Additive energy and a large sieve inequality for sparse sequences

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Abstract
We consider the large sieve inequality for sparse sequences of moduli and give a general result depending on the additive energy (both symmetric and asymmetric) of the sequence of moduli. For example, in the case of monomials $f(X) = X^k$ this allows us to improve, in some ranges of the parameters, the previous bounds of S. Baier and L. Zhao (2005), K. Halupczok (2012, 2015, 2018) and M. Munsch (2020). We also consider moduli defined by polynomials $f(X) \in \mathbb{Z}[X]$, Piatetski–Shapiro sequences and general convex sequences. We then apply our results to obtain a version of the Bombieri–Vinogradov theorem with Piatetski–Shapiro moduli improving the level of distribution of R. C. Baker (2014).

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1 | INTRODUCTION

1.1 | General setup

The large sieve, which originated in the work of Linnik [39], has become over the last decades an extremely powerful method in number theory, see [23, Chapter 9] and [34, Chapter 7]. More recently new variants of the large sieve over sparse sequences of moduli, such as squares, have appeared and found numerous applications in arithmetic problems of different flavours such as the distribution of primes in sparse progressions [3, 6–8], the existence of shifted primes divisible by a large square [41], the study of Fermat quotients [14, 51], elliptic curves [12, 52] and several others.

To formulate a general form of a large sieve inequality, we recall that a set of real numbers $\{x_k : k = 1, \ldots, K\}$ is called $\delta$-spaced modulo 1 if $\langle x_k - x_j \rangle \geq \delta$ for all $1 \leq j < k \leq K$, where $\langle x \rangle$ denotes the distance of a real number $x$ to its closest integer. Then by a result of Montgomery and Vaughan [45, Theorem 1] we have

$$\sum_{k=1}^{K} \left| \sum_{n=M+1}^{M+N} a_n e(x_k n) \right|^2 \leq (\delta^{-1} + N) \sum_{n=M+1}^{M+N} |a_n|^2,$$  \hspace{1cm} (1.1)

where $e(z) = \exp(2\pi iz)$ for $z \in \mathbb{C}$. See also [23, Theorem 9.1] or [34, Theorem 7.7].
The case when the \( \{x_k : k = 1, \ldots, K\} \) is the set of Farey fractions of order \( Q \), that is, \( \{a/q; \gcd(a, q) = 1, 1 \leq a < q, q \leq Q\} \), has always been of special interest due to the wealth of arithmetic applications, including the celebrated Bombieri–Vinogradov type theorem.

We now consider this question for the sequence of perfect \( k \)-powers for an integer \( k \geq 2 \). Similarly to the large sieve modulo squares used in [14] to study \( p \)-divisibility of Fermat quotients modulo \( p \), results of this type can be used to study the \( p^k \)-divisibility, and perhaps complement some results of Cochrane, De Silva and Pinner [20]. Furthermore, it is quite feasible that it can also embedded in the work of Matomäki [41] and Merikoski [42] (or in a weaker but more robust approach of Baier and Zhao [3]). In turn, this is expected to lead to showing the infinitude of primes \( p \) such that \( p - 1 \) is divisible by a large perfect \( k \)th power (rather than by a large perfect square as in [3, 41, 42]).

More generally, large sieve inequalities with any sparse sequence of moduli \( \{m_n\} \), which we also consider here, are expected to lead to results about shifted primes divisible by large divisors coming from the sequence \( \{m_n\} \), the most studied case being sequences of polynomial moduli. Indeed, the approach to shifted primes with large square divisors of Baier and Zhao [3] seems to extend to other sequences without appealing to their multiplicative properties (while the method of [41, 42] is more tuned to squares and perhaps other perfect powers).

Here we obtain such a result for Piatetski–Shapiro divisors, see Corollary 1.7 below. This approach has also been successfully used for sequences of multivariate polynomial moduli in a recent work of Halupczok and Munsch [31].

Another appearance of such large sieve can be found in a question of Erdös and Sárközy [22] about divisibility properties of sumsets. In the case of square-free numbers, Konyagin [38] has shown links between such problems and \( L^1 \)-norms of exponential sums considered by Balog and Ruzsa [11]. Most certainly these ideas extend to \( k \)-free numbers (that is, to integers which are not divisible by \( k \)th power of a prime).

Given a sequence \( a = \{a_n\} \) of complex numbers and positive integers \( k, M, N \) and \( Q \), we consider the sum

\[
\mathcal{S}_k(a; M, N, Q) = \sum_{q=1}^{Q} q^k \sum_{\substack{a=1 \\ \gcd(a, q) = 1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{q^k} n\right)\right|^2.
\]

Furthermore, given a polynomial \( f(T) \in \mathbb{Z}[T] \) with a positive leading coefficient, we consider more general sums with polynomial moduli \( f(q) \), which are defined as follows:

\[
\mathcal{S}_f(a; M, N, Q) = \sum_{q=1}^{Q} \sum_{\substack{a=1 \\ \gcd(a, f(q)) = 1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{f(q)} n\right)\right|^2,
\]

where without any loss of generality we always assume that \( f(q) \geq 1 \) for any integer \( q \geq 1 \).

For a general sequence of \( m = \{m_j\} \) of integers, we consider the sums

\[
\mathcal{S}(a, m; M, N, Q) = \sum_{j=1}^{Q} m_j \sum_{\substack{a=1 \\ \gcd(a, m_j) = 1}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{a}{m_j} n\right)\right|^2.
\]
A large sieve inequality is an estimate of the following kind:

\[ \mathcal{S}_k(a; M, N, Q) = O \left( \Delta_k(N, Q) \|a\|^2 \right) \]  

(1.2)

and similarly for \( \mathcal{S}_j(a; M, N, Q) \) and \( \mathcal{S}(a, m; M, N, Q) \), where

\[ \|a\| = \left( \sum_{n=M+1}^{M+N} |a_n|^2 \right)^{1/2} \]

and \( \Delta_k(N, Q) \) is some function of the parameters \( N \) and \( Q \) (which could both depend on \( k \)) and the implied constant may depend on \( k \).

In the simplest case \( k = 1 \), the bound

\[ \mathcal{S}_k(a; M, N, Q) \leq (Q^2 + N - 1)\|a\|^2 \]

is classical and in fact is a special case of the following general version of the large sieve inequality (1.1).

Here we are mostly interested in the case \( k \geq 2 \). For \( k = 2 \) the best known result is due to Baier and Zhao [4].

### 1.2 Previous results

We start with an observation due to Zhao [55], that the classical large sieve inequality (1.1) implies (1.2) with

\[ \Delta_k(N, Q) = \min \{ Q^{2k} + N, Q(Q^k + N) \}. \]  

(1.3)

Zhao [55] also conjectures that we can take

\[ \Delta_k(N, Q) = (Q^{k+1} + N)N^\omega(1) \]  

(1.4)

in (1.2), which is based on the heuristic that the fractions with power denominator are sufficiently regularly spaced. Note that this conjecture is non-trivial only for

\[ Q^k \leq N \leq Q^{2k} \]  

(1.5)

as otherwise it follows from (1.3). A recent result of Kerr [36] gives a version of the conjecture (1.4) with respect to the \( L^1 \)-norm.

Several authors have obtained improvements of (1.3) in the critical range (1.5), and we refer to [47] for a short survey and comparison of various bounds.

Firstly, Zhao [55, Theorem 3] has presented an inequality of type (1.2) with

\[ \Delta_k(N, Q) = Q^{k+1} + \left( NQ^{1-1/\kappa_k} + N^{1-1/\kappa_k}Q^{1+k/\kappa_k} \right)N^\omega(1), \]  

(1.6)
where
\[ \kappa_k = 2^{k-1}. \quad (1.7) \]

Baier and Zhao [2, Theorem 1] have shown that we can take
\[ \Delta_k(N, Q) = \left( Q^{k+1} + N + N^{1/2} Q^k \right) N^{o(1)}, \quad (1.8) \]
which improves (1.6) in the range
\[ Q^{2k-2+2k/\kappa_k} \leq N \leq Q^{2k}. \]

These results have been sharpened in a series of works of Halupczok [27–30], using the progress made on the Vinogradov mean value theorem by Bourgain, Demeter and Guth [13] and Wooley [53, 54]. Consequently, we can take
\[ \Delta_k(N, Q) = \left( Q^{k+1} + \min \{ A_k(Q, N), N^{1-\omega_k} Q^{1+(2k-1)\omega_k} \} \right) N^{o(1)} \quad (1.9) \]
with
\[ \omega_k = \frac{1}{(k-1)(k-2)+2} \]
and
\[ A_k(Q, N) = NQ^{1-1/(k(k-1))} + N^{1-1/(k(k-1))} Q^{k/(k-1)}. \]

In fact, one can use (1.9) to bound \( \mathfrak{S}_f(a; M, N, Q) \) in an analogue of (1.2) for any polynomial \( f \) of degree \( k \), see [30, Section 6].

Recently, Munsch [47] has further refined this estimate and obtained
\[ \Delta_k(N, Q) = Q^{(k+2)/(k+1)+o(1)} N^{1-1/(k(k+1))}. \quad (1.10) \]

We also note that in the special case of \( k = 3 \), Baier and Zhao [2, Theorem 2] have given the following estimate
\[ \mathfrak{S}_3(a; M, N, Q) \leq \left( Q^4 + \max \left\{ N^{9/10} Q^{6/5}, N Q^{6/7} \right\} \right) N^{o(1)}. \quad (1.11) \]

Finally we mention that we are not aware of any large sieve estimates with arbitrary sequences, that is, for \( \mathfrak{S}(a, m; M, N, Q) \) which depend on some additive properties of the sequence of moduli \( m \), in particular, on its additive energy as in this work, see (1.11) and (1.12) below.

### 1.3 New results

Let us introduce the following quantities. We define the additive energy of a finite set \( S \subseteq \mathbb{R} \) to be
\[ E^+(S) = \# \{(s_1, t_1, s_2, t_2) \in S^4 : s_1 + t_1 = s_2 + t_2 \} \quad (1.11) \]
and the “asymmetric” additive energy with respect to the parameter $h \in \mathbb{Z}$ to be

$$E^+_h(S) = \# \{(s_1, t_1, s_2, t_2) \in S^4 : s_1 + t_1 = s_2 + t_2 + h\}.$$  \hspace{1cm} (1.12)

It is also convenient to define

$$E^*_+(S) = \max_{h \neq 0} E^+_h(S).$$

In fact it is easy to show that $E^*_+(S) \leq E^+(S)$; however, for some sequences $E^*_+(S)$ is much smaller than $E^+(S)$, and in Theorem 1.4 we take advantage of this.

Now for a sequence $m = \{m_j\}$ of integers and any integer $Q \geq 1$, we denote by $m_Q = \{m_1, \ldots, m_Q\}$ the set of its first $Q$ elements. We now show that a variant of the ideas of [19, 24, 35] (rather than using results of [19, 24, 35] directly as in [47]) allows us to obtain a general result depending on the additive energies of the truncations of the sequence of moduli.

