Formation of solitary zonal structures via the modulational instability of drift waves

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Abstract
The dynamics of the radial envelope of a weak coherent drift-wave (DW) is approximately governed by a nonlinear Schrödinger equation (NLSE), which emerges as a limit of the modified Hasegawa–Mima equation (mHME). The NLSE has well-known soliton solutions, and its modulational instability (MI) can naturally generate solitary structures. In this paper, we demonstrate that this simple model can adequately describe the formation of solitary zonal structures in the mHME, but only when the amplitude of the coherent DW is relatively small. At larger amplitudes, the MI produces stationary zonal structures instead. Furthermore, we find that incoherent DWs with beam-like spectra can also be modulationally unstable to the formation of solitary or stationary zonal structures, depending on the beam intensity. Notably, we show that these DWs can be modeled as quantum-like particles (‘driftons’) within a recently developed phase-space (Wigner–Moyal) formulation, which intuitively depicts the solitary zonal structures as quasi-monochromatic drifton condensates. Quantum-like effects, such as diffraction, are essential to these condensates; hence, the latter cannot be described by wave-kinetic models that are based on the ray approximation.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In magnetically confined plasmas, radially propagating coherent structures are of great interest, as they can induce transport that is ballistic rather than diffusive. Examples include turbulence spreading [1–5] and avalanching [6–8], as well as the density ‘blobs’ in edge plasmas [9–12]. Recently, such structures have also been identified in gyrokinetic simulations of subcritical plasmas [13–15].

An arguably simplest model of radially propagating coherent structures considers the radial envelope dynamics of a weak coherent drift-wave (DW) without forcing and dissipation. It has been shown that the envelope approximately follows a nonlinear Schrödinger equation (NLSE) [16, 17], where the (cubic) nonlinearity originates from the quasilinear interaction between the primary DW and a secondary zonal-flow (ZF). The well-known soliton solution to the NLSE corresponds to a zonal structure that propagates radially at the DW group velocity. In particular, Guo et al [5] first studied this DW–ZF soliton in the context of turbulence spreading from linearly unstable regions to stable regions. Accordingly, they examined soliton formation due to the inhomogeneity of the linear growth rate. However, an intrinsic mechanism for the formation of solitary zonal structures is still needed, since events such as turbulence avalanching seem to be dominated by local physics [8]. One natural candidate is the modulational instability (MI) [16, 17], which is known to generate solitary structures in the NLSE. (Here, ‘solitary’ means propagating at a (roughly) constant speed while maintaining a (roughly) constant shape.) Nevertheless, the relevance of this
mechanism in the modified Hasegawa–Mima equation (mHME) [17, 18], the parent model of the NLSE, has remained unexplored.

In this paper, we explicitly demonstrate that the NLSE can adequately describe the formation of solitary zonal structures via the MI in the mHME, using both quasilinear and nonlinear simulations of the latter. However, these structures only emerge from primary DWs with relatively small amplitudes; at larger amplitudes, the MI produces stationary zonal structures instead. Then, as a generalization, we simulate the MI of incoherent DWs with beam-like spectra, using a recently developed Wigner–Moyal (WM) model of DW–ZF dynamics [19]. While the finite beam width has a stabilizing effect on the MI, the results are similar to those from coherent DWs. That is, with moderate beam intensity, solitary zonal structures are formed; as the intensity increases, the zonal structures become stationary.

One novelty of the WM model is that it treats DWs as quantum-like particles ('driftons') and facilitates analysis of the zonal structures from an instructive phase-space perspective. The Wigner function of the DW–ZF solitons show concentration of DW quanta in both position and momentum, which suggests that they are essentially the DW–ZF solitons. In turn, it also implies that these structures are not the same as ZFs. Instead, we will introduce the mHME and its quasilinear approximation. In section 4, we describe the WM model and the MI of general DW spectra. Most of our new results are presented in section 5, where we study solitary zonal structures and their formation from coherent DWs and incoherent DW spectra. Our results are summarized and discussed in section 6.

2. Basic model

2.1. Modified Hasegawa–Mima equation

In this paper, we study DWs within the mHME [17, 18], which is the simplest yet useful model that captures many basic effects of interest. (The mHME has been referred to as the generalized Hasegawa–Mima equation [23] or extended Hasegawa–Mima equation as well.) In a dimensionless form, the mHME can be written as

\[ \partial_t w + \mathbf{v} \cdot \nabla w - \beta \partial_y \varphi = 0, \]  
\[ w \doteq \nabla^2 \varphi - \tilde{\varphi}. \]  

(1a)

(1b)

It is a 2D model in slab geometry, with coordinates \( x \equiv (x, y) \) and a uniform magnetic field normal to the plane. The gradient of the plasma density \( n_0 \) is in the radial \( x \) direction, and is parameterized by a (positive) constant \( \beta = a/L_s \), where \( a \) is the system scale length and \( L_s = (\partial n_0/\partial x)^{-1} \) is the local scale length of the gradient. (The symbol \( \doteq \) denotes definitions.)

The ZF is in the poloidal \( y \) direction. Time \( t \) is normalized by the transit time \( a/c_s \), where \( c_s \) is the sound speed. Space is normalized by the ion sound radius \( \rho_i = c_s/L_i \), where \( L_i \) is the ion gyro-frequency. The electrostatic potential \( \varphi(t, x) \) is normalized by \( T_e\rho_i/(e\alpha) \), where \( e \) is the unit charge and \( T_e \) is the electron temperature. Accordingly, \( \mathbf{v} \doteq (\partial_x \varphi, \partial_y \varphi) \) is the \( \mathbf{E} \times \mathbf{B} \) velocity.

