Abstract

We study isomorphic universality of Banach spaces of a given density and a number of pairwise non-isomorphic models in the same class. We show that in the Cohen model the isomorphic universality number for Banach spaces of density $\aleph_1$ is $\aleph_2$, and analogous results are true for other cardinals (Theorem 1.15(1)) and that adding just one Cohen real to any model adds a Banach space of density $\aleph_1$ which does not embed into any such space in the ground model (Theorem 1.5). Moreover, such a Banach space can be chosen to be weakly compactly generated (Theorem 1.9), giving similar universality number calculations in the class of wcg spaces in the Cohen model. In another direction, we develop the framework of natural spaces to study isomorphic embeddings of Banach spaces and use it to show that a sufficient failure of the generalized continuum hypothesis implies that the universality number of Banach spaces of a given density under a certain kind of positive embeddings (very positive embeddings), is high (Theorem 4.8(1)), and similarly for the number of pairwise non-isomorphic models (Theorem 4.8(2)).

0 Introduction

We shall be concerned with the isomorphic embeddings of Banach spaces, and in particular with the universality number of this class. For a quasi-ordered class $\langle \mathcal{M}, \leq \rangle$, the universality number is defined as the smallest size of $\mathcal{N} \subseteq \mathcal{M}$ such that for every...
There is $M \in \mathcal{M}$ such that $M \leq N$. In Banach space theory we find many examples of classes whose universality numbers have been studied, with respect to isomorphic, isometric and other kinds of embeddings. Some specific examples of this are given in sections §2 and §4. Indeed, some of the most classical results in Banach space theory come from these considerations, such as the result by Banach and Mazur [1](p.185) that \( C([0,1]) \) is isometrically universal for all separable Banach spaces. The non-separable case is more complex. We shall note below that the question is only interesting in the context of the failure of the generalized continuum hypothesis, since GCH automatically gives one universal model for each uncountable density. The question has received considerable attention over the years, including some recent work such as that of Brech and Koszmider [3] who considered Banach spaces of density the continuum and proved that in the Cohen model for \( \aleph_2 \) many Cohen reals there is an isomorphically universal Banach space of density \( \text{c} = \aleph_2 \). In this paper we shall also consider Cohen and Cohen-like models, and we shall prove results complementary to those of Brech and Koszmider. Specifically, in the model they considered, we shall see that the isomorphic universality number for Banach spaces of density \( \aleph_1 \) is \( \aleph_2 \), and we shall also show that analogous results are true for other cardinals (Theorem 1.15(1)). We shall also see that adding just one Cohen real to any model adds a Banach space of density \( \aleph_1 \) which does not embed into any such space in the ground model (Theorem 1.5). We note that in [4] it is stated (pg. 1268), without proof, that Koszmider and Thompson noted that a version of the proof from [3] gives a model where there is no isomorphically universal Banach space of density \( \aleph_1 \). A further relation with the work of Brech and Koszmider is that in [4] they gave a model where \( 2^{\aleph_0} = \aleph_2 \) and the universality number for the class of weakly compactly generated Banach spaces of density \( \aleph_1 \) is \( \aleph_2 \), while we show that the same is true in the classical Cohen model for \( \aleph_2 \) Cohen reals (Theorem 1.16).

A question related to that of universality is that of the number of pairwise non-isomorphic models. This is a well studied question in model theory (see Shelah’s [15]) and it has received considerable attention in Banach space theory of separable Banach spaces, see Rosendal [14] for an excellent survey of this and related problems. In the non-separable context, there do not seem to be many results available. As a consequence of our non-universality results we show that in the Cohen model described above there is a family of \( 2^{\aleph_1} \) many Banach spaces of density \( \aleph_1 \) which are pairwise non-isomorphic, and the same is true for other cardinals (Theorem 1.15(2)) and wcg spaces (Theorem 1.16).

Moving away from forcing results, which give results valid in specific models of set theory, we would ideally like to establish results in the form of implications from cardinal arithmetic. An example is the above mentioned result that under GCH there is an isomorphically universal Banach spaces for every density. In the absence of GCH results of this type may be obtained by using the pcf theory. We explore this direction in §3 -§6, where we prove that for a special kind of embedding “very positive” (see Definition 4.4) a sufficient failure of GCH implies the nonexistence of universal Banach spaces in the class of spaces \( C(K) \) for \( K \) 0-dimensional, (Theorem 4.8 (1)) as well as the existence of
a large number of pairwise non-isomorphic models (Theorem 4.8 (2)). While the use of forcing methods in the isomorphic Banach space theory is not new, as explained above, to our knowledge no pcf results have previously been obtained in this context. Our result required a development of a new framework, which we call natural spaces. These are model-theoretic structures which we use to talk about the spaces of the form $C(K)$ indirectly. Model theory connected to pcf theory was used by Shelah and Usvyatsov [17] to study the isometric theory of Banach spaces and prove negative universality results about them. We review these results below. They can also serve as a motivation to a question that has interested us throughout, which is to what extent the universality number of the class of Banach spaces of given density depends on the kind of embedding considered. Let us now pass to some background results and notation. Throughout, $\kappa$ stands for an infinite cardinal.

By combining the Stone duality theorem, the fact that any Banach space $X$ is isometric to a subspace of $C(B_X^*)$ and that $B_X^*$ has a totally disconnected continuous preimage, Brech and Koszmider proved the following:

**Fact 0.1** [Fact 1.1, [3]] (1) The universality number of the class of Boolean algebras of size $\kappa$ is greater or equal to the universality number of the class of Banach spaces of density $\kappa$ with isometric embeddings, which is greater or equal than the universality number of the class of Banach spaces of density $\kappa$ with isomorphic embeddings.

(2) The class of spaces of the form $C(\text{St}(\mathcal{A}))$ for $\mathcal{A}$ a Boolean algebra of size $\kappa$ is isometrically universal for the class of Banach spaces of density $\kappa$, and in particular its universality number with either isometric or isomorphic embeddings is the same as the universality number of the whole class of Banach spaces of density $\kappa$.

Fact 0.1 is only interesting in the context of uncountable $\kappa$, since for $\kappa = \aleph_0$ we have a universal Boolean algebra as well as an isometrically universal Banach space, as explained above. On the other hand, it is known from the classical model theory (see [5] for saturated and special models) that in the presence of GCH there is a universal Boolean algebra at every uncountable cardinal, so the questions of universality for the above classes are interesting in the context of the failure of the relevant instances of GCH. Negative universality results for Boolean algebras are known to hold when GCH fails sufficiently by the work of Kojman and Shelah [12], and in Cohen-like extensions by the work of Shelah (see [12] for a proof). Shelah and Usvyatsov proved in [17] that in the models where the negative universality results that were obtained for Boolean algebras in [12] hold, the same negative universality results hold for Banach spaces under isometric embeddings. The smallest cardinal at which these results can apply is $\aleph_2$. For example, if $\lambda$ is a regular cardinal greater than $\aleph_1$ but smaller than $2^{\aleph_0}$ (so $2^{\aleph_0} \geq \aleph_3$) there is no universal under isometries Banach space of density $\lambda$.

On the basis of what is known in the literature and what we obtain here, it is interesting to note that no known result differentiates between the universality number of Banach spaces of a given density under isometries or under isomorphisms. Furthermore, it is not known how to differentiate them from the universality number of Boolean algebras.
Conjecture 0.2  The universality number of the class of Banach spaces of density $\kappa$ with isomorphic embeddings is the same as the universality number of the class of Boolean algebras of size $\kappa$.

It follows from the above discussion that Conjecture 0.2 would improve Fact 0.1 (1) and it would imply the negative universality results of Shelah and Usvyatsov. For all we know at this point, Conjecture 0.2 could be a theorem of ZFC, that is, it is not known to fail at any $\kappa$ even consistently. A particular case of Conjecture 0.2 is the following Conjecture 0.3, which summarises the most interesting case from the point of view of Banach space theory:

Conjecture 0.3  The universality number of the class of Banach spaces of density $\kappa$ with isomorphic embeddings is the same as the universality number of the class of Banach spaces of density $\kappa$ with isometric embeddings.

In Banach space theory one studies other kinds of embeddings but isomorphisms and isometries, an example is described in §2. It would be interesting to refine the above conjectures in other contexts, for example in the context of embeddings with a fixed Banach-Mazur diameter, see §2 for a discussion. We now finish the introduction by giving some background information for the readers less familiar with Banach space theory.

Definition 0.4  A Banach space is a normed vector space complete in the metric induced by the norm. A linear embedding $T : X \to Y$ between Banach spaces is an isometry if for every $x \in X$ we have $||x|| = ||Tx||$, where we use $Tx$ to denote $T(x)$. A linear embedding $T : X \to Y$ between Banach spaces is an isomorphism if there is a constant $D > 0$ such that for every $x \in X$ we have $\frac{1}{D}||x|| \leq ||Tx|| \leq D||x||$.

Remark 0.5  Every isometry is an isomorphism. An isomorphism is in particular an injective continuous function and in fact, a linear map $T$ is an isomorphism iff both $T$ and $T^{-1}$ are linear and continuous.

For $T$ an isomorphism we define $||T|| \overset{\text{def}}{=} \sup \{||T(f)|| : ||f|| = 1\}$ and we say that the Banach-Mazur diameter of $T$ is the quantity $||T|| \cdot ||T^{-1}||$.

Throughout the paper letters $\mathfrak{A}$ and $\mathfrak{B}$ will be used for Boolean algebras, $\kappa, \lambda$ for infinite cardinals and $K$ and $L$ for compact spaces. We shall write $\text{St}(\mathfrak{A})$ for the Stone space of a Boolean algebra $\mathfrak{A}$. Let us note that Fact 0.1 implies

Observation 0.6  The universality number of Banach spaces of density $\kappa$, under any kind of embeddings, is either 1 or $\geq \kappa^+$.

This is so because if for any $\alpha^* \in [1, \kappa^+]$ we had that $\{X_\alpha : \alpha < \alpha^*\}$ were a universal family of Banach spaces of density $\kappa$, then we could assume that each $X_\alpha = C(\text{St}(\mathfrak{A}_\alpha))$ for some Boolean algebras $\mathfrak{A}_\alpha$ of size $\kappa$. Therefore we could find a single algebra $\mathfrak{A}$ of size $\kappa$ such that all $\mathfrak{A}_\alpha$ embed into it $^2$ and hence then $C(\text{St}(\mathfrak{A}))$ would be a single universal Banach space of density $\kappa$.

$^2$simply by freely generating an algebra by a disjoint union of all $\mathfrak{A}_\alpha$. 

4
1 Universals in Cohen-like extensions

It is a well known theorem of Shelah (see [12], Appendix, for a proof) that in the extension obtained by adding a regular $\kappa \geq \lambda^{++}$ number of Cohen subsets to a regular $\lambda$ over a model of GCH, the universality number for models of size $\lambda^+$ for any unstable complete first order theory is $\kappa = 2^\lambda$, so the maximal possible. This in particular applies to Boolean algebras. We explore to what extent this can be adapted to the class of Banach spaces of density $\lambda$ by looking at variants of Cohen forcing. Our results show that Conjecture 0.2 holds in this special case, that is, in these circumstances we obtain the strongest possible negative universality result.

Let $\lambda = \lambda^{<\lambda}$. Fix a set $A = \{a_i : i < \lambda^+\}$ of indices which we shall assume forms an unbounded co-unbounded subset of $\lambda^+$.

**Definition 1.1** The forcing $\mathbb{P}(\lambda)$ consists of Boolean algebras $p$ generated on a subset of $\lambda^+$ by some subset $w_p$ of $A$ of size $< \lambda$ satisfying $p \cap A = w_p$. The ordering on $\mathbb{P}$ is given by $p \leq q$ if $p$ is embeddable as a subalgebra of $q$ and the embedding fixes $w_p$, where in our notation $q$ is the stronger condition.

Let us check some basic properties of the forcing $\mathbb{P}(\lambda)$, reminding the reader of the following notions:

**Definition 1.2**

1. A forcing notion $\mathbb{P}$ is said to be $\lambda^+$-stationary cc if for every set $\{p_i : i < \lambda^+\}$ of conditions in $\mathbb{P}$, there is a club $C$ and a regressive function $f$ on $C$ satisfying that for every $i, j < \lambda^+$ of cofinality $\lambda$ satisfying $f(i) = f(j)$, the conditions $p_i$ and $p_j$ are compatible.

2. A subset $Q$ of a partial order $P$ is directed if every two elements of $Q$ have an upper bound in $P$. A forcing notion $\mathbb{P}$ is $(< \lambda)$-directed closed if every directed $Q \subseteq \mathbb{P}$ of size $< \lambda$, has an upper bound in $\mathbb{P}$.

**Lemma 1.3**

1. $\mathbb{P}(\lambda)$ satisfies the $\lambda^+$-stationary cc and is $(< \lambda)$-directed closed.
2. $\mathbb{P}(\lambda)$ adds a Boolean algebra of size $\lambda^+$ generated by $\{a_i : i < \lambda^+\}$.

**Proof.** For part (1), suppose that $\{p_i : i < \lambda^+\}$ are conditions in $\mathbb{P}(\lambda)$. Let us denote by $w_i$ the set $w_{p_i}$ and let us consider it in its increasing enumeration. Let the isomorphism type of $w_i$ be determined by the order type of $w_i$ and the Boolean algebra equations satisfied between the elements of $w_i$, denote this by $t(w_i)$. Note that by $\lambda^{<\lambda} = \lambda$ the cardinality of the set $\mathcal{I}$ of isomorphism types is exactly $\lambda$, so let us fix a bijection $g : \mathcal{I} \times [\lambda^+]^{<\lambda} \to \lambda^+$. Note that there is a club $C$ of $\lambda^+ \setminus 1$ such that for every point $\gamma$ in $C$ of cofinality $\lambda$ we have $i < \gamma$ iff $w_i \subseteq \gamma$ and $g((\mathcal{I} \times \gamma) \subseteq \gamma$. Define $f$ on $C$ by letting $f(j) = g(t(w_j), w_j \cap j)$ for $j$ of cofinality $\lambda$ and 0 otherwise. Hence $f$ is regressive on $C$ and if $i < j$ both in $C$ have cofinality $\lambda$ and satisfy $f(i) = f(j)$ then we have that $w_i \cap i = w_j \cap j = w_i \cap w_j$ and that $p_i$ and $p_j$ satisfy the same equations on this intersection. Hence $p_i$ and $p_j$ are compatible.
For the closure, note that a family of conditions in \( P(\lambda) \) being \((< \lambda)\)-directed in particular implies that the conditions in any finite subfamily agree on the equations involving any common elements. Therefore the union of the family generates a Boolean algebra which is a common upper bound for the entire family.

