Fundamental Limits for Light Absorption and Scattering Induced by Cooperative Electromagnetic Coupling

Jean-Paul Hugonin1, Mondher Besbes1 and Philippe Ben-Abdallah1,*

1Laboratoire Charles Fabry, UMR 8501, Institut d’Optique, CNRS, Université Paris-Sud 11, 2, Avenue Augustin Fresnel, 91127 Palaiseau Cedex, France.

(Dated: March 16, 2015)

Absorption and scattering of electromagnetic waves by dielectric media are of fundamental importance in many branches of physics. Here we analytically derived the ultimate upper limits for the absorbed and scattered powers by any system of coupled particles and give sufficient conditions to reach these limits paving so a way for a rational design of optimal metamaterials.

PACS numbers: 42.25.Bs, 03.65.Nk, 78.67.Pt, 78.66.Sq

Understanding light absorption and scattering mechanisms is of prime importance in optical physics, photonics and plasmonics. Nowadays, the nanofabrication technologies provide tools to design artificial materials for engineering light-matter interactions. Plasmonic nanoparticles aggregates are, for instance, artificial structures which have been subject of intense researches 1, 2. The collective interactions of localized plasmon modes can give rise to pronounced Fano resonances 3, 4 or lead to electromagnetic induced transparency 5 or induced absorption 6 phenomena.

In this Rapid Communication, we analyze how strong light absorption and scattering can be in arbitrary sets of resonant particles when they are cooperatively coupled. This problem has been partly considered in the specific case of single objects 7, 13, objects with simple symmetry shapes 14, 15, periodic arrays of dipoles 16 or dilute distribution of scatterers 17. Here, we provide a general answer to this problem by deriving, in the framework of the linear classical electrodynamics, the fundamental upper bounds for light absorption and scattering for arbitrary distributions of resonant particles when potentially strong cooperative coupling mechanisms take place and when the field radiated by the multipolar orders of particles participate to the coupling. We also propose a general inverse strategy to reach these limits.

Let us start, for pedagogical reasons, by considering dipolar systems made with N dipoles of dipolar moment \( \mathbf{p} \) spatially distributed at the position \( \mathbf{r}_i \) in a transparent medium of permittivity \( \epsilon_h \) and highlighted by an incident field \( \mathbf{E}_{\text{inc}} \). The local electric field \( \mathbf{E}_{\text{loc}} \) measured at any point results from the superposition of external incident and scattered fields. Therefore it takes the self-consistent form 18, 19

\[
\mathbf{E}_{\text{loc}}(\mathbf{r}) = \mathbf{E}_{\text{inc}}(\mathbf{r}) + \omega^2 \mu_0 \sum_j \mathbf{G}_0(\mathbf{r}, \mathbf{r}_j) \mathbf{p}_j, \tag{1}
\]

where

\[
\mathbf{G}_0(\mathbf{r}, \mathbf{r'}) = \frac{\exp(i \mathbf{k} \cdot \mathbf{r})}{4 \pi \mathbf{r}} \times \left[ \mathbf{I} + \frac{3 - 3i \mathbf{k} \cdot \mathbf{p} - \mathbf{k}^2 \mathbf{p} \otimes \mathbf{p}}{k^2 \mathbf{p}} \right] \text{ is the free space Green tensor in the host material defined with the unit vector } \mathbf{\hat{p}} \equiv \mathbf{p}/|\mathbf{p}|, \mathbf{p} = \mathbf{r} - \mathbf{r}', \mathbf{k} \text{ is the wavevector in the host material while } \mathbf{I} \text{ denotes the unit dyadic tensor.}
\]

By introducing the vectorial fields \( \mathbf{p} = (\mathbf{p}(\mathbf{r}_1), ..., \mathbf{p}(\mathbf{r}_N)) \), \( \mathbf{E}_{\text{loc}} = (\mathbf{E}_{\text{loc}}(\mathbf{r}_1), ..., \mathbf{E}_{\text{loc}}(\mathbf{r}_N)) \), \( \mathbf{E}_{\text{inc}} = (\mathbf{E}_{\text{inc}}(\mathbf{r}_1), ..., \mathbf{E}_{\text{inc}}(\mathbf{r}_N)) \) and using the Poynting theorem 20, the total power dissipated in this system highlighted by a monochromatic incident field writes 21, 22

\[
P_{\text{abs}}(\omega) = -\frac{\omega}{2} \text{Im}[\mathbf{p}^\dagger(\omega) \cdot \mathbf{E}_{\text{loc}}(\omega)]. \tag{2}
\]