In the mHME, the definition of the generalized vorticity \( w \) (1b) involves separating the total \( \varphi \) into the zonal component \( \langle \varphi \rangle \) and non-zonal component \( \tilde{\varphi} \). The former is the ‘zonal average’ of \( \varphi, \langle \varphi \rangle \doteq \int \varphi/L_y \) (where \( L_y \) is the domain length in y), and corresponds to the ZF. The latter is the fluctuating component, \( \tilde{\varphi} \doteq \varphi - \langle \varphi \rangle \), and corresponds to DWs. The same notations apply to \( w \) and \( \mathbf{v} \) as well.

In contrast, in the original Hasegawa–Mima equation (oHME) [25], the generalized vorticity is defined as \( w \doteq \nabla^2 \varphi - \varphi \). The modification in the mHME is due to the finding that the zonal potential \( \langle \varphi \rangle \) does not contribute to the adiabatic electron response [26, 27]. The oHME is also called the Charney–Hasegawa–Mima equation for its equivalence to the Charney equation [28]. Meanwhile, with \( w \doteq \nabla^2 \varphi \), equation (1a) becomes equivalent to the barotropic vorticity equation [29]. Both the Charney equation and the barotropic vorticity equation are widely used in studies of geophysical fluids. The similarity between the mHME and these equations suggests that our results can, to an extent, be relevant to Rossby-wave turbulence in geophysics.

The mHME does not have a primary instability, i.e. an instability that generates DWs. Thus, external forcing is sometimes introduced as a proxy, and ad hoc dissipation must also be added to balance the energy input. However, due to the existence of the drift term (the linear term proportional to \( \beta \)), the mHME can support finite-amplitude DWs even in the absence of forcing. Then, ZFs can emerge from these DWs through a secondary instability, which is also known as the MI [16, 17] or the zonostrophic instability [30]. In this paper, we focus on this particular process and will not be concerned with the origin of the DWs. Instead, we will introduce finite-amplitude DWs via initial conditions, and exclude forcing and dissipation (except briefly in the end of section 5.3).

2.2. Quasilinear approximation

To proceed, let us separate equation (1a) into the non-zonal and zonal components, respectively:

\[ \partial_t w + \langle \mathbf{v} \rangle \cdot \nabla w + \tilde{\mathbf{v}} \cdot \nabla \langle w \rangle - \beta \partial_y \tilde{\varphi} = 0, \]  
\[ \partial_t \langle w \rangle + \tilde{\mathbf{v}} \cdot \nabla \langle w \rangle = 0. \]  

(2a)

(2b)

In the studies of ZF formation, it is common to assume that the nonlinearity on the right-hand side (rhs) of equation (2a) is weak, which can be physically interpreted as neglecting the eddy–eddy interactions between the DWs. This is called the quasilinear approximation, for it leads to two linear equations

\[ \partial_t \tilde{\mathbf{v}} + \langle \mathbf{v} \rangle \cdot \nabla \tilde{\mathbf{v}} = 0, \]  
\[ \partial_t \tilde{\varphi} + \langle \mathbf{v} \rangle \cdot \nabla \tilde{\varphi} = 0. \]  

(3a)

(3b)

In the equation (3a), the rhs is assumed to be negligible. However, the corresponding equation (3b) may not be so simple. In the following sections, we will show that the gradient of the potential \( \tilde{\varphi} \) is in fact mediated by the advection term in the equation (3a), which may be significant. In the simulations, we will impose a forcing term on the rhs of equation (2a), and use a nonlinear simulation to study ZF formation.
that are nonlinearly coupled:
\[
\begin{align*}
\partial_t \hat{w} + U \partial_x \hat{w} - (\beta + U^n) \partial_x \hat{\phi} &= 0, \\
\partial_t U - \partial_x (\beta \partial_x \hat{\phi}) &= 0.
\end{align*}
\] (3a, 3b)

For convenience, we introduce the ZF velocity \( U(t, x) \equiv \langle \psi \rangle = \partial_t \langle \phi \rangle \), with \( U^n = \partial_x^2 U \).

The quasilinear mHME (3) has been shown to reproduce many of the basic features of the original nonlinear system (2), at least qualitatively [30]. Hence, we consider the quasilinear approximation sufficient for our purposes, and adopt it throughout the rest of the paper (except in figures 4 and 5, where we briefly present some nonlinear simulation results as verifications for the corresponding quasilinear simulations).

The non-zonal component of the quasilinear mHME (3a) can also be written in the form of a Schrödinger equation for DW quanta (driftons)
\[
i \partial_t \hat{w} = \hat{H} \hat{w}.
\] (4)

Unlike the truly quantum Schrödinger equation, equation (4) does not contain \( \lambda \), so it is purely classical. (Likewise, \( |\hat{w}|^2 \) is an action density rather than number density.) Also, the Hamiltonian operator
\[
\hat{H} \equiv (\beta + U^n) \hat{p}_x^2 + \hat{U} \hat{p}_y,
\] (5)

is not entirely Hermitian, since \( \hat{U}^n \) does not commute with \( \hat{p}_x^2 \). Here, \( \hat{p} \equiv -i \nabla \) can be understood as the momentum (wave-vector) operator, and \( \hat{p}_x^2 \equiv \hat{p}_x^2 + \hat{p}_y^2 \) such that \( \hat{w} = -\hat{p}_x^2 \hat{\phi} \). Also, \( \hat{U} \equiv U(t, \hat{x}) \) and \( \hat{U}^n = U^n(t, \hat{x}) \), with \( \hat{x} \) being the position operator. This quantum-like formalism proves useful in deriving the NLSE that governs the envelope dynamics of coherent DWs (section 3), as well as the WM formulation that can describe incoherent DWs (section 4).

