ABSTRACT. In this paper, we consider the explicit bound for the second-order approximation of the quadratic variation of a general fractional Gaussian process $(G_t)_{t \geq 0}$. The second order mixed partial derivative of the covariance function $R(t, s) = \mathbb{E}[G_t G_s]$ can be decomposed into two parts, one of which coincides with that of fractional Brownian motion and the other of which is bounded by $(ts)^{H-1}$ up to a constant factor. This condition is valid for a class of continuous Gaussian processes that fails to be self-similar or have stationary increments. Under this assumption, we obtain the optimal Berry-Esséen bounds when $H \in (0, \frac{2}{3}]$ and the upper Berry-Esséen bounds when $H \in (\frac{2}{3}, \frac{3}{4}]$. As a by-product, we also show the almost sure central limit theorem (ASCLT) for the quadratic variation when $H \in (0, \frac{4}{5})$. The results extend that of [10] to the case of general Gaussian processes, improve the Berry-Esséen bounds and unify the proofs in [14], [1] and [7] for respectively the sub-fractional Brownian motion, the bi-fractional Brownian motion and the sub-bifractional Brownian motion.

Keywords: Malliavin calculus; Optimal Fourth Moment theorem; Berry-Esséen bounds; Gaussian process.

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1. INTRODUCTION

We are interested in the explicit bound for the second-order approximation of the quadratic variation of a general fractional Gaussian process $G = \{G_t : t \geq 0\}$ on $[0, T]$, defined as

$$Z_n := \sum_{k=0}^{n-1} \left[ (G_{k+1} - G_k)^2 - \mathbb{E}[(G_{k+1} - G_k)^2] \right].$$

Let the renormalization of $Z_n$ be

$$V_n := \frac{1}{\sigma_n} Z_n,$$

(1.1)
where $\sigma_n > 0$ is so that $\mathbb{E}[V_n^2] = 1$, i.e., $\sigma_n^2 := \mathbb{E}[Z_n^2]$.

When $G$ is the fractional Brownian motion $B^H$, by using Stein method and Malliavin calculus, Nourdin and Peccati [9, 10] derived explicit bounds for the total variation distance between the law of $V_n$ and the standard normal law $N$. From then on, the same problem was extended to some other fractional Gaussian processes such as the sub-fractional Brownian motion, the bi-fractional Brownian motion and the sub-bifractional Brownian motion in [14], [1] and [7] respectively.

We find that the above four types of fractional Gaussian processes are all special examples of the following general Gaussian process $G$. Let

$$R^B(s, t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad (1.2)$$

be the covariance function of the fractional Brownian motion $\{B^H(t), t \geq 0\}$.

**HYPOTHESIS 1.1.** For $H \in (0, 1)$, the covariance function $R(t, s) = \mathbb{E}[G_tG_s]$ satisfies that

1. for any $s \geq 0$, $R(0, s) = 0$.
2. for any fixed $s \in (0, T)$, $R(t, s)$ is continuous on $[0, T]$ and differentiable function with respect to $t$ on $(0, s) \cup (s, T)$, such that $\frac{\partial}{\partial t}R(t, s)$ is absolutely integrable.
3. for any fixed $t \in (0, T)$, the difference

$$\frac{\partial R(t, s)}{\partial t} - \frac{\partial R^B(t, s)}{\partial t}$$

is continuous on $[0, T]$, and it is differentiable with respect to $s$ on $(0, T)$ such that $\Psi(t, s)$, the partial derivative with respect to $s$ of the difference, satisfies

$$|\Psi(t, s)| \leq C'_H|ts|^{H-1}, \quad (1.3)$$

where the constants $H, C'_H \geq 0$ do not depend on $T$, and $R^B(t, s)$ is the covariance function of the fractional Brownian motion as in (1.2).

**Example 1.2.** The subfractional Brownian motion $\{S^H(t), t \geq 0\}$ with parameter $H \in (0, 1)$ has the covariance function

$$R(t, s) = s^{2H} + t^{2H} - \frac{1}{2}((s + t)^{2H} + |t - s|^{2H}),$$

which satisfies Hypothesis 1.1.
Example 1.3. The bi-fractional Brownian motion $\{B^{H',K}(t), t \geq 0\}$ with parameters $H', K \in (0,1)$ has the covariance function

$$R(t, s) = \frac{1}{2} \left( (s^{2H'} + t^{2H'})^K - |t - s|^{2H'K} \right),$$

which satisfies Hypothesis 1.1 when $H := H'K$.

Example 1.4. The generalized sub-fractional Brownian motion $S^{H',K}(t)$ with parameters $H' \in (0,1)$, $K \in [1,2)$ and $H'K \in (0,1)$ satisfies Hypothesis 1.1 when $H := H'K$. The covariance function is

$$R(t, s) = (s^{2H'} + t^{2H'})^K - \frac{1}{2} [(t + s)^{2H'K} + |t - s|^{2H'K}].$$

Notation 1. Given two deterministic numeric sequences $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}$, we use the following notations and definitions for respectively commensurability, equivalence:

$$a_n \asymp b_n \iff \exists c, C > 0 : cb_n \geq a_n \leq Cb_n, \text{ for } n \text{ large enough},$$

$$a_n \sim b_n \iff \exists c_n, C_n > 0 : \lim_{n \to \infty} c_n = \lim_{n \to \infty} C_n = 1, c_n b_n \leq a_n \leq C_n b_n, \text{ for } n \text{ large enough}.$$

Now we list our main results as follows:

Theorem 1.5. Let $N \sim N(0,1)$ and $V_n$ be given as in (1.1) and suppose Hypothesis 1.1 holds. When $H \in (0, \frac{2}{3})$,

$$d_{TV}(V_n, N) \asymp n^{-\frac{1}{2}}; \quad (1.4)$$

when $H = \frac{2}{3}$,

$$d_{TV}(V_n, N) \asymp n^{-\frac{1}{2}} \log^2 n; \quad (1.5)$$

when $H \in (\frac{2}{3}, \frac{3}{4})$, there exists a positive constant $c_H$ depending on $H$ such that for any $n \geq 1$,

$$d_{TV}(V_n, N) \leq c_H \times n^{\frac{1}{2}(4H-3)}; \quad (1.6)$$

when $H = \frac{3}{4}$, there exists a positive constant $c$ such that for any $n \geq 1$,

$$d_{TV}(V_n, N) \leq \frac{c}{(\log n)^{\frac{1}{2}}}; \quad (1.7)$$

Remark 1.6. (1) The above Berry-Esséen types bounds are more sharp than those obtained in [14], [1] and [7] for respectively the sub-fractional Brownian motion, the bi-fractional Brownian motion and the sub-bifractional Brownian motion.

