Two Approaches to Measurability Concept and Quantum Theory

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Abstract

In the present paper, in terms of the measurability concept introduced in the previous works of the author, a quantum theory is studied. Within the framework of this concept, several examples are considered using the Schrodinger picture; analogs of Fourier transformations from the momentum representation to the coordinate one and vice versa are constructed. It is shown how to produce a measurable analog of the Heisenberg picture. At the end of this paper the obtained results are used to substantiate another (more general) definition of the measurability concept that is not based on the Heisenberg Uncertainty Principle and its generalization, as it has been in the previous works of the author, and may be equally suitable for both non-relativistic and relativistic cases.

1 Introduction.

The present paper is a continuation of the previous works published by the author on the subject [1]–[5]. The main idea and target of these works is to construct a correct quantum theory and gravity in terms of the variations (increments) dependent on the existent energies. It is clear that such a theory should not involve infinitesimal space-time variations

$$dt, dx_i, i = 1, ..., 3.$$ (1)

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The main instrument in the above-mentioned articles was the measurability concept introduced in [2].

Within the framework of this concept, a theory becomes discrete but at low energies \( E \) far from the Planck energies \( E \ll E_p \) it becomes very close to the initial theory in the continuous space-time paradigm.

Quantum mechanics is studied in the present work in terms of the measurability notion. In Sections 2,3 the results earlier obtained by the author are elaborated as applied to non-relativistic and relativistic quantum theories. Section 4 presents a measurable analog of the wave function and the Schrödinger Equation as well as the main differential operators involved in quantum mechanics, in particular, the Laplace operator.

Measurable analogs of the momentum projection operator and the momentum angular projection operator are studied.

In terms of the measurability concept, analogs of Fourier transformations are constructed. It is shown how to produce a measurable analog of the Heisenberg picture.

At the end of the article, in Section 5, the obtained results are used to substantiate another (more general) definition of the measurability concept that is not based on the Heisenberg Uncertainty Principle and its generalizations, as it has been in the earlier works of the author, and may be equally suitable both for non-relativistic and relativistic cases.

### 2 Measurability Concept

#### 2.1 Primary Measurability in Nonrelativistic Case. Brief Consideration of the Principal Assumptions

In this Subsection we briefly recall the principal assumptions [2],[4],[5] that underlie further research.

According to Definition I. from [2],[4],[5] we call as primarily measurable variation any small variation (increment) \( \Delta x_i \) of any spatial coordinate \( x_i \) of the arbitrary point \( x_i, i = 1, ..., 3 \) in some space-time system \( R \) if
it may be realized in the form of the uncertainty (standard deviation) \( \Delta x_i \) when this coordinate is measured within the scope of Heisenberg’s Uncertainty Principle (HUP)\[6],[7].

Similarly, we call any small variation (increment) \( \tilde{\Delta} x_0 = \tilde{\Delta} t_0 \) by primarily measurable variation in the value of time if it may be realized in the form of the uncertainty (standard deviation) \( \Delta x_0 = \Delta t \) for pair “time-energy” \((t, E)\) when time is measured within the scope of Heisenberg’s Uncertainty Principle (HUP) too.

Next we introduce the following assumption:

**Supposition II.** There is the minimal length \( l_{\min} \) as a minimal measurement unit for all primarily measurable variations having the dimension of length, whereas the minimal time \( t_{\min} = l_{\min}/c \) as a minimal measurement unit for all quantities or primarily measurable variations (increments) having the dimension of time, where \( c \) is the speed of light.

According to HUP \( l_{\min} \) and \( t_{\min} \) lead to \( P_{\max} \) and \( E_{\max} \). For definiteness, we consider that \( E_{\max} \) and \( P_{\max} \) are the quantities on the order of the Planck quantities, then \( l_{\min} \) and \( t_{\min} \) are also on the order of Planck quantities \( l_{\min} \propto l_P, \ t_{\min} \propto t_P \). **Definition I.** and **Supposition II.** are quite natural in the sense that there are no physical principles with which they are inconsistent.

The combination of **Definition I.** and **Supposition II.** will be called the Principle of Bounded Primarily Measurable Space-Time Variations (Increments) or for short Principle of Bounded Space-Time Variations (Increments) with abbreviation (PBSTV).

As the minimal unit of measurement \( l_{\min} \) is available for all the primarily measurable variations \( \Delta L \) having the dimensions of length, the “Integrality Condition” (IC) is the case

\[
\Delta L = N_{\Delta L} l_{\min},
\]

where \( N_{\Delta L} > 0 \) is an integer number.

In a like manner the same “Integrality Condition” (IC) is the case for all the primarily measurable variations \( \Delta t \) having the dimensions of time.

And similar to Equation (2), we get for any time \( \Delta t \):

\[
\Delta t \equiv \Delta t(N_t) = N_{\Delta t} t_{\min},
\]

where \( N_{\Delta t} > 0 \) is an integer number.
where similarly \( N_{\Delta t} > 0 \) is an integer number too.

**Definition 1 (Primary or Elementary Measurability.)**

1. In accordance with PBSTV, let us define the quantity having the dimensions of length or time as **primarily (or elementarily) measurable**, when it satisfies the relation Equation (2) (and respectively Equation (3)).

2. Let us define any physical quantity **primarily (or elementarily) measurable**, when its value is consistent with points (1) of this Definition.

Here HUP is given for the nonrelativistic case. In the next subsection we consider the relativistic case for low energies \( E \ll E_P \) and show that for this case **Definition 1** of the (Primary Measurability) retains its meaning.

Further for convenience, we everywhere denote the minimal length \( l_{\text{min}} \neq 0 \) by \( \ell \) and \( t_{\text{min}} \neq 0 \) by \( \tau = \ell/c \).

### 2.2 Primary Measurability in Relativistic Case

In the Relativistic case HUP has its distinctive features ([8], Introduction). As known, in the relativistic case for **low energies** \( E \ll E_P \), when the total energy of a particle with the mass \( m \) and with the momentum \( p \) equals [9]:

\[
E = \sqrt{p^2c^2 + m^2c^4},
\]

(4)
a minimal value of \( \Delta x \) in the general case takes the form ([8], formula(1.3))

\[
\Delta q \approx \frac{ch}{E} = \frac{\hbar}{\sqrt{p^2 + m^2c^2}}.
\]

(5)

In this case, as well, nothing prevents the existence of the minimal length \( \ell \neq 0 \) and the minimal time \( \tau = \ell/c \) or execution of the conditions (2) and (3). Particularly, in the equation (2) for \( \Delta L = \Delta q \), due to the fact that \( E \ll E_P \), we have

\[
\Delta q = N_{\Delta q}\ell; \ N_{\Delta q} \gg 1.
\]

(6)

The formula (5) may be rewritten as

\[
E \approx \frac{ch}{N_{\Delta q}\ell}.
\]

(7)
Because $N_{\Delta q} \gg 1$ is an integer, in the general case the energy $E$ may vary almost continuously, similarly to the canonical theory with $\ell = 0$. A similar equation (7) in this case may be derived for the momentum $p$ from the right side of (5) too. It is clear that in the general case $p$ is also varying steadily. An analogue of (7) is easily obtained in the ultrarelativistic case ($E \approx p$) and in the rest frame of the particle mass ($E \approx mc^2$). It is obvious that, according to the above-mentioned equations, at low energies the picture is practically continuous.

In the relativistic case, at least at low energies $E \ll E_P$, Definition 1 of the (Primary Measurability) from the previous subsection is also meaningful, though within the framework of the Uncertainty Principle for a Relativistic System ([8],[9],Introduction).

3 Generalized Measurability

3.1 Generalized Measurability and Generalized Uncertainty Principle

Basic results of this Subsection are given in [2] and [3]. Further it is convenient to use the deformation parameter $\alpha_a$. This parameter has been introduced earlier in the papers [10],[11],[12]–[15] as a deformation parameter (in terms of paper [16]) on going from the canonical quantum mechanics to the quantum mechanics at Planck’s scales (Early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = \ell^2/a^2,$$

where $a$ is the primarily measuring scale. It is easily seen that the parameter $\alpha_a$ from Equation (8) is discrete as it is nothing else but

$$\alpha_a = \ell^2/a^2 = \frac{\ell^2}{N_a^2\ell^2} = \frac{1}{N_a^2},$$

for primarily measurable $a = N_a\ell$.

At the same time, from Equation (9) it is evident that $\alpha_a$ is irregularly discrete.
It should be noted that, physical quantities complying with Definition 1 won’t be enough for the research of physical systems. Indeed, such a variable as
\[ \alpha_{N_a\ell}(N_a\ell) = p(N_a)\frac{\ell}{\hbar} = \ell/N_a, \] (10)
(where \( \alpha_{N_a\ell} = \alpha_a \) is taken from formula (9) at \( a = N_a\ell \), and \( p(N_a) = \frac{\hbar}{N_a\ell} \) is the corresponding primarily measurable momentum), is fully expressed in terms only Primarily Measurable Quantities of Definition 1 and that’s why it may appear at any stage of calculations, but apparently doesn’t comply with Definition 1. That’s why it’s necessary to introduce the following definition generalizing Definition 1:

Definition 2. Generalized Measurability
We shall call any physical quantity as generalized-measurable or for simplicity measurable if any of its values may be obtained in terms of Primarily Measurable Quantities of Definition 1.

