Large Field Excursions from Dimensional (De)construction

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Abstract

An inflation model based on dimensional (de)construction of a massive gauge theory is proposed. The inflaton in this model is the “zero-mode” of a component of the massive gauge field in the (de)constructed extra dimensions. The inflaton potential originates from the gauge invariant Stueckelberg potential. At low energy, the field range of the inflaton is enhanced by a factor $N^{\frac{d}{2}}$ compared to the field range of the original fields in the model, where $d$ is the number of the (de)constructed extra dimensions and $N$ is the number of the lattice points in each (de)constructed dimension. This enhancement of the field range is used to achieve a trans-Planckian inflaton field excursion. The extension of the mechanism “excursions through KK modes” to the case of (de)constructed extra dimensions is also studied. The burst of particle productions by this mechanism may have observable consequences in a region of the model parameter space.
1 Introduction

Dimensional (de)construction\cite{1,2} provides a description of (latticized) extra dimensions purely in terms of 4D QFT. One of its important applications is a purely 4D construction\cite{3} of gauge-Higgs unification models\cite{4,5,6,7}. Most of the mechanisms for protecting the mass of the Higgs field against radiative corrections have also been used to make the inflaton potential sufficiently flat to realize slow-roll inflation. Therefore, it is natural to ask whether the (de)construction of extra-natural inflation\cite{8,9}, which incorporates the same mechanism used in gauge-Higgs unification to inflation, provides a viable model. However, it was already noticed in\cite{8} that large field inflation was difficult to achieve in (de)constructed extra-natural inflation. The main obstacle was that the trans-Planckian inflaton excursion requires the period of the inflaton potential $2\pi f_4$ to be much bigger than the (reduced) 4D Planck scale $M_P$: $2\pi f_4 \gg M_P$. However, in the (de)constructed model with one (de)constructed extra dimension, $f_4$ is related to a symmetry breaking scale $f$ in the model as $f_4 = f/\sqrt{N}$, where $N$ is the number of the lattice points in the (de)constructed dimension. As we do not have a valid description of the physics near the quantum gravity scale $M_P$, the model is restricted to have $f \ll M_P$, it follows that $f_4 \ll M_P$.

In this article, we consider the (de)construction of a massive gauge field theory. Due to the (de)constructed version of the Stueckelberg mechanism, the model can have gauge invariant potential for the gauge field in the (de)constructed extra dimensions. The inflaton is the “zero-mode” of a component of the gauge field in the (de)constructed extra dimensions. We consider a scenario in which the potential of the massive gauge field in
the (de)constructed extra dimensions dominates the inflaton potential, circumventing the problem mentioned above. Moreover, we find that the period of the inflaton potential is enhanced by a factor of $N^{d/2}$ compared with the potential for the original fields, where $d$ is the number of the (de)constructed extra dimensions and $N$ is the number of the lattice points in each (de)constructed dimension. The enhancement of the field range by a large number of fields in this model may be reminiscent of N-flation \cite{10}. However, the enhancement mechanisms are quite different in detail. In our model, the inflaton is a linear combination of the original fields, and the enhancement is due to the canonical normalization of many fields which have the same functional form of the potential due to the discrete translational symmetry on the lattice. On the other hand, in N-flation model, the inflaton was essentially the distance in the field space, and the enhancement of the field range is due to the Pythagoras theorem. In addition, in our model the (de)constructed version of KK reduction naturally selects the low energy degrees of freedom.

We also extend the rapid particle production mechanism of “excursions through KK modes” \cite{11} which periodically occurs as the inflaton travels a certain field distance, to the case in which the “KK” modes are the lattice counterpart of those in continuous extra dimensions.

The organization of this article is as follows: In Sec. 2, we introduce the (de)construction of a massive gauge theory with one (de)constructed extra dimension. The mass spectrum in a vacuum of this model (“KK” spectrum) is studied, followed by the construction of the low energy effective theory below the “KK” energy scale. Here, we derive one of our main results that the field range of the “zero-mode” of the gauge field in the (de)constructed direction is enhanced by the number of the lattice points. Then we generalize the model to the case with more (de)constructed extra dimensions. To better understand the field range enhancement mechanism in our model, we make a comparison with the corresponding model with continuous extra dimensions, and make clear the similarity and the difference. In Sec. 3 we construct a large field inflation model based on our (de)constructed massive gauge theory. The enhancement of the field range found in Sec. 2 is crucial for achieving a trans-Planckian inflaton field excursion. The rapid particle production mechanism “excursion through KK modes” is extended to the case of “KK” modes of the (de)constructed extra dimensions, and its cosmological consequences are analyzed. We conclude in Sec. 4 with summary and discussions. In Appendix A, we collect the relevant formulas of discrete Fourier transformation and also fix our convention.
2 (De)construction of a massive gauge theory

2.1 The action

We start with the following 4D action with one (de)constructed extra dimension:

\[ S_5 = \int d^4x \sum_{j=0}^{N-1} \left[ -\frac{1}{4} F_{\mu\nu}^{(j)} F_{\mu\nu}^{(j)} + \frac{f^2}{2} D_\mu U_{(j,j+1)} D^\mu U_{(j,j+1)}^\dagger + V_5(U_{(j,j+1)}, \theta_{(j)}, \theta_{(j+1)}) \\
+ D_\mu \chi_{(j)} D^\mu \chi_{(j)} - m^2 \chi_{(j)} \chi_{(j)} + f^2 \left( \gamma \chi_{(j)} U_{(j,j+1)} \chi_{(j+1)} + c.c. \right) + \ldots \right], \]

\[ (j = 0, 1, \cdots, N - 1 \text{ mod } N). \] (2.1)

The field \( U_{(j,j+1)}(x) \) takes value in \( U(1) \) and can be parametrized as

\[ U_{(j,j+1)}(x) = \exp \left[ i \frac{A_{(j,j+1)}(x)}{f} \right]. \] (2.2)

\( A_{(j,j+1)}(x) \) is an angular variable with the period \( 2\pi f \). The field \( \chi_{(j)} \) is a charged matter field which for simplicity we chose to be a scalar field. The action (2.1) has the product \( U(1)^N \) gauge symmetry. The gauge transformation generated by \( g_{(j)}(x) = e^{ig\alpha_{(j)}(x)} \) are given as

\[ A_{\mu,(j)}(x) \to A_{\mu,(j)}(x) - \partial_\mu \alpha_{(j)}(x), \] (2.3)

\[ U_{(j,j+1)}(x) \to g_{(j)}^{-1}(x) U_{(j,j+1)}(x) g_{(j+1)}(x), \] (2.4)

\[ \theta_{(j)}(x) \to \theta_{(j)}(x) + \alpha_{(j)}(x), \] (2.5)

\[ \chi_{(j)}(x) \to g_{(j)}^{-1}(x) \chi_{(j)}(x). \] (2.6)

The covariant derivatives in (2.1) are defined as

\[ D_\mu U_{(j,j+1)} = \partial_\mu U_{(j,j+1)} - ig A_{\mu,(j)} U_{(j,j+1)} + ig U_{(j,j+1)} A_{\mu,(j+1)}, \] (2.7)

\[ D_\mu \chi_{(j)} = \partial_\mu \chi_{(j)} - ig A_{\mu,(j)} \chi_{(j)}. \] (2.8)

The action (2.1) may arise as a low energy EFT of strongly coupled gauge theory below the chiral symmetry breaking scale [1], or as a low energy EFT of complex scalar fields which acquire vacuum expectation values [2]. In either case, the field \( A_{(j,j+1)}(x) \) in (2.2) is the Nambu-Goldstone boson arising from \( U(1) \) global symmetry breaking. In the meantime, the (de)constructed dimension has the same mathematical structure as the lattice in lattice gauge theory. In the terminology of lattice gauge theory, \( U_{(j,j+1)}(x) \) is the “link variable” on the lattice and \( A_{(j,j+1)}(x) \) is identified with the lattice gauge field in the
(de)constructed direction. Following the terminology of lattice gauge theory, we may define the “lattice spacing” \( a \) from (2.2) as

\[
a := \frac{1}{gf},
\]

since in lattice gauge theory the link variables are given as

\[
U_{(j,j+1)} = \exp \left[ i g A_{(j,j+1)} \right].
\]

We may also define the “radius” of the (de)constructed extra dimension \( L \) by

\[
L := \frac{Na}{2\pi} = \frac{N}{2\pi gf}.
\]

The field \( \theta_{(j)}(x) \) is a (de)constructed version of the Stueckelberg field which allows us to have a gauge invariant potential for the gauge field in the (de)constructed direction:

\[
V_S(A_{(j,j+1)}, \theta_{(j)}, \theta_{(j+1)}) = \sum_{p=0}^{\infty} i \tilde{V}_p (e^{i \theta_{(j)}} U_{(j,j+1)} e^{-i \theta_{(j+1)}})^p + c.c.
\]

\[
= \sum_{p=0}^{\infty} i \tilde{V}_p \exp ip \left[ \frac{A_{(j,j+1)}}{f} - (\theta_{(j+1)} - \theta_{(j)}) \right] + c.c.
\]

