New Successor Rules to Efficiently Produce Exponentially Many Binary de Bruijn Sequences

Zuling Chang, Martianus Frederic Ezerman, Pinhui Ke, and Qiang Wang

Abstract

We put forward new general criteria to design successor rules that generate binary de Bruijn sequences. Prior fast algorithms based on successor rules in the literature are then shown to be special instances. We implemented the criteria to join the cycles generated by a number of simple feedback shift registers (FSRs) of order $n$. These include the pure cycling register (PCR) and the pure summing register (PSR). For the PCR, we define a transitive relation on its cycles, based on their weights. We also extend the choices of conjugate states by using shift operations. For the PSR, we define three distinct transitive relations on its cycles, namely a run order, a necklace order, and a mixed order. Using the new orders, we propose numerous classes of successor rules. Each class efficiently generates a number, exponential in $n$, of binary de Bruijn sequences. Producing the next bit in each such sequence takes $O(n)$ memory and $O(n)$ time. We implemented computational routines to confirm the claims.

Index Terms

Binary de Bruijn sequence, cycle structure, order, pure cycling register, pure summing register, successor rule.

I. INTRODUCTION

A $2^n$-periodic binary sequence is a binary de Bruijn sequence of order $n$ if every binary $n$-tuple occurs exactly once within each period. There are $2^{2^n-1-n}$ such sequences [1]. They appear in many guises, drawing the attention of researchers from varied backgrounds and interests. Attractive qualities that include being balanced and having maximum period [2], [3] make these sequences applicable in coding and communication systems. A subclass with properly calibrated nonlinearity property, while satisfying other measures of complexity, can also be useful in cryptography.

Experts have been using tools from diverse branches of mathematics to study their generations and properties, see, e.g., the surveys in [4] and [5] for further details. Of enduring special interest are of course methods that excel in three measures: fast, with low memory requirement, and capable of generating a large number of sequences. Known constructions come with some trade-offs with respect to these measures. Notable examples include Lempel’s D-Morphism [6], an approach via preference functions described in [7] and in [8], greedy algorithms with specific preferences, e.g., in [9] and, more recently, in [10], as well as various fast generation proposals, e.g., those in [10] and in [11].

The most popular construction approach is the cycle joining method (CJM) [3]. It serves as the foundation of many techniques. A main drawback of the CJM, in its most general form, is the amount of computation to be done prior to actually generating the sequences. Given a feedback shift register, one must first determine its cycle structure before finding the conjugate pairs to build the so-called adjacency graph. Enumerating the spanning trees comes next. Once these general and involved steps have been properly done, then generating a sequence, either randomly or based on a predetermined rule, is very efficient in both time and memory. The main advantage, if carried out in full, is the large number of output sequences, as illustrated in [12, Table 3].

There are fast algorithms that can be seen as applications of the CJM on specially chosen conjugate pairs and designated initial states. They often produce a very limited number of de Bruijn sequences. One can generate a de Bruijn sequence, named the granddady in [10], in $O(n)$ time and $O(n)$ space per bit. A related de Bruijn sequence, named the grandmama, was built in [11]. Huang gave another early construction that joins the cycles of the complementing circulating register (CCR) in [13]. Etzion and Lempel proposed some algorithms to generate de Bruijn sequences based on the pure cycling register (PCR) and the pure summing register (PSR) in [14]. Their algorithms generate a number, exponential in $n$, of sequences at the expense of higher memory requirement.

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Jansen, Franx, and Boekee established a requirement to determine some conjugate pairs in [15], leading to another fast algorithm. In [16], Sawada, Williams, and Wong proposed a simple de Bruijn sequence construction, which is in fact a special case of the method in [15]. Gabric et al. generalized the last two works to form simple successor rule frameworks that yield more de Bruijn sequences in [17]. Further generalization to the constructions of $k$-ary de Bruijn sequences in [18] and [20] followed. Zhu et al. very recently in [19] proposed two efficient generic successor rules based on the properties of the feedback function $f(x_0, x_1, \ldots, x_{n-1}) = x_0 + x_1 + x_{n-1}$ for $n \geq 3$. Each rule produces at least $2^{n-3}$ binary de Bruijn sequences. They built upon the framework proposed in [17].

**Our Contributions**

1) Paying close attention to the approach in [15] and the series of works that lead to the recently presented framework in [20], we propose to generate de Bruijn sequences by using novel relations and orders on the cycles in combination with suitable successor rules.

2) We define new classes of successor rules and, then, prove that they generate, respectively, a number, exponential in $n$, of de Bruijn sequences. In particular, the number of generated sequences based on the PCR of order $n$ is

$$2(n-1)(n-2)\ldots1 = 2 \cdot (n-1)!$$

The cost to output the next bit is $O(n)$ time and $O(n)$ space. Nearly all known successor rules in the literature generate only a handful of de Bruijn sequences each. The few previously available approaches that can generate an exponential number of de Bruijn sequences require more space than the ones we are proposing.

3) We implemented the criteria on some simple FSRs, especially on the PCR and the PSR of order $n$. Based on the properties of their respective cycles, we define several relations. For the PCR we order the cycles by their weights and the states in the respective cycles by their positions relative to the necklaces. On the cycles produced by the PSR we define a run order, a necklace order, and a mixed order that combines the weight order and the necklace order.

Using the new relations, we design numerous successor rules to efficiently generate de Bruijn sequences. The exact number of output sequences can be determined for many classes of the rules. Given a current state, in most occasions, the next bit takes only $O(n)$ space and $O(n)$ time to generate. In a few other instances, the process can be made even faster. We also demonstrate the explicit derivation of the feedback functions of some of the resulting sequences.

4) Our results extend beyond providing a general formulation for already known fast algorithms that generate de Bruijn sequences by way of successor rules. The approach applies to any FSR. To remain efficient, one should focus on classes of FSRs whose cycles have periods which are linear in $n$. There are plenty of such FSRs around for further explorations.

A high level explanation of our approach is as follows. We begin with the set of cycles produced by any nonsingular feedback shift register. To join all of these cycles into a single cycle, i.e., to obtain a binary de Bruijn sequence, one needs to come up with a valid successor rule that assigns a uniquely identified state in one cycle to a uniquely identified state in another cycle and ensure that all of the cycles are joined in the end. If the cycles are represented by the vertices of an adjacency graph, then producing a de Bruijn sequence in the CJM corresponds to finding a spanning tree in the graph. The directed edges induced by a successor rule guide the actual process of generating the sequence. To certify that a successor rule can indeed yield a de Bruijn sequence we propose several new relations and orders on both the cycles and on the states in each cycle. These ensure the existence of spanning trees in the corresponding adjacency graphs. The relations and orders on the states are carefully chosen to guarantee that the next bit can be produced efficiently.

We collect preliminary notions and several useful known results in Section II. We present a new general criteria in Section III. Section IV shows how to apply the criteria on the cycles of the PCR, leading to scores of new successor rules to generate de Bruijn sequences. Section V gives a similar treatment on the PSR. The last section concludes this work by summarizing the contributions and listing some future directions.

### II. Preliminaries

#### A. Basic Definitions

An $n$-stage shift register is a circuit of $n$ consecutive storage units, each containing a bit. The circuit is clock-regulated, shifting the bit in each unit to the next stage as the clock pulses. A shift register generates a binary code if one adds a feedback loop that outputs a new bit $s_\ell$ based on the $n$ bits $s_0 = s_0, \ldots, s_{n-1}$, called an initial state of the register. The corresponding Boolean feedback function $f(s_0, s_1, \ldots, s_{n-1})$ outputs $s_\ell$ on input $s_0$. A feedback shift register (FSR) outputs a binary sequence $s = \{s_\ell\} = s_0, s_1, \ldots, s_n, \ldots$ that satisfies the recursive relation

$$s_{\ell+n} = f(s_\ell, s_{\ell+1}, \ldots, s_{\ell+n-1}) \text{ for } \ell = 0, 1, 2, \ldots$$

For $N \in \mathbb{N}$, if $s_{\ell+N} = s_\ell$ for all $\ell \geq 0$, then $s$ is $N$-periodic or with period $N$ and one writes $s = (s_0, s_1, s_2, \ldots, s_{N-1})$. The least among all periods of $s$ is called the least period of $s$. 
We say that \( s_i = s_{i1}, s_{i+1}, \ldots, s_{i+n-1} \) is the \( i \)th state of \( s \). Its predecessor is \( s_{i-1} \) while its successor is \( s_{i+1} \). For \( s \in \mathbb{F}_2 \), let \( \bar{s} := s+1 \in \mathbb{F}_2 \). Extending the definition to any binary vector or sequence \( s = s_0, s_1, \ldots, s_{n-1}, \ldots \), let \( \bar{s} := s_0, \overline{s_1}, \ldots, \overline{s_{n-1}}, \ldots \). An arbitrary state \( v = v_0, v_1, \ldots, v_{n-1} \) of \( s \) has 
\[
\hat{v} := \overline{v_0}, v_1, \ldots, v_{n-1} \quad \text{and} \quad \bar{v} := v_0, \ldots, v_{n-2}, \overline{v_{n-1}}
\]
as its conjugate state and companion state, respectively. Hence, \((v, \bar{v})\) is a conjugate pair and \((v, \hat{v})\) is a companion pair.

For any FSR, distinct initial states generate distinct sequences. There are \( 2^n \) distinct sequences generated from an FSR with feedback function \( f(x_0, x_1, \ldots, x_{n-1}) \). All these sequences are periodic if and only if \( f \) is nonsingular, i.e., \( f \) can be written as
\[
f(x_0, x_1, \ldots, x_{n-1}) = x_0 + h(x_1, \ldots, x_{n-1}),
\]
for some Boolean function \( h(x_1, \ldots, x_{n-1}) \) whose domain is \( \mathbb{F}_2^{n-1} \) \([3] \text{ p. } 116\). All feedback functions in this paper are nonsingular. An FSR is linear or an LFSR if its feedback function has the form
\[
f(x_0, x_1, \ldots, x_{n-1}) = c_0x_0 + c_1x_1 + \ldots + c_{n-1}x_{n-1}, \quad \text{with} \quad c_i \in \mathbb{F}_2,
\]
and its characteristic polynomial is
\[
f(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + 1 \in \mathbb{F}_2[x].
\]
Otherwise, it is nonlinear or an NLFSR. Further properties of LFSRs are treated in, e.g., [22] and [23].

