UNIVERSAL SIMPLE CURRENT VERTEX OPERATORS

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Abstract.
We construct a vertex operator realization for the simple current primary fields of WZW theories which are based on simply laced affine Lie algebras \( \mathfrak{g} \). This is achieved by employing an embedding of the integrable highest weight modules of \( \mathfrak{g} \) into the Fock space for a bosonic string compactified on the weight lattice of \( \mathfrak{g} \). Our vertex operators are universal in the sense that a single expression for the vertex operator holds simultaneously for all positive integral values of the level of \( \mathfrak{g} \).

\( ^x \) Heisenberg fellow
1 Simple currents

A simple current of a two-dimensional conformal field theory is a primary field \( \phi_J \) with quantum dimension 1. Simple currents can also be characterized by the property that their fusion product with any other primary field \( \phi \) consists of a single primary field \( \phi' = \phi_J \star \phi \). It follows in particular that the set of simple currents of a theory constitutes a subgroup of the fusion ring. While these characterizations of simple currents in terms of the fusion rules are both suggestive and elegant, they require to regard the primary fields as elements of an abstract algebraic structure, the fusion ring, and hence do not provide any concrete information about the action of the simple currents on conformal fields (via the operator product), respectively on the space \( \mathcal{H} \) of physical states. A more direct realization of simple currents in terms of the representations of the chiral algebra of the theory is therefore most desirable. For general conformal field theories such a formulation is so far not available. This is related to the fact that the fusion product of primary fields is not isomorphic to the tensor product of the representations of the chiral algebra that are carried by the primary fields, and can be regarded as another manifestation of the difficulties which lie behind the very concept of a conformal ‘field’.

In the case of WZW conformal field theories, the situation is different. For these theories the chiral algebra is generated by an untwisted affine Kac–Moody algebra \( \mathfrak{g} \), so that the well-developed representation theory of such Lie algebras allows one to analyse the fusion rules in considerable detail. In particular, it is known that – with the exception of an isolated case appearing for \( E_8 \) level two – the simple currents \( J \) correspond to certain symmetries \( \dot{\omega} \equiv \dot{\omega}_J \) of the Dynkin diagram of \( \mathfrak{g} \); more precisely, they are in one-to-one correspondence with the maximal abelian subgroup of those symmetries. As a consequence, it has been possible \[1\] to realize simple currents on the irreducible subspaces of \( \mathcal{H} \), i.e. on the integrable irreducible highest weight modules \( \mathcal{H}_\Lambda \) of \( \mathfrak{g} \), as certain linear maps

\[
\mathcal{T}_{\dot{\omega}} : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\dot{\omega} \star \Lambda}
\] (1.1)

associated to \( \dot{\omega} \), which are uniquely characterized (see the formulæ (6.1) and (6.2) below) by their action on the highest weight vector and by their commutation relations with the elements of \( \mathfrak{g} \).

The description of WZW simple currents through these maps \( \mathcal{T}_{\dot{\omega}} \) has the advantage of clearly displaying the underlying representation theoretic structures, and it also proves to be indispensable for various applications\[3\] of simple currents, e.g. to the resolution of field identification fixed points \[3, 4, 5\]. In particular, by collecting the linear maps \( \mathcal{T}_{\dot{\omega}} \) on all of the irreducible modules \( \mathcal{H}_\Lambda \) which belong to the spectrum of the theory, one arrives at a realization of the simple currents on the direct sum \( \bigoplus \mathcal{H}_\Lambda \) of these spaces; more specifically, one has to sum over all unitary irreducible highest weight modules with some fixed positive integral value \( \ell \) of the level, so that the direct sum \( \bigoplus \mathcal{H}_\Lambda \) is isomorphic to the (chiral) state space \( \mathcal{H} \equiv \mathcal{H}_{\mathfrak{g}, \ell} \) of the WZW theory based on \( \mathfrak{g} \) at level \( \ell \). However, in conformal field theory one thinks of the irreducible subspaces \( \mathcal{H}_\Lambda \subseteq \mathcal{H} \) as arising by the action of suitable field operators on the vacuum vector of the theory, whereas the approach of \[1\] is based entirely on the abstract description

\[1\] Another description\[2\] of WZW simple currents is in terms of twisted modules over a vertex operator algebra. While this does not seem to be particularly suited for describing their action on the space of physical states, it does allow one to discuss extensions of the chiral algebra by the simple currents.
of the $\mathfrak{g}$-modules $\mathcal{H}_\Lambda$ as quotients of Verma modules. Thus the realization of WZW simple currents through the maps (1.1) is still not as field-theoretic as one might wish.

In the present paper we obtain, for the case of simply-laced $\mathfrak{g}$, a much more concrete realization of WZW simple currents. To this end we embed the space $\mathcal{H}$ into the Fock space of a bosonic string that is completely compactified on the weight lattice of $\mathfrak{g}$ [3, 4, 5]. We can then identify unitary operators $U_\omega$ acting on this big space $\mathcal{F}$ which, when restricted to $\mathcal{H} \subset \mathcal{F}$, behave precisely as the linear maps $\hat{T}_\omega$ (1.1); in other words, we construct a map $\hat{T}_\omega : \mathcal{F} \to \mathcal{F}$ such that

$$\hat{T}_\omega|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}, \quad v \mapsto U_\omega v$$

and

$$\hat{T}_\omega|_{\mathcal{H}_\Lambda} = \mathcal{T}_\omega.$$ (1.3)

The action of $U_\omega$ on the vector space $\mathcal{F}$ induces in a natural manner an action on the string oscillators which operate on $\mathcal{F}$, namely as conjugation $x \mapsto U_\omega x U_\omega^{-1}$ of $x \in \mathcal{F}$ by $U_\omega$. Now the string oscillators (supplemented by appropriate cocycle operators) can be employed to construct a vertex operator representation of $\mathfrak{g}$, which by restriction to suitable subspaces of $\mathcal{F}$ provides in particular vertex operator realizations of the integrable irreducible highest weight representations $\pi_\Lambda$ of $\mathfrak{g}$. Moreover, the conjugation by $U_\omega$ supplies us with a realization of the WZW simple current as an automorphism of the algebra of string oscillators, which we can supplement in such a way that it also becomes an automorphism of the extension of that algebra by the cocycle operators. Combining these results we arrive at the desired realization of simple currents on the irreducible subspaces of $\mathcal{H}$. In fact, the so obtained realization of a simple current constitutes an inner automorphism of the oscillator (and cocycle) algebra; in contrast, when restricted to the affine algebra $\mathfrak{g}$, simple currents act via certain outer automorphisms (see formula (4.1)).

Besides providing a concrete action on $\mathcal{H}$, our construction has the additional benefit of being universal in the sense that a single expression for the vertex operator $U_\omega$ holds simultaneously for all irreducible submodules $\mathcal{H}_\Lambda$ of $\mathcal{H}$. Accordingly, it is justified to call $U_\omega$ the (universal) simple current vertex operator for the relevant simple current $J$. Furthermore, this expression for $U_\omega$ is in fact even valid simultaneously for all positive integral values of the level of $\mathfrak{g}$. (This is possible because the string Fock space $\mathcal{F}$ contains all the irreducible highest weight modules for all integrable highest $\mathfrak{g}$-weights $\Lambda$ [8].) In other words, our realization of the simple currents not only holds for each individual WZW theory based on an affine Lie algebra $\mathfrak{g}$ at some level $\ell$, but it applies in fact to the whole collection of all (unitary) WZW theories based on $\mathfrak{g}$ at all positive integral values of the level.

The rest of this paper is organized as follows. In sections 2 and 3 we collect those aspects of the covariant vertex operator construction and of the operation of the Lorentz group on the string Fock space and string oscillators, respectively, that are relevant to our arguments. Section 4 is devoted to a description of the pertinent features of the various automorphisms – of the Dynkin diagram, of the Lie algebra $\mathfrak{g}$, and of its weight space $\mathfrak{g}^\vee$ – which realize the action of a WZW simple current. In order to relate these structures to those discussed in the earlier sections, it is necessary to extend the action (by conjugation) of $U_\omega$ on the string oscillator algebra to include also the cocycle factors; this is achieved in section 5. In section 6 we put

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2 A description via inner automorphisms can also be achieved to a certain extent within the vertex operator algebra formulation of [6].
these results together, which finally allows us to demonstrate our main result, the identification (1.3) of the maps $T_\omega$ (1.1) in terms of the simple current vertex operators $U_\omega$. We conclude with a few short remarks on possible applications and open questions.

