Relativistic Gravity Theory And Related Tests
With A Mercury Orbiter Mission

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Abstract

Due to its relatively large eccentricity and proximity to the Sun, Mercury’s orbital motion provides one of the best solar-system tests of relativistic gravity. We emphasize the number of feasible relativistic gravity tests that can be performed within the context of the parameterized weak field and slow motion approximation - a useful framework for testing modern gravitational theories in the solar system. We discuss a new approximation method, which includes two Eddington parameters ($\gamma, \beta$), proposed for construction of the relativistic equations of motion of extended bodies. Within the present accuracy of radio measurements, we discuss the generalized Fermi-normal-like proper reference frame which is defined in the immediate vicinity of the extended compact bodies. Based on the Hermean-centric equations of motion of the spacecraft around the planet Mercury, we suggest a new test of the Strong Equivalence Principle. The corresponding experiment could be performed with the future Mercury Orbiter mission scheduled by the European Space Agency (ESA) for launch between 2006 and 2016. We discuss other relativistic effects including the perihelion advance, redshift and geodetic precession of the orbiter’s orbital plane about Mercury.

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1 Introduction

It is well known that some gravitational effects depend not only on the first or second derivatives of the gravitational potential, but also on the magnitude of this potential in the area where the gravitational experiments are performed. This is why, with the present level of experimental techniques, one should focus on locations with an intensive gravitational environment as the best place for conducting these studies. The planet Mercury plays a specific role in the history of modern gravitational physics. The successful explanation of the perihelion advance of its orbit has become one of the observational cornerstones for general relativity. Because of its proximity to the Sun, high eccentricity, and short orbital period, it offers a very interesting opportunity for the study of relativistic gravity. A spacecraft, placed in orbit about this planet, can provide the additional data necessary for dynamic tests of the principles of modern gravitational theories as well as the validation of the approximation methods used for analysis of the gravitational environment.

In this paper we analyze the relativistic gravitational experiments for the Mercury Orbiter mission which has been included by the European Space Agency as a cornerstone mission under the Horizon 2000 Plus program. The motivation for the research described here is to determine what scientific...
information may be obtained during this mission, how accurate these measurements can be, and what will be the significance of the knowledge obtained. As it is known, there are three different types of measurements that are used in spacecraft navigation: radiometric (range and Doppler), very-long baseline interferometry (VLBI), and optical [1]. In addition to navigation needs, the high precision Doppler, laser and radio range measurements of the velocity of and the distance to celestial bodies and spacecraft are presently the best ways to collect important information about relativistic gravity within the solar system. Combined with the technique of ground- and space-based VLBI, these methods provide us with a unique opportunity to explore the physical phenomena in our universe with very high precision. Most remarkable is that the accuracy of the modern VLBI observations is steadily increasing. Thus the delay residuals are presently of the order of 30-50 picoseconds (ps), which corresponds to an uncertainty in length of \( \sim 1 \) cm. Concerning the navigation of the interplanetary spacecraft, the short arcs of spacecraft range and Doppler measurements, reduced with Earth orientation information referred to the International Earth Rotation Service (IERS) celestial system, lead to a position determination in the extragalactic reference frame with an accuracy of order \( \sim 20 \) milliarcsecond (mas). At the same time VLBI observations of the spacecraft with respect to extragalactic radio-sources enable a direct measurement of one component of the spacecraft position in this extragalactic reference frame to an accuracy of about \( \sim 5 \) mas (2, 3). As a result, the use of such precise methods enables one to study the dynamics of celestial bodies and spacecraft with an unprecedented high accuracy. These data provide the necessary foundation for research of many scientific problems, such as:

(i). The construction of a dynamic inertial reference frame and a more precise definition of the orbital elements of the Sun, Earth, moon, planets and their satellites (4, 5-7).

(ii). The construction of a kinematic inertial reference frame, based on the observations of stars and quasars from spaceborne astronomical observatories (4, 6).

(iii). The construction of a precise ephemeris for the motion of bodies in the solar system to support reliable navigation in the solar system (1, 8). The construction of precise radio-star catalogs for the spacecraft astroorientation and navigation in outer space beyond the solar system.

(iv). The comparison of dynamic and kinematic inertial reference frames, based on the observations of spacecraft on the background of quasars, pulsars and radio-stars, as well as the verification of the zero-points of the coordinates in the inertial reference frame (3, 4, 5, 6).

(v). The experimental tests of the weak field and slow motion approximation (WFSMA) of modern theories of gravity (12, 13).

These studies will enrich our knowledge about our universe, its cosmological evolution, and the behavior of stellar systems in general. By presenting this list of problems, we would like to emphasize the importance of the Mercury Orbiter mission not only for the gravitational theory, but for many of the fundamental problems stated above. It should be noted that, besides offering the opportunity to study the gravitational environment in Mercury’s vicinity, this mission will also provide important data about the gravitational field of the Sun, Mercury’s magnetosphere and its interaction with the solar plasma (see more detailed analysis of these problems in 14), as well as enable one to verify the foundations of many recently proposed theories of relativistic reference frames (15-21). This mission provides a good opportunity for testing the principles of many modern approximation formalisms proposed for describing gravitational wave generation mechanisms (12, 22). It should be noted that a physically justified definition of the proper reference frame of the extended bodies (i.e. a complete multipolar solution of the gravitational N-body problem), in the considerably weak solar gravitational field, will allow us to better understand the physical processes in a stronger gravitational field regime.
This paper discusses the principles of a new approximation method developed for an astronomical relativistic N-body problem and analyzes the space gravitational experiments proposed for the future Mercury Orbiter mission. The outline of the paper is as follows. The next Section contains a brief introduction to the modern theoretical methods used to describe the motion of a system of N weakly interacting self-gravitating extended bodies and to analyze gravitational experiments within the solar system. We discuss problems associated with the traditional barycentric approach of the PPN formalism. In Section III we discuss the principles of a new iterative method for constructing the solution to the gravitational N-body problem in the WFSMA. In particular we discuss the physical and mathematical properties of the relativistic astronomical reference frames. In Section IV this method is applied to derive gravitational field solutions and coordinate transformations. Our derivations include two Eddington parameters ($\gamma, \beta$) which allow us to develop a new parametric theory of astronomical reference frames. Within the present accuracy of radio measurements, we discuss the generalized Fermi-normal-like proper reference frame, which is defined in the immediate vicinity of extended bodies. Thus, we have obtained the interesting result that although some terms in the Hermean-centric equations of motion of the spacecraft around the planet Mercury are zero for the case of general relativity, they may produce an observable effect in scalar-tensor theories. This allows us to propose a new test of the Strong Equivalence Principle (SEP), which is analyzed in Section V of the paper. Also in the fifth Section we discuss a number of relativistic gravitational experiments possible with the Mercury Orbiter. We present there both a quantitative and a qualitative analysis of the measurable effects such as Mercury’s perihelion advance, the precession phenomena of the Hermean orbital plane, and the redshift experiment. In Section VI we present the conclusions and recommendations for future gravitational experiments with the Mercury Orbiter mission. In order to make access to the basic results of this paper easier, we present some calculations in the Appendices.

2 Parametrized Post-Newtonian Gravity.

Metric theories of gravity have a special position among all the other possible theoretical models. The reason is that, independent of the many different principles at their foundations, the gravitational field in these theories affects matter directly through the metric tensor of the Riemann space-time $g_{mn}$, which is determined from the field equations of a particular theory of gravity. As a result, in contrast to Newtonian gravity, this tensor contains the properties of a particular gravitational theory as well as carrying the information about the gravitational field of the bodies. This property of the metric tensor enables one to analyze the motion of matter in one or another metric theory of gravity based only on the basic principles of modern theoretical physics.

Within the accuracy of modern experimental techniques, the WFSMA provides a useful starting point for testing the predictions of different metric theories of gravity in the solar system. Following Fock ([23], [24]), the perfect fluid is used most frequently as the model of matter distribution when describing the gravitational behavior of celestial bodies in this approximation. The density of the corresponding energy-momentum tensor $\tilde{T}^{mn}$ is as follows:

\[
\tilde{T}^{mn} = \sqrt{-g} \left( \rho (1 + \Pi) + p \right) u^m u^n - pg^{mn},
\]

where $\rho$ is mass density of the ideal fluid in coordinates of the co-moving reference frame, $u^k = dz^k/ds$ are the components of invariant four-velocity of a fluid element, and $p(\rho)$ is the isentropic pressure.

\footnote{In this paper the notations are the same as in [23]. In particular, the first three small latin letters $a, b, c$ number the bodies and run from 1 to $N$; the small latin letters $k, l, ... p$ run from 0 to 3 and greek letters $\alpha, \beta, \gamma, ...$ run from 1 to 3; the comma denotes a standard partial derivative and semicolon denotes a covariant derivative; repeat indices imply an Einstein rule of summation; round brackets surrounding indices denote symmetrization and square brackets denote anti-symmetrization.}
connected with \( \rho \) by an equation of state. The quantity \( \rho \Pi \) is the density of internal energy of an ideal fluid. The definition of \( \Pi \) is given by the equation based on the first law of thermodynamics ([3], [23], [24], [26], [27]):

\[
 u^n \left( \Pi_n + p \left( \frac{1}{\rho_n} \right) \right) = 0,
\]

(2)

where \( \hat{\rho} = \sqrt{-g_0^0} \) is the conserved mass density. Given the energy-momentum tensor, one may proceed to find the solutions of the gravitational field equations for a particular relativistic theory of gravity. The solution for an astronomical \( N \)-body problem is the one of most practical interest. In the following Sections we will discuss the properties of the solution of an isolated one-body problem as well as the features of construction of the general solution for the \( N \)-body problem in both barycentric and planeto-centric reference frames.

### 2.1 An Isolated One-Body Problem.

The solution for the isolated one-body problem in the WFSMA may be obtained from the linearized gravitational field equations of a particular theory under study. A perturbative gravitational field \( h^{mn}_{(0)} \), in this case, is characterized by the deviation of the density of the general Riemmanian metric tensor \( \sqrt{-g} g^{mn} \) from the background space-time \( \eta_{mn} \), which is considered to be a zero-th order approximation for a series of successive iterations:

\[
\sqrt{-g} g^{mn} - \sqrt{-\eta} \eta_{mn} = h^{mn}_{(0)},
\]

or equivalently

\[
g^{mn} = \eta_{mn} + h^{(0)}_{mn}.
\]

(3a)

In order to accumulate the features of many modern metric theories of gravity in one theoretical scheme, as well as to create a versatile mechanism to plan gravitational experiments and to analyze the data obtained, Nordtvedt and Will have proposed the parameterized post-Newtonian (PPN) formalism ([28]-[31]). This formalism allows one to describe the motion of celestial bodies for a wide class of metric theories of gravity within a common framework. The gravitational field in the PPN formalism is presumed to be generated by some isolated distribution of matter which is taken to be an ideal fluid Eq.(1). This field is represented by a sum of gravitational potentials with arbitrary coefficients: the PPN parameters. The two-parameter form of this tensor in four dimensions may be written as follows:

\[
h^{(0)}_{00} = -2U + 2(\beta - \tau)U^2 + 2\Psi + 2\tau(\Phi_2 - \Phi_w) + (1 - 2\nu)\chi_{00} + \mathcal{O}(c^{-6}),
\]

(3b)

\[
h^{(0)}_{\alpha \alpha} = (2\gamma + 2 - \nu - \tau)V_\alpha + (\nu + \tau)W_\alpha + \mathcal{O}(c^{-5}),
\]

(3c)

\[
h^{(0)}_{\alpha \beta} = 2\eta_{\alpha \beta}(\gamma - \tau)U - 2\tau U_{\alpha \beta} + \mathcal{O}(c^{-4}),
\]

(3d)

where \( \eta_{mn} \) is the Minkowski metric\(^5\) and the generalized gravitational potentials are given in Appendix A.

Besides the two Eddington parameters \( (\gamma, \beta) \), Eq.(3) contains two other parameters \( \nu \) and \( \tau \). The parameter \( \nu \) reflects the specific choice of the gauge conditions. For the standard PPN gauge it is given as \( \nu = \frac{1}{2} \), but for harmonic gauge conditions one should choose \( \nu = 0 \). The parameter \( \tau \) describes a possible pre-existing anisotropy of space-time and corresponds to different spatial coordinates, which may be chosen for modelling the experimental situation. For example, the case \( \tau = 0 \) corresponds to harmonic coordinates, while \( \tau = 1 \) corresponds to the standard (Schwarzschild) coordinates. A

\(^5\)For most of the non-radiative problems in solar system dynamics, this tensor usually is taken to be a flat Minkowski metric ([12], [13]).

\(^6\)The following metric convention (+ − − −) is used throughout as is the geometrical units \( \hbar = c = G = 1 \).
particular metric theory of gravity in this framework with a specific coordinate gauge \((\nu, \tau)\) may then be characterized by means of two PPN parameters \((\gamma, \beta)\), which are uniquely prescribed for each particular theory under study. In the standard PPN gauge (i.e. in the case when \(\nu = \frac{1}{2}, \tau = 0\)) these parameters have clear physical meaning. The parameter \(\gamma\) represents the measure of the curvature of the space-time created by the unit rest mass; the parameter \(\beta\) is the measure of the non-linearity of the law of superposition of the gravitational fields in the theory of gravity (or the measure of the metricity). Note that general relativity, when analyzed in standard PPN gauge, gives: \(\gamma = \beta = 1\), whereas, for the Brans-Dicke theory, one has \(\beta = 1, \gamma = \frac{1}{2 + \omega}\), where \(\omega\) is an unspecified dimensionless parameter of the theory.

The properties of an isolated one-body solution are well-known. It has been shown \([13, 12, 3]\) that for an isolated distribution of matter in WFSMA there exist a set of inertial reference frames and ten integrals of motion corresponding to ten conservation laws. Therefore, it is possible to consistently define the multipole moments characterizing the body under study. For practical purposes one chooses the inertial reference frame located in the center of mass of an isolated distribution of matter. By performing a power expansion of the potentials in terms of spherical harmonics, one may obtain the post-Newtonian set of ‘canonical’ parameters (such as unperturbed irreducible mass \(I^{(L)}_{a(0)}\) and current \(S^{(L)}_{a(0)}\) multipole moments\([2]\), generated by the inertially moving extended self-gravitating body \((A)\) under consideration:

\[
\begin{align*}
I^{(L)}_{a(0)} &= \left[ \int d^3 z^a (z^p_A) \right]^{STF}, \\
S^{(L)}_{a(0)} &= \left[ \epsilon_{\alpha\beta\gamma} \int d^3 z^a (z^p_A) z_a^{(a)} ... z_a^{(m)} \right]^{STF},
\end{align*}
\]

where \(\hat{P}^m_n\) are the components of the symmetric density of the energy-momentum tensor of matter and gravitational field taken jointly. As a result, the corresponding gravitational field \(h^{mn}_{(0)a}\) may be uniquely represented in the external domain as a functional depending on the set of these moments. Schematically this may be expressed as:

\[
\begin{align*}
h^{mn}_{(0)a} &= \mathcal{S}^{mn}(I^{(L)}_{a(0)}, S^{(L)}_{a(0)}),
\end{align*}
\]

where the functional dependence, in general, includes a non-local time dependence on the ‘past’ history\([2]\) of the moments \([2]\). However, by assuming that the internal processes in the body are adiabatic, one may neglect this non-local evolution. As a result, an external observer may uniquely establish the gravitational field of this body through determination of these multipole moments, for example, by studying the geodesic motion of the test particles in orbit around this distribution of matter \([3]\).

