The enclosure method for inverse obstacle scattering over a finite time interval: V. Using time-reversal invariance

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Abstract

The wave equation is time-reversal invariant. The enclosure method using a Neumann data generated by this invariance is introduced. The method yields the minimum sphere that is centered at a given arbitrary point and encloses an unknown obstacle embedded in a known bounded domain from a single point on the graph of the so-called response operator on the boundary of the domain over a finite time interval. The occurrence of the lacuna in the solution of the free space wave equation is positively used.

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1 Introduction

The so-called inverse obstacle problem is a typical problem in inverse problems community and the solution has several possibilities of applications for non-destructive testing, sonar, radar, to name a few. See [18] for a survey about the uniqueness and stability issue.

This paper is concerned with the reconstruction or extraction issue, in particular, its methodology. Succeeding to the previous studies about the time domain enclosure method for inverse obstacle problems governed by the wave developed in [11, 12, 13], we further continue to pursue various possibilities of the method itself. In [14] the author has introduced a new version of the time domain enclosure method for inverse obstacle scattering problems using the wave governed by the wave equation in a bounded domain over a finite time interval. The method employs the Neumann data generated by taking the normal derivative of a solution of the wave equation in the whole space on the boundary of the domain and yields the maximal sphere that is centered at an arbitrary given point outside the domain and its exterior encloses an unknown obstacle embedded in the domain. The point is: it makes use of a single point on the graph of the response operator associated with the wave equation in the domain. The aim of this paper is to add one more point to

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this new version of the enclosure method. To clarify the essence of the idea we consider
the same problem as [14].

First let us recall the problem. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^3 \) with \( C^2 \)-boundary. Let \( D \) be a nonempty bounded open set of \( \mathbb{R}^3 \) with \( C^2 \)-boundary such that \( \overline{D} \subset \Omega \) and \( \Omega \setminus \overline{D} \) is connected.

Given an arbitrary positive number \( T \) and \( f = f(x,t), \ (x,t) \in \partial \Omega \times [0, T] \), let \( u = u_f(x,t), \ (x,t) \in (\Omega \setminus \overline{D}) \times [0, T] \) denote the solution of the following initial boundary value problem for the classical wave equation

\[
\begin{cases}
(\partial_t^2 - \Delta)u = 0 & \text{in } (\Omega \setminus \overline{D}) \times [0, T], \\
u(x,0) = 0 & \text{in } \Omega \setminus \overline{D}, \\
\partial_t u(x,0) = 0 & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D \times [0, T], \\
\frac{\partial u}{\partial \nu} = f(x,t) & \text{on } \partial \Omega \times [0, T].
\end{cases}
\]

The problem considered in [14] is

**Problem.** Fix \( T \) (to be determined later). Assume that \( D \) is unknown. Find a suitable Neumann data \( f \) in such a way that the wave \( u_f \) on \( \partial \Omega \) over the time interval \([0, T]\) yields information about the geometry of \( D \).

What we found therein is: if \( f \) is given by the normal derivative of a special solution of the Cauchy problem for the classical wave equation in \( \mathbb{R}^3 \times [0, T] \) with special initial data supported on an arbitrary fixed ball outside \( \Omega \), then one can extract the distance of the ball to \( D \) provided, roughly speaking, \( T \) is large enough.

In this paper, we give another choice of the Neumann data that yields another information about the geometry of \( D \).

Let \( B \) be an open ball centered at \( p \in \mathbb{R}^3 \) with radius \( \eta \) and denote \( \chi_B \) its characteristic function. Define

\[
\Psi_B(x) = (\eta - |x - p|)\chi_B(x), \ x \in \mathbb{R}^3
\]

This function belongs to \( H^1(\mathbb{R}^3) \) and \( \text{supp } \Psi_B = \overline{B} \). Unlike [14], in this paper we do not make a restriction of the position \( B \) relative to \( \Omega \).

Let \( v = v(x,t) \) be the solution of the following Cauchy problem for the classical wave equation:

\[
\begin{cases}
(\partial_t^2 - \Delta)v = 0, & x \in \mathbb{R}^3, 0 < t < T \\
v(x,0) = 0, & x \in \mathbb{R}^3, \\
\partial_t v(x,0) = \Psi_B(x), & x \in \mathbb{R}^3.
\end{cases}
\]

It is well known that the solution \( v \) takes the form

\[
v(x,t) = \frac{1}{4\pi t} \int_{\partial B_t(x)} \Psi_B(y) dS_y,
\]

(1.3)
where
\[ B_t(x) = \{ y \in \mathbb{R}^3 | |y - x| < t \} . \]

From the form of (1.3) we see that
\[ \text{supp } v(\cdot, T) \cup \text{supp } \partial_t v(\cdot, T) \subset B_{T+\eta}(p) \] (1.4)
and
\[ \text{supp } v(\cdot, T) \cup \text{supp } \partial_t v(\cdot, T) \subset \mathbb{R}^3 \setminus B_{T-\eta}(p) , \] (1.5)
where \( B_{T\pm\eta}(p) = \{ x \in \mathbb{R}^3 | |x - p| < T \pm \eta \} \). In [14] we made use of (1.4) only, however, in this paper we make use of also the property (1.5) which is a quantitative expression of occurrence of lacuna (cf. [7]). It is a character of the wave equation in odd dimensions.

In this paper, we always choose \( T \) in such a way that \( \Omega \subset B_{T-\eta}(p) \), that is
\[ T - \eta \geq R_\Omega(p) , \] (1.6)
where \( B_{T-\eta}(p) = \{ x \in \mathbb{R}^3 | |x - p| < T - \eta \} \) and
\[ R_\Omega(p) = \sup_{x \in \Omega} |x - p| . \]

Define
\[ f_{B,T}(x,t) = \frac{\partial}{\partial \nu} v(x,T-t), \quad x \in \partial \Omega, \quad 0 \leq t \leq T . \] (1.7)
This is the special \( f \) mentioned above. Note that the property (1.5) and the time-reversal invariance of the wave equation yield the function \( v^*(x,t) = v(x,T-t) \) for \( x \in \Omega \) and \( 0 < t < T \) satisfies (1.1) with \( D = \emptyset \) and \( f = f_{B,T} \). Then, a combination of a standard lifting argument and the theory of \( C_0 \)-semigroups [21] enables us to solve (1.1) with \( f = f_{B,T} \) uniquely in the class
\[ C^2([0, T], L^2(\Omega \setminus \bar{D})) \cap C^1([0, T], H^1(\Omega \setminus \bar{D})) \cap C([0, T], H^2(\Omega \setminus \bar{D})) . \]

See [14] for this argument and [9] for the solvability of the reduced problems which are initial boundary value problems for hyperbolic equations with homogeneous boundary conditions.

Having the solution \( u = u_f \) of (1.1) with \( f = f_{B,T} \) given by (1.7) set
\[ w_{B,T}(x) = w_{B,T}(x, \tau) = \int_0^\tau e^{-\tau t} u_f(x,t)dt, \quad x \in \Omega \setminus \bar{D}, \quad \tau > 0 , \] (1.8)
and
\[ w^*_{B,T}(x) = w^*_{B,T}(x, \tau) = \int_0^\tau e^{-\tau t} v(x,T-t)dt, \quad x \in \mathbb{R}^3, \quad \tau > 0 . \] (1.9)
Define the indicator function
\[ I_{\partial \Omega}(\tau; B,T) = \int_{\partial \Omega} (w_{B,T} - w_{B,T}^*) \frac{\partial w_{B,T}^*}{\partial \nu} dS, \quad \tau > 0 . \] (1.10)
Define
\[ R_D(p) = \sup_{x \in D} |x - p|. \]

Note that again, in this paper we always assume that \( T \) satisfies (1.6).

