Field Theory Interpretation of $N = 2$ Stringy Instantons

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Abstract

We consider stringy instanton contributions to the prepotential of low-energy $N = 2$ theories engineered by D-brane set ups at orbifold and orientifold singularities. We show that such contributions can always be reproduced in a purely field theoretic UV completion. We perform the explicit check up to instanton number three, both for $Sp(0)$ and $U(1)$ stringy instantons, the latter being introduced for the purpose. We further argue that the UV completion that we propose, though weakly coupled, can be smoothly mapped to a gravity-dual inspired UV completion consisting of a cascade of baryonic root transitions.

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1 Introduction

A particular class of non-perturbative effects in string theory can be described by D-brane instantons, i.e. by the effects on the low-energy degrees of freedom caused by the presence of the world-volume of an Euclidean D-brane wrapped on some (compact) cycle of the internal geometry [1–4]. When the low-energy theory comprises gauge theory degrees of freedom, associated with spacetime-filling (ordinary) D-branes, the effects of D-brane instantons can sometimes be straightforwardly linked to gauge theory instantons. String theory actually provides in this case a somewhat simpler rationale for the ADHM construction [5], giving a physical interpretation for the many fermionic and bosonic zero modes that appear in that construction (see for instance [6,7]).

However, there are some instances where D-brane instantons provide non-perturbative corrections which cannot be ascribed to ordinary gauge theory instantons [8–10]. This kind of non-gauge instanton effects in gauge theories, derived in string theory by D-brane setups, have been dubbed stringy instantons (for a review, see [11]). As a rule of thumb, given a stack of spacetime-filling D-branes, gauge theory instantons correspond to D-brane instantons lying on top of it, while stringy instantons originate from D-brane instantons orthogonal to it in the compact space.

Corrections due to stringy instantons were studied mostly focusing on D-brane setups leading to gauge theories with $\mathcal{N} = 1$ supersymmetry, the instantons contributing to the superpotential of the theory. It was soon realized [15,16] that stringy instantons do actually contribute to the superpotential only in a few restricted cases, due to the generic presence of some extra neutral fermionic zero modes. A simple way to get rid of these embarrassing zero modes, devised in [15] in simple non-compact orbifold models but easily generalizable, is to project them out by introducing an orientifold. Another setting is to place a D-brane instanton on top of an additional single spacetime-filling D-brane [17] (see also [18,19]), so that the extra zero modes are soaked up just as in the usual gauge-instanton setup, except for the fact that for the $U(1)$ theory living on the extra brane there is no corresponding gauge theory instanton. We remark that in both of these cases of $\mathcal{N} = 1$ stringy instantons, only the single instanton contributes to the superpotential.

That stringy instantons are really stringy effects was put into question in [20]. There it was shown that, by embedding the low-energy theory in a cascade of Seiberg dualities (as is usual in quiver gauge theories arising at Calabi-Yau singularities), the stringy instanton contribution to the superpotential could alternatively be derived from known non-perturbative effects in the (strongly coupled) gauge theory one step up in the cascade.

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1 Analogous String theory instanton effects, so-called residual instantons, have been studied in [12,13] in the context of matrix models. For a discussion about the relation between stringy instantons and residual instantons see [14].
This kind of field theory interpretation of stringy instanton superpotential contributions was extended in [21,22] to virtually all cases, including both $Sp(0)$ (i.e. orientifolded) and $U(1)$ stringy instantons.

Stringy instanton contributions to four-dimensional $\mathcal{N} = 2$ gauge theories have relatively had little attention, possibly due to less phenomenological interest. They are however of clear theoretical interest. In [23,25] they were studied in a simple orientifold of an orbifold set up, thus pertaining to stringy instantons of the $Sp(0)$ kind (see also [15] for a very brief glance at the same effect). As it befits an $\mathcal{N} = 2$ set up, they contribute to the prepotential, and contrarily to the $\mathcal{N} = 1$ case, they contribute for every instanton number. In other words, in this instance multi-stringy instantons contribute and one can even hope to resum their contributions.

Once the existence of $\mathcal{N} = 2$ stringy instantons is established, it is legitimate to ask whether also for them a field theory interpretation can be given. Though from the point of view of string theory, $\mathcal{N} = 2$ stringy instantons are not too different from their $\mathcal{N} = 1$ cousins, in the field theoretic UV completion matters could be quite different. Indeed, the AdS/CFT correspondence suggests two rather different UV field theory completions. In the $\mathcal{N} = 1$ case the completion relied on the persistence of the superpotential through a Seiberg duality. In the $\mathcal{N} = 2$ case both the superpotential and the Seiberg duality are not applicable concepts anymore. As it was shown in [26] (see also [27]), a low energy D-brane set up emerges rather from a cascade of baryonic root transitions [28].

In this paper, we show that given a low-energy D-brane set up where $\mathcal{N} = 2$ stringy instantons contribute to the prepotential, there indeed is a UV completion such that the same contributions can be given a field theory interpretation, namely in terms of a prepotential derived from ordinary instanton localization techniques or from a Seiberg-Witten curve. We will check this up to three-instanton contributions, and both for $Sp(0)$ and $U(1)$ stringy instantons (which we introduce for the purpose). We also show that the UV completion, though it does not imply any strong coupling transition, can be smoothly related to a theory that appears higher up in a cascade of baryonic root transitions, thus making the proposed UV completion natural even as a quiver-inspired completion.

The paper is structured as follows. In Section 2 we present the simplest instance of an $\mathcal{N} = 2$ instanton, and give its field theory interpretation through a weakly coupled UV completion. Our aim is to present our main result devoid of all the technicalities involved in both pictures (stringy and field theoretic). In Section 3 we discuss in much more detail the $\mathcal{N} = 2$ orientifolded case, i.e. the $Sp(0)$ stringy instantons. Building on the results of [23,25] on the string side, we provide a matching of the two interpretations up to instanton number three, finding complete agreement. In Section 4 we introduce the $\mathcal{N} = 2$ version of the $U(1)$ stringy instantons in an orbifold set up, showing that they indeed contribute to the prepotential. We then proceed also in this case to exactly match the results to a specific UV completion. Moreover we discuss some simple generalizations of the $U(1)$
orbifolds, and the embedding of our procedure into duality cascades. In the Appendices we collect some technical results. In Appendix A we describe the introduction of masses in the symplectic case. In Appendix B we provide some details about the computations of the gauge instanton contributions using localization techniques. In Appendix C we discuss three different regularizations for the stringy instanton computation and, eventually, in Appendix D we illustrate the relation between the procedure we use in the main text and the $\mathcal{N} = 1$ set ups.

2 A simple $\mathcal{N} = 2$ stringy instanton and its field theory interpretation

The purpose of this section is to sketch in a very simple example the idea that we will then discuss in all detail and apply to all the occurrences of $\mathcal{N} = 2$ stringy instantons.

Our simplest example is provided by fractional D3-branes on an orientifold of the orbifold $\mathbb{C}^2/\mathbb{Z}_3$, see Figure 1 in the next Section and consider $N_1 = N'$ and $N_2 = N$. The four-dimensional $\mathcal{N} = 2$ quiver gauge theory associated to a general configuration is given by vector multiplets associated to a gauge group $Sp(N') \times U(N)$, hypermultiplets associated to bifundamental matter between the two gauge groups and a hypermultiplet in the symmetric representation attached to the unitary gauge group. When there are no fractional branes along the orientifold plane we have $N' = 0$ and the gauge theory reduces to a single $U(N)$ gauge group with a symmetric hypermultiplet.

We now consider the D-brane instantons. As for the spacetime-filling branes, there are two fractional D($-1$)-instantons, one on the node along the orientifold and one on the other node. The first would correspond to a gauge instanton of the $Sp(N')$ gauge group, with symmetry group $SO(k)$ for $k$ instantons [29], while the second is a gauge instanton for the $U(N)$ gauge group, with symmetry group $U(k')$. When $N' = 0$, the instantons on the first node appear as having a purely stringy origin, since the $Sp(0)$ gauge theory is empty. However, their $SO(k)$ symmetry group is such that the fermionic zero modes are just the right number in order to contribute to the prepotential.

Let us specialize to a single instanton on top of the $Sp(0)$ node. One can show [15,23,24] that as a result of integrating over all the fermionic and bosonic zero modes, except the four bosonic and the four fermionic zero modes corresponding to the integration over the chiral $\mathcal{N} = 2$ superspace, the following contribution to the prepotential is generated:

$$\mathcal{F}^s_{1-\text{inst}} = M_s^2 e^{2\pi i \tau_D} \det \Phi,$$

where $\Phi$ is the adjoint vector multiplet of the $U(N)$ gauge group, the superscript $s$ reminds us that we are dealing with a stringy instanton, $M_s = 1/\sqrt{\alpha'}$ is the string mass, and $\tau_D$ is related to the (stringy) volume of the cycle wrapped by the fractional D($-1$)-instanton.
This contribution cannot clearly be attributed neither to gauge instantons of the \( U(N) \) gauge group, nor obviously to ones of the \( Sp(0) \) gauge group. Nevertheless, string theory instructs us that if we are to consider this gauge theory within this framework, we have to take into account this instanton correction. Is the latter purely stringy in nature, then?

We now show that there is another, purely field theoretic, UV completion of the present gauge theory, that reproduces exactly the same instanton correction to the prepotential. Consider first the \( U(N) \) gauge group as weakly coupled, essentially providing for a global “flavor” symmetry. Actually, to be precise, in a \( Sp(N') \times U(N) \) quiver the \( Sp(N') \) gauge group has a total of \( N \) fundamental hypermultiplets. Starting then with the \( Sp(0) \times U(N) \) theory, we UV complete it with a \( Sp(M) \times U(N+M) \) theory,\(^2\) where \( M \) of the \( Sp(M) \) fundamental hypers have explicit mass terms \( m_i \), with \( i = 1 \ldots M \).

Now, we want to turn on vacuum expectation values to the adjoint scalar of the \( Sp(M) \) vector multiplet in such a way that, at low energies, the effective theory will contain no massless degrees of freedom except for a charged hypermultiplet for each one of the Cartan generators of \( Sp(M) \), and the spectator symmetric hyper of \( U(N) \). This low-energy theory is reminiscent, and indeed inspired to, the description of the quantum locus of the root of the baryonic branch for \( N = 2 \) gauge theories with flavors [28,30]. In order to obtain this spectrum, we have to choose each of the \( M \) masses \( m_i \) to be different from each other, and then tune the eigenvalues of the adjoint \( \Phi' \) of \( Sp(M) \) such that \( \phi'_i = m_i \). We can now compute the one instanton correction to the prepotential in the \( Sp(M) \) gauge theory in this particular vacuum, with flavor masses given as above and complemented by the diagonal values \( \phi \) of the scalar adjoint of \( U(N) \), which effectively act as masses for the remaining \( N \) light flavors.

One can use a direct ADHM construction [5] (see also [6]), possibly using the modern approach of [31–35]. Another approach is to determine the Seiberg-Witten curve of the theory, from which one extracts the one-instanton contribution to the prepotential [36–41].

In either way, one finds

\[
\mathcal{F}_{1-\text{inst}}^g = \frac{\Lambda^{M+2-N}}{\det m} \det \Phi, \tag{2.2}
\]

where again \( \Phi \) is the vector multiplet of the \( U(N) \) gauge group, whose lowest component is \( \phi \), the superscript “\( g \)” reminds us that we are dealing with a gauge instanton, \( \Lambda \) is the dynamical scale of the \( Sp(M) \) theory and we have defined \( \det m = \prod_i m_i \).

We find remarkable agreement between the expressions (2.1) and (2.2), being both a non-trivial prepotential term proportional to \( \det \Phi \). The constant of proportionality can be made to coincide by just tuning the arbitrary parameters of the field theory and stringy UV completions, i.e. \( m \) and \( \Lambda \) against \( M_s \) and \( \tau_D \). What we will show in the following, is that once this tuning is done for the one-instanton contribution, the two and three

\(^2\)We adopt the notation for which \( Sp(1) \simeq SU(2) \). Hence the fundamental representation of \( Sp(M) \) has \( 2M \) components.
gauge instanton contributions exactly coincide with the two and three stringy instanton contributions without the need of any further tuning.

This is our main result, on which we will elaborate in the rest of this paper. The stringy instanton contributions to the prepotential can be exactly reproduced by a simple field theory UV completion, involving massive flavors and a Higgsed gauge group, and going to a particular point of the moduli space reminiscent of the baryonic root.

3 Gauge vs stringy instantons in symplectic gauge theory

We consider the $\mathcal{N} = 2 \text{Sp}(N') \times U(N)$ model introduced in the preceding Section. As anticipated, the stringy and ordinary gauge instantons to be compared arise within the non-perturbative sectors of two distinct field theories. The ordinary instantons emerge in a gauge theory which represents an appropriate field theoretic UV completion of the theory affected by the stringy contributions. We henceforth refer to the former as the UV theory and to the latter as the IR theory.

In the following Subsections we first review the computation yielding the stringy instanton contributions in the IR theory. We then construct the UV gauge theory and compute the corresponding instanton corrections. Finally we successfully compare the instanton contributions in the two descriptions up to instanton number three.

3.1 Stringy instantons for $\mathcal{N} = 2 \text{Sp}(0) \times U(N)$ theory

The $\mathcal{N} = 2 \text{Sp}(0) \times U(N)$ gauge theory appears as the low-energy field theory description of the open string sector of the following set up.

The D-brane system lies on an orientifolded $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ orbifold background. Before considering the orientifold projection, the orbifold leads to a quiver with three independent, unitary gauge factors corresponding to the three irreducible representations of the $\mathbb{Z}_3$ orbifold group, each defining a fractional D3 brane. After enforcing the orientifold projection, the model is reduced to a two-node quiver with one unitary and one symplectic gauge factors. The symplectic factor arises from the projection of one of the original unitary nodes while the unitary factor of the projected model emerges from the identification of the two other unitary factors of the original quiver (see Figure 1). The original bi-fundamental matter associated to the string modes stretching between the two identified unitary nodes is projected down to matter in the symmetric representation of the single unitary factor of the projected quiver.

