COMPUTING SIMPLE MULTIPLE ZEROS OF POLYNOMIAL SYSTEMS

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Abstract. Given a polynomial system $f$ associated with a simple multiple zero $x$ of multiplicity $\mu$, we give a computable lower bound on the minimal distance between the simple multiple zero $x$ and other zeros of $f$. If $x$ is only given with limited accuracy, we propose a numerical criterion that $f$ is certified to have $\mu$ zeros (counting multiplicities) in a small ball around $x$. Furthermore, for simple double zeros and simple triple zeros whose Jacobian is of normalized form, we define modified Newton iterations and prove the quantified quadratic convergence when the starting point is close to the exact simple multiple zero. For simple multiple zeros of arbitrary multiplicity whose Jacobian matrix may not have a normalized form, we perform unitary transformations and modified Newton iterations, and prove its non-quantified quadratic convergence and its quantified convergence for simple triple zeros.

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1. Introduction

Consider an ideal \( I_f \) generated by a polynomial system \( f = \{f_1, \ldots, f_n\} \), where \( f_i \in \mathbb{C}[X_1, \ldots, X_n] \). An isolated zero of multiplicity \( \mu \) for \( f \) is a point \( x \in \mathbb{C}^n \) such that

1. \( f(x) = 0 \),
2. there exists a ball \( B(x, r) \) of radius \( r > 0 \) such that \( B(x, r) \cap f^{-1}(0) = \{x\} \),
3. \( \mu = \dim(\mathbb{C}[X]/Q_{f,x}) \),

where

\[
B(x, r) := \{y \in \mathbb{C}^n : \|y - x\| < r\},
\]

and \( Q_{f,x} \) is an isolated primary component of the ideal \( I_f \) whose associate prime is \( m_x = (X_1 - x_1, \ldots, X_n - x_n) \).

In [8], based on Rouché’s Theorem [1], the condition (3) is replaced by

(3a) a generic analytic \( g \) sufficiently close to \( f \) has \( m \) simple zeros in \( B(x, r) \).

We recall \( \alpha \)-theory below according to [2] and refer to [40, 38, 39, 41, 37, 44, 17] for more details. Let \( Df(x) \) denote the Jacobian matrix of \( f \) at \( x \). Suppose \( Df(x) \) is invertible, \( x \) is called a simple zero of \( f \). The Newton’s iteration is defined by

\[
N_f(x) = x - Df(x)^{-1} f(x).
\]

Shub and Smale [37] defined

\[
\gamma(f, x) = \sup_{k \geq 2} \left\| Df(x)^{k-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}},
\]

where \( D^k f \) denotes the \( k \)-th derivative of \( f \) which is a symmetric tensor whose components are the partial derivatives of \( f \) of order \( k \), \( \| \cdot \| \) denotes the classical operator norm.

According to [2] Theorem 1, if

\[
\|z - x\| \leq \frac{3 - \sqrt{7}}{2\gamma(f, x)},
\]

then Newton’s iterations starting at \( z \) will converge quadratically to the simple zero \( x \).

If \( y \) is another zero of \( f \), according to [2] Corollary 1, we have

\[
\|y - x\| \geq \frac{5 - \sqrt{17}}{4\gamma(f, x)},
\]

which separates the simple zero \( x \) from other zeros of \( f \).

Furthermore, according to [2] Theorem 2, if only a system \( f \) and a point \( x \) are given such that

\[
\alpha(f, x) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.157671,
\]

where \( \alpha(f, x) = \beta(f, x)\gamma(f, x) \) and

\[
\beta(f, x) = \|x - N_f(x)\| = \|Df(x)^{-1} f(x)\|,
\]

then Newton’s iterations starting at \( x \) will converge quadratically to a simple zero \( \xi \) of \( f \) and

\[
\|x - \xi\| \leq 2\beta(f, x).
\]
In [8], Dedieu and Shub gave quantitative results for simple double zeros satisfying \( f(x) = 0 \) and
(A) \( \dim \ker Df(x) = 1 \),
(B) \( D^2 f(x)(v, v) \notin \text{im} Df(x) \),
where \( \ker Df(x) \) is spanned by a unit vector \( v \in \mathbb{C}^n \). They generalized the definition of \( \gamma \) [1.2] to
\[
\gamma_2(f, x) = \max \left( 1, \sup_{k \geq 2} \left\| A(f, x, v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\| \right),
\]
where
\[
A(f, x, v) = Df(x) + \frac{1}{2} D^2 f(x)(v, \Pi_v),
\]
is a linear operator which is invertible at the simple double zero \( x \), and \( \Pi_v \) denotes the Hermitian projection onto the subspace \([v] \subset \mathbb{C}^n\).

In [8, Theorem 1], Dedieu and Shub also presented a lower bound for separating simple double zeros \( x \) from the other zeros \( y \) of \( f \),
\[
\| y - x \| \geq \frac{d}{2 \gamma_2(f, x)^2},
\]
where \( d \approx 0.2976 \) is a positive real root of
\[
\sqrt{1 - d^2} - 2d \sqrt{1 - d^2} - d^2 - d = 0.
\]

**Remark 1.** There are two typos in the statements of [8, Theorem 1] and [8, Lemma 4]: 1) the degree of \( \gamma_2 \) in (1.8) is 2 instead of 1 in [8, Theorem 1]; 2) the coefficient of the second \( \sqrt{1 - d^2} \) in (1.9) is \(-2d\) instead of \(-d\) in [8, Lemma 4].

In [8, Theorem 4], Dedieu and Shub showed that if the following criterion is satisfied at a given point \( x \) and a given vector \( v \)
\[
\| f(x) \| + \| Df(x)v \| \frac{d}{4 \gamma_2(f, x, v)^2} < \frac{d^3}{32 \gamma_2^2\| B(f, x, v)^{-1} \|},
\]
then \( f \) has two zeros in the ball of radius
\[
\frac{d}{4 \gamma_2(f, x)^2},
\]
avoid \( x \). Let us set
\[
B(f, x, v) = A(f, x, v) - L,
\]
where \( L(v) = Df(x)v, L(w) = 0 \) for \( w \in v^\perp \), and
\[
\gamma_2(f, x) = \max \left( 1, \sup_{k \geq 2} \left\| B(f, x, v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\| \right).
\]

**Our Contributions.** We generalize Dedieu and Shub’s quantitative results about simple double zeros to simple multiple zeros whose Jacobian matrix has corank one.

Let us recall our previous work on computing simple multiple zeros. Suppose \( x \) is an isolated singular zero of \( f \) satisfying \( \dim \ker Df(x) = 1 \). Let \( D_{f,x} \) denote the local dual space of an ideal \( I_f = (f_1, \ldots, f_n) \) at \( x \):
\[
D_{f,x} := \{ \Lambda \in \mathcal{D}_x \mid \Lambda(g) = 0, \forall g \in I_f \},
\]
where \( \mathfrak{D}_x = \text{span}_C \{ \mathfrak{d}_x^\alpha \} \) is the \( C \)-vector space generated by differential functionals \( \mathfrak{d}_x^\alpha \) of order \( \alpha \in \mathbb{N}^n \), see (2.1). Let \( \mu \) denote the multiplicity, then starting from \( \Lambda_0 = 1 \), and

\[
\Lambda_1 = d_1 + a_{1,2}d_2 + \cdots + a_{1,n}d_n,
\]

where \( d_1, \ldots, d_n \) are the first order differential functionals, we can construct

\[
\Lambda_k = \Delta_k + a_{k,2}d_2 + \cdots + a_{k,n}d_n,
\]

incrementally for \( k = 2, \ldots, \mu - 1 \) by formulas (2.6) and (2.8), s.t. \( \{ \Lambda_0, \Lambda_1, \ldots, \Lambda_{\mu-1} \} \) is a closed basis of the local dual space \( \mathcal{D}_{f,x} \). The method is efficient since the size of matrices involved in the computation is bounded by \( n \).

We generalize the definition of simple double zeros in [8]. A simple multiple zero \( x \) of \( f(x) = 0 \) and

(A) \( \dim \ker Df(x) = 1 \),
(B) \( \Delta_k(f) \in \text{im} Df(x) \), for \( k = 2, \ldots, \mu - 1 \),
(C) \( \Delta_\mu(f) \notin \text{im} Df(x) \).

Without loss of generality, we can assume that \( Df(x) \) has a normalized form:

\[
Df(x) = \begin{pmatrix} 0 & \hat{D}f(x) \end{pmatrix},
\]

where \( \hat{D}f(x) \) is the nonsingular Jacobian matrix of polynomials \( \hat{f} = \{ f_1, \ldots, f_{n-1} \} \) with respect to variables \( \hat{X} = \{ X_2, \ldots, X_n \} \). We will show in Section 2.3 that it is always possible to perform unitary transformations to \( f \) and variables \( X \) to obtain an equivalent polynomial system whose Jacobian matrix at the singular solution has the normalized form (1.14). This normalization step is similar to the reduction to one variable technique used in [10].

If \( x \) is a simple multiple zero of \( f \) of multiplicity \( \mu \) and \( Df(x) \) has the normalized form (1.14), then it is clear that

\[
\Delta_k(f) \in \text{im} Df(x) \Leftrightarrow \Delta_k(f_n) = 0,
\]

and the above (B) and (C) conditions can be simplified to

(B) \( \Delta_k(f_n) = 0 \), for \( k = 2, \ldots, \mu - 1 \),
(C) \( \Delta_\mu(f_n) \neq 0 \).

For a simple multiple zero \( x \) satisfying conditions (A),(B) and (C), we generalized the definition of \( \gamma_2 \) in (1.6) to

\[
\gamma_\mu = \gamma_\mu(f, x) = \max(\hat{\gamma}_\mu, \gamma_{\mu,n}),
\]

where

\[
\hat{\gamma}_\mu = \hat{\gamma}_\mu(f, x) = \max \left( 1, \sup_{k \geq 2} k! \left\| D\hat{f}(x)^{-1} \frac{Dk\hat{f}(x)}{k!} \right\|^{\frac{1}{k-1}} \right),
\]

and

\[
\gamma_{\mu,n} = \gamma_{\mu,n}(f, x) = \left( 1, \sup_{k \geq 2} \frac{1}{k!} \left\| \frac{Dk f_n(x)}{\Delta_\mu(f_n)} \right\|^{\frac{1}{k-1}} \right),
\]

where \( Dk \hat{f}(x) \) for \( k \geq 2 \) denote the partial derivatives of \( \hat{f} \) of order \( k \) with respect to \( X_1, X_2, \ldots, X_n \) evaluated at \( x \). We generalize main results in [8] to simple multiple zeros of higher multiplicities.
Firstly, in Theorem 5, we present a lower bound for separating simple multiple zeros \( x \) of multiplicity \( \mu \geq 2 \) from another zero \( y \) of \( f \),

\[
\|y - x\| \geq \frac{d}{2\gamma\mu(f, x)^\mu},
\]

where \( d \) is a positive real root of a univariate polynomial \( p(d) \) defined in (3.19). The explicit formulas of \( p(d) \) for multiplicity 2 and 3 are given in (3.24) and (3.3).

In Section 3.3, we also compare our local separation bound for simple double zeros with the one given in [8]. Although the smallest positive real root \( d \approx 0.2865 \) of (3.6) is smaller than \( d \approx 0.2976 \) given in [8], our value of \( \gamma_2 \) could be smaller too. Therefore, for some examples (see Example 1), our local separation bound still could be larger than the one given in [8].

Secondly, we define

\[
H_1 = \begin{pmatrix}
\frac{\partial f(x)}{\partial X_1} & 0 \\
\frac{\partial f_2(x)}{\partial X_1} & \frac{\partial f_3(x)}{\partial X_1}
\end{pmatrix},
\]

and tensors

\[
H_k = \begin{pmatrix}
0 & 0 \\
\Delta_k(f_n) & 0
\end{pmatrix} \otimes \underbrace{0 \times \cdots \times 0}_{k} \otimes \overbrace{n \times \cdots \times n \times (n-1)}^{n-k},
\]

\[
2 \leq k \leq \mu - 1,
\]

and polynomials

\[
g(X) = f(X) - f(x) - \sum_{1 \leq k \leq \mu - 1} H_k(X - x)^k.
\]

Let \( A \) be an invertible matrix

\[
A = \begin{pmatrix}
\sqrt{2} D f(x) & 0 \\
0 & \frac{1}{\sqrt{2}} \Delta_{\mu}(f_n)
\end{pmatrix}.
\]

In Theorem 8, we show that if

\[
\|f(x)\| + \sum_{1 \leq k \leq \mu - 1} \|H_k\| \left( \frac{d}{4\gamma\mu(g, x)^\mu} \right)^k < \frac{d^{\mu+1}}{2(4\gamma\mu(g, x)^\mu)^\mu} \|A^{-1}\|,
\]

then \( f \) has \( \mu \) zeros (counting multiplicities) in the ball of radius \( \frac{d}{4\gamma\mu(g, x)^\mu} \) around \( x \).

Thirdly, we design modified Newton iterations and extend the \( \gamma \)-theorem for simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form (1.14). Given an approximate zero \( z \) of \( f \) with an associated exact simple double zero \( \xi \), we show in Theorem 9 that when

\[
\|z - \xi\| < \frac{0.0318}{\gamma_2(f, \xi)^2},
\]

after \( k \) times of the modified Newton iteration defined in Algorithm 4.1 we have:

\[
\|N^k f(z) - \xi\| < \left( \frac{1}{2} \right)^{2^k - 1} \|z - \xi\|.
\]

Similarly, for an approximate zero \( z \) of \( f \) with an associated exact simple triple zero \( \xi \), we show in Theorem 10 that when

\[
\|z - \xi\| < \frac{0.0154}{\gamma_3(f, \xi)^3},
\]
after $k$ times of the modified Newton iteration defined in Algorithm 4.2 we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

Finally, for simple multiple zeros whose Jacobian matrix is of corank one but it does not have a normalized form (1.14), we apply the unitary transformations defined in Section 2.3 to obtain an equivalent polynomial system whose Jacobian matrix at the approximate simple multiple zeros $z$ has the normalized form approximately:

$$(1.23)\quad Df(z) = \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix},$$

where $\sigma_n$ is its smallest singular value and $\Sigma_{n-1}$ is a nonsingular diagonal matrix.

Then we perform the Newton iteration to refine the last $n-1$ variables. After the Newton iteration, we need to perform the unitary transformation again to ensure that the Jacobian matrix at the refined approximate solution satisfies (1.23). We define the modified Newton iterations based on our previous work in [24, Algorithm 1] to refine the first variable. We show in Theorem 11 for

$$\gamma_\mu(f, z) \|z - \xi\| < \frac{1}{2},$$

the refined singular solution $N_f(z)$ returned by Algorithm 4.3 satisfies

$$\|N_f(z) - \xi\| = O(\|z - \xi\|^2).$$

Furthermore, we show in Theorem 12 that for an approximate zero $z$ of a system $f$ associated to a simple triple zero $\xi$, when

$$\|z - \xi\| < \frac{0.0098 \gamma_3(f, \xi)^3}{\gamma_3(f, \xi)},$$

after $k$ times of the modified Newton iterations defined in Algorithm 4.3 we have:

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\|.$$

It is clear that the proof of Theorem 12 can be generalized to give a quantitative quadratic convergence of Algorithm 4.3 for simple multiple zeros of arbitrary higher multiplicities.

**Related Works.** Yakoubsohn [46] extended $\alpha$-theory to clusters of univariate polynomials and provided an algorithm to compute clusters of univariate polynomials [47]. Giusti, Lecerf, Salvy and Yakoubsohn [9] presented point estimate criteria for cluster location of analytic functions in the univariate case. They provided bounds on the diameter of the cluster which contains $\mu$ zeros (counting multiplicities) of $f$. They proposed an algorithm based on the Schröder iterations for approximating clusters and provided a stopping criterion which guarantees the algorithm converges to the cluster quadratically. In [10], they further generalized their results to locate and approximate clusters of zeros of analytic maps of embedding dimension one in the multivariate case. They reduced this particular multivariate case to univariate case via implicit theorem and deflation techniques. We are inspired by their technique of reduction to one variable but we try to avoid the use of implicitly known univariate analytic function. Dedieu and Shub [8] gave explicitly a lower bound
for separating simple double zeros $x$ from other zeros of $f$ depending on the approximate solution which guarantees the existence of a cluster of two zeros. Based on our previous work \cite{[23, 25]} on computing multiplicity structure of simple multiple zeros, we generalize Dedieu and Shub’s results and deal with simple multiple zeros with higher multiplicities. The proof of the non-quantified quadratic convergence \cite[Theorem 3.16]{[24]} of Algorithm 1 in \cite{[24]} has also been simplified.

There are other approaches to compute isolated multiple zeros or zero clusters, e.g., corrected Newton methods \cite{[33, 5, 6, 7, 14, 15, 34, 35, 29]}, deflation techniques \cite{[32, 48, 31, 20, 4, 21, 22, 45, 3, 36, 24, 27, 11, 26, 18, 16]}, we refer to \cite{[10, 16]} for excellent introductions of previous works on approximating multiple zeros.

**Structure of the Paper.** In Section 2, we recall some notations and show how to compute incrementally a closed basis of the local dual space of $I_f$ at a given multiple root $x$ of corank 1 and multiplicity $\mu$. In Section 3, we begin with explaining how to extend main results in \cite{[8]} to simple triple zeros. We present a lower bound for separating simple triple zeros from other zeros of $f$ and an explicit criterion that guarantees the existence of a cluster of three zeros of $f$ around the approximate singular solution $x$. Then we generalize these results to simple multiple zeros with arbitrary higher multiplicities. We also compare our local separation bound for simple double zeros with the one given in \cite{[8]}.

In Section 4, we design modified Newton iterations and extend the $\gamma$-theory for simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form. For a simple multiple zero of arbitrary large multiplicity whose Jacobian matrix does not have a normalized form, we perform unitary transformations and modified Newton iterations, and show non-quantified quadratic convergence of new algorithm. Furthermore, we show its quantified convergence for simple triple zeros.

## 2. Preliminaries

### 2.1. Local Dual Space.

Let $d_\alpha^\alpha : \mathbb{C}[X] \to \mathbb{C}$ denote the differential functional defined by

$$d_\alpha^\alpha (g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} (x), \quad \forall g \in \mathbb{C}[X],$$

where $x \in \mathbb{C}^n$ and $\alpha = [\alpha_1, \ldots, \alpha_n] \in \mathbb{N}^n$. We have

$$d_\alpha^\alpha ((X - x)^\beta) = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $I_f$ be an ideal generated by a polynomial system $f = \{f_1, \ldots, f_n\}$, where $f_i \in \mathbb{C}[X_1, \ldots, X_n]$. The local dual space of $I_f$ at a given isolated singular solution $x$ is a subspace $D_{f,x}$ of $D_x = \text{span}_\mathbb{C}\{d_\alpha^\alpha\}$ such that

$$D_{f,x} = \{ \Lambda \in D_x \mid \Lambda (g) = 0, \forall g \in I_f \}.$$

When the evaluation point $x$ is clear from the context, we write $d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ instead of $d_\alpha^\alpha$ for simplicity.

Let $D_{f,x}^{(k)}$ be the subspace of $D_{f,x}$ with differential functionals of orders bounded by $k$, we define

$$\text{breadth } \kappa = \dim \left( D_{f,x}^{(1)} \setminus D_{f,x}^{(0)} \right),$$

for excellent introductions of previous works on approximating multiple zeros.
(2) depth $\rho = \min \left\{ k \mid \dim \left( D_{f,x}^{(k+1)} \setminus D_{f,x}^{(k)} \right) = 0 \right\}$.

(3) multiplicity $\mu = \dim \left( D_{f,x}^{(\rho)} \right)$.

If $x$ is an isolated singular solution of $f$, then $1 \leq \kappa \leq n$ and $\rho < \mu < \infty$.

Let us introduce a morphism $\Phi_\sigma : D_x \to D_x$ which is an anti-differentiation operator defined by

$$
\Phi_\sigma(d_1^{\alpha_1} \cdots d_n^{\alpha_n}) = \left\{ \begin{array}{ll}
d_1^{\alpha_1+1} \cdots d_n^{\alpha_n}, & \text{if } \alpha_1 = \cdots = \alpha_{\sigma-1} = 0, \\
0, & \text{otherwise.}
\end{array} \right.
$$

Computing a closed basis of the local dual space is done essentially by matrix-kernel computations based on the stability property of $D_{f,x}$ [28, 33, 42, 44]:

(2.4) \quad \forall \Lambda \in D_{f,x}^{(k)} \Rightarrow \Phi_\sigma(\Lambda) \in D_{f,x}^{(k-1)}, \quad \sigma = 1, \ldots, n.

2.2. Simple Multiple Zeros. In this paper, we deal with simple multiple zeros satisfying $f(x) = 0$, $\dim D f(x) = 1$. It is also called breadth one singular zero in [4] as

(2.5) \quad \dim(D_{f,x}^{(k)} \setminus D_{f,x}^{(k-1)}) = 1, \quad k = 1, \ldots, \rho, \quad \rho = \mu - 1.

Therefore, the local dual space of $I_f$ at a given isolated simple singular solution $x$ is

$$
D_{f,x} = \text{span}_C \{ \Lambda_0, \Lambda_1, \ldots, \Lambda_{\mu-1} \},
$$

where $\text{deg}(\Lambda_k) = k$ and $\Lambda_0 = 1$. Suppose $\Lambda_1 = a_{1,1}d_1 + \cdots + a_{1,n}d_n$, without loss of generality, we assume $a_{1,1} = 1$. Let $\Psi_\sigma : D_x \to D_x$ be a differential operator defined by

$$
\Psi_\sigma(d_1^{\alpha_1} \cdots d_n^{\alpha_n}) = \left\{ \begin{array}{ll}
d_1^{\alpha_1+1} \cdots d_n^{\alpha_n}, & \text{if } \alpha_1 = \cdots = \alpha_{\sigma-1} = 0, \\
0, & \text{otherwise.}
\end{array} \right.
$$

For $k = 2, \ldots, \mu - 1$, by the stability property, we have

(2.6) \quad \left\{ \begin{array}{l}
\Phi_1(\Lambda_k) = a_{1,1}\Lambda_{k-1} + \cdots + a_{k-1,1}\Lambda_1 + a_{k,1}\Lambda_0, \\
\vdots \\
\Phi_n(\Lambda_k) = a_{1,n}\Lambda_{k-1} + \cdots + a_{k-1,n}\Lambda_1 + a_{k,n}\Lambda_0.
\end{array} \right.

