ELLIPSTIC CURVES AND A NEW CONSTRUCTION OF INTEGRABLE SYSTEMS

VLADIMIR DRAGOVIĆ AND BORISLAV GAJIĆ

Abstract. A class of elliptic curves with associated Lax matrices is considered. A family of dynamical systems on $e(3)$ parametrized by polynomial $a$ with above Lax matrices are constructed. Five cases from the family are selected by the condition of preserving the standard measure. Three of them are Hamiltonian. It is proved that two other cases are not Hamiltonian in the standard Poisson structure on $e(3)$. Integrability of all five cases is proven. Integration procedures are performed in all five cases. Separation of variables in Sklyanin sense is also given. A connection with Hess-Appel’rot system is established. A sort of separation of variables is suggested for the Hess-Appel’rot system.

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1. Introduction

Starting from a class of elliptic curves we construct a class of systems on the Lie algebra $e(3)$ by the Lax representation. Equations of motion depend on an arbitrary polynomial $a$ and have the form:

$$\dot{x} = \{x, H_1\} + a\{x, H_2\}$$

From the condition that the standard measure is preserved, we obtain five choices for $a$. In three cases, the systems are Hamiltonian. However, in two other cases they are not Hamiltonian in the standard Poisson structure on $e(3)$. Integrability in all five cases is performed. For all five cases classical integration procedure is presented. For the first three cases we gave an algebro-geometric integration procedure also. Using the Lax matrix, a
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2. LAX REPRESENTATIONS AND ELLIPTIC CURVES

Let us start from a class of elliptic curves

\[ \mu^2 = -p - \frac{a}{\lambda} - \frac{b}{\lambda^2} - \frac{c}{\lambda^3} - \frac{d}{\lambda^4}. \]

They have the following representation:

\[ \mu^2 = \frac{\omega^2(\lambda)}{\lambda^4} - 2\frac{\Delta(\lambda)\Delta^*(\lambda)}{\lambda^4}, \]

where

\[ \omega(\lambda) = -i(q\lambda^2 + x_1\lambda + y_1), \quad \Delta(\lambda) = x\lambda + y, \quad \Delta^*(\lambda) = \bar{x}\lambda + \bar{y}. \]

Such curves appear as spectral curves of Lax matrices of the form

\[ L(\lambda) = \begin{bmatrix} \frac{\omega(\lambda)}{\lambda^2} & \sqrt{2i}\frac{\Delta(\lambda)}{\lambda^2} \\ \sqrt{2i}\frac{\Delta^*(\lambda)}{\lambda^2} & -\frac{\omega(\lambda)}{\lambda^2} \end{bmatrix}. \]

We will consider a case that corresponds to real coefficients \(a, b, c, d\).

Let the functions \(M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3\) be generators of Lie algebra \(e(3)\) with Poisson structure defined by the relations

\[ \{M_i, M_j\} = -\epsilon_{ijk}M_k, \quad \{M_i, \Gamma_j\} = -\epsilon_{ijk}\Gamma_k, \quad i, j, k = 1, 2, 3. \]

Assume the following change of variables:

\[ y = \frac{1}{\sqrt{2}}(\beta\Gamma_1 - \alpha\Gamma_3 - i\Gamma_2), \quad x = \frac{1}{\sqrt{2}}(\beta M_1 - \alpha M_3 - iM_2), \]

\[ \bar{y} = \frac{1}{\sqrt{2}}(\beta\Gamma_1 - \alpha\Gamma_3 + i\Gamma_2), \quad \bar{x} = \frac{1}{\sqrt{2}}(\beta M_1 - \alpha M_3 + iM_2), \]

\[ y_1 = \alpha\Gamma_1 + \beta\Gamma_3, \quad x_1 = \alpha M_1 + \beta M_3, \quad q = I_2 \sqrt{x_0^2 + z_0^2}, \]

where \(\alpha = \frac{x_0}{\sqrt{x_0^2 + z_0^2}}, \beta = \frac{z_0}{\sqrt{x_0^2 + z_0^2}},\) and \(x_0, z_0, I_2\) are constants.

In terms of \(x, y, \bar{x}, \bar{y}, x_1, y_1\) the Poisson structure \([4]\) has the form

\[ \{x, y\} = 0, \quad \{\bar{x}, \bar{y}\} = 0, \quad \{x, x_1\} = ix, \quad \{\bar{x}, x_1\} = -i\bar{x}, \quad \{y, x_1\} = iy, \]

\[ \{\bar{y}, y_1\} = -i\bar{y}, \quad \{y_1, x_1\} = 0, \quad \{\bar{x}, y_1\} = -i\bar{x}, \quad \{x, y_1\} = iy, \quad \{y_1, y\} = 0, \]

\[ \{y_1, \bar{y}\} = 0, \quad \{x, \bar{x}\} = -i\bar{x}_1, \quad \{\bar{y}, y\} = 0, \quad \{x, y\} = -iy_1, \quad \{\bar{x}, y\} = iy_1. \]

To each elliptic curve \([2]\) we correspond a family of dynamical systems:

\[ \dot{L}(\lambda) = \frac{1}{2I_2} \left[ L(\lambda), \frac{\lambda^2L(\lambda) - a^2L(a)}{\lambda - a} \right]. \]

The matrix \(L(\lambda)\) in \([5]\) is of the form \([3]\) and \(a\) is an arbitrary polynomial in generators of algebra \(e(3)\).
Observe that matrices $L$ given by (3) satisfy
\begin{equation}
\left\{ \frac{1}{L}(\lambda), \frac{2}{L}(\mu) \right\} = \left[ r(\lambda - \mu), \frac{1}{L}(\lambda) + \frac{2}{L}(\mu) \right],
\end{equation}
where
\begin{align*}
\frac{1}{L}(\lambda) &= L(\lambda) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\frac{2}{L}(\mu) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes L(\mu),
\end{align*}
with permutation matrix as an $r$-matrix
\begin{equation*}
\frac{1}{r}(\lambda) = -\frac{1}{\lambda} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{equation*}

### 3. Equations of motion

The aim of this section is to provide the initial analysis of dynamical systems (5).

