NEW METHODS IN SPECTRAL THEORY OF $N$-BODY SCHRÖDINGER OPERATORS

T. ADACHI, K. ITAKURA, K. ITO, AND E. SKIBSTED

Abstract. We develop a new scheme of proofs for spectral theory of the $N$-body Schrödinger operators, reproducing and extending a series of sharp results under minimum conditions. Our main results include Rellich’s theorem, limiting absorption principle bounds, microlocal resolvent bounds, Hölder continuity of the resolvent and a microlocal Sommerfeld uniqueness result. We present a new proof of Rellich’s theorem which is unified with exponential decay estimates studied previously only for $L^2$-eigenfunctions. Each pair-potential is a sum of a long-range term with first order derivatives, a short-range term without derivatives and a singular term of operator- or form-bounded type, and the setup includes hard-core interaction. Our proofs consist of a systematic use of commutators with ‘zeroth order’ operators. In particular they do not rely on Mourre’s differential inequality technique.

Contents

1. Introduction 2
1.1. Setting 3
1.2. Results 6
2. Preliminaries: Operator $B$ 10
2.1. Notation 10
2.2. Functional calculus 11
2.3. Self-adjoint realization 13
2.4. First commutator 15
2.5. Second commutator 19
3. Proof of Rellich type theorems 22
3.1. Exponential decay estimates 22
3.2. Super-exponentially decaying eigenfunctions 27
4. Proof of LAP bounds 30
5. Proof of microlocal resolvent bounds and applications 32
5.1. Microlocal resolvent bounds 32
5.2. Applications 35
References 37

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1. Introduction

In this paper we introduce new methods establishing a series of sharp results in spectral theory of \( N \)-body Schrödinger operators. Using elementary functional calculus and mostly rather standard commutator arguments, we obtain Rellich’s theorem, LAP (Limiting Absorption Principle) bounds, microlocal resolvent bounds, Hölder continuity of the resolvent and a microlocal Sommerfeld uniqueness result. These are fundamental ingredients of the stationary scattering theory, which however is only poorly developed so far for \( N \geq 2 \), and particularly for \( N \geq 3 \). Moreover our results have interest of their own. Previously sharp, or Besov space, versions of the results were obtained only by sophisticated Mourre technology. In this paper we reformulate and refine the Mourre estimate in terms of a certain ‘zeroth order’ operator \( B \), and we prove ‘sharp results’ under natural and minimum assumptions. In fact each pair-potential is a sum of a long-range term with first order derivatives, a short-range term without derivatives and a singular term of operator- or form-bounded type. Hard-core interaction is also included (without additional complication).

We provide a unified treatment of Rellich’s theorem [IS2, Is] and exponential decay estimates of \( L^2 \)-eigenfunctions [FH] (thereby solving a problem stated in [IS2]). A sharp version of Rellich’s theorem for \( N \)-body operators was established only recently in [IS2]. However we extend it to an even stronger and more classical form. In addition, our proof of the sharp LAP bounds for \( N \)-body operators, which was first obtained by Jensen and Perry [JP], does not rely on Mourre’s differential inequality technique. This is in contrast to all the existing proofs we are aware of. Instead, an integral part of our proof of the LAP bounds consists of Rellich’s theorem, being to some extent similar to the proof by Agmon and Hörmander [AH, Hö] in the 1-body case. The precise setting will be presented in Section 1.1, and the main results in Section 1.2.

In the spectral theory of Schrödinger operators it has been an issue with a long history to achieve ‘minimum conditions’ on the pair-potentials. To our knowledge Lavine was the first who proved LAP in the 1-body case for a natural class of potentials by a commutator method [La1, La2]. After the discovery of the Mourre method [Mo1, Mo2] the question about minimum conditions, in particular the historically painful issue of inclusion of arbitrary short-range potentials, was raised again. This was particularly the case in the \( N \)-body setting where Mourre’s differential inequality technique, which involves a certain double commutator (arising from commutation with a certain ‘first order’ operator \( A \)) not obviously compatible with short-range potentials, was the only available method for a couple of decades. See [ABG1, ABG2, BGM1, BGM2, Ta] for studies of this problem. A similar study of the Mourre method for form-bounded potentials was done in [BMP] in the 1-body setting.

Inventing new techniques, we not only reproduce known LAP results for operator-bounded potentials, but also obtain new ones for form-bounded potentials; a brief comparison with the literature is given in Section 1.2. This being said we still have a ‘double commutator problem’, although in a disguised form in commutation with the operator \( B \). Most of our proofs use in a systematic way commutation with \( B \) (the same operator was used in [GIS], however not in a systematic way). It is a standing open problem to show the basic results of the stationary scattering theory
NEW METHODS IN SPECTRAL THEORY OF N-BODY SCHRODINGER OPERATORS

for N-body Schrödinger operators without some rudiment of such a commutator problem (this would require very different methods). A similar problem does not appear in time-dependent N-body scattering theory, see [De, Gr].

For rather different methods we refer to [Gér, GJ], which are also based on an elementary functional calculus with the Mourre estimate used as an input, although in less structured abstract contexts. The primary goal of these works is to demonstrate an alternative approach to Mourre’s differential inequality technique in their abstract settings. However, these papers do not contain the sharp LAP bounds, and more smoothness is required (due to multiple commutators). We would also like to mention that it was realized by Melrose and Vasy that the Mourre estimate combined with propagation of singularities in a certain calculus leads to LAP for a class of N-body Schrödinger operators, see [Va]. Our goals and techniques are different, again. We aim at showing sharp results by elementary methods and at a minimum cost.