Since \( A_{(j,j+1)} \) is a 4\( N \) angular variable with the period \( 2\pi f \), (2.12) provides the Fourier series expansion of the gauge invariant potential of \( A_{(j,j+1)} \). As in lattice field theory, we have imposed symmetry under the discrete translation on the action (2.1):

\[
A_{\mu(j)} \rightarrow A_{\mu(j+1)},
\]

\[
U_{(j,j+1)} \rightarrow U_{(j+1,j+2)},
\]

\[
\theta_{(j)} \rightarrow \theta_{(j+1)},
\]

\[
\chi_{(j)} \rightarrow \chi_{(j+1)}.
\]

On the other hand, there is no symmetry which mixes the continuous four space-time dimensions and the (de)constructed extra dimension, i.e. there is no (1+4)D Lorentz symmetry. As a consequence, the 4D gauge fields do not need to have Stueckelberg mass. The lack of (1+4)D Lorentz symmetry gives rise to interaction terms which do not exist in (1+4)D Lorentz symmetric models. Some of these terms can be suppressed by imposing the parity symmetry in the (de)constructed extra dimension:

\[
j \rightarrow N - j \pmod{N},
\]

\[
U_{(j,j+1)} \rightarrow U_{(N-j,N-(j+1))},
\]
where we have defined $U_{(j+1,j)} := U^\dagger_{(j,j+1)}$. The first line (2.17) should be regarded as the change of the label of the lattice points so that in (2.18) the lattice point label $j + 1$ is transformed to $N - (j + 1)$. The parity symmetry forbids terms which contain odd number of the lattice counterparts of the derivative. The parity symmetry also forbids the imaginary part of the parameter $\gamma$ in (2.1) and $\tilde{V}_p$ in (2.12). “…” in the action (2.1) represents higher dimensional operators which will not be relevant at low energy (more specifically, we will eventually be interested in the EFT below the “KK” energy scale $1/L$).

2.2 The mass spectrum

Gauge field (space-time components)

The mass-square matrix of the gauge fields in the vacuum $U_{(j,j+1)} = 1$ can be read off from the action (2.1):

$$M^2_g := g^2 f^2 K,$$

(2.19)

where $K$ is the $N \times N$ matrix:

$$K := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 2 & -1 \\ -1 & \cdots & -1 & 2 \end{pmatrix}.$$  

(2.20)

The problem of finding the eigenvalues and the corresponding eigenstates of the matrix $K$ given in (2.20) is a familiar one which appears in the system of coupled harmonic oscillators. The fact that the index $j$ is a periodic discrete variable motivates us to use the discrete Fourier transform (our convention for the discrete Fourier transform as well as relevant formulas are summarized in Appendix A):

$$A_{\mu(j)} = \frac{1}{\sqrt{N}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \tilde{A}_{\mu(n)} e^{i\frac{2\pi nj}{N}},$$

(2.21)

for odd $N$, and

$$A_{\mu(j)} = \frac{1}{\sqrt{N}} \sum_{n=-\left(\frac{N}{2}-1\right)}^{\frac{N}{2}-1} \tilde{A}_{\mu(n)} e^{i\frac{2\pi nj}{N}} + \frac{1}{\sqrt{N}} \tilde{A}_{\mu\left(\frac{N}{2}\right)} (-)^j,$$

(2.22)
for even $N$. Since $A_{\mu(j)}$ is a real variable, the Fourier coefficients satisfy the relation $\tilde{A}_{\mu(-n)} = \tilde{A}_{\mu(n)}$. Note that $\tilde{A}_{\mu(0)}$, and $\tilde{A}_{\mu(N)}$ in the case of even $N$, are real. The matrix multiplication of $K$ in (2.20) to $A_{\mu(j)}$ in (2.21) or (2.22) gives

$$\sum_{k=0}^{N-1} K_{jk} A_{\mu(k)} = 2A_{\mu(j)} - A_{\mu(j+1)} - A_{\mu(j-1)}$$

$$= \frac{1}{\sqrt{N}} \sum_n \tilde{A}_{\mu(n)} e^{i 2\pi n j} \left( 2 - e^{i 2\pi n} - e^{-i 2\pi n} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_n \tilde{A}_{\mu(n)} e^{i 2\pi n j} 4 \sin^2 \frac{\pi n}{N}, \quad (2.23)$$

where the sum over $n$ should be taken as in (A.1) or (A.2) depending on whether $N$ is odd or even. From (2.23) we find $N$ eigenvectors $A^{m.e.v.}_{\mu(n)}$ of the mass-square matrix (2.19), whose $j$-th component is given by

$$(A^{m.e.v.}_{\mu(n)})_j = \tilde{A}_{\mu(n)} e^{i 2\pi n j}. \quad (2.24)$$

The corresponding mass-square eigenvalue of $A^{m.e.v.}_{\mu(n)}$ is given by

$$M_{g(n)}^2 = 4g^2 f^2 \sin^2 \frac{\pi n}{N}. \quad (2.25)$$

(2.25) provides the lattice counterpart of the KK mass spectrum in the continuous extra dimension: In the formal continuous limit $N \to \infty$ and $a = 1/gf \to 0$ with $L$ fixed, the mass eigenvalues (2.25) reduce to

$$M_{g(n)}^2 \to 4g^2 f^2 \left( \frac{\pi n}{N} \right)^2 = \left( \frac{n}{L} \right)^2, \quad (2.26)$$

when $n \ll N$. (2.26) coincides with the KK mass spectrum from a compactification on a continuous circle with radius $L$. Note that the limit is formal, because the parameters are constrained as $g \lesssim 1$, $f \ll M_P$ in order for the model to be valid.

**Gauge field ((de)constructed extra dimensional component)**

To find the mass-square eigenvalues for the field $A_{(j,j+1)}$, as in the case of the gauge field (space-time components), we consider the discrete Fourier transform of $A_{(j,j+1)}$:

$$A_{(j,j+1)} = \frac{1}{\sqrt{N}} \sum_n \tilde{A}_{(n)} e^{i 2\pi n j}, \quad (2.27)$$

where the sum over $n$ is taken as in (A.1) for odd $N$ and as in (A.2) for even $N$.

We impose an analogue of the Lorenz gauge condition including the (de)constructed extra-dimension:

$$\partial_{\mu} \tilde{A}^\mu_{(n)} - i M_{g(n)} \tilde{A}_{(n)} = 0, \quad (2.28)$$

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where \( M_{g(n)} \) is the positive square root of \( M_{g(n)}^2 \) in (2.25). In this gauge, apart from the Stueckelberg mass term arising from the potential (2.12), the field \( \tilde{A}_{(n)} \) acquires the “KK” mass term. This “KK” contribution \( M_{\tilde{A}(n)}^{2KK} \) to the mass-square of the field \( \tilde{A}_n \) is given as

\[
M_{\tilde{A}(n)}^{2KK} = 4g^2 f^2 \sin^2 \frac{\pi n}{N}. \tag{2.29}
\]

### Matter field

Below the “KK” energy scale \( 1/L \), an appropriate EFT description is obtained by integrating out fields which have mass greater than \( 1/L \). Therefore we replace all \( A_{(j,j+1)} \) to its “zero-mode”, i.e. the \( n = 0 \) term in (2.27):

\[
A_{(j,j+1)} \bigg|_{\tilde{A}(n)=0 \text{ except } n=0} = \frac{1}{\sqrt{N}} \tilde{A}(0) := \frac{1}{\sqrt{N}} A, \tag{2.30}
\]

where the Fourier coefficient \( A = \tilde{A}(0) \) is expressed as (see Appendix A)

\[
A = \tilde{A}(0) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} A_{(j,j+1)}. \tag{2.31}
\]

The part of the action (2.1) which involves the matter fields \( \chi_{(j)}(x) \) can be rewritten as

\[
S_{\chi} = \int d^4x \left[ D_{\mu} \chi^\dagger_{(j)} D^\mu \chi_{(j)} - (m^2 - 2\gamma f^2) \chi^\dagger_{(j)} \chi_{(j)} - \gamma f^2 (U_{(j,j+1)} \chi_{(j+1)} - \chi_{(j)})^\dagger (U_{(j,j+1)} \chi_{(j+1)} - \chi_{(j)}) \right]. \tag{2.32}
\]

