Algebraic treatment of three-body problems

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Abstract

We discuss an algebraic treatment of three-body systems in terms of a $U(7)$ spectrum generating algebra. In particular, we develop the formalism for nonlinear configurations and present an algebraic description of vibrational and rotational excitations of symmetric ($X_3$) and asymmetric tops ($XY_2$ and $XYZ$). The relevant point group symmetry is incorporated exactly.

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1 Introduction

The study of few-body problems has played an important role in many fields of physics and chemistry 1. Over the years accurate methods have been developed to solve the few-body equations. The degree of sophistication required depends on the physical system, i.e. to solve the few-body problem in atomic physics requires a far higher accuracy than in hadronic physics.

In recent years, the development and application of algebraic methods to the many-body problem (e.g. collective excitations in nuclei 2 and molecules 3) has received considerable attention. In spectroscopic studies these algebraic methods provide a powerful tool to study symmetries and selection rules, to classify the basis states, and to calculate matrix elements. In this paper we discuss the application of algebraic methods to the few-body problem. Especially in the area of hadronic physics, which is that of strong interactions at low energies, for which exact solutions of QCD are unavailable, these methods may become very useful. The approach is based on the general criterion 4 to consider \( U(k+1) \) as a spectrum generating algebra for a bound-state problem with \( k \) degrees of freedom, and assigning all states to the symmetric representation \([N]\) of \( U(k+1) \). For the \( k = 5 \) quadrupole degrees of freedom in collective nuclei this led to the introduction of the \( U(6) \) interacting boson model 2 and for the \( k = 3 \) dipole degrees of freedom in diatomic molecules to the \( U(4) \) vibron model 3.

In a three-body system the dynamics is determined by the six degrees of freedom of two relative vectors, which in the algebraic approach leads to a \( U(7) \) spectrum generating algebra. This model was originally introduced for a system of three identical objects 5, 6. In this paper, we develop the general formalism for a system of three objects and discuss, in particular, an algebraic description of vibrational and rotational excitations of symmetric and asymmetric tops. Applications can be found in molecular physics (XYZ, XY\(_2\) and X\(_3\) molecules) and hadronic physics (qqq baryons).

2 Algebraic treatment of a three-body system

The internal motion of a three-body system can be described in terms of the relative Jacobi coordinates, \( \vec{\rho} \) and \( \vec{\lambda} \), which we choose as

\[
\vec{\rho} = \frac{1}{\sqrt{2}} (\vec{r}_1 - \vec{r}_2),
\]

\[
\vec{\lambda} = \frac{1}{\sqrt{m_1^2 + m_2^2 + (m_1 + m_2)^2}} \left[ m_1 \vec{r}_1 + m_2 \vec{r}_2 - (m_1 + m_2) \vec{r}_3 \right].
\]

Here \( m_i \) and \( \vec{r}_i \) denote the mass and coordinate of the \( i \)-th object. For three identical objects with equal masses, Eq. (1) reduces to the Jacobi coordinates used in 5. Instead of a formulation in terms of coordinates and momenta, we use the method of bosonic quantization in which we introduce a dipole boson with angular momentum and parity \( L^P = 1^- \) for each independent relative coordinate, and an
auxiliary scalar boson with $L^p = 0^+$

$$p^\dagger_{\rho,m}, p^\dagger_{\lambda,m}, s^\dagger \quad (m = -1, 0, 1) \ .$$

(2)

The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the total number of bosons $N = n_{\rho} + n_{\lambda} + n_s$ is conserved. This procedure leads to a spectrum generating algebra of $U(7)$ [5, 6]. For a system of interacting bosons the model space is spanned by the symmetric irreducible representation $[N]$ of $U(7)$, which contains the harmonic oscillator shells with $n = n_{\rho} + n_{\lambda} = 0, 1, \ldots, N$. The value of $N$ determines the size of the model space.

In case two or all three objects are identical, the Hamiltonian has to be invariant under the permutation group, $S_2$ or $S_3$, respectively. In the former case, in which we label the identical objects by 1 and 2, the permutation symmetry is determined by the transposition $P(12)$, and in the latter case by both $P(12)$ and the cyclic permutation $P(123)$ [7]. All other permutations can be expressed in terms of these two elementary ones. Algebraically, these operators can be transcribed as

$$P(12) = \exp\left\{ -i \pi \hat{n}_\rho \right\} ,$$
$$P(123) = \exp\left\{ -i \frac{2\pi}{3} \hat{F}_2 \right\} ,$$

(3)

with

$$\hat{n}_\rho = \sum_m p^\dagger_{\rho,m} p_{\rho,m} ,$$
$$\hat{F}_2 = -i \sum_m \left( p^\dagger_{\rho,m} p_{\lambda,m} - p^\dagger_{\lambda,m} p_{\rho,m} \right) .$$

(4)

