On the behaviour of Brauer $p$-dimensions under finitely-generated field extensions

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May 2, 2014

Abstract

Let $E$ be a field of absolute Brauer dimension $abrd(E)$, and $F/E$ a transcendental finitely-generated extension. This paper shows that the Brauer dimension $Brd(F)$ is infinite, if $abrd(E) = \infty$. When the absolute Brauer $p$-dimension $abrd_p(E)$ is infinite, for some prime number $p$, it proves that for each pair $(n, m)$ of integers with $n \geq m > 0$, there is a central division $F$-algebra of Schur index $p^n$ and exponent $p^m$. When $abrd_p(E) < \infty$, the paper obtains lower bounds on the Brauer $p$-dimension $Brd_p(F)$ in several important special cases. It concludes that if $q$ is prime or $q = 0$, then there exist characteristic $q$ fields $E_{q,k}$: $k \in \mathbb{N}$, such that $Brd(E_{q,k}) = k$ and $abrd_p(E_{q,k}) = \infty$, for every prime $p > \max\{2, q\}$. These results solve negatively a problem posed by Auel, Brusel, Garibaldi and Vishne in Transform. Groups 16, 219-264 (2011).

Keywords: Brauer group, Schur index, exponent, Brauer/absolute Brauer $p$-dimension, finitely-generated extension, valued field

MSC (2010): 16K20, 16K50 (primary); 12F20, 12J10, 16K40 (secondary).

1 Introduction

Let $E$ be a field, $s(E)$ the class of finite-dimensional associative central simple $E$-algebras, $d(E)$ the subclass of division algebras $D \in s(E)$, and for each $A \in s(E)$, let $[A]$ be the equivalence class of $A$ in the Brauer group $Br(E)$. It is known that $Br(E)$ is an abelian torsion group (cf. [26], Sect. 14.4), whence it decomposes into the direct sum of its $p$-components $Br(E)_p$, where $p$ runs across the set $\mathbb{P}$ of prime numbers. By Wedderburn’s structure theorem (see, e.g., [26], Sect. 3.5), each $A \in s(E)$ is isomorphic to the full matrix ring $M_n(D_A)$ of order $n$ over some $D_A \in d(E)$; the order $n$ is uniquely determined by $A$ and so is

*Throughout this paper, we write for brevity "FG-extension(s)" instead of "finitely-generated [field] extension(s)".
D_A, up to an E-isomorphism. This implies the dimension \([A: E]\) is a square of a positive integer \(\deg(A)\). The main numerical invariants of \(A\) are the degree \(\deg(A)\), the Schur index \(\text{ind}(A) = \deg(D_A)\), and the exponent \(\exp(A)\), i.e. the order of \([A]\) in \(\text{Br}(E)\). The following statements describe basic divisibility relations between \(\text{ind}(A)\) and \(\exp(A)\), and give an idea of their behaviour under the scalar extension map \(\text{Br}(E) \rightarrow \text{Br}(R)\), in case \(R/E\) is a field extension of finite degree \([R: E]\) (see, e.g., [36], Sects. 13.4, 14.4 and 15.2, and [11], Lemma 3.5):

\[(1.1)\] (a) \((\text{ind}(A), \exp(A))\) is a Brauer pair, i.e. \(\exp(A)\) divides \(\text{ind}(A)\) and is divisible by every \(p \in \mathbb{P}\) dividing \(\text{ind}(A)\).

(b) \(\text{ind}(A \otimes \overline{E} B)\) is divisible by l.c.m. \(\{\text{ind}(A), \text{ind}(B)\}\) and divides \(\text{ind}(A)\) \(\text{ind}(B)\), for each \(B \in s(E)\); in particular, if \(A, B \in d(E)\) and g.c.d. \(\{\text{ind}(A), \text{ind}(B)\} = 1\), then the tensor product \(A \otimes \overline{E} B\) lies in \(d(E)\).

(c) \(\text{ind}(A), \text{ind}(A \otimes \overline{E} R), \exp(A)\) and \(\exp(A \otimes \overline{E} R)\) divide \(\text{ind}(A \otimes \overline{E} R)[R: E], \text{ind}(A), \exp(A \otimes \overline{E} R)[R: E]\) and \(\exp(A)\), respectively.

Statements (1.1) (a), (b) imply Brauer’s Primary Tensor Product Decomposition Theorem, for any \(\Delta \in d(E)\) (cf. [36], Sect. 14.4), and (1.1) (a) fully describes general restrictions on index-exponent relations, in the following sense:

\[(1.2)\] Given a Brauer pair \((m', m) \in \mathbb{N}^2\), there is a field \(F\) with \(\text{ind}(D(D)), \exp(D) = (m', m)\), for some \(D \in d(F)\) (Brauer, see [36], Sect. 19.6). One may take as \(F\) any rational (i.e. purely transcendental) extension in infinitely many variables over any fixed field \(F_0\) (see also Corollary 1.10 and the comment after its proof).

As in [3], Sect. 4, we say that a field \(E\) is of finite Brauer \(p\)-dimension \(\text{Br}_p(E) = n\), for a fixed \(p \in \mathbb{P}\), if \(n\) is the least integer \(\geq 0\), for which \(\text{ind}(D) \leq \exp(D)^n\) whenever \(D \in d(E)\) and \([D] \in \text{Br}(E)_p\). If no such \(n\) exists, we set \(\text{Br}_p(E) = \infty\). The absolute Brauer \(p\)-dimension of \(E\) is defined as the supremum \(\text{abrd}_p(E) = \sup\{\text{Br}_p(R); R \in F(E)\}\), where \(F(E)\) is the set of finite extensions of \(E\) in a separable closure \(E_{\text{sep}}\). Clearly, \(\text{Br}_p(E) \leq \text{abrd}_p(E), p \in \mathbb{P}\). Note also that if \(E\) is a virtually perfect field, i.e. \(\text{char}(E) = 0\) or \(\text{char}(E) = q > 0\) and \(E\) is a finite extension of its subfield \(E^q\), then \(\text{Br}_p(E^q) \leq \text{abrd}_p(E), p \in \mathbb{P}\). Since in the case of \(\text{char}(E) = q > 0\), \([E^q : E^{q^e}] = [E : E^q]\) (cf. [25], Ch. VII, Sect. 7), this can be deduced from (1.1) (c) and Albert’s theory of \(q\)-algebras [11], Ch. VII, Theorem 28 (see also Lemma 1.1). Thus it becomes clear that the main result of [6] applies to locally finite-dimensional (abbr, LFD) central division \(E\)-algebras whenever \(E\) is virtually perfect and \(\text{abrd}_p(E) < \infty\), for each \(p \in \mathbb{P}\).

It is known that \(\text{Br}_p(E) = \text{abrd}_p(E) = 1\), for all \(p \in \mathbb{P}\), if \(E\) is a global or local field (cf. [37], (31.4) and (32.19)), or the function field of an algebraic surface defined over an algebraically closed field \(E_0\) [22], [27] (see also Remark [37]). The suprema \(\text{Br}(E) = \sup\{\text{Br}_p(E); p \in \mathbb{P}\}\) and \(\text{abrd}(E) = \sup\{\text{Br}_p(R); R \in F(E)\}\) are called a Brauer dimension and an absolute Brauer dimension of \(E\), respectively. In view of (1.1), the definition of \(\text{Br}(E)\) is the same as the one given in [3], Sect. 4. It has recently been proved [19], [35] (see also Lemmas [6.1] and [6.2], that \(\text{abrd}(K_m) < \infty\), provided \(m \in \mathbb{N}\) and \((K_m, v_m)\) is a complete \(m\)-discrete valued field, in the sense of [33], with a finite \(m\)-th residue field \(\hat{K}_m\).

The present research is devoted to the study of index-exponent relations over transcendental FG-extensions \(F\) of a field \(E\) and their dependence on \(\text{abrd}_p(E),\)
p ∈ P. It is motivated mainly by two questions concerning the dependence of 
Br(F) upon Br(E), stated as open problems in Section 4 of the survey [3].

2 The main results

Fields E with abrd_p(E) < ∞, for all p ∈ P, are singled out by Galois cohomology 
(see Remark [1,2]), and in the virtually perfect case, by the following result on 
their locally finite-dimensional central division LFD-algebras [6], [7]:

Proposition 2.1. Let E be a virtually perfect field with abrd_p(E) < ∞, for 
every p ∈ P, and let R be an associative central division LFD-algebra over E. 
Then R possesses an E-subalgebra ⃗R with the following properties:
(a) ⃗R decomposes into a tensor product ⊗_p∈P R_p, where ⊗ = ⊗_E, R_p ∈ d(E) 
and [R_p] ∈ Br(E)_p, for each p ∈ P;
(b) Finite-dimensional E-subalgebras of R are embeddable in ⃗R;
(c) ⃗R is isomorphic to R, if the dimension [R: E] is countably infinite.

Proposition 2.1 makes it possible to build a satisfactory structure theory of 
central division LFD-algebras over a virtually perfect field E with abrd_p(E) < ∞, 
p ∈ P (see [7], Sects. 4, 5, for the case where E is a global or local field). 
It would be of definite interest to know whether function fields of algebraic 
varieties over a global, local or algebraically closed field are of finite absolute 
Brauer dimensions. This draws our attention to the following open question:
(2.1) Is the class of fields E of finite absolute Brauer p-dimensions, for a fixed 
p ∈ P, p ≠ char(E), closed under the formation of FG-extensions?

The first main result of this paper is presented by the following theorem. 
It provides information on the behaviour of Brd_p(F), and on index-exponent 
relations in d(F), for an FG-extension F/E:

Theorem 2.2. Let E be a field, p ∈ P and F/E an FG-extension of transcendence 
degree trd(F/E) = κ ≥ 1. Then:
(a) Brd_p(F) ≥ abrd_p(E) + κ − 1, if abrd_p(E) < ∞ and F/E is rational;
(b) Brd_p(F) = ∞, if abrd_p(E) = ∞; in this case, for each (n, m) ∈ N^2 
with n ≥ m > 0, there exists D_{n,m} ∈ d(F), such that ind(D_{n,m}) = p^n 
and exp(D_{n,m}) = p^m;
(c) Brd_p(F) = ∞, provided p = char(E) and [E: E^p] = ∞; if char(E) = p 
and [E: E^p] = p^ν < ∞, then ν + κ − 1 ≤ Brd_p(F) ≤ abrd_p(F) ≤ ν + κ.

It is known (cf. [20], Ch. X) that each FG-extension F of a field E possesses 
a subfield F_0 that is rational over E with trd(F_0/E) = trd(F/E). This ensures that [F: F_0] < ∞, so (1.1) and Theorem 2.2 imply the following:
(2.2) If (2.1) has an affirmative answer, for some p ∈ P, p ≠ char(E), and 
each FG-extension F/E with trd(F/E) = κ ≥ 1, then there exists c_κ(p) ∈ N, 
depending on E, such that Brd_p(Φ) ≤ c_κ(p) whenever Φ/E is an FG-extension 
and trd(Φ/E) < κ. For example, this applies to c_κ(p) = Brd_p(E_κ), where E_κ/E 
is a rational FG-extension with trd(E_κ/E) = κ.

The second main result of this paper can be stated as follows:
Theorem 2.3. For each \( q \in \mathbb{P} \cup \{0\} \) and \( k \in \mathbb{N} \), there exists a field \( E_{q,k} \) with \( \text{char}(E_{q,k}) = q \), \( \text{Br}(E_{q,k}) = k \) and \( \text{abrd}_q(E_{q,k}) = \infty \), for all \( p \in \mathbb{P} \setminus \{q\} \), where \( P_q = \{2\} \) and \( P_q' = P_q \setminus \{q\} \), \( q \in \mathbb{P} \). Moreover, if \( q > 0 \), then \( E_{q,k} \) can be chosen so that \( [E_{q,k}:E_q] = \infty \).

Theorems 2.2 and 2.3 and statement (1.1) (b) imply the following:

(2.3) There exist fields \( E_k, k \in \mathbb{N} \), such that \( \text{char}(E_k) = 2 \), \( \text{Br}(E_k) = k \) and all Brauer pairs \((m', n') \in \mathbb{N}^2\) are index-exponent pairs over any transcendental FG-extension of \( E_k \).

It is not known whether (2.3) holds in any characteristic \( q \neq 2 \). This is closely related to the following open problem:

(2.4) Find whether there exists a field \( E \) containing a primitive \( p \)-th root of unity, for a given \( p \in \mathbb{P} \), such that \( \text{Brd}_p(E) < \text{abrd}_p(E) = \infty \).

Statement (1.1) (b) and Theorems 2.2 and 2.3 imply the validity of (2.3) in zero characteristic, for Brauer pairs of odd positive integers. When \( q > 2 \), they show that if \([E_{q,k}:E_q] = \infty \) and \( P_q' = P_q \setminus \{q\} \), then Brauer pairs \((m', n') \in \mathbb{N}^2\) not divisible by any \( p \in P_q' \) are index-exponent pairs over every transcendental FG-extension of \( E_{q,k} \). This solves in the negative Problem 4.4 of [3], proving (in the strongest presently known form) that the class of fields of finite Brauer dimensions is not closed under the formation of FG-extensions. As a whole, our research shows that (2.1) should replace Problem 4.4 in the list made in [3].

Theorem 2.2 (a) makes it easy to prove that the solution to [3], Problem 4.5, on the existence of a "good" definition of a dimension \( \text{dim}(E) < \infty \), for some fields \( E \), is negative whenever \( \text{abrd}(E) = \infty \) (see Corollary 5.4). It implies that if Problem 4.5 of [3] is solved affirmatively, for all FG-extensions \( F/E \), then each \( F \) satisfies, for all \( p \in \mathbb{P} \), the following stronger inequalities than those conjectured by (2.2) (see also Remark 5.6 Corollary 6.4 and [3], Sect. 4):

(2.5) \( \text{Br}(F) < \text{dim}(F), \text{abrd}(F) \leq \text{dim}(F) \) and \( \text{abrd}(F) \leq \text{Br}(E_{t+1}) \leq \text{abrd}(E) + t + c(E) \), for some integer \( c(E) \leq \text{dim}(E) - \text{abrd}(E) \), where \( t = \text{trd}(F/E) \), \( E_{t+1}/E \) is a rational extension and \( \text{trd}(E_{t+1}/E) = t + 1 \).