We also assume that the sequence of moduli $m = \{m_j\}$ satisfies the following additional regularity of growth hypothesis: there exists $\alpha > 0$ such that

$$m_j = j^{\alpha + o(1)}, \quad j \to \infty.$$  \hspace{1cm} (1.13)

**Theorem 1.1.** With $a = \{a_n\}$, $m = \{m_j\}$, $M$, $N$ and $Q$ as above and also satisfying (1.13) and $Q^\alpha \leq N \leq Q^{2\alpha}$, we have

$$\mathcal{G}(a, m; M, N, Q) \leq \left(NE^+(m_Q)^{1/4} + N^{3/4}Q^{\alpha/2}E^*_+(m_Q)^{1/4}\right)Q^{o(1)}\|a\|^2.$$  

Good bounds are known for the additive energy of a large class of sequences. For instance, for any convex sequence of moduli, that is, a sequence $m = \{m_j\}$ with

$$m_j - m_{j-1} < m_{j+1} - m_j, \quad j = 2, 3, \ldots,$$

using the general bound of Shkredov [50, Theorem 1], which asserts that

$$E^*_+(m_Q) \leq E^+(m_Q) \leq Q^{32/13 + o(1)}$$

(see also (2.1) below), we deduce the following result.

**Corollary 1.2.** Under the conditions of Theorem 1.1 and assuming that $m = \{m_j\}$ is a convex sequence, we have

$$\mathcal{G}(a, m; M, N, Q) \leq N^{3/4}Q^{\alpha/2 + 8/13}Q^{o(1)}\|a\|^2.$$  

We observe that the bound of Corollary 1.2 is superior to (1.3) (taken with $k = \alpha$) in the range $Q^{2\alpha - 20/13} \leq N \leq Q^{2\alpha - 32/39}$.

If more information is available about the sequence $m = \{m_j\}$, then one can also use stronger bounds from [15].
Furthermore, it follows immediately from the result of Robert and Sargos [49, Theorem 2] that for any fixed real $\alpha \neq 0, 1$, for the Piatetski–Shapiro sequence $m_j = \lfloor j^{\alpha} \rfloor$ we have

$$E^+_k(m_Q) \leq E^+(m_Q) \leq (Q^2 + Q^{4-\alpha})Q^{o(1)}. \quad (1.14)$$

**Corollary 1.3.** Under the conditions of Theorem 1.1 and assuming that $\mathbf{m} = \{m_j\}$ with $m_j = \lfloor j^{\alpha} \rfloor$ for any fixed real $\alpha \neq 0, 1$, we have

$$\mathcal{G}(\mathbf{a}, \mathbf{m}; M, N, Q) \leq N^{3/4} \left( Q^{(1+\alpha)/2} + Q^{1+\alpha/4} \right)Q^{o(1)} |\mathbf{a}|^2. \quad (1.15)$$

In Section 1.5, we show that Corollary 1.3 combined with the ideas of [3, 6, 7] lead to Bombieri–Vinogradov-type theorems for primes in progressions with Piatetski–Shapiro moduli (see also [10] for questions of similar flavour).

As a consequence of Theorem 1.1 we also obtain new bounds on $\mathcal{G}_k(\mathbf{a}; M, N, Q)$ for $k \geq 5$. Unfortunately the case of $k = 4$ is missing a substantial ingredient and so we have to exclude it. In the case of $k = 3$ our method works but does not improve previous results, see also Section 6.

**Theorem 1.4.** With $\{a_n\}, M, N$ and $Q$ as above and $k \geq 5$, we have

$$\mathcal{G}_k(\mathbf{a}; M, N, Q) \leq \left( N Q^{1/2} + N^{3/4} Q^{k/2+1/4+1/(2k^{1/2})} \right)Q^{o(1)} |\mathbf{a}|^2. \quad (1.16)$$

We now recall the definition (1.7). Our next result is essentially due to Zhao [55, Theorem 3], see (1.6) who presented it only for monomials. However, the approach undoubtedly works for any polynomial. However, since in [55] only a brief sketch of the proof of (1.6) is given, here we present a complete but slightly shorter proof, which uses a different technique and which we hope can find other applications. Finally, we formulate this bound in full generality for polynomial moduli (this can also be obtained via the method of [55]).

**Theorem 1.5.** Let $f(T) \in \mathbb{Z}[T]$ be of degree $k \geq 2$. With $\{a_n\}, M, N$ and $Q$ as above we have

$$\tilde{\mathcal{G}}_f(\mathbf{a}; M, N, Q) \leq \left( Q^{k+1} + \left( N Q^{1-1/\kappa_k} + N^{1-1/\kappa_k} Q^{1+k/\kappa_k} \right)Q^{o(1)} \right) |\mathbf{a}|^2. \quad (1.17)$$

1.4 | Comparison with previous results

As already mentioned, Theorem 1.1 has no predecessors, hence we only discuss Theorems 1.4 and 1.5.

To simplify the exposition, here we assume that all implied constants are absolute, while elsewhere in the paper they can depend on $k$.

The bound of Theorem 1.4 improves upon (1.10) when

$$N \geq Q^{\gamma_k},$$

for some $\gamma_k$ with $\gamma_k = 2k - 3 + O(k^{-1/2})$ as $k \to \infty$. 
Let us remark that the bound (1.10) improves (1.8) in the range

\[ Q^k \leq N \leq Q^{\lambda_k} \]

and improves (1.9) in the range

\[ Q^{k+1+2/(k-1)} \leq N \leq Q^{\mu_k}. \]

for some \( \lambda_k \) and \( \mu_k \) with

\[ \lambda_k = 2k - 2 + O(k^{-1}) \quad \text{and} \quad \mu_k = 2k - 1 + O(k^{-3}) \]

as \( k \to \infty \). Our bound is therefore superior to all previous bounds in the range

\[ Q^{\sigma_k} \leq N \leq Q^{\tau_k} \]

(1.15) for some \( \sigma_k \) and \( \tau_k \) with

\[ \sigma_k = 2k - 3 + O(k^{-1/2}) \quad \text{and} \quad \tau_k = 2k - 2 + O(k^{-1}), \]

as \( k \to \infty \). In particular, direct calculations show that for \( k \geq 7 \) we have \( \sigma_k < \tau_k \), and hence, the range (1.15) is not empty (unfortunately for \( k = 5, 6 \) we have \( \sigma_k \geq \tau_k \) and thus the range (1.15) is void).

We remark that after tedious but elementary calculations, one can easily get explicit expressions for \( \lambda_k, \mu_k, \sigma_k \) and \( \tau_k \).

### 1.5 Applications to primes in progressions with Piatetski–Shapiro moduli

Let us fix some \( \alpha > 1 \) and for a real \( R \geq 1 \) we consider the set

\[ S_{\alpha}(R) = \{ \lfloor j^\alpha \rfloor : j \in \mathbb{N} \} \cap [R, 2R]. \]

We further set

\[ M_{\alpha}(x; R) = \sum_{q \in S_{\alpha}(R)} \max_{\gcd(a,q)=1} |E(x, q, a)| \]

with

\[ E(x, q, a) = \sum_{n \equiv a \pmod{q}} \Lambda(n) - \frac{x}{\varphi(q)}, \]

where \( \Lambda(n) \) is the von Mangoldt function and \( \varphi(q) \) is Euler’s totient function.
where \( \varphi(q) \) is the Euler function and \( \Lambda(n) \) is the von Mangoldt function:

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n \text{ is a power of the prime } p, \\
0 & \text{otherwise.}
\end{cases}
\]

In the above notation we have the following version of the Bombieri–Vinogradov theorem for Piatetski–Shapiro moduli, which we derive combining the ideas and results of \([6, 7]\) with Theorem 1.1.

**Theorem 1.6.** For any fixed \( \alpha \) with \( 1 < \alpha < 9/4 \) and \( A > 0 \), we have

\[
M_\alpha(x; R) \leq \frac{\# S_\alpha(R)x}{RL^A},
\]

where \( L = \log x \) provided that \( R = x^{\delta} \) with some fixed \( \delta < \Phi(\alpha) \) and \( x \) is large enough, where

\[
\Phi(\alpha) = \begin{cases} 
3\alpha/(10\alpha - 4), & \text{for } 1 < \alpha < 26/23, \\
13/28, & \text{for } 26/23 \leq \alpha < 2, \\
13\alpha/(34\alpha - 12), & \text{for } 2 \leq \alpha < 23/11, \\
7\alpha/(20\alpha - 10), & \text{for } 23/11 \leq \alpha < 9/4.
\end{cases}
\]

Let \( \text{PS}_\alpha(n) \) be the largest divisor of \( n \in \mathbb{N} \) of the form \( \lfloor j^\alpha \rfloor \), \( j \in \mathbb{N} \). Repeating the argument of Baier and Zhao \([3, \text{Section 8}]\), we immediately obtain.

**Corollary 1.7.** Under the conditions of Theorem 1.6, for any fixed \( \delta < \Phi(\alpha) \) there are infinitely many primes \( p \) with

\[
\text{PS}_\alpha(p - 1) \geq p^\delta.
\]

The proof follows the same steps as the proof of \([3, \text{Theorem 5}]\) for \( p - 1 \) in \([3, \text{Section 8}]\) with the only difference that instead of using the asymptotic formula

\[
\sum_{y \leq j \leq 2y} \frac{1}{\varphi(j^2)} = \frac{3}{\pi^2 y} + O\left(y^{-2} \log y\right),
\]

we use the trivial lower bound

\[
\sum_{y \leq j \leq 2y} \frac{1}{\lfloor j^\alpha \rfloor} \geq \sum_{y \leq j \leq 2y} \frac{1}{j^\alpha} \geq 2^{-\alpha} y^{1-\alpha}.
\]

We note that \([7, \text{Theorem 1.3}]\) gives the bound of Theorem 1.6 for any \( \alpha \), provided that \( R \leq x^{9/20-\varepsilon} \) with any fixed \( \varepsilon > 0 \). Thus the novelty of Theorem 1.6 comes from the inequality \( \Phi(\alpha) > 9/20 \) for \( 1 < \alpha < 9/4 \), improving the previous level of distribution from \([7, \text{Theorem 1.3}]\).

Furthermore, based on the above we can always assume that

\[
x^{9/20-\varepsilon} \leq R \leq x^{1/2-\varepsilon}
\]

(1.17)

for some sufficiently small \( \varepsilon > 0 \).
2 | PREPARATIONS

2.1 | Notation and conventions

Throughout the paper, the notation $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq cV$ for some positive constant $c$, which depends on the degree $k$ and, where applies, on the coefficients of the polynomial $f$ and the real parameters $\alpha, \varepsilon$ and $A$ (in the proof of Theorem 1.6).

