Abstract In this paper, we define a new class of dynamic games played in large populations of anonymous agents. The behavior of agents in these games depends on a time-homogeneous type and a time-varying state, which are private to each agent and characterize their available actions and motifs. We consider finite type, state, and action spaces. On the individual agent level, the state evolves in discrete-time as the agent participates in interactions, in which the state transitions are affected by the agent’s individual action and the distribution of other agents’ states and actions. On the societal level, we consider that the agents form a continuum of mass and that interactions occur either synchronously or asynchronously, and derive models for the evolution of the agents’ state distribution. We characterize the stationary equilibrium as the solution concept in our games, which is a condition where all agents are playing their best response and the state distribution is stationary. At least one stationary equilibrium is guaranteed to exist in every dynamic population game. Our approach intersects with previous works on anonymous sequential games, mean-field games, and Markov decision evolutionary games, but it is novel in how we relate the dynamic setting to a classical, static population game setting. In particular, stationary equilibria can be reduced to standard Nash equilibria in classical population games. This simplifies the analysis of these games and inspires the formulation of an evolutionary model for the coupled dynamics of both the agents’ actions and states.

Keywords Population games · Stochastic games · Evolutionary dynamics

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1 Introduction

In a wide variety of fields, ranging from economics to biology and engineering, population games have become a standard model of strategic interplay in large societies of rational agents [1]. When the number of agents is large, the identity and behavior of the individual agent is not significant to the whole, and one can instead focus on the proportion of agents behaving in different ways. This enables the formulation of large-scale game-theoretic models that would otherwise be intractable.

In the classical population game, individual agents are grouped by their types. An agent’s type encodes all the characteristics relevant to their action: agents of the same type have the same motifs and are expected to act in the same way. While on the microscopic level agents of the same type could be acting differently, the individual differences average out on the macroscopic level, where it suffices to describe the behavior of agents via the probability distribution of their actions. Traditionally, the proportion of agents in the society of different types is considered to be fixed, and one only allows variation in the probability distribution of their actions.

It is natural, however, to expect that agents vary their action choices based on timely situational changes that both affect and are affected by their strategic decisions. A large firm might have a competitive edge over a small firm now, but the small firm can improve its future ability to compete through strategic investment [2]. A bird’s likelihood to contest in violent competition over prey depends on how hungry it is, and the success of such a contest affects how hungry it will be [3]. In a traffic scenario, a person’s sensitivity to delays can vary depending on if they are late for work, and could escalate as they get delayed further through poor strategic choices [4]. These examples highlight that the agents are often faced with a decision that is more complex than a “one-shot” problem, as they have time-varying states that are affected by their decision, in addition to their fixed types.

Game-theoretic settings involving time-varying states have classically been modeled by stochastic games [5]. In a stochastic game, a finite group of agents repeatedly interact in a sequence of games, which can vary from one time-step to the next. A global state observable by all agents characterizes which game they are playing now, and the combination of the agents’ actions affects which game they are likely to play next. This leads to a series of coupled Markov decision processes (MDPs) for the agents, which can be solved via dynamic programming techniques such as backward induction. Stochastic games are not suitable for a large population setting though. They suffer from the curse of dimensionality, and practical models are often limited to a few agents. Moreover, as the number of agents becomes large, it becomes unreasonable to expect
that the individual agent has perfect knowledge of the identities and states of
the opponents, and the standard population game assumption of anonymity
is needed.

This work can be viewed to lie at the interface between population games
and stochastic games. We utilize the macroscopic view of population games
which makes the study of interactions in large populations tractable, while
allowing individual agents to have time-varying states in addition to their
fixed types. As we review next, this kind of games has been studied in previ-
ous works under different classifications. We propose a novel formulation,
which we call dynamic population games. In contrast to previous works, our
formulation allows us to establish a reduction of our dynamic games to classi-
cal, static population games, which greatly simplifies their analysis and makes
them straightforward to apply in practice. In particular, we show that sta-
tionary equilibria in a dynamic population game coincide with standard Nash
equilibria in a suitably defined classical population game. To the extent of
our knowledge, this reduction has not been noticed before. The reduction is
enabled by the single-stage deviation principle: we consider that the agents
choose an action now which maximizes the discounted infinite horizon reward
of a single-stage deviation from the current social policy. This simplifies the
agents’ decision problem compared to the case in which they have to optimize
all of their future actions, and leads to the same optimal policies at the sta-
tionary equilibria. As one of the consequences of the reduction, we are able to
formulate a macroscopic model of the coupled evolution of both the agents’
actions and states, by adapting classical evolutionary dynamics to dynamic
population games.

1.1 Literature review

Dynamic interactions amongst many players have been studied under different
classifications, including anonymous sequential games, mean field games and
Markov decision evolutionary games.

Anonymous sequential games were first formulated by Jovanovic and Rosen-
thal [6]. Like in our work, they consider a setting with a continuum of agents
where the agents have individual state dynamics and stage rewards that are
a function of the distribution of agents’ states and actions in the society. The
payoffs of the agents are defined as the discounted infinite horizon rewards.
They introduce and show the existence of the stationary equilibrium, a condi-
tion where the state distribution is stationary, and simultaneously all agents
are playing optimally. Unlike in our work, they consider compact and continu-
ous state and action spaces and rely on measure-theoretic analysis. They also
define the equilibrium concept in terms of distributional strategies, which are
joint distributions over all the agents’ states and actions. This is a less natural
notion of strategies since they give a macroscopic description of the aggregate
behavior of agents rather than specifying how individual agents behave. Some
other works that use distributional strategies include Bergin and Bernhardt [7, 8], which additionally consider aggregate shocks on the agents’ states.

A departure from distributional strategies is in the work of Weintraub et al. [9], who define strategies as maps from the agents’ individual states, as well as the state distribution of others, to a randomized action. They also introduce the notion of oblivious strategies, which are maps from the agents’ individual states only to a randomized action, and the dependency on the state distribution of others is indirect through the agents’ best responses. Namely, they assume that the agents evaluate their discounted infinite horizon rewards at the long-run expected state distribution of others. We also follow this notion of oblivious strategies [4], but unlike in [9], we assume that the agents evaluate their discounted infinite horizon rewards at the current state distribution of others, rather than at the long-run expected distribution. This is less computationally demanding on the agents since they use the current information on the state distribution of others and do not have to predict its future evolution. Moreover, in many applications the long-run expected state distribution occurs too far in the future and/or is uninteresting. An example is strategic individual behavior during epidemics, where it is more interesting to study how the agents react to current levels of infections rather than the long-run expected state distribution, which can take very long to reach and is known to be typically infection-free (the epidemic eventually dies out if the recovered agents gain immunity) [10].

Perhaps most closely related to our work in the anonymous sequential games literature is Adlakha et al. [11], where the authors consider countable state spaces, as well as both finite and compact convex action spaces, and provide sufficient conditions for the existence of stationary equilibria. One of these sufficient conditions is that the state space is bounded and the action space is finite, which is the setting we consider as well. The main advantage of our approach in comparison to [11] is that we make an important simplification in the agents’ strategic decision model, namely, we consider that the agents seek to optimize single-stage deviations from the current strategy followed, rather than the whole strategy. We show that considering single-stage deviations is necessary and sufficient for the optimality of the strategy at the stationary equilibrium. This makes it straightforward to establish existence, and is also the key constituent in the reduction of stationary equilibria to standard Nash equilibria in classical population games.

Mean field games, introduced independently by Huang et al. [12] and Lasry and Lions [13], can be viewed as a continuous-time, state, and action space counterpart of our work. Inspired by large particle systems in statistical physics, mean field games model agents as interacting particles in a continuum of mass, where the rewards and dynamics of an individual particle are affected only by the mass effect of all other particles. The Nash certainty equivalence principle [12] is a condition where each individual particle is solving a

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1 We call our oblivious strategies policies.
stochastic optimal control problem which is parametrized by the mass effect, and the resulting control laws produce that same mass effect they were optimized for. We also model agents as particles in a continuum of mass, and our notion of the stationary equilibrium is conceptually similar to the Nash certainty equivalence principle. However, in our setting the individual agent dynamics follow discrete Markov chains rather than continuous Brownian motions, which are the standard stochastic processes considered in mean-field games. Therefore, our results do not generally apply to classical mean-field games, and vice versa.

