Bifurcation and Chaos Control for Prey Predator Model with Step Size in Discrete Time

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Abstract. The Lotka – Volterra systems served as a basis for the development and analysis of more realistic mathematical models of nonlinear interactions. A new form of discrete time 2-D prey predator model involving Lesile - Gower functional response with step size is proposed for discussion. Utilizing Euler scheme, discrete system is obtained from the continuous dynamical system. Dynamical consistency of the model which includes the existence and stability of the fixed points is investigated. Eigenvalues of Jacobian matrix are computed for corresponding fixed points. Analytical results illustrate rich dynamics and complexity of the model. The bifurcation theory is employed to study the existence of flip and Neimark-Sacker bifurcations. The chaos control of the discrete system is performed and numerical simulations are provided supporting the results.

Mathematics Subject Classification. [2010] : 34C23, 39A10, 39A30, 92D30.

Key words and phrases. : Prey - Predator System, Stability, Lesile - Gower functional response, Bifurcation.

1. Introduction
Among all mathematical models, prey-predator models have received much attention during the last few decades due to its wide range of applications. The dynamic relationship between predators and their preys have long been(and will continue to be) one of the dominant theme due to its universal prevalence and importance. Firstly, the prey-predator interaction model was described by Lotka (1925) and Volterra(1926). After them, more realistic prey-predator models were introduced by Holling and others. Moreover, there are many different kinds of prey-predator models in discussed mathematical ecology.

The predator-prey model is widely applied to deal with problems in real world. Extinction, stability of equilibrium, hopf bifurcation, limit cycle, chaotic behavior and so on, are many studies in the nonlinear dynamics of predator-prey models with different kinds of functional responses.

Research studies have mainly focused on continuous time prey-predator models with differential equations[5, 7, 8, 12, 16]. However the discrete time models governed by difference equations are more appropriate to describe the prey-predator relations. Ratio-dependent predator-prey model and the dynamical behaviors of discrete system by using the center manifold theorem is investigated by Chen et al. [2]. The dynamical behaviors of the discrete-time predator-prey bio-economic system by using new normal form of differential-algebraic system is investigated by Zhang et al. [17]. Wang and Li [15] revisited a discrete predator-prey model and proposed a lemma to study the system’s stability and bifurcation. Existence and stability of the fixed points of a discrete reduced Lorenz system by using center manifold theorem and bifurcation theory are established by Elabbasy et al.[3]. The resonance
and bifurcation in a discrete-time predator-prey system with Holling functional response are studied by Ghaziani et al. [4]. A reaction-diffusion neural network with delays and the stability and bifurcation of the networks are focused by Zhao et al. [18]. The dynamic characteristics of the discrete predator-prey model in two-dimensional parameter- spaces [6, 9, 10] are analyzed, and the numerical results show that the model exist many very interesting dynamic characteristics.

This article is organized as follows. We obtain a modified discrete predator-prey system with step size in section 2. In section 3, we investigate the existence of the fixed points. Stability of the model is discussed in section 4. Bifurcation theory and chaos control are derived in section 5 and 6. Numerical simulation is presented in section 7. The paper ends with conclusion in section 8 followed by relevant references.

2. Model with Leslie - Gower Functional Response
Let us consider a continuous time prey predator model with Leslie - Gower functional response

\[
\frac{dx}{dt} = rx(t)(1-x(t))-\alpha x(t)y(t)-hx(t)
\]

\[
\frac{dy}{dt} = y\left(\beta - \frac{\gamma y(t)}{x(t)}\right)
\]

where \( x \) and \( y \) represent the prey and predator densities respectively and \( r, \alpha, \beta, \gamma \) are positive constants.

We can obtain the discrete prey predator system by applying the forward Euler scheme to system (1)

\[
x(t+1) = x(t) + \delta \left( rx(t)(1-x(t))-\alpha x(t)y(t)-hx(t) \right)
\]

\[
y(t+1) = y(t) + \delta y\left( \beta - \frac{\gamma y(t)}{x(t)} \right)
\]

where \( \delta \) is the step size, \( r \) is intrinsic growth rate of prey, \( \alpha \) is coefficient of predation, \( h \) is harvesting rate of prey, \( \beta \) is the growth rate of predator and \( \gamma \) represents coefficient of conversion.