We also define $U \asymp V$ as an equivalent $U \ll V \ll U$.

For any quantity $V > 1$ we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of $V$ which satisfies $V^{-\varepsilon} \leq |U| \leq V^\varepsilon$ for any $\varepsilon > 0$, provided that $V$ is large enough. One additional advantage of using $V^{o(1)}$ is that it absorbs log $V$ and other similar quantities without changing the whole expression.

We also write $u \sim U$ means $U/2 < u \leq U$.

As we have mentioned, we always assume that $f(q) > 0$ for every positive integer $q$.

2.2 | Number of solutions to some asymmetric Diophantine equations

For integers $k, U \geq 1$, we introduce the set of powers

$$S_{U,k} = \{ u^k : 1 \leq u \leq U \}.$$ 

For any integer $h$, we seek a bound on $E^+_h(S_{U,k})$ which improves the essentially trivial estimate

$$E^+_h(S_{U,k}) \leq E^+(S_{U,k}) \leq U^{2+o(1)}. \quad (2.1)$$

It has been shown in [18] that a result of Marmon [40, Theorem 1.4] implies the following estimate.

Lemma 2.1. For a fixed $k \geq 2$ and uniformly over $h \neq 0$ we have

$$E^+_h(S_{U,k}) \leq U^{1+2/k^{1/2}+o(1)}.$$ 

Lemma 2.1 gives a non-trivial bound when $k \geq 5$. Unfortunately we do not have a non-trivial bound for $k = 4$. However, it is also shown in [18] that the classical argument of Hooley [33] gives a non-trivial bound for $k = 3$. We do not state it precisely because it does not imply a large sieve bound superior to the ones recalled in Section 1.2.

2.3 | Distribution of fractional parts and exponential sums

The following result is well known and can be found, for example, in [44, Chapter 1, Theorem 1] (which is a more precise form of the celebrated Erdös–Turán inequality).
Lemma 2.2. Let $\gamma_1, \ldots, \gamma_U$ be a sequence of $U$ points of the unit interval $[0,1]$. Then for any integer $H \geq 1$, and an interval $[\alpha, \beta] \subseteq [0,1]$, we have

$$\# \{ u = 1, \ldots, U : \gamma_u \in [\alpha, \beta] \} - U(\beta - \alpha) \ll \frac{U}{H} + \sum_{h=1}^{H} \left( \frac{1}{H} + \min\{\beta - \alpha, 1/h\} \right) \left| \sum_{u=1}^{U} e(h\gamma_u) \right|.$$ 

To use Lemma 2.2 we also need an estimate on exponential sums with polynomials, which is essentially due to Weyl, see [34, Proposition 8.2].

Lemma 2.3. Let $F(X) \in \mathbb{R}[X]$ be a polynomial of degree $k \geq 2$ with the leading coefficient $\vartheta \neq 0$. Then

$$\sum_{u=1}^{U} e(F(u)) \ll U^{1-k/2k-1} \left( \sum_{-U < \ell_1, \ldots, \ell_{k-1} < U} \min\{U, \langle \vartheta k! \ell_1 \ldots \ell_{k-1} \rangle^{-1} \} \right)^{1/2k-1},$$

where, as before, $\langle \xi \rangle = \min\{|\xi - k| : k \in \mathbb{Z}\}$ denotes the distance between a real $\xi$ and the closest integer.

2.4 Distribution in boxes and additive energy

Let us take a sequence of integers $m = \{m_j\}$ satisfying (1.13).

For integers $a$, $m$ with $\gcd(a, m) = 1$ and $U, V \geq 1$, we denote by $T_a(m; m, U, V)$ the number of solutions to the congruence

$$am_j \equiv v \pmod{m}, \quad 1 \leq j \leq U, \ |v| \leq V. \quad (2.2)$$

We now relate the congruence (2.2) to the additive energy of the sequence $m$.

Lemma 2.4. For any positive integers $U$ and $V$, uniformly over integers $a$ with $\gcd(a, m) = 1$, we have

$$T_a(m; m, U, V) \ll E^+(m_U)^{1/4} + \left( U^{\alpha+\omega(1)/m + 1} \right)^{1/4} V^{1/4} E^+(m_U)^{1/4}.$$ 

Proof. Observe that

$$T_a(m; m, U, V)^4 \leq W,$$

where $W$ is the number of solutions to the congruence

$$m_{j_1} + m_{j_2} - m_{j_3} - m_{j_4} \equiv a^{-1}w \pmod{m},$$

$$1 \leq j_1, j_2, j_3, j_4 \leq U, \ |w| \leq 4V.$$
Hence there is a set $\mathcal{Y} \subseteq \{0, ..., m - 1\}$ of cardinality $\# \mathcal{Y} = O(V)$, such that

$$T_a(m; m, U, V)^4 \leq \sum_{y \in \mathcal{Y}} W(y), \quad (2.3)$$

where $W(y)$ is the number of solutions to the congruence

$$m_{j_1} + m_{j_2} - m_{j_3} - m_{j_4} \equiv y \pmod{m},$$

$$1 \leq j_1, j_2, j_3, j_4 \leq U,$$

with a fixed $y$.

Clearly (2.4) implies that

$$m_{j_1} + m_{j_2} - m_{j_3} - m_{j_4} = y + mz$$

for some integer $z = O(U^{\alpha+o(1)}/m + 1)$. The contribution from the pairs $(y, z) = (0, 0)$ is obviously given by $E^+(mU)$. For other

$$O\left(\# \mathcal{Y}\left(U^{\alpha+o(1)}/m + 1\right)\right) = O\left(V\left(U^{\alpha+o(1)}/m + 1\right)\right)$$

admissible pairs we remark that (2.3) implies

$$T_a(m; m, U, V)^4 \ll E^+(mU) + V\left(U^{\alpha+o(1)}/m + 1\right)E^+(mU).$$

The result now follows. □

We remark that for Lemma 2.4 only the inequality $m_j \ll j^{\alpha+o(1)}$ matters; however, for our main results we need the full power of (1.13).

### 2.5 Polynomial values in small boxes

For integers $a, m$ with $\gcd(a, m) = 1$ and $U, V \geq 1$ and a polynomial $f(T) \in \mathbb{Z}[T]$, we denote by $\bar{T}_{a,f}(m; U, V)$ the number of solutions to the congruence

$$af(u) \equiv v \pmod{m}, \quad 1 \leq u \leq U, \ |v| \leq V.$$  

(2.5)

We also prove a new estimate on $\bar{T}_{a,f}(m; U, V)$ which uses the ideas of the proof of [17, Theorem 5], see also [37] for yet another approach in the case of prime moduli $m$. We now recall the definition (1.7).

**Lemma 2.5.** Let $f(T) \in \mathbb{Z}[T]$ be of degree $k \geq 2$. For any positive integers $U$ and $V$, uniformly over integers $a$ with $\gcd(a, m) = 1$, we have

$$\bar{T}_{a,f}(m; U, V) \ll \frac{UV}{m} + U^{1-1/\kappa_k} m^{o(1)} + U^{1-k/\kappa_k} V^{1/\kappa_k} m^{o(1)}.$$
Proof. Let \( T = \tilde{T}_{a,f}(m; U, V) \). Clearly we can assume that \( 1 \leq U, V < m \), as otherwise the result is trivial due to \( T \ll \min\{U, V\} \ll UV/m \).

We interpret the congruence (2.5) as a condition on fractional parts \( \{af(u)/m\}, u = 1, \ldots, U \). Applying Lemma 2.2 to the sequence of fractional parts \( \{af(u)/m\}, u = 1, \ldots, U \), with

\[
\alpha = 0, \quad \beta = V/m, \quad H = \lfloor m/V \rfloor,
\]

so that we have

\[
\frac{1}{H} + \min\{\beta - \alpha, 1/h\} \ll \frac{V}{m},
\]

for \( h = 1, \ldots, H \), we derive

\[
T \ll \frac{UV}{m} + \frac{V}{m} \sum_{h=1}^{H} \left| \sum_{u=1}^{U} e(ahf(u)/m) \right|.
\]

(2.6)

Therefore, by Lemma 2.3, we have

\[
T \ll \frac{UV}{m} + \frac{U^{1-k/k_k} V}{m} \sum_{h=1}^{H} \left( \sum_{-U<\ell_1<\cdots<\ell_{k-1}<U} \min \left\{ U, \left\langle \frac{ab}{m} k! \ell_1 \cdots \ell_{k-1} \right\rangle^{-1} \right\} \right)^{1/k_k},
\]

where \( b \) is the leading coefficient of \( f \). We remove the common divisors introducing

\[
a_0 = ab / \gcd(m, b) \quad \text{and} \quad m_0 = m / \gcd(m, b)
\]

and rewrite the last bound as

\[
T \ll \frac{UV}{m} + \frac{U^{1-k/k_k} V}{m} \sum_{h=1}^{H} W(h)^{1/k_k},
\]

where

\[
W(h) = \sum_{-U<\ell_1<\cdots<\ell_{k-1}<U} \min \left\{ U, \left\langle \frac{a_0}{m_0} k! \ell_1 \cdots \ell_{k-1} \right\rangle^{-1} \right\}.
\]

We observe that since \( f \) is a fixed polynomial, we have

\[
m_0 \asymp m.
\]

Firstly we estimate the contribution from the terms with

\[
\ell_1 \cdots \ell_{k-1} \equiv 0 \pmod{m_0}.
\]

Clearly the product \( z = \ell_1 \cdots \ell_{k-1} \) can take at most \( U^{k-1}/m_0 \) values. If \( z = 0 \), then we have at most \( (k-1)(2U)^{k-2} \) possibilities for \( (\ell_1, \ldots, \ell_{k-1}) \). For any other \( z \neq 0 \), from the well-known bound on
the divisor function, see [34, Equation (1.81)], we obtain $U^{o(1)}$ possibilities for $(\ell'_1, \ldots, \ell'_{k-1})$. Hence in total we have at most

$$(k - 1)(2U)^{k-2} + U^{k-1+o(1)}/m_0 \ll U^{k-2} + U^{k-1+o(1)}/m = U^{k-2+o(1)}$$

choices, and each of them contributes $U$ to the sum over $\ell'_1, \ldots, \ell'_{k-1}$.

