A delayed computer virus model with nonlinear incidence rate

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1. Introduction

The computer network has brought great convenience to our daily life. While enjoying the convenience from the computer network, we have to confront the threat of computer virus intrusions (Kumar, Mishra, & Panda, 2016; Ren & Xu, 2014). The file system is mostly attacked by the computer viruses which use system vulnerability to attack computers in networks. Based on Cybercrime-Report (2017), cybercriminal activity is one of the biggest challenges that humanity will face in the next two decades and Cybersecurity Ventures predicts that cybercrime will cost the world in excess of $6 trillion annually by 2021. Thus, conducting research on computer virus propagation dynamics is of considerable interest to elevate the computer network security. So far, a wide variety of models have been proposed to simulate the behaviour of computer virus throughout networks since the macroscopic model featuring the spread of computer viruses formulated by Kephart, White, and Chess (1993). In addition, researchers have paid attention to the combination of computer virus model and antivirus countermeasures such as quarantine and virus immunization, in order to analyse and counter against computer viruses. In Mishra, Srivasstava, and Mishra (2014), Khanh (2016), Nwokoye, Ozoegwu, and Ejiofor (2017), worm propagation models with quarantine in wireless sensor network to explore the spreading law of worms were investigated. Xiao et al. (2017) studied a worm propagation model with quarantine in mobile internet. Fatima, Ali, Ahmed, and Rafiq (2018) proposed a computer virus epidemic model with quarantine and infectivity in latent period. Mishra and Keshri (2013), Mishra and Tyagi (2014) formulated the malicious code propagation models with immunization to describe the dynamics of malicious code propagation in wireless sensor network. Worm propagation models with immunization were proposed by Nwokoye and Umeh (2017), Singh, Awasthi, Singh, and Srivastava (2018) who considered the communication radius and node density of wireless sensor network.

All the models mentioned above use the bilinear incidence rate to describe the transmission process of computer viruses, which is only suitable for the case that the proportion of the infected computers in a small network. To address the issue of bilinear incidence rate, Upadhyay, Kumari, and Misra (2017) proposed an Susceptible-Vaccinated-Exposed-Infectious-Recovered (SVEIR) computer virus model with nonlinear incidence rate and immunization. They also studied the stability and the persistence of the model. However, Upadhyay et al. (2017) neglected time delays in the process of computer virus propagation. Robustness of computer networks depends on their stability. If a computer network is stable then it will work properly. Time delays may lead to Hopf bifurcation and make the computer virus model become unstable, which may result in a crash of the entire computer network. Hence, it is important to know the critical point at which a computer virus model changes its stability. The analysis of Hopf bifurcation can ensure that the model is stable. To this end, Zhao, Zhang, and Upadhyay (2018) analysed effect of the time delay due to the period the antivirus software uses to clean the viruses on the SVEIR computer virus model with nonlinear
incidence rate proposed by Upadhyay et al. (2017). It should be pointed out that Zhao et al. (2018) omitted the latent period delay and the temporary immunity period delay in the SVEIR computer virus model. As is known to all, there is usually a latent period for the exposed nodes to obtain infectious capacity. On the other hand, the vaccinated nodes will lose their immunity when the new computer viruses appear and it needs a period to develop the new computer viruses. In Zhang and Bi (2015), Zhang and Bi investigated a delayed computer virus model with the effect of external computers and studied the Hopf bifurcation of the model by choosing the latent period delay as the bifurcation parameter. In Ren and Bi (2017), Ren et al. formulated a delayed SIRS (Susceptible-Infectious-Recovered-Susceptible) computer virus propagation model and analysed the existence of the Hopf bifurcation by choosing the different combinations of the latent period delay and the temporary immunity period delay as the bifurcation parameter. Subsequently, Muroya, Enatsu, and Li (2014) investigated global stability and permanence of the delayed SIRS computer virus propagation model formulated by Ren et al. (2012). Zhao and Bi (2017) also studied the Hopf bifurcation of the model by choosing the time delay due to the latent period and the time delay due to the latent period and the time delay due to the latency period delay in the SVEIR computer virus model. In Zhang et al. (2015), Zhang and Bi investigated a delayed computer virus propagation model with the latent period delay in the SVEIR computer virus model. In Zhang et al. (2017), Zhang et al. investigated a delayed computer virus propagation model with the latent period delay. In Upadhyay et al. (2017), Upadhyay et al. studied the Hopf bifurcation when the two delays are not equal. Motivated by the work about computer virus models with time delay in Zhao et al. (2018), Ren et al. (2012), Muroya et al. (2014), Zhao and Bi (2017), Zhao et al. (2018), we are concerned with another form of delayed SVEIR computer virus model including two delays based on the work in Upadhyay et al. (2017).

In this paper we consider a delayed SVEIR computer virus model with nonlinear incidence rate and immunization proposed by Upadhyay et al. (2017) as follows:

\[
\begin{align*}
\frac{dS(t)}{dt} &= A - \delta_0 S(t) - \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} + \eta V(t) - \mu S(t), \\
\frac{dE(t)}{dt} &= \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} - (\delta_0 + \delta_1)E(t), \\
\frac{dI(t)}{dt} &= \delta_1 E(t) - (\delta_0 + \delta_2 + \delta_3)I(t) - \frac{\beta I(t)}{l(t) + a}, \\
\frac{dR(t)}{dt} &= \delta_2 I(t) - \delta_0 R(t) + \frac{\beta I(t)}{l(t) + a}, \\
\frac{dV(t)}{dt} &= \mu S(t) - (\delta_0 + \eta)V(t).
\end{align*}
\]

in which the computers are divided into five subclasses: the susceptible computers (S), the exposed computers (E), the infectious computers (I), the vaccinated computers (V). S(t), E(t), I(t), R(t) and V(t) denote the numbers of S, E, I, R and V computers at time t, respectively. The detailed meanings of the bifurcating periodic solutions are obtained. In Section 4, a numerical simulation is presented to support our theoretical results. Finally in Section 5, we end our paper with a conclusion.

2. Preliminaries and model formulation

2.1. Preliminaries

i. Notations

The notations used in this paper are defined as follows if not otherwise stated. \( R^n \) denotes the set of real numbers and integers, respectively. \( R^n \) denotes n-dimensional space. \( C^1 \) denotes the set of functions which have continuous first derivative. \( C([-1, 0], R^n) \) denotes the set of continuous functions.

Let \( \dot{x}(t) = F(\mu_0, x_t) \) be a dynamical system. \( F(\mu_0, \varphi) \in C^1, \varphi \in C([-1, 0], R^n), \mu_0 \in R, F(\mu_0, 0) = 0 \) for each \( \mu_0 \in R \). Define the operator \( L(\mu_0) \varphi = F_{\varphi}(\mu_0, 0) \), in which \( F_{\varphi}(\mu_0, 0) \) is the derivative of \( F_{\varphi}(\mu_0, \varphi) \) at \( \varphi = 0 \).

ii. Hopf bifurcation theorem

Lemma 2.1 (Hassard, Kazarinoff, & Wan, 1981): For \( \mu_0 = 0 \), if the characteristic equation of the linear part of \( \dot{x}(t) = F(\mu_0, x_t) \) has a pair of purely imaginary roots \( \pm i \lambda_0 \) with \( \lambda_0 \neq 0 \), and no other root that is an integer multiple of \( i \lambda_0 \), and if the derivative of real part of characteristic root at \( \mu_0 = 0 \) is not equal to 0, that is \( \text{Re} \lambda'(0) \neq 0 \), then a Hopf bifurcation will occur around the equilibrium of \( \dot{x}(t) = F(\mu_0, x_t) \).

2.2. Model formulation

The SVEIR computer virus model with nonlinear incidence rate and immunization proposed by Upadhyay et al. (2017) is as follows:

\[
\begin{align*}
\frac{dS(t)}{dt} &= A - \delta_0 S(t) - \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} + \eta V(t) - \mu S(t), \\
\frac{dE(t)}{dt} &= \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} - (\delta_0 + \delta_1)E(t), \\
\frac{dI(t)}{dt} &= \delta_1 E(t) - (\delta_0 + \delta_2 + \delta_3)I(t) - \frac{\beta I(t)}{l(t) + a}, \\
\frac{dR(t)}{dt} &= \delta_2 I(t) - \delta_0 R(t) + \frac{\beta I(t)}{l(t) + a}, \\
\frac{dV(t)}{dt} &= \mu S(t) - (\delta_0 + \eta)V(t).
\end{align*}
\]
where

For this reason, we incorporate the latent period delay into system (1) and investigate the following computer virus model with two delays:

\[
\frac{dS(t)}{dt} = A - \delta_0 S(t) - \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} + \eta V(t - \tau_1) - \mu S(t),
\]

\[
\frac{dE(t)}{dt} = \frac{\alpha S(t)I(t)}{S(t) + I(t) + c} - \delta_0 E(t) - \delta_1 E(t - \tau_2),
\]

\[
\frac{dI(t)}{dt} = \delta_1 E(t - \tau_2) - (\delta_0 + \delta_2 + \delta_3)I(t) - \frac{\beta I(t)}{l(t) + a'},
\]

\[
\frac{dR(t)}{dt} = \delta_2 l(t) - \delta_0 R(t) + \frac{\beta I(t)}{l(t) + a'},
\]

\[
\frac{dV(t)}{dt} = \mu S(t) - \delta_0 V(t) - \eta V(t - \tau_1),
\]

where \( \tau_1 \) is the time delay due to the temporary immunity period and \( \tau_2 \) is the time delay due to the latent period.