From (2.32) we observe that the mass of the field \( \chi_{(j)} \) depends on the expectation value of \( A \). The mass-square matrix for the field \( \chi_{(j)} \) is given as

\[
M_{\chi}^2(A) = (m^2 - 2\gamma f^2) + \gamma f^2 K(A), \tag{2.33}
\]

where

\[
K(A) := \begin{pmatrix}
2 & -e^{i \frac{A}{\sqrt{N}}} & 0 & 0 & \cdots & -e^{-i \frac{A}{\sqrt{N}}}
\end{pmatrix}
\begin{pmatrix}
-e^{-i \frac{A}{\sqrt{N}}} & 2 & -e^{i \frac{A}{\sqrt{N}}} & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & -e^{-i \frac{A}{\sqrt{N}}} & 2 & -e^{i \frac{A}{\sqrt{N}}} & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 2 & -e^{i \frac{A}{\sqrt{N}}} & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
-e^{i \frac{A}{\sqrt{N}}} & \cdots & \cdots & \cdots & -e^{-i \frac{A}{\sqrt{N}}} & 2
\end{pmatrix} \tag{2.34}
\]

As before, to find the mass-square eigenvalues, we consider the discrete Fourier transform of \( \chi_{(j)} \):

\[
\chi_{(j)} = \frac{1}{\sqrt{N}} \sum_n \tilde{\chi}(n) e^{i \frac{2\pi nj}{N}}, \tag{2.35}
\]
where the sum over \(n\) is taken as in (A.1) or (A.2), according to whether \(N\) is odd or even. Multiplying the mass-square matrix to \(\chi(j)\), we obtain

\[
\sum_{k=0}^{N-1} K(A)_{jk} \chi(k) = 2\chi(j) - e^{i \frac{A}{\sqrt{N}} \chi(j+1)} - e^{-i \frac{A}{\sqrt{N}}} \chi(j-1)
\]

\[
= \frac{1}{\sqrt{N}} \sum_n \tilde{\chi}(n)e^{i \frac{2\pi nj}{N}} \left(2 - \exp \left[ i \left( \frac{2\pi n}{N} + \frac{A}{\sqrt{N} f} \right) \right] - \exp \left[ -i \left( \frac{2\pi n}{N} + \frac{A}{\sqrt{N} f} \right) \right] \right)
\]

\[
= \frac{1}{\sqrt{N}} \sum_n \tilde{\chi}(n)e^{i \frac{2\pi nj}{N}} 4\sin^2 \left[ \frac{1}{2\sqrt{N} f} \left( A + \frac{2\pi f n}{\sqrt{N}} \right) \right]
\]

\[
: = \frac{1}{\sqrt{N}} \sum_n \tilde{\chi}(n)e^{i \frac{2\pi nj}{N}} 4\sin^2 \left[ \frac{1}{2F} \left( A + 2\pi f_4 n \right) \right],
\]

where

\[
F := \sqrt{N} f,
\]

and

\[
f_4 := \frac{f}{\sqrt{N}}.
\]

From (2.36) we find the eigenvectors \(\chi_{m.e.v.}^{(n)}\) of the mass-square matrix (2.33), whose \(j\)-th component is given by

\[
(\tilde{\chi}_{m.e.v.})_j = \tilde{\chi}(n)e^{i \frac{2\pi nj}{N}}.
\]

The eigenvector \(\chi_{m.e.v.}^{(n)}\) has the eigenvalue \(M_{\chi(n)}^2(A)\) given by

\[
M_{\chi(n)}^2(A) = (m^2 - 2\gamma f^2) + M_{\chi(n,KK)}^2(A),
\]

where the “KK” mass spectrum of \(\chi\) is given by

\[
M_{\chi(n,KK)}^2(A) = 4\gamma f^2 \sin^2 \left( \frac{A + 2\pi f_4 n}{2F} \right).
\]

Unlike the real extra dimension whose radius appears the same for all the fields propagating in it, in (de)construction the “KK” mass spectrum of \(\chi\) given in (2.41) need not coincide with that of the gauge field even when \(A = 0\). The fields canonically normalized in \((1+3)D\) can have different coefficients for their “kinetic” term, i.e. the lattice counterpart of the kinetic term, in the (de)constructed extra dimension. In other words, different fields propagate in the (de)constructed extra dimension differently. This lack of universality in the (de)constructed extra dimension leads to the absence of Lorentz symmetry between the \((1+3)D\) space-time dimensions and the extra dimensions even in the formal continuum limit \(N \to \infty\). This breaking of universality/Lorentz symmetry in/involving the (de)constructed extra dimension is parametrized by the real parameter \(\gamma\). The universality in the (de)constructed extra dimension recovers at \(\gamma = g^2\), and only in this case their “KK” mass spectra coincide.
2.3 The low energy effective action below KK energy scale

Now we would like to have the low energy effective action of (2.1) which is suitable for describing the physics below the “KK” energy scale $1/L$. After integrating out the fields whose mass is above the “KK” energy scale $1/L$, we obtain the effective action $S_4$:

$$S_4 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu A \partial^\mu A + V^{(4)}_S(A) + \sum_n \left( D^{(4)}_\mu \tilde{\chi}^{\dagger}_n (D^{(4)}_\mu \tilde{\chi}_n - \tilde{\chi}^{\dagger}_n M_{\chi}^2 \tilde{\chi}_n) \right) \right]. \quad (2.42)$$

Here, the 4D gauge field $A_\mu$ is the “zero-mode” of the (de)constructed extra dimension:

$$A_\mu := \tilde{A}_\mu(0) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} A_{\mu(j)}. \quad (2.43)$$

The covariant derivative for the unbroken diagonal $U(1)$ is given by

$$D^{(4)}_\mu \tilde{\chi}_n = \partial_\mu \tilde{\chi}_n - ig_4 A_\mu \tilde{\chi}_n, \quad (2.44)$$

where the gauge coupling $g_4$ for the unbroken diagonal $U(1)$ is given by

$$g_4 := \frac{g}{\sqrt{N}}. \quad (2.45)$$

Since the mass of the charged fields $\tilde{\chi}_n$ (the range of $n$ is as in (A.1) or (A.2) depending on whether $N$ is odd or even) depend on the value of the field $A$, we do not know which of the field $\tilde{\chi}_n$ becomes light before knowing the value of $A$. Hence we keep all $\tilde{\chi}_n$ in the low energy effective action (2.42).

The (de)constructed version of the Stueckelberg field $\theta_{(j)}$ is expanded in discrete Fourier modes as

$$\theta_{(j)} = \frac{1}{\sqrt{N}} \sum_n \tilde{\theta}_n e^{2\pi nj/N} + \frac{2\pi w j}{N}. \quad (2.46)$$

Since $\theta_{(j)} (j = 0, 1, \cdots N - 1 \mod N)$ are angular variables: $\theta_{(j)} \sim \theta_{(j)} + 2\pi$, they can have a “winding number” $w$ which is an integer.\footnote{Those who are familiar with the Weak Gravity Conjecture (WGC) [12] might worry that by taking $N$ large, $g_4$ becomes too small so that the model is in tension with the conjecture that gravity is the weakest force. However, it has been shown that when WGC is respected at high energy, it is not violated by Higgs mechanism [13] [14] [15]. In the current case, if $g$ is not too small, the model would not have a tension with WGC.}

The Stueckelberg potential (2.12) leads
to the potential $V^{(4)}_S(A)$ for the “zero-mode” $A$:

$$V^{(4)}_S(A) = 2N \sum_{p=0}^{\infty} \tilde{V}_p \cos \left[ p \left( \frac{A}{\sqrt{Nf}} - \frac{2\pi w}{N} \right) \right] = 2N \sum_{p=0}^{\infty} \tilde{V}_p \cos \left[ \frac{p(A - 2\pi f w)}{F} \right].$$  \hfill (2.47)

(2.47) is one of our main results. Compared with the period $2\pi f$ of the original Stueckelberg potential (2.12), the period of the potential (2.47) enhanced by a factor $\sqrt{N}$. By taking $N$ large, the model may allow a large field excursion. We will discuss the application of this mechanism in cosmology in Sect. 3. In addition, the height of the potential $V^{(4)}_S$ is enhanced by a factor $N$ compared with the functional form of the original Stueckelberg potential (2.12).

