ON CYCLES FOR THE DOUBLING MAP WHICH ARE DISJOINT FROM AN INTERVAL

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Abstract. Let $T : [0, 1] \to [0, 1]$ be the doubling map and let $0 < a < b < 1$. We say that an integer $n \geq 3$ is bad for $(a, b)$ if all $n$-cycles for $T$ intersect $(a, b)$. Let $B(a, b)$ denote the set of all $n$ which are bad for $(a, b)$. In this paper we completely describe the sets:

$D_2 = \{ (a, b) : B(a, b) \text{ is finite} \}$
and

$D_3 = \{ (a, b) : B(a, b) = \emptyset \}$.

In particular, we show that if $b - a < \frac{1}{6}$, then $(a, b) \in D_2$, and if $b - a \leq \frac{2}{15}$, then $(a, b) \in D_3$, both constants being sharp.

1. Introduction and summary

Let $T : [0, 1] \to [0, 1]$ be the doubling map, i.e.,

$$T x = \begin{cases} 2x, & x \in [0, 1/2], \\ 2x - 1, & x \in (1/2, 1]. \end{cases}$$

Assume $0 < a < b < 1$ and put

$$J(a, b) = \{ x \in [0, 1] : T^n x \notin (a, b) \text{ for all } n \geq 0 \}.$$

In other words, $J(a, b)$ is the set of all $x$ whose $T$-orbits are disjoint from $(a, b)$. Thus, $T|_{J(a,b)}$ is what is usually referred to as an “open map” or a “map with a hole”. It is obvious that $\{0, 1\} \subset J(a, b)$.

It is intuitively clear that if $b - a$ is “small”, then $J(a, b)$ is “large” and vice versa. Such claims in their precise quantitative form have been obtained in the recent papers [4, 7]. Specifically, the following two sets have been described in [4]:

$$D_0 = \{ (a, b) \in (1/4, 1/2) \times (1/2, 3/4) : J(a, b) \neq \{0, 1\} \}$$
and

$$D_1 = \{ (a, b) \in (1/4, 1/2) \times (1/2, 3/4) : J(a, b) \text{ is uncountable} \}.$$

See Figure 6 on page 18 for $D_0$ and $D_1$. The reason why we can confine ourselves to $(a, b) \in (1/4, 1/2) \times (1/2, 3/4)$ without losing anything interesting is the following result:
Lemma 1.1. ([4, Lemma 1.1])

(i) If \( a < 1/4, b > 1/2 \) or \( a < 1/2, b > 3/4 \), then \( \mathcal{J}(a,b) = \{0, 1\} \).

(ii) If \( b < 1/2 \) or \( a > 1/2 \), then \( \dim_H \mathcal{J}(a,b) > 0 \).

In the present paper we will be interested in cycles (i.e. finite orbits) for \( T \) which are disjoint from an interval. Let us first introduce the following sets which are closely related to \( D_0 \) and \( D_1 \):

\[ D'_0 = \{ (a, b) \in (1/4, 1/2) \times (1/2, 3/4) : \exists \text{ a non-trivial cycle disjoint from } (a, b) \} \]

and

\[ D'_1 = \{ (a, b) \in (1/4, 1/2) \times (1/2, 3/4) : \exists \text{ infinitely many cycles disjoint from } (a, b) \} \]

(from here on by a “cycle” we will mean a prime \( T \)-cycle). In the definition of \( D'_0 \) we do not include the trivial 1-cycles of \{0\} or \{1\}. Note first that it is obvious that if \( (a, b) \notin D_0 \), then \( (a, b) \notin D'_0 \), whence \( D'_0 \subset D_0 \). On the other hand, if \( (a, b) \) is an interior point of \( D_0 \), then a cycle disjoint from \( (a, b) \) exists – see [4, Theorem 2.7]. Furthermore, the set \( \{ (a, a + 1/4) : a \in S \} \) is a subset of \( D_0 \) but it does not contain any cycles, and these are the only points on the boundary of \( D_0 \) with this property. (For the definition of \( S \) see Section 3.) Hence

\[ D'_0 \subsetneq D_0 \subsetneq \text{cl}(D'_0). \]

where \( \text{cl} \) is the closure of the set. In particular, the interiors of \( D_0 \) and \( D'_0 \) coincide. Similarly, in view of [4, Theorem 2.16],

\[ D'_1 \subsetneq D_1 \subsetneq \text{cl}(D'_1). \]

Thus, the sets \( D_0 \) (resp. \( D_1 \)) are “almost” the ones where for \((a, b)\) there are at least one (resp. infinitely many) disjoint cycles.

One may see this model as follows: take some large interval \((a, b)\) and begin shrinking it from both ends. At some point in time one gets a disjoint cycle, then infinitely many of those, and then (apparently!) for any \( n \) there will be an \( n \)-cycle disjoint from \((a, b)\). This is analogous to the famous “period three implies chaos” statement – see, e.g., [5]; in fact, it is more than just an analogy, it is a generalization.

More precisely, if we assume \( b = 1 - a \) (so our shrinking intervals are always symmetric about 1/2), then it follows from the main result of [1] that with the increase of \( a \) towards 1/2, the lengths of cycles disjoint from \((a, 1-a)\) appear in exactly the classical Sharkovskii order, with period 3 being indeed last to appear at \( a = 3/7 \). The case we consider in the present paper is, generally speaking, asymmetric, so our first goal will be to determine what curve in the plane \((a, b)\) is a natural analogue of 3/7 – see Remark [2.7] and Proposition [2.6] in Section 2.

Note first that \( n = 2 \) needs to be excluded, since there is only one 2-cycle, namely, \( \{1/3, 2/3\} \), so one can take \( (a, b) = (1/3 - \varepsilon, 1/3 + \varepsilon) \), and the 2-cycle is never disjoint from \((a, b)\), which is not particularly interesting.

To simplify our definitions, we say that an integer \( n \geq 3 \) is bad for \((a, b)\) if each \( n \)-cycle for \( T \) has a non-empty intersection with \((a, b)\). Let \( B(a, b) \) denote the set of all \( n \geq 3 \)
Figure 1. The set $D_3$ in $(0, 1) \times (0, 1)$ and $(1/4, 1/2) \times (1/2, 3/4)$ which are bad for $(a, b)$. Put

$$D_3 = \{(a, b) \in (0, 1) \times (0, 1) : B(a, b) = \emptyset\}.$$ 

Thus, $(3/7, 4/7) \in D_3$. Unlike the case of $D_0$ and $D_1$, there is some structure to $D_3$ outside of $(1/4, 1/2) \times (1/2, 3/4)$. We will show in Section 2 that this structure is very easily explained and all interesting structure will still lie within $(1/4, 1/2) \times (1/2, 3/4)$. So, although the set is defined on the larger range $(0, 1) \times (0, 1)$, we will quite often restrict our attention to the range $(1/4, 1/2) \times (1/2, 3/4)$. See Figure 1 for $D_3$, both on $(0, 1) \times (0, 1)$ and $(1/4, 1/2) \times (1/2, 3/4)$.