To obtain the results about the Hopf bifurcation of system (2), some assumptions are listed in the following for clarity.

**Assumption 2.1:** \( A_{00} > 0, A_{03}A_{04} > A_{02}, A_{02}(A_{01} + A_{03} A_{04}) > A_{01}A_{02} + A_{02}^2 \) and \( A_{00}A_{02}A_{03} + A_{01}A_{04}(A_{01} + 2A_{00}) > A_{00}^2 A_{02}^2 + A_{01}A_{02}^2 + A_{00}A_{03}A_{04} \), where

\[
A_{00} = A_0 + B_0 + C_0 + D_0, \quad A_{01} = A_1 + B_1 + C_1 + D_1, \quad A_{02} = A_2 + B_2 + C_2 + D_2, \quad A_{03} = A_3 + B_3 + C_3 + D_3, \quad A_{04} = A_4 + B_4 + C_4.
\]

with

\[
A_0 = -a_1 a_4 a_6 a_8 a_{10}, \quad A_1 = a_4 a_{10}(a_1 a_6 + a_1 a_8 + a_6 a_8) + a_1 a_6 a_8 (a_4 + a_{10}), \quad A_2 = -(a_1 a_6 a_8 + a_4 a_{10}(a_1 + a_6 + a_8)) + (a_4 + a_{10})(a_1 a_6 + a_1 a_8 + a_6 a_8), \quad A_3 = a_1 a_6 + a_1 a_8 + a_6 a_8 + a_4 a_{10} + (a_4 + a_{10})(a_1 + a_6 + a_8), \quad A_4 = -(a_1 + a_6 + a_8 + a_{10}), \quad B_0 = -a_4 a_6 a_8(a_1 b_2 + a_9 b_1), \quad B_1 = (a_1 b_2 + a_9 b_1)(a_4 a_6 + a_4 a_8 + a_6 a_8) + a_4 a_6 a_8 b_2, \quad B_2 = -((a_9 b_1 + a_1 b_2)(a_4 + a_6 + a_8) + b_2(a_4 a_6 + a_4 a_8 + a_6 a_8)), \quad B_3 = b_2(a_1 + a_4 + a_6 + a_8) - a_9 b_1, \quad B_4 = -b_2, \quad C_0 = a_8 a_{10}(a_1 a_5 c_2 - a_1 a_6 c_1 - a_2 a_3 c_2), \quad C_1 = a_1 a_5 a_8 c_1 + a_2 a_3 c_2(a_8 + a_{10}) - a_5 c_2(a_1 a_8 + a_1 a_{10} + a_8 a_{10}) + a_10 c_1(a_1 a_6 + a_1 a_8 + a_6 a_8), \quad C_2 = a_5 c_2(a_1 + a_8 + a_{10}) - c_1(a_1 a_6 + a_1 a_8 + a_6 a_8) - a_2 a_3 c_2 + a_10(a_1 + a_6 + a_8), \quad C_3 = c_1(a_1 + a_6 + a_8 + a_{10}), \quad C_4 = -c_1, \quad D_0 = a_5 a_8 b_2 c_2 + a_9 b_1(a_5 a_6 c_2 - a_6 a_8 c_1) - a_8 b_2(a_2 a_3 c_2 + a_1 a_6 c_1), \quad D_1 = b_2 c_1(a_1 a_6 + a_1 a_8 + a_6 a_8) + a_9 b_1 c_1(a_8 + a_8) - a_5 b_2 c_2(a_1 + a_8), \quad D_2 = b_2 c_1(a_1 + a_6 + a_8) + a_5 b_2 c_2 - a_9 b_1 c_1, \quad D_3 = b_2 c_1,
\]

and

\[
a_1 = -\left[ \frac{\delta_0 + \mu + \frac{a l_e (l_e + c)}{(S_e + l_e + c)^2}}{S_e + l_e + c} \right],
\]

\[
a_2 = -\frac{a S_e (S_e + c)}{(S_e + l_e + c)^2},
\]

\[
a_3 = \frac{a l_e (l_e + c)}{(S_e + l_e + c)^2}, \quad a_4 = -\delta_0,
\]

\[
a_5 = \frac{a S_e (S_e + c)}{(S_e + l_e + c)^2},
\]

\[
a_6 = -\left[ \frac{\delta_0 + \delta_2 + \delta_3 + \frac{a \beta}{l_e + a})^2}{l_e + a} \right],
\]

\[
a_7 = \delta_2 + \frac{a \beta}{(l_e + a)^2}, \quad a_8 = -\delta_0,
\]

\[
a_9 = \mu, \quad a_{10} = -\delta_0, \quad b_1 = \eta, \quad b_2 = -\eta, \quad c_1 = -\delta_1, \quad c_2 = \delta_1.
\]
The meanings and definitions of \(S_n\) and \(I_n\) can be found in the Section 4.

**Assumption 2.2:** Equation (3) has at least one positive root \(v_{10}\).

\[
v_{10}^5 + e_{14}v_{14}^4 + e_{13}v_{13}^3 + e_{12}v_{12} + e_{11}v_{11} + e_{10} = 0, \quad (3)
\]

where

\[
\begin{align*}
e_{10} &= A_{10}^2 - B_{10}^2, \\
e_{11} &= A_{11}^2 - 2A_{10}A_{12} + 2B_{10}B_{12} - B_{11}^2, \\
e_{12} &= A_{12}^2 - 2A_{11}A_{13} + 2A_{10}A_{14} - B_{12}^2 - 2B_{11}B_{13}, \\
e_{13} &= A_{13}^2 + 2A_{11} - 2A_{12}A_{14} + 2B_{12}B_{14} - B_{13}^2, \\
e_{14} &= A_{14}^2 - 2A_{13} - B_{14}^2.
\end{align*}
\]

with

\[
\begin{align*}
A_{10} &= A_0 + C_0, & A_{11} &= A_1 + C_1, & A_{12} &= A_2 + C_2, \\
A_{13} &= A_3 + C_3, & A_{14} &= A_4 + C_4, \\
B_{10} &= B_0 + D_0, & B_{11} &= B_1 + D_1, & B_{12} &= B_2 + D_2, \\
B_{13} &= B_3 + D_3, & B_{14} &= B_4.
\end{align*}
\]

**Assumption 2.3:** \(f_1(v_{10}) \neq 0\) where \(f_1(v_{11}) = v_{11}^5 + e_{14}v_{14}^4 + e_{13}v_{13}^3 + e_{12}v_{12} + e_{11}v_{11} + e_{10} \neq 0\).

**Assumption 2.4:** Equation (4) has at least one positive root \(v_{20}\).

\[
v_{20}^5 + e_{24}v_{24}^4 + e_{23}v_{23}^3 + e_{22}v_{22}^2 + e_{21}v_{21} + e_{20} = 0, \quad (4)
\]

where

\[
\begin{align*}
e_{20} &= A_{20}^2 - C_{20}, \\
e_{21} &= A_{21}^2 - 2A_{20}A_{22} + 2C_{20}C_{22} - C_{21}, \\
e_{22} &= A_{22}^2 - 2A_{21}A_{23} + 2A_{20}A_{24} - C_{22}^2 - 2C_{21}C_{23}, \\
e_{23} &= A_{23}^2 + 2A_{21} - 2A_{22}A_{24} + 2C_{22}C_{24} - C_{23}^2, \\
e_{24} &= A_{24}^2 - 2A_{23} - C_{24}^2.
\end{align*}
\]

with

\[
\begin{align*}
A_{20} &= A_0 + B_0, & A_{21} &= A_1 + B_1, \\
A_{22} &= A_2 + B_2, & A_{23} &= A_3 + B_3, & A_{24} &= A_4 + B_4, \\
C_{20} &= C_0 + D_0, & C_{21} &= C_1 + D_1, \\
C_{22} &= C_2 + D_2, & C_{23} &= C_3 + D_3, & C_{24} &= C_4.
\end{align*}
\]

**Assumption 2.5:** \(f_2(v_{20}) \neq 0\) where \(f_2(v_{21}) = v_{21}^5 + e_{24}v_{24}^4 + e_{23}v_{23}^3 + e_{22}v_{22}^2 + e_{21}v_{21} + e_{20} \neq 0\).

