Formal normal form of $A_k$ slow fast systems

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Abstract

An $A_k$ slow fast system is a particular type of singularly perturbed ODE. The corresponding slow manifold is defined by the critical points of a universal unfolding of an $A_k$ singularity. In this note we propose a formal normal form of $A_k$ slow fast systems.

1 Introduction

In this note we propose a formal normal form of a particular class of slow fast systems. A slow fast system (SFS) is a singularly perturbed ODE usually written as

$$
\begin{align*}
\dot{z} &= g(x, z, \epsilon) \\
\dot{z} &= g(x, z, \epsilon)
\end{align*}
$$

(1)

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ and $0 < \epsilon \ll 1$ is a small parameter, and where the over-dot denotes the derivative with respect to a time parameter $t$. Slow fast systems are often used as mathematical models of phenomena which occur in two time scales. Observe that as $\epsilon$ decreases, the time scale difference between $x$ and $z$ increases. A couple of classical examples of real life phenomena that were modeled by a SFS are the Zeeman’s heartbeat and nerve-impulse models [15]. For $\epsilon \neq 0$, we can define a new time parameter $\tau$ by $t = \epsilon \tau$. With this new time $\tau$ we can write (1) as

$$
\begin{align*}
\dot{x} &= \epsilon f(x, z, \epsilon) \\
\dot{z} &= g(x, z, \epsilon)
\end{align*}
$$

(2)

where the prime denotes derivative with respect to $\tau$. An important geometric object in the study of SFSs is the slow manifold which is defined by

$$
S = \{(x, z) \in \mathbb{R}^m \times \mathbb{R}^n \mid g(x, z, 0) = 0\}.
$$

(3)

When $\epsilon = 0$, the manifold $S$ serves as the phase space of (1) and as the set of equilibrium points of (2). In the rest of the document, we prefer to work with a SFS written as (2). Furthermore, to avoid working with an $\epsilon$-parameter family of vector fields as in (2), we plug-in into (2) the trivial equation $\dot{\epsilon} = 0$. To be more precise, we treat a $C^\infty$-smooth vector field defined as follows.

**Definition 1.1** ($A_k$ slow fast system). Let $k \in \mathbb{N}$ with $k \geq 2$. An $A_k$ slow fast system (for short $A_k$-SFS) is a vector field $X$ of the form

$$
X = \epsilon(1 + \epsilon f_1) \frac{\partial}{\partial x_1} + \sum_{i=1}^{k-1} \epsilon^2 f_i \frac{\partial}{\partial x_i} - (G_k - \epsilon f_k) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \epsilon}.
$$

(4)
where \( G_k = z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \) and where each \( f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon) \) is a \( C^\infty \)-smooth function vanishing at the origin.

**Remark 1.1.** The slow manifold associated to an \( A_k \)-SFS is defined by

\[
S = \left\{ (x, z) \in \mathbb{R}^k \mid z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0 \right\}.
\]

(5)

The manifold \( S \) can be regarded as the critical set of the universal unfolding of a smooth function with an \( A_k \) singularity at the origin [1, 3]. Hence the name \( A_k \)-SFS.

Observe that the origin is a non-hyperbolic equilibrium point of \( X \) and thus, it is not possible to study its local dynamics with the classical Geometric Singular Perturbation Theory [6]. In this case, a technique called blow-up [4, 5, 9] is usually applied to desingularize the SFS. This methodology has been successfully used in many cases, e.g. [2, 8, 10, 11, 13, 14], where many of these deal with an \( A_k \)-SFS with fixed \( k = 2 \) or \( k = 3 \). Briefly speaking, the blow-up technique consists in an appropriate change of coordinates under which the induced vector field is regular or has simpler singularities (hyperbolic or partially-hyperbolic). However, in this work we propose a normal form of \( A_k \)-SFS to be performed prior to the blow-up, see theorem 2.2. This normalization greatly simplifies the local analysis of \( A_k \)-SFSs as shown in [7, 8].

## 2 Formal normal form of an \( A_k \)-slow fast system

We regard the vector field \( X \) of definition 1.1 as \( X = F + P \), where \( F \) and \( P \) are smooth vector fields called “the principal part” and “the perturbation” respectively. That is

\[
F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0 \frac{\partial}{\partial x_i} - G_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \quad P = \sum_{i=1}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} + \varepsilon f_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.
\]

(6)

The idea of the rest of the document is motivated by [12]. In short, we want to formally simplify the expression of \( X \) by eliminating the perturbation \( P \). The terminology used below is that of [12].

