LOW DIMENSIONAL HOMOLOGY OF LINEAR GROUPS
OVER HENSEL LOCAL RINGS

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Abstract. We prove that if \( R \) is a Hensel local ring with infinite residue field \( k \), the natural map \( H_i(\text{GL}_n(R), \mathbb{Z}/p) \to H_i(\text{GL}_n(k), \mathbb{Z}/p) \) is an isomorphism for \( i \leq 3 \), \( p \neq \text{char } k \). This implies rigidity for \( H_i(\text{GL}_n) \), \( i \leq 3 \), which in turn implies the Friedlander–Milnor conjecture in positive characteristic in degrees \( \leq 3 \).

A fundamental question in the homology of linear groups is that of rigidity: given a smooth affine curve \( X \) over an algebraically closed field \( k \) and closed points \( x, y \) on \( X \), do the corresponding specialization homomorphisms

\[
\sigma_x, \sigma_y : H_*(G(k[X]), \mathbb{Z}/p) \to H_*(G(k), \mathbb{Z}/p)
\]

coincide? Here, \( G \) is a reductive algebraic group and \( p \) is a prime not equal to the characteristic of \( k \). The answer is yes when \( X \) is the affine line and \( G = \text{SL}_n, \text{GL}_n, \text{PGL}_n \) since the inclusion \( G(k) \to G(k[t]) \) induces an isomorphism

\[
H_*(G(k), \mathbb{Z}) \to H_*(G(k[t]), \mathbb{Z})
\]

(see \cite{3}) and the map is split by evaluation at any \( x \in \mathbb{A}^1 \).

Rigidity in algebraic \( K \)-theory has spectacular consequences, including the calculation of the \( K \)-theory of algebraically closed fields and the solution of the Friedlander–Milnor conjecture for the stable general linear group \( GL \). Similarly, a proof of rigidity for \( G(k[X]) \) would imply the Friedlander–Milnor conjecture for \( G \) (see Section \cite{5}).

Rigidity would follow if one could prove the following stronger result. Let \( X \) be a smooth curve over an algebraically closed field \( k \) and let \( x \) be a closed point on \( X \). Denote by \( \mathcal{O}_x^h \) the henselization of \( \mathcal{O}_x \).

Conjecture. The inclusion \( G(k) \to G(\mathcal{O}_x^h) \) induces an isomorphism

\[
H_*(G(k), \mathbb{Z}/p) \to H_*(G(\mathcal{O}_x^h), \mathbb{Z}/p)
\]

for \( p \neq \text{char } k \).

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In Section 5 we sketch a proof of rigidity assuming this conjecture (this argument is well-known). Our results will allow us to deduce the Friedlander–Milnor conjecture in positive characteristic for \( H_i(\text{GL}_n), i \leq 3 \). The conjecture is known to be true for \( H_3(\text{SL}_2) \) by the work of Sah [9].

In this note we study \( H^*(G(R), \mathbb{Z}/p) \) for \( G = \text{SL}_n \) and \( G = \text{GL}_n \), where \( R \) is a Hensel local \( k \)-algebra and \( k \) is an infinite field. Our results are far from complete. We prove the following result.

**Theorem.** The inclusion \( \text{GL}_n(k) \to \text{GL}_n(R) \) induces an isomorphism

\[
H_i(\text{GL}_n(k), \mathbb{Z}/p) \longrightarrow H_i(\text{GL}_n(R), \mathbb{Z}/p)
\]

for \( i \leq 3 \).

Note that this map is split injective for all \( i \); the only issue is surjectivity.

The expert reader by now will have noticed that this theorem contains only one new case, namely \( H_3(\text{GL}_2) \). The above map is in fact an isomorphism for \( i \leq n \) by a combination of Suslin’s stability theorem [8] and rigidity in \( K \)-theory. However, we show a bit more than the theorem states. We first prove that the map

\[
H_2(P\text{GL}_2(k), \mathbb{Z}/p) \longrightarrow H_2(P\text{GL}_2(R), \mathbb{Z}/p)
\]

is an isomorphism and use this to deduce the result for \( H_2(\text{GL}_2) \). Moreover, our approach is different from that used by Suslin [11] to compute \( H_*(\text{GL}(R), \mathbb{Z}/p) \). We then study Bloch groups to obtain the result for \( H_3(\text{GL}_2) \).

The corresponding statement for \( G = \text{SL}_n \) and \( i = 0, 1 \) is almost obvious. However, we provide an alternate proof for \( i = 1 \) in a special case as follows. Let \( F \) be any field and denote by \( m \) the maximal ideal \((t_1, t_2, \ldots, t_m)\) of \( F[[t_1, t_2, \ldots, t_m]]\). Consider the short exact sequence

\[
1 \longrightarrow C_{n,m} \longrightarrow \text{SL}_n(F[[t_1, t_2, \ldots, t_m]]) \longrightarrow \text{SL}_n(F) \longrightarrow 1. \tag{1}
\]

Here, the group \( C_{n,m} \) consists of those matrices which are congruent to the identity modulo \( m \). We have the following result.

**Proposition.** If \( n \geq 3 \), then

\[
H_1(C_{n,m}, \mathbb{Z}) = \bigoplus_{i=1}^{m} \text{sl}_n(F).
\]

When \( n = 2 \), assume further that \( \text{char } F \neq 2 \) and \( F \neq \mathbb{F}_3 \). Then

\[
H_1(C_{2,m}, \mathbb{Z}) = \bigoplus_{i=1}^{m} \text{sl}_2(F).
\]

When \( p \) is a prime distinct from the characteristic of \( k \), the proposition implies that \( H_1(C_{n,m}, \mathbb{Z}/p) = 0 \) unless \( n = 2, \text{char } k = 2 \) or \( F = \mathbb{F}_3 \). In this case, however, one can still conclude that \( H_1(C_{2,m}, \mathbb{Z}/p) = 0 \) by other
methods. One expects that $H_*(C_{n,m}, \mathbb{Z}/p)$ vanishes in all positive degrees. We provide evidence for this in Section 2.

