

SOME RESULTS ON SET COLORINGS OF DIRECTED TREES

NISHA REENA NAZARETH, LOLITA PRIYA CASTELINO*

Department of Mathematics, Shri Madhwa Vadiraja Institute of Technology and Management, Udupi-574115, INDIA

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Abstract. A set coloring of the digraph $D$ is an assignment (function) of distinct subsets of a finite set $X$ of colors to the vertices of the digraph, where the color of an arc, say $(u, v)$ is obtained by applying the set difference from the set assigned to the vertex $v$ to the set assigned to the vertex $u$ which are also distinct. A set coloring is called a strong set coloring if sets on the vertices and arcs are distinct and together form the set of all non empty subsets of $X$. A set coloring is called a proper set coloring if all the non empty subsets of $X$ are obtained on the arcs of $D$. A digraph is called a strongly set colorable (properly set colorable) if it admits a strong set coloring (proper set coloring).

In this paper we find some classes of directed trees which admit a strong set coloring and construction of strongly set colorable directed tree $\overrightarrow{T_n}$.

Keywords: set coloring; strong (proper)set coloring; digraphs.

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1. INTRODUCTION

In this paper, we consider only finite simple digraphs. For all notations we follow Harary [1].

The notion of set coloring of a graph has been introduced by Hegde [2] in 2009. Further Hegde and Sumana [4] determined the set coloring number of certain graphs.

*Corresponding author

E-mail address: plolita@yahoo.com

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The concept of set colorings of graph was then extended to digraphs by Hegde and Castelino [3].

**Definition 1.1.** Given a digraph \( D = (V, E) \) with a non empty set \( X \) of \( n \) colors and \( m \) arcs, a function \( f : V \to 2^X \) can be defined as the assignment of the colors \( f(v) \) to each of the vertices \( v \in V \) and given such a function \( f \) on the vertex set \( V \), we define \( f^* : E \to 2^X \) which assigns colors to the arcs \( e = uv \in E \), \( f^*(e) = f(v) - f(u) \).

A digraph \( D \) is said to be a set colorable if both \( f \) and \( f^* \) are injective functions. A digraph \( D \) is said to be properly set colorable if it is set colorable with \( f^*(E) = 2^X \setminus \emptyset \) and \( D \) is strongly set colorable if \( f(V) \cup f^*(E) = 2^X \setminus \emptyset \) and \( f(V) \cap f^*(E) = \emptyset \). They also determined the necessary condition for strong(proper) set colorings of digraphs.

**Definition 1.2.** Set coloring number [3], \( \sigma(D) \) of a digraph \( D \) is the least cardinality of a set \( X \) with respect to which \( D \) has a set coloring. Further, if \( f : V \to 2^X \) is a set coloring of \( D \) with \( |X| = \sigma(D) \) we call \( f \) an optimal set coloring of \( D \).

**Theorem 1.3.** [3] For any digraph \( D, \lceil \log_2(q + 1) \rceil \leq \sigma(D) \leq p - 1 \), where \( \lfloor x \rfloor \) denotes the least integer not less than the real \( x \), and bounds are best possible.

In this paper, we find the set coloring number of unipath, some classes of digraphs which admit a strong(proper) set coloring and construction of a strongly set colorable directed tree.

## 2. SET COLORING NUMBER OF A DIGRAPH

In this section we find set coloring number of unipath.

**Definition 2.1.** A oriented path is called unipath if \( id(v) = ed(v) = 1 \) for every vertex \( v \) except the first and last of the oriented path.

**Theorem 2.2.** Given any positive integer \( n \geq 2, \sigma(\overrightarrow{P_{2^n}}) > n \).

**Proof.** Let the vertices of \( \overrightarrow{P_{2^n}} \) be denoted by \( v_1, v_2, \ldots, v_{2^n} \) such that \( f^*(v_i, v_{i+1}) = f(v_i) - f(v_{i+1}) \), \( \forall (v_i, v_{i+1}) \in E(\overrightarrow{P_{2^n}}) \). Let us assume that there exist a set coloring \( (f, f^*) \) of \( \overrightarrow{P_{2^n}} \) with respect to a set \( X \) of \( |X| = n \) and both \( f \) and \( f^* \) are injective functions. That is sum of the number vertices and the number edges greater than \( 2^n \) which contradicts the fact that \( |X| = n \). Therefore \( \sigma(\overrightarrow{P_{2^n}}) > n \). \( \square \)
3. **Strongly (Properly) Set Colorable Directed Trees**

In this section we present some results on strong(proper) set colorings of some classes of directed trees.

