Is randomness near a black hole key for thermalization of its horizon?

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We introduce a possible theoretical model to understand the thermalization of black hole horizon. The near horizon Hamiltonian for a massless, chargeless particle is $xp$ type. This model possess the properties of randomness and can be chaotic. The particle in turn finds the horizon thermal when it interacts with it. We explicitly show this in the Schrodinger as well as in Heisenberg pictures by taking into account the time evolution of the system under this Hamiltonian. Hence we postulate that existing randomness near the horizon can be one of the potential candidates for explaining the black hole thermalization.

I. INTRODUCTION

Quantum mechanically black holes (BH) are thermal objects [1–3]. So far this is understood as a phenomena due to fact that the notion of uniqueness of vacuum does not comply with curved spacetimes. Therefore one observer’s vacuum may appear to be full of particles with respect to others. For instance the static observer at the asymptotic infinity finds particle in the vacuum of a freely falling observer, generally known as Hawking radiation. The horizon temperature $T$ comes out to be proportional to the BH surface gravity $κ$ [1, 2]. This event, since its prediction, has been a key to emerge as well as understand BH information paradox [4] and people thinks that the Hawking evaporated particles may carry the information of infalling objects inside the BH. Investigation so far on the basis of information theory created such possibility, but additionally develops several favourable as well as counter arguments starting from complementarity proposal [5] to Hayden-Preskill “scrambling time” [6, 7]. Till now we are far from the goal. We feel that the failure to achieve the complete success lies in the heart of the “lack of full (theoretical) understanding about the thermalization of horizon”.

In this paper, we show that the question of thermalization of horizon can be answered in a model, build out from a massless particle moving very near to the horizon. The Hamiltonian in near horizon regime for a massless and chargeless outgoing particle, moving in static, spherically symmetric (SSS) BH spacetime, is given by

$$H = κxp,$$

where $x = r − r_H$ with $r_H$ is the location of the horizon and $p$ is the momentum of particle corresponding to radial coordinate $r$. This was obtained in Eddington-Finkelstein (EF) outgoing null coordinates. In the above, the path has been chosen to be along the normal to null hypersurface, given by Eddington null coordinate $u = constant$, where $u = t − r^*$ with $r^*$ is the well known tortoise coordinate (see [8] for details on construction of (1)). The same can also be obtained in Painleve coordinates for SSS BH as well as Kerr BH, considering a particular trajectory [9–11]. Recently we explicitly showed that the trajectory of the particle feels a “local instability” which in turn causes temperature to the horizon at the semi-classical level [8, 11, 12].

Having this indication, we here investigate this model in a pure quantum mechanical way, particularly the time evolution of a wave function under (1). We observe that the later time wave function possess thermal features and $T$ is given by that of Hawking expression. This prediction is completely unexpected and new since the initial wave function is taken to be non-thermal and it is under unitary time evolution. The same has also been predicted in Heisenberg’s picture as well by studying the nature of out-of-time order correlation (OTOC) between the position and momentum of the particle. Thus we feel that this particular model has potential ability to reveal the underlying physical reason (theory) to describe the thermalization of BH horizon.

In search of the physical reason of the above observation we find from literature that the counting function of Riemann zeros (RZ) is identical in form with the same for $xp$ Hamiltonian eigenvalues [24, 25]. On the other hand statistics of RZ distribution is given by that of eigenvalues of a random matrix, which satisfies Gaussian unitary ensemble (GUE) distribution [26, 27]. To follow GUE distribution the Hamiltonian must be chaotic and devoid of time reversal symmetry [28]. Incidentally $xp$ Hamiltonian satisfies both these conditions [25] and hence it lies under the category of random matrix. We feel that this local randomness and chaotic nature is responsible for such thermalization and thereby can a potential physical reason of such phenomenon. More on this will come later.

We also observe that the present model can demonstrate the possibility of retrieving information from Hawking radiation. We find that to recover one bit of information (assuming one particle carries it) the minimum time required to be that given by Hayden and Preskill [6]. If so, then each bit must contains energy which is similar to standard equipartition of energy among each degrees of freedom (DOF). We also estimate the maximum number of bits (or DOF) a BH can emit by its later time wave function possesses thermal features and $T$ is given by that of Hawking expression. This prediction is completely unexpected and new since the initial wave function is taken to be non-thermal and it is under unitary time evolution. The same has also been predicted in Heisenberg’s picture as well by studying the nature of out-of-time order correlation (OTOC) between the position and momentum of the particle. Thus we feel that this particular model has potential ability to reveal the underlying physical reason (theory) to describe the thermalization of BH horizon.

