Uncertainty and certainty relations for quantum coherence with respect to design-structured POVMs

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The concept of quantum coherence and its possible use as a resource are currently the subject of active researches. Uncertainty and complementarity relations for quantum coherence allow one to study its changes with respect to other characteristics of the process of interest. Protocols of quantum information processing often use measurements that have a special inner structure. Quantum designs are considered as tools with a perspective of fruitful applications in quantum information processing. We obtain uncertainty and certainty relations for coherence averaged with respect to a set of design-structured POVMs of rank one. To characterize the degree of coherence, the relative entropy of coherence is utilized. The derived relations are exemplified with quantum designs in the two-dimensional Hilbert space.

Keywords: uncertainty principle, quantum coherence, quantum design, Shannon entropy

I. INTRODUCTION

The principle of state superposition plays a crucial role in quantum mechanics. Various aspects of the problem of coherence have found a considerable attention within historical development of modern physics. It is now clear that complete understanding of this problem could be reached only via a purely quantum approach [1, 2]. Efforts to build resource theory of quantum coherence are a part of more general attempt to resolve the strengths and limitations of non-classical correlations [3]. In order to implement efficient and fault-tolerant quantum computations, an impact of quantum coherence on manipulations with quantum register should be understood. Results of recent investigations have supported this conclusion [4–7].

Complementarity relations for quantum coherence were considered in [8, 9]. Uncertainty relations for quantum coherence were formulated in several ways [10–12]. Since the notion of coherence is basis dependent, complementarity and uncertainty relations of interest should take into account a character of utilized measurements. For measurements with a certain inner structure, we can often improve uncertainty bounds that follow from the results of general scope. Mutually unbiased bases are an especially important example of such measurements used for various purposes [13]. Symmetric informationally complete measurements give another very helpful tool [14]. Due to nice properties, quantum t-designs have been proposed for applications in quantum information science [15, 16].

The aim of this work is to examine uncertainty and certainty relations for quantum coherence with respect to POVMs assigned to a quantum design. The consideration is essentially based on new entropic uncertainty relations recently derived in [32]. The paper is organized as follows. In Sec. II the preliminary facts are recalled. The main findings are presented in Sec. III. The derived uncertainty and certainty relations are illustrated with qubit examples. In Sec. IV we conclude the paper with a summary of the obtained results.

II. DEFINITIONS AND NOTATION

In this section, we will present required material concerning the relative entropy of coherence and some properties of quantum designs. Let $\mathcal{H}$ be $d$-dimensional Hilbert space. We denote the space of linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and the set of positive semidefinite operators by $\mathcal{L}_+(\mathcal{H})$. By $\text{ran}(\mathcal{X})$, we mean the range of operator $\mathcal{X}$. The state of a quantum system is described by the density matrix $\rho \in \mathcal{L}_+(\mathcal{H})$ assumed to be normalized as $\text{tr}(\rho) = 1$. Then the von Neumann entropy is written as

$$S_1(\rho) := - \text{tr}(\rho \ln \rho). \quad (1)$$

A rigorous framework for the quantification of coherence was developed in [1]. To the given orthonormal basis $\mathcal{B} = \{ |b_j\rangle \}$, one assigns the set of all density matrices that are diagonal in this basis, viz.

$$\delta = \sum_{j=1}^d \delta_j |b_j\rangle \langle b_j|. \quad (2)$$

These density matrices form the set $\mathcal{I}_\mathcal{B}$ of states incoherent with respect to $\mathcal{B}$. To quantify the amount of coherence with respect to $\mathcal{B}$, one measures how far the given state is from states of $\mathcal{I}_\mathcal{B}$. The authors of [1] listed general conditions
for quantifiers of coherence. Additional conditions imposed on coherence measures were presented in [2]. The imposed conditions allow us to select proper candidates to quantify quantum coherence. In this paper, the relative entropy of coherence will be utilized.