3. Existence of Fixed Points
The system (2) has a trivial fixed point \((0,0)\), exclusion fixed point \(\left(\frac{r-h}{r},0\right)\) and co-existence fixed point \((x^*, y^*) = \left(\frac{r\gamma-h\gamma}{\alpha\beta+\gamma r}, \frac{r\beta-h\beta}{\alpha\beta+\gamma r}\right)\) provided \( r \neq h \). Here the trivial point \((0,0)\) is not considered for discussion since the Jacobian matrix does not exist for trivial point.

The Jacobian matrix \( J \) of the system (2) at fixed point \((x, y)\) is given by

\[
J(x,y) = \begin{bmatrix}
1 + \delta \left( r - 2rx - \alpha y - h \right) & -\delta \left( \alpha x \right) \\
\delta \left( \frac{\gamma y}{x} \right) & 1 + \delta \left( \beta - \frac{2\gamma y}{x} \right)
\end{bmatrix}.
\]

The characteristic equation is

\[
F(\lambda) = \lambda^2 - B\lambda + C = 0
\]

where \( B \) and \( C \) are the trace and determinant of \( J(x,y) \), expressed as
\[ B = 2 + \delta \left( 1 - 2x \right) r + \beta h - y \left( \alpha + \frac{2y r}{x} \right) \]
\[ C = \left( 1 + \delta \left( 1 - 2x \right) r - \alpha y - h \right) \left( 1 + \delta \left( \beta - \frac{2y r}{x} \right) \right) + \delta^2 \alpha \left( \frac{y r^2}{x} \right). \]

4. Stability of the Model.

We use the following lemma to analyze the stability of fixed points of system (2) which can be evaluated by the relations between roots and coefficients of a quadratic equation

**Lemma 4.1** [14] Let \( F(\lambda) = \lambda^2 - B\lambda + C \). Let \( \lambda_1 \) and \( \lambda_2 \) be two roots of the characteristic equation. Then

1. \( |\lambda_1| < 1 \) and \( |\lambda_2| < 1 \) if and only if \( F(-1) > 0 \) and \( C < 1 \) then it is a sink and it is locally asymptotically stable.
2. \( |\lambda_1| > 1 \) and \( |\lambda_2| > 1 \) if and only if \( F(-1) > 0 \) and \( C > 1 \) then it is a source and it is locally unstable.
3. \( |\lambda_1| < 1 \) and \( |\lambda_2| > 1 \) or \( (|\lambda_1| > 1 \text{ and } |\lambda_2| < 1) \) if and only if \( F(-1) < 0 \) then it is a saddle.
4. \( |\lambda_1| = 1 \) or \( |\lambda_2| = 1 \) if and only if \( B^2 - 4C < 0 \) and \( C = 1 \) then it is non-hyperbolic.

4.1 Exclusion Fixed Point

In order to find the exclusion fixed point, we take \( x \neq 0 \) in the system of equations (2). Hence

\[ E_1 = \left( \frac{r - h}{r}, 0 \right), \]

always exists.

**Theorem 4.2.** The fixed point \( E_1 \) of system (2) has at least four different topological types for all permissible values of parameters:

H-1. \( E_1 \) is asymptotically stable (sink) if one of the following conditions holds:
   (a) \( \Delta \geq 0 \) and \( r_1 < r < r_2 \),
   (b) \( \Delta < 0 \) and \( r < r_2 \).

H-2. \( E_1 \) is unstable (source) if one of the following conditions holds:
   (a) \( \Delta \geq 0 \) and \( r > \max \{ r_1, r_2 \} \),
   (b) \( \Delta < 0 \) and \( r > r_2 \).

H-3. \( E_1 \) is non-hyperbolic if one of the following conditions holds:
   (a) \( \Delta \geq 0 \) and \( r = r_1 \),
   (b) \( \Delta < 0 \) and \( r = r_2 \).

H-4. It is a saddle for other values of parameters except those values in H-1 to H-3, where \( r_1 = \frac{(2(h + \beta) + \delta \beta h) \delta + 4}{\delta (\delta \beta + 2)} \) and \( r_2 = \frac{(1 + \delta \beta) h + \beta}{1 + \delta \beta} \).