For three identical objects the eigenstates of the Hamiltonian are characterized by the irreducible representations of the $S_3$ permutation group [8]. However, in anticipation of the geometric analysis of the next sections, we use a labeling under the point group $D_3$ (which is isomorphic to $S_3$): namely $A_1$ and $A_2$ for the one-dimensional symmetric and antisymmetric representations, and $E$ for the two-dimensional mixed symmetry representation. The scalar boson $s^\dagger$ of Eq. (2) transforms as the symmetric representation $A_1$, whereas the two dipole bosons $p^\dagger_{\rho,m}$ and $p^\dagger_{\lambda,m}$ transform as the two components of the mixed symmetry representation, $E_\rho$ and $E_\lambda$, respectively. For two identical objects the Hamiltonian is invariant under $S_2$ (which is isomorphic to $D_2$), and the $D_3$ irreducible representations reduce to those of its $D_2$ subgroup as $A_1, E_\lambda \rightarrow A$ (symmetric) and $A_2, E_\rho \rightarrow B$ (antisymmetric).

All physical operators, such as the Hamiltonian and transition operators, are expressed in terms of the building blocks of Eq. (2). With the help of the transposition $P(12)$ and the cyclic permutation $P(123)$ we construct a set of generators of the algebra of $U(7)$ that transform as tensor operators under
the rotation group \( SO(3) \) and the point group chain \( D_3 \supset D_2 \)

\[
\begin{align*}
\hat{D}_{\rho,m} &= (p^\dagger_{\rho} s - s^\dagger \tilde{p}_\rho )_{m}^{(l)} (E_{\rho}, B) , \\
\hat{A}_{\rho,m} &= i (p^\dagger_{\rho} s + s^\dagger \tilde{p}_\rho )_{m}^{(l)} (E_{\rho}, B) , \\
\hat{G}^{(l)}_{\lambda,m} &= (p^\dagger_{\lambda} \tilde{p}_\lambda + p^\dagger_{\rho} \tilde{p}_\rho )_{m}^{(l)} (E_{\rho}, B) , \\
\hat{G}^{(l)}_{\lambda,m} &= (p^\dagger_{\rho} \tilde{p}_\rho - p^\dagger_{\lambda} \tilde{p}_\lambda )_{m}^{(l)} (E_{\rho}, B) , \\
\hat{n}_s &= s^\dagger s (A_1, A) , \\
\hat{G}^{(l)}_{A_2,m} &= i (p^\dagger_{\rho} \tilde{p}_\rho - p^\dagger_{\lambda} \tilde{p}_\lambda )_{m}^{(l)} (A_2, B) ,
\end{align*}
\]

with \( l = 0, 1, 2 \). Here \( \tilde{p}_{\rho,m} = (-1)^{1-m} p_{\rho,-m} \) and \( \tilde{p}_{\lambda,m} = (-1)^{1-m} p_{\lambda,-m} \). The \( U(7) \) Hamiltonian can be expressed in terms of scalar products of the operators of Eq. (5). The eigenvalues and corresponding eigenvectors can be obtained exactly by diagonalization in an appropriate basis.

### 3 Geometry and intrinsic-collective structure

A more intuitive geometric interpretation of algebraic Hamiltonians can be obtained by using mean-field techniques. The main ingredient is the introduction of coherent (or intrinsic) states as variational wave functions which for a system of bosons have the form of a condensate of \( N \) deformed bosons [9]. In the present case of \( U(7) \), the condensate can be parametrized as

\[
| N; r, \chi, \theta \rangle = \frac{1}{\sqrt{N!}} (b^\dagger)^N | 0 \rangle ,
\]

with

\[
b^\dagger_i = \left[ s^\dagger + r \cos \chi p^\dagger_{\lambda,x} + r \sin \chi (\cos \theta p^\dagger_{\rho,x} + \sin \theta p^\dagger_{\rho,y}) \right] / \sqrt{1 + r^2} .
\]

The geometry is chosen such that the \( xy \) plane is defined by \( \tilde{\rho} \) and \( \tilde{\lambda} \), with the \( x \)-axis along \( \tilde{\lambda} \) and the \( z \)-axis perpendicular to this plane. The two vectors \( \tilde{\rho} \) and \( \tilde{\lambda} \) are parametrized in terms of the three Euler angles which are associated with the orientation of the three-body system, and three internal coordinates which are taken as the two lengths of the vectors \( r_{\rho} \) and \( r_{\lambda} \), and their relative angle \( \theta \): \( \tilde{r}_{\lambda} \cdot \tilde{r}_{\rho} = r_{\lambda} r_{\rho} \cos \theta \). The two lengths are parametrized in terms of the hyperspherical radius \( r \) and the hyperangle \( \chi \): \( r_{\lambda} = r \cos \chi \) and \( r_{\rho} = r \sin \chi \). The hyperradius \( r \) is a measure of the size of the system, whereas the hyperangle \( \chi \) and the angle \( \theta \) determine its shape [10]. Since the Hamiltonian is rotationally invariant, there is no need to introduce the Euler angles in Eq. (5). The expectation value of the Hamiltonian in the coherent state defines its energy surface

\[
E_N (r, \chi, \theta) = \langle N; r, \chi, \theta | H | N; r, \chi, \theta \rangle .
\]

The minimum of the energy surface determines the equilibrium values \( (\bar{r}, \bar{\chi}, \bar{\theta}) \) of the geometric variables.

