ON A ORDER REDUCTION THEOREM

IN THE LAGRANGIAN FORMALISM

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Abstract

We provide a new proof of an important theorem in the Lagrangian formalism about necessary and sufficient conditions for a second-order variational system of equations to follow from a first-order Lagrangian.

1 Introduction

The importance of variational equations, i.e. equations which can be obtained from a variational principle, is well established in the physics literature. Practically all the equations of physical interest (electromagnetism, Yang-Mills, gravitation, string theory, etc.) are of this type. When faced with a system of partial differential equations (the number of the equations being equal to the number of the field components), one has to answer two questions: 1) if the equations are variational (at least locally); 2) how to choose the “simplest” Lagrangian i.e. of the lowest possible order.

The first question is answered completely by the so-called Anderson-Duchamp-Krupka \([\text{I}], \text{II}\) equations. These equations give necessary and sufficient conditions on a set of partial differential equations (the number of the equations being equal to the number of the field components) such that they can be derived from a (local) Lagrangian. These equations are nothing else but the generalization to classical field theory and to arbitrary order of the well-known Helmholtz-Sonin equations from particle mechanics and for second-order equations. The proof is based on the construction of an explicit Lagrangian associated to any set of variational equations, called Tonti Lagrangian. This Lagrangian is, in general, of the same order as the equations, so it is highly degenerated. The second question (called here the order-reduction problem) is not solved in complete generality, although a number of partial results and conjectures are available in the literature (see for instance \([\text{III}]\) and references cited there). It is clear that if the Lagrangian is of order \(r\) then the order

\[ r = \text{order of the system of equations} \]

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of the equations will be $s = 2r$ or less (if the Lagrangian is degenerated). One can conjecture that if the order of the equations is $s$ then one can choose a Lagrangian of minimal order $\left\lceil \frac{s}{2} \right\rceil + 1$. This conjecture is true in particle mechanics [4] but, in general, is false in classical field theory.

However, there exists a case which can be completely analyzed in the classical field theory framework, namely the case of second-order equations. Needless to say, most of the physical applications fall into this case. In [1] one can find the proof of the following statement: a second-order system of partial differential equations (the number of the equations being equal to the number of the field components) follows from a first-order Lagrangian if and only if it is at most linear in the second-order derivatives. This particular case is nevertheless, extremely important for physical applications. Indeed, in [3] and [4] one finds out the proof of the following facts: (i) every locally variational second-order system of partial derivative equations (the number of the equations being equal to the number of the field components) is a polynomial of maximal degree equal to the dimension of the space-time manifold in the second-order derivatives. (In fact, the dependence is through some special combinations called hyper-Jacobians [4]); (ii) a number of physically interesting groups of Noetherian symmetries (like gauge invariance or general transformation of coordinates in gravity) have the effect of reducing the dependence of the second-order derivatives to a dependence which is at most linear. So, in this case we are exactly in the particular case of the order-reduction theorem enunciated above.

The proof of this theorem from [1] is extremely long and based on complicated computations. Because results of this type are less known in the physical literature, we think that it is interesting to provide a rather elementary proof of the order-reduction statement from above. The idea of the proof is to use complete induction. This type of argument was intensively used in [3] and [4] and seems to be extremely useful for further generalizations. We feel that this line of argument might be the simplest in trying to analyze the general case.

In Section 2 we present the general formalism of jet-bundle extensions applied to variational problems and in section 3 we prove the order-reduction theorem.

\section{Jet Bundle Extensions and Variational Calculus}

Suppose that $\pi : Y \rightarrow X$ is a fibration of the manifold $Y$ (of dimension $N + n$) over the manifold $X$ (of dimension $n$).

If $x \in X, y \in Y$ and $\zeta, \zeta' : X \rightarrow Y$ are smooth sections such that $\zeta(x) = \zeta'(x) = y$ then we say that $\zeta$ is equivalent to $\zeta'$ if their partial derivatives up to order $r$ computed in an arbitrary chart $(U, \phi)$ around $x$ and $(V, \psi)$ around $y$ are identical. Then we denote the equivalence class of $\zeta$ by $j^r_{x,y} \zeta$ and the set of all such equivalence classes by $J^r_{x,y}(Y)$. Then the $r$-order jet bundle extension is by definition:

$$J^r_{x,y}(Y) = \cup J^r_{x,y}(Y). \quad (2.1)$$
We denote by $\pi^{s,t} : J^s_n(Y) \to J^t_n(Y)$ ($t \leq s \leq r$) the canonical projections. By convention $J^0_n(Y) \equiv Y$.