Let $a_{1,1} = 1$, $a_{k,1} = 0$ ($k = 2, \ldots, n$), the system [2.6] has a unique solution $\Lambda_k = \Delta_k + a_{k,2}d_2 + \cdots + a_{k,n}d_n$, where

(2.7) \quad \Delta_k = \sum_{\sigma=1}^{n} \Psi_\sigma(a_{1,\sigma}\Lambda_{k-1} + \cdots + a_{k-1,\sigma}\Lambda_1),

and $a_{k,2}, \ldots, a_{k,n}$ are determined by solving the linear system obtained from $\Lambda_k(f_i) = 0$, $i = 1, \ldots, n$:

(2.8) \quad \begin{pmatrix} d_2(f_1) & \cdots & d_n(f_1) \\ \vdots & \ddots & \vdots \\ d_2(f_n) & \cdots & d_n(f_n) \end{pmatrix} \begin{pmatrix} a_{k,2} \\ \vdots \\ a_{k,n} \end{pmatrix} = - \begin{pmatrix} \Delta_k(f_1) \\ \vdots \\ \Delta_k(f_n) \end{pmatrix}.

We refer to [23, 24, 25] for the justification of above arguments.

The following definition generalizes the simple double zero in [8].

**Definition 1.** Let $f : \mathbb{C}^n \to \mathbb{C}^n$, where $f_i \in \mathbb{C}[X]$ and suppose $f(x) = 0$. Then $x$ is a simple zero of multiplicity $\mu$ for $f$ if
(A) $\dim \ker Df(x) = 1$, 
(B) $\Delta_k(f) \in \text{im } Df(x)$, for $k = 2, \ldots, \mu - 1$, 
(C) $\Delta_\mu(f) \notin \text{im } Df(x)$.

In fact, for $\mu = 2$, suppose $\ker Df(x) = \text{span}_C \{ v \}$ with $\| v \| = 1$, then $\Lambda_1(f) = Df(x) \cdot v = v_1 d_1(f) + \cdots + v_n d_n(f)$ and

$$
\Delta_2(f) = \sum_{\sigma=1}^{n} \Psi_\sigma(v_\sigma \Lambda_1(f)) = \sum_{\sigma=1}^{n} \Psi_\sigma(v_\sigma(v_1 d_1 + \cdots + v_n d_n))(f) = \sum_{i>j} v_i v_j d_i d_j(f) + \sum v_i^2 d_i^2(f) = \frac{1}{2} D^2 f(x)(v, v).
$$

Hence, the condition $\Delta_2(f) \notin \text{im } Df(x)$ is equivalent to $D^2 f(x)(v, v) \notin \text{im } Df(x)$, the condition given for the simple double zero in \cite{8}.

2.3. Normalized Form. We show below that it is always possible to perform unitary transformations to obtain an equivalent polynomial system whose Jacobian matrix at the simple multiple zero has a normalized form.

**Definition 2.** For a polynomial function $f : \mathbb{C}^n \to \mathbb{C}^n$, where $f_i \in \mathbb{C}[X_1, \ldots, X_n]$, $Df(x)$ has a normalized form if

$$
Df(x) = \begin{pmatrix}
0 & D\hat{f}(x) \\
0 & 0
\end{pmatrix},
$$

$D\hat{f}(x)$ is the nonsingular Jacobian matrix of polynomials $\hat{f} = \{f_1, \ldots, f_{n-1}\}$ with respect to variables $X_2, \ldots, X_n$.

Let $Df(x) = U \cdot \begin{pmatrix}
\Sigma_{n-1} & 0 \\
0 & 0
\end{pmatrix} \cdot V^*$ be the singular value decomposition of $Df(x)$ of corank 1, where $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)$ are unitary matrices, $V^*$ is the Hermitian transpose of $V$, and $\Sigma_{n-1}$ is a nonsingular diagonal matrix. We can always assume that $Df(x)$ has a normalized form (2.9). Otherwise, let $g = U^* \cdot f(W \cdot X)$, where $W = (v_n, v_1, \ldots, v_{n-1})$ is also a unitary matrix. Suppose $x$ is a simple multiple zero of $f$ of multiplicity $\mu$, then $W^* x$ is a simple multiple zero of $g$ of multiplicity $\mu$ and the Jacobian matrix of $g$ at $W^* x$ has a normalized form:

$$
Dg(W^* x) = U^* \cdot Df(x) \cdot W
= U^* \cdot U \cdot \Sigma \cdot V^* \cdot W
= \begin{pmatrix}
\Sigma_{n-1} & 0 \\
0 & 0
\end{pmatrix} \cdot \begin{pmatrix}
0 & I_{n-1} \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \Sigma_{n-1} \\
0 & 0
\end{pmatrix}.
$$

Furthermore, suppose $y$ is another zero of $f$, then $W^* y$ is another zero of $g$, and the Euclidean distance between zeros $x$ and $y$ does not change under the unitary
transformation:

\[ \|W^*x - W^*y\| = \|W^*(x - y)\| = \|x - y\|. \]

If \( x \) is a simple multiple zero of multiplicity \( \mu \) for \( f \) and \( Df(x) \) has the normalized form (2.9). Then we have

\[ \text{im } Df(x) = \text{im } \left( \begin{array}{c} \hat{D}(x) \\ 0 \end{array} \right), \]

and

\[ \Delta_k(f) \in \text{im } Df(x) \iff \Delta_k(f_n) = 0. \]

The (B)(C) conditions can be simplified to check only the last polynomial \( f_n \):

(B) \( \Delta_k(f_n) = 0 \), for \( k = 2, \ldots, \mu - 1 \),
(C) \( \Delta_n(f_n) \neq 0 \).

The linear system (2.8) for getting the values of \( a_{k,2}, \ldots, a_{k,n} \) can be simplified to:

\[
\begin{pmatrix}
    d_2(f_1) & \cdots & d_n(f_1) \\
    \vdots & \ddots & \vdots \\
    d_2(f_{n-1}) & \cdots & d_n(f_{n-1})
\end{pmatrix}
\begin{pmatrix}
    a_{k,2} \\
    \vdots \\
    a_{k,n}
\end{pmatrix}
= - \begin{pmatrix}
    \Delta_k(f_1) \\
    \vdots \\
    \Delta_k(f_{n-1})
\end{pmatrix}.
\]

3. Local Separation Bound and Cluster Location

We begin with explaining how to extend main results in [8] to simple triple zeros. Then we generalize these results to simple multiple zeros with arbitrary higher multiplicities. We also compare our local separation bound for simple double zeros with the one given in [8].

3.1. Simple Triple Zeros. Let \( x \) be a simple triple zero of \( f \) and \( Df(x) \) has the normalized form (2.9), i.e.

\[
\frac{\partial f_i(x)}{\partial X_1} = 0, \quad \frac{\partial f_n(x)}{\partial X_i} = 0, \quad 1 \leq i \leq n
\]

and

\[ \Delta_2(f_n) = 0, \quad \Delta_3(f_n) \neq 0. \]

Let \( \Lambda_0 = 1, \Lambda_1 = d_1 \). By (2.7), we have

\[
\Delta_2 = \sum_{\sigma=1}^{n} \Psi(a_{1,\sigma}\Lambda_1) = d_1^2,
\]

and

\[ \Lambda_2 = d_1^2 + a_{2,2}d_2 + \cdots + a_{2,n}d_n, \]

where \( a_{2,2}, \ldots, a_{2,n} \) satisfy

\[
\begin{pmatrix}
    a_{2,2} \\
    \vdots \\
    a_{2,n}
\end{pmatrix}
= -D\hat{f}(x)^{-1}
\begin{pmatrix}
    \Delta_2(f_1) \\
    \vdots \\
    \Delta_2(f_{n-1})
\end{pmatrix}
= -D\hat{f}(x)^{-1}
\begin{pmatrix}
    d_1^2(f_1) \\
    \vdots \\
    d_1^2(f_{n-1})
\end{pmatrix},
\]
since \( d_2(f_n) = \cdots = d_n(f_n) = 0, \Delta_2(f_n) = 0 \) and the Jacobian \( D\hat{f}(x) \) of polynomials \( \hat{f} = \{f_1, \ldots, f_{n-1}\} \) with respect to variables \( \hat{X} = \{X_2, \ldots, X_n\} \) is invertible. Moreover, since \( a_{1,1} = 1, a_{2,1} = 0 \), we have

\[
\Delta_3 = \sum_{\sigma=1}^n \Psi_\sigma(a_{1,\sigma}A_2 + a_{2,\sigma}A_1)
\]

\[
= \Psi_1(A_2) + \sum_{\sigma=1}^n \Psi_\sigma(a_{2,\sigma}d_1)
\]

\[
= d_1^3 + a_{2,2}d_1d_2 + \cdots + a_{2,n}d_1d_n.
\]

\[
= d_1^3 + (d_1d_2, \ldots, d_1d_n) \cdot (-D\hat{f}(x)^{-1}) \cdot \begin{pmatrix} d_1^2(f_1) \\ \vdots \\ d_1^2(f_{n-1}) \end{pmatrix}.
\]

For simplicity, we use the following equivalent conditions

\[
\Delta_2(f_n) = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} = 0,
\]

\[
\Delta_3(f_n) = \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1^3} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} \cdot D\hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \neq 0.
\]

The definition of \( \gamma_3 \) has been given in (1.15):

\[
\gamma_3 = \gamma_3(f, x) = \max(\hat{\gamma}_3, \gamma_3, \gamma_3),
\]

where

\[
\hat{\gamma}_3 = \hat{\gamma}_3(f, x) = \max \left(1, \sup_{k \geq 2} \left\| D\hat{f}(x)^{-1} \frac{\partial^k \hat{f}(x)}{k!} \right\|^{\frac{1}{k}} \right),
\]

and

\[
\gamma_3, \gamma_3 = \gamma_3, \gamma_3(f, x) = \max \left(1, \sup_{k \geq 2} \left\| \frac{\partial^k f_n(x)}{k!} \right\|^{\frac{1}{k}} \right).
\]

For two nonzero vectors \( a, b \in \mathbb{C}^n \), we define their angle by

\[
d_p(a, b) = \arccos \frac{|a \cdot b|}{\|a\| \cdot \|b\|}.
\]

Let \( y \) be another vector in \( \mathbb{C}^n \) and \( y \neq x \) and define

\[
w = y - x = \begin{pmatrix} \zeta \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}.
\]

Let \( \varphi = d_p(v, y - x), v = (1, 0, \ldots, 0)^T \), then we have

\[
|\zeta| = \|w\| \cos \varphi, \quad \|\eta\| = \|w\| \sin \varphi.
\]

For \( k \geq 2 \), we use \( D^k f(x) \) to denote the partial derivatives of \( \hat{f} \) of order \( k \) with respect to \( X_1, X_2, \ldots, X_n \). We generalize main results in [3] to simple triple zeros.
Lemma 1. If $\hat{\gamma}_3(f, x)\|w\| \leq \frac{1}{2}$, then
$$\left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| \geq \|w\| \sin \varphi - 2\hat{\gamma}_3(f, x)\|w\|^2.$$ 

Proof. By Taylor's expansion of $\hat{f}(y)$ at $x$, and $\frac{\partial f(x)}{\partial x_i} = 0$, we have
$$\hat{f}(y) = \hat{f}(x) + D\hat{f}(x)\eta + \sum_{k \geq 2} \frac{D^k\hat{f}(x)(y-x)^k}{k!}.$$ 

Noticing that $\hat{f}(x) = 0$ and $D\hat{f}(x)$ is invertible, we have
$$\eta = D\hat{f}(x)^{-1}\hat{f}(y) - \sum_{k \geq 2} D\hat{f}(x)^{-1} \frac{D^k\hat{f}(x)(y-x)^k}{k!}.$$ 

By the triangle inequality, we have
$$\|w\| \sin \varphi = \|\eta\| \leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + \sum_{k \geq 2} \left\| D\hat{f}(x)^{-1} \frac{D^k\hat{f}(x)}{k!} \right\| \|y-x\|^k.$$ 

$$\leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + \sum_{k \geq 2} \hat{\gamma}_3(f, x)k^{-1}\|w\|^k.$$ 

$$\leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + 2\hat{\gamma}_3(f, x)\|w\|^2,$$

where the last inequality comes from the assumption that $\hat{\gamma}_3(f, x)\|w\| \leq \frac{1}{2}$. □

Let
$$(3.4) \quad \mathcal{A} = \begin{pmatrix} \sqrt{2}D\hat{f}(x) & 0 \\ 0 & \frac{1}{\sqrt{2}}\Delta_3(f_n) \end{pmatrix} \in \mathbb{C}^{n \times n},$$

since $D\hat{f}(x)$ is invertible and $\Delta_3(f_n) \neq 0$, we have
$$(3.5) \quad \mathcal{A}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}}D\hat{f}(x)^{-1} & 0 \\ 0 & \frac{\sqrt{2}}{\Delta_3(f_n)} \end{pmatrix}.$$ 

Lemma 2. If $\gamma_3(f, x)\|w\| \leq \frac{1}{2}$, then
$$\|\mathcal{A}^{-1}\hat{f}(y)\| \geq \frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^3 \sin^3 \varphi \|w\|^3 - 2\gamma_3^3 \|w\|^4}{1 + 2\cos \varphi + \sin \varphi}.$$ 

Proof. By Taylor’s expansion of $\hat{f}(y)$ at $x$, and $\frac{\partial f(x)}{\partial x_i} = 0$, we have
$$\eta = D\hat{f}(x)^{-1} \left( \hat{f}(y) - \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial X_2} \zeta^2 + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial X_2} \zeta \eta - \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2} \eta^2 - \sum_{k \geq 3} \frac{D^k\hat{f}(x)(y-x)^k}{k!} \right).$$ 

By expanding $f_n(y)$ at $x$ and $\frac{\partial f_n(x)}{\partial x_1} = \cdots = \frac{\partial f_n(x)}{\partial x_n} = \frac{\partial^2 f_n(x)}{\partial X^2} = 0$, we have:
$$f_n(y) = \left( \frac{\partial^2 f_n(x)}{\partial X_1 \partial X_2} \zeta \eta + \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_1 \partial X_3} \zeta^2 \eta \right) + \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial X_2 \partial X_3} \eta^3 + \sum_{k \geq 4} \frac{D^k f_n(x)(y-x)^k}{k!}.$$
Substituting one $\eta$ in $\frac{\partial^2 f_n(x)}{\partial X_1 \partial X} \zeta \eta$ and $\frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2} \eta^2$ by the expansion of $\eta$, as $\Delta_3(f_n) \neq 0$, we have

\[
\frac{1}{\Delta_3(f_n)} f_n(y) = \frac{1}{\Delta_3(f_n)} \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} D \hat{f}(x)^{-1} \hat{f}(y) \zeta + \frac{1}{\Delta_3(f_n)} \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2} D \hat{f}(x)^{-1} \hat{f}(y) \eta + \zeta^3 \\
+ \frac{1}{\Delta_3(f_n)} C_{1,2} \zeta^2 \eta + \frac{1}{\Delta_3(f_n)} C_{1,3} \zeta^2 \eta^2 + \frac{1}{\Delta_3(f_n)} C_{0,3} \eta^3 + \sum_{k \geq 4} \frac{1}{\Delta_3(f_n)} D^k f_n(x) (y - x)^k \\
+ \frac{1}{\Delta_3(f_n)} T_{1,0} \sum_{k \geq 3} D \hat{f}(x)^{-1} \frac{D^k \hat{f}(x) (y - x)^k}{k!} \zeta + \frac{1}{\Delta_3(f_n)} T_{0,1} \sum_{k \geq 3} D \hat{f}(x)^{-1} \frac{D^k \hat{f}(x) (y - x)^k}{k!} \eta.
\]

where

\[
C_{2,1} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} \cdot D \hat{f}(x)^{-1} \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial X} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2} \cdot D \hat{f}(x)^{-1} \frac{\partial^2 \hat{f}(x)}{\partial X^2}.
\]

\[
C_{1,2} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1 \partial X^2} - \frac{\partial^2 f_n(x)}{\partial X_1 \partial X^2} \cdot D \hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1 \partial X^2} \cdot D \hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2}.
\]

\[
C_{0,3} = \frac{1}{6} \frac{\partial^2 f_n(x)}{\partial X^3} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^3} \cdot D \hat{f}(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2},
\]

\[
T_{1,0} = -\frac{\partial^2 f_n(x)}{\partial X_1 \partial X},
\]

\[
T_{0,1} = -\frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2}.
\]

For the classical operator norm, we have the following inequalities for $i + j = k$:

\[
\left\| D^k \hat{f}(x) \right\| \leq \| D^k f_n(x) \|, \quad \left\| \frac{\partial^k f_n(x)}{\partial X_1 \partial X^j} \right\| \leq \| D^k f_n(x) \|.
\]

Therefore, by moving $\zeta^3$ to the left side and $\frac{1}{\Delta_3(f_n)} f_n(y)$ to the right side of the equation and applying the triangle inequalities, we have

\[
\left| \zeta^3 \right| \leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + 2 \gamma_{3,n} \left\| D \hat{f}(x)^{-1} \hat{f}(y) \right\| | \zeta | + \gamma_{3,n} \left\| D \hat{f}(x)^{-1} \hat{f}(y) \right\| | \eta | \\
+ (3 \gamma_{3,n}^2 + 2 \gamma_{3,n} + \gamma_{3,n} \gamma_{3,n} \gamma_{3,n}) | \zeta^2 | | \eta | \\
+ (3 \gamma_{3,n}^2 + 2 \gamma_{3,n} \gamma_{3,n} + \gamma_{3,n} \gamma_{3,n} + \gamma_{3,n} \gamma_{3,n}) | \zeta | | \eta |^2 + (\gamma_{3,n}^2 + \gamma_{3,n} \gamma_{3,n}) | \eta |^3 \\
+ \sum_{k \geq 4} \gamma_{3,n}^k \sum_{k \geq 3} \eta | w |^k \zeta | \eta |^k + \gamma_{3,n} \sum_{k \geq 3} \gamma_{3,n}^{k-1} | w |^k | \eta |^k \\
\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \left\| D \hat{f}(x)^{-1} \hat{f}(y) \right\| (2 \gamma_{3,n} | \zeta | + \gamma_{3,n} | \eta |) \\
+ (3 \gamma_{3,n}^2 + 5 \gamma_{3,n} \gamma_{3,n}) | \zeta^2 | | \eta | + (3 \gamma_{3,n}^2 + 4 \gamma_{3,n} \gamma_{3,n}) | \zeta | | \eta |^2 \\
+ (\gamma_{3,n}^2 + \gamma_{3,n} \gamma_{3,n}) | \zeta | | \eta |^3 + 2 \gamma_{3,n}^2 \left| \left| \left| w \right| \right|^4 \right| + 4 \gamma_{3,n} \gamma_{3,n} \left| \left| \left| w \right| \right|^3 \right| | \zeta | + 2 \gamma_{3,n} \gamma_{3,n} \left| \left| \left| w \right| \right|^2 \right| | \eta | \\
\leq \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \left\| D \hat{f}(x)^{-1} \hat{f}(y) \right\| (2 \gamma_{3,n} | \zeta | + \gamma_{3,n} | \eta |) + 8 \gamma_{3,n}^2 | \zeta^2 | | \eta | \\
+ 7 \gamma_{3,n}^2 | \left| \left| \left| \eta \right| \right|^2 \right| + 2 \gamma_{3,n}^3 | \left| \left| \left| \eta \right| \right|^3 \right| + 2 \gamma_{3,n}^3 | \left| \left| \left| \eta \right| \right|^4 \right| + 4 \gamma_{3,n} \gamma_{3,n} | \left| \left| \left| \eta \right| \right|^3 \right| | \zeta | + 2 \gamma_{3,n} \gamma_{3,n} | \left| \left| \left| \eta \right| \right|^2 \right| | \eta |.}
\]
Lemma 3. \(\theta\) be the positive root of the equation
\[
\|w\| \cos \varphi, \|\eta\| = \|w\| \sin \varphi, \text{ we have}
\|
\end{align*}
\begin{align*}
\|w\|^3 \cos^3 \varphi & \leq \frac{1}{\Delta_3(f_n)} f_n(y) + \|D\hat{f}(x)^{-1}\hat{f}(y)\| \gamma_3 n \|w\| (2 \cos \varphi + \sin \varphi) \\
& + 2 \gamma_3^3 \|w\|^3 \varphi + 2 \gamma_3^2 \|w\|^2 \cos \varphi \sin^2 \varphi + 8 \gamma_3^3 \|w\|^3 \cos^2 \varphi \sin \varphi \\
& + 2 \gamma_3^3 \|w\|^4 (1 + 2 \cos \varphi + \sin \varphi)
\end{align*}
\begin{align*}
\text{For } \varphi \in [0, \frac{\pi}{2}], \ 1 \leq 2 \cos \varphi + \sin \varphi \leq \sqrt{5}, \text{ we have}
\frac{\cos^3 \varphi - 8 \gamma_3^2 \cos^2 \varphi \sin^2 \varphi - 7 \gamma_3^3 \cos \varphi \sin^2 \varphi - 2 \gamma_3^3 \sin^3 \varphi}{1 + 2 \cos \varphi + \sin \varphi} \|w\|^3 - 2 \gamma_3^3 \|w\|^4 \\
\leq \frac{1}{1 + 2 \cos \varphi + \sin \varphi} \left| \frac{1}{\Delta_3(f_n)} f_n(y) \right| + \gamma_3 n w (2 \cos \varphi + \sin \varphi) \left| \left( D\hat{f}(x)^{-1}\hat{f}(y) \right) \right| \\
\leq \frac{1}{\Delta_3(f_n)} f_n(y) + \frac{\sqrt{5}}{2 + 2\sqrt{5}} \left| \left( D\hat{f}(x)^{-1}\hat{f}(y) \right) \right| \\
\leq \sqrt{2} \left| \left( \frac{\sqrt{5}}{2 + 2\sqrt{5}} \frac{\Delta_3(f_n)}{\Delta_3(f_n)} f_n(y) \right) \right| \\
\leq \sqrt{2} \left| \left( \frac{\sqrt{5}}{2 + 2\sqrt{5}} \frac{\Delta_3(f_n)}{\Delta_3(f_n)} f_n(y) \right) \right| \\
= \|A^{-1}f(y)\|.
\end{align*}
\[
0 \leq \varphi \leq \frac{\pi}{2} \text{ and } \|A^{-1}f(y)\| \geq \sqrt{2} \gamma_3 \|w\| \left( \frac{\sin \theta}{2 \gamma_3} - \|w\| \right),
\end{align*}
or
\[
0 \leq \varphi \leq \theta \text{ and } \|A^{-1}f(y)\| \geq 2 \gamma_3^3 \|w\|^3 \left( \frac{\sin \theta}{2 \gamma_3} - \|w\| \right).
\end{align*}
Proof. For \(\theta \leq \varphi \leq \frac{\pi}{2}\), by Lemma \[\] we have
\[
\sqrt{2} \|A^{-1}f(y)\| = \left| \left( D\hat{f}(x)^{-1}\hat{f}(y) \right) \right| \geq \left| D\hat{f}(x)^{-1}\hat{f}(y) \right|
\geq \|w\| \sin \theta - 2 \gamma_3 (f, x) \|w\|^2 \geq 2 \gamma_3 \|w\| \left( \frac{\sin \theta}{2 \gamma_3} - \|w\| \right).
For $0 \leq \varphi \leq \theta$, by Lemma \ref{lem:2}, we have
\[
\|A^{-1}f(y)\| \geq 2\gamma_3^3\|w\|^3 \left( \frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{2\gamma_3^3(1 + 2 \cos \varphi + \sin \varphi)} - \|w\| \right).
\]

Let
\[
h(\varphi) = \frac{\cos^3 \varphi - 8\gamma_3^2 \cos^2 \varphi \sin \varphi - 7\gamma_3^2 \cos \varphi \sin^2 \varphi - 2\gamma_3^2 \sin^3 \varphi}{2\gamma_3^3(1 + 2 \cos \varphi + \sin \varphi)}.
\]

We claim that
\[
h(\theta) \geq \sin \frac{\theta}{2\gamma_3}.
\]

To prove this claim, as $\sin \theta = \frac{d}{\gamma_3}$, it is sufficient to show that
\[
\left(1 - \frac{d^2}{\gamma_3^3}\right) - \frac{7d^2}{\gamma_3^3} \sqrt{1 - \frac{d^2}{\gamma_3^3}} - \frac{2d^3}{\gamma_3^3} - d - 2d \sqrt{1 - \frac{d^2}{\gamma_3^3} - \frac{d^2}{\gamma_3^3}} \geq 0.
\]

Since this function for $\gamma_3 \geq 1$, is increasing for any $d \in [0, \frac{1}{6}]$, similar to the proof of \cite[Lemma 4]{8}, it is sufficient to check this inequality for $\gamma_3 = 1$,
\[
(1 - 2d - 8d^2)\sqrt{1 - d^2} - 9d - d^2 + 6d^3 \geq 0.
\]

The smallest positive root of the equation (3.6) obtained by setting the above inequality to 0 is
\[
d \approx 0.08507,
\]
which lies in $[0, \frac{1}{6}]$. The claim $h(\theta) \geq \sin \frac{\theta}{2\gamma_3}$ follows.

