RATES OF CONVERGENCE FOR EMPirical SPECTRAL MEASURES: A SOFT APPROACH

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ABSTRACT. Understanding the limiting behavior of eigenvalues of random matrices is the central problem of random matrix theory. Classical limit results are known for many models, and there has been significant recent progress in obtaining more quantitative, non-asymptotic results. In this paper, we describe a systematic approach to bounding rates of convergence and proving tail inequalities for the empirical spectral measures of a wide variety of random matrix ensembles. We illustrate the approach by proving asymptotically almost sure rates of convergence of the empirical spectral measure in the following ensembles: Wigner matrices, Wishart matrices, Haar-distributed matrices from the compact classical groups, powers of Haar matrices, randomized sums and random compressions of Hermitian matrices, a random matrix model for the Hamiltonians of quantum spin glasses, and finally the complex Ginibre ensemble. Many of the results appeared previously and are being collected and described here as illustrations of the general method; however, some details (particularly in the Wigner and Wishart cases) are new.

Our approach makes use of techniques from probability in Banach spaces, in particular concentration of measure and bounds for suprema of stochastic processes, in combination with more classical tools from matrix analysis, approximation theory, and Fourier analysis. It is highly flexible, as evidenced by the broad list of examples. It is moreover based largely on “soft” methods, and involves little hard analysis.

The most fundamental problem in random matrix theory is to understand the limiting behavior of the empirical spectral distribution of large random matrices, as the size tends to infinity. The first result on this topic is the famous Wigner semi-circle law, the first version of which was proved by Wigner in 1955 [52, 53]. A random matrix is called a Wigner matrix if it is Hermitian, with independent entries on and above the diagonal. Wigner showed that, under some conditions on the distributions of the entries, the limiting empirical spectral measure of a (normalized) Wigner matrix is the semi-circular law $\rho_{sc}$.

Wigner’s first version of the semi-circle law gave convergence in expectation only; i.e., he showed that the expected number of eigenvalues of a Wigner matrix in an interval converged to the value predicted by the semi-circle law, as the size of the matrix tended to infinity. His second paper improved this to convergence “weakly in probability”. The analog for random unitary matrices, namely that their spectral measures converge to the uniform measure on the circle, seems intuitively obvious; surprisingly, convergence in mean and weak convergence in probability were not proved until nearly 40 years after Wigner’s original work [9].

While these results are fundamental, the limitations of limit theorems such as these are well known. Just as the Berry–Esseen theorem and Hoeffding-type inequalities provide real tools for applications where the classical central limit theorem only justifies heuristics, it is essential to improve the classical limit results of random matrix theory to quantitative approximation results which have content for large but finite random matrices. See [8, 49] for extended discussions of this so-called “non-asymptotic” random matrix theory and its applications.
In this paper, we describe a systematic approach to bounding rates of convergence and proving tail inequalities for the empirical spectral measures of a wide variety of random matrix ensembles. This approach makes use of techniques from probability in Banach spaces, in particular concentration of measure and bounds for suprema of stochastic processes, in combination with more classical tools from matrix analysis, approximation theory, and Fourier analysis. Our approach is highly flexible, and can be used for a wide variety of types of matrix ensembles, as we will demonstrate in the following sections. Moreover, it is based largely on “soft” methods, and involves little hard analysis. Our approach is restricted to settings in which there is a concentration of measure phenomenon; in this sense, it has rather different strengths than the methods used in, for example, [13, 17, 47] and many other works referred to in those papers. Those approaches achieve sharper results without requiring a measure concentration hypothesis, but they require many delicate estimates and are mainly restricted to random matrices constructed from independent random variables, whereas our methods have no independence requirements.

The following key observation, a consequence of the classical Hoffman–Wielandt inequality (see [2, Theorem VI.4.1]), underlies the approach.

**Lemma 1** (see [36, Lemma 2.3]). For an $n \times n$ normal matrix $M$ over $\mathbb{C}$, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues, and let $\mu_M$ denote the spectral measure of $M$; i.e.,

$$\mu_M := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}.$$ 

Then

(a) if $f : \mathbb{C} \rightarrow \mathbb{R}$ is 1-Lipschitz, then the map

$$M \mapsto \int f \, d\mu_M$$

is $\frac{1}{\sqrt{n}}$-Lipschitz, with respect to the Hilbert–Schmidt distance on the set of normal matrices; and

(b) if $\nu$ is any probability measure on $\mathbb{C}$ and $p \in [1, 2]$, the map

$$M \mapsto W_p(\mu_M, \nu)$$

is $\frac{1}{\sqrt{n}}$-Lipschitz.

Here $W_p$ denotes the $L^p$-Kantorovich (or Wasserstein) distance on probability measures on $\mathbb{C}$, defined by

$$W_p(\mu, \nu) = \left( \inf_{\pi} \int |x - y|^p \, d\pi(x, y) \right)^{1/p},$$

where the infimum ranges over probability measures $\pi$ on $\mathbb{C} \times \mathbb{C}$ with marginals $\mu$ and $\nu$. The Kantorovich–Rubinstein theorem (see [50, Theorem 1.14]) gives that

$$W_1(\mu, \nu) = \sup_{|f|_{L^1} \leq 1} \left( \int f \, d\mu - \int f \, d\nu \right),$$

where $|f|_{L^1}$ denotes the Lipschitz constant of $f$; this connects part (a) of Lemma 1 with estimates on $W_1$.

In many random matrix ensembles of interest there is a concentration of measure phenomenon, meaning that well-behaved functions are “essentially constant”, in the sense that
they are close to their means with high probability. A prototype is the following Gaussian concentration phenomenon (see [28]).

**Proposition 2.** If $F : \mathbb{R}^n \to \mathbb{R}$ is a $1$-Lipschitz function and $Z$ is a standard Gaussian random vector in $\mathbb{R}^n$, then

$$
P \left[ F(Z) - E F(Z) \geq t \right] \leq e^{-t^2/2}$$

for all $t > 0$.

Suppose now that $M$ is a random matrix satisfying such a concentration property. Lemma 1 means that one can obtain a bound on $W_p(\mu_M, \nu)$ which holds with high probability if one can bound $E W_p(\mu_M, \nu)$. That is, a bound on the expected distance to the limiting measure immediately implies an asymptotically almost sure bound. The tail estimates coming from measure concentration are typically exponential or better, and therefore imply almost sure convergence rates via the Borel–Cantelli lemma.

We are thus left with the problem of bounding the expected distance from the empirical spectral measure $\mu_M$ to some deterministic reference measure $\nu$. There are two different methods used for this step, depending on the properties of the ensemble:

1. **Eigenvalue rigidity.** In some ensembles, each of the (ordered) individual eigenvalues can be assigned a predicted location based on the limiting spectral measure for the ensemble, such that all (or at least many) eigenvalues concentrate strongly near these predicted locations. In this case $\nu$ is taken to be a discrete measure supported on those predicted locations, and the concentration allows one to easily estimate $E W_p(\mu_M, \nu)$.