In what follows, for simplicity, we will use the term Measurability instead of Generalized Measurability. It is evident that any primarily measurable quantity (PMQ) is measurable. Generally speaking, the contrary is not correct, as indicated by formula (10). It should be noted that Heisenberg’s Uncertainty Principle (HUP) [7] is fair at low energies \( E \ll E_P \). However it was shown that at the Planck scale a high-energy term must appear:
\[ \Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar} \] (11)
where \( l_p \) is the Planck length \( l_p^2 = G\hbar/c^3 \approx 1.6 \times 10^{-35} m \) and \( \alpha' \) is a constant. In [17] this term is derived from the string theory, in [18] it follows from the simple estimates of Newtonian gravity and quantum mechanics, in [19] it comes from the black hole physics, other methods can also be used [21, 20, 26]. Relation (11) is quadratic in \( \Delta p \)
\[ \alpha' l_p^2 (\Delta p)^2 - \hbar \Delta x \Delta p + \hbar^2 \leq 0 \] (12)
and therefore leads to the minimal length

\[ \Delta x_{\text{min}} = 2\sqrt{\alpha' l_p} \cong \ell \]  

(13)

Inequality (11) is called the Generalized Uncertainty Principle (GUP) in Quantum Theory.

Let us show that the generalized-measurable quantities are appeared from the Generalized Uncertainty Principle (GUP) [17]–[28] (formula (11)) that naturally leads to the minimal length \( \ell \) (13). Really solving inequality (11), in the case of equality we obtain the apparent formula

\[ \Delta p_{\pm} = \frac{(\Delta x \pm \sqrt{(\Delta x)^2 - 4\alpha' l_p^2})\hbar}{2\alpha' l_p^2}. \]  

(14)

Next, into this formula we substitute the right-hand part of formula (2) for primarily measurable \( L = x \). Considering (13), we can derive the following:

\[ \Delta p_{\pm} = \frac{(N\Delta x \pm \sqrt{(N\Delta x)^2 - 1})\hbar \ell}{2\ell^2} = \frac{2(N\Delta x \pm \sqrt{(N\Delta x)^2 - 1})\hbar}{\ell}. \]  

(15)

But it is evident that at low energies \( E \ll E_p; N\Delta x \gg 1 \) the plus sign in the numerator (15) leads to the contradiction as it results in very high (much greater than the Planck’s) values of \( \Delta p \). Because of this, it is necessary to select the minus sign in the numerator (15). Then, multiplying the left and right sides of (15) by the same number \( N\Delta x + \sqrt{N^2\Delta x^2 - 1} \), we get

\[ \Delta p = \frac{2\hbar}{(N\Delta x + \sqrt{N^2\Delta x^2 - 1})\ell}. \]  

(16)

\( \Delta p \) from formula (16) is the generalized-measurable quantity in the sense of Definition 2. However, it is clear that at low energies \( E \ll E_p \), i.e. for \( N\Delta x \gg 1 \), we have \( \sqrt{N^2\Delta x^2 - 1} \approx N\Delta x \). Moreover, we have

\[ \lim_{N\Delta x \to \infty} \sqrt{N^2\Delta x^2 - 1} = N\Delta x. \]  

(17)
Therefore, in this case (16) may be written as follows:

\[ \Delta p = \Delta p(N_{\Delta x}, HUP) = \frac{\hbar}{1/2(N_{\Delta x} + 1/\sqrt{N_{\Delta x}^2 - 1}) \ell} \approx \frac{\hbar}{N_{\Delta x} \ell} = \frac{\hbar}{\Delta x}; N_{\Delta x} \gg 1. \] (18)

in complete conformity with HUP. Besides, \( \Delta p = \Delta p(N_{\Delta x}, HUP) \), to a high accuracy, is a **primarily measurable** quantity in the sense of **Definition 1**.

And vice versa it is obvious that at high energies \( E \approx E_p \), i.e. for \( N_{\Delta x} \approx 1 \), there is no way to transform formula (16) and we can write

\[ \Delta p = \Delta p(N_{\Delta x}, GUP) = \frac{\hbar}{1/2(N_{\Delta x} + 1/\sqrt{N_{\Delta x}^2 - 1}) \ell}; N_{\Delta x} \approx 1. \] (19)

At the same time, \( \Delta p = \Delta p(N_{\Delta x}, GUP) \) is a **Generalized Measurable** quantity in the sense of **Definition 2**.

Thus, we have

\[ GUP \rightarrow HUP \] (20)

for

\[ (N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \] (21)

Also, we have

\[ \Delta p(N_{\Delta x}, GUP) \rightarrow \Delta p(N_{\Delta x}, HUP), \] (22)

where \( \Delta p(N_{\Delta x}, GUP) \) is taken from formula (19), whereas \( \Delta p(N_{\Delta x}, HUP) \) from formula (18).

**Comment 2*. From the above formulae it follows that, within GUP, the **primarily measurable** variations (quantities) are derived to a high accuracy from the **generalized-measurable** variations (quantities) only in the low-energy limit \( E \ll E_P \).

Next, within the scope of GUP, we can correct a value of the parameter \( \alpha_a \) from formula (9) substituting \( a \) for \( \Delta x \) in the expression \( 1/2(N_{\Delta x} + 1/\sqrt{N_{\Delta x}^2 - 1}) \ell \).
Then at low energies \(E \ll E_p\) we have the **primarily measurable** quantity \(\alpha_a(HUP)\)

\[
\alpha_a \doteq \alpha_a(HUP) = \frac{1}{[1/2(N_a + \sqrt{N_a^2 - 1})]^2} \approx \frac{1}{N_a^2}; N_a \gg 1, \tag{23}
\]

that corresponds, to a high accuracy, to the value from formula (9).

Accordingly, at high energies we have \(E \approx E_p\)

\[
\alpha_a \doteq \alpha_a(GUP) = \frac{1}{[1/2(N_a + \sqrt{N_a^2 - 1})]^2}; N_a \approx 1. \tag{24}
\]

When going from high energies \(E \approx E_p\) to low energies \(E \ll E_p\), we can write

\[
\alpha_a(GUP) \overset{(N_a \approx 1)\rightarrow (N_a \gg 1)}{\rightarrow} \alpha_a(HUP) \tag{25}
\]

in complete conformity to Comment 2*.

**Remark 3.1** What is the main difference between **Primarily Measurable Quantities (PMQ)** and **Generalized Measurable Quantities (GMQ)**? **PMQ** defines variables which may be obtained as a result of an immediate experiment. **GMQ** defines the variables which may be calculated based on **PMQ**, i.e. based on the data obtained in previous clause.

**Remark 3.2.** It is readily seen that a minimal value of \(N_a = 1\) is unattainable because in formula (19) we can obtain a value of the length \(l\) that is below the minimum \(l < \ell\) for the momenta and energies above the maximal ones, and that is impossible. Thus, we always have \(N_a \geq 2\). This fact was indicated in [10], [11], however, based on the other approach.

As follows from the above formula, the **generalized measurable** momenta at all energies are of the form

\[
p_{1/N} \doteq p(1/N, \ell), N \neq 0, \tag{26}
\]

where \(\ell = \kappa l_p\) up to the constant \(\kappa\) on the order of 1.

Therefore, \(p_{1/N}\) depends only on three fundamental constants \(c, \hbar, G\), the
constant $\kappa$, and the discrete parameter $1/N$. But for $N \gg 1$, i.e. at $E \ll E_p$, the mapping of $\tau : 1/N \Rightarrow p_1/N$ is actually continuous providing a high accuracy of coincidence between this discrete model and the initial continuous theory.

The main target of the author is to resolve a quantum theory and gravity in terms of the concept of primarily measurable quantities. As in this case these theories become discrete, in what follows we use the lattice representation.

3.2 Space and Momentum Lattices of Generalized Measurable Quantities, and $\alpha$ – lattice

In this subsection the results from [2],[4] are refined and supplemented. So, provided the minimal length $\ell$ exists, two lattices are naturally arising by virtue of the formulas from the previous subsection.

I. At low energies (LE) $E \ll E_{\text{max}} \propto E_P$, lattice of the space variation—$\text{Lat}_S[LE]$ representing, for sets integers $|N_w| \gg 1$ to within the known multiplicative constants, in accordance with the above formulas for each of the three space variables $w = x; y; z$.

$$\text{Lat}_S[LE] = (N_w \doteq \{N_x, N_y, N_z\}), |N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1.$$  \hspace{1cm} (27)

At high energies (HE) $E \rightarrow E_{\text{max}} \propto E_P$ to within the known multiplicative constants too in accordance with the formulas previous subsection we have the lattice $\text{Lat}_S[HE]$ for each of the three space variables $w = x; y; z$.

$$\text{Lat}_S[HE] \doteq (\pm 1/2[(N_w + \sqrt{N_w^2 - 1})]/2 \leq (N_w \doteq \{N_x, N_y, N_z\}) \approx 1.$$ \hspace{1cm} (28)

II. Next, let us define the lattice momentum variation $\text{Lat}_P$ as a set to obtain $(p_x, p_y, p_z)$ for low energies $E \ll E_P$, where all the components of the above sets conform to the space coordinates $(x, y, z)$, given by the corresponding formulae from the previous subsection.

From this it is inferred that, in analogy with point I of this subsection, within the known multiplicative constants, we have lattice $\text{Lat}_P[LE]$

$$\text{Lat}_P[LE] \doteq (\frac{1}{N_w}),$$ \hspace{1cm} (29)
where $N_w$ are integer numbers from Equation (27). In accordance with formulas (19), (28), the high-energy (HE) momentum lattice $Lat_P[HE]$ takes the form

$$\text{Lat}_P[HE] \doteq (\pm \frac{1}{2\sqrt{N_w^2 - 1}}), \quad \text{(30)}$$

where $N_w$ are integer numbers from Equation (28).

It is important to note the following. In the low-energy lattice $Lat_P[LE]$ all elements are varying very smoothly, enabling the approximation of a continuous theory. It is clear that lattices $Lat_S[LE]$ and $Lat_P[LE]$ are lattices of primarily measurable quantities, while the lattices $Lat_S[HE]$ and $Lat_P[HE]$ are lattices of the generalized measurable quantities.