Conventions. Throughout, $k$ denotes a field and $p$ is a prime distinct from the characteristic of $k$.

1. THE CONGRUENCE SUBGROUP

Recall that the group $C_{n,m}$ is the kernel of the natural map

$$SL_n(F[[t_1, t_2, \ldots, t_m]]) \xrightarrow{\text{mod } m} SL_n(F)$$

where $m$ is the ideal $(t_1, t_2, \ldots, t_m)$. Define a sequence of subgroups $C_{n,m}^i$ by

$$C_{n,m}^i = \{X \in SL_n(F[[t_1, t_2, \ldots, t_m]]) : X \equiv I \mod m^i\}.$$ 

If no confusion can result, we usually drop the subscripts from the notation. Note that each $C_i$ is a normal subgroup as it is the kernel of the map

$$SL_n(F[[t_1, t_2, \ldots, t_m]]) \rightarrow SL_n(F[[t_1, t_2, \ldots, t_m]])/m^i).$$

For each $i$, define a homomorphism

$$\rho_i : C_{n,m}^i \rightarrow \bigoplus_\lambda \mathfrak{sl}_n(F)$$

by

$$\rho_i(I + \sum_\lambda t_1^{k_1} t_2^{k_2} \cdots t_m^{k_m} X_\lambda + \cdots) = (X_\lambda)_\lambda$$

where $\lambda = (k_1, k_2, \ldots, k_m)$ is an $m$-partition of $i$ (i.e., $\lambda$ is an $m$-tuple of nonnegative integers whose sum is $i$) and $(X_\lambda)_\lambda$ denotes the element of $\mathfrak{sl}_n(F)$ given by the various $X_\lambda$. Note that $\rho_i$ is well-defined since the equation

$$1 = \det X \equiv 1 + \sum_\lambda t_1^{k_1} \cdots t_m^{k_m} \text{tr} X_\lambda \mod m^{i+1}$$

implies that $\text{tr} X_\lambda = 0$ for each partition $\lambda$. One checks easily that $\rho_i$ is a surjective group homomorphism with kernel $C_{n,m}^{i+1}$ (see [8] for an analogous result for congruence subgroups of $SL_n(\mathbb{Z})$). Hence, for each $i$ we have an isomorphism

$$C_{n,m}^i/C_{n,m}^{i+1} \cong \bigoplus_\lambda \mathfrak{sl}_n(F).$$

Remark. The above discussion remains valid for any simple algebraic group $G$. The quotients $C_i/C_i^{i+1}$ are direct sums of copies of the lie algebra $\mathfrak{g}$. We have chosen to work with $SL_n$ just to fix ideas.

Lemma 1.1. For each $i, j$, $[C_i, C_j] \subseteq C_{i+j}$.

Proof. This follows easily by writing out $XYX^{-1}Y^{-1}$ for $X \in C_i, Y \in C_j$ and noting that the inverse of $I + \sum Z_\lambda + \cdots$ is $I - \sum Z_\lambda + \cdots.$

For any group $G$, denote by $\Gamma^*$ the lower central series of $G$.

Corollary 1.2. For each $i$, $\Gamma_i \subseteq C_{n,m}^i$. 

Proof. The series $C_{n,m}^\ast$ is a descending central series and as such contains the lower central series.

We now recall the following theorem of Klingenberg [5].

**Theorem 1.3.** If $A$ is a local ring then for $n \geq 3$ the only normal subgroups of $SL_n(A)$ are the congruence subgroups. If $n = 2$, the same is true as long as the residue characteristic of $A$ is not 2 or the residue field is not $\mathbb{F}_3$.

**Corollary 1.4.** The filtration $C_{n,m}^\ast$ is the lower central series.

**Proof.** Note that the groups $\Gamma^i$ are characteristic subgroups of $C_{n,m}^\ast$ and hence are normal in $SL_n(F[[t_1, \ldots, t_m]])$. Since there are clearly elements in $\Gamma^i - C_{n,m}^i + 1$, it follows that $\Gamma^i = C_{n,m}^i$ for all $i$.

**Corollary 1.5.** For all $n \geq 3$, $H_1(C_{n,m}, \mathbb{Z}) = \bigoplus_{i=1}^m \mathfrak{s}\mathfrak{l}_n(F)$. If char $F \neq 2$ or $F \neq \mathbb{F}_3$, then $H_1(C_{2,m}, \mathbb{Z}) = \bigoplus_{i=1}^m \mathfrak{s}\mathfrak{l}_2(F)$.

**Proof.** Since $C_{n,m}^2 = \Gamma_{n,m}^2$, we have

\[
H_1(C_{n,m}, \mathbb{Z}) = C_{n,m}/\Gamma^2 = C_{n,m}/C_{n,m}^2 = \bigoplus_{i=1}^m \mathfrak{s}\mathfrak{l}_n(F).
\]

**Remark.** This corollary holds for power series rings over more general rings (for example, rings of the form $\mathbb{Z}[J^{-1}]$ for a set of primes $J$).