For density matrices \( \rho \) and \( \omega \), the quantum relative entropy is defined as [17]

\[
D_1(\rho|\omega) := \begin{cases} 
\text{tr}(\rho \ln \rho - \rho \ln \omega), & \text{if } \text{ran}(\rho) \subseteq \text{ran}(\omega), \\
+\infty, & \text{otherwise}.
\end{cases}
\]

(3)

Although the relative entropy cannot be treated as a metric, it is one of widely used measures of distinguishability of quantum states. Following [1], one defines the coherence measure

\[
C_1(\mathcal{B}|\rho) := \min_{\delta \in \mathcal{L}_e} D_1(\rho||\delta).
\]

(4)

The minimization procedure results in the formula [1]

\[
C_1(\mathcal{B}|\rho) = S_1(\rho_{\text{diag}}) - S_1(\rho),
\]

(5)

where the diagonal state

\[
\rho_{\text{diag}} := \text{diag}(\langle b_1|\rho|b_1\rangle, \ldots, \langle b_d|\rho|b_d\rangle).
\]

(6)

We can represent \( S_1(\rho_{\text{diag}}) \) as the Shannon entropy calculated with probabilities \( p_j(\mathcal{B}|\rho) = \langle b_j|\rho|b_j\rangle \):

\[
S_1(\rho_{\text{diag}}) = H_1(\mathcal{B}|\rho) := -\sum_{j=1}^d p_j(\mathcal{B}|\rho) \ln p_j(\mathcal{B}|\rho).
\]

(7)

For basic properties of [1], see the relevant sections of [1, 2]. The relative entropy of coherence seems to be the most justifiable measure. Together with [3], other quantum divergences were considered, including the quasi-entropies of Petz [18]. It is for this reason that we designate the considered entropic quantities by the subscript 1. Coherence quantifiers induced by quantum divergences of the Tsallis type were addressed in [19]. It turned out that such quantifiers do not have a simple form similar to [5]. Coherence monotones based on Rényi divergences were considered in [20–22].

The above definition is related to the case of orthonormal bases. In quantum information science, generalized quantum measurements are an indispensable tool. Let \( \mathcal{M} = \{M_j\} \) be a set of elements \( M_j \in \mathcal{L}_+(\mathcal{H}) \) satisfying the completeness relation

\[
\sum_j M_j = \mathbb{1}_d,
\]

(8)

where \( \mathbb{1}_d \) is the identity operator on \( \mathcal{H} \). These elements form a positive operator-valued measure (POVM) [17]. It is important that the number of possible outcomes can exceed the dimensionality. The question of quantifying quantum coherence beyond the case of orthonormal bases was addressed in [23–25]. In the following, we will deal only with rank-one POVMs, which play very important role in quantum information theory. One of the reasons for their utility was revealed by Davies [22]. Let \( \{\mu_j\}_{j=1}^N \) be a set of sub-normalized vectors that form a rank-one POVM with elements \( M_j = |\mu_j\rangle \langle \mu_j| \). By \( \mu_{ij} \), we mean \( i \)-th component of \( j \)-th vector \( |\mu_j\rangle \) in the calculation basis. Due to [3], rows of the \( d \times N \)-matrix \( [\mu_{ij}] \) are mutually orthogonal. By adding \( (N - d) \) new rows, this matrix can be converted into a unitary \( N \times N \)-matrix. Its columns denoted by \( |\tilde{\mu}_j\rangle \) form an orthonormal basis \( \tilde{\mathcal{B}} \) in \( N \) dimensions. As a block matrix, each column is now written as

\[
|\tilde{\mu}_j\rangle := \begin{pmatrix} \mu_{1j} \\ \vdots \\ \mu_{dj} \end{pmatrix}.
\]

(9)

To the original density matrix, we assign \( \tilde{\rho} = \text{diag}(\rho, 0) \), so that \( \langle \mu_j|\tilde{\rho}|\tilde{\mu}_j\rangle = \langle \mu_j|\rho|\mu_j\rangle \) for \( 1 \leq j \leq d \). Following [22], we characterize the amount of coherence of \( \rho \) with respect to \( \{ |\mu_j\rangle \}_{j=1}^N \) by the quantity

\[
C_1(\mathcal{M}|\rho) := C_1(\tilde{\mathcal{B}}|\tilde{\rho}) = H_1(\tilde{\mathcal{B}}|\tilde{\rho}) - S_1(\tilde{\rho}) = H_1(\mathcal{M}|\rho) - S_1(\rho).
\]

(10)