2. **Entropy methods.** If instead we set $\nu = E \mu_M$, then the Kantorovich–Rubinstein theorem implies that

$$W_1(\mu_M, \nu) = \sup_{|f|_L \leq 1} \left( \int f \, d\mu_M - E \int f \, d\mu_M \right),$$

so that $W_1(\mu_M, \nu)$ is the supremum of a centered stochastic process indexed by the unit ball of the space of Lipschitz functions on $\mathbb{C}$. In ensembles with a concentration phenomenon for Lipschitz functions, part (a) of Lemma 1 translates to an increment condition on this stochastic process, which gives a route to bounding its expected supremum via classical entropy methods.

Finally, it may still be necessary to estimate the distance from the measure $\nu$ to the limiting spectral measure for the random matrix ensemble. The techniques used to do this vary by the ensemble, but this is a more classical problem of convergence of a sequence of deterministic measures to a limit, and any of the many techniques for obtaining rates of convergence may be useful.

Applications of concentration of measure to random matrices date from at least as long ago as the 1970s; a version of the argument for the concentration of $W_1(\mu_M, \nu)$ essentially appears in the 2000 paper [22] of Guionnet and Zeitouni. See [8, 29, 48] for surveys of concentration methods in random matrix theory.

The method of eigenvalue rigidity to bound Kantorovich distances is particularly suited to situations in which the empirical spectrum is a determinantal point process; this was first observed in the work of Dallaporta [6, 7]. The entropy approach to random Kantorovich distances was introduced in the context of random projections in [33, 34]; it was first applied for empirical spectral measures in [35, 36]. A further abstraction was given by Ledoux [30].
Organization. The rest of this paper is a series of sections sketching some version of the program described above for a number of random matrix ensembles. Sections 1 and 2 discuss Wigner and Wishart matrices, combining eigenvalue rigidity arguments of Dallaporta [6, 7] with measure concentration. Section 3 discusses random matrices drawn uniformly from classical compact matrix groups, and Section 4 discusses powers of such matrices; both those sections follow [37] and also use the eigenvalue rigidity approach. The next three sections use the entropy method: Sections 5 and 6 discusses randomized sums and random compressions of Hermitian matrices, following [36], and Section 7 discusses Hamiltonians of quantum spin glasses, following [3]. Finally, Section 8, following [38], demonstrates in case of the complex Ginibre ensemble, how eigenvalue rigidity alone allows one to carry much of our program even without the use of a general concentration phenomenon together with Lemma 4.

1. Wigner matrices

In this section we outline how our approach can be applied to the most central model of random matrix theory, that of Wigner matrices. We begin with the most classical case: the Gaussian Unitary Ensemble (GUE). Let $\mathbf{M}_n$ be a random $n \times n$ Hermitian matrix, whose entries $\{\mathbf{M}_{jk} \mid 1 \leq j \leq k \leq n\}$ are independent random variables, such that each $\mathbf{M}_{jj}$ has a $\mathcal{N}(0, n^{-1})$ distribution, and each $\mathbf{M}_{jk}$ for $j < k$ has independent real and imaginary parts, each with a $\mathcal{N}(0, (2n)^{-1})$ distribution. Since $\mathbf{M}_n$ is Hermitian, it has real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Wigner’s theorem implies that the empirical spectral measure

$$\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j}$$

converges to the semicircle law $\rho_{sc}$. The following result quantifies this convergence.

**Theorem 3.** Let $\mathbf{M}_n$ be as above, and let $\mu_n$ denote its spectral measure. Then

(a) $\mathbb{E} W_2(\mu_n, \rho_{sc}) \leq C \frac{\sqrt{\log(n)}}{n},$

(b) $\mathbb{P} \left[ W_2(\mu_n, \rho_{sc}) \geq C \frac{\sqrt{\log(n)}}{n} + t \right] \leq e^{-nt^2/2}$ for all $t \geq 0$, and

(c) with probability 1, for sufficiently large $n$, $W_2(\mu_n, \rho_{sc}) \leq C' \frac{\sqrt{\log(n)}}{n}$.

Here and in what follows, symbols such as $c, C, C'$ denote constants which are independent of dimension.

Part (a) of Theorem 3 was proved by Dallaporta in [6] using the eigenvalue rigidity approach; the proof is outlined below.

Lemma 4 and the Gaussian concentration of measure property (Proposition 2), imply that if $F$ is a 1-Lipschitz function (with respect to the Hilbert–Schmidt distance) on the space of Hermitian matrices, then

$$\mathbb{P} \left[ F(\mathbf{M}_n) \geq \mathbb{E} F(\mathbf{M}_n) + t \right] \leq e^{-nt^2/2}$$

for all $t \geq 0$. This fact, together with part (b) of Lemma 4 and part (a) of Theorem 3, now imply part (b). Finally, part (c) follows from part (b) by the Borel–Cantelli lemma. So it remains only to prove part (a).
Define $\gamma_j \in \mathbb{R}$ such that $\rho_{sc}((\infty, \gamma_j]) = \frac{j}{n}$; this is the predicted location of the $j^{th}$ eigenvalue $\lambda_j$ of $M_n$. The discretization $\nu_n$ of the semi-circle law $\rho_{sc}$ is given by

$$\nu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{\gamma_j}.$$  

It can be shown that $W_2(\rho_{sc}, \nu_n) \leq \frac{C}{n}$. Furthermore, by the definition of $W_2$,

$$\mathbb{E}W_2^2(\mu_n, \nu_n) \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}|\lambda_j - \gamma_j|^2.$$  

This reduces the proof of part (a) to estimating the latter expectations.

It is a classical fact that the eigenvalues of the GUE form a determinantal point process with kernel

$$K_n(x, y) = \sum_{j=0}^{n} h_j(x)h_j(y)e^{-(x^2+y^2)/2},$$  

where the $h_j$ are the orthonormalized Hermite polynomials \cite[Section 6.2]{39}. (The reader is referred to \cite{24} for the definition of a determinantal point process.) The following is then a special case of some important general properties of determinantal point processes \cite[Theorem 7]{24}, \cite{23}.

**Proposition 4.** For each $x \in \mathbb{R}$, let $N_x$ denote the number of eigenvalues of $M_n$ which are less than or equal to $x$. Then

$$N_x \stackrel{d}{=} \sum_{i=1}^{n} \xi_i,$$

where the $\xi_i$ are independent $\{0, 1\}$-valued Bernoulli random variables. Moreover,

$$\mathbb{E}N_x = \int_{-\infty}^{x} K_n(u, u) \, du \quad \text{and} \quad \text{Var} N_x = \int_{-\infty}^{x} \int_{x}^{\infty} K_n(u, v)^2 \, du \, dv.$$

The first part of this result can be combined with the classical Bernstein inequality to deduce that for each $t > 0$,

$$\mathbb{P}[|N_x - \mathbb{E}N_x| > t] \leq 2 \exp\left(-\frac{t^2}{2\sigma_x^2 + t}\right),$$  

where $\sigma_x^2 = \text{Var} N_x$. Using estimates on $\mathbb{E}N_x$ due to Götze and Tikhomirov \cite{18} and on $\sigma_x^2$ due to Gustavsson \cite{23} (both of which can be deduced from the second part of Proposition 4), this implies that for $x \in (-2 + \delta, 2 - \delta)$,

$$\mathbb{P}[|N_x - n\rho_{sc}((-\infty, x])| > t + C] \leq 2 \exp\left(-\frac{t^2}{2c_\delta \log(n) + t}\right)$$  

for each $t \geq 0$. Combining this with the observation that

$$\mathbb{P} [\lambda_j > \gamma_j + t] = \mathbb{P} [N_{\gamma_j+t} < j],$$

one can deduce, upon integrating by parts, that

$$\mathbb{E}|\lambda_j - \gamma_j|^2 \leq C_\epsilon \frac{\log(n)}{n^2}.$$
for \( j \in [\epsilon n, (1-\epsilon)n] \). This provides the necessary estimates in the bulk of the spectrum. Dallaporta established similar but weaker bounds for the soft edge of the spectrum using essentially the last part of Proposition 4 and for the hard edge using tail estimates due to Ledoux and Rider [31]. This completes the proof of Theorem 3.