The corresponding condensate wave function \( | N; \bar{r}, \bar{\chi}, \bar{\theta} \rangle \) represents the equilibrium shape of the three-body system.
In order to analyze the vibrational and rotational excitations, it is convenient to split the Hamiltonian into an intrinsic (vibrational) and a collective (rotational) part \[11\]

\[ H = H_{\text{int}} + H_{\text{c}}. \] (9)

The intrinsic part, by definition, annihilates the equilibrium condensate and has, up to an overall constant, the same energy surface as the original Hamiltonian

\[ H_{\text{int}} |N; r, \chi, \theta \rangle = 0, \]

\[ \langle N; r, \chi, \theta | H_{\text{int}} |N; r, \chi, \theta \rangle = E_{N}(r, \chi, \theta) - E_{0}. \] (10)

For \( r > 0 \) the rotational symmetry is spontaneously broken. Although in this case the condensate of Eq. (6) is deformed, it is still an eigenstate of the rotationally invariant intrinsic Hamiltonian. Since \( H \) and \( H_{\text{int}} \) have the same energy surface, one can extract the normal modes of the system by carrying out a normal mode analysis on \( H_{\text{int}} \). The collective part is the remainder of the Hamiltonian \( H_{\text{c}} = H - H_{\text{int}} \).

Rather than starting from the most general algebraic Hamiltonian, analyzing its energy surface, determining its equilibrium shapes, and decomposing the Hamiltonian for each one of them into an intrinsic and collective part \[11\], here we start from a given equilibrium shape and construct the corresponding intrinsic and collective Hamiltonians to analyze the vibrational and rotational excitations. In this paper we discuss the nonlinear equilibrium shapes of three objects, see Fig. [1].

### 3.1 Intrinsic Hamiltonian

Nonlinear configurations are characterized by the equilibrium values of the geometric variables

\[ r = R > 0, \quad \chi = \beta, \quad \theta = \gamma \neq 0, \pi. \] (11)

The corresponding intrinsic Hamiltonian can be expressed in the form

\[ H_{\text{int}} = AP_{1}^{1}P_{1} + BP_{2}^{1}P_{2} + CP_{3}^{1}P_{3} \]

\[ + DP_{2}^{1}P_{1} + E(P_{3}^{1}P_{3} + P_{1}^{1}P_{1}) + F(P_{2}^{1}P_{3} + P_{3}^{1}P_{2}), \] (12)

with

\[ P_{1}^{1} = p_{p}^{\dagger}p_{p}^{\dagger} + p_{\lambda}^{\dagger}p_{\lambda}^{\dagger} - R^{2}s^{1}s^{1}, \]

\[ P_{2}^{1} = \sin^{2}\beta p_{\lambda}^{\dagger}p_{\lambda}^{\dagger} - \cos^{2}\beta p_{p}^{\dagger}p_{p}^{\dagger}, \]

\[ P_{3}^{1} = \sin(2\beta) p_{p}^{\dagger}p_{p}^{\dagger} - \cos\gamma \left( \sin^{2}\beta p_{\lambda}^{\dagger}p_{\lambda}^{\dagger} + \cos^{2}\beta p_{p}^{\dagger}p_{p}^{\dagger} \right). \] (13)
3.2 Collective Hamiltonian

By construction, the collective Hamiltonian has a completely flat (or structureless) energy surface, which does not depend on the geometric variables \((r, \chi, \theta)\). Its most general form can be found by examining the structure of the condensate boson which defines the energy surface. The condensate boson of Eq. (7) is related to the scalar boson \(s\) by a unitary transformation

\[
\hat{b}_c = U(r, \chi, \theta) s \hat{b}_s \quad \text{and} \quad U(r, \chi, \theta) = e^{-i\theta L_{rs}} e^{i\chi \hat{F}_2} e^{-i\alpha \hat{A}_s},
\]

with \(\cos \alpha = 1/\sqrt{1 + r^2}\) and \(\sin \alpha = r/\sqrt{1 + r^2}\). The operators \(\hat{L}_{\rho,m}\) and \(\hat{F}_2\) can be expressed in terms of the generators of Eq. (3) as \(\hat{L}_{\rho,m} = (\hat{G}^{(1)}_{\rho,1} + \hat{G}^{(1)}_{\rho,-1})/\sqrt{2}\) and \(\hat{F}_2 = -\sqrt{3} \hat{G}^{(0)}_{A_2}\). The generators of the transformation of Eq. (14) all belong to the \(SO(7)\) subgroup of \(U(7)\), which is generated by

\[
\hat{A}_{\rho,m}, \hat{A}_{\lambda,m}, \hat{G}^{(1)}_{\rho,m}, \hat{G}^{(1)}_{\lambda,m}, \hat{G}^{(1)}_{A_1,m}, \hat{G}^{(0)}_{A_2}, \hat{G}^{(2)}_{A_2,m}.
\]

