ALGORITHMS FOR OPTIMAL CONTROL OF HYBRID SYSTEMS
WITH SLIDING MOTION

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Abstract. This paper concerns two algorithms for solving optimal control problems with hybrid systems. The first algorithm aims at hybrid systems exhibiting sliding modes. The first algorithm has several features which distinguishes it from the other algorithms for problems described by hybrid systems. First of all, it can cope with hybrid systems which exhibit sliding modes. Secondly, the systems motion on the switching surface is described by index 2 differential–algebraic equations and that guarantees accurate tracking of the sliding motion surface. Thirdly, the gradients of the problems functionals are evaluated with the help of adjoint equations. The adjoint equations presented in the paper take into account sliding motion and exhibit jump conditions at transition times. We state optimality conditions in the form of the weak maximum principle for optimal control problems with hybrid systems exhibiting sliding modes and with piecewise differentiable controls. The second algorithm is for optimal control problems with hybrid systems which do not exhibit sliding motion. In the case of this algorithm we assume that control functions are $L_{\infty}$ measurable functions. For each algorithm, we show that every accumulation point of the sequence generated by the algorithm satisfies the weak maximum principle.

Key words. optimal control, hybrid systems, sliding modes, the weak maximum principle, algorithm for optimal control problem.

AMS subject classifications. 49J15, 49K15, 65K10, 34K34

1. Introduction. Hybrid systems are systems with mixed discrete-continuous dynamics ([31],[3]). The set of discrete states $Q$ consist of finite number of elements denoted by $q$. The admissible controls set $U$ consists of measurable functions $u : I \rightarrow U$ defined on a closed interval $I$ with the values in $U \subset \mathbb{R}^m$. The continuous dynamics in each discrete state is described by ordinary differential equations (ODEs)

$\begin{equation}
    x' = f(x,u)
\end{equation}$

or more generally by differential-algebraic equation (DAEs)

$\begin{equation}
    0 = F(x',x,u),
\end{equation}$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$. The transitions between discrete states are triggered when the condition of the form

$\begin{equation}
    \eta(x,u) \leq 0
\end{equation}$

stops to be satisfied, where $\eta : \mathbb{R}^n \times U \rightarrow \mathbb{R}$. The functions $\eta(x,u)$ are called guards. We restrict our analysis to systems with autonomous transitions and without state jumps during transitions.

The optimal control problem with a hybrid system has been considered in many papers. The necessary optimality conditions for a class of hybrid systems without state jumps have been first formulated in [32]. In [4] the variational methods have been used to formulate adjoint equations for systems with state jumps. The Pontryagin maximum principle for hybrid systems with state jumps has been formulated for

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several classes of hybrid systems in [27], [23], [24], [25], [28], [29], [30], [13]. In papers [23], [24], [25], [28], [30] also algorithms based on the hybrid maximum principle are discussed. Our paper does not consider optimal control problems in which switching times are decision variables ([22]).

In [17] we have introduced an algorithm for optimal control problems with hybrid systems described by higher index DAEs. In the paper we have assumed that we deal with discrete time differential–algebraic equations resulting from the system equations discretization by an implicit Runge–Kutta method.

In none of these papers an optimal control problem with hybrid systems exhibiting sliding modes has been considered. This paper fills that gap by considering hybrid systems whose motion can be on the switching manifold (transition guard). For these optimal control problems we formulate a weak version of the maximum principle provided that controls are represented by piecewise smooth controls. Furthermore, we propose an algorithm which generates a sequence of approximating optimal controls which converges to a control which satisfies optimality conditions stated by the weak maximum principle. The convergence is established under the assumption that continuous time system equations are given. In the accompanying papers ([18],[19]) we show that the algorithm convergence can still be attained if state trajectories are obtained by an implicit Runge–Kutta method provided that the integration procedure step sizes go to zero.

The main focus of the paper is on the problems with hybrid systems which can exhibit sliding motion. For these problems we propose algorithms which have global convergence properties. One algorithm is a first order method which is stated for problems with piecewise smooth controls. We also discuss the method which is intended for control problems with hybrid systems whose trajectories do not experience sliding motion. In the case of these optimal control problems we assume that control functions are elements of the space $L^\infty$.

Our convergence results are based on the trajectory sensitivity analysis of hybrid systems presented in the paper [20].

The computational approach to hybrid systems we advocate in this paper (and in [17], [18], [19]) assumes that for a given control function we attempt to follow true systems trajectories as close as possible by using integration procedures with a high order of convergence. It is in contrast to the approach which uses approximations to the discontinuous right-hand side for the differential equations or inclusions by a smooth right–hand side ([26], [9], [12]). The smoothing approach demonstrates that it can be used to find approximate solutions to optimal control problems with systems which exhibit sliding modes however there is still a lack of evidence that it is capable of efficiently finding optimal controls and state trajectories with high accuracy by choosing proper values of smoothing parameters.

2. Calculating the optimal control. In our work we consider hybrid systems with sliding modes. For the sake of simplicity let us consider a hybrid system with two discrete states collected in a set $Q = \{1, 2\}$. Let us assume that the invariant sets are $I(1) = \{x \in \mathbb{R}^n : h(x) \leq 0\}$ and $I(2) = \{x \in \mathbb{R}^n : h(x) \geq 0\}$ where $h : \mathbb{R}^n \to \mathbb{R}$. If the hybrid system starts its evolution from a discrete state $q = 1$ the continuous state evolves according to an equation $x' = f_1(x, u)$. At a transition time $t_t$ the continuous state trajectory reaches the boundary of an invariant set and we have $h(x(t_t)) = 0$. The first order condition which guarantees that the continuous state trajectory will leave the invariant set $I(1)$ is ([11])

\begin{equation}
    h_x(x(t_t))f_1(x(t_t), u(t_t)) > 0
\end{equation}
where $h^T(x)$ is the normal vector to a surface

$$\Sigma = \{ x \in \mathbb{R}^n : h(x) = 0 \}$$

at $x$. If at a transition time we also have

$$h_x(x(t_t))f_2(x(t_t), u(t_t)) > 0$$

then the discrete state changes from $q = 1$ to $q = 2$ and the continuous state continues the evolution according to the equation $x' = f_2(x, u)$. If at a transition time we have

$$h_x(x(t_t))f_2(x(t_t), u(t_t)) < 0$$

then both vector fields $f_1(x, u)$ and $f_2(x, u)$ point towards the surface $\Sigma$ and we face the sliding motion phenomenon ([7]).

The sliding motion can be handled with the concept of Filippov solutions ([8],[7]). We say that a continuous state trajectory is a Filippov solution of the considered hybrid system if the sliding motion can be handled with the concept of Filippov solutions ([8],[7]). We say that a continuous state trajectory is a Filippov solution of the considered hybrid system if

$$x' = \begin{cases} f_1(x, u) & \text{if } h(x) < 0 \\ \overline{\text{co}}(f_1(x, u), f_2(x, u)) & \text{if } h(x) = 0 \\ f_2(x, u) & \text{if } h(x) > 0 \end{cases}$$

for almost all $t$ from its domain of definition. Here, $\overline{\text{co}}(f_1(x, u), f_2(x, u))$ is the minimal closed convex set containing $f_1$ and $f_2$, that is

$$\overline{\text{co}}(f_1(x, u), f_2(x, u)) = \{ f_F \in \mathbb{R}^n : f_F = f_1 + \alpha (f_2 - f_1), \alpha \in [0,1] \}.$$

During the sliding motion the continuous state trajectory must stay in the surface $\Sigma$, so the condition

$$h_x(x)f_F(x, u) = 0$$

must be satisfied. From (2.5) and (2.6) it is easy to find the formula for $\alpha$ coefficient

$$\alpha(x, u) = \frac{h_x(x)f_1(x, u)}{h_x(x)(f_1(x, u) - f_2(x, u))}.$$

When dealing with hybrid systems we have to pay special attention to systems behavior at switching times $t_t$, on the left and the right of these points. To this end we define

$$u(t_t^-) = \lim_{t \to t_t, t < t_t} u(t), \quad u(t_t^+) = \lim_{t \to t_t, t > t_t} u(t),$$

and we assume that $u$ is continuous from the left which means that

$$u(t_t^-) = u(t_t) \quad \text{and} \quad u(t_t^+) \neq u(t_t)$$

in general.

This notation at a neighborhood of time $t_t$ apply also to other functions, for example

$$x(t_t^-) = \lim_{t \to t_t, t < t_t} x(t), \quad x(t_t^+) = \lim_{t \to t_t, t > t_t} x(t).$$
In this setting general conditions (2.1)-(2.3) for changing discrete states must be stated as follows:

\( h_x(x(t^-_j))f_1(x(t^-_j), u(t^-_j)) > 0 \)  \( (2.11) \)
\( h_x(x(t^+_j))f_2(x(t^+_j), u(t^+_j)) > 0 \)  \( (2.12) \)

and, in particular case when \( h \) is defined through the function \( \alpha(x,u) \) and, for example, the system transits to state \( q = 1 \)

\( \alpha(x(t^-_j), u(t^-_j)) = 0 \)  \( (2.13) \)
\( h_x(x(t^+_j))f_1(x(t^+_j), u(t^+_j)) < 0 \)  \( (2.14) \)
\( h_x(x(t^+_j))f_2(x(t^+_j), u(t^+_j)) < 0 \)  \( (2.15) \)

The introduced convention is also applied to integrals, for example

\[ \int_{t_0}^{t^-_j} p(x(t), u(t))dt = \lim_{t \to t^-_j, t < t^-_j} \int_{t_0}^{t} p(x(\tau), u(\tau))d\tau. \]

But taking piecewise smooth functions as controls requires also checking at each point \( t_j \) (see (3.5)) whether the system is still at a sliding mode after control changes at time \( t_j \), if not a new discrete state at this time must be determined. For lack of space we do take into account these nonautonomous switches times. However, our analysis could be easily extended with this respect under additional assumption (H2) iv stated in [20].

During the sliding motion the continuous state trajectory should stay in a set \( \Sigma \). To keep this condition satisfied during the numerical integration, we follow the approach proposed in [1] and integrate the differential-algebraic equations (DAEs)

\[ x' = f_F(x,u) + h(x)^T z \]  \( (2.16) \)
\[ 0 = h(x) \]  \( (2.17) \)

instead of ODEs

\[ x' = f_F(x,u). \]  \( (2.18) \)

It can be shown that at a sliding motion equations (2.16)-(2.17) and (2.18) are equivalent ([20]). We prefer using (2.16)-(2.17) instead of (2.18) for the reason of getting more stable numerical solution on the switching surface.

The sliding motion can be treated as the third discrete state with \( q = 3 \). Treating the sliding motion as another discrete state implies that guards functions \( h \) should depend also on control variables \( u \) and system equations should be given, in general, by implicit equations (2.16)-(2.17).

From now on the invariant set at a discrete state \( q \) will be defined as follows

\[ I(q) = \{(x,z,u) \in \mathbb{R}^{n+1+m} : \eta_q(x,u) \leq 0 \}, \]
\[ \eta_q : \mathbb{R}^n \times U \to \mathbb{R}^{n x q}. \]  \( (2.19) \)

and \( n x q \) denotes the number of functions defining the invariant set for state \( q \).

Furthermore, the function \( \mathcal{F} \) is related to the functions \( f_q \) and \( h_q \) defining, for a given discrete state \( q \), DAEs:

\[ x' = f_q(x,u) + (h_q)_x(x)^T z \]
\[ 0 = h_q(x). \]
In particular, we can have \( h_q(x) \equiv 0 \) for some \( q \) and then the guarding function \( \eta_q \) is not dependent on \( u \).

If we assume that a switching surface function \( \eta \) depends on \( u \) we need continuity of function \( \eta(x(t), u(t)) \) to locate uniquely a switching time \( t_i \). For that reason we assume that control functions \( u \) are piecewise smooth—another justification for the chosen admissible controls is given in the next Section 3 by referring to properties of the Filippov’s solution stated in [8].

Another implication of using switching functions dependent on \( u \) is the necessity of updating the discrete state \( q \) whenever control functions \( u \) exhibit jumps.

Consider the system

\[
\begin{align*}
(2.20) & \quad x' = f(x, u) + h_x(x)^T z = f_r(x, z, u) \\
(2.21) & \quad 0 = h(x),
\end{align*}
\]

on \([t_0, t_f]\). We assume that on some subintervals \([t_0, t_1]\) we can have \( h(x) \equiv 0 \) and in that case system (2.20)–(2.21) reduces to ODEs. Since we consider, for the simplicity of presentation, that our hybrid system can be only in three states \( q = 1, 2, 3 \) (and the third state reserved for a sliding motion) our state trajectory on the entire horizon \([t_0, t_f]\) will consist of trajectories of type \((x, z, u)\) on subintervals \( A_i^1, i \in I_s \), and of type \( x \) on subintervals \( A_i^2, i \in I_1 \) (if trajectory is defined by the function \( f_1 \)) and subintervals \( A_i^3, i \in I_2 \) (if trajectory is defined by the function \( f_2 \)). We have \( \cup_{i \in I_s} A_i^1 \cup_{i \in I_1} A_i^2 \cup_{i \in I_2} A_i^3 = [t_0, t_f] \).

Our approach is based on sensitivity analysis stated in [20] where it is shown that solutions to linearized equations of DAEs have properties similar to those exhibited by linearized equations to ordinary differential equations, provided that some conditions are met. Namely, if we denote by \((x, z)\) the solution of (2.20)–(2.21) for given \( u \), by \((x^d, z^d)\) the solution for the control \( u + d \), and by \((y^d, z^d)\) the solution to the linearized equations:

\[
\begin{align*}
(2.22) & \quad (y^d)' = f_s(x, u)y^d + (h_x(x)^T z) z^d = f_r(x, z, u)d \\
(2.23) & \quad 0 = h_x(x)y^d,
\end{align*}
\]

then the following will be satisfied

\[
\begin{align*}
(2.24) & \quad \|(x, z)\|_{L_\infty} \leq c_1 \\
(2.25) & \quad \|(x^d, z^d) - (x, z)\|_{L_\infty} \leq c_2 \|d\|_{L_\infty} \\
(2.26) & \quad \|(y^d, z^d)\|_{L_\infty} \leq c_3 \|d\|_{L_1}
\end{align*}
\]

for some positive constants \( c_1, c_2, c_3 \). Furthermore, for any \( \varepsilon \) satisfying \( \varepsilon_t > \varepsilon > 0 \), where \( \varepsilon_t \) is as specified in (H3), there exists function \( o : ([0, \infty) \to (0, \infty) \) such that \( \lim_{s \to 0^+} o(s)/s = 0 \) and

\[
\|x^d(t) - (x(t) + y^d(t))\| \leq o(\|d\|_{L_\infty}),
\]

\[
\forall t \in [t_i, t_{i+1} - \varepsilon] \bigcup \left[t_f - \varepsilon, t_f\right] \bigcup \bigcup_{i \in I_s(u) \setminus \{N_i(u)\}} [t_i, t_{i+1} - \varepsilon].
\]

Here, the points \( \{t_i\}_{i=1}^{N(u)} \) are as specified in the assumption (H3).

