On the applicability of approximations used in calculation of spectrum of Dark Matter particles produced in particle decays

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For the Warm Dark Matter candidates, the momentum distribution of particles becomes important, since it can be probed with observations of Lyman-α forest structures. We recall the calculation \[1\] of the spectrum in case of dark matter nonthermal production in decays of heavy particles emphasizing on the inherited applicability conditions, which are rather restrictive and sometimes ignored in literature.

I. INTRODUCTION

One of the major puzzles in physics as we know it at present—Dark Matter (DM) phenomenon—requires new massive electrically neutral collisionless particles stable at cosmological time-scale \[2\]. They must be produced in the early Universe before the plasma temperature \(T\) drops below 1 eV, since later cosmological stages definitely need them \[3\].

While the Universe expansion schedule is sensitive only to the total energy density associated with the new non-relativistic particles (and hence to their number density at a given particle mass), the evolution of spatial inhomogeneities of matter is also sensitive to the velocity distribution of the DM particles. Indeed, free streaming of the DM particles smooths out all the inhomogeneities smaller than the so-called free-streaming length \(l_{\text{f.s.}}\). The latter is of order \(l_{\text{f.s.}} \sim v \times l_H\), where \(v\) is the DM average velocity and \(l_H\) is the Hubble horizon size at a given time. In order for successful generation of the smallest observed primordial structures—dwarf galaxies—one needs \(v \lesssim 10^{-3}\) at the epoch of radiation-matter energy density equality, \(T \sim 1\) eV. This requirement defines the border line between faster and slower candidates named as Hot and Cold DM.

The candidates right at the border are called Warm DM, and the question about velocity distribution is mostly relevant for them. In fact, the Hot DM is disfavored by structure formation and may form only a small fraction of DM (precise amount depends on the velocity distribution). Then Cold DM candidates, like Weakly Interacting Massive Particles, are typically very slow at equality, and allow for formation of structures much smaller (and lighter) than the dwarf galaxies. These structures are expected to be starless and empty of baryons after reionization epoch and (partially) destroyed during subsequent formation of heavier structures. Yet if some remained, searches for gravitational lensing events in galaxies may (in principle) test them and reveal the structure abundance (the DM velocity distribution defines the size of smallest structures). Therefore, both Hot and Cold DM components suggest potential observables sensitive to the velocity distribution. However, this is the Warm DM case, where the corresponding observables have provide the most nontrivial constraints on the DM models, and they have been actively exploited in the literature.

The most promising observable for this task is the small structures in the Lyman-α forest \[4,5\]. Their studies have been already allowed for ruling out some models, e.g. (keV scale) sterile neutrino DM \[6\] produced nonthermally by oscillations of active neutrino in primordial plasma \[7\]. However, many other candidates are still valid (e.g. light gravitino \[8\], axino \[9\], etc.), which are mostly nonthermally produced, for a review see e.g. \[10\]. Moreover, even in the case of sterile neutrino DM various other mechanisms were proposed, such as resonant production \[11\], thermal production with subsequent dilution \[12\], production in decays of scalar particles \[13-15\], all leading to some ways to evade the present constraints from the Lyman-α \[16\].

To use the observations of Lyman-α forest structure in a particular model one must know the velocity distribution of the DM particles. In this note we focus our attention on the DM produced in the early Universe in decays of some particle, which we, following \[1\], denote as DDM. Several regimes are possible, corresponding to the DDM particle being in or out of thermal equilibrium. The case of DDM in thermal equilibrium due to annihilations channel in the SM particles, while also having a small decay branching ratio into the DM, is analyzed in \[13-14\]. The production happens mostly at temperature \(T \sim m_{\text{DM}}\) leading to the distribution which average momentum is slightly below the thermal one and later can be cooled further due to the decrease of degrees of freedom of the relativistic plasma in the expanding Universe \[13\]. This mechanism leads to the lower bound on the DM mass \(m_{\text{DM}} > 5.2\) keV following from analysis of \[16\]. Seemingly another situation corresponds to
the case of DDM decaying while being out of thermal equilibrium, \[1\] (in particular, this may correspond to the DDM itself produced in a non-thermal way). It was argued, that for sufficiently long living DDM, the majority of its decays happen when it is significantly non-relativistic, leading to a peculiar momentum distribution, strongly shifted towards low momenta. We show in this note, that the approximation of non-relativistic decay is actually valid only for the high energy part of the DM spectrum, while the low energy part is produced at earlier staged, when DDM still has non-negligible velocities. The result means that most cases of the DDM, decaying while out of thermal equilibrium, are also reduced (or effectively very close) to the situation described in \[13\] [14].