The vector field \( F \) is quasihomogeneous of type \( r = (k, k - 1, \ldots, 1, 2k - 1) \) and quasidegree \( k - 1 \) [1, 12]. From now on, we fix the type of quasihomogeneity \( r \). A quasihomogeneous object of type \( r \) will be called \( r \)-quasihomogeneous.

**Definition 2.1** (Good perturbation). Let \( F \) be an \( r \)-quasihomogeneous vector field of quasidegree \( k - 1 \). A good perturbation \( X \) of \( F \) is a smooth vector field \( X = F + P \), where \( P = P(x_1, \ldots, x_{k-1}, z, \varepsilon) \) satisfies the following conditions

- \( P \) is a smooth vector field of quasiorder greater than \( k - 1 \),
- \( P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \) with \( P|_{\varepsilon=0} = 0 \).

**Notation** By \( P_\delta \) we denote the space of \( r \)-quasihomogeneous polynomials (in \( k+1 \) variables) of quasidegree \( \delta \). By \( H_\gamma \) we denote the space of \( r \)-quasihomogeneous vector fields (in \( \mathbb{R}^{k+1} \)) of quasidegree \( \gamma \) and such that for all \( U \in H_\delta \) we have \( U = \sum_{i=1}^{k} U_i \frac{\partial}{\partial x_i} + 0 \frac{\partial}{\partial x_{k+1}} \). The formal series expansion of a function \( f \) is be denoted by \( \hat{f} \).
Definition 2.2 (The inner product $\langle \cdot , \cdot \rangle_{r, \delta}$ [12]). Let $x = (x_1, \ldots, x_n)$, and $s, q \in \mathbb{N}^n$. Let $f, g \in \mathcal{P}_\delta$, that is

$$f = \sum_{(r,s) = \delta} f_s x^s,$$

where $f_s \in \mathbb{R}$, $x^s = x_1^{s_1} \ldots x_n^{s_n}$; and similarly for $g$. Then the inner product $\langle \cdot , \cdot \rangle_{r, \delta}$ is defined as

$$\langle f, g \rangle_{r, \delta} = \sum_{(r,s) = \delta} f_s g_s (s!)^r,$$

where $(s!)^r = (s_1!)^{r_1} \ldots (s_n!)^{r_n}$, and where $(r, s)$ denotes the dot product $r \cdot s$. So for monomials one has

$$\langle x^s, x^q \rangle_{r, \delta} = \begin{cases} \frac{(s_1!)^{r_1} \ldots (s_n!)^{r_n}}{r!} & \text{if } s = q \text{ with } (s, r) = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly, for vector fields: let $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$, and $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$. Then

$$\langle X, Y \rangle_{r, \delta} = \sum_{i=1}^n \langle X_i, Y_i \rangle_{r, \delta + r_i}.$$  

Definition 2.3 (The operators $d$, $d^*$ and $\Box$ [12]). The operator $d : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma + k - 1}$ (associated to $F$) is defined by $d(U) = [F, U]$ for any $U \in \mathcal{H}_\gamma$, where $[\cdot, \cdot]$ denotes the Lie bracket. The operator $d^*$ is the adjoint operator of $d$ with respect to the inner product of definition 2.2. This is, given $U \in \mathcal{H}_\gamma$, $V \in \mathcal{H}_{\gamma + k - 1}$ we have

$$\langle d(U), V \rangle_{r, \gamma + k - 1} = \langle U, d^*(V) \rangle_{r, \gamma}$$

For any quasidegree $\beta > k - 1$, the self adjoint operator $\Box_\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta$ is defined by $\Box_\beta(U) = d d^*(U)$ for all $U \in \mathcal{H}_\beta$.

Definition 2.4 (Resonant vector field [12]).

- We say that a vector field $U \in \mathcal{H}_\beta$ is resonant if $U \in \ker \Box_\beta$.
- A formal vector field is called resonant if all its quasihomogeneous components are resonant.

Definition 2.5 (Normal Form [12]). A good perturbation $X = F + R$ of $F$ is a normal form with respect to $F$ if $R$ is resonant.

It is important to note the following.

Lemma 2.1. $\ker \Box_\beta = \ker d^*|_{\mathcal{H}_\beta}$.