For this reason, the collective Hamiltonian can be expressed in terms of the two-body part of scalar products of these generators and has the following \(D_3\) decomposition

\[
H_C = H_C^{(A_1)} + H_C^{(E_4)} + H_C^{(E_9)},
\]

where

\[
H_C^{(A_1)} = \kappa_1 : [\hat{A}_\rho \cdot \hat{A}_\rho + \hat{A}_\lambda \cdot \hat{A}_\lambda] + 2\kappa_2 : [\hat{G}^{(1)}_\rho \cdot \hat{G}^{(1)}_\rho + \hat{G}^{(1)}_\lambda \cdot \hat{G}^{(1)}_\lambda] + \kappa_3 : \hat{L}^{(1)} \cdot \hat{L}^{(1)} + \kappa_4 : \hat{F}_2^{(2)};
\]

\[
H_C^{(E_4)} = \kappa'_1 : [\hat{A}_\rho \cdot \hat{A}_\rho - \hat{A}_\lambda \cdot \hat{A}_\lambda] + 2\kappa'_2 : [\hat{G}^{(1)}_\rho \cdot \hat{G}^{(1)}_\rho - \hat{G}^{(1)}_\lambda \cdot \hat{G}^{(1)}_\lambda] + \kappa'_3 \sqrt{2} : [\hat{L}^{(1)} \cdot \hat{G}^{(1)}_\lambda + \hat{G}^{(1)}_\lambda \cdot \hat{L}^{(1)}];
\]

\[
H_C^{(E_9)} = \kappa''_1 : [\hat{A}_\rho \cdot \hat{A}_\lambda + \hat{A}_\lambda \cdot \hat{A}_\rho] + \kappa''_2 : [\hat{G}^{(1)}_\rho \cdot \hat{G}^{(1)}_\lambda + \hat{G}^{(1)}_\lambda \cdot \hat{G}^{(1)}_\rho] + \kappa''_3 \sqrt{2} : [\hat{L}^{(1)} \cdot \hat{G}^{(1)}_\rho + \hat{G}^{(1)}_\rho \cdot \hat{L}^{(1)}].
\]

Here \(\hat{L}^{(1)} = \sqrt{2} \hat{G}^{(1)}_{A_1,m}\) represents the angular momentum operator, and :: denotes normal ordering. Note that the \(A_1\) term \(\hat{G}^{(2)}_{A_2}, \hat{G}^{(2)}_{A_2}\) does not appear in Eq. (17) since it is not independent

\[
\hat{G}^{(2)}_{A_2} \cdot \hat{G}^{(2)}_{A_2} = \hat{G}^{(1)}_\rho \cdot \hat{G}^{(1)}_\rho + \hat{G}^{(1)}_\lambda \cdot \hat{G}^{(1)}_\lambda - \hat{G}^{(1)}_{A_1} \cdot \hat{G}^{(1)}_{A_1} + \hat{G}^{(0)}_{A_2} \cdot \hat{G}^{(0)}_{A_2}.
\]

The collective Hamiltonian shifts, splits and generally mixes the bands generated by \(H_{\text{int}}\).

4 Asymmetric tops

We first discuss nonlinear XYZ and XY2 configurations which correspond to asymmetric tops.
4.1 Nonlinear XYZ configurations

For a system of three different objects XYZ, there are no restrictions arising from the $C_s$ point group symmetry [12]. The equilibrium shape is given by Eq. (11), and hence all terms in the Hamiltonian of Eq. (4) are allowed. The intrinsic part is given by Eqs. (12) and (13), and the collective part by Eqs. (16) and (17). The normal modes can be obtained by carrying out a normal mode analysis on $H_{\text{int}}$. This is done by expressing it in terms of a deformed boson basis, which is spanned by the condensate boson of Eq. (8) with $r = R$, $\chi = \beta$ and $\theta = \gamma$ ($\neq 0, \pi$)

$$b_c^\dagger = \left[ s^\dagger + R \cos \beta p^\dagger_{\lambda,x} + R \sin \beta (\cos \gamma p^\dagger_{\rho,x} + \sin \gamma p^\dagger_{\rho,y}) \right] / \sqrt{1 + R^2} ,$$

and six additional orthonormal fluctuation bosons

$$b_u^\dagger = \left[ -R s^\dagger + \cos \beta p^\dagger_{\lambda,x} + \sin \beta (\cos \gamma p^\dagger_{\rho,x} + \sin \gamma p^\dagger_{\rho,y}) \right] / \sqrt{1 + R^2} ,$$