**Theorem 1.1.** (i) Let \( \eta \) satisfy
\[ \eta + 2R_D(p) > R_\Omega(p). \tag{1.11} \]
Then, there exists a positive number \( \tau_0 \) such that \( I_{\partial \Omega}(\tau; B, T) > 0 \) for all \( \tau \geq \tau_0 \) and we have
\[ \lim_{\tau \to \infty} \frac{1}{\tau} \log I_{\partial \Omega}(\tau; B, T) = -2 \{ (T - \eta) - R_D(p) \}. \tag{1.12} \]

(ii) If \( T > 2 \{ (T - \eta) - R_D(p) \}, \) then
\[ \lim_{\tau \to \infty} e^{\tau T} I_{\partial \Omega}(\tau; B, T) = \infty. \]

(iii) Assume instead of (1.6) the stronger condition
\[ T - \eta > R_\Omega(p). \tag{1.13} \]
If \( T < 2 \{ (T - \eta) - R_D(p) \}, \) then
\[ \lim_{\tau \to \infty} e^{\tau T} I_{\partial \Omega}(\tau; B, T) = 0. \]

Note that the indicator function \( I_{\partial \Omega}(\tau; B, T) \) can be computed from the wave field \( u_f \) on \( \partial \Omega \times ]0, T[ \) generated by the single Neumann data \( f = f_{B,T}. \) Thus formula (1.12) enables us to know the quantity \( R_D(p) \) which is the radius of the minimum sphere centered at \( p \) and enclosing \( D. \) The point \( p \) can be an arbitrary point in \( \mathbb{R}^3. \) We do not mind whether \( p \in D, p \in \Omega \setminus D \) or \( p \in \mathbb{R}^3 \setminus \Omega. \)

The condition (1.11) is equivalent to the condition
\[ T > \{ (T - \eta) - R_D(p) \} + (R_\Omega(p) - R_D(p)). \tag{1.14} \]

Under the assumption (1.6) we have
\[ 2 \{ (T - \eta) - R_D(p) \} = \{ (T - \eta) - R_D(p) \} + \{ (T - \eta) - R_D(p) \} \]
\[ \geq \{ (T - \eta) - R_D(p) \} + (R_\Omega(p) - R_D(p)). \]
Therefore, if \( T \) satisfies (1.6) and \( T > 2 \{ (T - \eta) - R_D(p) \}, \) then \( \eta \) satisfies (1.14) and hence (1.11). Thus, the assertion (ii) is a direct consequence of (i).

Summing up, we have obtained:
\[ \lim_{\tau \to \infty} e^{\tau T} I_{\partial \Omega}(\tau; B, T) = \begin{cases} 
\infty & \text{if } \eta + R_\Omega(p) \leq T < 2(\eta + R_D(p)), \\
0 & \text{if } T > 2(\eta + R_D(p))
\end{cases} \]
provided \( \eta \) satisfies (1.11). This criterion gives an alternative and qualitative characterization of \( R_D(p) \) instead of (1.12).
Note that, for all $T$ satisfying (1.6) we have
\[
\{(T - \eta) - R_D(p)\} + (R_\Omega(p) - R_D(p)) \\
\geq \inf \{|P - Q| + |Q - R| \mid P \in \partial B_{T - \eta}(p), Q \in \partial D, R \in \partial \Omega\}.
\]

(1.15)

This is proved as follows. First choose $Q \in \partial D$ such that $R_D(p) = |Q - p|$. Second choose $P \in \partial B_{T - \eta}(p)$ such that $Q$ is on the segment $[p, P]$. Thus we have $|P - Q| = (T - \eta) - R_D(p)$. Third choose $R' \in \partial B_{R_\Omega(p)}(p)$ such that $Q$ is on the segment $[p, R']$. We have $|Q - R'| = R_\Omega(p) - R_D(p)$. Then, one can find a point $R \in \partial \Omega$ on the segment $[Q, R']$. Then, we have $|Q - R'| \geq |Q - R|$ and thus
\[
\{(T - \eta) - R_D(p)\} + (R_\Omega(p) - R_D(p)) = |P - Q| + |Q - R'| \geq |P - Q| + |Q - R|.
\]

This yields the desired conclusion.

Note that the right-hand side on (1.15) gives the minimum length of the broken paths that strat at $P \in \partial B_{T - \eta}(p)$, reflect at $y \in \partial D$ and return to $R \in \partial \Omega$. Therefore condition (1.14) is quite natural, and so is (1.11).

If $D$ is large in the sense that $2R_D(p) \geq R_\Omega(p)$, then $\eta$ satisfying (1.11) can be arbitrary small. However, if $2R_D(p) < R_\Omega(p)$, then one has to choose a large $\eta$. The choice depends on a lower estimate of $R_D(p)$. This means that we need a-priori information about the size of $R_D(p)$ from below.

The main difference from [14] is the choice of the Neumann data $f$ in (1.1). Therin we restrict the location of $B$ to the outside of $\Omega$. Then the Neumann data in [14] is given by
\[
f_B(x, t) = \frac{\partial}{\partial \nu} v(x, t), \quad x \in \partial \Omega, \quad 0 \leq t \leq T,
\]
where $v$ is the solution of (1.2) with this restricted $B$. So in this case we have
\[
f_{B, T}(x, t) = f_B(x, T - t).
\]

That is, the Neumann data (1.7) plays the role of the time-reversal mirror [8] equipped on the boundary of $\partial \Omega$ for the wave generated by $f_B$ over the time interval $[0, T]$ in the case when $D = \emptyset$. We can generate a natural free wave in $\Omega$ which is emitted on $\partial \Omega$ possibly with some delay and goes to $B$.

The procedure for extracting $R_D(p)$ is explicit, direct and has the feature: in the processing of the signal we do not make use of the knowledge of the boundary condition. Note that in contrast to this, the so-called continuation procedure of the solutions of the governing equation close to obstacle makes use of the boundary condition of the obstacle in the procedure, such as that of [19] and also [5] which is a combination of a continuation method in the frequency domain and the Fourier transform.

A numerical method in [4] for a penetrable obstacle (embedded in the whole plane $\mathbb{R}^2$) is a combination of a time-reversed scattered wave field continuation method and an optimization method for unknown wave speed in the obstacle. To continue the scattered wave field from the obstacle, they choose a disc that encloses the obstacle and solve numerically a time-reversed initial boundary value problem for the original governing equation in an annulus like domain whose inner boundary is the boundary of the disc with a time reversed absorbing boundary condition. On the outer boundary of the domain
where the observed data are collected, the time-reversed scattered field is prescribed as another boundary condition. Using the computed scattered field in the annulus domain, they introduce an optimization problem with respect to the unknown wave speed in the obstacle. It seems that it is not clear whether their method can cover the case when the wave only propagate in a bounded domain not the whole space like our situation since in that case one has to consider the scattered wave not only from the obstacle but also from the outer boundary.

We mention an analytical approach due to Oksanen [20] which is based on the boundary control method [1]. Therein a similar inverse obstacle problem problem for the wave governed by the wave equation in a bounded domain or compact manifold with a boundary is considered. The approach therein enables us to compute the volume of a set as called the *domain of influence* which is closely related to an unknown obstacle embedded in the domain. Intuitively, in our Euclidean setting, it is the set of all points \( x \in \Omega \setminus \overline{D} \) such that the wave governed by the wave equation in \( \Omega \setminus \overline{D} \) generated at some point \( y_0 \) on \( \partial \Omega \) at \( t = 0 \) reaches at \( x \) within the time \( T(y_0) \), where \( T(y) \), \( y \in \partial \Omega \) is an arbitrary given continuous function with the values in \([0, T/2]\) and \( T(y) = 0 \) for \( y \in \partial \Omega \setminus \Gamma \); \( \Gamma \) is an arbitrary prescribed non empty open subset of \( \partial \Omega \). The computed volume yields some information about the location of the obstacle. The point is to construct a one parameter family of the Neumann data \( f \) in such a way that \( u_f(x, T/2) \) approximates the characteristic function of the domain of influence. The construction is reduced to solving an equation with a parameter written by the Tikhonov regularization of a linear operator on the boundary of the domain. The operator is written by using the local hyperbolic Neumann-to-Dirichlet operator and *time-reversal operation* on the boundary. It appears in Blagovestchenskii’s identity and is the base of the boundary control method (cf. [2]). The idea of the construction is closely related to the focusing wave approach developed for the wave speed determination problem, see [6, 3] and references therein. However, his result does not tell us what information about the unknown obstacle can be extracted from a single set of the Dirichlet and Neumann data. Note that in the crucial step of the proof for the justification of his method the unique continuation property of the governing equation is essential even in our simple situation. Our method together with the proof is free from the property, simple and rather elementary.