For an \( N \)-periodic sequence \( s \), the left shift operator \( L \) maps \((s_0, s_1, \ldots, s_{N-1}) \mapsto (s_1, s_2, \ldots, s_{N-1}, s_0)\), with the convention that \( L^0 \) fixes \( s \). The right shift operator \( R \) is defined analogously. The set
\[
|s| := \{|s, Ls, \ldots, L^{N-1}s\} = \{|s, Rs, \ldots, R^{N-1}s\}
\]
is a shift equivalent class. Sequences in the same shift equivalent class correspond to the same cycle in the state diagram of FSR [22]. We call a periodic sequence in a shift equivalent class a cycle. If an FSR with feedback function \( f \) generates \( r \) disjoint cycles \( C_1, C_2, \ldots, C_r \), then its cycle structure is
\[
\Omega(f) = \{C_1, C_2, \ldots, C_r\}.
\]
A cycle can also be viewed as a set of consecutive \( n \)-stage states in the corresponding periodic sequence. Since the cycles are disjoint, we can write
\[
\mathbb{F}_2^n = C_1 \cup C_2 \cup \ldots \cup C_r.
\]
When \( r = 1 \), the corresponding FSR is of maximal length and its output is a de Bruijn sequence of order \( n \).

The weight of an \( N \)-periodic cycle \( C \), denoted by \( wt(C) \), is
\[
\{|0 \leq j \leq N-1 : c_j = 1\}.
\]
Similarly, the weight of a state is the number of 1s in the state. The lexicographically least \( N \)-stage state in any \( N \)-periodic cycle is called its necklace. As discussed in, e.g., [24] and [17], there is a fast algorithm that determines whether or not a state is a necklace in \( O(N) \) time. In fact, one can efficiently sort all distinct states in \( C \). The standard Python implementation is timsort [25]. It was developed by Tim Peters based on McIlroy’s techniques in [26]. In the worst case, its space and time complexities are \( O(N) \) and \( O(N \log N) \) respectively.

A closely related proposal, by Buss and Knop, is in [27].

Given disjoint cycles \( C \) and \( C' \) in \( \Omega(f) \) with the property that some state \( v \) in \( C \) has its conjugate state \( \hat{v} \) in \( C' \), interchanging the successors of \( v \) and \( \hat{v} \) joins \( C \) and \( C' \) into a cycle whose feedback function is
\[
\hat{f} := f(x_0, x_1, \ldots, x_{n-1}) + \prod_{i=1}^{n-1} (x_i + \overline{x_i}).
\]
Similarly, if the companion states \( v \) and \( \bar{v} \) are in two distinct cycles, then interchanging their predecessors joins the two cycles. If this process can be continued until all cycles that form \( \Omega(f) \) merge into a single cycle, then we obtain a de Bruijn sequence. The CJM is, therefore, predicated upon knowing the cycle structure of \( \Omega(f) \) and is closely related to a graph associated to the FSR.

Given an FSR with feedback function \( f \), its adjacency graph \( G_f \), or simply \( G \) if \( f \) is clear, is an undirected multigraph whose vertices correspond to the cycles of \( \Omega(f) \). The number of edges between two vertices is the number of shared conjugate (or companion) pairs, with each edge labelled by a specific pair. It is well-known that there is a bijection between the set of spanning trees of \( G \) and the set of all inequivalent de Bruijn sequences constructible by the CJM on input \( f \).

We state the Generalized Chinese Remainder Theorem, which will be used as an enumeration tool in Section V.

**Theorem I**: [22] Section 2.4] Let \( m_1, \ldots, m_k \) be positive integers. For a set of integers \( a_1, \ldots, a_k \), the system of congruences
\[
\{x \equiv a_i \ (\text{mod } m_i) \text{ for all } i \in \{1, \ldots, k\}\}
\]
is solvable if and only if
\[
a_i \equiv a_j \ (\text{mod } \gcd(m_i, m_j)) \text{ for all } 1 \leq i \neq j \leq k.
\]
(3)
If the equivalence in (3) holds, then the solution is unique modulo lcm\((m_1, \ldots, m_k)\).
Algorithm 1. The correctness of the JFB Algorithm rests on the fact that the cycle representative in any cycle does not contain the all zero state \(0, \ldots, 0\). This ensures that we have a spanning tree and, hence, the resulting sequence must be de Bruijn.

Say that our focus is on the PSR since the corresponding results on the CSR become immediately apparent with the proper adjustment. The cycles of the PSR share some interesting properties. If \(C_{\text{PSR}}\) is any cycle generated by the PSR of order \(n\), then its least period divides \(n+1\). Hence, we can write it as an \((n+1)\)-periodic cycle, i.e., \(C_{\text{PSR}} := (c_0, c_1, \ldots, c_n)\). Notice that \(\text{wt}(C_{\text{PSR}})\) must be even. Let \(n \geq 2\). If \(n\) is odd, we can write \(n := 2n' - 1\). The number of distinct cycles in \(\Omega(f_{\text{PSR}})\) is

\[
Z_{n+1} = \frac{1}{2(n+1)} \left( \sum_{d|n'} \phi \left( \frac{n'}{d} \right) 2^{d^2} \right),
\]

where \(Z_{n+1}\) is computed based on (5). The number in (7) simplifies to \(\frac{1}{2} Z_{n+1}\) if \(n\) is even.

The complemented PSR, also known as the CSR, of order \(n\) is an LFSR with feedback function and characteristic polynomial

\[
f_{\text{CSR}}(x_0, x_1, \ldots, x_{n-1}) = \sum_{j=0}^{n-1} x_j.
\]

It assigns the next bit to be the complement of the bit produced by the feedback function \(f_{\text{PSR}}\) in (6), when given the same input. Hence, the least period of any cycle \(C_{\text{CSR}}\) divides \(n+1\) and the weight of any \(C_{\text{CSR}} := (e_0, e_1, \ldots, e_n)\) is odd.

Algorithm 1 Jansen-Franx-Boekee (JFB) Algorithm

| Line | Statement |
|------|-----------|
| 1 | if \(s_i = s_i, 0, \ldots, 0\) then |
| 2 | \(s_{i+1} \leftarrow 0, \ldots, 0, s_i + 1\) |
| 3 | else |
| 4 | if \(s_i, s_{i+1}, \ldots, s_{i+n-1}, 0\) or \(s_i, s_{i+1}, \ldots, s_{i+n-1}, 1\) is a cycle representative then |
| 5 | \(s_{i+1} \leftarrow s_i, s_{i+1}, \ldots, s_{i+n-1}, f(s_i, s_{i+1}, \ldots, s_{i+n-1}) + 1\) |
| 6 | else |
| 7 | \(s_{i+1} \leftarrow s_i, s_{i+1}, \ldots, s_{i+n-1}, f(s_i, s_{i+1}, \ldots, s_{i+n-1})\) |

The main task of keeping track of the cycle representatives in Algorithm 1 may require a lot of time if the least periods of the cycles are large. For cases where all cycles produced by a given FSR have small least periods, e.g., in the case of the PCR or the PSR of order \(n\), the algorithm generates de Bruijn sequences very efficiently. The space complexity is \(O(n)\) and the time complexity lies between \(O(n)\) and \(O(n \log n)\) to output the next bit.
Sawada et al. proposed a simple fast algorithm on the PCR to generate a de Bruijn sequence [16]. Their algorithm is a special case of the JFB Algorithm. Later, in [17], Gabric and the authors of [16] considered the PCR and the complemented PCR, also known as the CCR, and proposed several fast algorithms to generate de Bruijn sequences by ordering the cycles lexicographically according to their respective necklace and co-necklace. They replace the generating algorithm by some successor rule.

The general thinking behind the approach is as follows. Given an FSR with a feedback function $f(x_0, x_1, \ldots, x_{n-1})$, let $A$ label some condition which guarantees that the resulting sequence is de Bruijn. For any state $c := c_0, c_1, \ldots, c_{n-1}$, the successor rule assigns

$$
\rho_A(c) = \begin{cases} 
  f(c) + 1, & \text{if } c \text{ satisfies } A, \\
  f(c), & \text{otherwise}.
end{cases}
$$

The usual successor of $c$ is $c_1, \ldots, c_{n-1}, f(c_0, \ldots, c_{n-1})$. Every time $c$ satisfies Condition $A$, however, its successor is redefined to be $c_1, \ldots, c_{n-1}, f(c_0, \ldots, c_{n-1}) + 1$. The last bit of the successor is the complement of the last bit of the usual successor under the feedback function $f$. The basic idea of a successor rule is to determine spanning trees in $G_f$ by identifying a suitable Condition $A$. Seen in this light, the rule implements the CJM by assigning successors to carefully selected states.

We will devise numerous new successor rules to join the cycles produced by the PCR and the PSR of any order $n$. Known successor rules in the literature will subsequently be shown to be special instances of our more general results.

### III. New General Criteria for Successor Rules

New successor rules for de Bruijn sequences can be established by defining some relations or orders on the cycles of FSRs with special properties to construct spanning trees in $G_f$. This section proves a general criteria that such rules must meet. The criteria will be applied successfully, in latter sections, to the PCR and the PSR of any order $n$. The generality of the criteria allows for further studies to be conducted on the feasibility of using broader families of FSRs for fast generation of de Bruijn sequences.

We adopt set theoretic definitions and facts from [21]. Given $\Omega_f$, we define a binary relation $\prec$ on $\Omega_f := \{C_1, C_2, \ldots, C_r\}$ as a set of ordered pairs in $\Omega_f$. If $C \prec C$ for every $C \in \Omega_f$, then $\prec$ is said to be reflexive. Let $1 \leq i, j, k \leq r$. We say that $\prec$ is transitive if $C_i \prec C_j$ and $C_j \prec C_k$, together, imply $C_i \prec C_k$. It is symmetric if $C_i \prec C_j$ implies $C_j \prec C_i$ and antisymmetric if the validity of both $C_i \prec C_j$ and $C_j \prec C_i$ implies $C_i = C_j$.

The relation $\prec$ is called a preorder on $\Omega_f$ if it is reflexive and transitive. It becomes a partial order if it is an antisymmetric preorder. If $\prec$ is a partial order with either $C_i \prec C_j$ or $C_j \prec C_i$, for any $C_i$ and $C_j$, then it is a total order. A totally ordered set $\Omega_f$ is called a chain. Hence, we can now say that $\prec_{\text{lex}}$ defined in Subsection II-C is a total order on the corresponding chain $\Omega_f$.

**Theorem 2:** Given an FSR with feedback function $f$, let $\prec$ be a transitive relation on $\Omega(f) := \{C_1, C_2, \ldots, C_r\}$ and let $1 \leq i, j \leq r$.