Thus, the relative entropy of coherence is expressed in terms related to the original space \( \mathcal{H} \) solely. The authors of [24] showed that the relative entropy of coherence does not depend on the choice of a Naimark extension for its definition. The right-hand side of (10) should be modified, when we do not restrict a consideration to rank-one POVMs.
Let us recall some facts about quantum designs. In $d$-dimensional Hilbert space $\mathcal{H}$, we consider lines passing through the origin. These lines form a complex projective space $[15]$. Up to a phase factor, each line is represented by a unit vector $|\phi\rangle \in \mathcal{H}$. The set $\mathbb{D} = \{|\phi_k\rangle : \langle \phi_k | \phi_k \rangle = 1, \ k = 1, \ldots, K\}$ is a quantum $t$-design, when for all real polynomials $\mathcal{P}_t$ of degree at most $t$ it holds that

$$
\frac{1}{K} \sum_{k=1}^{K} \mathcal{P}_t\left( |\langle \phi_j | \phi_k \rangle|^2 \right) = \int \int d\mu(\psi) d\mu'(\psi') \mathcal{P}_t\left( |\langle \psi | \psi' \rangle|^2 \right). \quad (11)
$$

Here, $\mu(\psi)$ denotes the unique unitarily-invariant probability measure on the corresponding complex projective space $[13]$. It follows from the definition that each $t$-design is also a $s$-design with $s \leq t$. In general, $t$-designs in projective spaces were examined in $[27]$. Due to findings of the paper $[28]$, we know that quantum $t$-designs exist for all suitable $t$ and $d$. However, these results do not provide a common strategy to generate designs in all respective cases. In effect, there are several explicit examples to test a theoretical framework.

Quantum designs have interesting formal properties. Let $\Pi^{(t)}_{\text{sym}}$ be the projector onto the symmetric subspace of $\mathcal{H}^\otimes t$. The trace of $\Pi^{(t)}_{\text{sym}}$ gives dimensionality of this symmetric subspace. It holds that $[13]$

$$
\frac{1}{K} \sum_{k=1}^{K} |\phi_k\rangle \langle \phi_k |^{\otimes t} = \mathcal{D}^{(t)}_d \Pi^{(t)}_{\text{sym}}, \quad (12)
$$

where $\mathcal{D}^{(t)}_d$ denotes the inverse of $\text{tr}(\Pi^{(t)}_{\text{sym}})$, namely

$$
\mathcal{D}^{(t)}_d = \left( \frac{d + t - 1}{t} \right)^{-1} = \frac{t! (t-1)!}{(d+t-1)!}. \quad (13)
$$

At the given $t$, we can rewrite $\mathcal{D}^{(t)}_d$ for all positive integers $s \leq t$. Substituting $t = 1$ leads to the formula

$$
\frac{d}{K} \sum_{k=1}^{K} |\phi_k\rangle \langle \phi_k | = \mathds{1}_d. \quad (14)
$$

Thus, unit vectors $|\phi_k\rangle$ allow us to build to a resolution of the identity in $\mathcal{H}$. In principle, there may be several resolutions assigned to the given $t$-design. We cannot list all of them $a \ priori$, without an explicit analysis of $|\phi_k\rangle$. Obviously, one can take the complete set $\mathcal{E}$ consisting of operators

$$
\mathcal{E}_k = \frac{d}{K} |\phi_k\rangle \langle \phi_k |. \quad (15)
$$

Sometimes, $M$ rank-one POVMs $\left\{ \mathcal{E}^{(m)} \right\}_{m=1}^{M}$ can be assigned to the given quantum design. Each of them consist of $\ell$ operators of the form

$$
\mathcal{E}^{(m)}_j = \frac{d}{\ell} |\phi_j^{(m)}\rangle \langle \phi_j^{(m)} |. \quad (16)
$$

The integers $\ell$ and $M$ are connected by $K = \ell M$.

If the state of interest is described by density matrix $\rho$, then the probability of $j$-th outcome is equal to

$$
p_j(\mathcal{E}^{(m)} | \rho) = \frac{d}{\ell} \langle \phi_j^{(m)} | \rho | \phi_j^{(m)} \rangle. \quad (17)
$$

It follows from (12) that for all $s = 2, \ldots, t$ we have $[29]$

$$
\frac{1}{K} \sum_{k=1}^{K} \langle \phi_k | \rho | \phi_k \rangle^s = \mathcal{D}^{(s)}_d \text{tr}(\rho^{\otimes s} \Pi_{\text{sym}}^{(s)}). \quad (18)
$$

