Reliability-based tolerance redesign of mechanical assemblies using Jacobian-Torsor model

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Abstract
The purpose of this paper is to present a new method to redesign dimensional and geometric tolerances of mechanical assemblies at a lower cost and with higher reliability. A parametric Jacobian-Torsor model is proposed to conduct tolerance analysis of mechanical assembly. A reliability-based tolerance optimization model is established. Differing from previous studies of fixed process parameters, this research determines the optimal process variances of tolerances, which provide basis for the subsequent assembly tolerance redesign. By using the Lambert W function and the Lagrange multiplier method, the analytical solution of the parametric tolerance optimization model is obtained. A numerical example is presented to demonstrate the effectiveness of the model, while the results indicate that the total cost is reduced by 10.93% and assembly reliability improves by 2.12%. This study presents an efficient reliability-based tolerance optimization model. The proposed model of tolerance redesign can be used for mechanical assembly with a better economic effect and higher reliability.

Keywords
Jacobian-Torsor model, tolerance optimization, tolerance redesign, reliability, Lagrange multiplier method

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Introduction

The dimensions of manufactured components are inevitably different from the nominal ones. Three-dimensional tolerance analysis methods have become increasingly important tools in the field of tolerance analysis. Both dimensional and geometric aspects are taken into consideration in the tolerance analysis. Chen et al.1 briefly introduced four major 3D tolerance analysis models and made a comprehensive comparison and discussion over them. The Jacobian-Torsor model has been successfully applied for the deterministic and statistical tolerance analysis by Ghie.2 Polini and Corrado3 proposed an analytical approach, using the Jacobian model to carry out the tolerance analysis of mechanical assemblies. Khodaygan and Ghaderi4 presented a new efficient method for the tolerance-reliability analysis, based on First Order Reliability Method (FORM). Tolerance-reliability analysis is about estimating a defect probability, which quantifies the probability that the final assembly does not meet the functional requirements. The study on the tolerance-reliability analysis and tolerance redesign, using the parametric Jacobian-Torsor model, has not yet been carried out.

Tolerance design is a crucial step in product design. Recent developments in tolerance design heightened the necessity to consider reliability constraints. Kong et al.5 presented a tolerance design method to guide the quality improvement of helical springs, by considering the given reliability constraint, based on degradation performance. The process of determining the optimal tolerances is a trade-off between the total cost and the product performance. The total cost is the sum of manufacturing, inspection and rejection costs. In early references6 of tolerance optimization, only the manufacturing aspect was considered as the objective function of minimizing cost. Nowadays, quality loss has received more attention,7,8 while the quadratic quality loss function, proposed by Taguchi,9 is the most widely used model. The existing references10 studied the tolerance optimization model, mainly based on fixed process parameters, whereas optimal process parameters should be applied to reduce overall costs. Mahmood et al.11 presented an experimental approach, to investigate the effects on the geometrical properties of the benchmark component, considering the variation of the process parameter settings. Benanzer et al.12 performed research about the reliability-based design optimization of design variance, to identify critical tolerances. The reliability-based tolerance optimization model has become an indispensable part of tolerance redesigning.

Various tolerance optimization methods have been applied, to determine the optimal tolerance. Numerical optimization methods are employed, including genetic algorithm,13–15 particle swarm,16 Newton iteration method,17 scatter search,18 etc. The Lagrange multiplier method, as a classic method for constrained optimization problems, should be the first choice, as it can yield closed-form solution.19 Karush–Kuhn–Tucker (KKT) conditions extend the application of Lagrange multiplier method, from equality constraint to the inequality, while they are necessary conditions for the best solution of nonlinear programming. Ramesh Kumar et al.20 proposed an analytical method, to obtain a valid design by the Lagrange multiplier method, which integrates the Lambert W function. The
generalized KKT conditions for generalized Lagrange multiplier method are derived by Li.\cite{21} Satisfying the reliability-based inequality constraint, the assembly tolerance needs to be redesigned using optimal process parameters.

Based on the above ideas, a reliability-based tolerance redesign method, including the optimal process parameters, is proposed. The rest of this work is organized as follows: In section 2, the Jacobian model and Torsor representation are introduced, while two kinds of tolerance contributions are obtained. Section 3 presents the reliability and sensitivity analysis of assembly tolerance. The process variance optimization and tolerance optimization are carried out in section 4. Moreover, in section 5, the implementation and effectiveness of the proposed model are demonstrated. Section 6 includes the derived conclusions.