Thus we obtain

$$T \ll \frac{UV}{m} + \frac{U^{1-k/k_k}V}{m}H(U^{k-1+o(1)})^{1/k_k} + \frac{VU^{1-k/k_k}}{m}W,$$

where

$$W = \sum_{h=1}^{H} W_0(h)^{1/k_k}$$

with

$$W_0(h) = \sum_{0 < |\ell'_1|, \ldots, |\ell'_{k-1}| < U} \min \left\{ U, \left\langle \frac{a_0}{m_0} k!h\ell'_1 \ldots \ell'_{k-1} \right\rangle^{-1} \right\}.$$ 

Recalling the choice of $H$, we derive

$$T \ll \frac{UV}{m} + U^{1-1/k_k}m^{o(1)} + \frac{U^{1-k/k_k}V}{m}W. \quad (2.7)$$

The Hölder inequality implies the bound

$$W^{k}_k \ll H^{k_k-1} \sum_{h=1}^{H} W_0(h)$$

$$\ll H^{k_k-1} \sum_{h=1}^{H} \sum_{0 < |\ell'_1|, \ldots, |\ell'_{k-1}| < U} \min \left\{ U, \left\langle \frac{a_0}{m_0} k!h\ell'_1 \ldots \ell'_{k-1} \right\rangle^{-1} \right\}.$$ 

Collecting together the terms with the same value of

$$z = k!h\ell'_1 \ldots \ell'_{k-1} \not\equiv 0 \pmod{m_0}$$

and recalling the bound on the divisor function again, we conclude that

$$W^{k}_k \ll H^{k_k-1}m^{o(1)} \sum_{|z| < kHU^{k-1} \atop z \not\equiv 0 \pmod{m_0}} \min \left\{ U, \left\langle \frac{a_0}{m_0} z \right\rangle^{-1} \right\}.$$
Since the sequence $\langle a_0 z/m_0 \rangle$ is periodic with period $m_0$ and $HU^{k-1} \geq HU \geq m_0$, using that $\gcd(a_0, m_0) = 1$ we derive

$$W^\kappa_k \ll H^{\kappa_k-1} m^{o(1)} \left( \frac{HU^{k-1}}{m_0} + 1 \right) \sum_{z=1}^{m_0-1} \left\langle \frac{a_0}{m_0} \frac{z}{m_0} \right\rangle^{-1}$$

$$= H^{\kappa_k-1} m^{o(1)} \left( \frac{HU^{k-1}}{m_0} + 1 \right) \sum_{z=1}^{m_0-1} \left\langle \frac{z}{m_0} \right\rangle^{-1}$$

$$\leq H^{\kappa_k} U^{k-1} m^{o(1)} + H^{\kappa_k-1} m^{1+o(1)}.$$

Thus, recalling the choice of $H$, we derive

$$W \leq HU^{(k-1)/\kappa_k} m^{o(1)} + H^{1-1/\kappa_k} m^{1/\kappa_k + o(1)}$$

$$= U^{(k-1)/\kappa_k} V^{-1} m^{1+o(1)} + V^{-1+1/\kappa_k} m^{1+o(1)},$$

which after the substitution in (2.7) concludes the proof. \(\square\)

We remark that Halupczok [30] gives a different bound on the sum on the right-hand side of (2.6), based on the optimal form of the Vinogradov mean value theorem [13, 53, 54]. These bounds are used to derive (1.9), which in fact applies to any polynomial $f$ of degree $k$, see [30, Section 6].

2.6 Distribution of Farey fractions

Let us take a sequence of moduli $m = \{m_j\}$ satisfying (1.13). We introduce a subset of Farey fractions

$$S(m; Q) = \left\{ \frac{a}{m_j} : \gcd(a, m_j) = 1, 1 \leq a < m_j, Q \leq j \leq 2Q \right\}.$$ 

It is easy to remark that two distinct elements of $S(m; Q)$ are $1/Q^{2\alpha}$ spaced. Following a classical approach (see, for instance, [55]), we measure the spacings between these Farey fractions by the quantity

$$M(m; N, Q) = \max_{x \in S(m; Q)} \# \left\{ y \in S(m; Q) : \langle x - y \rangle < \frac{1}{2N} \right\}.$$

As noticed in [55], any good estimate on this quantity leads to an inequality of type (1.2). We prove the following bound.

**Lemma 2.6.** For any integers $N$ and $Q$ with (1.5), we have

$$M(m; N, Q) \leq E^+(m_Q)^{1/4} + N^{-1/4+o(1)} Q^{a/2} E^+(m_Q)^{1/4}.$$
Proof. Let \( x = a/m_k \) with \( \gcd(a, m_k) = 1 \). We would like to estimate the number of elements \( y = b/m_j \) with \( \gcd(b, m_j) = 1 \) such that

\[
\left\langle \frac{a}{m_k} - \frac{b}{m_j} \right\rangle = \frac{|am_j - bm_k|}{m_k m_j} < 1/(2N).
\]

We now count the number of pairs \((b, j)\) such that for \( z = am_j - bm_k \) we have \(|z| \leq Z\) for some \( Z = Q^{2\alpha + o(1)}/N \). This number does not exceed the number of pairs \((j, z)\) such that

\[
am_j \equiv z \pmod{m_k}, \quad j \leq 2Q, \quad |z| \leq Z.
\]

Applying Lemma 2.4 with parameters \( U = 2Q, V = Z, \) and \( m = m_k \) (thus by (1.13) we have \( U^\alpha \leq m^{1+o(1)} \)), we deduce the desired result. \( \square \)

Now, given a polynomial \( f(T) \in \mathbb{Z}[T] \) we denote

\[
\tilde{S}_f(Q) = \left\{ \frac{a}{f(q)} : \gcd(a, f(q)) = 1, \, 1 \leq a < f(q), \, Q \leq q \leq 2Q \right\}.
\]

and

\[
\tilde{M}_f(N, Q) = \max_{x \in \tilde{S}_f(Q)} \# \left\{ y \in S_f(Q) : \langle x - y \rangle < \frac{1}{2N} \right\}.
\]

**Lemma 2.7.** Let \( f(T) \in \mathbb{Z}[T] \) be of degree \( k \geq 2 \). For any integers \( N \) and \( Q \) with (1.5), we have

\[
\tilde{M}_f(N, Q) \ll Q^{k+1}N^{-1} + Q^{1 - 1/k + o(1)} + Q^{1 + k/k + o(1)}N^{-1/k}.
\]

**Proof.** We first observe

\[
f(q) \asymp q^k.
\]

Then proceeding as in the proof of Lemma 2.6 and using Lemma 2.5 instead of Lemma 2.4 we obtain the desired result. \( \square \)

3 | PROOFS OF LARGE SIEVE BOUNDS

3.1 | Proof of Theorem 1.1

Clearly it is enough to consider only a version of \( \mathcal{S}(a, m; M, N, Q) \) with summation of \( j \) over a dyadic interval, that is,

\[
\mathcal{S}(a, m; M, N, Q) = \sum_{j=Q}^{2Q} \sum_{a=1}^{q_j} \left| \sum_{n=M+1}^{N} a_n e \left( \frac{a}{m_j} \frac{m_j}{n} \right) \right|^2.
\]
We proceed similarly as in the proof of [47, Theorem 1.2] and arrive at

$$\mathfrak{G}(a, m; M, N, Q) \ll M(m; N, Q) N \sum_{n=M+1}^{M+N} |a_n|^2.$$  

The result follows then directly using Lemma 2.6.

### 3.2 Proof of Theorem 1.4

This is direct from Theorem 1.1, used together with (2.1) and Lemma 2.1.

### 3.3 Proof of Theorem 1.5

Again, we proceed similarly as in the proof of [47, Theorem 1.2] and arrive at

$$\mathfrak{G}_f(a; M, N, Q) \ll M_k(N, Q) N \sum_{n=M+1}^{M+N} |a_n|^2.$$  

We now apply Lemma 2.7 and derive the desired result.

### 4 SOME LARGE SIEVE ESTIMATES FOR PIATETSKI–SHAPIRO SEQUENCES

#### 4.1 Preliminaries

In order to use Theorem 1.1 in the proof of Theorem 1.6 we need to give a bound on the additive energy $E^+(t, R)$ of the sequence

$$S_{\alpha, t}(R) = \{ \lfloor \beta \rfloor / t : \lfloor \beta \rfloor \sim R, \ t \mid \lfloor \beta \rfloor \}.$$  

In fact we need it for every integer $t \in [1, R^{1/6}]$. The reason for the appearance of $t$ in our work is that a Dirichlet character modulo $\lfloor \beta \rfloor$ is induced by a primitive Dirichlet character modulo $\lfloor j^\alpha \rfloor / t$. However, the quantity $E^+(t, R)$ needs to be investigated only for

$$t \ll R^{1/6}. \quad (4.1)$$  

Firstly we estimate the cardinality of $S_{\alpha, t}(R)$.

**Lemma 4.1.** For $t \leq R^{1/6}$ we have

$$\# S_{\alpha, t}(R) \ll \frac{R^{1/\alpha}}{t} + R^{1/2}.$$  

Proof. We note first that (arguing as in [7, Equation (6.4)])

\[ \# S_{\alpha,t}(R) = \# \{ j \in \mathbb{N} : j^\alpha \sim R, \{ j^\alpha / t \} < 1 / t \} + O(1). \]

By the Erdös–Turán inequality, see Lemma 2.2, we have

\[ \# S_{\alpha,t}(R) - \frac{(2R)^{1/\alpha} - R^{1/\alpha}}{t} \ll 1 + \frac{R^{1/\alpha}}{t} + \sum_{1 \leq h \leq t} \frac{1}{h} |S_h|, \] (4.2)

where

\[ S_h = \sum_{j \sim R^{1/\alpha}} e(hj^\alpha / t). \]

To estimate the exponential sums \( S_h \) we apply Van der Corput’s inequality [25, Theorem 2.2] with

\[ \lambda = \frac{hR^{1-2/\alpha}}{t}, \]

which yields

\[ S_h \ll R^{1/\alpha} \lambda^{1/2} + \lambda^{-1/2} = \frac{h^{1/2}R^{1/2}}{t^{1/2}} + \frac{t^{1/2}R^{1/\alpha-1/2}}{h^{1/2}}. \]

Hence

\[ \sum_{1 \leq h \leq t} \frac{1}{h} |S_h| \ll R^{1/2} + t^{1/2}R^{1/\alpha-1/2}, \]

which after substitution in (4.2) implies

\[ \# S_{\alpha,t}(R) - \frac{(2R)^{1/\alpha} - R^{1/\alpha}}{t} \ll R^{1/2} + t^{1/2}R^{1/\alpha-1/2} + \frac{R^{1/\alpha}}{t}. \]

Since for \( t < R^{1/3} \) we have

\[ t^{1/2}R^{1/\alpha-1/2} \ll \frac{R^{1/\alpha}}{t} \]

the result now follows. \[ \square \]

Using the trivial bound

\[ E^+(t, R) \leq \left( \# S_{\alpha,t}(R) \right)^3, \]

we obtain from Lemma 4.1

\[ E^+(t, R) \ll \frac{R^{3/\alpha}}{t^3} + R^{3/2}. \] (4.3)
On the other hand, the bound (1.14) of Robert and Sargos [49, Theorem 2] implies

\[
E^+(t, R) \ll \begin{cases} 
R^{2/\alpha} , & \text{if } \alpha > 2 , \\
R^{4/\alpha - 1} , & \text{if } 2 \geq \alpha > 1 .
\end{cases}
\]  

(4.4)

\section{The case $1 < \alpha \leq 2$}

We may assume that $R > x^{9/20 - \varepsilon}$ in view of (1.17).