In order to specify the relations for differentials of \( x \) at a switching point \( t_i \), denoted by \( dx^d \), additional notation has to be introduced. Suppose that a control \( u \) is perturbed by \( d \), then a new switching point \( t_i^d \) will occur. We assume that before
switching of discrete states the system evolves according to equations (2.20–2.21) with f and h replaced by f_1 and h_1 respectively and after the switching the system is described by (2.20–2.21) with f_2 and h_2 instead of f and h. Then, we have

\begin{align}
(2.28) & \quad \left\| x_1^d(t^-_i) - x_1(t^-_i) - d(x_1)^d \right\| \leq o(\|d\|_{L^\infty}), \\
(2.29) & \quad \left\| x_2^d(t^+_i) - x_2(t^+_i) - d(x_2)^d \right\| \leq o(\|d\|_{L^\infty}),
\end{align}

for all u and d such that u + d ∈ U where \( d(x_1)^d = y_1^d(t^-_i) + f_1(x_1(t^-_i), z_1(t^-_i), u(t^-_i)) \times dt_1^d, x_1, x^d_1, z_1 \) are solutions to equations (2.20)–(2.21) with f = f_1, h = h_1, \( y_2^d \) are solutions to the corresponding linearized equations associated with equations (2.20)–(2.21), \( d(x_2)^d = y_2^d(t^+_i) + f_2(x_2(t^+_i), z_2(t^+_i), u(t^+_i)) dt_2^d, x_2, x^d_2, z_2 \) are solutions to equations (2.20)–(2.21) with f = f_2, h = h_2 and \( y_2^d \) are solutions to the corresponding linearized equations associated with equations (2.20)–(2.21). Here, \( dt_i^d \) is the differential of the switching point \( t_i \) and o are such that \( \lim_{s \to 0^+} |o(s)|/s = 0; dx^d \) and \( dx^d \) differentials correspond to systems behavior before and after the switching.

In order to present the conjectures under which (2.24)–(2.27) hold we introduce the function \( e(x, z, u) = h_2(x) f(x, u) + \|h_2(x)\|^2 z \). Then under (H1), (H2) and (H3) the relations (2.24)–(2.27) are satisfied.

(H1) \( U \) is a compact convex set in \( \mathbb{R}^n \), \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( h : \mathbb{R}^n \to \mathbb{R} \),

i) function \( f(\cdot, \cdot) \) is differentiable, \( f_x \) \( f_u \) are continuous and there exists \( 0 < k^1_j < +\infty \) such that

\begin{align}
(2.30) & \quad \|f_x(x, u)\| \leq k^1_j \\
\text{for all } (x, u) \in \mathbb{R}^n \times U,
\end{align}

ii) if \( h(x) \neq 0 \) then \( (x(t_0), z(t_0)) \) are such that

\begin{equation}
(2.31) \quad h(x(t_0)) = 0, \quad z(t_0) = 0, \quad h_x(x(t_0)) f(x(t_0), u(t_0)) = 0,
\end{equation}

\( h(\cdot) \) is twice differentiable and \( \nabla^2 h \) is continuous. Moreover, there exist \( \varepsilon > 0, 0 < k^1_f < +\infty, 0 < k^2_f < +\infty, 0 < k^3_f < +\infty, 0 < k^1_h < +\infty, 0 < k^2_h < +\infty, 0 < k^3_h < +\infty \) with the properties

\begin{align}
(2.32) & \quad \|f(x, u)\| \leq k^2_f \\
(2.33) & \quad \|f_u(x, u)\| \leq k^2_f \\
(2.34) & \quad \|h_x(x) f(x, u)\| \leq k^3_f \\
(2.35) & \quad \|e_{x, z, u}(\hat{x}, \hat{z}, \hat{u}) - e_{x, z, u}(x, z, u)\| \leq k^4_f \|(\hat{x}, \hat{z}, \hat{u}) - (x, z, u)\| \\
(2.36) & \quad k^1_k \leq \|h_x(x)\| \leq k^2_k \\
(2.37) & \quad \|\nabla^2 h(x)\| \leq k^3_k
\end{align}

for all \( (x, z, u), (\hat{x}, \hat{z}, \hat{u}) \) in \( \mathbb{R}^n \times \mathbb{R} \times U \) such that \( |h(x)| \leq \varepsilon, |h(\hat{x})| \leq \varepsilon \).

In (H1) it is assumed that \( U \) is a compact convex set. In order to show (2.24)–(2.27) it is sufficient to assume that \( U \) is bounded, but the convergence analysis which is based on (2.24)–(2.27) requires the stronger assumption.

(H2) i) Function \( \eta(\cdot, \cdot) \) is differentiable and there exist \( 0 < L_1 < +\infty \) and \( 0 < L_2 < +\infty \) such that

\begin{align}
(2.38) & \quad \|\eta_{x, u}(\hat{x}, \hat{u}) - \eta_{x, u}(x, u)\| \leq L_1 \|(\hat{x}, \hat{u}) - (x, u)\| \\
(2.39) & \quad \|f_x(\hat{x}, \hat{z}, \hat{u}) - f_x(x, z, u)\| \leq L_2 \|(\hat{x}, \hat{z}, \hat{u}) - (x, z, u)\|
\end{align}
for all \((x, z, u), (\hat{x}, \hat{z}, \hat{u})\) in \(\mathbb{R}^n \times \mathbb{R} \times U\).

ii) There exists \(\varepsilon > 0\) and \(0 \leq L_3 < +\infty\) such that for all switching points \(t_i\) (each switching time corresponds to some \(u \in U\)) and for all their perturbations \(t'_i\) triggered by perturbations \(d\) such that \(u + d \in U\), \(\|d\|_{L^\infty} \leq \varepsilon\), and for all \(\theta \in [0, 1]\) we have

\[
|\eta(x(\tau^{\theta,d}(t)), u(\tau^{\theta,d}(t)))| f_1(x(\tau^{\theta,d}(t)), z(\tau^{\theta,d}(t)), u(\tau^{\theta,d}(t)))
+ |\eta(x(\tau^{\theta,d}(t)), u(\tau^{\theta,d}(t)))| u'(\tau^{\theta,d}(t))| \geq L_3
\]

(2.40)

where

\[
\tau^{\theta,d}(t) = t + \theta(t'_i - t_i).
\]

(2.41)

iii) Perturbations \(d \in U - u\) are such that

\[
\|d\|_{L^\infty} \to 0 \Rightarrow \|d'\|_{L^\infty} \to 0.
\]

(2.42)

Notice that if system leaves discrete state \(q = 1, \) or \(q = 2\) then \(\eta = h\) and does not depend on \(u\).

We denote by \(N_t(u)\) the number of switching times triggered by a control \(u \in U\) and by \(I_t(u)\) the set of indices which indicates all switching points.

We make the hypothesis

(H3) There exists a nonnegative integer number \(I^\text{max}_t < +\infty\) and \(\varepsilon_t > 0\) such that

\[
N_t(u) \leq I^\text{max}_t,
\]

(2.43)

\[
t'_i \leq t_{i+1} - \varepsilon_t, \forall i \in I_t(u) \setminus \{N_t(u)\},
\]

(2.44)

\[
t^N_t(u) \leq t_f - \varepsilon_t
\]

(2.45)

for all \(u \in U\).

3. Optimal control problem with sliding modes and piecewise smooth controls. Taking into account the considerations and definitions presented in the previous section, in particular including the possibility of the system entering (and spending some time) in the sliding mode, we now turn attention to the optimal control problem of interest—(P):

\[
\min_{u} \phi(x(t_f))
\]

subject to the constraints

\[
\begin{cases}
    x' = f_1(x, u) & \text{if } q = 1 \\
    x' = f_2(x, u) & \text{if } q = 2 \\
    x' = f_F(x, u) + h(x)^T z & \text{if } q = 3
\end{cases}
\]

(3.2)

and the terminal constraints

\[
\begin{align}
    g^1_1(x(t_f)) &= 0 \forall i \in E \\
    g^1_2(x(t_f)) &\leq 0 \forall j \in I.
\end{align}
\]

(3.3)
We also assume that the initial state \( x(t_0) = x_0 \) is fixed.

Crucial assumption concerns the class of control functions. Our algorithm requires that control functions are piecewise smooth functions. The justification for that is given in [20] by referring to properties of Filippov’s solutions.

Thus the admissible control \( u \) is a function \( u \) from the set \( \mathcal{U} \), where

\[
\mathcal{U} = \{ u \in \mathcal{C}_m^N[T] : u(t) \in U \subset \mathbb{R}^m, \ t \in [t_0, t_f], \}
\]

\[
\mathcal{C}_m^N[T] = \left\{ u \in \mathcal{C}_m^2[T] : u(t) = \sum_{j=1}^{N} \psi_j(t)u_j^N(t), \right. \\
\left. \psi_j(t) = \begin{cases} 
1, & \text{if } t \in [t_{j-1}, t_j) \\
0, & \text{if } t \not\in [t_{j-1}, t_j) \end{cases}, \right.
\]

\( t_j = j t_f / N, \ j \in \{0, 1, \ldots, N\} \).

Regarding the control constraints we make the hypothesis:

\((\text{CC})\) The set of admissible controls \( \mathcal{U} \) is a convex compact subset of \( \mathcal{L}_m^2[T] \) and there exists \( 0 < L_d < \infty \) such that

\[
\|u' - v'\|_{\mathcal{L}^\infty} \leq L_d \|u - v\|_{\mathcal{L}^\infty} \ \forall u, v \in \mathcal{U}.
\]

For the lack of space we do not analyse in the paper under which conditions imposed on functions \( u_j^N \in \mathcal{C}_m^1[t_{j-1}, t_j] \) (constant, linear, quadratic, etc.) \((\text{CC})\) is satisfied.

The method we propose for solving the problem \((P)\) is based on an exact penalty function. By using an exact penalty function approach, instead of solving the problem \((P)\) we solve the problem \((P_e)\)

\[
\min_{u \in \mathcal{U}} \bar{F}_e(u)
\]

in which the exact penalty function \( \bar{F}_e(u) \) is defined as follows

\[
\bar{F}_e(u) = F_0(u) + c \max \left[ 0, \max_{i \in E} |g_i^1(u)|, \max_{j \in I} |g_j^2(u)| \right]
\]

and, since under our assumptions \( x \) uniquely depends on \( u \), we can write: \( F_0(u) = \phi(x^u(t_f)), \ g_i^1(u) = g_i^1(x^u(t_f)), \ i \in E, \ g_j^2(u) = g_j^2(x^u(t_f)), \ j \in I \).

For fixed \( c \) and \( u \) the direction finding subproblem, \( P_e(u) \), for the problem \((P_e)\) is:

\[
\min_{d \in \mathcal{D}, \beta \in \mathbb{R}} \left[ \langle \nabla F_0(u), d \rangle + c \beta + 1/2 \|d\|_{\mathcal{L}^2}^2 \right]
\]

subject to

\[
\begin{align*}
|g_i^1(u) + \langle \nabla g_i^1(u), d \rangle| & \leq \beta \ \forall i \in E \\
\bar{g}_j^2(u) + \langle \nabla \bar{g}_j^2(u), d \rangle & \leq \beta \ \forall j \in I.
\end{align*}
\]

Here (notice \((H3)\), part \(iii\)),

\[
\mathcal{D}_u \left\{ d \in \mathcal{C}_m^N[T] : d \in \mathcal{U} - u \right\}.
\]
The direction finding subproblem is based on first order approximations to the problem functionals, defined as follows

\[ \langle \nabla \bar{F}_0(u), d \rangle = \phi_x(x^u(t_f))y^x, d(t_f) \] (3.9)  
\[ \langle \nabla \bar{g}_i^1(u), d \rangle = (g_i^1)_x(x^u(t_f))y^x, d(t_f), \quad i \in E \] (3.10)  
\[ \langle \nabla \bar{g}_j^2(u), d \rangle = (g_j^2)_x(x^u(t_f))y^x, d(t_f), \quad j \in I. \] (3.11)

The subproblem can be reformulated as an optimization problem with the objective function which is strictly convex. The problem therefore has the unique solution \((\bar{d}, \bar{\beta})\). Since this solution depends on \(c\) and \(u\), we may define descent function \(\sigma_c(u)\) and penalty test function \(t_c(u)\) (see [16] for details), to be used to test optimality of a control \(u\) and to adjust \(c\), respectively, as

\[ \sigma_c(u) = \langle \nabla \bar{F}_0(u), d \rangle + c [\bar{\beta} - M(u)] \] (3.12)  
\[ t_c(u) = \sigma_c(u) + M(u)/c \] (3.13)

for given \(c > 0\) and \(u \in \mathcal{U}\). Here,

\[ M(u) = \max \left[ 0, \max_{i \in E} |\bar{g}_i^1(u)|, \max_{j \in I} |\bar{g}_j^2(u)| \right], \]

A general algorithm is as follows.

**Algorithm** Fix parameters: \(\gamma, \eta \in (0, 1), c^0 > 0, \kappa > 1\).

1. Choose the initial control \(u_0 \in \mathcal{U}\). Set \(k = 0, c_{-1} = c^0\).
2. Let \(c_k\) be the smallest number chosen from \(\{c_{k-1}, \kappa c_{k-1}, \kappa^2 c_{k-1}, \ldots\}\) such that the solution \((d_k, \beta_k)\) to the direction finding subproblem \(P_{c_k}(u_k)\) satisfies

\[ t_{c_k}(u_k) \leq 0. \] (3.14)

If \(t_{c_k}(u_k) = 0\) then STOP.
3. Let \(\alpha_k\) be the largest number chosen from the set \(\{1, \eta, \eta^2, \ldots, \}\) such that

\[ u_{k+1} = u_k + \alpha_k d_k \]

satisfies the relation

\[ \bar{F}_{c_k}(u_{k+1}) - \bar{F}_{c_k}(u_k) \leq \gamma \alpha_k \sigma_{c_k}(u_k). \]

Increase \(k\) by one. Go to Step 2.

Since

\[ \langle \nabla \bar{F}_0(u_k), d \rangle + c_k \beta + 1/2\|d\|_{L_2}^2 \leq c_k M(u_k), \]

which holds because \(0 \in \mathcal{U} - u_k\), we also have

\[ \langle \nabla \bar{F}_0(u_k), d \rangle + c_k [\beta - M(u_k)] \leq -1/2\|d\|_{L_2}^2 \leq 0 \] (3.15)

which implies that the descent function \(\sigma_{c_k}(u_k)\) is nonpositive valued at each iteration.
Algorithm generates a sequence of controls \( \{u_k\} \) and the corresponding sequence of penalty parameters \( \{c_k\} \) such that \( \{c_k\} \) is bounded and any accumulation point of \( \{u_k\} \) satisfies optimality conditions in the form of the weak maximum principle for the problem (P), i.e. \( \sigma_c(\tilde{u}) = 0 \) for the limit point \( \tilde{u} \) and the limit point of the sequence \( \{c_k\} \). But \( \sigma_c(\tilde{u}) = 0 \) implies that \( M(\tilde{u}) = 0 \) due to the definition (3.13) and since (3.14) holds.