Furthermore, the polynomial $h(\varphi)$ is non-negative and decreasing for
\[
\varphi \in [0, \theta], \quad \theta \in \left[0, \arcsin \frac{2}{\sqrt{5}}\right],
\]
as its numerator is decreasing and its numerator is increasing, and both are non-negative for $\varphi$ satisfying (3.8). Hence, we have
\[
h(\varphi) \geq h(\theta) \geq \sin \frac{\theta}{2\gamma_3},
\]
for $\varphi$ satisfying (3.8). Together with Lemma \ref{lem:2}, we have
\[
\|A^{-1}f(y)\| \geq 2\gamma_3^3\|w\|^3 \left( h(\varphi) - \|w\| \right) \geq 2\gamma_3^3\|w\|^3 \left( \frac{\sin \theta}{2\gamma_3} - \|w\| \right).
\]

Let $d \approx 0.08507$ be the smallest positive root of the equation (3.6). The following four theorems generalize the results in \cite{8} to simple triple zeros.

**Theorem 1.** Let $x$ be an isolated simple triple zero of the polynomial system $f$, and $y$ is another zero of $f$, then
\[
\|y - x\| \geq \frac{d}{2\gamma_3^3}.
\]

**Proof.** Since $f(y) = 0$, when $\gamma_3\|w\| \leq \frac{1}{2}$, by Lemma \ref{lem:3} and (3.7), we have
\[
\|y - x\| = \|w\| \geq \sin \frac{\theta}{2\gamma_3} = \frac{d}{2\gamma_3^3}.
\]

For $\gamma_3\|w\| \geq \frac{1}{2}$, the same conclusion holds as $\gamma_3 \geq 1$ and $d < 1$. \hfill \Box
Theorem 2. Let $x$ be an isolated simple triple zero of the polynomial system $f$, and $\|y - x\| \leq \frac{d}{4\gamma^3}$, then
\[
\|f(y)\| \geq \frac{d\|y - x\|^3}{2\|A^{-1}\|}.
\]

Proof. When
\[
\|w\| = \|y - x\| \leq \frac{d}{4\gamma^3} = \sin \theta 4\gamma^3,
\]
by Lemma 3 we have
\[
\|A^{-1}f(y)\| \geq 2\gamma^3\|w\|^3 \left(\frac{\sin \theta}{2\gamma^3} - \|w\|\right) \geq 2\gamma^3\|w\|^3 \frac{\sin \theta}{4\gamma^3} = 2\gamma^3\|w\|^3 \frac{d}{4\gamma^3} = \frac{d}{2}\|w\|^3.
\]

For $R > 0$, let us define
\[
(3.10) \quad d_R(f, g) = \max_{\|y - x\| \leq R} \|f(y) - g(y)\|.
\]

Theorem 3. Let $x$ be an isolated simple triple zero of the polynomial system $f$ and
\[
0 < R \leq \frac{d}{4\gamma^3},
\]
If
\[
d_R(f, g) < \frac{dR^3}{2\|A^{-1}\|},
\]
then the sum of the multiplicities of the zeros of $g$ in $B(x, R)$ is three.

Proof. By Theorem 2 for any $y$ such that $\|y - x\| = R$, we have
\[
\|f(y) - g(y)\| \leq d_R(f, g) < \frac{dR^3}{2\|A^{-1}\|} \leq \frac{d\|y - x\|^3}{2\|A^{-1}\|} \leq \|f(y)\|,
\]
by Rouché’s Theorem, $f$ and $g$ have the same number of zeros inside $B(x, R)$. By Theorem 11 when $R \leq \frac{d}{4\gamma^3}$, the only zero of $f$ in $B(x, R)$ is $x$. Therefore, $g$ has three zeros in $B(x, R)$. \qed

Given $f : \mathbb{C}^n \to \mathbb{C}^n$, $x \in \mathbb{C}^n$, such that $Df(x)$ is invertible, and
\[
\Delta_3(f_n) = \frac{1}{6} \partial^3 f_n(x) - \partial^2 f_n(x) Df_n(x)^{-1} \frac{1}{2} \partial^2 f(x) 2 \partial X_1^2 \neq 0.
\]
We define tensors
\[
H_1 = \left(\begin{array}{cc}
\frac{\partial^3 f_n(x)}{\partial X_1^3} & 0 \\
\frac{\partial^2 f_n(x)}{\partial X_1^2} & \frac{\partial f_n(x)}{\partial X_1} \\
\end{array}\right),
\]
\[
H_2 = \left(\begin{array}{c}
0 \\
\frac{1}{2} \partial^2 f_n(x) \\
\end{array}\right)_{0 \times n \times (n-1)}.
\]
and polynomials
\[
g(X) = f(X) - f(x) - H_1(X - x) - H_2(X - x)^2.
\]
Theorem 4. Let \( \gamma_3 = \gamma_3(g, x) \), if
\[
\|f(x)\| + \|H_1\| \frac{d}{4\gamma_3} + \|H_2\| \frac{d^2}{16\gamma_3^2} < \frac{d^4}{128\gamma_3^3 \|A^{-1}\|}
\]
then \( f \) has three zeros (counting multiplicities) in the ball of radius \( \frac{d}{4\gamma_3} \) around \( x \).

Proof. We have \( g(x) = 0 \),
\[
Dg(x) = Df(x) - H_1 = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix},
\]
Moreover, we have
\[
\Delta_2(g_n) = \frac{1}{2} \frac{\partial^2 g_n(x)}{\partial x_1^2} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial x_1^2} - \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial x_1^2} = 0,
\]
\[
\Delta_3(g_n) = \frac{1}{6} \frac{\partial^3 g_n(x)}{\partial x_1^3} = \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial x_1^3} \cdot Dg(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{g}(x)}{\partial x_1^2} = \frac{1}{6} \frac{\partial^3 f_n(x)}{\partial x_1^3} \cdot Df(x)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial x_1^2} \neq 0.
\]
Hence \( Dg(x) \) satisfies the normalized form, and \( x \) is a simple singular root of \( g \) with multiplicity three. Let \( R = \frac{d}{4\gamma_3}, \) we have
\[
d_R(g, f) = \max_{\|y-x\| \leq R} \|g(y) - f(y)\|
\leq \|f(x)\| + \|H_1\| R + \|H_2\| R^2
\leq \|f(x)\| + \|H_1\| \frac{d}{4\gamma_3} + \|H_2\| \frac{d^2}{16\gamma_3^2}.
\]
If
\[
\|f(x)\| + \|H_1\| \frac{d}{4\gamma_3} + \|H_2\| \frac{d^2}{16\gamma_3^2} < \frac{d^4}{128\gamma_3^3 \|A^{-1}\|},
\]
then
\[
d_R(g, f) < \frac{d^4}{128\gamma_3^3 \|A^{-1}\|} = \frac{dR^3}{2 \|A^{-1}\|}.
\]
By Theorem 4, the sum of the multiplicities of the zeros of \( f \) in \( B(x, R) \) is three. \( \square \)

Remark 2. The equality of \( \gamma_\mu(g, x) = \gamma_\mu(f, x) \) is true for \( \mu = 2 \) [8] Theorem 4. In Example 3, we show that \( \left\| \frac{1}{\Delta_2(f_2)} \cdot \frac{D^2 f_2(x)}{2} \right\| \neq \left\| \frac{1}{\Delta_2(g_2)} \cdot \frac{D^2 g_2(x)}{2} \right\|. \) Hence, \( \gamma_3,n(g, x) \) might be not equal to \( \gamma_3,n(f, x) \) if they are not equal to 1.

3.2. Simple Multiple Zeros. We generalize results in Section 3.1 to the simple multiple zeros of higher multiplicities.

Let \( f : \mathbb{C}^n \to \mathbb{C}^n \), and \( x \) be a simple zero of \( f \) of multiplicity \( \mu \), where \( Df(x) \) has the normalized form \( Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix}, D\hat{f}(x) \) is invertible and
\[
\Delta_k(f_n) = 0, \text{ for } k = 2, \ldots, \mu - 1, \quad \Delta_\mu(f_n) \neq 0.
\]
Let $y$ be another vector in $\mathbb{C}^n$ and $y \neq x$. Recall that $\varphi = d_p(v, y - x)$, $v = (1, 0, \ldots, 0)^T$ and $w = x - y = (\zeta, \eta_2, \ldots, \eta_n)^T$, $\eta = (\eta_2, \ldots, \eta_n)^T$, then we have $|\zeta| = \|w\| \sin \varphi$, $\|\eta\| = \|w\| \cos \varphi$. Let

$$A = \begin{pmatrix} \sqrt{2D\hat{f}(x)} & 0 \\ 0 & \frac{1}{\sqrt{2}} \Delta_{\mu}(f_n) \end{pmatrix},$$

and $\gamma_{\mu} = \max(\hat{\gamma}_{\mu}, \gamma_{\mu,n})$, where

$$\gamma_{\mu,n} = \gamma_{\mu,n}(f, x) = \max\left(1, \sup_{k \geq 2} \left\| \frac{1}{\Delta_{\mu}(f_n)} \frac{D^k f_n(x)}{k!} \right\|^\frac{1}{k-1} \right).$$

**Case 1:** For $\theta \leq \varphi \leq \frac{\pi}{2}$, assume that $\gamma_{\mu}\|w\| \leq \frac{1}{2}$. The Taylor's expansion of $\hat{f}(y)$ at $x$ is:

$$\hat{f}(y) = D\hat{f}(x)\eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2} \zeta^2 + \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial X} \zeta \eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2} \eta^2 + \sum_{k = 3} D^k \hat{f}(x)(y - x)^k.$$

By the triangle inequality, we have

$$\|w\| \sin \varphi = \|\eta\| \leq \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| + \sum_{k = 2} \hat{\gamma}_{\mu}(f, x)^{k-1} \|w\|^k$$

$$\leq \left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| + 2\hat{\gamma}_{\mu}(f, x)\|w\|^2.$$

Therefore, we have the following claim.

**Claim 1.** For $\theta \leq \varphi \leq \frac{\pi}{2}$, assume that $\gamma_{\mu}\|w\| \leq \frac{1}{2}$, we have

$$\left\| D\hat{f}(x)^{-1} \hat{f}(y) \right\| \geq 2\gamma_{\mu}\|w\| \left( \frac{\sin \theta}{2\gamma_{\mu}} - \|w\| \right),$$

and

$$\|w\| \geq \sin \frac{\varphi}{2\hat{\gamma}_{\mu}} \geq \sin \frac{\theta}{2\hat{\gamma}_{\mu}} \geq \sin \frac{\theta}{2\gamma_{\mu}}.$$

**Case 2:** For $0 \leq \varphi < \theta \leq \frac{\pi}{2}$, assume that $\gamma_{\mu}\|w\| \leq \frac{1}{2}$. The Taylor expansion of $f_n$ at $y$ is:

$$f_n(y) = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X_1^2} \zeta^2 + \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} \zeta \eta + \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2} \eta^2 + \cdots + \frac{1}{\mu!} \frac{\partial^\mu f_n(x)}{\partial X_1} \zeta^\mu$$

$$+ \frac{1}{(\mu - 1)!} \frac{\partial^\mu f_n(x)}{\partial X_1^{\mu-1}} \zeta^{\mu-1} \eta + \cdots + \frac{1}{\mu!} \frac{\partial^\mu f_n(x)}{\partial X^\mu} \eta^\mu + \sum_{k \geq \mu + 1} \frac{D^k f_n(x)(y - x)^k}{k!}.$$

The coefficient of the term $\zeta^i \eta^j$ in the Taylor expansion of $f_n$ is $\frac{1}{i! j!} \frac{\partial^{i+j} f_n(x)}{\partial X_1^i \partial X^j}$, whose norm divided by $\Delta_{\mu}(f_n)$ is bounded by

$$(3.12) \frac{(i + j)!}{i! j!} \gamma_{\mu+i+j-1}.$$

For the monomial $\zeta^i \eta^j$, $i + j < \mu$ and $j > 0$, after substituting the first $\eta$ in $\zeta^i \eta^j$ by

$$\eta = -D\hat{f}(x)^{-1} \left( \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X_1^2} \zeta^2 + \frac{\partial^2 \hat{f}(x)}{\partial X_1 \partial X} \zeta \eta + \frac{1}{2} \frac{\partial^2 \hat{f}(x)}{\partial X^2} \eta^2 \right) \cdots$$
where

\[ \text{Claim 2. We have } \Delta_t(f_n) = \Delta_t(\xi^2) + \cdots + C_{\mu} \Delta_t(\xi^\mu) + \sum_{i+j=\mu, j>0} C_{i,j} \Delta_t(\xi^i \eta^j) \]

Proof. Let us apply the differential functional \( \Delta_t \) to both sides of (3.14):

\[ \Delta_t(f_n) = C_2 \Delta_t(\xi^2) + \cdots + C_\mu \Delta_t(\xi^\mu) + \sum_{i+j=\mu, j>0} C_{i,j} \Delta_t(\xi^i \eta^j) \]
By the triangle inequality, we have

$$20 \leq \sum_{1 \leq i+j \leq \mu-2} T_{i,j} \cdot \left( \sum_{k \geq \mu+1-i-j} D\hat{f}(x)^{-1} \Delta_t \left( \frac{D^k\hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^j \right) \right)$$

$$- \sum_{1 \leq i+j \leq \mu-2} T_{i,j} D\hat{f}(x)^{-1} \Delta_t \left( \hat{f}(y) \zeta^i \eta^j \right) + \sum_{k \geq \mu+1} \Delta_t \left( \frac{D^k f_n(x)(y-x)^k}{k!} \right).$$

Based on (2.2), (2.4) and the fact that $d_1$ is the only differential monomial of the highest order $t$ in $\Delta_t$ and no other $d_\gamma$ with $s < t$ in $\Delta_t$, we derive for $2 \leq t \leq \mu$ that:

1. $\Delta_t(\zeta^s) = 1$ if $s = t$ and 0 otherwise;
2. $\Delta_t(\zeta^i \eta^j) = 0$ for $t \leq i + j = \mu$ and $j > 0$;
3. $\Delta_t \left( \frac{D^k f(x)(y-x)^k}{k!} \zeta^i \eta^j \right) = 0$ for $t \leq \mu < i + j + k$;
4. $\Delta_t \left( \hat{f}(y) \zeta^i \eta^j \right) = 0$ for $1 \leq i + j \leq \mu - 2$;
5. $\Delta_t \left( \frac{D^k f_n(x)(y-x)^k}{k!} \right) = 0$ for $t \leq k$.

Hence, we have $C_t = \Delta_t(f_n)$ for $t = 2, \ldots, \mu$. □

By Claim 2 and (3.13), we have

$$C_t = \Delta_t(f_n) = 0, \quad t = 2, \ldots, \mu - 1, \quad C_\mu = \Delta_\mu f_n \neq 0.$$

From (3.6) and (3.11), we obtain

$$\zeta^\mu = -\frac{1}{\Delta_\mu(f_n)} \sum_{i+j=\mu,j>0} C_{i,j} \zeta^i \eta^j - \frac{1}{\Delta_\mu(f_n)} \sum_{k \geq \mu+1} \frac{D^k f_n(x)(y-x)^k}{k!}$$

$$- \frac{1}{\Delta_\mu(f_n)} \sum_{1 \leq i+j \leq \mu-2} T_{i,j} \cdot \left( \sum_{k \geq \mu+1-i-j} D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)(y-x)^k}{k!} \zeta^i \eta^j \right)$$

$$+ \frac{1}{\Delta_\mu(f_n)} \sum_{1 \leq i+j \leq \mu-2} T_{i,j} D\hat{f}(x)^{-1} \hat{f}(y) \zeta^i \eta^j + \frac{1}{\Delta_\mu(f_n)} f_n(y).$$

By the triangle inequality, we have

$$|\zeta^\mu| \leq \sum_{i+j=\mu,j>0} \left| \frac{1}{\Delta_\mu(f_n)} C_{i,j} \right| |\zeta^i||\eta|^j + \sum_{k \geq \mu+1} \left| \frac{1}{\Delta_\mu(f_n)} \frac{D^k f_n(x)}{k!} \right| ||w||^k$$

$$+ \sum_{1 \leq i+j \leq \mu-2} \left| \frac{1}{\Delta_\mu(f_n)} T_{i,j} \right| \cdot \left( \sum_{k \geq \mu+1-i-j} \left| \frac{D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!}}{||w||^k ||\zeta^i|| ||\eta||^j} \right| \right)$$

$$+ \sum_{1 \leq i+j \leq \mu-2} \left| \frac{1}{\Delta_\mu(f_n)} T_{i,j} \right| \cdot \left| D\hat{f}(x)^{-1} \hat{f}(y) \right| ||\zeta^i|| ||\eta||^j + \frac{1}{\Delta_\mu(f_n)} f_n(y) \right|$$

$$\leq \sum_{i+j=\mu,j>0} c_{i,j} \gamma^{i+j-1}_{\mu} |\zeta^i||\eta|^j + \sum_{k \geq \mu+1} \gamma^{k-1}_{\mu,n} ||w||^k$$

$$+ \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma^{i+j} \cdot 2 \gamma^{i-j}_{\mu-1} ||w||^{\mu-i-j+1} ||\zeta^i|| ||\eta||^j$$

$$+ \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma^{i+j} \cdot ||D\hat{f}(x)^{-1} \hat{f}(y)|| ||\zeta^i|| ||\eta||^j + \frac{1}{\Delta_\mu(f_n)} f_n(y) \right|$$
where
\[
\|D\hat{f}(x)^{-1}\hat{f}(y)\| \leq \left\| D\hat{f}(x)^{-1}\hat{f}(y) \right\| + \frac{1}{\Delta_{p}(f_{n})}f_{n}(y).
\]

By \(|\xi| = \|w\| \sin \varphi, \|\eta\| = \|w\| \cos \varphi\), we have:

\[
\|w\|^\mu \cos^\mu \varphi \leq \sum_{i+j=\mu, j > 0} c_{i,j} \gamma_{\mu}^{i-1} \|w\|^\mu \cos^i \varphi \sin^j \varphi \\
+ \sum_{1 \leq i+j \leq \mu-2} 2t_{i,j} \gamma_{\mu}^{i+j} \|w\|^\mu+1 \cos^i \varphi \sin^j \varphi + 2\gamma_{\mu}^\mu \|w\|^\mu+1 \\
+ \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_{\mu}^{i+j} \|w\|^\mu+1 \cos^i \varphi \sin^j \varphi \cdot \|D\hat{f}(x)^{-1}\hat{f}(y)\| + \frac{1}{\Delta_{p}(f_{n})}f_{n}(y).
\]

Therefore, we have

\[
\begin{align*}
&\cos^\mu \varphi - \sum_{i+j=\mu, j > 0} c_{i,j} \gamma_{\mu}^{i-1} \cos^i \varphi \sin^j \varphi \\
&\quad \leq \frac{1}{1 + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \cos^i \varphi \sin^j \varphi} \left| \frac{1}{\Delta_{p}(f_{n})}f_{n}(y) \right| \\
&\quad + \frac{1}{1 + \sum_{1 \leq i+j \leq \mu-2} t_{i,j} \gamma_{\mu}^{i+j} \|w\|^\mu+1 \cos^i \varphi \sin^j \varphi} \left| \|D\hat{f}(x)^{-1}\hat{f}(y)\| \right| \\
&\quad \leq \frac{1}{\Delta_{p}(f_{n})}f_{n}(y) + \frac{1}{2} \left| D\hat{f}(x)^{-1}\hat{f}(y) \right| \\
&\quad \leq \|A^{-1}f(y)\|.
\end{align*}
\]

We have the following inequality:

\[
\|A^{-1}f(y)\| \geq h(\varphi) \cdot 2\gamma_{\mu}^\mu \|w\|^\mu - 2\gamma_{\mu}^\mu \|w\|^\mu+1.
\]

where

\[
h(\varphi) = \frac{\cos^\mu \varphi - \sum_{i+j=\mu, j > 0} c_{i,j} \gamma_{\mu}^{i-1} \cos^i \varphi \sin^j \varphi}{\sum_{1 \leq i+j \leq \mu-2} 2t_{i,0} \gamma_{\mu}^i + \sum_{1 \leq i+j \leq \mu-2, j > 0} 2t_{i,j} \gamma_{\mu}^{i+j} \cos^i \varphi \sin^j \varphi + 2\gamma_{\mu}^\mu}.
\]

Definition 3. We define \(d = \min(d_1, d_2, d_3)\), where

\[
d_1 = \sqrt{\frac{1}{\gamma_{\mu-1.1}^\mu + 1}}, \quad d_2 = \sqrt{\frac{1}{\mu - 1}},
\]

and \(d_3\) is the smallest positive real root of the polynomial

\[
p(d) = (1 - d^2)\psi - \sum_{i+j=\mu, j > 0} c_{i,j} d(1 - d^2)^\mu d^{i-1} \\
- d \left( \sum_{1 \leq i+j \leq \mu-2} t_{i,0} + \sum_{1 \leq i+j \leq \mu-2, j > 0} t_{i,j} (1 - d^2)^\mu d^{j+1} \right).
\]
In the sequel, we always assume that $d$ has the above definition.