1. Let there be a unique cycle $C$ with the property that $C \prec C'$ for any cycle $C' \neq C$, i.e., $C$ is the unique smallest cycle in $\Omega(f)$. Let $\rho$ be a successor rule that can be well-defined as follows. If any cycle $C_i \neq C$ contains a uniquely defined state whose successor can be assigned by $\rho$ to be a state in a cycle $C_j \neq C_i$ with $C_j \prec C_i$, then $\rho$ generates a de Bruijn sequence.

2. Let there be a unique cycle $C$ with the property that $C' \prec C$ for any cycle $C' \neq C$, i.e., $C$ is the unique largest cycle in $\Omega(f)$. Let $\rho$ be a successor rule that can be well-defined as follows. If any cycle $C_i \neq C$ contains a uniquely defined state whose successor can be assigned by $\rho$ to be a state in a cycle $C_j \neq C_i$ with $C_i \prec C_j$, then $\rho$ generates a de Bruijn sequence.

**Proof:** We prove the first case by constructing a rooted tree whose vertices are all of the cycles in $\Omega(f)$. This exhibits a spanning tree in the adjacency graph of the FSR according to the specified successor rule. The second case can be similarly argued.

Based on the condition set out in the first case, each $C_i \neq C$ contains a unique state whose assigned successor under $\rho$ is in $C_j \neq C_i$, revealing that $C_i$ and $C_j$ are adjacent. Since $C_j \prec C_i$, we direct the edge from $C_i$ to $C_j$. It is easy to check that, except for $C$ whose outdegree is 0, each vertex has outdegree 1. Since $\prec$ is transitive, there is a unique path from the vertex to $C$. We have thus built a spanning tree rooted at $C$.

**Remark 1:** Armed with Theorem 2, one easily verifies that the JFB Algorithm and the successor rules proposed in [17] are valid. In both references, the relation is a total order. In our present notation, [17] Theorem 3.5] says that a successor rule generates a de Bruijn sequence if $\Omega_f$ is a chain. The cycles, possibly with a relabelling of the indices, can be presented as $C_1 \prec C_2 \prec \cdots \prec C_r$. In each $C_i$, with $1 < i \leq r$, there exists a unique state whose successor can be defined to be a state in $C_j$ with $j < i$.

Known successor rules in the literature have so far been mostly based on the lexicographic order in $\Omega_f$ for a chosen $f$. A notable exception is the class of successor rules in [19, Section 4]. As we will soon see, many relations in the set of cycles that we are defining later do not constitute total orders, so [17] Theorem 3.5] cannot be used to prove the correctness of the
resulting successor rules directly. Theorem 2 relaxes the requirement and works as long as the relation is transitive. In the sequel we show that many alternatives to \( \prec_{\text{lex}} \) can be devised to efficiently generate de Bruijn sequences. The corresponding successor rules are simple to state and straightforward to validate. Thus, Theorem 2 can be viewed as the generalization of [17, Theorem 3.5].

There are two tasks to carry out in using Theorem 2. First, one must define a suitable transitive relation among the cycles to obtain the unique smallest or largest cycle \( C \). The second task is to determine the unique state in each cycle. A sensible approach is to designate a state \( v \) as the benchmark state in each cycle \( C \). We then uniquely define a state \( w \) in \( C \) with respect to the benchmark state. The cycle representative, i.e., the necklace in the PCR, is the most popular choice for \( v \). In this paper we mainly use the necklace as the benchmark state in each cycle.

The next two sections examine some FSRs whose cycles have small respective least periods. Based on the properties of their respective cycle structures, we define several relations or orders to come up with new successor rules that meet the criteria in Theorem 2.

### IV. Successor Rules from Pure Cycling Registers

This section applies the criteria in Theorem 2 to the PCR of any order \( n \). A good strategy is to consider the positions of the states in each cycle relative to its necklace by ordering the states in several distinct manners. This general route is chosen since we can check whether or not a state is a necklace in \( O(n) \) time and \( O(n) \) space. If the relative position of a state to the necklace is efficient to pinpoint, then the derived successor rule also runs efficiently.

#### A. The Weight Relation on the Pure Cycling Register

The cycles of the PCR share a nice property. All of the states in any cycle \( C \) are shift-equivalent and share the same weight \( \text{wt}(C) \). Hence, we can define a weight relation on the cycles based simply on their respective weights. For cycles \( C_i \neq C_j \), we say that \( C_i \prec_{\text{wt}} C_j \) if and only if \( \text{wt}(C_i) < \text{wt}(C_j) \).

The relation \( \prec_{\text{wt}} \) is not even a preorder, making it differs qualitatively from the lexicographic order.

**Example 1:** The PCR of order 6 generates \( C_1 := (001001) \) and \( C_2 := (000111) \). Lexicographically \( C_1 \succ_{\text{lex}} C_2 \) because the necklace 001001 in \( C_1 \) is lexicographically larger than the necklace 000111 in \( C_2 \). In the weight relation, however, \( C_1 \prec_{\text{wt}} C_2 \) since \( \text{wt}(C_1) = 2 < 3 = \text{wt}(C_2) \).

The following successor rules rely on the weight relation.

**Theorem 3:** For the PCR of order \( n \), if a successor rule \( \rho(x_0, x_1, \ldots, x_{n-1}) \) satisfies one of the following conditions, then it generates a de Bruijn sequence.

1. For any \( C_i \neq (0) \), the rule \( \rho \) exchanges the successor of a uniquely determined state \( v_i \in C_i \) with a state \( w_j \in C_j \), where \( C_j \prec_{\text{wt}} C_i \).
2. For any \( C_i \neq (1) \), the rule \( \rho \) exchanges the successor of a uniquely determined state \( v_i \in C_i \) with a state \( w_j \in C_j \), where \( C_i \prec_{\text{wt}} C_j \).

**Proof:** To prove the first case, note that \( (0) \prec_{\text{wt}} C_i \) for any \( C_i \neq (0) \) in \( \Omega(f_{\text{PCR}}) \). By the stated condition, \( C_i \) contains a unique state \( v_i \), such that its conjugate \( w_j := \hat{v}_i \) is in \( C_j \) and \( \text{wt}(C_j) < \text{wt}(C_i) \). The successor rule \( \rho \) satisfies the criteria in Theorem 2. The proof for the second case is similar. \( \blacksquare \)

Theorem 3 reduces the task to generate de Bruijn sequences by using \( \rho \) to performing one of two procedures. The first option is to find the uniquely determined state \( v_i \in C_i \neq (0) \) whose conjugate state \( \hat{v}_i \) is guaranteed to be in \( C_j \) with \( \text{wt}(C_j) < \text{wt}(C_i) \).

The second option is to find the uniquely determined state \( v_i \in C_i \neq (1) \) whose conjugate state \( \hat{v}_i \) is guaranteed to be in \( C_j \) with \( \text{wt}(C_j) > \text{wt}(C_i) \).

If, for every \( C_i \), its \( v_i \) can be determined quickly, then generating the de Bruijn sequence is efficient. Following the two cases in Theorem 3, the rule \( \rho \) comes in two forms. Let \( c := c_0, c_1, \ldots, c_{n-1} \).

First, let \( \mathcal{A} \) be

In \( C := (0, c_1, \ldots, c_{n-1}) \), the uniquely determined state \( v \) is \( 0, c_1, \ldots, c_{n-1} \). Its conjugate \( \hat{v} \) has \( \text{wt}(\hat{v}) > \text{wt}(v) \), which implies \( \hat{v} \) is in \( C' \) with \( C \prec_{\text{wt}} C' \).

It is then straightforward to confirm that the relevant requirement in Theorem 3 is met by

\[
\rho_{\mathcal{A}}(c) = \begin{cases} c_0, & \text{if } 0, c_1, \ldots, c_{n-1} \text{ satisfies } \mathcal{A}, \\ c_0, & \text{otherwise.} \end{cases}
\]  

Second, let \( \mathcal{B} \) be

In \( C := (c_1, \ldots, c_{n-1}, 1) \), the uniquely determined state \( v \) is \( c_1, \ldots, c_{n-1}, 1 \). Its companion \( \hat{v} \) has \( \text{wt}(\hat{v}) < \text{wt}(v) \), which means that \( \hat{v} \) is in \( C' \) with \( C \prec_{\text{wt}} C' \).

Hence, the successor rule

\[
\rho_{\mathcal{B}}(c) = \begin{cases} c_0, & \text{if } c_1, \ldots, c_{n-1}, 1 \text{ satisfies } \mathcal{B}, \\ c_0, & \text{otherwise.} \end{cases}
\]
If there is only one state whose last bit is 1 in \( w \) determined. Let \( w \) be the predecessor of \( v \) in the cycle \( v \). Since the necklace is unique, the state \( v \) itself if the cycle contains only a single state whose first bit is 0. Since the necklace is unique, the state \( v \) in the cycle \( (v) \neq (1) \) is uniquely determined. By how the successor rule is designed, the successor of \( v \) is therefore \( c_1, \ldots, c_{n-1}, 1 \). Its weight is \( wt(v) + 1 \). The criteria of Theorem \( 2 \) is met. Hence, the generated sequence is de Bruijn.

For \( v := 0, c_1, \ldots, c_{n-1} \), if there is an integer \( 1 \leq j < n \) such that \( j \) is the largest index for which \( c_j = 0 \), then \( u := 0, c_{j+1}, \ldots, c_{n-1}, 0, c_1, \ldots, c_{j-1} \). Otherwise, \( u := v \). The successor rule in the preceding paragraph simplifies to

\[
\rho \phi (c_0, c_1, \ldots, c_{n-1}) = \begin{cases} 
  c_0, & \text{if } u \text{ is a necklace}, \\
  c_0, & \text{otherwise}. 
\end{cases}
\]

Using \( \rho \phi \) in \( (12) \) and \( n = 6 \), we can choose \( c_0, c_1, \ldots, c_{n-1} \) as the uniquely determined state in each cycle \( (c_0, c_1, \ldots, c_{n-1}) \neq (1) \). Table \( 1 \) lists the states to choose from according to the weight of their respective cycles. It is then easy to construct the spanning tree in Figure \( 1 \). The directed edge from \( C_i \), for each \( 1 \leq i < 14 \), is labelled by the pair \( (v_i, \bar{v}_i) \). The resulting de Bruijn sequence is

\[
(00000010 10111000 11101100 10010110 11111100 11110100 11000011 01010001) .
\]

This sequence is distinct from the output of any previously known successor rule in the literature.

