Separability of Completely Symmetric States in Multipartite System

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(Dated: January 1, 2019)

Symmetry plays an important role in the field of quantum mechanics. We consider a subclass of symmetric quantum states in the multipartite system $N^\otimes d$, namely, the completely symmetric states, which are invariant under any index permutation. It was conjectured by L. Qian and D. Chu [arXiv:1810.03125] [quant-ph] that the completely symmetric states are separable if and only if it is a convex combination of symmetric pure product states. We prove that this conjecture is true for both the bipartite and multipartite cases. Further we prove that the completely symmetric state $\rho$ is separable if its rank is at most 5 or $N + 1$. For the states of rank 6 or $N + 2$, they are separable if and only if their range contain a product vector. We apply our results to a few widely useful states in quantum information, such as symmetric states, edge states, extreme states and nonnegative states. We also study the relation of CS states to Hankel and Toeplitz matrices.

PACS numbers: 03.67.-a, 03.65.Bz, 89.70.+c

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Quantum entanglement, first recognized by Einstein [1] and Schrödinger [2], plays a crucial role in the field of quantum computation and quantum information. It is key resources of quantum cryptography, quantum teleportation, and quantum key distribution [3]. Therefore, the question of whether a given quantum state is entangled or separable is of fundamental importance. For a given quantum state $\rho$ acting on the finite-dimensional Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, it is said to be separable if it can be written as a convex linear combination of pure product quantum states, i.e.,

$$\rho = \sum_i \lambda_i |x_i, y_i\rangle \langle x_i, y_i|,$$

where $\sum_i \lambda_i = 1$, $\lambda_i > 0$ and $|x_i\rangle$ and $|y_i\rangle$ are the pure states in the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Despite its wide-importance, to find an efficient method to solve this question in general is considered to be NP-hard [4, 5]. Nevertheless, some subclasses of quantum states have been studied. For example, the Positive-Partial-Transpose (PPT) states [6] are proved to be separable if $\text{dim}(\mathcal{H}_1) \cdot \text{dim}(\mathcal{H}_2) \leq 6$ [7, 8]. The Strong PPT (SPPT) states are considered [9, 10]. In particular it has been proved that all the $2 \otimes 4$ SPPT states are separable [11].

The low-rank states are also considered frequently. Kraus et al. [12] have proved that the $2 \otimes N$ supported PPT states of rank $N$ is separable. Horodecki et al. [13] extend the results to the general $M \otimes N$, states that is, all the supported $M \otimes N (M \leq N)$ PPT states of rank $N$ are separable. The results in the general multipartite system are discussed in Refs. [14, 15]. It tells that the supported $N_1 \otimes N_2 \cdots \otimes N_d (N_1 \leq N_d)$ PPT states of rank $N_d$ is separable if they are RFRP [16]. Next, it has been proven that multipartite PPT states of rank at most 3 are separable [17]. They also proved that the rank-4 PPT
states are separable if and only if their ranges contain at least one product vector. Further, the rank-4 entangled PPT states are necessarily supported in the two-qutrit or three-qubit subspaces. However, the separability problem regarding rank-5 states remains unknown. In this paper, we consider completely symmetric (CS) states of rank five. It turns out that these states are separable.

Due to the essential role of symmetry played in the field of quantum entanglement, it is of great importance to explore the properties of the symmetric states. A quantum state is called symmetric (bosonic) if it is invariant under the swap of particles in the symmetric system. It is known that PPT symmetric states of three-qubits are all separable [18]. Moreover, the four-qubit entangled PPT symmetric states have been characterized in Ref. [19]. Further, the existence of entangled PPT symmetric states of five and six qubits has been reported in Refs. [20, 21]. A well-known subclass of symmetric states, diagonal symmetric (DS) states, are proved to be separable if and only if it is PPT [22].

Within the symmetric states, the CS states are the main topic in this paper. They are invariant under any index permutation, so they possess a larger symmetry. We conduct a deep investigation into the properties of the CS states and characterization of the entanglement problem with the low-rank states. We prove that the CS states of rank at most 5 are separable in the multipartite system. We further show that the supported CS states of rank 6 and 7 are separable [18]. Moreover, the four-qubit entangled PPT symmetric states have been characterized in Ref. [19]. Further, the existence of entangled PPT symmetric states of five and six qubits has been reported in Refs. [20, 21]. A well-known subclass of symmetric states, diagonal symmetric (DS) states, are proved to be separable if and only if it is PPT [22].

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The pure state $\rho$ corresponds to the density matrix whose rank is one, i.e.,

$$\rho = |\phi\rangle\langle\phi|,$$

where $|\phi\rangle$ is a vector in the tensor space $\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_d$. In particular, $\rho$ is said to be a pure product state if

$$|\phi\rangle = |x_1, x_2, \ldots, x_d\rangle,$$

where $|x_i\rangle, i = 1, 2, \ldots, d$ are vectors in the spaces $\mathcal{H}_i$, respectively. Then a state $\rho$ is called separable if it is a convex combination of the pure product states. Otherwise, it is said to be entangled.

In the multipartite system, the symmetric (bosonic) space is the subspace which remains invariant under the exchange of the subsystems. Hence, a quantum state is said to be symmetric if its range is contained in the symmetric space. In this case, $\mathcal{H}_i = \mathcal{H}_l, i = 2, 3, \ldots, d$, denoted by $\mathcal{H}$. Particularly, for N-qubit systems, the following Dicke states formulate an orthonormal basis of the symmetric space:

$$|D_{d,k}\rangle = \frac{1}{\sqrt{C_k^d}} P_{\text{sym}}(|0\rangle^k \otimes |d-k\rangle),$$

where $P_{\text{sym}}$ is the projection onto the symmetric subspace, i.e.,

$$P_{\text{sym}} = \sum_{\pi \in S_d} U_{\pi},$$

the sum runs over all the permutation operators $U_{\pi}$ of the N-qubit system and $C_k^d = d! / (k! (d-k)!)$.

The diagonal symmetric state are defined as

$$\rho = \sum_{k=0}^d \lambda_k |D_{d,k}\rangle\langle D_{d,k}|,$$

which is a well-known symmetric state being studied frequently. And it is proved to be separable if it is PPT.

In Ref. [23], the authors considered a class of symmetric states, completely symmetric states (CS states) in the bipartite system. We shall generalize it to the multipartite system. Here the definition of completely symmetric states is given formally as follows.

Suppose $\rho$ is a quantum state in the d-partite system $\mathcal{H}^\otimes d$ and $\dim(\mathcal{H}) = N$, then it can be written as

$$\rho = \sum_{i_1,j_1,\ldots,i_d,j_d=0}^{N-1} \rho_{i_1,j_1,\ldots,i_d,j_d} |i_1,\ldots,i_d\rangle\langle j_1,\ldots,j_d|,$$

where $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ is a natural basis in the space $\mathcal{H}$. For simplicity, let $I_k=(i_1,i_2,\ldots,i_k)$, $J_k=(j_1,j_2,\ldots,j_k)$ be the k-dimensional multi-index, and $I=I_d$, $J=J_d$. Hence $\rho$ can be represented as

$$\rho = \sum_{I,J} \rho_{I,J} |I\rangle\langle J|,$$
Let $\rho$ be a quantum state as in Eq. (8). Then $\rho$ is said to be completely symmetric (CS) if

$$\rho_{\pi(i,j)} = \rho_{I,J},$$

for any index permutation $\pi$.

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for any index permutation $\pi$.

On the other hand, $\rho$ can be represented as a matrix with an $d$-level nested block structure [24],

$$\rho = \begin{bmatrix} \rho_{00} & \cdots & \rho_{0,N-1} \\ \vdots & \ddots & \vdots \\ \rho_{N-1,0} & \cdots & \rho_{N-1,N-1} \end{bmatrix},$$

where each block is an operator acting on the space $\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_d$, i.e.,

$$\rho_{ij} = \sum_{i_2,j_2,\ldots,i_d,j_d} \rho_{i,j,i_2,j_2,\ldots,i_d,j_d} |i_2,\ldots,i_d \rangle \langle j_2,\ldots,j_d|.$$  \hspace{1cm} (12)

Hence we have $\rho_{I,J} = (\rho_{i,J})_{i_2,j_2,\ldots,i_d,j_d} |i_2,\ldots,i_d \rangle \langle j_2,\ldots,j_d|$. The coefficients $\rho_{I,J}$ thus correspond to the density matrix $\rho$ in an intuitive way. Note that, we shall index from 0 instead of 1 in order to be consistent with the standard notations in quantum computation.

Let us, for the moment, restrict ourselves to supported states, i.e., the states are supported on $\mathcal{H}^\otimes d$ [12].

Denote by $A_1, A_2, \ldots, A_d$ the parties of each subsystem. For the CS states, it is easy to check that their reduced states are identical, that is,

$$\rho_{A_k} = \rho_{A_1}, \forall k > 1,$$

where

$$\rho_{A_k} = (\mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{k-1} \otimes \text{Tr} \otimes \mathbb{I}_{k+1} \otimes \cdots \otimes \mathbb{I}_d) \rho$$

$$= \sum_{i_k,j_k=0}^{N-1} \rho_{I,J} \sum_{i_2,j_2,\ldots,i_d,j_d} \rho_{i,J} |\phi \rangle \langle \phi|,$$

$$|\phi \rangle = |i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d \rangle.$$  \hspace{1cm} (14)

Specifically, denote by $\rho_{A_k}$ the reduced state after applying the trace operator on the $i$-th subsystem, where $i \in K$ and $K$ is an index subset. Then $\rho$ is said to be supported on $\mathcal{H}^\otimes d$ if and only if $\mathcal{R}(\rho_{A_{(2,n)}}) = \mathcal{H}$. Note that a CS state $\rho$ is necessarily a positive partial transposed (PPT) state.

Let us recall the definition of bi-separable and fully separable in the multipartite system.

Definition 2. Suppose $\rho$ is a state in the multipartite system $\mathcal{H}^\otimes d$. 