**Claim 3.** We have $h(\theta) \geq \frac{\sin(\theta)}{2\gamma_\mu}$, where $\sin \theta = \frac{d}{\gamma_\mu}$.

To prove this claim, substituting $\sin \varphi$ and $\cos \varphi$ in (3.18) by $\sin \frac{\theta}{2}$ and $\cos \frac{\theta}{2}$, we need to show

$$\cos \theta = \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{1/2},$$

we need to show

$$(3.20) \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{1/2} - c_{\mu-1,1}d \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{\frac{\mu-1}{2}}$$

$$- \sum_{i+j=\mu-1, j>0} d_i \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{\frac{i}{2}} d_j \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{\frac{j}{2}} \geq 0.$$

- The sum of the first two terms in (3.20) is non-negative and increasing in $\gamma_\mu$ for $\gamma_\mu \geq 1$ as it equals to

$$\left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{\frac{\mu-1}{2}} \left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} - c_{\mu-1,1}d \right),$$

and $d \leq d_1 = \sqrt{\frac{1}{c_{\mu-1,1}}}$.  

- The terms $\cos^i \varphi \sin^j \varphi$, $j > 0$ are increasing with respect to $\varphi$ for $\varphi \in [0, \arctan \sqrt{\frac{d}{d_1}}]$ since

$$(\cos^i \varphi \sin \varphi)' = \cos^{i-1} \varphi (\cos^2 \varphi - i \sin^2 \varphi) \geq 0.$$

Hence, for $1 \leq i + j \leq \mu - 1$, $j > 0$, $\left( 1 - \frac{d^2}{\gamma_\mu^2(\mu-1)} \right)^{\frac{j}{2}} \frac{d_j}{\gamma_\mu^2(\mu-1)}$, is decreasing in $\gamma_\mu$ for $d \leq d_1 = \sqrt{\frac{1}{c_{\mu-1,1}}}$.  

- The left side of (3.20) is a function which is increasing in $\gamma_\mu$ and it is sufficient to prove that the inequality is true when $\gamma_\mu = 1$:

$$p(d) = (1 - d^2)^{\frac{\mu-1}{2}} - \sum_{i+j=\mu, j>0} d_i \left( 1 - d^2 \right)^{\frac{i}{2}} d_j \left( 1 - d^2 \right)^{\frac{j}{2}} - d_i \left( 1 - d^2 \right)^{\frac{i}{2}} d_j \left( 1 - d^2 \right)^{\frac{j}{2}} \geq 0,$$

which is obvious as $p(d)$ is decreasing in $d$ for $0 \leq d \leq d_3$, $p(0) > 0$ and $d_3$ is the smallest real zero of $p(d) = 0$.

**Claim 4.** For $0 \leq \varphi \leq \theta$, $h(\varphi)$ is non-negative and decreasing for $0 \leq \varphi \leq \theta$ and

$$(3.21) h(\varphi) \geq h(\theta) \geq \frac{\sin \theta}{2\gamma_\mu}.$$
and

\[(3.22) \quad \| A^{-1} f(y) \| \geq 2\gamma^\mu \| w \|^\mu \left( \frac{\sin \theta}{2\gamma^\mu} - \| w \| \right). \]

Moreover, if \( y \) is another zero of \( f \), then

\[ \| w \| = \| y - x \| \geq \frac{\sin \theta}{2\gamma^\mu}. \]

For \( \varphi \in \left[ 0, \arctan \sqrt{\frac{\mu - 1}{\mu}} \right] \), since \( \cos \varphi \sin \varphi, \ i + j = \mu, \ j > 0 \) is increasing, the numerator of \( h(\varphi) \) is non-negative and decreasing, and the denominator of \( h(\varphi) \) is positive and increasing. Hence, \( h(\varphi) \) is non-negative and decreasing for \( 0 \leq \varphi \leq \theta \), we have \( (3.21) \). Moreover, by \( (3.17) \), we have

\[ \| A^{-1} f(y) \| \geq h(\varphi) \cdot 2\gamma^\mu \| w \|^{\mu} - 2\gamma^\mu \| w \|^{\mu+1} \geq 2\gamma^\mu \| w \|^{\mu} \left( \frac{\sin \theta}{2\gamma^\mu} - \| w \| \right). \]

**Theorem 5.** Let \( x \) be a simple multiple zero of \( f \) of multiplicity \( \mu \), and \( y \) be another zero of \( f \), then

\[ \| y - x \| \geq \frac{d}{2\gamma^\mu}. \]

**Proof.** By Claim 1 and Claim 4, we have

\[ \| w \| = \| y - x \| \geq \frac{\sin \theta}{2\gamma^\mu} = \frac{d}{2\gamma^\mu}, \]

since \( \sin \theta = \frac{d}{2\gamma^\mu}. \) \( \square \)

**Theorem 6.** Let \( x \) be a simple multiple zero of \( f \) of multiplicity \( \mu \), and \( \| y - x \| \leq \frac{d}{2\gamma^\mu} \), then we have

\[ \| f(y) \| \geq \frac{d\| y - x \|^{\mu}}{2\| A^{-1} \|}. \]

**Proof.** For \( \theta \leq \varphi \leq \frac{\pi}{2} \), by Claim 1 we can show that

\[ \| A^{-1} f(y) \| = \left\| \left( \frac{1}{\sqrt{2}} D\hat{f}(x)^{-1} \hat{f}(y) \right) \right\| \geq \frac{1}{\sqrt{2}} \| D\hat{f}(x)^{-1} \hat{f}(y) \| \]

\[ \geq \sqrt{2\gamma^\mu} \| w \| \left( \frac{\sin \theta}{2\gamma^\mu} - \| w \| \right). \]

For \( 0 \leq \varphi \leq \theta \), by Claim 1 we have

\[ \| A^{-1} f(y) \| \geq 2\gamma^\mu \| w \|^\mu \left( \frac{\sin \theta}{2\gamma^\mu} - \| w \| \right). \]

When \( \| w \| = \| y - x \| \leq \frac{d}{2\gamma^\mu} = \frac{\sin \theta}{4\gamma^\mu} \), we have

\[ \| A^{-1} f(y) \| \geq 2\gamma^\mu \| w \|^\mu \left( \frac{\sin \theta}{4\gamma^\mu} \right) = \frac{d\| y - x \|^{\mu}}{2}. \]

\( \square \)
Theorem 7. Let \( x \) be a simple multiple zero of \( f \) of multiplicity \( \mu \) and \[
0 < R \leq \frac{d}{4\gamma_\mu}.
\]
If \[
d_R(f,g) < \frac{dR^\mu}{2\|A^{-1}\|},
\]
then the sum of the multiplicities of the zeros of \( g \) in \( B_R(x) \) is \( \mu \).

Proof. By Theorem 6, for any \( y \) such that \( \|y - x\| = R \leq \frac{d}{4\gamma_\mu} \), we have \[
\|f(y) - g(y)\| \leq dR(f,g) < \frac{dR^\mu}{2\|A^{-1}\|} = \frac{d\|y - x\|^\mu}{2\|A^{-1}\|} \leq \|f(y)\|,
\]
by Rouché’s Theorem, \( f \) and \( g \) have the same number of zeros inside \( B_R(x) \). By Theorem 5, when \( R \leq \frac{d}{4\gamma_\mu} \), the only root of \( f \) in \( B_R(x) \) is \( x \). Therefore, \( g \) has \( \mu \) zeros in \( B_R(x) \).

Given \( f : \mathbb{C}^n \to \mathbb{C}^n \), \( x \in \mathbb{C}^n \), such that \( D\hat{f}(x) \) is invertible, and \( \Delta_\mu(f_n) \neq 0 \), we define tensors
\[
H_1 = \begin{pmatrix}
\frac{\partial f(x)}{\partial X} & 0 \\
\frac{\partial f_n(x)}{\partial X} & 0
\end{pmatrix}
\]
\[
H_k = \begin{pmatrix}
0 & 0 \\
\Delta_k(f_n) & 0
\end{pmatrix}_{0 \times n \times n \times (n-1)}, \quad 2 \leq k \leq \mu - 1,
\]
and polynomials
\[
g(X) = f(X) - f(x) - \sum_{1 \leq k \leq \mu - 1} H_k(X - x)^k.
\]

Theorem 8. Let \( \gamma_\mu = \gamma_\mu(g,x) \), if \[
(3.23) \quad \|f(x)\| + \sum_{1 \leq k \leq \mu - 1} \|H_k\| \left( \frac{d}{4\gamma_\mu} \right)^k < \frac{d^{\mu+1}}{2(4\gamma_\mu)\|A^{-1}\|},
\]
then \( f \) has \( \mu \) zeros (counting multiplicities) in the ball of radius \( \frac{d}{4\gamma_\mu} \) around \( x \).

Proof. We have \( g(x) = 0 \), \( Dg(x) = Df(x) - H_1 = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix} \). Moreover, \[
\Delta_k(g_n) = \Delta_k(f_n) - \Delta_k(f_n) = 0, \quad 2 \leq k \leq \mu - 1, \quad \Delta_\mu(g_n) = \Delta_\mu(f_n) \neq 0.
\]
Therefore, \( Dg(x) \) satisfies the normalized form, and \( x \) is a simple multiple root of \( g \) with multiplicity \( \mu \).

Let \( R = \frac{d}{4\gamma_\mu(g,x)} \), we have
\[
d_R(g,f) = \max_{\|y - x\| \leq R} \|g(y) - f(y)\|
= \max_{\|y - x\| \leq R} \|f(x) + \sum_{1 \leq k \leq \mu - 1} H_k(X - x)^k\|
\leq \|f(x)\| + \sum_{1 \leq k \leq \mu - 1} \|H_k\| R^k
\]
\[ \| f(x) \| + \sum_{1 \leq k \leq \mu - 1} \| H_k \| \left( \frac{d}{4 \gamma \mu} \right)^k. \]

If
\[ \| f(x) \| + \sum_{1 \leq k \leq \mu - 1} \| H_k \| \left( \frac{d}{4 \gamma \mu} \right)^k \leq \frac{d^{\mu + 1}}{2 (4 \gamma \mu)^\mu \| A^{-1} \|}, \]
then
\[ d_R(g, f) < \frac{d^{\mu + 1}}{2 (4 \gamma \mu)^\mu \| A^{-1} \|} = \frac{d R^\mu}{2 \| A^{-1} \|}. \]

By Theorem 7, we know that \( f \) has \( \mu \) zeros (counting multiplicities) in the ball of radius \( \frac{d}{4 \gamma \mu} \) around \( x \). \( \square \)

3.3. Re-examining Double Simple Zeros. In what follows, we assume \( x \) is a simple double zero of \( f \), \( Df(x) \) satisfies the normalized form, and \( y \) is another zero of \( f \). By expanding \( f_n(y) \) at \( x \) and \( \frac{\partial f_n(x)}{\partial X_i} = \cdots = \frac{\partial^2 f_n(x)}{\partial X_i \partial X_j} = 0 \), we have:
\[ f_n(y) = \frac{\partial^2 f_n(x)}{\partial X_1 \partial X} \zeta_\eta + \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2} \eta^2 + \sum_{k \geq 3} \frac{D^k f_n(x)(y - x)^k}{k!}. \]

The nonzero terms are
\[ C_{1,1} = \frac{\partial^2 f_n(x)}{\partial X_1 \partial X}, \quad C_{0,2} = \frac{1}{2} \frac{\partial^2 f_n(x)}{\partial X^2}, \quad c_{1,1} = 2, \quad c_{0,2} = 1. \]

Hence, \( p(d) \) has the following form:
\[ (3.24) \quad p(d) = 1 - 2d^2 - 2d \sqrt{1 - d^2} - d. \]

The smallest positive real root of \( p(d) \) is
\[ d \approx 0.2865. \]

Let \( y \) be another root of \( f \), by Theorem 5 we have
\[ \| y - x \| \geq \frac{d}{2 \gamma \mu^2} \]

Example 1. Suppose we are given polynomials:
\[ \begin{cases} f_1 = X_1^2 - \frac{1}{4} X_1 - \frac{1}{2} X_2, \\ f_2 = \frac{1}{2} X_1 X_2. \end{cases} \]

It is easy to check that \( x = (0, 0) \) is a simple double zero of \( f = \{ f_1, f_2 \} \), and \( (1/4, 0) \) is another zero of \( f \), and we have
\[ Df(x) = \left( \begin{array}{cc} -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{array} \right). \]

Let \( g(X) = f(W \cdot X) \), where \( W = \left( \begin{array}{cc} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{array} \right) \), then
\[ \begin{align*} g_1 &= \frac{4}{5} X_1^2 - \frac{4}{5} X_1 X_2 + \frac{1}{5} X_2^2 + \frac{\sqrt{5}}{4} X_2, \\ g_2 &= -\frac{4}{5} X_1^2 - \frac{3}{10} X_1 X_2 + \frac{1}{5} X_2^2. \end{align*} \]
and \( x = (0, 0) \) is a simple double zero of \( g \), \( y = \left( \frac{1}{2\sqrt{5}}, -\frac{1}{4\sqrt{5}} \right) \) is another zero. We have
\[
Dg(x) = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} \\ 0 & 0 \end{pmatrix},
\]
and \( \ker Dg(x) = \text{span}\{v\} \), where \( v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \),
\[
D^2g(x) = \begin{pmatrix} \left( \begin{array}{cc} \frac{8}{5} & -\frac{4}{10} \\ -\frac{4}{10} & \frac{8}{5} \end{array} \right) & \left( \begin{array}{cc} -\frac{4}{10} & \frac{2}{5} \\ \frac{2}{5} & \frac{8}{5} \end{array} \right) \end{pmatrix}.
\]
Since
\[
\frac{\partial g_1(x)}{\partial X_2} = \frac{\sqrt{5}}{4}, \quad 2 \frac{\partial^2 g_2}{\partial X^2} = \Delta_2 = -\frac{1}{5},
\]
we have
\[
\hat{\gamma}_2 = \max \left( 1, \left\| \left( \frac{\partial g_1(x)}{\partial X_2} \right)^{-1} \cdot \frac{D^2g_1(x)}{2} \right\| \right) = 4 \sqrt{5},
\]
and
\[
\gamma_{2,2} = \max \left( 1, \left\| \frac{1}{\Delta_2(g_2)} \cdot \frac{D^2g_2(x)}{2} \right\| \right) = 1.
\]
Therefore,
\[
\gamma_2 = \frac{4}{\sqrt{5}} \approx 1.7888,
\]
and the local separation bound we obtained according to Theorem 5 for \( \mu = 2 \) is
\[
\|y - x\| \geq \frac{d}{2\gamma_2} \geq 0.0447.
\]
Now let us estimate the local separation bound by the method in [8]. The invertible linear operator defined in [8] is
\[
A(f, x, v) = Df(x) + \frac{1}{2} D^2f(x)(v, \Pi_v),
\]
where \( v = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \), and
\[
D^2f(x) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.
\]
Let \( \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{C}^2 \), denote \( A = A(f, x, v) \), we have
\[
A\omega = Df(x)\omega + \frac{1}{2} D^2f(x)(v, \Pi_v, \omega) = \begin{pmatrix} -\frac{1}{1} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \omega_2 + \frac{2}{5\sqrt{5}} \omega_1 \\ -\frac{2}{5\sqrt{5}} \omega_1 + \frac{1}{5\sqrt{5}} \omega_2 \end{pmatrix}.
\]
We have
\[
A^{-1} \frac{D^2 f(x)}{2} = \left( \begin{array}{cc}
\frac{4}{8} - \frac{2(25+8\sqrt{5})}{10\sqrt{5}} & 0 \\
\frac{8}{3} - \frac{25+32\sqrt{5}}{20\sqrt{5}} & 0 \\
\end{array} \right).
\]

The computation of the norm of the tensor \(A^{-1} \frac{D^2 f(x)}{2}\) is quite challenge. However, using our SOS certificates for global optima of polynomials and rational functions \[19\], we can verify that
\[
\gamma_2(f, x) = \max \left(1, \left\| A^{-1} \frac{D^2 f(x)}{2} \right\| \right) \geq 3.1121.
\]

Therefore, the local separation bound computed by the method in \[8\] satisfies
\[
\frac{d}{2\gamma_2(f, x)^2} \leq 0.01546,
\]
for \(d \approx 0.2976\).

**Remark 3.** Although our \(d\) is smaller than the one obtained in \[8\], the value of \(\gamma_2\) computed by our method could be smaller too. Therefore, as shown by Example \[4\], we might get better local separation bound.

**Example 2.** Suppose we are given polynomials:
\[
\begin{align*}
f_1 &= \frac{64}{73} X_1^2 - \frac{48}{73} X_1 X_2 + \frac{9}{73} X_2^2 + \frac{\sqrt{73}}{12} X_2, \\
f_2 &= (8X_1 - 3X_2)^2(3X_1 + 8X_2).
\end{align*}
\]

Let \(x = (0, 0)\) be a simple triple zero of \(f = \{f_1, f_2\}\), and \(y = (\frac{2}{\sqrt{73}}, -\frac{3}{4\sqrt{73}})\) be another zero of \(f\), \(\|y - x\| = 0.25\).

We have
\[
Df(x) = \left( \begin{array}{c}
0 \\
\frac{\sqrt{73}}{12} \\
0 \\
0 \\
\end{array} \right),
\]
\[
\frac{\partial f_1(x)}{\partial X_2} = \frac{\sqrt{73}}{12}.
\]
\[
\Delta_3(f_2) = \frac{1}{6} \frac{\partial^3 f_2(x)}{\partial X_1^3} - \frac{\partial^2 f_2(x)}{\partial X_1 \partial X_2} \cdot \left( \frac{\partial f_1(x)}{\partial X_2} \right)^{-1} \frac{\partial^2 f_1(x)}{2 \partial X_1^2} = 192,
\]
\[
\hat{\gamma}_3(f, x) = \max \left(1, \left\| \left( \frac{\partial f_1}{\partial X_2} \right)^{-1} \cdot \frac{D^2 f_1(x)}{2} \right\| \right) = \frac{12}{\sqrt{73}},
\]
\[
\gamma_{3,2}(f, x) = \max_{2 \leq k \leq 3} \left(1, \left\| \frac{1}{\Delta_3(f_2)} \cdot \frac{D^k f_2(x)}{k!} \right\|^{\frac{1}{k}} \right) = \frac{\sqrt{73} \cdot 146^{\frac{2}{3}}}{24},
\]
\[
\gamma_3(f, x) = \max(\hat{\gamma}_3(f, x), \gamma_{3,2}(f, x)) = \frac{12}{\sqrt{73}} \approx 1.4045.
\]

By Theorem \[1\] the local separation bound we obtain is
\[
\|y - x\| \geq \frac{d}{2\gamma_3} \approx 0.01545.
\]
For an approximate solution \( x = (-4.1291 \cdot 10^{-8}, -2.9505 \cdot 10^{-8}) \) obtained after applying twice the modified Newton iterations defined by Algorithm 4.2 to an approximate zero \( x = (-0.01, 0.01) \), we have

\[
\gamma_3(g, x) = \max(\hat{\gamma}_3(g, x), \gamma_{3,2}(g, x)) = \frac{12}{\sqrt{73}} \approx 1.4045,
\]

and

\[
\| f(x) \| + \| H_1 \| \frac{d}{4\gamma_3^3} + \| H_2 \| \frac{d^2}{16\gamma_3^3} \approx 1.937364 \cdot 10^{-8} < \frac{d^4}{128\gamma_3^9 \| A^{-1} \|} \approx 1.937370 \cdot 10^{-8}.
\]

By Theorem 4, we can guarantee that \( f \) has three zeros (counting multiplicities) in the ball of radius \( \frac{d}{4\gamma_3^3} \approx 0.0076 \) around \( x \).

We notice that

\[
\left\| \frac{1}{\Delta_3(f_2)} \cdot \frac{D^2 f_2(z)}{2} \right\| = 0.00004935 \neq \left\| \frac{1}{\Delta_3(g_2)} \cdot \frac{D^2 g_2(z)}{2} \right\| = 0.00001467.
\]

4. Modified Newton Iterations

For simple double zeros and simple triple zeros whose Jacobian matrix has a normalized form \((2.9)\), we define modified Newton iterations and show the quantified quadratic convergence if the approximate zeros are near the exact singular zeros. For a simple multiple zero of arbitrary large multiplicity whose Jacobian matrix may not have a normalized form, we perform unitary transformations and modified Newton iterations based on our previous work in \([24]\), and show its non-quantified quadratic convergence for simple multiple zeros and the quantified convergence for simple triple zeros.

4.1. \( \gamma \)-theorem for Simple Double Zeros. Given an approximate zero \( z \) of \( f \) with associated simple double zero \( \xi \) such that \( D^2 f(\xi) \) is invertible and

\[
\frac{\partial f_i(\xi)}{\partial x_1} = 0, \quad \frac{\partial f_n(\xi)}{\partial x_1} = 0, \quad 1 \leq i \leq n, \quad \frac{\partial^2 f_n(\xi)}{\partial x_1^2} \neq 0,
\]

we aim to approximate \( \xi \) by applying modified Newton’s method to \( z \) and iterating \( k \) times such that \( \| N^k_f(z) - \xi \| < \epsilon \) for a given accuracy \( \epsilon \).