Denote

\[
S_t = \mathbb{G}(a, S_{\alpha,t}(R); M, N, (2R)^{1/\alpha} - R^{1/\alpha}).
\]

The bounds (4.3) and (4.4), together with Theorem 1.1, imply, respectively, the following two estimates on $S_t$:

\[
S_t \leq \left( \frac{R^{3/4\alpha}}{t^{3/4}} + \frac{R^{3/8}}{t^{1/2}} \right) \left( N + \frac{N^{3/4}R^{1/2}}{t^{1/2}} \right) R^{o(1)} \|a\|^2
\]  

(4.5)

and

\[
S_t \leq R^{1/\alpha - 1/4} \left( N + \frac{N^{3/4}R^{1/2}}{t^{1/2}} \right) R^{o(1)} \|a\|^2.
\]  

(4.6)

When $N \leq x^{3/5}$, we can absorb the term $N$ into $N^{3/4}R^{1/2}/t^{1/2}$. Indeed, $N^{1/4} < x^{3/20} \ll R^{1/2}/t^{1/2}$ since $R^{1/2}/t^{1/2} > R^{5/12} > x^{3/16}$. Thus, we get

\[
S_t \leq \left( \frac{N^{3/4}R^{3/4\alpha + 1/2}}{t^{5/4}} + \frac{N^{3/4}R^{7/8}}{t^{1/2}} \right) R^{o(1)} \|a\|^2
\]  

(4.7)

and

\[
S_t \leq \frac{R^{1/\alpha + 1/4}N^{3/4}}{t^{1/2}} R^{o(1)} \|a\|^2.
\]  

(4.8)

from (4.5) and (4.6), respectively.

Let $\mathcal{X}_q$ denote the set of all multiplicative characters modulo $q$ and let $\mathcal{X}_q^\ast$ be the set of non-principal characters, see [34, Chapter 3] for a background.

Let $\lambda > 0$, we now use the bounds (4.7) and (4.8) to estimate the sums

\[
T(c, \lambda) = \sum_{q \in S_2(R)} \sum_{\chi \in \mathcal{X}_q} \left| \sum_{n \leq N} c(n)\chi(n) \right|^2 C(\chi) - x^2,
\]  

(4.9)
where \( c_n = 0 \) for \( \gcd(n, q) > 1 \), the set \( S_\alpha(R) \) is defined by (1.16) and as usual \( C(\chi) \) is the conductor of the Dirichlet character \( \chi \). Indeed there is a primitive character to modulus \( \lfloor j^\alpha \rfloor \) such that

\[
\chi(n) = \begin{cases} 
\tilde{\chi}(n) & \text{if } \gcd(n, \lfloor j^\alpha \rfloor) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

We use Gallagher’s inequality [21, Equation (10), Chapter 27]. Discarding a factor \( \phi(R)/R \), for \( t \mid \lfloor j^\alpha \rfloor \) we have

\[
\sum_{\tilde{\chi} \in \chi^*_{\lfloor j^\alpha \rfloor/t}} \left| \sum_{n \leq N} c(n) \tilde{\chi}(n) \right|^2 \leq \sum_{a=1}^{\lfloor j^\alpha \rfloor/t} \gcd(a, \lfloor j^\alpha \rfloor/t) = 1 \left| \sum_{n=1}^N c_n \left( an \left\lfloor \frac{\lfloor j^\alpha \rfloor}{t} \right\rfloor \right) \right|^2. \tag{4.10}
\]

For those \( t = q/C(\chi) \) counted in \( T(c, \lambda) \) we have \( R x^{-\lambda} \ll t \ll R x^{-\lambda} \). Noticing that \( c_n \tilde{\chi}(n) = c_n \chi(n) \), we see that (4.10) yields

\[
T(c, \lambda) \ll \sum_{t \sim R x^{-\lambda}} \sum_{\lfloor j^\alpha \rfloor \sim R \lfloor j^\alpha \rfloor/t} \sum_{n \leq N} c_n \tilde{\chi}(n)^2 \leq \sum_{t \sim R x^{-\lambda}} \sum_{\lfloor j^\alpha \rfloor \sim R} \left| \sum_{a=1}^{\lfloor j^\alpha \rfloor/t} \gcd(a, \lfloor j^\alpha \rfloor/t) = 1 \left| \sum_{n=1}^N c_n \left( an \left\lfloor \frac{\lfloor j^\alpha \rfloor}{t} \right\rfloor \right) \right|^2. \tag{4.11}
\]

Let us assume the following condition

\[
N \leq x^{3/5}. \tag{4.12}
\]

Thus, by (4.7) and (4.8) we obtain the two estimates

\[
T(c, \lambda) \ll \sum_{t \sim R x^{-\lambda}} \left( \frac{N^{3/4} R^{3/4} x^{1/4} + 1/2}{t^{5/4}} + \frac{N^{3/4} R^{7/8}}{t^{1/2}} \right) \|c\|^2 R^{\omega(1)} \tag{4.13}
\]

and

\[
T(c, \lambda) \ll R^{\omega(1)} \sum_{t \sim R x^{-\lambda}} \frac{N^{3/4} R^{1/4} x^{1/4}}{t^{1/2}} \|c\|^2, \tag{4.14}
\]

\[
\ll N^{3/4} R^{1/4} x^{-\lambda/2} \|c\|^2. \tag{4.15}
\]
4.3 The case $\alpha > 2$

Here we have to adjust (4.8) replacing $1/\alpha + 1/4$ by $1/2\alpha + 1/2$. Thus

\[
T(c, \lambda) \ll R^{o(1)} \sum_{t \sim R x^{-\lambda}} N^{3/4} R^{1/2\alpha + 1/2} t^{1/2} ||c||^2
\]

\[
\ll N^{3/4} R^{1/2\alpha + 1 + o(1)} x^{-\lambda/2} ||c||^2.
\]

(4.14)

5 PROOF OF THEOREM 1.6

5.1 Preliminaries

In this section we assume that $\varepsilon$ is sufficiently small and write $\delta = \varepsilon^2$. We write

\[ R = x^\delta \]

and assume that (1.17) holds. That is,

\[ 9/20 - \varepsilon \leq \delta \leq 1/2 - \varepsilon. \]

Let $a^*$ (depending on $q$ and $x$) be chosen so that $\gcd(a^*, q) = 1$ and

\[ \max_{\gcd(a, q) = 1} |E(x, q, a)| = |E(x, q, a^*)|. \]

We define an “exceptional subset” $\mathcal{E}_\alpha(R)$ of $S_\alpha(R)$ below, and show that for any $A > 0$

\[ \sum_{q \in \mathcal{E}_\alpha(R)} |E(x, q, a^*)| \ll \frac{x \# S_\alpha(R)}{RL^A} \]

and

\[ \sum_{q \in S_\alpha(R) \setminus \mathcal{E}_\alpha(R)} |E(x, q, a^*)| \ll \frac{x \# S_\alpha(R)}{RL^A}. \]

For (5.1) it suffices to show that for any $A > 0$

\[ \# \mathcal{E}_\alpha(R) \ll \frac{\# S_\alpha(R)}{L^A}. \]

(5.3)

To see this, suppose that (5.3) holds. By the Brun–Titchmarsh theorem (see, for example, [34, Theorem 6.6] or [46, Theorem 3.9]), we have

\[ E(x, q, a^*) \ll \frac{x}{\varphi(q)} \ll \frac{x}{R} \log L \quad (q \in S_\alpha(R)). \]
Hence we see from (5.3) that or any \( A > 0 \), we have

\[
\sum_{q \in \mathcal{E}_\alpha(R)} |E(x, q, a^*)| \ll \frac{x}{R} \log \mathcal{L} \# \mathcal{E}_\alpha(R) \ll \frac{x \# S_\alpha(R)}{R \mathcal{L}^A}.
\]

We define \( \mathcal{E}_\alpha(R) \) by assigning \( q \in S_\alpha(R) \) to \( \mathcal{E}_\alpha(R) \) if for some Dirichlet character \( \chi \in \mathcal{X}_q^* \), the \( L \)-function \( L(s, \chi) \) has a zero \( \rho = \beta + \gamma i \) with

\[
(\beta, \gamma) \in \left[ 1 - \frac{\varepsilon}{144}, 1 \right) \times [-2R, 2R].
\]

By [7, Lemma 5.2], with the choice \( c_4 = \varepsilon/4 \), we have the bound (5.3). As a consequence of this definition, we have

\[
\sum_{n \leq N} \chi(n)n^{-\frac{1}{2}+it} \ll (|t| + 1)N^{\frac{1}{2}}x^{-3\delta}
\]

for \( q \in S_\alpha(R) \backslash \mathcal{E}_\alpha(R) \) and \( N \geq x^{\varepsilon/2} \), for any \( \chi \in \mathcal{X}_q^* \). The implied constant depends only on \( \alpha \) and \( \varepsilon \). To obtain this we argue as in [6, Lemma 5] followed by a partial summation as in [6, Lemma 6].

Before we begin the proof of (5.2), we assemble some results on mean and large values of Dirichlet polynomials.

**Lemma 5.1.** Let \( q \leq x \). Let \( a_n, n \sim N \), be complex numbers and let \( G = \sum_{n \sim N} |a_n|^2 \). We have

\[
\sum_{\chi \in \mathcal{X}_q^*} \left| \sum_{n \sim N} a_n \chi(n) \right|^2 \ll x^\delta (N + x^{\delta})G.
\]

**Proof.** The left-hand side of (5.5) is bounded by

\[
\Sigma = \sum_{r \mid q} \sum_{\substack{\chi \in \mathcal{X}_q^* \text{ primitive} \ \gcd(n, q) = 1}} \left| \sum_{n \sim N} a_n \chi(n) \right|^2.
\]

This can be bounded as

\[
\Sigma \ll \sum_{r \mid q} (N + x^\delta)G \ll x^\delta (N + x^\delta)G
\]

by [43, Theorem 6.2] and the bound on the divisor function, see [34, Equation (1.81)].

**Lemma 5.2.** Let \( q \), \( a_n \) and \( G \) be as in Lemma 5.1. We have, for \( V > 0 \),

\[
\# \left\{ \chi \in \mathcal{X}_q^* : C(\chi) \sim x^\delta, \sum_{n \sim N} a_n \chi(n) > V \right\} \ll x^{2\delta} (GNV^{-2} + x^\delta G^3 NV^{-6}).
\]


Proof. The left-hand side of (5.6) is bounded by

$$
\Sigma = \sum_{r \mid q} \sum_{N < n \leq 2N \atop \gcd(n, q) = 1} \left| a_n \overline{\chi}(n) \right| > V \}
$$

This can be bounded as

$$
\Sigma \ll \sum_{r \mid q} (GNV^{-2} + r^{1+\delta}G^3NV^{-6}) \ll x^{2\delta} (GNV^{-2} + x^\delta G^3NV^{-6})
$$

by [34, Theorem 9.18] and the bound on the divisor function, see [34, Equation (1.81)].