2 Covariant vertex operators

We consider a free relativistic string that is completely compactified on a torus in such a way that the string momenta are constrained to lie on the weight lattice $L_\mathfrak{g}$ of a simply laced affine Kac–Moody algebra $\mathfrak{g}$. Such a compactified string theory provides a covariant version \[6, 7, 8\] of the vertex operator construction of integrable highest weight representations of $\mathfrak{g}$. (In the more familiar non-covariant (Frenkel–Kac–Segal \[9, 10, 11\]) construction one compactifies on the root lattice of the horizontal subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ rather than on the affine weight lattice.) The state space of the theory is the Fock space $F$ that is generated from a collection of vacuum vectors $\Omega_\alpha$ with $\alpha \in L_\mathfrak{g}$ by applying the creation operators of the string oscillator algebra. Note that what is usually called the weight lattice of $\mathfrak{g}$ is not really a lattice, but a one-parameter family of lattices. Namely, it consists of the weights $\lambda = \tilde{\lambda} + c\delta$ where $\tilde{\lambda}$ is an integral linear combination of the fundamental weights $\Lambda_i$ of $\mathfrak{g}$ and $\delta$ is the so-called null root of $\mathfrak{g}$, and where $c$ is an arbitrary complex number. For the covariant vertex operator construction one rather considers only \textit{rational} values of $c$ \[8\], and accordingly from now on the notation $L_\mathfrak{g}$ will refer to the one-parameter family of lattices where the parameter is restricted to lie in $\mathbb{Q}$.

The string oscillators $a^\mu_m$ ($m \in \mathbb{Z}$), $p^\mu \equiv a^\mu_0$ and $q^\mu$ are the modes of the covariant Fubini–Veneziano coordinate and momentum fields

$$X^\mu(z) = q^\mu - ip^\mu \ln z + i \sum_{m \neq 0} \frac{1}{m} a^\mu_m z^{-m} \quad \text{and} \quad P^\mu(z) = i \frac{d}{dz} X^\mu(z). \quad (2.1)$$

They satisfy the Heisenberg commutation relations

$$[q^\mu, p^\nu] = i(G^{-1})^{\mu,\nu} 1, \quad [a^\mu_m, a^\nu_n] = m (G^{-1})^{\mu,\nu} \delta_{m+n,0} 1. \quad (2.2)$$

Here the labels $\mu, \nu$ take values in $\{0, 1, \ldots, r+1\}$, where $r$ is the rank of $\mathfrak{g}$, and the matrix $G$ in (2.2) denotes the metric on the weight space of $\mathfrak{g}$, which has Minkowskian signature.

We will denote the associative algebra that is generated freely by the unit $1$ and by the string oscillators $a^\mu_m$, $p^\mu$ and $q^\mu$ modulo the relations (2.2) by $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a $*$-algebra, i.e. is endowed with an involutive automorphism $^*$. The $*$-operation acts as $(p^\mu)^* = p^\mu$, $(q^\mu)^* = q^\mu$, $(a^\mu_m)^* = a^{-\mu}_{-m}$, i.e. in particular it exchanges creation operators ($a^\mu_m$ with $m<0$) with annihilation operators ($a^\mu_m$ with $m>0$).

As a basis of the weight space $\mathfrak{g}_0^*$ of $\mathfrak{g}$ we choose

$$\mathcal{B}_0 = \{\Lambda_{(0)}\} \cup \{\tilde{\Lambda}_{(i)} | i = 1, 2, \ldots, r\} \cup \{\delta\}, \quad (2.3)$$

where $\Lambda_{(i)}$ and $\tilde{\Lambda}_{(i)}$ indicate the fundamental weights of $\mathfrak{g}$ and of its horizontal subalgebra $\bar{\mathfrak{g}}$, respectively, and $\delta$ is the null root. In the basis (2.3), the metric $G$ on $\mathfrak{g}_0^*$ takes the form

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \tilde{G} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.4)$$
\[ G_{\mu,\nu} = \delta_{\mu,0} \delta_{\nu, r+1} + \delta_{\mu, r+1} \delta_{\nu,0} + \bar{G}_{\mu,\nu}, \]  
(2.5)

where \( \bar{G} \) is the (Euclidean) metric on the weight space \( \bar{\mathfrak{g}}^* \) of \( \bar{\mathfrak{g}} \), i.e. the inverse of the Cartan matrix of \( \bar{\mathfrak{g}} \) (thus the allowed range of labels of \( \bar{G} \) is \( \{1, 2, \ldots, r\} \), i.e. the last term in (2.5) should be interpreted as \( \bar{G}_{\mu,\nu} \equiv \sum_{r+1}^r \delta_{\mu,\nu} \bar{G}_{i,j} \)). We will write inner products with respect to the metric \( \bar{G} \) with a dot, \( \alpha \cdot \beta \equiv \sum_{r+1}^r G_{\mu,\nu} \alpha^\mu \beta^\nu \) for \( \alpha, \beta \in \mathfrak{g}^* \).

The covariant coordinate and momentum fields (2.1) can be employed to construct covariant vertex operators which, as compared to the ordinary vertex operator construction, have the advantage of commuting with the Virasoro constraints and hence being manifestly physical in the sense of string theory. Furthermore, this construction is applicable at arbitrary level \( \ell \) of the affine Lie algebra \( \mathfrak{g} \). Namely, for every integrable highest \( \mathfrak{g} \)-weight \( \Lambda \in \mathfrak{g}^* \), there exists \( [8] \) a subspace \( \mathcal{P}_\Lambda \) of the Fock space \( \mathcal{F} \) which carries a vertex operator representation \( \hat{\pi}_\Lambda: \mathfrak{g} \rightarrow \text{End} \mathcal{P}_\Lambda \) (2.6)

of the affine Lie algebra \( \mathfrak{g} \). This vertex operator representation, which was recently also exploited in \([12, 8]\), will be the basis of our construction of simple currents, and therefore we will now describe it in some detail.

The subspace \( \mathcal{P}_\Lambda \subset \mathcal{F} \) on which \( \hat{\pi}_\Lambda \) operates is defined as

\[ \mathcal{P}_\Lambda := \bigoplus_{\lambda \in \Xi(\Lambda)} \mathcal{P}(\lambda) \quad \text{with} \quad \mathcal{P}(\lambda) := \{ v \in \mathcal{F} \mid L_0 v = v, \ L_n v = 0 \text{ for all } n > 0, \ p^\mu v = \lambda^\mu v \} . \]  
(2.7)

Here the set \( \Xi(\Lambda) \) is the weight system of the irreducible highest weight module \( \mathcal{H}_\Lambda \) with highest weight \( \Lambda \), and \( L_m, m \in \mathbb{Z} \), are the string Virasoro operators, which are given by

\[ L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_n \cdot \alpha_{m-n} : ; \]  
(2.8)

they span a Virasoro algebra of central charge \( c = r+1 \).4 Moreover, every subspace \( \mathcal{P}(\lambda) \) of \( \mathcal{P}_\Lambda \) can be obtained by applying so-called DDF \([13]\) operators (which therefore constitute a spectrum generating algebra for the string) to a suitable tachyonic vacuum vector \( \Omega_a \), namely to one that is associated to the weight \( \lambda_0(\lambda) := \lambda + \ell^{-1}(1 - \frac{1}{2} \lambda \cdot \lambda) \delta \) (2.9)

Note that \( a(\lambda) \cdot a(\lambda) = 2 \) independently of \( \lambda \).