Let us specify \( \phi \) in \( (10) \) to be the state \( v \) is a necklace.

We obtain the PCR4 de Bruijn sequence in \( (17) \).

Let \( \phi \) be \( L^k \) a necklace with \( k \) being the smallest positive integer such that \( L^k v \) has 0 as its first bit.

The resulting sequence is the granddaddy.

One can also formulate specific successor rules based on \( (11) \). Let \( v := c_1, \ldots, c_{n-1}, 1 \) be a state in \( C \neq (0) \). Let \( \beta \) be \( L^k \) a necklace with \( k \) being the smallest positive integer such that \( L^k v \) has 1 as its last bit.

If there is only one state whose last bit is 1 in \( C \), then \( L^k v = v \). Since the necklace is unique, \( v \) in the cycle \( (v) \) is uniquely determined. Let \( w \) be the predecessor of \( v \) under \( f_{\text{PCR}} \). The rule \( \rho \beta \) assigns \( c_1, \ldots, c_{n-1}, 0 \), which has weight \( wt(v) - 1 \), as the successor of \( w \). Thus, by Theorem \( 2 \) the generated sequence is de Bruijn.

**Table I**

| \( i \) | \( wt(C_i) \) | \( v_i \in C_i := (v_i) \) | \( i \) | \( wt(C_i) \) | \( v_i \in C_i := (v_i) \) |
|---|---|---|---|---|---|
| 1 | 0 | 000000 | 8 | 3 | 010110 |
| 2 | 1 | 000010 | 9 | 3 | 011010 |
| 3 | 2 | 000110 | 10 | 4 | 011110 |
| 4 | 2 | 001010 | 11 | 4 | 011101 |
| 5 | 2 | 010010 | 12 | 4 | 011011 |
| 6 | 3 | 010101 | 13 | 5 | 011111 |
| 7 | 3 | 001110 |   |   |   |

**Fig. 1.** The spanning tree produced by the successor rule \( \rho \) in \( (12) \) when applied to the PCR of order 6. The cycles in gray, blue, and red are of the same weights 2, 3, and 4, respectively.
For \( v := c_1, \ldots, c_{n-1}, 1 \), if there is an integer \( 1 \leq j < n \) such that \( j \) is the least index that satisfies \( c_j = 1 \), then \( u := c_{j+1}, \ldots, c_{n-1}, 1, c_1, \ldots, c_{j-1}, 1 \). Otherwise, \( u := v \). The rule in the preceding paragraph becomes

\[
\rho_{\mathcal{B}}(c_0, c_1, \ldots, c_{n-1}) = \begin{cases} 
\bar{c}_0, & \text{if } u \text{ is a necklace,} \\
\bar{c}_0, & \text{otherwise.}
\end{cases}
\]

(13)

On the PCR of order 6, the resulting sequence is

\[
(00000010 01111000 01010001 10110001 00111000 10101111 11011101 01000011),
\]

which is again distinct from any that can be produced based on previously known successor rule.

Two more observations are worth mentioning. Let \( \mathcal{B} \) be

The state \( v \) is a necklace.

Then the output is the PCR3(J1) de Bruijn sequence in [17]. We obtain the grandmama when \( \mathcal{B} \) is

\( R^6 v \) is a necklace with \( k \) being the least positive integer such that \( R^6 v \) has 1 as its last bit.

\[ \square \]

Based on \( \mathcal{A} \) and \( \mathcal{B} \), valid successor rules can be easily formulated once we manage to determine a unique state whose first bit is 0, respectively, whose last bit is 1, in each \( C \neq (1) \), respectively, \( C \neq (0) \). There are numerous ways to do so if one sets aside the issue of efficiency. Let us consider valid successor rules designed based only on [16] on the PCR. A direct inspection on the list of \( C_i = (u_i) \) in Table II confirms that for \( n = 6 \) the number of resulting de Bruijn sequences is

\[
2^3 \cdot 3^3 \cdot 4^2 \cdot 5 = 17,280 \approx 2^{14}.
\]

When \( n = 7 \), the number is

\[
2^3 \cdot 3^5 \cdot 4^5 \cdot 5^3 \cdot 6 = 1,492,992,000 \approx 2^{30.475}.
\]

Taking the exhaustive approach incurs a steep penalty in memory requirement to store all qualified states in the corresponding cycles. Etzion and Lempel in [14] stored many, not all, qualified states to perform cycle joining by successor rules. Their construction generates a large number, exponential in the order \( n \), of de Bruijn sequences at the cost of raised memory demand.

B. Under the Shift Order

Imposing a shift order on the states in a given cycle yields a lot of feasible successor rules. We call a state whose first entry is 0 a leading zero state or an LZ state in short. Analogously, a state whose last entry is 1 is said to be an ending one state or an EO state.

The necklace in a given cycle \( (c_0, c_1, \ldots, c_{n-1}) \neq (1) \) must begin with 0, i.e., its necklace is an LZ state. Here we define a special left shift operator, denoted by \( L_{dz} \). Applied on a given LZ state \( v := 0, c_1, \ldots, c_{n-1} \) the operator \( L_{dz} \) outputs the first LZ state obtained by consecutive left shifts on \( v \). More formally, \( L_{dz} v := v \) if \( c_1, \ldots, c_{n-1} = 1, \ldots, 1 \). Otherwise, let \( 1 \leq j < n \) be the least index such that \( c_j = 0 \). Then

\[
L_{dz} v := 0, c_{j+1}, \ldots, c_{n-1}, 0, c_1, \ldots, c_{j-1}.
\]

Similarly, the necklace in any \( C \neq (0) \) must end with 1, i.e., it is an EO state. Given a state \( u := c_1, \ldots, c_{n-1}, 1 \), the special operator \( L_{eo} \) fixes \( u \) if \( c_1, \ldots, c_{n-1} := 0, \ldots, 0 \). Otherwise, let \( 1 \leq j < n \) be the least index such that \( c_j = 1 \). Then

\[
L_{eo} u := c_{j+1}, \ldots, c_{n-1}, 1, c_1, \ldots, c_{j-1}, 1.
\]
In other words, \( L_{eo} u \) is the first EO state found upon consecutive left shifts on \( u \).

For these two special operators, the convention is to let
\[
\begin{align*}
L_{iz}^0 v &= v, \\
L_{zo}^0 u &= u,
\text{ and } \\
L_{iz}^k v &= L_{iz}^{k-1}(L_{iz} v), \\
L_{zo}^k u &= L_{zo}^{k-1}(L_{zo} u),
\end{align*}
\]
for \( k > 0 \).

**Example 3:** For \( C = (001011) \) with \( v = 001011 \), we have \( L_{iz}^1 v = 010110 \) and \( L_{iz}^2 v = 011001 \), whereas \( L_{zo}^0 v = 011001 \) and \( L_{zo}^2 v = 100101 \).

Now we construct successor rules based on \( L_{iz} \) and \( L_{zo} \).

**Proposition 4:** With arbitrarily chosen \( 2 \leq t \leq n \), we let \( 1 = k_1 < k_2 < \cdots < k_t = n + 1 \) and \( k_{t-1} < n \). For a state \( c := c_0, c_1, \ldots, c_{n-1} \), let \( v := 0, c_1, \ldots, c_{n-1} \) and \( u := c_1, \ldots, c_{n-1}, 1 \). The following two successor rules generate de Bruijn sequences of order \( n \).

\[
\begin{align*}
\rho_{iz}(c) &= \begin{cases} 
\overline{c}_0, & \text{if } k_i \leq \text{wt}(v) < k_{i+1} \text{ for some } i \\
\text{and } L_{iz}^{k-1} v \text{ is a necklace,} \\
c_0, & \text{otherwise.}
\end{cases} \\
\rho_{zo}(c) &= \begin{cases} 
\overline{c}_0, & \text{if } k_i \leq \text{wt}(u) < k_{i+1} \text{ for some } i \\
\text{and } L_{zo}^{k-1} u \text{ is a necklace,} \\
c_0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

In **Proposition 4** we let \( k_t = n + 1 \) for consistency since \( \text{wt}(v) = n \) in \( C = (0) \) and \( \text{wt}(u) = n \) in \( C = (1) \). Each of these special cycles has only a single state. The reason to have \( k_t-1 < n \) is then clear. The correctness of **Proposition 4** comes from **Theorem 4** and the fact that the state satisfying the respective conditions in \( \rho_{iz} \) and \( \rho_{zo} \) is uniquely determined in the corresponding cycle. Examples of their output sequences are provided in Table II for \( n = 6 \).

**Proposition 5:** Each of the successor rules \( \rho_{iz} \) in (14) and \( \rho_{zo} \) in (15) generates \( 2^{n-2} \) de Bruijn sequences of order \( n \).

**Proof:** We supply the proof for \( \rho_{iz} \) in (14), the other case being similar to argue. For each \( 1 \leq \ell < n \), there exists at least one cycle of the PCR of order \( n \) having \( \ell \) distinct LZ states. To verify existence, one can, e.g., inspect the cycle
\[
(00 \ldots 011 \ldots 1)_{\ell} \text{ (for each } 1 \leq \ell < n).\]

On the other hand, taking all possible \( 2 \leq t \leq n \), there are \( 2^{n-2} \) distinct sets \( \{1 = k_1, k_2, \ldots, k_{t-1}, k_t = n + 1 \} \) with \( k_{t-1} < n \). Distinct sets provide distinct successor rules, producing \( 2^{n-2} \) inequivalent de Bruijn sequences in total.

We are not quite done yet. Here are two more general successor rules whose validity can be routinely checked.

