The inequivalence of thermodynamic ensembles

Renaud Parentani\(^1\)

The Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, ISRAEL

Abstract

The inequivalence of thermodynamical ensembles related by a Legendre transformation is manifest in self-gravitating systems and in black hole thermodynamics. Using the Poincaré’s method of the linear series, we describe the mathematical reasons which lead to this inequivalence which in turn induces a hierarchy of ensembles: the most stable ensemble describes the most isolated system. Moreover, we prove that one can obtain the degree of stability of all equilibrium configurations in any ensemble related by Legendre transformations to the most stable if one knows the degree of stability in the most stable ensemble.

\(^1\)Present address: Laboratoire de Physique Theorique de l’ Ecole Normale Superieure, 24 Rue Lhomond, 75005 Paris, France (E-mail: parenta@physique.ens.fr)
1. Introduction

In self-gravitating systems\textsuperscript{[1–5]} or in systems with highly degenerate spectra such as strings\textsuperscript{[6]} or black holes\textsuperscript{[7–14]}, it is well known that thermodynamical ensembles are generally inequivalent. This means that two ensembles related by a single Legendre transformation (or a Laplace transformation in Statistical Mechanics) become unstable in different situations. For instance there exist stable situations in microcanonical ensembles with negative specific heat. Surprisingly there also exist situations in which canonical ensembles with positive specific heat are unstable.

Thus, evidently, the heat capacity alone does not control the stability of those ensembles and one needs more reliable criteria. These can be provided, under specific conditions (see refs. 1, 2), by the Poincaré’s linear series method. This method offers the great advantage to work without having to solve the eigenvalue equation which control the quadratic fluctuations (i.e. without having to calculate the Poincaré’s coefficients of stability). The method provides also, as we will show, a useful guide to obtain the stability limits of any Legendre transformed ensemble when one knows the stability range of one ensemble and this, whether or not these ensembles are equivalent.

In this article we will study the origin as well as the generic properties of this inequivalence. Addressing the determination of stability criteria, we will first find the mathematical reason which leads to the breaking of the apparent “symmetry” between an ensemble and its Legendre transform. This is strange indeed in that if we apply a second Legendre transformation one is led back to the initial ensemble. This mathematical reason will also permit us to make contact with physics because the most stable ensemble will systematically describe the most isolated system. By isolated we mean an ensemble whose variable (called in this article control parameter) which characterizes it, is a conserved quantity of the hamiltonian of the system. On the contrary, non-isolated systems are characterized by control parameters kept fixed by the intervention of an external reservoir. Familiar examples of couples of control parameters are respectively: energy-temperature, particle number-chemical potential, volume-pressure, angular momentum-angular velocity.
We then demonstrate the following points:

1. The inequivalence comes from the fact that the less stable ensemble has a richer spectrum of fluctuations which contains one extra fluctuating quantity (or one more Poincaré stability coefficient).

2. From this fact one immediately concludes that when the corresponding eigenvalue of those extra fluctuations is positive it simply means that the number of unstable modes in the two ensembles is identical (This number can be zero in which case both ensembles are stable).

3. More remarkable is the fact that it is always this new eigenvalue which encodes the changes of stability of the less stable ensemble. This means that whenever the more stable system approaches instability (i.e. when its lowest eigenvalue tends to zero) an algebraic identity dictates that the new eigenvalue will cross zero before the lowest one of the more stable system.

Our analysis results in a qualification of the traditional criterion wherein the positivity of the specific heat was considered as a sufficient condition for the stability of the canonical ensemble. It provides also a timesaving procedure when, being aware that ensembles might be inequivalent, one wishes to determine stability ranges under various conditions\cite{4,5,14,15}. As an illustration of its power, we apply our analysis to the situation (recently studied by Kaburaki et al.\cite{10,11}) in which the angular momentum of rotating black holes has a stabilizing effect. We derive results on stability that are stronger than those previously published.

