Relativistic corrections to the central force problem in a generalized potential approach

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We present a novel technique to obtain relativistic corrections to the central force problem in the Lagrangian formulation, using a generalized potential energy function. We derive a general expression for a generalized potential energy function for all powers of the velocity, which when made a part of the regular classical Lagrangian can reproduce the correct (relativistic) force equation. We then go on to derive the Hamiltonian and estimate the corrections to the total energy of the system up to the fourth power in $|\vec{v}|/c$. We found that our work is able to provide a more comprehensive understanding of relativistic corrections to the central force results and provides corrections to both the kinetic and potential energy of the system. We employ our methodology to calculate relativistic corrections to the circular orbit under the gravitational force and also the first-order corrections to the ground state energy of the hydrogen atom using a semi-classical approach. Our predictions in both problems give reasonable agreement with the known results. Thus we feel that this work has pedagogical value and can be used by undergraduate students to better understand the central force and the relativistic corrections to it.

I. INTRODUCTION

The central force problem, in which the ordinary potential function depends only on the magnitude of the relative separation between the particles ($r$) is a very well studied problem in classical and quantum mechanics. The problem is particularly useful since central potentials are common in physics, e.g. the electrostatic Coulomb and the Newtonian gravitational potential. The two-body problem under the central potential is solved in any standard textbook of classical mechanics (e.g. [1]) to obtain the equation of orbit; the hydrogen atom problem in quantum mechanics, to yield expectation values of physical observables, etc. In particular, the total angular momentum, the total mechanical energy of the system are constants of motion. The standard approach in non-relativistic classical mechanics is to convert the two-body central force equation into an equivalent one-body problem. The classical Lagrangian for this holonomic system and the corresponding Jacobi Integral of this reduced system can be easily written, followed by solving the equation of orbit. The Lagrangian is identified as $\mathcal{L} = T - V$, and the Hamiltonian or Jacobi Integral as $\mathcal{H} = T + V$, with $T$ being the kinetic energy and $V$ the ordinary potential energy functions.

The Lagrangian and Hamiltonian formulations can be made compliant with a relativistic framework, which gives rise to the correct spatial equations of motion and the total energy as observed from a given inertial reference frame. Covariant Lagrangian and Hamiltonian formulations exist which treat space and time on a common footing as the generalized coordinates in a four-dimensional configuration space. This is indeed the correct relativistic formulation which is Lorentz invariant, but it is seen that the complexity of the equations is an issue, even for the simplest possible cases like that of a single particle. The true relativistic equation for the hydrogen atom is governed by the famous Dirac equation in relativistic quantum mechanics which offers excellent insight into the problem. In this paper, we have incorporated relativistic corrections to the central force problem by using a generalized velocity dependent potential energy setup. Our Lagrangian is of the form, $\mathcal{L} = T - \mathcal{V}$, with $T$ being the non-relativistic kinetic energy and $\mathcal{V}$ being the generalized potential energy function. The new Lagrangian constructed from the generalized potential energy function produces the correct relativistic equations of motion under the influence of central force. The Jacobian Integral or the Hamiltonian is then calculated from this modified Lagrangian using the standard prescription. While understanding the physics of these relativistic corrections, we see that our methodology provides a unified approach of the relativistic corrections to the (non-relativistic) central force. We employ our methodology to some physical problems, such as the ground state of the hydrogen atom semi-classically and orbits of celestial bodies. In these cases, particle speeds are semi-relativistic such that $|\vec{v}| < 0.1c$ and this permits us to retain maximally up to quartic power of $|\vec{v}|/c$, since higher powers have negligibly small contributions. This approach is interesting since we are able to avoid the non-trivial complexities in the covariant formulation and are able to successfully add relativistic corrections to the problem using the standard textbook approach.

A brief outline of the organization of the paper is as follows. In section 2, we add relativistic corrections to the attractive, inverse square central force to define a generalized force, which we use in section 3 to derive the generalized velocity dependent potential energy function. Section 4 develops the Lagrangian and Hamiltonian of
the system, using the generalized potential energy function of Section 3, and calculates the total energy of the system. In section 5, we show two model applications of our methodology, to the classical gravitationally bound, circular orbits and the other to find the first-order correction to the ground state energy of the hydrogen atom. We conclude in section 6 by summarizing our work.

