MULTI-TERM FRACTIONAL LINEAR EQUATIONS
MODELING OXYGEN SUBDIFFUSION THROUGH CAPILLARIES

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Abstract. For $0 < \nu_2 < \nu_1 \leq 1$, we analyze a linear integro-differential equation on the space-time cylinder $\Omega \times (0, T)$ in the unknown $u(x, t)$

$$D^{\nu_1}_t(\rho_1 u) - D^{\nu_2}_t(\rho_2 u) - L_1 u - K \ast L_2 u = f$$

where $D^{\nu}_t$ are the Caputo fractional derivatives, $\rho_i = \rho_i(x, t)$ with $\rho_i \geq \mu_0 > 0$, $L_i$ are uniform elliptic operators with time-dependent smooth coefficients, $K$ is a summable convolution kernel, and $f$ is an external force. Particular cases of this equation are the recently proposed advanced models of oxygen transport through capillaries. Under suitable conditions on the given data, the global classical solvability of the associated initial-boundary value problems is addressed. To this end, a special technique is needed, adapting the concept of a regularizer from the theory of parabolic equations. This allows us to remove the usual assumption about the nonnegativity of the kernel representing fractional derivatives. The problem is also investigated from the numerical point of view.

1. Introduction

Oxygen transport is a complex phenomenon including chemical reactions with hemoglobin, convective transport in red blood cells, diffusion and metabolic consumption [38, 39]. Convective oxygen in blood depends on active energy consuming processes generating flow in the circulation. Diffusion transport refers to the passive movement of oxygen down its concentration gradient across tissue barriers, including the alveolar-capillary membrane, and across the extracellular matrix between the tissue capillaries and individual cells to mitochondria. The amount of diffusive oxygen movement depends on the oxygen tension gradient and the diffusion distance, which is related to the tissue capillary density. The greater is the difference between capillary and cellular oxygen concentration and the shorter is the distance, the faster is the rate of diffusion [29, 50]. In abnormal body circulation, cells closer to the capillary at the venous end begin to suffer from hypoxia when perfusion levels drop to critically low values [6, 50]. The mechanisms controlling oxygen distribution involving a series of convective and diffusive processes are not yet completely understood [39].

There are actually some methods to measure the oxygen level, such as two-photon phosphorescence lifetime microscopy, that can be applied in vivo [30]. However, the existing techniques can hardly offer a complete spatial-temporal picture of the oxygen field on microscopic scales. Thus, analytic studying/theoretical modeling [32, 37, 38, 39] and numerical simulation [33, 42] are widely utilized in evaluating oxygen level and angiogenesis research. The classical Krogh cylinder model roughly describes the oxygen transport from blood vessels to tissues [27]. In particular, Krogh proposed that oxygen is transported in the tissue by passive diffusion driven by gradients of oxygen tension, and gave a simple geometrical model of the elementary tissue unit supplied by a single capillary. Coupled models for oxygen delivery, even in presence of a relatively complex vessel network structure, and a detailed description of the blood flow in the vessel network were also proposed in [7, 39]. Go [6] used a mathematical model for oxygen delivery through capillaries where the longitudinal diffusion of solute is neglected, and the diffusion and the consumption rate of oxygen are assumed to be the same everywhere, which is not the case in real situations [29]. The further paper [46] introduces a new advanced mathematical model for oxygen delivery through a capillary to tissue in both (transverse and longitudinal) directions. In this work, conveying...
oxygen from the capillary to the surrounding tissue is described by means of a subdiffusion equation containing two fractional derivatives in time, that is,
\[
D_t^\nu \mathbf{C} - \tau D_t^{\nu_2} \mathbf{C} = \operatorname{div}(\varrho \nabla \mathbf{C}) - k,
\]
with \(0 < \nu_2 < \nu_1 \leq 1\). The equation can also exhibit extra terms accounting for the presence of external forces, even in convolution form [37]. Here, \(\mathbf{C}\) is a function of space and time, representing the concentration of oxygen, \(\tau\) is the time lag in concentration of oxygen along the capillary, \(k\) is the rate of consumption per volume of tissue, and \(\varrho\) is the diffusion coefficient of oxygen, which possibly dependent on \(\mathbf{C}\). In particular, the term \(D_t^{\nu_1} \mathbf{C} - \tau D_t^{\nu_2} \mathbf{C}\) details the net diffusion of oxygen to all tissues. In the equation, the symbol \(D_t^{\theta}\) stands for the usual Caputo fractional derivative of order \(\theta \in (0, 1)\) with respect to time, defined as
\[
D_t^{\theta} h(t) = \frac{1}{\Gamma(1-\theta)} \frac{d}{dt} \int_0^t \frac{h(s) - h(0)}{(t-s)^\theta} ds = \frac{1}{\Gamma(1-\theta)} \int_0^t (t-s)^{-\theta} \frac{dh}{ds}(s) ds,
\]
where \(\Gamma\) is the Euler Gamma function, and the latter equality holds if \(h\) is an absolutely continuous function. In the limit cases \(\theta = 0\) and \(\theta = 1\), the Caputo fractional derivatives of \(h(t)\) boil down to \([h(t) - h(0)]\) and \(\frac{d}{dt} h(t)\), respectively.

In this paper, motivated by the discussion above, we focus on the analytical and the numerical study of initial-boundary value problems for evolution equations with multi-term fractional derivatives. Let \(\Omega \subset \mathbb{R}^n\), with \(n \geq 1\), be a bounded domain with smooth boundary \(\partial \Omega\), and let \(T > 0\) be an arbitrarily fixed final time. We denote
\[
\Omega_T = \Omega \times (0, T) \quad \text{and} \quad \partial \Omega_T = \partial \Omega \times [0, T].
\]
For \(0 < \nu_2 < \nu_1 \leq 1\), we consider the following non-autonomous multi-term subdiffusion equation with memory terms in the unknown function \(u = u(x,t) : \Omega_T \to \mathbb{R}\),
\[
D_t^{\nu_1}(\varrho_1 u) - D_t^{\nu_2}(\varrho_2 u) - \mathcal{L}_1 u - \mathcal{K} \ast \mathcal{L}_2 u = f,
\]
where the symbol \(\ast\) stands for the usual time-convolution product
\[
(\mathbf{h}_1 \ast \mathbf{h}_2)(t) = \int_0^t \mathbf{h}_1(t-s) \mathbf{h}_2(s) ds.
\]
Here, \(\varrho_i = \varrho_i(x,t)\) and \(f = f(x,t)\) are given functions, \(\mathcal{K}\) is a summable convolution kernel, and \(\mathcal{L}_i\) are linear elliptic operators of the second order with time-dependent coefficients, whose precise form will be given in Section 3 where we detail the general assumptions of our problem. The equation is supplemented with the initial condition
\[
u_1 \in \mathbb{R},
\]
and subject to the one of the following boundary conditions on \(\partial \Omega_T\):

(i) Dirichlet boundary condition (DBC)
\[
u_1 \in \mathbb{R},
\]
(ii) Boundary condition of the third kind (3BC)
\[
u_1 \in \mathbb{R},
\]
(iii) Fractional dynamic boundary condition (FDDBC)
\[
u_1 \in \mathbb{R},
\]
The functions \(\psi_i = \psi_i(x,t)\) are prescribed, as well as the summable memory kernel \(\mathcal{K}_0\), while \(\mathcal{M}_i\) are first-order differential operators, whose precise form, again, will be given in Section 3. It is then apparent that the aforementioned advanced models of oxygen transport through capillaries are just particular cases of our problem.

For last few decades, initial and initial-boundary value problems governed by subdiffusion with and without memory terms (i.e., [1,11] with \(\varrho_1 = 1\) and \(\varrho_2 = 0\)) have been extensively studied via various
approaches of contemporary analysis, such as the qualitative theory of differential equations and numerical calculus. With no claim of completeness, we recall a number of published results. Existence, uniqueness, regularity, lifetime behavior of mild, weak and strong solutions to linear and nonlinear initial-boundary value problems subject to Dirichlet or Neumann boundary conditions for evolution equations with single-term fractional derivatives in time were discussed in [18] and references therein. The $L_p$-theory for linear and semilinear subdiffusion equations was analyzed in [16] [51] [52], whereas for the solvability of the corresponding problems in smooth classes, we refer to [14] [20] [22] [25] [26] [32]. Concerning the mathematical treatment of fractional dynamic boundary conditions (with $K_0 = 0$, $q_1 = 1$, $q_2 = 0$), global and local solvability, regularity of solutions to linear and semilinear elliptic and parabolic operators were discussed in [50] [17] [19] [25] [17]. The physical interpretation of boundary conditions of this kind can be found in [9] [17].

Evolution equations containing the general integro-differential operator

$$\frac{\partial}{\partial t}(\mathcal{N} * u)(\cdot, t),$$

where $\mathcal{N}(t)$ is a nonnegative kernel, are studied in the papers [10] [41]. A particular case of this operator is the multi-term fractional derivative in time

$$\sum_{i=1}^{M} q_i D_{t}^{\nu_i} u,$$

with $0 < \nu_M < \ldots < \nu_1 < 1$ and $q_i \geq 0$. The Cauchy problem for a general diffusion equation on unbounded domains was discussed in detail in [18]. Existence and uniqueness along with a maximal principle for initial-boundary value problems were studied in [21] [21] [32] [35] [36]. Optimal decay estimates for equations on bounded domains and subject to the homogeneous Dirichlet boundary condition were examined in [40], which shows in particular that the decay pattern (e.g., exponential, algebraic or logarithmic) depends on the (positive) kernel $\mathcal{N}$. An initial value problem for a semilinear differential equation with a fractional operator of the form (1.6) was examined in [43], where local/global existence and uniqueness of solutions were established by exploiting the Schauder fixed point theorem. Finally, we quote [13] [37] [16] [53], where certain explicit and numerical solutions were constructed to the corresponding initial-boundary value problems to evolution equations with multi-term fractional derivatives with $q_i > 0$.

Coming to equation (1.1) and related problems, we point out two main differences with respect to the previous literature. The first is related to the presence of Caputo fractional derivatives of the product of two functions, that is, $q_1 u$ and $q_2 u$. Incidentally, we recall that the well-known Leibniz rule does not work in the case of fractional Caputo derivatives. The second difference is that the fractional derivatives appearing in (1.1), under certain assumptions on $q_1$ and $q_2$, can be represented in the form (1.6), but with a negative kernel. Indeed, [11] Lemma 4] tells us that, if $0 < \nu_2 < \nu_1 < 1$,

$$\frac{t^{-\nu_1}}{\Gamma(1-\nu_1)} - \frac{t^{-\nu_2}}{\Gamma(1-\nu_2)} < 0 \quad \text{whenever} \quad t > e^{-\gamma},$$

$\gamma$ being the Euler-Mascheroni constant, which in turn provides the relation

$$D_{t}^{\nu_1}(q_1 u) - D_{t}^{\nu_2}(q_2 u) = \frac{\partial}{\partial t}(\mathcal{N} * u)$$

for the kernel

$$\mathcal{N}(t) = q_1 \frac{t^{-\nu_1}}{\Gamma(1-\nu_1)} - q_2 \frac{t^{-\nu_2}}{\Gamma(1-\nu_2)},$$

which is negative for $t > e^{-\gamma}$ and $0 < q_1(x) \leq q_2(x)$, with $q_i$ time-independent. In fact, the nonnegativity of the kernel $\mathcal{N}$ is a key assumption in the previous works which is removed in our investigation.

The main goal of the present paper is the proof of the well-posedness and the regularity of a global classical solution to problems (1.1)-(1.5) in smooth classes for any fixed time $T$, without the assumption on the sign of the function $q_2 = q_2(x, t)$. This will be obtained by adapting the technique of a regularizer for parabolic equations [28] to the subdiffusion equation, so to establish the one-valued global classical solvability of (1.1)-(1.5). Our analysis is complemented by numerical simulations. It is also worth
observing that, once the linear case is fully understood, it is then possible to tackle the global classical solvability of boundary-value problems for nonlinear extensions of (1.1). This will be possibly the subject of future investigations.

**Outline of the paper.** In the next Section 2 we introduce the functional spaces and notations. The general assumptions are presented in Section 3. The main Theorem 4.1 is stated in Section 4. Section 5 is devoted to some auxiliary results concerning the properties of solutions to subdiffusion equations, which will play a key role in the investigation. In Section 6 we provide the proof of Theorem 4.1, combining some ideas from [28] with a priori estimates of the solutions. In the final Section 7 we study the equation from the numerical side.

2. **Functional Spaces and Notation**

Throughout this work, the symbol \( C \) will denote a generic positive constant, depending only on the structural quantities of the problem. We will carry out our analysis in the framework of the fractional Hölder spaces. To this end, in what follows we take two arbitrary (but fixed) parameters

\[
\alpha \in (0, 1) \quad \text{and} \quad \nu \in (0, 1].
\]

For any non-negative integer \( l \), any Banach space \((X, \| \cdot \|_X)\), and any \( p \geq 1 \) and \( s \geq 0 \), we consider the usual spaces

\[
C^s([0, T], X), \quad C^{l+\alpha}(\bar{\Omega}), \quad W^{l,p}(\Omega), \quad L_p(\Omega).
\]

Denoting for \( \beta \in (0, 1) \)

\[
\langle v \rangle^\beta_{x,\Omega_T} = \sup \left\{ \frac{|v(x_1, t) - v(x_2, t)|}{|x_1 - x_2|^\beta} : x_2 \neq x_1, \ x_1, x_2 \in \bar{\Omega}, \ t \in [0, T] \right\},
\]

\[
\langle v \rangle^\beta_{t,\Omega_T} = \sup \left\{ \frac{|v(x, t_1) - v(x, t_2)|}{|t_1 - t_2|^\beta} : t_2 \neq t_1, \ x \in \Omega, \ t_1, t_2 \in [0, T] \right\},
\]

we have the following definitions.

**Definition 2.1.** A function \( v = v(x, t) \) belongs to the class \( C^{l+\alpha, \frac{l+\alpha}{2} \nu}(\bar{\Omega}_T) \), for \( l = 0, 1, 2 \), if the function \( v \) together with its corresponding derivatives are continuous and the norms here below are finite:

\[
\|v\|_{C^{l+\alpha, \frac{l+\alpha}{2} \nu}(\bar{\Omega}_T)} = \|v\|_{C([0, T], C^{l+\alpha}(\bar{\Omega}))} + \sum_{|j|=0}^l \langle D^j_x v \rangle_{x,\Omega_T}^{(l+\alpha-|j|)\nu}, \ l = 0, 1,
\]

\[
\|v\|_{C^{2+\alpha, \frac{2+\alpha}{2} \nu}(\bar{\Omega}_T)} = \|v\|_{C([0, T], C^{2+\alpha}(\bar{\Omega}))} + \|D^2_t v\|_{C^{\alpha, \frac{\alpha}{2} \nu}(\bar{\Omega}_T)} + \sum_{|j|=1}^2 \langle D_x^j v \rangle_{x,\Omega_T}^{(2+\alpha-|j|)\nu}.
\]

In a similar way, for \( l = 0, 1, 2 \), we introduce the space \( C^{l+\alpha, \frac{l+\alpha}{2} \nu}(\partial \Omega_T) \). The properties of these spaces have been discussed in [22] Section 2. It is worth noting that, in the limiting case \( \nu = 1 \), the class \( C^{l+\alpha, \frac{l+\alpha}{2}} \) coincides with the usual parabolic Hölder space \( H^{l+\alpha, \frac{l+\alpha}{2}} \) (see e.g. [28], (1.10)-(1.12)).