So in this paper a possible theory is being proposed in
the context of randomness to understand the thermalization of horizon, which, as far as we know, was never been mentioned. Having this present new way of looking at the horizon thermality in a unified manner we feel that the present study and results have potential to achieve deeper understanding of this subject and may help in resolving the long standing information paradox problem, particularly in the context of information theory approach. Let us now proceed towards the calculation in support of our present claims and what we wanted to mean by thermalization.

II. TIME EVOLUTION OF A STATE AND THERMALIZATION

In order to perform the quantum mechanical analysis of (1), we first make it Hermitian. For this, using Weyl ordering, we express it in the following operator form:

\[ \hat{H} = \frac{\kappa}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) = \kappa \left( \hat{x}\hat{p} - \frac{i\hbar}{2} \right), \]  

where in the last step the commutator \([\hat{x}, \hat{p}] = i\hbar\) has been applied. Considering the unitary time evolution of an arbitrary quantum state \(\Psi(x,0)\), one can find the later time state as [31]

\[ \Psi(x,t) = e^{-\frac{\kappa}{2}t} \Psi(xe^{-\kappa t},0). \]  

Note that the later time quantum state suffers a dilation and this can lead to a mixing of positive and negative momenta in the evolved state, even though initial state is in particular momentum (positive or negative, not both). Due to this mixing the state becomes thermal in nature with respect to a relevant monochromatic plane wave. We shall now investigate this possibility for two simple one dimensional examples: (a) a plane wave and (b) a Gaussian wave packet.

(a) Plane wave: The initial wave function is forward propagating plane wave \(\Psi(x,0) \sim e^{ik_0x}\) with the momentum is defined as \(p_0 = \hbar k_0\). Later time this will evolve according to (3):

\[ \Psi(x,t) \sim e^{-\frac{\kappa}{2}t} \exp[ik_0xe^{-\kappa t}] . \]  

We shall now investigate whether the later time wave function (4) is thermal in nature. This can be shown by calculating the power spectrum of the wave, which is determined by taking the Fourier transform of the following form [32] (a justification, following [33, 34] is being presented in Appendix A):

\[ f(-\omega) \sim \int_{-\infty}^{+\infty} dt e^{-i\omega t} e^{-\frac{\kappa}{2}t} \exp[ik_0xe^{-\kappa t}] = \frac{1}{\kappa \sqrt{k_0\pi}} e^{i\left(\frac{\omega}{\kappa} - \frac{1}{2} \ln(k_0\pi)\right) - \frac{k_0}{2\pi} \Gamma\left(\frac{1}{2} + \frac{i\omega}{\kappa}\right)} \]  

where \(\omega > 0\). Utilizing the identity \(\Gamma\left(\frac{1}{2} + ia\right)^2 = \pi/cosh(\pi a)\) one finds

\[ |f(-\omega)|^2 \sim \frac{2}{\kappa^2k_0\pi} \frac{1}{e^{\frac{2\pi\omega}{\kappa}} + 1} , \]  

which is thermal in nature with the temperature is given by the Hawking expression

\[ T = \frac{\hbar\kappa}{2\pi} . \]  

In the above energy \(E = \hbar\omega\) has been taken.

It may be pointed out that expression (6) is considered to be thermal in the sense of usual Fermi distribution (further justification can be followed from [32–34]). Our obtained result is identical to Fermi distribution upto a pre-factor. The appearance of pre-factor is not new here. For black hole system, in original calculation, this usually arises due to the back scattering of the emitted particles by the effective potential which exists in between near horizon regime and infinity. Such a modification in the ideal thermal expression is usually known as grey-body factor. Therefore upto this factor one takes the result as thermal. In that sense we call (6) as thermal. In later part of this section and in the concluding section we will again comment on the realization of thermalization from (6).

