EXEMPLARY GENERALISED NETWORK DESCRIPTORS

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Abstract. In communication networks theory the concepts of networkness and network surplus have recently been defined. Together with transmission and betweenness centrality, they were based on the assumption of equal communication between vertices. Generalised versions of these four descriptors were presented, taking into account that communication between vertices $u$ and $v$ is decreasing as the distance between them is increasing. Therefore, we weight the quantity of communication by $\lambda d(u,v)$ where $\lambda \in (0,1)$. Extremal values of these descriptors are analysed.

1. Introduction

Complex networks are extensively used to model objects and their relations [2], [12]. Throughout this paper we consider the representation of a complex network as a simple connected graph $G = (V,E)$ and use standard graph–theoretical terminology [3].

Betweenness centrality is one of key concepts in the study of complex networks [8], [9] and it can be efficiently calculated by algorithm of Brandes [5].

For an edge $uv$, edge betweenness $b(uv)$ is defined in the following way:

$$b(uv) = \sum_{\{k,l\} \in \binom{V}{2}} \frac{s_{kl}^{uv}}{s^{kl}},$$

where $s_{kl}^{uv}$ is the number of shortest paths between vertices $k$ and $l$ that pass through the edge $uv$ and $s^{kl}$ is the total number of shortest paths between $k$ and $l$. 

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Betweenness centrality $c(u)$ of a vertex $u$ is the sum of edge betweennesses of all edges incident to $u$:

$$c(u) = \sum_{v \in [u]} b(uv),$$

where $[u]$ is the set of neighbours of vertex $u$. Note that this measure is closely related to, yet different from Freeman’s betweenness centrality defined in [6]:

$$b(u) = c(u) - n + 1.$$

In the context of the communication networks, betweenness centrality $c(u)$ can be interpreted as the quantity of communication processed by a vertex $u$ as stated in [13]. On the other hand, transmission of the vertex $u$ defined as

$$t(u) = \sum_{v \in V} d(u, v)$$

where $d(u, v)$ is the distance between vertices $u$ and $v$, can be interpreted as the cost of the vertex to the network [13].

Network surplus of the vertex $u$ (“added value” to the network provided by vertex $u$) is defined by $\nu(u) = c(u) - t(u)$. Another way to measure productivity of vertex $u$ is its networkness defined in [13] by $N(u) = c(u)/t(u)$ . Note that interpretation of the betweenness centrality as the amount of information processed by the vertex $u$ assumes that the quantity of the information exchanged by any two vertices is equal. This was amended in [1] by weighting the amount of communication by $d(u, v)^{\lambda}$ for some $\lambda < 0$, generalising the case $\lambda = -1$ introduced in [4]. Now we consider network descriptors based on the assumption that the amount of communication decreases as the distance between two vertices increases. Moreover, we assume that this amount is proportional to $\lambda^{d(u,v)}$ where $\lambda \in (0,1)$. We define:

$$t^c_\lambda(u) = \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^{d(u,v)},$$

$$b^c_\lambda(uv) = \sum_{(k,t) \in \binom{V}{2}} \frac{k^{kl}}{k^{kl}} \cdot \lambda^{d(u,v)},$$

$$c^c_\lambda(u) = \sum_{v \in [u]} \sum_{u} b^c_\lambda(uv)$$

Furthermore, we define:

$$N^c_\lambda(u) = \frac{c^c_\lambda(u)}{t^c_\lambda(u)},$$

$$\nu^c_\lambda(u) = c^c_\lambda(u) - t^c_\lambda(u).$$

This way, instead of assuming that the amount of communication between any two vertices is equal, we assume the amount of communication between vertices decays as their distance increases. The interpretations of newly defined descriptors are the same as before; we observe betweenness centrality of vertex $u$ as the amount of communication that passes through $u$, the transmission of vertex $u$ is interpreted as how much is the cost of getting the information to $u$, and networkness and network surplus are measures of cost-benefit ratio/difference of vertex $u$.

Analogously as in [13] we define:
Exponential generalised network descriptors

\[ mc^\lambda_G = \min \{ c^\lambda(u) : u \in V \} \]
\[ Mc^\lambda_G = \max \{ c^\lambda(u) : u \in V \} \]
\[ mt^\lambda_G = \min \{ t^\lambda(u) : u \in V \} \]
\[ Mt^\lambda_G = \max \{ t^\lambda(u) : u \in V \} \]
\[ mN^\lambda_G = \min \{ N^\lambda(u) : u \in V \} \]
\[ MN^\lambda_G = \max \{ N^\lambda(u) : u \in V \} \]
\[ m\nu^\lambda_G = \min \{ \nu^\lambda(u) : u \in V \} \]
\[ M\nu^\lambda_G = \max \{ \nu^\lambda(u) : u \in V \} \]

and we are interested in finding the lower and upper bounds of these values for all \( \lambda \in (0, 1) \). Our results can be summarized in the following way:

**Table 1. Extremal values of exponential generalised network descriptors**

| Descriptor | \( \lambda \in (0, 1) \) | Lower bound | Upper bound |
|------------|--------------------------|-------------|-------------|
| \( mt^\lambda_G \) | broom (starting vertex) | \( A_n \) | \( (n-1)\lambda \) |
| \( Mt^\lambda_G \) | open problem | \( B_n \) | \( (n-1)\lambda \) |
| \( mc^\lambda_G \) | path (end vertices) | \( \frac{\lambda^D - \lambda}{\lambda - 1} \) | \( (n-1)\lambda + \frac{1}{2}(n-2)\lambda^2 \) |
| \( Mc^\lambda_G \) | open problem | \( (n-1)\lambda \) | \( (n-1)\lambda + \frac{1}{2}(n-2)\lambda^2 \) |
| \( mN^\lambda_G \) | broom (starting vertex) | \( C_n \) | \( \frac{1}{2}(n-2)\lambda + 1 \) |
| \( MN^\lambda_G \) | vertex-transitive graph | \( D_n \) | \( 0 \) |
| \( m\nu^\lambda_G \) | broom (starting vertex) | \( D_n \) | \( \frac{1}{2}(n-1)(n-2)\lambda^2 \) |
| \( M\nu^\lambda_G \) | vertex-transitive graph | \( \frac{1}{2}(n-1)(n-2)\lambda^2 \) |