1. $\rho$ is bi-separable if there exists a bi-partition $A : B$ for $\{1,2,\ldots,d\}$ such that $\rho$ is separable in this bipartite system.

2. $\rho$ is said to be fully separable if it can be written as the convex combination of pure product states as in Eq. (1). For simplicity, a separable state will indicate the fully separable one in the multipartite case.

Now, we extend the S-separable states in Ref. [23] to the multipartite system. For the pure state $|\phi \rangle$ in the multipartite system $\mathcal{H}^\otimes d$, it can be represented by the natural basis

$$|\phi \rangle = \sum_I a_I |I \rangle, a_I \in \mathbb{R}. \hspace{1cm} (15)$$

Definition 3. Suppose $|\phi \rangle$ has the form as Eq. (15), then it is said to be symmetric if

$$a_{\pi(I)} = a_I \hspace{1cm} (16)$$

for any index permutation $\pi$.

Note that when the Schmidt rank of real pure state $|\phi \rangle$ is 1, then $|\phi \rangle$ can be written as

$$|\phi \rangle = \lambda |x \rangle^\otimes d, |x \rangle \in \mathbb{R}^N \hspace{1cm} (17)$$

which is called real symmetric pure product state.

For the mixed state, we have:

Definition 4. Suppose $\rho$ is a quantum state in $\mathcal{H}^\otimes d$ system. Then it is said to be symmetrically separable (S-separable) if it is a convex combination of real symmetric pure product states, i.e.,

$$\rho = \sum_{I=0}^{L-1} p_i |x_i \rangle^\otimes d \langle x_i |^\otimes d; x_i \in \mathbb{R}^N. \hspace{1cm} (18)$$

According to our results, we present the relation of the symmetric states and its subclasses by Figure 1 and 2. Here DSS and CSS indicate the diagonal symmetric states and completely symmetric states, respectively.
III. PROPERTIES OF CS STATES

In this section, we investigate the properties of CS and S-separable states.

Recall that in Ref. [23], the authors conjectured that all the CS states are separable if and only if they are S-separable. Now, we extend this conjecture into the multipartite system.

**Conjecture 1.** All the separable CS states in the multipartite system is S-separable.

We will prove that Conjecture 1 is true for any arbitrary separable CS states at the end of this section. To begin with, we show some properties of the CS states. Due to its special structure, it may admit some interesting features. One of them is that CS structure will preserve under some transforms.

The following lemma shows that the CS and S-separable states are invariant under the real invertible local operator (RILO) \( A \in \mathbb{R}^{N \times N} \).

**Lemma 1.** Suppose \( \rho \) is a quantum state in \( \mathcal{H}^{\otimes d} \) system and \( A \) is a RILO on \( \mathcal{H} \). Then

1. \( \rho \) is CS if and only if \( A^{\otimes d} \rho (A^{\otimes d})^T \) are CS.
2. \( \rho \) is S-separable if and only if \( A^{\otimes d} \rho (A^{\otimes d})^T \) are S-separable.

**Proof.** It suffices to prove only one side for the two conclusions. Otherwise, we can consider the state which is applied by the RILO \( (A^{-1})^{\otimes d} \).

First, we prove CS is invariant under the RILO \( A^{\otimes d} \). Suppose \( \rho \) is a CS states as in Eq. (7). Let \( \sigma = (A^{\otimes d}) \rho (A^{\otimes d})^T \) and

\[
A = \begin{bmatrix}
a_{00} & \cdots & a_{0,N-1} \\
\vdots & \ddots & \vdots \\
a_{N-1,0} & \cdots & a_{N-1,N-1}
\end{bmatrix}.
\]

we have,

\[
A|i\rangle = \sum_{i'=0}^{N-1} a_{i'i} |i'\rangle.
\]

Denote by \( I' = (i'_1, i'_2, \ldots, i'_d) \) and \( J' = (j'_1, j'_2, \ldots, j'_d) \).

After calculation, we have,

\[
\sigma = \sum_{I,J,I',J'} \rho_{I,J} \prod_{k,l=1}^{d} a_{i'_k} a_{j'_l} |I'\rangle \langle J'|
\]

\[
= \sum_{I',J'} \left( \sum_{I,J} \rho_{I,J} \prod_{k,l=1}^{d} a_{i'_k} a_{j'_l} \right) |I'\rangle \langle J'|
\]

\[
= \sum_{I',J'} \sigma_{I',J'} |I'\rangle \langle J'|
\]

where

\[
\sigma_{I',J'} = \sum_{I,J} \rho_{I,J} \prod_{k,l=1}^{d} a_{i'_k} a_{j'_l}.
\]

In order to prove \( \sigma \) is CS, we only need to prove \( \sigma_{I',J'} = \sigma_{\pi(I',J')} \) where \( \pi \) is an arbitrary index permutation. Note that

\[
\sigma_{\pi(I',J')} = \sum_{I,J} \rho_{\pi(I),\pi(J)} \prod_{k,l=1}^{d} a_{i'_k} a_{j'_l} = \sigma_{I',J'},
\]

which completes our proof.

It follows that symmetric separability is also invariant under the operation of \( A^{\otimes d} \) by applying \( A \) to each subsystem.

Actually, the CS structure will also hold under some conditions. For example, we have the following useful lemmas.

**Lemma 2.** Suppose \( \rho \) is a CS state in the multipartite system \( \mathcal{H}^{\otimes d} \). If \( d \) is even, and \( A : B \) is a bi-partition of the system \( \mathcal{H}^{\otimes d} \) with \( |A| = |B| \). Then \( \rho \) is also CS under the bi-partition \( A : B \).

Moreover, the reduced CS states still remain CS.

**Lemma 3.** Suppose \( \rho \) is a CS state in the multipartite system \( \mathcal{H}^{\otimes d} \). Then the reduced state \( \rho_{A_K} \) is CS, where \( K \) is a subset of \( \{1, 2, \ldots, d\} \).

**Proof.** It suffices to prove for the case \( K = \{d\} \), i.e., apply the trace operator to the last subsystem. Suppose

\[
\rho = \sum_{I,J} \rho_{I,J} |I,J\rangle \langle I,J|
\]
where $I = (i_1, i_2, \ldots, i_d)$ and $J = (j_1, j_2, \ldots, j_d)$. Then
\[ \rho_{A_d} = \sum_{I_d-1, J_d-1} \left( \sum_{k=0}^{N-1} \rho_{I_d-1,k,J_d-1,k} \right) |I_{d-1}\rangle\langle J_{k-1}|. \] (25)

It is easy to check that the coefficients $\sum_{k=0}^{N-1} \rho_{I_d-1,k,J_d-1,k}$ are also invariant under index permutation. ■

According to Proposition 13 in Ref. [25], the G-invariant states, i.e. invariant under partial transpose, is separable over complex if and only if it is separable over real. Note that the CS states are necessarily G-invariant, therefore, we have the following lemma, which enables us to consider the separability in real case.

**Lemma 4.** Suppose $\rho$ is a CS state. If $\rho$ is separable over the $\mathbb{C}$, then it is separable over $\mathbb{R}$, that is, $\rho$ can be written as a sum of real pure product states.

The conception of reducible states is introduced in Ref. [16], which enables us to consider the properties of irreducible components respectively. Recall the definition of reducible states:

**Definition 5.** Suppose $\rho$ is a state in the $d$-partite system $A_1 : A_2 : \ldots : A_d$. If
\[ \rho = \sigma + \delta, \] (26)
where $\sigma$ and $\delta$ are both quantum states in the $d$-partite system $A_1, A_2, \ldots, A_d$. Then $\rho$ is called $A_k$-reducible if
\[ \mathcal{R}(\sigma_{A_k}) \oplus \mathcal{R}(\delta_{A_k}) = \mathcal{R}(\rho_{A_k}). \] (27)
Otherwise, $\rho$ is said to be $A_k$-irreducible. Specifically, if $\sigma$ and $\delta$ are real operators, we say that $\rho$ is $A_k$-irreducible over real.

It is also proved in Ref. [16] that the invertible local operator (ILO) transforms the reducible states to reducible ones. Thus, we can always find an ILO such that $\sigma$ and $\delta$ have orthogonal local ranges. Moreover, if the two states are real, the ILO can also be chosen as real. It follows that:

**Lemma 5.** If $\rho = \sum_{i} \rho_i$ is a $A_k$-direct sum, i.e., the ranges of $(\rho_i)_{A_k}$ are linearly independent, then $\rho$ is separable (PPT) if and only if every $\rho_i$ is separable (PPT).

For the CS states, we also have following result.

**Lemma 6.** If $\rho = \sum_{i} \rho_i$ is a $A_k$-direct sum and $\rho_i$’s are real, then $\rho$ is CS (S-separable) if and only if every $\rho_i$ is CS (S-separable).

**Proof.** It only needs to prove the necessary part. Suppose $\rho = \sum_{i} \rho_i$ is a $A_k$-direct sum and every $\rho_i$ is real. First, we prove that every $\rho_i$ is CS. By applying a real ILO, we can assume
\[ \mathcal{R}((\rho_1)_{A_1}) = \text{span}\{0, \ldots, |r-1\}\}, \]
\[ \mathcal{R}((\rho_k)_{A_1}) = \text{span}\{|r\}, \ldots, |d-1\}\}, 2 \leq k \leq d. \] (28)

It suffices to prove
\[ \mathcal{R}((\rho_1)_{A_k}) \subset \text{span}\{0, |1\}, \ldots, |r-1\}, 2 \leq k \leq d. \] (29)

Suppose $\rho$ is written in the form of Eq. (8). Denote by $K$ the set $\{0, 1, \ldots, r-1\}$ and $L$ the set $\{r, r+1, \ldots, d\}$, then
\[ \rho_{I,J} = \begin{cases} \rho_{i,j} & \forall i_1, j_1 \in K, \\ 0 & \forall i_1 \in K, j_1 \in L, \\ 0 & \forall i_1 \in L, j_1 \in K. \end{cases} \] (30)

Then for all $I, J \in R^d$, we have
\[ (\rho_1)_{I,J} = \rho_{I,J}. \] (31)

Assume $i_1, j_1 \in I$ and there is an $i_n \in J, n \neq 1$. Then
\[ (\rho_1)_{I,J} = \rho_{I,J} \\ = \rho_{i_1, \ldots, i_{n-1}, j_1, j_2, \ldots, j_d} \] (32)

Similar discussion can be applied for $j_n \in J, n \neq 1$. Therefore, $\rho_1$ is the restriction of $\rho$ to the subspace $(R^r)^{\otimes d}$, hence it is CS.

