THE $W_{1+\infty}(gl_s)$–SYMMETRIES OF THE $s$–COMPONENT KP HIERARCHY

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ABSTRACT. Adler, Shiota and van Moerbeke obtained for the KP and Toda lattice hierarchies a formula which translates the action of the vertex operator on tau–functions to an action of a vertex operator of pseudo-differential operators on wave functions. This relates the additional symmetries of the KP and Toda lattice hierarchy to the $W_{1+\infty}$, respectively $W_{1+\infty} \times W_{1+\infty}$–algebra symmetries. In this paper we generalize the results to the $s$–component KP hierarchy. The vertex operators generate the algebra $W_{1+\infty}(gl_s)$, the matrix version of $W_{1+\infty}$. Since the Toda lattice hierarchy is equivalent to the 2–component KP hierarchy, the results of this paper uncover in that particular case a much richer structure than the one obtained by Adler, Shiota and van Moerbeke.

§0. Introduction.

The KP hierarchy is the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^n)_+, L],$$

for the first order pseudo-differential operator

$$L \equiv L(x,t) = \partial + u_1(x,t)\partial^{-1} + u_2(x,t)\partial^{-2} + \cdots,$$

here $\partial = \frac{\partial}{\partial x}$ and $t = (t_1, t_2, \ldots)$. It is well–known that $L$ dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(\tau, x, t) = 1 + a_1(x,t)\partial^{-1} + a_2(x,t)\partial^{-2} + \cdots$$

$$= \frac{\tau(x,t - [\partial^{-1}])}{\tau(x,t)},$$

where $\tau$ is the famous $\tau$–function, introduced by the Kyoto group [DJKM1-3] and $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots)$.

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The wave or Baker–Akhiezer function
\[ \Psi \equiv \Psi(\tau, x, t, z) = W(\tau, x, t, \partial)e^{zx}, \]
where
\[ W \equiv W(\tau, x, t, z) = P(\tau, x, t)e^{\xi(t)} \]
with \( \xi(t) = \sum_{k=1}^{\infty} t_k \partial^k \)
is an eigenfunction of \( L \), viz.,
\[ L \Psi = z \Psi \text{ and } \frac{\partial \Psi}{\partial t_k} = (L_k^+)^+ \Psi. \]
From this point of view, the introduction by Orlov and Schulman [OS] of another pseudo-differential operator
\[ M \equiv M(x, t) = W x W^{-1} \]
which action on \( \Psi \) amounts to
\[ M \Psi = \frac{\partial \Psi}{\partial z} \]
is rather natural.
Recently, Adler, Shiota and van Moerbeke [ASV1], [ASV2] proved a conjecture of Orlov and Schulman, viz., that there exists a relation between \((M^\ell L^{k+\ell})^{-} \) acting on \( \Psi \) and the generators \( W_k^{(\ell+1)} \sim -t^{k+\ell}(\frac{\partial}{\partial t})^\ell \) of the \( W_{1+\infty} \)-algebra acting on the \( \tau \)-function. More explicitly, let
\[ Y(\tau, y, w) = \sum_{\ell=0}^{\infty} \frac{(y - w)\ell}{\ell!} \sum_{k \in \mathbb{Z}} M^\ell L^{k+\ell} w^{-k-\ell-1} \]
be the generating series of the \( M^\ell L^{k+\ell} \) and let
\[ W(y, w) = \sum_{\ell=0}^{\infty} \frac{(y - w)\ell}{\ell!} \sum_{k \in \mathbb{Z}} W_k^{(\ell+1)} w^{-k-\ell-1} \]
be the vertex operator of the KP hierarchy, then one has the following formula [ASV1]:
\[ -Y(\tau, y, w) - \Psi(\tau, x, t, z) = \Psi(\tau, x, t, z)(e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial t}_k z^{-k}} - 1) \left( \frac{W(y, w)\tau(x, t)}{\tau(x, t)} \right). \]
Dickey gave another proof of this formula [D]. The “geometric interpretation” of this Adler–Shiota–van Moerbeke formula is as follows. The transformation \( e^{\lambda W(y, w)} = 1 + \lambda W(y, w) \) is a symmetric transformation or a kind of auto-Bäcklund transformation of the KP hierarchy. If one rewrites (0.2) as
\[ -\lambda Y(\tau, y, w) - \Psi(\tau, x, t, z) = \Psi(\tau, x, t, z)(e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial t}_k z^{-k}} - 1) \left( \frac{e^{\lambda W(y, w)}\tau(x, t)}{\tau(x, t)} \right), \]
then one easily sees that (0.2) is in fact a formula that relates this Bäcklund transformation to the so-called additional symmetries of the KP hierarchy. To be more precise, let $\sigma$ be the new solution of the KP hierarchy which one obtains from $\tau$ by this Bäcklund transformation, i.e., $\sigma = e^{\lambda W(y,w)}\tau$, then
\[
\Psi(\sigma, x, t, z) = \sigma(x, t)^{-1} e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial t_k} t_k^{-k}} \sigma(x, t) e^{\xi(t)} e^{zx}
\]
\[
= \frac{\tau(x, t)}{\sigma(x, t)} e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial t_k} t_k^{-k}} \left( \frac{e^{\lambda W(y,w)}\tau(x, t)}{\tau(x, t)} \right) \Psi(\tau, x, t, z)
\]
\[
= \Psi(\tau, x, t, z) + \frac{\tau(x, t)}{\tau(x, t)} (e^{-\sum_{k=1}^{\infty} \frac{\partial}{\partial t_k} t_k^{-k}} - 1) \left( \frac{e^{\lambda W(y,w)}\tau(x, t)}{\tau(x, t)} \right) \Psi(\tau, x, t, z).
\]
Now using (0.3) one obtains
\[
\Psi(\sigma, x, t, z) = \left( 1 - \lambda \frac{\tau(x, t)}{\sigma(x, t)} Y(\tau, y, w) \right) \Psi(\tau, x, t, z).
\]
Hence $Y(\tau, y, w)_-$ produces, as a consequence of formula (0.2), the Bäcklund transformation of the wave function corresponding to $\tau$.

Adler, Shiota and van Moerbeke also treated in [ASV2] the Toda lattice hierarchy of Ueno and Takasaki [UT] and showed that an analogous formula also holds. In their treatment they considered two vertex operators, each depending on a different time flow $t^{(j)} = (t^{(j)}_1, t^{(j)}_2, \ldots)$, $j = 1, 2$, of a form similar to that of (0.1). Hence The $W_{1+\infty}$–algebra of the KP hierarchy is replaced by two copies of this algebra.

Using $2 \times 2$–matrix pseudo–differential operators instead of infinite shift operators, one can show that the Toda lattice hierarchy is equivalent to the 2–component KP hierarchy as treated by Kac and the author in [KV]. In that case there are however more vertex operators than only the ones of the form (0.1), viz., one also has
\[
W^{(ab)}(y,w) = C^{(ab)}(y,w) \left( y - w \right)^{\sigma_{ab}} e^{yx(y-w) + \sum_{k=1}^{\infty} \left( t^{(a)}_k y - t^{(b)}_k w \right) k} e^{-\sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\partial}{\partial t^{(a)}_k} y - \frac{\partial}{\partial t^{(b)}_k} w \right) k},
\]
here $C^{(ab)}(y,w)$ are operators that act on the twisted group algebra of the root lattice of $sl_2$. A natural question now is: Are there also matrix pseudo–differential operators such that a formula as (0.2) hold for these $W^{(ab)}(y,w)$? In this paper we show that a similar result holds, not only for the 2–component KP hierarchy, but in general for the $s$–component KP hierarchy. One finds that the natural generalization of $W_{1+\infty}$ is not $(W_{1+\infty})^*$ but $W_{1+\infty}(gl_s)$, the central extension of the algebra of differential operators on $(\mathbb{C}[t, t^{-1}])^*$. Hence one can conclude that the results of [ASV2] for the Toda lattice hierarchy are not complete, but that the structure is richer.

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§1. $a_{\infty}$ and the KP hierarchy in the fermionic picture [KV].