□

Lemma 5.3. For $q \leq x$, $L \geq 1$, $t \in \mathbb{R}$ we have

$$
\# \left\{ \chi \in \chi_q^* : C(\chi) \sim x^\lambda, \left| \sum_{\ell \leq L} \chi(\ell') \ell^{-\frac{1}{2} - it} \right| \geq U \right\} \ll x^{1+6\delta} |s|^{1+\delta} U^{-4}.
$$

Proof. If $\chi \in \chi_q^*$ is induced by $\overline{\chi} \in \chi_r^*$, $r \sim x^\lambda$, then

$$
\sum_{\ell \leq L} \chi(\ell') \ell^{-\frac{1}{2} - it} = \sum_{\gcd(\ell', q) = 1} \overline{\chi}(\ell') \ell^{-\frac{1}{2} - it}
$$

$$
= \sum_{\ell \leq L} \left( \sum_{d \mid \ell} \mu(d) \right) \overline{\chi}(\ell') \ell^{-\frac{1}{2} - it}
$$

$$
= \sum_{d \mid q} \mu(d) \overline{\chi}(d) \sum_{k \leq L/d} \overline{\chi}(k) k^{-\frac{1}{2} - it}.
$$

We now see that if

$$
\left| \sum_{\ell \leq L} \chi(\ell') \ell^{-\frac{1}{2} - it} \right| \geq U,
$$

then

$$
\left| \sum_{k \leq L/d} \overline{\chi}(k) k^{-\frac{1}{2} - it} \right| \gg Ux^{-\delta}
$$

for some $d \mid q$. For a given $d$, the number of possible characters $\overline{\chi}$ is $O(|s|^{1+\delta}x^{1+\delta}(Ux^{-\delta})^{-4})$ by [43, Theorem 10.3]. Summing over $d \mid q$, we see that the number of possible characters $\overline{\chi}$ is $O(x^{1+6\delta} |s|^{1+\delta} U^{-4})$. Since $\overline{\chi}$ determines $\chi$, the result follows.

□
5.2 Application of Vaughan’s identity and Heath–Brown’s decomposition

We begin the proof of (5.2) by using Vaughan’s identity; see [21, Chapter 24]. Let \( Z = R^{\epsilon/4} \). Then

\[
\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n)
\]

with

\[
a_1(n) = \begin{cases} 
\Lambda(n) & \text{if } n \leq Z \\
0 & \text{if } n > Z,
\end{cases}
\]

\[
a_2(n) = -\sum_{m \mid d \mid r = n} \Lambda(m) \mu(d), \quad a_3(n) = \sum_{h \mid d \mid n} \mu(d) \log h
\]

and

\[
a_4(n) = -\sum_{m \mid r = n} \Lambda(m) \left( \sum_{d \mid r} \mu(d) \right).
\]

Let

\[
E_i(x, q, a) = \sum_{n \leq x} a_i(n) - \frac{1}{\varphi(q)} \sum_{n \leq x} a_i(n).
\]

For \( q \in S_\alpha(R) \), we have

\[
\sum_{i=1}^{4} E_i(x, q, a^*) = \psi(x; q, a^*) - \frac{1}{\varphi(q)} \sum_{n \leq x} \Lambda(n)
\]

\[
= \psi(x; q, a^*) - \frac{x}{\varphi(q)} + O\left(\frac{x L^{-A}}{R}\right)
\]

by the prime number theorem. Thus to prove (5.2) it suffices to show for \( 1 \leq i \leq 4 \) that for any \( A > 0 \)

\[
\sum_{q \in S_\alpha(R) \setminus \mathcal{E}_\alpha(R)} |E_i(x, q, a^*)| \ll \frac{x \# S_\alpha(R)}{R L^A}.
\] (5.7)

The case \( i = 1 \) of (5.7) is obvious from the Brun–Titchmarsh theorem (see, for example, [34, Theorem 6.6] or [46, Theorem 3.9]).

A partial summation, together with an elementary argument, gives that for any \( A > 0 \)

\[
E_3(x; q, a) \ll Z x^{\epsilon/2} \ll \frac{x}{R L^A},
\]

which yields (5.7) for \( i = 3 \).
For $i = 4$, we refer to [7, Section 6] for a proof of (5.7).
Hence, it remains to prove (5.7) for $i = 2$.

Let

$$I(x, q, a) = \sum_{m, n \leq Z \atop \gcd(mn, q) = 1} \Lambda(m) \mu(n) \left\{ \sum_{\ell \leq x/mn \atop \ell \equiv a \pmod{q}} 1 - \frac{x}{qmn} \right\}.$$ 

By the discussion on [6, p. 142], it suffices for the proof of (5.7) for $i = 2$ to show that for any $A > 0$ we have

$$\sum_{q \in S_{\alpha}(R) \setminus \mathcal{E}_{\alpha}(R)} |I(x, q, a^*)| \ll \frac{x \# S_{\alpha}(R)}{RL^A}.$$ 

To treat $I(x, q, a^*)$, we use Heath–Brown’s decomposition [32] of $\Lambda(m)$ and the variant, used, for example, in [5, Equation (2.3)], for the arithmetic function $\mu$. Taking $k = 3$ in both cases, we see that

$$\Lambda(m) = \sum_{(I_1, \ldots, I_6)} \sum_{m_i \in I_i \atop m_1 \cdots m_6 = m} (\log m_1) \mu(m_4) \mu(m_5) \mu(m_6) \quad (1 \leq m \leq Z)$$

and

$$\mu(n) = \sum_{(J_1, \ldots, J_5)} \sum_{n_j \in J_j \atop n_1 \cdots n_5 = n} \mu(n_3) \mu(n_4) \mu(n_5) \quad (1 \leq n \leq Z),$$

where the tuples of intervals $(I_1, \ldots, I_6)$ and $(J_1, \ldots, J_5)$ run through some families of cardinalities $O((\log Z)^6)$ and $O((\log Z)^5)$, respectively, with

$$I_i = (a_i, 2a_i], \quad i = 1, \ldots, 6, \quad \text{and} \quad J_j = (b_j, 2b_j], \quad j = 1, \ldots, 5,$$

such that

$$\prod_{i=1}^{6} a_i < Z, \quad \prod_{j=1}^{5} b_j < Z$$

and

$$2a_i \leq Z^{1/3}, \quad i = 4, 5, 6, \quad \text{and} \quad 2b_j \leq Z^{1/3}, \quad j = 3, 4, 5.$$ 

There are $O(L^6)$ tuples $(I_1, \ldots, I_6)$ and $O(L^5)$ tuples $(J_1, \ldots, J_5)$ in these expressions. Now write $\mu(m) = a(m) + b(m)$ with $a(m) = \max\{\mu(m), 0\}$. Define

$$r_0(x, q, a, d) = \sum_{\ell \equiv a \pmod{q} \atop \ell \equiv 0 \pmod{d}} 1 - \frac{x}{qd}.$$
We have

\[
I(x, q, a^*) = \sum_{m \leq Z, n \leq Z, \gcd(mn, q) = 1} \Lambda(m)\mu(n)r_0(x, q, a^*, mn)
\]

\[
= \sum_{(I_1, \ldots, I_6)} \sum_{(J_1, \ldots, J_5)} \sum_{m_i \in I_i, n_j \in J_j} \gcd(m_i n_i, q) = 1 \quad 1 \leq i \leq 6, 1 \leq j \leq 5
\]

\[
\prod_{i=4}^{6} (a(m_i) + b(m_i)) \prod_{j=3}^{5} (a(n_j) + b(n_j))
\]

\[
\cdot r_0(x, q, a^*, m_1 \ldots m_6 n_1 \ldots n_5) \log m_1.
\]

The last expression splits in an obvious way into \(O(L^{11})\) sums with an attached \(\pm\) sign, of the form

\[
\Phi(L_1, \ldots, L_{11}; q) = \sum_{\ell_i \sim L_i, \gcd(\ell_i, q) = 1} a_1(\ell_1) \cdots a_{11}(\ell_{11}) r_0(x, q, a^*, \ell_1 \ldots \ell_{11})
\]

with non-negative \(a_j(\ell_j)\), such that \(a_i(\ell_i)\) is identically 1 or identically \(\log \ell_i\) if \(2L_i > Z^{1/3}\); also

\[
\max\{L_1 \ldots L_6, L_7 \ldots L_{11}\} \ll Z.
\]

We now summarize our work so far.

**Lemma 5.4.** Let \(\Phi\) be as above. If for any \(A > 0\),

\[
\sum_{q \in S_2(R) \setminus \mathcal{E}_c(R)} |\Phi(L_1, \ldots, L_{11}; q)| \ll \frac{x \# S_2(R)}{RL^A}, \quad (5.8)
\]

then (5.2) holds.

Note that (5.8) is obvious if

\[
L_1 \ldots L_{11} \leq x^{9/20},
\]

by the argument used to deal with \(E_3\). We assume from now on that

\[
L_1 \ldots L_{11} > x^{9/20}. \quad (5.9)
\]
5.3 | Riesz’s means

In order to prove (5.8) we work with Riesz’s means; the advantage of the logarithmic weighting will become clear. Let us generalize \( r_0 \) by defining for \( k \geq 0 \),

\[
 r_k(x, q, a, d) = \frac{1}{k!} \sum_{\ell \leq x \atop \ell \equiv a \pmod{q}} \left( \log \frac{x}{\ell} \right)^k - \frac{x}{qd}.
\]

Let \( u_d \geq 0 \) be given \((D_1 < d \leq D)\) where \( D_1 \sim D \), \( D \leq x \) and suppose for some absolute constant \( B \) that

\[
 |u_d| \leq \tau(d)^B.
\]

Suppose further that \( 1 \leq k \leq 3 \) and we have a bound

\[
 \left| \sum_{q \in S_{\alpha}(R)} \sum_{D_1 < d \leq D} u_d r_k(x, q, a^*, d) \right| \ll \frac{x \# S_{\alpha}(R)}{RL^A}.
\]

Then, provided that \( A \) is sufficiently large,

\[
 \left| \sum_{q \in S_{\alpha}(R)} \sum_{D_1 < d \leq D} u_d r_{k-1}(x, q, a^*, d) \right| \ll \frac{x \# S_{\alpha}(R)}{RL^{A/3}}.
\]

See [6, p. 154], for the details of a similar deduction. Now we see that it suffices to prove that for any \( A > 0 \)

\[
 \left| \sum_{q \in S_{\alpha}(R)} |\Phi_4(L_1, \ldots, L_{11}; q)| \right| \ll \frac{x \# S_{\alpha}(R)}{RL^A},
\]

where

\[
 \Phi_4(L_1, \ldots, L_{11}; q) = \sum_{\ell \sim L_i, \gcd(\ell,q) = 1 \atop 1 \leq i \leq 11} a_1(\ell_1) \ldots a_{11}(\ell_{11}) r_4(x, q, a^*, \ell_1 \ldots \ell_{11}).
\]

We now convert this into a form that requires the counting of Dirichlet characters. We write \( r_4 \) in the form

\[
 r_4(x, q, a^*, d) = \frac{1}{24 \varphi(q)} \sum_{\chi \in \chi_q^*} \overline{\chi(a^*)} \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 - \frac{x}{qd}
\]

\[
 = \frac{1}{24 \varphi(q)} \sum_{\chi \in \chi_q^*} \overline{\chi(a^*)} \chi(d) \left( \log \frac{x}{bd} \right)^4 + O \left( \frac{x^3}{q} \right)
\]
for $\gcd(d, q) = 1$. We set

$$u_d = \sum_{\substack{d = e_1 \ldots e_{11} \\
\epsilon_i \sim L_i \gcd(\epsilon_i, q) = 1 \atop 1 \leq i \leq 11}} a_1(e_1) \ldots a_{11}(e_{11})$$ (5.10)

for $D_1 < d \leq D$ with $D = L_1 \ldots L_{11}$ and $D_1 = 2^{-11} D$.