II. RELATIVISTICALLY GENERALIZED CENTRAL FORCE

A central force acts along the position vector of a particle drawn from the center of force and depends only on the scalar distance \( r = |\vec{r}| \) from the fixed center of force. The convenient choice of this fixed point to be the origin of the coordinate system makes the central force look like,

\[
\vec{F}(\vec{r}) = F(r) \hat{\vec{r}}. \tag{1}
\]

In case of the non-relativistic central force, the force is derivable from an ordinary potential energy function \( V(r) \) such that,

\[
\vec{F}(\vec{r}) = F(r) \hat{\vec{r}} = -\nabla V(r). \tag{2}
\]

Consider two point particles of rest masses \( m_1 \) and \( m_2 \) having a position vector \( \vec{r}_1 \) and \( \vec{r}_2 \), respectively, moving under the influence of central force field, where the equation of motions can be decomposed into the relative \( \vec{r}(= \vec{r}_1 - \vec{r}_2) \) and center-of-mass (\( \vec{R} \)) coordinates, respectively. In the absence of any external force on this system, the center-of-mass (\( \vec{R} \)) of the system moves as a free particle. Thus in classical, non-relativistic mechanics, the two-body problem can be reduced to the dynamics of an equivalent one-body of a hypothetical mass point of reduced mass \( \mu = (m_1 m_2)/(m_1 + m_2) \) with the relative position vector \( \vec{r} \) from the origin, moving under the action of the internal interaction force. In general, the relativistic force equation for a particle of rest mass \( m_0 \) in classical mechanics can be written as (Einstein summation implied),

\[
F_i = m_{ij} \frac{d^2 x_j}{dt^2}, \tag{3}
\]

with \( m_{ij} \) being the relativistic mass tensor.

\[
m_{ij} = m_0 \gamma^3 \left[ \frac{\delta_{ij}}{\gamma^2} + \frac{\vec{v}_i \vec{v}_j}{c^2} \right]. \tag{4}
\]

In the above equation, symbols have their usual meanings with \( v_i \) being the \( i \)th component of the spatial velocity and \( \gamma \) being the Lorentz Factor \( \gamma = \left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{-1/2} \) and \( \delta_{ij} \) the Kronecker’s delta. To understand the relative importance of the two terms in the mass tensor (Eq. 4) to the total force, we rewrite the force equation of Eq. (3) in its vector notation,

\[
\vec{F} = \frac{d}{dt}(\gamma m_0 \vec{v}) = m_0 \left( \gamma \frac{d^2 \vec{r}}{dt^2} + \gamma^3 \frac{\vec{v}^2}{c^2} (\vec{v} \cdot \vec{a}) \right), \tag{5}
\]

where the first term in the acceleration \( (\gamma \frac{d^2 \vec{r}}{dt^2}) \) is the contribution of the diagonal term of the mass tensor and the second term \( (\gamma^3 \frac{\vec{v}^2}{c^2} (\vec{v} \cdot \vec{a}) ) \) represents the off-diagonal contribution which is a product of two terms: \( (\gamma^3 \frac{\vec{v}^2}{c^2} ) \) and \( (\vec{v} \cdot \vec{a}) \). The term \( (\gamma^3 \frac{\vec{v}^2}{c^2} ) \) is much smaller than unity since \( v/c < 1 \). The term \( (\vec{v} \cdot \vec{a}) \) is also very small - for elliptic orbits with small eccentricities, the velocity is nearly perpendicular to the acceleration and hence \( \vec{v} \cdot \vec{a} \) is close to zero. It becomes identically zero for the circular orbits, where the velocity is perpendicular to the acceleration. We can thus safely argue that the off-diagonal term in the acceleration is negligibly small compared to the diagonal term. Objects in our Solar System orbiting the Sun have very low eccentricities. For example, the Earth’s orbit is nearly circular with a small eccentricity of 0.017 and Pluto is the body with the largest mean eccentricity of 0.244. A simple calculation for the orbit of Pluto yields that the off-diagonal contribution of the force is \( \sim O(10^{-17}) \) smaller than the diagonal term. Also, on the other extreme (in the microscopic length scale), the relativistic Sommerfeld analysis predicts the low eccentricity for the elliptical orbits of the electron in the hydrogen atom. Thus, for an attractive central force,

\[
\vec{F}(r) = -k/r^2 \hat{\vec{r}} \quad k > 0, \tag{6}
\]

the force equation (5) for the reduced one-body problem, after neglecting the non-diagonal term, can therefore be written as

\[
\mu \vec{v} = Q \hat{\vec{r}}, \quad Q \equiv \frac{k}{r^2} \left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{1/2} \tag{7}
\]

which depends on both the relative separation \( r \) and the velocity \( \vec{v} \) of the particle. This re-arrangement allows us to write the force equation as typical Newtonian equation of motion and use the familiar non-relativistic Lagrangian Formulation with \( L = T - U \), \( (U \) being the generalized velocity dependent potential energy function) such that the Euler-Lagrange differential equation of motion can reproduce the correct force equation.