Algorithm 1 Modified Newton Iteration for Simple Double Zero

\begin{itemize}
  \item **Input:**
  \( f \): a polynomial system;
  \( z = (z_1, \hat{z}) \): an approximate simple double zero of \( f \);
\end{itemize}

\begin{itemize}
  \item **Output:**
  \( N_f(z) = (N_2(z_1), N_1(\hat{z})) \): a refined solution after one iteration;
\end{itemize}

\begin{itemize}
  \item 1. \( N_1(\hat{z}) \leftarrow \hat{z} - D^2 f(\hat{z})^{-1} f(\hat{z}) \);
  \item 2. \( \hat{y} \leftarrow N_1(\hat{z}) \);
  \item 3. \( z \leftarrow (z_1, \hat{y}) \);
  \item 4. \( N_2(z_1) \leftarrow z_1 - (\frac{\partial^2 f_n(z_1)}{\partial x_1^2})^{-1} \frac{\partial f_n(z_1)}{\partial x_1} \);
\end{itemize}
Definition 4. For an approximate zero $z$ of $f$ with associated simple double zero $\xi$, let $\gamma_2 = \gamma_2(f, \xi)$, $u = \gamma_2^2 \|z - \xi\|$, we define the following rational functions:

\[
\begin{align*}
    b_{2,1}(u) &= \frac{(1 - 2u)^2 u}{2(1 - 2u)^2 - 1}, \\
    b_{2,2}(u) &= \frac{u}{2(1 - 2u)^2 - 1}, \\
    b_{2,3}(u) &= \frac{u(32u^6 - 144u^5 + 272u^4 - 288u^3 + 174u^2 - 52u + 5)}{(24u^3 - 36u^2 + 18u - 1)(-1 + u)^3(8u^2 - 8u + 1)}, \\
    b_{2,4}(u) &= \frac{(-1 + 2u)^3(-2 + u)u}{(24u^3 - 36u^2 + 18u - 1)(-1 + u)^3(8u^2 - 8u + 1)}.
\end{align*}
\]

Theorem 9. Let $\xi$ be a simple double zero of $f$.

1. If $u < u_2 \approx 0.0418$, where $u_2$ is the smallest positive solution of the equation:

\[
2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 = 1,
\]

then the output of Algorithm 4.1 satisfies:

\[
\|N_f(z) - \xi\| < \|z - \xi\|.
\]

2. If $u < u_2' \approx 0.0318$, where $u_2'$ is the smallest positive solution of the equation:

\[
2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 = \frac{1}{4},
\]

then after applying $k$ times of the iteration defined in Algorithm 4.1, we have:

\[
\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k - 1} \|z - \xi\|.
\]

Lemma 4. For $u \leq u_2$ we have:

\[(4.1) \quad \left\|D\hat{f}(z)^{-1}D\hat{f}(\xi)\right\| \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1}.
\]

Proof. The Taylor’s expansions of $\hat{f}(z)$ and $D\hat{f}(z)$ at $\xi$ are

\[
\hat{f}(z) = D\hat{f}(\xi)(z - \xi) + \sum_{k=2}^{\infty} \sum_{i=0}^{k} \frac{1}{i!(k - i)!} \partial^k \hat{f}(\xi) \partial X^i \partial X^{k-i}(z_1 - \xi_1)^i(z - \xi)^{k-i},
\]

and

\[
D\hat{f}(z) = D\hat{f}(\xi) + \sum_{k=2}^{\infty} \sum_{i=0}^{k-1} \frac{1}{i!(k - i - 1)!} \partial^k \hat{f}(\xi) \partial X^i \partial X^{k-i}(z_1 - \xi_1)^i(z - \xi)^{k-i-1}.
\]

Since $D\hat{f}(\xi)^{-1}$ exists, we have

\[
D\hat{f}(\xi)^{-1}D\hat{f}(z)
= I_{n-1} + D\hat{f}(\xi)^{-1} \sum_{k=2}^{\infty} \sum_{i=0}^{k-1} \frac{1}{i!(k - i - 1)!} \partial^k \hat{f}(\xi) \partial X^i \partial X^{k-i}(z_1 - \xi_1)^i(z - \xi)^{k-i-1}
= I_{n-1} + B.
\]
Hence, we have

\[ \|B\| = \|D\hat{f}(\xi)^{-1}D\hat{f}(z) - I_{n-1}\| \leq \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{k!}{i!(k-i)!} \hat{\gamma}_2^{k-1}\|z - \xi\|^{k-1} \]

\[ = \sum_{k \geq 2} k \cdot 2^{k-1} (\hat{\gamma}_2\|z - \xi\|)^{k-1} \]

\[ \leq \frac{1}{(1 - 2\hat{\gamma}_2\|z - \xi\|)^2 - 1} \]

\[ \leq \frac{1}{(1 - 2u)^2 - 1}. \]

When \( u < u_2 \), \( \|B\| < 1 \), we have

\[ \left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| = \left\| (I_{n-1} + B)^{-1} \right\| \leq \sum_{k=0}^{\infty} \|B\|^k \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1} \]

\[ \square \]

**Lemma 5.** For \( u \leq u_2 \), we have:

\[ \left\| N_1(\hat{z}) - \hat{\xi} \right\| \leq \frac{\hat{\gamma}_2\|\hat{z} - \hat{\xi}\|^2}{[2(1 - 2u)^2 - 1](1 - u)} + \frac{(1 - 2u)^2\hat{\gamma}_2\|z_1 - \xi_1\|^2}{[2(1 - 2u)^2 - 1](1 - u)} \]

\[ \leq b_{2,1}(u)\|z_1 - \xi_1\| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|. \]

**Proof.**

\[ \left\| N_1(\hat{z}) - \hat{\xi} \right\| = \left\| \hat{z} - \hat{\xi} - D\hat{f}(z)^{-1}\hat{f}(z) \right\| \]

\[ = \left\| D\hat{f}(z)^{-1} \left[ D\hat{f}(z)(\hat{z} - \hat{\xi}) - \hat{f}(z) \right] \right\| \]

\[ \leq \left\| D\hat{f}(z)^{-1} \left[ D\hat{f}(\xi)(\hat{z} - \hat{\xi}) + \sum_{k \geq 2} \sum_{i=0}^{k-1} \frac{k!}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X^k_i} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i} \right] \right\| \]

\[ \leq \left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| \cdot \left\| D\hat{f}(\xi)^{-1} \sum_{k \geq 2} \sum_{i=0}^{k-2} \frac{k!}{i!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X^k_i} (z_1 - \xi_1)^i (\hat{z} - \hat{\xi})^{k-i} \right\| \]

\[ \leq \left\| D\hat{f}(z)^{-1}D\hat{f}(\xi) \right\| \cdot \left( \sum_{k \geq 2} \sum_{i=0}^{k-2} \frac{(k-i)k!}{i!(k-i)!} \hat{\gamma}_2^{k-1}\|z - \xi\|^{k-2}\|\hat{z} - \hat{\xi}\|^2 \right). \]
Remark 4. The Newton iteration defined by $N_1$ operator works for any simple multiple zero of multiplicity $\mu \geq 2$ whose Jacobian matrix has a normalized form. It is clear from the proofs of Lemma 4 and Lemma 5, if we set $u = \gamma^\mu_\mu \|z - \xi\|$, then the conclusions of both lemmas still hold.

Let $z = (z_1, \hat{y})$, where $\hat{y} = N_1(\hat{z})$, then we have the following Taylor's expansion of $f_n(z)$ at $\xi$:

$$f_n(z) = \sum_{k=2}^{\infty} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \left. \frac{\partial^k f_n(\xi)}{\partial X_i^k} \right|_{(z_1, \xi)} \hat{y}^{k-i}. $$

Lemma 6. When $u < u_2$, we have

$$\left( \frac{\partial^2 f_n(z)}{\partial X_i^2} \right)_{i=1}^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_i^2} \leq \frac{(2u - 1)^3}{24u^3 - 36u^2 + 18u - 1}.$$

Proof. We have

$$\left( \frac{\partial^2 f_n(\xi)}{\partial X_i^2} \right)_{i=1}^{-1} \frac{\partial^2 f_n(z)}{\partial X_i^2} = 1 + \sum_{k=3}^{\infty} \sum_{i=2}^{k} \frac{1}{i!(k-i)!} \left. \frac{\partial^k f_n(\xi)}{\partial X_i^k} \right|_{(z_1, \xi)} \hat{y}^{k-i}$$

$$= 1 + B,$$

where

$$|B| = \left| \left( \frac{\partial^2 f_n(\xi)}{\partial X_i^2} \right)_{i=1}^{-1} \frac{\partial^2 f_n(z)}{\partial X_i^2} - 1 \right|$$

$$\leq \sum_{k=3}^{\infty} \sum_{i=2}^{k} \frac{k!}{(i-2)!(k-i)!} \gamma_2, n \|z - \xi\|^{k-2}$$

$$= \sum_{k=3}^{\infty} k(k-1) \cdot 2^{k-2} \gamma_2, n \|z - \xi\|^{k-3} \gamma_2, n \|z - \xi\|.$$
\[
\frac{\partial^2 f_n(z)}{\partial X_1^2} - \frac{\partial^2 f_n(\xi)}{\partial X_1^2} - \frac{\partial f_n(z)}{\partial X_2^2} - \frac{\partial f_n(\xi)}{\partial X_2^2} = \left| (1+B)^{-1} \right| \leq \sum_{k=0}^{\infty} |B|^k \leq \frac{(2u - 1)^3}{24u^3 - 36u^2 + 18u - 1}.
\]

When \( u < u_2 \), \(|B| < 1\), we have
\[
\left| \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} - \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right) \right| \leq 16u^3 + 24u^2 - 12u.
\]

\( \Box \)

**Lemma 7.** When \( u < u_2 \), we have
\[
|N_2(z_1) - \xi_1| \leq b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)\|\hat{\zeta} - \hat{\xi}\|.
\]

**Proof.** For \( u < u_2 \), we have
\[
|N_2(z_1) - \xi_1| = z_1 - \xi_1 - \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial f_n(z)}{\partial X_1} \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \left( \frac{\partial f_n(z)}{\partial X_2^2} \right) z_1 - \xi_1
\]
\[
\leq \left| \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \left| \left( \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right)^{-1} \sum_{k \geq 2} \frac{1}{(k-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1 \partial X_{k-1}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i} \right|
\]
\[
+ \sum_{k \geq 2} k\gamma_{2,n}^k \|z - \xi\|^{k-2} \|\hat{y} - \hat{\xi}\|
\]
\[
\leq \left| \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \left( \sum_{k \geq 3} \frac{1}{(i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1 \partial X_{k-i}} (z_1 - \xi_1)^{i-1} (\hat{y} - \hat{\xi})^{k-i} \right)
\]
\[
+ \sum_{k \geq 2} k\gamma_{2,n}^k \|z - \xi\|^{k-2} \gamma_{2,n}^\gamma \|\hat{y} - \hat{\xi}\|
\]
\[
\leq \left| \left( \frac{\partial^2 f_n(z)}{\partial X_1^2} \right)^{-1} \frac{\partial^2 f_n(\xi)}{\partial X_1^2} \right| \left( \sum_{k \geq 3} \frac{1}{(i-1)!} \frac{\partial^k f_n(\xi)}{\partial X_1 \partial X_{k-i}} (\gamma_{2,n}^2 \|z - \xi\|)^{k-2} (z_1 - \xi_1) \right)
\]
\[
+ \sum_{k \geq 2} k\gamma_{2,n}^k \|z - \xi\|^{k-2} \gamma_{2,n}^\gamma \|\hat{y} - \hat{\xi}\|
\]
Proof. Now we can complete the proof of Theorem 9:

Then by Lemma 5 and Lemma 6, we have:

\[
\left| N_2(z_1) - \xi_1 \right| \\
\leq \frac{u(32u^6 - 144u^5 + 272u^4 - 288u^3 + 174u^2 - 52u + 5)}{(24u^3 - 36u^2 + 18u - 1)(-1 + u)^3(8u^2 - 8u + 1)} \left| \frac{z_1 - \xi_1}{u(4u - 3)} \right| + \frac{2 - u)}{(u - 1)^2(2u - 1)^3} \left| \frac{\hat{y} - \xi}{\hat{y} - \xi} \right|
\]

Then by Lemma 5 and Lemma 6, we have:

\[
\left| N_2(z_1) - \xi_1 \right| \\
\leq \frac{u(32u^6 - 144u^5 + 272u^4 - 288u^3 + 174u^2 - 52u + 5)}{(24u^3 - 36u^2 + 18u - 1)(-1 + u)^3(8u^2 - 8u + 1)} \left| \frac{z_1 - \xi_1}{u(4u - 3)} \right| + \frac{2 - u)}{(u - 1)^2(2u - 1)^3} \left| \frac{\hat{y} - \xi}{\hat{y} - \xi} \right|
\]

\[
=b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)||\hat{z} - \xi||.
\]

\[
\square
\]

Proof. Now we can complete the proof of Theorem 9

(1) For \(0 < u < u_2 \approx 0.0418\), we have

\[
2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 < 1, \quad 2b_{2,2}(u)^2 + 2b_{2,4}(u)^2 < 1.
\]

Hence, we have

\[
\left\| N_f(z) - \xi \right\|^2 \leq \left\| N_1(\hat{z}) - \hat{\xi} \right\|^2 + \left\| N_2(z_1) - \xi_1 \right\|^2 \\
\leq \left(b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)||\hat{z} - \hat{\xi}||\right)^2 + \left(b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)||\hat{z} - \hat{\xi}||\right)^2 \\
\leq \left(2b_{2,1}(u)^2 + 2b_{2,3}(u)^2\right)|z_1 - \xi_1|^2 + \left(2b_{2,2}(u)^2 + 2b_{2,4}(u)^2\right)||\hat{z} - \hat{\xi}||^2 \\
< \left\| z - \xi \right\|^2.
\]

(2) For \(0 < u < u_2' \approx 0.0318\), we have

\[
2b_{2,2}(u)^2 + 2b_{2,4}(u)^2 < 2b_{2,1}(u)^2 + 2b_{2,3}(u)^2 < \frac{1}{4}.
\]

Hence, we have

\[
\left\| N_f(z) - \xi \right\|^2 \leq \left\| N_1(\hat{z}) - \hat{\xi} \right\|^2 + \left\| N_2(z_1) - \xi_1 \right\|^2 \\
\leq \left(b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)||\hat{z} - \hat{\xi}||\right)^2 + \left(b_{2,3}(u)|z_1 - \xi_1| + b_{2,4}(u)||\hat{z} - \hat{\xi}||\right)^2 \\
\leq \left(2b_{2,1}(u)^2 + 2b_{2,3}(u)^2\right)|z_1 - \xi_1|^2 + \left(2b_{2,2}(u)^2 + 2b_{2,4}(u)^2\right)||\hat{z} - \hat{\xi}||^2 \\
\leq \left(2b_{2,1}(u)^2 + 2b_{2,3}(u)^2\right)||z - \xi||^2 \\
\leq \frac{1}{4} \left\| z - \xi \right\|^2.
\]

The following inequality is true for \(k = 1\):

\[
\left\| N_f^k(z) - \xi \right\| \leq \left(\frac{1}{2}\right)^{2k-1} \left\| z - \xi \right\|.
\]

For \(k \geq 2\), assume by induction that

\[
\left\| N_f^{k-1}(z) - \xi \right\| \leq \left(\frac{1}{2}\right)^{2k-1-1} \left\| z - \xi \right\|.
\]
Let \( u^{(k-1)} = \gamma_2^2 \left\| N_f^{k-1}(z) - \xi \right\| \). For \( 0 < u < u'_2, k \geq 2 \), we have \( u^{(k-1)} < u = \gamma_2^2 \| z - \xi \| \) and \( \frac{\sqrt{2b_{2,1}(u)x_2^2 + 2b_{2,3}(u)x_2^3}}{u} \) is increasing. Therefore, we have

\[
\left\| N_f^k(z) - \xi \right\| = \left\| N_f \left( N_f^{k-1}(z) - \xi \right) \right\| < \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{2,3}(u)^3}}{u} \left\| N_f^{k-1}(z) - \xi \right\|^2 < \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{2,3}(u)^3}}{u} \left( \frac{1}{2} \right)^{2k-2} \| z - \xi \|^2 = \left( \frac{1}{2} \right)^{2k-1} \| z - \xi \|. \]

\[ \square \]

4.2. \( \gamma \)-theorem for Simple Triple Zeros. Given an approximate zero \( z \) of \( f \) with associated simple triple zero \( \xi \) such that \( D\hat{f}(\xi) \) is invertible and

\[
\frac{\partial f_i(\xi)}{\partial X_1} = 0, \quad \frac{\partial f_i(\xi)}{\partial X_i} = 0, \quad 1 \leq i \leq n, \quad \frac{\partial^2 f_n(\xi)}{\partial X_1^2} = 0,
\]

but

\[
\Delta_3(f_n) = \frac{1}{6} \frac{\partial^3 f_n(X_1)}{\partial X_1^3} - \frac{\partial^2 f_n(X_1)}{\partial X_1 \partial X} \cdot D\hat{f}(\xi) - \frac{1}{2} \frac{\partial^2 \hat{f}(\xi)}{\partial X_1^2} \neq 0.
\]

We aim to approximate \( \xi \) by applying modified Newton’s method to \( z \) and iterating \( k \) times such that \( \left\| N_f^k(z) - \xi \right\| < \epsilon \) for a given accuracy \( \epsilon \).

Let us define the differential operator \( L_3 \),

\[
L_3(f_n)(z) = \frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_1^3} - \frac{\partial^2 f_n(z)}{\partial X_1 \partial X} \cdot D\hat{f}(z) - \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2},
\]

then we have \( L_3(f_n)(\xi) = \Delta_3(f_n) \). As \( \Delta_3(f_n) \neq 0 \), and \( z \) is near to \( \xi \), we can assume that \( L_3(f_n)(z) \neq 0 \). Moreover, we define the differential operator \( \Gamma_1 \) such that

\[
\Gamma_1(f_n)(z) = \frac{1}{6} \frac{\partial^2 f_n(z)}{\partial X_1^2} - \frac{\partial f_n(z)}{\partial X} \cdot D\hat{f}(z) - \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2}.
\]
Algorithm 2 Modified Newton Iteration for Simple Triple Zero

Input:
- $f$: a polynomial system;
- $z = (z_1, \hat{z})$: an approximate simple triple zero of $f$;

Output:
- $N_f(z) = (N_2(z_1), N_1(\hat{z}))$: a refined solution after one iteration;
- $N_1(\hat{z}) \leftarrow \hat{z} - D\hat{f}(\hat{z})^{-1}f(\hat{z})$;
- $\hat{y} \leftarrow N_1(\hat{z})$;
- $z \leftarrow (z_1, \hat{y})$;
- $N_2(z_1) \leftarrow z_1 - (L_3(f_n)(z))^{-1} \Gamma_1(f_n)(z)$;

Definition 5. For an approximate root $z$ of $\xi$, let $u = \gamma_3^3\|z - \xi\|$. We define the following rational functions:

$$a_2(u) = \frac{1}{2(1 - 2u)^2 - 1}(1 - 2u),$$

$$a_3(u) = \frac{2u - 1}{128u^9 - 384u^8 + 464u^7 - 320u^6 + 136u^5 - 30u + 1},$$

$$b_{3,3}(u) = \frac{-a_3(u)}{3(2u - 1)^4(8u^2 - 8u + 1)^2(u - 1)^4},$$

$$b_{3,4}(u) = \frac{a_3(u)(16u^6 - 72u^5 + 130u^4 - 106u^3 + 42u^2 - 9u)}{3(8u^2 - 8u + 1)^2(u - 1)^2(2u - 1)}.$$

Theorem 10. Let $\xi$ be a simple triple zero of $f$.

1. If $u < u_3 \approx 0.0222$, where $u_3$ is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 = 1,$$

then the output of Algorithm 4,3 satisfies:

$$\|N_f(z) - \xi\| < \|z - \xi\|.$$

2. If $u < u'_3 \approx 0.0154$, where $u'_3$ is the smallest positive solution of the equation:

$$2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 = \frac{1}{4},$$

then after $k$ times of iteration we have

$$\|N_f^k(z) - \xi\| < \left(\frac{1}{2}\right)^{2^k-1}\|z - \xi\|.$$

According to Remark 4, similar to Lemma 4 and Lemma 5 for $u \leq u_3 \approx 0.0222$, we have:

$$\left\|D\hat{f}(\hat{z})^{-1}D\hat{f}(\hat{z})\right\| \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1},$$

and

$$\|N_1(\hat{z}) - \xi\| \leq b_{2,1}(u)\|z_1 - \xi_1\| + b_{2,2}(u)\|\hat{z} - \xi\|. $$
Let $z = (z_1, \hat{y})$, where $\hat{y} = N_1(\hat{z})$, we have the following Taylor's expansion of $f_n(z)$ at $\xi$:

$$f_n(z) = \sum_{k \geq 2} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial X^{k-i}} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i}.$$ 

**Lemma 8.** When $u < u_3$, we have

$$\left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| \leq a_2(u)\gamma_3.$$ 

**Proof.** When $u < u_3$, we have

$$\left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) D\hat{f}(\xi)^{-1} \right\| \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1}.$$ 

Then, it is clear that

$$\left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2} \right\| = \left\| D\hat{f}(z)^{-1} \cdot \frac{1}{2} \sum_{k \geq 2} \sum_{i=2}^{k} \frac{1}{(i-2)!(k-i)!} \frac{\partial^k \hat{f}(\xi)}{\partial X_1^i \partial X^{k-i}} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \right\|$$

$$\leq \frac{1}{2} \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \sum_{k \geq 2} \sum_{i=2}^{k} \frac{k!}{(i-2)!(k-i)!} \hat{\gamma}_3^{i-2} \|z - \xi\|^2 \right)$$

$$= \frac{1}{2} \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \sum_{k \geq 2} k(k-1)(2\hat{\gamma}_3 \|z - \xi\|)^{k-2} \hat{\gamma}_3 \right)$$

$$\leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1} \cdot \frac{1}{(1 - 2u)^3} \hat{\gamma}_3 = a_2(u)\gamma_3.$$ 

**Lemma 9.** When $u < u_3$, we have

$$\| L_3(f_n)(z)^{-1} \Delta_3(f_n) \| \leq a_3(u).$$ 

**Proof.** By the Taylor’s expansion of $f_n$ at $\xi$, we have:

$$\frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_1^3} = \frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_1^3} + \frac{1}{6} \sum_{i=3}^{k} \sum_{i=3}^{k} \frac{1}{(i-3)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial X_1^i \partial X^{k-i}} (z_1 - \xi_1)^{i-3} (\hat{y} - \hat{\xi})^{k-i},$$

$$\frac{\partial^2 f_n(z)}{\partial X_1 \partial X} = \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial X} + \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial X} (z_1 - \xi_1) + \frac{\partial^3 f_n(\xi)}{\partial X_1 \partial X^2} (\hat{y} - \hat{\xi})$$
Proof. When Lemma 10.