**Parametric Jacobian-Torsor model**

In this section, Jacobian-Torsor (J-T) model is employed to establish the mathematical relationship between the assembly functional requirement and the function of spatial dimension chain. A probabilistic framework is employed, by assuming that dimensional and geometrical tolerances are modeled using random variables. In a parametric Jacobian-Torsor model, the aleatory uncertainty is also considered.

The J-T model is composed of the Jacobian matrix model and the Small Displacement Torsor (SDT) model. A Torsor model is a vector consisting of three rotational components and three translational components, which represent the position and orientation relative to a nominal position. It can be represented as: \([u, v, w, \alpha, \beta, \gamma]^T\), where \(u, v, \) and \(w\) are the small translation components on the \(x-, y-\) and \(z-\)axes respectively; \(\alpha, \beta, \) and \(\gamma\) are the small rotation components around the \(x-, y-\) and \(z-\)axes respectively. The Torsor model describes the small displacement variation of features with constrained torsors, which define the extreme limits in the 3D tolerance zone.

Jacobian matrixes describe the positions and orientations of the local reference frames in relation to the global reference frame, derived from the analysis of robot kinematic, in order to map velocities and displacements of joints to the end-effector coordinates system. The SDT of each functional element pair is computed to establish the stackup of functional requirements. There are two kinds of functional element pairs in an assembly: the interval functional elements pair and the contact functional elements pair. The difference is whether on the identical parts or not. The Jacobian matrix for both kinds of functional element pairs can be expressed as:

\[
[J]_{FEi} = \begin{bmatrix}
[R_0^i]_{3 \times 3} \cdot [R_{PTi}]_{3 \times 3} & [W^n_i]_{3 \times 3} \cdot [R_0^n]_{3 \times 3} \cdot [R_{PTi}]_{3 \times 3}
\end{bmatrix}
\]

where \([R_0^i]\) is the orientation matrix of the \(i^{th}\) reference frame relative to the global reference system; \([R_{PTi}]\) is a projection matrix to establish the projection of \(i^{th}\) local reference frame along the direction of tolerance analysis; \([W^n_i]\) is a Skew-symmetric position matrix representing the position vector between the \(i^{th}\) and \(n^{th}\) reference
frame (end point); $[R_0^i], [R_{PT_i}]$ reflects the leverage effect of $i^{th}$ rotation and displacement, relative to the functional requirement.

$$[W_i^n]_{3 \times 3} = \begin{bmatrix}
0 & d\alpha_i^n & -d\beta_i^n \\
-d\beta_i^n & 0 & d\gamma_i^n \\
d\gamma_i^n & -d\alpha_i^n & 0
\end{bmatrix}$$

(2)

where $dx_i^n = dx_n - dx_i$, $dy_i^n = dy_n - dy_i$, and $dz_i^n = dz_n - dz_i$, $n$ is the total number of dimensional and geometric tolerances.

Combining the advantages of Jacobian matrix model and Torsor model, the expression of the unified J-T model is as follows:

$$u v w a b g 266666664 \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} F_{E1} \cdots \begin{bmatrix} u_n \\ v_n \\ w_n \\ \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} F_{En} = \begin{bmatrix} u \\ v \\ w \\ \alpha \\ \beta \\ \gamma \end{bmatrix} F_R = C_{FE1} + \cdots C_{FEi} + \cdots C_{FEn} = C_{FR}$$

(3)

where $u, v, w, \alpha, \beta, \gamma$ and $u_i, v_i, w_i, \alpha_i, \beta_i, \gamma_i, i = 1...n$ are the random variables, which are associated with the tolerance value $t_i$. $C_{FEk}$ is the contribution of the FEk to the total tolerance $C_{FR}$.