It has been shown \([22, 12, 33]\) that for an isolated distribution of matter in WFSMA it is possible to consistently define the lowest conserved multipole moments, such as the rest mass of the body \(m_a\), it’s center of mass \(z_a^\alpha\), momentum \(p_a^\alpha\) and angular momentum \(S_a^{\alpha\beta}\). Thus the definitions for the mass \(m_a\) and coordinates of the center of mass of the body \(z_a^\alpha\) in any inertial reference frame are given by the formulae (for more detailed analysis see \([13, 12, 3]\) and references therein):

\[
\begin{align*}
m_a &= \int d^3 z^a \hat{P}^{00}(z^p), \\
\int d^3 z^a \hat{P}^{a0}(z^p). \quad z_a^\alpha(t) &= \frac{1}{m_a} \int d^3 z^a \hat{P}^{a\alpha}(z^p).
\end{align*}
\]

7To enable one to deal conveniently with sequences of many spatial indices, we shall use an abbreviated notation for ‘multi-indices’ where an upper-case letter in curly brackets denotes a multi-index, while the corresponding lower-case letter denotes its number of indices, for example: \(\{\mu\} := \mu_1\mu_2\ldots\mu_p\), \(\{\mu, \nu\} := \mu_1\mu_2\ldots\mu_p\). The explicit expression for the symmetric and trace-free (STF) part of the tensor \(T^\mu_{\{\nu\}}\) is given in \([13, 22, 33]\).

8Gravitational radiation problems are not within the scope of the present paper, and hence the set of multipole moments Eq.(4) are used for both tensor and scalar-tensor theories.
where the energy density \( \hat{\rho}^0(z^p) \) of the matter and the gravitational field is given by:

\[
\hat{\rho}^0(z^p) = \hat{\rho}_a \left( 1 + \Pi + \frac{1}{2} U - \frac{1}{2} \mathbf{v}_a \cdot \mathbf{v}_a + \mathcal{O}(c^{-4}) \right),
\]

(5b)

with \( \hat{\rho}_a \) being the conserved mass density defined by \( \hat{\rho}_a = \rho_a \sqrt{-g} u^0 \) and \( \mathbf{v}_a \) the velocity of the intrinsic motion of matter. In particular, the center of mass \( z_a^\alpha \) moves in space with a constant velocity along a straight line: \( z_a^\alpha (t) = p_a^\alpha \cdot t + k_a^\alpha \), where the constants \( p_a^\alpha = d z_a^\alpha / dt \) and \( k_a^\alpha \) are the body’s momentum and center of inertia, respectively. One may choose from the set of inertial reference frames the barycentric inertial one. In this frame the functions \( z_a^\alpha \) must equal zero for any moment of time. This condition may be satisfied by applying to the metric Eq.(3) the post-Galilean transformations \([39]\) where the constant velocity and displacement of origin should be selected in such a way that \( p_a^\alpha \) and \( k_a^\alpha \) equal zero (for details see \([13], [40]\)).

### 2.2 An Astronomical N-Body System.

By putting some restrictions on the shape and internal structure of the bodies, one can generalize the results presented above to the case of an isolated astronomical N-body system. Indeed, the assumption that the bodies possess only the lowest multipole moments (such as mass \( m_a \), intrinsic spin moment \( S_a^{\alpha \beta} \) and the quadrupole moment \( I_a^{\alpha \beta} \)) considerably simplifies the problem. The general solution with such an assumption is well-known (\([23]\), see \([11]\) and references therein). The main properties of the solution Eq.(3) applied to such a case are well established and widely in use in modern ephemerides astronomy. In order to analyze the motion of bodies in the solar system barycentric reference frame, one may obtain the restricted Lagrangian function \( L_N \) describing the motion of \( N \) self-gravitating bodies (\([13], [22]\)). Within the accuracy necessary for our future discussion of the gravitational experiments on the Mercury Orbiter mission, this simplified function may be presented in the following form:

\[
L_N = \frac{1}{2} \sum_a m_a v_a^\mu v_a^\mu \left( 1 - \frac{1}{4} v_a^\mu v_a^\mu \right) - \sum_a \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} \left( \frac{1}{2} + (3 + \gamma - 4\beta) E_a - (\gamma - \tau + \frac{1}{2}) v_a^\mu v_b^\mu + (\gamma - \tau + \frac{3}{4}) v_a^\mu v_b^\mu - (\gamma - \tau + \frac{1}{4}) n_{ab \lambda} n_{ab \mu} v_a^\lambda v_b^\mu + \tau (n_{ab \mu} v_a^\mu)^2 + \right.
\]

\[
+ \frac{n_{ab \lambda}}{r_{ab}} \left[ (\gamma + \frac{1}{2}) v_a^\mu - (\gamma + 1) v_b^\mu \right] \eta_a^{\lambda \mu} + n_{ab \lambda} n_{ab \mu} \frac{I_a^{\lambda \mu}}{r_{ab}^2} \right) + (\beta - \tau - \frac{1}{2}) \sum_a m_a \left( \sum_{b \neq a} \frac{m_b}{r_{ab}} \right)^2 - \eta_a^{\mu \nu} \nabla_a (\nabla_a z^a_\nu) - \frac{1}{r_{ab} r_{ac}} \right) \sum_{a} m_a \mathcal{O}(c^{-6}),
\]

(6)

where \( m_a \) is the isolated rest mass of a body \( (a) \), the vector \( r_a^\alpha \) is the barycentric radius-vector of this body, the vector \( r_{ab}^\alpha = r_b^\alpha - r_a^\alpha \) is the vector directed from body \( (a) \) to body \( (b) \), and the vector \( n_{ab}^\alpha = r_{ab}^\alpha / r_{ab} \) is the usual notation for the unit vector along this direction. It should be noted that Eq.(6) does not depend on the parameter \( \nu \), which confirms that this parameter is a gauge parameter only. The tensor \( I_a^{\mu \nu} \) is the quadrupole moment of body \( (a) \) defined as:

\[
I_a^{\mu \nu} = \frac{1}{2 m_a} \int_d^d \hat{z}_a^{\alpha \beta} \hat{\rho}_a (z_a^p) \left( 3 z_a^\mu z_a^\nu - \eta_a^{\mu \nu} z_a^\alpha z_a^\beta \right).
\]

(7)

The tensor \( S_a^{\mu \nu} \) is the body’s intrinsic spin moment which is given as:

\[
S_a^{\mu \nu} = \frac{1}{m_a} \int_d^d \hat{z}_a^\mu \hat{\rho}_a (z_a^p) \left( \mathcal{O}_a^{\mu \nu} - \eta_a^{\mu \nu} \right).
\]

(8)
where $\mathbf{v}_a^\mu$ is the velocity of the intrinsic motion of matter in body $(a)$. Finally, the quantity $E_a$ is the body’s gravitational binding energy:

$$E_a = \frac{1}{2m_a} \int \int a^3 \xi_a^\nu d^3 \xi_a^\mu \hat{\rho}_a(z_a^\nu) \hat{\rho}_a(z_a^\mu) \left| \frac{z_a^\nu - z_a^\mu}{\xi_a^\nu} \right|. \quad (9)$$

The corresponding equations of motion of the planet in the solar system barycentric reference frame are given in Appendix B. It should be noted here that in the present numerical algorithms for celestial mechanics problems (\cite{4}, \cite{13}, \cite{43}-\cite{45}) the bodies in the solar system are assumed to possess the lowest post-Newtonian multipole moments only. The corresponding barycentric inertial reference frame has been adopted for the fundamental planetary and lunar ephemeris (\cite{4}, \cite{46}). Moreover, the coordinate time of the solar barycentric (harmonic) reference frame is the $TDB$ time scale, which has been adopted by Fukushima \cite{11} and Standish \cite{47}.

However, it turns out that generalization of the results obtained for the one-body problem, to the problem of motion of an arbitrary $N$-body system is not quite straightforward. Thus, we point out that taking into account the presence of any non-vanishing internal multipole moments of an extended body, significantly changes its equations of motion due to coupling of the intrinsic multipole moments of the body to the surrounding gravitational field. For example, for a neutral monopole test particle, the external gravitational field completely defines the fiducial geodesic world line which this test body follows (\cite{13}, \cite{23}). On the other hand, the equations of motion for spinning bodies differ from these by additional terms due to the coupling of the body’s spin to the external gravitational field (\cite{48}-\cite{50}). As a result, one needs to present a post-Newtonian definition for the proper intrinsic multipole moments of the bodies in order to describe their interaction with the surrounding gravitational field and to obtain the corrections to the laws of motion and to the precession of extended bodies in this system (\cite{51}, \cite{52}). The fully relativistic definition of these moments may be given in the proper quasi-inertial reference frame only. Such a definition replaces Eq.\(4\) which was given in the rest-frame of the one-body problem\footnote{Note that, due to the breaking of the symmetry of the total Riemann space-time by realizing the $3 + 1$ split \cite{25}, these moments will not form tensor quantities with respect to general four-dimensional coordinate transformations. Instead, these quantities will behave as tensors under the sub-group of this total group of motion only, namely: the three-dimensional rotation. This is similar to the situation in classical electrodynamics, where electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields are not true vectors, but rather components of the $4 \times 4$ tensor of the electro-magnetic field $F_{\mu\nu} = (\mathbf{E} \otimes \mathbf{H})$ \cite{25}.}. In presenting these transformations one should also take into account that, due to the non-linear character of the gravitational interaction, these moments are expected to interact with external gravity, changing the state of motion of the body itself. Fock \cite{23} was the first to notice that in order to find the solution of the $global$ problem (the motion of the $N$-body system as a whole), the solution for the $local$ gravitational problem (in the body’s vicinity) is required. In addition, one must establish their correspondence by presenting the coordinate transformation by which the physical characteristics of motion and rotation are transformed from the coordinates of one reference frame to another. Thus, one must find the solutions to the three following problems (\cite{17}, \cite{34}):

(1). **The global problem:**

(i). We must construct the asymptotically inertial reference frame.

(ii). We must find the barycentric inertial reference frame for the system under study. This is primarily a problem of describing the $global$ translational motion of the bodies constituting the $N$-extended-body system (i.e. finding the geodesic structure of the space-time occupied by the whole system).

(2). **The local problem:**
(i). We must establish the properties of the gravitational environment in the proximity of each body in the system (i.e., finding the geodesic structure of the local region of the space-time in the body’s gravitational domain).

(ii). We must construct the local effective rest frame of each body.

(iii). We must study the internal motion of matter inside the bodies as well as establish their explicit multipolar structure and rotational motion.

(3). The theory of the reference frames:

(i). We must find a way to describe the mutual physical cross-interpretation of the results obtained for the above two problems (i.e., the mapping of the space-time).

Because the solutions to the first two problems will not be complete without presenting the rules of transforming between the global and the local reference frames chosen for such an analysis, the theory of the astronomical reference frames become inseparable from the problem of determination of the motion of the celestial bodies. From the other side, if one attempts to describe the global dynamics of the system of N arbitrarily shaped extended bodies, one will discover that even in WFSMA this solution will not be possible without an appropriate description of the gravitational environment in the immediate vicinity of the bodies (14, 40).

3 Gravitational N-body Problem in the WFSMA.

In this Section we will discuss the principles of a new iterative method for generating the solutions to an arbitrary N-body gravitational problem in the WFSMA of metric theories of gravity. In these theories one may choose any coordinates to describe the gravitational environment around the body under question. However non-optimal choice of these coordinates may cause unreasonable complications in the physical interpretations of the data obtained (see the related discussion in [40], [57]). Recently several different attempts have been made to improve the present solution to the N-body problem in WFSMA (see, for example, [15]-[17], [40], [53]-[54]). Note that although these methods represent a significant improvement in our understanding of the general problem, the present situation requires a more physically adequate approach to account for the difficulties in describing the relativistic motion of a system of extended bodies.

In the present paper we use an approach which has being developed to deal with the complications introduced by using a standard PPN formalism [21]. This new method is based on the Fock-Chandrasekhar approach for dealing with extended and arbitrarily shaped celestial bodies and is parameterized with the two Eddington parameters (γ, β). This formalism is based upon the construction of proper reference frames in the vicinities of each body in the system. Such frames are defined in the gravitational domain occupied by a particular body (b). One may expect that, in the immediate vicinity of this body, its proper gravitational field will dominate, while the existence of external gravity will manifest itself in the form of the tidal interaction only. Therefore, in the case of the WFSMA and in extreme proximity to the body under study, this proper reference frame should resemble an inertial frame and the solution Eq.(3) for an isolated one-body problem $h^{(0)}_{mn}$ should adequately represent the physical situation. However, if one decides to perform a physical experiment at some distance from the world-tube of the body, one should consider the existence of external gravity as well. This is true because external gravity plays a more significant role at large distances from the body.
3.1 Definition of a Physically Adequate Proper Reference Frame.

In order to construct a general solution for the N-body problem in a metric theory of gravity, let us make a few assumptions. First of all, we shall assume that the solution of the gravitational field equations $h_{mn}^{(0)}$ for an isolated unperturbed distribution of matter is known and it is given by the equations (3). With this assumption, we may construct the total solution of the global problem $g_{mn}$ in an arbitrary reference frame as a formal tensorial sum of the background space-time metric $\eta_{mn}$, the unperturbed solutions $h_{mn}^{(0)b}$ plus the gravitational interaction term $H_{mn}$. Thus, in the coordinates $x^p \equiv (x^a, x^b)$ of the barycentric inertial reference frame, one may search for the desired total solution in the following form \[22]\:

$$g_{mn}(x^p) = \eta_{mn}(x^p) + h_{mn}(x^p) = \eta_{mn}(x^p) + \sum_{b=1}^{N} \frac{\partial y^b_k}{\partial x^m} \frac{\partial y^b_l}{\partial x^n} h_{kl}^{(0)b}(y^b(x^p)) + H_{mn}(x^p),$$  \hspace{1cm} (10)

where the coordinate transformation functions $y^b_k = y^b_k(x^p)$ are yet to be determined. The interaction term $H_{mn}$ will be discussed below.

If the bodies in the system are compact and well separated, then, we may take into account that the mutual gravitational interaction between the bodies affects their distribution of matter through the metric tensor only. Therefore, without any loss of accuracy, we obtain the total energy-momentum tensor of the matter distribution in the system in the following form:

$$T_{mn}(x^s) = \sum_{b=1}^{N} \frac{\partial y^b_k}{\partial x^m} \frac{\partial y^b_l}{\partial x^n} T_{kl}^b(y_B(x^s)),$$  \hspace{1cm} (11)

where $T_{kl}^b$ is the energy-momentum tensor of a body ($b$) as seen by a co-moving observer.