II. GENERAL FORMALISM

In the model we have two sets of particles—the decaying one of mass \(M\) and the dark matter (stable one of mass \(m_{\text{DM}}\)) which is a product of a two body decay of the initial particle. Distribution of the particles over momentum \(f\) are normalized to the physical particle number density \(n\) in the expanding Universe with scale factor \(a\) as

\[
n = \frac{d^3p}{(2\pi)^3} f(pa) = \frac{d^3k}{(2\pi)^3 a^3} f(k) = \int_0^\infty \frac{k^2dk}{2\pi^2 a^3} f(k), \quad (1)
\]

with \(p\) and \(k = pa\) being physical and conformal 3-momenta, respectively. We may be sloppy of writing the conformal or physical momentum as an argument to \(f\). They can always be mapped on to another, one should just be careful for the solution of the kinetic equations, where the conformal momentum is always used. Also we often drop the time dependence where it does not lead to ambiguities. We use conformal \(\eta\) and physical time \(t\) (related by \(dt = ad\eta\)) interchangeably, which should be clear from notations). The normalization used corresponds to \(f(k)\) staying constant in time in the absence of interactions, and number density decreasing as \(n \propto 1/a^3\).

The equation for the DDM evolution (c.f. eq. (1) of [1])

\[
\frac{df_{\text{DDM}}(k_{\text{DDM}}, \eta)}{d\eta} = -\frac{aM}{\tau E_{\text{DDM}}} f_{\text{DDM}}(k_{\text{DDM}}, \eta), \quad (2)
\]

where \(\tau\) is the DDM lifetime. The equation for the DM is

\[
\frac{df_{\text{DM}}(k_{\text{DM}}, \eta)}{d\eta} = \frac{aM^2}{\tau E_{\text{DM}} p_{\text{DM}}^{PCM}} \int_{E_1}^{E_2} f_{\text{DDM}}(ap) dE
\]

\[
= \frac{a^3 M^2}{\tau k_{\text{DM}}^3} \int_{p_{\text{DM}} + m_{\text{DM}}^2 / 4p_{\text{DM}}}^{\infty} f_{\text{DDM}}(ap) dE, \quad (3)
\]

where \(p_{\text{DM}} \equiv k_{\text{DM}}/a\), \(p \equiv \sqrt{E^2 - M^2}\). There is overall coefficient 2 as compared to (2) of [1], because there it was assumed that only one of the two-body decay products is the DM, while we assume that both are DM.

III. ANALITICAL SOLUTIONS TO THE EQUATIONS IN CASE OF RADIATION DOMINATION AND CONSTANT \(g_\ast\)

Solution to eq. (2) at the radiation dominated stage with scale factor \(a = c\eta\) is (c.f. eq. [13])

\[
f_{\text{DDM}}(k, \eta) = f_{\text{DDM}}(k, \eta_0) \left( \frac{\eta + \sqrt{\eta^2 + \frac{k^2}{M^2c^2}}}{\eta_0 + \sqrt{\eta_0^2 + \frac{k^2}{M^2c^2}}} \right)^{-\frac{k^2}{2\gamma M^2}} \times e^{-\frac{k^2}{2\gamma M^2} \left( \eta \sqrt{\eta^2 + \frac{k^2}{M^2c^2}} - \eta_0 \sqrt{\eta_0^2 + \frac{k^2}{M^2c^2}} \right)}, \quad (4)
\]

where at \(\eta_0\) the particles freeze out or appear in the Universe through another mechanism, so that their spectrum is known to be \(f_{\text{DDM}}(k, \eta_0)\). The solution can be rewritten through the physical momenta \(p \equiv k/a\) and the Hubble parameter given by

\[
H \equiv \frac{da/d\eta}{a^2} = \frac{1}{c\eta^2}. \quad (5)
\]

In the limit of very nonrelativistic particles one obtains approximately from [1]

\[
f_{\text{DDM}}(k, \eta) = f_{\text{DDM}}(k, \eta_0) \times e^{-\frac{k^2}{2\gamma M^2} \left( \eta - \eta_0 \right)}, \quad (6)
\]

and at the next-to-leading order both for exponent (remains the same) and prefactor

\[
f_{\text{DDM}}(k, \eta) = f_{\text{DDM}}(k, \eta_0) \times \left( 1 + \frac{k^2}{a^2 M^2} \frac{1}{4\tau H} \log \frac{H_i}{H} \right) e^{-\frac{k^2}{2\gamma M^2} \left( \eta - \eta_0 \right)}. \quad (7)
\]

To solve [3] one must evaluate the upper \(E_2\) and lower \(E_1\) limits for the integration in the r.h.s. One gets approximately (in the relativistic limit, \(p_{\text{DM}} \gg m_{\text{DM}}\) )

\[
E_2 = p_{\text{DM}} + M^2_{\text{DM}} / 4p_{\text{DM}} \rightarrow \infty, \quad E_1 = p_{\text{DM}} + M^2 / 4p_{\text{DM}}.
\]