### 3. Coherent DWs

The quasilinear mHME (3) has an exact plane-wave solution with finite-amplitude, \( \hat{w} = \text{Re} \langle \psi \rangle e^{i k_x x - i \Omega t} \), where \( k \equiv (k_x, k_y) \) is the wave-vector and \( \Omega \equiv \pm k_z / \sqrt{2} \) is the DW frequency, with \( k_x^2 + k_y^2 = 1 + k_z^2 + k_y^2 \) and \( \psi_0 \) being a complex constant denoting the amplitude.

This primary-wave, when subject to large-scale modulations, can become unstable. One simplified way to study this MI is to consider the envelope dynamics of a coherent DW. The first study of such kind appears to be [31], which is based on the oHME and only considers poloidal modulations (‘streamers’). In [16], the mHME is employed, and radial and poloidal modulations are treated on the same footing. More comprehensive reviews of the envelope formalism can be found in [17, 22]. Also notably, related equations were later rediscovered independently in [5] (with over-simplified coefficients) and [32].

In all of these studies, it is noticed that the envelope equation can be approximated as a NLSE, which is well-known to have a MI. In section 3.1, we show how the NLSE follows naturally from our quantum-like formalism. In section 3.2, we re-derive the corresponding dispersion relation of the MI.

### 3.1. Nonlinear Schrödinger equation

Let us represent the Hamiltonian operator (5) as \( \hat{H} = \hat{H}_0 + \hat{H}_1 \), where \( \hat{H}_0 \) is the U-independent part and \( \hat{H}_1 \) scales linearly with \( U \). Since we focus on zonal structures, here we consider a coherent DW with radial modulation only, \( \hat{w} = \text{Re} \langle \psi(t, x) e^{i k_x x - i \Omega t} \rangle \). We assume that the envelope \( \psi \) is slow, i.e. \( |\partial_z \ln \psi| \ll |k| \), and also that \( |U| \) is small. Then, the Hamiltonian operator (5) can be approximated as
\[
\hat{H}_0 \approx \Omega + \frac{v_b}{2} \partial_x + (\chi / 2) \partial_x^2, \quad \hat{H}_1 \approx k_x U,
\] (6)

where \( v_b = \partial_x \Omega / \partial k_x \) is the radial group velocity and \( \chi = \partial^2 \Omega / \partial k_x^2 \), or explicitly
\[
v_b = -2 \frac{\lambda}{k} \frac{\psi^2}{k^4}, \quad \chi = \left( 2/3 \frac{\psi^2}{k^4} \right)(4k^4 - k^6).\n\] (7)

Equation (6) can be viewed as a special case of the Weyl expansion derived in [33] for an inhomogeneous medium. Additionally, the term proportional to \( U^n \) has been neglected because both \( |U| \) and \( \partial_x \psi \) are assumed small. Then, the resulting equation for \( \psi \) is
\[
i (\partial_t + v_b \partial_x) \psi \approx -(\chi / 2) \partial_x^2 \psi + k_x \psi U.
\] (8)

Meanwhile, using \( \varphi = -\sqrt{2} \psi / k \), we can approximate the ZF equation (3b) as
\[
\partial_t U \approx \partial_t \langle \partial_x \hat{w} \partial_x \hat{\phi} \rangle / k^4 \approx k_x k_y \partial_x |\psi|^2 / (2k^4),
\] (9)

where the factor 1/2 originates from zonal averaging. From equation (8), we can see that, to the leading order, the modulation propagates at the group velocity \( v_b \). Hence, we can assume that \( \partial_t \approx -v_b \partial_x \) in equation (9), and integrate in \( x \) to obtain
\[
U \approx -k_x k_y |\psi|^2 / (2k^4 v_b),
\] (10)

or more explicitly
\[
U \approx |\psi|^2 / (4 \beta).
\] (10)

Here, vanishing boundary condition in \( x \) is implied. For other boundary conditions (e.g. \( \varphi \) periodic in \( x \)), an additional integration constant may be needed on the rhs of equation (10) for consistency. (This constant can be easily removed by a Galilean transformation, however.) Also, with \( |U| \) assumed small, it is implied that the DW amplitude \( |\psi| \) should be small too. Substituting equations (10) into (8), we obtain
\[
i (\partial_t + v_b \partial_x) \psi \approx -(\chi / 2) \partial_x^2 \psi + k_x |\psi|^2 / (4 \beta).
\] (11)

Equation (11) has previously been derived (using somewhat different approaches) in [16, 17]. It has the form of a NLSE, or the Gross–Pitaevskii equation, so the structures it describes can be viewed as ‘driftron condensates’ (by analogy with the Bose–Einstein condensate). Namely, equation (11) shows that it is energetically favorable for driftions to be in a correlated state rather than have random phases. Also, equation (10) can be interpreted as an equation of state of the condensates, as it provides a local relation between the drifton density \( |\psi|^2 / 2 \) and another ‘thermodynamic’ property of the condensates, \( U \).
Figure 1. Sequences of (a) the DW envelope $\sqrt{\omega^2 + q^2}$ and (b) the ZF velocity $U[|\psi|^2/(4\beta)]$ obtained from NLSE (dashed) and quasilinear mHME (solid) simulations. From left to right, the snapshots are taken at $t = 0$, 40, ..., 480, respectively. The initial condition is a Gaussian envelope $\psi = 2\eta \sqrt{-\beta k_x/k_y} \exp(-\eta^2 x^2/2)$ with $\eta = 0.1$. We use the following parameters here and in all other figures throughout the paper: $\beta = 5$, $k_x = 0.3$, and $k_y = -0.3$.