**Corollary 1.6.** If $p$ is a prime different from the characteristic of $F$, then $H_1(C_{n,m}, \mathbb{Z}/p) = 0$.

**Proof.** This follows easily by considering the Hochschild–Serre spectral sequence associated to the extension (1).

**2. Conjectures about $H_\ast(C_{n,m}, \mathbb{Z}/p)$**

The isomorphism

\[
H_\ast(SL_n(F[[t_1, \ldots, t_m]]), \mathbb{Z}/p) \cong H_\ast(SL_n(F), \mathbb{Z}/p).
\]

would follow easily if one could prove the following.

**Conjecture 2.1.** For all $l > 0$, $H_l(C_{n,m}(k[[t_1, t_2, \ldots, t_m]]), \mathbb{Z}/p) = 0$.

We have just proved the case $l = 1$. We make the following observations.
2.1. **Continuous homology.** For each $i \geq 1$, denote by $L_{n,m}^i$ the kernel of the homomorphism

$$SL_n(k[t_1, \ldots, t_m]) \to SL_n(k[t_1, \ldots, t_m]/(t_1, \ldots, t_m)^i).$$

The group $C_{n,m}$ is the inverse limit of the nilpotent groups $L_{n,m}/L_{n,m}^i$. One checks that these groups have no homology with $\mathbb{Z}/p$ coefficients by noting that the successive graded quotients $L_{n,m}^j/L_{n,m}^{j+1}$ of $L_{n,m}/L_{n,m}^i$ are $k$-vector spaces; an iterated use of the Hochschild–Serre spectral sequence then shows that $L_{n,m}/L_{n,m}^i$ is $\mathbb{Z}/p$–acyclic. It follows that the “continuous homology”

$$\lim_{\leftarrow} H_{\cdot}(C_{n,m}/C_{n,m}^i, \mathbb{Z}/p) = \lim_{\leftarrow} H_{\cdot}(L_{n,m}/L_{n,m}^i, \mathbb{Z}/p)$$

vanishes and that

$$H_{\cdot}(SL_n(k[t_1, t_2, \ldots, t_m]/m^i), \mathbb{Z}/p) \cong H_{\cdot}(SL_n(k), \mathbb{Z}/p)$$

for all $i \geq 1$.

2.2. **Large acyclic subgroups.** When $m = 1$, it is easy to see that

$$H_{\cdot}(L_{n,1}, \mathbb{Z}/p) = 0$$

(see [1]) and the group $L_{n,1}$ is a subgroup of $C_{n,1}$.

2.3. **Product spaces.** The group $C_{n,m}$ is the subgroup of $\prod_i L_{n,m}/L_{n,m}^i$ consisting of coherent sequences. A theorem of P. Goerss [4] implies that this product is $\mathbb{Z}/p$–acyclic.

2.4. **Cosimplicial replacement.** We have seen that $C_{n,m}$ is the inverse limit of the nilpotent groups $L_{n,m}/L_{n,m}^i$. Let $X_i$ be a $K(L_{n,m}/L_{n,m}^i, 1)$ space and consider the tower of fibrations

$$X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots$$

The homotopy inverse limit of this tower is a $K(C_{n,m}, 1)$-space; denote it by $X$. Computing the homology of homotopy inverse limits is an important problem in homotopy theory, and has been studied intensely by Bousfield [2], Goerss [4], Shipley [10], and others.

The standard technique is to consider the cosimplicial replacement of the tower. This is a cosimplicial space $X^\bullet$ whose $n$th space $X^n$ is a certain product of the spaces $X_i$ of the tower (see Bousfield–Kan [1] XI, Sec. 5). Given a cosimplicial space $X^\bullet$, there is an associated homology spectral sequence

$$E^2_{m,1} = \pi^m H_1(X^\bullet; A)$$

for any abelian group $A$. Here, the term on the right is the $m$th cohomotopy of the cosimplicial abelian group $H_1(X^\bullet; A)$. The main problem is that of convergence of this spectral sequence. Under certain conditions, the spectral sequence converges to the homology of the total space of $X^\bullet$. In the case of the cosimplicial replacement of a tower of fibrations, the total space is the homotopy inverse limit of the tower.
In the case of $C_{n,m}$, we see that the cosimplicial replacement is particularly nice. Its spaces $X^n$ are certain products of the $X_i$ and by Goerss’ theorem each of these is $\mathbb{Z}/p$–acyclic. It follows that the associated homology spectral sequence has $E^2$–term

$$E^2_{m,t} = \begin{cases} \mathbb{Z}/p & (m,t) = (0,0) \\ 0 & (m,t) \neq (0,0). \end{cases}$$

The question is what, if anything, is this spectral sequence converging to? Unfortunately, all known convergence theorems do not apply to the situation at hand. Most of these require some hypotheses on the spaces $X^n$ such as nilpotency (which we do not have in this case—we would need the groups $L_{n,m}/L_{1,n,m}$ to have bounded nilpotence degree). A proof that the spectral sequence converges to the homology of the total space of $X^\bullet$ in this case would immediately show that $C_{n,m}$ is $\mathbb{Z}/p$–acyclic.

Given all of this, it is difficult to imagine that $C_{n,m}$ has any $\mathbb{Z}/p$ homology. Speaking heuristically, the group $C_{n,m}$ is built up by successively attaching copies of $\mathfrak{sl}_n(k)$ and since this group is $\mathbb{Z}/p$–acyclic, one would expect that $C_{n,m}$ is as well.

3. Spectral Sequences and the Second Homology Groups

Hereafter, $R$ denotes a Hensel local $k$-algebra, and $m$ denotes the maximal ideal of $R$. We have $R/m = k$.

We begin by considering the group $PGL_2(R)$. We will show that the inclusion $PGL_2(k) \to PGL_2(R)$ induces an isomorphism on second homology with coefficients in $\mathbb{Z}/p$.

