Global and exponential attractors for the Penrose-Fife system

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Abstract

The Penrose-Fife system for phase transitions is addressed. Dirichlet boundary conditions for the temperature are assumed. Existence of global and exponential attractors is proved. Differently from preceding contributions, here the energy balance equation is both singular at 0 and degenerate at \( \infty \). For this reason, the dissipativity of the associated dynamical process is not trivial and has to be proved rather carefully.

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1 Introduction

We consider here the thermodynamically consistent model for phase transitions proposed by Penrose and Fife in \([17, 18]\) and represented by the equations

\[
\begin{align*}
\vartheta_t + \lambda(\chi)_t + \text{div} \left( m(\vartheta) \nabla \frac{1}{\vartheta} \right) &= g, \quad (1.1) \\
\chi_t - \Delta \chi + W'(\chi) &= \lambda'(\chi) \left( -\frac{1}{\vartheta} + \frac{1}{\vartheta_c} \right). \quad (1.2)
\end{align*}
\]

The system above is settled in a smooth, bounded domain \( \Omega \subset \mathbb{R}^3 \), with boundary \( \Gamma \). The unknowns are the absolute temperature \( \vartheta > 0 \) and the order parameter \( \chi \). The smooth functions \( \lambda', m \) and \( W \) represent the latent heat, the thermal conductivity, and the potential associated to the local phase configuration, respectively, and \( \vartheta_c > 0 \) is a critical temperature. Finally, \( g \) is a volumic heat source. On the basis of physical considerations, the kinetic equation (1.2) is complemented, as usual, with no-flux (i.e., homogeneous Neumann) boundary conditions; instead, various types of meaningful boundary conditions can be associated with the energy balance equation (1.1). We shall consider here the Dirichlet boundary conditions.

As far as well-posedness is concerned, system (1.1)–(1.2) has been studied in a number of recent works, among which we quote \([4, 5, 7, 11, 14, 15, 29]\), under various assumptions on the data. The papers listed above also contain a much more comprehensive bibliography. Just a rapid survey of the literature suggests that, indeed, the choice of the boundary conditions for \( \vartheta \) can give rise to several different mathematical situations. In particular, the Dirichlet and Robin conditions seem easier to treat than the Neumann ones (cf., e.g., \([4, 11]\) for further comments), due to correspondingly higher coercivity. Another important factor is the expression of the thermal conductivity \( m \). Meaningful choices are given by (cf. \([5]\) for further comments)

\[
m(r) \sim m_0 r + m_\infty r^2, \quad m_0, m_\infty \in [0, \infty). \quad (1.3)
\]

In particular, \( m_0 = 0, m_\infty > 0 \) represents the Fourier heat conduction law, which appears to be the most difficult situation \([15]\) since equation (1.1), which is now linear in \( \vartheta \), is coupled with the singular relation (1.2). Instead, in the case \( m_0 > 0, m_\infty = 0 \), the well-posedness issue is simpler (cf. \([14, 29]\)); however, there is a lack of coercivity for large \( \vartheta \), which creates difficulties in the long-time analysis.
Finally, the probably simplest situation is that proposed in [5] (see also [6]), i.e., \( m_0, m_\infty > 0 \), since [14] maintains both the singular character at 0 and the coercivity at \( \infty \).

In view of these considerations, it is not surprising that the long time behavior of (1.1)–(1.2) is better understood when \( m_0, m_\infty > 0 \), and in this case the existence of the global attractor has been shown in [22] [23]. Indeed, testing (1.1) by \( \vartheta \) one readily gets a dissipative estimate for the temperature, which permits to construct a uniformly absorbing set and, consequently, the global attractor. Similar results are also obtained in [12] [13], where it is actually taken \( m_\infty = 0 \), but a term \( \mu_\infty \vartheta \), with \( \mu_\infty > 0 \), is added on the left hand side of (1.1), so that the system is still coercive in \( \vartheta \).

Speaking of the non-coercive case \( m_\infty = 0 \), up to our knowledge the only papers devoted to the large-times analysis of it are [27] (see also [28] for the conserved case) and [10]. In [27], the case of homogeneous Neumann conditions for both unknowns is addressed in one space dimension, and existence of a global attractor is shown in a proper phase space which takes into account the conservation (or dissipation) properties coming from the no-flux conditions. In [10], the (non-homogeneous) Dirichlet case is considered in three space dimensions and \( \omega \)-limits of single trajectories are studied. It is worth remarking that in both papers the external source \( g \) is taken equal to 0.

In the present work, we provide a further contribution to the analysis of the noncoercive case. Precisely, we assume \( m_0 > 0, m_\infty = 0 \), and take Dirichlet boundary conditions for \( \vartheta \) exactly as in [10]. For the resulting problem, we show existence of both global and exponential attractors. Comparing with [10], where the behavior of a single trajectory is investigated, here the proofs are very different and in several points more difficult. Indeed, determining attractors means to understand the behavior of bundles of trajectories, so that we need to find estimates which are uniform not only in time, but also with respect to initial data varying in a bounded set. We then try to minimize technicalities by making some restrictions on data. Namely, we take a constant latent heat (i.e., set \( \chi(\lambda) = \chi \)), set \( m(r) = 1 \) (i.e., \( m_0 = 1, m_\infty = 0 \)), let the critical temperature \( \vartheta_c \) be equal to 1, and correspondingly assume the Dirichlet condition \( \vartheta = \vartheta_c = 1 \) on the boundary. Actually, all these assumptions could by avoided by paying the price of some additional computations in the proofs. More restrictive is, instead, the assumption \( g = 0 \), which we take exactly as it was done in [10] [27]. We then end up with the system

\[
\vartheta_t + \chi_t - \Delta \left( -\frac{1}{\vartheta} \right) = 0, \quad \chi_t - \Delta \chi + W'(\chi) = 1 - \frac{1}{\vartheta},
\]

Being \( g = 0 \), (1.4)–(1.5) admits a Liapounov functional (and consequently a dissipation integral), and this information will be crucial to overcome the lack of coercivity in \( \vartheta \). Actually, the global attractor will be constructed by proving uniform boundedness and asymptotic compactness of single trajectories and taking advantage of the dissipation property. Although this procedure might seem straightforward, the proof presents a number of difficulties. First of all, we have to settle the problem in a phase space \( \mathcal{X} \) (cf. (2.13) below) where both \( \vartheta \) and \( \chi \) are bounded in sufficiently strong norms. The conditions we require on the initial data are in fact more restrictive than what is necessary, e.g., for the mere well-posedness. In particular, we cannot deal with completely general potentials \( W \). Namely, we are forced to assume \( W \) be a smooth function defined on the whole real line (like, e.g., the double well potential \( W(r) = (r^2 - 1)^2 \)), and, for instance, we cannot treat the singular potentials, i.e., those being identically \( +\infty \) outside a bounded interval, like the so-called logarithmic potential \( W(r) = (r+1) \log(r+1) + (1-r) \log(1-r) - \lambda r^2 \), where \( \lambda > 0 \). Moreover, we note that, in analogy with the coercive case \( m_\infty > 0 \) studied in [22], \( \mathcal{X} \) does not have a Banach structure, due to the nonlinear terms in the energy, but it is just a metric space. In this setting, the key point of our argument is the proof of a uniform time regularization property for the solutions, which, in our opinion, can constitute an interesting issue by itself. Namely, we can show that both \( \vartheta \) and \( u = \vartheta^{-1} \) are uniformly bounded for sufficiently large times, whereas this need not hold for the initial temperature \( \vartheta_0 \). Thus, (1.4) eventually loses both the singular and the degenerate character.

A further open problem to which we give a positive answer is the existence of exponential attractors for the system (1.4)–(1.5). This is shown by using the so-called method of \( \ell \)-trajectories (cf. [16] [19] [20] [21]). However, we cannot prove exponential attraction in the metric of \( \mathcal{X} \) (that keeps, in some way, a trace of the nonlinear terms), but are forced to work with a weaker norm, corresponding in fact to the only contractive estimate which seems to hold for system (1.4)–(1.5).
The rest of this paper is organized as follows. In the next Section 2, we present our hypotheses and state our main results. The proofs are collected in Section 3.

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2 Notation and main results

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain with boundary $\Gamma$. Let us set $H := L^2(\Omega)$ and denote by $(\cdot, \cdot)$ both the scalar product in $H$ and that in $H \times H$, and by $\| \cdot \|$ the induced norm. The symbol $\| \cdot \|_X$ indicates the norm in the generic Banach space $X$. Next, we set $V := H^1(\Omega)$, $V_0 := H^1_0(\Omega)$, and define

$$A : V \to V', \quad (Av, z)_0 := \int_{\Omega} \nabla v \cdot \nabla z, \quad \forall v, z \in V_0,$$

$$B : V \to V', \quad (Bv, z) := \int_{\Omega} (vz + \nabla v \cdot \nabla z), \quad \forall v, z \in V,$$  

$(\cdot, \cdot)_0$ and $(\cdot, \cdot)$ denoting the duality pairings between $V_0$ and $V_0' = H^{-1}(\Omega)$ and between $V$ and $V'$, respectively. It turns out that $A$ and $B$ are the Riesz operators associated to the standard norms in $V_0$ and $V$, respectively.