1.1. Setting. In this subsection we precisely formulate the setting of the paper. We work on a generalized N-body model with hard-cores. This is a natural generalization of the usual N-body model, and hopefully the terminologies used below would not need any motivation for the reader.

1.1.1. N-body Hamiltonian. Let \( X \) be a finite dimensional real inner product space, equipped with a finite family \( \{X_a\}_{a \in \mathcal{A}} \) of subspaces closed under intersection: For any \( a, b \in \mathcal{A} \) there exists \( c \in \mathcal{A} \) such that

\[
X_a \cap X_b = X_c.
\] (1.1)

The elements of \( \mathcal{A} \) are called cluster decompositions, and we order and write them as \( a \subset b \) if \( X_a \supset X_b \). It is assumed that there exist \( a_{\min}, a_{\max} \in \mathcal{A} \) such that

\[
X_{a_{\min}} = X, \quad X_{a_{\max}} = \{0\}.
\]

For a chain of cluster decompositions \( a_1 \subsetneq \cdots \subsetneq a_k \) the number \( k \) is called the length of the chain, and such a chain is said to connect \( a = a_1 \) and \( b = a_k \). For any \( a \in \mathcal{A} \) we denote the maximal length of all the chains connecting \( a \) and \( a_{\max} \) by \( \# a \):

\[
\# a = \max\{k \mid a = a_1 \subsetneq \cdots \subsetneq a_k = a_{\max}\}; \quad \# a_{\max} = 1.
\]

We say that the family \( \{X_a\}_{a \in \mathcal{A}} \) is of \( N \)-body type if \( \# a_{\min} = N + 1 \). To avoid confusion we remark that \( (N + 1) \) number of moving particles form an \( N \)-body system after separation of the center of mass.

Let \( X^a \subset X \) be the orthogonal complement of \( X_a \subset X \), and denote the associated orthogonal decomposition of \( x \in X \) by

\[
x = x^a \oplus x_a \in X^a \oplus X_a.
\]

The component \( x_a \) is called the inter-cluster coordinates, and \( x^a \) the internal coordinates. We note that the family \( \{X^a\}_{a \in \mathcal{A}} \) is closed under addition: For any \( a, b \in \mathcal{A} \) there exists \( c \in \mathcal{A} \) such that

\[
X^a + X^b = X^c,
\]

cf. (1.1). A real-valued measurable function \( V: X \to \mathbb{R} \) is called a potential of \( N \)-body type if there exist real-valued measurable functions \( V_a^c: X^a \to \mathbb{R} \) (i.e. pair-potentials) such that

\[
V(x) = \sum_{a \in \mathcal{A}} V_a(x^a) \quad \text{for} \quad x \in X.
\] (1.2)
Throughout the paper we may assume \( V_{a_{\min}} = 0 \) without loss of generality. We sometimes call \( V \) a soft potential compared to the following hard-cores: For each \( a \in \mathcal{A} \) let \( \Omega_a \subset X^a \) be a given non-empty open subset with \( X^a \setminus \Omega_a \) being compact, and we set
\[
\Omega = \bigcap_{a \in \mathcal{A}} (\Omega_a + X_a).
\] (1.3)

Note that by the non-emptiness assumption it automatically follows that \( \Omega_{a_{\min}} = X_{a_{\min}} = \{0\} \). The complement \( X \setminus \Omega \) corresponds to a region where particles cannot penetrate due to the existence of ‘hard-cores’.

Now we present conditions of the paper on these interactions. Throughout the paper we use the standard notation \( \langle y \rangle = (1 + |y|^2)^{1/2} \) for a vector or a complex number \( y \). We denote the space of bounded operators from a general Banach space \( X \) to another \( Y \) by \( \mathcal{L}(X,Y) \) and abbreviate \( \mathcal{L}(X) = \mathcal{L}(X,X) \). The space of compact operators from \( X \) to \( Y \) is denoted by \( \mathcal{C}(X,Y) \), and \( X^* \) denotes the dual space of \( X \).

**Condition 1.1.** Let \( \delta \in (0,1/2] \) be fixed. For each \( a \in \mathcal{A} \setminus \{a_{\min}\} \) there exists a splitting
\[
V_a = V_{a_{\text{lr}}}^a + V_{a_{\text{sr}}}^a + V_{a_{\text{si}}}^a
\]
into three real-valued measurable functions in \( \mathcal{C}(H_0^1(\Omega_a), H_0^1(\Omega_a)^*) \) such that

1. \( V_{a_{\text{lr}}}^a \) has first order distributional derivatives in \( L^1_{\text{loc}}(\Omega_a) \), and for any \( |\alpha| = 1 \)
\[
\langle x^\alpha \rangle^{1+2\delta} \partial^\alpha V_{a_{\text{lr}}}^a \in \mathcal{L}(H_0^1(\Omega_a), H_0^1(\Omega_a)^*) ;
\]
2. \( V_{a_{\text{sr}}}^a \) satisfies
\[
\langle x^\alpha \rangle^{1+2\delta} V_{a_{\text{sr}}}^a \in \mathcal{L}(H_0^1(\Omega_a), L^2(\Omega_a));
\] (1.4)
3. \( V_{a_{\text{si}}}^a \) vanishes outside a bounded subset of \( \Omega_a \).

In the case where hard-cores are absent, the following operator-bounded version is also available. It is almost identical with the condition of [ABG1].