The representation \( \hat{\pi}_\Lambda \) is of level \( \ell = \Lambda \cdot \delta \); it is defined on a Cartan–Weyl basis

\[ \mathcal{B}(\mathfrak{g}) = \{ K, D \} \cup \{ H_i^m \mid i = 1, 2, \ldots, r, \ m \in \mathbb{Z} \} \cup \{ E^a_m \mid \bar{\alpha} \text{ a } \bar{\mathfrak{g}} \text{–root}, \ m \in \mathbb{Z} \} \]  
(2.10)

\[ \begin{array}{l} \text{This Virasoro algebra is not related to the Virasoro algebra of the WZW conformal field theory associated to } \mathfrak{g}, \text{ which is determined through } \mathfrak{g} \text{ by the Sugawara construction and has a different, level dependent, central charge.} \\ \text{This formula actually demonstrates that as long as one works with a fixed value } \ell \text{ of the level, one needs not consider the full \text{ ‘lattice’ } L_\mathfrak{g}, \text{ where as noted above the coefficient of the null root } \delta \text{ can take arbitrary rational values } c, \text{ but can restrict } c \text{ to integral multiples of } (2K\ell)^{-1}, \text{ where } K \text{ is the maximal denominator of the entries of the matrix } G \text{ of } \bar{\mathfrak{g}}, \text{ so that one really deals with a proper lattice.} \end{array} \]
of \( g \) by

\[
\hat{\pi}_\Lambda(K) = \delta \cdot p, \quad \hat{\pi}_\Lambda(D) = \Lambda_{(0)} \cdot p,
\]

\[
\hat{\pi}_\Lambda(H^i_m) = \int \frac{dz}{2\pi i} \alpha^{(i)} \cdot P(z) \exp[i m\delta \cdot X(z)],
\]

\[
\hat{\pi}_\Lambda(E^\alpha_m) = \int \frac{dz}{2\pi i} \exp[i(\vec{\alpha} + m\delta) \cdot X(z)] \cdot c_\alpha.
\]

(2.11)

Here \( \alpha^{(i)} \) are the simple roots of \( g \) (which are orthonormal to the fundamental weights, \( \alpha^{(j)} \cdot \Lambda_{(i)} = \delta_{ij} \)), and \( c_\alpha \) are the cocycle operators which guarantee the proper sign factors in the commutation relations of the \( \hat{\pi}_\Lambda(E^\alpha_m) \). Implementing the relation

\[
\bar{\alpha}^{(i)} = \alpha^{(i)} \quad \text{for } i = 1, 2, ..., r, \quad \alpha^{(0)} = -\bar{\theta} + \delta
\]

(2.12)

between the simple roots \( \bar{\alpha}^{(i)} \) \( (i \in \{1, 2, ..., r\}) \) of \( \hat{g} \) and \( \alpha^{(i)} \) \( (i \in \{0, 1, ..., r\}) \) of \( g \) (\( \bar{\theta} \) denotes the highest root of \( \hat{g} \)), one learns in particular that the Chevalley generators of \( \hat{g} \) – that is, the generators \( H^i \equiv H^i_0, \ E^{\pm \alpha^{(i)}} \equiv E^{\pm \alpha^{(i)}}_0 \) for \( i \in \{1, 2, ..., r\} \), and \( H^0 := K - \sum_{i=1}^r a_j H^j_0, \ E^{\pm \alpha^{(0)}} := E^{\pm \theta}_0 \) which are associated to the simple \( g \)-roots – in the representation (2.6) read

\[
\hat{\pi}_\Lambda(H^i) = \alpha^{(i)} \cdot p, \quad \hat{\pi}_\Lambda(E^{\pm \alpha^{(i)}}) = \int \frac{dz}{2\pi i} \exp[\pm i\alpha^{(i)} \cdot X(z)]: \quad \text{for } i = 0, 1, ..., r.
\]

(2.13)

Moreover, one can check that these operators indeed commute with the Virasoro algebra, \([L_m, x] = 0 \) for all \( m \in \mathbb{Z} \) and all \( x \in \mathcal{B}(g) \).

The \( g \)-module \( \mathcal{P}_\Lambda \) (2.7) turns out to be highly reducible. It contains in particular an irreducible submodule that is isomorphic to (and will henceforth be identified with) the irreducible highest weight module \( \mathcal{H}_\Lambda \). By restriction to \( \mathcal{H}_\Lambda \subset \mathcal{P}_\Lambda \), \( \hat{\pi}_\Lambda \) defines a unitary irreducible highest weight representation

\[
\pi_\Lambda : \ g \to \text{End} \mathcal{H}_\Lambda
\]

(2.14)

of \( g \). The submodule \( \mathcal{H}_\Lambda \) of \( \mathcal{P}_\Lambda \) can be described as the highest weight module that is generated from a fixed vector

\[
v^\Lambda \equiv \Omega_{\alpha(\Lambda)} \in \mathcal{P}_\Lambda
\]

(2.15)

of weight \( \Lambda \) by application of (the enveloping algebra of) \( \hat{\pi}_\Lambda(g_-) \), with \( g_- = \text{span} \{ H^i_m, E^\alpha_m \mid m > 0 \} \cup \{ E^{\bar{\alpha}_0} \mid \bar{\alpha} < 0 \} \). It proves to be a crucial property of the Fock space vector \( v^\Lambda \in \mathcal{F} \) that it is in fact a vector in \( \mathcal{P}_\Lambda \), i.e. that it is a physical string ground state with weight \( \Lambda \) which satisfies

\[
L_0 v^\Lambda = v^\Lambda, \quad L_n v^\Lambda = 0 \quad \text{for } n > 0
\]

(2.16)

as well as

\[
p^\mu v^\Lambda = \Lambda^\mu v^\Lambda \quad \text{for } 1 \leq \mu \leq r + 1.
\]

(2.17)

\footnote{By \( a_i \) we denote the Coxeter labels of \( \hat{g} \), i.e. for \( i \in \{1, 2, ..., r\} \) they are the expansion coefficients of the highest root \( \bar{\theta} \) of \( \hat{g} \) with respect to the simple roots, while \( a_0 := 1 \).}
Namely, these properties are already sufficient to verify \( v_\Lambda^\Lambda \) both possesses the defining properties
\[
\begin{align*}
\hat{\pi}_\Lambda(K) v_\Lambda^\Lambda &= \ell v_\Lambda^\Lambda, \\
\hat{\pi}_\Lambda(H^i_0) v_\Lambda^\Lambda &= \Lambda^i v_\Lambda^\Lambda &\text{for } 1 \leq i \leq r, \\
\hat{\pi}_\Lambda(E_0^{(i)}) v_\Lambda^\Lambda &= 0 &\text{for } 1 \leq i \leq r, \\
\hat{\pi}_\Lambda(E_\bar{\theta}^{(i)}) v_\Lambda^\Lambda &= 0 &\text{for } 1 \leq i \leq r, \\
\end{align*}
\]
(2.18)
of a highest weight vector of highest weight \( \Lambda \) and satisfies the irreducibility (null vector) conditions
\[
\begin{align*}
(\hat{\pi}_\Lambda(E_0^{-\bar{\alpha}^{(i)}}))^{\Lambda^i+1} v_\Lambda^\Lambda &= 0 &\text{for } 1 \leq i \leq r, \\
(\hat{\pi}_\Lambda(E_{-\bar{\theta}}))^{\ell-\bar{\Lambda} \cdot \bar{\theta} + 1} v_\Lambda^\Lambda &= 0.
\end{align*}
\]
(2.19)
Note that for each integrable \( g \)-weight \( \Lambda \) the space \( P_\Lambda \) is by definition a subspace of the Fock space \( F \); but in fact even the direct sum \( \bigoplus_{\Lambda \in g^\delta: \Lambda \cdot \delta \in \mathbb{Z}_{>0}} P_\Lambda \) of these spaces for all integrable weights is contained in \( F \). Moreover, in the expressions (2.11) for the operators \( \hat{\pi}_\Lambda(x) \) no explicit reference to the highest weight \( \Lambda \) is necessary. Accordingly, these expressions supply us in fact with a unitary representation
\[
\hat{\pi}_\mathfrak{g} = \bigoplus_{\Lambda \in g^\delta: \Lambda \cdot \delta \in \mathbb{Z}_{>0}} \hat{\pi}_\Lambda
\]
(2.20)
of \( \mathfrak{g} \) on the direct sum of the spaces \( P_\Lambda \) for all integrable highest weights. By restriction, the representation (2.20) in turn also provides a representation
\[
\pi_{\mathfrak{g},\ell} = \bigoplus_{\Lambda \in g^\delta: \Lambda \cdot \delta = \ell} \pi_\Lambda
\]
(2.21)
of \( \mathfrak{g} \) on the direct sum \( \mathcal{H}_{\mathfrak{g},\ell} = \bigoplus_{\Lambda: \Lambda \cdot \delta = \ell} \mathcal{H}_\Lambda \) of the corresponding irreducible highest weight modules at any fixed value \( \ell \) of the level, and hence on the state space of the WZW theory based on \( \mathfrak{g} \) at level \( \ell \).

In short, for any fixed simply-laced affine Lie algebra \( \mathfrak{g} \) and arbitrary level \( \ell \in \mathbb{Z}_{>0} \) we can regard the WZW state space \( \mathcal{H} \equiv \mathcal{H}_{\mathfrak{g},\ell} \) as embedded into the string Fock space \( F \) and \( \pi_{\mathfrak{g},\ell} \) (2.21) as a level-\( \ell \) vertex operator representation of \( \mathfrak{g} \) on \( \mathcal{H} \).