**Proposition 6:** Let \( k \) be a nonnegative integer. For a state \( c := c_0, c_1, \ldots, c_{n-1} \), let \( v := 0, c_1, \ldots, c_{n-1} \) and \( u := c_1, \ldots, c_{n-1}, 1 \). The following successor rules generate de Bruijn sequences of order \( n \).

\[
\begin{align*}
\rho_1(c) &= \begin{cases} 
\overline{c}_0, & \text{if } L_{iz}^k v \text{ is a necklace,} \\
c_0, & \text{otherwise.}
\end{cases} \\
\rho_2(c) &= \begin{cases} 
\overline{c}_0, & \text{if } L_{zo}^k u \text{ is a necklace,} \\
c_0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

**Proposition 7:** The number of distinct de Bruijn sequences of order \( n \) produced by each of the rules in (16) and (17) is
\[
\begin{align*}
lcm(1,2,\ldots,n-1) &\geq (n-1) \left( \frac{n-2}{\frac{n}{2}} \right) \geq 2^{n-2}.
\end{align*}
\]

**Proof:** We supply the counting for the successor rule in (16). We know from the proof of **Proposition 5** that, for each \( 1 \leq \ell < n \), there exists at least one cycle of the PCR of order \( n \) having \( \ell \) distinct LZ states. For a given \( k \), we construct the system of congruences
\[
\{ k \equiv a_i \pmod{i} \} \text{ for } i \in \{1,2,\ldots,n-1\}.
\]

The number of resulting distinct de Bruijn sequences of order \( n \) is equal to the number of solvable systems of congruences in (19). The sequences are distinct because different nonempty subsets of \( \{a_1,\ldots,a_{n-1}\} \), whose corresponding systems are solvable, lead to different choices for the uniquely determined states in the respective cycles. By Generalized Chinese Remainder Theorem, the number is \( \text{lcm}(1,2,\ldots,n-1) \). From [29] Section 2 we get the lower bounds
\[
(n-1) \left( \frac{n-2}{\frac{n}{2}} \right) \geq 2^{n-2}.
\]

\[\square\]
Proposition 8 includes the constructions of de Bruijn sequences from the PCR of order \( n \) in [17] as special cases. Taking \( k \in \{0, 1, \text{lcm}(1, 2, \ldots, n-1) - 1\} \) in (16), for instance, outputs three sequences, namely the PCR4 in [17], granddaddy, and a sequence from \( \{17\} \), respectively. Using \( \{17\} \) with \( k \in \{0, 1, \text{lcm}(1, 2, \ldots, n-1) - 1\} \) yields the sequence PCR3 (J1) in [17], a sequence from \( \{13\} \), and grandmama, respectively.

Example 4: When \( n = 6 \), each successor rule in Proposition 8 yields 60 distinct de Bruijn sequences. Table III lists only 10 of the 60. We note their connection to known sequences to illustrate the generality of our approach.

For the successor rules in Propositions 4 and 8 generating the next bit means checking if a state is a cycle’s necklace by repeated simple left shifts. This can be done in \( O(n) \) time and \( O(n) \) space.

We generalize Proposition 8 to define more successor rules.

**Theorem 8:** Let \( g(k) \) be an arithmetic function
\[
g(k) : \{1, 2, \ldots, n\} \mapsto \{0, 1, \ldots, k-1\}. \tag{20}
\]
As before, for any \( c := c_0, c_1, \ldots, c_{n-1} \), let \( v := 0, c_1, \ldots, c_{n-1} \) and \( u := c_1, \ldots, c_{n-1}, 1 \). The following successor rules generate de Bruijn sequences of order \( n \).

\[
\rho_{Lz}^{g}(c) = \begin{cases} \tau_0, & \text{if } L_{Lz}^{g(wt(v))} \text{ v is a necklace}, \\ c_0, & \text{otherwise}. \end{cases} \tag{21}
\]

\[
\rho_{EO}^{g}(c) = \begin{cases} \tau_0, & \text{if } L_{EO}^{g(wt(u))} \text{ u is a necklace}, \\ c_0, & \text{otherwise}. \end{cases} \tag{22}
\]

For a cycle with \( 1 \leq \ell \leq n-1 \) distinct LZ states, there are \( \ell \) distinct ways to choose the uniquely determined state according to \( g(\ell) \). The counting for \( \ell \) distinct EO states is identical. It is then straightforward to confirm that each successor rule in Theorem 8 can generate \((n - 1)!\) distinct de Bruijn sequences of order \( n \) by using all possible \( g(\ell) \).

C. The Feedback Functions of the Resulting Sequences

Let \( x := x_0, x_1, \ldots, x_{n-1} \) be any state. We briefly discuss the feedback functions of the de Bruijn sequences produced earlier in this section. Their form is
\[
f(x) = \begin{cases} \tau_0, & \text{if } x \text{ satisfies the specified condition}, \\ x_0, & \text{otherwise}. \end{cases}
\]
Let $E$ be the set of states $v = v_0, v_1, \ldots, v_{n-1}$ such that each conjugate pair $(v, \hat{v})$ is used in generating the corresponding de Bruijn sequence. The feedback function of the resulting sequence is therefore

$$f(x_0, x_1, \ldots, x_{n-1}) = x_0 + h(x_1, \ldots, x_{n-1}),$$

with

$$h(x_1, \ldots, x_{n-1}) = \sum_{v \in E} \left( \prod_{1 \leq i < n} (x_i + v_i) \right).$$

(23)

Hence, determining $f$ requires computing $h$ such that

$$h(x_1, \ldots, x_{n-1}) = \begin{cases} 1, & \text{if } x_1, \ldots, x_{n-1} \text{ meets the condition}, \\ 0, & \text{otherwise}. \end{cases}$$

Since the resulting de Bruijn sequences come from joining all of the cycles in $\Omega(f_{\text{PCR}})$, we have $\text{wt}(h) = Z_n - 1$.

The following proposition will soon be useful.

**Proposition 9:** The feedback function of the successor rule

$$\rho(x) = \begin{cases} 1, & \text{if } x \text{ is a necklace}, \\ 0, & \text{otherwise} \end{cases}$$

is $f_{\rho}(x) := \prod_{i=1}^{n-1} f_i(x)$, (24)

where

$$f_i(x) = x_0 \cdot x_i + (x_0 + x_i) \cdot x_1 \cdot x_{i+1} + \ldots + (x_0 + x_i) \cdots (x_{n-2} + x_{i+n-2}) \cdot x_{n-1} \cdot x_{n-1+i}^\prime + (x_0 + x_i) \cdots (x_{n-1} + x_{n-1+i}).$$

**Proof:** The state $x := x_0, x_1, \ldots, x_{n-1}$ is a necklace if and only if it is lexicographically least in the set of all of its shifts. Let $x_i := x_i, x_{i+1}, \ldots, x_{i+n-1}$, where $1 \leq i < n$ and the subscripts are computed modulo $n$. The notation $\land$ stands for the logical AND. Then $x \preceq_{\text{lex}} x_i$ if and only if one and only one of the following conditions holds.

$$0 = x_0 < x_i = 1 \iff x_0 \cdot x_i = 1,$$

$$(x_0 = x_i) \land (x_1 < x_{i+1}) \iff (x_0 + x_i) \cdot x_1 \cdot x_{i+1} = 1,$$

$$\ldots, \ldots \iff \ldots, \ldots,$$

$$(x_0 = x_i) \land \ldots \land (x_{n-2} = x_{i+n-2}) \land (x_{n-1} < x_{i+n+1}) \iff (x_0 + x_i) \cdots (x_{n-2} + x_{i+n-2}) \cdot x_{n-1} \cdot x_{i+n-1} = 1,$$

$$(x_0 = x_i) \land \ldots \land (x_{n-2} = x_{i+n-2}) \land (x_{n-1} = x_{i+n+1}) \iff (x_0 + x_i) \cdots (x_{n-1} + x_{i+n-1}) = 1.$$  

Hence, $x \preceq_{\text{lex}} x_i$ if and only if $f_i = 1$ for all $1 \leq i < n$.

The next two corollaries to Proposition 9 give the respective feedback functions of the stated successor rules.

**Corollary 10:** Let $c := c_0, c_1, \ldots, c_{n-1}$. The feedback function of the de Bruijn sequence built by the successor rule

$$\rho(c) = \begin{cases} c_0, & \text{if } 0, c_1, \ldots, c_{n-1} \text{ is a necklace}, \\ c_0, & \text{otherwise}, \end{cases}$$

is $f(x_0, x_1, \ldots, x_{n-1}) = x_0 + f_{\rho}(0, x_1, \ldots, x_{n-1})$.

**Corollary 11:** Let $c := c_0, c_1, \ldots, c_{n-1}$. The feedback function of the successor rule

$$\rho(c) = \begin{cases} 1 + \sum_{i=0}^{n-1} c_i, & \text{if } c \text{ satisfies } B, \\ c_0, & \text{otherwise}, \end{cases}$$

is $f(x_0, x_1, \ldots, x_{n-1}) = x_0 + f_{\rho}(x_1, \ldots, x_{n-1}, 1)$.

The feedback function of the resulting de Bruijn sequence built by the other successor rules that we have discussed above can be inferred from a similar analysis on the corresponding Boolean logical operations. The details are omitted here.

**V. SUCCESSOR RULES FROM PURE SUMMING REGISTERS**

This section studies how to generate de Bruijn sequences by applying the CJM on the PSR of any order $n$. The strategy is to define several distinct relations or orders on the cycles before deploying them in constructing new successor rules. Let $B$ be a statement which guarantees that the resulting sequence is de Bruijn, given that the FSR is the PSR of order $n$. Hence, analogous to $\rho_A$ in (9), for any state $c := c_0, c_1, \ldots, c_{n-1}$, we define the successor rule $\rho_B$ by

$$\rho_B(c) = \begin{cases} 1 + \sum_{i=0}^{n-1} c_i, & \text{if } c \text{ satisfies } B, \\ \sum_{i=0}^{n-1} c_i, & \text{otherwise}. \end{cases}$$

(25)
A. The Run Order on the Pure Summing Register

For a binary periodic sequence, a run of \( k \) consecutive 0s preceded and followed by a 1 is a \textit{run of 0s of length} \( k \). A run of 1s of length \( k \) is defined analogously. The convention is to fix \( k = \infty \) and \( k = 0 \) as the respective lengths of runs of 0s in \((0)\) and in \((1)\). This subsection imposes a new order on the cycles of the PSR based on their runs of 0s.

For a given \( n \), let \( r \) be the number of cycles in \( \Omega_{\text{PSR}} \). Given cycles \( C_i \neq C_j \) with \( 1 \leq i, j \leq r \), we say that \( C_j \prec_{\text{rz}} C_i \) in the \textit{run order} if and only if the maximal length of runs of 0s in \( C_j \) is less than the maximal length of runs of 0s in \( C_i \). The subscript \( \text{rz} \) signifies that the arrangement is based on the \textit{run of zeros}. Strictly speaking, the run order is just a transitive relation since it is not necessary to define how \( C_i \) is related to itself.

\textbf{Theorem 12:} The following can serve as \( B \) to define \( \rho_{B} \) in Equation (25).

The \((n+1)\)-stage state \( c_1, c_{n-1}, 1 + \sum_{i=1}^{n-1} c_i, 1 \) is uniquely determined in the corresponding cycle that begins with a maximal length run of 0s.

