C[x_{n-1}, x_n]} \leq \frac{C}{N^4}. \quad (22)$$

Given this inequality in (21), we get the estimate

$$\varepsilon \|S_n'(x, \Phi) - P_n'(x)\|_{C[x_{n-1}, x_n]} \leq \frac{C}{N^3}, \quad n < \frac{N}{2}. \quad (22)$$

Now from the estimates (19), (22) we get

$$\varepsilon \|S_n'(x, \Phi) - \Phi'(x)\|_{C[x_{n-1}, x_n]} \leq \frac{C}{N^3}, \quad n < \frac{N}{2}. \quad (23)$$

Considering that for a regular component $q(x)$ with bounded derivatives, an estimate of the form (23) is valid [1], from (23) we obtain the required estimate (9).

It is similarly proved that

$$\varepsilon^2 \|S_n''(x, \Phi) - \Phi''(x)\|_{C[x_{n-1}, x_n]} \leq \frac{C}{N^2}, \quad n < \frac{N}{2}. \quad (24)$$
The estimate (24) is also valid for the regular component $q(x)$. This proves the estimate (12).

**The case** $n = N/2$. From (4) it follows

$$h_{N/2} = \frac{4\varepsilon}{\alpha} \ln \left[ 1 + \frac{K}{N\varepsilon} \right].$$

(25)

Given that $\varepsilon \leq 1/N$ and (25), from (15) we get

$$|\Phi(x) - P_n(x)| \leq \frac{C h_{N/2}^4}{\varepsilon^4} e^{-\alpha x_{N/2} - 1/\varepsilon} = C_1 \ln^4 \left( 1 + \frac{K}{N\varepsilon} \right) \left( \varepsilon + \frac{K}{N} \right)^4$$

\leq \frac{C_2}{N^4} \ln^4 \left( 1 + \frac{K}{N\varepsilon} \right).$$

(26)

Taking into account the estimate (6) for $n = N/2$ and (26), we have

$$|S_3(x, \Phi) - P_n(x)| \leq \frac{C}{N^4} \ln^4 \left( 1 + \frac{K}{N\varepsilon} \right).$$

(27)

Taking into account the inequality (21) and lemma 1 for $n = N/2$, we obtain

$$\varepsilon |S_3(x, \Phi) - P_n(x)| \leq \frac{C}{N^4} \ln^3 \left( 1 + \frac{K}{N\varepsilon} \right), \quad x \in [x_{N/2-1}, x_{N/2}].$$

(28)

Now we estimate $|\Phi'(x) - P_n'(x)|$. From (18) we have

$$|\Phi'(x) - P_n'(x)| \leq \frac{C}{\varepsilon} \ln^2 \left( 1 + \frac{K}{N\varepsilon} \right) \left( e^{-\alpha x_{n-3}/\varepsilon} - e^{-\alpha x_n/\varepsilon} \right).$$

Given that for $n = N/2$

$$e^{-\alpha x_n/\varepsilon} = \varepsilon^4, \quad e^{-\alpha x_{n-3}/\varepsilon} = \left( \varepsilon + \frac{C}{N} \right)^4,$$

under the condition $\varepsilon \leq 1/N$ we get

$$\varepsilon |\Phi'(x) - P_n'(x)| \leq \frac{C_2}{N^4} \ln^2 \left( 1 + \frac{K}{N\varepsilon} \right).$$

(29)

Given (28), (29) and the well-known estimate of errors on the regular component $q(x)$, we obtain the estimate (10).

Now we pass to the proof of the estimate (13).

We evaluate $\varepsilon^2 |\Phi''(x) - P_n''(x)|$ based on (20), (28). Considering (20), we have

$$h_n \| S_3''(x, \Phi) - P_n''(x) \|_{C[\lfloor x_{n-1} \rfloor, x_n]} \leq C \| S_3'(x, \Phi) - P_n'(x) \|_{C[\lfloor x_{n-1} \rfloor, x_n]}.$$  

(30)

Given the estimate (8) for the step $h_{N/2}$ and the resulting estimate (28), from (30) we have

$$\varepsilon^2 |S_3''(x, \Phi) - P_n''(x)| \leq \frac{C}{N^4} \ln^2 \left( 1 + \frac{K}{N\varepsilon} \right).$$

(31)

Similar to (29)

$$\varepsilon^2 |\Phi''(x) - P_n''(x)| \leq \frac{C}{N^4} \ln \left( 1 + \frac{K}{N\varepsilon} \right).$$

(32)

Given the estimates (31), (32) and the error on regular component $q(x)$, we obtain the estimate (13).
The case $n > N/2$. For $x \in [x_{n-1}, x_n]$, the derivatives of $u(x)$ are uniformly bounded due to the parameter $\sigma$ given according to (3). Moreover, the estimate (11) is proved by analogy with the case $n \leq N/2$. The estimate (14) is proved in the same way.