Consider the infinite dimensional complex Lie algebra $a_\infty := gl_\infty \bigoplus \mathbb{C} c$, where
\[
\overline{gl_\infty} = \{ a = (a_{ij})_{i,j\in \mathbb{Z}+\frac{1}{2}} | a_{ij} = 0 \text{ if } |i-j| >> 0 \},
\]
with Lie bracket defined by

\[ [a + \alpha c, b + \beta c] = ab - ba + \mu(a, b)c, \]

for \( a, b \in \mathfrak{gl}_\infty \) and \( \alpha, \beta \in \mathbb{C} \). Here \( \mu \) is the following 2–cocycle:

\[ \mu(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}(\theta(i) - \theta(j)), \]

with \( E_{ij} \) the matrix that has a 1 on the \((i, j)\)-th entry and zeros elsewhere and \( \theta : \mathbb{R} \to \mathbb{C} \) the function defined by

\[ \theta(i) := \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \leq 0. \end{cases} \]

Let \( C^\infty = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}v_j \) be the infinite dimensional complex vector space with fixed basis \( \{v_j\}_{j \in \mathbb{Z} + \frac{1}{2}} \). The Lie algebra \( a_\infty \) acts linearly on \( C^\infty \) via the usual formula:

\[ E_{ij}(v_k) = \delta_{jk}v_i. \]

We introduce, following [KP2], the corresponding semi-infinite wedge space \( F = \Lambda^\frac{1}{2}\infty C^\infty \), this is the vector space with a basis consisting of all semi-infinite monomials of the form \( v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \ldots \), where \( i_1 > i_2 > i_3 > \ldots \) and \( i_{\ell+1} = i_\ell - 1 \) for \( \ell \gg 0 \). In order to describe representations of the Lie algebra on this space, we find it convenient to define wedging and contracting operators \( \psi^-_j \) and \( \psi^+_j \) \((j \in \mathbb{Z} + \frac{1}{2})\) on \( F \) by

\[
\psi^-_j(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j = -i_s \text{ for some } s \\
(-1)^s v_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_s} \wedge v_{-j} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } i_s > -j > i_{s+1} \end{cases}
\]

\[
\psi^+_j(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s \\
(-1)^{s+1} v_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } j = i_s. \end{cases}
\]

These wedging and contracting operators satisfy the following relations \((i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)\):

\[ \psi^-_i \psi^\mu_j + \psi^\mu_j \psi^-_i = \delta_{\lambda, -\mu} \delta_{i, -j}, \]

hence they generate a Clifford algebra, which we denote by \( \mathcal{C} \).

Introduce the following elements of \( F \) \((m \in \mathbb{Z})\):

\[ |m\rangle = v_{m - \frac{1}{2}} \wedge v_{m - \frac{3}{2}} \wedge v_{m - \frac{5}{2}} \wedge \cdots. \]

It is clear that \( F \) is an irreducible \( \mathcal{C} \)-module such that \( \psi^\pm_j |0\rangle = 0 \) for \( j > 0 \). Define a representation \( \hat{r} \) of \( a_\infty \) on \( F \) by

\[ \hat{r}(E_{ij}) = :\psi^-_{-i}\psi^+_j:, \quad \hat{r}(c) = I, \]

where \( :\cdot: \) denotes the usual partial ordering notation.
where \( : \) stands for the normal ordered product defined in the usual way \((\lambda, \mu = + \text{ or } -)\):

\[
: \psi^\lambda_k \psi^\mu_\ell : = \begin{cases} 
\psi^\lambda_k \psi^\mu_\ell & \text{if } \ell \geq k \\
-\psi^\mu_\ell \psi^\lambda_k & \text{if } \ell < k.
\end{cases}
\]

Define the charge decomposition

\[
F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}
\]

by letting

\[
\text{charge}(|0\rangle) = 0 \text{ and charge}(\psi^\pm_j) = \pm 1.
\]

It is easy to see that each \(F^{(m)}\) is an irreducible \(a_\infty\)-highest weight module with highest weight vector \(|m\rangle\).

We are now able to define the KP hierarchy in the fermionic picture, it is the equation

\[
\sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^+ \tau \otimes \psi_{-k}^- \tau = 0,
\]

for \(\tau \in F^{(0)}\). One can prove (see e.g. [KP2] or [KR]) that this equation characterizes the group orbit of the vacuum vector \(|0\rangle\) for the group \(GL_\infty\). Since the group does not play an important role in this paper, we will not introduce it here.

§2. \(W_{1+\infty}(gl_s)\) as subalgebra of \(a_\infty\).

Let \(e_i, 1 \leq i \leq s\) be a basis of \(\mathbb{C}^s\). By identifying \((\mathbb{C}[t, t^{-1}])^s\) with \(\mathbb{C}^\infty\), we can embed \(W_{1+\infty}(gl_s)\) into \(a_\infty\). This, however, can be done in many different ways, the simplest one is the following. We put \((1 \leq a \leq s, j \in \mathbb{Z} + \frac{1}{2})\):

\[
v_j^{(a)} = v_{sj + \frac{1}{2}(s-2a+1)} = t^{-j} - \frac{a}{2} e_a,
\]

\[
\psi_j^{\pm(a)} = \psi_{sj + \frac{1}{2}(s-2a+1)}^{\pm}.
\]

Notice that with this relabeling we have: \(\psi_k^{\pm(a)}|0\rangle = 0\) for \(k > 0\). We introduce the generating series of the fermions, the so-called fermionic fields \((z \in \mathbb{C}^\times)\):

\[
\psi_j^{\pm(a)}(z) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^{\pm(a)} z^{-k - \frac{1}{2}}.
\]

We also rewrite the \(E_{jk}\)’s:

\[
E_{j,k}^{(ab)} = E_{sj + \frac{1}{2}(s-2a+1), sk + \frac{1}{2}(s-2b+1)}.
\]
then \( \hat{r}(E_{jk}^{(ab)}) =: \psi_{-j}^{-(a)} \psi_{k}^{+(b)} \).

We can associate to \((\mathbb{C}[t, t^{-1}])^s\) the Lie algebra of differential operators on this space, it has as basis the operators:

\[
-t^{k+\ell} \frac{\partial}{\partial t} \epsilon_{ij}, \quad \text{for } k \in \mathbb{Z}, \ell \in \mathbb{Z}_+, 1 \leq i, j \leq s.
\]

We will denote this Lie algebra by \( D(gl_s) \). We can embed this algebra via (2.1) into \( gl_\infty \) and also into \( a_\infty \), one finds

\[
(2.4) \quad -t^{k+\ell} \left( \frac{\partial}{\partial t} \epsilon_{ij} \right) \mapsto \sum_{m \in \mathbb{Z}} -m(m-1) \cdots (m-\ell+1) E_{-m-k-\frac{1}{2},-m-\frac{1}{2}}^{(ij)}.
\]

It is straightforward, but rather tedious, to calculate the corresponding 2–cocycle, the result is as follows (see also [KP1], [Ra] and [KRa]). Let \( f(t), g(t) \in \mathbb{C}[t, t^{-1}] \) and \( a, b \in gl_s \) then

\[
\mu(f(t) \left( \frac{\partial}{\partial t} \right)^\ell a, g(t) \left( \frac{\partial}{\partial t} \right)^m b) = \frac{\ell! m!}{(\ell + m + 1)!} \text{Res}_{t=0} dt f^{(m+1)}(t) g^{(\ell)}(t) \text{trace}(ab).
\]

Hence in this way we get a central extension \( W_{1+\infty}(gl_s) = D(gl_s) \oplus \mathbb{C}c \) of \( D(gl_s) \) with Lie bracket

\[
[f(t) \left( \frac{\partial}{\partial t} \right)^\ell a + \alpha c, g(t) \left( \frac{\partial}{\partial t} \right)^m b + \beta c] =
\]

\[
(2.5) \quad f(t) \left( \frac{\partial}{\partial t} \right)^\ell g(t) \left( \frac{\partial}{\partial t} \right)^m a b - g(t) \left( \frac{\partial}{\partial t} \right)^m f(t) \left( \frac{\partial}{\partial t} \right)^\ell b a + \mu(f(t) \left( \frac{\partial}{\partial t} \right)^\ell a, g(t) \left( \frac{\partial}{\partial t} \right)^m b)c.
\]