We will show in Section 2 that the boundary of $D_3$ is made up of a finite number of horizontal and vertical lines – see Figure F.

There is another important milestone in the Sharkovskii order, namely, the threshold below which all the even periods already exist but none of the odd ones does. In the symmetric model $b = 1 - a$ this milestone is $a = 5/12$ whose binary expansion is $01(10)\infty$ – this follows immediately from [1, Proposition 2.16], in which the critical values of $a$ for all the periods are computed. We introduce its natural analogue for the asymmetric case:

$$D_2 = \{(a, b) \in (1/4, 1/2) \times (1/2, 3/4) : B(a, b) \text{ is finite}\}.$$ 

Note that the restriction $(a, b) \in (1/4, 1/2) \times (1/2, 3/4)$ is, again, natural, since, as with $D_0$ and $D_1$, if $(a, b)$ contains $(1/4, 1/2)$ or $(1/2, 3/4)$, there cannot be any disjoint cycles for $(a, b)$. Also, if $b < 1/2$ or $a > 1/2$, then $B(a, b)$ is always finite (or empty). Indeed, let $b < 1/2$ (the case $a > 1/2$ is completely analogous); here one can take $x = \frac{2^n - 1}{2^n - 1}$ and it
Figure 2. The set $D_2$

is easy to check that it is a part of the $n$-cycle $\left\{ \frac{2^n-2}{2^n-1}, \frac{2^n-3}{2^n-1}, \frac{2^n-5}{2^n-1}, \frac{2^n-9}{2^n-1}, \ldots, \frac{2^{n-1}-1}{2^n-1} \right\}$ which lies in $[\frac{2^n-1}{2^n-1}, 1]$, which is disjoint from $(a, b)$ for all sufficiently large $n$.

Although most of the boundary of $D_2$ is made up of horizontal and vertical lines, it is in fact made up of an infinite number of horizontal and vertical lines (one associated to each rational number), creating a kind of Devil’s staircase. The precise structure of this set will be discussed in Section 3 (See Figure 2 as a shape of things to come.)

Thus, one can say that while [4] is about the initial part of the “asymmetric Sharkovskii order” which generalizes the usual period doubling in three different ways (see [4, Section 4.3] for a detailed exposition), the present paper is about the “final stretch” of such orders, which generalizes the usual sequence of odd numbers in the reverse order.

Thus, the main reason why we believe a detailed study of the sets $D_2$ and $D_3$ is interesting is the fact that these sets are cornerstones of the generalized Sharkovskii order, which appears to be an exciting object per se.

Regarding what happens “in between” – generalizing the range in the Sharkovskii order between getting all the powers of two and all the even numbers – note that $\partial D_1$ and $\partial D_2$ have a substantial intersection (see Remark 3.15 and Figure 6 below). This means that for
most patterns, shrinking \((a, b)\) results in simultaneously obtaining infinitely many disjoint cycles as well as finitely many bad \(n\).

Note, in [2], the authors studied for which \(n\) does the Lorentz-like map have an \(n\)-cycle. Our case is somewhat different, because we are also interested in avoiding holes.

2. The set \(D_3\)

In this section we will show that the boundary of \(D_3\) is composed of a finite number of horizontal and vertical lines. We will give a precise description for the locations of these lines.

For any \((w_1, w_2, \ldots) \in \{0, 1\}^\mathbb{N}\) put

\[x = \pi(w_1, w_2, \ldots) = \sum_{j=1}^\infty w_j 2^{-j},\]

i.e., the dyadic (binary) expansion of \(x\). From here on for the sake of notation we will not distinguish between the numbers in \([0, 1]\) and their dyadic expansions.

Since we plan to work closely with 0-1 words, we need some definitions and basic results from combinatorics on words – see [6, Chapter 2] for a detailed exposition. For any two finite words \(u = u_1 \ldots u_k\) and \(v = v_1 \ldots v_n\) we write \(uv\) for their concatenation \(u_1 \ldots u_k v_1 \ldots v_n\). In particular, \(u^m = u \ldots u\) \((m\) times\) and \(u^\infty = uuu\ldots = \lim_{n \to \infty} u^n\), where the limit is understood in the topology of coordinate-wise convergence. We will denote by \(u^*\) the set of words \(\{\lambda, u, u^2, u^3, u^4, \ldots\}\), where \(\lambda\) is the empty word.

From here on by a “word” we will mean a word whose letters are 0s and 1s. Let \(w\) be a finite or infinite word. We say that a finite or infinite word \(u\) is lexicographically smaller than a word \(v\) \((\text{notation: } u < v)\) if either \(u_1 < v_1\) or there exists \(n \geq 1\) such that \(u_i \equiv v_i\) for \(i = 1, \ldots, n\) and \(u_{n+1} < v_{n+1}\). We notice that if \(u < v\) then \(\pi(u) \leq \pi(v)\) with equality only if \(u = w01^\infty\) and \(v = w10^\infty\) for some finite word \(w\).

Recall \(D_3 := \{(a, b) : B(a, b) = \emptyset\}\) where \(B(a, b)\) is the set of bad \(n\) for \((a, b)\). We make two observations:

(i) if \((a, b) \not\in D_3\) then \((a - \delta, b + \varepsilon) \not\in D_3\) for any non-negative \(\varepsilon\) and \(\delta\);

(ii) if \((a, b) \in D_3\), then \((a + \delta, b - \varepsilon) \in D_3\) for any non-negative \(\varepsilon\) and \(\delta\).

We will show that \(D_3\) has a very simple structure, namely that \(D_3\)’s boundary is composed of finitely many horizontal and vertical lines.

**Definition 2.1.** We will say that \((a, b)\) is a **corner of \(D_3\)** if for \((a', b')\) sufficiently close to \((a, b)\) we have

- if \(a' > a\) or \(b' < b\) then \((a', b') \in D_3\);
- if \(a' < a\) and \(b' > b\) then \((a', b') \not\in D_3\).