**Definition 2.2.** For \( l = 0, 1, 2 \), we define \( C^{l+\alpha, \frac{l+\alpha}{2} \nu}(\bar{\Omega}_T) \) to be the space consisting of those functions \( v \in C^{l+\alpha, \frac{l+\alpha}{2} \nu}(\bar{\Omega}_T) \) satisfying the zero initial conditions:

\[
v|_{t=0} = 0 \quad \text{and} \quad D^m_{t...D^m_{t}} v|_{t=0} = 0, \ m = 0, \ldots, \left\lfloor \frac{l}{2} \right\rfloor,
\]

where \( \lfloor \cdot \rfloor \) denotes the floor function.

In a similar manner we introduce the space \( C^{l+\alpha, \frac{l+\alpha}{2} \nu} (\partial \Omega_T) \).
3. General Assumptions

We begin to state our general hypothesis on the structural terms appearing in the equation and in the boundary conditions.

**H1. Conditions on the fractional order of the derivatives:** We assume that

\[ \nu_1 \in (0, 1) \quad \text{and} \quad \nu_2 \in \begin{cases} (0, \frac{\nu_1(2-\alpha)}{2}) & \text{if either DBC or 3BC hold}, \\ (0, \frac{\nu_1(1-\alpha)}{2}) & \text{if FDBC holds}. \end{cases} \]

**H2. Conditions on the operators:** The operators appearing in (1.1), (1.4) and (1.5) read

\[
\mathcal{L}_1 = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t) \frac{\partial}{\partial x_i} + a_0(x,t), \\
\mathcal{L}_2 = \sum_{i,j=1}^{n} b_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + b_0(x,t),
\]

and

\[
\mathcal{M}_1 = \sum_{i=1}^{n} c_i(x,t) \frac{\partial}{\partial x_i} + c_0(x,t), \\
\mathcal{M}_2 = \sum_{i=1}^{n} d_i(x,t) \frac{\partial}{\partial x_i} + d_0(x,t).
\]

There exist constants \( \mu_2 > \mu_1 > 0, \mu_3 > 0 \) and \( \mu_0 > 0 \) such that

\[ \mu_1 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i \xi_j \leq \mu_2 |\xi|^2, \]

for any \((x,t,\xi) \in \Omega_T \times \mathbb{R}^n\);

\[ g_1(x,t) \geq \mu_0 > 0, \]

for any \((x,t) \in \Omega_T; \)

\[ \sum_{i=1}^{n} c_i(x,t)N_i(x) \geq \mu_3 > 0, \]

for any \((x,t) \in \partial \Omega_T, \) where \( N = \{N_1(x), ..., N_n(x)\} \) is the unit outward normal vector to \( \Omega. \)

**H3. Conditions on the coefficients:** For \( i,j = 1, ..., n, \)

\[ a_{ij}(x,t), a_i(x,t), a_0(x,t), b_{ij}(x,t), b_i(x,t), b_0(x,t) \in C^{0, \frac{\nu_1(1+\alpha)}{\alpha}}(\Omega_T), \]

and

\[ c_i(x,t), c_0(x,t), d_i(x,t), d_0(x,t) \in C^{1+\alpha, \frac{\nu_1+\alpha}{\alpha}}(\partial \Omega_T). \]

We assume that

\[
\begin{align*}
\mathcal{G}_1 & \in \begin{cases} C^{\nu_1}([0, T], C^1(\bar{\Omega})) & \text{in the DBC and 3BC cases,} \\
C^{\nu_2}([0, T], C^1(\bar{\Omega})) \cap C^{\gamma_3}([0, T], C^2(\partial \Omega)) & \text{in the FDBC case,} \end{cases} \\
\mathcal{G}_2 & \in \begin{cases} C^{\nu_2}([0, T], C^1(\bar{\Omega})) & \text{in the DBC and 3BC cases,} \\
C^{\gamma_1}([0, T], C^1(\bar{\Omega})) \cap C^{\gamma_4}([0, T], C^2(\partial \Omega)) & \text{in the FDBC case,} \end{cases}
\end{align*}
\]

where

\[ \gamma_0 > \frac{\nu_1(2+\alpha)}{2}, \quad \gamma_1 > \frac{\nu_2(2+\alpha)}{2}, \quad \gamma_3 > \frac{\nu_1(3+\alpha)}{2}, \quad \gamma_4 > \frac{\nu_2(3+\alpha)}{2}. \]

Besides, if \( \gamma_0 \) or/and \( \gamma_1 < 1; \) and \( \gamma_3 \) or/and \( \gamma_4 < 1, \) then we additionally require that \( D_t^{\gamma_0} \mathcal{G}_1 \) or/and \( D_t^{\gamma_1} \mathcal{G}_2 \in C^{0, \frac{\nu_1+\alpha}{\alpha}}(\Omega_T); \) and \( D_t^{\gamma_3} \mathcal{G}_1 \) or/and \( D_t^{\gamma_4} \mathcal{G}_2 \in C^{1+\alpha, \frac{\nu_1+\alpha}{\alpha}}(\partial \Omega_T) \cap C^{\gamma_1, \frac{\nu_1+\alpha}{\alpha}}(\Omega_T), \) respectively.
H4. Conditions on the given functions:
\[ \mathcal{K}(t), \mathcal{K}_0(t) \in L_1(0, T), \]
\[ u_0(x) \in C^{2+\alpha} (\Omega), \quad f(x, t) \in C^{\alpha, \frac{\alpha}{2}} (\Omega_T), \]
\[ \psi_1(x, t) \in C^{2+\alpha, \frac{2\alpha}{2+\alpha}} (\partial \Omega_T), \quad \psi_2(x, t), \psi_3(x, t) \in C^{1+\alpha, \frac{2\alpha}{2+\alpha}} (\partial \Omega_T). \]

H5. Compatibility conditions: The following compatibility conditions hold for every \( x \in \partial \Omega \) at the initial time \( t = 0 \):
\[ \psi_1(x, 0) = u_0(x), \quad D_{t}^\alpha \left( \frac{\partial \psi_1}{\partial t} \right)_{|t=0} - D_{t}^\alpha \left( \frac{\partial \psi_2}{\partial t} \right)_{|t=0} = L_1 u_0(x)_{|t=0} + f(x, 0), \]
if the DBC holds; and
\[ M_1 u_0(x)_{|t=0} = \psi_2(x, 0), \]
if the 3BC takes place; and in the case of FDBC
\[ L_1 u_0(x)_{|t=0} + f(x, 0) = \psi_2(x, 0) + M_1 u_0(x)_{|t=0}. \]

Assumption H2 on the coefficients \( c_i \) means that the vector \( c = \{c_1(x, t), \ldots, c_n(x, t)\} \) does not lie in the tangent plane to \( \partial \Omega \) at any point.

Remark 3.1. Thanks to Lemma 4.1 in [21], the equalities
\[ (K \ast L_2 u)(x, 0) = 0 \quad \text{and} \quad (K_0 \ast M_2 u)(x, 0) = 0 \]
hold for any \( u \in C^{2+\alpha, \frac{2\alpha}{2+\alpha}} (\partial \Omega_T) \) and any \( x \in \partial \Omega \). This explains the absence of the memory terms \((K \ast L_2 u)\) and \((K_0 \ast M_2 u)\) in the compatibility condition H5.

Remark 3.2. In the case of FDBC, the compatibility H5 and assumptions H3 and H4 provide the regularity
\[ (L_1 u_0|_{t=0} + f(x, 0)) \in C^{1+\alpha}(\partial \Omega). \]

4. Main Results

We are now ready to state our main result related to the global classical solvability of (1.1)-(1.5).

Theorem 4.1. Let \( T > 0 \) be fixed, \( \partial \Omega \in C^{2+\alpha} \), and let assumptions H1-H5 hold. Then equation (1.1) with the initial condition (1.2), subject to either boundary condition DBC, 3BC or FDBC admits a unique classical solution \( u = u(x, t) \) on \( \Omega_T \), satisfying the regularity \( u \in C^{2+\alpha, \frac{2\alpha}{2+\alpha}} (\Omega_T) \). Besides, this solution fulfills the estimate
\[
\|u\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\Omega_T)} + \|D_t^\alpha u\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)} \leq C \left\{ \|u_0\|_{C^{2+\alpha}(\Omega)} + \|\psi_1\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} + \|\psi_2\|_{C^{1+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} \right\} \quad \text{in the DBC case},
\]
\[
\|u\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\Omega_T)} + \|D_t^\alpha u\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)} \leq C \left\{ \|f\|_{C^{\alpha, \frac{\alpha}{2}}(\partial \Omega_T)} + \|u_0\|_{C^{2+\alpha}(\Omega)} + \|\psi_1\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} + \|\psi_3\|_{C^{1+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} \right\} \quad \text{in the 3BC case},
\]
while if the FDBC case holds then
\[
\|u\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\Omega_T)} + \|D_t^\alpha u\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)} \leq C \left\{ \|f\|_{C^{\alpha, \frac{\alpha}{2}}(\partial \Omega_T)} + \|u_0\|_{C^{2+\alpha}(\Omega)} + \|\psi_1\|_{C^{2+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} + \|\psi_3\|_{C^{1+\alpha, \frac{2\alpha}{2+\alpha}}(\partial \Omega_T)} \right\}.
\]

The generic constant \( C \) is independent of the right-hand sides of (1.1)-(1.5).

Indeed the positive constant \( C \) depends only on the Lebesgue measures of \( \Omega \) and its boundary \( \partial \Omega \), on the norm \( \|K\|_{L_1(0, T)} \), and on the norms of the coefficients of the operators \( L_i \) (as well as \( M_1, M_2 \) and \( \|K_0\|_{L_1(0, T)} \) in the case of 3BC and FDBC), and the corresponding norms of \( \varrho_1 \) and \( \varrho_2 \).

Remark 4.2. It is worth noting that our assumptions on the kernels \( K \) and \( K_0 \) include the case \( K = K_0 \equiv 0 \), meaning that the multi-term subdiffusion equation:
\[ D_t^\alpha (\varrho_1 u) - D_t^\alpha (\varrho_2 u) - L_1 u = f \]
fits in our analysis and is described by the theorem above.
Remark 4.3. Actually, assumptions H2, H3 on the coefficients $c_i$, $c_0$ and condition H4 on the right-hand side $\psi_2$ tell us that initial-value problem (1.1)-(1.2) subject to the Neumann boundary condition (NBC), that is,

$$
\frac{\partial u}{\partial N} + K_0 \ast \frac{\partial u}{\partial N} = \psi_2 \quad \text{on} \quad \partial \Omega_T,
$$
is just a particular case of problem (1.1), (1.2), (1.4) with $c = 0$.

Accordingly, the Caputo fractional derivative of the order $\theta$ is just a particular case of problem (1.1), (1.2), (1.4) with $c = 0$, $c_i = d_i = N_i(x)$, $i = 1, 2, \ldots, n$.

Thus, results of Theorem 4.1 extend to the NBC.

Remark 4.4. With inessential modification in the proofs, the very same results hold for the $M$-term fractional equations:

$$
D^\nu_i (\varrho_1 u) - \sum_{i=2}^{M} D^\nu_i (\varrho_i u) = \mathcal{L}_1 u - \int_{0}^{t} \mathcal{K}(t-s) \mathcal{L}_2 u(s) \, ds = f(x,t),
$$

$$
\varrho_1 D^\nu_i u - \sum_{i=2}^{M} \varrho_i D^\nu_i u = \mathcal{L}_1 u - \int_{0}^{t} \mathcal{K}(t-s) \mathcal{L}_2 u(s) \, ds = f(x,t).
$$

In these cases, we additionally assume that all $\nu_i$ and $\varrho_i$, $i = 3, \ldots, M$, have the properties of $\nu_2$ and $\varrho_2$ (see assumptions H1, H3), besides the second equality in compatibility conditions in the DBC case takes the form

$$
D^\nu_i (\varrho_1 \psi_1)_{t=0} - \sum_{i=2}^{M} D^\nu_i (\varrho_i \psi_1)_{t=0} = \mathcal{L}_1 u_0(x)_{t=0} + f(x,0)
$$
or

$$
\varrho_1(x,0) D^\nu_i (\psi_1)_{t=0} - \sum_{i=2}^{M} \varrho_i(x,0) D^\nu_i (\psi_1)_{t=0} = \mathcal{L}_1 u_0(x)_{t=0} + f(x,0),
$$
respectively.

Finally, we remark, that in the case of the second equation here, the regularity of the functions $\varrho_i$, $i = 1, \ldots, M$, can be relaxed. Namely, we need in $\varrho_i \in C^{\alpha, \alpha_0 \nu_i/2}(\bar{\Omega}_T)$ in the case of DBC or 3BC cases, while $\varrho_i \in C^{\alpha, \alpha_0 \nu_i/2}(\bar{\Omega}_T) \cap C^{1+\alpha, (1+\alpha) \nu_i/2}(\partial \bar{\Omega}_T)$ in FDBC case.

5. Technical Results

We recall some properties of fractional derivatives and integrals, along with several technical results that will be used in this article. In what follows, for any $\theta > 0$ we denote

$$
\omega_\theta(t) = \frac{t^{\theta-1}}{\Gamma(\theta)}.
$$

We define the fractional Riemann-Liouville integral and the derivative of order $\theta$ of a function $v = v(t)$ (possibly also depending on other variables) as

$$
I^\theta_0 v(t) = (\omega_\theta \ast v)(t), \quad \partial^\theta_0 v(t) = \frac{\partial^{[\theta]}}{\partial t^{[\theta]}}(\omega_{[\theta]-\theta} \ast v)(t),
$$
respectively, where $[\theta]$ is the ceiling function of $\theta$ (i.e. the smallest integer greater than or equal to $\theta$).