Let $A$ be a local ring with maximal ideal $m$ and residue field $F$, which we assume to be infinite. Recall that a column vector $v \in A^2$ is unimodular if the ideal generated by its entries is $A$ itself. Denote by $\overline{v}$ the vector in $F^2$ obtained by reducing the entries of $v$ modulo $m$. We say that a collection $v_1, v_2, \ldots, v_k$ of unimodular vectors is in general position if the collection $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k$ is in general position (i.e., the matrix determined by any pair of them is invertible).

Construct a simplicial set $S_\bullet$ as follows. The nondegenerate $p$–simplices are collections $v_0, v_1, \ldots, v_p$ of projective equivalence classes of unimodular vectors which are in general position (this amounts to saying that the $v_i$ are distinct closed points in $\mathbb{P}^1$). Denote by $C_\bullet(A)$ the associated simplicial chain complex. A proof of the following may be found in Nesterenko–Suslin [8].

**Lemma 3.1.** The augmented complex $C_\bullet(A) \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is acyclic.

Note that $PGL_2(A)$ acts transitively on $C_i(A)$ for $i \leq 2$ (i.e., $PGL_2(A)$ acts 3-transitively on points in $\mathbb{P}^1(A)$). Denote by 0 the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
by $\infty$ the vector $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$; and by 1 the vector $\left( \begin{array}{c} 1 \\ 1 \end{array} \right)$. Then each orbit of the action of $PGL_2(A)$ on $C_p(A)$ has a unique representative of the form $(0, \infty, 1, v_1, \ldots, v_{p-2})$. Since each $v_i$ is in general position with $0, \infty$ and 1, we see that both entries of $v_i$ must be units in $A$. Since we are using projective equivalence classes of vectors, we may then assume that the vector $v_i$ has the form $\left( \begin{array}{c} 1 \\ \alpha_i \end{array} \right)$ where $\alpha_i$ is a unit in $A$. Moreover, the $\alpha_i$ satisfy the additional conditions that $1 - \alpha_i \in A^\times$ (since $v_i$ is in general position with 1) and $\alpha_i - \alpha_j \in A^\times$ for $i \neq j$ (since $v_i$ is in general position with $v_j$).

Denote by $E^\bullet_\bullet(A)$ the spectral sequence associated to the action of the group $PGL_2(A)$ on $C_\bullet(A)$. This spectral sequence converges to the homology of $PGL_2(A)$ and has $E^1$–term

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma)$$

where $\Sigma_p$ is a set of representatives of the orbits of the action of $PGL_2(A)$ on $C_p(A)$ and $G_\sigma$ is the stabilizer of $\sigma$ in $PGL_2(A)$. Note that the stabilizers of the $PGL_2(A)$ action are

$$G_0 = B(A)/D(A), \quad G_{(0,\infty)} = T(A)/D(A), \quad G_{(0,\infty,1)} = D(A)/D(A)$$

where $B(A)$ is the upper triangular subgroup, $T(A)$ is the diagonal subgroup, and $D(A)$ is the subgroup of scalar matrices. The group $D(A)/D(A)$ is also the stabilizer of each orbit $(0, \infty, 1, v_1, \ldots, v_{p-2})$. Moreover, we have $H_\bullet(B(A)/D(A), \mathbb{Z}) \cong H_\bullet(T(A)/D(A), \mathbb{Z})$ (see [8]). It follows that our spectral sequence $E^\bullet_\bullet(A)$ has $E^1$–term (with $\mathbb{Z}/p$ coefficients)

$$\begin{array}{cccc}
H_\bullet(A^\times, \mathbb{Z}/p) & H_\bullet(A^\times, \mathbb{Z}/p) & H_\bullet(\{1\}, \mathbb{Z}/p) & \bigoplus_{\alpha \in A^\times} H_\bullet(\{1\}, \mathbb{Z}/p) \\
\end{array}$$

and converges to $H_\bullet(PGL_2(A), \mathbb{Z}/p)$.

Now consider the cases $A = k$ and $A = R$ where $k$ is an infinite field. The inclusion $PGL_2(k) \rightarrow PGL_2(R)$ induces an injective map of spectral sequences

$$E^\bullet_\bullet(k) \rightarrow E^\bullet_\bullet(R).$$

**Lemma 3.2.** The inclusion $k^\times \rightarrow R^\times$ induces an isomorphism

$$H_\bullet(k^\times, \mathbb{Z}/p) \rightarrow H_\bullet(R^\times, \mathbb{Z}/p).$$

**Proof.** If $G$ is an abelian group, we have an isomorphism

$$\bigwedge^\bullet_{\mathbb{Z}/p}(G \otimes \mathbb{Z}/p) \otimes \Gamma_\bullet(pG) \xrightarrow{\cong} H_\bullet(G, \mathbb{Z}/p),$$

where $\bigwedge^\bullet_{\mathbb{Z}/p}(G \otimes \mathbb{Z}/p)$ is the exterior algebra of $G \otimes \mathbb{Z}/p$. Then we have

$$H_\bullet(k^\times, \mathbb{Z}/p) \cong \bigwedge^\bullet_{\mathbb{Z}/p}(k^\times \otimes \mathbb{Z}/p) \otimes \Gamma_\bullet(pk^\times),$$

and

$$H_\bullet(R^\times, \mathbb{Z}/p) \cong \bigwedge^\bullet_{\mathbb{Z}/p}(R^\times \otimes \mathbb{Z}/p) \otimes \Gamma_\bullet(pr^\times).$$