Then we have:
\[
\Delta_3(f_n)^{-1}L_3(f_n)(z)
= 1 + \Delta_3(f_n)^{-1}[L_3(f_n)(z) - \Delta_3(f_n)]
= 1 + B,
\]

where
\[
\|B\| = \left\| \frac{1}{2} \frac{\partial^2 f_n(\xi)}{\partial x_1^2} (z_1 - \xi) + \frac{\partial^2 f_n(\xi)}{\partial x_1 \partial x^k} (z_1 - \xi)^{i-1} (\hat{y} - \hat{\xi}) \right\|
+ \frac{1}{6} \sum_{k \geq i=3} \frac{1}{(i-3)!(k-i)!} \frac{\partial^k f_n(\xi)}{\partial x_1^i \partial x^{k-i}} (z_1 - \xi)^{i-3} (\hat{y} - \hat{\xi})^{k-i}.
\]

By Lemma 8 we have
\[
\|B\| \leq 12a_2(u) \cdot \gamma_3^3 \cdot \|z - \xi\| + 2^{k-2} \cdot k \cdot (k - 1) \cdot (\gamma_3^2)^{k-2} \|z - \xi\|^{k-2}
\]
\[
+ \frac{1}{6} \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{k-3}{i-3} \cdot \gamma_3^3 \cdot \|z - \xi\| + 2^{k-3} \cdot (k - 1) \cdot (k - 2) \cdot (k - 3) \cdot (\gamma_3^3)^{k-3} \|z - \xi\|^{k-3}
\]
\[
\leq 12a_2(u) \cdot \gamma_3^3 \cdot \|z - \xi\| + 2^{k-2} \cdot k \cdot (k - 1) \cdot (\gamma_3^2)^{k-2} \|z - \xi\|^{k-2}
\]
\[
+ \frac{1}{6} \sum_{k \geq 4} 2^{k-3} \cdot (k - 1) \cdot (k - 2) \cdot (k - 3) \cdot (\gamma_3^3)^{k-3} \|z - \xi\|^{k-3}
\]
\[
\leq a_2(u) \left( 12u + \sum_{k \geq 4} k(k-1)2^{k-2}u^{k-2} \right)
+ \frac{1}{6} \sum_{k \geq 4} (k-1)(k-2)(k-3)2^{k-3}u^{k-3}
\]
\[
= \frac{2u(16u^2 - 20u + 7)}{(2u - 1)^4(8u^2 - 8u + 1)}.
\]

When \( u < u_3 \), \( \|B\| < 1 \), we have
\[
\|L_3(f_n)(z)^{-1}\Delta_3(f_n)\| \leq \|(1 + B)^{-1}\| \leq a_3(u).
\]

\( \square \)

**Lemma 10.** When \( u < u_3 \), we have
\[
|N_2(z_1) - \xi_1| \leq b_{3,3}(u)(u)|z_1 - \xi_1| + b_{3,4}(u)(u)\|\hat{z} - \hat{\xi}\|.
\]

**Proof.** We have
\[
|N_2(z_1) - \xi_1| = |z_1 - \xi_1 - [L_3(f_n)(z)]^{-1}\Gamma_1(f_n)(z)|
= |L_3(f_n)(z)^{-1}\{L_3(f_n)(z)(z_1 - \xi_1) - \Gamma_1(f_n)(z)\}|
\]
Furthermore, we have
\[ L_3(f_n)(z_1 - \xi_1) - \Gamma_1(f_n)(z) = |L_3(f_n)(z)|^{-1} \Delta_3(f_n) \left[ L_3(f_n)(z_1 - \xi_1) - \Gamma_1(f_n)(z) \right]. \]

From the Taylor expansions of \( \frac{\partial^3 f_n(y)}{\partial X^3} \), \( \frac{\partial^2 f_n(y)}{\partial X^2} \), \( \frac{\partial f_n(y)}{\partial X} \), \( \frac{\partial f_n(y)}{\partial X} \) at \( \xi \), we have
\[
L_3(f_n)(z_1 - \xi_1) - \Gamma_1(f_n)(z) = \left\{ \begin{array}{l}
- \frac{1}{6} \sum_{k \geq 4} \frac{1}{(k - 2)!} \frac{\partial^k f_n(\xi)}{\partial X^k} (\hat{y} - \hat{\xi})^k - \frac{1}{6} \sum_{k \geq 4} \frac{1}{(k - 2)!} \frac{\partial^k f_n(\xi)}{\partial X^k} (\hat{y} - \hat{\xi})^{k-2} \\
+ \sum_{k \geq 4} \frac{\partial f_n(y)}{\partial X} (z_1 - \xi_1)^i (\hat{y} - \hat{\xi})^{k-i} \\
- D\hat{f}(z)^{-1} \frac{\partial^2 \hat{f}(z)}{2 \partial X^2} (z_1 - \xi_1)^2 - \frac{\partial^2 f_n(\xi)}{\partial X^2} (\hat{y} - \hat{\xi})^2 + \frac{1}{2} \frac{\partial f_n(\xi)}{\partial X^3} (\hat{y} - \hat{\xi})^2 \\
+ \sum_{k \geq 4} \frac{1}{(k - 1)!} \frac{\partial^k f_n(\xi)}{\partial X^k} (\hat{y} - \hat{\xi})^{k-1} \\
- \sum_{k \geq 4} \frac{1}{(k - 1)!} \frac{\partial^k f_n(\xi)}{\partial X^k} (\hat{y} - \hat{\xi})^{k-1} \right\}.
\]

Then we have
\[
\left\| \frac{1}{|\Delta_3(f_n)|} \left( \frac{1}{|\Delta_3(f_n)|} \left( \frac{1}{\partial X^2} \right) (\hat{y} - \hat{\xi}) \right) + \left| D\hat{f}(z)^{-1} \frac{\partial^2 \hat{f}(z)}{2 \partial X^2} \right| \cdot \frac{1}{|\Delta_3(f_n)|} \frac{\partial^2 f_n(\xi)}{\partial X^2} \right\| \\
\leq \gamma_3^2 \cdot \| (\hat{y} - \hat{\xi}) \| + 2a_2(u) \gamma_3^2 \cdot \| (\hat{y} - \hat{\xi}) \| \\
\leq (1 + 2a_2(u)) \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \sum_{k \geq 2} \gamma_3^{k+1} \| z - \xi \|^{k-1} |z_1 - \xi_1| \\
+ \sum_{k \geq 2} (k \cdot 2^{k-1} - 2^k + 1) \gamma_3^{k+1} \| z - \xi \|^{k-1} \| \hat{z} - \hat{\xi} \| \right) \\
\leq (1 + 2a_2(u)) \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \sum_{k \geq 2} (\gamma_3^k \| z - \xi \|)^{k-1} |z_1 - \xi_1| \\
+ \sum_{k \geq 2} (k \cdot 2^{k-1} - 2^k + 1) (\gamma_3^k \| z - \xi \|)^{k-1} \| \hat{z} - \hat{\xi} \| \right) \\
\leq \frac{u(2u - 1)^2(1 + 2a_2(u))}{(8u^2 - 8u + 1)(1 - u)} |z_1 - \xi_1| + \frac{u(1 + 2a_2(u))}{(8u^2 - 8u + 1)(1 - u)} \| \hat{z} - \hat{\xi} \|. 
\]

Furthermore, we have
\[
\left\| \frac{1}{|\Delta_3(f_n)|} \left( \frac{1}{|\Delta_3(f_n)|} \left( \frac{1}{\partial X^2} \right) (\hat{y} - \hat{\xi}) \right) \right\| \\
\leq \frac{1}{6} \sum_{k \geq 4} \sum_{i = 3}^{k} \frac{i - 3}{(i - 2)!} \frac{\partial^k f_n(\xi)}{\partial X^k} (z_1 - \xi_1)^{i-2} (\hat{y} - \hat{\xi})^{k-i} \\
\leq \frac{1}{6} \sum_{k \geq 4} \sum_{i = 3}^{k} \frac{k!(i - 3)!}{(i - 2)!} \gamma_3^{k-3} \| z - \xi \|^{k-3} |z_1 - \xi_1|. 
\]
\begin{equation}
\leq \frac{1}{6} \sum_{k \geq 4} \sum_{i=3}^{k} \left( \frac{k-3}{i-3} \right) \cdot k \cdot (k-1) \cdot (k-2) \cdot \gamma_3^{3k-9} \cdot ||z - \xi||^{k-3} \cdot |z_1 - \xi_1|
\end{equation}

\begin{equation}
\leq \frac{1}{6} \sum_{k \geq 4} k \cdot (k-1) \cdot (k-2) \cdot 2^{k-3} \cdot (\gamma_3^{3} ||z - \xi||)^{k-3} |z_1 - \xi_1|
\end{equation}

\begin{equation}
\leq \frac{1}{6} \sum_{k \geq 4} k \cdot (k-1) \cdot (k-2) \cdot 2^{k-3} \cdot u^{k-3} |z_1 - \xi_1|
\end{equation}

\begin{equation}
= \frac{-8u(2u^3 - 4u^2 + 3u - 1)}{(2u - 1)^4} |z_1 - \xi_1|.
\end{equation}

We have

\begin{equation}
\frac{1}{|\Delta_3(f_n)|} \left| \frac{1}{6} \sum_{k \geq 4} \frac{\partial^k f_n(\xi)}{(k-2)!} \frac{\partial X_i}{\partial X_k} \frac{\partial^k (\tilde{y} - \tilde{\xi})^{k-2}}{||z - \xi||^{k-3} ||\tilde{y} - \tilde{\xi}||} \right|
\end{equation}

\begin{equation}
\leq \frac{1}{6} \sum_{k \geq 4} k(k-1) \gamma_3^{k-1} ||z - \xi||^{k-3} ||\tilde{y} - \tilde{\xi}||
\end{equation}

\begin{equation}
\leq \frac{-u(3u^2 - 8u + 6)}{3(u - 1)^3} ||\tilde{y} - \tilde{\xi}||
\end{equation}

\begin{equation}
\leq \frac{(2u - 1)^2 u^2 (3u^2 - 8u + 6)}{3(8u^2 - 8u + 1)(u - 1)^4} |z_1 - \xi_1| + \frac{u^2 (3u^2 - 8u + 6)}{3(8u^2 - 8u + 1)(u - 1)^4} |\tilde{z} - \tilde{\xi}|
\end{equation}

By Lemma 8 we have

\begin{equation}
\left| D\tilde{f}(z)^{-1} \frac{\partial^2 \tilde{f}(z)}{\partial X_i^2} \right| \left| \frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial X_i^2 \partial X} (z_1 - \xi_1)^2 \right| \leq 3a_2(u) \gamma_3^3 ||z - \xi|| |z_1 - \xi_1| \leq 3a_2(u) |z_1 - \xi_1|.
\end{equation}

By Lemma 8 we have

\begin{equation}
\left| D\tilde{f}(z)^{-1} \frac{\partial^3 \tilde{f}(z)}{\partial X_i^3} \right| \left| \frac{1}{2} \frac{\partial^3 f_n(\xi)}{\partial X_i^3} (\tilde{y} - \tilde{\xi})^2 \right|
\end{equation}

\begin{equation}
\leq 3a_2(u) \gamma_3^3 ||z - \xi|| ||\tilde{y} - \tilde{\xi}||
\end{equation}

\begin{equation}
\leq \frac{3a_2(u) u^2}{2(1 - u)^2 - 1} |\tilde{z} - \tilde{\xi}| + \frac{3a_2(u) u^2 (1 - 2u)^2}{2(1 - u)^2 - 1} |z_1 - \xi_1|.
\end{equation}

We have

\begin{equation}
\left| D\tilde{f}(z)^{-1} \frac{\partial^2 \tilde{f}(z)}{\partial X_i^2} \right| \left| \frac{1}{6} \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{i - 1}{i!(k - i - 1)!} \frac{\partial^k f_n(\xi)}{\partial X_i^k \partial X_k} (z_1 - \xi_1)^i (\tilde{y} - \tilde{\xi})^{k-i-1} \right|
\end{equation}

\begin{equation}
\leq a_2(u) \sum_{k \geq 4} \sum_{i=1}^{k-1} \frac{k! \cdot (i - 1)}{i!(k - i - 1)!} \gamma_3^k ||z - \xi||^{k-2} |z_1 - \xi_1|
\end{equation}

\begin{equation}
\leq a_2(u) \sum_{k \geq 4} (k - 1) \cdot (k - 3) \cdot 2^k \cdot u^{k-2} |z_1 - \xi_1|
\end{equation}
\[
\leq \frac{16u^2(2u - 3)a_2(u)}{(u - 1)^3}|z_1 - \xi_1|.
\]

By Lemma 8, we have
\[
\left\| Df(z) \left( \frac{-1}{2} \frac{\partial^2 f(z)}{\partial X_1^2} \right) \right\| \leq a_2(u) \sum_{k \geq 4} k^k \frac{k!}{(k - 1)!} \frac{a_2(u)}{(u - 1)^3} \left\| z - \xi \right\|^{k-2} \left\| \tilde{y} - \tilde{\xi} \right\|
\leq -u^2(3u - 4)a_2(u)\left\| z - \xi \right\|^{k-2} \left\| \tilde{y} - \tilde{\xi} \right\|
\leq \frac{(2u - 1)^2u^3(3u - 4)a_2(u)}{(8u^2 - 8u + 1)(u - 1)^3} |z_1 - \xi_1| + \frac{u^3(3u - 4)a_2(u)}{(8u^2 - 8u + 1)(u - 1)^3} \left\| \tilde{z} - \tilde{\xi} \right\|.
\]

Finally, by Lemma 9 and the above estimations, we have
\[
|N_2(z_1) - \xi_1| \\
\leq a_3(u) \cdot \left( \left( \frac{u(2u - 1)^2(1 + 2a_2(u))}{(8u^2 - 8u + 1)(u - 1)^3} + \frac{16u^2(2u - 3)a_2(u)}{(u - 1)^3} \right) + \frac{-8u(2u^2 - 4u^2 + 3u - 1)}{(2u - 1)^4} + \frac{3u^2(1 - 2u)^2a_2(u)}{2(1 - 2u)^2 - 1}(1 - u) + 3ua_2(u) + \frac{(2u - 1)^2u^2(3u^2 - 8u + 6)}{3(8u^2 - 8u + 1)(u - 1)^3} + \frac{(u(1 + 2a_2(u))}{(8u^2 - 8u + 1)(u - 1)^3} + \frac{3u^2a_2(u)}{2(1 - 2u)^2 - 1}(1 - u) + \frac{u^3(3u - 4)a_2(u)}{(8u^2 - 8u + 1)(u - 1)^3} \cdot \left\| \tilde{z} - \tilde{\xi} \right\| \right)
= b_{3,3}(u)|z_1 - \xi_1| + b_{3,4}(u)\left\| \tilde{z} - \tilde{\xi} \right\|.
\]

**Proof.** Now we can complete the proof of Theorem 10.

1. For \( u < u_3 \approx 0.0222 \), it is true that
\[
2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 < 1, \quad 2b_{2,2}(u)^2 + 2b_{3,4}(u)^2 < 1.
\]

Therefore, we have
\[
\| N_f(z) - \xi \|^2 \\
\leq \left\| N_1(z) - \tilde{\xi} \right\|^2 + \left\| N_2(z_1) - \xi_1 \right\|^2 \\
\leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\left\| \tilde{z} - \tilde{\xi} \right\|)^2 + (b_{3,3}(u)|z_1 - \xi_1| + b_{3,4}(u)\left\| \tilde{z} - \tilde{\xi} \right\|^2 \\
\leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{3,4}(u)^2)\left\| \tilde{z} - \tilde{\xi} \right\|^2 \\
< \| z - \xi \|^2.
\]

2. For \( u < u_3' \approx 0.0154 \), it is true that
\[
2b_{2,2}(u)^2 + 2b_{3,4}(u)^2 < 2b_{2,1}(u)^2 + 2b_{3,3}(u)^2 < \frac{1}{4}.
\]
Hence, we have
\[ \|N_f(z) - \xi\| \leq \left\| N_1(\hat{z}) - \hat{\xi} \right\|^2 + |N_2(z) - \xi_1|^2 \]
\[ \leq (b_{2,1}(u)|z_1 - \xi_1| + b_{2,2}(u)\|\hat{z} - \hat{\xi}\|)^2 + (b_{3,3}(u)|z_1 - \xi_1| + b_{3,4}(u)\|\hat{z} - \hat{\xi}\|)^2 \]
\[ \leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2)|z_1 - \xi_1|^2 + (2b_{2,2}(u)^2 + 2b_{3,4}(u)^2)\|\hat{z} - \hat{\xi}\|^2 \]
\[ \leq (2b_{2,1}(u)^2 + 2b_{3,3}(u)^2)\|z - \xi\|^2 \]
\[ \leq \frac{1}{4} \|z - \xi\|^2. \]
Hence, the following inequality is true for \( k = 1 \):
\[ \|N_f^k(z) - \xi\| < \left( \frac{1}{2} \right)^{2^{k-1}} \|z - \xi\|. \]

For \( k \geq 2 \), assume by induction that
\[ \|N_f^{k-1}(z) - \xi\| < \left( \frac{1}{2} \right)^{2^{k-1}-1} \|z - \xi\|. \]
Let \( u^{(k-1)} = \gamma_3^3 \|N_f^{k-1}(z) - \xi\| \). For \( 0 < u < u_3 \), we have \( u^{(k-1)} < u = \gamma_3^3 \|z - \xi\| \) and \( \sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2} \gamma_3^2 \) is increasing. Therefore, we have
\[ \|N_f^k(z) - \xi\| \]
\[ = \left\| N_f \left( N_f^{k-1}(z) \right) - \xi \right\| \]
\[ < \frac{\sqrt{2b_{2,1}(u^{(k-1)})^2 + 2b_{3,3}(u^{(k-1)})^2} \gamma_3^3}{u^{(k-1)}} \left\| N_f^{k-1}(z) - \xi \right\|^2 \]
\[ < \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2} \gamma_3^3}{u} \left\| N_f^{k-1}(z) - \xi \right\|^2 \]
\[ < \frac{\sqrt{2b_{2,1}(u)^2 + 2b_{3,3}(u)^2} \gamma_3^3}{u} \left( \frac{1}{2} \right)^{2^{k-2}} \|z - \xi\|^2 \]
\[ = \left( \frac{1}{2} \right)^{2^{k-1}} \|z - \xi\|. \]

\[ \square \]

4.3. Simple Multiple Zeros. For simple double zeros and simple triple zeros of \( f \), we have defined modified Newton iterations based on the first, second and third order differential operators computed at the approximate solutions, and provided quantified criterions to guarantee its quadratic convergence. Although it is possible to extend the modified Newton iterations defined in Algorithm 4.1, 4.2 to simple multiple zeros of higher multiplicities, the iterations are only defined for systems whose Jacobian matrix at the exact multiple zero has a normalized form (2.9), they might be of limited applications.

In order to refining an approximate simple singular zero whose Jacobian matrix has corank one but it does not have a normalized form (2.9), in Algorithm 4.3 we perform the unitary transformations to both variables and equations defined at the approximate simple singular solutions, then we define the modified Newton
iterations based on our previous work in [24]. We show firstly its non-quantified quadratic convergence for simple multiple zeros of higher multiplicities, and then its quantified convergence for simple triple zero.

**Algorithm 3** Modified Newton Iteration for Simple Multiple Zeros

**Input:**
- \( f \): a polynomial system;
- \( z \): an approximate simple multiple zero;
- \( \mu \): the multiplicity;

**Output:**
- \( N_f(z) \): a refined solution after one iteration;

1. \( Df(z) = U \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} V^*, W_1 = (v_n, v_1, \ldots, v_{n-1}) \);
2. \( f(X) \leftarrow U^* \cdot f(W_1 \cdot X), \quad z \leftarrow W_1^* z \);
3. \( N_1(f, \hat{z}) \leftarrow \hat{z} - D\hat{f}(z)^{-1} \hat{f}(z), \quad y = (y_1, \hat{y}) \leftarrow (z_1, N_1(f, \hat{z})) \);
4. \( Df(z) = U \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} V^*, W_1 = (v_n, v_1, \ldots, v_{n-1}) \);
5. \( g(X) \leftarrow U^* \cdot f(W_1 \cdot X), \quad w = (w_1, \hat{w}) \leftarrow W_1^* y \);
6. \( N_2(g_n, w) \leftarrow w_1 - \frac{1}{n} \Delta_{\mu}(g_n)^{-1} \Delta_{\mu-1}(g_n), \quad x = (x_1, \hat{x}) \leftarrow (N_2(g_n, w), \hat{w}) \);
7. \( N_f(z) \leftarrow W_1 \cdot W_1^* x \).

**Theorem 11.** Given an approximate zero \( z \) of a system \( f \) associated to a simple multiple zero \( \xi \) of multiplicity \( \mu \) and satisfying \( f(\xi) = 0 \), \( \dim \ker Df(\xi) = 1 \). Suppose

\[
\hat{\gamma}_\mu(f, z) \|z - \xi\| < \frac{1}{2},
\]

where

\[
\hat{\gamma}_\mu(f, z) = \max \left\{ 1, \sup_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \right\},
\]

then the refined singular solution \( N_f(z) \) returned by Algorithm 4.3 satisfies

\[
\|N_f(z) - \xi\| = O(\|z - \xi\|^2).
\]

In what follows, we give quantitative analysis of the convergency of the first five steps in Algorithm 4.3. For Step 6, we show its non-quantified quadratic convergence first, and then show its quantified convergency for simple triple zeros, which can be generalized naturally to simple multiple zeros of higher multiplicities.

In the second step of Algorithm 4.3, we perform the unitary transformations to both variables and equations according to the singular value decomposition of the Jacobian matrix \( Df(z) \). Since \( \hat{\gamma}_\mu(f, z) \) and the Euclidean distance between zeros \( \xi \) and \( z \) do not change under the unitary transformation, in what follows, for simplicity, after the first two steps, we use the same notations for \( f, \xi, z \), i.e.,

\[
\xi \leftarrow W_1^* \xi, \quad z \leftarrow W_1^* z, \quad f(X) \leftarrow U^* \cdot f(W_1 \cdot X).
\]

**Claim 5.** After the first two steps in Algorithm 4.3, we have

\[
Df(z) = \begin{pmatrix} 0 & \Sigma_{n-1} \\ \sigma_n & 0 \end{pmatrix},
\]
where $\Sigma_{n-1}$ is a nonsingular diagonal matrix. Moreover, we have $\sigma_n \leq L \|z - \xi\|$, where $L$ is the Lipschitz constant of the function $Df(X)$.

