NONPERTURBATIVE MODEL OF LIOUVILLE GRAVITY

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ABSTRACT

We obtain nonperturbative results in the framework of continuous Liouville theory. In particular, we express the specific heat $Z$ of pure gravity in terms of an expansion of integrals on moduli spaces of punctured Riemann spheres. The integrands are written in terms of the Liouville action. We show that $Z$ satisfies the Painlevé I.

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1. In this paper we introduce models of Liouville theory in the continuum which are based on the Riemann sphere with punctures. The models include pure gravity. In particular we will show that

\[ Z(t) = t^{-12} \sum_{k=4}^{\infty} t^{5k} \int_{\overline{\mathcal{M}}_{0,k}} \left( i \partial \overline{\partial} S_{cl}^{(k)} \right)^{k-4} \wedge \omega^{F_0} - \frac{t^3}{2} \]  

is the specific heat of pure gravity, namely \( Z \) satisfies the Painlevé I

\[ Z^2(t) - \frac{1}{3} Z''(t) = t. \]  

\( S_{cl}^{(k)} \) in (1) denotes the classical Liouville action on the \( k \)-punctured Riemann sphere. The class \( [\omega^{F_0}] \) is the Poincaré dual of a divisor on the compactified moduli space \( \overline{\mathcal{M}}_{0,k} \) which is given in terms of the \( (2k-8) \)-cycles defining the Deligne-Knudsen-Mumford boundary of \( \overline{\mathcal{M}}_{0,k} \). The basic tools to obtain (1) are classical Liouville theory and intersection theory.

This result reproduces in the continuum the well-known result obtained in the matrix model approach to pure gravity \([\text{1}]\). For reviews on matrix models and 2D gravity see \([\text{2}]\).

2. The problems arising in the continuum formulation of Liouville gravity \([\text{3,4,5}]\) are essentially:

a. To evaluate Liouville correlators on Riemann surfaces of genus \( h > 2 \);

b. To perform the integration on moduli spaces;

c. To recover nonperturbative results from the topological expansion.

Results from matrix models and topological gravity show that these aspects are strictly related with the structure of \( \mathcal{M}_h \equiv \mathcal{M}_{h,0} \) (we denote by \( \mathcal{M}_{h,n} \) the moduli spaces of Riemann surfaces of genus \( h \) and \( n \) punctures). In particular, it turns out that the Liouville action is the Kähler potential for the natural (Weil-Petersson) metric on the moduli space. Also CFT is strictly related with the geometry of moduli space. For example, the Mumford isomorphism

\[ \lambda_n = \det \mathrm{ind} \overline{\nabla}_n \sim \lambda_1^{c_n}, \quad c_n = 6n^2 - 6n + 1, \]

where \( \lambda_n = \det \mathrm{ind} \overline{\nabla}_n \) are the determinant line bundles, connects geometrical properties of \( \mathcal{M}_h \) with the central charge \( d = -2c_n \) of a weight \( n \), \( b-c \) system (notice that \( d \leq 1 \)). Actually, the bosonization of \( b-c \) systems can be used to reproduce the Coulomb gas formulation of \( d \leq 1 \) conformal matter. For \( d > 1 \) it is not possible to represent conformal matter by a \( b-c \) system. In this case one can consider the \( \beta-\gamma \) system of weight \( n \) whose central charge
is $2c_n$. However, the representation of the $\beta$-$\gamma$ system in terms of free fields is a long-standing problem which seems related to the $d = 1$ barrier. These aspects indicate that there is a connection between the barrier and the Mumford isomorphism. This is related to a similar structure considered in [3] in the framework of the geometrical formulation of 2D gravity [3,7] where representing elliptic and parabolic Liouville operators by means of a scalar field constrains the conformal matter to be in the sector $d \leq 1$.