In addition to the action (2.42), we should include the following “Wilson loop operator”

$$W(A) = \prod_{j=0}^{N-1} U_{(j,j+1)} = \exp \left[ i \sum_{j=0}^{N-1} A_{(j,j+1)} \right] = \exp \left[ i \frac{\sqrt{N} \tilde{A}_0}{f} \right] = \exp \left[ i \frac{A}{f_4} \right],$$ \hfill (2.48)

where $f_4$ is defined in (2.38). The Wilson loop operator is generated in the 1-PI effective action at 1-loop level as

$$c_{WL} f^2 \Lambda^2 \left( \frac{\gamma f^2}{\Lambda^2} \right)^N W(A) + c.c.,$$ \hfill (2.49)

where $\Lambda = 4\pi f$ is the UV energy scale at which the perturbative expansion of the EFT (2.42) breaks down. The natural magnitude of the coefficient $c_{WL}$ is $O(1)$. $c_{WL}$ is real as we have imposed a symmetry under the parity transformation (2.17), (2.18). The Wilson loop operator (2.48) contributes to the potential of the “zero-mode” $A$:

$$V_{WL}(A) = 2c_{WL} f^2 \Lambda^2 \left( \frac{\gamma f^2}{\Lambda^2} \right)^N \cos \left( \frac{A}{f_4} \right).$$ \hfill (2.50)

It is important to notice that the “KK” mass term of the charged fields $\tilde{\chi}(n)$ summed over the “KK” modes $n$:

$$\sum_n \tilde{\chi}^\dagger_{(n)} M_{KK}^2 (A) \tilde{\chi}_{(n)} ,$$ \hfill (2.51)

is invariant under the following transformation:

$$A \rightarrow A + 2\pi f_4 , \hfill (2.52)$$
$$\tilde{\chi}(n) \rightarrow \tilde{\chi}(n+1) . \hfill (2.53)$$
The invariance of the mass term (2.51) under the transformation (2.52), (2.53) originates from the $U(1)^N$ gauge symmetry. To understand it, first notice that the gauge parameters $\alpha(j) (j = 0, 1, \cdots, N - 1 \mod N)$ are angular variables:

$$\alpha(j) \sim \alpha(j) + \frac{2\pi}{g}.$$  \hspace{1cm} (2.54)

The following choice of the gauge parameter is a legitimate gauge transformation due to (2.54):

$$\alpha(j) = \frac{2\pi j}{gN}.$$  \hspace{1cm} (2.55)

The gauge transformation (2.55) leads to the transformation (2.52), (2.53). Note that $f_4$ appearing in (2.52) is the same as the one appearing in the Wilson loop potential (2.50), because its periodicity also follows from the same gauge transformation by (2.55) and its gauge invariance. Also note that the period $f_4$ becomes smaller as we take $N$ large (provided that we do not scale $f$ with $N$).

From (2.40), we observe that in the case $m^2 - 2\gamma f^2 = 0$, the field $\tilde{\chi}(n)$ becomes massless when $A + 2\pi f_4 n = 0$\footnote{More precisely, when $A + 2\pi f_4 n$ equals integer multiple of $2\pi F$. However, we will be mostly interested in the field range of $A$ much smaller than $2\pi F$.} where $n$ is an integer in the range specified in (A.1) or (A.2). This has interesting consequences in cosmology, which we explore in Sec. 3. Notice that small $m^2 - 2\gamma f^2$ is natural in the sense of ’t Hooft [16], as it can be protected by the approximate shift symmetry in the action (2.1) (which may be clearer from (2.32)):

$$\chi(j) \to \chi(j) + c \quad \text{for all } j,$$

where $c$ is a complex constant. The shift symmetry is a good symmetry when the gauge coupling $g$ and the coupling $\gamma$ are also small. Note that these parameters are related as $\gamma = g^2$ at the point of physical interest where the universality of the (de)constructed extra dimensions recovers, as explained in the end of Sec. 2.2.

2.4 Generalization to higher (de)constructed extra dimensions

Below, we generalize the model to have more (de)constructed extra dimensions. Let us consider a $d$-dimensional periodic lattice (a lattice on a $d$-dimensional torus) with $N_I$ ($I = 1, 2, \cdots, d$) lattice points in the $I$-th direction. The action we consider is a
straightforward generalization of (2.1):

\[ S_{(4+d)} = \int d^4 x \sum_j \left[ -\frac{1}{4} F_{\mu\nu(j)} F^{\mu\nu(j)} + D_\mu \chi_{(j)}^\dagger D^\mu \chi_{(j)} - m^2 \chi_{(j)}^\dagger \chi_{(j)} \right] 
+ \sum_{I=1}^d \left\{ \frac{f_I^2}{2} D_\mu U_{(j,j+\vec{e}_I)}^I D^{\mu} U_{(j,j+\vec{e}_I)}^I + \mathcal{V}_S^{(4+d)}(A_{(j,j+\vec{e}_I)}^I, \theta_{(j)}, \theta_{(j+\vec{e}_I)}) \right. 
+ f_I^2 \left( \gamma_I \chi_{(j)}^\dagger U_{(j,j+\vec{e}_I)}^I \chi_{(j+\vec{e}_I)} + \text{c.c.} \right) \left. \right\} + \ldots \right],
\]

The \( I \)-th component of \( \vec{n} \) and \( \vec{j} \) are denoted as \( n_I \) and \( j_I \), respectively. The link variable can be parametrized as

\[ U_{(j,j+\vec{e}_I)}^I = \exp \left[ i \frac{A_{(j,j+\vec{e}_I)}^I}{f_I} \right], \]

where the \( d \)-dimensional vector \( \vec{j} \) parametrizes the lattice points and \( \vec{e}_I \) is a vector whose \( J \)-th component is given by \( \delta_{IJ} \). There are \( d \) components of the gauge field in the (de)constructed directions, \( A_{(j,j+\vec{e}_I)}^I (I = 1, 2, \ldots, d) \). The Stueckelberg potential is given as

\[ \mathcal{V}_S^{(4+d)}(A_{(j,j+\vec{e}_I)}^I, \theta_{(j)}, \theta_{(j+\vec{e}_I)}) = \sum_{p_1=0}^\infty \sum_{p_2=0}^\infty \cdots \sum_{p_d=0}^\infty \tilde{V}_S(p_1, p_2, \ldots, p_d) \exp \left[ i \sum_{I=1}^d p_I \left( \frac{A_{(j,j+\vec{e}_I)}^I}{f_I} - (\theta_{j+\vec{e}_I} - \theta_{(j)}) \right) \right] + \text{c.c.} \]

As before, we consider the discrete Fourier transform

\[ A_{(j,j+\vec{e}_I)}^I = \frac{1}{\prod_{I=1}^d N_I^{1/2}} \sum_{\vec{n}} \tilde{A}_{(\vec{n})}^I e^{i \sum_{I=1}^d \frac{2\pi n_I j_I}{N_I}}. \]

Here, the sum over \( \vec{n} \), we apply our discrete Fourier transform convention (A.1) or (A.2) to each component \( n_I \).

We will eventually be interested in one of the components of \( A^I \) in the application to a single field inflation model to be discussed in Sect. 3. To have such a single field inflation model, we choose a potential (2.59) in such a way that except for \( I = 1 \) all the components have mass larger than the lowest of the “KK” energy scales \( 1/L_I := 2\pi g f_I/N_I \). Then below the lowest “KK” energy scale we can set \( A_{(j,j+\vec{e}_I)}^I = 0 \) except for \( I = 1 \) in the potential (2.59). We can also set \( \theta_{(j)} = 0 \), as the winding number can be set to zero by a choice of
gauge. We regard this $A^I_{(\vec{j},\vec{j}+\vec{e}_I)} = 0$ ($I \neq 1$), $\theta_{(\vec{j})} = 0$ slice of $V^{(4+d)}_S(A^I_{(\vec{j},\vec{j}+\vec{e}_I)}, \theta_{(\vec{j})}, \theta_{(\vec{j}+\vec{e}_I)})$ as a function of $A^I_{(\vec{j},\vec{j}+\vec{e}_I)}$ which we call $V_1(A^I_{(\vec{j},\vec{j}+\vec{e}_I)})$, and consider its Fourier series expansion:

\[ V_1(A^I_{(\vec{j},\vec{j}+\vec{e}_I)}) : = \sum_{p_1=0}^{\infty} \tilde{V}_1(p_1) \exp \left[ ip_1 \left( \frac{A^I_{(\vec{j},\vec{j}+\vec{e}_I)}}{f_1} \right) \right] + c.c. \quad (2.61) \]

We can further set $\tilde{A}_1^I(\vec{n}) = 0$ except for the “zero-mode” $\vec{n} = \vec{0}$, which we call $A$:

\[ A := \tilde{A}_1^I(\vec{0}) = \frac{1}{\prod_{I=1}^{d} N_I^{1/2}} \sum_{\vec{j}} A^I_{(\vec{j},\vec{j}+\vec{e}_I)} \quad (2.62) \]

\[ A^I_{(\vec{j},\vec{j}+\vec{e}_I)} \bigg|_{\vec{n}(\sigma)=0 \text{ except } \vec{n} = \vec{0}} = \frac{1}{\prod_{I=1}^{d} N_I^{1/2}} \tilde{A}_1^I(\vec{0}) = \frac{1}{\prod_{I=1}^{d} N_I^{1/2}} A. \quad (2.63) \]

Substituting (2.63) to (2.61), we obtain

\[ V^{(4)}_S(A) : = \sum_{\vec{j}} V_1 \left( A^I_{(\vec{j},\vec{j}+\vec{e}_I)} \bigg|_{\vec{n}(\sigma)=0 \text{ except } \vec{n} = \vec{0}} \right) = \left( \prod_{I=1}^{d} N_I \right) \sum_{p_1} \tilde{V}_1(p_1) \exp \left[ ip_1 \left( \frac{A}{f_1} \right) \right] + c.c. \quad (2.64) \]

where we have defined

\[ F := f_1 \prod_{I=1}^{d} N_I^{1/2}. \quad (2.65) \]

We observe that compared with the period $2\pi f_1$ of the original potential (2.61), the period $2\pi F$ of the potential (2.64) is enhanced by a factor $\prod_{I=1}^{d} N_I^{1/2}$. The height of the potential (2.64) is also enhanced compared with that in the original potential (2.61) by a factor $\prod_{I=1}^{d} N_I$.