The terms \( A_n, B_n, C_n \) and \( D_n \) from Table 1 represent the following expressions:

\[
A_n = \min_{1 \leq D \leq n-1} \frac{\lambda[1-(D+1)\lambda^D+D\lambda^{D+1}]}{(\lambda-1)^2} + (n-D-1)D \cdot \lambda^D,
\]
\[
B_n = \max_{1 \leq D \leq n-1} \frac{\lambda[1-(D+1)\lambda^D+D\lambda^{D+1}]}{(\lambda-1)^2} + (n-D-1)D \cdot \lambda^D,
\]
\[
C_n = \min_{1 \leq D \leq n-1} \frac{\lambda^{D+1} \left( \frac{\lambda^D - \lambda}{\lambda - 1} \right)}{D! \cdot \left( \frac{\lambda^D - \lambda}{\lambda - 1} \right)^{D+1}},
\]

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Theorem 3.1. For each graph \( B \) of the path is called starting vertex \( S \) pendant vertex of star broom the following definition. A \( D_k, l \) for a given pair of vertices \( (u, v) \) do not hold in general case.

Remark 1. Upper bounds marked (*) were stated and proven for \( \lambda \in (0, \frac{1}{2}) \). They do not hold in general case.

2. Connection between \( t^e_\lambda \) and \( c^e_\lambda \)

As in the papers [1] and [13], \( t^e_\lambda \) can be considered as the cost of the vertex to the network and \( c^e_\lambda \) can be considered as the quantity of communication processed by the same vertex. Let us prove that the sum of these quantities is equal.

Theorem 2.1. For each graph \( G \), it holds:

\[
\sum_{u \in V} t^e_\lambda(u) = \sum_{k, l \in V} d(k, l) \cdot \lambda^{d(k, l)} = \sum_{u \in V} c^e_\lambda(u).
\]

Proof. First equality holds by definition of transmission. Next, it holds

\[
\sum_{u \in V} c^e_\lambda(u) = \sum_{u \in V} \sum_{u \in V \in [u]} \sum_{k, l \in V} \sum_{k, l \in V} \lambda^{d(k, l)} = \sum_{u \in V} \sum_{u \in V \in [u]} \lambda^{d(k, l)} = \sum_{u \in V} \sum_{v \in [u]} s^{kl}_{uv}.
\]

For a given pair of vertices \((k, l)\), \( \sum_{u \in V \in [u]} s^{kl}_{uv} \) is number of pairs \((u, v)\) such that \( d(u, v) = 1 \) and that a shortest path between \( k \) and \( l \) passes through the edge \( uv \). The length of each of the \( s^{kl} \) shortest paths from \( k \) to \( l \) is \( d(k, l) \) and therefore on each such path we can choose \( d(k, l) \) pairs \( \{u, v\} \) such that \( d(u, v) = 1 \). Thus,

\[
\sum_{u \in V \in [u]} \sum_{v \in [u]} s^{kl}_{uv} = d(k, l) \cdot s^{kl}.
\]

Finally, we have

\[
\sum_{u \in V} t^e_\lambda(u) = \sum_{k, l \in V} \lambda^{d(k, l)} \cdot d(k, l) \cdot s^{kl} = \sum_{k, l \in V} d(k, l) \cdot \lambda^{d(k, l)}. \quad \square
\]

3. Transmission

Before concentrating on the lower and upper bounds for transmission, we need the following definition. A broom \( B_{n, k} \) is a graph obtained by identification of a pendant vertex of star \( S_{k+1} \) and an end–vertex of path \( P_{n-k} \). The other end–vertex of the path is called starting vertex of the broom. In particular, \( B_{n-1, 2} = P_n \) and \( B_{1, n} = S_n \). For minimal transmission let us prove:

Theorem 3.1. For each graph \( G \) with \( n \) vertices, it holds

\[
(1) \quad \min_{1 \leq D \leq n-1} \frac{\lambda [1 - (D + 1) \lambda^D + D \lambda^{D+1}]}{(\lambda - 1)^2} + (n - D - 1)(\lambda^D - D \cdot \lambda^D) \leq m t^e_\lambda(G).
\]

The lower bound is reached for a broom (in its starting vertex).

Proof. Let \( G \) be a graph for which the minimum \( m t^e_\lambda(G) \) is attained and let \( u \) be a vertex of the graph for which \( t^e_\lambda(u) = m t^e_\lambda(G) \). Let \( v_D \) be a vertex which is farthest away from \( u \) and let \( S = u v_1 v_2 \ldots v_D \) be a shortest path from \( u \) to \( v_D \). Furthermore,
let $k = n - D - 1$ and let $W = \{w_1, w_2, ..., w_k\}$ be a set of all vertices that do not lie on the path $S$. Since $d(u, v_i) = i$ for all $i \in \{1, 2, ..., D\}$ we have:

$$mt^*_\lambda(G) = \lambda^t(u) = \sum_{i=1}^D i \cdot \lambda^i + \sum_{w \in W} d(u, w) \cdot \lambda^{d(u, w)}.$$  

For positive numbers $a_1, a_2, ..., a_n$ such that $a = \min{\{a_1, a_2, ..., a_n\}}$, it holds

$$\sum_{i=1}^n a_i \geq \sum_{i=1}^n a = n \cdot a,$$

so

$$mt^*_\lambda(G) = \sum_{i=1}^D i \cdot \lambda^i + \sum_{w \in W} d(u, w) \cdot \lambda^{d(u, w)} \geq \sum_{i=1}^D i \cdot \lambda^i + (n - D - 1) \cdot x \cdot \lambda^x$$

where $x = d(u, q)$ for some $q \in W$ for which the expression $d(u, q) \cdot \lambda^{d(u, q)}$ has minimal value.