The case $\varepsilon \geq 1/N$.

By analogy with the case $\varepsilon \leq 1/N$ we have

$$|\Phi(x) - P_n(x)| \leq \frac{C}{N^4} (N - Kn + K)^4 \ln^4 \left[1 + \frac{K}{N - Kn}\right]. \quad (33)$$

Given the condition $\varepsilon \geq 1/N$, from (33) we get that for some constant $C$

$$|\Phi(x) - P_n(x)| \leq \frac{C}{N^4}, \quad x \in [x_{n-1}, x_n], \ n \leq N/2. \quad (34)$$

From (18) we obtain

$$\varepsilon|\Phi'(x) - P'_n(x)| \leq \frac{C}{N^3} (N - Kn + K)^3 \ln^3 \left[1 + \frac{K}{N - Kn}\right].$$

From this inequality follows

$$\varepsilon|\Phi'(x) - P'_n(x)| \leq \frac{C}{N^3}, \ x \in [x_{n-1}, x_n], \ n \leq N/2. \quad (35)$$

Given (6) for $n \leq N/2$ and (34), we obtain

$$|S_3(x, \Phi) - P_n(x)| \leq C/N^4, \ x \in [x_{n-1}, x_n], \ n \leq N/2. \quad (36)$$

Given the inequalities (20), (36), we have

$$h_n|S'_3(x, \Phi) - P'_n(x)| \leq C/N^4, \ x \in [x_{n-1}, x_n], \ n \leq N/2. \quad (37)$$

From (5) it follows that $h_n \geq C \varepsilon/N$. Now from (37) we obtain

$$\varepsilon|S'_3(x, \Phi) - P'_n(x)| \leq C/N^3, \ x \in [x_{n-1}, x_n], \ n \leq N/2. \quad (38)$$

Considering (35), (38), we obtain

$$\varepsilon|S'_3(x, \Phi) - \Phi'(x)| \leq C/N^3, \ x \in [x_{n-1}, x_n], \ n \leq N/2.$$

The following estimate is proved similarly

$$\varepsilon^2|S''_3(x, \Phi) - \Phi''(x)| \leq C/N^2, \ x \in [x_{n-1}, x_n], \ n \leq N/2.$$

Taking into account the contribution to the error of the regular component $q(x)$, we obtain the statement of the theorem in the case $\varepsilon \geq 1/N$. The theorem is proved.

4. Numerical results

We define a function of the form (1)

$$u(x) = \cos \frac{\pi x}{2} + e^{-x/\varepsilon}, \ x \in [0, 1], \ \Phi(x) = e^{-x/\varepsilon}, \ \varepsilon \in (0, 1].$$

In tables, $e - m$ stands for $10^{-m}$.
Tables 1–3 show the relative error

$$\Delta_{N,\varepsilon} = \varepsilon \max_{n,j} \left| S'(\tilde{x}_{n,j}, u) - u'(\tilde{x}_{n,j}) \right|,$$

when calculating the first derivative of the function $u(x)$ based on the spline $S_3(x, u)$ in cases of a uniform grid, Shishkin and Bakhvalov meshes. Here $\tilde{x}_{n,j}$ are nodes of the condensed mesh obtained from the mesh $\Omega$ by division of each grid interval $[x_{n-1}, x_n]$ into 10 equal parts.

The numerical order of accuracy $M_{N,\varepsilon}$ was calculated by the formula

$$M_{N,\varepsilon} = \log_2 \frac{\Delta_{N,\varepsilon}}{\Delta_{2N,\varepsilon}}.$$  

The results of calculations on the Bakhvalov mesh for all values of $\varepsilon$ and $N$ correspond to the third order of accuracy.

The calculated order of accuracy $M_{N,\varepsilon}$ on the Shishkin mesh, according to Table 2, corresponds to the estimate

$$\varepsilon |S'_3(x, u) - u'(x)| \leq C \frac{\ln^3 N}{N^3}.$$  

This estimate was obtained in [4].

The results in Table 1 show that it is unacceptable to use a cubic spline on a uniform mesh to calculate the derivative.