(2) We do not know how to obtain the optimal bound in the case of $H \in (\frac{2}{3}, \frac{3}{4}]$. 

(3) In the same vein, we can also extend Theorem 1.5 to the pth Hermite variation with \( p > 2 \).

As a by-product of Theorem 1.5, we have the ASCLT of the sequence \((V_n)_{n \geq 1}\).

**Theorem 1.7.** If \( H \in (0, \frac{3}{4}] \) then the sequence \((V_n)_{n \geq 1}\) satisfies the ASCLT. In other words, for any bounded and continuous function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \), we have almost surely,

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(V_k) \overset{a.s.}{\longrightarrow} \mathbb{E}(\varphi(N))
\]

as \( n \rightarrow \infty \), where \( N \sim N(0,1) \).

### 2. Preliminary

In this section, we describe some basic facts on stochastic calculus with respect to the Gaussian process, for more complete presentation on the subject can be found in [6].

Denote \( G = \{G_t, t \in [0,T]\} \) as a continuous centered Gaussian process with \( G_0 = 0 \) and the covariance function

\[
\mathbb{E}(G_tG_s) = R(s, t), \ s, t \in [0,T],
\]

defined on a complete probability space \((\Omega, \mathcal{F}, P)\), where the \( \mathcal{F} \) is generated by the Gaussian family \( G \). Suppose in addition that the covariance function \( R \) is continuous. Let \( \mathcal{E} \) denote the space of all real valued step functions on \([0, T]\). The Hilbert space \( \mathcal{H} \) is defined as the closure of \( \mathcal{E} \) endowed with the inner product

\[
\langle 1_{[a,b)}, 1_{[c,d]} \rangle_{\mathcal{H}} = \mathbb{E}((G_b - G_a)(G_d - G_c)),
\]

where \( 1_{[a,b)} \) is the indicator function of the interval \([a, b)\). With abuse of notation, we also denote \( G = \{G(h), h \in \mathcal{H}\} \) as the isonormal Gaussian process on the probability space, indexed by the elements in the Hilbert space \( \mathcal{H} \). Then \( G \) is a Gaussian family of random variables such that

\[
\mathbb{E}(G) = \mathbb{E}(G(h)) = 0, \ \mathbb{E}(G(g)G(h)) = \langle g, h \rangle_{\mathcal{H}}, \ \forall g, h \in \mathcal{H}.
\]

**Notation 2.** Let \( R^B(t,s) \) be the covariance function of the fractional Brownian motion as in (1.2). \( V_{[0,T]} \) denote the set of bounded variation functions on \([0, T]\).
For functions \( f, g \in \mathcal{V}_{[0,T]} \), we define two products as
\[
\langle f, g \rangle_{\mathcal{H}_1} = - \int_{[0,T]^2} f(t) \frac{\partial R^B(t,s)}{\partial t} d\nu_g(ds),
\]
\[
\langle f, g \rangle_{\mathcal{H}_2} = C'_H \int_{[0,T]^2} |f(t)g(s)| (ts)^{H-1} dt ds.
\]

(2.4)

The following proposition is an extension of [6, Theorem 2.3] and [5, Proposition 2.2], which gives the inner products representation of the Hilbert space \( \mathfrak{H} \):

**Proposition 2.1.** \( \mathcal{V}_{[0,T]} \) is dense in \( \mathfrak{H} \) and we have
\[
\langle f, g \rangle_{\mathfrak{H}} = \int_{[0,T]^2} R(t,s) \nu_f(dt) \nu_g(ds), \quad \forall f, g \in \mathcal{V}_{[0,T]},
\]
where \( \nu_g \) is the restriction to \(([0,T], \mathcal{B}([0,T]))\) of the Lebesgue-Stieljes signed measure associated with \( g^0 \) defined as
\[
g^0(x) = \begin{cases} 
g(x), & \text{if } x \in [0,T), \
0, & \text{otherwise.}
\end{cases}
\]

Furthermore, if the covariance function \( R(t,s) \) satisfies Hypothesis 1.1, then
\[
\langle f, g \rangle_{\mathfrak{H}} = - \int_{[0,T]^2} f(t) \frac{\partial R(t,s)}{\partial t} d\nu_g(ds), \quad \forall f, g \in \mathcal{V}_{[0,T]}.
\]

(2.6)

In particular, we have
\[
|\langle f, g \rangle_{\mathfrak{H}} - \langle f, g \rangle_{\mathcal{H}_1}| \leq \langle f, g \rangle_{\mathcal{H}_2}, \quad \forall f, g \in \mathcal{V}_{[0,T]}.
\]

(2.7)

**Remark 2.2.** When \( H \in (\frac{1}{2}, 1) \), Hypothesis 1.1 (3) and Lemma 3.1 imply that the identity (2.6) can be rewritten as
\[
\langle f, g \rangle_{\mathfrak{H}} = \int_{[0,T]^2} f(t)g(s) \frac{\partial^2 R(t,s)}{\partial t \partial s} dt ds, \quad \forall f, g \in \mathcal{V}_{[0,T]}.
\]

(2.8)

In this case, the inequality (2.7) has obtained from (2.8) in [4]. When \( H \in (0, \frac{1}{2}) \), it is well known that both \( \frac{\partial^2}{\partial t \partial s} R(t,s) \) and \( \frac{\partial^2}{\partial t \partial s} R^B(t,s) \) are not absolutely integrable. But the absolute integrability of their difference makes the key inequality (2.7) still valid.