The $\mathcal{N} = 2 \text{Sp}(0) \times U(N)$ gauge theory corresponds to a configuration with $N$ fractional D3 branes on the unitary node and none on the symplectic node. However, we
Figure 1: Quiver diagram for the orientifolded $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ orbifold. The fields $Q$, $\tilde{Q}$ transform in the bifundamental representation of the two gauge nodes while the fields $S$, $\tilde{S}$ transform in the symmetric representation of $U(N_2)$.

consider the latter node to be populated by $D(-1)$-instantons. Since the instanton branes are located on a different node with respect to the D3 branes, they correspond to stringy instantons.

The $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ orientifold background preserves $\mathcal{N} = 2$ supersymmetry and the direct instantonic computations are performed within the localization framework for stringy instantons [42]. From the explicit evaluation of the instanton partition functions, one is able to compute directly the stringy instanton contributions to the prepotential of the theory order by order in the instanton number. The entire computation was performed in detail in [23–25] of which we simply report the results of use here.

The stringy, non-perturbative correction to the prepotential is of the form

$$F^s = \sum_{k=1}^{\infty} \mathcal{F}^s_k = \sum_{k=1}^{\infty} \left( M_8^{2-N} e^{2\pi i \tau D} \right)^k \mathcal{F}^s_k, \quad (3.1)$$

where the superscript $s$ stands for “stringy” and $\mathcal{F}^s_k$ is the $k$-th stringy instanton contri-
The explicit contributions at instanton level $k = 1, 2$ and 3 are respectively:

$$F_1^s = (-1)^N N_1 \det(\Phi_2),$$  \hspace{1cm} (3.2)
$$F_2^s = \frac{N_1^2}{8} e_{N-1}(\Phi_2^2),$$ \hspace{1cm} (3.3)
$$F_3^s = (-1)^N \frac{N_1^3}{12} \det(\Phi_2) e_{N-2}(\Phi_2^2);$$ \hspace{1cm} (3.4)

where $N_1$ is an overall numerical normalization, $\Phi_2$ represents the scalar belonging to the gauge vector multiplet of the $U(N)$ node and eventually $e_i(X)$ is the $i$-th symmetric polynomial defined as follows

$$e_i(X) = e_i(X_1,...,X_N) = \sum_{1 \leq j_1 < j_2 < ... < j_i \leq N} X_{j_1} X_{j_2} ... X_{j_i},$$ \hspace{1cm} (3.5)

where the $X_i$ are the eigenvalues of $X$.

We note here that the results above can be reproduced using the approach of \cite{31–35} in the case of $Sp(N')$ gauge theories with fundamental matter, taking however the final expression to be valid even after continuation to the unphysical case of $Sp(0)$.

What we are left now to do, is to show that the expressions (3.2)–(3.4) can be reproduced by an instanton computation in a physically sensible (i.e. asymptotically free) $Sp(N')$ gauge theory. The non-trivial task is to match both the field-dependence of each $k$-instanton contribution, and their normalization which implies just a single constant for all contributions.

### 3.2 UV and IR symplectic theories

We now describe the gauge theory UV completion of the $\mathcal{N} = 2$ $Sp(0) \times U(N)$ model. Such UV completion is given by the $Sp(N_c) \times U(N_f)$ theory (namely the model corresponding to the quiver depicted in Figure 1 with $N_1 = N_c$ and $N_2 = N_f$) considered at a specific point in the moduli space. Indeed, in order to break completely the $Sp(N_c)$ gauge group, we consider the following diagonal VEV assignment

$$\Phi_1 = (x_1 M, ... , x_{N_c} M),$$ \hspace{1cm} (3.6)

where $\Phi_1$ is now the vector whose components parametrize the $N_c$-dimensional Cartan subgroup of $Sp(N_c)$; $M$ represents a mass scale and the $x$’s are $N_c$ numbers of order one inserted to break the degeneracy among the components of $\Phi_1$. The VEV’s \cite{36} break the original $Sp(N_c)$ gauge group down to $U(1)^{N_c}$.

\footnote{In \cite{24} the 3-instanton prepotential lacks the factor $(-1)^N$; we repeated and corrected the computation obtaining \cite{34}}
We assign masses to $N_c$ out of $N_f$ flavors requiring the existence of $N_c$ massless hypermultiplets, each of them charged under one of the broken $U(1)^{N_c}$ gauge factors (see Appendix A). The mass matrix is then diagonal,

$$
\hat{M} = \left( x_1 M, ..., x_{N_c} M, 0, ..., 0 \right) .
$$

(3.7)

This choice guarantees the existence of a Higgs branch for each of the $U(1)^{N_c}$ factors. The mass assignment preserves a $U(N)$ symmetry with

$$
N = N_f - N_c .
$$

(3.8)

We can parametrize the small masses of the remaining flavors with $N$ fields $\phi_{2,i}$,

$$
\hat{M} = \left( x_1 M, ..., x_{N_c} M, -\phi_{2,1}, ..., -\phi_{2,N} \right) .
$$

(3.9)

The fields $\phi_{2,i}$ are the eigenvalues of a field $\Phi_2$ in the adjoint representation of the preserved $U(N)$ flavor group, where we have chosen the negative signs for later convenience.

The IR theory is obtained considering the UV model with the VEV’s (3.6) and masses (3.9) in the large $M$ limit. The IR theory has therefore an $Sp(0) \times U(N)$ symmetry.

### 3.3 Ordinary gauge instantons in $Sp$ gauge theories

The ordinary gauge instanton contributions to an $\mathcal{N} = 2$ $Sp(N_c)$ gauge theory with matter can be computed following two different approaches: either by direct calculation through localization techniques [34, 35] or by studying the corresponding Seiberg-Witten curve [41]. The two possibilities are equivalent and we adopt the former approach. In Appendix B we briefly sketch the direct calculation described in [34,35] to which we refer for any further detail.

The gauge instanton contribution to the prepotential can be expressed as

$$
\mathcal{F}^g = \sum_{k=1}^{\infty} \mathcal{F}_k^g = \sum_{k=1}^{\infty} F_k^g \Lambda^{kb_0}
$$

(3.10)

where $\Lambda$ is the strong coupling scale of the $Sp(N_c)$ gauge group, the superscript $g$ reminds us that we are considering gauge instantons, $\mathcal{F}_k^g$ is the $k$-th gauge instanton contribution, and the 1-loop coefficient of the $\beta$-function is given by

$$
b_0 = 2N_c + 2 - N_f .
$$

(3.11)
We report here the first three gauge instanton corrections to the prepotential

\[ F^g_1 = -\frac{1}{2} \left( \prod_{l=1}^{N_c} \frac{1}{\phi_{1,l}^2} \right) \left( \prod_{j=1}^{N_f} m_j \right), \]  

(3.12)

\[ F^g_2 = -\frac{1}{16} \left[ \left( \sum_{l=1}^{N_c} \frac{S_l(\phi_{1,l})}{\phi_{1,l}^4} \right) + \frac{1}{4} \frac{\partial^2 T(t)}{\partial t^2} \bigg|_{t=0} \right], \]  

(3.13)

\[ F^g_3 = -\frac{1}{16} \left( \prod_{l=1}^{N_c} \frac{1}{\phi_{1,l}^2} \right) \left( \prod_{j=1}^{N_f} m_j \right) \left[ \left( \sum_{l=1}^{N_c} \frac{S_{1,l}(\phi_{1,l})}{\phi_{1,l}^2} \right) + \frac{1}{144} \frac{\partial^4 T(t)}{\partial t^4} \bigg|_{t=0} \right], \]  

(3.14)

where we have defined

\[ S_l(t) = \frac{1}{4\phi_{1,l}} t \prod_{k \neq l} \frac{1}{(t^2 - \phi_{1,k}^2)^2} \prod_{j=1}^{N_f} (m_j^2 - t^2), \]  

(3.15)

\[ T(t) = \frac{1}{P(t)^2} \prod_{j=1}^{N_f} (m_j^2 - t^2), \]  

\[ P(t) = \prod_{l=1}^{N_c} (t^2 - \phi_{1,l}^2). \]  

(3.16)

Note that here \( \phi_{1,l} \) denotes the entries of the adjoint field \( \Phi_1 \) given in (3.6) and \( m_j \) denotes the entries of the mass array \( \hat{M} \) given in (3.9). The latter includes the diagonal part of the field \( \Phi_2 \).

### 3.4 Comparing gauge and stringy instantons

In this Subsection we show that the gauge instanton and stringy instanton computations give the same contribution to the prepotential at least up to 3 instantons: namely \( \mathcal{F}_k^s = \mathcal{F}_k^g \) for \( k = 1, 2, 3 \).

In order to compare the gauge instanton contributions in the UV theory with the stringy instanton contributions in the IR theory, we first evaluate the gauge instanton corrections to the UV theory with VEV and mass assignments (3.6), (3.9) and then study the large \( M \) limit. Notice once again that the mass scale \( M \) appears both in the VEV’s and masses.

Plugging the VEV’s (3.6) and the masses (3.9) into the residue function \( S \) defined explicitly in (3.15), we observe that

\[ S_l(\phi_{1,l}) = 0, \]  

(3.17)

for any \( \phi_{1,l} \) with \( l = 1, ..., N_c \); indeed, the last product in (3.15) contains always a vanishing factor. As a consequence, all the terms in \( S_l(\phi_{1,l}) \) contained in the expression for the prepotential (3.13) and (3.14) vanish too.
3.4.1 1-instanton contribution

We start comparing the one stringy instanton contribution with the one gauge instanton contribution.

We consider the expression for the 1-instanton prepotential given in (3.12) plugging into it the VEV’s and masses (3.6), (3.9),

\[ F_1^g = -\frac{1}{2} \frac{(-1)^N \phi_{2,1} \cdots \phi_{2,N}}{(x_1 \cdots x_{N_c}) M_{N_c}^{N_c}} = -\frac{1}{2} \frac{(-1)^N \det(\Phi_2)}{\det(x) M_{N_c}^{N_c}}, \tag{3.18} \]

where we have introduced the following notation

\[ \det(\Phi_2) = \phi_{2,1} \cdots \phi_{2,N}, \tag{3.19} \]
\[ \det(x) = x_1 \cdots x_{N_c}. \tag{3.20} \]

Note that the dimensionful prefactor of the gauge instanton prepotential, including the determinant \( \det(x) M_{N_c}^{N_c} \), can be combined in an IR scale

\[ \Lambda_{IR}^{2-N} \equiv \frac{\Lambda_{UV}^{2N_c+2-N_f}}{\det(x) M_{N_c}^{N_c}}. \tag{3.21} \]

The \( k = 1 \) gauge instanton correction (3.18) to the prepotential can be exactly matched with the stringy instanton one (3.2). Indeed, we identify the IR theory scale with the string parameter \( M_s^{2-N} e^{2\pi i \tau_D} \), and we fix the overall normalization constant \( \mathcal{N}_1 \) as follows

\[ \Lambda_{IR}^{2-N} = M_s^{2-N} e^{2\pi i \tau_D} \mathcal{N}_1 = -\frac{1}{2}. \tag{3.22} \]

In this way, the \( k = 1 \) gauge instanton contribution (3.18) equals the 1-instanton stringy contribution (3.2), namely

\[ M_s^{2-N} e^{2\pi i \tau_D} F_1^s = \Lambda_{UV}^{2N_c+2-N_f} F_1^g, \tag{3.23} \]

or equivalently

\[ F_1^s = F_1^g. \tag{3.24} \]

3.4.2 2-instanton contribution

The comparison of the 1-instanton contributions has allowed us to match completely the scales and the normalization factor \( \mathcal{N}_1 \) (which could actually be reabsorbed into a redefinition of the stringy scale). Now that we have no parameters left to fix, let us compare the two instanton computations.
Plugging the VEV and mass assignment (3.6), (3.9) into the 2-instanton prepotential (3.13), we simplify $F_g^2$ to

$$F_g^2 = -\frac{1}{64} \frac{\partial^2 T(t)}{\partial t^2} \bigg|_{t=0},$$

(3.25)

indeed, as already observed, the terms containing the residue function $S_l(\phi_{1,l})$ vanish. We then compute the second derivative of the $T$ function defined in (3.16), inserting the VEV’s (3.6) and the masses (3.9); for large values of $M$ we obtain

$$F_g^2 = -\frac{1}{64} \sum_{k=Nc+1}^{Nf} \prod_{j \neq k} m_j^2 + ...$$

(3.26)

where the dots indicate the omission of terms which are subleading in the large $M$ limit.

We consider again the matching of the scale (3.21) and its identification with the string parameter (3.22) as already done for the 1-instanton contribution, including the fixed value for the overall numerical normalization constant $N_1$. With this mapping, the $k = 2$ gauge instanton contribution to the UV theory then equals the 2-instanton stringy contribution to the IR model (3.3),

$$\left( M_s^{2-N} e^{2\pi i \tau D} \right)^2 F^s_2 = \Lambda_{UV}^{2(2Nc+2-Nf)} F_g^2,$$

(3.27)

or equivalently

$$F^s_2 = F_g^2.$$  

(3.28)

3.4.3 3-instanton contribution

Let us now pass to the three instantons case.

We insert the masses (3.9) and the VEV’s (3.6) into the 3-instanton prepotential (3.14); in this case as well only the term in $T$ will contribute, so we have

$$F_g^3 = -\frac{1}{16} \left( \prod_{l=1}^{Nc} \frac{1}{\phi_{1,l}^2} \right) \left( \prod_{j=1}^{Nf} m_j \right) \frac{1}{144} \frac{\partial^4 T(t)}{\partial t^4} \bigg|_{t=0}. $$

(3.29)

Using the definition of the function $T(t)$ given in (3.16), we compute the relevant terms...
in its fourth derivative and, in the limit of large $M$, we finally obtain

\[
F_3^g = -\frac{1}{16 \cdot 12} \left( \prod_{l=1}^{N_c} \frac{1}{\phi_{1,l}^6} \right) \left( \prod_{h=1}^{N_f} m_h \right) \left( \sum_{k=N_c+1}^{N_f} \sum_{i \neq k}^{N_f} \prod_{j \neq k, j \neq i}^{N_f} m_j^2 \right) + \ldots
\]

\[
= -\frac{1}{16 \cdot 12} (-1)^N \det(\Phi_2) M^{3N_c} \det(x)^3 \frac{2}{e_{N-2}} (\phi_{2,1}^2, \ldots, \phi_{2,N}^2) + \ldots
\]

\[
= -\frac{1}{8 \cdot 12} (-1)^N \det(\Phi_2) M^{3N_c} \det(x)^3 e_{N-2} (\Phi_2^2, \ldots, \Phi_2^N) + \ldots
\]

where again the dots indicate terms which are suppressed in the large $M$ limit.