We will expand the space lattice $Lat_S[LE]$ to space-time lattice $Lat_{S-T}[LE]$:

$$\text{Lat}_{S-T}[LE] \doteq (N_w, N_t), \quad N_w = \{N_x, N_y, N_z\}, \quad |N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1, |N_t| \gg 1 \quad \text{(31)}$$

Now primarily lattice $Lat_{S-T}[LE]$ will be replaced with “$\alpha$–lattice”, measurable space-time quantities, which will be denoted by $Lat_{S-T}^\alpha[LE]$:

$$\text{Lat}_{S-T}^\alpha[LE] \doteq (\alpha N_w \ell, \alpha N_t \tau) = (\frac{\ell^2}{\hbar} p(N_w), \frac{\ell^2}{\hbar} p(N_t)) = (\frac{\ell}{N_w}, \frac{\tau}{N_t}). \quad \text{(32)}$$

In the last formula by the variable $\alpha N_t \tau$ we mean the parameter $\alpha$ corresponding to the length $(N_t \tau)c$:

$$\alpha N_t \tau \doteq \alpha (N_t \tau)c. \quad \text{(33)}$$

And $p(N_w)$ it is taken from formula (10), where $N_t$ corresponds formula (32). As low energies $E \ll E_P$ are discussed, $\alpha N_w \ell$ in this formula is consistent with the corresponding parameter from formula (23):

$$\alpha N_w \ell = \alpha N_w \ell(HUP) \quad \text{(34)}$$
As it was mentioned in the previous section, in the low-energy $E \ll E_{max} \propto E_P$ all elements of sublattice $Lat_{P-E}[LE]$ are varying very smoothly enabling the approximation of a continuous theory. It is similar to the low-energy part of the $Lat_{S-T}^\alpha[LE]$ of lattice $Lat_{S-T}^\alpha$ will vary very smoothly:

$$Lat_{S-T}^\alpha[LE] = \left(\frac{\ell}{N_w}, \frac{\tau}{N_t}\right); |N_x| \gg 1, |N_y| \gg 1, |N_z| \gg 1, |N_t| \gg 1.$$  (35)

In Section 5 of [2] three following cases were selected:

(a) “Quantum Consideration, Low Energies”:

$$1 \ll |N_w| \leq \tilde{N}, 1 \ll |N_t| \leq \hat{N}$$

(b) “Quantum Consideration, High Energies”:

$$|N_w| \approx 1, |N_t| \approx 1;$$

(c) “Classical Picture”:

$$|N_w| \to \infty, |N_t| \to \infty.$$

Here $\tilde{N}, \hat{N}$ is a cutoff parameters, defined by the current task [2] and corrected in this paper.

Let us for three space coordinates $x_i; i = 1, 2, 3$ we introduce the following notation:

$$\Delta(x_i) \equiv \tilde{\Delta}[\alpha_{N_{\Delta x_i}}] = \alpha_{N_{\Delta x_i}}(N_{\Delta x_i}) = \ell/N_{\Delta x_i};$$

$$\frac{\Delta_{N_{\Delta x_i}}[F(x_i)]}{\Delta(x_i)} = \frac{F(x_i + \Delta(x_i)) - F(x_i)}{\Delta(x_i)},$$  (36)

where $F(x_i)$ is ”measurable” function, i.e function represented in terms of measurable quantities.
Then function $\Delta_{x_i}[F(x_i)]/\Delta(x_i)$ is "measurable" function too. It’s evident that
\[
\lim_{|\Delta x_i| \to \infty} \frac{\Delta_{x_i}[F(x_i)]}{\Delta(x_i)} = \lim_{\Delta(x_i) \to 0} \frac{\Delta_{x_i}[F(x_i)]}{\Delta(x_i)} = \frac{\partial F}{\partial x_i}. \tag{37}
\]
Thus, we can define a measurable analog of a vectorial gradient $\nabla$
\[
\nabla_{N\Delta x_i} \equiv \left\{ \frac{\Delta_{N\Delta x_i}[F(x_i)]}{\Delta(x_i)} \right\} \tag{38}
\]
and a measurable analog of the Laplace operator
\[
\Delta_{(N\Delta x_i)} \equiv \nabla_{N\Delta x_i} \nabla_{N\Delta x_i} \equiv \sum_i \frac{\Delta^2_{N\Delta x_i}}{\Delta(x_i)^2} \tag{39}
\]
Respectively, for time $x_0 = t$ we have:
\[
\Delta(t) = \Delta[t\alpha_{N\Delta t}] = \alpha_{N\Delta t}(N\Delta t \tau) = \tau/N\Delta t;
\]
\[
\frac{\Delta_{N\Delta t}[F(t)]}{\Delta(t)} = \frac{F(t + \Delta(t)) - F(t)}{\Delta(t)}, \tag{40}
\]
then
\[
\lim_{|N\Delta t| \to \infty} \frac{\Delta_{N\Delta t}[F(t)]}{\Delta(t)} = \lim_{\Delta(t) \to 0} \frac{\Delta_{N\Delta t}[F(t)]}{\Delta(t)} = \frac{dF}{dt}. \tag{41}
\]
We shall designate for momenta $p_i; i = 1, 2, 3$
\[
\Delta p_i = \frac{\hbar}{N\Delta x_i \ell};
\]
\[
\Delta p_i F(p_i) \equiv \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} = \frac{F(p_i + \frac{\hbar}{N\Delta x_i \ell}) - F(p_i)}{\frac{\hbar}{N\Delta x_i \ell}}. \tag{42}
\]
From where similarly (37) we get
\[
\lim_{|N\Delta x_i| \to \infty} \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} = \lim_{|N\Delta x_i| \to \infty} \frac{F(p_i + \frac{\hbar}{N\Delta x_i \ell}) - F(p_i)}{\frac{\hbar}{N\Delta x_i \ell}} =
\]
\[
= \lim_{\Delta p_i \to 0} \frac{F(p_i + \Delta p_i) - F(p_i)}{\Delta p_i} = \frac{\partial F}{\partial p_i}. \tag{43}
\]
Therefore, at low energies \( E \ll E_P \), i.e. at \( |N_{\Delta x_i}| \gg 1; |N_{\Delta t}| \gg 1, i = 1, ..., 3 \) on passage to the limit (37), (41), (43) we can obtain from "measurable" functions the partial derivatives like in the case of continuous space-time. That is, the partial derivatives of "measurable" functions can be considered as "measurable" functions with any given precision. In this case the infinitesimal space-time variations (1) are appearing in the limit from measurable quantities too

\[
(\alpha_{N_{\Delta t}\tau} N_{\Delta t}\tau = \frac{\tau}{N_{\Delta t}} = p_{N_{\Delta t}} \frac{\ell^2}{\hbar^2} \xrightarrow{N_{\Delta t} \to \infty} dt),
\]

\[
(\alpha_{N_{\Delta x_i}\ell} N_{\Delta x_i}\ell = \frac{\ell}{N_{\Delta x_i}} = p_{N_{\Delta x_i}} \frac{\ell^2}{\hbar} \xrightarrow{N_{\Delta x_i} \to \infty} dx_i, 1 = 1, ..., 3. \quad (44)
\]

**Remark 3.2.1**

As mentioned above, we suppose that the energies \( E \) are low, i.e. \( E \ll E_P \). So far it has been connived that all the numbers \( N_{\Delta x_i}, N_{\Delta t} \) are integers giving rise to the primarily measurable space-time quantities \( N_{\Delta x_i}, N_{\Delta t} \). Now this restriction is lifted because, unless it is specially noted otherwise, we assume that \( N_{\Delta x_i}, N_{\Delta t} \) are generalized measurable (or simply measurable) quantities. At that, due to the fact that the energies \( E \) are low \( E \ll E_P \), the following condition is still true:

\[
|N_{\Delta x_i}| \gg 1; |N_{\Delta t}| \gg 1, i = 1, ..., 3. \quad (45)
\]

Therefore, in formula (44) the momenta \( p_{N_{\Delta x_i}}, p_{N_{\Delta t}} \) from this point onwards are generalized measurable quantities. Evidently, a good example of such momenta is an exact rather than approximate value of the quantity from equation (18)

\[
p_{N_{\Delta x_i}} = \frac{\hbar}{1/2(N_{\Delta x_i} + \sqrt{N_{\Delta x_i}^2 - 1})} \ell; N_{\Delta x_i} \gg 1. \quad (46)
\]

Besides, if \( N_{\Delta x_i} \ell \) and \( N_{\Delta t}\tau \) are measurable quantities, then the numeric coefficients \( N_{\Delta x_i} \) and \( N_{\Delta t} \) are also measurable quantities.
In this case any **measurable** triplet \( N_q = \{N_{\Delta x_i}\}, |N_{\Delta x_i}| \gg 1, i = 1, \ldots, 3 \) corresponds to the small **measurable** momentum \( p_{N_q} = \{p_{N_{\Delta x_i}}\} \), with the components \( p_{N_{\Delta x_i}}, |p_{N_{\Delta x_i}}| \ll P_{pl} \):

\[
N_{\Delta x_i} \xrightarrow{p} p_{N_{\Delta x_i}} = \frac{\hbar}{N_{\Delta x_i} \ell}.
\]

(47)

And, vice versa, any small **measurable** momentum \( p_q \) with the nonzero components \( p_q = \{p_i\}; 0 \neq |p_i| \ll P_{pl} \) corresponds to the **measurable** triplet \( N_q = \{N_{\Delta x_i}\}, |N_{\Delta x_i}| \gg 1, i = 1, \ldots, 3, \) satisfying the condition (45):

\[
p_i \xrightarrow{x} N_{\Delta x_i} = \frac{\hbar}{p_i \ell}.
\]

(48)

For simplicity, instead of \( N_{\Delta x_\mu} \), we use \( N_{x_\mu}, \mu = 0, \ldots, 3 \).