### 3 Lorentz transformations

The weight space \( g^\delta \) naturally carries an action of the Lorentz group \( O(r+1,1) \). The elements of \( O(r+1,1) \) are \( (r+1) \times (r+1) \)-matrices \( M \equiv (M^\mu_{\nu}) \) which are orthogonal with respect to the metric (2.4), i.e. satisfy \( M^T G M = G \). Furthermore, the string Fock space \( F \) carries a unitary representation of the discrete subgroup \( O_L(r+1,1) \) of the Lorentz group \( O(r+1,1) \) that leaves the weight lattice \( L_\mathfrak{g} \) of \( \mathfrak{g} \) invariant. For orthochronous Lorentz transformations it is given by
\[
O_L^{\uparrow}(r+1,1) \ni M \mapsto U \equiv U(M) := D_M \exp(\mathbf{i} m \cdot \mathcal{L}).
\]
(3.1)

\[^{6}\text{In the metric used here, the subgroup } O^{\uparrow}(r+1,1) \text{ of orthochronous Lorentz transformations is characterized by } M^0_0 + M^r_{r+1} - M^r_{r+1} - M^r_{r+1} \geq 2. \]
Here
\[ \mathcal{L}^{\mu,\nu} := \frac{1}{2} \left( q^\mu p^\nu - q^\nu p^\mu - i \sum_{n \neq 0} \frac{1}{n} \left( \alpha^\mu_n \alpha^\nu_n - \alpha^\nu_n \alpha^\mu_n \right) \right) = (\mathcal{L}^{\mu,\nu})^* \] (3.2)
generate infinitesimal Lorentz transformations on the Fock space \( \mathcal{F} \), \( m \cdot \mathcal{L} \equiv \sum_{\mu,\nu=0}^{r+1} (Gm)_{\mu,\nu} \mathcal{L}^{\mu,\nu} \), and \( m \) is an element of the Lie algebra \( \mathfrak{so}(r+1,1) \) of \( \text{O}(r+1,1) \) such that
\[ M = D_M \exp(m) \] (3.3)
is a coset decomposition of \( M \) into the product of a suitable matrix \( D_M \) with \( \det D_M = \det M \) and a proper orthochronous Lorentz transformation \( \exp(m) \); when \( \det M = 1 \), \( D_M \) is just the unit matrix, while for \( \det M = -1 \), \( D_M = -1 \) for odd \( r \) and \( D_M = \frac{1}{2^n} \) for even \( r \). (Note that \( m \in \mathfrak{so}(r+1,1) \) satisfies \( Gm G^{-1} = -m^t \), or in other words, \( Gm \) and \( m G^{-1} \) are antisymmetric matrices.)

The operation (3.1) of \( \text{O}(r+1,1) \) on \( \mathcal{F} \) extends to an action of Lorentz transformations on the oscillator algebra \( \mathfrak{a} \) by conjugation \( \text{ad}_U \),
\[ \mathfrak{a} \ni x \mapsto \text{ad}_U(x) := U x U^{-1} \] (3.4)
which by the definition of \( U \) is an inner automorphism of \( \mathfrak{a} \). From the Heisenberg commutation relations (2.2) for the string oscillator modes we deduce that operators with an upper index \( \mu \) transform as Lorentz vectors. For example, we have \( [p^\mu, m \cdot \mathcal{L}] = -i (mp)^\mu \) so that \( p^\mu U = U(Mp)^\mu \), and hence
\[ \text{ad}_U(p^\mu) = (M^{-1}p)^\mu, \quad \text{ad}_U(p_\mu) = (p M)_\mu. \] (3.5)
It follows e.g. that the Virasoro operators (2.8) are Lorentz singlets,
\[ \text{ad}_U(L_m) = L_m. \] (3.6)
Also, the inner product of any operator \( x^\mu \in \mathfrak{a} \) that is a Lorentz vector with a weight \( \lambda \in \mathfrak{g}_c^* \) transforms as
\[ \text{ad}_U(\lambda \cdot x) = \sum_\mu \lambda^\mu (xM)_\mu = \sum_\mu (M\lambda)^\mu x_\mu = (M\lambda) \cdot x, \] (3.7)
i.e. the action on the operator can be absorbed in a transformation
\[ \lambda^\mu \mapsto (M\lambda)^\mu \] (3.8)
of the vector \( \lambda \). In particular, when \( M \) leaves the null root \( \delta \in \mathfrak{g}_c^* \) invariant, i.e. \( M\delta = \delta \), then \( \delta \cdot x \) is invariant under \( \text{ad}_U \).

4 Affine Lie algebras

As already mentioned above, a simple current \( J \) of a WZW theory based on an affine Lie algebra \( \mathfrak{g} \) corresponds (except for an isolated case occurring for \( E_8 \) level two) to a symmetry of the Dynkin diagram of \( \mathfrak{g} \). Such a symmetry can be described as a permutation \( \dot{\omega} \equiv \dot{\omega}_J \)

\footnote{The bar in the notation \( \bar{\delta}_\cdot \) means that the allowed range of the indices of the Kronecker delta is from 1 to \( r \), analogously as for the \( \bar{G} \)-part of the quadratic form matrix \( G \) (2.4) of \( \bar{\mathfrak{g}} \).}
of the index set \( \{0, 1, 2, \ldots, r\} \) which leaves the Cartan matrix \( A \) of \( \mathfrak{g} \) invariant in the sense that \( A^\omega \hat{\omega}j = A^i j \) for all \( i, j \). To such a permutation, in turn, there is associated in natural manner a distinguished outer automorphism \( \omega \) of \( \mathfrak{g} \). On the Cartan-Weyl basis \([2],[10]\) of \( \mathfrak{g} \), this automorphism \( \omega \) acts as \([1\text{eqs. (6.7), (6.8)}]\)

\[
\begin{align*}
\omega(K) &= K, \\
\omega(H_i^m) &= H_{m^i}, \\
\omega(E_{m}^\alpha) &= \eta_\alpha \, E_{m+\alpha}^{\omega\bar{\alpha}}
\end{align*}
\]  

\((m \in \mathbb{Z}, i \in \{1, 2, \ldots, r\}, \bar{\alpha} \text{ a } \mathfrak{g}\text{-root}) \text{ and}
\]

\[
\omega(D) = D - \frac{1}{2} \sum_{i=1}^{r} \bar{G}_{0,i} \alpha\omega K + \sum_{i=1}^{r} \bar{G}_{0,i} \alpha\omega H^i_0.
\]  

\((4.1)\)

In \([1\text{eq. (6.55)}]\), \( m_\alpha \) denotes the integer \( m_\alpha := \bar{\alpha} \cdot \bar{\Lambda}(\omega^{-10}) \)

\((4.3)\)

(which can actually only take the values 0 or \( \pm 1 \)), while

\[
\bar{\omega}^* \bar{\alpha} := -m_\alpha \bar{\theta} + \sum_{i \neq \omega^{-10}}^r n_i \bar{\alpha}^{\omega i} \quad \text{for} \quad \bar{\alpha} = \sum_{i=1}^{r} n_i \bar{\alpha}^{(i)}.
\]  

\((4.4)\)

Moreover, \( \eta_\alpha \in \{\pm 1\} \) are certain sign factors, which are completely characterized by the following properties. First, \( \eta_{\alpha(i)} = 1 \) for all \( i = 1, 2, \ldots, r \). And second, whenever \( \bar{\alpha}, \bar{\beta} \) and \( \bar{\alpha} + \bar{\beta} \) are \( \mathfrak{g}\)-roots, then

\[
\eta_{\bar{\alpha}} \, \eta_{\bar{\beta}} \, \eta_{\bar{\alpha} + \bar{\beta}} = e_{\bar{\alpha}, \bar{\beta}} \, e_{\bar{\omega^* \bar{\alpha}, \bar{\omega^* \bar{\beta}}}}.
\]  

\((4.5)\)

where \( e_{\bar{\alpha}, \bar{\beta}} \) is the two-cocycle which constitutes the structure constants in the Lie bracket relations \([E^\alpha, E^\beta] = e_{\bar{\alpha}, \bar{\beta}} E^{\alpha + \beta} \) of \( \mathfrak{g} \).