It can be shown that the set \( \mathcal{U} \) is compact (since we assume that \( U \) is compact) and this fact can be used to prove the convergence of Algorithm without referring to relaxed controls (cf. [16]). However, we still need to introduce two additional hypotheses. The first one concerns the functions defining the objective and constraints:

(H4) \( \phi, g^1_i, i \in E, g^2_j, j \in I \) are continuously differentiable functions.

The second one is related to a constraint qualification which is needed to prove the convergence of Algorithm. To this end we first introduce the set

\[ D = \{ d \in C^N_m[T] : \exists u \in \mathcal{U} \text{ such that } u + d \in \mathcal{U} \} \]

and the set

\[ \mathcal{F}(u) = \left\{ d \in D : \max_{j \in I} \langle \nabla \tilde{g}^2_j(u), d \rangle < 0 \right\}. \]

Then the constraint qualification condition takes the form (CQ) for each \( u \in \mathcal{U}, \mathcal{F}(u) \neq \emptyset \), and in the case \( E \neq \emptyset \) we have

\[ 0 \in \text{interior} \mathcal{E}(u) \]

where

\[ \mathcal{E}(u) = \left\{ \{ (\nabla \tilde{g}^1_i, d) \} : i \in E \in \mathbb{R}^{|E|} : d \in \mathcal{F}(u) \right\}. \]

The constraint qualification (CQ) is similar to these stated in [15] and [10], if we observe that it must take into account the constraint \( u \in \mathcal{U} \). We will show that under the constraint qualification (CQ) any limit point of the sequence \( \{u_k\} \) generated by Algorithm—\( \bar{u} \in \mathcal{U} \) will satisfy the conditions (NC):

(NC) :

\[ 0 \leq \min_{d \in \mathcal{D}_a} \phi_x(\bar{x}(t_f)) y^{x,d}(t_f) \]

subject to the constraints

\[ g^1_i(\bar{x}(t_f)) + (g^1_i)_x(\bar{x}(t_f)) y^{x,d}(t_f) = 0, \quad i \in E \]

\[ g^2_j(\bar{x}(t_f)) + (g^2_j)_x(\bar{x}(t_f)) y^{x,d}(t_f) \leq 0, \quad j \in I_{0,\bar{u}} \]

together with \( g^1_i(\bar{x}(t_f)) = 0, \quad i \in E, \quad g^2_j(\bar{x}(t_f)) \leq 0, \quad j \in I \). Here,

\[ I_{\varepsilon,u} = \left\{ j \in I : g^2_j(u) \geq \max_{j \in I} g^2_j(u) - \varepsilon \right\}, \quad \varepsilon \geq 0. \]

Under stated assumptions Algorithm is globally convergent in the following sense.

**Theorem 3.1.** Assume that data for (P) satisfies hypotheses (H1), (H2), (H3), (H4), (CC) and (CQ). Let \( \{u_k\} \) be a sequence of controls generated by Algorithm and let \( \{c_k\} \) be a sequence of the Corresponding penalty parameters. Then
i) \{c_k\} is a bounded sequence

\[
\lim_{k \to \infty} \sigma_{c_k}(u_k) = 0, \quad \lim_{k \to \infty} M(u_k) = 0.
\]

ii) Let \(\bar{u}\) be the limit point of the sequence \{u_k\} and \(\bar{x}\) the trajectory corresponding to \(\bar{u}\), then the pair \((\bar{x}, \bar{u})\) satisfies (NC).

The proof of Theorem 3.1 follows the lines of the proof of Theorem 5.1 stated in [21], however for the completeness of presentation it is provided in Appendix D. The proof heavily uses the implications of the constraint qualification (CQ) stated in the following lemma and on the properties of solutions to the system equations and solutions to their linearized equations \((x, z)\), \((y^{z,d}, y^{z,d})\) expressed by (2.24)–(2.27).

**Lemma 3.1.** Assume (H1), (H2), (H3), (H4), (CC) and (CQ). For any control \(\bar{u} \in \mathcal{U}\) there exist a neighborhood \(\mathcal{O}(\bar{u})\) of \(\bar{u}\), \(K_1 > 0\) and \(K_2 > 0\) with the following properties: given any \(u \in \mathcal{U}\) such that \(u \in \mathcal{O}(\bar{u})\) there exists \(v \in \mathcal{U}\) such that

\[
\max_{i \in I} |\tilde{g}_i^1(u) + \langle \nabla \tilde{g}_i^1(u), v - u \rangle| - \max_{i \in I} |\tilde{g}_i^1(u)| \leq -K_1 M(u) \tag{3.21}
\]

\[
\max_{j \in J} \langle \nabla \tilde{g}_j^2(u), v - u \rangle \leq -K_1 M(u) \tag{3.22}
\]

and \(\|v - u\|_{\infty} \leq K_2 M(u)\) \tag{3.23}

The role of (CQ) is to ensure uniform boundedness of penalty parameter values. Using the Lemma 3.1, following the analysis presented in [15] and [10], one can show that there exists \(0 < \bar{c} < \infty\) such that for any \(c \geq \bar{c}\) a local solution solution to the problem \((P_c)\) is a local solution to the problem \((P)\) (see the analysis given in Appendix B). The proof Lemma 3.1 is given in Appendix A.

The Step 2) of Algorithm is used to approximate the value \(\bar{c}\). In particular, if (CQ) is satisfied one can prove the following proposition which is needed in order to demonstrate the feasibilities of Steps 2–3 of Algorithm.

**Proposition 3.1.** (i) Assume that hypotheses (H1), (H2), (H3), (H4), (CC) and (CQ) are satisfied. Then for any \(u \in \mathcal{U}\) there exists \(\bar{c} > 0\) such that for all \(c > \bar{c}\)

\[
t_c(u) \leq 0.
\]

(ii) Assume that hypotheses (H1), (H2), (H3) and (H4) are satisfied. Then for any \(u \in \mathcal{U}\) and \(c > 0\) such that \(\sigma_c(u) < 0\) there exists \(\bar{c} > 0\) such that if \(\alpha \in [0, \bar{c}]\) then

\[
\bar{F}_c(\bar{u}) - \bar{F}_c(u) \leq \gamma \alpha \sigma_c(u)
\]

where \(\bar{u} = u + \alpha d\), \((d, \beta)\) is the solution to the direction finding subproblem corresponding to \(c\) and \(u\) and \(\sigma_c(u)\) is defined by (3.12).

Since for \(\bar{c} \leq 1\) \(\bar{u} \in \mathcal{U}\) part (ii) of the proposition states that \(\sigma_c(u) = 0\) \((d \equiv 0, \beta = 0)\) are the necessary optimality conditions for the problem \((P_c)\). If, in addition the trajectory \(x\) corresponding to \(u\) fulfills the terminal constraints, these optimality conditions are equivalent to (NC). The proof of Proposition 3.1 is given in Appendix C.
4. Adjoint equations. The efficient implementation of Algorithm needs computationally tractable algorithm for solving the direction finding subproblem $P_c(u)$ which at every iteration requires the evaluation of $\langle \nabla F_0(u), d \rangle$, $\langle \nabla g^i_j(u), d \rangle$, $i \in E$, $\langle \nabla g^i_j(u), d \rangle$, $j \in I$. If we do that by using solutions to the linearized equations then each update of $d$ would request solving these equations. If we use adjoint equations the evaluation comes down to the calculations of scalar products since $\nabla F_0(u)$, $\nabla g^i_j(u)$, $i \in E$, $\nabla g^i_j(u)$, $j \in I$ are available prior to solving $P_c(u)$. It is sufficient to show how $\nabla F_0(u)$ could be calculated with the help of adjoint equations since the adjoint equations for the other functionals will differ only by terminal conditions.

We formulate the adjoint equations for two cases, they are derived on the basis of a variational approach presented in [4] and by taking into account our results on solutions to linearized equations of ordinary differential equations and differential–algebraic equations of the special form arising when a system remains in the sliding mode.

Case 1). In the first case we assume that in the time interval $[t_0, t_1]$ the system evolves according to

\begin{equation}
\dot{x} = f_1(x, u).
\end{equation}

At a transition time $t_1$ the continuous state trajectory meets the switching surface such that the following condition holds

\begin{equation}
h(x(t_1)) = 0.
\end{equation}

After the transition the system evolves according to DAEs

\begin{align}
\dot{x} &= f_F(x, u) + h^T_x(x)z \\
0 &= h(x)
\end{align}

up to an ending time $t_f$.

To derive the adjoint equations we construct the following augmented functional

\[
\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi(x(t_f)) + \pi h(x(t_1)) + \int^{t_1}_{t_0} \lambda^T_f(t) \left( \dot{x}(t) - f_1(x(t), u(t)) \right) dt \\
&+ \int^{t_1}_{t_1^+} \left[ \lambda^T_f(t) \left( \dot{x}(t) - f_F(x(t), u(t)) - h^T_x(x(t))z(t) \right) + \lambda^T_h(t)h(x(t)) \right] dt.
\]

Suppose now that we calculate the variation of the augmented functional with respect to $(x, z)$ only.

\[
d\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi_x(x(t_f))y_x^d(t_f) + \pi h_x(x(t_1))dx(t_1) + \\
\lambda^T_f(t_1^+) \left( x'(t_1^+) - f_1(x(t_1^+), u(t_1^+)) \right) dt_1 + \\
+ d \left[ \int^{t_1^+}_{t_0} \lambda^T_f(t) \left( \dot{x}(t) - f_1(x(t), u(t)) \right) dt \right] \\
- \lambda^T_f(t_1^+) \left( x'(t_1^+) - f_F(x(t_1^+), u(t_1^+)) - h^T_x(x(t_1^+))z(t_1^+) \right) dt_1 - \lambda^T_h(t_1^+)h(x(t_1^+))dt_1 \\
+ d \left[ \int^{t_1^+}_{t_1^+} \left[ \lambda^T_f(t) \left( \dot{x}(t) - f_F(x(t), u(t)) - h^T_x(x(t))z(t) \right) + \lambda^T_h(t)h(x(t)) \right] dt \right].
\]
By integrating by parts the formulas \( \int \lambda_f(t)x(t)dt \) (since \( \lambda_f \) is sufficiently smooth) we obtain
\[
\begin{align*}
\frac{d\Phi(x, z, u, \lambda_f, \lambda_h, \pi)}{dt} &= \phi_x(x(t))y^{x.d}(t) + \pi h_x(x(t))dx(t) + \\
&+ \lambda_f^T(t^-) (x'(t^-) - f_1(x(t^-), u(t^-))) dt_t + d \left[ \lambda_f^T(t)x(t) \right]_{t_0}^{t^-} - \\
&- \lambda_f^T(t^+) (x'(t^+) - f_2(x(t^+), u(t^+)) - h_x^T(x(t^+))z(t^+)) dt_t - \\
&- \lambda_h^T(t^+) h(x(t^+)) dt_t + d \left[ \lambda_f^T(t)x(t) \right]_{t^+}^{t^-} - \\
&- \lambda_f^T(t^-) h(x(t^-)) dt_t + d \left[ \lambda_f^T(t)x(t) \right]_{t_0}^{t^-}.
\end{align*}
\]

Expanding further the variations (also with respect to \( u \)) and taking into account the initial conditions of the linearized equations we obtain
\[
\begin{align*}
\frac{d\Phi(x, z, u, \lambda_f, \lambda_h, \pi)}{dt} &= \phi_x(x(t))y^{x.d}(t) + \pi h_x(x(t))dx(t) + \\
&+ \lambda_f^T(t^-) x'(t^-) dt_t - \lambda_f^T(t^-) f_1(x(t^-), u(t^-)) dt_t + \lambda_f^T(t^-) y^{x.d}(t^-) - \\
&- \int_{t_0}^{t^-} \left[ (\lambda_f^T)'(t)y^{x.d}(t) + \lambda_f^T(f_1)_x(x(t), u(t))y^{x.d}(t) \right] dt + \\
&+ \lambda_f^T(t^-) (f_1)_u(x(t), u(t)) dt_t - \lambda_f^T(t^-) x'(t^-) dt_t + \\
&+ \lambda_f^T(t_0^+) f_2(x(t_0^+), u(t_0^+)) dt_t + \lambda_f^T(t_0^+) h_x^T(x(t_0^+))z(t_0^+)) dt_t - \\
&- \lambda_h^T(t_0^+) h(x(t_0^+)) dt_t + \lambda_f^T(t_0^+) y^{x.d}(t_0^+) - \\
&- \lambda_f^T(t^-) h(x(t^-)) dt_t + d \left[ \lambda_f^T(t)x(t) \right]_{t_0}^{t^-}.
\end{align*}
\]

Now, if we take into account the formula (2.28)–(2.29) for the differential of \( x(t_i) \), and rearrange the components with respect to differentials \( dx(t_i), dt_t \) and variations \( y^{x.d}(t_f), y^{x.d}(t), y^{x.d}(t), dt_t \) and variations
\[
\begin{align*}
\frac{d\Phi(x, z, u, \lambda_f, \lambda_h, \pi)}{dt} &= \phi_x(x(t))y^{x.d}(t) + (\pi h_x(x(t)) + \\
&+ \lambda_f^T(t^-) x'(t^-) dt_t + (\lambda_f^T(t^-) f_1(x(t^-), u(t^-))) + \\
&+ \lambda_f^T(t_0^+) f_2(x(t_0^+), u(t_0^+)) + \lambda_f^T(t_0^+) h_x^T(x(t_0^+))z(t_0^+)) dt_t - \\
&- \lambda_h^T(t_0^+) h(x(t_0^+)) dt_t + d \left[ \lambda_f^T(t)x(t) \right]_{t_0}^{t^-} - \\
&- \int_{t_0}^{t^-} \left[ (\lambda_f^T)'(t)y^{x.d}(t) + \lambda_f^T(f_1)_x(x(t), u(t))y^{x.d}(t) + \\
&+ \lambda_f^T(t)(f_2)_u(x(t), u(t)) dt_t + \lambda_f^T(t) h_x^T(x(t))y^{x.d}(t) + \\
&+ \lambda_f^T(t) h(x(t)) dt_t + \lambda_f^T(t_0^+) h(x(t_0^+))y^{x.d}(t_0^+) \right] dt_t.
\end{align*}
\]

We can now state conditions for adjoint equations such that the expressions with differentials \( dx(t_i), dt_t \) and variations \( y^{x.d}(t_f), y^{x.d}(t), y^{x.d}(t) \) disappear and only
the coefficients with variation \(d(t)\) remain. These components will state a formula to efficiently calculate the gradient of \(\phi(x(t_f))\) with respect to controls.