**Proof.** Jacobian matrix of the system (2) evaluated \( E_1 \) is

\[ J(E_1) = \begin{bmatrix} 1 - \delta (r - h) & -\delta \alpha \left( \frac{r - h}{r} \right) \\ 0 & 1 + \delta \beta \end{bmatrix}. \] (3)

The characteristic equation of \( J(E_1) \) is
\( F(\lambda) = \lambda^2 - (2 - \delta(r - h - \beta))\lambda + 1 - \delta(r - h - \beta) - \delta^2 \beta(r - h) = 0. \)

Therefore, the eigenvalues of \( J(E_i) \) are \( \lambda_{1,2} = \frac{[-2 - \delta(r - h - \beta) \pm \sqrt{\Delta}]}{2}, \) where \( \Delta = (r - h + \beta)^2. \)

Using Lemma 4.1, we have the necessary and sufficient condition for local stability of the exclusion fixed point \( E_i \) is satisfied has the condition H-1 to H-3 holds.

4.2 Coexistence fixed point

In order to find the coexistence fixed point, we take \( x \neq 0 \) and \( y \neq 0 \) in the system of equations (2).

Hence we obtain \( E_2 = (x^*, y^*) = \left( \frac{\gamma(r-h)}{\alpha \beta + r \gamma}, \frac{\beta(r-h)}{\alpha \beta + r \gamma} \right), \) which exist if \( r > h \).

**Theorem 4.3.** The co-existence fixed point \( E_2 \) of system (2) has at least four different topological types for all permissible values of parameters:

G-1. \( E_2 \) is sink if one of the following conditions holds:

(a) \( M^2 < 4N \) and \( 0 < \delta < \delta_2 \),

(b) \( M^2 \geq 4N \) and \( 0 < \delta < \delta_1 \)

G-2. \( E_2 \) is source if one of the following conditions holds:

(a) \( M^2 < 4N \) and \( \delta > \delta_2 \)

(b) \( M^2 \geq 4N \) and \( \delta > \delta_1 \)

G-3. \( E_2 \) is non-hyperbolic if one of the following conditions holds:

(a) \( M^2 > 4N \) and \( \delta = \delta_1 \) or \( \delta_2 \),

(b) \( M^2 < 4N \) and \( \delta = \delta_2 \).

G-4. \( E_2 \) is saddle if one of the following condition hold:

(a) \( M^2 \geq 4N \) and \( \delta_1 < \delta < \delta_3 \).

where \( \delta_i = \frac{-M - \sqrt{M^2 - 4N}}{N}, \delta_2 = \frac{-M}{N} \) and \( \delta_3 = \frac{-M + \sqrt{M^2 - 4N}}{N}. \)

**Proof.** Jacobian matrix of the system (2) evaluated \( E_2 \) is

\[
J(E_2) = \begin{bmatrix}
1 + \delta a_{11} & -\delta a_{12} \\
\delta a_{21} & 1 + \delta a_{22}
\end{bmatrix},
\]

where \( a_{11} = r - 2rx^* - \alpha y^* - h, \ a_{12} = \alpha x^*, \ a_{21} = \gamma \left( \frac{y^*}{x} \right)^2 \) and \( a_{22} = \beta - \frac{2\gamma y^*}{x}. \)

The characteristic equation of \( J(E_2) \) is

\[
F(\lambda) = \lambda^2 - S_1 \lambda + S_2 = 0,
\]

where \( S_1 = 2 + \delta M \) and \( S_2 = 1 + \delta M + \delta^2 N \) such that \( M = a_{11} + a_{22}, \ N = a_{11}a_{22} + a_{12}a_{21}. \) Now, applying Lemma 4.1, the coexistence steady state \( E_2 \) of system (2) is locally asymptotically stable if the conditions G-1 to G-3 are satisfied. For G-4 if \( M^2 > 4N, \) then equation (5) has two real roots. And if \( F(-1) = 0, \) i.e;

\[
F(-1) = 1 + (\delta M + 2) + (\delta^2 N + \delta M + 1) = \delta^2 N + 2\delta M + 4 = 0.
\]
By simple calculation, we can get \( \delta = \delta_1 \) or \( \delta = \delta_2 \). On the other hand, the eigenvalues \( \lambda_{1,2} \) are complex roots if \((\delta M + 2)^2 - 4(\delta^2 N + \delta M + 1) < 0\) which leads to \( M^2 < 4N \).

