A class of entangling gates for trapped atomic ions is studied and the use of numeric optimization techniques to create a wide range of fast, error-robust gate constructions is demonstrated. A numeric optimization framework is introduced targeting maximally- and partially-entangling operations on ion pairs, multi-ion registers, multi-ion subsets of large registers, and parallel operations within a single register. Ions are assumed to be individually addressed, permitting optimization over amplitude- and phase-modulated controls. Calculations and simulations demonstrate that the inclusion of modulation of the difference phase for the bichromatic drive used in the Mølmer–Sørensen gate permits approximately time-optimal control across a range of gate configurations, and when suitably combined with analytic constraints can also provide robustness against key experimental sources of error. The impact of experimental constraints such as bounds on coupling rates or modulation band-limits on achievable performance is further demonstrated. Using a custom optimization engine based on TensorFlow, for optimizations on ion registers up to 20 ions, time-to-solution of order tens of minutes using a local-instance laptop is also demonstrated, allowing computational access to system-scales relevant to near-term trapped-ion devices.

1. Introduction

Quantum computing requires a universal set of high-fidelity gates that are fast, robust and scalable. In trapped ion systems, entangling gates are mediated by shared motional modes that are coupled to the qubit states through atom-light interactions. High-fidelity entangling gates many orders of magnitude faster than decoherence timescales have been demonstrated, and research continues on faster gate times while maintaining high fidelities. The dynamics for Mølmer–Sørensen-type operations is derived for the regime where gate timescales are slow compared to the trapping period; however, errors arising from approaching this timescale can be mitigated by careful control. Two-qubit Mølmer–Sørensen gates have been implemented with infidelity on the order of $10^{-3}$. Recently, a range of control protocols has been introduced to expand the functionality of these gates via modulation of the laser fields mediating the spin-motional interaction. For instance, Mølmer–Sørensen gates have demonstrated tremendous flexibility, permitting parallel couplings within large registers and using overlapping pairs via control modulation. Moreover, the addition of control permits the introduction of noise and drift-robustness even in complex multi-ion settings. The general theory for the controlled dynamics of Mølmer–Sørensen-type operations is well established; however, special cases are typically explored in theory and experiment to simplify implementation or to make dynamical control more tractable. The most common dynamical control method employs modulation of the amplitude (with fixed phase) of the control drives. This restriction to a real drive permits deeper analytical treatment of the gate conditions and reduces the degrees of freedom required for numerical optimization. Accordingly, amplitude-modulated gates have been successfully implemented in a range of experiments including the execution of five parallel pairwise interactions within an 11-ion chain, or in achieving a many-body 12-of-12 qubit gate. Executing parallel gates of this nature is essential for algorithmic scalability when employing large or even mesoscale ion registers. On the other hand, complex drives have been demonstrated experimentally using phase-modulation or laser-detuning modulation. However, these have generally been limited to demonstrations with smaller registers, in part due to the challenge of efficient gate construction within a large control space.

In this paper, we demonstrate computational access to a general control framework leveraging modulation of complex control drives, and apply this framework to efficiently achieve a range of optimized error-robust gates in large ion registers. Framing the gate conditions in this way allows numeric optimization of amplitude- and phase-modulated controls optimized for
each individually-addressed ion within a register. This accommodates many-qubit and parallel operations within a single register, where the target relative phases \(\psi_{jk}\) between each pair of ions \(j\) and \(k\) can be freely specified while ensuring qubit-motional decoupling. We first present the theoretical framework for Mølmer–Sørensen-type operations, including introduction of an operational fidelity measure and error-robustness conditions. Next, we pose the numeric optimization problem subject to a variety of hardware-motivated constraints, before presenting results derived from a custom TensorFlow-based optimization package. We demonstrate a range of high-fidelity, error-robust, and scalable control solutions for parallel many-body operations within ion-chains up to 20 ions. Comparative analysis reveals that the specific inclusion of a complex drive provides access to otherwise unachievable entangling-gate fidelities and reduced gate-durations for a broad range of laser detunings. Finally, we study the scaling of computational resources and control parameters required to obtain solutions for different-length ion chains.