Upon considering the scale matching (3.21) and the mapping (3.22) as in the $k = 1$ case, the $k = 3$ gauge instanton contribution to the UV theory equals the stringy 3-instanton contribution (3.4) computed directly in the IR theory,

\[
(M_s^{2-N} e^{2\pi i R_D})^3 F_3^s = \Lambda_{UV}^{3(2N_c+2-N_f)} F_3^g,
\]

that is

\[
F_3^s = F_3^g.
\]

We hope to have convinced the reader of the matching of the gauge and stringy computation at any instanton number $k$. In Section 4.4, we will give some additional arguments to support this conjecture.

4 Gauge vs stringy instantons in unitary gauge theory

In this Section we consider a different kind of low-energy theories that have stringy instanton corrections, namely theories with a $U(1)$ gauge group. In this case, the gauge group is not completely absent as in the $Sp(0)$ case, however it is Abelian and does not support gauge instantons.

We thus consider $\mathcal{N} = 2$ SQCD quiver gauge theories realized as worldvolume theories of fractional D3-branes probing an orbifold singularity. We first compute the contribution to the prepotential in a $U(1) \times U(N)$ gauge theory arising from D($-1$)-instantons located at the $U(1)$ node. We then describe how this configuration can be realized as the IR limit of a UV completed field theory considered at a specific point on the moduli space. The ordinary gauge theory instanton corrections in this UV field theory are studied and successfully matched to the stringy D($-1$)-instanton corrections we compute directly in the IR theory. We conclude the Section by discussing how the connection between the IR and UV theory can be naturally understood in terms of a duality cascade.
4.1 D-instantons in $\mathcal{N} = 2$ Abelian gauge theories

Consider an $\mathcal{N} = 2 U(1) \times U(N)$ gauge theory arising from fractional D3-branes at a $\mathbb{C}^2/\mathbb{Z}_h$ singularity, with $h > 2^4$ where the fractional D3-brane rank assignment has been chosen to be $(1, N, 0, \cdots, 0)$. See Figure 2. We are interested in the effect of fractional D($-1$)-instantons with charge $k$, located at the $U(1)$ node of the quiver. The spectrum of instanton zero modes, corresponding to the massless modes of the open strings with at least one endpoint attached to the D($-1$)-instantons, and their interactions, captured by the moduli action, can be obtained straightforwardly by starting from a D3/D($-1$) system \cite{45,46}, or a D9/D5 system \cite{6}, in flat space and performing an orbifold projection, see e.g. \cite{47} for details concerning the zero mode spectrum and the orbifold projection. This $U(1) \times U(N)$ quiver gauge theory with the $k$ fractional D($-1$)-instantons located at the $U(1)$ node and the associated zero modes are depicted in Figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{quiver.png}
\caption{$\mathbb{C}^2/\mathbb{Z}_h \times \mathbb{C}$ quiver diagram with $k$ instantons placed at node 1.}
\end{figure}

\footnote{The calculations and results in this section trivially carry over to the case where $h = 2$ with the only modification that, due to the $\mathbb{C}^2/\mathbb{Z}_2$ quiver structure, the gauge node under consideration is subject to a doubling of the number of flavors; for further details see Subsection 4.5.}
The zero modes in the neutral sector, consisting of those that do not transform under the gauge group of the fractional D3-branes, arise from the open strings that have both their endpoints on the D(−1)-instantons. This sector involves the modes $a^\mu, M^\alpha$ and $M'^\alpha$, which are traceless $k \times k$ matrices. The traces of these modes correspond to the broken supertranslation generators and they play the role of the $\mathcal{N} = 2$ superspace coordinates $x^\mu, \theta^\alpha$ and $\theta'^\alpha$. The neutral sector also contains the modes $\lambda^\dot{\alpha}, \lambda'^\dot{\alpha}, D^c$ and $\chi$ which are (non-traceless) $k \times k$ matrices. The mass dimension and the number of Chan-Paton degrees of freedom of all the zero modes are summarized in Table 1.

|        | $x$  | $\theta$ | $a$  | $\chi$ | $M$      | $\lambda$ | $D$ | $\omega$ | $\mu$ | $\alpha, \beta$ |
|--------|------|----------|------|--------|----------|------------|-----|----------|-------|-----------------|
| Mass dim. | -1   | -1/2     | -1   | 1      | -1/2     | 3/2        | 2   | -1       | -1/2  | -1/2            |
| CP d.o.f. | 1    | 1        | $k^2-1$ | $k^2$  | $k^2-1$  | $k^2$      | $k^2$| $k$      | $k$   | $kN$            |

Table 1: The mass dimension and the number of Chan-Paton degrees of freedom of the instanton zero modes.

The zero modes in the charged sector arise from the open strings stretching between the D(−1)-instantons and the fractional D3-branes. Specifically, from the strings stretch-
ing to the single fractional D3-branes at node 1, we obtain the bosonic modes $\omega_\alpha$ and the fermionic modes $\mu, \mu'$ (with the conjugate modes $\overline{\omega}_\alpha, \overline{\mu}, \overline{\mu}'$) and they all have $k$ components. Furthermore, from the strings stretching between the D$(-1)$-instantons and the $N$ fractional D3-branes at node 2, we obtain the fermionic modes $\alpha$ and $\beta$ and their Chan-Paton factors are $N \times k$ and $k \times N$ matrices, respectively.

The 0-dimensional worldvolume moduli action for the $k$ D$(-1)$-instantons for this configuration is given by

$$S^k_{\text{moduli}} = \text{tr}_k \{ S_1 + S_2 + S_3 + S_4 \} ,$$

with

$$S_1 = -i D^c \left( \overline{\omega}^\alpha (\tau^c)^{\beta} \omega_\beta + i \eta^c_{\mu\nu} [a^\mu, a^\nu] \right) + i \left( \overline{\omega}_{\alpha} + \overline{\omega}_\alpha + \sigma^\mu_{\beta \alpha} [M^\beta, a_\mu] \right) \lambda^\alpha$$

$$+ i \left( \overline{\mu}_{\alpha} + \overline{\mu}_\alpha + \sigma^\mu_{\beta \alpha} [M^\beta, a_\mu] \right) \lambda'^\alpha - [a_\mu, \lambda'^\alpha] [a^\mu, \lambda] - i \frac{1}{2} M^\alpha [\lambda^\dagger, M'_\alpha]$$

$$S_2 = \frac{1}{2} \overline{\omega}_{\alpha} (\chi^\dagger \chi + \chi_{\lambda}^\dagger) + i \left( \overline{\mu}^\mu \mu - \overline{\mu}' \mu' \chi^\dagger + \beta (\Phi_2 - \Phi_1) \alpha + \beta \alpha \chi \right)$$

$$S_3 = \frac{1}{2 g_0^2} \left( D_\alpha^2 - \lambda_{\alpha} \chi, \lambda^\dagger \alpha \right)$$

$$S_4 = \frac{1}{2} \overline{\omega}_{\alpha} (\widetilde{Q}^\dagger \widetilde{Q} + Q^\dagger Q)^{\alpha} \omega^{\alpha} - \frac{i}{2} \left( \beta \overline{Q}^\mu - \overline{\mu} Q \alpha + \beta Q^\dagger \mu' - \overline{\mu}' Q^\dagger \alpha \right)$$

where $\tau^c, \eta^c_{\mu\nu}$ and $\sigma^\mu_c$ are the Pauli matrices, the 't Hooft symbols and one of the off-diagonal Weyl blocks in the four-dimensional gamma-matrices, respectively, see [4,6] for their precise definitions. We have used the freedom to shift the modes $\chi$ by (the VEV of) $\Phi_1 = \Phi_1 1_k$. Note that, while the node 2 adjoint scalar $\Phi_2$ is an $N \times N$ matrix (whose diagonal components are henceforth denoted with $\phi_{2,i}$), the node 1 scalar $\Phi_1$ has only one single component (henceforth denoted with $\phi_1$).

The $S_3$-part of (4.1) contains the 0-dimensional (dimensionful) coupling constant $g_0^2 = g_s / \alpha'^2$, where $g_s$ and $\sqrt{\alpha'}$ are the string coupling and the string scale, respectively. In the field theory limit we send $\alpha' \to 0$ while keeping $g_s$ fixed, implying that the action $S_3$ vanishes and the modes $D_\alpha^c, \lambda_\alpha$ and $\lambda'^_\alpha$ in the $S_1$-part of (4.1) become Lagrange multipliers enforcing the bosonic and fermionic ADHM constraints [7].

From the mass dimensions of the moduli fields, given in Table 1, we see that the dimension of the measure of the corresponding moduli space integral is $k(N-2)$, implying the presence of a dimensionful prefactor $M_s^{\frac{k(2-N)}{2}}$, where $M_s = 1/\sqrt{\alpha'}$. By factoring out the part of the measure involving the zero modes corresponding to the center of mass motion of the D$(-1)$-instantons, which provide the integration measure of the chiral $\mathcal{N} = 2$ superspace, we obtain the centered moduli space integral in the form of a contribution from the charge $k$ D$(-1)$-instanton sector to the prepotential of the worldvolume gauge
theory of the fractional D3-branes,
\[
\mathcal{F}_k^s \propto M_k^{(2-N)} e^{2\pi i k \tau_D} \int \text{d} \{a M M' \lambda \lambda' D \chi \omega \mu \mu' \alpha \beta\} e^{-S_{\text{moduli}}} + \ldots ,
\]
(4.2)
where \(-2\pi i \tau_D\) is the complexified instanton action for a fractional D\((-1)\)-instanton at the \(U(1)\) node.

4.1.1 The \(k = 1\) prepotential

Let us start by explicitly evaluating the prepotential contribution \((4.2)\) for a single \(k = 1\) D\((-1)\)-instanton at the \(U(1)\) node in the quiver shown in Figure 8. Since we are interested in the Coulomb branch of the theory we take the VEV’s of the hypermultiplet scalars \(Q\) and \(\tilde{Q}\) to be vanishing, implying the vanishing of the \(S_4\)-part in \((4.1)\). Moreover, we take the field theory limit in which the \(S_3\)-part of \((4.1)\) also vanishes. Thus, we are left with a moduli action given by the following two parts,
\[
S_1 = -i D^\alpha \overline{\omega} \delta_\alpha^\beta \omega_\beta + i (\overline{\mu} \omega_\alpha + \overline{\omega} \mu') \lambda^\alpha + i (\overline{\mu'} \omega_\alpha + \overline{\omega} \mu) \lambda'^\alpha
\]
(4.3)
\[
S_2 = \overline{\sigma} \alpha \chi \chi + i 2 (\overline{\mu} \mu - \overline{\mu'} \mu') + \beta (\Phi_2 - \Phi_1 + \chi) \alpha .
\]
(4.4)
Since the \(\lambda\)-modes appear linearly in the moduli action we begin by integrating over them, which brings down a factor,
\[
\mu \mu' \mu \mu' (\overline{\omega} \omega_\alpha)^2 .
\]
(4.5)
This factor is used when integrating out all the 4 modes \(\mu, \mu', \overline{\mu}\) and \(\overline{\mu}'\). Therefore, in order to get a non-vanishing contribution to the prepotential, the second and third term in \((4.4)\) should be expanded only to zeroth order and hence we discard them. The integrals over the charged fermion zero modes \(\alpha\) and \(\beta\) make use of the last term in \((4.4)\) and bring down a determinant,
\[
\det (\Phi_2 - \Phi_1 \mathbf{1}_N + \chi \mathbf{1}_N) .
\]
(4.6)
Since this determinant depends holomorphically on \(\chi\) (and not on \(\overline{\chi}\)), and since there is no term in the remaining moduli action that can be brought down in order to compensate for the complex phase of the \(\chi\)’s, the integration over \(\chi\) and \(\overline{\chi}\) will only give a non-vanishing result for terms arising from the determinant \((4.6)\) which do not contain any power of \(\chi\). Hence, we can remove \(\chi\) from \((4.6)\) and perform the Gaussian integrals over \(\chi\) and \(\overline{\chi}\), which yield an inverse factor of \(\overline{\omega} \omega_\alpha\).
By combining these results we conclude that the $k=1$ stringy instanton contribution to the prepotential has the following form,

$$F_s^1 = \mathcal{N}_1 M_s e^{2\pi i \tau} \det (\Phi_2 - \Phi_1 \mathbf{1}_N) \times \mathcal{I}$$  \hspace{1cm} (4.7)

where $\mathcal{N}_1$ is an overall normalization constant and the remaining bosonic integral is given by

$$\mathcal{I} = \int d^2 \omega_\alpha d^2 \tilde{\omega}^\delta d^3 D^c \tilde{\omega}^\delta \omega_\alpha e^{-iD^c \tilde{\omega}^\delta (r^c)^\delta_{\beta \beta}}.$$  \hspace{1cm} (4.8)

The integral $\mathcal{I}$ is a dimensionless number and our remaining task is to show that this number is non-zero. As the integral (4.8) stands, it is not well-defined and therefore we are required to regularize it. We will evaluate (4.8) using 3 different regularization methods. The key feature of all these regularization methods is to introduce a dimensionful parameter which keeps the instanton from shrinking to zero size by smoothing out the corresponding singularity in the instanton moduli space.