The Poisson bracket (4) is degenerative. As it is well known, there are two Casimir functions:
\begin{equation}
F_1 = M_1 \Gamma_1 + M_2 \Gamma_2 + M_3 \Gamma_3, \quad F_2 = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2.
\end{equation}
Thus, a symplectic leaf, defined by conditions $F_1 = c_1, F_2 = c_2$ is a four-dimensional manifold. For integrability in Liouville sense on $e(3)$, one first integral more beside the Hamiltonian is necessary. On the other hand, if a system is not Hamiltonian, generally speaking, five first integrals of motion for integrability in quadratures are required. But, if a nonhamiltonian system has an invariant measure, then, according to the Jacobi theorem, for integrability one needs only four first integrals of motion.

Equations (5) can be rewritten in the form
\begin{align*}
\dot{M}_1 &= z_0 \Gamma_2 + a z_0 M_2, \\
\dot{M}_2 &= x_0 \Gamma_3 - z_0 \Gamma_1 + a (x_0 M_3 - z_0 M_1), \\
\dot{M}_3 &= -x_0 \Gamma_2 - a x_0 M_2, \\
\dot{\Gamma}_1 &= \frac{\Gamma_2 M_3 - \Gamma_3 M_2}{I_2} + a z_0 \Gamma_2, \\
\dot{\Gamma}_2 &= \frac{\Gamma_3 M_1 - \Gamma_1 M_3}{I_2} + a (x_0 \Gamma_3 - z_0 \Gamma_1), \\
\dot{\Gamma}_3 &= \frac{\Gamma_1 M_2 - \Gamma_2 M_1}{I_2} - a x_0 \Gamma_2.
\end{align*}

We have the following Proposition.

**Proposition 1.** System (8) can be rewritten as:
\begin{align*}
\dot{M}_i &= \{ M_i, H_1 \} + a \{ M_i, H_2 \}, \\
\dot{\Gamma}_i &= \{ \Gamma_i, H_1 \} + a \{ \Gamma_i, H_2 \}, \quad i = 1, 2, 3,
\end{align*}
where
\[ H_1 = \frac{M_1^2 + M_2^2 + M_3^2}{2I_2} + (x_0 \Gamma_1 + z_0 \Gamma_3), \quad H_2 = x_0 M_1 + z_0 M_3. \]

For a general polynomial \( a \), the system (8) is neither Hamiltonian in the Poisson structure (4), nor preserves the standard measure. Simple criterion for preserving the standard measure is given by:

**Proposition 2.** The system (8) preserves the standard measure if and only if the polynomial \( a \) satisfies the condition:
\[ \{ a, x_0 M_1 + z_0 M_3 \} = 0. \]

**Proof.** The divergence of the vector field in (8) is equal to \( \{ a, x_0 M_1 + z_0 M_3 \} \). □

As a consequence of Proposition 2 we have:

**Proposition 3.** In the following five cases, the standard measure is preserved

(i): if the polynomial \( a \) is a Casimir function: \( a = M_1 \Gamma_1 + M_2 \Gamma_2 + M_3 \Gamma_3 \);
(ii): if the polynomial \( a \) is a Casimir function: \( a = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \);
(iii): if \( a = x_0 M_1 + z_0 M_3 \);
(iv): if \( a = x_0 \Gamma_1 + z_0 \Gamma_3 \);
(v): if \( a = M_2^1 + M_2^2 + M_3^2 \).

**Theorem 1.** If \( x_0 \neq 0 \), or \( z_0 \neq 0 \), in the first three cases given above, the systems are Hamiltonian, while in the fourth and the fifth case, the systems are not Hamiltonian in the Poisson structure (4).

**Proof.** If \( a \) is a Casimir function, for an arbitrary function \( f \) we have
\[ \{ f, H_1 \} + a \{ f, H_2 \} = \{ f, H_1 + aH_2 \}. \]
Hence, in the first two cases the systems are Hamiltonian with Hamiltonian functions
\[ H = H_1 + aH_2. \]
In the third case, since \( a = H_2 \), we have
\[ \{ x^i, H_1 \} + H_2 \{ x^i, H_2 \} = \{ x^i, H_1 + \frac{H_2^2}{2} \}, \]
where \( x^i, i = 1, ..., 6 \) are coordinates \( M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3 \). Thus, the system is also Hamiltonian with the Hamiltonian function
\[ H = H_1 + \frac{H_2^2}{2}. \]
The fourth and the fifth cases are more complicated. For a system (8) to be Hamiltonian, the polynomial \( a \) needs to satisfy
\[ a \{ x^i, H_2 \} = \{ x^i, H \} \]
for some function $H$. We get the condition

$$\sum_{j=1}^{6} \{x^i, x^j\} \left( a \frac{\partial H_2}{\partial x^j} - \frac{\partial H}{\partial x^j} \right) = 0. \tag{10}$$

