ON THE FLOW MAP OF THE BENJAMIN-ONO EQUATION ON THE TORUS

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Abstract. We prove that for any $0 < s < 1/2$, the Benjamin–Ono equation on the torus is globally in time $C^0$–well-posed on the Sobolev space $H^{-s}(T, \mathbb{R})$, in the sense that the solution map, which is known to be defined for smooth data, continuously extends to $H^{-s}(T, \mathbb{R})$. The solution map does not extend continuously to $H^{-s}(T, \mathbb{R})$ with $s > 1/2$. Hence the critical Sobolev exponent $s_c = -1/2$ of the Benjamin–Ono equation is the threshold for well-posedness on the torus. The obtained solutions are almost periodic in time. Furthermore, we prove that the traveling wave solutions of the Benjamin–Ono equation on the torus are orbitally stable in $H^{-s}(T, \mathbb{R})$ for any $0 \leq s < 1/2$.

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Date: December 5, 2019.

2010 Mathematics Subject Classification. 37K15 primary, 47B35 secondary.
Key words and phrases. Benjamin–Ono equation, well-posedness, critical Sobolev exponent, almost periodicity of solutions, orbital stability of traveling waves.

We would like to warmly thank J.C. Saut for very valuable discussions and for making us aware of many references, in particular [4]. We also thank T. Oh for bringing reference [5] to our attention. T.K. partially supported by the Swiss National Science Foundation. P.T. partially supported by the Simons Foundation, Award #526907.
1. Introduction

In this paper we consider the Benjamin-Ono (BO) equation on the torus,

\begin{equation}
\partial_t v = H \partial_x^2 v - \partial_x (v^2), \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R},
\end{equation}

where \( v \equiv v(t, x) \) is real valued and \( H \) denotes the Hilbert transform, defined for \( f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx} \), \( \hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \), by

\[
Hf(x) := \sum_{n \in \mathbb{Z}} -i \text{sign}(n) \hat{f}(n) e^{inx}
\]

with \( \text{sign}(\pm n) := \pm 1 \) for any \( n \geq 1 \), whereas \( \text{sign}(0) := 0 \). This pseudo-differential equation (ΨDE) in one space dimension has been introduced by Benjamin \[6\] and Ono \[26\] to model long, uni-directional internal gravity waves in a two-layer fluid. It has been extensively studied, both on the real line \( \mathbb{R} \) and on the torus \( \mathbb{T} \). For an excellent survey, including the derivation of (1), we refer to the recent article by Saut \[28\].

Our aim is to study low regularity solutions of the BO equation on \( \mathbb{T} \). To state our results, we first need to review some classical results on the well-posedness problem of (1). Based on work of Saut \[27\], Abelouhab, Bona, Felland, and Saut proved in \[1\] that for any \( s \geq 3/2 \), equation (1) is globally in time well-posed on the Sobolev space \( H^s \equiv H^s(\mathbb{T}, \mathbb{R}) \) (endowed with the standard norm \( \| \cdot \|_s \), defined by (5) below), meaning the following:

- **(S1) Existence and uniqueness of classical solutions:** For any initial data \( v_0 \in H^s_* \), there exists a unique curve \( v : \mathbb{R} \to H^s_* \) in \( C(\mathbb{R}, H^s_*) \cap C^1(\mathbb{R}, H^{s-2}_*) \) so that \( v(0) = v_0 \) and for any \( t \in \mathbb{R} \), equation (1) is satisfied in \( H^{s-2}_* \). (Since \( H^s_* \) is an algebra, one has \( \partial_x v(t)^2 \in H^{s-1}_* \) for any time \( t \in \mathbb{R} \).)

- **(S2) Continuity of solution map:** The solution map \( S : H^s_* \to C(\mathbb{R}, H^s_*) \) is continuous, meaning that for any \( v_0 \in H^s_* \), \( T > 0 \), and \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that for any \( w_0 \in H^s_* \) with \( \| w_0 - v_0 \|_s < \delta \), the solutions \( w(t) = S(t, w_0) \) and \( v(t) = S(t, v_0) \) of (1) with initial data \( w(0) = w_0 \) and, respectively, \( v(0) = v_0 \) satisfy \( \sup_{|t| \leq T} \| w(t) - v(t) \|_s \leq \varepsilon \).

In a straightforward way one verifies that

\begin{equation}
\mathcal{H}^{-1}(v) := \langle v | 1 \rangle, \quad \mathcal{H}^{(0)}(v) := \frac{1}{2} \langle v | v \rangle
\end{equation}

are integrals of the above solutions of (1), referred to as classical solutions, where \( \langle \cdot | \cdot \rangle \) denotes the \( L^2 \)-inner product,

\begin{equation}
\langle f | g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x) dx.
\end{equation}
In particular it follows that for any $c \in \mathbb{R}$ and any $s \geq 3/2$, the affine space $H_{r,c}^{s}$ is left invariant by $S$ where for any $\sigma \in \mathbb{R}$

\[(4) \quad H_{r,c}^{\sigma} := \{ w \in H_{r}^{\sigma} : \langle w \rangle 1 = c \}.
\]

In the sequel, further progress has been made on the well-posedness of (1) on Sobolev spaces of low regularity. The best results so far in this direction were obtained by Molinet by using the gauge transformation introduced by Tao [30]. Molinet’s results in [22] (cf. also [24]) imply that the solution map $S$, introduced in (S2) above, continuously extends to any Sobolev space $H_{r}^{s}$ with $0 \leq s \leq 3/2$. More precisely, for any such $s$, $S : H_{r}^{s} \to C(\mathbb{R}, H_{r}^{s})$ is continuous and for any $v_{0} \in H_{r}^{s}$, $S(t, v_{0})$ satisfies equation (1) in $H_{r}^{s-2}$. The fact that $S$ continuously extends to $L_{t}^{2} \equiv H_{r}^{0}$, $S : L_{t}^{2} \to C(\mathbb{R}, L_{r}^{2})$, can also be deduced by methods recently developed in [12]. Furthermore, one infers from [12] that any solution $S(t, v_{0})$ with initial data $v_{0} \in L_{r}^{2}$ can be approximated in $C(\mathbb{R}, L_{r}^{2})$ by solutions of (1) which are rational functions of $\cos x, \sin x$. In this paper we will refer to these solutions as rational solutions.

In this paper we show that the BO equation is well-posed in the Sobolev space $H_{r}^{-s}$ for any $0 < s < 1/2$ and that this result is sharp. i.e., that the critical Sobolev exponent $s_{c} = -1/2$ is the threshold for well-posedness. Since the nonlinear term $\partial_{x}v^{2}$ in equation (1) is not well-defined for elements in $H_{r}^{-s}$, we first need to define what we mean by a solution of (1) in such a space.

**Definition 1.** Let $s \geq 0$. A continuous curve $\gamma : \mathbb{R} \to H_{r}^{-s}$ with $\gamma(0) = v_{0}$ for a given $v_{0} \in H_{r}^{-s}$, is called a global in time solution of the BO equation in $H_{r}^{-s}$ with initial data $v_{0}$ if for any sequence $(v_{0}^{(k)})_{k \geq 1}$ in $H_{r}^{s}$ with $s > 3/2$, which converges to $v_{0}$ in $H_{r}^{-s}$, the corresponding sequence of classical solutions $S(\cdot, v_{0}^{(k)})$ converges to $\gamma$ in $C(\mathbb{R}, H_{r}^{-s})$. The solution $\gamma$ is denoted by $S(\cdot, v_{0})$.

We remark that for any $v_{0} \in L_{r}^{2}$, the solution $S(\cdot, v_{0})$ in the sense of Definition 1 coincides with the solution obtained by Molinet in [22].

**Definition 2.** Let $s \geq 0$. Equation (1) is said to be globally $C^{0}$-well-posed in $H_{r}^{-s}$ if the following holds:

- (i) For any $v_{0} \in H_{r}^{-s}$, there exists a global in time solution of (1) with initial data $v_{0}$ in the sense of Definition 1.
- (ii) The solution map $S : H_{r}^{-s} \to C(\mathbb{R}, H_{r}^{-s})$ is continuous, i.e. satisfies (S2).

Our main results are the following ones:

**Theorem 1.** For any $0 \leq s < 1/2$, the Benjamin-Ono equation is globally $C^{0}$-well-posed on $H_{r}^{-s}$ in the sense of Definition 2. For any $c \in \mathbb{R}$, $t \in \mathbb{R}$, the flow map $S^{t} = S(t, \cdot)$ leaves the affine space $H_{r,c}^{s}$ introduced in (4), invariant.
Furthermore, there exists an integral $I_s: H^{-s}_r \to \mathbb{R}_{\geq 0}$ of $(1)$ satisfying
\[ \|v\|_{-s} \leq I_s(v), \quad \forall v \in H^{-s}_r. \]
In particular, one has
\[ \sup_{t \in \mathbb{R}} \| S(t, v_0) \|_{-s} \leq I_s(v_0), \quad \forall v_0 \in H^{-s}_r. \]

**Remark 1.** (i) Note that global $C^0$–wellposedness implies the group property $S^{t_1} \circ S^{t_2} = S^{t_1+t_2}$. Consequently, $S^t$ is a homeomorphism of $H^{-s}$.

(ii) Since by $(2)$, the $L^2$–norm is an integral of $(1)$, $I_s$ in the case $s = 0$ can be chosen as $I_0(v) := \|v\|_2^2$. The definition of $I_s$ for $0 < s < 1/2$ can be found in Remark 6 in Section 2.

(iii) By the Rellich compactness theorem, $S^t$ is also weakly sequentially continuous, in particular on $L^2_{r,c}$. Note that this contradicts the result of [23]. Very recently, however, an error in the proof of this statement has been found, leading to the withdrawal of the paper (cf. arXiv:0811.0505). A proof of this weak continuity property was indeed the starting point of the present paper.

**Remark 2.** It was already observed in [5] that the solution map $S$ does not continuously extend to $H^{-s}_r$ with $s > 1/2$. More precisely, for any $c \in \mathbb{R}$, there exists a sequence $(v_0^{(k)})_{k \geq 1}$ in $\cap_{n \geq 0} H^0_{r,c}$, which for any $s > 1/2$ converges to an element $v_0$ in $H^{-s}_r$ so that for any $t \neq 0$, the sequence $S(t, v_0^{(k)})$ does not even converge to a distribution on $\mathbb{T}$ in the sense of distributions. Since the methods developed in this paper allow us to give a short proof of this result, we include it in Section 6.

One of the key ingredients of our proof of Theorem 1 are explicit formulas for the frequencies of the Benjamin-Ono equation, defined by $(9)$ below. They are not only used to prove the global wellposedness results for $(1)$, but at the same time allow to obtain the following qualitative properties of solutions of $(1)$.

**Theorem 2.** For any $v_0 \in H^{-s}_r$ with $0 < s < 1/2$ and $c \in \mathbb{R}$, the solution $t \mapsto S(t, v_0)$ has the following properties:

(i) The orbit $\{S(t, v_0) : t \in \mathbb{R}\}$ is relatively compact in $H^{-s}_r$.

(ii) The solution $t \mapsto S(t, v_0)$ is almost periodic in $H^{-s}_r$.

**Remark 3.** For $s = 0$, results corresponding to the ones of Theorem 2 have been obtained in [12].

In [3], Amick&Toland characterized the travelling wave solutions of $(1)$, originally found by Benjamin [6]. It was shown in [12, Appendix B] that they coincide with the so called one gap solutions, described explicitly in [12]. Note that one gap potentials are rational solutions of $(1)$ and evolve in $H^s_\alpha$ for any $s \geq 0$. In [4, Section 5.1] Angulo Pava&Natali proved that every travelling wave solution of $(1)$ is orbitally stable in
Our newly developed methods allow to complement their result as follows:

**Theorem 3.** Every travelling wave solution of the BO equation is orbitally stable in $H^{-s}_r$ for any $0 \leq s < 1/2$.

Let us comment on the novelty of our results.

1. A straightforward computation shows that $s_c = -1/2$ is the critical Sobolev exponent of the Benjamin-Ono equation. Hence Theorem 1 and Remark 1(iv) imply that the threshold of well-posedness of (1) on the scale of Sobolev spaces $H^s_r$ is given by the critical Sobolev exponent $s_c$.

2. In a recent, very interesting paper [29], Talbut proved by the method of perturbation determinants, developed for the KdV and the NLS equations by Killip, Visan, and Zhang in [19], that for any $0 < s < 1/2$, there exists a constant $C_s > 0$, only depending on $s$, so that any sufficiently smooth solution $t \to v(t)$ of (1) satisfies the estimate

$$
\sup_{t \in \mathbb{R}} \|v(t)\|^{-s} \leq C_s \left(1 + \|v(0)\|^{-s} \right)^{\frac{2}{1-2s}} \|v(0)\|^{-s}.
$$

Note that our method allows to prove that the solution map $S$ continuously extends to $H^{-s}_r$. Actually, it allows to achieve much more by constructing a nonlinear Fourier transform, also referred to as Birkhoff map (cf. Section 2), which is also of great use for proving Theorem 2 and Theorem 3. The integral $I_s$ of Theorem 1(iv) is tailored to show that for any $0 < s < 1/2$, the Birkhoff map $\Phi : H^{-s}_{r,0} \to h^{1/2-s}_+$ is onto (cf. Theorem 5).

For recent work on a priori bounds for Sobolev norms of smooth solutions of the KdV equation and/or the NLS equation in 1d and their applications to the initial value problem of these equations see also [18], [20].

3. Using a probabilistic approach developed by Tzvetkov and Visciglia [32], Y. Deng [10] proved wellposedness result for the BO equation on the torus for almost every data with respect to a measure which is supported by $H^{-\varepsilon}_r$ for any $\varepsilon > 0$ and has the property that $L^2_r$ has measure 0. Our result provides a deterministic framework for these solutions.