This means that the transmission will be minimal if all the vertices in $W$ are equally distant from $u$. We will prove that, in that case, $x = D$. Suppose the contrary. Let us observe graph $G'$ which is obtained by removing vertex $v_D$ and connecting it to $v_{D-1}$. Transmission in $G'$ will be smaller than in $G$ which is a contradiction, hence $x = D$. We conclude that one of the graphs for which minimal transmission is attained is indeed a broom, i.e. all the vertices in $W$ are directly connected to $v_{D-1}$. 

![Figure 1. A broom that minimizes $mt^*_\lambda(G)$.

Under certain assumptions, we can reduce the case $D \in \{1, ..., n - 1\}$ to the case when $D \in \{1, n - 1, \lfloor D_{\min} \rfloor, \lceil D_{\min} \rceil\}$. Let us prove:

**Theorem 3.2.** Function $f(D) = \sum_{i=1}^D i \cdot \lambda^i + (n - D - 1)D \cdot \lambda^D$ has local minimum for

$$D_1 = \frac{2 - 2\lambda + [1 + (\lambda - 1)n] \ln \lambda + \sqrt{3\lambda}}{2(\lambda - 1) \ln \lambda}$$

and local maximum for

$$D_2 = \frac{2 - 2\lambda + [1 + (\lambda - 1)n] \ln \lambda - \sqrt{3\lambda}}{2(\lambda - 1) \ln \lambda}$$
where

\[ S_\lambda = 4 \left( \lambda - 1 \right)^2 + \left[ (n - 1)^2 + \lambda^2 n^2 - 2\lambda \left( n^2 - n + 2 \right) \right] \ln \lambda, \]

if \( D_1, D_2 \in \mathbb{R} \).

**Proof.** The problem reduces to finding minimum (maximum) for the function

\[ f(D) = \sum_{i=1}^{D} i \cdot \lambda^i + (n - D - 1)D \cdot \lambda^D. \]

Deriving the function and simplifying it gives us

\[ f'(D) = \frac{\lambda D}{(\lambda - 1)^2} \left( AD^2 + BD + C \right); \]

where:

\[ A = 2\lambda \ln \lambda - \ln \lambda - \lambda^2 \ln \lambda; \]
\[ B = 4\lambda - 2 - 2\lambda^2 + \ln \lambda (\lambda - 1 + n - 2\lambda n + \lambda^2 n); \]
\[ C = \lambda + n - 1 - 2\lambda n + \lambda^2 n - \lambda \ln \lambda. \]

Stationary points are

\[ D_1 = \frac{2 - 2\lambda + [1 + (\lambda - 1)n] \ln \lambda + \sqrt{S_\lambda}}{2(\lambda - 1)\ln \lambda} \]

and

\[ D_2 = \frac{2 - 2\lambda + [1 + (\lambda - 1)n] \ln \lambda - \sqrt{S_\lambda}}{2(\lambda - 1)\ln \lambda} \]

where

\[ S_\lambda = 4 \left( \lambda - 1 \right)^2 + \left[ (n - 1)^2 + \lambda^2 n^2 - 2\lambda \left( n^2 - n + 2 \right) \right] \ln \lambda. \]

Let us analyze \( f'(D) \). Since \( \frac{\lambda D}{(\lambda - 1)^2} \) is always positive, whether the function \( f(D) \) is increasing or decreasing depends on the second-degree polynomial. The leading coefficient \( 2\lambda \ln \lambda - \ln \lambda - \lambda^2 \ln \lambda > 0 \) for \( \lambda \in (0,1) \). We conclude that, under the assumption that \( D_1, D_2 \in \mathbb{R} \), the function \( f(D) \) has minimal value for \( D_1 \) and maximal value for \( D_2 \).

**Remark 2.** Let us denote \( D_{\text{min}} = D_1 \). If \( D_{\text{min}} \in [1, n - 1] \) and real, than the minimum of expression \( \sum_{i=1}^{D} i \cdot \lambda^i + (n - D - 1)D \cdot \lambda^D \) will be reached for some \( D \in \{ 1, n - 1, \lfloor D_{\text{min}} \rfloor, \lceil D_{\text{min}} \rceil \} \). Otherwise, it will be reached for \( D \in \{ 1, n - 1 \} \). This remark can simplify the calculation of the lower bound in Theorem 3.1.

Now, let us concentrate on upper bound. We were able to find it in a special case when \( \lambda \in (0, \frac{1}{2}) \).

**Theorem 3.3.** For each graph \( G \) with \( n \) vertices and for \( \lambda \in (0, \frac{1}{2}) \), it holds

\[ \text{mt}_e^\lambda(G) \leq (n - 1) \cdot \lambda. \]

The lower bound is reached for any vertex of a complete graph.
Proof. Let $G$ be a graph for which the maximum $mt^i\lambda(G)$ is attained and let $u$ be a vertex of the graph for which $t^i\lambda(u) = mt^i\lambda(G)$. It holds:

$$mt^i\lambda(G) = t^i\lambda(u) = \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^d(u, v) \leq \sum_{v \in V \setminus \{u\}} \lambda = (n - 1) \cdot \lambda.$$  

The inequality holds since, for $\lambda \in \langle 0, \frac{1}{2} \rangle$ function $f(x) = x^\lambda$ is decreasing on integer interval $[1, \infty]$. The equality holds for a complete graph since $d(u, v) = 1$ for all $u, v \in V$.

