HYPERBOLIC RELATIVELY HYPERBOLIC GRAPHS AND DISC GRAPHS

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ABSTRACT. We show that a relatively hyperbolic graph with uniformly hyperbolic peripheral subgraphs is hyperbolic. As an application, we show that the disc graph and the electrified disc graph of a handlebody $H$ of genus $g \geq 2$ are hyperbolic, and we determine their Gromov boundaries.

1. Introduction

Consider a connected metric graph $G$ in which a family $H = \{H_c | c \in C\}$ of complete connected subgraphs has been specified. Here $C$ is a countable, finite or empty index set. The graph $G$ is hyperbolic relative to the family $H$ if the following properties are satisfied.

Define the $H$-electrification $E_G$ of $G$ to be the graph which is obtained from $G$ by adding for every $c \in C$ a new vertex $v_c$ which is connected to each vertex $x \in H_c$ by an edge and which is not connected to any other vertex. We require that the graph $E_G$ is hyperbolic in the sense of Gromov and that moreover a property called bounded penetration holds true (see [F98] for perhaps the first formulation of this property). We refer to [ST12] for a consolidation of the various notions of relative hyperbolicity found in the literature.

If $G$ is a hyperbolic metric graph and if $H$ is a family of disjoint connected uniformly quasi-convex subgraphs of $G$ then $G$ is hyperbolic relative to $H$. This fact is probably folklore; implicitly it was worked out in a slightly modified form in [KR12].

Vice versa, Farb showed in [F98] that if $G$ is the Cayley graph of a finitely generated group and if the graphs $H_c$ are $\delta$-hyperbolic for a number $\delta > 0$ not depending on $c \in C$ then $G$ is hyperbolic. In [BF06] it is noted that using a result of Bowditch [Bw91], the argument in [F98] can be extended to arbitrary (possibly locally infinite) relatively hyperbolic metric graphs.

Our first goal is to give a different and self-contained proof of this result which gives effective estimates for the hyperbolicity constant as well as explicit control on uniform quasi-geodesics. We show

Theorem 1. Let $G$ be a metric graph which is hyperbolic relative to a family $H = \{H_c | c \in C\}$ of complete connected subgraphs. If there is a number $\delta > 0$ such that each of the graphs $H_c$ is $\delta$-hyperbolic then $G$ is hyperbolic. Moreover, the subgraphs $H_c$ $(c \in C)$ are uniformly quasi-convex.

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The control we obtain allows to use the result inductively. Moreover, the Gromov boundary of $G$ can easily be determined from the Gromov boundaries of $EDG$ and the Gromov boundaries of the quasi-convex subgraphs $H_c$.

We next discuss applications of Theorem 1.

Let $S$ be a closed surface of genus $g \geq 2$. For a number $k < g$ define the graph of non-separating $k$-multicurves to be the following metric graph $NC(k)$. Vertices are $k$-tuples of simple closed curves on $S$ which cut $S$ into a single connected component. Two such multicurves $c_1, c_2$ are connected by an edge if $c_1 \cup c_2$ is a non-separating multicurve with $k + 1$ components. In [H13a] we used Theorem 1 to show

**Theorem 2.** For $k < g/2 + 1$ the graph $NC(k)$ is hyperbolic.

We also observed that the bound $k < g/2 + 1$ is sharp. The same argument applies to the graph of non-separating multi-curves on a surface with punctures.

In this article we use Theorem 1 to investigate the geometry of graphs of discs in a handlebody. A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold $H$ which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of $g$ circles. Its boundary $\partial H$ is an oriented surface of genus $g$.

An **essential disc** in $H$ is a properly embedded disc $\left( (D, \partial D) \subset (H, \partial H) \right)$ whose boundary $\partial D$ is an essential simple closed curve in $\partial H$.

A subsurface $X$ of the compact surface $\partial H$ is called **essential** if it is a complementary component of an embedded multicurve in $\partial H$. Note that the complement of a non-separating simple closed curve in $\partial H$ is essential in this sense, i.e. the inclusion $X \rightarrow \partial H$ need not induce in injection on fundamental groups.

Define a connected essential subsurface $X$ of the boundary $\partial H$ of $H$ to be **thick** if the following properties hold true.

1. Every disc intersects $X$.
2. $X$ is filled by boundaries of discs.

The boundary surface $\partial H$ of $H$ is thick. An example of a proper thick subsurface of $\partial H$ is the complement in $\partial H$ of a suitably chosen simple closed curve which is not discbounding.

**Definition 1.** Let $X \subset \partial H$ be a thick subsurface. The **electrified disc graph** of $X$ is the graph $EDG(X)$ whose vertices are isotopy classes of essential discs in $H$ with boundary in $X$. Two vertices $D_1, D_2$ are connected by an edge of length one if there is an essential simple closed curve in $X$ which can be realized disjointly from both $\partial D_1, \partial D_2$.

If $X = \partial H$ then we call $EDG(X)$ the **electrified disc graph** of $H$. Using Theorem 1 we show

**Theorem 3.** The electrified disc graph $EDG(X)$ of a thick subsurface $X \subset \partial H$ of the boundary $\partial H$ of a handlebody $H$ of genus $g \geq 2$ is hyperbolic.

For the investigation of the **handlebody group**, i.e. the group of isotopy classes of homeomorphisms of $H$, a more natural graph to consider is the so-called disc graph which is defined as follows.

**Definition 2.** The disc graph $DG$ of $H$ is the graph whose vertices are isotopy classes of essential discs in $H$. Two such discs are connected by an edge of length one if and only if they can be realized disjointly.
Since for any two disjoint essential simple closed curves $c, d$ on $\partial H$ there is a simple closed curve on $\partial H$ which can be realized disjointly from $c, d$ (e.g. one of the curves $c, d$), the electrified disc graph is obtained from the disc graph by adding some edges. This observation allows to apply Theorem 1inductively to the graphs $\mathcal{EDG}(X)$ where $X$ passes through the thick subsurfaces of $\partial H$ and deduce in a bottom-up inductive procedure hyperbolicity of the disc graph from hyperbolicity of the electrified disc graph. In this way we obtain a new, completely combinatorial and significantly simpler proof of the following result which was first established by Masur and Schleimer [MS13].

**Theorem 4.** The disc graph $\mathcal{DG}$ of a handlebody $H$ of genus $g \geq 2$ is hyperbolic.

We also determine the Gromov boundary of the disc graph. Namely, recall from [K99, H06] that the Gromov boundary of the curve graph of an essential subsurface $X$ of $\partial H$ can be identified with the space of minimal geodesic laminations $\lambda$ in $X$ which fill $X$, i.e. are such that every essential simple closed curve in $X$ has non-trivial intersection with $\lambda$. The Gromov topology on this space of geodesic laminations is the coarse Hausdorff topology which can be defined as follows. A sequence $\lambda_i$ converges to $\lambda$ if and only if every limit in the usual Hausdorff topology of a subsequence of $\lambda_i$ contains $\lambda$ as a sublamination. Notice that the coarse Hausdorff topology is defined on the entire space $L(\partial H)$ of geodesic laminations on $\partial H$, however it is not Hausdorff.

We observe that for every thick subsurface $X$ of $\partial H$ the Gromov boundary $\partial \mathcal{EDG}(X)$ of the electrified disc graph $\mathcal{EDG}(X)$ can be identified with a subspace of the space of topological laminations on $X$, equipped with the coarse Hausdorff topology. Moreover we show

**Theorem 5.** The Gromov boundary $\partial \mathcal{DG}$ of the disc graph equals the subspace

$$\partial \mathcal{DG} = \cup_X \partial \mathcal{EDG}(X) \subset L(\partial H)$$

equipped with the coarse Hausdorff topology. The union is over all thick subsurfaces $X$ of $\partial H$.

There is no analog of this result for handlebodies with spots, i.e. with marked points on the boundary. Indeed, we showed in [H13b] that the disc graph of a handlebody with one or two spots on the boundary is not hyperbolic. The electrified disc graph is not hyperbolic for handlebodies with one spot on the boundary, and the same holds true for sphere graphs.

The organization of this paper is as follows. In Section 2 we show Theorem 1. Section 3 discusses some relative version of results from [H11]. In Section 4, we show the second part of Theorem 4 and the proof of the first part as well as of Theorem 5 is contained in Section 5.

## 2. Hyperbolic thinnings of hyperbolic graphs

In this section we show Theorem 1 from the introduction. Consider a (not necessarily locally finite) metric graph $\mathcal{G}$ (i.e. edges have length one) and a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs, where $\mathcal{C}$ is any countable, finite or empty index set.

Define the $\mathcal{H}$-electrification of $\mathcal{G}$ to be the metric graph $(\mathcal{E}\mathcal{G}, d_\mathcal{G})$ which is obtained from $\mathcal{G}$ by adding vertices and edges as follows. For each $c \in \mathcal{C}$ there is a unique
vertex \( v_c \in \mathcal{E}G - G \). This vertex is connected with each of the vertices of \( H_c \) by a single edge of length one, and it is not connected with any other vertex.

In the sequel all parametrized paths \( \gamma \) in \( G \) or \( \mathcal{E}G \) are supposed to be simplicial. This means that the image of every integer is a vertex, and the image of an integral interval \([k, k+1]\) is an edge or a single vertex.

Call a simplicial path \( \gamma \) in \( \mathcal{E}G \) efficient if for every \( c \in \mathcal{C} \) we have \( \gamma(k) = v_c \) for at most one \( k \). Note that if \( \gamma \) is an efficient simplicial path in \( \mathcal{E}G \) which passes through \( \gamma(k) = v_c \) for some \( c \in \mathcal{C} \) then \( \gamma(k-1) \in H_c, \gamma(k+1) \in H_c \).

The following definition is an adaptation of a definition from [F98].

**Definition 2.1.** The family \( \mathcal{H} \) has the bounded penetration property if for every \( L > 0 \) there is a number \( p(L) > 2r \) with the following property. Let \( \gamma \) be an efficient \( L \)-quasi-geodesic in \( \mathcal{E}G \), let \( c \in \mathcal{C} \) and let \( k \in \mathbb{Z} \) be such that \( \gamma(k) = v_c \). If the distance in \( H_c \) between \( \gamma(k-1) \) and \( \gamma(k+1) \) is at least \( p(L) \) then every efficient \( L \)-quasi-geodesic \( \gamma' \) in \( \mathcal{E}G \) with the same endpoints as \( \gamma \) passes through \( v_c \). Moreover, if \( k' \in \mathbb{Z} \) is such that \( \gamma'(k') = v_c \) then the distance in \( H_c \) between \( \gamma(k-1), \gamma'(k'-1) \) and between \( \gamma(k+1), \gamma'(k'+1) \) is at most \( p(L) \).

The definition of relative hyperbolicity for a graph below is taken from [Si12] where it is shown to be equivalent to other definitions of relative hyperbolicity found in the literature.

**Definition 2.2.** Let \( \mathcal{H} \) be a family of complete connected subgraphs of a metric graph \( G \). The graph \( G \) is hyperbolic relative to \( \mathcal{H} \) if the \( \mathcal{H} \)-electrification of \( G \) is hyperbolic and if moreover \( \mathcal{H} \) has the bounded penetration property.

From now on we always consider a metric graph \( G \) which is hyperbolic relative to a family \( \mathcal{H} = \{ H_c \mid c \in \mathcal{C} \} \) of complete connected subgraphs.

We say that the family \( \mathcal{H} \) is \( r \)-bounded for a number \( r > 0 \) if \( \text{diam}(H_c \cap H_d) \leq r \) for \( c \neq d \in \mathcal{C} \) where the diameter is taken with respect to the intrinsic path metric on \( H_c \) and \( H_d \). A family which is \( r \)-bounded for some \( r > 0 \) is simply called bounded.

The following is a consequence of the main theorem of [Si12] (the equivalence of definition RH0 and RH2).

**Proposition 2.3.** If \( G \) is hyperbolic relative to the family \( \mathcal{H} \) then \( \mathcal{H} \) is bounded.

Let \( \mathcal{H} \) be as in Definition 2.1. Define an enlargement \( \hat{\gamma} \) of an efficient simplicial \( L \)-quasi-geodesic \( \gamma : [0, n] \to \mathcal{E}G \) with endpoints \( \gamma(0), \gamma(n) \in G \) as follows. Let \( 0 < k_1 < \cdots < k_s < n \) be those points such that \( \gamma(k_i) = v_{c_i} \) for some \( c_i \in \mathcal{C} \). Then \( \gamma(k_i-1), \gamma(k_i+1) \in H_{c_i} \). For each \( i \leq s \) replace \( \gamma[k_i-1, k_i+1] \) by a simplicial geodesic in \( H_{c_i} \) with the same endpoints.