It is easy to see that $\rho_1$ is S-separable if $\rho$ is S-separable by taking the restriction of $\rho$ to the subspace $(R^r)^{\otimes d}$. This completes our proof. ■

It should be noted that $\rho$ is $A_k$-reducible over real if and only if it is $A_n$-reducible ($n \neq k$) over real. From now on, the CS state $\rho$ is said to be reducible over real if it is $A_k$-reducible over real for some $k$.

To prove another feature of CS states, we first introduce the a useful result about the semidefinite positive operators.

**Lemma 7.** Suppose $A$ is a semidefinite positive operator, i.e., satisfies the condition (2). Then
1. $A_{ii} \geq 0, \forall i$,
2. $A_{ij} = A_{ji} = 0, \forall j$ if $A_{ii} = 0$ for some $i$.

With this lemma, we can prove that the CS states are reducible over real if and only if their reduced states are reducible when $d \geq 3$.

**Lemma 8.** Suppose $\rho$ is a CS state in the $N^{\otimes d}$ ($d \geq 3$) system. Then $\rho$ is reducible over real if and only if $\text{Tr}_k(\rho)$ is reducible over real for some $k$, where $\text{Tr}_k$ is the partial trace that applies the trace to the $k$-th subsystem.

**Proof.** Without the loss of generality, we can assume that $k = 1$. It is easy to see that $\rho$ is reducible over real implies that of $\text{Tr}_1(\rho)$. To prove the inverse direction, suppose $\text{Tr}_1(\rho)$ is reducible.

In fact $\text{Tr}_1(\rho) = \sum_{i=0}^{N-1} \rho_{ii}$ according to the representation as in Eq. (11). Since $\rho$ is semidefinite positive, the diagonal blocks $\rho_{ii}$ is semidefinite positive as well. Suppose
\[ \text{Tr}_1(\rho) = \sigma + \delta, \] (33)
where the local range of $\sigma$ equals span\{0, 1, \ldots, |r-1}\} and the local range of $\delta$ equals span\{|r, |r+1, \ldots, |d-1\}\}. It follows that

$$\sum_{i_1 \neq j_1} \rho_{i_1,j_1} = 0, \forall i_2 < r, i_3 \geq r, \forall i_3, \ldots, i_d, J.$$  \hspace{1cm} (34)

Note that for a semidefinite positive operator, the diagonal entries are all nonnegative. Let $J = I$, then we have

$$\sum_{i_1 = 0}^{N-1} \rho_{i_1,I} = 0, \forall i_2 < r, i_3 \geq r, \forall i_3, \ldots, i_d = 0, 1, \ldots, d-1,$$

which implies that

$$\rho_{i_1,I} = 0, \forall i_2 < r, i_3 \geq r, \forall i_3, i_4, \ldots, i_d = 0, 1, \ldots, d-1.$$  \hspace{1cm} (35)

According to Lemma 10, we have

$$\rho_{i_1,I} = 0,$$  \hspace{1cm} (37)

for $\forall i_2 < r, i_3 \geq r, \forall i_1, i_3, i_4, \ldots, i_d = 0, 1, \ldots, d-1, \forall J$. Since $\rho$ is CS, we have

$$\rho_{i_1,J} = 0, \forall i_1 < r, j_1 \geq r \text{ or } i_1 \geq r, j_1 < r.$$  \hspace{1cm} (38)

That is $\rho_{i,j} = 0$, if $i < r, j \geq r$ or $i \geq r, j < r$. Hence $\rho$ is reducible over real according to the definition. $\blacksquare$

Note that by Lemma 3, the reduced CS states remain CS, therefore, $\text{Tr}_1(\rho)$ is reducible over real if and only if $\text{Tr}_2(\text{Tr}_1 \rho)$ is reducible over real. Hence we have the following corollary.

**Corollary 9.** Suppose $\rho$ is a CS state in the $A_1 : A_2 : \ldots : A_d$ system with $d \geq 3$. Then $\rho$ is reducible over real if and only if $\rho_{A_k}$ is reducible over real, where $K$ is a subset of $\{1, 2, \ldots, d\}$ with $|K| \geq 2$.

It is ready to prove that Conjecture 1 holds for arbitrary multipartite system. First, consider the simplest case where $\rho$ is a pure CS state.

**Lemma 10.** Any CS pure states are S-separable.

**Proof.** Note that any quantum state is assumed to be semidefinite positive. Suppose that $\rho$ is a CS pure state in the $H^{\otimes d}$ system. Hence it can be written as

$$\rho = |x_1, \ldots, x_d\rangle\langle x_1, \ldots, x_d|,$$  \hspace{1cm} (39)

where $x_i, i = 1, 2, \ldots, d$ are unit vectors in $H$. If we take the trace to $d-2$ subsystems, then the reduced bipartite state is also CS by Lemma 3. According to the results for bipartite system in Ref. [23], $x_i$ and $x_j$ must be linearly dependent for any $i \neq j$. Hence, $\rho$ is S-separable. $\blacksquare$

Next, consider the general bipartite case. Denote by $S$ the subspace spanned by vectors $\{|i,j\} + |j,i\}$.

**Lemma 11.** Suppose $\rho$ is a separable CS state in the bipartite system $H \otimes H$. Then $\mathcal{R}(\rho) \subset S$.

**Proof.** For any state $\rho$, it can be written as

$$\rho = \sum_{ijkl=0}^N \rho_{ijkl}|i,k\rangle\langle j,l|.$$  \hspace{1cm} (40)

Since $\rho$ is CS,

$$\rho_{ijkl} = \rho_{klij},$$  \hspace{1cm} (41)

which implies that

$$\rho = \sum_{ijkl=0}^N \rho_{ijkl}|k,i\rangle\langle j,l|.$$  \hspace{1cm} (42)

Add Eq. (40) and Eq. (42), we have

$$\rho = \frac{1}{2} \left( \sum_{ijkl=0}^{N-1} \rho_{ijkl}|i,k\rangle\langle j,l| + \sum_{ijkl=0}^{N-1} \rho_{ijkl}|k,i\rangle\langle j,l| \right),$$  \hspace{1cm} (43)

$$= \frac{1}{2} \sum_{ijkl=0}^{N-1} \rho_{ijkl}(|i,k\rangle + |k,i\rangle)(j,l|),$$

which completes our proof. $\blacksquare$

Note that the pure states in the space $S$ are invariant under the exchange of particles, we have the following lemma.

**Lemma 12.** Suppose $|x,y\rangle$ is a product vector contained in the space $S$, then $|y\rangle = \lambda|x\rangle$ for some $\lambda \in \mathbb{C}$ if $|x\rangle \in \mathbb{C}^N$.

If $|x,y\rangle$ is real, then $\lambda$ is real in the above lemma.

The following observation shows that any pure state in $S$ can be written as the linear combination of symmetric pure product states.

**Lemma 13.** Suppose $|\psi\rangle$ is a symmetric bipartite pure state. Then

(i) $|\psi\rangle = \sum_i |a_i, a_i\rangle$ where the $|a_i\rangle$’s are orthogonal states.

(ii) if $|\psi\rangle$ is real then the $|a_i\rangle$’s in (i) can be chosen as real states up to the imaginary unit $i$. That is, $|\psi\rangle = \sum_i (-1)^{c_i}|a_i, a_i\rangle$ where $c_i = 0$ or 1 and $|a_i\rangle$ is real.

**Proof.** (i) Since $|\psi\rangle$ is a symmetric bipartite pure state, we have

$$|\psi\rangle = \sum_{i,j} m_{i,j}|i,j\rangle$$  \hspace{1cm} (44)

with $m_{i,j} = m_{j,i}$ for any $i, j$. So $M := (m_{i,j})$ is a symmetric matrix. Using Takagi’s factorization we have

$$M = UDU^T$$  \hspace{1cm} (45)
for a unitary operator $U$ and a diagonal semidefinite positive operator $D = \text{diag}(d_1, d_2, \cdots)$. We have

$$|\psi\rangle = I \otimes M \sum_i |ii\rangle$$  \hspace{1cm} (46)

$$= I \otimes UDU^T \sum_i |ii\rangle$$  \hspace{1cm} (47)

$$= U \otimes UD \sum_i |ii\rangle$$  \hspace{1cm} (48)

$$= \sum_i d_i |i\rangle \otimes U |i\rangle$$  \hspace{1cm} (49)

$$= \sum_i |a_i\rangle \otimes |a_i\rangle,$$  \hspace{1cm} (50)

where $|a_i\rangle = \sqrt{d_i} |i\rangle$. So assertion (i) holds.

(ii) Since $M$ is a real symmetric matrix, using the eigen decomposition, we can choose $U$ as an orthogonal matrix, and $D$ as a real diagonal matrix in Eq.(45). Then we repeat the proof of assertion (i). So assertion (ii) holds.

With the above lemma, we can then prove Conjecture 1 is true for arbitrary multipartite system.

Theorem 14. Conjecture 1 is true in arbitrary multipartite system.

Proof. Suppose $\rho$ is a separable CS states in the multipartite system $\mathcal{H} \otimes d$. Then it admits a decomposition

$$\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|,$$  \hspace{1cm} (51)

where $|\phi_k\rangle$ is a product vector

$$|\phi_k\rangle = |x_{k1}, x_{k2}, \ldots, x_{kd}\rangle.$$  \hspace{1cm} (52)

According to Lemma 4, we can assume that $|x_{ki}\rangle \in \mathbb{R}^N$.