In the following we give the details for the case $r = 2$. If $\phi = (x^\mu), \mu = 1, \ldots, n$ and $\psi = (x^\mu, \psi^A), A = 1, \ldots, N$ are two chart systems adapted to the fiber bundle structure, then one can extend it to $J^2_n(S) : (V^2, \psi^2)$ where $V^2 \equiv (\pi^{2,0})^{-1}(V)$ and $\psi^2 = (x^\mu, \psi^A, \psi^A_\mu, \psi^A_{\mu\nu})$ where $\mu \leq \nu$.

The definition of the last two coordinates are:

$$
\psi^A_\mu(j^2_n \zeta) \equiv \partial_\mu \psi^A \circ (x) \circ \phi^{-1}(\phi(x)) \quad (2.2)
$$

$$
\psi^A_{\mu\nu}(j^2_n \zeta) \equiv \partial_\mu \partial_\nu \psi^A \circ (x) \circ \phi^{-1}(\phi(x)). \quad (2.3)
$$

It is convenient to extend $\psi^A_{\mu\nu}$ to all couples $\mu\nu$ by symmetry: we denote

$$
\{\mu\nu\} = \begin{cases} 
\mu\nu & \text{for } \mu \leq \nu \\
\nu\mu & \text{for } \nu \leq \mu
\end{cases}; \quad (2.4)
$$

then $\psi^A_{\mu\nu} \equiv \psi^A_{\{\mu\nu\}}$. Now we define the differential operators:

$$
\partial^\mu A \equiv \frac{\partial}{\partial \psi^A_\mu}; \quad \partial^\mu_{\mu\nu} \equiv \frac{\partial}{\partial \psi^A_{\mu\nu}} \times \begin{cases} 
1 & \text{if } \mu = \nu \\
1/2 & \text{if } \mu \neq \nu
\end{cases} \quad (2.5)
$$

and the total derivative operators

$$
D_\mu \equiv \frac{\partial}{\partial x^\mu} + \psi^A \partial^\mu A + \psi^A_{\mu\nu} \partial^\mu_{\mu\nu}. \quad (2.6)
$$

Suppose that $T_A, (A = 1, \ldots, N)$ are some smooth functions on $J^2_n(Y)$ i.e. they depend on $(x^\mu, \psi^A, \psi^A_\mu, \psi^A_{\mu\nu})$.

One calls the $n + 1$-form

$$
T = T_A \, d\psi^A \wedge dx^1 \wedge \cdots \wedge dx^n \quad (2.7)
$$
a second-order differential equation. One says that $T$ is locally variational if there exists a locally defined function $L$ on $J^2_n(Y)$ such that:

$$
T_A = \mathcal{E}_A(L) \equiv (\partial_A - D_\mu \partial^\mu_A + D_\mu D_\nu \partial^\mu_{\mu\nu}) \, L \quad (2.8)
$$

One calls $L$ a local Lagrangian and:

$$
L \equiv L \, dx^1 \wedge \cdots \wedge dx^n \quad (2.9)
$$
a local Lagrange form.

If the differential equation $T$ is constructed as above then we denote it by $E(L)$. A local Lagrangian is called a total divergence if it is of the form:

$$
\mathcal{L} = D_\mu V^\mu. \quad (2.10)
$$
One can check that in this case we have:

\[ E(L) = 0. \]  

(2.11)

The content of the local variationality conditions ADK is expressed by a set of partial derivative equations on the components \( \mathcal{T}_A \) which can be found in \([5]\) (see eqs. (3.1)-(3.3) there). As we have said in the introduction, we will be concerned here with the case when \( \mathcal{T}_A \) are of the form:

\[ \mathcal{T}_A = t_{\mu\nu}^{AB} \psi_{\mu}^A + t_A. \]  

(2.12)

with \( t_{\mu\nu}^{AB} \) and \( t_A \) smooth functions depending on \((x^\mu, \psi^A, \psi^A_\nu)\) and one can suppose that we have the symmetry property:

\[ t_{\mu\nu}^{AB} = t_{\nu\mu}^{BA}. \]  

(2.13)

The ADK equations are in this case:

\[ t_{\mu\nu}^{AB} = t_{BA}^{\mu\nu}, \]  

(2.14)

\[ \partial_A^{\mu} t_B + \partial_B^{\mu} t_A = 2 \frac{\delta}{\delta x^\nu} t_{AB}^{\mu\nu}, \]  