The natural framework to investigate the aspects considered above is the theory of uniformization of Riemann surfaces where Liouville theory plays a crucial role. Actually, in [8] it has been shown that the Liouville action appears in the correlators (intersection numbers) of topological gravity [9]. The relationships between Liouville theory, matrix models and topological gravity suggest that it is possible to extend the above Liouville-topological gravity relationship by recovering the nonperturbative results of matrix models by continuum Liouville theory. In our model we will reduce all aspects concerning higher genus contributions to punctured spheres. The reduction to punctured sphere has been considered also by V.G. Knizhnik who expressed the sum of the genus expansion as a CFT on an arbitrary $N$-sheet covering of the Riemann sphere with branch points. For each branch point he associated a vertex operator and proposed to express the infinite sum on all genus ($h \geq 2$) as the limit for $N \to \infty$ of a ‘nonperturbative’ partition function [10].

A natural way to get punctured spheres is by pinching all handles of a compact Riemann surface. Degenerate (singular) surfaces belong to the boundary of moduli spaces. These singularities play a fundamental role in the evaluation of relevant integrals (intersection theory). The fact that the classical Liouville action is the Kähler potential for the Weil-Petersson metric and the structure of the boundary of moduli space suggest to consider integrals on $\overline{M}_h$ in the framework of the Duistermaat-Heckman integration formula [11]. The final result should be a sum of integrals $Z_n^F$ on the moduli space of punctured Riemann spheres $\mathcal{M}_{0,n} = \left(\hat{C} \backslash \Delta_n\right) / Symm(n) \times PSL(2, \mathbb{C})$ with the integrands involving the Liouville action. These remarks indicate that a theory à la Friedan-Shenker [12] can be concretely formulated to recover nonperturbative results in the continuum formulation. In this paper, we do not consider points a-c separately, rather we state the final solution finding the explicit form of the integrals $Z_n^F$ on $\overline{M}_{0,n}$.

The reduction to punctured sphere is particularly evident in topological field theory coupled to 2D gravity where higher genus contributions to the free energy $\langle 1 \rangle_h$ can be written in terms of the sphere amplitudes of the puncture operator $P$ [9,13].
observables of the theory are the primary fields \( \mathcal{O}_\alpha (\alpha = 0, 1, \ldots, N - 1, \mathcal{O}_0 \text{ is the identity operator}) \) and their gravitational descendents \( \sigma_n (\mathcal{O}_\alpha), n = 1, 2, \ldots \). In the coupled system \( \mathcal{O}_0 \) becomes non-trivial and it is identified with \( P \). Denoting by \( \mathcal{L}_0 \) the minimal Lagrangian, the more general one is \( \mathcal{L} = \mathcal{L}_0 + \sum_{n, \alpha} t_{n, \alpha} \sigma_n (\mathcal{O}_\alpha), \sigma_0 (\mathcal{O}_\alpha) \equiv \mathcal{O}_\alpha \), where \( t_{n, \alpha} \) are coupling constants. With this definition one can compute correlation functions with an insertion of \( \sigma_k \) just by differentiating \( \langle 1 \rangle_h \) with respect to \( t_k \). Thus in general

\[
\langle \sigma_{d_1} (\mathcal{O}_{\alpha_1}) \cdots \sigma_{d_n} (\mathcal{O}_{\alpha_n}) \rangle_h = \frac{\partial}{\partial t_{d_1, \alpha_1}} \cdots \frac{\partial}{\partial t_{d_n, \alpha_n}} \langle 1 \rangle_h.
\]

Therefore \( \langle 1 \rangle_h \) is the crucial quantity to compute. By means of KdV recursion relations

\[
\langle \sigma_1 (P) \rangle_h = 2\langle P^4 \rangle_{h-1} + \frac{1}{2} \sum_{h'=0}^h \langle P^2 \rangle_{h'} \langle P^2 \rangle_{h-h'},
\]

it is possible \(^3\) to express \( \langle 1 \rangle_h \) as a sum of terms of the form \( \langle P^{n_1} \rangle_0 \cdots \langle P^{n_j} \rangle_0 / \langle P^3 \rangle_0^{h+j-1} \) for \( 1 \leq j \leq 3h - 3 \) with the constraint \( \sum_{k=1}^j n_k = 3(j + h - 1) \).