**Definition 2.2.** We will say that \((a, b)\) is an **anti-corner of \(D_3\)** if for \((a', b')\) sufficiently close to \((a, b)\) we have

- If \(a' > a\) and \(b' < b\) then \((a', b') \in D_3\);
- If \(a' < a\) or \(b' > b\) then \((a', b') \not\in D_3\).
Figure 3. A corner (left) and an anti-corner (right)

See Figure 3 for a sketch. It is worth noting that we make no comment on $a' = a$ or $b' = b$. Although $D_3$ is closed, $D_2$ is open, and we wish to reuse the definition of corner later on for $D_2$. It is not clear a priori that $D_3$ will have corners and anti-corners as we have defined them. For example $\{(x, y) : x > y\}$ does not have either. We will show that in fact $D_3$ is made up of a finite number of corners and anti-corners. Further we will show that these corners and anti-corners completely describes $D_3$.

Theorem 2.3. There are 9 corners of $D_3$. They are

$$(a_1, b_1) = \left(\frac{2}{3}, \frac{3}{3}\right), \quad (a_2, b_2) = \left(\frac{20}{33}, \frac{32}{33}\right), \quad (a_3, b_3) = \left(\frac{10}{31}, \frac{16}{31}\right),$$

$$(a_4, b_4) = \left(\frac{2}{3}, \frac{15}{31}\right), \quad (a_5, b_5) = \left(\frac{2}{3}, \frac{7}{31}\right), \quad (a_6, b_6) = \left(\frac{15}{37}, \frac{37}{37}\right),$$

$$(a_7, b_7) = \left(\frac{15}{31}, \frac{21}{31}\right), \quad (a_8, b_8) = \left(\frac{31}{63}, \frac{43}{63}\right), \quad (a_9, b_9) = \left(\frac{7}{3}, \frac{9}{3}\right).$$

Letting $a_0 = 0$ and $b_{10} = 1$, and the $a_i$ and $b_i$ as above, there are 10 anti-corners of $D_3$. They are:

$$(a_0, b_1), (a_1, b_2), (a_2, b_3), (a_3, b_4), (a_4, b_5), (a_5, b_6), (a_6, b_7), (a_7, b_8), (a_8, b_9), (a_9, b_{10}).$$

Corollary 2.4. If $b - a \leq \frac{2}{13}$, then $(a, b) \in D_3$. On the other hand, if $(a, b) \in D_3$, then $b - a \leq \frac{3}{7}$.

Proof. This follows by noticing that $\min(b_i - a_i) = b_4 - a_4 = \frac{2}{13}$ and $\max(b_i+1 - a_i) = 3 - \frac{3}{7}$. \hfill $\square$

Proof of Theorem 2.3. We will show

- $(a_i - \varepsilon, b_i + \varepsilon) \notin D_3$ for all $\varepsilon > 0$, by explicitly finding an $n$ such that all $n$-cycles intersect $(a_i - \varepsilon, b_i + \varepsilon)$.
- $(a_i, b_{i+1}) \in D_3$ by explicitly giving for each $n \geq 3$ an $n$-cycle that avoids $(a_i, b_{i+1})$. 
These two claims, combined with the observations $[\text{ii}]$ and $[\text{iii}]$ on page 4, prove Theorem 2.3. To see this, assume that we have shown for some fixed $i$ the

(a) $(a_i - \varepsilon, b_i + \varepsilon) \not\in D_3$,
(b) $(a_i, b_{i+1}) \in D_3$.
(c) $(a_{i+1} - \varepsilon, b_{i+1} + \varepsilon) \not\in D_3$ and

We see from (b) above, and observation $[\text{ii}]$ that for all $a' > a_i$ and $b' < b_{i+1}$ that $(a', b') \in D_3$. We see from (m) above, and observation $[\text{i}]$ that for all $a' < a_i$ and $b' > b_i$ that $(a', b') \not\in D_3$. As $b_i < b_{i+1}$, this shows that for $b'$ sufficiently close to $b_i$ and $a' < a_i$ that $(a', b') \not\in D_3$. Similarly, we see that for $a'$ sufficiently close to $a_i$ and $b' > b_{i+1}$ we have that $(a', b') \not\in D_3$. This shows that $(a_i, b_{i+1})$ is an anti-corner.

In a similar fashion, the two claims at the start of the proof would show that each $(a_i, b_i)$ is a corner.

We will show that $D_3$ has a very simple structure, namely that $D_3$’s boundary is composed of finitely many horizontal and vertical lines. To see this we see that if we show all of the points $(a_i, b_i)$ are corners and $(a_i, b_{i+1})$ are anti-corners, we see that the line from $(a_i, b_i)$ to $(a_i, b_{i+1})$ is on the boundary of $D_3$. Similarly the line from $(a_i, b_i)$ to $(a_{i-1}, b_i)$ is also on the boundary of $D_3$. This in turn shows that there cannot be any other corners or anti-corners, which proves the result.

The first part is demonstrated in Table 2.1. Here we give $(a_i, b_i)$, the $n$ for which all $n$-cycles intersect $(a_i - \varepsilon, b_i + \varepsilon)$. For $i = 1, 2, \ldots, 5$ we also give all of the $n$-cycles. For each $n$-cycle we indicate in bold the term in the orbit that intersects $(a_i - \varepsilon, b_i + \varepsilon)$. Note that in some cases there are multiple terms.

Consider, for instance, the special case of showing $(\frac{2}{7} - \varepsilon, \frac{3}{7} + \varepsilon) \not\in D_3$. We see that there are only two different 3-cycles. One of these 3-cycles contains $\frac{2}{7}$ and the other contains $\frac{3}{7}$. Hence this interval will always contain a 3-cycle and hence is not in $D_3$.

To see that $(a_i, b_{i+1})$ is in $D_3$ we must show, for all $n \geq 3$, how to construct an $n$-cycle that avoids $(a_i, b_{i+1})$. These results are summarized in Table 2.2. We will consider only one of these cases in detail, all of the rest are equivalent. The second half of Table 2.2 comes by replacing all of the 0s with 1s and all of the 1s with 0s in the first half of the table.

Consider the special case of finding a 7-cycle that avoids $(a_1, b_2) = (\frac{2}{7}, \frac{32}{63})$. We see that $(0100100)^\infty$, a special case of $(010(010)^*0)^\infty$, is a 7-cycle. We see that the 7 terms in the orbit of $(0100100)^\infty$ are

$(0100100)^\infty, (1001000)^\infty, (0010001)^\infty, (0100010)^\infty, (1000100)^\infty, (0001001)^\infty, (0010010)^\infty$.

By looking at the dyadic expansions, we see that the first, third, fourth, sixth and seventh term are all strictly less that $\frac{2}{7}$, whereas the second and fifth term are strictly larger than $\frac{32}{63}$.

This proves the result that $(a_i, b_i)$ for $i = 1, 2, \ldots, 9$, form the corners of $D_3$ and $(a_i, b_{i+1})$ for $i = 0, 1, \ldots, 9$ form the anti-corners.