(2.15)

\[ \partial_C t_{AB}^{\mu\nu} + \partial_B t_{AC}^{\mu\nu} = \partial_A t_{BC}^{\mu\nu} + \partial_C t_{AB}^{\mu\nu}, \]  

(2.16)

\[ 2\partial_B t_A - \frac{\delta}{\delta x^\nu} \partial_B t_A = A \leftrightarrow B, \]  

(2.17)

\[ 4\partial_B t_{AC}^{\mu\nu} - 2 \frac{\delta}{\delta x^\rho} \partial_B t_{AC}^{\rho\mu} - (\partial_B^{\mu} \partial_C^{\nu} + \partial_B^{\nu} \partial_C^{\mu}) t_A = A \leftrightarrow B, \]  

(2.18)

\[ (\partial_B^{\rho} \partial_C^{\sigma} + \partial_B^{\sigma} \partial_C^{\rho}) t_{AD}^{\rho\sigma} = (\partial_A^{\rho} \partial_D^{\sigma} + \partial_A^{\sigma} \partial_D^{\rho}) t_{BC}^{\mu\nu}, \]  

(2.19)

where:

\[ \frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \psi^A \partial_A^{\mu}. \]  

(2.20)

If the equations (2.13)-(2.19) are satisfied then the differential equation \( T \) can be derived from a Lagrangian. This Lagrangian is highly non-unique. A possible choice is the Tonti expression which is in our case:

\[ L = L_0 + L_1 \]  

(2.21)

where

\[ L_0 \equiv \int_0^1 \psi^A t_A \circ \chi \lambda d\lambda, \]  

(2.22)
\[ \mathcal{L}_1 \equiv \psi^A_{\mu\nu} \mathcal{L}^{\mu\nu}_A \]  
(2.23)

with

\[ \mathcal{L}^{\mu\nu}_A = \int_0^1 \lambda \psi^B t^{\mu\nu}_{AB} \circ \chi d\lambda \]  
(2.24)

and

\[ \chi\lambda(x^\mu, \psi^A, \psi^A_\nu) = (x^\mu, \lambda \psi^A, \lambda \psi^A_\nu). \]  
(2.25)

Notice that \( \mathcal{L} \) is a second-order Lagrangian. The purpose of this paper is to prove that one can find an equivalent Lagrangian which is of first order.

Let us note in closing that from (2.16) and (2.19) one obtains that the functions \( \mathcal{L}^{\mu\nu}_A \) defined above satisfy the following equations:

\[ \partial^\rho_B \mathcal{L}^{\mu\nu}_A + \partial^\mu_B \mathcal{L}^{\rho\nu}_A + \partial^\nu_B \mathcal{L}^{\mu\rho}_A = A \leftrightarrow B \]  
(2.26)

and

\[ \partial^\rho_B (\partial^\nu_C \mathcal{L}^{\mu^\sigma}_A + \partial^\nu_C \mathcal{L}^{\rho^\sigma}_A) + \partial^\nu_C (\partial^\mu_B \mathcal{L}^{\rho^\sigma}_A + \partial^\mu_B \mathcal{L}^{\sigma^\rho}_A) - (\partial^\mu_B \partial^\nu_C + \partial^\nu_B \partial^\mu_C) \mathcal{L}^{\sigma^\rho}_B - \partial^\rho_C \partial^\nu_C \mathcal{L}^{\mu^\rho}_A = 0. \]  
(2.27)

**Remark 1** In [1] these two equations are obtained in a different way: one computes \( E_A(\mathcal{L}) \) with \( \mathcal{L} \) given by (2.21)-(2.25) and imposes a structure of the type (2.12).

## 3 The Order Reduction Theorem

In this section we will prove by induction the following theorem:

**Theorem 1** \((T_n)\) Suppose that the functions \( \mathcal{L}^{\mu\nu}_A \) depending on \((x^\mu, \psi^A, \psi^A_\nu)\) verify the equations (2.26) and (2.27). Then there exists a set of functions \( V^\mu \) depending on the same variables such that:

\[ \mathcal{L}^{\mu\nu}_A = \partial^\mu AV^\nu + \partial^\nu V^\mu. \]  
(3.1)

Before starting the proof of theorem \((T_n)\) let us note the following corollary:

**Theorem 2** \((C_n)\) If the functions \( t^{\mu\nu}_{AB} \) verify the system of equations (2.13), (2.14), (2.16) and (2.19) then there exists a function \( \mathcal{L} \) depending on \((x^\mu, \psi^A, \psi^A_\nu)\) such that:

\[ t^{\mu\nu}_{AB} = -\frac{1}{2} (\partial^\mu_B \partial^\nu_A + \partial^\nu_B \partial^\mu_A) \mathcal{L}. \]  
(3.2)
Proof of the corollary:

Define $L_1$ as in (2.23)-(2.25). Because (2.16) and (2.19) are valid we also have (2.20) and (2.27). Applying $(T_n)$ we obtain the functions $V^\mu$ such that (3.1) is true.