2. Operation Dynamics and Measures

Mølmer–Sørensen-type multi-qubit operations employ bichromatic lasers which produce beatnotes detuned above and below the qubit transition frequency in order to achieve a generalized pairwise coupling, as shown in Figure 1b for the specific example case of \(^{171}\)Yb\(^+\) ions. The laser detuning established by the Raman beatnote is kept close to the excitation frequencies of the motional modes used to couple the internal qubit states (Figure 1b). By Mølmer–Sørensen-type operations we denote generalized forms of this operation that couple arbitrary pairs of qubits according to the unitary evolution operator

\[
U_t = e^{i\sum_{p=1}^{N} \omega_p \hat{a}_p^{\dagger} \hat{a}_p \Delta t} e^{i\sum_{j=1}^{N} \frac{1}{2} \sigma_{x,j} \Delta t}
\]

Here \(j\) and \(k\) are indices over ions, \(\psi_{jk}\) is the target pairwise entangling phase and \(\sigma_{x,j}\) is the Pauli \(X\)-operator acting on ion \(j\). A maximally entangling pairwise gate would have entangling phase \(\psi_{jk} = \pi/4\) in this formulation. In the following section, we present a Hamiltonian-level description of the interactions and then frame the implementation subject to user-configurable constraints on the target operation.

2.1. Mølmer–Sørensen Dynamics

We model the control problem for this gate, beginning with a conventional Hamiltonian description of the coupled dynamics of the internal and motional degrees of freedom for trapped ions, written

\[
H_0 = \sum_{p=1}^{N} \hbar \nu_p \left( a_p^{\dagger} a_p + \frac{1}{2} \right) + \sum_{j=1}^{N} \frac{\hbar \omega_{0j}}{2} \sigma_{x,j}
\]

where motional mode \(p\) has frequency \(\nu_p\), and the \(N\) trapped ions have an internal qubit transition at frequency \(\omega_{0j}\). We denote Pauli \(k\)-operators for ion \(j\) as \(\sigma_{x,k,j}\).

In the rotating frame with respect to \(H_0\), the interaction Hamiltonian for Mølmer–Sørensen-type operations is given by

\[
H_I(t) = \hbar \sum_{j=1}^{N} \sum_{p=1}^{M} \left( -\beta_j^p(t) a_p^{\dagger} + \beta_j^p(t) a_p \right)
\]

The term coupling ion \(j\) to motional mode \(p\) is given by

\[
\beta_j^p(t) = \eta_p^j \gamma(t) \frac{1}{2} e^{i\phi(t)}
\]

where \(\eta_p^j \equiv \eta_p b_{j,p}^{(0)}\) with \(\eta_p\) the Lamb–Dicke parameter and ion-mode participation eigenvectors \(b_{j,p}^{(0)}\). The relative detuning from the \(p\)th mode is \(\delta_p(t) = \nu_p - \Omega(t)\) with the laser frequency detuned by \(\Omega(t)\) from the qubit transition \(\omega_{0j}\). We represent the complex drive \(\gamma(t) = \Omega(t) e^{i\phi(t)}\), with Rabi frequency \(\Omega(t)\) and phase \(\phi(t)\).

The interaction Hamiltonian is valid when several approximations hold. First, it is necessary that phase-space (or equivalently ion) displacements remain small, such that

\[
\langle (kx_j)^2 \rangle_{x_{max}} \ll 1 \quad \forall j, t
\]
where \( x_t \) is the displacement operator for ion \( j \), \( k \) is the addressing radiation wavevector, \( p_{\text{ion}} \) is the motional state of the ions, and where the angle brackets \( \langle \cdot \rangle \) denote an expectation value in the state \( \rho \). Note that \( k_x = \sum \eta^a_j (a_p + a_p^\dagger) \). Second, the protocol involves a pair (or pairs) of laser frequencies that we denote with subscripts \( a \) and \( b \): the laser pair has opposite detunings \( \delta_{ij}(t) = \delta(t) \), \( \delta_{ik}(t) = -\delta(t) \), and phases \( \phi_{ij}(t) = \phi(t) \), \( \phi_{ik}(t) = -\phi(t) + \pi \). Finally, detunings \( \delta(t) \) should be close to \( \nu_p \), such that a rotating wave approximation eliminates carrier transitions.

The dynamical equations may be generalized to accommodate individual drives for different ions, moving beyond the shared expression \( \gamma(t) \) in Equation (4). For ion-dependent complex drives, we transform \( \gamma(t) \) to \( \gamma_j(t) \) for the \( j \)th ion, with corresponding transformations \( \phi_{ij}(t) \to \phi_{ij}(t) \) and \( \Omega_j(t) \to \Omega_j(t) \).

As highlighted in Figure 1b, this ion-specific complex drive, induced by the Raman lasers, represents the key control knob within our possession. This is the parameter over which we will perform optimization as outlined in Section 3. In this manuscript, we fix the laser detuning in order to facilitate the numeric optimization described in Section 3, though in principle this parameter may be transformed in the same way.