More closely related to our setting are works that study mean field games with finite state and action spaces, which were pioneered by Gomes et al. [14, 15], and include follow up works by Doncel et al. [16] and Neumann [17]. The first models by Gomes et al. [14,15] were restrictive because they assume that the agents can directly control their state transitions, irrespective of the environment. This was generalized by Doncel et al. [16] who parametrized the state transitions by a finite set of actions, and further considered the individual state transitions and stage rewards to be functions of the state distribution, but not the others’ actions. For the individual state dynamics, [16] considers either continuous-time controlled Markov chains or synchronous discrete-time Markov chains, where in the latter the game is played simultaneously by all the agents at each time instant, leading to discrete-time societal state dynamics. We also consider synchronous discrete-time Markov chains, and additionally asynchronous discrete-time Markov chains, where the discrete-time instances of the different agents occur at rings of independent Poisson alarm clocks with constant (but possibly type-dependent) rates. This leads to continuous-time societal state dynamics when aggregated. We believe that the asynchronous model is more natural in the large population setting, where the actions of many players are rarely synchronized.

Another difference of our work to [16] is that they focus on equilibria in non-stationary strategies. In that regard, Neumann [17] is more closely related to our work, who considers a similar model as [16] for the continuous-time case, with a focus on equilibria in stationary strategies. However the continuous-time Markov chain model considered there leads to different formulations. We argue that from the individual agent point of view, many decision processes are more natural to model as discrete events rather than continuous dynamics.

Markov decision evolutionary games, introduced by Altman and Hayel [18] and studied by Flesch et al. [19] under the name of evolutionary stochastic games, are an extension of classical evolutionary games [20] to the case where agents have time-varying states. In the classical evolutionary game, a symmetric two-player matrix game is played repeatedly between randomly matched agents in a large population. The agents are characterized by the strategy that they play, which in the evolutionary games literature is commonly referred to as the type or sub-population, with the notion that the agents can choose (or evolve) their types. We remark that in our work, we make an explicit distinction between types, which are immutable characteristics of the agents, and
actions, which are our equivalent of the notion of types in evolutionary games. Markov decision evolutionary games further consider that the matrix games played in every interaction are not the same, but vary according to states of the agents, where the state space is finite. The states follow Markov chains which are affected by the gameplay. This is similar to our setting, however we highlight two main differences. First, we do not only consider matching in matrix games, but any sort of games played by individual agents against the field of opponents. Second, Markov decision evolutionary games consider equilibria in mixed strategies, whereas we consider equilibria in behavioral strategies. By mixed strategies it is meant that the population consists of a mixture of agents who follow deterministic maps from their states to their actions. This essentially reduces these games to classical evolutionary games where the strategies are the deterministic state to action maps, and the payoffs are the expected average stage rewards over the lifetime of the agents. This approach is tractable only for small state spaces though as the number of deterministic state to action maps is exponential in the number of states. On the other hand, under the behavioral strategies we consider all agents in the society follow the same state to action map, with a possible randomization over the actions. This leads to a tractable formulation since the randomization occurs over the actions state-wise, rather than over the exponentially large set of deterministic state to action maps. It also leads to a more natural treatment of future discounting, since it is not clear how to assess the discounted infinite horizon reward of a state to action map without accounting for initial conditions.

A distinguishing feature in evolutionary game theory is that its focus is not only placed on characterizing equilibrium conditions and proving their existence, but also on if and how the equilibrium is reached. The main enabler is the notion of evolutionary dynamics, which model the evolution of the agents’ strategies throughout time. The classical example is the replicator dynamics, which model how the proportions of agents following the different strategies grow or shrink as a consequence of them observing a higher or lower payoff than the average payoff in the society (where payoffs are often referred to as fitness in the evolutionary context, in relation to the Darwinian theory of evolution). Evolutionary dynamics enrich the game theoretic notion of the Nash equilibrium, allowing to study its stability, dependence on initial conditions, and incorporating assumptions on the rationality and information capabilities of the agents in the process of reaching it.

However, the setting where the agents have individual state dynamics in addition to dynamics of their strategy choices is largely unexplored in the literature. To the extent of our knowledge, the only effort to formalize a cou-

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2 We only require that the dependency of state transitions and stage rewards on the field is continuous.

3 Under a common assumption of irreducibility of the state Markov chains, this difficulty is not present when the agents are infinitely future sighted since they can assess their average rewards at the unique stationary state distribution induced by their strategy. If agents have a shortsightedness though, the initial state must be specified in order to assess the payoff of a strategy, since early states contribute more to the payoff than future states.
pled evolutionary model of both the agents’ states and decision making is the recent work by Brunetti et al. [22]. Inspired by the reduction of stationary equilibria in dynamic population games to standard Nash equilibria in classical population game, we are also able to formulate such a model, which has two main differences to the approach in [22]. First, in line with the literature on Markov decision evolutionary games, [22] considers mixed rather than behavioral strategies, which as argued earlier leads to tractability problems for large state spaces. Second, [22] gives an abstract form of the policy-state dynamics, and characterizes them explicitly only for a two-state two-strategy game with the replicator evolutionary model. In contrast, our model can adapt all the existing classes of evolutionary dynamics without restrictions on the sizes of the state and action spaces.

1.2 Outline

We start by recalling preliminaries from classical population games in Section 2. We then describe the dynamic population game setting in Section 3. In Section 4, we introduce the stationary equilibrium as the solution concept of dynamic population games and prove its existence. This is followed by our main contribution in Section 5, where we show a reduction of stationary equilibria to standard Nash equilibria in classical population games. As a consequence of this reduction, we formulate an evolutionary policy-state dynamics model in Section 6. In Section 7, we conclude with a discussion and point to existing and potential applications as well as future directions of research.

2 Preliminaries: Classical population games

We recall the setting and the main elements of classical population games, following the formalism by Sandholm [11], with a few notational deviations.

Population games consider a society \( \mathcal{N} = \{ 1, \ldots, n \} \) of anonymous agents, with \( n \) large such that the agents approximately form a continuum of mass. The agents belong to one of a finite number of populations

\[ \rho \in \mathcal{P} = \{ 1, \ldots, n_\rho \}. \]

The mass of agents in each population is a static parameter \( m_\rho \).

A strategic scenario is considered where agents of population \( \rho \) can play one of a finite number of actions \( a \in \mathcal{A}_\rho \). On the macroscopic level, the distribution of the action choices of agents of population \( \rho \) is described by the population state \( \chi_\rho \), where \( \chi_\rho[a] \in [0, m_\rho] \) is the mass of agents playing action \( a \in \mathcal{A}_\rho \). The population state satisfies

\[ \chi_\rho \in X_\rho = \left\{ \chi_\rho \in \mathbb{R}^{\left| \mathcal{A}_\rho \right|} \middle| \sum_{a \in \mathcal{A}_\rho} \chi_\rho[a] = m_\rho \right\}. \]
There are two classical interpretations of how the population state can be derived from the microscopic behavior of the agents:

1. Agents of the same population are heterogeneous in their behavior. Different agents deterministically play different actions, with the mass of agents playing the different actions specified by $\chi_\rho$.

2. Agents of the same population are homogeneous in their behavior, but they play randomized rather than deterministic actions. In particular, they randomize their actions according to the same probability distribution $\chi_\rho/m_\rho$, which is often referred to as the mixed strategy of the agents.

The concatenation of the population states for all populations is the social state $\chi = (\chi_1, \ldots, \chi_n) \in X = \prod_{\rho \in \mathcal{P}} X_\rho$, which gives a complete macroscopic description of the agents’ behavior in the society.

The payoff to an agent of population $\rho$ playing action $a$ is a continuous function in the social state $F_\rho[a] : X \to \mathbb{R}$, and the agents seek to maximize their payoff. The population game is fully specified by the payoff functions $F_\rho[a](\chi)$ for all populations $\rho$ and actions $a \in A_\rho$. These are grouped per population $\rho$ in a vector of payoffs to all actions $F_\rho(\chi) \in \mathbb{R}^{|A_\rho|}$, and the concatenation of those vectors for all populations is simply the payoff vector $F(\chi) = (F_1(\chi), \ldots, F_n(\chi)) \in \prod_{\rho \in \mathcal{P}} \mathbb{R}^{|A_\rho|}$, which compactly denotes the population game.

The best response of agents in population $\rho$ at the social state $\chi$ is the set valued correspondence

$$B_\rho(\chi) = \left\{ \sigma \in X_\rho \mid \forall \sigma' \in X_\rho, \sum_{a \in A_\rho} (\sigma[a] - \sigma'[a]) F_\rho[a](\chi) \geq 0 \right\}. \tag{1}$$

The best response is the set of population states that place positive mass only on the action(s) maximizing $F_\rho(\chi)$. Alternatively, $B_\rho(\chi)/m_\rho$ can be understood as the set of mixed strategies that maximize the expected payoff of agents in population $\rho$.