(b) Gaussian wave packet: Consider the initial wave packet propagating along positive \(x\)-direction as

\[ \Psi(x,0) \sim e^{ik_0x - \frac{x^2}{2\sigma^2}} . \]  

Then at time \(t\) it will evolve to

\[ \Psi(x,t) \sim e^{-\frac{\kappa}{2}t} \exp \left[ ik_0x e^{-\kappa t} - \frac{x^2e^{-2\kappa t}}{2\sigma^2} \right] . \]  

Fourier transformation of (9) yields

\[ f(-\omega) = \frac{1}{\kappa} 2^{-\frac{\kappa}{2} + \frac{i\omega}{2\kappa}} \left( \frac{x^2}{d^2} \right)^{-\frac{\kappa}{2} + \frac{i\omega}{2\kappa}} \times \left\{ \frac{x}{d^2} \Gamma\left(\kappa + 2i\omega\right)^{1} F_1\left[\kappa + 2i\omega; \frac{1}{4\kappa}; -\frac{d^2k_0^2}{2}\right] 
\right.

\[ + \frac{i\omega}{2\kappa} \Gamma\left(\kappa + 3i\omega; \frac{3}{4}; -\frac{d^2k_0^2}{2}\right) \right\} . \]  

where \(1_F(a; b; c)\) is the Kummer confluent hypergeometric function. Like earlier, one needs to compute \(|f(-\omega)|^2\) from the above expression. But this will not provide any suitable analytical form to investigate. Therefore whether the above one corresponds to thermal distribution, we plot \(\omega^2|f(-\omega)|^2\) as a function of \(\omega\) in Fig. 1. It is found the distribution is thermal in nature in the sense that the feature is exactly similar to usual Fermi distribution (more precisely to \(\omega^2\) times the Fermi occupation number). An analytical expression for temperature is not possible to obtain here. But a numerical comparison between these two must yields the required expression.
In this scenario we at least observed that the later time wave function occupies thermal property with respect to monochromatic waves like $e^{-i\omega t}$. It confirms that the time evaluated wave packet turns out to be thermal with respect to plane monochromatic wave, although it was not initially.

In the above discussion it must be noted that the later time wave functions (4) and (9) have been constructed by using unitary time evolution under the Hamiltonian (1). Surprisingly the final wave functions become thermal, even if the initial is not. Since the time evolution is done by the unitary transformation, we feel that such is completely due to the nature of our particular Hamiltonian (1) which has been provided by the horizon in its vicinity. Hence we state that the horizon shows its presence by thermalizing a quantum system and in reverse makes the system to feel BH as a thermal object. This is a very non-trivial observation and reveals more compare to the original field theoretical calculation based on the idea of non-uniqueness of vacuum in curved spacetime.

We just saw that the later time wave function yields non-vanishing Bogoliubov coefficient like quantity $f(\omega)$ (called as mixing coefficient) which quantifies the mixing of positive and negative frequencies (or momentum here) instantaneous monochromatic plane waves (see Appendix A for justification of this comment). In this sense we call this thermalization mechanism as relative thermalization with respect to plane monochromatic waves. A more direct evidence for demanding the present one as a case of relative thermalization is being given in Appendix B. Here by calculating Bogoliubov coefficients between the eigenstates of Hamiltonian for a free particle (e.g. in absence of gravity which can be chosen in the asymptotically infinite radial distance) and those of Hamiltonian (1) we show that the eigenstates of $\kappa \sigma \pi$ Hamiltonian appear to be thermal in nature with respect to the other one. Later we will again make comments on the issue of thermalization.

### III. THERMALIZATION FROM “HORIZON AS A SCRAMBLER”

Now we will show that the similar thermalization can be addressed at the operator evolution level. The OTOC $A(t) = \langle |\langle \hat{x}(t), \hat{p}(0) \rangle|^2 | \rangle$ can be evaluated as follows. $\hat{x}(t)$ is obtained by using Heisenberg’s equation of motion:

$$\frac{d\hat{x}(t)}{dt} = \frac{1}{i\hbar} [\hat{x}(t), \hat{H}] .$$

(11)

Using the form for $\hat{H}$ from the first equality of (2) and the commutator $[\hat{x}, \hat{p}] = i\hbar$ in (11) one finds $\hat{x}(t) = \hat{x}(0)e^{\kappa t}$. Hence the OTOC is given by

$$A(t) = \hbar^2 e^{2\kappa t} ,$$

(12)

where we have used the fact that $| \rangle$ is a normalised state. Note that OTOC increases exponentially with time. This shows that BH horizon acts as scrambler and the sensitivity is controlled by Lyapunov exponent $\lambda_L = \kappa$. In this context it may be mentioned that in a many body system the upper limit of $\lambda_L$ is controlled by the system’s temperature \([35]: \lambda_L \leq (2\pi T)/\hbar\). In this case the dual BH geometry incorporates temperature, given by the saturation value. This led to conjecture that the system with positive $\lambda_L$ incorporates temperature in the semi-classical regime \([15]\). Therefore our above analysis provides the concept of temperature which is again given by (7). It shows that the horizon acts as scrambler on the particle motion and consequently generates temperature at the quantum (semi-classical) level. We feel that such feature of horizon is responsible to perceive BH as a thermal object.