$$b_v^\dagger = \sin \beta p^\dagger_{\lambda,x} - \cos \beta (\cos \gamma p^\dagger_{\rho,x} + \sin \gamma p^\dagger_{\rho,y}) ,$$

$$b_w^\dagger = \sin \beta p^\dagger_{\lambda,y} + \cos \beta (\sin \gamma p^\dagger_{\rho,x} - \cos \gamma p^\dagger_{\rho,y}) ,$$

$$b_1^\dagger = p^\dagger_{\rho,z} ,$$

$$b_2^\dagger = p^\dagger_{\lambda,z} ,$$

$$b_3^\dagger = \cos \beta p^\dagger_{\lambda,y} - \sin \beta (\sin \gamma p^\dagger_{\rho,x} - \cos \gamma p^\dagger_{\rho,y}) .$$

The mean-field Hamiltonian $B H_{\text{int}}$ is obtained by the usual Bogoliubov treatment of bosonic systems which amounts to replacing the condensate boson operators $b_c^\dagger$ and $b_c$ by $\sqrt{N}$ and keeping only terms to leading order in $N$ [11]

$$B H_{\text{int}} = \epsilon_u b_u^\dagger b_u + \epsilon_v b_v^\dagger b_v + \epsilon_w b_w^\dagger b_w + \epsilon_{uv}(b_u^\dagger b_v + b_v^\dagger b_u)$$

$$+ \epsilon_{uw}(b_u^\dagger b_w + b_w^\dagger b_u) + \epsilon_{vw}(b_v^\dagger b_w + b_w^\dagger b_v) ,$$

with

$$\epsilon_u = 4ANR^2 ,$$

$$\epsilon_v = BNR^2 \sin^2(2\beta)/(1 + R^2) ,$$

$$\epsilon_w = CNR^2 \sin^2(2\beta) \sin^2 \gamma/(1 + R^2) ,$$

$$\epsilon_{uv} = 2DNR^2 \sin(2\beta)/\sqrt{1 + R^2} ,$$

$$\epsilon_{uw} = 2ENR^2 \sin(2\beta) \sin \gamma/\sqrt{1 + R^2} ,$$

$$\epsilon_{vw} = FNR^2 \sin^2(2\beta) \sin \gamma/(1 + R^2) .$$

The spontaneously broken rotational symmetry ($R > 0$) ensures that only the vibrational bosons $b_i^\dagger$ ($i = u, v, w$) show up in $B H_{\text{int}}$. The intrinsic modes of the system which correspond to small oscillations
about the minimum of the energy surface can be obtained by diagonalizing $^B H_{\text{int}}$. The bosons $b_i^\dagger$ ($i = 1, 2, 3$) are associated with rotations of the equilibrium condensate, and hence can be identified with the Goldstone modes of the spontaneously broken rotational symmetry.

The mean-field or Bogoliubov image of the collective Hamiltonian has the form

\[
^B H_c = \eta_u (b_u^\dagger - b_u)^2 + \eta_v (b_v^\dagger - b_v)^2 + \eta_w (b_w^\dagger - b_w)^2 \\
+ \eta_1 (b_1^\dagger - b_1)^2 + \eta_2 (b_2^\dagger - b_2)^2 + \eta_3 (b_3^\dagger - b_3)^2 \\
+ \eta_{uv} (b_u^\dagger - b_u) (b_v^\dagger - b_v) + \eta_{uw} (b_u^\dagger - b_u) (b_w^\dagger - b_w) + \eta_{vw} (b_v^\dagger - b_v) (b_w^\dagger - b_w) \\
+ \eta_{12} (b_1^\dagger - b_1) (b_2^\dagger - b_2) + \eta_{3u} (b_3^\dagger - b_3) (b_u^\dagger - b_u) + \eta_{3w} (b_3^\dagger - b_3) (b_w^\dagger - b_w) ,
\]

(23)

with

\[
\eta_u = -N \left\{ \kappa_1 - \kappa'_1 \cos(2\beta) + \kappa''_1 \cos \gamma \sin(2\beta) \right\} ,
\]

\[
\eta_v = -N \left\{ \kappa_1 + R^2 \left[ \kappa_2 \sin^2 \gamma + \kappa_4 \cos^2 \gamma \right] \right\} / (1 + R^2) \\
+ N \kappa''_1 \cos \gamma \sin(2\beta) / (1 + R^2) ,
\]

\[
\eta_w = -N \left\{ \kappa_1 + R^2 \left[ \kappa_2 \left( \cos^2 \gamma \cos^2(2\beta) + \sin^2(2\beta) \right) + \kappa_4 \sin^2 \gamma \cos^2(2\beta) \right] \right\} / (1 + R^2) \\
+ N \kappa''_1 \cos \gamma \sin(2\beta) / (1 + R^2) ,
\]