In the first regularization we allow the hypermultiplet scalars to acquire a non-vanishing VEV, such that $\tilde{Q}^\dagger \tilde{Q} + QQ^\dagger = |v|^2 \neq 0$\footnote{See, for example, [17] for an $\mathcal{N} = 1$ configuration in which this regularization was used.}. This implies that the $S_4$-part in (4.1) is now non-vanishing and should be added to (4.4). Note that all the terms in the $S_4$-part in (4.1) that involve fermion zero modes are irrelevant in this configuration since they contain one of the modes $\mu, \mu'$, $\bar{\mu}$ or $\bar{\mu}'$, which have already been soaked up. By taking this non-vanishing VEV into account, the integral (4.8) becomes well-defined and gives the following finite result,

$$\mathcal{I}_v = \int d^2 \omega_\alpha d^2 \tilde{\omega}^\delta d^3 D^c \tilde{\omega}^\delta \omega_\alpha e^{-iD^c \tilde{\omega}^\delta (r^c)^\delta_{\beta \beta} \frac{1}{2} |v|^2 \tilde{\omega}^\delta \omega_\alpha} = 8\pi^4$$  \hspace{1cm} (4.9)

where the explicit steps of the evaluation are given in Appendix C.1. As expected, the result in (4.9) is independent of the VEV $v$ we deformed the integral (4.8) with.

A common way to regularize instanton partition functions is to consider the gauge theory to be defined on a non-commutative space. In such theories, the ADHM constraints are deformed by the presence of a non-commutativity parameter $\xi$. One consequence of this deformation is that even gauge theories with an Abelian $U(1)$ gauge group allow for non-trivial instanton configurations [48]. In terms of the instanton moduli action, the non-commutative deformation amounts to simply adding a term $iD^c \xi \delta_{c3}$ to (4.3), where $\delta_{c3}$ is a Kronecker delta. This procedure renders the integral (4.8) well-defined and we obtain

$$\mathcal{I}_\xi = \int d^2 \omega_\alpha d^2 \tilde{\omega}^\delta d^3 D^c \tilde{\omega}^\delta \omega_\alpha e^{-iD^c \tilde{\omega}^\delta (r^c)^\delta_{\beta \beta} \omega_\alpha + iD^c \xi \delta_{c3}} = 8\pi^4$$  \hspace{1cm} (4.10)

where the intermediate steps are provided in Appendix C.2.

$$\int d^2 \omega_\alpha d^2 \tilde{\omega}^\delta d^3 D^c \tilde{\omega}^\delta \omega_\alpha e^{iD^c \tilde{\omega}^\delta (r^c)^\delta_{\beta \beta} \omega_\alpha} = 8\pi^4$$

$$\int d^2 \omega_\alpha d^2 \tilde{\omega}^\delta d^3 D^c \tilde{\omega}^\delta \omega_\alpha e^{-iD^c \tilde{\omega}^\delta (r^c)^\delta_{\beta \beta} \omega_\alpha} = 8\pi^4$$
In the third way of regularizing, we refrain from taking the field theory limit by keeping $\alpha'$, and hence $g_0$, finite. This implies that the $S_3$-part of (4.1), which for $k = 1$ reduces to only the first term in $S_3$, no longer vanishes. By adding the term $D^2 c/(2g_0^2)$ to (4.3), the integral (4.8) is again well-defined and gives the result,

$$I_{\alpha'} = \int d^2 \omega \bar{\omega} D^2 \omega e^{-iD^2 \bar{\omega} \omega_{\alpha} \omega_{\beta} - \frac{i}{2g_0^2} (D^c)^2} = 8\pi^4$$

(4.11)

where the details are given in Appendix C.3.

We conclude that the 3 regularization methods indeed yield the same non-vanishing number, $I_v = I_\xi = I_{\alpha'} = 8\pi^4$ which can be absorbed in the string scale in (4.7). The fact that these regularized integrals are independent of their corresponding dimensionful parameter is simply a consequence of the invariance of (4.8) under rescalings in terms of a parameter $k$, e.g. $D^c \rightarrow k^2 D^c$, $\omega \rightarrow (1/k) \omega$ and $\bar{\omega} \rightarrow (1/k) \bar{\omega}$. Hence, the dimensionful parameters are rescaled away by choosing $k = v$ in (4.9), $k = 1/\sqrt{\xi}$ in (4.10) and $k = g_0$ in (4.11).

4.1.2 The $k > 1$ prepotentials

We have seen that a D($-1$)-instanton on the $U(1)$ node of the quiver gauge theory in Figure 3 gives a non-vanishing contribution to the prepotential of the form displayed in (4.7). In particular, we saw that using a finite VEV for the hypermultiplet scalars or a finite value of $\alpha'$ was equivalent to using the non-commutative regularization. The latter regularization is the one employed, for example, by Nekrasov in [31] when deriving the instanton contributions to the prepotential in a $U(N_c) \mathcal{N} = 2$ gauge theory with $N_f$ flavors. This equivalence among regularizations motivates us to relate the stringy D($-1$)-instanton contribution in (4.7) to the Nekrasov instanton formulae, in the same way as we noted in the $Sp(0)$ case that the stringy instanton expressions were completely equivalent to the continuation to $N_c = 0$ of the $Sp(N_c)$ expressions obtained in gauge theory.

Let us observe first that the Nekrasov formulae are for gauge instanton contributions in a non-Abelian $U(N_c)$ theory where $N_c > 1$. We thus assume that the stringy instanton corrections are just given by the formulae obtained in [31] extended to the Abelian case where $N_c = 1$ and $N_f = N$. The “stringy” character of such contributions consists in having instanton corrections in an Abelian (commutative) theory.

Further motivation to the extension of Nekrasov’s formulae to the Abelian $N_c = 1$ case as a mean to obtain stringy instanton results can be based on the analysis of the associated

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6This regularization was used, for instance, in [45] for an $\mathcal{N} = 4$ configuration and in [39] for an $\mathcal{N} = 1$ configuration.

7As usual in D-brane set ups, the gauge group is really $U(N_c)$ rather than $SU(N_c)$, though the Abelian factor is usually discarded. Note that the construction of [31] is also based on a $U(N_c)$ gauge group.
brane constructions. Indeed, \( k \) D\((-1)\)-instantons lying on top of a single D3-brane yield an instantonic configuration that, although stringy from the field theory viewpoint, possesses the same moduli structure as an ordinary instanton.

The generic form of the prepotential in the \( U(N_c) \) \( N = 2 \) gauge theory is given by \( (3.10) \) but where \( \Lambda \) now is the strong coupling scale of the \( U(N_c) \) gauge group and the 1-loop coefficient of the \( \beta \)-function is given by \( b_0 = 2N_c - N_f \). The instanton corrections to the prepotential up to \( k = 3 \) can be expressed as follows \[31\],

\[
F_1^g = \sum_{u=1}^{N_c} S_u
\]

\[
F_2^g = \sum_{u \neq v}^{N_c} \frac{S_u S_v}{\phi_{uv}^2} + \frac{1}{4} \sum_{u=1}^{N_c} S_u \frac{\partial^2 S_u}{\partial \phi_u^2}
\]

\[
F_3^g = \frac{1}{36} \sum_{u=1}^{N_c} S_u \left[ \frac{\partial^4 S_u}{\partial \phi_u^4} + 2 \frac{\partial S_u}{\partial \phi_u} \frac{\partial^3 S_u}{\partial \phi_u^3} + 3 \frac{\partial^2 S_u}{\partial \phi_u^2} \frac{\partial^2 S_u}{\partial \phi_u^2} \right] + \frac{5 S_u}{2} \frac{\partial S_u}{\partial \phi_u} + \frac{\partial^2 S_u}{\partial \phi_u^2}
\]

\[
+ \sum_{u \neq v}^{N_c} S_u S_v \sum_{v \neq w}^{N_c} \frac{2 S_u S_v S_w}{3 \phi_{uv}^2 \phi_{vw}^2 \phi_{uw}^2} \left[ \phi_{uv}^2 + \phi_{vw}^2 + \phi_{wu}^2 \right]
\]

where \( \phi_u \) indicates the \( u \)-th diagonal component of the adjoint field \( \phi \) and \( \phi_{uv} = \phi_u - \phi_v \). Moreover, the function \( S_u = S_u(\phi_u) \) for a unitary gauge group is given by,

\[
S_u(\phi_u) = \frac{\prod_{j=1}^{N_f} (\phi_u + m_j)}{\prod_{v \neq u}^{N_c} \phi_{uv}^2}.
\]

In order to compare first the \( k = 1 \) prepotential in \( (4.12) \) with the D\((-1)\)-instanton stringy contribution in \( (4.7) \), we have to set \( N_c = 1 \) and \( N_f = N \), such that \( \phi_u \rightarrow \phi \) and \( S_u \rightarrow S = \prod_{j=1}^{N} (\phi + m_j) \). Moreover, by mapping \( \phi \rightarrow -\phi \) and \( m_j \rightarrow \phi_{2j} \) we obtain,

\[
F_1^s = \Lambda^{b_0} \prod_{j=1}^{N} (\phi_{2j} - \phi_1)
\]

which indeed has the same form as \( (4.7) \) (when we diagonalize the matrix \( (\Phi_2 - \Phi_1 1_N) \) in \( (4.7) \) and express it in terms of its eigenvalues); we also identify \( M_s^{N} e^{2\pi i \tau D} \equiv \Lambda^{b_0} \), where \( b_0 = 2 - N \) and we have taken \( N_1 = 1 \). Note that this is not the 1-loop beta
function coefficient for an Abelian gauge theory, hence \( \Lambda \) above is not the scale of the Landau pole of this IR free theory, but an independent scale. However, it is precisely the 1-loop beta function coefficient one would expect for a non-commutative \( \mathcal{N} = 2 \) Abelian gauge theory \[50,51\], in agreement with the fact that the formulæ in \[31\], from which we extracted (4.14), were obtained for a non-commutative \( U(N_c) \) gauge theory.

To get a feeling that one indeed obtains a non-vanishing result for (4.2) for a general value of \( k \), we now perform a simple fermion zero mode counting, in order to check that the \( D(-1) \)-instanton moduli space integral allows us to integrate out all the fermion zero modes. As before, we begin by considering the Coulomb branch in the field theory limit, i.e. when \( S_3 \) and \( S_4 \) in (4.1) are vanishing. In this limit, integrating over the modes \( \lambda \) and \( \lambda' \) gives rise to the 4\( k^2 \) fermionic ADHM constraints,

\[
\prod_{\alpha,i,j} (\overline{\omega}_\alpha + \overline{\sigma}_\alpha \mu + \sigma^n_{\alpha,i,j} [M^\alpha, a_\mu])^2_i \prod_{\beta,m,n} (\overline{\omega}_\beta + \overline{\sigma}_\beta \mu' + \sigma^n_{\beta,m,n} [M^\beta, a_\mu])^n_m
\]  

(4.15)

with \( i, j, m, n = 1 \ldots k \). Since there are \( 4k^2 - 4 \) of the \( M, M' \) modes and \( 4k \) of the \( \mu, \mu', \overline{\mu}, \overline{\mu}' \) modes, after using the \( 4k^2 \) constraints, there will remain \( 4k - 4 \) modes of the set \( \{M, M', \mu, \mu', \overline{\mu}, \overline{\mu}'\} \). We can use the terms \( M^\alpha \chi^\dagger, M' \mu^\dagger \) and \( \overline{\mu} \mu'^\dagger \chi^\dagger \) in the \( S_1 \) and \( S_2 \)-parts of (4.1) in order to integrate out these remaining \( 4k - 4 \) modes in this set, bringing down a factor of \( (\chi^\dagger)^{2k-2} \). In order to cancel the complex phase of this factor, we can use the term \( \beta \alpha \chi \) in the \( S_2 \)-part of (4.1) and hence \( 4k - 4 \) modes of the set \( \{\alpha, \beta\} \). Since there are \( 2Nk \) of this latter set, there will still remain \( 2Nk - 4k + 4 \) of these modes. Under the simplifying assumption that all the fluctuations \( \phi_{2,j} \) are equal to a single field \( \phi_2 \), if we use the term \( \beta (\Phi_2 - \Phi_1) \alpha \) in the \( S_2 \) part in order to integrate out these remaining modes, we bring down a factor,

\[
(\phi_2 - \phi_1)^{Nk - 2k + 2} = \frac{(\phi_2 - \phi_1)^N}{(\phi_2 - \phi_1)^{2k-2}} \tag{4.16}
\]

where, in the numerator of the RHS, we see the function in (4.13) for \( N_c = 1 \) and \( N_f = N \),

\[
S(\phi) = \prod_{j=1}^N (\phi + m_j) \rightarrow (\phi_2 - \phi_1)^N, \tag{4.17}
\]

appearing to the power \( k \), in agreement with \( k = 1, 2, 3 \) prepotentials in (4.12). Note that the dimension of (4.16) is in agreement with the dimensionful prefactor \( M_s^{k(2 - N)} \) of the centered moduli space measure in (4.2).

In summary, we have argued that the evaluation of the stringy \( D(-1) \)-instanton moduli space integral (4.2) is equivalent to the Nekrasov evaluation \[31\], or even identical if we use the non-commutative regularization. We thus report the explicit stringy results up
to \( k = 3 \) computed by means of Nekrasov formulae extended to \( N_c = 1 \); this amounts to performing the following substitutions in (4.12),

\[
\phi \to -\phi_1, \quad m_j \to \phi_2,j.
\]  

Eventually, the stringy contributions to the prepotential for \( k = 1, 2, 3 \) are

\[
\mathcal{F}_1^s = \Lambda^{b_0} \det(\Phi_2 - \Phi_1),
\]

\[
\mathcal{F}_2^s = \frac{\Lambda^{2b_0}}{2} \det(\Phi_2 - \Phi_1) e_{N-2}(\Phi_2 - \Phi_1),
\]

\[
\mathcal{F}_3^s = \frac{\Lambda^{3b_0}}{3} \det(\Phi_2 - \Phi_1) \left( 2 \det(\Phi_2 - \Phi_1) e_{N-4}(\Phi_2 - \Phi_1) + e_{N-1}(\Phi_2 - \Phi_1) e_{N-3}(\Phi_2 - \Phi_1) + (e_{N-2}(\Phi_2 - \Phi_1))^2 \right),
\]

where the \( e_i \) where defined in (3.5). Considering again all the fluctuations \( \phi_{2,j} \) to be equal to \( \phi_2 \) we have the simpler expressions

\[
\mathcal{F}_1^s = \Lambda^{b_0} (\phi_2 - \phi_1)^N,
\]

\[
\mathcal{F}_2^s = \frac{\Lambda^{2b_0}}{4} N(N-1) (\phi_2 - \phi_1)^{2N-2},
\]

\[
\mathcal{F}_3^s = \frac{\Lambda^{3b_0}}{6} N(N-1)^3 (\phi_2 - \phi_1)^{3N-4}.
\]

In the following Subsections we are going to compare these stringy instanton contributions against the corrections introduced by ordinary gauge instantons in a different gauge theory representing a specific UV completion of the \( \mathcal{N} = 2 U(1) \times U(N) \) model at hand.