In the non-relativistic limit [3], when all decaying particles DDM are (almost) at rest, it is reasonable to assume that their distribution function is

\[
f_{\text{DDM}}(k, \eta) = F \frac{2\pi^2 \delta(k)}{k^2}, \quad (8)
\]

where the normalization is fixed by [1], and

\[
F(\eta) = F_i \times e^{-\frac{k^2}{2\gamma M^2} \left( \eta - \eta_i \right)} \equiv F_i \times e^{-\frac{\eta - \eta_i}{\gamma}}. \quad (9)
\]

To avoid singularities at \(E = E_1\) in (8), it is convenient to regularize (8) as

\[
f_{\text{DDM}}(k) = F \frac{2\pi^2 \delta(k - k_i)}{k^2}. \quad (10)
\]
This is rather physical, as it assumes that the DM particles are not exactly at rest, but really move with some small conformal momentum $\kappa$. In this approximation the collision integral in (3) can be taken easily

$$\int_{\kappa_{DDM}}^{\eta} f_{DDM}(\kappa) d\kappa = \frac{2\pi^2 F}{a^{3} M} \delta(\kappa_{DDM} - \frac{M}{2}). \quad (11)$$

Note, that this normalization means, that at the moment $\eta$ of DDM freezeout its concentration is given by $\eta_{DDM}(\eta_i) = F_i/a^3$.

Using eq. (11) one reduces the eq. (3) to

$$\frac{df_{DM}(\kappa)}{d\eta} = \frac{4\pi^2 F}{\tau a^2 M^2} \delta\left(\frac{k}{a} - \frac{M}{2}\right) \cdot \frac{16\pi^2 F}{\tau a^2 M^2} \delta\left(\frac{k}{a} - \frac{M}{2}\right) \quad (12)$$

and can be easily integrated for each $k$ individually. The meaning is simple—the $\delta$-function gives the moment, $\eta = \eta_*$, when the particle with properly rescaled 3-momentum

$$p = k/a = a_*/a \quad \frac{a_*/a}{a^3} = \frac{M}{2} \quad (13)$$

was created, so

$$f_{DM}(\kappa) = \frac{16\pi^2 F(\eta_*)}{\tau a^2 M^2}. \quad (14)$$

The conformal time (for some given number of d.o.f. encoded in $c_*$) is $\eta_* = a_*/c_*$ and (13) implies $\eta_* = 2k/(c_* M)$. Then for the Hubble parameter one obtains from (5)

$$H_* = \frac{1}{c_* H_*^2} = \frac{c_* M^2}{4k^2} = \frac{c_* H M^2}{4p^2}. \quad (15)$$

Using (9) we get finally

$$f_{DM}(\kappa) = \frac{32\pi^2 F_1}{\tau M^5} \frac{1}{a_*/a^3 H_*} \times e^{-\frac{\tau\kappa}{2\eta_*}} \quad (16)$$

$$= \frac{1}{a^{3} \tau M^2} \frac{1}{c_* H} \frac{1}{p} \times e^{-\frac{\tau\kappa}{2\eta_*}},$$

which precisely coincides with the results from [1].

IV. APPLICABILITY OF THE NONRELATIVISTIC APPROXIMATION FORMULA (16)

The formula (16) is valid as far as the approximation (11) was applicable. This puts two bounds on the momenta. The rather trivial and not extremely important for most considerations is the upper bound, which kicks in the region where the spectrum is anyway significantly (exponentially) suppressed. Above the bound the result is just completely cut-off,

$$p < M/2, \quad f(p > M/2) = 0. \quad (17)$$

The more interesting bound modifies the spectrum for the low values of momenta. This is not a hard cut-off, just the suppression of the spectrum. This happens because the formula (16) is obtained in the assumption that at the moment $\eta_*$ the DDM particle was non-relativistic, or

$$p_{DM} = M/2 \gg \langle p_{DDM} \rangle |_{\eta = \eta_*}. \quad (18)$$

If the shape of the DDM momentum distribution $f_{DDM}$ has the same maximum as those of the thermal bath, then $\langle p_{DDM} \rangle$ is applicable only for

$$p \gg \sqrt{M/M_p} \sim T \left(\frac{g}{g_*}\right)^{1/3}. \quad (19)$$

So the cold part of the distribution is not grasped by (16), and analysis beyond the non-relativistic approximation (11) is required.

Account for these limits changes the calculations of the average velocity made in literature (e.g. [1, 17–21]) by integrating with (16) for all the momentum unconstrained. The average velocity is usually adopted in estimates of the free streaming length important for the small scale structure formation and tested with Lyman-$\alpha$ forest data. The most cold dark matter may be obtained for decays of DDM which just start to be nonrelativistic, so $M \sim \langle p_{DDM} \rangle$ as considered in [14]. For intermediate situation one can read out the average momentum of the results in [15].

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1 Of course, if the spectrum $f_{DDM}$ is on its own significantly non-thermal for some reason, this bound may not be applied immediately.
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