In order to prove that $\rho$ is S-separable, it suffices to prove $|x_{ki}\rangle$ and $|x_{kj}\rangle$ are identical up to a real scalar multiplication.

Take trace to $d-2$ parties except the $i, j$-th subsystem, the reduced state

$$\sigma = \sum_i \lambda_i |x_{ki}, x_{kj}\rangle \langle x_{ki}, x_{kj}|$$  \hspace{1cm} (53)

is still CS and $|x_{ki}, x_{kj}\rangle \in S$. By Lemma 12, $|x_{kj}\rangle = \lambda_{ki} |x_{ki}\rangle$ for some real constant $\lambda_{ki}$. Therefore, $\rho$ is S-separable, which leads to the validity of Conjecture 1.

IV. SEPARABILITY OF CS STATES

In this section, we consider the entanglement problem of the CS states. In general, this problem is regarded as NP-hard. Nevertheless, some results have been obtained with respect to the low-rank states both in the bipartite and multipartite system. As described in the introduction, any PPT states of rank at most 4 are separable except in the $3 \otimes 3$ and $2 \otimes 2 \otimes 2$ case. For the certain two cases, they are separable if and only if they have a product vector in their range. It is also proved that any supported $M \otimes N$ states of rank $N$ are separable. However, in the multipartite system, the supported $N_1 \otimes N_2 \cdots \otimes N_d (N_1 \leq N_2 \leq \cdots \leq N_d)$ state $\rho$ of rank $N_d$ is separable only if there exists a product vector $|\phi\rangle = |x_1, x_2, \ldots, x_{d-1}\rangle$ such that

$$\text{rank}(\langle \phi | \rho | \phi \rangle) = N_d.$$  \hspace{1cm} (54)

In this part, we will develop stronger results about the entanglement problem of CS states. In addition, all the separable CS states are necessarily S-separable. First, for the CS states, we prove that the full separability is equivalent to bi-separability.

Let $\pi_0$ be the periodic index permutation

$$\pi_0(1, 2, \ldots, d) \Rightarrow (2, 3, \ldots, d, 1).$$  \hspace{1cm} (55)

Let $\mathcal{F}$ be the subspace spanned by the vectors which is invariant under the periodic permutation:

$$\mathcal{F} = \{ \phi \in \mathbb{R}^N : \pi_0^k(|\phi\rangle) = |\phi\rangle, \forall k > 0 \},$$  \hspace{1cm} (56)

where

$$\pi_0^k = \pi_0 \circ \pi_0 \circ \cdots \circ \pi_0.$$  \hspace{1cm} (57)

Lemma 15. The range of CS states is contained in $\mathcal{F}$.

Proof. Suppose $\rho$ is a CS states as in Eq. (8). Note that $\rho$ is CS, hence

$$\rho\pi_0^k(I), J = \rho_{I, J}, \forall k > 0.$$  \hspace{1cm} (58)

Therefore,

$$\rho = \frac{1}{d} \sum_{k=0}^{d-1} \left( \sum_{I, J} \rho_{\pi_0^k(I), J} |I\rangle \langle J| \right),$$  \hspace{1cm} (59)

Since $\sum_{k=0}^{d-1} |\pi_0^k(I)\rangle$ is contained in $\mathcal{F}$, it follows that $\mathcal{R}(\rho) \subset \mathcal{F}$.

Theorem 16. The CS states are fully separable if and only if they are bi-separable.
Suppose that $\rho$ is a CS state in the multipartite system $H^{\otimes d}$. It only needs to prove that the bi-separability implies the full separability. Assume $\rho$ is bi-separable under the bipartition $1, \ldots, k : k + 1, \ldots, d$.

According to Lemma 4, $\rho$ can be written as

$$\rho = \sum_i \lambda_i |\varphi_i\rangle \langle \varphi_i|,$$

where $\lambda_i > 0, |\varphi_i\rangle = |\phi_i, \psi_i\rangle, |\phi_i\rangle \in \mathbb{R}^{N_k}, |\psi_i\rangle \in \mathbb{R}^{N_d - k}$.

By Lemma 15, each $|\varphi_i\rangle$ is a symmetric pure state, which is invariant under the periodic permutation $\pi_0$. Hence $|\varphi_i\rangle$ should also be separable under the bipartition $\{2, 3, \ldots, k + 1\} : \{k + 2, \ldots, d, 1\}$.

Take the trace to the $k + 1, k + 2, \ldots, d$-th parties, then $|\phi_i\rangle$ is a separable pure product vector under the partition $\{2, 3, \ldots, k\} : 1$. That is, $|\phi_i\rangle$ can be written as

$$|\phi_i\rangle = |x_i, \phi_i^{(1)}\rangle,$$

where $|x_i\rangle \in \mathbb{R}^N$ and $|\phi_i^{(1)}\rangle \in \mathbb{R}^{N^{k-1}}$.

Apply the similar discussion to $\text{Tr}_1(\rho)$, we can conclude that $\varphi$ is fully separable, which completes our proof.

With Theorem 16 and Theorem 14, any bi-separable CS states are necessarily S-separable. The following lemma describes the product vectors in the range or kernel of CS states.

**Lemma 17.** Suppose $\rho$ is a bipartite CS state, the product vectors $|a, a\rangle \in \mathcal{R}(\rho)$ and $|b, c\rangle \in \ker \rho$. Then (i) $|a^*, a^*\rangle \in \mathcal{R}(\rho)$; (ii) $|b^*, c\rangle, |b, c^*\rangle, |b^*, c^*\rangle \in \ker \rho$.

**Proof.** (i) The assertion follows from the fact that $\rho$ is real.

(ii) Note that $\rho$ is unchanged under the partial transpose operators, hence, we have

$$0 = \langle b, c|\rho|b, c\rangle = \langle b^*, c|\rho|b^*, c\rangle$$

$$= \langle b, c^*|\rho|b, c^*\rangle = \langle b^*, c^*|\rho|b^*, c^*\rangle. \quad (62)$$

So assertion (ii) holds.

**A. Multi-qubit state**

**Theorem 18.** All the multi-qubit CS states are separable.

**Proof.** In order to prove this result, we claim that any $2 \otimes N$ G-invariant quantum states are separable.

Suppose $\rho$ is a G-invariant quantum state supported in the system $2 \otimes N$. Since $\rho$ is semidefinite positive, $\text{rank}(\rho) \geq N$.

If $\text{rank}(\rho) = N$, it is separable according to the result in Ref. [12] since G-invariant states are necessarily PPT.

Assume $\text{rank}(\rho) > N$. Then any G-invariant states are separable if for any G-invariant states of rank at least $N + 1$ contains a real product vector in its range.

Suppose the kernel of $\rho$ is spanned by the vectors $|f_i\rangle = |0, f_i0\rangle + |1, f_i1\rangle$. Let

$$|x\rangle = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, x_0, x_1 \in \mathbb{R}. \quad (63)$$

The range of $\rho$ contains a real product vector if and only if the following equations has a real solution:

$$(x_0F_0 + x_1F_1)y = 0, \quad (64)$$

where

$$F_1 = \begin{bmatrix} |f_{i1}\rangle \\ |f_{i2}\rangle \\ \vdots \\ |f_{i1}\rangle \end{bmatrix}, \quad (65)$$

$$F_2 = \begin{bmatrix} |f_{i1}\rangle \\ |f_{i2}\rangle \\ \vdots \\ |f_{i1}\rangle \end{bmatrix},$$

$$r = 2N - \text{rank}(\rho) \leq N - 1.$$

Since $r < N$, the above linear system (64) has a real solution for any real number $x_1, x_2$.

Suppose $|x_1, y_1\rangle$ is a real product vector in the range of $\rho$, there exists a positive number $\lambda_1$ such that

$$\rho^{(1)} = \rho - \lambda_1|x_1, y_1\rangle \langle x_1, y_1| > 0, \quad (66)$$

and $\text{rank}(\rho^{(1)}) = \text{rank}(\rho) - 1$. By the deduction on the ranks, we can then conclude $\rho$ is separable.

Note that an arbitrary $d$-qubit CS state under the bi-partition $1 : \{2, 3, \ldots, d\}$ is G-invariant, hence bi-separable. According to Theorem 16, it is separable.

**B. Two-qutrit state**

Using the Segre variety, we have the following lemma for bipartite system.

**Lemma 19.** Suppose $n \geq \frac{N(N-1)}{2} + 1$, then any $n$-dimensional subspace of $S$ contains a complex product vector.

**Theorem 20.** In the bipartite system, any CS states are separable if $N \leq 3$.

**Proof.** It has been proved for the case $N \leq 2$ in Theorem 18. We need only consider the two-qutrit system. Suppose $\rho$ is a $3 \otimes 3$ CS state. If $\text{rank}(\rho) \leq 3$, then it is separable according to the results in Refs. [16]. If $\text{rank}(\rho) = 4$, then $\rho$ is separable if and only if its range contains a product vector. It follows from Lemma 19 that $\rho$ is separable.
Assume $\text{rank}(\rho) \geq 5$, we claim that its range must contains a real pure product vector $|x, x\rangle$. In fact, this claim will lead to the separability of $\rho$ of any rank.