## 4 Quantum Mechanics in Terms of Measurable Quantities

### 4.1 General Remarks on Wavefunction Representation

Now any coordinate \( u \) from the set \( q = (x, y, z) \in \mathbb{R}^3 \) and some **measurable** quantity \( N_u \ell; |N_u| \gg 1 \) we can correlate with the **measurable** quantity \( \Delta_{N_u}(u) = \ell/N_u \), and \( N_q = \{N_x, N_y, N_z\} \) – with the **measurable** product

\[
\Delta_{N_q}(q) = |\Delta_{N_x}(x) \cdot \Delta_{N_y}(y) \cdot \Delta_{N_z}(z)| = \frac{\ell^3}{|N_x N_y N_z|}.
\]

(49)

Then it is clear that, for **measurability** of the wave function, \( \Psi(q) \), \( \Psi(q) \) is determined in terms of the **measurability** concept of the spatial coordinates \( q \), (i.e. all variations of \( q \) are **measurable**). We can find the quantity

\[
|\Psi(q)|^2 \Delta_{N_q}(q)
\]

(50)

which is the probability that the measurement performed for the system will give the coordinate value in the **measurable** volume element \( \Delta_{N_q}(q) \).
of the configuration space.

Then the known condition for the total probability in a continuous case

\[ \int |\Psi(q)|^2 dq = 1 \quad (51) \]

is replaced, with any preassigned accuracy, by the condition

\[ \sum_q |\Psi(q)|^2 \Delta_{N_q}(q) = 1. \quad (52) \]

Actually, due to equation (44), the measurable volume element \( \Delta_{N_q}(q) \) of the configuration space may be considered as arbitrarily close to \( dq \), meaning that the measurable element \( q + \Delta_{N_q}(q) \) may be considered arbitrarily close to the nonmeasurable element \( q + dq \).

It is obvious that a set of measurable functions forms the space, where the integrals of a continuous theory, if any, are replaced by the corresponding sums over the measurable quantities, and \( dq \) is replaced by \( \Delta_{N_q}(q) \). In the limit of high \( |N_q| \), this space is arbitrarily close to the corresponding Hilbert space of a continuous theory.

In particular, the normalization condition for the measurable eigenfunction \( \Psi_n \) of the given measurable physical quantity \( f \) is varying from continuous to the measurable representation in the following way:

\( \left( \int |\Psi_n|^2 dq = 1 \right) \mapsto \left( \sum_q |\Psi_n|^2 \Delta_{N_q}(q) = 1 \right). \quad (53) \)

Similarly, we have

\[ \int \Psi \Psi^* dq \mapsto \sum_q \Psi \Psi^* \Delta_{N_q}(q). \quad (54) \]

It is seen that, for the spaces of measurable functions, we can redefine all the principal properties of the canonical quantum mechanics, superposition principle, properties of the operators and of their spectra, replacing the integrals by the corresponding sums and \( dq \) – by \( \Delta_{N_q}(q) \) (as in the formula (53), (54)).
4.2 Schrödinger Equation and Other Equations of Quantum Mechanics in ”Measurable” Format

4.2.1 Schrödinger Equation for Free Particle

Let us consider the Schrödinger Equation \[7\] in terms of measurable quantities. As it is shown in the formula (44) taking into account Remark 3.2.1 at low energies \(E \ll E_P\) (i.e. \(|N_{x_\mu}| \gg 1\)), the infinitesimal space-time variations \(dx_\mu, \mu = 0, ..., 3\) occur in the limit of \(|N_{x_\mu}| \to \infty\) from the measurable momenta \(p_{N_{x_i}}, (p_{N_{t_i}})\) multiplied by the constant \(\ell^2/\hbar, (\ell^2/\hbar)\) and are nothing else but \(\ell/N_{x_i}, \tau/N_t\).

Therefore, in all the cases we assume that the following conditions are met: \(|N_{x_i}| \gg 1, |N_t| \gg 1; i = 1, ..., 3\).

Then a measurable \(N_t\)-analog of the derivative for the measurable wave function \(\Psi(t)\) in the continuous case is nothing else but

\[
\Delta_{N_t}[\Psi(t)]/\Delta(t) = \Psi(t + \tau/N_t) - \Psi(t)/\tau/N_t, \tag{55}
\]

and a measurable \(N_t\)-analog of the Schrödinger Equation

\[
d\Psi(t)/dt = 1/i\hbar \hat{H}\Psi(t) \tag{56}
\]

is as follows:

\[
\Delta_{N_t}[\Psi(t)]/\Delta(t) = \Psi(t + \tau/N_t) - \Psi(t)/\tau/N_t = 1/i\hbar \hat{H}_{meas}\Psi(t). \tag{57}
\]

Here \(\hat{H}_{meas}\) is some measurable analog of the Hamiltonian \(\hat{H}\) in the continuous case, i.e., \(\hat{H}_{meas} – \) operator expressed in terms of measurable values.

Let us consider an example of the Schrödinger Equation for a free particle \[7\]

\[
i\hbar \frac{\partial}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \Delta \Psi(r, t), \tag{58}
\]

where \(\Delta \equiv \nabla \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is the Laplace operator and \(m\) – mass of the particle.
The formula \( (39) \) has been initially considered for the case of integer \( N_{x_i}, |N_{x_i}| \gg 1 \). However, due to Remark 3.2.1, it is still valid for any measurable numbers \( N_{x_i}, |N_{x_i}| \gg 1 \).

From this formula it directly follows that

\[
\lim_{|N_{x_i}| \to \infty} \Delta(N_{x_i}) = \Delta.
\]

(59)

Then, because of the condition \( |N_t| \gg 1, |N_{x_i}| \gg 1 \), we can infer that a measurable analog of the Schrödinger Equation \( (58) \)

\[
\hbar \frac{\Delta N_{x_i}}{\Delta(t)} \Psi(r, t) = -\frac{\hbar^2}{2m} \Delta(N_{x_i}) \Psi(r, t),
\]

(60)

at rather high but finite \( |N_t|, |N_{x_i}| \), is congruent with the Schrödinger Equation in the continuous case to any preset accuracy.

Similarly, from the formula for a measurable value of the momentum at low energies \( E \ll E_P \)

\[
p_{N_{x_i}} = \frac{\hbar}{N_{x_i} \ell}
\]

(61)

as well as the equation \( (38) \) for a measurable analog of the vectorial gradient \( \nabla_{N_{x_i}} \) and the equations \( (36), (37) \) it follows that the correspondence rule in the measurable case

\[
P_{N_{x_i}} \hat{=} P_{N_{q}} \mapsto \frac{\hbar}{\ell} \nabla_{N_{q}}
\]

(62)

can, to any preset accuracy, reproduce the correspondence rule in the continuous case

\[
p \mapsto \frac{\hbar}{\ell} \nabla.
\]

(63)

In a similar way for a measurable value of the energy

\[
E_{N_{q}} = \frac{p_{N_{q}}^2}{2m} = \frac{p_{N_{x}}^2 + p_{N_{y}}^2 + p_{N_{z}}^2}{2m}
\]

(64)
the correspondence

\[ E_{N_q} \mapsto i\hbar \frac{\Delta N_q}{\Delta(t)} \]  

(65)

reproduces the correspondence

\[ E \mapsto i\hbar \frac{\partial}{\partial t} \]  

(66)

of a continuous theory.

So, in terms of measurable quantities we can derive a discrete model arbitrary close the initial continuous theory.

From this it follows that the measurable wave function \( \Psi_{\text{meas}}(r, t, N_q, N_t) \)

of the form

\[ \Psi_{\text{meas}}(r, t, N_q, N_t) = A \exp\{i(\frac{p}{\hbar} \cdot r - E \cdot r)\}, \]  

(67)

where \( r \) and \( t \) – measurable, reproduces the corresponding wave function \( \Psi(r, t) \) in the continuous case \([7]\) to a high accuracy.

A particular example was given in preceding sections of the text. It is obvious that it allows for more general conclusions.

The measurable analog \( \hat{H}_{\text{meas}} \) of the Hamiltonian \( \hat{H} \) from the equation (57) in the general case should be of the following form:

\[ \hat{H}_{\text{meas}} = \hat{H}_{\text{meas}}(N_q), \]  

(68)

where \( N_q \) is measurable and

\[ \lim_{|N_q| \to \infty} \hat{H}_{\text{meas}} = \hat{H}. \]  

(69)

As we have

\[ \lim_{|N_t| \to \infty} \frac{\Delta N_t[\Psi(t)]}{\Delta(t)} = \frac{d\Psi(t)}{dt}, \]  

(70)

then in the general case, in passage to the limit at \( |N_q| \to \infty, |N_t| \to \infty \), from a measurable analog of the Schrodinger equation (57) we can get the
Schrodinger equation (56) in the continuous picture. At that we can suppose that all variables, including time \( t \), influencing the wave function \( \psi \) are measurable quantities. A similar supposition is correct for the Hamiltonian \( \hat{H}_{\text{meas}} \) as well.

Without loss of generality, we can assume that the values of \( |N_q| \gg 1 \) are high enough so that the measurable Hamiltonian analog \( \hat{H}_{\text{meas}} \) be equal to the Hamiltonian in the continuous case to a high accuracy

\[
\hat{H}_{\text{meas}} = \hat{H}
\]  
(71)

Then, at the fixed \( N_t \), high in absolute value, and at measurable \( \psi \), a measurable analog of the Schrodinger equation (57) may be solved recurrently as follows:

\[
\Psi(t + \tau/N_t) = \left( \frac{\tau}{iN_t \hbar} \hat{H} + 1 \right) \Psi(t).
\]  
(72)

Taking some measurable quantity \( \psi(t) \) (possibly \( t = 0 \)) as a reference point and first substituting it into the right side (72), and then repeating this procedure for the value of \( \Psi(t + \tau/N_t) \), already calculated in the left side, sufficiently many times, we can get the function \( \Psi(t + \Delta t) \) for arbitrary \( \Delta t = K \tau/N_t \), where \( K \) is a natural number. It is clear that, if \( N_t \) – integer number, then primary measurable variations in this series correspond to only the integer \( K \) divisible by \( N_t \), i.e. \( K = MN_t \), where \( M \) – integer number. But, as the energies are low (\( E \ll E_P \)), we also have \( |M| \gg 1 \).