For a number \( k > 0 \) define a subset \( Z \) of the metric graph \( G \) to be \( k \)-quasi-convex if any geodesic with both endpoints in \( Z \) is contained in the \( k \)-neighborhood of \( Z \). In particular, up to perhaps increasing the number \( k \), any two points in \( Z \) can be connected in \( Z \) by a (not necessarily continuous) path which is a \( k \)-quasi-geodesic in \( G \). The goal of this section is to show

**Theorem 2.4.** Let \( G \) be a metric graph which is hyperbolic relative to a family \( \mathcal{H} = \{ H_c \mid c \in \mathcal{C} \} \) of complete connected subgraphs. If there is a number \( \delta > 0 \) such that each of the graphs \( H_c \) is \( \delta \)-hyperbolic then \( G \) is hyperbolic. Enlargements of geodesics in \( \mathcal{E}G \) are uniform quasi-geodesics in \( G \). The subgraphs \( H_c \) are uniformly quasi-convex.
For the remainder of this section we assume that \( \mathcal{G} \) is a graph which is hyperbolic relative to a family \( \mathcal{H} \) of complete connected \( \delta \)-hyperbolic subgraphs.

For a number \( R > 2r \) call \( c \in \mathcal{C} \) \( R \)-wide for an efficient \( L \)-quasi-geodesic \( \gamma \) in \( \mathcal{E}\mathcal{G} \) if the following holds true. There is some \( k \in \mathbb{Z} \) such that \( \gamma(k) = v_c \), and the distance between \( \gamma(k-1), \gamma(k+1) \) in \( H_c \) is at least \( R \). Note that since \( \mathcal{H} \) is \( r \)-bounded, \( c \) is uniquely determined by \( \gamma(k-1), \gamma(k+1) \). If \( R = p(L) \) is as in Definition 2.1 then we simply say that \( c \) is wide.

**Lemma 2.5.** Let \( L \geq 1 \) and let \( \gamma_1, \gamma_2 \) be two efficient \( L \)-quasi-geodesics in \( \mathcal{E}C \) with the same endpoints. If \( c \in \mathcal{C} \) is \( 3p(L) \)-wide for \( \gamma_1 \) then \( c \) is wide for \( \gamma_2 \).

**Proof.** By definition, if \( c \) is \( 3p(L) \)-wide for \( \gamma_1 \) then there is some \( k \) so that \( \gamma_1(k) = v_c \) and that the distance in \( H_c \) between \( \gamma_1(k-1) \) and \( \gamma_1(k+1) \) is at least \( 3p(L) \). Since \( \gamma_2 \) is an efficient \( L \)-quasi-geodesic with the same endpoints as \( \gamma_1 \), by the bounded penetration property there is some \( k' \) so that \( \gamma_2(k') = v_c \), moreover the distance in \( H_c \) between \( \gamma_1(k-1) \) and \( \gamma_2(k') \) and between \( \gamma_1(k+1) \) and \( \gamma_2(k' + 1) \) is at most \( p(L) \). Thus by the triangle inequality, the distance in \( H_c \) between \( \gamma_2(k') \) and \( \gamma_2(k' + 1) \) is at least \( p(L) \) which is what we wanted to show. \( \square \)

Define the **Hausdorff distance** between two closed subsets \( A, B \) of a metric space to be the infimum of the numbers \( b > 0 \) such that \( A \) is contained in the \( b \)-neighborhood of \( B \) and \( B \) is contained in the \( b \)-neighborhood of \( A \).

**Lemma 2.6.** For every \( L > 0 \) there is a number \( \kappa(L) > 0 \) with the following property. Let \( \gamma_1, \gamma_2 \) be two efficient simplicial \( L \)-quasi-geodesics in \( \mathcal{E}\mathcal{G} \) connecting the same points in \( \mathcal{G} \), with enlargements \( \hat{\gamma}_1, \hat{\gamma}_2 \). Then the Hausdorff distance in \( \mathcal{G} \) between the images of \( \hat{\gamma}_1, \hat{\gamma}_2 \) is at most \( \kappa(L) \).

**Proof.** Let \( \gamma : [0, n] \to \mathcal{E}\mathcal{G} \) be an efficient simplicial \( L \)-quasi-geodesic with endpoints \( \gamma(0), \gamma(n) \in \mathcal{G} \). Let \( R > p(L) \) and assume that \( c \in \mathcal{C} \) is not \( R \)-wide for \( \gamma \). If there is some \( u \in \{1, \ldots, n-1\} \) such that \( \gamma(u) = v_c \) then \( \gamma(u-1), \gamma(u+1) \in H_c \). Since \( c \) is not \( R \)-wide for \( \gamma \), \( \gamma(u-1) \) can be connected to \( \gamma(u+1) \) by an arc in \( H_c \) of length at most \( R \). In particular, if no \( c \in \mathcal{C} \) is \( R \)-wide for \( \gamma \) then an enlargement \( \hat{\gamma} \) of \( \gamma \) is an \( L \)-quasi-geodesic in \( \mathcal{E}\mathcal{G} \) for a universal constant \( \hat{L} = \hat{L}(L, R) > 0 \). Then \( \hat{\gamma} \) is a \( \hat{L} \)-quasi-geodesic in \( \mathcal{G} \) as well (note that the inclusion \( \mathcal{G} \to \mathcal{E}\mathcal{G} \) is 1-Lipschitz).

Let \( \gamma_i : [0, n_i] \to \mathcal{E}\mathcal{G} \) be efficient \( L \)-quasi-geodesics \((i = 1, 2)\) with the same endpoints in \( \mathcal{G} \). Assume that no \( c \in \mathcal{C} \) is wide for \( \gamma_1 \). By Lemma 2.5 no \( c \in \mathcal{C} \) is \( R = 3p(L) \)-wide for \( \gamma_2 \). Let \( \hat{\gamma}_i \) be an enlargement of \( \gamma_i \). By the above discussion, the arcs \( \hat{\gamma}_i \) are \( \hat{L} = \hat{L}(L, 3p(L)) \)-quasi-geodesics in \( \mathcal{E}\mathcal{G} \). In particular, by hyperbolicity of \( \mathcal{E}\mathcal{G} \), the Hausdorff distance in \( \mathcal{E}\mathcal{G} \) between the images of \( \hat{\gamma}_i \) is bounded from above by a constant \( b - 1 > 0 \) only depending on \( L \) and \( R \).

We have to show that the Hausdorff distance in \( \mathcal{G} \) between these images is also uniformly bounded. For this let \( x = \hat{\gamma}_1(u) \) be any vertex on \( \hat{\gamma}_1 \) and let \( y = \hat{\gamma}_2(w) \) be a vertex on \( \hat{\gamma}_2 \) of minimal distance in \( \mathcal{E}\mathcal{G} \) to \( x \). Then \( d_{\mathcal{E}}(x, y) \leq b \) (here as before, \( d_{\mathcal{E}} \) is the distance in \( \mathcal{E}\mathcal{G} \), and we let \( d \) be the distance in \( \mathcal{G} \)). Let \( \zeta \) be a geodesic in \( \mathcal{E}\mathcal{G} \) connecting \( x \) to \( y \). Since \( y \) is a vertex on \( \hat{\gamma}_2 \) of minimal distance to \( x \), \( \zeta \) intersects \( \hat{\gamma}_2 \) only at its endpoints.

We claim that there is a universal constant \( \chi > 0 \) such that no \( c \in \mathcal{C} \) is \( \chi \)-wide for \( \zeta \). Namely, since \( \hat{\gamma}_1 \) does not pass through any of the special vertices in \( \mathcal{E}\mathcal{G} \), the concatenation \( \xi = \zeta \circ \hat{\gamma}_1[0, u] \) is efficient. Thus \( \xi \) is an efficient \( L' \)-quasi-geodesic in \( \mathcal{E}\mathcal{G} \) with the same endpoints as \( \hat{\gamma}_2[0, w] \) where \( L' > \hat{L} > L \) only depends on \( L \).
Hence by the bounded penetration property, if \( c \in C \) is \( p(L') \)-wide for \( \zeta \) then the \( L \)-quasi-geodesic \( \hat{\gamma}_2[0, w] \) passes through the vertex \( v_c \) which is a contradiction.

As a consequence of the above discussion, the length of an enlargement of \( \zeta \) is bounded from above by a fixed multiple of \( d_G(\hat{\gamma}_1(u), \hat{\gamma}_2(w)) \), i.e. it is uniformly bounded. This shows that \( d(\hat{\gamma}_1(u), \hat{\gamma}_2(w)) \) is uniformly bounded. As a consequence, the image of \( \hat{\gamma}_1 \) is contained in a neighborhood of uniformly bounded diameter in \( G \) of the image of \( \hat{\gamma}_2 \). Exchanging \( \gamma_1 \) and \( \gamma_2 \) we conclude that indeed the Hausdorff distance in \( G \) between the images of the enlargements \( \hat{\gamma}_1, \hat{\gamma}_2 \) is bounded by a number only depending on \( L \).

Now let \( \gamma_j : [0, n_j] \to \mathcal{EG} \) be arbitrary efficient \( L \)-quasi-geodesics \( (j = 1, 2) \) connecting the same points in \( G \). Then there are numbers \( 0 < u_1 < \cdots < u_k < n_j \) such that for every \( i \leq k \), \( \gamma_1(u_i) = v_{c_i} \), where \( c_i \in C \) is wide for \( \gamma_1 \), and there are no other wide points for \( \gamma_1 \). Put \( u_0 = -1 \) and \( u_{k+1} = n_j + 1 \).

By the bounded penetration property, there are numbers \( w_i \in \{1, \ldots, n_2 - 1\} \) such that \( \gamma_2(w_i) = \gamma_1(u_i) = v_{c_i} \) for all \( i \). Moreover, the distance in \( H_{c_i} \) between \( \gamma_1(u_i-1) \) and \( \gamma_2(w_i-1) \) and between \( \gamma_1(u_i+1) \) and \( \gamma_2(w_i+1) \) is at most \( p(L) \). Since \( \gamma_1, \gamma_2 \) are \( L \)-quasi-geodesics by assumption, we may assume that the special vertices \( v_{c_i} \) appear along \( \gamma_2 \) in the same order as along \( \gamma_1 \), i.e. that \( 0 < w_1 < \cdots < w_k < n_2 \). Put \( w_0 = -1 \) and \( w_{k+1} = n_2 + 1 \).

For each \( i \leq k \), define a simplicial edge path \( \zeta_i : [a_i, a_{i+1}] \to \mathcal{EG} \) connecting \( \zeta_i(a_i) = \gamma_1(u_i + 1) \in H_{c_i} \) to \( \zeta_i(a_{i+1}) = \gamma_1(u_i + 1 - 1) \in H_{c_{i+1}} \) as the concatenation of the following three arcs. A geodesic in \( H_{c_i} \) connecting \( \gamma_1(u_i + 1) \) to \( \gamma_2(w_i + 1) \) (whose length is at most \( p(L) \)), the arc \( \gamma_2[w_i + 1, w_{i+1} - 1] \) and a geodesic in \( H_{c_{i+1}} \) connecting \( \gamma_2(w_{i+1} - 1) \) to \( \gamma_2(u_{i+1} - 1) \). Let moreover \( \eta_i = \gamma_1[u_i + 1, n_j + 1 - 1] \) \((i \geq 0)\). Then \( \eta_i, \zeta_i \) are efficient uniform quasi-geodesics in \( \mathcal{EG} \) with the same endpoints, and \( \eta_i \) does not have wide points.

For each \( i \) let \( \hat{\nu}_i \) be an enlargement of the arc \( \nu_i = \gamma_2[w_i + 1, w_{i+1} - 1] \). By construction, there is an enlargement \( \hat{\zeta}_i \) of the efficient quasi-geodesic \( \zeta_i \) which contains \( \hat{\nu}_i \) as a subarc and whose Hausdorff distance in \( G \) to \( \hat{\nu}_i \) is uniformly bounded. Let \( \hat{\eta}_i \) be an enlargement of \( \eta_i \). Then \( \hat{\zeta}_i, \hat{\eta}_i \) are enlargements of the efficient uniform quasi-geodesics \( \zeta_i, \eta_i \) in \( \mathcal{EG} \) with the same endpoints, and \( \hat{\eta}_i \) does not have wide points. Therefore by the first part of this proof, the Hausdorff distance in \( G \) between \( \hat{\eta}_i \) and \( \hat{\zeta}_i \) is uniformly bounded. Hence the Hausdorff distance between \( \hat{\eta}_i \) and \( \hat{\nu}_i \) is uniformly bounded as well.

There is an enlargement \( \hat{\gamma}_1 \) of \( \gamma_1 \) which can be represented as

\[
\hat{\gamma}_1 = \hat{\eta}_k \circ \sigma_k \circ \cdots \circ \sigma_1 \circ \hat{\nu}_0
\]

where for each \( i \), \( \sigma_i \) is a geodesic in \( H_{c_i} \) connecting \( \gamma_1(u_i - 1) \) to \( \gamma_1(u_i + 1) \). Similarly, there is an enlargement \( \hat{\gamma}_2 \) of \( \gamma_2 \) which can be represented as

\[
\hat{\gamma}_2 = \hat{\nu}_k \circ \tau_k \circ \cdots \circ \tau_1 \circ \hat{\nu}_0
\]

where for each \( i \), \( \tau_i \) is a geodesic in \( H_{c_i} \) connecting \( \gamma_2(w_i - 1) \) to \( \gamma_2(w_i + 1) \).