Combining (17) with (18) then gives

$$
\sum_{m=1}^{M} \sum_{j=1}^{\ell} p_j(\mathcal{E}^{(m)} | \rho)^s = \left( \frac{d}{\ell} \right)^s \sum_{k=1}^{K} \langle \phi_k | \rho | \phi_k \rangle^s = K \ell^{-s} d^s \mathcal{D}^{(s)}_d \text{tr}(\rho^{\otimes s} \Pi_{\text{sym}}^{(s)}). \quad (19)
$$
When a single POVM is assigned, one has $\ell = K$ and
\[
\sum_{k=1}^{K} p_k (\mathcal{E} | \rho)^s = K^{1-s} d^s D_d^{(s)} \text{tr}(\rho^\otimes s \Pi_{\text{sym}}^{(s)}).
\] (20)

The authors of $[29, 30]$ have answered the question how to express $\text{tr}(\rho^\otimes s \Pi_{\text{sym}}^{(s)})$ as a sum of monomials of the moments of $\rho$. Of course, complexity of such expressions increases with growth of $s$. To avoid bulky expressions in the following, we introduce the two quantities
\[
\tilde{\beta}^{(s)}_\ell (\rho) = \ell^{1-s} d^s D_d^{(s)} \text{tr}(\rho^\otimes s \Pi_{\text{sym}}^{(s)}),
\]
(21)
\[
\tilde{\beta}^{(s)} (\rho) = K^{1-s} d^s D_d^{(s)} \text{tr}(\rho^\otimes s \Pi_{\text{sym}}^{(s)}).
\]
(22)
The latter is obtained from (21) by taking $\ell = K$. The formulas (19) and (20) impose restrictions on measurement statistics for design-structured POVMs. Uncertainty relations in terms of generalized entropies were considered in $[29, 31]$. The paper $[31]$ actually develops the idea originally proposed in $[33]$. Uncertainty and certainty relations in terms of the corresponding Shannon entropies are derived in $[32]$.

III. UNCERTAINTY AND CERTAINTY RELATIONS FOR QUANTUM COHERENCE

In this section, we will derive uncertainty and certainty relations for the relative entropy of coherence averaged with respect to a set of design-structured POVMs. The consideration is essentially based on the results of $[32]$. The principal idea is to use effectively restrictions of the form (19). Before presenting the desired complementarity relations, some definitions from $[32]$ will be recalled. First, we introduce two families of coefficients, namely
\[
a^{(1)}_n = \sum_{r=1}^{n-1} \frac{1}{r}, \quad a^{(s)} = (-1)^{s-1} \sum_{r=s-1}^{n-1} \frac{1}{r} \binom{r}{s-1} \quad (2 \leq s \leq n),
\]
(23)
\[
b^{(1)}_n = \sum_{r=2}^{n-1} \frac{1}{r}, \quad b^{(s)} = \frac{(-1)^{s-1}}{s} \sum_{r=s-1}^{n-1} \frac{1}{r} \binom{r}{s-1} \quad (2 \leq s \leq n).
\]
(24)

These coefficients follow from truncated expansions according to the Taylor formula $[32]$. Another two families of coefficients are represented in terms of coefficients of $n$-th shifted Chebyshev polynomials
\[
c^{(s)}_n = (-1)^{n+s} 2^{2s-1} \left[ 2 \binom{n+s}{n-s} - \binom{n+s-1}{n-s} \right].
\]
(25)

They are used to obtain expansions with flexible coefficients. Such expansions were introduced and motivated by Lanczos $[34]$. Further, we define the coefficients $[32]
\[
\tilde{a}^{(1)}_n = \frac{(-1)^n}{2 n^2} \sum_{s=2}^{n} \frac{c^{(s)}_n}{s-1}, \quad \tilde{a}^{(s)} = \frac{(-1)^{n+1}}{2 n^2} \frac{c^{(s)}_n}{s-1} \quad (2 \leq s \leq n),
\]
(26)
\[
\tilde{b}^{(0)}_n = 1 - \sum_{s=1}^{n} \frac{\tilde{a}^{(s)}_n}{s}, \quad \tilde{b}^{(1)}_n = \tilde{a}^{(1)}_n - 1, \quad \tilde{b}^{(s)}_n = \frac{\tilde{a}^{(s)}_n}{s} \quad (2 \leq s \leq n).
\]
(27)

The following result will be proved in $[31]$. Let $\Upsilon^{(t)}_{K-1} (\beta)$ denote the maximal real root of the algebraic equation
\[
(1 - \Upsilon)^t + (K - 1)^{t-1} \Upsilon^t = (K - 1)^{t-1} \beta.
\]
(28)

For all $k = 1, \ldots, K$ we have $[31]
\[
p_k (\mathcal{E} | \rho) \leq \Upsilon^{(t)}_{K-1} (\tilde{\beta}^{(t)} (\rho)).
\]
(29)