**Proof.** The first claim and the identity (2.5) are rephrased from Theorem 2.3 of [6]. Hypothesis 1.1 (2) and Lemma 3.1 imply the inner products representation (2.6).
Finally, it follows from the identity (2.6) and Notation 2 that
\[
\langle f, g \rangle_{H_1} - \langle f, g \rangle_{H} = \int_{[0,T]^2} f(t) \left[ \frac{\partial R(t,s)}{\partial t} - \frac{\partial R^B(t,s)}{\partial t} \right] d\nu_g(ds).
\]
By the fundamental theorem of calculus (see Proposition 1.6.41 of [13]), Hypothesis 1.1 (1) and (3) imply that when \( s \neq t \),
\[
\frac{\partial R(t,s)}{\partial t} - \frac{\partial R^B(t,s)}{\partial t} = \int_0^s \Psi(t,r)dr.
\] (2.9)
Hence, Lemma 3.1 implies that
\[
\langle f, g \rangle_{H_1} - \langle f, g \rangle_{H} = -\int_{[0,T]} f(t) d\int_{[0,T]} \left[ \frac{\partial R(t,s)}{\partial t} - \frac{\partial R^B(t,s)}{\partial t} \right] \nu_g(ds)
= \int_{[0,T]} f(t) d\int_{[0,T]} g(s)\Psi(t,s)ds,
\]
which implies the inequality (2.7) since \( \Psi(t,s) \) satisfies the inequality (1.3). □

Denote \( \mathcal{F}_{\otimes p} \) and \( \mathcal{F}_{\odot p} \) as the \( p \)th tensor product and the \( p \)th symmetric tensor product of the Hilbert space \( \mathcal{F} \). Let \( H_p \) be the \( p \)th Wiener chaos with respect to \( G \). It is defined as the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_p(G(h)) : h \in \mathcal{F}, \| h \|_\mathcal{F} = 1 \} \), where \( H_p \) is the \( p \)th Hermite polynomial defined by
\[
H_p(x) = (-1)^p e^{\frac{x^2}{2}} \frac{d^p}{dx^p} e^{-\frac{x^2}{2}}, \quad p \geq 1,
\]
and \( H_0(x) = 1 \). We have the identity \( I_p(h_{\otimes p}) = H_p(G(h)) \) for any \( h \in \mathcal{F} \) with \( \| h \|_\mathcal{F} = 1 \) where \( I_p(\cdot) \) is the \( p \)th multiple Wiener-Itô integral. Then the map \( I_p \) provides a linear isometry between \( \mathcal{F}_{\otimes p} \) (equipped with the norm \( \sqrt{p!}\| \cdot \|_{\mathcal{F}_{\otimes p}} \)) and \( H_p \). Here \( \mathcal{H}_0 = \mathbb{R} \) and \( I_0(x) = x \) by convention.

The following Theorem 2.3, known as the optimal fourth moment theorem, provides exact rates of convergence in total variation distance between a multiple Wiener-Itô integral and a normal distribution (see [12, 3]).

**Theorem 2.3.** Let \( N \sim N(0,1) \) be a standard Gaussian random variable. Let \( \{ F_n : n \geq 1 \} \) be a sequence of random variables living in the \( p \)th multiple Wiener-Itô integral with unit variance. If \( \lim_{n \to \infty} \mathbb{E}[F_n^4] = 3 \), then there exist two finite constants \( 0 < c < C \) (possibly depending on \( p \) and on the sequence \( \{ F_n \} \), but not on \( n \)) such that the following estimate in total variation holds for every \( n \):
\[
cM(F_n) \leq d_{TV}(F_n, N) \leq C M(F_n),
\]
where

\[ M(F_n) := \max \{ |\mathbb{E}[F_n^3]|, |\mathbb{E}[F_n^4] - 3| \}. \]

The quantities \( \kappa_3(F_n) := \mathbb{E}(F_n^3) \) and \( \kappa_4(F_n) := \mathbb{E}[F_n^4] - 3 \) are called the 3rd and 4th cumulants of \( F_n \). That \( \kappa_3(F_n) \) coincides with the third moment is because \( F_n \) is centered. Moreover, \( \kappa_4(F_n) \) is strictly positive (see [11, 8]).

The following theorem is used to show the ASCLT.

**Theorem 2.4** ([2]). Let \( p \geq 2 \) be an integer, and let \( F_n = I_q(f_n) \), with \( f_n \in H_0 \cap \mathcal{F}^p \).

Assume that for all \( n \), and that \( F_n \xrightarrow{\text{law}} N(0,1) \) as \( n \to \infty \). If the two following conditions are satisfied

\[
(1) \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{\infty} \frac{1}{|k|} \| f_k \otimes_r f_k \|_{\mathcal{F}^{2q(r)}} < \infty, \quad \text{for every } 1 \leq r \leq p - 1,
\]

\[
(2) \sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{\infty} \frac{|\langle f_k, f_l \rangle_{\mathcal{F}^p}|}{kl} < \infty.
\]

Then \( \{ F_n : n \geq 1 \} \) satisfies an ASCLT.

Denote \( \rho(r) := \frac{1}{2} (|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}) \), \( r \in \mathbb{Z} \).

It is well-known that for any \( r \neq 0 \), \( \rho(r) \) is positive when \( H \in (\frac{1}{2}, 1) \) and is negative when \( H \in (0, \frac{1}{2}) \). It behaves asymptotically as \( |r| \to \infty \),

\[ \rho(r) \sim H(2H - 1) |r|^{2H-2}. \]

In particular, when \( H > \frac{1}{2} \), for \( |r| \) large enough,

\[ \rho(r) \geq H(H - \frac{1}{2})(1 + |r|)^{2H-2}. \]

The following proposition is cited from [9, p.74].

**Proposition 2.5.** If \( H \in (0, \frac{3}{4}) \) then

\[ \lim_{n \to \infty} \frac{2}{n} \sum_{i,j=0}^{n-1} \rho^2(i-j) = 2 \sum_{r \in \mathbb{Z}} \rho^2(r) := \sigma^2; \quad (2.11) \]

and if \( H = \frac{3}{4} \) then

\[ \lim_{n \to \infty} \frac{2}{n \log n} \sum_{i,j=0}^{n-1} \rho^2(i-j) = \frac{9}{16}. \]
The following propositions are cited from Proposition 6.6 and Proposition 6.7 of [3] respectively. For the case of $H = \frac{2}{3}$, please refer to [8].