The tolerance percentage contributions of each functional element are calculated as:

$$PC_{FEi} = \frac{C_{FEi}}{C_{FR}}$$

(4)

The tolerance percentage contributions of each tolerance $t_i$ to the total tolerance $C_{FR}$ can be calculated as:

$$PC_{ii} = \frac{C_{ii}}{C_{FR}}$$

(5)

**Tolerance-reliability and sensitivity analysis**

Uncertainty is usually decomposed into aleatory uncertainty (e.g. variability in material properties) and epistemic uncertainty (e.g. uncertainty arising from measuring procedures or possible human errors). The role of these uncertainty sources, along with their potential for reduction, is extremely difficult to detect and qualify. The focus will be on the uncertainty, stemming from both part tolerances and process parameters. Dimensional and geometrical tolerances are modeled by random variables. Finally, the assembly quality can be quantitatively measured by the degree of reliability $R$ based on the parametric Jacobian-Torsor model. The
reliability of assembly tolerance refers to the probability that the assembly tolerance falls within the maximum allowable range, in the specified direction. The tolerance of functional requirement ($t_{FR}(t)$) and the quality requirement ($t_0$) are both considered. It is assumed that the components of SDT are independent and follow normal distributions, $P_i(t_i) = N(\mu_i, \sigma_i), i = 1...n$. $P_i(t_i)$ indicates the probability density function of the corresponding tolerance $t_i$.

The limit state function of tolerance requirement can be defined as follows:

$$g(t) = \frac{t_0}{C_0} - |t_{FR}(t)|$$

where $t_{FR}(t)$ is a function of the $n$ independent variables $t_1, t_2...t_n$. $g(t) = 0$ indicates the critical state of assembly tolerance; $g(t)>0$ represents the safe state; $g(t)<0$ refers to the failure state and the assembly tolerance does not fulfill the quality requirement.

In order to control the assembly quality of functional requirement, the degree of reliability $R$ of the mechanical assembly is presented as:

$$R = P(g(t) \geq 0)$$

The relation as expressed in equation (7), can be transferred into the standard normal space.

$$R = P(|t_{FR}(t)| \leq t_0) = 2\Phi\left(\frac{t_0 - \mu_{FR}}{\sigma_{FR}}\right) - 1$$

The different sensitivities of dimensional and geometrical tolerances $t_i$ can be ranked according to their contribution to the tolerance of functional requirement $t_{FR}$, which can help to reduce the uncertainty and improve the assembly reliability. The partial derivative of the model response function in relation to the input variable, at the nominal value point, is written as follows:

$$S_i = \frac{\partial C_{FR}(t_1, ...t_i, ...t_n)}{\partial t_i}$$

where $S_i$ indicates the sensitivity of the tolerance $t_i$ to the total tolerance $C_{FR}(t_1, ...t_i, ...t_n)$.

**Tolerance optimization and redesign**

The process parameter optimization (process variance) is conditional on the assembly reliability requirement and involves a trade-off between the quality control and the relevant costs. The optimal setting of process parameters is determined using Lagrange multiplier method and considering the quality loss and process cost, simultaneously. KKT-conditions extend the application of Lagrange multiplier method and the candidate solution is the optimal solution, when the KKT
conditions are satisfied. Based on the analytical solution of optimal process variance, tolerance optimization model is employed to get the optimal tolerances.

**Parameter optimization**

The problem of parameter optimization involves the process variance. The process mean can be determined according to the design requirements. Due to inherent variability of mechanical components, it is necessary to select the optimal parameter by minimizing the total cost. The quality loss function of the assembly is given by\(^{23}\):

\[ L_1(t_i) = k t_i^2, \quad i = 1...n \]  

where \( k \) is the quality loss coefficient, \( t_i \) is the \( i \)th assembly tolerance and follows normal distribution with mean \( \mu_i \) and variance \( \sigma_i \). The expected quality loss is expressed as follows:

\[ E(L_1(t_i)) = \int_{-\infty}^{+\infty} k t_i^2 f(t_i) dt_i \sqrt{b^2 - 4ac} \quad i = 1...n \]  

where \( L_1(t_i) \) is the quality loss of the \( i \)th assembly tolerance and \( f(t_i) \) is the probability density function of \( t_i \).

The process cost can be expressed by the empirical linear model by Chase et al.\(^{24}\):

\[ C_{M1i} = c_0 + c_1 \mu_i + c_2 \sigma_i + c_3 \mu_i \sigma_i \]  

where \( c_0, c_1, c_2, \) and \( c_3 \) are the regression coefficients of empirical model, described by polynomial fitting and associated with the manufacturing costs.