A Nash equilibrium is a social state $\chi$ which satisfies

$$\chi_\rho \in B_\rho(\chi), \forall \rho \in \mathcal{P}, \tag{2}$$

i.e., at the Nash equilibrium, the population states in $\chi$ are a best response to $\chi$ for all the populations. It is therefore considered to be a stationary state of the game, since no single agent will have an incentive to deviate from it. At least one Nash equilibrium is guaranteed to exist for every population game \[\Pi, \text{Theorem 2.1.1}\].

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\[\text{4 It is straightforward to verify that if all agents randomize their actions according to the same mixed strategy } \chi_\rho/m_\rho, \text{ the resulting macroscopic distribution of action choices coincides with } \chi_\rho.\]
3 Dynamic population games

As in classical population games, we consider a society $\mathcal{N} = \{1, \ldots, n\}$ of anonymous agents, with $n$ large such that agents approximately form a continuum of mass. Moreover, we consider that the agents have static immutable types, which are the equivalent notion of populations in classical population games, and additionally they have time-varying states, whose dynamics are affected by the gameplay.

3.1 Types, states, state distributions

Individual agents are characterized by a fixed private type

$$\tau \in \mathcal{T} = \{1, \ldots, n_\tau\},$$

which represents immutable characteristics of the agents. The distribution of agent types is the parameter

$$g \in \Delta(\mathcal{T}) = \left\{g \in \mathbb{R}_{+}^{n_\tau} \mid \sum_{\tau \in \mathcal{T}} g_\tau = 1 \right\},$$

where $g_\tau$ specifies the proportion of agents of type $\tau$.

The individual agent interacts with the society at discrete-time instances, at which it is further characterized by a private state

$$x \in \mathcal{X} = \{1, \ldots, n_x\},$$

in addition to its type $\tau$. The state $x$ varies from one time instant to the next, either through an exogenous process or as a consequence of the interaction.

The distribution of agents having type $\tau$ and being in the different states $x \in \mathcal{X}$ is given by the time-varying state distribution

$$d_\tau \in \mathcal{D}_\tau = \left\{d_\tau \in \mathbb{R}_{+}^{n_x} \mid \sum_{x \in \mathcal{X}} d_\tau(x) = g_\tau \right\},$$

where $d_\tau(x)$ denotes the proportion of agents of type $\tau$ and state $x$. The set $\mathcal{D}_\tau$ respects the total proportion of agents in type $\tau$, given by $g_\tau$. The concatenation of the state distributions of all types is the type-state distribution

$$d = (d_1, \ldots, d_n) \in \mathcal{D} = \prod_{\tau \in \mathcal{T}} \mathcal{D}_\tau,$$

which gives the time-varying joint distribution over agent types and states.
3.2 Actions and policies

In an interaction, an agent of type $\tau$ and state $x$ can choose an action

$$a \in \mathcal{A}_{\tau,x} \subseteq \mathcal{A} = \{1, \ldots, n_a\}.$$  

The type-state specific sets $\mathcal{A}_{\tau,x}$ allow agents of different types and states to have access to different actions, and the set $\mathcal{A}$ is the union of all the type-state specific sets.

We envision a dynamic game where the agents do not face a one-shot decision problem, and specify their behavior using policies, which map their states to a randomization over the actions

$$\pi_{\tau} : \mathcal{X} \rightarrow \Delta(\mathcal{A}_{\tau,x}).$$

We use the subscript $\tau$ to distinguish between the policies of different types. We write the probability distribution over the actions induced by $\pi_{\tau}$ on state $x \in \mathcal{X}$ as

$$\pi_{\tau}[\cdot | x] \in \Delta(\mathcal{A}_{\tau,x}),$$

and the probability of choosing a particular action $a$ as

$$\pi_{\tau}[a | x] \in [0, 1].$$

The interpretation of policies in our setting is that agents of the same type are homogeneous in their behavior: given their state they choose their action according to the same probability distribution $\pi_{\tau}[\cdot | x]$. This is similar to interpretation II of population states in classical population games, under which agents of the same population behave homogeneously according to the same randomization scheme (see Section 2). One can instead consider that different agents of the same type follow different deterministic policies, in a manner similar to interpretation I of population states in classical population games. This however becomes intractable in practice, since the space of deterministic policies is exponential in the number of states (its size is $O(n_a^n)$).

The set of policies of type $\tau$ is denoted by $\Pi_{\tau}$, and the concatenation of the policies of all types is

$$\pi = (\pi_1, \ldots, \pi_{n_{\tau}}) \in \Pi = \prod_{\tau \in \mathcal{T}} \Pi_{\tau},$$

simply referred to as the policy.

3.3 Social state

In an analogous terminology to classical population games, we refer to the pair $(\pi, d) \in \Pi \times \mathcal{D}$ as the social state. This gives a macroscopic description of the distribution of the agents’ types and states, as well as how they behave. Note that the social state is time-varying in our setting, due to the agents’ state dynamics, described next.
3.4 State dynamics

3.4.1 Individual state transitions

We first take a microscopic view and characterize how the state of the individual agent evolves through time. We consider discrete-time stochastic dynamics for the individual agent, specified by the state transition function

\[ p_{\tau}[x^+ | x, a](\pi, d). \] (4)

After participating in an interaction, an agent of type \( \tau \) and state \( x \) having played action \( a \) will transition to state \( x^+ \) with a probability that also depends on the social state \((\pi, d)\). For each tuple \((\tau, x, a) \in T \times X \times A_{\tau,x}, \) \( p_{\tau}[\cdot | x, a] : H \times D \to \Delta(X) \) specifies a probability distribution over the agent’s next states as a function of the social state. Note that these transitions are mass-preserving; the agent can transition from one state to another, but must remain in the population. The state transitions are fully specified by at most \( n_{\tau} n_{\tau}^2 n_a \) functions of the social state.

Assumption 1 (State transition continuity) For all \((\tau, x, a) \in T \times X \times A_{\tau,x}, \) the state transition function \( p_{\tau}[\cdot | x, a](\pi, d) \) is continuous in the social state \((\pi, d)\).

A continuous state transition model is natural in practice. In many cases, the dependence on the social state arises in the form of expectations, as the agent draws a random member of the population to interact with.

When the agents of type \( \tau \) follow the homogeneous policy \( \pi_{\tau}, \) their states follow a Markov chain characterized by the stochastic matrix

\[ P_{\tau}[x^+ | x](\pi, d) = \sum_{a \in A_{\tau,x}} \pi_{\tau}[a | x] p_{\tau}[x^+ | \tau, x, a](\pi, d). \] (5)

Note that this Markov chain is not time-homogeneous, due to the dependency on the time-varying social state \((\pi, d)\). However, it becomes time-homogeneous if \((\pi, d)\) is stationary (see the equilibrium definition in Section 4).

3.4.2 Societal state dynamics

We now take a macroscopic view and characterize how the type-state distribution \( d \) evolves through time. For this purpose, we consider either a synchronous or an asynchronous interaction model.

Synchronous interaction model: The agents have a global interaction clock, i.e., at every discrete-time index all the agents interact once. This leads to the following discrete-time updates

\[ d_{\tau}^+ = d_{\tau} P_{\tau}(\pi, d) := W^d_{\tau}(\pi, d), \forall \tau \in T. \] (6)

A setting where agents can enter and/or exit the system can be readily modelled through a special non-participation state.
Asynchronous interaction model: The agents have individual interaction clocks, whose ticks follow independent Poisson processes. The rates of the Poisson processes are $\delta_\tau > 0$, which can vary per type. This leads to the following continuous-time dynamics

$$\dot{d}_\tau = \delta_\tau (d_\tau P_\tau(\pi, d) - d_\tau) := W^\tau(\pi, d), \forall \tau \in \mathcal{T}, \quad \text{(7)}$$

which are derived in Appendix A. The derivation tracks the expected change in the number of agents in type-state $(\tau, x)$ over an infinitesimal time period $dt$ when the agents interact asynchronously, and works most naturally when the interactions occur between individual agents and the field. In settings where the interactions take form of matching between a group of agents, some additional care is needed to ensure that agents find matching partners when their clock ticks.

The trajectories generated by either the synchronous model (6) or the continuous model (7) remain in the set of permissible distributions (3), as we show next. We will use the following standard result for stochastic matrices.