Next we introduce the Wilson loop operator wrapping around the $I = 1$ direction:

\[ W_1(A) : = \exp \left[ i \sum_{j=0}^{N_I-1} \frac{A^I_{(j\vec{e}_1,(j+1)\vec{e}_1)}}{f_1} \right] = \exp \left[ i \frac{N_I A}{f_1 \prod_{I=1}^{d} N_I^{1/2}} \right], \quad (2.66) \]

where

\[ f_4 := \frac{f_1 \prod_{I=1}^{d} N_I^{1/2}}{N_1}. \quad (2.67) \]
Thus the Wilson-loop potential has a period $2\pi f_4$.

Next we study charged scalar fields $\chi_{(j)}$. The discrete Fourier transform of the charged field $\chi_{(j)}$ is given by

$$\chi_{(j)} = \frac{1}{\prod_{l=1}^{d} N_l^{1/2}} \sum_{\vec{n}} \tilde{\chi}_{(\vec{n})} e^{i \sum_{l=1}^{d} \frac{2\pi n_l j_l}{N_l}}. \quad (2.68)$$

Below the “KK” energy scale, we can set $\tilde{\chi}_{(\vec{n})} = 0$ except for $\vec{n} = (n_1, 0, \cdots, 0)$. Then $N_1$ eigenvalues of the mass-square matrix of $\chi$ are given as

$$M_{\chi(n_1)}^2 = (m^2 - 2 \sum_{l=1}^{d} \gamma_l f^2_l) + M_{\chi(n_1)}^{2KK}, \quad (2.69)$$

where the “KK” contribution to the mass-square is given by

$$M_{\chi(n_1)}^{2KK} = 4 \gamma_1 f_1^2 \sin^2 \left( \frac{A + 2\pi f_4 n_1}{2F} \right). \quad (2.70)$$

### 2.5 Comparison with continuous extra dimensions

It will be instructive to make a comparison with a $(4 + d)$D massive gauge theory compactified on a $d$-dimensional torus. For the purpose of illustration, we only write down the terms in the action of the massive gauge theory which are relevant for understanding the correspondence:

$$S_{(4+d)}^c = \int d^4 x \int d^d y \left[ \frac{1}{2} \partial_\mu A^1(x, \vec{y}) \partial^\mu A^1(x, \vec{y}) - 2V_{(4+d)} \cos \left( \frac{A^1}{F_c} \right) \right] + \ldots, \quad (2.71)$$

where $x$ is the 4-coordinate of uncompactified space-time and $\vec{y}$ is the coordinate vector on the $d$-dimensional torus. $A^1(x, \vec{y})$ is the 1st of the $d$ components of the massive gauge field in the compactified directions. $f^c$ is a constant with mass-dimension $(d + 2)/2$ and $V_{(4+d)}$ is a constant with mass-dimension $4 + d$. The potential of the massive gauge field in (2.71) is to be compared with that in (2.61) with all $\tilde{V}_{1}(p_1)$ set to zero except for $p_1 = 1$:

$$V_1(A^1_{(uj+\vec{e}_l)}) = \tilde{V}_1(p_1 = 1) \exp \left[ i \left( \frac{A^1_{(uj+\vec{e}_l)}}{f_1} \right) \right] + c.c. = 2\tilde{V}_1(p_1 = 1) \cos \left( \frac{A^1_{(uj+\vec{e}_l)}}{f_1} \right). \quad (2.72)$$

Consequently (2.64) becomes

$$V^{(4)}_S(A) = \left( \prod_{l=1}^{d} N_l \right) 2\tilde{V}_1(p_1 = 1) \cos \left( \frac{A}{F} \right) = 2\tilde{V}_1 \cos \left( \frac{A}{F} \right), \quad (2.73)$$
where
\[ V_4 := \left( \prod_{I=1}^{d} N_I \right) \tilde{V}_1 (p_1 = 1). \] (2.74)

For simplicity, we assume that the compactification radii to be the same \( L \) in all compactified directions. With a bit of abuse of notation, we use the same symbols for both the (de)constructed model and the model with continuous extra dimensions when the correspondence of the quantity is clear, like \( L \) here, \( A, g_4, f_4 \) and \( F \), and \( V_4 \) to be introduced shortly. The correspondence between the model with the (de)constructed extra dimensions and that with the continuous extra dimensions are summarized in Table. Then the \((4+d)D\) (reduced) Planck scale and the 4D (reduced) Planck scale \( M_P \) are related as
\[ M_P^2 = (2\pi L)^{d} M_{(4+d)}^{2+d}. \] (2.75)

It is natural to assume that \( f^c \) is sub-Planckian:\footnote{Our viewpoint is that firstly this proposal is still a conjecture, and secondly even if the field domain in the original EFT is restricted to be sub-Planckian, there are mechanisms to obtain a trans-Planckian field range at low energy, such as axion monodromy [18, 19, 20] or as we discuss in this article. There is no obvious reason why the distance conjecture should constrain such enhancements of effective field range.}
\[ f^c \lesssim M_{(4+d)}^{d+2}. \] (2.76)

The Fourier expansion of the field \( A^1(x, \vec{y}) \) is given as
\[ A^1(x, \vec{y}) = \frac{1}{(2\pi L)^{d/2}} \sum_{\vec{n}} \tilde{A}^1(x, \vec{n}) e^{i\vec{n} \cdot \vec{y}}. \] (2.77)

The Fourier coefficients are given as
\[ \tilde{A}^1(x, \vec{n}) = \frac{1}{(2\pi L)^{d/2}} \int d^d y \tilde{A}^1(x, \vec{y}) e^{-i\vec{n} \cdot \vec{y}}. \] (2.78)

From (2.78), when \( \tilde{A}^1(x, \vec{n}) \) is set to zero except for the zero-mode \( \vec{n} = \vec{0} \),
\[ A^1(x, \vec{y}) \bigg|_{\tilde{A}^1(x, \vec{n}) = 0 \text{ except } \vec{n} = \vec{0}} = \frac{1}{(2\pi L)^{d/2}} \tilde{A}^1(x, \vec{n} = \vec{0}), \] (2.79)
we obtain 4D action \( S_4 \) at the classical level:
\[ S_4 = \int d^4 x \left[ \frac{1}{2} \partial_{\mu} A \partial^{\mu} A - 2 V_4 \cos \left( \frac{A}{F} \right) \right] + \ldots , \] (2.80)

Here we are not claiming that \((f^c)^{\frac{d}{d+2}} > M_{(4+d)}\) is fundamentally inconsistent. It has been proposed that if an EFT can be embedded in string theory, the field range is restricted by the Planck scale: \((f^c)^{\frac{d}{d+2}} < M_{(4+d)}\) (the distance conjecture [17]).
where
\[ A(x) := \tilde{A}^1(x, \tilde{0}), \quad (2.81) \]
and
\[ V_4 := (2\pi L)^d V_{(4+d)}, \quad F := f^c (2\pi L)^{d/2}. \quad (2.82) \]

For the sake of comparison, in the (de)constructed model we take all \( d \) directions to be the same, \( f_I = f, \ N_I = N \) and \( \gamma_I = \gamma \ (I = 1, 2, \cdots, d) \).

Comparing the 4D potential with the original \((4+d)D\) potential, we observe that:
1. The overall height of the potential is multiplied by the factor \((2\pi L)^d\).
2. The overall shape of the potential is elongated in the field direction by the factor \((2\pi L)^{d/2}\).

On the other hand, in the model with (de)constructed extra dimensions, by comparing the original potential \([2.61]\) with the final potential \([2.64]\) we observe that:
1. The overall height of the potential is multiplied by the factor \(N^d\).
2. The overall shape of the potential is elongated in the field direction by the factor \(N^{d/2}\).