Concerning the formula \([4.2]\), it must be noted that the convention for the derivation of \( \bar{\omega} \) chosen here (and in \([8]\)) differs from the one used in \([1]\): in \([1]\) the derivation was taken to be

\[
D^{[i]} := D - \frac{1}{4N} \sum_{m=1}^{N-1} \bar{G}_{m,0} \alpha m_0 K + \frac{1}{N} \sum_{i=1}^{r} \sum_{m=1}^{N-1} \bar{G}_{m} \alpha \omega m, i H^i_0.
\]  

\((4.6)\)

where \( N \) is the order of \( \bar{\omega} \) (and hence of \( \omega \)). With the help of the identity \([1\text{eq. (6.55)}]\])

\[
\sum_{m=0}^{N-1} G_{m,0,0} = \frac{1}{2}Na_i \bar{G}_{0,0} \quad \text{for} \quad i = 1, 2, \ldots, r
\]  

\((4.7)\)

it follows from \([4.1] \) and \([4.2]\) that \( D^{[i]} \) transforms as

\[
\omega(D^{[i]}) = D^{[i]} + \xi_0 K + \sum_{i=1}^{r} \xi_i H^i_0,
\]  

\((4.8)\)

\footnote{Note that \( \eta_{\bar{\alpha}} \, \eta_{\bar{\beta}} \, \eta_{\bar{\alpha} + \bar{\beta}} = \eta_{\bar{\alpha} + \bar{\beta}} / \eta_{\bar{\alpha}} \eta_{\bar{\beta}} \) provides a two-coboundary, so that \( \hat{\bar{\alpha}} \, \hat{\bar{\beta}} := e_{\bar{\omega^* \bar{\alpha}, \bar{\omega^* \bar{\beta}}}} \) supplies an equivalent collection of structure constants.}
where \( \xi_i \) are the linear combinations
\[
\xi_i := \bar{G}_{\omega_0,i} - \frac{1}{2} a_i \bar{G}_{\omega_{-1},\omega_{-10}} + \frac{1}{N} \sum_{m=1}^{N-1} (\bar{G}_{\omega_m,\omega_{-1}i} - \bar{G}_{\omega_m,0,i}), \quad i \in \{0, 1, \ldots, r\}
\]
of elements of the metric \( \bar{G} \); in particular, \( \xi_0 = \frac{1}{N} \sum_{m=1}^{N-1} \bar{G}_{\omega_m,\omega_{-10}} - \frac{1}{2} \bar{G}_{\omega_{-10},\omega_{-10}} \). By direct calculation one checks that in fact \( \xi_i = 0 \) for all \( i = 0, 1, \ldots, r \) (for \( i = 0 \) this was already observed in [1, eq. (6.56)], while for \( i \neq 0 \) and order \( N = 2 \) it is a special case of the identity (4.7)). Thus the derivation \( D^{[1]} \) is in fact \( \omega \)-invariant,
\[
\omega(D^{[1]}) = D^{[1]}
\]
(in [1] \( D^{[1]} \) was constructed via this property).

We will also need various properties of the map \( \bar{\omega}^* \). First, the relation (4.4) can also be written as
\[
\bar{\omega}^* \bar{\alpha} = \omega^* \bar{\alpha} - m_\delta \delta,
\]
where \( \omega^* \) is the linear map that is induced by \( \omega \big|_{\mathfrak{g}_0} \) on the weight space \( \mathfrak{g}_0^* \) of \( \mathfrak{g} \) according to
\[
(\omega^* \lambda)(\omega^{-1}(x)) = \lambda(x)
\]
for all \( \lambda \in \mathfrak{g}_0^* \) and all \( x \in \mathfrak{g}_0 \). On the fundamental \( \mathfrak{g} \)-weights, this map \( \omega^* \) acts as [1]
\[
\omega^*(\Lambda(i)) = \Lambda(\bar{\omega}i) + \left(\bar{G}_{\omega_{-10,i}} - \frac{1}{2} a_i \bar{G}_{\omega_{-1},\omega_{-10}}\right) \delta \quad \text{for } i = 0, 1, \ldots, r,
\]
and on the simple roots and the null root as
\[
\omega^*(\alpha(i)) = \alpha(\bar{\omega}i) \quad \text{for } i = 0, 1, \ldots, r
\]
and
\[
\omega^*(\delta) = \delta,
\]
respectively. Moreover, for every \( \bar{\mathfrak{g}} \)-root \( \bar{\alpha} \), also \( \bar{\omega}^* \bar{\alpha} \) is a \( \bar{\mathfrak{g}} \)-root; in particular, from (4.14) we have
\[
\bar{\omega}^*(\bar{\alpha}(i)) = \begin{cases} 
-\bar{\theta} & \text{for } i = \bar{\omega}^{-1}0, \\
\bar{\alpha}(\bar{\omega}i) & \text{else}.
\end{cases}
\]
Both \( \omega^* \) and \( \bar{\omega}^* \) are isometries (on the weight spaces \( \mathfrak{g}_0^* \) and \( \bar{\mathfrak{g}}_0^* \), respectively); this implies e.g. that [1, eq. (6.39)]
\[
\bar{G}_{\omega_{i,j}} - \bar{G}_{i,j} = a_i \bar{G}_{\omega_{0},\omega_{j}} + a_j \bar{G}_{\omega_{0},\omega_{i}} - a_i a_j \bar{G}_{\omega_{0},\omega_{0}}
\]
for \( i, j \neq \bar{\omega}^{-1}0 \), and also that the vectors (2.9) obey
\[
\omega^* a(\lambda) = a(\omega^* \lambda)
\]
for all \( \mathfrak{g} \)-weights \( \lambda \).

When the \( \mathfrak{g} \)-weights are expressed in terms of the basis (2.3) of \( \mathfrak{g}_0^* \), then the map \( \omega^* \) is implemented by multiplication of the components with respect to this basis with an \((r+1) \times (r+1)\)-matrix \( M \equiv M_\omega \); the entries of \( M \) read
\[
M^{\mu}_\nu = \delta_{\omega_{\nu},\mu} + (\delta_{\nu,0} - a_\nu) \delta_{\mu,0} + (\bar{G}_{\omega_{-10},\nu} - \frac{1}{2} \bar{G}_{\omega_{-1},\omega_{-10}} \delta_{\nu,0}) \delta_{\mu,r+1} + \delta_{\mu,0} \delta_{\nu,0} + \delta_{\mu,r+1} \delta_{\nu,r+1}.
\]
Here for $\nu \in \{0, 1, \ldots, r\}$ the numbers $a_\nu$ are the Coxeter labels of $g$ and $a_{r+1} := 0$.

As an illustration, we display the matrix $M$ for the case of the ‘basic’ order-$N$ automorphism of $g = \mathfrak{sl}(N)$, for which the permutation $\omega$ acts as $i \mapsto i+1 \mod N$, for the values $N = 2$ and $N = 3$:

$$M_{(N=2)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1/4 & 1/2 & 1 \end{pmatrix}, \quad M_{(N=3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ -1/3 & 2/3 & 1/3 & 1 \end{pmatrix}.$$ (4.20)

We also note that for all $N$, the determinant of these matrices is

$$\det M = (-1)^{N+1},$$ (4.21)

and that

$$M_{(N=2)} = -\exp(m_{(N=2)}) \quad \text{with} \quad m = i\pi \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & -1/4 & -1 \end{pmatrix}. $$ (4.22)

The following general properties of the matrix $M$ – valid for arbitrary untwisted affine Lie algebra $g$ (i.e. even for non-simply laced ones) and for arbitrary simple current symmetries $\omega$ of the Dynkin diagram of $g$ – can be directly deduced from its definition. First, $M_\delta = \delta$, in accordance with (4.13). Second, $M$ is unipotent; its order equals the order $N$ of $\omega$:

$$M^N = \mathbb{1}. $$ (4.23)

And third, $M \in O_L^+(r+1,1)$; more precisely, $M$ is a Lorentz transformation, it is orthochronous, and and it maps the weight lattice $L_g$ to itself.

That $M$ is orthochronous follows from $M_0^0 + M_{r+1} + M_{r+1}^+ + M_0^+ = 2 + \tilde{G}_{\omega,0,\omega,0}/2$. To deduce that $M \in O(r+1,1)$, i.e. that it satisfies $M^t GM = G$ with the metric $G$ given by (2.4), one proceeds as follows. With the help of the identity (4.17) one derives from the explicit expressions (4.19) and (2.5) that

$$(M^t GM - G)_{\mu,\nu} = \delta_{\mu,0} (\tilde{G}_{\omega,0,\omega,0} + \tilde{G}_{\omega,0,\omega,0} - a_\nu \tilde{G}_{\omega,0,\omega,0})$$
$$+ \delta_{\nu,0} (\tilde{G}_{\omega,0,\omega,0} + \tilde{G}_{\omega,0,\omega,0} - a_\mu \tilde{G}_{\omega,0,\omega,0})$$
$$+ \delta_{\mu,0} \delta_{\nu,0} (\tilde{G}_{\omega,0,\omega,0} - \tilde{G}_{\omega,0,\omega,0}). $$ (4.24)

For order $N = 2$, the last term on the right hand side is manifestly zero, while the other terms vanish owing to the identity (4.7). For larger order, it is again easily seen that the last term on the right hand side vanishes, namely as a consequence of the fact that in these cases the fundamental weights $\Lambda_{\omega,0}$ and $\Lambda_{\omega,-10}$ are each others conjugates; that the other terms vanish as well can be checked case by case.