Let us start with components that are present under the integrals. We want to zero the following components

\[
\left((\lambda_f^T)'(t) + \lambda_f^T(t)(f_1)_x(x(t), u(t)))\right)y^{x,d}(t), \quad t \in [t_0, t_i^-],
\]

\[
\left((\lambda_f^T)'(t) + \lambda_f^T(t)(f_2)_x(x(t), u(t))\right) + \lambda_f^T(t)(h_x^T(x(t)))z(t)x_x \quad t \in [t_i^+, t_f],
\]

\[
\lambda_f^T(t)h_x^T(x(t))y^{x,d}(t), \quad t \in [t_i^+, t_f].
\]

We achieve that by assuming that

\[
(4.5) \quad (\lambda_f^T)'(t) = -\lambda_f^T(t)(f_1)_x(x(t), u(t)), \quad t \in [t_0, t_i^-]
\]

and

\[
(4.6) \quad (\lambda_f^T)'(t) = -\lambda_f^T(t)(f_2)_x(x(t), u(t)) - \lambda_f^T(t)(h_x^T(x(t)))z(t)x_x + \lambda_h(t)h_x(x(t)) \quad t \in [t_i^+, t_f]
\]

(4.5) is a system of ODEs with respect to the adjoint variable \(\lambda_f(t)\). (4.6)-(4.7) form a system of DAEs with respect to the adjoint variables \(\lambda_f(t)\) and \(\lambda_h(t)\). The DAEs are of index 2 which can be shown by considering the differentiation of (4.7)

\[
0 = (\lambda_f^T(t)h_x^T(x(t)))' = (\lambda_f^T)'(t)h_x^T(x(t)) + \lambda_f^T(t)(h_x^T(x(t)))'
\]

\[
= (-\lambda_f^T(t)(f_2)_x(x(t), u(t)) - \lambda_f^T(t)(h_x^T(x(t)))z(t)x_x + \lambda_h(t)h_x(x(t))) h_x^T(x(t)) + \lambda_f^T(t)(h_x^T(x(t)))'
\]

The algebraic variable \(\lambda_f^T(t)\) is multiplied by \(h_x(x(t))h_x^T(x(t))\) which is nonzero by the assumption (H1). This means that we can evaluate \(\lambda_h(t)\) by using the above equation which proves that the DAEs are index 2 equations. One can also show that under (H1) a solution to the equations (4.6)-(4.7), and to the equations (4.5), are such that \(\lambda_f\) is an absolutely continuous functions. This fact justifies the use of the integration by parts formula in our derivations.

Let us consider the component

\[
(\phi_x(x(t_f)) + \lambda_f^T(t_f))y^{x,d}(t_f).
\]

At the end of the time interval the trajectory evolves according to DAEs (4.6)-(4.7). The trajectory, after variation of controls, stays within the hyperplane defined by \(h(x(t)) = 0\) which means that the variation \(y^{x,d}(t_f)\) is always orthogonal to \(h_x(x(t_f))\). Therefore, the component disappears if the following condition is satisfied:

\[
\phi_x(x(t_f)) + \lambda_f^T(t_f) = \nu h_x(x(t_f)).
\]

for some real number \(\nu\).

The DAEs (4.6)-(4.7) have to be consistently initialized. To provide the consistent endpoint conditions for the adjoint variables \(\lambda_f\), \(\lambda_h\) we follow the approach presented
in [5] and solve the following system of equations at time $t_f$ for the variables $\lambda_f, \lambda_h, \nu$

$$h^T_x(x(t_f)) = \phi^T_x(x(t_f)) + \lambda_f(t_f)$$
0 = h_x(x(t_f))\lambda_f(t_f)
0 = (h_x(x(t_f)))'\lambda_f(t_f) - h_x(x(t_f)) (f_F)_T(x(t_f), u(t_f))\lambda_f(t_f) - h_x(x(t_f)) (h^T_x(x(t_f))) z(t_f)\lambda_f(t_f) + h_x(x(t_f)) h^T_x(x(t_f)) \lambda_h(t_f).$$

The solution to these equations is (due to (H1)): $\nu = h_x \phi^T_x / \|h_x\|^2$, $\lambda_f = h_x \phi^T_x h^T_x / \|h_x\|^2 - \phi^T_x$, $\lambda_h = \lambda^T_f \left( (f_F)_x + (h_x z)_x \right) / \|h_x\|^2$, where all functions are evaluated at $(x(t_f), z(t_f), u(t_f))$.

At the transition time $t_t$, the adjoint variable $\lambda_f$ undergoes a jump. To calculate the value of $\lambda_f(t^-)$ the following system of equations have to be solved for the variables $\lambda_f(t^-)$, $\pi$

$$\lambda_f(t^-) = \lambda_f(t^+) - \pi h^T_x(x(t_t))$$
$$\lambda^T_f(t^-)f_1(x(t^-), u(t^-)) = \lambda^T_f(t^+)f_F(x(t^+), u(t^+)) + \lambda^T_f(t^+) h^T_x(x(t^+)) z(t^+) - \lambda^T_h(t^+) h(x(t^+)).$$

The solution to these equations is (due to (H2)): $\pi = ((\lambda^T_f)^T f_1(-) - \lambda^T_f)^T f_F(+ - \lambda^T_f)^T h_x(z^+) + \lambda^T_h h(+))/h_x(+ f_1(-)$, $\lambda_f = \lambda_f(t^-) - \pi h_x(z^+)$. where $\lambda_f$, etc. is a a function evaluated at $t^+_t$ (according to our notation (2.8)), $\lambda_f$ at $t^-_t$, function $f_F(+$, etc. is as evaluated at $(x(t^+_t), u(t^+_t))$, function $f_1(-$, etc. at $(x(t^-_t), u(t^-_t))$.

Once we solve the adjoint equations we obtain the adjoint variables $\lambda_f$ and $\lambda_h$.

Eventually we are in a position to calculate the first variation of the cost function $\phi(t_f)$ with respect to a control function variation $d$ as follows

$$d\phi(x(t_f)) = d\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \langle F_0(u), d \rangle = -\int_{t_0}^{t_f} \lambda^T_f(t) f_1(x(t), u(t))d(t)dt - \int_{t^+_t}^{t_f} \lambda^T_f(t) f_F(x(t), u(t))d(t)dt.$$ 

**Case 2.** In the second case we assume that in the time interval $[t_0, t_t]$ the system evolves according to DAEs (17)-(18). At a transition time $t_t$ the continuous state trajectory leaves the switching surface with the condition $a(x(t_t), u(t_t)) = 0$ satisfied—see (10) (from now on we will use $\eta(x, u)$ instead of $-a(x, u)$). After the transition the system evolves according to the equation $x' = f_1(x, u)$ up to an ending time $t_f$ which we assume is a varying parameter.

To derive the adjoint equations we construct the following augmented functional

$$\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi(x(t_f)) + \pi \eta(x(t^-_t), u(t^-_t)) + \int_{t_0}^{t_f} \left[ \lambda^T_f(t) (x'(t) - f_F(x(t), u(t)) - h^T_x(x(t)) z(t)) + \lambda^T_h(t) h(x(t)) \right] dt + \int_{t^+_t}^{t_f} \lambda^T_f(t) (x'(t) - f_1(x(t), u(t))) dt.$$
We now calculate the variation of the augmented functional

\[ d\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi_x(x(t_f))dx(t_f) + \pi\eta_x(x(t^*_f), u(t^*_f))dx(t_i) + \pi\eta_u(x(t^*_f), u(t^*_f))du(t^*_f) + \lambda^T_f(t^*_f) \left( x'(t^*_f) - f_F(x(t^*_f), u(t^*_f)) - h^T_x(x(t^*_f))z(t^*_f) \right) dt + \lambda^T_h(t^*_f)h(x(t^*_f))dt_i + \]

\[
d\left[ \int_{t_0}^{t_f} \left[ \lambda^T_f(t) \left( x'(t) - f_F(x(t), u(t)) - h^T_x(x(t))z(t) \right) + \lambda^T_h(t)h(x(t)) \right] dt \right]
\]

By taking into account the fact that \( dx(t_f) = y^{x,d}(t_f) \) and by integrating by parts the formulas \( \int \lambda_f(t)x(t)dt \) we obtain

\[ d\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi_x(x(t_f))y^{x,d}(t_f) + \pi\eta_x(x(t^*_f), u(t^*_f))dx(t_i) + \pi\eta_u(x(t^*_f), u(t^*_f))du(t^*_f) + \lambda^T_f(t^*_f) \left( x'(t^*_f) - f_F(x(t^*_f), u(t^*_f)) - h^T_x(x(t^*_f))z(t^*_f) \right) dt_i + \lambda^T_h(t^*_f)h(x(t^*_f))dt_i + \]

\[
d\left[ \int_{t_0}^{t_f} \left[ (\lambda^T_f)'(t)x(t) + \lambda^T_f(t) \left( f_F(x(t), u(t)) + h^T_x(x(t))z(t) \right) \right] dt \right]
\]

Expanding further the variations and taking into account the initial conditions of the linearized equations we obtain

\[ d\Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \phi_x(x(t_f))y^{x,d}(t_f) + \pi\eta_x(x(t^*_f), u(t^*_f))dx(t_i) + \pi\eta_u(x(t^*_f), u(t^*_f))du(t^*_f) + \lambda^T_f(t^*_f) \left( x'(t^*_f) - f_F(x(t^*_f), u(t^*_f)) - h^T_x(x(t^*_f))z(t^*_f) \right) dt_i + \lambda^T_h(t^*_f)h(x(t^*_f))dt_i + \]

\[
\lambda^T_f(t^*_i)y^{x,d}(t^*_i) - \int_{t_0}^{t_i} \left[ (\lambda^T_f)'(t)y^{x,d}(t) + \lambda^T_f(t)(f_F)_x(x(t), u(t))y^{x,d}(t) \right] dt + \lambda^T_f(t)(f_F)_u(x(t), u(t))dt(t) + \lambda^T_f(t)h^T_x(x(t))y^{x,d}(t) + \lambda^T_f(t)(h^T_x(x(t))z(t))y^{x,d}(t)
\]

\[ - \lambda^T_f(t)h^T_x(x(t))y^{x,d}(t) \ dt - \lambda^T_f(t^*_i)x'(t^*_i)dt(t) + \lambda^T_f(t^*_i)h^T_x(x(t^*_i))z(t^*_i)dt_i + \lambda^T_h(t^*_i)h(x(t^*_i))dt_i + \lambda^T_f(t^*_i)(f_F)_u(x(t^*_i), u(t^*_i))dt_i + \lambda^T_f(t^*_i)(f_F)_x(x(t^*_i), u(t^*_i))y^{x,d}(t^*_i) - \lambda^T_f(t^*_i)y^{x,d}(t^*_i) \ dt + \lambda^T_f(t^*_i)h^T_x(x(t^*_i))y^{x,d}(t^*_i) \ dt + \lambda^T_f(t^*_i)((\lambda^T_f)'(t)y^{x,d}(t) + \lambda^T_f(t)(f_F)_x(x(t), u(t))y^{x,d}(t)) \ dt \]

Now we can utilize the formula for the differential \( dx(t_i) \), \( du(t_i) \) \((du(t^*_i) = d(t^*_i) + u(t^*_i)dt_i)\) and rearrange the components with respect to differentials \( dx(t_i), du(t_i), \)
To this end we want to zero the following components

\[
\begin{align*}
\int_{t_0}^{t_f} & \left( (\lambda^T_x)'(t) + \lambda^T_x(t)(f_F)_x(x(t), u(t)) + \\
& \lambda^T_x(t) h^T_x(x(t)) z(t) \right) dx(t) - \lambda^T_x(t_0) h^T_x(x(t_0)) z(t_0), \quad t \in [t_0, t_f] \\
\int_{t_0}^{t_f} & \left( (\lambda^T_x)'(t) + \lambda^T_x(t)(f_1)_x(x(t), u(t)) \right) dx(t) - \lambda^T_x(t_0) h^T_x(x(t_0)) z(t_0), \quad t \in [t_0, t_f].
\end{align*}
\]

We achieve that by assuming that

\[
\begin{align*}
(\lambda^T_x)'(t) &= -\lambda^T_x(t)(f_F)_x(x(t), u(t)) - \lambda^T_x(t) h^T_x(x(t)) z(t), \quad t \in [t_0, t_f] \\
0 &= \lambda^T_x(t) h^T_x(x(t)), \quad t \in [t_0, t_f]
\end{align*}
\]

and

\[(\lambda^T_x)'(t) = -\lambda^T_x(t)(f_1)_x(x(t), u(t)), \quad t \in [t_0, t_f].\]

with the terminal condition

\[
\lambda_f(t_f) = -\phi^T_x(x(t_f)).
\]

To calculate the consistent values of \(\lambda_f(t_0^-)\) and \(\lambda_h(t_0^-)\) the following system of equations have to be solved for the variables \(\lambda_f(t_0^-), \lambda_h(t_0^-), \pi, \nu_t \)

\[
\begin{align*}
\lambda_f(t_0^-) - \lambda_f(t_0^+) + \pi \eta_t(x(t_0^-), u(t_0^-)) &= \nu_t h^T_x(x(t_0^-)) \\
\lambda^T_f(t_0^-) f_F(x(t_0^-), u(t_0^-)) + \lambda^T_x(t_0^-) h^T_x(x(t_0^-)) z(t_0^-)
\end{align*}
\]

\[
\begin{align*}
-\lambda^T_x(t_0^-) h(x(t_0^-)) - \pi \eta_t(x(t_0^-), u(t_0^-)) u(t_0^-) = \lambda^T_x(t_0^+) f_1(x(t_0^+), u(t_0^+))
\end{align*}
\]
Suppose that we can formulate the weak maximum principle for the considered problem. Having the equation (4.15), that $h_x(x(t_i^-)) (f_{P}(t_i^-), u(t_i^-)) + h_x(x(t_i^-)) \frac{d}{dt} (x(t_i^-), u(t_i^-)) = 0$ to obtain (which is possible to (H2))

$$\pi = \lambda_f(t_i^+)^T \left( f_{P}(x(t_i^-), u(t_i^-)) + h_x(x(t_i^-)) \frac{d}{dt} (x(t_i^-), u(t_i^-)) \right) / \|h_x(x(t_i^-))\|^2.$$ 

Next, we multiply (4.14) by $h_x(x(t_i^-))$ and take into account (4.16) to obtain (which is possible to (H1))

$$\nu_i = h_x(x(t_i^-)) \left( \lambda_f(t_i^+)^T + \pi \eta_x(x(t_i^-), u(t_i^-)) \right) \left/ \|h_x(x(t_i^-))\|^2 \right.$$ 

Having $\pi$ and $\nu_i$ we can evaluate then

$$\lambda_f(t_i^-) = \lambda_f(t_i^+) - \pi \eta_x^T(x(t_i^-), u(t_i^-)) + \nu_i h_x^T(x(t_i^-))$$

and eventually

$$\lambda^T(t_i^-) = \lambda_f(t_i^+)^T \left( \left( f_{P}(x(t_i^-), u(t_i^-)) + h_x(x(t_i^-)) \frac{d}{dt} (x(t_i^-), u(t_i^-)) \right) h_x^T(x(t_i^-)) \right) \left/ \|h_x(x(t_i^-))\|^2 \right.$$ 

