Mixed mode oscillations in a conceptual climate model

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Abstract

Much work has been done on relaxation oscillations and other simple oscillators in conceptual climate models. However, the oscillatory patterns in climate data are often more complicated than what can be described by such mechanisms. This paper examines complex oscillatory behavior in climate data through the lens of mixed-mode oscillations. As a case study, a conceptual climate model with governing equations for global mean temperature, atmospheric carbon, and oceanic carbon is analyzed. The nondimensionalized model is a fast/slow system with one fast variable (corresponding to temperature) and two slow variables (corresponding to the two carbon stores). Geometric singular perturbation theory is used to demonstrate the existence of a folded node singularity. A parameter regime is found in which (singular) trajectories that pass through the folded node are returned to the singular funnel in the limiting case where $\epsilon = 0$. In this parameter regime, the model has a stable periodic orbit of type $1^s$ for some $s > 0$. To our knowledge, it is the first conceptual climate model demonstrated to have the capability to produce an MMO pattern.

Keywords: mixed-mode oscillations, MMO, fast/slow, folded node, glacial cycle, energy balance, paleoclimate

1. Introduction

There has been a significant amount of research aimed at explaining oscillations in various historical periods of the climate system. Crucifix surveys some of the work on oscillators found in conceptual climate models in \cite{6}. Maasch and Saltzman have a series of papers on the Mid-Pleistocene transition, a change from oscillations with a dominant period of 40 kyr to oscillations with a dominant period of 100 kyr \cite{21, 22, 23}. Paillard and Parrenin also seek to explain the Mid-Pleistocene transition and the glacial-interglacial cycles of the late Pleistocene, with a discontinuous and piecewise linear model \cite{17}. Their work, and the work of Hogg \cite{8}, use Milankovitch forcing—changes in solar forcing due to variation in the Earth’s orbit—to generate oscillations. However the vast majority of research on oscillations in climate data has focused on relaxation oscillations or some other mechanism that only explains oscillations of a single amplitude \cite{21, 22, 23}.

Looking at Figure 1, each 100 kyr cycle contains a sharp increase leading into the interglacial period (denoted by the red spikes). This relaxation behavior clearly indicates the existence of

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multiple time-scales in the underlying problem. There are also smaller, structured oscillations in the glacial state that are repeated in each 100 kyr cycle. The presence of the large relaxation oscillation and the small amplitude oscillations suggests that these are mixed-mode oscillations (MMOs)—a pattern of $L_1$ large amplitude oscillations (LAOs) followed by $s_1$ small amplitude oscillations (SAOs), then $L_2$ large spikes, $s_2$ small cycles, and so on. The sequence $L_1^{s_1}L_2^{s_2}L_3^{s_3} \ldots$ is known as the MMO signature. A reasonable expectation for a model that claims to explain the 100 kyr glacial-interglacial cycles would be to explain the largest of the SAOs—i.e., the largest cycles that do not enter the interglacial state.

This chapter tests the scientific hypothesis that oscillatory behavior in climate data can be interpreted as MMOs, and we take the data in Figure 1 as a case study. Desroches et al. survey the mechanisms that can produce MMOs in systems with multiple time-scales [7]. From the data set shown in Figure 1 we know that the underlying model has a multiple time-scale structure. If we want to find MMOs, the model must have at least three state variables. Assuming we can find a global time-scale splitting, there are three distinct ways to have a 3D model with multiple time scales: (a) 1 fast, 2 slow; (b) 2 fast, 1 slow; and (c) 1 fast, 1 intermediate, 1 slow (i.e., a three-time-scale model). Each of these options can create MMOs through different mechanisms. Models with 1 fast and 2 slow variables can create MMOs through a folded node or folded saddle-node with a global return mechanism that repeatedly sends trajectories near the singularities. Models with 1 slow and 2 fast variables can create MMOs through a delayed Hopf mechanism that also

![Figure 1: Temperature record depicting interglacial periods](image.png)
requires a global return. MMOs in three time-scale models are reminiscent of MMOs due to a folded saddle-node type II—where one of the equilibria is a folded singularity —although the amplitudes of the SAOs are more pronounced in this case.

To verify our hypothesis, we have to strike a delicate balance. The model needs to be complex enough to exhibit the desired behavior, but if it is too complex we will be unable to prove that it does so. It is clear from [5] that temperature and atmospheric carbon should be two variables in any model that describes glacial-interglacial cycles. We consider a physical, conceptual model that incorporates these two components of the climate system as well as oceanic carbon. Since this approach has never been used in a climate-based model, our desire is that the analysis is clear enough to replicate. This is a major reason for our choice of such a simplistic 3D model. Indeed, we omit time-dependent forcing such as Milankovitch cycles, leaving these effects to future work. Even so, a minimal model is able to provide insight into key mechanisms behind the MMOs. We include oceanic carbon as the third variable because the model was able to produce MMOs. However, we were unable to find MMOs in other minimal models with, for example, deep ocean temperature.

We know from the data shown in Figure 1 that temperature \((T)\) shows relaxation behavior, so we can assume that \(T\) is a fast variable within our conceptual model. The main task is to obtain the “global” time-scale separation between the temperature evolution and the evolution of the carbon equations denoted by \(\epsilon_1\) and \(\epsilon_2\). In general, a time-scale separation can be revealed through dimensional analysis. The process should relate a small parameter \(\epsilon_i\) to physical parameters of the dimensional model. In applications such as neuroscience, it is often possible to get a handle on the “smallness” of the \(\epsilon_i\) because there are accepted values or ranges for many of the physical parameters. Unfortunately, parameters in paleoclimate models are not as constrained. We rely on the intuition of physicists, geologists, and atmospheric scientists to determine a reasonable separation of time-scales.

While it may be unsettling to not have a more concrete argument, the ambiguity regarding parameter values—and even the governing equations—allows more freedom. With this in mind we take a different approach than is often sought in the paleoclimate literature. In the vast majority of climate science papers, the authors simulate models with judiciously chosen parameters. Our approach is different in that we assume nothing about any parameters except that they are physically meaningful. Then, through the analysis, we find conditions under which the model behaves qualitatively like the data. The idea is not to pinpoint specific parameter values, but to find a range of possible parameters. There are two advantages to this approach. First, the parameter range can be used to constrain (or maybe constrain further) previous parameter estimates, which may tell us something previously unknown about the climate system. It can be used to inform parameter choices for large simulations. Second, a parameter range is useful to eliminate options. That is, if the only parameter range which produces the correct qualitative behavior is entirely unreasonable, the model needs to be changed.