Table 4 shows the relative error

$$\Delta_{N,\varepsilon} = \varepsilon^2 \max_{n,j} \left| S''(\tilde{x}_{n,j}, u) - u''(\tilde{x}_{n,j}) \right|$$

on the Bakhvalov mesh. Results of calculations for all values of $\varepsilon$ and $N$ correspond to the relation $M_{N,\varepsilon} \approx 2$, which is not lower than in the error estimates obtained in Theorem 2.

Table 1. The error in calculating the first derivative based on the cubic spline $S_3(x, u)$ on a uniform grid

| $\varepsilon$ | 16  | 32  | 64  | 128 | 256 | 512  |
|--------------|-----|-----|-----|-----|-----|------|
| $N$          |     |     |     |     |     |      |
| 1            | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| 10^{-1}      | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| 10^{-2}      | 8.85e-1 | 2.59e-1 | 5.36e-2 | 8.59e-3 | 1.20e-3 | 1.58e-4 |
| 10^{-3}      | 1.22e+1 | 6.09   | 2.92  | 1.23  | 4.00e-1 | 9.21e-2 |
| 10^{-4}      | 1.22e+2 | 6.09+1 | 3.05e+1 | 1.53e+1 | 7.63  | 3.73  |

Table 2. The error in calculating the first derivative based on the cubic spline $S_3(x, u)$ on the Shishkin mesh

| $\varepsilon$ | 16  | 32  | 64  | 128 | 256 | 512  |
|--------------|-----|-----|-----|-----|-----|------|
| $N$          |     |     |     |     |     |      |
| 1            | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| 10^{-1}      | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| 10^{-2}      | 1.35e-1 | 2.45e-2 | 3.65e-3 | 4.94e-4 | 6.41e-5 | 8.15e-6 |
| 10^{-3}      | 3.16e-1 | 6.86e-2 | 1.13e-2 | 1.60e-3 | 2.12e-4 | 2.73e-5 |
| 10^{-4}      | 5.37e-1 | 1.35e-1 | 2.45e-2 | 3.65e-3 | 4.94e-4 | 6.41e-5 |
| 10^{-5}      | 7.78e-1 | 2.19e-1 | 4.36e-2 | 6.83e-3 | 9.46e-4 | 1.24e-4 |
Table 3. The error in calculating the first derivative based on the cubic spline $S_3(x, u)$ on the Bakhvalov mesh

| $\varepsilon$ | $N$  |
|---------------|------|
|               | 16   | 32   | 64   | 128  | 256  | 512  |
| 1             | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| 10^{-1}       | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| 10^{-2}       | 2.78e-3 | 3.42e-4 | 4.25e-5 | 5.29e-6 | 6.60e-7 | 8.24e-8 |
| 10^{-3}       | 2.86e-3 | 3.52e-4 | 4.36e-5 | 5.43e-6 | 6.78e-7 | 8.47e-8 |
| 10^{-4}       | 2.87e-3 | 3.53e-4 | 4.37e-5 | 5.45e-6 | 6.80e-7 | 8.49e-8 |

Table 4. The error in calculating the second derivative based on the cubic spline $S_3(x, u)$ on the Bakhvalov mesh

| $\varepsilon$ | $N$  |
|---------------|------|
|               | 16   | 32   | 64   | 128  | 256  | 512  |
| 1             | 2.15e-3 | 5.37e-4 | 1.34e-4 | 3.36e-5 | 8.41e-6 | 2.10e-6 |
| 10^{-1}       | 2.50e-2 | 6.91e-3 | 1.81e-3 | 4.64e-4 | 1.17e-4 | 2.95e-5 |
| 10^{-2}       | 1.81e-2 | 4.64e-3 | 1.17e-3 | 2.96e-4 | 7.42e-5 | 1.86e-5 |
| 10^{-3}       | 1.84e-2 | 4.72e-3 | 1.20e-3 | 3.01e-4 | 7.55e-5 | 1.89e-5 |
| 10^{-4}       | 1.84e-2 | 4.73e-3 | 1.20e-3 | 3.02e-4 | 7.56e-5 | 1.89e-5 |

5. Conclusion

The problem of approximating a function and its derivatives in the presence of an exponential boundary layer based on the use of a cubic spline is considered. It is assumed that the function is given by its values in the nodes of the grid. In the case of the Bakhvalov grid, error estimates are obtained taking into account uniformity in a small parameter. The results of numerical experiments are consistent with the obtained error estimates.

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