In Figure 1, we compare the evolution of an initially Gaussian envelope in numerical simulations of the NLSE and the quasilinear mHME. The good agreement between the solutions confirms that the former is a reasonable approximation of the latter. All of our simulations using configuration-space models (the NLSE, the quasilinear mHME, and the nonlinear mHME) are pseudo-spectral, dealiased, and performed on periodic domains.

3.2. Modulational instability

The NLSE (11) has an exact homogeneous solution $\psi = \psi_0 \exp[-i k_x |\psi_0|^2 t/(4\beta)]$. The frequency $\omega$ and wave-number $q$ of a linear perturbation on this solution satisfy the following dispersion relation [17]

$$\left(\omega - q k_x \right)^2 = \frac{\lambda^2 q^4}{4} \left(1 + \frac{k_x |\psi_0|^2}{\beta \lambda q^2} \right).$$

When $\beta k_x \propto 4k_x^2 - k_y^2 < 0$, the frequency is complex for small $q$ and the solution is linearly unstable, with the wave-number of the fastest-growing mode given by $q_{\text{max}} = |\psi_0| \sqrt{-k_y/(2\beta \lambda)}$. This is the well-known MI of the NLSE, arising from the interplay of diffraction and self-focusing (the first and second terms on the rhs of equation (11), respectively).

The NLSE (11) offers an intuitive perspective on the MI of coherent DWs. However, as an approximate model, it is restricted to slow modulations and small $|\psi_0|$. When $|\psi_0|$ is large, $q_{\text{max}}$ can be comparable or larger than $k_x$, which is inconsistent with the underlying assumption of the NLSE. In addition, the NLSE only applies to primary waves with non-zero radial group velocity $v_g \propto k_x k_y$. While in this paper we focus on such waves for this very feature, primary waves with $k_x = 0$ are also of interest, since they correspond to the fastest-growing modes in some primary instabilities, particularly, ion-temperature-gradient modes [34, 35].

In fact, there are more general approaches to deriving the dispersion relation of the MI. One way is to employ the four-mode truncation (4MT) method. As the name suggests, the 4MT is a truncation of the mHME in spectral representation by only keeping four modes: a primary-wave with wave-vector $k$, a modulation with wave-vector $q$, and two side-bands with wave-vectors $k \pm q$. In general, the modulation does not have to be purely radial. For example, the MI with a purely radial primary-wave (a ZF) and a purely poloidal modulation is a tertiary instability of the ZF [36, 37]. Detailed discussions on the 4MT can be found in [38–40]. Meanwhile, for purely radial modulations with $q = (q, 0)$, which we focus on in this paper, the MI of coherent DWs can be considered as a special case of the MI of general DW spectra, which is discussed in section 4.

4. Drift-wave ensembles

Equation (4), along with equation (3b), governs the quasilinear dynamics of a single realization of DWs (in quantum mechanical terms, a ‘pure state’). However, due to the incoherent nature of DW turbulence, it is useful to consider the dynamics of an ensemble of DWs statistically. This is equivalent to studying the von Neumann equation that follows from the Schrödinger equation (4), which can describe the dynamics of ‘mixed states’. In double-configuration-space representation, this leads to the theory of second-order cumulant expansion (CE2), which has been widely used in geophysical fluids (e.g. [30, 41–43]) and subsequently introduced to plasma physics [44, 45]. A mathematically equivalent yet physically more intuitive alternative to the CE2 model is the phase-space representation of the von Neumann equation. This leads to the WM model of DW–ZF dynamics, which we introduce in section 4.1. The WM model can describe the MI of general DW spectra, which is presented in section 4.2.

4.1. WM formulation

The WM model of DW–ZF dynamics was first derived in [19]. The derivation starts from the quasilinear mHME (3) and leads to the following equations:

$$\partial_t W = \{H, W\} + \{\Gamma, W\},$$

(13a)

$$\partial_t U = \partial_p \int \frac{d^2 p}{(2\pi)^2} \psi^* p_y W^* \frac{1}{p_y^2}.$$  

(13b)

Here, $p = (p_x, p_y)$ is the coordinate in the DW momentum (wave-vector) space, and $W(t, x, p)$ is the zonal-averaged Wigner function [46]. For a single realization of $\tilde{\psi}(t, x)$, $W$ can be written as

$$W \equiv \left\langle \int d^2 s e^{-ip_x s} \tilde{\psi}\left(t, x + \frac{s}{2}\right) \tilde{\psi}\left(t, x - \frac{s}{2}\right) \right\rangle.$$  

(14)

For an ensemble of realizations, the zonal average (again, denoted by the angle bracket) can be regarded as an ensemble
average. The Wigner function \( W \) is a quasi-probability distribution of driftons, and the ZF velocity \( U \) serves as a collective field through which they interact. Since \( \tilde{\omega} \) is real (unlike in quantum mechanics where the wave functions are complex), the DW Wigner function has a unique parity in \( p \) that \( W(t, x, -p) = W(t, x, p) \). This implies that driftons come in pairs, i.e. each drifton with wave-vector \( p \) has a twin with wave-vector \(-p\).

The specific dynamics of the driftons is governed by

\[
\mathcal{H} = \beta p_y/\beta_y^2 + p_y U + \left[ U''(p_x, p_y/\beta_y^2) \right]/2, \tag{15a}
\]

\[
\Gamma = \left[ U''(p_x, p_y/\beta_y^2) \right]/2, \tag{15b}
\]

which are the Hermitian and anti-Hermitian parts of the Hamiltonian, respectively. Here, the Moyal star product \( A(x, p) \star B(x, p) \equiv A \exp(i\hat{L}/2)B \) [47], where the Janus operator \( \hat{L} \) is defined as \( A\hat{L}B \equiv (\partial_x A) \cdot (\partial_p B) - (\partial_p A) \cdot (\partial_x B) \). The Moyal sine and cosine brackets are given by [\( [A, B] \) \( \equiv 2A \sin(\hat{L}/2)B \) and \( [A, B] \) \( \equiv 2A \cos(\hat{L}/2)B \), respectively. A detailed derivation of equation (13) and a review of the Weyl calculus can be found in [19].