The expected total cost of the assembly is expressed as follows:

\[ E(TC_1) = \sum_{i=1}^{n} (E(L(t_i))) + E(C_{M1i})) = \sum_{i=1}^{n} (k \sigma_i^2 + c_0 + c_1 \mu_i + c_2 \sigma_i + c_3 \mu_i \sigma_i) \]  

Considering the assembly tolerance reliability constraint, the optimal value of \( \sigma^* \) can be determined by Lagrange multiplier method:

\[ \min E(TC_1) \]  

\[ \text{s.t. } P(g(t) \geq 0) \geq R_0 \]

The Lagrange function, considering inequality constraint, can be expressed as follows:

\[ L_1 = E(TC_1) + \lambda (R_0 - P(g(t) \geq 0)) \]  

where \( \lambda \) is the Lagrange multiplier.

The necessary conditions to get the optimal solution can be provided according to Karush–Kuhn–Tucker conditions\(^{25}\).
\[
\begin{aligned}
\frac{\partial L}{\partial \lambda} &= 0 \\
\lambda h(t_i^*) &= 0 \\
\lambda &\geq 0 \\
i &= 1..n
\end{aligned}
\]  

(16)

Then, the analytical solution of optimal process variance can be obtained under the constraint of assembly reliability.

**Tolerance optimization**

The tolerance \(t_i\) can be defined as an interval, whose upper limits \(U_i\) and lower limits \(L_i\) can be expressed as \(\mu_i + \delta_i \sigma_i\) and \(\mu_i - \delta_i \sigma_i\), respectively. \(\mu\) can be obtained according to the design requirements and \(\sigma_i\) is the optimal process variance of \(i^{th}\) tolerance; \(\delta_i\) represents the deviation range of \(i^{th}\) tolerance from the mean. The limit of some assembly tolerances is 0, while the tolerance can be defined as \(\mu_i / C_0\) and 0, or 0 and \(\mu_i + \delta_i \sigma_i\). When the quality characteristics fall within the range of tolerance limits, the expected quality loss can be given as follows:

\[
E(L_2(t_i)) = \int_{L_i}^{U_i} k t_i^2 f(t_i) dt_i \quad i = 1..n
\]  

(17)

where \(k\) is a positive quality loss coefficient, which can be determined from the information on losses, relating to exceeding the tolerance given by the customer. \(f(t_i)\) denotes the probability density function of the tolerance \(t_i\). \(U_i\) and \(L_i\) represent the upper and lower limit, respectively.

Letting \(z_i = (t_i - \mu_i) / \sigma_i\), the equation can be transferred to the standard normal space, where \(\sigma_i\) is the optimal process variance for the \(i^{th}\) assembly tolerance.

In case the \(i^{th}\) quality characteristic falls outside the range of tolerance limits, the components are unacceptable and will increase the costs. The expected cost of falling within the rejection range is defined as follows:

\[
E(C_R) = C_R \left( \int_{-\infty}^{L_i} f(t_i) dt_i + \int_{U}^{+\infty} f(t_i) dt_i \right)
\]  

(18)

where \(C_R\) denotes the rejection unit cost, when the quality characteristic falls within the rejection range.

The manufacturing cost is associated with the range of tolerance. A tight assembly tolerance can increase the respective manufacturing cost. The range of tolerance can be written in terms of \(U_i\) and \(L_i\),

\[
T_i = U_i - L_i = (\mu_i + \delta_i \sigma_i) - (\mu_i - \delta_i \sigma_i) = 2\delta_i \sigma_i
\]  

(19)

where \(\delta_i\) is positive coefficient and represents the number of standard deviations from the middle of tolerance range to the \(i^{th}\) specification limit.
The manufacturing cost function is usually modeled as a first-order form,

$$C_{M2} = a_0 + a_1 T_i + \varepsilon$$

(20)

where $\varepsilon$ is the least-squares error; $a_0, a_1$ are the regression coefficients of the empirical cost model. The expectation of manufacturing cost can be expressed as:

$$E(C_{M2}) = a_0 + 2a_1 \delta_i \sigma_i$$

(21)

The expectation of total cost is established as follows:

$$E(TC_2) = \sum_{i=1}^{N} (E(L_2(t_2)) + E(C_R) + E(C_{M2}))$$

$$= \sum_{i=1}^{N} (k\sigma_i^2(2\Phi(\delta_i) - 2\delta_i\phi(\delta_i) - 1) + 2C_R(1 - \Phi(\delta_i)) + a_0 + 2a_1 \delta_i \sigma_i)$$