Let \( \delta = \delta_2 \), we get

\[
\lambda_{1,2} = \frac{(\delta h + 2) \pm i\delta \sqrt{4N - M^2}}{2}
\]

then equation (5) has two conjugate eigenvalues and the modulus of each of them equals to one.

5. Bifurcation Analysis

Bifurcation analysis is an interesting and fruitful topic to investigate the topological nature of the model. Bifurcation occurs only when a certain parameter value called bifurcation parameter value passes through a critical value and the stability of a fixed point changes. In this section, Flip and Neimark-Sacker bifurcation behaviors of the exclusion and co-existence fixed points \( E_{1,2} \) of system (2) are investigated.

5.1 Flip Bifurcation of (2).

First, choose \( r \) as the bifurcation parameter for discussing Flip bifurcation. The exclusion fixed point \( E_1 \) undergoes Flip bifurcation when one of the eigenvalues of Jacobian matrix is \(-1\) at a fixed point and another eigenvalue is neither 1 nor -1. The Jacobian matrix of system (2) at \( E_1 \) is shown in equation (3). The characteristic polynomial of (3) is

\[
F(\lambda) = \lambda^2 - (2 - \delta(r - h - \beta))\lambda + 1 - \delta(r - h - \beta) - \delta^2 \beta(r - h) = 0
\]

By Theorem 4.2, if \( \Delta \geq 0 \) and \( r = r_1 \), then the eigenvalues of the flip bifurcation \( E_1 \) are

\[
\lambda_{1,2} = \frac{2 - \delta(r - h - \beta)}{2} \pm \delta \sqrt{\Delta}
\]

We summarize the above analysis into the following theorem.

**Theorem 5.1** The exclusion fixed point \( E_1 \) loses its stability, via a Flip bifurcation when \( \Delta \geq 0 \), \( r = r_1 \), \( \lambda_1 = -1 \) and \( \lambda_2 = 1 - \delta(r - h) \neq \pm 1 \).

5.2 Neimark-Sacker Bifurcation of (2).

This section analyzes Neimark-Sacker bifurcation by using bifurcation theory for the choice of \( \delta \) as a bifurcation parameter. Corresponding to the co-existence fixed point \( E_2 \) Neimark-Sacker bifurcation occurs when two eigenvalues of the Jacobian matrix at a fixed point are a pair of complex conjugate numbers with module one. The Jacobian matrix of system (2) at the coexistence fixed point \( E_2 \) is shown in equation (4). The characteristic polynomial of (4) is

\[
F(\lambda) = \lambda^2 - S_1\lambda + S_2 = 0,
\]

By Theorem 4.3, if \( M^2 - 4N < 0 \) and \( \delta^2 = \delta_2 \), then the eigenvalues \( E_2 \) are

\[
\lambda_{1,2} = \frac{(\delta h + 2) \pm i\delta \sqrt{4N - M^2}}{2}
\]

Analyzing the above conditions for Neimark-Sacker bifurcation, we state the theorem:

**Theorem 5.2** The co-existence fixed point \( E_2 \) loses its stability and system (2) undergoes a Neimark-Sacker bifurcation when \( M^2 - 4N < 0 \) and \( \delta^2 = \delta_2 \), and
6. Chaos Control

This section presents two control strategies in order to move the unstable periodic orbits or the chaotic orbits towards the stable one. First, linear feedback control method is applied to system (2). For this, the controller of system (2) is defined by

\[
x_{i+1} = x_i + \delta \left( r x_i [1 - x_i] - \alpha x_i y_i - h x_i \right) + S_i
\]

\[
y_{i+1} = y_i + \delta \left( \beta y_i - \frac{y_i^2}{x_i} \right)
\]

where \( S_i = -p_1 (x_i - x^*) - p_2 (y_i - y^*) \) is the feedback controlling force, \( p_{i,2} \) stands for the feedback gains and \((x^*, y^*)\) be an exclusion fixed point of system (2). The Jacobian matrix of system (6) evaluated at exclusion fixed point \( E \) is

\[
J_i(E) = \begin{bmatrix}
1 - \delta (r - h) - p_1 & -h \alpha \left( \frac{r - h}{r} \right) - p_2 \\
0 & 1 + \delta \beta
\end{bmatrix}.
\]