Using relation (1) and the reciprocity principle 23, this expression can be recasted in term of the incident field

\[
P_{\text{abs}}(\omega) = -\mathbf{p}^\dagger \cdot \left( \frac{\omega^3 \mu_0}{2} [\text{Im} \mathbf{G}] \right) \cdot \mathbf{p} + \text{Re}[\mathbf{p}^\dagger \cdot \frac{\omega}{2} \mathbf{E}_{\text{inc}}]. \tag{3}
\]

where we have set the \( N \times N \) block matrix of component \( \mathbf{G} = \mathbf{G}_0(\mathbf{r}_i, \mathbf{r}_j) \). Hence, the maximal power than a set of dipoles can dissipate under an external highlighting reads (see Supplemental Material 24)

\[
P_{\text{abs}}^{\max}(\omega) = \frac{1}{8 \omega \mu_0} \mathbf{E}_{\text{inc}}^\dagger [\text{Im} \mathbf{G}]^{-1} \mathbf{E}_{\text{inc}}. \tag{4}
\]

This expression is the first result of the Letter. It represents the fundamental limit for light absorption by any system of dipoles which collectively interact. This expression depends only on the geometric configuration and on the characteristics of incident field. It immediately follows from it that the maximal absorption cross-section for a set of dipoles reads

\[
\sigma_{\text{abs}}^{\max} = \frac{c}{4 \omega} \frac{\mathbf{E}_{\text{inc}}^\dagger [\text{Im} \mathbf{G}]^{-1} \mathbf{E}_{\text{inc}}}{\sqrt{\epsilon_h |\mathbf{E}_{\text{inc}}|^2}}. \tag{5}
\]

This result could seem at first view counter intuitive. Indeed, with resonant particles we could expect a blowup mechanism for light absorption allowing to go beyond this limit. However, this intuition is wrong. To be convinced of this let us simply examine the case of an isolated particle of polarizability \( \alpha = \alpha' + i \alpha'' \). In this simple case, the dissipated power writes 16

\[
P_{\text{abs}}(\alpha', \alpha'') = \frac{\omega}{2 \mu_0 c^2} |\mathbf{E}_{\text{inc}}|^2 \left( \alpha'' - \frac{\omega^3}{6 \pi c^3} |\alpha|^2 \right). \tag{6}
\]
Then, a straightforward calculation shows that the optimal polarizability is \( \alpha_{\text{opt}} = \frac{3i\pi}{4} \) and it leads to the maximal dissipated power \( P_{\text{abs}}^{\text{max, sing}}(\omega) = \frac{3}{4}\frac{\pi c}{\omega^3\mu_0\rho_{\text{abs}}} |E_{\text{inc}}|^2 \) which exactly corresponds (using \( \mu_0 = \frac{1}{6\pi} \)) to the value predicted by relation (4).

Similarly, we can maximize the power

\[
P_{\text{scat}} = \mathbf{p} \cdot (\frac{\omega^3\mu_0}{2}|ImG|) \cdot \mathbf{p}
\]

scattered by a set of dipoles under the constraint \( P_{\text{abs}} \geq 0 \) to find (see Supplemental Material [24])

\[
P_{\text{scat}}^{\text{max}} = \frac{1}{2i\mu_0} |E_{\text{inc}}| |ImG|^{-1} E_{\text{inc}}.
\]

Comparing this expression to relation (4) we see that the maximal power than a set of dipoles can scatter is four times equal to the maximal power it can absorb.

