Recovering coherence from decoherence: a method of quantum state reconstruction

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We present a feasible scheme for reconstructing the quantum state of a field prepared inside a lossy cavity. Quantum coherences are normally destroyed by dissipation, but we show that at zero temperature we are able to retrieve enough information about the initial state, making possible to recover its Wigner function as well as other quasiprobabilities. We provide a numerical simulation of a Schrödinger cat state reconstruction.

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I. INTRODUCTION

The reconstruction of quantum states is a central topic in quantum optics and related fields [1,2]. During the past years, several techniques have been developed, for instance, the direct sampling of the density matrix of a signal mode in multiphoton optical homodyne tomography [3], tomographic reconstruction by unbalanced homodyning [4], cascaded homodyning [5] and reconstruction via photocounting [6]. There are also proposals of measurement of the electromagnetic field inside a cavity [7] as well as the vibrational state of an ion in a trap [8]. The full reconstruction of nonclassical field states [9] as well as of (motional) states of an ion [10] have been already experimentally accomplished. The quantum state reconstruction is normally achieved through a finite set of either field homodyne measurements, or selective measurement of atomic states [7] in the case of cavities. This makes possible to construct a quasidistribution (such as the Wigner function) which constitutes an alternative representation of the quantum state of the field.

Nevertheless, in real experiments, the presence of noise and dissipation has normally destructive effects. In fact, as it has been already pointed out, the reconstruction schemes themselves also indicate loss of coherence in quantum systems [11] regarding this subject, a scheme for compensation of losses in quantum-state measurements has been already proposed [11], and the relation between losses and s-parameterized quasiprobability distributions has been already pointed out in [12]. The scheme on loss compensation in [11] applies to photodetector losses, and consists essentially of a mathematical inversion formula expressing the initial density matrix in terms of the decayed one. Our scheme, as discussed in [13], involves a physical process that actually enables us to store information about all the quantum coherences of the initial state in the diagonal elements (photon distribution) of the density matrix of a transformed state. By storing this information in the diagonal elements, it becomes much more robust under dissipation, allowing us to recover the Wigner function of the initial field state in a time scale of the order of the energy decay time that is, of course, much longer than the extremely fast decoherence time scale that is normally associated with the dissipation of quantum coherences.

We consider a single mode high-Q cavity where we suppose that a (nonclassical) field state $\hat{\rho}(0)$ is previously prepared. The first step of our method consists in driving the generated quantum state by a coherent pulse. The reconstruction of the field state may be accomplished after turning-off the driving field, i.e., at a time in which the cavity field has already suffered decay. We use the fact that by displacing the initial state (even while it is decaying) we make its quantum coherences robust enough to allow its experimental determination, at a later time, despite dissipation. We then show that the evolution of the cavity field is such that it directly yields the Wigner function of the initial nonclassical field simply by measuring the photon number distribution of the displaced field. For that we make direct use of the series representation of quasiprobability distributions [14]. A numerical simulation of our method is presented, and we take into account the action of dissipation while driving the initial field.

This manuscript is organized as follows: in Sec. II we discuss, taking into account losses, the process of displacement of the initial field. In Sec. III we show how to reconstruct the initial cavity field after allowing the displaced field to

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decay. In Sec. IV we present a simulation of the reconstruction of a Schrödinger cat state. In Sec. V we summarize our conclusions.

II. DRIVING THE INITIAL FIELD

We assume that the initial nonclassical field $\hat{\rho}(0)$ is prepared inside a high $Q$ cavity. The master equation in the interaction picture for the reduced density operator $\hat{\rho}$ relative to a driven cavity mode, taking into account cavity losses at zero temperature and under the Born-Markov approximation is given by [15]

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}_d, \hat{\rho}] + \frac{\gamma}{2} \left( 2\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger \right),$$  (1)

with

$$\hat{H}_d = \frac{i}{\hbar}(\alpha^*\hat{a} - \alpha\hat{a}^\dagger),$$  (2)

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators, $\gamma$ the (cavity) decay constant and $\alpha$ the amplitude of the driving field.