Proof. Let $\alpha = k - 1$, then $d : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma + \alpha}$ and $d^* : \mathcal{H}_{\gamma + \alpha} \to \mathcal{H}_\gamma$. Due to the fact that $d^*$ is the adjoint of $d$, we have the decomposition $\mathcal{H}_\gamma = \text{Im} d^*|_{\mathcal{H}_{\gamma + \alpha}} \oplus \ker d|_{\mathcal{H}_\gamma}$. Now let $U \in \mathcal{H}_{\gamma + \alpha} = \mathcal{H}_\beta$, then $\Box_\beta(U) = d d^*(U) = 0$ if and only if $d^* U \in \ker d$. Furthermore, $d^* U \in \text{Im} d^*$. That is $d^* U \in \text{Im} d^* \cap \ker d$. However $\text{Im} d^*$ and $\ker d$ are orthogonal. Then $\Box_\beta(U) = 0$ if and only if $d^* U = 0$.

We now recall a result of [12] (Proposition 4.4), we only adapt it for the present context.
**Theorem 2.1** (Formal normal form [12]). Let \( X = F + P \) be a good perturbation of \( F \) as in definition 2.1. Then there exists a formal diffeomorphism \( \Phi \) such that \( \Phi \) conjugates \( X \) to a vector field \( F + R \), where \( R \) is a resonant formal vector field in the sense of definition 2.4.

Finally, we present our result. In short, we prove that the resonant vector field \( R \) in theorem 2.1 associated to \( F \) be a good perturbation of the vector field \( F \).

**Theorem 2.2.** Let \( X = F + P \) be a good perturbation of the vector field

\[
F = \varepsilon \frac{\partial}{\partial z} + \sum_{i=1}^{k-1} 0 \frac{\partial}{\partial x_i} - \left( z^k + \sum_{j=1}^{k-1} x_j z^{j-1} \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \quad (12)
\]

Then, there exists a formal diffeomorphism \( \Phi \) that conjugates \( X \) with \( F \), this is \( \Phi \ast X = F \).

**Proof.** From theorem 2.1 and lemma 2.1 we will show that if \( P \in \ker d^r \mid_{k \geq k} \), then \( P = 0 \). Let us start by rewriting \( d^r(P) \) in a more workable format.

To simplify the notation, let \( \alpha \geq k, \beta \in \mathcal{H}_\alpha, \beta = \alpha - k + 1 \), and let \( x = (x_1, \ldots, x_{k-1}, \varepsilon) = (x_1, \ldots, x_{k-1}, x_k, x_{k+1}) \). If \( D \) is an operator, its adjoint with respect to the inner product definition 2.2 is always denoted as \( D^* \).

We start with the inner product (definition 2.2)

\[
\langle d(Q), P \rangle_{r,\alpha} = \langle Q, d^r(P) \rangle_{r,\beta}. \quad (13)
\]

We can write \( d(Q) = \sum_{i=1}^{k+1} F_i Q_i - Q(F_i) \), where \( F_i = \sum_{j=1}^{k+1} F_{ij} \frac{\partial Q}{\partial x_j} \) and similarly for \( Q(F_i) \), then

\[
\langle d(Q), P \rangle_{r,\alpha} = \sum_{i=1}^{k+1} \langle F_i, P_i \rangle_{r,\alpha} = \sum_{i=1}^{k+1} \langle F_i, P_i \rangle_{r,\alpha + r_i} - \langle Q(F_i), P_i \rangle_{r,\alpha + r_i}
\]

\[
= \sum_{i=1}^{k+1} \langle Q, F^*(P_i) \rangle_{r,\beta + r_i} - \langle Q(F_i), P_i \rangle_{r,\alpha + r_i} = \sum_{i=1}^{k+1} \langle Q, F^*(P_i) \rangle_{r,\beta + r_i} - \sum_{j=1}^{k+1} \langle Q, (\frac{\partial F_i}{\partial x_j})^* (P_j) \rangle_{r,\alpha + r_i}
\]

\[
= \sum_{i=1}^{k+1} \langle Q, F^*(P_i) \rangle - \sum_{j=1}^{k+1} \langle \frac{\partial F_i}{\partial x_j}^* (P_j) \rangle_{r,\alpha + r_i}
\]