The reduction to the punctured sphere arises also in the evaluation of \( \text{Vol}_{WP} (\mathcal{M}_{h,n}) \). Indeed, at least in some cases, there is a relationship between \( \mathcal{M}_{h,n} \), \( \mathcal{M}_{0,n+3h} \) and their volumes\(^4\). The first example is the geometric isomorphism \(^{14} \) \( \mathcal{M}_{1,1} \cong \mathcal{M}_{0,4} \), and

\[
\text{Vol}_{WP} (\mathcal{M}_{1,1}) = 2\text{Vol}_{WP} (\mathcal{M}_{0,4}). \tag{3}
\]

To understand this result it is sufficient to recall that the \( \wp \)-function enters in the expression of the uniformizing connection of the once punctured torus \( \Sigma_{1,1} \) (note that \( \wp \) is a solution of the KdV equation)

\[
T_{\Sigma_{1,1}} = \frac{1}{2} (\wp(\tau, z) + c(\tau)),
\]

where \( c(\tau) \) is the accessory parameter for \( \Sigma_{1,1} \). Eq.(3) follows from the fact that \( T_{\Sigma_{1,1}} \) is strictly related to the uniformizing connection \( T_{\Sigma_{0,4}} \) of the Riemann sphere with four punctures since \( \wp \) maps \( \Sigma_{1,1} \) two-to-one onto the four punctured Riemann sphere. Let us notice that another isomorphism is \(^{15} \) \( \mathcal{M}_{2,0} \cong \mathcal{M}_{0,6} \).

There is another way to understand why punctured spheres play a crucial role in 2D gravity. The point is to notice that equal size triangulated Riemann surfaces considered in matrix models can be realized in terms of thrice punctured spheres\(^6\). This aspect is related

\(^1\) The space \( \mathcal{M}_{h,n} \) is not affine for \( h > 2 \). Conversely the space \( \mathcal{M}_{0,k} \) is finitely covered by the affine space \( V^{(k)} \) defined in \(^6\). Thus for \( h > 2 \) there are not geometrical isomorphisms between \( \mathcal{M}_{h,n} \) and \( \mathcal{M}_{0,n+3h} \). However, in principle, nothing exclude the possibility to express \( \text{Vol}_{WP} (\mathcal{M}_{h,n}) \) in terms of \( \text{Vol}_{WP} (\mathcal{M}_{0,n+3h}) \).
to arithmetic surfaces theory \[16,17\]. In this context one should investigate whether this kind of surfaces have some suitable symmetry to define antiholomorphic involution. This question is important in order to investigate Osterwalder-Schrader positivity. This is connected with the problem of defining the adjoint in higher genus. On the sphere it can be done thanks to the natural antinvolution \(z \rightarrow \bar{z}^{-1}\). In higher genus this problem has been solved only on a Schottky double where there is a natural antinvolution \[18\]. Recently Harvey and González Diéz \[19\] have considered loci of curves which are prime Galois covering of the sphere. In particular they considered the important case of Riemann surfaces admitting non-trivial automorphisms and showed that there is a birational isomorphism between a subset of the moduli space \(\mathcal{M}_h\) and \(V^{(n)}\) (defined in (3)).