In particular, for $\theta \in (0,1)$

$$
\partial^\theta_0 v(t) = \frac{\partial}{\partial t}(\omega_{1-\theta} \ast v)(t).
$$

Accordingly, the Caputo fractional derivative of the order $\theta \in (0,1)$ reads

$$
D^\theta_0 v(t) = \frac{\partial}{\partial t}(\omega_{1-\theta} \ast v)(t) - \omega_{1-\theta}(t)v(0) = \partial^\theta_0 v(t) - \omega_{1-\theta}(t)v(0),
$$
provided that both derivatives exist.
Our first assertion playing a key point in the proof of Theorem 4.1 describes the regularity of lower fractional derivatives in time $D_t^\beta w$, with $0 < \beta < \nu \leq 1$, in the case when $w \in C^{2+\alpha,\frac{2+\alpha}{2}}_0(\Omega_T)$. To this end, we define

$$
\Omega' = \Omega \cap B_r(x^0), \quad \Omega'_T = \Omega' \times (0,T), \quad \partial \Omega' = \partial \Omega \cap \partial \Omega',
$$

where $B_r(x^0)$ is the ball centered at a point $x^0 \in \Omega$ of radius $r$. Then we introduce the functions

$$
\xi = \xi(x) \in C_0^\infty(\mathbb{R}^n), \quad \xi \in [0,1], \quad \xi = \begin{cases} 1, & x \in \Omega', \\ 0, & x \in \mathbb{R}^n \setminus \Omega_{2r}, \end{cases}
$$

and

$$
\mathcal{J}_\theta(t) = \mathcal{J}_\theta(t;w_1, w_2) = \int_0^t \frac{[w_2(\cdot,t) - w_2(\cdot,s)] [w_1(\cdot,s) - w_1(\cdot,0)]}{(t-s)^{\theta+1}} ds
$$

where $\theta \in (0,1)$, $w_1$ and $w_2$ are some given smooth functions.

**Lemma 5.1.** Let $x^0 \in \Omega$ be arbitrarily fixed, let $0 < \beta < \nu \leq 1$. We assume that

$$
w_1 \in C^{2+\alpha,\frac{2+\alpha}{2}}_0(\Omega_T) \quad \text{and} \quad w_2 \in C([0,T], C^1(\Omega)),
$$

with $\gamma \geq \frac{2+\alpha}{2} \nu$, and we set

$$
\delta = \min\{1, \gamma\}.
$$

If $\gamma < 1$ we additionally require $D_t^\nu w_2 \in C^{\alpha,\alpha/2}(\Omega_T)$. Then, for any $\beta \in (0, \frac{2+\alpha}{2} \nu)$ and any $\tau \in (0,T]$, the following estimates hold:

i: 

$$
\|w_2 \xi D_t^\beta w_1\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C [\tau^{\delta-\beta+\nu/2} + \tau^{\beta} + \nu \alpha/2] \|D_t^\nu w_1\|_{C^{\alpha,\alpha/2}(\Omega_T)}.
$$

ii: If $w_2(x,0) = 0$ then

$$
\|w_2 \xi D_t^\beta w_1\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C [\tau^{\delta-\nu/2} + \tau^{\beta} + \tau^{\nu}\|D_t^\nu w_1\|_{C^{\alpha,\alpha/2}(\Omega_T)}.
$$

iii: 

$$
\|\mathcal{J}_\beta(t)\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C \tau^{\delta-\beta+\nu(1-\alpha)/2} \|\tau^{\nu/2} + \tau^{\alpha} + \tau^{\nu+1}\|_{C^{0,\alpha/2}(\Omega_T)}
$$

and

$$
\|\mathcal{J}_\nu(t)\|_{C^{\alpha,\alpha/2}(\Omega_T)} \leq C \tau^{\delta-\nu(1+\alpha)/2} \|\tau^{\nu/2} + \tau^{\alpha} + \tau^{\nu+1}\|_{C^{0,\alpha/2}(\Omega_T)}
$$

The positive quantity $C$ depends only on $\nu, \beta, T$, the Lebesgue measure of $\Omega$ and the norm of $w_2$.

**Proof.** We start with the evaluation of the term $D_t^\beta w_1$. To this end, appealing to representation (10.34) in [24], we deduce that

$$
w_2 \xi D_t^\beta w_1 = i_1 + i_2,
$$

where we put

$$
i_1 = i_1(x,t) = \xi(x) w_2(x,t) \frac{\tau^{\nu-\beta}}{\Gamma(1+\nu-\beta)} D_t^\nu w_1(x,t),
$$

$$
i_2 = i_2(x,t) = \xi(x) w_2(x,t) \int_0^t \frac{(t-s)^{\nu-\beta-1}}{\Gamma(\nu-\beta)} D_t^\nu w_1(x,t) - D_t^\nu w_1(x,s) ds.
$$

We estimate the norms of $i_1$ and $i_2$ separately.
As for \( \|i_1\|_{C^{\alpha/2}(\Omega_t)} \), the desired bound is a simple consequence of the following easily verified relations:

\[
\sup_{\Omega_t} |i_1| \leq C T^{\nu - \beta} \sup_{\Omega_t} |w_2| \sup_{\Omega_t} |D'_t w_1|,
\]

\[
(i_1)_{x,\Omega_t}^{(\alpha)} \leq T^{\nu - \beta} \left[ (w_2)_{x,\Omega_t}^{(\alpha)} \sup_{\Omega_t} |D'_t w_1| + \sup_{\Omega_t} |w_2| \langle D'_t w_1 \rangle_{x,\Omega_t}^{(\alpha)} \right]
\leq C |T^{1 - \alpha} + T^{\nu - \beta} + T^{\nu - \beta} \|D'_t w_1\|_{C([0,\tau],C^{\alpha}(\Omega))},
\]

and

\[
(i_1)_{t,\Omega_t}^{(\alpha/2)} \leq C T^{\nu - \beta + \alpha/2} \langle w_2 \rangle_{t,\Omega_t}^{(\delta)} \|D'_t w_1\|_{C(\Omega_t)} + \langle T^{\nu - \beta} D'_t w_1 \rangle_{t,\Omega_t}^{(\alpha/2)} \|w_2\|_{C(\Omega_t)}
\leq C T^{\nu - \beta - \alpha/2} \|1 + T^{\alpha/2} + T^{\delta} \|w_2\|_{C^1([0,T],C^{\alpha}(\Omega))} \|D'_t w_1\|_{C^{\alpha/2}(\Omega_t)}.
\]

Coming to \( \|i_2\|_{C^{\alpha/2}(\Omega_t)} \), we easily find

\[
\|i_2\|_{C(\Omega_t)} + \langle i_2 \rangle_{t,\Omega_t}^{(\alpha)} \leq C T^{\nu - \beta} \left[ T^{\alpha/2} \langle w_2 \rangle_{t,\Omega_t}^{(\alpha/2)} + \langle D'_t w_1 \rangle_{x,\Omega_t}^{(\alpha)} \right] \|w_2\|_{C^1([0,T],C^{\alpha}(\Omega))}.
\]

In order to complete the estimate of \( i_2 \), hence establishing point (i), we are left to examine the difference \( |i_2(x,t_2) - i_2(x,t_1)| \). To this end, assuming \( t_2 > t_1 \) and setting \( \Delta t = t_2 - t_1 \), we have

\[
|i_2(x,t_2) - i_2(x,t_1)| \leq \sum_{j=1}^4 i_{2,j},
\]

where

\[
i_{2,1} = \xi |w_2(x,t_2) - w_2(x,t_1)| \int_0^{t_2} s^{\nu - \beta - 1} \frac{1}{\Gamma(\nu - \beta)} \|D'_t w_1(x,t_2 - s) - D'_t w_1(x,t_2)\| ds,
\]

\[
i_{2,2} = \xi |w_2(x,t_1)| \int_0^{t_1} s^{\nu - \beta - 1} \frac{1}{\Gamma(\nu - \beta)} \|D'_t w_1(x,t_2 - s) - D'_t w_1(x,t_1 - s)\| ds,
\]

\[
i_{2,3} = \xi |w_2(x,t_1)| \|D'_t w_1(x,t_2) - D'_t w_1(x,t_1)\| \int_0^{t_1} s^{\nu - \beta - 1} \frac{1}{\Gamma(\nu - \beta)} ds,
\]

\[
i_{2,4} = \xi |w_2(x,t_1)| \int_0^{t_2} s^{\nu - \beta - 1} \frac{1}{\Gamma(\nu - \beta)} \|D'_t w_1(x,t_2 - s) - D'_t w_1(x,t_2)\| ds.
\]

Exploiting the smoothness of the functions \( w_2 \) and \( D'_t w_1 \), and taking into account of the relation between \( \nu \) and \( \beta \), we arrive at the sought estimate for \( (i_2)_{t,\Omega_t}^{(\alpha/2)} \).

To verify statement (ii), it is worth noting that the assumptions on \( w_2 \) (i.e. \( w_2(x,0) = 0 \)) provide the estimate

\[
\|w_2\|_{C^{\alpha/2}(\Omega_t)} \leq C T^{\delta - \alpha/2} \left[ T^{\alpha/2} + 1 + T^{1 - \alpha} T^{\alpha/2} \right] \|w_2\|_{C^1([0,T],C^{\alpha}(\Omega))}.
\]

Collecting this bound with the regularity of \( D'_t w_1 \), the desired claim follows.

Coming to (iii), we restrict ourselves to the verification of the first inequality, for the second one is deduced in a similar manner. Straightforward calculations lead to the relations

\[
\sup_{\Omega_t} |\xi \hat{A}_\beta(t)| \leq C T^{\delta - \beta + \nu} \langle w_2 \rangle_{t,\Omega_t}^{(\delta)} \|D'_t w_1\|_{C(\Omega_t)},
\]

\[
\left| \xi(x_2) \hat{A}_\beta(t) |_{x=x_1} - \xi(x_1) \hat{A}_\beta(t) |_{x=x_2} \right| \leq C T^{-\alpha} T^{\delta - \beta + \nu} \|D'_t w_1\|_{C(\Omega_t)} \langle w_2 \rangle_{t,\Omega_t}^{(\delta)} |x_1 - x_2|^\alpha + C \left| \hat{A}_\beta(t) |_{x=x_1} - \hat{A}_\beta(t) |_{x=x_2} \right|,
\]

where \( \hat{A}_\beta(t) \) is a function defined by

\[
\hat{A}_\beta(t) = \int_0^t s^{\nu - \beta - 1} \frac{1}{\Gamma(\nu - \beta)} \|D'_t w_1(x,t_2 - s) - D'_t w_1(x,t_2)\| ds.
\]
and

\[ |\mathfrak{J}_\beta(t)|_{x=x_1} - |\mathfrak{J}_\beta(t)|_{x=x_2} | \leq C \left\{ |x_2 - x_1| (D_x w_1)_{t,\Omega_c}^{(1+\nu)} \right\} \int_0^t \frac{|w_2(x_2, s) - w_2(x_2, t)| s \frac{\nu}{1+\nu}}{(t-s)^{1+\beta}} ds \\
+ \int_0^t \frac{|w_2(x_1, t) - w_2(x_2, t) + w_2(x_2, s) - w_2(x_1, s)| s^\nu \|D_t^\nu w_1\|_{C(\Omega_c)} ds}{(t-s)^{1+\beta}} \right\} \\
\leq C |x_1 - x_2|^\alpha r^{\delta - \beta + \frac{\nu(1+\nu)}{2}} (1 + r^{1-\alpha}) \|w_2\|_{C^\alpha([0,T], C^\nu(\tilde{\Omega}))} \|w_1\|_{C^{2+\alpha, 2+\alpha/(1+\nu)}(\Omega_c)}.
\]

which in turn entail

\[ \|\xi_{\mathfrak{J}_\beta(t)}\|_{C([0,\tau], C^\alpha(\tilde{\Omega}))} \leq C r^{\delta - \beta + \frac{\nu(1+\nu)}{2}} (r^{1-\alpha} + \tau^{1-\alpha} r^{1+\alpha} + 1) \|w_1\|_{C^{2+\alpha, 2+\alpha/(1+\nu)}(\tilde{\Omega})} \|w_2\|_{C^\nu([0,T], C^\alpha(\Omega))}. \tag{5.1} \]

Thus, taking into account Definition 2.1, we will complete the proof of the first bound in (iii) if we obtain the corresponding estimate of the seminorm \( \langle \mathfrak{J}_\beta(t) \rangle_{t \Omega_c} \). To this end, assuming \( T \geq t_2 > t_1 \geq 0 \), let us define

\[ \Delta t = t_2 - t_1. \]

If \( \Delta t \geq t_1/2 \), it is apparent that

\[ |\mathfrak{J}_\beta(t_2) - \mathfrak{J}_\beta(t_1)| \leq C \sum_{j=1}^3 b_j, \]

where

\[ b_1 = \int_0^{t_1} \frac{|w_2(x, t_1) - w_2(x, s)| |w_1(x, s) - w_1(x, 0)|}{(t_1 - s)^{1+\beta}} ds, \]
\[ b_2 = \int_0^{t_1} \frac{|w_2(x, t_2) - w_2(x, s)| |w_1(x, s) - w_1(x, 0)|}{(t_2 - s)^{1+\beta}} ds, \]
\[ b_3 = \int_{t_1}^{t_2} \frac{|w_2(x, t_2) - w_2(x, s)| |w_1(x, s) - w_1(x, 0)|}{(t_2 - s)^{1+\beta}} ds. \]

Then, appealing to the smoothness of the functions \( w_1 \) and \( w_2 \), we conclude that

\[ b_1 \leq C \int_0^{t_1} (t_1 - s)^{\delta - \beta - 1 + \nu} ds \|w_2\|_{C(\Omega_c)} \|D_t^\nu w_1\|_{C(\Omega_c)} \]
\[ \leq C (\Delta t)^{\nu/2} r^{\delta + \nu - \beta - \alpha \nu/2} \|D_t^\nu w_1\|_{C(\Omega_c)} \|w_2\|_{C(\Omega_c)}, \]

and

\[ b_3 \leq \|D_t^\nu w_1\|_{C(\Omega_c)} \|w_2\|_{C(\Omega_c)} \int_{t_1}^{t_2} (t_2 - s)^{\delta - 1 + \beta} ds \]
\[ \leq C r^{\delta + \nu - \beta - \alpha \nu/2} (\Delta t)^{\nu/2} \|D_t^\nu w_1\|_{C(\Omega_c)} \|w_2\|_{C^\nu(\Omega_c)}. \]

The estimate for \( b_2 \) is analogous to the one of \( b_1 \). Taking into account (5.1), this yields the desired bound in (iii) when \( \Delta t \geq t_1/2 \). If instead \( \Delta t < t_1/2 \), we rewrite the difference as

\[ |\mathfrak{J}_\beta(t_2) - \mathfrak{J}_\beta(t_1)| \leq C \sum_{j=1}^4 a_j, \tag{5.2} \]
Due to the properties of the functions $w_2$ and $w_1$, and exploiting the mean-value theorem in the evaluation of the term $a_3$, we end up with

$$|J_\beta(t_2) - J_\beta(t_1)| \leq C(w_2, t_{\Omega T}) \|D_t^\alpha w_1\|_{C(\Omega_T)} (\Delta t)^{\alpha/2} \|D_t^{\rho} + \rho^{\alpha/2} \|_{C^2(\partial \Omega_T)},$$

which completes the argument. \qed

Recasting the same proof, we immediately obtain

**Lemma 5.2.** Let the assumptions of Lemma 5.1 hold. Besides, let $\gamma > \frac{(3+\alpha)\nu}{2}$ and

$$D_t^\nu w_1 \in C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T) \quad \text{and} \quad w_2 \in C^\alpha([0, T], C^2(\partial \Omega)).$$

If $\gamma < 1$ we also require $D_t^\nu w_2 \in C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T)$. Then, for $\beta \in (0, \frac{\nu(1-\alpha)}{2})$, with $\delta$ as above, the following estimates hold:

i: $\|D_t^\nu w_1\|_{C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T)} \leq C(t_2 - t_1)^{\gamma - \beta}$

ii: If $w_2(x, 0) = 0$ then

$$\|w_2 D_t^\nu w_1\|_{C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T)} \leq C(t_2 - t_1)^{\gamma - \beta}$$

iii: $\|\xi J_\beta(t)\|_{C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T)} \leq C(t_2 - t_1)^{\gamma - \beta}$

Again, $C$ depends only on $\nu, \beta, T$, the Lebesgue measure of $\Omega$ and the norm of $w_2$.