(22)

Based on the optimal process tolerance variances $\sigma_i^*$, the Lagrange multiplier method can be applied to obtain the optimal $\delta_i^* i = 1...n$.

$$\frac{\partial E(TC_2)}{\partial \delta_i} = 2a_1 \sigma_i - 2C_R \phi(\delta_i) + 2k\delta_i^2 \phi(\delta) = 0$$

(23)

The closed-form solution of $\delta_i^*$ expressed by the Lambert W function is given by:

$$\delta_i^* = \sqrt{-2LambertW\left(\frac{a_1\sqrt{2\pi} \cdot e^{\frac{C_R}{2k\sigma_i^*2}}}{2k\sigma_i^*}\right) + \frac{C_R}{k\sigma_i^*2}}$$

(24)

Satisfying the following inequation, the closed-form solutions $\delta_i^*$ are the global minimum of the total cost.

$$\frac{\partial^2 E(TC_2)}{\partial \delta_i^2} = 2k\delta_i\phi(\delta_i)\left(\frac{C_R}{k} + 2\delta_i^2\left(1 - \frac{\delta_i^2}{2}\right)\right) > 0$$

(25)

**Example verification**

In this section, a numerical example is employed to demonstrate the application of tolerance optimization redesign model. The assembly composed of three components is illustrated in Figure 1. The functional requirement of this assembly is the vertical dimension of the centering pin. The corresponding dimensional and geometric tolerances are shown in Figure 2.

Figure 1 shows that, there are six effective geometric features in the assembly, which give base to the established local coordinates frames. The contact between
the surfaces of column and base is assumed to be perfect. The connection graph of the local coordinates frames is constructed in Figure 3. The connection graph contains three interval functional element pairs and two contact functional element pairs. The details of unified J-T model are listed in Table 1. The torsors are associated with the tolerance $t_i$. The final representation of the parametric J-T model is calculated as:
Figure 3. Assembly connection graph.

Table 1. Details of torsors.

| Torsors | Constraints |
|---------|-------------|
| \( \tau_{1/0} \) | \( \alpha_1 = t_2/100, \beta_1 = t_2/80, w_1 = t_1 \) |
| \( \tau_{3/2} \) | \( \alpha_2 = \gamma_2 = t_4/50, u_2 = t_4, w_2 = t_3 \) |
| \( \tau_{4/3} \) | \( \alpha_3 = \gamma_3 = (t_5-t_6)/50, u_3 = w_3 = (t_5-t_6)/2 \) |
| \( \tau_{5/4} \) | \( \alpha_4 = \gamma_4 = t_7/30, u_4 = w_4 = t_7/2 \) |

\[
\begin{bmatrix}
  u \\
v \\
w \\
\alpha \\
\beta \\
\gamma
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 55 & -110 \\
  0 & 1 & 0 & -55 & 0 & 0 \\
  0 & 0 & 1 & 110 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & -85 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 85 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  0 \\
  u_2 \\
  u_3 \\
  u_4 \\
  w_1 \\
  w_2 \\
  w_3 \\
  w_4 \\
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3 \\
  \alpha_4 \\
  \beta_1 \\
  \gamma_2 \\
  \gamma_3 \\
  \gamma_4
\end{bmatrix}
\]
The assigned initial tolerance values are listed in the Table 2. Using equations (4) and (5), the tolerance percentage contributions of each functional element and tolerance $t_i$ are calculated. The derived tolerance percentage contribution pies are illustrated in Figure 4.

Based on the experimental observations, the dimensional and geometric tolerances are modeled in the normal distribution. The obtained data of normal distribution can be expressed as follows:

$$t_1 : N(15, 0.04)$$
$$t_2 : N(15, 0.05)$$
$$t_3 : N(55, 0.14)$$
$$t_4 : N(55, 0.13)$$
$$t_5 : N(20, 0.06)$$
$$t_6 : N(20, 0.07)$$
$$t_7 : N(15, 0.04)$$

The LSF of assessing the quality requirement can be considered as follows:

$$g(t) = 1.5 - |t_{FR}(t)|$$ (28)
Based on the proposed method, the computed reliability value is 95.10%, which does not satisfy the assembly reliability requirement of 96.00%. It is necessary to conduct the reliability-based tolerance optimization and allowance. According to equation (9), the sensitivities of different tolerances to the assembly tolerance are shown in the Figure 5.