Finally, we point out that, in [17] an extraction formula of \( R_D(p) \) is given when \( D \) is an inclusion embedded in a homogeneous isotropic conductive medium and the governing equation of the signal propagating inside the medium is given by a heat equation. It is easy to see that the result therein also covers the cavity case treated in [16]. The data used therein is the Neumann-to-Dirichlet map in the time domain acting on the special Neumann data having the separation of variables form

\[
\varphi(t) \frac{\partial v_r}{\partial \nu}(x; p),
\]

where \( p \) is an arbitrary point in \( \mathbb{R}^3 \), say \( \varphi(t) \sim t^m \) as \( t \downarrow 0 \) with an integer \( m \) and

\[
v_r(x; p) = \begin{cases} \sinh \sqrt{\tau} |x - p|, & x \in \mathbb{R}^3 \setminus \{p\}, \\ \frac{1}{|x - p|}, & x \in \mathbb{R}^3 \setminus \{p\}, \\ \sqrt{\tau}, & x = p. \end{cases}
\]

(1.16)
Since $v_\tau$ depends on $\tau$, in this sense, the data to determine $R_D(p)$ for a fixed $p$ is infinitely many. In this sense the result shares the same spirit as a typical result in the classical enclosure method [10] which employs infinitely many observation data. However, note that the normal derivative of $v_\tau$ blows up as $\tau \to \infty$. It should be emphasized that the Neumann data $f_{B,T}$ given by (1.7) is independent of such a parameter which causes the blowing up. At the present time the author does not know whether there exists a suitable Neumann data depending only $p$ or a ball centered at $p$ with a small radius that yields $R_D(p)$ for inverse obstacle problems governed by the heat equation. The main obstruction is the lack of time-reversal invariance and that of the occurrence of lacuna for the fundamental solution.

A brief outline of this paper is as follows. Theorem 1.1 is proved in Section 2. The proof starts with describing the decomposition formula of the indication function. Using the formula together with a lemma concerning with an upper bound for the second term in the formula, we reduce the problem to deriving estimates of the energy integral for $w_{B,T}$ as $\tau \to \infty$ from above and below. For the purpose, using the time domain expression (1.3) of $v$, we explicitly write the leading profile of $w_{B,T}$ in $B_{T-\eta}(p)$ as $\tau \to \infty$ down as stated in Lemma 2.2. This is the key point of the proof of Theorem 1.1. The proof of Lemma 2.2 is given in Section 3. Since the proof requires explicit forms of some volume integrals, we give their derivation in Appendix.

2 Proof of Theorem 1.1.

In this section, for simplicity of description, we always write

$$w = w_{B,T}, \quad w^* = w_{B,T}^*, \quad R = w - w^*.$$  

It is a due course to have the following decomposition formula (see Proposition 2.1 in [14]).

**Proposition 2.1.** We have

$$I_{\partial \Omega}(\tau; B, T) = J_*(\tau) + E(\tau) + R(\tau),$$  

(2.1)

where

$$J_*(\tau) = \int_D (|\nabla w^*|^2 + \tau^2 |w^*|^2) \, dx,$$  

(2.2)

$$E(\tau) = \int_{\Omega \setminus D} (|\nabla R|^2 + \tau^2 |R|^2) \, dx,$$  

(2.3)

$$R(\tau) = e^{-\tau T} \left\{ \int_D F_0 w^* \, dx + \int_{\Omega \setminus D} FR \, dx + \int_{\Omega \setminus D} (F_0 - F) w^* \, dx \right\},$$  

(2.4)

$$F = F(x, \tau) = \partial_\tau u_f(x, T) + \tau u_f(x, T), \quad x \in \Omega \setminus \overline{D}$$  

(2.5)

and

$$F_0(x) = -\Psi_B(x), \quad x \in \mathbb{R}^3.$$  

(2.6)

Note that the proof of (2.1) is based on the two facts.
First it follows from (1.1) and (1.8) that \( w \) satisfies
\[
\begin{cases}
(\Delta - \tau^2)w = e^{-\tau T}F & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial w}{\partial \nu} = \frac{\partial w^*}{\partial \nu} & \text{on } \partial \Omega, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial D.
\end{cases}
\]  
(2.7)

This is the same as before.

Second from (1.2) and (1.9) we have:
\[
(\Delta - \tau^2)w^* - e^{-\tau T}F_0 = \partial_t v(x, T) - \tau v(x, T), \quad x \in \mathbb{R}^3.
\]  
(2.8)

Then, from (1.5) and assumption (1.6) we see that \( w^* \) satisfies
\[
(\Delta - \tau^2)w^* = e^{-\tau T}F_0, \quad x \in \Omega.
\]  
(2.9)

Using (2.7) and (2.9) together with integration by parts we obtain (2.1).

Similarly to Lemma 2.2 in [14], we have

**Lemma 2.1 (Dominance estimate).** We have
\[
E(\tau) = O(\tau^2 J_*(\tau) + \tau^2 e^{-2\tau T})
\]  
(2.10)
as \( \tau \to \infty \).

The point is: \( f = f_{B,T} \) is independent of \( \tau \) and thus (2.5) gives \( \|F\|_{L^2(\Omega, \overline{D})} = O(\tau) \). This together with (2.6) gives \( \|F - F_0\|_{L^2(\Omega, \overline{D})} = O(\tau) \) same as in the proof of Lemma 2.2 in [14].

The next task is to give an upper bound on \( R(\tau) \) and the upper and lower estimates on \( J_*(\tau) \). For the purpose we study the local behaviour of \( w^* \).

Changing the role of \( F_0 \) and \( \partial_t v(x, T) - \tau v(x, T) \) on (2.8), we have
\[
(\Delta - \tau^2)w^* + (\tau v(x, T) - \partial_t v(x, T)) = e^{-\tau T}F_0, \quad x \in \mathbb{R}^3.
\]  
This is the different point from [14]. As can be seen below the term \( \tau v(x, T) - \partial_t v(x, T) \) plays a role of the main source.

The \( w^* \) takes the form
\[
w^* = w_{1}^* + e^{-\tau T}w_{R}^*,
\]  
(2.11)
where
\[
w_{1}^*(x, \tau) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} (\tau v(y, T) - \partial_t v(y, T))dy, \quad x \in \mathbb{R}^3
\]  
(2.12)
and \( w_{R}^* \) satisfies
\[
(\Delta - \tau^2)w_{R}^* + \Psi_B = 0, \quad x \in \mathbb{R}^3.
\]  
By integration by parts we have immediately, as \( \tau \to \infty \),
\[
\tau \|w_{R}^*\|_{L^2(\mathbb{R}^3)} + \|\nabla w_{R}^*\|_{L^2(\mathbb{R}^3)} = O(1).
\]  
(2.13)
Thus, to clarify the behaviour of $w^*$ in $B_{T-\eta}(p)$ it suffices to study that of $w^*_1$. Noting (1.4) and (1.5), we prepare two lemmas in which the first one yields an explicit form of $w^*_1(x, \tau)$ for $x \in B_{T-\eta}(p)$ and the second its upper and lower estimates.

**Lemma 2.2.** Let $T > \eta$. We have

$$
\frac{\tau^2}{4\pi} \int_{B_{T+\eta}(p) \setminus B_{T-\eta}(p)} \frac{e^{-\tau|x-y|}}{|x-y|} (\tau v(y, T) - \partial_t v(y, T)) \, dy
$$

$$
eq e^{-\tau(T-\eta)} \mathcal{H}(\tau; T, \eta) \frac{\sinh \tau|x-p|}{|x-p|}
$$

for all $x \in B_{T-\eta}(p) \setminus \{p\}$, where

$$\mathcal{H}(\tau; T, \eta) = \tau^{-1} \left( \eta + O(\tau^{-1}) \right).$$

For the proof of Lemma 2.2 see Section 3. It is a chain of a careful explicit computation by using the speciality of the form.

Note that the function

$$\frac{\sinh \tau|x-p|}{|x-p|}, \quad x \in \mathbb{R}^3 \setminus \{p\}$$

has a unique extension to the whole space as a smooth function and satisfies the modified Helmholtz equation $$(\Delta - \tau^2) v = 0$$ in the whole space. More precisely the function coincides with $v_{\tau^2}(x; p)$ which is given by (1.16) with $\tau$ replaced with $\tau^2$. In the following lemma we continue to use this notation to denote its extension.