Finally, to see that $M$ maps $L_g$ to itself one must first note that the relation between $g$-weights $\lambda$ and the associated vectors $a(\lambda)$ (2.9) is not one-to-one, but rather $a(\lambda + c\delta) = a(\lambda)$ for all $c \in \mathbb{C}$. Moreover, the isomorphism class of the $g$-module $\mathcal{H}_\Lambda$ also depends on the highest weight $\Lambda \in L_g$ only modulo $\mathbb{C}\delta$. Taken together, this implies that in the construction above we can prescribe the $\delta$-components of the weights $\Lambda$ in the construction of section 2 as we like. In particular we can choose $\Lambda$ in such a way that its coefficient in $\delta$-direction is an integral multiple of $(2K\ell)^{-1}$. Doing so, comparison with the formula (4.13) (see also footnote 3) shows that together with $\Lambda$, also $M\Lambda$ lies on the lattice $L_g$. 

11
5  Cocycle factors

We would like to describe the action of the conjugation ad$_U$ (3.4) for Lorentz transformations which leave the null root $\delta$ invariant not only on the string oscillators, but also on the vertex operators (2.11). It follows immediately from the result (3.7) that on the generators $K$, $D$ and $H^i_m$ of the vertex operator representation $\hat{\pi}_\Lambda$ this action reads

$$\text{ad}_U \circ \hat{\pi}_\Lambda(K) = \hat{\pi}_\Lambda(K),$$
$$\text{ad}_U \circ \hat{\pi}_\Lambda(D) = (M\Lambda(0)) \cdot p,$$
$$\text{ad}_U \circ \hat{\pi}_\Lambda(H^i_m) = \oint \frac{dz}{2\pi i} (M\alpha^{(i)}) \cdot P(z) \exp[im\delta \cdot X(z)].$$

(5.1)

On the other hand, the action on the generators $E^\alpha_m$ requires more work. Using the identity $c^2_{\bar{\alpha}} = 1$ that is satisfied by the cocycle operators $c_{\bar{\alpha}}$, we can write

$$\text{ad}_U \circ \hat{\pi}_\Lambda(E^\alpha_m) = \hat{\pi}_\Lambda(E^{M\bar{\alpha}}_m) \cdot c_{M\bar{\alpha}} \text{ad}_U(c_{\bar{\alpha}}).$$

(5.2)

Thus in order to determine the action on the generators $E^\alpha_m$ we need to know the action of ad$_U$ on the cocycle operators. This action, however, is not determined yet, because the cocycle operators constitute an additional input which is not provided by the string oscillators.

In other words, it is necessary to extend the definition of ad$_U$ from the algebra $\mathfrak{g}$ to its extension by the cocycle operators. We will not attempt to achieve this in full generality, but be content to define the action in the case of the specific Lorentz transformations (4.19), i.e. for ad$_U = \text{ad}_{U_\omega}$. To this end we recall (see e.g. appendix B of [14]) that the cocycle operators for all $\bar{\alpha}$-roots (and more generally, for every element of the root lattice of $\bar{\mathfrak{g}}$) can be described in terms of certain gamma matrices $\gamma_i \equiv \gamma_{\bar{\alpha}(i)}$ associated to the simple roots of $\bar{\mathfrak{g}}$. We will define an action of ad$_{U_\omega}$ on these gamma matrices, which then immediately supplies the action on the cocycle operators.

The gamma matrices satisfy the relations

$$\gamma_i \gamma_j = (-1)^{A_{ij}} \gamma_j \gamma_i \quad \text{for } i, j = 1, 2, \ldots, r$$

(5.3)

(here $A$ is the Cartan matrix of $\mathfrak{g}$, and we have used that $\bar{\alpha}^{(i)} \cdot \bar{\alpha}^{(j)} = A^{i,j}$ for $i, j = 1, 2, \ldots, r$).

The extension to arbitrary roots (respectively elements of the root lattice of $\bar{\mathfrak{g}}$) $\bar{\alpha}$ is based on the prescription

$$\gamma_{\bar{\alpha}} := \epsilon_{\bar{\alpha}} (\gamma_1)^{n_1} (\gamma_2)^{n_2} \cdots (\gamma_r)^{n_r} \quad \text{when } \bar{\alpha} = \sum_{i=1}^r n_i \bar{\alpha}^{(i)},$$

(5.4)

which implies $\gamma_{\bar{\alpha}} \gamma_{\bar{\beta}} = (-1)^{\bar{\alpha} \cdot \bar{\beta}} \gamma_{\bar{\beta}} \gamma_{\bar{\alpha}}$. Here $\epsilon_{\bar{\alpha}}$ is a sign factor which for any $\bar{\alpha}$ can be prescribed arbitrarily; in particular, it can be chosen in such a way that

$$\gamma_{\bar{\alpha}} \gamma_{\bar{\beta}} = \epsilon_{\bar{\alpha},\bar{\beta}} \gamma_{\bar{\alpha} + \bar{\beta}}$$

(5.5)

Another case which can be treated is the one where the Lorentz transformation is an element of the translation subgroup of the Weyl group of $\mathfrak{g}$. In that case it is consistent to require that the cocycle operators are ad$_U$-invariant [8].
for all $\alpha, \beta$, where $e_i$ is the two-cocycle appearing in formula (4.3), and we will stick to this choice from now on.

We now define $\text{ad}_{U_\omega}$ on the associative algebra $\mathcal{C}$ that is freely generated by the gamma matrices $\gamma_i$ ($i = 1, 2, \ldots, r$) modulo the relations (5.3) as follows. We set

$$\text{ad}_{U_\omega}(\gamma_i) := \gamma_i \eta_i \equiv \gamma_i \omega^* \bar{\alpha}(i) \quad \text{for} \quad i = 0, 1, \ldots, r,$$

(5.6)

where we introduced the convention that

$$\gamma_i = \left\{ \begin{array}{ll} \gamma_{\bar{\alpha}(i)} & \text{for} \quad i = 1, 2, \ldots, r, \\ \gamma_{-\bar{\theta}} & \text{for} \quad i = 0, \end{array} \right.$$  

(5.7)

and we impose the homomorphism property

$$\text{ad}_{U_\omega}(\gamma_{\alpha} \gamma_{\beta}) = \text{ad}_{U_\omega}(\gamma_{\alpha}) \text{ad}_{U_\omega}(\gamma_{\beta}).$$

(5.8)

In order for the definition (5.6) to be consistent and to consistently extend to the whole algebra $\mathcal{C}$, it is necessary and sufficient that, first, for $i, j = 1, 2, \ldots, r$ (5.6) is compatible with the relations (5.3) of $\mathcal{C}$, and second, that the definition of $\text{ad}_{U_\omega}(\gamma_0)$ is compatible with (5.4). These requirements are indeed satisfied: The first follows as

$$\text{ad}_{U_\omega}(\gamma_i \gamma_j) = \gamma_i \omega_j \gamma_{ij} = (-1)^{A_{i\bar{\alpha},j\bar{\alpha}}} \gamma_i \omega_j \gamma_{\alpha} = (-1)^{A_{i\bar{\alpha},j\bar{\alpha}}} \text{ad}_{U_\omega}(\gamma_j \gamma_i)$$

(5.9)

by the fundamental property $A_{i\bar{\alpha},j\bar{\alpha}} = A_{i\bar{\alpha},j\bar{\alpha}}$ of the permutation $\omega$. The second property is a direct consequence of the fact that the relation (5.3) continues to hold for $i = 0$, which in turn follows from $-\sum_{i=1}^r a_i A_{i\bar{\alpha}} = A_{0\bar{\alpha}}$ (recall that $\bar{\alpha} = \sum_{i=1}^r a_i \bar{\alpha}(i)$).