Denote by $|\phi_i\rangle, i = 0, 1, 2, 3, 4$ the 5 linearly independent vectors which are contained in the range of $\rho$. On the other hand, the $\mathcal{R}(\rho)$ can be represented by the following vectors:

$$
\begin{align*}
|\phi_0\rangle &= |0, 0\rangle, \\
|\phi_1\rangle &= |1, 1\rangle, \\
|\phi_2\rangle &= |2, 2\rangle, \\
|\phi_3\rangle &= |0, 1 \rangle + |1, 0\rangle, \\
|\phi_4\rangle &= |0, 2 \rangle + |2, 0\rangle, \\
|\phi_5\rangle &= |1, 2 \rangle + |2, 1\rangle.
\end{align*}
$$

(67)

Hence, we have

$$
|\phi_i\rangle = \sum_{j=0}^{5} a_{ij} |\phi_j\rangle, \quad i = 0, 1, \ldots, 4, a_{ij} \in \mathbb{R}. 
$$

(68)

Since $|\phi_i\rangle, i = 0, \ldots, 4$, $|\phi_0\rangle$, and $|\phi_1\rangle$ are linearly dependent, there exists a linear combination such that

$$
\sum_{i=0}^{5} b_i |\phi_i\rangle = c_0 |\phi_0\rangle + d_0 |\phi_1\rangle \in \mathcal{R}(\rho), b_i \in \mathbb{R}. 
$$

(69)

If $c_0 d_0 = 0$, there we can conclude that $\mathcal{R}(\rho)$ contains a real product vector, which completes our proof. Otherwise, we can assume $c_0 = 1$. Denote by $|\psi_0\rangle = |\phi_0\rangle + d_0 |\phi_1\rangle$. Similarly, we have the following vectors in the range of $\rho$,

$$
\begin{align*}
|\psi_1\rangle &= |\phi_0\rangle + d_1 |\phi_2\rangle, \\
|\psi_2\rangle &= |\phi_1\rangle + d_2 |\phi_2\rangle, \\
|\psi_3\rangle &= |\phi_3\rangle + e_0 |\phi_1\rangle, \\
|\psi_4\rangle &= |\phi_5\rangle + e_1 |\phi_1\rangle, \\
|\psi_5\rangle &= |\phi_4\rangle + e_2 |\phi_2\rangle.
\end{align*}
$$

(70)

We can assume that at least one of $d_0, d_1, d_2$ is greater than 0. Otherwise $\psi_0 - d_0 \psi_2 = |0, 0\rangle - d_0 |2, 2\rangle$ is a vector in $\mathcal{R}(\rho)$ which satisfies the requirement.

For simplicity, assume $d_0 > 0$. Then the two vectors

$$
\psi_0 = |0, 0\rangle + d_0 |1, 1\rangle, \quad \psi_3 = |0, 1\rangle + |1, 0\rangle + e_0 |1, 1\rangle, 
$$

(71)

are in the range of $\rho$. Note that by the Schmidt rank of pure states, the vector $x|\psi_0\rangle + |\psi_3\rangle$ is a produce vector if and only if

$$
\text{det} \begin{bmatrix} x & 1 \\ 1 & e_0 x + d_0 \end{bmatrix} = e_0 x^2 + d_0 x - 1 = 0.
$$

(72)

Since $e_0 > 0$, there exists a real solution, denoted by $x_0$, of the above equation. Hence $x_0 |\psi_0\rangle + |\psi_3\rangle$ is a real product vector which is contained in the range of $\rho$, which completes our proof.

C. $\text{rank}(\rho) = N$

**Theorem 21.** Suppose $\rho$ is a CS state supported in the multipartite system $N \otimes d$. If $\text{rank}(\rho) = N$, then $\rho$ is separable.

**Proof.** Consider the bi-partition $1 : \{2, \ldots, d\}$, then $\rho$ is supported in the system $N \otimes R, R \geq N$. According to the result in Ref. [13], it is separable. By Theorem 16, it is fully separable, hence $S$-separable. 

For the CS states, the condition of separability is weaker than what presented in Ref. [26].

D. $\text{rank}(\rho) = N + 1$

Let us recall a useful result in Ref. [27]. Denote by $A, B$ the parties for each subsystem. We shall refer to $\rho \oplus_B \sigma$ in the sense that the range of $\rho_B$ and $\sigma_B$ have no intersection except the zero vector.

**Lemma 22.** Suppose $\rho$ is an $M \otimes N (2 < M \leq N)$ PPT state of rank $N + 1$, then $\rho$ is either a sum of $N + 1$ pure products states or

$$
\rho = \rho_1 \oplus_B \rho_2 \oplus_B \cdots \oplus_B \rho_k \oplus_B \sigma,
$$

(73)

where $\rho_i$’s are pure product states and $\sigma$ is a bipartite PPT state of rank $N + 1 - k$.

**Lemma 23.** Any $N \otimes N$ CS states of rank $N + 1$ are separable.

**Proof.** Suppose $\rho$ is an $N \otimes N$ CS state of rank $N + 1$, which is necessarily PPT. To prove the assertion, it suffices to assume $N > 2$. By Lemma 22, we obtain that

$$
\rho = \rho_1 \oplus_B \cdots \oplus_B \rho_k \oplus_B \sigma,
$$

(74)

where $\rho_i$’s are pure product states.

Since $\rho$ is CS, one can show that every $\rho_i$ is real symmetric product state. So $\sigma$ is CS as well.

Hence, $\sigma$ is a bipartite state of rank$\text{rank}(\sigma_B) = N - k$ and $\text{rank}(\sigma) = N - k + 1$. Since $\sigma$ is CS, it is symmetric. So $\text{rank}(\sigma) = N - k$.

We prove the assertion by contradiction. If $\rho$ is entangled, Theorem 45 in Ref. [27] says that $\text{rank}(\sigma_A) = 2$ or 3. It means that $\text{rank}(\sigma) \leq 4$. We have a contradiction with the proven fact that every CS state of rank at most four is separable. This completes the proof.

**Theorem 24.** Any $N \otimes d$ CS states of rank $N + 1$ are separable.

**Proof.** Consider the bi-partition $A_1 : A_2, \ldots, A_d$. We have that

$$
N \leq \text{rank}(\text{Tr}_1 \rho) \leq N + 1.
$$

(75)

If $\text{rank}(\text{Tr}_1 \rho) = N + 1$, then $\rho$ is bi-separable by Lemma 22, which is separable according to Theorem 16.
If \( \text{rank}(\text{Tr}_1 \rho) = N, N > 3 \), by Lemma 22, we can find a reducible components \( \rho_1 \) such that
\[
\rho = \rho_1 + \sigma,
\]
where \( \rho_1 \) is a CS pure product state and \( \sigma \) is of rank \( N \) CS state supported in the \((N - 1) \otimes d\) space. Apply the similar discussion recursively on \( \sigma \), hence we can assume \( \sigma \) is a \( 3 \otimes d \) rank-4 CS state, which is separable according to Theorem 26 in the next subsection.

If \( \text{rank}(\text{Tr}_1 \rho) = N, N \leq 2 \), it is separable by Theorem 18.

Above all, any \( N \otimes d \) CS states of rank \( N + 1 \) are separable.

A corollary about the rank-(\( N + 2 \)) CS states follows from Theorem 24.

**Corollary 25.** Suppose \( \rho \) is a CS state supported in the multipartite system \( N \otimes d \). If \( \text{rank}(\rho) = N + 2 \), then it is separable if and only if \( \mathcal{R}(\rho) \) contains a real product vector.

**E. \( \text{rank}(\rho) \leq 5 \)**

Before all else, we prove all the rank-4 states are separable.

**Theorem 26.** Any CS states of rank at most 4 in the multipartite system are separable.

**Proof.** First, we consider the bipartite system.

Suppose \( \rho \) is a CS state in the bipartite system \( \mathcal{H} \otimes \mathcal{H} \). In order to prove that \( \rho \) is S-separable, by Theorem 14, it only need to prove \( \rho \) is separable.

Note that a CS state is necessarily PPT. In addition, it was proved that the PPT states of rank at most 3 are separable in Refs. [16].

It remains to prove it when the rank is 4. According to the result in Ref. [16], only the 3 \( \otimes 3 \) state could be entangled and it is separable if and only if its range contains a product vector.

By Lemma 19, the rank-4 states in \( 3 \otimes 3 \) must contains a product vector. Therefore, \( \rho \) is separable.

Finally, we consider the general multipartite system.

By the results in Ref. [17], all the PPT states of rank at most 3 are separable. And the rank 4 PPT states are separable except in the \( 3 \otimes 3 \) and \( 2 \otimes 2 \otimes 2 \) cases. It thus only to prove that the \( 2 \otimes 2 \otimes 2 \) rank-4 state is separable. It follows by Theorem 18 that our proof completes. □

Furthermore, we can prove that any rank-5 CS states are separable.

**Theorem 27.** Any CS states of rank at most 5 are separable in the multipartite system.

**Proof.** By Theorem 26, it suffices to prove the rank-5 CS states are separable.

Suppose \( \rho \) is a rank-5 CS state supported in the system \( N \otimes d \). It is easy to show that \( N \leq 5 \).

If \( N = 5 \), by Theorem 21, \( \rho \) is S-separable.

If \( N = 4 \), consider the bi-partition \( A_1 : A_2, \ldots, A_d \).

Note that \( 4 \leq \text{rank}(\text{Tr}_1\rho) \leq 5 \). If \( \text{rank}(\text{Tr}_1\rho) = 5 \), \( \rho \) is bi-separable under this bi-partition. If \( \text{rank}(\text{Tr}_1\rho) = 4 \), \( \rho \) is a \( 4 \otimes 4 \) rank-5 state under this partition. By Lemma 22, there exist a real product vector such \( |x, y\rangle \in \mathcal{R}(\rho) \), where \( |x\rangle \in \mathcal{H} \) and \( |y\rangle \in \otimes^{d-1} \mathcal{H} \). Since \( |x, y\rangle \in \mathcal{F} \), where \( \mathcal{F} \) is a periodic symmetric subspace as defined in Eq. (56), \( |x, y\rangle \) can be written as \( \gamma|x\rangle \otimes y \). Therefore, there exists a positive constant \( \lambda \) such that \( \rho - \lambda|x\rangle \otimes y|y\rangle \) is CS, semidefinite positive and of rank four, which is separable by Theorem 26. Hence \( \rho \) is separable.

If \( N = 3 \), consider the bi-partition \( A_1 : A_2, \ldots, A_d \).

Note that \( 3 \leq \text{rank}(\text{Tr}_1\rho) \leq 5 \). rank(Tr_1\rho) = 5, \( \rho \) is bi-separable by Theorem 21 and hence fully separable.