(24)

and

\[
\eta_1 = -N \left\{ \kappa_1 + R^2 \left[ \kappa_2 + \kappa_3 \sin^2 \beta \right] \right\} / (1 + R^2) \\
+ N \kappa''_1 \cos \gamma \sin \beta \cos \beta / (1 + R^2) ,
\]

\[
\eta_2 = -N \left\{ \kappa_1 + R^2 \left[ \kappa_2 + \kappa_3 \cos^2 \beta \right] \right\} / (1 + R^2) \\
- N \kappa''_1 \cos \gamma \sin \beta \cos \beta / (1 + R^2) ,
\]

\[
\eta_3 = -N \left\{ \kappa_1 + R^2 \left[ \kappa_2 \left( \cos^2 \gamma \sin^2(2\beta) + \cos^2(2\beta) \right) + \kappa_3 + \kappa_4 \sin^2 \gamma \sin^2(2\beta) \right] \right\} / (1 + R^2) \\
+ N \kappa''_1 \cos \gamma \sin \beta \cos \beta / (1 + R^2) ,
\]

(25)

The only nonvanishing coupling terms are $\eta_{uw}, \eta_{uw}, \eta_{uv}, \eta_{12}, \eta_{3u}$ and $\eta_{3w}$

\[
\eta_{uv} = 2N \left\{ \kappa'_1 \sin(2\beta) + \kappa''_1 \cos \gamma \cos(2\beta) \right\} / \sqrt{1 + R^2} ,
\]

\[
\eta_{uw} = -2N \kappa''_1 \sin \gamma / \sqrt{1 + R^2} ,
\]

8
\[ \eta_{vw} = -NR^2 [\kappa_2 - \kappa_4] \sin(2\gamma) \cos(2\beta) / (1 + R^2) \]
\[-NR^2 \kappa'_2 \sin(2\gamma) \cos(2\beta) / (1 + R^2) \]
\[-NR^2 \kappa''_2 \sin \gamma \sin(2\beta) / (1 + R^2) , \] (26)

and

\[ \eta_{12} = -NR^2 \kappa_3 \cos \gamma \sin(2\beta) / (1 + R^2) \]
\[-2NR^2 \kappa'_2 \cos \gamma \sin(2\beta) / (1 + R^2) \]
\[-N \{ 2\kappa''_1 - R^2 [\kappa''_2 \cos(2\beta) + 2\kappa''_3] \} / (1 + R^2) , \]
\[ \eta_{3w} = -NR^2 [2\kappa'_2 - \kappa'_4] \sin(2\gamma) \sin(2\beta) / (1 + R^2) \]
\[+NR^2 \kappa'_2 \sin(2\gamma) \sin(2\beta) / (1 + R^2) \]
\[+N \{ 2\kappa''_1 + R^2 [-\kappa''_2 \cos(2\beta) + 2\kappa''_3] \} \sin \gamma / (1 + R^2) , \]
\[ \eta_{3w} = -NR^2 [\kappa_2 - \kappa_4] \sin(4\beta) \sin^2 \gamma / (1 + R^2) \]
\[+N \{ 2\kappa'_1 + R^2 [2\kappa'_2 \cos(2\beta)(1 + \cos^2 \gamma) + 2\kappa'_3] \} \sin(2\beta) / (1 + R^2) \]
\[+N \{ 2\kappa''_1 \cos(2\beta) + R^2 [-\kappa''_2 \cos(4\beta) + 2\kappa''_3 \cos(2\beta)] \} \cos \gamma / (1 + R^2) . \] (27)

4.2 Nonlinear XY\(_2\) configurations

For nonlinear XY\(_2\) configurations the relevant point group is \( C_{2v} \). If we label the identical objects by 1 and 2, the Jacobi coordinates of Eq. (4) with \( m_1 = m_2 \) become perpendicular. The corresponding equilibrium shape is

\[ \tau = R > 0 , \quad \chi = \beta , \quad \theta = \gamma = \pi/2 . \] (28)

The permutation symmetry associated with the interchange of the two identical objects \( S_2 \) (isomorphic to \( D_2 \)) is determined by the transposition \( P(12) \). This excludes from the Hamiltonian all terms that change the number of \( p_\rho \) bosons by an odd number.

Accordingly, the intrinsic Hamiltonian for nonlinear XY\(_2\) configurations is given by Eqs. (12) and (13) with the shape variables of Eq. (28), and

\[ E = F = 0 . \] (29)

The normal modes are obtained as before by carrying out a mean-field analysis of the intrinsic Hamiltonian

\[ B_H^{\text{int}} = \epsilon_u b_u^\dagger b_u + \epsilon_v b_v^\dagger b_v + \epsilon_w b_w^\dagger b_w + \epsilon_{uv}(b_u^\dagger b_v + b_v^\dagger b_u) . \] (30)

In this case, the two radial modes, \( u \) and \( v \), are decoupled from the angular mode \( w \). The normal radial modes are obtained by diagonalizing a \( 2 \times 2 \) matrix, and correspond to the symmetric and antisymmetric
stretching modes. This diagonalization is the algebraic analogue of the orthogonal transformation of Delves coordinates (i.e. mass-scaled Jacobi coordinates) to the normal coordinates in the small displacement limit \(13\).