**Definition 2.5.** We say that $n \geq 3$ is an exit period if there exists a continuous family of intervals $(a_\alpha, b_\alpha)_{\alpha \in [a_0, a_1]}$ such that
Proof. It is clear from Definition 2.5 that \((a_{\alpha}, b_{\alpha})\) must belong to the boundary of \(D_3\). It follows from the proof of Theorem 2.3 that for any \(\varepsilon > 0\) there exists \((a, b)\) at a distance \(\varepsilon\) from \(\partial D_3\) such that \(B(a, b) \subset \{3, 4, 5, 6\}\). Hence \(EP \subset \{3, 4, 5, 6\}\).
To prove that \( \{3, 4, 5, 6\} \subseteq EP \), it suffices to show that for any \( n \in \{3, 4, 5, 6\} \) the corresponding corner is indeed \( (a_\alpha, b_\alpha) \) for some family of intervals satisfying Definition 2.5. This is a simple check; for instance, for the interval \( \left(\frac{20}{63}, \frac{32}{63}\right) \) we have that any 6-cycle which does not include its endpoints contains two consecutive 1s in its dyadic expansions and thus, must intersect it. Hence \( 6 \in EP \). The cases of \( n = 3, 4, 5 \) are similar. \( \square \)

Remark 2.7. The only symmetric point on the boundary of \( D_3 \), \( \left(\frac{3}{7}, \frac{4}{7}\right) \), corresponds to the appearance of period 3 in the classical Sharkovskii order – see Introduction. We see thus that \( \partial D_3 \) can be perceived as a generalization of period 3 to our asymmetric case.

3. The set \( D_2 \)

Similar to the boundary of set \( D_3 \), the boundary of the set \( D_2 \) consists of horizontal and vertical line segments. Unlike \( D_3 \) though, the boundary of this set consists of an infinite
number of such segments. Definitions 2.1 of corners remains. Definition 2.2 of anti-corners is not relevant in this case (although this is not immediately obvious). If we consider a horizontal line in $D_3$, we see that the right end of this line is a corner and the left end of this line is an anti-corner. In the case of $D_2$, the right end of this line is again a corner and the left end of this line is a limit of corners, and comes from a kind of Devil’s staircase construction. We will make this rigorous in Proposition 3.11.

Before discussing the result in detail, we must first introduce some additional notation from the combinatorics on words. We say that a finite word $w$ is a factor of $w$ if there exists $k$ such that $u = w_k \ldots w_{k+n}$ for some $n \geq 0$. For a finite word $w$ let $|w|$ stand for its length and $|w|_1$ stand for the number of 1s in $w$. The 1-ratio of $w$ is defined as $|w|_1/|w|$. For an infinite word $w_1 w_2 \ldots$ the 1-ratio is defined as $\lim_{n \to \infty} |w_1 \ldots w_n|/n$ (if it exists).

We say that a finite or infinite word $w$ is balanced if for any $n \geq 1$ and any two factors $u, v$ of $w$ of length $n$ we have $||u|_1 - |v|_1| \leq 1$. A finite word $w$ is cyclically balanced if all of its cyclic permutations are balanced. (And therefore, $w^\infty$ is balanced.) It is well known that if $u$ and $v$ are two cyclically balanced words with $|u| = |v| = q$ and $|u|_1 = |v|_1 = p$ and $\gcd(p, q) = 1$, then $u$ is a cyclic permutation of $v$. Thus, there are only $q$ distinct cyclically balanced words of length $q$ with $p$ 1s.

A finite word $w$ which begins with 0 is called 0-max if it is larger than any of its cyclic permutations beginning with 0. A finite word is called 1-min if it is smaller than any if its cyclic permutations beginning with 1. Similarly, an infinite word $w = w_1 w_2 \ldots$ with $w_1 = 0$ is 0-max if $(w_{k+1}, w_{k+2}, \ldots) \prec w$ for any $k \geq 1$ such that $w_{k+1} = 0$. An infinite word $w = w_1 w_2 \ldots$ with $w_1 = 1$ is 1-min if $(w_{k+1}, w_{k+2}, \ldots) \succ w$ for any $k \geq 1$ such that $w_{k+1} = 1$.

For any $r = p/q \in \mathbb{Q} \cap (0, 1)$ we define two words as follows: $s(r)$ is the lexicographically largest cyclically balanced word of length $q$ with 1-ratio $r$ beginning with 0, and $t(r)$ is the lexicographically smallest cyclically balanced word of length $q$ with 1-ratio $r$ beginning with 1. In particular, $s(r)$ is 0-max and $t(r)$ is 1-min.

Note that there is an explicit way to construct $s(r)$ and $t(r)$ for any given $r$. Namely, let $r = p/q \leq 1/2$ have a continued fraction expansion $[d_1 + 1, \ldots, d_n]$ with $d_n \geq 2$ and $d_1 \geq 1$ (in view of $r \leq 1/2$). We define the sequence of 0-1 words given by $r$ as follows: $u_{-1} = 1, u_0 = 0, u_{k+1} = u_k d_{k+1} + u_{k-1}$, $k \geq 0$. The word $u_n$ has length $q$ and is called the $n$th standard word given by $r$. Given an irrational $\gamma \in (0, 1/2)$ with the continued fraction expansion $\gamma = [d_1 + 1, d_2, \ldots]$, the word $u_\infty$ defined as the limit of the $u_n$ is called the characteristic word given by $\gamma$.

Let $w_1 \ldots w_q := u_n$. Then

$$s(r) = 01w_1 \ldots w_{q-2}, \quad t(r) = 10w_1 \ldots w_{q-2}. \quad (3.1)$$

For $r \in \mathbb{Q} \cap (1/2, 1)$ we have $s(r) = \overline{t(1-r)}$, $t(r) = \overline{s(1-r)}$, where $\overline{0} = 1, \overline{1} = 0$, and $w_1 w_2 = w_1 w_2$ for any two words $w_1, w_2$.

**Example 3.1.** We have $s(2/5) = 01010, \quad t(2/5) = 10010, \quad s(3/5) = 01101, \quad t(3/5) = 10101.$
Lemma 3.2. Assume \( r_1 = p_1/q_1 \) and \( r_2 = p_2/q_2 \) to be Farey neighbours with \( r_1 < r_2 \) and \( r_2 \leq 1/2 \), i.e., with \( p_2q_1 - p_1q_2 = 1 \). Put

\[
r_3 := \frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2},
\]

i.e., \( r_3 \) is the mediant of \( r_1 \) and \( r_2 \). Put \( s_i = s(r_i), t_i = t(r_i) \) for \( i = 1, 2, 3 \). Then we have

\[
s_3 = s_2s_1, \tag{3.2}
\]

\[
t_3 = t_1t_2, \tag{3.3}
\]

\[
s_3 = s_1t_2 \tag{3.4}
\]

and

\[
t_3 = t_2s_1. \tag{3.5}
\]

Proof. Let the continued fraction expansion for \( r_1 \) be \([d_1, 1, d_{m-1}, d_m]\), in which case \( m \) is even and \( r_2 = [d_1 + 1, \ldots, d_{m-1}] \). Then, as is well known, the continued fraction expansion of their mediant \( r_3 \) is \([d_1 + 1, \ldots, d_{m-1}, d_m + 1]\).