Section 2

Our goal is to compare the stability limits of two ensembles related by a single Legendre transformation. The demonstration of the three points mentioned in the Introduction proceeds as follows. We start by the analysis of a simplified mathematical abstraction in which all the members of the first ensemble are states parametrized by a single variable, denoted by $x$, at a fixed control parameter $s$. Subsequently the applicability of the mathematical exercise to physical situations including many degrees of freedom (as in thermodynamics) is discussed.
In the first ensemble, equilibrium situations and stability conditions are provided by a potential function $F(x; s)$. The Legendre transformation to a new potential $G(x; t)$ with control parameter $t$, is given by:

$$G(x; t) = F(x; S(x; t)) - tS(x; t)$$  \hspace{1cm} (1)

where $S(x; t)$ is the solution (unique at fixed $x$) of:

$$t = \partial_s F(x; s)$$  \hspace{1cm} (2)

$\partial$ designates partial derivative with all other quantities kept fixed. The equilibrium configurations, $X_i(s) \quad i = 1, \cdots n$, are the solutions of:

$$\partial_x F(x; s) = 0$$  \hspace{1cm} (3)

Those linear series of equilibrium configurations coincide with the solutions of

$$\partial_x G(x; t) = 0$$  \hspace{1cm} (4)

by virtue of eq(2). At this point, the equivalence of the two ensembles is manifest since $F$ is the Legendre transform potential of $G$, with the role of $s$ and $t$ interchanged, and since they determine the same equilibrium configurations. (We shall adopt the notation $F$-ensemble when the description of the states is given by $F(x; s)$ and $G$-ensemble for the other).

A dissymmetry between the two is introduced by formulating a necessary and sufficient condition for the stability of equilibrium configurations of one ensemble, say the $F$-ensemble:

$$\partial_x^2 F(x = X_i(s); s) < 0$$  \hspace{1cm} (5)

(*i.e.* that $F$ be a maximum). Thus any change of stability, in the $F$-ensemble, will occur if and only if $\partial_x^2 F$ crosses zero.

---

1For the reader who wants to have in mind a specific example, $s$ may be interpreted as the total energy $E$ (then by working at fixed $s$ one has the microcanonical ensemble), the variable $x$ viewed as the energy repartition between two subsystems and the potential $F(x; s)$ may be interpreted as the entropy out of equilibrium when the variable $x$ is not at its equilibrium value $X_i(s)$. Then $t$ is the inverse temperature and the potential $-G$ is the free energy divided by the temperature. The reader might usefully consult ref. 19 to see in detail how this method apply to situations containing a black hole surrounded by radiation.
Contrariwise, in the $G$-ensemble, at fixed $t$, $s$ fluctuates as well around the solution of eq(2): $S(x, t)$. Thus further conditions must be met to fulfill stability (i.e. $G$ has to be a true maximum and not only a saddle point in order to insure that the new fluctuations be bounded). Since those new conditions need not be met even when eq(5) is satisfied, the $G$-ensemble is the less stable one. Our purpose is now to prove that those new conditions inevitably restrict the range of stability of equilibrium situations in the $G$-ensemble. (At the end of the paper, we shall discuss the “opposite” problem, that is, how, when one’s consideration begins with the knowledge of the stability limits of the less stable ensemble, one may compute the extension of the stability range for the more stable ensemble.)

We will first review how changes of stability, in the $F$-ensemble, manifest themselves using Poincaré method. The reader unfamiliar with this method may usefully consult refs [1,2,19]. In addition, we are using the same notations and conventions. The reason for this review is that it affords an instructive illustrative example of how the topological behavior of the linear series in the vicinity of a critical point inevitably restricts the stability range of the Legendre transformed ensemble.

The idea of the method is to construct the linear series, locus of solutions of eq(3), as a function of $s$: $x = X_i(s)$. Poincaré theorem states that stability changes may occur only at a bifurcation (i.e. when two series intersect) or at a turning point (i.e. when two series merge into each other). Bifurcations occur because of an “excess” of symmetry and can be removed by the slightest modification of the potential $F(x; s)$\textsuperscript{[2,16]}. Hence, only the behavior of the series near a turning point is studied here.