For part (2), note that if \( p \in P \) and \( a_i \) is not in \( w_p \), then we can extend \( p \) to \( q \) which is freely generated by \( w_p \cup \{a_i\} \), except for the equations present in \( p \). Hence the set \( D_i = \{ p : a_i \in w_p \} \) is dense. Note that if \( p, q \in G \) satisfy \( p \cap A = q \cap A \), then \( p \) and \( q \) are isomorphic over \( w_p \). Let \( H \) be a subset of \( G \) which is obtained by taking one representative of each isomorphism class with the induced ordering, hence \( H \) is still a filter in \( P \) and it intersects every \( D_i \). By the downward closure of \( G \) it follows that for every \( F \) a subset of \( A \) of size \(< \lambda \), there is exactly one element \( p_F \) of \( H \) with \( w_{p_F} = F \).

(Note that no new bounded subsets of \( \lambda \) are added by \( P \), since \( 1 \) holds). Now note that \( \{p_F : F \in [\lambda]^{<\lambda}\} \) with the relation of embeddability over \( A \) form an inverse system of Boolean algebras directed by \(([\lambda]^{<\lambda}, \subseteq)\). Let \( B \) be the inverse limit of this system (see [9] for an explicit construction of this object), so \( B \) is generated by \( \{a_\alpha : \alpha < \lambda \} \) and it embeds every element of \( H \). (We shall call this algebra “the” generic algebra. It is unique up to isomorphism). \( \star_{1.3} \)

Note that in the case of \( \lambda = \aleph_0 \), we in particular have that the forcing is ccc.

**Theorem 1.4** If \( \mathfrak{A} \) is the generic Boolean algebra for some \( P(\lambda) \), then there is no Banach space \( X_* \) in the ground model such that \( C(\text{St}(\mathfrak{A})) \) isomorphically embeds into \( X_* \) in the extension by \( P(\lambda) \).

**Proof.** Let us fix a \( \lambda \). Suppose for a contradiction that there is an \( X_* \) and embedding \( T \) in the extension contradicting the statement of the theorem. Let \( p^* \) force that \( T \) is an isomorphic embedding and, without loss of generality \( p^* \) decides a positive constant \( c \) such that \( 1/c \cdot ||x|| \leq ||Tx|| \leq c \cdot ||x|| \) for every \( x \). Let \( n_* \) be a positive integer such that \( c < n_* \). For each \( i < \lambda^+ \) let \( p_i \geq p^* \) be such that \( a_i \in w_{p_i} \), and without loss of generality \( p_i \) decides the value \( x_i = T(\chi_{[a_i]}) \). Now let us perform a similar argument as in the proof of the chain condition, and in particular by passing to a subfamily of the same cardinality we can assume that the sets \( w_{p_i} \) form a \( \Delta \)-system with a root \( w^* \). Since \( a_i \in w_{p_i} \), we can assume that \( w_i \neq w^* \), and in particular that \( a_i \in w_i \setminus w^* \). Let us take distinct \( i_0, i_1, \ldots i_{n_2} \). Since there are no equations in \( p_i \) or \( p_j \) that connect \( a_i \) and \( a_j \) for \( i \neq j \), we can find two conditions \( q' \) and \( q'' \) which both extend \( p_{i_0}, \ldots, p_{n_2} \) and such that in \( q' \) we have \( a_{i_0} \leq \ldots \leq a_{i_{n_2}} \) and in \( q' \) we have that \( a_{i_0}, \ldots, a_{i_{n_2}} \) are pairwise disjoint. Hence \( q' \) forces that \( \chi_{[a_{i_0}]} + \ldots \chi_{[a_{i_{n_2}}]} \) has norm \( n_*^2 + 1 \) and \( q'' \) forces that it has norm 1. Therefore the vector \( x_{i_0} + \ldots x_{i_{n_2}} \) must have norm both \( > n_* \) and \( < n_* \) in \( X_* \), which is impossible. \( \star_{1.4} \)

The ideas in the proof of Theorem 1.4 which use the possibility to extend Cohen-like conditions in incomparable ways are present in the proof by Brech and Koszmider ([3]; Theorem 3.2) that \( l^\infty/c_0 \) is not a universal Banach space for density \( c \) in the extension.
by \(\aleph_2\) Cohen reals, and the earlier proof by Džamonja and Shelah ([7], Theorem 3.4) that in the model considered there there is no universal normed vector space over \(\mathbb{Q}\) of size \(\aleph_1\) under isomorphic vector space embeddings. In fact in the case of \(\lambda = \aleph_0\) we can expand on these ideas to prove the following Theorem 1.5. We shall use a simplified \((\omega, 1)\)-morass constructed in ZFC by Velleman in [18]\(^3\).

**Theorem 1.5** *Forcing with one Cohen real adds a Boolean algebra \(A\) of size \(\aleph_1\) such that in the extension the Banach space \(C(\text{St}(A))\) does not isomorphically embed into any space \(X_\ast\) in the ground model of density \(\aleph_1\).*

**Proof.** Let us recall that a simplified \((\omega, 1)\)-morass is a system \(\langle \theta_\alpha : \alpha \leq \omega, \alpha < \omega \rangle\), \(\langle F_{\alpha,\beta} : \alpha < \beta \leq \omega \rangle\) such that

1. for \(\alpha < \omega\), \(\theta_\alpha\) is a finite number \(> 0\), and \(\theta_\omega = \omega_1\),
2. for \(\alpha < \beta < \omega\), \(F_{\alpha,\beta}\) is a finite set of order preserving functions from \(\theta_\alpha\) to \(\theta_\beta\),
3. \(F_{\alpha,\omega}\) is a set of order preserving functions from \(\theta_\alpha\) to \(\omega_1\) such that \(\bigcup_{f \in F_{\alpha,\omega}} f^\ast \theta_\alpha = \omega_1\),
4. for all \(\alpha < \beta < \gamma \leq \omega\) we have that \(F_{\alpha,\gamma} = \{ f \circ g : g \in F_{\alpha,\beta} \text{ and } f \in F_{\beta,\gamma} \}\),
5. \(F_{\alpha,\alpha+1}\) always contains the identity function \(id_\alpha\) on \(\theta_\alpha\) and either this is all, or \(F_{\alpha,\alpha+1} = \{id_\alpha, h_\alpha\}\) for some \(h_\alpha\) such that there is a splitting point \(\beta\) with \(h_\alpha \upharpoonright \beta = id_\alpha \upharpoonright \beta\) and \(h_\alpha(\beta) > \theta_\alpha\),
6. for every \(\beta_0, \beta_1 < \omega\) and \(f_l \in F_{\beta_l,\omega}\) for \(l < 2\) there is \(\gamma < \omega\) with \(\beta_0, \beta_1 < \gamma\), function \(g \in F_{\gamma,\omega}\) and \(f_l' \in F_{\beta_l,\gamma}\) such that \(f_l = g \circ f_l'\) for \(l < 2\).

For future use in the proof we shall need the following:

**Observation 1.6** *The sequence \(\langle \theta_\alpha \rangle_{\alpha < \omega}\) is a non-decreasing sequence diverging to \(\infty\).*

To see this, notice that the fact that \(id_{\theta_\alpha} \in F_{\alpha,\alpha+1}\) guarantees that \(\theta_\alpha \leq \theta_{\alpha+1}\). For the rest, suppose that \(\langle \theta_\alpha \rangle_{\alpha < \omega}\) is bounded by some \(k < \omega\). This implies that there is \(\alpha < \omega\) such that for all \(\gamma \geq \alpha\) we have \(\theta_{\gamma+1} = \theta_\gamma \leq k\) and \(F_{\gamma,\gamma+1} = \{id_\gamma\}\). This implies by (4) that \(F_{\alpha,\omega} = \{id_\alpha\}\), in contradiction with (3).

Fixing a simplified \((\omega, 1)\)-morass as in the above definitions, we shall define a Boolean algebra \(A\) as follows. By induction on \(\alpha \leq \omega\) we define a Boolean algebra \(A_\alpha\) on a subset of \(\theta_\alpha \cup (\omega_1 \times \{0\})\) generated by \(\{i : i < \theta_\alpha\}\), and at the end we shall have \(A = A_\omega\). The first requirement of the induction will be:

(i) if \(\beta < \alpha\) and \(f \in F_{\beta,\alpha}\) then \(f\) gives rise to a Boolean algebra embedding.

\(^3\)In a related work, Velleman in [19] showed that an effect of a Cohen real on such a morass is to add a Souslin tree.
In order to guarantee this, we first define by induction on $\aleph_1$, and then it suffices to define $j$ natural numbers in the case $\aleph_1$ is uncountable, and then we shall guarantee that for all natural numbers $n_\ast \geq 3$ and for all $i : \omega_1 \to \omega_1$ there are $i_0, i_1, \ldots, i_{n_\ast}$ such that:

\begin{align*}
\text{if } ||z_{j(i_0)} + z_{j(i_1)} + \ldots + z_{j(i_{n_\ast})}|| < n_\ast - 1, \text{ then } i_0, i_1, \ldots, i_{n_\ast} \text{ are disjoint in } A \text{ and } \\
\text{if } ||z_{j(i_0)} + z_{j(i_1)} + \ldots + z_{j(i_{n_\ast})}|| \geq n_\ast - 1, \text{ then } i_0 \alpha i_1 \ldots <_\alpha i_{n_\ast}.
\end{align*}

To see that our algebra, once constructed, has the required properties, suppose that $T$ is an isomorphic embedding of $C(\text{St}(A))$ into some $X_\ast$ as above and that (*) holds for a dense set $\{z_i : i < \omega_1\}$ of $X_\ast$. Let $x_i = T(\chi[i])$ for $i < \omega_1$ and let $n_\ast \geq 3$ be large enough that for each $x \in C(\text{St}(A))$ we have that

\[
\frac{1}{n_\ast} ||x|| < ||T(x)|| < n_\ast ||x||.
\]

For each $i < \omega_1$ let us choose $j(i)$ such that $||x_i - z_{j(i)}|| < \frac{1}{n_\ast + 1}$. Let $i_0, i_1, \ldots, i_{n_\ast}$ be as guaranteed by (*). Suppose first that $||z_{j(i_0)} + z_{j(i_1)} + \ldots + z_{j(i_{n_\ast})}|| < n_\ast - 1$. Then in the case $||z_{j(i_0)} + z_{j(i_1)} + \ldots + z_{j(i_{n_\ast})}|| \geq n_\ast - 1$ we have that

\[
||x_{i_0} + x_{i_1} + \ldots + x_{i_{n_\ast}}|| \leq ||z_{j(i_0)} + z_{j(i_1)} + \ldots + z_{j(i_{n_\ast})}|| + \sum_{k \leq n_\ast} ||x_{i_k} - z_{j(i_k)}|| < n_\ast - 1 + \frac{n_\ast^2 + 1}{n_\ast + 1} = n_\ast,
\]

yet $\frac{1}{n_\ast} ||\chi[i_0] + \ldots + \chi[i_{n_\ast}]|| = \frac{n_\ast^2 + 1}{n_\ast} > n_\ast$, in contradiction with the choice of $n_\ast$. The other case is similar.

Now we claim that to guarantee the condition (*) for any fixed $X_\ast$ and $\{z_i : i < \omega_1\}$, it suffices to assure that for all

\[
n_\ast \geq 3, A \in [\omega_1]^{\omega_1}, j_0 : A \to \omega_1 \in V \text{ there are } i_0, i_1, \ldots, i_{n_\ast} \in A \text{ exemplifying (*)}.\]

For this, let us recall that for every $j : \omega_1 \to \omega_1$ in $V[G]$, there is $A \in [\omega_1]^{\omega_1}$ in $V$ and $j_0 : A \to \omega_1$ in $V$ such that $j \upharpoonright A = j_0$. [Namely, if $j$ is a name for a function $j : \omega_1 \to \omega_1$ then for every $\alpha < \omega_1$ there is $p \in G$ deciding the value $j(\alpha)$. Since the forcing notion is countable, there is $p^* \in G$ such that the set $A$ of all $\alpha$ for which $p$ decides the value of $\alpha$ is uncountable, and then it suffices to define $j_0$ on $A$ by $j_0(\alpha) = \beta$ iff $p$ forces $j(\alpha) = \beta$.]

In order to guarantee this, we first define by induction on $n$ an increasing sequence of natural numbers $\alpha_n$ such that $\alpha_0 = 0$ and $\theta_{\alpha_{n+1}} > 2\theta_{\alpha_n} + n$. In order to formulate the second requirement of the inductive construction we need a lemma.