Figure 4 shows that the tolerances of $t_5$ and $t_6$ have the most significant sensitivity, while the variance of $t_7$ has the least influence on the tolerance of functional requirement. There is a negative correlation between the tolerance of functional requirement and $t_6$, while positive correlations are observed between the tolerances of $t_1$–$t_5$, $t_7$, and $t_{FR}$.

Due to the deviation from the quality requirement, the total quality loss function is described as the sum of the quadratic functions $\sum_{i=1}^{7} E(L(t_i))$, where $k$ and $T$ are 800 and 50, respectively. When the inspections are not conducted, the empirical linear model, associated with manufacturing costs, is described by the polynomial in equation (13), where $c_0$, $c_1$, $c_2$, and $c_3$ are 3, $-6.1$, $-3.1$, and $-3.85$, respectively. In the case where actual inspections are conducted, the manufacturing costs are expressed as in equation (20), where $a_0$ and $a_1$ are 5 and $-0.2$, respectively. When the dimensions of components fall outside the limits of $L_i$ and $U_i$, the costs of rejection range are calculated by equation (18), where $C_R$ is set to be three in this specific assembly.

The objective function is as follows:

$$E(TC_1) = \sum_{i=1}^{n} (800\sigma_i^2 + 3 - 6.1\mu_i - 3.1\sigma_i - 3.85\mu_i\sigma_i)$$

(29)

Subject to the following:

$$\Phi\left(\frac{1.5 - \mu_{FR}}{\sigma_{FR}}\right) \geq 0.98$$

(30)
According to the KKT-conditions, the necessary conditions are expressed as:

\[
\begin{align*}
\frac{\partial E(TC_1)}{\partial \sigma_i} &= 0 \\
\lambda \left( \Phi \left( \frac{1.5 - \mu_{FR}}{\sigma_{FR}} \right) - 0.98 \right) &= 0 \\
\lambda &\geq 0
\end{align*}
\]

\[i = 1..n\]  

The optimal solution \(\sigma_i^*\) can be calculated as follows: \(\sigma_1^* = 0.038, \sigma_2^* = 0.038, \sigma_3^* = 0.1343, \sigma_4^* = 0.1343, \sigma_5^* = 0.0501, \sigma_6^* = 0.0501, \sigma_7^* = 0.038\), while the results satisfy the condition of reliability, which can also be illustrated in the Figure 6. Considering the variation of process variance as \(\sigma_i\), the expected quality loss, the process cost and the total costs of the assembly can be demonstrated in Figure 7.

The solution of the analytical method is consistent with the results in Figure 6, while the lowest point of the curve represents the minimum costs. The illustration in Figure 6 validates the process variance optimization based on assembly reliability.

Based on the variation of process variance \(\sigma_i\), the expectation of total cost can be calculated as follows:
\[ E(TC_2) = \sum_{i=1}^{N} (800\sigma_i^2(2\Phi(\delta_i) - 2\delta_i\phi(\delta_i) - 1) + 6(1 - \Phi(\delta_i)) + 5 - 0.4\delta_i\sigma_i) \] (32)

The plots of \( E(TC_2) \), \( E(L_2(t_i)) \), \( E(C_R(t_i)) \), and \( E(C_M(t_i)) \) with respect to \( \delta_i \) are shown as follows:

Using equations (23) and (25), the following expressions are calculated as:

\[ \frac{\partial E(TC_2)}{\partial \delta_i} = 2a_1\sigma_i - 2C_R\phi(\delta_i) + 2k\delta_i^2\phi(\delta) \] (33)

\[ \frac{\partial^2 E(TC_2)}{\partial \delta_i^2} = 1600\delta_i\phi(\delta_i)\left(\frac{3}{800} + 2\delta_i^2\left(1 - \frac{\delta_i^2}{2}\right)\right) \] (34)

Based on the solution of equation (24), the optimal \( \delta_i \) are obtained as:

\( \delta_1^* = 1.6294 \), \( \delta_2^* = 1.6294 \), \( \delta_3^* = 0.4617 \), \( \delta_4^* = 0.4617 \), \( \delta_5^* = 1.2341 \), \( \delta_6^* = 1.2341 \), \( \delta_7 = 1.6294 \), which can be illustrated in Figure 7. The optimal tolerance limits \( U_i \) and \( L_i \) are listed in Table 3.