![Figure 1: Maximal absorption cross-section of a pair of electric dipoles enlightened (a) in a transversal direction (TE and TM polarization) and (b) in a longitudinal direction. In this case, the power dissipated by the first dipole is plotted in blue dot dashed line while the power dissipated by the second dipole is plotted in red dashed line. At subwavelength separation distance the first dipole is amplifying. All curves are normalized by the maximal cross-section of (resp. the power dissipated by) a single dipole.(c) Poynting vector streamlines around a dimer of SiC nanoparticles optimized to reach the maximal absorption marked by the green disk in Fig. 1(a) for a distance \( d = 7.3\mu m \) and a wavelength \( \lambda = 10.8\mu m \). The optimal radius is 0.805\mu m. The electric field inside the particles is two order larger than the unitary incident field.](image)

The above analysis gives the ultimate limits that any dipolar system can reach given a spatial configuration. Surprisingly, these limits are independent of the material properties. However, hereafter we show that, in practical point of view, these values can be reached by using appropriate nanoparticles. To this end, let us consider the general relation \( \mathbf{p} = \alpha \mathbf{E}_{\text{loc}}^{\text{abs}} \) between the generalized polarizability of dipoles and the regularized version of local field defined as \( \mathbf{E}_{\text{loc}}^{\text{abs}}(\mathbf{r}_i) = \mathbf{E}_{\text{inc}}(\mathbf{r}_i) + \omega^2\mu_0 \sum_{j \neq i} G_0(\mathbf{r}_i, \mathbf{r}_j) \mathbf{p}_j \) and let us assume, for clarity reasons, that all polarizability tensors are diagonal (i.e. \( \alpha = \text{diag}(\alpha_{ix}, \alpha_{iy}, \alpha_{iz}) \) with \( \alpha_i = \text{diag}(\alpha_{ix}, \alpha_{iy}, \alpha_{iz}) \)). Then, by inverting this relation using the optimal dipolar moments we find for \( i = 1, ..., N \) and \( \beta = x, y, z \)

\[
\alpha_{i,\beta} = \frac{\mathbf{p}_{\text{opt}, i, \beta}}{E_{\text{inc}, i, \beta} + \omega^2\mu_0 \sum_{j \neq i} G_0(\mathbf{r}_i, \mathbf{r}_j) \mathbf{p}_{\text{opt}, j, \beta}].
\]

Here, we must emphasize that, in principle, those optimal polarizabilities do not necessary correspond to polarizabilities of lossy media so that, generally speaking, the optimal absorption can be achieved by combining dipoles with gain to lossy dipoles.

Let us now examine the usefull case of a pair of dipoles (separation distance \( d \)) along the x-axis: \( \text{Im} G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{pmatrix} \) and \( \text{Im} G_0(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0\end{pmatrix} \) with \( a = \frac{\sin(kd)}{2\pi d} \) \( \frac{1}{kd} - \cos(kd) \) \( \frac{1}{kd} \) and \( b = \frac{\sin(kd)}{2\pi d} \) \( \frac{k^2d^2-1}{k^2d^2} + \frac{\cos(kd)}{2\pi d} \) \( \frac{1}{kd} \) so that \( |ImG|^2 = \begin{pmatrix} V & W \\ W & V\end{pmatrix} \)

with \( V = 6\pi k \left( \frac{1}{k^2-(6\pi)^2a^2} \right) \) \( \frac{1}{k^2-(6\pi)^2b^2} \) 0 0

\( W = -(6\pi)^2 \left( \frac{1}{k^2-(6\pi)^2a^2} \right) \) \( \frac{b}{k^2-(6\pi)^2b^2} \) 0 0

and \( \mathbf{E}_{\text{inc}} = \frac{\mathbf{E}_{\text{inc}, i, \beta}}{E_{\text{inc}, i, \beta}} \)

For an incident field of magnitude \( E_{\text{inc}} \) orthogonal to the axis linking the two dipoles, we have for TE waves \( P_{\text{abs}}^{\text{max}, \perp, \text{TE}} = \frac{3\pi}{2\mu_0} \frac{E_{\text{inc}}^2}{k+6\pi b} \) while for TM waves \( P_{\text{abs}}^{\text{max}, \perp, \text{TM}} = \frac{3\pi}{2\mu_0} \frac{E_{\text{inc}}^2}{k+6\pi a} \). This result is plotted in Fig. 1-a. When the separation distance becomes sufficiently large then \( b \sim a \sim 0 \) and we see that the power absorbed by the pair is twice the power absorbed by an isolated dipole. On the contrary, close to the contact, \( a \sim \frac{1}{6\pi d} |kd - (kd)^2| \) and \( b \sim \frac{1}{6\pi d} |kd - (kd)^2| \) so that \( P_{\text{abs}}^{\text{max}, \perp, \text{TE, TM}} = P_{\text{abs}}^{\text{max, sing}} \). Between these two extreme regimes the maximum is an oscillating function with respect to the separation distance as illustrated in Fig.1. in the case of a silicone carbide (SiC) nanoparticles dimer. In this case, the radius of SiC particles has been optimized from the optimal polarizability [9] using the method described in [13] to reach the ultimate absorption shown by the green disk in Fig.1(a) at a separation distance \( d = 7.3\mu m \) for a wavelength \( \lambda = 10.8\mu m \) (\( \epsilon_{\text{SiC}} = -2.53 + 0.18i \)). It clearly appears Fig. 1-c that the dissipation of the energy of incident field is due to a strong enhancement of field inside the particles.