Since we have the representation \( \hat{r} \) of \( a_\infty \), we find that

\[
\hat{r}(-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell \epsilon_{ab}) = \sum_{m \in \mathbb{Z}} m(m-1) \cdots (m-\ell+1) : \psi_{-m-\frac{1}{2}}^{+(a)} \psi_{m+k+\frac{1}{2}}^{-(b)} :.
\]

In terms of the fermionic fields (2.2), we find

\[
(2.6) \quad \sum_{k \in \mathbb{Z}} \hat{r}(-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell \epsilon_{ab}) z^{-k-\ell-1} = : \frac{\partial^\ell \psi^{+(a)}(z)}{\partial z^\ell} \psi^{-b}(z) :.
\]

Now define

\[
W_k^{(ab, \ell+1)} := \hat{r}(-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell \epsilon_{ab}),
\]

\[
(2.7) \quad W^{(ab, \ell+1)}(z) = \sum_{k \in \mathbb{Z}} W_k^{(ab, \ell+1)} z^{-k-\ell-1},
\]
then
\[
W^{(ab)}(y, z) := \psi^{+(a)}(y)\psi^{-(b)}(z) :
\]
\[
= \sum_{\ell=0}^{\infty} \frac{(y - z)^\ell}{\ell!} W^{(ab, \ell + 1)}(z)
\]
(2.8)
\[
= \sum_{\ell=0}^{\infty} \frac{(y - z)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} W^{(ab, \ell + 1)}_k z^{-k - \ell - 1}.
\]

§3. The \(s\)-component boson-fermion correspondence.

Using a bosonization one can rewrite (1.9) as a system of partial differential equations and express the basis elements of \(W_{1+\infty}(gl_s)\) in terms of vertex operators. We begin by introduce bosonic fields \((1 \leq i \leq s)\):

\[
\alpha^{(i)}(z) \equiv \sum_{k \in \mathbb{Z}} \alpha^{(i)}_k z^{-k - 1} \overset{def}{=} \psi^{-(i)}(z)\psi^{+(i)}(z) :,
\]
(3.1)

Since \(\alpha^{(i)}(z) = W^{(ii,1)}(z)\), one easily checks that the operators \(\alpha^{(i)}_k\) satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by \(\mathfrak{a}\):

\[
[\alpha^{(i)}_k, \alpha^{(j)}_\ell] = k \delta_{ij} \delta_{k, -\ell},
\]
(3.2)

and one has

\[
\alpha^{(i)}_k | m \rangle = 0 \text{ for } k > 0.
\]
(3.3)

It is easy to see that restricted to \(\hat{gl}_s\), which is the subalgebra generated by the elements \(W^{(ij,1)}_k, F^{(0)}\) is its basic highest weight representation (see [K, Chapter 12]).

We will now describe the \(s\)-component boson-fermion correspondence (see [KV]). Let \(L\) be a lattice with a basis \(\delta_1, \ldots, \delta_s\) over \(\mathbb{Z}\) and the symmetric bilinear form \((\delta_i|\delta_j) = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker symbol. Let

\[
\varepsilon_{ij} = \begin{cases} 
-1 & \text{if } i > j \\
1 & \text{if } i \leq j.
\end{cases}
\]
(3.4)

Define a bimultiplicative function \(\varepsilon : L \times L \to \{\pm 1\}\) by letting

\[
\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}.
\]
(3.4)

Let \(\delta = \delta_1 + \ldots + \delta_s, Q = \{\gamma \in L| (\delta|\gamma) = 0\}, \Delta = \{\alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, s, i \neq j\}\). Of course \(Q\) is the root lattice of \(sl_s(\mathbb{C})\), the set \(\Delta\) being the root system.
Consider the vector space $\mathbb{C}[L]$ with basis $e^\gamma$, $\gamma \in L$, and the following twisted group algebra product:

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}. \quad (3.6)$$

Let $\mathbb{C}[t]$ be the space of polynomials in indeterminates $t = \{t^{(i)}_k\}$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots, s$ and denote by $B = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[L]$ be the tensor product of these algebras. Then the $s$-component boson-fermion correspondence is the vector space isomorphism

$$\sigma : F \sim B, \quad (3.7)$$

given by $\sigma(\mid 0\rangle) = 1$ and

$$\sigma \psi^{\pm(a)}(z)\sigma^{-1} = e^{\pm \delta_a z \pm \delta_a} \exp(\pm \sum_{k=1}^{\infty} t^{(a)}_k z^k) \exp(\pm \sum_{k=1}^{\infty} \frac{\partial}{\partial t^{(a)}_k} z^{-k}), \quad (3.8)$$

where

$$\delta_a (p(t) \otimes e^\gamma) = (\delta_a | \gamma) p(t) \otimes e^\gamma. \quad (3.9)$$

The transported charge then is as follows:

$$\text{charge}(p(t) \otimes e^\gamma) = (\delta | \gamma).$$

We denote the transported charge decomposition by $B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}$, then the transported action of the operators $\alpha^{(i)}_m$ is given by

$$\left\{ \begin{array}{ll}
\sigma \alpha^{(j)}_{-m} \sigma^{-1}(p(t) \otimes e^\gamma) = mt^{(j)}_m p(t) \otimes e^\gamma, & \text{if } m > 0, \\
\sigma \alpha^{(j)}_m \sigma^{-1}(p(t) \otimes e^\gamma) = \frac{\partial p(t)}{\partial t^{(a)}_m} \otimes e^\gamma, & \text{if } m > 0, \\
\sigma \alpha^{(j)}_0 \sigma^{-1}(p(t) \otimes e^\gamma) = (\delta_j | \gamma) p(t) \otimes e^\gamma. & 
\end{array} \right. \quad (3.10)$$

If one substitutes (3.8) into (2.8), one obtains the following vertex operator expression for the generating series of the fields $W^{(ab,\ell+1)}(z)$:

$$W^{(ab)}(y, z) = \frac{1}{(y - z)^{\delta_{ab}}}(X^{(ab)}(y, z) - \delta_{ab}), \quad \text{where}$$

$$X^{(ab)}(y, z) : =$$

$$\epsilon(\delta_a, \delta_b) e^{\delta_a - \delta_b y \delta_a z - \delta_b} \exp(\sum_{k=1}^{\infty} (t^{(a)}_k y^k - t^{(b)}_k z^k)) \exp(- \sum_{k=1}^{\infty} \frac{\partial}{\partial t^{(a)}_k} y^{-k} - \frac{\partial}{\partial t^{(b)}_k} z^{-k}).$$

The fields for which $\ell = 0$, give the vertex operator realization of the homogeneous realization of $\hat{gl}_s$, which was first found by Frenkel and Kac [FK] and independently by Segal [Se] (see also [TV] for more details).
Using the isomorphism \( \sigma \) we can reformulate the KP hierarchy (1.9) in the bosonic picture. We start by observing that (1.9) can be rewritten as follows:

\[
\text{Res}_{z=0} \frac{dz}{z} \sum_{j=1}^{s} \psi^{+(j)}(z) \tau \otimes \psi^{-(j)}(z) = 0, \quad \tau \in F^{(0)}.
\]

Notice that for \( \tau \in F^{(0)} \), \( \sigma(\tau) = \sum_{\gamma \in Q} \tau_{\gamma}(t) e^{\gamma} \). Here and further we write \( \tau_{\gamma}(t) e^{\gamma} \) for \( \tau_{\gamma}(t) \otimes e^{\gamma} \). Using (3.8), equation (3.12) turns under \( \sigma \otimes \sigma : F \otimes F \rightarrow \mathbb{C}[t, t'] \otimes (\mathbb{C}[L] \otimes \mathbb{C}[L']) \) into the following set of equations: for all \( \alpha, \beta \in L \) such that \( (\alpha \mid \delta) = -(\beta \mid \delta) = 1 \) we have:

\[
\text{Res}_{z=0} \left( \sum_{j=1}^{s} \varepsilon(\delta_j, \alpha - \beta) z^{(\delta_j \mid \alpha - \beta - 2\delta_j)} \right) \times \exp\left( \sum_{k=1}^{\infty} (t_{k}^{(j)} - s_{k}^{(j)}) z^k \right) \exp\left( -\sum_{k=1}^{\infty} \left( \frac{\partial}{\partial t_{k}^{(j)}} - \frac{\partial}{\partial s_{k}^{(j)}} \right) z^{-k} \right) \\
\tau_{\alpha - \delta_j}(t)(e^{\alpha}) \tau_{\beta + \delta_j}(t')(e^{\beta})' = 0.
\]

Notice that if \( s = 2 \), the set of equations (3.13) are equivalent (for more general \( \tau \)) to the Toda lattice hierarchy of Ueno and Takasaki [UT]. For this reason, we assume from now on that \( \tau = \sum \tau_{\alpha} e^{\alpha} \) is any solution of (3.13). Hence the results of this paper also hold for an (alternative) definition of the Toda lattice hierarchy.