**Definition 3.1.** A $n$-centipede $\overrightarrow{C_n}$ is a directed tree obtained by joining each vertex of the unipath to a pendent vertex whose in degree is zero.

**Theorem 3.2.** A directed centipede tree $\overrightarrow{C_n}$ is strongly set colorable if and only if $n = 2^k - 1$, where $k = 2, 3, 4$.

**Proof.** A directed centipede $\overrightarrow{C_n}$ has $n$ vertices and $n - 1$ arcs. Let $\overrightarrow{C_n}$ be strongly set colorable directed tree with respect to a set $X$ having $k$ colors. Then $|V(\overrightarrow{C_n})| + |E(\overrightarrow{C_n})| = 2^k - 1 \Rightarrow n + (n - 1) = 2^k - 1 \Rightarrow n = 2^k - 1$.

Conversely, let $\overrightarrow{C_n}$ be a directed tree such that $n = 2^k - 1$. Let $X = \{1, 2, 3, ..., k\}$. Also let $X_1 = \{1, 2, 3, ..., k\}$, the full set of $X$ and $X_2$ is a subset containing $k - 1$ elements of $X$ which doesn’t contain the element $a, a \in X$. Then assign the set $X_1$ to the sink of $\overrightarrow{C_n}$, that is vertex $v$ of $\overrightarrow{C_n}$ where $od(v) = 0$. Also assign the set $X_2$ to the vertex say $u$ adjacent to $v$ and $id(u) = 0$ and the remaining subsets of $X$ to the $n - 2$ vertices of $\overrightarrow{C_n}$. Then one can observe that the elements on the arcs are also subsets of $X$ and together form the set of all nonempty subsets of $X$. Hence $\overrightarrow{C_n}$ is strongly set colorable. \[\square\]

**Remark 3.3.** A directed centipede tree $\overrightarrow{C_n}$ is not strongly set colorable if and only if $n = 2^k - 1$, where $k > 4$.

**Theorem 3.4.** A directed centipede tree $\overrightarrow{C_n}$ is properly set colorable if and only if $n = 2^k$, where $k = 2, 3, 4$.

**Proof.** A directed centipede $\overrightarrow{C_n}$ has $n$ vertices and $n - 1$ arcs. Let $\overrightarrow{C_n}$ be properly set colorable directed tree with respect to a set $X$ having $k$ colors. Then $|E(\overrightarrow{C_n})| = 2^k - 1 \Rightarrow (n - 1) = 2^k - 1 \Rightarrow n = 2^k$.

Conversely, let $\overrightarrow{C_n}$ be a directed tree such that $n = 2^k$. Let $X = \{1, 2, 3, ..., k\}$. Also let $X_1 = \{1, 2, 3, ..., k\}$, the full set of $X$, assigned to the sink i.e., vertex $v$ of $\overrightarrow{C_n}$ where $od(v) = 0$ and assign empty set to the source i.e., vertex $u$ of $\overrightarrow{C_n}$ where $id(u)$. Let $X_2$ is a subset containing $k - 1$ elements of $X$ which doesn’t contain the element $a, a \in X$. Also assign the set $X_2$ to the
vertex say $v_1$ adjacent to $v$ and the remaining subsets of $X$ to the $n - 2$ vertices of $\overrightarrow{Cn}$. Then one can observe that the elements on the arcs are also subsets of $X$ and form the set of all nonempty subsets of $X$. Hence $\overrightarrow{Cn}$ is properly set colorable. □

**Remark 3.5.** A directed centipede tree $\overrightarrow{Cn}$ is not properly set colorable if and only if $n = 2^k$, where $k > 4$.