Since \( m \) is even, the standard words \( u_m \) and \( u_m' \) which correspond to \( r_1 \) and \( r_3 \) respectively, end with 10, while \( u_{m-1} \) which corresponds to \( r_2 \) ends with 01. Denote \( u_m = v_m 10, u_{m-1} = v_{m-1} 01 \) and \( u_m' = v_m' 10 \). We have \( u_m = u_{m-1} u_{m-2} \) and \( u_m' = v_{m-1} u_{m-2} = u_{m-1} u_m \), whence

\[
v_m' 10 = v_{m-1} 01 v_m 10 = v_{m-1} s_1 10.
\]

Prepending 01 as a prefix for both sides of this equation and deleting the suffix 10, we obtain \((3.2)\). Replacing the first two symbols 01 with 10 yields \((3.5)\).

To prove \((3.3)\), notice that \((u_{m-1}, u_m)\) is a standard pair (see [6, Chapter 2.2.1]), whence \( u_{m-1} u_m \) and \( u_m u_{m-1} \) differ only by the last two symbols (cf. Proposition 2.2.2(iii))]. Therefore,

\[
v_{m-1} 01 v_m = v_m 10 v_{m-1}.
\]

Prepending 10 as a prefix for both sides of the equation, we obtain \( t_2 s_1 = t_1 t_2 \), which in view of \((3.5)\) yields \((3.3)\). Replacing the first two symbols 01 with 10 in \((3.3)\) yields \((3.4)\).

The case of \( r_1 = [d_1 + 1, \ldots, d_{m-1}] \) and \( r_2 = [d_1 + 1, \ldots, d_{m-1}, d_m] \) implies that \( m \) is odd and is treated similarly, so we omit the proof.

\[
\square
\]

Corollary 3.3. We have for \( r_n \) and \( r \) in \( \mathbb{Q} \) that

\[
\lim_{r_n \uparrow r} s(r_n) = s(r) \uparrow r, \quad \lim_{r_n \downarrow r} t(r_n) = t(r) \downarrow r,
\]

\[
\lim_{r_n \uparrow r} t(r_n) = t(r) \uparrow r, \quad \lim_{r_n \downarrow r} t(r_n) = t(r) \downarrow r.
\]

Here \( r_n \uparrow r \) is the one-sided limit from the left, and \( r_n \downarrow r \) the one-sided limit from the right.

Proof. We observe that the map \( r \to s(r_n) \) and \( r \to t(r_n) \) are both strictly monotonically increasing. Hence, it is sufficient to show this result on a subsequence of \( r_n \to r \) from either above or below. Let \( r_0 \) be a Farey neighbour of \( r \). Without loss of generality assume \( r_0 < r \).
Define recursively $r_n = r \oplus r_{n-1}$. We see that $s(r_n) := s(r)s(r_{n-1}) = s(r)^n s(r_0) \to s(r)^\infty$ and similarly $t(r_n) \to t(r)s(r)^\infty$. The other cases are similar. \hfill \square

Given $r = p/q \in (0,1)$, put $s = s(r), t = t(r)$. We know that $s^\infty$ and $t^\infty$ belong to the same $q$-cycle. Furthermore, it follows from [7, Corollary 3.6] that this is the only cycle in $J(s^\infty, t^\infty)$.

Remark 3.4. Note that the notation in [7, Corollary 3.6] differs from the present paper. Specifically, $\alpha_r$ in [7] stands for $s(r)^\infty$ and $\gamma_r$ for $t(r)^\infty$, where $r = p/q$. The $T$-orbit of $\alpha_r$ is denoted by $O(p/q)$.

Let $\{s, t\}^\omega$ denote the set of infinite words which are concatenations of $s$ and $t$ together with its shifts. That is,
\[
\{s, t\}^\omega = \{T^k a_1 a_2 a_3 \ldots | k \geq 0, a_i \in \{s, t\}\}
\]

Lemma 3.5. We have
\[
J(st^\infty, ts^\infty) \cap (s^\infty, t^\infty) \subset \{s, t\}^\omega \subset J(st^\infty, ts^\infty).
\]

Proof. Suppose $x \in (s^\infty, t^\infty) \setminus (st^\infty, ts^\infty)$. Then either $x \in (s^\infty, st^\infty)$ or in $(ts^\infty, t^\infty)$. Both cases are similar, so let us assume the former. Then the dyadic expansion of $x$ begins with $s$. If $T^q x \notin (st^\infty, ts^\infty)$, then again, its dyadic expansion begins with either $s$ or $t$, etc. – see Figure 4 (Note that $T^q$ acts on the dyadic expansions as the shift by $q$ symbols.) This proves $J(st^\infty, ts^\infty) \cap (s^\infty, t^\infty) \subset \{s, t\}^\omega$.

Now let us show that
\[
\{s, t\}^\omega \subset J(st^\infty, ts^\infty).
\]
Let $w \in \{s,t\}^{\omega'}$. It suffices to show that for any $j \geq 0$ such that $w_{j+1} = 0$ we have $(w_{j+1}, w_{j+2}, \ldots) \prec st^\infty$ and for any $j$ such that $w_{j+1} = 1$ we have $ts^\infty \prec (w_{j+1}, w_{j+2}, \ldots)$. Both claims are similar, so we will only prove the first.

Let $s = s_1 \ldots s_q$ and $t = t_1 \ldots t_q$. We first consider the case $q = 2$. Here $r = 1/2, s = 01$ and $t = 10$. It is a simple check that the $0$-max of $(s, t)^{\omega'}$ is $01(10)^\infty = st^\infty$, and the $1$-min is $ts^\infty$. This yields (3.6) for $q = 2$.