The unitary operator resulting from Equation (3) can be written as time-ordered infinitesimal (state-dependent) displacement operators, from which (up to global-phase terms) we obtain

\[
U(t) = \exp \left( \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_{j,k} B_j(t) + i \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \phi_{jk}(t) + \phi_{kj}(t) \right) \sigma_{j,k} \sigma_{j,k} \right)
\]

The protocol involves a pair (or pairs) of laser frequencies that we denote with subscripts \( a \) and \( b \): the laser pair has opposite detunings \( \delta_{ij}(t) = \delta(t) \), \( \delta_{ik}(t) = -\delta(t) \), and phases \( \phi_{ij}(t) = \phi(t) \), \( \phi_{ik}(t) = -\phi(t) + \pi \). Finally, detunings \( \delta(t) \) should be close to \( \nu_p \), such that a rotating wave approximation eliminates carrier transitions.

Using this basis, we write the equations for entangling phase accumulation (Equation (8)) and motional displacement (Equation (9)) as

\[
\eta^a_j \alpha^s_j(t) = \eta^b_j \int_0^t dt' \frac{\gamma_j(t')}{2} e^{\phi_{jk} t'}
\]

for a gate of duration \( r \). Next, we require elimination of qubit-motional entanglement at the completion of the operation. The residual qubit-motional entanglement is eliminated by ensuring that

\[
\alpha^a_j(r) = 0 \quad \forall j, p
\]

2.2 Target Operations and Fidelity Metrics

Our specific target is the achievement of high-fidelity operations under arbitrary couplings between ions within an \( N \)-ion register. These couplings can be achieved for an individual pair, for multiple pairs in parallel, or as many-body (\( M \)-of-\( N \)) operations, respectively, as depicted in Figure 1a. This introduces two control targets in our problem. First, we desire arbitrary and specifiable relative phases between ions \( j \) and \( k \). Referring to the target unitary in Equation (1), we thus require that the acquired phases satisfy

\[
\phi_{jk}(t) + \phi_{kj}(t) = \psi_{jk}
\]

where \( \psi_{jk} \) is the phase offset added between the \( j \)th and \( k \)th ion. This introduces two control targets in our problem. First, we desire arbitrary and specifiable relative phases between ions \( j \) and \( k \). Referring to the target unitary in Equation (1), we thus require that the acquired phases satisfy

\[
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Second, the protocol involves a pair (or pairs) of laser frequencies that we denote with subscripts \( a \) and \( b \): the laser pair has opposite detunings \( \delta_{ij}(t) = \delta(t) \), \( \delta_{ik}(t) = -\delta(t) \), and phases \( \phi_{ij}(t) = \phi(t) \), \( \phi_{ik}(t) = -\phi(t) + \pi \). Finally, detunings \( \delta(t) \) should be close to \( \nu_p \), such that a rotating wave approximation eliminates carrier transitions.

Using this basis, we rewrite the equations for entangling phase accumulation (Equation (8)) and motional displacement (Equation (9)) as

\[
\phi_{jk}(t) = \psi_{jk} = \sum_{k=1}^{N} \sum_{m=1}^{N} \chi_{kl}(t) \Omega_{kl} e^{i\phi_{jk}}
\]

where segment \( k \) is defined over the interval \( A_k = [t_k, t_{k+1}] \), and \( \chi_{kl} \) is the indicator function that takes a value of 1 for \( t \in A_k \) and 0 otherwise. The operation begins at \( t_k = 0 \) and finishes at \( t_{k+1} = r \). Note that one may follow the same procedure using different basis functions such that the control degrees of freedom are time-independent.

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\[
\phi_{jk}(t) = \psi_{jk} = \sum_{k=1}^{N} \sum_{m=1}^{N} \chi_{kl}(t) \Omega_{kl} e^{i\phi_{jk}}
\]
where $n$ and $m$ are ion indices, $\mathbf{u}_k$ is the vector of controls such that element $k$ is the $k$th piecewise segment value $\gamma_{n,k}$, and entry $p$ of the vector $\alpha_n$ is $a_{\alpha}^p$. The matrices $M$ and $P_{mn}$ have elements given by

$$M_{p,k} = \frac{1}{2} \int_{\Delta_i} dt \ e^{i\gamma_{p,k}}$$

(18)

$$p_{k,l}^{mn} = \sum_{p=1}^{M} \eta_{n,p}^{m,n} \int_{\Delta_i} dt \ e^{i\gamma_{p,k-l}} \int_{A_i}^{A_f} dt_2 \ e^{-i\gamma_{p,k-l}}$$

(19)

respectively. Here $p$ is an index over motional modes, and $k$ and $l$ are indices over segments.