\footnote{Reduced mass does not have a simple definition in relativistic theory. However, we could treat the heavier mass as a source of static field and also the motion of the heavier mass can be taken as non-relativistic to a very good approximation. This justifies the use of an equivalent one-body having a reduced mass \( \mu \).}
Under the central force, the motion is planar and we can use two-dimensional polar coordinates \((r, \theta)\) to describe the motion of the reduced particle. Thus, the velocity of the reduced particle can be resolved into its radial and tangential components as,

\[
|\dot{v}|^2 = \dot{r}^2 + r^2 \dot{\theta}^2.
\]

Therefore the force equation (7) can be decomposed into radial and tangential components:

\[
\mu \ddot{r} - \mu r \ddot{\theta}^2 = Q \quad (10)
\]

\[
\mu r \ddot{\theta} + 2 \mu \dot{r} \dot{\theta} = 0.
\]

We can clearly see from Eq. (11) that the total angular momentum \(L\) is a constant of motion such that,

\[
\frac{dL}{dt} = \frac{d}{dt} (\mu r^2 \dot{\theta}) = 0. \quad (12)
\]

Hence, we can eliminate \(\dot{\theta}\) in terms of \(L\) and \(r\), so that the planar problem again be reduced to the radial \((r)\) problem only.

For \(|\vec{v}|/c < 1\), \((\vec{v} = \vec{r})\), we can write the generalized force by expanding the Lorentz factor in powers of \(|\vec{v}|/c\)

using a binomial expansion,

\[
Q = -\frac{k}{r^2} \left[ \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{|\vec{v}|}{c} \right)^{2p} \right], \quad (13)
\]

where \(\frac{1}{p!}\) is the binomial coefficient of \(1/2\) and \(p\). Due to the conservation of angular momentum, the velocity \(|\vec{v}|\) becomes

\[
|\vec{v}| = \frac{L^2}{\mu r^3} \left( 1 + \frac{\mu^2 r^2}{L^2} \dot{r}^2 \right), \quad (14)
\]

so we further expand \((|\vec{v}|/c)^{2p}\) using Eq. (13) in Eq. (13) in terms of a power series in the radial speed \(\dot{r}\) only. Therefore the generalized force becomes

\[
Q = k \left[ \sum_{p=0}^{\infty} \sum_{q=0}^{p} B_{p,q}(L, \mu) v^{2(q-p-i)} (\dot{\theta})^{2q} \right], \quad (15)
\]

where we have defined \(B_{p,q}(L, \mu)\) as the expansion coefficient depending on \(L\) and \(\mu\) given by,

\[
B_{p,q}(L, \mu) = \frac{(-1)^{p+1}}{e^{2p}} \left( \frac{1}{p!} \right)^{q} \left( \frac{L}{\mu} \right)^{2(p-q)} \quad (16)
\]

It is proper to mention here that the generalized force for a relativistic particle of rest mass \(\mu\) contains all powers in \(|\vec{v}|/c\). In this paper, we have limited ourselves upto the quartic power in \(|\vec{v}|/c\) since higher order terms will have negligibly small contribution in the semi-relativistic regime. Interested readers are encouraged to explore the contribution of higher powers to the central force. Thus, up to fourth power in \(|\vec{v}|/c\), the generalized force becomes

\[
Q = -\frac{k}{r^2} + \frac{k L^2}{2 \mu^2 c^2 r^4} + \frac{k L^4}{8 \mu^2 c^4 r^6} + \frac{k \dot{r}^2}{2 c^2 r^2} + \frac{k L^2 \dot{r}^2}{4 \mu^2 c^2 r^4} + \frac{1}{8} \frac{k}{c^4} r^2, \quad (17)
\]

where the first term is the usual inverse square attractive force. Both the second and third terms represent the relativistic corrections to the (non-relativistic) central force and allow for the coupling of the angular momentum component with the central force component. However, in the third term higher powers of angular momentum couples with the central force. Later we will see that these two terms in the generalized force will contribute to the (relativistic) corrections of the potential energy of the system. The fourth and fifth terms which depend on the square of the radial speed and also couple the angular momentum of the system with the central force, will represent quadratic and quartic power corrections to the kinetic energy of the system respectively. The sixth term is a higher order kinetic correction to the central force of the system depending on the fourth power of the radial speed.

We add a further simplification here - building on the fact that \((\vec{v} \cdot \hat{a})\) (in Eq. 5) is negligibly small for closed orbits of smaller eccentricity. It can be shown that the term \((\vec{v} \cdot \hat{a})\), which is a measure of the eccentricity, is proportional to the radial velocity, \(\dot{r}\), so for a circular orbit, \(\dot{r} = 0\) identically, and for nearly circular orbits, \(\dot{r}\) is small. Out of the three terms in the expansion having fourth power in \(|\vec{v}|/c\) in the generalized force, we neglect only the term having the fourth power of the radial speed \(\sim \dot{r}^4/c^4\) but retain the contribution of the fourth power of tangential speed and a mixed biquadratic term in radial and tangential velocities. Thus the generalized force of Eq. (17) is simplified into

\[
Q = -\frac{k}{r^2} + \frac{k L^2}{2 \mu^2 c^2 r^4} + \frac{k L^4}{8 \mu^2 c^4 r^6} + \frac{1}{2} \frac{k \dot{r}^2}{c^2 r^2} + \frac{k L^2 \dot{r}^2}{4 \mu^2 c^2 r^4}, \quad (18)
\]

which will make the radial momentum (Eq. 8) linear in radial velocity, \(\dot{r}\) and presents a welcome simplification to construct the Hamiltonian from the Lagrangian without loosing much on the physics of the problem.