**Figure 1.** Properly and strongly set colorable directed centipede $\overrightarrow{Cn}$

**Definition 3.6.** Let $\overrightarrow{S_k}$ be a directed star with $k$ vertices such that $od(w) = 0$. The directed banana tree $\overrightarrow{B}(n,k)$ is a directed tree obtained by joining one leaf of each $n$ copies of a $k$-star $\overrightarrow{S_k}$ to a single vertex $w_0$ where $od(w_0) = 0$.

**Theorem 3.7.** For any positive integer $r \geq 1$ a directed banana tree $\overrightarrow{B}(n,k)$ is strongly set colorable if and only if $n = 2^r \backslash 2 - 1$ and $k = 2^r \backslash 2 + 1$.

**Proof.** A directed banana tree $\overrightarrow{B}(n,k)$ has $n$ vertices and $n - 1$ arcs. Let $\overrightarrow{B}(n,k)$ be strongly set colorable with respect to a set $X$ having $k$ colors. Then $|V(\overrightarrow{B}(n,k))| + |E(\overrightarrow{B}(n,k))| = 2^k - 1 \Rightarrow n + (n - 1) = 2^k - 1 \Rightarrow n = 2^{k-1}$.
Conversely, let $w_0$ be the root vertex of $\overrightarrow{B}(n,k)$ and $w_1, w_2, w_3, \ldots, w_k$ be the central vertices of the $k$-stars joining the central vertex. Let $u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,n}$ denote the pendant vertices joining $w_i, 1 \leq i \leq k$. Let $X$ be a non-empty set with $|X| = k$. Let $X_1$ be the full set of $X$ and $X_2$ be the $(k-1)$-element set of $X$. Then we define a mapping $f : V(\overrightarrow{B}(n,k)) \cup E(\overrightarrow{B}(n,k)) \rightarrow 2^X \setminus \emptyset$ as follows $f(w_0) = X_1$, $f(w_i) = A$, where $A \subseteq X_2$ and $f(u_{i,j}) = B$, where $B$ is the remaining $(k-1)$-element sets and $(k-2)$-element sets. Since $A$ and $B$ are disjoint, vertices are assigned by the distinct subsets. Therefore the mapping $f$ is injective. Hence the $\overrightarrow{B}(n,k)$ is strongly set colorable.

**Definition 3.8.** A lobster $Lb$ is a tree in which all the vertices are within the distance 2 of a central path. A directed lobster $\overrightarrow{Lb}$ is a oriented tree which gives a directed caterpillar when all its pendant vertices are deleted.

**Theorem 3.9.** A directed lobster $\overrightarrow{Lb}$ is strongly set colorable if and only if $n = 2^{k-1}, k \geq 5$. 

**Figure 2.** Strong set colorable directed banana tree $\overrightarrow{B}(7,9)$. 
Proof. A directed Lobster $\overrightarrow{Lb}$ has $n$ vertices and $n - 1$ arcs. Let $\overrightarrow{Lb}$ be strongly set colorable with respect to a set $X$ having $k$ colors. Then $|V(\overrightarrow{Lb})| + |E(\overrightarrow{Lb})| = 2^k - 1 \Rightarrow n + (n - 1) = 2^k - 1 \Rightarrow n = 2^{k-1}$.

Conversely, let $\overrightarrow{Lb}$ be a directed lobster such that $n = 2^{k-1}$. Let $v$ be the central vertex of $\overrightarrow{Lb}$ and $od(v) = 0$. Let $X = \{1, 2, ..., k\}$, the full set of $X$. Let $P$ be the longest path from $v$. Then assign $(k - 1)$-elements subsets of $X$ say, $A$ to the vertices of $P$. Let $N_1$ be the set of all vertices which are at a distance one from $P$. Then assign remaining $(k - 1)$-elements subsets of $X$ together with $(k - 2)$-elements subsets of $X$ other than $A$ say, $B$ to the vertices of $N_1$. Let $N_2$ be the set of all pendant vertices of $\overrightarrow{Lb}$. Then assign $(k - 2)$-elements subsets of $X$ together with the remaining subsets of $X$ other than $A$ and $B$ say, $C$ to the vertices of $N_2$. Assign all the subsets of $X$ which contains the element $a$, except the singleton set $a$, to the remaining vertices of $\overrightarrow{Lb}$. Hence $\overrightarrow{Lb}$ is strongly set colorable. \qed

\textbf{Figure 3.} Strong set colorable directed Lobster $\overrightarrow{Lb}$.

