Uniqueness of the minimizer
Let $X \in \mathbb{R}^{n \times p}$, $y \in \mathbb{R}^n$ and $\| \cdot \|$ be a norm on $\mathbb{R}^p$ whose unit ball is a polytope.

$$S_{X,\|\|}(y) := \arg\min_{b \in \mathbb{R}^p} \frac{1}{2}\|y - Xb\|_2^2 + \|b\|.$$  

Note that $S_{X,\|\|}(y) \neq \emptyset$ but $S_{X,\|\|}(y)$ is not always a singleton. Theorem 1 provides a condition on $X$ so that whenever $y$ the set $S_{X,\|\|}(y)$ is a singleton. We remind that for a norm $\| \cdot \|$ on $\mathbb{R}^p$, the dual norm $\| \cdot \|_*$ is defined by $\|x\|_* = \sup_{s \in \mathbb{R}^p : \|s\| \leq 1} s^Tx$.

**Theorem 1** Let $X \in \mathbb{R}^{n \times p}$ and $\| \cdot \|$ be a norm for which the unit ball $B$ is polytope. There exists $y \in \mathbb{R}^n$ for which $S_{X,\|\|}(y)$ is not a singleton if and only if row$(X)$ intersects a face of the unit ball of the dual norm $B^*$ whose dimension is $< \text{dim}(\ker(X))$.

When $X = (1\ 0)$, the figure below illustrates that the set $S_{X,bp}(y)$ is not a singleton by plotting the contour lines of the function $\phi(b_1, b_2) = 0.5(2 - b_1)^2 + \max\{|b_1|, |b_2|\}$.

![Figure](https://via.placeholder.com/150)

When $X = (1\ 1)$, the figure below illustrates that $S_{X,\|\|_\infty}(2)$ is a singleton through the contour lines of the function $\phi(b_1, b_2) = 0.5(2 - b_1 - b_2)^2 + \max\{|b_1|, |b_2|\}$.

![Figure](https://via.placeholder.com/150)

The set $S_{X,bp}(y)$ of BP minimizers is defined as

$$S_{X,bp}(y) = \arg\min \|b\|_1 \text{ subject to } Xb = y.$$  

**Theorem 2** Let $X \in \mathbb{R}^{n \times p}$. There exists $y \in \text{col}(X)$ for which $S_{X,bp}(y)$ is not a singleton if and only if row$(X)$ intersects a face of the unit cube $[-1, 1]^p$ whose dimension is $< \text{dim}(\ker(X))$.

Accessible sign vectors for LASSO and BP

**Definition 1** Let $X \in \mathbb{R}^{n \times p}$, $\sigma \in \{-1, 0, 1\}^p$, and $\lambda > 0$. We say that $\sigma$ is an accessible sign vector for LASSO (or BP), if there exists $y \in \mathbb{R}^n$ and $\beta \in S_{X,\|\|_1}(y)$ (or there exists $y \in \text{col}(X)$ and $\beta \in S_{X,bp}(y)$), such that $\text{sign}(\beta) = \sigma$.

$$F(\sigma) = E_1 \times \cdots \times E_p \text{ with } E_j = \begin{cases} \{\sigma_j\} & |\sigma_j| = 1 \\ [-1, 1] & \sigma_j = 0 \end{cases}.$$  

**Theorem 3** Let $X \in \mathbb{R}^{n \times p}$, $\lambda > 0$ and $\sigma \in \{-1, 0, 1\}^p$.

**Geometrical characterization:** The sign vector $\sigma$ is accessible for LASSO (or BP) if and only if row$(X)$ intersects the face $F(\sigma)$.

**Analytical characterization:** The sign vector $\sigma$ is accessible for LASSO (or BP) if and only if the following implication holds: $Xb = X\sigma \Rightarrow \|b\|_1 \geq \|\sigma\|_1$.

In the high-dimensional linear regression model, the accessibility condition is actually a necessary and sufficient for sign recovery by thresholded LASSO and by thresholded BP (Tardivel and Bogdan) and by thresholded justice pursuit (Descloux et al.).