Dyadically dissecting the values of the conductor $C(\chi)$, and noting that $D < Z^2 < x^{1-\varepsilon/2}$, it suffices to show that for any $A > 0$

$$\sum_{q \in \mathcal{S}_\alpha(R) \setminus \mathcal{E}_\alpha(R)} \sum_{\chi \in \mathcal{X}_q} \left| \sum_{D_1 < d \leq D} u_d \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 \right| \lesssim \frac{x \# \mathcal{S}_\alpha(R)}{L^A}$$ (5.11)

whenever $0 \leq \lambda \leq \vartheta$.

We now use the integral formula

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} y^s \frac{ds}{s^5} = \begin{cases} \frac{1}{24} (\log y)^4 & \text{if } y \geq 1 \\ 0 & \text{if } 0 < y < 1 \end{cases}$$

(see [46, p. 143]). This gives

$$\frac{1}{24} \sum_{D_1 < d \leq D} u_d \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} x^s \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} B(s, \chi) \frac{ds}{s^5}$$

with

$$B(s, \chi) = \sum_{b \leq x/D_1} \chi(b) b^{-s}.$$ 

It follows that

$$\sum_{q \in \mathcal{S}_\alpha(R) \setminus \mathcal{E}_\alpha(R)} \sum_{\chi \in \mathcal{X}_q} \left| \sum_{D_1 < d \leq D} u_d \chi(d) \sum_{b \leq x/d} \chi(b) \left( \log \frac{x}{bd} \right)^4 \right|$$

$$\ll x^{1/2} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \sum_{q \in \mathcal{S}_\alpha(R) \setminus \mathcal{E}_\alpha(R)} q^s ds$$
Thus, in order to prove (5.11) it suffices to show that for \( \text{Re}(s) = \frac{1}{2} \), for any \( A > 0 \) we have

\[
\sum_{q \in S_\alpha \setminus \mathcal{E}_\alpha} \left| \sum_{\chi \in \chi_q} \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} \right| |B(s, \chi)| \frac{|ds|}{|s|^5}. 
\]

(5.12)

\[ \ll \#S_\alpha \langle R \rangle |s|^3 x^{1/2} \mathcal{L}^{-A}. \]

5.4 A key result

To deal with “small” \( \lambda \), we prove a result that is a variant of [6, Proposition 1].

Lemma 5.5. Let \( M_1, \ldots, M_{11} \in [1, x] \) and suppose that

\[
M = \prod_{i=1}^{6} M_i \ll x^{\theta + \varepsilon/4}, \quad N = \prod_{i=7}^{11} M_i \ll x^{\theta + \varepsilon/4}. 
\]

Let \( a_i(m), m \sim M_i \), satisfy

\[ |a_i(m)| \leq \log 2m, \quad 1 \leq i \leq 11, \quad m \sim M_i. \]

We further set

\[
M_i(s, \chi) = \sum_{m_i \sim M_i} a_i(m) \chi(m) m^{-s} \quad \text{and} \quad L = \frac{x}{M_1 \ldots M_{11}}, \quad B(s, \chi) = \sum_{n \leq L} \chi(n) n^{-s}. 
\]

Let \( \text{Re}(s) = 1/2 \) and

\[
\lambda \leq \min \left\{ \frac{9}{20}, \frac{5}{6} (1 - \theta) \right\} - \varepsilon. \quad (5.13)
\]

Let \( q \in S_\alpha \setminus \mathcal{E}_\alpha \). Then for any \( A > 0 \)

\[
\sum_{\chi \in \chi_q} \left| B(s, \chi) \prod_{i=1}^{11} M_i(s, \chi) \right| \ll |s|^3 x^{1/2} \mathcal{L}^{-A}. \quad (5.14)
\]
Proof. Let
\[ M(s, \chi) = \prod_{i=1}^{6} M_i(s, \chi), \quad N(s, \chi) = \prod_{i=7}^{11} M_i(s, \chi). \]

We have
\[ M(s, \chi) \ll M^{1/2} L^{11}, \quad N(s, \chi) \ll N^{1/2} L^{11}, \quad B(s, \chi) \ll L^{1/2}. \]

Thus the characters \( \chi \in \mathcal{X}_q^* \) for which one of these three Dirichlet polynomials has absolute value less than \( x^{-1} \) can be neglected. We partition the remaining characters with \( O(L^3) \) sets \( A_q(U, V, W) \) defined by the inequalities.

\[ U < |B(s, \chi)| \ll 2U, \quad V < |M(s, \chi)| < 2V, \quad W < |N(s, \chi)| \ll 2W, \]

where \( U \ll L^{1/2}, \quad V \ll M^{1/2} L^{11}, \quad W \ll N^{1/2} L^{11} \). To prove (5.14) it suffices to show that for any \( A > 0 \)
\[ UVW |A_q(U, V, W)| \ll |s|^3 x^{1/2} L^{-A}. \]

From Lemmas 5.1, 5.2 and 5.3 applied to \( B(s, \chi), M(s, \chi), N(s, \chi), B(s, \chi)^2 \) we obtain
\[ |A_q(U, V, W)| \ll |s|^2 Px^5, \]

where
\[ P = \min \left\{ \frac{M + x^\lambda}{V^2}, \frac{N + x^\lambda}{W^2}, \frac{x^\lambda}{U^4}, \frac{M}{V^2} + \frac{x^\lambda M}{V^6}, \frac{N}{W^2} + \frac{x^\lambda N}{W^6}, \frac{L^2}{U^4} + \frac{x^\lambda L^2}{U^{12}} \right\}. \]

Thus it suffices to show that
\[ UVWP \ll |s|x^{1/2-2\delta}. \]

We consider four cases depending on the size of the parameters.

**Case 1.** \( P \leq 2V^{-2} M, \quad P \leq 2W^{-2} N \). We apply (5.4); we have \( L \gg x^\delta/2 \). Since \( q \in S_\alpha(R) \setminus E_\alpha(R) \), we have
\[ U \ll |s| L^{1/2} x^{-3\delta} \]

and
\[ UVWP \leq 2UVW \min\{V^{-2} M, W^{-2} N\} \leq 2U(MN)^{1/2} \ll |s|x^{1/2} L^{11} x^{-3\delta} \ll |s|x^{1/2-2\delta}. \]
Case 2. $P > 2V^{-2}M, P > 2W^{-2}N$. We proceed as in Case 2 of [6, p. 145] with $Q$ replaced by $x^\lambda$. We obtain

$$P \ll (UW)^{-1}(x^{1/4+3\lambda/32} + x^{1/20+\lambda}) \ll (UW)^{-1}x^{1/2-2\delta}$$

since $\lambda \leq \frac{9}{20} - \varepsilon$.

Case 3. $P > 2V^{-2}M, P \leq 2W^{-2}N$.

We proceed as in Case 3 of [6, p. 145], again with $Q$ replaced by $x^\lambda$. We obtain

$$P \ll (UW)^{-1}(x^{1/8+7\lambda/16}N^{3/8} + x^{1/12+\lambda/2}N^{5/12}) \ll (UW)^{-1}x^{1/2-2\delta}$$

since $N \leq x^{\vartheta+\varepsilon/4}$ and $\lambda \leq \frac{5}{6}(1 - \vartheta) - \varepsilon$.

Case 4. $P > 2W^{-2}N, P \leq 2V^{-2}M$. We proceed as in Case 3, interchanging the roles of $M$ and $N$.

This completes the proof. $\square$

Summing over $q \in S_\alpha(R) \setminus E_\alpha(R)$, we see that Lemma 5.5 implies the bound (5.12) whenever (5.13) holds. Hence Lemma 5.4 holds in this case and (5.2) follows.

It remains to show that (5.12) holds whenever

$$\min\left\{\frac{9}{20}, \frac{5}{6}(1 - \vartheta)\right\} - \varepsilon \leq \lambda \leq \vartheta. \tag{5.15}$$

Before turning to the proof of this case, let us make the following remark. In order to apply the large sieve bounds (4.12), (4.13) and (4.14), we need (4.1). Note that if $t \geq R^{1/6}$, then $x^\lambda \leq R^{5/6}$, or $\lambda \leq \frac{5\vartheta}{6}$. Recalling that $\vartheta \leq 1/2 - \varepsilon$, we deduce that

$$\frac{5\vartheta}{6} \leq \min\left\{\frac{9}{20}, \frac{5}{6}(1 - \vartheta)\right\} - \varepsilon.$$

Hence the result for $\lambda$ in this range follow from Lemma 5.5 and we can assume in the following that $t \leq R^{1/6}$.

5.5 Conclusion of the proof of Theorem 1.6

Lemma 5.6. Let

$$K(s, \chi) = \sum_{n \sim K} a_n \chi(n)n^{-s} \quad \text{and} \quad H(s, \chi) = \sum_{m \sim H} b_m \chi(m)m^{-s}$$

with $a_n = x^{o(1)}$, $b_m = x^{o(1)}$, $a_n = b_n = 0$ for $(n, q) > 1$, $H \ll K \ll x^{3/5}$, $HK \ll x$. Let $\text{Re}(s) = 1/2$ and define

$$U(H, K, \lambda) = \sum_{q \in S_\alpha(R)} \sum_{\chi \in \chi_q^0} |H(s, \chi)K(s, \chi)|.$$
Then for $1 < \alpha \leq 2$ we have

$$U(H, K, \lambda) \ll x^{3/8 + o(1)} (R^{3/16 + \frac{1}{\alpha} + \frac{1}{4}} x^{\lambda/4} + R^{11/8} x^{-\lambda/2})$$

(5.16)

and

$$U(H, K, \lambda) \ll x^{3/8 - \lambda/2 + o(1)} R^{1/\alpha + 3/4}.$$  

(5.17)

For $\alpha > 2$, we have

$$U(H, K, \lambda) \ll x^{3/8 - \lambda/2 + o(1)} R^{1/2\alpha + 1}.$$  

(5.18)

Proof. The three upper bounds (5.16), (5.17) and (5.18) are obtained by applying the Cauchy inequality in combination with (4.12), (4.13) and (4.14), respectively (note that $H \leq K \leq x^{3/5}$, so the condition (4.11) is fulfilled).