Let us analyse the lower bound for $Mt^i\lambda(G)$. We find it only in the special case of 2–connected graph for $\lambda \in \langle 0, \frac{1}{2} \rangle$. Further we conjecture

**Conjecture 1.** For each graph $G$ with $n \geq 3$ vertices and for $\lambda \in \langle 0, \frac{1}{2} \rangle$, it holds that

$$
\begin{align*}
\sqrt[n]{\frac{\sqrt[n]{\lambda(2\sqrt[n]{\lambda+(n-1)\lambda^2}-(n-1)\lambda^n)}}}{(n-1)^2}}, & \text{ n odd} \\
\frac{1}{2} n \lambda^\frac{n}{2} + \sqrt[n]{\frac{\sqrt[n]{\lambda(2\sqrt[n]{\lambda+(n-1)\lambda^2}-(n-1)\lambda^n)}}}{(n-1)^2}}, & \text{ n even}
\end{align*}
\leq Mt^i\lambda(G)
$$

The equality holds for a cycle.

**Remark 3.** The previous conjecture is true in the special case when $G$ is a 2–connected graph. To prove this we need the following lemma.

**Lemma 3.4.** Let $n \geq 3$. Let $\lambda \in \langle 0, \frac{1}{2} \rangle$ and let $S$ be a set of sequences $(x_1, x_2, ..., x_{[n/2]}) \in \mathbb{N}_{[n/2]}$ such that $x_1 + x_2 + ... + x_{[n/2]} = n - 1$ and there exists $k \in \{1, ..., [n/2]\}$ such that $x_i \geq 2$ for each $i \leq k$ and $x_i = 0$ for each $i > k$. Let $S'$ be the set of sequences in $S$ of the form $(x_1, x_2, ..., x_{[n/2]})$ such that there is $k \in \{1, ..., [n/2]\}$ such that $x_k \in \{0, 1\}$, $x_i = 2$ for each $1 \leq i < k$ and $x_i = 0$ for each $i > k$.

Let $T_n$ be defined by

$$T_n(x_1, x_2, ..., x_{[n/2]}) = \sum_{i=1}^{[n/2]} x_i \cdot i \cdot \lambda^i.$$  

Then

$$\min \{T_n(s) : s \in S\} = \min \{T_n(s) : s \in S'\}.$$  

Furthermore, minimal value of $T_n$ in $S'$ is:

$$\begin{align*}
\begin{cases}
2 \cdot \sum_{i=1}^{n-1} i \cdot \lambda^i, & n \text{ odd} \\
2 \cdot \sum_{i=1}^{n-2} i \cdot \lambda^i + \left(\frac{n}{2}\right) \cdot \lambda^\frac{n}{2}, & n \text{ even}
\end{cases}
\end{align*}$$

Proof. Suppose to the contrary. Let $(x_1, x_2, ..., x_{[n/2]}) \notin S'$ minimize $T_n$ in $S$. Then there is $l$ such that $x_l > 2$. Note that $l < [n/2]$. Then,

$$(x_1, x_2, ..., x_l - 1, x_{l+1} + 1, x_{l+2}, ..., x_{[n/2]}) \in S.$$  

It follows that

$$0 \geq T_n(x_1, x_2, ..., x_{[n/2]}) - T_n(x_1, x_2, ..., x_l - 1, x_{l+1} + 1, x_{l+2}, ..., x_{[n/2]}) = l\lambda^l - (l + 1)\lambda^{l+1} > 0,$$
which is a contradiction. So the sequence \( s \in S' \) that minimizes \( T_n \) is \((2, 2, \ldots, 2, 0)\) for \( n \) odd, and \((2, 2, \ldots, 2, 1)\) for \( n \) even. It can be easily seen that the value of \( T_n \) for those sequences is \( 2 \cdot \sum_{i=1}^{\frac{n-1}{2}} i \cdot \lambda^i = \frac{\sqrt{\lambda}}{(\lambda - 1)^2} \left[ (\lambda^2 - n\lambda + 2) - (\lambda^2 + n\lambda + 2) \right] \) in the first case, and \( 2 \cdot \sum_{i=1}^{\frac{n-2}{2}} i \cdot \lambda^i + \left( \frac{\sqrt{\lambda}}{(\lambda - 1)^2} \right) \cdot \lambda^2 = \frac{1}{2} n \lambda^2 + \frac{\sqrt{\lambda}}{(\lambda - 1)^2} \left[ (\lambda^2 + n\lambda - 2) - (\lambda^2 - n\lambda - 2) \right] \) in the second case.

**Proof of Remark 3.** Let us denote the left–hand side of the inequality (4) by \( \text{cyc}_\lambda(n) \) and assume the contrary–that there exists a 2–connected graph \( G \) with \( n \) vertices such that \( M_{T\lambda}^*(G) < \text{cyc}_\lambda(n) \). This implies that \( t\lambda^*_n(u) < \text{cyc}_\lambda(n) \), for all \( u \in V \). Therefore:

\[
\sum_{u \in V} t\lambda^*_n(u) < n \cdot \text{cyc}_\lambda(n),
\]

and thus there exists \( w \in V \) such that \( t\lambda^*_n(w) < \text{cyc}_\lambda(n) \).

Let \( w_1 \) be the vertex that is farthest from \( w \) and let \( d(w, w_1) = D \). Since \( G \) is 2-connected, it holds that for every \( d < D \) there are at least 2 vertices on a distance \( d \) from \( w \). From that it is easily seen that \( D \leq \left\lfloor \frac{n}{2} \right\rfloor \). Let us denote with \( x_i \) the number of vertices on a distance \( i \) from \( w \), and let us observe the sequence \((x_1, x_2, \ldots, x_{\lfloor n/2 \rfloor})\). This sequence is obviously in \( S \) defined in Lemma 3.4. It follows \( \text{cyc}_\lambda(n) \leq t\lambda^*_n(w) \) which is a contradiction.