**Assumption 2.6:** Equation (5) has at least one positive root \(\omega_0\).

\[
(M_{31}^2(\omega) + M_{32}(\omega) + M_{34}(\omega))^2 - (M_{33}(\omega) + M_{35}(\omega) + M_{36}(\omega))^2 = 0, \quad (5)
\]

where

\[
\begin{align*}
M_{31}(\omega) &= A_{34}\omega^4 - (A_{32} - D_{32})\omega^2 + A_{30} - D_{30}, \\
M_{32}(\omega) &= \omega^5 - (A_{33} + D_{32})\omega^3 + (A_{31} + D_{31})\omega, \\
M_{33}(\omega) &= A_{34}\omega^4 - (A_{32} + D_{32})\omega^2 + A_{30} + D_{30}, \\
M_{34}(\omega) &= \omega^5 - (A_{33} - D_{33})\omega^3 + (A_{31} - D_{31})\omega, \\
M_{35}(\omega) &= B_{33}\omega^3 - B_{31}\omega, \\
M_{36}(\omega) &= B_{32}\omega^2 - B_{34}\omega^4 - B_{30}.
\end{align*}
\]

with

\[
\begin{align*}
A_{30} &= A_0, & A_{31} &= A_1, & A_{32} &= A_2, \\
A_{33} &= A_3, & A_{34} &= A_4, \\
B_{30} &= B_0 + C_0, & B_{31} &= B_1 + C_1, & B_{32} &= B_2 + C_2, \\
B_{33} &= B_3 + C_3, & B_{34} &= B_4 + C_4, \\
D_{30} &= D_0, & D_{31} &= D_1, & D_{32} &= D_2, & D_{33} &= D_3.
\end{align*}
\]

**Assumption 2.7:** \(U_{3R}V_{3R} + U_{3I}V_{3I} \neq 0\), where

\[
U_{3R} = [5\omega_0^4 - 3(A_{33} + D_{33}\omega_0^2) + A_{31} + D_{31}] \cos \tau_0\omega_0 \\
+ [2(A_{32} - D_{32})\omega_0 - 4A_{34}\omega_0^3] \sin \tau_0\omega_0 + 3B_{33}\omega_0^2.
\]

\[
U_{3I} = [5\omega_0^4 - 3(A_{33} - D_{33}\omega_0^2) + A_{31} - D_{31}] \cos \tau_0\omega_0 \\
+ [2(A_{32} + D_{32})\omega_0 - 4A_{34}\omega_0^3] \sin \tau_0\omega_0 + 2B_{32} - 4B_{34}\omega_0^3.
\]

\[
V_{3R} = [(D_{30} - A_{30})\omega_0 - (D_{32} - A_{32})\omega_0^3 - A_{34}\omega_0^5] \sin \tau_0\omega_0 \\
+ [(D_{33} + A_{33})\omega_0^4 - (D_{31} + A_{31})\omega_0^2 + \omega_0^6] \cos \tau_0\omega_0,
\]

\[
V_{3I} = [(D_{30} + A_{30})\omega_0 - (D_{32} + A_{32})\omega_0^3 + A_{34}\omega_0^5] \cos \tau_0\omega_0 \\
- [(D_{33} - A_{33})\omega_0^4 - (D_{31} - A_{31})\omega_0^2 + \omega_0^6] \sin \tau_0\omega_0.
\]

**Assumption 2.8:** Equation (6) has at least one positive root \(\omega_{2+}\).

\[
f_{40}(\omega_2) + f_{41}(\omega_2) \cos \tau_1\omega_2 + f_{42}(\omega_2) \sin \tau_1\omega_2 = 0, \quad (6)
\]
where

\[ f_{40}(\omega_2) = \omega_2^4 + (A_2^2 - 2A_3 + B_4^2 - C_4^2)\omega_2^2 \\
+ (A_2^2 - 2A_4A_2 + 2A_1 + B_3^2 - 2B_2B_4) \\
- C_2^2 - D_2^2 + 2C_4^2)\omega_2^2 \\
+ (A_3^2 + 2A_4A_2 - 2A_4A_3 \\
- 2B_1B_3 - C_2^2 - 2C_0C_4 \\
+ 2B_0B_4 + 2C_1C_3 + 2D_1D_3 - D_2^2)\omega_2^2 \\
+ (A_2^2 - 2A_0A_2 + B_1^2 - 2B_0B_2 - C_1^2 \\
+ 2C_0C_2 - D_1^2 + 2D_0D_2)\omega_2^2 \\
+ A_0^2 + B_0^2 - C_0^2 - D_0^2, \]

\[ f_{41}(\omega_2) = 2((A_4B_4 - B_3)\omega_2^2 + (A_3B_3 + B_1) \\
- A_2B_4 - A_4B_2 - C_3D_3 + C_4D_2)\omega_2^6 \\
+ (A_2B_2 - A_3B_1 - A_3B_3 + A_4B_0 + A_0B_4 - C_4D_0 \\
- C_2D_2 + C_3D_3 + C_4D_3 - 1)\omega_2^4 \\
+ (A_1B_1 - A_2B_0 - A_0B_2 - C_1D_1 \\
+ C_2D_0 + C_3D_2)\omega_2^4 + A_0B_0, \]

\[ f_{42}(\omega_2) = 2((4B_4 - B_3)\omega_2^2 + (A_3B_3 + B_2 + C_4D_3)\omega_2^6 \\
+ (A_2B_2 + A_4B_1 - A_3B_2 - B_0 - A_1B_4 \\
- C_4D_1 - C_2D_3 + C_3D_2)\omega_2^4 \\
+ (A_0B_0 - A_3B_1 - A_0B_2 - C_1B_2 - C_1D_2 \\
- C_3D_0 + C_2D_1 + C_0D_3)\omega_2^4 \\
+ (A_0B_1 - A_1B_0 - C_0D_1 - C_1D_0)\omega_2^2). \]

**Assumption 2.9:** \( U_{4R}V_{4R} + U_4V_{4R} \neq 0, \) where

\[ U_{4R} = (2C_2\omega_{2s} - 4C_4\omega_{2s}^3) \cos \tau_{2s}\omega_{2s} \\
+ (C_1 - 3C_3\omega_{2s}^2) \cos \tau_{2s}\omega_{2s} \\
+ [\tau_1D_3\omega_{2s}^3 + (2D_2 - \tau_1D_1)\omega_{2s}^2] \\
\times (\sin \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s} \\
+ \cos \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s} \\
- \sin \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s}) \\
+ [D_1 - \tau_1D_0 - (3D_1 - \tau_1D_2)\omega_{2s}^2] \\
\times (\cos \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s} \\
- \sin \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s}) \\
+ [(4B_4 - B_1)\omega_{2s}^3 - (2B_2 - \tau_1B_1)\omega_{2s}^2] \sin \tau_{1\omega_{2s}} \\
+ [\tau_1B_0 - B_1] \cos \tau_{1\omega_{2s}} \\
+ S_\omega_{2s}^2 - 3A_2A_3^2 + A_1, \]

\[ U_{4I} = (2C_2\omega_{2s} - 4C_4\omega_{2s}^3) \cos \tau_{2s}\omega_{2s} \\
- (C_1 - 3C_3\omega_{2s}^2) \sin \tau_{2s}\omega_{2s} \\
+ [\tau_1D_3\omega_{2s}^3 + (2D_2 - \tau_1D_1)\omega_{2s}^2] \\
\times (\cos \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s} \\
- \sin \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s}) \\
- [D_1 - \tau_1D_0 - (3D_1 - \tau_1D_2)\omega_{2s}^2] \\
\times (\sin \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s} \\
- \cos \tau_{1\omega_{2s}} \cos \tau_{2s}\omega_{2s}) \\
+ [(4B_4 - B_1)\omega_{2s}^3 - (2B_2 - \tau_1B_1)\omega_{2s}^2] \sin \tau_{1\omega_{2s}} \\
+ [\tau_1B_0 - B_1] \cos \tau_{1\omega_{2s}} \\
+ S_\omega_{2s}^2 - 3A_2A_3^2 + A_1. \]