III. THE RELATIVISTIC GENERALIZED POTENTIAL

We now construct a Lagrangian function \(L(r, \dot{r}, t)\) of the non-relativistic, standard form of \(L = T - U\), such that Eq. (10) can be reproduced through the Euler-Lagrange equation of motion satisfying,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0, \quad (19)
\]

and the right hand side of the above equation is zero since there are no non-potential forces acting on the sys-
The kinetic energy function $T$ is chosen to have the Newtonian, non-relativistic form,

$$T(r, \dot{r}, t) = \frac{1}{2}m|\dot{r}|^2,$$  \hspace{1cm} (20)

and $U \equiv U(\vec{r}, \dot{r}, t)$ is the generalized potential energy function. The potential energy function $U$ will be chosen such that the generalized force $Q$ should be derivable from $U$ as,

$$Q = \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{r}} \right) - \frac{\partial U}{\partial r}.$$  \hspace{1cm} (21)

Since the corrected force also depends on the radial velocity $\dot{r}$ in addition to $r$, hence an ordinary potential $V(r) = -k/r$ will not be sufficient to describe the generalized force. At this stage, we can expect that the generalized potential $U(r, \dot{r}, t)$ contains only even powers of $\dot{r}$, hence an ordinary potential $V(r)$ will not be sufficient to describe the generalized force expansion of Eq. (15) contains only even terms of the force of Eq. (15) in equal powers of $\dot{r}$, which are independent of time explicitly. This allows welcome time dependence in the generalized force equation as well. The generalized potential $Q$ of Eq. (15) is independent of the particle’s radial acceleration $\ddot{r}$, as is the case with most naturally occurring forces in nature and hence the sum of the second term in Eq. (23) must result up to zero, we get,

$$\sum_{n=0}^{\infty} (2n)(2n-1)h_{2n}(r) v^{2n-2} = 0,$$  \hspace{1cm} (24)

and thus, the generalized force can be expressed in a power series form as follows,

$$Q = \sum_{n=0}^{\infty} (2n-1) \frac{\partial h_{2n}(r)}{\partial r} v^{2n}.$$  \hspace{1cm} (25)

We now compare the coefficients in Eq. (25) with the terms of the force of Eq. (15) in equal powers of $v$, which gives us a general expression for $h_{2n}(r)$,

$$h_{2n}(r) = \frac{k}{2n-1} \sum_{p=n}^{\infty} B_{p,n}(L, \mu) \frac{r^{2(n-p)-1}}{2(n-p)-1} + A_{2n},$$  \hspace{1cm} (26)

with $A_{2n}$ being the constants of integration, to be determined by boundary conditions. Thus, we have constructed a general expression for the generalized potential $U(r, v)$ which when plugged in Eq. (21) will give the exact expansion for the corrected force. To determine the integration constants $A_{2n}$, we use the usual boundary condition as employed in standard analysis of the central force. As $r \to \infty$, the force field dies down to zero. In this limit, the potential goes to zero and the first term ($r$-dependent) in the expansion coefficients $h_{2n}(r) = 0$ vanishes, we get,

$$\lim_{r \to \infty} U(r, v) = 0 = \sum_{n=0}^{\infty} A_{2n} v^{2n}.$$  \hspace{1cm} (27)

Since this holds for any arbitrary radial velocity $v$ of the particle, we find all $A_{2n}$ to be zero identically, for all $n$,

$$A_{2n} = 0.$$  \hspace{1cm} (28)

By comparing with the relativistic force equation Eq. (15), we obtain the expansion coefficients as

$$h_0(r) = -\frac{k}{r} + \frac{kL^2}{6\mu^2 c^2 r^3} + \frac{kL^4}{40\mu^4 c^4 r^5} + A_0,$$  \hspace{1cm} (29)

$$h_2(r) = -\frac{k}{2c^2 r} + \frac{kL^2}{12\mu^2 c^4 r^3} + A_2.$$  \hspace{1cm} (30)

Thus, up to fourth power in $|\vec{v}|/c$, the generalized potential energy can be written as,

$$U(r, v) = -\frac{k}{r} + \frac{kL^2}{6\mu^2 c^2 r^3} + \frac{kL^4}{40\mu^4 c^4 r^5} + \frac{k v^2}{2c^2 r} - \frac{kL^2 v^2}{12\mu^2 c^4 r^3}.$$  \hspace{1cm} (31)

We see an ordinary potential correction coming from $h_0(r)$ which tends to couple the angular momentum with the central force constant $k$. There are additional “kinetic” corrections proportional to the square of the radial velocity arising from $h_2(r)$ and these will modify the kinetic energy of the system, as we shall see later. The contribution of these terms will be discussed in detail in the next section.