Our hypotheses on the potential $W$ are the following:

$$W \in C^2(\mathbb{R}; \mathbb{R}), \quad W'(0) = 0, \quad \lim_{r \to -\infty} W'(r)r = +\infty,$$

$$\exists \lambda \geq 0 : \quad W''(r) \geq -\lambda \forall r \in \mathbb{R}. \quad (2.3) \quad \text{(2.4)}$$

In particular, by the latter assumption, $\beta(r) := W'(r) + \lambda r$ is increasingly monotone. Next, considering $B$, with a small abuse of notation, as a strictly positive unbounded linear operator on $H$ with domain $D(B) = \{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial \Omega \}$, we can take real powers of $B$ and set $\mathcal{V}_{2s} := D(B^s)$, endowed with the graph norm $\| v \|_s := \| B^s v \|$. Note that $\mathcal{V}_1 = V$. The variational formulation of system (2.1)–(2.5) takes then the form

$$\vartheta_t + \chi_t + A \left( 1 - \frac{1}{\vartheta} \right) = 0, \quad \text{in } V_0', \quad \text{(2.5)}$$

$$\chi_t + B \chi + W'(\chi) = 1 - \frac{1}{\vartheta}, \quad \text{in } V', \quad \text{(2.6)}$$

(in order to get the Riesz map $B$, $\chi$ has been added and subtracted from the left hand side, and $-\chi$ has been included in $W'$). Next, we define the associated energy functional as:

$$\mathcal{E} = \mathcal{E}(\vartheta, \chi) := \int_{\Omega} \left( \vartheta - \log \vartheta + \frac{1}{2}|\chi|^2 + \frac{1}{2}|
abla \chi|^2 + W(\chi) \right). \quad \text{(2.7)}$$

We immediately observe that $\mathcal{E}$ is finite and bounded from below on the “energy space”

$$\mathcal{X}_\mathcal{E} := \{ (\vartheta, \chi) : \vartheta \in L^1(\Omega), \quad \vartheta > 0 \text{ a.e. in } \Omega, \quad \log \vartheta \in L^1(\Omega), \quad \chi \in V, \quad W(\chi) \in L^1(\Omega) \}. \quad \text{(2.8)}$$

Nevertheless, due to the lack of coercivity (and consequently of compactness) in $\vartheta$ (the finiteness of energy only implies that $\vartheta \in L^1(\Omega)$), no existence result is known, up to our knowledge, for data lying just in $\mathcal{X}_\mathcal{E}$. Namely, noting as Problem (P) the coupling of (2.5)–(2.6) (intended to hold for a.e. value of time in $(0, \infty)$) with the initial condition

$$\vartheta|_{t=0} = \vartheta_0, \quad \chi|_{t=0} = \chi_0, \quad \text{a.e. in } \Omega, \quad \text{(2.9)}$$

we have the following result, proved in [10, Thm. 2.1] (see also [11, Prop. 2.1]):
Theorem 2.1. Let \((2.3) - (2.4)\) hold and let \((\vartheta_0, \chi_0) \in \mathcal{X}_\varepsilon\). Let, in addition \(\vartheta_0 \in L^p(\Omega)\) for some \(p > 6/5\). Then, there exists one and only one couple \((\vartheta, \chi)\) solving Problem (P) and such that

\[
\vartheta \in H^1(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)), \quad \vartheta > 0 \quad \text{a.e. in } \Omega \times (0, T),
\]

\[
(1 - 1/\vartheta) \in L^2(0, T; V_0),
\]

\[
\chi \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; H^2(\Omega)),
\]

hold for all \(T > 0\). Such a couple will be called a “solution” in the sequel.

Since we need to control uniformly in time the “large values” of the temperature, we have to ask a bit more summability on \(\vartheta_0\) and a bit more regularity on \(\chi_0\). Correspondingly, we will also get some more regularity than \((2.10) - (2.12)\). Namely, we set

\[
\mathcal{X} := \{ (\vartheta, \chi) \in \mathcal{X}_\varepsilon : \vartheta \in L^p(\Omega), \ \chi \in \mathcal{V}_{3+} \},
\]

where we assume that

\[
\varepsilon \in (0, 1), \quad p > 3.
\]

Actually, we need \(\varepsilon > 0\) in order to ensure that \(\chi\) stays in \(L^\infty(\Omega)\), while the higher summability of \(\vartheta_0\) seems necessary to get a uniform in time estimate for \(\vartheta(t)\).

We remark that the set \(\mathcal{X}\), which of course has no linear structure, can be endowed with a complete metric which makes it a suitable phase space for the associated dynamical process. As in \([22]\) (see also \([24, 26]\)), we can take

\[
d_\mathcal{X}((\vartheta_1, \chi_1), (\vartheta_2, \chi_2)) := \| \vartheta_1 - \vartheta_2 \|_{L^p(\Omega)} + \| \chi_1 - \chi_2 \|_{3+}
+ \| \log^+ \vartheta_1 - \log^+ \vartheta_2 \|_{L^1(\Omega)} + \| \beta(\chi_1) - \beta(\chi_2) \|_{L^1(\Omega)},
\]

where \((-)^+\) denotes negative part (notice, however, that the latter term could be omitted since it is dominated by the second one due to \((2.3)\) and the continuous embedding \(\mathcal{V}_{3+} \subset L^\infty(\Omega)\)). Correspondingly, we take initial data such that

\[
(\vartheta_0, \chi_0) \in \mathcal{X}.
\]

In the sequel, we will denote by \(S(t)\) the semigroup operator associating to \((\vartheta_0, \chi_0)\) the corresponding solution evaluated at time \(t\). The proof that \(S(\cdot)\) fulfills the usual properties of a continuous semigroup on \(\mathcal{X}\) is more or less standard and can be carried out along the lines, e.g., of \([22]\) Sec. 4. Hence, we omit the details. Instead, we focus on regularization properties of \(S(t)\). The key step of our investigation is the following

Theorem 2.2. Let \((2.3) - (2.4), (2.14)\) hold and let \(B\) be a set of initial data bounded in \(\mathcal{X}\). More precisely, let \(D_0\) stand for the \(d_\mathcal{X}\)-radius of the set, namely

\[
D_0 := \sup_{(\vartheta_0, \chi_0) \in B} d_\mathcal{X}((\vartheta_0, \chi_0), (1, 0)).
\]

Then, letting \((\vartheta, \chi) \in B\) and \((\vartheta(t), \chi(t)) := S(t)(\vartheta_0, \chi_0)\), there exist a time \(T_\infty > 0\) and a constant \(Q_\infty\) depending only on \(D_0\), such that, for all \(t \geq T_\infty\), there holds

\[
\| \vartheta(t) \|_{V \cap L^\infty(\Omega)} + \| \vartheta^{-1}(t) \|_{V \cap L^\infty(\Omega)} \leq Q_\infty,
\]

\[
\| \chi(t) \|_{H^2(\Omega)} \leq Q_\infty.
\]

Remark 2.3. Suitably modifying the proofs, one could show that any strictly positive time could be taken as \(T_\infty\). We omit the proof of this fact since it would involve further technical complications. We just notice that the quantity \(Q_\infty\) in \((2.18) - (2.19)\) would then depend on \(T_\infty\) and explode as \(T_\infty \searrow 0\).

Notice that the bounds \((2.18) - (2.19)\) are somehow weaker than a true dissipative estimate. Nevertheless, they will suffice for the proof of our main result (for the definition of the global attractor we refer to the monograph \([30]\)).
Theorem 2.4. Let the assumptions of Theorem 2.2 hold. Then, the semigroup \( S(\cdot) \) associated with Problem (P) admits the global attractor \( A \), which is compact in \( X \). More precisely,

\[
\exists c_A > 0 : \| \vartheta \|_{V \cap L^\infty(\Omega)} + \| \vartheta^{-1} \|_{V \cap L^\infty(\Omega)} + \| \chi \|_{H^2(\Omega)} \leq c_A \forall (\vartheta, \chi) \in \mathcal{A}. \tag{2.20}
\]

Finally, we can prove existence of an exponential attractor:

Theorem 2.5. Let the assumptions of Theorem 2.2 hold. Then, the semigroup \( S(\cdot) \) associated with Problem (P) admits an exponential attractor \( M \). More precisely, \( M \) is a compact set of \( X \), which has finite fractal dimension in \( V' \times H \), such that for any bounded set \( B \subset X \) there holds

\[
\text{dist}(S(t)B, M) \leq Q(\mathcal{D}_0)e^{-\kappa t}, \quad \forall t \geq 0,
\]

where \( \text{dist} \) represents the unilateral Hausdorff distance of sets with respect to the (product) norm in \( V' \times H \), \( \kappa > 0 \) is independent of \( B \), \( Q \) is a monotone function, and \( \mathcal{D}_0 \) is the \( X \)-radius of \( B \) given by (2.17).