### 4.2 UV and IR unitary theories

In order to interpret the stringy instantons in terms of ordinary gauge instantons, we must complete the \( U(1) \times U(N) \) theory in the ultraviolet. The \( U(1) \) gauge group can be naturally interpreted as resulting from a Higgsing of a larger gauge group \( U(N_c) \). The Higgsing procedure corresponds to considering a point in the moduli space where the scalar field \( \Phi_1 \) transforming in the adjoint representation of the gauge group acquires a non-trivial VEV. In general, a complete breaking of the original \( U(N_c) \) gauge group leads to a \( U(1)^N \) low-energy theory. In order to be able to neglect \( N_c - 1 \) out of the \( N_c \) \( U(1) \) factors of the low-energy theory, we need to have a Higgs branch for each of them. This can be achieved by identifying a specific point in the moduli space for \( \Phi_1 \) and assigning specific masses to the flavors [28].
To be concrete, the UV theory we consider is still described by the $\mathbb{C}^2/\mathbb{Z}_h$ quiver diagram introduced in the previous Subsection, but now the rank assignment is

$$(P + 1, N, 0, 0, ..., P)_h$$

and then the symmetry group is given by

$$U(P + 1) \times U(N) \times U(P),$$

where we have introduced one positive parameter $P$; the UV completion (4.26) can be thought of as resulting from the addition of $P$ fractional branes on nodes 1 and $h$ of the quiver. For later convenience, we also define

$$N_c = P + 1, \quad N_f = P + N,$$

which represent the number of “colors” and “flavors” of the UV theory.

In the moduli space of the UV theory, we consider the specific point where the adjoint field $\Phi_1$ breaks completely the gauge symmetry by acquiring the following non-degenerate VEV

$$\Phi_1 = (-x_1 M, \ldots, -x_{N_c-1} M, -\phi_1)_{N_c},$$

where $x_i$ are $N_c - 1$ different numbers of order one which break the gauge group $U(N_c)$ down to $U(1)^{N_c-1} \times U(1)$, and $\phi_1$ is a field fluctuation associated to the last $U(1)$ factor. We also add a diagonal mass term $m$ for the $N_f$ flavors. It can be explicitly checked that, in order to have a Higgs branch at low energy for each of the $U(1)^{N_c-1}$ factors, we have to assign the non-trivial masses

$$m = (x_1 M, \ldots, x_{N_c-1} M, 0, \ldots, 0)_{N_f};$$

the unbroken, low-energy flavor group is therefore $U(N)$.

The mass assignment (4.29) can be interpreted in terms of a specific non-vanishing VEV for the adjoint field $\Phi_h$ associated to the $h$-th node of the quiver, namely

$$\Phi_h = (x_1 M, \ldots, x_{N_c-1} M)_P.$$

The field $\Phi_2$, which can be associated to the last $N$ components in the mass array (4.29), does not acquire any VEV; we parametrize its components as follows:

$$\Phi_2 = (\phi_{2,1}, \ldots, \phi_{2,N})_N.$$

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Taking these fluctuations into account in the mass array we obtain

\[ m = (x_1 M, \ldots, x_{N_c-1} M, \phi_{2,1}, \ldots, \phi_{2,N_f})_{N_f} \].

(4.32)

In the next Subsections we will consider all \( \phi_{2,i} \) to be equal and we will consequently drop the label \( i \).

To summarize, the \( U(1) \times U(N) \) IR theory is re-obtained by considering the large mass limit \( M \to \infty \) in the UV theory (4.26) at the point of the moduli space specified by (4.28), (4.30) and (4.31).

### 4.3 Comparing gauge and stringy instantons

In this Subsection we show that the gauge instanton computation exactly reproduces the stringy instanton results we have presented in Subsection 4.1, at least up to three instantons.

Recalling Eq. (4.27) relating \( N \) and \( P \) (introduced in (4.25)) to \( N_c \) and \( N_f \), the one-loop \( \beta \)-function coefficient for the UV theory is

\[ b_0 = 2N_c - N_f = P - N + 2 \].

(4.33)

We require asymptotic freedom for the UV theory, namely \( b_0 > 0 \). In this regime, in order to compute the instanton contributions to the prepotential of the UV theory, we can use ordinary instanton techniques. In particular, the ordinary gauge instanton corrections to the prepotential of an asymptotically free \( N = 2, SU(N_c) \) gauge theory with \( N_f \) flavors have been explicitly computed in [31] using localization techniques, and for instance in [36] using the Seiberg-Witten curve.

#### 4.3.1 1-instanton contribution

In order to obtain the prepotential instanton contribution on the moduli space point that we have identified in the previous section, we first evaluate the \( S_k(\phi_k) \) function (4.13) on that point. We replace \( \phi_k \) with the components of \( \Phi_1 \) in (4.28) and the mass array as in (4.32) with all \( \phi_{2,i} = \phi_2 \). The function \( S_k(\phi_k) \) simplifies to

\[ S_k(\phi_k) = \delta_{k,N_c} \frac{(\phi_2 - \phi_1)^N \prod_{j=1}^{N_c-1} (x_j M - \phi_1)}{\prod_{l=1}^{N_c-1} (x_l M - \phi_1)^2} \],

(4.34)

where, according to the VEV assignment (4.28), we have \( \phi_{N_c} = -\phi_1 \). In the limit of large mass \( M \), the leading term of \( S_k(\phi_k) \) is

\[ S_k(\phi_k) = \delta_{k,N_c} \frac{(\phi_2 - \phi_1)^N}{\det M} + O(M^{-N_c}) \]

(4.35)
where
\[ \det M = \prod_{i=1}^{P} x_i M . \] (4.36)

Recalling Eq. (4.12), the 1-instanton contribution to the prepotential is
\[ F^g_1 = \Lambda^b_0 F^g_1 = \Lambda^b_0 \sum_{k=1}^{N_c} S_k(\phi_k) = \Lambda^b_0 \frac{(\phi_2 - \phi_1)^N}{\det M} . \] (4.37)

Eventually we have
\[ F^s_1 = F^g_1 , \] (4.38)
or, said otherwise, the gauge instanton contribution (4.37) coincides with the stringy instanton contribution (4.22) if we consider the matching of scales between the UV, the IR theory and the string theory parameters as
\[ M^2_s e^{2\pi i\tau_D} = \Lambda^b_0_{\text{IR}} = \Lambda^b_0_{\text{UV}} \frac{\det M}{N} . \] (4.39)

This accounts for the presence of the factor \( \det M \) in the expression (4.37) for the prepotential in the gauge theory computation.

### 4.3.2 2-instanton contribution

The \( k = 2 \) gauge instanton contribution to the prepotential is given in (4.12) and, if we consider the masses (4.32) (with all the fluctuations equal to \( \phi_2 \)) and the VEV’s (4.28), the first sum is vanishing; this follows from the combination of the facts that the sum runs on \( u \neq v \) and that \( S_k(\phi_k) \propto \delta_{N_c,k} \). We have then
\[ F^g_2 = \frac{1}{4} S_{N_c}(\phi_{N_c}) \left. \frac{\partial^2}{\partial \phi_k^2} S_{N_c}(\phi_k) \right|_{\phi_k = \phi_{N_c}} . \] (4.40)

The \( n \)-th derivative of \( S_{N_c}(\phi_k) \) for \( n \leq N \) evaluated at \( \phi_k = \phi_{N_c} \) is given by
\[ \left. \frac{\partial^n}{\partial X_n} S_{N_c}(\phi_k) \right|_{\phi_k = \phi_{N_c}} = \frac{N!}{(N-n)!} \frac{(\phi_2 - \phi_1)^{N-n}}{\det M} + O(M^{-N_c}) . \] (4.41)

Using this general formula in (4.40), the \( k = 2 \) gauge instanton prepotential becomes
\[ F^g_2 = \frac{1}{4} N(N-1) \frac{(\phi_2 - \phi_1)^{2N-2}}{\det M^2} + O(M^{-2N_c+1}) , \] (4.42)

Once again this corresponds to the stringy instanton contribution at level \( k = 2 \) if we use the matching of scale (4.39):
\[ F^s_2 = F^g_2 . \] (4.43)
4.3.3 3-instanton contribution

We consider the 3-instanton contribution to the prepotential given in \((4.12)\), with masses \((4.32)\) (again with all fluctuations equal to \(\phi_2\)) and VEV’s \((4.28)\). As already happened at level \(k = 2\), the delta function structure of the function \(S_u(\phi_u)\) simplifies the expressions and the \(k = 3\) prepotential \((4.12)\) can be rewritten as follows:

\[
F^g_3 = \frac{1}{36} \sum_{u=1}^{N_c} S_u \left[ \frac{\partial^4 S_u}{\partial \phi_u^4} + 2 \frac{\partial S_u}{\partial \phi_u} \frac{\partial^3 S_u}{\partial \phi_u^3} + 3 \frac{\partial^2 S_u}{\partial \phi_u^2} \frac{\partial^2 S_u}{\partial \phi_u^2} \right],
\]

where the symbol \(S_u\) is a compact way to indicate \(S_{N_c}(\phi_k)\). Recalling the expression \((4.41)\) for the \(n\)-th derivative of the function \(S_{N_c}(\phi_k)\), we have

\[
F^g_3 = \frac{N(N - 1)^3}{6} \frac{(\phi_2 - \phi_1)^{3N-4}}{\det M^3} + O(M^{-3N_c+2})
\]

which once again is equivalent to the stringy instanton contribution by using the matching of scale \((4.39)\), namely

\[
F^s_3 = F^g_3.
\]

At this point we hope to have provided convincing evidence of the matching between the gauge and stringy instanton computations. Actually, there is a simple argument that suggests that the matching can be extended to any order in the instanton number. Indeed, the expression for \(S_k(\phi_k)\) given in \((4.35)\) reduces, at leading order when \(M\) is large, to the expression \((4.17)\) for \(S\) in the \(N_c = 1\), “stringy” case, up to the \(\det M\) factor. The expression from which any prepotential contribution is computed is then exactly the same, and so are the results.

4.4 UV and IR unitary theories through duality cascade

In the previous Sections we have seen that computations of the gauge instanton and stringy instanton contributions to the prepotential agree, order by order in the instanton number, at least up to three instantons. In this Subsection we would like to give a slightly broader picture of the equivalence of the two computations.

It is a well know fact that the prepotential \(F\) of an \(\mathcal{N} = 2\) theory is a holomorphic function of the fields and of its parameters. For this reason, \(F\) is supposed not to jump when we move inside the moduli space of the theory, even if, in different points of the moduli space, the microscopic origin of the effective prepotential could be different. The continuity of the prepotential is the underlying mechanism that allows the reinterpretation of the stringy instantons as gauge theory instantons. Indeed, the flows we have used until now to pass from the UV theory, where we can do gauge theory instanton computations,
to the IR theory, where we are required to do stringy instanton computations, imply sending some vacuum expectation values $M$ to large values, and the prepotential should be continuous in this process. The flows we use in Subsections 3.2 and 4.2 break completely the non-Abelian gauge group and change the nature of the D-brane instantons; these are interpreted as gauge theory instantons in the UV theory and as stringy instantons in the IR theory. However, due to the holomorphicity of the prepotential, the two computations should give the same results, which they do.

The flows that we have considered until now, assuming $M \gg \Lambda$, do not pass through any strongly coupled region and hence they allow us to map gauge and stringy instantonic computations order by order. However, the quiver gauge theories we are considering are the same ones that appear in the far IR of the field theory side of many examples of AdS/CFT correspondence. This embedding of the field theory inside AdS/CFT correspondence provides a privileged flow, called the duality cascade, that can be deduced from the dual supergravity solutions. This is a well known phenomenon in $\mathcal{N} = 1$ field theories where the flow takes place intrinsically at strong coupling and it can be thought of as proceeding through a series of Seiberg dualities [52]. The continuity of the superpotential along this flow was nicely used in [19] to infer the equivalence of the superpotential contribution among theories related by Seiberg dualities; in addition, in the particular case of the last step in the cascade, [19] shows the equivalence between the gauge theory non-perturbative contribution to the superpotential of the theory before the last step of the cascade and the stringy instantonic contribution to the superpotential of the theory after the last step of the cascade.

In the specific cases of unitary groups, like the one considered in this Section, it is known that a similar cascade occurs also for $\mathcal{N} = 2$ theories [26, 27]. In the $\mathcal{N} = 2$ case, the flow proceeds through a series of baryonic root transitions [28], and it reproduces the flow described by the supergravity solution. The baryonic root transition is a strong coupling effect of some $\mathcal{N} = 2$ theories that reproduces the same numerology of the Seiberg duality of $\mathcal{N} = 1$ theories. Namely, $\mathcal{N} = 2$ SQCD with $SU(N_c)$ gauge group and $N_f$ fundamental hypermultiplets has, when asymptotically free (i.e. $2N_c - N_f > 0$), a quantum modification of the moduli space that splits the intersection of the Higgs and Coulomb branches of the theory. At the baryonic root, namely the point in the quantum moduli space where the baryonic branch (i.e. the branch where the gauge group is fully Higgsed) intersects the Coulomb branch, quantum effects force the adjoint field $\Phi$ in the vector multiplet to acquire the expectation value:

$$\Phi = \Lambda(0, ..., 0, \omega, ..., \omega^{2N_c-N_f}) \, ,$$

where $\Lambda$ is the strong coupling scale of the theory and $\omega$ is the $(2N_c-N_f)$-th root of unity. At this specific point of the moduli space, the effective theory is an $\mathcal{N} = 2$ SQCD with $SU(N_f-N_c) \times U(1)^{2N_c-N_f}$ gauge group, $N_f$ fundamental hypermultiplets and $2N_c-N_f$
hypermultiplets which are singlets under the non-Abelian gauge group, but each one is charged under one of the $U(1)$ factors. Let us observe that in this case the UV theory is UV free, while the IR theory is IR free and the numerology of the rank of the color groups and the number of flavors is exactly the same as for Seiberg duality in $\mathcal{N} = 1$ SQCD with $SU(N_c)$ gauge group and $N_f$ fundamental flavors. Note that the VEVs (4.47) are due to the strongly coupled dynamics of the theory. A flow at the baryonic root point of the moduli space, or close to it, is hence intrinsically at strong coupling.