\textbf{Proof:} Recall that the length of run of 0s in \((0)\) is \( \infty \). By Theorem 2, it suffices to show that for each \( C_i \neq (0) \), there is a uniquely determined state whose conjugate state is in \( C_j \), with \( C_j \succ_{\text{rz}} C_i \). Any nonzero cycle

\[
C_i := \left( 1, c_1, \ldots, c_{n-1}, 1 + \sum_{i=1}^{n-1} c_i \right)
\]

has at least one state with the property that a maximal length run of 0s starts at \( c_1 \). Suppose that \( 1, c_1, \ldots, c_{n-1} \) has been uniquely identified. Then its conjugate, namely \( 0, c_1, \ldots, c_{n-1} \), must be in

\[
C_j := \left( 0, c_1, \ldots, c_{n-1}, 1 + \sum_{i=1}^{n-1} c_i \right)
\]

with a larger maximal length run of 0s. Thus, \( C_j \succ_{\text{rz}} C_i \).

\textbf{Remark 2:} The run order here is well-defined for arbitrary FSRs. We can use it to generate de Bruijn sequences by joining the cycles of an arbitrary FSR based on a similarly defined successor rules. Efficiency is another matter altogether since the cycle structure may be harder to manage if the choice of the FSR is not done judiciously.

How to efficiently determine a unique state in a nonzero \( C \in \Omega_{\text{PSR}} \) of order \( n \)? An \((n+1)\)-stage state that starts with a run of 0s with maximal length must exist because the necklace satisfies this condition. If \( C = (1) \), which happens whenever \( n \) is odd, then the maximal length of a run of 0s is understood to be 0 and we use 1, \ldots, 1 as the required \((n+1)\)-stage. Suppose that \( v \) is an \((n+1)\)-stage state that starts with a maximal length run of 0s. A new operator \( L_{\text{rz}} \) on \( v \) is defined such that \( L_{\text{rz}} v \) is the next state that also starts with a maximal length run of 0s obtained by repeated applications of the left shift \( L \) on \( v \). Let \( L_{\text{rz}}^k v = L_{\text{rz}}^{k-1}(L_{\text{rz}} v) \) for any positive integer \( k \).

We can now propose several distinct successor rules simply by specifying how to uniquely determine the \((n+1)\)-stage state in Theorem 12.

\textbf{Proposition 13:} Let \( k \) be a nonnegative integer. For a given state \( c := c_0, c_1, \ldots, c_{n-1}, 1 \), let

\[
v := c_1, \ldots, c_{n-1}, 1 + \sum_{i=1}^{n-1} c_i, 1.
\]

We call \( v \) a \textit{nice state} if it starts with a maximal length run of 0s and \( L_{\text{rz}}^k v \) is a necklace. Then the successor rule

\[
\rho_{\text{nice}}(c) = \begin{cases} 
1 + \sum_{i=0}^{n-1} c_i, & \text{if } v \text{ is nice,} \\
\sum_{i=0}^{n-1} c_i, & \text{otherwise,}
\end{cases}
\]

generates de Bruijn sequences of order \( n \).

If \( k = 0 \) in Proposition 13 then the rule simplifies to

\[
\rho_{\text{nice}}(c) = \begin{cases} 
1 + \sum_{i=0}^{n-1} c_i, & \text{if } v \text{ is a necklace,} \\
\sum_{i=0}^{n-1} c_i, & \text{otherwise.}
\end{cases}
\]

\textbf{Example 5:} We label the 10 cycles generated by the PSR of order 6 as

\[
C_1 = (0111111), \quad C_2 = (0101011), \quad C_3 = (0011101), \quad C_4 = (0011011), \quad C_5 = (0010111),
\]

\[
C_6 = (0001111), \quad C_7 = (0001001), \quad C_8 = (0000101), \quad C_9 = (0000011), \quad C_{10} = (0000000).
\]

All, except for \( C_{10} \), have least period 7. In the run order \( C_{10} \) is the largest. The maximal lengths of runs of 0s in \( C_8 \) and \( C_4 \) are 4 and 2 respectively, implying \( C_4 \not\sim_{\text{rz}} C_8 \).

The maximal length of run of 0s in \( C_2 \) is 1 with 3 distinct 7-stage states that start with 0, namely \( v = 0101011, L_{\text{rz}} v = 0101101, \) and \( L_{\text{rz}}^2 v = 0110101, \) since \( L_{\text{rz}}^3 v = v \). Each of these 3 states can be chosen to be the uniquely determined state. Each \( C_i \), for \( i \not\in \{2, 10\} \), has only one choice for a state that starts with the respective longest run of 0s. Proposition 13 yields three distinct de Bruijn sequences which are presented in Table IV. The spanning tree when \( v \) is chosen, \( i.e., \) when \( c_0, c_1, \ldots, c_5 = 101010 \) is given in Figure 2.
The Necklace Order on the Pure Summing Register

This subsection presents another general method to construct successor rules which can generate de Bruijn sequences based on the PSR of order \( n \). Given \( n \), each cycle in \( \Omega_{\text{PSR}} \) is \((n+1)\)-periodic and, hence, can be written as \((c_0,c_1,\ldots,c_n)\). We define a new total order, which we name the \textit{necklace order} denoted by \( \prec_{\text{nk}} \) on the cycles. Given \( C_i \neq C_j \), we say \( C_i \prec_{\text{nk}} C_j \) if and only if the necklace of \( C_i \) is lexicographically less than that of \( C_j \).

The companion state of \( C_i \) is \( C_i^\perp \), the necklace in \( C_i \) whose conjugate is in \( C_i \). Let \( 0 = (\underbrace{0,\ldots,0}_t,\underbrace{1,\ldots,1}_t) \). The edge label from \( C_i \) to \( C_j \) is the only EO state in its cycle. Our task is to determine a state in \( C_i \) whose companion state is in \( C_j \) with \( C_j \prec_{\text{nk}} C_i \).

\textbf{Lemma 14:} For a given \( n \), let \( C_i := (0,1,\ldots,c_{n-1},1) \) and \( C_j := (c_0,\ldots,c_{n-1},0) \) be adjacent nonzero cycles in \( \Omega_{\text{PSR}} \).

1) If \( 0 \) is the unique EO state, that is, it is the necklace in \( C_i \), then \( C_j \prec_{\text{nk}} C_i \).

2) If there are two or more EO states in \( C_i \) and the necklace in \( C_i \) is not \( c_0,1,\ldots,c_{n-1},1 \), then \( C_j \prec_{\text{nk}} C_i \).

\textbf{Proof:} In the first case, \( C_j \) is either (1) or has the form

\[
(0,\ldots,0,1,\ldots,0,1).
\]

If \( C_i = (1) \), then \( C_j = (0,1,\ldots,1,0) \), making \( C_j \prec_{\text{nk}} C_i \). If \( C_i \neq (1) \), then it is easy to confirm that the maximal length of the run of 0s in \( C_j \) is larger than that of \( C_i \). Thus, \( C_j \prec_{\text{nk}} C_i \) as well.

For the second case, suppose that the necklace of \( C_i \) is of the form

\[
c_{t},\ldots,c_{n-1},1,c_0,\ldots,c_{t-1}
\]

for some positive integer \( t \). Then there exists a state in \( C_j \) with the form

\[
c_{t},\ldots,c_{n-1},0,c_0,\ldots,c_{t-1}
\]

which is lexicographically less than the necklace of \( C_i \). Thus, \( C_j \prec_{\text{nk}} C_i \).

In a similar manner we can prove the following lemma, which will be used in the proof of Theorem 19.

\textbf{Lemma 15:} For a given \( n \), let \( C_i := (0,c_1,\ldots,c_{n-1},1) \), with \( c_1,\ldots,c_{n-1} \neq 1,\ldots,1 \), and \( C_j := (1,c_1,\ldots,c_{n-1},0) \) be adjacent nonzero cycles in \( \Omega_{\text{PSR}} \).

1) If \( 0,c_1,\ldots,c_{n-1},1 \) is the unique LZ state, that is, it is the necklace in \( C_i \), then \( C_j \prec_{\text{nk}} C_i \).

2) If there are two or more LZ states in \( C_i \) and the necklace in \( C_i \) is not \( 0,c_1,\ldots,c_{n-1},1 \), then \( C_j \prec_{\text{nk}} C_i \).

Combining Theorem 2 and Lemma 14 results in the next theorem.

\textbf{Theorem 16:} Let \( c := c_0,1,\ldots,c_{n-1} \) and \( v := 1 + \sum_{t=1}^{n-1} c_1,\ldots,c_{n-1},1 \). The successor rule \( \rho_B \) defined as

\[
\rho_B(c) = \begin{cases} 
1 + \sum_{t=0}^{n-1} c_t, & \text{if } v \text{ satisfies Condition B}, \\
\sum_{t=0}^{n-1} c_t, & \text{otherwise},
\end{cases}
\]

(28)
generates a de Bruijn sequence. Condition B consists of any of the following two requirements.

1) The state \( v \) is the only EO state in its cycle.
2) Among the two or more EO states in the cycle, \( v \) is not the necklace, but it can be uniquely identified.

Theorem 16 allows us to generate \( 3^5 \cdot 5 = 1,215 \) distinct de Bruijn sequences from the PSR of order \( n = 6 \). Some of the successor rules, however, are rather ad hoc and not easy to generate. The next proposition gives a simple rule to efficiently and uniquely identify an EO state which is not a necklace by using the operator \( L_{eo} \) defined earlier.
Proposition 17: Let \( k \) be a positive integer. Given a state \( c := c_0, c_1, \ldots, c_{n-1} \), let \( v := 1 + \sum_{i=0}^{n-1} c_i, c_1, \ldots, c_{n-1}, 1 \). The following rule generates a de Bruijn sequence of order \( n \).

\[
\rho_k(c) = \begin{cases} 
1 + \sum_{i=0}^{n-1} c_i, & \text{if } L_{eo}^k v \neq v \text{ and } L_{eo}^k v \text{ is a necklace}, \\
1 + \sum_{i=0}^{n-1} c_i, & \text{if } L_{eo}^k v = v \text{ and } L_{eo} v \text{ is a necklace}, \\
\sum_{i=0}^{n-1} c_i, & \text{otherwise}.
\end{cases}
\] (29)

Proof: Assume \( L_{eo}^k v = v \) and \( L_{eo} v \) is the necklace in \( C \). If a cycle \( C \) contains only one EO state, then \( v \) must be this state and it is the necklace, which is uniquely determined. If \( C \) has \( \ell \geq 1 \) distinct EO states, then we note that \( k \ell v = v \) if and only if \( \ell \mid k \). If \( L_{eo}^k v = v \) and \( L_{eo} v \) is the necklace, then \( v \) is not the necklace, but \( v \) is uniquely determined. If \( L_{eo}^k v \neq v \) and \( L_{eo} v \) is the necklace, then \( v \) is uniquely determined, although it cannot be the necklace. Thus, by Theorem 16 the rule \( \rho_k \) in (29) generates a de Bruijn sequence.