One may establish the properties of the solution Eq.(10) with respect to an arbitrary coordinate transformation simply by applying the basic rules of tensorial coordinate transformations. In particular, in the coordinates $y^b_k \equiv (y^b_a, y^b_p)$ of an arbitrary proper reference frame this tensor will take the following form:

$$g^a_{mn}(y^b_k) = \frac{\partial x^k}{\partial y^a} \frac{\partial x^l}{\partial y^n} g_{kl}(x^s(y^b_k)) = \frac{\partial x^k}{\partial y^a} \frac{\partial x^l}{\partial y^n} \eta_{kl}(x^s(y^b_k)) + h_{mn}^{(0)a}(y^b_a) + \sum_{b \neq a} \frac{\partial y^b_k}{\partial y^a} \frac{\partial y^b_l}{\partial y^n} h_{kl}^{(0)b}(y^b_k(y^b_p)) + \frac{\partial x^k}{\partial y^a} \frac{\partial x^l}{\partial y^n} H_{kl}(x^s(y^b_k)).$$  \hspace{1cm} (12)

The expression for $T_{mn}(y^b_k)$ could be obtained analogously from that given by Eq.(11). To complete the formulation of the perturbative scheme we need to introduce the procedure for constructing the solutions for the various unknown functions entering Eqs.(10)-(12), including the four functions of the coordinate transformations $y^b_m = y^b_m(y^p_k)$ and the interaction term $H_{mn}$.

\[10\] As a partial result of the representation Eq.(11) one can see that the Newtonian mass density $\rho_b$ of a particular body ($b$) is defined in a sense of a three-dimensional Dirac delta-function. Thus in the body’s proper compact-support volume one will have: $\rho_b = m_b \delta(y^b_k)$, so that

$$\int_{V_b} d^3y_b \rho_b(y^b_k) = m_b \delta_{ab},$$

where $\delta_{ab}$ is the three-dimensional Kronecker symbol ($\delta^a_b = \delta_{ab}$; $\delta_{ab} = 1$ for $a = b$ and $= 0$, for $a \neq b$). Then in any proper reference frame the total density $\overline{\rho}$ of the whole N-body system will be given by the expression $\overline{\rho}(y^p_k) = \sum_b \rho_b(y^p_k)$. This representation allows one to distinguish between the microscopic and macroscopic (or integral) descriptions of the physical processes and, hence, provides correct relativistic treatment of the problem of motion of an astronomical N-body system.
For the four functions of the coordinate transformations \( y_b^m = y_b^m(y_a^p) \) we use the general post-Newtonian coordinate transformations [21] which connect the coordinates \((x^p)\) of the barycentric inertial reference frame to those \((y_a^p)\) of a proper quasi-inertial reference frame of an arbitrary body \((a)\). The general form of these relations may be given as follows:

\[
\begin{align*}
    x^0 &= y_a^0 + c^{-2} K_a(y_a^0, y_a^0) + c^{-4} L_a(y_a^0, y_a^0) + \mathcal{O}(c^{-6}), \\
    x^\alpha &= y_a^\alpha + y_a^\alpha(y_a^0) + c^{-2} Q_a^\alpha(y_a^0, y_a^0) + \mathcal{O}(c^{-4}),
\end{align*}
\]

(13a)

(13b)

where the barycentric radius-vector \( r_a^\alpha \) of the body \((a)\) in the coordinates of the proper reference frame is decomposed into Newtonian and post-Newtonian parts and is given as follows:

\[
\begin{align*}
    \langle r_a^\alpha(y_a^p) \rangle_a &= y_a^\alpha(y_a^0) + \frac{1}{m_a c^2} \int_a d^3 y_a^\nu \hat{t}^{00}(y_a^p) Q_a^\alpha(y_a^0, y_a^0) + \mathcal{O}(c^{-6}),
\end{align*}
\]

(14a)

with

\[
    m_a = \int_a d^3 y_a^\nu \hat{t}^{00}(y_a^p) + \mathcal{O}(c^{-6}),
\]

(14b)

where \( \hat{t}^{00}(y_a^p) \) is the component of the conserved density of the energy-momentum tensor of matter, inertia and gravitational field taken jointly. The notation \( \langle \cdot \rangle_a \) applied to any function \( f(y_a^p) \) means an averaging over body \((a)\)’s volume as follows:

\[
\begin{align*}
    \langle f(y_a^p) \rangle_a \equiv \hat{f}(y_a^0) &= \frac{1}{m_a} \int_a d^3 y_a^\nu \hat{t}^{00}(y_a^p) f(y_a^0, y_a^0).
\end{align*}
\]

(15)

In order to complete the formulation of the perturbative scheme, we need to introduce the procedure for constructing the solutions for the interaction term \( \mathcal{H}_{mn} \) and for the post-Newtonian transformation functions \( K_a, Q_a^\alpha \) and \( L_a \) which are still unspecified. The way to construct the solution for the interaction term \( \mathcal{H}_{mn} \) is quite straightforward; it is sufficient to require that the metric tensor in the form of Eq.(10) or Eq.(12) will be the explicit solution of the gravitational field equations in the corresponding reference frame. These solutions are assumed to satisfy the covariant hamilton de Donder gauge, which, for an arbitrary reference frame, may be written as follows:

\[
D_n^b(\sqrt{-g_b} g_b^{mn}(y_b^p)) = 0,
\]

(16a)

where \( D_n^b \) is the covariant derivative with respect to the metric \( \eta_{mn}^b(y_b^p) \) of the inertial Riemann-flat \( (R_{nmkl}^b(\eta_{mn}^b(y_b^p)) = 0) \) space-time in these coordinates\[1\]. For most of the practically interesting problems in the WFMS in quasi-Cartesian coordinates this metric may be represented as the sum of two tensors: the Minkowski metric \( \eta_{mn}^{(0)} \) and the field of inertia \( \phi_{mn} \):

\[
\eta_{mn}^b(y_b^p) = \frac{\partial x^k}{\partial y_b^m} \frac{\partial x^l}{\partial y_b^n} \eta_{kl}(x^s(y_b^p)) = \eta_{mn}^{(0)} + \phi_{mn}^b(y_b^p).
\]

(16b)

Note that the term \( \phi_{mn} \) appears to be parameterized by the coordinate transformation functions \( K_a, L_a \) and \( Q_a^\alpha \) defined in Eq.(13); thus we have \( \phi_{mn}(y_a^p) = \phi_{mn}[K_a, L_a, Q_a^\alpha] \), a formulation which will be referred to as the \( K\mathcal{L}Q \) parameterization in the WFMS.

\[1\]In Cartesian coordinates of the inertial Galilean reference frame \((x^p)\) the flat metric \( \eta_{mn} \) can be chosen as \( \eta_{mn}^{(0)} = \text{diag}(1, -1, -1, -1) \), so that the Christoffel symbols \( \Gamma_{mn}^{(0)} = 0 \) all vanish and conditions Eq.(16a) take the more familiar form of the harmonic conditions

\[
\partial_a(\sqrt{-g} g^{mn}) = 0,
\]

which are equivalent to setting \( \nu = \tau = 0 \) in the Eq.(3).
The search for the general solution for $H_{mn}(x^p)$ is performed in a barycentric inertial reference frame ($x^p$), which is singled out by the Fock-Sommerfeld’s boundary conditions imposed on $h_{mn}$ and $\partial_k h_{mn}$ (given by Eq.(10)):

$$\lim_{r \to \infty} \left( h_{mn}(x^p); \ r \left[ \frac{\partial}{\partial x^i} h_{mn}(x^p) + \frac{\partial}{\partial p} h_{mn}(x^p) \right] \right) \to 0,$$

$$t + \frac{r}{c} = \text{const},$$

(17a)

where $r^2 = -\eta_{\mu\nu} x^\mu x^\nu$. These conditions define the asymptotically Minkowskian space-time in a weak sense, consistent with the absence of any flux of gravitational radiation falling on the system from the external universe [12]. Moreover, one assumes that there exists such a quantity $h_{mn}^{\text{max}} = \text{const}$ (for the solar system this constant is of order $10^{-5}$) for which the condition

$$h_{mn}(x^p) < h_{mn}^{\text{max}},$$

(17b)

should be satisfied for each point ($\vec{x}$) inside the system. Note that any distribution of matter is considered isolated, if conditions Eq.(17) are fulfilled in any inertial reference frame.

The advantage of using the imposed conditions Eq.(17) is that it gives us the opportunity to determine the interaction term $H_{mn}(x^p)$ in a unique way. It should be stressed that the corresponding solution $g_{mn}(x^p)$ in the barycentric inertial reference frame resembles the form of the solution for an isolated one-body problem Eq.(3). The only change that should be made is to take into account the number of the bodies in the system: $\rho \to \sum_b \rho_b$, where $\rho_b$ is the compact-support mass density of a body ($b$) in the system. However, both the interaction term $H_{mn}$ and the total solution for the metric tensor $g_{mn}$ in the coordinates ($y^a$) appear to be ‘parameterized’ by the arbitrary functions $K_a, L_a, Q_{\alpha a}$. This result reflects the covariance of the gravitational field equations as well as the well-defined transformation properties of the gauge conditions Eq.(16) used to derive the total solution. This arbitrariness suggests that one could choose any form of these functions in order to describe the dynamics of the extended bodies in the system. However, as we noted earlier, a disadvantageous choice of the proper reference frame (or, equivalently, the functions $K_a, L_a$ and $Q_{\alpha a}$) might cause an unreasonable complication in the future physical interpretations of the results obtained.

### 3.2 Principles of Construction of the Proper Reference Frame.

In this Section we will present a way to find the transformation functions necessary for constructing a proper reference frame with well defined properties. As one can see from the expressions (12), in the WFSMA the main contribution to the geometrical properties of the proper reference frame in the body’s immediate vicinity comes from its own gravitational field $h_{mn}^{(0)a}$. Then, based on the Principle of Equivalence, the external gravitational influence should vanish at least to first order in the spatial coordinates ($x^a$). The proper reference frame, constructed this way, should resemble the properties of a quasi-inertial (or Lorentzian) reference frame and, as such, will be well suited for discussing the physical experiments. Note that the tensors $h_{mn}^{(0)b}$ and $H_{mn}$ represent the real gravitational field which no coordinate transformation can eliminate everywhere in the system. In the case of a massive monopole body, one can eliminate the influence of the external field on the body’s world-line only. However, for an arbitrarily shaped extended body, the coupling of the body’s intrinsic multipole moments to the surrounding gravitational field changes the physical picture significantly. This means that the definition of the proper reference frame for the extended body must take into account this non-linear gravitational coupling.

In order to suggest the procedure for the choice of the coordinate transformations to the proper reference frame with well established properties, let us discuss the general structure of the solution
given by expression (12). Thus, in the expressions for \( g^a_{mn} \) one may easily separate the four physically different terms. These terms are:

(i). The Riemann-flat contribution of the field of inertia \( \phi^a_{mn} \) given by expression (16b).

(ii). The contribution of the body’s own gravitational field \( h^{(0)a}_{mn} \).

(iii). The term due to the non-linear interaction of the proper gravitational field with an external field. This contribution is due to the Newtonian potential and the potential \( \Phi_2 \) in the expressions (3). These interaction terms show up as the coupling of the body’s intrinsic multipole moments with the external field.

(iv). The term describing the field of the external sources of gravity. This term comes from the transformed solutions \( h^{(0)b}_{mn} \) and the interaction term \( \mathcal{H}_{mn} \).

The first contribution depends on the external field in the gravitational domain occupied by the body \((a)\) and appears to be ‘parametrized’ by the transformation functions Eq.(13). Note that for any choice of these functions, the obtained metric \( g^a_{mn} \), by the way it was constructed, satisfies the gravitational field equations of the specific metric theory of gravity under study. Furthermore, based on the properties of the proper reference frame discussed above, one may expect that the functions \( \mathcal{K}_a, \mathcal{L}_a \) and \( \mathcal{Q}^a_b \) should form a background Riemann-flat inertial space-time \( \eta^0_{mn} \) in this reference frame which will be tangent to the total gravitational field in the vicinity of body \((a)\)’s world-line \( \gamma_a \). Moreover, the difference of these fields should vanish to first order with respect to the spatial coordinates \((i.e.\) the ‘external’ dipole moment equals zero \([53]\)). These conditions, applied to moving test particles, are known as Fermi conditions \([35], [64], [65]\). We have extended the applicability of these conditions to the case of a system composed of \( N \) arbitrarily shaped extended celestial bodies.

In order to obtain the functions \( \mathcal{K}_a, \mathcal{L}_a \) and \( \mathcal{Q}^a_b \) for the coordinate transformation Eq.(13) we will introduce an iterative procedure which will be based on a multipole power expansion with respect to the unperturbed spherical harmonics. To demonstrate the use of these conditions, let us denote \( H^a_{mn}(y^p_a) \) as the local gravitational field, \( i.e. \) the field which is formed from contributions (ii) and (iii) above. The metric tensor in the local region in this case can be represented by the expression:

\[
g^a_{mn}(y^p_a) = \eta^0_{mn} + H^a_{mn}(y^p_a). \tag{16a}
\]

Then the generalized Fermi conditions in the local region of body \((a)\) (or in the immediate vicinity of its world-line \( \gamma_a \)) may be imposed on this local metric tensor by the following equations:

(iii). The term due to the non-linear interaction of the proper gravitational field with an external field. This contribution is due to the Newtonian potential and the potential \( \Phi_2 \) in the expressions (3). These interaction terms show up as the coupling of the body’s intrinsic multipole moments with the external field.

\[
\lim_{\gamma \to \gamma_a} g^a_{mn}(y^p_a) = g^{(loc)}_{mn}(y^p_a)_{\gamma_a}, \tag{18a}
\]

\[
\lim_{\gamma \to \gamma_a} \Gamma^k_{mn}(y^p_a) = \Gamma^{(loc)}_{mn}(y^p_a)_{\gamma_a}, \tag{18b}
\]

where \( \gamma \) is the world-line of the point of interest and the quantities \( \Gamma^{(loc)}_{mn}(y^p_a) \) are the Christoffel symbols calculated with respect to the local gravitational field \( g^{(loc)}_{mn}(y^p_a) \). Application of these conditions will determine the functions \( \mathcal{K}_a, \mathcal{L}_a, \mathcal{Q}^a_b \) which are as yet unknown. Moreover, this procedure will enable us to derive the second-order ordinary differential equations for the functions \( y^a_0(y^0_a) \) and \( \mathcal{Q}^a_b(y^0_a, 0) \), or, in other words, to determine the equations of the perturbed motion of the center of the local field in the vicinity of body \((a)\).

The relations Eq.(18) summarize our expectations based on the Equivalence Principle about the local gravitational environment of the self-gravitating bodies. By making use of these equations, we will be able to separate the local gravitational field from the external field in the immediate vicinity of the bodies. However, these conditions only allow us to determine the transformation functions for the free-falling massive monopoles \((i.e. \) only up to the second order with respect to the spatial coordinates).
The transformation functions Eq.(13) in this case will depend only on the unperturbed contributions of the external gravitational potentials $U_b$ and $V_b^\alpha$ and their first derivatives taken on the world-line of body $(a)$. The results obtained will not account for the contribution of the multipolar interaction of the proper gravity with the external field in the volume of the extended body. This accuracy is sufficient for taking into account the terms describing the interaction of the intrinsic quadrupole moments of the bodies with the surrounding gravitational field, but some more general condition, in addition to Eq.(18), must be applied in order to account for the higher multipole structure of the bodies.

The conditions Eq.(18), however, enable one to obtain the complete solution for the Newtonian function $K_a$. Functions $\mathcal{L}_a$ and $Q^a_\alpha$ may be defined up to the second order with respect to the spatial point separation, namely $\mathcal{L}_a, Q^a_\alpha \sim O((y^\lambda_\alpha)^3)$, so the arbitrariness of higher orders ($k \geq 3$) in the spatial point separation will remain in the transformation. In order to get the corrections to these functions up to $k^{th}$ order ($k \geq 3$) with respect to the powers of the spatial coordinate $y^\lambda_\alpha$, one should use conditions which contain the spatial derivatives of the metric tensor to order $(k-1)$. The mathematical methods of modern theoretical physics generally consider local geometrical quantities only and involve second order differential equations. These equations alone may not be very helpful for constructing the remaining terms in functions $\mathcal{L}_a, Q^a_\alpha$ up to the order $k \geq 2$. However, following Synge [3], one may apply additional geometrical constructions, such as properties of the Riemann tensor and the Fermi-Walker transport law [3, 64-67]. Another possibility is to postulate the existence of so called ‘external multipole moments’ (13, 17, 36, 37). However, those moments are defined through vacuum solutions of the Hilbert-Einstein field equations of general relativity in an inertial reference frame, while the influence of external sources of gravity are ignored. The fact of defining the moments in this way is essentially equivalent to defining the structure of the proper reference frame for the body under question.