In the fourth case, when $a = x_0 \Gamma_1 + z_0 \Gamma_3$, the system of linear partial differential equation (10) becomes

$$\begin{align*}
M_3 \frac{\partial H}{\partial M_2} - M_2 \frac{\partial H}{\partial M_3} + x_0 z_0 \Gamma_1 M_2 + z_0^2 \Gamma_3 M_2 + \Gamma_3 \frac{\partial H}{\partial \Gamma_2} - \Gamma_2 \frac{\partial H}{\partial \Gamma_3} &= 0, \\
M_1 \frac{\partial H}{\partial M_3} - M_3 \frac{\partial H}{\partial M_1} + x_0^2 \Gamma_1 M_3 + x_0 z_0 \Gamma_3 M_3 \\
- x_0 z_0 \Gamma_1 M_1 - z_0^2 \Gamma_3 M_1 - \Gamma_2 \frac{\partial H}{\partial \Gamma_1} + \Gamma_1 \frac{\partial H}{\partial \Gamma_3} &= 0, \\
M_2 \frac{\partial H}{\partial M_1} - M_1 \frac{\partial H}{\partial M_2} - x_0^2 \Gamma_1 M_2 - x_0 z_0 \Gamma_3 M_2 + \Gamma_2 \frac{\partial H}{\partial \Gamma_1} - \Gamma_1 \frac{\partial H}{\partial \Gamma_2} &= 0, \\
\Gamma_3 \frac{\partial H}{\partial M_2} - \Gamma_2 \frac{\partial H}{\partial M_3} + x_0 z_0 \Gamma_1 \Gamma_2 + z_0^2 \Gamma_3 \Gamma_2 &= 0, \\
\Gamma_1 \frac{\partial H}{\partial M_3} - \Gamma_3 \frac{\partial H}{\partial M_1} + x_0^2 \Gamma_1 \Gamma_3 + x_0 z_0 \Gamma_3^2 - x_0 z_0 \Gamma_1^2 - z_0^2 \Gamma_3 \Gamma_1 &= 0, \\
\Gamma_2 \frac{\partial H}{\partial M_1} - \Gamma_1 \frac{\partial H}{\partial M_2} - x_0^2 \Gamma_1 \Gamma_2 - x_0 z_0 \Gamma_3 \Gamma_2 &= 0. \tag{11}
\end{align*}$$

It is easy to see that the system (11) consists of only four independent equations

$$\begin{align*}
\Gamma_2 \frac{\partial H}{\partial M_1} &= \Gamma_1 \frac{\partial H}{\partial M_2} + x_0^2 \Gamma_1 \Gamma_2 + x_0 z_0 \Gamma_2 \Gamma_3, \\
\Gamma_2 \frac{\partial H}{\partial M_3} &= \Gamma_3 \frac{\partial H}{\partial M_2} + x_0 z_0 \Gamma_1 \Gamma_2 + z_0^2 \Gamma_2 \Gamma_3, \\
\Gamma_2 \frac{\partial H}{\partial \Gamma_1} &= \Gamma_1 \frac{\partial H}{\partial \Gamma_2} + M_1 \frac{\partial H}{\partial M_2} - \Gamma_1 M_2 \frac{\partial H}{\partial M_2}, \\
\Gamma_2 \frac{\partial H}{\partial \Gamma_3} &= \Gamma_3 \frac{\partial H}{\partial \Gamma_2} + M_3 \frac{\partial H}{\partial M_2} - \Gamma_3 M_2 \frac{\partial H}{\partial M_2}. \tag{12}
\end{align*}$$

From the first and the third equation (12), one gets

$$\frac{\partial^2 H}{\partial \Gamma_1 \partial M_1} - \frac{\partial^2 H}{\partial M_1 \partial \Gamma_1} = x_0^2.$$

If $x_0 \neq 0$, the condition (10) is not satisfied. Consequently, the system is not Hamiltonian. In the case when $z_0 \neq 0$ one gets contradiction by observing that $\frac{\partial^2 H}{\partial \Gamma_3 \partial M_3} - \frac{\partial^2 H}{\partial M_3 \partial \Gamma_3} \neq 0$. 
Following the same procedure given below, in the fifth case, when \( a = M_1^2 + M_2^2 + M_3^2 \) system (10) becomes (13)

\[
\begin{align*}
M_1 \frac{\partial H}{\partial M_2} - z_0 M_2 (M_1^2 + M_2^2 + M_3^2) + \Gamma_3 \frac{\partial H}{\partial \Gamma_2} - \Gamma_2 \frac{\partial H}{\partial \Gamma_3} &= 0, \\
M_2 \frac{\partial H}{\partial M_1} - M_1 \frac{\partial H}{\partial M_3} + x_0 M_3 (M_1^2 + M_2^2 + M_3^2) - z_0 M_1 (M_1^2 + M_2^2 + M_3^2) - \Gamma_3 \frac{\partial H}{\partial \Gamma_1} + \Gamma_1 \frac{\partial H}{\partial \Gamma_2} &= 0, \\
M_3 \frac{\partial H}{\partial M_1} - M_1 \frac{\partial H}{\partial M_2} + \Gamma_2 \frac{\partial H}{\partial \Gamma_1} - \Gamma_1 \frac{\partial H}{\partial \Gamma_2} &= 0.
\end{align*}
\]

System (13) has four independent equations

\[
\begin{align*}
\Gamma_2 \frac{\partial H}{\partial M_1} &= \Gamma_1 \frac{\partial H}{\partial M_3} + x_0 \Gamma_2 (M_1^2 + M_2^2 + M_3^2), \\
\Gamma_2 \frac{\partial H}{\partial M_3} &= \Gamma_3 \frac{\partial H}{\partial M_2} + z_0 \Gamma_2 (M_1^2 + M_2^2 + M_3^2), \\
\Gamma_2 \frac{\partial H}{\partial \Gamma_1} &= \Gamma_1 \frac{\partial H}{\partial \Gamma_2} + M_1 \frac{\partial H}{\partial M_2} - \Gamma_1 M_2 \frac{\partial H}{\partial M_2}, \\
\Gamma_2 \frac{\partial H}{\partial \Gamma_3} &= \Gamma_3 \frac{\partial H}{\partial \Gamma_2} + M_3 \frac{\partial H}{\partial M_2} - \Gamma_3 M_2 \frac{\partial H}{\partial M_2}.
\end{align*}
\]

From the first and the third equation (14), one get

\[
\frac{\partial^2 H}{\partial \Gamma_1 \partial M_1} - \frac{\partial^2 H}{\partial M_1 \partial \Gamma_1} = 2x_0 M_2 \Gamma_2 (M_1 \Gamma_2 - M_2 \Gamma_1).
\]

Consequently, in the fifth case system is not Hamiltonian. \(\square\)

Regarding integrability of given five cases, we have simple Proposition.