To obtain control solutions, we apply a custom gradient-based optimization engine\[26\] built on TensorFlow to minimize the operation error. To this end we minimize the cost function $C$ defined as

$$C = \sum_{k,j} (\epsilon_{\alpha,k})^2 + \sum_{j=1}^{N} \sum_{k=1}^{M} |a_{\alpha}^p(\tau)|^2$$

(20)

The terms included, $(\epsilon_{\alpha,k})^2$ and $|a_{\alpha}^p(\tau)|^2$, are proportional to the lowest-order infidelity contributions, for each mode $p$ and ion-pair $(j,k)$. Minimizing this simpler functional form provides better performance than using the full functional form of the infidelity. Note that there is freedom in the choice of cost function, for instance to specify coefficients that match the lowest-order infidelity terms exactly, which similarly provides fast optimization and high fidelities. Using Equations (16) and (17), we obtain quadratic and linear expressions for $\epsilon_{\alpha,k}$ and $a_{\alpha}^p(\tau)$ in terms of our control degrees of freedom, respectively.

Given our control and optimization framework, we may impose additional physical constraints on the free variables as part of the optimization problem. This includes bounding the rate-of-change of the drive phase and amplitude (Figure 3, and corresponding to band limits in hardware), fixing the phase or amplitude (Figure 4), sharing the same drive parameters between arbitrary ions in the chain (Figure 4), or incorporating generic linear-time-invariant filters on control transmission.\[26,29\]

### 3.2. Integration of Error-Robustness

We can analyze gate error-robustness, and reduce the error-susceptibility of optimized controls, by modeling the impact of common noise terms on the dynamic evolution of the system. Here, we focus on several different error processes that are commonly encountered in laboratory environments, ranging from trap instability and laser frequency drift to systematic timing errors.

We begin with dephasing errors, which can arise from imperfect calibration or drift in the motional mode frequencies $\nu_p$ as trapping potentials frequently vary in time.\[9,16\] This mode frequency error can be written as $\nu_p \to \nu_p + \epsilon_p$, which becomes a shift in the relative detuning $\delta_p \to \delta_p + \epsilon_p$. This type of error impacts the mode closure:

$$\bar{a}_{\alpha}^p(\tau) = \int_{0}^{t} dt \frac{\gamma(\nu_{p,x})}{2} e^{i\phi_{p,x} t}$$

(21)

In order to compensate the effect of quasi-static noise on mode trajectory closure to first order, we require

$$0 = \frac{\bar{a}_{\alpha}^p(\tau)}{de_p} \bigg|_{\epsilon_p=0} = i \int_{0}^{t} dt \frac{\gamma(\nu_{p,x})}{2} e^{i\phi_{p,x} t}$$

(22)

$$= ir_{\alpha}^p(\tau) - i \int_{0}^{t} dt \frac{\gamma(\nu_{p,x})}{2} e^{i\phi_{p,x} t}$$

(23)

The term proportional to $r_{\alpha}^p(\tau)$ is set to zero in the usual motional conditions for an operation, and the integral over $r_{\alpha}^p(t)$ can be set to zero as an additional robustness condition, as in ref. [9]. Since $r_{\alpha}^p(t)$ is proportional to the displacement of ion $j$ in mode $p$ at time $t$, this condition is equivalent to setting the center of mass of ion $j$'s contribution to mode $p$'s phase space trajectory to zero. When the center of mass is set to zero for each ion's contributions to phase space trajectories, the residual motion condition (trajectory closure) can be satisfied by enforcing symmetry in the controls as described in ref. [21]. This work found that robustness to both quasi-static and zero-mean fluctuating dephasing noise processes can be obtained by setting the center of mass of each motional mode's phase space trajectory to zero.

Dephasing errors can also impact the acquired entangling phase, as determined by Equation (8). The relative detuning error that arises in $\rho_{\alpha}^p$ conveniently cancels; however, the entangling phase contains $\eta_{n,p}$, which is rescaled by the error $\nu_p \to \nu_p + \epsilon_p$. This produces a first-order entangling-phase error with the scaling factor $\epsilon_p / \nu_p$, which is dominated by motional errors in our calculations. We thus focus on robustness to residual state-motional entanglement induced by dephasing errors.