The most natural approach to define the desirable properties of the proper quasi-inertial reference frames for the system of extended and deformable bodies is to study the motion of this system in an arbitrary $KLQ$-parametrized frame. There exist two different ways to do this, namely: (i) to study the infinitesimal motion of each element of the body, or (ii) to study the motion of a whole body with respect to an accelerated frame attached, say, to the center of inertia of the local fields of matter, inertia, and gravity. In our method we will use the second way and will study the dynamics of the body in its own reference frame. Our analysis will be directed toward finding the functions $K_a, \mathcal{L}_a$ and $Q^a_\alpha$ with the condition that the Riemann-flat inertial space-time $g^\mu_\nu(y^\alpha_a)$ corresponding to these functions will be tangent to the total Riemann metric $g^\mu_\nu(y^\alpha_0)$ of the entire system in the body’s vicinity. Physically, one expects that this inertial space-time will produce a ‘fictitious’ (or inertial) force with the density $\delta F_{KLQ}$ acting on the body in its proper reference frame. At the same time, the body is under influence of the overall real force due to the local fields of matter and gravity with the density $f_0$. Thus, the condition for finding the transformation functions $K_a, \mathcal{L}_a$ and $Q^a_\alpha$ is conceptually simple: the difference between these two densities $\delta F = \oint_f y^\lambda_\alpha \mathcal{F} = \oint_a d^3 x_\alpha y_\alpha (\delta f_{KLQ}) = 0$. (19)

Note that the notion of ‘the center of mass’ in this case loses its practical value, and one should substitute instead ‘the local center of inertia’. Thus, the force $\delta F_{KLQ}$ should provide the overall static equilibrium for the body under consideration in the local center of inertia, which is defined for all three fields present in the immediate vicinity of the body, namely: matter, inertia and gravity. Let us mention here that in practice it is not possible to separate these two forces $f_0$ and $\delta F_{KLQ}$ from each other. Fortunately, it is possible to obtain at once the difference between them $\mathcal{F}$. This will considerably simplify the further analysis [21].
In order to construct the necessary solution for the functions $\mathcal{K}_a, \mathcal{L}_a$ and $\mathcal{Q}_a^\alpha$ in a way that will be valid for a wide class of metric theories of gravity, one must first analyze the conservation laws in an arbitrary $\mathcal{KLQ}$-parameterized reference frame. This could be done based on the conservation law for the density of the total energy-momentum tensor $\hat{T}^{mn}$ of the whole isolated $N$-body system:

$$\nabla_a^\alpha \hat{T}^{mn}(y_a^\alpha) = 0,$$

where $\nabla_a^\alpha$ is the covariant derivative with respect to total Riemann metric $g_a^{mn}(y_a^\nu)$ in these coordinates. Then, by using a standard technique of integration with Killing vectors, one will have to integrate this equation over the compact volume of the body \textit{(a)}, and one can obtain the equations of motion of the extended body \textit{(13), [24], [26]}. Then the necessary conditions, equivalent to those of Eq.(18), may be formulated as the requirement that the translational motion of the extended bodies vanish in their own reference frames. This corresponds to the following conditions applied to the dipole mass moment $\vec{m}_a \equiv I_a^{(1)}$:

$$\frac{d^2 \vec{m}_a}{dy_a^{\beta \gamma}} = \frac{d\vec{m}_a}{dy_a^{\alpha}} = \vec{m}_a = 0,$$

where the quantity $\vec{m}_a$ is calculated based on the total energy-momentum tensor matter, inertia and gravitational field taken jointly (similar to condition of Eq.(4)). These conditions may also be presented in a different form. Indeed, if we require that the total momentum $\vec{P}_a$ of the local fields of matter, inertia and gravity in the vicinity of the extended body vanish, we will have the following physically equivalent condition:

$$\frac{d\vec{P}_a}{dy_a^{\alpha}} = \vec{P}_a = 0.$$

These conditions finalize the formulation of the basic principles of construction of the relativistic theory of celestial reference frames in the WFSMA. In the next Section we will present the results obtained when these principles are applied.

### 4 Properties of the Proper Reference Frame.

In this Section we will present the basic results obtained for the relativistic coordinate transformations in the WFSMA. We will present the results for the transformation functions Eq.(13) as well as the solution for the metric tensor in the coordinates of the proper reference frame.

#### 4.1 Coordinate Transformations and Metric Tensor.

By taking into account all the conditions presented in the previous Section, one can obtain the set of differential equations on the transformation functions $\mathcal{K}_a, \mathcal{Q}_a^\alpha$ and $\mathcal{L}_a$. The solutions to these equations can be given as follows:

$$\mathcal{K}_a(y_a^0, y_a^\nu) = \int dt' \left( \sum_{b \neq a} \langle U_b \rangle_a - \frac{1}{2} v_a^\nu v_a^\nu \right) - v_a^\nu \cdot y_a^\nu + O(c^{-4}) y_a^0,$$

$$\mathcal{Q}_a^\alpha(y_a^0, y_a^\nu) = -\gamma \sum_{b \neq a} \left( y_a^\alpha y_a^\beta \cdot \langle \partial_{\nu} U_b \rangle_a - \frac{1}{2} y_a^\beta y_a^\alpha \langle \partial^\alpha U_b \rangle_a + y_a^\alpha \langle U_b \rangle_a \right) +$$

$$+ y_a^\alpha \int dt' \left( \frac{1}{2} \langle \partial^\alpha \nu_b \rangle_a + (\gamma + 1) \sum_{b \neq a} \left[ \langle \partial^\alpha \nu_b \rangle_a + \langle \partial^\alpha \nu_b \rangle_a \right] \right) -$$
\[ -\frac{1}{2} v_{a0}^\alpha v_{a0}^\beta y_{a\beta} + w_{a0}^\alpha(y_a^0) + \sum_{l \geq 3} Q_{a\{L\}}^\alpha(y_A^0) \cdot y_{a\{L\}}^\alpha + \mathcal{O}(|y_a^\nu|^{k+1}) + \mathcal{O}(e^{-4})y_a^\alpha, \quad (22b) \]

\[ L_\alpha(y_a^0, y_a^\nu) = \sum_{b \neq a} \left( \frac{1}{2} \gamma y_{a\beta} y_{b\beta} \frac{\partial}{\partial y_a^\nu} \langle U_b \rangle_a - (\gamma + 1) y_a^\lambda y_b^\lambda \cdot \left[ \langle \partial_{\lambda} V_{b\beta} \rangle_a + \langle v_{b\beta} \partial_{\lambda} U_b \rangle_a \right] + + \gamma v_{a0} \left[ y_a^\lambda \lambda(y_a^\nu) - y_a^\nu \langle \partial_{\lambda} U_b \rangle_a \right] + + y_{a\lambda} v_{a0} \beta \int dt \left( \frac{1}{2} y_{a\lambda} y_{a\beta} + (\gamma + 1) \sum_{b \neq a} \left[ \langle \partial_{\lambda} V_{b\beta} \rangle_a + \langle v_{b\beta} \partial_{\lambda} U_b \rangle_a \right] \right) + + y_{a\lambda} \left[ (\gamma + 1) v_{a\lambda} \sum_{b \neq a} \langle U_b \rangle_a - 2(\gamma + 1) \sum_{b \neq a} \langle V_{b\lambda} \rangle_a - w_{a0} \rangle \right] - - \int dt \left[ \sum_{b \neq a} \langle W_b \rangle_a + \frac{1}{2} \sum_{b \neq a} \langle U_b \rangle_a - \frac{1}{2} v_{a0} \beta v_{a0} \rangle \right] \right] + + \sum_{l \geq 3} L_{a\{L\}}(y_a^0, y_a^\{L\}) + \mathcal{O}(|y_a^\nu|^{k+1}) + \mathcal{O}(e^{-6}), \quad (22c) \]

where the dot over the function \( w_{a0}^\alpha \) denotes the regular time derivative. Note that in the case of the free-falling structureless test particle with conserved mass density given by \( \hat{\rho}_0(y_a^0, y_a^\nu) = m_0 \delta(y_a^\nu) \), the functions Eq.(22) correspond to the coordinate transformations to the proper reference frame defined on the geodesic world line of this particle. The time-dependent functions \( Q_{a\{L\}}^\alpha \) and \( L_{a\{L\}} \) in the expressions Eq.(22) are the contributions coming from the higher multipoles \( (l \geq 3) \) (both mass and current induced ones) of the external gravitational field generated by the bodies \( (b \neq a) \) in the system. These functions enable one to take into account the geometrical features of the proper reference frame with respect to three-dimensional spatial rotation. The form of these functions may be chosen arbitrarily. This freedom enables one to choose any coordinate dependence for the terms with \( l \geq 3 \) in order to describe the motion of the highest multipoles. Moreover, one may show that, even though the total solution to the metric tensor \( g_{mn}(x^p) \) in the barycentric inertial reference frame resembles the form of the one-body solution Eq.(3), if one will expresses this solution through the proper multipole moments of the bodies, it will contain the contributions from the functions \( Q_{a\{L\}}^\alpha(y_a^0) \) and \( L_{a\{L\}}(y_a^0) \).

Within the accuracy necessary for future analysis, we present the equations for both time-dependent functions \( y_{a0}^\alpha \) and \( w_{a0}^\alpha \) written with respect to time \( y_a^0 \) of the proper reference frame. Thus the Newtonian acceleration of body \( a \) with respect to the barycentric reference frame may be described as follows

\[ a_{a0}^\alpha(y_a^0) = -\eta_{\alpha\mu} \sum_{b \neq a} \langle \partial_{\mu} U_b \rangle_a + \mathcal{O}(e^{-4}) = -\eta_{\alpha\mu} \sum_{b \neq a} \frac{1}{m_a} \int d^3y_{a\nu} \hat{\rho}_a(y_a^\nu) \frac{\partial U_b(y_a^\nu)}{\partial y_{a\nu}} + \mathcal{O}(e^{-4}). \quad (23a) \]

Whereas the post-Newtonian part of the acceleration may be represented by the expression

\[ w_{a0}^\alpha(y_a^0) = \sum_{b \neq a} \left( \eta_{\alpha\mu} \langle \partial_{\mu} W_b \rangle_a + v_{a0}^\alpha \frac{\partial}{\partial y_{a\nu}} \langle U_b \rangle_a - 2(\gamma + 1) \frac{\partial}{\partial y_{a\nu}} \langle V_{b\lambda} \rangle_a \right) - \frac{1}{2} v_{a0}^\alpha v_{a0}^\beta \langle U_b \rangle_a - \langle \partial^\alpha \Sigma \left( \Pi - \frac{1}{2} (1 + 2\gamma) v_{\mu} v_{\mu} + \frac{3p}{p} + (2\beta + 2\gamma - 1) T \right) \rangle_a + \]

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For practical purposes one may find the value of the surface integrals in the expression Eq.(23) by performing an iteration procedure. It can be shown that the lowest multipole moments of the bodies will not contribute to this surface integration. However, the general results will fully depend on the non-linear interaction of the intrinsic multipole moments with the external gravity in the local region at the vicinity of the body under consideration. This additional iterative option will make all the results obtained with the proposed formalism easy to use in practical applications.

The expressions Eq.(23) are the two parts of the force necessary to keep the body \((a)\) in its orbit (world tube) in the N-body system. These expressions are written in terms of proper time, and if one performs the coordinate transformation from the coordinates \((y_{a}^{P})\) to those of \((x_{P})\) for all the functions and potentials entering both equations (23), and takes into account the lowest intrinsic multipole moments of the bodies only, one obtains the simplified equations of motion for the extended bodies Eqs.(B1-B7) written in the coordinates \((x_{P})\) of the barycentric inertial reference frame.

The transformations Eqs.(13), (22)-(23) are the generalization of the Poincare’ group for the case of motion of extended self-gravitating bodies. Note, that Chandrasekhar and Contopoulos [39] obtained the coordinate transformations with the limitation of not violating the form-invariancy of the metric tensor. This ensured that the equations of motion would preserve their form. However, in our case, the transformation of coordinates to the proper reference frame connected with body \((a)\), were obtained with the generalized Fermi conditions Eq.(18) and Eq.(21), and hence they changed the metric tensor considerably. The most notable contribution to the equations of motion of the test particle orbiting this body comes from the gravitational field of body \((a)\) itself. The influence of the external sources of gravity presents itself in the form of the tidal terms only. This can be seen from the metric tensor \(g_{mn}^{a}\) in the coordinates \((y_{a}^{P})\), which may be obtained in the following form:

\[
g_{00}^{a}(y_{a}^{P}) = 1 - 2U + 2W + y_{a}^{\mu}y_{a}^{\nu} \cdot \left[ \gamma \eta_{\mu\beta} a_{aa\lambda}a_{a0} - (2\gamma - 1) a_{aa\mu}a_{a0\beta} + \right.
\]

\[
+ \sum_{b \neq a} \frac{\partial}{\partial y_{a}^{\mu}} \left( \gamma \eta_{\mu\beta} a_{aa\lambda}a_{a0} \right) \left( U_{b} \right)_{a} - (\gamma + 1) \left[ \left( \partial_{(\mu}V_{\beta)} \right)_{a} + \left( v_{(\beta}a_{\mu)} \right)_{a} \right] + \]

\[
+ 2 \sum_{l \geq 3} \left[ \partial_{0}L_{a\{l\}}(y_{a}^{0}) + v_{a0\beta}\partial_{0}Q_{a\{l\}}^{\beta}(y_{a}^{0}) \right] \cdot y_{a}^{(L)} + O(|y_{a}^{P}|^{k+1}) + O(c^{-6}),
\]

\[
g_{0a}^{a}(y_{a}^{P}) = 2(\gamma + 1) \eta_{\alpha\gamma} \nabla^{\gamma} + \frac{1}{2} \gamma \left( \eta_{\alpha\gamma} \delta_{a}^{\gamma} - \eta_{a\gamma} \delta_{a}^{\gamma} \right) y_{a}^{\mu}y_{a}^{\lambda} \cdot \dot{a}_{a0\gamma} + \]

\[
\frac{1}{5} \dot{a}_{a0\mu} \int d^{3}y_{a} \rho_{a} \left( y_{a}^{\alpha} y_{a}^{\beta} + \frac{1}{2} \gamma \dot{a}_{a0\mu} y_{a}^{\alpha} y_{a}^{\beta} \right) - \frac{1}{5} \dot{a}_{a0\mu} \int d^{3}y_{a} \rho_{a} \left( y_{a}^{\alpha} y_{a}^{\beta} + \frac{1}{2} \gamma \dot{a}_{a0\mu} y_{a}^{\alpha} y_{a}^{\beta} \right) - \frac{1}{2} \delta_{a\beta} \partial y_{a}^{\alpha} \left( \frac{\partial U}{\partial y_{a}^{\alpha}} \right) + \frac{1}{2} \delta_{a\beta} \left( \frac{\partial U}{\partial y_{a}^{\alpha}} \right)^{2} - \int d^{3}y_{a} \rho_{a} \sum_{l \geq 3} \left[ \partial_{0}Q_{a\{l\}}^{\alpha}(y_{a}^{0}) \partial_{0\mu}y_{a}^{(L)} + 2 \partial_{0}Q_{a\{l\}}^{\alpha}(y_{a}^{0}) \partial_{\mu}y_{a}^{(L)} + \right.
\]

\[
+ (\dot{\rho}_{a} \gamma^{\mu} \nu^{\lambda} - \gamma^{\mu\nu}) \sum_{l \geq 3} Q_{a\{l\}}^{\alpha}(y_{a}^{0}) \partial_{\mu}y_{a}^{(L)} - \dot{\rho}_{a} \frac{\partial U(y_{a}^{P})}{\partial y_{a}^{\alpha}} \sum_{l \geq 3} Q_{a\{l\}}^{\alpha}(y_{a}^{0}) \partial_{\mu}y_{a}^{(L)} + O(c^{-6}).
\]
\[ + \sum_{l \geq 3} \left[ \eta_{\alpha \lambda} \partial_0 Q^\lambda_{a(L)}(y^0_a) + \left( L_{a(L)}(y^0_a) + v_{a0} \cdot Q^\beta_{a(L)}(y^0_a) \right) \frac{\partial}{\partial y^\alpha_a} \right] \cdot y^L_a + \mathcal{O}(|y^r_A|^{k+1}) + \mathcal{O}(c^{-5}), \tag{24b} \]

\[ y^0_{a\beta}(y^p_a) = \eta_{\alpha \beta} \left( 1 + 2\gamma U \right) + \]

\[ + \sum_{l \geq 3} \left[ \eta_{\alpha \lambda} Q^\lambda_{a(L)}(y^0_a) \frac{\partial}{\partial y^\beta_a} + \eta_{\beta \lambda} Q^\lambda_{a(L)}(y^0_a) \frac{\partial}{\partial y^\alpha_a} \right] \cdot y^L_a + \mathcal{O}(|y^r_A|^{k+1}) + \mathcal{O}(c^{-4}), \tag{24c} \]

where the total gravitational potential \( U \) at the vicinity of body \((a)\) is composed of the local Newtonian potential generated by the body \((a)\) itself and the tidal gravitational potential produced by external sources of gravity:

\[ U(y^p_a) = U_a(y^p_a) + \sum_{b \neq a} U_b(y^p_b) - \frac{\partial K_a(y^0_a)}{\partial y^0_a} - \frac{1}{2} \hat{v}_{a\nu} v_{a\mu} = \]

\[ = \sum_b U_b(y^p_b) - \sum_{b \neq a} \left( y^\beta_a \frac{\partial U_b}{\partial y^\alpha_a} \right) + \left\langle U_b \right\rangle_a + \mathcal{O}(c^{-4}), \tag{25} \]

with the Newtonian acceleration of body \((a)\) given by Eq.(23a). This potential is the solution of the Poisson equation in coordinates \((y^p_a)\), namely:

\[ \eta_{\mu \lambda} \frac{\partial^2}{\partial y^\mu_a \partial y^\lambda_a} U(y^p_a) = 4\pi \sum_b \hat{\rho}_b(y^p_b)(1 + \mathcal{O}(c^{-2})). \tag{26a} \]