**Lemma 1.7** Let $n < \omega$. Then there is a set $A_\ast \subseteq \theta_{\alpha_{n+1}}$ of size $n + 1$ such that for any $\beta \leq \alpha_n$ and $f \in F_{\beta, \alpha_{n+1}}$, the intersection $A_\ast \cap f^{\beta} \theta_{\beta}$ is empty.
Proof of the Lemma. Requirement (4) in the definition of the morass shows that it suffices to work with $\beta = \alpha_n$ as for $\beta < \alpha_n$ we have $\bigcup \{h^*\theta_{\beta} : h \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}\} \subseteq \{f^*\theta_{\alpha_n} : f \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}\}$. The proof proceeds on a backward induction on the size $k$ of $\alpha_{n+1} - \alpha_n \geq 1$. In the case $k = 1$ we have that $|\mathcal{F}_{\alpha_n, \alpha_{n+1}}| \leq 2$ and for each $f \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}$ we have $|f^*\theta_{\alpha_n}| \leq \theta_{\alpha_n}$ and therefore $\theta_{\alpha_{n+1}} \setminus \bigcup_{f \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}} f^*\theta_{\alpha_n} \geq 2\theta_{\alpha_n} > n$, so we can take $A = \theta_{\alpha_{n+1}} \setminus \bigcup_{f \in \mathcal{F}_{\alpha_n, \alpha_{n+1}}} f^*\theta_{\alpha_n}$. For the step $k + 1$, let $\beta = \alpha_n + 1$ and then it suffices to again apply (4) from the definition of a morass. \(\star_{1.7}\)

Our second requirement of the induction will be as follows, where $r$ is the generic Cohen real, viewed as a function from $\omega$ to 2:

(iii) Let $n < \omega$ and suppose that $\mathfrak{A}_{\alpha_n}$ has been defined. Then $\mathfrak{A}_{\alpha_{n+1}}$ is generated freely over $\mathfrak{A}_{\alpha_n}$ except for the equations induced by requirement (i) and the requirement that $i_0, i_1, \ldots, i_n$ are disjoint in $\mathfrak{A}_{\alpha_{n+1}}$ if $r(n) = 0$ and $i_0 <_{\alpha_{n+1}} i_1 <_{\alpha_{n+1}} \ldots <_{\alpha_{n+1}} i_n$ if $r(n) = 1$, where $\{i_0, i_1, \ldots, i_n\}$ is the increasing enumeration of the first $n + 1$ elements of $A_n$. For $\beta \in (\alpha_n, \alpha_{n+1})$ we let $\mathfrak{A}_\beta$ be the Boolean span in $\mathfrak{A}_{\alpha_{n+1}}$ of $\theta_\beta$.

Given Lemma 1.7, it is clear that conditions (i) and (ii) can be met in a simple inductive construction of $\mathfrak{A}_\alpha$, as there will be no possible contradiction between (i) and the requirements that (ii) puts on the elements of $A_n$. Let us now show that the resulting algebra is as required. Hence let us fix $\alpha_n, X_*, \{z_i : i < \omega_1\}, A$ and $j_0$ as in (**). Is it then sufficient to observe that the following set is dense in the Cohen forcing (note that the set is in the ground model):

$$\{p : (\exists n \in \text{dom}(p))(n \geq n_*^2 \text{ and } p(n) = 0 \iff \|z_{j_0(i_0)} + z_{j_0(i_1)} + \ldots + z_{j_0(i_{n_*^2})}\| < n_* - 1\},$$

where $\{i_0, i_1, \ldots, i_{n_*^2}\}$ is the increasing enumeration of the first $n_*^2 + 1$ elements of $A_n$.

\(\star_{1.5}\)

We note in the following Theorem 1.9 that the construction from Theorem 1.5 can be refined so that the resulting Banach space is weakly compactly generated. This give rise to Theorem 1.16 which shows that in the Cohen model for $\aleph_2$ Cohen reals the universality number for wcg Banach spaces of density $\aleph_1$ is $\aleph_2$. A model for this was obtained by Brech and Koszmider in [4] using a ready-made forcing. Theorem 1.16 answers a question raised by Koszmider (private communication). The reader not interested in wcg spaces can jump directly to after the proof of Theorem 1.9, those who proceed to read Theorem 1.9 will be assumed to know what wcg spaces are. Let us however recall one definition:

**Definition 1.8** A subset $C$ of a Boolean algebra $\mathfrak{A}$ has the nice property if for no finite $F \subseteq C$ do we have $\bigvee F = 1$. A Boolean algebra $\mathfrak{A}$ is a $c$-algebra iff there is a family $\{B_n : n < \omega\}$ of pairwise disjoint antichains of $\mathfrak{A}$ whose union has the nice property and generates $\mathfrak{A}$.

Bell showed in [7] that the Stone space of a $c$-algebra $\mathfrak{A}$ is a uniform Eberlein compact and hence $C(\text{St}(\mathfrak{A}))$ is a wcg space, see [4] for details. We prove:
Theorem 1.9 Forcing with one Cohen real adds a $c$-algebra $\mathfrak{A}$ of size $\aleph_1$ such that in the extension the Banach space $C(\text{St}(\mathfrak{A}))$ does not isomorphically embed into any space $C(\text{St}(\mathfrak{B}))$, where $\mathfrak{B}$ is any Boolean algebra in the ground model of size $\aleph_1$.

Proof. Fixing a simplified $(\omega, 1)$-morass as in the proof of Theorem 1.5, we shall define a Boolean algebra $\mathfrak{A}$ as follows. By induction on $\alpha \leq \omega$ we define a Boolean algebra $\mathfrak{A}_\alpha$ on a subset of $\theta_\alpha \cup (\omega_1 \times \{0\})$ generated by $\{i : i < \theta_\alpha\}$ freely except for the requirements in the next sentence, and at the end we shall have $\mathfrak{A} = \mathfrak{A}_\omega$. At the same time we shall define pairwise disjoint antichains $\langle B^n_\alpha : n < n_\alpha \rangle$ such that $\bigcup_{n < n_\alpha} B^n_\alpha = \{i : i < \theta_\alpha\}$ and this set has the nice property. The basic requirements of the induction will be:

(i) if $\beta < \alpha$ and $f \in \mathcal{F}_{\beta, \alpha}$ then $f$ gives rise to a Boolean algebra embedding.

(ii) if $\beta < \alpha$ then $\theta_\alpha < \theta_\beta \implies n_\beta < n_\alpha$ and if $f \in \mathcal{F}_{\beta, \alpha}$ and $i \in B^n_\alpha$ then $f(i) \in B^n_\beta$.

Requirement (i) guarantees that the final object $\mathfrak{A}$ is indeed a well defined Boolean algebra. The second requirement guarantees that the algebra obtained is a $c$-algebra, as will become clear later. As in the proof of Theorem 1.5 and using the same notation as there, we shall guarantee the requirement (**) if $\beta < \alpha$ then $\theta_\alpha < \theta_\beta \implies n_\beta < n_\alpha$ and if $f \in \mathcal{F}_{\beta, \alpha}$ and $i \in B^n_\alpha$ then $f(i) \in B^n_\beta$.

Our final requirement of the induction will be as follows, where $r$ is the generic Cohen real, viewed as a function from $\omega$ to $2$:

(iii) Let $n < \omega$ and suppose that $\mathfrak{A}_{\alpha_n}$ has been defined. Then $\mathfrak{A}_{\alpha_{n+1}}$ is generated freely over $\mathfrak{A}_{\alpha_n}$ except for the equations induced by requirements (i) and (ii) and the requirement that $i_0, i_1, \ldots, i_n$ are disjoint in $\mathfrak{A}_{\alpha_{n+1}}$ if $r(n) = 0$ and $i_0 <_{\alpha_{n+1}} i_1 <_{\alpha_{n+1}} \ldots <_{\alpha_{n+1}} i_n$ if $r(n) = 1$, where $\{i_0, i_1, \ldots, i_n\}$ is the increasing enumeration of the first $n + 1$ elements of $A_n$. For $\beta \in (\alpha_n, \alpha_{n+1})$ we let $\mathfrak{A}_\beta$ be the Boolean span in $\mathfrak{A}_{\alpha_{n+1}}$ of $\theta_\beta$.

Given Lemma 1.7, it is clear that conditions (i) and (ii) can be met in a simple inductive construction of $\mathfrak{A}_\alpha$, as there will be no possible contradiction between (i) and the requirements that (ii) puts on the elements of $A_n$. Let us comment on how to meet the requirement (iii), say at stage $n + 1$. We have chosen $A_n$ so that it elements are not in the image of any $B^n_\beta$ for any $\beta < \alpha_{n+1}$ and any $m < n_\beta$. Therefore we are free to either set $n_{\alpha_{n+1}} = n_\alpha + 1$ and put all elements of $A_n$ into one antichain, if $r(n) = 0$, or set $n_{\alpha_{n+1}} = n_\alpha + |A_n|$ and put each element of $A_n$ into a new antichain, requiring $i_0 <_{\alpha_{n+1}} i_1 <_{\alpha_{n+1}} \ldots <_{\alpha_{n+1}} i_n$ if $r(n) = 1$. There are no problems in generating $\mathfrak{A}_{\alpha_{n+1}}$ freely except for these requirements and those dictated by (i)-(iii) as the requirements are not contradictory.

Lemma 1.10 Suppose that the requirements (i)-(iii) from above are satisfied. Then letting for $n < \omega$, $B^n = \bigcup_{\alpha \leq \omega, n < \alpha} B^n_\alpha$, we obtain disjoint antichains which have the nice property in $\mathfrak{A}$ and $\mathfrak{A}$ is generated by their union.
Proof. (of the Lemma) As in the proof of Theorem 1.5, (i) guarantees that $\mathfrak{A}$ is a Boolean algebra. Requirement (ii) guarantees that $B_n$s are disjoint antichains. The set $B = \bigcup_{n<\omega} B_n$ has the nice property since this property is verified by finite subsets of $B_n$. In other words, if $B$ does not have the nice property, then there is a finite $F \subseteq B$ with $\forall F = 1$. There must be $\alpha < \omega$ such that $F \subseteq \bigcup_{n<\alpha} B_n$ already holds in $\mathfrak{A}_\alpha$. However, this is in contradiction to the fact that $\mathfrak{A}_\alpha$ is generated freely except for the equations required in (i)-(iii). It is also clear that $B$ generates $\mathfrak{A}$. ⭐1.10

The proof that (***) holds is exactly the same as in the proof of Theorem 1.5. ⭐1.9

We would now like to extend Theorems 1.4 and 1.5 by using “classical tricks” with iterations to obtain the non-existence of an isomorphically universal Banach space of density $\lambda^+$ in an iterated extension. It turns out that the cases $\lambda = \aleph_0$ and larger $\lambda$ are different. Let us consider the former case first.

Scenario. Let $P$ denote $P(\omega_1)$ and let $Q$ be the iteration of $\omega_2$ steps of $P$, say over a model of GCH. Then in the extension by $Q$ we have that $2^{\aleph_0} = \aleph_2$ and the new reals are added throughout the iteration. We try to argue that the universality number for the class of Banach spaces of density $\aleph_1$ is $\aleph_2$, so we try for a contradiction. Suppose that the number is $\leq \aleph_1$, then by Observation 0.6 there is actually a single universal element, say $X_\ast$. We can by Fact 0.1 assume that $X_\ast = C(St(\mathfrak{A}))$ for some Boolean algebra $\mathfrak{A}_\ast$ of size $\aleph_1$. By standard arguments, we can find an intermediate universe in the iteration, call it $V$, which contains $\mathfrak{A}_\ast$. If we could say that $C(St(\mathfrak{A}))$ is also in $V$, then we could apply Theorem 1.4 to conclude that for the generic Boolean algebra $\mathfrak{B}$ added to $V$, $C(St(\mathfrak{B}))$ does not embed into $C(St(\mathfrak{A}))$ and we would be done. The problem is that since we keep adding reals, $C(St(\mathfrak{A}))$ keeps changing from $V$ to the final universe and hence there is no contradiction. We resolve this difficulty by a use of simple functions with rational coefficients. 4.

Theorem 1.11 Let $\kappa$ be a cardinal with $\text{cf}(\kappa) \geq \aleph_2$ and let $G$ be a generic for the iteration of length $\kappa$ with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ by finite conditions over a model $V$ of GCH, or the forcing to add $\kappa$ many Cohen reals over a model $V$ of GCH. Then in $V[G]$ the universality number of the class of Banach spaces of density $\aleph_1$ under isomorphisms is $\kappa$.

Proof. For simplicity, let us show that there is no single universal element, the proof in the case of $< \kappa$ universals being very similar. Suppose that $C(St(\mathfrak{A}))$ is a universal Banach space of density $\aleph_1$ and without loss of generality, assume that $\mathfrak{A}$ is in the ground model. Let $p^*$ force $\hat{T}$ to be an isomorphic embedding from the $C$ of the Stone space of the first generic algebra to $C(St(\mathfrak{A}))$ with $||T|| \leq c < n_*$. Let $\varepsilon > 0$ be small enough, precisely $\varepsilon < \min \left\{ \frac{n_* - c}{n_*}, \frac{n_*^2 + 1 - cn_*}{cn_*^2 + 1} \right\}$. For $i < \omega_1$ let $p_i \geq p^*$ force that $h_i$ is a

4A preliminary version of this proof was stated for isometric embeddings and upon hearing about it, Saharon Shelah suggested that it should work for the isomorphic embeddings as well.
simple function with rational coefficients (so \( h_i \in \mathbf{V} \)) satisfying \( ||T(\chi_{[a_i]}) - h_i|| < \varepsilon \). We proceed similarly to the proof of Theorem 1.4, and without loss of generality assume that \( a_i \in \text{dom}(p_i(0)) \), for every \( i \). Now do several \( \Delta \)-system arguments as standard, and take “clean” \( i_0 < i_1 < \ldots < i_{n^2} \), hence \( \bigcup_{k=0}^{n^2} p_{i_k} \) does not decide any relation between \( a_{i_k} \)’s. If \( ||\Sigma_{k=0}^{n^2} h_{i_k}|| \geq n^* \), it follows by the choice of \( \varepsilon \) that there cannot be an extension of \( \bigcup_{k=0}^{n^2} p_{i_k} \) forcing \( a_{i_k} \)’s and to be disjoint, a contradiction, and if \( ||\Sigma_{k=0}^{n^2} h_{i_k}|| < n^* \) then it follows that there cannot be an extension of \( \bigcup_{k=0}^{n^2} p_{i_k} \) forcing \( a_{i_k} \)’s to be increasing with \( k \), again a contradiction. \( \star \)

In the situation in which we add a generic Boolean algebra of cardinality \( \lambda^+ \geq \aleph_2 \) with conditions of size \( < \lambda \), the forcing is countably closed and therefore it does not add any new reals. Hence the attempted argument of proving the analogue of Theorem 1.11 for isomorphic embeddings works in this case. However, to make this into a theorem we need an iteration theorem for the forcing, in order to make sure that the cardinals are preserved. Paper [6] reviews various known forcing axioms and building up on them proves the following Theorem 1.12, which is perfectly suitable for our purposes.

