Poly: An abundant categorical setting for mode-dependent dynamics

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Abstract

Dynamical systems—by which we mean machines that take time-varying input, change their state, and produce output—can be wired together to form more complex systems. Previous work has shown how to allow collections of machines to reconfigure their wiring diagram dynamically, based on their collective state. This notion was called “mode dependence”, and while the framework was compositional (forming an operad of re-wiring diagrams and algebra of mode-dependent dynamical systems on it), the formulation itself was more “creative” than it was natural.

In this paper we show that the theory of mode-dependent dynamical systems can be more naturally recast within the category \( \text{Poly} \) of polynomial functors. This category is almost superlatively abundant in its structure: for example, it has four interacting monoidal structures \((+, \times, \otimes, \circ)\), two of which \((\times, \otimes)\) are monoidal closed, and the comonoids for \(\circ\) are precisely categories in the usual sense. We discuss how the various structures in \( \text{Poly} \) show up in the theory of dynamical systems. We also show that the usual coalgebraic formalism for dynamical systems takes place within \( \text{Poly} \). Indeed one can see coalgebras as special dynamical systems—ones that do not record their history—formally analogous to contractible groupoids as special categories.

1 Introduction

We propose the category \( \text{Poly} \) of polynomial functors on \( \text{Set} \) as a setting in which to model very general sorts of dynamics and interaction. Let’s back up and say what exactly it is that we’re generalizing.

A wiring diagram can be used to specify a fixed communication pattern between systems:

\[
\begin{array}{c}
\text{System} \\
A \\
\text{Controller} \\
B \\
\text{Plant} \\
C \\
\end{array}
\]

Shown here, the plant—say a power plant or a car—is a dynamical system that receives input of type \( A \) from the outside world and input of type \( B \) from the controller, and it produces output of type \( C \); this in turn is fed both to the outside world and to the controller. Given these fixed sets \( A, B, C \), we will see shortly that the two interior boxes
and one exterior box shown in (1) can be faithfully represented by polynomials in one variable \( y \), as follows:

\[
\begin{align*}
\text{Plant} &= Cy^{AB} \\
\text{Controller} &= By^C \\
\text{System} &= Cy^A.
\end{align*}
\]

Observe that in each case the output type is the coefficient on \( y \), and the input type is the exponent on \( y \). In Section 3.3 we will see that the wiring diagram (1) itself, as well as the interacting dynamics, can be represented by morphisms involving these polynomials.

### 1.1 Introduction to mode-dependence

Notice that the polynomials in (2) are monomials; it is this we want to generalize. By using more general polynomials such as \( \text{Robot} = y^{A_1A_2} + y + By \), we can create a system for which the input-output types are not fixed:

\[
\begin{align*}
\left\{ \begin{array}{c}
\text{Robot} \\
\text{accepting inputs}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\text{Robot} \\
\text{non-interacting}
\end{array} \right\} \\
\left\{ \begin{array}{c}
\text{Robot} \\
\text{producing output}
\end{array} \right\}
\end{align*}
\]

What we discuss in this paper are dynamical systems whose interfaces change in time, and similarly where the wiring diagram connecting the systems changes in time. These changes will be based on the internal states of the systems—say robots—involved.

The real world is filled with instances of systems with time-varying input-output patterns. The network topology—the way that the system wires up—changes based on both internal and environmental contexts. Consider the following situations:

1. When too much force is applied to a material, bonds can break;

2. A company may change its supplier at any time;

3. When someone assembles a machine, their own outputs dictate the connection pattern of the machine’s components.

We will discuss (3) and (4) further in Example 3.5. In each of the above cases the wiring diagram—the connection pattern—changes based on the states (position, decision-making, environmental context, etc.) of some or all the systems involved. In [ST17] this was called mode-dependence; the goal of that article was to create an operadic framework in which mode-dependent dynamics and communication could be specified compositionally. While successful, the presentation was fairly ad hoc. The purpose of the present
paper is to explain that the category Poly provides an abundant setting in which to work quite naturally with mode-dependent dynamics.