By satisfying the requirement that the body is at rest in its proper reference frame, we derive the following results:

\[ \left\langle U \right\rangle_a = \int_a d^3 y^\mu_a \hat{\rho}_a(y^p_a) U(y^p_a) = \int_a d^3 y^\mu_a \hat{\rho}_a(y^p_a) U_a(y^p_a) = 2E_a, \tag{26b} \]

\[ \left\langle \frac{\partial U}{\partial y^\mu_a} \right\rangle_a = \int_a d^3 y^\mu_a \hat{\rho}_a(y^p_a) \frac{\partial U(y^p_a)}{\partial y^\mu_a} = 0. \]

The quantity \( V^\alpha(y^p_a) \) in the expressions Eq.(24) is the total vector-potential produced by all the bodies in the system represented in the coordinates \((y^p_a)\) of the reference frame:

\[ V^\alpha(y^p_a) = \sum_b V^\alpha_b(y^p_b) - \sum_{b \neq a} \left( \hat{\theta}_a \left[ \left\langle \partial_\mu V^\alpha_b \right\rangle_a + \left\langle \nu^\alpha_\mu \partial_\mu U_b \right\rangle_a \right] + \left\langle V^\alpha_b \right\rangle_a \right) + \]

\[ + \frac{1}{10} \left( 3y^\alpha_a y^\lambda_a - \eta^\alpha_\lambda y^\mu_a y^\mu_a \right) \hat{a}_{a\alpha\lambda} + \mathcal{O}(c^{-4}). \tag{27} \]

This potential satisfies the equation

\[ \eta_{\mu \lambda} \frac{\partial^2}{\partial y^\mu_a \partial y^\lambda_a} V^\alpha(y^p_a) = -4\pi \sum_b \hat{\rho}_b(y^p_b) v^\alpha(y^p_b)(1 + \mathcal{O}(c^{-2})). \tag{28} \]
The form of the solution for $\nabla V^{\alpha}$ Eq.(27) is chosen in such a way that it satisfies the Newtonian equation of continuity written in the proper reference frame: $\partial_0 \rho_0(y_a^p) + \partial_\mu [\rho_0(y_a^p) \nu^\mu(y_a^p)] = O(c^{-3})$, which provides the following Newtonian-like relation between these quantities:

$$\frac{\partial \rho}{\partial y_a^0} = \frac{\partial \nabla V^\mu}{\partial y_a^\mu}.$$

Another quantity we have introduced in the formulae Eq.(24) is $W(y_a^p)$. This is the post-Newtonian contribution to the component $g_{00}$ of the effective metric tensor in the coordinates $(y_a^p)$ of the proper reference frame and it is given as:

$$W(y_a^p) = \sum_b W_b(y_b^p(y_a^p)) - \sum_{b \neq a} \left( y_a^\mu \left( \frac{\partial W_b}{\partial y_a^\mu} \right)_a + \left( W_b \right)_a \right) + O(c^{-6}),$$

(29)

with the expressions for the functions $W_a$ and $W_b$ given as follows:

$$W_a(y_a^p) = \beta U_a^2(y_a^p) + \Psi_a(y_a^p) + 2a_0^a \cdot \frac{\partial}{\partial y_a^0} \chi_a(y_a^p) + \frac{1}{2} \frac{\partial^2}{\partial y_a^0 \partial y_a^0} \chi_a(y_a^p) +$$

$$+ \sum_{b \neq a} \left( 2 \beta U_a(y_a^p) U_b(y_a^p) - (3\gamma + 1 - 2\beta) \int_a \frac{d^3 y_b^\rho}{|y_b^\rho - y_a^\rho|} \rho_a(y_a^0, y_b^\rho) U_b(y_a^0, y_b^\rho) \right) +$$

$$+ \sum_{l \geq 3} Q_a^l \left( L \right)(y_a^0) \int_a d^3 y_a^\rho \rho_a(y_a^0, y_a^\rho) \frac{\partial}{\partial y_a^0} \left[ y_a^L - y_a^L \right] + O(|y_a^{\nu}|^{k+1}) + O(c^{-6}).$$

(30a)

$$W_b(y_b^p) = \beta U_b(y_b^p) \sum_{c \neq a} U_c(y_c^p) + \Psi_b(y_b^p) + 2a_0^b \cdot \frac{\partial}{\partial y_a^0} \chi_b(y_b^p) + \frac{1}{2} \frac{\partial^2}{\partial y_a^0 \partial y_a^0} \chi_b(y_b^p) -$$

$$- (3\gamma + 1 - 2\beta) \int_b \frac{d^3 y_a^\rho}{|y_a^\rho - y_b^\rho|} \rho_b(y_a^0, y_a^\rho) \sum_{\nu} U_b(y_a^0, y_a^\nu) +$$

$$+ \sum_{l \geq 3} Q_b^l \left( L \right)(y_b^0) \int_a d^3 y_a^\rho \rho_b(y_a^0, y_a^\nu) \left( y_b^L - y_b^L \right) +$$

$$+ O(|y_a^{\nu}|^{k+1}) + O(c^{-6}).$$

(30b)

The functions $W_a$ and $W_b$ fully represent the non-linearity of the total post-Newtonian gravitational field in the proper reference frame. These functions contain the contributions of two sorts: (i) the gravitational field produced by the external $(b \neq a)$ bodies in the system, and (ii) the field of inertia caused by the accelerated (the terms with $a_0^a$) and non-geodesic motion (due to the coupling of the proper multipole moments of the body $(a)$ with the external gravitational field and the self-action contributions both given by the terms with $Q_a^L \left( L \right)$ in the expressions Eq.(30)) of the proper reference frame.

4.2 The Fermi Normal Coordinates and the Equations of the Spacecraft Motion.

The expressions Eq.(24) are the general two-parametric solution for the field equations of general relativity and scalar-tensor theories of gravity in WFSMA. These expressions satisfy the generalized Fermi conditions Eq.(18) at the immediate vicinity of body $(a)$. This solution reflects the geometrical features of the proper reference frame with respect to the special properties of the motion of the $k$-th
multipoles of the extended bodies. In order to find the unknown functions $Q^α_α(y^0_a)$ and $L_α(y^0_a)$ up to the $k$-th $(k \geq 3)$ order, one should use the conditions which will contain the spatial derivatives from the metric tensor to the $(k - 1)$ order. Thus, following Synge [63] in addition to the Fermi-Walker transport law [35], one may apply the Fermi-Walker transport law [35]. Another possible method is to use the ‘external’ multipole moments as defined for the gravitational wave theory ([36]-[38]). However, one can show that in WFSMA the functions $Q^α_α(y^0_a)$ and $L_α(y^0_a)$ may be chosen in such a way that the metric tensor Eq.(24) in a proper reference frame will take the form corresponding to any of these multipolar expansions. One can show that this happens because of the ‘external’ multipole moments in the transformation functions, and that this simply corresponds to the choice of coordinates of the reference frame with specific dynamical properties. Thus, for example, one can construct the proper reference frame for the Fermi normal coordinates [64]. Note that these conditions require corrections up to the third order with respect to the spatial coordinates in the transformation functions $Q^α_α$ and $L_α$ ([32], [50], [66]-[69]). The necessary corrections to the functions Eq.(22) have the following form [33]:

$$Q^α_α(y^0_a) = 6 \sum b \not= a \ y^α_b y^β_a \partial_α y^β_a \cdot \left( \eta^{γλ} \eta_{μν} \left( \partial_β U_b^γ \right)_a - 2 \delta^γ_β \left( \partial_μ^2 U_b^γ \right)_a \right) + \mathcal{O}(y^α_a |^4),$$ (31a)

$$L_α(y^0_a) = \sum b \not= a \ y^α_b y^β_a \partial_α y^β_a \cdot \left( \gamma \eta_{μν} \partial_γ U_b^φ \right)_a - 2(γ + 1) \left( \partial_μ^2 V_β \right)_a - γ v_{ασ} \left[ \eta^{σλ} \eta_{μν} \left( \partial_β^2 U_b^λ \right)_a - 2 \delta^γ_β \left( \partial_μ^2 U_b^γ \right)_a \right] + \mathcal{O}(y^α_a |^4).$$ (31b)

Given this form for the corrections, the metric tensor Eq.(24) will take the form corresponding to generalized Fermi normal coordinates chosen in the proper reference frame:

$$g^F_{α0}(y^0_a) = 1 - 2u(y^0_a) + 2w(y^0_a) +$$

$$+ \left( \sum b \not= a \left[ \left( \partial_μ^2 W_b^γ \right)_a + \partial_μ \left[ \gamma \eta_{μν} \partial_γ U_b^φ \right]_a \right] - (γ + 1) \left( \partial_μ V_b^σ \right)_a \right) +$$

$$+ γ \eta_{μν} a_{αλ} a^λ_α - (2γ - 1) a_{αμ} a_{αν} \right) \cdot y^α_b y^β_a + \mathcal{O}(c^{-6}) + \mathcal{O}(y^α_a |^3),$$ (32a)

$$g^F_{αβ}(y^0_a) = 2(γ + 1) \eta_{αβ} V^σ_a(y^0_a) + \frac{2}{3} \left( γ \left( \eta_{αμ} a_{αν} - \eta_{αν} a_{αμ} \right) +$$

$$+ (γ + 1) \sum b \not= a \left[ \eta_λ \left( \partial^2_μ V^λ_b \right)_a - \eta_β \left( \partial^2_μ V^β_b \right)_a \right] \right) \cdot y^α_b y^β_a + \mathcal{O}(y^α_a |^3) + \mathcal{O}(c^{-5}),$$ (32b)

$$g^F_{αβ}(y^0_a) = \eta_{αβ} \left( 1 + 2γ U_b^γ(y^0_a) \right) + \frac{1}{3} \sum b \not= a \left[ \eta_λ \left( \partial^2_μ U_b^λ \right)_a + \eta_μ \left( \partial^2_μ U_b^μ \right)_a \right] +$$

$$- \eta_β \left( \partial^2_μ U_b^β \right)_a - \eta_μ \left( \partial^2_μ U_b^μ \right)_a \right) \cdot y^α_b y^β_a + \mathcal{O}(y^α_a |^3) + \mathcal{O}(c^{-4}),$$ (32c)

with the corresponding equation for $a^α_α$ given by Eq.(23a). Thus we have obtained the form of the metric tensor in Fermi normal coordinates and the coordinate transformations leading to this form. These transformations are defined up to the third order with respect to the spatial coordinates. This

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12 With $δ^α_α$ is being the Kronecker symbol and $η^{μν} η_{λν} = δ^α_λ$.  

19
accuracy is sufficient to analyze the motion of a spacecraft in orbit around Mercury consistent with parameterized relativistic gravity.

We now obtain the equations of the spacecraft motion in a Hermean-centric reference frame. To do this, we consider a Riemann space-time whose metric coincides with the metric of N moving extended bodies. We study the motion of a point body in the neighborhood of body \((a)\). The expression for the acceleration of the point body \(a^α_{(0)}\) can be obtained in two ways: either by using the equations of geodesics of Riemann space-time \(du^α/\partial y^β = 0\) or by computing the acceleration of the center of mass of the extended body and then letting all quantities characterizing its internal structure and proper gravitational field tend to zero. In either case one obtains the same result. While the full derivation of the equations of motion is given in the Appendix C, we present here the restricted version of the equations \((C4)\) which is consistent with the expected accuracy for ESA’s Mercury Orbiter mission. This limited accuracy permits us to completely neglect contributions proportional to the spatial coordinates \(y^α_a\). The planeto-centric equations of satellite motion around Mercury can be represented by a series in \(1/|y_{ba0}|\) as follows

\[
a^α_{(0)} = -\eta^αμ \left( \frac{\partial U_μ}{\partial y^α} + \sum_{b\neq a} \left( \frac{\partial U_b}{\partial y^α} - \left( \frac{\partial U_b}{\partial y^α} \right)_a \right) \right) + \delta_a a^α_{(0)} + \\
+ \sum_{b\neq a} \left( (4\beta - 3\gamma - 1) \frac{y_{ba0}}{y^2} + 2(\beta - 1) \frac{y_{ba0}}{y} \frac{N^μ_{ba0}}{y} \right) \left( \frac{\partial U_b}{\partial y^α} \right) + \frac{y_{ba0}}{y} N^μ_{ba0} (\delta^α + n^α n^μ) + \\
+ \frac{y_{ba0}}{y} P^μλ \left[ (2\beta + \frac{5}{3}\gamma) \eta^μn^λ + (\beta - \frac{1}{6}) n^α n^ε n^λ \right] \right) + O(|y^α_a|) + O(c^{-6}),
\]

(33)

where \(P^αλ = \eta^αλ + 3N^α_{ba0} N^λ_{ba0}\) is the polarizing operator, subscript \((a)\) denotes the planet Mercury and the post-Newtonian acceleration \(\delta_a a^α_{(0)}\) is due to the gravitational field of Mercury only. This term is not new \([7]\) and it is given by the expression \((C5a)\). Note that many relativistic terms in the geodetic equations have canceled out and, as a result, the equations of motion of a spacecraft around Mercury takes a very simple form Eq.(33). At this point we have all the necessary equations in order to discuss gravitational experiments with the future Mercury Orbiter mission.

5 Gravitational Experiment for Post 2000 Missions

Mercury is the closest to the Sun of all the planets of the terrestrial group and because of its unique location and orbital parameters, it is well suited to relativistic gravitational experiments. The short period of its solar orbit allows experiments over several orbital revolutions and its high eccentricity and inclination allow various effects to be well separated. In this Section we will discuss the possible gravitational experiments for the Mercury Orbiter mission. Analysis performed in this Section is directed towards the future mission, so we will show which relativistic effects may be measured and how accurately.

By means of a topographic Legendre expansion complete through the second degree and order, the systematic error in Mercury radar ranging has been reduced significantly \([7]\). However, a Mercury Orbiter is required before significant improvements in relativity tests become possible. Currently, the precession rate of Mercury’s perihelion, in excess of the 530 arcsec per century \(('/cy)\) from planetary perturbations, is 43.13 \(('/cy)\) with a realistic standard error of 0.14 \(('/cy)\) \([2]\). After taking into account a small excess precession from solar oblateness, the later authors find that this result is consistent with general relativity. Pitjeva \([3]\) has obtained a similar result but with a smaller estimated error of 0.052 \(('/cy)\). Similarly, attempts to detect a time variation in the gravitational constant \(G\) using
Mercury’s orbital motion have been unsuccessful, again consistent with general relativity. The current result is \( \dot{G}/G = (4.7 \pm 4.7) \times 10^{-12} \text{ yr}^{-1} \).