**Theorem 1.12 (Cummings, Džamonja, Magidor, Morgan and Shelah)** Let \( \lambda = \lambda^{<\lambda} \). Then, the iterations with \( (< \lambda) \)-supports of \( (< \lambda) \)-closed stationary \( \lambda^+\text{-cc} \) forcing which is countably parallel-closed, are \( (< \lambda) \)-closed stationary \( \lambda^+\text{-cc} \).

Here, the property of countable parallel-closure is defined as follows:

**Definition 1.13** Two increasing sequences \( \langle p_i : i < \omega \rangle \) and \( \langle q_i : i < \omega \rangle \) of conditions in a forcing \( P \) are said to be pointwise compatible if for each \( i < \omega \) the conditions \( p_i, q_i \) are compatible. The forcing \( P \) is said to be countably parallel-closed if for every two \( \omega \)-sequences of pointwise compatible conditions as above, there is a common upper bound to \( \{ p_i, q_i : i < \omega \} \) in \( P \).

**Lemma 1.14** The forcing \( P(\lambda) \) is countably parallel-closed.

**Proof.** Suppose that \( \langle p_i : i < \omega \rangle \) and \( \langle q_i : i < \omega \rangle \) are pairwise compatible increasing sequences. In particular this means that, on the one hand, each \( p_i, q_i \) agree on their intersections, and on the other hand that \( \bigcup_{i<\omega} p_i \) and \( \bigcup_{i<\omega} q_i \) each form a Boolean algebra. Furthermore \( \bigcup_{i<\omega} p_i \) and \( \bigcup_{i<\omega} q_i \) agree on their intersection, and hence their union can be used to generate a Boolean algebra, which will then be a common upper bound to the two sequences. \( \star \)

**Theorem 1.15** Let \( \lambda = \lambda^{<\lambda} \), \( \text{cf}(\kappa) \geq \lambda^{++} \) and let \( G \) be a generic for the iteration with \( (< \lambda) \)-supports of length \( \kappa \) of the forcing to add the generic Boolean algebra of size \( \lambda^+ \) by conditions of size \( < \lambda \) over a model of GCH, or in the case of \( \lambda = \aleph_0 \) the forcing to add \( \kappa \) many Cohen reals. Then in \( V[G] \):
(1) the universality number of the class of Banach spaces of density $\lambda^+$ under isomorphisms is $\kappa = 2^{\lambda^+}$.

(2) there are $2^{\lambda^+}$ many pairwise non-isomorphic Banach spaces of density $\lambda^+$.

Proof. (1) For the case of $\lambda = \aleph_0$ we use Theorem 1.11. In the case of uncountable $\lambda$, it follows from Lemma 1.3, Lemma 1.14 and Theorem 1.12 that the iteration of the forcing $\mathbb{P}(\lambda)$ described in the statement of the Theorem, preserves cardinals. By the countable closure of the forcing no reals are added, and hence the argument described in the Scenario, with $\lambda^+$ in place of $\aleph_1$, works out to give the desired conclusion. (2) The claim is that the Banach spaces $C(\text{St}(\mathcal{A}))$ added at the individual steps of the iteration are not pairwise isomorphic. To prove this, we only need to notice that an embedding of a Banach space into another is determined by the restriction of an embedding onto a dense set, hence in our case a set of size $\lambda^+$, and then to argue as in (1). ★1.15

Using the same reasoning as in Theorem 1.15 with $\lambda = \aleph_0$ and relating this to Theorem 1.9 we obtain

**Theorem 1.16** In the Cohen model for $2^{\aleph_0} = \kappa$, where $\text{cf}(\kappa) \geq \aleph_2$, the universality number of the wcg Banach spaces is at least $\text{cf} \kappa$ and there are $\text{cf}(\kappa)$ pairwise non-isomorphic wcg Banach spaces.

## 2 Isometries and isomorphisms of small Banach-Mazur diameter

As we have seen in §1, the Cohen-like extensions do not distinguish the universality number of Banach spaces of a fixed density under isomorphisms or under isometries, the subject of Conjecture 0.3. This short section gives evidence that some forms of that conjecture are just true in $\text{ZFC}$ and serves as an interlude to motivate study of other kinds of embedding but isometries and isomorphisms. The results will be easy to a Banach space theorist as they use a well known theorem in the subject, due to Jarosz in [10].

**Theorem 2.1 (Jarosz)** Suppose that $T : C(K) \to C(L)$ is an isomorphic embedding such that $\|T\| \cdot \|T^{-1}\| < 2$. Then there is a closed subspace $L_1 \subseteq L$ which can be continuously mapped onto $K$.

We can use Jarosz’s theorem to reduce the question of the existence of isometrically universal Banach spaces or more generally, universals under isomorphisms of small Banach-Mazur diameter to that of Boolean algebras and their ideals. Let us recall the following
Theorem 2.4 Suppose that $L$ maps onto $K$, then $C(K)$ isometrically embeds into $C(L)$.

Proof. Let $F$ be a continuous mapping from $L$ that maps onto $K$. Defining for $f \in C(K)$ $T(f)$ by $Tf(x) = f(F(x))$ for $x \in L$ gives the desired isometry. $\star_{2.2}$

We shall also use the following

Definition 2.3 Let $\mathcal{M}$ be a family of Boolean algebras and $\mathcal{N} \subseteq \mathcal{M}$. We say that $\mathcal{N}$ is weakly universal for $\mathcal{M}$ if every element of $\mathcal{M}$ embeds into a factor of an element of $\mathcal{N}$.

Theorem 2.4 For every $\kappa$, the following numbers are the same

- the universality number of the class of Banach spaces of density $\kappa$ with isomorphic embeddings of the Banach-Mazur diameter $< 2$,
- the smallest size of a weakly universal family of Boolean algebras of size $\kappa$ and
- the universality number of the class of Banach spaces of density $\kappa$ under isometries.

Proof. Let $\kappa$ be fixed. Suppose that $\{X_\alpha : \alpha < \alpha^*\}$ are universal under isomorphisms of the Banach-Mazur diameter $< 2$ in the class of Banach spaces of density $\kappa$. By the introductory remarks, it follows that we can assume that for every $\alpha$ there is a Boolean algebra $\mathfrak{A}_\alpha$ of size $\kappa$ such that $X_\alpha = C(St(\mathfrak{A}_\alpha))$. The closed subspaces of $St(\mathfrak{A}_\alpha)$ are of the form $St(\mathfrak{A}_\alpha/I)$ where $I$ is an ideal in $\mathfrak{A}_\alpha$. Let $\mathfrak{A}$ be a Boolean algebra of size $\kappa$ and let $K$ be it Stone space. Hence $C(K)$ embeds with a Banach-Mazur distance $< 2$ in some $X_\alpha$. By Jarosz’s theorem, there is a closed subspace $L$ of $St(\mathfrak{A}_\alpha)$ such that $L$ maps continuously onto $K$. Let $I$ be an ideal on $\mathfrak{A}_\alpha$ such that $L = St(\mathfrak{A}_\alpha/I)$. It follows that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_\alpha/I$. Hence every Boolean algebra of size $\kappa$ embeds in a factor of some $\{\mathfrak{A}_\alpha : \alpha < \alpha^*\}$.

On the other hand suppose that $\{\mathfrak{A}_\alpha : \alpha < \alpha^*\}$ is a family of Boolean algebras of size $\kappa$ such that every Boolean algebra of size $\kappa$ embeds in a factor of some $\mathfrak{A}_\alpha$. Let $X_\alpha = C(St(\mathfrak{A}_\alpha))$ and we claim that $\{X_\alpha : \alpha < \alpha^*\}$ are universal under isometries in the class of Banach spaces of density $\kappa$. It suffices to show that for every Boolean algebra $\mathfrak{A}$ of size $\kappa$, $C(St(\mathfrak{A}))$ isometrically embeds into some $X_\alpha$. Given such $\mathfrak{A}$, let $\alpha < \alpha^*$ be such that for some ideal $I$ on $\mathfrak{A}$ we have that $\mathfrak{A}$ embeds into $\mathfrak{A}_\alpha/I$. Hence $St(\mathfrak{A})$ is a continuous image of the $St(\mathfrak{A}_\alpha/I)$, which is a closed subspace $L$ of $X_\alpha$. By Observation 2.2, $C(St(\mathfrak{A}))$ embeds isometrically into $C(L)$. Let $B$ a basis of $C(L)$ as a vector space. By Tietze’s extension theorem, every function $f \in B$ can be extended to a function $g \in X_\alpha = C(St(\mathfrak{A}_\alpha))$ of the same norm. Let this association be called $T$. Now extend $T$ to a linear embedding from $C(L)$, which finally shows that $C(L)$ and hence $C(K)$ embeds isomorphically into $X_\alpha$. As the universality number of the class of Banach spaces of density $\kappa$ under isometries is clearly $\geq$ the universality number of the class of Banach spaces of density $\kappa$ with isomorphic embeddings of the Banach-Mazur diameter $< 2$, we obtain the desired equality. $\star_{2.4}$

$^5$ of course, we are simply spelling out an application of the Hahn-Banach theorem
3 Natural spaces of functions

As we have seen in §2, there is a good motivation to study other kinds of embeddings of Banach spaces but isometries and isomorphisms. Sticking to the spaces of the form $C(K)$, among the classically studied isomorphic embeddings are those that preserve multiplication, or the ones that preserve the pointwise order of functions. It is known for either one of them (Gelfand and Kolmogorov [8] for the former and Kaplansky [11] for the latter) that if they are onto they actually characterize the topological structure of the space, that is if $T : C(K) \to C(L)$ is an onto embedding which either preserves multiplication or the pointwise order, then $K$ and $L$ are homeomorphic. We shall show that in moving from the order preserving onto assumption just a small bit, we no longer have the preservation of the homeomorphic structure, but under the assumption that GCH fails sufficiently, we do have a large number of pairwise nonisomorphic spaces and a large universality number. The same methods can be applied to study embeddings with some amount of preservation of multiplication, which we shall not do here. Our methods will involve a combination of model theory, set theory and Banach space theory. In this section we introduce a simple model-theoretic structure which will be used to achieve that mixture of methods.

Suppose that $\mathfrak{A}$ is a Boolean algebra. We shall associate to it a simple structure whose role is to represent the space $C(K)$, where $K$ is the Stone space of $\mathfrak{A}$, $K = St(\mathfrak{A})$. The idea is as follows. We are interested in the set of all simple functions with rational coefficients defined on $K$, so functions of the type $\sum_{i \leq n} q_i \chi_{[a_i]}$, where each $q_i$ is rational, $a_i \in \mathfrak{A}$ and $[a_i]$ denotes the basic clopen set in $K$ determined by $a_i$. Every element of $C(K)$ is a limit of a sequence of such functions, since the limits of such sequences form exactly the class of Lebesgue integrable functions, which of course includes $C(K)$. Let us then consider the vector space freely generated by $\mathfrak{A}$ over $\mathbb{Q}$, call it $V = V(\mathfrak{A})$. Hence every simple function on $K$ with rational coefficients corresponds uniquely to an element of $V$, via an identification of each $a \in \mathfrak{A}$ with $\chi_{[a]}$. Using coordinatwise addition and scalar multiplication the product $W = V^\omega$ becomes a vector space. Any function $f$ in $C(K)$ can be identified with an element of this vector space, namely a sequence of simple rational functions whose limit is $f$, and hence $C(K)$ can be identified with a subset of $W$.

To encapsulate this discussion we shall work with vector spaces with rational coefficients and with two distinguished unary predicates $C, C_0$ satisfying $C_0 \subseteq C$. With our motivation in mind, we shall call them function spaces. If such a space $(V, C, C_0)$ is the space of sequences of simple rational functions over a Stone space $K = St(\mathfrak{A})$ and $C, C_0$ correspond respectively to the set of such sequences which converge or converge to 0, then we call $(V, C, C_0)$ a natural space and we denote it by $N(\mathfrak{A})$. In spaces of the form $N(\mathfrak{A})$ for an element $\bar{f}$ of $C^N(\mathfrak{A})$ we define $||\bar{f}||$ as the norm in $C(St(\mathfrak{A}))$ of the limit $f$ of $\bar{f}$. If $\phi$ is an embedding between $N(\mathfrak{A})$ and $N(\mathfrak{B})$ we shall say that $D > 0$ is a constant of the embedding if for every $\bar{f}$ of $C^N(\mathfrak{B})$ we have that $\frac{1}{D} \cdot ||\bar{f}|| \leq ||\phi(\bar{f})|| \leq D \cdot ||\bar{f}||$. Not

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This vector space figures in [3] with the notation $C_Q(\mathfrak{A})$ and is considered in a different context.
every embedding has such a constant, but we shall only work with the ones which do.

We shall mostly be interested in a specific case of the representation of continuous functions as limits of simple functions, given by the following observation:

**Lemma 3.1** Suppose that $K = \text{St}(\mathfrak{A})$ is the Stone space of a Boolean algebra $\mathfrak{A}$ and let $f \geq 0$ be a function in $C(K)$ with $\|f\| \leq D^*$ for some $D^* > 0$. Then there is a sequence $\langle f_n : n < \omega \rangle$ of simple functions, where each $f_n$ is of the form $\Sigma_{i \leq n} q_i \chi_{[a_i]}$, with each $q_i$ rational in $(0, D^*]$ and $a_i \in \mathfrak{A}$, such that $f = \lim f_n$.