The real symmetric counterpart of the GUE is the Gaussian Orthogonal Ensemble (GOE), whose entries \( \{M_{nk} | 1 \leq j, k \leq n\} \) are independent real random variables, such that each \( M_{jj} \) has a \( N(0, n^{-1}) \) distribution, and each \( M_{jk} \) for \( j < k \) has a \( N(0, (\sqrt{2n})^{-1}) \) distribution. The spectrum of the GOE does not form a determinantal point process, but a close distributional relationship between the eigenvalue counting functions of the GOE and GUE was found in [16, 41]. Using this, Dallaporta showed that part [a] of Theorem 3 also applies to the GOE. Part [b] then follows from the Gaussian concentration of measure property as before, and part [c] from the Borel–Cantelli lemma.

To move beyond the Gaussian setting, Dallaporta invokes the Tao–Vu four moment theorem [46, 45] and a localization theorem due to Erdős, Yau, and Yin [14] to extend Theorem 3(a) to random matrices with somewhat more general entries. The proofs of these results involve the kind of hard analysis which it is our purpose to avoid in this paper. However, it is straightforward, under appropriate hypotheses, to extend the measure concentration argument for part [b] of Theorem 3 and we indicate briefly how this is done.

A probability measure \( \mu \) on \( \mathbb{R} \) is said to satisfy a quadratic transportation cost inequality (QTCI) with constant \( C > 0 \) if

\[
W_2(\mu, \nu) \leq \sqrt{CH(\nu|\mu)}
\]

for any probability measure \( \nu \) which is absolutely continuous with respect to \( \mu \), where \( H(\nu|\mu) \) denotes relative entropy.

**Proposition 5** (see [28, Chapter 6]). Suppose that \( X_1, \ldots, X_n \) are independent random variables whose distributions each satisfy a QTCI with constant \( C \). If \( F : \mathbb{R}^n \to \mathbb{R} \) is a 1-Lipschitz function, then

\[
P\left[F(X) - \mathbb{E}F(X) \geq t\right] \leq e^{-t^2/C}
\]

for all \( t > 0 \).

A QTCI is the most general possible hypothesis which implies subgaussian tail decay, independent of \( n \), for Lipschitz functions of independent random variables; see [19]. It holds in particular for any distribution satisfying a logarithmic Sobolev inequality, including Gaussian distributions, or a distribution with a density on a finite interval bounded above and below by positive constants. Using Dallaporta’s arguments for part [a] and substituting Proposition 5 in place of the Gaussian concentration phenomenon, we arrive at the following generalization of Theorem 3.

**Theorem 6.** Let \( M_n \) be a random Hermitian matrix whose entries satisfy each of the following:

- The random variables \( \{\text{Re } M_{jk}\}_{1 \leq j \leq k \leq n} \) and \( \{\text{Im } M_{jk}\}_{1 \leq j < k \leq n} \) are all independent.
- The first four moments of each of these random variables is the same as for the GUE (respectively, GOE).
- Each of these random variables satisfies a QTCI with constant \( cn^{-1/2} \).

Let \( \mu_n \) denote the spectral measure of \( M_n \). Then
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(a) \( \mathbb{E} W_2(\mu_n, \rho_{sc}) \leq C \frac{\sqrt{\log(n)}}{n} \),

(b) \( \mathbb{P} \left[ W_2(\mu_n, \rho_{sc}) \geq C \frac{\sqrt{\log(n)}}{n} + t \right] \leq e^{-cn^2t^2} \) for all \( t \geq 0 \), and

(c) with probability 1, for sufficiently large \( n \), \( W_2(\mu_n, \rho_{sc}) \leq C' \frac{\sqrt{\log(n)}}{n} \).

As mentioned above, a QTCI is a minimal assumption to reach exactly this result by these methods. A weaker and more classical assumption would be a Poincaré inequality, which implies subexponential decay for Lipschitz functions, and is the most general hypothesis implying any decay independent of \( n \); see [20] and the references therein. If the third condition in Theorem 6 is replaced by the assumption of a Poincaré inequality with constant \( cn^{-1/2} \), then the same kind of argument leads to an almost sure convergence rate of order \( \frac{\log(n)}{n} \); we omit the details.

2. Wishart matrices

In this section we apply the strategy described in the introduction to Wishart matrices (i.e., random sample covariance matrices). Let \( m \geq n \), and let \( X \) be an \( m \times n \) random matrix with i.i.d. entries, and define the Hermitian positive-semidefinite random matrix

\[
S_{m,n} := \frac{1}{m} X^* X.
\]

We denote the eigenvalues of \( S_{m,n} \) by \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \) and the empirical spectral measure by

\[
\mu_{m,n} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}.
\]

It was first proved in [32] that, under some moment conditions, if \( \frac{n}{m} \to \rho > 0 \) as \( n, m \to \infty \), then \( \mu_{m,n} \) converges to the Marchenko–Pastur law \( \mu_\rho \) with parameter \( \rho \), with compactly supported density given by

\[
f_\rho(x) = \frac{1}{2\pi x} \sqrt{(b_\rho - x)(x - a_\rho)},
\]

on \((a_\rho, b_\rho)\), with \( a_\rho = (1 - \sqrt{\rho})^2 \) and \( b_\rho = (1 + \sqrt{\rho})^2 \). The following result quantifies this convergence for many distributions.

Theorem 7. Suppose that for each \( n \), \( 0 < c \leq \frac{n}{m} \leq 1 \), and that \( X \) is an \( m \times n \) random matrix whose entries satisfy each of the following:

- The random variables \( \{\text{Re} \, X_{jk}\}_{1 \leq j \leq m}^{1 \leq k \leq n} \) and \( \{\text{Im} \, X_{jk}\}_{1 \leq j \leq m}^{1 \leq k \leq n} \) are all independent.
- The first four moments of each of these random variables are the same as for a standard complex (respectively, real) normal random variable.
- Each of these random variables satisfies a QTCI with constant \( C \).

Let \( \rho = \frac{n}{m} \) and let \( \mu_{m,n} \) denote the spectral measure of \( S_{m,n} = \frac{1}{m} X^* X \). Then

(a) \( \mathbb{E} W_2(\mu_{m,n}, \mu_\rho) \leq C \frac{\sqrt{\log(n)}}{n} \),

(b) \( \mathbb{P} \left[ W_2(\mu_{m,n}, \mu_\rho) \geq C \frac{\sqrt{\log(n)}}{n} + t \right] \leq e^{-cm \min(nt^2, \sqrt{nt})} \) for all \( t \geq c \frac{\sqrt{\log(n)}}{n} \), and
with probability 1, for sufficiently large $n$, $W_2(\mu_{m,n}, \mu_\rho) \leq C' \sqrt{\log(n)/n}$. 