**Proposition 4.**

(a) A function \( F \) is a first integral of equations (8) if it satisfies

\[
\dot{F} = \{F, H_1\} + a\{F, H_2\} = 0.
\]

(b) The Casimir functions \( F_1 \) and \( F_2 \) and functions \( H_1 \) and \( H_2 \) are integrals of system (8) for any polynomial \( a \).

Finally we have:

**Theorem 2.** The system (8) in cases (i)-(iii) is completely integrable in Liouville sense. In cases (iv) and (v), system (8) is integrable in quadratures.
4. Algebro-geometric integration procedure of the systems

We pass to an algebro-geometric integration procedure, based on construction of the Baker-Akhiezer vector-function. For more details about the Baker-Akhiezer functions see [7, 8, 9].

The dynamical systems have Lax representation

\[ \dot{L}(\lambda) = [L(\lambda), A(\lambda)], \]

with \( L(\lambda) \) given by (3) and

\[ A(\lambda) = \frac{\lambda^2 L(\lambda) - a^2 L(a)}{2I_2(\lambda - a)}. \]

The corresponding spectral curve \( \Gamma \) is elliptic curve (2).

As usual, we consider the following eigenvalue problem

\[
\begin{align*}
\left( \frac{d}{dt} + A(\lambda) \right) \Psi(t, P) &= 0, \\
L(\lambda) \Psi(t, P) &= \mu \Psi(t, P),
\end{align*}
\]

with a normalization

\[
\Psi^1(0, P) + \Psi^2(0, P) = 1,
\]

where \( P = (\lambda, \mu) \) is a point on the spectral curve \( \Gamma \).

Let us denote by \( \Phi(t, \lambda) \) the fundamental solution of the system

\[
\left( \frac{d}{dt} + A(\lambda) \right) \Phi(t, \lambda) = 0,
\]

normalized by the condition \( \Phi(0, \lambda) = 1 \).

Let us denote by \( \infty^+ \) and \( \infty^- \) the two points on the curve \( \Gamma \) over \( \lambda = \infty \), with \( \mu = iI_2\sqrt{x_0^2 + z_0^2} \) and \( \mu = -iI_2\sqrt{x_0^2 + z_0^2} \) respectively.

Proposition 5. If the polynomial \( a \) is a first integral of motion, then the vector-function \( \Psi(t, P) \) satisfies the following conditions:

(a) In the affine part of the curve \( \Gamma \), the vector-function \( \Psi(t, P) \) has two time independent poles, and each of the components \( \Psi^1(t, P) \) and \( \Psi^2(t, P) \) has one zero.

(b) At the points \( \infty^+ \) and \( \infty^- \), the functions \( \Psi^1 \) and \( \Psi^2 \) have essential singularities with the following asymptotics:

\[
\begin{align*}
\Psi^1(t, P) &= \begin{cases} 
     e^{\frac{1}{2} \left( \sqrt{x_0^2 + z_0^2} (\lambda + a) + \frac{\pi}{2} \right) t} \left( 1 + O(\frac{1}{\lambda}) \right), & P \to \infty^- \\
     e^{-\frac{1}{2} \left( \sqrt{x_0^2 + z_0^2} (\lambda + a) + \frac{\pi}{2} \right) t} \left( O(\frac{1}{\lambda}) \right), & P \to \infty^+
\end{cases}, \\
\Psi^2(t, P) &= \begin{cases} 
     e^{\frac{1}{2} \left( \sqrt{x_0^2 + z_0^2} (\lambda + a) + \frac{\pi}{2} \right) t} \left( O(\frac{1}{\lambda}) \right), & P \to \infty^- \\
     e^{-\frac{1}{2} \left( \sqrt{x_0^2 + z_0^2} (\lambda + a) + \frac{\pi}{2} \right) t} \left( 1 + O(\frac{1}{\lambda}) \right), & P \to \infty^+
\end{cases}.
\end{align*}
\]
(c) The asymptotics have the form

\[
\Psi^1(t, P) = e^{-\frac{i}{2} \left( \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) + \frac{\lambda x_1}{2} \right)} \left( \frac{x}{I_2 \sqrt{2} \sqrt{x_0^2 + z_0^2}} \frac{1}{\lambda} + O(1/\lambda^2) \right), \quad P \rightarrow \infty^+
\]

\[
\Psi^2(t, P) = e^{\frac{i}{2} \left( \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) + \frac{\lambda x_1}{2} \right)} \left( -\frac{\bar{x}}{I_2 \sqrt{2} \sqrt{x_0^2 + z_0^2}} \frac{1}{\lambda} + O(1/\lambda^2) \right), \quad P \rightarrow \infty^-
\]

Proof. Since

\[
\Psi(t, P) = \Phi(t, \lambda) \Psi(0, P), \quad \Phi(0, \lambda) = 1,
\]

the poles of \( \Psi(t, P) \) coincides with poles of \( \Psi(0, P) \) (\( \Phi(t, \lambda) \) is holomorphic), so they are time-independent. Using normalization (16), and (15) one can calculate that \( \Psi(0, P) \) has two poles \( P_1 \) and \( P_2 \) in the affine part of \( \Gamma \). One can also conclude, using \( d \ln \Psi_1 \) and \( d \ln \Psi_2 \) that each of the components \( \Psi_1 \) and \( \Psi_2 \) has a one zero. This proves part (a).