**Lemma 2.3.** Let $U$ be an arbitrary bounded open subset of $\mathbb{R}^3$ and $p$ an arbitrary point in $\mathbb{R}^3$. Set $R_U(p) = \sup_{x \in U} |x-p|$. Then

(i) There exists a real number $\mu_1$ such that, as $\tau \to \infty$

$$
\int_U v_{\tau^2}(x; p)^2 \, dx + \int_U \|
abla v_{\tau^2}(x; p)\|^2 \, dx = O(\tau^{2\mu_1} e^{2\tau R_U(p)}).
$$

(ii) Assume that $\partial U$ is Lipschitz. There exist positive numbers $C$ and $\tau_0$ and a real number $\mu_2$ such that

$$
\tau^{2\mu_2} e^{-2\tau R_U(p)} \int_U v_{\tau^2}(x; p)^2 \, dx \geq C
$$

for all $\tau \geq \tau_0$.

Since $U \subset B_{R_U(p)}(p)$, the proof of Lemma 2.3 (i) can be done by replacing $U$ with the ball $B_{R_U(p)}(p)$ and using the polar coordinates around $p$. The proof of Lemma 2.3 (ii) can be done by using the same argument for the proof of Lemma 6 in [15]. The point of the argument is to find a subdomain $\bar{U}$ of $U$ such that $R_{\bar{U}}(p) = R_U(p)$ and $|x-p| \geq R_U(p)/2$ for all $x \in \bar{U}$. For the purpose the Lipschitz regularity of $\partial U$ is enough. By these reasons we omit to describe the proof of Lemma 2.3. Note that the concrete values of $\mu_1$ and $\mu_2$ are not essential in this paper as same as [15], and other papers for the time domain enclosure methods.

From (1.4), (1.5), the expression (2.12) and Lemma 2.2 one gets an explicit asymptotic form of $w^*_1$ in $B_{T-\eta}(p)$. Then, from Lemma 2.3 together with (2.11) and (2.13) we immediately obtain
Lemma 2.4 (Propagation estimate). Let $U$ be an arbitrary bounded open subset of $\mathbb{R}^3$ such that $U \subset B_{T-\eta}(p)$, that is
\[ T - \eta \geq R_U(p). \tag{2.14} \]

(i) There exist a real number $\mu_3$ such that, as $\tau \to \infty$
\[ \tau \|
abla w^*\|_{L^2(U)} + \|w^*\|_{L^2(U)} = O(\tau^{\mu_3} e^{-\tau(T-\eta)} e^{\tau R_U(p)} + \tau^2 e^{-\tau T}). \]

(ii) If $\partial U$ be Lipschitz, then there exist positive numbers $\tau_0$ and $C$ such that
\[ \tau^{\mu_2+3} e^{-\tau R_U(p)} \|w^*\|_{L^2(U)} \geq C \tag{2.15} \]
for all $\tau \geq \tau_0$, where $\mu_2$ is the same as that of Lemma 2.3 (ii).

Using the facts $\|F\|_{L^2(\Omega, D)} = O(\tau)$, $\|F\|_{L^2(D)} = O(1)$ and $\|F - F_0\|_{L^2(\Omega, D)} = O(\tau)$ together with (2.2), (2.3) and (2.10) we have, as $\tau \to \infty$
\[
\begin{cases}
\int_{\Omega \setminus D} FRd\tau = O(\tau \cdot \tau^{-1} E(\tau)^{1/2}) = O(E(\tau)^{1/2}) = O(\tau J_*(\tau)^{1/2} + \tau e^{-\tau T}), \\
\int_D F_0 w^* d\tau = O(\tau^{-1} J_*(\tau)^{1/2}).
\end{cases}
\]

Applying Lemma 2.4 (i) to the case $U = \Omega$, we obtain
\[ \|w^*\|_{L^2(\Omega, D)} = O(\tau^{\mu_3-1} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)} + \tau e^{-\tau T}) \]
and this thus yields
\[ \int_{\Omega \setminus D} (F_0 - F) w^* d\tau = O(\tau^{\mu_3} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)} + \tau^2 e^{-\tau T}). \]

Moreover, from Lemma 2.4 (i) in the case $U = D$, we obtain
\[ J_*(\tau) = O(\tau^{2\mu_3-2\tau(T-\eta)} e^{2\tau R_D(p)} + \tau^4 e^{-2\tau T}). \tag{2.16} \]

From these, we obtain
\[ \mathcal{R}(\tau) \]
\[ = O(e^{-\tau T} (\tau J_*(\tau)^{1/2} + \tau e^{-\tau T})) + O(\tau^{\mu_3} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)} + \tau^2 e^{-2\tau T}) \]
\[ = O(\tau^{\mu_3} \{ \tau (\mu_3 e^{-\tau(T-\eta)} e^{\tau R_D(p)} + \tau^2 e^{-\tau T}) + \tau e^{-\tau T} \}) + O(\tau^{\mu_3} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)} + \tau^2 e^{-2\tau T}) \]
\[ = O(\tau^{\mu_3+1} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_D(p)} + \tau^2 e^{-2\tau T} + \tau^{\mu_3} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)}). \tag{2.17} \]

Thus
\[ e^{2\tau(T-\eta)} e^{-2\tau R_D(p)} \mathcal{R}(\tau) \]
\[ = e^{2\tau(T-\eta)} e^{-2\tau R_D(p)} O(\tau^{\mu_3+1} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_D(p)} + \tau^2 e^{-2\tau T} + \tau^{\mu_3} e^{-\tau T} e^{-\tau(T-\eta)} e^{\tau R_\Omega(p)}) \]
\[ = O(\tau^{\mu_3+1} e^{-\tau T} e^{-\tau R_D(p)} + \tau^2 e^{-2\tau T} e^{-2\tau R_D(p)} + \tau^{\mu_3} e^{-\tau T} e^{-(\eta+2\tau R_D(p)-R_\Omega(p)))}. \tag{2.18} \]
Now we are ready to describe the proof of Theorem 1.1 (i). Let \( \eta \) satisfies the condition (1.11). Then (2.18) yields
\[
e^{2\tau(T-\eta)}e^{-2\tau R_D(p)}R(\tau) = O(\tau^{-\infty}).
\]  
(2.19)
Thus from this, (2.1), (2.10) and (2.16) we obtain
\[
I_{\partial \Omega}(\tau; B, T) = O(\tau^{2+2\mu_3}e^{-2\tau(T-\eta)}e^{2\tau R_D(p)} + \tau^6e^{-2\tau T})
\]
(2.20)
Moreover, from (2.1), (2.2) and (2.19) we have
\[
I_{\partial \Omega}(\tau; B, T) = O(\tau^{2+2\mu_3}e^{-2\tau(T-\eta)}e^{2\tau R_D(p)(1 + \tau^{6-2\mu_3}e^{-\tau(\eta + R_D(p))})})
\]
\[
= O(\tau^{2+2\mu_3}e^{-2\tau(T-\eta)}e^{2\tau R_D(p)}).
\]
Moreover, from (2.1), (2.2) and (2.19) we have
\[
e^{2\tau(T-\eta)}e^{-2\tau R_D(p)}I_{\partial \Omega}(\tau; B, T) \geq e^{2\tau(T-\eta)}e^{-2\tau R_D(p)}J_\omega(\tau) + O(\tau^{-\infty})
\]
\[
\geq \tau^2 e^{2\tau(T-\eta)}e^{-2\tau R_D(p)}\|w\|_{L^2(D)}^2 + O(\tau^{-\infty}).
\]
Since (1.6) implies (2.14) with \( U = D \), from Lemma 2.4 (ii) in the case when \( U = D \) and writing
\[
\tau^2 = \tau^{-2(\mu_2+2)}\tau^{2(\mu_2+3)},
\]
one can conclude that: there exist positive numbers \( C \) and \( \tau_0 \) such that
\[
\tau^{2(\mu_2+2)}e^{2\tau(T-\eta)}e^{-2\tau R_D(p)}I_{\partial \Omega}(\tau; B, T) \geq C
\]  
(2.21)
for all \( \tau \geq \tau_0 \). A combination of (2.20) and (2.21) ensures that the assertion (i) is valid.