If \( \text{rank}(\text{Tr}_1\rho) = 4 \), \( \rho \) is S-separable by Theorem 26. Moreover, it is a sum 4 CS pure product states:
\[
\text{Tr}_1 \rho = \sum_{i=0}^{3} |x_i\rangle \langle x_i| \otimes d - 1, (x_i|R \rangle \langle R|) = 5, \text{Tr}_1^H \mathcal{F} = 4, \text{Tr}_1^{H \otimes d} |x\rangle \langle x| \otimes d - 1, \]

where \( |x_i\rangle \in \mathbb{R}^3 \). Since \( \text{Tr}_{1,2} \rho \) is of rank at least 3, we can assume \( |x_i\rangle, i = 0, 1, 2 \) are linearly independent. If \( |x_3\rangle \) is a linear combination of two vectors of \( |x_0\rangle, |x_1\rangle, |x_2\rangle \), then it is reducible over real, which implies that \( \rho \) is also reducible over real. Hence \( \rho \) is separable. Therefore, we can further assume that any three vectors of \( |x_i\rangle \) are linearly independent. It implies that there exists a real ILO such that
\[
(A \otimes (d-1))(\text{Tr}_1 \rho))(A^T) \otimes (d-1)) = \sum_{i=0}^{2} \gamma_i |i\rangle \langle i| \otimes d - 1 + \gamma_3 |x\rangle \langle x| \otimes d - 1, \]

where \( |x\rangle = |0\rangle + |1\rangle + |2\rangle \) and \( \gamma_i > 0, i = 0, 1, 2, 3 \).

For simplicity, we substitute \( \rho \) with \( (A \otimes d) \rho (A^T) \otimes d \) but still denote this state by \( \rho \). Suppose \( |\phi\rangle \) is a vector in the range of \( \rho \), then it must has the form
\[
|\phi\rangle = \sum_{i=0}^{2} |a_i\rangle |i\rangle \otimes d - 1 + |a_3\rangle |x\rangle \otimes d - 1. \]

Since \( |\phi\rangle \) is in the space \( \mathcal{F} \), the periodic subspace, we have
\[
\sum_{i=0}^{2} |a_i\rangle |i\rangle \otimes d - 1 + |a_3\rangle |x\rangle \otimes d - 1 = \sum_{i=0}^{2} |i, a_i\rangle |i\rangle \otimes d - 2 + |x, a_3\rangle |x\rangle \otimes d - 2. \]

By solving the above system of equations, we have
\[
|a_i\rangle \propto |i\rangle, |a_3\rangle \propto |x\rangle. \]
That is, $|\phi\rangle$ has the following form
\[
|\phi\rangle = \sum_{i=0}^{2} \lambda_i |\phi_i\rangle \otimes d + \lambda_3 |x\rangle \otimes d,
\] (82)
where $\lambda_i \in \mathbb{R}$. However, the range spanned by these vectors is rank at most 4, which leads to a contradiction.

If rank$(Tr_1 \rho) = 3$, then the reduced states is a $3 \otimes 3$ rank 3 states, which is a sum of three real pure product states according to the results in Ref. [13]. It follows that $Tr_1 \rho$ is reducible over real. By Corollary 9, $\rho$ is also reducible over real. Note that for all the irreducible components, they are CS and of rank at most 4, hence separable. It has been prove that for $N = 2$, the CS state is S-separable and it is trivial case when $N = 1$. Therefore, $\rho$ is separable in this case.

Above all, any multipartite CS states of rank at most 5 are separable.

\[\text{F. rank}(\rho) = 6\]

We have proved that many CS states of low ranks are separable. Unfortunately, not all CS states are separable although they have a special structure. Here, we construct a rank-6 entangled CS states as follows. Let
\[
\rho = \sum_{i=0}^{6} \lambda_i |\phi_i\rangle \langle \phi_i|, \lambda_i > 0,
\] (83)
where
\[
|\phi_i\rangle = |x_i, x_i\rangle, \text{ for } i = 0, 1, \ldots, 6,
\] (84)
and
\[
|x_i\rangle = |i\rangle, \text{ for } i = 0, 1, 2, 3
\]
\[
|x_4\rangle = |0\rangle + |1\rangle + |2\rangle + |3\rangle,
\]
\[
|x_5\rangle = |0\rangle + 2|1\rangle + 3|2\rangle + 4|3\rangle,
\]
\[
|x_6\rangle = |0\rangle - 2|1\rangle + 3|2\rangle - 4|3\rangle.
\] (85)

We claim that the range of $\sigma$ contains exactly 8 real symmetric product vectors, that is $|\phi_i\rangle, i = 0, 1, \ldots, 6$ and $|\phi_7\rangle = |x_7, x_7\rangle$, where
\[
|x_7\rangle = |0\rangle - \frac{8}{3}|1\rangle + |2\rangle - \frac{8}{3}|3\rangle.
\] (86)

In order to be concise, we append the proof in Appendix A.

Note that there exists a positive constant $\lambda$ such that
\[
\rho = \sigma - \lambda |\phi_7\rangle \langle \phi_7| \geq 0
\] (87)
has rank 6. We claim that $\rho$ is entangled. Otherwise, assume $\rho$ is separable, and thus S-separable. Since $\mathcal{R}(\rho) \subset \mathcal{R}(\sigma)$, the symmetric product vector contained in $\rho$ must only be $|\phi_0\rangle, \ldots, |\phi_6\rangle$. Note that the rank of $\rho$ is 6, without the loss of generality, we can hence assume that $\rho = \sum_{i=0}^{5} \gamma_i |\phi_i\rangle \langle \phi_i|$. Therefore $\sigma = \sum_{i=0}^{5} \gamma_i |\phi_i\rangle \langle \phi_i| + \lambda |\phi_7\rangle \langle \phi_7|$. By Eq. (83), we have,
\[
\sum_{i=0}^{5} (\gamma_i - \lambda_i) |\phi_i\rangle \langle \phi_i| + \lambda_6 |\phi_6\rangle \langle \phi_6| + \lambda |\phi_7\rangle \langle \phi_7| = 0. \] (88)

Note that the operators $|\phi_i\rangle \langle \phi_i|, i = 0, 1, \ldots, 7$ are linearly independent, hence we have $\lambda_6 = 0$, which is a contradiction. Therefore, $\rho$ must be entangled.

In fact, for rank 6 bipartite CS states, we have the following sufficient and necessary condition:

**Lemma 28.** Suppose $\rho$ is CS state of rank six in the bipartite system. Then $\rho$ is separable if and only if the range of $\rho$ has a product state.

**Proof.** Suppose $|x, x\rangle \in \mathcal{R}(\rho)$. The assertion holds if $|x\rangle$ is real. We assume that $|x\rangle$ is not proportional to a real vector. Up to a real ILO, we may assume that $|x\rangle = |0\rangle + i|1\rangle$, and $\rho$ is still a CS state of rank six. Hence $|01\rangle + |10\rangle, |00\rangle - |11\rangle \in \mathcal{R}(\rho)$. Let
\[
H = (|01\rangle + |10\rangle)(\langle 01| + \langle 10|)
- (|00\rangle - |11\rangle)(\langle 00| - \langle 11|)
\] (89)
be a Hermitian matrix. One can verify that $H$ is a CS matrix. So there exists a small enough $\epsilon > 0$ such that $\sigma = \rho + \epsilon H = (\rho + \epsilon H)^T \geq 0$. Note that $\sigma$ is a CS state, $\mathcal{R}(\sigma) \subset \mathcal{R}(\rho)$, and $\sigma$ is not proportional to $\rho$. Hence, there exists a large enough $\epsilon_1 > 0$ such that $\alpha = \rho - \epsilon_1 \sigma \geq 0$ is a CS state of rank at most five. We have proven that $\rho$ is separable, so $\alpha$ is a convex sum of real pure product states. Hence $\mathcal{R}(\rho)$ has a real pure product states, and $\rho$ is separable.

For the arbitrary multipartite system, we have the following sufficient and necessary condition. We omit the proof as it is similar to the previous ones.

**Lemma 29.** Suppose $\rho$ is CS state of rank six in the multipartite system. Then $\rho$ is separable if and only if the range of $\rho$ has a real product vector.

Using Lemma 28, one can show that for $2n$-partite $\rho$ Lemma 29 still holds if the condition “real” is removed. However, further exploration needs to know whether it still holds when $\rho$ is a $(2n + 1)$-partite CS state.

**V. APPLICATIONS**

In this section, we investigate the application of our results to a few widely useful states in quantum information. In Sec. VA, we show that all multipartite PPT symmetric states of rank at most 4 are separable. We also construct a sufficient condition by which an arbitrary multiqubit symmetric state is separable. In Sec. VB, we show that the CS state in (87) is an extreme and edge PP entangled state. The property is unique for the above
CS state, as we also construct a non-extreme and non-edge \( 4 \times 4 \) PPT entangled states of rank six, and it is invariant under partial transpose. Further in Sec. \( \text{VC} \), we show that the entangled state in (87) become non-negative states by choosing suitable coefficients \( \lambda_i \)'s. We investigate its geometric measure of entanglement, and its asymptotic version. In Sec. \( \text{VD} \), we highlight the relation between the symmetric states and the Hankel and Toeplitz matrices. Our results show that the CS states are intimately connected with the fundamentals of quantum information and matrix theory.

A. Symmetric states

Note that CS states are a subclass of symmetric states. In this subsection, we will study the properties of the symmetric states. By Lemma 19, we have the following result.

**Corollary 30.** All multipartite PPT symmetric states of rank at most 4 are separable.

**Proof.** Suppose \( \rho \) is a separable state of rank at most 4. By the results in Ref. [17], \( \rho \) is separable except \( \rho \) is supported in \( 3 \otimes 3 \) or \( 2 \otimes 2 \otimes 2 \). For the former case, by Lemma 19, \( \rho \) contains a product vector in its range, hence separable. For the later case, it is separable according to the results in Ref. [18]. \( \blacksquare \)

From the discussion of Theorem 18, the results can be generalized as

**Corollary 31.** Suppose \( \rho \) is symmetric state and \( \rho = \rho^{T_1} \) in multi-qutrit system, then \( \rho \) is separable, where \( T_1 \) is the partial transpose to the first subsystem.