\textbf{Theorem 3.10.} Every properly set colorable directed tree is strongly set colorable.

\textit{Proof.} Let $\overrightarrow{T_n}$ be a properly set colorable tree with proper set coloring with proper coloring $f$ with respect to a set $X$ of cardinality $m$. Let $X' = X \cup \{x\}$. Since $\overrightarrow{T_n}$ is properly set colorable, all the subsets of $X$ are assigned to the vertices and $f(\overrightarrow{T_n}) = \{f(v) : v \in V(\overrightarrow{T_n})\} = 2^X$ and $f^*(\overrightarrow{T_n}) = \{f(e) : e \in E(\overrightarrow{T_n})\} = 2^X \setminus \emptyset$.

Define a function $F : V(\overrightarrow{T_n}) \rightarrow 2^{X'}$ by $F(v) = f(v) \cup \{x\}$ for all $v \in V(\overrightarrow{T_n})$. Since $f$ and $f^*$ are injective, $F$ and $F^*$ are also injective. Also $F(V(\overrightarrow{T_n})) \cap F^*(E(\overrightarrow{T_n})) = \emptyset$. 


Since \( f(\overrightarrow{T_n}) = 2^X \) and \( f^*(\overrightarrow{T_n}) = 2^X \setminus \emptyset \), we get \( F(\overrightarrow{T_n}) = 2^{X'} - 2^X \) and \( F^*(\overrightarrow{T_n}) = 2^X \setminus \emptyset \). That is, \( f^*(\overrightarrow{T_n}) = F^*(\overrightarrow{T_n}) \). Further, \( |F(\overrightarrow{T_n})| = 2^{|X'|} - 2^{|X|} = 2^{m+1} - 2^m = 2^m(2 - 1) = 2^m \) and \( |F^*(\overrightarrow{T_n})| = 2^m - 1 \).

Therefore \( |F(\overrightarrow{T_n})| + |F^*(\overrightarrow{T_n})| = 2^m + 2^m - 1 = 2^{m+1} - 1 = 2^{|X'|} - 1 \). This implies that \( F \) is a strong set coloring of \( \overrightarrow{T_n} \).

\( \square \)

4. CONSTRUCTION OF STRONGLY SET COLORABLE DIRECTED TREE BY ADDING AN ARC TO THE ROOT VERTEX OF A COMPLETE BINARY TREE

Given below is a construction of a strongly set colored directed tree.

**Definition 3.11.** A directed tree \( \overrightarrow{T_n} \) with \( n \) vertices is said to be multi-scale if an arc is added to the root vertex of a binary tree, where the outdegree of the root vertex and indegree of all the pendant vertices of \( \overrightarrow{T_n} \) is 0.

Next, we give the construction of an infinite family of strongly set colorable directed tree by adding an arc to the root vertex of a binary tree.

Let \( X_1 \) be a non-empty set with \( |X_1| = m_1 \), where \( m_1 = 3 \) is a positive integer. Consider \( K_{1,2^{m_1-1}} = \overrightarrow{T_0}(m_1) \) with \( \text{od}(v) = 0 \), where \( v \) is the central vertex. Let \( v_1, v_2, \ldots, v_{2^{m_1-1}-1} \) be the pendant vertices of \( \overrightarrow{T_0}(m_1) \). We define a mapping \( f_1 : V(\overrightarrow{T_0}(m_1)) \to 2^{X_1} \) as follows:

\[
\begin{align*}
  f_1(v) &= \{X_1\} \\
  f_1(v_i) &= A_r, \quad \text{where} \ A_r \text{ is a} (m_1 - 1) \text{element subset of} \ X_1 \text{ for} \ i = 1, 2, \ldots, 2^{m_1-1} - 1. \\
  f_1(v_{2^{m_1-1}-1}) &= X_1.
\end{align*}
\]

Clearly, \( f_1 \) and \( f_1^* \) are injective functions. Let \( X_2 \) be a set of cardinality \( m_2 \), where \( m_2 > m_1 \).