For each \( i \) the distance in \( H_{c_i} \) between \( \gamma_1(u_i - 1) \) and \( \gamma_2(w_i - 1) \) is at most \( p(L) \), and the same holds true for the distance between \( \gamma_1(u_i + 1) \) and \( \gamma_2(w_i + 1) \). Since \( H_{c_i} \) is \( \delta \)-hyperbolic for a constant \( \delta > 0 \) not depending on \( c_i \), the Hausdorff distance in \( H_{c_i} \) between any two geodesics connecting \( \gamma_1(u_i - 1) \) to \( \gamma_1(u_i + 1) \) and connecting \( \gamma_2(w_i - 1) \) to \( \gamma_2(w_i + 1) \) is uniformly bounded. Together with the above discussion, this shows the lemma. \( \square \)
Let for the moment $X$ be an arbitrary geodesic metric space. Assume that for every pair of points $x, y \in X$ there is a fixed choice of a path $\rho_{x,y}$ connecting $x$ to $y$. The thin triangle property for this family of paths states that there is a universal number $C > 0$ so that for any triple $x, y, z$ of points in $X$, the image of $\rho_{x,y}$ is contained in the $C$-neighborhood of the union of the images of $\rho_{y,z}, \rho_{z,x}$.

For two vertices $x, y \in G$ let $\rho_{x,y}$ be an enlargement of a geodesic in $\mathcal{E}G$ connecting $x$ to $y$. We have

**Proposition 2.7.** The thin triangle inequality property holds true for the paths $\rho_{x,y}$.

**Proof.** Let $x_1, x_2, x_3$ be three vertices in $G$ and for $i = 1, 2, 3$ let $\gamma_i : [0, n_i] \to \mathcal{E}G$ be a geodesic connecting $x_i$ to $x_{i+1}$.

By hyperbolicity of $\mathcal{E}G$ there is a number $L > 0$ not depending on the points $x_i$, and there is a vertex $y \in \mathcal{E}G$ with the following property. For $i = 1, 2, 3$ let $\beta_i : [0, p_i] \to \mathcal{E}G$ be a geodesic in $\mathcal{E}G$ connecting $x_i$ to $y$. Then for all $i$, $\alpha_i = \beta_{i+1}^{-1} \circ \beta_i$ is an $L$-quasi-geodesic connecting $x_i$ to $x_{i+1}$.

We claim that without loss of generality we may assume that the quasi-geodesics $\alpha_i$ are efficient. Namely, since the arcs $\beta_i$ are geodesics, they do not backtrack. Thus if $\alpha_1$ is not efficient then there is a common point $y$ on $\beta_1$ and $\beta_2$. Let $s_1 < p_1$ be the smallest number so that $\beta_1(s_1) = \beta_2(s_2)$ for some $s_2 \in \beta_2[0, p_2]$. Then the distance between $y$ and $\beta_1(s_1)$ $(i = 1, 2)$ is uniformly bounded, and $\alpha_1 = (\beta_2[0, s_2])^{-1} \circ \beta_1[0, s_1]$ is an efficient $L$-quasi-geodesic connecting $x_1$ to $x_2$. Replace $y$ by $\beta_1(s_1)$, replace $\beta_1$ by $\tilde{\beta}_1 = \beta_1[0, s_1]$ $(i = 1, 2)$ and replace $\beta_3$ by a geodesic $\tilde{\beta}_3 : [0, \tilde{p}_3] \to \mathcal{E}G$ connecting $x_3$ to $\beta_1(s_1)$. Thus up to increasing the number $L$ by a uniformly bounded amount we may assume that the quasi-geodesic $\alpha_1$ is efficient.

Assume from now on that $\beta_1, \beta_2, \beta_3$ are such that the quasi-geodesic $\alpha_1 = \beta_2^{-1} \circ \beta_1$ is efficient. Using the notation from the second paragraph of this proof, if there is some $s < p_3$ such that $\beta_3(s)$ is contained in $\alpha_1$ then let $s_3$ be the smallest number with this property. Replace the point $y = \beta_3(p_3)$ be $\beta_3(s_3)$, replace $\beta_3$ by $\beta_3[0, s_3]$ and for $i = 1, 2$ replace $\beta_i$ by the subarc of $\alpha_1$ connecting $x_i$ to $\beta_3(s_3)$. With this construction, up to increasing the number $L$ by a uniformly bounded amount and perhaps replacing $\beta_1, \beta_2$ by uniform quasi-geodesics rather than geodesics we may assume that all three quasi-geodesics $\tilde{\alpha}_i = \beta_{i+1}^{-1} \circ \beta_i$ $(i = 1, 2, 3)$ are efficient.

Resuming notation, assume from now on that the quasi-geodesics $\alpha_i$ are efficient. By Lemma 2.6 the Hausdorff distance between an enlargement of the geodesic $\gamma_i$ and any choice of an enlargement of the efficient uniform quasi-geodesic $\alpha_i$ with the same endpoints is uniformly bounded. Thus it suffices to show the thin triangle property for enlargements of the quasi-geodesics $\alpha_i$.

If $y \in G$ then an enlargement of the quasi-geodesic $\alpha_i$ is the concatenation of an enlargement of the quasi-geodesic $\beta_i$ and an enlargement of the quasi-geodesic $\beta_{i+1}^{-1}$ which have endpoints in $G$. Hence in this case the thin triangle property follows once more from Lemma 2.6.

If $y = v_c$ for some $c \in C$ then we distinguish two cases.

*Case 1: $c \in C$ is wide for each of the quasi-geodesics $\alpha_i$.*

Recall that $y = \beta_i(p_i)$. By hyperbolicity of $H_c$, there is a number $R > 0$ not depending on $c$ such that for all $i \in \{1, 2, 3\}$ the image of any geodesic in $H_c$ connecting $\beta_i(p_i - 1)$ to $\beta_{i+1}(p_{i+1} - 1)$ is contained in the $R$-neighborhood of the union of the images of any two geodesics connecting $\beta_j(p_j - 1)$ to $\beta_{j+1}(p_{j+1} - 1)$ for
$j \neq i$ and where indices are taken modulo three. In other words, the thin triangle property holds true for such geodesics.

Now let $\hat{\alpha}_i$ be an enlargement of $\alpha_i$ and let $\zeta_i$ be the subarc of $\hat{\alpha}_i$ which connects $\hat{\beta}_i(p_i-1)$ to $\hat{\beta}_{i+1}(p_{i+1}-1)$. By the definition of an enlargement, $\zeta_i$ is a geodesic in $H_c$. Thus by the discussion in the previous paragraph and by the fact that we may use the same enlargement of the arc $\hat{\beta}_{i+1}[0,p_{i+1}-1]$ for the construction of an enlargement of $\alpha_i$ and $\alpha_{i+1}$, the thin triangle property holds true for some suitable choice and hence any choice of an enlargement of the quasi-geodesics $\alpha_i$ which is what we wanted to show.

Case 2: For at least one $i$, $c \in C$ is not wide for the quasi-geodesic $\alpha_i$.

Assume that this holds true for the quasi-geodesic $\alpha_1$. Then the distance in $H_c$ between $\beta_1(p_1-1)$ and $\beta_2(p_2-1)$ is uniformly bounded (depending on the quasi-geodesic constant for $\alpha_1$). Replace the point $y$ by $\beta_1(p_1-1)$, replace the quasi-geodesic $\beta_1$ by $\tilde{\beta}_1 = \beta_1[0,p_1-1]$, replace the quasi-geodesic $\beta_2$ by the concatenation $\tilde{\beta}_2 = \beta_2(p_2-1)$ with a geodesic in $H_c$ connecting $\beta_2(p_2-1)$ to $\beta_1(p_1-1)$, and replace the geodesic $\beta_3$ by the concatenation $\tilde{\beta}_3 = \beta_3$ with the edge connecting $v_c$ to $\beta_1(p_1-1)$. The resulting arcs $\tilde{\beta}_i$ are efficient uniform quasi-geodesics in $\mathcal{E}G$, and they connect the points $x_i$ to $y \in G$. Moreover, the quasi-geodesics $\tilde{\beta}_{i+1}^{-1} \circ \tilde{\beta}_i$ are efficient as well and hence we are done by the above proof for the case $y \in G$.

Now we are ready to show

**Corollary 2.8.** $\mathcal{G}$ is hyperbolic. Enlargements of geodesics in $\mathcal{E}G$ are uniform quasi-geodesics in $\mathcal{G}$.

**Proof.** For any pair $(x,y)$ of vertices in $\mathcal{G}$ let $\eta_{x,y}$ be a reparametrization on $[0,1]$ of the path $\rho_{x,y}$. By Proposition 3.5 of [H07], it suffices to show that there is some $n > 0$ such that the paths $\eta_{x,y}$ have the following properties (where $d$ is the distance in $\mathcal{G}$).

1. If $d(x,y) \leq 1$ then the diameter of $\eta_{x,y}[0,1]$ is at most $n$.
2. For $x,y$ and $0 \leq s < t < 1$ the Hausdorff distance between $\eta_{x,y}[s,t]$ and $\eta_{x,y}(s),\eta_{x,y}(t)[0,1]$ is at most $n$.
3. For all vertices $x,y,z$ the set $\eta_{x,y}[0,1]$ is contained in the $n$-neighborhood of $\eta_{x,z}[0,1] \cup \eta_{y,z}[0,1]$.

Properties 1) and 2) above are immediate from Lemma [2.6]. The thin triangle property 3) follows from Proposition [2.7].

The following corollary is an immediate consequence of Corollary [2.8].

**Corollary 2.9.** There is a number $k > 0$ such that each of the subgraphs $H_c$ ($c \in C$) is $k$-quasi-convex.

We complete this section with a calculation of the Gromov boundary of $\mathcal{G}$.

Let as before $\mathcal{E}G$ be the $\mathcal{H}$-electrification of $\mathcal{G}$. Denote by $\partial \mathcal{E}G$ be the Gromov boundary of $\mathcal{E}G$. For each $c \in C$ let moreover $\partial H_c$ be the Gromov boundary of $H_c$. We equip

$$\partial \mathcal{G} = \partial \mathcal{E}G \cup_c \partial H_c$$

with a topology which is determined by describing for each point $\xi \in \partial \mathcal{G}$ a neighborhood basis as follows.

Let first $\xi \in \partial \mathcal{E}G$. Let $L > 1$ be such that every point $x \in \mathcal{G}$ can be connected in $\mathcal{E}G$ to every point $\zeta \in \partial \mathcal{E}G$ by an $L$-quasi-geodesic.
Let $\delta_G$ be the Gromov metric on $\partial E\mathcal{G}$ based at a fixed point $x \in \mathcal{G}$. For $\epsilon > 0$ let $\mathcal{C}_\epsilon(x)$ be the collection of all $x \in \mathcal{C}$ such that there is a $L$-quasi-geodesic $\gamma$ connecting $x$ to a point in the $\epsilon$-neighborhood of $x$ in $(\partial E\mathcal{G}, \delta_G)$ whose tail $\gamma[-\log \epsilon, \infty)$ passes through the $p(L)$-neighborhood of $x$. Define $B_\epsilon(\xi) \subset \partial \mathcal{G}$ by

$$B_\epsilon(\xi) = \{ \xi \subset \partial E\mathcal{G}, \delta_G(\xi, \xi) < \epsilon \} \cup \bigcup_{c \in \mathcal{C}_\epsilon(x)} \partial H_c.$$ 

Clearly we have $\bigcap_{\epsilon > 0} B_\epsilon(\xi) = \xi$. Moreover, changing the basepoint $x$ yields an equivalent neighborhood basis.

If $c \in \mathcal{C}$ and $\xi \in \partial H_c$ then choose a basepoint $x \in H_c$ and an $L$-quasi-geodesic $\gamma : [0, \infty) \to H_c$ connecting $\gamma(0) = x$ to $\xi$. For $\epsilon > 0$ let $\mathcal{C}_\epsilon(x)$ be the collection of all $d \in \mathcal{C}$ such that a geodesic in $E\mathcal{G}$ connecting $x$ to $v_d$ passes through the $p(L)$-neighborhood of $\gamma[-\log \epsilon, \infty)$ in $H_c$. Note that this makes sense since the vertex $v_c$ is only connected to vertices in $H_c$ by an edge.

Let moreover $\hat{B}_\epsilon$ be the set of all $\beta \in \partial E\mathcal{G}$ such that an $L$-quasi-geodesic in $E\mathcal{G}$ connecting $x$ to $\beta$ passes through the $p(L)$-neighborhood of $\gamma[-\log \epsilon, \infty)$ in $H_c$. Define

$$B_\epsilon(\xi) = \hat{B}_\epsilon \cup \bigcup_{d \in \mathcal{C}_\epsilon(x)} \partial H_d.$$ 

By the bounded penetration property, this makes sense. Declare the family of sets $B_\epsilon(\xi)$ to be a neighborhood basis of $\xi \in \partial \mathcal{G}$. We have

**Proposition 2.10.** $\partial \mathcal{G}$ is the Gromov boundary of $\mathcal{G}$. 

**Proof.** For a number $L > 1$ define an unparametrized $L$-quasi-geodesic in the graph $E\mathcal{G}$ to be a path $\eta : [0, \infty) \to E\mathcal{G}$ with the following property. There is some $n \in (0, \infty)$ and there is an increasing homeomorphism $\rho : [0, n) \to [0, \infty)$ such that $\eta \circ \rho$ is an $L$-quasi-geodesic in $E\mathcal{G}$.