When the dimer is lighted along its axis (see Fig. 1-b) both dipoles receive a phase shifted incident field.
Let \( \varphi = kd \) be that phase shift and let us denote by \( E_{inc} \) the magnitude of incident field on the first dipole. Then , it is straightforward to see that 
\[
P_{abs}^{max,||} = \frac{3}{2} E_{inc}^2 \frac{\pi}{\omega \mu_0} \frac{k^{-6}\text{sec}^2(\varphi)}{k^2-(6\pi)^2b^2}.
\]
In this case, at close separation distance 
\[
P_{abs}^{max,\perp} = \frac{3}{2} P_{abs}^{max,\text{sing}}.
\]
This value is the maximal power a dimer can dissipate in this configuration. At first sight, this result seems to be paradoxal because one dipole is in the shadow of the second. But when we examine the optimal losses per particle in near-field regime (i.e. \( \frac{d}{\lambda} < 1 \)) we see Fig. 1-b that one particle dissipates the incident energy while the second not. On the contrary, the latter is purely amplifying. In this regime of strong coupling, the higher are the overall losses, the higher is the gain in the first particle. This exaltation of losses is driven by a cooperative coupling mechanism between a lossy and a particle with gain. Even when the first particle is lossless (at \( \frac{d}{\lambda} \approx 0.195 \) in Fig. 1-b), the power which is dissipated in the second particle and therefore the overall dissipated power is more than three time larger than the maximal power than a single isolated dipole can dissipate. As shown in Figs. 2 similar results can be observed with linear chains of dipoles. The spatial distribution of losses inside such chains illuminated transversally and along the axis are plotted in Figs. 2-a and 2-b in the case of N=10 particles regularly spaced. In the longitudinal lighting case, we observe Fig. 2-c an almost symmetrical structure that emerges from the optimization process with half of the chain which is made of dissipative particles and the rest of the chain which is made of amplifying particles. As for the variation of the maximum absorption cross-section \( \sigma_{abs}^{max} \) with respect to the number N of dipoles inside the chain, the numerical simulations predict that it increases (Fig. 2-c) linearly as \( \sigma_{abs}^{max} \sim 2N+1 \) showing so that apparently there is no upper bound for long chains provide had hoc optical properties are used.

In two dimensional networks (Fig. 3-a-and 3-c) we get qualitatively the same behaviors than in chains. In contrast, in three dimensional networks the optimal power is almost independent on the separation distance between the dipoles even in near-field regime where the electromagnetic couplings are strong. One can speculate that both the exaltation and inhibition mechanisms annihilate each other. However, the detailed investigation of three-dimensional dipolar networks goes far beyond the objective of the present work and it is in itself a problem which would necessitate a specific study.

It clearly appears in Fig. 3, that the absorption cross-section of the whole system can, in near-field regime, be much larger than the apparent area of domain (shown by the green dashed lines) on which the dipoles are dispersed. The numerical simulations show that, at subwavelength separation distances, the cross-section can be more than one order of magnitude larger than the maximal cross-section of a single dipole. This result outlines the importance of collective couplings in this regime and show its strong potential for a variety of problem such as for instance, the design of subwavelength superabsorbers. Nevertheless, it is worth to emphasize that to get an absorption cross-section which is much larger than the apparent support of dipoles, active media must generally be inserted inside the network. When we impose all dipoles to be dissipative we observe that the cross-section drastically decreases with the separation distance to become comparable with the maximal cross-section of a single dipole (shown by the horizontal blue dashed lined in Fig. 3 ) near the contact.