The proof of Theorem 2.2 is based on Merkur’ev’s theorem concerning central division algebras of prime exponent [31], Sect. 4, Theorem 2, and on a characterization of fields of finite absolute Brauer \( p \)-dimensions generalizing Albert’s theorem [1], Ch. XI, Theorem 3. It strongly relies on results of valuation theory, like theorems of Grunwald-Hasse-Wang type, Morandi’s theorem on tensor products of valued division algebras [33], Theorem 1, lifting theorems over Henselian (valued) fields and Ostrowski’s theorem. Theorem 2.3 is proved by applying a standard method of realizing profinite groups as Galois groups [47], and using a construction of Henselian fields with prescribed properties of their value groups, residue fields and finite extensions. Our proof also relies on the Mel’nikov-Tavgen’ theorem [30] and the theory of maximally complete fields (see (3.6) and [17], Sects. 4.2 and 18.4). In addition, we use a formula for \( \text{Brd}_p(K) \), where \( K \) is a field with a Henselian valuation \( v \) whose residue field \( \hat{K} \) satisfies the conditions \( \text{Brd}_p(\hat{K}) = 0 \) and \( \text{char}(\hat{K}) \neq p \) (see Lemmas 6.1 and 6.2). The flexibility of this approach enables one to obtain the following results:
(2.6) (a) There exists a field $E_1$ with $abrdd(E_1) = \infty$, $abrdd_p(E_1) < \infty$, $p \in \mathbb{P}$, and $Brd(L_1) < \infty$, for every finite extension $L_1/E_1$; (b) for any integer $n \geq 2$, there is a Galois extension $L_n/E_n$, such that $[L_n: E_n] = n$, $Brd_p(L_n) = \infty$, for all $p \in \mathbb{P}$, $p \equiv 1(mod\, n)$, and $Brd(M_n) < \infty$, provided that $M_n$ is an extension of $E$ in $L_n, sep$ not including $L_n$.

Our basic notation and terminology are standard, as used in [9]. For any field $K$ with a Krull valuation $v$, unless stated otherwise, we denote by $O_v(K)$, $\hat{K}$ and $v(K)$ the valuation ring, the residue field and the value group of $(K,v)$, respectively; $v(K)$ is supposed to be an additively written totally ordered abelian group. The union $\mathbb{N} \cup \{0, \infty\}$ is denoted by $\mathbb{N}_\infty$ and is regarded as an ordered extension of the set $\mathbb{N} \cup \{0\}$ with a maximal element $\infty$. As usual, $\mathbb{Z}$ stands for the additive group of integers, $\mathbb{Z}_p$, $p \in \mathbb{P}$, are the additive groups of $p$-adic integers, and $[r]$ is the integral part of any real number $r \geq 0$. We write $I(\Lambda'/\Lambda)$ for the set of intermediate fields of a field extension $\Lambda'/\Lambda$, and $Br(\Lambda'/\Lambda)$ for the relative Brauer group of $\Lambda'/\Lambda$. By a $\Lambda$-valuation of $\Lambda'$, we mean a Krull valuation $v$, such that $v(\lambda) = 0$, for all $\lambda \in \Lambda^*$. Given a field $E$ and $p \in \mathbb{P}$, $E(p)$ denotes the maximal $p$-extension of $E$ in $E_{sep}$, and $r_p(E)$ - the rank of the Galois group $G(E(p)/E)$ as a pro-$p$-group $(r_p(E) = 0$, if $E(p) = E$). Brauer groups are considered to be additively written. Galois groups are viewed as profinite with respect to the Krull topology, and by a homomorphism of profinite groups, we mean a continuous one. We refer the reader to [17], [21], [26], [36] and [40], for any missing definitions concerning valuation theory, field extensions, simple algebras, Brauer groups and Galois cohomology.

Here is an overview of the rest of the paper: Section 3 includes preliminaries used in the sequel. Theorem 2.2 is proved in Sections 4 and 5. Statement (2.6) and Theorem 2.3 are proved in Section 6. Our proofs contain results of independent interest, such as Theorem 6.6, Lemma 4.3 and a formula for $abrdd_p(K)$, for a Henselian field $(K,v)$ with $\text{char}(\hat{K}) \neq p$ and an absolute Galois group $G(\hat{K}_{sep}/\hat{K})$ of $p$-cohomological dimension $cd_p(G_{\hat{K}}) \leq 1$ (for the case of $p = \text{char}(K)$ and $(K,v)$ maximally complete, see Remark 4.4). In Section 7 we give an alternative proof of Theorem 2.3 for $q = 0$, which shows that the answer to (2.1) will be affirmative, if this is the case in zero characteristic.

3 Preliminaries on valuation theory and fields with prescribed absolute Galois groups

The results of this Section are known and will often be used without an explicit reference. We begin with a lemma essentially due to Saltman [38].

Lemma 3.1. Let $(K,v)$ be a height 1 valued field, $K_v$ a Henselization of $K$ in $K_{sep}$ relative to $v$, and $\Delta_v \in d(K_v)$ an algebra of exponent $p \in \mathbb{P}$. Then there exists $\Delta \in d(K)$ with $\exp(\Delta) = p$ and $[\Delta \otimes_K K_v] = [\Delta_v]$.

Proof. By [31], Sect. 4, Theorem 2, $\Delta_v$ is Brauer equivalent to a tensor product of degree $p$ algebras from $d(K_v)$, so one may consider only the case of $\deg(\Delta_v) = p$. Then, by Saltman’s theorem (cf. [38]), there exists $\Delta \in d(K)$, such that $\deg(\Delta) = p$ and $\Delta \otimes_K K_v$ is $K_v$-isomorphic to $\Delta_v$, which proves Lemma 3.1. □
In what follows, we shall use the fact that the Henselization \( K_v \) of a field \( K \) with a valuation \( v \) of height 1 is separably closed in the completion of \( K \) relative to the topology induced by \( v \) (cf. [17], Theorem 15.3.5 and Sect. 18.3). For example, our next lemma is a consequence of Galois theory, this fact and Lorenz-Roquette’s valuation-theoretic generalization of Grunwald-Wang’s theorem (cf. [23], Ch. VIII, Theorem 4, and [29], page 176 and Theorems 1 and 2).

**Lemma 3.2.** Let \( F \) be a field, \( S = \{v_1, \ldots, v_s\} \) a finite set of non-equivalent height 1 valuations of \( F \), and for each index \( j \), let \( F_{v_j} \) be a Henselization of \( K \) in \( K_{\text{sep}} \) relative to \( v_j \), and \( L_j/F_{v_j} \) a cyclic field extension of degree \( p^e_j \), for some \( p \in \mathbb{P} \) and \( e_j \in \mathbb{N} \). Let \( \mu = \max\{\mu_1, \ldots, \mu_s\} \), and in the case of \( p = 2 \) and \( \text{char}(F) = 0 \), suppose that the extension \( F(\delta_\mu)/F \) is cyclic, where \( \delta_\mu \in F_{\text{sep}} \) is a primitive \( 2^{\mu} \)-th root of unity. Then there is a cyclic field extension \( L/F \) of degree \( p^e \), whose Henselization \( L_{v_j} \) is \( F_{v_j} \)-isomorphic to \( L_j \), where \( v_j \) is a valuation of \( L \) extending \( v_j \), for \( j = 1, \ldots, s \).

Assume that \( K = K_v \), or equivalently, that \( (K, v) \) is a Henselian field, i.e. \( v \) is a Krull valuation on \( K \), which extends uniquely, up to an equivalence, to a valuation \( v_L \) on each algebraic extension \( L/K \). Put \( v(L) = v_L(L) \) and denote by \( \hat{L} \) the residue field of \((L, v_L)\). It is known that \( \hat{L}/\hat{K} \) is an algebraic extension and \( v(K) \) is a subgroup of \( v(L) \). When \( [L: K] \) is finite, Ostrowski’s theorem states the following (cf. [17], Theorem 17.2.1):

\[
(3.1) \quad \hat{L}/\hat{K}[e(L/K)] \text{ divides } [L: K] \quad \text{and} \quad [L: K][\hat{L}/\hat{K}] \divides e(L/K) - 1 \text{ is not divisible by any } p \in \mathbb{P} \text{ different from char}(\hat{K}), e(L/K) \text{ being the index of } v(K) \text{ in } v(L); \text{ in particular, if char}(\hat{K}) \nmid [L: K], \text{ then } [L: K] = [\hat{L}: \hat{K}]e(L/K).
\]

Statement (3.1) and the Henselity of \( v \) imply the following:

\[
(3.2) \quad \text{The quotient groups } v(K)/pv(K) \text{ and } v(L)/pv(L) \text{ are isomorphic, if } p \in \mathbb{P} \text{ and } L/K \text{ is a finite extension. When char}(\hat{K}) \nmid [L: K], \text{ the natural embedding of } K \text{ into } L \text{ induces canonically an isomorphism } v(K)/pv(K) \cong v(L)/pv(L).
\]

A finite extension \( R/K \) is said to be defectless, if \( [R: K] = [\hat{R}: \hat{K}]e(R/K) \). It is called inertial, if \( [R: K] = [\hat{R}: \hat{K}] \) and \( \hat{R} \) is separable over \( \hat{K} \). We say that \( R/K \) is totally ramified, if \( [R: K] = e(R/K); \) \( R/K \) is called tamely ramified, if \( \hat{R}/\hat{K} \) is separable and \( \text{char}(\hat{K}) \nmid e(R/K) \). The Henselity of \( v \) ensures that the compositum \( K_{ur} \) of inertial extensions of \( K \) in \( K_{\text{sep}} \) has the following properties:

\[
(3.3) \quad (a) \quad v(K_{ur}) = v(K) \text{ and finite extensions of } K \text{ in } K_{ur} \text{ are inertial;}
\]

\[
(b) \quad K_{ur}/K \text{ is a Galois extension, } \hat{K}_{ur} \cong \hat{K}_{\text{sep}} \text{ over } \hat{K}, \mathcal{G}(K_{ur}/K) \cong \mathcal{G}, \text{ and the natural mapping of } I(K_{ur}/K) \text{ into } I(\hat{K}_{\text{sep}}/\hat{K}) \text{ is bijective.}
\]

Recall that the compositum \( K_{tr} \) of tamely ramified extensions of \( K \) in \( K_{\text{sep}} \) is a Galois extension of \( K \) with \( v(K_{tr}) = pv(K_{tr}) \), for every \( p \in \mathbb{P} \) not equal to \( \text{char}(\hat{K}) \). It is therefore clear from (3.1) that if \( K_{tr} \neq K_{\text{sep}} \), then char(\( K_{tr} \)) = \( q \neq 0 \) and \( \mathcal{G}_{K_{tr}} \) is a pro-\( q \)-group. When this holds, it follows from (3.3) and Galois cohomology (cf. [29], Ch. II, 2.2) that \( cd_p(\mathcal{G}(K_{tr}/K)) \leq 1 \). Hence, by [40], Ch. I, Proposition 16, there is a closed subgroup \( \mathcal{H} \leq \mathcal{G}_K \), such that \( \mathcal{G}_{K_{tr}} \cap \mathcal{H} = \{1\} \) and \( \mathcal{H} \cong \mathcal{G}(K_{tr}/K) \). In view of Galois theory and the Mel’nikov-Tavgen’ theorem [30], these results imply in the case of \( \text{char}(\hat{K}) = q > 0 \) the existence of a field \( K' \in I(K_{\text{sep}}/K) \) satisfying the following conditions:
Lemma 3.3. Let $K$, $K'$, $K_{tr}$, $K_{sep}$ and $K_{sep} \cong K_{tr} \otimes_K K'$ over $K$; the field $\bar{K}'$ is a perfect closure of $\bar{K}$, finite extensions of $K$ in $K'$ are of $q$-primary degrees, $K_{sep} = K_{tr}'$, $v(K') = qv(K')$, and the natural embedding of $K$ into $K'$ induces isomorphisms $v(K)/pv(K) \cong v(K')/pv(K')$, $p \in \mathbb{P} \setminus \{q\}$.

Assume as above that $(K, v)$ is Henselian. Then each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation $v_{\Delta}$ extending $v$ so that the value group $v(\Delta)$ of $(\Delta, v_\Delta)$ is totally ordered and abelian (cf. [15] and [18]). It is known that $v(K)$ is a subgroup of $v(\Delta)$ of index $e(\Delta/K) \leq [\Delta: K]$, and the residue division ring $\Delta$ of $(\Delta, v_\Delta)$ is a $\bar{K}$-algebra. Moreover, by the Ostrowski-Draxl theorem [13], $[\Delta: K]$ is divisible by $e(\Delta/K)[\Delta': \bar{K}]$, and in case char($\bar{K}$) $| [\Delta: K]$, $[\Delta: K] = e(\Delta/K)[\Delta': \bar{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D: K] = [\bar{D}: \bar{K}]$ and $\bar{D} \in d(\bar{K})$. Similarly to inertial extensions, the defined algebras have a lifting property described by the following result (see [21], Theorem 2.8):

(3.4) $K' \cap K_{tr} = K$, $K'K_{tr} = K_{sep}$ and $K_{sep} \cong K_{tr} \otimes_K K'$ over $K$; the field $\bar{K}'$ is a perfect closure of $\bar{K}$, finite extensions of $K$ in $K'$ are of $q$-primary degrees, $K_{sep} = K_{tr}'$, $v(K') = qv(K')$, and the natural embedding of $K$ into $K'$ induces isomorphisms $v(K)/pv(K) \cong v(K')/pv(K')$, $p \in \mathbb{P} \setminus \{q\}$.

Assume as above that $(K, v)$ is Henselian. Then each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation $v_{\Delta}$ extending $v$ so that the value group $v(\Delta)$ of $(\Delta, v_\Delta)$ is totally ordered and abelian (cf. [15] and [18]). It is known that $v(K)$ is a subgroup of $v(\Delta)$ of index $e(\Delta/K) \leq [\Delta: K]$, and the residue division ring $\Delta$ of $(\Delta, v_\Delta)$ is a $\bar{K}$-algebra. Moreover, by the Ostrowski-Draxl theorem [13], $[\Delta: K]$ is divisible by $e(\Delta/K)[\Delta': \bar{K}]$, and in case char($\bar{K}$) $| [\Delta: K]$, $[\Delta: K] = e(\Delta/K)[\Delta': \bar{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D: K] = [\bar{D}: \bar{K}]$ and $\bar{D} \in d(\bar{K})$. Similarly to inertial extensions, the defined algebras have a lifting property described by the following result (see [21], Theorem 2.8):

(3.5) (a) Each $\bar{D} \in d(\bar{K})$ has an inertial lift over $K$, i.e. there is $D \in d(K)$ inertial over $K$ with $\bar{D} = D$; $D$ is uniquely determined by $\bar{D}$, up-to a $K$-isomorphism.

(b) The set $\text{IBr}(K) = \{[I] \in \text{Br}(K) : I \in d(K) \text{ is inertial} \}$ is a subgroup of $\text{Br}(K)$; the canonical mapping $\text{IBr}(K) \to \text{Br}(K)$ is an isomorphism.