Remark 2.6. As noted in the Introduction, we use the (rather weak) topology of \( V' \times H \) since it seems difficult to prove a contractive estimate in a better norm. Further comments will be given at the end of the proof (cf. Remark 3.6 below).

3 Proofs

In what follows, the symbols \( c, \kappa, \) and \( c_i, i \geq 0 \), will denote positive constants depending on \( W, \Omega \), and independent of the initial datum and of time. The values of \( c \) and \( \kappa \) are allowed to vary even within the same line. Moreover, \( Q : \mathbb{R}^+ \to \mathbb{R}^+ \) denotes a generic monotone function. Capital letters like \( C \) or \( C_i \) will be used to indicate constant which have other dependencies (in most cases, on the initial datum). Finally, the symbol \( c_\Omega \) will denote some embedding constants depending only on the set \( \Omega \).

Proof of Theorem 2.2. The basic idea to prove the uniform bounds (2.18)–(2.19) is to combine an estimate in a small interval \([0, T_0]\), where \( T_0 \) depends on \( \mathcal{D}_0 \), with a further uniform estimate holding on \([T_0, \infty)\). This procedure requires a number of steps, which are carried out below. Notice that some parts of the procedure might have a formal character in the present regularity setting (e.g., test functions could be not regular enough). However, all the procedure could be standardly made rigorous by working on some approximation and then passing to the limit (notice that the solution is known to be unique). We omit the details of this straightforward argument, for brevity.

First estimate. We start by deriving the energy estimate. Testing (2.5) by \( 1 - 1/\vartheta \), we have

\[
\frac{d}{dt} \int_{\Omega} (\vartheta - \log \vartheta) + \left\| 1 - \frac{1}{\vartheta} \right\|_{V_0}^2 = -\left\langle \chi_t, 1 - \frac{1}{\vartheta} \right\rangle. \tag{3.1}
\]

Next, multiplying (2.6) by \( \chi_t \), we obtain

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} |\chi|^2 + \frac{1}{2} |\nabla \chi|^2 + W(\chi) \right) + \| \chi_t \|^2 = \left\langle \chi_t, 1 - \frac{1}{\vartheta} \right\rangle, \tag{3.2}
\]

whence, summing (3.1) and (3.2) and recalling (2.7),

\[
\frac{d}{dt} \mathcal{E} + \left\| 1 - \frac{1}{\vartheta} \right\|_{V_0}^2 + \| \chi_t \|^2 = 0. \tag{3.3}
\]

In particular, integrating from 0 to an arbitrary \( t > 0 \) we have

\[
\mathcal{E}(t) + \int_0^t \left( \left\| 1 - \frac{1}{\vartheta} \right\|_{V_0}^2 + \| \chi_t \|^2 \right) \leq \mathcal{E}(0) \leq Q(\mathcal{D}_0). \tag{3.4}
\]

Second estimate. We test (2.6) by \( 2B^{1/2} \chi_t \). Thanks to \( \epsilon < 1 \) and using Poincaré’s and Young’s inequalities, we obtain

\[
\frac{d}{dt} \| \chi \|_{2^{1/2}}^2 + \| \chi_t \|_{2^{1/2}}^2 \leq c \left\| 1 - \frac{1}{\vartheta} - W'(\chi) \right\|_{2^{1/2}}^2 \leq c_1 \left\| 1 - \frac{1}{\vartheta} \right\|_{V_0}^2 + c \| W'(\chi) \|_{V}^2. \tag{3.5}
\]
Then, we note that, by (2.3) and the continuous embedding \( \mathcal{V}_{2+\epsilon} \subset L^\infty(\Omega) \),

\[
\|W'(\chi)\|_{2}^{2} = \|W'(\chi)\|^{2} + \int_{\Omega} |W''(\chi)\nabla\chi|^{2} \leq (1 + \|\nabla\chi\|^{2})Q(\|\chi\|_{L^\infty(\Omega)}) \leq Q(\|\chi\|_{L^\infty(\Omega)}^{2}).
\]

(3.6)

Next, let us compute (3.5) plus \( c_{1} \times (3.6) \). Using also (3.6), we arrive at

\[
\frac{d}{dt} \left[ \|\chi\|_{2+\epsilon}^{2} + c_{1}E \right] + \|\chi_{t}\|_{2+\epsilon}^{2} \leq Q(\|\chi\|_{2+\epsilon}^{2}).
\]

(3.7)

Thus, noting as \( \Psi \) the quantity in square brackets on the left hand side, and using the comparison principle for ODE’s, it follows that there exists a time \( T_{0} > 0 \), depending on \( D_{0} \) in a monotonically decreasing way, such that

\[
\|\Psi\|_{L^{\infty}(0,T_{0})} \leq Q(D_{0}),
\]

(3.8)

whence, integrating (3.7) in time over \( (0,T_{0}) \), and recalling (3.4),

\[
\|\chi\|_{L^{\infty}(0,T_{0};\mathcal{V}_{2+\epsilon})} + \|\chi_{t}\|_{L^{2}(0,T_{0};\mathcal{V}_{2+\epsilon})} \leq Q(D_{0}).
\]

(3.9)

In particular, by the continuous embedding \( \mathcal{V}_{2+\epsilon} \subset L^{3}(\Omega) \), we have

\[
\|\chi_{t}\|_{L^{2}(0,T_{0};L^{3}(\Omega))} \leq Q(D_{0}).
\]

(3.10)

**Third estimate.** Let us note that, since \( \vartheta \) solves (2.5), it is \( \vartheta > 0 \) a.e. in \( \Omega \times (0,\infty) \) and \( \vartheta = 1 \) a.e. on \( \Gamma \times (0,\infty) \). Thus, we can test (2.5) by \( \vartheta^{p-1} - 1 \) (given by (2.14)), which (at least in an approximation) lies in \( \mathcal{V}_{0} \) for a.e. \( t \in (0,\infty) \). We then get

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{p} \vartheta^{p} - \vartheta \right) + \frac{4(p-1)}{(p-2)^{2}} \|\nabla \vartheta^{\frac{p-2}{2}}\|^{2} \leq - \int_{\Omega} (\vartheta^{p-1} - 1) \chi_{t}
\]

(3.11)

and we estimate the right hand side as follows:

\[
- \int_{\Omega} (\vartheta^{p-1} - 1) \chi_{t} \leq \|\chi_{t}\|_{L^{3}(\Omega)} \|\vartheta^{\frac{p-2}{2}}\|_{L^{6}(\Omega)} \|\vartheta^{\frac{p}{2}}\| + c(1 + \|\chi_{t}\|^{2})
\]

\[
\leq \sigma \|\vartheta^{\frac{p-2}{2}}\|_{L^{6}(\Omega)}^{2} + c_{\vartheta} \|\chi_{t}\|_{L^{3}(\Omega)} \|\vartheta^{\frac{p}{2}}\|^{2} + c(1 + \|\chi_{t}\|^{2})
\]

\[
\leq \sigma \|\vartheta^{\frac{p-2}{2}}\|_{L^{6}(\Omega)}^{2} + \sigma \|\chi_{t}\|_{L^{3}(\Omega)} \|\vartheta\|_{L^{p}(\Omega)}^{2} + c(1 + \|\chi_{t}\|_{L^{3}(\Omega)}^{2}),
\]

(3.12)

where \( \sigma > 0 \) denotes a “small” constant, independent of \( p \), to be chosen at the end, and correspondingly \( c_{\vartheta} > 0 \) depends on the same quantities as the generic \( c \) and, additionally, on the final choice of \( \sigma \). In fact, passing from row to row, we allow \( \sigma \) to “incorporate” embedding constants. We used here the continuous embedding \( V \subset L^{6}(\Omega) \) and the Young and Poincaré inequalities. Although \( p \) is a fixed value (cf. (2.14)), here and below we emphasize the dependence on \( p \) of the estimates, since they will be repeatedly repeated with different exponents.