We define the superoperators $\hat{R}$ and $\hat{L}$ by their action on the density operator [16]

$$\hat{R}\hat{\rho} = (\alpha^*\hat{a} - \alpha\hat{a}^\dagger)\hat{\rho} - \hat{\rho}(\alpha^*\hat{a} - \alpha\hat{a}^\dagger),$$  (3)

and

$$\hat{L}\hat{\rho} = \gamma\hat{a}\hat{a}^\dagger - \frac{\gamma}{2} \left( \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger \right).$$  (4)

It is not difficult to show that

$$[\hat{R}, \hat{L}]\hat{\rho} = \frac{\gamma}{2}\hat{R}\hat{\rho},$$  (5)

and the formal solution of Eq. (1) can then be written as [17]

$$\hat{\rho}(t) = \exp\left( (\hat{R} + \hat{L})t \right) \hat{\rho}(0) = \exp(\hat{L}t)\exp\left[ -2\frac{\hat{R}}{\gamma}(1 - e^{\gamma t/2}) \right] \hat{\rho}(0).$$  (6)

After driving the initial field during a time $t_d$, the resulting field density operator will read

$$\hat{\rho}(t_d) = e^{\hat{L}t_d} \hat{\rho}(0),$$  (7)

where

$$\hat{\rho}(0) = \hat{D}^\dagger(\beta)\hat{\rho}(0)\hat{D}(\beta),$$  (8)

and with

$$\beta = 2\alpha \frac{1 - e^{-\gamma t_d/2}}{\gamma}.$$  (9)

This means that if we drive the initial field while it decays, during a time $t_d$, this is equivalent to having the field driven by a coherent field with an effective amplitude $\beta$ given in Eq. (9).

III. THE RECONSTRUCTION METHOD

The driving of the initial field is carried out during a time $t_d$. This procedure will enable us to obtain information about all the elements of the initial density matrix from the diagonal elements of the time-evolved displaced density matrix only. As diagonal elements decay much slower than off-diagonal ones, information about the initial state stored this way becomes robust enough to withstand the decoherence process. We will now show how this robustness can be used to obtain the Wigner function of the initial state after it has already started to dissipate. Once the injection of
the coherent pulse is completed, the cavity field is left to decay, so that its dynamics will be governed by the master equation in Eq. (9) without the first (driving) term in its right-hand-side. Therefore, the cavity field density operator will be, at a time \( t \), given by

\[
\hat{\rho}_\beta(t) = e^{(\hat{J} + \hat{L})t} \hat{\rho}_\beta(0),
\]

with

\[
\hat{J} \hat{\rho} = \gamma \hat{a} \hat{\rho} \hat{a}^\dagger, \quad \hat{L} \hat{\rho} = -\frac{\gamma}{2} (\hat{a}^\dagger \hat{a} \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{a}).
\]

The next step is to calculate the diagonal matrix elements of \( \hat{\rho}_\beta(t) \) = \( \exp \left[(\hat{J} + \hat{L})t \right] \hat{\rho}_\beta \) in the number state basis, or

\[
\langle m | \hat{\rho}_\beta(t) | m \rangle = \frac{e^{-\gamma m t}}{q^m} \sum_{n=0}^{\infty} q^n \begin{pmatrix} n \\ m \end{pmatrix} \langle n | \hat{\rho}_\beta(0) | n \rangle,
\]

where \( q = 1 - e^{-\gamma t} \).

Now we multiply those matrix elements by powers of the function

\[
\chi(t) = 1 - 2 e^{\gamma t}.
\]

If we sum the resulting expression over \( m \), we obtain the following simple sum

\[
F_W = \frac{2}{\pi} \sum_{m=0}^{\infty} \chi^m(t) \langle m | \hat{\rho}_\beta(t) | m \rangle = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \langle n | \hat{D}^\dagger(\beta) \hat{\rho}(0) \hat{D}(\beta) | n \rangle.
\]

The expression in Eq. (14) is exactly the Wigner function corresponding to \( \hat{\rho} \) (the initial field state) \[14\] at the point specified by the complex amplitude \( \beta \). Therefore we need simply to measure the diagonal elements of the displaced cavity field \( P_m(\beta; t) = \langle m | \hat{\rho}_\beta(t) | m \rangle \) for a range of \( \beta \)'s, the transformation in Eq. (14) in order to obtain the Wigner function of the initial state for this range. We note that after performing the sum, the time-dependence cancels out completely, leaving us a constant Wigner function, as it should be. Therefore the initial state may be reconstructed, at least in principle, at an arbitrary later time. In practice, however, the decay of the field energy will impose a limitation on the times during which we will be able to measure the \( P_m \)'s.