4. A first version of this paper appeared on arXiv in September 2019 – see [13].

**Outline of proofs.** The key idea is to construct for any $0 \leq s < 1/2$, globally defined canonical coordinates on $H^{-s}_{r,0}$ with the property that when expressed in these coordinates, equation (1) can be solved by quadrature. Such coordinates are referred to as nonlinear Fourier coefficients or Birkhoff coordinates and the corresponding transformation, denoted by $\Phi$, as Birkhoff map. Such a map was constructed.
on $L^2_{r,0} \equiv H^0_{r,0}$ in our previous work [12]. In this paper we show that it can be continuously extended to the Sobolev spaces $H^{-s}_{r,0}$ for any $0 < s < 1/2$. For this purpose we develop a new approach for studying the Lax operator appearing in the Lax pair formulation of (1).

In Section 2, we state our results on the extension of Birkhoff map $\Phi$ (cf. Theorem 5) and discuss first applications. All these results are proved in Section 3 and Section 4. In Section 5, we study the solution map $S_B$ corresponding to the system of equations, obtained when expressing (1) in Birkhoff coordinates. The main point is to show that the frequencies of the BO equation, which have been computed in our previous work [12] on $L^2_{r,0}$, continuously extend to $H^{-s}_{r,0}$ for any $0 < s < 1/2$. These results are then used to study the solution map $S$ of (1). In the same section we also introduce the solution map $S_c$, related to the equation (1) when considered in the affine space $H^s_{r,c}$ and study the solution map $S_{c,B}$, obtained by expressing $S_c$ in Birkhoff coordinates. With all these preparations done, we then prove our main results, stated in Theorem 1 – Theorem 3, in Section 6. We remark that the proof of Theorem 2 uses the same arguments as the one of Theorem 2 in [12], stating corresponding results for solutions of (1) with initial data in $H^0_{r,0}$. In order to be comprehensive and since the proof is short, we included it.

Related work. By similar methods, results on global wellposedness of the type stated in Theorem 1 have been obtained for other integrable PDEs such as the KdV, the KdV2, the mKdV, and the defocusing NLS equations. In addition, a detailed analysis of the frequencies of these equations allowed to prove in addition to the wellposedness results qualitative properties of solutions of these equations, among them properties corresponding to the ones stated in Theorem 2 – see e.g. [16],[17], [14], [15]. Very recently, new global wellposedness results for the KdV equation on the line were obtained by Killip, Visan, and Zhang in [18] by using the integrable structure of the KdV equation in a novel way. By the same method, the authors also prove global wellposedness results for the KdV equation on the torus (of the type stated in Theorem 1) and for the NLS equation.

Let us comment on the principal differences between the KdV equation and the Benjamin–Ono equation, when viewed as integrable systems with a Lax pair formulation. One of the main differences is that the Lax operator $L$ associated with the Benjamin–Ono equation is non-local. As a result, the spectral analysis of $L$ is of a quite different nature than the one of the Lax operator of the KdV equation, given by the Hill operator and hence being a differential operator of order two. One of the consequences of $L$ being nonlocal is that the study of the regularity of the Birkhoff map and of its restrictions to the scale of Sobolev spaces $H^s_{r,0}$, $s \geq 0$, is much more involved than in the case of the Birkhoff map of the KdV equation. We plan to address this issue
in future work. A second principal difference is that the BO frequencies are affine functionals of the symplectic actions whereas the KdV frequencies are transcendental functionals of such actions, making it much more difficult to extend them to Sobolev spaces of functions of low regularity or Sobolev spaces of distributions. A third major difference concerns finite gap potentials: in the case of the BO equation, finite gap potentials are finite sums of Poisson kernels, whereas in the case of the KdV equation, such potentials are given in terms of theta functions.

**Notation.**
By and large, we will use the notation established in [12]. In particular, the $H^s$–norm of an element $v$ in the Sobolev space $H^s \equiv H^s(T, \mathbb{C})$, $s \in \mathbb{R}$, will be denoted by $\|v\|_s$. It is defined by

$$\|v\|_s = \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\hat{v}(n)|^2 \right)^{1/2}, \quad \langle n \rangle = \max\{1, |n|\}.$$  

For $\|v\|_0$, we usually write $\|v\|$. By $\langle \cdot | \cdot \rangle$, we will also denote the extension of the $L^2$–inner product, introduced in (3), to $H^{-s} \times H^s$, $s \in \mathbb{R}$, by duality. By $H_+$ we denote the Hardy space, consisting of elements $f \in L^2(T, \mathbb{C}) \equiv H^0$ with the property that $\hat{f}(n) = 0$ for any $n < 0$. More generally, for any $s \in \mathbb{R}$, $H^+_s$ denotes the subspace of $H^s$, consisting of elements $f \in H^s$ with the property that $\hat{f}(n) = 0$ for any $n < 0$.

**2. The Birkhoff map**

In this section we present our results on Birkhoff coordinates which will be a key ingredient of the proofs of Theorem 1 – Theorem 3. We begin by reviewing the results on Birkhoff coordinates proved in [12]. Recall that on appropriate Sobolev spaces, (1) can be written in Hamiltonian form

$$\partial_t u = \partial_x (\nabla H(u)), \quad H(u) := \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{2} (|\partial_x|^1/2 u)^2 \right. - \left. \frac{1}{3} u^3 \right) dx$$

where $|\partial_x|^{1/2}$ is the square root of the Fourier multiplier operator $|\partial_x|$ given by

$$|\partial_x| f(x) = \sum_{n \in \mathbb{Z}} |n| \hat{f}(n) e^{inx}.$$  

Note that the $L^2$–gradient $\nabla H$ of $H$ can be computed to be $|\partial_x|u - u^2$ and that $\partial_x \nabla H$ is the Hamiltonian vector field corresponding to the Gardner bracket, defined for any two functionals $F, G : H^0_+ \to \mathbb{R}$ with sufficiently regular $L^2$–gradients by

$$\{F, G\} := \frac{1}{2\pi} \int_0^{2\pi} (\partial_x \nabla F) \nabla G dx.$$
In [12], it is shown that (1) admits global Birkhoff coordinates and hence is an integrable ΨDE in the strongest possible sense. To state this result in more detail, we first introduce some notation. For any subset \( J \subset \mathbb{N}_0 := \mathbb{Z}_{\geq 0} \) and any \( s \in \mathbb{R} \), \( h^s(J) \equiv h^s(J, \mathbb{C}) \) denotes the weighted \( \ell^2 \)-sequence space
\[
h^s(J) = \{(z_n)_{n \in J} \subset \mathbb{C} : \| (z_n)_{n \in J} \|_s < \infty \}
\]
where
\[
\| (z_n)_{n \in J} \|_s := \left( \sum_{n \in J} (n)2^s |z_n|^2 \right)^{1/2}, \quad \langle n \rangle := \max\{1, |n|\}.
\]
By \( h^s(J, \mathbb{R}) \), we denote the real subspace of \( h^s(J, \mathbb{C}) \), consisting of real sequences \((z_n)_{n \in J}\). In case where \( J = \mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\} \) we write \( h^s_r \) instead of \( h^s(\mathbb{N}) \). If \( s = 0 \), we also write \( \ell^2 \) instead of \( h^0 \) and \( \ell^2_r \) instead of \( h^0_r \). In the sequel, we view \( h^s_r \) as the \( \mathbb{R} \)-Hilbert space \( h^s(\mathbb{R}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R}) \) by identifying a sequence \((z_n)_{n \in \mathbb{N}} \in h^s_r \) with the pair of sequences \((\text{Re} z_n)_{n \in \mathbb{N}}, (\text{Im} z_n)_{n \in \mathbb{N}}\) in \( h^s(\mathbb{R}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R}) \). We recall that \( L^2_r = H^0_r \) and \( L^2_{r,0} = H^0_{r,0} \). The following result was proved in [12]:

**Theorem 4. ([12, Theorem 1])** There exists a homeomorphism
\[
\Phi : L^2_{r,0} \rightarrow h^{1/2}_r, \quad u \mapsto (\zeta_n(u))_{n \geq 1}
\]
such that the following holds:

(B1) For any \( n \geq 1 \), \( \zeta_n : L^2_{r,0} \rightarrow \mathbb{C} \) is real analytic.

(B2) The Poisson brackets between the coordinate functions \( \zeta_n \) are well-defined and for any \( n, k \geq 1 \),
\[
\{\zeta_n, \overline{\zeta_k}\} = -i \delta_{nk}, \quad \{\zeta_n, \zeta_k\} = 0.
\]
It implies that the functionals \( |\zeta_n|^2, n \geq 1 \), pairwise Poisson commute,
\[
\{|\zeta_n|^2, |\zeta_k|^2\} = 0, \quad \forall n, k \geq 1.
\]

(B3) On its domain of definition, \( \mathcal{H} \circ \Phi^{-1} \) is a (real analytic) function, which only depends on the actions \( |\zeta_n|^2, n \geq 1 \). As a consequence, for any \( n \geq 1 \), \( |\zeta_n|^2 \) is an integral of \( \mathcal{H} \circ \Phi^{-1} \), \( \{\mathcal{H} \circ \Phi^{-1}, |\zeta_n|^2\} = 0 \). The coordinates \( \zeta_n \), \( n \geq 1 \), are referred to as complex Birkhoff coordinates and the functionals \( |\zeta_n|^2, n \geq 1 \), as action variables.

**Remark 4.** (i) When restricted to submanifolds of finite gap potentials (cf. [12, Definition 2.2]), the map \( \Phi \) is a canonical, real analytic diffeomorphism onto corresponding Euclidean spaces – see [12, Theorem 3] for details.

(ii) For any bounded subset \( B \) of \( L^2_{r,0} \), the image \( \Phi(B) \) by \( \Phi \) is bounded in \( h^{1/2}_r \). This is a direct consequence of the trace formula, saying that
for any $u \in L^2_{r,0}$ (cf. [12, Proposition 3.1]),
\begin{equation}
\|u\|^2 = 2 \sum_{n=1}^{\infty} n|\zeta_n|^2 .
\end{equation}

Theorem 4 together with Remark 4(i) can be used to solve the initial value problem of (1) in $L^2_{r,0}$. Indeed, by approximating a given initial data in $L^2_{r,0}$ by finite gap potentials (cf. [12, Definition 2.2]), one concludes from [12, Theorem 3] and Theorem 4 that equation (1), when expressed in the Birkhoff coordinates $\zeta = (\zeta_n)_{n \geq 1}$, reads
\begin{equation}
\partial_t \zeta_n = \{H \circ \Phi^{-1}, \zeta_n\} = i\omega_n \zeta_n, \quad \forall n \geq 1 ,
\end{equation}
where $\omega_n$, $n \geq 1$, are the BO frequencies,
\begin{equation}
\omega_n = \partial_{|\zeta_n|^2} H \circ \Phi^{-1} .
\end{equation}
Since the frequencies only depend on the actions $|\zeta_k|^2$, $k \geq 1$, they are conserved and hence (8) can be solved by quadrature,
\begin{equation}
\zeta_n(t) = \zeta_n(0) e^{i\omega_n(\zeta(0))t}, \quad t \in \mathbb{R}, \quad n \geq 1 .
\end{equation}
By [12, Proposition 8.1]), $\mathcal{H}_B := H \circ \Phi^{-1}$ can be computed as
\begin{equation}
\mathcal{H}_B(\zeta) := \sum_{k=1}^{\infty} k^2 |\zeta_k|^2 - \sum_{k=1}^{\infty} \left( \sum_{p=k}^{\infty} |\zeta_p|^2 \right)^2 ,
\end{equation}
implying that the frequencies, defined by (9), are given by
\begin{equation}
\omega_n(\zeta) = n^2 - 2 \sum_{k=1}^{\infty} \min(n,k)|\zeta_k|^2, \quad \forall n \geq 1 .
\end{equation}
Remarkably, for any $n \geq 1$, $\omega_n$ depends linearly on the actions $|\zeta_k|^2$, $k \geq 1$. Furthermore, while the Hamiltonian $\mathcal{H}_B$ is defined on $h^1_+$, the frequencies $\omega_n$, $n \geq 1$, given by (12) for $\zeta \in h^1_+$, extend to bounded functionals on $\ell^2_+$,
\begin{equation}
\omega_n : \ell^2_+ \to \mathbb{R}, \quad \zeta = (\zeta_k)_{k \geq 1} \mapsto \omega_n(\zeta) .
\end{equation}
We will prove that the restriction $S_0$ of the solution map of (1) to $L^2_{r,0}$, when expressed in Birkhoff coordinates,
\begin{equation*}
S_B : h^{1/2}_+ \to C(\mathbb{R}, h^{1/2}_+), \quad \zeta(0) \mapsto (\zeta_n(0) e^{i\omega_n(\zeta(0))t})_{n \geq 1}
\end{equation*}
is continuous – see Proposition 3 in Section 5. By Theorem 4, $\Phi : L^2_{r,0} \to h^{1/2}_+$ and its inverse $\Phi^{-1} : h^{1/2}_+ \to L^2_{r,0}$ are continuous. Since
\begin{equation*}
S_0 = \Phi^{-1} S_B \Phi : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0}), \quad u(0) \mapsto \Phi^{-1} S_B(t, \Phi(u(0)))
\end{equation*}
it follows that $S_0 : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0})$ is continuous as well. We remark that for any $u(0) \in L^2_{r,0}$, the solution $t \mapsto S(t, u(0))$ can be approximated in $L^2_{r,0}$ by classical solutions of equation (1) (cf. Remark 4(ii)) and thus coincides with the solution, obtained by Molinet in [22] (cf. also [24]).
Starting point of the proof of Theorem 1 is formula (52) in Subsection 5.3. We will show that it extends to the Sobolev spaces $H_{r,0}^{-s}$ for any $0 < s < 1/2$. A key ingredient to prove Theorem 1 is therefore the following result on the extension of the Birkhoff map $\Phi$ to $H_{r,0}^{-s}$ for any $0 < s < 1/2$:

**Theorem 5. (Extension of $\Phi$.)** For any $0 < s < 1/2$, the map $\Phi$ of Theorem 4 admits an extension, also denoted by $\Phi$,

$$\Phi : H_{r,0}^{-s} \rightarrow h_{+}^{1/2-s}, \quad u \mapsto \Phi(u) := (\zeta_n(u))_{n \geq 1},$$

so that the following holds:

(i) $\Phi$ is a homeomorphism.

(ii) There exists an increasing function $F_s : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ so that

$$\|u\|_{-s} \leq F_s(\|\Phi(u)\|_{1/2-s}), \quad \forall u \in H_{r,0}^{-s}.$$ 

(iii) $\Phi$ and its inverse map bounded subsets to bounded subsets.