This scrambling leads to the distribution of the system information among all possible eigenstates of it. After certain time scale, known as scrambling time, the distribution will be such that the average value of any observable is always near to the thermal equilibrium (microcanonical ensamble average) value. This sometimes known as quantum ergodicity. This is the main reason why the horizon incorporates thermality. We will shortly discuss this more elaborately. The time scale is determined by setting $A(t) \sim \mathcal{O}(1)$. From (12) we find the scrambling time as

$$t_s \sim \left| \frac{1}{2\kappa} \ln \frac{1}{\hbar^2} \right| = \left| \frac{1}{\kappa} \ln \hbar \right| .$$

(13)

This is sometime called as Ehrenfest time. Classical to quantum correspondence principle tells that $t_s \rightarrow \infty$ as $\hbar \rightarrow 0$. This is because to obtain classical ergodicity one needs to wait for sufficiently long time. This is exactly happening for the above one as well and hence our predicted scrambling time is consistent with the well celebrated correspondence principle. Now for Schwarzschild BH we have $\kappa = 1/(4M)$, where $M$ is the mass of BH and so $t_s \sim M/\hbar$. The above can also be expressed in terms of $T$ as $t_s \sim \hbar/2\pi T$.
So far we observed that relative thermalization of horizon is occurring due to evolution of system under the Hamiltonian (1) (either in Schrodinger or in Heisenberg picture). Such is due to the spacial characteristic feature of the distribution of eigenvalues of (1). The general observation is – counting function of function μ(x) and the same for xp Hamiltonian eigenvalues are identical \([24, 25]\). On the other side the spacing distribution of RZ is given by that of eigenvalues of a random matrix, especially GUE distribution \([26, 27]\). In order to follow GUE distribution the Hamiltonian must be chaotic and devoid of time reversal symmetry \([28]\). As argued in \([25]\), xp Hamiltonian is quite a good choice to satisfy these conditions and hence it lies under the category of random matrix. Since the initial momentum space. On other side at \(t > 0\) region. In this sense it covers less “DOF” compared to its actual size at \(t = 0\). Moreover, the probability density \(|Ω(x, t)|^2 \sim e^{-κt}\), which can be interpreted as diagonal element of density matrix in position basis (i.e. \(ρ_{xx} = ⟨x|Ω⟩⟨Ω|x⟩\)), is suppressed for \(t \neq 0\) and leading to more and more “loss of information” for this particular eigenbasis as \(t\) increases. Moreover here the whole system is composed of two objects: BH and particle; and they are interacting through (1). Note that the particle wave function (e.g. the plane wave here) is evolved under this Hamiltonian, not by its own free particle Hamiltonian. Additionally, the particle is not isolated from the environment (like black hole here) as it is in constant interaction with BH. This BH provides the local randomness in the particle’s trajectory and in turn at the quantum level the particle finds the horizon as thermal object. This explains the physical reason why both levels of evolution – wave function as well as operator – leads to (relative) thermalization and thereby providing a strong mechanism for thermal behaviour of horizon.

Before moving forward we want to show that the eigenstates of our Hamiltonian (1) are indeed thermal in nature with respect to the arbitrary state of a system. Choosing the basis as eigenstates of (1), an arbitrary state can be expanded as \(|Φ⟩ = \sum_E c_E |E⟩\). Here \(|c_E|^2 = |⟨E|Φ⟩|^2\) is the probability of finding \(|Φ⟩\) in eigenstate \(|E⟩\) with energy \(E\). Using position kets \(|x⟩\) as basis one finds

\[
c_E = \int_0^{∞} dx Φ(x)u_E^*(x), \tag{14}
\]

where \(u_E(x)\) is the normalized eigenfunction of (1) in position representation, which is given by (see Section VII of [8])

\[
u_E(x) = \frac{1}{\sqrt{2πℏκ}}(x)^{-\frac{1}{2}} e^{\frac{−iE}{ℏκ}}. \tag{15}
\]