The outline of the paper is as follows: In section 2 we set up the model and provide relevant background from the paleoclimate literature. Then we nondimensionalize the model and discuss assumptions on some of the parameters. We analyze the dimensionless model in section 3, with a focus on finding conditions for MMOs. We conclude with a discussion in section 4.
2. Setting up the Model

We start with a model of the form

\[
\frac{dT}{dt} = \frac{1}{C_p}[Q(1 - \alpha(T)) - (B_0 - B_1A + B_2T)] \tag{1}
\]

\[
\frac{dA}{dt} = B_3[P(T - T^\ast)^2 - B_4 - A] - (L + B_5A - B_6H) \tag{2}
\]

\[
\frac{dH}{dt} = L + B_5A - B_6H. \tag{3}
\]

\(T\) is globally averaged temperature in degrees Celsius, \(A\) is PgC (Petagrams of Carbon) in the atmosphere, and \(H\) is PgC in the mixed layer of the ocean. Often atmospheric carbon is discussed as carbon concentration in the atmosphere in ppm (parts per million) [4]. However when discussing land-atmosphere flux, as we will do, it makes sense to discuss carbon in terms of mass, hence the choice of PgC [26]. Equation (1) is a minor variant of the standard global energy balance equation due to Budyko [3] and Sellers [24]. \(C_p\) is planetary heat capacity and \(Q\) is the total incoming solar radiation—abbreviated “insolation” in the climate literature—that reaches the Earth. The albedo, \(\alpha(T)\) is the proportion of this incoming shortwave radiation that is immediately reflected back to space. The functional dependence of albedo on temperature will be explored shortly. Since whatever radiation is not reflected must be absorbed, the quantity \(Q(1 - \alpha(T))\) is the absorbed insolation. The \((B_0 - B_1A + B_2T)\) term is the linearized outgoing longwave radiation. The amount of heat radiated by a blackbody is proportional to its temperature to the fourth power \((T^4)\). Due to atmospheric greenhouse gases, however, the Earth does not radiate heat as a perfect blackbody. It is a standard practice in paleoclimate to linearize the outgoing radiation term [27] as

\[\tilde{A} + BT,\]  \tag{4}

where the dependence on atmospheric greenhouse gases is implicitly built into \(\tilde{A}\).

While the climate system is far more complicated than merely a stable temperature, one can think of a climate as a fixed point of equation (1). These fixed points occur precisely when there is an energy balance, i.e. when the absorbed insolation is equal to the outgoing longwave radiation. Clearly, the values of \(T\) for which fixed points occur depend on the nonlinear albedo function, \(\alpha(T)\).

| Variable/Parameter | Unit     | Parameter | Unit             |
|-------------------|----------|-----------|------------------|
| \(t\)             | yr       | \(B_0\)   | \(W m^{-2}\)    |
| \(T\)             | K (Kelvin) | \(B_1\)  | \(W m^{-2} PgC^{-1}\) |
| \(A\)             | PgC      | \(B_2\)   | \(W m^{-2} K^{-1}\) |
| \(H\)             | PgC      | \(B_3\)   | \(yr^{-1}\)     |
| \(C_p\)           | \(JK^{-1}m^{-2}\) | \(B_4\)  | \(PgC yr^{-1}\) |
| \(Q\)             | \(W m^{-2}\) | \(B_5\)  | \(yr^{-1}\)     |
| \(\alpha_1\)      | 1        | \(B_6\)   | \(PgC yr^{-1}\) |
| \(\alpha_2\)      | 1        | \(P\)     | \(PgCK^2\)      |
| \(T^\ast, \tilde{T}\) | K       | \(L\)     | \(PgC yr^{-1}\) |

Table 1: Summary of the parameters, variables and their units.
In general, the albedo function is unknown and the subject of current research. What is known is that ice reflects much more radiation than land or water, so the average albedo of a cold Earth should be much higher than that of a warm Earth. However, many other factors such as clouds and vegetation affect the albedo as well, albeit in a manner that is not entirely understood. For the purposes of paleoclimate models, the albedo is often taken to be a step function [1, 27] or a piecewise linear ramp function [9]. When smoothness of the vector field is required, a hyperbolic tangent may be used [28]. Since our analysis will require continuous derivatives, we have taken

$$\alpha(T) = \frac{\alpha_M + \alpha_m}{2} - \frac{\alpha_M - \alpha_m}{2} \tanh \left( \frac{T - \tilde{T}}{D} \right),$$

where $\alpha_M$ and $\alpha_m$ are the maximum and minimum planetary albedos, respectively, and $\tilde{T}$ is half the activation temperature of $\alpha$. Figure 2 plots the absorbed radiation (for $\alpha(T)$ modeled with a hyperbolic tangent) and outgoing radiation. A quick examination of Figure 2 shows that there are two stable climates ($T_w$, $T_c$) and an intermediate unstable climate ($T_m$). The standard energy balance equation that leads to (1) is one of the paradigmatic sources of bistability in conceptual climate models. This bistability plays an important role in demonstrating the capability for MMOs.

As mentioned previously, the outgoing radiation term is usually linearized as in (4). Our variation, $(B_0 - B_1 A + B_2 T)$, explicitly includes a dependence of outgoing radiation on atmospheric greenhouse gases $A$—also known as climate sensitivity. This allows us to include $A$ as a state variable that evolves according to (2). Equation (2) can be decomposed into two terms: the land-atmosphere flux,

$$B_3[P(T - T_s)^2 - B_4 - A],$$

and the ocean-atmosphere flux

$$L + B_5 A - B_6 H.$$

Notice that the ocean-atmosphere flux is balanced in equation (3). That is, whatever carbon is outgassed from (or absorbed by) the ocean must be transferred to (or from) the atmosphere. While the ocean has numerous carbon reservoirs of various sizes (e.g. the mixed layer and the deep ocean
we assume that the carbon exchange between atmosphere and ocean follows a simple linear

We assume that the carbon exchange between atmosphere and ocean follows a simple linear equation as described in [25]. The air-sea exchange of carbon is temperature dependent [14]; in our model this means that the coefficients \( B_5 \) and \( B_6 \) may depend on temperature, but for the sake of simplicity, we suppress this dependence.

The terrestrial, or land-atmosphere, flux depends on plants (among other things) [15]. Therefore, the carbon drawdown is most efficient at \( T^* \), the temperature at which \( \text{CO}_2 \)-absorbing life is most prolific. As a testament to how difficult it is to pin down actual parameter values for climate models, especially in paleoclimate problems, we point to the literature with regards to \( T^* \). In [15], Lenton and Huntingford state that \( T^* \) should be a temperature in the warm, interglacial state (i.e. \( T^* \approx T_w \)). That is, carbon drawdown should be most effective during the large spikes in Figure 1. However in [16], Lovelock calls the interglacial states “fevers,” suggesting that carbon drawdown is most efficient in the climate’s “natural,” glacial state (i.e. \( T^* \approx T_c \)).

The reason there is no governing equation for terrestrial PgC is that the total carbon content of the system should be conserved. While there can be subdivisions within them [26], we are considering three carbon stores: atmosphere, land, and ocean. Since there is a conserved quantity, only the two governing equations are needed.