**Condition 1.2.** Let \( \delta \in (0,1/2] \) be fixed. For each \( a \in \mathcal{A} \setminus \{a_{\min}\} \) hard-core interaction is absent, i.e. \( \Omega_a = X^a \), and there exists a splitting
\[
V_a = V_{a_{\text{lr}}}^a + V_{a_{\text{sr}}}^a
\]
into two real-valued measurable functions in \( V_a \in \mathcal{C}(H^2(X^a), L^2(X^a)) \) such that

1. \( V_{a_{\text{lr}}}^a \) has first order distributional derivatives in \( L^1_{\text{loc}}(X^a) \), and for any \( |\alpha| = 1 \)
\[
\langle x^\alpha \rangle^{1+2\delta} \partial^\alpha V_{a_{\text{lr}}}^a \in \mathcal{L}(H^2(X^a), H^{-1}(X^a));
\]
2. \( V_{a_{\text{sr}}}^a \) satisfies
\[
\langle x^\alpha \rangle^{1+2\delta} V_{a_{\text{sr}}}^a \in \mathcal{L}(H^2(X^a), L^2(X^a)).
\]

**Remark.** It is easily checked that the proof of the Mourre estimate in [IS1] works under either Condition 1.1 or Condition 1.2, see (2.19), (2.20) and Lemma 2.10 below. In this paper the Mourre estimate, or Lemma 2.10, is used as an ‘input’, and we will not give a proof of it.

For an \( N \)-body potential (1.2) and an exterior region (1.3) satisfying Conditions 1.1 or 1.2 we define the generalized \( N \)-body Hamiltonian \( H \) as
\[
H = H_0 + V; \quad H_0 = -\frac{1}{2} \Delta, \quad \text{on } \mathcal{H} = L^2(\Omega).
\]
Here $\Delta$ is the Laplace–Beltrami operator associated with the inner product on $\mathbf{X}$, and we impose the Dirichlet boundary condition on $\partial \Omega$. More precisely, $H$ is defined as the self-adjoint operator associated with the closed quadratic form $\tilde{H}$:

$$\langle \tilde{H} \psi \rangle = \frac{1}{2} \langle p \psi, p \psi \rangle + \langle \psi, V \psi \rangle \quad \text{for} \quad \psi \in Q(\tilde{H}) = H^1_0(\Omega),$$

where $p = -i \nabla$. Note that $V$ is infinitesimally $H^1_0$-small in the form sense. For later use let us be more careful about the domain $\mathcal{D}(H)$. Note that the above quadratic form $\tilde{H}$ may be considered as a bounded operator $H^1_0(\Omega) \to (H^1_0(\Omega))^*$. Then the self-adjoint operator $H$ is realized by letting

$$H = \tilde{H}|_{\mathcal{D}(H)}; \quad \mathcal{D}(H) = \{ \psi \in H^1_0(\Omega) \mid \tilde{H} \psi \in \mathcal{H} \}. \quad (1.5)$$

This is exactly the definition of the Friedrichs extension, see e.g. [Yo, proof of Theorem XI.7.2]. Note in particular that, if Condition 1.2 is adopted, then we have

$$\mathcal{D}(H) = \mathcal{D}(H_0) = H^2(\mathbf{X}) \supset C^\infty_c(\mathbf{X}). \quad (1.6)$$

We henceforth denote the quadratic form $\tilde{H}$ on $H^1_0(\Omega)$, or the bounded operator $\tilde{H}: H^1_0(\Omega) \to (H^1_0(\Omega))^*$, simply by $H$ if there is no confusion.

### 1.1.2. Sub-Hamiltonians.

We recall the definition of the sub-Hamiltonian $H^a$ associated with a cluster decomposition $a \in \mathcal{A}$. For $a = a_{\min}$, noting that $V^a_{\min} = 0$ and $\Omega^a_{\min} = \{0\}$, we define

$$H^{a_{\min}} = 0 \quad \text{on} \quad \mathcal{H}^{a_{\min}} = L^2(\Omega^{a_{\min}}) = \mathbb{C}.$$  

For $a \neq a_{\min}$, since $\{\mathbf{X}_b \cap \mathbf{X}^a\}_{b \subset a}$ forms a family of subspaces of $(\#a - 1)$-body type in $\mathbf{X}^a$, we can consider, similarly to the full Hamiltonian $H$,

$$V^a(x^a) = \sum_{b \subset a} V_b(x^b), \quad \Omega^a = \bigcap_{b \subset a} [\Omega_b + (\mathbf{X}_b \cap \mathbf{X}^a)].$$

Then we define the associated sub-Hamiltonian $H^a$ as

$$H^a = -\frac{1}{2} \Delta x^a + V^a \quad \text{on} \quad \mathcal{H}^a = L^2(\Omega^a)$$

with the Dirichlet boundary condition on $\partial \Omega^a$. We remark that in particular

$$V^{a_{\max}} = V, \quad \Omega^{a_{\max}} = \Omega, \quad H^{a_{\max}} = H, \quad \mathcal{H}^{a_{\max}} = \mathcal{H}.$$

The thresholds of $H$ are the eigenvalues of sub-Hamiltonians $H^a$, $a \in \mathcal{A} \setminus \{a_{\max}\}$. We set

$$\mathcal{T}(H) = \bigcup \{ \sigma_{pp}(H^a) \mid a \in \mathcal{A} \setminus \{a_{\max}\} \}.$$  