Similarly, the collective Hamiltonian is given by Eqs. (16) and (17) with

\[
H^{(E_r)}_C = 0 .
\]

(31)

Its Bogoliubov image has the form

\[
B H_C = \eta_u (b^\dagger_u - b_u)^2 + \eta_v (b^\dagger_v - b_v)^2 + \eta_w (b^\dagger_w - b_w)^2 \\
+ \eta_1 (b^\dagger_1 - b_1)^2 + \eta_2 (b^\dagger_2 - b_2)^2 + \eta_3 (b^\dagger_3 - b_3)^2 \\
+ \eta_{1w} (b^\dagger_1 - b_1) (b^\dagger_w - b_w) + \eta_{3w} (b^\dagger_3 - b_3) (b^\dagger_w - b_w) .
\]

(32)

In addition to the \(\eta\) terms, the only coupling terms present are \(\eta_{uw}\) and \(\eta_{3w}\). In comparison with the nonlinear XYZ configuration, the coupling terms \(\eta_{uw}, \eta_{vw}, \eta_{12}\) and \(\eta_{3v}\) are missing. This is a consequence of the different point group symmetries.

## 5 Oblate symmetric top

Nonlinear \(X_3\) configurations have the equilibrium shape of an equilateral triangle, which is characterized by two orthogonal Jacobi coordinates of equal length. The geometric configuration has one 3-fold symmetry axis, three 2-fold symmetry axes perpendicular to it and one reflection plane. The corresponding point group is \(D_{3h}\) whose classification is equivalent to that of its subgroup \(D_3\) (which is isomorphic to the permutation group of three identical objects \(S_3\)) and parity. In comparison with nonlinear \(XY_2\) configurations, for three identical objects the Jacobi coordinates not only are perpendicular, but also have equal lengths. The equilibrium shape is then characterized by

\[
\tau = R > 0 , \quad \chi = \beta = \pi/4 , \quad \theta = \gamma = \pi/2 .
\]

(33)

The permutation symmetry associated with the interchange of the three identical objects \(S_3\) (isomorphic to \(D_3\)) is determined by the transposition \(P(12)\) and the cyclic permutation \(P(123)\). This excludes from the Hamiltonian all terms that are not scalar with respect to \(S_3\). The \(S_3\) invariant Hamiltonian (or mass operator) is discussed in more detail in \(12\). Here we briefly present the results that are relevant to the present article.

The intrinsic Hamiltonian for nonlinear \(X_3\) configurations is given by Eqs. (12) and (13) with the shape variables of Eq. (33), and

\[
B = C , \quad D = E = F = 0 .
\]

(34)
As before, the normal modes are obtained by a mean-field analysis

\[ ^B H_{\text{int}} = \epsilon_u b_u^\dagger b_u + \epsilon_v (b_v^\dagger b_v + b_w^\dagger b_w) . \] (35)

In this case, the two radial modes, \( u \) and \( v \), and the angular mode \( w \) are decoupled, and correspond to the fundamental vibrations (see Fig. 1) of the configuration of Fig. 1. The first term represents the symmetric stretching mode \( u \), whereas the second term corresponds to a degenerate doublet \( \epsilon_v = \epsilon_w \) of the antisymmetric stretching mode \( v \) and the bending mode \( w \). This is in agreement with the point-group classification of the fundamental vibrations of a symmetric-top \( X_3 \) configuration [12].

The collective Hamiltonian is given by Eqs. (16) and (17) with

\[ H_{\text{c}}^{(E_{\lambda})} = H_{\text{c}}^{(E_{\rho})} = 0 , \] (36)

and its Bogoliubov image has the form

\[ ^B H_{\text{c}} = \eta_u (b_u^\dagger - b_u)^2 + \eta_v [(b_v^\dagger - b_v)^2 + (b_w^\dagger - b_w)^2] \]

\[ + \eta_1 [(b_1^\dagger - b_1)^2 + (b_2^\dagger - b_2)^2] + \eta_3 (b_3^\dagger - b_3)^2 . \] (37)

The terms which depend on the vibrational bosons \( b_i^\dagger (i = u, v, w) \) cause a shift in the vibrational frequencies, whereas the terms involving the rotational bosons \( b_i^\dagger (i = 1, 2, 3) \) are responsible for in-band rotational splitting. The equality of the moments of inertia about the \( x \) and \( y \) axis \( (\eta_1 = \eta_2) \), and the equality of the coefficients \( \eta_v = \eta_w \) reflect the axial symmetry of the underlying shape. In comparison with the nonlinear XYZ and XY\( _2 \) configurations all coupling terms \( \eta_{ij} \) are missing.