In order to find asymptotic of \( \Psi^1 \) at \( \infty^+ \) and \( \infty^- \) one needs to consider

\[
(\ln(\Psi^1))^\prime = \frac{\Psi^1}{\Psi^1} = A_{11} - A_{12} \frac{\Psi^2}{\Psi_1}.
\]

Since \( \Psi^2 = \frac{\mu - L_{11}}{L_{12}} = \frac{L_{21}}{\mu - L_{22}} \), and using asymptotics

\[
\mu = i \left[ I_2 \sqrt{x_0^2 + z_0^2} \frac{x_1}{\lambda} + O(1/\lambda) \right], \quad P \rightarrow \infty^+,
\]

and

\[
\mu = -i \left[ I_2 \sqrt{x_0^2 + z_0^2} \frac{x_1}{\lambda} + O(1/\lambda) \right], \quad P \rightarrow \infty^-,
\]

we have:

\[
(\ln(\Psi^1))^\prime = \frac{i}{2} \sqrt{x_0^2 + z_0^2} \left( \lambda - a \right) - \frac{i x_1}{2 I_2} + \frac{i}{2} \sqrt{x_0^2 + z_0^2} y \frac{x}{x} + O(1/\lambda), \quad P \rightarrow \infty^+,
\]

and

\[
(\ln(\Psi^1))^\prime = \frac{i}{2} \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) + \frac{i x_1}{2 I_2} + O(1/\lambda), \quad P \rightarrow \infty^-.
\]

Since

\[
\frac{\dot{x}}{x} = ia \sqrt{x_0^2 + z_0^2} + i \sqrt{x_0^2 + z_0^2} \frac{y}{x},
\]

we finally get:

\[
(\ln(\Psi^1))^\prime = \frac{i}{2} \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) - \frac{i x_1}{2 I_2} + \frac{\dot{x}}{x} + O(1/\lambda), \quad P \rightarrow \infty^+,
\]

and

\[
(\ln(\Psi^1))^\prime = \frac{i}{2} \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) + \frac{i x_1}{2 I_2} + O(1/\lambda), \quad P \rightarrow \infty^-.
\]

Similarly, for the function \( \Psi^2 \) we have:

\[
(\ln(\Psi^2))^\prime = \frac{i}{2} \sqrt{x_0^2 + z_0^2} \left( \lambda + a \right) - \frac{i x_1}{2 I_2} + O(1/\lambda), \quad P \rightarrow \infty^+,
\]
and

$$(\ln(\Psi^2))' = \frac{i}{2} \sqrt{x_0^2 + z_0^2} (\lambda + a) + \frac{ix_1}{2I_2} + \frac{\dot{x}}{x} + O(1/\lambda), \quad P \to \infty^-.$$  

This proves parts (b) and (c). □

**Remark 1.** The vector-function $\Psi(t, P)$ satisfies standard conditions of the 2-point Baker-Akhiezer function. From the parts (a) and (b) of Proposition 5 we can construct $\Psi(t, P)$. Using the part (c), one can reconstruct $x(t)$ (and $\dot{x}(t)$) from the Baker-Akhiezer function.

**Remark 2.** All formulae from the proof of Proposition 5 are still valid even if the polynomial $a$ is not an integral of motion. But in that case statements (b) and (c) of the Proposition are not valid any more.

Now we will give explicit formulae for the Baker-Akhiezer function in terms of the Jacobi theta-function $\theta$. Explicit formulas are similar to the Lagrange case (see [11]).

Let us fix the canonical basis of cycles $A$ and $B$ on $\Gamma$ $(A \cdot B = 1)$, and let $\omega$ be the holomorphic differential normalized by the conditions

$$\oint_A \omega = 2i\pi, \quad \oint_B \omega = \tau.$$  

Theta-function $\theta_{11}(z|\tau)$ is defined by the relation

$$\theta_{11}(z|\tau) = \sum_{n=-\infty}^{\infty} \exp \left[ \frac{1}{2} \tau(n + \frac{1}{2}) + (z + i\pi)(n + \frac{1}{2}) \right]$$

and satisfies

$$\theta_{11}(0) = 0, \quad \theta_{11}(z + 2\pi i) = -\theta_{11}(z), \quad \theta_{11}(z + \tau) = -\exp(-z - \frac{1}{2}\tau)\theta_{11}(z).$$

Let $\Omega^+$ and $\Omega^-$ be differentials of the second kind with principal parts $-\frac{1}{2} \sqrt{x_0^2 + z_0^2} d\lambda$ and $+\frac{1}{2} \sqrt{x_0^2 + z_0^2} d\lambda$ at $\infty^+$ and at $\infty^-$ respectively, normalized by the condition that $A$-periods are zero. Let us introduce differential $\Omega = \Omega^+ + \Omega^-$. We will denote by $U$ the $B$-period of differential $\Omega$, and by $c^+$ and $c^-$ the constants:

$$\int_{P_0}^P \Omega = -\frac{i}{2} \sqrt{x_0^2 + z_0^2} \lambda + c^+ + O(1/\lambda), \quad P \to \infty^+$$

$$\int_{P_0}^P \Omega = +\frac{i}{2} \sqrt{x_0^2 + z_0^2} \lambda + c^- + O(1/\lambda), \quad P \to \infty^-.$$  