In an effort to simplify calculations, we will first translate \( T \) by \( \tilde{T} \), giving the albedo function odd symmetry about the vertical axis. We introduce the variable

\[
S = T - \tilde{T}
\]
as well as the parameter

\[
S^*_r = T^*_r - \tilde{T}. 
\]

In terms of the variables \( S, A, H \), the system (1)-(3) becomes

\[
\frac{dS}{dt} = \frac{1}{C_p} \left[ Q(1 - \alpha(S)) - (B_0 + B_2\tilde{T} - B_1A + B_2S) \right] 
\]

\[
\frac{dA}{dt} = B_3[P(S - S^*_r)^2 - B_4 - A] - (L + B_5A - B_6H) 
\]

\[
\frac{dH}{dt} = L + B_5A - B_6H, 
\]

where

\[
\alpha(S) = \frac{\alpha_M + \alpha_m}{2} - \frac{\alpha_M - \alpha_m}{2} \tanh \left( \frac{S}{D} \right). 
\]

Secondly, we observe that the RHS of (6) can be reasonably approximated by a cubic (see Figure 3). Thus we simplify the system (6)-(8) to

\[
\frac{dS}{dt} = \frac{1}{C_p} \left[ B_1A - \frac{Q(\alpha_M - \alpha_m)}{6D^3} S^3 + \left( \frac{Q(\alpha_M - \alpha_m)}{2D} - B_2 \right) S + K \right] 
\]

\[
\frac{dA}{dt} = B_3[P(S - S^*_r)^2 - B_4 - A] - (L + B_5A - B_6H) 
\]

\[
\frac{dH}{dt} = L + B_5A - B_6H, 
\]

where

\[
K = Q \left( 1 - \frac{\alpha_M + \alpha_m}{2} \right) - (B_0 + B_2\tilde{T}), 
\]

and the RHS of (10)-(12) are all polynomials.
Thirdly, based on the observation made in Figure 1, the model (10)-(12) should evolve on multiple time-scales. Such a separation of time-scales can only be identified in a dimensionless model. Therefore, we define the dimensionless quantities

\[ x = \frac{S}{S_0}, \quad y = \frac{A}{A_0}, \quad z = \frac{H}{H_0}, \quad \text{and} \quad s = \frac{t}{t_0} \]

where

\[ S_0 = \left( \frac{D^3(Q(\alpha_M - \alpha_m) - 4DB_2)}{Q(\alpha_M - \alpha_m)B_1} \right)^{\frac{1}{2}} \]

\[ A_0 = \left( \frac{D^3(Q(\alpha_M - \alpha_m) - 4DB_2)^3}{Q(\alpha_M - \alpha_m)B_1^3} \right)^{\frac{1}{2}} \]

\[ H_0 = \frac{B_5}{B_6} \left( \frac{D^3(Q(\alpha_M - \alpha_m) - 4DB_2)^3}{Q(\alpha_M - \alpha_m)B_1^3} \right)^{\frac{1}{2}} \]

\[ t_0 = \frac{1}{B_5} \]

Then equations (10)-(12) become

\[ \epsilon \dot{x} = y - x^3 + 3x - k \quad \text{(13)} \]
\[ \dot{y} = p(x - a)^2 - b - my - (\lambda + y) + z \quad \text{(14)} \]
\[ \dot{z} = r(\lambda + y - z) \quad \text{(15)} \]

where the dot (\( \dot{\cdot} \)) denotes \( \frac{d}{dt} \). The new dimensionless parameters relate to the physical parameters of equations (10)-(12) in the following way:

\[ k = \frac{K}{B_1A_0}, \quad p = \frac{B_3PS_0^2}{B_5A_0}, \quad a = \frac{S_0}{S_0}, \quad b = \frac{B_3B_4}{B_5A_0} \]
\[ m = \frac{B_3}{B_5}, \quad \lambda = \frac{L}{B_5A_0}, \quad r = \frac{B_6}{B_5}, \quad \text{and} \quad \epsilon = \frac{B_5C_pS_0}{B_1A_0} \]
Any time-scale separation is determined by $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon r$. As mentioned earlier, parameter values in paleoclimate problems are the subject of some debate. In accordance with our observation based on Figure 1, we assume that temperature evolves on a faster time-scale than carbon, implying $0 < \epsilon \ll 1$. This assumption is further supported by [3]. If $r = O(1)$ we have 1 fast and 2 slow variables, and if $r \ll 1$ we are in the three time-scale case. Depending on which parameters hold the key to having $0 < \epsilon \ll 1$, other parameters (e.g. $p$, $b$, or $\lambda$) may be small as well. Again, our approach is to assume as little as possible about the parameters, so we will keep this in mind as we perform the analysis. Figure 4 depicts examples of MMO orbits produced by the model.

Remark 1. The dimensionless form of the model is a variant of the Koper model, an electrochemical model that is known to exhibit MMOs [11, 13]. Many other models in chemistry and neuroscience also demonstrate MMOs (e.g. the Hodgkin-Huxley equations [20]). Indeed, many mechanisms in other areas such as mass balance in chemical reactions or gated ion channels in neural models behave similarly to certain climate mechanisms such as conservation of mass or exchange of carbon dioxide across the ocean-atmosphere surface.

3. Analyzing the System

In this section we will analyze the system (13)-(15). We assume that the system is singularly perturbed with singular perturbation parameter $\epsilon$. We also assume that $r = O(\epsilon^n)$ where $n = 0$ or $n = 1$ (although fractional powers may be acceptable as well). Hence we are using a 2 slow/1 fast approach that allows for the case where $r = O(\epsilon)$. We will comment on the case where $r$ is small when appropriate. The method of analyzing singularly perturbed systems—geometric singular perturbation theory (GSP)—was first developed by Fenichel. We will introduce the important concepts from GSP as required by the analysis. For further details on the theory, we direct the reader to the survey by Jones [10].

The following quantities will appear often in our calculations, so we define

$$h(x) = x^3 - 3x + k,$$
$$f(x) = p(x - a)^2 - b,$$

as well as

$$F(x, y) = y - h(x).$$
3.1. The Layer Problem

To begin the analysis, we rescale the time variable $s$ by $\epsilon^{-1}$ to obtain the system

$$x' = y - x^3 + 3x - k$$
$$y' = \epsilon(p(x-a)^2 - b - my - (\lambda + y) + z)$$
$$z' = \epsilon r(\lambda + y - z),$$

where the prime (') denotes $d/d\tau$ and $\tau = \epsilon^{-1}s$. As long as $\epsilon > 0$, the new system (16)-(18) is equivalent to (13)-(15) in the sense that the paths of trajectories are unchanged—they are merely traced with different speeds. However, in the singular limit (i.e. as $\epsilon \to 0$) the systems are different.

When $\epsilon = 0$, the system (16)-(18) becomes

$$x' = F(x, y)$$
$$y' = 0$$
$$z' = 0,$$

which is called the layer problem. Notice that the dynamics in the $y$ and $z$ directions are trivial. The critical manifold,

$$M_0 = \{F(x, y) = 0\} = \{y = h(x)\},$$

is the set of critical points of the layer problem. $M_0$ is attracting (resp. repelling) whenever $F_x < 0$ (resp. $F_x > 0$), which corresponds to the $x$-values where the cubic $h(x)$ is increasing (resp. decreasing). A simple calculation shows $h'(x) = 0$ when $x = \pm 1$, so $M_0$ is attracting on the outer branches where $|x| > 1$, repelling on the middle branch where $|x| < 1$ and folded at $x = \pm 1$. To make this more explicit, $M_0$ is 'S'-shaped with two attracting branches

$$M_A^\pm = \{\pm x > 1\}$$

and a repelling branch

$$M_R = \{-1 < x < 1\}.$$

The attracting and repelling branches are separated by the folds

$$L^\pm = \{x = \pm 1\}.$$

At the folds $L^\pm$, the critical manifold is degenerate and the basic GSP theory for normally hyperbolic critical manifolds breaks down. As is so often the case, the scientifically and mathematically interesting behavior arises where the standard theory does not apply. In our case, the folds allow for more complicated dynamics such as relaxation oscillations or MMOs.

