Realisations of $GL_{p,q}(2)$ quantum group and its coloured extension through a novel Hopf algebra with five generators

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Running Title: A new Hopf algebra related to $GL_{p,q}(2)$ quantum group

Abstract

A novel Hopf algebra ($\tilde{G}_{r,s}$), depending on two deformation parameters and five generators, has been constructed. This $\tilde{G}_{r,s}$ Hopf algebra might be considered as some quantisation of classical $GL(2) \otimes GL(1)$ group, which contains the standard $GL_q(2)$ quantum group (with $q = r^{-1}$) as a Hopf subalgebra. However, we interestingly observe that the two parameter deformed $GL_{p,q}(2)$ quantum group can also be realised through the generators of this $\tilde{G}_{r,s}$ algebra, provided the sets of deformation parameters $p, q$ and $r, s$ are related to each other in a particular fashion. Subsequently we construct the invariant noncommutative planes associated with $\tilde{G}_{r,s}$ algebra and show how the two well known Manin planes corresponding to $GL_{p,q}(2)$ quantum group can easily be reproduced through such construction. Finally we consider the ‘coloured’ extension of $GL_{p,q}(2)$ quantum group as well as corresponding Manin planes and explore their intimate connection with the ‘coloured’ extension of $\tilde{G}_{r,s}$ Hopf structure.

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1. Introduction

Quantum groups and related algebras [1-4], which have originated from the study of integrable models, are now finding their applications in diverse branches of physics and mathematics [5-8]. Since the underlying symmetry of many physical systems are governed by the quantum group structures, it is expected that their representations would also play a significant role in determining the behaviour of such systems. This is one of the prime reasons why a lot of work has been done in recent years for building up the representations of quantised universal enveloping algebras [6,9]. However, it may be noted that the representations of their dual objects, i.e. quantum groups, have not received that much attention and yet under rapid development [10-14]. Still lesser amount of progress has been made for the case of multiparameter quantum groups [15], which are more difficult to handle due to the presence of multiple deformation parameters in the commutation relations. So it is natural to hope that a better understanding on some common origin of single and multiparameter quantum groups would be helpful for building up their representation theory.

Moreover, as has been found recently [16], it is possible to construct a ‘colour’ parameter dependent quantum group which interestingly reproduces the standard $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups as its subalgebras, at the monochromatic limit of colour parameters. The invariant Manin planes corresponding to this infinite dimensional quantum group would also depend explicitly on the colour parameters. Consequently this coloured quantum group might be looked as some further extension of the two parameter deformed $GL_{p,q}(2)$ case, and one may expect that similar generalisations can be done for other multiparameter quantum groups. However, though having many nice properties, such coloured algebras look rather cumbersome in comparison with their standard counterparts. So for obtaining a deeper insight about these algebras and constructing their representations, it is useful to enquire whether they can be connected to some other algebras with relatively simpler structure.
The purpose of the present article is to shed some light on the above issues, by focussing on the simplest \( GL_q(2) \) and \( GL_{p,q}(2) \) quantum groups as well as their coloured extension. In our investigation we are able to find a novel Hopf algebra ( denoted by \( \tilde{G}_{r,s} \) ) depending on two deformation parameters and five generators. Curiously, the first four generators of such Hopf algebra form a Hopf subalgebra, which coincides exactly with the single parameter dependent \( GL_q(2) \) quantum group when \( q = r^{-1} \). However, it turns out that, that the two parameter deformed \( GL_{p,q}(2) \) quantum group can also be realised through the generators of this \( \tilde{G}_{r,s} \) Hopf algebra, provided the sets of deformation parameters \( p, q \) and \( r, s \) are related to each other in a particular fashion. So this new algebra with five generators is found to be connected in a strange way to both \( GL_q(2) \) and \( GL_{p,q}(2) \) quantum groups. In sec.2 we introduce this Hopf structure and explore its relation with \( GL_{p,q}(2) \) quantum group.

Generators of \( \tilde{G}_{r,s} \) Hopf algebra can interestingly be arranged in a \((3 \times 3)\) matrix form, which satisfies the quantum Yang-Baxter equation for a certain choice of braid group representation. So this Hopf algebra might also be considered as a quantum group, and one may naturally ask whether there exists any invariant noncommutative plane related to such structure. In sec.3 we investigate on this problem and able to find such quantum planes possessing some curious properties. For example, in contrast to the case of a usual quantum group where one gets only two Manin planes, in the present case we obtain as many as eleven such invariant noncommutative planes. Moreover the algebra of corresponding coordinates contain a free parameter, which cannot be determined from the associated \( \tilde{G}_{r,s} \) structure. We also discuss in sec.3 how some composite functions of these noncommuting coordinates are able to reproduce the two well known Manin planes related to the \( GL_{p,q}(2) \) quantum group. Finally in sec.4 we consider the colour parameter dependent extensions of \( GL_{p,q}(2) \) and \( \tilde{G}_{r,s} \) quantum groups, and then discuss their close connection to each other. Sec. 5 is the concluding section.
2. Realisation of $GL_{p,q}(2)$ quantum group through a novel Hopf algebra

Before coming to the main results of this section let us summarise briefly some basic properties of the $GL_{p,q}(2)$ quantum group. This quantum group might be generated by four elements $a$, $b$, $c$ and $d$ satisfying the relations [17,18]

\begin{align*}
ab &= p \ ba , \quad ac = q \ ca , \quad db = q^{-1}bd , \quad dc = p^{-1}cd , \\
bc &= \frac{q}{p} cb , \quad [a,d] = (p-q^{-1}) bc ,
\end{align*}

(2.1)

where the deformation parameters $p$ and $q$ are some nonzero complex numbers. For the particular case $p = q$, (2.1) reduces to the algebra of one parameter deformed $GL_q(2)$ quantum group. As it is well known, quantum groups can in general be constructed from the solutions of quantum Yang-Baxter equation (QYBE) given by [3]

\[ R T_1 T_2 = T_2 T_1 R , \quad (2.2) \]

where $R$ is a nonsingular $(N^2 \times N^2)$-dimensional matrix with usual $c$-number valued elements, $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$ and noncommuting elements of $T$-matrix are identified with the generators of a quantum group. Due to the associativity of QYBE (2.2), $R$-matrix satisfies the spectral parameterless Yang-Baxter equation (YBE) given by