To perform the integration in (14) analytically, we choose a simple test wave function \(Φ(x) \sim e^{−ik_0x}\). Using all these in (14) one easily obtains

\[
|c_E|^2 = \frac{1}{ℏκk_0} \frac{1}{e^{\frac{πE}{ℏκ}} + 1}. \tag{16}
\]

Now for the ideal Fermi particles we know that the probability of finding \(n\) number of particles in state \(E\) is given by (see section 6.3 of [36])

\[
P_E(n) = \begin{cases} 1− < n_E >, & \text{for } n = 0 \\ < n_E >, & \text{for } n = 1, \end{cases} \tag{17}
\]

where

\[
< n_E > = \frac{1}{e^{\frac{E}{κ}} + 1}. \tag{18}
\]

This shows that the probability distribution of each eigenstates of our Hamiltonian (1) with respect to the test wave function is thermal (up to a pre-factor) if one compares with the probability of finding \(n = 1\) number of particles in energy state \(E\) for Fermions. The role of time evolution, as done earlier, was just to reveal this thermal property of each energy eigenstates.

Below, in this framework, we will discuss the possibility of information retrieval from Hawking radiation.

IV. INFORMATION RETRIEVING FROM HAWKING RADIATION

The general picture of Hawking radiation is described in the following way. Near the horizon there are always particle-antiparticle pairs. One of the member of this pairs, namely anti-particle, remains within BH while the other one (the particle) escapes from the horizon barrier to reach at infinity. Escape through the horizon barrier is possible only at quantum level and so the Hawking radiation is a quantum phenomenon. Here we consider that Hawking radiation from BH occurs only when the particle and antiparticle, within a pair, are at least Planck length \(l_p\) order distant apart. Here \(l_p\) is taken as it is the possible minimum physical distance. By this we meant the antiparticle remains within the horizon and the particle is visible as Hawking radiated one when it travels at least \(l_p\) from the horizon in radial direction. Below this, it is still a part of BH.

Now just after coming out of the horizon, particle will see the horizon and so its motion will be dominated by Hamiltonian (1) if it is massless and chargeless (like photon) and as long as it is very near to the horizon. In this
case the radial trajectory is given by \( x = \frac{1}{\kappa} e^{\kappa t} \) [8–11]. Note that, as stated earlier, it has been obtained in EF and Painlevé coordinates. The SSS BH metric in EF is given by (see [8] for details)

\[
ds^2 = -f(r)dt^2 + 2(1 - f(r))dt\,dr + (2 - f(r))dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(19)

\( f(r_H) = 0 \) determines the horizon location and near to this we have \( f(r) \approx 2\kappa x \). Then the physical radial distance is given by \( r_{phy} = \int_{r_H}^r \sqrt{2 - f(r)} dr \), which at the leading order in the near horizon regime yields

\[
r_{phy} \sim \int_0^x dx(1 - \frac{\kappa x}{2}) \approx x.
\]  

(20)

Similarly in Painlevé also the physical radial distance is given by (20). Therefore we conclude that the nature of the physical radial distance, in the near horizon regime, characterized by \( x = \frac{1}{\kappa} e^{\kappa t} \). With this we now want to calculate the minimum time taken by the particle to be identified as Hawking particle assuming that the dynamics is driven by (1), even within distance \( t_p \). Since the minimum distance needs to be \( x = t_p \), estimated the minimum time is then given by

\[
t_{min} \sim \frac{1}{\kappa} \ln(l_p\kappa).
\]  

(21)

We may call this as the minimum time one needs to spend near the horizon to get one bit of information in the form of massless particle as Hawking radiation. Interestingly, for Schwarzschild BH we have \( t_{min} \sim M \ln(M/l_p) \) which is identical to Hayden-Preskill time [6].

Now let us estimate how much minimum energy the observer will register within time \( t_{min} \), who is detecting the Hawking particle. Uncertainty relation between energy and time will give this estimation. For each particle this will be given by

\[
E_{min} \sim \frac{\hbar}{t_{min}} = \frac{\hbar\kappa}{\ln(l_p\kappa)} = \frac{2\pi T}{\ln(l_p\kappa)}.
\]  

(22)

It implies that each radiated particle carries above minimum amount of energy or in other words, at least this energy is required to create one Hawking particle. In this situation if we assume that the whole BH of energy \( E \) is radiated by this type of particle, then the maximum number emitted particles will be

\[
N_{max} \sim \frac{E}{E_{min}} = \frac{E}{\frac{2\pi T}{\ln(l_p\kappa)}}.
\]  

(23)

Next use of Gibbs-Duhem like relation \( 2TS = E \) for BH, known as Smarr formula [37–40], yields

\[
N_{max} \sim \frac{S}{\pi} \frac{\ln(l_p\kappa)}{\kappa}.
\]  

(24)

which implies that the maximum number of emitted particles is proportional to BH entropy.