3. The relation between Liouville and uniformization theory of Riemann surfaces arises in considering the Liouville equation

\[
\partial_z \partial_{\bar{z}} \varphi_{cl} = \frac{e^{\varphi_{cl}}}{2},
\]

which is uniquely satisfied by the Poincaré metric (i.e. the metric with Gaussian curvature \(-1\)). This metric can be written in terms of the inverse of the uniformizing map \(J_H\), that is \(e^{\varphi_{cl}} = \left| J_H^{-1} \right|^2 \left( \text{Im} \ J_H^{-1} \right)^2\), \(J_H : H \rightarrow \Sigma \cong H/\Gamma\) where \(H\) is the upper half-plane and \(\Gamma\) a Fuchsian group. Let us introduce the \(n\)-punctured sphere \(\Sigma = \hat{C} \setminus \{z_1, \ldots, z_n\}\), \(\hat{C} \equiv \mathbb{C} \cup \{\infty\}\). Its moduli space is the space of classes of isomorphic \(\Sigma\)'s, that is

\[
\mathcal{M}_{0,n} = \{(z_1, \ldots, z_n) \in \hat{C}^n | z_j \neq z_k \text{ for } j \neq k\}/\text{Symm}(n) \times \text{PSL}(2, \mathbb{C}),
\]

where \(\text{Symm}(n)\) acts by permuting \(\{z_1, \ldots, z_n\}\) whereas \(\text{PSL}(2, \mathbb{C})\) acts by linear fractional transformations. By \(\text{PSL}(2, \mathbb{C})\) we can recover the ‘standard normalization’: \(z_{n-1} = 0\), \(z_{n-2} = 1\) and \(z_n = \infty\). Furthermore, without loss of generality, we assume that \(w_{n-2} = 0\), \(w_{n-1} = 1\) and \(w_n = \infty\). For the classical Liouville tensor we have

\[
T^F(z) = \sum_{k=1}^{n-1} \left( \frac{1}{2(z - z_k)^2} + \frac{c_k}{z - z_k} \right), \quad \lim_{z \rightarrow \infty} T^F(z) = \frac{1}{2z^2} + \frac{c_n}{z^3} + O \left( \frac{1}{|z|^4} \right),
\]

with the following constraints on the \textit{accessory parameters}

\[
\sum_{k=1}^{n-1} c_k = 0, \quad \sum_{k=1}^{n-1} c_k z_k = 1 - n/2, \quad \sum_{k=1}^{n-1} z_k(1 + c_k z_k) = c_n.
\]

The \(c_k\)'s are functions on

\[
V^{(n)} = \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} | z_j \neq 0, 1; z_j \neq z_k, \text{ for } j \neq k\}.
\]
Note that
\[ \mathcal{M}_{0,n} \cong V^{(n)}/\text{Symm}(n), \]
where the action of \( \text{Symm}(n) \) on \( V^{(n)} \) is defined by comparing (7) with (2).

Let us now consider the compactification divisor (in the sense of Deligne-Knudsen-Mumford) \( D = \overline{V}^{(n)} \setminus V^{(n)} \). This divisor decomposes in the sum of divisors \( D_1, \ldots, D_{\lfloor n/2 \rfloor - 1} \) which are subvarieties of real dimension \( 2n - 8 \). The locus \( D_k \) consists of surfaces that split, on removal of the node, into two Riemann spheres with \( k + 2 \) and \( n - k \) punctures. In particular \( D_k \) consists of \( C(k) \) copies of the space \( \overline{V}^{(k+2)} \times \overline{V}^{(n-k)} \) where \( C(k) = \binom{n}{k+1} \) for \( k = 1, \ldots, \frac{n-1}{2} - 1 \), with the exception that for \( n \) even \( C(\frac{n}{2}) = \frac{1}{2} \binom{n}{n/2} \). It turns out that the image of the \( D_k \)'s, provide a basis in \( H_{2n-8}(\overline{V}_{0,n}, \mathbb{R}) \).