Dephasing noise can also arise from errors in the laser-pair detunings such that $\delta_p \neq \delta_{p0}$. The gate dynamics can be rederived with this detuning asymmetry, as shown in the Supporting Information, where we find that the robustness conditions derived above also provide robustness to relative detuning noise.

We next consider systematic timing errors such that the control pulses are scaled by a uniform factor $(1 + \epsilon_p)$. The impact of this on the mode closure condition is obtained by evaluating

$$\bar{a}_{\alpha}^p(t) = (1 + \epsilon_p) \int_{0}^{t} dt \frac{\gamma(\nu_{p,x})}{2} e^{i\phi_{p,x} t}$$

(24)

Transforming $t \to t' \equiv t/(1 + \epsilon_p)$ we then obtain

$$\bar{a}_{\alpha}^p(t') = \int_{0}^{t'} dt' (1 + \epsilon_p) \frac{\gamma(\nu_{p,x})}{2} e^{i\phi_{p,x} t'}$$

(25)

showing the impact on mode closure is proportional to Equation (21) with a dephasing shift $\epsilon_p = \epsilon_p \delta_p$. This equivalence means that a control scheme satisfying the dephasing robustness conditions to a given order is also robust to timing errors to that same order.
To apply error-robust optimization with respect to these noise sources, we require that the residual phase space displacements are zero as in Equation (11), and that the integral (or center of mass) of each phase-space trajectory is zero. The center of mass conditions can be written in a linear form with respect to the controls as
\[
0 = Ru_n \quad \forall n \tag{26}
\]
\[
R_{p,k} = \int_0^T dt_1 \int_{\Delta(t_1,0)}^{\Delta(t_1,T)} dt_1 e^{i\phi_1} \tag{27}
\]
where \( n \) is an index over ions, and the matrix elements \( R_{p,k} \) of \( R \) are defined in the second equation above. Here \( p \) is an index over motional modes, and \( k \) is an index over segments. If these conditions are satisfied, the closure of the phase space trajectories (satisfying the residual displacement conditions) can also be enforced by imposing symmetry in the drives across the temporal midpoint of the gate operation.\(^{21}\) For piecewise-constant drives with variable amplitude and phase, the symmetry can take the form
\[
\Delta \phi_{j,n+1} = \Delta \phi_{j,n-(n+1)}
\]
\[
\Omega_{S,n} = \Omega_{S,n-n} \tag{28}
\]
where \( \Delta \phi_{j,n} \) is \( (\phi_{j,n} - \phi_{j,n-1}) \), \( \Omega_{S,n} \) and \( \phi_{j,n} \) are the fixed amplitude and phase for the \( j \)th ion over the \( n \)th drive segment, and \( S \) is the number of segments in the drive. We note that the number of segments can be set independently for different ion-specific drives, as each drive modulation pattern is reflected individually to satisfy the symmetry conditions. We thus achieve error-robust solutions using a combination of symmetry and numerical optimization approaches.

We have derived and implemented robustness conditions particularly for laser-detuning or equivalent noise sources; one may alternatively consider laser amplitude fluctuations. Previous work\(^{21}\) has demonstrated that in the ensemble average, zero-mean temporally fluctuating processes may be suppressed by the same prescription, but sensitivity to quasi-static errors in individual gates remains. This is evident when considering Equation (8) for the entangling phase, which shows that quasi-static errors of the form \( \Omega(t) \rightarrow \Omega(t) \) directly induce the acquired entangling-phase rescaling \( \phi_{jk}(t) \rightarrow s^2 \phi_{jk}(t) \). Such entangling-phase errors dominate fidelity contributions arising from residual motional entanglement and remain the subject of future work.

4. Optimization Results and Performance Analysis

An example optimization highlighting various capabilities of this framework is presented in Figure 2. Here, we optimize two parallel asymmetric gates on a chain of 20 ions, as shown in the schematic in Figure 2a, which achieve infidelity \( I = 1.8 \times 10^{-7} \). The first gate (“Gate 1”) is a maximally-entangling two-qubit gate between ions 0 and 3 in the chain (indexing from 0). The second gate (“Gate 2”) is a four-qubit gate on ions 2, 5, 6, and 10 that prepares user-defined relative phases between different sub-pairs in steps of \( \pi/10 \). These choices of relative phases are configurable and were chosen arbitrarily to highlight the freedom inherent in the optimization. Figure 2b displays the optimized drive for ion 0; the control for each ion varies rapidly between discretized time segments, exploiting the full parameter space afforded to the optimizer in achieving the target performance. Controls for other ions are similar in overall appearance but vary in their detailed prescription.