Let $x \in \zeta$ and let $p > 1$ be sufficiently large that $x$ can be connected to every point in the Gromov boundary of $G$ by a $p$-quasi-geodesic ray in $G$. Let $\gamma : [0, \infty) \to \mathcal{G}$ be such a simplicial $p$-quasi-geodesic ray. We claim that there is a number $p' > 1$ such that $\gamma$ viewed as a path in $E\mathcal{G}$ is an unparametrized $p'$-quasi-geodesic in $E\mathcal{G}$. Namely, for each $i > 0$ let $\zeta_i$ be an enlargement of a geodesic in $E\mathcal{G}$ with endpoints $\gamma(0), \gamma(i)$. Then there is a number $L > 1$ such that $\zeta_i$ is an $L$-quasi-geodesic in $\mathcal{G}$. By hyperbolicity, the Hausdorff distance in $G$ between $\gamma[0, i]$ and the image of $\zeta_i$ is uniformly bounded. Then the same holds true if this Hausdorff distance is measured with respect to the distance in $E\mathcal{G}$. Thus the Hausdorff distance in $E\mathcal{G}$ between $\gamma[0, i]$ and a geodesic with the same endpoints is uniformly bounded. Since $i > 0$ was arbitrary, this implies that $\gamma$ is an unparametrized $p'$-quasi-geodesic in $E\mathcal{G}$ for a number $p' > 0$ only depending on $p$.

As a consequence, if the diameter of $\gamma[0, \infty)$ in $E\mathcal{G}$ is infinite then up to parametrization, $\gamma[0, \infty)$ is a $p'$-quasi-geodesic ray in $E\mathcal{G}$ and hence it converges as $i \to \infty$ to a point $\xi \in \partial \mathcal{G}$.

Now assume that the diameter of $\gamma[0, \infty)$ in $E\mathcal{G}$ is finite. By Corollary 2.9 there is a number $M > 0$ not depending on $c$ with the following properties.

(1) If $x, y \in \mathcal{G}$ are any two vertices and if $c \in \mathcal{C}$ is such that the distance in $H_c$ of some shortest distance projection of $x, y$ into $H_c$ is at least $M$ then a geodesic connecting $x$ to $y$ in $E\mathcal{G}$ passes through the special vertex $v_c$ defined by $c$. 


(2) Let $\gamma : [0, \ell] \to G$ be a $p$-quasi-geodesic. If there is some $k \leq \ell$ and some $c \in C$ such that the distance in $H_c$ of some shortest distance projection of $\gamma(0), \gamma(k)$ into $H_c$ is at least $2M$ then for each $\ell \geq k$ the distance in $H_c$ of a shortest distance projection of $\gamma(0), \gamma(\ell)$ into $H_c$ is at least $M$.

For $k > 0$ let $C_1(k)$ (or $C_2(k)$) be the set of all $c \in C$ so that the distance in $H_c$ between a shortest distance projection of $\gamma(0), \gamma(k)$ into $H_c$ is at least $M$ (or $2M$).

By property (2) above, for $\ell \geq k$ we have $C_2(\ell) \subset C_1(k)$.

The diameter of the image of any simplicial geodesic in $\mathcal{E}G$ equals the length of the geodesic and hence it is bounded from below by the number of special vertices it passes through. Since the diameter of $\gamma[0, \infty)$ in $\mathcal{E}G$ is finite by assumption, by property (1) the cardinality of $C_1(k)$ is bounded independent of $k$.

By property (2) above, we deduce that there is some $k_0 > 0$ so that $C_2(\ell) \subset C_1(k_0)$ for all $\ell \geq k_0$. Since the diameter of $\gamma[k_0, \infty)$ in $\mathcal{E}G$ is finite, it follows that there is some $c \in C$ so that $\gamma[k_0, \infty)$ is contained in a uniformly bounded neighborhood of $H_c$. Now $\gamma$ is a $p$-quasi-geodesic in $\mathcal{E}G$ and the embedding $H_c \to \mathcal{E}G$ is a quasi-isometry. Thus by hyperbolicity, there is a quasi-geodesic $\zeta$ in $H_c$ whose Hausdorff distance to $\gamma[k_0, \infty)$ is bounded. From Corollary 2.8 we conclude that $\gamma$ converges as $j \to \infty$ to some $\mu \in \partial H_c$.

To summarize, there is a map $\Lambda$ from the Gromov boundary of $G$ into $\partial G$. Corollary 2.8 and the above discussion shows that $\Lambda$ is a bijection. The description of the topology on the boundary of $G$ as the topology described above for $\partial G$ is now immediate from the description of the Gromov boundary of a hyperbolic metric graph.

\[\square\]

3. Thick subsurfaces

In this section we consider a handlebody $H$ of genus $g \geq 2$. By a disc in $H$ we mean an essential disc in $H$.

Two discs $D_1, D_2 \subset H$ are in normal position if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that discs are in normal position; this can be achieved by modifying one of the two discs with an isotopy.

As in the introduction, call a connected essential subsurface $X$ of $\partial H$ thick if the following conditions are satisfied.

1. Every disc intersects $X$.
2. $X$ is filled by boundaries of discs.

An example of a thick subsurface is the complement in $\partial H$ of a suitably chosen simple closed curve which is not discbounding. The entire boundary surface $\partial H$ is thick as well.

For a thick subsurface $X$ of $\partial H$ define $\mathcal{E}DG(X)$ to be the graph whose vertices are discs with boundary contained in $X$. By property (1) in the definition of a thick subsurface, the boundary of each such vertex is an essential simple closed curve in $X$. Two such discs $D, E$ are connected by an edge of length one if and only if there is an essential simple closed curve $\gamma$ in $X$ which can be realized disjointly from both $D, E$ (e.g. the boundary of $D$ if the discs $D, E$ are disjoint).

Denote by $d_{\mathcal{E}, X}$ the distance in $\mathcal{E}DG(X)$. The disc graph $\mathcal{D}G(X)$ of $X$ is defined in the obvious way, and we denote by $d_{\mathcal{D}, X}$ its distance function.
In the sequel we always assume that all curves and multicurves on \( X \subseteq \partial H \) are essential. For two simple closed multicurves \( c, d \) on \( \partial H \) let \( \iota(c, d) \) be the geometric intersection number between \( c, d \).

The following lemma \[\text{(MM04)}\] implies that for every thick subsurface \( X \) of \( \partial H \) the graph \( DG(X) \) is connected. For its proof and later use, let \( D, E \) be discs in minimal position. Define an outer component of \( E \) with respect to \( D \) which is a disc whose boundary consists of a single subarc of \( \partial E \) and a single subarc \( \alpha \) of \( D \). The arc \( \alpha \) intersects the boundary of \( D \) precisely at its endpoints. Surgery of \( D \) at this outer component \( E \) replaces \( D \) by the union of \( E \) with one of the two components of \( D - \alpha \) (compare e.g. \[\text{MM04, H11}\]).

**Lemma 3.1.** Let \( X \subseteq \partial H \) be a thick subsurface. Let \( D, E \subseteq H \) be discs with boundaries in \( X \). Then \( D \) can be connected to a disc \( E' \) which is disjoint from \( E \) by at most \( \iota(\partial D, \partial E)/2 \) simple surgeries. In particular,

\[
d_{\mathcal{D},X}(D, E) \leq \iota(\partial D, \partial E)/2 + 1.
\]

**Proof.** Let \( D, E \) be two discs in normal position with boundary in \( X \). Assume that \( D, E \) are not disjoint. Then there is an outer component of \( E-D \). A disc \( D' \) obtained by simple surgery of \( D \) at this component is essential in \( \partial H \) and its boundary is contained in \( X \), i.e. \( D' \in \mathcal{EDG}(X) \). Moreover, \( D' \) is disjoint from \( D \), i.e. we have \( d_{\mathcal{D},X}(D', D) = 1 \), and

\[
\iota(\partial D', \partial E) \leq \iota(\partial D, \partial E) - 2.
\]

The lemma now follows by induction on \( \iota(\partial D, \partial E) \). \( \square \)

**Remark 3.2.** Lemma \[3.1\] implies that a thick subsurface \( X \) of \( \partial H \) can not be a four-holed sphere or a one-holed torus. Namely, if \( X \) is a four-holed sphere or a one-holed torus and if \( X \) contains the boundaries of two distinct discs \( D, E \) then these discs intersect. Surgery of \( D \) at an outer component of \( E-D \) results in an essential disc \( D' \) which up to homotopy is disjoint from the disc \( D \) and whose boundary is contained in \( X \). Since any two essential simple closed curves in \( X \) intersect, the boundary of \( D' \) is peripheral in \( X \) which violates the assumption that no boundary component of \( X \) is discbunding.

A simple closed multicurve \( \gamma \) in a thick subsurface \( X \) of \( \partial H \) is called discbusting if \( \gamma \) intersects every disc with boundary in \( X \).

Consider an oriented I-bundle \( J(F) \) over a compact (not necessarily orientable) surface \( F \) with (not necessarily connected) boundary \( \partial F \). The boundary \( \partial J(F) \) of \( J(F) \) decomposes into the horizontal boundary and the vertical boundary. The vertical boundary is the interior of the restriction of the I-bundle to \( \partial F \) and consists of a collection of pairwise disjoint open incompressible annuli. The horizontal boundary is the complement of the vertical boundary in \( \partial J(F) \).

For a given boundary component \( \alpha \) of \( F \), the union of the horizontal boundary of \( J(F) \) with the I-bundle over \( \alpha \) is a compact connected orientable surface \( F_\alpha \subseteq \partial J(F) \). The boundary of \( F_\alpha \) is empty if and only if the boundary of \( F \) is connected. If the boundary of \( F \) is not connected then \( F_\alpha \) is properly contained in the boundary \( \partial J(F) \) of \( J(F) \). The complement \( \partial J(F) - F_\alpha \) is a union of incompressible annuli.

**Definition 3.3.** An I-bundle generator in a thick subsurface \( X \subseteq \partial H \) is an essential simple closed curve \( \gamma \subseteq X \) with the following property. There is a compact surface \( F \) with non-empty boundary \( \partial F \), there is a boundary component \( \alpha \) of \( \partial F \),
and there is an orientation preserving embedding $\Psi$ of the oriented $I$-bundle $J(F)$ over $F$ into $H$ which maps $\alpha$ to $\gamma$ and which maps $F_\alpha$ onto the complement in $X$ of a tubular neighborhood of the boundary $\partial X$ of $X$.

We call the surface $F$ the base of the $I$-bundle generated by $\gamma$.

**Example 3.4.**

1) An orientable $I$-bundle over an orientable base is a trivial bundle. Thus if $\partial H$ admits an $I$-bundle generator $\gamma$ with orientable base surface $F$ then the genus $g$ of $\partial H$ is even and equals twice the genus of $F$. The $I$-bundle over every essential arc in $F$ with endpoints in $\partial F$ is an embedded disc in $H$. There is an orientation reversing involution $\Phi : H \to H$ whose fixed point set intersects $\partial H$ precisely in $\gamma$. This involution acts as a reflection in the fiber. The union of any essential arc $\beta$ in $F$ with endpoints in $\partial F$ with its image under $\Phi$ is the boundary of a disc in $H$ (there is a small abuse of notation here since the fixed point set of $\Phi$ intersects $\partial H$ in a subset of the fibre over $\partial F$). This disc is just the $I$-bundle over the arc $\beta$.

2) Let $F$ be an oriented surface of genus $k \geq 1$ with two boundary components. The oriented $I$-bundle $J(F) = F \times [0, 1]$ over $F$ is homeomorphic to a handlebody $H$ of genus $2k + 1$. A boundary component $\beta$ of $F$ is neither discbounding nor discbusting in $H$, and the subsurface $X = \partial H - \beta \subset \partial H$ is thick. The second boundary component $\gamma$ of $F$ intersects every disc with boundary in $X$ and is an $I$-bundle generator for $X$ whose base is the surface $F$. The image of $F \times [0, 1] = J(F)$ under the embedding $J(F) \to H$ is the complement of a neighborhood of $\beta$ in $H$ which is homeomorphic to a solid torus.

3) Let $F$ be the connected sum of $g$ copies of the real projective plane with a disc. The orientable $I$-bundle over $F$ is a handlebody $H$ of genus $g$. The vertical boundary of the $I$-bundle is an annulus whose core curve $\gamma$ is non-separating. The complement of the annulus is the two-sheeted orientation cover of $F$. The $I$-bundle over any simple arc in $F$ with both endpoints on the boundary of $F$ is an embedded disc in $H$.

4) Let $\gamma$ be a non-separating $I$-bundle generator for a proper thick subsurface $X$ of $\partial H$, with base $F$. Then $F$ is non-orientable. Up to isotopy, the thick subsurface $X$ of $\partial H$ is the intersection of the boundary $\partial J(F)$ of the bundle $J(F) \subset H$ with $\partial H$. It can be obtained from the orientation cover $\tilde{F}$ of $F$ by gluing an annulus to the two preimages of the preferred boundary component $\alpha$ of $F$ with a homeomorphism which reverses the boundary orientations. The $I$-bundle over every essential arc $\beta$ in $F$ with endpoints on $\alpha$ is a disc in $H$. Its boundary is the preimage of $\beta$ in $F_\alpha \subset \partial J(F)$, viewed as the orientation cover of $F$ (here we use the same small abuse of terminology as before).

