$p$-ADIC $q$-EXPANSION OF ALTERNATING SUMS OF POWERS

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ABSTRACT. In this paper, we give an explicit $p$-adic expansion of

$$\sum_{j=1 \atop (j,p)=1}^{np} \frac{(-1)^j}{[j]_q^r}$$

as a power series in $n$. The coefficients are values of $p$-adic $q$-l-function for $q$-Euler numbers.

§1. INTRODUCTION

Let $p$ be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf.[1, 4, 6, 10]. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, defined over the $p$-adic number field, that serve as $p$-adic equivalents of the Dirichlet $L$-series, cf.[10, 11].

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These $p$-adic $L$-functions interpolate the values

$$L_p(1-n, \chi) = \frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n, \chi_n}, \text{ for } n \in \mathbb{N} = \{1, 2, \ldots, \},$$

where $B_{n, \chi}$ denote the $n$th generalized Bernoulli numbers associated with the primitive Dirichlet character $\chi$, and $\chi_n = \chi w^{-n}$, with $w$ the Teichmüller character, cf. [8, 10].

In [10], L. C. Washington have proved the below interesting formula:

$$\sum_{j=1}^{np} \frac{1}{j^r} = -\sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k L_p(r + k, w^{1-k-r}), \text{ where } \binom{-r}{k} \text{ is binomial coefficient.}$$

To give the $q$-extension of the above Washington result, author derived the sums of powers of consecutive $q$-integers as follows:

$$\sum_{l=0}^{n-1} \frac{q^l}{[l]_q} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} q^m l \beta_l[n]_q^{m-l} + \frac{1}{m} (q^m - 1) \beta_m, \text{ see [6, 7],}$$

where $\beta_m$ are $q$-Bernoulli numbers. By using (*), we gave an explicit $p$-adic expansion

$$\sum_{j=1}^{np} \frac{q^j}{[j]_q} = \sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k L_{p, q}(r + k, w^{1-r-k})$$

$$- (q - 1) \sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k T_{p, q}(r + k, w^{1-r-k}) - (q - 1) \sum_{a=1}^{p-1} B_{p, q}^{(n)}(r, a : F),$$

where $L_{p, q}(s, \chi)$ is $p$-adic $q$-$L$-function (see [7]). Indeed, this is a $q$-extension result due to Washington, corresponding to the case $q = 1$, see [10]. For a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N,$$

$$X_1 = \mathbb{Z}_p, X^* = \bigcup_{0 < a < dp, (a, p) = 1} a + dp \mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{p^N} \},$$
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^N \), (cf.[3, 4, 9]). We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), and write \( f \in UD(\mathbb{Z}_p) \), if the difference quotients 
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]
have a limit \( f'(a) \) as \( (x, y) \to (a, a) \), cf.[3].

For \( f \in UD(\mathbb{Z}_p) \), let us begin with the expression
\[
\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p), \quad \text{cf.[1, 3, 4, 7, 8, 9],}
\]
which represents a \( q \)-analogue of Riemann sums for \( f \). The integral of \( f \) on \( \mathbb{Z}_p \) is defined as the limit of those sums (as \( n \to \infty \)) if this limit exists. The \( q \)-Volkenborn integral of a function \( f \in UD(\mathbb{Z}_p) \) is defined by
\[
I_q(f) = \int_X f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x) q^x, \quad \text{cf. [2, 3].}
\]

It is well known that the familiar Euler polynomials \( E_n(z) \) are defined by means of the following generating function:
\[
F(z, t) = \frac{2}{e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \quad \text{cf.[1, 5].}
\]

We note that, by substituting \( z = 0 \), \( E_n(0) = E_n \) are the familiar \( n \)-th Euler numbers. Over five decades ago, Carlitz defined \( q \)-extension of Euler numbers and polynomials, cf.[1, 4, 5]. Recently, author gave another construction of \( q \)-Euler numbers and polynomials (see [1, 5, 9]). By using author’s \( q \)-Euler numbers and polynomials, we gave the alternating sums of powers of consecutive \( q \)-integers as follows: For \( m \geq 1 \), we have
\[
2 \sum_{l=0}^{n-1} (-1)^l [l]_q^m = (-1)^{n+1} \sum_{l=0}^{m-1} \binom{m}{l} q^n E_{l,q}[n]_q^{m-l} + ((-1)^{n+1} q^{nm} + 1) E_{m,q},
\]
where \( E_{l,q} \) are \( q \)-Euler numbers (see [5] ). From this result, we can study the \( p \)-adic interpolating function for \( q \)-Euler numbers and sums of powers due to author [7]. Throughout this paper, we use the below notation:
\[
[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 - q} = 1 - q + q^2 - q^3 + \cdots + (-q)^{x-1}, \quad \text{cf.[5, 9].}
\]
Note that when \( p \) is prime \([p]_q\) is an irreducible polynomial in \( Q[q] \). Furthermore, this means that \( Q[q]/[p]_q \) is a field and consequently rational functions \( r(q)/s(q) \) are well defined \( \mod [p]_q \) if \( (r(q), s(q)) = 1 \). In a recent paper \([5]\) the author constructed the new \( q \)-extensions of Euler numbers and polynomials. In Section 2, we introduce the \( q \)-extension of Euler numbers and polynomials. In Section 3 we construct a new \( q \)-extension of Dirichlet’s type \( l \)-function which interpolates the \( q \)-extension of generalized Euler numbers attached to \( \chi \) at negative integers. The values of this function at negative integers are algebraic, hence may be regarded as lying in an extension of \( Q_p \).