The embedding in a duality cascade of the $U(1) \times U(N)$ theory which we have considered in Subsection 4.1 can be done, for example, in the following way: consider a set of $(2N, N, 2N - 1)$ fractional branes at the tip of the $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ singularity, and perform a series of baryonic root transitions on the nodes 3, 1, 3 in the order just specified. This yields the following sequence of cascading theories,

$$
(2N, N, 2N - 1) \rightarrow (2N, N, N + 1) \rightarrow (1, N, N + 1) \rightarrow (1, N, 0). \quad (4.48)
$$

This sequence could be thought as arising from the last three steps of an infinite cascade originated by an appropriate choice of fractional branes in the UV theory, in exactly the same way as the cascading theory of [52] in the case of the conifold, or the cascading theory of [26,27] for the $\mathbb{C}^2/\mathbb{Z}_2$ singularity. In our $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ case, in analogy with $\mathbb{C}^2/\mathbb{Z}_2$, every step in the duality cascade corresponds to the coupling of one of the gauge groups becoming very large; this triggers the need to perform a baryonic root transition in order to continue the flow at lower energies using a dual weakly coupled description.

Due to the strongly coupled nature of this flow, we are not justified to follow the computation order by order in the instanton number, and hence we should not be able to compare $k$-instantons of the theory at the $i$-th step of the cascade with $k$-instantons of the theory at the $(i - 1)$-th step of the cascade. In particular, we should not be able to compare the gauge instanton computation of the theory just before the last step in the cascade and the stringy instanton computation of the theory after the last step in the cascade. However, the full prepotential should be continuous along the cascade. In other terms, to follow the natural flow suggested by AdS/CFT correspondence, we should know the prepotential contributions at all instanton numbers for both theories. On the contrary, in the weakly coupled flows we have chosen in the previous Sections, it was enough to compare the prepotentials order by order in the instanton number. It is nevertheless interesting to observe that the theory at the first step in the “AdS/CFT” cascade of Eq. (4.48), is exactly the UV theory we considered in Subsection 4.2, namely $U(P + 1) \times U(N) \times U(P)$ (with $P = 2N - 1$). The flow we discussed in that Subsection is controlled by the vacuum expectation value of $P$ diagonal components of the adjoint field of the first node, and the VEV of the $P$ diagonal components of the adjoint field of the third node (see eqs (4.30, 4.31)); these latter play the role of the masses for the hypermultiplets considered in Subsection 4.2.
In [28] it is shown that the particular weakly coupled flows we considered in the previous Sections lead exactly to the same IR effective theory reached after the last step in duality cascades such as the one in (4.48). Hence we have the same UV completion for the IR theory as the one suggested by the AdS/CFT correspondence, but, while the flow we followed is weakly coupled, the AdS/CFT flow passes through strongly coupled regions. The link between the “AdS/CFT flow” and the one we considered in Subsection 4.2 is explicitly shown in the cartoon in Figure 4.

Figure 4: The last steps of an AdS/CFT inspired duality cascade and the interpolating weak coupling flow we use in the paper.

The two different flows are controlled by different choices for the relations among the VEV of the adjoint hypermultiplets and the strong coupling scales of the gauge factors. As the prepotential is holomorphic, this choice should not induce any jump in the prepotential itself. This observation based on holomorphicity closes the circle: although we take an alternative journey that avoids the strong coupling regions of the flow, we are actually computing and comparing non-perturbative effects at the various steps of the duality cascade associated to the natural flow provided by the AdS/CFT embedding of the theory. Since we perform the comparison avoiding strongly coupled regions, we can proceed order
by order; in other terms, we are allowed to map the gauge and the stringy instantons at every instanton number without needing to compute the full prepotential.

4.5 **Comments on** $\mathbb{C}^2/\mathbb{Z}_2$ **and generalizations to** $\mathbb{C}^2/\mathbb{Z}_h$

In this Subsection we focus on the stringy instanton contributions to unitary gauge theories in both the particular example of $\mathbb{C}^2/\mathbb{Z}_2$ and a simple generalization of the $\mathbb{C}^2/\mathbb{Z}_h$ set up.

The $\mathbb{C}^2/\mathbb{Z}_2$ quiver is special due to a doubling of the number of flavors. Actually, the matter lines stretching between the two nodes are double lines. In this case, although the computation proceeds in a similar way as the one we have already presented, the final result is slightly different. Indeed, let us consider the example where the rank assignment is $(1, N)$: the number of colors is $N_c = 1$, but the number of flavors is $N_f = 2N$. Eventually, the stringy 1-instanton contribution is:

$$F_{s1} = M_2^{2-2N} e^{2\pi i\tau_D} \det(\Phi_2 - \Phi_1 1_N)^2. \tag{4.49}$$

In analogy with the computations we have already described, it is possible to show that the stringy one-instanton contribution is reproduced by the gauge one-instanton in the UV theory $(1 + 2P, N + P)$, i.e. where here $N_c = 1 + 2P$ and $N_f = 2(N + P)$.

A set up similar to the one that we have just analyzed for the $\mathbb{C}^2/\mathbb{Z}_2$ case provides a simple generalization for the generic $\mathbb{C}^2/\mathbb{Z}_h$ orbifold. Indeed, considering the rank assignment $(N_1, 1, N_3, 0, ..., 0)$, the stringy 1-instanton contribution at node 2 for this theory is:

$$F_{s2} = M_2^{2-N_1-N_3} e^{2\pi i\tau_D} \det(\Phi_2 - \Phi_1 1_{N_1}) \det(\Phi_3 - \Phi_2 1_{N_3}). \tag{4.50}$$

Again, applying the same procedure we have previously used, it is possible to show that the stringy instanton contribution (4.50) corresponds to the gauge 1-instanton contribution of the UV theory $(N_1 + P, 1 + 2P, N_3 + P, 0, ..., 0)$.

5 **Conclusions**

We believe that we have closed the argument on the existence of truly stringy instanton effects in four-dimensional theories engineered on the world-volume of branes: the stringy instanton effects can be reconducted to field theory instantonic effects. We showed this for $\mathcal{N} = 2$ theories in this paper. It was already shown for $\mathcal{N} = 1$ theories that stringy instanton effects could be ascribed to non-perturbative effects, though generally strongly

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8In this example, we note that the mass matrix that allows us to Higgs completely the $U(N_c)$ gauge group cannot be encoded in the adjoint field of the $U(N_f/2)$ group of the second node.
coupled. Actually, it is easy to argue that also in $\mathcal{N} = 1$ set ups, stringy instanton effects can be related to gauge theory instantons in a weakly coupled way by embedding into larger gauge theories, in complete analogy with the prescription presented in this paper. We discuss this issue in Appendix D.

In [42] (see also [53]), stringy instanton effects were computed in D($-1$)/D7 brane set ups and successfully mapped to the heterotic side up to instanton number 5. The complete sums of stringy instanton corrections were obtained in [54] via 1-loop calculations of D-particles on the T-dual side and also of Horava-Witten BPS states on the M-theory side. Since we have argued that stringy instanton effects can be realized in terms of ordinary gauge instanton effects in UV completed gauge theories, this suggests direct relations between the various dual string theory perspectives and the Seiberg-Witten curve, as well as the Nekrasov formulae, for the corresponding gauge theory. As a future direction it would be interesting to explore these relations further.

Another direction is to consider the resummation of the instanton series, in order to give a field theory interpretation of the resulting expression for the prepotential. Such an expression could be checked against instanton expansions also in theories that are related by UV to IR flows passing through intermediate strongly coupled regions, such as the cascade described in Subsection 4.4.

As a concluding remark, we hope that with this paper we have underscored the deep interplay that occurs between instantonic effects in string theory and in gauge theory, and the sort of complementary equivalence that exists between the two pictures.

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A Masses for the symplectic quiver

Using $\mathcal{N} = 1$ notation, the superpotential of the $\mathcal{N} = 2$ $U(N_1) \times U(N_2) \times U(N_3)$ gauge theory associated to the $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ orbifold is given by

$$
\Phi^3_{(11)}(\Phi^1_{(12)}\Phi^2_{(21)} - \Phi^2_{(13)}\Phi^1_{(31)}) - \Phi^3_{(22)}(\Phi^2_{(21)}\Phi^1_{(12)} - \Phi^1_{(23)}\Phi^2_{(32)}) + \Phi^3_{(33)}(\Phi^1_{(13)}\Phi^2_{(13)} - \Phi^2_{(32)}\Phi^1_{(23)})
$$

(A.1)

Considering the orientifold specified in [23], we have the following constraints

$$
\begin{align*}
\Phi^3_{(11)} &= \epsilon (\Phi^3_{(11)})^T \epsilon , \\
\Phi^3_{(22)} &= -\epsilon (\Phi^3_{(33)})^T , \\
\Phi^1_{(12)} &= -\epsilon (\Phi^1_{(31)})^T , \\
\Phi^1_{(23)} &= (\Phi^1_{(23)})^T , \\
\Phi^2_{(13)} &= \epsilon (\Phi^2_{(21)})^T , \\
\Phi^2_{(32)} &= (\Phi^2_{(32)})^T .
\end{align*}
$$

(A.2)

The implementation of the orientifold projection reduces the superpotential (A.1) to

$$
2 \Phi^3_{(11)}\Phi^1_{(12)}\Phi^2_{(21)} - 2 \Phi^3_{(22)}(\Phi^2_{(21)}\Phi^1_{(12)} - \Phi^1_{(23)}\Phi^2_{(32)}) .
$$

(A.5)

We introduce here the following nomenclature

$$
\Phi = \Phi^3_{(11)} , \quad \mathcal{M} = \Phi^3_{(22)} , \quad X = \Phi^1_{(12)} , \quad Y = \Phi^2_{(21)} ,
$$

(A.6)

and we rewrite the part of the superpotential involving the node 1 as follows,

$$
\Phi^I_J X^\alpha_J X^\alpha_I - \mathcal{M}_{\alpha\beta} Y^\beta_J X^\alpha_I ,
$$

(A.7)

where the indexes $I, J$ run over $1, ..., 2N_c$ and $\alpha, \beta$ run over $1, ..., N_f$; referring to the quiver depicted in Figure 1 we are here adopting $N_1 = N_c$ and $N_2 = N_f$. We split the gauge indices as follows

$$
I = \underbrace{1, ..., N_c}_{I} , \underbrace{N_c + 1, ..., 2N_c}_{i} ;
$$

(A.8)

the matrix $\Phi$ is accordingly decomposed in four blocks

$$
\Phi^I_J = \begin{pmatrix} \Phi^I_J & \Phi^I_J \\ \Phi^I_J & \Phi^I_J \end{pmatrix} .
$$

(A.9)

We further take

$$
\Phi^I_J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \Phi ,
$$

(A.10)

---

9We remind the reader that, in our notation, $Sp(N_c)$ has rank $N_c$ and its fundamental representation has dimension $2N_c$. 

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with
\[ \Phi = \text{diag}(\phi_1, ..., \phi_{N_c}) , \] (A.11)
so that
\[ \Phi_i^j = \phi_i \delta_i^j , \Phi_i^i = \Phi_i^i = 0 , \Phi_i^j = -\phi_i \delta_i^j . \] (A.12)

We can rewrite (A.7) as follows
\[ \left( \Phi_I^j \delta_\alpha^\beta - \delta_I^j \mathcal{M}_\alpha^\beta \right) X_I^\alpha Y_J^\beta \] (A.13)
where
\[ \delta_I^j = \left( \begin{array}{c} \delta_i^j \\ 0 \end{array} \right) . \] (A.14)

We then assign diagonal masses to the flavors,
\[ \mathcal{M}_\alpha^\beta = m_\alpha \delta_\alpha^\beta . \] (A.15)

Eventually, the superpotential (A.7) is given by
\[ (\phi_i - m_\alpha) \left( X_i^{\alpha} Y_{\alpha i} + X_i^{\alpha} Y_{\alpha} \right) . \] (A.16)

### B Gauge theory \( Sp(N_c) \) instanton computation

In this Appendix we would like to give some details about the procedure to obtain the \( k \)-th order gauge instanton contribution to the prepotential.

The non-perturbative partition function corresponding to the \( k \)-th instanton sector can be written as
\[ Z_k(\Phi; E_1, E_2) = \int d\vec{\chi} z_k(\vec{\chi}; \Phi; E_1, E_2) , \] (B.1)
where \( \vec{\chi} \) represents the array of neutral moduli corresponding to the Cartan basis of the instanton group, \( \Phi \) is the adjoint scalar field belonging to the vector gauge supermultiplet and \( E_1, E_2 \) are the graviphoton background flux parameters.\(^{10} \) The function

---

\( ^{10} \) A technical though important ingredient which is essential to employ localization techniques is the so-called \( \Omega \)-deformation; it essentially amounts to introducing a non-vanishing background flux for the graviphoton.\(^ {13} \) The presence of a non-null graviphoton flux regularizes the integral over the instanton moduli parameterizing the instanton center position in superspace. The graviphoton background corresponds to a closed RR flux and introduces a non-zero curvature for the superspace. The final results obtained through localization, namely the instanton partition function and the instanton contribution to the prepotential, are well defined also in the limit of zero graviphoton flux, i.e. in flat superspace. On the computational level, the graviphoton background flux is parametrized by two quantities, namely \( E_1 \) and \( E_2 \), corresponding to the two Cartan directions of the 4-dimensional Lorentz group. The instanton contributions in flat superspace are obtained considering the limit \( E_1, E_2 \to 0 \).
$z_k(\vec{\chi}; \Phi; E_1, E_2)$ encodes the result of the integration over all the instanton moduli but $\vec{\chi}$ of the exponentiated instanton action multiplied by the Vandermonde factor (introduced when considering the matrix $\chi$ along its Cartan, see for instance [23]).