**Remark 5.** Notice that (i) and (iii) combined with the Rellich compactness theorem imply that for $0 \leq s < 1/2$, the map $\Phi : H_{r,0}^{-s} \rightarrow h_{+}^{1/2-s}$ and its inverse $\Phi^{-1} : h_{+}^{1/2-s} \rightarrow H_{r,0}^{-s}$ are weakly sequentially continuous on $H_{r,0}^{-s}$.

**Remark 6.** The above a priori bound for $\|u\|_{-s}$ can be extended to the space $H_{r}^{-s}$ as follows

$$\|v\|_{-s} \leq F_s(\|\Phi(v-[v])\|_{1/2-s}) + \|[v]\|, \quad [v] = (v[1]), \quad \forall v \in H_{r}^{-s}.$$ 

For any $0 < s < 1/2$, the integral $I_s$ in Theorem 1(iv) is defined as

$$I_s(v) := F_s(\|\Phi(v-[v])\|_{1/2-s}) + \|[v]\|.$$ 

**Ideas of the proof of Theorem 5.** At the heart of the proof of Theorem 1 in [12] is the Lax operator $L_u$, appearing in the Lax pair formulation in [25] (cf. also [8], [9], [11])

$$\partial_t L_u = [B_u, L_u]$$

of (1) – see [12, Appendix A] for a review. For any given $u \in L_r^2$, the operator $L_u$ is the first order operator acting on the Hardy space $H_+$,

$$L_u := -i\partial_x - T_u, \quad T_u(\cdot) := \Pi(u \cdot)$$

where $\Pi$ is the orthogonal projector of $L^2$ onto $H_+$ and $T_u$ is the Toeplitz operator with symbol $u$,

$$H_+ := \{f \in L^2 : \hat{f}(n) = 0 \ \forall n < 0\}.$$ 

The operator $L_u$ is self-adjoint with domain $H_+^1 := H^1 \cap H_+$, bounded from below, and has a compact resolvent. Its spectrum consists of eigenvalues which bounded from below. When listed in increasing order they form a sequence, satisfying

$$\lambda_0 \leq \lambda_1 \leq \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$
For our purposes, the most important properties of the spectrum of $L_u$ are that the eigenvalues are conserved along the flow of (1) and that they are all simple. More precisely, one has

\[ \gamma_n := \lambda_n - \lambda_{n-1} - 1 \geq 0, \quad \forall n \geq 1. \tag{14} \]

The nonnegative number $\gamma_n$ is referred to as the $n$th gap of the spectrum $\text{spec}(L_u)$ of $L_u$. – see [12, Appendix C] for an explanation of this terminology. For any $n \geq 1$, the complex Birkhoff coordinate $\zeta_n$ of Theorem 4 is related to $\gamma_n$ by $|\zeta_n|^2 = \gamma_n$ whereas its phase is defined in terms of an appropriately normalized eigenfunction $f_n$ of $L_u$, corresponding to the eigenvalue $\lambda_n$.

A key step for the proof of Theorem 5 is to show that for any $u \in H^{-s}_r$ with $0 < s < 1/2$, the Lax operator $L_u$ can be defined as a self-adjoint operator with domain included in $H^{1-s}_r$ and that its spectrum has properties similar to the ones described above in the case where $u \in L^2_r$. In particular, the inequality (14) continues to hold. Since the proof of Theorem 5 requires several steps, it is split up into two parts, corresponding to Section 3 and Section 4.

A straightforward application of Theorem 5 is the following result on isospectral potentials. To state it, we need to introduce some additional notation. For any $\zeta \in h^{1/2-s}$, define

\[ \text{Tor}(\zeta) := \{ z \in h^{1/2-s} : |z_n| = |\zeta_n| \quad \forall n \geq 1 \}. \tag{15} \]

where as above, $\text{spec}(L_u)$ denotes the spectrum of the Lax operator $L_u := -i \partial_x - T_u$. The spectrum of $L_u$ continues to be characterized in terms of its gaps $\gamma_n$, $n \geq 1$, (cf. (14)) and the extended Birkhoff coordinates continue to satisfy $|\zeta_n|^2 = \gamma_n$, $n \geq 1$. An immediate consequence of Theorem 5 then is that [12, Corollary 8.1] extends as follows:

**Corollary 1.** For any $u \in H^{-s}_{r,0}$ with $0 < s < 1/2$,

\[ \Phi(\text{Iso}(u)) = \text{Tor}(\Phi(u)). \]

Hence by the continuity of $\Phi^{-1}$, $\text{Iso}(u)$ is a compact, connected subset of $H^{-s}_{r,0}$.

### 3. Extension of $\Phi$. Part 1

In this section we prove the first part of Theorem 5, which we state as a separate result:

**Proposition 1.** (Extension of $\Phi$. Part 1) For any $0 < s < 1/2$, the following holds:

(i) For any $n \geq 1$, the formula in [12, (4.1)] of the Birkhoff coordinate
\( \zeta_n : L^2_{r,0} \to \mathbb{C} \) extends to \( H_{r,0}^{-s} \) and for any \( u \in H_{r,0}^{-s} \), \((\zeta_n(u))_{n \geq 1} \) is in \( h_{+}^{1/2-s} \). The extension of the map \( \Phi \) of Theorem 4, also denoted by \( \Phi \),
\[
\Phi : H_{r,0}^{-s} \to h_{+}^{1/2-s}, \quad u \mapsto \Phi(u) := (\zeta_n(u))_{n \geq 1},
\]
maps bounded subsets of \( H_{r,0}^{-s} \) to bounded subsets of \( h_{+}^{1/2-s} \).

(iii) \( \Phi \) is sequentially weakly continuous and one-to-one.

First we need to establish some auxiliary results related to the Lax operator \( L_u \).

**Lemma 1.** Let \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \). Then for any \( f, g \in h_{+}^{1/2} \), the following estimates hold:

(i) There exists a constant \( C_{1,s} \) such that
\[
\|fg\|_s \leq C_{1,s}^2 \|f\|_\sigma \|g\|_\sigma, \quad \sigma := (1/2 + s)/2.
\]

(ii) The expression \( \langle u|f\rangle \) is well defined and satisfies the estimate
\[
|\langle u|f\rangle| \leq \frac{1}{2} \|f\|_{1/2}^2 + \eta_\varepsilon(\|u\|_{-s}) \|f\|^2,
\]
where
\[
\eta_\varepsilon(\|u\|_{-s}) := \|u\|_{-s}(2(1 + \|u\|_{-s}))^\alpha C_{2,s}^2, \quad \alpha := \frac{1 + 2s}{1 - 2s}
\]
and \( C_{2,s} > 0 \) is a constant, only depending on \( s \).

**Proof.** (i) Estimate (16) is obtained from standard estimates of multiplication (cf. e.g. [2, Exercise II.A.5], [7, Theorem 2.82, Theorem 2.85]). (ii) By item (i), \( \langle u|f\rangle \) is well defined by duality and satisfies
\[
|\langle u|f\rangle| \leq \|u\|_{-s} \|f\|_{1/2} \|f\|_{1/2-s} \leq \|u\|_{-s} C_{1,s}^2 \|f\|^2.
\]
In order to estimate \( \|f\|_{1/2}^2 \), note that by interpolation one has \( \|f\|_\sigma \leq \|f\|_{1/2} \|f\|_{1/2+s} \) and hence
\[
C_{1,s} \|f\|_\sigma \leq \|f\|_{1/2} \|f\|_{1/2+s} \left(C_{2,s} \|f\|\right)^{1/2-s}
\]
for some constant \( C_{2,s} > 0 \). Young’s inequality then yields for any \( \varepsilon > 0 \)
\[
(C_{1,s} \|f\|_\sigma)^2 \leq \varepsilon \|f\|_{1/2}^2 + \varepsilon^{-\alpha} \left(C_{2,s} \|f\|\right)^2, \quad \alpha = \frac{1 + 2s}{1 - 2s}.
\]
Estimate (17) then follows from (20) by choosing \( \varepsilon = (2(1 + \|u\|_{-s}))^{-1} \).
\( \square \)

Note that estimate (17) implies that the sesquilinear form \( \langle Tu_f|g\rangle \) on \( h_{+}^{1/2} \), obtained from the Toeplitz operator \( T_u f := \Pi(u f) \) with symbol \( u \in L^2_{r,0} \), can be defined for any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \) by setting \( \langle Tu_f|g\rangle := \langle u|g\rangle \) and that it is bounded. For any \( u \in H_{r,0}^{-s} \), we then define the sesquilinear form \( Q_u^s \) on \( h_{+}^{1/2} \) as follows
\[
Q_u^s(f, g) := \langle -i\partial_x f|g\rangle - \langle Tu_f|g\rangle + (1 + \eta_\varepsilon(\|u\|_{-s})) \langle f|g\rangle
\]
where \( \eta_u(\|u\|_{-s}) \) is given by (18). The following lemma says that the quadratic form \( Q_u^+(f, f) \) is equivalent to \( \|f\|_{1/2}^2 \). More precisely, the following holds.

**Lemma 2.** For any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \), \( Q_u^+ \) is a positive, sesquilinear form, satisfying

\[
\frac{1}{2} \|f\|_{1/2}^2 \leq Q_u^+(f, f) \leq (3 + 2\eta_u(\|u\|_{-s})) \|f\|_{1/2}^2, \quad \forall f \in H_+^{1/2}.
\]

**Proof.** (i) Using that \( u \) is real valued, one verifies that \( Q_u^+ \) is sesquilinear. The claimed estimates are obtained from (17) as follows: since \( \langle n \rangle \leq 1 + |n| \) one has \( \|f\|_{3/2}^2 \leq \langle -i\partial_x f, f \rangle + \|f\|_2^2 \), and hence by (17),

\[
|\langle Tu f | f \rangle| \leq \frac{1}{2} \langle -i\partial_x f, f \rangle + \left( \frac{1}{2} + \eta_u(\|u\|_{-s}) \right) \|f\|_2^2.
\]

By the definition (21), the claimed estimates then follow. In particular, the lower bound for \( Q_u^+(f, f) \) shows that \( Q_u^+ \) is positive. \( \square \)

Denote by \( \langle f | g \rangle_{1/2} \equiv \langle f | g \rangle_{H_+^{1/2}} \) the inner product, corresponding to the norm \( \|f\|_{1/2} \). It is given by

\[
\langle f | g \rangle_{1/2} = \sum_{n \geq 0} \overline{(n) f(n) \hat{g}(n)}, \quad \forall f, g \in H_+^{1/2}.
\]

Furthermore, denote by \( D : H_+^t \to H_+^{t-1} \) and \( \langle D \rangle : H_+^t \to H_+^{t-1} \), \( t \in \mathbb{R} \), the Fourier multipliers, defined for \( f \in H_+^t \) with Fourier series \( f = \sum_{n=0}^{\infty} \hat{f}(n) e^{inx} \) by

\[
D f := -i \partial_x f = \sum_{n=0}^{\infty} n \hat{f}(n) e^{inx}, \quad \langle D \rangle f := \sum_{n=0}^{\infty} \langle n \rangle \hat{f}(n) e^{inx}.
\]

**Lemma 3.** For any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \), there exists a bounded linear isomorphism \( A_u : H_+^{1/2} \to H_+^{1/2} \) so that

\[
\langle A_u f | g \rangle_{1/2} = Q_u^+(f, g), \quad \forall f, g \in H_+^{1/2}.
\]

The operator \( A_u \) has the following properties:
(i) \( A_u \) and its inverse \( A_u^{-1} \) are symmetric, i.e., for any \( f, g \in H_+^{1/2} \),

\[
\langle A_u f | g \rangle_{1/2} = \langle f | A_u g \rangle_{1/2}, \quad \langle A_u^{-1} f | g \rangle_{1/2} = \langle f | A_u^{-1} g \rangle_{1/2}.
\]

(ii) The linear isomorphism \( B_u \), given by the composition

\[
B_u := \langle D \rangle A_u : H_+^{1/2} \to H_+^{1/2}
\]

satisfies

\[
Q_u^+(f, g) = \langle B_u f | g \rangle, \quad \forall f, g \in H_+^{1/2}.
\]

The operator norm of \( B_u \) and the one of its inverse can be bounded uniformly on bounded subsets of elements \( u \) in \( H_{r,0}^{-s} \).
Proof. By Lemma 2, the sesquilinear form $Q_u^+$ is an inner product on $H_{+}^{1/2}$, equivalent to the inner product $\langle \cdot | \cdot \rangle_{1/2}$. Hence by the theorem of Fréchet-Riesz, for any $g \in H_{+}^{1/2}$, there exists a unique element in $H_{+}^{1/2}$, which we denote by $A_u g$, so that

$$\langle A_u g | f \rangle_{1/2} = Q_u^+(g, f), \quad \forall f \in H_{+}^{1/2}.$$ 

Invitation for special issue in honor of Tony Bloch in Journal of Geometric Mechanics Then $A_u : H_{+}^{1/2} \to H_{+}^{1/2}$ is a linear, injective operator, which by Lemma 2 is bounded, i.e., for any $f, g \in H_{+}^{1/2}$,

$$|\langle A_u g | f \rangle_{1/2}| = |Q_u^+(g, f)| \leq Q_u^+(g, g)^{1/2} Q_u^+(f, f)^{1/2} \leq (3 + 2\eta_s(||u||_2)) ||g||_{1/2} ||f||_{1/2},$$

implying that $||A_u g||_{1/2} \leq (3 + 2\eta_s(||u||_2)) ||g||_{1/2}$.