More specifically we consider the particular situation where one has two equilibrium solutions of eq(3) for $s$ smaller than a certain maximum denoted $s_f$. These solutions merge into each other as $s$ tends to $s_f$ and do not exist for $s$ greater than $s_f$. (This situation corresponds to the case $i$ (Fig. 2) of

\textsuperscript{2}We are using the thermodynamical terminology even though the present analysis applies as well to mechanical stability, thus by ‘fluctuates’ we mean that we have to consider neighboring solutions around the equilibrium one in ordre to determine stability. In the same spirit by ‘fluctuations of $x$’ we shall designate the quantity $(-\partial^2 F)^{-1/2}$ which is the RMS fluctuations of $x$ in a statistical ensemble weighted by $e^F$. We refer again to ref. 19 for explicit examples in which the notions of stability and fluctuations are defined more concretely.
the classification of ref.1) The series of solutions \( x = X_1(s) \) describes stable equilibrium situations for which eq(5) is satisfied, and the series \( x = X_2(s) \) describes unstable situations. The method then proceeds by evaluating the second total derivative along those series:

\[
\frac{d^2 F_a}{ds^2} = \partial_s^2 F_a + \partial_x \partial_s F_a \frac{dX_a}{ds}
\]  \( \tag{6} \)

where \( F_a \equiv F(X_a(s), s) \) and \( a = 1, 2 \). With the help of the following identity

\[
\frac{d}{ds} (\partial_x F_a) = 0 = \partial_x \partial_s F_a + \partial_x^2 F_a \frac{dX_a}{ds}
\]  \( \tag{7} \)

eq(6) becomes

\[
\frac{d^2 F_a}{ds^2} = \partial_s^2 F_a - (\partial_x \partial_s F_a)^2 / \partial_x^2 F_a
\]  \( \tag{8} \)

Hence the change of stability at \( s = s_f \), where \( \partial_x^2 F_a = 0 \), is manifested by the divergence of this second derivative unless

\[
\partial_x \partial_s F_a(s = s_f) = 0
\]  \( \tag{9} \)

(We do not consider this degenerate possibility, which leads to a bifurcation, since it has been proven\(^2,16\) that the slightest modification of \( F \) will destroy the simultaneous vanishing of \( \partial_x^2 F_a \) and \( \partial_s \partial_x F_a \)). We emphasize the role and the physical meaning of eq(7). If \( \partial_x \partial_s F_a \neq 0 \), one has

\[
\frac{dX_a}{ds} \sim (\partial_x^2 F_a)^{-1}
\]  \( \tag{10} \)

which means that near the instability, the evolution of the equilibrium value \( X_a(s) \) (or the mean value in thermodynamical ensembles) is entirely controlled by the growing of the fluctuations. Hence, the knowledge of \( X_a(s) \) is sufficient to determine the stability changes without computing separately the fluctuations themselves.

Furthermore since

\[
\frac{d^2 F_a}{ds^2} \xrightarrow{s \to s_f} (-) (\partial_x^2 F_a)^{-1} \quad \rightarrow +\infty \text{ along } X_1(s) \\
\rightarrow -\infty \text{ along } X_2(s)
\]  \( \tag{11} \)

the linear series 1, describing stable situations, lies inevitably below (in the \( s - t \) plane) the series 2, as depicted in Fig.1.
Before examining what are the consequences of this unique behavior, we briefly discuss what could have happened in more complex situations.

1. For ensembles with many $x$ variables, the linear series 1 could describe already unstable situations with $n$ negative modes (where $n < \text{number of degrees of freedom described by the } x \text{ variables}$). Nevertheless by going anti clock wise along the curve of Fig.1. one still increases the number of unstable modes each time $\partial_x F$ has a vertical tangent.

2. In the cases where one has many parameters $s$, each of them or any non singular function of them can be used to study the changes of stability of the $F$-ensemble, leading to identical predictions because the vertical tangent only occurs when $\partial^2_x F$ vanishes. But if one wants to study the stability of a particular Legendre transformed ensemble and compare it to the stability of the initial ensemble, it is appropriate to select the parameter which is used to define the Legendre transformation. (In the present case with only one parameter, we are automatically in such a situation.)

We now examine the stability in the $G$-ensemble. We first redefine in a slightly different manner the Legendre transformation, eq(1), to see clearly how the transformation from $F$ to $G$ enlarges the spectrum of fluctuations by one eigenvalue\(^3\) (in our case we have two fluctuating quantities $x$ and $s$) and why the new fluctuations always encode the changes of stability of the $G$-ensemble.

\[
G(x, s; t) = F(x; s) - st
\]  

(12)

Equilibrium configurations are now provided by

\[
\begin{align*}
\partial_x G &= 0 = \partial_x F \\
\partial_s G &= 0 = \partial_x F - t
\end{align*}
\]  

(13) (14)

Eq(13) leads identically to eq(3) since now the only $x$ dependence of $G$ is through $F$. Eq(14) connects $s$ with its Legendre-conjugate $t$ as in eq(2).