For a thick subsurface $X$ of $\partial H$ let $SDG(X)$ be the graph whose vertices are discs with boundaries contained in $X$ and where two such discs $D, E$ are connected by an edge of length one if one of the following two possibilities is satisfied.

1. There is an essential simple closed curve $\alpha \subset X$ (i.e. which is essential as a curve in the subsurface $X$ of $\partial H$) which is disjoint from $D \cup E$ (for example, $\partial D$ if $D, E$ are disjoint).
2. There is an $I$-bundle generator $\gamma \subset X$ which intersects both $D, E$ in precisely two points.

We denote by $d_{S,X}$ the distance in $SDG(X)$. If $X = \partial H$ then we simply write $d_S$ instead of $d_{S,\partial H}$. 

The following was proved in [H11] in the case $X = \partial H$. The proof of the result carries over to an arbitrary thick subsurface without modification.

**Proposition 3.5.** Let $X \subset \partial H$ be a thick subsurface. The vertex inclusion defines a quasi-isometric embedding of $SDG(X)$ into the curve graph of $X$. In particular, $SDG(X)$ is $\delta$-hyperbolic for a number $\delta > 0$ only depending on the genus of $H$.

4. **Hyperbolicity of the electrified disc graph**

As in Section 3, we consider a handlebody $H$ of genus $g \geq 2$, with boundary $\partial H$. The goal of this section is to use Theorem 1 to show hyperbolicity of the electrified disc graph $EDG(X)$ of a thick subsurface $X$ of $\partial H$. We also determine the Gromov boundary of $EDG(X)$.

Thus let $X \subset \partial H$ be a thick subsurface. Recall that $X$ is connected, and by the remark after Lemma 3.1, $X$ is distinct from a sphere with at most four holes and from a torus with a single hole. Denote by $d_{CG,X}$ the distance in the curve graph $CG$ of $X$, by $d_{SDG,X}$ the distance in the graph $SDG(X)$ and by $d_{E,X}$ the distance in the electrified disc graph $EDG(X)$ of $X$.

If $X$ does not contain any $I$-bundle generator then $EDG(X) = SDG(X)$ and there is nothing to show. Thus assume that there is an $I$-bundle generator $\gamma \subset X$. Let $E(\gamma) \subset EDG(X)$ be the complete subgraph of $EDG(X)$ whose vertices are discs intersecting $\gamma$ in precisely two points. Define $E = \{E(\gamma) \mid \gamma \}$ where $\gamma$ runs through all $I$-bundle generators in $X$. By definition, $SDG(X)$ is $2$-quasi-isometric to the $E$-electrification of $EDG(X)$. Thus by Theorem 2.4 to show hyperbolicity of $EDG(X)$ it suffices to show that each of the graphs $E(\gamma)$ is $\delta$-hyperbolic for a universal number $\delta > 0$ and that the bounded penetration property holds true.

We begin with establishing hyperbolicity of the graphs $E(\gamma)$. To this end, for a compact (not necessarily orientable) surface $F$ with boundary $\partial F$ and for a fixed boundary component $\alpha$ of $F$, define the electrified arc graph $C'_F(\alpha)$ as follows. Vertices of $C'_F(\alpha)$ are essential embedded arcs in $F$ with both endpoints on $\alpha$ or essential simple closed curves in $F$. Two such arcs or curves are connected by an edge of length one if they are disjoint or if they are disjoint from a common essential simple closed curve. If $F$ is non-orientable, then we require that an essential simple closed curve does not bound a Moebius band in $F$.

The following statement is well known but hard to find in the literature. We give a proof for completeness.

**Lemma 4.1.** Let $F$ be a compact surface with boundary $\partial F$. Assume that $F$ is not a sphere with at most three holes or a projective plane with at most three holes. Let $\alpha$ be a boundary circle of $F$. Then $C'_F(\alpha)$ is 4-quasi-isometric to the curve graph of $F$.

**Proof.** Define the arc and curve graph $A_F(\alpha)$ of $F$ to be the graph whose vertices are arcs with endpoints on $\alpha$ or essential simple closed curves in $F$. Two such arcs or curves are connected by an edge of length one if they can be realized disjointly.

Consider first the case that $F$ either is a one-holed torus, a one-holed Klein bottle, a four holed sphere or a four-holed projective plane. In this case two simple closed curves in $F$ are connected by an edge in the curve graph of $F$ if they intersect in the minimal number of points (one or two). Let $\beta$ be an essential simple closed curve in $F$. Cutting $F$ open along $\beta$ yields a three-holed sphere (if $F$ is a one-holed torus or a one-holed Klein bottle), the disjoint union of two three holed spheres (if
$F$ is a four-holed sphere) or the disjoint union of a three holed sphere and a three holed projective plane (if $F$ is a four-holed projective plane).

Thus there is a unique essential arc $\Lambda(\beta) \subset F$ with endpoints on $\alpha$ which is disjoint from $\beta$. The distance between two essential simple closed curves $\beta, \gamma$ in the curve graph of $F$ equals one if and only if the arcs $\Lambda(\beta), \Lambda(\gamma)$ are disjoint. This means that the map $\Lambda$ which associates to a simple closed curve $\beta$ in $F$ the unique arc $\Lambda(\beta)$ with endpoints on $\alpha$ which is disjoint from $\beta$ defines an isometry of the curve graph of $F$ onto the arc graph of $(F, \alpha)$. This arc graph is the complete subgraph of $\mathcal{A}(F, \alpha)$ whose vertex set consists of arcs with endpoints on $\alpha$. Moreover, in the special case at hand, this arc graph is just the graph $C'(F, \alpha)$. This yields the statement of the lemma for one-holed tori, one-holed projective planes, four holed spheres and four-holed projective planes.

Now assume that the surface $F$ is such that two vertices in the curve graph of $F$ are connected by an edge if they can be realized disjointly. Then for any two disjoint essential simple closed curves $\beta, \gamma$ in $F$ there is an essential arc with endpoints on $\alpha$ which is disjoint from both $\beta, \gamma$. In particular, for every simplicial path $c$ in the arc and curve graph $\mathcal{A}(F, \alpha)$ connecting two vertices in $\mathcal{A}(F, \alpha)$ which are arcs with endpoints on $\alpha$, there is a path of at most double length in $C'(F, \alpha)$ connecting the same endpoints. This path can be obtained from $c$ as follows. If $c(i), c(i + 1)$ are both simple closed curves then replace $c[i, i + 1]$ by a simplicial path in $\mathcal{A}(F, \alpha)$ of length 2 with the same endpoints whose midpoint is an arc disjoint from $c(i), c(i + 1)$. In the resulting path, a simple closed curve $\beta \subset F$ is adjacent to two arcs disjoint from $\beta$ and hence we can view this path as a path in $C'(F, \alpha)$. Thus the vertex inclusion $C'(F, \alpha) \to \mathcal{A}(F, \alpha)$ is a quasi-isometry.

We are left with showing that $\mathcal{A}(F, \rho)$ is quasi-isometric to the curve graph of $F$. However, this is well known, and the proof will be omitted. □

A thick subsurface $X$ of $\partial H$ is not a four-holed sphere. Thus if $\gamma$ is a separating $I$-bundle generator for $X$ then the base of the $I$-bundle either has positive genus or is a sphere with at least four holes. Similarly, if $\gamma$ is a non-separating $I$-bundle generator for $X$ then we may assume that the base $F$ of the $I$-bundle is not a projective plane with three holes. Namely, if $F$ is a projective plane with three holes and if $\alpha$ is a distinguished boundary component of $F$ then there is up to homotopy a unique essential arc $\beta$ in $F$ with boundary on $\alpha$. The $I$-bundle over $\beta$ is the unique disc in the oriented $I$-bundle over $F$ which intersects the curve $\alpha$ in precisely two points.

We use Lemma 4.1 to verify hyperbolicity of the subgraphs $\mathcal{E}(\gamma)$. For the formulation of the following lemma, for an $I$-bundle generator $\gamma$ in a thick subsurface $X$ of $\partial H$, with base surface $F$, denote again by $\gamma$ the distinguished boundary component of $F$. A disc $D \subset H$ with boundary $\partial D \subset X$ which intersects $\gamma$ in precisely two points is an $I$-bundle over a simple arc $\beta \subset F$ with boundary on $\gamma$. We call $\beta$ the projection of $\partial D$ to $F$. With these notations we show.

**Lemma 4.2.** Let $X \subset \partial H$ be a thick subsurface and let $\gamma$ be an $I$-bundle generator in $X$, with base surface $F$. Then the map which associates to a disc $D \in \mathcal{E}(\gamma)$ the projection of $\partial D$ to $F$ extends to a 2-quasi-isometry of $\mathcal{E}(\gamma)$ onto the electrified arc graph $C'(F, \gamma)$ of $F$.

**Proof.** Let $\gamma$ be an $I$-bundle generator in $X$ and let $F$ be the base surface of the $I$-bundle generated by $\gamma$. Let $V$ be the oriented $I$-bundle over $F$ as in the definition
of an $I$-bundle generator and let $\Psi : V \rightarrow H$ be a corresponding embedding. Up to isotopy, we have $\Psi(\partial V) \cap \partial H = X$. There is an orientation reversing bundle involution $\Phi$ of $V$ which exchanges the endpoints of the fibres. The involution preserves $X \subset \partial V$ and the curve $\gamma$. The quotient of $X$ under this involution equals the base surface $F$ of the $I$-bundle. The projection of $\gamma$ is the distinguished boundary component of $F$, again denoted by $\gamma$.

Up to isotopy, if the boundary $\partial D$ of a disc $D$ in $H$ is contained in $X$ and intersects the curve $\gamma$ in precisely two points then $\partial D$ is invariant under the involution $\Phi$. Thus the map $\Theta : V(C'(F, \gamma)) \rightarrow V(\mathcal{E}(\gamma))$ which associates to an arc $\beta$ in $F$ with endpoints on $\gamma$ the $I$-bundle over $\beta$ is a bijection. Here $V(C'(F, \gamma))$ (or $V(\mathcal{E}(\gamma))$) is the set of vertices of $C'(F, \gamma)$ (or $\mathcal{E}(\gamma)$).

If $\alpha, \beta \in V(C'(F, \gamma))$ are connected by an edge then either $\alpha, \beta$ are disjoint and so are $\Theta(\alpha), \Theta(\beta)$, or $\alpha, \beta$ are disjoint from an essential simple closed curve $\rho$ in $F$ and therefore the discs $\Theta(\alpha), \Theta(\beta)$ are disjoint from $\rho \subset X$. Thus $\Theta$ extends to a 1-Lipschitz map $C'(F, \gamma) \rightarrow \mathcal{E}(\gamma)$.

We are left with showing that $\Theta^{-1} : V(\mathcal{E}(\gamma)) \rightarrow V(C'(F, \gamma))$ is 2-Lipschitz where $V(C(\gamma))$ and $V(C'(F, \gamma))$ are equipped with the restriction of the metric on $C(\gamma), C'(F, \gamma)$. To this end let $\alpha, \beta \in V(C'(F, \gamma))$ be such that $\Theta(\alpha), \Theta(\beta)$ are connected by an edge in $\mathcal{E}(\gamma)$. If $\Theta(\alpha), \Theta(\beta)$ are disjoint then the same holds true for $\alpha, \beta$ and hence $\alpha, \beta$ are connected by an edge in $C'(F, \gamma)$.

Otherwise $\Theta(\alpha), \Theta(\beta)$ are disjoint from an essential simple closed curve $\rho$.

The boundaries $\partial \Theta(\alpha), \partial \Theta(\beta)$ of the discs $\Theta(\alpha), \Theta(\beta)$ are invariant under the involution $\Phi$ and therefore $\partial \Theta(\alpha) \cup \partial \Theta(\beta)$ is disjoint from $\rho \cup \Phi(\rho)$. As a consequence, the projection of $\rho$ to the base surface $F$ is a union of essential arcs with boundary on $\gamma$ and closed curves (not necessarily simple) which are disjoint from $\alpha \cup \beta$. Then either there is a simple arc in $F$ with endpoints on $\gamma$ which is disjoint from $\alpha \cup \beta$, or there is an essential simple closed curve in $F$ which is disjoint from $\alpha \cup \beta$. Thus the distance in $C'(F, \gamma)$ between $\alpha \cup \beta$ is at most two. The lemma follows.

From Lemma 4.2, Lemma 4.1 and hyperbolicity of the curve graph of $X$ ([MM99], and [BF07] for the curve graph of a non-orientable surface) we immediately obtain

**Corollary 4.3.** There is a number $\delta > 0$ such that each of the graphs $\mathcal{E}(\gamma)$ is $\delta$-hyperbolic.

Note that the number $\delta > 0$ in the statement of the corollary only depends on $H$ but not on $X$. In fact, the main result of [HPW13] together with Lemma 4.2 shows that it can even be chosen independent of $H$.