We enumerate the number of distinct de Bruijn sequences from \( \rho_k \) in (29) based on the number of valid values that \( k \) can take.

Proposition 18: The number of de Bruijn sequences of order \( n \geq 2 \) that can be generated from Proposition 17 is

\[
lcm(2, 4, \ldots, 2[n/2]) - 1 \geq 2^{[n/2]} - 1.
\] (30)

Proof: Since the proposition clearly holds for \( n \in \{2, 3\} \), we treat \( n \geq 4 \).

Let the numbers of distinct EO states in the cycles generated by the PSR of a fixed order \( n \geq 2 \) be listed as \( b_1, b_2, \ldots, b_t \), with \( b_j > 1 \) for \( 1 \leq j \leq t \). We exclude \( b_j \in \{0, 1\} \) for obvious reason. By the properties of the PSR, for each \( 1 \leq i \leq [n/2] \), there exists some \( 1 \leq s \leq t \) such that \( 2i = b_s \). Since \( wt(C_{PSR}) \) is even, if \( b_j > 1 \) is odd, for some \( 1 \leq j \leq t \), then \( b_j < [n/2] \) and \( b_j \mid 2i \), for some \( i \). Hence,

\[
lcm(b_1, b_2, \ldots, b_t) = lcm(2, 4, \ldots, 2[n/2]).
\]

For a given \( k \), we construct a vector of integers

\[
(a_1', a_2', \ldots, a_t') \text{ with } a_j' \equiv k \pmod{b_j} \text{ for all } 1 \leq j \leq t.
\] (31)

By Theorem 1 the corresponding vectors \((a_1', a_2', \ldots, a_t')\) in (31) are distinct for distinct choices of \( k \), as \( k \) ranges from 1 to \( lcm(b_1, b_2, \ldots, b_t) \). Thus, the number of such vectors is \( lcm(2, 4, \ldots, 2[n/2]) \).

From each vector \((a_1', a_2', \ldots, a_t')\), we construct a new vector

\[
(a_1, a_2, \ldots, a_t) \text{ with } a_j = \begin{cases} 
 a_j', & \text{if } a_j' \neq 0, \\
 1, & \text{if } a_j' = 0,
\end{cases} \text{ for all } 1 \leq j \leq t.
\] (32)

Following the rule \( \rho_k \) in (29), the uniquely determined state \( v \) in the cycle \( C_{PSR} \) with \( b_j \) distinct EO states satisfies

\( L_{eo}^k v \) is the necklace in \( C_{PSR} \).

Thus, distinct vectors \((a_1, a_2, \ldots, a_t)\) in (32) yield distinct de Bruijn sequences. We now enumerate the number of distinct vectors \((a_1, a_2, \ldots, a_t)\) in (32).

If \( k = 1 \) or \( k = lcm(b_1, b_2, \ldots, b_t) = lcm(2, 4, \ldots, 2[n/2]) \), then \((a_1, a_2, \ldots, a_t) = (1, 1, \ldots, 1)\). Conversely, we claim that the only \( k \), with \( 1 < k < lcm(2, 4, \ldots, 2[n/2]) \), that satisfies \((a_1, a_2, \ldots, a_t) = (1, 1, \ldots, 1)\) is \( k = 1 \). The condition implies that \( a_j' \equiv k \pmod{b_j} \) \( \forall 1 \leq j \leq t \). By Theorem 1 if \( b_j > 1 \) is odd, then \( k \pmod{b_j} = (k \pmod{2b_j}) \pmod{b_j} \), because \( b_j \mid (k \pmod{b_j}) - k \pmod{2b_j} \). Similarly, \( k \pmod{2j} = k \pmod{2i} \), because \( 2 \mid (k \pmod{2j}) - k \pmod{2i} \). Having

\[
k \pmod{b_1} = \ldots = k \pmod{b_t} = 1 \text{ forces } k = 1.
\]

Let \( 1 < k < lcm(2, 4, \ldots, 2[n/2]) \) and \( 1 \leq \ell \leq t \). Since \( k \neq 1 \), there exists some \( \ell \) such that \( a_\ell = a_\ell' = k \pmod{b_\ell} > 1 \). If \( b_\ell \) is odd, then it is immediate to confirm that \( k \pmod{2b_\ell} > 1 \). Without loss of generality, we can assume that \( b_\ell \) is even. We construct a bijection between \( \{(a_1', a_2', \ldots, a_\ell')\} \) and \( \{(a_1, a_2, \ldots, a_\ell)\} \) by showing that each possible vector \((a_1, a_2, \ldots, a_\ell) \neq (1, 1, \ldots, 1)\) uniquely determines a vector \((a_1', a_2', \ldots, a_\ell')\). If \( a_\ell' > 1 \) for all \( j \), then we are done. Otherwise, if \( a_\ell = 1 \), then \( a_\ell' \in \{0, 1\} \). In this case, if \( b_\ell \) is odd, then we consider \( b_\ell = 2b_\ell' \) and \( a_\ell' \). If \( a_\ell' = a_\ell > 1 \), then \( a_\ell' \) is fixed because \( b_\ell \mid (a_\ell' - a_\ell) \). If \( a_\ell = 1 \), then since \( b_\ell \) is even and \( a_\ell = a_\ell' = k \pmod{b_\ell} > 1 \), we can determine \( a_\ell' \) from the fact that \( 2 \mid (a_\ell' - a_\ell) \). Once this is done, \( a_\ell' \) is accordingly identified.

The results that we have established imply that, for \( 1 \leq k < lcm(2, 4, \ldots, 2[n/2]) \), distinct choices for \( k \) yield distinct de Bruijn sequences upon applying the rule \( \rho_k \) in (29). Thus, the number of distinct de Bruijn sequences is

\[
lcm(2, 4, \ldots, 2[n/2]) - 1 = 2 \cdot lcm(1, 2, \ldots, [n/2]) - 1 \geq 2^{[n/2]} - 1.
\] (33)

The lower bound comes from a result in Section 2.

One can slightly modify the condition in Proposition 17 to generate even more de Bruijn sequences. When \( L_{eo}^k v = v \), there are some quite obvious ways to determine the unique state in the corresponding cycle. The details can be worked out routinely and, hence, are omitted here.
Another approach is to simplify $\rho_k$ by restricting the choices for $k$. Given $c := c_0, c_1, \ldots, c_{n-1}$, one constructs a new $(n+1)$-stage state $d_0, d_1, \ldots, d_{n-1}, 1$ such that

$$d_0 = 1 + \sum_{i=1}^{n-1} c_i$$

and $d_i = c_i$ for $i \in \{1, \ldots, n-1\}$.

Let $0 \leq s, t < n$ be, respectively, the largest and smallest indices such that $d_s = d_t = 1$. Such indices exist since the state contains at least two 1s. Let

$$v_s := d_{s+1}, \ldots, d_{n-1}, 1, d_0, \ldots, d_s$$

and

$$v_t := d_{t+1}, \ldots, d_{n-1}, 1, d_0, \ldots, d_t.$$

We then define the successor rule

$$\rho_s(c) = \ \begin{cases} 
1 + \sum_{i=0}^{n-1} c_i, & \text{if } v_s \text{ is a necklace}, \\
\sum_{i=0}^{n-1} c_i, & \text{otherwise}. 
\end{cases} \quad (34)$$

The rule $\rho_t$ is exactly the same with $v_t$ replacing $v_s$.

**Example 6:** For $n = 6$, Table IV lists 11 generated distinct de Bruijn sequences from Proposition 17. Sequences from $k = 1$ and $k = 11$ are also those ruled by $\rho_x$ and $\rho_y$, respectively. □

**C. The Mixed Order on the Pure Summing Register**

This subsection imposes another new order on the cycles of the PSR of order $n$ to define novel successor rules.

We start with a new total order, called the *mixed order* and denoted by $\prec_{\text{mix}}$, on the cycles of the PSR by *combining the necklace order and the weight relation*. We say that $C_j \prec_{\text{mix}} C_i$, if and only if they satisfy one of the following conditions.

1) $\text{wt}(C_j) > \text{wt}(C_i)$.
2) $\text{wt}(C_j) = \text{wt}(C_i)$ and, in their necklace order, $C_j \prec_{\text{nk}} C_i$.

It is rather typical in the construction of rooted spanning trees that adjacent cycles are chosen to follow lexicographically decreasing or increasing patterns. Adjacent cycles under our mixed order do not usually obey a lexicographic pattern. This sets our successor rules apart from those formulated in the spirit of the work by Sawada *et al.* in [18].

### Table IV

| $k$ | de Bruijn sequences based on Equation (26) | Notes |
|-----|------------------------------------------|-------|
| 0   | (0000001000011010100011010000110111100010001101010010110001) | Also from Equation (27) |
| 1   | (0000001000011010100011010000110111100010001101010010110001) | |
| 2   | (0000001000011010100011010000110111100010001101010010110001) | |

| $k$ | de Bruijn sequences based on Equation (29) | Notes |
|-----|------------------------------------------|-------|
| 1   | (0000001111111001110110110110110110100111011011011010110001) | Equation (24) on index $t$ |
| 2   | (0000001100010101010101010101010101010101010101010101010001) | |
| 3   | (0000001100010101010101010101010101010101010101010101010001) | |
| 4   | (0000001100010101010101010101010101010101010101010101010001) | |
| 5   | (0000001100010101010101010101010101010101010101010101010001) | |
| 6   | (0000001100010101010101010101010101010101010101010101010001) | |
| 7   | (0000001100010101010101010101010101010101010101010101010001) | |
| 8   | (0000001100010101010101010101010101010101010101010101010001) | |
| 9   | (0000001100010101010101010101010101010101010101010101010001) | |
| 10  | (0000001100010101010101010101010101010101010101010101010001) | |
| 11  | (0000001100010101010101010101010101010101010101010101010001) | |

For $n = 6$, Table IV lists 11 generated distinct de Bruijn sequences from Proposition 17. Sequences from $k = 1$ and $k = 11$ are also those ruled by $\rho_x$ and $\rho_y$, respectively.
Example 7: In terms of weight, the 10 cycles generated by the PSR of order 6 have the following distribution. The weight of \((0)\) is clearly 0. There are three cycles of weight 2, five cycles of weight 4 and one cycle of weight 6. Given in increasing mixed order, the cycles are

\[
(0111111) \prec_{\text{mix}} (00011111) \prec_{\text{mix}} (00101111) \prec_{\text{mix}} (0011101) \prec_{\text{mix}} (0011101) \prec_{\text{mix}} (0101011) \prec_{\text{mix}} (00001111) \prec_{\text{mix}} (0000101) \prec_{\text{mix}} (00010101) \prec_{\text{mix}} (00010011) \prec_{\text{mix}} (00010011) \prec_{\text{max}} (0).
\]

One confirms by inspection that the mixed order differs from the other orders that we have applied on these 10 cycles.