Adding \( 2 \times (3.11) \) (where the term on the right hand side is split via Young’s inequality) to (3.11), multiplying the result by \( p \), and taking \( \sigma \) small enough, we then obtain

\[
\frac{d}{dt} \int_{\Omega} \left[ \vartheta^{p} + p(\vartheta - 2 \log \vartheta) \right] + \kappa \|\nabla \vartheta^{\frac{p-2}{2}}\|^{2} \leq cp + p^{2}c_{2}\|\chi_{t}\|_{L^{3}(\Omega)}^{2} (1 + \|\vartheta\|_{L^{p}(\Omega)}^{2})
\]

(3.13)

for some \( c_{2} > 0 \). Now, let us set

\[
m := c_{2}\|\chi_{t}\|_{L^{3}(\Omega)}^{2},
\]

so that, by (3.10), \( \|m\|_{L^{2}(0,T_{0})} \leq Q(D_{0}). \)

(3.14)

Defining \( Y \) as 1 plus the integral on the left hand side of (3.13), we then have

\[
\frac{d}{dt} Y + \kappa \|\nabla \vartheta^{\frac{p-2}{2}}\|^{2} \leq cp + p^{2}m (1 + \|\vartheta\|_{L^{p}(\Omega)}^{2}) \leq cp + p^{2}m Y,
\]

(3.15)
whence, recalling (3.10) and (2.13) and using Gronwall’s Lemma,
\[ \| \vartheta \|_{L^\infty(0,T_0;L^p(\Omega))} \leq Q(\mathbb{D}_0). \] (3.16)

**Fourth estimate.** We test (2.5) by \( 2t\vartheta/\vartheta^2 \); next, we differentiate (2.6) in time and test the result by \( 2t\chi_t \). Taking the sum and noting that two terms cancel, we get
\[ \frac{d}{dt} (t\|\chi_t\|^2 + \|\nabla \frac{1}{\vartheta}\|^2) + 2t \int_\Omega \frac{\vartheta^2}{\vartheta^2} + 2t\|\chi_t\|_V^2 \leq (1 + 2\lambda t)\|\chi_t\|^2 + \|\nabla \frac{1}{\vartheta}\|^2. \] (3.17)
Then, integrating over \((0, T_0)\) and using (3.4), we obtain
\[ \|\chi_t(T_0)\|^2 + \|\nabla \frac{1}{\vartheta(T_0)}\|^2 \leq Q(\mathbb{D}_0)(1 + \frac{1}{T_0}) \leq Q(\mathbb{D}_0); \] (3.18)
actually, \((T_0)^{-1}\) depends increasingly on \(\mathbb{D}_0\).

**Fifth estimate.** Up to now, we got uniform bounds in the (small) time interval \([0, T_0]\). Our aim is now to get uniform estimates on \([T_0, \infty)\). First, we essentially repeat the previous estimate, but without the weight \(t\). This gives, of course,
\[ \frac{d}{dt} \left( \|\chi_t\|^2 + \|\nabla \frac{1}{\vartheta}\|^2 \right) + 2 \int_\Omega \frac{\vartheta^2}{\vartheta^2} + 2\|\chi_t\|_V^2 \leq 2\lambda\|\chi_t\|^2, \] (3.19)
whence, integrating over \((T_0, t)\) for arbitrary \(t \geq T_0\),
\[ \|\chi_t(t)\|^2 + \|\nabla \frac{1}{\vartheta(t)}\|^2 \leq 2\int_{T_0}^t \left( \|\vartheta\|_1 \|\chi_t(s)\|_V^2 + \|\vartheta\|_1 \|\chi_t(s)\|_V^2 \right) ds \leq \|\chi_t(T_0)\|^2 + \|\nabla \frac{1}{\vartheta(T_0)}\|^2 + 2\lambda \int_{T_0}^t \|\chi_t(s)\|^2 ds \leq Q(\mathbb{D}_0), \] (3.20)
where we used (3.18) and (3.4) to control the terms on the right hand side.

**Sixth estimate.** We repeat the Third estimate restarting from \(T_0\). Let us notice that, by (3.14), (3.20) and the continuous embedding \(V \subset L^6(\Omega)\),
\[ m = c_2\|\chi_t\|_{L^\infty(\Omega)}^2 \text{ satisfies } M := \|m\|_{L^2(0,\infty)} \leq Q(\mathbb{D}_0). \] (3.21)
Thus, by the continuous embedding \(V \subset L^6(\Omega)\), (3.15) takes the form
\[ \frac{d}{dt} \mathcal{Y} + \kappa \|\vartheta\|_{L^{3p-6}(\Omega)}^{p-2} \leq cp + p^2 m \mathcal{Y}. \] (3.22)
Let us now notice that, thanks to \(p > 3\), we can write
\[ \|\vartheta\|_{L^p(\Omega)}^{p-2} = \left( \int_\Omega \vartheta^p \right)^{\frac{p-2}{p}} \leq \left( \|\vartheta^p\|_{L^{\frac{3p-6}{p}}(\Omega)} \|1\|_{L^{\frac{3p-6}{p}}(\Omega)} \right)^{\frac{p-2}{p}} \right) \leq c_0 \|\vartheta\|_{L^{3p-6}(\Omega)}^{\frac{2p-6}{3p-6}} \] (3.23)
Moreover, we have
\[ \|\vartheta\|_{L^{p}(\Omega)}^{p-2} = \left( \mathcal{Y} - \frac{1}{p} \int_\Omega (\vartheta - 2 \log \vartheta) \right)^{\frac{p-2}{p}} \geq c\mathcal{Y}^{\frac{p-2}{p}} - \kappa \mathcal{Y} - pQ(\mathbb{D}_0). \] (3.24)
Thus, (3.22) gives
\[ \frac{d}{dt} \mathcal{Y} + \kappa \mathcal{Y}^{\frac{p-2}{p}} \leq p^2 m \mathcal{Y} + pQ(\mathbb{D}_0), \] (3.25)
so that, for \(\mathcal{H} := \log \mathcal{Y}\),
\[ \frac{d}{dt} \mathcal{H} + c \frac{2m}{\kappa} \left[ (\kappa - pQ(\mathbb{D}_0)e^{-\frac{(p-2)\mathcal{H}}{p-2}}) \right] \leq p^2 m. \] (3.26)
Noting as $\Sigma$ the quantity in square brackets, an easy computation shows that

$$\Sigma \geq 0 \Leftrightarrow H \geq \frac{p}{p-2} \log \left(\frac{p}{\kappa} Q(D_0)\right) =: \zeta. \quad (3.27)$$

Consequently, it is not difficult to obtain

$$\|H\|_{L^\infty(T_0, \infty)} \leq \max \left\{ \zeta, \mathcal{H}(T_0) \right\} + p^2 \int_0^\infty m(s) \, ds, \quad (3.28)$$

so that, being by (3.27) and (3.16),

$$\exp(\zeta) \leq pQ(D_0), \quad \exp(\mathcal{H}(T_0)) = \mathcal{Y}(T_0) \leq pQ(D_0), \quad (3.29)$$

and using (3.21), we readily get

$$\|\vartheta\|_{L^\infty(T_0, \infty; L^p(\Omega))} \leq \|\mathcal{Y}\|_{L^\infty(T_0, \infty)} \leq \exp \left( \max \{\zeta, \mathcal{H}(T_0)\} \right) \exp \left( p^2 M \right) \leq pQ(D_0) \exp \left( p^2 M \right), \quad (3.30)$$

whence, clearly,

$$\|\vartheta\|_{L^\infty(T_0, \infty; L^p(\Omega))} \leq Q(D_0). \quad (3.31)$$

Suitable time integrations of (3.22) permit us collect what we have proved so far in a

**Lemma 3.1.** Under the assumptions of Theorem 2.2, there exist a time $T_0 > 0$ and a quantity $Q_0 > 0$, both depending on $D_0$, such that, for all $t \geq T \geq T_0$,

$$\|\vartheta(t)\|_{L^p(\Omega)} + \int_t^{t+1} \|\vartheta(s)\|_{L^p(\Omega)}^2 \, ds \leq Q_0, \quad (3.32)$$

$$\int_t^T \|\vartheta(s)\|_{L^p(\Omega)}^2 \, ds \leq Q_0 + Q_0(t - T). \quad (3.33)$$

Our next aim is to extend (3.32) and (3.33) to any finite exponent. Namely, we have

**Lemma 3.2.** Under the assumptions of Theorem 2.2, for all $q \in [p, \infty)$ there exist a time $T_q > 0$ and a quantity $Q_q > 0$, both depending on $D_0$ and $q$, such that, for all $t \geq T \geq T_q$,

$$\|\vartheta(t)\|_{L^q(\Omega)}^q + \int_t^{t+1} \|\vartheta(s)\|_{L^{q-\delta}(\Omega)}^{q-\delta} \, ds \leq Q_q, \quad (3.34)$$

$$\int_t^T \|\vartheta(s)\|_{L^{q-\delta}(\Omega)}^{q-\delta} \, ds \leq Q_q + Q_q(t - T). \quad (3.35)$$

**Proof.** It suffices to iterate finitely many times the procedure in the Sixth Estimate. Namely, setting $p_0 := p$, we observe that, by (3.32) and interpolation, there follows

$$\sup_{t \in [T_0, \infty)} \|\vartheta\|_{L^{p_i}(t, t+1; L^{p_i}(\Omega))} \leq Q(D_0), \quad (3.36)$$

where we have set (for $i = 1$, at least in the meanwhile)

$$p_i := \frac{5p_{i-1} - 6}{3}. \quad (3.37)$$

Note that $p_1 > p_0$ since $p_0 > 3$. Then, we repeat the argument leading to (3.22), but with $p_1$ in place of $p$. This gives (for $i = 1$ and with obvious meaning of $\mathcal{Y}_i$)

$$\frac{d}{dt}\mathcal{Y}_i + c\|\vartheta\|_{L^{p_{i-1}-\delta}(\Omega)}^{p_{i-1}-\delta} \leq c \mathcal{Y}_i + p_i^2 m \mathcal{Y}_i. \quad (3.38)$$

Noting that both $m$ and $\mathcal{Y}_i$ are summable on time intervals of finite length thanks to (3.21) and, respectively, (3.36), we can use the uniform Gronwall Lemma (cf., e.g., [30, Lemma III.1.1]), that gives

$$\|\vartheta(t)\|_{L^{p_i}(\Omega)} \leq \mathcal{Y}_i(t) \leq Q_i(D_0) \quad \forall t \geq T_i := T_{i-1} + 1, \quad (3.39)$$
with obvious meaning of $Q_i(D_0)$. Thus, suitable integrations in time of (3.38) give the analogue of (3.32) and (3.33), with $p_1$ in place of $q$. To get (3.34) and (3.35), it then suffices to proceed by iteration on $i$ until $p_i$ is larger than $q$. Notice that since a finite number of steps is sufficient, we do not have to take care of the dependence on $i$ of the quantities $Q_i$ and $T_i$ (both, in fact, would explode if infinite iterations were needed). The proof of the Lemma is concluded.