The following lemma plays a crucial role in the proof of Theorem 2.8.

Lemma 3.3. Let $K_0$ be a perfect field of characteristic $q \geq 0$, and let $n(p)$: $p \in \mathbb{P}$, be a sequence with terms in $\mathbb{N}_\infty$. Then there exists a Henselian field $(K, v)$ with char($K$) = $q$ and $\bar{K} = K_0$, such that the group $v(K)/pv(K)$ has dimension $n(p)$ as a vector space over the field $\mathbb{F}_p$ with $p$ elements, for each $p \in \mathbb{P}$. Moreover, if $q > 0$, then $K$ can be chosen so that its finite extensions be defectless relative to $v$, and $[K: K'] = q^n(p)$ in case $n(q) < \infty$.

Proof. Let $K_\infty$ be an extension of $K_0$ obtained as the union $K_\infty = \cup_{n \in \mathbb{N}} K_n$ of iterated formal (Laurent) power series fields, defined inductively by the rule $K_0 = K_{n-1}(x_n)$, $n \in \mathbb{N}$. Denote by $\omega_n$ the standard $K_0$-valuation of $K_n$ with $\omega_n(K_n) = \mathbb{Z}^n$, for each $n \in \mathbb{N}$. Here $\mathbb{Z}^n$, $n \in \mathbb{N}$, are viewed as ordered groups with respect to the inverse lexicographic ordering. Let $\omega$ be the natural valuation of $K_\infty$ extending $\omega_n$, for every $n$. Clearly, $K_0$ is the residue field of $(K_\infty, \omega)$ and $\omega(K_\infty)$ equals the union $\mathbb{Z}_\infty = \cup_{n \in \mathbb{N}} \mathbb{Z}^n$, considered with its unique ordering inducing the noted orderings on $\mathbb{Z}^n$, for all $n \in \mathbb{N}$. It is well-known (cf. [17], Sects. 4.2 and 18.4) that the valuations $\omega_n$, $n \in \mathbb{N}$, are Henselian, which implies that $\omega$ is of the same kind. Fix an algebraic closure $\overline{K}_\infty$ of $K_\infty$, a divisible hull $\omega(K_\infty)$ of $\omega(K_\infty)$, and for each $R \in I(\overline{K}_\infty/K_\infty)$, let $\omega_R$ be the unique valuation of $R$ extending $\omega$ so that $\omega(R) = \omega_R(R)$ be an ordered subgroup of $\omega(K_\infty)$. Clearly, the valuations $\omega_R$, $R \in I(\overline{K}_\infty/K_\infty)$, are Henselian. Note also that finite extensions of $K_n$ are defectless relative to $\omega_n$, for each $n \in \mathbb{N}$ (cf. [17], Theorem 18.4.1, and [16], Theorem 31.21). In addition, it is not difficult to see that each finite extension $K'_n$ of $K_\infty$ possesses a subfield $K'_n$ including $K_\infty$ so that $K'_\infty = K'_n K_\infty$ and $[K'_n: K_\infty] = [K'_n: K_\infty]$, for some index $n$. These observations show that if $n(p) = \infty$, for every $p \in \mathbb{P}$, then it suffices, for the proof of Lemma 3.3, to take as $(K, v)$ the valued field $(K_\infty, \omega)$. 

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Henceforth, we assume that the set \( P = \{ p \in \mathbb{P} : n(p) < \infty \} \) is nonempty. For any \( p \in P \) and each index \( n > n(p) \), let \( \Sigma_{p,n} = \{ X_{p,n,m} : m \in \mathbb{N} \} \) be a subset of \( K_\infty \), such that \( Y^p_{p,n,1} = X_n \) and \( Y^p_{p,n,m} = Y^p_{p,n,m-1} \), \( m \geq 2 \). Put \( \Sigma = \cup_{p \in P} \Sigma_p \), where \( \Sigma_p = \cup_{n=n(p)+1}^{\infty} \Sigma_{p,n} \), for each \( p \in P \). Denote by \( \tilde{K} \) the extension of \( K_\infty \) generated by \( \Sigma \), and by \( \tilde{K}_{\text{sep}} \) the separable closure of \( \tilde{K} \) in \( K_\infty \). It is easily verified that finite extensions of \( K_\infty \) in \( K \) are totally ramified, and for each \( p \in P \), \( n(p) \) equals the dimension of \( \omega(K)/\nu(K) \) as an \( \mathbb{F}_p \)-vector space. In view of (3.1), this means that \( (\tilde{K}, \omega_{\tilde{K}}) \) has the property required by Lemma 3.3 in the case where \( q = 0 \) or \( q > 0 \) and \( n(q) = \infty \). Suppose now that \( q > 0 \) and \( n(q) < \infty \). Then, by (3.4), there exists \( \Theta_0 \in I(K_\infty/K) \), such that \( \Theta_0 \cap \tilde{K}_{\text{tr}} = \tilde{K} \) and \( \Theta_0 \tilde{K}_{\text{tr}} = \tilde{K}_{\text{sep}} \). Let \( \Theta \) be the perfect closure of \( \Theta_0 \) in \( K_\infty \). As \( K_0 \) is perfect, (3.4) and the basic theory of algebraic extensions (cf. [26], Ch. VII, Proposition 12) imply that \( \Theta \) is perfect with \( \Theta_0 \), \( \omega(\Theta) = \omega(\Theta_0) = q\omega(\Theta_0) \), and for each \( p \in \mathbb{P} \setminus \{ q \} \), \( \omega(\Theta)/p\omega(\Theta) \) has dimension \( n(p) \) over \( \mathbb{F}_p \). Thus Lemma 3.3 is proved in the case where \( n(q) = 0 \).

It remains to consider the case of \( 0 < n(q) < \infty \). Let \( n(q) = n \), \( \Theta_n \) be an iterated formal power series field in \( n \) variables over \( \Theta, \kappa \) the standard \( \mathbb{Z}^n \)-valued \( \Theta \)-valuation of \( \Theta_n \), and \( w \) the valuation of \( \Theta_n \) extending \( \omega_q \) so that \( \omega(\Theta) \) be an isolated subgroup of \( w(\Theta_n) \), \( w(\Theta_n) \) the direct sum \( \omega(\Theta) \oplus \kappa(\Theta_n) \), and \( \kappa \) be induced canonically by \( w \) and \( \omega(\Theta) \) (cf. [17], Sect. 4.2). Then [17], Theorem 18.1.2, and [46], Theorem 32.15, imply \( \omega \) inherits the Henselity of \( \omega \) and \( \kappa \). Applying (3.1), [17], Theorem 18.4.1, and [46], Theorem 31.21, and using the fact that \( \Theta \) is perfect with \( \Theta_{\text{sep}} = \Theta_{\text{tr}} \), one concludes that finite extensions of \( \Theta_n \) are defectless relative to both \( \kappa \) and \( w \). In addition, it is easy to see that \( n \) equals the \( \mathbb{F}_p \)-dimension of \( \kappa(\Theta_n)/p\kappa(\Theta_n) \), for \( p \in \mathbb{P} \). Let now \( K \) be a maximal extension of \( \Theta_n \) in \( \Theta_n_{\text{sep}} \) with respect to the property that finite extensions of \( \Theta_n \) in \( K \) have degrees not divisible by \( q \) and are totally ramified over \( \Theta_n \) relative to \( \kappa \). Then \( [K:K^q] = q^n, \kappa(K) = p\kappa(K), p \in \mathbb{P} \setminus \{ q \} \), and it follows from (3.2), [1], (1.2), and the preceding observation that the natural embedding of \( \Theta_n \) into \( K \) induces an isomorphism \( \kappa(\Theta_n)/q\kappa(\Theta_n) \cong \kappa(K)/q\kappa(K) \). These results and the obtained properties of \( (\Theta, \omega_q) \) indicate that \( \kappa(K) \cong v(K)/\omega(K) \) and \( v(K) \) has the properties required by Lemma 3.3, where \( v = v_K \). They also imply finite extensions of \( K \) are defectless relative to \( v \), so our proof is complete.

The following classical results (see [16], Theorems 31.21, 31.22 and 31.24, and page 483) show that the valued field \( (K,v) \) in Lemma 3.3 can be chosen among maximally complete fields. Before stating them, note that maximal completeness is characterized by the nonexistence of improper extensions of \( (K,v) \), i.e. of valued extensions \((\Lambda,\lambda)\) with \( \Lambda \neq K, \lambda(\Lambda) = v(K) \) and \( \Lambda = \hat{K} \):

(3.6) (a) Each nontrivially valued field \((L,w)\) has an immediate extension \((L',w')\) which is maximally complete;

(b) If \((L,w)\) is maximally complete, then so are its valued finite extensions;

(c) Maximally complete fields are Henselian and their finite extensions are defectless.

**Remark 3.4.** Under the hypotheses of Lemma 3.3, suppose that \( q > 0 \) and \( 0 < n(q) = n < \infty \), fix \( m \in \mathbb{N}_\infty \) so that \( m \geq n \), and take \((K,v), \Theta, \Theta_j, j = 1, \ldots, n, \kappa \) and \( \omega \) as in the proof of the lemma. Let \( \Theta_n = \Lambda((Z_n)) \), where
Λ = Θ, if n = 1, and Λ = Θ_{n-1}, otherwise. It is known that (Θ,κ) is maximally complete (cf. [7], Theorem 18.4.1) and \( \text{trd}(\Theta/\Lambda(Z_n)) = \infty \) (see [3], page 2 and further references there). Fix a rational extension \( \Omega_m \) of \( \Lambda(Z_n) \) in \( \Theta \) so that \( \text{trd}(\Omega_m/\Lambda(Z_n)) = m - n \), if \( m - n \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \), and \( m \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \). It is easily verified (cf. [26], Ch. VII, Sect. 7) that \( [\Omega_m: \Omega_n] = q^n \), if \( m - n \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \), and \( m \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \). When \( \omega \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \), and \( m \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \), it is easily verified (cf. [26], Ch. VII, Sect. 7) that \( [\Omega_m: \Omega_n] = q^n \), if \( m - n \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \), and \( m \) is divisible by \( \mu \), \( \mu \) being a primitive \( \mu \)-th root of unity in \( \Theta \). We conclude this Section with two lemmas which contain the main Galois-theoretic ingredients of our proofs of (2.6) (b) and Theorem 2.3. For a proof of the following lemma, we refer the reader to [11], Lemma 3.2.

Lemma 3.5. Let \( c_p : p \in \mathbb{P}, \ p \notdiv \mu - 1 \), for each \( p \), \( \mu > 1 \), and let \( \mathbb{P} \) is a subset of \( \mathbb{P} \) including the set of those \( \pi \in \mathbb{P} \), for which there is \( p | \pi \notdiv \mu \), such that \( c_p \notdiv \mu \), divisible by \( \mu \). Then there exists a field \( E_0 \) with \( \text{char}(E_0) = 0 \), \( \mathcal{G}_{E_0} \) isomorphic to the topological group product \( \mathbb{Z}_p \notdiv \mu \), for any finite extension \( R_0 \) of \( \mathcal{G}_{E_0} \), for any finite extension \( R_0 \) of \( \mathcal{G}_{E_0} \). For a proof of the following lemma, we refer the reader to [11], Lemma 3.2.

Lemma 3.6. Assume that \( E_0 \) is a field, such that \( \mathcal{G}_{E_0} \) is of cohomological dimension \( \text{cd}(\mathcal{G}_{E_0}) \leq 1 \), and let \( G \) be a profinite group with \( \text{cd}(G) \leq 1 \) and \( \text{cd}(G) = 0 \) whenever \( p \notdiv \mu \) and \( \text{cd}(\mathcal{G}_{E_0}) \notdiv \mu \). Then there exists a field extension \( E/E_0 \), such that \( \mathcal{G}_{E} \) is algebraically closed in \( E \) and \( \mathcal{G}_{E} \) is isomorphic to the topological group product \( \mathcal{G}_{E_0} \times G \).

Proof. It is known (cf. [17]) that \( E_0 \) has extensions \( R \) and \( R' \), such that \( R'/E_0 \) is rational, \( R \in I(R'/E_0) \) and \( R'/R \) is Galois with \( \mathcal{G}(R'/R) \cong G \). Identifying \( E_0, \text{sep} \) with its \( E_0 \)-isomorphic copy in \( R', \text{sep} \), and observing that \( E_0 \) is algebraically closed in \( R' \), one obtains that \( E_0, \text{sep} \supseteq R'/R \) is Galois with \( \mathcal{G}(E_0, \text{sep} R'/R) \cong \mathcal{G}_{E_0} \times G \). In view of the assumptions on \( E_0, \text{sep} \) and \( G \), this yields \( \text{cd}(\mathcal{G}(E_0, \text{sep} R'/R)) = 1 \), which means that \( \mathcal{G}(E_0, \text{sep} R'/R) \) is a projective profinite group (cf. [10], Ch. I, 5.9). Hence, by Galois theory, there is a field \( E \in I(R', \text{sep})/R \), such that \( E_0, \text{sep} R'/E = R'/R \) and \( E_0, \text{sep} R'/R \cong E \times G \). This shows that \( E_0 \) is algebraically closed in \( E \) and \( \mathcal{G}_{E} \cong \mathcal{G}(E_0, \text{sep} R'/R) \cong \mathcal{G}_{E_0} \times G \), which proves Lemma 3.6. \( \square \)

4 Proof of Theorem 2.2 (a) and (c)

The study of Brauer p-dimensions of FG-extensions of a field \( E \) relies on the following lemma which characterizes the condition abrd_p(E) \( \leq \mu \), for a given \( \mu \in \mathbb{N} \). When \( E \) is virtually perfect, the lemma is in fact equivalent to [35], Lemma 1.1, and in case \( \mu = 1 \), it restates Theorem 3 of [1], Ch. XI.