Similarly, by the theorem of Fréchet-Riesz, for any $h \in H_{+}^{1/2}$, there exists a unique element in $H_{+}^{1/2}$, which we denote by $E_u h$, so that

$$\langle h | f \rangle_{1/2} = Q_u^+(E_u h, f), \quad \forall f \in H_{+}^{1/2}.$$ 

Then $E_u : H_{+}^{1/2} \to H_{+}^{1/2}$ is a linear, injective operator, which by Lemma 2 is bounded, i.e.,

$$\frac{1}{2} \|E_u h\|_{1/2}^2 \leq Q_u^+(E_u h, E_u h) = \langle h | E_u h \rangle_{1/2} \leq \|h\|_{1/2} \|E_u h\|_{1/2},$$

implying that $\|E_u h\|_{1/2} \leq 2 \|h\|_{1/2}$. Note that $A_u(E_u h) = h$ and hence $E_u$ is the inverse of $A_u$. Therefore, $A_u : H_{+}^{1/2} \to H_{+}^{1/2}$ is a bounded linear isomorphism. Next we show item $(i)$. For any $f, g \in H_{+}^{1/2}$,

$$\langle g | A_u f \rangle_{1/2} = \frac{\langle A_u f | g \rangle_{1/2}}{Q_u^+(f, f)} = Q_u^+(g, f) = \langle A_u g | f \rangle_{1/2}.$$ 

The symmetry of $A_u^{-1}$ is proved in the same way. Towards item $(ii)$, note that for any $f, g \in H_{+}^{1/2}$, $\langle f | g \rangle_{1/2} = \langle \langle D \rangle f | g \rangle$ and therefore

$$\langle A_u g | f \rangle_{1/2} = \langle \langle D \rangle A_u g | f \rangle,$$

implying that the operator $B_u = \langle \langle D \rangle A_u : H_{+}^{1/2} \to H_{+}^{-1/2}$ is a bounded linear isomorphism and that

$$\langle B_u g | f \rangle = Q_u^+(g, f), \quad \forall g, f \in H_{+}^{1/2}.$$ 

The last statement of $(ii)$ follows from Lemma 2. \qed

We denote by $L_u^+$ the restriction of $B_u$ to $\text{dom}(L_u^+)$, defined as

$$\text{dom}(L_u^+) := \{ g \in H_{+}^{1/2} : B_u g \in H_+ \}.$$ 

We view $L_u^+$ as an unbounded linear operator on $H_+$ and write $L_u^+ : \text{dom}(L_u^+) \to H_+$. 

Flow map of the BO equation 14
Lemma 4. For any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \), the following holds:

(i) \( \text{dom}(L_u^+) \) is a dense subspace of \( H^{1/2}_+ \) and hence of \( H_+ \).
(ii) \( L_u^+ : \text{dom}(L_u^+) \to H_+ \) is bijective and the right inverse of \( L_u^+ \), \( (L_u^+)^{-1} : H_+ \to H_+^\prime \), is compact. Hence \( L_u^+ \) has discrete spectrum.
(iii) \( (L_u^+)^{-1} \) is symmetric and \( L_u^+ \) is self-adjoint and positive.

Proof. (i) Since \( H_+ \) is a dense subspace of \( H_+^{1/2} \) and \( B_u^{-1} : H_+^{1/2} \to H_+^{1/2} \) is a linear isomorphism, \( \text{dom}(L_u^+) = B_u^{-1}(H_+) \) is a dense subspace of \( H_+^{1/2} \), and hence also of \( H_+ \).

(ii) Since \( L_u^+ \) is the restriction of the linear isomorphism \( B_u \), it is one-to-one. By the definition of \( L_u^+ \), it is onto. The right inverse of \( L_u^+ \), denoted by \( (L_u^+)^{-1} \), is given by the composition \( \iota \circ B_u^{-1}|_{H_+} \), where \( \iota : H_+^{1/2} \to H_+ \) is the standard embedding which by Sobolev’s embedding theorem is compact. It then follows that \( (L_u^+)^{-1} : H_+ \to H_+^\prime \) is compact as well.

(iii) For any \( f, g \in H_+ \)

\[
\langle (L_u^+)^{-1} f | g \rangle = \langle A_u^{-1} \langle D \rangle^{-1} f | g \rangle = \langle A_u^{-1} \langle D \rangle^{-1} f | \langle D \rangle^{-1} g \rangle_{1/2}.
\]

By Lemma 3, \( A_u^{-1} \) is symmetric with respect to the \( H_+^{1/2} \)-inner product. Hence

\[
\langle (L_u^+)^{-1} f | g \rangle = \langle (D)^{-1} f | A_u^{-1} \langle D \rangle^{-1} g \rangle_{1/2} = \langle (L_u^+)^{-1} g \rangle,
\]

showing that \( (L_u^+)^{-1} \) is symmetric. Since in addition, \( (L_u^+)^{-1} \) is bounded it is also self-adjoint. By Lemma 2 it then follows that

\[
\langle L_u^+ f | f \rangle = \langle (D) A_u f | f \rangle = \langle A_u f | f \rangle_{1/2} = Q_u^+(f, f) \geq \frac{1}{2} \|f\|^2_{1/2},
\]

implying that \( L_u^+ \) is a positive operator. \( \Box \)

We now define for any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \), the operator \( L_u \) as a linear operator with domain \( \text{dom}(L_u) := \text{dom}(L_u^+) \) by setting

\[
L_u := L_u^+ + (1 + \eta_u(\|u\|_s)) : \text{dom}(L_u) \to H_+.
\]

Lemma 4 yields the following

Corollary 2. For any \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \), the operator \( L_u : \text{dom}(L_u) \to H_+ \) is densely defined, self-adjoint, bounded from below, and has discrete spectrum. It thus admits an \( L^2 \)-normalized basis of eigenfunctions, contained in \( \text{dom}(L_u) \) and hence in \( H_+^{1/2} \).

Remark 7. Let \( u \in H_{r,0}^{-s} \) with \( 0 \leq s < 1/2 \) be given. Since \( \text{dom}(L_u^+) \) is dense in \( H_+^{1/2} \) and \( L_u^+ \) is the restriction of \( B_u : H_+^{1/2} \to H_+^{1/2} \) to \( \text{dom}(L_u^+) \), the symmetry

\[
\langle L_u^+ f | g \rangle = \langle f | L_u^+ g \rangle, \quad \forall f, g \in \text{dom}(L_u^+)
\]

can be extended by a straightforward density argument as follows

\[
\langle B_u f | g \rangle = \langle f | B_u g \rangle, \quad \forall f, g \in H_+^{1/2}.
\]
Note that for any \( f, g \in H^{1/2}_+ \), \( \langle B_u f | g \rangle = \langle (D) A_u f | g \rangle = \langle A_u f | g \rangle_{1/2} \) and hence by (21),
\[
\langle B_u f | g \rangle = Q_u^+(f, g) = \langle D f - T_u f + (1 + \eta_s(\|u\|_{-s}))f | g \rangle,
\]
yielding the following identity in \( H^{-1/2}_+ \),
\[
(22) \quad B_u f = D f - T_u f + (1 + \eta_s(\|u\|_{-s}))f, \quad \forall f \in H^{1/2}_+.
\]
Given \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \), let us consider the restriction of \( B_u \) to \( H^{-s}_{r,0} \).

**Lemma 5.** Let \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \). Then \( B_u(H^{-s}_{1-r}) = H^{-s}_{1-r} \) and the restriction \( B_u|_{H^{-s}_{1-r}} : H^{1-s}_{1-r} \to H^{-s}_{1-r} \) is a linear isomorphism. The operator norm of \( B_u|_{H^{-s}_{1-r}} \) and the one of its inverse are bounded uniformly on bounded subsets of elements \( u \in H^{-s}_{r,0} \).

**Proof.** Since \( 1 - s > 1/2 \), \( H^{1-s} \) acts by multiplication on itself and on \( L^2 \), hence by interpolation on \( H^r \) for \( 0 \leq r \leq 1 - s \). By duality, it also acts on \( H^{-r} \), in particular with \( r = s \). This implies that \( B_u|_{H^{1-s}_{1-r}} : H^{1-s}_{1-r} \to H^{-s}_{1-r} \) is bounded. Being the restriction of an injective operator, it is injective as well. Let us prove that \( B_u|_{H^{1-s}_{1-r}} \) has \( H^{-s}_{1-r} \) as its image.

To this end consider an arbitrary element \( h \in H^{-s}_{1-r} \). We need to show that the solution \( f \in H^{1/2}_+ \) of \( B_u f = h \) is actually in \( H^{1-s}_{1-r} \). Write
\[
(23) \quad D f = h + (1 + \eta_s(\|u\|_{-s}))f + T_u f.
\]
Note that \( h + (1 + \eta_s(\|u\|_{-s}))f \) is in \( H^{-s}_{1-r} \) and it remains to study \( T_u f \).

By Lemma 1(i) one infers that for any \( g \in H^2_+ \), with \( \sigma = (1/2 + s)/2 \),
\[
|\langle T_u f | g \rangle| = |\langle u | g h \rangle| \leq \|u\|_{1-r} \|g h\|_s \leq C^2_1 \|u\|_{1-r} \|g\|_s \|f\|_\sigma,
\]

implying that \( T_u f \in H^{-s}_{1-r} \) and hence by (23), \( f \in H^{1-s}_{1-r} \). Since \( 1 - \sigma > 1/2 \), we argue as at the beginning of the proof to infer that \( T_u f \in H^{-s}_{1-r} \). Thus applying (23) once more we conclude that \( f \in H^{1-s}_{1-r} \). This shows that \( B_u|_{H^{1-s}_{1-r}} : H^{1-s}_{1-r} \to H^{-s}_{1-r} \) is onto. Going through the arguments of the proof one verifies that the operator norm of \( B_u|_{H^{1-s}_{1-r}} \) and the one of its inverse are bounded uniformly on bounded subsets of elements \( u \in H^{-s}_{r,0} \). This completes the proof of the lemma. \( \square \)

**Corollary 3.** For any \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \), \( \text{dom}(L_u^+) \subset H^{1-s}_+ \).

In particular, any eigenfunction of \( L_u^+ \) (and hence of \( L_u \)) is in \( H^{1-s}_+ \).

**Proof.** Since \( H_+ \subset H^{1-s}_+ \), one has \( B_u^{-1}(H_+) \subset B_u^{-1}(H^{1-s}_+) \) and hence by Lemma 5, \( \text{dom}(L_u^+) = B_u^{-1}(H_+) \subset H^{1-s}_+ \). \( \square \)

With the results obtained so far, it is straightforward to verify that many of the results of [12] extend to the case where \( u \in H^{-s}_{r,0} \). More precisely, let \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \). We already know that the
spectrum of $L_u$ is discrete, bounded from below, and real. When listed in increasing order and with their multiplicities, the eigenvalues of $L_u$ satisfy $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. Arguing as in the proof of \cite[Proposition 2.1]{12}, one verifies that $\lambda_n \geq \lambda_{n-1} + 1$, $n \geq 1$, and following \cite[(2.10)]{12} we define

$$\gamma_n(u) := \lambda_n - \lambda_{n-1} - 1 \geq 0.$$  

It then follows that for any $n \geq 1$,

$$\lambda_n = n + \lambda_0 + \sum_{k=1}^{n} \gamma_k \geq n + \lambda_0.$$  

Since \cite[Lemma 2.1, Lemma 2.2]{12} continue to hold for $u \in H_{r,0}^{-s}$, we can introduce eigenfunctions $f_n(x,u)$ of $L_u$, corresponding to the eigenvalues $\lambda_n$, which are normalized as in \cite[Definition 2.1]{12}. The identities \cite[(2.13)]{12} continue to hold,

$$\lambda_n \langle 1 | f_n \rangle = -\langle u | f_n \rangle$$  

as does \cite[Lemma 2.4]{12}, stating that $\gamma_n = 0$ if and only if $\langle 1 | f_n \rangle = 0$. Furthermore, the definition \cite[(3.1)]{12} of the generating function $\mathcal{H}_\lambda(u)$ extends to the case where $u \in H_{r,0}^{-s}$ with $0 < s < 1/2$,

$$\mathcal{H}_\lambda : H_{r,0}^{-s} \rightarrow \mathbb{C}, u \mapsto \langle (L_u + \lambda)^{-1} 1 | 1 \rangle.$$  

and so do the identity \cite[(3.2)]{12}, the product representation of $\mathcal{H}_\lambda(u)$, stated in \cite[Proposition 3.1(i)]{12},

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left( 1 - \frac{\gamma_n}{\lambda_n + \lambda} \right),$$  

and the one for $|\langle 1 | f_n \rangle|^2$, $n \geq 1$, given in \cite[Corollary 3.1]{12},

$$|\langle 1 | f_n \rangle|^2 = \gamma_n \kappa_n, \quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right).$$  

The product representation (26) then yields the identity (cf. \cite[Proposition 3.1(ii)]{12} and its proof),

$$-\lambda_0(u) = \sum_{n=1}^{\infty} \gamma_n(u).$$  

Since $\gamma_n(u) \geq 0$ for any $n \geq 1$, one infers that for any $u \in H_{r,0}^{-s}$ with $0 \leq s < 1/2$, the sequence $(\gamma_n(u))_{n \geq 1}$ is in $\ell^1_+ \equiv \ell^1(\mathbb{N}, \mathbb{R})$ and

$$\lambda_n(u) = n - \sum_{k=n+1}^{\infty} \gamma_k(u) \leq n.$$  

By (21), Lemma 2, and (22), we infer that $-\lambda_0 \leq \frac{1}{2} + \eta_s(\|u\|_{-s})$, yielding, when combined with (28) and (29), the estimate

$$n - \frac{1}{2} - \eta_s(\|u\|_{-s}) \leq \lambda_n(u) \leq n, \quad \forall n \geq 0.$$
In a next step we want consider the linear isomorphism
\[ B_{u;1-s} = B_u|_{H^{1-s}_+} : H^{1-s}_+ \rightarrow H^{-s}_+ \]
on the scale of Sobolev spaces. By duality, \( B_{u;1-s} \) extends as a bounded linear isomorphism, \( B_{u,s} : H^s_+ \rightarrow H^{-1+s}_+ \) and hence by complex interpolation, for any \( s \leq t \leq 1-s \), the restriction of \( B_{u,s} \) to \( H^t_+ \) gives also rise to a bounded linear isomorphism, \( B_{u,t} : H^t_+ \rightarrow H^{-1+t}_+ \). All these operators satisfy the same bound as \( B_{u;1-s} \) (cf. Lemma 5). To state our next result, it is convenient to introduce the notation \( N_0 := \mathbb{Z}_{\geq 0} \). Recall that \( h^t(N_0) = h^t(N_0, \mathbb{C}), t \in \mathbb{R} \), and that we write \( \ell^2(N_0) \) instead of \( h^0(N_0) \).