Follows Theorem 16, for symmetric states, we also have the following result.

**Corollary 32.** The multipartite symmetric states are separable if and only if bi-separable.

Apply the similar discussion in Theorem 24 for symmetric states with substituting real symmetric vector with complex symmetric vector, we can prove the rank-\( N \) and rank-(\( N + 1 \)) symmetric states are separable.

**Corollary 33.** Any multipartite symmetric states supported in the \( N \otimes 4 \) space of rank at most \( N + 1 \) are separable.

B. Edge and extreme PPT entangled states

The edge and extreme PPT entangled states help understand the structure of PPT entangled states and separability problem. Thus their construction is an important problem in quantum information. So far, the progress towards the problem is little due to mathematical difficulty.

It follows from Lemma 28 that the range of the \( 4 \times 4 \) PPT entangled state \( \rho \) of rank six has no product vector. So it is an edge PPT entangled state. An example is the state in (87).

We further claim that every \( \rho \) is an extreme PPT entangled state [27]. We prove the claim by contradiction. Suppose some \( \rho \) is not extreme, so \( \rho = \alpha + \beta \), where \( \alpha \) and \( \beta \) are both PPT entangled states, and neither of them is proportional to \( \rho \). Hence \( \mathcal{R}(\alpha), \mathcal{R}(\beta) \subseteq \mathcal{R}(\rho) \). We may choose a large enough \( x > 0 \) such that \( \sigma := \rho - x \alpha \) is a PPT state of rank at most five. Since \( \mathcal{R}(\sigma) \subseteq \mathcal{R}(\rho) \), the former does not have any product vector. Therefore, \( \sigma \) is a PPT entangled state. The only possibility is that, \( \sigma \) is a two-qutrit state of rank four. It is a contradiction with Lemma 30. Consequently, \( \rho \) is an extreme PPT entangled state. We thus conclude the above findings as follows.

**Lemma 34.** Every \( 4 \otimes 4 \) CS entangled state of rank six is an edge and extreme state.

We emphasize that the above fact is a unique property for CS entangled states of rank six. Indeed, there exist non-extreme and non-edge \( 4 \times 4 \) PPT entangled states of rank six, which is invariant under partial transpose. We construct an example as follows. Let \( \alpha = [C_0, C_1, C_2] \) where

\[
C_0 = \begin{bmatrix}
0 & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, C_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & -1/d
\end{bmatrix}, C_2 = \begin{bmatrix}
0 & -1/b & 0 \\
0 & 1 & -c \\
-d & 0 & 0
\end{bmatrix},
\]

where \( a \in \mathbb{R}, a \neq 0 \) and \( b, c, d > 0 \). It has been shown in Theorem 23 of Ref. [28] that \( \alpha \) is a two-qutrit PPT entangled state of rank four, and \( \alpha^{T_1} = \alpha \). Let \( \beta = P\alpha P^\dagger \) where \( P = I_3 \otimes (|4\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|) \). So \( \rho = \alpha + \epsilon \beta \) with small enough \( \epsilon > 0 \) is \( 4 \times 4 \) PPT entangled state of rank six, and \( \rho^{T_1} = \rho \). By definition, \( \rho \) is not extreme.

One can further show there exist product vectors in the range of \( \rho \). The six different row vectors of \( [C_0, C_1, C_2] \) and \( [C_0, C_1, C_2]^\dagger \) are as follows.

\[
\begin{align*}
\gamma_1 &= a(1, 2) + b(1, 3) - \frac{1}{b}(3, 2), \\
\gamma_2 &= (1, 3) + c(2, 3) + (3, 2), \\
\gamma_3 &= (2, 3) + (3, 1) - c(3, 2), \\
\gamma_4 &= (2, 1) - \frac{1}{d}(2, 3) + d(3, 1), \\
\gamma_5 &= (2, 3) + (3, 4) - c(3, 2), \\
\gamma_6 &= (2, 4) - \frac{1}{d}(2, 3) + d(3, 4).
\end{align*}
\]

It is equivalent to determining whether there exist product vectors in the space \( \text{span}\{\gamma_1, \gamma_2, \ldots, \gamma_6\} \). One can verify

\[
\gamma_3 - \gamma_5 + \gamma_4 - \gamma_6 = (2| + (1 + d)(3))(1| - 4). \quad (92)
\]

Therefore, the range of \( \rho \) has product vectors. By definition, \( \rho \) is not edge.
C. Nonnegative states

The multipartite quantum state \( \rho \) is nonnegative if all entries in its density matrix are real and nonnegative. Many well-known states are indeed nonnegative states. Further, nonnegative states satisfy the additivity property of the geometric measure of entanglement (GME), which is an extensively useful multipartite entanglement measure. It is defined as

\[
G(\rho) := - \log_2 \max_{a_1, \ldots, a_d} \langle a_1, \ldots, a_d | \rho | a_1, \ldots, a_d \rangle, 
\]

where \( |a_1, \ldots, a_d \rangle \) is a normalized product state. Since \( \rho \) is nonnegative, it is known that \( |a_1, \ldots, a_d \rangle \) can be chosen as nonnegative states too. See more details on page 12 of Ref. [29]. Hence the asymptotic GME (AGME)

\[
G^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} G(\rho^\otimes n)
\]
is equal to \( G(\rho) \).

In our case, the entangled CS state \( \rho \) in (87) is nonnegative if we choose \( \lambda_2 = \lambda_6 \) and small enough \( \lambda > 0 \). So it satisfies the facts in the last paragraph. It constructs a novel example of entangled states whose GME is additive. In the following we compute the GME of the bipartite state \( \rho \) in Eq. (87), and thus its AGME.

It follows from Ref. [29] that we can choose \( |a_1\rangle = \cdots = |a_n\rangle \) whose entries are all real and nonnegative. Using the Lagrange multiplier, for the case \( d = 2 \), the solution of the optimization problem

\[
\max_{\|a\|=1} \langle a, a | \rho | a, a \rangle
\]
satisfies the KKT condition

\[
\langle a | \rho | a, a \rangle = \mu |a\rangle, \quad a \in \mathbb{R}^d, \quad \|a\| = 1.
\]
The Perron-Frobenius theorem on the largest Z-eigenvalue of nonnegative tensor in Ref. [30] says that if all entries of \( |a\rangle \) are positive in Eq. (95), then the corresponding \( \mu \) equals the maximal value of the optimization problem (94).

In particular, if \( \lambda_i \) satisfy the following condition,

\[
\lambda_0 = \lambda_3 + 3040 \lambda_5 - \frac{11000}{81} \lambda_7,
\]

\[
\lambda_1 = \lambda_3 + 2016 \lambda_5,
\]

\[
\lambda_2 = \lambda_3 + 1054 \lambda_5 - \frac{11000}{81} \lambda_7,
\]

then \( a = \frac{1}{3} (1, 1, 1)^T \) and \( \mu = \frac{\lambda_6}{3} + 16 \lambda_4 + 248 \lambda_5 + \frac{250}{27} \lambda_7 \) satisfy the condition (95). Hence, we have

\[
G(\rho) = G^\infty(\rho) = - \log_2 \left( \frac{\lambda_6}{4} + 16 \lambda_4 + 248 \lambda_5 + \frac{250}{27} \lambda_7 \right).
\]

D. Connection to Hankel and Toeplitz matrices

In this subsection, we will consider some classes of symmetric states. In the multi-qubit system, the symmetric states can be represented by the Dicke basis

\[
|\rho\rangle = \sum_{i,j=0}^d m_{ij}|D_{d,i}\rangle\langle D_{d,j}|.
\]

Denote by \( M_\rho \) the \( (d+1) \times (d+1) \) matrix whose \((i,j)\)-th entry is \( m_{ij} \). Hence, the corresponding semidefinite positive matrix \( M_\rho \) can be used to represent the symmetric states which has \( \frac{(d+1)(d+2)}{2} \) degree of freedom.

In particular, \( \rho \) is a DS state if \( M_\rho \) is a semidefinite positive diagonal matrix. Moreover, \( \rho \) is a CS state, if \( M_\rho \) is a Hankel matrix, which is an extensively useful multipartite entanglement

\[
M_\rho = \begin{bmatrix}
    a_0 & a_1 & a_2 & \cdots & a_d \\
    a_1 & a_2 & a_3 & \cdots & a_{d-1} \\
    a_2 & a_3 & a_4 & \cdots & a_{d-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_d & a_{d-1} & a_{d-2} & \cdots & a_1
\end{bmatrix}.
\]

Equivalently,

\[
\rho = \sum_{k=0}^d a_k \left( \sum_{i+j=k} |D_{d,i}\rangle\langle D_{d,j}| \right).
\]

By our results, if \( M_\rho \) is semidefinite positive, then \( \rho \) is S-separable. Hence, we have the following result about the Hankel matrix.