Comparing (14) to \( \langle Q, d^r(P) \rangle_{r,\beta} \) we can write

\[
d^r(P) = \begin{pmatrix}
\frac{\partial F_i}{\partial x_1}^* & \frac{\partial F_i}{\partial x_2}^* & \cdots & \frac{\partial F_i}{\partial x_k}^* \\
\frac{\partial F_i}{\partial x_1} & \frac{\partial F_i}{\partial x_2} & \cdots & \frac{\partial F_i}{\partial x_k} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\partial F_i}{\partial x_{k+1}} & -\frac{\partial F_i}{\partial x_{k+1}} & \cdots & \frac{\partial F_i}{\partial x_{k+1}}^*
\end{pmatrix} \begin{pmatrix}
P_1 \\
P_2 \\
\vdots \\
P_{k+1}
\end{pmatrix}
\]

Plugging the expressions of \( F \) and \( P \) into (15) we get

\[
d^r(P) = \begin{pmatrix}
F^* & 0 & \cdots & 0 & 1 & 0 \\
0 & F^* & \cdots & 0 & 0 & \varepsilon \\
0 & 0 & \cdots & F^* & (z^{k-1})^* & 0 \\
0 & 0 & \cdots & 0 & F^* + Z^* & 0 \\
-1 & 0 & \cdots & 0 & 0 & F^*
\end{pmatrix} \begin{pmatrix}
P_1 \\
P_2 \\
\vdots \\
P_{k+1}
\end{pmatrix} = 0. \quad (16)
\]
where \( Z^* = \left( k z^{k-1} + \sum_{i=2}^{k-1} (i-1)x_i z^{i-2} \right)^* \). Now note that (16) implies \( F^*(P_j) = 0 \) for all \( j = 2, \ldots, k-1 \) and \( P_1 = P_k = 0 \).

**Remark 2.1.** For \( k = 2 \), the result is trivial: we have \( F = \varepsilon \frac{\partial}{\partial x_1} - (z^2 + x_1) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial z} \), and therefore \( d^*(P) = 0 \) is written as

\[
d^*(P) = \begin{bmatrix} F^* & 1 & 0 \\ 0 & F^* + 2z^* & 0 \\ -1 & 0 & F^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ 0 \end{bmatrix} = 0, \tag{17}
\]

which immediately implies \( P_1 = P_2 = 0 \).

Now, we study \( F^*(P_j) = 0 \). Recall that \( P = P(x_1, \ldots, x_{k-1}, z, \varepsilon) \) is not any vector field, but it has the property that \( P(x_1, \ldots, x_{k-1}, z, 0) = 0 \). That is, we can write

\[
P = \sum_{i=1}^{k-1} \varepsilon P_i \frac{\partial}{\partial x_i} + \varepsilon \tilde{P}_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \tag{18}
\]

where \( \tilde{P}_j \in \mathcal{P}_{\alpha+ r_j - 2k+1} \). This is because the (quasihomogeneous) weight of \( \varepsilon \) is \( 2k - 1 \). Now, since it is complicated to work with the adjoint, we first rewrite the problem \( F^*(\varepsilon \tilde{P}_j) = 0 \). We then prove that \( F^*(\varepsilon \tilde{P}_j) = 0 \) implies \( \tilde{P}_j = 0 \).

Note that \( F^*(\varepsilon \tilde{P}_j) = 0 \) is equivalent to \( \langle Q, F^*(\varepsilon \tilde{P}_j) \rangle_{\alpha+ r_j - k+1} = 0 \) for all \( Q \in \mathcal{P}_{\beta+ r_j} \). Next, we use the definition of \( F^* \) that is

\[
\langle Q, F^*(\varepsilon \tilde{P}_j) \rangle_{r, \beta+ r_j} = (F(Q), \varepsilon \tilde{P}_j)_{r, \alpha+ r_j} = 0. \tag{19}
\]

We will now show that if \( \langle F(Q), \varepsilon \tilde{P}_j \rangle_{r, \alpha+ r_j} = 0 \) for all \( Q \in \mathcal{P}_{\beta+ r_j} \), then \( \tilde{P}_j = 0 \). Note that by (19), this is the same as proving that \( F^*(\varepsilon \tilde{P}_j) = 0 \) implies \( \tilde{P}_j = 0 \).