We therefore look for a \( p \)-adic function which agrees with at negative integers. The purpose of this paper is to construct the new \( q \)-extension of generalized Euler numbers attached to \( \chi \) due to author and prove the existence of a specific \( p \)-adic interpolating function which interpolate the \( q \)-extension of generalized Bernoulli polynomials at negative integer. Finally, we give an explicit \( p \)-adic \( q \)-expansion

\[
\sum_{j=1}^{np} \frac{(-1)^j}{[j]_q^r},
\]

as a power series in \( n \). The coefficients are values of \( p \)-adic \( q \)-\( l \)-function for \( q \)-Euler numbers.

2. Preliminaries

For any non-negative integer \( m \), the \( q \)-Euler numbers, \( E_{m,q} \), were represented by

\[
(2) \quad 2 \int_{z_p} q^{-x} [x]_q^m d\mu_q(x) = E_{m,q} = 2 \left( \frac{1}{1 - q} \right)^m \sum_{i=0}^{m} \left( \frac{m}{i} \right) (-1)^i \frac{1}{1 + q^i}, \text{ see } [9].
\]

Note that \( \lim_{q \to 1} E_{m,q} = E_m \). From Eq.(2), we can derive the below generating function:

\[
(3) \quad F_q(t) = 2e^{1+q} \sum_{j=0}^{\infty} \frac{1}{1 + q^j} (-1)^j \frac{1}{1 - q^j} \frac{t^j}{j!} = \sum_{j=0}^{\infty} E_{n,q} \frac{t^n}{n!}.
\]

By using \( p \)-adic \( q \)-integral, we can also consider the \( q \)-Euler polynomials, \( E_{n,q}(x) \), as follows:

\[
(4) \quad E_{n,q}(x) = 2 \int_{z_p} q^{-t} [x + t]_q^n d\mu_q(t) = 2 \left( \frac{1}{1 - q} \right)^n \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{(-q^x)^k}{1 + q^k}, \text{ see } [5, 9].
\]
Note that

\( E_{n,q}(x) = \frac{2}{[2]_q} \int_{\mathbb{Z}_p} ([x]_q + q^x[t]_q)^n d\mu_{-q}(x) = \sum_{j=0}^{n} \binom{n}{j} q^{ix} E_{j,q}[x]^{n-j}. \)

By (4), we easily see that

\( \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = F_q(x, t) = 2e^{\frac{-t}{q}} \sum_{j=0}^{\infty} (-1)^j q^{ix} \left( \frac{1}{1-q} \right)^j \frac{t^j}{j!}. \)

From (6), we derive

\( F_q(x, t) = 2 \sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \)

3. On the \( q \)-analogue of Hurwitz’s type \( \zeta \)-function associated with \( q \)-Euler numbers

In this section, we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). It is easy to see that

\[ E_{n,q}(x) = [m]_q^{m-1} \sum_{a=0}^{m-1} (-1)^a E_{n,q}^m \left( \frac{a+x}{m} \right), \text{ see [1, 5]}, \]

where \( m \) is odd positive integer. From (7), we can easily derive the below formula:

\( E_{k,q}(x) = \frac{d^k}{dt^k} F_q(x, t)|_{t=0} = 2 \sum_{n=0}^{\infty} (-1)^n [n+x]_q^k. \)

Thus, we can consider a \( q \)-\( \zeta \)-function which interpolates \( q \)-Euler numbers at negative integer as follows:

**Definition 1.** For \( s \in \mathbb{C} \), define

\[ \zeta_{E,q}(s, x) = 2 \sum_{m=1}^{\infty} \frac{(-1)^n}{[n+x]_q^s}. \]

Note that \( \zeta_{E,q}(s, x) \) is meromorphic function in whole complex plane.