The extension of the prescription (5.6) to the whole algebra $\mathcal{C}$ is achieved by implementing the property (5.8). We find that this leads to the formula

$$\text{ad}_{U_\omega}(\gamma_{\alpha}) = \eta_{\bar{\alpha}} \gamma_{\omega^* \bar{\alpha}}$$

(5.10)

for all $\bar{\alpha}$-roots $\bar{\alpha}$, where $\eta_{\bar{\alpha}}$ is the sign factor that was introduced before the relation (1.3). (This is most conveniently proven by induction on the height of $\bar{\alpha}$, i.e. on the number $\sum_{i=1}^r n_i$ of simple roots ‘contained’ in $\bar{\alpha} = \sum_{i=1}^r n_i \bar{\alpha}(i)$. Namely, for $\bar{\alpha}$ of height larger than one, one can write $\bar{\alpha} = \bar{\beta} + \bar{\alpha}(i)$ for some $i \neq \omega^{-1} \bar{\alpha}(i)$. Then the height of $\bar{\beta}$ is smaller than the one of $\bar{\alpha}$, so that by the induction assumption we have $\text{ad}_{U_\omega}(\gamma_{\beta}) = \eta_{\bar{\beta}} \gamma_{\omega^* \bar{\beta}}$. Hence

$$\text{ad}_{U_\omega}(\gamma_{\alpha}) = e_{\bar{\beta}, \bar{\alpha}(i)} \text{ad}_{U_\omega}(\gamma_{\beta} \gamma_{\alpha}) = e_{\bar{\beta}, \bar{\alpha}(i)} \eta_{\bar{\beta}} \gamma_{\omega^* \bar{\beta}} \gamma_{\alpha} = e_{\bar{\beta}, \bar{\alpha}(i)} \eta_{\bar{\beta}} e_{\omega^* \bar{\beta}, \bar{\alpha}(i)} \eta_{\bar{\alpha}} e_{\omega^* \bar{\alpha}, \bar{\alpha}(i)} \gamma_{\omega^* \alpha},$$

(5.11)

which by $\eta_{\bar{\alpha}(i)} = 1$ and by the identity (1.5) satisfied by the signs $\eta_{\bar{\alpha}}$ reduces to the equality (5.10).)

When the cocycle operators are defined via the gamma matrices as in [14], then the result (5.10) translates into an analogous formula for the cocycle operators themselves:

$$\text{ad}_{U_\omega}(c_{\bar{\alpha}}) = \eta_{\bar{\alpha}} c_{\omega^* \bar{\alpha}}.$$ 

(5.12)

It is straightforward to check that this prescription is indeed compatible with all properties of the cocycle operators. For instance, their basic property reads

$$e^{\bar{\alpha} q} c_{\bar{\alpha}} e^{\bar{\beta} q} c_{\bar{\beta}} = e_{\bar{\alpha} \bar{\beta}} e^{(\bar{\alpha} + \bar{\beta}) q} c_{\bar{\alpha} + \bar{\beta}};$$

(5.13)
upon application of \( \text{ad}_{\omega} \), the left hand side becomes
\[
\eta_\alpha \eta_\beta e^{\omega^* \alpha \cdot q} c_{\omega^* \alpha} e^{\omega^* \beta \cdot q} c_{\omega^* \beta} = \eta_\alpha \eta_\beta e^{\omega^* (\alpha + \beta) \cdot q} c_{\omega^* (\alpha + \beta)} ,
\] (5.14)
and by (4.5) this coincides with what is obtained by acting with \( \text{ad}_{\omega} \) on the right hand side.

We can now come back to our original problem to compute \( \text{ad}_{\omega} \circ \hat{\pi}_A(E_m^\alpha) \). Combining the formulæ (5.2) and (5.12) we obtain (recall that \( M\bar{\alpha} = \omega^* \bar{\alpha} \))
\[
\text{ad}_{\omega} \circ \hat{\pi}_A(E_m^\alpha) = \eta_\alpha \hat{\pi}_A(E_m^{M\bar{\alpha}}) .
\] (5.15)

6 The simple current vertex operators

Putting the results of the previous sections together, we are finally in a position to obtain the interpretation of the unitary operators \( U_\omega \) as vertex operators for simple currents. As already mentioned in the introduction, in WZW theories one can realize simple currents by the linear maps \( T_\omega \) between the irreducible highest weight modules \( \mathcal{H}_\Lambda \) and \( \mathcal{H}_{\omega^* \Lambda} \) of \( \mathfrak{g} \). As the notation \( \mathcal{S}_\omega \) indicates, this simple current map corresponds to the outer automorphism \( \omega \) of \( \mathfrak{g} \). In fact, the map \( \mathcal{S}_\omega \) is completely characterized by its action on the highest weight vector of \( \mathcal{H}_\Lambda \), which reads
\[
\mathcal{S}_\omega : \; v_\Lambda \mapsto v_{\omega^* \Lambda} ,
\] (6.1)
and by the \( \omega \)-twining property which says that
\[
\mathcal{S}_\omega \circ \pi_\Lambda(x) = \pi_{\omega^* \Lambda}(\omega(x)) \circ \mathcal{S}_\omega
\] (6.2)
for all \( x \in \mathfrak{g} \), where \( \pi_\Lambda \) denotes the irreducible highest weight representation (2.14) of \( \mathfrak{g} \). Because of (6.2), \( \mathcal{S}_\omega \) was called an \( \omega \)-twining map in [1].

We will now demonstrate:

The \( \omega \)-twining map \( \mathcal{S}_\omega \) between irreducible highest weight modules \( \mathcal{H}_\Lambda \) and \( \mathcal{H}_{\omega^* \Lambda} \) can be realized as the unitary vertex operator \( U_\omega \equiv U(M) \) of the Lorentz transformation \( M \equiv M_\omega \) on the string Fock space \( \mathcal{F} \).

More precisely, by restriction of the linear map
\[
\hat{\mathcal{S}}_\omega : \; \mathcal{F} \to \mathcal{F}
\]
\[
v \mapsto U_\omega v
\] (6.3)
on \( \mathcal{F} \) to the subspace \( \mathcal{H}_\Lambda \subseteq \mathcal{P}_\Lambda \subseteq \mathcal{F} \) one reproduces the map \( \mathcal{S}_\omega \).

To prove this statement, we show that the map (6.3) satisfies the analogues
\[
\hat{\mathcal{S}}_\omega(v_\Lambda) = v_{\omega^* \Lambda}
\] (6.4)
and
\[
\hat{\mathcal{S}}_\omega \circ \hat{\pi}_A(x) = \hat{\pi}_{\omega^* \Lambda}(\omega(x)) \circ \hat{\mathcal{S}}_\omega
\] (6.5)
of (5.1) and (5.2), and that these properties are preserved by the restriction to irreducible modules.
To verify (6.3), we first observe that the formulæ (5.1) and (5.15) hold for $M \equiv M_\omega$, and that we can rewrite those equations as
\[
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(D) = (\omega^* \Lambda(0)) \cdot p, \\
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(K) = \hat{\pi}_{\omega^* \Lambda}(K), \\
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(H^i_m) = \oint \frac{dz}{2\pi i} (\omega^* \alpha(i)) \cdot P(z) \exp[im\delta \cdot X(z)] = \hat{\pi}_{\omega^* \Lambda}(H^i_m), \\
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(E^\alpha_{m,\alpha}) = \eta_{\alpha} \hat{\pi}_{\omega^* \Lambda}(E^{\omega^* \alpha}_{m,\alpha}).
\] (6.6) (6.7) (6.8) (6.9)

Here we recalled that in the vertex operator representation (2.11) no explicit reference to the highest weight $\Lambda$ is made and that $M \Lambda = \omega^* \Lambda$. In addition, in order to arrive at (6.8) we implemented the identity (4.12) for $\omega^* \alpha(i)$, and for (6.9) we used the fact that according to the definitions (2.11) of $E^\alpha_{m,\alpha}$ and (4.3) of the integer $m_{\alpha}$ we have to identify $E^M_{\alpha,\alpha} \equiv E^{\omega^* \alpha}_{m} = E^{\omega^* \alpha}_{m,\alpha} = E^{\omega^* \alpha}_{m + m_{\alpha}}$. (6.10)

Also note that in the first place the right hand side of (5.8) strictly makes sense only for $i \neq \omega^{-1}0$; however, it extends to the case $i = \omega^{-1}0$ (and, likewise, to $i = 0$) when one adopts the convention to write $H^0_m := \delta_{m,0} K - \sum_{j=1}^r a_j H^j_m$. (6.11)

The results (5.8) and (5.9) tell us in particular that on the Chevalley generators of $g$ we simply have
\[
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(H^i) = \hat{\pi}_{\omega^* \Lambda}(H^i), \quad \text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(E^{\pm \alpha(i)}_{m,\alpha}) = \hat{\pi}_{\omega^* \Lambda}(E^{\pm \alpha}_{m,\alpha}) \quad \text{for all } i = 0, 1, \ldots, r.
\] (6.12)