Remark 3: There is another useful variant of the mixed order. We can say, alternatively, that \(C_j \prec_{\text{mix}} C_i\) if and only if they satisfy one of the following conditions.

1) \(\text{wt}(C_j) < \text{wt}(C_i)\),
2) \(\text{wt}(C_j) = \text{wt}(C_i)\) and, in the necklace order, \(C_j \prec_{\text{nk}} C_i\).

This alternative also works in the next theorem, with the mechanism adjusted accordingly.

We give a sufficient condition based on the PSR of order \(n\).

Theorem 19: In the PSR of order \(n\), let \(c := c_0, c_1, \ldots, c_{n-1}\) be any given state and \(v := 0, c_1, \ldots, c_{n-1}\). The successor rule \(\rho_B\) in (25) can be validly defined as follows.

1) If there are two consecutive 0s in \(C := (0, c_1, \ldots, c_{n-1}, \sum_{i=1}^{n-1} c_i)\), the sum \(\sum_{i=1}^{n-1} c_i = 0\), and \(v\) can be uniquely determined, then \(\rho_B(c) := 1 + \sum_{i=0}^{n-1} c_i\). 

2) If there are no two consecutive 0s in \(C := (0, c_1, \ldots, c_{n-1}, \sum_{i=1}^{n-1} c_i) \neq (0^a)\) and exactly one of the followings holds, then \(\rho_B(c) := 1 + \sum_{i=0}^{n-1} c_i\).

a) \(v\) is the only one LZ state.

b) When there are two or more LZ states in \(C\), \(v\) is uniquely identified and \(0, c_1, \ldots, c_{n-1}, 1\) is not the necklace.

3) In all other cases, \(\rho_B(c) := \sum_{i=0}^{n-1} c_i\).

Proof: The \((n+1)\)-periodic cycle with the least mixed order is

\[
C_1 := \begin{cases} 
(0, 1, 1, \ldots, 1), & \text{if } n \text{ is even,} \\
(1, 1, 1, \ldots, 1), & \text{if } n \text{ is odd.}
\end{cases}
\]

By Theorem 2 and by the definition of \(\rho_B\), it suffices to show that there exists a unique state in each \(C \neq C_1\) whose successor, as determined by \(\rho_B\), is contained in another cycle \(C'\) with \(C' \prec_{\text{mix}} C\). It is clear that \(C\) has at least one LZ state.

Let \(C\) be any cycle of the PSR that is not equal to \(C_1\). Let there be two consecutive 0s in \(C\). Without loss of generality, let \(C := (0, c_1, \ldots, c_{n-1}, \sum_{i=1}^{n-1} c_i)\) such that \(\sum_{i=1}^{n-1} c_i = 0\), and \(v := 0, c_1, \ldots, c_{n-1}\) is the uniquely determined state. By the definition of \(\rho_B\), the successor of \(v\) is \(c_1, \ldots, c_{n-1}, 1\), which is in cycle \(C' := (1, c_1, \ldots, c_{n-1}, 1)\). In this case, \(\text{wt}(C) < \text{wt}(C')\) and, thus, \(C' \prec_{\text{nk}} C\), as desired.

If there are no two consecutive 0s in \(C\), then we can assume \(C := (0, c_1, \ldots, c_{n-1}, 1)\). If \(C\) has only one LZ state, then the successor of \(v\) is \(c_1, \ldots, c_{n-1}, 0\), by how \(\rho_B\) is defined. If \(C\) has multiple LZ states, then we choose \(v\) to be a uniquely determined LZ state such that \(c_1, \ldots, c_{n-1}, 1\) is not the necklace. Again, by the definition of \(\rho_B\), its successor is \(c_1, \ldots, c_{n-1}, 0\).

In both cases, the successor is in another cycle \(C' := (1, c_1, \ldots, c_{n-1}, 0)\) with \(\text{wt}(C) = \text{wt}(C')\). By Lemma 15 we have \(C' \prec_{\text{mix}} C\), as required. By Theorem 2 \(\rho_B\) generates a de Bruijn sequence.

Checking whether a cycle has two consecutive 0s is very fast. Theorem 19 implies that, if a cycle has two or more LZ states and the sum of the next \(n-1\) bits in each of the states is 0, then distinct LZ states lead to different successor rules. We supply several ways to uniquely identify qualified states.

If a given cycle has two consecutive 0s, then its necklace begins with two 0s and all \((n+1)\)-stage states that begin with two 0s are obtainable by the shift operations on the necklace. We define a new operator, denoted by \(L_{dz}\). The subscript \(dz\) stands for double zeros to indicate that the operator is applicable on any state \(v := 0, c_1, \ldots, c_{n-1}\) that starts with two 0s. The state \(L_{dz}v\) is the first state with two leading 0s obtained by repeated application of the left shift \(L\) on \(v\). We now propose a new successor rule based on the mixed order.

Proposition 20: Let \(k\) be a nonnegative integer. For any state \(c := c_0, c_1, \ldots, c_{n-1}\), let \(C := (0, c_1, \ldots, c_{n-1}, \sum_{i=1}^{n-1} c_i)\). Then

\[
\rho_{dz}(c) = \begin{cases} 
1 + \sum_{i=0}^{n-1} c_i, & \text{if } c \text{ satisfies Condition } X, \\
\sum_{i=1}^{n-1} c_i, & \text{otherwise,}
\end{cases}
\]

generates a de Bruijn sequence. Condition \(X\) consists of any of the following two requirements.

1) The cycle \(C\) has two consecutive 0s, the sum \(\sum_{i=1}^{n-1} c_i = 0\), and \(L_{dz}^k v\) is a necklace, where \(v := 0, c_1, \ldots, c_{n-1}\).

2) The cycle \(C \neq (0, 1, 1, \ldots, 1)\) has no two consecutive 0s and its necklace is \(0, c_{j+1}, \ldots, c_{n-1}, 1, 0, c_1, \ldots, c_{j-1}\), with \(1 \leq j < n\) being the least positive integer for which \(c_j = 0\).

Proof: We prove this proposition by showing the successor rule \(\rho_{dz}\) satisfies the conditions in Theorem 19 that a uniquely desired state is identified in each \(C \neq C_1\), where \(C_1\) is given in (35).
If $C$ has two consecutive 0s and $\sum_{i=1}^{n-1} c_i = 0$, then $C \neq C_1$ and $v := 0, 0, c_1, \ldots, c_{n-1}$ such that $L_{I_{-2}}^v$ is a necklace is uniquely determined. Consequently the state $0, c_1, \ldots, c_{n-1}$ is uniquely determined and its conjugate state is in $C' := (1, 1, c_1, \ldots, c_{n-1})$, which has larger weight and hence, $C \prec_{\text{mix}} C$.

If $C \neq C_1$ has no two consecutive 0s, then $C := (0, c_1, \ldots, c_{n-1}, 1)$ must contain at least two 0s. Hence, there is $j > 0$ such that $c_j = 0$. Because $0, c_{j+1}, \ldots, c_{n-1}, 1, 0, c_1, \ldots, c_{j-1}$ is the necklace, $0, c_1, \ldots, c_{n-1}$ in $C$ is uniquely determined and, by Lemma 15 its conjugate state must be in another cycle $C'$ with $C' \preceq_{\text{nk}} C$. Thus, $C' \prec_{\text{mix}} C$.

\textbf{Remark 4:} There are many other ways to determine the unique state for the two cases in Statement X of Proposition 20. Here is an example. If $C$ contains two consecutive 0s, then we can use the method proposed in Subsection V-B to determine the unique state and, hence, to provide new successor rules.

\textbf{Proposition 21:} The successor rules $\rho_{I_{-2}}$ in \textbf{(36)}, identified with different valid choices for $k$, generate distinct de Bruijn sequences in total.

\textbf{Proof:} It suffices to count the number of two consecutive 0s in the cycles that contain consecutive 0s. For each $2 \leq i < n$, there exists at least one cycle that contains $i$ consecutive 0s. For example, when $i = n - 1$, the cycle $(0, \ldots, 0, 1, 1)$ has a run of 0s of length $n - 1$. For each $1 \leq j \leq n - 2$, there is at least one cycle that has $j$ distinct states, each of which can be declared to be the uniquely determined state. The same reasoning we used in proving Proposition 7 tells us that each $k \leq 0 < \lcm(1, 2, \ldots, n - 2)$ yields a de Bruijn sequence. Different values for $k$ produce distinct sequences based on the corresponding successor rules. The lower bounds in \textbf{(37)} comes from \textbf{(29)}.

\textbf{Example 8:} Table \textbf{IV} contains 12 distinct de Bruijn sequences produced by using Proposition 20 with $n = 6$. 

\section{VI. Conclusions}

In this paper, we have proposed a general design criteria for feasible successor rules. They perform the cycle joining method to output binary de Bruijn sequences. The focus of our demonstration is on their efficacy and efficiency when applied to two classes of simple FSRs. These are the pure cycling register (PCR) and the pure summing register (PSR) of any order.

Our approach is versatile. It goes beyond the often explored route of relying on the lexicographic ordering of the cycles. We have shown that many transitive relations can also be used to put the cycles in some order. We have enumerated the respective output sizes of various specific successor rules that can be validly defined based on the general criteria. A straightforward complexity analysis has confirmed that generating the next bit in each resulting sequence is efficient.

We assert that the criteria we propose here can be applied to all nonsingular FSRs. If a chosen FSR has cycles with small least periods, then the complexity to produce the next bit can be kept low. Interested readers are invited to come up with feasible successor rules for their favourite FSRs. We intend to do the same and to further look into, among others, the cryptographic properties of the binary de Bruijn sequences produced by more carefully designed successor rules.

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