The WM equation (13) can be understood as a kinetic model treating driftons as quantum-like particles with finite 'de Broglie' wavelengths. As such, it captures 'full-wave' effects missing in wave-kinetic models of DW–ZF dynamics based on the ray approximation [20, 48], which treat driftions as point particles with zero wavelengths. While the wave-kinetic models prove useful in some scenarios [49–51], they are insufficient for the problems that we study in this paper. A detailed discussion on the limitations of the wave-kinetic models is presented in appendix.

### 4.2. Modulational instability

In the WM model, a statistically homogeneous equilibrium can be described by a Wigner function \( W(\mathbf{p}) \), which can be interpreted as a DW spectrum. Linearizing equation (13) about \( W(\mathbf{p}) \) leads to the following dispersion relation of the MI \([19, 36]\)

\[
\omega = \sum_\pm \int \frac{dp_y}{(2\pi)^2} W(\mathbf{p}) \left[ \frac{\partial}{\partial p_y} \left( p_x \pm \frac{q}{2} \right) \right] \left[ 1 - \frac{q^2}{\beta_y^2} \right], \tag{16}
\]

where \( \beta_y^2 \equiv 1 + (p_x \pm q)^2 + p_y^2 \). The MI of a monochromatic DW \( \tilde{\omega} = \text{Re}(\psi_0 e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}) \) can be considered as a special case. Using equation (14), we can obtain the corresponding spectrum of this DW

\[
W(\mathbf{p}) = \pi^2|\psi_0|^2[\delta(\mathbf{p} + \mathbf{k}) + \delta(\mathbf{p} - \mathbf{k})]. \tag{17}
\]

Substituting equation (17) into (16) leads to the dispersion relation of the MI of this monochromatic DW

\[
\sum_\pm \left[ k_\perp^2 \omega + 2\beta q_c(k_c \pm q/2) \right] = |\psi_0|^2 q^2 k_\parallel^2 (k^2 - q^2)(4k_\parallel^2 - q^2 - q^2)/2. \tag{18}
\]

Note that this dispersion relation agrees with that given by the 4MT method [38, 39], and is the exact dispersion relation of the quasilinear mHME (3).

It is instructive to compare the exact dispersion relation (18) with the approximate one (12) obtained from the NLSE. In figure 2, we show both dispersion relations (for a given \( k \)) with multiple values of \( |\psi_0| \). It can be seen that the agreement between the dispersion relations is better for small \( |\psi_0| \) and small \( q \), which is consistent with the fact that the NLSE is derived based on the assumptions of slow modulation and small DW amplitude. Still, in figure 2, the NLSE appears to be a reasonable approximation of the quasilinear mHME even for relatively large \( |\psi_0| \), in terms of capturing the linear MI. (This is not the case in the nonlinear stage of the MI, as we will discuss in section 5.) For comparison, figure 2 also shows the dispersion relations obtained from a wave-kinetic model based on the ray approximation (overviewed in appendix), which do not accurately approximate the exact ones. The reason for the discrepancy is that the wave-kinetic model misses essential full-wave (quantum-like) effects, particularly diffraction.

From figure 2, we can see that the unstable modulations have finite real frequencies due to the DW group velocity, and hence propagate while growing. This feature has largely been overlooked in the past, possibly because zonal structures are typically perceived as (quasi-) stationary. Nevertheless, as we will show in section 5, the propagation of these zonal modes can have consequences in the nonlinear stage of the MI, leading to the formation of solitary zonal structures.

### 5. Solitary zonal structures

#### 5.1. Drift-wave–zonal-flow soliton

Since equation (11) is of the same form as the well-known NLSE, it has the usual soliton solution

\[
\psi = 2\eta \sqrt{\frac{\partial \chi}{\beta_y}} \exp(i\chi \eta^2 t/2) \cosh[\eta(x - \psi t)]. \tag{19}
\]
Here, $\eta$ is a free parameter that can be regarded as the soliton inverse width. With $\tilde{\omega} = \text{Re}[\psi(t, x)e^{i(k_0 - \eta^2 t^2)/2}]$ and the equation of state (10), equation (19) translates to an approximate soliton solution to the quasilinear mHME (3):

$$\tilde{\omega} = 2\eta \sqrt{-\frac{\partial^2}{\partial k_y^2} \cos[k \cdot x - (\Omega - \chi \eta^2/2)t]} \cosh[\eta(x - \nu t)]$$

$$U = \frac{-\eta^2 \chi}{k_y \cosh^2[\eta(x - \nu t)]}$$

To our knowledge, this DW–ZF soliton was first explicitly discussed in [5] and then in [32], even though the NLSE (11) that governs the DW envelope dynamics had been derived earlier [16, 17]. It is fundamentally different from the vortex-pair solution called ‘modon’ [52, 53], which is a 2D structure that propagates poloidally, whereas the DW–ZF soliton is an essentially 1D structure that propagates radially.

