Uniqueness of the mass in the radiating regime

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Abstract

The usual approaches to the definition of energy give an ambiguous result for the energy of fields in the radiating regime. We show that for a massless scalar field in Minkowski space–time the definition may be rendered unambiguous by adding the requirement that the energy cannot increase in retarded time. We present a similar theorem for the gravitational field, proved elsewhere, which establishes that the Trautman–Bondi energy is the unique (up to a multiplicative factor) functional, within a natural class, which is monotonic in time for all solutions of the vacuum Einstein equations admitting a smooth “piece” of conformal null infinity $\mathcal{J}$.

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Consider a Lagrangian theory of fields $\phi^A$ defined on a manifold $M$ with a Lagrange function density

$$L = L[\phi^A, \partial_\mu \phi^A, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_k} \phi^A],$$

(1)

for some $k \in \mathbb{N}$, where $\partial_\mu$ denotes partial differentiation with respect to $x^\mu$. Suppose further that there exists a function $t$ on $M$ such that $M$ can be decomposed as $\mathbb{R} \times \Sigma$, where $\Sigma \equiv \{ t = 0 \}$ is a hypersurface in $M$ and the vector $\partial/\partial t$ is tangent to the $\mathbb{R}$ factor. The proof of the Noether theorem, as presented e.g. in [Section 10.1] of Ref. 1, shows that the vector density

$$E^\lambda = -L X^\lambda + X^\mu \sum_{\ell=0}^{k-1} \phi^A_{,\alpha_1 \ldots \alpha_\ell \mu} \times \sum_{j=0}^{k-\ell-1} (-1)^j \partial_{\gamma_1} \ldots \partial_{\gamma_j} \left( \frac{\partial L}{\partial \phi^A, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_k} \phi^A} \right)$$

(2)

has vanishing divergence, $E^\lambda,\lambda = 0$, when the fields $\phi^A$ are sufficiently smooth and satisfy the variational equations associated with a sufficiently smooth $L$ (cf. also Ref. 2). (This is in any case easily seen by calculating the divergence of the right-hand-side of eq. (2).) Here $\phi^A_{,\alpha_1 \ldots \alpha_\ell} = \partial_{\alpha_1} \ldots \partial_{\alpha_\ell} \phi^A$, and $X^\mu \partial_\mu = \partial_\mu$. In theories in which $L$ depends only upon $\phi^A$ and its first derivatives, it is customary to define the total energy associated with the hypersurface $\Sigma$ by the formula

$$E(\Sigma) = \int_\Sigma E^\lambda dS_\lambda,$$

(3)

with $dS_\lambda = \partial_\lambda \ldots dx^0 \wedge \ldots dx^3$, where $\ldots$ denotes contraction. By extrapolation one can also use (3) to define an “energy” for higher order theories. Now it is well known that the addition to $L$ of a functional of the form

$$\partial_\lambda (Y^\lambda[\phi^A, \partial_\alpha \phi^A, \ldots, \partial_{\alpha_1} \ldots \partial_{\alpha_{k-1}} \phi^A]) ,$$

(4)

where $k$ is as in (4), does not affect the field equations. Such a change of the Lagrange function will change $E(\Sigma)$ by a boundary integral (see, e.g., Ref. 1 for an explicit formula for $\Delta E^\mu^\lambda$):

$$E(\Sigma) \rightarrow \hat{E}(\Sigma) = E(\Sigma) + \int_{\partial \Sigma} \Delta E^\mu_\lambda dS^\mu_\lambda ,$$

(5)

where $S_{\alpha \beta} = \partial_\alpha \ldots \partial_\beta \ldots dx^0 \wedge \ldots \wedge dx^3$. If $\partial \Sigma$ is a “sphere at infinity” the integral over $\partial \Sigma$ has of course to be understood by a limiting process. Unless the boundary conditions at $\partial \Sigma$ force all such boundary integrals to give a zero contribution, if one wants to define energy using this framework one has to have a criterion for choosing a “best” functional, within the class of all functionals obtainable in this way. The vanishing of such boundary integrals will

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1Here we adopt the standard point of view, that the field equations are obtained by requiring the action to be stationary with respect to all compactly supported variations (cf. e.g. Ref. 3 for a discussion of problems that might arise when this requirement is not enforced).
not occur in several cases of interest, including a massless scalar field and general relativity in the radiation regime.