**Proposition 6.** The Baker-Akhiezer functions are given by

$$\Psi^1(t, P) = c_1 \exp \left[ \left( \int_{P_0}^P \Omega - c^- + \frac{i}{2} a + \frac{i x_1}{2T_2} \right)t \right] \theta_{11}(A(P + \infty^+ - P_1 - P_2) + tU) \quad \theta_{11}(A(\infty^+ + \infty^- - P_1 - P_2) + tU).$$

$$\Psi^2(t, P) = c_2 \exp \left[ \left( \int_{P_0}^P \Omega - c^- - \frac{i}{2} a - \frac{i x_1}{2T_2} \right)t \right] \theta_{11}(A(P + \infty^- - P_1 - P_2) + tU) \quad \theta_{11}(A(\infty^+ + \infty^- - P_1 - P_2) + tU).$$
where constants \( c_1 \) and \( c_2 \) are
\[
\begin{align*}
  c_1 &= \frac{\theta_{11}(\mathcal{A}(P - \infty))\theta_{11}(\mathcal{A}(\infty - P))\theta_{11}(\mathcal{A}(\infty - P))}{\theta_{11}(\mathcal{A}(\infty - \infty))\theta_{11}(\mathcal{A}(P - P))\theta_{11}(\mathcal{A}(P - P))}, \\
  c_2 &= \frac{\theta_{11}(\mathcal{A}(P - \infty))\theta_{11}(\mathcal{A}(\infty + P))\theta_{11}(\mathcal{A}(\infty + P))}{\theta_{11}(\mathcal{A}(\infty + \infty))\theta_{11}(\mathcal{A}(P - P))\theta_{11}(\mathcal{A}(P - P))},
\end{align*}
\]
and \( \mathcal{A} \) is the Abel map, and \( P_1 \) and \( P_2 \) are the poles of the function \( \Psi \).

**Proof.** The proof is based on general theory (see [7, 8, 9, 11]).

\[\square\]

5. **Classical integration of the systems**

We are going to show that in all considered five cases integration of the systems can be reduced to elliptic integrals.

Let us change coordinates by the rotation of \( xOz \) plane:
\[
\begin{align*}
  X_1 &= \alpha M_1 + \beta M_3 \\
  X_2 &= M_2 \\
  X_3 &= -\beta M_1 + \alpha M_3 \\
  Y_1 &= \alpha \Gamma_1 + \beta \Gamma_3 \\
  Y_2 &= \Gamma_2 \\
  Y_3 &= -\beta \Gamma_1 + \alpha \Gamma_3
\end{align*}
\]

Differential equations (8) of motion become
\[
\begin{align*}
  \dot{X}_1 &= 0 \\
  \dot{X}_2 &= \sqrt{x_0^2 + z_0^2}(Y_3 + aX_3) \\
  \dot{X}_3 &= -\sqrt{x_0^2 + z_0^2}(Y_2 + aX_2) \\
  \dot{Y}_1 &= \frac{1}{I_2}(X_3Y_2 - X_2Y_3) \\
  \dot{Y}_2 &= \frac{1}{I_2}(X_1Y_3 - X_3Y_1) + a\sqrt{x_0^2 + z_0^2}Y_3 \\
  \dot{Y}_3 &= \frac{1}{I_2}(X_2Y_1 - X_1Y_2) - a\sqrt{x_0^2 + z_0^2}Y_2.
\end{align*}
\]

(17)

The first integrals are
\[
\begin{align*}
  F_1 &= X_1Y_1 + X_2Y_2 + X_3Y_3 = c_1 \\
  F_2 &= Y_1^2 + Y_2^2 + Y_3^2 = c_2 \\
  H_1 &= \frac{X_1^2 + X_2^2 + X_3^2}{2I_2\sqrt{x_0^2 + z_0^2}} + Y_1 = d_1 \\
  H_2 &= X_1 = d_2.
\end{align*}
\]

(18)

From (17) we have:
\[
\begin{align*}
  \dot{X}_2^2 + \dot{X}_3^2 &= (x_0^2 + z_0^2)\left[(Y_2^2 + Y_3^2) + a^2(X_2^2 + X_3^2) + 2a(X_2Y_2 + X_3Y_3)\right] \\
  \dot{X}_2X_3 - \dot{X}_3X_2 &= \sqrt{x_0^2 + z_0^2}\left[X_2Y_2 + X_3Y_3 + a(X_3^2 + X_2^2)\right].
\end{align*}
\]
Using the first integrals \[18\], one gets
\[
\dot{X}_2^2 + \dot{X}_3^2 = \left[ c_2 - Y_1^2 + a^2(X_2^2 + X_3^2) + 2a(c_1 - d_2Y_1) \right]
\]
\[
\dot{X}_2X_3 - \dot{X}_3X_2 = \sqrt{x_0^2 + z_0^2} \left[ c_1 - d_2Y_1 + a(X_3^2 + X_2^2) \right].
\]
Introducing polar coordinates
\[
X_2 = \rho \cos \sigma, \quad X_3 = \rho \sin \sigma
\]
the last equations become
\[
\dot{\rho}^2 + \rho^2 \dot{\sigma}^2 = \left( x_0^2 + z_0^2 \right) \left[ c_2 - Y_1^2 + a^2 \rho^2 + 2a(c_1 - d_2Y_1) \right] - \rho^2 \dot{\sigma} = \sqrt{x_0^2 + z_0^2} \left[ c_1 - d_2Y_1 + a\rho \right].
\]
Let us subtract the square of the second equation multiplied by 4 from the first equation multiplied by \(4\rho^2\). After simplifying one gets
\[
\left[ \frac{d}{dt}(\rho^2) \right]^2 = 4(x_0^2 + z_0^2) \left[ \rho^2(c_2 - Y_1^2) - (c_1 - d_2Y_1)^2 \right].
\]
Finally, using
\[
Y_1 = d_1 - \frac{d_2^2 + \rho^2}{2I_2 \sqrt{x_0^2 + z_0^2}}
\]
and denoting \(\rho^2 = u\), we get
\[
\dot{u}^2 = -\frac{u^3}{I_2^2} - Bu^2 - Cu - D
\]
where
\[
B = \frac{-4A\sqrt{x_0^2 + z_0^2}}{I_2} + \frac{d_2^2}{I_2^2}
\]
\[
C = 4(x_0^2 + z_0^2) \left( A^2 - c_2 + \frac{d_2(c_1 - d_2A)}{I_2 \sqrt{x_0^2 + z_0^2}} \right)
\]
\[
D = 4(x_0^2 + z_0^2)(c_1 - d_2A)^2
\]
\[
A = d_1 - \frac{d_2^2}{2I_2 \sqrt{x_0^2 + z_0^2}}.
\]
So, the following proposition is proved:

**Proposition 7.** The function \( u(t) \) is an elliptic function of time.

Let us remark that \( u \) (and consequently \( \rho \)) does not depend of a choice of the polynomial \( a \).

Having \( u(t) \) as a known function of time, one can find \( \rho(t) \) as a known function of time. In order to reconstruct \( X_2 \) and \( X_3 \), one needs to find \( \sigma \) as a function of time.
From the second equation of (19), using (20), we have
\[
\dot{\sigma} = -\frac{1}{\rho^2(t)} \sqrt{x_0^2 + z_0^2} \left[ c_1 - d_2 \left( d_1 - \frac{d_2^2 + \rho^2(t)}{2I_2 \sqrt{x_0^2 + z_0^2}} \right) + a \rho^2(t) \right]
\]
The right hand side of the last equation is a function of time and of the polynomial \(a\). When \(a\) is a first integral of motion (in the first three cases defined above) e.q. when \(a = \text{const}\), then right hand side of the last equation is a known function of time and one can find \(\sigma\) by quadratures. In the fourth case
\[
a = x_0 \Gamma_1 + z_0 \Gamma_3 = \sqrt{x_0^2 + z_0^2} Y_1 = \sqrt{x_0^2 + z_0^2} d_1 - \frac{d_2^2 + \rho^2(t)}{2I_2 \sqrt{x_0^2 + z_0^2}}
\]
So, in this case \(a(t)\) is a known function of time and one can find \(\sigma\) by solving a differential equation. Similarly, in the fifth case
\[
a = M_1^2 + M_2^2 + M_3^2 = X_1^2 + X_2^2 + X_3^2 = d_2^2 + \rho^2(t)
\]
is again a known function of time and a differential equation for determining \(\sigma\) can be solved. Knowing \(\rho\) and \(\sigma\) as functions of time, one can reconstruct \(X_2\) and \(X_3\). From (20), one finds \(Y_1\) as a function of time. Finally, using the differential equation for \(Y_1\) from (17), and the second first integral \(F_2 = c_2\) from (18) one can reconstruct \(Y_2\) and \(Y_3\).

Two elliptic curves appeared here. The first one \(\Gamma\), has been defined by the equation (1), and it was the curve from which we started. The other one \(\Gamma'\), given by

(22)
\[
v^2 = -\frac{u^3}{I_2^2} - Bu^2 - Cu - D
\]
corresponds to the solution of the differential equation (21). A natural question is how these two curves are related. We have the following proposition:

Proposition 8. The elliptic curves \(\Gamma\), defined by the equation (1) and \(\Gamma'\)
defined by (22) are isomorphic.

Proof. A direct calculation gives that \(j\)-invariant of both curves are the same, so they are isomorphic.

6. Separation of variables

One of the oldest methods in the theory of integrable dynamical systems is separation of variables. Originally, this method was built in order to find exact solutions of the Hamilton-Jacobi equations. In the middle of 1990’s, Sklyanin in his celebrated paper [16], introduced a new concept of separation of variables which was related to modern techniques in the theory of integrable systems such as the inverse scattering method and the Lax representation.
If a completely integrable system on \(2n\)-dimensional symplectic manifold is given with \(n\) functionally independent commuting first integrals of motion \(F_1, \ldots, F_n\), then variables \(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n\) are separated if they are canonical e.q.

\[
\{\lambda_i, \mu_j\} = \delta_{ij}, \quad \{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0, \quad i, j = 1 \ldots n,
\]

and if there exist \(n\) relations \(\Phi_i\) such that

\[
(23) \quad \Phi_i(\lambda_i, \mu_i, F_1, \ldots, F_n) = 0, \quad i = 1, \ldots, n.
\]

The Sklyanin magic recipe gives separation variables \(\lambda_i\) as poles of the properly normalized Baker-Akhiezer function. The canonically conjugated variables are corresponding eigenvalues of the Lax matrix \(L(\lambda_i)\) (for details see [16]).

In the case of algebra \(e(3)\), since symplectic leaves are four-dimensional, one needs to find four separation variables \(\lambda_1, \lambda_2, \mu_1, \mu_2\).