We are left with the verification of the bounded penetration property. To this end recall from [MM00] the definition of a subsurface projection. Namely, let again $X \subset \partial H$ be a thick subsurface and let $Y \subset X$ be an essential, open connected subsurface which is distinct from $X$, a three-holed sphere and an annulus. We call such a subsurface $Y$ a proper subsurface of $X$. The arc and curve graph $\mathcal{AC}(Y)$ of $Y$ (here we do not specify a boundary component) is the graph whose vertices are isotopy classes of arcs with endpoints on $\partial Y$ or essential simple closed curves in $Y$, and two such vertices are connected by an edge of length one if they can be realized disjointly. The vertex inclusion of the curve graph of $Y$ into the arc and curve graph is a quasi-isometry [MM00].
There is a projection $\pi_Y$ of the curve graph $\mathcal{CG}(X)$ of $X$ into the space of subsets of $\mathcal{AC}(Y)$ which associates to a simple closed curve in $X$ the homotopy classes of its intersection components with $Y$. For every simple closed multicurve $c$, the diameter of $\pi_Y(c)$ in $\mathcal{AC}(Y)$ is at most one. If $c$ can be realized disjointly from $Y$ then $\pi_Y(c) = \emptyset$.

As before, call a path $\rho$ in a metric graph $G$ simplicial if $c$ maps each interval $[k, k+1]$ (where $k \in \mathbb{Z}$) isometrically onto an edge of $G$. The following lemma is a version of Theorem 3.1 of [MM00].

**Lemma 4.4.** For every number $L > 1$ there is a number $\xi(L) > 0$ with the following property. Let $Y$ be a proper subsurface of $X$ and let $\rho$ be a simplicial path in $\mathcal{CG}(X)$ which is an $L$-quasi-geodesic. If $\pi_Y(v) \neq \emptyset$ for every vertex $v$ on $\rho$ then

$$\text{diam}(\pi_Y(\rho)) < \xi(L).$$

Moreover, $\xi(L) \leq ML^3 + M$ for a universal constant $M > 0$.

**Proof.** By hyperbolicity, for every $L > 1$ there is a number $n(L) > 0$ so that for every $L$-quasi-geodesic $\rho$ in $\mathcal{CG}(X)$ of finite length, the Hausdorff distance between the image of $\rho$ and the image of a geodesic $\rho'$ with the same endpoints does not exceed $n(L)$. Indeed, there is a number $k > 0$ only depending on the hyperbolicity constant for $\mathcal{CG}(X)$ such that we can choose $n(L) = kL^2$ (Proposition III.H.1.7 in [BH99]).

Now let $Y \subset X$ be a proper subsurface. By Theorem 3.1 of [MM00], there is a number $M > 0$ with the following property. If $\zeta$ is any simplicial geodesic in $\mathcal{CG}(X)$ and if $\pi_Y(\zeta(s)) \neq \emptyset$ for all $s \in \mathbb{Z}$ in the domain of $\zeta$ then

$$\text{diam}(\pi_Y(\zeta)) \leq M.$$ 

Let $L > 1$, let $\rho : [0, k] \to \mathcal{CG}(X)$ be a simplicial path which is an $L$-quasi-geodesic and assume that

$$\text{diam}(\pi_Y(\rho(0) \cup \rho(k))) \geq 2M + L(2n(L) + 4).$$

Our goal is to show that $\rho$ passes through the set $A \subset \mathcal{CG}(X)$ of all essential simple closed curves in $X - Y$. The diameter of $A$ in $\mathcal{CG}(X)$ is at most two.

To this end let $\rho'$ be a simplicial geodesic in $\mathcal{CG}(X)$ with the same endpoints as $\rho$. Theorem 3.1 of [MM00] shows that there is some $u \in \mathbb{Z}$ such that $\rho'(u) \in A$. Then $\rho$ passes through the $n(L)$-neighborhood of $A$.

Let $s + 1 \leq t - 1$ be the smallest and the biggest number, respectively, so that $\rho(s + 1), \rho(t - 1)$ are contained in the $n(L)$-neighborhood of $A$. Then $\rho[0, s]$ (or $\rho[t, k]$) is contained in the complement of the $n(L)$-neighborhood of $A$. Since $\rho$ is an $L$-quasi-geodesic, a geodesic connecting $\rho(0)$ to $\rho(s)$ (or connecting $\rho(t)$ to $\rho(k)$) is contained in the $n(L)$-neighborhood of $\rho[0, s]$ (or of $\rho[t, k]$) and hence it does not pass through $A$. In particular,

$$\text{diam}(\pi_Y(\rho(0) \cup \rho(s))) \leq M \text{ and } \text{diam}(\pi_Y(\rho(t) \cup \rho(k))) \leq M.$$ 

As a consequence, we have

$$\text{(2)} \quad \text{diam}(\pi_Y(\rho(s) \cup \rho(t))) \geq L(2n(L) + 4).$$

Since $d_{\mathcal{CG},X}(\rho(s + 1), A) \leq n(L)$ and $d_{\mathcal{CG},X}(\rho(t - 1), A) \leq n(L)$ and since the diameter of $A$ is at most 2, we obtain $d_{\mathcal{CG},X}(\rho(s), \rho(t)) \leq 2n(L) + 2$. Now $\rho$ is a simplicial $L$-quasi-geodesic in $\mathcal{CG}(X)$ and hence the length $t - s$ of $\rho[s, t]$ is at most $L(2n(L) + 2) + L = L(2n(L) + 3)$. For all $\ell \in \mathbb{Z}$ the curves $\rho(\ell), \rho(\ell + 1)$
are disjoint and therefore if \( \rho(\ell), \rho(\ell + 1) \) both intersect \( Y \) then the diameter of \( \pi_Y(\rho(\ell) \cup \rho(\ell + 1)) \) is at most one. Thus if \( \rho(\ell) \) intersects \( Y \) for all \( \ell \)
\[
\text{diam}(\pi_Y(\rho(s) \cup \rho(t))) \leq L(2n(L) + 3).
\]
This contradicts inequality 2 and completes the proof of the lemma. \( \square \)

For simplicity of notation, in the remainder of this section we identify discs in \( H \) with their boundaries. In other words, for a thick subsurface \( X \) of \( \partial H \) we view the vertex sets of the graphs \( SDG(X), EDG(X) \) as subsets of the vertex set of the curve graph \( CG(X) \) of \( X \).

Let \( SDG_0(X) \) be the \( \mathcal{E} \)-electrification of \( EDG(X) \). For each \( I \)-bundle generator \( \gamma \) in \( X \), the graph \( SDG_0(X) \) contains a special vertex \( v_\gamma \). The vertex set of \( SDG_0(X) \) is the union of the set of all discbounding simple closed curves in \( X \) with the set \( \{v_\gamma \mid \gamma \} \). In particular, there is a natural vertex inclusion \( \mathcal{V}(SDG_0(X)) \to CG(X) \) which maps the special vertex \( v_\gamma \) to the simple closed curve \( \gamma \). Since \( SDG(X) \) is quasi-isometric to the \( \mathcal{E} \)-electrification of \( EDG(X) \), Proposition 3.5 shows that this vertex inclusion extends to a quasi-isometric embedding \( SDG_0(X) \to CG(X) \).

Now we are ready to show

**Lemma 4.5.** For every thick subsurface \( X \) of \( \partial H \) the family \( \mathcal{E} \) has the bounded penetration property.

**Proof.** Let \( L \geq 1 \) and let \( \rho : [0, n] \to SDG_0(X) \) be an efficient simplicial \( L \)-quasi-geodesic. Let \( \tilde{\rho} \) be a simplicial arc in \( CG(X) \) which is obtained from \( \rho \) as follows.

A vertex \( \rho(j) \) in \( SDG_0(X) \) which is not one of the special vertices \( v_\gamma \) also defines a vertex in \( CG(X) \). If \( \rho(j), \rho(j + 1) \) are two such vertices which are connected in \( SDG_0(X) \) by an edge then they are connected in \( EDG(X) \subset SDG_0(X) \) by an edge. By the definition of the electrified disc graph, this means that there is a simple closed curve \( \alpha \) in \( X \) which is disjoint from \( \rho(j) \cup \rho(j + 1) \). Thus \( \rho(j) \) and \( \rho(j + 1) \) can be connected in \( CG(X) \) by an edge path of length at most two.

Similarly, if \( \rho(j) = v_\gamma \) for an \( I \)-bundle generator \( \gamma \) in \( X \), then \( \rho(j - 1), \rho(j + 1) \in EDG(X) \), moreover \( \rho(j - 1), \rho(j + 1) \) intersect \( \gamma \) in precisely two points. Replace \( \rho(j - 1, j + 1) \) by an edge path in \( CG(X) \) with the same endpoints of length at most four which passes through \( \gamma \). The arc \( \tilde{\rho} \) constructed in this way from \( \rho \) is a uniform quasi-geodesic in \( CG(X) \) which passes through any \( I \)-bundle generator \( \gamma \) at most once, and it passes through \( \gamma \) if and only if it passes through a simple closed curve which is disjoint from \( \gamma \).

Let \( \gamma \) be a separating \( I \)-bundle generator in \( X \). Then \( X - \gamma \) has two homeomorphic components \( X_1, X_2 \). Denote by \( d_{AC,X} \), the distance in the arc and curve graph of \( X_i \) \((i = 1, 2)\). Every simple closed curve \( \alpha \) in \( X \) which has an essential intersection with \( \gamma \) projects to a collection of arcs \( \alpha_1, \alpha_2 \) in \( X_1, X_2 \). If \( \beta \) is another such curve then define
\[
d_{AC(X-\gamma)}(\alpha, \beta) = \min\{d_{AC(X_1)}(\alpha_1, \beta_1), d_{AC(X_2)}(\alpha_2, \beta_2)\}.
\]
Thus if \( \pi^\gamma : CG(X) \to AC(X - \gamma) = AC(X_1) \cup AC(X_2) \) denotes the subsurface projection then by Lemma 4.4 there is a number \( M(L) > 0 \) with the following property.

Let again \( \rho : [0, n] \to SDG_0(X) \) be a simplicial \( L \)-quasi-geodesic. If
\[
d_{AC(X-\gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(n))) \geq M(L)
\]
then there is some \( k_0 \in \mathbb{Z} \) such that \( \rho(k_0) = \gamma \). Equivalently, there is some \( k < n \) such that \( \rho(k) = v_\gamma \). Moreover,

\[
d_{\mathcal{AC}(X, i)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(k - 1))) \leq M(L) \quad (i = 1, 2),
\]

and similarly

\[
d_{\mathcal{AC}(X, i)}(\pi^\gamma(\rho(k + 1)), \pi^\gamma(\rho(n))) \leq M(L) \quad (i = 1, 2).
\]

As a consequence, if \( \rho': \mathbb{R} \to \mathcal{S}_{\mathcal{DG}}_0(X) \) is another efficient quasi-geodesic with the same endpoints, then there is some \( k' < n' \) such that \( \rho'(k') = v_\gamma \), and

\[
d_{\mathcal{AC}(X - \gamma)}(\pi^\gamma(\rho(k - 1)), \pi^\gamma(\rho'(k' - 1))) \leq 2M(L),
\]

\[
d_{\mathcal{AC}(X - \gamma)}(\pi^\gamma(\rho(k + 1)), \pi^\gamma(\rho'(k' + 1))) \leq 2M(L).
\]

Lemma 4.2 and Lemma 4.4 now show that the distance in \( \mathcal{E}(\gamma) \) between \( \rho(k - 1), \rho'(k' - 1) \) and between \( \rho(k + 1), \rho'(k' + 1) \) is uniformly bounded. In particular, the bounded penetration property holds true for the subgraph \( \mathcal{E}(\gamma) \) and for quasi-geodesics connecting two discs whose boundaries have projections of large diameter into \( X - \gamma \).

On the other hand, if \( \rho: \mathbb{R} \to \mathcal{S}_{\mathcal{DG}}_0(X) \) is any efficient \( \mathcal{L} \)-quasi-geodesic and if \( \rho(k) = v_\gamma \) for some \( I \)-bundle generator \( \gamma \) then using once more Lemma 4.4 we conclude that

\[
d_{\mathcal{AC}(X - \gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(k - 1))) \leq M(L).
\]

Therefore the reasoning in the previous paragraph shows that whenever the distance in \( \mathcal{E}(\gamma) \) between \( \rho(k - 1), \rho(k + 1) \) is sufficiently large then

\[
d_{\mathcal{AC}(X - \gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(n))) \geq M(L).
\]

In other words, the conclusion in the previous paragraph holds true, and the bounded penetration property for separating \( I \)-bundle generators follows.

Now assume that \( \gamma \) is non-separating. Let \( \pi^\gamma: \mathcal{CG}(X) \to \mathcal{AC}(X - \gamma) \) be the subsurface projection. Using the notations from the beginning of this proof, if the distance in \( \mathcal{AC}(X - \gamma) \) between \( \pi^\gamma(\rho(0)) \) and \( \pi^\gamma(\rho(n)) \) is at least \( M(L) \) then there is some \( k \) so that \( \rho(k) = v_\gamma \). Moreover, we have \( \rho(k - 1) \in \mathcal{E}(\gamma), \rho(k + 1) \in \mathcal{E}(\gamma) \).