3.2. The Reduced Problem

The layer problem, which describes the fast dynamics off the critical manifold, was obtained by considering the $\epsilon = 0$ limit of equations (16)-(18). The dynamics on the critical manifold, or slow dynamics, are obtained by looking at the system (13)-(15) as $\epsilon \to 0$. In the singular limit, the system becomes

$$0 = y - x^3 + 3x - k$$
$$\dot{y} = p(x-a)^2 - b - my - (\lambda + y) + z$$
$$\dot{z} = r(\lambda + y - z).$$

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The first equation \( (19) \) provides an algebraic condition for a manifold on which this new system is defined. The critical manifold \( M_0 \) is precisely the set that satisfies the algebraic condition, so equations \( (20)-(21) \) describe the dynamics on the manifold \( M_0 \). The two dimensional dynamical system \( (20)-(21) \) is called the \textit{reduced problem}. Two equations are required to describe the dynamics on the 2D surface \( M_0 \), and with the algebraic condition we should be able to (locally) write the vector field of the reduced problem in terms of only the variables \( y \) and \( z \). Since \( F_x = 0 \) for some points on \( M_0 \), we cannot universally write \( x \) as a function of \( y \) and \( z \) on \( M_0 \)—the function would have to be defined separately on each branch. However, \( F_y \equiv 1 \), so we can write \( y \) as a function of \( x \) (namely, \( y = h(x) \)) on \( M_0 \). In order to exploit this functional dependence, we would like to formulate the reduced problem in terms of \( x \) and \( z \) instead of \( y \) and \( z \). This is done by differentiating the algebraic condition in \( (19) \), and substituting it for \( \dot{y} \) equation \( (20) \). Doing so produces

\[
-F_x \dot{x} = F_y \dot{y} + F_z \dot{z} = r(\lambda + y - z).
\]

Substitution provides

\[
\begin{align*}
    h'(x) \dot{x} &= f(x) - (m + 1)h(x) - \lambda + z \\
    \dot{z} &= r(\lambda + h(x) - z),
\end{align*}
\]

and now the reduced problem is formulated as two equations in terms of two variables. System \( (22) \) has three different types of singularities:

- ordinary singularities—these are equilibria of the full system \( (13)-(15) \)
- regular fold points—also known as jump points, and
- folded singularities—isolated points along \( L^\pm \) where the reduced flow changes orientation.

Since \( h'(\pm 1) = 0 \), \( (22) \) describes a singular system. We can rescale the time variable \( s \) by \( h'(x) \) to obtain the desingularized system

\[
\begin{align*}
    \dot{x} &= f(x) - (m + 1)h(x) - \lambda + z \\
    \dot{z} &= rh'(x)(\lambda + h(x) - z).
\end{align*}
\]

The rescaling reverses trajectories on \( M_R \) because this is precisely the set where \( h'(x) < 0 \), but the benefit of being able to define dynamics on the folds outweighs the cost. From system \( (23) \), it is now easy to classify the different singularities of the reduced problem. Ordinary singularities occur where \( z = h(x) + \lambda \) and \( f(x) = mh(x) \). Folded singularities are equilibria of \( (23) \) where \( \dot{z} = 0 \) as a result of the rescaling. That is, folded singularities occur where \( h'(x) = 0, \lambda + h(x) - z \not= 0 \), and \( \dot{x} = 0 \). When \( h'(x) = 0 \), but \( \dot{x} \neq 0 \), we have regular fold points.

3.3. Canard Induced MMOs

Folded nodes (as well as folded saddle-nodes) can produce MMOs with a suitable global return mechanism. We establish a global return mechanism by constructing a singular periodic orbit \( \Gamma \), consisting of heteroclinic orbits of the layer problem and a segment on each of those stable branches \( M^+_A \). The heteroclinic orbits of the layer problem take trajectories from a fold \( L^\pm \) to its projection \( P(L^\pm) \) on the opposite stable branch. An example of a singular periodic orbit \( \Gamma \) is shown in Figure 5. Assuming there is a folded node on \( L^- \) (wlog), we can construct \( \Gamma \) by following the fast fiber from the node to the stable branch \( M^+_A \). From there, the trajectory follows the slow flow on \( M^+_A \) as described by \( (23) \), until it reaches the fold \( L^+ \). If it reaches \( L^+ \) at a jump point, we follow the fast fiber back to \( M_A^- \). We want the landing point on \( M_A^- \) to be in the singular funnel—that is, the
Figure 5: Example of a singular periodic orbit \( \Gamma \) for \( a = 0.8, b = 3, k = 4, r = 1, m = 1, \) and \( \lambda = 1. \)

region bounded by the strong stable trajectory \( \gamma_s \) and the fold \( L^- \) that contains the weak stable trajectory \( \gamma_w \). The trajectory \( \gamma_s \) is also called the strong canard (denoted ‘SC’ in Figure 5a), and it forms the boundary of the funnel. The region is called a funnel because all trajectories in the region get funneled through the folded node. Therefore, any singular orbit from the folded node which returns to the funnel will necessarily be a singular periodic orbit.

A node (in a 2D system) has two real eigenvalues of the same sign, a weak eigenvalue \( \mu_w \) and a strong eigenvalue \( \mu_s \) such that \(|\mu_w| < |\mu_s|\). The ratio of these eigenvalues

\[
\mu = \frac{\mu_w}{\mu_s} < 1,
\]

is important in determining the number of small-amplitude oscillations in the MMO signature. This is made explicit in the following theorem due to Brøns et al [2] that provides conditions under which a system has a stable MMO orbit.

**Theorem 1.** Suppose that the following assumptions hold in a fast/slow system,

(A1) \( 0 < \epsilon \ll 1 \) is sufficiently small with \( \epsilon^{1/2} \ll \mu \)

(A2) the critical manifold is ‘S’-shaped, i.e. \( M_0 = M^-_A \cup L^- \cup M_R \cup L^+ \cup M^+_A \)

(A3) there is a (stable) folded node \( N \) on (wlog) \( L^- \),

(A4) there is a singular periodic orbit \( \Gamma \) such that \( \Gamma \cap M^-_A \) lies in the interior of the singular funnel to \( N \), and

(A5) \( \Gamma \) crosses \( L_{\pm} \) transversally.

Then there exists a stable periodic orbit of MMO type \( 1^s \), where

\[
s = \left\lfloor \frac{(1 + \mu)}{2\mu} \right\rfloor, \tag{24}
\]

and the right-hand side of (24) denotes the the greatest integer less than \((1 + \mu)/(2\mu)\).
In [12], Krupa and Wechselberger show that the folded node theory still applies in the parameter regime where \( \mu = \mathcal{O}(\epsilon^{1/2}) \) if the global return mechanism is still in tact (i.e. \( \Gamma \cap M_{\Gamma}^- \) lies in the interior of the singular funnel). Note that in this parameter regime, the MMO signature can be more complicated. Figure 13 depicts a few of the more interesting MMO patterns generated by \((13)-(15)\) when \( \mu = \mathcal{O}(\epsilon^{1/2}) \).

The remainder of this section will focus on finding conditions on the parameters of equations \((13)-(15)\) so that the system satisfies \((A1)-(A5)\). Since \( \mu \) is calculated in the singular limit, we can always choose \( \epsilon \) small enough to satisfy condition \((A1)\). Also, we have already discussed the ‘S’-shape of the critical manifold, demonstrating that condition \((A2)\) is satisfied. The next task will be find conditions so that equations \((13)-(15)\) have a folded node singularity.