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} , \quad (2.3) \]

where $R_{ij}$ acts on the direct product of three vector spaces and is nontrivial only on the $i$th and $j$th spaces (e.g. $R_{12} = R \otimes 1$ etc.). All commutation relations in (2.1), corresponding to the $GL_{p,q}(2)$ quantum group, can easily be generated from QYBE (2.2) by taking the $T$-matrix in the form [18]

\[ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad (2.4) \]

and the related $R$-matrix satisfying YBE (2.3) as

\[ R = \sqrt{\frac{p}{q}} \begin{pmatrix} p^{-1} & 1 - q \\ 0 & qp^{-1} \end{pmatrix} \begin{pmatrix} 1 & p^{-1} - q \\ 0 & p^{-1} \end{pmatrix} . \quad (2.5) \]
The inverse of $T$-matrix (2.4), defined through the relation $TT^{-1} = T^{-1}T = 1$, may be given by

$$T^{-1} = \delta^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix},$$

(2.6)

where $\delta^{-1}\delta = \delta\delta^{-1} = 1$, and $\delta$ is the quantum determinant:

$$\delta = ad - qcb.$$  

(2.7)

However, in contrast to the case of one parameter deformed $GL_q(2)$ group, the above quantum determinant is no longer a central element of $GL_{p,q}(2)$ algebra and by using (2.1) it is easy to see that $\delta$ obeys the commutation relations

$$a\delta = \delta a, \quad b\delta = q^{-1}p\delta b, \quad c\delta = q^{-1}p\delta c, \quad d\delta = \delta d.$$  

(2.8)

The coproduct ($\Delta$) for $GL_{p,q}(2)$ Hopf algebra may be obtained by substituting the form of $T$-matrix (2.4) in the expression $\Delta T = T \otimes T$, where the symbol $\otimes$ signifies ordinary matrix multiplication with tensor multiplication of algebra. The other Hopf structures, like co-unit ($\epsilon$) and antipode ($K$) for quantum groups, might in general be expressed as

$$\epsilon(T_{ij}) = \delta_{ij}, \quad K(T) = T^{-1}$$

(2.9)

and for the particular case of $GL_{p,q}(2)$ quantum group they may be explicitly given by

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$

$$K(a) = \delta^{-1}d, \quad K(b) = -q^{-1}\delta^{-1}b, \quad K(c) = -q^{-1}\delta^{-1}c, \quad K(d) = \delta^{-1}a.$$  

(2.10)

Here it is assumed that the inverse of quantum determinant, satisfying the relations $\Delta(\delta^{-1}) = \delta^{-1} \otimes \delta^{-1}, \quad \epsilon(\delta^{-1}) = 1, \quad K(\delta^{-1}) = \delta$, is included in the algebra.

Now for constructing a new realisation of this $GL_{p,q}(2)$ quantum group, we consider in the following another Hopf algebra $\tilde{G}_{r,s}$ containing five generators $A, B, C, D$ and $F$. These generators satisfy the algebraic relations

$$AB = r^{-1}BA, \quad AC = r^{-1}CA, \quad DB = rBD, \quad DC = rCD,$$

$$BC = CB, \quad [A,D] = (r^{-1} - r)BC,$$

(2.11a)
AF = FA, BF = s^{-1} FB, CF = s FC, DF = FD, \quad \text{(2.11b)}

where the two deformation parameters \( r \) and \( s \) are arbitrary nonzero complex numbers.

Notice that the elements \( A, B, C \) and \( D \) of \( \tilde{G}_{r,s} \) algebra form a subalgebra, which obeys the commutation relations (2.11a) and exactly coincides with the single parameter deformed \( GL_q(2) \) algebra when \( q = r^{-1} \). It is easy to see that, again quite similar to the \( GL_q(2) \) case, the Casimir operator of algebra (2.11) would be given by

\[ \mathcal{D} = AD - r^{-1} BC. \quad \text{(2.12)} \]

The coproduct for this \( \tilde{G}_{r,s} \) structure may be written as

\[
\begin{align*}
\Delta(A) &= A \otimes A + B \otimes C, \\
\Delta(B) &= A \otimes B + B \otimes D, \\
\Delta(C) &= C \otimes A + D \otimes C, \\
\Delta(D) &= C \otimes B + D \otimes D, \\
\Delta(F) &= F \otimes F.
\end{align*}
\quad \text{(2.13)}
\]

Finally, other Hopf relations like co-unit and antipode for \( \tilde{G}_{r,s} \) will take the form

\[
\begin{align*}
\epsilon(A) &= \epsilon(D) = \epsilon(F) = 1, \quad \epsilon(B) = \epsilon(C) = 0, \\
K(A) &= \mathcal{D}^{-1}D, \\
K(B) &= -r\mathcal{D}^{-1}B, \\
K(C) &= -r^{-1}\mathcal{D}^{-1}C, \\
K(D) &= \mathcal{D}^{-1}A, \\
K(F) &= F^{-1},
\end{align*}
\quad \text{(2.14)}
\]

where \( \mathcal{D}^{-1} \) is included in the algebra and it satisfies the relations like \( \Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1} \otimes \mathcal{D}^{-1} \), \( \epsilon(\mathcal{D}^{-1}) = 1 \), \( K(\mathcal{D}^{-1}) = \mathcal{D} \). Now by using the expressions (2.11)-(2.14), one can easily define a consistent Hopf structure through the polynomials made of \( \tilde{G}_{r,s} \) generators, which would satisfy all axioms [1,19] of Hopf algebra:

\[
\begin{align*}
m(id \otimes m) &= m(m \otimes id), \\
(id \otimes \Delta)\Delta &= (\Delta \otimes id)\Delta, \quad \text{(2.15a,b)} \\
(id \otimes \epsilon)\Delta &= (\epsilon \otimes id)\Delta = id, \\
m(id \otimes K)\Delta &= m(K \otimes id)\Delta = 1 \cdot \epsilon, \quad \text{(2.15c,d)} \\
\Delta(xy) &= \Delta(x)\Delta(y), \\
\epsilon(xy) &= \epsilon(x)\epsilon(y), \\
K(xy) &= K(y)K(x),
\end{align*}
\quad \text{(2.15e,f,g)}
\]