\(^3\) We recall that in a canonical ensemble the total energy $E$ (here $s$) of the system fluctuates due to the contact with the heat-reservoir.
These two equations furnish: \( x = X_a(t) = X_a(S_a(t)) \), where \( S_a(t) \) are the two values of the solution of eq(14) evaluated at the equilibrium situations \( x = X_a \).

The range of stability is obtained as previously by computing the second derivative of \( G \), with respect to \( t \), along the linear series \( x = X_a(t) \), see eqs(6-11)

\[
\frac{d^2 G_a}{dt^2} = \partial_t^2 G_a + \partial_x\partial_t G_a \frac{dX_a}{dt} + \partial_s\partial_t G_a \frac{dS_a}{dt}
\]

(15)

where \( G_a \equiv G(X_a(t), S_a(t); t) \). Since by construction in eq(12) one has

\[
\partial_t^2 G = 0, \quad \partial_t\partial_s G = 0, \quad \partial_s\partial_t G = -1
\]

(16)

one finds

\[
\frac{d^2 G_a}{dt^2} = -\frac{dS_a}{dt}
\]

\[
= -\left( \partial_s^2 F_a - (\partial_x\partial_s F_a)^2 / \partial_x^2 F_a \right)^{-1}
\]

\[
= - \left( \frac{d^2 F_a}{ds^2} \right)^{-1}
\]

\[
= - \left( \frac{dT_a}{ds} \right)^{-1}
\]

(17)

where \( T_a(s) = \partial_s F_a \) and where the second equality follows, as for the \( F \)-ensemble (see eq(7)), from

\[
\frac{d}{dt} (\partial_x G_a) = 0 = \partial_x^2 F_a \frac{dX_a}{dt} + \partial_s\partial_x F_a \frac{dS_a}{dt}
\]

\[
\frac{d}{dt} (\partial_s G_a) = 0 = -1 + \partial_x\partial_s F_a \frac{dX_a}{dt} + \partial_s^2 F_a \frac{dS_a}{dt}
\]

(18)

We thereby recover the well-known identity, eq(17), between first total derivatives. The reasons that we prove this identity are the following:

1. By the very structure of the Legendre transformation which implies eqs(16), we see that the fluctuations of \( s \) (i.e. \( (-d^2 F/ds^2)^{-1/2} \)) alone control the vertical tangents of \( \partial_t G_a = S_a(t) \). But they also control, as we will prove, the changes of stability in the \( G \)-ensemble.
2. These new fluctuations encode in a very specific way the previous fluctuations of $x$ (see eq.(17)) and one has still to impose eq.(5) in order to have stable equilibrium (even though, there will be no new unstable mode, in the $G$-ensemble, when $\partial_x^2 F_a$ will become negative, see the Appendix).

From eq.(17) we obtain that changes of stability in the $G$-ensemble may occur when $d^2 F_a/d^2 s = 0$ and that (since only maxima describe stable equilibrium) the more stable series is the one with $d^2 F_a/d^2 s < 0$, i.e. see eq.(8) when:

$$\partial_s^2 F_a < (\partial_x \partial_s F_a)^2 / \partial_x^2 F_a$$

(19)

Since $\partial_x^2 F_a$ is negative along the stable linear series 1 and tends to $-\infty$ as $s$ tends to $s_f$, the inequality (19) has to break down before $s_f$ at some maximum point $s = s_g$. (We assume that $F$ has regular second derivatives). Hence, since $d^2 F_a/ds^2$ crosses zero before $\partial_x^2 F_a$, the new fluctuations will always control the changes of stability of $G$. This proves the third and the last point mentioned in the introduction.

One can visualize this ordering by examining the behavior of the linear series in the vicinity of $s_f$ and $s_g$. Only two situations may occur. Either eq(19) is never satisfied and the $G$-ensemble is always unstable with one negative mode (case I of Fig.2. An example of this case is provided by a Black Hole in contact with a heat bath,[7]), or eq(19) is satisfied until $s = s_g$ and the $G$-ensemble is stable up to that point (case II, see the example of star cluster in ref. 1).

