KINETIC THEORY OF NON-ABELIAN PLASMA:
A NON-MINIMAL MODEL

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Abstract

The self-consistent system of master equations describing the kinetics of a relativistic non-Abelian plasma, influenced by curvature interactions, is formulated. Non-minimal (curvature induced) coupling is shown to modify all the subsystems of the model: the gauge field equations, the gravity field equations and the kinetic equation for colored particles.

1 Introduction

Relativistic kinetic theory of non-Abelian plasma in its minimal version is developed in detail (see, e.g., reviews [1, 2] for references). This theory is a direct non-Abelian generalization of the kinetic theory of electrically charged relativistic particles and focuses on the problems of quark-gluon plasma dynamics [3]. Taking into account numerous astrophysical and cosmological applications of the quark-gluon plasma theory, it is reasonable to discuss the non-minimal extension of the model describing colored multi-particle systems in a strong gravitational field. Non-minimal extension means that the Riemann, Ricci tensors and Ricci scalar appear in the kinetic equations, in the gauge field equations and in the gravity field equations as extra terms, vanishing in the absence of tidal (curvature induced) interactions (see, e.g., [4, 5]). The non-minimal extension of the gauge field equations was discussed in many papers, and a number of approaches have been proposed. One of them is connected with a dimensional reduction of the Gauss-Bonnet action [6]. We use a non-Abelian generalization of the non-minimal Einstein-Maxwell theory along the lines proposed by Drummond and Hathrell for the linear electrodynamics [7]. Based on the results of the paper [8], we considered in [9, 10, 11] a three-parameter gauge-invariant non-minimal Einstein-Yang-Mills model, linear in curvature. The same idea is used here to form the subsystem of non-minimally extended gauge field equations and gravity field equations. The generalization of the kinetic equation for the non-Abelian plasma, presented here, is based on three key elements. The first one is the theory of extended phase space (see, e.g., [12, 13, 14, 15, 11, 2, 16]). The second element is a four-vector of the (non-minimal) self-interaction force, changing the momentum of a colored particle. This force enters the equation of particle dynamics and is a subject of phenomenological modeling. The third element is the scalar (non-minimal) force-like term, which predetermines the evolution of color charges. This quantity appears in the equation of charge dynamics, the generalization of the Wong equation [17], and is modelled here phenomenologically.

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2 Kinetic equation on the extended phase space

2.1 Distribution function

The extended phase space in the non-minimal non-Abelian \((SU(n))\)-symmetric kinetic theory is \((7 + n^2)\)-dimensional, i.e., is based on four spacetime coordinates \(x^i\), four-vector of particle momentum \(p^k\) and \(n^2 - 1\) color charges \(Q^{(a)}\) treated as stochastic scalar variables, the distribution function being of the form

\[
f\left(x^i, p^k, Q^{(a)}\right) \delta \left(p_n p^n - m^2 c^2\right) \delta \left(G_{(a)(b)} Q^{(a)} Q^{(b)} - \alpha_2\right) \delta \left(d_{(a)(b)(c)} Q^{(a)} Q^{(b)} Q^{(c)} - \alpha_3\right) \ldots
\]

The Latin indices in parentheses stand for the group ones taking \(n^2 - 1\) values, the symmetric tensor \(G_{(a)(b)}\), defined as

\[
G_{(a)(b)} \equiv 2 \text{Tr} \ t_{(a)} t_{(b)},
\]

plays a role of a metric in the group space. \(t_{(a)}\) are the Hermitian traceless generators of \(SU(n)\) group, they satisfy the relations

\[
[t_{(a)}, t_{(b)}] = if_{(c)}^{(a)(b)} t_{(c)},
\]

where \(f_{(a)(b)}^{(c)}\) are the structure constants of the \(SU(n)\) group. The first delta-function in \((1)\) guaranties that the particle momentum lies on the mass-shell. The second one prescribes the square of color charges to be constant and equal to \(\alpha_2\). When we deal with \(SU(n)\) group and \(n > 2\), the third order Casimir invariant, \(d_{(a)(b)(c)} Q^{(a)} Q^{(b)} Q^{(c)}\), has to be constant, and the third delta-function appears in \((1)\), etc. The completely symmetric coefficients \(d_{(a)(b)(c)}\) are defined by

\[
\{t_{(a)}, t_{(b)}\} \equiv t_{(a)} t_{(b)} + t_{(b)} t_{(a)} = \frac{1}{n} \delta_{(a)(b)} I + d_{(a)(b)}^{(c)} t_{(c)},
\]