If $q \geq 3$, then $s_k = t_k, 3 \leq k \leq q$. Since $s \prec t$, it suffices to show that

$$s_{j+1} \ldots s_q(t_1 \ldots t_q)^\infty \prec st^\infty$$

provided $s_{j+1} = 0$. Since $s$ is $0$-max, we have $s_{j+1} \ldots s_q \preceq s_1 \ldots s_{q-j}$; furthermore, if we have an equality, then $s_{q-j+1} = 1$ ([7, Lemma 5.1]). Thus, if $s_{j+1} \ldots s_q \prec s_1 \ldots s_{q-j}$, we are done. Otherwise, (3.7) will follow from

$$t_1 \ldots t_q \prec s_{q-j+1} \ldots s_q t_1 \ldots t_{q-j}.$$ 

Consider first the case $j = q - 1$. Here (3.8) turns into

$$t_2 \ldots t_q \prec t_3 \ldots t_q t_1.$$ 

Since $t$ is $1$-min and $s$ is its cyclic permutation, we have that any of the cyclic permutations of $s$ which begins with $1$ is lexicographically larger than $t$. Hence $t_1 \ldots t_q \prec s_2 \ldots s_q s_1$. By noticing that $t_1 = s_2 = 1, t_k = s_k$ for $k = 3, 4, \ldots q$ and $s_1 = 0 < t_1 = 1$ we get $t_2 \ldots t_q \prec s_3 \ldots s_q s_1 \prec t_3 \ldots t_q t_1$ as required. Hence $t_1 \ldots t_q \prec s_2 \ldots s_q s_1$, which implies (3.9), as $s_1 < t_1$.

Now assume $j \leq q - 2$. Since $s_k = t_k, 3 \leq k \leq q$, this is equivalent to

$$t_1 \ldots t_q \prec t_{q-j+1} \ldots t_q t_1 \ldots t_{q-j},$$

which follows from $t_{q-j+1} = s_{q-j+1} = 1$ and $t$ being $1$-min.

\textbf{Corollary 3.6.} We have

$$\dim_H \mathcal{J}(st^\infty, ts^\infty) = \frac{1}{q}.$$ 

\textbf{Proof.} It follows from [7, Corollary 3.6] that $\mathcal{J}(s^\infty, t^\infty)$ is a countable set. When $x \in \mathcal{J}(st^\infty, ts^\infty) \cap [s^\infty, t^\infty]$, we know that its dyadic expansion is a concatenation of the blocks $s$ and $t$. This means that the topological entropy of $T$ restricted to $\mathcal{J}(st^\infty, ts^\infty)$ is $\frac{1}{q} \log 2$, whence the claim follows from the well known formula

$$\text{Hausdorff dimension} = \text{topological entropy/Lyapunov exponent}$$

(see, e.g., the seminal paper [3]) and the fact that the Lyapunov exponent of $T$ is equal to $\log 2$. \hfill \Box

\textbf{Remark 3.7.} It is important to state the exact logical dependence between results in [7] and the present paper. Namely, Lemma [3,2] $\rightarrow$ [7, Lemma 2.5] $\rightarrow$ [7, Corollary 3.6] $\rightarrow$ Corollary 3.6.

Now we are ready to prove the main result of this section.
Theorem 3.8. Let \( s := s(r), t := t(r) \) for \( r \in \mathbb{Q} \cap (0,1) \). For all \( r \in \mathbb{Q} \cap (0,1) \) we have that \((s^\infty, ts^\infty)\), \((st^\infty, ts^\infty)\) and \((st^\infty, t^\infty)\) are on the boundary of \( D_2 \). In particular, \((st^\infty, ts^\infty)\) is a corner point. Furthermore, \((s^\infty, ts^\infty)\) and \((st^\infty, t^\infty)\) are the limit points of corner points.

Proof. We will first show that \((st^\infty, ts^\infty)\) is on the boundary of \( D_2 \). Then, using Corollary 3.3 we get that \((s^\infty, ts^\infty)\) and \((st^\infty, t^\infty)\) are also on the boundary of \( D_2 \) by noting that

\[
\lim_{r_n \uparrow r} (s(r_n) t(r_n)^\infty, t(r_n) s(r_n)^\infty) = (s(r)^\infty, t(r) s(r)^\infty)
\]

and

\[
\lim_{r_n \uparrow r} (s(r_n) t(r_n)^\infty, t(r_n) s(r_n)^\infty) = (s(r) t(r)^\infty, t(r)^\infty).
\]

This then implies that \((st^\infty, ts^\infty)\) is a corner point.

Let us prove first that \( (st^\infty, ts^\infty) \) is on the boundary of \( D_2 \). Note first that by Lemma 3.5, \( J(st^\infty, ts^\infty) \) contains only cycles whose lengths are multiples of \( q \), we have that \( (st^\infty, ts^\infty) \notin D_2 \). Consequently, \( (st^\infty - \varepsilon, ts^\infty + \varepsilon) \notin D_2 \) for any \( \varepsilon > 0 \).

To see for arbitrarily small \( \varepsilon > 0 \) that \( (st^\infty + \varepsilon, ts^\infty) \) and \( (st^\infty, ts^\infty - \varepsilon) \) are in \( D_2 \) we must show that there exists an \( N \) (dependent on \( \varepsilon \)) such that for all \( n > N \) there exists an \( n \)-cycle that is disjoint from \( (st^\infty, ts^\infty - \varepsilon) \) and an \( n \)-cycle that is disjoint from \( (st^\infty + \varepsilon, ts^\infty) \).

We will show the first case only, the second case is symmetric.

Since \( s \) and \( t \) are cyclic permutations of each other, we can write \( s = uv \) and \( t = vu \). More precisely, (3.4) and (3.5) yield explicit \( u \) and \( v \) with \( \gcd(|u|, |s|) = \gcd(|v|, |s|) = 1 \). We will first show that the orbit of \( w := (ut^m)^\infty \) is disjoint from \( (st^\infty, ts^\infty - \varepsilon) \) for all sufficiently large \( m \).

Let \( q = |s| = |t| \), where \( r = p/q \) and \( j = |u| \). Write

\[
w = (ut^m)^\infty = (u(vu)^m)^\infty = (uv)(uv)(uv) \ldots uu \ldots
\]

\[
\text{(3.10)}
\]

\[
= ss \ldots s uu \ldots
\]

\[
= u(vu)(vu)(vu)(vu)(vu) \ldots uu \ldots
\]

\[
\text{(3.11)}
\]

Let as usual, \( \text{dist}(x, y) = 2^{-\min\{j \geq 1 \mid x_j \neq y_j\}} \) for any pair \( x, y \in \{0,1\}^N \). Letting \( x = x_1 x_2 \ldots, y = y_1 y_2 \ldots, \in \{0,1\}^N \) we see that \( | \sum x_i/2^i - \sum y_i/2^i | \leq 2 \text{dist}(x, y) \).