Introduce new vertices \( u_1, u_1, u_1, \ldots, u_{1,k_1} \), where \( k_1 = 2^{m_2-1} - 2^{m_1-1} \). Join each pair of these vertices to \( v_2, v_3, \ldots, v_{2^{m_1-1}-1} \) such that \( id((u_1,k_1)) = 0 \). Let the resulting directed tree be denoted by \( \overrightarrow{T_1}(m_2) \) and define the mapping \( f_2 : V(\overrightarrow{T_1}(m_2)) \to 2^{X_2} \) as follows:

\[
\begin{align*}
  f_2(v) &= \{X_1\} \cup \{m_2\} = A. \\
  f_2(v_i) &= A_r \cup \{m_2\} = A' \text{ for} \ i = 1, 2, \ldots, 2^{m_1-1} - 1. \\
  f_2(v_{2^{m_1-1}-1}) &= X_1 \cup \{m_2\} = X_2. \\
  f_2(u_{i,k_1}) &= B_r, \quad B_r \subset X_2, B_r \neq A'_r \text{ for} \ i' = 1, 2, \ldots, k_1. \\
  f_2(u_{1,k_1}) &= X_2 - \{x_0\} = B.
\end{align*}
\]
Let $f^*_2 : E(\overrightarrow{T}_1'(m_2)) \rightarrow 2^{X_2}$ denote the induced edge function defined by $f^*_2(u,v) = f_2(v) - f_2(u)$ where $(u,v) \in E(\overrightarrow{T}_1'(m_2))$. Then one can easily verify that both $f_2$ and $f^*_2$ are injective functions and hence $\overrightarrow{T}_1'(m_2)$ is strongly set colorable.

Let $X_3$ be a set of cardinality $m_3$, where $m_3 > m_2 > m_1$.

Introduce $u_{2,1}, u_{2,2}, \ldots, u_{1,k_2}$, where $k_2 = 2^{m_3-1} - 2^{m_2-1}$, join two of them to $u_{1,1}, u_{1,2}, \ldots, u_{1,k_1}$ where $id(u_{1,k_1}) = 0$. Let the resulting directed tree be $\overrightarrow{T}_2'(m_3)$. Define the mapping $f_3 : V(\overrightarrow{T}_2'(m_3)) \rightarrow 2^{X_3}$ as follows:

$f_3(v) = X \cup \{m_3\} = X_3$.

$f_3(v_i) = A'_r \cup \{m_3\} = A''_r$, $i < 2^{m_1-1} - 1$.

$f_3(v_{2^{m_1-1}-1}) = X_2$.

$f_3(u_{1,i'}) = B'_r \cup \{m_3\} = B''_r$, $i' = 1, 2, \ldots, k_1$.

$f_3(u_{1,k_1}) = X_3$.

$f_3(u_{2,i''}) = C_r, C_r \subset X_3$, $C_r \neq A''_r$, $C_r \neq B'_r$ and $m_3 \in C_r$ for $i'' = 1, 2, \ldots, k_2$.

$f_3(u_{2,k_2}) = X_3 - \{x_0\} = C$.

Let $f^*_3 : E(\overrightarrow{T}_2'(m_3)) \rightarrow 2^{X_3}$ denote the induced edge function defined by $f^*_3(u,v) = f_3(v) - f_3(u)$, where $(u,v) \in E(\overrightarrow{T}_1'(m_3))$. Then one can easily verify that both $f_3$ and $f^*_3$ are injective functions and hence $\overrightarrow{T}_2'(m_3)$ is strongly set colorable.

We can continue this procedure indefinitely to obtain the strongly set colorable directed tree at the $n^{th}$ step where $m_n > m_{n-1} > \ldots > m_3 > m_2 > m_1$ are chosen arbitrarily.

**Figure 4.** Strong set colorable digraph of complete oriented binary tree $\overrightarrow{T}_2'(m_3)$. 
The author(s) declare that there is no conflict of interests.

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