So far, we have only considered couplings between particles described by simple dipoles. Hereafter we consider the most general situation where higher order modes (i.e. multipoles) are taken into account [26]. The electromagnetic field inside a medium of refractive index \( n_b \) can be expressed in term of ingoing (\( -\) ) and outgoing (\( +\)) vector spherical wave functions (which form a complete basis)

\[
\psi_{pq}^\pm = \begin{pmatrix} E_{pq}^b H_{pq}^c \\ H_{pq}^c E_{pq}^b \end{pmatrix}
\]

where we have adopted the usual convention for the multipolar index \((m, n)\) which are replaced by a single index \( p = n(n+1) + m \) and where \( q \) set the polarization state (i.e. \( q = 1 \) for \( TE \) waves and \( q = 2 \) for \( TM \) waves). The outgoing wave functions \( \psi_{pq}^- \) are solutions of Maxwell’s
duce the following real spherical harmonic functions

\[ \psi_{pq} = \sum_{j} T_{ij}^{p,q} a_j^{p,q} \]

As for the local field it takes the form

\[ \psi_{loc}(\mathbf{r}) = \sum_{pq} \chi_{pq}^{\pm}(\mathbf{r} - \mathbf{r}_i) + \sum_{pq} A_{pq}^{\pm} \cdot \mathbf{r} - \mathbf{r}_i \]

where the second term of rhs is the total scattered field which corresponds to incoming field scattered by all particles in direction of the \( i^{th} \) scatterer and the outgoing field diffracted out this particle. Since the incoming field is related to all outgoing fields by a linear relation of the type \( a_i^{pq} = \sum_{j} T_{ij}^{p,q} a_j^{p,q} \), where the \( T_{ij}^{p,q} \) are the components of a translation operator (a propagator) between the \( i^{th} \) and the \( j^{th} \) particle, the local field takes the form

\[
\psi_{loc}(\mathbf{r}) = \\
\sum_{pq} \chi_{pq}^{\pm}(\mathbf{r} - \mathbf{r}_i)\left(\frac{1}{2}a_{inc,i}^{pq} + \frac{1}{2} \sum_{j} T_{ij}^{p,q} A_{j}^{p,q} + A_{i}^{pq}\right) \\
+ \sum_{pq} \chi_{pq}^{-}(\mathbf{r} - \mathbf{r}_i)\left(\frac{1}{2}a_{inc,i}^{pq} + \frac{1}{2} \sum_{j} T_{ij}^{p,q} A_{j}^{p,q}\right).
\]

(15)

Then, using the orthogonality relations for the functions \( \chi_{pq}^{\pm} \), the power dissipated in each particle can be rewritten as the net flux

\[
P_{abs,i} = \sum_{pq} \left[ \left| A_{i}^{pq}\right|^2 + \sum_{j} \left| T_{ij}^{p,q} A_{j}^{p,q}\right|^2 \right]
\]

(16)

across a surface surrounding the particle. By developing this expression we get

\[
P_{abs,\omega} = \sum_{i} \left[ \left| A_{i}^{pq}\right|^2 + \sum_{j} \left| T_{ij}^{p,q} A_{j}^{p,q}\right|^2 \right] + \sum_{i} \sum_{j} T_{ij}^{p,q} A_{j}^{p,q}\]

(17)

and the power scattered by it

\[
P_{scat,\omega} = A^{\dagger} \mathbf{V} A + \text{Re}[A^{\dagger} \mathbf{B}]
\]

(18)

In these two expressions, \( \mathbf{A} \) and \( \mathbf{B} \) are, in a system of \( N \) scatterers, the block vectors defined with the subvectors \( \mathbf{A}_i \) and \( \mathbf{B}_i = (a_{inc,i}^{pq}) \) while \( \mathbf{V} = \mathbf{I} + \frac{\mathbf{U} + \mathbf{U}^{\dagger}}{2} \) is the block matrix defined with the sub-blocks elements \( \mathbf{U}_{ij} = T_{ij} \). Following the same reasoning as used for a set of dipoles we obtain the upper bounds for the dissipated and the scattered powers inside any arbitrary system of particles which are collectively coupled (see Supplemental Material [24]).