Strictly speaking, part (c) does not, as stated, imply almost sure convergence of $\mu_{m,n}$, since $\rho$ and hence $\mu_\rho$ itself depends on $n$. However, if $\rho = \rho(n)$ has a limiting value $\rho^*$ as $n \to \infty$ (as in the original Marchenko–Pastur result), then the measures $\mu_\rho$ converge to $\mu_{\rho^*}$. This convergence can easily be quantified, but we will not pursue the details here.

**Proof.** Part (a) was proved by Dallaporta in [7], by the same methods as in Theorem 6(a) discussed in the last section. First, when the entries of $X$ are complex normal random variables (in which $S_{m,n}$ is the unitary Laguerre ensemble), the eigenvalues of $S_{m,n}$ form a determinantal point process. This implies an analogue of Proposition 4, from which eigenvalue rigidity results can be deduced, leading to the estimate in part (a) in this case.

The result is extended to real Gaussian random matrices using interlacing results, and to more general distributions using versions of the four moment theorem for Wishart random matrices. The reader is referred to [7] for the details.

The proof of part (b) is more complicated than in the previous section, because the random matrix $S_{m,n}$ depends quadratically on the independent entries of $X$. However, we can still apply the machinery of measure concentration by using the fact that $S_{m,n}$ possesses local Lipschitz behavior, combined with a truncation argument. Indeed, if $X, Y$ are $m \times n$ matrices over $\mathbb{C}$, then

$$\frac{1}{m} X^* X - \frac{1}{m} Y^* Y \leq \frac{1}{m} \|X^*(X - Y)\|_{HS} + \frac{1}{m} \|(X^* - Y^*)Y\|_{HS} \leq \frac{1}{m} (\|X\|_{op} + \|Y\|_{op}) \|X - Y\|_{HS},$$

where we have used the facts that both the Hilbert–Schmidt norm $\|\cdot\|_{HS}$ and the operator norm $\|\cdot\|_{op}$ are invariant under conjugation and transposition, and that $\|AB\|_{HS} \leq \|A\|_{op} \|B\|_{HS}$.

Thus, for a given $K > 0$, the function $X \mapsto \frac{1}{m} X^* X$ is $\frac{2K}{\sqrt{m}}$-Lipschitz on $\{X \in M_{m,n}(\mathbb{C}) \mid \|X\|_{op} \leq K \sqrt{m}\}$, and so by Lemma [15] the function $F : X \mapsto W_2(\mu_{m,n}, \mu_\rho)$ is $\frac{2K}{\sqrt{mn}}$-Lipschitz on this set. We can therefore extend $F$ to a $\frac{2K}{\sqrt{mn}}$-Lipschitz function $\tilde{F} : M_{m,n}(\mathbb{C}) \to \mathbb{R}$ (cf. [15] Theorem 3.1.2); we may moreover assume that $\tilde{F}(X) \geq 0$ and

$$\sup_{X \in M_{m,n}(\mathbb{C})} \|X\|_{op} \leq K \sqrt{m}$$

Proposition 5 now allows us to control $\tilde{F}(X)$ and $\|X\|_{op}$, which are both Lipschitz functions of $X$.

First, an elementary discretization argument using Proposition 5 (cf. [19] Theorem 5.39), or alternatively Lemma [15] below) shows that

$$P[\|X\|_{op} > K \sqrt{m}] \leq 2e^{-cm}$$

for some $K, c > 0$. We will use this $K$ in the following.
Next, Proposition 5 implies that
\[ P[\tilde{F}(X) > t] \leq C e^{-cmnt^2} \]
as long as \( t \geq 2\mathbb{E}\tilde{F}(X) \). Now
\[ \mathbb{E}\tilde{F}(X) = \mathbb{E}W_2(\mu_{m,n}, \mu_\rho) + \mathbb{E}\left( (\tilde{F}(X) - W_2(\mu_{m,n}, \mu_\rho)) \mathbb{I}_{\|X\|_{op} > K\sqrt{m}} \right) \]
\[ \leq C \frac{\sqrt{\log(n)}}{n} + \left( \sup_{\|X\|_{op} \leq K\sqrt{m}} W_2(\mu_{m,n}, \mu_\rho) \right) \mathbb{P}[\|X\|_{op} > K\sqrt{m}] \]
by part (a) and (3). Since \( \mu_\rho \) is supported on \([a_\rho, b_\rho]\), and \( \mu_{m,n} \) is supported on \([0, \frac{1}{m} \|XX^*\|_{op}] = [0, \frac{1}{m} \|X\|_{op}]\),
\[ \sup_{\|X\|_{op} \leq K\sqrt{m}} W_2(\mu_{m,n}, \mu_\rho) \leq \max\{b_\rho, K^2\} \leq C, \]
and so by (4) and (6),
\[ \mathbb{E}\tilde{F}(X) \leq C \frac{\sqrt{\log(n)}}{n} + C e^{-cm} \leq C' \frac{\sqrt{\log(n)}}{n}. \]
Finally, we have
\[ P[W_2(\mu_{m,n}, \mu_\rho) > t] \leq P[W_2(\mu_{m,n}, \mu_\rho) > t, \|X\|_{op} \leq K\sqrt{m}] + P[\|X\|_{op} > K\sqrt{m}] \]
\[ \leq P[\tilde{F}(X) > t] + P[\|X\|_{op} > K\sqrt{m}] \]
\[ \leq C' e^{-cmnt^2} \]
for \( c_1 \frac{\sqrt{\log(n)}}{n} \leq t \leq c_2 \frac{\sqrt{n}}{n} \) by (2) and (3). We omit the details of the similar argument to obtain a subexponential bound for \( t > c_2 \frac{\sqrt{n}}{n} \). This concludes the proof of part (b).

Part (c) follows as before using the Borel–Cantelli lemma.

An alternative approach to quantifying the limiting behavior of the spectrum of Wishart matrices is to consider the singular values \( 0 \leq \sigma_1 \leq \cdots \leq \sigma_n \) of \( \frac{1}{\sqrt{m}}X \); that is, \( \sigma_j = \sqrt{\lambda_j} \).Lemma 1 can be applied directly in that context, by using the fact that the eigenvalues of the Hermitian matrix \( \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \) are \( \{ \pm \sigma_j \} \). However, if one is ultimately interested in the eigenvalues \( \{ \lambda_j \} \), then translating the resulting concentration estimates to eigenvalues ends up requiring the same kind of analysis carried out above.

3. Uniform random matrices from the compact classical groups

Each of the compact classical matrix groups \( O(n), SO(n), U(n), SU(n), Sp(2n) \) possesses a uniform (Haar) probability measure which is invariant under translation by a fixed group element. Each of these uniform measures possesses a concentration of measure property making it amenable to the program laid out in the introduction; moreover, the eigenvalues of a random matrix from any of these groups is a determinantal point process, meaning that the eigenvalue rigidity approach used in Section 1 applies here as well. The limiting empirical spectral measure for all of these groups is the uniform probability measure on the
circle, as first shown in [9]. This convergence is quantified in the following result, proved in [37].