As a consequence, the curves \( \rho(k - 1), \rho(k + 1) \) are invariant under the orientation reversing involution \( \varphi \) of \( X \) which preserves \( \gamma \) and extends to an involution of the \( I \)-bundle defined by \( \gamma \).

Let \( F \) be the base of the \( I \)-bundle defined by \( \gamma \) and let \( \alpha, \beta \in C'(F, \gamma) \) be the projections of \( \rho(k - 1), \rho(k + 1) \). By Lemma 4.2 the distance in \( \mathcal{E}(\gamma) \) between \( \rho(k - 1), \rho(k + 1) \) is uniformly equivalent to the distance in \( C'(F, \gamma) \) between \( \alpha, \beta \).

Since \( \rho(k - 1), \rho(k + 1) \) are invariant under the involution \( \varphi \), the main result of [RS09] shows that this distance is also uniformly equivalent to the distance between \( \pi^\gamma(\rho(k - 1)) \) and \( \pi^\gamma(\rho(k + 1)) \) in \( \mathcal{AC}(X - \gamma) \).

In particular, if \( \rho' \) is any other efficient \( \mathcal{L} \)-quasi-geodesic in \( \mathcal{S}_{\mathcal{DG}}_0(X) \) with the same endpoints, then there is some \( k' \) with \( \rho(k') = v_\gamma \), and the distance in \( \mathcal{E}(\gamma) \) between \( \rho(k - 1), \rho'(k' - 1) \) and between \( \rho(k + 1) \) and \( \rho'(k' + 1) \) is uniformly bounded.

The bounded penetration property follows in this case.

Finally, as in the case of a separating \( I \)-bundle generator, this argument can be inverted. Together this completes the proof of the lemma.

\[\square\]

We can now apply Theorem 2.4 to conclude
Corollary 4.6. For every thick subsurface $X$ of $\partial H$, the graph $\mathcal{EDG}(X)$ is $\delta$-hyperbolic for a number $\delta > 0$ not depending on $X$. There is a number $k > 0$ such that for every $I$-bundle generator $\gamma$ in $X$, the subgraph $\mathcal{E}(\gamma)$ of $\mathcal{EDG}(X)$ is $k$-quasi-convex.

In the remainder of this section, we specialize to the case $X = \partial H$. We begin with establishing a distance estimate for the electrified disc graph $\mathcal{EDG} = \mathcal{EDG}(\partial H)$.

If $\gamma$ is an $I$-bundle generator in $\partial H$ then let $\pi_\gamma$ be the subsurface projection of a simple closed curve in $\partial H$ into the arc and curve-graph of $\partial H - \gamma$.

For a subset $A$ of a metric space $Y$ and a number $C > 0$ define $\text{diam}(A)$ to be the diameter of $A$ if this diameter is at least $C$ and let $\text{diam}(A) = 0$ otherwise.

The notation $\asymp$ means equality up to a universal multiplicative constant.

Corollary 4.7. Let $H$ be a handlebody of genus $g \geq 2$. Then there is a number $C > 0$ such that
\[
\text{d}_\mathcal{E}(D, E) \asymp \text{d}_\mathcal{CG}(\partial D, \partial E) + \sum_\gamma \text{diam}(\pi_\gamma(\partial D \cup \partial E))C
\]
where $\gamma$ passes through all $I$-bundle generators on $\partial H$.

Proof. Let $\mathcal{SDG}_0$ be the $\mathcal{E}$-electrification of $\mathcal{EDG}$. For an $I$-bundle generator $\gamma$ in $\partial H$ denote by $v_\gamma$ the special vertex in $\mathcal{SDG}_0$ defined by $\gamma$.

Let $\rho : [0, k] \to \mathcal{SDG}_0$ be a geodesic. By Corollary 2.8 and Corollary 4.6, an enlargement $\hat{\rho}$ of $\rho$ is a uniform quasi-geodesic in $\mathcal{EDG}$. Thus it suffices to show that the length of $\hat{\rho}$ is uniformly comparable to the right hand side of the formula in the corollary.

By Proposition 3.5 there is a number $L > 1$ such that a simplicial arc $\tilde{\rho}$ in $\mathcal{CG}$ constructed from $\rho$ as in the proof of Lemma 4.5 is an $L$-quasi-geodesic in the curve graph $\mathcal{CG}$ of $\partial H$. Lemma 4.5 shows that if $\hat{\rho}$ is an enlargement of $\rho$ then the diameter of the intersection of $\hat{\rho}$ with $\mathcal{E}(c)$ equals the diameter of $\pi_{\partial H - c}(\gamma(0) \cup \gamma(k))$ up to a universal multiplicative and additive constant. This is what we wanted to show.

We complete this section with determining the Gromov boundary of the electrified disc graph of $H$. To this end let $H$ be a handlebody of genus $g \geq 2$. Let $\mathcal{L}$ be the space of all geodesic laminations on $\partial H$ equipped with the coarse Hausdorff topology $[H_06]$. In this topology, a sequence of laminations $\lambda_i$ converges to $\lambda$ if every accumulation point of $(\lambda_i)$ in the usual Hausdorff topology for compact subsets of $\partial H$ contains $\lambda$ as a sublamination. Let

$\mathcal{H} \subset \mathcal{L}$

be the subspace of all minimal laminations which fill up $\partial H$, i.e. such that complementary components are simply connected, and which are limits in the coarse Hausdorff topology of disbounding simple closed curves.

For an $I$-bundle generator $\gamma$ let $\partial \mathcal{E}(\gamma) \subset \mathcal{L}$ be the set of all geodesic laminations which consist of two minimal components filling up $\partial H - \gamma$ and which are limits in the coarse Hausdorff topology of boundaries of discs contained in $\mathcal{E}(\gamma)$. Each lamination $\mu \in \mathcal{E}(\gamma)$ is invariant under the orientation reversing involution $\Phi_\gamma$ of $\partial H$ which fixes $\gamma$ pointwise and exchanges the endpoints of the fibres of the defining $I$-bundle.
Define
\[ \partial \mathcal{EDG} = \partial \mathcal{H} \cup \bigcup_{\gamma} \partial \mathcal{E}(\gamma) \subset \mathcal{L} \]
where the union is over all \(I\)-bundle generators \(\gamma \subset \partial \mathcal{H}\). Then \(\partial \mathcal{EDG}\) is a \(\text{Map}(\mathcal{H})\)-space.

Proposition 2.10 can now be applied to show

**Proposition 4.8.** The Gromov boundary of \(\mathcal{EDG}\) can naturally be identified with \(\partial \mathcal{EDG}\).

**Proof.** We show first that the subspace \(\partial \mathcal{EDG}\) of \(\mathcal{L}\) is Hausdorff.

A point \(\lambda \in \partial \mathcal{EDG}\) either is a minimal geodesic lamination which fills up \(\partial \mathcal{H}\), or it is a geodesic lamination with two minimal components which fill up \(\partial \mathcal{H} - \gamma\) for some \(I\)-bundle generator \(\gamma\). Let \(\nu \neq \lambda\) be another such lamination. We claim that \(\nu\) and \(\lambda\) intersect. This means that for some fixed hyperbolic metric on \(\partial \mathcal{H}\), the geodesic representatives of \(\nu, \lambda\) intersect transversely.

If either \(\nu\) or \(\lambda\) fills up \(\partial \mathcal{H}\) (i.e. if the complementary components of \(\nu, \lambda\) are simply connected) then this is obvious. Otherwise \(\nu\) fills up the complement of an \(I\)-bundle generator \(\gamma\), and \(\lambda\) fills up the complement of an \(I\)-bundle generator \(\gamma'\). Now the simple closed curve \(\gamma\) is the only minimal geodesic lamination which does not intersect \(\nu\) and which is distinct from a component of \(\nu\). The lamination \(\lambda\) consists of two minimal components which are not simple closed curves and therefore the geodesic laminations \(\nu, \lambda\) indeed intersect.

Since \(\nu, \lambda \in \partial \mathcal{EDG}\) intersect, by the definition of the coarse Hausdorff topology there are neighborhoods \(U\) of \(\lambda\), \(V\) of \(\nu\) in \(\mathcal{L}\) so that any two laminations \(\nu' \in U, \nu' \in V\) intersect. In particular, the neighborhoods \(U, V\) are disjoint. This shows that \(\partial \mathcal{EDG}\) is Hausdorff.

Proposition 2.10 shows that there is a natural bijection between \(\partial \mathcal{EDG}\) and the Gromov boundary of \(\mathcal{EDG}\). That this bijection is in fact a homeomorphism follows from the description the Gromov boundary of the curve graph of \(\partial \mathcal{H}\) as discussed in \([K99, H06]\) and Proposition 2.10.

To be more precise, let \(\gamma\) be a separating \(I\)-bundle generator for \(\partial \mathcal{H}\). The orientation reversing involution \(\Phi\) of the \(I\)-bundle determined by \(\gamma\) restricts to a homeomorphism of \(\partial \mathcal{H} - \gamma\) which exchanges the two components of \(\partial \mathcal{H} - \gamma\). By Lemma 4.1 and Lemma 4.2, the graph \(\mathcal{E}(\gamma)\) can be identified with the graph of all simple closed curves \(\alpha\) in \(\mathcal{X}\) which intersect \(\gamma\) in precisely in two points and are invariant under \(\Phi\). Thus by \([K99, H06]\), the Gromov boundary of \(\mathcal{E}(\gamma)\) has a natural identification with the space of all \(\Phi\)-invariant geodesic laminations which consist of two minimal components, each of which fills a component of \(\partial \mathcal{H} - \gamma\). The topology on this space is the coarse Hausdorff topology. A similar description is valid for the Gromov boundary of \(\mathcal{E}(\gamma)\) where \(\gamma\) is an orientation reversing \(I\)-bundle generator.

Proposition 2.10 shows that the Gromov boundaries of the subspaces \(\mathcal{E}(\gamma)\) are embedded subspaces of the Gromov boundary of \(\mathcal{EDG}\). The Gromov boundary \(\mathcal{H}\) of \(\mathcal{SDG}\) is embedded in the Gromov boundary of \(\mathcal{EDG}\) as well. For every \(\xi \in \mathcal{H}\), a neighborhood basis of \(\xi\) in the Gromov boundary of \(\mathcal{EDG}\) consists of sets which are unions of a neighborhood of \(\xi\) in \(\mathcal{H}\) with sets \(\partial \mathcal{E}(\gamma)\) where the curves \(\gamma\) are contained in a neighborhood of \(\xi\) in \(\mathcal{CG} \cup \partial \mathcal{CG}\). By the description of neighborhood bases of \(\xi\) in \(\mathcal{CG} \cup \partial \mathcal{CG}\) as neighborhoods of \(\xi\) in lamination space, equipped with the coarse Hausdorff topology \([K99, H06]\), this completes the proof of the proposition. \(\square\)
5. Hyperbolicity of the disc graph

In this section we use Corollary 4.6 and Theorem 1 to give a new and simpler proof of the following result of Masur and Schleimer [MS13].

**Theorem 5.1.** The disc graph of a handlebody is hyperbolic.

The argument consists in an inductive application of Theorem 1 to electrified disc graphs of thick subsurfaces of $\partial H$. For technical reasons we slightly weaken the definition of a thick subsurface of $\partial H$ as follows.

Define a connected properly embedded subsurface $X$ of $\partial H$ to be visible if every discs intersects $X$ and if moreover $X$ contains the boundary of at least one disc. Thus a thick subsurface is visible, but a visible subsurface may not be filled by boundaries of discs and hence may not be thick. Note that if $X$ is visible then the electrified disc graph $\mathcal{EDG}(X)$ of $X$ is defined. However, if $X$ is not thick then its diameter equals one.

Let $\mathcal{DG}(X)$ be the disc graph of the visible subsurface $X$. Its vertices are discs with boundary in $X$, and two such discs are connected by an edge of length one if they are disjoint. The next lemma shows that if $X$ is a visible five holed sphere or two holed torus then $\mathcal{DG}(X)$ is hyperbolic.

**Lemma 5.2.** Let $X \subset \partial H$ be a visible subsurface which is a five-holed sphere or a two-holed torus. Then the vertex inclusion $\mathcal{DG}(X) \to \mathcal{EDG}(X)$ is a quasi-isometry.

**Proof.** Let $X \subset \partial H$ be a visible subsurface. Let $\rho : [0, k] \to \mathcal{EDG}(X)$ be a geodesic. By modifying $\rho$ while increasing its length by at most a factor of two we may assume that for each $i$, either $\rho(i)$ is disjoint from $\rho(i + 1)$, or there is an essential simple closed curve in $X$ which is not discbounding and which is disjoint from both $\rho(i), \rho(i + 1)$, but there is no discbounding curve in $X$ disjoint from both $\rho(i), \rho(i + 1)$.

If $X$ is a five-holed sphere then every simple closed curve $\gamma$ in $X$ is separating, and $X - \gamma$ is the disjoint union of a four holed sphere $X_1$ and a three holed sphere. Any two essential simple closed curves $\alpha, \beta$ in $X$ which are disjoint from $\gamma$ are contained in $X_1$. If $\gamma \subset X$ is not discbounding then $X_1$ is a four holed sphere whose boundary components are not discbounding. If $\gamma$ is disjoint from a discbounding simple closed curve then $X_1$ contains the boundary of a disc. By Remark 3.2, $X_1$ contains the boundary of precisely one disc. This implies that for all $i$ the disc $\rho(i)$ is disjoint from $\rho(i + 1)$ and therefore $\rho$ is in fact a simplicial path in $\mathcal{DG}(X)$.