Once we solve the adjoint equations we obtain the adjoint variables $\lambda_f$ and $\lambda_h$. Now we can calculate the first variation of a cost function $\phi(t_f)$ with respect to a control function variation $d$ as follows

$$d \phi(x(t_f)) = d \Phi(x, z, u, \lambda_f, \lambda_h, \pi) = \pi \eta_u(x(t_i), u(t_i^-))d(t_i^-)$$

$$- \int_{t_i}^{t_f} \lambda^T(t)(f_{P})(u, t)dt - \int_{t_i}^{t_f} \lambda^T(t)(f_{I})(u, t)dt dt.$$ 

5. The weak maximum principle. On the basis of the defined adjoint equations we can formulate the weak maximum principle for the considered problem. Suppose that $\bar{u}$ is the problem solution. Then, as shown in Theorem 3.1, $\bar{u}$ will satisfy conditions (3.17)–(3.19) together with $q^i_1(x^i(t_j)) = 0, i \in E, q^j_2(x^j(t_j)) \leq 0, j \in I$. The weak maximum principle for the problem (P) can assume a quite complicated form depending on the number of switching points triggered by the optimal control $\bar{u}$. In order to exemplify the possible conditions stated by the weak maximum principle consider the case when optimal control $\bar{u}$ forces the system behavior denoted as Case 1) in Section 4. We call the necessary optimality conditions for that case (NC13).
(NC\textsuperscript{13}): There exist: nonnegative numbers $\alpha_j^2$, $j \in I$, numbers $\alpha_i^1$, $i \in E$ such that $\sum_{i \in E} |\alpha_i^1| + \sum_{j \in I} \alpha_j^2 \neq 0$; piecewise differentiable function $\lambda_f$; piecewise continuous function $\lambda_h$, such that the following hold:

(i) terminal conditions ($\lambda_f(t_f)$, $\lambda_h(t_f)$ and $\nu$ are evaluated from equations below)

\[-\lambda_f(t_f) = \phi^T_{\pi}(\bar{x}(t_f)) + \nu h^T_{\pi}(\bar{x}(t_f)) + \sum_{i \in E} \alpha_i^1 (g_i^1)_{\pi}^T(\bar{x}(t_f)) + \sum_{j \in I} \alpha_j^2 (g_j^2)_{\pi}^T(\bar{x}(t_f))\]

\[0 = h_{\pi}(\bar{x}(t_f)) \lambda_f(t_f)\]

\[0 = (h_{\pi}(\bar{x}(t_f)))' \lambda_f(t_f) - h_{\pi}(\bar{x}(t_f)) (f_{\pi})_{\pi}^T(\bar{x}(t_f), \bar{u}(t_f)) \lambda_f(t_f) - h_{\pi}(\bar{x}(t_f)) (h_{\pi}^T(\bar{x}(t_f)) \bar{z}(t_f))_{\pi}^T \lambda_f(t_f) + h_{\pi}(\bar{x}(t_f)) h_{\pi}^T(\bar{x}(t_f)) \lambda_h(t_f);\]

(ii) adjoint equations

for $t \in [t_i, t_f]$

\[\lambda'_f = - (f_{\pi})_{\pi}^T(\bar{x}, \bar{u}) \lambda_f - (h_{\pi}^T(\bar{x}) \bar{z})_{\pi}^T \lambda_f + h_{\pi}^T(\bar{x}) \lambda_h\]

\[0 = h_{\pi}(\bar{x}) \lambda_f\]

for $t \in [t_0, t_i]$

\[\lambda'_f = -(f_i)_{\pi}^T(\bar{x}, \bar{u}) \lambda_f;\]

(iii) jump conditions ($\lambda_f(t^-_f)$ and $\pi$ are evaluated from the equations below)

\[\lambda_f(t^-_f) = \lambda_f(t^+_f) - \pi h^T_{\pi}(\bar{x}(t^-_f))\]

\[\lambda_f^T(t^-_f) h_{\pi}(\bar{x}(t^-_f), \bar{u}(t^-_f)) = \lambda_f^T(t^+_f) h_{\pi}(\bar{x}(t^+_f), \bar{u}(t^+_f)) + \lambda_f^T(t^+_f) h_{\pi}^T(\bar{x}(t^+_f)) \bar{z}(t^+_f) - \lambda_h^T(t^+_f) h(\bar{x}(t^+_f));\]

(iv) the weak maximum principle

\[H^{13}(\bar{x}, \bar{u}, \lambda_f, t_i, u) \leq H^{13}(\bar{x}, \bar{u}, \lambda_f, t_i, \bar{u}), \forall \in U\]

where $H^{13}(\bar{x}, \bar{u}, \lambda_f, t_i, u) = \int_{t_0}^{t_i} \lambda_f^T(t) (f_1)_u (\bar{x}(t), \bar{u}(t)) \times u(t) dt + \int_{t_i}^{t_f} \lambda_f^T(t) (f_2)_u (\bar{x}(t), \bar{u}(t)) u(t) dt$.

(v) complementarity conditions

\[\alpha_j^2 = 0, \text{ if } j \notin I_0, \bar{u}.\]

The necessary optimality conditions (NC\textsuperscript{13}) are stated for the case when the optimal trajectory ($\bar{x}, \bar{z}$) corresponding to the optimal control $\bar{u}$ is such that to the switching point $t_i$ the system evolution is according to $f_1$, and then the movement takes on the sliding surface until $t_f$. For that case we can prove the theorem which is the restatement of part iii) of Theorem 3.1.

Theorem 5.1. Assume that data for (P) satisfies hypotheses (H1), (H2), (H3), (H4), (CC) and (CQ). Let $\{u_k\}$ be a sequence of controls generated by Algorithm and let $\{c_k\}$ be a sequence of the corresponding penalty parameters. Then

(iii) if $\bar{u}$ is the limit point of $\{u_k\}$, $\bar{x}$ its state corresponding trajectory and suppose that for $\bar{u}$ Case 1–3) applies, then the pair ($\bar{x}, \bar{u}$) satisfies the constraints and the conditions (NC\textsuperscript{13}).
Proof. As shown in the proof of Theorem 5.1 (Stage 2–Dualization) in [21] the conclusions of the proof of Theorem 3.1 can be expressed by

\[
\min_{d \in \mathcal{D}_a} \max_{\gamma \in \mathcal{K}} \Psi(d, \gamma) = 0
\]

where

\[
\mathcal{K} = \left\{ \gamma = (\alpha_0, \{\alpha_i^1\}_{i \in E}, \{\alpha_j^2\}_{j \in I}) \in \mathbb{R}^{1 + |E| + |I|} : \alpha_0 \geq 0, \alpha_j^2 \geq 0, j \in I, \alpha_0 + \sum_{i \in E} |\alpha_i^1| + \sum_{j \in I} \alpha_j^2 = 1, \alpha_j^2 = 0 \text{ if } j \notin I_{0,a} \right\}
\]

and

\[
\Psi(d, \gamma) := \alpha_0 \langle \nabla F_0(\bar{u}), d \rangle + c \left( \sum_{i \in E} \alpha_i^1 \langle \nabla g_i^1(\bar{u}), d \rangle + \sum_{j \in I_{0,a}} \alpha_j^2 \langle \nabla g_j^2(\bar{u}), d \rangle \right).
\]

\(\Psi(\cdot, \gamma)\) is a linear function on \(C_\nu^N[T]\) of which \(\mathcal{D}_a\) is a convex subset. \(\Psi(d, \cdot)\) is a bounded linear map and \(\mathcal{K}\) is a compact convex set with respect to the product topology of \(\mathbb{R}^{1 + |E| + |I|}\). It follows from the minimax theorem ([2]) that there exists some nonzero \(\bar{\gamma} \in \mathcal{K}\) such that

\[
(5.1) \quad \min_{d \in \mathcal{D}_a} \max_{\gamma \in \mathcal{K}} \Psi(d, \gamma) = \min_{d \in \mathcal{D}_a} \Psi(d, \bar{\gamma}) = 0,
\]

with \(\bar{\gamma} = (\bar{\alpha}_0, \{\bar{\alpha}_i^1\}_{i \in E}, \{\bar{\alpha}_j^2\}_{j \in I})\).

Since the constraint qualification (CQ) holds we can easily show that \(\bar{\alpha}_0 \neq 0\).

The adjoint equations have been derived for the functional \(\bar{F}_0(u)\), however similar analysis could be carried out for the functional

\[
H(u) = \bar{F}_0(u) + c \left( \sum_{i \in E} \alpha_i^1 \bar{g}_i^1(u) + \sum_{j \in I} \alpha_j^2 \bar{g}_j^2(u) \right),
\]

then by taking \(\alpha_i^1 = c\alpha_i^1, i \in E, \alpha_j^2 = c\alpha_j^2, j \in I\) (notice that \(c > 0\)).

The other cases than Case 1) can be analyzed, with respect to necessary optimality conditions, in a similar way. We do not enumerate these cases because they are numerous, e.g.: the system behaves first according to function \(f_2\), then, until the end it is in the sliding mode; the system starts in the sliding mode before moving to the mode with dynamics described by \(f_1\) or \(f_2\); there are several discrete modes the system assumes on the interval \([t_0, t_f]\).

In particular, in Case 2)—the system starts in the sliding mode and at the time \(t_i\) it switches to the discrete state \(q = 1\), the necessary optimality conditions will shape as follows (we assume that the switching function for leaving the sliding mode is given by \(\eta(x, u)\)).

\((\text{NC31})\): There exist: nonnegative numbers \(\alpha_j^2, j \in I\), numbers \(\alpha_i^1, i \in E\) such that \(\sum_{i \in E} |\alpha_i^1| + \sum_{j \in I} \alpha_j^2 \neq 0\); piecewise smooth function \(\lambda_f\); piecewise continuous function \(\lambda_h\), such that the following hold:

(i) terminal conditions

\[
-\lambda_f(t_f) = \phi_z^T(\bar{x}(t_f)) + \sum_{i \in E} \alpha_i^1 \left( g_i^1 \right)_x^T (\bar{x}(t_f)) + \sum_{j \in I} \alpha_j^2 \left( g_j^2 \right)_x^T (\bar{x}(t_f))
\]
\( \text{(ii) adjoint equations} \)

for \( t \in [t_e, t_f] \)

\[
\lambda_f^x = -(f_1)_{x}^T(\bar{x}, \bar{u})\lambda_f;
\]

for \( t \in [t_0, t_e] \)

\[
\lambda_f^x = -(f_F)_{x}^T(\bar{x}, \bar{u})\lambda_f - (h_{x}^T(\bar{x})z)_{x}^T\lambda_f + h_{x}^T(\bar{x})h_f
\]

\[
0 = h_x(\bar{x})\lambda_f
\]

\( \text{(iii) jump conditions} (\lambda_f(t^-_i), \nu_i \text{ and } \pi \text{ are evaluated from the equations below}) \)

\[
\lambda_f(t^-_i) - \lambda_f(t^+_i) + \pi_\eta^T(x(t_i), u(t_i)) = \nu_i h^T_x(x(t^-_i))
\]

\[
\lambda_f^x(t^+_i)F_f(x(t^+_i), u(t^+_i)) + \lambda^x_f(t^+_i)h^z_f(x(t^+_i))z(t^+_i)
\]

\[
\lambda_f(t^-_i) - \lambda_f(t^+_i)h(x(t^-_i)) = \lambda_f^x(t^+_i)f_1(x(t^+_i), u(t^+_i))
\]

and

\[
0 = h_x(x(t^-_i))\lambda_f(t^-_i)
\]

\[
0 = (h_x(x(t^-_i)))^T\lambda_f(t^-_i) - h_x(x(t^-_i))(f_F)_{x}^T(x(t^-_i), u(t^-_i))\lambda_f(t^-_i) - h_x(x(t^-_i))(h^z_f(x(t^-_i))z(t^-_i))_{x}^T\lambda_f(t^-_i) + h_x(x(t^-_i))h^z_f(x(t^-_i))\lambda_h(t^-_i).
\]

\( \text{(iv) the weak maximum principle} \)

\[
H^{31}(\bar{x}, \bar{u}, \lambda_f, t_e, \pi, u) \leq H^{31}(\bar{x}, \bar{u}, \lambda_f, t_e, \pi, \bar{u}), \ \forall \in U
\]

where \( H^{31}(\bar{x}, \bar{u}, \lambda_f, t_e, \pi, u) = \int_{t_0}^{t_e} \lambda_f^x(t) (f_F)_{x}^T(\bar{x}(t), \bar{u}(t))u(t)dt + \int_{t_e}^{t_f} \lambda_f^x(t) (f_1)_{x}^T(\bar{x}(t), \bar{u}(t))u(t)dt - \pi_\eta u(\bar{x}(t^-_i), \bar{u}(t^-_i))u(t^-_i). \)

\( \text{(v) complementarity conditions} \)

\[
\alpha_j^2 = 0, \text{ if } j \notin I_{0, \bar{u}}.
\]

6. Hybrid systems without sliding motion. The process of finding optimal solutions to control problems with hybrid systems significantly simplifies when trajectories generated during the process do not contain sections with sliding modes. In this case we assume that our state trajectory on the entire horizon \([t_0, t_f]\) will consist of trajectories of type of type \(x\) only, i.e. we have

\[
[t_0, t_f] = \cup_{i \in I_1} A_1^1 \cup_{i \in I_2} A_2^2.
\]

For the simplicity of presentation consider the case when the system on time subinterval \([t_0, t_e]\) evolves according to equations

\[
x' = f_1(x, u),
\]

and then, on the subinterval \([t_e, t_f]\), according to

\[
x' = f_2(x, u),
\]
Due to our previous remarks we do not have to restrict the class of admissible control functions to piecewise smooth functions, and we can take as $\mathcal{U}$ the set

$$\mathcal{U} = \{ u \in L^2_m[\mathcal{I}] : u(t) \in U, \text{ a.e. on } \mathcal{T}\},$$

(6.3)

For this set of admissible controls, and due to our assumption that sliding mode does not occur, the relations (2.24)–(2.27) hold for $x$, $x^d$ and $y^{x,d}$ on the subintervals $[t_0, t_i]$ and $[t_i, t_f]$.