In figure 3, we show snapshots of a DW–ZF soliton in both configuration-space (1-a) and phase-space (1-b). The Wigner function $W$, obtained using equation (14), is concentrated in both position and momentum, illustrating that the soliton is a quasi-monochromatic drifton condensate. It is also shown that the contours of $W$ do not coincide with those of the Hamiltonian in the moving frame $\mathcal{H}_m \cong \mathcal{H} - \nu \mathcal{P}_x$. This distinguishes the DW–ZF soliton from the BGK-type structures obtained from wave-kinetic models based on the ray approximation (appendix), where $W$ is a function of $\mathcal{H}_m$ only [54–57]. After all, the DW–ZF soliton is a result of the balance between self-focusing and diffraction, and the latter is a quantum-like effect missing in wave-kinetic models [22]. Therefore, wave-kinetic models cannot describe the DW–ZF soliton (20), even though the requirement that its envelope be slow may seem consistent with the ray approximation.

Also shown in figure 3 (row 2) are snapshots of two superposing solitons with wave-vectors $(k_x, k_y)$ and $(-k_x, k_y)$, respectively. Accordingly, these solitons have opposing group velocities and hence counter-propagate. Upon colliding, they tunnel through each other. The striations between the drifton condensates in (2-b) are signatures of the quantum superposition of macroscopically distinct states, i.e. ‘cat states’ [58, 59].

In principle, $\eta \ll |k_y|$ is required for the soliton solution (20) to stand in the quasilinear mHME, since the NLSE is derived under the assumptions of slow envelope modulation and small DW amplitude. In practice, however, we find that the solitary behavior of this solution is quite robust, and extends even to $\eta \sim |k_y|$. In figure 4 (row 1), we show the spatial-temporal evolution of the DW envelope from quasilinear mHME simulations initialized with equation (20) for various values of $\eta$. As $\eta$ is doubled from (1-a) to (1-b), the zonal structure keeps propagating much like a soliton, while some small-amplitude structures emerge and the speed decreases slightly. As $\eta$ is tripled and quadrupled in (1-c) and (1-d), respectively, the zonal structure gradually breaks down and eventually stops propagating.

In figure 4 (row 2), we also show corresponding results from nonlinear mHME simulations to verify the quasilinear simulations. The same qualitative features can be observed: the solitary behavior is robust at relatively small $\eta$, while the propagation eventually stops as $\eta$ keeps increasing. Admittedly, at larger $\eta$, the propagation stops more quickly in the nonlinear simulations. Nonetheless, for solitary structures with smaller $\eta$, which we focus on in this paper, the quasilinear approximation proves reasonable and acceptable.

Having confirmed that the mHME indeed allows for solitary zonal structures, next, we demonstrate how they can spontaneously form via the MI, from either coherent DWS (section 5.2) or incoherent DW spectra (section 5.3). These are the main results of this paper.

5.2. Zonal structures from coherent DWS

In figure 5, we present numerical simulations of the MI, through the nonlinear stage, using three different models: the NLSE, the quasilinear mHME, and the nonlinear mHME. The initial states are chosen to be primary waves with various amplitudes $|\psi_0|$ and random perturbations on top to seed the instability. The amplitudes of the perturbations are the same in all simulations.

In the NLSE simulations (row 1), for all values of $|\psi_0|$, the MI leads to the formation of zonal structures that behave approximately as solitons, notwithstanding the increasingly apparent nonlinear oscillations and interactions with increasing $|\psi_0|$. The wavenumbers of the structures are consistent with the fastest-growing wavenumbers in figure 2. The amplitudes of the structures increase with $|\psi_0|$ while the widths decrease, which is also consistent with the properties of the soliton solution (19). Note that in (1-c) and (1-d), the amplitudes of the structures are already comparable to those that stop propagating in figures 4(c) and (d), respectively. The implication is that in the corresponding mHME simulations, we may not be able to observe these solitary zonal structures.
Indeed, this can be seen in both the quasilinear and nonlinear simulations in figure 5. As $\psi_0$ increases (column (b)), the nonlinear interactions between the zonal structures begin to disrupt their propagation. At even larger $|\psi_0|$ (columns (c) and (d)) the zonal structures cease to propagate quickly after they develop, in contrast to the NLSE cases (1-c) and (1-d)).
qualitatively similar, demonstrating the sufficiency of the quasilinear approximation for our purposes.

In summary, in the nonlinear stage of the MI of coherent DWs, the NLSE does not properly approximate the (quasi-linear) mHME when the primary-wave amplitude is relatively large. Only when the primary-wave amplitude is relatively small can propagating zonal structures similar to the DW–ZF solitons form.

5.3. Zonal structures from incoherent DWs

As discussed in section 4.2, the MI of a monochromatic DW, which is studied in section 5.2, is equivalent to that of the delta-shaped DW spectrum (17) in the WM model. Now, let us consider a slightly more general case, where the DW spectrum has a finite width in $p_x$. Specifically, we adopt

$$\mathcal{W}(p) = \pi^2|\psi_0|^2 \sum_{\pm} \delta(p_x \pm k_x) f_s(p_x \pm k_x),$$

(21)

with $f_s(p) \equiv \exp[-p^2/(2\sigma^2)]/\sqrt{2\pi}\sigma$. Here, we keep the distribution in $p_x$ as delta functions for simplicity. The justification is that within the quasilinear approximation, $p_x$ is a constant of motion, and the coupling in $p_x$ is weak. For convenience, we refer to the spectrum (21) as ‘2-beam’ for it consists of a pair of quasi-monochromatic drifiton beams. (As mentioned in section 4.1, drifitons come in pairs.) Equation (17) is reproduced as the limit of equation (21) as the beam width $\sigma \to 0$.