Now we postulate that each emitted particle contains one bit information of BH. Then the estimated maximum number of bits, representing BH, is given by (24) and therefore each bit contains energy, given by (22). Note that (24) is proportional to BH entropy which depends on horizon area in Einstein’s gravity. It indicates that all the information of BH are contained on the surface of it. Interestingly, the above one is consistent with an earlier result, given by Padmanabhan [29], to estimate the number of surface DOF of a BH based on equipartition of energy. Then time for complete evaporation is \( t_{min} N_{max} \sim (S/\kappa)(\ln(l_p\kappa))^2/\pi \) which is for Schwarzschild BH turns out to be \( \sim M^3 \). Note that this time scale is of the order that predicted by Page [30].

We conclude this discussion by mentioning the recent developments in Page curve in the context of information paradox problem. Page curve [41, 42] suggests that for the unitarity, initially the entanglement entropy should be equal to Hawking radiation and must increase till a certain time. At this time, known as “Page time”, both black hole and radiation entropies become equal. After that entanglement entropy decreases and always equal to black hole entropy. The reason for this nature of Page curve has been recently elaborately described in [43, 44] in the context of gravitational point of view using Ryu-Takayanagi (RT) formula [45]. In early time the RT surface includes all the interior to form pure state and the early Hawking radiation leads to increase in entanglement entropy. Later time this surface moves near to the horizon and providing the decrease in entanglement entropy. In this portion, i.e. after Page time, to reconstruct one bit information which was thrown to black hole one needs to wait a time given by scrambling time as predicted earlier. The present discussion, although being based on very basic semi-classical approach, is capable of reproducing several related old results and moreover makes a connection to them with earlier results, done in a completely different angle, like given in (23) and (24). The positive side of our model is it not only provides a way of understanding the horizon thermalization, but also goes beyond that. It may be noted that the number of bits, as estimated in (24), depends on black hole entropy. Therefore we feel that the results obtained in this section, in the light of Page curve, are much more suitable after Page time regime.

V. CONCLUSIONS

In this paper we showed that the time evolved system thermalizes itself under the \( xp \) type Hamiltonian. Since the near horizon massless, chargeless particle follows this model, we concluded that the phenomena of horizon (relative) thermalization can be understood within this model. The temperature is derived to be that of the Hawking expression. We discussed that since \( xp \) Hamiltonian is in random matrix category, its eigenvalues follows the GUE. Therefore we conclude that the horizon
thermalization is a particular case of existing randomness in the near horizon region, thereby we are providing a distinct idea to understand the underlying physical reason for perceiving BH as a thermal object. Note that the obtained spectrum (6) is Fermionic in nature. This may be due to the spacial form of our Hamiltonian which contains first order derivative with respect to $x$ as one writes the Schrodinger type equation for $H \sim xp$. Such character is very similar to Dirac equation for spin half particle. But it may be speculative and hence to get conclusive statement further understanding about the $xp$ Hamiltonian is necessary.

Our Hamiltonian (1) represents a single-particle and integrable system. Therefore, although (1) is random in nature, the obtained thermalization can not be described by Gibbs ensemble (GE). In this regard it may be pointed out that a possible connected mechanism can be based on generalized Gibbs ensemble (GGE). GGE is well applicable for an integrable few-body system, but does not describe a genuine thermalization. But whether the present thermalization can be described by GGE is still illusive at this moment. It needs further investigation and analysis to find a concrete answer in this direction. On the other hand, the thermalization in the present case is being determined by the existing idea for the black holes. Till date this is being investigated by calculating the Bogoliubov coefficients between mode functions for the two observers. It is found that one observer’s positive frequency mode appears to be mixture of positive and negative frequency modes of the other observer. In that case the modulus square of Bogoliubov coefficient $\beta$ takes the form of usual thermal distribution (Bose-Einstein for scalar modes and Fermionic for Dirac mode) upto a grey-body factor. In that case comparison between $|\beta|^2$ with usual one concludes that one observer’s mode is thermal with respect to other one’s and therefore the thermalization of horizon is an observer dependent phenomenon. In the present analysis identical idea has been adopted. Therefore as explained in the main text, we call this thermalization as relative thermalization. Mainly the time evolved wave function by the Hamiltonian (1) or its eigenstates appear to be thermal in nature with respect to certain wave function. Now since (1) has random property, we feel that this randomness is responsible for such mixing and thereby yields relative thermalization. But whether it is a genuine thermalization in the sense of GE is still an open question. We are working in this direction and as far as we know, till date it is not well understood. Hope, on the basis of this analysis, the present model will be able to enlighten this direction.