Proof. According to the chain rule, we have

$$Df(z) = U^* \cdot U \cdot \left( \begin{array}{cc} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{array} \right) \cdot V^* \cdot W_r,$$

$$= \left( \begin{array}{cc} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{array} \right) \cdot \left( \begin{array}{cc} 0 & I_{n-1} \\ 1 & 0 \end{array} \right)$$

Furthermore, since $\dim \ker Df(\xi) = 1$, the following perturbation theorem about the singular values can be found in [12, 13],

$$\sigma_n \leq \|Df(z) - Df(\xi)\| \leq L \|z - \xi\|.$$ 

\[ \square \]

Claim 6. After running the first three steps in Algorithm 4.3, suppose $\tilde{\gamma}_\mu(f, z) \|z - \xi\| < \frac{1}{2}$, we have

$$\|\hat{y} - \hat{\xi}\| \leq \frac{1}{1 - \tilde{\gamma}_\mu(f, z)} \|\xi - z\| \hat{\gamma}_\mu(f, z) \|\xi - z\|^2,$$

and

$$\|\hat{f}(y)\| \leq \frac{4 \|\hat{f}(z)\|}{1 - 2\tilde{\gamma}_\mu(f, z)} \|\xi - z\| \hat{\gamma}_\mu(f, z) \|\xi - z\|^2,$$

where

$$y = (y_1, \hat{y}) \leftarrow (z_1, N_1(\hat{f}, \hat{\xi})).$$

Proof. According to Claim 5 we have $\frac{\partial \hat{f}(z)}{\partial x_1} = 0$ and $D\hat{f}(z) = \Sigma_{n-1}$ is an invertible diagonal matrix. Therefore, by the Taylor expansion of $\hat{f}$ at $z$, we have

$$0 = \hat{f}(\xi) = \hat{f}(z) + D\hat{f}(z)(\hat{\xi} - \hat{z}) + \sum_{k \geq 2} \frac{D^k \hat{f}(z)}{k!} (\xi - z)^k,$$

$$0 = D\hat{f}(z)^{-1} \hat{f}(z) + \hat{\xi} - \hat{z} + \sum_{k \geq 2} D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} (\xi - z)^k.$$

Then we have

$$\|N_1(\hat{f}, z) - \hat{\xi}\| \leq \sum_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \|\xi - z\|^k$$

$$\leq \sum_{k \geq 2} \tilde{\gamma}_\mu(f, z) k^{k-1} \|\xi - z\|^k$$

$$\leq \hat{\gamma}_\mu(f, z) \|\xi - z\|^2 \sum_{k \geq 2} \hat{\gamma}_\mu(f, z) k^{k-2} \|\xi - z\|^{k-2}$$

$$\leq \frac{1}{1 - \tilde{\gamma}_\mu(f, z)} \|\xi - z\| \hat{\gamma}_\mu(f, z) \|\xi - z\|^2.$$
Furthermore, we have

\[
\| \hat{f}(y) \| = \left\| \hat{f}(z) + D\hat{f}(z)(\hat{y} - \hat{z}) + \sum_{k \geq 2} \frac{D^k \hat{f}(z)}{k!} (y - z)^k \right\| \\
\leq \| D\hat{f}(z) \| \sum_{k \geq 2} \left\| D\hat{f}(z)^{-1} \frac{D^k \hat{f}(z)}{k!} \right\| \| y - z \|^k \\
\leq \| D\hat{f}(z) \| \left\| \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-1} \right\| \| y - z \|^k \\
\leq \| D\hat{f}(z) \| \| \hat{\gamma}_\mu(f, z) \| y - z \|^2 \sum_{k \geq 2} \hat{\gamma}_\mu(f, z)^{k-2} \| y - z \|^k \\
\leq \frac{4}{1 - \hat{\gamma}_\mu(f, z)\| \xi - z \|} \hat{\gamma}_\mu(f, z) \| \xi - z \|^2,
\]

where

\[
\| y - z \| = \| \hat{y} - \hat{z} \| \leq \| \hat{y} - \hat{\xi} \| + \| \hat{\xi} - \hat{z} \| \leq \frac{\hat{\gamma}_\mu(f, z) \| \xi - z \|}{1 - \hat{\gamma}_\mu(f, z)\| \xi - z \|} \| \xi - z \| + \xi - z \| \leq 2\| \xi - z \|,
\]

and

\[
\hat{\gamma}_\mu(f, z) \geq 1.
\]

Let \( \text{span}_C\{v_n(z)\}, \text{span}_C\{u_n(z)\} \) be a pair of singular subspaces of \( Df(z) \) corresponding to its smallest singular value \( \sigma_n \), and \( \delta = \sigma_{n-1} - \sigma_n = O(1) \). If

\[
(4.8) \quad \| Df(y) - Df(z) \|_F \leq \frac{\delta}{5},
\]

which could be satisfied in general since \( y \) is close to \( z \) and \( \delta = O(1) \), then according to [13] Theorem 8.6.5] or [43] Theorem 6.4, we have

\[
(4.9) \quad \| v_n(y) - v_n(z) \|_F \leq 4 \frac{\| Df(y) - Df(z) \|_F}{\delta} \leq 4 \frac{L\| y - z \|}{\delta} \leq 8 \frac{L\| \xi - z \|}{\delta},
\]

and

\[
(4.10) \quad \| u_n(y) - u_n(z) \|_F \leq 4 \frac{\| Df(y) - Df(z) \|_F}{\delta} \leq 4 \frac{L\| y - z \|}{\delta} \leq 8 \frac{L\| \xi - z \|}{\delta},
\]

where \( \text{span}_C\{v_n(y)\}, \text{span}_C\{u_n(y)\} \) is a pair of singular subspaces of \( Df(y) \) corresponding to its smallest singular value.

In what follows, for the sake of simplicity, we always assume \( 1.8 \) is satisfied and set

\[
L \leftarrow \frac{8L}{\delta}.
\]

According to [14], we know that \( v_n(z) = (1, 0, \ldots, 0)^T \) and \( u_n(z) = (0, \ldots, 0, 1) \) generate a pair of singular subspaces of \( Df(z) \) corresponding to its smallest singular value \( \sigma_n \).

Let

\[
W_1 = (v_n(y), v_1(y), \ldots, v_{n-1}(y)) = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},
\]

and \( v_n(y) = (W_1, W_3)^T \), by (4.9), we have

\[
(4.11) \quad |W_1| \geq 1 - L\| \xi - z \|, \quad \| W_3 \| \leq L\| \xi - z \|.
\]
Since $W_1$ is a unitary matrix, we have
\begin{equation}
\|W_2\| \leq L\|\xi - z\|, \quad \|W_4\| \leq \|W_1\| = 1.
\end{equation}

Let $U = \left( \begin{array}{cc} U_1 & U_2 \\ U_3 & U_4 \end{array} \right)$ and $u_n(y) = (U_3, U_4)$, by (4.10), we have
\begin{equation}
\|U_3^\ast\| \leq L\|\xi - z\|.
\end{equation}

It is clear that Step 4 and Step 5 in Algorithm 4.3 are used to normalize the Jacobian matrix at the approximate solution $\hat{y}$ after running Step 4 and Step 5 in Algorithm 4.3, we have
\begin{equation}
\|U_3^\ast\| \leq L\|\xi - z\|.
\end{equation}

Claim 7. After running Step 4 and Step 5 in Algorithm 4.3, we have
\begin{equation}
\|\hat{w} - \hat{\xi}\| \leq L\|\xi - z\|^2 + \frac{1}{1 - \hat{\gamma}_\mu(f, z)}\hat{\gamma}_\mu(f, z)\|\xi - z\|^2,
\end{equation}
and
\begin{equation}
\|g(w)\| \leq \frac{4\|D\hat{f}(z)\|\|\hat{\gamma}_\mu(f, z)\|\xi - z\|^2 + lL\|\xi - z\|^2,}
\end{equation}
where
\begin{equation}
\hat{\xi} \leftarrow W_4^\ast \cdot \xi, \quad g(X) \leftarrow U^\ast \cdot f(W_1 \cdot X), \quad w = (w_1, \hat{w}) \leftarrow W_4^\ast y,
\end{equation}
and $l$ is the Lipschitz constant of the function $f_n(X)$.

Proof. By (4.5) and (4.12), we have
\begin{align*}
\|\hat{w} - \hat{\xi}\| &= \|W_2^\ast(y_1 - \xi_1) + W_4^\ast(\hat{y} - \hat{\xi})\| \\
&\leq L\|\xi - z\|^2 + \|\hat{w} - \hat{\xi}\|
\end{align*}

By (4.6) and (4.13), we have
\begin{align*}
\|g(w)\| &= \|U_1^\ast \hat{f}(y) + U_2^\ast f_n(y)\| \leq \|\hat{f}(y)\| + L\|\xi - z\||f_n(y)|
\end{align*}
\begin{align*}
&\leq \frac{4\|D\hat{f}(z)\|\|\hat{\gamma}_\mu(f, z)\|\|\xi - z\|^2 + lL\|\xi - z\|^2,}
\end{align*}
\[\square\]

Let $\Delta_k$ and $\Lambda_k$ be differential functionals calculated by (2.7) and (2.11) incrementally from $\Lambda_1 = d_1$ until $\Delta_\mu(g_n) = O(1)$. It should be noted that $d_1^\ast$ is the only differential monomial of the highest order $k$ in $\Delta_k$ and no other $d_s^\ast$ with $s < k$ in $\Delta_k$.

Claim 8. After running the first six steps in Algorithm 4.3, we have
\begin{equation}
\|x_1 - \zeta_1\| = O(\|\xi - z\|^2),
\end{equation}
where $x = (x_1, \hat{x}) \leftarrow (N_2(g_n, w), \hat{w})$, $\zeta \leftarrow W_4^\ast \cdot \xi$. 

Proof. It is straightforward to check that \( w \) is a simple multiple zero of the system
\[
\begin{cases}
\hat{h}(X) = \hat{g}(X) - \hat{g}(w), \\
h_n(X) = g_n(X) - g_n(w) - \sum_{k=1}^{\mu-1} \Delta_k(g_n)(X_1 - w_1)^k.
\end{cases}
\]
with multiplicity \( \mu \) and \( Dg(w) \) is of normalized form (2.9) (the construction is very similar to Theorem 9). Thus, by (3.14), we have
\[
h_n(X) = - \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} \cdot D\hat{h}(w)^{-1} \hat{h}(X)(X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1}
+ \Delta_\mu(h_n)(X_1 - w_1)^\mu + \sum_{i+j=\mu,j>0} C_{i,j} (X_1 - w_1)^i(\hat{X} - \hat{w})^j + \sum_{k=1}^{\mu-1} \frac{D^k h_n(w)(X - w)^k}{k!}
+ \sum_{1 \leq i+j-1 \leq \mu-2} T_{i,j-1} \left( \sum_{k+i+j-1 \geq \mu+1} D\hat{h}(w)^{-1} \frac{D^k \hat{h}(w)(X - w)^k}{k!} (X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1} \right),
\]
Let \( g_n(X) = h_n(X) + g_n(w) + \sum_{k=1}^{\mu-1} \Delta_k(g_n)(X_1 - w_1)^k \) and \( \{1, \bar{\Lambda}_1, \ldots, \bar{\Lambda}_{\mu-1}\} \) be a reduced basis of \( D_{g,\zeta} \),
\[
0 = \bar{\Lambda}_{\mu-1}(g_n) = \bar{\Lambda}_\mu(h_n) + \Delta_\mu(h_n) = \mu \Delta_\mu(h_n)(\zeta_1 - w_1) + \Delta_\mu(g_n) + O(\|\xi - z\|^2)
= \mu \Delta_\mu(g_n)(\zeta_1 - w_1) + \Delta_\mu(g_n) + O(\|\xi - z\|^2)
= -\mu \Delta_\mu(g_n)(N_2(w_1) - \zeta_1) + O(\|\xi - z\|^2),
\]
because of the following facts:
- for the term #1, \( \bar{\Lambda}_{\mu-1} \left( \hat{h}(X)(X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1} \right) = O(\|\xi - z\|^2) \) for \( 1 \leq i+j-1 \leq \mu-2 \), since \( \bar{\Lambda}_{\mu-1} \in D_{g,\zeta} \), \( \hat{h}(X) = \hat{g}(X) - \hat{g}(w) \) and \( \|\hat{g}(w)\| = O(\|\xi - z\|^2) \) by (1.15);
- for the term #2, \( \bar{\Lambda}_{\mu-1} \left( (X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1} \right) = O(\|\xi - z\|^2) \), since \( j > 0 \), \( i+j-1 = \mu-1 \) and \( \|\hat{w} - \zeta\| = O(\|\xi - z\|^2) \) by (1.14);
- for the term #3, \( \bar{\Lambda}_{\mu-1} ((X - w)^k) = O(\|\xi - z\|^2) \) for \( k - (\mu - 1) \geq 2 \);
- for the term #4, \( \bar{\Lambda}_{\mu-1} ((X - w)^k(X_1 - w_1)^i(\hat{X} - \hat{w})^{j-1}) = O(\|\xi - z\|^2) \) for \( k + i+j-1 = (\mu - 1) \geq 2 \).
Moreover, we have \( \Delta_\mu(g_n) = O(1) \). Therefore, after running Step 6, we have
\[
\|x_1 - \zeta_1\| = \|N_2(w_1) - \zeta_1\| = O(\|\xi - z\|^2).
\]
\( \square \)

Now we are ready to prove Theorem 11.

Proof. Both \( W_1 \) and \( W_2 \) are unitary matrices, according to (4.3) and Claim 14, we have
\[
\|W_1 \cdot W_2 \cdot x - \xi\| = \left\| \left( \begin{array}{c} x_1 - W_1^* \cdot W_2^* \cdot \zeta_1 \\ \bar{x} - W_1^* \cdot W_2^* \cdot \zeta_1 \end{array} \right) \right\| = O(\|\xi - z\|^2).
\]
\( \square \)
Remark 5. Theorem [77] can be combined with Theorem [8] to provide an algorithm for computing an certified ball which contains a simple singular solution of \( f \). After running the first six steps in Algorithm [4.3], if the condition \((7.28)\) is satisfied by \( g \) at \( x \), then according to Theorem [8], it has \( \mu \) zeros (counting multiplicities) in the ball of radius \( r = \frac{1}{137} \) around \( x \), which indicates that the input polynomial system \( f \) has \( \mu \) zeros (counting multiplicities) in the ball \( B(N_f(z), r) \).

The difficulty of giving a quantified quadratic convergence of Step 6 is due to the complicate expression of the polynomial \( h_n(X) \). In order to avoid awkward large expressions, in what follows, we only show the proof of the quantitative version of Claim [8] for simple triple zeros. It is clear from proofs given below that there is no significant obstacle to extend the quantified quadratic convergence proof of the Algorithm [4,3] for simple triple zeros to simple multiple zeros of higher multiplicities. This can also be observed by our analysis in Section [5.2] which generalizes results in Section [5.1] for simple triple zeros to simple multiple zeros of higher multiplicities.

Definition 6. Let \( u = \max\{\gamma_3(f, \xi)^3\|\xi - z\|, L\gamma_3(f, \xi)^2\|\xi - z\|\} \). We define the following rational functions:

\[
\begin{align*}
  l_1(u) &= \frac{(1 - 2u)^2}{(2(1 - 2u)^2 - 1) \cdot (1 - u)^3}, \\
l_2(u) &= \frac{(2u - 1)^6}{(128u^6 - 384u^5 + 480u^4 - 336u^3 + 140u^2 - 32u + 1) \cdot (1 - u)^3}, \\
l_3(u) &= \sqrt{1 + \left( \frac{l_1u}{1 - l_1u} \right)^2}, \\
b_1(u) &= u + \frac{l_1u}{1 - l_1u}, \\
b_2(u) &= \frac{(16l_1^2l_3^2u^4 + (16l_1^2l_3^2 + 16l_1l_3)u^3 + (16l_1l_3 + 4)u^2 - 4u + 1)^2}{(1 - 2u)^2(1 - 2l_1l_3u)^2} \\
  &\quad \cdot \left[ \left( \frac{l_2}{3} + \frac{17l_2l_3^2}{3} \right) u + \frac{7l_2}{3} u^2 + \frac{4l_2^2}{3} u^3 + \left( \frac{l_2}{3} + \frac{7l_2u}{3} + \frac{8l_2^2u}{3} \right) \right] \frac{l_1u}{1 - l_1u} \\
  &\quad + \frac{4l_2^2u}{3} \left( \frac{l_1u}{1 - l_1u} \right)^2 + l_2 \cdot \frac{l_1l_3^2}{3} u \frac{l_1l_3}{3} u - 8l_2^2 \frac{l_3^2}{3} u \left( 2l_2l_3^3 \frac{l_3^2}{3} u^3 + (6l_3^2 - 14l_3)u^2 + (4l_3 - 8l_2l_3)u + 3 \right) \\
  &\quad + \frac{2l_2^2l_3^2}{3} u \left( (16l_1^2l_3^2u^3 + (4l_1^2l_3^2 - 20l_1l_3)u^2 + (6 - 6l_1l_3)u + 3 \right) \right) \\
  &\quad + \frac{l_2^2}{3} \left( u + \frac{l_1u}{1 - l_1u} \right) \left( \frac{2(-4l_1^2l_3^2u^2 + 4l_1l_3u)^2}{(1 - 2u)(1 - 2l_1l_3u)^2} + \frac{7(-4l_1^2l_3^2u^2 + 4l_1l_3u)}{(1 - 2u)(1 - 2l_1l_3u)^2} + 8 \right). 
\end{align*}
\]

Theorem 12. Given an approximate zero \( z \) of a system \( f \) associated to a simple triple zero \( \xi \) of multiplicity 3 and satisfying \( f(\xi) = 0 \), \( \dim \ker Df(\xi) = 1 \).

(1) If \( u < u_3 \approx 0.0137 \), where \( u_3 \) is the smallest positive solution of the equation:

\[
b_1(u)^2 + b_2(u)^2 = 1,
\]
then the output of Algorithm 4.3 satisfies:
\[ \|N_f(z) - \xi\| < \|z - \xi\|. \]

(2) If \( u < u'_3 \approx 0.0098 \), where \( u'_3 \) is the smallest positive solution of the equation:
\[ b_1(u)^2 + b_2(u)^2 = \frac{1}{4}. \]

then after \( k \) times of iteration we have
\[ \|N_f^k(z) - \xi\| < \left( \frac{1}{2} \right)^{2^k-1} \|z - \xi\|. \]

The proof of Theorem 12 is based on the following facts.

**Claim 9.** When \( u \leq u_3 \), we have
\[ \gamma_3(f, z) \leq l_1 \cdot \gamma_3(f, \xi), \]
\[ \gamma_{3,n}(f, z) \leq l_2 \cdot \gamma_3(f, \xi). \]

**Proof.** For \( k \geq 2 \), by the Taylor expansion of \( D^k \hat{f}(z) \) at \( \xi \), we have
\[
\left\| D\hat{f}(z)^{-1} D^k \hat{f}(z) \right\| \\
\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \left\| D\hat{f}(\xi)^{-1} \left( \frac{D^k \hat{f}(\xi)}{k!} + \sum_{i \geq 1} \frac{D^{k+i} \hat{f}(\xi)}{k! i!} (\xi - z)^i \right) \right\| \\
\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \left( \gamma_3(f, \xi)^{k-1} + \sum_{i \geq 1} \frac{(k+i)!}{k! i!} \gamma_3(f, \xi)^{k+i-1} \|\xi - z\|^i \right) \\
\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \gamma_3(f, \xi)^{k-1} \left( 1 - \gamma_3(f, \xi)^k \|\xi - z\|^k \right)^{1/k} \\
\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \frac{1}{1 - u} \right)^{k+1} \gamma_3(f, \xi)^{k-1}.
\]

Then by the definition of \( \gamma_3 \) and the above inequalities, we have
\[ \gamma_3(f, z) \leq \max_{k \geq 2} \left\| D\hat{f}(z)^{-1} D^k \hat{f}(z) \right\| \frac{1}{k!} \\
\leq \max_{k \geq 2} \left( \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \left( \frac{1}{1 - u} \right)^{k+1} \gamma_3(f, \xi) \right) \\
\leq \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \cdot \left( \frac{1}{1 - u} \right)^3 \gamma_3(f, \xi).
\]

According to Remark 11 for \( u \leq u_3 \approx 0.0137 \), we can show that:
\[ \left\| D\hat{f}(z)^{-1} D\hat{f}(\xi) \right\| \leq \frac{(1 - 2u)^2}{2(1 - 2u)^2 - 1}, \]

therefore (4.18) holds by (4.20) and Definition 6.
Similar to the proof of inequality (4.20), we have

\[
\gamma_{3,n}(f,z) \leq \left\| \Delta_3(f_n)(z)^{-1} \Delta_3(f_n)(\xi) \right\| \cdot \left( \frac{1}{1 - u} \right)^3 \gamma_3(f,\xi).
\]

To prove (4.19), we notice that:

\[
\Delta_3(f_n)(z)^{-1} \Delta_3(f_n)(\xi) = (1 + (\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1))^{-1}.
\]

By the Taylor expansion of \( \frac{\partial^3 f(z)}{\partial X_i^3} \) and \( \frac{\partial^2 f(z)}{\partial X_i \partial X} \) at \( \xi \), we have:

\[
\Delta_3(f_n)(z) = \frac{1}{6} \frac{\partial^3 f_n(z)}{\partial X_i^3} + a_{2,z} \cdot \frac{\partial^2 f_n(z)}{\partial X_1 \partial X} = \frac{1}{6} \frac{\partial^3 f_n(\xi)}{\partial X_i^3} + a_{2,z} \cdot \frac{\partial^2 f_n(\xi)}{\partial X_1 \partial X} + \sum_{k=1}^{k=1} i! (k-i)! \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{k+2} \partial X_1 \partial X} (\xi - \xi)^{k+2} + \sum_{k=1}^{k=1} i! (k-i)! \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{k+2} \partial X_1 \partial X} (\xi - \xi)^{k+2}.
\]