For the upper bound let us prove:

**Theorem 3.5.** For each graph \( G \) with \( n \) vertices, it holds

\[
M_{T\lambda}^*(G) \leq \max_{1 \leq D \leq n-1} \frac{\lambda \left[ 1 - (D + 1)\lambda^D + D\lambda^{D+1} \right]}{(\lambda - 1)^2} + (n - D - 1)D \cdot \lambda^D.
\]

The equality hold for a broom (in its starting vertex).

**Proof.** Let \( G \) be a graph for which the maximum \( M_{T\lambda}^*(G) \) is attained and let \( u \) be a vertex of the graph for which \( t\lambda^*_n(u) = M_{T\lambda}^*(G) \). Let \( v_D \) be a vertex which is farthest away from \( u \) and let \( S = u v_1 v_2 \ldots v_D \) be a shortest path from \( u \) to \( v_D \). Furthermore, let \( k = n - D - 1 \) and let \( W = \{w_1, w_2, \ldots, w_k\} \) be a set of all vertices that do not lie on the path \( S \). Since \( d(u, v_i) = i \) for all \( i \in \{1, 2, \ldots, D\} \) we have:

\[
M_{T\lambda}^*(G) = t\lambda^*_n(u) = \sum_{i=1}^{D} i \cdot \lambda^i + \sum_{w \in W} d(u, w) \cdot \lambda^{d(u, w)}.
\]

For positive numbers \( a_1, a_2, \ldots, a_n \) such that \( a = \max\{a_1, a_2, \ldots, a_n\} \), it holds

\[
\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a = n \cdot a,
\]

so

\[
M_{T\lambda}^*(G) = \sum_{i=1}^{D} i \cdot \lambda^i + \sum_{w \in W} d(u, w) \cdot \lambda^{d(u, w)} \leq \sum_{i=1}^{D} i \cdot \lambda^i + (n - D - 1)x \cdot \lambda^x
\]

where \( x \leq D \) such that \( x = d(u, q) \), for some \( q \in V(G) \backslash \{u\} \) for which the expression \( d(u, q) \lambda^{d(u, q)} \) has maximal value.
This means that the transmission will be maximal if all the vertices in \( W \) are equally distant from \( u \). We will prove that this holds when \( x = D \). In other words, the graph for which the maximal transmission is attained is a broom.

Let us assume that this is not the case, that instead \( x \leq D - 1 \). Let us observe graph \( G' \) obtained from \( G \) by removing vertex \( v_D \) and connecting it to \( v_{x-1} \). The distance between vertices \( u \) and \( v_D \) is \( x \) in \( G' \), so the value \( t_x^\lambda(u) \) in \( G' \) is:

\[
t_x^\lambda(u) = \sum_{i=1}^{D-1} i\lambda^i + (n - D)x\lambda^x.
\]

This value is greater then \( M_{t_x^\lambda}(G) \), because augend \( D\lambda^D \) is replaced with \( x\lambda^x \), and by the definition of \( x \), the expression \( a\lambda^x \) is greatest for \( x \), for all \( a \in \{1, ..., D\} \). So we’ve obtained graph \( G' \) such that \( M_{t_x^\lambda}(G') > M_{t_x^\lambda}(G) \), for \( x \leq D - 1 \), which is a contradiction. We conclude that \( x = D \) must hold and hence the desired graph is a broom. The length of the broom will depend on the value of \( \lambda \), it will be the number \( a \in \{1, ..., n - 1\} \) for which the expression \( a\lambda^a \), and hence the expression

\[
\sum_{i=1}^{a} i\lambda^i + (n - a - 1)a\lambda^a
\]

is the greatest. \( \Box \)

**Remark 4.** Let us denote \( D_2 \) in Theorem 3.2 as \( D_{\text{max}} \). If \( D_{\text{max}} \in \{1, n - 1\} \) and real, than the maximum of expression \( \sum_{i=1}^{D} i \cdot \lambda^i + (n - D - 1)D \cdot \lambda^D \) will be reached for some \( D \in \{1, n - 1, [D_{\text{max}}], \lfloor D_{\text{max}} \rfloor \} \). Otherwise, it will be reached for \( D \in \{1, n - 1\} \). This remark can simplify the calculation of the lower bound in Theorem 3.5.

### 4. Betweenness centrality

**Lemma 4.1.** For all \( \lambda \in (0, 1) \) and for a given integer \( n \), among all graphs with \( n \) vertices, any graph \( G \) for which maximum \( c_\lambda(G) \) is obtained is a tree.

**Proof.** Let \( G \) be a graph such that \( c_\lambda(G) \) is maximal and let \( u \) be a vertex for which maximal centrality is reached. We will prove that \( G \) is a tree.

Suppose that is not the case. Let us observe Dijkstra spanning tree \( G' \) that is obtained as follows: starting from vertex \( u \), in each step we choose a vertex \( v \) that is closest to \( u \) (the distance between \( u \) and \( v \) is minimal) and is still outside the tree. Since \( G' \) is a tree, it holds that \( s_{kl}^\lambda = 1 \) for each \( k, l \in V \) that are connected by a path passing through the edge \( uv \). From the way \( G' \) was obtained, it is obvious that the distances between \( u \) and \( v \), for every \( v \in V \), will remain the same. This means that \( c_\lambda(G) \) is greater in \( G' \) than in \( G \) which contradicts our assumption. \( \Box \)

**Lemma 4.2.** For each graph \( G \) with \( n \) vertices and for \( \lambda \in (0, 1) \), it holds

\[
\sum_{v \in V \setminus \{u\}} \lambda^{d(u,v)} \geq \sum_{i=1}^{n-1} \lambda^i = \frac{\lambda^D - \lambda}{\lambda - 1}.
\]