As an example, consider a scalar field \( \phi \) in the Minkowski space–time, with \( \Sigma = \{ t = 0 \} \). Assume that \( \phi \) satisfies the rather strong fall–off conditions on \( \Sigma \)

\[
\partial_{a_1} \ldots \partial_{a_j} \phi = o(r^{-2}), \quad 0 \leq j \leq k - 1 ,
\]

where \( k \) is the integer appearing in (1). In this case the boundary integral in (3) will vanish for all smooth \( Y^\mu \)'s, as considered in eq. (4). This shows that eq. (3) leads to a well–defined notion of energy on this space of fields (whatever the Lagrange function \( \mathcal{L} \)), as long as the volume integral there converges. (That will be the case if, e.g., \( \mathcal{L} \) has no linear terms in \( \phi \) and its derivatives.)

Consider, next, a scalar field in Minkowski space–time, with \( \Sigma \) being a hyperboloid, \( t = \sqrt{1 + x^2 + y^2 + z^2} \). Suppose further that in Minkowski coordinates \( \mathcal{L} = \frac{1}{2} \eta^\mu\nu \partial_\mu \phi \partial_\nu \phi \), so that the field equations read

\[
\Box \phi = 0 .
\]

In that case the boundary conditions (6) would be incompatible with the asymptotic behavior of those solutions of eq. (7) which are obtained by evolving compactly supported data on \( \{ t = 0 \} \) (see eq. (8) below). Thus, even for scalar fields in Minkowski space–time, a supplementary condition singling out a preferred \( E^\lambda \) is needed in the radiation regime.

Now for various field theories on the Minkowski background, including the scalar field, one can impose some further conditions on \( E^\lambda \) which render it unique \( \Box \phi \), such as Lorentz covariance, and dependence upon the first derivatives of the field only. The extension of that analysis to the gravitational field has been carried out in \( \Box \phi \), and it also leads to a unique \( E^\lambda \) (namely the one obtained from the so–called “Einstein energy–momentum pseudo–tensor”), within the class of objects considered. While this is certainly an interesting observation, the restrictions imposed in that last paper on the energy–momentum pseudo–tensor of the gravitational field are much more restrictive than is desirable. Thus it seems of interest to find a more natural criterion, which would encompass both general relativity and field theories on a Minkowski background, and which would single out a preferred expression for energy in the radiation regime. In this letter we wish to point out that the requirement of monotonicity of energy in retarded time allows one to single out an energy expression in a unique way, within a natural class of “energies”. Let us start with the case of a massless scalar field in Minkowski space–time. The variational formalism described above leads one to consider functionals of the form

\[
H[\phi, t] = E(\Sigma_t) + \int_{\partial \Sigma_t} H^{\alpha \beta} dS_{\alpha \beta} ,
\]

\[
E(\Sigma_t) = \int_{\Sigma_t} T^\mu_\nu X^\nu dS_\mu , \quad X^\nu \partial_\nu = \partial_t ,
\]

where \( H^{\alpha \beta} \) is a twice continuously differentiable function of \( \phi(x), \partial_{a_1} \phi(x), \ldots, \partial_{a_1} \ldots \partial_{a_n} \phi(x) \), for some \( n \). The indices \( \alpha \) refer to Minkowski coordinates, and in Minkowski coordinates the \( H^{\alpha \beta} \)'s depend upon the coordinates through the fields only. Here and throughout the \( \Sigma_t \)'s are unit hyperboloids in Minkowski space–time: \( \Sigma_t = \{ x^0 = t + \)
\[
\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2},
\]
\(T^\mu_\nu\) is the standard energy–momentum tensor for the scalar field (with the normalization determined by eq. (2)), \(\frac{\partial}{\partial \Sigma_t} H^{\alpha\beta} dS_{\alpha\beta}\) is understood as a limit as \(R\) tends to infinity of integrals on coordinate balls of radius \(R\) included in \(\Sigma_t\).