**Proposition 2.6.** We have

$$
\sum_{j,k,l=0}^{n-1} \rho(j-k)\rho(k-l)\rho(j-l) \approx \begin{cases} 
  n, & \text{if } H \in (0, \frac{2}{3}), \\
  n\log^2 n, & \text{if } H = \frac{2}{3}, \\
  n^6H^{-3}, & \text{if } H \in (\frac{2}{3}, \frac{4}{3}].
\end{cases}
$$

and

$$
\sum_{i,j,k,l=0}^{n-1} \rho(i-j)\rho(i-k)\rho(k-l)\rho(j-l) \approx \begin{cases} 
  n, & \text{if } H \in (0, \frac{2}{3}), \\
  n\log n, & \text{if } H = \frac{2}{3}, \\
  n^{8H^{-4}}, & \text{if } H \in (\frac{2}{3}, \frac{3}{4}].
\end{cases}
$$

### 3. Proof of Theorems 1.5 and 1.7.

We will discuss exclusively the case $H \neq \frac{1}{2}$ since the case $H = \frac{1}{2}$ is easy. First, we need two technical lemmas. The first one is a variant of the classical integration by parts formula. It is well known that when $f, \phi : \mathbb{R} \to \mathbb{R}$ are monotone non-decreasing and continuous functions, then

$$
-\int_{[a,b]} f \, d\phi = \int_{[a,b]} \phi \, df + f(a)\phi(a) - f(b)\phi(b) \tag{3.1}
$$

for any compact interval $[a, b]$ [13, p.160]. If $f$ are continuously differentiable on $[a, b]$, we formally have the following expression (see (10) of [5]):

$$
\text{d}(f \cdot 1_{[a,b]}(\cdot)) = \left[f'(t)1_{[a,b]}(t) + f(t)(\delta_a(t) - \delta_b(t))\right]dt,
$$

This expression suggests that we can take the terms $f(a)\phi(a) - f(b)\phi(b)$ in the right hand side of (3.1) as parts of the measure $\nu_f$ for conveniences. This is what the following lemma to do.

**Lemma 3.1.** (Integration by parts formula) Let $[a, b]$ be a compact interval of positive length, let $\phi : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $(a, b)$, such that $\phi'$ is absolutely integrable. For any $f \in \mathcal{V}_{[a,b]}$, we have

$$
-\int_{[a,b]} f(t)\phi'(t)dt = \int_{[a,b]} \phi(t)\nu_f(dt), \tag{3.2}
$$
where \( \nu_f \) is given as in Proposition 2.1, i.e., \( \nu_f \) is the restriction to \(([a, b], \mathcal{B}([a, b]))\) of the Lebesgue-Stieltjes signed measure associated with \( f^0 \) defined as

\[
f^0(x) = \begin{cases} f(x), & \text{if } x \in [a, b), \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** The proof is similar to that of [5, Proposition 2.2]. We establish this in stages. We first deal with the case when \( f \) is a step functions on \([a, b)\) of the form

\[f = \sum_{j=0}^{N-1} f_j \mathbf{1}_{[t_j, t_{j+1})},\]

where \( \{a = t_0 < t_1 < \cdots < t_N = b\} \) is a partition of \([a, b)\) and \( f_j \in \mathbb{R} \). The corresponding signed measure is [6, p.1123]

\[\nu_f = \sum_{j=1}^{N-1} (f_j - f_{j-1})\delta_{t_j} + f(a+)\delta_0 - f(b-)\delta_T.\]

By the fundamental theorem of calculus again, the assumption of \( \phi \) implies that

\[\int_{[t_j, t_{j+1})} \phi'(t) dt = \phi(t_{j+1}) - \phi(t_j).\]

Hence, the following formula of integration by parts hold:

\[- \int_{[a, b]} f(t)\phi'(t) dt = \sum_{j=0}^{N-1} f_j [\phi(t_j) - \phi(t_{j+1})] = \sum_{j=1}^{N-1} (f_j - f_{j-1})\phi(t_j) + f_0\phi(t_0) - f_{N-1}\phi(t_N) = \int_{[a, b]} \phi(t)\nu_f(dt). \quad (3.3)\]

Now we assume that \( f \) is a right continuous monotone non-decreasing function on \([a, b)\). For any sequence partitions \( \pi_n = \{a = t_0^n < t_1^n < \cdots < t_{k_n}^n = b\} \) such that \( \pi_n \subset \pi_{n+1} \) and \( |\pi_n| \to 0 \) as \( n \to \infty \), consider

\[f_n = \sum_{j=0}^{k_n-1} f(t_j^n) \mathbf{1}_{[t_j^n, t_{j+1}^n)},\]
which is uniform bounded since $f$ is bounded. It is clear that the sequence of signed measures $\nu_{f_n}$ converges weakly to $\nu_f$ [6]. Hence, we have

\[
\int_{[a,b]} \phi(t) \nu_f(dt) = \lim_{n \to \infty} \int_{[a,b]} \phi(t) \nu_{f_n}(dt)
\]

\[
= - \lim_{n \to \infty} \int_{[a,b]} f_n(t) \phi'(t) dt \quad \text{(by (3.3))}
\]

\[
= - \int_{[a,b]} f(t) \phi'(t) dt,
\]

where the last line is from Lebesgue’s dominated theorem since $f_n$ is uniform bounded and $\phi'$ are absolutely integrable.