The adjoint equations for the equations (6.1)–(6.2) can be derived in a way described in Section 4. To this end we define the functional

$$\Phi(x, z, u, \lambda_f, \pi) = \phi(x(t_f)) + \pi h(x(t_i)) +$$

$$\int_{t_0}^{t_i} \lambda_f^T(t) (x'(t) - f_1(x(t), u(t))) dt + \int_{t_i}^{t_f} \lambda_f^T(t) (x'(t) - f_2(x(t), u(t))) dt,$$

and after applying arguments as in Section 4, we arrive at the equations for adjoint variable $\lambda_f$:

$$\lambda_f^T(t) = -\lambda_f^T(t) (f_1)_{x}(x(t), u(t)), \quad t \in [t_0, t_i]$$

(6.4)

and

$$\lambda_f^T(t) = \lambda_f^T(t) (f_2)_{x}(x(t), u(t)), \quad t \in (t_i, t_f].$$

(6.5)

The transversality condition for the equations (6.4)–(6.5) is

$$\phi^T_\pi (x(t_f)) + \lambda_f(t_f) = 0$$

(6.6)

and the jump conditions at the switching time $t_i$ are

$$\lambda_f(t_i^-) = \lambda_f(t_i^+) - \pi h^T_\pi(x(t_i))$$

$$\lambda_f^T(t_i^-) f_1(x(t_i^-), u(t_i^-)) = \lambda_f^T(t_i^+) f_2(x(t_i^+), u(t_i^+)).$$

(6.7)

Then the first variation of the cost function $\phi(t_f)$ with respect to a control function variation $d$ is follows

$$d\phi(x(t_f)) = d\Phi(x, z, u, \lambda_f, \pi) =$$

$$- \int_{t_0}^{t_i} \lambda_f^T(t) (f_1)_{u}(x(t), u(t)) d(t) dt - \int_{t_i}^{t_f} \lambda_f^T(t) (f_2)_{u}(x(t), u(t)) d(t) dt.$$

Having adjoint equations for the simplest possible case of control problems which hybrid systems do not exhibit sliding modes we can formulate necessary optimality conditions for the case—(NC12).

(\text{NC12}) There exist: nonnegative numbers $\alpha_j^2$, $j \in I$, numbers $\alpha_i^1$, $i \in E$ such that $\sum_{i \in E} |\alpha_i^1| + \sum_{j \in I} \alpha_j^2 \neq 0$; number $\pi$; absolutely continuous function $\lambda_f$, such that the following hold:

(i) terminal conditions

$$\lambda_f(t_f) = -\phi^T_\pi (x(t_f))$$
(ii) adjoint equations
for \( t \in (t_i, t_f) \)
\[
\lambda_f^T = - (f_2)^T_x (\bar{x}, \bar{u}) \lambda_f
\]
for \( t \in [t_0, t_i] \)
\[
\lambda_f^T = - (f_1)^T_x (\bar{x}, \bar{u}) \lambda_f;
\]
(iii) jump conditions \( (\lambda_f(t^-_i) \) and \( \pi \) are the solutions to the algebraic equations stated below)
\[
\lambda_f(t^-_i) = \lambda_f(t^+_i) - \pi h^T_x (\bar{x}(t^-_i))
\]
\[
\lambda^T_f(t^-_i) f_1 (\bar{x}(t^-_i), \bar{u}(t^-_i)) = \lambda^T_f(t^+_i) 2_f (\bar{x}(t^+_i), \bar{u}(t^+_i))
\]
(iv) the weak maximum principle
for \( t \in (t_i, t_f) \)
\[
\lambda^T_f(t) (f_2)_u (\bar{x}(t), \bar{u}(t))u \leq \lambda^T_f(t) (f_2)_u (\bar{x}(t), \bar{u}(t))\bar{u}(t)
\]
for \( t \in [t_0, t_i] \)
\[
\lambda^T_f(t) (f_1)_u (\bar{x}(t), \bar{u}(t))u \leq \lambda^T_f(t) (f_1)_u (\bar{x}(t), \bar{u}(t))\bar{u}(t)
\]
for all \( u \in U \);
(v) complementarity conditions
\[
\alpha^2_j = 0, \text{ if } j \notin I_{0, \bar{u}}.
\]

Since we elaborate on the case of optimal control problems with hybrid systems which do not involve sliding modes we have to introduce the assumption which excludes the possibility of system going into the sliding mode. The assumption concerns switching points \( t_i \) and is postulated instead of \( (H2) \).

\( (H2') \) Function \( h(\cdot) \) is differentiable and there exist \( 0 < L_1 < +\infty \) and \( 0 < L_2 < +\infty \) such that
\[
\| h_x (\bar{x}) - h_x (x) \| \leq L_1 \| \bar{x} - x \|,
\]
\[
| h_x (\bar{x}) f_i (\bar{x}, \bar{u}) - h_x (x) f_i (x, u) | \leq L_2 \| (\bar{x}, \bar{u}) - (x, u) \|,
\]
for all \( (x, u), (\bar{x}, \bar{u}) \) in \( \mathbb{R}^n \times U \).

For any \( u \in U \) and any switching point \( t_i \) the following limits exist
\[
\lim_{t \rightarrow t^-_i, t < t_i} u(t), \lim_{t \rightarrow t^+_i, t > t_i} u(t)
\]
(and are denoted by \( u(t^-_i) \) and \( u(t^+_i) \) respectively).

There exists \( \varepsilon > 0 \) and \( 0 < L_3 < +\infty \) such that for all switching points \( t_i \) (each switching time corresponds to some \( u \in U \)) and for all their perturbations \( t_i^d \) triggered by perturbations \( d \) such that \( u + d \in U, \| d \|_{\mathbb{L}^2} \leq \varepsilon \), and for all \( \theta \in [0, 1] \) we have
\[
h_x (x(\tau^{0,d}(t_i))) f_i (x(\tau^{0,d}(t_i)), u(\tau^{0,d}(t_i))) \geq L_3,
\]
or
\[
h_x (x(\tau^{0,d}(t_i))) f_i (x(\tau^{0,d}(t_i)), u(\tau^{0,d}(t_i))) \leq -L_3,
\]
for \( i = 1, 2 \), where
\[
\tau^{0,d}(t_i) = t_i + \theta(t_i^d - t_i).
\]
The hypothesis \((H_2')\) which replaces \((H_2)\) does not request the condition \((2.42)\) for perturbations \(d\). This is the consequence of having the switching function \(\eta = h\) which does not depend on controls. That simplification spreads through the definitions which concern perturbations \(d\), for example, the definitions of \(D_u\) and \(D\), which should be updated in order to work with the broader sets of perturbations, do not contain the requirement \((2.42)\). Theorem 6.1 can be proved along the lines of the proof of Theorem 3.1, Theorem 5.1 and with sets \(D_u, D\) modified in this way.

**Theorem 6.1.** Assume that data for \((P)\) satisfies hypotheses \((H_1), (H_2'), (H_3), (H_4)\) and \((CQ)\). Let \(\{u_k\}\) be a sequence of controls generated by Algorithm and let \(\{c_k\}\) be a sequence of the corresponding penalty parameters. Then

i) \(\{c_k\}\) is a bounded sequence

ii) \(\lim_{k \to \infty} M(u_k) = 0\).

iii) if \(\bar{u}\) is the limit point of \(\{u_k\}\), \(\bar{x}\) its state corresponding trajectory and suppose that for \(\bar{u}\) the case of this section applies, then the pair \((\bar{x}, \bar{u})\) satisfies the necessary optimality conditions \((NC_{12})\).

**Proof.** Following analysis applied in the proof of Theorem 5.1 we can show that if \(\bar{u}\) is a local minimizer to the problem \((P)\) and the assumptions of the theorem hold then

\[
H^{12}(\bar{x}, \bar{u}, \lambda_f, t_t, u) \leq H^{12}(\bar{x}, \bar{u}, \lambda_f, t_t, \bar{u}), \quad \forall \in U
\]

where

\[
H^{12}(\bar{x}, \bar{u}, \lambda_f, t_t, u) = \int_{t_0}^{t_t} \lambda_f^T(t) (f_1) \bar{x}(t) \bar{u}(t) \bar{u}(t) dt + \int_{t_0}^{t_t} \lambda_f^T(t) (f_2) \bar{x}(t) \bar{u}(t) u(t) dt.
\]

(6.16)

Referring to the mean value theorem for integrals and taking into account the definition of the admissible set of controls \((6.3)\) (the set \(U\) is bounded) we come to the conditions \((6.8)-(6.9)\)

7. **Conclusions.** The paper presents the computational approach to hybrid optimal control problems with sliding modes. It seems to be the first method for optimal control problems with hybrid systems which can exhibit sliding modes. We show that under the assumption that controls are represented by piecewise constant functions the presented method is globally convergent in the sense that every accumulation point of a sequence generated by the method satisfies necessary optimality conditions in the form of the weak maximum principle. In the accompanying papers ([18],[19]) we present the implementable version of our computational method by relaxing the condition that continuous state trajectories are available. In these papers we assume that system equations are integrated by an implicit Runge–Kutta method and we show that the global convergence of the implementable version of our algorithm is preserved provided that the integration procedure step sizes converge to zero.

The algorithms presented in the paper are first order methods since they are based on linearizations of functions defining our optimal control problem. We can provide second order versions of these algorithms by applying an SQP approach in
which second order terms in the functions expansions are placed in a quadratic term of the objective function of \( P_c(u) \)—then instead of using the term \( 1/2\|d\|_{L^2}^2 \) we use \( 1/2d^T Hd \). \( H \) represents the approximation to the Hessian of the Lagrange function of the problem. Of course the second order approach can be applied if the optimal control problem can be stated as the problem in the Euclidean space, for example when control functions are piecewise constant functions ([16]). The convergence analysis presented in this paper can be used to prove global convergence of the second order method provided that matrices \( H \) used at iterations are uniformly positive definite and bounded.

**Appendix A. Proof of Lemma 3.1.**

Proof. Take any \( u \in U \). Let \( r > 0 \) be a number such that \( M(u) < r \)
for all \( u \in U \). We deduce from (CQ) that there is a simplex in \( E(u) \subset \mathbb{R}^{n_E} \) \((n_E = |E|)\) with vertices \( \{e_j\}_{j=0}^{n_E} \) which contains 0 as an interior point. By definition of \( E(u) \), there exist \( d_0, \ldots, d_{n_E} \in D \) and \( \delta > 0 \) such that for \( j = 0, \ldots, n_E \)
\[
\{\langle \nabla \bar{g}_i^1(u), d_j \rangle \}_{i \in E} = e_j, \quad \max_{i \in I} \langle \nabla \bar{g}_i^2(u), d_j \rangle \leq -\delta.
\]

Let \( (\lambda_0, \lambda_1, \ldots, \lambda_{n_E}) \) be the barycentric coordinates of 0 w.r.t. the vertices \( e_j \) of the simplex, i.e.
\[
0 = \sum_{j=0}^{n_E} \lambda_j e_j = \nabla \bar{g}^1(u) \circ \sum_{j=0}^{n_E} \lambda_j d_j.
\]
Here
\[
\nabla \bar{g}^1(u) \circ d \ := \ \{\langle \nabla \bar{g}_i^1(u), d \rangle \}_{i \in E}.
\]
We shall also write
\[
\bar{g}^1(u) \ := \ \{\bar{g}_i^1(u) \}_{i \in E}.
\]

Since the vertices are in general position and 0 is an interior point, the \( \lambda_i \)'s are all positive and we may find \( \delta_1 > 0 \) such that
\[
\left( \lambda_0 - \sum_{j=1}^{n_E} \alpha_j, \lambda_1 + \alpha_1, \ldots, \lambda_{n_E} + \alpha_{n_E} \right) \\
\in \left\{ \gamma \in \mathbb{R}^{n_E+1} : \ \gamma_j \geq 0 \ \forall j, \ \sum_{j=0}^{n_E} \gamma_j = 1 \right\}
\]
whenever \( \alpha \in B(0, \delta_1) \subset \mathbb{R}^{n_E} \). \( B(0, \delta_1) \) is a ball with radius \( \delta_1 \). Furthermore, the \( n_E \times n_E \) matrix \( P(u) \) defined by
\[
(A.1) \quad P(u)\alpha := \sum_{j=1}^{n_E} \nabla \bar{g}_i^1(u) \circ \alpha_j (d_j - d_0)
\]
is invertible for \( u = \hat{u} \), from the definition of \( d_j \), \( j = 1, \ldots, n_E \).

In consequence of hypothesis (CQ) and in view of the continuity properties of the mapping \( u \rightarrow y^{x.d} \) for fixed \( d \) we may choose a neighborhood \( \mathcal{O}(\hat{u}) \) of \( \hat{u} \) in \( U \) and numbers \( \hat{r} \geq r \) and \( \delta_2 \in (0, \hat{r}^{-1}] \) such that for any \( u \in \mathcal{U} \) satisfying \( u \in \mathcal{O}(\hat{u}) \)

\[
\begin{align*}
(i) & \quad \max_{i \in I} \langle \nabla \tilde{g}^2(u), v_j - u \rangle \leq -\delta/2 \quad \forall j \\
(ii) & \quad P(u) \text{ is invertible} \\
(iii) & \quad \left\| P(u)^{-1} \nabla \tilde{g}^l(u) \circ \left( \sum_{j=0}^{n_E} \lambda_j v_j \right) - u \right\| \leq \delta_1/2 \\
(iv) & \quad \delta_2 \| P(u)^{-1} \| n_E^{1/2} \leq \delta_1/2.
\end{align*}
\]

(In (iv) the norm is the Frobenius norm. Here the controls \( v_j \in U, j = 0, \ldots, n_E \) are defined to be

\[ v_j(t) := u(t) + d_j(t, u(t)). \]

Now suppose that the control \( u \) is not feasible. Set

\[ \alpha = P(u)^{-1} \left[ -\nabla \tilde{g}^3(u) \circ \left( \sum_{j=0}^{n_E} \lambda_j v_j \right) - u - \delta_2 M(u)^{-1} \tilde{g}^l(u) \right]. \]

(A.2)

Notice that, by properties (iii) and (iv), \( \| \alpha \| \leq \delta_1 \). Also set

\[ \hat{v} = v_0 + \sum_{j=1}^{n_E} (\lambda_j + \alpha_j)(v_j - v_0). \]

Because \( \| \alpha \| \leq \delta_1 \) we have that \( \hat{v} \in U \). Finally we define \( v \) to be

\[ v = u + (M(u)/\hat{r}) (\hat{v} - u). \]

Since \( M(u)/\hat{r} \leq 1 \), it follows that \( v \in U \). We now verify that this control function has the required properties.

Notice first that

(A.3) \[ \| v - u \|_{L^\infty} \leq (2d/\hat{r}) M(u), \]

where \( d \) is a bound on the norms of elements in \( U \).