These results are illustrated in Fig. 4 in the specific case of a dimer of nanoparticles. We note that, similarly to the results obtained for single objects [8], the maximal cross-section is an increasing function with the number of multipolar orders. In fact the number of resonant modes of the whole system increases with the multipoles order creating so new channels for dissipating or scatter light.

In conclusion, we have analytically derived the ultimate limits for light absorption and scattering by a system of \( N \) point resonant multipoles for a given geometry
Figure 4: Maximal absorption cross-section for a dimer of nanoparticles versus the separation distance. The green dashed line corresponds to the dipolar (electric) case while the blue, purple and black lines stand for the contribution of multipolar orders one (dipolar electric+magnetic), two and three, respectively.

Figure 5: Illustration of two optimization problems. On the bottom, the paraboloid corresponds to all physically admissible dissipated powers inside a system over the space of configuration (brown plane). The orange area on the upper part of this paraboloid surface corresponds to lossy systems while the lower part correspond to amplifying systems. The blue sphere on the top of this surface denotes the maximal dissipated power while its orthogonal projection on the configuration plane stands by a blue cross marks the optimal generalized polarizability. The upper part of this mapping illustrates the maximization of the scattered power. The green paraboloid stands for all physically admissible scattered powers. The optimal generalized polarizability under the constraint of a positive dissipated power is shown by the red sphere.

**Supplemental material**

In this supplemental document we give supplementary informations concerning the derivation of the optimal power absorbed or scattered by a set of optical resonators. The derivation is general and can be used either in the dipolar or the multipolar case.

I. Maximal absorption

The maximal power which can be absorbed by such a system can be found by solving the following constrained optimal problem. Find $X_{\text{opt}}$,

$$\begin{array}{l}
\mathcal{P}_{\text{abs}}(X_{\text{opt}}) \rightarrow \max \{ \mathcal{P}_{\text{abs}}(X) \}
\end{array}
\begin{array}{l}
\text{with} \quad \mathcal{P}_{\text{abs}}(X_{\text{opt}}) \geq 0
\end{array}
$$

(20)

where, according to relation (3) the absorbed power writes under the form $\mathcal{P}_{\text{abs}} = -A^\dagger \cdot \epsilon \cdot A + \text{Re} \{ A^\dagger \cdot B \}$ with $A = p$, $V = \frac{\epsilon_0}{2} \mu [\text{Im} G]$ and $B = \frac{\epsilon_0}{2} E_{\text{inc}}$ in the dipolar case. By decomposing this expression with respect to the real and imaginary parts of those vectors we have equivalently

$$\begin{array}{l}
\mathcal{P}_{\text{abs}}(\omega) = -\frac{1}{2} X^\dagger \cdot A \cdot X + B^\dagger \cdot X
\end{array}
$$

(21)

with $A = 2 \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$, $B = \begin{pmatrix} \text{Re} [B] \\ \text{Im} [B] \end{pmatrix}$ and $X = \begin{pmatrix} \text{Re} [A] \\ \text{Im} [A] \end{pmatrix}$.

By construction $A$ is a symmetric matrix so that expression (21) is a quadratic polynomial. Hence, the maximal dissipated power can be obtained by solving constrained optimal problem (1) using the classical Kuhn-Tucker (KT) conditions (see S. Boyd and L. Vandenberghe, Convex Optimization (Cambridge University Press, Cambridge, 2004). In the present case, this is
equivalent to cancel the gradient \( \frac{dP}{dX} = -A \cdot X + B \) so that
\[
X_{opt} = A^{-1}B = \frac{1}{2} \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \cdot \begin{pmatrix} Re[|B|] \\ Im[|B|] \end{pmatrix} \tag{22}
\]
It follows from the definition of vector \( X \) that the optimal generalized dipole vector field reads
\[
A_{opt} = \frac{1}{2} V^{-1} B. \tag{23}
\]
By inserting this expression into relation (3) of the Letter we get the maximal absorption
\[
P_{abs}^{max}(\omega) = \frac{1}{4} B^\dagger V^{-1} B \tag{24}
\]
which is equivalent, according to the definition of different vectors and tensor fields to
\[
P_{abs}^{max}(\omega) = \frac{1}{8 \omega \mu_0} E_{inc}^\dagger |Im \mathcal{G}|^{-1} E_{inc}. \tag{25}
\]