In order to define the equations (3.13) in terms of formal pseudo-differential operators it will be convenient to replace \( t_{k}^{(j)} \) by \( t_{k}^{(j)} + \delta_{k,1} x \) and to introduce the notations

\[
\xi^{(j)}(t, z) = \sum_{i=1}^{\infty} t_{i}^{(j)} z^i, \quad \xi^{(j)}(x, t, z) = z x + \xi^{(j)}(t, z)
\]

and

\[
\eta^{(j)}(t, z) = \sum_{i=1}^{\infty} \frac{\partial}{\partial t_{i}^{(j)}} z^{-i}.
\]

Next we replace \( \alpha \) resp. \( \beta \) by \( \alpha + \delta_i \) and \( \beta - \delta_k \) then for all \( \alpha, \beta \in Q \) and \( 1 \leq i, k \leq s \) (3.13) turns into

\[
\text{Res}_{z=0} \left( \sum_{j=1}^{s} \varepsilon(\delta_j, \alpha + \delta_i - \beta + \delta_k) z^{(\delta_j \mid \alpha - \beta + \delta_i + \delta_k - 2\delta_j)} e^{\xi^{(j)}(x, t, z)} - \xi^{(j)}(x', t', z) \right) \\
\times e^{-\eta^{(j)}(t, z) + \eta^{(j)}(t', z)} \tau_{\alpha + \delta_i - \delta_j}(x, t)(e^{\alpha + \delta_j}) \tau_{\beta + \delta_j - \delta_k}(x', t')(e^{\beta - \delta_k})' = 0.
\]

§4. The algebra of formal pseudo-differential operators and the \( s \)-component KP hierarchy as a dynamical system.
We proceed now to rewrite the formulation (3.14) of the s-component KP hierarchy in terms of formal pseudo-differential operators, generalizing the results of [DJKM1-3]. For more details see [KV]. For each $\alpha \in \operatorname{supp} \tau := \{ \alpha \in Q | \tau = \sum_{\alpha \in Q} \tau_{\alpha} e^{\alpha}, \tau_{\alpha} \neq 0 \}$ we define the (matrix valued) functions
\[ \Psi^\pm(\alpha, z) \equiv \Psi^\pm(\tau_{\alpha}, x, t, z) = (\Psi^\pm_{ij}(\tau_{\alpha}, x, t, z))_{i,j=1}^s \]
as follows:
\[ \Psi^\pm_{ij}(\tau_{\alpha}, x, t, z) \overset{\text{def}}{=} \mathcal{E}(\delta_j, \alpha + \delta_i) z^{(\xi^{(j)})(x, t, z)} e^{\mp \xi^{(j)}(x, t, z)} \tau_{\alpha \mp (\delta_i - \delta_j)}(x, t)/\tau_{\alpha}(x, t) \\
= \mathcal{E}(\delta_j, \alpha + \delta_i) z^{(\xi^{(j)})(x, t, z)} \tau_{\alpha \mp (\delta_i - \delta_j)}(x, t(k) - \delta_{jk}[z^{-1}])/\tau_{\alpha}(x, t) e^{\pm \xi^{(j)}(x, t, z)}, \]
where $[w] = (w, w^2, w^3, \ldots)$ It is easy to see that equation (3.14) is equivalent to the following bilinear identity:
\[ \text{Res}_{z=0} \Psi^+(\tau_{\alpha}, x, t, z)^t \Psi^-(\tau_{\beta}, x', t', z) dz = 0 \text{ for all } \alpha, \beta \in Q. \]
Define $s \times s$ matrices $S^{\pm(m)}(\tau_{\alpha}, x, t)$ by the following generating series (cf. (4.2)):
\[ \sum_{m=0}^{\infty} S^{\pm(m)}_{ij}(\tau_{\alpha}, x, t)(\pm z)^{-m} = \mathcal{E}(\delta_j, \alpha) e^{\mp \xi^{(j)}(x, t, z)} \tau_{\alpha \mp (\delta_i - \delta_j)}(x, t)/\tau_{\alpha}(x, t) \\
= \mathcal{E}(\delta_j, \alpha) e^{\mp \xi^{(j)}(x, t, z)} \tau_{\alpha \mp (\delta_i - \delta_j)}(x, t(k) - \delta_{jk}[z^{-1}])/\tau_{\alpha}(x, t). \]
We see from (4.2) that $\Psi^\pm(\tau_{\alpha}, x, t, z)$ can be written in the following form:
\[ \Psi^\pm(\tau_{\alpha}, x, t, z) = (\sum_{m=0}^{\infty} S^{\pm(m)}(\tau_{\alpha}, x, t) R^\pm(\alpha, \pm z)(\pm z)^{-m}) e^{\pm \xi(x, t, z)}, \]
where
\[ R^\pm(\alpha, z) = \sum_{i=1}^{s} \mathcal{E}(\delta_i, \alpha) e^{\pm \xi^{(i)}(x, t, z)^{\pm(\delta_i | \alpha)}}, \]
as before $e_{ij}$ stands for the $s \times s$ matrix whose $(i, j)$ entry is 1 and all other entries are zero. Let
\[ \partial = \frac{\partial}{\partial x}, \]
we can now rewrite $\Psi^\pm(\tau_{\alpha}, x, t, z)$ in terms of formal pseudo-differential operators, define
\[ e^{\pm \xi(t, \pm \partial)} = \sum_{j=1}^{s} e^{\pm \xi^{(j)}(t, \pm \partial)} e_{jj} \]
\[ P^\pm(\alpha) \equiv P^\pm(\tau_{\alpha}, x, t, \partial) = I_s + \sum_{m=1}^{\infty} S^{\pm(m)}(\tau_{\alpha}, x, t) \partial^{-m}, \]
\[ R^\pm(\alpha) \equiv R^\pm(\alpha, \partial) \text{ and } W^\pm(\alpha) \equiv W^\pm(\tau_{\alpha}, x, t, z) = P^\pm(\alpha) R^\pm(\alpha) e^{\pm \xi(t, \pm \partial)} \]
then:
\begin{equation}
(4.8) \quad \Psi^\pm(\tau_\alpha, x, t, z) = W^\pm(\alpha)e^{\pm xx} = P^\pm(\alpha)R^\pm(\alpha)e^{\pm(t, \partial) e^{\pm xx}}.
\end{equation}

As usual one denotes the differential part of \(P(x, t, \partial)\) by \(P_+(x, t, \partial) = \sum_{j \geq 0} P_j(x, t)\partial^j\), and writes \(P_− = P − P_+\). The linear anti-involution \(*\) is defined by the following formula:
\begin{equation}
(4.9) \quad (\sum_j P_j \partial^j)^* = \sum_j (-\partial)^j \circ^t P_j.
\end{equation}

Here and further \(^tP\) stands for the transpose of the matrix \(P\). Then one has the following fundamental lemma:

**Lemma 4.1.** Let \(P, Q\) be two formal pseudo–differential operators, then
\begin{equation}
(PQ^\ast)_− = \sum_{i < 0} R_i \partial^i
\end{equation}
if and only if
\begin{equation}
\text{Res}_{z=0} dz(P(x, \partial)e^{zx}) \, \langle(Q(x', \partial')e^{−zx'}) = \sum_{i < 0} R_i(x) \frac{(x − x')^{−i−1}}{(-i − 1)!}.
\end{equation}