3. Main results

By a direct computation, we know that if \([\alpha \delta_1 - (\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3)](l_1 + a) > \beta\delta_1(\delta_0 + \delta_1),\) then system (2) has a viral equilibrium \(P_0(S_s, E_s, I_s, R_s, V_s),\) where

\[
S_s = \frac{[(\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3)](l_1 + a)}{\beta\delta_1(\delta_0 + \delta_1)}, \\
E_s = \frac{\delta_2l_1}{\delta_0(\delta_0 + \delta_1)}, \\
R_s = \frac{\delta_2l_1}{\delta_0(\delta_0 + \delta_1)}, \\
V_s = \frac{\mu[(\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3)](l_1 + a)}{\beta\delta_1(\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3)}, \]
and \( I_a \) is the positive root of Equation (7)

\[
k_3 l^3 + k_2 l^2 + k_1 l + k_0 = 0, \quad (7)
\]

where

\[
k_0 = \delta_1 U_0 c - U_1 c(\delta_0 + \eta) + \delta_1 a c[U_2 - U_2(\delta_0 + \eta)] + A a e \delta_1(\delta_0 + \eta)(U_0 a - U_0),
\]

\[
k_1 = U_0 a^2 \delta_1 - e^2 \delta_1(\delta_0 + \eta)(U_1 \delta_1 + U_3 U_5) + \delta_1(\delta_0 + \eta)(U_2 - U_2(\delta_0 + \eta)) + A(\delta_0 + \eta)(U_4 U_5 \delta_1 - U_5 \beta(\delta_0 + \delta_1)) + (\delta_0 + \eta)(2A U_3 \delta_1 - U_4 \delta_1 + U_4 \beta(\delta_0 + \delta_1)),
\]

\[
k_2 = \delta_1(2U_0 a + U_2) + \delta_1(\delta_0 + \eta)(U_3 + U_4 U_5 - U_2) - (\delta_0 + \eta)(U_3 \beta(\delta_0 + \delta_1) + \delta_1(U_1 + U_3 U_5)),
\]

\[
k_3 = \delta_1 U_6 - \delta_1(\delta_0 + \eta)(U_1 + U_3 U_5),
\]

with

\[
U_1 = (\delta_0 + \mu)(\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3),
\]

\[
U_2 = \beta \delta_1(\delta_0 + \mu)(\delta_0 + \delta_1),
\]

\[
U_3 = \alpha \delta_1 - (\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3),
\]

\[
U_4 = \beta \delta_1(\delta_0 + \delta_1),
\]

\[
U_5 = \frac{(\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3)}{\delta_1},
\]

\[
U_6 = \eta \mu (\delta_0 + \delta_1)(\delta_0 + \delta_2 + \delta_3),
\]

\[
U_7 = \eta \mu \beta \delta_1(\delta_0 + \delta_1).
\]

The characteristic equation of system (2) at \( P_0(S_0, E_0, I_0, R_0, V_0) \) is

\[
\lambda^5 + A_4 \lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0
+ (B_4 \lambda^4 + B_3 \lambda^3 + B_2 \lambda^2 + B_1 \lambda + B_0) e^{-\lambda \tau_1}
+ (C_4 \lambda^4 + C_3 \lambda^3 + C_2 \lambda^2 + C_1 \lambda + C_0) e^{-\lambda \tau_2}
+ (D_3 \lambda^3 + D_2 \lambda^2 + D_1 \lambda + D_0) e^{-\lambda (\tau_1 + \tau_2)} = 0. \quad (8)
\]

**Remark 3.1:** By the Routh-Hurwitz criteria in Hassard et al. (1981) we know that system (2) is locally asymptotically stable when \( \tau_1 = \tau_2 = 0 \) under Assumption 2.1.

**Theorem 3.1:** For \( \tau_1 > 0 \) and \( \tau_2 = 0 \), under Assumptions 2.1–2.3, the viral equilibrium \( P_0(S_0, E_0, I_0, R_0, V_0) \) is locally asymptotically stable when \( \tau_1 \in [0, \tau_{10}) \); a Hopf bifurcation occurs at \( P_0(S_0, E_0, I_0, R_0, V_0) \) when \( \tau_1 = \tau_{10} \) and a family of periodic solutions bifurcate from \( P_0(S_0, E_0, I_0, R_0, V_0) \) near \( \tau_1 = \tau_{10} \), and \( \tau_{10} \) is defined as in Equation (12).

**Proof:** When \( \tau_1 > 0 \) and \( \tau_2 = 0 \), Equation (8) reduces to

\[
\lambda^5 + A_4 \lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0
+ (B_4 \lambda^4 + B_3 \lambda^3 + B_2 \lambda^2 + B_1 \lambda + B_0) e^{-\lambda \tau_1} = 0.
\]

Let \( \lambda = \omega_1(\omega_1 > 0) \) be the root of Equation (9), then

\[
(B_{11} \omega_1 - B_{13} \omega_1^3) \sin \tau_1 \omega_1
+ (B_{14} \omega_1^4 - B_{12} \omega_1^2 + B_{10}) \cos \tau_1 \omega_1 = A_{12} \omega_1^2 - A_{14} \omega_1^4 - A_{10},
\]

\[
(B_{11} \omega_1 - B_{13} \omega_1^3) \cos \tau_1 \omega_1
- (B_{14} \omega_1^4 - B_{12} \omega_1^2 + B_{10}) \sin \tau_1 \omega_1 = A_{13} \omega_1^3 - \omega_1^5 - A_{11} \omega_1.
\]

Eliminating trigonometric functions from Equation (10), we obtain the following equation

\[
\omega_1^{10} + e_{14} \omega_1^4 + e_{13} \omega_1^6 + e_{12} \omega_1^4 + e_{11} \omega_1^2 + e_{10} = 0, \quad (11)
\]

Let \( \nu_1 = \omega_1^2 \), then Equation (11) becomes Equation (3). Thus, under Assumption 2.2, Equation (11) has a positive root \( \omega_{10} = \sqrt{\nu_{10}} \). For \( \omega_{10} \), we have

\[
\tau_{10} = \frac{1}{\omega_{10}} \times \arccos \left( \frac{g_{11}(\omega_{10})}{g_{12}(\omega_{10})} \right), \quad (12)
\]

where

\[
g_{11}(\omega_{10}) = (B_{13} - A_{14} B_{14}) \omega_{10}^8
+ (A_{13} B_{13} - B_{11} + A_{12} B_{14} + A_{14} B_{12}) \omega_{10}^6
+ (A_{11} B_{13} + A_{13} B_{11} - A_{10} B_{14}) \omega_{10}^4
+ A_{12} B_{12} - A_{14} B_{10} \omega_{10}^2
+ (A_{10} B_{12} + A_{12} B_{10} - A_{11} B_{11}) \omega_{10}^2 - A_{10} B_{10},
\]

\[
g_{12}(\omega_{10}) = B_{14} \omega_{10}^8 + (B_{13}^2 - 2 B_{12} B_{14}) \omega_{10}^6
+ (B_{12}^2 + 2 B_{10} B_{14} + 2 B_{11} B_{13}) \omega_{10}^4
+ (B_{11}^2 - 2 B_{10} B_{12}) \omega_{10}^2 + B_{10}.
\]

Also, differentiating Equation (9) with respect to \( \tau_1 \), we get

\[
\left[ \frac{d\lambda}{d\tau_1} \right]^{-1} = -\frac{5 \lambda^4 + 4 A_4 \lambda^3 + 3 A_3 \lambda^2 + 2 A_2 \lambda + A_1}{\lambda^5 + A_4 \lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0}
+ \frac{4 B_4 \lambda^3 + 3 B_3 \lambda^2 + 2 B_2 \lambda + B_1}{\lambda (B_4 \lambda^4 + B_3 \lambda^3 + B_2 \lambda^2 + B_1 \lambda + B_0)} \times \frac{\tau_1}{\lambda}.
\]
Further, we have

\[
\Re \left( \frac{d\lambda}{d\tau_1} \right)_{\tau_1=\tau_{10}}^{-1} = \frac{f'_1(v_{10})}{(B_{11}w_{10} - B_{13}w_{10}^3)^2 + (B_{14}w_{10}^4 - B_{12}w_{10}^2 + B_{10})^2}.
\]

Hence, under Assumption 2.3 we can conclude that the transversality condition is satisfied. Based on the Hopf bifurcation theorem in Lemma 3.1, we have Theorem 3.1 and the proof is completed.

**Theorem 3.2:** For \( \tau_1 = 0 \) and \( \tau_2 > 0 \), under Assumptions 2.1, 2.4, 2.5, the viral equilibrium \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) is locally asymptotically stable when \( \tau_2 \leq \tau_{20} \); a Hopf bifurcation occurs at \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) when \( \tau_2 = \tau_{20} \) and a family of periodic solutions bifurcate from \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) near \( \tau_2 = \tau_{20} \), and \( \tau_{20} \) is defined as in Equation (15).