Before analyzing convergence of functionals (8) we need to specify the class of fields \(\phi\) of interest. Consider solutions of (7) which have smooth compactly supported initial data on the hyperplane \(\{x^0 = 0\}\), where \(x^0\) is a standard Minkowski coordinate. Using conformal covariance of eq. (7) (cf., e.g., arguments in Ref. [7]) it can be shown that there will exist smooth functions \(c(u, \theta, \phi)\) and \(d(u, \theta, \phi)\) defined on \((-\infty, \infty) \times S^2\) such that

\[
\phi(u, r, \theta, \phi) - \frac{c(u, \theta, \phi)}{r} - \frac{d(u, \theta, \phi)}{r^2} = O(r^{-3}),
\]
(9)

with \(u = x^0 - r\). Moreover (9) is preserved under differentiation in the obvious way. (The hypothesis that the initial data are compactly supported is not necessary, and is made only to avoid unnecessary technical discussions.) In what follows we will only consider solutions of (7) satisfying (9). For our purposes it is important to emphasize that given arbitrary functions \(c\) on, say \([u_0 - 1, u_0 + 1] \times S^2\) and \(d_0\) on \(S^2\) there exists a solution of the wave equation defined on Minkowski space–time such that (9) holds, with

\[
\frac{\partial d}{\partial u} = -\frac{1}{2} \Delta_2 c, \quad d(u_0, \theta, \phi) = d_0(\theta, \phi),
\]
(10)

Here \(\Delta_2\) denotes the Laplace operator on \(S^2\) with the standard round metric. (Eq. (10) is obtained by inserting the expansion (9) in (7)). This assertion is easily proved using again the conformal covariance of eq. (7). We claim the following:

**Theorem 1** Let \(H\) be as described above and suppose that (8) converges for all solutions of the wave equation satisfying (9). If \(\frac{dH}{dt} \leq 0\), then for all such \(\phi\)’s

\[
\int_{\partial \Sigma_t} H^{\alpha\beta} dS_{\alpha\beta} = 0,
\]

so that the numerical value of \(H\) equals the standard canonical energy.

**Proof:** We can Taylor expand \(H^{\alpha\beta}\) at \(\phi = 0\) up to second order to obtain

\[
H^{\alpha\beta} = H^{\alpha\beta}\big|_{\phi=0} + \sum_{0 \leq |I| \leq k} \frac{\partial H^{\alpha\beta}}{\partial \phi_I} \big|_{\phi=0} \phi_I + \sum_{0 \leq |I|, |J| \leq k} \frac{\partial^2 H^{\alpha\beta}}{\partial \phi_I \partial \phi_J} \big|_{\phi=0} \phi_I \phi_J + r^{\alpha\beta},
\]
(11)

where we use the symbol \(\phi_I\) to denote objects of the form \(\partial_{\alpha_1} \ldots \partial_{\alpha_\ell} \phi\), with \(|I| = |(\alpha_1, \ldots, \alpha_\ell)| = \ell\). By hypothesis \(H^{\alpha\beta}\big|_{\phi=0}\) depends only upon the metric and its derivatives, so in Minkowski coordinates the coefficients \(H^{\alpha\beta}\big|_{\phi=0}, \frac{\partial H^{\alpha\beta}}{\partial \phi_I} \big|_{\phi=0}\) etc. in (11) are constants.
By well known properties of Taylor expansions and by (9) we have \( r^{\alpha \beta} = o(r^{-2}) \), so that \( r^{\alpha \beta} \) will not contribute to \( H \) in the limit \( r \to \infty \). In Minkowski coordinate systems (9) can be rewritten as

\[
\frac{\partial}{\partial x^\alpha_1} \ldots \frac{\partial}{\partial x^\alpha_j} \phi = \frac{c^{(j)}}{r} n_{\alpha_1} \ldots n_{\alpha_j} + \frac{L_{\alpha_1 \ldots \alpha_j}}{r^2} + O(r^{-3}),
\]

(12)

\[
\frac{\partial}{\partial x^\alpha_1} \ldots \frac{\partial}{\partial x^\alpha_j} d = \frac{d^{(j)}}{r} n_{\alpha_1} \ldots n_{\alpha_j} + O(r^{-3}),
\]

(13)

where \( c^{(m)}(t, r, \theta, \phi) = \frac{\partial^m}{\partial u^m} (c(u, \theta, \phi)) \bigg|_{u=t-r} \), similarly for \( d^{(m)} \), and \( n_\mu = (1, -\mathbf{x}^i, r) \). Here \( L_{\alpha_1 \ldots \alpha_j} \) is a linear functional of \( c \) and its \( u \) derivatives up to order \( j-1 \). Inserting (9) in (11) and making use of (12)–(13) might produce several terms which do not obviously converge, but those have to cancel out or integrate out to zero by our hypothesis of convergence of (8). It then follows that (8) can be rewritten as

\[
H = E(\Sigma_t) + \int_{S^2} \hat{h}(c, c^{(1)}, \ldots, c^{(k)}, d^{(1)}, \ldots, d^{(k)}, \theta, \phi) \, d^2 \mu,
\]