By using Definition 1 and Eq.(8), we obtain the following:
Proposition 2. For any positive integer \( k \), we have

\[
\zeta_{E,q}(-k, x) = E_{k,q}(x).
\]

Let \( \chi \) be the Dirichlet character with conductor \( f \in \mathbb{N} \). Then we define the generalized \( q \)-Euler numbers attached to \( \chi \) as

\[
F_{q,\chi}(t) = 2\sum_{n=0}^{\infty} e^{[n]_q t} \chi(n)(-1)^n = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}.
\]

Note that

\[
E_{n,\chi,q} = [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{m,q,f} \left( \frac{a}{f} \right), \text{ where } f(= \text{odd}) \in \mathbb{N}.
\]

By (9), we easily see that

\[
\frac{d^k}{dt^k} F_{q,\chi}(t)|_{t=0} = E_{k,\chi,q} = 2\sum_{n=1}^{\infty} \chi(n)(-1)^n [n]_q^k
\]

Definition 3. For \( s \in \mathbb{C} \), we define Dirichlet’s type \( l \)-function as follows:

\[
l_q(s, \chi) = 2\sum_{n=1}^{\infty} \chi(n)(-1)^n \frac{[n]_q^s}{n^s}.
\]

From (11) and Definition 3, we can derive the below theorem.

Theorem 4. For \( k \geq 1 \), we have

\[
l_q(-k, \chi) = E_{k,\chi,q}.
\]

In [5], it was known that

\[
2\sum_{l=0}^{n-1} (-1)^l [l]_q^m = ((-1)^{n+1} q^n E_{m,q}(n) + E_{m,q}), \text{ where } m, n \in \mathbb{N}.
\]
From (4) and (12), we derive

\begin{equation}
2 \sum_{l=0}^{n-1} (-1)^l l^m q^n l_q^m = (-1)^{n+1} \sum_{l=0}^{m-1} \binom{m}{l} q^{nl} E_{l,q}[n]_q^m + ((-1)^{n+1} q^{nm} + 1) E_{m,q}.
\end{equation}

Let \( s \) be a complex variable, and let \( a \) and \( F(= \text{odd}) \) be the integers with \( 0 < a < F \).

We now consider the partial \( q \)-zeta function as follows:

\begin{equation}
H_q(s, a : F) = \sum_{m \equiv a(F) \mod{m} > 0} \frac{(-1)^m}{[m]_q} = (-1)^a \frac{[F]_q^{-s}}{2} \zeta_{E,q^F}(s, \frac{a}{F}).
\end{equation}

For \( n \in \mathbb{N} \), we note that \( H_q(-n, a : F) = (-1)^n q^a [F]_q^n E_{n,q^F}(\frac{a}{F}) \). Let \( \chi \) be the Dirichlet’s character with conductor \( F(= \text{odd}) \). Then we have

\begin{equation}
l_q(s, \chi) = 2 \sum_{a=1}^{F} \chi(a) H_q(s, a : F).
\end{equation}

The function \( H_q(s, a : F) \) will be called the \( q \)-extension of partial zeta function which interpolates \( q \)-Euler polynomials at negative integers. The values of \( l_q(s, \chi) \) at negative integers are algebraic, hence may be regarded as lying in an extension of \( \mathbb{Q}_p \). We therefore look for a \( p \)-adic function which agrees with \( l_q(s, \chi) \) at the negative integers in Section 4.

§4. \( p \)-ADIC \( q \)-FUNCTIONS AND SUMS OF POWERS

We define \( < x > = q^{\frac{[x]}{w(x)}} \), where \( w(x) \) is the Teichmüller character. When \( F(= \text{odd}) \) is multiple of \( p \) and \((a, p) = 1\), we define a \( p \)-adic analogue of (14) as follows:

\begin{equation}
H_{p,q}(s, a : F) = \frac{(-1)^a}{2} q^{sa} \sum_{j=0}^{\infty} \binom{[F]_q}{a}^j E_{j,q^F}, \text{ for } s \in \mathbb{Z}_p.
\end{equation}
Thus, we note that

\[
H_{p,q}(-n, a : F) = \frac{(-1)^a}{2} < a >^n \sum_{j=0}^{\infty} \binom{n}{j} q^ja \left( \frac{[F]_q}{[a]_q} \right)^j E_{j,q^F}.
\]

\[
= \frac{(-1)^a}{2} w^{-n}(a)[F]_q^n E_{n,q^F}(\frac{a}{F}) = w^{-n}(a)H_q(-n, a : F), \text{ for } n \in \mathbb{N}.
\]