The corresponding characteristic equation of \( J_i(E) \) is

\[
\lambda^2 - (2 - \delta (r - h - \beta) - p_i) \lambda + 1 - \delta (r - h - \beta) - \delta^2 \beta (r - h) - p_1 (1 + \delta \beta) = 0.
\]  

(7)

Let \( \lambda_1, \lambda_2 \) are the eigenvalues of (7), then we have

\[
\lambda_1 \lambda_2 = 1 - \delta (r - h - \beta) - \delta^2 \beta (r - h) - p_1 [1 + \delta \beta]
\]

(8)

The lines of marginal stability are evaluated by solving \( \lambda_1 = \pm 1 \) and \( \lambda_1 \lambda_2 = 1 \). These restrictions guarantee that the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) have absolute value less than 1. Suppose that \( \lambda_1 \lambda_2 = 1 \), then equation (8) implies that

\[
l_1 : p_1 (1 + \delta \beta) = \delta (\beta - (1 + \delta \beta) (r - h))
\]

Suppose that \( \lambda_1 = 1 \), then equation (7) yields

\[
l_2 : p_1 + \delta (r - h) = 0
\]

or \( \lambda_1 = -1 \), then equation (7) yields

\[
l_2 : p_1 [\delta \beta + 2] = 4 - \delta [2 (r - h - \beta) + \delta \beta (r - h)]
\]

The stable eigenvalues lie within the triangular region bounded by the lines \( l_1, l_2 \) and \( l_3 \).

Next, in order to control the chaos produced by Neimark-Sacker bifurcation in system (2), the authors introduce hybrid control strategy. Assuming that system (2) undergoes Neimark-Sacker bifurcation at coexistence fixed point \( E \), then corresponding controlled system is expressed as

\[
x_{i1} = \phi x_i + \delta \left( r x_i (1 - x_i) - \alpha x_i y_i - h x_i \right) + (1 - \phi) x_i
\]

\[
y_{i1} = \phi y_i + \delta \left( \beta y_i - \frac{y_i^2}{x_i} \right) + (1 - \phi) y_i
\]

(9)

where \( 0 < \phi < 1 \).

Controlling strategy in (9) is a combination of both parameter perturbation and feedback control. Moreover, by proper choice of controlled parameter \( \phi \), the Neimark-Sacker bifurcation of the fixed
point $E_2$ of controlled system (9) can be advanced (delayed) or even completely eliminated. The Jacobian matrix of (9) evaluated at the coexistence fixed point $E_2$ is

$$J_2(E_2) = \begin{bmatrix} 1 + \delta \phi a_{11} & -\delta \phi a_{12} \\ \delta \phi a_{21} & 1 + \delta \phi a_{22} \end{bmatrix}. \quad (10)$$

Moreover, the characteristic equation of Jacobian matrix of the controlled system (9) is given by

$$\lambda^2 - (2 + \delta \phi M)\lambda + (1 + \delta \phi M + \delta^2 \phi^2 N) = 0.$$ 

The following results provides conditions for local asymptotic stability of coexistence steady state $E_2$ of the controlled system (9).

**Theorem 6.1** The coexistence fixed point $E_2$ of the controlled system (9) is locally asymptotically stable if

$$|2 + \delta \phi M| < 2 + \delta \phi M + \delta^2 \phi^2 N < 2.$$ 

### 7. Numerical Simulations

In this section, using the computer algebraic software (CAS), bifurcation diagrams of the system (2) are plotted in particular ranges.