Therefore, we have

\[
\| \Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1 \| \leq \sum_{k=1}^{k=1} i! (k-i)! \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{k+2} \partial X_1 \partial X} (\xi - \xi)^{k+2} + \sum_{k=1}^{k=1} i! (k-i)! \frac{\partial^{k+3} f_n(\xi)}{\partial X_1^{k+2} \partial X_1 \partial X} (\xi - \xi)^{k+2}.
\]

where

\[
a_{2,z} = D\hat{f}(z)^{-1} \frac{1}{2} \frac{\partial^2 \hat{f}(z)}{\partial X_1^2}.
\]
\[ \leq \gamma_3(f, \xi) + \sum_{k \geq 1} \frac{(k + 2)!}{2 \cdot i!(k - i)!} \gamma_3(f, \xi)^k \| \xi - z \|^k \]

\[ \leq \gamma_3(f, \xi) + \frac{1}{2} \sum_{k \geq 1} (k + 2)(k + 1)2^k \gamma_3(f, \xi)^k \| \xi - z \|^k \]

\[ \leq \gamma_3(f, \xi) + \frac{-2u(4u^2 - 6u + 3)}{(2u - 1)^3}. \]

By (4.22), we have:

\[ \| \Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1 \| \leq -8u(2u^3 - 4u^2 + 3u - 1) + \frac{-2u(4u^2 - 6u + 3)}{(2u - 1)^3} \cdot \frac{-4u(4u^2 - 6u + 3)}{(2u - 1)^3} + \frac{\sum_{k \geq 1} (k + 2)(k + 1)2^k \gamma_{3,n}(f, \xi)^k \| \xi - z \|^k}{(2u - 1)^6} \]

\[ \leq -4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5) \]

When \( u \leq u_3 \),

\[ \frac{-4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5)}{(2u - 1)^6} < 1, \]

then we have

\[ \| \Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(\xi) \| = \| (1 + (\Delta_3(f_n)(\xi)^{-1} \Delta_3(f_n)(z) - 1))^{-1} \| \leq \frac{1}{1 - \frac{-4u(16u^5 - 48u^4 + 60u^3 - 44u^2 + 20u - 5)}{(2u - 1)^6}} = \frac{(2u - 1)^6}{(128u^6 - 384u^5 + 480u^4 - 336u^3 + 140u^2 - 32u + 1)}. \]

Hence, the inequality (4.19) holds. \( \square \)

We notice that after running first five steps in Algorithm 4.3, we have

\[ \xi \leftarrow W_4^* \xi, \quad g(X) \leftarrow U^* \cdot f(W_4^* X), \quad W_4^* y, \quad y \leftarrow (z_1, N_1(\hat{f}, \hat{z})). \]

For unitary matrices \( U \) and \( W_4 \), we have \( \gamma_3(f, \xi) = \gamma_3(U^* \cdot f, W_4^* \cdot \xi) \) and the following inequalities hold:

\[ \gamma_3(g, w) \leq l_1 \cdot \gamma_3(g, \xi) = l_1 \cdot \gamma_3(f, \xi), \]

\[ \gamma_{3,n}(g, w) \leq l_2 \cdot \gamma_3(g, \xi) = l_2 \cdot \gamma_3(f, \xi). \]

Claim 10. When \( u \leq u_3 \), we have

\[ (4.23) \quad \| w - \xi \| \leq l_3 \cdot \| z - \xi \|. \]

Proof. It is clear that

\[ \| w - \xi \| = \| W_4^* (y - \xi) \| = \| y - \xi \|. \]

By Claim 10 and Claim 9, when \( u \leq u_3 \), we have:

\[ \| \hat{y} - \hat{\xi} \| \leq \frac{1}{1 - \gamma_3(f, z)\| \xi - z \|} \gamma_3(f, z)\| \xi - z \|^2 \]
By the definition of $\hat{\gamma}$, it is clear that
\[ \|w - \xi\| = \|y - \xi\| \leq \sqrt{1 + \left(\frac{l_1 u}{1 - l_1 u}\right)^2} \|\xi - z\| = l_3 \cdot \|z - \xi\|. \]

\[ \square \]

In Claim 8 we have shown that
\[ \|x_1 - \zeta_1\| = O(\|\xi - z\|^2). \]

Below, we give a quantitative version of Claim 8 for the simple triple zero case.

Following the proof of Claim 8, we consider the following system:

\[
\begin{align*}
\dot{h}(X) &= \hat{g}(X) - \hat{g}(w) \\
\dot{h}_n(X) &= g_n(X) - g_n(w) - \sum_{k=1}^{2} \Delta_k(g_n)(X_1 - w_1)^k
\end{align*}
\]

$w$ is a simple triple zero of $h(X)$ whose Jacobian matrix is of normalized form. The polynomial $h_n(X)$ can be written as:

\[
\begin{align*}
h_n(X) &= -\sum_{i+j=1} T_{i,j} \cdot D\hat{h}(w)^{-1}\hat{h}(X)(X_1 - w_1)^i(\hat{X} - \hat{w})^j \\
&\quad + \Delta_3(h_n)(X_1 - w_1)^3 + \sum_{i+j=3, j>0} C_{i,j}(X_1 - w_1)^i(\hat{X} - \hat{w})^j \\
&\quad + \sum_{k \geq 4} \frac{D^k h_n(w)(X - w)^k}{k!} \\
&\quad + \sum_{i+j=1} T_{i,j} \cdot \sum_{k \geq 1} D\hat{h}(w)^{-1} \frac{D^k \hat{h}(w)(X - w)^k}{k!} (X_1 - w_1)^j(\hat{X} - \hat{w})^j \\
&\Delta \triangleq \Delta_3(h_n)(X_1 - w_1)^3 + B.
\end{align*}
\]

It is clear that $D\hat{h}(w) = D\hat{g}(w)$, $D^kg(w) = D^kg(w)$ for $k \geq 3$,

\[
\begin{align*}
C_{2,1} &= \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial X_1^2} - \frac{\partial^2 g_n(w)}{\partial X_1 \partial X} \cdot D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X \partial X} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial X^2} - D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X^2}, \\
C_{1,2} &= \frac{1}{2} \frac{\partial^3 g_n(w)}{\partial X_1 \partial X^2} - \frac{\partial^2 g_n(w)}{\partial X_1 \partial X} \cdot D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X \partial X} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial X^2}, \\
C_{0,3} &= \frac{1}{6} \frac{\partial^3 g_n(w)}{\partial X^3} - \frac{1}{2} \frac{\partial^2 g_n(w)}{\partial X^2} \cdot D\hat{g}(w)^{-1} \frac{\partial^2 \hat{g}(w)}{\partial X^2}, \\
T_{1,0} &= -\frac{\partial^2 g_n(w)}{\partial X_1 \partial X}, \\
T_{0,1} &= -\frac{\partial^2 g_n(w)}{2 \partial X^2}.
\end{align*}
\]

By the definition of $\hat{\gamma}_3(g, w)$ and $\gamma_{3,n}$, we also have the following facts:

\[
\|\Delta_3(g_n)^{-1}C_{2,1}\| \leq 3\gamma_{3,n}(g, w)^2 + 2\gamma_{3,n}(g, w) \cdot 2\hat{\gamma}_3(g, w) + \gamma_{3,n}(g, w) \cdot \hat{\gamma}_3(g, w) \leq 8\gamma_3(g, w)^2,
\]
Claim 11. We have

\[ N_2(g_n, w) - \zeta_1 = \frac{1}{3} \Delta_3(g_n)^{-1} \bar{\Lambda}_2(B). \]
Proof. As $\bar{A}_2 = \Delta_2 + a_2 d_2$, applying $\bar{A}_2$ on both sides of the equation:

$$g_n(X) = h_n(X) + g_n(w) + \sum_{k=1}^{2} \Delta_k(g_n)(X_1 - w_1)^k,$$

we have:

$$0 = \bar{A}_2(g_n) = \bar{A}_2(h_n) + \Delta_2(g_n) = 3\Delta_3(h_n)(\zeta_1 - w_1) + \Delta_2(g_n) + \bar{A}_2(B) = 3\Delta_3(g_n)(\zeta_1 - w_1) + \Delta_2(g_n) + \bar{A}_2(B).$$

Therefore, we have

$$N_2(g_n, w) - \zeta_1 = w_1 - \frac{1}{3}\Delta_3(g_n)^{-1}\Delta_2(g_n) - \zeta_1 = \frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2(B).$$

\[\square\]

**Claim 12.** When $u \leq u_3$, we have

$$|x_1 - \zeta_1| \leq b_2(u)\|z - \xi\|.$$

Proof. By Claim [11] and above arguments, we have

$$\left|N_2(g_n, w) - \zeta_1\right| = \frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2(B)$$

$$= \left|\frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2 \left(T_{1,0} \cdot D\hat{h}(w)^{-1}\hat{h}(X)(X_1 - w_1) + T_{0,1} \cdot D\hat{h}(w)^{-1}\hat{h}(X)(\hat{X} - \hat{w})\right)\right|$$

$$+ \left|\frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2 \left(\sum_{i+j=3, j>0} C_{i,j}(X_1 - w_1)^i(\hat{X} - \hat{w})^j\right)\right|$$

$$+ \left|\frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2 \left(\sum_{k \geq 4} \frac{D^k h_n(w)(X - w)^k}{k!}\right)\right|$$

$$+ \left|\frac{1}{3}\Delta_3(g_n)^{-1}\bar{A}_2 \left(\sum_{i+j=1} T_{i,j} \cdot \sum_{k \geq 3} D\hat{h}(w)^{-1}\frac{D^k \hat{h}(w)(X - w)^k}{k!}(X_1 - w_1)^i(\hat{X} - \hat{w})^j\right)\right|$$

$$= \left|\frac{1}{3}\left|\Delta_3(g_n)^{-1}T_{0,1}\right| \cdot \left|D\hat{h}(w)^{-1}\hat{h}(\zeta)\right| \cdot |a_2|\right.$$
\[
+ \frac{1}{3} ||a_2|| \sum_{k \geq 4} \sum_{i = 0}^{k-1} \frac{(k - i)}{l!(k - i)!} \left| \Delta_3(g_n)^{-1} \frac{\partial h_n(w)}{\partial X^i X^{k-i}} \right| \left| \zeta - w_1^i \right| \left| \hat{\zeta} - \hat{w} \right|^{k-i-1}
\]
\[
+ \frac{1}{3} \sum_{i+j=1} \left| \Delta_3(g_n)^{-1} T_{i,j} \right| \cdot \sum_{k \geq 3} \left( \frac{1}{2} \sum_{l=0}^{k} \frac{l(k-l)}{l!(k-l)!} \left| \Delta_3(h(w))^{-1} \frac{\partial \hat{h}(w)}{\partial X^i X^{k-i}} \right| \left| \zeta - w_1^l \right| \left| \hat{\zeta} - \hat{w} \right|^{k-l} \right)
\]
\[
\leq \frac{1}{3} \gamma_3(g, w) \left( ||\zeta - \hat{w}|| + \frac{1}{1 - \gamma_3(g, w)||\zeta - w||} \gamma_3(g, w)||\zeta - w||^2 \right) ||a_2||
\]
\[
+ \frac{1}{3} \left( 3 \gamma_3(g, w)^2 + 7 \gamma_3(g, w)^2 ||a_1|| + 2 \gamma_3(g, w)^2 ||a_2|| \right) ||\hat{\zeta} - \hat{w}||
\]
\[
+ \frac{1}{3} \gamma_3(g, w)^2 ||a_2|| ||\zeta - w_1|| ||\hat{\zeta} - \hat{w}|| + \frac{4}{3} \gamma_3(g, w)^2 ||a_2|| ||\zeta - \hat{w}||^2
\]
\[
+ \frac{17}{15} \gamma_3(g, w)||\zeta - w||^2 ||a_2||
\]
\[
+ \frac{1}{6} \sum_{k \geq 4} \sum_{i = 2}^{k-1} \frac{i(i-1)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} ||\zeta - w_1||^{i-2} ||\hat{\zeta} - \hat{w}||^{k-i}
\]
\[
+ \frac{1}{3} ||a_1|| \sum_{k \geq 4} \sum_{i = 1}^{k-1} \frac{i(k-i)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} ||\zeta - w_1||^{i-1} ||\hat{\zeta} - \hat{w}||^{k-i-1}
\]
\[
+ \frac{1}{6} ||a_1||^2 \sum_{k \geq 4} \sum_{i = 0}^{k-2} \frac{(k-i)(k-i-1)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} ||\zeta - w_1||^{i} ||\hat{\zeta} - \hat{w}||^{k-i-2}
\]
\[
+ \frac{1}{3} ||a_2|| \sum_{k \geq 4} \sum_{i = 0}^{k-1} \frac{(k-i)k!}{i!(k-i)!} \gamma_3(g, w)^{k-1} ||\zeta - w_1||^{i} ||\hat{\zeta} - \hat{w}||^{k-i-1}
\]
\[
+ \gamma_3(g, w) \cdot \sum_{k \geq 3} \left( \frac{1}{2} \sum_{l=2}^{k} \frac{l(l-1)k!}{l!(k-l)!} \right) \gamma_3(g, w)^{k-1} \left| \zeta - w \right|^2 ||a_2||
\]
\[
+ ||a_1|| \sum_{l=1}^{k-1} \frac{l(k-l)k!}{l!(k-l)!} \gamma_3(g, w)^{k-1} \left| \zeta - w \right|^2 ||a_2||
\]
\[
+ \frac{1}{2} ||a_1||^2 \sum_{l=0}^{k-2} \frac{(k-l)(k-l-1)k!}{l!(k-l)!} \gamma_3(g, w)^{k-1} \left| \zeta - w \right|^2 ||a_2||
\]
\[
+ ||a_2|| \sum_{l=0}^{k-1} \frac{(k-l)k!}{l!(k-l)!} \gamma_3(g, w)^{k-1} \left| \zeta - w \right|^2 ||a_2||
\]
\begin{align*}
&= \frac{1}{3} \gamma_3(g, w) \left( \|\zeta - \hat{w}\| + \frac{1}{1 - \gamma_3(g, w)} \|\zeta - w\| \right) \|a_2\|
&+ \frac{1}{3} \left( 8\gamma_3(g, w)^2 + 7\gamma_3(g, w)^2 \|a_1\| + 2\gamma_3(g, w)^2 \|a_1\|^2 \right) \|\zeta - \hat{w}\|
&+ \frac{7}{3} \gamma_3(g, w)^2 \|a_2\| \|\zeta - w\| + \frac{4}{3} \gamma_3(g, w)^2 \|a_2\| \|\zeta - \hat{w}\|^2
&+ \frac{17}{3} \gamma_3(g, w) \|\zeta - w\|^2 \|a_2\|
&+ \frac{1}{6} \sum_{k \geq 4} 2^{k-2} k (k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2}
&+ \frac{1}{3} \|a_1\| \sum_{k \geq 4} 2^{k-2} k (k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2}
&+ \frac{1}{6} \|a_1\|^2 \sum_{k \geq 4} 2^{k-2} k (k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-2}
&+ \frac{1}{3} \|a_2\| \sum_{k \geq 4} 2^{k-1} k \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-1}
&+ \frac{1}{2} \gamma_3(g, w) \sum_{k \geq 3} (1 + 2 \|a_1\| + \|a_1\|^2) 2^{k-2} k (k-1) \gamma_3(g, w)^{k-1} \|\zeta - w\|^{k-1}
&+ \gamma_3(g, w) \sum_{k \geq 3} \|a_2\| 2^{k-1} k \gamma_3(g, w)^{k-1} \|\zeta - w\|^k
\leq \frac{l_2}{3} (1 + 2 \|a_1\| + \|a_1\|^2) \left( u + \frac{l_1 u}{1 - l_1 u} \right) \|\xi - z\|
&+ \frac{l_2}{3} (1 + 2 \|a_1\| + \|a_1\|^2) \cdot \frac{l_1 l_3^2 u}{1 - l_1 l_3 u} \|\xi - z\|
&+ \frac{l_2}{3} (8 + 7 \|a_1\| + 2 \|a_1\|^2) \left( u + \frac{l_1 u}{1 - l_1 u} \right) \|\xi - z\|
&+ \frac{7l_2^2}{3} (1 + 2 \|a_1\| + \|a_1\|^2) \left( u^2 + \frac{l_1 u^2}{1 - l_1 u} \right) \|\xi - z\|
&+ \frac{4l_2^2}{3} (1 + 2 \|a_1\| + \|a_1\|^2) u \left( u + \frac{l_1 u}{1 - l_1 u} \right)^2 \|\xi - z\|
&+ \frac{17l_2}{3} (1 + 2 \|a_1\| + \|a_1\|^2) l_2^2 u \|\xi - z\|
&+ \frac{1}{6} (1 + 2 \|a_1\| + \|a_1\|^2) \cdot \frac{8l_2^3 l_3^2 u (12l_2^3 l_3^2 u^2 - 16l_2 l_3 u + 6)}{(1 - 2l_2 l_3 u)^3} \|\xi - z\|
&+ \frac{1}{3} (1 + 2 \|a_1\| + \|a_1\|^2) u \cdot \frac{8l_2^3 l_3^2 u (4 - 6l_2 l_3 u)}{(1 - 2l_2 l_3 u)^2} \|\xi - z\|
&+ \frac{l_2}{2} (1 + 2 \|a_1\| + \|a_1\|^2) \cdot \frac{4l_2^3 l_3^2 u (4l_2^3 l_3^2 u^2 - 6l_2 l_3 u + 3)}{(1 - 2l_2 l_3 u)^3} \|\xi - z\|
&+ (1 + 2 \|a_1\| + \|a_1\|^2) l_2 u \cdot \frac{4l_2^3 l_3^2 u (3 - 4l_2 l_3 u)}{(1 - 2l_2 l_3 u)^2} \|\xi - z\|
&= b_2(u) \|z - \xi\|.
\end{align*}
When \( u < u_3 \approx 0.0137 \), by Claim 12 and Definition 6 we have:
\[
\| \hat{x} - \hat{\zeta} \| \leq L \| \xi - z \|^2 + \frac{1}{1 - \gamma_3(f, z)} \| \xi - z \| \gamma_3(f, z) \| \xi - z \|^2
\]
\[
\leq \left( u + \frac{l_1 u}{1 - l_1 u} \right) \| z - \xi \|
= b_1(u) \| z - \xi \|.
\]
Now we can complete the proof for Theorem 12.

**Proof.** (1) When \( u < u_3 \approx 0.0137 \), \( b_1(u)^2 + b_2(u)^2 < 1 \), by Claim 12 we have
\[
\| N_f(z) - \xi \| = \| W_1 \cdot W_1 \cdot (x - \zeta) \|
= \| x - \zeta \|
= \sqrt{|x_1 - \zeta_1|^2 + \| \hat{x} - \hat{\zeta} \|^2}
\]
\[
\leq \sqrt{b_1(u)^2 + b_2(u)^2} \| z - \xi \|
< \| z - \xi \|.
\]
(2) When \( u < u_3' \approx 0.0098 \), \( b_1(u)^2 + b_2(u)^2 < \frac{1}{4} \), by Claim 12 we have
\[
\| N_f(z) - \xi \| = \| W_1 \cdot W_1 \cdot (x - \zeta) \|
= \| x - \zeta \|
= \sqrt{|x_1 - \zeta_1|^2 + \| \hat{x} - \hat{\zeta} \|^2}
\]
\[
\leq \sqrt{b_1(u)^2 + b_2(u)^2} \| z - \xi \|
< \frac{1}{2} \| z - \xi \|.
\]
Hence, the following inequality is true for \( k = 1 \):
\[
\| N_f(x) - \xi \| < \left( \frac{1}{2} \right)^{2^{k-1}} \| z - \xi \|.
\]
For \( k \geq 2 \), assume by induction that
\[
\| N_f^{k-1}(z) - \xi \| < \left( \frac{1}{2} \right)^{2^{k-1-1}} \| z - \xi \|.
\]
Let \( u^{(k-1)} = \gamma_3(f, \xi)^3 \| N_f^{k-1}(z) - \xi \|. \) For \( 0 < u < u_3' \), we have \( u^{(k-1)} < u \) and \( \sqrt{b_1(u)^2 + b_2(u)^2} \) is increasing. Therefore, by induction we have
\[
\| N_f^k(z) - \xi \|
< \sqrt{\frac{b_1(u^{(k-1)})^2 + b_2(u^{(k-1)})^2 \gamma_3(f, \xi)^3}{u^{(k-1)}}} \| N_f^{k-1}(z) - \xi \|\|^2
< \frac{b_1(u)^2 + b_2(u)^2 \gamma_3(f, \xi)^3}{u} \| N_f^{k-1}(z) - \xi \|\|^2
< \frac{b_1(u)^2 + b_2(u)^2 \gamma_3(f, \xi)^3}{u} \left( \frac{1}{2} \right)^{2^{k-2}} \| z - \xi \|^2
\]
\[ \leq \left( \frac{1}{2} \right)^{k-1} \| z - \xi \|. \]

\[\square\]

**Remark 6.** The unitary transformations in Algorithm 4.3 may convert a sparse polynomial system into a dense polynomial system, therefore, the computations of the modified Newton iterations may become more costly. Hence, in our implementation, we use the chain rule to avoid storing or computing with dense polynomial systems obtained after performing unitary transformations. For example, suppose \( g(X) = U^* \cdot f(W_1 \cdot X) \), then we have
\[
(4.25) \quad Dg(X) = U^* \cdot Df(W_1 \cdot X) \cdot W_1.
\]
Let \( y = W_1^* z \), we have
\[
(4.26) \quad Dg(y) = U^* \cdot Df(z) \cdot W_1.
\]
Instead of evaluating \( Dg(X) \) at \( y \), we evaluate \( Df(X) \) at \( z \) and then perform matrix multiplications, which avoids storing and computing the dense system \( Dg(X) \). Similarly, as we have already demonstrated in [23 Example 3.1], instead of computing and storing dense differential functionals \( \Delta_k \) and \( \Lambda_k \), we compute polynomials \( L_k(g) \) and \( P_k(g) \) as
\[
(4.27) \quad P_k(g) = \sum_{j=1}^{k-1} \frac{j}{k} \cdot D(L_{k-j}(g)) \cdot a_j \quad \text{and} \quad L_k(g) = P_k(g) + Dg \cdot a_k,
\]
where \( L_k \) and \( P_k \) are differential operators corresponding to \( \Delta_k \) and \( \Lambda_k \) respectively, \( a_1 = (1,0,\ldots,0)^T \) and \( a_k = (0,a_k,2,\ldots,a_k,n)^T \). The polynomial system \( P_k(g) \) and the value \( \Delta_k(g) \) can be obtained by applying the chain rule \((4.24)\) and \((4.26)\) recursively to \((4.27)\).

**Remark 7.** The Maple codes of three Algorithms and test results are available via request.

Although the algorithms and proofs of quadratic convergence given in the paper are for polynomial systems with exact simple multiple zeros, examples are given to demonstrate that our algorithms are also applicable to analytic systems and polynomial systems with a cluster of simple roots.

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