We have from (A.1) and (A.2) that

\[
\begin{align*}
P(u)\alpha & = \nabla \tilde{g}^1(u) \circ \sum_{j=1}^{n_E} \alpha_j (v_j - v_0) \\
& = -\nabla \tilde{g}^1(u) \circ \left( \sum_{j=1}^{n_E} \lambda_j (v_j - v_0) + v_0 - u \right) - \delta_2 M(u)^{-1} \tilde{g}^l(u).
\end{align*}
\]

By definition of \( \hat{v} \),

\[ \nabla \tilde{g}^1(u) \circ (\hat{v} - u) = -\delta_2 M(u)^{-1} \tilde{g}^l(u). \]
But then
\[ \nabla g^1(u) \circ (v - u) = -(\delta_2/\hat{r})\bar{g}^1(u). \]
Since \( \delta_2/\hat{r} \leq 1 \), it follows that
\[ (A.4) \quad \max_{i \in E} |\bar{g}^1_i(u) + \langle \nabla \bar{g}^1_i(u), v - u \rangle| - \max_{i \in E} |\bar{g}^1_i(u)| \leq -(\delta_2/\hat{r})M(u). \]

We deduce from property (i) that
\[ \left\langle \nabla \bar{g}^2_j(u), v_0 + \sum_{i=1}^{n_E} (\lambda_i + \alpha_i) (v_i - v_0) - u \right\rangle \leq -\delta/2, \quad \forall j \in I. \]
It follows that
\[ \left\langle \nabla \bar{g}^2_j(u), v - u \right\rangle \leq (M(u)) (-\delta/2) \quad \forall j \in I. \]
Since \( M(u)/\hat{r} \leq 1 \) we deduce that
\[ \min [0, \bar{g}^2_j(u)] + \left\langle \nabla \bar{g}^2_j(u), v - u \right\rangle \leq -(\delta/2\hat{r})M(u) \quad \forall j \in I. \]
(A.5)

Surveying inequalities (A.3)–(A.5), we see that \( v \) satisfies all relevant conditions for completion of the proof, when we set \( K_1 = \min\{\delta_2, \delta/(2\hat{r})\} \) and \( K_2 = 2d/\hat{r} \), numbers whose magnitudes do not depend on our choice of \( u \).

Appendix B. Exact penalty function \((P_c)\).

**Theorem B.1.** Assume that data for \((P)\) satisfies hypotheses \((H1), (H2), (H3), (H4)\) and \((CC)\). Suppose that \( \bar{u} \) is a strict local minimum of the problem \((P)\) (which means that \( \bar{u} \) is feasible with respect to all constraints, in particular \( \bar{u} \in \mathcal{U} \)) and that at the point \( \bar{u} \) constraint qualification \((CQ)\) holds. Then there exists an \( \bar{c} > 0 \) such that for all \( c \geq \bar{c}, \bar{u} \) is a local minimum of \((P_c)\) on \( \mathcal{U} \).

**Proof.** For the simplicity of presentation we assume that \( I = \emptyset \). The proof goes along the lines of the proof of Theorem 4.4 in [10].

Suppose that \( \bar{u} \) is the strict local minimum of the problem \((P)\) on the set \( \mathcal{N}(\bar{u}, \bar{\varepsilon}) \cap \mathcal{U}, \bar{\varepsilon} > 0 \). According to Theorem in [14] for sufficiently large values of \( c \) there exist \( \varepsilon(c) > 0 \) and \( u(c) \) such that \( u(c) \) is a local minimum of the problem \((P_c)\) on the set \( \mathcal{N}(\bar{u}, \varepsilon(c)) \cap \mathcal{U} \). Moreover it follows from the theorem that \( \varepsilon(c) \to c \to \infty 0 \) so we can choose \( c \) sufficiently large so that we will have \( \varepsilon(c) \leq \bar{\varepsilon} \). Suppose that \( u(c) \) is feasible with respect to the constraints \( \bar{g}^1_i(u) = 0, i \in E \). Then we will have
\[ \hat{F}_c(u(c)) = \hat{F}_0(u(c)) \leq \hat{F}_c(\bar{u}), \]
which means that we must have \( u(c) = \bar{u} \) since \( \bar{u} \) is a strict local minimizer on the set \( \mathcal{N}(\bar{u}, \varepsilon(c)) \cap \mathcal{U} \) so it is the only minimizer in the set. This means \( \bar{u} \) is the local minimizer for the problem \((P_c)\) on the set \( \mathcal{N}(\bar{u}, \varepsilon(c)) \cap \mathcal{U} \).

Therefore, we have to show that for sufficiently large values of \( c \) vectors \( u(c) \) is feasible with respect to the constraints \( \bar{g}^1_i(u) = 0, i \in E \). Suppose that it is not true, thus for any \( c \to \infty \) there exists an index \( i_c \in E \) such that \( \bar{g}^1_{i_c}(u(c)) \neq 0 \). Since at \( \bar{u} \) \((CQ)\) holds there exists a neighborhood \( \mathcal{N}(\bar{u}, \varepsilon_1), \varepsilon_1 > 0 \) such that for any \( u \in \mathcal{N}(\bar{u}, \varepsilon_1) \) there exists \( v \in \mathcal{U} \) and \( K_1 > 0 \) with the property
\[ \left\langle \nabla \bar{g}^1_i(u), v - u \right\rangle = -K_1\bar{g}^1_i(u)/M(u), \quad i \in E. \]
Suppose that $\tilde{i}_e(u) \in E$ is such that $|\tilde{g}_i^1(u)| = M(u)$ (notice that we have assumed that $I = \emptyset$). Then we have

$$\langle \nabla \tilde{g}_i^1(u), v - u \rangle = \begin{cases} -K_1 & \text{if } i = \tilde{i}_e(u) \text{ and } \tilde{g}_i^1(u) = M(u) \\ K_1 & \text{if } i = \tilde{i}_e(u) \text{ and } \tilde{g}_i^1(u) = -M(u) \\ -K_1|\tilde{g}_i^1(u)|/M(u) & \text{if } \tilde{g}_i^1(u) \geq 0 \\ K_1\tilde{g}_i^1(u)/M(u) & \text{if } \tilde{g}_i^1(u) < 0 \end{cases}.$$

For sufficiently large $c u(c) \in \mathcal{N}(\bar{u}, \varepsilon_1)$ and $u(c)$ is not feasible with respect to constraints $\tilde{g}_i^1(u) = 0$, $i \in E$. From Lemma 3.1 for any $c$ there exists $v \in U$ such that the directional derivative of $\bar{F}_c(u)$ in the direction $v - u$ can be evaluated according to the formula (Danskin’s formula, see, for example, Theorem 10.22 in [6])

$$D\bar{F}_c(u; v - u) = \langle \nabla \bar{F}_0(u), v - u \rangle + c \max \left[ \max \limits_{i \in I_+(u)} \langle \nabla \tilde{g}_i^1(u), v - u \rangle, \right. \left. \vphantom{\max} \right. \max \limits_{i \in I_-(u)} \left[ -\langle \nabla \tilde{g}_i^1(u), v - u \rangle \right], \right.$$

where

$$I(u) = \{i \in E : |\tilde{g}_i^1(u)| = M(u)\}$$

$$I_+(u) = \{i \in I(u) : \tilde{g}_i^1(u) = M(u)\}$$

$$I_-(u) = \{i \in I(u) : \tilde{g}_i^1(u) = -M(u)\}$$

(notice that $I_+(u) \cap I_-(u) = \emptyset$ since $M(u) > 0$).

From (B.1) it follows that

$$D\bar{F}_c(u; v - u) = \langle \nabla \bar{F}_0(u), v - u \rangle - K_1c \leq \|\bar{F}_0(u)\|\|v - u\| - K_1c.$$  

Furthermore, for sufficiently large values of $c$ we have that $u(c) \in \mathcal{N}(\bar{u}, \varepsilon_1)$ and at the same time, from (B.1),

$$D\bar{F}_c(u(c); v - u(c)) < 0$$

and, since $v \in U$, (B.2) contradicts the assumption that $u(c)$ is a local minimizer of the problem $(\text{P}_c)$. It means that for sufficiently large values of $c u(c)$ is feasible with respect to the constraints $\tilde{g}_i^1(u) = 0$, $i \in E$. \hfill \Box

Using parts of the proof of Theorem B.1 we are able to prove the following theorem.

**Theorem B.2.** Assume that data for $(\text{P})$ satisfies hypotheses $(\text{H1})$, $(\text{H2})$, $(\text{H3})$, $(\text{H4})$, $(\text{CC})$ and $(\text{CQ})$ hold. Then there exists an $\tilde{c} > 0$ such that if $\bar{u}$ is a strict local minimum point of the problem $(\text{P}_c)$ then $\bar{u}$ is also a strict local minimum point of the problem $(\text{P})$.

**Proof.** In the proof of Theorem B.1 we showed that under constraint qualification, if $\bar{u}$ is a strict local minimum point of the problem $(\text{P}_c)$ it is also a feasible point as far as constraints of the problem $(\text{P})$ are concerned. It means that we have

$$\bar{F}_0(\bar{u}) = \bar{F}_c(\bar{u}) < \bar{F}_c(u), \quad \forall u \in \mathcal{N}(\bar{u}, \varepsilon(c)) \cap U, \quad u \neq \bar{u}, \quad c \geq \tilde{c},$$

and
which implies that
\[(B.3) \quad \bar{F}_0(\bar{u}) = \bar{F}_c(\bar{u}) < \bar{F}_c(u) = \bar{F}_0(u), \quad \forall u \in \mathcal{N}(\bar{u}, \varepsilon(c)) \cap \mathcal{U} \cap \mathcal{G}, \quad u \neq \bar{u}, \quad c \geq \check{c},\]
where \(\mathcal{G} = \{ u \in C^\infty_{\text{ad}}(T) : \bar{g}_1^i(u) = 0, \ i \in E, \ \bar{g}_2^j(u) = 0, \ j \in I \} \). But \((B.3)\) states that \(\bar{u}\) is a strict local minimum for the problem \((\mathcal{P})\).
\[\square\]

**Appendix C. Proof of Proposition 3.1.**

**Proof.** (i) Fix \(u \in \mathcal{U}\), we must find \(\hat{c} > 0\) such that, if \(c > \hat{c}\) then \(t_c(u) \leq 0\). If \(M(u) = 0\) then of course \(t_c(u) \leq 0\) for any \(c > 0\). If \(M(u) > 0\) then according to \(Lemma 3.1\) there exists \(\bar{d} \in \mathcal{U} - u\) such that, if we set \(\varepsilon = K_1 M(u) > 0\) with \(K_1\) as in \(Lemma 3.1\), then
\[
\max_{i \in I} \left[ \left| \bar{g}_1^i(u) \right| + \left| \nabla \bar{g}_1^i(u), d \right| - M(u) \right] \leq \max_{i \in E} \left[ \left| \bar{g}_1^i(u) \right| + \left| \nabla \bar{g}_1^i(u), d \right| - \left| \bar{g}_1^i(u) \right| \right] \leq -\varepsilon,
\]
and
\[
\max_{j \in I} \left[ \left| \bar{g}_2^j(u) \right| + \left| \nabla \bar{g}_2^j(u), d \right| - M(u) \right] \leq \max_{j \in I} \left[ \left| \bar{g}_2^j(u) \right| + \left| \nabla \bar{g}_2^j(u), d \right| - \bar{g}_2^j(u) \right] \leq -\varepsilon
\]
and
\[
\theta(u) < -\varepsilon.
\]
Here
\[
\theta(u) = \max \left[ \max_{i \in E} \left| \bar{g}_1^i(u) \right| + \left| \nabla \bar{g}_1^i(u), d \right| - M(u), \right]
\]
\[
\max_{j \in I} \left[ \left| \bar{g}_2^j(u) \right| + \left| \nabla \bar{g}_2^j(u), d \right| - M(u) \right].
\]
Because \(\sigma_c(u) = \left( \nabla \bar{F}_0(u), d \right) + c \left[ \bar{\beta} - M(u) \right] \), from the definition of \(t_c\) and \((2.24) - (2.26)\), we get
\[
t_c(u) \leq W + c \theta(u) + M(u)/c,
\]
where \(W = \max \left[ 0, \left( \nabla \bar{F}_0(u), d \right) \right] \). It follows that \(t_c(u) \leq 0\) for any \(c > \hat{c}\) where
\[
\hat{c} := \max \left[ 1, \frac{W + M(u)}{-\theta(u)} \right].
\]

(ii) Take \(u \in \mathcal{U}\) and \(c > 0\) such that \(\sigma_c(u) < 0\). Let \((d, \beta)\) be the solution to \(P_c(u)\). Since \(\sigma_c(u) \neq 0\), it follows that \(d \neq 0\).

We deduce from the differentiability properties of \(\phi, g_1^1, g_2^2\), and \((2.27)\) that there exists \(o : [0, \infty) \rightarrow [0, \infty)\) such that \(s^{-1} o(s) \rightarrow 0\) as \(s \downarrow 0\) and the following three inequalities are valid for any \(\alpha \in [0, 1]\):

\[
\bar{F}_c(u + \alpha d) - \bar{F}_c(u) \leq \alpha \left( \nabla \bar{F}_0(u), d \right) + c \max_{\alpha \in E} \left[ \left| \bar{g}_1^i(u) \right| + \alpha \left( \nabla \bar{g}_1^i(u), d \right) \right],
\]
\[
(C.1) \quad \max_{j \in I} \left( \bar{g}_2^j(u) + \alpha \left( \nabla \bar{g}_2^j(u), d \right) \right) - \max_{\alpha \in E} \left[ \left| \bar{g}_1^i(u) \right|, \max_{\alpha \in I} \bar{g}_2^j(u) \right] + o(\alpha).
\]

By convexity of the functions \(e \rightarrow \max_{\alpha \in E} \left| \bar{g}_1^i(u) + \alpha \left( \nabla \bar{g}_1^i(u), e \right) \right|, e \rightarrow \max_{\alpha \in I} \left( \bar{g}_2^j(u) + \alpha \left( \nabla \bar{g}_2^j(u), e \right) \right)\):

\[
\max_{\alpha \in E} \left( \bar{g}_1^i(u) + \alpha \left( \nabla \bar{g}_1^i(u), d \right) \right) - M(u_k) \leq \alpha \left[ \max_{\alpha \in E} \left| \bar{g}_1^i(u) + \alpha \left( \nabla \bar{g}_1^i(u), d \right) \right| - M(u_k) \right],
\]
\[
\max_{j \in I} \left( \bar{g}_2^j(u) + \alpha \left( \nabla \bar{g}_2^j(u), d \right) \right) - M(u_k) \leq \alpha \left[ \max_{j \in I} \left( \bar{g}_2^j(u) + \alpha \left( \nabla \bar{g}_2^j(u), d \right) \right) - M(u_k) \right].
\]
From inequality (C.1) then
\[ F_c(u + ad) - F_c(u) \leq \alpha \left\{ \langle \nabla F_0(u), d \rangle + c \max_{i \in E} \max_{j} |\nabla g_i^j(u)| + \langle \nabla g_i^j(u), d \rangle \right\}, \]
\[
\max_{j \in I} \left( \bar{g}_j^2(u) + \langle \nabla \bar{g}_j^2(u), d \rangle \right) - cM(u) \right\} + o(\alpha) \leq \alpha \sigma_c(u) + o(\alpha).
\]

It follows that
\[ (C.2) \quad F_c(u + ad) - F_c(u) \leq \alpha \gamma \sigma_c(u) \quad \forall \alpha \in [0, \alpha_1], \]
where \( \alpha_1 > 0 \) is such that \( o(\beta) \leq \beta(\gamma - 1)\sigma_c(u) \) for all \( \beta \in [0, \alpha_1] \).