The next step in our scheme is to measure the field photon number distribution \( P_m(\beta; t) \). In a cavity, particularly in the microwave regime where there are no photodetectors available, experimentalists have been forced to use atoms instead to probe the intra-cavity field. One way of determining \( P_m \) is by injecting atoms into the cavity and measuring their population inversion as they exit after an interaction time \( \tau \) much shorter than the cavity decay time. It is convenient in this case to use three-level atoms in a cascade configuration with the upper and the lower level having the same parity and satisfying the two-photon resonance condition. The population inversion in this case is \[18\]

\[
W(\alpha; t + \tau) = \sum_{n=0}^{\infty} P_n(\alpha; t) \left[ \frac{\Gamma_n}{\delta_n^2} + \frac{(n+1)(n+2)}{\delta_n^2} \cos \left(2 \delta_n \lambda \tau \right) \right],
\]

where \( \Gamma_n = (\Delta + \chi(n+1))/2, \delta_n^2 = \Gamma_n^2 + \lambda^2 (n+1)(n+2), \Delta \) is the atom-field detuning, \( \chi \) is the Stark shift coefficient, and \( \lambda \) is the atom-field coupling constant. Now we take \( \Delta = 0 \) (two-photon resonance condition) and \( \chi = 0 \). It is a good approximation to make \([(n+1)(n+2)]^{1/2} \approx n + 3/2\), so that the population inversion reduces to

\[
W(\alpha; t + \tau) = \sum_{n=0}^{\infty} P_n(\alpha; t) \cos \left(2n + 3 \right) \lambda \tau \right).
\]

This represents the atomic response to the displaced field. In order to obtain \( P_m \) from a family of measured population inversions, we need to invert the Fourier series in Eq. (14), or

\[
P_m(\beta; t) = \frac{2 \lambda}{\pi} \int_0^{\tau_{\text{max}}} d\tau W(t + \tau) \cos \left(2m + 3 \right) \lambda \tau \right).
\]

We need a maximum interaction time \( \tau_{\text{max}} = \pi / \lambda \) much shorter than the cavity decay time, and this condition implies that we must be in the strong-coupling regime, i.e. \( \lambda \gg \gamma \). Of course after some time the atoms sent through the
cavity will not be able to respond anymore to the decaying field, hampering our reconstruction scheme. Nevertheless this happens at a time scale of \( t_d \approx 1/\lambda \), which is normally much longer than a typical decoherence time.

Our scheme is easily extended to other \((s\text{-parametrized} \ [19])\) quasi-probability distributions, which also may be expressed as a series \([14]\).

\[
F(\beta; s) = -\frac{2}{\pi(s-1)} \sum_{n=0}^{\infty} \left(\frac{s+1}{s-1}\right)^n \langle n| \hat{\rho}_\beta |n\rangle.
\]

For this we have to multiply the photon number distribution of the displaced (and dissipated) field by a generalization of the function in Eq. (13), or

\[
\chi(s; t) = 1 + \frac{2e^{\gamma t}}{s-1}.
\]

This increases our possibilities of measuring quantum states. We have available several quasiprobabilities, and we may choose a more convenient function depending on the particular conditions of a reconstruction experiment.

### IV. RECONSTRUCTION OF A SCHRÖDINGER CAT STATE

We show now how our method may be applied to a specific case, e.g., to a reconstruction of a Schrödinger cat state, represented by a quantum superposition of two coherent states having distinct amplitudes, \(|\alpha\rangle\) and \(|-\alpha\rangle\). The density operator corresponding to such state is

\[
\hat{\rho}(0) = \mathcal{N} \left[ |\alpha\rangle \langle \alpha| + |-\alpha\rangle \langle -\alpha| + e^{i\phi} |\alpha\rangle \langle -\alpha| + e^{-i\phi} | -\alpha\rangle \langle \alpha| \right],
\]

where \(\phi\) is a relative phase and \(\mathcal{N}\) a normalization constant. Schrödinger cat states are very fragile under dissipation \([21]\), and therefore are specially suitable for exemplifying our scheme.