We now construct the \( p \)-adic analytic function which interpolates \( q \)-Euler number at negative integer as follows:

\[
l_{p,q}(s, \chi) = 2 \sum_{a=1}^{F} \chi(a)H_{p,q}(s, a : F).
\]

In [5, 9], it was known that

\[
E_{k,\chi,q} = \frac{2}{[2]_q^f} \int_X \chi(x)[x]^k q^{-x} d\mu_{-q}(x), \text{ for } k \in \mathbb{N}.
\]

For \( f(= \text{ odd}) \in \mathbb{N} \), we note that

\[
E_{n,\chi,q} = [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{n,q^f}(\frac{a}{f}).
\]

Thus, we have

\[
l_{p,q}(-n, \chi) = 2 \sum_{a=1}^{F} \chi(a)H_{p,q}(-n, a : F) = \frac{2}{[2]_q^f} \int_X \chi w^{-n}(x)[x]^n q^{-x} d\mu_{-q}(x)
\]

\[
= E_{n,\chi w^{-n},q} - [p]_q^n \chi w^{-n}(p)E_{n,\chi w^{-n},q^p}.
\]

In fact,

\[
l_{p,q}(s, \chi) = 2 \sum_{a=1}^{F} (-1)^a < a >^{-s} \chi(a) \sum_{j=0}^{\infty} \binom{-s}{j} q^ja \left( \frac{[F]_q}{[a]_q} \right)^j E_{j,q^F}, \text{ for } s \in \mathbb{Z}_p.
\]
This is a $p$-adic analytic function and has the following properties for $\chi = w^t$:

\begin{equation}
l_{p,q}(-n, w^t) = E_{n,q} - [p]_q^n E_{n,q^p}, \quad \text{where } n \equiv t \pmod{p-1},
\end{equation}

\begin{equation}
l_{p,q}(s,t) \in \mathbb{Z}_p \text{ for all } s \in \mathbb{Z}_p \text{ when } t \equiv 0 \pmod{p-1}.
\end{equation}

If $t \equiv 0 \pmod{p-1}$, then $l_{p,q}(s_1, w^t) \equiv l_{p,q}(s_2, w^t) \pmod{p}$ for all $s_1, s_2 \in \mathbb{Z}_p$, $l_{p,q}(k, w^t) \equiv l_{p,q}(k + p, w^t) \pmod{p}$. It is easy to see that

\begin{equation}
\frac{1}{r+k-1} \binom{-r}{k} \binom{1-r-k}{j} = -\frac{1}{j+k} \binom{-r}{k+j-1} \binom{k+j}{j},
\end{equation}

for all positive integers $r, j, k$ with $j, k \geq 0$, $j + k > 0$, and $r \neq 1 - k$. Thus, we note that

\begin{equation}
\frac{1}{r+k-1} \binom{-r}{k} \binom{1-r-k}{j} = \frac{1}{r-1} \binom{-r+1}{k+j} \binom{k+j}{j}.
\end{equation}

From (22) and (22-1), we derive

\begin{equation}
\frac{r}{r+k} \binom{-r-1}{k} \binom{-r-k}{j} = \binom{-r}{k+j} \binom{k+j}{j}.
\end{equation}

By using (13), we see that

\begin{equation}
\sum_{l=0}^{n-1} (-1)^{Fl+a} \frac{F_{l+a}}{[Fl+a]_q} = \sum_{l=0}^{n-1} (-1)^l (-1)^a ([a]_q + q^a [F]_q [l]_q^p)^{-r}

= -\sum_{s=1}^{\infty} [a]_q^{-r} \binom{[F]_q}{[a]_q}^s q^{as} (-1)^a \binom{-r}{s} \left( -\frac{1}{2} \right)^{n-1} \sum_{l=0}^{s-1} \binom{s}{l} q^{nFl} E_{l,q^F} [n]_q^{s-l}

- \sum_{s=1}^{\infty} [a]_q^{-r} \binom{[F]_q}{[a]_q}^s q^{as} (-1)^a \binom{-r}{s} \frac{(-1)^n q^{Fk} - 1}{2} E_{s,q^F} + \frac{1 - (-1)^n [a]_q^{-r}}{2} (-1)^a.
\end{equation}