In figure 6 (row 1), we present numerical simulations of the MI of the 2-beam spectrum (21) with random perturbations. These simulations employ the spectral representation of the WM equation (13) derived in the appendix of [19]. The simulation domain is periodic in $x$. In contrast to figure 5 (2-a), there is no instability at small $|\psi_0|$ (1-a), which demonstrates the stabilizing effect of the finite beam width $\sigma$. Still, as the effective amplitude $|\psi_0|$ increases, the system becomes modulationally unstable, and the corresponding features are qualitatively similar to the quasilinear mHME simulations in figure 5 (row 2). With moderate $|\psi_0|$ (1-b), solitary zonal structures emerge. When $|\psi_0|$ is increased further ((1-c) and (1-d)), the zonal structures stop propagating and eventually become stationary.

As the next generalization, we consider the MI of a DW spectrum that has two pairs of quasi-monochromatic drifiton beams

$$\mathcal{W}(p) = \pi^2|\psi_0|^2 \sum_{\pm} \delta(p_x \pm k_x) [f_s(p_x \pm k_x) + f_s(p_x \pm k_x)],$$

(22)

For convenience, we refer to the spectrum (22) as ‘4-beam’. In the limit as $\sigma \to 0$, this spectrum corresponds to the mixed state of two plane waves with amplitude $|\psi_0|$ and
wave-vectors \((k_x, k_y)\) and \((-k_x, k_y)\), respectively. In this limit, the dispersion relation of the MI reads [19]

\[
[\epsilon(k_x^2 \omega^2 - 4 \beta^2 q^2 k_y^2) (k_x \pm q/2)]^2
= |\psi_0|^2 q^2 k_y^2 (k_x^2 - q^2) (4k_x^2 - k_y^2 - q^2) x [4k_x^2 \omega^2 + \beta^2 q^2 k_y^2 (4k_x^2 - q^2)].
\]

In figure 6 (row 2), we show numerical simulations of the MI of the 4-beam spectrum (22) with random perturbations. On one hand, we observe many features common with the 2-beam case (row 1): the system is stable at small \(|\psi_0|\) (2-a); solitary zonal structures form as \(|\psi_0|\) increases (2-b); when \(|\psi_0|\) is even larger, the zonal structures become stationary (2-c) and (2-d). On the other hand, unlike in the 2-beam case, counter-propagating solitary zonal structures emerge. This is due to the fact that the group velocities of the two drifton beams have opposite signs. The implication of the 4-beam case is that solitary zonal structures can also form via the MI of multiple (pairs of) quasi-monochromatic drifton beams.

Furthermore, we model the 4-beam case with some external forcing \(F\) added to the rhs of equation (13a). To balance the energy input, we also add frictional damping \(-2\mu W\) and \(-\mu U\) to the rhs of equations (13a) and (13b), respectively [19]. In order to compare directly with the unforced 4-beam case above, we choose \(F = 2\nu\psi_0 V(p)\), with \(W\) given by the 4-beam spectrum (22), such that the corresponding homogeneous equilibrium is \(W = F/(2\mu) = \tilde{W}\). The simulations are initialized with such equilibria in place and random perturbations on top, and the results are shown in figure 6 (row 3). Many qualitative features of the unforced case (row 2) are reproduced here: stability at small \(|\psi_0|\) (3-a), solitary zonal structures at moderate \(|\psi_0|\) (3-b), and stationary zonal structures at large \(|\psi_0|\) (3-d). Hence, we conclude that the formation of solitary zonal structures is still possible even with forcing and dissipation, provided that the forcing spectrum consists of quasi-monochromatic peaks, producing quasi-monochromatic distributions of driftons. That being said, the forced case visibly differs from the unforced case in that the former has larger amplitudes of DWs between the zonal structures, which is maintained by the homogeneous production of driftons by the external forcing. Another distinction is that the solitary zonal structures in the forced case (3-b) are more coherent than those in the unforced case (2-b), which also owes to the fact that the external forcing tends to keep the DWs quasi-monochromatic.

5.4. Phase-space structures

It is instructive to examine these zonal structures in phase-space by studying snapshots of the Wigner function. Figure 3(1-b) illustrates that a DW–ZF soliton is a quasi-monochromatic drifton condensate that is concentrated in both space and momentum (at some non-zero \(p_x\)). For comparison, the phase-space snapshots in figure 7 (row 1) are taken from solitary zonal structures in various simulations. They all reveal the presence of trains of such condensates located at \(p_x \approx (\pm)k_x\). In the quasilinear mHME simulation (1-a), the striations between the drifton condensates are signatures of 'cat states', akin to those in figure 3(2-b). In the 4-beam cases ((1-c) and (1-d)), the phase-space snapshots show two trains of drifton condensates with opposing group velocities, also similar to figure 3(2-b), such that the zonal structures counter-propagate.
In contrast, the phase-space snapshots in figure 7 (row 2) are taken from stationary zonal structures in different simulations. Accordingly, in these cases, the DW quanta are mostly localized at \( p_x = 0 \), which is consistent with the stationarity of the zonal structures. Finally, we note that the phase-space structures are more coherent in the forced cases (column (d)) than in the unforced cases (column (c)), due to the homogeneous quasi-monochromatic external forcing applied.

6. Summary and discussion

In this paper, we first consider coherent DWs, by re-deriving the NLSE that approximately governs their radial envelope dynamics, and subsequently, its dispersion relation of the MI and soliton solution. Using mHME simulations, both quasi-linear and nonlinear, we validate the NLSE and the DW soliton solution. Then, we demonstrate that the NLSE can adequately describe the spontaneous generation of solitary zonal structures in the mHME, which takes place in the nonlinear stage of the MI, but only when the amplitude of the primary DW is relatively small. Otherwise, stationary zonal structures are formed instead.

Next, we consider the MI of incoherent DW spectra, using the recently developed WM model. We show that DW spectra that consist of quasi-monochromatic drift beams can also be modulationally unstable to the formation of solitary zonal structures, but only when the beam intensity is moderate. At higher intensity, the zonal structures become stationary, similar to the case with coherent DWs. Meanwhile, due to the stabilizing effect of the finite beam width, the system becomes stable to modulations at lower beam intensity.