Lemma 4.1. Let \( E \) be a field, \( p \in \mathbb{P} \) and \( \mu \in \mathbb{N} \). Then abrd_p(E) \( \leq \mu \) if and only if, for each \( E' \in \mathcal{F}(E) \), \( \text{ind}(\Delta) \leq p^\mu \) whenever \( \Delta \in d(E') \) and \( \exp(\Delta) = p \).
Proof. The left-to-right implication is obvious, so we prove only the converse one. Fix a field $E' \in \text{Fe}(E)$ and an algebra $\Delta' \in d(E')$ with $\text{exp}(\Delta') = p^n$, for some $n \in \mathbb{N}$. We show that $\text{ind}(\Delta') | p^{\nu_\mu}$. The assertion is clear, if $n = 1$, so we assume that $n \geq 2$. Take $\Delta \in d(E')$ so that $|\Delta| = p^{n-1}[\Delta']$, and let $Y$ be a maximal subfield of $\Delta$. It is well-known that $[Y: E'] = \text{ind}(\Delta)$ and $Y$ can be chosen so as to be separable over $E'$ (see [30], Sect. 13.5). Therefore, our assumptions indicate that $[Y: E'] | p^n$. Note also that, by the choice of $\Delta$, $\Delta' \otimes_F Y \in s(Y)$ and $\text{exp}(\Delta' \otimes_F Y) = p^{n-1}$. These remarks and a standard inductive argument lead to the conclusion that it suffices to prove the divisibility $\text{ind}(\Delta') | p^{\nu_\mu}$, provided $\text{ind}(\Delta' \otimes_F Y) | p^{(n-1)\mu}$. Fix $\Delta'_Y \in d(Y)$ so that $[\Delta'_Y] = [\Delta' \otimes_F Y]$, and take a maximal subfield $Y'$ of $\Delta'_Y$. Then $[Y': E'] = \text{ind}(\Delta' \otimes_F Y).[Y: E']$, which implies $[Y': E'] | p^{n\mu}$. Observing finally that $[\Delta'_Y] \in \text{Br}(Y'/E')$ (cf. [30], Sects. 9.4 and 13.1), one obtains that $\text{ind}(\Delta') | [Y': E'] | p^{\nu_\mu}$, so Lemma 4.1 is proved.

Remark 4.2. Note that a field $E$ satisfies $\text{abrd}_p(E) < \infty$, for some $p \in \mathbb{P}$, if and only if there exists $c_p(E) \in \mathbb{N}$, such that each $A_R \in s(R)$ with $\text{exp}(A_R) = p$ is Brauer equivalent to a tensor product of $c_p(E)$ algebras from $s(R)$ of degree $p$, where $R$ ranges over $\text{Fe}(E_p)$ and $E_p$ is the fixed field of a Sylow $p$-subgroup $G_p$ of $G_E$. Since $E_p$ contains a primitive $p$-th root of unity unless $p = \text{char}(E)$, this can be deduced from Lemma 4.1 and "quantitative" versions of [32], (16.1), and [77], Ch. VII, Theorem 28 (see [42], page 506, and [77], respectively). When $\text{abrd}_p(E) < \infty$ and $p \neq \text{char}(E)$, $c_p(E)$ is in fact a cohomological invariant of $G_p$ (cf. [32], (11.5)). As noted in [23], the Bloch-Kato Conjecture, proved in [44], implies that if $\text{abrd}_p(E) < \infty$, then $\alpha_p(G_E) < \infty$ unless $E$ is formally real and $p = 2$ (see also [26], Ch. XI, Sect. 2, and [44], Ch. I, 3.3).

Let now $F/E$ be a transcendental FG-extension and $F_0 \in I(F/E)$ a rational extension of $E$ with $\text{trd}(F_0/E) = d_0(F/E) = t$. Clearly, an ordering on a fixed transcendence basis of $F_0/E$ gives rise to a height $t$ $E$-valuation $v_0$ of $F_0$ with $v_0(F_0) = \mathbb{Z}^t$ and $\hat{F}_0 = E$. Considering any prolongation of $v_0$ on $F$, and taking into account that $[F: F_0] < \infty$, one obtains the following:

(4.1) $F$ has an $E$-valuation $v$ of height $t$, such that $v(F) \cong \mathbb{Z}^t$ and $\hat{F}$ is a finite extension of $E$; in particular, $v(F)/pv(F)$ is a group of order $p^t$, for every $p \in \mathbb{P}$.

When $\text{char}(E) = p$, (4.1) implies $[\hat{F}: \hat{F}^p] = [E: E^p]$, so the former assertion of Theorem 2.2 (c) can be deduced from the following lemma.

Lemma 4.3. Let $(K, v)$ be a valued field with $\text{char}(K) = q > 0$ and $v(K) \neq qv(K)$, and let $\tau(q)$ be the $F_q$-dimension of $v(K)/qv(K)$. Then:

(a) For each $\pi \in K^*$ with $v(\pi) \notin qv(K)$, there are degree $q$ extensions $L_m$ of $K$ in $K(q)$, $m \in \mathbb{N}$, such that the compositum $M_m = L_1 \ldots L_m$ has a unique valuation $v_m$ extending $v$, up-to an equivalence, $(M_m, v_m)/(K, v)$ is totally ramified. $[M_m: K] = q^m$ and $v(\pi) \in q^m v_m(M_m)$, for each $m$;

(b) Given an integer $n \geq 2$, there exists $T_n \in d(K)$ with $\text{exp}(T_n) = q$ and $\text{ind}(T_n) = q^{n-1}$ except, possibly, if $\tau(q) < \infty$ and $[\hat{K}: \hat{K}^q] < q^{n-\tau(q)}$.
Proof. It suffices to consider the special case of $v(\pi) < 0$. Fix a Henselization $(K_v, v)$ of $(K, v)$, put $p(K_v) = \{u^q - u : u \in K_v\}$, and for each $m \in \mathbb{N}$, denote by $L_m$ the root field in $K_{\text{sep}}$ over $K$ of the polynomial $f_m(X) = X^q - X - \pi_m$, where $\pi_m = p^1 + q^m$. Also, let $\mathbb{F}$ be the prime subfield of $K$, $\Phi = \mathbb{F}(\pi)$, $\omega$ the valuation of $\Phi$ induced by $v$, and $(\Phi_v, \hat{\omega})$ a Henselization of $(\Phi, \omega)$, such that $\Phi_v \subseteq K_v$ and $\hat{\omega}$ extends $\omega$ (the existence of $(\Phi_v, \hat{\omega})$ follows from [17], Theorem 15.3.5). Identifying $K_v$ with its $K$-isomorphic copy in $K_{\text{sep}}$, put $L_m = L_m K_v$ and $M'_m = M_m K_v$, for every index $m$. It is easily verified that $p(K_v)$ is an $\mathbb{F}$-subspace of $K_v$ and $\hat{\omega}(u - v) \in q^m K_v$, for every $u, v \in K_v$ with $\hat{\omega}(u) < 0$. As $\hat{\omega}(K_v) = \omega(K_v)$, this observation and the choice of $\pi$ implies that the cosets $\pi_m + p(K_v)$, $m \in \mathbb{N}$, are linearly independent over $\mathbb{F}$. In view of the Artin-Schreier theorem and Galois theory (cf. [26], Ch. VIII, Sect. 6), this implies $f_m(X)$ is irreducible over $K_v$, $L_m/K_v$ and $M'_m/K_v$ are cyclic extensions of degree $q$, $M'_m/K_v$ and $M_m/K_v$ are abelian, and $[M'_m : K_v] = [M_m : K] = q^m$, for each $m \in \mathbb{N}$. Moreover, our argument proves that degree $q$ extensions of $K_v$ in the compositum of the fields $L_m$, $m \in \mathbb{N}$, are cyclic and totally ramified over $K_v$. At the same time, it follows from the Henselity of $\hat{\omega}$ and the equality $\hat{K}_v = \hat{K}$ that $M'_m$ contains as a subfield an inertial lift over $K_v$ of the separable closure of $\hat{K}$ in $M'_m$. Thus it turns out that $M'_m/\hat{K}_v$ is purely inseparable. When $v$ is discrete and $\hat{K}$ is perfect, the obtained results imply the assertions of Lemma 4.3(a), since finite extensions of $K_v$ in $K_{\text{sep}}$ are defectless (relative to $\hat{\omega}$) [33], Proposition 2.2.

To prove Lemma 4.3(a) in general it remains to be seen that, for any fixed $m \in \mathbb{N}$, $M_m$ has a unique, up-to an equivalence, valuation $v_m$ extending $v$, $(M_m, v_m)/(K, v)$ is totally ramified and $v(\pi) \in q^n v(M_m)$. The extendability of $v$ to a valuation $v_m$ of $M_m$ is well-known (cf. [26], Ch. XII, Sect. 4), so our assertions can be deduced from the concluding one, the equality $[M_m : K] = [M_m : K_v : K_v] = q^m$ and statement (3.1). Our proof also relies on the fact that $(\Phi, \omega)$ is a discrete valued field and $\Phi/\mathbb{F}$ is a finite extension (see [5], Ch. II, Lemma 3.1, or [17], Example 4.1.3); in particular, $\hat{\Phi}$ is perfect. Let now $\Psi_m \in I(K_{\text{sep}}/\Phi)$ be the root field of $f_m(X)$ over $\Phi$. Then $L_m = \Psi_m K_v$, $[\Psi_m : \Phi] = q$, $M_m = \Theta_m K_v$ and $[\Theta_m : \Phi] = q^m$, where $\Theta_m = \Psi_1 \ldots \Psi_m$. Therefore, $\Theta_m \Psi_m / \Phi$ is totally ramified modulo $\hat{\omega}$. Equivalently, the integral closure of $O_\Phi(\Psi)$ in $O_\Phi$ contains a primitive element $t'_m$ of $\Theta_m / \Phi$, whose minimal polynomial $\theta_m(X)$ over $O_\Phi(\Phi)$ is Eisensteinian (cf. [5], Ch. I, Theorem 6.1, and [26], Ch. XII, Sects. 2, 3 and 6). Hence, $\omega$ has a unique prolongation $\omega_m$ on $\Theta_m$, up-to an equivalence, $\omega(t_m) \notin q^n \omega(t_m')$ and $q^n \omega_m(t'_m) = \omega(t_m)$, where $t_m$ is the free term of $\theta_m(X)$. As $\pi \in \Phi$, $v(\pi) \notin qv(K)$ and $\Theta_m / \Phi$ is a Galois extension, this implies $t'_m$ is a primitive element of $M_m/K_v$ and $M'_m/K_v$, $q^n v_m(t'_m) = v(t_m) = \omega(t_m)$ and $v(\pi) \in q^n v_m(M_m)$, which completes the proof of Lemma 4.3(a).

We prove Lemma 4.3(b). Put $\pi_1 = \pi$ and suppose that there exist elements $\pi_j \in K^*, j = 2, \ldots, n$, and an integer $\mu \leq n$, such that the cosets $v(\pi_i) + qv(K)$, $i = 1, \ldots, \mu$, are linearly independent over $\mathbb{F}$, and in case $\mu < n$, $v(\pi_n) = 0$ and the residue classes $\bar{\pi}_u, u = \mu + 1, \ldots, n$, generate an extension of $K^q$ of degree $q^{n-\mu}$. Fix a generator $\lambda_m$ of $G(L_m/K)$, for each $m \in \mathbb{N}$, denote by $T_m$ the $K$-algebra $\otimes_{j=1}^{n-1}(L_{j-1}/K, \lambda_{j-1}, \pi_j)$, where $\otimes = \otimes_K$, and put $T'_m = T_n \otimes_K K_v$. We show that $T_n \in d(K)$ (whence $\exp(T_n) = q$ and $\text{Ind}(T_n) = q^{n-1}$). Clearly, there is a $K_v$-isomorphism $\Delta_{j} : \otimes_{j=2}^{n-1}(L_{j-1}/K, \lambda_{j-1}, \pi_j)$, where $\otimes = \otimes_K$, and $\Delta_{j}$ is the unique $K_v$-automorphism of $L_{j-1}$ extending $\lambda_{j-1}$, for each $j$. Therefore, it suffices for the proof of Lemma 4.3(b) to show that $T'_n \in d(K_v)$. Since $K_v$
and \( L'_n, m \in \mathbb{N} \), are related as required by Lemma 4.3 (a), this amounts to proving that \( T_n \in d(K) \), for \((K, v)\) Henselian. Suppose first that \( n = 2 \). As \( L_1/K \) is totally ramified, it follows from the Henselity of \( v \) that \( v(l) \in qv(L_1) \), for every element \( l \) of the norm group \( N(L_1/K) \). One also concludes that if \( l \in N(L_1/K) \) and \( v_L(l) = 0 \), then \( \bar{K}^q \) contains the residue class \( \bar{l} \). These observations prove that \( \pi_2 \notin N(L_1/K) \), so it follows from [36], Sect. 15.1, Proposition b, that \( T_2 \in d(K) \). Henceforth, we assume that \( n \geq 3 \) and view all value groups considered in the rest of the proof as (ordered) subgroups of a fixed divisible hull of \( v(K) \). Note that the centralizer \( C_n \) of \( L_n \) in \( T_n \) is \( L_n \)-isomorphic to \( T_{n-1} \otimes_K L_n \) and \( \otimes_{j=2}^{\infty} (L_{j-1} L_n, \lambda_{j-1}, \pi_j) \), where \( \otimes = \otimes_{L_n} \) and \( \lambda_{j-1} \) is the unique \( L_n \)-automorphism of \( L_{j-1} L_n \) extending \( \lambda_{j-1} \), for each index \( j \). Therefore, using (3.1) and Lemma 4.3 (a), one obtains by a standard inductive argument that it suffices to prove that \( T_n \in d(K) \), provided \( C_n \in d(L_n) \).

Denote by \( w_n \) the valuation of \( C_n \) extending \( v_{L_n} \), and by \( \hat{C}_n \) its residue division ring. It follows from the Ostrowski-Draixl theorem that \( w_n(C_n) \) equals the sum of \( v(M_n) \) and the group generated by \( q^{-1}v(\pi_i) \), \( i = 2, \ldots, n-1 \). Similarly, it is proved that \( \hat{C}_n/K \) is a field extension and \( \hat{C}_n \subseteq \bar{K} \). One also sees that \( \hat{C}_n \neq \bar{K} \) if and only if \( \mu < n-1 \), and such being the case, \( \hat{C}_n : \bar{K} = q^{n-1-\mu} \) and \( \pi_u \in \hat{C}_n^\mu, u = \mu + 1, \ldots, n-1 \). These results show that \( v(\pi_u) \notin qw_n(C_n) \), if \( \mu = n-1 \) and \( \pi_u \notin \hat{C}_n \) when \( \mu < n-1 \). Let now \( \hat{\lambda} \) be the \( K \)-automorphism of \( C_n \) extending both \( \lambda_n \) and the identity of the natural \( K \)-isomorphic copy of \( T_{n-1} \) in \( C_n \), and let \( t_n = \prod_{k=0}^{\infty} \lambda_n^m(t_n) \), for each \( t_n \in C_n \). Then, by Skolem-Noether's theorem (cf. [36], Sect. 12.6), \( \hat{\lambda} \) is induced by an inner \( K \)-automorphism of \( T_n \). This implies \( \hat{w}_n(t_n) = w_n(\hat{\lambda}_n(t_n)) \) and \( \hat{w}_n(t_n') \in qw_n(C_n) \), for all \( t_n \in C_n \), and also indicates that \( \hat{\lambda}^n \) is \( \bar{K}^q \) when \( \hat{w}_n(t_n) = 0 \). It is now easy to see that \( t_n' \neq \pi_n \), for any \( t_n \in C_n \), so it follows from [1], Ch. XI, Theorems 11 and 12, that \( T_n \in d(K) \). Lemma 4.3 is proved.