For the interested reader we present in the appendix the (more traditional) analysis of the stability of the $G$-ensemble when any reference to $s$ has been eliminated, through the use of eq(2), before looking at equilibrium. The result one obtains is that the quadratic fluctuations of $x$ now evaluated at fixed $t$, rather than fixed $s$, encode automatically the determinant factor $d^2 F_a/ds^2$ hence leading again to the same range of stability. In addition we present in this appendix how, starting from $G$, one recovers the correct enlarged range of stability for the $F$-ensemble by an inverse Legendre transformation. The mathematical reason for which one increases the stability range, by going back from $G$ to $F$, is directly related to the fact that when both ensemble are stable one has
This inevitable discrepancy of signs has the following physical interpretation. Parameters of the $s$ type, like energy, angular momentum or volume represent conserved quantities of the Hamiltonian of an isolated system and have to be treated as the variables $x$ (stability upon extremization, eq(14), requires the same sign for the curvature, see eq(19)). In contrast parameters of the $t$ type, like temperature, angular velocity or pressure, are kept fixed by the intervention of an external “reservoir”. Furthermore they enter only in a linear way into the state function (see eq(12)) and play an identical role to Lagrange multipliers. The fact that their extrema correspond to minima, eq(20), translates in Statistical Mechanics into the fact that inverse Laplace transformations behave like Fourier transformations\cite{15}. Thus their integration, due to their linear dependence, leads to a Dirac $\delta$-function which reduces by one the fluctuating quantities.

In thermodynamics, in complex situations, one deals with many control parameters. The ensemble with all its parameters being of the $s$ type is the most stable one and describes the most isolated system. The stability of this ensemble does not refer to any other ensemble and the extremization of its state function is a rephrasing of the Second Principle of Thermodynamics\cite{4}. Contrariwise the stability of all other ensembles is subject not only to additional necessary conditions (like the positivity of the specific heat, see eq(19)) but also to the stability of the most stable ensemble, eq(5).

Application to rotating black holes
To illustrate the power and the simplicity of our analysis we consider the situation in which one has a rotating black hole in contact with a heat bath. Working at fixed angular momentum, it has been noted that there exists a temperature at which the specific heat changes sign\cite{8,9,10}. The question is then: is the equilibrium, in the canonical ensemble, stable or only less unstable when the specific heat is positive?

The problem of the stability of a rotating black hole was addressed in ref(10). In the microcanonical ensemble (i.e. at fixed mass $M$, fixed angular momentum $J$ and with the entropy $S$ as the relevant state function), it was

\[
\frac{d^2 F_a}{ds^2} < 0, \quad \frac{d^2 G_a}{dt^2} = -\left(\frac{d^2 F_a}{ds^2}\right)^{-1} > 0 \quad (20)
\]
stated that the absence of vertical tangent in the plot $\beta = \partial_M S$ versus $M$ indicates that there is no change of stability when one varies the ratio $J/M^2$ (see Fig. 3). Then, since it had been proven that isolated Schwarzschild black holes are stable\cite{17}, it was correctly concluded that all isolated rotating black holes are stable\footnote{Nevertheless, since the entropy $S(M, J)$ is already an extremized quantity with respect to the unknown variables “$x$” which describe the microscopical states of the black hole, one has to assume that a simultaneous vanishing (see eq(9) and the associated discussion) which could hide changes of stability does not occur.}

In the canonical ensemble, by virtue of this analysis and our statement 2 (see Introduction), one immediately concludes that the flip of sign of the specific heat at $\beta = \beta_c$ (see Fig. 3) does correspond to a real change of stability and that all equilibrium situations along the linear series with the smaller mass are stable. This conclusion is stronger that the one given in ref.(10) because the authors did not take into account the fact that the instability of the Schwarzschild black hole in contact with a heat bath is due to a single instable mode.

\section*{Appendix A}

Two points will be discussed in this appendix. The first one is the recovering of the changes of stability of the $G$-ensemble when all dependence of $s$ has been eliminated prior looking at equilibrium configurations. The second point concerns the extension of the stability range when, starting from $G$, one wishes to analyze the stability of the $F$-ensemble.