**Proof.** Let \( G \) be a graph with \( n \) vertices and let \( u, v_1, v_2, ..., v_D \) be the longest path in \( G \), i.e. \( D = \text{diam}(G) \). Let \( W \) be the set of all vertices that do not lie on this
path from $u$ to $v_D$. The observed sum for graph $G$ is

$$\sum_{i=1}^{D} \lambda^i + \sum_{w \in W} \lambda^d(u,w).$$

It holds $d(u,w)_G \leq D$, for all $w \in W$, and $\lambda \in (0,1)$. Let us consider graph $G'$ which is obtained by removing any vertex $w$ in $W$ and connecting in to vertex $v_D$, so that $d(u,w)_{G'} = D + 1$. For $G'$ the observed sum is smaller than the sum in $G$, because one augend $\lambda^d(u,w)_G$, for some $w \in W$, is changed to $\lambda^{D+1}$. By repeating this process, we conclude that the desired sum will be minimal if graph $G$ is a path, i.e., it holds

$$\sum_{v \in V \setminus \{u\}} \lambda^d(u,v) \geq \sum_{i=1}^{n-1} \lambda^i = \frac{\lambda^D - \lambda}{\lambda - 1}. \quad \square$$

Note that the quantity $\sum_{v \in V \setminus \{u\}} \lambda^d(u,v)$ is known as the decay centrality (see [7], [10], [11]).

**Theorem 4.3.** For each graph $G$ with $n$ vertices, it holds

$$\frac{\lambda^D - \lambda}{\lambda - 1} \leq mc^\lambda(G).$$

The lower bound is reached for a path (in its end–vertex).

**Proof.** Let us prove the lower bound. Let $G$ be a graph for which $mc^\lambda(G)$ is minimal and let $u$ be a vertex such that $c^\lambda(u) = mc^\lambda(G)$. Using Lemma 4.2, it holds

$$mc^\lambda(G) \geq \sum_{v \in V \setminus \{u\}} \lambda^d(u,v) \geq \sum_{i=1}^{n-1} \lambda^i = \frac{\lambda^D - \lambda}{\lambda - 1}. \quad \square$$

For the upper bound we solve the problem for $\lambda \in (0, \frac{1}{2})$.

**Theorem 4.4.** For each graph $G$ with $n$ vertices and for $\lambda \in (0, \frac{1}{2})$, it holds

$$mc^\lambda(G) \leq (n - 1) \cdot \frac{\lambda}{2}.$$  

The upper bound is reached for a complete graph (in any of its vertices).

**Proof.** Using Theorem 2.1, we can bound the average centrality of all vertices in the following way:

$$\frac{1}{n} \sum_{u \in V} c^\lambda(u) = \frac{1}{n} \sum_{k,l \in V} d(k,l) \cdot \lambda^d(k,l) \leq \frac{1}{n} \sum_{k,l \in V} \lambda = \frac{1}{n} n(n - 1) \cdot \lambda = (n - 1) \cdot \lambda.$$ 

Since minimal centrality is smaller than or equal to the average centrality, the claim is proven. The equality holds for a complete graph since $d(k,l) = 1$ for any two vertices $k,l \in V$. \quad \square

**Theorem 4.5.** For each graph $G$ with $n$ vertices, it holds

$$Mc^\lambda(G) \leq (n - 1)[\lambda + \frac{1}{2}(n - 1) \cdot \lambda^2].$$

The equality holds for a star (in its central vertex).
Proof. Using Lemma 4.1, we conclude that the desired graph is a tree. Let $G$ be a tree such that $c^*_G(v)$ is maximal and let $u$ be a vertex for which maximal centrality is reached. Let $P$ be a set of all unordered pairs of vertices $v, w \in V \setminus \{u\}$ such that the shortest path from $v$ to $w$ passes through $u$. It holds:

$$Mc^*_G(G) = \sum_{v \in [u]} \sum_{k,l \in V} \frac{s_{kl}^v}{s_{kl}} \lambda^{d(k,l)} = \sum_{v \in V \setminus \{u\}} \lambda^{d(u,v)} + \sum_{\{v,w\} \in P} \lambda^{d(v,w)}$$

$$\leq (n-1) \cdot \lambda + \frac{1}{2} (n-1)(n-2) \cdot \lambda^2$$

$$= (n-1)[\lambda + \frac{1}{2} (n-2) \cdot \lambda^2].$$

The maximal centrality is reached for the central vertex of a star since all vertices $v \in V \setminus \{u\}$ are directly connected to $u$ and for all vertices $v, w \in V \setminus \{u\}$, it holds that $d(v, w) = 2$. \hfill \qed

5. Networkness

In paper [1] it has been proven that:

**Lemma 5.1.** For positive numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, the following holds:

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq \min_{b_i} \left\{ \frac{a_i}{b_i} \right\}.$$

Using this, let us prove:

**Theorem 5.2.** For each graph $G$ with $n$ vertices, it holds

$$\min_{2 \leq D \leq n-1} \frac{\lambda^D + \frac{1}{n-D} \left( \frac{\lambda^D - \lambda}{\lambda - 1} \right)}{D \lambda^D + \frac{1}{n-D} \left[ \frac{\lambda^D - D \lambda^{D+1} + (D-1) \lambda^{D+1}}{(\lambda - 1)^2} \right]} \leq mN^*_\lambda(G) \leq 1.$$

The lower bound is reached for a broom (in its starting vertex) and the upper bound is reached for any vertex of a vertex-transitive graph.