By (3.10), we have \( \text{dist}(T^i w, T^i s^\infty) \leq 2^{-mq} \) for all \( i \leq j \). Furthermore, since \( w = s^m uu \ldots \) and \( s^\infty = s^m uu \ldots \), we have \( T^i w \prec T^i s^\infty \) for all \( i \leq j \). Lemma 3.5 yields \( s^\infty \in J(st^\infty, ts^\infty) \), whence for \( m \) sufficiently large the first \( j \) terms in the orbit of \( w \) are disjoint from \((st^\infty, ts^\infty - \varepsilon)\).
By (3.11), we have dist\((T^i w, T^{i-j}(tm^\infty s^\infty))\) \(\leq 2^{-mq}\) and \(T^i w \prec T^{i-j}(tm^\infty s^\infty)\) if \(j + 1 \leq i \leq mq + j\). Again, by Lemma 3.3, \(tm^\infty s^\infty \in J(st^\infty, ts^\infty)\), whence for \(m\) sufficiently large the \(j + 1\)-st term to the \(mq\)-th in the orbit of \(w\) are disjoint from \((st^\infty, ts^\infty - \varepsilon)\).

Thus, we have proved that for \(m\) sufficiently large the orbit of \(w = (ut^m)^\infty\) (whose length is \(mq + j\)) is disjoint from \((st^\infty, ts^\infty - \varepsilon)\). Now we will show for all \(\ell \in \{0, 1, \ldots, q - 1\}\) there exists a word \(W\) of length \(mq + \ell\) (for all \(m\) sufficiently large) such that the orbit of \(W\) is disjoint from \((st^\infty, ts^\infty - \varepsilon)\).

Since \(j\) is coprime with \(q\), for all \(\ell\) there exists a \(k\) such that \(\ell \equiv kj \mod q\). We now let \(m_1, m_2, \ldots, m_k\) be sufficiently large, and distinct, such that each \((u(vu)^{m_1})^\infty\) is disjoint from \((st^\infty, ts^\infty - \varepsilon)\). Consider
\[
W := (u(vu)^{m_1}u(vu)^{m_2} \ldots u(vu)^{m_k})^\infty.
\]

The same argument as before shows that the orbit of \(W\) is disjoint from \((st^\infty, ts^\infty - \varepsilon)\). Furthermore, \(|W| \equiv \ell \mod q\), which concludes the proof of the first part of the claim.

This shows that all neighbourhoods of \((st^\infty, ts^\infty)\) have points in \(D_2\) and points not in \(D_2\), and hence it is a boundary point.

To see it is a corner point, we consider \(r_n \uparrow r\) and notice that \((s(r_n)t(r_n)^\infty, t(r_n)s(r_n)) \rightarrow (s(r)^\infty, t(r)s(r)^\infty)\) must also be on the boundary. Similarly \((st^\infty, ts^\infty)\) is a boundary point. This implies that \((st^\infty, ts^\infty)\) is a corner point. \(\Box\)

For a sketch of \(D_2\) see Figure 6. For the purposes of that diagram, let \(p(r) = (s(r)t(r)^\infty, t(r)s(r)^\infty)\) and \(p'(r) = (s(r)^\infty, t(r)s(r)^\infty)\). We notice that visually \(p(n/(2n + 1)) \rightarrow p'(1/2)\), as proven theoretically in Theorem 3.8.

**Corollary 3.9.** If \(b - a < \frac{1}{6}\), then \((a, b) \in D_2\), and the constant \(\frac{1}{6}\) cannot be improved. If \((a, b) \in D_2\) then \(b - a < \frac{1}{4}\) and the constant \(\frac{1}{4}\) cannot be improved.

**Proof.** As usual, let \(t = t(r)\) and \(s = s(r)\) for some \(r\). We have
\[
\inf\{b - a : (a, b) \in D_2\} = \min\{b - a : (a, b)\ \text{is a corner of} \ D_2\} = ts^\infty - st^\infty = ts^\infty - s^\infty - (st^\infty - s^\infty)
\]
\[
= t - s - 2^{-q}(t^\infty - s^\infty) = \frac{1}{4} - 2^{-q}(t^\infty - s^\infty)
\]
\[
= \frac{1}{4} - \frac{1}{4(2^q - 1)} = \frac{2q - 2}{4(2^q - 1)},
\]
and its minimum is attained at \(q = 2\) and is equal to \(\frac{1}{6}\). Clearly, \(\frac{1}{6}\) cannot be improved, in view of \(\left(\frac{5}{12}, \frac{7}{12}\right)\) being a corner of \(D_2\).

We similarly have
\[
\sup\{b - a : (a, b) \in D_2\} = \max\{b - a : (a, b)\ \text{is the left endpoint of a horizontal line}\} = ts^\infty - s^\infty = t - s
\]
\[
= \frac{1}{4}.
\]
\(\Box\)
It is interesting to note that the minimum occurs in only one location whereas the maximum occurs in infinitely many places.

**Remark 3.10.** Combined with the results of [4], we now have four sharp constants \( c_0 = \frac{1}{2}, \quad c_1 = \frac{1}{2} \prod_{n=1}^{\infty} (1 - 2^{-2n}) \approx 0.175092, \quad c_2 = \frac{1}{6} \) and \( c_3 = \frac{1}{3} \) such that if \( b - a < c_j \), then \((a, b) \in D_j \) for \( j = 0, 1, 2, 3 \).

Put for \( \frac{1}{4} \leq a \leq \frac{1}{2} \),

\[ \kappa(a) = \sup \{ b : \mathcal{J}(a, b) \text{ contains infinitely many cycles} \}. \]

**Proposition 3.11.** We have

\[ \left\{ (a, b) \in \partial D_2 : \frac{1}{4} < a \leq \frac{1}{2} \right\} = \text{cl} \left( \left\{ (a, \kappa(a)) : \frac{1}{4} < a \leq \frac{1}{2} \right\} \cup \left\{ (\kappa(a), a) : \frac{1}{4} < a \leq \frac{1}{2} \right\} \right), \]

where \( \kappa(a) \)

- (i) is non-decreasing;
- (ii) is constant almost everywhere;
- (iii) has jump discontinuities at \( a = s(r)t(r)^\infty \) for every \( r \in \mathbb{Q} \cap (0, 1) \).

**Remark 3.12.** This is a kind of Devil’s staircase. The traditional Devil’s staircase is continuous non-decreasing function that is constant almost everywhere. Here the first restriction of continuity is relaxed, allowing instead jump discontinuities at \( t(r)s(r)^\infty \).

**Proof.** Item (i) is obvious from the definition of \( \kappa \); item (iii) follows from the fact that \((s(r)t(r)^\infty, t(r)s(r)^\infty)\) and \((s(r)t(r)^\infty, t(r)^\infty)\) are both on the boundary of \( D_2 \) (see Theorem 3.8), so we have a jump. Let us prove (ii).