When we say that Poly is abundant, we mean that it is exceptionally rich in structure, and that structure is highly relevant to dynamical systems. Here are some of the features of this category:

1. Poly has coproducts and products, $+, \times$, the usual sum and product of polynomials.
2. Poly has two additional monoidal structures: $\otimes$ and $\circ$.
3. Poly has two monoidal closed structures: for $\times$ (cartesian closure) and $\otimes$.
4. Poly has a duoidal structure: $(\circ) \otimes (\circ) \to (\otimes) \circ (\otimes)$.
5. Poly has all small limits and is extensive.
6. Poly has two orthogonal factorization systems (epi/mono and vertical/cartesian).
7. Poly admits a monoidal bifibration $Poly \to Set$ with $\otimes \mapsto \times$.
8. Poly admits an adjoint quadruple with $Set$ and an adjoint pair with $Set^{op}$.
9. Comonoids in $(Poly, \circ)$ are precisely categories in the usual sense.

In Section 2 we will introduce Poly and many of its interesting features. In Section 3, we will discuss how these features relate to dynamical systems.

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2 Introduction to Poly

2.1 Polynomial functors

Notation 2.1. We usually denote sets with upper-case letters $A, B$, etc.; the exception is ordinals: we denote the $n$th ordinal by $n = \{1, \ldots, n\}$. We denote functions between sets—including elements of sets—using upright letters $f: A \to B$ and $a \in A$.

All polynomials discussed here have a single variable, always $y$; in particular $y$ itself is a polynomial. Coefficients and exponents of polynomials are arbitrary sets, e.g. $\mathbb{N}y^{\mathbb{R}} + 3$ is a polynomial. Every set $A$ will also be a polynomial, namely a constant. We denote generic polynomials with lower-case letters $p, q$, etc.

Recall that a representable functor $Set \to Set$ is one of the form $Set(A, -)$ for a set $A$. We denote this functor by $y^A : Set \to Set$ and say it is represented by $A \in Set$. For example $y^3$ is represented by 3 and $y^3(2) \cong 8$. As $A$ varies we obtain the contravariant Yoneda embedding.

Classically, a polynomial $p$ in one variable with set coefficients is a function $p(y) = A_n y^n + \cdots + A_1 y^1 + A_0 y^0$ with each $A_i \in \mathbb{N}$. In category theory this is often generalized to allow for infinitely many terms and infinite exponents; e.g. we consider the following to be a polynomial

$$p(y) = \sum_{i \in I} y^{A_i}$$
for arbitrary small sets $I$ and $A$. We can think of such a $p$ as a functor $\text{Set} \to \text{Set}$; it sends a set $X \in \text{Ob}(\text{Set})$ to the coproduct, over $i \in I$, of the set $X^{A_i}$ of functions $A_i \to X$, or equivalently the $A_i$-fold product of $X$ with itself. The result is covariantly functorial in $X$.

Considered this way, $p$ is called a polynomial functor; polynomial functors sit inside of the category of all functors $\text{Set} \to \text{Set}$ as a full subcategory, namely the one spanned by coproducts of representables.

**Definition 2.2.** The category $\text{Poly}$ has polynomial functors $p(y)$ as in (5) as objects and natural transformations between them as morphisms.

In $\text{Poly}$, products distribute over coproducts
\[
\left( \sum_i p_i \right) \times q \cong \sum_i (p_i \times q).
\]
In fact, $\text{Poly}$ can be characterized as the free category that has both coproducts and also products that distribute over coproducts. $\text{Poly}$ is also equivalent to the Grothendieck construction of the canonical functor $\text{Set}^{op} \to \text{Cat}$ sending each object to the corresponding slice category $A \mapsto \text{Set}/A$ and sending $f: B \to A$ to pullback along $f$.