Metric tests utilizing a *Mercury Orbiter* have been studied both at JPL and at the Joint Institute for Laboratory Astrophysics (JILA) and the University of Colorado. The JPL studies, conducted in the 1970’s, assumed that orbiter tracking could provide daily measurements (normal points) between the Earth and Mercury centers of mass with a 10 m standard error. A covariance analysis was performed utilizing a 16-parameter model consisting of six orbital elements for Mercury and Earth respectively, the relativistic gravity parameters \( \beta \) and \( \gamma \), the solar quadrupole moment \( J_{2\odot} \), and the conversion factor AU between unit distance (Astronomical Unit) and the distance in meters between Earth and Mercury. It was assumed that no other systematic effects were present, and that the normal-point residuals after removal of the 16-parameter model would be white and Gaussian. The total data interval, assumed equal to two years, corresponded to 730 measurements. Under the assumed random distribution of data, the error on the mean Earth-Mercury distance was \( 10/\sqrt{730} = 37 \text{ cm} \). The JPL studies showed, based on a covariance analysis, that the primary metric relativity result from a *Mercury Orbiter* mission would be the determination of the parameter \( \gamma \), which describes the amount of spatial curvature caused by solar gravitation. The standard error was 0.0006, about a factor of two improvement over the Viking Lander determination. This accuracy reflected the effect of spatial curvature on the propagation of the ranging signal and also its effect on Mercury’s orbit, in particular the precession of the perihelion. The error in the metric parameter \( \beta \) and the error in the solar \( J_{2\odot} \) were competitive with current results, but not significantly better.

Within the last five years, a more detailed covariance analysis by the JILA group assumed 6 cm ranging accuracy over a data interval of two years, but with only 40 independent measurements of range. Unmodeled systematic errors were accounted for with a modified worst-case error analysis. Even so, the JILA group concluded that a two-order of magnitude improvement was possible in the perihelion advance, the relativistic time delay, and a possible time variation in the gravitational constant \( G \) as measured in atomic units. However, the particular orbit proposed by ESA for its 2000 Plus mission was not analyzed. It is almost certain that the potential of the ESA mission lies somewhere between the rather pessimistic JPL error analysis and the JILA analysis of an orbiter mission more nearly optimized for relativity testing.

In order to study the relativistic effects in the motion of the *Mercury Orbiter* satellite, we separate these effects into the three following groups:

(i). The effects due to Mercury’s motion with respect to the solar system barycentric reference frame.

(ii). Effects in the satellite’s motion with respect to the Hermean-centric reference frame.

(iii). Effects due to the dragging of the inertial frames.

The effects of the first group are standard and all of them may be obtained directly from the Lagrangian function Eq.(6) or from the equations of motion Eqs.(B1-B7). The effects of the second group can be discussed based on the equations (33). And finally, the effects of the last group can be discussed based on the coordinate transformation rules given by Eq.(22). In the last case, however, we employ a simplified version of these transformations, due to the limited expected accuracy (~ 1 m) of the Mercury ranging data. Thus, in the future discussion we will use the following expression for the temporal components:

\[
\begin{align*}
x^0(y^\mu_a, y^\mu_a) &= y^0_a + c^2 \left( \int y^0_a \left[ \sum_{b \neq a} m_b \frac{y^\lambda_b}{y^\lambda_a} \left( 1 + (I^\lambda_a + I^\lambda_b) \frac{N^\lambda_a \cdot N^\lambda_b}{y^\lambda_a} \right) - \right] \right. \\
& \quad \left. - \frac{1}{2} v^\rho_a \mu \nu_a \left[ dt' - v^\rho_a \nu_a y^\mu_a \right] + O(c^{-4}) \right) 
\end{align*}
\] (34a)
where $I_{c}^{\mu
u}$ represents the intrinsic quadrupole moments Eq.(7) of the bodies. The corresponding expression for the spatial components of the coordinate transformation is given by:

$$x^{\alpha}(y^{0}, y^{\mu}) = y^{\alpha}_{a}(y^{0}) + y^{\alpha}_{a} + c^{-2}\left(\int y^{0}_{a} \Omega^{\alpha\mu}(t') dt' - \frac{1}{2} y^{\alpha}_{a} y^{\mu}_{a} - \gamma y^{\alpha}_{a} \sum_{b \neq a} \frac{mb}{y_{ba}} - \right)$$

$$- \sum_{b \neq a} \frac{mb}{y_{ba}} \left[y^{\alpha}_{a} y^{\beta}_{a} N^{\mu}_{ba} - \frac{1}{2} y^{\alpha}_{a} y^{\mu}_{a} N^{\alpha}_{ba} - w^{\alpha}_{a}(y^{0})\right] + O(|y_{a}|^3) + O(e^{-4}),$$

(34b)

with the precession angular velocity tensor $\Omega^{\alpha\beta}$ given as follows:

$$\Omega^{\alpha\beta}(y^{0}) = \sum_{b \neq a} \left[(\gamma + \frac{1}{2}) \frac{mb}{y_{ba}} N^{[\alpha\beta]}_{ba} - (\gamma + 1) \frac{mb}{y_{ba}} N^{[\alpha\beta]}_{ba} +

+(\gamma + 1) \frac{mb}{2y_{ba}} \left(S^{[\alpha\beta]}_{\lambda} + S^{[\beta\lambda]}_{a}\right)\right],$$

(34c)

where $S^{\mu\nu}_{c}$ is the intrinsic spin moments Eq.(8) of the bodies.

### 5.1 Mercury’s Perihelion Advance.

Based on Mercury’s barycentric equations of motion one can study the phenomenon of Mercury’s perihelion advance. The secular trend in Mercury’s perihelion advance depends on the linear combination of the PPN parameters $\gamma$ and $\beta$ and the solar quadrupole coefficient $J_{2\odot}$ (I3, 75, 76):

$$\dot{\pi} = (2 + 2\gamma - \beta) \frac{\mu_{\odot} n_{M}}{a_{M}(1 - e_{M}^{2})} + \frac{3}{4} \left(\frac{R_{\odot}}{a_{M}}\right)^{2} \frac{J_{2\odot} n_{M}}{(1 - e_{M}^{2})^{2}} (3\cos^{2} i_{M} - 1), \ \"/cy$$

(35a)

where $a_{M}, n_{M}, i_{M}$ and $e_{M}$ are the mean distance, mean motion, inclination and eccentricity of Mercury’s orbit. The parameters $\mu_{\odot}$ and $R_{\odot}$ are the solar gravitational constant and radius respectively.

For Mercury’s orbital parameters one obtains:

$$\dot{\pi} = 42^{\prime}98 \left[\frac{1}{3} (2 + 2\gamma - \beta) + 0.296 \times J_{2\odot} \times 10^{4}\right], \ \"/cy$$

(35b)

Thus, the accuracy of the relativity tests on the Mercury Orbiter mission will depend on our knowledge of the solar gravity field. The major source of uncertainty in these measurements is the solar quadrupole moment $J_{2\odot}$. As evidenced by the oblateness of the photosphere and perturbations in frequencies of solar oscillations, the internal structure of the Sun is slightly aspherical. The amount of this asphericity is uncertain. It has been suggested that it could be significantly larger than calculated for a simply rotating star, and that the internal rotation rate varies with the solar cycle. Solar oscillation data suggest that most of the Sun rotates slightly slower than the surface except possibly for a more rapidly rotating core. An independent measurement of $J_{2\odot}$ performed with the Mercury Orbiter would provide a valuable direct confirmation of the indirect helioseismology value $(2 \pm 0.2) \times 10^{-7}$. Furthermore, there are suggestions of a rapidly rotating core, but helioseismology determinations are limited by uncertainties at depths below 0.4 solar radii.

The Mercury Orbiter will help us understand this asphericity and independently will enable us to gain some important data on the properties of the solar interior and the features of it’s rotational motion. Preliminary analysis of a Mercury Orbiter mission suggests that $J_{2\odot}$ would be measurable to at best $\sim 10^{-9}$ or about 1% of the expected $J_{2\odot}$ value. This should be compared with the present 10% solar oscillation determination.

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13 It should be noted that the Mercury Orbiter itself, being placed in orbit around Mercury, will experience the phenomenon of periapse advance as well. However, we expect that uncertainties in Mercury’s gravity field will mask the relativistic precession, at least at the level of interest for ruling out alternative gravitational theories.
5.2 The Redshift Experiment.

Another important experiment that could be performed on a Mercury Orbiter mission is a test of the solar gravitational redshift. This would require a stable frequency standard to be flown on the spacecraft. The experiment would provide a fundamental test of the theory of general relativity and the Equivalence Principle upon which it and other metric theories of gravity are based ([13], [81]). Because in general relativity the gravitational redshift of an oscillator or clock depends upon its proximity to a massive body (or more precisely the size of the Newtonian potential at its location), a frequency standard at the location of Mercury would experience a large, measurable redshift due to the Sun. With the result for the function \( K_a \) given by Eq.(22a) and Eq.(34a) in hand, one can obtain the corresponding Newtonian proper frequency variation between the barycentric standard of time and that of the satellite (the terms with the magnitude up to \( 10^{-12} \)), given as:

\[
\frac{dx^0}{dy^0(0)} = 1 + \frac{\mu_\odot}{R_M} + \frac{\mu_M}{y(0)} + \frac{1}{2c^2}(\vec{v}_M + \vec{v}(0))^2 - \frac{\mu_\odot}{R_M^3}(\vec{R}_M \vec{y}(0)) + O(c^{-4}),
\]

(36a)

where \( (y^0_0, \vec{y}(0)) \) are the four-coordinates of the spacecraft in the Hermean-centric reference frame and \( \vec{v}(0) \) is the spacecraft orbital velocity. One can see that the eccentricity of Mercury’s orbit would be highly effective in varying the solar potential at the clock, thereby producing a distinguishing signature in the redshift. The anticipated frequency variation between perihelion and aphelion is to first-order in the eccentricity:

\[
(\frac{\delta f}{f_0})_{e_M} = \frac{2\mu_\odot e_M}{a_M}.
\]

(36b)

This contribution is quite considerable and is calculated to be \( (\delta f/f_0)_{e_M} = 1.1 \times 10^{-8} \). Its magnitude, for instance, at a radio-wave length \( \lambda_0 = 3\,\text{cm} \) \( (f_0 = 10\,\text{GHz}) \) is \( (\delta f)_{e_M} = 110\,\text{Hz} \). We would also benefit from the short orbital period of Mercury, which would permit the redshift signature of the Sun to be measured several times over the duration of the mission. If the spacecraft tracking and modelling are of sufficient precision to determine the spacecraft position relative to the sun to 100m (a conservative estimate) then a frequency standard with \( 10^{-15} \) fractional frequency stability \( \delta f/f = 10^{-15} \) would be able to measure the redshift to 1 part in \( 10^7 \) or better. This stability is within the capability of proposed spaceborne trapped-ion [82] or H-maser clocks ([83], [84]).

5.3 The SEP violation effect.

Besides the Nordtvedt effect (for more details see [14] and [85]), there exists an interesting possibility for testing the SEP violation effect by studying spacecraft motion in orbit around Mercury. The corresponding equation of motion is given by Eq.(33). As one can see, the two terms in the second line of this equation vanish for general relativity, but for scalar-tensor theories, they become responsible for small deviations of the spacecraft motion from the fiducial geodesic. Both of these effects, if they exist, are due to non-linear coupling of the gravitational field of Mercury to external gravity. They come from the expression for \( W_a \) given by Eq.(28a), which is the local post-Newtonian contribution to the \( g_{00} \) component of the metric tensor in the proper reference frame.

The first of these terms may be interpreted as a dependence of the locally measured gravitational constant on the external gravitational environment and may be expressed in the vicinity of body \( (a) \) as follows:

\[
G_a = G_0 \left[ 1 - (4\beta - 3\gamma - 1) \sum_{b \neq a} \frac{m_b}{y_{ba0}} \right].
\]

(37)
In the case of a satellite around Mercury, the main contribution to this effect comes from the Sun. Because of the high eccentricity of Mercury’s orbit, the periodic changing of the sun’s local gravitational potential may produce an observable effect, which can be modeled by a periodic time variation in the effective local gravitational constant:

$$\left[ \frac{\dot{G}}{G} \right]_{\text{period}} = (4\beta - 3\gamma - 1) \left[ \frac{\mu_{\odot}}{a_{M}(1 - e_{M}^{2})} \right]^{\frac{3}{2}} c e_{M} \sin \phi(t) \left( 1 + e_{M} \cos \phi(t) \right)^{2},$$

which gives the following estimate for this effect on Mercury’s orbit:

$$\left[ \frac{\dot{G}}{G} \right]_{\text{period}} \approx (4\beta - 3\gamma - 1) \times 1.52 \times 10^{-7} \sin \phi(t) \quad \text{yr}^{-1}. \quad (39)$$

Note that this effect Eq.(39) is fundamentally different from that introduced by Dirac’s hypothesis of possible time dependence of the gravitational constant [7]. As one can see from expression (39), the characteristic time in this case is Mercury’s sidereal period. This short period may be considered as an advantage from the experimental point of view. In addition, the results of the redshift experiment could help in confident studies of this effect. Recently a different combination of the post-Newtonian parameters in the Nordtvedt effect, $\eta = 4\beta - \gamma - 3$, was measured at $\eta \lesssim 10^{-4}$ [3]. This means that, in order to obtain the comparable accuracy for the combination of parameters Eq.(39), one should perform the Mercury gravimetric measurements on the level no less precise than $[\dot{G}/G]_{\text{period}} \approx 10^{-11} \text{ yr}^{-1}$. Recently a group at the University of Colorado has analyzed a number of gravitational experiments possible with future Mercury missions [74]. Using a modified worst case error analysis, this group suggests that after one year of ranging between Earth and Mercury (and assuming a 6 cm rms error), the fractional accuracy of determination of the sun’s gravitational constant $m_{\odot}G$ is expected to be of order $\sim 2.1 \times 10^{-11}$. Moreover, even higher accuracy could be achieved with a Mercury lander as proposed by [74]. This suggests that the experiment for determination of the effect Eq.(39) may be feasible with the Mercury Orbiter mission.

Another interesting effect on the satellite’s orbit may be derived from the Eq.(33) in the form of following acceleration term:

$$\delta \vec{a}_{(0)\text{SEP}} = 2(\beta - 1) \frac{m_{M}m_{\odot}}{yR_{M}^{2}} (\vec{N}_{M} - \vec{n} \vec{N}_{M}),$$

where $R_{M}$ is Mercury’s heliocentric radius-vector and $\vec{N}_{M}$ is the unit vector along this direction. This effect is very small for the orbit proposed for ESA’s Mercury Orbiter mission. However, one can show that there exist two resonant orbits for a satellite around Mercury, either with the orbital frequency $\omega_{(0)}$ equal to Mercury’s sidereal frequency $\omega_{M}$: $\omega_{(0)} \approx \omega_{M}$ or at one third of this frequency $\omega_{(0)} \approx \omega_{M}/3$. For these resonant orbits, the corresponding experiment could provide an independent direct test of the parameter $\beta$.