Next, we define:

$$L' \equiv L_1 - 2D_\mu V^\mu$$

(3.3)

and prove that $L'$ does not depend on the second-order derivatives $\psi^A_{\mu\nu}$ i.e. is a first-order Lagrangian. Moreover, because $D_\mu V^\mu$ is a trivial Lagrangian we have:

$$E_A(L_1) = E_A(L').$$

(3.4)

So we have:

$$T_A = E_A(L) = E_A(L_0) + E_A(L_1) = E_A(L_0) + E_A(L') = E_A(L_0 + L')$$

(3.5)

so we have (3.2) with $L \rightarrow L_0 + L'$ which is of first order.

We start now the induction proof of $(T_n)$. The assertion $(T_1)$ is elementary: from (2.26) we have in this case:

$$\partial_B L_A = \partial_A L_B$$

(3.6)

and using Frobenius theorem one obtains a (local) function $V$, depending on the variables $(x^\mu, \psi^A, \psi^A_\nu)$ such that:

$$L_A = 2\partial_A V$$

(3.7)

i.e. we have (3.1) for $n = 1$.

Suppose that $(T_n)$ is true: we prove $(T_{n+1})$. It is convenient to assume that the $n + 1$ indices run from 0 to $n$; then the indices from 1 to $n$ are denoted by latin letters $i, j, \ldots$ and the indices from 0 to $n$ by greek letters $\alpha, \beta, \ldots$

We divide the proof in a number of steps.

(i) We take in (2.26) $\mu = \nu = \rho = 0$ and we have:

$$\partial_B L^0_A = \partial_A L^0_B$$

(3.8)

and Frobenius theorem provides us with a function $V^0$ of $(x^\mu, \psi^A, \psi^A_\nu)$ such that:

$$\partial^0_A V^0 = \frac{1}{2} L^0_A$$

(3.9)

i.e. we have (3.1) for $\mu = \nu = 0$. One can give an explicit expression for $V^0$ using a well-known homotopy formula:

$$V^0 = \frac{1}{2} \int_0^1 \psi^C_0 L^0_C \circ \phi_s ds + \tilde{V}^0.$$  

(3.10)

Here
\[\phi_s(x^\mu, \psi^A, \psi^A_\nu) = (x^\mu, \psi^A, s\psi^A_0, \psi^A_i)\]  
\[\text{(3.11)}\]

and \(\tilde{V}^0\) is a function independent of \(\psi^A_0\) i.e.
\[\partial^0_A \tilde{V}^0 = 0.\]  
\[\text{(3.12)}\]

For the moment \(\tilde{V}^0\) is restricted only by this relation.

(ii) Let us define now the functions:
\[T_{AB}^{ij} \equiv \partial^i_B \mathcal{L}^0_A + \partial^j_B \mathcal{L}^{0i}_A - \partial^0_A \mathcal{L}^{ij}_B.\]  
\[\text{(3.13)}\]

We will prove that \(\tilde{V}^0\) can be chosen such that \(V^0\) fulfills, beside (3.9) the equation:
\[(\partial^i_A \partial^j_B + \partial^j_A \partial^i_B) V^0 = T_{AB}^{ij}.\]  
\[\text{(3.14)}\]

Indeed, if we introduce here \(V^0\) given by (3.10) we obtain that \(\tilde{V}^0\) verifies an equation of the type:
\[(\partial^i_A \partial^j_B + \partial^j_A \partial^i_B) \tilde{V}^0 = \tilde{T}_{AB}^{ij}\]  
\[\text{(3.15)}\]

with
\[\tilde{T}_{AB}^{ij} \equiv T_{AB}^{ij} - \frac{1}{2} \int_0^1 \psi_0^C \left[ (\partial^i_A \partial^j_B + \partial^j_A \partial^i_B) \mathcal{L}^{0i}_C \right] \circ \phi_s ds.\]  
\[\text{(3.16)}\]