**Notation 2.3.** We denote the product of polynomials by juxtaposition or sometimes $\cdot$, i.e.
\[
pq := p \times q = p \cdot q.
\]
For any set $A$ we denote the $A$-fold repeated product of $p$ by $p^A := \prod_{a \in A} p$; in particular $p^1 \cong p$ and $p^0 \cong 1$. The representable $y^n$ is indeed the $n$-fold repeated product of $y$.

For any polynomial $p$, the set $p(1)$ has particular importance; it can be identified with the set of representable summands (pure-power terms) $y^k$ in $p$. For example if $p = y^2 + 3y + 2$ then $p(1) = 6$ corresponding to the six representable summands in $p = y^2 + y^1 + y^1 + y^1 + y^0 + y^0$. We will denote the representing object for the $i$th representable summand of $p$ by $p_i$, i.e.
\[
p = \sum_{i \in p(1)} y^{p_i}.
\]
There are many ways to think about polynomials, and one becomes more versatile by being able to use different representations for different purposes. So far we have been writing polynomials in the typical algebraic style, but one can also represent them as bundles, as forests of corollas, or as dependent types.

**Algebraic**
\[
y^2 + 3y + 2
\]
**Bundle**
\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\pi \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
**Corolla forest**
\[
\begin{array}{c}
\bigvee \\
\uparrow \\
\uparrow \\
\uparrow \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

Given a bundle $\pi: E \to B$, and element $b \in B$, we denote the fiber $\pi^{-1}(b)$ by $E_b$. We will refer to elements of $B$ as positions and elements of $E_b$ as the directions in position $b$. From the algebraic viewpoint, a position is a ‘pure-power’, or representable summand, and the associated direction-type is its ‘exponent’ or representing object; from the tree viewpoint, a position is a root and the associated directions are its leaves.

Polynomials can be implemented in a dependently typed programming language, such as Idris. Here is a specification of the type for polynomials:
2.2 Morphisms of polynomials, concretely

As mentioned, the morphisms between polynomials are the natural transformations. As easy as this is to state—and as much as it gives us confidence in the reasonableness of the definition—it can be useful to have a more hands-on understanding of the morphisms.

By the Yoneda lemma, a morphism \( y^A \to y^B \) can be identified with a function \( B \to A \). One can prove that \( + \) is the coproduct in \( \text{Poly} \) and \( \times \) is the product. Thus \( y^2 + 3y + 2 \) is a product of \( y + 1 \) and \( y + 2 \), and it is a coproduct of \( y^2 + 1 \) and \( 3y + 1 \). This also holds for infinite sums and products: the usual algebraic operations coincide with the categorical operations. From this, and the fact that coproducts of functors \( \text{Set} \to \text{Set} \) are taken pointwise, we obtain the following formula for the set of morphisms \( p \to q \):

\[
\text{Poly}(p, q) \cong \prod_{i \in p} \sum_{j \in q(i)} p_i^q_i.
\]

For each representable summand of \( p \)—i.e. position of \( p \)—choose a representable summand of \( q \) and give a function from the representing object (exponent) in \( q \) back to the representing object (exponent) in \( p \). Thus for example \( \text{Poly}(y^2 + 3y + 2, y^3 + 1) \equiv (2^5 + 1)(1^5 + 1)(1^5 + 1)(0^5 + 1)(0^5 + 1) \).

In terms of bundles, a morphism \( E \to B \) to \( E' \to B' \) consists of a pair \( (f, f^\sharp) \) as shown:

\[
\begin{array}{c}
E \xleftarrow{\quad f \quad} B \times_B E' \\
\downarrow \quad \downarrow \quad \downarrow \\
B \xrightarrow{\quad f \quad} B'
\end{array}
\]

This will be the most convenient way to write morphisms of polynomials; we further denote by \( f^\sharp_i \) the map on fibers \( E'(f(p)) \to E(p) \). We refer to \( f \) as the on-positions function and \( f^\sharp \) as the on-directions function. This way of thinking about morphisms of polynomials extends readily to Idris:

```idris
record Lens (dom : Poly) (cod : Poly) where
  constructor MkLens
    onPos : position dom -> position cod
    onDir : (i : position dom) -> direction cod j -> direction dom i
      where j = onPos i
```