Since all explicit dependence of $s$ has been eliminated, one has to analyze the stability of $x$ fluctuations at fixed $t$ rather than fixed $s$. We have nevertheless to verify that the fluctuations of $s$ at fixed $x$ described by $\partial_s^2 F$ are bound (i.e. $\partial_s^2 F < 0$. We assume here that this is satisfied). The state function which control the $x$ fluctuations is (see eq(1)):

$$G(x; t) = F(x; S(s; t)) - tS(x; t) \quad (A. 1)$$

where $S(x; t)$ is the solution of eq(2). Equilibrium situations are furnished (see eqs(3,4)) by:
\[ \partial_x G|_t = 0 = \partial_x F|_s \]  
(A. 2)

by virtue of eq(2). (In this appendix, in order to prevent any quibble we specify by a vertical line which quantity is kept fixed upon derivation.)

Stable equilibrium requires \( G \) to be a maximum (see eqs(5,17,19))

\[ \partial^2_x G|_t < 0 \]  
(A. 3)

This condition reads

\[ \partial^2_x G|_t = \partial^2_x F|_s + \partial_s (\partial_x F|_s)|_x (dS_a/dx)|_t \]  
(A. 4)

where \( S_a(t) = S(X_a(t), t) \). \( (dS_a/dx)|_t \) can be obtained by taking the total differential of eq(2):

\[ \partial_x (\partial_s F|_x)|_s dx + \partial^2_s F|_x ds = dt = 0 \]  
(A. 5)

hence

\[ \frac{dS}{dx}|_t = -\frac{(\partial_x \partial_s F)}{(\partial^2_s F)} \]

Then eq(A.4) becomes

\[ \partial^2_x G_a = \partial^2_x F_a - (\partial_x \partial_s F_a)^2 / \partial^2_s F_a \]
\[ = \partial^2_x F_a \left( \frac{d^2 F_a}{ds^2} \right) (\partial^2_s F_a)^{-1} \]  
(A. 6)

where we have used eq(8). By virtue of the analysis which follows eq(19) and which indicates that \( d^2 F_a/ds^2 \) crosses always zero before \( \partial^2_x F_a \), one recovers that the changes of stability of the \( G \)-ensemble occur when \( d^2 F_a/ds^2 \) vanishes. Furthermore, eq(A.6) indicates that the vanishing of \( \partial^2_x F_a \) at \( s = s_f \) does not lead to the vanishing of \( \partial^2_x G_a \) because of the divergence of \( d^2 F_a/ds^2 \) at that point (see eq(11)). Finally, by an analysis similar to the one given after eq(18) one can easily show that the second negative eigenvalue of the \( G \)-ensemble will always appear after the first one of the \( F \)-ensemble, thereby reinforcing the point 1 (see Introduction) by introducing a well defined ordering.
The second point of this appendix concerns the following problem: having obtained the limit of the stable configurations for the less stable ensemble (when A.6 vanishes) how can one evaluate the extended range of stable configurations in the $F$-ensemble? This problem is by no means purely academical: Under certain circumstances the evaluation of quadratic fluctuations can be performed easily only in, say, the grand canonical ensemble. Then two questions arise when one wants to use those results into the evaluation of the stability limits for more stable ensembles (see refs. [4,5,15], where this problem was encountered and discussed). The first one concerns the possibility of extending the evaluation of the fluctuations outside the stability domain of the ensemble in which they were computed. The second question is: how the instability of the initial ensemble will manifest itself into the expression of the stability of the fluctuations in a more stable ensemble? Answering those questions can be achieved in many different ways. We only sketch two possible ways using our simple mathematical abstraction.

The most pedestrian way takes the lines of the analysis presented from eq(12) to eq(19): one introduces a third fluctuating quantity (i.e. $t$) and one finds that the role of its fluctuations is to suppress completely the fluctuations of $s$ thereby leading back to the “initial” $F$-ensemble with its single fluctuating variable: $x$. (This procedure corresponds to the second (noted II) diagonalization scheme of ref. [15]).

A second illuminating method proceeds, as shown above (see eq(A.6)), by expressing the quadratic fluctuations of $x$ at fixed $s$ in terms of the fluctuations evaluated at fixed $t$. One has:

$$\partial_x^2 F_a = \partial_x^2 G_a \left( \frac{d^2 G_a}{dt^2} \right) (\partial_t^2 G_a)^{-1} \quad (A.7)$$