where \( m \) denotes the multiplication operation ( \( m(x \otimes y) = xy \) ) and \( id \) is the identity transformation. Moreover, the elements \( A, B, C \) and \( D \) of \( \tilde{G}_{r,s} \) evidently form a Hopf subalgebra, which coincides with the \( GL_q(2) \) quantum group.
Next we attempt to explore whether there exists any connection between $GL_{p,q}(2)$ quantum group and the newly defined $\tilde{G}_{r,s}$ Hopf algebra, both of which contain two deformation parameters. For this purpose we propose a simple realisation of $GL_{p,q}(2)$ generators through the elements of $\tilde{G}_{r,s}$ algebra as

$$a = F^N A, \quad b = F^N B, \quad c = F^N C, \quad d = F^N D,$$

(2.16a)

where $N$ is any fixed nonzero integer. Notice that the above relations can also be written in a convenient matrix form given by

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} = F^N
\begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}.
$$

(2.16b)

Now by using the $\tilde{G}_{r,s}$ algebra (2.11a,b) it is not difficult to verify that (2.16) indeed gives us a realisation of $GL_{p,q}(2)$ algebra (2.1), provided the two sets of deformation parameters $(p, q)$ and $(r, s)$ are related through the equations

$$p = r^{-1} s^N, \quad q = r^{-1} s^{-N}.$$

(2.17)

Observe that when $N \neq 0$, one can easily invert the above equations to find out the values of deformation parameters $(r, s)$ for any given value of $(p, q)$ and consequently realisation like (2.16) is always possible. Moreover by using (2.16) and the coproduct for $\tilde{G}_{r,s}$ algebra (2.13) along with its homeomorphism property (2.15e), one can easily recover the standard coproduct for $GL_{p,q}(2)$ quantum group. Finally, it is also possible to get back the co-unit and antipode for $GL_{p,q}(2)$ (2.10), with the help of relations (2.14), (2.15f,g) and (2.16). Thus we see that the full Hopf algebra structure related to $GL_{p,q}(2)$ quantum group can in fact be reproduced through the realisation (2.16).

Though we shall not consider here about the representations of $GL_{p,q}(2)$ quantum group, let us make some remarks on how the realisation like (2.16) might be useful in this context. As we have already observed, in contrast to $GL_{p,q}(2)$ algebra (2.1) where parameters $p$ and $q$ both appear in the commutation relations, a large subalgebra of $\tilde{G}_{r,s}$
depends on a single deformation parameter. So it might be relatively easier to construct first the representations of \( \tilde{G}_{r,s} \) algebra and then use the realisation (2.16) for obtaining the representations of \( GL_{p,q}(2) \) quantum group. Moreover, it is interesting to notice that the mapping (2.17) from \((r,s)\)-plane to \((p,q)\)-plane depends on the choice of integer \(N\). So by taking different values of \(N\), a single point on the \((r,s)\)-plane can be mapped over infinite number of discrete points, which would satisfy the conditions like

\[
p_N q_N = r^{-2}, \quad \frac{p_{N+1}}{p_N} = s, \quad \frac{q_{N+1}}{q_N} = s^{-1},
\]

(2.18)

where \((p_N,q_N)\) denotes a point on the \((p,q)\)-plane corresponding to the \(N\)-th mapping. As evident from eqn. (2.18), all these discrete points will lie on a hyperbola and their density in a given region would be controlled by the parameter \(s\). Therefore we find that from the representation of \( \tilde{G}_{r,s} \) algebra for a particular value of deformation parameters \((r,s)\), one can build up the representations of \( GL_{p,q}(2) \) quantum group for infinitely many discrete values of corresponding deforming parameters. Thus there exists a rather interesting structural connection among the \( GL_{p,q}(2) \) representations for these discrete values of \((p,q)\) parameters. Furthermore from eqn. (2.18) we see that by taking the limit \(s \to 1\), for any fixed value of \(r\), one can increase the density of such discrete points arbitrarily. So the properties of \( \tilde{G}_{r,s} \) algebra at the neighbourhood of \(s = 1\) line may have some particular significance in the context of \( GL_{p,q}(2) \) representations. In the next section we shall explore further the close connection between \( \tilde{G}_{r,s} \) and \( GL_{p,q}(2) \) structures, by studying the corresponding noncommutative planes.

3. Quantum plane for \( \tilde{G}_{r,s} \) algebra

For constructing invariant quantum planes [20-21] associated with the \( \tilde{G}_{r,s} \) Hopf algebra, we first try to cast this algebra in the form of a quantum group and define a \((3 \times 3)\) \(T\)-matrix through the corresponding generators as

\[
T = \begin{pmatrix}
A & B & 0 \\
C & D & 0 \\
0 & 0 & F
\end{pmatrix}.
\]

(3.1)
Now it is easy to verify that, for the above form $T$-matrix and the choice of $(9 \times 9) R$-matrix given by

$$ R = r \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} f_{ij} \cdot e_{ii} \otimes e_{jj} + (r - r^{-1}) \sum_{i < j} e_{ij} \otimes e_{ji}, \quad (3.2) $$

where $i, j \in [1, 3]$, $f_{12} = f_{23} = 1$, $f_{13} = s$ and $f_{ij} = f^{-1}_{ji}$, QYBE (2.2) is able to reproduce all relations of $\tilde{G}_{r,s}$ algebra (2.11). Moreover the $R$-matrix (3.2) turns out to be a solution of the spectral parameterless YBE (2.3). So we find that the associative $\tilde{G}_{r,s}$ algebra can in fact be written in the form of a quantum group. Furthermore, by taking the $T$-matrix as (3.1) and using the expression of coproduct for quantum groups: $\Delta T = T \otimes T$, one can easily recover the coproduct relations (2.13) corresponding to $\tilde{G}_{r,s}$ algebra. Finally, by constructing the inverse of $T$-matrix (3.1) as

$$ T^{-1} = \begin{pmatrix} D^{-1}D & -rD^{-1}B & 0 \\ -r^{-1}D^{-1}C & D^{-1}A & 0 \\ 0 & 0 & F^{-1} \end{pmatrix} $$

and using the expression (2.9), one can also get back the related counit and antipode structures (2.14).