The equation (3.17) involves in the left-hand side a function \(\tilde{V}^0\) which does not depend on \(\psi^A_0\) (see (3.12)). The same must be true for the right-hand side. Indeed, using (3.8) and (2.27) for \(\mu = i, \nu = j, \rho = \sigma = 0\) one easily proves that:
\[\partial^0_D \tilde{T}_{AB}^{ij} = 0.\]  
\[\text{(3.17)}\]

So, the system (3.15) makes sense. We now note that this system is exactly of the same type as in the corollary \((C_n)\). So, one has to check if the integrability conditions of the type (2.13), (2.14), (2.16) and (2.19) hold and one is entitled to apply the induction procedure. Indeed we have:
- (2.13): directly from the definition (3.13)
- (2.14): from (2.26) with \(\mu = i, \nu = j, \rho = 0\)
- (2.16): by direct computation using (2.26) and (2.27)
- (2.19): again by direct computation using (2.27).

So, applying the induction hypothesis and the implication \((T_n) \Rightarrow (C_n)\) one gets that the system (3.15) has a solution. We will suppose from now on that \(V^0\) is fixed (non-uniquely however) by (3.9) and (3.14).

(iii) We now define the functions:
\[G_A^i \equiv \mathcal{L}^{0i}_A - \partial^i_A V^0.\]  
\[\text{(3.18)}\]
One proves using (2.27) that:

$$
\partial_B^0 G_A^i = \partial_B^0 G_B^i.
$$

(3.19)

Using Frobenius theorem one obtains the existence of a set of (local) functions $V^i$ such that:

$$
\partial_A^0 V^i = G_A^i.
$$

(3.20)

Explicitly, one has a formula of the same type as (3.10):

$$
V^i = \int_0^i \psi^A_0 G_A^i \circ \psi_s + \tilde{V}^i
$$

(3.21)

where:

$$
\partial_A^0 \tilde{V}^i = 0.
$$

(3.22)

From (3.18) and (3.20) we note that it follows:

$$
\partial_A^0 V^i + \partial_i^0 V^0 = \mathcal{L}_A^{0i}
$$

(3.23)

i.e. (3.1) for $\mu = i$, $\nu = 0$.

(iv) Let us define the functions:

$$
\tilde{\mathcal{L}}_A^{ij} \equiv \mathcal{L}_A^{ij} - \int_0^1 \psi_0^C (\partial_A^j G_C^i + \partial_A^i G_C^j) \circ \phi_s ds.
$$

(3.24)

One can check by direct computation that:

$$
\partial_A^0 \tilde{\mathcal{L}}_A^{ij} = 0.
$$

(3.25)

Moreover, it follows from (2.26) and (2.27) that $\tilde{\mathcal{L}}_A^{ij}$ verifies integrability equations of the same type as $\mathcal{L}_A^{ij}$. So, applying the induction procedure we obtain a set of functions $\tilde{V}^i$ depending on $(x^\mu, \psi^A, \psi^A_i)$ such that:

$$
\tilde{\mathcal{L}}_A^{ij} = \partial_A^i \tilde{V}^j + \partial_A^j \tilde{V}^i.
$$

(3.26)

It is natural to take in (3.21) $\tilde{V}^i$ exactly the solution of the system above. Then (3.24) implies:

$$
\mathcal{L}_A^{ij} = \partial_A^i V^j + \partial_A^j V^i
$$

(3.27)

i.e. we have (3.1) for $\mu = i$, $\nu = j$. This finishes the induction.

Finally we note that the same idea used in proving $(T_n) \Rightarrow (C_n)$ proves the order-reduction theorem i.e. if $T_A$ given by (2.12) is locally variational, then one can find a first-order Lagrangian $\mathcal{L}$ such that $T_A = \mathcal{E}_A(\mathcal{L})$.

In particular we have (3.2) and:
\[ t_A = \partial_A \mathcal{L} - \frac{\delta}{\delta x^\mu} \partial^\mu_A \mathcal{L}. \] (3.28)

**Remark 2** The function \( \mathcal{L} \) is determined by (3.2) up to an expression of the following form:

\[ \mathcal{L} = \sum_{k=0}^n \frac{1}{k!} C_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k} \prod_{I=0}^k \psi_{\mu_i}^{A_i} \] (3.29)

where the functions \( C_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k} \) are independent of the first order-derivatives and are completely antisymmetric in the indices \( \mu_1,\ldots,\mu_k \) and in the indices \( A_1,\ldots,A_k \).

This is the first step in deriving the most general expression of a trivial first-order Lagrangian (see [3]).

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