**Lemma 1** Let $\sigma \in \mathcal{D}_\tau$ and $P$ be a stochastic matrix. Then $z = \sigma P \in \mathcal{D}_\tau$.

**Proof** Since all elements of $\sigma$ and $P$ are non-negative, it immediately follows that all elements of $z$ are non-negative as well. Moreover,

$$\sum_{x \in \mathcal{X}} z[x] = \sum_{x, x' \in \mathcal{X}} \sigma[x'] P[x | x'] = \sum_{x' \in \mathcal{X}} \sigma[x'] \sum_{x \in \mathcal{X}} P[x | x'] = \sum_{x' \in \mathcal{X}} \sigma[x'] = g_\tau.$$ 

Therefore, $z \in \mathcal{D}_\tau$. \qed

As an immediate consequence of Lemma 1, trajectories of the synchronous model (6) that start in $\mathcal{D}_\tau$ will always remain in $\mathcal{D}_\tau$, since $P_\tau(\pi, d)$ is a stochastic matrix for all $(\pi, d) \in \Pi \times \mathcal{D}$. For the asynchronous model (7), we have the following proposition.

**Proposition 1** The set $\mathcal{D}_\tau$ is invariant under the dynamics (7).

**Proof** It suffices to show that the vector field $W^\tau(\pi, d)$ lies in the tangent cone $TD_\tau(d_\tau)$ for all social states $(\pi, d) \in \Pi \times \mathcal{D}$; if it does, the direction of motion is always admissible and a trajectory starting in $\mathcal{D}_\tau$ will remain in $\mathcal{D}_\tau$. Since $\mathcal{D}_\tau$ is a polytope, the tangent cone can be characterized by [1] p. 40

$$TD_\tau(d_\tau) = \{ y \in \mathbb{R}^n | \exists z \in \mathcal{D}_\tau, \epsilon \geq 0 : y = \epsilon(z - d_\tau) \}.$$ 

(8)

By choosing $y = W^\tau(\pi, d) = \delta_\tau (d_\tau P_\tau(\pi, d) - d_\tau)$, $z = d_\tau P_\tau(\pi, d) \in \mathcal{D}_\tau$ (by Lemma 1), and $\epsilon = \delta_\tau > 0$, it is straightforward to verify that $W^\tau(\pi, d) \in TD_\tau(d_\tau)$ always holds. \qed
3.5 Rewards

At every interaction, an agent of type $\tau$ and state $x$ playing action $a$ observes a stage reward that also depends on the social state $(\pi, d)$, which is specified by the reward function

$$r_{\tau}[x, a](\pi, d).$$

(9)

For each tuple $(\tau, x, a) \in T \times X \times A_{\tau,x}$, $r_{\tau}[x, a] : \Pi \times D \rightarrow \mathbb{R}$ specifies the agent’s reward as a function of the social state. The rewards are fully specified by at most $n_{\tau} n_x n_a$ functions of the social state.

**Assumption 2 (Rewards continuity)** For all $(\tau, x, a) \in T \times X \times A_{\tau,x}$, the stage reward function $r_{\tau}[x, a](\pi, d)$ is continuous in the social state $(\pi, d)$.

This is analogous to Assumption 1 for the state transitions.

3.6 Strategic decisions

The state transitions (4) and rewards (9) constitute the building blocks of a Markov decision process (MDP) which is parametrized in the social state $(\pi, d)$. In our strategic decision model, we consider that the agents seek to maximize their discounted infinite horizon rewards with type-dependent future discount factors $\alpha_{\tau} \in [0, 1)$, and according to the following principle.

**Single-stage deviation principle:** Given the current social state $(\pi, d)$, agents make decisions on what action to play when in state $x$ based on the benefits of single-stage deviations from $\pi_{\tau}$, the homogeneous policy dictated by the social state for their type. They consider that at the following stages the social state $(\pi, d)$ will be the same and they will follow $\pi_{\tau}$.

We now derive the payoff structure of the agents, based on which they make their strategic decisions. Given the current social state $(\pi, d)$, the expected stage reward of an agent of type $\tau$ following $\pi_{\tau}$ when in state $x$ is

$$R_{\tau}[x](\pi, d) = \sum_{a \in A_{\tau,x}} \pi[a \mid x] r_{\tau}[x, a](\pi, d),$$

(10)

and its expected discounted infinite horizon reward is recursively defined as

$$V_{\tau}[x](\pi, d) = R_{\tau}[x](\pi, d) + \alpha_{\tau} \sum_{x^+ \in X} P_{\tau}[x^+ \mid x](\pi, d) V_{\tau}[x^+](\pi, d),$$

or, equivalently in vector form,

$$V_{\tau}(\pi, d) = (I - \alpha_{\tau} P_{\tau}(\pi, d))^{-1} R_{\tau}(\pi, d).$$

(11)
Note that (11) is continuous in the social state \((\pi, d)\), as \(I - \alpha_\tau P_{\tau}(\pi, d)\) is guaranteed to be invertible for \(\alpha_\tau \in [0, 1)\) (here, \(I\) is the identity matrix of appropriate dimensions).

While an agent can compute the expected discounted rewards \(V_{\tau}(\pi, d)\) for the current social state \((\pi, d)\), the policy of its type \(\pi_\tau\) may not be optimal. As per the single-stage deviation principle, the agent chooses an action \(a\) when in state \(x\) according to the single-stage deviation rewards given by

\[
Q_{\tau}[x, a](\pi, d) = r_{\tau}[x, a](\pi, d) + \alpha_\tau \sum_{x^+ \in X} p_{\tau}[x^+ | x, a](\pi, d) V_{\tau}[x^+](\pi, d).
\]

The single stage deviation rewards \(Q_{\tau}[x, a](\pi, d)\) can be viewed as the dynamic population games equivalent of the payoffs \(F_{\rho}[a](\chi)\) in classical population games (see Section 2). This connection is further explored in Section 5 where we show a reduction of stationary equilibria in dynamic population games to standard Nash equilibria in classical population games.

3.7 Complete definition of dynamic population games

We are now ready to state a formal definition of dynamic population games, which includes all of its elementary building blocks.

**Definition 1 (Dynamic population game)** A dynamic population game is a tuple \(G = \langle \mathcal{T}, X, A, g, D, \Pi, p, r, \alpha, \delta \rangle\), where

- \(\mathcal{T}\) is the finite set of agent types, with cardinality \(n_\tau\),
- \(X\) is the finite set of agent states, with cardinality \(n_x\),
- \(A\) is the finite set of actions, with cardinality \(n_a\). For type-state \((\tau, x)\), the available actions are a subset \(A_{\tau,x} \subseteq A\).
- \(g \in \Delta(\mathcal{T})\) is the type distribution, with \(g_\tau\) specifying the proportion of agents of type \(\tau\),
- \(D\) is the set of joint type-state distributions, with time-varying elements \(d \in D\). The current proportion of agents in type-state \((\tau, x)\) is \(d_\tau[x]\), and \(d_\tau\) satisfies \(\sum_{x \in X} d_\tau[x] = g_\tau\),
- \(\Pi\) is the set of policies, with elements \(\pi \in \Pi\). For the type-state \((\tau, x)\), \(\pi_\tau[· | x]\) specifies the probability distribution over the actions \(a \in A_{\tau,x}\).
- The pair \((\pi, d) \in \Pi \times D\) is the social state,
- \(p\) is the collection of state transition functions, with elements \(p_{\tau}[x^+ | x, a](\pi, d)\), continuous in the social state. The game is fully specified by at most \(n_\tau n_x^2 n_a\) such functions,
- \(r\) is the collection of stage reward functions, with elements \(r_{\tau}[x, a](\pi, d)\), continuous in the social state. The game is fully specified by at most \(n_\tau n_x n_a\) such functions,
- \(\alpha \in [0, 1)^n_\tau\) is the vector of future discount factors, with \(\alpha_\tau\) the future discount factor of type \(\tau\),
- \(\delta \in (0, \infty)^n_\tau\) is the vector of interaction rates, with \(\delta_\tau\) the interaction rate of type \(\tau\). This is used in the asynchronous interaction model.
Definition 1 includes all the elements required to fully specify a dynamic population game. From these elements, one can further derive the following useful quantities

- $P_\tau(\pi, d)$, given in (5), is the stochastic matrix describing the state dynamics of agents of type $\tau$ following $\pi_\tau$.
- $R_\tau(\pi, d)$, given in (10), is the expected stage reward vector of agents of type $\tau$ following $\pi_\tau$.
- $V_\tau(\pi, d)$, given in (11), is the expected discounted infinite horizon reward vector of agents of type $\tau$ following $\pi_\tau$.
- $Q_\tau[x, a](\pi, d)$, given in (12), is the single-stage deviation reward to an agent of type-state $(\tau, x)$ playing current action $a$ instead of following $\pi_\tau$, which the agent seeks to maximize.