**Proof.** By multiplying by a constant if necessary, we can assume that $D^* = 1$. Functions of the form $\Sigma_{i \leq n} r_i \chi_{[a_i]}$, with each $r_i$ real, contain the constant function 1, form an algebra and separate the points of $K$, hence by the Stone-Weierstrass theorem they form a dense subset of $C(K)$. Notice that every function $\Sigma_{i \leq n} r_i \chi_{[a_i]}$ can be, by changing the coefficients and the sets $a_i$ if necessary, represented in the form where all $[a_i]$s are pairwise disjoint, so we can without loss of generality work only with such functions. Given $\varepsilon > 0$ and $\Sigma_{i \leq n} r_i \chi_{[a_i]}$ with $a_i$s disjoint, we can find for $i \leq n$ rational numbers $q_i$ with $|q_i - r_i| < \varepsilon$, hence the function $\Sigma_{i \leq n} r_i \chi_{[a_i]}$ is approximated within $\varepsilon$ by $\Sigma_{i \leq n} q_i \chi_{[a_i]}$, showing that also functions with rational coefficients and sijoint $a_i$s are dense. Now given $f \geq 0$ a function in $C(K)$ with $\|f\| \leq 1$ and $\varepsilon > 0$, let $\Sigma_{i \leq n} q_i \chi_{[a_i]}$ be a function with rational coefficients and disjoint $a_i$s satisfying $\|f - \Sigma_{i \leq n} q_i \chi_{[a_i]}\| < \varepsilon$, recalling that the $\|\|\|$ in $C(K)$ is the supremum norm. Define now:

$$s_i = \begin{cases} q_i & \text{if } q_i \in [0, 1], \\ 0 & \text{if } q_i < 0, \\ 1 & \text{if } q_i > 1, \end{cases}$$

and consider the function $\Sigma_{i \leq n} s_i \chi_{[a_i]}$. We claim that $\|f - \Sigma_{i \leq n} s_i \chi_{[a_i]}\| < \varepsilon$. By the assumptions that $a_i$s are disjoint, for any $x$ there is at most one $i = i(x)$ such that $x \in [a_i]$. If $x \notin \bigcup_{i \leq n} [a_i]$ or $s_i(x) = q_i(x)$ then $\|f(x) - \Sigma_{i \leq n} s_i \chi_{[a_i]}(x)\| = \|f(x) - \Sigma_{i \leq n} q_i \chi_{[a_i]}(x)\| < \varepsilon$. If $x \in [a_i]$ and $q_i < 0$ then $\|f(x) - s_i\| = f(x) - f(x) - q_i < \varepsilon$ as $f(x) \geq 0$. If $x \in [a_i]$ and $q_i > 1$ then $\|f(x) - s_i\| = 1 - f(x) < q_i - f(x) < \varepsilon$ as $f(x) \leq 1$, which finishes the proof. $\star_{3.1}$

**Definition 3.2** Suppose that $\bar{f} = \langle f_n : n > \omega \rangle$ is a sequence in $N(\mathfrak{A})$ and suppose that $\bar{f}' = \langle f'_n : n > \omega \rangle$ was obtained by first replacing each $f_n$ with an equivalent function $\Sigma_{i \leq n} q_i \chi_{[a_i]}$ with disjoint $a_i$s and then replacing the coefficients $q_i$ by $s_i$ using the procedure described in the proof of Lemma 3.1. We say that $\bar{f}'$ is a top-up of $\bar{f}$.

**Corollary 3.3** Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and let $\mathcal{A}$ denote the linear subspace of $C^N(\mathfrak{A})$ spanned by the functions whose rational coefficients are in $[0, 1]$. Then for every $\bar{f} = \langle f_n : n < \omega \rangle$ in $C^N(\mathfrak{A})$ there is $\bar{g} \in \mathcal{A}$ with $\lim_n f_n = \lim_n g_n$. 

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Proof. Let \( f = \lim_n f_n \), hence \( f \) can be written as \( f = f^+ - f^- \) where \( f^+ = \max\{f, 0\} \) and \( f^- = \min\{f, 0\} \) are both continuous and positive. Therefore, by the closure of \( A \) under linear combinations it suffices to prove the Corollary in the case of \( f \geq 0 \). Let \( D^* = ||f|| \), and we now apply Lemma 3.1. \( \star_{3.3} \)

The point of these definitions is the connection between the embeddability of the class of spaces of the form \( C(\text{St}(A)) \) and the class of function spaces. Namely, we have the following

**Theorem 3.4** Suppose that \( A \) and \( B \) are Boolean algebras and let \( A \) denote the linear subspace of \( C^N(\mathbb{A}) \) spanned by the functions whose rational coefficients are in \([0, 1]\). Then if there is an isomorphic embedding \( T \) from \( C(\text{St}(A)) \) to \( C(\text{St}(B)) \) then there is an isomorphic embedding \( \phi \) from \( A \) to \( N(B) \) satisfying that for every \( \bar{f} = \langle f_n : n < \omega \rangle \) in \( A \), if \( f = \lim_n f_n \), then \( \lim_n \phi(f)_n = T(f) \).

Proof. Let \( T : C(\text{St}(A)) \rightarrow C(\text{St}(B)) \) be an isomorphic embedding, so \( ||T|| < \infty \). We intend to define an isomorphic embedding \( \phi \) from \( A \) to \( N(B) \). By linearity it is sufficient to work with the basis of \( A \), so denote by \( A' \) the sequences of simple rational functions whose coefficients are in \([0, 1]\). Let us use the notation \( \pi_n \) for the projection on the \( n \)-th coordinate. First we define the action of \( \phi \) on those \( f \in A' \) which have the property that there is at most one \( n \) such that \( \pi_n(f) \) is not the identity zero function, and then \( \pi_n(f) \) is a function of the form \( \chi[a] \) for some \( a \in A \). If there is no such \( n \) with \( a \neq 0 \) then we let \( \phi(f) \) be the element of \( N(B) \) whose all projections \( \pi_n \) are zero. Otherwise, let \( n \) be such that \( \pi_n(f) \neq 0 \) and consider \( T(\pi_n(f)) \), which is well defined. We have no reason to believe that \( T(\pi_n(f)) \) is a simple function with rational coefficients. However, there is a function \( F(\pi_n(f)) \) which is a simple function with rational coefficients and whose distance to \( T(\pi_n(f)) \) in \( C(\text{St}(B)) \) is less than \( \frac{1}{2n+1} \). We define \( \phi(f) \) to be the unique element \( g \) of \( N(B) \) such that the only \( \pi_m(g) \) which is not identically zero is \( \pi_n(g) \) and \( \pi_n(g) = F(\pi_n(f)) \).

Now suppose that \( \bar{f} \in A' \) is such that for exactly one \( n \), \( \pi_n(\bar{f}) \) is not identity zero and \( \pi_n(\bar{f}) = \sum_{i=0}^m g_i \chi[a_i] \) for some \( g_0, \ldots, g_m \in A \) and some rational \( q_0, \ldots, q_m \) in \([0, 1]\). For \( i < m \) let \( f_i \) be the element of \( N(A) \) whose \( n \)-th projection is \( \chi[a_i] \) and all other projections are identity zero. Hence we have already defined \( \phi(f_i) \) and we let \( \phi(f) = \sum_{i=m}^m q_i \phi(f_i) \).

Finally suppose that \( f = \langle f_n : n < \omega \rangle \) is any element of \( A' \). Therefore for every \( n \) we have already defined \( g_n = \phi(\langle 0, \ldots, f_n, 0, \ldots \rangle) \), where \( f_n \) is on the \( n \)-th coordinate. Let \( \phi(f) = \langle g_n : n < \omega \rangle \). Hence we have defined a linear embedding of \( A' \) to \( N(B) \). We extend this embedding to \( A \) by linearity. We need to check that this embedding preserves \( C \) and \( C_0 \). So suppose that \( \bar{f} = \langle f_n : n < \omega \rangle \) is in \( C^N(\mathbb{A}) \cap A \) and let \( \bar{g} = \langle g_n : n < \omega \rangle \) be its image under \( \phi \). We shall show that \( \bar{g} \in C^N(\mathbb{B}) \) by showing that it is a Cauchy sequence. Let \( n, m < \omega \). We shall consider \( ||g_n - g_m|| \). Clearly it suffices to assume that \( \bar{f} \in A' \). Let \( f_n = \sum_{i \leq k} q_i \chi[a_i] \) and \( f_m = \sum_{j \leq l} r_j \chi[b_j] \). We have

\[
||g_n - g_m|| = ||\phi(f_n) - \phi(f_m)|| \leq ||\phi(f_n) - T(f_n)|| + ||T(f_n) - T(f_m)|| + ||T(f_m) - \phi(f_m)||
\]

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\[ \leq \sum_{i \leq k} |q_i| \|\phi(\chi_{[a_i]}) - T(\chi_{[a_i]})\| + \|T(f_n - f_m)\| + \sum_{j \leq t} |r_j| \|\phi(\chi_{[b_j]}) - T(\chi_{[b_j]})\| \]
\[ \leq \frac{(k + 1)}{2^{n+1}} + \|T\| \cdot |f_n - f_m| + \frac{(l + 1)}{2^{m+1}}, \]
which goes to 0 as \( n, m \to \infty. \)

At the end suppose that \( \bar{f} = \langle f_n : n < \omega \rangle \) is in \( C_0(\omega^2) \) and let \( \bar{g} = \langle g_n : n < \omega \rangle \) be its image under \( \phi \). By the definition of \( \phi \) we have that \( \|g_n\| \leq \|T(f_n)\| + \frac{(n + 1)}{2^{n+1}} \). Since \( \|T\| < \infty \) we have that \( \lim_{n \to \infty} \|T(f_n)\| = 0 \), so in conclusion, \( \lim_{n \to \infty} \|g_n\| = 0. \)

\[ \star_{3.4} \]

4 Invariants for the natural spaces and very positive embeddings

We shall now adapt the Kojman-Shelah method of invariants [12], to the natural spaces and a specific kind of isomorphic embeddings between Banach spaces, which we call very positive embeddings (see Definition 4.4). From this point on we assume that \( \lambda \) is a regular uncountable cardinal.

**Definition 4.1** (1) Suppose that \( M \) is a model of size \( \lambda \). A filtration of \( M \) is a continuous increasing sequence \( \langle M_\alpha : \alpha < \lambda \rangle \) of elementary submodels of \( M \), each of size \( < \lambda \).

(2) For a regular cardinal \( \theta < \lambda \) we use the notation \( S_\theta^\lambda \) for \( \{ \alpha < \lambda : \text{cf}(\alpha) = \theta \} \).

(3) A club guessing sequence on \( S_\theta^\lambda \) is a sequence \( \langle C_\delta : \delta \in S_\theta^\lambda \rangle \) such that each \( C_\delta \) is a club in \( \delta \), and for every club \( E \subseteq \lambda \) there is \( \delta \) such that \( C_\delta \subseteq E \).

**Observation 4.2** Suppose that \( \theta > \aleph_1 \) and there is a club guessing sequence \( \langle C_\delta : \delta \in S_\theta^\lambda \rangle \). Then there is a club guessing sequence \( \langle D_\delta : \delta \in S_\theta^\lambda \rangle \) such that for all \( i < \theta \)

\[ \text{cf}(i) \neq \omega \implies \text{cf}(\alpha_\delta^i) \neq \omega, \]  

where \( \langle \alpha_\delta^i : i < \theta \rangle \) is the increasing enumeration of \( D_\delta \), for each \( \delta \).

**Proof.** First of all notice that by passing to subsets if necessary we can without loss of generality assume that each \( C_\delta \) has order type \( \theta \). Given \( \delta \), let \( C'_\delta \) consists of the points of \( C_\delta \) of cofinality \( > \omega \) and let \( D_\delta \) be the closure of \( C'_\delta \) in \( \delta \). Since \( \theta > \aleph_1 \) we have that \( C'_\delta \) is unbounded in \( \delta \), so it is clear that \( D_\delta \) is a club of \( \delta \) and since we have \( D_\delta \subseteq C_\delta \), we obtain that the resulting sequence is a club guessing sequence on \( S_\theta^\lambda \). It also follows that \( \text{otp}(D_\delta) = \theta \), so the increasing enumeration as claimed exists. \( \star_{4.2} \)

The main definition we need is the definition of the invariant. Let us suppose that \( \theta > \aleph_1 \) is regular and that \( \langle D_\delta : \delta \in S_\theta^\lambda \rangle \) is a club guessing sequence with an increasing enumeration \( \langle \alpha_\delta^i : i < \theta \rangle \) of \( D_\delta \), for each \( \delta \) and satisfying the requirement (1). This

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sequence will be fixed throughout. The existence of such a sequence will be discussed at the end of the section but for the moment let us say that Shelah (see Theorem 4.7) proved that such a sequence exist in many circumstances, notably for any \( \lambda \) regular \( \geq \theta^++ \).

**Definition 4.3** Suppose that \( \mathfrak{A} \) is a Boolean algebra of size \( \lambda \), \( \bar{\mathfrak{A}} = \langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle \) a filtration of \( \mathfrak{A} \), that \( \delta \in S^\theta_\omega \) and that \( \bar{f} \in C^{N(\mathfrak{A}) \setminus N(\mathfrak{A}_\delta)} \). An ordinal \( i \in S^\theta_\omega \) is an element of the invariant \( \text{inv}_{\mathfrak{A},\delta}(\bar{f}) \) iff there is \( \bar{f}' \in C^{N(\mathfrak{A}_{\delta+1})} \) such that for every \( \bar{g} \in C^{N(\mathfrak{A}_{\delta+1})} \) we have

\[
0 \leq \lim_n g_n \leq |\lim_n f_n - \lim_n f'_n| \implies \bar{g} \in C_0. \tag{2}
\]

We shall be interested in the kind of embeddings between Banach spaces which will allow us to define appropriate \( \phi \) which preserve the invariants, see the Preservation Lemma 4.5. We have succeeded to do this in the case of a special kind of positive embeddings, as defined in the following definition.

**Definition 4.4** We say that an isomorphic embedding \( T : C(K) \to C(L) \) is very positive if the following requirements hold:

(i) \( g \geq 0 \implies Tg \geq 0 \) (positivity),

(ii) for every \( g \in C(L) \setminus \{0\} \) with \( 0 \leq g \), there is \( h \geq 0 \) with \( 0 \leq Th \leq g \) and \( h \neq 0 \),

(iii) if \( 0 \leq Th \leq Tf, h, f \neq 0 \) and \( h \neq f \geq 0 \) then there is \( s \geq 0 \) definable from \( h \) with \( 0 \leq s \leq f \).

We do not know if very positive embeddings were studied in the literature but clearly, one kind of embedding that is very positive, is an order preserving onto embedding. In this case we have Kaplansky’s theorem [11] mentioned above, which shows that in the presence of such an embedding from \( C(K) \) to \( C(L) \) we have that \( K \) and \( L \) are homeomorphic. We show in the example in §5 that the analogue is not true for very positive embeddings, not even when they are assumed to be onto. In particular the question of the number of pairwise nonisomorphic by very positive embeddings spaces of the form \( C(\text{St}(\mathfrak{A})) \) does not reduce to the well studied and understood question of the number of pairwise nonisomorphic Boolean algebras of a given cardinality (which for any infinite \( \kappa \) is always equal to \( 2^\kappa \), see Shelah’s [15]).