It is worth noting at this point that the most general Hopf algebra which can be generated from the $R$-matrix (3.2), by assuming all elements of corresponding $(3 \times 3)$ $T$-matrix to be nonzero objects, is a multiparameter dependent extension of $GL_q(3)$ quantum group. Then putting by hand the additional restriction on such general $T$-matrix that some of its elements are equal to zero, one can reproduce the $\tilde{G}_{r,s}$ structure and so this Hopf algebra with five generators might be interpreted as a quotient of multiparameter deformed $GL_q(3)$ quantum group. However it is important to notice that the elements of general $T$-matrix, which have been put to zero in the above mentioned case, actually form a biideal of multiparameter deformed $GL_q(3)$ quantum group and due to this crucial reason one gets here a consistent Hopf structure through the quotient procedure. A different kind of quotients, for some other quantum groups, have been studied very recently in the context of braided Heisenberg algebra [22].
There exists another interesting way to look at the $\tilde{G}_{r,s}$ structure, by considering it a two parameter dependent quantisation of classical $GL(2) \otimes GL(1)$ group. Evidently, the first four generators of $\tilde{G}_{r,s}$, i.e. $A$, $B$, $C$, $D$ correspond to $GL(2)$ group at the classical level and the remaining generator $F$ is related to $GL(1)$ group. Moreover, while the parameter $r$ arises from the deformation of $GL(2)$ group, the other parameter $s$ curiously enters through the nontrivial cross commutation relations between $GL_r(2)$ and $GL(1)$ group elements at the quantum level. Similar type of quantisations, for the particular case of some semisimple Lie groups, has been considered in ref.23 and subsequently used to construct the deformation of wellknown Lorentz algebra.

Now to find out the invariant Manin planes associated with $\tilde{G}_{r,s}$ quantum group, we take a vector $\vec{X} = (X_1, X_2, X_3)$ having three noncommuting components and define the action of $T$-matrix (3.1) on this vector as

$$
\vec{X}' = T \vec{X} ; \quad \begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & F \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.
$$

Through explicit calculation we find that there exist eleven such sets of commutation relations and two among these sets may be given by

$$(3.4a)$$

$$X_1X_2 = r^{-1} X_2X_1, \quad X_2X_3 = k X_3X_2, \quad X_1X_3 = ks^{-1} X_3X_1,$$

and

$$(3.4b)$$

$$X_1^2 = X_2^2 = 0, \quad X_1X_2 = -r X_2X_1, \quad X_2X_3 = k X_3X_2, \quad X_1X_3 = ks^{-1} X_3X_1,$$

respectively, where $k$ is an arbitrary number. Thus the relations (3.4a) and (3.4b) provide us two Manin planes related to the $\tilde{G}_{r,s}$ quantum group. To obtain other Manin planes
one may notice that each of the four sets of relations given by

\begin{align*}
1) \quad & X_3^2 = 0, \quad 2) \quad X_2 X_3 = k X_3 X_2, \quad X_1 X_3 = ks^{-1} X_3 X_1, \\
3) \quad & X_1 X_2 = r^{-1} X_2 X_1, \quad 4) \quad X_1^2 = X_2^2 = 0, \quad X_1 X_2 = -r X_2 X_1,
\end{align*}

would also remain invariant under the action of $T$-matrix (3.1). However, it is interesting to observe further that, no smaller subset of any of these four set of relations is able to generate independently some other invariant noncommutative plane. Consequently, the four noncommutative planes related to these sets are ‘irreducible’ in nature. Now by taking advantage of the block diagonal form of $T$-matrix (3.1), it is not difficult to prove that all possible consistent combinations of these four sets of relations, would also give us new invariant quantum planes. For example, by combining the sets 2) and 3) one can easily reproduce the earlier relation (3.4a). By taking various combinations of the sets 1) to 4), we can similarly generate all of the eleven Manin planes related to $\tilde{G}_{r,s}$ structure. Evidently, in any of such combinations the mutually inconsistent sets 3) and 4) would not appear simultaneously. So this procedure of deriving composite Manin planes from the four fundamental blocks, is very similar to the construction of reducible representations of group theory by taking direct sum of its irreducible representations. It may also be noted that the noncommutative planes related to the usual quantum groups do not depend on any extra parameter, apart from those which are already present in the corresponding algebra. But in the set of commutation relations (3.4a,b) and in some other similar sets, we interestingly find the existence of free parameter $k$, which cannot be fixed through our quantum group relations (2.11).

After finding out the noncommutative planes related to $\tilde{G}_{r,s}$ structure, we like to explore in the following their connection with the Manin planes associated with $GL_{p,q}(2)$ quantum group. For this purpose we denote the two dimensional quantum plane related to the latter case as $\vec{x} = (x_1, x_2)$, on which the $T$-matrix (2.4) would act like

\begin{equation}
\begin{pmatrix}
  x'_1 \\
  x'_2
\end{pmatrix} = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}.
\end{equation}

(3.5)
Next we propose a simple realisation of this two dimensional $\vec{x}$ plane through previously discussed three dimensional $\vec{X}$ plane associated with the $\tilde{G}_{r,s}$ quantum group:

$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} = X_3^N \begin{pmatrix}
  X_1 \\
  X_2
\end{pmatrix},
$$

(3.6)

where as before $N$ is a fixed nonzero integer. Substituting now the expressions (3.6) and (2.16b) in the r.h.s. of (3.5) and using subsequently the relation (3.3) we get

$$
\begin{pmatrix}
  x'_1 \\
  x'_2
\end{pmatrix} = F^N \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix} \cdot X_3^N \begin{pmatrix}
  X_1 \\
  X_2
\end{pmatrix} = (FX_3)^N \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix} \begin{pmatrix}
  X_1 \\
  X_2
\end{pmatrix} = X_3'^N \begin{pmatrix}
  X'_1 \\
  X'_2
\end{pmatrix}.
$$