\(I\) being the matrix-unity. Using the complete and incomplete measures

\[
dQ \equiv dQ^{(1)} dQ^{(2)} \ldots dQ^{(n^2 - 1)}, \quad dQ/dQ^{(b)} \equiv dQ^{(1)} \ldots dQ^{(b-1)} dQ^{(b+1)} dQ^{(n^2 - 1)},
\]

one can define, respectively, the averaged and specific distribution functions:

\[
f \left(x^i, p^k\right) = \int dQ f \left(x^i, p^k, Q^{(a)}\right), \quad f_{(b)} \left(x^i, p^k, Q^{(b)}\right) = \int dQ/dQ^{(b)} f \left(x^i, p^k, Q^{(a)}\right).
\]

2.2 Kinetic equation and its characteristics

The distribution function is considered to be a solution of the kinetic equation

\[
\frac{p^i}{mc} \hat{\nabla}_i f + \frac{\partial}{\partial p^i} \left(F^i f\right) + \frac{\partial}{\partial Q^{(a)}} \left(H^{(a)} f\right) = \mathcal{J}.
\]

Here \(\hat{\nabla}_i\) is the Cartan derivative \([13]\)

\[
\hat{\nabla}_i \equiv \nabla_i - \Gamma^k_{ip} \frac{\partial}{\partial p^k}, \quad \hat{\nabla}_i p^k = 0, \quad \hat{\nabla}_i Q^{(a)} = 0.
\]
\( \nabla_i \) is the covariant derivative and \( \Gamma^k_{ij} \) are the Christoffel symbols, associated with the spacetime metric \( g_{ik} \). \( J \) symbolizes the collision integral, the four-vector \( F^i \) is a generalised force, depending on coordinates, particle momentum and color charges, the scalar function \( H^{(a)} \) is a force-like term predetermining the color charge evolution. The physical sense of the terms \( F^i \) and \( H^{(a)} \) can be clarified using the characteristic equations for the kinetic equation (7):

\[
\frac{dx^i}{ds} = \frac{p^i}{mc}, \quad \frac{dp^i}{ds} + \frac{1}{mc} \Gamma^i_{kl} p^k p^l = F^i, \quad \frac{d}{ds} Q^{(a)} = H^{(a)}.
\] (9)

As a consequence of (1), the force \( F^i \) is orthogonal to the particle momentum, i.e., \( F^i p^i = 0 \). This four-vector can be modelled as a sum of two forces

\[
F^i = \frac{g}{mc} Q^{(a)} F^{ik}_{(a)} p_k + G^i_{(\text{self})}.
\] (10)

The first part is clearly the non-Abelian generalization of the Lorentz force in \( U(1) \) - electrodynamics, and \( F^{ik}_{(a)} \) is the strength of the Yang-Mills field. The second part includes the self-interaction forces of different types. We will discuss their structure below. Analogously, due to (1) the following relations hold:

\[
G_{(a)(b)} H^{(a)} Q^{(b)} = 0, \quad d_{(a)(b)(c)} H^{(a)} Q^{(b)} Q^{(c)} = 0, ...
\] (11)

giving the key for modeling of the functions \( H^{(a)} \).

### 2.3 Macroscopic moments and balance equations

The standard procedure of subsequent integration of the kinetic equation yields the balance equations. The first one is the conservation law for the particle number:

\[
\nabla_i N^i(x) = 0, \quad N^i(x) \equiv \int dP dQ f p^i, \quad dP \equiv \sqrt{-g} d^4 p.
\] (12)

The second balance equation

\[
\nabla_k T^{ik}(x) = mc \int dP dQ f F^i, \quad T^{ik}(x) \equiv \int dP dQ f p^i p^k
\] (13)

relates to the evolution of the particle stress-energy tensor under the influence of the force \( F^i \). The equations

\[
\nabla_k T^{k(a)}(x) = \int dP dQ f H^{(a)}, \quad T^{k(a)}(x) \equiv \frac{1}{mc} \int dP dQ Q^{(a)} p^k f
\] (14)

describe the evolution of the color currents \( T^{k(a)} \); they become conservation laws if the integral of \( H^{(a)} \) vanishes. Finally, the balance of entropy is regulated by the equation

\[
\sigma(x) \equiv \nabla_i S^i = k_B mc^2 \int dP f \left[ \frac{\partial F^i}{\partial p^i} + \frac{\partial H^{(a)}}{\partial Q^{(a)}} \right],
\] (15)

\[
S^i(x) \equiv -k_B c \int dP dQ f p^i \left[ \ln h^* f - 1 \right],
\] (16)

where \( \sigma(x) \) is the entropy production scalar.
3 Non-minimal field equations