**IV. LAGRANGIAN AND HAMILTONIAN FORMULATIONS**

We now proceed to build a Lagrangian and following which, a Hamiltonian of the relativistic system under central force. As suggested earlier, we will use a Lagrangian function $L(\vec{r}, \dot{r}, t) (= T-U)$, with the relativistically corrected generalized potential $U$ found in the previous section. With such a form, we see that the Euler-Lagrange
In the limit \( \Gamma \ll 1 \), the rest mass energy with the binding energy of the system. In many physical situations, the binding energy of the system, while in the opposite relativistically since the rest mass energy is much larger compared to the binding energy of the system, and has dimensions of energy. It is here we see the usefulness of neglecting the term proportional to fourth power in radial speed in Eq. (18). It allows us to keep the radial momentum linear in the radial speed. Relativistic corrections can be clubbed into a single pre-factor which corrects the non-relativistic momentum \( \mu v \). This is in line with our motivation to use a Newtonian framework for formulating relativistic corrections to the central force. Thus the canonical momentum can be expressed in a succinct form by allowing comparison with the classical, non-relativistic momentum as,

\[
p_r = \left[ 1 + \frac{\Gamma}{r} \left( 1 + \frac{L^2}{6 \mu^2 c^4 r^2} \right) \right] \mu v = \chi(r) \mu v .
\]  

(34)

In the above equation, we have substituted \( \Gamma \) for \( k/(\mu c^2) \), which is an interaction parameter, useful in better understanding the physics of the problem. The parameter \( \Gamma \) is an indicative of the relative strength of the central force coupling to the rest mass energy \( \mu c^2 \) of the system, and has dimensions of length. It allows us to compare the rest mass energy with the binding energy of the system. In the limit \( \Gamma \ll 1 \), the problem can be treated non-relativistically since the rest mass energy is much larger than the binding energy of the system, while in the opposite limit, the treatment is required to be relativistic, since the binding energy is comparable to the rest mass energy of the system. In many physical situations, \( \mu c^2 \) is very large compared to the binding energy of the system, for example, an electron in the hydrogen atom, the rest mass energy (\( \sim 511 \) keV) is much larger than its binding energy (\( \sim 10 \) keV) so that the state properties can be understood through the non-relativistic Schrödinger wave equation. The entire pre-factor has been labeled as \( \chi(r) \) to allow expressions to have a condensed form. The angular momentum \( p_\theta \), canonically conjugate to the \( \theta \) coordinate is a constant of motion as we had seen earlier and labeled it as \( L \). The Hamiltonian \( \mathcal{H} \) of the system can now be calculated as a Legendre dual transform of the Lagrangian \[12\]

\[
\mathcal{H}(\vec{r}, \vec{p}, t) = p_r \dot{r} + p_\theta \dot{\theta} - \mathcal{L} .
\]  

(35)

Again eliminating \( \dot{\theta} \) using the fact that the angular momentum \( p_\theta \equiv L \) is a constant of motion, we can express the Hamiltonian in terms of the radial coordinate and its conjugate momentum exclusively,

\[
\mathcal{H} = \frac{p_r^2}{2 \mu \chi(r)} + \frac{L^2}{2 \mu r^2} - k + \frac{\Gamma L^2}{r} + \frac{\Gamma L^4}{40 \mu^3 c^2 r^5} .
\]  

(36)

Since the Hamiltonian does not depend on time explicitly, so it can be identified as the total energy \( E \) of the system, which is a constant of motion. Thus, the total mechanical energy of a system is seen to have relativistic correction to both the kinetic and potential energy terms. It can readily be verified that in the \( \Gamma \to 0 \) limit, we recover the non-relativistic results. In the next section, we present two model applications of this relativistically corrected central force formulation to physical systems under the influence of the central force.

V. MODEL APPLICATIONS

We now present model applications of the relativistic central force formulation using a generalized potential to the classical circular orbit in celestial mechanics under the gravitational force and the first order energy correction to the ground state of the hydrogen atom in quantum mechanics. These applications can be used as a pedagogical tool in the undergraduate classroom to understand the implications of relativistic corrections to the central force. These can be seen as approximations that bring out the physics of the problem without having to indulge non-trivial mathematics.