**Proof of the latter assertion of Theorem 2.2 (c).** Assume that \( F/E \) is an FG-extension, such that \( \text{char}(E) = p \), \( [E:F^p] = p^\nu < \infty \) and \( \text{trd}(F/E) = t \geq 1 \). This implies \( [F:F^p] = p^{\nu+1} \), so it follows from Lemma 4.1 and [1], Ch. VII, Theorem 28, that \( \text{Brd}_p(F) \leq \nu + t \). At the same time, it is clear from (4.1) and Lemma 4.3 that there exists \( \Delta \in d(F) \) with \( \exp(\Delta) = p \) and \( \text{ind}(\Delta) = p^{\nu+1} \), which yields \( \text{Brd}_p(F) \geq \nu + t - 1 \) and so completes our proof.

**Remark 4.4.** Let \((K, v)\) be a maximally complete virtually perfect field with \( \bar{K} \) perfect, \( \text{char}(K) = q > 0 \) and \([K: K^q] = q^n > 1 \). Then it follows from Lemma 4.3 and [1], Ch. VII, Theorem 28, that \( n - 1 \leq \text{Brd}_q(K) \leq n \). Hence, by (3.1), (3.3), Theorem 1, and [2], Theorem 3.3, \( \text{Brd}_q(K) = n - 1 \) if and only if \( r_q(\bar{K}) < n \). Since, by (3.6) (b), \((L, v_L)\) is maximally complete, and \([L: L^p] = q^n \), for every \( L \in \text{Fe}(K) \) (cf. [20], Ch. VII, Sect. 7), this enables one to deduce from [28], Theorem 2, Galois cohomology and the Nielsen-Shreier formula for open subgroups of free pro-q-groups (cf. [14], Ch. I, 4.2, and Ch. II, 2.2) that \( \text{abrd}_q(K) = n - 1 \) if and only if \( \text{cd}_q(G_{\bar{K}}) = 0 \) or the Sylow pro-q-subgroups of \( G_{\bar{K}} \) are isomorphic to \( \mathbb{Z}_q \).

Our next lemma is implied by (3.5), Lemma 3.1 and the immediacy of Henselizations of valued fields (cf. [17], Theorems 15.2.2 and 15.3.5).
Lemma 4.5. Let $E$ be a field, $F = E(X)$ a rational extension of $E$ with $\text{trd}(F/E) = 1$, $f(X) \in E[X]$ an irreducible polynomial over $E$, $M$ an extension of $E$ generated by a root of $f$ in $E_{\text{sep}}$, $v$ a discrete $E$-valuation of $F$ with a uniform element $f$, and $(F_v, \bar{v})$ a Henselization of $(F, v)$. Also, let $\bar{D} \in d(M)$ be an algebra of exponent $p \in \mathbb{P}$. Then $M$ is $E$-isomorphic to the residue field of $(F, v)$ and $(F_v, \bar{v})$, and there exists $\mathcal{D} \in d(\bar{F})$ with $\text{exp}(\mathcal{D}) = p$ and $[D \otimes_F \bar{F}_v] = [\mathcal{D}]$, where $D' \in d(F_v)$ is an inertial lift of $\bar{D}$ over $F_v$.

Proof of Theorem 2.2 (a). Let $\text{abrd}_p(E) = \lambda \in \mathbb{N}$ and $F = E(X_1, \ldots, X_n)$. Then, by Lemma 4.4 there exists $M \in \text{Fe}(E)$, such that $d(M)$ contains an algebra $\hat{\Delta}$ with $\text{exp}(\hat{\Delta}) = p$ and $\text{ind}(\hat{\Delta}) = p^\lambda$. We show that there is $\Delta \in d(F)$ with $\text{exp}(\Delta) = p$ and $\text{ind}(\Delta) \geq p^{\lambda + \kappa - 1}$. Suppose first that $\kappa = 1$, take a primitive element $\alpha$ of $M/E$, and denote by $f(X_1)$ its minimal monic polynomial over $E$. Attach to $f$ a discrete valuation $v$ of $F$ and fix $(F_v, \bar{v})$ as in Lemma 4.5. Then, by Lemma 3.1 there exists $\Delta_1 \in d(F)$ with $[\Delta_1 \otimes_F \bar{F}_v] = [\hat{\Delta}]$, in $\text{Br}(F_v)$, where $\hat{\Delta}$ is an inertial lift of $\Delta$ over $F_v$. Since $\Delta \in d(\bar{F}_v)$, $\text{exp}(\Delta) = p$ and $\text{ind}(\Delta) = p^\lambda$, this indicates that $p^\lambda \mid \text{ind}(\Delta_1)$, which proves Theorem 2.2 (a) when $\kappa = 1$. In addition, Lemma 3.2 implies that there exist infinitely many degree $p$ cyclic extensions of $F$ in $\bar{F}_v$. Hence, $F_v$ contains a subfield $R_\kappa$ of $F$ with $\mathcal{G}(R_\kappa/F)$ of order $p^{\kappa - 1}$ and exponent $p$. When $\text{ind}(\Delta_1) = p^\kappa$, this makes it easy to deduce the existence of $\Delta$, for an arbitrary $\kappa$, from (4.1) (with a ground field $E(X_1)$ instead of $E$) and [33], Theorem 1, or else, by repeatedly using the Proposition in [30], Sect. 19.6. It remains to consider the case where $\kappa \geq 2$ and there exists $D_1 \in d(E(X_1))$ with $\text{exp}(D_1) = p$ and $\text{ind}(D_1) = p^{\lambda_1} > p^\lambda$. It is easily verified that $D_1 \otimes_{E(X_1)} E(X_1)((X_2)) \subseteq d(E(X_1)((X_2)))$, and it follows from Lemma 3.2 that there are infinitely many degree $p$ cyclic extensions of $E(X_1, X_2)$ in $E(X_1)((X_2))$. As in the case of $\kappa = 1$, this enables one to prove the existence of $\Delta' \in d(F)$ with $\text{exp}(\Delta') = p$ and $\text{ind}(\Delta') = p^{\lambda + \kappa - 2} \geq p^{\lambda + \kappa - 1}$. Thus Theorem 2.2 (a) is proved.

Corollary 4.6. Let $E$ be a field and $F/E$ a rational extension with $\text{trd}(F/E) = \infty$. Then $\text{Brd}_p(F) = \infty$, for every $p \in \mathbb{P}$.

Proof. This follows from Theorem 2.2 (a) and the fact that, for any rational field extension $F'/F$ with $\text{trd}(F'/F) = 2$, there is an $E$-isomorphism $F \cong F'$, whence $\text{Brd}_p(F) = \text{Brd}_p(F')$ for each $p \in \mathbb{P}$.

Let $E$ be a field with $\text{abrd}_p(E) = \infty$, $p \in \mathbb{P}$, and let $F/E$ be a transcendental FG-extension. Then it follows from (1.1) (b), (c) and Theorem 2.2 (b) that Brauer pairs $(m, n) \in \mathbb{N}^2$ are index-exponent pairs over $F$. Therefore, Corollary 4.6 with its proof implies the latter assertion of (1.2).

Alternatively, it follows from Galois theory, Lemmas 3.2, 4.5 and basic theory of valuation prolongations that $r_p(\Phi) = \infty$, $p \in \mathbb{P}$, for every transcendental FG-extension $\Phi/E$. Hence, by [15] and Witt’s lemma (cf. [13], Sect. 15, Lemma 2), finite abelian groups are realizable as Galois groups over $\Phi$, so both parts of (1.2) can be proved by the method used in [30], Sect. 19.6.
Proposition 4.7. Let $F/E$ be an FG-extension with $\text{trd}(F/E) = t \geq 1$ and $\text{abrd}_p(E) < \infty$, $p \in P$, for some subset $P \subseteq \mathbb{P}$. Then $P$ possesses a finite subset $P(F/E)$, such that $\text{Brd}_p(F) \geq \text{abrd}_p(E) + t - 1$, $p \in P \setminus P(F/E)$.

Proof. It follows from (1.1) (c) and Theorem 2.2 (a) that one may take as $P(F/E)$ the set of divisors of $[F : F_0]$ lying in $P$, for some rational extension $F_0$ of $E$ in $F$ with $\text{trd}(F_0/E) = t$.

Example 4.8. There exist field extensions $F/E$ satisfying the conditions of Proposition 4.7, for $P = \mathbb{P}$, such that $P(F/E)$ is nonempty. For instance, let $E$ be a real closed field, $\Phi$ the function field of the Brauer-Severi variety attached to the symbol E-algebra $A = A_{-1}(-1, -1; E)$, and $F/\Phi$ a finite field extension with $\sqrt{-1} \notin F$. Then $\text{abrd}(F) = 0 < \text{abrd}_1(E) = 1$ (see the example in [10]) and $\text{abrd}_p(E) = 0$, $p > 2$, which implies $P(F/E) = \{2\}$ and $P = \mathbb{P}$.

5 Proof of Theorem 2.2 (b)

The former claim of Theorem 2.2 (b) is implied by the following lemma.

Lemma 5.1. Let $K$ be a field with $\text{abrd}_p(K) = \infty$, for some $p \in \mathbb{P}$, and let $F/K$ be an FG-extension with $\text{trd}(F/K) \geq 1$. Then there exist $D_\nu \in d(F)$, $\nu \in \mathbb{N}$, such that $\exp(D_\nu) = p$ and $\text{ind}(D_\nu) \geq p^{\nu}$.

Proof. Statement (1.1) (c) implies the class of fields $\Phi$ with $\text{abrd}_p(\Phi) = \infty$ is closed under the formation of finite extensions. Since $K$ has a rational extension $F_0$ in $F$ with $\text{trd}(F_0/K) = \text{trd}(F/K)$, whence $[F : F_0] < \infty$, this shows that it is sufficient to prove Lemma 5.1 in the case of $F = F_0$. Note also that $\text{ind}(T_0 \otimes_K F_0) = \text{ind}(T_0)$ and $\exp(T_0 \otimes_K F_0) = \exp(T_0)$, for each $T_0 \in d(K)$, so one may assume, for the proof, that $F = F_0$ and $\text{trd}(F/K) = 1$. It follows from Lemma 11 and the equality $\text{abrd}_p(K) = \infty$ that there are $M_\nu \in \text{Fe}(K)$ and $D_\nu \in d(M_\nu)$, $\nu \in \mathbb{N}$, with $\exp(D_\nu) = p$ and $\text{ind}(D_\nu) \geq p^{\nu}$, for each index $\nu$. Hence, by Lemmas 4.1 and 5.1 there exist a discrete $K$-valuation $v_\nu$ of $F$, and an algebra $D_\nu \in d(F)$, such that the residue field of $(F, v_\nu)$ is $K$-isomorphic to $M_\nu$, $\exp(D_\nu) = p$, and $[D_\nu \otimes_F F_\nu] = [D_\nu']$, where $D_\nu'$ is an inertial lift of $D_\nu$ over $F_\nu$. This implies $\text{ind}(D_\nu) = \text{ind}(D_\nu')$, $\nu \in \mathbb{N}$, proving Lemma 5.1.

To prove the latter part of Theorem 2.2 (b) we need the following lemma.

Lemma 5.2. Let $A$, $B$ and $C$ be algebras over a field $F$, such that $A, B, C \in s(F)$, $A = B \otimes_F C$, $\exp(C) = p \in \mathbb{P}$, and $\exp(B) = \text{ind}(B) = p^m$, for some $m \in \mathbb{N}$. Assume that $\text{ind}(A) = p^n > p^m$ and $k$ is an integer with $m < k \leq n$. Then there exists $T_k \in s(F)$ with $\exp(T_k) = p^m$ and $\text{ind}(T_k) = p^k$. 

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Proof. When \( k = n \), there is nothing to prove, so we assume that \( k < n \). By [31, Sect. 4, Theorem 2, [C] = [\Delta_1 \otimes_F \cdots \otimes_F \Delta_\nu], \) where \( \nu \in \mathbb{N} \) and for each index \( j \), \( \Delta_j \in d(F) \) and \( \text{ind}(\Delta_j) = p \). Put \( T_j = B \otimes_F (\Delta_1 \otimes_F \cdots \otimes_F \Delta_j) \) and \( t_j = \deg(T_j)/\text{ind}(T_j), \) \( j = 1, \ldots, \nu \), and let \( S(A) \) be the set of those \( j \), for which \( \text{ind}(T_j) \geq p^k \). Clearly, \( S(A) \neq \emptyset \) and the set \( S_0(A) = \{ i \in S(A) : t_i \leq t_j, j \in S(A) \} \) contains a minimal index \( \gamma \). The conditions of Lemma 5.2 ensure that \( \exp(T_j) = p^m \), so \( \text{ind}(T_j) = p^m(j) \), where \( m(j) \in \mathbb{N} \), for each \( j \in S(A) \). We show that \( \text{ind}(T_i) = p^k \). If \( \gamma = 1 \), then (1.1) (c) and the inequality \( m < k \) imply \( k = m + 1 \) and \( \text{ind}(T_1) = p^k \), as claimed. Suppose now that \( \gamma \geq 2 \). Then it follows from (1.1) (b) that \( \text{ind}(T_i) = \text{ind}(T_{i+1}), p^\mu \), for some \( \mu \in \{-1, 0, 1\} \). The possibility that \( \mu \neq 1 \) is ruled out, since it contradicts the fact that \( \gamma \in S_0(A) \). This yields \( \text{ind}(T_i) = \text{ind}(T_{i-1}), p \) and \( t_i = t_{i-1} \). As \( \gamma \) is minimal in \( S_0(A) \), it is now easy to see that \( \text{ind}(T_{i-1}) = p^{k-u}, u = 0, 1, \) which proves Lemma 5.2. \( \Box \)

The conditions of Lemma 5.2 are fulfilled, for each \( m \in \mathbb{N} \) and infinitely many integers \( u > m \), if \( \text{char}(E) = p, \) then \( E \) is not virtually perfect and \( F/E \) satisfies the conditions of Theorem 2.2. Since, by Witt’s lemma, cyclic \( p \)-extensions of \( F \) are realizable as intermediate fields of \( \mathbb{Z}_p \)-extensions of \( F \), this can be obtained by applying (1.1) (b), (4.1) and Lemma 4.3, together with general properties of cyclic \( F \)-algebras, see [32, Sect. 15.1, Corollary b and Proposition b. Thus Theorem 2.2 is proved in the case of \( p = \text{char}(E) \). For the proof of the latter assertion of Theorem 2.2, when \( p \neq \text{char}(E) \), we need the following lemma.