**Proof.** Let \(y = x − x', P(x, z) = \sum P_i(x)z^i\) and \(Q(x, z) = \sum Q_i(x)(−z)^i\), then
\begin{equation}
\text{Res}_{z=0} dz\left(P(x, \partial)e^{zx}\right) \, \langle\left(Q(x', \partial')e^{−zx'}\right) = \\
\text{Res}_{z=0} dz P(x, z) \left(\sum_{k \geq 0} \frac{(-1)^k \partial^k tQ(x, −z)}{k!} y^k e_{zy}\right) = \\
\text{Res}_{z=0} dz \left(\sum_{k, \ell \geq 0, i, j} \frac{(-1)^k}{k!} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} y^{k+\ell} \frac{\ell!}{\ell!} z^{i+j+\ell}\right) = \\
\sum_{k \geq 0, i+j\leq -1} \frac{(-1)^k}{k!(−i − j − 1)!} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} y^{k−i−j−1} = \\
\sum_{k \geq 0, i+j\leq -1} \binom{i+j}{k} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} y^{k−i−j−1} \frac{1}{(k−i−j−1)!}.
\end{equation}

Next we calculate
\begin{equation}
(P(x, \partial)Q^\ast(x, \partial))_{−} = (\sum_{i,j} P_i(x) \partial^{i+j} tQ_j(x))_{−}
\end{equation}
\begin{equation}
= (\sum_{k \geq 0, i, j} \binom{i+j}{k} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} \partial^{i+j−k})_{−}
\end{equation}
\begin{equation}
= (\sum_{k \geq 0, i+j−k < 0} \binom{i+j}{k} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} \partial^{i+j−k})_{−}
\end{equation}
\begin{equation}
= \sum_{k \geq 0, i+j\leq -1} \binom{i+j}{k} P_i(x) \frac{\partial^k tQ_j(x)}{\partial x^k} \partial^{i+j−k},
\end{equation}
here we have used the fact that \((i+j)/k = 0\) if \(i + j > 0\) and \(i + j - k < 0\). Now comparing (4.10) and (4.11) gives the desired result. \(\Box\)

Using this Lemma one deduces the following
\[
(W^+ (\tau_\alpha, x, t, \partial) W^- (\tau_\beta, x, t', \partial))_- = 0.
\]

By putting \(t = t'\), one proves in a similar way as in [KV] that given \(\beta \in \text{supp} \tau\), all the pseudo-differential operators \(P^\pm (\alpha), \alpha \in \text{supp} \tau\), are completely determined by \(P^+ (\beta)\) from the following equations
\[
R^-(\alpha)^{-1} = R^+(\alpha)^*, \tag{4.13}
\]
\[
P^- (\alpha) = (P^+(\alpha)^*)^{-1}, \tag{4.14}
\]
\[
(P^+(\alpha) R^+ (\alpha - \beta) P^+(\beta)^{-1})_- = 0 \text{ for all } \alpha, \beta \in \text{supp} \tau. \tag{4.15}
\]

This and the above lemma can be used to prove the following proposition which will be crucial later on. Adler, Shiota and van Moerbeke stated this proposition in the 1-component case [ASV2].

**Proposition 4.2.** Let \(\Psi^\pm (\alpha, x, t, z)\) satisfy the bilinear identity (4.3) for \(\beta = \alpha\) and let \(Q(x, t, \partial)\) be an arbitrary pseudo-differential operator. Then \(Q\) is a differential operator if and only if
\[
\text{Res}_{z=0} dz Q(x, t, \partial) \Psi^+(\tau_\alpha, x, t, z) \Psi^-(\tau_\alpha, x', t', z) = 0 \tag{4.16}
\]

**Proof.** Suppose that \(Q\) is a differential operator, then since by lemma 4.1 \(W^+ (\tau_\alpha, x, t, \partial) W^- (\tau_\alpha, x, t', \partial)\) is a differential operator
\[
(Q(x, t, \partial) W^+ (\tau_\alpha, x, t, \partial) W^- (\tau_\alpha, x, t', \partial))_- = 0. \tag{4.17}
\]

Conversely, suppose (4.16) holds, then again by Lemma , (4.17) holds. Now put \(t = t'\), and use (4.13-14), then one deduces that \(Q(x, t, \partial)_- = 0. \) \(\Box\)

In [KV] Victor Kac and the author also showed the following

**Proposition 4.3.** Consider \(\Psi^+ (\tau_\alpha, x, t, z)\) and \(\Psi^- (\tau_\alpha, x, t, z)\), \(\alpha \in Q\), of the form (4.8), then the bilinear identity (4.3) for all \(\alpha, \beta \in \text{supp} \tau\) is equivalent to the Sato equation:
\[
\frac{\partial P^+(\alpha)}{\partial t^{(j)}} = -(P^+(\alpha) e_{jj} \partial^k P^+(\alpha)^{-1})_- P^+(\alpha), \tag{4.18}
\]

for each \(\alpha \in \text{supp} \tau\) and the matching conditions (4.13-15) for all \(\alpha, \beta \in \text{supp} \tau\).

As a consequence of (4.18) one obtains for each \(W^+ (\alpha)\):
\[
\frac{\partial W^+ (\alpha)}{\partial t^{(j)}} = (P^+(\alpha) e_{jj} \partial^k P^+(\alpha)^{-1})_+ W^+ (\alpha) = (W^+ (\alpha) e_{jj} \partial^k W^+(\alpha)^{-1})_+ W^+ (\alpha)
\]
Remark 4.4. (i) It is our purpose to describe the general operators

\[ L(\alpha) = W^+(\alpha)\partial W^+(\alpha)^{-1} = P^+(\alpha)\partial P^+(\alpha)^{-1}, \]

\[ \Gamma = x + \sum_{a=0}^{s} \sum_{k=0}^{\infty} kt_k^{(a)} \partial^{k-1} e_{aa} \]

(4.20)

\[ M(\alpha) = W^+(\alpha)xW^+(\alpha)^{-1} = P^+(\alpha)\Gamma P^+(\alpha)^{-1}, \]

\[ N(\alpha, \beta) = W^+(\alpha)W^+(\beta)^{-1} = P^+(\alpha)R^+(\alpha - \beta)P^+(\beta)^{-1}, \]

\[ \Delta^{(ij)}(\alpha) = e^{\xi(t, \partial)} e_{ij} e^{-\xi(t, \partial)} \]

\[ C^{(ij)}(\alpha) = W^+(\alpha)e_{ij}W^+(\alpha)^{-1} = P^+(\alpha)\Delta^{(ij)} P^+(\alpha)^{-1}, \]

\[ B_m^{(ij)}(\alpha) = (W^+(\alpha)e_{ij}\partial^m W^+(\alpha)^{-1})_+ = (P^+(\alpha)e_{ij}\partial^m P^+(\alpha)^{-1})_+. \]

Here we write \( x \) for \( xI_s \). Denote by \( C^{(i)}(\alpha) = C^{(ii)}(\alpha) \) and \( B_m^{(i)}(\alpha) = B_m^{(ii)}(\alpha) \). Then

\[ [L(\alpha), M(\alpha)] = I_s, \quad \sum_{i=1}^{s} C^{(i)}(\alpha) = I_s, \quad C^{(i)}(\alpha)L(\alpha) = L(\alpha)C^{(i)}(\alpha), \]

(4.21)

\[ C^{(ij)}(\alpha)C^{(k\ell)}(\alpha) = \delta_{jk} C^{(i\ell)}(\alpha), \quad L(\alpha)N(\alpha, \beta) = N(\alpha, \beta)L(\beta), \]

\[ M(\alpha)N(\alpha, \beta) = N(\alpha, \beta)M(\beta), \quad C^{(ij)}(\alpha)N(\alpha, \beta) = N(\alpha, \beta)C^{(ij)}(\beta), \]

\[ N(\alpha, \beta)N(\beta, \gamma) = N(\alpha, \gamma). \]

Remark 4.4. (i) It is our purpose to describe the general operators

\[ Y_k^{(ab, \ell+1)}(\alpha, \beta) = W^+(\alpha)x^\ell \partial^{k+\ell} e_{ab}W^+(\beta)^{-1}. \]

One can express them in the operators defined in (4.20), viz.,

\[ Y_k^{(ab, \ell+1)}(\alpha, \beta) = M(\alpha)^\ell L(\alpha)^{k+\ell} C^{(ab)}(\alpha)N(\alpha, \beta). \]

(ii) Notice that (4.14) is equivalent to \( N(\alpha, \beta)_- = 0 \), so from now on we will assume that \( N(\alpha, \beta) \) is a differential operator.