We now use Lemma 5.6 to obtain (5.12) when (5.15) holds. In fact, in this part of the argument we do not need to discard $\mathcal{E}_\alpha(R)$ which has only been used in the Case 1 of the proof of Lemma 5.5.

It suffices to prove (5.12) with $B(s, \chi)$ replaced by

$$B_1(s, \chi) = \sum_{\ell \sim L_1} \chi(\ell) e^{-s\ell},$$

with $1 \leq L_1 \leq L$. Using the shape of the coefficients $u_d$ given by (5.10), we factorize

$$B_1(s, \chi) \sum_{D_1 < d \leq D} u_d \chi(d) d^{-s} = N_1(s, \chi) \ldots N_{12}(s, \chi),$$

where

$$N_i(s, \chi) = \sum_{n \sim N_i} c_{i,n} \chi(n) n^{-s}, \quad N_1 \geq \ldots \geq N_{12}.$$

Remark that these coefficients $c_{i,n}$ are identically 1, or identically $\log n$, if $N_i > Z^{1/3}$. Write $N_i = x^{\beta_i}$. We recall that we may suppose

$$\beta_1 + \ldots + \beta_{12} \geq x^{9/20}$$

(see (5.9)).

Suppose first that $\beta_1 + \beta_2 > 3/5$. We write

$$N_0(s, \chi) = N_3(s, \chi) \ldots N_{12}(s, \chi)$$

and define

$$A(U_0, U_1, U_2) = \{ \chi \in \mathcal{X}_q^* : q \in S_\alpha(R), C(\chi) \sim x^\lambda, \quad U_j < |N_j(s, \chi)| \leq 2U_j, \ j = 0, 1, 2 \}.$$
Arguing as in the proof of Lemma 5.5, it suffices to show that for any $A > 0$,

$$U_0 U_1 U_2 \# A(U_0, U_1, U_2) \ll R^{1/\alpha} |s|^{1+\delta} x^{\delta+\delta} U_1^{-4}.$$  \hspace{1cm} (5.19)

Since $N_1 \geq x^{3/10} > Z^{1/3}$, we have

$$\# A(U_0, U_1, U_2) \ll R^{1/\alpha} |s|^{1+\delta} x^{\delta+\delta} U_1^{-4}$$

from Lemma 5.3 (and, if needed, a partial summation). Next,

$$\# A(U_0, U_1, U_2) \ll R^{1/\alpha} |s|^{1+\delta} x^{\delta+\delta} U_2^{-4}$$

from Lemma 5.3 (if $N_2 > Z^{1/3}$) and Lemma 5.1 applied to $N_2(s, \chi)^2$, if $N_2 \leq Z^{1/3}$. We have also

$$\# A(U_0, U_1, U_2) \ll R^{1/\alpha} x^{\delta+\delta} U_0^{-2}$$

from Lemma 5.1, since $N_0 \ll x^{3/5} \ll x^{\delta}$. Hence

$$\# A(U_0, U_1, U_2) \ll R^{1/\alpha} |s|^{1+\delta} x^{\delta+\delta} (U_1^{-4})^{1/4} (U_2^{-4})^{1/4} (U_0^{-2})^{1/2},$$

and (5.19) follows at once.

Now suppose that $\beta_1 + \beta_2 \leq 3/5$. For some integer $k$, $2 \leq k \leq 12$, we have, by an elementary argument,

$$x^{2/5} \ll \prod_{j=1}^{k} N_j \ll x^{3/5}$$

(compare with [6, Lemma 14]). We now apply Lemma 5.6 with

$$H(s, \chi) = \prod_{j \leq k} N_j(s, \chi) \quad \text{and} \quad K(s, \chi) = \prod_{k<j \leq 12} N_j(s, \chi).$$

Suppose first that $1 < \alpha \leq 2$. We see that (5.17) yields the desired bound

$$U(H, K, \lambda) \ll x^{1/2-\delta} R^{1/\alpha}$$  \hspace{1cm} (5.20)

if

$$\frac{\lambda}{2} > -\frac{1}{8} + \frac{3\delta}{4} + 2\delta,$$

that is,

$$\lambda > \frac{3\delta}{2} - \frac{1}{4} + 4\delta.$$  \hspace{1cm} (5.21)
Suppose that \( \vartheta \leq 13/28 - \varepsilon \). Then

\[
\frac{3\vartheta}{2} - \frac{1}{4} + 4\delta \leq \min \left\{ \frac{9}{20}, \frac{5}{6}(1 - \vartheta) \right\} - \varepsilon
\]

and (5.21) is a consequence of our hypothesis (5.15). This completes the proof of Theorem 1.6 in the case

\[
\frac{26}{23} \leq \alpha < 2.
\]

Now suppose that \( 1 < \alpha < 26/23 \). We get (5.20) from (5.16) provided that

\[
\frac{3}{8} + \frac{\lambda}{4} + \left( \frac{3}{4\alpha} + \frac{1}{4} \right)\vartheta \leq \frac{1}{2} + \frac{\vartheta}{\alpha} - 2\delta
\]

and

\[
\frac{3}{8} - \frac{\lambda}{2} + \frac{11\vartheta}{8} \leq \frac{1}{2} + \frac{\vartheta}{\alpha} - 2\delta.
\]

This gives an interval of \( \lambda \) in which we obtain (5.20), namely

\[
\vartheta \left( \frac{11}{4} - \frac{2}{\alpha} \right) - \frac{1}{4} + 4\delta \leq \lambda \leq \frac{1}{2} - \vartheta \left( 1 - \frac{1}{\alpha} \right) - 8\delta.
\]

Recalling (5.21), we see that for a constant \( \vartheta_0 \), all \( \vartheta \leq \vartheta_0 \) are admissible if

\[
\vartheta_0 \left( \frac{11}{4} - \frac{2}{\alpha} \right) - \frac{1}{4} < \min \left\{ \frac{9}{20}, \frac{5}{6}(1 - \vartheta_0) \right\}
\]

and

\[
\frac{1}{2} - \vartheta_0 \left( 1 - \frac{1}{\alpha} \right) > \frac{3\vartheta_0}{2} - \frac{1}{4}.
\]

The second of these conditions is equivalent to

\[
\vartheta_0 < \frac{3\alpha}{10\alpha - 4}, \tag{5.22}
\]

and one can verify that (5.22) implies the first condition, namely

\[
\vartheta_0 < \min \left\{ \frac{14\alpha}{55\alpha - 40}, \frac{13\alpha}{43\alpha - 24} \right\}.
\]

This completes the proof of Theorem 1.6 for \( 1 < \alpha < 26/23 \).

Now suppose that \( \alpha > 2 \). We obtain the desired bound (5.20) from (5.18) provided that

\[
\frac{3}{8} - \frac{\lambda}{2} + \vartheta \left( \frac{1}{2\alpha} + 1 \right) < \frac{1}{2} + \frac{\vartheta}{\alpha} - 2\delta,
\]
that is,

\[
\lambda > \vartheta \left( 2 - \frac{1}{\alpha} \right) - \frac{1}{4} + 4\delta.
\]

This is a consequence of (5.15) if

\[
\vartheta \left( 2 - \frac{1}{\alpha} \right) - \frac{1}{4} \leq \min \left\{ \frac{9}{20}, \frac{5}{6} (1 - \vartheta) \right\} - 2\varepsilon.
\]

It suffices if

\[
\vartheta \leq \min \left\{ \frac{13\alpha}{34\alpha - 12}, \frac{7\alpha}{20\alpha - 10} \right\} - 2\varepsilon.
\]

Theorem 1.6 now also follows for \( \alpha > 2 \), and the proof is complete.

### 6 | COMMENTS

As we have mentioned, our method also works for \( k = 3 \) if one uses a modification of a result of Hooley [33] given in [18]. Any improvements on that result can potentially make our approach more competitive for \( k = 3 \) as well. Furthermore, it is quite feasible that the method of Browning [16] can be used to obtain a version of Lemma 2.1 for general polynomials and thus enable our method of proof of Theorem 1.4 to work for general polynomial moduli.

Considering only prime moduli, we can extend the level of distribution of Theorem 1.6 up to \( x^{1/2} \). Indeed, we remark that if we assume that for some \( \alpha_0 \), for \( \alpha < \alpha_0 \) the number of primes \( p \) of the form \( p = \lfloor j^\alpha \rfloor \in [R, 2R] \) is of right order of magnitude, that is of cardinality \( R^{1+o(1)} \) we can replace the set \( S_\alpha(R) \) defined by (1.16) by the following set of primes:

\[
\bar{S}_\alpha(R) = \{ \lfloor j^\alpha \rfloor \text{ prime : } j \in \mathbb{N} \} \cap [R, 2R).
\]

Then for the corresponding analogues \( \bar{T}(\mathbf{c}, \lambda) \) of the sums \( T(\mathbf{c}, \lambda) \) defined in (4.9) we only have terms with \( t = 1 \) giving

\[
\bar{T}(\mathbf{c}, \lambda) \leq R^{1/\alpha + 1/4 + o(1)} N^{3/4} \| \mathbf{c} \|^2
\]

(as in (4.8) taken with \( t = 1 \)). In turn, for the following analogue

\[
\bar{U}(H, K) = \sum_{p \in \bar{S}_\alpha(R)} \sum_{\chi \in \chi_p^*} |H(1/2 + it, \chi)K(1/2 + it, \chi)|
\]

of \( U(H, K, \lambda) \) in Lemma 5.6 this leads to

\[
\bar{U}(H, K) \leq x^{3/8} R^{1/\alpha + 1/4 + o(1)} \leq x^{1/2 - \varepsilon/5} R^{1/\alpha},
\]

provided that \( R \leq x^{1/2 - \varepsilon} \), which is what required for our purpose.
By the result of Rivat and Wu [48] we can take

$$\alpha_0 = \frac{243}{205} = 1.1853 \ldots$$

In particular for $\alpha < \alpha_0$ we obtain a version of Corollary 1.7 with any fixed $\delta < 1/2$.

Using integers of the form $\lfloor j^\alpha \rfloor$ without small prime divisors, as, for example, in [1, 9 26], one could derive other versions of Theorem 1.6 with moduli restricted to subsequences of Piatetski–Shapiro integers.

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