Then, because of

\[
\left( \frac{\tau}{iN_t \hbar} \hat{H} + 1 \right) = \hat{U}(\tau/N_t),
\]  
(73)

we obtain

\[
\frac{1}{\hbar} \hat{H} = \frac{\hat{U}(\tau/N_t) - 1}{\tau/N_t}.
\]  
(74)

Next, assuming that \( U(0) = 1 \) and considering (57)

\[
\frac{\Delta N_t [\hat{U}(t')]}{\Delta(t)} = \frac{\hat{U}(t' + \tau/N_t) - \hat{U}(t')}{\tau/N_t},
\]  
(75)
we have
\[ \frac{\Delta_{N_t} [\hat{U}(t')]}{\Delta(t)}|_{t'=0} = \frac{1}{i\hbar} \hat{H}, \] (76)
in strict conformity with the well-known formula in the continuous case
\[ \hat{H} = i\hbar \frac{d\hat{U}(t')}{dt'}|_{t'=0}. \] (77)

The operator \( \hat{U}(t') \) satisfying the equations (73)–(76) we denote as \( \hat{U}_{N_t} \).

From all the above formula it is trivial that
\[ \Psi(t + \tau/N_t) = \hat{U}(\tau/N_t)\Psi(t). \] (78)

The presented calculations are easily generalized to non-autonomous systems when the Hamiltonian \( \hat{H}, (\hat{H}_{\text{meas}}) \) depends on time \( t \), i.e. \( \hat{H} = \hat{H}(t) \) and the condition (71) is met. In this case, again assuming that all values (operators and the wave function) are measurable quantities depending on time, we have
\[ \frac{\Delta_{N_t}}{\Delta(t)}\Psi(t) = \frac{\Delta_{N_t}[\hat{U}(t + \tau', t)]}{\Delta(\tau')}|_{(\Delta(\tau')=\tau/N_t)}\Psi(t) = \frac{1}{i\hbar} \hat{H}(t)\Psi(t), \]
\[ \hat{H}(t) = i\hbar \frac{\Delta_{N_t}[\hat{U}(t + \tau', t)]}{\Delta(\tau')}|_{\Delta(\tau')=\tau/N_t} \] (79)

It is clear that in the suggested formalism one can reproduce all the basic formulas of the continuous case replacing \( dt \) by \( \tau/N_t \), in particular
\[ \hat{U}^{\dagger}(t + \tau/N_t, t) = (\hat{1} + \frac{\tau}{N_t} \frac{\hat{H}}{i\hbar} + o(\frac{\tau}{N_t}))^{\dagger} = \hat{1} - \frac{\tau}{N_t} \frac{\hat{H}^{\dagger}}{i\hbar} + o(\frac{\tau}{N_t}) = \]
\[ = \hat{U}^{-1}(t + \tau/N_t, t) = (\hat{1} + \frac{\tau}{N_t} \frac{\hat{H}}{i\hbar} + o(\frac{\tau}{N_t}))^{-1} = \hat{1} - \frac{\tau}{N_t} \frac{\hat{H}}{i\hbar} + o(\frac{\tau}{N_t}). \] (80)

What is the essence of replacing \( dt \) by \( \tau/N_t \) and of going from continuous to the discrete picture in terms of measurable quantities? By the author’s
opinion, the main point is that the following **Hypothesis** is valid:

at low energies $E \ll E_P$, i.e. at $|N| \gg 1$, for any wave function $\Psi(t)$ there exists the integer $N(\psi), |N(\psi)| \gg 1$ dependent on $\Psi(t)$, with unimprovable approximation of the Schrodinger equation (56) by the discrete equation (57). Of course, the condition $1 \ll |N| \leq |N(\psi)|$ is satisfied.

### 4.2.2 Linear Momentum Operator

It is known that a problem in eigenvalues and eigenfunctions of the momentum projection $\hat{p}_{x_i}$ onto the coordinate $x_i$ in the case of continuous space-time is reduced to the differential equation [29]

$$-\imath \hbar \frac{\partial \Psi(x_i)}{\partial x_i} = p_{x_i} \Psi(x_i).$$

One can find continuous single-valued and bounded solutions of this equation for all real values of $p_{x_i}$ in the interval $-\infty < p_{x_i} < \infty$ with the eigenfunctions

$$\Psi_p(x_i) = A \exp(\frac{p_{x_i}}{\hbar} x_i).$$

Thus, we have one eigenfunction (no degeneracy) for each eigenvalue $p_{x_i} = p$. As stated above, in the measurable case under consideration in the left side of (82), for some fixed measurable $|N_{x_i}| \gg 1$, the replacement operation is used

$$\frac{\partial}{\partial x_i} \mapsto \frac{\Delta_{N_{x_i}}}{\Delta(x_i)}$$

and the eigenvalues $p_{N_{x_i}}$ of the operator $\hat{p}_{x_i}$ become discrete $N_{x_i}$

$$p_{N_{x_i}} = \frac{\hbar}{N_{x_i} \ell}, |N_{x_i}| \gg 1.$$

But, due to the condition $|N_{x_i}| \gg 1$, we obtain a discrete spectrum of the operator $\hat{p}_{x_i}$ that is nearly continuous.
Taking into account that at sufficiently high $|N_{x_i}|$, within any preset accuracy, we have

$$\frac{\Delta N_{x_i}}{\Delta(x_i)} = \frac{\partial}{\partial x_i}$$

and considering the formula (84), we can get an analog of formula (82) in the case under study

$$\Psi_{p_{N_{x_i}}}(x_i) = Aexp(i \frac{x_i}{N_{x_i} \ell}).$$

As seen, for fixed $x_i$, the corresponding discrete set of eigenfunctions also varies almost continuously.

It should be noted that the condition $-\infty < p_{x_i} < \infty$ in this case is incorrect because

$$((p_{x_i} = p_{N_{x_i}}) \rightarrow \pm \infty) \equiv (|N_{x_i}| \rightarrow 1),$$

that contradicts the condition $|N_{x_i}| \gg 1$.

However, for a real problem, the abstract condition $|N_{x_i}| \gg 1$ is always replaced by the specific condition

$$|N_{x_i}| \geq N_\ast \gg 1.$$  

Then the condition $-\infty < p_{x_i} < \infty$ for the continuous case is replaced in the case under study by the condition $p_{-N_\ast} \leq p_{x_i} \leq p_{N_\ast}$, with the distinguished point $p_{x_i} = 0$ apparently not belonging to the equation (84) at finite $N_{x_i}$.

It is clear that the case $N_{x_i} = \pm \infty$ associated with the point $p_{x_i} = 0$ is degenerate and hence, if we limit ourselves to finite $N_{x_i}$, the condition of (88) should be replaced with the condition

$$N_\ast \geq |N_{x_i}| \geq N_\ast \gg 1.$$  

Then, respectively, we have $p_{x_i} \in [p_{-N_\ast}, p_{-N_\ast}] \bigcup [p_{N_\ast}, p_{N_\ast}]$.

Next we denote with $\Delta_{N_\ast,N_\ast} (p_{x_i})$ the integration of the intervals

$$\Delta_{N_\ast,N_\ast} (p_{x_i}) = [p_{-N_\ast}, p_{-N_\ast}] \bigcup [p_{N_\ast}, p_{N_\ast}],$$

and use $\Delta_{N_\ast} (p)$ for the following:

$$\Delta_{N_\ast,N_\ast} (p) = \prod_i \Delta_{N_\ast,N_\ast} (p_{x_i}).$$
4.2.3 \( z \)-component of the Angular Momentum \( \hat{L}_z \)

In the conventional quantum mechanics a problem in eigenvalues and eigenfunctions of the angular momentum operator \( \hat{L}_z \)

\[
\hat{L}_z = -i\hbar(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})
\]

(92)

is reduced to solution of the differential equation \[29\]

\[
-i\hbar \frac{\partial \Psi(\phi)}{\partial \phi} = L_z \Psi(\phi),
\]

(93)

where \( 0 \leq \phi \leq 2\pi \).

In the considered case we can suppose that \( \phi = \phi(x, y, z) \) – measurable function of the variables \( x, y, z \) that in the continuous case has well-defined partial derivatives for each of them.

It is obvious that by substitution into the formula \(37\) for \( F(x_i) = \phi(x, y, z) \) we immediately get

\[
\lim_{|N\Delta x_i| \to \infty} \frac{\Delta N\Delta x_i [\phi(x, y, z)]}{\Delta (x_i)} = \lim_{\Delta(x_i) \to 0} \frac{\Delta N\Delta x_i [\phi(x, y, z)]}{\Delta (x_i)} = \frac{\partial \phi}{\partial x_i}.
\]

(94)

From whence it directly follows – there exists the measurable function \( \Delta \Psi/\Delta \phi \) so that

\[
\lim_{\Delta \phi \to 0} \frac{\Delta \Psi}{\Delta \phi} = \lim_{|N\Delta x_i| \to \infty} \frac{\Delta \Psi}{\Delta \phi} = \frac{\partial \Psi}{\partial \phi},
\]

(95)

where \( \Delta \phi(x_i) = \sum_i (\phi(x_i + \Delta x_i) - \phi(x_i)) \) and measurable increments of \( \Delta x_i \) are taken from the formula \(36\).

Considering that, for sufficiently high \( |N_{x_i}| \), with a high accuracy we have \( \Delta N_{x_i}/\Delta (x_i) = \partial/\partial x_i \) and \( \Delta \Psi(\phi)/\Delta \phi = \partial \Psi(\phi)/\partial \phi \), it is concluded that the equation \(93\) to a high accuracy may be used in the measurable case as well, when regarding \( \phi(x, y, z) \) the measurable function of a measurable set of the coordinates \( \{x, y, z\} \).

Then a solution for \(93\) is given by the exponent

\[
\Psi(\phi) = A exp(i \frac{L_z}{\hbar} \phi),
\]

(96)
where \( \phi = \phi(x, y, z) \) - measurable function of the measurable variables \( x, y, z \).