(3.7)

Comparing now the eqns. (3.6) and (3.7) one may interestingly observe that, the transformed vectors $\vec{x}'$ and $\vec{X}'$ are related to each other in exactly same way as their original counterparts $\vec{x}$ and $\vec{X}$. Now for $\tilde{G}_{r,s}$ invariant quantum planes, the form of commutation relations among the components of $\vec{X}'$ must be identical with that of $\vec{X}$. So from the structural similarity of eqns. (3.6) and (3.7) it follows that, the form of commutation relations among the components of vector $\vec{x}'$ will also be identical with that of vector $\vec{x}$. Consequently, by using the realisation (3.6) and the commutation relations corresponding to a $\tilde{G}_{r,s}$ invariant plane, if we are able to derive some bilinear relations for the components of vector $\vec{x}$, then that would automatically provide us an invariant Manin plane related to $GL_{p,q}(2)$ quantum group. For example, by using (3.4a) and (3.4b) respectively, along with (3.6), we will get two sets of commutation relations:

$$
x_1x_2 = r^{-1}s^{-N} x_2x_1; \quad x_1^2 = x_2^2 = 0, \quad x_1x_2 = -\frac{1}{r^{-1}s^N}x_2x_1.
$$

(3.8a, b)

Notice that the above relations are free from the extra parameter $k$, though it was present in the original commutation relations (3.4a,b). Moreover, by using (2.17), one can easily rewrite (3.8a,b) in terms of familiar $p$, $q$ variables and this would exactly reproduce the two well known Manin planes [18] related to $GL_{p,q}(2)$ quantum group. It may also be noted that if one starts with other invariant quantum planes corresponding to $\tilde{G}_{r,s}$ structure,
then by using corresponding commutation relations and realisation (3.6) it would not be possible to derive some nontrivial bilinear relation for the components of vector $\vec{x}$. So these cases do not lead us to any new $GL_{p,q}(2)$ invariant plane.

Thus from the above analysis we find an interesting alternative method for constructing the noncommutative planes associated with $GL_{p,q}(2)$ quantum group, through that of the $\tilde{G}_{r,s}$ quantum group. In the next section our aim is to discuss how a simple modification of this $\tilde{G}_{r,s}$ structure also provides a very convenient and natural basis, for realising the ‘coloured’ extension of $GL_{p,q}(2)$ quantum group.

4. Coloured extension of $GL_{p,q}(2)$ and $\tilde{G}_{r,s}$ quantum groups

There exists an intriguing possibility of extending the standard quantum group relations, by parametrising the corresponding generators through some continuous ‘colour’ variables and redefining the associated algebra, co-product etc. in such a way that all Hopf algebra properties would still remain preserved. In analogy with the standard cases, the algebra for these coloured quantum groups might be constructed by slightly modifying the form of QYBE (2.2) as [3,16]

$$R^{(\lambda,\mu)} T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R^{(\lambda,\mu)}, \quad (4.1)$$

where $\lambda, \mu$ are the colour parameters. Notice that the usual $T$-matrix in (2.2) has been replaced by $T(\lambda)$ or $T(\mu)$ in the above expression and the same $ij$-th element of $T(\lambda)$ and $T(\mu)$ matrices, i.e. $T_{ij}(\lambda)$ and $T_{ij}(\mu)$, would correspond to different generators when $\lambda \neq \mu$. So this type of algebra practically depends on infinite number of generators, in contrast to finite dimensional cases encountered in the previous sections. However due to the structure of QYBE (4.1), generators of only two different colours can at most appear simultaneously in the algebraic relations. Moreover the subalgebra formed by the elements of any particular colour should coincide with some usual quantum group related algebra, since at the monochromatic limit $\lambda = \mu$ (4.1) effectively reduces to the more common form
of QYBE (2.2). The coproduct of algebra (4.1) may be given by the simple relation

$$\Delta T(\lambda) = T(\lambda) \otimes T(\lambda),$$

(4.2)

where the symbol $\otimes$ signifies ordinary matrix multiplication with tensor multiplication of algebra.

Due to the associativity of QYBE (4.1), $R^{(\lambda,\mu)}$ satisfies the YBE like

$$R_{12}^{+(\lambda,\mu)} R_{13}^{+(\lambda,\gamma)} R_{23}^{+(\mu,\gamma)} = R_{23}^{+(\mu,\gamma)} R_{13}^{+(\lambda,\gamma)} R_{12}^{+(\lambda,\mu)},$$

(4.3)

which naturally reduces to eqn. (2.3) at the limit $\lambda = \mu = \gamma$. We shall further assume that the $R^{(\lambda,\mu)}$-matrix satisfying (4.3) would be upper or lower triangular in form, in analogy with the structure of $R$-matrices related to the standard quantum groups. This type of $R^{(\lambda,\mu)}$-matrices are also closely connected with the coloured braid group representations, which have attracted much attentions in recent years [24-28].

A special form of $(4 \times 4)$ $R^{(\lambda,\mu)}$-matrix, which can be derived by taking the fundamental representation of universal $\mathcal{R}$-matrix associated with the $U_q(gl(2))$ quantised algebra [26], may be given by

$$R^{(\lambda,\mu)} = \begin{pmatrix} t^{1-(\lambda-\mu)} & t^{\lambda+\mu} & (t - t^{-1}) & t^{1+(\lambda-\mu)} \\ 0 & t^{-\lambda+\mu} & t^{-1} & t^{-(\lambda+\mu)} \\ t^{1-(\lambda-\mu)} & t^{-\lambda+\mu} & 0 & t^{1+(\lambda-\mu)} \end{pmatrix}.$$  

(4.4)

The above solution of YBE (4.3) is particularly interesting, since at the monochromatic limit $\lambda = \mu = \Lambda$ it reduces exactly to the $R$-matrix (2.5) associated with $GL_{p,q}(2)$ quantum group, when the parameters $p$ and $q$ are given through the relations

$$p = t^{-1+2\Lambda}, \quad q = t^{-(1+2\Lambda)}.$$  

(4.5)