**Theorem 8.** Let $M_n$ be uniformly distributed in any of $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(2n)$, and let $\mu_n$ denote its spectral measure. Let $\mu$ denote the uniform probability measure on the unit circle $S^1 \subseteq \mathbb{C}$. Then

(a) $E W_2(\mu_n, \mu) \leq C \sqrt{\frac{\log(n)}{n}}$

(b) $P \left[ W_2(\mu_n, \mu) \geq C \sqrt{\frac{\log(n)}{n}} + t \right] \leq e^{-cn^2t^2}$, and

(c) with probability 1, for sufficiently large $n$, $W_2(\mu_n, \mu) \leq C \sqrt{\frac{\log(n)}{n}}$.

We briefly sketch the proof below; for full details, see [37]. Part (a) is proved using the eigenvalue rigidity approach described in Section 1 for the GUE. We first order the eigenvalues of $M_n$ as $\{e^{i\theta_j}\}_{1 \leq j \leq n}$ with $0 \leq \theta_1 \leq \cdots \leq \theta_n < 2\pi$, and define the discretization $\nu_n$ of $\mu$ by

$$\nu_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{e^{i2\pi j/n}}.$$ 

It is easy to show that $W_2(\mu, \nu_n) \leq \frac{C}{n}$, and by the definition of $W_2$, 

$$E W_2^2(\mu_n, \nu_n) \leq \frac{1}{n} \sum_{j=1}^{n} E \left| e^{i\theta_j} - e^{i2\pi j/n} \right|^2 \leq \frac{1}{n} \sum_{j=1}^{n} E \left| \theta_j - \frac{2\pi j}{n} \right|^2,$$

so that part (a) can be proved by estimating the latter expectations.

For these estimates, as for the GUE, one can make use of the determinantal structure of the eigenvalue processes of uniformly distributed random matrices. For the case of the unitary group $U(n)$, the eigenvalue angles $\{\theta_j\}$ form a determinantal point process on $[0, 2\pi)$ with kernel 

$$K_n := \frac{\sin \left( \frac{n(x-y)}{2} \right)}{\sin \left( \frac{x-y}{2} \right)};$$

this was first proved by Dyson [11]. The determinantal structure provides an analogue of Proposition 4.

**Proposition 9.** For each $0 \leq x < 2\pi$, let $N_x$ denote the number of eigenvalues $e^{i\theta_j}$ of $M_n \in U(n)$ such that $\theta_j \leq x$. Then

$$N_x = \sum_{i=1}^{n} \xi_i,$$

where the $\xi_i$ are independent $\{0,1\}$-valued Bernoulli random variables.

Moreover,

$$EN_x = \int_0^x K_n(u,u) \, du \quad \text{and} \quad \text{Var} N_x = \int_0^x \int_x^{2\pi} K_n(u,v)^2 \, du \, dv.$$
Appropriately modified versions of Proposition 9 hold for the other groups as well, due to determinantal structures in those contexts identified by Katz and Sarnak [26].

Using (9), one can estimate $\mathbb{E}N_x$ and $\text{Var} N_x$, and then use (8) and Bernstein’s inequality to deduce that

$$\mathbb{P} \left[ N_x - \frac{nx}{2\pi} > t + C \right] \leq 2 \exp \left( -\frac{t^2}{2c \log(n) + t} \right)$$

for all $t > 0$. Combining this with the observation that

$$\mathbb{P} \left[ \theta_j > \frac{2\pi j}{n} + t \right] = \mathbb{P} \left[ N_{\frac{2\pi j}{n} + t} < j \right],$$

one can deduce, upon integrating by parts, that

$$\mathbb{E} \left| \theta_j - \frac{2\pi j}{n} \right|^2 \leq C \frac{n \log(n)}{n^2},$$

for each $j$, which completes the proof of part (a). Observe that this is made slightly simpler than the proof of Theorem 4(a) for the GUE by the fact that all of the eigenvalues of a unitary matrix behave like “bulk” eigenvalues.

Part (b) of Theorem 8 follows from part (a) and the following concentration of measure property of the uniform measure on the compact classical groups. (There is an additional subtlety in dealing with the two components of $O(n)$, which can be handled by conditioning on $\det M_n$.)

**Proposition 10.** Let $G_n$ be one of $SO(n)$, $SU(n)$, $U(n)$, or $S\mathbb{P}(2n)$, and let $F : G_n \to \mathbb{R}$ be 1-Lipschitz, with respect to either the Hilbert–Schmidt distance or the geodesic distance on $G_n$. Let $M_n$ be a uniformly distributed random matrix in $G_n$. Then

$$\mathbb{P} \left[ |F(M_n) - \mathbb{E}F(M_n)| > t \right] \leq e^{-cnt^2}$$

for every $t > 0$.

For $SO(n)$, $SU(n)$, and $S\mathbb{P}(2n)$, this property goes back to the work of Gromov and Milman [21]; for the precise version stated here see [1, Section 4.4]. For $U(n)$ (which was not covered by the results of [21] because its Ricci tensor is degenerate), the concentration in Proposition 10 was proved in [37].

Finally, part (c) follows from part (b) via the Borel-Cantelli lemma, thus completing the proof of Theorem 8.

### 4. Powers of uniform random matrices

The approach used with random matrices from the compact classical groups in the previous section can be readily generalized to powers of such matrices, as follows.

**Theorem 11.** Let $M_n$ be uniformly distributed in any of $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $S\mathbb{P}(2n)$. Let $m \geq 1$, and let $\mu_{m,n}$ denote the spectral measure of $M_n^m$. Let $\mu$ denote the uniform probability measure on the unit circle $S^1 \subseteq \mathbb{C}$. There are universal constants $C, c$ such that

(a) $\mathbb{E}W_2(\mu_{m,n}, \mu) \leq C \sqrt{m \left( \log \left( \frac{n}{m} \right) + 1 \right)}$,
(b) \[ P \left[ W_2(\mu_{m,n}, \mu) \geq C \sqrt{\frac{m \left( \log \left( \frac{n}{m} \right) + 1 \right)}{n} + t} \right] \leq e^{-cn^2t^2}, \] and

(c) with probability 1, for sufficiently large \( n \), \( W_2(\mu_{m,n}, \mu) \leq C \sqrt{\frac{m \left( \log \left( \frac{n}{m} \right) + 1 \right)}{n}} \).

In fact, the same proof works for \( m > 1 \) as in the previous section, because of the following result of Rains \[42\]. The result is stated in the unitary case for simplicity, but analogous results hold in the other compact classical matrix groups.

**Proposition 12.** Let \( m \leq n \) be fixed. If \( M_n \) is uniformly distributed in \( U(n) \), the eigenvalues of \( M_n^m \) are distributed as those of \( m \) independent uniform unitary matrices of sizes \( \left\lfloor \frac{n}{m} \right\rfloor := \max \{ k \in \mathbb{N} \mid k \leq \frac{n}{m} \} \) and \( \left\lceil \frac{n}{m} \right\rceil := \min \{ k \in \mathbb{N} \mid k \geq \frac{n}{m} \} \), such that the sum of the sizes of the matrices is \( n \).