Proof. Using Theorem 2.1 and Lemma 5.1, it holds

$$\min_{u \in V} \left\{ \frac{c^*_\lambda(u)}{t^*_\lambda(u)} : u \in V \right\} \leq \frac{\sum_{u \in V} c^*_\lambda(u)}{\sum_{u \in V} t^*_\lambda(u)} \leq \frac{\sum_{k,l \in V} d(k,l) \cdot \lambda^{d(k,l)}}{\sum_{k,l \in V} d(k,l) \cdot \lambda^{d(k,l)}} = 1.$$

Let us prove the lower bound. Let $G$ be a graph for which the minimum of $mN^*_\lambda(G)$ is attained and let $u$ be a vertex of the graph for which $N^*_\lambda(u) = mN^*_\lambda(G)$. It holds

$$N^*_\lambda(G) = \frac{c^*_\lambda(G)}{t^*_\lambda(G)} \geq \frac{\sum_{v \in V \setminus \{u\}} \lambda^{d(u,v)}}{\sum_{v \in V \setminus \{u\}} d(u, v) \lambda^{d(u,v)}}$$

because $u$ certainly lies on every shortest path between itself and every other vertex $v$. Now let $v_D$ be a vertex which is farthest away from $u$ and let $S = w_1 v_2 \ldots v_D$ be a shortest path from $u$ to $v_D$. Furthermore, let $k = n - D - 1$, let $\{w_1, \ldots, w_k\} = V \setminus \{u, v_1, \ldots, v_D\}$ be set of all vertices that do not lie on the path $S$ and let $W = \{w_1, w_2, \ldots, w_k, v_D\}$. 

Advances in Mathematics of Communications  Volume 13, No. 3 (2019), 405–420
Because \( d(u, v_i) = i \) for all \( i \in \{1, 2, ..., D\} \), we have:

\[
N^x(u) \geq \frac{1}{\lambda} \sum_{v \in V \setminus \{u\}} \lambda^d(u, v) \lambda^d(u, v) = \frac{D}{\lambda} \sum_{i=1}^{D} \sum_{i=1}^{D} \lambda^d(u, w_i) \lambda^d(u, w_i).
\]

The last expression in (5) can be written as

\[
\sum_{v \in W} \left( \lambda^d(u, v) + \frac{1}{n-D} \sum_{i=1}^{D-1} \lambda^i \right)
\]

\[
\sum_{v \in W} \left( d(u, v) \lambda^d(u, v) + \frac{1}{n-D} \sum_{i=1}^{D-1} i \lambda^i \right).
\]

Using Lemma 5.1, the minimum of expression (6) is

\[
\frac{\lambda x + \frac{1}{n-D} \sum_{i=1}^{D-1} \lambda^i}{x \lambda x + \frac{1}{n-D} \sum_{i=1}^{D-1} i \lambda^i},
\]

where \( x = d(u, q) \) for some \( q \in W \) for which the expression (6) has minimal value. This minimum is obtained if and only if ratio \( \frac{a_i}{b_i} \) is constant for all \( i \in \{1, 2, ..., n\} \) and one way to achieve this is that \( d(u, w_i) = D \) for all \( i \in \{1, 2, ..., k\} \). This is possible if \( w_i \) is directly connected to \( v_{D-1} \) for all \( i \leq k \), which is true when \( G \) is a broom.

**Theorem 5.3.** For each graph \( G \) with \( n \) vertices, it holds

\[
1 \leq MN^x(G) \leq \frac{1}{2} (n-2) \cdot \lambda + 1.
\]

The lower bound is reached for any vertex of a vertex-transitive graph and the upper bound is reached for a star (in its central vertex).

**Proof.** First, let us prove the lower bound. Using Theorem 2.1, since maximum is greater than or equal to average, we have

\[
\max \left\{ \frac{c^*_i(u)}{t^*_i(u)} : u \in V \right\} \geq \frac{1}{n} \sum_{u \in V} c^*_i(u) = \frac{1}{n} \sum_{u \in V} t^*_i(u) = \frac{1}{n} \sum_{k, l \in V} d(k, l) \cdot \lambda^d(k, l) = 1.
\]

Now, let us prove the upper bound. From Lemma 4.1 we conclude that the graph that maximizes \( MN^x(G) \) is a tree. Namely, since networkness is defined as quotient of betweenness centrality and transmission, we want the numerator to be maximal. That holds when the graph \( G \) is a tree. We can assume that the graph used in the denominator is also a tree. If this is not the case, we can repeat the construction of \( G' \) in Lemma 4.1 to obtain a tree in which the distances between \( u \) and all the other vertices remain the same, thus, transmission stays the same.

Let \( u \in V \) be a vertex that maximizes networkness. Let \( P \) be a set of all unordered pairs of vertices \( v, w \in V \setminus \{u\} \) such that the shortest path between \( v \)
and \( w \) passes through \( u \). It holds:

\[
N_{\lambda}^{e}(u) = \sum_{k,l \in V} \lambda^{d(k,l)} \frac{d(u,v) \cdot \lambda^{d(u,v)}}{\sum_{v \in V \setminus \{u\}} d(u,v) \cdot \lambda^{d(u,v)}} \leq \sum_{\{v,w\} \in P} \lambda^{d(v,u) + d(u,w)} + \sum_{v \in V \setminus \{u\}} \lambda^{d(u,v)}
\]

\[
= \frac{1}{2} \sum_{v \in V \setminus \{u\}} d(u,v) \cdot \lambda^{d(u,v)} \leq \frac{1}{2} (n - 2) \cdot \lambda + 1.
\]

Simple calculation show that equality holds for a central vertex of a star.

6. Network surplus

**Theorem 6.1.** For each graph \( G \) with \( n \) vertices, it holds

\[
\min_{1 \leq D \leq n - 1} \lambda \left[ D \lambda^D - \lambda - (D - 1) \lambda^{D+1} \right] + (n - D - 1)(\lambda^D - D \lambda^D) \leq m \nu_{\lambda}^{e}(G)
\]

and

\[
m \nu_{\lambda}^{e}(G) \leq 0.
\]

The lower bound is reached for broom (in its starting vertex) and the upper bound is reached for a vertex-transitive graph (in any of its vertices).