It is known that under Conditions 1.1 or 1.2 that the set $\mathcal{T}(H)$ is closed and at most countable. Moreover the set of non-threshold eigenvalues is discrete in $\mathbb{R} \setminus \mathcal{T}(H)$, and it can only accumulate at points in $\mathcal{T}(H)$ from below. See Lemma 2.10, Remark 3.2 and [FH, IS1, Pe]. By the so-called HVZ theorem the essential spectrum of $H$ is given by the formula

$$\sigma_{\text{ess}}(H) = \left( \min \mathcal{T}(H), \infty \right),$$

cf. [RS, Theorem XIII.17].
1.1.3. Unique continuation property. Most of the results of the paper depend on the unique continuation property. Due to singularities of pair-potentials $V_a$ and the hard-cores $X^a \setminus \Omega_a$ this property does not necessarily hold for our Hamiltonian $H$. In this paper we are going to assume this property rather than imposing technical sufficient conditions on $V_a$ and $\Omega_a$. 

For any $a \in A \setminus \{a_{\text{min}}\}$ we introduce the locally $H^1_a$ space by

$$H^1_{0,\text{loc}}(\Omega^a) = \{ \psi \in L^2_{\text{loc}}(\Omega^a) \mid \chi \psi \in H^1_0(\Omega^a) \text{ for any } \chi \in C^\infty_c(X^a) \}.$$ 

Then, since $H^a : H^1_{0,\text{loc}}(\Omega^a) \rightarrow (H^1_0(\Omega^a))^* \subset \mathcal{D}'(\Omega)$ is a local operator, it naturally extends as $H^a : H^1_{0,\text{loc}}(\Omega^a) \rightarrow \mathcal{D}'(\Omega^a)$ by using a partition of unity. The vector $H^a \psi$ for $\psi \in H^1_{0,\text{loc}}(\Omega^a)$ may be referred to as ‘$H^a \psi$ in the distributional sense’. We call a function $\phi \in H^1_{0,\text{loc}}(\Omega^a)$ a generalized Dirichlet eigenfunction for $H^a$ with eigenvalue $E \in \mathbb{C}$, if it satisfies

$$H^a \phi = E \phi$$

in the distributional sense.

**Condition 1.3.** The unique continuation property holds for $H^a$ for all $a \neq a_{\text{min}}$: If $\phi \in H^1_{0,\text{loc}}(\Omega^a)$ is a generalized Dirichlet eigenfunction for $H^a$ and $\phi = 0$ on a non-empty open subset of $\Omega^a$, then $\phi = 0$ on $\Omega^a$.

For a particular result on the unique continuation property for $N$-body Schrödinger operators (without hard-core interaction) we refer to [Geo]. To our knowledge this property is not well understood in the $N$-body case.

1.2. Results. In this subsection we state all the main results of the paper.

1.2.1. Rellich type theorems. Let us first recall the definitions of the Besov spaces associated with the multiplication operator $|x|$ on $\mathcal{H}$. Set

$$F_0 = F\left(\{ x \in \Omega \mid |x| < 1 \}\right),$$

$$F_n = F\left(\{ x \in \Omega \mid 2^{n-1} \leq |x| < 2^n \}\right) \text{ for } n = 1, 2, \ldots,$$

where $F(S)$ is the sharp characteristic function of any given subset $S \subset \Omega$. The Besov spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}_0^*$ are defined as

$$\mathcal{B} = \{ \psi \in L^2_{\text{loc}}(\Omega) \mid \| \psi \|_\mathcal{B} < \infty \}, \quad \| \psi \|_\mathcal{B} = \sum_{n=0}^{\infty} 2^{n/2} \| F_n \psi \|_{\mathcal{H}},$$

$$\mathcal{B}^* = \{ \psi \in L^2_{\text{loc}}(\Omega) \mid \| \psi \|_{\mathcal{B}^*} < \infty \}, \quad \| \psi \|_{\mathcal{B}^*} = \sup_{n \geq 0} 2^{-n/2} \| F_n \psi \|_{\mathcal{H}},$$

$$\mathcal{B}_0^* = \left\{ \psi \in \mathcal{B}^* \mid \lim_{n \to \infty} 2^{-n/2} \| F_n \psi \|_{\mathcal{H}} = 0 \right\},$$

respectively. Denote the standard weighted $L^2$ spaces by

$$L^2_s = \langle x \rangle^{-s} L^2(\Omega) \text{ for } s \in \mathbb{R}, \quad L^2_{-\infty} = \bigcup_{s \in \mathbb{R}} L^2_s, \quad L^2_\infty = \bigcap_{s \in \mathbb{R}} L^2_s.$$ 

Then note that for any $s > 1/2$

$$L^2_s \subseteq \mathcal{B} \subseteq L^2_{1/2} \subseteq H \subseteq L^2_{-1/2} \subseteq \mathcal{B}^* \subseteq \mathcal{B}_0^* \subseteq L^2_{-s}. \tag{1.7}$$

Now we have Rellich type theorems on the following form extending [IS1, IS2].
Theorem 1.4. Suppose Condition 1.1 or Condition 1.2. Let \( \phi \in \mathcal{B}_0^* \cap H^1_{0,\text{loc}}(\Omega) \), \( E \in \mathbb{R} \) and \( \rho \geq 0 \), and assume that

\[
(H - E)\phi(x) = 0 \quad \text{for} \quad |x| > \rho \quad \text{in the distributional sense.}
\]

Set \( \alpha_0 = \sup \{ \alpha \geq 0 \mid e^{\alpha|x|}\phi \in \mathcal{B}_0^* \} \in [0, \infty] \). Then

\[
E + \frac{1}{2} \alpha_0^2 \in \mathcal{T}(H) \cup \{\infty\}. \tag{1.8}
\]