**Example 7.1** We take $\delta \in [0.85, 0.97]$, $\alpha = 1.59$, $\beta = 1.58$, $r = 2.79$, $\gamma = 1.79$ and $h = 0.59$ with initial values $x_0 = 0.4$, $y_0 = 0.3$. Then we get $M^2 - 4N = -4.6626 < 0$ and $\delta = \delta_2 = 0.87561$. The coefficients of system (2) satisfy Theorem 5.2. By calculation, we have at $\delta = \delta_2 = 0.87561$, the coexistence fixed point is $E_2 = (0.5246, 0.4631)$. The characteristic polynomial evaluated at $E_2$ is given by

$$F(\lambda) = \lambda^2 + 0.6651\lambda + 1 = 0. \quad (11)$$

The roots of (11) are $\lambda_{1,2} = -0.3325 \pm 0.9430$ with $|\lambda_{1,2}| = 1$. Hence, according to Theorem 5.2, the conditions of Neimark-Sacker bifurcation are obtained for the coexistence fixed point $E_2$ at the bifurcation critical value $\delta_2$. From figure 1(a) and 1(b), it is observed that coexistence fixed point of system (2) is locally asymptotically stable for $\delta < \delta_2 = 0.87561$, and loses its stability at $\delta = \delta_2 = 0.87561$ and attracting different invariant cycles appear for $\delta$ in the range of $[0.86, 0.97]$, see figure 2(b) and figure 2(c). The various phase plane diagrams of $\delta$ corresponding to figure 1 are plotted in Figure 2, to illustrate these observations. When $\delta = 0.87$, the quasi-periodic orbits appear and increasing the value of $\delta$, the circles breakdown and also seen that the attracting chaotic sets are plotted in Figure 2(g) - Figure 2(i) to illustrate these observations.
Figure 1: Neimark-Sacker bifurcation diagrams of system (2) in $(\delta-x)$ and $(\delta-y)$ planes

**Example 7.2** Let $\alpha = 1.59$, $\beta = 1.58$, $r = 2.79$, $\gamma = 1.79$ and $h = 0.59$ and with initial values $(x_0, y_0) = (0.4, 0.3)$, then the first example shows that system (2) undergoes Neimark-Sacker bifurcation as $\delta$ varies in $[0.85, 0.97]$. Moreover, figure 2 shows that a closed invariant circle appears at $\delta = 0.876$ enclosing this unstable coexistence fixed point $E_2 = (0.5246, 0.4631)$. For these parametric values, the controlled system (9) can be written as

$$
x_{t+1} = x_t + \phi \delta (rx_t(1-x_t) - \alpha x_t y_t - hx_t)
y_{t+1} = y_t + \phi \delta (\beta y_t - \frac{\gamma y_t^2}{x_t})
$$

(12)

where $\alpha = 1.59$, $\beta = 1.58$, $r = 2.79$, $\gamma = 1.79$, $h = 0.59$ and $0 < \phi < 1$. Then Jacobian matrix of controlled system (12) evaluated at $E_2$ is

$$
J_z(E_2) = \begin{bmatrix}
1-1.2822\phi & -0.7307\phi \\
1.2217\phi & 1-1.3841\phi
\end{bmatrix}.
$$

(13)

The characteristic polynomial of (13) is given by

$$
F(\lambda) = \lambda^2 - (2-2.6663\phi)\lambda + 2.6674\phi^2 - 2.6663\phi + 1 = 0.
$$

(14)

Then, the roots of (14) lie in the unit open disk if and only if $0 < \phi < 0.99990$. Moreover, the plots for $x_t, y_t$ of the controlled system (12) are in figure 4 with $\phi = 0.98$. From figure 4(a) and 4(b), it is clear that the positive steady state $E_2$ is stable.
Figure 2: Phase portraits for various values of $\delta$ corresponding to Figure 1

Figure 3: (a) Plots of $x_t$ and $y_t$ for system (2), (b) Phase portrait for system (2)
Figure 4: (a) Plots of $x_t$ and $y_t$ for controlled system (9), (b) Phase portrait for controlled system (9)

8. Conclusion

This article investigated new form of discrete time 2-D prey predator model involving Lesile - Gower functional response with step size. Utilizing Euler scheme, discrete system is obtained from the continuous dynamical system. Dynamical consistency of the model which includes the existence and stability of the fixed points is investigated. Eigenvalues of Jacobian matrix are computed for non-negative biologically meaningful fixed points. Analytical results illustrate the rich dynamics and complexity of the model. The bifurcation theory is employed to study the existence of flip and Neimark-Sacker bifurcations. The chaos control of the discrete system is performed and numerical simulations are provided supporting the results.

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