II. Maximal scattering

The maximal power which can be scattered by an arbitrary system can be found by solving the following constrained optimal problem. Find \( X_{opt} \),
\[
\begin{cases}
P_{scat}(X_{opt}) \rightarrow \text{Max} \{P_{scat}(X)\} \\
\text{with} \quad P_{abs}(X_{opt}) \geq 0
\end{cases} \tag{26}
\]
where \( P_{scat} = A^\dagger V A \). Using the matrix \( A \) and the vector \( B \) introduced above the scattered power writes equivalently \( P_{scat} = \frac{1}{2} X^\dagger A \cdot X \). Then, to solve the constrained optimal problem (6) we introduce the Lagrangian \( \mathcal{L}(X, \lambda) = \frac{1}{2} X^\dagger A \cdot X + \lambda P_{abs}(X) \) where \( \lambda > 0 \) denotes a positive KT-multiplier [7]. Thus, according to the KT conditions, the optimal solution must cancel the gradient \( \nabla \mathcal{L} = A \cdot X + \lambda (B - A \cdot X) \) of this Lagrangian. This condition implies that \( X = \frac{1}{2 \lambda} A^{-1} B \). Beside, the constraint \( P_{abs} = 0 \) gives the optimal KT multiplier \( \lambda = 2 \) so that case \( X_{opt} = 2 A^{-1} B \). It follows from the definition of vector \( X \) that the optimal dipole vector field reads
\[
A_{opt} = V^{-1} B. \tag{27}
\]
Finally, by inserting this expression into the relation (6) of the Letter we get
\[
P_{scat}^{max} = B^\dagger V^{-1} B \tag{28}
\]
which is equivalent to
\[
P_{scat}^{max} = \frac{1}{2 \omega \mu_0} E_{inc}^\dagger |Im \mathcal{G}|^{-1} E_{inc}. \tag{29}
\]

III. A simple alternative for the derivation of optimum

Here below we give another demonstration for the optimal powers absorbed or scattered by a N-body system following a purely algebraic approach. By construction \( V \) is a positive definite real matrix. Thus the following scalars
\[
Q_1 = (A^\dagger - \frac{1}{2} B^\dagger V^{-1}) \cdot V \cdot (A - \frac{1}{2} V^{-1} B) \tag{30}
\]
and
\[
Q_2 = (A^\dagger - B^\dagger V^{-1}) \cdot V \cdot (A - V^{-1} B) \tag{31}
\]
are positive. From these expressions the absorbed and the scattered power given in Eqs. (3) and (6) of the Letter write
\[
P_{abs} = \frac{1}{4} B^\dagger V^{-1} B - Q_1 \tag{32}
\]
and
\[
P_{scat} = B^\dagger V^{-1} B - Q_2 - 2 P_{abs}, \tag{33}
\]
respectively. According to (32) \( P_{abs} \) is bounded by \( P_{abs}^{max} = \frac{1}{4} B^\dagger V^{-1} B \) and from the definition of \( Q_1 \) we see that this value is reach when \( A_{opt} = \frac{1}{2} V^{-1} B \). Following a similar reasoning for the scattered power we have from (33) \( P_{scat}^{max} \leq B^\dagger V^{-1} B \). Then, by choosing \( A = V^{-1} B \) we see, according to expression (34), that \( Q_2 = 0 \) and \( P_{abs} = 0 \) and we reach the upper limit for the scattered power.

* Electronic address: pba@institutoptique.fr

[1] J. A. Fan, C. H. Wu, K. Bao, J. M. Bao, R. Bardhan, N. J. Halas, V. N. Manoharan, P. Nordlander, G. Shvets, F. Capasso, Self-Assembled Plasmonic Nanoparticle Clusters, Science 328 (5982), 1135 (2010)

[2] J. B. Lassiter, H. Sobhani, J. A. Fan, J. Kundu, F. Capasso, P. Nordlander and N. J. Halas, Nano Lett., 10 (8), 3184 (2010)

[3] N. Liu, S. Kaiser and H. Giessen, Magnetoinductive and electroinductive coupling in plasmonic metamaterial molecules, Adv. Mater., 20 (23) 4521 (2008).