A similar property holds also for the inverse temperature:

**Lemma 3.3.** Setting $u := \vartheta^{-1}$, under the assumptions of Theorem 2.2 for all $q \in [1, \infty)$ there exist a time $T_q > 0$ and a quantity $Q_q > 0$, both depending on $D_0$ and $q$ (and possibly larger from those in the previous Lemma) such that, for all $t \geq T \geq T_q$,

$$
\|u(t)\|_{L^q(\Omega)}^q + \int_t^{t+1} \|u(s)\|_{L^{q+4+6}(\Omega)}^{q+4+6} \, ds \leq Q_q,
$$

and we claim that, for any $\nu$ and $\nu'$, the continuous embedding $V \subset L^6(\Omega)$. Then, we proceed essentially as in the Third estimate, i.e., for a generic $q \geq 6$, we multiply (3.25) by $1 - u^{q+1}$. In place of (3.11), we get

$$
\frac{d}{dt} \int_\Omega \left( \frac{1}{q} u^q + \vartheta \right) + \frac{4(q+1)}{q(q+2)} \|\nabla u^{q+2}\|^2 \leq -\int_\Omega (1 - u^{q+1})\chi_t,
$$

so that, estimating the right hand side as in (3.12), we infer

$$
\frac{d}{dt} Z_q + c\|u\|_{L^{q+4+6}(\Omega)}^{q+4+6} \leq c q + q^2 m Z_q, \quad \text{where } Z_q := \int_\Omega (u^q + q\vartheta)
$$

and $m$ is as in (3.14) (possibly for a different value of $c_0$). At this point, noticing that the exponents are even better than in (3.22), the proof can be completed by mimicking the arguments in the Sixth estimate and in the proof of Lemma 3.2.

**Lemma 3.4.** Under the assumptions of Theorem 2.2 there exist a time $T_\star$ and a quantity $Q_\star$, both depending on $D_0$, such that, for all $t \geq T_\star$,

$$
\|\chi_t(t)\|_{L^{24/5}(\Omega)}^2 \leq Q_\star.
$$

**Proof.** The exponent $24/5$ in (3.41) is chosen just for later convenience. In fact, (3.41) can be proved for any exponent strictly smaller than 6. Differentiating in time (2.6), we have

$$
\chi_{tt} + B\chi_t = \Phi := -W''(\chi)\chi_t + \frac{\vartheta_t}{\vartheta^2}
$$

and we claim that, for any $\nu \in (0, 1)$, we can choose $T_\nu > 0$ and $Q_\nu > 0$, both depending only on $D_0$ and $\nu$, such that

$$
\int_{T_\nu}^\infty \|\Phi(s)\|_{H^{-\nu}(\Omega)} \, ds \leq Q(D_0).
$$

Actually, recalling (3.20) and applying standard regularity results to (2.6) (seen here as a time-dependent family of elliptic equations), it follows that

$$
\|\chi(t)\|_{H^2(\Omega)} \leq Q(D_0) \quad \text{for all } t \geq T_0.
$$

Thus, by (2.3), the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$, and (3.4),

$$
\int_{T_0}^\infty \|W''(\chi(s))\chi_t(s)\|^2 \, ds \leq Q(D_0).
$$

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Analogously, using the first integral bound in (3.20), the bound of the first term in (3.40) with \( q \) sufficiently large (depending on \( \nu \)), and elementary interpolation, it is not difficult to get, for some \( T_\nu > 0 \) and \( Q_\nu > 0 \),

\[
\int_{T_\nu}^{\infty} \left\| \frac{\partial \chi}{\partial t}(s) \right\|_{L^{\frac{n+2}{n}}(\Omega)}^2 \, ds \leq Q_\nu(D_0).
\]  

(3.49)

Thus, thanks to (3.38), (3.39) and the continuous embedding \( L^{\frac{n}{n+2}}(\Omega) \subset H^{-\nu}(\Omega) \), we see that (3.40) holds for any \( T_\nu \geq T'_\nu \). Now, let us observe that, by the bound of the second integral term in (3.20), \( T_\nu \in [T'_\nu, T'_\nu + 1] \) can be chosen such that

\[
\| \chi_t(T_\nu) \|_{H^{1-\nu}(\Omega)}^2 \leq c \| \chi_t(T_\nu) \|_V^2 \leq Q(D_0).
\]  

(3.50)

Then, applying the standard linear parabolic Hilbert theory to the equation (3.45) on the time interval \( [T_\nu, \infty) \) and with the initial condition \( \chi_t(T_\nu) \), and using (3.40), we have (possibly for a different value of \( Q_\nu \))

\[
\| \chi_t \|_{C^0([T_\nu, \infty); H^{1-\nu}(\Omega))} + \| \chi_t \|_{L^2(T_\nu, \infty; H^{2-\nu}(\Omega))} \leq Q_\nu(D_0),
\]  

(3.51)

whence the assertion follows from the continuous embedding \( H^{1-\nu}(\Omega) \subset L^{24/5}(\Omega) \), which holds for \( \nu \) small enough.

**End of proof of Theorem 2.2.** We use a modified Alikakos-Moser iteration scheme similar to that in [13], but suitably adapted in order to obtain time regularization effects. Similar procedures have been proved to be effective in other recent papers, cf. [8] [22].

As a first step, we come back to (3.11), where the exponent \( p \) is substituted by a number \( q_i \) to be chosen later. Since we need infinitely many iterations, now the right hand side has to be estimated more carefully. Namely, we have

\[
- \int_{\Omega} (q^{q_i-1} - 1) \chi_t \leq \| \chi_t \|_{L^{24/5}(\Omega)} \left\| \frac{\partial^{2q_i-2}}{\partial t^{2q_i-2}} \|_{L^6(\Omega)} \right\|_{L^{24/5}(\Omega)} + c \left( 1 + \| \chi_t \| \right)^2
\]

\[
\leq \frac{\sigma}{q_i} \left\| \nabla \partial^{\frac{q_i-2}{q_i}} \right\|^2 + \frac{\sigma}{q_i} + c_{\sigma} q_i Q_\star \| \|_{L^{4q_i/5}(\Omega)} + C_0,
\]  

(3.52)

where \( Q_\star \) is exactly the same quantity as in (3.41) and the constant \( C_0 \) depends on \( D_0 \) and is independent of \( q_i \). Thus, possibly modifying \( C_0 \), in place of (3.22) we get

\[
\frac{d}{dt} \mathcal{Y}_i + \kappa \| \partial \|_{L^{4q_i/5}(\Omega)} \leq c q_i^3 Q_\star \| \|_{L^{4q_i/5}(\Omega)} + C_0 q_i,
\]  

(3.53)

where it is worth noting that

\[
\| \partial \|_{L^{4q_i/5}(\Omega)} \leq \mathcal{Y}_i := 1 + \int_{\Omega} [\partial^{q_i} + q_i (\partial - 2 \log \partial)] \leq \| \partial \|_{L^{4q_i/5}(\Omega)} + C_1 q_i,
\]  

(3.54)

where \( C_1 \) is another quantity depending only on \( D_0 \). Moreover, (3.52) gives

\[
\frac{d}{dt} \mathcal{Y}_i + \kappa \| \partial \|_{L^{4q_i/5}(\Omega)} \leq c q_i^4 Q_\star \| \|_{L^{12q_i/5}(\Omega)} \| \|_{L^{4q_i/5}(\Omega)} + C_0 q_i,
\]  

(3.55)

provided that \( 12q_i/5 - 6 \geq 4q_i/5 \), which is true for all \( i \in \mathbb{N} \) if \( q_0 \) is large enough and we set

\[
q_i = \frac{5}{4} q_{i-1}, \quad \forall i \geq 1.
\]  

(3.56)

The choice (3.56) permits to rewrite (3.55) in the form

\[
\frac{d}{dt} \mathcal{Y}_i + \kappa \| \|_{L^{4q_i/5}(\Omega)} \leq c_3 q_i^4 Q_\star \mathcal{Y}_i^{-1} \| \|_{L^{4q_i/5}(\Omega)} + C_0 q_i,
\]  

(3.57)

for some \( c_3 > 0 \). Let us now define

\[
\tau_i := \frac{2}{q_i}, \quad \text{so that} \quad \tau_\infty := \sum_{i=1}^{\infty} \tau_i = \sum_{i=1}^{\infty} \frac{2}{q_i} < \infty.
\]  

(3.58)
Moreover, let us take $T_*$ as in Lemma 3.3 and set $T_\infty := T_* + \tau_\infty$. We now aim to show that for all fixed $t \geq T_\infty$ the bounds (2.18), (2.19) are satisfied. More precisely, we will limit ourselves to prove the $L^\infty$-bound of $\vartheta$ in (2.18). Indeed, as noted in the proof of Lemma 3.3, the argument to prove the $L^\infty$-bound of $u$ is similar and even simpler; moreover, the $V$-bounds are consequence of the $L^\infty$-bounds and of (3.20); finally, (2.19) is already known from (3.47).