### 3.4. Folded Node Conditions

The data in Figure 1 show small amplitude oscillations occurring at low temperatures, so we seek parameters for which \((22)\) has a stable folded node along the lower fold \( L^- \).

**Lemma 2.** Define

\[
\delta = f(-1) - mh(-1) = p(a + 1)^2 - b - m(k + 2).
\]

Assume the parameters of the system \((13)-(15)\) satisfy

1. \( p > 0 \),
2. \( a > -1 \),
3. \( \delta > 0 \), and
4. \( p^2(a + 1)^2 - 6r\delta > 0 \).

Then there is a folded node at \((-1, z_-)\) where

\[
z_- = 2 + k + \lambda - \delta.
\]

**Proof.** The linearization of \((23)\) at any fixed point \((x_0, z_0)\) is

\[
J(x_0, z_0) = \begin{pmatrix}
\frac{f'(x_0) - (m + 1)h'(x_0)}{r[(h'(x_0))^2 + h''(x_0)(\lambda + h(x_0) - z_0)]} & 1 \\
-6r[f(-1) - mh(-1)] & -r h'(x_0)
\end{pmatrix}.
\]

There is a folded singularity at \((-1, z_-)\) where

\[
z_- = (m + 1)h(-1) + \lambda - f(-1).
\]

Since \( h'(-1) = 0 \), we have the linearization

\[
J(-1, z_-) = \begin{pmatrix}
f'(-1) & 1 \\
-6r[f(-1) - mh(-1)] & 0
\end{pmatrix}.
\]

For \((-1, z_-)\) to be a stable folded node, \( J(-1, z_-) \) must satisfy three conditions:

1. \( \text{Tr}(J(-1, z_-)) < 0 \),
2. \( \text{det}(J(-1, z_-)) > 0 \), and
3. \( [\text{Tr}(J(-1, z_-))]^2 - 4 \text{det}(J(-1, z_-)) > 0 \).
The requirement on the trace implies that $f'(-1) < 0$, or $p(-1 - a) < 0$. Assuming $p > 0$, we arrive at condition (b) $a > -1$. The requirement on the determinant gives us $6r[f(-1) - mh(-1)] > 0$. Since $r > 0$, we will have $\det(J(-1, z_-)) > 0$ whenever $\delta = f(-1) - mh(-1) > 0$. That is precisely condition (c). Conditions (a)-(c) are enough to guarantee that the folded equilibrium is stable, but they do not distinguish between stable a stable node or a stable focus. This is determined by the discriminant condition, which is satisfied if

$$p^2(a + 1)^2 - 6r[f(-1) - mh(-1)] = p^2(a + 1)^2 - 6r\delta > 0.$$ 

Note that $|\delta|$ is precisely the distance along the fold from the node to the intersection of the true $z$ nullcline with the fold at $x = -1$. If $\delta > 0$, which is required by condition (b), then the node lies under the $z$ nullcline on $M_0$. That is, if $z_n$ is the intersection of the $z$ nullcline with the fold (i.e., $z_n = h(-1) + \lambda$), then $z_n > z_-$ with $z_n = z_- + \delta$. As we will see, the parameter $r$ will not appear in the remaining calculations. Thus we strive to find conditions on $\delta$, $a$, and $p$. Choosing values that satisfy those conditions, (c) then provides an upper bound on $r$.

**Remark 2.** In each of the limiting cases $r \to 0$ and $\delta \to 0$, the Jacobian [27] will have a zero eigenvalue and the system will have a folded saddle-node of type II. Near the $r = 0$ limit we are in the three time-scale case with a global three time-scale separation. Near the $\delta = 0$ limit, we have a local three time-scale split at the folded singularity. In either case, the ratio of eigenvalues $\mu$ will be small, so near the saddle-node limit, we use the theory for $\mu = O(\epsilon^{1/2})$.

Having found conditions for a folded node, it remains to be shown that these conditions are consistent with a return mechanism satisfying (A4) and (A5) from Theorem 1. As indicated by (A4), the singular funnel is a vital component of the global return mechanism. Typically, the functionality of the return mechanism is demonstrated numerically [13, 19]. This is done by choosing a set of reasonable parameters and then varying one parameter until the MMO orbit disappears. For example we set parameters to have the following values: $p = 3$, $b = 2.1$, $k = 4$, $r = 1$, $m = 1$, $\lambda = 1$, and we vary $a$. If $a \approx 0.643$, we have that $\delta \approx 0$ and we are in the SN-II case. From this value we increase $a$ to 0.8 to obtain the singular orbit in Figure 5. Figure 6 shows stable MMO time series for these parameters away from the singular limit. Continuing to increase $a$, we see that when $a \approx 0.823$, the singular periodic orbit lands on the strong canard. For $a \geq 0.824$, there is no MMO orbit since the return mechanism misses no longer sends trajectories into the funnel.

Figure 6: Time series for different values of $\epsilon$ when $a = 0.8$, $p = 3$, $b = 2.1$, $k = 4$, $r = 1$, $m = 1$, and $\lambda = 1$. 

(a) Time series for $\epsilon = 0.01$. 
(b) Time series for $\epsilon = 0.05$. 

The requirement on the trace implies that $f'(-1) < 0$, or $p(-1 - a) < 0$. Assuming $p > 0$, we arrive at condition (b) $a > -1$. The requirement on the determinant gives us $6r[f(-1) - mh(-1)] > 0$. Since $r > 0$, we will have $\det(J(-1, z_-)) > 0$ whenever $\delta = f(-1) - mh(-1) > 0$. That is precisely condition (c). Conditions (a)-(c) are enough to guarantee that the folded equilibrium is stable, but they do not distinguish between stable a stable node or a stable focus. This is determined by the discriminant condition, which is satisfied if

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**Remark 2.** In each of the limiting cases $r \to 0$ and $\delta \to 0$, the Jacobian [27] will have a zero eigenvalue and the system will have a folded saddle-node of type II. Near the $r = 0$ limit we are in the three time-scale case with a global three time-scale separation. Near the $\delta = 0$ limit, we have a local three time-scale split at the folded singularity. In either case, the ratio of eigenvalues $\mu$ will be small, so near the saddle-node limit, we use the theory for $\mu = O(\epsilon^{1/2})$.

Having found conditions for a folded node, it remains to be shown that these conditions are consistent with a return mechanism satisfying (A4) and (A5) from Theorem 1. As indicated by (A4), the singular funnel is a vital component of the global return mechanism. Typically, the functionality of the return mechanism is demonstrated numerically [13, 19]. This is done by choosing a set of reasonable parameters and then varying one parameter until the MMO orbit disappears. For example we set parameters to have the following values: $p = 3$, $b = 2.1$, $k = 4$, $r = 1$, $m = 1$, $\lambda = 1$, and we vary $a$. If $a \approx 0.643$, we have that $\delta \approx 0$ and we are in the SN-II case. From this value we increase $a$ to 0.8 to obtain the singular orbit in Figure 5. Figure 6 shows stable MMO time series for these parameters away from the singular limit. Continuing to increase $a$, we see that when $a \approx 0.823$, the singular periodic orbit lands on the strong canard. For $a \geq 0.824$, there is no MMO orbit since the return mechanism misses no longer sends trajectories into the funnel.
In the following subsections, we use approximations to obtain analytical results and prove our main result. The strategy is to linearly approximate strong canard and show that the approximation lies within the funnel. Then we find conditions under which the return mechanism lands in the approximated funnel region.