Noticing that \(L\) is proportional to \(N\) when \(f\) and \(g\) do not scale with \(N\) (see \([2.11]\)) in the model with (de)constructed extra dimensions, we observe that there is a parallel between the continuous extra dimensions and (de)constructed dimensions. The correspondence between them are summarized in Table. 1.

However, in spite of the enhancement in the field range, in the case of continuous extra dimensions the field range cannot exceed 4D Planck scale if the field range in higher dimensional theory is bounded by higher dimensional Planck scale:

\[
F = f^c (2\pi L)^{d/2} \lesssim M_{(4+d)}^{d/2} (2\pi L)^{d/2} = \left( \frac{M_P^2}{(2\pi L)^d} \right)^{\frac{1}{d+2}} (2\pi L)^{d/2} = M_P. \quad (2.84)
\]

This is the place where the difference between the model with continuous extra dimensions and that with (de)constructed extra dimensions comes in. Since the (de)constructed model is purely a 4D QFT, there is no notion of higher dimensional Planck scale. Thus unlike \(f^c\) in the model with continuous extra dimensions which may be bounded from above by the \((4+d)D\) (reduced) Planck scale \(M_{(4+d)}\), the symmetry breaking scale \(f\) is bounded from the above only by the 4D (reduced) Planck scale \(M_P\). This allows the effective field range to be trans-Planckian in models with (de)constructed extra dimensions if we take \(N\) sufficiently large. Also note that one cannot take the continuum limit \(N \to \infty\) in (de)constructed model. The gauge coupling \(g\) should also be bounded from the above as \(g \lesssim 1\), otherwise the perturbative description of the action \([2.1]\) is invalid. Then from \([2.11]\), \(N \to \infty\) leads to the decompactification \(L \to \infty\), which is not acceptable.
The period of the Wilson loop operator wrapping around $y^1$ direction and the period of the mass term of the charged field is determined by the gauge transformation

$$\alpha(\vec{y}) = \frac{y^1}{g_{(4+d)}L}, \quad (2.85)$$

which leads to the gauge equivalence

$$A^1(\vec{y}) \sim A^1(\vec{y}) + \frac{1}{g_{(4+d)}L}. \quad (2.86)$$

From (2.77), (2.86) leads to the gauge equivalence of the zero-mode of $A^1(\vec{y})$:

$$\frac{1}{(2\pi L)^{\frac{d}{2}}} A \sim \frac{1}{(2\pi L)^{\frac{d}{2}}} A + \frac{1}{g_{(4+d)}L}, \quad (2.87)$$

which can be rewritten as

$$A \sim A + 2\pi f_4, \quad (2.88)$$

where $f_4$ is given by

$$f_4 = \frac{(2\pi L)^{\frac{d}{2}-1}}{g_{(4+d)}} = \frac{1}{g_4 2\pi L}. \quad (2.89)$$

Again we observe that the $L$ dependence of $f_4$ in the model with continuous extra dimensions is in parallel with the $N$ dependence of $f_4$ in the (de)constructed model.
Table 1: The correspondence between the quantities in (de)constructed extra dimensions and those in continuous extra dimensions. The correspondence is such that the quantities in the left column goes to the quantities in the right column in the formal continuum limit $N \to \infty$, $a \to 0$ with $N a = 2\pi L$ fixed. Here, $a := 1/gf$ in terms of the original parameters in 4D QFT.

| (De)constructed | Continuous |
|-----------------|------------|
| $2\pi L = Na = \frac{N}{gf}$ | $2\pi L$ |
| $a^{-\frac{d}{2}} A^1_{(\tilde{j},\tilde{\epsilon})}$ | $A^1(\tilde{y})$ |
| $A = \frac{1}{N^{\frac{d}{2}}} \sum_j A^1_{(\tilde{j},\tilde{\epsilon})}$ | $A = \frac{1}{(2\pi L)^{\frac{d}{2}}} \int d^d y A^1(y)$ |
| $a^{\frac{d}{2}} g$ | $g_{(4+d)}$ |
| $g_4 = \frac{g}{N^{\frac{d}{2}}}$ | $g_4 = \frac{g_{(4+d)}}{(2\pi L)^{\frac{d}{2}}}$ |
| $a^{-\frac{d}{2}} f$ | $f^c$ |
| $f_4 = f N^{\frac{d}{2}}$ | $f_4 = \frac{(2\pi L)^{\frac{d}{2}}}{g_{(4+d)}}$ |
| $F = f N^{\frac{d}{2}}$ | $F = f^c (2\pi L)^{\frac{d}{2}}$ |
| $a^{-d} \tilde{V}_1(p_1 = 1)$ | $V_{(4+d)}$ |
| $V_4 = N^{\frac{d}{2}} \tilde{V}_1(p_1 = 1)$ | $V_4 = (2\pi L)^d V_{(4+d)}$ |

3 An explicit inflation model and comparison with the observations

In this section we construct a single field slow-roll inflation model in which the field $A$, the “zero-mode” of a component of the gauge field in (de)constructed extra dimensions, plays the role of the inflaton. For simplicity, we consider the case in which all $d$ directions in the (de)constructed extra dimensions look the same, $f_I = f$, $N_I = N$ and $\gamma_I = \gamma$.
$\ldots$  

We consider the case where the Stueckelberg potential (3.5) dominates over the Wilson loop potential (2.50):

$$V_{WL}(A) \ll V_S(A),$$  \hspace{1cm} (3.1)

$$V'_{WL}(A) \ll V'_S(A).$$  \hspace{1cm} (3.2)

(3.1) and (3.2) respectively give

$$f \lesssim 2 \times 10^{-3} (4\pi)^{\frac{N}{2}} M_P,$$  \hspace{1cm} (3.3)

$$f \lesssim 1 \times 10^{-4} (4\pi)^{\frac{N}{2}} \frac{N^2_d}{M_P}.$$  \hspace{1cm} (3.4)

Here, we have used $c_{WL} \sim \mathcal{O}(1)$ and $\Lambda = 4\pi f$. With a moderately small lattice size $N \sim 6$ and above, (3.3) and (3.4) do not give further constraint once $f \ll M_P$ is assumed.

So far, the Stueckelberg potential (2.64) was an arbitrary symmetric function of $A$ in the field range $-\pi F \leq A < \pi F$. Now, as an example, we assume that the inflation took place in the region of the potential which is approximately linear:

$$V_S(A) = \lambda M^3_P |A - A_0|,$$  \hspace{1cm} (3.5)

where $A_0$ is a positive constant with $A_0 \ll \pi F$. Furthermore, without loss of generality, we assume that the inflation took place in the region $A > A_0$. We define $\phi = A - A_0$ and then the inflaton potential can be written in the field range of our interest as

$$V(\phi) = \lambda M^3_P \phi.$$  \hspace{1cm} (3.6)

The linear inflaton potential is compatible with the latest Planck 2018 results [21]. From the inflaton potential (3.6), we obtain the slow-roll parameters as

$$\epsilon(\phi) := \frac{M^2_P}{2} \left(\frac{V'}{V}\right)^2 = \frac{M^2_P}{2\phi^2},$$  \hspace{1cm} (3.7)

$$\eta(\phi) := M^2_P \frac{V''}{V} = 0.$$  \hspace{1cm} (3.8)

The number of e-folds is given as

$$N(\phi) \simeq \int_{\phi_{end}}^\phi d\phi \frac{V}{M^2_P V'} = \frac{1}{M^2_P} \left[\frac{\phi^2}{2}\right]^{\phi}_{\phi_{end}},$$  \hspace{1cm} (3.9)

where we define the inflaton field value $\phi_{end}$ at the end of the slow-roll inflation by the condition $\epsilon(\phi_{end}) = 1$, which gives

$$\phi_{end} = \frac{M_P}{\sqrt{2}}.$$  \hspace{1cm} (3.10)
We choose the number of e-folds at the pivot scale $0.05 \text{ Mpc}^{-1}$ as

$$N(\phi_*) = 50,$$  \hspace{1cm} (3.11)

where $*$ denotes the value at the pivot scale. From (3.9) we obtain

$$\phi_* \simeq 10M_P.$$  \hspace{1cm} (3.12)

The slow-roll parameters at the pivot scale are given as

$$\epsilon(\phi_*) \simeq 5.0 \times 10^{-3},$$  \hspace{1cm} (3.13)

$$\eta(\phi_*) = 0.$$  \hspace{1cm} (3.14)

The scalar power spectrum is given as

$$P_s \simeq \frac{V(\phi_*)}{24\pi^2M_P^2\epsilon(\phi_*)} = 2.2 \times 10^{-9},$$  \hspace{1cm} (3.15)

where the value on the right hand side is the COBE normalization. Substituting (3.6), (3.12), (3.13) in (3.15) the value of the parameter $\lambda$ is fixed as

$$\lambda \simeq 2.6 \times 10^{-10}.$$  \hspace{1cm} (3.16)