Suppose now that one increases $s$, starting from stable configurations for $G$ (i.e. for $s < s_g$, see Fig. 2, where $\partial_x^2 F_a < 0$). When $s$ crosses $s_g$ both $\partial_x^2 G_a$ and $d^2 G_a/dt^2$ flip sign but their product is perfectly well defined. Hence $\partial_x^2 F_a$ remains negative and will vanish only when $d^2 G_a/dt^2$ vanishes. Eq(A.7) proves that the evaluation of the $x$ fluctuations in the $F$-ensemble (controlled by $\partial_x^2 F_a$) can be computed safely in the $G$-ensemble even in situations which correspond to unstable equilibrium in $G$. (This simultaneous vanishing, at $s = s_g$, is precisely what was encountered in ref. 4, and subsequently analyzed in refs. 5 and 15. It leads to the first (I) diagonalization scheme of ref. 15.)
Acknowledgements

We would like to thank J. Katz for his careful explanations of the Poincaré’s method and for many useful remarks. We also thank R. Brout for interesting discussions about the physical content of the conclusions reached in this article.
References

1. J. Katz, Mon. Not. R. Soc. Lond. **183** (1978) 765.
2. J. Katz, Mon. Not. R. Soc. Lond. **189** (1979) 817.
3. D. Lynden-Bell and R. Wood, Mon. Not. R. Soc. Lond. **138** (1968) 495.
4. G. Horwitz and J. Katz, Astroph. J. **211** (1977) 226.
5. G. Horwitz and J. Katz, Astroph. J. **222** (1978) 941.
6. Y. Aharonov, F. Englert and J. Orloff, Phys. Lett. **B199** (1987) 366.
7. S. Hawking, Phys. Rev. **D13** (1976) 191.
8. P. Davies, Proc. R. Soc. Lond. **A353** (1977) 499.
9. B. Schumaker, W. Miller and W. Zurek, Phys. Rev. **D46** (1992) 1416.
10. O. Kaburaki, I. Okamoto and J. Katz, Phys. Rev. **D47** (1992) 2234.
11. J. Katz, I. Okamoto and O. Kaburaki, Class. Quant. Grav. **10** (1993) 1323.
12. J. York, Phys. Rev. **D33** (1986) 2092.
13. R. Balbinot and A. Barletta, Class. Quant. Grav. **6** (1989) 195, 203.
14. G. Comer, Class. Quant. Grav. **9** (1992) 947.
15. G. Horwitz, Commun. Math. Phys. **89** (1983) 117.
16. J. M. T. Tompson, Phil. Trans. R. Soc. Lond. **292** (1979) 1386.
17. C. Misner, K. Thorne and J. Wheeler, Gravitation. Freeman (1973).
18. H. Callan, Thermodynamics. Wiley International (1960).
19. R. Parentani, J. Katz and I. Okamoto, Thermodynamics of a black hole in a cavity (1994) [gr-qc/9410015](https://arxiv.org/abs/gr-qc/9410015)
Figure captions

Fig. 1. Plot of the equilibrium curve $T_a(s) \equiv \partial_s F_a(s)$, evaluated along the linear series $X_1(s)$ and $X_2(s)$, in the vicinity of the turning point $s = s_g$. The series $X_1(s)$ describe stable equilibrium configurations and lie inevitably under the series $X_2(s)$. At $s = s_g$, the fluctuations of $x$ are unbound ($\partial_x^2 F_a$ vanishes) and this manifests itself by a vertical tangent of $T_a(s)$. This can be understood by extending the definition of the parameter $t$ away from equilibrium: $t(x,s) \equiv \partial_s F$, and letting it fluctuate around $T_a(s)$ according to the $x$ fluctuations[19].

Fig. 2. This is the same plot as the one of Fig. 1. The two possible behaviors of $T_a(s)$ are displayed. Along the linear series (1,I) the $G$-ensemble is always unstable with one negative eigenvalue. Along (1,II) the $G$-ensemble is stable for $s < s_g$ (or $t < t_g$). The horizontal tangent of $dS_a/dt$ at $t = t_g$, indicates that the fluctuations of the variable $s$ are infinite. This should be contrasted with the parametrical fluctuations of $t$ in the $F$-ensemble.

Fig.3. Plot of the equilibrium curve $\beta = \partial_M S(J,M)$ versus the mass $M$, at fixed angular momentum $J$ and describing a Kerr black hole[7,9]. At $\beta = \beta_c$, $\partial_M \beta$ flips sign. Below $M_c$, in the canonical ensemble, the black hole is in stable equilibrium with the heat bath. This situation corresponds to the case II of Fig. 2, with $s_f$ rejected at infinity.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9410017v1