As pointed out in Remark 1.2 the assertion (ii) is a direct consequence of (i). Thus it suffices to prove (iii). Instead of (2.19) which is a consequence of the assumption (1.11), we go back to (2.17). Then, we have
\[
e^{\tau T}R(\tau) = O(\tau^{\mu_3+1}e^{-\tau(T-\eta)}e^{\tau R_D(p)} + \tau^2e^{-\tau T} + \tau^{\mu_3}e^{-\tau(T-\eta)}e^{\tau R_D(p)}).
\]
Note that \( T < 2\{(T - \eta) - R_D(p)\} \) implies that \( T - \eta > R_D(p) \). Thus, under the assumption (1.13) which is stronger than (1.6), we conclude
\[
e^{\tau T}R(\tau) = O(\tau^{-\infty}).
\]
Now from this, (2.1), (2.10) and (2.16) we obtain
\[
e^{\tau T}I_{\partial \Omega}(\tau; B, T) = O(\tau^{2\mu_3+2}e^{\tau T}e^{-2\tau(T-\eta)}e^{2\tau R_D(p)}) + O(\tau^{-\infty}).
\]
Since \( T < 2\{(T - \eta) - R_D(p)\} \), we conclude
\[
e^{\tau T}I_{\partial \Omega}(\tau; B, T) = O(\tau^{-\infty}).
\]
This completes the proof of Theorem 1.1.
3 Proof of Lemma 2.2

First we compute the value of \( v(x, T) \) together with \( \partial_t v(x, T) \) at \( x \in B_{T+\eta}(p) \setminus B_{T-\eta}(p) \).

**Proposition 3.1.**

(i) If \(|x - p| - t| < \eta \) and \( \eta < |x - p| + t \), then we have

\[
\begin{align*}
\left\{ \begin{array}{l}
v(x, t) = \frac{1}{2} \left\{ \frac{\eta^3}{6|x - p|} - \frac{\eta(|x - p| - t)^2}{2|x - p|} + \frac{|x - p| - t|^3}{3|x - p|} \right\}, \\
\partial_t v(x, t) = \frac{|x - p| - t}{2|x - p|} (\eta - |x - p| - t) \cdot \end{array} \right. \\
\end{align*}
\]

(ii) If \(|x - p| + t < \eta\), then we have

\[
\begin{align*}
\left\{ \begin{array}{l}
v(x, t) = \eta t - \frac{1}{6|x - p|} \left\{ (|x - p| + t)^3 - |x - p| - t|^3 \right\}, \\
\partial_t v(x, t) = \eta - \frac{1}{2|x - p|} \left\{ (|x - p| + t)^2 + (|x - p| - t)|x - p| - t \right\}. \\
\end{array} \right. \\
\end{align*}
\]

**Proof.** Write (1.3) as

\[
v(x, t) = \frac{t}{4\pi} \int_{S(x; B)} (\eta - |(x + t\omega) - p|) d\omega, \tag{3.1}
\]

where

\[ S(x; B) = \{ \omega \in S^2 \mid |(x + t\omega) - p| < \eta \}. \]

The inequality \(|(x + t\omega) - p| < \eta\) for \( \omega \in S^2 \) is equivalent to

\[
\omega \cdot \frac{p - x}{|p - x|} > \frac{|p - x|^2 + t^2 - \eta^2}{2t|p - x|}. \\
\]

First consider the case when \(||x - p| - t| < \eta\) and \(|x - p| + t > \eta\). In this case we have

\[-1 < \frac{|p - x|^2 + t^2 - \eta^2}{2t|p - x|} < 1. \]

Define

\[ \phi_0 = \arccos \frac{|p - x|^2 + t^2 - \eta^2}{2t|p - x|}. \]

Then, one can write all the points \( \omega \in S(x; B) \) in terms of the polar coordinates:

\[ \omega = \sin \phi \left( \cos \theta \mathbf{b} + \sin \theta \mathbf{c} \right) + \cos \phi \frac{p - x}{|p - x|}, \]

where \( 0 \leq \theta \leq 2\pi, 0 \leq \phi < \phi_0 \); unit vectors \( \mathbf{b} \) and \( \mathbf{c} \) are parallel and satisfy

\[ \mathbf{b} \times \mathbf{c} = \frac{p - x}{|p - x|}. \]
Thus, (3.1) becomes
\[ v(x, t) = \frac{t}{2} \int_0^{\phi_0} \sin \phi \left( \eta - \sqrt{|x - p|^2 + t^2 - 2t|x - p| \cos \phi} \right) d\phi. \]  
(3.2)

Here we have
\[ \int_0^{\phi_0} \sin \phi d\phi = 1 - \cos \phi_0 \]
\[ = 1 - \frac{|x - p|^2 + t^2 - \eta^2}{2t|x - p|} \]
\[ = \frac{\eta^2 - (|x - p| - t)^2}{2t|x - p|} \]

and
\[ \int_0^{\phi_0} \sqrt{|x - p|^2 + t^2 - 2t|x - p| \cos \phi} \sin \phi d\phi \]
\[ = \frac{1}{3t|x - p|} (\sqrt{|x - p|^2 + t^2 - 2t|x - p| \cos \phi_0})^3|_{\phi_0=0} \]
\[ = \frac{1}{3t|x - p|} \left\{ (\sqrt{|x - p|^2 + t^2 - 2t|x - p| \cos \phi_0})^3 - ||x - p| - t|^3 \right\} \]
\[ = \frac{1}{3t|x - p|} (\eta^3 - ||x - p| - t|^3). \]

Thus from (3.2) we have
\[ v(x, t) = \frac{t}{2} \cdot \eta \cdot \frac{\eta^2 - (|x - p| - t)^2}{2t|x - p|} - \frac{t}{2} \cdot \frac{1}{3t|x - p|} (\eta^3 - ||x - p| - t|^3). \]

We have
\[ \partial_t(||x - p| - t|^3) = -3(||x - p| - t)||x - p| - t|. \]  
(3.3)

Thus, one gets
\[ \partial_t v(x, t) = \frac{\eta(|x - p| - t)}{2|x - p|} - \frac{(|x - p| - t)||x - p| - t|}{2|x - p|}. \]

This yields the desired conclusion (i).

Next consider the case when $|x - p| + t < \eta$. We see that
\[ \frac{|p - x|^2 + t^2 - \eta^2}{2t|p - x|} < -1. \]

Thus, $S(x; B) = S^2$ and using the same polar coordinates as above with $\phi_0 = \pi$ we have
\[ v(x, t) = \frac{t}{2} \cdot 2\eta - \frac{t}{2} \cdot \frac{1}{3t|x - p|} \left\{ (|x - p| + t)^3 - ||x - p| - t|^3 \right\}. \]
Using (3.3), we have
\[ \partial_t v(x, t) = \eta - \frac{1}{6|x-p|} \left\{ 3(|x-p| + t)^2 + 3(|x-p| - t)||x-p|-t| \right\}. \]

This yields the desired formula (ii).

\[ \square \]

**Remark 3.1.** (a) Let \(||x-p|-t|<\eta\) and \(\eta<|x-p|+t\). Then, we have
\[ |(|x-p|-t) - \eta| < 2t. \]

Thus, from Proposition 3.1 (i) we have \(\lim_{t \to 0} \partial_t v(x, t) = 0\).

(b) Let \(|x-p|<\eta\). Then, for all \(t > 0\) with \(t < \eta - |x-p|\) we have \(|x-p| + t < \eta\).

Then form Proposition 3.1 (ii) we obtain \(\lim_{t \to 0} \partial_t v(x, t) = \eta - |x-p|\).

Thus, it suffices to compute the integrals
\[ I_j(x; R_1, R_2) = \frac{1}{4\pi} \int_{B_{R_2}(p) \setminus B_{R_1}(p)} \frac{e^{-|x-y|}}{|x-y|} |y - p|^j dy, \quad x \in B_{R_1}(p) \]
for \(j = -1, 0, 1, 2\) and \(R_2 > R_1\).