**Lemma 6.** Let \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \) and let \( (f_n)_{n \geq 0} \) be the basis of \( L^2_+ \), consisting of eigenfunctions of \( L_0 \) with \( f_n, n \geq 0 \), corresponding to the eigenvalue \( \lambda_n \) and normalized as in [12, Definition 2.1]. Then for any \(-1+s \leq t \leq 1-s\),
\[ K_{ud} : H^t_+ \rightarrow h^t(N_0), f \mapsto (\langle f| f_n \rangle)_{n \geq 0} \]
is a linear isomorphism. In particular, for \( f = \Pi u \in H^{-s}_+ \), one obtains that \( (\Pi u| f_n)_{n \geq 0} \in h^{-s}(N_0) \). The operator norm of \( K_{ud} \) and the one of its inverse can be uniformly bounded for \(-1+s \leq t \leq 1-s\) and for \( u \) in a bounded subset of \( H^{-s}_{r,0} \).

**Proof.** We claim that the sequence \( (\tilde{f}_n)_{n \geq 0} \), defined by
\[ \tilde{f}_n = \frac{f_n}{(\lambda_n + 1 + \eta_0(\|u\|_{-s}))^{1/2}}, \]
is an orthonormal basis of the Hilbert space \( H^{1/2}_+ \), endowed with the inner product \( Q^+_u \). Indeed, for any \( n \geq 0 \) and any \( g \in H^{1/2}_+ \), one has
\[ Q^+_u(\tilde{f}_n, g) = \langle L^+_u \tilde{f}_n | g \rangle = (\lambda_n + 1 + \eta_0(\|u\|_{-s}))^{1/2} \langle f_n | g \rangle. \]
As a consequence, for any \( n, m \geq 0 \), \( Q^+_u(\tilde{f}_n, \tilde{f}_m) = \delta_{nm} \) and the orthogonal complement of the subspace of \( H^{1/2}_+ \), spanned by \( (\tilde{f}_n)_{n \geq 0} \), is the trivial vector space \( \{0\} \), showing that \( (\tilde{f}_n)_{n \geq 0} \) is an orthonormal basis of \( H^{1/2}_+ \). In view of (30), we then conclude that
\[ K_{u;1/2} : H^{1/2}_+ \rightarrow h^{1/2}(N_0), f \mapsto (\langle f| f_n \rangle)_{n \geq 0} \]
is a linear isomorphism. Its inverse is given by
\[ K^{-1}_{u;1/2} : h^{1/2}(N_0) \rightarrow H^{1/2}_+, (z_n)_{n \geq 0} \mapsto f := \sum_{n=0}^{\infty} z_n f_n. \]
By interpolation we infer that for any \( 0 \leq t \leq 1/2 \), \( K_{ut} : H^t_+ \rightarrow h^t(N_0) \) is a linear isomorphism. Taking the transpose of \( K^{-1}_{u;1/2} \) it then follows that for any \( 0 \leq t \leq 1/2 \),
\[ K_{u;1-t} : H^{-1}_+ \rightarrow h^{-1}(N_0), f \mapsto (\langle f| f_n \rangle)_{n \geq 0}, \]
is also a linear isomorphism. It remains to discuss the remaining range of \( t \), stated in the lemma. By Lemma 5, the restriction of \( B_u^{-1} \) to \( H_+^{1-s} \) gives rise to a linear isomorphism \( B_{u;1-s}^{-1} : H_+^{1-s} \rightarrow H_+^{1-s} \). For any \( f \in H_+^{1-s} \), one then has

\[
B_{u;1-s}^{-1}f = \sum_{n=0}^{\infty} \frac{\langle f | f_n \rangle}{\lambda_n + 1 + \eta_s(\|u\|_s)} f_n.
\]

Since by our considerations above, \( (\langle f | f_n \rangle)_{n \geq 0} \in h^{-s}(\mathbb{N}_0) \) one concludes that the sequence \( \left( \frac{\langle f | f_n \rangle}{\lambda_n + 1 + \eta_s(\|u\|_s)} \right)_{n \geq 0} \) is in \( h^{1-s}(\mathbb{N}_0) \). Conversely, assume that \( (z_n)_{n \geq 0} \in h^{1-s}(\mathbb{N}_0) \). Then \( (\lambda_n + 1 + \eta_s(\|u\|_s))z_n \) is in \( h^{-s}(\mathbb{N}_0) \). Hence by the considerations above on \( K_{u;1-s} \), there exists \( g \in H_+^{1-s} \) so that

\[
\langle g | f_n \rangle = (\lambda_n + 1 + \eta_s(\|u\|_s))z_n , \quad \forall n \geq 0 .
\]

Hence

\[
g = \sum_{n=0}^{\infty} z_n (\lambda_n + 1 + \eta_s(\|u\|_s)) f_n = \sum_{n=0}^{\infty} z_n B_u f_n
\]

and \( f := B_u^{-1} g \) is in \( H_+^{1-s} \) and satisfies \( f = \sum_{n=0}^{\infty} z_n f_n \). Altogether we have thus proved that

\[
K_{u;1-s} : H_+^{1-s} \rightarrow h^{1-s}(\mathbb{N}_0) , \ f \mapsto (\langle f | f_n \rangle)_{n \geq 0} ,
\]

is a linear isomorphism. Interpolating between \( K_{u;1-s} \) and \( K_{u;1-s} \) and between the adjoints of their inverses shows that for any \(-1 + s \leq t \leq 1 - s\),

\[
K_{u;t} : H^t \rightarrow h^t(\mathbb{N}_0) , \ f \mapsto (\langle f | f_n \rangle)_{n \geq 0}
\]

is a linear isomorphism. Going through the arguments of the proof one verifies that the operator norm of \( K_{u;t} \) and the one of its inverse can be uniformly bounded for \(-1 + s \leq t \leq 1 - s\) and for bounded subsets of elements \( u \in H^{-s}_{r,0} \).

With these preparations done, we can now prove Proposition 1(i).

**Proof of Proposition 1(i).** Let \( u \in H^{-s}_{r,0} \) with \( 0 \leq s < 1/2 \). By (27), one has for any \( n \geq 1 \),

\[
|\langle 1 | f_n \rangle|^2 = \gamma_n \kappa_n , \quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right).
\]

Note that the infinite product is absolutely convergent since the sequence \( (\gamma_n(u))_{n \geq 1} \) is in \( \ell^1_+ \) (cf. (28)). Furthermore, since

\[
1 - \frac{\gamma_p}{\lambda_p - \lambda_n} = \frac{\lambda_p - 1 - \lambda_n}{\lambda_p - \lambda_n} > 0 , \quad \forall p \neq n
\]
it follows that $\kappa_n > 0$ for any $n \geq 1$. Hence, the formula [12, (4.1)] of the Birkhoff coordinates $\zeta_n(u)$, $n \geq 1$, defined for $u \in L^2_{r,0}$,

$$
(31) \quad \zeta_n(u) = \frac{1}{\sqrt{\kappa_n(u)}} \langle 1|f_n(\cdot, u) \rangle,
$$
extends to $H^{-s}_{r,0}$. By (24) one has (cf. also [12, (2.13)])

$$
\lambda_n \langle 1|f_n \rangle = -\langle u|f_n \rangle = -\langle \Pi u|f_n \rangle.
$$
Since by Lemma 6, $(\langle \Pi u|f_n \rangle)_{n \geq 0} \in h^{-s}(\mathbb{N}_0)$ and by (30)

$$
\eta_n \langle \Pi u|f_n \rangle = n - \frac{1}{2} - \eta_n(\|u\|) \leq \lambda_n(u) \leq n, \quad \forall n \geq 0,
$$
one concludes that

$$
(\langle 1|f_n \rangle)_{n \geq 1} \in h^{1-s}_+, \quad \kappa^{-1/2}_n = \sqrt{n} + o(1)
$$
and hence $(\zeta_n(u))_{n \geq 1} \in h^{1/2-s}_+$. In summary, we have proved that for any $0 < s < 1/2$, the Birkhoff map $\Phi : L^2_{r,0} \rightarrow h^{1/2}_+$ of Theorem 4 extends to a map

$$
H^{-s}_{r,0} \rightarrow h^{1/2-s}_+, u \mapsto (\zeta_n(u))_{n \geq 1},
$$
which we again denote by $\Phi$. Going through the arguments of the proof one verifies that $\Phi$ maps bounded subsets of $H^{-s}_{r,0}$ into bounded subsets of $h^{1/2-s}_+$.

To show Proposition 1(ii) we first need to prove some additional auxilary results. By (25), the generating function is defined as

$$
\mathcal{H}_\lambda : H^{-s}_{r,0} \rightarrow \mathbb{C}, u \mapsto \langle (L_u + \lambda)^{-1}11 \rangle.
$$
For any given $u \in H^{-s}_{r,0}$, $\mathcal{H}_\lambda(u)$ is a meromorphic function in $\lambda \in \mathbb{C}$ with possible poles at the eigenvalues of $L_u$ and satisfies (cf. (26))

$$
(32) \quad \mathcal{H}_\lambda(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^\infty \left(1 - \frac{\gamma_n}{\lambda_n + \lambda}\right).
$$

**Lemma 7.** For any $0 \leq s < 1/2$, the following holds:

(i) For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\mathcal{H}_\lambda : H^{-s}_{r,0} \rightarrow \mathbb{C}$ is sequentially weakly continuous.

(ii) $(\sqrt{\gamma_n})_{n \geq 1} : H^{-s}_{r,0} \rightarrow h^{1/2-s}_+$ is sequentially weakly continuous. In particular, for any $n \geq 0$, $\lambda_n : H^{-s}_{r,0} \rightarrow \mathbb{R}$ is sequentially weakly continuous.

**Proof.** (i) Let $(u^{(k)})_{k \geq 1}$ be a sequence in $H^{-s}_{r,0}$ with $u^{(k)} \rightharpoonup u$ weakly in $H^{-s}_{r,0}$ as $k \rightarrow \infty$. By the definition of $\zeta_n(u)$ (cf. (27) – (31)) one has $|\zeta_n(u)|^2 = \gamma_n(u)$. Since by Proposition 1(i), $\Phi$ maps bounded subsets
of $H^{1/2-s}_{r,0}$ to bounded subsets of $H^{1/2-s}_+$, there exists $M > 0$ so that for any $k \geq 1$
\[ \|u\|, \|u^{(k)}\| \leq M \quad \sum_{n=1}^{\infty} n^{1-2s} \gamma_n(u), \quad \sum_{n=1}^{\infty} n^{1-2s} \gamma_n(u^{(k)}) \leq M. \]

By passing to a subsequence, if needed, we may assume that

\[(\gamma_n(u^{(k)})^{1/2})_{n \geq 1} \rightharpoonup (\rho_n^{1/2})_{n \geq 1} \]

weakly in $H^{1/2-s}(\mathbb{N}, \mathbb{R})$ where $\rho_n \geq 0$ for any $n \geq 1$. It then follows that $(\gamma_n(u^{(k)}))_{n \geq 1} \rightharpoonup (\rho_n)_{n \geq 1}$ strongly in $\ell^1(\mathbb{N}, \mathbb{R})$. Define

\[ \nu_n := n - \sum_{p=n+1}^{\infty} \rho_p, \quad \forall n \geq 0. \]

Then for any $n \geq 1$, $\nu_n = \nu_{n-1} + 1 + \rho_n$ and $\lambda_n(u^{(k)}) \rightarrow \nu_n$ uniformly in $n \geq 0$. Since $L_{u^{(k)}} \geq \lambda_0(\nu^{(k)})$ we infer that there exists $c > | - \nu_0 + 1|$ so that for any $k \geq 1$ and $\lambda \geq c$,

\[ L_{u^{(k)}} + \lambda : H^{1-s} \rightarrow H^{1-s}_+ \]

is a linear isomorphism whose inverse is bounded uniformly in $k$. Therefore

\[ w^{(k)}_\lambda := (L_{u^{(k)}} + \lambda)^{-1}[1], \quad \forall k \geq 1, \]

is a well-defined, bounded sequence in $H^{1-s}_+$. Let us choose an arbitrary countable subset $\Lambda \subset [c, \infty)$ with one cluster point. By a diagonal procedure, we extract a subsequence of $(w^{(k)}_\lambda)_{k \geq 1}$, again denoted by $(w^{(k)}_\lambda)_{k \geq 1}$, so that for every $\lambda \in \Lambda$, the sequence $(w^{(k)}_\lambda)$ converges weakly in $H^{1-s}_+$ to some element $v_\lambda \in H^{1-s}_+$. By Rellich's theorem

\[ (L_{u^{(k)}} + \lambda)w^{(k)}_\lambda \rightharpoonup (L_u + \lambda)v_\lambda \]

weakly in $H^{1-s}_+$ as $k \rightarrow \infty$. Since by definition, $(L_{u^{(k)}} + \lambda)w^{(k)}_\lambda = 1$ for any $k \geq 1$, it follows that for any $\lambda \in \Lambda$, $(L_u + \lambda)v_\lambda = 1$ and thus by the definition of the generating function

\[ \mathcal{H}_\lambda(u^{(k)}) = \langle w^{(k)}_\lambda | 1 \rangle \rightarrow \langle v_\lambda | 1 \rangle = \mathcal{H}_\lambda(u), \quad \forall \lambda \in \Lambda. \]

Since $\mathcal{H}_\lambda(u^{(k)})$ and $\mathcal{H}_\lambda(u)$ are meromorphic functions whose poles are on the real axis, it follows that the convergence holds for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This proves item (i).