In the case of the punctured Riemann sphere eq. (4) follows from the Liouville action
\[ S^{(n)} = \lim_{r \to 0} \left[ \int_{\Sigma_r} (\partial_z \varphi \partial_{\bar{z}} \varphi + e^\varphi) + 2\pi (n \log r + 2(n-2) \log |\log r|) \right], \]
where \( \Sigma_r = \Sigma \setminus \left( \bigcup_{i=1}^{n-1} \{ z \mid |z - z_i| < r \} \cup \{ z \mid |z| > r^{-1} \} \right) \). This action, evaluated on the classical solution, is the Kähler potential for the Weil-Petersson two-form on \( V^{(n)} \)
\[ \omega^{(n)}_{WP} = \frac{i}{2} \partial \bar{\partial} S^{(n)}_3 = -i\pi \sum_{j,k=1}^{n-3} \frac{\partial c_k}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_k. \]

Let us consider the volume of moduli space of punctured Riemann spheres
\[ \text{Vol}_{WP}(\mathcal{M}_{0,n}) = \frac{1}{(n-3)!} \int_{\mathcal{M}_{0,n}} \omega^{(n)}_{WP}^{n-3} = \frac{1}{(n-3)!} \left[ \omega^{(n)}_{WP} \right]^{n-3} \cap \left[ \overline{\mathcal{M}}_{0,n} \right] . \]

Recently it has been shown that
\[ \text{Vol}_{WP}(\mathcal{M}_{0,n}) = \frac{1}{n!} \text{Vol}_{WP} \left( V^{(n)} \right) = \frac{\pi^{2(n-3)} V_n}{n!(n-3)!}, \quad n \geq 4, \]
where \( V_n = \pi^{2(3-n)} \left[ \omega^{(n)}_{WP} \right]^{n-3} \cap \left[ \overline{V}^{(n)} \right] \) satisfies the recursion relations
\[ V_3 = 1, \quad V_n = \frac{1}{2} \sum_{k=1}^{n-3} \frac{n-k-2}{n-1} \binom{n}{k+1} \binom{n-4}{k-1} V_{k+2} V_{n-k}, \quad n \geq 4. \]

Remarkably the basic tools in the computation of the volumes are classical Liouville theory and intersection theory.

4. We now consider the differential equation associated with (9). First of all we define
\[ a_k = \frac{V_k}{(k-1)((k-3)!)^2}, \quad k \geq 3, \]
where
so that (9) becomes

\[ a_3 = 1/2, \quad a_n = \frac{1}{2} \frac{n(n-2)}{(n-1)(n-3)} \sum_{k=1}^{n-3} a_{k+2} a_{n-k}, \quad n \geq 4. \]  

(11)

Eq. (11) is equivalent to the differential equation

\[ g'' = \frac{g^2 t - gg' + g't}{t(t - g)}, \]  

(12)

where \( g(t) = \sum_{k=3}^{\infty} a_k t^{k-1} \). Notice that by (8)

\[ g(t) = \sum_{k=3}^{\infty} \frac{k(k-2)}{(k-3)!} t^{k-1} \int_{\mathcal{M}_{0,k}} \left( \frac{i\partial \bar{\partial} S_{cl}^{(k)}}{2\pi^2} \right)^{k-3}, \]  

(13)

where \( \int_{\mathcal{M}_{0,1}} 1 = \frac{1}{6} \). Function \( g(t) \) resembles a topological expansion of string theory. Furthermore the structure of eq.(12) resembles the Painlevé I. These remarks indicate that it is possible to recover the specific heat of pure gravity in the continuum. Actually, we will recover the Painlevé I by classical Liouville theory. In particular we will get the recursion relations for the Painlevé I by performing a suitable modification of the Weil-Petersson volume form \( \omega_{WP}^{(n)} \). Remarkably, as we will show, it is possible to perform the substitution

\[ \omega_{WP}^{(n)} \rightarrow \omega_{WP}^{(n-4)} \wedge \omega^F, \]  

in (13) without changing the general structure of (11); that is we will get recursion relations of the following structure

\[ A_n = C(n) \sum_{k=1}^{n-3} A_{k+2} A_{n-k}, \quad n \geq 4. \]  

(14)