The results are listed below which are the direct consequence of Proposition A in Appendix:
\[ I_j(x; R_1, R_2) = \frac{1}{r^2} H_j(\tau; R_1, R_2) \frac{\sinh \tau|x-p|}{|x-p|}, \quad x \in B_{R_1}(p) \setminus \{p\}, \quad (3.4) \]
where
\[
\begin{align*}
H_{-1}(\tau; R_1, R_2) &= e^{-\tau R_1} - e^{-\tau R_2}, \\
H_0(\tau; R_1, R_2) &= \left( R_1 + \frac{1}{\tau} \right) e^{-\tau R_1} - \left( R_2 + \frac{1}{\tau} \right) e^{-\tau R_2}, \\
H_1(\tau; R_1, R_2) &= \left( R_1^2 + \frac{2}{\tau} R_1 + \frac{2}{\tau^2} \right) e^{-\tau R_1} - \left( R_2^2 + \frac{2}{\tau} R_2 + \frac{2}{\tau^2} \right) e^{-\tau R_2}, \\
H_2(\tau; R_1, R_2) &= \left( R_1^3 + \frac{3}{\tau} R_1^2 + \frac{6}{\tau^2} R_1 + \frac{6}{\tau^3} \right) e^{-\tau R_1} - \left( R_2^3 + \frac{3}{\tau} R_2^2 + \frac{6}{\tau^2} R_2 + \frac{6}{\tau^3} \right) e^{-\tau R_2}.
\end{align*}
\]

From Proposition 3.1 and (3.4) we obtain

**Proposition 3.2.** Let \(T > \eta\). We have the expression
\[
\frac{1}{4\pi} \int_{B_{T+\eta}(p) \setminus B_{T-\eta}(p)} \frac{e^{-|x-y|}}{|x-y|} (\tau v(y, T) - \partial_t v(y, T)) dy
= \frac{1}{r^2} (\mathcal{H}_+(\tau; T, \eta) + \mathcal{H}_-(\tau; T, \eta)) \frac{\sinh \tau|x-p|}{|x-p|}.
\]
for all \( x \in B_{T-\eta}(p) \setminus \{ p \} \), where

\[
\mathcal{H}_+(\tau; T, \eta) \nonumber
\]

\[
= \left\{ \frac{1}{12} \tau(\eta - 2T)(\eta + T)^2 + \frac{1}{2} T(\eta + T) \right\} H_{-1}(\tau; T, T + \eta) \\
+ \frac{1}{2} \tau T(\eta + T) - \frac{1}{2} (\eta + 2T) \right\} H_0(\tau; T, T + \eta) \\
+ \left\{ \frac{1}{4} \tau (\eta + 2T) + \frac{1}{2} \right\} H_1(\tau; T, T + \eta) + \frac{1}{6} \tau H_2(\tau; T, T + \eta)
\]

and

\[
\mathcal{H}_-(\tau; T, \eta) \nonumber
\]

\[
= \left\{ \frac{1}{12} \tau(\eta + 2T)(\eta - T)^2 + \frac{1}{2} T(\eta - T) \right\} H_{-1}(\tau; T - \eta, T) \\
+ \frac{1}{2} \tau T(\eta - T) - \frac{1}{2} (\eta - 2T) \right\} H_0(\tau; T - \eta, T) \\
+ \left\{ \frac{1}{4} \tau (\eta - 2T) - \frac{1}{2} \right\} H_1(\tau; T - \eta, T) - \frac{1}{6} \tau H_2(\tau; T - \eta, T).
\]

**Proof.** Divide the integrand \( B_{T+\eta}(p) \setminus B_{T-\eta}(p) \) as

\[
B_{T+\eta}(p) \setminus B_{T-\eta}(p) = B_1 \cup B_2,
\]

where

\[
B_1 = \{ y \in \mathbb{R}^3 \mid T \leq |y - p| < T + \eta \} = B_{T+\eta}(p) \setminus B_T(p)
\]

and

\[
B_2 = \{ y \in \mathbb{R}^3 \mid T - \eta < |y - p| \leq T \} = B_T(p) \setminus B_{T-\eta}(p).
\]

Since \( T > \eta \), Proposition 3.1 yields

\[
v(y, T) = \left\{ \begin{array}{ll}
\frac{1}{2} \left\{ \frac{\eta^3}{6|y - p|} - \frac{\eta(|y - p| - T)^2}{2|y - p|} + \frac{(|y - p| - T)^3}{3|y - p|} \right\}, & y \in B_1 \\
\frac{1}{2} \left\{ \frac{\eta^3}{6|y - p|} - \frac{\eta(|y - p| - T)^2}{2|y - p|} - \frac{(|y - p| - T)^3}{3|y - p|} \right\}, & y \in B_2
\end{array} \right.
\]

and

\[
\partial_t v(y, T) = \left\{ \begin{array}{ll}
\frac{|y - p| - T}{2|y - p|} \{ \eta - (|y - p| - T) \}, & y \in B_1, \\
\frac{|y - p| - T}{2|y - p|} \{ \eta + (|y - p| - T) \}, & y \in B_2.
\end{array} \right.
\]

Thus one gets:
\[(a)\text{ for } y \in B_1\]
\[v(y, T) = \frac{1}{12}(\eta - 2T)(\eta + T)^2 \cdot \frac{1}{|y - p|} + \frac{1}{2}T(\eta + T) - \frac{1}{4}(\eta + 2T)|y - p| + \frac{1}{6}|y - p|^2\]
and
\[
\partial_t v(y, T) = -\frac{1}{2}T(\eta + T) \cdot \frac{1}{|y - p|} + \frac{1}{2}(\eta + 2T) - \frac{1}{2}|y - p|;\]

\[(b)\text{ for } y \in B_2\]
\[v(y, T) = \frac{1}{12}(\eta + 2T)(\eta - T)^2 \cdot \frac{1}{|y - p|} + \frac{1}{2}T(\eta - T) - \frac{1}{4}(\eta - 2T)|y - p| - \frac{1}{6}|y - p|^2\]
and
\[
\partial_t v(y, T) = -\frac{1}{2}T(\eta - T) \cdot \frac{1}{|y - p|} + \frac{1}{2}(\eta - 2T) + \frac{1}{2}|y - p|.
\]

Therefore we have, for \(y \in B_1\)
\[
\tau v(y, T) - \partial_t v(y, T)
\]
\[
= \left\{ \frac{1}{12}\tau(\eta - 2T)(\eta + T)^2 + \frac{1}{2}T(\eta + T) \right\} \cdot \frac{1}{|y - p|} + \left\{ \frac{1}{2}\tau T(\eta + T) - \frac{1}{2}(\eta + 2T) \right\}
\]
\[
+ \left\{ -\frac{1}{4}\tau(\eta + 2T) + \frac{1}{2} \right\} |y - p| + \frac{1}{6}\tau|y - p|^2
\]
and for \(y \in B_2\)
\[
\tau v(y, T) - \partial_t v(y, T)
\]
\[
= \left\{ \frac{1}{12}\tau(\eta + 2T)(\eta - T)^2 + \frac{1}{2}T(\eta - T) \right\} \cdot \frac{1}{|y - p|} + \left\{ \frac{1}{2}\tau T(\eta - T) - \frac{1}{2}(\eta - 2T) \right\}
\]
\[
+ \left\{ -\frac{1}{4}\tau(\eta - 2T) - \frac{1}{2} \right\} |y - p| - \frac{1}{6}\tau|y - p|^2.
\]

Let \(x \in B_{T-\eta}(p) \setminus \{p\}\). Using (3.4) in the case when \(R_1 = T, R_2 = T + \eta\), we have
\[
\frac{1}{4\pi} \int_{B_1} \frac{e^{-\tau|x-y|}}{|x - y|} (\tau v(y, T) - \partial_t v(y, T))dy = \frac{1}{\tau^2} \mathcal{H}_+(\tau; T, \eta) \frac{\sinh \tau|x - p|}{|x - p|}.
\]

Using (3.4) in the case when \(R_1 = T - \eta, R_2 = T\), we have
\[
\frac{1}{4\pi} \int_{B_2} \frac{e^{-\tau|x-y|}}{|x - y|} (\tau v(y, T) - \partial_t v(y, T))dy = \frac{1}{\tau^2} \mathcal{H}_-(\tau; T, \eta) \frac{\sinh \tau|x - p|}{|x - p|}.
\]

From these we obtain the desired formula.

\[\blacksquare\]

**Proposition 3.3.** We have
\[
\begin{align*}
\mathcal{H}_+(\tau; T, \eta) &= f_\tau(T)e^{-\tau T} - f_\tau(T + \eta)e^{-(T + \eta)}, \\
\mathcal{H}_-(\tau; T, \eta) &= g_\tau(T - \eta)e^{-\tau(T - \eta)} - g_\tau(T)e^{-\tau T},
\end{align*}
\]
where
\[
\begin{align*}
 f_\tau(\xi) &= \frac{\tau}{6} \xi^3 + \left\{ 1 - \frac{\tau}{4}(\eta + 2T) \right\} \xi^2 + \left\{ \frac{1}{2}\tau T(\eta + T) - (\eta + 2T) + \frac{2}{\tau} \right\} \xi \\
&+ \left\{ \frac{1}{12}\tau(\eta - 2T)(\eta + T)^2 + T(\eta + T) - \frac{\eta + 2T}{\tau} + \frac{2}{\tau^2} \right\} \\

g_\tau(\xi) &= -\frac{\tau}{6} \xi^3 - \left\{ 1 + \frac{\tau}{4}(\eta - 2T) \right\} \xi^2 + \left\{ \frac{1}{2}\tau T(\eta - T) - (\eta - 2T) - \frac{2}{\tau} \right\} \xi \\
&+ \left\{ \frac{1}{12}\tau(\eta + 2T)(\eta - T)^2 + T(\eta - T) - \frac{\eta - 2T}{\tau} - \frac{2}{\tau^2} \right\}.
\end{align*}
\]