Let us now make a further assumption on \( \lambda \):

\[
\kappa < \lambda \implies \kappa^{\aleph_0} < \lambda. \tag{3}
\]

**Lemma 4.5** (Preservation Lemma) Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be Boolean algebras of size \( \lambda \) and suppose that \( T : C(\text{St}(\mathfrak{A})) \to C(\text{St}(\mathfrak{B})) \) is a very positive embedding. Let \( \bar{\mathfrak{A}} \) and \( \bar{\mathfrak{B}} \) be any filtrations of \( \mathfrak{A} \) and \( \mathfrak{B} \) respectively and let \( \mathfrak{A}' \) denote the linear subspace of \( C^{N(\mathfrak{A})} \) spanned by the set \( \mathfrak{A}' \) of sequences of functions whose rational coefficients are in \([0,1]\).
If $\phi : \mathcal{A} \to N(\mathfrak{B})$ is an isomorphic embedding satisfying that $\lim \phi(\bar{f}) = T(\lim(\bar{f}))$ for every $\bar{f} \in \mathcal{A}$, then there is a club $E$ of $\lambda$ such that for every $\delta$ with $D_\delta \subseteq E$ and for every $\bar{f} \in \mathcal{A}' \setminus N(\mathfrak{A}_\delta)$ with $0 \leq \lim_n f_n$ and $||\lim_n f_n|| = 1$ we have that

$$\inv_{\mathfrak{A}', \delta}(\bar{f}) = \inv_{\mathfrak{B}, \delta}(\phi(\bar{f})).$$

**Proof.** We may assume that the underlying set of $\mathfrak{A}$ and $\mathfrak{B}$ is the ordinal $\lambda$. Let us define a model $M$ with the universe two disjoint copies of the $\omega$-sequences of the simple functions on $\lambda$ with rational coefficients, interpreted as the elements of $N(\mathfrak{A})$ and $N(\mathfrak{B})$, all the symbols of $N(\mathfrak{A})$ and $N(\mathfrak{B})$ with interpretations induced from these models, and the symbols $\mathcal{A}$, $\mathcal{A}'$ and $\phi$. By the assumption (3), there is a club $E$ of $\lambda$ such that for every $\delta \in E$ of cofinality not $\omega$ we have that $M$ restricted to the sequences whose ordinal coefficients are $< \delta$ is an elementary submodel of $M$ and that is has universe corresponding to $N(\mathfrak{A}_\delta) \cup N(\mathfrak{B}_\delta)$. Let us denote the latter model by $M \upharpoonright \delta$.

Suppose now that $D_\delta \subseteq E$. Choose $\bar{f} \in C^{N(\mathfrak{A}')} \setminus N(\mathfrak{A}_\delta)$ with $0 \leq \lim_n f_n$ and $||\lim_n f_n|| = 1$. Let $f = \lim f_n$. By the choice of $\delta$ we have that $\phi(\bar{f}) \in C^{N(\mathfrak{A}')} \setminus N(\mathfrak{B}_\delta)$.

Suppose first that $i \in \inv_{\mathfrak{A}', \delta}(\bar{f})$ and let $\bar{f}' \in C^{N(\mathfrak{A}'_{\alpha_\delta^i + 1})}$ demonstrate this. Let $f' \overset{\text{def}}{=} \lim f'_n$, which is well defined as $\bar{f}' \in C^{N(\mathfrak{A}')}$. Notice that the requirement (2) will hold if we replace $\bar{f}'$ by any top-up $\bar{f}''$ (see Definition 3.2) as $0 \leq f' \leq 1$ and hence for all $x$ we have $|f(x) - f''(x)| \leq |f(x) - f'(x)|$. Since the top-up process is definable in $M \upharpoonright \alpha_\delta^i$, we may assume that $0 \leq f'_n \leq 1$ for all $n$ and $0 \leq f' \leq 1$. By Lemma 3.1 applied within $\mathcal{M}_{\alpha_\delta^i}$ we can assume that $\bar{f} \in \mathcal{A}$. By the choice of $\phi$ we have that $\lim \phi(\bar{f}') = T(f')$ and similarly $0 \leq \lim \phi(\bar{f}')$. By the fact that $D_\delta \subseteq E$ and since $\phi$ is an isomorphism we have that $\phi(\bar{f}') \in C^{N(\mathfrak{B}_{\alpha_\delta^i + 1})}$. We would like to use $\phi(\bar{f}')$ to witness that $i \in \inv_{\mathfrak{B}, \delta}(\phi(\bar{f}))$, so let us try. By the choice of $\phi$ we have that $\lim \phi(\bar{f}') = T(f')$ and similarly $0 \leq \lim \phi(\bar{f}')$.

It remains to check the property (2) of $\phi(\bar{f}')$.

Suppose for a contradiction that there is $\bar{g} \in N(\mathfrak{B}_{\alpha_\delta^i})$ such that $0 \leq g \overset{\text{def}}{=} \lim_n g_n \leq |Tf - Tf'|$ but that $\bar{g} \notin C_0$. Applying (ii) we can find $h \geq 0$ with $0 \leq Th \leq g$ and $h \neq 0$. By Corollary 3.3 we can assume that there is $\bar{h} \in \mathcal{A}$ with $h = \lim_n h_n$ and hence $Th = \lim_n \phi(\bar{h})$. Translating (ii) into the terms of $\phi$ and applying the elementarity of $M \upharpoonright \alpha_\delta^i$ we can assume that $\bar{h} \in N(\mathfrak{A}_{\alpha_\delta^i})$. Now we apply (iii) to find $s \geq 0$, $s \neq 0$ definable from $\bar{h}$ and satisfying $s \leq |f - f'|$. Being definable from $\bar{h}$, $s$ has an approximation $\bar{s}$ with $\bar{s}$ definable from $\bar{h}$, hence $\bar{s} \in N(\mathfrak{A}_{\alpha_\delta^i})$. By top-up if necessary as in Lemma 3.1 and in the above paragraph, we can assume that every element $s_n$ in $\bar{s}$ satisfies $s_n \geq 0$, therefore $\bar{s}$ contradicts the choice of $\bar{f}'$.

Now let us prove the other direction of the desired equality. Let $i \in \inv_{\mathfrak{B}, \delta}(\phi(\bar{f}))$ as exemplified by some $\bar{g} \in N(\mathfrak{B}_{\alpha_\delta^i + 1})$. As in the previous paragraphs, we can assume that $0 \leq g \overset{\text{def}}{=} \lim_n g_n$ and hence by (ii) we can assume that for some $f' \geq 0$ we have $0 \leq Tf' \leq g$ and by the argument as above we can assume that there is $\bar{f} = \langle f' : n < \omega \rangle \in N(\mathfrak{A}_{\alpha_\delta^i + 1}) \cap \mathcal{A}$ such that $\lim f'_n = f'$. Now we claim that $\bar{f}'$ exemplifies that $i \in \inv_{\mathfrak{A}, \delta}(\bar{f})$. Suppose for a contradiction that $\langle h_n : n \in \omega \rangle \in \mathfrak{N}(\mathfrak{A}_{\alpha_\delta^i}) \setminus C_0$ and $0 \leq \lim h_n \leq |\lim f_n - \lim f'_n|$. Let $h = \lim h_n$. As before, we can assume that $\bar{h} \in \mathcal{A}$ and

$$h_n = \lim f_n - \lim f'_n.$$
and each $h_n \geq 0$. So by the positivity we have $0 \leq Th \leq T(|f - f'|) = |Tf - Tf'|$, by the choice of $\phi$ we have that $\phi(\bar{h}) = Th$, by elementarity we have $\phi(\bar{h}) \in N(\mathcal{B}_{\alpha_i^i}) \setminus C_0$. It follows that $\phi(\bar{h})$ contradicts $i \in \text{inv}_{\mathcal{B}_{\alpha_i}^i}(\phi(\bar{f}))$. \begin{star} 4.5 \end{star}

The next task is to construct lots of Boolean algebras $\mathfrak{A}$ with different invariants for $N(\mathfrak{A})$ and then to use the Preservation Lemma to show that no fixed $N(\mathfrak{B})$ can embed them all.

**Lemma 4.6** (Construction Lemma) Suppose that $\theta^+ < \lambda$. Then the club guessing sequence $\langle D_\delta : \delta \in S^\theta_\lambda \rangle$ can be chosen so that for any $A \subseteq \theta$ which is a closed set of limit ordinals, there is a Boolean algebra $\mathfrak{A} = \mathfrak{A}[A]$, a filtration $\mathfrak{A}$ of $\mathfrak{A}$ and a club $E$ of $\lambda$ such that for every $\delta \in E$ there is $\bar{f} \in \mathfrak{A} \setminus N(\mathfrak{A}_{\delta})$ with $\text{inv}_{\mathfrak{A},\delta}(\bar{f}) = A \cap S^\theta_\delta \neq \omega$ and $||\lim f_n|| = 1$.

The proof of this lemma is presented in the §6. The following theorem of Shelah will be used in the proof of the Construction Lemma as well as in the proof of Theorem 4.8.

**Theorem 4.7** (Shelah, [16], 1.4.) Let $\theta < \lambda$ be two regular cardinals with $\theta^+ < \lambda$. Then there is a stationary set $S \subseteq S^\theta_\lambda$ and sequences $\langle c_\delta : \delta \in S \rangle$, $\langle P_\alpha : \alpha < \lambda \rangle$ such that:

- $\text{otp}(c_\delta) = \theta$ and $\text{sup}(c_\delta) = \delta$,
- for every club $E$ of $\lambda$ there is $\delta \in S$ with $c_\delta \subseteq E$,
- $P_\alpha \subseteq P(\alpha)$ and $|P_\alpha| < \lambda$,
- if $\alpha$ is a non-accumulation point of $c_\delta$ then $c_\delta \cap \alpha \in \bigcup_{\alpha' < \alpha} P_{\alpha'}$.
- the non-accumulation points of every $c_\delta$ are successor ordinals$^7$.

Now we present the main theorem of this part of the paper.

**Theorem 4.8** Suppose that $\theta$ and $\lambda$ are two regular cardinals with $\aleph_2 \leq \theta < \theta^+ < \lambda$, and that $(\forall \kappa < \lambda) \kappa^{\aleph_0} < \lambda$. Then

1. the minimal number of spaces of the form $C(\text{St}(\mathfrak{A}))$ of density $\lambda$ needed to embed all Banach spaces of the form $C(\text{St}(\mathfrak{B}))$ of density $\lambda$ very positively is $2^\theta$. In particular, if $2^\theta > \lambda$ then there is no very-positively universal space $C(\text{St}(\mathfrak{A}))$ of density $\lambda$.

2. if $2^\theta > \lambda$ then there are at least $\text{cf}(2^\theta)$ pairwise non-very positively isomorphic Banach spaces $C(K)$ of density $\lambda$.

$^7$Claim 1.4. in [16] does not state this property explicitly, but it follows from the first line of the proof of that Claim.
Proof. Fix sequences \( \langle c_\delta : \delta \in S \subseteq S^\lambda_0 \rangle \) and \( \langle P_\alpha : \alpha < \lambda \rangle \) as guaranteed by Theorem 4.7. Notice that \( \langle c_\delta : \delta \in S \rangle \) satisfies that with \( \langle \alpha_i^0 : i < \theta \rangle \) being the increasing enumeration of \( c_\delta \), we have that \( \text{cf}(i) \neq \omega \implies \text{cf}(\alpha_i^0) \neq \omega \). Hence letting \( D_\delta = c_\delta \) for \( \delta \in S \) and \( D_\delta \) an arbitrary club of \( \delta \) of order type \( \theta \) satisfying \( \text{cf}(i) \neq \omega \implies \text{cf}(\alpha_i^0) \neq \omega \) for \( \delta \in S^\lambda_0 \setminus S \), the sequence can be used in the context of the Preservation Lemma 4.5. It can also be used in the context of the Construction Lemma 4.6. Let us therefore find the Boolean algebras \( \mathfrak{A}[A] \) as described in the statement of the Construction Lemma. Notice that there are \( 2^\theta \) many different choices for \( A \cap S^\theta_\omega \).

1. Suppose for a contradiction that there is a family \( \{ C(\text{St}(\mathfrak{A}_\alpha)) : \alpha < \alpha^* \} \) for some \( \alpha^* < 2^\theta \) for some algebras \( \mathfrak{A}_\alpha \) of size \( \lambda \) which is very positively universal for all \( C(\text{St}(\mathfrak{A})) \) for Boolean algebras \( \mathfrak{A} \) of size \( \lambda \). Notice that the assumptions we have made on \( \lambda \) imply that \( \lambda^{\aleph_0} = \lambda \) so the size of each \( N(\mathfrak{A}_\alpha) \) is \( \lambda \). Let \( \mathcal{F} \) be the family of all subsets of \( \theta \) that appear as invariants of elements of \( \bigcup_{\alpha < \alpha^*} N(\mathfrak{A}_\alpha) \), hence the size of \( \mathcal{F} \) is \( < 2^\theta \), and in particular there is \( A \subseteq \theta \) a closed set of limit ordinals such that \( A \cap S^\theta_\omega \notin \mathcal{F} \). Let \( \mathfrak{A} = \mathfrak{A}[A] \). Suppose that \( T \) is a very positive embedding of \( C(\text{St}(\mathfrak{A})) \) into some \( C(\text{St}(\mathfrak{A}_\alpha)) \) and let \( \phi \) be an embedding of \( N(\mathfrak{A}) \) into \( N(\mathfrak{A}_\alpha) \) satisfying that for every \( \bar{f} \in A \cap C^{N(\mathfrak{A})} \) we have \( \phi(\bar{f}) = T(f) \), which exists by Theorem 3.4. Let \( E_0 \) be a club of \( \lambda \) as guaranteed by the Preservation Lemma and \( E_1 \) a club of \( \lambda \) as guaranteed by the Construction Lemma, let \( E = E_0 \cap E_1 \) and suppose that \( \delta \) is such that \( D_\delta \subseteq E \). Then by the choice of \( E_0 \) there is \( \bar{f} \) in \( N(\mathfrak{A}) \) whose invariant is \( A \cap S^\theta_\omega \), but then \( \phi(\bar{f}) \) also has invariant \( A \cap S^\theta_\omega \), by the choice of \( E_1 \) and we have a contradiction with the choice of \( A \).