**Remark 5.3.** It is apparent that the estimates of the terms $\|\xi J_\beta\|_{C^{1+\alpha, \frac{1+\alpha}{\nu}}(\partial \Omega_T)}$ and $\|\xi J_\beta\|_{C^{1+\alpha, (1+\alpha)/2}(\partial \Omega_T)}$ in points (iii) of both lemmas above hold within weaker assumptions on $\gamma$, namely, $\gamma > \frac{3+\alpha}{2}$, and $\gamma > \frac{3+\alpha}{2}$, respectively.
Remark 5.4. The following estimates are simple consequences of Lemma 5.1

\[ \|D_\nu (w_2w_1)\|_{C^\alpha, \frac{\alpha}{2}(\partial_\Omega)} + \|D_\nu (w_2w_1)\|_{C^\alpha, \frac{\alpha}{2}(\partial_\Omega)} + \|D_\nu w_1\|_{C^\alpha, \frac{\alpha}{2}(\partial_\Omega)} \]

\[ \leq C\|D_\nu w_1\|_{C^\alpha, \frac{\alpha}{2}(\partial_\Omega)}, \]

\[ \|D_\nu (w_2w_1)\|_{C^{1+\alpha, \frac{\alpha}{2}}(\partial_\Omega)} + \|D_\nu (w_2w_1)\|_{C^{1+\alpha, \frac{\alpha}{2}}(\partial_\Omega)} + \|D_\nu w_1\|_{C^{1+\alpha, \frac{\alpha}{2}}(\partial_\Omega)} \]

\[ \leq C\|D_\nu w_1\|_{C^{1+\alpha, \frac{\alpha}{2}}(\partial_\Omega)} + \|w_1\|_{C^{2+\alpha, \frac{\alpha}{2}}(\partial_\Omega)} \],

and

\[ J_\beta (0; w_1, w_2) = J_\nu (0; w_1, w_2) = 0. \]

Here the positive constant \( C \) depends only on \( T \), the Lebesgue measure of \( \Omega \), and the norm of \( w_2 \).

We complete this section by discussing the properties of the solution to initial and initial-boundary value problems for a certain subdiffusion equation, which will be the key point in the construction of a regularizer to the linear problems (1.1)-(1.5). To this end, we denote

\[ \mathbb{R}^n_+ = \{ x : (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0 \} \quad \text{and} \quad \mathbb{R}^n_+T = \mathbb{R}^n_+ \times (0, T). \]

Let the function \( v_i = v_i(x, t) \) solve the problems

\[ \begin{cases} D_\nu v_i - \Delta v_i = F_0(x, t) & \text{in } \mathbb{R}^n, \\ v_i(x, 0) = v_{i,0}(x) & \text{in } \mathbb{R}^n, \end{cases} \quad (5.3) \]

where \( F_0 \) and \( v_{i,0} \) are some given functions; and for \( i = 2, 3, 4 \),

\[ \begin{cases} D_\nu v_i - \Delta v_i = 0 & \text{in } \mathbb{R}^n_+, \\ v_i(x, 0) = 0 & \text{in } \mathbb{R}^n_+, \\ v_i(x, t) \to 0 & \text{if } |x| \to +\infty, \end{cases} \quad (5.4) \]

with one of the following boundary conditions:

\[ v_2(x, t) = F_1(x, t) \quad \text{on } \partial \mathbb{R}^n_+, \quad (5.5) \]

\[ \sum_{i=1}^n c_i \frac{\partial v_3}{\partial x_i} = F_2(x, t) \quad \text{on } \partial \mathbb{R}^n_+, \quad (5.6) \]

\[ D_\nu v_4 - \sum_{i=1}^n c_i \frac{\partial v_4}{\partial x_i} = F_3(x, t) \quad \text{on } \partial \mathbb{R}^n_+, \quad (5.7) \]

where \( F_i \) are given functions, and \( c_1, ..., c_n \) are constants with \( c_n \neq 0 \). The classical solvability of problems (5.3)-(5.7) with \( \nu \in (0, 1) \) has been discussed in the one-dimensional case in [25-26], and in the multi-dimensional case in [19-20]. As for \( \nu = 1 \), these problems are analyzed in [28, Section 4]. We subsume these results in a lemma.

Lemma 5.5. Let \( c_n \neq 0 \), or \( c_n > 0 \) in the case of the fractional dynamic boundary condition (5.7), let \( v_{1,0} \in C^{2+\alpha} (\mathbb{R}^n) \), and let

\[ F_0 \in C^{\alpha, \alpha/2} (\bar{\mathbb{R}}^n), \quad F_1 \in C^{2+\alpha, \alpha/2} (\partial \mathbb{R}^n_+), \quad F_2, F_3 \in C^{1+\alpha, \alpha/2} (\partial \mathbb{R}^n_+). \]

Assume also that there exists a positive number \( r_0 \) such that

\[ v_{1,0}(x), F_0(x, t), F_i(x, t) \equiv 0, \quad \text{if } |x| > r_0, \quad t \in [0, T]. \]
Then, there are unique classical solutions \( v_i(x,t) \) to problems (5.3)-(5.7). In addition, the following estimates hold:
\[
\begin{align*}
\|v_1\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega})} & \leq C \left[ \|v_1\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega})} + \|F_0\|_{C^{2+\alpha, \frac{2}{2}}(\bar{\Omega})} \right], \\
\|v_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega})} & \leq C \|F_1\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\partial \Omega^m,T)}, \\
\|v_3\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega})} & \leq C \|F_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\partial \Omega^m,T)}, \\
\|v_4\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega})} + \|D_tv_4\|_{C^{1+\alpha, \frac{2+\alpha}{2}}(\partial \Omega^m,T)} & \leq C \|F_3\|_{C^{1+\alpha, \frac{2+\alpha}{2}}(\partial \Omega^m,T)}.
\end{align*}
\]

Here the generic constant \( C \) is independent of the right-hand sides in (5.3)-(5.7).

6. Proof of Theorem 4.1

The strategy of the proof is based on the construction of a regularizer (see [28, Section 4]), and it consists of fourth main steps. In the first one, we build a special covering of the domain \( \Omega \). Next, assuming the additional hypotheses on the right-hand sides
\[
u_0(x) = 0, \quad x \in \tilde{\Omega}, \quad f(x,0) = 0, \quad x \in \tilde{\Omega}, \quad \psi_1(x,0) = 0, \quad x \in \partial \Omega, \tag{6.1}
\]
which, in particular, give
\[
u(x,0) = 0, \quad x \in \tilde{\Omega}, \tag{6.2}
\]
we freeze the coefficients of the operators \( L_1 \) and \( M_1 \), and, by exploiting the properties of the solutions of the so-called model problems (5.3)-(5.7), we construct a regularizer, i.e., the inverse operator of (1.1)-(1.5) in the case of a small time interval \( t \in [0,T] \). After that, we discuss how to extend the obtained results to the whole time interval \([0,T]\). Finally, we show how to reduce (1.1)-(1.5) in the general case to the special one related with assumption (6.1), in other words, we discuss the reduction of problems (1.1)-(1.5) to the problems with homogenous initial data (6.2). In our analysis, we focus on the case \( \nu_1 \in (0,1) \), whereas the case \( \nu_1 = 1 \) is examined either with similar or simpler arguments, due to the equivalent definitions of Caputo fractional derivatives.

6.1. Step I: Covering of the domain \( \Omega \). For an arbitrarily fixed \( \lambda > 0 \), it is always possible to find a finite collection of points \( x^m \in \tilde{\Omega} \) along with sets
\[
\omega^m = B_{\lambda^2}(x^m) \cap \tilde{\Omega} \quad \text{and} \quad \Omega^m = B_{\lambda}(x^m) \cap \tilde{\Omega},
\]
satisfying the following properties:

(i) \( \bigcup_m \omega^m = \bigcup_m \Omega^m = \tilde{\Omega} \);

(ii) there exists a number \( N_0 \), independent of \( \lambda \), such that the intersection of any \( N_0 + 1 \) distinct \( \Omega^m \) (and consequently any \( N_0 + 1 \) distinct \( \omega^m \)) is empty.

Notice that, by construction,
\[
x^m \in \omega^m \subset \overline{\omega^m} \subset \Omega^m \subset \tilde{\Omega}.
\]
Moreover, we partition the sets of indexes \( m \) into the disjoint union \( \mathcal{M} \cup \mathcal{N} \), by setting
\[
m \in \mathcal{M} \quad \text{if} \quad \Omega^m \cap \partial \Omega = \emptyset \quad \text{and} \quad m \in \mathcal{N} \quad \text{if} \quad \omega^m \cap \partial \Omega \neq \emptyset.
\]

In the sequel, let us denote \( \partial \Omega^m = \partial \Omega \cap B_{\lambda}(x^m) \). Let \( \xi^m = \xi^m(x) : \Omega \to [0,1] \) be a smooth function possessing the following properties: \( \xi^m \in (0,1) \) if \( x \in \Omega^m \setminus \omega^m \) and
\[
\xi^m = \begin{cases}
1, & \text{if} \quad x \in \omega^m, \\
0, & \text{if} \quad x \in \Omega \setminus \omega^m,
\end{cases}
\]
\[
|D_x^j \xi^m| \leq C \lambda^{-|j|}, \quad |j| \geq 1, \quad 1 \leq \sum_m (\xi^m)^2 \leq N_0.
\]
Then, we define the function
\[
\eta^m = \frac{\xi^m}{\sum_j (\xi^j)^2}. \tag{6.3}
\]
Due to the properties of the function $\xi^m$, we see that $\eta^m$ vanishes for $x \in \Omega \setminus \overline{\Omega^m}$, and $|D^j_\tau \eta^m| \leq C\lambda^{-|j|}$. Thus, the product $\eta^m \xi^m$ defines a partition of unity via the formula

$$\sum_m \eta^m \xi^m = 1.$$  

At this point, we define the local coordinate systems connected with each point $x^m$, $m \in \mathfrak{R}$. For each $m \in \mathfrak{R}$, the point $x^m$ will be the origin of a local coordinate system. Let $\partial \Omega$ be described by $y_n = \tilde{\delta}^m(y_1, \ldots, y_{n-1})$ in a small vicinity of each point $x^m$, $m \in \mathfrak{R}$, and

$$y = \mathfrak{B}^m(x - x^m), \quad \left| \frac{\partial \tilde{\delta}^m}{\partial y_i} \right| \leq C \lambda, \; i = 1, 2, \ldots, n - 1,$$

where $\mathfrak{B}^m = (b^m_{ij})_{i,j=1,\ldots,n}$ is an orthogonal matrix with elements $b^m_{ij}$, and $(b^m_{ij})^{-1}$ is an element of the inverse matrix to $\mathfrak{B}^m$. To obtain the local “flatness” of the boundary, we make the change of variables

$$z_i = y_i, \quad z_n = y_n - \tilde{\delta}^m(y_1, \ldots, y_{n-1}), \quad i = 1, 2, \ldots, n - 1, \; m \in \mathfrak{R}.$$

Hence, we have built the mapping $Z_m$ which connects the original variable $x = (x_1, \ldots, x_n)$ with the new variable $z = (z_1, \ldots, z_n)$ in a neighborhood of each point $x^m$, $m \in \mathfrak{R}$ via relations:

$$x = Z_m(z) \quad \text{and} \quad z = Z_m^{-1}(x).$$

Next, we introduce the following norms in the spaces $C^{l+\alpha, \frac{1}{2} + \beta \nu_1}T \Omega_T$, $l = 0, 1, 2$, which are related with the covering $\{\Omega^m\}$:

$$\{v\}^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T} = \sup \|v\|^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T)}.$$

We now state a lemma, which subsumes Propositions 4.5-4.7 in our previous work [21], in order to describe the properties of these norms. To this end, for an arbitrarily given $0 < \kappa < 1$, we define

$$\tau = \lambda^{2/\nu_1} \kappa, \quad (6.4)$$

such that $\tau \in (0, T]$. Then we consider (any) function $\Phi_m(x)$ (defined in $\Omega^m$) such that

$$|D^j_\tau \Phi_m(x)| \leq C\lambda^{-|j|}, \quad 0 \leq |j| \leq 2,$$

along with (any) function $\tilde{v}(x, t)$ of the form

$$\tilde{v}(x, t) = \sum_{m \in \mathfrak{R}} v^m(x, t),$$

for some $v^m \in C^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega^m_T}$, $l = 0, 1, 2$, with $v^m$ vanishing outside $\Omega^m$.

**Lemma 6.1.** Let (6.4) hold. Then for any $v \in C^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T}$, $l = 0, 1, 2$, we have the following relations:

$$\{v\}^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T} \leq \|v\|^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T} \leq C \{v\}^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T},$$

$$\|\Phi_m v\|^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega^m_T} \leq \sup_{m \in \mathfrak{R}} \|v\|^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega^m_T};$$

$$\{\tilde{v}\}^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega_T} \leq C \sup_{m \in \mathfrak{R}} \|v^m\|^{l+\alpha, \frac{1}{2} + \beta \nu_1 T \Omega^m_T}.$$

Here the positive constant $C$ is independent of $\lambda$ and $\tau$.

**6.2. Step II: Construction of a regularizer for (6.1)**. We aim to construct the inverse operator for problem (6.1), (6.2) and (6.3), i.e., in FDBC case. The analysis of the remaining cases (1.3) and (1.4) are performed in similar manner. First, we recall that assumption H4, H5 and (6.1) imply

$$f(x, t) \in C^{\alpha, \frac{1}{2}}_0(\Omega_T), \quad \psi_3 \in C^{1+\alpha, \frac{1}{2} + \beta \nu_1 T \partial \Omega_T}.$$

For the sake of convenience, we rewrite problem (1.1), (6.2) and (1.3) in the compact form

$$L u = F, \quad F = \{f, \psi_3\}. \quad (6.6)$$

Here, $L$ is the linear operator acting as

$$L u = \{Au, A_1 u|_{\partial \Omega_T}\},$$

where $A$ and $A_1$ are the operators defined by

$$Au = \mathfrak{F} u, \quad A_1 u|_{\partial \Omega_T} = \mathfrak{G} u|_{\partial \Omega_T}.$$
Lemma 6.3. Let $F$ be the left-hand side of (1.1), while $A_1$ is the left-hand side of (1.5). For $m \in \mathcal{M} \cup \mathcal{N}$, we set

$$a_{ij}^m = a_{ij}(x^m,0), \quad \varphi_1^m = \varphi_1(x^m,0), \quad c_i^m = c_i(x^m,0), \quad f^m(x,t) = \xi^m(x)f(x,t),$$

and, for $m \in \mathcal{N}$,

$$\tilde{f}^m(z,t) = f^m(x,t)|_{x = z_m(z)}, \quad \tilde{\psi}^m(z,t) = \xi^m(x)\psi_3(x,t)|_{x = z_m(z)},$$

with $\xi^m$, $Z_m(z)$, $\mathcal{M}$, $\mathcal{N}$ as in Subsection [6.1].