So it is expected that the solution (4.4) would lead to a ‘coloured’ quantum group, whose generators corresponding to any particular colour will reproduce the $GL_{p,q}(2)$ algebra as
a subalgebra. Such coloured extension of $GL_{p,q}(2)$ quantum group may be constructed by taking the corresponding $(2 \times 2)$ $T(\lambda)$-matrix as \[ T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \] and inserting this $T(\lambda)$ along with $R^{(\lambda,\mu)}$ (4.4) in the QYBE (4.1). By expressing then QYBE in elementwise form we would get a large number of independent algebraic relations, which may be grouped together in two different sets. The first set is formed through the relations

\begin{align*}
    a(\lambda)b(\mu) &= t^{-1+2\lambda} b(\mu)a(\lambda), \\
    a(\lambda)c(\mu) &= t^{-1-2\lambda} c(\mu)a(\lambda), \quad (4.7a,b) \\
    d(\lambda)b(\mu) &= t^{1+2\lambda} b(\mu)d(\lambda), \\
    d(\lambda)c(\mu) &= t^{1-2\lambda} c(\mu)d(\lambda), \quad (4.7c,d) \\
    b(\lambda)c(\mu) &= t^{-2(\lambda+\mu)} c(\mu)b(\lambda), \\
    [a(\lambda),d(\mu)] &= -(t-t^{-1})t^{\lambda+\mu} b(\mu)c(\lambda), \quad (4.7e,f)
\end{align*}

and the second set contains the remaining independent relations given by

\begin{align*}
    a(\lambda)b(\mu) &= t^{\lambda-\mu} a(\mu)b(\lambda), \\
    a(\lambda)c(\mu) &= t^{-\lambda+\mu} a(\mu)c(\lambda), \quad (4.8a,b) \\
    d(\lambda)b(\mu) &= t^{\lambda-\mu} d(\mu)b(\lambda), \\
    d(\lambda)c(\mu) &= t^{-\lambda+\mu} d(\mu)c(\lambda), \quad (4.8c,d) \\
    b(\lambda)c(\mu) &= b(\mu)c(\lambda), \\
    a(\lambda)d(\mu) &= a(\mu)d(\lambda), \\
    a(\lambda)a(\mu) &= a(\mu)a(\lambda), \quad (4.8e,f,g) \\
    b(\lambda)b(\mu) &= t^{2(\lambda-\mu)}b(\mu)b(\lambda), \\
    c(\lambda)c(\mu) &= t^{-2(\lambda-\mu)}c(\mu)c(\lambda), \\
    d(\lambda)d(\mu) &= d(\mu)d(\lambda). \quad (4.8h,i,j)
\end{align*}

So the above two sets define explicitly the algebra of the present coloured quantum group.

Now for finding out the commutation relations among the elements of a particular colour, one have to take the $\lambda = \mu = \Lambda$ limit for both of the sets (4.7) and (4.8). By taking this limit for the set (4.7) we find that it would exactly reproduce the $GL_{p,q}(2)$ algebra (2.1), when the values of $p$, $q$ parameters are given through the expression (4.5). On the other hand, all relations in the second set (4.8) would become trivial at this $\lambda = \mu$ limit. These
results demonstrate that the $GL_{p,q}(2)$ algebra can be reproduced as a subalgebra, from the monochromatic limit of our coloured algebra. Moreover the relation (4.5) shows that, by selecting different values of the monochromatic limit $\Lambda$, one can also change the values of deformation parameters $p$ and $q$. So this coloured algebra actually contains infinite number of such $GL_{p,q}(2)$ subalgebras, with continuously varying values of corresponding deformation parameters.

The coproduct for this extended algebra might be obtained by simply substituting the $(2 \times 2)$ $T(\lambda)$-matrix (4.6) in the general expression (4.2). One can also define the quantum determinant corresponding to the $T(\lambda)$-matrix (4.6) as

$$\delta(\lambda) = a(\lambda)d(\lambda) - t^{1+2\lambda} c(\lambda)b(\lambda).$$

By using now (4.7) and (4.8) it is not difficult to verify that the above quantum determinant satisfies the relations like

$$a(\lambda)\delta(\mu) = \delta(\mu)a(\lambda), \quad b(\lambda)\delta(\mu) = t^{-4\mu}\delta(\mu)b(\lambda), \quad c(\lambda)\delta(\mu) = t^{4\mu}\delta(\mu)c(\lambda), \quad (4.10a, b, c)$$

$$d(\lambda)\delta(\mu) = \delta(\mu)d(\lambda), \quad \delta(\lambda)\delta(\mu) = \delta(\mu)\delta(\lambda). \quad (4.10d, e)$$

From the above expression we see that $\delta(\mu)$ does not commute with all elements of $T(\lambda)$ unless $\mu = 0$, and so only $\delta(0)$ would be a Casimir operator of full coloured algebra. However the relation (4.10e) reveals the interesting fact that the determinants associated with different colours form a set of mutually commuting operators. By assuming now that the inverse of $\delta(\lambda)$ exists for all values of $\lambda$, one can write down the inverse of the $T(\lambda)$-matrix (4.6) as

$$T(\lambda)^{-1} = \delta(\lambda)^{-1} \begin{pmatrix} d(\lambda) & -t^{1+2\lambda}b(\lambda) \\ -t^{-1-2\lambda}c(\lambda) & a(\lambda) \end{pmatrix}.$$
with $\Delta (\delta (\lambda)^{-1}) = \delta (\lambda)^{-1} \otimes \delta (\lambda)^{-1}$, $\epsilon (\delta (\lambda)^{-1}) = 1$, $k (\delta (\lambda)^{-1}) = \delta (\lambda)$, and it is easy to check that they would satisfy all conditions of a Hopf algebra.