Similarly, if $X$ is a two-holed torus then a simple closed curve $\gamma$ in $X$ either is non-separating and $X - \gamma$ is a four holed sphere, or $\gamma$ is separating and $\gamma$ decomposes $X$ into a three holed sphere and a one holed torus. Using again Remark 3.2, a one holed torus whose boundary is not discbounding contains the boundary of at most one disc. Thus the argument in the previous paragraph is valid in this situation as well and shows the lemma. $\square$

From now on let $X$ be a visible subsurface of $\partial H$ which is not a sphere with at most five holes or a torus with at most two holes. Our next goal is to show hyperbolicity of a graph $\mathcal{EDG}(2, X)$ whose vertices are isotopy classes of essential discs with boundary in $X$, and which can be obtained from $\mathcal{DG}(X)$ by adding edges and can be obtained from $\mathcal{EDG}(X)$ by removing edges. Namely, two discs $D, E$ are connected in $\mathcal{EDG}(2)$ by an edge of length one if either $D, E$ are disjoint
or if $\partial D, \partial E$ are disjoint from an essential multicurve $\beta \subset \partial X$ consisting of two components which are not freely homotopic.

Call a simple closed curve $\gamma$ in $X$ admissible if $\gamma$ has the following properties.

(1) $\gamma$ is neither discbounding nor discbusting.

(2) Either $\gamma$ is non-separating or $\gamma$ decomposes $X$ into a three-holed sphere $X_1$ and a visible second component $X_2$.

By assumption, $X$ is distinct from a sphere with at most five holes and a torus with at most two holes. We claim that if $\gamma \subset X$ is an admissible simple closed curve and if $\eta$ is any other simple closed curve then a tubular neighborhood of $\gamma \cup \eta$ contains an essential simple closed curve which is disjoint from $\gamma$.

To see this observe that if $\gamma \cap \eta = \emptyset$ then we may choose $\eta$ to be such a curve. If $\gamma \cap \eta \neq \emptyset$ let $\eta_0$ be a component of $\eta - \gamma$. In the case that $\gamma$ is separating we require that $\eta_0$ is not contained in the three-holed sphere component of $X - \gamma$. Then $\eta_0$ is contained in a component of $X - \gamma$ which neither is a three holed sphere nor a one holed torus. As a consequence, one of the boundary components of a tubular neighborhood of $\gamma \cup \eta_0$ is an essential simple closed curve in $X$ distinct from $\gamma$.

For an admissible simple closed curve $\gamma$ in $X$ define $\mathcal{H}(\gamma)$ to be the complete subgraph of $\mathcal{EDG}(2, X)$ whose vertex set consists of all discs which are disjoint from $\gamma$.

**Lemma 5.3.** $\mathcal{H}(\gamma)$ is isometric to $\mathcal{EDG}(X - \gamma)$.

**Proof.** A disc $D \in \mathcal{H}(\gamma)$ is disjoint from $\gamma$. Thus $D$ defines a vertex in $\mathcal{EDG}(X - \gamma)$. Two discs $D, E \in \mathcal{H}(\gamma)$ are connected by an edge in $\mathcal{EDG}(2, X)$ if and only if either they are disjoint or if there is a pair $\beta_1, \beta_2$ of disjoint essential simple closed curves in $X$ which are disjoint from both $D, E$.

If one of the curves $\beta_1, \beta_2$, say the curve $\beta_1$, is disjoint from $\gamma$, then by definition, $D, E$ viewed as vertices in $\mathcal{EDG}(X - \gamma)$ are connected by an edge. Otherwise by the remark preceding the lemma, there is an essential simple closed curve contained in a tubular neighborhood of $\gamma \cup \beta_1$ which is disjoint from $\gamma, D, E$ and once again, $D, E$ are connected by an edge in $\mathcal{EDG}(X - \gamma)$.

As a consequence, the vertex inclusion $\mathcal{H}(\gamma) \to \mathcal{EDG}(X - \gamma)$ extends to a 1-Lipschitz embedding. By definition, this embedding is surjective on vertices. By definition, any two vertices which are connected in $\mathcal{EDG}(X - \gamma)$ by an edge are also connected in $\mathcal{H}(\gamma)$ by an edge. In other words, the 1-Lipschitz embedding $\mathcal{H}(\gamma) \to \mathcal{EDG}(X - \gamma)$ is in fact an isometry. \qed

Lemma 5.3 and Corollary 1.6 imply that there is a number $\delta > 0$ so that each of the graphs $\mathcal{H}(\gamma)$ is $\delta$-hyperbolic.

Let $\mathcal{H} = \{\mathcal{H}(\gamma) \mid \gamma\}$ be the family of all these subgraphs of $\mathcal{EDG}(2, X)$ where $\gamma$ passes through all admissible curves in $X$. Our goal is to apply Theorem 2.4 to the family $\mathcal{H}$ of complete subgraphs of $\mathcal{EDG}(2, X)$. We first note.

**Lemma 5.4.** $\mathcal{EDG}(X)$ is quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{EDG}(2, X)$.

**Proof.** Let $\mathcal{G}$ be the $\mathcal{H}$-electrification of $\mathcal{EDG}(2, X)$. We show first that the vertex inclusion $\mathcal{EDG}(X) \to \mathcal{G}$ is coarsely Lipschitz.

To this end let $D, E$ are any two vertices in $\mathcal{EDG}(X)$ which are connected by an edge. Then either $D, E$ are disjoint, of they are disjoint from a common essential simple closed curve $\gamma$ in $X$.\end{document}
If $D, E$ are disjoint then $D, E$ viewed as vertices in $\mathcal{EDG}(2, X)$ are connected by an edge in $\mathcal{EDG}(2, X)$ as well.

Now assume that $D, E$ are disjoint from a common essential simple closed curve $\gamma$ in $X$. If $\gamma$ either is admissible or discbounding, then by the definition of the $H$-electrification of $\mathcal{EDG}(2, X)$, their distance in $\mathcal{G}$ is at most two. On the other hand, if $\gamma$ is neither admissible nor discbounding, then $\gamma$ is a separating simple closed curve in $X$. The surface $X - \gamma$ is a disjoint union of essential surfaces $X_1, X_2$ which are distinct from three-holed spheres. The boundaries of $D, E$ are contained in $X_1 \cup X_2$.

If $\partial D, \partial E$ are contained in distinct components of $X - \gamma$ then $D, E$ are disjoint and hence $D, E$ are connected by an edge in $\mathcal{EDG}(2, X)$. If $\partial D, \partial E$ are contained in the same component of $X - \gamma$, say in $X_1$, then the second component $X_2$ contains an essential simple closed curve $\eta$, and $\partial D, \partial E$ are disjoint from the multi-curve $\gamma \cup \eta$ with two components. Once more, this implies that $D, E$ are connected in $\mathcal{EDG}(2, X)$ by an edge. As a consequence, the vertex inclusion $\mathcal{EDG}(X) \to G$ is indeed coarsely Lipschitz.

That this inclusion is in fact a quasi-isometry is immediate from the definitions. Namely, if $\gamma \subset X$ is admissible then by the definition of $\mathcal{EDG}(X)$, any two vertices in $H(\gamma)$ are connected in $\mathcal{EDG}(X)$ by an edge. 

We use Lemma 5.4 and Theorem 2.4 to show hyperbolicity of $\mathcal{EDG}(2, X)$.

**Corollary 5.5.** $\mathcal{EDG}(2, X)$ is hyperbolic. Enlargements of geodesics in $\mathcal{EDG}(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(2, X)$.

**Proof.** It suffices to show that the family $\mathcal{H} = \{ H(\gamma) \mid \gamma \}$ satisfies the assumptions in Theorem 2.4.

For an admissible simple closed curve $\gamma \subset X$, $\delta$-hyperbolicity of $H(\gamma)$ for a number $\delta > 0$ not depending on $\gamma$ follows from Corollary 5.3 and Corollary 4.6.

To show the bounded penetration property, recall that enlargements of geodesics in $\mathcal{SDG}(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(X)$. Let $\gamma$ be an admissible simple closed curve and let $X_1$ be the component of $X - \gamma$ which is not a three-holed sphere. By Lemma 4.3, Lemma 4.5 and the proof of Corollary 4.6, an enlargement of a geodesic in $\mathcal{SDG}(X)$ passes through two points of large distance in $H(\gamma) = \mathcal{EDG}(X - \gamma)$ if and only if one of the following two possibilities holds true.

1. The diameter of the subsurface projection of the endpoints into the arc and curve graph of $X_1$ is large.
2. There is an $I$-bundle generator $\beta \subset X_1$ such that the diameter of the subsurface projection of the endpoints into the arc and curve graph of $X_1 - \beta$ is large.

From this the bounded penetration property follows as in the proof of Lemma 4.5. 

**Proof of Theorem 5.7.** For $k \geq 1$ define $\mathcal{EDG}(k)$ to be the graph whose vertex set is the set of all discs in $H$ and where two such discs are connected by an edge of length one if and only if either they are disjoint or they are both disjoint from a multicurve in $\partial H$ with a least $k$ components. Note that if $k$ equals the cardinality of a pants decomposition for $\partial H$ then $\mathcal{EDG}(k)$ equals the disc graph of $H$.

We show by induction on $k$ the following. The graph $\mathcal{EDG}(k)$ is hyperbolic, and there is a collection $\mathcal{H}$ of complete hyperbolic subgraphs of $\mathcal{EDG}(k)$ which satisfies
the hypothesis in Theorem 2.4 and such that the $H$-electrification of $EDG(k)$ is naturally quasi-isometric to $EDG(k - 1)$. In particular, enlargements of quasi-geodesics in $EDG(k - 1)$ are uniform quasi-geodesics in $EDG(k)$.

The case $k = 1$ is just Corollary 4.6 and the case $k = 2$ was shown in Corollary 5.5. Thus assume that the claim holds true for $k - 1 \in [2, 3g - 3)$. Let $D, E$ be any two discs in $H$. Let $\rho$ be a geodesic in $EDG(k - 1)$ connecting $D$ to $E$.

Let $i \geq 0$ be such that the discs $\rho(i), \rho(i + 1)$ are not disjoint. Then they are disjoint from a multicurve $\alpha$ in $\partial H$ with at least $k - 1$ components. We may assume that none of the components of $\alpha$ is discbounding. Since $\partial \rho(i), \partial \rho(i + 1)$ intersect they are contained in the same component $X$ of $\partial H - \alpha$. Then $X$ is a visible subsurface of $\partial H$.

If either $\partial H - X$ contains a multicurve with $k$ components or if $X - (\partial \rho(i) \cup \partial \rho(i + 1))$ contains an essential simple closed curve then $\rho(i)$ is connected to $\rho(i + 1)$ by an edge in $EDG(k)$. Otherwise replace the edge $\rho[i, i + 1]$ in $EDG(k - 1)$ by a geodesic $\rho_k^i$ in $EDG(X)$ with the same endpoints. The concatenation of these arcs is a curve $\rho_k$ which is an enlargement of $\gamma$. For all $j$, the discs $\gamma_k(j), \gamma_k(j + 1)$ either are disjoint or disjoint from a multicurve containing at least $k$ components which are not discbounding.

This process stops in the moment the component $X$ of $\partial H - \alpha$ is a five-holed sphere or a two-holed torus since by Lemma 5.2, for such a surface the vertex inclusion $DG(X) \to EDG(X)$ is a quasi-isometry. □

For a thick subsurface $Y$ of $\partial H$ denote as before by $\pi_Y$ the subsurface projection of simple closed curves into the arc and curve graph of $Y$. If $\gamma$ is an $I$-bundle generator in a thick subsurface $Y$ then let $\pi_\gamma$ be the subsurface projection into $Y - \gamma$.

The following corollary is now immediate from our construction. It was earlier obtained by Masur and Schleimer (Theorem 19.9 of [MS13]).

**Corollary 5.6.** There is a number $C > 0$ such that

$$d_D(D, E) \approx \sum_Y \text{diam}(\pi_Y(E \cup D))_C + \sum_\gamma \text{diam}(\pi_\gamma(E \cup D))_C$$

where $Y$ passes through all thick subsurfaces of $\partial H$, where $\gamma$ passes through all $I$-bundle generators in thick subsurfaces of $\partial H$, and the diameter is taken in the arc and curve graph.

For a thick subsurface $Y$ of $\partial H$ let $\partial EDG(Y)$ be the Gromov boundary of $EDG(Y)$. Define

$$\partial DG = \cup_Y \partial EDG(Y) \subset L$$

where the union is over all thick subsurfaces of $\partial H$ and where this union is viewed as a subspace of $L$, i.e. it is equipped with the coarse Hausdorff topology. The proof of the following statement is completely analogous to the proof of Proposition 4.8 and will be omitted.

**Corollary 5.7.** $\partial DG$ is the Gromov boundary of $DG$.

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