5.4 The Precession Phenomena.

In addition to the perihelion advance, while constructing the Hermean proper reference frame, one should take into account several precession phenomena included in the transformation function $Q_{a}$ and associated with the angular momentum of the bodies. As one may see directly from Eq.(226) and

\footnote{Note that this combination of PPN parameters differs from that for a similar effect presented in [13]. The reason for this is that, in our case the transformations in the form of Eq.(22) enable us to define the physically adequate transformation rules for the metric tensor between the barycentric and planeto-centric reference frames and, hence, to obtain the correct and complete equations of geodesic motion Eq.(33) in the Hermean-centric reference frame.}
Eq.(34b), besides the obvious special relativistic contributions, the post-Newtonian transformation of the spatial coordinates contains terms due to the non-perturbative influence of the gravitational field. This non-Lotenzian behavior of the post-Newtonian transformations was discussed first by [39] for the case of post-Galilean transformations. Our derivations differ from the latter by taking into account the acceleration of the proper reference frame and by including the infinitesimal precession of the coordinate axes with the angular velocity tensor $\Omega_M^{\alpha \beta}$, given by expression (34c) as follows:

$$\Omega_M^{\alpha \beta} = \sum_{b \neq M} \left[ \left( \gamma + \frac{1}{2} \right) \frac{m_b}{y_{bM_0}} N_{bM_0}^{[\alpha \beta]} \right] M_0 - \left( \gamma + 1 \right) \frac{m_b}{y_{bM_0}} N_{bM_0}^{[\alpha \beta]} +$$

$$+ \left( \gamma + 1 \right) \frac{m_b}{2y_{bM_0}} P_{\lambda}^{[\alpha \beta]} (S_M^{\beta \lambda} + S_b^{\beta \lambda}). \quad (41)$$

where the summation is performed over the other bodies of the solar system. This expression re-derives and generalizes the result for the precession of the spin of a gyroscope $\vec{s}_0$ attached to a test body orbiting a gravitating primary. Previously this result was obtained from the theory of Fermi-Walker transport [13]. Indeed, in accord with Eq.(34b), this spin (or coordinate axes of a proper Hermean reference frame) will precess with respect to a distant standard of rest such as quasars or distant galaxies. The motion of the spin vector of a gyroscope can be described by the relation:

$$\frac{d\vec{s}_0}{dt} = [\vec{\Omega}_M \times \vec{s}_0]. \quad (42)$$

By keeping the leading contributions only and neglecting the influence of the Mercury’s intrinsic spin moment, we obtain from the expression (41) the angular velocity $\vec{\Omega}_M$ in the following form:

$$\vec{\Omega}_M = \left( \gamma + \frac{1}{2} \right) \frac{\mu_\odot}{R_M^3} [\vec{R}_M \times \vec{v}_{M_0}] - \left( \gamma + 1 \right) \frac{\mu_\odot}{R_M^3} [\vec{R}_M \times \vec{v}_\odot] +$$

$$+ \left( \gamma + 1 \right) \frac{\mu_\odot}{2R_M^3} \left( \vec{S}_\odot - 3(\vec{S}_\odot \vec{N}_M)\vec{N}_M \right), \quad (43)$$

where $\vec{v}_{M_0}$ and $\vec{v}_\odot$ are Mercury’s and the Sun’s barycentric orbital velocities and $\vec{S}_\odot$ is the solar intrinsic spin moment.

The first term in Eq.(43) is known as geodetic precession [86]. This term arises in any non-homogeneous gravitational field because of the parallel transport of a direction defined by $\vec{s}_0$. It can be viewed as spin precession caused by a coupling between the particle velocity $\vec{v}_{M_0}$ and the static part of the space-time geometry. For Mercury orbiting the Sun this precession has the form:

$$\vec{\Omega}_G = \left( \gamma + \frac{1}{2} \right) \frac{\mu_\odot}{R_M^3} [\vec{R}_M \times \vec{v}_{M_0}). \quad (44)$$

This effect could be studied for the Mercury Orbiter, which, being placed in orbit around Mercury is in effect a gyroscope orbiting the Sun. Thus, if we introduce the angular momentum per unit mass, $\vec{L} = \vec{R}_M \times \vec{v}_{M_0}$, of Mercury in solar orbit, the equation (44) shows that $\vec{\Omega}_G$ is directed along the pole of the ecliptic, in the direction of $\vec{L}$. The vector $\vec{\Omega}_G$ has a constant part

$$\vec{\Omega}_0 = \frac{1}{2} \left( 1 + 2\gamma \right) \frac{\mu_\odot \omega_M}{a_M} \omega_M t = \frac{1 + 2\gamma}{3} \cdot 0.205 \, "/\text{yr}, \quad (45a)$$

with a significant correction due to the eccentricity $e_M$ of the Mercury’s orbit,

$$\vec{\Omega}_1 \cos \omega_M t = \frac{3}{2} \left( 1 + 2\gamma \right) \frac{\mu_\odot \omega_M}{a_M} e_M \cos \omega_M t_0 = \frac{1 + 2\gamma}{3} \cdot 0.126 \cos \omega_M t_0 \, "/\text{yr}, \quad (45b)$$

25
where $\omega_M$ is Mercury’s siderial frequency, $t_0$ is reckoned from a perihelion passage; $a_M$ is the semimajor axis of Mercury’s orbit.

Geodetic precession has been studied for the motion of lunar perigee and its existence was first confirmed with an accuracy of 10% [87]. Two other groups have analyzed the lunar laser-ranging data more completely to estimate the deviation of the lunar orbit from the predictions of general relativity ([88], [89]). Geodetic precession has been confirmed within a standard deviation of 2%. The precession of the orbital plane proposed for ESA’s *Mercury orbiter* (periherm at 400 km altitude, apher at 16,800 km, period 13.45 hr and latitude of periherm at +30 deg) would include a contribution of order 0.205 $''$/yr from the geodetic precession. We recommend that this precession be included in future studies of the *Mercury Orbiter* mission.

The third term in expression (43) is known as Lense-Thirring precession $\vec{\Omega}_{LT}$. This term gives the relativistic precession of the gyroscope’s spin $\vec{s}_0$ caused by the intrinsic angular momentum $\vec{S}$ of the central body. This effect is responsible for a small perturbation in the orbits of artificial satellites around the Earth ([90], [91]). However, our preliminary studies indicate that this effect is so small for the satellite’s orbit around Mercury that will be masked by uncertainties in the orbit’s inclination.

### 6 Discussion

The use of tracking data from orbiters for relativity tests requires the consideration of more error sources than for landers. This is because of the need to convert from the measured earth-spacecraft distance to the desired earth-planet distance. This involves determining the orbit of the spacecraft about the planetary center of mass, which requires solving from the tracking data for a number of spatial harmonics of the gravitational field, solving for radiation pressure, and other non-gravitational forces. Firing of attitude control jets which produce unbalanced forces are of particular concern. The orbit determination of the Mars orbiter *Mariner 9*, for example, was substantially affected not only by these factors, but also by the fact that the spacecraft was placed on a 12 hr period orbit with low periapsis ([92], [93]). Thus, in order to precisely describe the motion of the *Mercury Orbiter* relative to Earth, one must solve two problems, namely: (i) the problem of the satellite motion about Mercury’s center of mass in the Hermean-centric frame, and (ii) the relative motion of the both planets - Earth and Mercury - in the solar system barycentric reference frame. Our analysis provides a framework for the complete solution of these two problems in terms of the corresponding differential equations.

The formalism presented in this paper addresses the general problem of radio tracking of a *Mercury Orbiter*. We have presented the Hermean-centric equations of the satellite motion, the barycentric equations of the planet’s motion in the solar system barycentric reference frame, and the coordinate transformations which link these equations together. In particular, our analysis shows that in a proper Hermean-centric reference frame, the equations of the satellite motion depend on Mercury’s gravitational field only. This set of equations is available in the form of the motion of test bodies in the isolated gravitational one body problem. The existence of the external gravitational field manifests itself in the form of the usual tidal forces, but it also determines the dynamic properties of the constructed Hermean-centric proper reference frame. Within the accuracy expected for the future *Mercury Orbiter* mission, one can completely neglect the post-Newtonian tidal terms. However, while constructing this reference frame, we derived terms smaller than the expected accuracy of future experiments. Indeed, the last term in equation (33) is due to the coordinate transformation to the Fermi-normal-like reference frame in the planet’s vicinity. One can neglect this term for solar system motion. However, if one applies the results to problems of motion within a more intensive gravitational environment, this term can play a significant role. The application of the results obtained here to problems of motion of double pulsars is currently under study and will be reported later.
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Appendix A: Generalized Gravitational Potentials.

The generalized gravitational potentials for the non-radiative problems in the WFSMA are given as in (13):

\[ U(z^p) = \int \frac{d^3 z^p(z^p)}{|z^p - z^p'|}, \quad V^\alpha(z^p) = - \int d^3 z^p(z^p)v^\alpha(z^p), \]

\[ W^\alpha(z^p) = \int d^3 z^p(z^p)\rho(z^p)(z^p - z^p'), \quad P^\alpha(z^p) = - \int d^3 z^p(z^p)|z^p - z^p'|, \]

\[ A(z^p) = \int d^3 z^p(z^p)\rho(z^p)(\frac{z^p - z^p')^2}{|z^p - z^p'|^3}, \quad \chi(z^p) = - \int d^3 z^p(z^p)|z^p - z^p'|, \]

\[ U^\alpha\beta(z^p) = \int d^3 z^p(z^p)\rho(z^p)(z^\alpha - z^\alpha')(z^\beta - z^\beta'), \quad \Psi(z^p) = -(\gamma + 1)\Phi_1 - (3\gamma + 1 - 2\beta)\Phi_2 - 3\gamma\Phi_3, \]

where the other potentials are given as follows:

\[ \Phi_1(z^p) = - \int d^3 z^p(z^p)\rho(z^p)\rho(z^p), \quad \Phi_2(z^p) = \int d^3 z^p(z^p)^2 \rho(z^p)U(z^p), \]

\[ \Phi_3(z^p) = \int d^3 z^p(z^p)\Pi(z^p), \quad \Phi_4(z^p) = \int d^3 z^p(z^p)^2\rho(z^p)\rho(z^p), \]

\[ \Phi_{ab}(z^p) = \int \int d^3 z^p d^3 z^p\rho(z^p)\rho(z^p)(z^\beta - z^\beta')(z^\gamma - z^\gamma') \frac{(z_{ab} - z_{ab}')}{|z^p - z^p'|^3} - \frac{(z_{ab} - z_{ab}')}{|z^p - z^p'|^3}. \]

In order to indicate the functional dependence in the potentials introduced above, we have used the following notation: \((z^p) \equiv (z^0, z^\nu)\). Then for any function \(f\) one will have: \(f(z^p) = f(z^0, z^\nu)\) and \(f(z^p) = f(z^0, z^\nu)\).

Appendix B: Simplified Barycentric Equations of Motion.

In this Appendix we will present the simplified barycentric equations of motion corresponding the Lagrangian Eq.(6). Assuming that bodies in the system possess the lowest intrinsic multipole moments only, one can obtain the corresponding simplified equations of motion. Thus, with the help of the expressions (6), for an arbitrary body \((a)\) these equations will read as follows:

\[ \ddot{r}^\alpha_a = \sum_{b \neq a} M_b \frac{\ddot{r}^\alpha_a}{r_{ab}} + \sum_{b \neq a} \frac{m_b}{r_{ab}} \left[ \frac{A^\alpha_{ab} + B^\alpha_{ab}}{r_{ab}} + \frac{C^\alpha_{ab}}{r_{ab}} - \frac{\ddot{r}^\alpha_{ab}}{r_{ab}} \left( (2\beta + 2\gamma - 2\tau + 1)m_a + (2\beta + 2\gamma - 2\tau)m_b \right) \right] + \]

\[ - \frac{\ddot{r}^\alpha_{ab}}{r_{ab}} \left( (2\beta + 2\gamma - 2\tau + 1)m_a + (2\beta + 2\gamma - 2\tau)m_b \right) \]

\[ - \frac{\ddot{r}^\alpha_{ab}}{r_{ab}} \left( (2\beta + 2\gamma - 2\tau + 1)m_a + (2\beta + 2\gamma - 2\tau)m_b \right) \]
where, in order to account for the influence of the gravitational binding energy \( E_b \), we have introduced the passive gravitational rest mass \( M_b \) \( [23], [13] \) as follows

\[
M_b = m_b \left( 1 + (3 + \gamma - 4\beta)E_b + O(c^{-4}) \right) .
\]  

(B2)

The unit vector \( n_{ab} \) must also be corrected using the gravitational binding energy and the tensor of the quadrupole moment \( I_{a}^{\alpha\beta} \) of the body (a) under question:

\[
\hat{n}_{ab}^\alpha = n_{ab}^\alpha \left( 1 + (3 + \gamma - 4\beta)E_a + 5n_{ab\lambda}n_{ab\mu} \frac{I_{a}^{\lambda\mu}}{r_{ab}} \right) + 2n_{ab\beta} \frac{I_{a}^{\alpha\beta}}{r_{ab}} + O(c^{-4}) .
\]  

(B3)

The term \( A_{ab}^\alpha \) in the expression (B1) is the orbital term which is given as follows:

\[
A_{ab}^\alpha = v_{ab}^\alpha n_{ab\lambda} \left( v_{a}^\lambda - (2\gamma - 2\tau + 1)v_{ab}^\lambda \right) + \\
+ n_{ab}^\alpha (v_{a\lambda} v_{a}^\lambda - (\gamma + 1 + \tau)v_{ab\lambda} v_{ab}^\lambda - 3\tau(n_{ab\lambda} v_{ab}^\lambda)^2 - \frac{3}{2}(n_{ab\lambda} v_{b}^\lambda)^2) .
\]  

(B4)

The spin-orbital term \( B_{ab}^\alpha \) has the form:

\[
B_{ab}^\alpha = \left( 3 + 2\gamma \right) v_{ab\lambda} (S_a^{\alpha\lambda} + S_b^{\alpha\lambda}) + \frac{1}{2} v_{a\lambda} (S_a^{\alpha\lambda} - S_b^{\alpha\lambda}) + \\
+ \frac{3}{2} (1 + 2\gamma) n_{ab\lambda} v_{ab\beta} \left[ n_{ab}^\beta (S_a^{\alpha\lambda} + S_b^{\alpha\lambda}) - n_{ab}^\alpha (S_a^{\beta\lambda} + S_b^{\beta\lambda}) \right] + \\
+ \frac{3}{2} n_{ab\lambda} \left[ n_{ab}^\alpha (v_{a\beta} S_b^{\beta\lambda} - v_{b\beta} S_a^{\beta\lambda}) + n_{ab\beta} v_{ab}^\beta S_a^{\alpha\lambda} \right] .
\]  

(B5)

The term \( C_{ab}^\alpha \) is caused by the oblateness of the bodies in the system:

\[
C_{ab}^\alpha = 2n_{ab\beta} I_{b}^{\alpha\beta} + 5n_{ab}^\alpha n_{ab\lambda} n_{ab\mu} I_{b}^{\lambda\mu} .
\]  

(B6)

And, finally, the contribution \( D_{abc}^\alpha \) to the equations of motion Eq.(B1) of body (a), caused by the interaction of the other planets \( \left( b \neq a, c \neq a, b \right) \) with each other is presented as:

\[
D_{abc}^\alpha = \frac{n_{ab}^\alpha}{r_{ab}^3} \left( \frac{1}{2} - 2\beta \right) \frac{1}{r_{bc}} - 2(\beta + \gamma) \frac{1}{r_{ac}} + \\
+ \frac{\Pi^{\alpha\lambda}}{r_{ab}^3} (n_{bc\lambda} + n_{ca\lambda}) + \frac{n_{ab\lambda} \Lambda^{\alpha\lambda}}{r_{ab}^3} \frac{1}{r_{ac}} + \frac{1}{2} (1 + 2\tau) \frac{n_{bc\lambda} \Lambda^{\alpha\lambda}}{r_{bc}^3} \frac{1}{r_{ac}} + 2(1 + \gamma) \frac{n_{bc\alpha}}{r_{bc}^3} \frac{1}{r_{ab}^2} .
\]  

(B7)

where \( \Lambda^{\mu\nu} = \eta^{\mu\nu} + n_{ab}^\mu n_{ab}^\nu \) and \( \Pi^{\mu\nu} = \eta^{\mu\nu} + 3n_{ab}^\mu n_{ab}^\nu \).
expected accuracy of the orbit determination for the future Mercury Orbiter mission, one may neglect the post-Newtonian perturbations caused by the other planets on the Mercury’s orbital motion in the terms $A_{ab}^\alpha, B_{ab}^\alpha, C_{ab}^\alpha, D_{ab}^\alpha$, leaving only the solar contribution. The set of the resulting equations now defines the properties of the motion of the Mercury in the solar system barycentric reference frame.

Appendix C: Equations of the Spacecraft Motion.