As a consequence, if \( N_x \) is the number of eigenvalues of \( M_n^m \) lying in the arc from 1 to \( e^{ix} \), then

\[ N_x = \sum_{j=0}^{m-1} N_x^j, \]

where the \( N_x^j \) are the counting functions of \( m \) independent random matrices, each uniformly distributed in \( U \left( \left\lfloor \frac{n}{m} \right\rfloor \right) \) or \( U \left( \left\lceil \frac{n}{m} \right\rceil \right) \). In particular, by Proposition 9, \( N_x \) is equal in distribution to a sum of independent Bernoulli random variables, and its mean and variance can be estimated using the available estimates for the individual summands established in the previous section. One can thus again apply Bernstein’s inequality to obtain eigenvalue rigidity, leading to a bound on \( \mathbb{E} W_2(\mu_{m,n}, \mu) \).

Crucially, the concentration phenomenon on the compact classical groups tensorizes in a dimension-free way: the product of uniform measure on the \( m \) smaller unitary groups above has the same concentration property as any one of those groups. This is a consequence of the fact that the uniform measures on the compact classical groups satisfy logarithmic Sobolev inequalities; see [1, Section 4.4] and the Appendix of [37]. This allows for the full program laid out in the introduction to be carried out in this case, yielding Theorem 11 above.

**5. Randomized sums**

In this section we show how our approach can be applied to randomized sums of Hermitian matrices. In this and the following two sections, we no longer have a determinantal structure allowing us to use eigenvalue rigidity. Instead we will use entropy methods to bound the expected distance between the empirical spectral measure and its mean.

Let \( A_n \) and \( B_n \) be fixed \( n \times n \) Hermitian matrices, and let \( U_n \in U(n) \) be uniformly distributed. Define

\[ M_n := U_n A_n U_n^* + B_n; \]

the random matrix \( M_n \) is the so-called randomized sum of \( A_n \) and \( B_n \). This random matrix model has been studied at some length in free probability theory: the limiting spectral measure was studied first by Voiculescu [51] and Speicher [43], who showed that if \( \{A_n\} \) and \( \{B_n\} \) have limiting spectral distributions \( \mu_A \) and \( \mu_B \) respectively, then the limiting spectral distribution of \( M_n \) is given by the free convolution \( \mu_A \boxplus \mu_B \).
The following sharpening of this convergence is a special case of Theorem 3.8 and Corollary 3.9 of [36]; we present below a slightly simplified version of the argument from that paper.

**Theorem 13.** In the setting above, let \( \mu_n \) denote the empirical spectral measure of \( M_n \), and let \( \nu_n = \mathbb{E}\mu_n \). Then

\[
(a) \quad \mathbb{E} W_1(\mu_n, \nu_n) \leq C \frac{\|A_n\|_{op}^{2/3} (\|A_n\|_{op} + \|B_n\|_{op})^{1/3}}{n^{2/3}},
\]

\[
(b) \quad \mathbb{P} \left[ W_1(\mu_n, \nu_n) \geq C \frac{\|A_n\|_{op}^{2/3} (\|A_n\|_{op} + \|B_n\|_{op})^{1/3}}{n^{2/3}} + t \right] \leq e^{-cn^2 t^2/\|A_n\|_{op}^2}, \text{ and}
\]

\[
(c) \quad \text{with probability } 1, \text{ for sufficiently large } n,
\]

\[
W_1(\mu_n, \nu_n) \leq C' \frac{\|A_n\|_{op}^{2/3} (\|A_n\|_{op} + \|B_n\|_{op})^{1/3}}{n^{2/3} - 2/3}.
\]

In the most typical situations of interest, \( \|A_n\|_{op} \) and \( \|B_n\|_{op} \) are bounded independently of \( n \). If \( \{A_n\} \) and \( \{B_n\} \) have limiting spectral distributions \( \mu_A \) and \( \mu_B \) respectively, then the rate of convergence of the (deterministic) measures \( \nu_n \) to \( \mu_A \oplus \mu_B \) will depend strongly on the sequences \( \{A_n\} \) and \( \{B_n\} \); we will not address that question here.

The Lipschitz property which is a crucial ingredient of our approach to prove Theorem 13 is provided by the following lemma.

**Lemma 14.** For each 1-Lipschitz function \( f : \mathbb{R} \to \mathbb{R} \), the maps

\[
U_n \mapsto \int f \, d\mu_n \quad \text{and} \quad U_n \mapsto W_1(\mu_n, \nu_n)
\]

are \( 2\|A_n\|_{op}^{\sqrt{n}} \)-Lipschitz on \( U(n) \).

**Proof.** Let \( A \) and \( B \) be \( n \times n \) Hermitian matrices, and let \( U, V \in U(n) \). Then it is straightforward to show that

\[
\left\| (UAU^* + B) - (VAV^* + B) \right\|_{HS} \leq 2 \|A\|_{op} \left\| U - V \right\|_{HS}
\]

(see [36, Lemma 3.2]). The lemma now follows by Lemma 11. \( \square \)

Part (b) of Theorem 13 now follows from part (a) using Lemma 14 and the concentration of measure phenomenon for \( U(n) \) (Proposition 10), and part (c) follows as usual by the Borel–Cantelli lemma. It remains to prove part (a), as mentioned above, this is done using entropy techniques for bounding the supremum of a stochastic process.

The following lemma summarizes what is needed here. This fact is well-known to experts, but we were not able to find an explicit statement in the literature.

**Lemma 15.** Suppose that \( (V, \|\cdot\|) \) be a finite-dimensional normed space with unit ball \( \mathcal{B}(V) \), and that \( \{X_v \mid v \in V\} \) is a family of centered random variables such that

\[
\mathbb{P}[|X_u - X_v| \geq t] \leq 2e^{-t^2/K^2\|u-v\|^2}
\]