A. Relativistic Correction to the Classical Circular Orbit

We shall now be considering a gravitationally bound circular orbit and calculate a relativistic correction to it. Before we calculate the correction, we assert that in case of planetary orbits in the solar system, \( |\vec{v}|/c < 10^{-4} \), so we retain only in quadratic powers in \( |\vec{v}|/c \). This would also allow us to give a closed form to the relativistic correction to the radius of the circular orbit. In this case, the central force constant \( k = G m_1 m_2 \), where \( G \) is the universal gravitational constant. As is customary in non-relativistic central force problems, we define an effective potential \( V_{\text{eff,NR}} \) (NR stands for non-relativistic), which compares the effect of the (attractive) central potential and the (repulsive) centrifugal terms\[13\], as

\[
V_{\text{eff,NR}}(r) = \frac{-G m_1 m_2}{r} + \frac{L^2}{2 \mu r^2} .
\]  

(37)

On similar lines, we can define a relativistic effective potential for the central force which has an additional corrective term,

\[
V_{\text{eff,rel}}(r) = \frac{-G m_1 m_2}{r} + \frac{L^2}{2 \mu r^2} + \frac{\Gamma L^2}{6 \mu r^3} .
\]  

(38)
\[ V_{\text{eff,rel}}(r) = V_{\text{eff,NR}}(r) + \frac{\Gamma L^2}{6\mu c^3}, \] (39)

such that the energy of the system can now be compactly written as (where ‘rel’ stands for relativistic), in terms of effective potential

\[ E = \frac{1}{2}\mu \left( 1 + \frac{\Gamma}{r} \right) v^2 + V_{\text{eff,rel}}. \] (40)

We can now solve for the radial speed \( \dot{r} \) by re-arranging terms in Eq. (40),

\[ \dot{r}^2 = \frac{2}{\mu} \left( 1 + \frac{\Gamma}{r} \right)^{-1} (E - V_{\text{eff,rel}}(r)). \] (41)

It is known that that bound orbits, under the classical central force problem of an inverse square attractive potential, occur when \( \min(V_{\text{eff}}) \leq E < 0 \). A special case occurs when \( E = \min(V_{\text{eff}}) \) such that \( \dot{r} = 0 \) and the resultant orbit is circular. Non-relativistically, a circular orbit occurs when \( E = \min(V_{\text{eff,NR}}) \) at a radius \( r_0 \),

\[ r_0 = \frac{L^2}{\mu k} \equiv 2\eta, \] (42)

where, we have defined \( \eta \) for mathematical convenience. Relativistically, we will now give an estimate of the radius of the circular orbit for a given value of \( L \). We observe that

\[ (1 + \frac{\Gamma}{r})^{-1} \neq 0, \hbox{ hence, } \dot{r} = 0 \] for a circular orbit occurs only when \( E = \min(V_{\text{eff,rel}}) \). Let the minimum of \( V_{\text{eff,rel}} \) occur at \( r = r_\ast \), such that,

\[ \left( \frac{dV_{\text{eff,rel}}}{dr} \right)_{r_\ast} = 0 \ \hbox{and} \ \left( \frac{d^2V_{\text{eff,rel}}}{dr^2} \right)_{r_\ast} > 0. \] (43)

The stationarity condition of Eq. (43) can be solved to yield the following quadratic equation in \( r \),

\[ \dot{r}^2 - 2\eta r - \Gamma \eta = 0, \] (44)

The radius of the relativistic circular orbit is found to be,

\[ r_\ast = \frac{r_0}{2} \left[ 1 + \left( \frac{1 + \Gamma}{\eta} \right)^{1/2} \right] \equiv r_0 \delta, \] (45)

where we have defined \( \delta \) as a parameter to compare the circular radii for the non-relativistic and relativistic cases, for a given \( L \) and coupling \( k \),

\[ \delta = \frac{1}{2} \left[ 1 + \left( 1 + \frac{2k^2}{L^2 c^2} \right)^{1/2} \right]. \] (46)

It is readily seen that \( \delta > 1 \), thus, the radius of the circular orbit with the relativistic correction is more than without it, \( r_\ast > r_0 \). The presence of the additional positive term in the relativistic effective potential, \((\Gamma L^2/6\mu c^3)\) tends to flatten the potential curve at larger radii and can be understood as the reason for the swelling of the circular orbit. It can easily be calculated that the energy of the circular orbit with the relativistic correction \( E_\ast \) is,

\[ E_\ast = \frac{E_0}{\delta} + \frac{\mu k^2}{2L^2 \delta} \left( \frac{1}{3} + \frac{k^2}{3\delta^2 L^2 c^2} - 1 \right), \] (47)

where \( E_0 \) is the energy of the non-relativistic circular orbit with the same angular momentum,

\[ E_0 = V_{\text{eff,NR}}(r_0) = -\frac{\mu k^2}{2L^2}. \] (48)

Thus, we have calculated the radius and energy of a circular orbit under the given central force with the relativistic correction incorporated and have presented our results in a comparative way with the non-relativistic ones for a given value of \( L \).