In a real experiment, just after having prepared the field in the state in Eq. (20) by one of the conventional methods, (see reference [22] for instance), the cavity will be driven by a coherent field, say, of amplitude \(\alpha_0\). After that, at a time \(t_0\), \(N\) conveniently prepared three-level atoms should cross the cavity, so that we are able to assign a particular population inversion \(W_{\alpha_0}(t_0) = P_\alpha(t_0) - P_{-\alpha}(t_0)\) to that time. These sort of measurements for the same driving field amplitude \(\alpha_0\) will be repeated for a range of times \(t_0, t_1, t_2 \ldots t = t_0 + \pi/\lambda\), so that we obtain a set of values for the population inversion in that time interval. Now through numerical integration (see Eq. (17)), we obtain the required photon number distribution \(P_m(\alpha_0, t)\). This is exactly what we need in our reconstruction scheme. The next step is to multiply \(P_m\) by the terms \(\chi(t)^m = (1 - 2e^{\gamma t})^m\) and sum over \(m\). This directly yields, according to our scheme, the value of the Wigner function of the initial field in Eq. (20) at the point \(\alpha_0 = \alpha_0 + i\gamma t\) in phase space. We remark that the convergence of the series in Eq. (14) is guaranteed, because the photon number distributions of physical states normally decreases very quickly as \(m\) increases, and the statistical errors also become small simply because we are dealing with diagonal elements of the density operator in the number state basis \([11]\). We need then to repeat this procedure \(M\) times, for different values of the driving field amplitude \(\alpha_1, \alpha_2 \ldots \alpha_M\), to be able to cover enough points in phase-space and obtain the whole Wigner function of the original field.

We have produced a numerical simulation of the above mentioned steps. Our simulation is going to be illustrated by the Wigner functions themselves. At \(t = 0\), a Schrödinger cat state (Eq. (20)) is generated within the cavity. Its corresponding Wigner function is shown in Fig. 1, where we note the characteristic interference structure. After the field has decayed, say, at \(t = 0.1/\gamma\), the loss of coherence is indicated by the significant reduction of the interference structure, as seen in Fig. 2. Dissipation brings out the initially pure state very close to a statistical mixture \([21]\). We may instead drive the cavity with a coherent pulse (duration \(t_d\)), at \(t = 0\), in order to start our reconstruction procedure. After following the steps described above, we obtain the reconstructed Wigner function, as shown in Fig. 3, which is essentially the one in Fig. 1. We note that both peaks as well as the interference structure characteristic of a Schrödinger cat state are entirely preserved. In our simulation we have considered fluctuations that might be present during the measurement of the atomic inversion. There might be experimental errors from various sources, such as fluctuations in the amplitude of the driving field as well as in the generated nonclassical state, which would cause distortions in the atomic inversion. In Fig. 4 we show the atomic inversion as a function of time and for a given value of the driving field amplitude \(\beta = (0,2)\). We have obtained the field’s Wigner function shown in Fig. 3 departing from a family of those “distorted” atomic inversions, for different values of \(\beta\).
V. CONCLUSION

In conclusion, we have presented a method for reconstructing the Wigner function of an initial nonclassical state at times when the field would have normally lost its quantum coherence. In particular, even at times such that the Wigner function would have lost its negativity, reflecting the decoherence process. A crucial step in our approach is the driving of the initial field immediately after preparation, which stores quantum coherences in the diagonal elements of the time evolved displaced density matrix, making them robust. We have therefore shown that the initial displacement transfers the robustness of a coherent state [20] against dissipation to any initial state, allowing the full reconstruction of the field state under less than ideal conditions.

A natural application of our method would be the measurement of quantum states in cavities, where dissipation is difficult to avoid. Moreover, the application of the driving pulse at different times after the generation of a field state, would allow the “snapshooting” of the Wigner function as the state is dissipated. This means that valuable information about the (mixed) quantum state as well as about the decay mechanism itself could be retrieved while it suffers decay. The possibility of reconstructing quantum states even in the presence of dissipation may be also relevant for applications in quantum computing. Loss of coherence associated to dissipation is likely to occur in those devices, and our method could be used, for instance, as a scheme to refresh the state of a quantum computer [23] in order to minimize the destructive action of the environment.
FIG. 1. Wigner function of a Schrödinger cat state with $\alpha = 2$ and $\phi = 0$ at $t = 0$.

FIG. 2. Wigner function of a Schrödinger cat state with $\alpha = 2$ and $\phi = 0$ at $t = 0/\gamma$.

FIG. 3. Wigner function of a Schrödinger cat state reconstructed at $t = 0.1/\gamma$, with $\alpha = 2$ and $\phi = 0$. The Wigner function at $t = 0$ has been recovered (see Fig. 1).
FIG. 4. “Measured” atomic inversion (dashed line) and “exact” atomic inversion of (continuous line) the displaced cavity field as a function of time for a displacement amplitude $\beta = (0, 2)$. 
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