3.5. Estimate of the Funnel

Assuming the node conditions (a)-(d) from Lemma 2 are met, the folded singularity will have a strong stable eigenvalue (eigenvector) and a weak stable eigenvalue (eigenvector). Let \( \mu_{s,w} \) be the eigenvalues, where \( s \) and \( w \) denote strong and weak, respectively. Then

\[ \mu_s < \mu_w < 0. \]

Also let \((x_{s,w}, z_{s,w})\) denote the corresponding eigenvector. A simple computation shows the slope of the eigenvector

\[ m_i = \frac{z_i}{x_i} = \frac{-6r\delta}{\mu_i} > 0, \]

where \( i \) can be either \( s \) or \( w \). Then we have the following relationships

\[ 0 < m_s < m_w < -f'(-1), \]

where \(-f'(-1)\) is the slope of the \( x \) nullcline at the node. Recall that the singular funnel is the region bounded by the fold \( L^- \) and the strong canard \( \gamma_s \) (the trajectory that approaches the node with slope \( m_s \)) that contains the weak canard. In our case, locally near the folded node, the funnel will lie below the strong canard. Following \( \gamma_s \) away from the node in reverse time, we see that if \( \gamma_s \) intersects the \( x \) nullcline, it will turn down and to the right until it intersects the fold \( L^- \). We want to avoid this situation since it effectively precludes a global return mechanism. However, if \( \gamma_s \) intersects the \( z \)-nullcline, then it will continue up and to the left in reverse time as in Figure 7. The following lemma provides conditions under which \( \gamma_s \) lies entirely above its tangent line at the node, allowing us use a linear approximation to find a lower bound for the intersection of \( \gamma_s \) with \( P(L^+) \).

**Lemma 3.** Let equations (13)-(15) satisfy the conditions (a)-(d) of Lemma 2. Furthermore assume

(e) \[ \frac{2p^2(a+1)^2}{\delta} + 2pa - 6(m+1) < 0, \]

(f) \[ p(a+1) > 2. \]

Let \( m_s \) denote the slope of the strong eigenvector to the node. That is

\[ m_s = \frac{6r\delta}{-\mu_s} = \frac{6r\delta}{|\mu_s|}. \]

Then the strong canard \( \gamma_s \) is tangent to the line \( z = m_s(x+1) + z_- \) at \( x = -1 \), and lies above the line for \( x < -1 \).

The method of proof is to show that \( \gamma_s \), thought of as \( z = \gamma_s(x) \), is concave up at the node \((-1, z_-)\). This shows that \( \gamma_s \) lies above the line

\[ z - m_s(x+1) = z_- \]

near the folded node. We will then consider the direction of the vector field along the line to show that \( \gamma_s \) remains above the line.
Figure 7: A lower bound for the edge of the funnel.

Proof. The $z$ coordinate of the strong canard tends to $z_-$ as $x \to -1$, however since the point $(-1, z_-)$ is a node, there are many trajectories that do so. The strong canard can be characterized as the trajectory whose slope tends to $m_s$ as $x \to -1$. That is

$$\lim_{x \to -1} \frac{dz}{dx} = \frac{6r\delta}{|\mu_s|}.$$ 

The concavity of the strong canard determines whether it approaches its tangent line from above or below. We begin with the first derivative,

$$\frac{dz}{dx} = \frac{r h'(x) (\lambda + h(x) - z)}{f(x) - (m + 1)h(x) - \lambda + z}.$$ 

To assist us in the calculations, we define

$$\eta(x) = f(x) - (m + 1)h(x) - \lambda + z$$
$$\phi(x) = \lambda + h(x) - z$$

noting that

$$\phi(-1) = \delta, \quad \phi'(-1) = -\frac{6r\delta}{|\mu_s|}, \quad \eta(-1) = 0,$$

$$h'(-1) = 0, \quad h''(-1) = -6, \quad h'''(x) = 6$$

and

$$\lim_{x \to -1} \frac{h'(x)}{\eta(x)} = \frac{6}{|\mu_s|}.$$ 

Now, we use the quotient rule to obtain:

$$\frac{d^2z}{dx^2} = \frac{r}{(\eta(x))^2} \left[ \eta(x) \left( h''(x) \left( h'(x) - \frac{dz}{dx} \right) + \phi(x)h''(x) \right) - h'(x)\phi(x)\eta'(x) \right].$$
In particular, we are interested in
\[ L = \lim_{x \to -1} \frac{d^2 z}{dx^2}. \]

Using L'Hopital's rule, we see
\[ L = \lim_{x \to -1} \left[ \frac{r}{2\eta(x)\eta'(x)} \right. \]
\[ \cdot \left[ \eta(x) \left( h'(x) \left( h''(x) - \frac{dz}{dx} \right) + h''(x) \left( h'(x) - \frac{dz}{dx} \right) + \phi(x)h'''(x) + h''(x)\phi(x) \right) \right. \]
\[ + \eta'(x) \left( h'(x) \left( h'(x) - \frac{dz}{dx} \right) + \phi(x)h''(x) \right) - \eta'(x) \phi(x)h''(x) \]
\[ - h'(x) \left( \phi(x) \left( f''(x) - (m + 1)h''(x) + \frac{dz}{dx} \right) + \phi'(x)\eta'(x) \right) \left] \right], \]
which simplifies to
\[ L = \frac{3r}{\eta'(-1)} \left( \frac{12r\delta}{|\mu_s|} + \delta \right) - \frac{3r}{|\mu_s|} \left( \frac{6r\delta}{|\mu_s|} \right) - \frac{3r}{|\mu_s|} \left( \frac{\delta(2p + 6(m + 1) + L)}{\eta'(-1)} - \frac{6r\delta}{|\mu_s|} \right). \]

Simplifying further and gathering the \( L \) terms on one side gives
\[ \frac{\eta'(-1)|\mu_s| + 3r\delta}{3r\delta} L = 12r + |\mu_s| - 2p - 6(m + 1). \] (28)

Using the node conditions—specifically the bound on \( r \) from condition (d)—we can show the coefficient of \( L \) is negative since
\[ \eta'(-1)|\mu_s| + 3r\delta = \left( -2p(a + 1) + \frac{6r\delta}{|\mu_s|} \right) \mu_s + 3r\delta \]
\[ = 9r\delta - 2p(a + 1)|\mu_s| \]
\[ < \frac{3}{2}p^2(a + 1)^2 - 2p^2(a + 1)^2 - 2p(a + 1)\sqrt{p^2(a + 1)^2 - 6r\delta} \]
\[ < 0. \]

Since we are looking for a lower bound on the edge of the singular funnel, we want the strong canard to lie above its tangent line at the node. So, we want \( L > 0 \), which happens when the right-hand side of (28) is negative. That is, we want
\[ 12r + |\mu_s| - 2p - 6(m + 1) < 0. \] (29)

Using the bound for \( r \) again as well as the estimate \( |\mu_s| < 2p(a + 1) \), we see that (29) will be true if
\[ \frac{2p^2(a + 1)^2}{\delta} + 2pa - 6(m + 1) < 0. \] (30)

Therefore condition (e) implies that \( \gamma_s \) lies above the line \( z - m_s(x + 1) = z_- \) near the node.