In addition, within this framework, we also discussed the possibility of information collection from the Hawking radiated particles. Although the analysis is based on certain assumptions, but this heuristic approach is capable of giving important information of BH radiation which are the subject of investigation within the information theory approach [6]. Our estimated time to collect one bit of information for an outside observer, very close to the horizon, nicely matches with that predicted by Hayden and Preskill [6, 7]. Moreover, the minimum energy of each bit is quite consistent with the equipartition of energy (see Eq. (22)). Finally the estimated number of bits in the complete evaporation of a BH came out to be very much identical to the predicted one in literature [29], while the total evaporation time scale matches with Page’s estimation [30].

We now conclude with the statement that we here found a possible underlying physical theory which can explain the thermalization of horizon within the $xp$ Hamiltonian model. This is the first attempt, so far as we know, to explain such phenomenon within in a concrete theoretical background. Moreover, we showed that this model is very effective in explaining various important information about the BH evaporation in a unified way. Hope this present findings will help to understand this paradigm more in future.

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Appendix A: Mixing of positive and negative frequencies

This analysis has a similarity with Bogolyubov coefficient calculation. For details please follow Ref. [34] (also see section 14.3 of [33]). Since every details are there the reader may be requested to go through these refs. These types of analysis, related to black holes, are very well known in literature. A brief analysis is as follows. If the later time wave function can be expressed as mixture of positive and negative frequencies, then we must have

\[\Psi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \alpha(e^{-i\omega t}) + \beta(e^{i\omega t}) \right],\]  

(A1)

with \(\omega > 0\). The modes \(e^{\pm i\omega t}\) are interpreted as instantaneous monochromatic plane waves in a comoving frame with respect to time \(t\). This can also be expressed as (using \(\omega \rightarrow -\omega\) in the second term)

\[\Psi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega)e^{-i\omega t},\]  

(A2)

with the identifications \(\alpha = f(\omega)\) and \(\beta = f(-\omega)\). These \(\alpha\) and \(\beta\) are known as mixing coefficients. These are similar to Bogoliubov coefficients, but not same. Now in the light of (A2) we have

\[\alpha = f(\omega) = \int_{-\infty}^{\infty} dt\Psi(t)e^{i\omega t};\]  

(A3)

\[\beta = f(-\omega) = \int_{-\infty}^{\infty} dt\Psi(t)e^{-i\omega t};\]  

(A4)

Note that if \(\beta\) vanishes, then there is not mixing. Otherwise the mixing of both positive and negative frequency modes is there and in the light of Bogolyubov analysis \(|\beta|^{2}\) can be treated as power spectrum. In our paper we exactly calculated this in Eqs. (6) and (10) and found that the spectrum is thermal in nature, just like what happened in Eq. (39) of Ref. [34]. Under this logic we argue that the later wave function, with respect to instantaneous monochromatic plane wave, possess thermal nature. Therefore we call this as relative thermalization.

Appendix B: Relative thermalization: non-vanishing Bogoliubov coefficient \(\beta\)

The Hamiltonian for a relativistic outgoing massless particle, in absence of black hole (i.e. particle is free), is given by \(H_{f} = p\) and so its outgoing eigenstate is of the form \(\Psi_{k_{0}} = Ne^{ik_{0}x}\), where \(k_{0} = p/\hbar\). Here \(N\) is the normalization constant. On the other hand the eigenstates of Hamiltonian (1) are given by (15). The eigenstate \(\Psi_{k_{0}}\) can now be expressed as a linear combination of the eigenstates (15) in the following form:

\[\Psi_{k_{0}}(x) = \sum_{E} \left( \alpha_{k_{0}E}u_{E}(x) + \beta_{k_{0}E}u_{E}^{*}(x) \right),\]  

(B1)

where \(\alpha_{k_{0}E}\) and \(\beta_{k_{0}E}\) are known as Bogoliubov coefficients. Since \(u_{E}^{*}(x)\) corresponds to negative energy, the non-vanishing value of \(\beta_{k_{0}E}\) yields the mixing of positive and negative energies in \(\Psi_{k_{0}}\). Whether \(u_{E}(x)\) is thermal with respect to \(\Psi_{k_{0}}(x)\) is determined by the value of \(\beta_{k_{0}E}\) (see section 3.2 of [46] for the justification). Therefore we will now calculate this coefficient.