In so doing the eigenfunctions for a discrete spectrum \( L_z = \hbar m; m = 0, \pm 1, \pm 2, \ldots \) of the operator \( \hat{L}_z \), as in the continuous case, are given by

\[
\Psi_m(\phi) = (2\pi)^{-1/2}e^{im\phi},
\]

(97)

where \( \phi \) is a measurable quantity.

However, at the normalization condition the integral in the continuous case [29] is replaced by the sum

\[
\left( \int_0^{2\pi} |\Psi_m|^2 d\phi = 1 \right) \Rightarrow \left( \sum_{0 \leq \phi \leq 2\pi} |\Psi_m|^2 \Delta(\phi) = 1 \right),
\]

(98)

where \( \Delta(\phi) \) is taken from the formula (95).

### 4.3 Position and Momentum Representations, and Fourier Transform in Terms of Measurability

Now, using the formulas from the previous sections, let us study quantum representations and the Fourier transform in terms of the concept of measurable quantities. A scalar product in the position representation in the continuous case is given by the equality [7, 30]:

\[
(\varphi_1, \varphi_2) = \int_{R^3} \varphi_1^*(x)\varphi_2(x)dx.
\]

(99)

Both the coordinate \( x_j \) and momentum \( p_j \) \((j = 1, 2, 3)\) operators in the position representation are introduced by [7]

\[
x_j\varphi(x) = x_j\varphi(x),
\]

\[
p_j\varphi(x) = -i\hbar \frac{\partial}{\partial x_j}\varphi(x).
\]

(100)

According to the formula (100), we have \( x = q \) and hence the integral from the equation (100) is replaced by the sum

\[
(\varphi_1, \varphi_2)_{meas} = \sum_{x \in R^3} \varphi_1^* x_2 \Delta_N(x),
\]

(101)
where \( x \) – measurable coordinates.

It is clear that the passage to the limit takes place

\[
\lim_{N_{x_j} \to \infty} (\varphi_1, \varphi_2)_{\text{meas}} = (\varphi_1, \varphi_2),
\]

where \( \{N_{x_j}\} = N_q \) due to the equation (49) and at sufficiently high \( \{N_{x_j}\} = N_q \) with a high precision we have

\[
(\varphi_1, \varphi_2)_{\text{meas}} = (\varphi_1, \varphi_2).
\]

In the considered case the first formula from (100) is valid for all measurable values in the left and right sides, while the second one is replaced by

\[
\mathbf{p}_{N_{x_j}} \varphi(x) = -i\hbar \frac{\Delta N_{x_j}}{\Delta(x_j)} \varphi(x) = -i\hbar \frac{\varphi(x_{i\neq j}, x_j + \ell/N_{x_j}) - \varphi(x)}{\ell/N_{x_j}}.
\]

Here \( \mathbf{p}_{N_{x_j}} \) – j-th measurable component of the momentum taking the form

\[
\mathbf{p}_{N_{x_j}} = \frac{\hbar}{N_{x_j} \ell}.
\]

and the function \( \varphi(x_{i\neq j}, x_j + \ell/N_{x_j}) \) differs from \( \varphi(x) \) only by the “shift” to \( \ell/N_{x_j} \) in the j-component.

As follows from the formulae given above and, in particular, the formula (37), in this low-energy case when \( E \ll E_P \), i.e. at \( |N_{x_j}| \gg 1 \), to a high accuracy we have

\[
\frac{\Delta N_{x_j}}{\Delta(x_j)} = \frac{\partial}{\partial x_j}.
\]

Then, due to the formulae (104)–(106), in the low-energy case \( E \ll E_P \) for measurable quantities within a high accuracy we get

\[
[x, \mathbf{p}] \varphi(x) = \mathbf{x} \cdot \varphi(x) - \mathbf{p} \cdot \varphi(x) = i\hbar \varphi(x).
\]
In the momentum representation for the continuous picture we have

\[ x_j \cdot \varphi(p) = i \hbar \frac{\partial}{\partial p_j} \varphi(p), \]
\[ p_j \cdot \varphi(p) = p_j \varphi(p). \]  \hspace{1cm} (108)

In the measurable case the second equation (108) for measurable momenta remains unchanged. According to the formulae (42) and (43), in the measurable case in the first equation from (108) the replacement is performed

\[ \frac{\partial}{\partial p_j} \mapsto \frac{\Delta p_j}{\Delta p_j}, \]  \hspace{1cm} (109)

where

\[ p_j \equiv p_{N_{x_j}} = \frac{\hbar}{N_{x_j} \ell}; \]
\[ \frac{\Delta p_j \varphi(p)}{\Delta p_j} \equiv \frac{\varphi(p + p_j) - \varphi(p)}{p_j} = \frac{\varphi(p + \frac{\hbar}{N_{x_j} \ell}) - \varphi(p)}{\frac{\hbar}{N_{x_j} \ell}}; \]  \hspace{1cm} (110)

and \( \varphi(p + p_j) \) differs from \( \varphi(p) \) by \( p_j \) only in the \( j \)-th component. Then from the expression (113), due to the fact that \(|N_{x_j}| \gg 1\), with a high accuracy we get

\[ \frac{\Delta p_j}{\Delta p_j} = \frac{\partial}{\partial p_j}. \]  \hspace{1cm} (111)

Now let us consider \([x, p] \varphi(p)\) in the momentum representation. Taking into account the formula (111), we have

\[ [x_j, p_j] \varphi(p) = x_j p_j \cdot \varphi(p) - p_j x_j \cdot \varphi(p) = \]
\[ = i \hbar (\varphi(p) + p_j \frac{\varphi(p + p_j) - \varphi(p)}{p_j} - p_j \frac{\varphi(p + p_j) - \varphi(p)}{p_j}) = \]
\[ = i \hbar \varphi(p). \]  \hspace{1cm} (112)
Thus, the expressions (106)–(112) show that the commutator
\[
[x_i, p_j] = i \delta_{ij} \hbar
\]
in the measurable case occurs both in the position and momentum representations.

In the continuous picture the Fourier transform is of the following form [30]:
\[
\varphi(x) = \left(\frac{1}{2\pi \hbar}\right)^{3/2} \int_{\mathbb{R}^3} e^{i \frac{p \cdot x}{\hbar}} \varphi(p) dp.
\]  
(114)

And the operator \( p_j \) applied to the formula (114) gives [30]
\[
p_j \varphi(x) = -i \hbar \frac{\partial}{\partial x_j} \varphi(x) = -i \hbar \frac{1}{2\pi \hbar}^{3/2} \int_{\mathbb{R}^3} e^{i \frac{p \cdot x}{\hbar}} \varphi(p) dp =
\]
\[
= \left(\frac{1}{2\pi \hbar}\right)^{3/2} \int_{\mathbb{R}^3} e^{i \frac{p \cdot x}{\hbar}} p_j \varphi(p) dp.
\]  
(115)

However, as indicated in the formulae (87),(88), in the considered measurable case at low energies the values of \(|p|\) are bounded, therefore \(p\) fills not the whole space of \(\mathbb{R}^3\), belonging only to its part \(\Delta_{N_*,N^*}(p)\) (formula (91)).

That is why the integral in the equation (114) should be replaced by the sum
\[
\varphi_{meas}(x) = \left(\frac{1}{2\pi \hbar}\right)^{3/2} \sum_{p \in \Delta_{N_*,N^*}(p)} e^{i \frac{p \cdot x}{\hbar}} \varphi_{meas}(p) \Delta_p(p_{N_*)},
\]  
(116)

where \(x, p\) and \(\varphi_{meas}(p)\) are measurable quantities. So, we have
\[
\Delta_p(p_{N_*)} = \prod_j p_{N_*)},
\]  
(117)

where \(p_{N_*)} \) is taken from the equation (110).

As \(|N_*)| \gg 1\), then in the limit \(|N_*)| \to \infty\) the sum in the right side of the equation (116) is replaced by the integral and, to a high accuracy, we get
\[
\left(\frac{1}{2\pi \hbar}\right)^{3/2} \int_{\Delta_{N_*,N^*}(p)} e^{i \frac{p \cdot x}{\hbar}} \varphi(p) dp =
\]
\[
= \left(\frac{1}{2\pi \hbar}\right)^{3/2} \sum_{p \in \Delta_{N_*,N^*}(p)} e^{i \frac{p \cdot x}{\hbar}} \varphi_{meas}(p) \Delta_p(p_{N_*)}.
\]  
(118)
It should be noted that in this case the domain of the function varies only for the momenta. Due to the above-mentioned equations, it is narrowing \( \{ p \in \mathbb{R}^3 \} \mapsto \{ p \in \Delta_{N_x, N_y}(p) \} \). For the coordinates, it remains \( \{ x \in \mathbb{R}^3 \} \).

The function \( \varphi(p) \) in the continuous case is of the following form \([30]\):

\[
\varphi(p) = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \int_{\mathbb{R}^3} e^{-\frac{i}{\hbar} p x} \varphi(x) dx.
\]

(119)

As the domain in the position representation remains the same \( \{ x \in \mathbb{R}^3 \} \), then for the measurable case \( \varphi_{\text{meas}}(p) \) takes the form

\[
\varphi_{\text{meas}}(p) = \left( \frac{1}{2\pi \hbar} \right)^{3/2} \sum_{\mathbb{R}^3} e^{-\frac{i}{\hbar} p x} \varphi_{\text{meas}}(x) \Delta_{N_x}(x),
\]

(120)

where \( x = q \) from the formula \((49)\), i.e.

\[
\Delta_{N_x}(x) = \prod_j \Delta_{N_{x_j}}(x_j) = \frac{\ell^3}{N_x N_y N_z}.
\]

(121)

In this case, due to the condition \( |N_{x_j}| \gg 1 \), we have

\[
\left( \frac{1}{2\pi \hbar} \right)^{3/2} \int_{\mathbb{R}^3} e^{-\frac{i}{\hbar} p x} \varphi(x) dx \approx \left( \frac{1}{2\pi \hbar} \right)^{3/2} \sum_{\mathbb{R}^3} e^{-\frac{i}{\hbar} p x} \varphi_{\text{meas}}(x) \Delta_{N_x}(x),
\]

(122)

where all values in the right side of \((122)\) are measurable.