(ii) We apply item (i) (and its proof) as follows. As mentioned above, $\lambda_n(u^{(k)}) \rightarrow \rho_n$, uniformly in $n \geq 0$. By the proof of item (i) one has for any $c \leq \lambda < \infty$,

\[ \mathcal{H}_\lambda(u^{(k)}) \rightarrow \frac{1}{\nu_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\rho_n}{\nu_n + \lambda}\right) \]
and we conclude that for any $\lambda \in \Lambda$

$$
\frac{1}{\lambda_0(u) + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n(u)}{\lambda_n(u) + \lambda}\right) = \mathcal{H}_\lambda(u) = \frac{1}{\nu_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\rho_n}{\nu_n + \lambda}\right).
$$

Since $\mathcal{H}_\lambda(u)$ and infinite product are meromorphic functions in $\lambda$, the functions are equal. In particular, they have the same zeroes and the same poles. Since the sequences $(\lambda_n(u))_{n \geq 0}$ and $(\nu_n(u))_{n \geq 0}$ are both listed in increasing order it follows that $\lambda_n(u) = \nu_n$ for any $n \geq 0$, implying that for any $n \geq 1$,

$$
\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1 = \nu_n - \nu_{n-1} - 1 = \rho_n.
$$

By (33) we then conclude that

$$
\left(\gamma_n(u^{(k)})^{1/2}\right)_{n \geq 1} \rightarrow \left(\gamma_n(u)^{1/2}\right)_{n \geq 1}
$$

weakly in $h^{1/2-s}(\mathbb{N}, \mathbb{R})$. \qed

**Corollary 4.** For any $0 \leq s < 1/2$ and $n \geq 1$, the functional $\kappa_n : H_{r,0}^{-s} \rightarrow \mathbb{R}$, introduced in (27), is sequentially weakly continuous.

**Proof.** Let $(u^{(k)})_{k \geq 1}$ be a sequence in $H_{r,0}^{-s}$ with $u^{(k)} \rightharpoonup u$ weakly in $H_{r,0}^{-s}$ as $k \rightarrow \infty$. By (29), one has for any $p < n$,

$$
\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n - \sum_{j=p+1}^{n} \gamma_j(u^{(k)})
$$

whereas for $p > n$

$$
\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n + \sum_{j=n+1}^{p} \gamma_j(u^{(k)}).
$$

By Lemma 7, one then concludes that

$$
\lim_{k \rightarrow \infty} \left(\lambda_p(u^{(k)}) - \lambda_n(u^{(k)})\right) - \left(\lambda_p(u) - \lambda_n(u)\right) = 0
$$

uniformly in $p, n \geq 0$. By the product formula (27) for $\kappa_n$ it then follows that for any $n \geq 1$

$$
\lim_{k \rightarrow \infty} \kappa_n(u^{(k)}) = \kappa_n(u).
$$

\qed

Furthermore, we need to prove the following lemma concerning the eigenfunctions $f_n(\cdot, u)$, $n \geq 0$, of $L_u$.

**Lemma 8.** Given $0 \leq s < 1/2$, $M > 0$ and $n \geq 0$, there exists a constant $C_{s,M,n} \geq 1$ so that for any $u \in H_{r,0}^{-s}$ with $\|u\|_{-s} \leq M$ and any $n \geq 0$

$$
\|f_n(\cdot, u)\|_{1-s} \leq C_{s,M,n}.
$$
Proof. By the normalisation of \(f_n\), \(\|f_n\| = 1\). Since \(f_n\) is an eigenfunction, corresponding to the eigenvalue \(\lambda_n\), one has
\[-i\partial_x f_n = L_u f_n + T_u f_n = \lambda_n f_n + T_u f_n,
\]
implies that
\[
\|\partial_x f_n\|_{-s} \leq |\lambda_n| + \|T_u f_n\|_{-s}.
\]
Note that by the estimates (30),
\[
|\lambda_n| \leq \max\{n, |\lambda_0|\} \leq n + |\lambda_0|,
\]
where \(\eta_n(\|u\|_{-s})\) is given by (18). Furthermore, since \(\sigma = (1/2 + s)/2\) (cf. (16)) one has \(1 - s > 1 - \sigma > 1/2\), implying that \(H_+^{1-\sigma}\) acts on \(H_+^t\) for every \(t\) in the open interval \((-1 - \sigma, 1 - \sigma)\). Hence
\[
\|T_u f_n\|_{-s} \leq C_s \|u\|_{-s} \|f_n\|_{1-\sigma}.
\]
Using interpolation and Young’s inequality (cf. (19), (20)), (37) yields an estimate, which together with (35) and (36) leads to the claimed estimate (34).

With these preparations done, we can now prove Proposition 1(ii).

Proof of Proposition 1(ii). First we prove that for any \(0 \leq s < 1/2\), \(\Phi : H_{r,0}^{-s} \rightarrow h_+^{1/2-s}\) is sequentially weakly continuous: assume that \((u^{(k)})_{k\geq 1}\) is a sequence in \(H_{r,0}^{-s}\) with \(u^{(k)} \rightharpoonup u\) weakly in \(H_{r,0}^{-s}\) as \(k \to \infty\). Let \(\zeta^{(k)} := \Phi(u^{(k)})\) and \(\zeta := \Phi(u)\). Since \((u^{(k)})_{k\geq 1}\) is bounded in \(H_{r,0}^{-s}\) and \(\Phi\) maps bounded subsets of \(H_{r,0}^{-s}\) to bounded subsets of \(h_+^{1/2-s}\), the sequence \((\zeta^{(k)})_{k\geq 1}\) is bounded in \(h_+^{1/2-s}\). To show that \(\zeta^{(k)} \rightharpoonup \zeta\) weakly in \(h_+^{1/2}\), it then suffices to prove that for any \(n \geq 1\),
\[
\lim_{k \to \infty} \zeta_n^{(k)} = \zeta_n.
\]
By the definition of the Birkhoff coordinates (31),
\[
\zeta_n^{(k)} = \langle 1|f_n^{(k)}(\cdot)/\kappa_n^{(k)} \rangle^{1/2} \text{ where } \kappa_n^{(k)} := \kappa_n(u^{(k)}) \text{ and } f_n^{(k)} := f_n(\cdot, u^{(k)}).
\]
By Corollary 4, \(\lim_{k \to \infty} \kappa_n^{(k)} = \kappa_n\) and by Lemma 8, saying that for any \(n \geq 0\), \(\|f_n\|_{1-s}\) is uniformly bounded on bounded subsets of \(H_{r,0}^{-s}\),
\[
\lim_{k \to \infty} \langle 1|f_n^{(k)} \rangle = \langle 1|f_n \rangle \text{ where } \kappa_n := \kappa_n(u) \text{ and } f_n := f_n(\cdot, u). \]
This implies that \(\lim_{k \to \infty} \zeta_n^{(k)} = \zeta_n\) for any \(n \geq 1\).

It remains to show that for any \(0 < s < 1/2\), \(\Phi : H_{r,0}^{-s} \rightarrow h_+^{1/2-s}\) is one-to-one. In the case where \(u \in L^2_{r,0}\), it was verified in the proof of [12, Proposition 4.2] that the Fourier coefficients \(\hat{u}(k), k \geq 1\), of \(u\) can be explicitly expressed in terms of the components \(\zeta_n(u)\) of the sequence \(\zeta(u) = \Phi(u)\). These formulas continue to hold for \(u \in H_{r,0}^{-s}\). This completes the proof of Proposition 1(ii).

4. Extension of \(\Phi\). Part 2

In this section we prove the second part of Theorem 5, which we again state as a separate proposition.
Proposition 2. (Extension of Φ. Part 2) For any $0 < s < 1/2$, the map $Φ : H_+^{-s} → h_+^{1/2-s}$ has the following additional properties:

(i) The inverse image of $Φ$ of any bounded subset of $h_+^{1/2-s}$ is a bounded subset in $H_+^{-s}$.

(ii) $Φ$ is onto and the inverse map $Φ^{-1} : h_+^{1/2-s} → H_+^{-s}$ is sequentially weakly continuous.

(iii) For any $0 < s < 1/2$, the Birkhoff map $Φ : H_+^{-s} → h_+^{1/2-s}$ and its inverse $Φ^{-1} : h_+^{1/2-s} → H_+^{-s}$ are continuous.

Remark 8. As mentioned in Remark 5, the map $Φ : L^2_{r,0} → h_+^{1/2}$ and its inverse $Φ^{-1} : h_+^{1/2} → L^2_{r,0}$ are sequentially weakly continuous.

Proof of Proposition 2(i). Let $0 < s < 1/2$ and $u ∈ H_+^{-s}$. Recall that by Corollary 2, $L_u$ is a self-adjoint operator with domain $\text{dom}(L_u) ⊂ H_+$, has discrete spectrum and is bounded from below. Thus $L_u - λ_0(u) + 1 ≥ 1$ where $λ_0(u)$ denotes the smallest eigenvalue of $L_u$. By the considerations in Section 3 (cf. Lemma 5), $L_u$ extends to a bounded operator $L_u : H_+^{1/2} → H_+^{-1/2}$ and satisfies

$$⟨L_u f, f⟩ = ⟨Df, f⟩ - ⟨u|f|^2⟩, \quad ∀ f ∈ H_+^{1/2}.$$ 

By Lemma 1(i) one has $|⟨u|f|^2⟩| ≤ C_{1,s}^2∥u∥_{-s}∥f∥_{1/2}^2$ for any $f ∈ H_+^{1/2}$ and hence

$$∥f∥^2 ≤ ⟨(L_u - λ_0(u) + 1)f, f⟩ ≤ ⟨Df, f⟩ + C_{1,s}^2∥u∥_{-s}∥f∥_{1/2}^2 + (-λ_0(u) + 1)∥f∥^2,$$

yielding the estimate

$$∥f∥^2 ≤ ⟨(L_u - λ_0(u) + 1)f, f⟩ ≤ M_u ∥f∥_{1/2}^2$$

where

$$M_u := C_{1,s}^2∥u∥_{-s} + (2 - λ_0(u)) .$$

(38)

To shorten notation, we will for the remainder of the proof no longer indicate the dependence of spectral quantities such as $λ_n$ or $γ_n$ on $u$ whenever appropriate. The square root of the operator $L_u - λ_0 + 1$,

$$R_u := (L_u - λ_0 + 1)^{1/2} : H_+^{1/2} → H_+ ,$$

can then be defined in terms of the basis $f_n ≡ f_n(⋅, u)$, $n ≥ 0$, of eigenfunctions of $L_u$ in a standard way as follows: By Lemma 6, any $f ∈ H_+^{1/2}$ has an expansion of the form $f = \sum_{n=0}^∞ ⟨f|f_n⟩ f_n$ where $((⟨f|f_n⟩)_{n≥0}$ is a sequence in $h_+^{1/2}(N_0)$. $R_u f$ is then defined as

$$R_u f := \sum_{n=0}^∞ (λ_n - λ_0 + 1)^{1/2} ⟨f|f_n⟩ f_n,$$
Since \((\lambda_n - \lambda_0 + 1)^{1/2} \sim \sqrt{n}\) (cf. (30)) one has
\[
((\lambda_n - \lambda_0 + 1)^{1/2} (f | f_n))^{1/2})_{n \geq 0} \in \ell^2(\mathbb{N}_0)
\]
implying that \(R_u f \in H_+\) (cf. Lemma 6). Note that
\[
\|f\|^2 \leq \langle R_u f | R_u f \rangle = \langle R_u^2 f | f \rangle \leq M_u \|f\|^2_{1/2}, \quad \forall f \in H_+^{1/2},
\]
and that \(R_u\) is a positive self-adjoint operator when viewed as an operator with domain \(H_+^{1/2}\), acting on \(H_+\). By complex interpolation (cf. e.g. [31, Section 1.4]) one then concludes that for any \(0 \leq \theta \leq 1\)
\[
R_u^\theta : H_+^{\theta/2} \to H_+, \quad \|R_u^\theta f\|^2 \leq M_u^\theta \|f\|^2_{\theta/2}, \quad \forall f \in H_+^{\theta/2}.
\]
Since by duality,
\[
R_u^\theta : H_+ \to H_+^{-\theta/2}, \quad \|R_u^\theta g\|^2_{-\theta/2} \leq M_u^\theta \|g\|^2, \quad \forall g \in H_+,
\]
one infers, using that \(R_u^\theta : H_+ \to H_+^{-\theta/2}\) is boundedly invertible, that for any \(f \in H_+^{-\theta/2}\),
\[
\|f\|^2_{-\theta/2} \leq M_u^\theta \|R_u^{-\theta} f\|^2, \quad R_u^{-\theta} := (R_u^\theta)^{-1}.
\]
Applying the latter inequality to \(f = \Pi u\) and \(\theta = 2s\) and using that \(\Pi u = \sum_{n=1}^\infty (\Pi u | f_n) f_n\) and \(\langle \Pi u | f_n \rangle = -\lambda_n \langle 1 | f_n \rangle\) one sees that
\[
\frac{1}{2} \|u\|_{-s}^2 = \|\Pi u\|_{-s}^2 \leq M_u^{2s} \Sigma
\]
where
\[
\Sigma := \sum_{n=1}^\infty \lambda_n^2 (\lambda_n - \lambda_0 + 1)^{-2s} \langle 1 | f_n \rangle^2.
\]
We would like to deduce from (39) an estimate of \(\|u\|_{-s}\) in terms of the \(\gamma_n\)’s. Let us first consider \(M_u^{2s}\). By (38) one has
\[
M_u^{2s} = 2^{2s} \max \left\{ (C_{1,s}^2 \|u\|_{-s})^{2s}, (2 - \lambda_0(u))^{2s} \right\},
\]
yielding
\[
M_u^{2s} \leq (\|u\|_{-s}^2) \|2C_{1,s}^2\|^{2s} + (2(2 - \lambda_0(u)))^{2s}.
\]
Applying Young’s inequality with \(1/p = s\), \(1/q = 1 - s\) one obtains
\[
(\|u\|_{-s}^2)^s (2C_{1,s}^2)^{2s} \Sigma \leq \frac{1}{4} \|u\|_{-s}^2 + \left( (4C_{1,s}^2)\Sigma \right)^{1/(1-s)},
\]
which when combined with (39) and (40), leads to
\[
\frac{1}{4} \|u\|_{-s}^2 \leq \left( (4C_{1,s}^2)\Sigma \right)^{1/(1-s)} + (2(2 - \lambda_0(u)))^{2s} \Sigma.
\]
The latter estimate is of the form
\[
\|u\|_{-s}^2 \leq C_{3,s} \Sigma^{1/(1-s)} + C_{4,s} (2 - \lambda_0(u))^{2s} \Sigma,
\]
where \( C_{s, 3}, C_{4, s} > 0 \) are constants, only depending on \( s \). Next let us turn to \( \Sigma = \sum_{n=1}^{\infty} \lambda_n^2 (\lambda_n - \lambda_0 + 1)^{-2s} |\langle 1 | f_n \rangle|^2 \). Since

\[
\lambda_n = n - \sum_{k=n+1}^{\infty} \gamma_k , \quad |\langle 1 | f_n \rangle|^2 = \gamma_n \kappa_n ,
\]

and

\[
\kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) ,
\]

the series \( \Sigma \) can be expressed in terms of the \( \gamma_n \)'s. To obtain a bound for \( \Sigma \) it remains to estimate the \( \kappa_n \)'s. Note that

\[
\prod_{p \neq n} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) \leq \prod_{p < n} \left( 1 + \frac{\gamma_p}{\lambda_n - \lambda_p} \right) \leq e^{\sum_{p=1}^{n} \gamma_p} \leq e^{-\lambda_0} .
\]