Use of the normalization condition on \(u_{E}(x)\) yields

\[\beta_{k_{0}E} = \int_{0}^{\infty} dx\Psi_{k_{0}}(x)u_{E}(x).\]  

(B2)

Substitution of the respective eigenstates in the above we obtain

\[\beta_{k_{0}E} = \frac{N}{\sqrt{2\pi\hbar k_0}} \int_{0}^{\infty} e^{ik_{0}x}(-\frac{1}{2} + \frac{iE}{\hbar})\]  

\[= \frac{N}{\sqrt{2\pi\hbar k_0}} e^{(\frac{1}{2} + \frac{iE}{\hbar})\ln(-ik_{0})} - \frac{1}{2} \Gamma\left(\frac{1}{2} + \frac{iE}{\hbar}\right)\]  

\[= \frac{N}{\sqrt{2\pi\hbar k_0}} e^{(\frac{1}{2} - \frac{E\ln\hbar}{\hbar})} - \frac{1}{2} \Gamma\left(\frac{1}{2} + \frac{iE}{\hbar}\right).\]  

(B3)
To evaluate the integration we used the following formula

$$\int_0^\infty dx \ x^{s-1} e^{-bx} = e^{-s \ln b} \Gamma(s),$$  \hspace{1cm} (B4)

with \( \text{Re}(b) > 0 \) and \( \text{Re}(s) > 0 \). In order to apply the above formula we identified \( s = (1/2) + i(E/\hbar \kappa) \) and chose \( b = -i k_0 + \epsilon \) with \( \epsilon > 0 \) to make the integration convergent. At the end \( \epsilon \to 0 \) limit has been taken. Then one finds

$$|\beta_{k_0E}|^2 = \frac{N^2}{\hbar k_0 \kappa} \frac{1}{e^\frac{2\pi E}{\hbar \kappa} + 1}.$$  \hspace{1cm} (B5)

Non-vanishing of this value signifies that the mixing has happened and \( u_E \) appears to be thermal with respect to \( \Psi_{k_0} \). One may see that \( u_E \) will feel a temperature given by Hawking expression \( T = \hbar \kappa / 2\pi \). It may be mentioned that \( (B5) \) gives the probability of finding negative energy state \( u_E^\ast \).

Similarly one can find the probability of finding positive energy state \( u_E \) by calculating \( |\alpha_{k_0E}|^2 \). This is found out to be as

$$|\alpha_{k_0E}|^2 = \left| \int_0^\infty dx \Psi_{k_0}(x) u^*_E(x) \right|^2$$

$$= \frac{N^2}{\hbar k_0 \kappa} \frac{e^\frac{2\pi E}{\hbar \kappa}}{e^\frac{2\pi E}{\hbar \kappa} + 1}.$$  \hspace{1cm} (B6)

The ratio \( |\beta_{k_0E}|^2/|\alpha_{k_0E}|^2 \) then provides the information about the temperature [47] seen by the particle with Hamiltonian (1) for energy \( E \). This is given by

$$\frac{|\beta_{k_0E}|^2}{|\alpha_{k_0E}|^2} = e^{-\frac{2\pi E}{\hbar \kappa}}.$$  \hspace{1cm} (B7)

Comparing with the Boltzmann factor one identifies the temperature as \( T = \hbar \kappa / 2\pi \). Note that this has been obtained with respect to state \( \Psi_{k_0} \) and therefore the eigenstate of (1) appears to be thermal when one compares with \( \Psi_{k_0} \). Moreover if the mixing was not there then \( \beta_{k_0E} \) must vanish and in that case the above ratio also vanishes. So here the thermalization is due to the occurrence of this mixing.

Here we found that even if individual eigenstates are not showing any thermal property within their own territory, whereas one state is appeared to be thermal in nature with respect to other state. Therefore we call this mechanism as relative thermalization. This is happening to eigenstates of (1) which incorporates a random property and therefore we argue that such randomness (provided by horizon) may be the underlying reason for horizon appears to be thermal with respect to this particle.