3.1 Gauge field

The Yang-Mills field strength \( F_{mn} \) and the Yang-Mills potential \( A_i \) are considered to take values in the Lie algebra of the gauge group \( SU(n) \) [18]:

\[
F_{mn} = -i G t_{(a)} F^{(a)}_{mn}, \quad A_{m} = -i G t_{(a)} A^{(a)}_{m},
\]

where the real fields \( F^{(a)}_{mn} \) and \( A^{(a)}_{m} \) are connected as follows

\[
F^{(a)}_{mn} = \nabla_m A^{(a)}_n - \nabla_n A^{(a)}_m + G f^{(a)}_{(b)(c)} A^{(b)}_m A^{(c)}_n.
\]

The tensor \( F^{(a)}_{ik} \) satisfies the relation

\[
\hat{D}^k F^{(a)}_{ik} = 0, \quad F^{(a)}_{ik} \equiv \frac{1}{2} \varepsilon^{ikls} F_{ls}^{(a)},
\]

where \( \varepsilon^{ikls} = \frac{1}{\sqrt{-g}} E^{ikls} \) is the Levi-Civita tensor, \( E^{ikls} \) is the completely antisymmetric symbol with \( E^{0123} = 1 \). The gauge invariant derivative \( \hat{D}^k \) is defined as

\[
\hat{D}^k T^{(a)\cdots (d)} = \nabla^k T^{(a)\cdots (d)} + G f^{(a)}_{(b)(c)} A^{(b)}_m T^{(c)\cdots (d)} - G f^{(c)}_{(b)(d)} A^{(b)}_m T^{(a)\cdots (d)} + \cdots,
\]

where \( T^{(a)\cdots (d)} \) is arbitrary tensor in the group space. The Yang-Mills field strength \( F^{(a)}_{ik} \) is considered to be a solution of the non-minimally extended master equations for the gauge field. In the three-parameter non-minimal Einstein-Yang-Mills model [9, 10, 11] these equations read

\[
\hat{D}^k \left[ F^{(a)}_{ik} + \mathcal{R}^{ikmn} F^{(a)}_{mn} \right] = T^{(a)}_{ik},
\]

where the color current \( T^{(a)}_{ik} \) is given by (14), and the susceptibility tensor \( \mathcal{R}^{ikmn} \) is

\[
\mathcal{R}^{ikmn} = \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn}.
\]

Here \( q_1, q_2 \) and \( q_3 \) are the constants of non-minimal coupling of the gauge and gravity fields, \( R^{ikmn} \) is the Riemann tensor, \( R^{ik} \) is the Ricci tensor, \( R \) is the Ricci scalar.

3.2 Gravitational field

In the three-parameter non-minimal theory, linear in curvature, the equations for the gravity field take the standard form (see [8, 9, 10, 11])

\[
\left( R_{ik} - \frac{1}{2} R g_{ik} \right) = \Lambda g_{ik} + \kappa \left( T_{ik}^{(YM)} + T_{ik}^{(NM)} \right).
\]

The stress-energy tensor of pure Yang-Mills field is given by the term \( T_{ik}^{(YM)} \):

\[
T_{ik}^{(YM)} \equiv \frac{1}{4} g_{ik} F_{mn}^{(a)} F_{mn}^{(a)} - F_{in}^{(a)} F_{kn}^{(a)}.
\]
The term $T_{ik}$ denotes a stress-energy tensor of color particles, given by (13). The non-minimal contribution $T_{ik}^{(NM)}$ is presented by the decomposition

$$T_{ik}^{(NM)} = q_1 T_{ik}^{(I)} + q_2 T_{ik}^{(II)} + q_3 T_{ik}^{(III)}, \quad (25)$$

$$T_{ik}^{(I)} = RT_{ik}^{(YM)} - \frac{1}{2} R_{ik} F_{mn}^{(a)} F_{mn}^{(a)} + \frac{1}{2} [\hat{D}_i \hat{D}_k - g_{ik} \hat{D}_j \hat{D}_l] [F_{mn}^{(a)} F_{mn}^{(a)}], \quad (26)$$

$$T_{ik}^{(II)} = -\frac{1}{2} g_{ik} \left[ \hat{D}_m \hat{D}_i \left( F_{mn}^{(a)} F_{ln}^{(a)} \right) - R_{lm} F_{mn}^{(a)} F_{ln}^{(a)} \right] -$$

$$- F_{ln}^{(a)} (R_{ik} F_{kn}^{(a)} + R_{kl} F_{in}^{(a)}) - R_{mn} F_{in}^{(a)} F_{kn}^{(a)} - \frac{1}{2} \hat{D}_m \hat{D}_i \left( F_{ln}^{(a)} F_{kn}^{(a)} \right) +$$