For $s \in \mathbb{Z}_p$, we define the below $T$-Euler polynomials:

\begin{equation}T_{n,q}(s, a : F) = (-1)^a < a >^{-s} \sum_{k=1}^{\infty} \binom{-s}{k} \frac{a}{[F]_q^k q^{ak}} \frac{(-1)^n q^{nk}}{2} E_{k,q^F}.
\end{equation}
Note that \( \lim_{q \to 1} T_{n,q}(s, a : F) = 0 \), if \( n \) is even positive integer. From (23) and (24), we derive

\[
\sum_{l=0}^{n-1} (-1)^{Fl+a} [Fl + a]_q^r
\]

\[
= - \sum_{s=1}^{\infty} \left( -\frac{r}{s} \right) [a]_q^{-r} \left( \frac{[F]_q}{[a]_q} \right)^s \left( -q^s \right)^a \sum_{l=0}^{s-1} \left( \frac{s}{l} \right) q^{nFl} E_{l,q^F}[n]_{q^F}^{s-l}
\]

\[- \frac{w^{-r}(a)}{2} T_{n,q}(r, a : F), \text{ where } n \text{ is positive even integer .}
\]

Let \( n \) be positive even integer. Then, we evaluate the right side of Eq.(26) as follows:

\[
\sum_{s=1}^{\infty} \left( -\frac{r}{s} \right) [a]_q^{-r} \left( \frac{[F]_q}{[a]_q} \right)^s \left( -q^s \right)^a \sum_{l=0}^{s-1} \left( \frac{s}{l} \right) q^{nFl} E_{l,q^F}[n]_{q^F}^{s-l}
\]

\[
= \sum_{k=1}^{\infty} \frac{r}{r+k} \left( -\frac{r-1}{k} \right) [a]_q^{-k-r} q^{a(k-1)} n[F]_q^{k} E_{l,q^F}[n]_{q^F}^{s-l}
\]

It is easy to check that

\[
q^{nFl} = \sum_{j=0}^{l} \binom{l}{j} [nF]_q^{j}(q-1)^j = 1 + \sum_{j=1}^{l} \binom{l}{j} [nF]_q^{j}(q-1)^j.
\]

Let

\[
K_{p,q}(s, a : F) = \frac{(-1)^a}{2} < a > -s \sum_{l=1}^{\infty} \left( \frac{-s}{l} \right) q^{al} \left( \frac{[F]_q}{[a]_q} \right)^l E_{l,q^F} \sum_{j=1}^{l} \binom{l}{j} [nF]_q^{j}(q-1)^j.
\]

Note that \( \lim_{q \to 1} K_{p,q}(s, a; F) = 0 \). For \( F = p, r \in \mathbb{N} \), we see that

\[
2 \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^{a+pl}}{[a + pl]_q^r} = 2 \sum_{j=1}^{np} \frac{(-1)^j}{[j]_q^r}.
\]

For \( s \in \mathbb{Z}_p \), we define \( p \)-adic analytically continued function on \( \mathbb{Z}_p \) as

\[
K_{p,q}(s, \chi) = 2 \sum_{a=1}^{p-1} \chi(a) K_{p,q}(s, a : F),
\]

\[
T_{p,q}(s, \chi) = 2 \sum_{a=1}^{p-1} \chi(a) T_{n,q}(s, a : F), \text{ where } k, n \geq 1 .
\]
From (24)-(31), we derive
\[
2 \sum_{j=1 \atop (j,p)=1}^{np} \frac{(-1)^j}{[j]_q^r} = - \sum_{k=1}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (-1)^n [pn]_q^k l_{p,q}(r + k, w^{-r-k}) \\
- \sum_{k=1}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (-1)^n [pn]_q^k K_{p,q}(r + k, w^{-r-k}) - T_{p,q}(r, w^{-r}).
\]

Therefore we obtain the following theorem:

**Theorem 5.** Let $p$ be an odd prime and let $n \geq 1$ be positive even integer. Then we have
\[
(32) \\
2 \sum_{j=1 \atop (j,p)=1}^{np} \frac{(-1)^j}{[j]_q^r} = - \sum_{k=1}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (-1)^n [pn]_q^k l_{p,q}(r + k, w^{-r-k}) \\
- \sum_{k=1}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (-1)^n [pn]_q^k K_{p,q}(r + k, w^{-r-k}) - T_{p,q}(r, w^{-r}), \text{ where } r \geq 1.
\]

For $q = 1$ in (32), we have
\[
2 \sum_{j=1 \atop (j,p)=1}^{np} \frac{(-1)^j}{j^r} = - \sum_{k=1}^{\infty} \frac{r}{k+r} \binom{-r-1}{k} (-1)^n (pn)_p^k l_{p}(r + k, w^{-r-k}),
\]
where $n$ is positive even integer.

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