The performance of the gate can be explored visually through simulation of both the phase-space motional dynamics (Figure 2c) and entangling phase for different ion pairs as a function of time during the gate (Figure 2d–f). As expected, for two transverse motional modes illustrated here, both make a complex excursion in phase space before returning to the origin at the end of the gate, indicating efficient qubit-motional decoupling. Similarly, we observe that the pairwise entangling dynamics for both Gates 1 and 2 achieve target phases for each pair, and qubit pairs
Fidelity is 3 \( \Delta \Omega \). The bound-rate of change for both the modulus and phase:

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Date hardware constraints the optimization may include effects of dephasing noise for “Robust” and “Standard” (non-robust) optimized controls. We illustrate these capabilities for a 2-of-2 ion maximally-entangling gate with shared addressing in **Figure 3**. A key consideration in implementation is the response time of either RF signal generators or optical modulators employed in gate implementation, the experimental impact of which was treated in ref. [21]. In order to accommodate hardware constraints the optimization may include effective band limits implemented through a number of filtering techniques.

In **Figure 3a**, we illustrate one example hardware-compatible constraint based on limiting the time-derivative of the modulation profile, which we term a bound-slew-rate control. [26] The bound-slew-rate control waveform achieves an infidelity \( 3.7 \times 10^{-9} \), despite the substantial differences in allowable waveform relative to the unconstrained solution presented in Figure 3c. Phase space trajectories for the respective controls are displayed in Figure 3b and Figure 3d, and reflect the limit on allowable modulation bandwidth through smoothing of the trajectories in Figure 3b. A variety of other smoothing filters could be considered, and are compatible with the optimization engine as described in ref. [29], including arbitrary linear time-invariant filters which capture measured modulator responses.

We demonstrate the error-robustness of these and two additional 2-of-2 gate optimizations using conventional analytic techniques in robust control. [26] First, for both the bound-slew-rate and unbounded optimizations we calculate the filter functions for gate variants designed to either simply enact the target gate or include robustness to detuning noise. The filter function serves as a proxy measure for noise admittance as a function of noise frequency, and is experimentally validated for single-qubit gates [30] and multi-qubit Mølmer–Sørensen gates [12]. A robust control will suppress noise at low frequencies, by contrast the standard controls exhibit broadband noise susceptibility up to a frequency commensurate with the inverse gate time.

Similarly, we evaluate the robustness of the gates to quasi-static detuning errors (Figure 3f) via calculation of gate infidelity in the presence of fixed detuning offsets. Recall that the dynamics depends on the relative detunings \( \delta = \nu_i - \delta \), such that a laser 

detuning offset \( \epsilon \) is equivalent to mode frequency offsets of the same magnitude (up to a small correction in the Lamb–Dicke parameter). In Figure 3f we see that as a function of offsets from ideal laser settings (no detuning), gate infidelity will increase at varying rates depending on the specifics of gate construction. The range of laser detuning over which infidelity remains low serves as an effective measure of error-robustness. The standard control solutions (orange) both exhibit a narrow range of detunings allowing high-fidelity implementation. By contrast, the robust solutions exhibit a broad domain of flat infidelity around zero detuning error, indicating that small drifts will not substantially degrade operational fidelity. These results hold with or without bounds on the slew-rates for the controls. The effective reduction of detuning-induced infidelity using our robust methodology is also displayed for 2-of-5 qubit gates in **Figure 4a**, for different control schemes. The robust controls in Figure 4a are less susceptible to noise than their counterparts in Figure 3f due to a faster (50 \( \mu s \)) gate duration.

The demonstrations above have shown optimized controls utilizing complex drives, where both the amplitude and phase are modulated in time with the aim of achieving low gate infidelities. We now highlight the applicability of this methodology in achieving high-fidelity, short-time gates.