The first problem is to find a suitable expansion for the Painlevé I field such that the structure of the associated recursion relation be the same of (14). Remarkably this expansion exists, namely

\[ f(t) = t^{-12} \sum_{k=3}^{\infty} d_k t^{5k}. \]  

(15)

It is interesting that in searching the expansion reproducing the general structure of (11), which is a result obtained from continuous Liouville theory, one obtains an expansion involving only positive powers of \( t \). With this expansion the Painlevé I

\[ f^2(t) - \frac{1}{3} f''(t) = t, \]  

(16)
is equivalent to the recursion relations\footnote{Notice that \((-1)^k d_k\) is positive.}
\[
d_n = \frac{3}{(12 - 5n)(13 - 5n)} \sum_{k=1}^{n-3} d_{k+2} d_{n-k}, \quad d_3 = -1/2, \tag{17}
\]
which has the same structure of (11). We now investigate on the possible volume forms reproducing (17). To understand which kind of modification to \(\omega^{(n)}_{WP}\) can be performed without changing the basic structure of (11) we recall basic steps in \cite{21} to obtain (9).

Let \(D_{WP}\) be the \((2n - 8)\)-cycle dual to the Weil-Petersson class \([\omega^{(n)}_{WP}]\). To compute the volumes it is useful to expand \(D_{WP}\) in terms of the divisors \(D_k\) in the boundary of moduli space. It turns out that \cite{21}
\[
D_{WP} = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n - k - 2) D_k. \tag{18}
\]

Let us set
\[
\tilde{V}_n = \pi^{2(n-3)} V_n = \left[\omega^{(n)}_{WP}\right]^{n-3} \cap \left[\nabla^{(n)}\right] = \left[\omega^{(n)}_{WP}\right]^{n-4} \cap \left(\left[\omega^{(n)}_{WP}\right] \cap \left[\nabla^{(n)}\right]\right).
\]

On the other hand \([\omega^{(n)}_{WP}] \cap \left[\nabla^{(n)}\right] = D_{WP} \cdot \nabla^{(n)} = D_{WP}\), so that by (18)
\[
\tilde{V}_n = \left[\omega^{(n)}_{WP}\right]^{n-4} \cap \left[D_{WP}\right] = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n - k - 2) \left[\omega^{(n)}_{WP}\right]^{n-4} \cap \left[D_k\right].
\]

Since \(D_k\) consists of \(C(k)\) copies of the space \(\nabla^{(k+2)} \times \nabla^{(n-k)}\), we have
\[
\tilde{V}_n = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n - k - 2) C(k) \left[\omega^{(n)}_{WP}\right]^{n-4} \cap \left[\nabla^{(k+2)} \times \nabla^{(n-k)}\right].
\]

Finally, since \cite{21}
\[
\left[\omega^{(n)}_{WP}\right]^{n-4} \cap \left[\nabla^{(k+2)} \times \nabla^{(n-k)}\right] = \left[\omega^{(k+2)}_{WP} + \omega^{(n-k)}_{WP}\right]^{n-4} \cap \left[\nabla^{(k+2)} \times \nabla^{(n-k)}\right], \tag{19}
\]
it follows that
\[
\tilde{V}_3 = 1, \quad \tilde{V}_n = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n - k - 2) C(k) \binom{n-4}{k-1} \tilde{V}_{k+2} \tilde{V}_{n-k}, \quad n \geq 4,
\]
which coincides with (9).
We now introduce the divisor

\[ D^F = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n-k-2) F(n,k) D_k, \]  

(20)

where \( F(n,k) \) is a function to be determined. Let \( [\omega^F] \) be the Poincaré dual to \( D^F \) and define

\[ Z^F_n = \int_{\mathcal{M}_{0,n}} \omega_{WP}^{(n)} n^{-4} \wedge \omega^F = \int_{\mathcal{M}_{0,n}} \left( \frac{i\partial\partial S^{(n)}_{cl}}{2} \right)^{n^{-4}} \wedge \omega^F, \quad n \geq 4. \]  