Proposition 4.5. If for every \( \alpha, \beta \in Q \) the formal pseudo-differential operators \( L(\alpha), M(\alpha), C^{(ij)}(\alpha) \) and the differential operators \( N(\alpha, \beta) \) satisfy conditions (4.21) and if the equations

(4.22)

\[
\begin{align*}
L(\alpha)P^+(\alpha)R^+(\alpha) &= P^+(\alpha)R^+(\alpha)\partial \quad \text{(or equivalently } L(\alpha)P^+(\alpha) = P^+(\alpha)\partial) \\
M(\alpha)P^+(\alpha)R^+(\alpha) &= P^+(\alpha)R^+(\alpha)\Gamma \\
N(\alpha, \beta)P^+(\beta)R^+(\beta) &= P^+(\alpha)R^+(\alpha) \\
C^{(ij)}(\alpha)P^+(\alpha)R^+(\alpha) &= P^+(\alpha)R^+(\alpha)\Delta^{(ij)} \\
\frac{\partial P^+(\alpha)}{\partial t_k^{(ij)}} &= -(C^{(ij)}(\alpha)L(\alpha)^k)_-P^+(\alpha)
\end{align*}
\]
have a solution $P^+(\alpha)$ of the form (4.7), then the differential operators $B_k^{(j)}(\alpha)$ satisfies the following conditions:

$$\begin{align*}
\frac{\partial L(\alpha)}{\partial t_k^{(j)}} &= [B_k^{(j)}(\alpha), L(\alpha)], \\
\frac{\partial M(\alpha)}{\partial t_k^{(j)}} &= [B_k^{(j)}(\alpha), M(\alpha)], \\
\frac{\partial N(\alpha, \beta)}{\partial t_k^{(j)}} &= B_k^{(j)}(\alpha)N(\alpha, \beta) - N(\alpha, \beta)B_k^{(j)}(\beta), \\
\frac{\partial C^{(i)}(\alpha)}{\partial t_k^{(j)}} &= [B_k^{(j)}(\alpha), C^{(i)}(\alpha)],
\end{align*}$$

(4.23)

Now (4.23) implies that

$$\frac{\partial Y^{(ab,\ell+1)}_m(\alpha, \beta)}{\partial t_k^{(j)}} = B_k^{(j)}(\alpha)Y^{(ab,\ell+1)}_m(\alpha, \beta) - Y^{(ab,\ell+1)}_m(\alpha, \beta)B_k^{(j)}(\beta)$$

Notice that the conditions (4.22) are equivalent to one of the following conditions:

$$\begin{align*}
L(\alpha)W^+(\alpha) &= W^+(\alpha)\partial, \\
M(\alpha)W^+(\alpha) &= W^+(\alpha)x, \\
N(\alpha, \beta)W^+(\beta) &= W^+(\alpha), \\
C^{(ij)}(\alpha)W^+(\alpha) &= W^+(\alpha)e_{ij}, \\
\frac{\partial W^+(\alpha)}{\partial t_k^{(j)}} &= B_k^{(j)}(\alpha)W^+(\alpha), \\
\frac{\partial \Psi^+(\alpha, z)}{\partial t_k^{(j)}} &= B_k^{(j)}(\alpha)\Psi^+(\alpha, z).
\end{align*}$$

(4.24)

§5 The Adler–Shiota–van Moerbeke formula.

Fix $\alpha, \beta \in Q$ and recall

$$(5.1) \quad Y^{(ab,\ell+1)}_k(\alpha, \beta) \equiv Y^{(ab,\ell+1)}_k(\tau_\alpha, \tau_\beta) = W^+(\alpha)x^\ell \partial^k \epsilon_{ab}W^+(\beta)^{-1},$$

then define

$$\begin{align*}
Y^{(ab)}(\alpha, \beta, y, w) &\equiv Y^{(ab)}(\tau_\alpha, \tau_\beta, y, w) = \sum_{\ell=0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1} Y^{(ab,\ell+1)}_k(\alpha, \beta) \\
&= \sum_{\ell=0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1} M(\alpha)^\ell L(\alpha)^{k+\ell} C^{(ab)}(\alpha)N(\alpha, \beta).
\end{align*}$$

(5.2)

We write $Y^{(ab,\ell+1)}_k(\alpha)$ and $Y^{(ab)}(\alpha, y, w) \equiv Y^{(ab)}(\tau_\alpha, y, w)$ for respectively $Y^{(ab,\ell+1)}_k(\alpha, \alpha)$, $Y^{(ab)}(\alpha, \alpha, y, w)$, then

$$\begin{align*}
Y^{(ab)}(\alpha, y, w) &= \sum_{\ell=0}^{\infty} \frac{(y - w)^\ell}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1} M(\alpha)^\ell L(\alpha)^{k+\ell} C^{(ab)}(\alpha).
\end{align*}$$

One deduces the following
Proposition 5.1.

\begin{equation}
(5.3) \quad Y^{(ab)}(\alpha, \beta, y, w)_- = \Psi^+(\alpha, y) e_{ab} \partial^{-1} t \Psi^-(\beta, w)
\end{equation}

Proof. First, notice that

\begin{align*}
(W^+(\alpha)x^{\ell} \partial^{k+\ell} e_{ab} W^+(\beta)^{-1})_- &= \sum_{j=1}^{\infty} \partial^{-j} \text{Res}_0 \partial^{j-1} W^+(\alpha)x^{\ell} \partial^{k+\ell} e_{ab} W^+(\beta)^{-1} \\
&= \sum_{j=1}^{\infty} \partial^{-j} \text{Res}_{z=0} dz (\partial^{j-1} W^+(\alpha)x^{\ell} \partial^{k+\ell} e_{ab}) t (W^- (\beta)e^{-zx}) \\
&= \text{Res}_{z=0} dz \sum_{j=1}^{\infty} \partial^{-j} [(\frac{\partial}{\partial x})^{j-1} (z^{k+\ell} \frac{\partial^{j} \Psi^+(\alpha, z)}{\partial z^{\ell}})] e_{ab} t \Psi^-(\beta, z) \\
&= \text{Res}_{z=0} dz z^{k+\ell} \frac{\partial^{j} \Psi^+(\alpha, z)}{\partial z^{\ell}} e_{ab} \partial^{-1} t \Psi^-(\beta, z)
\end{align*}

Hence

\begin{align*}
Y^{(ab)}(\alpha, \beta, y, w)_- &= \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1} \text{Res}_{z=0} dz z^{k+\ell} \frac{\partial^{j} \Psi^+(\alpha, z)}{\partial z^{\ell}} e_{ab} \partial^{-1} t \Psi^-(\beta, z) \\
&= \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \frac{\partial^{\ell} \Psi^+(\alpha, w)}{\partial w^\ell} e_{ab} \partial^{-1} t \Psi^-(\beta, w) \\
&= \Psi^+(\alpha, y) e_{ab} \partial^{-1} t \Psi^-(\beta, w) \quad \square
\end{align*}

Proposition 5.1 was obtained in the 1–component case by Dickey [D].

Next we calculate

\begin{align*}
Y^{(ab)}(\alpha, \beta, y, w) \Psi^+(\beta, z) &= \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \sum_{k \in \mathbb{Z}} w^{-k-\ell-1} \text{Res}_{z=0} dz z^{k+\ell} e_{ab} e^{zx} \\
&= W^+(\alpha) \delta(w, z) e^{(y-w)x} e_{ab} \\
&= W^+(\alpha) \delta(w, z) e^{(z+y-w)x} e_{ab} \\
&= W^+(\alpha) \delta(w, z) e^{yx} e_{ab} \\
&= \delta(w, z) \Psi^+(\alpha, y) e_{ab},
\end{align*}

where \(\delta(w, z) = \sum_{n \in \mathbb{Z}} w^{-n} z^{n-1}\).