The reason for the name lens comes from the following.
Example 2.4 (Bimorphic lenses). In [Hed18], Hedges defines the category of bimorphic lenses to have objects given by pairs of sets \((A, B)\) and morphisms (called lenses) from \((A, B)\) to \((A', B')\) defined by a pair of maps \(A \rightarrow A'\) and \(A \times B' \rightarrow B\). It is straightforward to check that Hedges’ category of bimorphic lenses is equivalent to the full subcategory of \(\text{Poly}\) spanned by the monomials \(B^A\).

Monomials \(B^A\) will play a special role in the theory of this paper, namely they correspond to interfaces that have fixed inputs \((A)\) and outputs \((B)\), e.g. as seen in (1).

The category \(\text{Poly}\) has all small limits. Suppose given a small category \(J\) and functor \(p: J \rightarrow \text{Poly}\), and for each \(j\), let \(p^i\) denote the corresponding polynomial. The limit \(\lim_{i \in I} \ p^i\) has positions given by the limit \(\lim_{i \in I} p^i(1)\) of positions, and for each such position \((i')_{i \in I}\), where \(i' \in p^i(1)\), the set of directions there is given by the colimit \(\text{colim}_{i \in I} \ p^i_{i'}\) of directions.

We note two orthogonal factorization systems on \(\text{Poly}\): (epi/mono) and (vertical/cartesian). The first is straightforward (e.g. epimorphisms of polynomials are surjective on positions and injective on directions). More interestingly, the functor \(p \mapsto p(1)\) is a monoidal \(*\)-bifibration in the sense of [Shu08, Definition 12.1]. Indeed, if \(B = p(1)\) and we have a function \(f: A \rightarrow B\), we can take the pullback of polynomials

\[
\begin{array}{c}
A \times_B p \\
\downarrow \quad \downarrow \quad f \\
A \\
\end{array}
\]

Thus we obtain a fibration \(\text{Poly} \rightarrow \text{Set}\), with its attendant vertical/cartesian factorization system. Moreover each functor \(f^*: \text{Poly}_B \rightarrow \text{Poly}_A\) has both a left adjoint \(f_l\) and a right adjoint \(f_r\), and both \(f_l\) and \(f^*\) interact well with \(\otimes\). In fact, identifying \(\text{Poly}^{\text{op}}\) with \(\text{Set}^{A}\), the functors \(\text{Set}^l \rightarrow \text{Set}^l\) arising from multivariate polynomials \(I \leftarrow E \xrightarrow{g} B \xrightarrow{h} J\) as in [GK12] can be represented using the \(*\)-bifibration structure, namely as \((h, g, f^*)^{\text{op}}\).

2.3 Adjunctions with \(\text{Set}\) and \(\text{Set}^{\text{op}}\)

It is useful to note that \(\text{Poly}\) contains two copies of \(\text{Set}\) and a copy of \(\text{Set}^{\text{op}}\), namely as the constant polynomials \(A\), the linear polynomials \(A_B\), and the representables \(y^A\). Indeed there is an adjoint quadruple and an adjoint pair as follows, labeled by where they send objects \(A \in \text{Set}, p \in \text{Poly}\):

\[
\begin{array}{cccc}
\text{Set} & \xleftarrow{\text{A}} & \xrightarrow{\text{Poly}} & \text{Set}^{\text{op}} \\
\text{Set} & \xrightarrow{\text{A}} & \xleftarrow{\text{Poly}} & \text{Set}^{\text{op}} \\
\end{array}
\]

All of the functors out of \(\text{Set}\) and \(\text{Set}^{\text{op}}\) shown in (6) are fully faithful, and the rightmost adjoint \(p \mapsto p(0)\) preserves coproducts. The functor \(\Gamma\) is given by global sections: \(\Gamma p := \text{Poly}(p, y) \cong \prod_{i \in p(1)} \ P_i\).