For $m \in \mathcal{M} \cup \mathcal{N}$, $\tau \in (0,T]$ and $\lambda$ as in (6.4), we define the functions $u^m(x,t)$ to be the solutions to the following problems: if $m \in \mathcal{M}$, then

$$\begin{aligned}
\{ & \varphi_1^m D_t^\nu_1 u^m - \sum_{i,j=1}^n a_{ij}^m \frac{\partial^2 u^m}{\partial x_i \partial x_j} = f^m(x,t) \quad \text{in } \mathbb{R}_+^n, \\
& u^m(x,0) = 0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
(6.7)$$

while, for $m \in \mathcal{N}$,

$$u^m(x,t) = \tilde{u}^m(z,t)|_{z = z_m^{-1}(x)},$$

where $\tilde{u}^m$ solves the initial-boundary value problem

$$\begin{aligned}
\{ & \varphi_1^m D_t^\nu_1 \tilde{u}^m - \sum_{i,j=1}^n a_{ij}^m \frac{\partial^2 \tilde{u}^m}{\partial z_i \partial z_j} = \tilde{f}^m(z,t) \quad \text{in } \mathbb{R}_+^n, \\
& \varphi_1^m D_t^\nu_1 \tilde{u}^m - \sum_{i=1}^n \varphi_1^m \frac{\partial \tilde{u}^m}{\partial z_i} = \tilde{\psi}^m(z,t) \quad \text{on } \partial \mathbb{R}_+^{n-1}, \\
& \tilde{u}^m(z,0) = 0 \quad \text{in } \mathbb{R}_+^n,
\end{aligned}
(6.8)$$

At this point, we define the space

$$\mathcal{H} = \{ u : u \in C_0^{2+\alpha,2+\alpha}(\bar{\Omega}_\tau), \quad D_t^\nu_1 u \in C_0^{1+\alpha,1+\alpha}(\partial \Omega_\tau) \},$$

normed by

$$\|u\|_{\mathcal{H}} = \|u\|_{C_0^{2+\alpha,2+\alpha}(\bar{\Omega}_\tau)} + \|D_t^\nu_1 u\|_{C_0^{1+\alpha,1+\alpha}(\partial \Omega_\tau)},$$

together with the product space

$$\mathcal{H}_0 = C_0^{\alpha,\alpha}(\bar{\Omega}_\tau) \times C_0^{1+\alpha,1+\alpha}(\partial \Omega_\tau),$$

normed by

$$\|(f,\psi_3)\|_{\mathcal{H}_0} = \|f\|_{C_0^{\alpha,\alpha}(\bar{\Omega}_\tau)} + \|\psi_3\|_{C_0^{1+\alpha,1+\alpha}(\partial \Omega_\tau)}.$$

We are now in the position to give the definition of a regularizer.

Definition 6.2. Let $\tau \in (0,T]$. An operator $\mathcal{R} : \mathcal{H}_0 \to \mathcal{H}$ is called a regularizer on the time-interval $[0,\tau]$, if

$$\mathcal{R}(f,\psi_3) = \sum_{m \in \mathcal{M} \cup \mathcal{N}} \eta^m(x)w^m(x,t),$$

where the functions $\eta^m(x)$ and $w^m(x,t)$ are defined in (6.3) and (6.7)-(6.8), respectively.

The following result details the main properties of $\mathcal{R}$, allowing us eventually to construct the inverse of $L$.

Lemma 6.3. Let $\tau \in (0,T]$ satisfy (6.4). We assume that the hypotheses of Theorem [2.1] and (6.1) hold. Then, for any $F \in \mathcal{H}_0$ and $u \in \mathcal{H}$ the following hold:

(i) $\mathcal{R}$ is a bounded operator:

$$\|\mathcal{R}F\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}_0},$$

where the positive constant $C$ is independent of $\lambda$ and $\tau$. 


(ii) There exist operators $\mathfrak{T}_1 : \mathcal{H}_0 \to \mathcal{H}_0$ and $\mathfrak{T}_2 : \mathcal{H} \to \mathcal{H}$ such that the decompositions

$$L\mathfrak{R} F = F + \mathfrak{T}_1 F \quad \text{and} \quad \mathfrak{R} \mathfrak{R} u = u + \mathfrak{T}_2 u$$

hold, and

$$\|\mathfrak{T}_1 F\|_{\mathcal{H}_0} \leq \frac{1}{2} \|F\|_{\mathcal{H}_0} \quad \text{and} \quad \|\mathfrak{T}_2 u\|_{\mathcal{H}} \leq \frac{1}{2} \|u\|_{\mathcal{H}}.$$ 

Proof. It is worth noting that the results of Lemma 5.3 are valid in the case of problems (6.7) and (6.8). Then, collecting [21] Proposition 4.4 with Lemmas 5.1, 5.2, 5.3 and Remark 5.4 we end up with the estimates:

$$\|\mathfrak{R} F\|_{\mathcal{H}} \leq C \left( \sup_{m \in \mathfrak{N}, \Omega \in \mathfrak{M}} \|u^m\|_{C^{\alpha, \frac{3}{2}+\alpha}(\Omega_\nu(\Omega_\nu^0))} + \sup_{m \in \mathfrak{N}} \|D_t^{\tau_1} u^m\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega_\nu^0)} \right)$$

$$\leq C \left( \sup_{m \in \mathfrak{N}, \Omega \in \mathfrak{M}} \|f\mathfrak{L}^m\|_{C^{\alpha, \frac{3}{2}+\alpha}(\Omega_\nu(\Omega_\nu^0))} + \sup_{m \in \mathfrak{N}} \|\mathfrak{S}_m \mathfrak{L}^m\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega_\nu^0)} \right)$$

$$\leq C \|F\|_{\mathcal{H}_0},$$

where $C$ is independent of $\lambda$ and $\tau$. The last inequality is just (i). Now we verify (ii). Here, we limit ourselves to deal with $\mathfrak{T}_1$, being the other case completely analogous. The definition of the operator $L$ together with (6.3) allow us to conclude that

$$L\mathfrak{R} F = \{ A\mathfrak{R} F, A_1\mathfrak{R} F \}_{\partial \Omega},$$

with

$$A\mathfrak{R} F = A^0\mathfrak{R} F + A^1\mathfrak{R} F \quad \text{and} \quad A_1\mathfrak{R} F |_{\partial \Omega} = A^0_1\mathfrak{R} F + A^1_1\mathfrak{R} F,$$

where we set

$$A^0\mathfrak{R} F = \left\{ \sum_{m} \mathfrak{L}_m D_t^{\tau_1} u^m \eta^m(x) - \mathfrak{L}_1\mathfrak{R} F - \mathfrak{L}_2\mathfrak{R} F + \mathfrak{S}_0 \mathfrak{L}_1\mathfrak{R} F + \mathfrak{S}_0 \mathfrak{L}_2\mathfrak{R} F, \quad m \in \mathfrak{M}, \right\}$$

$$A^1\mathfrak{R} F = \left\{ D_t^{\tau_1} \mathfrak{L}_1\mathfrak{R} F - D_t^{\tau_2} \mathfrak{L}_2\mathfrak{R} F \right\},$$

$$A^0_1\mathfrak{R} F = \left\{ \sum_{m} \mathfrak{L}_m D_t^{\tau_1} \tilde{u}^m(z, t)_{z=\tilde{z}_m^{-1}(x)} - \mathfrak{M}_1\mathfrak{R} F + \mathfrak{M}_2\mathfrak{R} F \right\} |_{\partial \Omega},$$

$$A^1_1\mathfrak{R} F = \left\{ D_t^{\tau_1} \mathfrak{L}_1\mathfrak{R} F - \sum_{m} \mathfrak{L}_m \mathfrak{S}_m D_t^{\tau_1} \tilde{u}^m(z, t)_{z=\tilde{z}_m^{-1}(x)} - D_t^{\tau_2} \mathfrak{L}_2\mathfrak{R} F \right\} |_{\partial \Omega},$$

Then, Lemma 5.2 in [21] and Theorem 2 in [20] tell us that

$$A^0\mathfrak{R} F = f + \mathfrak{T}_1^0\mathfrak{R} F \quad \text{and} \quad A^1\mathfrak{R} F = \psi_1 + \mathfrak{T}_1^1\mathfrak{R} F,$$

where

$$\|\mathfrak{T}_1^1\mathfrak{R} F\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega)} \leq \frac{1}{8} \|F\|_{\mathcal{H}_0}$$

$$\text{and} \quad \|\mathfrak{T}_1^2\mathfrak{R} F\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega)} \leq \frac{1}{8} \|F\|_{\mathcal{H}_0}, \quad \text{provided that} \quad \lambda \quad \text{and} \quad \tau \quad \text{comply with} \quad (6.3).$$

Hence, we are left to prove the estimates

$$\|A^1\mathfrak{R} F\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega)} \leq \frac{1}{8} \|F\|_{\mathcal{H}_0}$$

$$\text{and} \quad \|A^1_1\mathfrak{R} F\|_{C^{\alpha, \frac{3}{2}+\alpha}(\partial \Omega)} \leq \frac{1}{8} \|F\|_{\mathcal{H}_0}.$$ 

Indeed, point (ii) for $\mathfrak{T}_1$ immediately follows from representation of $L\mathfrak{R} F$ and estimates (6.10)–(6.11), implying that

$$\mathfrak{T}_1\mathfrak{R} F = \{ \mathfrak{T}_1^1\mathfrak{R} F, \mathfrak{T}_1^2\mathfrak{R} F + A^1_1\mathfrak{R} F \} \quad \text{and} \quad \|\mathfrak{T}_1\mathfrak{R} F\|_{\mathcal{H}_0} \leq \frac{1}{2} \|F\|_{\mathcal{H}_0}.$$
Concerning the first inequality in (6.11), we treat the case $m \in \mathcal{M}$ (the case $m \notin \mathcal{M}$ being similar). Appealing to Corollary 3.1 in [23], and keeping in mind that we have null initial data, we have

$$
D_t^m (\varrho_1 \mathcal{R} F) - D_t^m (\varrho_2 \mathcal{R} F) - \sum_{m \in \mathcal{M}} \varrho_1^m D_t^m u_m(x,t) \eta^m \\
\begin{aligned}
&= \sum_{m \in \mathcal{M}} \{[\varrho_1 - \varrho_1^m] \eta^m D_t^m u_m(x,t) + \frac{\nu}{1 - \nu_2} \eta^m \mathcal{J}_{\nu_2}(t; u^m, \varrho_1) \\
&\quad - \frac{\nu_2}{1 - \nu_2} \eta^m \mathcal{J}_{\nu_2}(t; u^m, \varrho_2) - \varrho_2 \eta^m D_t^m u_m(x,t) \}. 
\end{aligned}
$$

On account of the properties of the functions $\varrho_1$ and $\varrho_2$ (see H3), and exploiting Lemmas 5.1 and 6.1 along with Remark 5.3 to evaluate the terms in the right-hand sides of the equality above, we conclude that

$$
\|D_t^m (\varrho_1 \mathcal{R} F) - D_t^m (\varrho_2 \mathcal{R} F) - \sum_{m \in \mathcal{M}} \varrho_1^m D_t^m u_m(x,t) \eta^m \|_{C^{\alpha, \alpha \nu} / 2(\Omega_\tau)} \\
\leq C \left[ r \delta_{\alpha^2 - 2} + \kappa_\alpha + \kappa_\alpha r^2 - \nu_1 \right] \|\mathcal{R} F\|_H \\
\leq C \left[ r \delta_{\alpha^2 - 2} + \kappa_\alpha + \kappa_\alpha r^2 - \nu_1 \right] \|\mathcal{R} F\|_H_0,
$$

where $\delta_0 = \min\{1, \gamma_0\}$ and $\delta_1 = \min\{1, \gamma_1\}$. The constant $C$ is independent of $L$ and $\tau$, and depends only on the norms of $\varrho_1$, $\varrho_2$, the Lebesgue measure of $\Omega$ and $T$. Thanks to the relation between $\nu_1$ and $\nu_2$ (see H1), and assumption H3 on $\gamma_0$ and $\gamma_1$, the last two estimates provide the first inequality in (6.11). The second one follows by recasting the arguments above, but using Lemma 5.2 in place of Lemma 5.1.

Coming to construction of the inverse of $\mathcal{L}$, we note that Lemma 6.3 ensures the existence of the bounded operators $(I + \mathcal{F}_1)^{-1}$ and $(I + \mathcal{F}_2)^{-1}$ ($I$ is the identity in the respective spaces), therefore,

$$
\mathcal{L}(I + \mathcal{F}_1)^{-1} = (I + \mathcal{F}_2)^{-1} = \mathcal{L}^{-1} : \mathcal{H}_0 \to \mathcal{H}.
$$

Accordingly, the unique solution of (6.12) is given by

$$
u = \mathcal{L}^{-1} (f, \varphi_3) \quad \text{for} \quad t \in [0, \tau]. \quad (6.12)
$$

The estimate of the norm $\mathcal{L}^{-1}$ follows from the estimates of Lemma 6.3. In summary, we have verified Theorem 4.1 (in the case of (6.1)) for a small time interval $[0, \tau]$. 