**Figure 3.** Controls, dynamics, and robustness under different constraints for a 2-of-2 qubit maximally-entangling gate. The same 512-segment drive addresses both ions, with \( \Omega_{\text{max}}/2\pi = 100 \text{ kHz} \). a) Robust drive with a bound rate of change for both the modulus and phase: \( \Delta \Omega / 2\pi \leq 10 \text{ kHz} \), \( \Delta \phi \leq \pi / 8 \) between segments with duration \( \approx 0.6 \mu s \). The operational infidelity is \( 3.7 \times 10^{-9} \). b) Phase space trajectories for the center-of-mass mode of each axis under the robust, physically constrained drive in (a). A cross marks the endpoint of each trajectory at the origin. c) Unconstrained optimized drive for the operation, with operational infidelity \( 1.5 \times 10^{-12} \). d) Phase space trajectories for the center-of-mass mode of each axis under the unconstrained drive in (c). e) Filter functions displaying susceptibility to dephasing noise, as described in the main text. f) Quasi-static scans of dephasing noise for “Robust” and “Standard” (non-robust) optimized control solutions, with and without bound slew rates.
Control solution analysis for a 2-of-5 qubit maximally-entangling gate with a shared (128-) 64-segment (robust) drive addressing ions 0 and 1. a) Quasi-static scans of dephasing noise for instances of different control schemes: standard and robust amplitude-modulated (AM), phase-modulated (PM), amplitude- and phase-modulated (AM + PM) schemes. Robust gate infidelities overlap near zero for the displayed detunings (the dashed lines are superimposed). b) Infidelity with maximum Rabi rate \( \Omega_{\text{max}} \); this upper bound on \( \Omega \) is applied to each control optimization. The curves (error bars) display the mean (standard deviation) over ten runs of the optimizer, where each run involves five optimization instances. For context, a primitive square pulse providing maximally-entangled phase for 2-of-2 qubits (and otherwise matching parameters) requires amplitude \( \Omega/2\pi = 0.824 \text{ MHz} \); however, this primitive pulse results in residual motional entanglement. In (a) and (b) the detuning and gate duration are fixed at \( \delta = 1.365 \text{ MHz} \) and \( \tau = 50 \mu s \), respectively. c–f) Optimized control infidelities for scans over laser detuning \( \delta \) and gate time \( \tau \). Different control configurations are displayed in each subfigure. In (a) and (c–f), \( \Omega_{\text{max}}/2\pi = 1 \text{ MHz} \).

High-fidelity control solutions can be achieved for different gate time and laser detuning domains depending on the degrees of freedom in the control. As an example these domains are displayed in Figure 4c–f for a maximally-entangling 2-of-5 qubit gate, using different modulation protocols: amplitude-modulated (AM), phase-modulated (PM), amplitude- and phase-modulated (AM + PM) and robust phase-modulated controls (Robust PM). Here, dark regions represent high-fidelity gate implementations that have been found by the optimizer while light regions show gate implementations exhibiting larger errors. As expected, as gate durations decrease it becomes more challenging for the optimizer to find high-fidelity solutions, and below a certain threshold no high-fidelity gates may be achieved for a fixed maximum Rabi rate. In our calculations, we find that both the high-fidelity domain and its boundary for AM controls routinely exhibit substantial structure yielding an approximate minimum-gate-duration threshold nearly 50% larger than gate constructions incorporating phase modulation. In the latter cases the optimal gate duration (for a given target infidelity) is reduced but also appears to depend only weakly on the choice of detuning. It is interesting that AM + PM controls have a slightly reduced low-infidelity domain compared with the PM case despite being a super-set (any valid PM control is also a valid AM + PM control); this may simply be a manifestation of an underconstrained optimization problem exhibiting local minima. Finally, we note that despite the reflection of controls (using twice as many segments) required to ensure robustness we observe only a marginal change in the threshold gate-duration before achieving high-fidelity gates when incorporating a robustness constraint.

Another practical consideration for gate implementation is the drive power requirement of a given scheme. In Figure 4b, we display the achievable infidelity for a 2-of-5 qubit gate and different modulation schemes as a function of the permitted maximum drive power. Again, we observe that the solution incorporating only AM is most restrictive; the optimized controls require higher drive power to reach infidelity below any given threshold. The addition of phase modulation reduces required drive power by approximately 2x, whether used on its own or in combination with amplitude modulation. In all cases we have considered, the addition of robustness constraints increases the maximum drive-power requirements (10–20%) and limits the best achievable infidelity due to the segment number in the controls. In the presence of noise, however, the lower susceptibility of the robust solution can quickly outweigh this advantage of the ideal Standard operations (as displayed in Figure 4a). An additional motivation for using lower-power schemes is that Rabi frequencies reaching \( \Omega/2\pi \approx 1 \text{ MHz} \) can lead to significant infidelity contributions induced by carrier transitions. If necessary, these infidelity terms can be minimized by introducing cost terms similar to the motional and relative phase constraints. Constraints that are linear and quadratic in the control can represent the carrier transition terms to second order from the Magnus expansion of the unitary evolution; this approach is used for carrier suppression in ref. [9].