for every \( t \geq 0 \). Then

\[
\mathbb{E} \sup_{v \in \mathcal{B}(V)} X_v \leq CK \sqrt{\dim V}.
\]
Lemma 5.2]) shows that
\[ N(13) \]
Lemma 14 and Proposition 10 imply that so that Lip\(0\) then Dudley’s bound yields
\[ X(11) \]
By the Kantorovich–Rubinstein theorem, thus by (12),
\[ W_i(\mu_n, \nu_n) = \sup \{ X_f : f \in \mathcal{B}(\text{Lip}_0) \}. \]
Lemma [44] and Proposition [10] imply that
\[ \mathbb{P} [ |X_f - X_g| \geq t] = \mathbb{P} [ |X_f| \geq t] \leq 2 \exp \left[ -\frac{cn^2 t^2}{\|A_n\|_\infty^2 |f|_\infty^2} \right]. \]
We would like to appeal to Lemma [15] but unfortunately, Lip\(0\) is infinite-dimensional. We can get around this problem with an additional approximation argument.
Observing that \(\mu_n\) is supported on \([-\|M_n\|_\infty, \|M_n\|_\infty]\) and \(\|M_n\|_\infty \leq \|A_n\|_\infty + \|B_n\|_\infty\), we begin by replacing Lip\(0\) with
\[ \text{Lip}_0([-R, R]) := \{ f : [-R, R] \to \mathbb{R} | |f|_L < \infty \text{ and } f(0) = 0 \}, \]
with \(R = \|A_n\|_\infty + \|B_n\|_\infty\), for (11), (12), and (13) above. Now for an integer \(m \geq 1\), let Lip\(0^n([-R, R])\) be the 2\(m\)-dimensional space of piecewise affine functions \(f \in \text{Lip}_0([-R, R])\) such that \(f\) is affine on each interval \([-R + \frac{(k-1)R}{m}, -R + \frac{kR}{m}]\) for \(k = 1, \ldots, 2m\). Given \(f \in \text{Lip}_0([-R, R])\), there is a unique function \(g \in \text{Lip}_0^n([-R, R])\) such that \(g(\frac{jR}{m}) = f(\frac{jR}{m})\) for each integer \(j \in [-m, m]\); and this \(g\) satisfies
\[ |g|_L \leq |f|_L \quad \text{and} \quad \|f - g\|_\infty \leq \frac{|f|_L R}{2m}. \]
Thus by (12),
\[ W_1(\mu_n, \nu_n) \leq \frac{R}{2m} + \sup \{ X_g : g \in \mathcal{B}(\text{Lip}_0^n([-R, R])) \}. \]
Now by (13) and Lemma [15]
\[ \mathbb{E} W_1(\mu_n, \nu_n) \leq \frac{R}{2m} + C \frac{\|A_n\|_\infty \sqrt{m}}{n}. \]
Part (a) now follows by optimizing over \(m\). This completes the proof of Theorem 13.
An additional conditioning argument allows one to consider the case that $A_n$ and $B_n$ are themselves random matrices in Theorem 13, assuming a concentration of measure property for their distributions. We refer to [36] for details.

It seems that the entropy method does not usually result in sharp rates; for example, in [36], we used the entropy approach for Wigner and Haar-distributed matrices, and the results were not as strong as those in Sections 1 and 3. On the other hand, the entropy method is more widely applicable than the determinantal point process methods which yielded the results of Sections 1 and 3. In addition to the randomized sums treated in this section, we show in Sections 6 and 7 how the entropy method can be used for random compressions and for the Hamiltonians of quantum spin glasses. The paper [36] also used the entropy approach to prove convergence rates for the empirical spectral measures of the circular orthogonal ensemble and the circular symplectic ensemble, which we have omitted from this paper.

6. Random compressions

Let $A_n$ be a fixed $n \times n$ Hermitian (respectively, real symmetric) matrix, and let $U_n$ be uniformly distributed in $U(n)$ (respectively, $O(n)$). Let $P_k$ denote the projection of $\mathbb{C}^n$ (respectively $\mathbb{R}^n$) onto the span of the first $k$ standard basis vectors. Finally, define a random matrix $M_n$ by

$$M := P_k U_n A_n U_n^* P_k^*.$$  

Then $M_n$ is a compression of $A_n$ to a random $k$-dimensional subspace. In the case that $\{A_n\}_{n \in \mathbb{N}}$ has a limiting spectral distribution and $\frac{k}{n} \to \alpha$, the limiting spectral distribution of $M_n$ can be determined using techniques of free probability (see [43]); the limit is given by a free-convolution power related to the limiting spectral distribution of $A_n$ and the value $\alpha$.

For this random matrix model, the program laid out in the introduction produces the following (cf. Theorem 3.5 and Corollary 3.6 in [36]).

**Theorem 16.** In the setting above, let $\mu_n$ denote the empirical spectral distribution of $M_n$, and let $\nu_n = E\mu_n$. Then

(a) $E W_1(\mu_n, \nu_n) \leq C \frac{\|A_n\|_{\text{op}}}{((kn)^{1/3})},$

(b) $P \left[ W_1(\mu_n, \nu_n) \geq C \frac{\|A_n\|_{\text{op}}}{(kn)^{1/3}} + t \right] \leq e^{-ckn^2/\|A_n\|_{\text{op}}^2},$ and

(c) with probability 1, for sufficiently large $n$, $W_1(\mu_n, \nu_n) \leq C' \|A_n\|_{\text{op}} (kn)^{-1/3}$.

The proof is essentially identical to the one in the previous section; the $k$-dependence in the bounds is a consequence of the fact that $k$, not $n$, is the size of the matrix when Lemma 1 is applied. As with Theorem 13 an additional conditioning argument allows one to consider the case that $A_n$ is random, with distribution satisfying a concentration of measure property.

7. Hamiltonians of quantum spin glasses

In this section we consider the following random matrix model for the Hamiltonian of a quantum spin glass: let $\{Z_{a,b,j}\}_{1 \leq a,b \leq 3}^{1 \leq j \leq n}$ be independent standard Gaussian random variables,
and define the $2^n \times 2^n$ random Hermitian matrix $H_n$ by

$$H_n := \frac{1}{\sqrt{9n}} \sum_{j=1}^{n} \sum_{a,b=1}^{3} Z_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)},$$

where for $1 \leq a \leq 3$,

$$\sigma_j^{(a)} := I_n^\otimes (j-1) \otimes \sigma^{(a)} \otimes I_2^\otimes (n-j),$$

with $I_2$ denoting the $2 \times 2$ identity matrix, $\sigma^{(a)}$ denoting the $2 \times 2$ matrices

$$\sigma^{(1)} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^{(2)} := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^{(3)} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the labeling cyclic so that $\sigma_{n+1} := \sigma_1^{(b)}$. The random matrix $H_n$ acts on the space $(\mathbb{C}^2)\otimes^n$ of $n$ distinguishable qubits; the specific structure of $H_n$ above corresponds to nearest-neighbor interaction on a circle of qubits.

If $\mu_n$ denotes the empirical spectral measure of $H_n$, then the ensemble average $\nu_n = \mathbb{E}\mu_n$ is known in this context as the density of states measure $\mu_n^{\text{DOS}}$. Recently, Keating, Linden and Wells [27] showed that $\mu_n^{\text{DOS}}$ converges weakly to Gaussian, as $n \to \infty$; i.e., they showed that the empirical spectral measure of $H_n$ converges to Gaussian in expectation. The paper [27] gives a similar treatment for more general collections of (still independent) coupling coefficients, and more general coupling geometries than that of nearest-neighbor interactions. In more recent work, Erdős and Schröder [12] have considered still more general coupling geometries, and found a sharp transition in the limiting behavior of the density of states measure depending on the size of the maximum degree of the underlying graph, relative to its number of edges.

The following result, essentially proved in [3], quantifies this convergence.

**Theorem 17.** Let $\mu_n$ be the spectral measure of $H_n$ and let $\gamma$ denote the standard Gaussian distribution on $\mathbb{R}$. Then

(a) $\mathbb{E}W_1(\mu_n, \gamma) \leq C' n^{-1/6}$,

(b) $\mathbb{P} \left[ W_1(\mu_n, \gamma) \geq C' n^{-1/6} + t \right] \leq e^{-9nt^2/2}$, and

(c) with probability 1, for all sufficiently large $n$,

$$W_1(\mu_n, \gamma) \leq C'' n^{-1/6}.$$

Because the coefficients $Z_{a,b,j}$ in (15) are taken to be i.i.d. Gaussian random variables, the Gaussian concentration of measure phenomenon (Proposition 2) can be combined with Lemma 1 to carry out a version of the approach used in the cases of random sums and random compressions (Sections 5 and 6). The following lemma provides the necessary link between Lemma 1 and Proposition 2 for this random matrix model.