(14)

for some functional \( \hat{h} \), smooth in all its arguments, with \( d^2 \mu = \sin \theta \, d\theta \, d\phi \). Moreover \( \hat{h} \) is linear in \( d \) and its \( u \)–derivatives. Eq. (10) allows one to eliminate the \( u \)–derivatives of \( d \) in terms of derivatives of \( c \), so that (14) can be rewritten as

\[
H = E(\Sigma_t) + \int_{S^2} (h(c, c^{(1)}, \ldots, c^{(k)}, \theta, \phi) + \alpha(d, \theta, \phi)) \, d^2 \mu,
\]

(15)

for some functionals \( h \) and \( \alpha \), with \( \alpha \) linear in \( d \). The \( u \)–derivative of (15) gives

\[
\frac{dH}{dt} = \int_{S^2} (-c^{(1)})^2 + \frac{\delta h}{\delta c} c^{(1)} + \ldots + \frac{\delta h}{\delta c^{(k+1)}} c^{(k+1)} + \frac{\delta \alpha}{\delta d} d^{(1)} \, d^2 \mu,
\]

Since \( c^{(k+1)} \) is an arbitrary function on \( S^2 \) at fixed \( c, \ldots, c^{(k)} \) and \( d_0 \), we can choose it so that \( \frac{dH}{dt} \leq 0 \) unless \( \frac{\delta h}{\delta c^{(k)}} = 0, k \geq 1 \). A suitable redefinition of \( h \) leads to

\[
H(t) = E(\Sigma_t) + \int_{S^2} (h(c, \theta, \phi) + \alpha(d, \theta, \phi)) \, d^2 \mu,
\]

\[
\frac{dH}{dt} = \int_{S^2} (-c^{(1)} + \frac{\delta h}{\delta c} c^{(1)} - \frac{1}{2} \frac{\delta \alpha}{\delta d} \Delta_2 c) \, d^2 \mu,
\]

(16)

Consider now solutions of the wave equation with \( c^{(1)}(u = u_0) = 0 \). In this case (16) and arbitrariness of \( c(u = u_0) \) imply that \( dH/dt \) will be non–positive if and only if \( \Delta_2 (\frac{\delta \alpha}{\delta d}) = 0 \), which forces \( \frac{\delta \alpha}{\delta d} \) to be a constant. We note that for any constant \( a \) the integral

\[
\int_{S^2} a \, d^2 \mu,
\]

(17)

is a constant of motion (see also [8, Sect. 8.2]. However, integrals of the form (17) cannot arise in the class of functionals considered here. Indeed, the identity (13) shows that all the
terms which would give a non–vanishing contribution to $H$ and which contain derivatives of $d$ contain at least one $u$ derivative of $d$. Then the only possible term which would contain $d$ would come from the term
\[
\int_{S^2} \left. \frac{\partial H^{\alpha \beta}}{\partial \phi} \right|_{\phi=0} \phi dS_{\alpha \beta} = \int_{S^2} \left. \frac{\partial H^{rt}}{\partial \phi} \right|_{\phi=0} (cr + d) d^2 \mu ,
\]
which for generic $c$ diverges when $r$ goes to infinity, unless identically vanishing. We thus obtain $\delta a / \delta d = 0$. The right hand side of eq. (16) can be made positive by choosing $c^{(1)}(u = u_0) = \frac{1}{2} \delta h / \delta c$, unless $\delta h / \delta c = 0$, and our claim follows.