Theorem 1.5. Suppose Condition 1.1. Let \( \phi \in \mathcal{B}_0^* \cap H^1_{0,\text{loc}}(\Omega) \), \( E \in \mathbb{R} \) and \( \rho \geq 0 \), and assume that

\[
(H - E)\phi(x) = 0 \quad \text{for} \quad |x| > \rho \quad \text{in the distributional sense.}
\]

Suppose \( \sup \{ \alpha \geq 0 \mid e^{\alpha|x|}\phi \in \mathcal{B}_0^* \} = \infty \). Then there exists \( \rho' \geq 0 \) such that

\[
\phi(x) = 0 \quad \text{for} \quad |x| > \rho'. \tag{1.9}
\]

The combination of Theorems 1.4 and 1.5 extends the classical Rellich theorem for \( N = 1 \) and \( E > 0 \). We refer to [Is] and the references therein for an account of the history of Rellich's theorem. With the unique continuation property we can extend the classical Rellich theorem to any \( N \), see 3) below.

Corollary 1.6. Suppose Conditions 1.1 and 1.3.

1) Let \( \phi \in \mathcal{B}_0^* \cap H^1_{0,\text{loc}}(\Omega) \) be a generalized Dirichlet eigenfunction for \( H \) with real eigenvalue \( E \in \mathbb{R} \), and set

\[
\alpha_0 = \sup \{ \alpha \geq 0 \mid e^{\alpha|x|} \phi \in \mathcal{B}_0^* \} \in [0, \infty].
\]

Then \( E + \frac{1}{2} \alpha_0^2 \in \mathcal{T}(H) \cup \{\infty\} \), and if \( \alpha_0 = \infty \) the function \( \phi = 0 \) on \( \Omega \).

2) There are no positive thresholds for \( H \), and there are no nonzero generalized Dirichlet eigenfunctions for \( H \) in \( \mathcal{B}_0^* \) with a positive eigenvalue.

3) Let \( \phi \in \mathcal{B}_0^* \cap H^1_{0,\text{loc}}(\Omega) \), \( E > 0 \) and \( \rho \geq 0 \), and assume that

\[
(H - E)\phi(x) = 0 \quad \text{for} \quad |x| > \rho \quad \text{in the distributional sense.}
\]

Then there exists \( \rho' \geq 0 \) such that \( \phi(x) = 0 \quad \text{for} \quad |x| > \rho' \).

Proof. The assertions follow from Theorems 1.4 and 1.5. Note that 2) needs an induction argument on \( N \), cf. [IS1]. □

Using the proof of Theorem 1.4 we can extend the result of [Pe], or [HuS, Theorem 6.11], showing that the set of non-threshold eigenvalues of \( H \) can accumulate only at points in \( \mathcal{T}(H) \) from below, see Lemma 3.1 and Remark 3.2.

1.2.2. LAP bounds. Next we present the sharp LAP bounds. For any interval \( I \subset \mathbb{R} \) let us set

\[
I_{\pm} := \{ z \in \mathbb{C} \mid \text{Re} \, z \in I, \ 0 < \pm \text{Im} \, z \leq 1 \}.
\]

Theorem 1.7. Suppose Condition 1.1 or Condition 1.2. Let \( I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H)) \) be a compact interval. Then there exists \( C > 0 \) such that for all \( z \in I_{\pm} \) and \( \psi \in \mathcal{B} \)

\[
\| R(z)\psi \|_{\mathcal{B}} + \| pR(z)\psi \|_{\mathcal{B}} \leq C\| \psi \|_{\mathcal{B}}. \tag{1.10}
\]
The LAP bound stated in Theorem 1.7 is new in that they allow form-bounded
local singularities (by Condition 1.1). To our knowledge the LAP bounds for formbounded singularities have been obtained only for $N = 1$ and with $\Omega = \mathbb{R}^3$, see [BMP]. See also [ABG2, pp. 270–271] for a review of the LAP bounds.

The weighted $L^2$ space version of the LAP bounds, proven in [ABG1, Ta], are essentially consequences of (1.7) and (1.10) since Condition 1.2 is almost identical with the conditions of [ABG1, Ta]. Similarly our result includes the Besov space refinements of [BGM1, BGM2] (at least essentially). In conclusion, Theorem 1.7 extends previous results for the usual $N$-body Schrödinger operators without hardcores.

We also mention that the LAP bounds for hard-core models were considered in [BGS] with some regularity conditions on the obstacles and with operator-bounded local singularities. Our result is more general, again.

1.2.3. Rescaled Graf function. To state microlocal resolvent bounds we introduce certain (rescaled) operators $A_R, B_R$. They are defined in terms of a function $r_1 \in C^\infty(X)$ with the following properties. We will not verify the existence of such $r_1$, but only enumerate the properties required in this paper. Let us denote the gradient vector field and the Hessian of $r_1^2/2$ by

$$\tilde{\omega}_1 = \frac{1}{2} \nabla r_1^2, \quad \tilde{h}_1 = \frac{1}{2} \text{Hess} r_1^2,$$

respectively. It is standard to identify the tangent space of $X$ at each $x \in X$ with $X$ itself. Then we may consider $\tilde{\omega}_1(x) \in X$ for each $x \in X$. Similarly, using the inner product structure on $X$, we may consider $\tilde{h}_1(x) : X \to X$ as a linear map for each $x \in X$. Let $N_0 = \{0\} \cup \mathbb{N}$ with $\mathbb{N} = \{1, 2, \ldots\}$. We assume that there exist $r_1 \in C^\infty(X)$ and a smooth partition of unity $\{\eta_{1,a}\}_{a \in A}$ on $X$ obeying:

1. There exist $c, C > 0$ such that for any $a, b \in A$ with $a \not\subset b$ and $x \in \text{supp} \eta_{1,b}$
   $$|x^a| \geq c, \quad |x^b| \leq C; \quad (1.11)$$

2. There exists $C' > 0$ such that for any $x \in X$
   $$r_1(x) \geq 1, \quad |r_1(x) - |x|| \leq C'; \quad (1.12)$$

3. There exists $c' > 0$ such that for any $a \in A$ and $x \in X$ with $|x^a| \leq c'$
   $$\tilde{\omega}_1^a(x) = 0; \quad (1.13)$$

4. For any $x, y \in X$
   $$\langle y, \tilde{h}_1(x)y \rangle \geq \sum_{a \in A} \eta_{1,a}(x) |y_a|^2;$$

5. For any $\alpha \in \mathbb{N}_0^{\text{dim}X}$ and $k \in \mathbb{N}_0$ there exists $C_{\alpha k} > 0$ such that for any $x \in X$
   $$\sum_{a \in A} |\partial^\alpha \eta_{1,a}(x)| + |\partial^\alpha (x \cdot \nabla)^k (\tilde{\omega}_1(x) - x)| \leq C_{\alpha k}. \quad (1.14)$$

For a construction of such $r_1$ and $\{\eta_{1,a}\}_{a \in A}$ we refer to [Gr], see also [De, Sk]. We note that (1.12) in fact is verified from the other properties. The former bound of (1.12) always holds if we add a large positive constant to $r_1$, and the latter follows by integrating (1.14) with $\alpha = 0$ and $k = 0$.

Now we set for large $R \geq 1$

$$r_R(x) = Rr_1(x/R), \quad \tilde{\omega}_R = \frac{1}{R} \nabla r_R^2, \quad \omega_R = \text{grad} r_R, \quad \eta_R(x) = \eta_1(x/R),$$
and define the self-adjoint operators $A_R, B_R$ on $\mathcal{H}$ as
\[
A_R = \frac{1}{2}(\tilde{\omega}_R \cdot p + p \cdot \tilde{\omega}_R), \quad B_R = \frac{1}{2}(\omega_R \cdot p + p \cdot \omega_R); \quad p = -i\nabla,
\]
respectively. The vector field $\tilde{\omega}_R$ is called the rescaled Graf vector field, and $A_R$ is a conjugate operator in the Mourre theory, cf. [Sk, IS1]. We remark that, however, the operator $A_R$ is only auxiliarily used in this paper, or we can actually remove it completely from this paper. Instead, $B_R$ plays a central role in our theory. We will investigate its properties in Section 2.

In the sequel we will often suppress the dependence on the parameter $R \geq 1$ of the above quantities, writing simply
\[
r = r_R, \quad \tilde{\omega} = \tilde{\omega}_R, \quad \omega = \omega_R, \quad A = A_R, \quad B = B_R, \quad \eta = \eta_R.
\]

1.2.4. Microlocal resolvent bounds and applications. Now we present some microlocal resolvent bounds. Define a function $d : \mathbb{R} \to \mathbb{R}$ as
\[
d(\lambda) = \begin{cases} \inf \{ \lambda - \tau \mid \tau \in \mathcal{T}(H) \cap (-\infty, \lambda] \} & \text{if } \mathcal{T}(H) \cap (-\infty, \lambda] \neq \emptyset, \\ 1 & \text{if } \mathcal{T}(H) \cap (-\infty, \lambda] = \emptyset,
\end{cases}
\]
and introduce for any $\lambda \in \mathbb{R}$ and $I \subset \mathbb{R}$
\[
\gamma(\lambda) = \sqrt{2d(\lambda)}, \quad \gamma_-(I) = \inf_{\lambda \in I} \gamma(\lambda) = \inf_{\lambda \in I} \sqrt{2d(\lambda)}.
\]

With $\delta \in (0, 1/2]$ from Condition 1.1 or 1.2 let
\[
\kappa = \delta/(1 + 2\delta) \in (0, 1/4].
\]

**Theorem 1.8.** Suppose Conditions 1.1 or 1.2. Let $I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H))$ be a compact interval, and take both $R \geq 1$ and $\tilde{\gamma} > 0$ sufficiently large. Then for any $\beta \in (0, \kappa)$ and $F \in C^\infty_c(\mathbb{R})$ with
\[
\text{supp } F \subset (-\infty, \gamma_-(I)) \cup (\tilde{\gamma}, \infty) \quad \text{and} \quad F' \in C^\infty_c(\mathbb{R}),
\]
there exists $C > 0$ such that for all $z \in I_\pm$ and $\psi \in L^2_{1/2+\beta}$
\[
\|F(\pm B)R(z)\psi\|_{L^2_{1/2+\beta}} \leq C\|\psi\|_{L^2_{1/2+\beta}},
\]
respectively.

**Remarks.** (1) The critical exponent $\kappa$ in Theorem 1.8 and Corollaries 1.9 and 1.10 below, which is worse than $\delta$, comes from an optimization in the proof of Lemma 2.13 under Condition 1.1. Under Condition 1.2 we can actually choose $\kappa = \delta$.