In this Appendix we will study the motion of the spacecraft around the planet Mercury. In order to obtain the Hermean-centric equations of the satellite motion we will write out the equations of geodesics to the required degree of accuracy. For $n = \alpha$ we have:

$$\frac{du^\alpha}{ds} + \Gamma_{0\beta}^\alpha u^0 u^\beta + 2\Gamma_{0\beta}^\gamma u^0 u^\gamma + \Gamma_{\mu\beta} u^\mu u^\beta = \mathcal{O}(c^{-6}).$$  \hspace{1cm} (C1)

We consider the metric tensor of Riemann space-time to be given by the expressions (30) in this case. It is then possible to find the connection components of this space-time needed for subsequent computations:

$$\Gamma_{0\beta}(y_a^\mu) = \eta^{\alpha\lambda} \left[ \frac{\partial U}{\partial y_\lambda^a} - \frac{\partial W_a}{\partial y_\lambda^a} - \gamma \frac{\partial U^2_a}{\partial y_\lambda^a} \right] + 2(\gamma + 1) \frac{\partial V_a^\alpha}{\partial y_\lambda^a} +$$

$$+ (2\gamma - 1) a_{a\alpha} a_{a\lambda} - \gamma \delta_{\mu}^\alpha \cdot a_{a\lambda} - \eta^{\alpha\beta} \sum_{b\neq a} \left[ \left( \delta_{\alpha}^\gamma W_b \right)_a + 2\gamma U_a \left( \delta_{\gamma}^\beta U_b \right)_a \right] +$$

$$+ \sum_{b\neq a} \frac{\partial}{\partial y_a^\mu} \left[ (\gamma + 1) \left( \delta_{\alpha}^\gamma V_b \right)_a - \delta_{\gamma}^\beta \frac{\partial}{\partial y_a^\mu} (U_b)_a \right] \cdot y_b^\mu + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-5}),$$  \hspace{1cm} (C2a)

$$\Gamma_{0\beta}(y_a^\mu) = \gamma \delta_{\mu}^\alpha \frac{\partial U_a}{\partial y_\lambda^a} + (\gamma + 1) \delta_{\beta}^a \frac{\partial V_a^\alpha}{\partial y_\lambda^a} + \gamma \eta_{\beta\mu} \partial_{a0\lambda} y_a^\alpha +$$

$$+ (\gamma + 1) \sum_{b\neq a} \left[ \left( \delta_{\beta}^\gamma V_b \right)_a - \delta_{\alpha}^\gamma \delta_{\beta}^\gamma V_b \right] \cdot y_b^\mu + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-5}),$$  \hspace{1cm} (C2b)

$$\Gamma_{0\beta}(y_a^\mu) = \gamma \left( \delta_{\beta}^\alpha \frac{\partial U_a}{\partial y_\lambda^a} + \delta_{\beta}^\omega \frac{\partial U_a}{\partial y_\lambda^a} - \eta_{\beta\omega} \eta_{\alpha}^{\lambda} \frac{\partial U_a}{\partial y_\lambda^a} \right) +$$

$$+ \frac{1}{3} \gamma \sum_{b\neq a} \left[ \eta_{\beta\mu} \left( \delta_{\alpha}^\gamma \partial_{a} U_b \right)_a + \eta_{\beta\mu} \left( \delta_{\alpha}^\gamma \partial_{a} U_b \right)_a + \delta_{\gamma}^\alpha \left( \delta_{\beta}^\gamma U_b \right)_a + \delta_{\gamma}^\beta \left( \partial_{\beta} U_b \right)_a -$$

$$- 2\eta_{\beta\omega} \left( \delta_{\alpha}^\gamma \partial_{\beta} U_b \right)_a - 2\delta_{\alpha}^\beta \left( \partial_{\beta} U_b \right)_a \right] \cdot y_b^\mu + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-4}).$$  \hspace{1cm} (C2c)

To reduce the equation of geodesic motion Eq.(C1) we shall use these expressions above and the definition for the four-vector of velocity in the form:

$$u^\alpha = \frac{dy_a^\alpha}{dy_b^0} \left( g_{00} + 2g_{0\mu} u^\mu + g_{\mu\lambda} u^\mu u^\lambda \right)^{-1/2}.$$

Then by taking into account that $d/ds = u^0 d/dy_b^0$ (with the components of the three-dimensional velocity vector of the point body denoted as $v_{(0)} = dy_a^\alpha/dy_b^0$) and by using the Newtonian equation of motion of a point body as:

$$a_{(0)}^\alpha = \frac{dv_{(0)}^\alpha}{dy_b^0} = -\eta_{\alpha\mu} \frac{\partial U}{\partial y_a^\mu} + \mathcal{O}(c^{-4}),$$

29
we may make the following simplification:

\[ v_0^\alpha \frac{d \ln u^0}{dy_a^0} = v_0^\alpha \left( \frac{\partial U}{\partial y_a^0} + 2 v_0^\mu \frac{\partial U}{\partial y_a^\mu} + O(c^{-5}) \right). \]

Substituting this relation into the equations of motion (C1), we find the acceleration \( a_0^\alpha \) of the point body:

\[
\begin{align*}
  a_0^\alpha &= -\eta^{\alpha \mu} \frac{\partial U}{\partial y_a^\mu} (1 - 2 \gamma U_a) + \partial^\alpha W_a + \sum_{b \neq a} \left( \partial^\alpha \partial_\mu W_b \right) \cdot y_a^\mu - \\
  &- (2\gamma + 1) v_0^\alpha \frac{\partial U_a}{\partial y_a^0} - 2(\gamma + 1) \frac{\partial V_0^\alpha}{\partial y_a^0} - 2(\gamma + 1) v_0(\mu) \left[ \partial^\mu V_0^\alpha - \partial^\alpha V_0^\mu \right] + \\
  &+ \gamma v_0(\mu) v_0^\mu \left( \partial^\alpha U_a + \frac{2}{3} \sum_{b \neq a} \left( \partial_\alpha \partial_\mu U_b \right) \cdot y_a^\mu \right) - \\
  &- v_0^\alpha v_0^\lambda \left( 2(\gamma + 1) \partial_\lambda U_a + \frac{2}{3} (\gamma + 3) \sum_{b \neq a} \left( \partial_\lambda \partial_\mu U_b \right) \cdot y_a^\mu \right) + \\
  &+ y_a^\mu \left( \gamma \delta_\mu a_0 a_0 a_0^\lambda - (2\gamma - 1) a_0^\alpha a_0 \mu + 2\gamma v_0^\lambda [\eta \lambda \mu a_0^\alpha - \delta_\mu a_0 a_0^\lambda] + \\
  &+ \sum_{b \neq a} \left( \frac{\partial}{\partial y_a^\mu} \left[ \gamma \delta_\mu b_0 b_0 \langle U_b \rangle - (\gamma + 1) \langle \partial_\mu V_b^\alpha \rangle \right] + \\
  &+ 2(\gamma + 1) v_0^\lambda \left( \langle \partial^\alpha \partial_\mu V_0^\lambda \rangle \cdot y_a^\mu \right) \right) \right) + O(|y_a^\nu|^2) + O(c^{-6}).
\end{align*}
\]

(C3)

By expanding all the potentials in Eq. (C3) in power series of \( 1/y_{ba0} \) and retaining terms with \( \sim y^\alpha/|y_{ba0}| \) only to the required accuracy, we then obtain:

\[
\begin{align*}
  a_0^\alpha &= -\eta^{\alpha \mu} \frac{\partial U}{\partial y_a^\mu} + \delta_a a_0^\alpha + \delta_{ab} a_0^\alpha + \delta_{ba} a_0^\alpha + \delta_{bc} a_0^\alpha + O(|y_a^\nu|^2) + O(c^{-6}),
\end{align*}
\]

(C4)

where the post-Newtonian acceleration \( \delta_a a_0^\alpha \) due to the gravitational field of the body \( (a) \) only, may be given as

\[
\begin{align*}
  \delta_a a_0^\alpha &= 2(\gamma + 3) U_a \partial^\alpha U_a - (\gamma + \frac{1}{2}) \partial^\alpha \Phi_{1a} + (2\beta - \frac{3}{2}) \partial^\alpha \Phi_{2a} + \\
  &+ (1 - \gamma) \partial^\alpha \Phi_{4a} + \frac{1}{2} \partial^\alpha A_a - (2\gamma + 1) v_0^\alpha \partial_\mu V_0^\mu + \gamma v_0(\mu) v_0^\mu \partial^\alpha U_a - \\
  &- 2(\gamma + 1) v_0(\mu) v_0^\mu \partial_\mu U_a - 2(\gamma + 1) v_0(\mu) \left[ \partial^\mu V_0^\alpha - \partial^\alpha V_0^\mu \right] - \\
  &- 2(\gamma + 1) \int_A d^3 y_a^\rho \tilde{\rho}_a v_a^\tau \partial_\alpha U_a \left( \frac{y_\alpha^\nu - y_a^\nu}{y_a^\mu - y_a^\nu} \right) - \frac{1}{2} (4\gamma + 3) \int_A d^3 y_a^\nu \frac{\tilde{\rho}_a \partial^\alpha U_a}{|y_a^\nu - y_a^\mu|} + \\
  &+ \frac{1}{2} \int_A d^3 y_a^\nu \tilde{\rho}_a \partial_\mu U_a \left( \frac{y_a^\alpha - y_a^\alpha}{y_a^\mu - y_a^\nu} \right) \frac{1}{|y_a^\nu - y_a^\mu|} + O(c^{-6}).
\end{align*}
\]

(C5a)
The term $\delta_{ab}a_{(0)}^a$ is the acceleration due to the interaction of the gravitational field of the extended body $(a)$ with the external gravitation in the N-body system:

$$\delta_{ab}a_{(0)}^a = \sum_{b \neq a} \left( (4\beta - 3\gamma - 1) \frac{m_am_b}{y_{ba}} \frac{n_a^\mu}{y^2} + 2(\beta - 1) \frac{m_am_b}{y_{ba}} \frac{N_{ba}^\mu}{y^2} (\delta_\mu^a + n^\alpha n_\mu) + \right.$$  

$$+ \frac{m_am_b}{y_{ba}} \mathcal{P}_\lambda \left( (2\beta + 5\beta) \eta^\alpha n^\lambda + (\beta - 1) n^\alpha n^\lambda \right) +$$  

$$+ \frac{m_am_b}{y_{ba}} \left( (2\beta + \gamma - 1) \delta^\alpha_\mu + 2(3\beta + \gamma - 1) N_{ba}^\alpha N_{ba_\mu} \right) y_a^\mu +$$  

$$+ 3\beta \frac{m_am_b}{y_{ba}} |y_a^\mu| n^\nu n^\lambda \left( 2\delta^\alpha_\nu N_{ba_\alpha} + N_{ba}^\alpha (\eta_\nu + 5N_{ba_\alpha} N_{ba_\lambda}) \right) + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-6}). \tag{C5b}$$

Note that the combination of the post-Newtonian parameters in the first term of the expression (C5b) differs from that for the well known Nordtvedt effect ([33, 29]). This may provide an independent test for the parameters involved. The reason that our third term in this expression differs from the analogous term derived in [29] is that, in order to obtain this result Eqs.(C4-C5), we used the consistent definitions for the conserved mass density in the proper reference frame. Moreover, in constructing the Fermi normal coordinates previous authors used the incomplete expressions for the spatial coordinate transformations, which are differing from Eq.(31a). Note that if one decide to use our definitions, the result cited above will take the form of Eq.(C5b). The next term $\delta_{bc}a_{(0)}^a$ is the post-Newtonian acceleration caused by the other bodies in the system on the orbit of the body $(a)$ (the effect of the post-Newtonian tidal forces):

$$\delta_{bc}a_{(0)}^a = \sum_{b \neq a} y_a^\mu \left( - \frac{3}{2} \frac{m_b^2}{y_{ba}} \left( 1 + \frac{2}{3} \frac{14\gamma - 4\beta - 7}{42\gamma - 16\beta - 17} \right) + \right.$$  

$$+ \frac{m_b}{y_{ba}} \mathcal{P}_\mu \left[ (\gamma + 1) v_{ba_\lambda} v_{ba}^\lambda + \frac{3}{2} (v_{ba_\lambda} N_{ba}^\lambda)^2 + (4\beta - \gamma) E_b \right] +$$  

$$+ \frac{m_b}{y_{ba}} \mathcal{P}_\lambda \left[ (\delta^\alpha_\nu v_{ba_\alpha} v_{ba}^\nu - v_{ba_\alpha} v_{ba_\nu} \delta^\lambda_\nu - v_{ba_\lambda} v_{ba_\mu} \eta_{\lambda\mu}) - \right.$$  

$$- \frac{m_b}{y_{ba}} \left( \delta^\alpha_\nu \eta_\lambda - 3N_{ba_\alpha} N_{ba_\mu} N_{ba_\lambda} \right) v_{ba_\lambda} v_{ba}^\lambda +$$  

$$+ \frac{m_b}{y_{ba}} \mathcal{P}_\lambda \left( \frac{2}{3} \eta_{(0)} v_{(0)}^\beta \eta_{(0)}^\alpha \sigma^\lambda_\mu - \frac{2}{3} (\gamma + 3) v_{(0)}^\alpha v_{(0)}^\beta \sigma^\lambda_\mu + \right.$$  

$$+ \frac{2}{3} \eta_{(0)} v_{(0)}^\lambda \left[ \delta_\mu^a \sigma^\beta_\nu - \eta_{\beta\mu} \eta^{\alpha\nu} \right] + 2 \eta_{(0)} v_{(0)}^\lambda v_{(0)}^\alpha \delta^\beta_\mu -$$  

$$+ 2(\gamma + 1) v_{(0)}^\beta \delta^\lambda_\mu v_{ba_\beta} \eta_{\alpha\gamma} - v_{ba_\alpha} \delta^\beta_\mu \right] + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-6}). \tag{C5c}$$

Finally, the last term in the expression (C4) $\delta_{bc}a_{(0)}^a$ is the contribution to the equation of motion of the non-linear gravitational interaction of the external bodies with each other given as follows:

$$\delta_{bc}a_{(0)}^a = - \sum_{b \neq c \neq a} y_a^\mu \left( \frac{m_bm_c}{y_{ba} y_{ca}} \left[ (3\gamma - 1) \mathcal{P}_\mu^a N_{ba}^\lambda N_{ca}^\lambda + \right. \right.$$  

$$\left. - 2(\gamma + 1) v_{(0)}^\gamma v_{(0)}^\lambda v_{ba_\gamma} \delta_{(0)}^\beta_\mu - \eta_{\beta\mu} \eta^{\alpha\gamma} - \right. \left. v_{ba_\alpha} \delta^\beta_\mu \right) + \mathcal{O}(|y_a^\mu|^2) + \mathcal{O}(c^{-6}). \tag{C5d}$$
Thus, the equations presented in this Appendix are represent the motion of a test body in the Fermi-normal-like coordinates chosen in the proper reference frame of a body \((a)\). Together with the coordinate transformations Eq.(22) this is the general solution of the gravitational \(N\)-body problem.

### Appendix D: Astrophysical Parameters Used in the Paper.

In this Appendix we present the astrophysical parameters used in the calculations of the gravitational effects for the *Mercury Orbiter* mission in Section V:

- **Solar radius**: \(R_\odot = 695,980\) km,
- **Solar gravitational constant**: \(\mu_\odot = c^{-2}GM_\odot = 1.4766\) km,
- **Solar quadrupole coefficient** (Brown *et al.*, 1989): \(J_{2\odot} = (1.7 \pm 0.17) \times 10^{-7},\)
- **Solar rotation period**: \(\tau_\odot = 25.36\) days,
- **Mercury’s mean distance**: \(a_M = 0.3870984\) AU = \(57.91 \times 10^6\) km,
- **Mercury’s radius**: \(R_M = 2,439\) km,
- **Mercury’s gravitational constant**: \(\mu_M = c^{-2}GM_M = 1.695 \times 10^{-7}\mu_\odot,\)
- **Mercury’s sidereal period**: \(T_M = 0.241\) yr = \(87.96\) days,
- **Mercury’s rotational period**: \(\tau_M = 59.7\) days,
- **Eccentricity of Mercury’s orbit**: \(e_M = 0.20561421,\)
- **Jupiter’s gravitational constant**: \(\mu_J = 9.547 \times 10^{-4}\mu_\odot,\)
- **Jupiter’s sidereal period**: \(T_J = 11.865\) yr,
- **Astronomical Unit**: \(AU = 1.49597892(1) \times 10^{13}\) cm.

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