**Appendix D. Proof of Theorem 3.1.**

Proof. (i) Let \( \{u_k\} \) be the sequence generated by Algorithm and let \( \{c_k\} \) be the corresponding penalty parameters. Let \( \{k_l\} \) be the sequence of index values at which the penalty parameter increases. By extracting a further subsequence (we do not relabel) we can arrange that the sequence \( \{u_{k_l}\} \) has a limit point \( \bar{u} \in U \) because \( U \) is compact. We shall find a number \( \hat{c} < \infty \) such that for sufficiently large \( k_l \), \( c_{k_l} > \hat{c} \) implies \( \sigma_{c_{k_l}}(u_{k_l}) \leq -M(u_{k_l})/c_{k_l} \). This contradicts our assumption that the penalty parameter increases along the subsequence. So we may conclude that \( \{c_k\} \) is bounded.

Fix \( k_l \) such that \( u_{k_l} \in O(\bar{u}) \), where \( O(\bar{u}) \) is the neighborhood of \( \bar{u} \) as specified in Lemma 3.1. From the minimizing property of \( \sigma_{c_{k_l}}(u_{k_l}) \) we deduce
\[
\sigma_{c_{k_l}}(u_{k_l}) \leq \langle \nabla F_0(u_{k_l}), v_{k_l} - u_{k_l} \rangle + c_{k_l} \max_{i \in E} \max_{j} |\nabla g_i^j(u_{k_l}) + \langle \nabla g_i^j(u_{k_l}), v_{k_l} - u_{k_l} \rangle| - M(u_{k_l}),
\]
\[ (D.1) \quad \max_{j \in I} \left( \bar{g}_j^2(u_{k_l}) + \langle \nabla \bar{g}_j^2(u_{k_l}), v_{k_l} - u_{k_l} \rangle \right) - M(u_{k_l}) \]
for a control function \( v_{k_l} \) satisfying conditions (3.21)–(3.23) of Lemma 3.1 in which \( v_{k_l} \), \( u_{k_l} \) replace \( v \), \( u \) respectively. It follows
\[
\sigma_{c_{k_l}}(u_{k_l}) \leq |\langle \nabla F_0(u_{k_l}), v_{k_l} - u_{k_l} \rangle| - c_{k_l} K_1 M(u_{k_l}).
\]
Since the control constraint \( U \) is bounded and in view of (2.24)–(2.26) there exists \( r > 0 \) (independent of \( k_l \)) such that
\[
|\langle \nabla F_0(u_{k_l}), v_{k_l} - u_{k_l} \rangle| \leq r\|v_{k_l} - u_{k_l}\|_{L^\infty}.
\]
Hence
\[
\sigma_{c_{k_l}}(u_{k_l}) \leq -(c_{k_l} K_1 - r K_2) M(u_{k_l}).
\]
We conclude that
\[
\sigma_{c_{k_l}}(u_{k_l}) \leq -M(u_{k_l})/c_{k_l}
\]
if \( c_{k_l} \geq \hat{c} \), where \( \hat{c} \) is the smallest positive number \( c \) which satisfies the equation:
\[
c^2 K_1 - c(r K_2) \geq 1.
\]

(ii) and (iii) Let \( \{u_k\} \) be an infinite sequence generated by Algorithm. We must show that \( \lim_{k \to \infty} \sigma_{c_{k_l}}(u_{k_l}) = 0 \) and, if a convergent subsequence of \( \{u_k\} \) has a limit point \( \bar{u} \in U \), that conditions (NC\(^1\)) are satisfied at \( \bar{u} \).
Stage 1. (Convergence analysis) Since the $c_k$’s are bounded and can increase only by multiples of $e^0$, we must have $c_k = c$ for all $k \geq k_0$, for some $k_0$ and $c > 0$. In view of the manner in which $u_k$’s are constructed, we have

$$\bar{F}_c(u_{k+1}) - \bar{F}_c(u_k) \leq \gamma \alpha_k \sigma_c(u_k)$$

for all $k \geq k_0$. This means that, for all $j \geq 1$, $k \geq k_0$

$$\bar{F}_c(u_{k+j}) - \bar{F}_c(u_k) \leq \gamma \sum_{i=0}^{j-1} \alpha_{k+i} \sigma_c(u_{k+i}).$$

(D.2)

Since $\{\bar{F}_c(u_k)\}$ is a bounded sequence and $\alpha_k \sigma_c(u_k)$ are nonpositive, we conclude

$$\alpha_k \sigma_c(u_k) \to 0 \text{ as } k \to \infty.$$ (D.3)

Since $\sigma_c(u)$ is bounded as $u$ ranges over $\mathcal{U}$ we can arrange by a subsequence extraction (we do not relabel) that

$$\sigma_c(u_k) \to \sigma \text{ for some } \sigma \leq 0.$$

We claim that $\sigma = 0$. To show this, suppose to the contrary that $\sigma < 0$. Then by (D.3) $\alpha_k \to 0$.

We must have

$$\bar{F}_c(u_k + \eta^{-1} \alpha_k d_k) - \bar{F}_c(u_k) > \gamma \eta^{-1} \alpha_k \sigma_c(u_k)$$

for all $k$ sufficiently large.

However $\|\alpha_k d_k\|_{L^\infty} \to 0$ as $k \to \infty$ (it follows from the presence of term $1/2\|d\|^2_{L^2}$ in the objective function of the direction finding subproblem).

It follows now from (2.24)–(2.27) and Propositions 2.1.5–2.1.6 stated in [16] that there exists a function $o : [0, \infty) \to [0, \infty)$ such that $s^{-1}o(s) \to 0$ as $s \downarrow 0$ and

$$\max_{i \in I} |\bar{g}_i^1(u_k + \eta^{-1} \alpha_k d_k)| \leq \max_{i \in I} |\bar{g}_i^1(u_k) + \langle \nabla \bar{g}_i^1(u_k), \eta^{-1} \alpha_k d_k \rangle| + o(\eta^{-1} \alpha_k \|d_k\|_{L^\infty}),$$

(D.4)

$$\max_{j \in I} \bar{g}_j^2(u_k + \eta^{-1} \alpha_k d_k) \leq \max_{j \in I} [\bar{g}_j^2(u_k) + \langle \nabla \bar{g}_j^2(u_k), \eta^{-1} \alpha_k d_k \rangle] + o(\eta^{-1} \alpha_k \|d_k\|_{L^\infty}),$$

(D.5)

and

$$\bar{F}_c(u_k + \eta^{-1} \alpha_k d_k) - \bar{F}_c(u_k) \leq \langle \nabla \bar{F}_0(u_k), \eta^{-1} \alpha_k d_k \rangle + c \max_{i \in E} |\bar{g}_i^1(u_k) + \langle \nabla \bar{g}_i^1(u_k), \eta^{-1} \alpha_k d_k \rangle| - M(u_k),$$

(D.6)

$$\max_{j \in I} \bar{g}_j^2(u_k) + \langle \nabla \bar{g}_j^2(u_k), \eta^{-1} \alpha_k d_k \rangle - M(u_k) + o(\eta^{-1} \alpha_k \|d_k\|_{L^\infty}).$$

By convexity of the functions $e \to \max_{i \in E} |\bar{g}_i^1(u) + \langle \nabla \bar{g}_i^1(u), e \rangle|$, $e \to \max_{j \in I} (\bar{g}_j^2(u)$
+ ⟨∇g_j^2(u), e⟩:

\[
\max_{i \in I} g_i^1(u_k) + \langle \nabla g_i^1(u_k), \eta^{-1} \alpha_k d_k \rangle - M(u_k)
\]

(D.7)

\[
\leq \eta^{-1} \alpha_k \left( \max_{i \in E} g_i^1(u_k) + \langle \nabla g_i^1(u_k), d_k \rangle - M(u_k) \right),
\]

\[
\max_{j \in I} g_j^2(u_k) + \langle \nabla g_j^2(u_k), \eta^{-1} \alpha_k d_k \rangle - M(u_k)
\]

(D.8)

\[
\leq \eta^{-1} \alpha_k \left( \max_{j \in I} g_j^2(u_k) + \langle \nabla g_j^2(u_k), d_k \rangle - M(u_k) \right),
\]

Combining inequalities (D.4)–(D.8), noting the definition of \( \sigma_c(u_k) \) and the fact that \( (d_k, \beta_k) \) solves \( P_c(u_k) \), and dividing across the resulting inequality by \( \alpha_k \) we arrive at

\[
\eta^{-1} \sigma_c(u_k) + \alpha_k^{-1} \sigma \| d_k \|_{L^\infty} \geq \eta^{-1} \gamma \sigma_c(u_k).
\]

We get \( \eta^{-1} \sigma \geq \eta^{-1} \gamma \sigma \) in the limit. But this implies \( \gamma \geq 1 \), since \( \sigma < 0 \) by assumption. From this contradiction we conclude the validity of \( \sigma = 0 \). Assertion (ii) of the theorem follows from the definition of \( t_k \) and part (i).

Let \( \{u_k\} \) be a convergent subsequence with the limit point \( \bar{u} \in U \). We must show that conditions (NC\textsuperscript{13}) are satisfied at \( \bar{u} \). First we establish that \( \bar{u} \) is feasible for \( P \) and, for some \( c > 0 \),

\[
0 \leq \left[ \langle \nabla F_0(\bar{u}), d \rangle + c \max_{i \in E} \frac{\max_{j \in I} \langle \nabla g_j^1(\bar{u}), d \rangle, \max_{j \in I} \langle \nabla g_j^2(\bar{u}), d \rangle}{} \right]
\]

for all \( d \in D_{\bar{u}} \).

Since \( \{u_k\} \to \bar{u} \) we know that \( x_k = x(u_k) \to \bar{x} = \bar{u} \) uniformly. Because the penalty parameter is not updated for \( k \) sufficiently large we have

\[
\sigma_c(u_k) \leq -M(u_k)/c.
\]

But we have shown that \( \sigma_c(u_k) \to 0 \) as \( k \to \infty \). It follows now from the fact that \( u_k \to \bar{u} \in U \) that in the limit

\[
\max_{i \in E} g_i^1(\bar{u}) = 0, \quad \max_{j \in I} g_j^2(\bar{u}) \leq 0.
\]

We have established that \( \bar{u} \) is feasible for \( P \).

Since \( \sigma_c(u_k) \to 0 \) and \( M(u_k) \to 0 \) we must also have \( d_k \to 0 \). If it does not happen then we would come to the contradiction with (3.15). That implies that we also must have \( \beta_k \to 0 \) and that means that at \( \bar{u} \) (3.18)–(3.19) must hold.

Now choose any sequence \( \rho_k \downarrow 0, \rho_k \leq 1 \) for all \( k \) such that

\[
\rho_k^{-1} \sigma_c(u_k) \to 0 \quad \text{as} \quad k \to \infty.
\]

Choose any \( d \in D_{\bar{u}} \). By the convexity of \( U \), \( \rho_k d_k \in D_{u_k} \) for each \( k \).

By definition of \( \sigma_c \) then

\[
\sigma_c(u_k) \leq \phi_x(x(u_k(t_f)))y^{x_k, \rho_k d_k}(t_f) +
\]

\[
c \max_{i \in E} \left[ g_i^1(x(u_k(t_f)) + (g_i^1)_x(x(u_k(t_f)))y^{x_k, \rho_k d_k}(t_f) - M(u_k),
\]

\[
\max_{j \in I} \left[ g_j^2(x(u_k(t_f)) + (g_j^2)_x(x(u_k(t_f)))y^{x_k, \rho_k d_k}(t_f) - M(u_k) \right].
\]

(D.11)
Fix $\hat{\varepsilon} > 0$. Since $\rho_k \downarrow 0$ (and consequently $y^{x_k, \rho_k d_k} \to 0$ uniformly), we have:
\[
\begin{align*}
\max_{j \in I} \left[ g_j^2(x^{u_k}(t_f)) + (g_j^2)_x(x^{u_k}(t_f)) y^{x_k, \rho_k d_k}(t_f) \right] \\
\leq \max_{I_{x, u}} \left[ g_j^2(x^{u_k}(t_f)) + (g_j^2)_x(x^{u_k}(t_f)) y^{x_k, \rho_k d_k}(t_f) \right],
\end{align*}
\]
for all $k$ sufficiently large. Here, $I_{x, u} = \{ i \in I : \, \bar{g}_i^2(u) \geq \max_{j \in I} \{ \bar{g}_j^2(u) \} - \varepsilon \}$. Inserting these inequalities into (D.11), noting that $y^{x_k, \rho_k d_k} = \rho_k y^{x_k, d_k}$, dividing across by $\rho_k$ and passing to the limit with the help of (D.10) and continuity of $F_0(\cdot)$, $u \to (\nabla F_0(u), d)$, etc, we obtain
\[
0 \leq \phi_{x}(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) + c \max_{i \in E} \left[ (g_i^1)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) \right],
\]
(D.12)
\[
\max_{j \in I_{x, u}} (g_j^2)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f).
\]
This inequality is valid for each $\hat{\varepsilon} > 0$ and $d \in D_u$.

Again choose arbitrary $d \in D_u$ and take $\varepsilon_k \downarrow 0$. For each $k$ let $(g_j^2)_x(x^{\bar{u}}(t_f))$. $y^{\bar{x}, d}(t_f)$ achieves its maximum over $I_{x_k, \bar{u}}$ at $j = j_k$. Then
\[
0 \leq \phi_{x}(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) + c \max_{i \in E} \left[ (g_i^1)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) \right],
\]
(D.13)
\[
(g_j^2)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f).
\]
Extract a subsequence (we do not relabel) such that $j_k = \bar{j}$ for all $k$ and $t_k \to \bar{t}$ for some index value $\bar{j}$. By continuity of the functions involved $\bar{j} \in I_{\bar{u}, \bar{u}}$ and (D.13) is valid with $\bar{j}$.

We have arrived at
\[
0 \leq \phi_{x}(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) + c \max_{i \in E} \left[ (g_i^1)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f) \right],
\]
(D.14)
\[
\max_{j \in I_{\bar{u}, \bar{u}}} (g_j^2)_x(x^{\bar{u}}(t_f)) y^{\bar{x}, d}(t_f).
\]
Since $\sigma_{c_k}(u_k) \to 0$, from (3.15) we must also have $d_k \to 0$. $\{c_k\}$ is bounded and $M(u_k) \to 0$ then (3.15) also implies that $\beta_k \to 0$. Therefore, the inequality (D.14), which holds for all $d \in D_u$, in particular for $(d \equiv 0) \in D_u$, is what we set out to prove.

Finally, we must attend to the case when Algorithm generates a finite sequence which terminates at a control $u_k = \bar{u}$ satisfying the stopping criterion. This case is dealt with by applying the preceding arguments to the infinite sequence of controls obtained by ‘filling in’ with repetitions of the following control $u_k$.

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