**Lemma 18.** Let $x = \{x_{a,b,j}\} \in \mathbb{R}^{9n}$ (with, say, lexicographic ordering), and assume that $n \geq 3$. Define $H_n(x)$ by

$$H_n(x) := \frac{1}{3\sqrt{n}} \sum_{a,b=1}^{3} \sum_{j=1}^{n} x_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)}.$$

Then the map $x \mapsto H_n$ is $\frac{2n/2}{3\sqrt{n}}$-Lipschitz.
Lemma 18 and Lemma 1(b) together show that
\[ x \mapsto W_1(\mu_n, \gamma) \]
is a \( \frac{1}{3\sqrt{n}} \)-Lipschitz function of \( x \). Part (b) of Theorem 17 then follows from part (a) and Proposition 2, and part (c) follows by the Borel–Cantelli lemma.

The proof of part (a) has two main components. First, \( W_1(\mu_n, E\mu_n) \) is estimated via the approach used in Sections 5 and 6: Lemma 18, Lemma 1(a), and Proposition 2 show that the stochastic process
\[ X_f := \int f \, d\mu_n - E \int f \, d\mu_n \]
satisfies a subgaussian increment condition as in Lemma 15, which can then be used to show that
\[ EW_1(\mu_n, E\mu_n) \leq \frac{C}{n^{1/6}}. \]

Second, the convergence in expectation proved in [27] was done via a pointwise estimate of the difference between the characteristic functions of \( E\mu_n \) and \( \gamma \); this estimate can be parlayed into an estimate on \( W_1(\mu_n, \gamma) \) via Fourier analysis. This is carried out in detail in [3] for the bounded-Lipschitz distance; a similar argument shows that
\[ W_1(E\mu_n, \gamma) \leq \frac{C}{n^{1/6}}, \]
completing the proof of Theorem 17.

8. The complex Ginibre ensemble

Let \( G_n \) be an \( n \times n \) random matrix with i.i.d. standard complex Gaussian entries; \( G_n \) is said to belong to the complex Ginibre ensemble. It was first established by Mehta that if \( \mu_n \) is the empirical spectral measure of \( \frac{1}{\sqrt{n}} G_n \), then as \( n \to \infty \), \( \mu_n \) converges to the circular law; i.e., to the uniform measure \( \mu \) on the unit disc \( D := \{ z \in \mathbb{C} \mid |z| \leq 1 \} \).

This is the one ensemble we treat in which the general concentration of measure approach does not apply. The issue is that while there is a concentration phenomenon for the i.i.d. Gaussian entries of \( G_n \), the spectral measure of a nonnormal matrix (\( G_n \) is nonnormal with probability 1) is not a Lipschitz function of the matrix. Nevertheless, the eigenvalue process of \( G_n \) is a determinantal point process, and so some of the techniques used above are still available. We sketch the basic idea below; full details can be found in [38].

The eigenvalues of \( G_n \) form a determinantal point process on \( \mathbb{C} \) with the kernel
\[ K(z, w) = \frac{1}{\pi} e^{-((|z|^2+|w|^2)/2} \sum_{k=0}^{n-1} \frac{(z \bar{w})^k}{k!}. \]

This means that in principle, the determinantal approach to eigenvalue rigidity used in the case of the GUE (Section 1) and of the compact classical groups (Section 3) can be used for this model. A challenge, however, is the lack of an obvious order on the eigenvalues of an arbitrary matrix over \( \mathbb{C} \); without one, there is no hope to assign predicted locations around which the individual eigenvalues concentrate. We therefore impose an order on \( \mathbb{C} \) which is well-adapted for our purposes; we refer to this as the spiral order. Specifically, the linear order \( \prec \) on \( \mathbb{C} \) is defined by making 0 initial, and for nonzero \( w, z \in \mathbb{C} \), we declare \( w \prec z \) if any of the following holds:

- \( \lfloor \sqrt{n} |w| \rfloor < \lfloor \sqrt{n} |z| \rfloor \).
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• ⌊ √ \text{n} \mid w \mid ⌋ = ⌊ √ \text{n} \mid z \mid ⌋ and \arg w < \arg z.
• ⌊ √ \text{n} \mid w \mid ⌋ = ⌊ √ \text{n} \mid z \mid ⌋, \arg w = \arg z, and \mid w \mid \geq \mid z \mid.

Here we are using the convention that \arg z \in (0, 2\pi].

We order the eigenvalues according to \prec: first the eigenvalues in the disc of radius \frac{1}{\sqrt{n}} are listed in order of increasing argument, then the ones in the annulus with inner radius \frac{1}{\sqrt{n}} and outer radius \frac{2}{\sqrt{n}} in order of increasing argument, and so on. We then define predicted locations \tilde{\lambda}_j for (most of) the eigenvalues based on the spiral order: \tilde{\lambda}_1 = 0, \{\tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4\} are \frac{1}{\sqrt{n}} times the 3\text{rd} roots of unity (in increasing order with respect to \prec), the next five are \frac{2}{\sqrt{n}} times the 5\text{th} roots of unity, and so on. Letting \nu_n denote the normalized counting measure supported on the \{\tilde{\lambda}_j\}, it is easy to show that

W_2(\nu_n, \mu) \leq C \frac{n}{\sqrt{n}}.

(One can deduce a slightly tighter bound for \text{EW}_p(\mu_n, \nu_n) for 1 \leq p < 2, and a weaker one for p > 2.)

In this setting we cannot argue that the concentration of \text{W}_1(\mu_n, \mu) is immediate from general concentration properties of the ensemble, but the eigenvalue rigidity itself can be used as a substitute. Indeed,

W_2(\mu_n, \nu_n)^2 \leq \frac{1}{n} \sum_{j=1}^{n} \mid \lambda_j - \tilde{\lambda}_j \mid^2,

and so

\text{P} \left[ W_2(\mu_n, \nu_n)^2 > t \right] \leq \text{P} \left[ \sum_{j=1}^{n} \mid \lambda_j - \tilde{\lambda}_j \mid^2 > nt \right] \leq \sum_{j=1}^{n} \text{P} \left[ \mid \lambda_j - \tilde{\lambda}_j \mid^2 > t \right].

For most of the eigenvalues the eigenvalue rigidity about \tilde{\lambda}_j is strong enough to bound this quite sharply; as before, for about \sqrt{n \log(n)} of the largest eigenvalues a more trivial bound is used. Since this approach does not produce a particularly clean tail inequality for \text{W}_2(\mu_n, \nu_n), we will instead simply state the almost-sure convergence rate which follows by the Borel–Cantelli lemma.

**Theorem 19.** Let \mu_n denote the empirical spectral measure of \frac{1}{\sqrt{n}} G_n, and let \mu denote the uniform measure on the unit disc in \mathbb{C}. Then with probability 1, for sufficiently large n,

W_2(\mu_n, \mu) \leq C \frac{\sqrt{\log n}}{n^{1/4}}.
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