Finally, it is well known that every function of bounded variation is the difference of two bounded monotone non-decreasing function and that the value of $f$ at its points of discontinuity are irrelevant for the purposes of determining the Lebesgue-Stieltjes measure $\nu_f$ [13]. Hence, (3.2) is valid for any $f \in V_{[a,b]}$. □

**Lemma 3.2.** Let $v_1, \ldots, v_l$ be positive. There is a positive constant $c$ depending on $v_1, \ldots, v_l$ such that when $r \in \mathbb{N}$ is large enough,

\[
\sum_{r_i \in \mathbb{N}, \sum_{i=1}^l r_i < r} r_1 v_1^{-1} r_2 v_2^{-1} \cdots r_l v_l^{-1} \leq c \times r^{\sum_{i=1}^l v_i}.
\] (3.4)

**Remark 3.3.** When $v_1, \ldots, v_l$ are negative, the following inequality is trivial: there is a positive constant $c$ depending on $v_1, \ldots, v_l$ such that when $r \in \mathbb{N}$ is large enough,

\[
\sum_{r_i \in \mathbb{N}, \sum_{i=1}^l r_i < r} r_1 v_1^{-1} r_2 v_2^{-1} \cdots r_l v_l^{-1} \leq c < \infty.
\] (3.5)

**Proof.** Let $v_0 > 0$. It is well-known that on the standard simplex in $\mathbb{R}^l$:

\[
T^l := \left\{ (x_1, \ldots, x_l) : x_i \geq 0, \sum_{i=1}^l x_i \leq 1 \right\},
\]

the following integral converges:

\[
\int_{T^l} x_1^{v_1-1} x_2^{v_2-1} \cdots x_l^{v_l-1} (1 - x_1 - \cdots - x_l)^{v_0-1} dx = \frac{\prod_{i=0}^l \Gamma(v_i)}{\Gamma(\sum_{i=0}^l v_i)}
\]

where $\Gamma(\cdot)$ the Gamma function. The change of variables implies that there is a positive constant $c$ depending on $v_0, v_1, \ldots, v_l$ such that when $r \in \mathbb{N}$ is large
enough,
\[
\sum_{r_i \in \mathbb{N}, \sum_{i=1}^l r_i < r} r_1^{v_1-1} r_2^{v_2-1} \cdots r_l^{v_l-1} (r - r_1 - \cdots - r_l)^{v_0-1} \leq c \times r^{-1+\sum_{i=0}^l v_i}.
\]

Especially, when \( v_0 = 1 \), the above inequality collapses to (3.4). \( \Box \)

Without any loss of generality, we suppose for simplicity that \( C_H = 1 \) in this section. Denote
\[
\begin{align*}
    \theta(i, j) &:= \mathbb{E}\left[ (G_{i+1} - G_i)(G_{j+1} - G_j) \right], \\
    \gamma(i, j) &:= \theta(i, j) - \rho(i - j).
\end{align*}
\]
(3.6)

**Proposition 3.4.** Under Hypothesis 1.1, there exists a positive constant \( c \) such that for any \( n \geq 1 \),
\[
\frac{1}{n} \sum_{i,j=0}^{n-1} \gamma(i, j)^2 \leq c \times n^{(4H-3)(-1)}.
\]
(3.7)

Hence, we have that as \( n \to \infty \), when \( H \in (0, \frac{3}{4}) \),
\[
\frac{1}{n} \sum_{i,j=0}^{n-1} \gamma(i, j)^2 \to 0;
\]

and when \( H = \frac{3}{4} \),
\[
\frac{1}{n \log n} \sum_{i,j=0}^{n-1} \gamma(i, j)^2 \to 0.
\]

**Proof.** It is clear that we need only show the inequality (3.7) holds. It follows (2.2), the definition of the inner product, that
\[
\theta(i, j) = \mathbb{E}\left[ (G_{i+1} - G_i)(G_{j+1} - G_j) \right] = \langle \mathbb{I}_{[i,i+1)}, \mathbb{I}_{[j,j+1)} \rangle_{H},
\]
and
\[
\rho(i - j) = \mathbb{E}\left[ (B_{i+1}^H - B_i^H)(B_{j+1}^H - B_j^H) \right] = \langle \mathbb{I}_{[i,i+1)}, \mathbb{I}_{[j,j+1)} \rangle_{H_1},
\]
where \( B^H \) is the fractional Brownian motion with Hurst index \( H \). The inequality (2.7) implies that
\[
|\gamma(i, j)| = |\theta(i, j) - \rho(i - j)| \leq \int_{[0,T]^2} \mathbb{I}_{[i,i+1)}(r_1)\mathbb{I}_{[j,j+1)}(r_2) (r_1 r_2)^{H-1}dr_1 dr_2
= \frac{1}{H^2} [(i+1)^H - i^H] [(j+1)^H - j^H].
\]
(3.8)
Hence, as \( n \to \infty \),
\[
\frac{1}{n} \sum_{i,j=0}^{n-1} \gamma(i,j)^2 \leq \frac{1}{nH^4} \sum_{i,j=0}^{n-1} [(i+1)^H - i^H]^2 [(j+1)^H - j^H]^2
\]
\[
= \frac{1}{n} \left( \frac{1}{H^2} \sum_{i=0}^{n-1} [(i+1)^H - i^H]^2 \right)^2.
\]
(3.9)

It is clear that the function \( f(u) = (1+u)^H \) in \( u \in [0, \infty) \) is concave, i.e., \( f''(u) \leq 0 \), which implies that for any \( u \geq 0 \), \( (1+u)^H \leq 1 + Hu \). Hence, for any \( i \geq 1 \),
\[
(i+1)^H - i^H = i^H \left[ (1 + \frac{1}{i})^H - 1 \right] \leq H \times i^{H-1}.
\]
(3.10)

We have that there exists a positive constant \( c \) independent on \( n \) such that
\[
\frac{1}{H^2} \sum_{i=0}^{n-1} [(i+1)^H - i^H]^2 \leq \frac{1}{H^2} + \sum_{i=1}^{n} i^{2(H-1)} \leq c \times n^{(2H-1)\nu_0},
\]
(3.11)
please refer to Lemma 6.3 of [9].