**Proposition 9.** For the Hamiltonian systems (i), (ii), (iii) defined in Proposition 3 the separation variables are

\[
(24) \quad \lambda_1 = \frac{y}{x}, \quad \mu_1 = i \left( I_2 \sqrt{x_0^2 + z_0^2} - \frac{(\alpha M_1 + \beta M_3)x}{y} + \frac{(\alpha \Gamma_1 + \beta \Gamma_3)x^2}{y^2} \right),
\]

\[
\lambda_2 = x, \quad \mu_2 = -\frac{i}{x}(\alpha M_1 + \beta M_3).
\]

Corresponding separation relations are:

\[
(25) \quad \mu_1^2 = \frac{\omega(-\lambda_1)^2 - 2\Delta(-\lambda_1)\Delta^*(-\lambda_1)}{\lambda_1^2}, \quad \lambda_2\mu_2 = -iH_2 = \text{const}.
\]

**Proof.** Following [16] [14], the separation variables for the systems (3) satisfy the equation

\[
(26) \quad (1 \ 0) \text{adj}(L(\lambda) - \mu \cdot 1) = 0,
\]

which corresponds to the standard normalization \(\tilde{a}_0 = (1 \ 0)\). From (26) one gets only one pair of separation variables \(\lambda_1, \mu_1\). The other two variables \(\lambda_2, \mu_2\) are calculated from the asymptotics of the matrix \(L(\lambda)\) when \(\lambda\) goes to infinity.

By direct calculations we see that variables are canonical. \(\square\)

In terms of \(H_1, H_2, F_1, F_2\), the separation relations can be rewritten in the form:

\[
\mu_1^2 = \frac{1}{\lambda_1^2} \left( -I_2^2(x_0^2 + z_0^2)\lambda_1^4 + 2I_2 \sqrt{x_0^2 + z_0^2} H_2 \lambda_1^3 - 2I_2 H_1 \lambda_1^2 + 2F_1 \lambda_1 - F_2 \right),
\]

\[
H_2 = i\lambda_2\mu_2.
\]
7. Connection with Hess-Appel’rot system

In this section we find a sort of separation of variables for the Hess-Appel’rot case of motion of heavy rigid body about fixed point.

The equations of motion are the Euler-Poisson equations:

\[
\dot{M} = M \times \Omega + \Gamma \times \chi, \quad \dot{\Gamma} = \Gamma \times \Omega.
\]

Here \(M\) is the kinetic momentum vector, \(\Omega\) is the vector of angular velocity, \(\Gamma\) is the unit vertical vector, and \(\chi = (x_0, y_0, z_0)\) is the radius vector of the center of masses. Connection between \(M\) and \(\Omega\) is given by \(M = I \Omega\), where \(I\) is the inertia operator, and we can assume that \(I = \text{diag}(I_1, I_2, I_3)\).

Equations (27) are Hamiltonian on the Lie algebra \(\mathfrak{e}(3)\) in the standard Poisson structure given by (4) with the Hamiltonian function:

\[
H_{RB} = \frac{1}{2} \langle M, \Omega \rangle + \langle \Gamma, \Omega \rangle.
\]

Hence, for integrability in the Liouville sense of equations (27), one needs one first integral of motion more.

The Hess-Appel’rot case of rigid body is introduced by Hess and Appel’rot (see [1, 13]). It is defined by conditions

\[
y_0 = 0, \quad x_0 \sqrt{I_1(I_2 - I_3)} + z_0 \sqrt{I_3(I_1 - I_2)} = 0.
\]

Instead of the fourth integral, in the Hess-Appel’rot case there is an invariant relation

\[
x_0 M_1 + z_0 M_3 = 0.
\]

In [4] (see also [5]) a Lax representation for the Hess-Appel’rot system was found. The spectral curve is reducible. One component is a rational curve, and the other one is an elliptic curve. In order to find a sort of separation of variables for this specific situation, we give first another Lax representation. It is based on following observation:

**Proposition 10.** On hypersurface (30) the equations of the Hess-Appel’rot system are equivalent to the Lax representation (3) with \(a = \frac{\alpha \Omega_1 + \beta \Omega_2}{\sqrt{x_0^2 + z_0^2}}\).

This Lax representation for the Hess-Appel’rot system was given in [10]. The Lax representation of the same form for the Lagrange top was constructed in [15].

As a consequence of Propositions 10 and 9, we have:

**Proposition 11.** The separation variables for the Hess-Appel’rot system are

\[
\begin{align*}
\lambda_1 &= \frac{y}{x}, \quad \mu_1 = i \left( I_2 \sqrt{x_0^2 + z_0^2} - \frac{(\alpha M_1 + \beta M_3)x}{y} + \frac{(\alpha \Gamma_1 + \beta \Gamma_3)x^2}{y^2} \right), \\
\lambda_2 &= x, \quad \mu_2 = -\frac{i}{x}(\alpha M_1 + \beta M_3).
\end{align*}
\]
Corresponding separation relations are:
\[
\mu_2^2 = \frac{\omega (-\lambda_1)^2 - 2\Delta (-\lambda_1) \Delta^* (-\lambda_1)}{\lambda_1^4}, \quad \mu_2 = 0.
\]

In terms of \( H_1, H_2, F_1, F_2 \) separation relations can be rewritten in the form:
\[
\mu_2^2 = \frac{1}{\lambda_1^4} \left( -I_2^2 (x_0^2 + z_0^2) \lambda_1^4 + 2I_2 \sqrt{x_0^2 + z_0^2} H_2 \lambda_1^3 - 2I_2 H_1 \lambda_1^2 + 2F_1 \lambda_1 - F_2 \right),
\]
\[
i\lambda_2^2 \mu_2 = 0.
\]

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Vladimir Dragović
MATHEMATICAL INSTITUTE SANU
Kneza Mihaila 36, 11000 Belgrade, Serbia

and

Mathematical Physics Group, UNIVERSITY OF LISBON, PORTUGAL
e-mail: vladad@mi.sanu.ac.rs

Borislav Gajić
MATHEMATICAL INSTITUTE SANU
Kneza Mihaila 36, 11000 Belgrade, Serbia

e-mail: gajab@mi.sanu.ac.rs