**Proof.** Let us prove the upper bound. Using Theorem 2.1, we have:

\[
m \nu_{\lambda}^{e}(u) = \min \{ c_{\lambda}^{e}(u) - t_{\lambda}^{e}(u) : u \in V \} \leq \frac{1}{n} \left( \sum_{u \in V} (c_{\lambda}^{e}(u) - t_{\lambda}^{e}(u)) \right)
\]

\[
= \frac{1}{n} \left( \sum_{u \in V} c_{\lambda}^{e}(u) - \sum_{u \in V} t_{\lambda}^{e}(u) \right) = 0.
\]

The first inequality holds since minimum is smaller than or equal to the average. For the lower bound, let us suppose \( G \) is a graph for which the minimum \( m \nu_{\lambda}^{e}(G) \) is attained an let \( u \) be a vertex of the graph for which the minimum is attained. Let \( v_{D} \) be a vertex which is farthest away from \( u \) and let \( S = uv_{1}v_{2}...v_{d} \) be a shortest path from \( u \) to \( v_{D} \). Furthermore, let \( k = n - D - 1 \) and let \( W = \{ w_{1}, w_{2}, ..., w_{k} \} \) be a set of all vertices that do not lie on the path \( S \). Since \( d(u,v_{i}) = i \) for all
For each graph $G$, we have:

$$mν_κ^e(u) = c_κ^e(u) - t_κ^e(u)$$

$$\geq \sum_{i=1}^{D} \lambda^i + \sum_{w \in W} \lambda^d(u, w) - \sum_{i=1}^{D} i \cdot \lambda^i - \sum_{w \in W} d(u, w) \lambda^d(u, w)$$

$$\geq \sum_{i=1}^{D} (\lambda^i - i \cdot \lambda^i) + \sum_{w \in W} \lambda^d(u, w) [1 - d(u, w)]$$

$$\geq \sum_{i=1}^{D} (\lambda^i - i \cdot \lambda^i) + (n - D - 1) \lambda^2 (1 - x).$$

where $x = d(u, q)$ for some $q \in W$ which minimizes the expression $\lambda^d(u, q) [1 - d(u, q)]$.

This means that all the vertices in $W$ are equally away from $u$. As proven in Theorem 3.5, in this case, it holds $d(u, w) = D$ for all $w \in W$, i.e., all the vertices in $W$ are directly connected to $v_{D-1}$. We conclude that one of the graphs for which the lower bound is reached is a broom.

**Theorem 6.2.** For each graph $G$ with $n$ vertices, it holds

$$0 \leq Mν_κ^e(G) \leq \frac{1}{2} (n - 1)(n - 2) \cdot \lambda^2.$$

The lower bound is reached for any vertex a vertex-transitive graph and the upper bound is reached for a a star (in its central vertex).

**Proof.** For lower bound, we have, again using Theorem 2.1

$$Mν_κ^e(u) = \max \{ c_κ^e(u) - t_κ^e(u) : u \in V \} \geq \frac{1}{n} \left( \sum_{u \in V} (c_κ^e(u) - t_κ^e(u)) \right)$$

$$= \frac{1}{n} \left( \sum_{u \in V} c_κ^e(u) - \sum_{u \in V} t_κ^e(u) \right) = 0.$$

The first inequality holds because maximum is greater than or equal to the average. Now, let us prove the upper bound. From Lemma 4.1 it is obvious that the desired graph is a tree. Let $G$ be a graph for which $Mν_κ^e(G)$ is maximal and let $u \in V$ be a vertex such that $ν_κ^e(u) = Mν_κ^e(G)$. Let $P$ be a set of all unordered pairs of vertices $v, w \in V \setminus \{u\}$ such that the shortest path between $v$ and $w$ passes through $u$. It holds:

$$mν_κ^e(u) = c_κ^e(u) - t_κ^e(u) \leq \sum_{P} \lambda^d(k, l) - \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^d(u, v)$$

$$\leq \sum_{\{v, w\} \in P} \lambda^d(v, u) + d(u, w) + \sum_{v \in V \setminus \{u\}} \lambda^d(u, v) - \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^d(u, v)$$

$$\leq \sum_{\{v, w\} \in P} \lambda^2 + \sum_{v \in V \setminus \{u\}} \lambda^d(u, v) - \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^d(u, v)$$

$$\leq \frac{1}{2} (n - 1)(n - 2) \cdot \lambda^2 + \sum_{v \in V \setminus \{u\}} \lambda^d(u, v) - \sum_{v \in V \setminus \{u\}} d(u, v) \cdot \lambda^d(u, v)$$
\[ \frac{1}{2} (n - 1)(n - 2) \cdot \lambda^2. \]

\section*{Discussion and Conclusions}

Transmission and betweenness centrality are well known concepts in communication networks theory. Based on them, new concepts of networkness and network surplus have been defined [13]. They include the assumption of equal communication between vertices. Based on a new assumption that communication decreases as the distance between vertices increases, generalised network descriptors were presented. In [1] the amount of communication was weighted by \( d(u, v)^{\lambda} \) where \( \lambda < 0 \). In this paper we wanted to explore a more radical assumption, so we weighted the amount of communication by \( \lambda^{d(u,v)} \) where \( \lambda \in (0, 1) \). We have defined and analyzed exponential generalised network descriptors. Extremal values of these descriptors and graphs which they are obtained for can be found in Table 1. Lower bounds for \( Mt_{\lambda}^{e}(G) \) and \( Mc_{\lambda}^{e}(G) \) remain open.

Further research of these descriptors may include applying them to real-life networks. It would be interesting to obtain the data about some real social network, in which informations circulate regularly, calculate the values of exponential network descriptors presented in this paper and compare them to the values of generalized descriptors from [1] and the regular descriptors that assume equal amount of communication between vertices. Besides that, the idea of dampening factor \( \lambda^{d(u,v)} \), for \( \lambda \in (0, 1) \), can be used to model simulations about information flow in networks.

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