Finally, we explore the performance-scaling of the optimization framework we employ with the number of qubits, considering both time-to-solution and minimum achievable infidelity. In this scaling analysis, we perform optimizations using chains up to 20 ions in length given state-of-the-art experimental capabilities,[31] and execute code using like-for-like local-instance hardware (a standard consumer grade laptop). These two metrics are presented in Figure 5 for the optimization of two parallel pairwise gates implemented within ion chains of different lengths. We observe that a single gate-optimization calculation may be completed via local-instance code execution in \( \lesssim 50 \text{ min} \) for the longest 20-ion chain considered here. Parallelization using cloud-compute infrastructure has been shown to reduce the absolute time-to-solution by a variable factor depending on the structure of the optimization, but reported up to 10x in previous tests[26] when leveraging GPU support for complex tasks. Calculations require \( \lesssim 10 \text{ min} \) in total runtime up to \( \approx 14 \text{ ions} \) for the gate configurations treated here. As expected, the addition of symmetry constraints in robust optimizations adds only a small overhead for chains \( \lesssim 12 \text{ ions} \) in length, with an approximate doubling of runtime for longer chains. We find that within the range of parameters considered runtimes also scale approximately linearly with segment number. In all cases (except for the 19- and
2133MHzLPDDR3). Pro (2019) using CPU (Processor: 1.4 GHz Intel Core i5; Memory: 8 GB (see Supporting Information). Calculations were performed on a MacBook Pro (2019) using CPU (Processor: 1.4 GHz Intel Core i5; Memory: 8 GB 2133 MHz LPDDR3).

Figure 5. Scaling of a) calculation time and b) infidelity with ion chain length for two simultaneously maximally entangling gates on ions (0,1) and (2,3) in the chain, using the AM+PM control scheme. Individual drives were applied to each ion for an operation duration of 300 µs, with Ω_{max}/2π = 100 kHz. A single optimization involves selecting the best performing solution from five realizations of the computation. We then repeat this process ten times and average these results for each data point, while error bars give the standard deviation. The data points for 16 ions in the chain have been omitted as the chosen trap frequencies give a geometric anomaly for the calculation time and fidelity with this ion number (see Supporting Information). Calculations were performed on a MacBook Pro (2019) using CPU (Processor: 1.4 GHz Intel Core i5; Memory: 8 GB 2133 MHz LPDDR3).

20-ion chains) achieved infidelities are \( \approx 10^{-7} \). For the longest chains, the availability of 64 unconstrained drive segments (128 for the robust case) over which the optimization is performed appears insufficient to obtain a baseline infidelity equivalent to that achieved for shorter ion chains.

5. Conclusions and Outlook

This work addressed the problem of achieving reconfigurable, high-fidelity multi-qubit gates in large trapped-ion registers. By framing the problem of obtaining target quantum gates using complex drives, and exploiting computationally efficient numeric optimization, we obtain the most flexible control solutions reported in the literature to the best of our knowledge. Specifically, the control solutions we demonstrate employ both phase and amplitude modulation on the mediating laser field implementing a Mølmer–Sørensen interaction. Numerically optimized solutions enact high-fidelity multi-body and parallel operations by development of a cost function which includes both motional decoupling and achievement of target pairwise entangling phases. We have realized solutions on chains of up to 20 ions in this work, demonstrated the ability to engineer robustness to common sources of laser noise (and equivalently trap-frequency drifts or miscalibrations), and incorporated common constraints on modulation hardware into the optimization procedure. These highly configurable operations exhibit faster gate times (or lower power requirements) than controls with only amplitude modulation, and time-to-solution remains manageable for standard computational resources available in consumer laptops.

Implementation of quantum logic gates in large trapped-ion registers requires that the gate constructions be fast, flexible, high-fidelity, and scalable in order to leverage the benefits of trapped-ion hardware. This work has contributed to each of these desiderata, while maintaining a focus on addressing practical implementation challenges. We are excited to extend this framework to incorporate new forms of error robustness including nonlinearities in modulator response, laser-amplitude fluctuations, and laser crosstalk. The software-configurable nature of interactions in trapped-ion quantum computers \([12]\) makes them an ideal target for advanced numeric optimization techniques and we look forward to continuing to advance the utility of quantum optimal control techniques in this hardware platform.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

optimized quantum gates, quantum computing, quantum control, trapped ions

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