Lemma 5.3. Let \( K \) be a field and \( F/K \) an FG-extension with \( \text{trd}(F/K) = 1 \). Then, for each \( p \in \mathbb{P} \) different from \( \text{char}(K) \), there exist non-equivalent discrete \( K \)-valuations \( \nu_m \) of \( F \), \( m \in \mathbb{N} \), satisfying the following:

(1) For any \( m \in \mathbb{N} \), \( (F, \nu_m) \) possesses a totally ramified extension \( (F_m, w_m) \), such that \( F_m \in I(F_{\text{sep}}/F) \), \( F_m/F \) is cyclic and \( \{F_m : F\} = p^m \);

(2) The valued fields \( (F_m, w_m) \) can be chosen so that \( F_{m'} \cap F_m = F, m' \neq m \).

Proof. Let \( X \in F \) be a transcendental element over \( K \). Then \( F/K(X) \) is a finite extension, and the separable closure of \( K(X) \) in \( F \) is unramified relative to every discrete \( K \)-valuation of \( K(X) \), with at most finitely many exceptions (up-to an equivalence, see [3, Ch. I, Sect. 5]). This reduces the proof of Lemma 5.3 to the special case of \( F = K(X) \). For each \( m \in \mathbb{N} \), let \( \delta_m \in F_{\text{sep}} \) be a primitive \( p^m \)-th root of unity, \( K_m = K(\delta_m) \), \( f_m(X) \in K[X] \) the minimal polynomial of \( \delta_m \) over \( K \), and \( \rho_m \) a discrete \( K \)-valuation of \( F \) with a uniform element \( f_m \). Clearly, the valuations \( \rho_m, m \in \mathbb{N} \), are pairwise non-equivalent. Also, it is well-known (see [26], Ch. V, Theorem 6; Ch. VIII, Sect. 3, and [20], Ch. 4, Sect. 1) that if \( m', \bar{m} \in \mathbb{N} \), then the extension \( K_{m'}(\delta_{m'/})/K_{m'} \) are cyclic except, possibly, in the case where \( m' = 1, \bar{m} > 2, p = 2, \text{char}(K) = 0 \) and \( \delta_2 \notin K \). Denote by \( \nu_m \) the valuation \( \rho_{m+1} \), for each \( m, \) if \( p = 2 \), \( \text{char}(K) = 0 \) and \( \delta_2 \notin K \), and put \( \nu_m = \rho_m, m \in \mathbb{N} \), otherwise. Since \( p \neq \text{char}(K) \), and by Lemma 4.5, \( K_m \) is \( K \)-isomorphic to the residue field of \( (F, \rho_m) \), we have \( \delta_m \in F_{\nu_m} \), where \( F_{\nu_m} \) is a Henselization of \( F \) in \( F_{\text{sep}} \) relative to \( \nu_m \). This enables one to deduce from Kummer theory that \( F_{\nu_m} \) possesses a totally ramified cyclic extension \( L_{\nu_m} \) of degree \( p^m \). Furthermore, it follows from the choice of \( \nu_m \) and the observation on the extensions \( K_{m'}(\delta_{m'})/K_{m'} \) that \( F_{m'}(\delta_{m'})/F_{m'} \) are cyclic, for all pairs \( m', m \in \mathbb{N} \). Hence, by the generalized Grunwald-Wang
theorem (cf. [29], Theorems 1 (ii) and 2) and the note preceding the statement of Lemma 5.3 there exist totally ramified extensions \((F_m, w_m)/(F, v_m), m \in \mathbb{N}\), such that \(F_m \in I(F_{\text{sep}}/F), F_m/F\) is cyclic with \([F_m : F] = p^m\), for each \(m\), and in case \(m \geq 2, F_m/F\) is unramified relative to \(v_1, \ldots, v_{m-1}\). This ensures that \(F_m^{m'} \cap F_m = F, m' \neq m\), and so completes the proof of Lemma 5.3.

\[\square\]

**Proof of the latter statement of Theorem 2.2 (b).** Let \(\text{abrd}_p(E) = \infty\), for some \(p \in \mathbb{P}\). In view of (1.1) (b), Lemmas 8.1 5.1 and 5.2, it is sufficient to show that there exists \(A_m \in d(F)\) with \(\exp(A_m) = \text{ind}(A_m) = p^m\), for any fixed \(m \in \mathbb{N}\). As in the proof of Lemma 5.1 our considerations reduce to the special case of \(\text{trd}(F/K) = 1\). Analyzing this proof, one obtains that there is \(M \in \text{Fe}(E)\), such that \((d(M)\) contains a cyclic \(M\)-algebra \(A_1\) of degree \(p\), and when \(p \neq \text{char}(E)\), \(M\) contains a primitive \(p^m\)-th root of unity \(\delta_m\). Note further that \(M\) can be chosen so as to be \(E\)-isomorphic to the residue field \(\bar{F}\) of \(F\) relative to some discrete \(E\)-valuation \(v\). In view of Kummer theory (see [26], Ch. VIII, Sect. 6) and Witt’s lemma, the assumptions on \(M\) ensure that each degree \(p\) cyclic extension \(Y_1\) of \(M\) lies in \(\text{I}(Y_m/M)\), for some degree \(p^m\) cyclic extension \(Y_m/M\). Suppose now that \(Y_1\) embeds in \(A_1\) as an \(M\)-subalgebra, fix a generator \(\tau_1\) of \(G(Y_1/M)\) and an automorphism \(\tau_m\) of \(Y_m\) extending \(\tau_1\). Then \(\tilde{A}_1\) is isomorphic to the cyclic \(M\)-algebra \((Y_1/M, \tau_1, \tilde{\beta})\), for some \(\tilde{\beta} \in M^*, \tau_m\) generates \(G(Y_m/M)\), the \(M\)-algebra \(\tilde{A}_m = (Y_m/M, \tau_m, \tilde{\beta})\) lies in \(s(M)\), and we have \(p^{m-1}[A_m] = [\tilde{A}_1]\) (cf. [30], Sect. 15.1, Corollary b). Therefore, \(\tilde{A}_m \in d(M)\) and \(\text{ind}(\tilde{A}_m) = \exp(\tilde{A}_m) = p^m\). Assume now that \((F, v)\) has a valued extension \((L, v_L)\), such that \(L/F\) is cyclic, \([L : F] = p^m\) and the residue field of \((L, v_L)\) is \(E\)-isomorphic to \(Y_m\). Then \(G(L/F) \cong G(Y_m/M)\) and for each generator \(\sigma\) of \(G(L/F)\) and pre-image \(\beta\) of \(\tilde{\beta}\) in \(O_{\sigma}(F)\), the algebra \(A_m = (L/F, \sigma, \beta)\) lies in \(d(F)\) (see [30], Sect. 15.1, Proposition b, and [24], Theorem 5.6). Note also that \(\text{ind}(A_m) = \exp(A_m) = p^m\) and \(\sigma\) can be chosen so that \(A_m \otimes_F E_{\nu}\) be an inertial lift of \(\tilde{A}_m\) over \(E_{\nu}\). When \(p > 2\), this completes the proof of Theorem 2.2 (b), since Lemma 8.1 guarantees in this case the existence of a valued extension \((L, v_L)\) of \((F, v)\) with the above-noted properties.

Similarly, one concludes that if \(p = 2\), then it suffices to prove Theorem 2.2 (b), provided \(\text{char}(E) = 0\) and \(G(E(\delta_m)/E)\) is noncyclic, where \(\delta_m\) is a primitive \(2^m\)-th root of unity in \(E_{\text{sep}}\). This implies the group \(E_1'/E_1^{2^m}\) has exponent \(2^m\), for each \(\nu \in \mathbb{N}, E_1 \in \text{Fe}(E)\) (cf. [26], Ch. VIII, Sects. 3 and 9). Take a valued extension \((F_m, w_m)/(F, v_m)\) as required by Lemma 5.3 and denote by \(\tilde{F}_m\) the residue field of \((F, v_m)\). Fix a generator \(\psi_m\) of \(G(F_m/F)\) and an element \(\tilde{\beta}_m \in \tilde{F}_m^{*-1}\) so that \(\tilde{\beta}_m^{2^m} \notin \tilde{F}_m^{2^m}\), and put \(A_m = (F_m/F, \psi_m, \beta_m)\), for some pre-image \(\beta_m\) of \(\tilde{\beta}_m\) in \(O_{\psi_m}(F)\). As \((F_m, w_m)/(F, v_m)\) is totally ramified, \(w_m\) is uniquely determined by \(v_m, u\) to an equivalence. Therefore, \(w_m(\lambda_m) = w_m(\psi_m(\lambda_m))\), for all \(\lambda_m \in F_m\), and when \(w_m(\lambda_m) = 0\), \(\tilde{F}_m^{2^m}\) contains the residue class of the norm \(N_{F_m/F}(\lambda_m)\). Now it follows from [30], Sect. 15.1, Proposition b that \(A_m \in d(F)\) and \(\text{ind}(A_m) = \exp(A_m) = 2^m\), so Theorem 2.2 is proved.

**Corollary 5.4.** Let \(E\) be a field with \(\text{abrd}(E) = \infty\). Then \(\text{Brd}(F) = \infty\), for every transcendental \(F\)-extension \(F/E\).
Proof. The equality $\text{abrd}(E) = \infty$ means that either $\text{abrd}_p(E) = \infty$, for some $p' \in \mathcal{P}$, or $\text{abrd}_p(E)$, $p \in \mathcal{P}$, is an unbounded number sequence. In view of Theorem 2.2 (b) and Proposition 4.7, this proves our assertion. 

Corollary 5.4 shows that a field $E$ satisfies $\text{abrd}(E) < \infty$, if its FG-extensions have finite dimensions, in the sense of [3], Sect. 4. In view of (2.6) (a), this proves that Problem 4.4 of [3] is solved, generally, in the negative, even when finite extensions of $E$ have finite Brauer dimensions. Statements (2.6) also imply that both cases pointed out in the proof of Corollary 5.4 can be realized.

Corollary 5.5. Let $F$ be a rational extension of an algebraically closed field $F_0$. Then $\text{trd}(F/F_0) = n < \infty$ if and only if each Brauer pair $(m,n) \in \mathbb{N}^2$ is realizable as an index-exponent pair over $F$.

Proof. Assume that $\text{trd}(F/F_0) = n < \infty$. Then finite extensions of $F$ are $C_n$-fields, by Lang-Tsen’s theorem [29], so Lemma 4.4 and 32, (16.10), imply $\text{Brd}_2(F) \leq \text{abrd}_2(F) < 2^{n-1}$. In view of (1.2), this completes our proof.

Theorem 2.2 and Example 4.8 lead naturally to the question of whether $\text{Brd}_p(F) \geq k + \text{trd}(F/E)$, provided that $F/E$ is an FG-extension and $\text{Brd}_p(F') = k < \infty$, $F' \in \text{Fe}(E)$, for a given $p \in \mathcal{P}$. Our next result gives an affirmative answer to this question in several frequently used special cases:

Proposition 5.6. Let $E$ be a field and $F$ an FG-extension of $E$ with $\text{trd}(F/E) = n > 0$. Suppose that there exists $M \in \text{Fe}(E)$ satisfying the following condition, for some $p \in \mathcal{P}$ and $k \in \mathbb{N}$:

(c) For each $M' \in \text{Fe}(M)$, there are $D' \in d(M')$ and $L' \in I(M'(p)/M')$, such that $\exp(D') = [L': M'] = p$, ind$(D') = p^k$ and $D' \otimes_M L' \in d(L')$.

Then there exist $D \in d(F)$, such that $\exp(D) = p$ and ind$(D) \geq p^{k+n};$ in particular, $\text{Brd}_p(F) \geq k + n$.

Proposition 5.6 is proved along the lines drawn in the proofs of Theorem 2.2 (a) and (b), so we omit the details. Note only that if $n \geq 2$ or $k = 1$, then $D$ can be chosen so that $D \otimes_F F_v \in d(F_v)$, $[D \otimes_F F_v] \in \text{Br}(F_v, m/F_v)$ and $p^{n-1} \mid e(D \otimes_F F_v/F_v) \mid p^n$, for some $E$-valuation $v$ of $F$ with $\mathbb{Z}^{n-1} \leq v(F) \leq \mathbb{Z}^n$.

Remark 5.7. Condition (c) of Proposition 5.6 is fulfilled, for $k = 1 = \text{abrd}(E)$ and any $p \in \mathcal{P}$, if $E$ is a global field or an FG-extension of an algebraically closed field $E'_0$, with $\text{trd}(E/E'_0) = 2$. It also holds when $k = 1$, $p \in \mathcal{P}$ and $E$ is an FG-extension of a perfect PAC-field $E_0$ with $\text{trd}(E/E_0) = 1 = \text{cd}_p(E_0)$ (see [10], Sect. 3, [30], Sect. 19.3, and the proof of [17], Proposition 5.1). In these cases, it can be deduced from (3.1) and [33], Theorem 1, that the power series fields $E_m = E((X_1)) \ldots ((X_m))$, $m \in \mathbb{N}$, satisfy (c), for $k = 1 + m = \text{abrd}_p(E_m)$ (cf. [28], Appendix A, or [11], (5.2) and Proposition 5.1). In addition, the conclusion of Proposition 5.6 is valid, if $E$ is a local field, $k = 1$ and $p \in \mathcal{P}$, although (c) is then violated, for every $p$ (see Proposition 7.3 with its proof, and the appendices to [39] and [2], Ch. VI, Sect. 1).
For a proof of the concluding result of this Section, we refer the reader to [10]. When $F'/E$ is a rational extension and $r_p(E) \geq \text{trd}(F/E)$, this result is contained in [34]. Combined with Lemma 3.2 it implies Nakayama’s inequalities $\text{Br}_p(F') \geq \text{trd}(F'/E') - 1$, $p' \in \mathbb{P}$, for any FG-extension $F'/E'$.

**Proposition 5.8.** Let $F/E$ be an FG-extension with $\text{trd}(F/E) = n \geq 1$ and $\text{cd}_p(G_E) \neq 0$, for some $p \in \mathbb{P}$. Then $\text{Br}_p(F) \geq n$ except, possibly, if $p = 2$, the $\text{Sylow$pro-2$-subgroups of } G_E$ are of order 2, and $F$ is a nonreal field.