In addition, using the WM formulation, we compare the solitary zonal structures formed via the MI with the DW–ZF solitons in phase-space. This approach enables extraction of information that can be obscure in configuration-space, especially when the DW spectrum has multiple quasi-monochromatic peaks. (As a data analysis tool alone, it can be straightforwardly applied to systems that are more complicated than the mHME.) The phase-space distributions of DW quanta show common features of quasi-monochromatic drift soliton condensates, which suggests that the MI-induced solitary zonal structures are essentially the DW–ZF solitons. These structures cannot be described by wave-kinetic models that are based on the ray approximation, which neglect critical quantum-like effects such as diffraction. In contrast, the WM model retains these effects, subsumes both the NLSE and the wave-kinetic models, and hence can support these solitary zonal structures.

It is worthwhile to comment on the relevance of our results to Rossby-wave turbulence in geophysics. [17], the envelope dynamics within the oHME is also discussed, and the equation of state is more complicated than our equation (10); namely, it is not a local but an integral equation. Hence, the applicability of our results to the Charney equation remains to be further investigated. Meanwhile, our results are readily applicable to the barotropic vorticity equation, and the only adaption needed is to replace \( \hat{\mathbf{p}} \) with \( \hat{\mathbf{p}}^2 \), and similarly for \( \hat{\mathbf{k}}^2 \). In the future, we plan to investigate the effect of background shear flows on the DW–ZF solitons, motivated by the radially propagating coherent structures recently identified in gyrokinetic simulations of subcritical plasmas [13–15]. In particular, the structures in [15] (figure 3(b) therein) seem close to quasi-monochromatic, much similar to our figure 3 (1-a). The (normalized) sizes and amplitudes of those structures are also comparable to those of the DW–ZF solitons discussed in this paper. However, to properly account for subcriticality would be a challenge, and it is possible that one needs to resort to more sophisticated models with primary instabilities, such as the (modified) Hasegawa–Wakatani equations [60, 61].

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Appendix. Limitations of wave-kinetic models

Wave-kinetic models of DW–ZF dynamics invoke the ray (geometrical-optics) approximation, i.e. assume that the DW wavelength is negligible compared with the ZF wavelength. In this case, the WM model (13) reduces to

\[
\partial_t W = \{H, W\} + 2\Gamma W, \quad (A1a)
\]

\[
\partial_t U = \partial_t \int \frac{d^2 p}{(2\pi)^2} \frac{p_x p_y}{\hat{\mathbf{p}}^2} W, \quad (A1b)
\]

with the Poisson bracket \( \{A, B\} = \langle \partial_x A \rangle \cdot \langle \partial_y B \rangle - \langle \partial_y A \rangle \cdot \langle \partial_x B \rangle \). The Hermitian and anti-Hermitian parts of the Hamiltonian are given by, respectively [19]

\[
\mathcal{H} = (\beta + U) p_x / \hat{\mathbf{p}}^2 + p_y U, \quad (A2a)
\]

\[
\Gamma = -U p_x p_y / p^4. \quad (A2b)
\]

This model was first derived as the geometrical-optics limit of the CE2 equations [48]. Following [36, 50], we refer to it as the improved wave-kinetic equation (iWKE). The dispersion relation of the MI in the iWKE reads [36, 48]

\[
1 = \int \frac{d^2 p}{(2\pi)^2} W(\mathbf{p}) - q_x^2 (p_x^2 - q_y^2)(p_y^2 - 4q_x^2) / (\omega p^4 + 2/\beta q_x p_y)^2. \quad (A3)
\]

By substituting the Wigner function (17) into equation (A3), we obtain the iWKE dispersion relation of the MI of a monochromatic DW

\[
(\omega - q_y^2) = |\psi_0|^2 k_y q^2 (1 - q_x^2 / k^2) / (4\beta). \quad (A4)
\]
The tWKE dispersion relation (A4) is plotted in figure 2 for comparison with the NLSE dispersion relation (12) and the WM dispersion relation (18). To better understand the discrepancy between the two, let us also consider the traditional wave-kinetic equation (tWKE) [20]. The tWKE differs from the iWKE by further neglecting the high-order derivatives of U in the Hamiltonian (A2), such that

$$\mathcal{H} = \beta p_x p_x + p_y U,$$

$$\Gamma = 0.$$  \hfill (A5a)

Accordingly, the tWKE dispersion relation for the M1 can be obtained by simply neglecting the factor $q^2/k^2$ in equation (A4):

$$\omega - \frac{q_y q_y}{k} = \frac{\beta 0^2 k q^2}{4 \beta}.$$ \hfill (A5b)

This dispersion relation diverges at large $q$, which is typical for the tWKE. Such divergence can give rise to unphysical grid-scale ZFs in numerical simulations, as shown in [19, 48], where more detailed discussions on the differences between the tWKE, the iWKE, and the WM (CE2) model can be found.

Despite this caveat, the tWKE dispersion relation (A6) warrants direct comparison with the NLSE dispersion relation (12), since both models neglect the same high-order derivatives of $U$. The difference is the $\chi^{-4}/4$ term. This term is due to diffraction, which is not included in the ray approximation that the tWKE (and iWKE) is based on, but retained in the quasi-optical approximation that the NLSE invokes. Likewise, it is the balance of diffraction and self-focusing that determines the size of the solitary zonal structures discussed in section 5. In summary, because of the absence of full-wave (quantum-like) effects such as diffraction, wave-kinetic models based on the ray approximation do not properly capture the MI or the solitary zonal structures discussed in the main text.

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