Let us now turn our attention to those gravitational fields which are asymptotically Minkowskian in light–like directions. An appropriate mathematical framework here is that of space–like hypersurfaces which intersect the future null infinity $J^+$ in a compact cross–section $K$. For such field configurations it is widely accepted that the “correct” definition of energy of a gravitating system is that given by Freud [9], Trautman [10,11], Bondi et al. [12], and Sachs [13], which henceforth will be called the Trautman–Bondi (TB) energy. There have been various attempts to exhibit a privileged role of that expression as compared with many alternative ones ( [14–17,6,18–23], to quote a few), but the papers known to us have failed, for reasons sometimes closely related to the ones described above, to give a completely unambiguous prescription about how to define energy at $J^+$. (We make some more comments about that in [4], cf. also [24].) In a way rather similar to that for the case of the scalar field described above, in [4] we show that the TB energy is, up to a multiplicative constant $\alpha \in \mathbb{R}$, the only functional of the gravitational field, in a certain natural class of functionals, which is monotonic in time for all vacuum field configurations which admit (a piece of) a smooth null infinity $J^+$. More precisely, in [4] we show the following:

**Theorem 2** Let $H$ be a functional of the form
\[
H[g, u] = \int_{S^2(u)} H^{\alpha \beta}(g_{\mu \nu}, g_{\mu \nu, \sigma}, \ldots, g_{\mu \nu, \sigma_1 \ldots \sigma_k}) dS_{\alpha \beta},
\]
where the $H^{\alpha \beta}$ are twice differentiable functions of their arguments, and the integral over $S^2(u)$ is understood as a limit as $\rho$ goes to infinity of integrals over the spheres $t = u + \rho, r = \rho$. Suppose that $H$ is monotonic in $u$ for all vacuum metrics $g_{\mu \nu}$ for which $H$ is finite, provided that $g_{\mu \nu}$ satisfies
\[
g_{\mu \nu} = \eta_{\mu \nu} + \frac{h^1_{\mu \nu}(u, \theta, \phi)}{r} + \frac{h^2_{\mu \nu}(u, \theta, \phi)}{r^2} + o(r^{-2}) ,
\]
\[
\partial_{\sigma_1} \ldots \partial_{\sigma_i}(g_{\mu \nu} - \frac{h^1_{\mu \nu}(u, \theta, \phi)}{r} - \frac{h^2_{\mu \nu}(u, \theta, \phi)}{r^2}) = o(r^{-2}),
\]
with $1 \leq i \leq k$, for some $C^k$ functions $h^a_{\mu \nu}(u, \theta, \phi)$, $a = 1, 2$. If $H$ is invariant under passive BMS super-translations, then the numerical value of $H$ equals (up to a proportionality constant) the Trautman–Bondi mass.

Some comments are in order. First, the volume integral $E(\Sigma_t)$ which was present in (8) does not occur in (18), because the Trautman–Bondi mass is itself a boundary integral. Next, Theorem 2 imposes the further requirement of passive BMS invariance, which did not occur in the scalar field case. This requirement arises as follows: recall that the coordinate
systems in which the metric satisfies (13) are, roughly speaking, defined only modulo BMS transformations. Then the requirement of passive BMS invariance is the rather reasonable requirement that the concept of energy be independent of the coordinate system chosen to measure this energy. We note that we believe that the requirement of monotonicity forces the energy to be invariant under (passive) super–translations, but we have not succeeded in proving this so far.

The proof of Theorem 2 is similar to the proof for the scalar field presented here, but technically rather more involved. A key ingredient of the proof is the Friedrich–Kannar [25–27] construction of space–times “having a piece of $\mathcal{J}$”.

It is natural to ask why the Newman–Penrose constants of motion [28], or the logarithmic constants of motion of [29], do not occur in our results of [4]. These quantities are excluded by the hypothesis that the boundary integrand $H^{\alpha\beta}$ which appears in the integrals we consider depends on the coordinates only through the fields. The Newman–Penrose constants could be obtained as integrals of the form (2) (cf. e.g. [30]) if explicit $r^2$ factors were allowed in $H^{\alpha\beta}$. Similarly logarithmic constants could occur as integrals of the form (2) if explicit $1/\ln r$ or $r^{+1}\ln^{-3} r$ factors were allowed there.

Let us finally mention that one can set up a Hamiltonian framework in a phase space which consists of Cauchy data on hyperboloids together with values of the fields on appropriate parts of Scri to describe the dynamics in the radiation regime [31]. Unsurprisingly, the Hamiltonians one obtains in such a formalism are again not unique, but the non–uniqueness can be controlled in a very precise way. The Trautman–Bondi mass turns out to be a Hamiltonian, and an appropriate version of the uniqueness Theorem 2 can be used to single out the TB mass amongst the family of all possible Hamiltonians. In the Hamiltonian framework the freedom of multiplying the functional by a constant disappears.
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