4 Solution concept: the stationary equilibrium

The solution concept that we propose in dynamic population games is the stationary equilibrium, a condition where all agents are playing an optimal policy and in addition, the type-state distribution is stationary. Other notions of equilibrium have been considered in previous works [6,9,11,16], and include equilibria in non-stationary policies (i.e., policies that depend on the current discrete-time index) and policies with explicit dependency on the social state (i.e., maps from the agent’s own state and the social state to randomized actions, also referred to as non-oblivious policies [9]). We are of the opinion that the stationary equilibrium notion considered here is the most useful in practice due to its simplicity to compute and follow by the individual agent. We are also able to show a reduction of these equilibria to standard Nash equilibria in classical population games, which is the subject of Section 5.

Recall that the agents face a MDP that is not time-homogeneous due to the dependency on the time-varying social state $(\pi, d)$, which complicates the analysis of optimal policies. However, if we can characterize conditions under which the social state is stationary, the MDP becomes time-homogeneous, and standard results regarding optimal policies apply.

For the type-state distribution $d$, when the agents follow the homogeneous policy $\pi$, a stationary distribution $d \in \mathcal{D}$ (denoted with boldface) satisfies

$$d_\tau = d_\tau P_\tau(\pi, d), \forall \tau \in \mathcal{T}. \quad (13)$$

Notice that this coincides with the rest points of both the discrete-time synchronous societal state dynamics (4) and the continuous-time asynchronous variant (4), as expected. Traditionally, the stationary distribution is solved for by finding the left eigenvector(s) corresponding to the eigenvalue 1 of the stochastic matrix $P_\tau$, however, here the matrix is itself a function of the distribution, so we do not have a simple eigenvector problem.

For the policy $\pi$, we introduce the notion of best response.
Definition 2 (Best response) The best response of an agent in type-state $(\tau, x)$ at the social state $(\pi, d)$ is the set-valued correspondence $B_{\tau,x} : \Pi \times \mathcal{D} \Rightarrow \Delta(A_{\tau,x})$ given by

$$B_{\tau,x}(\pi, d) \in \left\{ \sigma \in \Delta(A_{\tau,x}) \mid \forall \sigma' \in \Delta(A_{\tau,x}), \sum_{a \in A_{\tau,x}} (\sigma[a] - \sigma'[a]) Q_{\tau}[x,a](\pi, d) \geq 0 \right\}. \quad (14)$$

This chooses a probability distribution over the actions $\sigma$ which maximizes the single-stage deviation reward $Q_{\tau}[x,a](\pi, d)$, given in (12). We are now ready to formally state the definition of the stationary equilibrium.

Definition 3 (Stationary equilibrium) A stationary equilibrium is a social state $(\pi, d) \in \Pi \times \mathcal{D}$ which satisfies

$$\pi_{\tau}[\cdot \mid x] \in B_{\tau,x}(\pi, d), \forall (\tau, x) \in T \times X, \quad (SE.1)$$

$$d_{\tau} = d_{\tau} P_{\tau}(\pi, d), \forall \tau \in T. \quad (SE.2)$$

At the stationary equilibrium, it is a best response for agents of all types to not deviate from the policy $\pi_{\tau}$ in a single stage at all states $(SE.1)$, and additionally, the type-state distribution $d$ is stationary $(SE.2)$.

Proposition 2 At the stationary equilibrium $(\pi, d)$, no agents have an incentive to deviate from the policy of their type $\pi_{\tau}$.

Proof We hypothesize that $(\pi, d)$ is stationary, and show that no agents have an incentive to deviate from the policy of their type $\pi_{\tau}$, which together with $(SE.2)$ supports the hypothesis and completes the proof. At the stationary $(\pi, d)$, the discounted MDP faced by agents of an arbitrary type $\tau$ is time-homogeneous. Let us apply the well-known policy iteration algorithm to compute an optimal policy of this MDP, initialized at $\pi_{\tau}$. Performing one policy iteration, we get that the state transition matrix when following $\pi_{\tau}$ is

$$P_{\tau}[x^+ \mid x](\pi, d) = \sum_{a \in A_{\tau,x}} \pi_{\tau}[a \mid x] p_{\tau}[x^+ \mid \tau, x, a](\pi, d), \quad (15)$$

the expected stage rewards are

$$R_{\tau}[x](\pi, d) = \sum_{a \in A_{\tau,x}} \pi[a \mid x] r[x,a](\pi, d), \quad (16)$$

and the expected discounted infinite horizon rewards are

$$V_{\tau}(\pi, d) = (I - \alpha_{\tau} P_{\tau}(\pi, d))^{-1} R_{\tau}(\pi, d). \quad (17)$$

To improve the policy, we need to maximize

$$Q_{\tau}[x,a](\pi, d) = r_{\tau}[x,a](\pi, d) + \alpha_{\tau} \sum_{x^+ \in X} p_{\tau}[x^+ \mid x,a](\pi, d) V_{\tau}[x^+](\pi, d), \quad (18)$$
for all states \( x \in X \). But (15), (16), (17), (18) coincide with the definitions (5), (9), (11), (12), respectively, at \((\pi, d)\). Therefore such an improvement is not possible by the best response condition (SE.1). Thus, the policy iteration algorithm terminates immediately with \( \pi_\tau \), which is well-known to be optimal [23, Proposition 7.3.1], and agents of type \( \tau \) indeed have no incentive to deviate from \( \pi_\tau \). Since \( \tau \) was arbitrary, this holds for all types.

It is straightforward to see that if it is optimal to not deviate from \( \pi_\tau \), it will necessarily satisfy (SE.1), i.e., it will be optimal to not deviate from it in a single stage at all states. Proposition 2 further asserts that (SE.1) is sufficient for optimality, i.e., if for a policy \( \pi_\tau \) it is optimal to not deviate in a single stage at all states, then it is optimal to not deviate from \( \pi_\tau \) in any number of stages. Conditions (SE.1)–(SE.2) are therefore a full characterization of equilibrium policy-stationary distribution pairs, and there is no loss in considering single-stage deviations at the equilibrium. There is significant gain, however, since the individual agent’s decision problem is much simpler than if they were to solve an entire MDP. Among others, this makes it straightforward to establish the existence of stationary equilibria, which we turn to next.

It is convenient to combine conditions (SE.1)–(SE.2) as the fixed point of a single correspondence. Let \( SE : \mathcal{P} \times D \ni \mathcal{P} \times D \) be defined as

\[
SE(\pi, d) = \left\{ (\gamma, z) \in \mathcal{P} \times D \mid \gamma_\tau[\cdot \mid x] \in B_{\tau,x}(\pi, d), \forall (\tau, x) \in \mathcal{T} \times X, \right. \\
\left. z_\tau = d_\tau P_\tau(\pi, d), \forall \tau \in \mathcal{T} \right\}
\]

Then, a stationary equilibrium satisfies

\[
(\pi, d) \in SE(\pi, d).
\]

Theorem 1 There exists at least one stationary equilibrium \((\pi, d)\) for every dynamic population game.

Proof The proof is a straightforward application of Kakutani’s fixed point theorem for the map \( SE(\pi, d) \), which requires that:

– The set \( \mathcal{P} \times D \) is convex, compact and nonempty, which holds since it is the Cartesian product of simplices.
– \( SE(\pi, d) \subseteq \mathcal{P} \times D \), which holds by definition of the best response (14) and Lemma 1
– The set-valued correspondence \( SE(\pi, d) \) is convex.

This holds for \( B_{\tau,x}(\pi, d) \) since all convex mixtures of elements in \( B_{\tau,x}(\pi, d) \) are randomizations over the actions maximizing the single-stage deviation reward at \((\tau, x)\), and thus also lie in \( B_{\tau,x}(\pi, d) \). The map \( d_\tau P_\tau(\pi, d) \) is convex because it is a singleton. Therefore, \( SE(\pi, d) \), the Cartesian product of the above sets, is convex.
– The set-valued correspondence \( SE(\pi, d) \) is upper hemicontinuous and nonempty.

This holds for \( B_{\tau,x}(\pi, d) \) by the continuity of \( Q_\tau[x,a](\pi, d) \) in \((\pi, d)\) for all tuples \((\tau, x, a) \in \mathcal{T} \times X \times A_{\tau,x} \), and Berge’s maximum theorem.
The map \( d, P_\tau(\pi, d) \) is single-valued and continuous by the continuity of \( P_\tau(\pi, d) \) in \((\pi, d)\), thereby also upper hemicontinuous and nonempty. Therefore, \( SE(\pi, d) \), the Cartesian product of the above sets, is upper hemicontinuous and nonempty.