**Proof:** When \( \tau_1 = 0 \), \( \tau_2 > 0 \), Equation (8) becomes

\[
\lambda^5 + A_{24}\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20} + (C_{24}\lambda^4 + C_{23}\lambda^3 + C_{22}\lambda^2 + C_{21}\lambda + C_{20})e^{-\lambda \tau_2} = 0.
\]

(13)

Let \( \lambda = i\omega_2 (\omega_2 > 0) \) be the root of Equation (13). Then, we can obtain the following equation with respect to \( \omega_2 \):

\[
\omega_2^{10} + e_{24}\omega_2^8 + e_{23}\omega_2^6 + e_{22}\omega_2^4 + e_{21}\omega_2^2 + e_{20} = 0,
\]

(14)

Let \( \omega_2 = \omega_2^2 \), then Equation (14) becomes Equation (4). Under Assumption 2.4, Equation (14) has a positive root \( \omega_{20} = \sqrt{\omega_2} \). For \( \omega_{20} \), one can obtain

\[
\tau_{20} = \frac{1}{\omega_{20}} \times \arccos \left( \frac{g_{21}(\omega_{20})}{g_{22}(\omega_{20})} \right),
\]

(15)

where

\[
g_{21}(\omega_{20}) = (C_{23} - A_{24}C_{24})\omega_{20}^8 + (A_{23}C_{23} - C_{21} + A_{22}C_{24} + A_{24}C_{22})\omega_{20}^6 + (A_{21}C_{23} + A_{23}C_{21} - A_{20}C_{24} - A_{12}B_{12} - A_{24}C_{20})\omega_{20}^4 + (A_{20}C_{22} + A_{22}C_{20} - A_{21}C_{21})\omega_{20}^2 - A_{20}C_{20},
\]

\[
g_{22}(\omega_{20}) = C_{14}^2\omega_{20}^8 + (C_{22}^2 - 2C_{22}C_{24})\omega_{20}^6 + (C_{22}^2 + 2C_{20}C_{24} + 2C_{21}C_{23})\omega_{20}^4 + (C_{21} - 2C_{20}C_{22})\omega_{20}^2 + C_{22}^2.
\]

Further,

\[
\Re \left( \frac{d\lambda}{d\tau_2} \right)_{\tau_2=\tau_{20}}^{-1} = \frac{f'_2(v_{20})}{(C_{21}\omega_{20} - C_{23}\omega_{20}^3)^2 + (C_{24}\omega_{20}^4 - C_{22}\omega_{20}^2 + C_{20})^2}.
\]

Thus, under Assumption 2.5 we can conclude that the transversality condition is satisfied. Based on the Hopf bifurcation theorem in Lemma 3.1, we have Theorem 3.2 and the proof is completed.

**Theorem 3.3:** For \( \tau_1 = \tau_2 = \tau > 0 \), under Assumptions 2.1, 2.6, 2.7, the viral equilibrium \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) is locally asymptotically stable when \( \tau \in [0, \tau_0) \); a Hopf bifurcation occurs at \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) when \( \tau = \tau_0 \) and a family of periodic solutions bifurcate from \( P_\lambda(S_e, E_v, I_v, R_v, V_v) \) near \( \tau = \tau_0 \), and \( \tau_0 \) is defined as in Equation (21).

**Proof:** When \( \tau_1 = \tau_2 = \tau > 0 \), Equation (8) becomes

\[
\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} + (B_{34}\lambda^4 + B_{33}\lambda^3 + B_{32}\lambda^2 + B_{31}\lambda + B_{30})e^{-\lambda \tau} + (D_{33}\lambda^3 + D_{32}\lambda^2 + D_{31}\lambda + D_{30})e^{-2\lambda \tau} = 0.
\]

(16)

Multiplying by \( e^{\lambda \tau} \), Equation (16) yields

\[
B_{34}\lambda^4 + B_{33}\lambda^3 + B_{32}\lambda^2 + B_{31}\lambda + B_{30} + (\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30})e^{\lambda \tau} + (D_{33}\lambda^3 + D_{32}\lambda^2 + D_{31}\lambda + D_{30})e^{-\lambda \tau} = 0.
\]

(17)

Let \( \lambda = i\omega (\omega > 0) \) be the root of Equation (17) and substituting this into Equation (17), we get

\[
M_{31}(\omega) \sin \tau \omega + M_{32}(\omega) \cos \tau \omega = M_{35}(\omega),
\]

\[
M_{33}(\omega) \cos \tau \omega - M_{34}(\omega) \sin \tau \omega = M_{36}(\omega),
\]

(18)

Then,

\[
\cos \tau \omega = \frac{M_{31}(\omega) \times M_{36}(\omega) + M_{34}(\omega) \times M_{35}(\omega)}{M_{31}(\omega) \times M_{33}(\omega) + M_{32}(\omega) \times M_{34}(\omega)},
\]

\[
\sin \tau \omega = \frac{M_{33}(\omega) \times M_{35}(\omega) - M_{32}(\omega) \times M_{36}(\omega)}{M_{31}(\omega) \times M_{33}(\omega) + M_{32}(\omega) \times M_{34}(\omega)}.
\]

(19)

(20)

Further, we can get Equation (5). Under Assumption 2.6, Equation (5) has a positive root \( \omega_0 \) such that Equation (17)
has a pair of purely imaginary roots $\pm i\omega_0$. For $\omega_0$, we have
\[
\tau_0 = \frac{1}{\omega_0} \times \arccos \left( \frac{M_{31}(\omega_0) \times M_{36}(\omega_0)}{M_{34}(\omega_0) \times M_{35}(\omega_0)} \right) .
\]  
(21)

Taking the derivative of $\lambda$ with respect to $\tau$ in Equation (17), we obtain
\[
\left[ \frac{d\lambda}{d\tau} \right]^{-1} = \frac{F_{31}(\lambda)}{F_{32}(\lambda)} - \frac{\tau}{\lambda},
\]
where
\[
F_{31}(\lambda) = 4B_{34}\lambda^3 + 3B_{33}\lambda^2 + 2B_{32}\lambda + B_{31} + (5\lambda^4 + 4A_{34}\lambda^3 + 3A_{33}\lambda^2 + 2A_{32}\lambda + A_{31})e^{\lambda \tau} + (3D_{33}\lambda^2 + 2D_{32}\lambda + D_{31})e^{-\lambda \tau},
\]
\[
F_{32}(\lambda) = \lambda(D_{33}\lambda^3 + D_{32}\lambda^2 + D_{31}\lambda + D_{30})e^{-\lambda \tau} - \lambda(\lambda^5 + A_{34}\lambda^3 + A_{33}\lambda^2 + A_{32}\lambda + A_{31} + A_{30})e^{\lambda \tau}.
\]
Thus,
\[
\text{Re} \left[ \frac{d\lambda}{d\tau} \right]_{\tau = \tau_0} = \frac{U_{38}V_{3R} + U_{33}V_{3I}}{V_{3R}^2 + V_{3I}^2}.
\]
Clearly, under Assumption 2.7, we know that $\text{Re} [d\lambda/d\tau]_{\tau = \tau_0} \neq 0$. Based on the Hopf bifurcation theorem in Lemma 3.1, we have Theorem 3.3 and the proof is completed.

**Theorem 3.4:** For $\tau_1 \in (0, \tau_{10})$ and $\tau_2 > 0$, under Assumptions 2.1, 2.8, 2.9, the viral equilibrium $P_\infty(S_\infty, E_\infty, I_\infty, R_\infty, V_\infty)$ is locally asymptotically stable when $\tau_2 \in [0, \tau_2^*)$; a Hopf bifurcation occurs at $P_\infty(S_\infty, E_\infty, I_\infty, R_\infty, V_\infty)$ when $\tau_2 = \tau_2^*$ and a family of periodic solutions bifurcate from $P_\infty(S_\infty, E_\infty, I_\infty, R_\infty, V_\infty)$ near $\tau_2 = \tau_2^*$, and $\tau_2$ is defined as in Equation (23).