2. Consider the family \( \mathcal{G} = \{ C(\text{St}(\mathfrak{A}[A])) : A \) a closed set of limit ordinals in \( \theta \} \). By the argument in 1. for every \( A \), the set \( \{ B \) a closed set of limit ordinals in \( \theta : C(\text{St}(\mathfrak{A}[B])) \) embeds very positively into \( C(\text{St}(\mathfrak{A}[A])) \} \) has size \( < 2^\theta \), so clearly every \( C(\text{St}(\mathfrak{A}[A])) \) is very positively isomorphic with \( < 2^\theta \) many \( C(\text{St}(\mathfrak{A}[B])) \). Hence we can choose \( (2^\theta) \) pairwise non very positively isomorphic elements of \( \mathcal{G} \) by a simple induction.

\[ \star 4.8 \]

An example of circumstances when Theorem 4.8 applies is when

\[ \theta = \aleph_1, \lambda = \aleph_3, 2^{\aleph_0} = \aleph_1 \text{ but } 2^{\aleph_1} \geq \aleph_4. \]

5 Example

We give an example of two 0-dimensional spaces \( K \) and \( L \) which are not homeomorphic yet they admit a very-positive isomorphism onto. The example itself was constructed by Grzegorz Plebanek in \([13]\), Example 5.3 when considering positive onto isomorphisms.

Let \( K \) consist of two disjoint convergent sequences \( \langle x_n : n < \omega \rangle \) with \( \lim_n x_n = x \) and \( \langle y_n : n < \omega \rangle \) with \( \lim_n y_n = y \neq x \) and let \( L \) consist of a single convergent sequence \( \langle z_n : n < \omega \rangle \) with \( \lim_n z_n = z \). Define \( T : C(K) \to C(L) \) by letting for all \( n \)

\[ Tf(z_0) = f(y), \]

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$$Tf(z_{2n-1}) = \frac{f(x_n) + f(y)}{2}, \quad Tf(z_{2n+2}) = \frac{f(x) + f(y_n)}{2}$$

Plebanek shows that $T$ is a positive isomorphism onto $C(L)$ and moreover he calculates the inverse $S = T^{-1}$ which is given by

$$Sh(y) = h(z_0), Sh(x) = 2h(z) - h(z_0),$$

$$Sh(x_n) = 2h(z_{2n-1}) - h(z_0), Sh(y_n) = 2h(z_{2n}) - 2h(z) + h(z_0) \text{ for } n \geq 1.$$ 

We shall show that $T$ is a very positive embedding. Considering property (ii) of Definition 4.4, suppose that $g \in C(L) \setminus \{0\}$ with $0 \leq g$, we need to find $h \geq 0$ with $0 \leq Th \leq g$ and $h \neq 0$. Let $t$ in $C(L)$ be such that $t(z_0) = 0$ and for $n \geq 1$ the sequence $\{t(z_n)\}_n$ is non-negative and not identically 0, converges to 0 and is bounded by the sequence $\{g(z_n)\}_n$. It is possible to find such a sequence since $g \geq 0$ and $g \neq 0$. Let $t(z) = 0$, hence $t \in C(L)$. Let $h = St$, and we note from the definition of $S$ that $h \geq 0$ and we have $Th = t \leq g$.

For the property (iii), we shall have an existential proof of the existence of the $s$ as required. Let $S$ be the family of all non-negative functions in $C(K)$ for which there is exactly one point with non-zero value, and on that point the value is equal to that of $h$. Each element of $S$ is clearly definable from $h$. We claim that some $s \in S$ can be chosen to demonstrate (iii). Namely, since we do not have $Tf \leq Th$, we cannot have $f \leq h$ by positivity. Hence there is some value $w$ with $h(w) < f(w)$, and by the continuity of the functions $h$ and $f$, there must be some such $w \in \{x_n, y_n : n \in \omega\}$. Then letting $s(w) = h(w)$ and $s(v) = 0$ for $v \neq w$ gives a function in $S$ and we have $0 \leq s \leq f$ and $0 \neq s$.

### 6 Proof of the Construction Lemma

We present a proof of Lemma 4.6. Let $S \subseteq S_\theta^\lambda$ and sequences $\langle c_\delta : \delta \in S \rangle$, $\langle P_\alpha : \alpha < \lambda \rangle$ be as in the statement of Theorem 4.7, while $\langle D_\delta : \delta \in S_\theta^\lambda \rangle$ is such that $D_\delta = c_\delta$ for $\delta \in S$. For all the definitions of invariants we use here, the value of the invariant is the same with respect to $\langle c_\delta : \delta \in S \rangle$ as it is with respect to $\langle D_\delta : \delta \in S_\theta^\lambda \rangle$ so we shall not make a difference between the two. We start with a construction lemma for a certain family of linear orders, as obtained by Kojman and Shelah in [12]. Let us give their definition of the invariants of linear orders:

**Definition 6.1** Suppose that $L$ is a linear order with the universe $\lambda$ and $\Sigma = \langle L_\delta : \delta < \lambda \rangle$ is a filtration of $L$. Then for every $\delta \in S$ such that the universe of $L_\delta$ is $\delta$ we define

$$\text{inv}_{\Sigma, \delta}(\delta) \overset{\text{def}}{=} \{i < \theta : (\exists \delta' \in (\alpha_i^\delta, \alpha_{i+1}^\delta))(\forall x \in L_{\alpha_i^\delta}) x \leq_L \delta \iff x \leq_L \delta'\}.$$ 

Lemma 3.7 in [12] proves that under the assumptions we have stated, for every closed set $A$ of limit ordinals in $\theta$ there is a linear order $L[A]$ with universe $\lambda$ and a filtration
\( \mathfrak{L}[A] = \langle L_\delta[A] : \delta < \lambda \rangle \) of \( L[A] \) such that for every \( \delta \in S \) with \( L_\delta[A] = \delta \) we have \( \text{inv}_{\mathfrak{L}[A],\delta}(\delta) = A^8 \).

The idea of our proof is to transform the Kojman-Shelah construction first into a construction of a family of Boolean algebras of size \( \lambda \) and then to use these Boolean algebras to define natural spaces of functions with appropriate invariants.

**Definition 6.2** Suppose that \( \mathfrak{A} \) is a Boolean algebra with the set of generators \( \{ a_\alpha : \alpha < \lambda \} \) and \( \mathfrak{A} = \langle \mathfrak{A}_\delta : \delta < \lambda \rangle \) is a filtration of \( \mathfrak{A} \), while \( \delta \in S \) is such that \( \mathfrak{A}_\delta \) is generated by \( \{ a_\alpha : \alpha < \delta \} \). We define

\[
\text{inv}_{\mathfrak{A},\delta}(a_\delta) \overset{\text{def}}{=} \{ i < \theta : (\exists \delta' \in (a_i^\delta,a_i^\delta])((\forall \alpha < a_i^\delta) a_\alpha \cap a_\delta = a_\alpha \cap a_\delta') \mod \mathfrak{A}_{a_i^\delta} \text{ and } a_\alpha^c \cap a_\delta = a_\alpha^c \cap a_\delta') \mod \mathfrak{A}_{a_i^\delta} \},
\]

where \( a = b \mod \mathfrak{A}_{a_i^\delta} \) means that for any element \( w \) of \( \mathfrak{A}_{a_i^\delta} \) we have \( w \leq a \) iff \( w \leq b \).

**Definition 6.3** Suppose that \( L \) is a linear order with universe \( \lambda \). We define a Boolean algebra \( \mathfrak{A}[L] \) as being generated by \( \{ a_\alpha : \alpha < \lambda \} \) freely except for the equations

\[
a_\delta \leq a_\varepsilon \iff \delta \leq L \varepsilon. \tag{4}
\]

Since the equations in (4) are finitely consistent with the axioms of a Boolean algebra it follows from the compactness theorem that the algebra \( \mathfrak{A}[L] \) is well defined. Now we shall see a translation between the calculation of the invariants of the linear orders and the associated Boolean algebras.

**Sublemma 6.4** Let \( L \) be a linear order on \( \lambda \) and \( \mathfrak{A}[L] \) the algebra associated to \( L \) as per Definition 6.3. Let \( L \) and \( \mathfrak{A} \) be any filtrations of \( L \) and \( \mathfrak{A}[L] \) respectively. Then there is a club \( E \) such that for every \( \delta \in S \cap E \) we have

\[
\text{inv}_{\mathfrak{A},\delta}(a_\delta) = \text{inv}_{\mathfrak{A},\delta}(a_\delta)
\]

and moreover, for any \( i \in \text{inv}_{\mathfrak{A},\delta}(a_\delta) \), this is exemplified by \( \delta' \) iff \( i \in \text{inv}_{\mathfrak{A},\delta}(a_\delta) \) is exemplified by \( a_{\delta'} \).

**Proof.** Let \( E \) be the club of \( \delta \) such that the universe of \( L_\delta \) is \( \delta \), \( L_\delta \) is an elementary submodel of \( L \), \( \mathfrak{A}[L_\delta] \) is generated by \( \{ a_\alpha : \alpha < \delta \} \) and is an elementary submodel of \( \mathfrak{A} \). Suppose that \( \delta \in E \cap S \).

First suppose that \( i \in \text{inv}_{\mathfrak{A},\delta}(\delta) \) as exemplified by \( \delta' \). Let \( \alpha < a_i^\delta \), we need to prove \( a_\alpha \cap a_\delta = a_\alpha \cap a_\delta' \mod \mathfrak{A}_{a_i^\delta} \) and \( a_\alpha^c \cap a_\delta = a_\alpha^c \cap a_\delta' \mod \mathfrak{A}_{a_i^\delta} \).

Case 1. \( \alpha < L \delta \). Hence by the choice of \( \delta' \) we have \( \alpha < L \delta' \) and \( a_\alpha \leq a_\delta, a_\alpha \leq a_\delta' \). Therefore \( a_\alpha \cap a_\delta = a_\alpha \cap a_\delta' = a_\alpha \).

---

\(^8\)Lemma 3.7 in [12] also states the assumption \( 2^\theta > \lambda \), but that assumption is not used in the proof, it is only needed for the final result in [12].
Suppose that \( z > 0 \) is in \( \mathfrak{A}_{\alpha S} \) and satisfies \( z \leq a_{\alpha c}^S \cap a_\delta \). By the Disjunctive Normal Form for Boolean algebras, we can assume that \( z = \bigvee_{i \leq n} \bigwedge_{j < k_i} a_{\beta(i,j)}^{l(i,j)} \) for some \( l(i,j) \in \{0,1\} \) and \( \beta(i,j) < \alpha \). It suffices to prove that for every \( i \) we have \( \bigwedge_{j < k_i} a_{\beta(i,j)}^{l(i,j)} \leq a_{\alpha c}^S \cap a_\delta' \).

Fix an \( i \) and without loss of generality assume that \( \bigwedge_{j < k_i} a_{\beta(i,j)}^{l(i,j)} > 0 \), as otherwise the conclusion is trivial.

Let \( A_1 = \{ j \leq k_i : \beta(i,j) = l \} \), for \( l \in \{0,1\} \). Let \( \beta_1 \) be the \( L \)-minimal element of \( A_1 \), hence \( \bigwedge_{j \in A_1} a_{\beta(i,j)}^{l(i,j)} = a_{\beta_1} \). Let \( \beta_0 \) be the \( L \)-maximal element of \( A_0 \), hence \( \bigwedge_{j \in A_0} a_{\beta(i,j)}^{l(i,j)} = \bigwedge_{j \leq k_i} a_{\beta(i,j)}^{l(i,j)} = a_{\beta_0} \). In conclusion, \( \bigwedge_{j < k_i} a_{\beta(i,j)}^{l(i,j)} = a_{\beta_0} \cap a_{\beta_1} \).

Since we have assumed that \( \bigwedge_{j < k_i} a_{\beta(i,j)}^{l(i,j)} > 0 \), we cannot have \( a_{\beta_0} \leq a_{\beta_1} \), equivalently \( \beta_1 \leq L \beta_0 \). Hence we have \( \beta_0 < L \beta_1 \). Similarly, since \( a_{\beta_1} \cap a_{\alpha c}^S > 0 \) we can conclude that \( \alpha < L \beta_1 \). Finally, if we had \( \beta_0 < L \beta_1 \), then we would obtain \( a_{\beta_0}^c \leq a_\delta^c \), in contradiction with \( 0 < a_{\beta_0} \cap a_{\beta_1} \leq a_\delta \), and therefore \( \beta_0 < L \beta_1 \).

Suppose now that \( \delta < L \beta_1 \). Therefore \( a_{\beta_1} \cap a_{\delta}^c \neq 0 \). On the other hand, \( a_{\beta_0}^c \cap a_{\beta_1} \cap a_{\delta}^c = a_{\delta} \cap a_{\beta_1} \cap a_{\delta}^c = 0 \) and hence we must have \( a_{\beta_0}^c \cap a_{\beta_1} \cap a_{\delta} > 0 \), which, taking into account \( \beta_0 < L \beta_1 \) gives that \( a_{\beta_0} \cap a_{\delta}^c > 0 \) and hence \( \delta < L \beta_0 \), a contradiction. Hence we have \( \beta_1 < L \delta \). By the choice of \( \delta' \) we have \( \beta_1 < L \delta' \) and hence \( a_{\beta_1} \leq a_{\delta'} \) and in particular \( a_{\delta_0}^c \cap a_{\beta_1} \leq a_{\delta'} \), as required. Since the roles of \( \delta \) and \( \delta' \) in this proof were symmetric we can prove in the same way that for any \( z > 0 \) is in \( \mathfrak{A}_{\alpha S} \) which satisfies \( z \leq a_{\alpha c}^S \cap a_\delta \) we also have \( z \leq a_{\alpha c}^S \cap a_\delta \).