The scalar spectral index $n_s$ is given as

$$n_s \simeq 1 - 6\epsilon(\phi_*) + 2\eta(\phi_*) = 0.97.$$  \hspace{1cm} (3.17)

The tensor-to-scalar ratio $r_*$ at the pivot scale is given as

$$r_* \simeq 16\epsilon(\phi_*) = 8.0 \times 10^{-2}.$$  \hspace{1cm} (3.18)

From the slow-roll approximation of the Friedmann equation,

$$H(\phi)^2 \simeq \frac{V(\phi)}{3M_P^2},$$  \hspace{1cm} (3.19)

we obtain the Hubble scale when the pivot scale exited the horizon:

$$H(\phi_*) \simeq 7.2 \times 10^{13}\text{ GeV}.$$  \hspace{1cm} (3.20)

Now, in order to achieve the trans-Planckian inflaton excursion (3.12), the lattice size $N$ need to be large enough so that the field range $2\pi fN^{\frac{1}{2}}$ can accommodate the excursion. This condition requires the lowest value of $N$ as

$$N \gg (3.9 \times 10^2)^{\frac{2}{3}} \times \left(\frac{f}{1.0 \times 10^{16}\text{ GeV}}\right)^{-\frac{2}{3}}.$$  \hspace{1cm} (3.21)
The constraint becomes weaker for larger \( d \), in which case a small value of \( f \) may be accommodated.

In order for the inflaton EFT (2.42) to be valid during inflation, the “KK” energy scale must be above the Hubble scale at the time of inflation. This condition gives an upper bound on the lattice size \( N \):

\[
N < 8.8 \times 10^2 \times \left( \frac{g}{1.0} \right) \left( \frac{f}{1.0 \times 10^{16} \text{ GeV}} \right). \tag{3.22}
\]

When \( d = 1 \) with \( g \simeq 1 \), in order for the lower bound (3.21) to be below the upper bound (3.22), \( f \) needs to be as large as \( \simeq 6 \times 10^{16} \) GeV, which is getting close to the bound \( \Lambda = 4\pi f \ll M_P \). The constraints are mild for \( d \geq 2 \).

The \( A \) dependent mass (2.40) of the charged field \( \tilde{\chi}(n) \) may have an interesting cosmological consequence, if \( m^2 - 2\gamma f^2 \ll H^2 \) during inflation: Whenever the inflaton \( A \) crosses the value \( A = -2\pi f_4 n \) the field \( \tilde{\chi}(n) \) becomes almost massless, leading to a burst of productions of \( \tilde{\chi}(n) \) particles, which may leave observable features in the anisotropy of the CMB radiation [22, 23, 24, 25]. This is an extension of the mechanism “excursions through KK modes” found in [11] to the case where the KK modes are those of the (de)constructed extra dimensions.

Near \( A = -2\pi f_4 n \), the mass term of \( \tilde{\chi}(n) \) can be approximated as

\[
\tilde{\chi}^\dagger(n) M_{\tilde{\chi}(n)}^2 A \tilde{\chi}(n) \simeq \tilde{\chi}^\dagger(n) \sqrt{\gamma} \left( \frac{A + 2\pi f_4 n}{N^d} \right)^2 \tilde{\chi}(n). \tag{3.23}
\]

Here, we have assumed \( m^2 - 2\gamma f^2 \ll H^2 \). (3.23) is the same interaction studied in [23, 24, 25]. From the latest analytic result of [25], the contribution of the rapid particle production due to the interaction (3.23) to the power spectrum \( \delta P_s \) is given by

\[
\delta := \frac{\delta P_s}{P_s} \simeq 2 \times 300 \left( \frac{\sqrt{\gamma}}{N^d} \right)^{7/2}, \tag{3.24}
\]

where the factor 2 in the right hand side came from the fact that the complex field \( \chi(n) \) has two real degrees of freedom. Below, we restrict ourselves to the case close to the universality restoration point \( \gamma \approx g^2 \) (see the explanation below (2.40)) in order to make a direct comparison with the model with continuous extra dimension [11]. As a crude upper bound on \( \delta \), we assume that the contribution of the particle production to the power spectrum does not exceed that of the inflaton.\footnote{When the distances between the peaks of the primordial features in the power spectrum are small, i.e. \( \Delta_i \ll 1 \) where \( \Delta_i \) is defined in (3.27), observations will not be able to resolve each peak [24]. In such a case it may become harder to distinguish the primordial features in the power spectrum from the almost scale invariant power spectrum.} As a rough criterion for the detectability...
of the primordial feature in near future, we require that the amplitude of the feature to be more than a percent. These requirements,
\[ 0.01 < \delta < 1, \quad (3.25) \]
give
\[ (6.2 \, g)^{\frac{3}{4}} < N < (2.3 \times 10 \, g)^{\frac{3}{4}}. \quad (3.26) \]
It is natural to have the value of the gauge coupling in the range \( 0.1 \lesssim g \lesssim 1 \). Then, the upper bound on \( N \) in (3.26) is quite tight when \( d \geq 2 \). However, note that the upper bound of (3.26) came from the condition that the primordial feature is within the reach of near future detection. Thus when \( N \) is smaller than this bound, the primordial feature may not be detectable in the near future, but the model is still valid as a single field slow-roll inflation model.

The number of e-folds from the \( i \)-th peak to the \( (i+1) \)-th peak is given by
\[
\Delta_i := \ln \left( \frac{k_{i+1}}{k_i} \right)
= N(\phi(k_i)) - N(\phi(k_{i+1}))
= N(\phi(k_i)) - N(\phi(k_i) - 2\pi f_4)
\approx \frac{dN}{d\phi}(\phi(k_i))2\pi f_4
= \frac{\phi(k_i)}{M_P^2}2\pi f_4.
\quad (3.27)
\]
Remembering (2.67), (3.27) can be rewritten as
\[
\Delta_i = 2.6 \times 10^{-1} \times \left( \frac{\phi(k_i)}{10 M_P} \right) \left( \frac{f}{1.0 \times 10^{16} \text{GeV}} \right) N^{\frac{d}{2} - 1}. \quad (3.28)
\]
Notice that for \( d \geq 3 \) a larger lattice size \( N \) leads to larger intervals between the peaks of the primordial features.

4 Summary and discussions

In this article, we constructed a massive gauge field theory with (de)constructed extra dimensions. One of our main results was that the effective field range of the “zero-mode” of a component of the massive gauge field in the (de)constructed extra dimensions was enhanced by a factor \( N^{\frac{d}{2}} \), where \( d \) was the number of the (de)constructed extra dimensions and \( N \) was the number of the lattice points in each (de)constructed direction. We applied this mechanism of field range enhancement in a large field inflation model. We obtained constraints on the model parameters from the latest CMB observations. We also extended
the rapid particle production mechanism “excursion through KK modes” to the case with “KK” modes of the (de)constructed extra dimensions. The cosmological consequences of this mechanism were also studied and compared with the CMB data.

In this article we focused on the case in which the Stueckelberg potential dominates over the Wilson loop potential, (3.1) and (3.2). However, it should be noted that the enhancement of the period also occurs in the Wilson loop potential for $d \geq 3$, as shown in (2.67). Therefore, for $d \geq 3$ we can have a large field inflation model in which the Stueckelberg potential is sub-dominant or even absent. This model is a (de)constructed version of the original extra-natural inflation.

We observed in Sec. 2.5 that in a model with continuous extra dimensions, the field range cannot exceed the 4D Planck scale $M_P$ if the original field range in the higher dimensional theory is below the higher dimensional Planck scale. The purely 4D nature of the (de)construction circumvented this constraint. Those familiar with string theory might worry that if our (de)constructed model is to be realized in string theory, the lattice of the (de)constructed space may be embedded in real space-time in which closed strings propagate. Then the circumferences of the periodic lattice may coincide with the circumferences of the real continuous extra dimensions, and the obstacle for achieving an effective trans-Planckian field range may reappear in the (de)constructed model. While this is a valid concern, we feel it is still too early to make a conclusion on this issue, since our current understanding of string vacua is still very limited. For example, it is not clear whether (de)constructed space always need to be embedded in real space-time in string theory. We leave this interesting issue to future investigations.

Dimensional (de)construction is not the only way to have purely 4D QFT description of extra dimensions. For example, a gauge-Higgs unification model from spontaneously created fuzzy extra-dimensions was proposed in [26]. It will also be interesting to explore inflation models based on fuzzy extra dimensions.