**Proof:** We take $\tau_2$ as the bifurcation parameter when $\tau_1 \in (0, \tau_{10})$. Let $\lambda = i\omega_2^* (\omega_2^* > 0)$ be the root of Equation (8). For convenience, we still denote $\omega_2^*$ as $\omega_2$. Then,
\[
M_{41}(\omega_2) \sin \tau_2 \omega_2 + M_{42}(\omega_2) \cos \tau_2 \omega_2 = M_{43}(\omega_2),
\]
\[
M_{41}(\omega_2) \cos \tau_2 \omega_2 - M_{42}(\omega_2) \sin \tau_2 \omega_2 = M_{44}(\omega_2),
\]
with
\[
M_{41}(\omega_2) = C_1 \omega_2 - C_3 \omega_2^3 + (D_1 \omega_2 - D_3 \omega_2^3) \cos \tau_1 \omega_2 - (D_0 - D_2 \omega_2^3) \sin \tau_1 \omega_2,
\]
\[
M_{42}(\omega_2) = C_4 \omega_2^4 - C_2 \omega_2^2 + C_0 + (D_1 \omega_2 - D_3 \omega_2^3) \sin \tau_1 \omega_2 + (D_0 - D_2 \omega_2^3) \cos \tau_1 \omega_2.
\]
\[
M_{43}(\omega_2) = (B_3 \omega_2^3 - B_1 \omega_2) \sin \tau_1 \omega_2 - (B_4 \omega_2^4 - B_2 \omega_2^2 + B_0) \cos \tau_1 \omega_2 + A_2 \omega_2^2 - A_4 \omega_2^4 - A_0,
\]
\[
M_{44}(\omega_2) = (B_3 \omega_2^3 - B_1 \omega_2) \cos \tau_1 \omega_2 + (B_4 \omega_2^4 - B_2 \omega_2^2 + B_0) \sin \tau_1 \omega_2 + A_3 \omega_2^3 - A_5 \omega_2^5 - A_1 \omega_2.
\]
Eliminating trigonometric functions $\cos \tau_2 \omega_2$ and $\sin \tau_2 \omega_2$ from above equations, we can obtain Equation (6). Under Assumption 2.8, there exists a positive root $\omega_2 > 0$ such that Equation (8) has a pair of purely imaginary roots $\pm i\omega_2^*$. For $\omega_2^*$, we have
\[
\tau_{2^*} = \frac{1}{\omega_2^*} \times \arccos \left( \frac{M_{41}(\omega_2^*) \times M_{44}(\omega_2^*)}{M_{43}(\omega_2^*)} \right).
\]
(23)

Taking the derivative of $\lambda$ with respect to $\tau$ in Equation (8), we have
\[
\left[ \frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{F_{41}(\lambda) - \frac{\tau_2}{\lambda}}{F_{42}(\lambda) - \frac{\tau_2}{\lambda}},
\]
where
\[
F_{41}(\lambda) = (\tau_1 B_4 \lambda^4 - (4B_4 - \tau_1 B_3) \lambda^3 - (3B_4 - \tau_1 B_2) \lambda^2 - (2B_2 - \tau_1 B_1) \lambda + \tau_1 B_0 - B_1)e^{-\lambda \tau_1} + (4C_4 \lambda^3 + 3C_3 \lambda^2 + 2C_2 \lambda + C_1)e^{-\lambda \tau_2} + ((3D_3 - \tau_1 D_2) \lambda^2 - \tau_1 D_3 \lambda^3 + (2D_2 - \tau_1 D_1) \lambda + D_1 - \tau_1 D_0)e^{-\lambda (\tau_1 + \tau_2)} + 5\lambda^4 + 4A_{44} \lambda^3 + 3A_{33} \lambda^2 + 2A_{22} \lambda + A_{11},
\]
\[
F_{42}(\lambda) = \lambda(C_4 \lambda^3 + C_3 \lambda^2 + C_2 \lambda + C_1 + C_0)e^{-\lambda \tau_2} + \lambda(D_3 \lambda^2 + D_2 \lambda + D_1 \lambda + D_0)e^{-\lambda (\tau_1 + \tau_2)}.
\]
Then, we obtain
\[
\text{Re} \left[ \frac{d\lambda}{d\tau_2} \right]_{\tau = \tau_{2^*}} = \frac{U_{48}V_{4R} + U_{44}V_{4I}}{V_{4R}^2 + V_{4I}^2}.
\]
Clearly, under Assumption 2.9, $\text{Re} [d\lambda/d\tau_2]_{\tau = \tau_{2^*}} \neq 0$. Based on the Hopf bifurcation theorem in Lemma 3.1, we have Theorem 3.4 and the proof is completed.

**Theorem 3.5:** For system (2), the direction, stability and period of the bifurcating periodic solutions of system from the viral equilibrium $P_\infty(S_\infty, E_\infty, I_\infty, R_\infty, V_\infty)$ when $\tau_1 = \tau_1^* \in (0, \tau_{10})$ and $\tau_2 > 0$ can be described as follows: if $\mu_2 > 0 (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical); if $\beta_2 < 0 (\beta_2 > 0)$, then the bifurcated periodic solutions are stable (unstable); if $T_2 > 0 (T_2 < 0)$, then the period
of the bifurcated periodic solutions increases (decreases). Expressions of $\mu_2, \beta_2$ and $T_2$ are defined by Equation (24).

$$C_1(0) = \frac{i}{2\tau_2a\omega_2} \left( g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{20}|^2}{3} \right) + \frac{g_{21}}{2}$$

$$\mu_2 = - \frac{\text{Re}[C_1(0)]}{\text{Re}(\lambda'(\tau_2))},$$

$$\beta_2 = 2\text{Re}[C_1(0)],$$

$$T_2 = - \frac{\text{Im}[C_1(0)] + \mu_2\text{Im}(\lambda'(\tau_2))}{\tau_2a\omega_2}.$$  \hfill (24)

**Proof:** Throughout this section, we assume that $\tau_{1*} < \tau_2$. Define the space of continuous real-valued functions as $C = C([-1, 0], \mathbb{R}^5)$. Let $u_1(t) = S(t) - S_e, u_2(t) = E(t) - E_e, u_3(t) = \rho(t) - \rho_e, u_4(t) = R(t) - R_e, u_5(t) = V(t) - V_e$, and rescale the time delay by $t \rightarrow t/\tau_2$. Setting $\tau_2 = \tau_{2*} + \rho, \rho \in \mathbb{R}$, then the Hopf bifurcation occurs at $\rho = 0$. Thus, system (2) can be transformed into

$$\dot{u}(t) = L_\rho u(t) + F(\rho, u(t)), \tag{25}$$

where $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))^T \in \mathbb{R}^5$, $u(t) = u(t + \theta) \in C$ and $L_\rho : C \rightarrow \mathbb{R}^5$, $F(\rho, u(t)) \rightarrow \mathbb{R}^5$ are given respectively by

$$L_\rho \phi = (\tau_2 + \rho) \left( A_{\text{max}} \phi(0) + B_{\text{max}} \phi \left( - \frac{\tau_{1*}}{\tau_2} \right) \right) + C_{\text{max}} \phi(-1),$$

$$F(\rho, \phi) = (F_1, F_2, F_3, F_4, 0)$$

with

$$A_{\text{max}} = \begin{pmatrix}
    a_1 & 0 & a_2 & 0 & 0 \\
    a_3 & a_4 & a_5 & 0 & 0 \\
    0 & 0 & a_6 & 0 & 0 \\
    0 & 0 & a_7 & a_8 & 0 \\
    a_9 & 0 & 0 & 0 & a_{10}
\end{pmatrix},$$

$$B_{\text{max}} = \begin{pmatrix}
    0 & 0 & 0 & 0 & b_1 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & b_2
\end{pmatrix},$$

$$C_{\text{max}} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & c_1 & 0 & 0 & 0 \\
    0 & c_2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and

$$F_1 = a_16 \phi_1^2(0) + a_{17} \phi_2^2(0) + a_{18} \phi_1(0) \phi_3(0) + a_{19} \phi_2(0) \phi_3(0) + a_{110} \phi_1(0) \phi_3^2(0) + a_{111} \phi_3^3(0) + a_{112} \phi_3^3(0) + \cdots,$$

$$F_2 = a_{24} \phi_1(0) + a_{25} \phi_2^2(0) + a_{26} \phi_1(0) \phi_3(0) + a_{27} \phi_1(0) \phi_3^2(0) + a_{28} \phi_1(0) \phi_3^3(0) + a_{29} \phi_3(0) + a_{30} \phi_3^2(0) + \cdots,$$

$$F_3 = a_{34} \phi_3^2(0) + a_{35} \phi_3^3(0) + \cdots,$$

$$F_4 = a_{45} \phi_3^3(0) + a_{46} \phi_3^3(0) + \cdots.$$

$$a_{16} = \frac{\alpha I_s l_s c + c}{(S_e + l_e + c)^3}, \quad a_{17} = \frac{\alpha S_e (S_e + c)}{(S_e + l_e + c)^3},$$

$$a_{18} = - \frac{2\alpha I_s l_s c + c}{(S_e + l_e + c)^3},$$

$$a_{19} = \frac{2\alpha l_s (2S_e - l_s) + 2\alpha c (S_e + c)}{(S_e + l_e + c)^3},$$

$$a_{20} = \frac{2\alpha S_e (2I_s - l_s) + 2\alpha c (I_s + c)}{(S_e + l_e + c)^3},$$

$$a_{21} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{22} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{23} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{24} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{25} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{26} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{27} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{28} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{29} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{30} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{31} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{32} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{33} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{34} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{35} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4},$$

$$a_{36} = \frac{-\alpha l_s (l_s + c)}{(S_e + l_e + c)^4}.$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \phi)$ whose components are of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\rho \phi = \int_{-1}^{\theta} d\eta(\theta, \phi) \phi(\theta), \quad \text{for } \phi \in C. \tag{26}$$