It follows that a fixed point \((\pi, d)\) satisfying \((\pi, d) \in SE(\pi, d)\) exists, and is a stationary equilibrium by definition. \(\square\)

We conclude this section with the remark that the existence of at least one stationary equilibrium does not guarantee that such equilibrium will be attractive for the social state dynamics. In fact, it is possible to construct simple dynamic population games where a unique stationary equilibrium exists but cannot be reached, given an initial type-state distribution. Consider, for example, a dynamic population game with one type \( \tau \) in which the stochastic matrix \( P_\tau \) does not depend on the social state \((\pi, d)\), and is periodic. Under the synchronous interaction model \(6\), if the initial state distribution is not the stationary equilibrium distribution (which is necessarily the stationary distribution of \( P_\tau \)), it will oscillate forever. The topic of reachability and attractiveness of the stationary equilibria is an important direction that we will explore in dedicated future work.

5 Reduction to classical population games

Every classical population game, as reviewed in Section 2, is a dynamic population game; one where the state space \( X \) is simply a singleton, and the types \( \tau \) are the populations \( \rho \) of the classical population game. It turns out that a connection exists in the opposite direction as well. Namely, we show that for every dynamic population game, one can define a classical population game whose Nash equilibria are in one-to-one correspondence with the stationary equilibria of the dynamic population game.

Given a dynamic population game \( G \), we define a classical population game \( F^G \) with the following high-level description. Each type-state pair \((\tau, x)\) in the dynamic population game is assigned to a static population \( \rho'_{\tau,x} \), with the payoffs of that population equalling the single-stage deviation rewards at \((\tau, x)\). Additionally, each type \( \tau \) in the dynamic population game is assigned to a population \( \rho''_\tau \), whose population state coincides with the state distribution of \( \tau \). The payoffs of that population is constructed in a manner that leads the state distribution to stationarity when the agents act rationally, as we show in Theorem 2.

More precisely, \( F^G \) is constructed as follows:

1. For each \((\tau, x)\) \(\in T \times X\), we define a corresponding population \( \rho'_{\tau,x} := (\tau, x) \).
2. Additionally, for each type \( \tau \) \(\in T\), we define a corresponding population \( \rho''_\tau := \tau \).
3. The masses of populations \( \rho'_{\tau,x} \) are all one, i.e.,
   \[
m_{\rho'_{\tau,x}} := 1, \forall \rho'_{\tau,x}.
\]
The masses of populations $\rho'_\tau$ equal the proportion of agents in the corresponding types, i.e.,
\[ m_{\rho'_\tau} := g_\tau, \forall \rho'_\tau. \]

The action sets of populations $\rho'_\tau,x$ are the action sets of the corresponding type-state pairs, i.e.,
\[ A_{\rho'_\tau,x} := A_{\tau,x}, \forall \rho'_\tau,x. \]

The action sets of populations $\rho''_\tau$ all equal the set of states, i.e.,
\[ A_{\rho''_\tau} := \mathcal{X}, \forall \rho''_\tau. \]

The population state of population $\rho'_\tau,x$ is defined as the probability distribution over the actions of the corresponding type-state pair dictated by the policy, i.e.,
\[ \chi_{\rho'_\tau,x} := \pi_{\tau}[\cdot | x], \forall \rho'_\tau,x. \]

The population state of population $\rho''_\tau$ is defined as the state distribution of the corresponding type $\tau$, i.e.,
\[ \chi_{\rho''_\tau} := d_{\tau}, \forall \rho''_\tau. \]

The social state $\chi$ in $F^\mathcal{G}$ is the social state $(\pi, d)$ in $\mathcal{G}$, by construction.

The payoffs of populations $\rho'_\tau,x$ are the single-stage deviation rewards of the corresponding type-state pairs, i.e.,
\[ F^\mathcal{G}_{\rho'_\tau,x}(\chi) := Q_\tau[x,a](\pi, d), \forall \rho'_\tau,x, \forall a. \]

The payoff vectors of populations $\rho''_\tau$ are the vector fields of the asynchronous societal state dynamics (7), i.e.,
\[ F^\mathcal{G}_{\rho''_\tau}(\chi) := W^\mathcal{G}_{\tau}(\pi, d) = \delta_\tau (d, P_\tau(\pi, d) - d_\tau), \forall \rho''_\tau. \]

Recall that $Q_\tau[x,a](\pi, d)$ and $W^\mathcal{G}_{\tau}(\pi, d)$ are continuous in the social state $(\pi, d)$, and therefore the payoff vector $F^\mathcal{G}(\chi)$ is continuous in $\chi$, as needed in classical population games. We are now ready to state our main result.

**Theorem 2** The social state $(\pi, d)$ is a stationary equilibrium of the dynamic population game $\mathcal{G}$ if and only if $\chi = (\pi, d)$ is a Nash equilibrium of the population game $F^\mathcal{G}$.

**Proof** Let us write the Nash equilibrium condition (2) for $F^\mathcal{G}$ explicitly, as
\[ \chi_{\rho'_\tau,x} \in B_{\rho'_\tau,x}(\chi), \forall \rho'_\tau,x \in \mathcal{T} \times \mathcal{X}, \tag{19} \]
\[ \chi_{\rho''_\tau} \in B_{\rho''_\tau}(\chi), \forall \rho''_\tau \in \mathcal{T}. \tag{20} \]

It is straightforward to verify that condition (19) is the same as condition (SE.1) of the stationary equilibrium of $\mathcal{G}$, which we get immediately by construction of $F^\mathcal{G}$. Therefore, in order to prove the one-to-one correspondence between the stationary equilibria of $\mathcal{G}$ and the Nash equilibria of $F^\mathcal{G}$, we must
show that condition (20) is the same as the stationarity condition (SE.2) for \( G \).

We proceed by writing (20) for an arbitrary type \( \tau \in T \) in vector form as

\[
\chi_{\rho''} \in B_{\rho''}(\chi) \Leftrightarrow d_{\tau} \in B_{\rho''}(\pi, d)
\]
\[
\Leftrightarrow \forall \sigma' \in D_{\tau}, W_{\rho''}(\pi, d) (d_{\tau} - \sigma')^\top \geq 0,
\]
\[
\Leftrightarrow \forall \sigma' \in D_{\tau}, \delta_{\tau} (d_{\tau} P_{\tau}(\pi, d) - d_{\tau}) (d_{\tau} - \sigma')^\top \geq 0,
\]
\[
\Leftrightarrow \forall \sigma' \in D_{\tau}, (d_{\tau} P_{\tau}(\pi, d) - d_{\tau}) (d_{\tau} - \sigma')^\top \geq 0. \quad (21)
\]

On the other hand, condition (SE.2) for type \( \tau \) can be written as

\[
d_{\tau} P_{\tau}(\pi, d) - d_{\tau} = 0. \quad (22)
\]

Note that (22) trivially implies (21) (which holds with equality). We show that (21) implies (22) by contradiction. Suppose that (21) holds but \( d_{\tau} P_{\tau}(\pi, d) - d_{\tau} \neq 0 \). Then, one can choose \( \sigma' = d_{\tau} P_{\tau}(\pi, d) \in D_{\tau} \) (by Lemma 1) such that

\[
(d_{\tau} P_{\tau}(\pi, d) - d_{\tau})(d_{\tau} - \sigma')^\top = (\sigma' - d_{\tau})^\top (d_{\tau} - \sigma')^\top = -(d_{\tau} - \sigma')(d_{\tau} - \sigma')^\top < 0,
\]

a contradiction. So \( d_{\tau} P_{\tau}(\pi, d) - d_{\tau} = 0 \) is the only way to satisfy (21). Since the type \( \tau \) was arbitrary, this holds for all types, and (20) holds if and only if (SE.2) holds.

As an immediate consequence of Theorem 2, we have:

**Corollary 1** There exists at least one stationary equilibrium \((\pi, d)\) for every dynamic population game.

This is a re-statement of Theorem 1 which we already proved. Theorem 2 provides an alternative proof, since there exists at least one Nash equilibrium for the constructed population game \( \mathcal{F}^G \) [1, Theorem 2.1.1], which coincides with a stationary equilibrium of the dynamic population game \( G \).

### 6 Evolutionary policy-state dynamics

In classical population games, **evolutionary dynamics** provide a rich family of models of how the behavior of agents in the society evolves under the game-theoretic setting considered. On the other hand, the setting where agents have state dynamics in addition to the evolutionary decision dynamics is mostly unexplored in the literature. The reduction of stationary equilibria in dynamic population games to Nash equilibria in classical population games provides an avenue to adapt classical evolutionary dynamics to the setting with dynamic states, which we explore in this section. We first review evolutionary dynamics

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6 Note that elements of \( D_{\tau} \) are row vectors in this convention.
in classical population games in Section 6.1 and then propose a model for the coupled policy-state evolutionary dynamics in dynamic population games in Section 6.2, which is inspired by the reduction. A discussion of this model follows in the concluding remarks of Section 7.