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A Discrete Fourier Transform

Let us consider a cyclically ordered $N$ points labeled by $j$ ($j = 0, 1, \cdots, N-1$ (mod $N$)). Consider a variable $f_j$ which has a value on each point. We use the following convention for the discrete Fourier expansion of the variable $f_j$:

$$f_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \tilde{f}_n e^{i2\pi nj/N} \quad (N: \text{odd}). \quad (A.1)$$

$$f_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \tilde{f}_n e^{i2\pi nj/N} + \frac{1}{\sqrt{N}} \tilde{f}_N (-)^j \quad (N: \text{even}). \quad (A.2)$$

Our convention is convenient since when applied in (de)construction each “KK” mode is canonically normalized.

When $f_j$ is a real variable, $\tilde{f}_n^* = \tilde{f}_n$. The orthogonality of the exponential function:

$$\sum_{j=0}^{N-1} \left(e^{i2\pi nj/N}\right)^* e^{i2\pi n_2j/N} = N\delta_{n_1n_2}, \quad (A.3)$$

leads to the following formula for the discrete Fourier coefficient:

$$\tilde{f}_n = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j e^{-i2\pi nj/N}. \quad (A.4)$$

We would like to have a formula for the discrete counterpart of the dimensional reduction. For this purpose, let us first consider

$$\sum_{j=0}^{N-1} (f_j)^m = \sum_{j=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_n \tilde{f}_n e^{i2\pi nj/N}\right)^m$$

$$= \frac{1}{N^{m/2}} \sum_{j=0}^{N-1} \sum_n \cdots \sum_{n_m} \tilde{f}_{n_1} \cdots \tilde{f}_{n_m} \exp \left[i \sum_{\ell=1}^{m} \frac{2\pi n_{\ell}j}{N} \right]$$

$$= \frac{N}{N^{m/2}} \sum_{n_1} \cdots \sum_{n_m} \tilde{f}_{n_1} \cdots \tilde{f}_{n_m} \delta_{n_1+n_2+\cdots+n_m=0 \text{ mod } N}. \quad (A.5)$$

In (A.5) the sum over $n_\ell$ ($\ell = 1, 2, \cdots, m$) is taken as in (A.1) or (A.2) depending on whether $N$ is odd or even.

In the discrete version of the dimensional reduction, we set all the discrete Fourier coefficients except for the “zero-mode” to zero: $\tilde{f}_k = 0$ for $k \neq 0$ in (A.1) or (A.2). In this case, (A.5) becomes

$$\sum_{j=0}^{N-1} (f_j)^m \bigg|_{\tilde{f}_k=0 \text{ except } k=0} = N \left(\frac{\tilde{f}_0}{\sqrt{N}}\right)^m. \quad (A.6)$$
Now suppose that a function $V(x)$ has a Taylor series expansion around $x = 0$:

$$V(x) = \sum_{m=0}^{\infty} \frac{V^{(m)}(0)}{m!} x^m,$$  \hspace{1cm} (A.7)

where $V^{(m)}(0)$ denotes the $m$-th derivative of the function $V(x)$ at $x = 0$. Consider a field theory on the discrete points with a potential of the form $V(f_j)$. Then from (A.6) the discrete dimensional reduction of this potential is given as

$$\left. \sum_{j=0}^{N-1} V(f_j) \right|_{\tilde{f}_k=0 \text{ except } k=0} = N \sum_{m=0}^{\infty} \frac{V^{(m)}(0)}{m!} \left( \frac{\tilde{f}_0}{\sqrt{N}} \right)^m.$$  \hspace{1cm} (A.8)

References

[1] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, “(De)constructing dimensions,” *Phys. Rev. Lett.* **86** (2001) 4757–4761, arXiv:hep-th/0104005 [hep-th].

[2] C. T. Hill, S. Pokorski, and J. Wang, “Gauge Invariant Effective Lagrangian for Kaluza-Klein Modes,” *Phys. Rev.* **D64** (2001) 105005, arXiv:hep-th/0104035 [hep-th].

[3] N. Arkani-Hamed, A. G. Cohen, and H. Georgi, “Electroweak symmetry breaking from dimensional deconstruction,” *Phys. Lett.* **B513** (2001) 232–240, arXiv:hep-ph/0105239 [hep-ph].

[4] Y. Hosotani, “Dynamical Mass Generation by Compact Extra Dimensions,” *Phys. Lett.* **126B** (1983) 309–313.

[5] Y. Hosotani, “Dynamics of Nonintegrable Phases and Gauge Symmetry Breaking,” *Annals Phys.* **190** (1989) 233.

[6] A. T. Davies and A. McLachlan, “Congruency Class Effects in the Hosotani Model,” *Nucl. Phys.* **B317** (1989) 237.

[7] H. Hatanaka, T. Inami, and C. S. Lim, “The Gauge hierarchy problem and higher dimensional gauge theories,” *Mod. Phys. Lett.* **A13** (1998) 2601–2612, arXiv:hep-th/9805067 [hep-th].

[8] N. Arkani-Hamed, H.-C. Cheng, P. Creminelli, and L. Randall, “Extra natural inflation,” *Phys. Rev. Lett.* **90** (2003) 221302, arXiv:hep-th/0301218 [hep-th].

[9] D. E. Kaplan and N. J. Weiner, “Little inflatons and gauge inflation,” *JCAP* **0402** (2004) 005, arXiv:hep-ph/0302014 [hep-ph].

25
[10] S. Dimopoulos, S. Kachru, J. McGreevy, and J. G. Wacker, “N-flation,” *JCAP* **0808** (2008) 003, arXiv:hep-th/0507205 [hep-th].

[11] K. Furuuchi, “Excursions through KK modes,” *JCAP* **1607** no. 07, (2016) 008, arXiv:1512.04684 [hep-th].

[12] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, “The String landscape, black holes and gravity as the weakest force,” *JHEP* **06** (2007) 060, arXiv:hep-th/0601001 [hep-th].

[13] P. Saraswat, “Weak gravity conjecture and effective field theory,” *Phys. Rev.* **D95** no. 2, (2017) 025013, arXiv:1608.06951 [hep-th].

[14] K. Furuuchi, “Weak Gravity Conjecture From Low Energy Observers’ Perspective,” *Fortsch. Phys.* **66** no. 10, (2018) 1800016, arXiv:1712.01302 [hep-th].

[15] B. Heidenreich, M. Reece, and T. Rudelius, “The Weak Gravity Conjecture and Emergence from an Ultraviolet Cutoff,” *Eur. Phys. J.* **C78** no. 4, (2018) 337, arXiv:1712.01868 [hep-th].

[16] G. ’t Hooft, “Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking,” *NATO Sci. Ser. B* **59** (1980) 135–157.

[17] H. Ooguri and C. Vafa, “On the Geometry of the String Landscape and the Swampland,” *Nucl. Phys.* **B766** (2007) 21–33, arXiv:hep-th/0605264 [hep-th].

[18] J. E. Kim, H. P. Nilles, and M. Peloso, “Completing natural inflation,” *JCAP* **0501** (2005) 005, arXiv:hep-ph/0409138 [hep-ph].

[19] E. Silverstein and A. Westphal, “Monodromy in the CMB: Gravity Waves and String Inflation,” *Phys. Rev.* **D78** (2008) 106003, arXiv:0803.3085 [hep-th].

[20] L. McAllister, E. Silverstein, and A. Westphal, “Gravity Waves and Linear Inflation from Axion Monodromy,” *Phys. Rev.* **D82** (2010) 046003, arXiv:0808.0706 [hep-th].

[21] *Planck* Collaboration, Y. Akrami *et al.*, “Planck 2018 results. X. Constraints on inflation,” arXiv:1807.06211 [astro-ph.CO].

[22] D. J. H. Chung, E. W. Kolb, A. Riotto, and I. I. Tkachev, “Probing Planckian physics: Resonant production of particles during inflation and features in the primordial power spectrum,” *Phys. Rev.* **D62** (2000) 043508, arXiv:hep-ph/9910437 [hep-ph].

26
[23] N. Barnaby, Z. Huang, L. Kofman, and D. Pogosyan, “Cosmological Fluctuations from Infra-Red Cascading During Inflation,” *Phys. Rev.* **D80** (2009) 043501, arXiv:0902.0615 [hep-th].

[24] N. Barnaby and Z. Huang, “Particle Production During Inflation: Observational Constraints and Signatures,” *Phys. Rev.* **D80** (2009) 126018, arXiv:0909.0751 [astro-ph.CO].

[25] L. Pearce, M. Peloso, and L. Sorbo, “Resonant particle production during inflation: a full analytical study,” *JCAP* **1705** no. 05, (2017) 054, arXiv:1702.07661 [astro-ph.CO].

[26] K. Furuuchi, T. Inami, and K. Okuyama, “Gauge-Higgs Unification In Spontaneously Created Fuzzy Extra Dimensions,” *JHEP* **11** (2011) 006, arXiv:1108.4462 [hep-ph].