By plugging the above inequality into (3.9), we obtain the desired inequality (3.7).

\[\square\]

**Proposition 3.5.** Recall that \( \sigma^2_n := \mathbb{E}[Z_n^2] \) and \( \sigma^2 \) is given as in (2.11). Under Hypothesis 1.1, we have
(i) when \( H \in (0, \frac{3}{4}) \), as \( n \to \infty \),
\[
\frac{\sigma^2_n}{n} \to \sigma^2.
\]
(3.12)
(ii) when \( H = \frac{3}{4} \), as \( n \to \infty \),
\[
\frac{\sigma^2_n}{n \log n} \to \frac{9}{16}.
\]
(3.13)

**Proof.** By the definition of multiple Wiener-Itô integrals, we rewrite \( Z_n \) as follows:
\[
Z_n = I_2(g_n),
\]
(3.14)
where
\[
g_n = \sum_{i=0}^{n-1} I_{[i,i+1)}^2.
\]
(3.15)

By Itô's isometry, we have
\[
\sigma^2_n = \mathbb{E}[Z_n^2] = 2 \|g_n\|_\mathcal{B}^2 = 2 \sum_{i,j=0}^{n-1} \langle 1_{[i,i+1)}, 1_{[j,j+1)} \rangle_\mathcal{B}^2 = 2 \sum_{i,j=0}^{n-1} \theta(i,j)^2.
\]
(3.16)
It is clear that the identity (3.6) implies that
\[
\frac{1}{n} \sum_{i,j=0}^{n-1} \theta(i,j)^2 = \frac{1}{n} \sum_{i,j=0}^{n-1} \rho^2(i-j) + \frac{1}{n} \sum_{i,j=0}^{n-1} \gamma^2(i,j) + \frac{2}{n} \sum_{i,j=0}^{n-1} \rho(i-j) \gamma(i,j). \tag{3.17}
\]
The Cauchy-Schwarz inequality implies that the third term satisfies that as \(n \to \infty\),
\[
2 \left( \frac{1}{n} \sum_{i,j=0}^{n-1} \rho^2(i-j) \times \frac{1}{n} \sum_{i,j=0}^{n-1} \gamma(i,j)^2 \right)^{\frac{1}{2}} \to 0,
\]
where in the last line we have used Propositions 2.5 and 3.4. By plugging this limit into the identity (3.17) and using Propositions 2.5 and 3.4 again, we obtain the desired limit (3.12).

In the similar vein, the desired limit (3.13) holds. □

**Proposition 3.6.** Let \(\theta(i,j), \gamma(i,j), \rho(r)\) be given as in (3.6). When \(H \in (0, 1)\), there exists a positive constant \(c\) such that for all \(n \geq 1\)
\[
\left| \sum_{j,k,l=0}^{n-1} \gamma(j,k) \gamma(k,l) \gamma(j,l) \right| \leq c \times n^{(6H-3)\vee 0},
\]
\[
\left| \sum_{j,k,l=0}^{n-1} \gamma(j,k) \gamma(k,l) \rho(j-l) \right| \leq c \times n^{(6H-3)\vee 0}, \tag{3.18}
\]
\[
\left| \sum_{j,k,l=0}^{n-1} \gamma(j,k) \rho(k-l) \rho(j-l) \right| \leq c \times n^{(6H-3)\vee 0}. \tag{3.19}
\]

**Proof.** It follows from the inequalities (3.8) and (3.11) that
\[
\left| \sum_{j,k,l=0}^{n-1} \gamma(j,k) \gamma(k,l) \gamma(j,l) \right| \leq \left( \frac{1}{H^2} \sum_{j=0}^{n-1} [(j+1)^H - j^H]^2 \right)^{\frac{3}{2}} \leq c \times n^{(6H-3)\vee 0}.
\]

Similarly, we have
\[
\left| \sum_{j,k,l=0}^{n-1} \gamma(j,k) \gamma(k,l) \rho(j-l) \right| \leq c \times n^{(6H-3)\vee 0}.
\[ \leq \sum_{j,k,l=0}^{n-1} |\gamma(j, k)\gamma(k, l)\rho(j - l)| \]

\[ \leq cn^{(2H-1)\nu_0} \sum_{j,l=0}^{n-1} [(j + 1)^H - j^H] [(l + 1)^H - l^H] |\rho(j - l)|. \] (3.20)

In the above summation, when \( j = l \), it is clear that

\[ \sum_{j=0}^{n-1} [(j + 1)^H - j^H] [(l + 1)^H - l^H] |\rho(j - l)| = \sum_{j=0}^{n-1} [(j + 1)^H - j^H]^2 \leq cn^{(2H-1)\nu_0}. \]

When \( j = 0 < l \), we have

\[ \sum_{j=0}^{l-1} [(j + 1)^H - j^H] [(l + 1)^H - l^H] |\rho(j - l)| = \sum_{l=1}^{n-1} [(l + 1)^H - l^H] \rho(l) \]

\[ \leq c \sum_{l=1}^{n-1} l^{H-1}l^{2H-2} \]

\[ \leq c \times n^{(3H-2)\nu_0}, \]

where in the last line we use Lemma 6.3 of [9]. The symmetry, the inequality (3.10), the monotonicity of the power function \( f(x) = x^\nu \) with \( x > 0, \nu < 0 \), and the change of variable \( k = l - j \) imply that the other terms are less than:

\[ 2 \times \sum_{0 < j < l \leq n-1} [(j + 1)^H - j^H] [(l + 1)^H - l^H] |\rho(j - l)| \]

\[ \leq c \times \sum_{0 < j < l \leq n-1} j^{H-1}l^{H-1}(l - j)^{2H-2} \]

\[ \leq c \times \sum_{j,k\in\mathbb{N},j+k<n} j^{2H-2}k^{2H-2} \leq c \times n^{(4H-2)\nu_0}. \]

where the last line is from Lemma 3.2. Plugging the above three estimates into (3.20), we obtain the inequality (3.18).