In the presence of matter, the integrand function (B.1) admits the following factorization

$$z_k(\vec{\chi}; \Phi; E_1, E_2) = z_k^{(\text{pure gauge})}(\vec{\chi}; \Phi; E_1, E_2) \cdot z_k^{(\text{matter})}(\vec{\chi}; \Phi; E_1, E_2). \quad (B.2)$$

In the case of $Sp(N)$ gauge group, the ordinary $k$-instanton symmetry group is $SO(k)$ whose rank is $k/2$; this means that, at level $k$, we have $k/2$ Cartan directions parameterized by $\vec{\chi}$. Referring to [34, 35], the pure gauge factor in (B.2) is given by

$$z_k^{(\text{pure gauge})}(\vec{\chi}; \Phi; E_1, E_2) = \left(\frac{-1}{2^{k/2} + \delta} \right)^{k/2} \frac{(E_1 + E_2)}{E_1 E_2} \frac{\Delta(0) \Delta(E_1 + E_2)}{\Delta(E_1) \Delta(E_2)} \prod_{i=1}^{[k/2]} P[\chi_i - (E_1 + E_2)/2] P[\chi_i + (E_1 + E_2)/2] \left(1 - \frac{1}{E_1 E_2} \prod_{i=1}^{[k/2]} \chi_i^2 (\chi_i^2 - (E_1 + E_2)^2) \prod_{i=1}^{[k/2]} (\chi_i^2 - E_1^2)(\chi_i^2 - E_2^2) \right)^{\delta} \quad (B.3)$$

where we have defined

$$\Delta(x) = \prod_{1 \leq i < j \leq [k/2]} \left[ (\chi_i + \chi_j)^2 - x^2 \right] \left[ (\chi_i - \chi_j)^2 - x^2 \right], \quad (B.4)$$

$$P[x] = \prod_{i=1}^{N} (x^2 - \phi_i^2), \quad (B.5)$$

and $\delta = 0$ when $k$ is even, while $\delta = 1$ when $k$ is odd.

In the presence of $N_f$ fundamental flavors, the matter factor in $z_k$ is given by (see again [34,35])

$$z_k^{(\text{matter})}(\vec{\chi}; \Phi; E_1, E_2) = \prod_{j=1}^{N_f} \left( m_j - \frac{1}{2} (E_1 + E_2) \right) \prod_{i=1}^{[k/2]} \left[ \left( m_j - \frac{1}{2} (E_1 + E_2) \right)^2 - \chi_i^2 \right]. \quad (B.6)$$

Eventually, in order to obtain the partition functions $Z_k$ in (B.1), we have to integrate over the instanton moduli $\vec{\chi}$; following Nekrasov’s prescription, the integration variables $\chi_i$ must be complexified one at a time and the integral itself has to be performed via the residues method. Nekrasov’s prescription requires that the singularities encountered along the real $\chi_i$ axes be slightly moved away from it assigning appropriate, small complex
displacements; moreover, the integration contours are closed at infinity even though the integrand function does not go to zero at large distances from the origin. Even though not completely motivated \textit{a priori}, Nekrasov’s prescription has been widely tested and proved to lead to correct results whenever alternative methods are available.

We skip the detail of the $\tilde{\chi}$ integration and jump to the final results. The relation between the non-perturbative prepotential and the instanton partition function is (see for instance [44])

$$F^g = (E_1 E_2) \log Z .$$

Both sides of (B.7) can be expanded according to the topological charge $k$ obtaining

$$F^g_1 = (E_1 E_2) Z_1 ,$$

$$F^g_2 = (E_1 E_2) Z_2 - \frac{(F^g_1)^2}{2(E_1 E_2)} ,$$

$$F^g_3 = (E_1 E_2) Z_3 - \frac{F^g_1 F^g_2}{(E_1 E_2)^2} - \frac{(F^g_1)^3}{6(E_1 E_2)^2} .$$

Finally, the explicit results up to instanton number three are reported in Eqs. (3.12)–(3.16), where they are already expressed in the flat spacetime limit of vanishing graviphoton background, $E_1, E_2 \to 0$.

## C Instanton partition function regularizations

In this Appendix we evaluate the integral (4.8) by using three different regularizations and show that they give the same result.

### C.1 VEV regularization

\[
I_v = \int d^2 \omega \, d^2 \bar{\omega} \, d^3 D^c \, \bar{\omega} \, \omega \, e^{-4D^c \bar{\omega} (r^c)_\alpha M_\alpha - \frac{1}{2} |v|^2 \bar{\omega} \omega}
\]

\[
= \int d^2 \omega \, d^2 \bar{\omega} \, d^3 D^c \left( -\frac{\partial}{\partial M_1} - \frac{\partial}{\partial M_4} \right) \exp \left\{ -(\bar{\omega}^1, \bar{\omega}^2) \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\}
\]

\[
= 4\pi^2 \int d^3 D^c \left( -\frac{\partial}{\partial M_1} - \frac{\partial}{\partial M_4} \right) \frac{1}{M_1 M_4 - M_2 M_3}
\]

\[
= 4\pi^2 \int d^3 D^c \sqrt{|v|^2 + |D^c|^2} = 16\pi^3 \int_0^\infty dD \, D^2 \sqrt{|v|^2 + D^2}^2 = 32\pi^3 \int_0^\infty d\tilde{D} \, \tilde{D}^2 \frac{1}{(1 + \tilde{D}^2)^2} = 8\pi^4 ,
\]

(C.1)
where $M_1 = iD^3 + |v|^2/2$, $M_2 = i(D^1 + iD^2)$, $M_3 = i(D^1 - iD^2)$, $M_4 = -iD^3 + |v|^2/2$, we have rewritten the $D^*$ variables in terms of spherical coordinates and we have performed the rescaling $\bar{D} = D/(4|v|^2)$.

C.2 Non-commutative regularization

\[
\mathcal{I}_\xi = \int d^2\omega_\alpha d^2\bar{\omega}^\alpha d^3D^c \bar{\omega}^\alpha e^{-iD^c\bar{\omega}^\beta (r^c)^\beta_\alpha (\omega_\beta + iD^c\xi_{\alpha\beta})}
= (2\pi)^3 \int d^2\omega_\alpha d^2\bar{\omega}^\alpha \prod_c \delta \left(\bar{\omega}^\beta (r^c)^\beta_\alpha \omega_\beta - D^c\xi_{\alpha\beta}\right)
= (2\pi)^3 \int d^4y \left(\bar{y} \cdot \hat{\bar{y}}\right) \delta(y_1y_3 + y_2y_4) \delta(y_1y_4 - y_2y_3) \delta \left(y_1^2 + y_2^2 - y_3^2 - y_4^2 - \xi\right)
= (2\pi)^3 \int d^3y \left(\bar{y}^1 + \bar{y}^2 + \bar{y}^3 + \frac{y_1^2 y_3^2}{y_2^2}\right) \delta(y_3) \delta \left(y_1^2 + y_2^2 + y_3^2 + \frac{y_1^2 y_3^2}{y_2^2} - \xi\right)
= (2\pi)^3 \int r\, dr\, d\theta \frac{\delta (r - \sqrt{\xi})}{2r} = 8\pi^4,
\]

where we have rewritten the $\omega_\alpha$ and $\bar{\omega}^\alpha$ variables first in terms of real coordinates, $\omega_1 = y_1 + iy_2$ and $\omega_2 = y_3 + iy_4$ (note that there is a factor of 4 from the Jacobian) and then in terms of polar ones.

C.3 $\alpha'$ regularization

\[
\mathcal{I}_{\alpha'} = \int d^2\omega_\alpha d^2\bar{\omega}^\alpha d^3D^c \bar{\omega}^\alpha e^{-iD^c\bar{\omega}^\beta (r^c)^\beta_\alpha \omega_\beta - \frac{1}{2\sqrt{\alpha'}} (D^c)^2}
= 4 \int d^3D^c d^4y \left(\bar{y} \cdot \hat{\bar{y}}\right) e^{-2i(y_1y_3 + y_2y_4)D^1 - \frac{1}{\sqrt{\alpha'}} (D^1)^2} \times e^{-2i(y_1y_4 - y_2y_3)D^2 - \frac{1}{\sqrt{\alpha'}} (D^2)^2} e^{-i(y_1^2 + y_2^2 - y_3^2 - y_4^2)D^3 - \frac{1}{\sqrt{\alpha'}} (D^3)^2}
= 4 (2\pi g_0^2)^{\frac{3}{2}} \int d^4y \left(\bar{y} \cdot \hat{\bar{y}}\right) e^{-\frac{3}{4}(\bar{y} \bar{y})} = 4 (2\pi g_0^2)^{\frac{3}{2}} \text{(volS}^3\text{)} \int d\rho\, \rho\, e^{-\frac{3}{4}\rho^2} = 8\pi^4.
\]

D Analogy with the $\mathcal{N} = 1$ case

The correspondence between stringy instantons and gauge instantons in $\mathcal{N} = 1$ gauge theories has been already explored in [19–22]. We here review in a simple quiver-like
example some aspects which clarify the analogy with the $\mathcal{N} = 2$ case that we have studied in this paper. Indeed, also in the $\mathcal{N} = 1$ case, the stringy instanton in the IR theory can be understood as an ordinary gauge instanton in a UV completion of the theory. The flow between the UV and the IR theory involves Higgsing of the gauge group and at the same time integrating out some flavors.

Consider the quiver in the Figure 5 with $N_1 = 1$, $N_2 = N$ and with superpotential

$$W = \hat{X} q_{21} q_{12} .$$

A D($-1$) stringy instanton on node 1 gives the following contribution to the superpotential, up to a numerical factor,

$$W_{\text{inst}}^s = M_s^{3-N} e^{2\pi i \tau_D} \det \hat{X} .$$

The same contribution can be obtained through a proper gauge instanton computation as follows. We consider the UV completion of the previous theory as the $U(N_1 = 1 + P) \times U(N_2 = N + P)$ quiver gauge theory with superpotential

$$W = X_{22} Q_{21} Q_{12} + m^2 Z ,$$

where we have split the field $X$ as

$$X_{22} = \begin{pmatrix} Z & Y \\ Y & \hat{X} \end{pmatrix} .$$
Here $Z$ is a $P \times P$ square matrix and $\hat{X}$ is an $N \times N$ square matrix. On the vacuum of this theory, the first $P$ entries of the fields $Q_{21}$ and $Q_{12}$ get a VEV equal to $m$. The gauge group $U(1 + P)$ is Higgsed down to $U(1)$ and the flavors are reduced from $N + P$ to $N$. The low energy scale is defined as

$$\Lambda_{IR}^{3-N} = \frac{\Lambda_{UV}^{3+2P-N}}{m^{2P}}. \tag{D.5}$$

The IR theory is the one discussed above with superpotential (D.1), where $q_{12}$ and $q_{21}$ are the remaining parts of $Q_{21}$ and $Q_{12}$ that do not get a VEV.

Now we compute the gauge instanton contribution to the superpotential in the UV theory. The instanton action for one instanton placed on node 1 is

$$S_{\text{inst}} = S_1 + S_2 + S_W \tag{D.6}$$

$$S_1 = i(\bar{\mu}_1 \omega_{\dot{\alpha}1} + \bar{\omega}_{\dot{\alpha}1} \mu_1) \lambda_{\dot{\alpha}} - i D^c(\bar{\omega}_{\dot{\alpha}1} \tau^c \omega_{\dot{\alpha}1})$$

$$S_2 = \bar{\omega}_1 Q_{21}^\dagger Q_{21} \omega_1 + \bar{\omega}_1 Q_{12}^\dagger Q_{12} \omega_1 + i \bar{\mu}_1 Q_{12}^\dagger \mu_1 - i \bar{\mu}_1 Q_{21}^\dagger \mu_21$$

$$S_W = -i \bar{\mu}_1 X_{22} \mu_21.$$ 

The integral over the instanton moduli space gives

$$Z = \Lambda_{UV}^{3+2P-N} \int d\mu_1 d\bar{\mu}_1 d\bar{\omega}_1 d\omega_1 d\lambda d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\bar{\mu}_1^{N_2} d\mu_1^{N_2} e^{-S_{\text{inst}}}. \tag{D.7}$$

The measure $d\mu_1 d\bar{\mu}_1$ is interpreted as the $N = 1$ superspace measure, hence giving the following contribution to the superpotential

$$W_{\text{inst}}^g = \Lambda_{UV}^{3+2P-N} \int dDd\omega_1 d\bar{\omega}_1 d\lambda d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\bar{\mu}_1^{N_2} d\mu_1^{N_2} e^{-S_{\text{inst}}} \tag{D.8}.$$ 