Complementarity relations for the relative entropy of coherence with respect to design-structured POVMs are posed as follows.
Proposition 1 Let \( M \) rank-one POVMs \( \mathcal{E}(m) \), each with \( \ell \) elements of the form \( \{\phi_k\} \), be assigned to a quantum \( t \)-design \( \mathbb{D} = \{ |\phi_k\rangle \}^K_{k=1} \) in \( d \) dimensions. It then holds that

\[
\sum_{s=1}^{t} a_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho) \leq \frac{1}{M} \sum_{m=1}^{M} C_1(\mathcal{E}(m)|\rho) \leq \frac{Y_t}{t} + \sum_{s=1}^{t} b_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho),
\]

\[
\sum_{s=1}^{n} \tilde{a}_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho) \leq \frac{1}{M} \sum_{m=1}^{M} C_1(\mathcal{E}(m)|\rho) \leq \frac{\bar{b}_s^{(0)}}{\ell} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho).
\]

Here \( Y = \min\{ M \mathcal{Y}_{K-1}(\mathcal{E}(m)|\rho) \}, n = \min\{t, 15\} \) and the coefficients are respectively defined by \( \{25\} - \{27\} \).

Proof. Under the same preconditions, it was proved in \( \{32\} \) that

\[
\sum_{s=1}^{t} a_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho) \leq \frac{1}{M} \sum_{m=1}^{M} H_1(\mathcal{E}(m)|\rho) \leq \frac{Y_t}{t} + \sum_{s=1}^{t} b_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y,
\]

\[
\sum_{s=1}^{n} \tilde{a}_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y \leq \frac{1}{M} \sum_{m=1}^{M} H_1(\mathcal{E}(m)|\rho) \leq \frac{\bar{b}_s^{(0)}}{\ell} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y.
\]

Combining \( \{10\} \) with \( \{32\} \) and \( \{33\} \) leads to \( \{30\} \) and \( \{31\} \), respectively.

The statement of Proposition 1 provides two-sided estimates on the averaged relative entropy of coherence taken with respect to a set of design-structured POVMs. It was found in \( \{32\} \) that the two-sided estimates \( \{32\} \) and \( \{33\} \) give better results in application to single assigned POVM. In the case of rank-one POVM \( \mathcal{E} \) with \( K \) elements of the form \( \{15\} \), the results \( \{30\} \) and \( \{31\} \) read as

\[
\sum_{s=1}^{t} a_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho) \leq C_1(\mathcal{E}|\rho) \leq \frac{Y_t}{t} + \sum_{s=1}^{t} b_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho),
\]

\[
\sum_{s=1}^{n} \tilde{a}_s^{(s)} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho) \leq C_1(\mathcal{E}|\rho) \leq \frac{\bar{b}_s^{(0)}}{\ell} \log \frac{1}{p_s^{(s)}}(\rho) - \log Y - S(\rho),
\]

where \( Y = \mathcal{Y}_{K-1}(\mathcal{E}(m)|\rho) \). For the maximally mixed state \( \rho_s = \mathbb{1}_d/d \), the above bounds are saturated. In very deed, the two-sided estimates \( \{32\} \) and \( \{33\} \) are saturated here \( \{32\} \).

We shall apply the above complementarity relations to concrete quantum designs in two dimensions. The authors of \( \{29\} \) gave a short description of these designs in terms of components of the Bloch vector. To represent each \(|\phi_k\rangle\), the Bloch vector comes to one of vertices of some polyhedron. The qubit density matrix is characterized by its minimal eigenvalue \( \lambda \). We also restrict a consideration to the case of single assigned POVM. To avoid bulky legends on figures, the following notation will be used. By “LT-estimate” and “UT-estimate”, we mean the left- and right-hand sides of \( \{34\} \), respectively. They are based on approximation by polynomials, whose coefficients are due to the Taylor scheme. The terms “LCh-estimate” and “UChe-estimate” respectively refer to the left- and right-hand sides of \( \{35\} \). These estimates use polynomials with coefficients linked to coefficients of the shifted Chebyshev polynomials.