B. Relativistic Correction to the Ground State
Energy of the Hydrogen Atom in Vacuum

In this section, we present a first order correction to the energy of the ground state of the hydrogen atom in vacuum due to relativistic corrections in the Hamiltonian. We assume that the proton is infinitely heavy compared to the electron and can therefore be assumed to be stationary. Thus, the electron feels an electrostatic Coulomb force of attraction and in vacuum, \( k = e^2/(4\pi \epsilon_0) \) with \( e \) being the electronic charge and \( \epsilon_0 \) the permittivity of free space. Contrast to the analysis of celestial circular orbits in the previous section V A, where we retained only quadratic powers in \( |\vec{r}|/c \), here we consider corrections till fourth power in \( |\vec{r}|/c \). Thus the Hamiltonian of the hydrogen atom can be written from Eq. (50),

\[ \mathcal{H}_{\text{rel}} = \frac{p^2}{2\mu \chi(r)} + \frac{L^2}{2\mu r^2} - \frac{k}{r} + \frac{\Gamma L^2}{6\mu r^3} + \frac{\Gamma L^4}{40\mu c^2 r^5}, \] (49)

It is worth mentioning at this stage that the usual relativistic kinetic energy in standard texts involves expanding the relativistic kinetic energy in powers of \( p/(\mu c) \) and retaining terms till the fourth power of \( \mu \). We later argue as to why both approaches, even though retaining the same power in their respective expansions do not result in the same numerical value of the corrections. They, however match in order of magnitude, as expected. The \( r \) dependent function \( \chi(r) \) can be re-written as,

\[ \chi(r) = \left[ 1 + \frac{r}{L} \left( 1 + \frac{L^2}{6\mu c^2 r^3} \right) \right] \left( 1 + \frac{\Gamma \alpha(r)}{r} \right), \] (50)

where we have defined \( \alpha(r) \) for convenience. Readers’ attention is brought to the fact that the fourth term in the relativistic Hamiltonian, \( kL^2/(6\mu c^2 r^3) \) resembles the spin-orbit\([13]\) coupling term in the hydrogen atom problem. It is very interesting to note that a unified generalized potential correction to the central force Hamiltonian automatically provides a fore-runner for spin-orbit
like terms, which essentially work to couple the central force with the angular momentum of the system. In particular, the spin-orbit term in the hydrogen atom involves the coupling of the electron intrinsic spin angular momentum with the orbital magnetic field of the proton seen by the electron. Our generalized potential approach predicts a similar form of the coupling of the central force with the total angular momentum of the system as a relativistic phenomenon, and may be understood as a pedagogical explanation for coupling of the spin and orbital angular momentum of the system with the central force. Our treatment here would be semi-classical, whereby we would use the classical Hamiltonian derived in the earlier sections, to find a quantum mechanical expectation value of the first order correction in the ground state energy. The non-relativistic, unperturbed Hamiltonian for the hydrogen atom \(H_{NR}\) (NR stands for non-relativistic) can be easily written as,

\[
H_{NR}(r, p_r) = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} - \frac{e^2}{4\pi\varepsilon_0 r}.
\]

As expected, Eq. (51) corresponds to substituting \(\Gamma = 0\) in Eq. (49). In case of the hydrogen atom, we see that \(\Gamma \simeq 2.81 \times 10^{-13}\)m, while the most probable radius in the ground state is the Bohr radius \(a_0(\approx 5.29 \times 10^{-11}\)m\).\[16\]

As a first approximation for the ground state, we replace \(r\) in the kinetic term in Eq. (49) by the Bohr radius \(a_0\), to ease the calculation. This is justified, since for the ground state of the hydrogen atom, the variance in the observable corresponding to \(r\) is small. Since \(\Gamma/a_0 \ll 1\), we can expand \(\chi(r)\) in the kinetic term binomially and retain terms up to first order in \(\Gamma/a_0\). The idea is to treat the relativistic correction to the Hamiltonian as a small perturbation since \(\Gamma\alpha(a_0)/a_0 \ll 1\), and calculate the expectation value of the first order perturbation to the ground state wave energy of the hydrogen atom by treating \(r\) and \(p_r\) as corresponding quantum mechanical operators. The relativistic correction is treated as a perturbation \((\hat{H}')\) to the non-relativistic Hamiltonian \(H_{NR}\),

\[
H_{rel} = H_{NR} + \hat{H}'(r, p_r): \hat{H}' << H_{NR}.
\]

and correspondingly the relativistic energy correction may be added to \(E^{(0)}\) (the unperturbed, non-relativistic ground state energy of the hydrogen atom),

\[
E = E^{(0)} + E_r^{(1)},
\]

where \(E_r^{(1)}\) is the first order correction to the energy due to relativistic effects. After replacement of \(r\) by \(a_0\) in the kinetic term and retaining up to first power in \(\Gamma\alpha(a_0)/a_0\), we replace the physical observable \(r\) and \(p_r\) with their corresponding quantum mechanical operators and write an operator for the perturbation in the Hamiltonian,

\[
\hat{H}'(r, p_r) = \left(-\frac{\Gamma\alpha(a_0)}{a_0}\right) \frac{\hat{p}_r^2}{2\mu} + \frac{\Gamma L^2}{6\mu r^3} + \frac{\Gamma \hat{L}^4}{40\mu^4\varepsilon_0^2 r^5}.
\]