Finally, we have

\[ \left| \sum_{j,k,l=0}^{n-1} \gamma(j, k)\rho(k - l)\rho(j - l) \right| \]

\[ \leq \sum_{j,k,l=0}^{n-1} |\gamma(j, k)\rho(k - l)\rho(j - l)| \]
\[
\leq \frac{1}{H^2} \sum_{j,k,l=0}^{n-1} \left[ (j+1)^H - j^H \right] \left[ (k+1)^H - k^H \right] \left| \rho(k-l\rho(j-l) \right| . \tag{3.21}
\]

In the similar vein, we have that in the summation the contribution of all the terms such that \( j = k \) or \( k = l \) or \( j = 0 \) or \( k = 0 \) or \( l = 0 \) are negligible to compare with \( n^{(6H-3)\vee 0} \). The symmetry implies that other terms are less than:

\[
2 \times \sum_{0 < j < k < n, l \neq j, k} |\gamma(j,k)\rho(k-l)\rho(j-l)| \leq c \times \sum_{0 < j < k < n, l \neq j, k} j^{H-1}k^{H-1}|k-l|^{2H-2}|j-l|^{2H-2}.
\]

According to the distinct orders of \( j, k, l \), we do the change of variables \( a = j, k - j = b, l - k = c \) when \( 0 < j < k < l \), or \( a = j, l - j = b, k - l = c \) when \( 0 < j < l < k \), or \( a = l, j - l = b, k - j = c \) when \( 0 < l < j < k \), and then by the monotonicity of the power function again, we have

\[
\sum_{0 < j < k < n, l \neq j, k} j^{H-1}k^{H-1}|k-l|^{2H-2}|j-l|^{2H-2} \leq 3 \times \sum_{a,b,c \in \mathbb{N}, a+b+c < n} a^{2H-2}b^{2H-2}c^{2H-2} \leq c \times n^{(6H-3)\vee 0},
\]

where the last line is from Lemma 3.2. Taking the above three inequalities together, we obtain the desired (3.19).

In the same way, we can show the following proposition.

**Proposition 3.7.** Let \( \theta(i,j), \gamma(i,j), \rho(r) \) be given as in (3.6). When \( H \in (0, 1) \), there exists a positive constant \( c \) such that for all \( n \geq 1 \)

\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\gamma(i,k)\gamma(k,l) \right| \leq c \times n^{(8H-4)\vee 0},
\]

\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\gamma(i,k)\rho(k-l) \right| \leq c \times n^{(8H-4)\vee 0}, \tag{3.22}
\]

\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\rho(k-l) \right| \leq c \times n^{(8H-4)\vee 0}. \tag{3.23}
\]
\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\rho(i-k)\rho(k-l)\rho(j-l) \right| \leq c \times n^{(8H-4)\vee 0}.
\]

**Proof.** Similarly, we have
\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\gamma(i,k,\gamma(k,l))\gamma(j,l) \right| \leq \sum_{i,j,k,l=0}^{n-1} |\gamma(j,k)\gamma(i,k)\gamma(k,l)\gamma(j,l)|
\]
\[
\leq \left( \frac{1}{H^2} \sum_{j=0}^{n-1} [(j+1)^H - j^H]^2 \right)^4
\]
\[
\leq c \times n^{(8H-4)\vee 0},
\]
and
\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\gamma(i,k)\gamma(k,l)\rho(j-l) \right|
\]
\[
\leq \sum_{i,j,k,l=0}^{n-1} |\gamma(i,j)\gamma(i,k)\gamma(k,l)\rho(j-l)|
\]
\[
\leq c \times n^{(4H-2)\vee 0} \sum_{j,l=0}^{n-1} [(j+1)^H - j^H] [(l+1)^H - l^H] |\rho(j-l)|.
\]
\[
\leq c \times n^{(8H-4)\vee 0},
\]
where in the last line we have used the proof of the inequality (3.18), please refer to (3.20). Next, we have
\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\gamma(i,k)\rho(k-l)\rho(j-l) \right|
\]
\[
\leq \sum_{i,j,k,l=0}^{n-1} |\gamma(i,j)\gamma(i,k)\rho(k-l)\rho(j-l)|
\]
\[
\leq c \times n^{(2H-1)\vee 0} \sum_{j,k,l=0}^{n-1} [(j+1)^H - j^H] [(l+1)^H - l^H] |\rho(k-l)\rho(j-l)|
\]
\[
\leq c \times n^{(8H-4)\vee 0},
\]
where in the last line we have used the proof of the inequality (3.19), please refer to (3.21).
Finally, we have
\[
\left| \sum_{i,j,k,l=0}^{n-1} \gamma(i,j)\rho(i-k)\rho(k-l)\rho(j-l) \right| \\
\leq \sum_{i,j,k,l=0}^{n-1} |\gamma(i,j)\rho(i-k)\rho(k-l)\rho(j-l)| \\
\leq \frac{1}{H^2} \sum_{i,j,k,l=0}^{n-1} \left| (i+1)^H - i^H \right| \left| (j+1)^H - j^H \right| |\rho(i-k)\rho(k-l)\rho(j-l)|.
\]

It is easy to show that when any two index of \(i, j, k, l\) are equal or any index vanishes, the contribution to the sum are negligible to compare with \(n^{(8H-4)\vee 0}\). The symmetry implies that other terms is less than
\[
4 \times \sum_{0<i<j<n, 0<k<l<n, i,j\neq k,l} i^{H-1}j^{H-1} |i-k|^{2H-2} |k-l|^{2H-2} |j-l|^{2H-2}.
\]

According to the distinct orders of \(i, j, k, l\), we do the change of variables \(a = i, j - i = b, k - j = c, l - k = d\) when \(0 < i < j < k < l\), or \(a = i, k - i = b, j - k = c, j - l = d\) when \(0 < i < k < j < l\), or \(a = k, i - k = b, j - i = c, l - j = d\) when \(0 < k < i < j < l\), or \(a = k, l - k = b, i - l = c, j - i = d\) when \(0 < k < i < j < l\), and then by the monotonicity of the power function again, we have
\[
\sum_{0<i<j<n, 0<k<l<n, i,j\neq k,l} j^{H-1}k^{H-1} |k-l|^{2H-2} |j-l|^{2H-2} \\
\leq 6 \sum_{a,b,c,d\in\mathbb{N}, a+b+c+d<n} a^{2H-2}b^{2H-2}c^{2H-2}d^{2H-2} \\
\leq c \times n^{(8H-4)\vee 0},
\]
where the last line is from Lemma 3.2.