(2) If higher commutators were available like in [GIS], similar bounds would hold for powers of the resolvent. However, we have at most second commutators available under Conditions 1.1 or 1.2, cf. Lemma 2.13. For the same reason we need $\beta$ to be small.

The first application of the above results is Hölder continuity of the resolvent, and in particular the LAP.

**Corollary 1.9.** Suppose Conditions 1.1 or 1.2. Let $I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H))$ be a compact interval, and let $s > 1/2$ and $\beta \in (0, \min\{\kappa, s - 1/2\})$. Then there exists $C > 0$ such that for all $k \in \{0, 1\}$, $z \in I_\pm$ and $z' \in I_\pm$, respectively,
\[
\|p^k R(z) - p^k R(z')\|_{L^2_{1/2+\beta}} \leq C|z - z'|^\beta.
\]
In particular, for any $E \in \mathbb{I}$ and $s > 1/2$ the following boundary values exist:

$$p^k R(E \pm i0) := \lim_{\epsilon \to 0^\pm} p^k R(E \pm i\epsilon) \quad \text{in} \ L^2_s(L^2_{-s}),$$

respectively. The same boundary values are realized (in an extended form) as

$$p^k R(E \pm i0) = s-w^*\lim_{\epsilon \to 0^+} p^k R(E \pm i\epsilon) \quad \text{in} \ L(B, B^*),$$

respectively.

The second application is a microlocal Sommerfeld uniqueness result, which characterizes the limiting resolvents $R(E \pm i0)$ by the Helmholtz equation and microlocal radiation conditions. Given $\psi \in L^2_{\text{loc}}(\Omega)$, we say a function $\phi \in H^1_{0,\text{loc}}(\Omega)$ is a generalized Dirichlet solution to $(H - E)\phi = \psi$, if it satisfies

$$(H - E)\phi = \psi \quad \text{in the distributional sense.}$$

**Corollary 1.10.** Suppose Conditions 1.1 or 1.2. Let $E \in \mathbb{R} \setminus (\sigma_{\text{pp}}(H) \cup \mathcal{T}(H))$, and take $R \geq 1$ sufficiently large. Let $\psi \in r^{-\beta} B$ with $\beta \in [0, \kappa)$. Then $\phi = R(E \pm i0) \psi \in B^* \cap H^1_{0,\text{loc}}(\Omega)$ satisfies

1. $\phi$ is a generalized Dirichlet solution to $(H - E)\phi = \psi$,
2. there exists $\gamma > 0$ such that for any $F \in C^\infty(\mathbb{R})$ with

$$\text{supp } F \subset (\infty, \gamma(E)) \cup (\gamma, \infty) \quad \text{and} \quad F' \in C^\infty_c(\mathbb{R}),$$

the functions $F(\pm B)\phi$ belong to $r^{-\beta} B^*_0$,

respectively. Conversely, if $\phi' \in L^2_s \cap H^1_{0,\text{loc}}(\Omega)$ satisfies

1'. $\phi'$ is a generalized Dirichlet solution to $(H - E)\phi' = \psi$,
2'. there exists $\gamma > 0$ such that for any $F \in C^\infty(\mathbb{R})$ with

$$\text{supp } F \subset (\infty, \gamma) \quad \text{and} \quad F' \in C^\infty_c(\mathbb{R}),$$

the functions $F(\pm B)\phi'$ belong to $B^*_0$,

then $\phi' = R(E \pm i0) \psi$, respectively.

2. Preliminaries: Operator $B$

In this section we provide various preliminaries needed for the proofs of our main results. In Section 2.1 we introduce notation frequently used in the later arguments. Section 2.2 includes the Helffer–Sjöstrand formula and its direct application to compute commutators. In Section 2.3 we formulate the self-adjoint realization of the conjugate operator $B$ from (1.15). We investigate the first commutator $i[H, B]$ in Section 2.4, and the second commutator $i[H, B]$, $B$ in Section 2.5. The control of the second commutator $i[i[H, B], B]$ is a key, and Section 2.5 is one of the most technical parts of the paper. Remark that for the absense of positive $L^2$-eigenvalues [IS1] only a first commutator was needed.

2.1. Notation. This is a short subsection devoted to some notation only.

Let $T$ be an linear operator on $H = L^2_s(\Omega)$ such that $T, T^* : L^2_{-s} \to L^2_{-s}$, and let $t \in \mathbb{R}$. Then we say that $T$ is an operator of order $t$, if for each $s \in \mathbb{R}$ the restriction $T|_{L^2_{-s}}$ extends to an operator $T_s \in \mathcal{L}(L^2_s, L^2_{-s})$. Alternatively stated, for any $R \geq 1$ and with $r = r_R$

$$\|r^{s-t}T r^{-s} f\| \leq C_s \|f\| \quad \text{for all} \ f \in L^2_s.$$
Note (for consistency) that $T_s$ extends the restriction $T_{|\mathcal{D}(T)\cap L^2}$: If $T$ is of order $t$, we write
\[ T = O(r^t). \] (2.1)
Note also that, if $T = O(r^t)$ and $S = O(r^s)$, then $T^* = O(r^t)$ and $TS = O(r^{t+s})$.