For each \(A \in \text{Set}\) the functor \(\text{Poly} \rightarrow \text{Set}\) given by \(q \mapsto q(A)\) has a left adjoint, namely \(B \mapsto B^A\); we saw this for the cases \(A \cong 0, 1\) in (6). Using \(p := y^A\) and the Yoneda lemma, this generalizes to a two-variable adjunction \(\text{Set} \times \text{Poly} \rightarrow \text{Poly}\):

\[
\text{Poly}(Ap, q) \cong \text{Poly}(p, q^A) \cong \text{Set}(A, \text{Poly}(p, q)).
\]
2.4 Monoidal structures on Poly

We have already mentioned two monoidal structures on Poly, namely coproduct \((+, 0)\) and product \((\times, 1)\). They are given by the following formulas:

\[
p + q = \sum_{i \in p(1)} y^p_i + \sum_{j \in q(1)} y^q_j \quad \text{and} \quad p \times q = \sum_{i \in p(1)} \sum_{j \in q(1)} y^{p_i + q_j}.
\]  

These form a distributive category. The product monoidal structure is closed—Poly is cartesian closed—and we denote this closure operation by exponentiation:

\[
q^p \cong \prod_{i \in p(1)} q \circ (p_i + y).
\]

Thus for example \((y^2 + 3y + 2)^{y^5 + y^4} \cong ((5 + y^2)(2)(5 + y) + 2)(4 + y^2)(2)(4 + y) + 2)\). The constant-polynomials functor \(\text{Set} \to \text{Poly}\) is cartesian closed.

In terms of bundles, the coproduct is given by disjoint union, and product is given by adding fibers (though the formula is reminiscent of adding fractions):

\[
\begin{pmatrix}
E \\
\downarrow \quad B
\end{pmatrix} + \begin{pmatrix}
E' \\
\downarrow \quad B'
\end{pmatrix} \cong \begin{pmatrix}
E + E' \\
\downarrow \quad B + B'
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
E \\
\downarrow \quad B
\end{pmatrix} \times \begin{pmatrix}
E' \\
\downarrow \quad B'
\end{pmatrix} \cong \begin{pmatrix}
E \times B' + B \times E' \\
\downarrow \quad B \times B'
\end{pmatrix}.
\]

In terms of forests, coproduct (undrawn) is given by disjoint union and product is given by multiplying the roots and adding the leaves. Here is a picture of \((y + 1)(y + 2) \cong y^2 + 3y + 2\):

\[
\begin{array}{ccc}
\bullet \\
\downarrow \\
\bullet
\end{array} \times \begin{array}{ccc}
\bullet \\
\downarrow \\
\bullet \quad \bullet
\end{array} \cong \begin{array}{ccc}
\bullet \\
\downarrow \\
\bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]

There are two more monoidal structures on Poly; one is symmetric and is denoted \((\otimes, y)\), and the other is not symmetric and is denoted \((\circ, y)\). We first discuss \(\otimes\). In terms of polynomials, it is given by the Dirichlet product\(^1\)

\[
p \otimes q = \sum_{i \in p(1)} \sum_{j \in q(1)} y^{p_i q_j},
\]

which we invite the reader to compare with \(\times\) from (8). For example \((y^3 + y) \otimes (y^2 + y^0) \cong y^6 + y^2 + 2y^0\). Like \(\times\), the Dirichlet product \(\otimes\) distributes over \(+\). In terms of bundles, Dirichlet product is straightforward:

\[
\begin{pmatrix}
E \\
\downarrow \\
B
\end{pmatrix} \otimes \begin{pmatrix}
E' \\
\downarrow \\
B'
\end{pmatrix} \cong \begin{pmatrix}
E \times E' \\
\downarrow \\
B \times B'
\end{pmatrix}.
\]