First, we consider estimates on coherence with respect to POVM assigned to the 3-design with \( K = 6 \) vertices of octahedron. In Fig. 1 we plot the two lower estimates and the two upper ones as functions of \( \lambda \). The relative entropy of coherence changes in sufficiently restricted diapason. By two dotted lines, we also show values of the relative entropy of coherence for two orientations of the Bloch vector. Let the three coordinate axes pass through vertices of octahedron. One of the used orientations is along a coordinate axis, whereas the other lies in a coordinate plane along quadrant bisector. The two-sided estimate \( \{34\} \) is better for states sufficiently close to the maximally mixed one. The formula \( \{35\} \) gives a stronger bound for pure states and states with moderate mixedness. Near the right least point \( \lambda = 1/2 \), all the curves converge at one point.

The following example concerns the 5-design with \( K = 12 \) vertices forming an icosahedron. The four estimates on the relative entropy of coherence versus \( \lambda \) are shown in Fig. 2. Of course, possible changes of the relative entropy of coherence lie in enough limited diapason. Similarly to the previous example, the two-sided estimate \( \{34\} \) is insufficient for pure states and states with low mixedness. One the other hand, we see wider region in which the result \( \{34\} \) leads to stronger bounds. By two dotted lines, we also show values of the relative entropy of coherence for two orientations of the Bloch vector. Let the \( z \)-axis pass through two opposite vertices and form symmetry axis of icosahedron. Let the \( x \)-axis pass so that one of inclined edges lies in the \( zx \)-plane. One of the used orientations is along the \( z \)-axis,
Estimates on the relative entropy of coherence versus $\lambda$ for the 3-design with 6 vertices.

FIG. 1: Estimates on the relative entropy of coherence versus $\lambda$ for the 3-design with 6 vertices.

Estimates on the relative entropy of coherence versus $\lambda$ for the 5-design with 12 vertices.

FIG. 2: Estimates on the relative entropy of coherence versus $\lambda$ for the 5-design with 12 vertices.

whereas the other is along positive part of the $x$-axis. In this example, the two dotted lines are closer than in Fig. 1. Again, all the curves converge at one point for $\lambda = 1/2$.

Finally, we address the 5-design with $K = 30$ vertices forming an icosidodecahedron. In Fig. 4, the two lower estimates and the two upper ones are shown as functions of $\lambda$. Due to increasing number of vertices, the relative entropy of coherence generally takes larger values than in the two previous examples. Nevertheless, main observations remain the same. By two dotted lines, one shows values for two orientations of the Bloch vector. Let the $z$-axis pass through two opposite pentagons, and let the $x$-axis pass through one of equatorial vertices. One of the used orientations is along the $z$-axis, whereas the other is along positive part of the $x$-axis. Here, the dotted lines are almost indistinguishable. It witnesses that some improvements of uncertainty and certainty bounds may be therein. However, such improvements would give a correction in few percents. This question deserves further investigations.

With growth of $t$, we have seen narrowing of the range in which the relative entropy of coherence varies. It is a reflection of increasing number of imposed restrictions. Applying the two-sided estimates \((34)\) and \((35)\) to other
examples of quantum designs in two dimensions support the above observations. For states with moderate mixedness, the formula (35) should be preferred. Recall that it is based on power expansions with flexible coefficients. For sufficiently mixed states, the two-sided estimate (34) provides better results. This estimate follows from truncated expansions according to the Taylor formula. The interval \( \lambda \in [0, 1/2] \) is divided into two parts unequal generally. The former is where the two-sided estimate (35) is stronger, and the latter is a domain for the use of (34). The second part becomes wider when \( t \) grows.

IV. CONCLUSIONS

We have considered uncertainty bounds on the relative entropy of coherence averaged with respect to a set of design-structured POVMs. The concept of coherence in purely quantum formulation is currently the subject of active researches. It is also well known that generalized quantum measurements are an indispensable tool in quantum information science. One of principal properties of such measurements is that the number of possible outcomes can exceed the system dimensionality. Among generalized measurements, rank-one POVMs form an especially important class. A quantum design consists of unit vectors that enjoy a list of formal properties. In particular, rank-one POVMs can be built of these vectors. Hence, the question of characterizing coherence quantifiers with respect to design-structured POVMs is interesting for several reasons. The derived relations are mainly based on novel uncertainty and certainty relations derived recently in [32]. The inner structure of assigned POVMs leads to sufficiently strong restrictions on measurement statistics. On the other hand, such restrictions allow one to estimate the relative entropy of coherence from below as well as from above. As a result, the relative entropy of coherence with respect to design-structured POVMs varies in enough narrow diapason. The considered examples also allow one to compare two ways to construct two-sided estimates on the entropic functions of interest.

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