Next, we want to show that it remains above the line moving away from \( L^- \) in reverse time. To do so we consider the vector field on lines of the form
\[ C = z - m_s x. \]
In particular, we look for conditions such that

\[ \dot{C}|_{C=z_-} \leq 0. \]  

(31)

When this happens, \( \gamma_s \) must be repelled away above the line in reverse time. Obviously, \( \dot{C} = 0 \) at the node \((-1, z_-)\). When

\[ p(a + 1) > 2, \]

then \( \dot{C}|_{C=z_-} \) is increasing as a function of \( x \) and the condition in (31) is satisfied. Thus conditions (e) and (f) together ensure that \( \gamma_s \) lies above the line \( z - m_s(x + 1) = z_- \) on \( M^-_A \).

We define \( z_* \) to be the intersection of the line \( x = -2 \) (i.e. \( P(L^+) \)) with the linear approximation of the funnel, \( z - m_s(x + 1) = z_- \) as shown in Figure 7. Lemma 3 ensures that \( z_* \) lies in the interior of the funnel. As we construct the singular periodic orbit, \( z_* \) provides a target for trajectories returning from \( M^+_A \).

3.6. Singular Periodic Orbit

We now seek conditions so that a singular orbit leaves the folded node, lands on \( M^+_A \) along \( P(L^-) \), follows a trajectory of the reduced problem towards \( L^+ \), crosses \( L^+ \) transversely, and returns to \( M^-_A \) on \( P(L^+) \) below \( z_* \). Singularities on \( L^+ \) and \( M^+_A \) will play a major role in determining conditions that guarantee the existence of the singular periodic orbit.

We will define \( z_+ \) to be the \( z \) coordinate of the folded singularity on \( L^+ \), so

\[ z_+ = (m + 1)h(1) + \lambda - f(1) \]
\[ = (m + 1)(k - 2) + \lambda - p(1 - a)^2 + b. \]

If \( z_+ \) lies above the \( z \) nullcline, then there will be a region where trajectories cross \( L^+ \) transversely as depicted in Figure 8.

**Lemma 4.** Let equations (13) - (15) satisfy the conditions of Lemmas 2 and 3. Furthermore, assume \( \delta < 4 \) and

(g) \( 4(ap - m) - \delta > 0 \)

(h) \( \Delta_\delta(a, p, m) < 0, \)

where

\[ \Delta_\delta(a, p, m) = p^2(-3m + 2ap)^2 - 4m(-3m + 2ap)^3 \]
\[ + 4p^3(-\delta - 2m + p + 2ap) - 18mp(-3m + 2ap)(-\delta - 2m + p + 2ap) \]
\[ - 27m^2(-\delta - 2m + p + 2ap)^2. \]  

(32)

(33)

Then the singular orbit from the folded node will land on \( P(L^-) \subset M^-_A \), follow a trajectory of the reduced problem (23), and cross the fold \( L^+ \).

**Remark 3.** The condition that \( \delta < 4 \) will be replaced with a stricter condition in Lemma 5 to ensure that the singular orbit returns to the funnel.
Figure 8: Position of nullclines in the singular limit when $a = 0.8$, $p = 3$, $b = 2.1$, $k = 4$, $r = 1$, $m = 1$, and $\lambda = 1$. The blue curve denotes the $x$ nullcline, and the black curve denotes the $z$ nullcline. The blue point is the landing point of the singular orbit from the node. The region $R$ between the nullclines on $M^+_A$ is locally positively invariant.

**Proof.** The intersection of the $z$ nullcline with $L^+$ occurs at $z = h(1) + \lambda$. Therefore, the region $R$ between the nullclines on $M^+_A$ will be locally positively invariant if

$$z_+ = (m + 1)h(1) + \lambda - f(1) > h(1) + \lambda,$$

which happens if and only if

$$0 < mh(1) - f(1) \quad \iff \quad 0 < m(k - 2) - p(1 - a)^2 + b \quad \iff \quad 0 < 4(ap - m) - \delta.$$

Thus condition (g) gives us that the nullclines are aligned as in Figure 8 along $L^+$, and the positively invariant region exists. Any trajectory that enters $R$ can only escape by crossing $L^+$. Next, we show that the singular orbit from the node enters $R$.

The assumption that $\delta < 4$ ensures that $z_- > h(1) + \lambda$. This is because the fast fiber from the folded node on $L^-$ lands on $P(L^-) \subset M^+_A$ exactly the distance $\delta$ below the $z$ nullcline. At the landing point (denoted by a blue dot in Figure 8) the vector field of (23) points up and to the left. If the $x$ nullcline lies above the $z$ nullcline, then the trajectory will continue up and to the left until it enters $R$. Condition (g) implies that the $x$ nullcline lies above the $z$ nullcline at the fold. Thus, the only way for the nullclines to switch their orientation is for them to intersect, creating a true equilibrium of (23). The nullclines intersect wherever the curves $z = h(x) + \lambda$ and $z = (m + 1)h(x) + \lambda - f(x)$ intersect. That is, intersections occur whenever

$$mh(x) - f(x) = 0.$$
Note that \( mh(x) - f(x) \) is a cubic. Therefore, the number of zeroes of \( mh(x) - f(x) \) is determined by the cubic discriminant, which is precisely the quantity \( \Delta \).

If \( \Delta_\delta < 0 \) there is only one intersection, but if \( \Delta_\delta > 0 \) there are three. Condition (c) implies the \( x \) nullcline lies below the \( z \) nullcline on \( L^- \) (i.e. where \( x = -1 \)), and condition (g) implies the \( x \) nullcline lies above the \( z \) nullcline on \( L^+ \) (i.e. where \( x = +1 \)). By the Intermediate Value Theorem, the nullclines will intersect for some \( x \) such that \(-1 < x < 1\). Therefore, the conditions (g) and (h) prevent there from being an intersection on either stable branch of \( M_0 \). This implies a singular trajectory through the folded node will cross \( L^+ \).

**Remark 4.** In fact, the condition \( \delta > 0 \) precludes true equilibria on \( M^-_A \). This can be seen by comparing the slopes of the \( x \) and \( z \) nullclines on \( M^-_A \). The \( x \) nullcline is the curve \( z = (m + 1)h(x) - \lambda - f(x) \), so it has slope

\[
\frac{dz}{dx} = (m + 1)h'(x) - f'(x) = (m + 1)h'(x) - 2p(x - a) > (m + 1)h'(x),
\]

since \( x \leq -1 \) on \( M^-_A \). Meanwhile, the \( z \) nullcline is given by the equation \( z = h(x) + \lambda \) which has slope

\[
\frac{dz}{dx} = h'(x).
\]

Since an equilibrium is precisely the intersection of these curves, any equilibrium on \( M^-_A \) will result in the \( x \) nullcline crossing the fold above the \( z \) nullcline, implying \( \delta < 0 \).