Hence

$$H_\bullet(k^\times, \mathbb{Z}/p) \cong H_\bullet(R^\times, \mathbb{Z}/p).$$
where $pG$ is the subgroup of $G$ killed by $p$ and $\Gamma_\bullet$ is a divided power algebra. The lemma will follow if we can show that
\[ k^\times \otimes \mathbb{Z}/p \cong R^\times \otimes \mathbb{Z}/p \quad \text{and} \quad p^{k^\times} \cong p^{R^\times}. \]

We have a natural homomorphism
\[ \pi : R^\times \rightarrow k^\times/(k^\times)^p \]
given by composing reduction modulo $m$ with the projection $k^\times \rightarrow k^\times/(k^\times)^p$. This map is clearly surjective; we need only check that the kernel coincides with $(R^\times)^p$. Suppose $x \in R^\times$ maps to 0 under $\pi$, and consider the equation $t^p - x = 0$ in $R[t]$. Since $\pi(x) = 0$, the reduced equation has a solution in $k$. Since $R$ is Henselian and $p$ is invertible in $R$, there exists a $y \in R$ with $y^p - x = 0$. That is, $x \in (R^\times)^p$.

Note that $p^{k^\times}$ is the subgroup of $p$th roots of unity in $k$. A similar application of Hensel’s Lemma shows that this coincides with $p^{R^\times}$. This completes the proof of the lemma.

**Corollary 3.3.** For each $n$, the natural map
\[ H_1(GL_n(k), \mathbb{Z}/p) \rightarrow H_1(GL_n(R), \mathbb{Z}/p) \]
is an isomorphism.

**Corollary 3.4.** The map of spectral sequences
\[ E_{p,q}^1(k) \rightarrow E_{p,q}^1(R) \]
is an isomorphism for $p \leq 2$.

**Corollary 3.5.** The natural map
\[ H_2(PGL_2(k), \mathbb{Z}/p) \rightarrow H_2(PGL_2(R), \mathbb{Z}/p) \]
is an isomorphism.

**Proof.** The $E^2$–terms of the spectral sequences look as follows:

\[
E^2 : \begin{array}{ccc}
E^2_{0,2} & E^2_{1,2} & 0 \\
0 & 0 & 0 \\
\mathbb{Z}/p & 0 & 0 & E^2_{3,0}
\end{array}
\]

By considering the commutative diagram
\[
\begin{array}{ccc}
E^3_{3,0}(k) & \xrightarrow{d^1} & E^2_{0,2}(k) \\
\downarrow & & \downarrow \\
E^3_{3,0}(R) & \xrightarrow{d^1} & E^2_{0,2}(R)
\end{array} \rightarrow H_2(PGL_2(k), \mathbb{Z}/p) \rightarrow 0
\]

we see that the map
\[ H_2(PGL_2(k), \mathbb{Z}/p) \rightarrow H_2(PGL_2(R), \mathbb{Z}/p) \]
is an isomorphism.
Corollary 3.6. The natural map
\[ H_2(GL_2(k), \mathbb{Z}/p) \to H_2(GL_2(R), \mathbb{Z}/p) \]
is an isomorphism.

Proof. The Hochschild–Serre spectral sequences associated to the extensions
\[
1 \to k^\times \to GL_2(k) \to PGL_2(k) \to 1 \\
1 \to R^\times \to GL_2(R) \to PGL_2(R) \to 1
\]
are isomorphic at \( E_{p,q}^2 \) for \( 0 \leq p \leq 2, 0 \leq q \leq 2 \). This yields a commutative diagram
\[
\begin{array}{ccc}
H_3(PGL_2(k), \mathbb{Z}/p) & \xrightarrow{d_1^3} & E_{0,2}^3(k) \\
\downarrow & & \downarrow \\
H_3(PGL_2(R), \mathbb{Z}/p) & \xrightarrow{d_1^3} & E_{0,2}^3(R) \\
\cdots & \xrightarrow{\sim} & H_2(PGL_2(k), \mathbb{Z}/p) \\
\cdots & \xrightarrow{\sim} & H_2(PGL_2(R), \mathbb{Z}/p)
\end{array}
\]
The result follows from the Five Lemma.

Corollary 3.7. The natural map
\[ H_2(GL_n(k), \mathbb{Z}/p) \to H_2(GL_n(R), \mathbb{Z}/p) \]
is an isomorphism for all \( n \).

Proof. Consider the commutative diagram
\[
\begin{array}{ccc}
H_2(GL_2(k), \mathbb{Z}/p) & \xrightarrow{\sim} & H_2(GL_2(R), \mathbb{Z}/p) \\
\downarrow & & \downarrow \\
H_2(GL_n(k), \mathbb{Z}/p) & \longrightarrow & H_2(GL_n(R), \mathbb{Z}/p)
\end{array}
\]
The vertical arrows are isomorphisms by Suslin’s stability theorem \[8\].

Remark. The last two corollaries actually follow from the corresponding statement for \( H_2(GL) \) \[11\]. Our approach provides an alternate proof of this fact and also has the advantage of proving the corresponding statement for \( PGL_2 \).

Remark. Since the map \( E_{p,q}^1(k) \to E_{p,q}^1(R) \) is an isomorphism for \( p \leq 2 \), to show that \( H_\bullet(PGL_2(k), \mathbb{Z}/p) \cong H_\bullet(PGL_2(R), \mathbb{Z}/p) \) it would suffice to show that the chain complexes
\[
D_\bullet(k) = E_{3,0}^1(k) \leftarrow E_{4,0}^1(k) \leftarrow \cdots
\]
and
\[
D_\bullet(R) = E_{3,0}^1(R) \leftarrow E_{4,0}^1(R) \leftarrow \cdots
\]
are quasi-isomorphic. One way to do this is to write down a contracting homotopy for the quotient complex \( Q_\bullet = D_\bullet(R)/D_\bullet(k) \). Note that the complex \( Q_\bullet \) must be acyclic in the case \( R = k[t_1, \ldots, t_m]/m^l \) for each \( l \geq 2 \).
(see 2.1). One would hope to be able to write down a contracting homotopy which is independent of \(l\) (or more generally, which is independent of the fact that \(R\) is a truncated polynomial ring, but is really just a local \(k\)-algebra). However, this seems to be a very difficult question (even in the simplest case \(m = 1, l = 2\)). In fact, so far we have been unable to write down a contracting map in the first case \(Q_0 \to Q_1\).