Since \( (\lambda_n - \lambda_0)^{-1} = (n + \sum_{k=1}^{n} \gamma_k)^{-1} \leq n^{-1} \), it then follows that

\[
0 < \kappa_n \leq \frac{e^{-\lambda_0}}{n} , \quad \forall n \geq 1.
\]

Combining the estimates above we get

\[
\Sigma \leq e^{-\lambda_0} \sum_{n=1}^{\infty} \lambda_n^2 n^{-2s-1} \gamma_n .
\]

By splitting the sum \( \Sigma \) into two parts, \( \Sigma = \sum_{n < -\lambda_0(u)} + \sum_{n \geq -\lambda_0(u)} \) and taking into account that \( 0 \leq \lambda_n \leq n \) for any \( n \geq -\lambda_0 \) and \( |\lambda_n| \leq -\lambda_0 \) for any \( 1 \leq n < -\lambda_0 \), one has

\[
\Sigma \leq (1 - \lambda_0)^2 e^{-\lambda_0} \sum_{n=1}^{\infty} n^{1-2s} \gamma_n .
\]

Together with the estimate (42) this shows that the inverse image by \( \Phi \) of any bounded subset of sequences in \( h^{1/2-s} \) is bounded in \( H^{r, 0} \). \( \square \)

**Proof of Proposition 2(ii).** First we prove that for any \( 0 < s < 1/2 \), \( \Phi : H^{-s}_{r, 0} \rightarrow h^{1/2-s} \) is onto. Given \( z = (z_n)_{n \geq 1} \in h^{1/2-s}_+ \), consider the sequence \( \zeta^{(k)} = (\zeta_n^{(k)})_{n \geq 1} \), defined for any \( k \geq 1 \) by

\[
\zeta_n^{(k)} = z_n \quad \forall 1 \leq n \leq k , \quad \zeta_n^{(k)} = 0 \quad \forall n > k .
\]

Clearly \( \zeta^{(k)} \rightarrow z \) strongly in \( h^{1/2-s} \). Since for any \( k \geq 1 \), \( \zeta^{(k)} \in h^{1/2}_+ \), Theorem 4 implies that there exists a unique element \( u^{(k)} \in L^2_{r, 0} \) with \( \Phi(u^{(k)}) = \zeta^{(k)} \). By Proposition 2(i), \( \sup_{k \geq 1} \| u^{(k)} \|_{-s} < \infty \). Choose a weakly convergent subsequence \( (u^{(k_j)})_{j \geq 1} \) of \( (u^{(k)})_{k \geq 1} \) and denote its weak limit in \( H^{-s}_{r, 0} \) by \( u \). Since by Proposition 1, \( \Phi : H^{-s}_{r, 0} \rightarrow h^{1/2-s}_+ \) is sequentially weakly continuous, \( \Phi(u^{(k_j)}) \rightarrow \Phi(u) \) weakly in \( h^{1/2-s}_+ \). On the other hand, \( \Phi(u^{(k_j)}) = \zeta^{(k_j)} \rightarrow z \) strongly in \( h^{1/2-s}_+ \), implying that \( \Phi(u) = z \). This shows that \( \Phi \) is onto.
It remains to prove that for any $0 \leq s < 1/2$, $\Phi^{-1}$ is sequentially weakly continuous. Assume that $(\zeta^{(k)})_{k \geq 1}$ is a sequence in $h^{1/2-s}$, weakly converging to $\zeta \in h^{1/2-s}$. Let $u^{(k)} := \Phi^{-1}(\zeta^{(k)})$. By Proposition 2(i) (in the case $0 < s < 1/2$) and Remark 4(ii) (in the case $s = 0$), $(u^{(k)})_{k \geq 1}$ is a bounded sequence in $H^{-s}_{r,0}$ and thus admits a weakly convergent subsequence $(u^{(k)})_{j \geq 1}$. Denote its limit in $H^{-s}_{r,0}$ by $u$. Since by Proposition 1, $\Phi$ is sequentially weakly continuous, $\Phi(u^{(k)}) \rightharpoonup \Phi(u)$ weakly in $h^{1/2-s}$. On the other hand, by assumption, $\Phi(u^{(k)}) = \zeta^{(k)} \rightharpoonup \zeta$ and hence $u = \Phi^{-1}(\zeta)$ and $u$ is independent of the chosen subsequence $(u^{(k)})_{j \geq 1}$. This shows that $\Phi^{-1}(\zeta^{(k)}) \rightharpoonup \Phi^{-1}(\zeta)$ weakly in $H^{-s}_{r,0}$. □

**Proof of Proposition 2(iii).** By Proposition 1, $\Phi : H^{-s} \rightarrow h^{1/2-s}_{+}$ is sequentially weakly continuous for any $0 \leq s < 1/2$. To show that this map is continuous it then suffices to prove that the image $\Phi(A)$ of any relatively compact subset $A$ of $H^{-s}_{r,0}$ is relatively compact in $h^{1/2-s}_{+}$. For any given $\varepsilon > 0$, choose $N = N_{\varepsilon} \geq 1$ and $R \equiv R_{\varepsilon} > 0$ as in Lemma 9, stated below. Decompose $u \in A$ as $u = u_{N} + u_{\perp}$ where

$$u_{N} := \sum_{0 < |n| \leq N_{\varepsilon}} \hat{u}(n) e^{inx}, \quad u_{\perp} := \sum_{|n| > N_{\varepsilon}} \hat{u}(n) e^{inx}.$$ 

By Lemma 9, $\|u_{N}\| < R_{\varepsilon}$ and $\|u_{\perp}\|_{-s} < \varepsilon$. By Lemma 6, applied with $\theta = -s$, one has

$$K_{u:-s}(\Pi u) = K_{u:-s}(\Pi u_{N}) + K_{u:-s}(\Pi u_{\perp}) \in h^{-s}(N_{0})$$

where $K_{u:-s}(\Pi u_{N}) = K_{u,0}(\Pi u_{N})$ since $\Pi u_{N} \in H_{+}$. Lemma 6 then implies that there exists $C_{A} > 0$, independent of $u \in A$, so that

$$\|K_{u,0}(\Pi u_{N})\| \leq C_{A} R_{\varepsilon}, \quad \|K_{u:-s}(\Pi u_{\perp})\|_{-s} \leq C_{A} \varepsilon.$$ 

Since $\varepsilon > 0$ can be chosen arbitrarily small, it then follows by Lemma 9 that $K_{u:-s}(\Pi(A))$ is relatively compact in $h^{-s}(N_{0})$. Since by definition

$$\left(K_{u:-s}(\Pi u)\right)_{n} = \langle \Pi u | f_{n}(\cdot, u) \rangle, \quad \forall n \geq 0,$$

and since by (24),

$$\zeta_{n}(u) \simeq \frac{1}{\sqrt{n}} \langle \Pi u | f_{n}(\cdot, u) \rangle \quad \text{as} \quad n \to \infty$$

uniformly with respect to $u \in A$, it follows that $\Phi(A)$ is relatively compact in $h^{1/2-s}_{+}$.

Now let us turn to $\Phi^{-1}$. By Proposition 2(ii), $\Phi^{-1} : h^{1/2-s}_{+} \rightarrow H^{-s}_{r,0}$ is sequentially weakly continuous. To show that this map is continuous it then suffices to prove that the image $\Phi^{-1}(B)$ of any relatively compact subset $B$ of $h^{1/2-s}_{+}$ is relatively compact in $H^{-s}_{r,0}$. By the same arguments as above one sees that $\Phi^{-1} : h^{1/2-s}_{+} \rightarrow H^{-s}_{r,0}$ is also continuous. □
It remains to state Lemma 9, used in the proof of Proposition 2(iii). It concerns the well known characterization of relatively compact subsets of $H_{r,0}^s$ in terms of the Fourier expansion $u(x) = \sum_{n \neq 0} \hat{u}(n) e^{inx}$ of an element $u$ in $H_{r,0}^{-s}$.

**Lemma 9.** Let $0 < s < 1/2$ and $A \subset H_{r,0}^{-s}$. Then $A$ is relatively compact in $H_{r,0}^{-s}$ if and only if for any $\varepsilon > 0$, there exist $N_\varepsilon \geq 1$ and $R_\varepsilon > 0$ so that for any $u \in A$,

$$(\sum_{|n| > N_\varepsilon} |n|^{-2s} |\hat{u}(n)|^2)^{1/2} < \varepsilon, \quad (\sum_{0 < |n| \leq N_\varepsilon} |\hat{u}(n)|^2)^{1/2} < R_\varepsilon.$$  

The latter conditions characterize relatively compact subsets of $h^{-s}(\mathbb{N}_0)$.

**Proof of Theorem 5.** The claimed statements follow from Proposition 1 and Proposition 2. In particular, item (ii) of Theorem 5 follows from Proposition 2(i). $\square$

## 5. Solution Maps $S_0$, $S_B$ and $S_c$, $S_{c,B}$

In this section we provide results related to the solution map of (1), which will be used to prove Theorem 1 in the subsequent section.

### 5.1. Solution Map $S_B$ and its Extension.

First we study the map $S_B$, defined in Section 2 on $h_+^{1/2}$. Recall that by (12) – (13), the $n$th frequency of (1) is a real valued map defined on $\ell_2^+$ by

$$\omega_n(\zeta) := n^2 - 2 \sum_{k=1}^{n} k|\zeta_k|^2 - 2n \sum_{k=n+1}^{\infty} |\zeta_k|^2.$$  

For any $0 < s \leq 1/2$, the map $S_B$ naturally extends to $h_+^{1/2-s}$, mapping initial data $\zeta(0) \in h_+^{1/2-s}$ to the curve

$$S_B(\cdot, \zeta(0)) : \mathbb{R} \to h_+^{1/2-s}, \quad t \mapsto S_B(t, \zeta(0)) := (\zeta_n(0)e^{\omega_n(t)})_{n \geq 1}.$$  

We first record the following properties of the frequencies.

**Lemma 10.** (i) For any $n \geq 1$, $\omega_n : \ell_2^+ \to \mathbb{R}$ is continuous and

$$|\omega_n(\zeta) - n^2| \leq 2n\|\zeta\|_0^2, \quad \forall \zeta \in \ell_2^+; \quad |\omega_n(\zeta) - n^2| \leq 2\|\zeta\|_{1/2}^2, \quad \forall \zeta \in h_+^{1/2}.$$  

(ii) For any $0 \leq s < 1/2$, $\omega_n : h_+^{1/2-s} \to \mathbb{R}$ is sequentially weakly continuous.

**Proof.** Item (i) follows in a straightforward way from the formula (12) of $\omega_n$. Since for any $0 \leq s < 1/2$, $h_+^{1/2-s}$ compactly embeds into $\ell_2^+$, item (ii) follows from (i). $\square$

From Lemma 10 one infers the following properties of $S_B$. We leave the easy proof to the reader.
Proposition 3. For any \(0 \leq s \leq 1/2\), the following holds:

(i) For any initial data \(\zeta(0) \in h^{1/2-s}_+\), the curve

\[
\mathbb{R} \to h^{1/2-s}_+, \ t \mapsto S_B(t, \zeta(0))
\]

is continuous.

(ii) For any \(T > 0\),

\[
S_B : h^{1/2-s}_+ \to C([-T, T], h^{1/2-s}_+), \ \zeta(0) \mapsto S_B(\cdot, \zeta(0)),
\]

is continuous. For any \(t \in \mathbb{R}\),

\[
S_B^t : h^{1/2-s}_+ \to h^{1/2-s}_+, \ \zeta(0) \mapsto S_B(t, \zeta(0)),
\]

is a homeomorphism.

5.2. Solution map \(S_0\) and its extension. Recall that in Section 2 we introduced the solution map \(S_0\) of (1) on the subspace space \(L^2_{r,0}\) of \(L^2\), consisting of elements in \(L^2\) with average 0, in terms of the Birkhoff map \(\Phi\),

\[
S_0 = \Phi^{-1}S_B\Phi : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0}).
\]

Theorem 5 will now be applied to prove the following result about the extension of \(S_0\) to the Sobolev space \(H^{s}_{r,0}\) with \(0 < s < 1/2\), consisting of elements in \(H^s\) with average zero. It will be used in Section 6 to prove Theorem 1.

Proposition 4. For any \(0 \leq s < 1/2\), the following holds:

(i) The Benjamin-Ono equation is globally \(C^0\)-well-posed on \(H^s_{r,0}\).

(ii) There exists an increasing function \(F_s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) so that

\[
\|u\|_{-s} \leq F_s(\|\Phi(u)\|_{1/2-s}), \ \forall u \in H^s_{r,0}.
\]

In particular, for any initial data \(u(0) \in H^s_{r,0}\),

\[
\sup_{t \in \mathbb{R}} \|S_0^t(u(0))\|_{-s} \leq F_s(\|\Phi(u(0))\|_{1/2-s}).
\]

Remark 9. (i) By the trace formula (7), for any \(u(0) \in L^2_{r,0}\), estimate (46) can be improved as follows,

\[
\|u(t)\| = \sqrt{2}\|\Phi(u(0))\|_{1/2} = \|u(0)\|, \ \forall t \in \mathbb{R}.
\]

(ii) We refer to the comments of Theorem 1 in Section 1 for a discussion of the recent results of Talbut [29], related to (46).

Proof. Statement (i) follows from the corresponding statements for \(S_B\) in Proposition 3 and the continuity properties of \(\Phi\) and \(\Phi^{-1}\) stated in Theorem 5.