Thus, we assume that $t \geq T_\infty$ is fixed and set $S_0 := t - \tau_\infty$, so that $S_0 \geq T_*$. We shall now work on the interval $[t - \tau_\infty, t - \tau_\infty] = [S_0, S_0 + 2\tau_\infty]$. Using Lemma 3.2 we can also assume $q_0$ as large as we want, so that there exists a quantity $R_0$, depending only on $D_0$ and on the choice of $\tau_\infty$, such that the bound

$$\|\vartheta_0\|_{L^\infty(S_0; s)} + \kappa \int_{S_0}^s \|\vartheta\|_{L^{q_0 - 2}(\Omega)}^{q_0 - 2} \leq R_0,$$

(3.59)

where $\vartheta_0$ is defined as in (3.54) with $i = 0$, holds uniformly w.r.t. $s \in [S_0, S_0 + 2\tau_\infty]$. Thus, taking $i = 1$, integrating (3.57) in time over $(S_1, s)$, where $S_1 \in [S_0, S_0 + \tau_1] = [S_0, S_0 + 1]$ will be chosen later and $s$ is a generic point in $[S_1, S_1 + 2\tau_\infty]$ so that $s - S_1 \leq 2\tau_\infty$, we have

$$\mathcal{Y}_1(s) + \kappa \int_{S_1}^s \|\vartheta\|_{L^{q_0 - 2}(\Omega)}^{q_0 - 2} \leq \mathcal{Y}_1(S_1) + \frac{c_3}{\kappa} \left(\frac{5}{4}\right)^2 q_0^2 \|\vartheta\|_{L^{q_0 - 2}(\Omega)}^{q_0 - 2} + \frac{5}{4} C_0 q_0 (2\tau_\infty)$$

(3.60)

and we notice that also the first term on the right hand side can be estimated. Indeed, by the latter of (3.59), $S_1 \in [S_0, S_0 + \tau_1]$ can be chosen such that

$$\|\vartheta(S_1)\|_{L^{q_0 - 2}(\Omega)}^{q_0 - 2} \leq \frac{1}{\kappa} \int_{S_0}^{S_0 + \tau_1} \|\vartheta\|_{L^{q_0 - 2}(\Omega)} \leq \frac{1}{\kappa} \int_{S_0}^{S_0 + \tau_1} \|\vartheta\|_{L^{q_0 - 2}(\Omega)} \leq c_4 \frac{R_0}{\tau_1} = c_4 R_0,$$

(3.61)

for some $c_4 > 0$. We used that $\tau_1 = 1$. Recalling (3.54), as a consequence we obtain

$$\mathcal{Y}_1(S_1) \leq \|\vartheta(S_1)\|_{L^{q_0 - 2}(\Omega)} \leq q_1 C_1 \leq \left(\left(\frac{R_0}{\tau_1}\right)^{\frac{5}{4}} q_0 C_1 \right) + \frac{5}{4} q_0 C_1 = (c_4 R_0)^{\frac{q_0}{q_0 - 2}} + \frac{5}{4} q_0 C_1.$$  

(3.62)

Then, setting for $i \geq 0$

$$\eta_i := \frac{q_i}{q_i - 2} \geq \frac{5q_i + 8}{5q_i},$$

(3.63)

and collecting (3.60)-(3.62), we obtain that for all $s \in [S_1, S_0 + 2\tau_\infty]$,

$$\mathcal{Y}_1(s) + \kappa \int_{S_1}^s \|\vartheta\|_{L^{q_0 - 2}(\Omega)} \leq R_0 \left(\left(\frac{c_4}{16}\right)^{i} q_0 \left(\frac{25}{16}\right) \left(\frac{c_3}{\kappa} q_0^2 Q_0 \right) + \frac{5}{4} q_0 (2\tau_\infty C_0 + C_1)\right):= R_i.$$  

(3.64)

At this point we can proceed by iteration and observe that, as the procedure is repeated, the main modification comes from a term $T_i^{-1} = i^2$ additionally appearing on the right hand side of the $i$-analogue of (3.61). Suitably modifying the procedure, (3.64) takes the new form

$$\|\vartheta\|_{L^\infty(S_i; s)} + \kappa \int_{S_i}^s \|\vartheta\|_{L^{q_0 - 2}(\Omega)} \leq R_i, \quad \forall s \in [S_i, S_0 + 2\tau_\infty],$$

(3.65)

where we point out that $\kappa$, which comes from (3.57) and, in fact, from (3.22), is independent of $i$. Moreover, $S_i$ is a suitable point in $[S_{i-1}, S_{i-1} + \tau_1]$ and $R_i$ is given by

$$R_i = R_i^{\frac{q_i}{q_i - 1}} \left(\left(\frac{c_4}{16}\right)^{i} q_0 \left(\frac{25}{16}\right) \left(\frac{c_3}{\kappa} q_0^2 Q_0 \right) + \frac{5}{4} q_0 (2\tau_\infty C_0 + C_1)\right) \leq \left(\left(\frac{c_4}{16}\right)^{i} \left(\frac{25}{16}\right)^{i} \left(\frac{c_3}{\kappa} q_0^2 Q_0 \right) + \frac{5}{4} q_0 (2\tau_\infty C_0 + C_1)\right),$$

(3.66)

for some $K > 0$ depending on $D_0$ and the choice of $\tau_\infty$, and independent of $i$. Consequently, $R_i$ is estimated in terms of $R_0$ by

$$R_i \leq R_0^{\frac{q_i}{q_i - 1}} A_i^{\frac{q_i}{q_i - 1}} A_{i-1}^{\frac{q_i}{q_i - 1}} A_{i-2}^{\frac{q_i}{q_i - 1}} \cdots = R_0^{\frac{q_i}{q_i - 1}} \prod_{k=0}^{i-1} A_{k+1}^{\frac{q_i}{q_i - 1}} \prod_{k=0}^{i-1} \eta_k,$$

(3.67)
where it is intended that the latter productory is 1 in the case \( j = i \). Passing to the logarithm and observing that
\[
\prod_{k=1}^{\infty} \eta_k < \infty,
\]
it is then easy to verify that
\[
\limsup_{i \to \infty} R_i^i = \limsup_{i \to \infty} R_i^{(\frac{4}{5})} \leq Q(\mathbb{D}_0) < \infty.
\]

Thus, coming back to (3.67), recalling (3.54), and noting that the sequence \( S_i \) converges to a point \( S_\infty \) such that \( S_0 \leq S_\infty \leq t = S_0 + \tau_\infty \leq S_0 + 2\tau_\infty \), we finally infer that
\[
\limsup_{i \to \infty} \| \vartheta(s) \|_{L^\infty(\Omega)} \leq \limsup_{i \to \infty} R_i^{\vartheta} \leq Q(\mathbb{D}_0) \quad \forall s \in [S_\infty, S_0 + 2\tau_\infty].
\]

In particular, this holds for \( s = t \) and it is worth remarking once more that the latter quantity \( Q(\mathbb{D}_0) \) is independent of \( t \in [T_* + \tau_\infty, \infty) \). Actually, it depends on time only through the choice of the sequence \( \tau_\infty \), and not on the choice of \( S_0 \geq T_* \), i.e., of \( t \). The proof of the first of (2.18) and of the Theorem is complete. \( \blacksquare \)

**Proof of Theorem 2.4** We start noticing that \( \mathcal{E} \), defined in (2.7), is a Liapunov functional for Problem (P). Namely, the following conditions (cf., e.g., [3, Sec. 5]) hold:

(L1) \( \mathcal{E} \) is continuous on \( \mathcal{X} \) (recall (2.13));

(L2) \( \mathcal{E} \) is nonincreasing along solution trajectories;

(L3) if \( S(t)w = w \) for some \( t > 0 \) and \( w \in \mathcal{X} \), then \( w \) belongs to the set \( \mathbb{E}_0 \) of equilibrium points of the semigroup (and consequently it identifies a stationary solution).