It is not known whether an FG-extension $F/E$ with $\text{trd}(F/E) = n \geq 3$ satisfies $\text{abrd}_p(F) = \text{Brd}_p(F) = n - 1$, provided that $p \in \mathbb{P}$, $\text{cd}_p(G_E) = 0$, and $E$ is perfect in the case where $p = \text{char}(E)$. It follows from (1.1) (c) that this question is equivalent to the Standard Conjecture on $F/E$ (stated by Colliot-Thélène, see [28] and [27], Sect. 1) when $E$ is algebraically closed. The question is also open in the case excluded by Proposition 5.8. Note further that the Lang-Tsen theorem and [32], (16.10), attract interest in the problem of whether $\text{Brd}_p(F) < p^{n-1}$ when $E$ is algebraically closed. If this holds, for every $p \in \mathbb{P}$, one would have $\text{abrd}_p(F) < \infty$, $p \in \mathbb{P}$, which would considerably extend the applicability of Proposition 2.1. Finally, it follows that if $\text{Brd}_p(F) \leq p^{n-2}$ and $p^{n-2}$ is a sufficiently exact upper bound for $\text{Brd}_p(F)$ and each $p$, then this would solve negatively [3]. Problem 4.5, by showing that $\text{Br}(F) = \infty$ whenever $n \geq 3$.

### 6 Proof of Theorem 2.3

The proof of Theorem 2.3 is based on the following two lemmas, the former of which is a special case of [11], Theorem 2.3.

**Lemma 6.1.** Let $(K,v)$ be a Henselian field with $\text{Br}(\hat{K})_p = \{0\}$, for some $p \in \mathbb{P}$, $p \neq \text{char}(\hat{K})$, and let $\tau(p)$ be the $\mathbb{F}_p$-dimension of $v(K)/pv(K)$, $\varepsilon_p \in \hat{K}_{\text{sep}}$ a primitive $p$-th root of unity, and $m_p = \min\{\tau(p), \tau_p(\hat{K})\}$. Then:

(a) $\text{Brd}_p(K) = \infty$ if and only if $m_p = \infty$ or $\tau(p) = \infty$ and $\varepsilon_p \in \hat{K}$;

(b) When $\text{Brd}_p(K) < \infty$, it is determined by the formula $\text{Brd}_p(K) = u_p$, where $u_p = \lfloor(\tau(p) + m_p)/2\rfloor$ if $\varepsilon_p \in \hat{K}$, and $u_p = m_p$, otherwise.

**Lemma 6.2.** In the setting of Lemma 6.1, let $\text{cd}_p(G_{\hat{K}}) \leq 1$, $G_p$ be a Sylow pro-$p$-subgroup of $G_{\hat{K}}$, and $t_p$ the rank of $G_p$ as a pro-$p$-groups. Then:

(a) $\text{abrd}_p(K) = \infty$ if and only if $\tau(p) = \infty$;

(b) $\text{abrd}_p(K) = \tau(p)$, provided that $t_p \geq 2$ and $\tau(p) < \infty$;

(c) If $t_p \leq 1$ and $\tau(p) < \infty$, then $\text{abrd}_p(K) = \lfloor(\tau(p) + t_p)/2\rfloor$.

**Proof.** Let $K_p$ be the fixed field of some Sylow pro-$p$-subgroup of $G_{\hat{K}}$. Then it follows from (1.1) (c) and [9] (1.2), that $\text{abrd}_p(K_p) = \text{abrd}_p(K)$. Also, it is clear from (3.1), (3.2), (3.3) and Galois theory that $v(K_p)/pv(K_p) \cong v(K)/pv(K)$ and $\hat{K}_p$ is $\hat{K}$-isomorphic to the fixed field of $G_p$. Thus our proof reduces to the case of $K_{\text{sep}} = K_p$. Then $G_{\hat{K}} = G_p$, $\text{abrd}_p(\hat{K}) = 0$ (cf. [10], Ch. II, 3.1) and $\varepsilon_p \in \hat{K}$,
so Lemma 6.2 (a) follows from Lemma 6.1. We assume further that $\tau(p) < \infty$ and $K_{\text{sep}} = K_p$. In view of (3.2) and Lemma 6.1 this implies abrd$_p(K) \leq \tau(p)$. As $\text{cd}_q(G_k) \leq 1$, one obtains from (3.3), [20] and Galois cohomology (cf. [10], Ch. I, 4.2) that $G_k$ and $G(K_{ur}/K)$ are free pro-$p$-groups of rank $t_p$. Suppose first that $t_p \geq 2$. Then it follows from Galois theory and Nielsen-Schreier’s formula for open subgroups of free pro-$p$-groups that, for each $m \in \mathbb{N}$, there exists $T_m \in \text{Fe}(K) \cap R(K_{ur}/K)$ with $r_p(T_m) \geq m$. Hence, by (3.3) and Lemma 6.1 (b), $\text{Brd}_p(T_m) = \tau(p)$, $m > \tau(p)$, so Lemma 6.2 (b) is proved.

Let now $t_p \leq 1$ and $T \in \text{Fe}(K)$. Then (3.2), (3.3), [43], Theorem 2, and Nielsen-Schreier’s formula imply $v(T)/pv(T) \cong v(K)/pv(K)$ and $r_p(T) = t_p$, so Lemma 6.1 yields $\text{Brd}_p(T) = \lfloor (\tau(p) + t_p)/2 \rfloor$, proving Lemma 6.2 (c). □

Our next result, applied to the field $K_0 = \mathbb{F}_q$, proves Theorem 2.3 in the case where $q > 2$ or $(q, k) = (2, 1)$. In view of (3.6) (a), Remark 4.4 and Lemmas 6.1 and 6.2 it describes the sequences $(\text{Brd}_p(K), \text{abrd}_p(K))$, $p \in \mathbb{P}$, when $(K, v)$ runs across the class of maximally complete fields with finite residue fields.

Proposition 6.3. Assume that $K_0$ is a finite field with $q^n$ elements, where $q = \text{char}(K_0)$, put $P_{q,m} = \{ p \in \mathbb{P} : p \mid q^{(q^n - 1)} \}$, and let $c_p, p \in \mathbb{P}$, be a sequence of elements of $\mathbb{N}_\infty$. Then:

(a) There exists a Henselian field $(E, \psi)$ with $\text{char}(E) = q$, $E = K_0$, $\text{Brd}_p(E) = c_q$ and $\text{abrd}_p(E) = c_p$, for each $p \in \mathbb{P}$; this ensures that $\text{Brd}_p(E) \leq 1$ when $p \in \mathbb{P} \setminus P_{q,m}$, $\text{Brd}_p(E) = c_p, p \in P_{q,m}$, and $\text{Brd}_p(E) \neq 0$ in case $c_p \neq 0$;

(b) If $0 < c_q \neq \infty$, then $(E, \psi)$ can be chosen so that $[E : E^n] = q^{1+c_q}$ and finite extensions of $E$ be defectless;

(c) When $c_q = 0$, $(E, \psi)$ can be chosen so that $E_{tr} = E_{\text{sep}}$ and either $[E : E^n] = \infty$ or $E$ is a perfect field.

Proof. Let $n(p) \in \mathbb{N}_\infty : p \in \mathbb{P}$, be a sequence, such that $n(p) = \infty$, provided $c(p) = \infty$, and $2c_p - 1 \leq n(p) \leq 2c_p$ in case $c(p) < \infty$ and $p \neq q$. Let also $(K, v)$ be a Henselian field, defined as in the proof of Lemma 5.3, with $\text{char}(K) = q$. $\tilde{K} = K_0$, and $v(K)/pv(K)$ having $\mathbb{F}_q$-dimension $n(p)$, for each $p \in \mathbb{P}$. Then it follows from Remark 4.4 and Lemmas 6.1, 6.2 that if $c(q) = \infty$ or $2 \leq n(q) = c_q + 1 < \infty$ and $[K : K^n] = q^{c(q)}$, then the valued field $(E, \psi) = (K, v)$ has the properties required by Proposition 6.3 (a) and (b). When $n(q) = c_q = 0$ and $(E, \psi) = (K, v)$, one sees that $E_{\text{sep}} = E_{tr}$, $E$ is perfect and $(E, \psi)$ is chosen in accordance with Proposition 6.3 (a). It remains for us to complete the proof of Proposition 6.3 (c). Assume that $c_q = 0$ and $n(q) = \infty$, take $E \in I(K_{\text{sep}}/K)$ so that $EK_{tr} = K_{\text{sep}}$ and $E \cap K_{tr} = K$, and put $\psi = v_E$. Then, by (3.4), $v(E)/pv(E) \cong v(K)/pv(K)$, $p \in \mathbb{P} \setminus \{q\}$, $E_0 = K_0$ and $G_E \cong G(K_{tr}/K)$. This implies the Sylow pro-$q$-subgroups of $G_E$ are isomorphic to $\mathbb{Z}_q$. $E_{\text{tr}} = K_{\text{sep}}$ and $\psi(E) = q\psi(E)$. Applying (3.1) and [43], Theorem 3.1, one obtains further that if there exist $E' \in \text{Fe}(E)$ and $D' \in d(E')$ with $\text{ind}(D') = q$, then $D'/E'$ must be inertial. Since, however, $\tilde{E} = K_0$, whence $\text{Br}(\tilde{E'}) = \{0\}$, for every $\tilde{E'} \in \text{Fe}(\tilde{E})$, this observation and [31], Sect. 4, Theorem 2, prove in fact that $\text{Br}(E_1) = \{0\}$, for $E_1 \in \text{Fe}(E)$, i.e. $\text{abrd}_p(E) = 0 = c_q$. As $n(q) = \infty$ and $E \in I(K_{\text{sep}}/K)$, it follows that $[E : E^n] = [K : K^n] = \infty$, which completes our proof. □
Our next result proves Theorem 6.3 in the case of $q = 2 \leq k$.

**Corollary 6.4.** In the setting of Proposition 6.3, if $0 < c(q) < \infty$, then there are Henselian fields $(E_{m},\psi_{m})$, $m \in \mathbb{N}_{\infty}$, $m > c(q)$, with the following properties:

(a) $\text{char}(E_{m}) = q_{\ast}$, $[E_{m} : E] = q^{m}$, $\psi_{m}(E_{m}) = \psi_{m'}(E_{m'})$, where $m' = 1 + c(q)$, $\text{Brd}_{q}(E_{m}) = c_{q}$, and $\text{abrd}_{q}(E_{m}) = c_{p}$, $p \in \mathbb{P}$, for every index $m$;

(b) $[E_{\infty} : E_{\infty}] = \infty$ and $[E_{m} : E_{m}] = q^{m}$, provided that $c(q) < m < \infty$.

**Proof.** Fix a Henselian field $(K, v)$ as in the proof of Proposition 6.3 (a) and (b), denote by $\Theta$ the subfield of $K$ defined in the proof of Lemma 3.3, and put $n = n(q) = m'$ and $(E_{m}, \psi_{m}) = (K_{n}, \psi_{m})$, $m' \leq m < \infty$, where $(K_{n}, \psi_{m})$ is a valued subfield of $(K, v)$, taken as in Remark 3.4 for each $m$. We show that $(E_{m}, \psi_{m})$, $m' < m < \infty$, have the properties required by Corollary 6.4. In view of Remark 3.4, Proposition 6.3 and Lemmas 6.1 and 6.2, it suffices to prove that $\text{Brd}_{q}(K_{m}') = c_{q}$, where $m$ is an arbitrary index and $K_{m}' \in \text{Fe}(K_{m})$. Take $\Theta_{n-1}$, $\Theta = \Theta_{n-1}((\mathbb{Z}_{q}))$, $\kappa$, $\Theta_{m}$ and $\Theta_{m}'$ as in Remark 3.4 and denote by $\theta$ the valuation of $\Theta_{m}$ induced by $\kappa$ and the maximal isolated subgroup of $\kappa(\Theta_{m})$. Clearly, $\Theta$ is a Henselian $\Theta_{n-1}$-valuation with $\Theta_{n-1} = \theta(\Theta_{n}) = \mathbb{Z}$ and $\theta(\mathbb{Z}_{q}) = 1$. Also, the valuation $\theta_{m}$ of $\Theta_{m}$ induced by $\theta$ is Henselian and discrete. $\Theta_{m}$ is a completion of $\Omega_{m}'$ with respect to $\theta_{m}$, and $\theta$ extends $\theta_{m}$ continuously on $\Theta_{n}$. This shows that $\Theta_{n-1} = (\Theta_{n}, \theta)$ and the group $\text{ind}(D_{m} \otimes \gamma_{m}, \Theta_{n}) = \text{ind}(D_{m})$, for each $D_{m} \in d(\Omega_{m}')$ (see [26], Ch. XII, Sect. 5, and [12], Theorem 2). Hence, $\text{Brd}_{q}(\Omega_{m}') \leq \text{Brd}_{q}(\Theta_{n})$. Observe now that $(\Theta_{n}, \theta)$ is maximally complete, $[\Theta_{n} : \Omega_{n}] = q^{m}$, and by the proofs of Proposition 6.3 (a), (b) and Lemma 6.3, $\text{Brd}_{q}(\Theta_{n}) = n - 1 = c_{q}$. On the other hand, it follows from Lemma 4.3 applied to $(\Omega_{m}', \theta_{m})$, that $\text{Brd}_{q}(\Omega_{m}') \geq c_{q}$, so we have $\text{Brd}_{q}(\Omega_{m}') = c_{q}$. As finite valued extensions of $(\Theta_{n}, \kappa)$ are maximally complete, this argument leads to the conclusion that $\text{Brd}_{q}(\Omega_{m}') = c_{q}$, for all $\Omega_{m}' \in \text{Fe}(\Omega_{m}')$ (see Remark 4.1 and apply e.g., [26], Ch. XII, Proposition 6, to a finite valued extension of $(\Omega_{m}', \theta_{m})$). It is now easy to deduce from (1.1) (c) and [26], (1.2), that $\text{Brd}_{q}(Y) \leq c_{q}$, $Y \in I(\Omega_{m}', \theta_{m})$. Since $K_{m}/\Omega_{m}$ is a separable extension, it remains to be seen that $\text{Brd}_{q}(K_{m}') \geq c_{q}$ for any fixed $K_{m}' \in \text{Fe}(K_{m})$. Let $\kappa_{m}$ be the valuation of $K_{m}$ induced by $\kappa$. Then $\kappa_{m}$ is Henselian and it follows from (3.2) and Remark 3.4 that $(K, \kappa_{m})/(K_{m}, \kappa_{m})$ is immediate and the group $\kappa_{m}(K_{m}')/q_{m}(K_{m}')$ has order $q^{m}$. These results and Lemma 4.3 yield $\text{Brd}_{q}(K_{m}') \geq c_{q}$, so Corollary 6.4 is proved.