6.1 Preliminaries: Evolutionary dynamics in classical population games

When coupled to a population game $F(\chi)$, evolutionary dynamics model the evolution of the social state $\chi$ as the agents strategically adapt their action choices in repeated runs of $F(\chi)$. We refer the reader to [1, Chapters 4–6] for a comprehensive coverage of evolutionary dynamics, and summarize the main elements here. Under this model, agents get to revise their action choices according to a specified revision protocol. In this process, the agents are typically assumed to have inertia and to be myopic. Inertia means that the agents keep on playing the same action until they get a sporadic opportunity to revise it, and myopia means that the agents do not look ahead when performing these revisions, i.e., they only consider the current social state and do not take into account its future evolution.

In particular, the agents of population $\rho$ get asynchronous revision opportunities at rings of independent Poisson alarm clocks with rate $\eta_\rho > 0$. When an agent’s clock rings, they switch from their current action $a$ to a new action $a^+$, according to a probability

$$\zeta_\rho[a^+ \mid a](F_\rho(\chi), \chi_\rho) \in [0,1].$$

The probability distribution

$$\zeta_\rho[\cdot \mid a] : \mathbb{R}^{|A_\rho|} \times X_\rho \to \Delta(A_\rho)$$

is called the revision protocol of population $\rho$. Naturally, it is a function of the payoffs to the different actions $F_\rho(\chi)$ and (possibly) the current population state $\chi_\rho$. We recall some examples of revision protocols shortly. This revision process leads to the following expected motion of the population states $\chi_\rho$

$$\dot{\chi}_\rho[a] = \eta_\rho \left( \sum_{a' \in A_\rho} \chi_\rho[a'] \zeta_\rho[a \mid a'](F_\rho(\chi), \chi_\rho) - \chi_\rho[a] \right), \forall \rho \in \mathcal{P}, a \in A_\rho.$$  

(23)

This is compactly written as

$$\dot{\chi}_\rho = \eta_\rho H_\rho(F_\rho(\chi), \chi_\rho), \forall \rho \in \mathcal{P},$$  

(24)

One can also consider a synchronous revision model where all the agents revise their actions together, leading to discrete-time dynamics in (23). However, the asynchronous model is the standard in the literature (see [1]). Under this model, population states admit the interpretation that different agents of the population deterministically follow different actions, until revised.
where

\[ H_\rho: \mathbb{R}^{|A_\rho|} \times X_\rho \rightarrow \mathbb{R}^{|A_\rho|} \]

is the vector field with \( H_\rho[a](F_\rho(\chi), \chi_\rho) \) defined as the right hand side of (23). This vector field is classically referred to as the mean dynamic of the population state \( \chi_\rho \). The following is a desirable property for the mean dynamic.

**Definition 4 (Nash stationarity)** The mean dynamic \( H_\rho(F_\rho(\chi), \chi_\rho) \) is said to satisfy Nash stationarity if

\[ H_\rho(F_\rho(\chi), \chi_\rho) = 0 \iff \chi_\rho \in B_\rho(\chi). \quad \text{(NS)} \]

It is straightforward to verify that if the mean dynamics of all populations \( \rho \in \mathcal{P} \) satisfy Nash stationarity, then a social state \( \chi \) is a rest point of (24) if and only if it is a Nash equilibrium of the game \( F(\chi) \). Note that Nash stationarity is not always satisfied, as exemplified by the well-known replicator dynamic, for which Nash equilibria are necessarily rest points, but there could exist rest points that are not equilibria [1, Section 5.4.6].

We now recall some examples of revision protocols, along with the resulting mean dynamics.

**Example 1 (Best response dynamic)** When an agent gets a revision opportunity, it switches to the best response action with probability 1, i.e.,

\[ \zeta_\rho[a^+ | a'](F_\rho(\chi)) = \begin{cases} 1 & \text{if } a^+ = \arg \max_{a'} F_\rho(a')(\chi), \\ 0 & \text{otherwise.} \end{cases} \quad (25) \]

Here, we assume that the best response action is unique; if there are multiple best response actions then the agent can switch to any mixture of them according to a prescribed tie-breaking rule \(^8\). The resulting mean dynamic is

\[ \dot{\chi}_\rho = \eta_\rho (B_\rho(\chi) - \chi_\rho), \quad (26) \]

where \( B_\rho(\chi) \) is the best response correspondence with the prescribed tie-breaking rule. The best response dynamic (26) can be viewed as an asynchronous, continuous-time variant of the well-known iterated best response algorithm. Note that it satisfies Nash stationarity [1, Theorem 6.1.4].

**Example 2 (Projection dynamic)** The motivation of the projection dynamic is to follow the population payoff vector \( F_\rho(\chi) \) in the motion of the population state \( \chi_\rho \), since \( F_\rho(\chi) \) points in the direction of actions that achieve high payoffs. This occurs by projecting \( F_\rho(\chi) \) into the feasible set of motion directions characterized by the tangent cone \( TX_\rho(\chi_\rho) \), i.e.,

\[ \dot{\chi}_\rho = \eta_\rho \operatorname{proj}_{TX_\rho(\chi_\rho)} F_\rho(\chi). \quad (27) \]

Refer to [1, Section 6.3.4] for examples of revision protocols that yield the projection dynamic. Note that it also satisfies Nash stationarity [1, Theorem 6.3.5].

---

\(^8\) If a tie-breaking rule is not specified, the best response correspondence is multi-valued and we have a differential inclusion instead of a differential equation in (26). Alternatively, one can consider the perturbed best response dynamic [1, Section 6.2], in which the switch probabilities are smoothed via the logit choice function.
6.2 Evolutionary policy-state dynamics in dynamic population games

We restrict our attention to the case of asynchronous interactions and propose the following model for the coupled policy-state evolutionary dynamics in dynamic population games:

\[
\begin{align*}
\dot{\pi}_\tau[\cdot | x] &= \eta_\tau H_\tau(Q_\tau[x,\cdot](\pi, d), \pi_\tau[\cdot | x]), \forall (\tau, x) \in T \times X, \\
\dot{d}_\tau &= \delta_\tau (d_\tau P_\tau(\pi, d) - d_\tau), \forall \tau \in T,
\end{align*}
\]

where \( Q_\tau[x,\cdot](\pi, d) \in \mathbb{R}^{A_{\tau,x}} \) denotes the vector of single-stage deviation rewards for the different actions at type-state \((\tau, x)\) and social state \((\pi, d)\). This model has the following intuitive interpretation. The agents interact asynchronously and follow the current policy of their type \(\pi_\tau\) in their interactions. This yields the continuous-time societal state dynamics (7) in (EV.2). The policy \(\pi_\tau\) is revised in an independent process according to (EV.1), which resembles the classical mean dynamics (24). The rate of revisions \(\eta_\tau\) can be viewed as a parameter that controls the speed of the policy revisions with respect to the speed of the state dynamics, and can vary across the types, representing differences in the responsiveness of the agents. One generally expects that \(\eta_\tau \leq \delta_\tau\), i.e., revisions occur at a slower or equal timescale when compared to the state dynamics. Notice that we also allow different types to follow different mean dynamics \(H_\tau\).

**Proposition 3** Let \(F^G\) be the reduced population game of dynamic population game \(G\). Suppose that in \(F^G\) populations \(\rho'_{\tau,x}\) follow mean dynamics \(H_\tau\) with rates \(\eta_\tau\), respectively per \(\tau\), and populations \(\rho''_\tau\) follow the projection dynamic with unit rates. Then the evolutionary dynamics (24) of \(F^G\) coincide with the evolutionary policy-state dynamics (EV.1)–(EV.2) of \(G\).

**Proof** Recall that in \(F^G\), populations \(\rho'_{\tau,x}\) are defined such that \(\chi_{\rho'_{\tau,x}} = \pi_\tau[\cdot | x]\) and \(F^G_{\rho'_{\tau,x}}(\chi) = Q_\tau[x,\cdot](\pi, d)\). It immediately follows that for populations \(\rho'_{\tau,x}\), (24) coincides with (EV.1). Moreover, populations \(\rho''_\tau\) are defined such that \(\chi_{\rho''_\tau} = d_\tau\) and \(F^G_{\rho''_\tau}(\chi) = W^c_\tau(\pi, d) = \delta_\tau (d_\tau P_\tau(\pi, d) - d_\tau)\). Under the projection dynamic (24), the projection operator is redundant since \(W^c_\tau(\pi, d)\) always lies in the tangent cone \(TD_\tau(d_\tau)\) (see proof of Proposition 1). This straightforwardly yields the continuous-time societal state dynamics (EV.2) as the evolutionary dynamics (24) of populations \(\rho''_\tau\).