**Lemma 35.** Suppose \( M \) is an \( N \times N \) Hankel matrix, then it is semidefinite positive if and only if has the following decomposition

\[
M = \sum_{i=1}^r \lambda_i z_i z_i^T,
\]

where \( r = \text{rank}(M) \), \( \lambda_i > 0 \), and

\[
z_i = \begin{bmatrix} t_i^0 \\ t_i^1 \\ t_i^2 \\ \vdots \\ t_i^{N-1} \end{bmatrix}, \quad t_i \in \mathbb{R}.
\]

**Proof.** Let \( V \) be a diagonal matrix \( V = \text{diag}(\sqrt{C_{d-1}^0}, \sqrt{C_{d-1}^1}, \cdots, \sqrt{C_{d-1}^{d-1}}) \), then \( V M V^T \) is also a Hankel matrix. If \( M \) is semidefinite positive, \( V M V^T \) is also semidefinite positive. We can construct a (N-1)-qubit CS state \( \rho \) according to Theorem 18, \( \rho \)
is separable. That is, it can be written as

$$\rho = \sum_{i=1}^{r} \lambda_i |x_i\rangle \langle x_i|^\otimes N-1, x \in \mathbb{R}^2,$$  \hspace{1cm} \text{(103)}

where \( r = \text{rank}(\rho) \). If \(|x_i\rangle \not\propto |0\rangle\), we can assume \(|x_i\rangle = |1\rangle + t_i|0\rangle\). Then

$$\langle 0 | + t_i |1\rangle \rangle ^{(N-1)} = \sum_{j=0}^{N-1} t_i^j \sqrt{C_{N-1}} |D_{N-1,j}j\rangle.$$  \hspace{1cm} \text{(104)}

If \(|x_i\rangle = |0\rangle\), then \(|x_i\rangle ^{(N-1)} = |D_{N-1,N-1}\rangle\). Therefore,

$$VMV^T = \sum_i Vz_i z_i^T V^T,$$  \hspace{1cm} \text{(105)}

where \( z_i \) has the form in Eq. (102). The proof completes by cancel \( V \) on the above equation. \hfill \Box

It is also of interest to consider some other classes of symmetric states. In particular, if \( M_\rho \) is a Toeplitz matrix,

$$M_\rho = \begin{bmatrix} a_0 & a_1 & \cdots & a_d \\ a_1 & a_1 & \cdots & \vdots \\ \vdots & \cdots & \ddots & a_1 \\ a_d & a_1 & \cdots & a_0 \end{bmatrix}.$$  \hspace{1cm} \text{(106)}

Equivalently,

$$\rho = \sum_{k=0}^{d} a_k \left( \sum_{|i-j|=k} |D_{d,i}\rangle \langle D_{d,j}| \right).$$  \hspace{1cm} \text{(107)}

The Hankel matrix is closely related to the Toeplitz matrix, in fact, a Hankel matrix is an upside-down Toeplitz matrix. We conjecture that these states are separable.

**Conjecture 2.** Suppose \( \rho \) is multi-qubit symmetric states. If the corresponding \( M_\rho \) is Toeplitz, then \( \rho \) is separable.

**VI. CONCLUSION**

We have studied the property of completely symmetric (CS) states in the multipartite system.

First, we have proved that the CS states admit some nice features. For example, if \( \rho \) is separable if and only it is bi-separable or S-separable. Moreover, it is reducible over real if and only the reduced states are reducible over real for \( d \)-partite \((d > 2)\) system. In particular, the CS states are separable if and only if every real irreducible components are separable, which makes the separability problem easier to solve.

Second, we distinguished the separability of some low-rank CS states. It turns out that all the multi-qubit and two-qutrit CS states are separable. Moreover, all the CS states of rank at most \( 5 \) or \( N+1 \) are separable both in the bipartite and multipartite system. As for the rank-6 or \( N+2 \) CS states, they are separable if and only their range contains a real product vector. However, there exists a CS entangled state of rank 6. So the CS structure cannot guarantee the separability, and the further exploration is needed.

Third, we showed some application of CS states to some widely useful states in quantum information. For example the symmetric states, edge and extreme states, nonnegative states. We also have suggested the connection between the separability to the decomposition of Hermitian matrices, such as the Hankel and Toplitz matrices.

There are many problems one may consider in the future study of entanglement.

1. Is Conjecture 2 true?

2. Are multi-qutrit CS states separable?

3. What is the relation between the separability and local ranks for CS states? Does the Conjecture 24 hold for the CS states in Ref. [17]?

4. For a CS state, if all the reduced states are separable, can we conclude that the state is separable?

5. How is to find an entanglement witness for a given entangled CS state?

6. Can the symmetric extension criterion detect the separability of CS states completely? Is the corresponding numerical algorithm practical? Can we find a variation of symmetric extension criterion to fit the structure of CS states?

7. The CS states correspond to the supersymmetric tensor in tensor analysis, and the tensor rank of two copies of special symmetric tensors has been considered recently [31]. Is it possible to compute the tensor rank of two copies of some CS states?

**ACKNOWLEDGMENTS**

LC and YS were supported by the NNSF of China (Grant No. 11871089), and the Fundamental Research Funds for the Central Universities (Grant Nos. KG12040501, ZG216S1810 and ZG226S18C1).
Appendix A: Real Product vectors in $\mathcal{R}(\sigma)$ of (83)

Let $\sigma$ is the CS state in Eq. (83). Suppose the subspace $P$ is the range of $\sigma$, then it is spanned by the following vectors:

$$|\phi_i\rangle = |x_i, x_i\rangle, \text{ for } i = 0, 1, \ldots, 6 \quad (A1)$$

and

$$|x_i\rangle = |i\rangle, i = 0, 1, 2, 3$$
$$|x_4\rangle = |0\rangle + |1\rangle + |2\rangle + |3\rangle, \quad (A2)$$
$$|x_5\rangle = |0\rangle + 2|1\rangle + 3|2\rangle + 4|3\rangle,$$
$$|x_6\rangle = |0\rangle - 2|1\rangle + 3|2\rangle - 4|3\rangle.$$

In this appendix, we want to find that $P$ contains exactly 8 real product vectors. The extra real product vector is $|\phi_7\rangle = |x_7, x_7\rangle$, where

$$|x_7\rangle = |0\rangle - \frac{8}{3}|1\rangle + \frac{8}{3}|3\rangle. \quad (A3)$$

Proof. Suppose

$$v = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}. \quad (A4)$$

If $|w\rangle = |v, v\rangle$ is a linear combination of $|\phi_i\rangle, i = 0, 1, \ldots, 6$, then the matrix

$$\begin{bmatrix} \phi_0 & \phi_1 & \cdots & v \end{bmatrix} \quad (A5)$$

must be of rank 7. Remove the repeated rows, we have,

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & a_0^2 \\
0 & 1 & 0 & 0 & 1 & 4 & a_2^2 \\
0 & 0 & 1 & 0 & 1 & 9 & a_2^2 \\
0 & 0 & 0 & 1 & 1 & 6 & a_2^2 \\
0 & 0 & 0 & 0 & 1 & 2 & -2 a_{01} \\
0 & 0 & 0 & 0 & 0 & 3 & 3 a_{02} \\
0 & 0 & 0 & 0 & 0 & 1 & 4 - 4 a_{03} \\
0 & 0 & 0 & 0 & 0 & 1 & 6 - 6 a_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 8 a_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 12 - 12 a_{23}
\end{bmatrix} \quad (A6)$$

is of rank 7. It follows the submatrix

$$\begin{bmatrix}
1 & 2 & -2 a_{01} \\
1 & 3 & 3 a_{02} \\
1 & 4 & -4 a_{03} \\
1 & 6 & -6 a_{12} \\
1 & 8 & 8 a_{13} \\
1 & 12 & -12 a_{23}
\end{bmatrix} \quad (A7)$$

is of rank 3. Which is equivalent to

$$\begin{bmatrix}
1 & 5 & a_0 a_2 - a_0 a_1 \\
2 & -2 & a_0 a_3 - a_0 a_1 \\
4 & -4 & a_1 a_2 - a_0 a_1 \\
6 & 10 & a_1 a_3 - a_0 a_1 \\
10 & -10 & a_2 a_3 - a_0 a_1
\end{bmatrix} \quad (A8)$$
is of rank 2. Furthermore,

\[
\begin{pmatrix}
-12 & a_0a_3 + a_0a_1 - 2a_0a_2 \\
-24 & a_1a_2 + 3a_0a_1 - 4a_0a_2 \\
-20 & a_1a_3 + 5a_0a_1 - 6a_0a_2 \\
-60 & a_2a_3 + 9a_0a_1 - 10a_0a_2
\end{pmatrix}
\]

is of rank 1. Hence

\[
\frac{a_0a_3 + a_0a_1 - 2a_0a_2}{3} = \frac{a_1a_2 + 3a_0a_1 - 4a_0a_2}{6} = \frac{a_1a_3 + 5a_0a_1 - 6a_0a_2}{5} = \frac{a_2a_3 + 9a_0a_1 - 10a_0a_2}{15}.
\]

(A9)

If \(a_0 \neq 0\), we can assume that \(a_0 = 1\). Then Eq. (A10) implies that

\[
\begin{align*}
2a_3 &= a_1a_2 + a_1, \\
5a_3 &= a_2a_3 + 4a_1, \\
3a_1a_3 &= a_2a_3 - 6a_1 + 8a_2.
\end{align*}
\]

(A11)

If \(a_1 = 0\), then \(a_2 = a_3 = 0\), where \(|w⟩ = |φ_0⟩\). Solving Eq. (A11), we have

\[a_2 = 3 \text{ or } a_2 = 1.\]

(A12)

If \(a_2 = 3\), by Eq. (A11), \(a_1 = ±2, a_3 = ±4\). Then \(|w⟩ = |φ_5⟩\) or \(|φ_6⟩\).

If \(a_2 = 1\), by Eq. (A11), we have \(a_1 = 1 \text{ or } -\frac{8}{3}\). If \(a_1 = 1\), we have \(a_3 = 1\), where \(|w⟩ = |φ_4⟩\). If \(a_1 = -\frac{8}{3}\), then \(a_3 = a_1\), which coincides with Eq. (A3).

If \(a_0 = 0, a_1 \neq 0\), we can assume that \(a_1 = 1\), by Eq. (A10), we have

\[
0 = \frac{a_2}{6} = \frac{a_3}{5}.
\]

(A13)

It follows that \(|w⟩ = |φ_1⟩\). It remains to consider the case \(a_0 = a_1 = 0\), we assume \(a_2 = 1\), otherwise, \(|φ_7⟩ = |φ_3⟩\).

By Eq. (A10), we have \(a_3 = 0\), i.e., \(|w⟩ = |φ_2⟩\).

Above all, the subspace \(P\) contains exactly 8 real product vectors, that is \(|φ_0⟩, |φ_1⟩, \ldots, |φ_7⟩\).