\(^1\)The reason for the name \textit{Dirichlet} is that if one replaces polynomials with Dirichlet series by reversing each summand \(y^A\) to \(A^y\), the result is the usual product. For example

\[
(3^y + 2^y) \times (4^y + 0^y) \cong 12^y + 8^y + 2 \times 0^y
\]

See [SM20] for more on the connection between Dirichlet series and polynomials.
2.5 Comonoids for $\circ$ are categories

In terms of forests, one multiplies roots and for each pair, multiplies the leaves:

The Dirichlet monoidal structure is closed as well and its formula is similar to that in (9). We denote this closure operation (internal hom) using brackets:

$$[p, q] \equiv \prod_{i \in \mathbb{P}(1)} q \circ (p_i y).$$

(11)

Thus for example $[(y^5 + y^4, y^2 + 3y + 2)] \equiv ((5y)^2 + 3(5y) + 2) \cdot ((4y)^2 + 3(4y) + 2)$. The last monoidal structure we discuss, $(\circ, y)$, was already used above in (??). It is the usual composition of polynomials, both algebraically and as functors; e.g. $(y^2 + y) \circ (y^3 + 1) \equiv y^6 + 3y^5 + 2$. Thinking of $p$ as a functor, its evaluation at a set $A$ is $p \circ A$.

The most computationally useful formula for $p \circ q$ is probably the following:

$$p \circ q \equiv \sum_{i \in \mathbb{P}(1)} \prod_{d \in \mathbb{P}(1)} \sum_{j \in \mathbb{Q}(1)} \prod_{e \in \mathbb{Q}(1)} y.$$  

(12)

In terms of forests, $p \circ q$ is obtained by adding up all ways to adjoin trees in $q$ to leaves in $p$. For example, here is $(y^2 + y) \circ (y^3 + 1)$:

$$
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \circ 
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \Rightarrow 
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array}
\end{array}
$$

(13)

More precisely, the monoidal operation $\circ$ collapses the trees in (13) to mere corollas:

$$
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \circ 
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \Rightarrow 
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\text{•} \\
\end{array} \text{•} \\
\begin{array}{c}
\text{•} \\
\end{array}
\end{array}
$$

The composition product $\circ$ is duoidal over $\otimes$ in the sense that there is a natural map

$$(p_1 \circ p_2) \otimes (q_1 \circ q_2) \rightarrow (p_1 \otimes q_1) \circ (p_2 \otimes q_2),$$

(14)

satisfying the usual axioms. Both $+$ and $\times$ commute with $\circ$ on the left

$$(pq + r) \circ s \equiv (p \circ s)(q \circ s) + (r \circ s).$$

2.5 Comonoids for $\circ$ are categories

A comonoid in the (nonsymmetric) monoidal category $(\text{Poly}, \circ, y)$ is a tuple $(C, \epsilon, \delta)$, where $C$ is a polynomial, and $\epsilon: C \rightarrow y$ and $\delta: C \rightarrow C \circ C$ are morphisms of polynomials, such that the usual diagrams commute. Using (14), we can lift the Dirichlet product on polynomials to a monoidal structure $(\otimes, y)$ on comonoids.

One of the most surprising aspects of $\text{Poly}$ is that the comonoids for $\circ$—polynomial comonads on $\text{Set}$—are categories in the usual sense! This requires a calculation (see

\textit{2We use upper case to denote the polynomials that underlie comonoids.}
We next discuss how structures available in Poly describe phenomena in dynamical systems.