(21)

An important aspect of (21) is that we can use the recursion relations (11) to obtain non-perturbative results. This possibility is based on the obvious, but important fact, that \( [\omega_{WP}^{(n)} n^{-3} \cap \nabla^{(n)}] = [\omega_{WP}^{(n)} n^{-4} \cap [D_{WP} \implies \text{that the general structure of (11) (the same of (17)) is unchanged under the substitution} \omega_{WP}^{(n)} n^{-3} \cap \nabla^{(n)} = \omega_{WP}^{(n)} n^{-4} \cap \omega^F. \) To see this note that

\[ Z^F_n = \frac{1}{n!} [\omega_{WP}^{(n)} n^{-4} \cap [D^F] = \frac{\pi^2}{(n-1)n!} \sum_{k=1}^{[n/2]-1} F(n,k) k(n-k-2) [\omega_{WP}^{(n)} n^{-4} \cap [D_k], \]  

(22)

On the other hand by (19)

\[ \sum_{k=1}^{[n/2]-1} F(n,k) k(n-k-2) [\omega_{WP}^{(n)} n^{-4} \cap [D_k] = \frac{1}{2} \sum_{k=1}^{n-3} F(n,k) \frac{k(n-k-2)}{n-1} \left( \frac{n}{k+1} \right) \left( \frac{n-4}{k-1} \right) V_{k+2} V_{n-k}, \]  

(23)

and by (10)

\[ Z^F_n = \frac{\pi^2 (n-4)!}{2(n-1)} \sum_{k=1}^{n-3} F(n,k) a_{k+2} a_{n-k}, \quad n \geq 4. \]  

(24)

Let us define the ‘Liouville F-models’

\[ Z^{F,\alpha}(x) = x^{-\alpha} \sum_{k=3}^{\infty} x^k Z^F_k, \]  

(25)

where \( x \) is the coupling constant. These models are classified by \( \alpha, F(n,k) \) and \( Z^F_3 \). We now show that \( Z^{F,\alpha}(x) \) includes pure gravity. In fact, putting

\[ Z(t) = Z^{F_0,\alpha}(t^5), \quad Z_{3}^{F_0} = -1/2, \quad \alpha_0 = 12/5, \]  

(26)

where

\[ F_0(n,k) = \frac{6}{\pi^2 (12-5n)(13-5n)(n-4)!} \frac{Z_{k+2}^0 Z_{n-k}^0}{a_{k+2} a_{n-k}}, \]  

(27)
we have, by (24) and (27),
\[ Z_{F_0}^3 = -\frac{1}{2}, \quad Z_{F_0}^n = \frac{3}{(12 - 5n)(13 - 5n)} \sum_{k=1}^{n-3} Z_{F_0}^{k+2} Z_{F_0}^{n-k}, \quad n \geq 4, \tag{28} \]
so that by (15)-(17)
\[ Z(t) = t^{-12} \sum_{k=4}^{\infty} t^{5k} \int_{M_0,k} \left( \frac{i\partial \partial S^{(k)}}{2} \right)^{k-4} \wedge \omega^F - \frac{t^3}{2}, \tag{29} \]
satisfies the Painlevé I
\[ Z^2(t) - \frac{1}{3} Z''(t) = t. \tag{30} \]

5. In conclusion we have introduced a class of Liouville models by defining a suitable $D^F$ divisor. These Liouville $F$-models (LFM) include pure gravity. In this context we recall that the Liouville action arises also in the correlators of topological gravity [8].

Punctures correspond to real points on the boundary of the upper half-plane. Correspondingly one can define hyperelliptic Riemann surfaces. In the case of infinite genus one gets the McKean-Trubowitz [22] model which is related to matrix model. This suggests a nonperturbative formulation on $H$ with the image of punctures related to the eigenvalues of the Hermitian matrix models. In the discrete version of this approach one should be able to connect this formulation with the ideas at the basis of [23].

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