[4] N. Liu, H. Liu, S. Zhu and H. Giessen, Stereometamaterials, Nature Photonics 3, 157 (2009)

[5] N. Liu, L. Langguth, T. Weiss, J. Kästel, M. Fleischhauer, T. Pfau and H. Giessen, Plasmonic analogue of electromagnetically induced transparency at the Drude damping limit, Nature Materials 8, 758 (2009)

[6] R. Taubert, M. Hentschel, J. Kästel and H. Giessen, Classical analog of electromagnetically induced absorption in plasmonics, Nano Lett., 12 (3), 1367 (2012)

[7] C. F. Bohren and D. R. Huffman, Absorption and scattering of light by small particles (John Wiley & Sons, New York, 1983).

[8] Z. Ruan and S. Fan, "Superscattering of light from subwavelength nanostructures", Phys. Rev. Lett. 105, 013901 (2010).
[9] Z. Ruan and S. Fan, "Design of subwavelength superscattering nanospheres", Appl. Phys. Lett. 98, 043101 (2011).
[10] L. Verslegers, Z. Yu, Z. Ruan, P. B. Catrysse and S. Fan, "From electromagnetically induced transparency to superscattering with a single structure: a coupled-mode theory for doubly resonant structures", Phys. Rev. Lett. 108, 083902 (2012).
[11] R. Fleury, J. Soric, and A. Alù, "Physical bounds on absorption and scattering for cloaked sensors", Phys. Rev. B 89, 045122 (2014).
[12] N. M. Estakhri and A. Alù, "Minimum-scattering superabsorbers", Phys. Rev. B 89, 121416(R) (2014).
[13] V. Grigoriev, N. Bonod, J. Wenger and B. Stout, ACS Photonics, ACS Photonics DOI: 10.1021/ph500456w (2015).
[14] B. S. Luk’yanchuk, A. E. Miroshnichenko, M. I. Troyan, Y. S. Kivshar and A. R. Khokhlov, "Paradoxes in laser heating of plasmonic nanoparticles", New J. Phys. 14, 093022 (2012).
[15] W. Qiu, B. G. Delacy, S. G. Johnson, J. D. Joannopoulos and M. Soljacic, "Optimization of broadband optical response of multilayer nanospheres", Opt. Express, 20, 18494 (2012).
[16] S. Tretyakov, "Maximizing absorption and scattering by dipole particles", Plasmonics, 9, 935 (2014).
[17] O. D. Miller, C. W. Hsu, M. T. H. Reid, W. Qiu, B. G. DeLacy, J. D. Joannopoulos, M. Soljacic and S. G. Johnson, "Fundamental limits to extinction by metallic nanoparticles", Phys. Rev. Lett. 112, 123903 (2014).
[18] E. M. Purcell and C. R. Pennypacker, "Scattering and absorption of light by nonspherical dielectric grains," Astrophys. J. 186, 705 (1973).
[19] B. T. Draine and P. J. Flateau, "Discrete-dipole approximation for periodic targets: theory and tests," J. Opt. Soc. Am. A. 25, 2693 (2008).
[20] J. D. Jackson, Classical Electrodynamics, third edition, John Wiley (1999).
[21] P. Ben-Abdallah, S.-A. Diehs, and K. Joulain, "Many-body radiative heat transfer theory," Phys. Rev. Lett. 107, 114301 (2011).
[22] R. Messina, M. Tschikin, S.-A. Diehs, and P. Ben-Abdallah, "Fluctuation-electrodynamic theory and dynamics of heat transfer in systems of multiple dipoles," Phys. Rev. B 88, 104307 (2013).
[23] L. Landau, E. Lifshitz, and L. Pitaevskii, Electromagnetics of Continuous Media (Pergamon, Oxford, 1984).
[24] See EPAPS Document No. [number will be inserted by publisher] for a derivation of maximal power absorbed or scattered by a set of dipoles. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.
[25] Handbook of Optical Constants of Solids, edited by E. Palik (Academic Press, New York, 1998).
[26] M. Langlais, J.-P. Hugonin, M. Besbes and Philippe Ben-Abdallah, Cooperative electromagnetic interactions between nanoparticles for solar energy harvesting, Optics Express, 22, S3, A577 (2014).
[27] B. Stout, J.-C. Auuger, and J. Lafait, "A transfer matrix approach to local field calculations in multiple-scattering problems", J. of Modern Optics, 49(13) (2002).