**Proof of Theorem 1.5.** We will discuss exclusively the case \(H \in (0, \frac{3}{4})\) since the case \(H = \frac{3}{4}\) is similar. Recall (3.14) and (3.15), the expressions of \(Z_n\) and \(g_n\). Denote
\[
F_n := \frac{Z_n}{\sqrt{n}} = \frac{I_2(g_n)}{\sqrt{n}}.
\]
First, the identities (6.2-6.3) of \[3\] imply that

\[
\kappa_3(F_n) = \mathbb{E}[F_n^3] = \frac{8}{n^2} \sum_{j,k,l=0}^{n-1} \theta(j,k)\theta(k,l)\theta(j,l)
\]

(3.25)

\[
\kappa_4(F_n) = \mathbb{E}[F_n^4] - 3\mathbb{E}[F_n^2]^2 = \frac{48}{n^2} \left\| g_n \otimes_1 g_n \right\|_{H^2}^2 = \frac{48}{n^2} \sum_{j,k,l=0}^{n-1} \theta(i,j)\theta(j,k)\theta(k,l)\theta(j,l)
\]

(3.26)

The symmetry and (3.6) imply that

\[
\sum_{j,k,l=0}^{n-1} \theta(j,k)\theta(k,l)\theta(j,l) = \sum_{j,k,l=0}^{n-1} \rho(j-k)\rho(k-l)\rho(j-l) + \sum_{j,k,l=0}^{n-1} \gamma(j,k)\gamma(k,l)\gamma(j,l)
\]

\[
+ 3 \sum_{j,k,l=0}^{n-1} \gamma(j,k)\gamma(k,l)\rho(j-l) + 3 \sum_{j,k,l=0}^{n-1} \gamma(j,k)\rho(k-l)\rho(j-l)
\]

Rearranging, and using Proposition 3.6 we have that there exists a positive constant \(c\) such that

\[
\left| \kappa_3(F_n) - \frac{8}{n^2} \sum_{j,k,l=0}^{n-1} \rho(j,k)\rho(k,l)\rho(j,l) \right| \leq c \times n^{(6H-3)\vee 0 - \frac{3}{2}},
\]

which together with Proposition 2.6, implies that when \(H \in (0, \frac{2}{3})\),

\[
\kappa_3(F_n) \asymp n^{-\frac{1}{2}};
\]

(3.27)

and when \(H = \frac{2}{3}\),

\[
\kappa_3(F_n) \asymp n^{-\frac{1}{2}} \log^2 n;
\]

(3.28)

and when \(H \in (\frac{2}{3}, \frac{3}{4})\),

\[
\left| \kappa_3(F_n) \right| \leq n^{\frac{3}{2}(4H-3)};
\]

(3.29)

In the same vein, Proposition 3.7 implies that there exists a positive constant \(c\) such that

\[
\left| \kappa_4(F_n) - \frac{48}{n^2} \sum_{i,j,k,l=0}^{n-1} \rho(i-j)\rho(i-k)\rho(k-l)\rho(j-l) \right| \leq c \times n^{(8H-6)\vee 0},
\]

which together with Proposition 2.6, implies that when \(H \in (0, \frac{5}{8})\),

\[
\kappa_4(F_n) \asymp n^{-1};
\]

(3.30)
and when $H = \frac{5}{8}$,
\[
\kappa_4(F_n) \asymp n^{-1} \log^3 n;
\]
and when $H \in (\frac{5}{8}, \frac{3}{4})$,
\[
|\kappa_4(F_n)| \leq n^{8H-6},
\]
Combining (3.27)-(3.32) with Theorem 2.3 and Propositions 2.6, 3.5, we obtain the desired result since
\[
\kappa_4(V_n) = \frac{n^2}{\sigma_n^2} \kappa_3(F_n), \quad \kappa_4(V_n) = \frac{n^2}{\sigma_n^4} \kappa_4(F_n).
\]

**Proof of Theorem 1.7.** From Theorem 1.5, $V_n$ satisfies the CLT. Hence, we need only to check the conditions (1) and (2) of Theorem 2.4 are valid. We will discuss exclusively the case $H \in (0, \frac{3}{4})$ since the case $H = \frac{3}{4}$ is similar.

Recall $V_n = I_2(f_n)$ where
\[
f_n = \frac{1}{\sigma_n} g_n = \sqrt{n} \frac{g_n}{\sigma_n} \sqrt{n},
\]
which together with (3.26) and Proposition 3.5 implies that
\[
\|f_n \otimes_1 f_n\|_{\mathcal{B} \otimes 2}^2 \leq c \times \frac{1}{n^2} \|g_n \otimes_1 g_n\|_{\mathcal{B} \otimes 2}^2
\]
\[
\leq c \times \begin{cases} 
\frac{1}{n^2}, & \text{if } H \in (0, \frac{5}{8}), \\
\frac{(\log n)^3}{n^2}, & \text{if } H = \frac{5}{8}, \\
n^8H-6, & \text{if } H \in (\frac{5}{8}, \frac{3}{4}). 
\end{cases}
\]

Hence, the condition (1) of Theorem 2.4 is valid.

To check the condition (2) of Theorem 2.4, first noting that $E[V_n^2] = 2 \|f_n\|_{\mathcal{B} \otimes 2}^2 = 1$, we need only to show that when $0 < k < l$, the following inequality holds:
\[
|\langle f_k, f_l \rangle_{\mathcal{B} \otimes 2}| \leq c \times \left[ \frac{k}{l} + (kl)^{(2H-1)\sqrt{0-\frac{1}{2}}} \right].
\]

In fact, we have
\[
|\langle f_k, f_l \rangle_{\mathcal{B} \otimes 2}| \leq c \times \frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \theta^2(i, j)
\]

\[ \leq c \times \frac{2}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \left[ \gamma^2(i, j) + \rho^2(i - j) \right]. \quad (3.35) \]

[2, Theorem 5.1] implies that
\[ \frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho^2(i - j) \leq c \times \sqrt{\frac{k}{l}}. \]

It follows from the inequality (3.8) that
\[
\frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \gamma^2(i, j) \leq c \times \frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} [(i + 1)^H - i^H]^2 \times \sum_{j=0}^{l-1} [(j + 1)^H - j^H]^2 \\
\leq c \times (kl)^{(2H-1) \vee 0 - \frac{1}{2}},
\]

where in the last line we have used the inequality (3.11). Plugging the above two inequalities into (3.35), we obtain the desired (3.34). Hence, the ASCLT holds when \( H \in (0, \frac{3}{4}) \).

\[ \blacksquare \]

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