The above analysis presents a clear interpretation of the various interaction terms in the intrinsic and collective Hamiltonians. The two terms in \( H_{\text{int}} \) proportional to \( A \) and \( B = C \) contribute only to the frequencies of the vibrational modes. The \( \kappa_3 \) and \( \kappa_4 \) terms in \( H_{\text{c}} \) contribute only to the rotational modes, whereas the \( \kappa_1 \) and \( \kappa_2 \) terms both effect the vibrational and the rotational modes, and hence represent vibration-rotation couplings. The equality of the frequencies of the antisymmetric stretching and the bending modes, and the equality of the moments of inertia about the \( x \) and \( y \) axis, show that the above Hamiltonians correspond to an oblate symmetric top with the threefold symmetry axis along the \( z \)-axis. In [3] it was shown that \( U(7) \) provides a complete classification of the vibrational and rotational excitations of the oblate symmetric top.

6 Summary and conclusions

We have presented an algebraic treatment of three-body problems. The relative motion of the three-body system is treated by the method of bosonic quantization, which for the two relative vectors gives rise to a \( U(7) \) spectrum generating algebra. The model space is spanned by the symmetric irreducible
representation $[N]$ of $U(7)$. In particular, we have considered nonlinear XYZ, XY$_2$ and X$_3$ configurations. The $U(7)$ interaction terms have been interpreted geometrically by means of coherent states and a normal mode analysis. We have studied the effect of the permutation symmetry among the identical particles on the coupling between the vibrational and rotational modes. For XYZ and XY$_2$ the geometry is that of an asymmetric top, whereas for X$_3$ that of an oblate top. The ensuing algebraic treatment of the oblate top has found useful applications both in molecular physics (X$_3$ molecules [6]) and hadronic physics (nonstrange $qqq$ baryons [3, 4]).

In conclusion, we have shown that $U(7)$ which was previously introduced to describe X$_3$ configurations [3, 4], also provides a spectrum generating algebra for XY$_2$ and XYZ configurations. The latter are important to describe nonlinear molecules and strange baryons. In this respect, $U(7)$ provides a unified treatment of vibrational and rotational excitations of both symmetric and asymmetric tops.

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References

[1] See e.g. Proceedings of the Thirteenth International Conference on Few Body Problem in Physics, Adelaide, S.A., Australia, Eds. I.R. Afnan and R.T. Cahill, Nucl. Phys. A 543 (1992); Proceedings of the 15th European Conference on Few-Body Problems in Physics, Peñiscola, Spain, Ed. R. Guardiola, Few-Body Systems, Supplement 8, Springer Verlag, 1996.

[2] A. Arima and F. Iachello, Phys. Rev. Lett. 35, 1069 (1975); for a review see F. Iachello and A. Arima, ‘The interacting boson model’, Cambridge University Press, 1987.

[3] F. Iachello, Chem. Phys. Lett. 78, 581 (1981); for a review see F. Iachello and R.D. Levine, ‘Algebraic theory of molecules’, Oxford University Press, 1995.

[4] F. Iachello, Nucl. Phys. A 560, 23 (1993).

[5] R. Bijker, F. Iachello and A. Leviatan, Ann. Phys. (N.Y.) 236, 69 (1994).

[6] R. Bijker, A.E.L. Dieperink, A. Leviatan, Phys. Rev. A 52, 2786 (1995).

[7] P. Kramer and M. Moshinsky, Nucl. Phys. 82, 241 (1966).

[8] A.N. Mitra, A. Sharma and B. Mitra-Sodermark, Few-Body Systems 19, 145 (1995).

[9] For a general review on coherent states see e.g.: W.M. Zhang, D.H. Feng and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990); For an application to the interacting boson model see e.g.: A. Bohr and B.R. Mottelson, Physica Scripta 22, 468 (1980); J.N. Ginocchio and M.W. Kirson, Phys. Rev. Lett. 44, 1744 (1980); A.E.L. Dieperink, O. Scholten and F. Iachello, Phys. Rev. Lett. 44, 1747 (1980).

[10] See e.g. L.M. Delves, Nucl. Phys. 9, 391 (1959); F.T. Smith, Phys. Rev. 120, 1058 (1960); J.L. Ballot and M. Fabre de la Ripelle, Ann. Phys. (N.Y.) 127, 62 (1980).

[11] M.W. Kirson and A. Leviatan, Phys. Rev. Lett. 55, 2846 (1985).

[12] G. Herzberg, ‘The spectra and structures of simple free radicals’, Dover, 1971.

[13] J.M. Bowman, J. Zuniga and A. Wierzbicki, J. Chem. Phys. 90, 2708 (1989).

[14] R. Bijker, F. Iachello and A. Leviatan, Phys. Rev. C 54, 1935 (1996); R. Bijker, F. Iachello and A. Leviatan, Phys. Rev. D 55, 2862 (1997).
Figure 1: Geometry of a three-body system.

Figure 2: Schematic representation of the normal vibrations of a nonlinear $X_3$ configuration. The Jacobi coordinates are indicated by the dotted lines.