---

\[ C_0 \times_{D_0} D_1 \rightarrow C_0 \]
\[ C_0 \times_{D_0} D_1 \rightarrow C_1 \]
\[ C_0 \times_{D_0} D_1 \rightarrow C_0 \]
\[ C_0 \times_{D_0} D_1 \rightarrow C_1 \]
\[ C_1 \times_{D_0} D_1 \rightarrow C_1 \times_{D_0} C_0 \]
\[ C_0 \times_{D_0} D_1 \rightarrow C_1 \times_{D_0} C_1 \]
3.1 Dynamical systems in Poly

By a fixed-interface \((A, B)\)-dynamical system, we mean a Moore machine, i.e. a function \(r: S \to B\) (called readout), and a function \(u: A \times S \to S\) (called update). Given an initial state \(s_0 \in S\), a Moore machine lets us transform any stream \((a_0, a_1, \ldots)\) of \(A\)'s into a stream of \(B\)'s by repeatedly updating the state:

\[ s_{n+1} = u(a_n, s_n), \quad b_n = r(s_n). \]

**Proposition 3.1.** Let \(S, A, B\) be sets. The following are equivalent:

1. Moore machines with inputs \(A\), outputs \(B\),
2. coalgebras for the polynomial functor \(By^A\),
3. morphisms in \(\text{Poly}\) of the form \(S^p \to By^A\).

The second and third perspectives easily generalize to replacing \(By^A\) with an arbitrary polynomial. We prefer the third because it allows us to remain within the category \(\text{Poly}\), which has such abundant structure. Recall from Example 2.7 that \(S^p\) can be given the structure of a comonoid in \((\text{Poly}, \circ, y)\), corresponding under Theorem 2.6 to the contractible groupoid on \(S\).

**Definition 3.2.** A mode-dependent dynamical system consists of a comonoid \((C, \epsilon, \delta)\) in \((\text{Poly}, \circ)\) together with a morphism \(f: C \to p\) for some polynomial \(p\). Here \(C\) is called the state system \(p\) is called the interface and \(f\) is called the dynamics.

Note that given such a morphism \(f: C \to p\), the comonoid structure on \(C\) gives us a canonical morphism \(\delta^{n-1}: C \to C^{\circ n}\) for each \(n\), where \(\delta^{-1} = \epsilon\) and \(\delta^0 = \text{id}\). Since \(\circ\) is monoidal, we also have a map \(f^{\circ n}: C^{\circ n} \to p^{\circ n}\), and composing we obtain

\[ C \to C^{\circ n} \to p^{\circ n}. \]

Thus each \(i \in C(1)\) is endowed with an element of \(p^{\circ n}(1)\), which by (12) can be understood as a length-\(n\) strategy

\[ p^{\circ n}(1) \cong \sum_{i_1 \in p(1)} \prod_{i_1 \in p(1)} \sum_{i_2 \in p(2)} \prod_{i_2 \in p(2)} \cdots \sum_{i_n \in p(n)} \prod_{i_n \in p(n)} 1. \]

It is a choice of a position (move by ‘player’) in \(i_1 \in p(1)\), and for every direction there (move by ‘opponent’) \(d_1 \in p(1)\), a choice of position \(i_2 \in p(1)\), etc. Thus a sort of game is inherent in the dynamical system itself; it would be interesting to explore a relationship between this and open economic game theory [Gha+16].

But the map \(C \to p^{\circ n}\) does not only give a mapping on positions; it says that for every \(n\) choices of directions—each dependent on the last—in \(p\), there is a choice of direction, i.e. morphism, in the comonoid/category \(C\). Thus the history of play is encoded as a morphism in \(C\). In the case of coalgebras, where \(C = S^p\) is simply a contractible groupoid, there is no information encoded in this history of play, except for its final destination.

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\[ \text{For these readers who are more accustomed to coalgebras, note that one can take the limit of the on-}\]

positions functions \(C(1) \to p^{\circ n}(1)\), as \(n\) increases; this induces the usual map from \(C(1)\) to the terminal coalgebra of \(p\). In \(\text{Poly}\) one represents this by a right adjoint \(\text{Poly} \to \text{Comon(Poly)}\) to the forgetful functor. That is, \(C \to p\) induces a comonoid morphism \(C \to \text{CoF}(p)\) to the cofree comonoid on \(p\), which itself is given by the limit \(1 \leftarrow y \cdot p(1) \leftarrow y \cdot p(y \cdot p(1)) \leftarrow \cdots\) in \(\text{Poly}\); its set of positions \(\text{CoF}(p)(1)\) is again the terminal coalgebra on \(p\).
3.2 Products of interfaces

The product of polynomials allows one to overlay two different interfaces on the same state system. That is, given dynamical systems \( C \to p \) and \( C \to q \), there is a unique dynamical system \( C \to pq \). This is quite useful for dynamical systems, as we now show.