Lemma 4 allows for the possibility that the folded singularity \((1, z_+)\) is also a folded node. If we consider the Jacobian at the point \((1, z_+)\), we see that condition (g) implies \( \det(J(1, z_+)) > 0 \). Therefore, the stability of the folded singularity depends on \( f'(1) \). To exclude the possibility of SAOs along \( L^+ \), we want to avoid the case where \((1, z_+)\) is a stable folded node. If \( f'(1) > 0 \), then the folded singularity will be unstable. Requiring \( f'(-1) < 0 < f'(1) \) implies that \( p > 0 \) and \(-1 < a < 1\). We update condition (b) from Lemma 2 accordingly, so we now have

\[
(b) \quad -1 < a < 1.
\]

Finally, we need to find conditions so that the singular trajectory from the folded node returns to the funnel. This will show that we in fact have a singular periodic orbit.

**Lemma 5.** Let equations (13)-(15) satisfy the conditions (a)-(h) from Lemmas 2-4. Additionally, suppose the equations satisfy

\[
(i) \quad 4(m + 4) - 5ap - p > 0.
\]

Then there is a singular periodic orbit \( \Gamma \).

**Proof.** By Lemma 2, we know the system will have a folded node singularity. Let \( \Gamma \) be the singular trajectory consisting of the fast fiber of the layer problem from the singular node to \( P(L^-) \). By Lemma 4, we know that the trajectory will follow the slow flow on \( M^-_A \) until it crosses \( L_+ \). Furthermore, we know that \( z_+ \) is an upper bound on the \( z \) coordinate of the intersection. If \( z_+ < z_\ast \), then \( \Gamma \) will land in the singular funnel upon leaving \( L^+ \). Direct calculation shows that \( z_+ < z_\ast \) precisely when \( 4(m + 4) - 5ap - p > 0 \).
Figure 9: MMO orbit for $\epsilon = 0.001$, $a = 0.91$, $p = 1.05$, $b = 0.31$, $k = 2.2$, $r = 0.3$, $\lambda = 1$, and $m = 0.6$. With these parameters $\delta = 1$.

3.7. Main Result

**Theorem 6.** Suppose the parameters of the system (13)-(15) satisfy the conditions (a)-(i). Then, for $\epsilon$ sufficiently small, the system will have a stable periodic orbit of MMO-type $1^s$ for some $s > 0$.

**Proof.** Lemmas 2-5 show that these conditions satisfy the assumptions of Theorem 1. Figure 10 depicts a portion of parameter space that satisfies conditions (a)-(i) in Theorem 6. These conditions place restrictions on $a$, $p$, $m$, and $r$ explicitly, as well as $b$ and $k$ through the restrictions on $\delta$. However, there are no restrictions on $\lambda$. Additionally, Figure 9 shows the time series for $x$ for a trajectory satisfying the conditions of Theorem 6.

3.8. Extending the Parameter Regime

While the conditions (a)-(i) in Theorem 6 are sufficient, they are not all necessary conditions for the model to exhibit MMOs. In fact, they are rather strict. This is a direct consequence of linearly approximating the funnel to obtain conditions analytically. Figure 11 depicts the portion of phase space satisfying only the conditions of Theorem 6 that do not relate to the linear approximation of the funnel. However, not all parameters from the region pictured in Figure 11 will produce MMO orbits.

The stable periodic orbits (of some MMO type) outside of the parameter regime described by Theorem 6, such as the one in Figure 6, can be much more complicated as a result of the return mechanism projecting the singular periodic orbit closer to the boundary of the funnel (i.e., closer to the strong canard $\gamma_s$). The behavior in this regime is also described by Brøns et al in [2]. For the parameters that generate the orbit in Figure 6, $\mu \approx 0.1010$ and $s = 5$. However, the MMO signature for the orbit is $1^2$. This is because the return mechanism sends the trajectory near the boundary of the funnel in the singular limit.

4. Discussion

We have found sufficient conditions such that the system (13)-(15) has a stable periodic orbit with MMO signature $1^s$. To our knowledge, this is the first climate-based model that has been analyzed to demonstrate MMOs. The dimensionless model is a variant of the Koper model with
Figure 10: Parameters in $apm$-space for $\delta = 1.3$ that satisfy conditions (a)-(i) from Theorem 6.
Figure 11: Parameters in $apm$-space for $\delta = 1.3$ that satisfy conditions (a)-(d), (g), and (h) from Theorem 6.
(a) Attracting periodic orbit in the 3D phase space.

(b) The attracting periodic orbit shown with the critical manifold.

(c) Model output for $x$ for the trajectory in Figure 12a.

Figure 12: MMOs for $\epsilon = 0.1$, $a = 0.8$, $p = 3$, $b = 2.1$, $k = 4$, $r = 1$, $m = 1$, and $\lambda = 1$. 
an added nonlinearity. As with the standard Koper model, the model has an ‘S’-shaped critical manifold and a parameter regime with both a folded node and global return mechanism. Although the additional nonlinearity in the model does not factor into obtaining a folded node, nonlinear effects play a significant role in determining the shape of the funnel, and consequently the return mechanism. From a mathematical standpoint, it is significant that the additional nonlinearity does not destroy the functionality of the model to produce an MMO pattern.

We are able to find the conditions in Theorem 6 analytically, which is a rarity for MMO problems. Although it is nice to have an analytical proof, the approach excludes a significant region of parameter space where MMOs can be found. Numerically approximating the strong canard, the standard practice for demonstrating MMOs, helps provide a more complete picture of the parameter regime that produces MMOs. The method relies on varying one parameter at a time, making it difficult to actually plot the complete region.

The time series in Figures 1 and 12c are qualitatively similar in that they both contain large oscillations followed by a series of smaller amplitude oscillations. Note that $\epsilon = 0.1$ in Figure 12 is not truly small. In Figures 4 and 6 we plotted analogous trajectories with smaller $\epsilon$ (keeping the other parameters fixed). Figure 5 depicts the singular orbit for all of these examples. We focus on the “nice” MMO in Figure 12 because of its similarity to Figure 1. Since $\epsilon$ is larger, we are able to see the small amplitude oscillations clearly.

The model can give us some insight about the climate system. The physical implication of the requirement that $-1 < a < 1$ is that CO$_2$ drawdown due to terrestrial mechanisms is most efficient at a temperature somewhere between the stable glacial and interglacial states. Through the requirements on $\delta$ we learn about the relationship between $b$, $m$, and $k$. This relates the amount of CO$_2$ removed from the atmosphere when the planet is most efficient at doing so ($b$), the ratio of the timescales of the land-atmosphere carbon flux to that of the ocean-atmosphere exchange ($m$), and the minimum/maximum values of atmospheric carbon ($k$). Finally, $r$ tells us something about the proportion of carbon in the atmosphere to carbon in the ocean required for the ocean to switch from absorbing to outgassing. If $r$ is large, we will no longer have a folded node. It may be the case that $r \ll 1$, which puts us near the folded saddle-node limit and allows for more complicated behavior. Some simulations with $r \ll 1$ are shown in Figure 13.

The analysis required to show MMOs due to a folded node assumes a separation of time scales and (at least) two slow variables. As mentioned in the introduction and Section 2, it is often difficult to determine exactly which parameters are small enough to perform this analysis. Here we rely on the wisdom of climate scientists. It may be that changes in atmospheric greenhouse gases happen on a similar timescale to temperature. Figure 12 depicts the case where there is only a
marginal time-scale separation and we still see MMOs.

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