**Case 2.** \( \alpha < L \delta \), so \( \alpha < L \delta' \) by the choice of \( \delta' \). We have \( a_{\delta}^c \geq a_{\alpha c}^S \), so \( a_{\alpha c}^S \cap a_\delta = 0 \) and similarly \( a_{\alpha c}^S \cap a_\delta' = 0 \). We also have \( a_{\alpha c}^S \cap a_\delta = a_{\delta} \) and similarly for \( \delta' \), hence we need to prove that \( a_{\delta} = a_{\delta'} \) mod \( \mathfrak{A}_{\alpha S} \). As in Case 1, it suffices to show that for every \( \beta_0, \beta_1 < a_{\delta} \) with \( 0 < a_{\beta_0} \cap a_{\beta_1} \leq a_{\delta} \), we have \( a_{\beta_0} \cap a_{\beta_1} \leq a_{\delta} \) (the equality cannot occur), and vice versa. Let us start with the forward direction. As before, from \( 0 < a_{\beta_0} \cap a_{\beta_1} \) we conclude \( \beta_0 < L \beta_1 \). Also, if \( \beta_0 > L \delta \) then we have \( a_{\beta_0}^c \leq a_\delta^c \), contradicting that \( a_{\beta_0} \cap a_\delta > 0 \). Hence \( \beta_0 < L \delta \).

If \( \beta_1 < L \delta \) then \( \beta_1 < L \delta' \) so \( a_{\beta_1} \leq a_\delta \) and hence \( a_{\beta_0} \cap a_{\beta_1} \leq a_\delta' \), as required. So assume that \( \delta < L \beta_1 \). Hence \( a_{\beta_1} \geq a_\delta \) and so \( a_{\beta_0} \cap a_{\beta_1} \geq a_{\beta_0} \cap a_\delta \), a contradiction. This finishes the proof of the forward direction, and the other direction follows from the symmetry of the roles of \( \delta \) and \( \delta' \) in the proof.

Now suppose that \( i \in \text{inv}_{\mathfrak{A},S}(a_\delta) \) as exemplified by \( a_\delta \). Let \( \alpha < a_{\delta}^S \), we need to prove \( \alpha < L \delta \) \( \iff \alpha < L \delta' \). If \( \alpha < L \delta \) then \( a_\alpha < a_\delta \) hence \( a_\alpha < a_\delta \) by the assumption, and hence \( \alpha < L \delta' \) by the definition of \( \mathfrak{A}[L] \). The other direction follows by symmetry. \( \star_{6.4} \)

**Sublemma 6.5** Let \( \mathfrak{A}[L] \) be one of the algebras described in the above and \( \mathfrak{A} \) its filtration. Then there is a club \( E \) of \( \lambda \) such that for every \( \delta \in S \) with \( D_\delta \subseteq E \), we have that

\[
\text{inv}_{\mathfrak{A},S}(\chi_{[a_\delta]}) = \text{inv}_{\mathfrak{A},S}(a_\delta)
\]

and moreover, for any \( i \in \text{inv}_{\mathfrak{A},S}(\chi_{[a_\delta]}) \), this is exemplified by \( \chi_{[a_i]} \) iff \( i \in \text{inv}_{\mathfrak{A},S}(a_\delta) \) is exemplified by \( a_\delta \). Here, the invariant on the left refers to the invariant in the natural
space \( N(\mathfrak{A}) \) and the invariant on the right to the invariant in the algebra \( \mathfrak{A} \). The notation \( \chi_{[a]} \) is used for the sequence \( \langle \chi_{[a]}, \chi_{[a]}, \chi_{[a]}, \ldots \rangle \) in \( C^{N(\mathfrak{A})} \).

**Proof.** Let \( \mathfrak{M}^* \) be a model consisting of \( L, \mathfrak{A} \), two disjoint copies of the \( \omega \)-sequences of the simple functions on \( \lambda \) with rational coefficients, interpreted as the elements of \( N(\mathfrak{A}) \) and all the symbols of \( N(\mathfrak{A}) \) with induced interpretations induced from these models. Recall the assumption that \( \forall \kappa < \lambda \) we have \( \kappa^{\aleph_0} < \lambda \) and notice that it implies that there is a club \( E_0 \) of \( \lambda \) such that for every \( \delta \in E_0 \) of cofinality \( > \aleph_0 \), the model \( \mathfrak{M}^* \upharpoonright \delta \) is \( \aleph_1 \)-saturated in \( \mathfrak{M}^* \), that is it realizes all the types with countably many parameters in \( \mathfrak{M}^* \upharpoonright \delta \) which are realized in \( \mathfrak{M}^* \). Let \( \bar{L} \) be any filtration of \( L \), let \( E \subseteq E_0 \) be a club witnessing Sublemma 6.4, and let \( \delta \in S \) be such that \( D_\delta \subseteq E \).

Suppose \( i \in \text{inv}_{\mathfrak{A}, \delta}(a_\delta) \) as exemplified by \( a_{\delta'} \) but \( \bar{g} \geq 0 \) with \( \bar{g} \in C^{N(\mathfrak{A}, a_{\delta'})} \setminus C_0 \) and the limit \( g \) of \( \bar{g} \) satisfies \( g \leq |\chi_{[a_\delta]} - \chi_{[a']_\delta}] = \chi_{[a_\delta \Delta a_{\delta'}]} \). By topping up if necessary (see Definition 3.2) we may assume that each \( g_n \geq 0 \), and by throwing away unnecessary elements of \( \bar{g} \) we may assume that every \( g_n \neq 0 \). We can then assume that for each \( n \) there are pairwise disjoint \( \{ b_{n}^0, \ldots, b_{n}^k \} \in \mathfrak{A}_{\delta^j} \) and \( q_n^i \leq k_n \) such that \( g_n = \Sigma_{i \leq k_n} q_n^i \chi_{[b_n^i]} \). Since \( ||g_n - \chi_{[a_\delta \Delta a_{\delta'}]}|| \to 0 \), there has to be a \( [b_n^i] \) with a non-empty intersection with \( [a_{\delta_{\delta}}] \). By applying the Disjunctive Normal form, we can assume that

\[
\bar{b}_i = \bigwedge_{j \leq m} \bigwedge_{\alpha \leq \alpha_m} a_{\beta(m), \alpha} \text{ for some } l(m, \alpha) \in \{0, 1\} \text{ and } \beta(m, \alpha) < \alpha^*_\delta.
\]

Therefore there is \( j \leq m \) such that \( \bigwedge_{\alpha \leq \alpha_m} [a_{\beta(m), \alpha}] \cap [a_{\delta_{\delta}}] \neq \emptyset \). Then we have that \( \bigwedge_{\alpha \leq \alpha_m} [a_{\beta(m), \alpha}] \cap [a_{\delta_{\delta}}] \). From the choice of \( E \), using Sublemma 6.4 we have that for \( R \in \{ < L, > L \} \), \( \beta R \delta \) iff \( \beta R \delta' \). We go through a case analysis like in the proof of Sublemma 6.4. If \( \beta < L \delta \) then we have \( \beta < L \delta' \) so \( [a_{\beta_{\delta}} \cap [a_{\delta}] = [a_{\delta}] \cap [a_{\delta}] \), a contradiction. If \( \beta > L \delta \) then \( [a_{\delta'}] \cap [a_{\delta}] \neq \emptyset \). Therefore \( i \in \text{inv}_{\mathfrak{A}, \delta}(\chi_{[a_{\delta}]} \).

**Claim 6.6** Suppose that \( i \in \text{inv}_{\mathfrak{A}, \delta}(\chi_{[a_{\delta}]} \) as exemplified by some \( \bar{f} \). Without loss of generality we can assume that \( \bar{f} = \chi_{[a_{\delta}]} \) for some \( \delta' \).

**Proof of the Claim.** First let us notice that if \( f = \lim_n f_n \) then for \( f' = \min\{f, 1\} \) we have \( |\chi_{[a_{\delta}]} - f'| \leq |\chi_{[a_{\delta}]} - f| \) so we can without loss of generality assume that \( f \leq 1 \).

Similarly we can assume that \( f \geq 0 \), and then by applying a similar logic we can also assume that \( 0 \leq f_n \leq 1 \) for all \( n \) and that \( f_n \neq 0 \). Each \( f_n \) is a simple function with rational coefficients defined on (without loss of generality) disjoint basic clopen sets of the form \( [a_{l_\beta}^j] \) where \( \beta < \alpha^*_j + 1 \) and \( l_\beta < 2 \). Let \( \{\beta_n : n < \omega\} \) enumerate all the relevant \( \beta \). For each \( n \) and \( R \in \{ < L, > L \} \) let \( j_R^n \) be the truth value of \( \text{"} a_{\beta_n} \Delta R \delta'' \text{"} \). Consider the following sentence with parameters \( \bar{f}, \alpha^*_\delta \) and the elements of \( \{\beta_n : n < \omega\} \): there is \( \beta \) such that

- for all \( \bar{g} \in N(\mathfrak{A}, a_{\delta'}) \) if \( 0 \leq \lim_n g_n \leq |\lim_n f_n - \chi_{[a_{\delta}]}| \), we have \( \bar{g} \in C^0 \),
- for all \( n \) and \( R \in \{ < L, > L \} \) we have \( a_{\beta_n} \Delta R \delta'' \) iff \( j_R^n = 1 \).
This sentence is true as exemplified by $\delta$, so by the choice of $E_0$ it is true in $\mathfrak{M}^* \upharpoonright \alpha_{i+1}$, say as exemplified by $\delta'$. Let us note that in $L$ we have $\delta < L \delta'$ or $\delta' < L \delta$, let us assume that $\delta' < L \delta$, as the other case is symmetric. We claim that $\chi_{[a']_i}$ exemplifies that

$$i \in \text{inv}_{\vec{a}, \vec{b}}(\chi_{[a]_i}).$$

If not, we can find $\bar{g} \in C^{N(\alpha, \delta)} \setminus C^0$ with $0 \leq \bar{g}$ and $g \overset{\text{def}}{=} \lim_n g_n \leq |\chi_{[a]_i} - \chi_{[a']_i}| = \chi_{[a]_i \Delta a'_i} \leq 1$. By the triangle inequality it follows that

$$g \leq |\chi_{[a]_i} - f| + |f - \chi_{[a']_i}|.$$

We have that for every $x$ both $|\chi_{[a]_i} - f|(x)$ and $|\chi_{[a]_i} - f|(x)$ are equal to $f(x)$ if $x \in [a'_{\delta}]$ and $1 - f(x)$ if $x \in [a_{\delta'}]$. The possible difference is on $[a_{\delta} \setminus a_{\delta'})$, where the former function is equal to $1 - f(x)$ and the latter to $f(x)$. We now claim that $f$ is constant on $[a_{\delta} \setminus a_{\delta'})$.

Clearly, it suffices to show that each $f_n$ is constant on $[a_{\delta} \setminus a_{\delta'})$, and by the choice of $\{\beta_n : n < \omega\}$, it suffices to show that for each $l < 2$ and each $n$, $\chi_{[a_{\delta/n}]}$ is constant on $[a_{\delta} \setminus a_{\delta'})$. Let $\beta = \beta_n$ for some $n$. If $\beta \leq L \delta'$ then $a_{\beta} \leq a_{\delta'}$ so $\chi_{[a]_i}$ is constantly 0 on $[a_{\delta} \setminus a_{\delta'})$. In addition, we have $a_{\delta}^c_{\beta} \geq a_{\delta}^c_{\beta'} \geq (a_{\delta} \setminus a_{\delta'})$ so $\chi_{[a_{\delta}]}$ is constantly 1 on $[a_{\delta} \setminus a_{\delta'})$. If $\beta \geq L \delta'$ then $\beta \geq L \delta$ by the choice of $\delta'$ so $a_{\beta} \geq a_{\delta}$ and hence $\chi_{[a]_i}$ is constantly 1 on $[a_{\delta} \setminus a_{\delta'})$. In addition, $a_{\beta}^c \leq a_{\delta}^c$ so $\chi_{[a_{\delta}]}$ is constantly 0 on $[a_{\delta} \setminus a_{\delta'})$, and the statement is proved.

Let $\varepsilon \in [0, 1]$ be such that $f$ is constantly $\varepsilon$ on $[a_{\delta} \setminus a_{\delta'})$. Say $\varepsilon \leq 1/2$ as the other case is symmetric. Hence $\max\{f, 1 - f\}$ on $a_{\delta} \setminus a_{\delta'}$ is $1 - f$ and in particular we have

$$0 \leq g \leq |\chi_{[a]_i} - f| + |f - \chi_{[a']_i}| \leq 2 \cdot |\chi_{[a]_i} - f|$$

and hence $1/2 \cdot g$ is a function which contradicts that $\bar{f}$ exemplifies that $i \in \text{inv}_{\vec{a}, \vec{b}}(\chi_{[a]_i})$. A contradiction and hence $\chi_{[a']_i}$ exemplifies that $i \in \text{inv}_{\vec{a}, \vec{b}}(\chi_{[a]_i})$ as required. ★6.6

Now suppose that $i \in \text{inv}_{\vec{a}, \vec{b}}(\chi_{[a]_i})$ and assume without loss of generality by Claim 6.6 that $\bar{f} = \chi_{[a']_i}$ for some $\delta'$. We need to prove that $i \in \text{inv}_{\vec{a}, \vec{b}}(a_{\delta})$ as exemplified by $a_{\delta'}$. But indeed, if for some $\beta < a_{\delta}^c$ and $l < 2$ we have that $a_{\delta}^c \cap a_{\delta} \neq a_{\delta} \cap a_{\delta'}$ mod $\mathfrak{W}^{\omega_l}$, then there is $w > 0$ in $\mathfrak{W}^{\omega_l}$ with $w \leq a_{\delta} \Delta a_{\delta'}$ and then clearly the sequence $\langle \chi_{[w], \chi_{[w]}, \ldots} \rangle$ is not in $C^0$ and is below $\langle \chi_{[a_{\delta} \Delta a_{\delta'}], \chi_{[a_{\delta} \Delta a_{\delta'}]}, \ldots} \rangle$, a contradiction. ★6.5

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