4. Bloch Groups and Third Homology

We now turn our attention to the computation of \(H_3(GL_2(R), \mathbb{Z}/p)\). The case \(R = k\) was treated fully by Suslin in his beautiful paper [12]. We recall the following theorem.

**Theorem 4.1.** There are exact sequences

\[
H_3(GM_2(k), \mathbb{Z}) \to H_3(GL_2(k), \mathbb{Z}) \to B(k) \to 0
\]

and

\[
\pi_3^0(BGM(k)^+) \to K_3(k) \to B(k) \to 0.
\]

Here, \(B(k)\) denotes the Bloch group of \(k\), \(GM(k)\) denotes the group of monomial matrices over \(k\), and \(\pi_3^0(BGM(k)^+)\) denotes the kernel of the canonical map

\[
\pi_3(BGM(k)^+) \to \pi_3(B\Sigma^+)
\]

induced by the projection \(GM(k) \to \Sigma\) where \(\Sigma\) is the infinite symmetric group \((\Sigma = \bigcup_{n \geq 2} \Sigma_n)\).

The amazing fact is that the above theorem holds for any local \(k\)-algebra, as we now describe.

For any local ring \(A\) with infinite residue field \(k\), denote by \(D(A)\) the free abelian group with basis \([x]\), where \(x, 1-x \in A^x\) and define a homomorphism

\[
\phi : D(A) \to A^x \otimes A^x
\]

by \(\phi([x]) = x \otimes (1-x)\). Denote by \(\sigma\) the involution of \(A^x \otimes A^x\) given by \(\sigma(x \otimes y) = -y \otimes x\), and by \((A^x \otimes A^x)_\sigma\) the quotient of \(A^x \otimes A^x\) by the action of \(\sigma\). Define a group \(p(A)\) by

\[
p(A) = D(A)/([x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]).
\]

One checks easily (see Lemma 1.1 of [12]) that \(\phi\) induces a homomorphism \(\overline{\phi} : p(A) \to (A^x \otimes A^x)_\sigma\). By definition, the Bloch group, \(B(A)\), is the kernel of \(\overline{\phi}\).

Consider the action of \(GL_2(A)\) on the chain complex \(C_\bullet(A)\) of the previous section. The resulting spectral sequence is studied by Suslin in Section 2 of [12]. The main result of that section is the existence of the exact sequence

\[
H_3(GM_2(k), \mathbb{Z}) \to H_3(GL_2(k), \mathbb{Z}) \to B(k) \to 0
\]

(Theorem 2.1). The proof of Theorem 2.1 goes through word for word with \(k\) replaced by \(A\). Hence we have the following result.
Proposition 4.2. There is an exact sequence

$$H_3(GM_2(A), \mathbb{Z}) \rightarrow H_3(GL_2(A), \mathbb{Z}) \rightarrow B(A) \rightarrow 0.$$ 

To construct the second exact sequence, Suslin considers the action of $GL_3(k)$ on the chain complex $C^2_\bullet(k)$, where each $C^2_p(k)$ is the free abelian group on $(p + 1)$-tuples of points in general position in $\mathbb{P}^2(k)$. The construction works equally well over $A$ if we recall that closed points in $\mathbb{P}^2(A)$ consist of projective equivalence classes of unimodular vectors in $A^3$. The resulting complex $C^2_\bullet(A)$ is acyclic and the proofs of all the results in Section 3 of [12] go through with one minor modification. Lemma 3.4 of [12] asserts that the restriction of the homomorphism $\partial : H_3(GL_3(k)) \rightarrow B(k)$ to $H_3(GL_2(k))$ coincides with the homomorphism above. The proof makes use of the projection $\mathbb{P}^2(k) \rightarrow \mathbb{P}^1(k)$ centered at $(0,0,1)$, which is not well-defined for $\mathbb{P}^2(A)$. However, a careful reading of the proof shows that one need only construct maps among the various $(C^\bullet)_GL_\alpha(k)$ under consideration. The projection $\mathbb{P}^2(A) \rightarrow \mathbb{P}^1(A)$ is well-defined on these coinvariants since each orbit has a unique representative for which each entry of each vector is a unit in $A$. This allows the proof of Lemma 3.4 to go through for $A$.

The results in Section 4 and those through Proposition 5.1 in Section 5 may be applied word for word to $A$. The end result is the following.

Proposition 4.3 (cf. Prop. 5.1, [12]). There is an exact sequence

$$\pi_3^0(BGM(A)^+) \rightarrow \tilde{K}_3(A) \rightarrow B(A) \rightarrow 0.$$ 

Remark. The rest of the proofs in Section 5 of [12] do not go through for $A$ as stated. Thus, we cannot immediately get the exact sequence

$$0 \rightarrow \text{Tor}(\mu(A), \mu(A)) \rightarrow \tilde{K}_3(A) \rightarrow B(A) \rightarrow 0.$$ 

However, since $\mu(R) = \mu(k)$ for a Hensel local $k$-algebra $R$ (see Lemma 3.2) and $K^M_\bullet(R) \otimes \mathbb{Z}/p = K^M_\bullet(k) \otimes \mathbb{Z}/p$, we see that the exact sequence exists for $R$ after tensoring with $\mathbb{Z}/p$ (see Proposition 4.4 below). Moreover, the above results hold even more generally (e.g., semilocal rings with infinite residue fields).