Furthermore, by noticing that $a_{\omega^0} = 1$, so that $\omega^* \Lambda(0) - \Lambda(0) = \Lambda(\omega - \omega^0) - a_{\omega^0} \Lambda(0) = \sum_{i=1}^r \bar{c}_{\omega^0,i} \alpha(i)$, and using the identity $\xi_i = 0$ for the numbers $\xi_i$ defined in (4.9), one deduces from the results (6.6) to (6.8) that the linear combination $D^{(i)}$ of generators of $g$ obeys
\[
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(D^{(i)}) = \Lambda(0) \cdot p = \hat{\pi}_{\omega^* \Lambda}(D^{(i)}).
\] (6.13)

Now comparing the formulæ (6.7) to (6.9) and (6.13) with the action (4.1) and (4.10) of the automorphism $\omega$ of $g$, we conclude that the equality
\[
\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(x) = \hat{\pi}_{\omega^* \Lambda}(\omega(x))
\] (6.14)
holds for all $x \in \{K, D^{(i)}\} \cup \{H^i_m | i = 1, 2, \ldots, r, m \in \mathbb{Z}\} \cup \{E^\alpha_{m,\alpha} | \bar{\alpha} a \text{ g-root}, m \in \mathbb{Z}\}$, and hence by linearity of $\omega$ for all $x \in g$. Rewriting this identity as
\[
U_\omega \hat{\pi}_\Lambda(x) = (\text{ad}_{U_\omega} \circ \hat{\pi}_\Lambda(x)) U_\omega = \hat{\pi}_{\omega^* \Lambda}(\omega(x)) U_\omega,
\] (6.15)
we learn that the map $\hat{\pi}_\omega$ (6.3) indeed satisfies the analogue (6.5) of (5.2).

Next we recall that the vector $v^A_\Lambda \in H_\Lambda$ lies in the subspace $P_\Lambda$ of $F$ and is characterized by the properties (2.10) and (2.17). Combining the first of those properties with (3.6) we obtain
\[
L_0 U_\omega v^A_\Lambda = U_\omega v^A_\Lambda \quad \text{and} \quad L_n U_\omega v^A_\Lambda = 0 \quad \text{for } n > 0,
\] (6.16)
while making use of (6.3) it follows that
\[ p^\mu U_\omega v_A^\lambda = U_\omega (M p)^\mu v_A^\lambda = (M \Lambda)^\mu U_\omega v_A^\lambda = (\omega^* \Lambda)^\mu U_\omega v_A^\lambda. \] (6.17)

Thus we can conclude that the vector \( U_\omega v_A^\lambda \in \mathcal{F} \) lies in fact in \( \mathcal{P}_{\omega^* \Lambda} \), i.e. is a physical ground state with \( g \)-weight \( \omega^* \Lambda \); in particular, taking also into account that the associated vectors \( a(\lambda) \) satisfy (4.18), \( U_\omega v_A^\lambda \) can be regarded as the highest weight vector \( v_\omega^* \Lambda \) of the irreducible submodule \( \mathcal{H}_{\omega^* \Lambda} \subset \mathcal{P}_{\omega^* \Lambda} \). Hence (6.4) is satisfied, as claimed. (It is straightforward to check directly that the highest weight vector properties hold. Namely, it follows immediately from (6.15) that
\[ \hat{\pi}_{\omega^* \Lambda}(x) U_\omega v_A^\lambda = U_\omega \hat{\pi}_A(\omega^{-1}(x)) v_A^\lambda \] (6.18)
for all \( x \in g \). In particular, we have
\[ \hat{\pi}_{\omega^* \Lambda}(H^i) U_\omega v_A^\lambda = U_\omega \hat{\pi}_A(H_0^{\omega^{-1}i}) v_A^\lambda = \Lambda^{\omega^{-1}i} U_\omega v_A^\lambda \equiv (\omega^* \Lambda)^i U_\omega v_A^\lambda \quad \text{for } 0 \leq i \leq r, \] \[ \hat{\pi}_{\omega^* \Lambda}(E^{\alpha(i)}) U_\omega v_A^\lambda = U_\omega \hat{\pi}_A(E^{\alpha(\omega^{-1}i)}) v_A^\lambda = 0 \quad \text{for } 0 \leq i \leq r, \] (6.19)
which shows that \( U_\omega v_A^\lambda \) possesses the defining properties of a highest weight vector of weight \( \omega^* \Lambda \). Similarly, by
\[ (\hat{\pi}_{\omega^* \Lambda}(E^{-\alpha(i)}))^{(\omega^* \Lambda)^i+1} U_\omega v_A^\lambda = U_\omega (\hat{\pi}_A(E^{-\alpha(\omega^{-1}i)}))^{\Lambda^{\omega^{-1}i}+1} v_A^\lambda = 0 \quad \text{for all } i = 0, 1, \ldots, r \] (6.20)
one verifies that the irreducibility (null vector) conditions are satisfied.)

Finally, it is clear that the properties (6.4) and (6.5) survive the restriction to the irreducible submodules \( \mathcal{H}_{\Lambda} \) of \( \mathcal{P}_{\Lambda} \). More precisely, because of \( v_A^\lambda \in \mathcal{H}_{\Lambda} \), (6.1) is immediately implied by (6.4), while the results (6.7) – (6.9) and (6.13) show that \( (\hat{\pi}_\omega \circ \hat{\pi}_{\Lambda}(\cdot))|_{\mathcal{H}_{\omega^* \Lambda}} = \hat{\pi}_{\omega} \circ \hat{\pi}_{\Lambda}(\cdot) \) and \( (\hat{\pi}_{\omega^* \Lambda}(\cdot) \circ \hat{\pi}_\omega)|_{\mathcal{H}_{\omega^* \Lambda}} = \pi_{\omega^* \Lambda}(\cdot) \circ \hat{\pi}_\omega \), so that upon identifying \( \hat{\pi}_\omega := \hat{\pi}_\omega|_{\mathcal{H}_{\Lambda}} \) as in (1.3), the property (6.3) reproduces (6.2). This finally concludes our proof.

### 7 Outlook

In this paper we have presented a realization of the simple currents of a simply-laced WZW conformal field theory as vertex operators acting on the Fock space of a compactified string theory. Besides providing for the first time a concrete field-theoretic description of these primary fields, this construction may also have applications in other areas. For example, by studying the explicit form of the bases of the irreducible highest weight modules of \( g \) in terms of the string oscillators, one could extract informations about the structure of the operator product algebra. Also, by organizing the whole string Fock space \( \mathcal{F} \) into orbits with respect to the action of the simple current automorphisms \( \hat{\pi}_\omega \) and using the connection (6.3) between the compactified string theory and the root multiplicities of hyperbolic Kac–Moody algebras, it should be possible to find relations between the multiplicities of roots which correspond to different affine Lie algebra modules.

It is also interesting to note that to the simple current symmetries \( \hat{\omega} \) of the Dynkin diagram of \( g \) there are not only associated the outer automorphisms \( \omega \) (6.1) of \( g \), but in fact isomorphism classes of outer modulo inner automorphisms. The inner automorphisms of \( g \) correspond to the
Weyl group $W$ of $\mathfrak{g}$. Now usually the abelian subgroup of $W$ that is isomorphic to the coroot lattice is realized in a way rather different from the form (1.1) of the outer automorphisms considered here. However, as described in detail in [8], these Weyl transformations can also be described as elements of the subgroup $O^+_L(r+1,1)$ of the Lorentz group $O(r+1,1)$, and hence on precisely the same footing as the outer automorphisms. This should prove useful in applications to coset conformal field theories, where in order to study the field identification procedure, it is necessary (except for generalized diagonal cosets [3]) to deal also with inner automorphisms of the relevant affine subalgebra of $\mathfrak{g}$.

Finally we should mention that in our discussions we have followed the habit of talking about ‘the’ highest weight vector of a highest weight $\mathfrak{g}$-module $\mathcal{H}_\Lambda$. Strictly speaking, the highest weight vector is unique only up to a scalar multiple. For our construction this is quite irrelevant as long as the weight $\omega^*\Lambda$ is different from $\Lambda$. In contrast, when $\Lambda$ is a fixed point of the simple current, i.e. when $\omega^*\Lambda = \Lambda$ (which for integrable weights is only possible for a proper subset of the allowed levels $\ell$ of $\mathfrak{g}$), then $\hat{T}_\omega$ is an endomorphism of $\mathcal{P}_\Lambda$ and there is room for an arbitrary phase. To get a complete picture also for such modules will therefore require further study.

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