6.3. **Step III: Extension of the solution to whole interval** $[\tau, T]$. The next goal is to extend the solution found in Step I to the intervals $[\tau, 2\tau]$, $[2\tau, 3\tau]$ and so on, so to cover the whole $[\tau, T]$. Again, we shall give the details only for the (most difficult) case FDBC. First, we set

$$
\Phi(x, t) = \begin{cases} 
D_t^m u(x, t) - \Delta u(x, t), & t \in [0, \tau], \ x \in \Omega, \\
|D_t^m u(x, t)|_{t=\tau}, & t \in [\tau, 2\tau], \ x \in \Omega,
\end{cases}
$$

and

$$
\Psi(x, t) = \begin{cases} 
D_t^m u(x, t) - \frac{\gamma}{\partial N}, & t \in [0, \tau], \ x \in \partial\Omega, \\
|D_t^m u(x, t)|_{t=\tau}, & t \in [\tau, 2\tau], \ x \in \partial\Omega.
\end{cases}
$$

The results of Step II tell us that

$$
\|\Phi\|_{C^{\alpha, \alpha \nu}(\Omega_{2\tau})} \leq C \|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_{\tau})} + \|\varphi_3\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\partial\Omega_{\tau})},
$$

and

$$
\|\Psi\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\partial\Omega_{2\tau})} \leq C \|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\Omega_{\tau})} + \|\varphi_3\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\partial\Omega_{\tau})}.
$$
After that, we define the function \( v = v(x, t) \) to be the solution of the initial-boundary problem
\[
\begin{align*}
\mathbf{D}_t^\nu v - \Delta v &= \Phi \quad \text{in} \quad \Omega_{2\tau}, \\
\mathbf{D}_t^\nu v - \frac{\partial^\nu v}{\partial \mathbf{n}} &= \Psi \quad \text{on} \quad \partial\Omega_{2\tau}, \\
v(x, 0) &= 0 \quad \text{in} \quad \bar{\Omega}.
\end{align*}
\tag{6.13}
\]
In light of the regularity of the right-hand side in \( \text{(6.13)} \) and the compatibility conditions \( \text{H5} \), together with the requirement \( \text{(6.1)} \), we can apply Theorem 2 in \cite{19} to \( \text{(6.13)} \), so to get the existence of a unique classical solution \( v \) satisfying the properties:
\[
v \in \mathcal{C}^{2+\alpha, \frac{\nu}{2\nu+1}}(\bar{\Omega}_{2\tau}), \quad \mathbf{D}_t^\nu v \in \mathcal{C}^{1+\alpha, \frac{\nu}{2\nu+1}}(\partial\Omega_{2\tau}),
\]
and
\[
v(x, t) = u(x, t) \quad \text{for} \quad (x, t) \in \bar{\Omega}_\tau.
\]
Now we are ready to look for the solution of \( \text{(1.1)}, \text{(6.2)}, \text{(1.5)} \) for \( t \in [0, 2\tau] \) in the form
\[
u(x, t) = U(x, t) + v(x, t),
\]
where the unknown function \( U(x, t) \) solves the problem
\[
\begin{align*}
\mathbf{D}_t^\nu (\varrho_1 U) - \mathbf{D}_t^\nu (\varrho_2 U) - \mathcal{L}_1 U - \mathcal{K} * \mathcal{L}_2 U &= f^* \quad \text{in} \quad \Omega_{2\tau}, \\
\mathbf{D}_t^\nu (\varrho_1 U) - \mathbf{D}_t^\nu (\varrho_2 U) - \mathcal{M}_1 U + \mathcal{K}_0 * \mathcal{M}_2 U &= \psi^* \quad \text{on} \quad \partial\Omega_{2\tau}, \\
U(x, 0) &= 0 \quad \text{in} \quad \bar{\Omega}.
\end{align*}
\tag{6.14}
\]
Here we set
\[
f^* = f - \mathbf{D}_t^\nu (\varrho_1 v) + \mathbf{D}_t^\nu (\varrho_2 v) + \mathcal{L}_1 v + \mathcal{K} * \mathcal{L}_2 v,
\]
\[
\psi^* = \psi_3 - \mathbf{D}_t^\nu (\varrho_1 v) + \mathbf{D}_t^\nu (\varrho_2 v) + \mathcal{M}_1 v - \mathcal{K}_0 * \mathcal{M}_2 v.
\]
Collecting properties of \( v \) with assumptions \( \text{H3}, \text{H4}, \text{(6.1)} \), and exploiting Remark 5.4 and \cite[Lemma 4.1]{21}, we arrive at the relations:
\[
\psi^* \in \mathcal{C}^{1+\alpha, \frac{\nu}{2\nu+1}}(\partial\Omega_{2\tau}), \quad f^* \in \mathcal{C}^{2+\alpha, \frac{\nu}{2\nu+1}}(\bar{\Omega}_{2\tau}),
\tag{6.15}
\]
and
\[
\psi^* \equiv 0 \quad x \in \partial\Omega, \quad f^* \equiv 0 \quad x \in \bar{\Omega}, \quad t \in [0, \tau].
\tag{6.16}
\]
In particular, appealing to the results of Step II, \( \text{(6.10)} \) tell us that
\[
U(x, t) = 0 \quad \text{for} \quad x \in \bar{\Omega}_\tau.
\tag{6.17}
\]
Finally, let us introduce the new time-variable
\[
\sigma = t - \tau \in [-\tau, \tau]
\]
in problem \( \text{(6.14)} \), and for every function \( \zeta \) appearing in the sequel we denote
\[
\tilde{\zeta}(x, \sigma) = \zeta(x, \sigma + \tau).
\]
and we call \( \bar{\mathcal{L}}_i \) and \( \bar{\mathcal{M}}_i \) the operators \( \mathcal{L}_i \) and \( \mathcal{M}_i \), respectively, with the \( \text{bar} \) coefficients. It is easy to verify that the coefficients of \( \bar{\mathcal{L}}_i, \bar{\mathcal{M}}_i \), and the functions \( \bar{\psi}, \bar{f}, \bar{\varrho}_1, \bar{\varrho}_2 \) meet the requirements of Theorem 4.1. Besides, relations \( \text{(6.15)-(6.17)} \) provide
\[
\bar{U} = \bar{\psi} = \bar{f} = 0 \quad \text{if} \quad \sigma \in [-\tau, 0],
\]
and
\[
\mathbf{D}_t^\nu \bar{U} = \mathbf{D}_t^\nu U, \quad \mathbf{D}_t^2 \bar{U} = \mathbf{D}_t^2 U, \quad \text{if} \quad \sigma \in [-\tau, \tau], \ t \in [0, \tau].
\]
It is worth noting that the latter two equalities above are examined in \cite{25} \( (3.111) \). Moreover, recasting the arguments in \cite{21} p.441, we conclude that
\[
(\mathcal{K} * \mathcal{L}_2 U)(x, t) = (\mathcal{K} * \mathcal{L}_2 \bar{U})(x, \sigma) \quad \text{and} \quad (\mathcal{K}_0 * \mathcal{M}_2 U)(x, t) = (\mathcal{K}_0 * \mathcal{M}_2 \bar{U})(x, \sigma).
\]
In order to rewrite problem (6.14) in the new variable, we are left to recalculate the terms: \( D_t^{\nu_1}(g_1 U) \) and \( D_t^{\nu_2}(g_2 U) \). Keeping in mind the homogenous initial condition and equality (6.17), we deduce that

\[
\Gamma (1 - \nu_1) D_t^{\nu_1}(g_1 U)(x, t) = \frac{\partial}{\partial t} \int_0^t g_1(x, s) U(x, s) ds + \frac{\partial}{\partial \sigma} \int_0^\sigma g_1(x, z + \tau) U(x, z + \tau) dz = \frac{\partial}{\partial \sigma} \int_0^\sigma \tilde{g}_1(x, z) U(x, z) dz + \Gamma (1 - \nu_1) D_\sigma^{\nu_1}(\tilde{g}_1 \tilde{U})(x, \sigma).
\]

Similar calculations entail the equality

\[
\Gamma (1 - \nu_2) D_t^{\nu_2}(g_2 U)(x, t) = D_\sigma^{\nu_2}(\tilde{g}_2 \tilde{U})(x, \sigma).
\]

As a result, we can rewrite problem (6.14) in the variable \( \sigma \) as

\[
\begin{cases}
D_\sigma^{\nu_1}(\tilde{g}_1 \tilde{U}) - D_\sigma^{\nu_2}(\tilde{g}_2 \tilde{U}) - \tilde{L}_1 \tilde{U} - \mathcal{K}_s \mathcal{L}_2 \tilde{U} = \tilde{f}(x, \sigma) & \text{in } \Sigma, \\
\tilde{U}(x, 0) = 0 & \text{in } \tilde{\Omega},
\end{cases}
\]

Recasting the arguments of Step II for this problem, we immediately draw the one-to-one classical solvability in \( C^{2+\alpha, \frac{2+\alpha}{\nu_1}} \) for \( \alpha \in [0, \tau], \) i.e., \( t \in [0, 2\tau] \). Other words, we have extended the solution \( u(x, t) \) from \([0, \tau]\) to \([\tau, 2\tau]\) if (6.14) holds. By the same token, we repeat this procedure to continue the constructed solution on the intervals \([i\tau, (i+1)\tau], \) \( i = 2, 3, \ldots \), until the whole interval \([0, T]\) is exhausted. This allows us to get the classical solution \( u(x, t) \) on \([0, T]\), satisfying the inequalities stated in Theorem 4.1. This completes the proof of Theorem 4.1 under the additional assumption (6.1).

**Remark 6.4.** In order to continue the solution \( u(x, t) \) from \([0, \tau]\) to \([\tau, T]\) in the DBC or 3BC cases, the initial-boundary value problem (6.15) is replaced by the initial-value problem:

\[
\begin{cases}
D_t^{\nu_1} v - \Delta v = \tilde{\Phi} & \text{in } \mathbb{R}_2^n, \\
v(x, 0) = 0 & \text{on } \mathbb{R}^n,
\end{cases}
\]

and

\[
\tilde{\Phi} = \begin{cases}
D_t^{\nu_1} \tilde{u} - \Delta \tilde{u}, & x \in \mathbb{R}^n, \ t \in [0, \tau], \\
D_t^{\nu_2} \tilde{u} - \Delta \tilde{u}, & x \in \mathbb{R}^n, \ t \in [\tau, 2\tau].
\end{cases}
\]

### 6.4 Step IV: Removing restriction (6.14).

To complete the proof of Theorem 4.1 we just need to remove the additional assumption (6.1). Again, we shall only focus on the FDBC case. Define

\[
\mathcal{W} = \mathcal{W}(x) \text{ and } \mathcal{U} = \mathcal{U}(x, t),
\]

and let \( \mathcal{U} = \mathcal{U}(x, t) \) be the solution to the problem

\[
\begin{cases}
\rho_1 D_t^{\nu_1} \mathcal{U} - \mathcal{L}_1 \mathcal{U} - \mathcal{W} = f(x, t) & (x, t) \in \Omega, \\
\rho_2 D_t^{\nu_2} \mathcal{U} - \mathcal{M}_1 \mathcal{U} - \mathcal{W} = \psi_3(x, t) & (x, t) \in \partial \Omega,
\end{cases}
\]

(6.18)

Indeed, the assumptions of Theorem 4.1 (see H1-H5) allow us to exploit [19] Theorem 2 and Remark 5.4 yielding the one-valued classical solvability of (6.18), along with the inequality

\[
\|\mathcal{U}\|_{c^{2+\alpha, \frac{2+\alpha}{\nu_1}(\partial \Omega)}} + \|D_\nu^{\nu_1}(\mathcal{U})\|_{c^{1+\alpha, \frac{1+\alpha}{\nu_1}(\partial \Omega)}} + \|D_\nu^{\nu_2}(\mathcal{U})\|_{c^{1+\alpha, \frac{1+\alpha}{\nu_2}(\partial \Omega)}} + \|D_\nu^{\nu_2}(\mathcal{U})\|_{c^{1+\alpha, \frac{1+\alpha}{\nu_2}(\partial \Omega)}} \leq C\|u_0\|_{c^{2+\alpha}(\Omega)} + \|f\|_{c^{1+\alpha, \frac{1+\alpha}{\nu_1}(\partial \Omega)}} + \|\psi_3\|_{c^{1+\alpha, \frac{1+\alpha}{\nu_2}(\partial \Omega)}}.
\]

(6.19)

Collecting this estimate with formula (10.34) in [24], and applying Corollary 3.1 in [23] and Remark 5.4 we obtain

\[
\|D_t^{\nu_1}(\rho_1 U) - D_t^{\nu_2}(\rho_2 U)\|_{t=0} = \rho_1(x, 0) D_t^{\nu_1} \mathcal{U}_{t=0} - \mathcal{W}(x).
\]

(6.20)
Then, coming to the original problem (1.1), (1.2), (1.3), we look for a solution of the form
\[ u(x, t) = \mathcal{V}(x, t) + \Omega(x, t), \]
where the new unknown \( \mathcal{V} = \mathcal{V}(x, t) \) solves the problem
\[
\begin{aligned}
&\mathbf{D}\mathcal{V}(\varrho_1 \mathcal{V}) - \mathbf{D}\mathcal{V}(\varrho_2 \mathcal{V}) - \mathcal{L}_1 \mathcal{V} + \mathcal{K} * \mathcal{L}_2 \mathcal{V} = \tilde{\mathbf{F}} \quad \text{in} \quad \Omega_T, \\
&\mathbf{D}\mathcal{V}(\varrho_1 \mathcal{V}) - \mathbf{D}\mathcal{V}(\varrho_2 \mathcal{V}) - \mathcal{M}_1 \mathcal{V} + \mathcal{K}_0 * \mathcal{M}_2 \mathcal{V} = \tilde{\mathbf{F}}_1 \quad \text{on} \quad \partial \Omega_T, \\
&\mathcal{V}(x, 0) = 0 \quad \text{in} \quad \tilde{\Omega}.
\end{aligned}
\] (6.21)

Here we set
\[ \tilde{\mathcal{F}} = f - \mathbf{D}\mathcal{V}(\varrho_1 \mathcal{U}) + \mathbf{D}\mathcal{V}(\varrho_2 \mathcal{U}) + \mathcal{L}_1 \mathcal{U} + \mathcal{K} * \mathcal{L}_2 \mathcal{U}, \]
\[ \tilde{\mathcal{F}}_1 = \psi_3 - \mathbf{D}\mathcal{V}(\varrho_1 \mathcal{U}) + \mathbf{D}\mathcal{V}(\varrho_2 \mathcal{U}) + \mathcal{M}_1 \mathcal{U} - \mathcal{K}_0 * \mathcal{M}_2 \mathcal{U}. \]

Relations (6.18)-(6.21) and Remarks 3.1 and 3.2 readily yield
\[
\| \tilde{\mathcal{F}} \|_{C^{1,1/2}(\Omega_T)} + \| \tilde{\mathcal{F}}_1 \|_{C^{1,1/2}(\partial \Omega_T)} \leq C \left[ \| u_0 \|_{C^{2+\alpha}(\Omega)} + \| f \|_{C^{1+\alpha}(\Omega)} \right],
\]
and
\[ \tilde{\mathcal{F}}(x, 0) = 0 \quad x \in \tilde{\Omega} \quad \text{and} \quad \tilde{\mathcal{F}}_1(x, 0) = 0 \quad x \in \partial \Omega, \]
which tell us that the right-hand sides of (6.21) meet the additional requirement (6.1). Thus, recasting the arguments of Steps I-III in the case of problem (6.21), and taking into account the representation of \( u(x, t) \) and (6.19)-(6.20), we complete the proof of the theorem in the FDBC case, without the restriction (6.1).

For the DBC or the 3BC cases, such a restriction is removed in a similar manner, but replacing problem (6.18) by
\[
\begin{aligned}
\varrho_1 \mathbf{D}\mathcal{V}(\varrho_1 \mathcal{U}) - \mathcal{L}_1 \mathcal{U} - \mathcal{W} = f(x, t) \quad (x, t) \in \Omega_T, \\
\mathcal{U}(x, 0) = u_0(x) \quad x \in \tilde{\Omega},
\end{aligned}
\]
subject either to the Dirichlet or the Neumann boundary condition. The proof of Theorem 4.1 is now finished.