We now compare the exact sequences for $k$ and $R$, where $R$ is a Hensel local $k$-algebra. Let $p$ be a prime distinct from the characteristic of $k$. Since tensor product is right exact, we have a commutative diagram

$$\begin{array}{cccccc}
\pi_3^0(BGM(k)^+) \otimes \mathbb{Z}/p & \rightarrow & K_3(k) \otimes \mathbb{Z}/p & \rightarrow & B(k) \otimes \mathbb{Z}/p & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_3^0(BGM(R)^+) \otimes \mathbb{Z}/p & \rightarrow & K_3(R) \otimes \mathbb{Z}/p & \rightarrow & B(R) \otimes \mathbb{Z}/p & \rightarrow & 0.
\end{array}$$

The map $B(k) \otimes \mathbb{Z}/p \rightarrow B(R) \otimes \mathbb{Z}/p$ is injective and the map $K_3(k) \otimes \mathbb{Z}/p \rightarrow K_3(R) \otimes \mathbb{Z}/p$ is an isomorphism [11]. A simple diagram chase gives the following result.
Proposition 4.4. The natural map
\[ B(k) \otimes \mathbb{Z}/p \longrightarrow B(R) \otimes \mathbb{Z}/p \]
is an isomorphism.

We are now in a position to compute \( H_3(GL_2(R), \mathbb{Z}/p) \).

Proposition 4.5. The natural map
\[ H_3(GL_2(k)) \otimes \mathbb{Z}/p \longrightarrow H_3(GL_2(R)) \otimes \mathbb{Z}/p \]
is an isomorphism.

Proof. Consider the commutative diagram
\[
\begin{array}{c}
H_3(GL_2(k)) \otimes \mathbb{Z}/p \\
\downarrow \\
H_3(GL_2(R)) \otimes \mathbb{Z}/p
\end{array}
\]
\[
\begin{array}{c}
\longrightarrow B(k) \otimes \mathbb{Z}/p \\
\downarrow \\
\longrightarrow B(R) \otimes \mathbb{Z}/p
\end{array}
\]
\[ 0 \]

The map \( H_3(GL_2(k)) \otimes \mathbb{Z}/p \rightarrow H_3(GL_2(R)) \otimes \mathbb{Z}/p \) is an isomorphism since we have an isomorphism
\[ H_\bullet(GL_2(k), \mathbb{Z}/p) \longrightarrow H_\bullet(GL_2(R), \mathbb{Z}/p). \]
The latter isomorphism is proved by considering the Hochschild–Serre spectral sequences associated to the extension
\[ 1 \longrightarrow T \longrightarrow GM_2 \longrightarrow \Sigma_2 \longrightarrow 1 \]
and noting that \( H_\bullet(T(k), \mathbb{Z}/p) \cong H_\bullet(T(R), \mathbb{Z}/p) \). By the Five Lemma, the map
\[ H_3(GL_2(k)) \otimes \mathbb{Z}/p \longrightarrow H_3(GL_2(R)) \otimes \mathbb{Z}/p \]
is an isomorphism.

Corollary 4.6. The natural map
\[ H_3(GL_2(k), \mathbb{Z}/p) \longrightarrow H_3(GL_2(R), \mathbb{Z}/p) \]
is an isomorphism.

Proof. Consider the commutative diagram of universal coefficient sequences
\[
\begin{array}{c}
H_3(GL_2(k)) \otimes \mathbb{Z}/p \\
\downarrow \\
H_3(GL_2(R)) \otimes \mathbb{Z}/p
\end{array}
\]
\[
\begin{array}{c}
\longrightarrow pH_2(GL_2(k)) \\
\downarrow \\
\longrightarrow pH_2(GL_2(R))
\end{array}
\]
The last map is an isomorphism by stability for \( H_2(GL_n) \) and the corresponding statement for \( H_2(GL) \). The result now follows from the Five Lemma.
5. Unstable Rigidity and the Friedlander–Milnor Conjecture

We now apply our results to the study of the Friedlander–Milnor conjecture in low degree. I thank Andrei Suslin for showing me the proofs of Propositions 5.1 and 5.4 below; these are well-known to the experts.

**Proposition 5.1.** Let $X$ be a smooth affine curve over an algebraically closed field $k$ and let $x, y$ be closed points on $X$. Suppose that for all $z \in X$, the inclusion $G(k) \to G(O^h_x)$ induces an isomorphism on homology with $\mathbb{Z}/p$ coefficients. Then the specialization homomorphisms

$$s_x, s_y : H_\bullet(G(k[X]), \mathbb{Z}/p) \to H_\bullet(G(k), \mathbb{Z}/p)$$

coincide.