In fact, we choose

$$\eta(\theta, \phi) = \begin{cases}
    (\tau_2 + \rho) \left( A_{\text{max}} + B_{\text{max}} + C_{\text{max}} \right), & \theta = 0, \\
    (\tau_2 + \rho) (A_{\text{max}} + B_{\text{max}} + C_{\text{max}}), & \theta \in \left[ -\frac{\tau_{1*}}{\tau_2}, 0 \right), \\
    (\tau_2 + \rho) C_{\text{max}}, & \theta \in \left( -1, -\frac{\tau_{1*}}{\tau_2} \right), \\
    0, & \theta = -1.
\end{cases} \tag{27}$$

with $\delta(\theta)$ is the Dirac delta function.
For \( \phi \in C([-1, 0], R^5) \), define

\[
A(\phi) = \begin{cases} 
\frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\
\int_{-1}^{0} d\eta(\theta, \phi) \phi(\theta), & \theta = 0,
\end{cases}
\]

and

\[
R(\phi) = \begin{cases} 
0, & -1 \leq \theta < 0, \\
F(\phi, \phi), & \theta = 0.
\end{cases}
\]

Then system (25) becomes

\[
\dot{u}(t) = A(\phi)u(t) + R(\phi)u(t).
\]

For \( \psi \in C^1([0, 1], R^5) \), define

\[
A^*(\psi) = \begin{cases} 
-\frac{d\psi(s)}{ds}, & 0 < s \leq 1, \\
\int_{-1}^{0} d\eta^T(s, 0) \psi(-s), & s = 0,
\end{cases}
\]

and a bilinear inner product

\[
\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0) \phi(0) - \int_{\theta=1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,
\]

where \( \eta(\theta) = \eta(\theta, 0) \).

Then, \( A(\phi) \) and \( A^* \) are adjoint operators. Since \( \pm i\omega_{2a} T_{2a} \) are the eigenvalues of \( A(\phi) \) and \( A^* \), respectively. We need to compute eigenvectors of \( A(\phi) \) and \( A^* \) corresponding to \( +i\omega_{2a} T_{2a} \) and \( -i\omega_{2a} T_{2a} \), respectively. Let \( q(\theta) = (1, q_2, q_3, q_4, q_5)^T e^{i\theta T_{2a}, 0} \) and \( \rho^*(s) = (1, q_2^*, q_3^*, q_4^*, q_5^*)^T e^{i2 \omega_{2a} s, 0} \) be the eigenvectors for \( A(\phi) \) and \( A^* \) corresponding to \( +i\omega_{2a} T_{2a} \) and \( -i\omega_{2a} T_{2a} \), respectively. Then, we have

\[
\rho_2 = \frac{a_{21} + a_{22} \rho_3}{i\omega_0 - a_{22}}, \quad \rho_3 = \frac{i\omega_0 - a_{11}}{a_{13}} - \frac{a_{15} a_{51}}{a_{13}(i\omega_0 - a_{55})},
\]

\[
\rho_4 = \frac{-a_{32} e^{-i\omega_0 \rho_3}}{i\omega_0 - a_{44}}, \quad \rho_5 = \frac{a_{51}}{i\omega_0 - a_{55}},
\]

\[
\rho_2^* = \frac{-a_{21} a_{51}}{a_{21} (i\omega_0 + a_{55})}, \quad \rho_5^* = \frac{i\omega_0 + a_{11}}{a_{21}},
\]

\[
\rho_3^* = \frac{(i\omega_0 + a_{22}) \rho_2}{a_{32}}, \quad \rho_2^* = -\frac{a_{15}}{i\omega_0 + a_{55}},
\]

\[
\rho_4^* = -\frac{(i\omega_0 + a_{33} + b_{33} e^{-i\omega_0 \rho_3}) \rho_3}{b_{43} e^{-i\omega_0 \rho_3}}.
\]

From Equation (29), we get

\[
\bar{D} = [1 + \rho_2 \rho_2^* + \rho_3 \rho_3^* + \rho_4 \rho_4^* + \rho_5 \rho_5^* + \tau_0 e^{-i\omega_0 \rho_3} (b_{33} \rho_3^* + b_{43} \rho_4^*)]^{-1},
\]

such that \( \langle \rho^*, \rho \rangle = 1 \) and \( \langle \rho^*, \rho \rangle = 0 \).

Next, based on the algorithm in Hassard et al. (1981) and the similar computation process as that in Bianca, Ferrara, and Guerrini (2013), Upadhyay and Agrawal (2016), Upadhyay and Agrawal (2015), we obtain

\[
g_{20} = 2 \tau_2 \tilde{D} [a_{16} + a_{17} q_2^2 + a_{18} q_3
\]

\[
+ \tilde{q}_2^2 (a_{24} + a_{25} q_3^2 + a_{26} q_3) + (a_{34} \bar{q}_3^2 + a_{45} \bar{q}_4^* \bar{q}_4^2 e^{-2i \tau_{2a} q_2^2})],
\]

\[
g_{11} = \tau_2 \tilde{D} [2 a_{16} + 2 a_{17} q_3 q_3 + 2 a_{18} e^{i \tau_{2a} q_3}]
\]

\[
+ q_2^2 (2 a_{24} + 2 a_{25} q_3 q_3 + 2 a_{26} e^{i \tau_{2a} q_3}) + 2 (a_{34} \bar{q}_3^2 + a_{45} \bar{q}_4^* \bar{q}_4^2),
\]

\[
g_{21} = 2 \tau_2 \tilde{D} [a_{16} (2 W_{11}^{(1)} (0) + W_{20}^{(1)} (0))
\]

\[
+ a_{17} (2 W_{11}^{(3)} (0) q_3 + W_{20}^{(3)} (0) q_3) + a_{18} (W_{11}^{(1)} (0) q_3 + \frac{1}{2} W_{20}^{(1)} (0) q_3)
\]

\[
+ a_{19} (q_3 + 2 q_3) + a_{110} (q_3^2 + 2 q_3 q_3)
\]

\[
+ 3 a_{111} + 3 a_{112} q_3^2 \bar{q}_3
\]

\[
+ q_2^2 (a_{24} (2 W_{11}^{(1)} (0) + W_{20}^{(1)} (0))
\]

\[
+ a_{25} (2 W_{11}^{(3)} (0) q_3 + W_{20}^{(3)} (0) q_3)
\]

\[
+ a_{26} (W_{11}^{(1)} (0) q_3 + \frac{1}{2} W_{20}^{(1)} (0) q_3)
\]

\[
+ W_{11}^{(3)} (0) + \frac{1}{2} W_{20}^{(3)} (0))
\]

\[
+ a_{27} (q_3 + 2 q_3) + a_{28} (q_3^2 + 2 q_3 q_3)
\]

\[
+ 3 a_{29} + 3 a_{210} q_3 q_3
\]

\[
+ q_2^2 (a_{34} (2 W_{11}^{(3)} (0) q_3 e^{-i \tau_{2a} q_2^2} + W_{20}^{(3)} (0) q_3 e^{i \tau_{2a} q_2^2})
\]

\[
+ 3 a_{35} q_3^2 q_3 e^{-i \tau_{2a} q_2^2} + q_2^2 (a_{45} (2 W_{11}^{(3)} (0) q_3 e^{-i \tau_{2a} q_2^2} + W_{20}^{(3)} (0) q_3 e^{i \tau_{2a} q_2^2})
\]

\[
+ 3 a_{46} q_3^2 q_3 e^{-i \tau_{2a} q_2^2}),
\]

with

\[
W_{20}(\theta) = \frac{ig_{20} \phi(0)}{\tau_{2a} \omega_{2a}} e^{i \tau_{2a} \omega_{2a} \theta}
\]

\[
+ \frac{ig_{21} \phi(0)}{3 \tau_{2a} \omega_{2a}} e^{-i \tau_{2a} \omega_{2a} \theta} + E_1 e^{i \tau_{2a} \omega_{2a} \theta},
\]

\[
W_{11}(\theta) = -\frac{ig_{11} \phi(0)}{\tau_{2a} \omega_{2a}} e^{i \tau_{2a} \omega_{2a} \theta} + \frac{ig_{11} \phi(0)}{\tau_{2a} \omega_{2a}} e^{-i \tau_{2a} \omega_{2a} \theta} + E_2.
\]
$E_1$ and $E_2$ can be obtained by the following two equations

$$E_1 = 2 \begin{pmatrix} a_1^* & 0 & -a_2 \\ -a_3 & a_2^* & -a_5 \\ 0 & -c_2 e^{-2iτ_{2},ω_2} & a_3^* \\ 0 & 0 & -a_7 \\ -a_9 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \\ E_1^{(4)} \\ 0 \end{pmatrix}$$