7. Numerical Simulations

Once the well-posedness of the problem is established by our main Theorem 4.1 (see also Remark 4.4), one might like to find explicit solutions. To this aim, we implement a numerical scheme, and we apply it to a number of cases. In the forthcoming Examples 7.1-7.4 we consider a one dimensional domain \( \Omega \), whereas the last Example 7.5 is set in a two-dimensional domain. In particular, we examine the case
\[
\mathbf{D}\mathcal{V}(\varrho_1 \mathcal{U}) - \mathbf{D}\mathcal{V}(\varrho_2 \mathcal{U}) = - \frac{\partial}{\partial t}(\mathcal{N} * u),
\]
where the kernel
\[ \mathcal{N}(x) = \varrho_1(x)\omega_{1-\nu_1}(t) - \varrho_2(x)\omega_{1-\nu_2}(t) \]
is possibly nonpositive, as in Example 7.2.

Coming to Examples 7.1-7.4 we focus on the initial-boundary value problem in the one-dimensional domain \( \Omega = (0, 1) \)
\[
\begin{aligned}
\varrho_1(x) \mathbf{D}\mathcal{V}(u_t - \mathfrak{a}(x, t) \mathcal{D} u + \mathcal{D}(x, t) \frac{\partial u}{\partial x} - (\mathcal{K} * b \frac{\partial u}{\partial x}) = f(x, t) \quad \text{in} \quad \Omega_T, \\
u(x, 0) = u_0(x), \quad x \in [0, 1], \\
\mathcal{C}_1 \frac{\partial u}{\partial x}(0, t) + \mathcal{C}_2 u(0, t) = \varphi_1(t), \quad t \in [0, T], \\
\mathcal{C}_3 \frac{\partial u}{\partial x}(1, t) + \mathcal{C}_4 u(1, t) = \varphi_2(t), \quad t \in [0, T].
\end{aligned}
\] (7.1)

We introduce the space-time mesh with nodes
\[ x_k = kh, \quad \sigma_j = j \sigma, \quad k = 0, 1, \ldots, K, \quad j = 0, 1, \ldots, J, \quad h = L/K, \quad \sigma = T/J. \]
For these examples, we actually take \( L = 1, \quad K = 10^3 \) and \( J = 10^2 \). Denoting the finite-difference approximation of the function \( u \) at the point \((x_k, \sigma_j)\) by \( u_k^j \), and calling

\[
a_k^{j+1} = a(x_k, \sigma_{j+1}), \quad \sigma_k^{j+1} = 0(x_k, \sigma_{j+1}), \quad b_k^j = b(x_k, \sigma_j),
\]

\[
K_{m,j} = \int_{\sigma_m}^{\sigma_{m+1}} K(\sigma_{j+1} - s)ds, \quad \rho_m = (-1)^m \left( \nu_1 \right)_m, \quad \tilde{\rho}_m = (-1)^m \left( \nu_2 \right)_m,
\]

we approximate the differential equation in (7.1) at each time level \( \sigma_{j+1} \), so to obtain the finite-difference scheme

\[
\xi \frac{\partial}{\partial t} \left( \frac{\partial^\nu u}{\partial \sigma^\nu} \right)_{\sigma_{j+1}} = \frac{1}{h^2} \sum_{m=0}^{j+1} \left( b_k^{j+1 \nu} u_k^{j+1} - u_k^{j+1} \right) + \frac{\partial_{2,k}^{j+1 \nu}}{h^2} \left( u_k^{j+1} - u_k^{j+1} \right)
\]

\[
= \sum_{m=0}^{j} \left( b_k^{m \nu} u_k^{m} - 2u_k^{m} + u_k^{m+1} \right) + \frac{\partial_{2,k}^{m \nu}}{h^2} \left( u_k^{m+1} - u_k^{m+1} \right) \right) K_{m,j} + f(x_k, \sigma_{j+1}),
\]

for

\[
k = 1, \ldots, K - 1 \quad \text{and} \quad j = 0, 1, \ldots, J - 1.
\]

Here, the derivatives \( u_x \) and \( u_{xx} \) are approximated by the second-order finite-difference formulas; the trapezoid-rule is employed to approximate the integrals in the sum (see [24])

\[
\sum_{m=0}^{j} \int_{\sigma_m}^{\sigma_{m+1}} K(\sigma_{j+1} - s)b(x, s)u_{xx}(x, s)ds;
\]

and the Grünwald-Letnikov formula [11] is applied to approximate the fractional derivatives \( D_x^{\nu_1} u \) and \( D_t^{\nu_2} u \). It is worth noting that an improvement in the accuracy of the approximation of the fractional derivatives is achieved here by the Richardson extrapolation, see [4]. Finally, two fictitious mesh points outside the spatial domain to approximate the derivatives in the boundary conditions with the second order of accuracy are exploited (see, e.g., [24]). Further improvement in the accuracy of calculations may be reached by resorting to finite element methods [12, 44, 45], albeit we do not have the possibility to pursue this direction further here.

In all our examples, including in the 2-dimensional case treated later in Example 7.5, we can exhibit the exact solution \( u \), and the absolute error

\[
\mathbf{J} = \max |u - u_N|
\]

between \( u \) and the numerical solution \( u_N \), where the maximum is taken over all the grid points in the space-time mesh, is listed in Tables [I, 11].

**Example 7.1.** Consider problem (7.1) with \( T = 0.1 \) and

\[
K(t) = t^{-1/3}, \quad a(x, t) = \cos(\pi x/4) + t,
\]

\[
0(x, t) = x + t, \quad b(x, t) = t^{1/3} + \sin(\pi x),
\]

\[
\varphi_1(x) = 1 + x^2, \quad c_1 = c_3 = 1, \quad c_2 = c_4 = 0,
\]

\[
\varphi_2(t) = 0, \quad u_0(x) = \cos(\pi x),
\]

\[
f(x, t) = \pi^2 \left( \cos \frac{\pi x}{4} + t + \frac{3t^{2/3} \sin(\pi x)}{2} + \frac{t\pi}{3 \sin(\pi/3)} \right) \cos(\pi x)
\]

\[
-(x + t)\pi \sin(\pi x) - \frac{\varphi_2(x, t) \nu_1 - \nu_2}{\Gamma(1 + \nu_1 - \nu_2)} + 1 + x^2.
\]

As for the function \( \varphi_2(x, t) \), we have two options:
Table 1. Values of \( \sigma \) in Example 7.1 \( \rho_2(x, t) = 1 + (t + 1) (x + 0.01), \nu_2 = \nu_1 / 2 \).

| \nu_1  | \sigma     |
|--------|-----------|
| 0.1    | 1.6544e-02|
| 0.2    | 4.2775e-03|
| 0.3    | 2.1238e-03|
| 0.4    | 1.0632e-03|
| 0.5    | 5.1204e-04|
| 0.6    | 2.3459e-04|
| 0.7    | 9.9984e-05|
| 0.8    | 3.8166e-05|
| 0.9    | 2.1979e-05|

Table 2. Values of \( \sigma \) in Example 7.1 \( \rho_2(x, t) = (x - 1/2)^3, \nu_2 = \nu_1 / 3 \).

| \nu_1  | \sigma     |
|--------|-----------|
| 0.1    | 8.7910e-04|
| 0.2    | 1.4009e-04|
| 0.3    | 3.6891e-04|
| 0.4    | 3.5521e-04|
| 0.5    | 2.4190e-04|
| 0.6    | 1.3600e-04|
| 0.7    | 6.5636e-05|
| 0.8    | 2.6783e-05|
| 0.9    | 1.1683e-05|

Figure 1. Exact and numerical solutions in Example 7.1 at \( t = 0.1, \nu_1 = 0.1, \nu_2 = 0.05, \rho_2(x, t) = 1 + (t + 1)(x + 0.01) \).

(i) \( \rho_2(x, t) = 1 + (t + 1)(x + 0.01) \) if \( \nu_2 = \nu_1 / 2 \) with \( \nu_1 \) listed in Table 1
(ii) \( \rho_2(x, t) = (x - 0.5)^3 \) if \( \nu_2 = \nu_1 / 3 \) with \( \nu_1 \) listed in Table 2

It is easy to verify that the function

\[
u(x, t) = \cos(\pi x) + \frac{t^{\nu_1}}{\Gamma(1 + \nu_1)}
\]

the solves initial-boundary value problem 7.1 with the parameters specified above. The outcomes of this example (the absolute errors and the plot of numerical and analytical solutions) are given in Figure 1, Tables 1 and 2.
Example 7.2. In this test we examine \( \rho_2(x,t) = 0.5 \) and \( \rho_2(x,t) = 2.2 \) for \( T = 0.1 \) and \( T = 0.7 \), the remaining parameters being as in Example 7.1. The corresponding results are reported in Table 3. In Figures 2 and 3 we plot the kernel \( N \) for the different choice of parameters. Note that \( N \) changes its sign in the considered time period.
Example 7.3. Consider problem (7.1) with $T = 1$ and
\[
\begin{align*}
a(x, t) &= 1, \quad d(x, t) = 0, \quad b(x, t) = 1, \\
g_1(x) &= 1 + x, \quad g_2(x, t) = t \sin(2\pi x), \\
K(t) &= \frac{t^{\nu_2}}{\Gamma(1 - \nu_2)}, \quad c_1 = c_3 = 1, \quad c_2 = c_4 = 0, \\
u_0(x) &= \cos(\pi x), \quad \varphi_1(t) = \varphi_2(t) = 0, \\
f(x, t) &= \cos(\pi x) \left\{ (1 + x) \Gamma(1 + \nu_1) + \pi^2(1 + t^{\nu_1}) + \pi^2t(1 + \Gamma(1 + \nu_1)) + \frac{1 + x + \pi^2}{\Gamma(2 - \nu_2)} t^{1-\nu_1} \\
&\quad + \frac{\pi^2}{\Gamma(3 - \nu_1)} t^{2-\nu_1} - \left( \frac{t^{2-\nu_2}}{\Gamma(2 - \nu_2)} + \frac{\Gamma(1 + \nu_1) t^{1+\nu_1-\nu_2}}{\Gamma(1 + \nu_1 - \nu_2)} \right) \sin(2\pi x) \right\}.
\end{align*}
\]
Here, the analytic solution reads
\[
\begin{align*}
\frac{u(x, t) = [1 + t + t^{\nu_1}] \cos(\pi x).}
\end{align*}
\]
The outcomes of this example are listed in Table 4.

**Table 4. Values of $\mathcal{J}$ in Example 7.3 $\nu_2 = \nu_1/2$.**

| $\nu_1$ | $\mathcal{J}$   |
|-------|----------------|
| 0.15  | 7.4473e-04    |
| 0.25  | 1.2041e-03    |
| 0.35  | 1.1158e-03    |
| 0.45  | 6.5545e-04    |
| 0.55  | 2.5780e-04    |
| 0.65  | 2.1395e-04    |
| 0.75  | 2.6327e-04    |
| 0.85  | 2.7676e-04    |
| 0.95  | 1.7288e-04    |

Example 7.4. Consider problem (7.1) with $T = 1$ and
\[
\begin{align*}
a(x, t) &= (x + 1)(t + 1), \quad d(x, t) = x \sin t, \quad b(x, t) = 0, \\
g_2(x, t) &= t \cos(2\pi x), \quad g_1(x) = 2 + \sin(2\pi x), \\
K(t) &= 0, \quad c_1 = c_3 = 1, \quad c_2 = -2, \quad c_4 = 0, \\
u_0(x) &= 2x - x^2, \quad \varphi_1(t) = 2E_{\nu_1}(t^{\nu_1}), \quad \varphi_2(t) = 0, \\
f(x, t) &= E_{\nu_1}(t^{\nu_1}) \left\{ (2x - x^2)(2 + \sin(2\pi x)) + 2(x + 1)(t + 1) + x(2 - 2x) \sin t \right\} \\
&\quad - t^{1-\nu_2} \cos(2\pi x)(2x - x^2)\left(E_{\nu_1,1-\nu_2}(t^{\nu_1}) - 1/\Gamma(1 - \nu_2)\right),
\end{align*}
\]
whose exact solution is
\[
\begin{align*}
\frac{u(x, t) = [2x - x^2] E_{\nu_1}(t^{\nu_1}).}
\end{align*}
\]
The outcomes of this example are listed in Table 5.
Table 5. Values of \( \gamma \) in Example 7.4; \( \nu_2 = \nu_1/2 \).

| \( \nu_1 \) | \( \gamma \) |
|---|---|
| 0.1 | 4.6741e-03 |
| 0.2 | 3.3408e-03 |
| 0.3 | 1.9065e-03 |
| 0.4 | 8.0956e-04 |
| 0.5 | 3.3009e-04 |
| 0.6 | 2.2661e-04 |
| 0.7 | 1.7038e-04 |
| 0.8 | 1.1417e-04 |
| 0.9 | 4.6430e-05 |

Figure 4. Exact and numerical solutions in Example 7.4 at \( t = T \), \( \nu_1 = 0.9 \), \( \nu_2 = 0.45 \).

Our last test is set in the two-dimensional domain \( \Omega = (0, L_x) \times (0, L_y) \). Let us briefly describe the finite-difference scheme exploited in this case. We rewrite (1.1), (1.2), (1.4) in the more suitable form

\[
\begin{align*}
\rho_1(x,y)D_t^{\nu_1} u - \rho_2(x,y,t)D_t^{\nu_2} u &= a^1(x,y,t) \frac{\partial^2 u}{\partial y^2} - a^2(x,y,t) \frac{\partial^2 u}{\partial x^2} \\
&+ b^1(x,y,t) \frac{\partial u}{\partial x} + b^2(x,y,t) \frac{\partial u}{\partial y} - (\mathcal{K} * [b^1 \frac{\partial^2 u}{\partial x^2} + b^2 \frac{\partial^2 u}{\partial y^2}]) = f(x,y,t) \quad \text{in } \Omega_T, \\
u(x,y,0) &= u_0(x,y), \quad (x,y) \in \bar{\Omega}, \\
u(0,y,t) &= u(L_x,y,t) = 0, \quad t \in [0,T], \quad y \in [0,L_y], \\
\frac{\partial u}{\partial y}(x,0,t) &= \frac{\partial u}{\partial y}(x,L_y,t) = 0, \quad t \in [0,T], \quad x \in [0,L_x],
\end{align*}
\]

and we introduce the space-time mesh with nodes

\[
x_k = kh_x, \quad y_l = lh_y, \quad \sigma_j = j\sigma, \quad k = 0,1,\ldots,K_x, \quad l = 0,1,\ldots,K_y, \quad j = 0,1,\ldots,J, \\
h_x = L_x/K_x, \quad h_y = L_y/K_y, \quad \sigma = T/J.
\]
At each time level \( \sigma_{j+1} \), we approximate the differential equation in (7.2) via the finite-difference scheme

\[
q_{1,k,l}^{-1} \sum_{m=0}^{j+1} \left[ u_{k,l}^{j+1-m} - u_0(x_k, y_l) \right] \rho_m - \frac{\rho_{j+1}}{h^2} \sum_{m=0}^{j+1} \left[ u_{k,l}^{j+1-m} - u_0(x_k, y_l) \right] \rho_m
\]

\[
= - \frac{\rho_{j+1}}{h^2} \left[ u_{k+1,l}^{j+1} - 2u_{k,l}^{j+1} + u_{k-1,l}^{j+1} \right] + \frac{\rho_{j+1}}{h^2} \left( u_{k+1,l}^{j+1} - u_{k-1,l}^{j+1} \right)
\]

\[
= \sum_{m=0}^{j} \left[ \frac{\rho_{j+1}}{h^2} \left( u_{k+1,l}^{j+1} - 2u_{k,l}^{j+1} + u_{k-1,l}^{j+1} \right) + \frac{\rho_{j+1}}{h^2} \left( u_{k+1,l}^{j+1} - u_{k-1,l}^{j+1} \right) \right]
\]

for \( k = 1, \ldots, K_x - 1, \quad l = 1, \ldots, K_y - 1, \quad j = 0, 1, \ldots, J - 1. \)

Here we called \( u_{k,l}^j \) the finite-difference approximation of the function \( u \) at the point \((x_k, y_l, \sigma_j)\), and

\[
\rho_{j+1} = \rho_1 \sum_{m=0}^{j} \left( \frac{\rho_{j+1}}{h^2} \left( u_{k+1,l}^{j+1} - 2u_{k,l}^{j+1} + u_{k-1,l}^{j+1} \right) + \frac{\rho_{j+1}}{h^2} \left( u_{k+1,l}^{j+1} - u_{k-1,l}^{j+1} \right) \right)
\]

\[
\sigma_1 = \sigma_1(x_k, y_l, \sigma_j), \quad \sigma_2 = \sigma_2(x_k, y_l, \sigma_j),
\]

while \( \rho_0, \rho_m, \rho_{j+1} \) are defined as in the one-dimensional case.