We use the ground state of the hydrogen atom to obtain the expectation value of the first order energy correction to the system as,

\[
E_r^{(1)} = -\frac{\Gamma}{a_0}\left(1 + \frac{l(l+1)\hbar^2}{6\mu^2\varepsilon_0^2 a_0^2}\right)\left(E^{(0)} + \frac{e^2}{4\pi\varepsilon_0 a_0}\right) + \frac{\Gamma(l+1)\hbar^2}{6\mu}\left(\frac{1}{r^3}\right) + \frac{\Gamma^2(l+1)^2\hbar^4}{40\mu^4\varepsilon_0^2}\left(\frac{1}{r^5}\right) \tag{55}
\]

The expectation values of powers of \((1/r)\) can be computed from the famous Kramer’s relation (or the second Pasternack relation\[17, 18\])

A numerical comparison is in place here. The unperturbed ground state energy \(E^{(0)} = -13.6\)eV, while standard texts\[19\] quote a first order correction its kinetic energy solely to be \(-8.99 \times 10^{-4}\)eV by expanding the relativistic kinetic energy in powers of \(p/\mu c\) and retaining up to fourth power in \(p\). The correction due to the fine-structure which includes both kinetic and spin-orbit corrections is quoted to be \(-1.81 \times 10^{-4}\)eV. The relativistic correction due to our Hamiltonian, using the generalized potential is found to be from Eq. (55) to be \(-2.41 \times 10^{-4}\)eV. Thus, we see that our prediction matches more closely to the fine-structure value which includes corrections to the kinetic energy and also incorporates corrections due to angular momenta coupling to the central force. On utilizing the Schrödinger’s Equation to find out the expectation value of \(p^4\) in the standard approach\[20\] we notice that it splits into terms containing \(1/r\) and \(1/r^2\). This can also be seen in the kinetic correction term (first term) in our Hamiltonian of Eq. (49), which when expanded in powers of \(\Gamma/r\) yields similar expressions as in the standard textbook approach. Thus, we stress on the similarity of structure of the kinetic energy corrections in both the approaches.

Our methodology of building a generalized potential energy is able to give both a kinetic correction and in addition generates terms which correct the potential energy of the system as well. Corrections to the potential energy of a system due to relativistic effects are not very surprising. A famous motivation for their existence is the Darwin term\[21\] which occurs naturally in Dirac equation and involves corrections to the effective potential at the nucleus in the hydrogen atom. Our Hamiltonian splits into terms independent of \(\hat{r}\) and also terms depending on various powers of \(\hat{r}\). It allows for coupling of the central force with the angular momentum of the system (spin and orbital) and produces corresponding corrections to the system, which match closely to the predicted value of

\^3 In the standard approach, the first order energy correction is proportional to \(<p^3>\) and is given by,

\[
E_r^{(1)} = -\frac{1}{2mc^2}\left(<(E-V)^3>\right)
= -\frac{1}{2mc^2}\left[E_s^3 + 2E_n \frac{e^2}{4\pi\varepsilon_0} \left(\frac{1}{r}\right) + \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \left(\frac{1}{r^2}\right)\right] \tag{56}
\]
the fine-structure corrections. Our unified methodology which aims to reproduce the force equation using a suitable generalized potential energy and consequently study the corrections it produces in the Hamiltonian. Thus, our approach is different from the regular kinetic correction in texts and leads to a different Hamiltonian which adds additional correction terms (to the potential energy) due to coupling of the angular momentum of the system with the central force and hence the two numerical values are not expected to match exactly. Hence, using a generalized potential and a semi-classical approach, we are able to predict the correct order of the correction in the energy due to a relativistic correction, which can be seen as a model application for the relativistically corrected Hamiltonian.

VI. CONCLUSION

In our novel approach to introduce relativistic corrections to the central force problem, we have derived a generalized potential energy function which depends both on the position and velocity of the two-body system. The Lagrangian constructed has a familiar classical form, only with the replacement of the ordinary potential with the generalized counterpart derived in the paper. Our methodology employs a power series expansion in all powers in $|\vec{v}|/c$ for the generalized potential energy. The new Lagrangian is able to reproduce the relativistic force equation for the particle under the influence of the central force. Thus, we did not have to indulge in a covariant formulation (Lorentz invariance), while we used a simplistic Lagrangian methodology to extract the basic physics of the problem at hand. The Hamiltonian of the system can easily be constructed and used to calculate relativistic corrections to the total energy of the system. We employ our model to derive and understand the relativistic corrections in a central force, namely (i) to the circular orbits in celestial mechanics and (ii) to the ground state Bohr energy of the hydrogen atom and found a reasonable match with known results.

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