**Proof.** (Sketch) Observe that

$$H_\bullet(G(O^h_x), \mathbb{Z}/p) = \lim_{\rightarrow} H_\bullet(G(Y), \mathbb{Z}/p).$$

Let $\alpha \in H_\bullet(G(k[X]), \mathbb{Z}/p)$. Denote by $\alpha(x)$ the image of $\alpha$ under the specialization map $s_x$. Note that since $H_\bullet(GL_n(k), \mathbb{Z}/p)$ is a direct summand of $H_\bullet(G(k[X]), \mathbb{Z}/p)$, the class $\alpha(x)$ also lies in the latter group. Consider $\alpha - \alpha(x)$. This class has specialization 0 and maps to 0 in $H_\bullet(G(O^h_x), \mathbb{Z}/p)$. It follows that there is a curve $Y$ which is étale over $X$ such that $\alpha - \alpha(x)$ maps to 0 in $H_\bullet(G(k[Y]), \mathbb{Z}/p)$. Denote by $\alpha_Y$ the image of $\alpha$ in $H_\bullet(G(k[Y]), \mathbb{Z}/p)$. Then we have that $\alpha_Y = \alpha(x)$, which is a constant class. It follows that the specialization of $\alpha_Y$ at all points of $Y$ is the same and hence there is an affine open neighborhood of $x \in X$ on which the specialization of $\alpha$ is constant. Now if $y \in X$, find an affine neighborhood of $y$ on which the specialization is constant. Since these two neighborhoods have nontrivial intersection, we see that $\alpha(x) = \alpha(y)$; i.e., the homomorphisms $s_x$ and $s_y$ agree.

**Corollary 5.2.** If $X$ is a smooth affine curve and $x, y$ are closed points on $X$, then the specialization homomorphisms

$$s_x, s_y : H_i(GL_n(k[X]), \mathbb{Z}/p) \to H_i(GL_n(k), \mathbb{Z}/p)$$

coincide for $i \leq 3$.

**Proof.** By Corollary 4.6, $H_i(GL_n(O^h_x), \mathbb{Z}/p) \cong H_i(GL_n(k), \mathbb{Z}/p)$ for $i \leq 3$. The result follows.

Recall the Friedlander–Milnor conjecture [3]. Suppose that $G$ is a reductive group scheme over an algebraically closed field $k$. Denote by $BG_k$ the simplicial classifying scheme of $G$ and by $BG(k)$ the classifying space of the discrete group $G(k)$ of $k$-rational points of $G$.

**Conjecture 5.3.** The comparison map

$$H_\bullet\ell(BG_k, \mathbb{Z}/p) \to H^\bullet(BG(k), \mathbb{Z}/p)$$

is an isomorphism.
This is known to hold when $k = \overline{\mathbb{F}}_l$ [3], and for the stable general linear group $GL_n$ [11]. A few other special cases are known [4].

Rigidity implies the following result.

**Proposition 5.4.** Let $k$ be an algebraically closed field and denote by $K$ the algebraic closure of $k(T)$. Then if rigidity holds, the natural homomorphism

$$i_* : H_\bullet(G(k), \mathbb{Z}/p) \rightarrow H_\bullet(G(K), \mathbb{Z}/p)$$

is an isomorphism.

**Proof.** (Sketch) Injectivity does not require rigidity. Indeed, $K = \lim \rightarrow k[C]$, where $C$ ranges over the smooth affine curves over $k$, and each map $H_\bullet(G(k[C]), \mathbb{Z}/p) \rightarrow H_\bullet(G(k[C]), \mathbb{Z}/p)$ is split injective. It follows that $i_*$ is injective (but not necessarily split).

To prove surjectivity, note that each class in $H_\bullet(G(K), \mathbb{Z}/p)$ comes from some $\alpha \in H_\bullet(G(k[C]), \mathbb{Z}/p)$, where $C$ is a smooth affine curve over $k$. Denote by $C_K$ the curve $C \times_{\text{Spec } k} \text{Spec } K$ and by $x$ the point of $C_K$ given by the inclusions $k[C] \hookrightarrow k(C) \hookrightarrow K$. Consider the class $\alpha_K \in H_\bullet(G(K[C]), \mathbb{Z}/p)$ arising from the homomorphism

$$H_\bullet(G(k[C]), \mathbb{Z}/p) \rightarrow H_\bullet(G(K[C]), \mathbb{Z}/p)$$

and denote by $\alpha_K(x)$ the image of $\alpha$ in $H_\bullet(G(K), \mathbb{Z}/p)$. Choose a rational point $y : \text{Spec } k \rightarrow C$. This yields another map $y : \text{Spec } K \rightarrow C$. Rigidity implies that $\alpha_K(x) = \alpha_K(y)$ and hence $\alpha_K(x) = \alpha(y)$ (since $y$ factors through $\text{Spec } k$) so that $\alpha$ arises from a class in $H_\bullet(G(k), \mathbb{Z}/p)$. That is, the map $i_*$ is surjective. \hfill \Box

**Corollary 5.5.** If rigidity holds and $\text{char } k > 0$, then the Friedlander–Milnor conjecture is true for $G$.

**Proof.** Note that if $k$ is an algebraically closed field of characteristic $l > 0$, then $k$ is a union of transcendental extensions of $\overline{\mathbb{F}}_l$. It follows that the natural map

$$H_\bullet(G(k), \mathbb{Z}/p) \rightarrow H_\bullet(G(\overline{\mathbb{F}}_l), \mathbb{Z}/p)$$

is an isomorphism. The result follows by considering the commutative diagram

$$
\begin{array}{ccc}
H^\bullet_{\text{et}}(BG_k, \mathbb{Z}/p) & \rightarrow & H^\bullet(BG(k), \mathbb{Z}/p) \\
\downarrow & & \downarrow \\
H^\bullet_{\text{et}}(BG(\overline{\mathbb{F}}_l), \mathbb{Z}/p) & \rightarrow & H^\bullet(BG(\overline{\mathbb{F}}_l), \mathbb{Z}/p)
\end{array}
$$

where the left vertical arrow is an isomorphism by the proper base change theorem for étale cohomology and the bottom arrow is an isomorphism by [3]. \hfill \Box

**Corollary 5.6.** The Friedlander–Milnor conjecture holds in positive characteristic for $H_i(GL_n)$, $i \leq 3$. \hfill \Box
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