Define

\begin{align*}
X^{(ab)}(y, w) &= X^{(ab)}(y, w)e^{(y-w)x}, \\
W^{(ab)}(y, w) &= \sum_{\ell=0}^{\infty} \frac{(y - w)^{\ell}}{\ell!} \sum_{k \in \mathbb{Z}} W^{(ab, \ell+1)}(\alpha) w^{-k-\ell-1} \\
&= (y - w)^{-\delta_{ab}} (X^{(ab)}(y, w) - \delta_{ab}),
\end{align*}
then
\[ \mathbb{W}^{(ab)}(y, w) =: \psi^{+(a)}(y)\psi^{-(b)}(w) : e^{(y-w)x}. \]

It is straightforward that the \( \mathbb{W}^{(ab, \ell+1)}_k \) have the same commutation relations as the \( \mathbb{W}^{(ab, \ell+1)}_k \), since we have only replaced all \( t_1^{(j)} \) by \( t_1^{(j)} + x \) and kept \( \frac{\partial}{\partial t_1^{(j)}} \) unchanged in the vertex operator (3.11) of \( \mathbb{W}^{(ab)}(y, w) \) (\( \frac{\partial}{\partial x} \) does not appear in this expression).

One has the following

**Lemma 5.2.**
\[ \mathbb{X}^{(ab)}(y, w)\psi^{+(j)}(z)e^{xz} = \delta_{bj}(y-w)\delta_{ab}\delta(w, z)\psi^{+(a)}(y)e^{yx} + \psi^{+(j)}(z)e^{xz}\mathbb{X}^{(ab)}(y, w). \]

**Proof.** Let \( \gamma \in Q \), we calculate
\[ \mathbb{X}^{(ab)}(y, w)\psi^{+(j)}(z)e^{xz}\tau_\gamma(x, t)e^\gamma = \]
\[ = \delta_{bj}(y-w)\delta_{ab}\delta(w, z)y(\delta_{a} | \gamma + \delta_{j})(y-w)(\delta_{b} | \gamma)\tau_\gamma e^{\gamma + \delta_{a}} \]
\[ = \delta_{bj}(y-w)\delta_{ab}\delta(w, z)y(\delta_{a} | \gamma + \delta_{j})(y-w)(\delta_{b} | \gamma)\tau_\gamma e^{\gamma + \delta_{a}} \]
\[ + \delta_{bj}(y-w)\delta_{ab}\delta(w, z)y(\delta_{a} | \gamma + \delta_{j})(y-w)(\delta_{b} | \gamma)\tau_\gamma e^{\gamma + \delta_{a}} \]
\[ = \{ \delta_{bj}(y-w)\delta_{ab}\delta(w, z)\psi^{+(a)}(y)e^{yx} + \psi^{+(j)}(z)e^{xz}\mathbb{X}^{(ab)}(y, w) \}\tau_\gamma e^{\gamma}. \]

Recall the bilinear identity (3.14) with \( \alpha \) replaced by \( \alpha + \delta_b - \delta_a \) in a slightly different version, viz.,
\[ \text{Res}_{z=0}dz \sum_{j=1}^s \psi^{+(j)}(z)e^{xz}\tau_{\alpha + \delta_i + \delta_b - \delta_j - \delta_a}(x, t)e^{\alpha + \delta_i + \delta_b - \delta_j - \delta_a} \]
\[ = \psi^{-(j)'}(z)e^{-xz'}\tau_{\beta + \delta_j - \delta_k}(x', t')(e^{\beta + \delta_j - \delta_k})' = 0. \]

Let \( \mathbb{X}^{(ab)}(y, w) \) act on this identity, then using Lemma 5.2 one obtains:
\[ \text{Res}_{z=0}dz\{ (y-w)\delta_{ab}\delta(w, z)\psi^{+(a)}(y)e^{xz}\tau_{\alpha + \delta_i - \delta_a}(x, t)e^{\alpha + \delta_i - \delta_a} \psi^{-(a)'}(z) \]
\[ + \sum_{j=1}^s \psi^{+(j)}(z)e^{xz}\mathbb{X}^{(ab)}(y, w)\tau_{\alpha + \delta_i + \delta_b - \delta_j - \delta_a}(x, t)e^{\alpha + \delta_i + \delta_b - \delta_j - \delta_a} \psi^{-(j)'}(z) \]
\[ \times e^{-xz'}\tau_{\beta + \delta_j - \delta_k}(x', t')(e^{\beta + \delta_j - \delta_k})' = 0. \]
Now divide by $\tau_\alpha(x,t)\tau_\beta(x',t')$ and remove the factors $e^{\alpha+\delta_i}$ and $(e^{\beta-\delta_k})'$. Notice that by doing this, the action of $\mathcal{X}^{(ab)}(y, w)$ is no longer well–defined, for that reason we introduce $\hat{X}^{(ab)}(y, w)$ by

$$
\hat{X}^{(ab)}(y, w)\tau_\gamma(x, t) = \epsilon(\delta_a, \delta_b)\epsilon(\delta_a - \delta_b, \gamma) y^{(\delta_a|\gamma)} w^{-(\delta_a|\gamma)} e^{\xi_\gamma(x.t,y) - \xi_\gamma(x.t,w)} e^{-\eta_\gamma(t.y) + \eta_\gamma(t.w)} \tau_\gamma(x, t).
$$

and

$$
\hat{W}^{(ab)}(y, w) = \sum_{\ell=0}^{\infty} \frac{(y-w)^\ell}{\ell!} \sum_{k}\hat{W}^{(ab,\ell+1)}_{k} w^{-k-\ell-1}
$$

\begin{equation}
(5.7)
\end{equation}

Then (5.6) turns into

$$
\text{Res}_{z=0}dz\{(y-w)^{\delta_{ab}}\delta(w,z)\Psi^+_i(\alpha, y)\Psi^-_{kb}(\beta, z)'
+ \sum_{j=1}^{s} e^{-\eta^{(j)}(t,z)} \left( \frac{\hat{X}^{(ab)}(y, w)\tau_{\alpha+\delta_i+\delta_b-\delta_j-\delta_a}(x, t)}{\tau_{\alpha+\delta_i-\delta_j}(x, t)} \right) \Psi^+_j(\alpha, z)\Psi^-_{kj}(\beta, z) \} = 0.
$$

Using (5.4) one obtains

$$
\text{Res}_{z=0}dz\{e_{ii}(y-w)^{\delta_{ab}} Y^{(ab)}(\alpha, y, w)\Psi^+(\alpha, z)
+ \sum_{j=1}^{s} e^{-\eta^{(j)}(t,z)} \left( \frac{\hat{X}^{(ab)}(y, w)\tau_{\alpha+\delta_i+\delta_b-\delta_j-\delta_a}(x, t)}{\tau_{\alpha+\delta_i-\delta_j}(x, t)} \right) \Psi^+(\alpha, z)e_{jj} \} = 0.
$$

Now notice that

$$
e^{-\eta^{(j)}(t,z)} \left( \frac{\hat{X}^{(ab)}(y, w)\tau_{\alpha+\delta_i+\delta_b-\delta_j-\delta_a}(x, t)}{\tau_{\alpha+\delta_i-\delta_j}(x, t)} \right) \Psi^+(\alpha, z)e_{jj} = \sum_{k=0}^{\infty} c_{jk} L(\alpha)^{-k} C^{(j)}(\alpha) \Psi^+(\alpha, z)
$$

\begin{align*}
&= \left\{ \left( \sum_{k=1}^{\infty} c_{jk} L(\alpha)^{-k} C^{(j)}(\alpha) \right) + \frac{\hat{X}^{(ab)}(y, w)\tau_{\alpha+\delta_i+\delta_b-\delta_j-\delta_a}(x, t)}{\tau_{\alpha+\delta_i-\delta_j}(x, t)} e_{jj} \right\} \Psi^+(\alpha, z),
\end{align*}

hence

$$
\text{Res}_{z=0}dz e_{ii} \left( y-w)^{\delta_{ab}} Y^{(ab)}(\alpha, y, w) + \sum_{j=1}^{s} \sum_{k=1}^{\infty} c_{jk} L(\alpha)^{-k} C^{(j)}(\alpha) \right) \Psi^+(\alpha, z)t\Psi^-(\beta, z) = 0.
$$