Proof. First note that we have the relationship:
\[
\begin{align*}
 H_0(\tau; R_1, R_2) &= R_1 e^{-\tau R_1} - R_2 e^{-\tau R_2} + \frac{1}{\tau} H_{-1}(\tau; R_1, R_2), \\
 H_1(\tau; R_1, R_2) &= R_1^2 e^{-\tau R_1} - R_2^2 e^{-\tau R_2} + \frac{2}{\tau} H_0(\tau; R_1, R_2), \\
 H_2(\tau; R_1, R_2) &= R_1^3 e^{-\tau R_1} - R_2^3 e^{-\tau R_2} + \frac{3}{\tau} H_1(\tau; R_1, R_2).
\end{align*}
\]
(3.7)

Let \( R_1 = T \) and \( R_2 = T + \eta \). Substituting the expression of \( H_2(\tau; R_1, R_2) \) in terms of \( H_1(\tau; R_1, R_2) \) in (3.7) into (3.5), we have
\[
\mathcal{H}_+(\tau; T, \eta)
\]
\[
= \left\{ \frac{1}{12}\tau(\eta - 2T)(\eta + T)^2 + \frac{1}{2}T(\eta + T) \right\} H_{-1}(\tau; T, T + \eta) \\
+ \left\{ \frac{1}{2}\tau T(\eta + T) - \frac{1}{2}(\eta + 2T) \right\} H_0(\tau; T, T + \eta) \\
+ \left\{ -\tau \frac{1}{4}(\eta + 2T) + \frac{1}{2} \right\} H_1(\tau; T, T + \eta) \\
+ \frac{1}{6} \tau \left\{ R_1^3 e^{-\tau R_1} - R_2^3 e^{-\tau R_2} + \frac{3}{\tau} H_1(\tau; T, T + \eta) \right\} \\
= \left\{ \frac{1}{12}\tau(\eta - 2T)(\eta + T)^2 + \frac{1}{2}T(\eta + T) \right\} H_{-1}(\tau; T, T + \eta) \\
+ \left\{ \frac{1}{2}\tau T(\eta + T) - \frac{1}{2}(\eta + 2T) \right\} H_0(\tau; T, T + \eta) \\
+ \left\{ 1 - \tau \frac{1}{4}(\eta + 2T) \right\} H_1(\tau; T, T + \eta) \\
+ \frac{1}{6} \tau (R_1^3 e^{-\tau R_1} - R_2^3 e^{-\tau R_2}).
\]
Continuing this procedure step by step by using the relationship (3.7) until eliminating all the terms $H_j(\tau; R_1, R_2)$, $j = 1, 0$ and finally substituting the explicit form of $H_{-1}(\tau; R_1, R_2)$ into the resulted form, we obtain

$$
\mathcal{H}_+(\tau; T, \eta)
= \left\{ \frac{1}{12} \tau (\eta - 2T)(\eta + T)^2 + T(\eta + T) - \frac{\eta + 2T}{\tau} + \frac{2}{\tau^2} \right\} (e^{-\tau R_1} - e^{-\tau R_2})
+ \left\{ \frac{1}{2} \tau T(\eta + T) - (\eta + 2T) + \frac{2}{\tau} \right\} \left\{ R_1 e^{-\tau R_1} - R_2 e^{-\tau R_2} \right\}
+ \left\{ 1 - \tau \frac{1}{4}(\eta + 2T) \right\} \left( R_1^2 e^{-\tau R_1} - R_2^2 e^{-\tau R_2} \right) + \frac{1}{6} \tau (R_1^3 e^{-\tau R_1} - R_2^3 e^{-\tau R_2}).
$$

Making order of this right-hand side, we obtain the desired expression for $\mathcal{H}_+(\tau; T, \eta)$.

Next Let $R_1 = T - \eta$ and $R_2 = T$. Applying the same procedure based on the relationship (3.7) to the right-hand side on (3.6), we obtain

$$
\mathcal{H}_-(\tau; T, \eta)
= \left\{ \frac{1}{12} \tau (\eta + 2T)(\eta - T)^2 + T(\eta - T) - \frac{\eta - 2T}{\tau} - \frac{2}{\tau^2} \right\} (e^{-\tau R_1} - e^{-\tau R_2})
+ \left\{ \frac{1}{2} \tau T(\eta - T) - (\eta - 2T) - \frac{2}{\tau} \right\} \left\{ R_1 e^{-\tau R_1} - R_2 e^{-\tau R_2} \right\}
+ \left\{ -1 - \tau \frac{1}{4}(\eta - 2T) \right\} \left( R_1^2 e^{-\tau R_1} - R_2^2 e^{-\tau R_2} \right) - \frac{1}{6} \tau (R_1^3 e^{-\tau R_1} - R_2^3 e^{-\tau R_2}).
$$

This yields the desired expression for $\mathcal{H}_-(\tau; T, \eta)$.

\[ \square \]

From Proposition 3.3 we have

$$
\mathcal{H}_+(\tau; T, \eta) + \mathcal{H}_-(\tau; T, \eta)
= g_\tau(T - \eta)e^{-\tau(T - \eta)} + (f_\tau(T) - g_\tau(T))e^{-\tau T} - f_\tau(T + \eta)e^{-\tau(T + \eta)}.
$$
Moreover, set $\xi = T - \eta$. We have

$$g_\tau(T - \eta)$$

$$= -\frac{\tau}{6} \xi^3 - \left\{ 1 + \frac{\tau}{4}(-\xi - T) \right\} \xi^2 + \left\{ -\frac{1}{2} \tau T \xi - (-\xi - T) - \frac{2}{\tau} \right\} \xi$$

$$+ \left\{ \frac{1}{12} \tau (-\xi + 3T) \xi^2 - T \xi - \frac{-\xi - T}{\tau} - \frac{2}{\tau^2} \right\}$$

$$= -\frac{\tau}{6} \xi^3 - \xi^2 + \frac{\tau}{4} \xi^3 + \frac{\tau T}{4} \xi^2 - \frac{\tau T}{2} \xi^2 + T \xi - \frac{2 \xi}{\tau}$$

$$- \frac{\tau}{12} \xi^3 + \frac{\tau T}{4} \xi^2 - T \xi + \frac{\xi}{\tau} + \frac{T}{\tau} - \frac{2}{\tau^2}$$

$$= \frac{T}{\tau} - \frac{\xi}{\tau} - \frac{2}{\tau^2}$$

$$= \frac{\eta}{\tau} - \frac{2}{\tau^2}$$

$$= \frac{1}{\tau} \left( \eta - \frac{2}{\tau} \right).$$

This yields

$$e^{\tau(T - \eta)} (\mathcal{H}_+(\tau; T, \eta) + \mathcal{H}_-(\tau; T, \eta))$$

$$= g_\tau(T - \eta) + O(\tau e^{-\tau \eta})$$

$$= \frac{1}{\tau} (\eta + O(\tau^{-1})).$$

This completes the proof of Lemma 2.2.