Thus, the equations \((116)\) and \((120)\) are analogues of direct and inverse Fourier transforms in terms of measurable quantities or, better to say, of the measurable direct and inverse Fourier transforms.

In this formalism we can easily derive a measurable analog of the equation \((115)\) by the replacement of \( p_j \mapsto p_{N_{x_j}}, \partial / \partial x_j \mapsto \Delta_{N_{x_j}} / \Delta(x_j), \varphi(x) \mapsto \varphi_{\text{meas}}(x) \) and \( \int_{\mathbb{R}^3} \mapsto \sum_{\Delta_{N_x}(p)} \).

Similarly, by the adequate replacement in the measurable variant it is possible to get an analog of the correspondence

\[
x_j, \varphi(p) \mapsto i\hbar \frac{\partial}{\partial p_j} \varphi(p)
\]

(123)
in the continuous picture.

It is necessary to make some important comments.

Commentary 4.3.

4.3.1. As we consider the minimal length $\ell$ and the minimal time $\tau$ at Plank’s level $\ell \propto l_p, \tau \propto t_p$, the use of the measurable quantities $\ell/N_{x_i}; i = 1, ..., 3$ and $\tau/N_t$ at $|N_{x_i}| \gg 1, |N_t| \gg 1$ as a substitution for $dx_i, dt$ in the continuous case is absolutely correct and justified. Indeed, as in this case $\ell$ is on the order of $\approx 10^{-33} \text{cm}$, then $\ell/N_{x_i}$ is on the order of $\approx 10^{-33} - \log|N_{x_i}| \text{cm}$, without doubt being beyond any computational accuracy. A similar statement is true for $\tau/N_t$ as well, where $\tau$ is on the order of the Plank time $t_p$, i.e. $\approx 10^{-44} \text{sec}$. For this reason, it is correct to use in the continuous case $pN_{x_i}$ instead of $dp_i$ and $\Delta_{N_{x_i}}/\Delta(x_i), \Delta_{N_t}/\Delta(t), \Delta_{p_i}/\Delta p_i$ instead of $\partial/\partial x_i, \partial/\partial t, \partial/\partial p_i$, respectively.

4.3.2. For generality, in Remark 3.2.1 we have supposed that $N_{x_i}, N_t$ are the generalized measurable quantities. As $|N_{x_i}| \gg 1, |N_t| \gg 1$, without loss of generality, we can regard $N_{x_i}$ and $N_t$ as primarily measurable quantities. It is clear that

$$[N_{x_i}] \leq N_{x_i} \leq [N_{x_i}] + 1,$$  \hspace{1cm} (124)

where $[N]$ defines the integer part of $N$. Then $|N_{x_i}|^{-1}$ falls within the interval with the finite points $|[N_{x_i}]|^{-1}$ and $|[N_{x_i}] + 1|^{-1}$ (which of the numbers is greater or smaller, depends on a sign of $N_{x_i}$). In any case we have $|N_{x_i}^{-1} - [N_{x_i}]^{-1}| \leq |([N_{x_i}] + 1)^{-1} - [N_{x_i}]^{-1}| = |([N_{x_i}] + 1)[N_{x_i}]|^{-1}$.

Actually, the difference between $\ell/N_{x_i}$ and $\ell/[N_{x_i}]$ (or, respectively, between $\Delta_{N_{x_i}}/\Delta(x_i)$ and $\Delta_{[N_{x_i}]}/\Delta(x_i)$, and so on) is insignificant. Similar computations are correct for $\tau/N_t$ and $\tau/[N_t]$ as well.

4.3.3a. It is important: despite the fact that in the measurable case we have analogues of the direct and inverse Fourier transforms specified by the equations (116) and (120), the difference between the position and momentum representations in this case is significant. Indeed, for the first the
domain is represented by the whole three-dimensional space $R^3$, whereas for the second the domain represents a particular part of the finite dimensions $\Delta_{N_x,N^*}(p)$, "cut out" from the three-dimensional space $\Delta_{N_x,N^*}(p) \subset R^3$.

4.3.3b. A significant difference between the position and momentum representations in the **measurable** case is associated with their different nature in this formalism. In principle, the position representation in this case is formed similar to the continuous case. The momentum representation in the **measurable** case, as follows from the formulas **Remark 3.2.1**, is formed on the basis of **measurable variations** in the coordinate representation. As, to within a multiplicative constant, $\ell$ agrees with $l_p$ and $p_{N_N}$ – with $\ell/N_x$ (formula (11)), the measures on summation in the **measurable** case in equations (116) and (120) for the momentum and position spaces also agree to within the multiplicative constant

$$\Delta_{N_x}(x) = \frac{\ell^6}{\hbar^3} \Delta_p(p_{N_N}). \quad (125)$$

4.3.4. Note that the above-mentioned formalism used to study the Schrödinger picture in terms of **measurability** may be applied to the Heisenberg picture too [7],[30]. Indeed, in the paradigm of the continuous space and time a motion equation for the Heisenberg operators $\hat{L}(t)$ is as follows [7],[30]:

$$\frac{d\hat{L}(t)}{dt} = \frac{\partial\hat{L}(t)}{\partial t} + [\hat{H}, \hat{L}(t)], \quad (126)$$

where $\hat{H}$ – Hamiltonian and $[\hat{H}, \hat{L}(t)] = \frac{1}{\hbar} (\hat{L}(t)\hat{H} - \hat{H}\hat{L}(t))$–quantum Poisson bracket [30].

In the **measurable** case the quantum Poisson bracket preserves its form for the enclosed **measurable** quantities. $\partial\hat{L}(t)/\partial t$ is replaced by $\Delta_{N_i}[\hat{L}(t)]/\Delta(t)$, where the operator $\Delta_{N_i}[\hat{L}(t)]/\Delta(t)$ may be obtained from equation (75) due to replacement of $\tilde{U}(t')$ by $\hat{L}(t)$ at $|N_i| \gg 1$.

Then an analogue of (126) in the **measurable** case is given by

$$\frac{\Delta_{N_i}[\hat{L}(t)]}{\Delta(t)} = \frac{\Delta_{N_i}[\hat{L}(t)]}{\Delta(t)} + [\hat{H}, \hat{L}(t)]. \quad (127)$$
It is clear that
\[
\lim_{|N_t| \to \infty} \frac{\Delta N_t[\hat{L}(t)]}{\Delta(t)} = \frac{d\hat{L}(t)}{dt}.
\] (128)

Thus, at sufficiently high $|N_t|$, equation (127) agrees with equation (126) to a high accuracy.

5 More General Definition of Measurability

Proceeding from all the above, the author suggests another definition of \textbf{measurability} that is more general than the initial one.

As before, we begin with a particular minimal (universal) unit for measurement of the length $\ell$ corresponding to some maximal energy $E_\ell = \frac{h c}{\ell}$ and a universal unit for measurement of time $\tau = \ell/c$. Without loss of generality, we can consider $\ell$ and $\tau$ at Plank’s level, i.e. $\ell = \kappa \oplus p = \kappa t$, where the numerical constant $\kappa$ is on the order of 1. Consequently, we have $E_\ell \propto E_p$ with the corresponding proportionality factor.

Note that $\ell$ and $\tau$ are referred to as ”minimal” and ”universal” units of measurement because in our case this is actually true.

Now consider in the space of momenta $P$ the domain defined by the conditions

\[
P = \{p_{x_i}\}, i = 1, \ldots, 3; P_{pl} \gg |p_{x_i}| \neq 0,
\] (129)

where $P_{pl}$–Plank momentum. Then we can easily calculate the numerical coefficients $N_{x_i}$ as follows:

\[
N_{x_i} = \frac{h}{p_{x_i} \ell}, \quad \text{or}
\] (130)

\[
p_{x_i} = p_{N_{x_i}} = \frac{h}{N_{x_i} \ell}
\]

\[
|N_{x_i}| \gg 1,
\]

where the last row of the equation (130) is given by the formula (129).
Definition 1*
1*.1 The momenta \( p \) given by the formula (129) are called primarily measurable when all \( N_{x_i} \) from the equation (130) are integer numbers.
1*.2 Any variation in \( \Delta x_i \) for the coordinates \( x_i \) and \( \Delta t \) of the time \( t \) at the energies \( E \ll E_p \) is considered primarily measurable if
\[
\Delta x_i = N_{x_i} \ell, \Delta t = N_t \tau,
\] (131)
where \( N_{x_i} \) satisfies the condition 1*.1 and \( |N_t| \gg 1 \) – integer number.
1*.3 Let us define any physical quantity as primary or elementary measurable at low energies \( E \ll E_p \) when its value is consistent with points 1*.1 and 1*.2 of this Definition.

For convenience, we denote a domain of the momenta satisfying the conditions (129) (or (130)) in terms of \( P_{LE} \).

In Commentary 4.3.2 it is shown that, since the energies are low \( E \ll E_p \) (\(|N_{x_i}| \gg 1 \)), primary measurable momenta are sufficient to find the whole domain of momenta \( P_{LE} \).

This means that in the indicated domain a discrete set of primary measurable momenta \( p_{N_{x_i}}; i = 1, \ldots, 3 \) (where \( N_{x_i} \) – integer number and \( |N_{x_i}| \gg 1 \)), varies almost continuously, practically covering the whole domain.

That is why further \( P_{LE} \) is associated with the domain of primary measurable momenta, satisfying the conditions of the formula (129) (or (130)).

Then boundaries of the domain \( P_{LE} \) are determined by the condition (89) for each coordinate
\[
N^* \geq |N_{x_i}| \geq N_\ast \gg 1,
\]
where high natural numbers \( N^*, N_\ast \) are determined by the problem at hand.