Now using Proposition 4.2 for $\beta = \alpha$, one obtains

$$
e_{ii}(y-w)^{\delta_{ab}} Y^{(ab)}(\alpha, y, w) + \sum_{j=1}^{s} \left( \sum_{k=0}^{\infty} c_{jk} L(\alpha)^{-k} C^{(j)}(\alpha) \right) = 0.
$$
Hence

\[-e_{ii}(y - w)^\delta_{ab} Y^{(ab)}(\alpha, y, w) \Psi^+(\alpha, z) =
\]
\[
\sum_{j=1}^s e_{ii} e^{-\eta^{(j)}(t, z)} \left( \frac{\hat{X}^{(ab)}(y, w) \tau_{\alpha+\delta_i} - \delta_j (x, t)}{\tau_{\alpha+\delta_i} - \delta_j (x, t)} \right) \Psi^+(\alpha, z) e_{jj} \]

- \frac{\hat{X}^{(ab)}(y, w) \tau_{\alpha+\delta_i} - \delta_a (x, t)}{\tau_{\alpha} (x, t)} e_{jj} \Psi^+(\alpha, z).

So we obtain the following generalization of the Adler–Shio-ta–van Moerbeke formula

**Theorem 5.3.**

\[(y - w)^\delta_{ab} (-Y^{(ab)}(\alpha, y, w) - \Psi^+(\alpha, z))_{ij} =
\]
\[
\{e^{-\eta^{(j)}(t, z)} \left( \frac{\hat{X}^{(ab)}(y, w) \tau_{\alpha+\delta_i} - \delta_j (x, t)}{\tau_{\alpha+\delta_i} - \delta_j (x, t)} \right) - \frac{\hat{X}^{(ab)}(y, w) \tau_{\alpha+\delta_b} - \delta_a (x, t)}{\tau_{\alpha} (x, t)} \} \Psi^+_{ij}(\alpha, z).
\]

In a similar way as in the introduction the operator \(e^{\lambda X^{(ab)}(y, w)} = 1 + \lambda X^{(ab)}(y, w)\) is an auto-\(\text{Bäcklund}\) transformation of the \(s\)-component KP hierarchy (see also [KV]). Now let

\[\sigma(x, t) = \sum_{\gamma \in Q} \sigma_\gamma (x, t) e^{\gamma} = e^{\lambda X^{(ab)}(y, w)} \sum_{\gamma \in Q} \tau_\gamma (x, t) e^{\gamma} = \sum_{\gamma \in Q} (\tau_\gamma (x, t) + \lambda \hat{X}^{(ab)}(y, w) \tau_{\gamma+\delta_b} - \delta_a) e^{\gamma},\]

then (5.8) is equal to

\[-\lambda (y - w)^\delta_{ab} (Y^{(ab)}(\tau_\alpha, y, w) - \Psi^+(\tau_\alpha, x, t, z))_{ij} =
\]
\[
\{e^{-\eta^{(j)}(t, z)} \left( \frac{\sigma_{\alpha+\delta_i} - \delta_j (x, t)}{\tau_{\alpha+\delta_i} - \delta_j (x, t)} \right) - \frac{\sigma_\alpha (x, t)}{\tau_\alpha (x, t)} \} \Psi^+_{ij}(\tau_\alpha, x, t, z).
\]

So

\[\Psi^+_{ij}(\sigma_\alpha, x, t, z) = \frac{e^{-\eta^{(j)}(t, z)} \sigma_{\alpha+\delta_i} - \delta_j (x, t)}{\sigma_\alpha (x, t)} e^{\xi^{(j)}(x, t, z)} \]

\[= \frac{\tau_\alpha (x, t)}{\sigma_\alpha (x, t)} e^{-\eta^{(j)}(t, z)} \left( \frac{\sigma_{\alpha+\delta_i} - \delta_j (x, t)}{\tau_{\alpha+\delta_i} - \delta_j (x, t)} \right) \Psi^+_{ij}(\tau_\alpha, x, t, z) \]

\[= \Psi^+_{ij}(\sigma_\alpha, x, t, z) + \frac{\tau_\alpha (x, t)}{\sigma_\alpha (x, t)} (e^{-\eta^{(j)}(t, z)} - 1) \left( \frac{\sigma_{\alpha+\delta_i} - \delta_j (x, t)}{\tau_{\alpha+\delta_i} - \delta_j (x, t)} \right) \Psi^+_{ij}(\tau_\alpha, x, t, z) \]

\[= \Psi^+_{ij}(\sigma_\alpha, x, t, z) - \lambda \frac{\tau_\alpha (x, t)}{\sigma_\alpha (x, t)} (y - w)^\delta_{ab} (Y^{(ab)}(\tau_\alpha, y, w) - \Psi^+(\tau_\alpha, x, t, z))_{ij} \]

\[= \left( 1 - \lambda \frac{\tau_\alpha (x, t)}{\sigma_\alpha (x, t)} (y - w)^\delta_{ab} Y^{(ab)}(\tau_\alpha, y, w) - \Psi^+(\tau_\alpha, x, t, z) \right)_{ij}
\]

and we obtain the following consequence of Theorem 5.3:
Corollary 5.4. Let \( \tau(x, t) \) be a solution of the s–component KP hierarchy, then \( \sigma(x, t) = e^{X^{(ab)}(y, w)} \tau(x, t) \) is a new solution of this hierarchy and the wave functions are related by

\[
\Psi^+_{ij}(\sigma_\alpha, x, t, z) = \left( 1 - \lambda \frac{\tau_\alpha(x, t)}{\sigma_\alpha(x, t)} (y - w)^{\delta_{ab}Y^{(ab)}(\tau_\alpha, y, w)_-} \right) \Psi^+(\tau_\alpha, x, t, z)_{ij}
\]

Hence (5.8) relates this Bäcklund transformation of the s–component KP hierarchy acting on the \( \tau \)–function to a “Bäcklund transformation” on the wave function.

Since the left–hand–side of (5.8) is also equal to

\[
\{ e^{-\eta^{(j)}(t, z)} \left( \frac{\hat{X}^{(ab)}(y, w) - \delta_{ab}}{\tau_\alpha(x, t)} \right) \}
\]

we have the following

**Corollary 5.5.**

\[
(-M(\alpha)^{\ell}L(\alpha)^{k+\ell}C^{(ab)}(\alpha))_+ - \Psi^+(\alpha, z))_{ij} =
\]

\[
\{ e^{-\eta^{(j)}(t, z)} \left( \frac{\hat{W}_k^{(ab,\ell+1)} \tau_\alpha(x, t)}{\tau_\alpha(x, t)} \right) \}
\]

Proof. Compare in (5.8) the expansions for the vertex operators \( Y^{(ab)}(\alpha, y, w) \) as in (5.2) and for \( \hat{W}^{(ab)}(y, w) \) as in (5.7).

As an application of Corollary 5.5 we see that if

\[
\sum_{a=1}^{s} \sum_{\ell=0}^{\infty} \sum_{k \in \mathbb{Z}} c_{a\ell k} (M(\alpha)^{\ell}L(\alpha)^{k+\ell}C^{(ab)}(\alpha))_- = 0,
\]

one finds that for any \( 1 \leq j \leq s \)

\[
(e^{-\eta^{(j)}(t, z)} - 1) \left( \sum_{a=1}^{s} \sum_{\ell=0}^{\infty} \sum_{k \in \mathbb{Z}} c_{a\ell k} \hat{W}_k^{(a,\ell+1)} \tau_\alpha(x, t) \right) = 0,
\]

hence

\[
\sum_{a=1}^{s} \sum_{\ell=0}^{\infty} \sum_{k \in \mathbb{Z}} c_{a\ell k} \hat{W}_k^{(a,\ell+1)} \tau_\alpha(x, t) e^\alpha = \text{constant } \tau_\alpha(x, t) e^\alpha.
\]

Thus Corollary 5.5 provides an alternative proof of Theorem 6.5 of [V].
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