Indeed, (L1) is obvious since \( \mathcal{X} \) is endowed with the metric (2.15); (L2) is a simple consequence of the energy equality (3.3); finally, (L3) still follows from (3.3) by noticing that if \( \chi_t = 0 \) and \( \vartheta^{-1} = 1 \) then it is also \( \vartheta_t = 0 \) by comparison in (2.5). It is worth remarking that \( w = (\vartheta, \chi) \in \mathcal{X} \) is a stationary point of \( S(\cdot) \) if and only if \( \vartheta \equiv 1 \) in \( \Omega \) and \( \chi \) solves
\[
B\chi + W'(\chi) = 0, \quad \text{in} \ V'.
\]

It is well-known that, due to nonconvexity of \( W \), (3.71) can have infinitely many solutions, so that the structure of \( \omega \)-limits of solutions to (P) and, a fortiori, of attractors, is nontrivial. Nevertheless, by maximum principle arguments and standard elliptic regularity theorems (cf., e.g., [1]), it is easy to prove that the projection of \( \mathbb{E}_0 \) on the second component \( \chi \) is bounded at least in \( W^{2,\zeta}(\Omega) \) for all \( \zeta \in [1, \infty) \) (actually, using bootstrap arguments, more could be said depending on the smoothness of \( W \), but we are not interested in maximal regularity here).

Thus, let \( B_0 \) be the neighbourhood of \( \mathbb{E}_0 \) of radius 1 in the metric of \( \mathcal{X} \). Then, a simple and direct contradiction argument (cf. [3, Thm. 5.1]) shows that \( B_0 \) is pointwise absorbing for \( S(\cdot) \), i.e., given a solution \( w \) to (P) with initial datum in \( \mathcal{X} \), there exists a time \( T_w \) such that \( w(t) \in B_0 \) for all \( t \geq T_w \). Thus, being \( S(\cdot) \) asymptotically compact (i.e., \( S(\cdot) \) eventually maps \( \mathcal{X} \)-bounded sets of initial data into relatively compact sets) thanks to (2.18)–(2.19), we deduce existence of the global attractor \( \mathcal{A} \) by means, e.g., of [3, Thm. 3.3].

To complete the proof, we have to show that (2.20) holds. To do this, it suffices to notice that, as a consequence of the existence of \( \mathcal{A} \), \( S(\cdot) \) admits a \( \mathcal{X} \)-bounded and uniformly absorbing set \( B_1 \). Namely, for every \( \mathcal{X} \)-bounded set \( B \) there exists \( T_B \) such that \( S(t)B \subset B_1 \) for all \( t \geq T_B \). We then notice that, by Theorem 2.2 for sufficiently large \( t \), \( S(t) \) maps \( B_1 \) into a set \( B_2 \) which is bounded in the same sense as (2.20). Thus, \( B_2 \) is absorbing because \( B_1 \) is absorbing, and, consequently, the bound (2.20) holds also for the attractor \( \mathcal{A} \) which is the \( \omega \)-limit of \( B_2 \). The proof is concluded. \( \blacksquare \)

**Proof of Theorem 2.5** Let us recall the basic uniqueness estimate for system (2.5)–(2.6). Let \( (\vartheta_1, \chi_1), \ i = 1, 2 \), be a couple of solutions to (P) and set \( (\vartheta, \chi) := (\vartheta_1, \chi_1) - (\vartheta_2, \chi_2) \). Set also \( e_i := \vartheta_i + \chi_i, \ i = 1, 2 \), and \( e := e_1 - e_2 \) (the new variable has the physical meaning of enthalpy). Write
the differences of (2.5) and (2.6) for \( i = 1, 2 \) and test them, respectively, by \( A^{-1}e \) and by \( \lambda \). Taking the sum, noting that two terms cancel, and using (2.24), we then obtain

\[
\frac{d}{dt}(\|e\|_{H^{-1}(\Omega)}^2 + \|\lambda\|^2) + 2\int_{\Omega} \left( -\frac{1}{\vartheta_1} + \frac{1}{\vartheta_2} \right) \vartheta + 2\|\lambda\|_V^2 \leq 2\lambda\|\lambda\|_H^2. \tag{3.72}
\]

Now, let us restrict ourselves to consider only initial data lying in a suitable absorbing set. Namely, we take the absorbing set \( \mathcal{B}_2 \) defined above and set

\[
\mathcal{B}_3 := \{ \cup_{t \geq T_2} S(t)\mathcal{B}_2 \}, \tag{3.73}
\]

where \( T_2 > 0 \) is such that \( \mathcal{B}_2 \) absorbs itself for \( t \geq T_2 \) and we have taken what we will call the sequential weak star closure in \( \mathcal{W} \). Namely, we define the set \( \mathcal{W} \) as

\[
\mathcal{W} := \{ (\vartheta, \chi) \in (V \cap L^\infty(\Omega)) \times H^2(\Omega) \colon \vartheta > 0 \text{ a.e.}, \text{ and } u = \vartheta^{-1} \in V \cap L^\infty(\Omega) \} \tag{3.74}
\]

and we intend that a point \( w = (\vartheta, \chi) \) belongs to \( \mathcal{B}_3 \) iff there exist two sequences \( \{w_n\} = \{(\vartheta_n, \chi_n)\} \subset \mathcal{B}_2 \) and \( \{t_n\} \subset [T_2, \infty) \) such that, as \( n \to \infty \), \( S(t_n)w_n := (\vartheta_n(t_n), \chi_n(t_n)) \) satisfies

\[
\vartheta_n(t_n) \to \vartheta, \quad \vartheta_n^{-1}(t_n) \to \vartheta^{-1}, \quad \text{weakly-* in } V \cap L^\infty(\Omega) \quad \text{and} \quad \chi_n(t_n) \to \chi, \quad \text{weakly in } H^2(\Omega). \tag{3.75}
\]

Of course, this weak star convergence of \( \mathcal{W} \) is associated to a suitable (Hausdorff) topology, which we note as the “weak star topology”, or simply the “topology” of \( \mathcal{W} \). Instead, when we speak, e.g., of the \( (H \times V) \)-norm of an element \( w = (\vartheta, \chi) \in \mathcal{W} \), we will just mean \( (\|\vartheta\|^2 + \|\chi\|_V^2)^{1/2} \) so that we are neglecting, in fact, the behavior of \( u = \vartheta^{-1} \). Thus, the generic element of \( \mathcal{W} \) is seen just as a couple; however, the “weak star convergence” defined in (3.75) and the related topology take also the additional variable \( u \) into account.

Next, it is worth noticing that, by construction, \( \mathcal{B}_3 \) is positively invariant (i.e. \( S(\tau)\mathcal{B}_3 \subset \mathcal{B}_3 \) for all \( \tau \geq 0 \), sequentially weakly star closed in \( \mathcal{W} \), and contained in the \( \mathcal{W} \)-sequential weak star closure of \( \mathcal{B}_2 \), so that, in particular, there holds (cf. (2.18)–(2.19))

\[
\|\vartheta\|_{V \cap L^\infty(\Omega)} + \|\vartheta^{-1}\|_{V \cap L^\infty(\Omega)} + \|\chi\|_{H^2(\Omega)} \leq c_3 \tag{3.76}
\]

for all \( (\vartheta, \chi) \in \mathcal{B}_3 \) and for some constant \( c_3 > 0 \). Consequently, we have

\[
2\int_{\Omega} \left( -\frac{1}{\vartheta_1} + \frac{1}{\vartheta_2} \right) \vartheta \geq c_5 \|\vartheta\|^2 \quad \forall \vartheta \in \Pi_1(\mathcal{B}_3). \tag{3.77}
\]

where \( \Pi_1 \) is the projection on the first component and \( c_5 \) suitably depends on \( c_3 \).

We now refer to the so-called method of \( t \)-trajectories (cf. [19] [19] [20] [21]). To do this, let us take \( t > 0 \) and define the set \( \mathcal{W}_t \) of \( t \)-trajectories of (P) simply as the set of the solutions whose initial datum lies in \( \mathcal{B}_3 \), restricted to the time interval \((0, t)\). Thus, as before, solutions are seen as couples \( (\vartheta, \chi) \) and the behavior of \( u = \vartheta^{-1} \) is not considered; nevertheless, since the elements of \( \mathcal{W}_t \) take values in \( \mathcal{B}_3 \) (recall that \( \mathcal{B}_3 \) is positively invariant), they satisfy (3.76) uniformly in time. The set \( \mathcal{W}_t \) is endowed with the norm of \( L^2(0, t; H \times V) \) (the reason for a choice of such a weak metric is in estimate (3.72)). Let us notice, however, that, if \( \{w_n\} \subset \mathcal{W}_t \) tends to some limit \( w \) (strongly) in \( L^2(0, t; H \times V) \), then also \( w \) lies in \( \mathcal{W}_t \) (in other words, \( \mathcal{W}_t \) is complete in the chosen metric). Indeed, by construction, for all \( n \) there holds

\[
w_{0,n} := w_n(0) = \lim_{k \to \infty} S(t_k^n)w_{0,n}^k, \quad \text{where } w_{0,n}^k \in \mathcal{B}_2, \tag{3.78}
\]