The choice of the number \( N^* \) is of particular importance. If \( N^* < \infty \), then it is clear that the studied momenta fall within the domain \( P_{LE} \). Assuming the condition \( N^* = \infty \), to \( P_{LE} \) for every coordinate \( x_i \) we should add “improper” (or “singular”) point \( p_{x_i} = 0 \) (these cases are called degenerate).

In any case, for each coordinate \( x_i \), the boundaries of \( P_{LE} \) are of the form:
\[
p_{N^*} \leq |p_{N_{x_i}}| \leq p_{N_\ast}.
\] (132)
For definiteness, we denote $P_{LE}$, having the boundaries specified by the formula (132), in terms of $P_{LE}[N^*, N_*]$. It is obvious that in this formalism small increments for any component $p_{N_{x_i}}$ of the momentum $p \in P_{LE}$ are values of the momentum $p_{N_{x_i}'}$, so that $|N_{x_i}'| > |N_{x_i}|$. And then, incrementing $|N_{x_i}'|$, we can obtain arbitrary small increments for the momenta $p \in P_{LE}$.

In this case it is correct to define a “measurable partial derivative” with respect to the momentum $p_{N_{x_i}}$ specified in equations (42) and (43) in terms of $\Delta p_{N_{x_i}}/\Delta p_{N_{x_i}}$. As shown by the equations (42) and (43) and in the previous paragraph, at sufficiently high $|N_{x_i}|$, with any predetermined accuracy, we have the equality $\Delta p_{N_{x_i}} = \partial \partial p_{N_{x_i}}$ (for example, formula (111)).

It is clear that primary measurable variations in $\Delta x_i$ for the coordinates $x_i$ and in $\Delta t$ for the time $t$ given in point 1.2 of Definition 1* could hardly play a role of small spatial and temporal variations. Still, based on the equation (44) and its applications in subsequent parts of the text, we can state that the space and time quantities

$$\frac{\tau}{N_t} = p_{N_t} \frac{\ell^2}{\hbar}, \quad \frac{\ell}{N_{x_i}} = p_{N_{x_i}} \frac{\ell^2}{\hbar}, \quad 1 = 1, \ldots, 3$$

(133)

are small and, see (44), they may be arbitrary small for sufficiently high values of $|N_{x_i}|, |N_t|$. Here $p_{N_{x_i}}, p_{N_t}$—corresponding primarily measurable momenta.

Of course, due to point 1.2 of Definition 1, the space and time quantities $\tau/N_t, \ell/N_{x_i}$ are not primary measurable despite the fact that they, to within a constant factor, are equal to primarily measurable momenta.

Therefore, it seems expedient to introduce the following definition:

**Definition 2*.(Generalized Measurability at Low Energies).**

Any physical quantity at low energies $E \ll E_p$ may be called generalized measurable or, for simplicity, measurable if any of its values may be obtained in terms of Primary Measurable Quantities specified by Definition 1*.
Lifting the restriction of $P_{pl} \gg |p_x|$ in the equation (129) or, similarly, of $|N_{x_i}| \gg 1$ in the formula (130), i.e. considering the momenta space $p$ at all energy scales

$$p = \{p_{x_i}\}, i = 1, \ldots, 3; |p_{x_i}| \neq 0;$$

$$N_{x_i} = \frac{\hbar}{p_{x_i} \ell}, \text{or}$$

$$p_{x_i} = p_{N_{x_i}} = \frac{\hbar}{N_{x_i} \ell};$$

$$1 \leq |N_{x_i}| < \infty, \text{or} \quad E \leq E_\ell,$$

we can introduce the following definition.

**Definition 3* (Primary and Generalized Measurability at All Energy Scales).**

**3*.1.** The momenta $p$, set by the formula (134), are referred to as **primarily measurable**, if all the numbers $N_{x_i}$ from this formula (134) are integers.

**3*.2.** Any variation $\Delta x_i$ in the coordinates $x_i$ and $\Delta t$ in the time $t$ at all energy scales $E \leq E_\ell$ are referred to as **primary measurable** if

$$\Delta x_i = N_{x_i} \ell, \Delta t = N_t \tau,$$

where $N_{x_i}$ satisfies the condition **3*.1** and the integer $N_t$ falls within the interval $1 \leq |N_t| < \infty$.

**3*.3.** We define any physical quantity as **primary or elementary measurable** at all energies scales $E \leq E_\ell$ when its value is consistent with points **3*.1** and **3*.2** of this Definition.

**3*.4.** Finally, we define any physical quantity at all energy scales $E \leq E_\ell$ as **generalized measurable** or, for simplicity, **measurable** if any of its values may be obtained in terms of **Primarily Measurable Quantities** specified by points **3*.1–3*.3** of **Definition 3*.**

The "improper" points associated with $|N_{x_i}| = \infty$ and $|N_t| = \infty$ may be introduced into equation (134) and into **Definition 3*", respectively, as
in the case of low energies. It has been shown that the **Primary Measurable Momenta** nearly cover the whole momenta domain $P_{LE}$ at low energies $E \ll E_p$ (or identically $E \ll E_\ell$). However, this is no longer the case at all the energy scales $E \leq E_\ell$.

Therefore, the main target of the author is to construct a quantum theory at all energy scales $E \leq E_\ell$ in terms of **measurable** (or identically **primary measurable**) quantities from Definition 3*.

In this theory the values of the physical quantity $G$ may be represented by the numerical function $F$ in the following way:

$$G = F(N_{xi}, N_t, \ell) = F(N_{xi}, N_t, G, h, c, \kappa),$$  

(136)

where $N_{xi}, N_t$—integers from the formulae (134), (135) and $G, h, c$ are fundamental constants. The last equality in (136) is determined by the fact that $\ell = \kappa l_p$ and $l_p = \sqrt{Gh/c^3}$.

If $N_{xi} \neq 0, N_t \neq 0$ (nondegenerate case), then it is clear that (136) can be rewritten as follows:

$$G = F(N_{xi}, N_t, \ell) = \tilde{F}(\left(N_{xi}\right)^{-1}, \left(N_t\right)^{-1}, \ell)$$  

(137)

Then at low energies $E \ll E_p$, i.e. at $|N_{xi}| \gg 1, |N_t| \gg 1$, the function $\tilde{F}$ is a function of the variables changing practically continuously, though these variables cover a discrete set of values. Naturally, it is assumed that $\tilde{F}$ varies smoothly (i.e. practically continuously). As a result, we get a model, discrete in nature, capable to reproduce the well-known theory in continuous space-time to a high accuracy, as it has been stated above.

Obviously, at low energies $E \ll E_p$ the formula (137) is as follows:

$$G = F(N_{xi}, N_t, \ell) = \tilde{F}(\left(N_{xi}\right)^{-1}, \left(N_t\right)^{-1}, \ell) = \tilde{F}(p_{N_{xi}}, p_{Ntc}, \ell),$$  

(138)

where $p_{N_{xi}}, p_{Ntc}$ are **primary measurable** momenta from formula (44).

It should be noted that our approach to the concept **measurability**, as set forth in this Section, is considerably more general than in Sections 2, 3 for two reasons:
a) it is not directly related to the Heisenberg Uncertainty Principle and its generalizations;

b) it can be successfully used both for the nonrelativistic [7] and relativistic cases [31].

6 Final Comments and Further Prospects

6.1. Thus, for all the energy scales, we can derive a model still to be constructed) that is dependent on the same discrete parameters and, at low energies $E$ far from the Planck energies $E \ll E_p$, is very close to the initial theory, reproducing all the main results of the canonical quantum theory in continuous space-time to a high accuracy. At high (Planck’s) energies $E \approx E_p$ this discrete model is liable to give new results. By the author’s opinion, the model is deprived of the principal drawbacks of the canonical quantum theory—ultraviolet and infrared divergences [31]. Being finite at all orders of a perturbation theory, it requires no renormalization procedures [31].

6.2. As shown by the formula (44) and (133), measurable analogs of small and infinitesimal space-time quantities are equal (up to constants) to the primary measurable momenta. This allows us to state for gravity [32] the same problem as for a quantum theory in paragraph 6.1.: to construct a measurable model of gravity depending on the same discrete parameters $N_{x_i}, N_t$, that is practically continuous and ”very close” to General Relativity at low energies $E \ll E_p$, giving the correct quantum theory without ultraviolet divergences at high energies $E \approx E_p, (E \approx E_\ell)$. However, the words ”very close” in the last paragraph don’t mean that there is an ideal correspondence between the above-mentioned model and General Relativity [32]. The author assumes that the desired model should have no ”nonphysical” solutions of General Relativity (for example, those involving the Closed Time-like Curves (CTC) [33–36]).
6.3. At the present time each of the involved theories (Quantum Theory and Gravity) considered within the scope of continuous space-time is represented differently at low \( E \ll E_p \) and at high \( E \approx E_p \) energies. We can summarize points 6.1. and 6.2. as follows.

In the **measurable** format, both these theories (quantum theory and gravity) represent a unified theory at all energies scales \( E \leq E_\ell \). The word "unified" means that at all the energy scales they should be determined by the same discrete set of the parameters \( N_{x_i}, N_t \) and by the constants \( G, \hbar, c, \kappa \).

The main problem in this case is associated with a correct definition and computation of the functions \( F \) and \( \tilde{F} \) from formula (136)–(138). In Subsection 3.1, within the scope of the Generalized Uncertainty Principle, we have already found the function \( F \) for all **measurable** momenta \( p_{i, \text{meas}}; i = 1, \ldots, 3 \) at all the energy scales \( E \leq E_\ell \) by the formula (18), (19):

\[
p_{i, \text{meas}} = F(N_{x_i}, \ell) = \frac{\hbar}{1/2(N_{x_i} + \sqrt{N_{x_i}^2 - 1})\ell}.
\]

(139)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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