**Example 3.3.** Consider two four-state dynamical systems \( 4y^4 \to \mathbb{R} y^{\{r,b\}} \) and \( 4y^4 \to \mathbb{R} y^{\{g\}} \), each of which gives outputs in \( \mathbb{R} \); we think of \( r, b, g \) as red, blue, and green, respectively. We can draw such morphisms as labeled transition systems, e.g.

![Labeled Transition System](image)

Each bullet refers to a state, is labeled by its output position in \( \mathbb{R} \), and has a unique emanating arrow for each sort of input (red and blue, or green), indicating how that state is updated upon encountering said input.

The universal property of products provides a unique way to put these systems together to obtain a morphism \( 4y^4 \to (\mathbb{R} y^{\{r,b\}} \times \mathbb{R} y^{\{g\}}) \). With the examples above, it looks like this:

![Wiring Diagram](image)

Thus the intuitively obvious act of overlaying these dynamical systems falls out of the mathematics, in particular the universal property of products \( \times \) in \( \text{Poly} \). This works for non-monomial (context-dependent) interfaces as well.

3.3 Wiring diagrams and mode-dependence

The Dirichlet product \((10)\) of polynomials and comonoids allows us to juxtapose dynamical systems in an environment. That is, given dynamical systems \( C_1 \to p_1 \) and \( C_2 \to p_2 \), we can form a new dynamical system \( (C_1 \otimes C_2) \to (p_1 \otimes p_2) \).

**Example 3.4 (Wiring diagrams).** Suppose given a wiring diagram such as that in \((1)\); as mentioned in \((2)\), the interfaces of the controller and plant are the polynomials \( By^C \) and \( Cy^{AB} \), and that of the total system is \( Cy \). All of these are monomials, meaning that the directions do not depend on the positions; this allows us to think of positions as outputs and directions as inputs, drawn on the right and left of boxes respectively. The wiring diagram \((1)\) itself is syntax for a morphism

\[
By^C \otimes Cy^{AB} \to Cy^A.
\] (15)

On positions the required map \( BC \to C \) is the projection, and on directions the required map \( BCA \to CAB \) is the obvious symmetry.

**Example 3.5 (Mode-dependent wiring diagrams).** In \((3)\) we depicted a company \( C \) changing its supplier of widgets \( W \), based on \( C \)'s internal state. The company was shown with
no output wires, but in fact it has two positions corresponding to choosing supplier 1 or supplier 2. Let’s redraw it to emphasize its change of position:

The company has interface $2y^W$, and the each supplier has interface $Wy$; let’s take the total system interface (undrawn) to be the closed system $y$. Then this mode-dependent wiring diagram is just a map $2y^W \otimes Wy \otimes Wy \to y$. Its on-positions function $2W^2 \to 1$ is uniquely determined, and its on-directions function $2W^2 \to W$ is the evaluation. In other words, the company’s position determines which supplier from which it receives widgets.

Similarly we could say that the person in (4) has interface $2y$, the units have interfaces $Xy$ and $y^X$ respectively, and the whole system is closed; that is, the diagram represents a morphism $2y \otimes Xy \otimes y^X \to y$. We did not mention but need that unit B has a default value, say $x_0 \in X$, for when its input wire is unattached. The morphism $2Xy^X \to y$ is uniquely determined on positions, and on directions it is given by cases $(1, x) \mapsto x_0$ and $(2, x) \mapsto x$.

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