By the above, we may define \( \kappa \) at \( s(r)t(r)^\infty \) either as \( t(r)s(r)^\infty \) or \( t(r)^\infty \) as both will give the same result in the closure. Following [7], we introduce the set \( \mathcal{S} \) which is defined as a union of the points in \((1/4, 5/12)\) whose dyadic expansion is of the form \( 01w \), where \( w \) is the characteristic word for some irrational \( \gamma \in (0, 1/2) \) (an uncountable set) and a countable set \( \bigcup_{r \in (0, 1/2) \cap \mathbb{Q}} \{ s(r)^\infty, s(r)t(r)^\infty \} \). This set is related to the exceptional set \( \mathcal{E} \) for our staircase in the following way (see [7, Section 3]):

\[ \left( \frac{1}{4}, \frac{5}{12} \right) \setminus \bigcup_{r \in (0, 1/2) \cap \mathbb{Q}} (s(r)^\infty, s(r)t(r)^\infty) = \mathcal{S} \cap \left( \frac{1}{4}, \frac{5}{12} \right). \]

Consequently,

\[ \mathcal{E} = \left( \frac{1}{4}, \frac{5}{12} \right) \setminus \bigcup_{r \in (0, 1/2) \cap \mathbb{Q}} [s(r)^\infty, s(r)t(r)^\infty] = \mathcal{S} \setminus \bigcup_{r \in (0, 1/2) \cap \mathbb{Q}} \{ s(r)^\infty, s(r)t(r)^\infty \} \cap \left( \frac{1}{4}, \frac{5}{12} \right). \]

The result now follows from the fact that \( \mathcal{S} \) has zero measure. (In fact, even zero Hausdorff dimension – see [7, Corollary 3.12].) \[ \square \]
Since the length of the plateau given by $r = p/q$ is of order $\frac{1}{4} \cdot 2^{-q}$, one can say colloquially that the exceptional set corresponds to the case $q \to +\infty$.

**Corollary 3.13.** The set $D_2$ is open. Consequently, $\partial D_2 \cap \partial D_3 = \emptyset$.

**Proof.** Fix $r$ and denote $s = s(r), t = t(r)$. As mentioned in the proof of Theorem 3.8, $(st^\infty, ts^\infty) \notin D_2$, whence the same is true for the plateaus, i.e., $([s^\infty, st^\infty] \times \{ts^\infty\}) \cap D_2 = \emptyset$ and $(\{st^\infty\} \times [ts^\infty, t^\infty]) \cap D_2 = \emptyset$ otherwise. As for the exceptional set, it is known that for any given irrational $\gamma \in (0, 1/2)$ its characteristic word is aperiodic (see, e.g., [6, Chapter 2]), whence this set cannot contain any cycles.

The second claim is a direct consequence of $D_2$ being open, $D_3$ being closed and $D_3 \subset D_2$. \qed

**Remark 3.14.** Figure 6 suggests that the two closest points on the boundaries of $D_2$ and $D_3$ are the anti-corner $(10/31, 8/15)$ of $D_3$ and the corner $(9/28, 15/28)$ of $D_2$, whence the distance between $\partial D_2$ and $\partial D_3$ is equal to $\frac{\sqrt{1186}}{13020} \approx 0.002645$. We leave this a conjecture for the interested reader.
Remark 3.15. It is interesting to compare the boundaries of $D_0, D_1$ and $D_2$. It follows from \cite{4} Proposition 2.6 that $\partial D_0 \cap (1/4, 5/12)$ is also the graph of a function of $a$, denoted by $\phi$, which has exactly the same plateau regions as $\kappa$. However, $\phi(a) \equiv t(r)\infty$ on $[s(r)\infty, s(r)t(r)\infty]$, whereas, as we know, $\kappa(a) \equiv t(r)s(r)\infty$ on the same segment, which is strictly less. As $q$ grows, these values tend to the same limit, whence $\phi$ and $\kappa$ coincide on $\mathcal{E}$.

For $D_1$ the corresponding function, $\chi$, is also a kind of Devil’s staircase on $(1/4, 5/12)$, however, it has a significantly more complicated set of plateau regions. Nonetheless, it follows from \cite{4} Theorem 2.13 that $\chi(a) \equiv \kappa(a)$ on $[s(r)\infty, s(r)t(r)s(r)\infty]$. In particular,

$$\partial D_0 \cap \partial D_1 \cap \partial D_2 = \left\{ \left( a, a + \frac{1}{4} \right) : a \in \mathcal{E} \right\}.$$  

See Figure 6 for hints of more details.

Finally, we would like to describe all possibilities for a “final stretch”, i.e., all possible sequences in $B(a, b)$ when we descend from $\partial D_2$ towards $\partial D_3$. By definition, when $(a, b) \in$
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Put \( \mathbb{N}_L = \{L, L+1, L+2, \ldots \} \) for any \( L \geq 3 \). Note first that if \( (a, b) \) is in the exceptional set \( \mathcal{E} \), then \( B(a, b) = \mathbb{N}_3 \), since \( J(a, b) \) does not contain any cycles.

Assume first that \( (a, b) \) is on a horizontal plateau \( [s^\infty, st^\infty] \times \{ts^\infty\} \) for some \( r = p/q \) and \( s = s(r), t = t(r) \). If \( a = s^\infty \), then, as we know, \( J(a, b) \) contains only a \( q \)-cycle; in fact, the same is true for all \( a \in [s^\infty, sts^\infty] \), since for any of those there exists \( k \geq 1 \) such that \( T^kq(a) \in [st^\infty, ts^\infty] \) (see Figure 4), which implies \( J(a, b) = J(s^\infty, t^\infty) \).

Now assume \( a \in (sts^\infty, st^\infty) \). There exists \( N \) such that \( a > (sts^{n-2})^\infty \) for all \( n \geq N \). We claim that if \( a > (sts^{n-2})^\infty \), then the orbit of \( (sts^{n-2})^\infty \) is contained in \( J((sts^{n-2})^\infty, ts^\infty) \), which follows from \( (sts^{n-2})^\infty \) being a 0-max – a claim which is proved in a way similar to the proof of (3.6) (using (3.7)), so we leave it to the interested reader.

This implies that \( J(a, b) \) contains cycles of all sufficiently large lengths which are multiples of \( q \). In view of Lemma 3.5 \( J(a, b) \) does not contain any cycle of length \( qn + j \) for \( j \neq 0 \). The case of vertical plateaus is analogous, so we omit it.

Thus, we have two essentially different possibilities for a “final stretch”:

(i) \( B(a, b) \cap \mathbb{N}_L = \mathbb{N}_L \) for some \( L \geq 3 \);
(ii) \( B(a, b) \cap \mathbb{N}_L = (\mathbb{N}_L \setminus q\mathbb{N}) \) for some \( L \geq 3 \) and some \( q \geq 2 \).

The classical Sharkovskii order corresponds to the second case with \( q = 2 \).

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