A Space-Time Approach for the Time-Domain Simulation in a Rotating Reference Frame

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Abstract—We approach the discretisation of Maxwell’s equations directly in space-time without making any non-relativistic assumptions with the particular focus on simulations in rotating reference frames. As a research example we study Sagnac’s effect in a rotating ring resonator. After the discretisation, we express the numerical scheme in a form resembling 3D FIT with leapfrog. We compare the stability and convergence properties of two 4D approaches, namely FIT and FEM, both using Whitney interpolation.

I. INTRODUCTION

Concerning Clifford’s Geometric Algebra, we use the nomenclature and notation of [5], which we also perceive as a comprehensive introduction to this mathematical formalism. Maxwell’s equations [5, Eq. (7.39)] read

\[ \nabla \wedge F = 0, \quad \nabla \cdot G = J, \]

where F (or G) is the (dual) Faraday bivector, and J the space-time current vector.

The bivectors F and G are related by constitutive material mapping \( \xi \) defined implicitly by

\[ G = \xi(F). \]  

(2)

For linear, isotropic media \( \xi \) is a space-time generalisation of the well known constitutive equations. To derive an explicit formula for \( \xi \), we start with

\[ \begin{align*}
\vec{D} &= \varepsilon \vec{E} \\
\vec{H} &= \nu \vec{B}
\end{align*} \]

(3)

where \( \varepsilon, \nu, \vec{E}, \vec{H}, \vec{D}, \vec{B} \) are electric permittivity, magnetic reluctance, electric and magnetic field strengths, and fluxes, respectively. The resulting expression is given by

\[ \xi(F) = \frac{1}{2} \left[ \left( \varepsilon + \frac{\nu}{c^2} \right) F - \left( \varepsilon - \frac{\nu}{c^2} \right) u Fu \right], \]

with \( u \) the four-velocity vector of the material, and \( c \) the speed of light in vacuum.

II. MOTION AS MESH’S GEOMETRY

Motion of the system is modelled by specifying Lagrangian placement map \( p_t(\vec{r}_{\text{ref}}) \), which gives a space-time position (at time \( t \)) of a particle with initial position \( \vec{r}_{\text{ref}} \). The cylindrical coordinates of \( \vec{r}_{\text{ref}} \) are denoted by \( r, \varphi, z \). For a system rotating with constant angular velocity \( \Omega \) around \( z \)-axis we use

\[ p_t(\vec{r}_{\text{ref}}) = [t, r \cos(\theta), r \sin(\theta), z], \]

(5)

with \( \theta = \varphi + \tanh (r \Omega/c) ct/r \).

Fig. 1. Sketch of a space-time mesh used in simulation. The thick blue line in the left image is mapped to horizontal lines in the right figure.

We extrude a 3D mesh by applying \( p_t \) to nodes’ positions for all time steps \( t_i \) and connecting the 4D nodes stemming from the same 3D node; see Fig. 1. By \( K^i_n \) (respectively \( \tilde{K}^i_n \)) we denote the \( i \)-th \( n \)-dimensional element of the (barycentric dual) mesh.
III. DISCRETIZATION

Maxwell’s equations \([1]\) are discretised by applying their integral form to the primal/dual mesh pair, i.e.,
\[
\oint_{\partial K_i^j} (d^2 x) \cdot F = 0, \quad \oint_{\partial K_i^j} (d^2 x) \wedge G = \int_{K_i^j} (d^3 x) \wedge J,
\]
and introducing scalar DoFs (Degrees of Freedom) on the primal mesh
\[
f_j := \int_{K_i^j} (d^2 x) \cdot F,
\]
and analogously for the dual mesh. Next, we rename and renumber the DoFs on the primal mesh as follows
\[
f_j := \begin{cases} 
  e_t^{n+1/2} & \text{if } K_j^l \text{ is timelike}, \\
  b_m^n & \text{if } K_j^l \text{ is spacelike},
\end{cases}
\]
where the relation between the indices \(j, l, m, n\) is as follows (see also Fig. 2). If the facet \(K_j^l\) is timelike, it is the edge \(l\) in the reference mesh \(M^2\) extruded in time via the placement map \(p_t\) such that \(t \in [t_n, t_{n+1}]\). Similarly, if the facet \(K_j^2\) is spacelike, then it is the image of a facet with index \(m\) in \(M^3\) under the map \(p_t\) with \(t = t_n\). Analogously on the dual mesh, we split \(g_j\) into \(d^{n+1/2}\) and \(h^n\).

![Diagram of relation between indices](image)

**A. Material Equations**

A discrete equivalent of \(\xi\) in \([4]\) is a material matrix \(M_\xi\) relating discrete equivalents of \(F\) and \(G\), i.e.,
\[
g_i = \sum_j [M_\xi]_{ij} f_j.
\]

In both methods described below, we use \(n\)-vector valued Whitney interpolating functions \(N_i^n\) associated with \(i\)-th \(n\)-dimensional element of the mesh to derive explicit expressions for \(M_\xi\).

**B. Resulting System of Linear Equations**

We have split space-time material matrix and DoFs into their 3D counterparts in a way, that the obtained numerical scheme resembles 3D FIT with leapfrog, i.e.,
\[
\begin{align*}
  n &= M_{\nu b} b^{n-1} + M_{\nu e} e^{n-1/2} + M_{\nu b} b^n + M_{\nu e} e^{n+1/2} + M_{\nu b} b^{n+1} + M_{\nu e} e^{n+1/2} + M_{\nu b} b^{n+1} \\
  d^{n+1/2} &= d^{n-1/2} + \bar{C} h^n \\
  e^{n+1/2} &= M_{\nu e}^{-1} \left[ d^{n+1/2} - M_{\nu b} b^n - M_{\nu b} b^{n+1} \right] \\
  b^{n+1} &= b^n + e^{n+1/2}.
\end{align*}
\]
We would like to note, that if \( \Omega = 0 \) then all material matrices except \( M_{ee} \) and \( M_{eb} \) vanish, and on Cartesian grid the 4D FIT is naturally related to 3D FIT with leapfrog.