Proposition 6.3, Corollary 6.4 and the triviality of Brauer groups of separably closed fields show that the system of pairs $(\text{Brd}_{q}(E_{q}), [E_{q} : E_{q}^{1}])$, where $E_{q}$ ranges over the class of fields of characteristic $q > 0$, takes all values admissible by [11], Ch. VII, Theorem 28. As to our next result, it proves Theorem 2.3 in the case of $q = 0$. This result is of independent interest and considerably facilitates the application of Theorem 2.2. In view of (2.4), its proof relies on the existence of a field $E_{0}$, such that $\text{char}(E_{0}) = 0$, $G_{E_{0}} \cong \mathbb{Z}_{2}$ and $E_{0}$ does not contain a primitive $p$-th root of unity, for any odd $p \in \mathbb{P}$ [8], Example 1.3. The existence of $E_{0}$ can also be deduced from Lemma 3.5 applied to $P = \{2\}$ and the case in which $c_{p} > 1$ and $c_{p}$ is a 2-primary divisor of $p - 1$, for each $p \in \mathbb{P} \setminus \{2\}$.
Theorem 6.5. Given a sequence \((b_p, a_p) \in \mathbb{N}^2_\infty\); \(p \in \mathbb{P}\), with \(b_2 = a_2\) and \(b_p \leq a_p, p > 2\), there exists a Henselian field \((K, v)\), such that \(\text{char}(\mathring{K}) = 0, \mathring{G}_K\) is pronilpotent, \(\text{cd}(\mathring{G}_K) \leq 1\), and \((\text{Br}d_p(K), \text{abrd}_p(K)) = (b_p, a_p), p \in \mathbb{P}\).

Proof. Let \(G\) be a pronilpotent group with \(\text{cd}(G) = 1, G_p\) the Sylow pro-

Remark 6.6. Proposition 6.3 and Theorem 6.5 imply that, for each triple \((q, k, k')\) with \(q \in \mathbb{P} \cup \{0\}, k, k' \in \mathbb{N}\) and \(k \leq k'\), there exists a field \(E_0\), such that \(\text{char}(E_0) = q\), \(\text{Br}d(E_0) = k\) and \(\text{abrd}(E_0) = k'\). In view of (2.5), this en-

Note also that such a solvability would imply that the numbers \(c(E), E \in A\), in (2.5), depend on the choice of \(E\) and may be arbitrarily large. Indeed, let \(C\) be an algebraically closed field and \(C_\nu = C((X_1)) \ldots ((X_\nu))\), where \(\nu \in \mathbb{N}\). Then any \(\mathbb{F}_p\)-extension \(F/C_\nu\) with \(\text{trd}(F/C_\nu) = 1\) has a \(C\)-valuation \(\nu_F\), such that \(\text{trd}(\mathring{F}/C) = 1\) and \(\text{trd}(\mathring{F}/C) = 2\nu_F\) (argue as in the proof of Proposition 7.3). Therefore, \(\nu_F(\mathring{F}) = \infty\), for all \(p \in \mathbb{P}\), which enables one to deduce from (2.3), (2.6) (a), that \(\text{Br}d_p(F) = \text{abrd}_p(F) = \nu_F\), \(p \in \mathbb{P}\) and \(p \neq \text{char}(C)\) (see (2.3), page 37, for more details in the case where \(F/C_\nu\) is rational). This, combined with (2.5) and Theorem 2.3 (a), requires that \(\text{dim}(C_\nu) \geq \nu_F\) (cf. also (2.5), (b), (c)). At the same time, it follows from Lemma 4.7 that if \(\text{char}(C) = 0\), then \(\text{Br}(C_\nu) = [\nu_F/2];\) hence, by (2.6), \(c(C_\nu) \geq [\nu_F/2] - 1, \nu \in \mathbb{N}\).

When \(e_p \in \mathbb{N}, p \in \mathbb{P}\), is an unbounded sequence, the fields \(E\) singled out by Proposition 6.3 have the properties required by (2.6) (a). As to (2.6) (b), it is implied by Lemmas 5.5, 5.6, and our next result. Before presenting it, note that (2.6) (b) supplements Theorems 2.2 and 2.3, showing that the class of fields of finite Brauer dimensions is not closed under taking finite extensions. This fact as well as Lemmas 6.1 and 6.2 motivate interest in Problem 2.4.

Corollary 6.7. In the setting of Lemma 6.1, let \(\mathring{K}\) be a quasifinite field with \(\text{char}(\mathring{K}) = 0\) and \(\varepsilon_p \notin \mathring{K}\), for any \(p \in \mathbb{P}\), and let \(U_n\ be the degree \(n\ extension\ of \ K\ in \ K_n\), for a fixed integer \(n \geq 2\). Suppose that \(P_n = \{p_n \in \mathbb{P}: n | p_n - 1\}, [\mathring{K}(\varepsilon_{p_n})]: \mathring{K} = n, \ for \ all \ p_n \in P_n\), and the sequence \(\tau(p): p \in \mathbb{P}\), satisfies the condition \(\tau(p) = \infty\ if \ and \ only \ if \ p \in P_n\). Then a field \(L \in \text{Fe}(K)\) has \(p\)-dimensions \(\text{Br}d_p(L) < \infty, p \in \mathbb{P}\), if and only if \(U_n \notin I(L/K)\).
Proof. It follows from Lemma 6.2 and our assumptions that if \( p \notin P_n \), then \( \text{Brd}_p(L) \leq \text{abrd}_p(K) < \infty \). When \( p \in P_n \) and \( L \in \text{Fe}(K) \), one sees that \( L \) contains a primitive \( p \)-th root of unity if and only if \( U_n \subseteq L \), which reduces our assertion to a consequence of Lemma 6.1.

\[ \square \]

7 Reduction of (2.1) to the case of \( \text{char}(E) = 0 \)

Assume that \( U = E_{0,1} \) is a field with the properties required by Theorem 2.3 such that the Sylow pro-2-subgroups of \( \hat{G}_U \) are isomorphic to \( \mathbb{Z}_2 \). Then the existence of fields \( E_{0,k}, k \in \mathbb{N} \setminus \{1\}, \) claimed by Theorem 2.3 can be proved alternatively by applying Lemma 5.3 to \( K_0 = U \), \( n(2) = 2k \) and \( n(p) = 0, p > 2 \), for each index \( k \). The existence of \( U \) is proved in two steps. First, we obtain a perfect field \( \hat{U} \) with \( \text{char}(\hat{U}) = 2 \) by applying Proposition 6.3 and Lemma 3.3 to \( K_0 = \mathbb{F}_2, n(2) = 0 \) and \( n(p) = \infty, p > 2 \). Next we apply to \( \hat{U} \) the following result (a considerable part of which is contained in [17], Corollary 22.2.3, and [27], 4.1.2) and so prove the existence of a field \( U \) with the desired properties.

Proposition 7.1. Let \( E \) be a field of characteristic \( q > 0 \) and \( F/E \) an FG-extension. Then there exists an FG-extension \( L/E' \) satisfying the following:

(a) \( \text{char}(E') = 0 \), \( \hat{G}_{E'} \cong \hat{G}_E \) and \( \text{trd}(L/E') = \text{trd}(F/E) \);

(b) \( \text{Brd}_p(L) \geq \text{Brd}_p(F), \text{abrd}_p(L) \geq \text{abrd}_p(F), \text{Brd}_p(E') = \text{Brd}_p(E) \) and \( \text{abrd}_p(E') = \text{abrd}_p(E), \) for each \( p \in \mathbb{P}, p \neq q \).

Assume that \( E \) is perfect, i.e. there is a Henselian field \( (K,v) \) with \( \text{char}(K) = 0 \) and \( \hat{K} \cong \mathbb{Z} \), which can be chosen so that \( v(K) = \mathbb{Z} \) and \( v(q) = 1 \). Moreover, it follows from (3.4), (30) and Galois theory (see also the proof of 17), Corollary 22.2.3) that there is \( E' \in I(K_{\text{sep}}/K) \), such that \( E' \cap K_{\text{ur}} = K \) and \( E'K_{\text{ur}} = K_{\text{sep}} \). This ensures that \( v(E') = \mathbb{Q}, \hat{E}' = \hat{K} = E \) and \( E'_{\text{sep}} = E_{\text{sep}} = K_{\text{sep}} \). Hence, by (3.3) and (3.5), \( \hat{G}_{E'} \cong \hat{G}_E, \text{Brd}_p(E') = \text{Brd}_p(E) \) and \( \text{abrd}_p(E) = \text{abrd}_p(E'), p \in \mathbb{P} \setminus \{q\} \). Observe that, since \( E \) is perfect, \( F/E \) is separably generated, i.e. there is \( F_0 \in I(F/E) \), such that \( F_0/E \) is rational and \( F \in \text{Fe}(F_0) \) (cf. 26, Ch. X). Note further that each rational extension \( L_0 \) of \( E' \) with \( \text{trd}(L_0/E') = \text{trd}(F_0/E) \) has a restricted Gauss valuation \( \omega_0 \) extending \( v_{E'} \) with \( L_0 = F_0 \) (cf. 17, Example 4.3.2). Fixing \( (L_0, \omega_0) \), one can take its valued extension \( (L, \omega) \) so that \( L_0 \cong L \oplus_{L_0} L_0, \omega_0 \) is an inertial lift of \( F \) over \( L_0, \omega_0 \). This yields \( \omega(L) = \omega_0(L_0) = \mathbb{Q}, L \cong F \) over \( F_0, [L: L_0] = [F: F_0] \) and \( \text{trd}(L/K) = \text{trd}(F/E) \). It also becomes clear that,
for each $F' \in \text{Fe}(F)$, there exists a valued extension $(L', \omega')$ of $(L, \omega)$ with $[L': L] = [F': F]$ and $\hat{L}' \cong F'$. Observing now that $L'/E'$, $F' \in \text{Fe}(F)$, are FG-extensions, applying (3.3) and (3.5) to a Henselization $L'_{\omega'}$, for any admissible $F'$, and using Lemmas 3.1 and 4.1 one concludes that $\text{Brd}_p(L') \geq \text{Brd}_p(F')$ and abrd$_p(L) \geq \text{abrd}_p(F)$, for all $p \in \mathbb{P} \setminus \{q\}$. Proposition 7.4 is proved.

**Remark 7.2.** Given a class $C$ of profinite groups and some $n \in \mathbb{N}$, Proposition 7.4 implies that (2.1) will have an affirmative answer, for FG-extensions $F/E$ with $G_E \in C$ and trd$(F/E) \leq n$, if this holds when char$(E) = 0$. Hence, de Jong’s theorem [22], Sect. 3, shows that if $n = 2$, char$(E) = q > 0$ and $E_{sep} = E$, then $\text{ind}(D) = \exp(D)$, for all $D \in d(F)$ with $q \mid \text{ind}(D)$. In view of (1.1), Lemma 4.1 and [37], Sect. 4, Theorem 2, this reduces the proof of [27], Theorem 4.2.2.3, to the claim that $D_1 \otimes_F D_2 \notin d(F)$, for any $D_i \in d(F)$ with $\text{ind}(D_i) = q^i$, $i = 1, 2$.

The proofs of Proposition 7.4 and of our concluding result demonstrate the applicability of restricted Gauss valuations to problems of finding lower bounds on $\text{Brd}_p(F)$, for FG-extensions $F$ of valued fields $E$ with abrd$_p(E) < \infty$:

**Proposition 7.3.** Let $E$ be a local field and $F/E$ an FG-extension. Then $\text{Brd}_p(F) \geq 1 + \text{trd}(F/E)$, for every $p \in \mathbb{P}$.

**Proof.** As $\text{Brd}_p(F) = 1$ when trd$(F/E) = 0$, we assume that trd$(F/E) = n \geq 1$. We show that, for each $p \in \mathbb{P}$, there exists $D_p \in d(F)$, such that $\exp(D_p) = p$, $\text{ind}(D_p) = p^{n+1}$ and $D_p$ decomposes into a tensor product of cyclic division $F$-algebras of degree $p$. Let $\omega$ be the standard discrete valuation of $E$, $\hat{E}$ its residue field, and $F_0$ a rational extension of $E$ in $F$ with trd$(F_0/E) = n$. Considering a discrete restricted Gauss valuation of $F_0$ extending $\omega$, and its prolongations on $F$, one obtains that $F$ has a discrete valuation $\nu$ extending $\omega$, such that $\hat{F}$ is an FG-extension of $\hat{E}$ with trd$(\hat{F}/\hat{E}) = n$. Hence, by the proof of Proposition 5.3 given in [10], there exist $\Delta'_p \in d(\hat{F})$ and a degree $p$ cyclic extension $L'_p/\hat{F}$, such that $\Delta'_p \otimes_{\hat{F}} L'_p \in d(L'_p)$, $\exp(\Delta'_p) = p$, $\text{ind}(\Delta'_p) = p^{n+1}$ and $\Delta'_p$ is a tensor product of cyclic division $\hat{F}$-algebras of degree $p$. Given a Henselization $(F_v, \nu)$ of $(F, \nu)$, Lemma 3.1 implies the existence of $\Delta_p \in d(F_v)$, such that $\Delta_p \otimes_F F_v \in d(F_v)$ is an inertial lift of $\Delta'_p$ over $F_v$. Also, by Lemma 4.2 there is a degree $p$ cyclic extension $L_p/F$ with $L_p \otimes_F F_v$ an inertial lift of $L'_p$ over $F_v$. Fix a generator $\sigma$ of $G(L_p/F)$, take a uniform element $\beta$ of $(F, \nu)$, and put $D_p = \Delta_p \otimes_F (L_p/F, \sigma, \beta)$. Then it follows from (3.1) and [39], Theorem 1, that $D_p \in d(F)$, $\exp(D_p) = p$, $\text{ind}(D_p) = p^{n+1}$ and $D_p \otimes_F F_v \in d(F_v)$, so Proposition 7.3 is proved.

Note finally that if $E$ is a local field, $F/E$ is an FG-extension and trd$(F/E) = 1$, then $\text{Brd}_p(F) = 2$, for every $p \in \mathbb{P}$. When $p = \text{char}(E)$, this is implied by Proposition 7.3 and Theorem 4.2.2.3, and for a proof in the case of $p \neq \text{char}(E)$, we refer the reader to [35], Theorems 1 and 3, [39] and [28], Corollary 1.4.

**Acknowledgements.** The concluding part of this research was done during my visit to Tokai University, Hiratsuka, Japan, in 2012. I would like to thank my host-professor Junzo Watanabe, the colleagues at the Department of Mathematics, and Mrs. Yoko Kinoshita and her team for their genuine hospitality.
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