We focus now on the fermionic integrations. The integral over $\lambda$ can be done using the $S_1$ part of the action and gives the usual ADHM constraints

$$W_{\text{inst}}^g \propto \Lambda_{UV}^{3+2P-N} \int d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\bar{\mu}_1^{N_2} d\mu_1^{N_2} (\bar{\mu}_1 \omega_1 + \bar{\omega}_1 \mu_1)^2 e^{-S_2-S_W}; \tag{D.9}$$

this soaks up only one of each of the fermionic zero modes $\mu_1$ and $\bar{\mu}_1$. Now, we saturate the remaining $(N_1 - 1) \mu_1$ and $(N_1 - 1) \bar{\mu}_1$ fermionic zero modes by using the action $S_2$

$$W_{\text{inst}}^g \propto \Lambda_{UV}^{3+2P-N} \int d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\bar{\mu}_1^{N_2} d\mu_1^{N_2} (\bar{\mu}_1 \omega_1 + \bar{\omega}_1 \mu_1)^2 \cdot (\bar{\mu}_1 Q_{12}^\dagger \mu_1)^{N_1-1} (\bar{\mu}_1 Q_{21}^\dagger \mu_21)^{N_1-1} e^{-S_W}. \tag{D.10}$$
This also saturates $N_1 - 1$ of $\mu_{12}$ and $N_1 - 1$ of $\mu_{21}$ fermionic zero modes. Finally, we saturate the remaining $N_2 - N_1 + 1$ of $\mu_{12}$ and $N_2 - N_1 + 1$ of $\mu_{21}$ fermionic zero modes by using the action $S_W$, obtaining

$$W_{\text{inst}}^g \propto \Lambda_{UV}^{3+2P-N} \int d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\mu_{12}^{N_2} d\bar{\mu}_{12}^{N_2} (\bar{\mu}_1 \omega_1 + \bar{\omega}_1 \mu_1)^2 \cdot$$

$$\cdot (\bar{\mu}_{12} Q_{12}^\dagger \mu_{12})^{N_1-1} (\bar{\mu}_{12} Q_{21}^\dagger \mu_{21})^{N_1-1} (\bar{\mu}_{12} X_{22} \mu_{21})^{N_2-N_1+1}.$$  \hfill (D.11)

This procedure saturates all the fermionic zero modes and can give a non vanishing result. Note that we can split the integration in two independent pieces

$$W_{\text{inst}}^g \propto \int d\mu_1^{N_1} d\bar{\mu}_1^{N_1} d\mu_{12}^{N_2} d\bar{\mu}_{12}^{N_2} (\bar{\mu}_1 \omega_1 + \bar{\omega}_1 \mu_1)^2 \cdot$$

$$\Lambda_{UV}^{3+2P-N} \int d\mu_{12}^{N_2-N_1+1} d\mu_{21}^{N_2-N_1+1} (\bar{\mu}_{12} X_{22} \mu_{21})^{N_2-N_1+1}.$$  \hfill (D.12)

The first one, re-including also the bosonic integration, is the same as the one discussed in \[15\] and gives the same result. The second part gives a sub-determinant of the field $X_{22}$. Note that in the vacuum only the first $N_1 - 1 = P$ components of $Q_{12}$ and $Q_{21}$ get a VEV. So in order to get a non-vanishing contribution we consider to have selected precisely those components in the first line of (D.12); as a consequence in the second line only the $\hat{X}$ part of the field $X_{22}$ is involved (remember that $N_2 - N_1 + 1 = N$).

We conclude that the instanton contribution to the superpotential is

$$W_{\text{inst}}^g = \frac{\Lambda_{UV}^{3+2P-N}}{\langle Q_{12} \rangle^{N_1-1} \langle Q_{12} \rangle^{N_1-1}} \det \hat{X} = \frac{\Lambda_{UV}^{3+2P-N}}{m^{2N_1-2}} \det \hat{X},$$  \hfill (D.13)

where here in the second passage we have inserted the VEVs for the fields $Q_{12}$ and $Q_{21}$. Remembering that $N_1 = 1 + P$, $N_2 = P + N$, and matching the string scale to the low energy scale (D.5) as

$$M_s^{3-N} e^{2\pi i r P} = \Lambda_{IR}^{3-N},$$  \hfill (D.14)

we recover exactly the contribution (D.2).

This procedure provides a gauge theory interpretation of the stringy instantons in $\mathcal{N} = 1$ theories which is analogous to the approach we have adopted for the $\mathcal{N} = 2$ case.

Note that the instanton computation leading to the contribution (D.2) for an s-confining gauge theory has been discussed in [55] by interpreting the low energy description of the $N_f = N_c + 1$ theory as a completely Higgsed $SU(2)$ magnetic theory with $N_f + 1$ flavors. Here we have extended the computation to an arbitrary number $P$ of integrated-in flavors in order to make a connection with the $\mathcal{N} = 2$ case.
References

[1] E. Witten, “Small instantons in string theory,” Nucl. Phys. B 460 (1996) 541 [hep-th/9511030].

[2] M. R. Douglas, “Branes within branes,” In *Cargese 1997, Strings, branes and dualities* 267-275 [hep-th/9512077].

[3] O. J. Ganor, “A Note on zeros of superpotentials in F theory,” Nucl. Phys. B 499 (1997) 55 [hep-th/9612077].

[4] M. B. Green and M. Gutperle, “Effects of D instantons,” Nucl. Phys. B 498 (1997) 195 [hep-th/9701093].

[5] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, “Construction of instantons,” Phys. Lett. A 65, 185 (1978).

[6] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, “The Calculus of many instantons,” Phys. Rept. 371 (2002) 231 [hep-th/0206063].

[7] M. Bianchi, S. Kovacs and G. Rossi, “Instantons and Supersymmetry,” Lect. Notes Phys. 737 (2008) 303 [hep-th/0703142 [HEP-TH]].

[8] R. Blumenhagen, M. Cvetic and T. Weigand, “Spacetime instanton corrections in 4D string vacua: The Seesaw mechanism for D-Brane models,” Nucl. Phys. B 771 (2007) 113 [hep-th/0609191].

[9] L. E. Ibanez and A. M. Uranga, “Neutrino Majorana Masses from String Theory Instanton Effects,” JHEP 0703 (2007) 052 [hep-th/0609213].

[10] B. Florea, S. Kachru, J. McGreevy and N. Saulina, “Stringy Instantons and Quiver Gauge Theories,” JHEP 0705 (2007) 024 [hep-th/0610003].

[11] R. Blumenhagen, M. Cvetic, S. Kachru and T. Weigand, “D-Brane Instantons in Type II Orientifolds,” Ann. Rev. Nucl. Part. Sci. 59 (2009) 269 [arXiv:0902.3251 [hep-th]].

[12] M. Aganagic, K. A. Intriligator, C. Vafa and N. P. Warner, “The Glueball superpotential,” Adv. Theor. Math. Phys. 7 (2004) 1045 [hep-th/0304271].

[13] K. A. Intriligator, P. Kraus, A. V. Ryzhov, M. Shigemori and C. Vafa, “On low rank classical groups in string theory, gauge theory and matrix models,” Nucl. Phys. B 682 (2004) 45 [hep-th/0311181].
[14] I. Garcia-Etxebarria, “D-brane instantons and matrix models,” JHEP 0907 (2009) 017 [arXiv:0810.1482 [hep-th]].

[15] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda and C. Petersson, “Stringy instantons at orbifold singularities,” JHEP 0706 (2007) 067 [arXiv:0704.0262 [hep-th]].

[16] M. Bianchi, F. Fucito and J. F. Morales, “D-brane instantons on the T**6 / Z(3) orientifold,” JHEP 0707 (2007) 038 [arXiv:0704.0784 [hep-th]].

[17] C. Petersson, “Superpotentials From Stringy Instantons Without Orientifolds,” JHEP 0805 (2008) 078 [arXiv:0711.1837 [hep-th]].

[18] M. Aganagic, C. Beem and S. Kachru, “Geometric transitions and dynamical SUSY breaking,” Nucl. Phys. B 796 (2008) 1 [arXiv:0709.4277 [hep-th]].

[19] I. Garcia-Etxebarria and A. M. Uranga, “Non-perturbative superpotentials across lines of marginal stability,” JHEP 0801 (2008) 033 [arXiv:0711.1430 [hep-th]].

[20] O. Aharony and S. Kachru, “Stringy Instantons and Cascading Quivers,” JHEP 0709 (2007) 060 [arXiv:0707.3126 [hep-th]].

[21] D. Krefl, “A Gauge theory analog of some ’stringy’ D-instantons,” Phys. Rev. D 78 (2008) 066004 [arXiv:0803.2829 [hep-th]].

[22] A. Amariti, L. Girardello and A. Mariotti, “Stringy Instantons as Strong Dynamics,” JHEP 0811 (2008) 041 [arXiv:0809.3432 [hep-th]].

[23] H. Ghorbani, D. Musso and A. Lerda, “Stringy instanton effects in N=2 gauge theories,” JHEP 1103 (2011) 052 [arXiv:1012.1122 [hep-th]].

[24] H. Ghorbani and D. Musso, “Stringy Instantons in SU(N) N=2 Non-Conformal Gauge Theories,” JHEP 1112 (2011) 070 [arXiv:1111.0842 [hep-th]].

[25] D. Musso, “D-branes and Non-Perturbative Quantum Field Theory: Stringy Instantons and Strongly Coupled Spintronics,” arXiv:1210.5600 [hep-th].

[26] F. Benini, M. Bertolini, C. Closset and S. Cremonesi, “The N=2 cascade revisited and the enhancon bearings,” Phys. Rev. D 79 (2009) 066012 [arXiv:0811.2207 [hep-th]].

[27] S. Cremonesi, “Transmutation of N=2 fractional D3 branes into twisted sector fluxes,” J. Phys. A 42 [arXiv:0904.2277 [hep-th]].
[28] P. C. Argyres, M. R. Plesser and N. Seiberg, “The Moduli space of vacua of N=2 SUSY QCD and duality in N=1 SUSY QCD,” Nucl. Phys. B 471 (1996) 159 [hep-th/9603042].

[29] E. G. Gimon and J. Polchinski, “Consistency conditions for orientifolds and d-manifolds,” Phys. Rev. D 54 (1996) 1667 [hep-th/9601038].

[30] P. C. Argyres, M. R. Plesser and A. D. Shapere, “N=2 moduli spaces and N=1 dualities for SO(n(c)) and USp(2n(c)) superQCD,” Nucl. Phys. B 483 (1997) 172 [hep-th/9608129].

[31] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. 7 (2004) 831 [hep-th/0206161].

[32] M. Marino and N. Wyllard, “A Note on instanton counting for N=2 gauge theories with classical gauge groups,” JHEP 0405 (2004) 021 [hep-th/0404125].

[33] N. Nekrasov and S. Shadchin, “ABCD of instantons,” Commun. Math. Phys. 252 (2004) 359 [hep-th/0404225].

[34] S. Shadchin, “Saddle point equations in Seiberg-Witten theory,” JHEP 0410 (2004) 033 [hep-th/0408066].

[35] S. Shadchin, “On certain aspects of string theory/gauge theory correspondence,” hep-th/0502180.

[36] E. D’Hoker, I. M. Krichever and D. H. Phong, “The Effective prepotential of N=2 supersymmetric SU(N(c)) gauge theories,” Nucl. Phys. B 489 (1997) 179 [hep-th/9609041].

[37] E. D’Hoker, I. M. Krichever and D. H. Phong, “The Effective prepotential of N=2 supersymmetric SO(N(c)) and Sp(N(c)) gauge theories,” Nucl. Phys. B 489 (1997) 211 [hep-th/9609145].

[38] J. D. Edelstein, M. Marino and J. Mas, “Whitham hierarchies, instanton corrections and soft supersymmetry breaking in N=2 SU(N) superYang-Mills theory,” Nucl. Phys. B 541 (1999) 671 [hep-th/9805172].

[39] J. D. Edelstein, M. Gomez-Reino and J. Mas, “Instanton corrections in N=2 supersymmetric theories with classical gauge groups and fundamental matter hypermultiplets,” Nucl. Phys. B 561 (1999) 273 [hep-th/9904087].
[40] G. Chan and E. D’Hoker, “Instanton recursion relations for the effective prepotential in N=2 superYang-Mills,” Nucl. Phys. B 564 (2000) 503 [hep-th/9906193].

[41] I. P. Ennes, C. Lozano, S. G. Naculich and H. J. Schnitzer, “Elliptic models and M theory,” Nucl. Phys. B 576, 313 (2000) [hep-th/9912133].

[42] M. Billo, L. Ferro, M. Frau, L. Gallot, A. Lerda and I. Pesando, “Exotic instanton counting and heterotic/type I-prime duality,” JHEP 0907 (2009) 092 [arXiv:0905.4586 [hep-th]].

[43] M. Billo, M. Frau, F. Fucito and A. Lerda, “Instanton calculus in R-R background and the topological string,” JHEP 0611 (2006) 012 [hep-th/0606013].

[44] M. Billo, M. Frau, F. Fucito, A. Lerda, J. F. Morales and R. Poghossian, “Stringy instanton corrections to N=2 gauge couplings,” JHEP 1005, 107 (2010) [arXiv:1002.4322 [hep-th]].

[45] M. B. Green and M. Gutperle, “D instanton induced interactions on a D3-brane,” JHEP 0002 (2000) 014 [hep-th/0002011].

[46] M. Billo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, “Classical gauge instantons from open strings,” JHEP 0302 (2003) 045 [hep-th/0211250].

[47] R. Argurio, G. Ferretti and C. Petersson, “Instantons and Toric Quiver Gauge Theories,” JHEP 0807 (2008) 123 [arXiv:0803.2041 [hep-th]].

[48] N. Nekrasov and A. S. Schwarz, “Instantons on noncommutative R**4 and (2,0) superconformal six-dimensional theory,” Commun. Math. Phys. 198 (1998) 689 [hep-th/9802068].

[49] G. Ferretti and C. Petersson, “Non-Perturbative Effects on a Fractional D3-Brane,” JHEP 0903 (2009) 040 [arXiv:0901.1182 [hep-th]].

[50] C. P. Martin and D. Sanchez-Ruiz, “The one loop UV divergent structure of U(1) Yang-Mills theory on noncommutative R**4,” Phys. Rev. Lett. 83 (1999) 476 [hep-th/9903077].

[51] V. V. Khoze and G. Travaglini, “Wilsonian effective actions and the IR / UV mixing in noncommutative gauge theories,” JHEP 0101 (2001) 026 [hep-th/0011218].

[52] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities,” JHEP 0008 (2000) 052 [hep-th/0007191].
[53] F. Fucito, J. F. Morales and R. Poghossian, “Exotic prepotentials from D(-1)D7 dynamics,” JHEP 0910 (2009) 041 [arXiv:0906.3802 [hep-th]].

[54] C. Petersson, P. Soler and A. M. Uranga, “D-instanton and polyinstanton effects from type I’ D0-brane loops,” JHEP 1006 (2010) 089 [arXiv:1001.3390 [hep-th]].

[55] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435 (1995) 129 [hep-th/9411149].