**Remark 3.2.** Similarly to the derivation of (3.8), one gets

$$\begin{cases} 
  f_\tau(T) = \frac{\tau}{12} \eta^3 - \frac{\eta}{\tau} + \frac{2}{\tau^2}, \\
  f_\tau(T + \eta) = \frac{1}{\tau} \left( \eta + \frac{2}{\tau} \right), \\
  g_\tau(T) = \frac{\tau}{12} \xi^3 - \frac{\eta}{\tau} - \frac{2}{\tau^2}
\end{cases}$$

and thus

$$\mathcal{H}_+(\tau, T, \eta) + \mathcal{H}_-(\tau; T, \eta)$$

$$= \frac{4}{\tau^2} e^{-\tau T} - \frac{1}{\tau} \left( \eta + \frac{2}{\tau} \right) e^{-\tau(T + \eta)} + \frac{1}{\tau} \left( \eta - \frac{2}{\tau} \right) e^{-\tau(T - \eta)}.$$

However, we do not need this explicit formula for the present purpose.
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4 Appendix

In this appendix we give an explicit computation result for the potential

\[ v_j(x) = \int_B \frac{e^{-\tau|x-y|}}{|x-y|} |y|^j dy, \quad x \in B, \]

where \( B = \{y \in \mathbb{R}^3 | |y| < \eta\} \) with \( \eta > 0 \) and \( j = -1, 0, 1, 2 \).

**Proposition A.** For all \( x \in B \setminus \{0\} \) we have

\[
\begin{align*}
 v_{-1}(x) &= \frac{4\pi}{\tau^2} \left( \frac{1 - e^{-\tau|x|}}{|x|} - e^{-\tau\eta} \frac{\sinh \tau|x|}{|x|} \right), \\
v_0(x) &= \frac{4\pi}{\tau^2} \left\{ 1 - \left( \eta + \frac{1}{\tau} \right) e^{-\tau\eta} \frac{\sinh \tau|x|}{|x|} \right\}, \\
v_1(x) &= \frac{4\pi}{\tau^2} \left\{ |x| + \frac{2}{\tau^2} \frac{1 - e^{-\tau|x|}}{|x|} - e^{-\tau\eta} \left( \eta^2 + \frac{2}{\tau} \eta + \frac{2}{\tau^2} \right) \frac{\sinh \tau|x|}{|x|} \right\}, \\
v_2(x) &= \frac{4\pi}{\tau^2} \left\{ |x|^2 + \frac{6}{\tau^2} - e^{-\tau\eta} \left( \eta^2 + \frac{3\eta^2}{\tau} + \frac{6\eta}{\tau^2} + \frac{6}{\tau^3} \right) \frac{\sinh \tau|x|}{|x|} \right\}.
\end{align*}
\]

**Proof.** The change of variables \( y = r\omega (0 < r < \eta, \omega \in S^2) \) and a rotation give us

\[
v_j(x) = \int_0^\eta r^{2+j} dr \int_{S^2} \frac{e^{-\tau|x-r\omega|}}{|x-r\omega|} d\omega = \int_0^\eta r^{2+j} dr \int_{S^2} \frac{e^{-\tau||x||e_3-r\omega|}}{|x|e_3-r\omega|} d\omega = \int_0^\eta r^{2+j} dr \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \frac{e^{-\tau\sqrt{|x|^2-2r|x|\cos \varphi + r^2}}}{\sqrt{|x|^2-2r|x|\cos \varphi + r^2}} = 2\pi \int_0^\eta Q(|x|, r)r^{2+j} dr,
\]

where

\[
Q(\xi, r) = \int_0^\pi \frac{e^{-\tau\sqrt{\xi^2-2r\xi \cos \varphi + r^2}}}{\sqrt{\xi^2-2r\xi \cos \varphi + r^2}} \sin \varphi d\varphi, \quad 0 \leq \xi < \eta, \quad 0 < r < \eta.
\]
Fix $\xi \in [0, \eta]$ and $r \in [0, \eta]$. The change of variable
\begin{equation}
    s = \sqrt{\xi^2 - 2r \xi \cos \varphi + r^2}, \quad \varphi \in [0, \pi]
\end{equation}
gives
\begin{equation}
    s^2 = \xi^2 - 2r \xi \cos \varphi + r^2
\end{equation}
and
\begin{equation}
    sds = r \xi \sin \varphi d\varphi.
\end{equation}

Hence we have
\begin{equation}
    Q(\xi, r) = \frac{1}{r \xi} \int_{|\xi-r|}^{\xi+r} e^{-\tau s} ds
\end{equation}
\begin{equation}
    = -\frac{1}{r \xi \tau} \left( e^{-\tau(\xi+r)} - e^{-\tau|\xi-r|} \right).
\end{equation}

Therefore we obtain
\begin{equation}
    v_j(x) = 2\pi \int_0^\eta Q(|x|, r) r^{2+j} dr
\end{equation}
\begin{equation}
    = \frac{2\pi}{\xi \tau} \int_0^\eta \left( e^{-\tau|\xi-r|} - e^{-\tau(\xi+r)} \right) r^{1+j} dr |_{\xi=|x|}.
\end{equation}

Thus everything is reduced to computing the integral
\begin{equation}
    K_j = \int_0^\eta \left( e^{-\tau|\xi-r|} - e^{-\tau(\xi+r)} \right) r^{1+j} dr, \quad j = -1, 0, 1, 2.
\end{equation}

A direct computation yields
\begin{equation}
\begin{aligned}
    \int_0^\eta e^{-\tau|\xi-r|} dr &= \frac{2}{\tau} - \frac{e^{-\tau \xi}}{\tau} - \frac{e^{-\tau(\eta-\xi)}}{\tau}, \\
    \int_0^\eta e^{-\tau|\xi-r|} r dr &= \frac{2\xi}{\tau} - \frac{e^{-\tau \xi}}{\tau^2} - \frac{e^{\tau(\xi-\eta)}}{\tau} \left( \eta + \frac{1}{\tau} \right), \\
    \int_0^\eta e^{-\tau|\xi-r|} r^2 dr &= \frac{1}{\tau^3} \{ (2\tau^2 \xi^2 + 4) - 2e^{-\tau \xi} = \tau^2 \eta^2 + 2\tau \eta + 2 \} e^{-\tau(\eta-\xi)} , \\
    \int_0^\eta e^{-\tau|\xi-r|} r^3 dr &= \frac{2\xi^3}{\tau} - \frac{1}{\tau} e^{-\tau(\eta-\xi)} \eta^3 + \frac{6}{\tau^4} e^{-\tau \xi} - \frac{3}{\tau^4} \{ (\tau^2 \eta^2 + 2\tau \eta + 2) e^{-\tau(\eta-\xi)} - 4\tau \xi \}.
\end{aligned}
\end{equation}

And also we have
\begin{equation}
\begin{aligned}
    \int_0^\eta e^{-\tau(\xi+r)} dr &= \frac{e^{-\tau \xi}}{\tau} - \frac{e^{-\tau(\xi+\eta)}}{\tau}, \\
    \int_0^\eta e^{-\tau(\xi+r)} r dr &= \frac{e^{-\tau \xi}}{\tau^2} - \frac{e^{-\tau(\xi+\eta)}}{\tau} \left( \eta + \frac{1}{\tau} \right), \\
    \int_0^\eta e^{-\tau(\xi+r)} r^2 dr &= \frac{1}{\tau^3} e^{-\tau \xi} \left\{ -e^{-\tau \eta} (\tau^2 \eta^2 + 2\tau \eta + 2) + 2 \right\}, \\
    \int_0^\eta e^{-\tau(\xi+r)} r^3 dr &= -\frac{1}{\tau} \eta^3 e^{-\tau(\xi+\eta)} + \frac{3}{\tau^4} e^{-\tau \xi} \left\{ -e^{-\tau \eta} (\tau^2 \eta^2 + 2\tau \eta + 2) + 2 \right\}.
\end{aligned}
\end{equation}
From these we obtain

\[
\begin{align*}
K_{-1} &= \frac{2}{\tau} (1 - e^{-\tau \xi} - e^{-\tau \eta} \sinh \tau \xi), \\
K_0 &= \frac{2}{\tau} \left\{ \xi - \left( \eta + \frac{1}{\tau} \right) e^{-\tau \eta} \sinh \tau \xi \right\}, \\
K_1 &= \frac{2}{\tau^3} \left\{ (\tau^2 \xi^2 + 2) - 2e^{-\tau \xi} - (\tau^2 \eta^2 + 2 \tau \eta + 2)e^{-\tau \eta} \sinh \tau \xi \right\}, \\
K_2 &= \frac{2\xi^3}{\tau} + \frac{12}{\tau^3} \xi - \frac{2}{\tau} \left( \eta^3 + \frac{3\eta^2}{\tau} + \frac{6\eta}{\tau^2} + \frac{6}{\tau^3} \right) e^{-\tau \eta} \sinh \tau \xi.
\end{align*}
\]

Substituting these into (A.1), we obtain the desired formulae.

\[
\square
\]

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