Figure 8 shows the first derivative and the second derivative of \( E(TC_2) \) with respect to \( \delta_i \). It can be observed that, the \( \frac{\partial E(TC_2)}{\partial \delta_i} \) is at its maximum, when the \( \frac{\partial^2 E(TC_2)}{\partial \delta_i^2} \) becomes 0, while the \( \frac{\partial E(TC_2)}{\partial \delta_i} \) becomes 0 when \( \delta_i \) is at the optimal point. It is evident that, the plots are in good agreement with the closed-form solutions.

Based on the optimal solutions, the comparison of the original to the optimal costs is presented in Figure 9. The cost of each tolerance \( t_i \) appears reduced and the optimal costs are also obtained. Using equation (7), the assembly reliability is improved from 95.10% to 97.22%. The detailed comparison of original and optimal data is reported in Table 3.

The obtained results show that, the proposed tolerance-reliability analysis and tolerance optimization can achieve lower cost and higher reliability. Based on equation (27), the original tolerance of functional requirement is calculated as \((-0.9245, + 0.9245)\), while the optimal tolerance of functional requirement obtains improved assembly tolerance of \((-0.6318, + 0.6318)\). Consequently, the proposed tolerance redesign model can produce ideal results.

**Conclusion**

This study presents a reliability-based tolerance optimization model. The Jacobian-Torsor model is applied to assembly tolerance analysis and analytical method is employed to obtain the closed-form solutions of the optimal tolerance. The torsors are represented by random variables, associated with assembly tolerances, which is the premise of assembly tolerance reliability analysis. The proposed tolerance optimization model is superior to the models, based on the assumption that process parameters are fixed. The optimal process variance and tolerance are considered
separately, while the total costs, including quality loss, rejection cost, manufacturing cost, are treated as the objective function of parameter and tolerance optimization. Subject to the constraint of reliability-based inequality, the KKT conditions are employed to get a closed-form solution. Finally, the tolerance reliability

Figure 7. Plots of the effect of $\delta_i$ on $E(\text{TC}_2)$.

Table 3. The comparison of original and optimal data.

| $t_i$ | $\mu$ | $\delta_i$ | $\delta$ | $t$ | $\delta$ | $\text{cost}$ | $\text{cost}$ | $R$ | $R$
|-------|-------|-----------|----------|-----|----------|--------------|--------------|-----|-----
| $t_1$ | 15    | 0.04      | 0.038    | $\pm 0.1$ | $\pm 0.062$ | 50.354       | 44.852       | 95.10% | 97.22% |
| $t_2$ | 15    | 0.05      | 0.038    | 0.125     | 0.062     |              |              |       |       |
| $t_3$ | 55    | 0.14      | 0.1343   | $\pm 0.14$ | $\pm 0.0623$ |              |              |       |       |
| $t_4$ | 55    | 0.13      | 0.1343   | 0.13      | 0.0619    |              |              |       |       |
| $t_5$ | 20    | 0.06      | 0.0501   | (0, 0.06) | (0, 0.0618) |              |              |       |       |
| $t_6$ | 20    | 0.07      | 0.0501   | (-0.07, 0) | (-0.0618, 0) |              |              |       |       |
| $t_7$ | 15    | 0.04      | 0.038    | 0.08      | 0.0621    |              |              |       |       |
Figure 8. Plots of $\partial E(TC_2)/\partial \delta_i$ and $\partial^2 E(TC_2)/\partial \delta_i^2$ with respect to $\delta_i$.

Figure 9. Comparison plots of original and optimal costs.
satisfies the assembly requirement, while the tolerance redesign of a mechanical assembly is realized at a reduced total cost. It will provide a more practical method for designers to deal with the tolerance redesign of mechanical assemblies.

However, there are limitations to the proposed method, such as the fact that it cannot solve the partial parallel chain problem for mechanical assembly and the assumption that the tolerances follow the normal distribution. In the future, the assembly tolerance reliability of partial parallel connections will be considered to obtain the optimal tolerance, and a non-probabilistic interval uncertainty method could be employed for tolerance reliability analysis, instead of the normal distribution.

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