**Example 7.5.** We analyze (7.2) with \( L_x = L_y = T = 1 \) and

\[
a_1 = \cos \frac{\pi x}{4} \cos \frac{\pi y}{4} + t, \quad a_2 = 2 \cos \frac{\pi x}{4} \cos \frac{\pi y}{4} + 2t,
\]

\[
b_1 = x + y + 1, \quad b_2 = 3 - x - y, \quad d_1 = x + y + t, \quad d_2 = x + y - t,
\]

\[
K = \frac{t^{-\nu_1}}{\Gamma(1 - \nu_1)}, \quad q_1 = 1 + x^2 + y^2, \quad q_2 = 1 + (t + 1)(x + y + 0.01), \quad u_0 = \sin(\pi x) \cos(\pi y),
\]

\[
f = \left( 1 + t^2 \right) \left( \Gamma(1 + \nu_1) + \frac{t^{1-\nu_1}}{\Gamma(2 - \nu_1)} \right) - \left[ 1 + (t + 1)(x + y + 0.01) \right] \left( \frac{t^{1-\nu_2}}{\Gamma(2 - \nu_2)} + \frac{t^{1-\nu_2} - t^{1-\nu_2} \Gamma(1 + \nu_1)}{\Gamma(3 - \nu_1)} \right)
\]

\[
+ 3\pi^2(1 + t + t') \left[ t + \cos \frac{\pi x}{4} \cos \frac{\pi y}{4} + 4\pi^2 \left[ t\Gamma(1 + \nu_1) + \frac{t^{1-\nu_2}}{\Gamma(2 - \nu_2)} - \frac{t^{1-\nu_2} \Gamma(1 + \nu_1)}{\Gamma(3 - \nu_1)} \right] \sin(\pi x) \cos(\pi y)
\]

\[
+ \pi(1 + t + t')(x + y + t \cos \pi x + t \sin \pi x) \right]
\]

The function

\[
u(x,y) = \frac{1}{1 + t + t'} \sin(\pi x) \cos(\pi y)
\]

solves the initial-boundary value problem (7.2) for this choice of parameters. In our numerical calculations, we set \( K_x = K_y = J = 10^2 \). Table 6 reports the results for various values \( \nu_1 \), while Figure 5 plots the corresponding numerical solution at \( \nu_1 = 0.5 \).

8. Conclusion

In this paper, we propose an approach to study the well-posedness of initial-boundary value problems subject to various type of boundary conditions for multi-term fractional derivatives. Our method is particularly efficient when the multi-term derivatives can be represented in the form \( \frac{d}{dt} (\mathcal{N} * u) \), for some nonpositive kernel \( \mathcal{N} \). We find sufficient conditions on the orders of the fractional derivatives, providing the one-valued classical solvability in the smooth classes. Our theoretical result are confirmed by the computational outcomes, and the numerical examples witness the high accuracy and efficacy of the proposed numerical schemes. A possible further development of this research regards the inverse
Table 6. Values of $\lambda$ in Example 7.5 $\nu_2 = \nu_1/2$.

| $\nu_1$ | $\lambda$ |
|---------|-----------|
| 0.1     | 6.4793e-04 |
| 0.2     | 7.8800e-04 |
| 0.3     | 5.6016e-04 |
| 0.4     | 3.4389e-04 |
| 0.5     | 3.1859e-04 |
| 0.6     | 3.2473e-04 |
| 0.7     | 3.3240e-04 |
| 0.8     | 3.0207e-04 |
| 0.9     | 1.8727e-04 |

Figure 5. Numerical solution in Example 7.5 at $t = T$, $\nu_1 = 0.5$, $\nu_2 = 0.25$.

problem related with the identification of the parameters in the model of oxygen subdiffusion through capillaries. Also, the complete knowledge of the linear case is a starting point for the investigation of the corresponding nonlinear equations, including equations with degenerate coefficients.

References

[1] P. Clément, G. Gripenberg, S.O. Londen, Schauder estimates for equations and continuous interpolation spaces, J. Differential Equa., 196 (2004) 418–447.
[2] V. Daftardar-Gejji, S. Bhalekar, Boundary value problems for multi-term fractional differential equations, J. Math. Anal. Appl., 345 (2008) 754–765.
[3] M. D'Ovidio, Fractional boundary value problems and elastic sticky Brownian motions, arXiv:2205.04162, 2022.
[4] K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, Algorithms for the fractional calculus: A selection of numerical methods, Comput. Methods Appl. Mech. Engrg., 194 (2005) 743–773; DOI: 10.1016/j.cma.2004.06.006.
[5] C.G. Gal, M. Warma, Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions, Evol. Equa., Control Theory, 5 (2016) 61–103.
[6] J.G. Go, Oxygen delivery through capillaries, Math. Biosci., 208 (2007) 166–176.
[7] D. Goldman, Theoretical models of microvascular oxygen transport to tissue, Microcirculation, 15 (2008) 795–811.
[8] D. Goldman, A.S. Popel, A computational study of the effect of vasomotion on oxygen transport from capillary networks, J. Theor. Biology, 209 (2001) 189–199.
[9] G.R. Goldstein, Derivation and physical interpretation of general boundary conditions, Adv. Differenti. Equa., 11 (2006) 457–480.
[10] J. Janno, Determination of the order of fractional derivatives and a kernel in an inverse problem for a generalized time fractional diffusion equation, Electron. J. Differential Equa., 2016 (2016) 1–28.
[11] J. Janno, N. Kinash, Reconstruction of an order of derivative and a source term in a fractional diffusion equation from final measurements, Inverse Problems, 34 (2018) 02507.
[12] B. Jin, R. Lazarov, Z. Zhou, Numerical methods for time-fractional evolution equations with nonsmooth data: A concise overview, Comput. Methods Appl. Mech. Engrg., 346 (2019) 332–358.

[13] A.S. Joujchi, M.H. Derakhshan, H.R. Marasi, An efficient hybrid numerical method for multi-term time fractional partial differential equations in fluid mechanics with convergence and error analysis, Commun. Nonlinear Sci. Numer. Simulations, 114 (2022) 106620.

[14] J. Kemppainen, K. Ruotsalainen, Boundary integral solution of the time-fractional diffusion equation, Integr. Equ. Oper. Theory, 64 (2009) 239–249.

[15] J. Kemppainen, J. Siljander, V. Vergara, R. Zacher, Decay estimates for time-fractional and other nonlocal in time subdiffusion equations in R^d, Math. Ann., 366 (2016) 941–979.

[16] I. Kim, H-H. Kim, S. Lim, On L_p(L_q)–theory for the time fractional evolution equations with variable coefficients, Advances Math., 306 (2017) 123–176.

[17] M. Kirane, N. Tatar, Absence of local and global solutions to an elliptic system with time-fractional dynamical boundary conditions, Siberian Math. J., 48 (2007) 477–488.

[18] A. Kochubei, General fractional calculus, evolution equations, and renewal processes, Integr. Equ. Oper. Theory, 71 (2011) 583–600.

[19] M. Krasnoschok, Time-fractional diffusion equation with dynamical boundary condition, Fractional Diff. Calculus, 6 (2016) 151–178.

[20] M.V. Krasnoschok, Solvability in Hölder space of an initial boundary value problem for the time-fractional diffusion equation, J. Math. Phys. Anal. Geometry, 12 (2016) 48–77.

[21] M. Krasnoschok, V. Pata, N. Vasylyeva, Solvability of linear boundary value problems for subdiffusion equation with memory, J. Integral Equations Appl., 30 (2018) 417–445.

[22] M. Krasnoschok, V. Pata, N. Vasylyeva, Semilinear subdiffusion with memory in multidimensional domains, Mathematische Nachrichten, 292 (2019) 1490–1513.

[23] M. Krasnoschok, V. Pata, S.V. Siryk, N. Vasylyeva, Equivalent definitions of Caputo derivatives and applications to subdiffusion equations, Dynamics of PDE, 17 (2020) 383–402.

[24] M. Krasnoschok, S. Pereverzyev, S.V. Siryk, N. Vasylyeva, Regularized reconstruction of the order in semilinear subdiffusion with memory, (In: Cheng J., Lu S., Yamamoto M. (Eds.) Inverse Problems and Related Topics ICIP2 2018), Springer Proceedings in Mathematics&Statistics, 310 (2020) 205–236, doi:10.1007/978-981-15-1592-7-10.

[25] M. Krasnoschok, N. Vasylyeva, On a solvability of a nonlinear fractional reaction-diffusion system in the Hölder spaces, Nonlinear Studies, 20 (2013) 589–619.

[26] M. Krasnoschok, N. Vasylyeva, Existence and uniqueness of the solutions for some initial-boundary value problems with the fractional dynamic boundary condition, International J. Part. Diff. Equa., 2013 (2013) ID 796430.

[27] A. Krogh, Oxygen diffusive shunts under conditions of heterogeneous oxygen delivery, J. Physiol., 52 (1919) 409–415.

[28] O.A. Ladyzhenskaia, V.A. Solonnikov, N.N. Ural’tseva, Linear and quasilinear parabolic equations, Academic Press, New York, 1968.

[29] R.M. Leach, D.F. Treacher, ABC of oxygen. Oxygen transport-2. Tissue hypoxia, Clinical Rev. BMJ, 317 (1998) 1370–1373.

[30] J. Lecoq, A. Parpaleix, E. Roussakis, M. Ducros, Y.G. House, S.A. Vinogradov, et al., Simultaneous two photon imaging of oxygen and blood flow in deep cerebral vessels, Nature Medicine., 17 (2011) 893–901.

[31] Z. Li, Y. Liu, M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients, Appl. Math. Comput., 257 (2015) 381–397.

[32] Y. Liu, M. Yamamoto, Uniqueness of orders and parameters in multi-term time-fractional diffusion equations by in exact date, arXiv:2206.02108v1, 2022.

[33] D. Liu, N. Wood, N. Witt, A. Hughes, S. Thom, X. Xu, Computational analysis of oxygen transport in the retinal arterial network, Current Eye Res., 34 (2009) 945–956.

[34] C. Lizama, G.M. Guérékata, Bounded mild solutions for semilinear integro-differential equations, J. Integral Equa. Appl., 5 (1993) 75-78.

[35] Y. Luchko, M. Yamamoto, General time-fractional diffusion equation: Some uniqueness and existence results for the initial-boundary value problems, Fract. Calc. Appl. Anal., 19 (2016) 676–695.

[36] Y. Luchko, A. Suzuki, M. Yamamoto, On the maximum principle for the multi-term fractional transport equation, J. Math. Anal. Appl., 505 (2022) 125579.

[37] V.F. Marales-Delgado, J.F. Gómez-Aguilar, K.M. Saad, M.A. Khan, P. Agarwal, Analytical solution for oxygen diffusion from capillary to tissues involving external force effects: A fractional calculus approach, Physica A, 523 (2019) 48–65.

[38] A.S. Popel, Oxygen diffusive shunts under conditions of heterogeneous oxygen delivery, J. Theor. Biol., 96 (1982) 533-541.

[39] A.S. Popel, Theory of oxygen transport to tissues, Crit. Rev. Biomed. Eng., 17 (1989) 257–321.

[40] T. Sandev, I.M. Sokolov, R. Metzler, A. Chechkin, Beyond monofractional kinetics, Chaos, Solitons&Fractals, 102 (2017) 210–217.

[41] T. Sandev, R. Metzler, A. Chechkin, From continuous time random walks to the generalized diffusion equation, Fract. Calc. Appl. Anal., 21 (2018) 19–28.

[42] T.W. Secomb, R. Hsu, E.Y. Park, M.W. Dewhirst, Green’s function methods for analysis of oxygen delivery to tissue by microvascular networks, Annals Biomed. Engineering, 32 (2004) 1519–1529.
[43] C-S. Sin, Well-posedness of general Caputo-type fractional differential equations, Fract. Calc. Appl. Anal., 21 (2018) 819–832.
[44] S.V. Siryk, A note on the application of the Guermond-Pasquetti mass lumping correction technique for convection-diffusion problems, J. Comput. Phys., 376 (2019) 1273–1291; DOI: 10.1016/j.jcp.2018.10.016.
[45] S.V. Siryk, Analysis of lumped approximations in the finite-element method for convection-diffusion problems, Cybernetics and Systems Analysis, 49 No. 5, (2013) 774–784; DOI: 10.1007/s10559-013-9565-5
[46] V. Srivastava, K.N. Rai, A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues, Math. Comput. Modelling, 51 (2010) 616–624.
[47] N. Vasylyeva, L. Vynnytska, On multidimensional moving boundary problem governed by anomalous diffusion: analytical and numerical study, NoDEA: Nonlinear Differ. Equa. Appl., 22 (2015) 543–577.
[48] V. Vergara, R. Zacher, Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods, SIAM J. Math. Anal., 47 (2015) 210–239.
[49] V. Vergara, R. Zacher, Stability, instability and blowup for time fractional and other nonlocal in time semilinear subdiffusion equations, J. Evol. Equa., 17 (2017) 599–626.
[50] C.Y. Wang, J.B. Bassingthwaighte, Capillary supply regions, Math. Biosci., 173 (2001) 103.
[51] M. Yamamoto, Fractional derivatives and time-fractional ordinary differential equations in $L_p$–space, arXiv:2201.07094v1, 2022.
[52] R. Zacher, Maximal regularity of type $L_p$ for abstract parabolic Volterra equations, J. Evol. Equa., 5 (2005) 79–103.
[53] J. Zhang, F. Liu, Z. Lin, V. Anh, Analytical and numerical solutions of a multi-term time-fractional Burgers fluid model, Appl. Math. Comput., 356 (2019) 1–22.

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