Start by choosing an element \( x^q \) of the basis of \( \mathcal{P}_{\beta+ r_j} \), this is

\[
x^q = x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{q_{k+1}}, \quad (r, q) = \beta + r_j. \tag{20}
\]

Then we have

\[
F(x^q) = q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}-1} z^{q_k} \varepsilon^{q_{k+1}+1} - \left( \frac{k}{2} + \sum_{i=1}^{k-1} x_i z^{i-2} \right) q_k x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k-1} \varepsilon^{q_{k+1}}. \tag{21}
\]

Let us write \( \varepsilon \tilde{P}_j \in \mathcal{P}_{\alpha+ r_j} \) as

\[
\varepsilon \tilde{P}_j = \varepsilon \sum_{(r, p) = \alpha+ r_j - 2k+1} a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon^{p_{k+1}}, \tag{22}
\]

where \( a_p \in \mathbb{R} \). We now proceed by recursion on the exponent of \( \varepsilon \). Let \( q_{k+1} = 0 \), then the inner product \( \langle F(Q), \varepsilon \tilde{P}_j \rangle_{\alpha+ r_j} \) has only one term since \( F(Q) \) has only one monomial containing \( \varepsilon \). That is

\[
\langle F(Q), \varepsilon \tilde{P}_j \rangle_{\alpha+ r_j} \varepsilon^{q_{k+1}} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}-1} z^{q_k} \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r, \alpha+ r_j} = 0. \tag{23}
\]

We naturally consider \( q_1 > 0 \). If \( q_1 = 0 \), then the equality is automatically satisfied. Recalling the definition 2.2 of the inner product, the equality (23) means that

\[
\langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}-1} z^{q_k} \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r, \alpha+ r_j} = q_1 a_p \frac{(q_1)!}{(\alpha + r_j)!} = 0, \tag{24}
\]
and therefore from (23) we have
\[ a_p = a_{q_1-1,p_2,\ldots,p_k,1} = 0, \]  
for all \( q_1 > 0, p_2, \ldots, p_k \geq 0 \) (naturally, also satisfying the degree condition \((r,p) = \alpha + r_j\)).  
Next, let \( q_{k+1} = 1 \). Then
\[ F(x^q) = q_1x_1^{q_1-1}\cdots x_k^{q_k-1}z^{q_k}e^2 - \left( z^k + \sum_{i=1}^{k-1} x_1^i z^{i-1} \right) q_kx_1^{q_1}x_2^{q_2}z^{q_k-1}e. \]  
Once again, the inner product \( \langle F(Q), \varepsilon \hat{P}_j \rangle \) has only one term, now this is due to the fact that all coefficients \( a_p \) of monomials containing \( e \) are zero due to (25). Then
\[ \langle F(Q), \varepsilon \hat{P}_j \rangle \eta_{q_k+1} = 1 = \langle q_1x_1^{q_1-1}\cdots x_k^{q_k-1}z^{q_k}e^2, \varepsilon a_{p_1}x_1^{p_1}x_2^{p_2}z^{p_k}e \rangle r, \alpha + r_j = 0. \]  
Therefore, similarly as above, we have the condition
\[ a_p = a_{q_1-1,p_2,\ldots,p_k,2} = 0, \]  
for all \( q_1 > 0, p_2, \ldots, p_k \geq 0 \) (naturally, also satisfying the degree condition \((r,p) = \alpha + r_j\)). By recursion arguments, assume \( q_{k+1} = n \) and that all the coefficients
\[ a_p = a_{p_1,p_2,\ldots,p_k,m} = 0, \quad \forall m \leq n. \]  
Then again the inner product \( \langle F(Q), \varepsilon \hat{P}_j \rangle \) has only one term, namely
\[ \langle F(Q), \varepsilon \hat{P}_j \rangle \eta_{q_k+1} = \langle q_1x_1^{q_1-1}\cdots x_k^{q_k-1}z^{q_k}e^{n+1}, \varepsilon a_{p_1}x_1^{p_1}x_2^{p_2}z^{p_k}e^n \rangle r, \alpha + r_j = 0. \]  
The latter then implies
\[ a_p = a_{q_1-1,p_2,\ldots,p_k,n+1} = 0. \]  
This finishes the proof of \( \langle F(Q), \varepsilon \hat{P}_j \rangle = 0 \) implies \( \hat{P}_j = 0 \). \( \square \)

**Remark 2.2.** Theorem 2.2 together with Borel’s lemma [3], imply that an \( A_k\)-SFS \( X = F + P \) is smoothly conjugate to a smooth vector field \( Y = F + H \) where \( H \) is flat at the origin. The benefits of this normal form are exploited in [7, 8].

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