(ii) By Theorem 5 there exists an increasing function \(F_s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) so that for any \(u \in H^s_{r,0}\), \(\|u\|_{-s} \leq F_s(\|\Phi(u)\|_{1/2-s})\). Since the norm of \(h^{1/2-s}\) is left invariant by the flow \(S_B^t\), it follows that for any initial data \(u(0) \in H^s_{r,0}\), one has \(\sup_{t \in \mathbb{R}} \|S_B^t(u(0))\|_{-s} \leq F_s(\|\Phi(u(0))\|_{1/2-s})\). □
5.3. Solution map $S_c$. Next we introduce the solution map $S_c$ where $c$ is a real parameter. Let $v(t, x)$ be a solution of (1) with initial data $v(0) \in H^s_r$ and $s > 3/2$, satisfying the properties (S1) and (S2) stated in Section 1. By the uniqueness property in (S1), it then follows that

\begin{equation}
(47) \quad v(t, x) = u(t, x - 2ct) + c, \quad c = \langle v(0) \rangle_1
\end{equation}

where $u \in C(\mathbb{R}, H^s_{r,0}) \cap C^1(\mathbb{R}, H^{s-2}_{r,0})$ is the solution of the initial value problem

\begin{equation}
(48) \quad \partial_t u = H\partial_x^2 u - \partial_x(u^2), \quad u(0) = v(0) - \langle v(0) \rangle_1,
\end{equation}

satisfying (S1) and (S2). It then follows that $w(t, x) := u(t, x - 2ct)$ satisfies $w(0) = u(0)$ and

\begin{equation}
(49) \quad \partial_t w = H\partial_x^2 w - \partial_x(w^2) + 2c\partial_x w.
\end{equation}

By (47), the solution map of (49), denoted by $S_c$, is related to the solution map $S$ of (1) (cf. property (S2) stated in Section 1) by

\begin{equation}
(50) \quad S(t, v(0)) = S_{v(0)}(t, v(0)) + [v(0)], \quad [v(0)] := \langle v(0) \rangle_1.
\end{equation}

In particular, for any $s > 3/2$,

\begin{equation}
(51) \quad S_c : H^s_{r,0} \to C(\mathbb{R}, H^s_{r,0}), w(0) \mapsto S_c(\cdot, w(0))
\end{equation}

is well defined and continuous. Molinet’s results in [22] (cf. also [24]) imply that the solution map $S_c$ continuously extends to any Sobolev space $H^s_{r,0}$ with $0 \leq s \leq 3/2$. More precisely, for any such $s$, $S_c : H^s_{r,0} \to C(\mathbb{R}, H^s_{r,0})$ is continuous and for any $v_0 \in H^s_{r,0}$, $S_c(t, w_0)$ satisfies equation (1) in $H^{s-2}_{r,0}$.

5.4. Solution map $S_{c,B}$ and its extension. Arguing as in Section 2, we use Theorem 4, to express the solution map $S_{c,B}$, corresponding to the equation (49) in Birkhoff coordinates. Note that (49) is Hamiltonian, $\partial_t w = \partial_x \nabla H_c$, with Hamiltonian

$\mathcal{H}_c : H^s_{r,0} \to \mathbb{R}$, \hspace{1cm} $\mathcal{H}_c(w) = \mathcal{H}(w) + 2c\mathcal{H}^{(0)}(w)$

where by (2), $\mathcal{H}^{(0)}(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} w^2 dx$. Since by Parseval’s formula, derived in [12, Proposition 3.1],

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} w^2 dx = \sum_{n=1}^{\infty} n|\zeta_n|^2$$

one has

$$\mathcal{H}_{c,B}(\zeta) := \mathcal{H}_c(\Phi^{-1}(\zeta)) = \mathcal{H}_B(\zeta) + 2c \sum_{n=1}^{\infty} n|\zeta_n|^2,$$

implying that the corresponding frequencies $\omega_{c,n}$, $n \geq 1$, are given by

$$\omega_{c,n}(\zeta) = \partial|\zeta_n|^2 \mathcal{H}_{c,B}(\zeta) = \omega_n(\zeta) + 2cn.$$
For any \( c \in \mathbb{R} \), denote by \( S_{c,B} \) the solution map of (49) when expressed in Birkhoff coordinates,

\[
S_{c,B} : h^{1/2} \to C(\mathbb{R}, h^{1/2}) , \zeta(0) \mapsto [t \mapsto (\zeta_n(0)e^{it\omega_{c,n}(\zeta(0))})_{n \geq 1}].
\]

Note that \( \omega_{0,n} = \omega_n \) and hence \( S_{0,B} = S_B \). Using the same arguments as in the proof of Proposition 3 one obtains the following

**Corollary 5.** The statements of Proposition 3 continue to hold for \( S_{c,B} \) with \( c \in \mathbb{R} \) arbitrary.

5.5. **Extension of the solution map \( S_c \).** Above, we introduced the solution map \( S_c \) on the subspace space \( L^2_{r,0} \). One infers from (50) that

\[
S_c = \Phi^{-1}S_{c,B}\Phi : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0}).
\]

Using the same arguments as in the proof of Proposition 4 one infers from Corollary 5 the following results, concerning the extension of \( S_c \) to the Sobolev space \( H^{-s}_{r,0} \) with \( 0 < s < 1/2 \).

**Corollary 6.** The statements of Proposition 4 continue to hold for \( S_{c,B} \) with \( c \in \mathbb{R} \) arbitrary.

6. **Proofs of the main results**

**Proof of Theorem 1.** Theorem 1 is a straightforward consequence of Proposition 4 and Corollary 6. \( \square \)

**Proof of Remark 2 (Illposedness of (1) in \( H^{-s} \) for \( s > 1/2 \)).** Since the general case can be proved by the same arguments we consider only the case \( c = 0 \). Let \( t \neq 0 \). To show that the solution map \( S^t \) cannot be extended to \( H^{-s}_{r,0} \) for \( s > 1/2 \), we will study one gap solutions. Without further reference, we use notations and results from [12, Appendix B], where one gap potentials have been analyzed. Consider the following family of one gap potentials of average zero, parametrized by \( 0 \leq q < 1 \),

\[
u_0,q(x) = 2\text{Re}(qe^{ix}/(1 - qe^{ix})), \quad 0 < q < 1.
\]

The gaps \( \gamma_n(u_{0,q}) \), \( n \geq 1 \), of \( u_{0,q} \) can be computed as

\[
\gamma_{1,q} := \gamma_1(u_{0,q}) = q^2/(1 - q^2), \quad \gamma_n(u_{0,q}) = 0, \quad \forall n \geq 2.
\]

The frequency \( \omega_{1,q} := \omega_1(u_{0,q}) \) is thus given by (cf. (12))

\[
\omega_{1,q} = 1 - 2\gamma_{1,q} = \frac{1 - 3q^2}{1 - q^2}.
\]

The one gap solution, also referred to as travelling wave solution, of the BO equation with initial data \( u_{0,q} \) is then given by

\[
u_q(t,x) = u_{0,q}(x + \omega_{1,q}t), \quad \forall t \in \mathbb{R}.
\]

Note that for any \( s > 1/2 \),

\[
\lim_{q \to 1} u_{0,q} = 2\text{Re}(\sum_{k=1}^{\infty} e^{ikx}) = \delta_0 - 1
\]
strongly in $H_{r,0}^{-s}$ where $\delta_0$ denotes the periodic Dirac $\delta$—distribution, centered at 0. Since $\omega_{1,q} \to -\infty$ as $q \to 1$, it follows that for any $t \neq 0$, $u_q(t, \cdot)$ does not converge in the sense of distributions as $q \to 1$. □

Proof of Theorem 2. We argue similarly as in the proof of [12, Theorem 2]. Since the case $c \neq 0$ is proved be the same arguments we only consider the case $c = 0$. Let $u_0 \in H_{r,0}^{-s}$ with $0 \leq s < 1/2$ and let $u(t) := S_0(t, u_0)$. By formula (44), $\zeta(t) := S_B(t, \Phi(u_0))$ evolves on the torus $\text{Tor} (\Phi(u_0))$, defined by (15).

(i) Since $\text{Tor} (\Phi(u_0))$ is compact in $h_+^{1/2-s}$ and $\Phi^{-1} : h_+^{1/2-s} \to H_{r,0}^{-s}$ is continuous, $\{ u(t) : t \in \mathbb{R} \}$ is relatively compact in $H_{r,0}^{-s}$.

(ii) In order to prove that $t \mapsto u(t)$ is almost periodic, we appeal to Bochner’s characterization of such functions (cf. e.g. [21]): a bounded continuous function $f : \mathbb{R} \to X$ with values in a Banach space $X$ is almost periodic if and only if the set $\{ f_{\tau}, \tau \in \mathbb{R} \}$ of functions defined by $f_{\tau}(t) := f(t + \tau)$ is relatively compact in the space $C_b(\mathbb{R}, X)$ of bounded continuous functions on $\mathbb{R}$ with values in $X$. Since $\Phi : H_{r,0}^{-s} \to h_+^{1/2-s}$ is a homeomorphism, in the case at hand, it suffices to prove that for every sequence $(\tau_k)_{k \geq 1}$ of real numbers, the sequence $f_{\tau_k}(t) := \Phi(u(t + \tau_k))$, $k \geq 1$, in $C_b(\mathbb{R}, h_+^{1/2-s})$ admits a subsequence which converges uniformly in $C_b(\mathbb{R}, h_+^{1/2-s})$. Notice that

$$f_{\tau_k}(t) = (\zeta_n(u(0))e^{i\omega_n(t+\tau_k)})_{n \geq 1}.$$

By Cantor’s diagonal process and since the circle is compact, there exists a subsequence of $(\tau_k)_{k \geq 1}$, again denoted by $(\tau_k)_{k \geq 1}$, so that for any $n \geq 1$, $\lim_{k \to \infty} e^{i\omega_n \tau_k}$ exists, implying that the sequence of functions $f_{\tau_k}$ converges uniformly in $C_b(\mathbb{R}, h_+^{1/2-s})$. □

Proof of Theorem 3. Since the general case can be proved by the same arguments we consider only the case $c = 0$. By [12, Proposition B.1], the travelling wave solutions of the BO equation on $\mathbb{T}$ coincide with the one gap solutions. Without further reference, we use notations and results from [12, Appendix B], where one gap potentials have been analyzed. Let $u_0$ be an arbitrary one gap potential. Then $u_0$ is $C^\infty$—smooth and there exists $N \geq 1$ so that $\gamma_N(u_0) > 0$ and $\gamma_n(u_0) = 0$ for any $n \neq N$. Furthermore, the orbit of the corresponding one gap solution is given by $\{ u_0(\cdot + \tau) : \tau \in \mathbb{R} \}$. Let $0 \leq s < 1/2$. It is to prove that for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any $v(0) \in H_{r,0}^{-s}$ with $\| v(0) - u_0 \|_{-s} < \delta$ one has

$$\sup_{t \in \mathbb{R}} \inf_{\tau \in \mathbb{R}} \| v(t) - u_0(\cdot + \tau) \|_{-s} < \varepsilon .$$

To prove the latter statement, we argue by contradiction. Assume that there exists $\varepsilon > 0$, a sequence $(v^{(k)}(0))_{k \geq 1}$ in $H_{r,0}^{-s}$, and a sequence
that inf \( \tau \in \mathbb{R} \) \( \|v^{(k)}(t_k) - u_0(\cdot + \tau)\|_{-s} \geq \varepsilon \), \( \forall k \geq 1 \), \( \lim_{k \to \infty} \|v^{(k)}(0) - u_0\|_{-s} = 0 \).

Since \( A := \{v^{(k)}(0) \mid k \geq 1\} \cup \{u_0\} \) is compact in \( H_{r,0}^{-s} \) and \( \Phi \) is continuous, \( \Phi(A) \) is compact in \( h_{1/2-s} \) and
\[
\lim_{k \to \infty} \|\Phi(v^{(k)}(0)) - \Phi(u_0)\|_{1/2-s} = 0.
\]

It means that
\[
\lim_{k \to \infty} \sum_{n=1}^{\infty} n^{1-2s}|\zeta_n(v^{(k)}(0)) - \zeta_n(u_0)|^2 = 0.
\]

Note that for any \( k \geq 1 \),
\[
\zeta_n(v^{(k)}(t_k)) = \zeta_n(v^{(k)}(0)) e^{it_n\omega_n(v^{(k)}(0))}, \quad \forall n \geq 1
\]
and \( \zeta_n(u(t_k)) = \zeta_n(u_0) = 0 \) for any \( n \neq N \). Hence
\[
\lim_{k \to \infty} \sum_{n \neq N} n^{1-2s}|\zeta_n(v^{(k)}(t_k))|^2 = 0,
\]
and since \( |\zeta_N(v^{(k)}(t_k))| = |\zeta_N(v^{(k)}(0))| \) one has
\[
\lim_{k \to \infty} \left| |\zeta_N(v^{(k)}(t_k))| - |\zeta_N(u_0)| \right| = 0,
\]
implying that \( \sup_{k \geq 1} |\zeta_N(v^{(k)}(t_k))| < \infty \). It thus follows that the subset \( \{v^{(k)}(t_k) \mid k \geq 1\} \) is relatively compact in \( h_{1/2-s} \) and hence \( \{v^{(k)}(t_k) : k \geq 1\} \) relatively compact in \( H_{r,0}^{-s} \). Choose a subsequence \( (v^{(k)}(t_k))_{j \geq 1} \) which converges in \( H_{r,0}^{-s} \) and denote its limit by \( w \in H_{r,0}^{-s} \).

By (54)–(55) one infers that there exists \( \theta \in \mathbb{R} \) so that
\[
\zeta_n(w) = 0, \quad \forall n \neq N, \quad \zeta_N(w) = \zeta_N(u_0) e^{i\theta}.
\]
As a consequence, \( w(x) = u_0(x + \theta/N) \), contradicting the assumption that \( \inf_{\tau \in \mathbb{R}} \|v^{(k)}(t_k) - u_0(\cdot + \tau)\|_{-s} \geq \varepsilon \) for any \( k \geq 1 \). \( \square \)

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