**Theorem 3** Suppose that \(H_\tau\) satisfies Nash stationarity for all \(\tau \in T\). Then, a social state \((\pi, d)\) is a rest point of (EV.1)–(EV.2) if and only if it is a stationary equilibrium of the dynamic population game \(G\).

**Proof** As per Proposition 3 (EV.1)–(EV.2) coincide with the mean dynamics of the reduced population game \(F^G\), with populations \(\rho''_\tau\) following the projection dynamic, which satisfies Nash stationarity. The mean dynamics of the remaining populations \(\rho'_{\tau,x}\) are \(H_\tau\), respectively per \(\tau\), and satisfy Nash stationarity as well by assumption. Thus, rest points of (EV.1)–(EV.2) are
necessarily and sufficiently Nash equilibria of $F^G$, which are necessarily and sufficiently stationary equilibria of $G$, by Theorem 2.

7 Discussion and Conclusion

In this work, we introduced dynamic population games, a class of games that are played in large populations where the agents are characterized by static, immutable types as well as time-varying states, whose dynamics are affected by the gameplay. We defined the elementary building blocks of these games (summarized in Section 3.7), and developed the core machinery to analyze them. We hope that our formulation will assist practitioners in developing practical dynamic population game models in a straightforward, yet rigorous, manner. In what follows, we discuss some of the main features of these games.

**Stationary equilibria:** An important quantity in our setting is the social state, which is the pair of the policy followed by the agents and the distribution of the agents’ types and states. This is a time-varying quantity due to the state dynamics, and also because the policy could be changing as the agents adapt their strategic behavior. We focus on characterizing conditions for stationary equilibria, which are social states that can be considered stationary points of the game, since at those points all agents act optimally and the type-state distribution is stationary. We show that a stationary equilibrium is guaranteed to exist for every dynamic population game, motivating its use as a solution concept.

**Single-stage deviation principle:** In our setting, individual agents face a discounted Markov decision process (MDP) that is parametrized in the social state, and is therefore not time-homogeneous. Under the single-stage deviation principle, we consider that the agents do not attempt to fully solve the MDP starting at their current state, but rather optimize the benefits of a single-stage deviation from the policy dictated by the social state. This means that they consider that the social state will not change in the future and that they will follow its policy in the upcoming stages. We prove that the single-stage deviation principle is not restrictive at the stationary equilibrium, where the social state is indeed stationary, and the equilibrium policy is optimal if and only if it is optimal to not deviate from it in a single stage at all states.

An important consequence of the single-stage deviation principle is that it allows for the reduction of stationary equilibria in dynamic population games to standard Nash equilibria in a suitably defined classical population game. In this classical population game, populations correspond to type-state pairs, and payoffs in the classical sense correspond to single-stage deviation rewards. Additional populations are considered whose payoffs are defined in a way that leads to stationary type-state distributions under rational gameplay. This reduction is considered an important contribution of our work, since it opens the
avenue to adapt analysis tools of classical population games to our dynamic setting.

While the single-stage deviation principle is not restrictive at the stationary equilibrium, care must be taken in its application off the equilibrium, where the assumption that the social state is stationary does not hold. It could be reasonable however to consider that the future evolution of the social state is too difficult to predict by the agents, who use the current social state as a surrogate for future states. Another difficulty is that while it is reasonable to expect that the agents have limited lookahead (see [23, Sections 6.3]), it is likely that they simultaneously optimize their actions for more than a single-stage deviation. The empirical validation of the single-stage deviation principle as well as the extension to multi-stage deviations are interesting directions for future work.

Evolutionary policy-state dynamics: We propose an evolutionary model for the coupled policy-state dynamics in dynamic population games, which has close connections to classical evolutionary dynamics as applied to the reduced classical population game. This evolutionary model provides a starting point to a largely unexplored research problem, by considering the evolution in settings where agents have time-varying states. However, two difficulties arise in the interpretation of our evolutionary model. The first difficulty is related to the single-stage deviation principle, which is violated off the stationary equilibrium, as previously discussed. The second difficulty is related to the interpretation of population states in classical evolutionary dynamics as to describe that agents deterministically follow heterogeneous actions, whereas the randomized policies considered in dynamic population games are homogeneous. Therefore, the microscopic interpretation that action revisions occur asynchronously for different agents (e.g., when they interact) does not transfer well to our model, where the policy dynamics are to be interpreted as to occur across the whole population. We believe however that a decentralized policy update scheme can be derived based on a microscopic model of the joint interaction-strategic decision process of the agents, which will be the subject of future work.

Applications: Dynamic population games are natural to model many real-world problems, and have already seen applications in autonomous mobility [4] and epidemic modelling [10]. In the autonomous mobility domain [4], a dynamic population game was used to model a complete karma economy which enables the fair and efficient allocation of shared road infrastructure amongst self-interested vehicles in a decentralized manner. In the epidemic modelling domain [10], dynamic population games allowed for the straightforward extension of classical SAIR epidemic models to consider strategic agents that actively adapt their behavior in response to the epidemic threat, by for example reducing the size of their social interactions or moving to less infectious zones. There, the evolutionary policy-state dynamics model plays an important role in studying the off-equilibrium evolution of the epidemic spread. We further
suspect that dynamic population games can find many applications in \textit{evolutionary biology}, where it is natural to consider that species have time-varying states that affect and are affected by their behavior, such as the state of their hunger \cite{3}. It is natural to consider applications in \textit{economics} as well, including for example the dynamic oligopoly models typically studied using stochastic games or anonymous sequential games (see \cite{11} and references therein). We are of the opinion that the reduction of dynamic population games to classical population games will make such models more easily accessible, as the reduction eliminates much of the complexity associated with the dynamics.

A Derivation of the asynchronous state dynamics

We will use the (large) number of agents \( n \), which is normalized at the end of the derivation. Considering that the current social state is \((\pi, d)\), the number of agents of type \( \tau \) and state \( x \) in the current time is

\[
 n \ d_\tau \left[ x \right].
\]

(28)

We will characterize how this quantity is expected to change over the next \( dt \) seconds. Each agent of type \( \tau \) interacts on rings of Poisson alarm clocks with rate \( \delta_\tau \), so they are expected to interact \( \delta_\tau \ dt \) times in the next \( dt \) seconds. If we choose \( dt \) small enough, such that no single agent interacts more than once, then out of the \( n \ d_\tau \left[ x \right] \) agents of type \( \tau \) and state \( x \), the number who are expected to interact in the next \( dt \) seconds is

\[
 n \ d_\tau \left[ x \right] \delta_\tau \ dt,
\]

and the number who are expected to not interact is \( n \ d_\tau \left[ x \right] \left(1 - \delta_\tau \ dt\right)\). Note that we can choose \( dt \) small enough such that not all agents are expected to interact, i.e., \( \delta_\tau \ dt < 1 \).

Out of the \( n \ d_\tau \left[ x \right] \delta_\tau \ dt \) agents of type \( \tau \) and state \( x \) who are expected to interact in the next \( dt \) seconds, the number who are expected to transition to state \( x^+ \) is

\[
 n \ d_\tau \left[ x \right] \delta_\tau \ dt \ P_\tau \left[ x^+ \mid x \right](\pi, d) \ dt,
\]

where we assume that \( P_\tau \left[ x^+ \mid x \right](\pi, d) \) is constant over \( dt \), since the change in the social state \((\pi, d)\) is small in this infinitesimal time period. Note that agents from states \( x' \in X \) will be transitioning into state \( x \) in the next \( dt \) seconds in a similar manner. It follows that the expected number of agents of type \( \tau \) and state \( x \) after \( dt \) seconds is

\[
 n \ d_\tau \left[ x \right] \left(1 - \delta_\tau \ dt\right) + \sum_{x' \in X} n \ d_\tau \left[ x' \right] \ P_\tau \left[ x \mid x' \right](\pi, d) \ dt.
\]

(29)

The left term is the number of agents of type \( \tau \) and state \( x \) who are expected to not interact in the next \( dt \) seconds (and hence not change their state), and the right term is the number of agents of type \( \tau \) and all states \( x' \in X \) that are expected to interact and transition into \( x \) (or, in the case of \( x' = x \), stay in \( x \)). Subtracting the initial number \( 28 \) from \( 29 \), normalizing by \( n \), and eliminating the time differential \( dt \) yields the expected rate of change of the proportion of agents of type \( \tau \) and state \( x \)

\[
 \dot{d}_\tau \left[ x \right] = \delta_\tau \left( \sum_{x' \in X} d_\tau \left[ x' \right] P_\tau \left[ x \mid x' \right](\pi, d) - d_\tau \left[ x \right] \right).
\]

In vector form, this is \( 32 \) given in Section 3.4.2.
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