One of the most interesting features of the above described coloured extension of $GL_{p,q}(2)$ quantum group is the existence of corresponding noncommutative planes [16]. Such quantum planes may be constructed by inserting the colour parameter to the original two dimensional $GL_{p,q}(2)$ quantum plane: $\vec{x}(\lambda) = (x_1(\lambda), x_2(\lambda))$. So the coordinates of this infinite dimensional quantum plane not only depend on a discrete set of indices, but also on a continuously variable colour parameter. With the help of $T(\lambda)$-matrix (4.6), one may define the transformation on these coloured coordinates as

$$
\begin{pmatrix}
  x'_1(\lambda) \\
  x'_2(\lambda)
\end{pmatrix} =
\begin{pmatrix}
  a(\lambda) & b(\lambda) \\
  c(\lambda) & d(\lambda)
\end{pmatrix}
\begin{pmatrix}
  x_1(\lambda) \\
  x_2(\lambda)
\end{pmatrix}.
$$

(4.12)

Next, in analogy with the standard cases, we assume that the matrix elements of $T(\lambda)$ would commute with $x_i(\mu)$ for all vaules of $\lambda, \mu$, and using the algebra (4.7), (4.8) try to find out the commutation relations between the coordinates of different colours which will be invariant under the above transformation. Being guided by the form of QYBE (4.1), we also make the simplifying assumption that the coordinates of only two different colours can appear simultaneously in these commutation relations. Now by somewhat lengthy but straightforward calculation we find that there exist two such sets of invariant relations given by

$$
x_1(\lambda)x_1(\mu) = t^{\lambda-\mu} x_1(\mu)x_1(\lambda), \quad x_1(\lambda)x_2(\mu) = t^{-(1+\lambda+\mu)} x_2(\mu)x_1(\lambda),
$$

$$
x_1(\lambda)x_2(\mu) = x_1(\mu)x_2(\lambda), \quad x_2(\lambda)x_2(\mu) = t^{-\lambda+\mu} x_2(\mu)x_2(\lambda),
$$

(4.13)

and

$$
x_1(\lambda)x_1(\mu) = x_2(\lambda)x_2(\mu) = 0,
$$

$$
x_1(\lambda)x_2(\mu) = - t^{1-\lambda-\mu} x_2(\mu)x_1(\lambda), \quad x_1(\lambda)x_2(\mu) = x_1(\mu)x_2(\lambda).
$$

(4.14)

It is worth noting that at the monochromatic limit $\lambda = \mu$, (4.13) and (4.14) recover the expressions (3.8a) and (3.8b) respectively, after a trivial transformation of the related
parameters. So the two coloured quantum planes, defined through the relations (4.13) and (4.14), contain the standard \( q \)-plane and its exterior plane corresponding to \( GL_{p,q}(2) \) quantum group as their subspaces.

In spite of the fact that the algebra defined through the relations (4.7) and (4.8) exhibit many interesting properties, it is dependent on the colour parameters in a quite complicated way. So for having some deeper insight into this kind of algebraic structure, it is helpful to investigate whether it can be realised through some other algebra of comparatively simpler form. For this purpose we propose a simple coloured extension of \( \tilde{G}_{r,s} \) Hopf algebra discussed in sec.2, by keeping its first four generators \( A, B, C, D \) to be independent of the colour parameters, but replacing the fifth generator \( F \) by the continuous set of generators \( F(s) \):

\[
AB = r^{-1} BA, \quad AC = r^{-1} CA, \quad DB = r BD, \quad DC = r CD,
\]
\[
BC = CB, \quad [A, D] = (r^{-1} - r)BC, \quad [F(s), F(s')] = 0,
\]
\[
AF(s) = F(s)A, \quad BF(s) = s^{-1}F(s)B, \quad CF(s) = s F(s)C, \quad DF(s) = F(s)D,
\]

where the colour parameters \( s \) and \( s' \) may take arbitrary values. One can easily check that the associativity property holds for the above algebra. Moreover, it is possible to define the corresponding coproduct, antipode etc., in close analogy with the \( \tilde{G}_{r,s} \) case, which would obey all axioms of a Hopf algebra. Genetators of the algebra (4.15) can also be cast in the form of a coloured quantum group satisfying QYBE (4.1), when the parameters \( \lambda, \mu \) are replaced by the present colour parameters \( s, s' \). The \( T(s) \)-matrix might be defined for this case by simply substituting the element \( F \) with \( F(s) \) in the expression (3.1) and the associated \( R^{(s,s')} \)-matrix would be a coloured generalisation of the \( R \)-matrix (3.2), which we are not giving here explicitly. From the structure of \( T(s) \)-matrix it is evident that, the noncommuting plane corresponding to this coloured quantum group would be composed of the components \( X_1, X_2, X_3(s) \) and these components will be transformed like

\[
X_1' = AX_1 + BX_2, \quad X_2' = CX_1 + DX_2, \quad X_3(s)' = F(s)X_3(s).
\]
Now by using the algebra (4.15) one can find that, just like the case of $\tilde{G}_{r,s}$ quantum group, there exists eleven invariant noncommutative planes in the present case and two among them may be given through the commutation relations

$$X_1X_2 = r^{-1}X_2X_1, \ X_2X_3(s) = kX_3(s)X_2, \ X_1X_3(s) = ks^{-1}X_3(s)X_1, \ [X_3(s), X_3(s')] = 0$$

(4.17a)

and

$$X_1^2 = X_2^2 = 0, \ X_1X_2 = -rX_2X_1,$$

$$X_2X_3(s) = kX_3(s)X_2, \ X_1X_3(s) = ks^{-1}X_3(s)X_1, \ [X_3(s), X_3(s')] = 0$$

(4.17b)

respectively, where $k$ is an arbitrary number. Other invariant quantum planes can also be derived easily, by following a procedure similar to which has been described in sec.3. However these quantum planes will not be relevant for our purpose of making connection with the coloured extension of $GL_{p,q}(2)$ quantum group and so we are not giving their explicit expression.