\( t_k^n \geq T_2 \) for all \( k \) and \( n \), and the limit is intended in the topology of \( \mathcal{W} \) (cf. (3.75)). In particular, both the above \( k \)-limit and the \( n \)-limit \( w_{0,n} \to w_0 \) (the latter thanks to (3.76)) hold in \( \mathcal{X} \), which is a metric space. Thus, we can extract a diagonal subsequence such that

\[
\lim_{j \to \infty} S(t_{k_j}^n)w_{0,n_j}^{k_j} \to w_0 := w(0), \quad \text{in } \mathcal{X}, \tag{3.79}
\]

and, again by uniform validity of (3.70), weakly star in \( \mathcal{W} \). This means that \( w_0 \in \mathcal{B}_3 \) and \( w \in \mathcal{W}_t \), as desired.
Let us now integrate (3.72) over \( (\tau, 2\ell) \), where \( \tau \) is a generic point in \([0, \ell]\). We then get

\[
\|e(2\ell)\|^2_{H^{-1}(\Omega)} + \|\chi(2\ell)\|^2 + \int_{\tau}^{2\ell} (c_5 \|\vartheta(s)\|^2 + 2\|\chi(s)\|^2) \, ds \leq \|e(\tau)\|^2_{H^{-1}(\Omega)} + \|\chi(\tau)\|^2 + 2\lambda \int_{\tau}^{2\ell} \|\chi(s)\|^2 \, ds. \tag{3.80}
\]

Integrating the above relation with respect to \( \tau \in (0, \ell) \), we obtain

\[
\ell \|e(2\ell)\|^2_{H^{-1}(\Omega)} + \ell \|\chi(2\ell)\|^2 + \ell \int_{0}^{\ell} (c_5 \|\vartheta(s)\|^2 + 2\|\chi(s)\|^2) \, ds \leq \int_{0}^{\ell} c_6 (\|\vartheta(\tau)\|^2_{H^{-1}(\Omega)} + \|\chi(\tau)\|^2) \, d\tau + 2\lambda \ell \int_{0}^{2\ell} \|\chi(s)\|^2 \, ds. \tag{3.81}
\]

Now, let us use the following straightforward fact (see, e.g., \[19, \text{Lemma 3.2}\]):

**Lemma 3.5.** Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{W} \) a Banach space such that \( \mathcal{H} \) is compactly embedded into \( \mathcal{W} \). Then, for any \( \gamma > 0 \) there exist a finite-dimensional orthonormal projector \( P : \mathcal{H} \to \mathcal{H} \) and a positive constant \( k \), both depending on \( \gamma \) and such that, for all \( z \in \mathcal{H} \),

\[
\|z\|^2_{\mathcal{W}} \leq \gamma \|z\|^2_{\mathcal{H}} + k \|Pz\|^2_{\mathcal{H}}. \tag{3.82}
\]

We apply here the Lemma to \( z = \chi \) with \( \mathcal{H} = V \) and \( \mathcal{W} = H \) and to \( z = \vartheta \) with \( \mathcal{H} = H \) and \( \mathcal{W} = V'^* = H^{-1}(\Omega) \). Then, introducing the time shift operator \( \mathcal{L} \), given by \( \mathcal{L} : v(\cdot) \mapsto v(\cdot + \ell) \) (where \( v \) is a generic function of time), and dividing (3.81) by \( \ell \), we obtain

\[
c_5 \|\mathcal{L}\vartheta\|^2_{L^2(0,\ell;H)} + 2\|\mathcal{L}\chi\|^2_{L^2(0,\ell;V)} \leq c_6 \ell^{-1} \left( \|\vartheta\|^2_{L^2(0,\ell;H)} + \|\chi\|^2_{L^2(0,\ell;V)} \right) + 2\gamma \lambda \left( \|\mathcal{L}\chi\|^2_{L^2(0,\ell;V)} + \|\chi\|^2_{L^2(0,\ell;V)} \right) + 2k\lambda \left( \|P\mathcal{L}\chi\|^2_{L^2(0,\ell;V)} + \|P\chi\|^2_{L^2(0,\ell;V)} \right), \tag{3.83}
\]

whence, recalling the notation \( w := (\vartheta, \chi) \) and rearranging,

\[
\min \left\{ c_5, 2 - 2\gamma \lambda \right\} \|\mathcal{L}\vartheta\|^2_{L^2(0,\ell;H \times V)} \leq \left( \frac{c_6}{\ell} + 2\gamma \lambda \right) \|w\|^2_{L^2(0,\ell;H \times V)} + \epsilon \left( \|Pw\|^2_{L^2(0,\ell;H \times V)} + \|P\mathcal{L}\vartheta\|^2_{L^2(0,\ell;H \times V)} \right), \tag{3.84}
\]

where \( \epsilon \) depends on \( \gamma, \lambda, \ell \) and all the other constants. Being not restrictive to assume \( c_5 \leq 1 \), it is clear that we can divide the above by \( c_5 \) and choose \( \ell \) large enough and \( \gamma \) small enough to obtain (clearly for a different value of \( \epsilon \))

\[
\|\mathcal{L}\vartheta\|^2_{L^2(0,\ell;H \times V)} \leq \frac{1}{\epsilon} \|w\|^2_{L^2(0,\ell;H \times V)} + \epsilon \left( \|Pw\|^2_{L^2(0,\ell;H \times V)} + \|P\mathcal{L}\vartheta\|^2_{L^2(0,\ell;H \times V)} \right). \tag{3.85}
\]

Consequently, the semigroup \( S(\cdot) \) enjoys the **generalized squeezing property** introduced in \[19, \text{Def. 3.1}\] on the set \( \mathcal{B}_3 \). Recalling \[20, \text{Lemma 2.2}\], we then infer that the discrete dynamical system on \( \mathcal{W}_\ell \) generated by \( \mathcal{L} \) admits an exponential attractor \( \mathcal{M}_{\text{discr}} \).

To conclude, we have to prove that, in fact, we can build the exponential attractor for the original semigroup \( S \). Here, however, we have to pass from the \( H \times V \) to the weaker \( V'^* \times H \)-topology (we recall that \( V'_0 = H^{-1}(\Omega) \)). Actually, we can observe that the following properties hold:

(M1) The evaluation map \( e : \mathcal{W}_\ell \to V'_0 \times H \) given by \( e : w \mapsto \vartheta \ell \) is Lipschitz continuous. To see this, it suffices to multiply (3.72) by \( t \) and integrate in \( dt \) between 0 and \( \ell \). Notice that, more precisely, Lipschitz continuity still holds as \( \mathcal{W}_\ell \) is endowed with the weaker \( L^2(0, \ell; V'_0 \times H) \)-norm;

(M2) The map \( S(t) \) is uniformly Lipschitz continuous on \([0, \ell]\) in the sense that

\[
\|S(t)w_1 - S(t)w_2\|_{H^{-1}(\Omega) \times H} \leq c(\ell) \|w_1 - w_2\|_{H^{-1}(\Omega) \times H}, \quad \forall w_1, w_2 \in \mathcal{B}_3 \text{ and } \forall t \in [0, \ell]. \tag{3.86}
\]

This is easily shown by integrating once more (3.72) over \((0, t)\) and using the Gronwall Lemma.
where the constants $C$ depend on the “radius” of $B_3$ in $W$ (i.e. on $\epsilon_3$, cf. (3.70)) and the latter estimate is a consequence of the regularity properties (2.10)–(2.12). Thus, $\ell$-trajectories in $W_\ell$ are uniformly Hölder continuous in time (notice that this even holds in the $H \times V$-norm). Then, properties (M1)–(M3) allow us to apply, e.g., [16, Thm. 2.6], which states that there exists a set $M$ which is compact and has finite fractal dimension in $V_0'$ and therefore compact in $\mathcal{X}$. To conclude the proof of Theorem 2.5 we have to show that $M$ attracts exponentially fast any bounded $B \subset \mathcal{X}$. Actually, this is true since $B_3$ is uniformly absorbing (so that it exponentially attracts $B$) and one can use the contractive estimate (3.72) and the transitivity property of exponential attraction proved in [9, Thm. 5.1]. Notice in particular that the constant $\kappa$ in (2.21) can be taken independent of $B$ since $\lambda$ on the right hand side of (3.72) is also independent of $B$ (cf. [9, (5.1)])).

**Remark 3.6.** We can see that exponential attraction still holds in the (stronger) $V_0' \times V$-norm. This requires to show (M1) and (M2) with respect to that topology. To do this, we can write the difference of (2.6) and test it by $\chi_t = \chi_{1,t} - \chi_{2,t}$. Standard manipulations then lead to

$$
\frac{d}{dt}\|\chi\|_V^2 + \|\chi_t\|^2 \leq c_7 \left(\|\chi\|^2 + \|\vartheta\|^2\right),
$$

(3.88)

where $\vartheta = \vartheta_1 - \vartheta_2$ and $c_7$ depends on the “$W$-radius” of $B_3$. Then, multiplying (3.88) by $c_5/2c_7$ ($c_5$ being as in (3.77)) and summing to (3.72), we get a contractive estimate in the desired topology. On the contrary, at least in the three-dimensional case, it seems more difficult to obtain an $H$-contraction estimate for $\vartheta$ (i.e. to pass to the $H \times V$-norm). Actually one could test the difference of (2.5) by $\vartheta$. However, even knowing the boundedness (3.70), getting a control of the term involving the Laplacian seems out of reach.

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