$$E_2 = - \begin{pmatrix} a_1 & 0 & a_2 & 0 & b_1 \\ a_3 & a_4 + c_1 & a_5 & 0 & 0 \\ 0 & c_2 & a_6 & 0 & 0 \\ 0 & 0 & a_7 & a_8 & 0 \\ a_9 & 0 & 0 & 0 & a_{10} + b_2 \end{pmatrix}^{-1} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \\ E_2^{(4)} \\ 0 \end{pmatrix}$$

where

$$a_1^* = 2iω_{2\ast} - a_1,$$
$$a_2^* = 2iω_0 - a_4 - c_1 e^{-2iτ_{2},ω_2},$$
$$a_3^* = 2iω_0 - a_6,$$
$$a_4^* = 2iω_0 - a_8,$$
$$a_5^* = 2iω_0 - a_{10} - b_2 e^{-2iτ_{1},ω_2},$$

and

$$E_1^{(1)} = a_{16} + a_{17} q_3^2 + a_{18} q_3,$$
$$E_1^{(2)} = a_{24} + a_{25} q_3^2 + a_{26} q_3,$$
$$E_1^{(3)} = a_{34} q_3^2 e^{-2iτ_{2},ω_2}, E_1^{(4)} = a_{45} q_3^2 e^{-2iτ_{2},ω_2},$$
$$E_2^{(1)} = 2a_{16} + 2a_{17} q_3 q_3 + 2a_{18} Re(q_3),$$
$$E_2^{(2)} = 2a_{24} + 2a_{25} q_3 q_3 + 2a_{26} Re(q_3),$$
$$E_2^{(3)} = 2a_{34} q_3 q_3, E_2^{(4)} = 2a_{45} q_3 q_3.$$

Then, one can obtain the expressions of $g_{20}, g_{11}, g_{02}$ and $g_{21}$. Further, Equation (24) can be obtained. Thus, we have Theorem 3.5 and the proof is completed.

4. Numerical simulation

In this section, we present a numerical simulation to validate the obtained main results in this paper. By extracting some values from Upadhyay et al. (2017) and considering the conditions for the existence of the Hopf bifurcation, we consider the following special case of system (2):

$$\frac{dS(t)}{dt} = 2 - 0.02S(t) - \frac{0.27S(t)I(t)}{S(t) + I(t) + 0.01} + 0.2V(t - τ_1) - 0.0035S(t),$$
$$\frac{dE(t)}{dt} = \frac{0.27S(t)I(t)}{S(t) + I(t) + 0.01} - 0.02E(t) - 0.2E(t - τ_2),$$
$$\frac{dI(t)}{dt} = 0.2E(t - τ_2) - 0.095I(t) - \frac{0.003(t)}{I(t) + 0.4},$$
$$\frac{dR(t)}{dt} = 0.045I(t) - 0.02R(t) + \frac{0.003(t)}{I(t) + 0.4},$$
$$\frac{dV(t)}{dt} = 0.003S(t) - 0.02V(t) - 0.2V(t - τ_1),$$

from which we obtain the unique viral equilibrium $P_s(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$.

For $τ_1 > 0$, $τ_2 = 0$. We can get $ω_{10} = 1.0325$, $τ_{10} = 8.3515$. Based on Theorem 1, we know that the viral equilibrium $P_s(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$ is locally asymptotically stable when $τ_1 ∈ [0, τ_{10})$. The dynamic behaviour can be shown as in Figures 1 and 2 and in this case, the propagation of the computer virus in system (30) can be predicted and controlled easily. Further, system (30) undergoes a Hopf bifurcation at the viral equilibrium $P_s(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$ when $τ_1 = τ_{10}$ and a family of bifurcating periodic solutions bifurcate from $P_s(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$, which can be illustrated in Figures 3 and 4 and in this case the propagation of the computer virus is out of control. Similarly, we have $ω_{20} = 0.8692$, $τ_{20} = 9.5354$. The corresponding phase plots are drawn as shown in Figures 5–8.

For $τ_1 = τ_2 = τ > 0$. We obtain $ω_0 = 0.5864$, $τ_0 = 8.2806$. From Theorem 3, we know that the viral equilibrium $P_a(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$ is locally asymptotically stable when the delay $τ$ increases from zero to $τ_0$. Once the delay $τ$ passes through $τ_0$, the viral equilibrium $P_a(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$ will lose its stability and a Hopf bifurcation occurs. This property can be illustrated in Figures 9–12.

For $τ_1 = 3.5 ∈ (0, τ_{10})$ and $τ_2 > 0$. We obtain $ω_{2\ast} = 0.3796$, $τ_{20} = 9.8986$. Thus, the viral equilibrium $P_a(10.8619, 8.3082, 17.1832, 38.8068, 0.1481)$ is locally asymptotically stable when $τ_2 ∈ [0, τ_{2\ast})$ and unstable when $τ_2$ passes through $τ_{2\ast}$, which can be depicted by the numerical simulations in Figures 13–16. In addition, we obtain $λ'(τ_{2\ast}) = 19.5886 + 2.3054i$ and $C(0) = -61.7440 + 25.8147i$ by some complex computations. Thus, we have $μ_2 = 3.1520 > 0$, $β_2 = -123.4880 < 0$ and $T_2 = -8.8544 < 0$. Thus, we can conclude that the Hopf bifurcation with respect to $τ_2$ and $τ_1 = 3.5 ∈ (0, τ_{10})$
**Figure 1.** Dynamic behaviour of system (30): projection on $S$–$E$–$I$ with $\tau_1 = 8.0515 < \tau_{10}$.

**Figure 2.** Dynamic behaviour of system (30): projection on $I$–$R$–$V$ with $\tau_1 = 8.0515 < \tau_{10}$.

**Figure 3.** Dynamic behaviour of system (30): projection on $S$–$E$–$I$ with $\tau_1 = 8.5175 > \tau_{10}$.
Figure 4. Dynamic behaviour of system (30): projection on I–R–V with $\tau_1 = 8.5175 > \tau_{10}$.

Figure 5. Dynamic behaviour of system (30): projection on S–E–I with $\tau_2 = 9.4662 < \tau_{20}$.

Figure 6. Dynamic behaviour of system (30): projection on I–R–V with $\tau_2 = 9.4662 < \tau_{20}$. 
**Figure 7.** Dynamic behaviour of system (30): projection on S–E–I with $\tau_2 = 10.5075 > \tau_{20}$.

**Figure 8.** Dynamic behaviour of system (30): projection on I–R–V with $\tau_2 = 10.5075 > \tau_{20}$.

**Figure 9.** Dynamic behaviour of system (30): projection on S–E–I with $\tau = 7.8822 < \tau_0$. 
Figure 10. Dynamic behaviour of system (30): projection on I–R–V with $\tau = 7.8822 < \tau_0$.

Figure 11. Dynamic behaviour of system (30): projection on S–E–I with $\tau = 8.4752 > \tau_0$.

Figure 12. Dynamic behaviour of system (30): projection on I–R–V with $\tau = 8.4752 > \tau_0$. 
**Figure 13.** Dynamic behaviour of system (30): projection on $S$–$E$–$I$ with $\tau_1 = 3.5$ and $\tau_2 = 9.7167 < \tau_{2*}$.

**Figure 14.** Dynamic behaviour of system (30): projection on $I$–$R$–$V$ with $\tau_1 = 3.5$ and $\tau_2 = 9.7167 < \tau_{2*}$.

**Figure 15.** Dynamic behaviour of system (30): projection on $S$–$E$–$I$ with $\tau_1 = 3.5$ and $\tau_2 = 10.4685 > \tau_{2*}$. 
is supercritical; the bifurcating periodic solutions are stable and decrease.

5. Conclusion

A delayed SVEIR computer virus model with nonlinear incidence rate was investigated in this paper. First, conditions guaranteeing the local stability of the viral equilibrium and the existence of the Hopf bifurcation were obtained by choosing the different combination of the two delays. Then, the properties of the Hopf bifurcation are studied by applying a method based on the centre manifold theorem and normal form theory. Finally, numerical results have been presented to validate of the theoretical analysis. It has shown that the propagation of the viruses can be controlled when the value of the delay is below the corresponding critical value. However, a Hopf bifurcation occurs when the value of the delay passes through the corresponding critical value, which means that computers of the five classes in the model may coexist in an oscillatory mode under some conditions and the viruses will be out of control in this case. Through the simulation, we found that the time delay caused by the temporary immunity period and the time delay caused by the latent period play different roles. The effect of the time delay caused by the temporary immunity period is more significant because the corresponding critical value is smaller when we consider it in isolation.

Acknowledgements

The authors are grateful to the editor and the anonymous referees for their valuable comments and suggestions on the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This research was supported by the Natural Science Foundation of the Higher Education Institutions of Anhui Province [grant numbers KJ2018A0437, KJ2019A0655, KJ2019A0656, KJ2019A0662].

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