Thus from the above discussion we see that $\tilde{G}_{r,s}$ quantum group admits its coloured extension in a very natural way and the associated Manin planes can also be obtained by slightly modifying the relations of $\tilde{G}_{r,s}$ invariant planes. However, the situation is widely different for the case of coloured $GL_{p,q}(2)$ group, since the corresponding algebra and Manin planes are much more complicated in form in comparison with their standard counterparts. Nevertheless we interestingly find that, similar to the realisation (2.16) related to the standard cases, one can build up a realisation of the coloured extension of $GL_{p,q}(2)$ algebra ((4.7),(4.8)) through that of the $\tilde{G}_{r,s}$ algebra (4.15) as

$$\begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix} = F(s) \begin{pmatrix} A & s^{-\frac{1}{2}}B \\ s^{\frac{1}{2}}C & D \end{pmatrix},$$

(4.18)

where we have taken $t = r$ as the relation among fixed deformation parameters and $t^{2\lambda} = s$ as the relation among variable colour parameters corresponding to these two algebras. Note that it is also possible to get a slightly more general form of above realisation, which would depend on a fixed nonzero integer $N$ analogous to the expression
Since the coloured $\tilde{G}_{r,s}$ algebra is much simpler in form in comparison with the coloured extension of $GL_{p,q}(2)$ algebra, one may expect that realisations like (4.18) would be useful for constructing the representations corresponding to the latter case. However, in the following, we shall restrict ourselves by just showing how the realisation (4.18) gives us a convenient shortcut way of constructing the Manin planes (4.13), (4.14) related to coloured $GL_{p,q}(2)$ algebra. So, in analogy with the approach adapted in sec.3, we propose a relation between the noncommuting coordinates associated with coloured $GL_{p,q}(2)$ and $\tilde{G}_{r,s}$ algebras as

$$\begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \end{pmatrix} = X_3(s) \begin{pmatrix} s^{-\frac{1}{4}}X_1 \\ s^{\frac{1}{4}}X_2 \end{pmatrix},$$

where $s = t^{2\lambda}$. Substituting now the expressions (4.18) and (4.19) in the r.h.s. of (4.12) and using then the transformation law (4.16) we find

$$\begin{pmatrix} x'_1(\lambda) \\ x'_2(\lambda) \end{pmatrix} = F(s)X_3(s) \begin{pmatrix} s^{-\frac{1}{4}}(AX_1 + BX_2) \\ s^{\frac{1}{4}}(CX_1 + DX_2) \end{pmatrix} = X'_3(s) \begin{pmatrix} s^{-\frac{1}{4}}X'_1 \\ s^{\frac{1}{4}}X'_2 \end{pmatrix}.$$  

Comparing the eqns. (4.19) and (4.20) it may be observed that the transformed vectors $\vec{x}'(\lambda)$ and $\vec{X}'(s)$ are related to each other in exactly same way as their original counterparts $\vec{x}(\lambda)$ and $\vec{X}(s)$. So we may conclude that, if the vector $\vec{X}(s)$ corresponds to some invariant quantum plane associated with the coloured $\tilde{G}_{r,s}$ algebra, and by using the realisation (4.19) we are able to derive some nontrivial bilinear relations among the components of vectors $\vec{x}(\lambda)$ and $\vec{x}(\mu)$, then those binary relations would automatically define an invariant Manin plane related to the coloured extension of $GL_{p,q}(2)$ quantum group. By using now the expressions (4.17a) and (4.17b) respectively, along with the realisation (4.19), we interestingly get two such sets of bilinear relations and these two sets exactly coincide with our previous expressions (4.13) and (4.14). Thus we find here an elegant alternative way of constructing the Manin planes related to the coloured extension of $GL_{p,q}(2)$ quantum group, since in this method we do not have to directly deal with the complicated algebraic relations (4.7) and (4.8).
5. Concluding remarks

In this article we have attempted to find some common origin of single and multiparameter deformed $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups. Such attempt led us to construct a novel Hopf algebra $\tilde{G}_{r,s}$, which depends on two deformation parameters and five generators. This $\tilde{G}_{r,s}$ Hopf algebra might be considered as a quantisation of classical $GL(2) \otimes GL(1)$ group, and contains the standard $GL_q(2)$ quantum group (with $q = r^{-1}$) as a Hopf subalgebra. On the other hand, the two parameter deformed $GL_{p,q}(2)$ quantum group can also be realised through the generators of this $\tilde{G}_{r,s}$ algebra, provided the set of deformation parameters $p$, $q$ and $r$, $s$ are related to each other in a particular fashion. Thus we interestingly find that both of these $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups can be generated from the same $\tilde{G}_{r,s}$ structure, by taking different combinations of its generators.

Subsequently we try to construct the Manin planes related to the $\tilde{G}_{r,s}$ quantum group. In contrast to the case of a usual quantum group where one gets only two Manin planes, in the present case we find as many as eleven such invariant planes. The block diagonal nature of corresponding $T$-matrix seems to be responsible for the existence of so many invariant noncommutative planes. By using some of these invariant planes, along with the connection between $GL_{p,q}(2)$ and $\tilde{G}_{r,s}$ quantum groups, we also find an alternative way of constructing the wellknown Manin planes related to $GL_{p,q}(2)$ case.

The realisation of $GL_{p,q}(2)$ algebra through the $\tilde{G}_{r,s}$ generators becomes very useful in the context of corresponding ‘coloured’ extensions. As we have discussed in sec.4, it is possible to construct an infinite dimensional Hopf algebra by inserting colour parameters in the original $GL_{p,q}(2)$ generators. One can also find the ‘coloured’ Manin planes associated with such extension of $GL_{p,q}(2)$ quantum group. Though having many interesting properties, this coloured $GL_{p,q}(2)$ algebra is rather complicated in form and to find out the corresponding Manin planes one has to go through straightforward but lengthy calculations. On the other hand, it turns out that there exists a quite simple coloured extension of our $\tilde{G}_{r,s}$ algebra and corresponding invariant quantum planes. Furthermore we find that,
similar to the case of their standard counterparts, it is possible to give a realisation of coloured $GL_{p,q}(2)$ algebra through the generators of coloured $\tilde{G}_{r,s}$ algebra. By exploiting such realisation we also present an elegant alternative method for constructing the Manin planes associated with the coloured $GL_{p,q}(2)$ algebra.

It seems that the relations established in this article, among some simplest type of quantum groups, would be applicable to more general contexts. For example, one may start with a multiparameter deformed version of $GL_q(N)$ quantum group and investigate whether it can also be realised through another Hopf algebra, which has some extra generators but possibly contains a core subalgebra with comparatively simpler structure. This type of realisations of multiparameter quantum groups might be useful for constructing their representations. Moreover such realisations may lead to the coloured extensions of associated quantum groups and Manin planes in a very natural way.

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