$$+ \frac{1}{2} \hat{D}_l \left[ \hat{D}_i \left( F_{ln}^{(a)} F_{kn}^{(a)} \right) + \hat{D}_k \left( F_{ln}^{(a)} F_{in}^{(a)} \right) \right], \quad (27)$$

$$T_{ik}^{(III)} = \frac{1}{4} g_{ik} R_{mnls}^{(a)} F_{mn}^{(a)} F_{ls}^{(a)} - \frac{3}{4} F_{ls}^{(a)} \left( F_{i(a)}^{n} R_{knls} + F_{k(a)}^{n} R_{nlms} \right) -$$

$$- \frac{1}{2} \hat{D}_m \hat{D}_n \left( F_{i(a)}^{n} F_{k(a)}^{m} + F_{k(a)}^{n} F_{i(a)}^{m} \right). \quad (28)$$

The conservation law is valid for the total stress-energy tensor:

$$\nabla^k \left( T_{ik}^{(YM)} + T_{ik} + T_{ik}^{(NM)} \right) = 0, \quad (29)$$

this fact can be verified directly, using the balance equations and Bianchi identities.

## 4 Non-minimal color forces

We obtained a self-consistent non-minimally extended evolutionary model for the system of colored particles, which interact by gauge and gravitational fields. This model is clearly a non-Abelian non-minimal generalization of the Einstein-Maxwell-Vlasov model, which is investigated in detail by the Kazan Gravitational Group in 1975-2005 (see historical review in this issue). The system is non-linear and self-consistent. For instance, in order to find the Yang-Mills field strength $F_{ik}^{(a)}$ we should solve the equations (21), in which the color currents $T^{k(a)}$ are macroscopic moments of the distribution function $f$. The latter is a solution of kinetic equation (7), which contains, in its turn, the tensor $F_{ik}^{(a)}$, which we search for. Thus, the kinetic equation becomes implicitly non-minimal due to the curvature coupling of gravitational and gauge fields. Let us now discuss a new aspect in the non-minimal extension of the theory of non-Abelian plasma, namely, the explicit non-minimal generalization of the force-like structures appeared in the kinetic equation.

### 4.1 Non-minimal self-force $G_{(self)}^{i}$

The simplest reconstruction of this self-force, orthogonal to the particle four-momentum, yields

$$G_{(self)}^{i} = \left( \lambda_1 R^k U^k + \lambda_2 p^k p^m - p^i p_k \right) + \lambda_3 R_{kmm} p^k U^m p^n + ... \quad (30)$$
by analogy with the decomposition of the susceptibility tensor (22). Here $U^k$ is a four-vector of macroscopic velocity of the multi-particle colored system. This time-like four-vector can be obtained by the Eckart receipt as $U^k = N^k/\sqrt{N^m N_m}$. It is important that the self-force contains not only the microscopic particle momentum $p^k$, but the macroscopic quantity $U^k(x)$, as well. This explain the word ”self-interaction” in the force indication. If we take into account the particle spin, the modeling possibilities of this force may be extended significantly (see, e.g., [19, 20]).

4.2 Non-minimal term $H^{(a)}$

We suggest the following non-minimal phenomenological generalization of the Wong equations [17]

$$\frac{\dot{D}}{ds} Q^{(a)} \equiv \frac{d}{ds} Q^{(a)} + \frac{g}{mc} f^{(a)}_{(b)(c)} p^k A^{(b)}_k Q^{(c)} = H^{(a)}_{(NM)}. \quad (31)$$

In the minimal theory the term $H^{(a)}_{(NM)}$ vanishes, and we cover the standard case (see [1] [2] [21] [22]). The non-minimal term $H^{(a)}_{(NM)}$ can be modelled along the line of the decomposition of the susceptibility tensor (22) as

$$H^{(a)}_{(NM)} = \left[ \delta^{(a)}_{(b)} - \frac{Q^{(a)}_{(b)}}{Q^{(c)}_{(c)}} \right] F_{mn}^{(b)} \left[ \omega_1 R^m p^n U^n + \omega_2 R^m_s (p^n U^s + p^s U^n) + \omega_3 R^{mn}_{ls} p^l U^s + \ldots \right]. \quad (32)$$

The equation (31) for the evolution of a color charge is gauge-covariant and contains the susceptibility-like tensorial structures, which are adopted in the non-minimal Einstein-Maxwell theory [23]-[27].

To conclude it is worth stressing that the presented model of evolution of the non-minimal non-Abelian plasma is so far a general scheme to be supplemented with exact solutions. We hope to discuss the corresponding solutions of cosmological type, solutions with pp-wave and spherical symmetries in future papers.

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