**IV. STABILITY**

Starting from (17) we derive the recursive formula

\[
\begin{bmatrix}
    b^{n+1} \\
    e^{n+1/2} \\
    b^n
\end{bmatrix} = U^n
\begin{bmatrix}
    b^1 \\
    e^{1/2} \\
    b^0
\end{bmatrix},
\]

where \( U \) is the update matrix

\[
U := \begin{bmatrix}
1 + CM^{-1} \gamma & CM^{-1} \beta & CM^{-1} \alpha \\
M^{-1} \gamma & M^{-1} \beta & M^{-1} \alpha \\
0 & 0 & 0
\end{bmatrix},
\]

where 0 and 1 above are zero and identity matrices of proper dimensions and

\[
M := M_{ee} - \tilde{C} M_{ve} - \tilde{C} M_{vb} C + M_{eb} C
\]

\[
\alpha := M_{eb} + \tilde{C} M_{vb}
\]

\[
\beta := M_{ee} + \tilde{C} M_{vb}
\]

\[
\gamma := \tilde{C} M_{vb} - \tilde{C} M_{vb} - M_{eb}.
\]

The solution vector \([b^n, e^{n-1/2}]\) will stay bounded if the modulus of all eigenvalues \( \lambda \) of \( U \) is less or equal to unity, \( |\lambda| \leq 1 \). For both methods considered, the greatest \( |\lambda| \) calculated on exemplary meshes used in simulations is at most 1.

As another test of stability, we initialise the solver with randomly generated initial values and observe that the norm of the solution does not grow in time.

**V. CONVERGENCE**

In order to verify the convergence of the scheme, we simulate the first six, \( m = 0 \ldots 5 \), eigenmodes of the ring resonator structure studied in [1] Section IV, and compare the rotation induced frequency shift \( \delta \omega_{\text{sim}} \) extracted from our numerical simulation with the non-relativistic semi-analytic approximation, [1] Eq. (4.5), \( \delta \omega_{\text{anal}} = m \Omega \). As depicted in Fig. 3 both approaches agree well in a non-relativistic regime. However, in relativistic cases the relative difference

\[
\eta := \frac{\delta \omega_{\text{sim}} - \delta \omega_{\text{anal}}}{|\delta \omega_{\text{anal}}|}
\]

between two approaches becomes significant, see Table I. We expect our solution to be correct as we have not made any non-relativistic assumptions as opposed to [1].

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![Graph](image_url)

**Fig. 3.** Comparison of analytical (solid lines) and numerical (points) frequency shifts in case of non-relativistic velocities.

**TABLE I**

| \( m \) | \( \eta_{\text{max}} / \text{c} \) |
|---|---|
| 1 | 95.7% 58.1% 2.2% 0.0% 0.2% 3.2% |
| 2 | 95.8% 58.5% 2.8% 0.9% 0.9% 1.0% |
| 3 | 95.8% 59.2% 2.6% 2.0% 3.9% 4.4% |
| 4 | 96.0% 60.6% 4.6% 1.3% 1.3% 2.4% |
| 5 | 96.2% 62.2% 5.0% 8.2% 8.2% 2.6% |

**A. 3D Wave Simulation without Rotation**

The requirements of the convergence proof in [2] are neither fulfilled by \( M_{\xi} \) nor by \( M_{\xi} \) (due to asymmetry). Since this is the feature of the proposed method itself, rather than its space-time extension, we focus now on 3D wave-simulation in the non-rotating ring resonator. We investigate the convergence by comparing results obtained using 4D FIT material matrix \( M_{\xi} \), with the ones using \( M_{\xi} \), for which the proof [2] holds. We use temporal \( L^2 \) norm of a function of time \( w : t \mapsto w(t) \)

\[
\|w\|_2 := \sqrt{\int_{t_{\text{min}}}^{t_{\text{max}}} dt \left[w(t)\right]^2},
\]

(21)

to define the relative difference

\[
\text{diff}(E_2^\pm, E_2^\pm) := \frac{\|E_2^\pm(\bar{r}_{\text{sample}}) - E_1^\pm(\bar{r}_{\text{sample}})\|_2}{\|E_1^\pm(\bar{r}_{\text{sample}})\|_2},
\]

(22)

where \( E_1^\pm(\bar{r}_{\text{sample}}) \) is the time signal at the sample point \( \bar{r}_{\text{sample}} \), obtained via interpolation of DoFs calculated using method and mesh indexed as 1.

The results for a (non-)orthogonal mesh, left (right) in Fig. 4 are depicted in top (bottom) of Fig. 5.
VI. CONCLUSION

We have applied space-time discretisation in two flavours, namely 4D FIT and FEM, without making any non-relativistic approximations. As a verification we recovered with good accuracy the rotation induced frequency shifts of the rotating ring resonator predicted by an alternative non-relativistic approach. The material matrix of our proposed extension of FIT is symmetrised to avoid instabilities. With taking only the symmetric part of the material matrix, the convergence is not guaranteed. However, we investigated numerically that FIT converges to the same solution as FEM, which is known to converge.

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