Define the Sobolev spaces $\mathcal{H}^s$ of order $s \in \mathbb{R}$ associated with $H$ as
\[ \mathcal{H}^s = (H + E_{\text{min}} + 1)^{-s/2} \mathcal{H}; \quad E_{\text{min}} = \min \sigma(H). \] (2.2)

We note that
\[ \mathcal{H}^1 = Q(H) = H^1_0(\Omega), \quad \mathcal{H}^2 = \mathcal{D}(H), \quad \mathcal{H}^{-1} = (H^1_0(\Omega))^*, \quad \mathcal{H}^{-2} = (\mathcal{H}^2)^*. \]

Let $\chi \in C^\infty(\mathbb{R})$ be a real-valued function such that
\[ \chi(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ 0 & \text{for } t \geq 2, \end{cases} \quad \chi' \leq 0, \quad \text{and } \chi^{1/2}, |\chi'|^{1/2} \in C^\infty(\mathbb{R}). \] (2.3)

We then define smooth cut-off functions $\chi_m, \tilde{\chi}_m, \chi_{m,n} \in C^\infty(\mathbb{X})$ for $n > m \geq 0$ and $R \geq 1$ as
\[ \chi_m = \chi(r/2^m), \quad \tilde{\chi}_m = 1 - \chi_m, \quad \chi_{m,n} = \tilde{\chi}_m \chi_n. \] (2.4)

Here $r = r_R$ in fact depends on $R \geq 1$, but the dependence on $R$ is suppressed.

Next we construct a smooth sign function $\zeta \in C^\infty(\mathbb{R})$. The approach of [IS2] was based largely on such a function, but our construction below is a slightly simplified one. Choose a function $\zeta_i \in C^\infty(\mathbb{R})$ such that
\[ \zeta_i'(b) \geq 0 \quad \text{for } b \in \mathbb{R}, \quad \sqrt{\zeta_i} \in C^\infty_c(\mathbb{R}), \quad \zeta_1(b) = \begin{cases} -1 & \text{for } b \leq -1, \\ 2b & \text{for } -1/4 \leq b \leq 1/4, \\ 1 & \text{for } b \geq 1, \end{cases} \]
and let $\zeta \in C^\infty(\mathbb{R})$ be defined as
\[ \zeta(b) = \zeta_i(b) = \zeta_i(b/\epsilon) \quad \text{for } \epsilon \in (0, 1), \quad b \in \mathbb{R}. \]
We will use the following elementary properties of $\zeta$ (we omit the proof).

**Lemma 2.1.** For any $\epsilon \in (0, 1)$ the above function $\zeta \in C^\infty(\mathbb{R})$ satisfies that
\[ \zeta'(b) \geq 0 \quad \text{for } b \in \mathbb{R}, \quad \zeta'(b) = 0 \quad \text{for } |b| \geq \epsilon, \quad \sqrt{\zeta} \in C^\infty_c(\mathbb{R}), \]
and, in addition, that for any $c > 0$ and $b \in \mathbb{R}$
\[ b\zeta(b) + c\zeta'(b) \geq \min\{\epsilon/8, 2c/\epsilon\}. \]

**2.2. Functional calculus.** Here we present the Helffer–Sjöstrand formula to write down functions of self-adjoint operators, and its application to commutators. The following results are abstract and time-independent versions of similar results in the literature, typically more sophisticated ones. We omit most of the proofs, and only refer e.g. to [HeS, Lemma 3.5], [GIS, Section 2] or [DG, Appendix C].

For any $t \in \mathbb{R}$ we set
\[ \mathcal{F}^t = \{ f \in C^\infty(\mathbb{R}) \mid |f^{(k)}(x)| \leq C_k \langle x \rangle^{t-k} \text{ for any } k \in \mathbb{N}_0 \text{ and } x \in \mathbb{R} \}. \]
It is known that for any $f \in \mathcal{F}^t$, $t \in \mathbb{R}$, there always exists an almost analytic extension $\tilde{f} \in C^\infty(\mathbb{C})$ such that
\[ \tilde{f}_{|\mathbb{R}} = f, \quad |\tilde{f}(z)| \leq C\langle z \rangle^t, \quad |(\tilde{f})'(z)| \leq C_k \text{Im} z^{k-1} \langle z \rangle^{t-k-1} \text{ for any } k \in \mathbb{N}_0. \]
Here one can choose $\tilde{f} \in C^\infty_c(\mathbb{C})$ if $f \in C^\infty_c(\mathbb{R})$. 
Lemma 2.2. Let $T$ be a self-adjoint operator on $\mathcal{H}$, and let $f \in \mathcal{F}^t$ with $t \in \mathbb{R}$. Take an almost analytic extension $\hat{f} \in C^\infty(\mathbb{C})$ of $f$, and set
\[
\mathrm{d}\mu_f(z) = \pi^{-1}(\partial \hat{f})(z) \, \mathrm{d}u \, \mathrm{d}v; \quad z = u + iv. \tag{2.5a}
\]
Then for any $k \in \mathbb{N}_0$ with $k > t$ the operator $f^{(k)}(T) \in \mathcal{L}(\mathcal{H})$ is expressed as
\[
f^{(k)}(T) = (-1)^k k! \int_{\mathbb{C}} (T - z)^{-k-1} \, \mathrm{d}\mu_f(z). \tag{2.5b}
\]

The expression (2.5b) is the well known Helffer–Sjőstrand formula. We omit its verification. The formula is useful when we compute and bound commutators. In general there are several variations of the definition of a commutator, and in this paper we do not fix a particular one. It will be clear from the context in what sense we will be considering a commutator. Typically, for symmetric operators $T, S$, we first define $i[T, S]$ as the symmetric quadratic form
\[
(i[T, S])\psi = 2(\mathrm{Im}(ST))\psi = i(T\psi, S\psi) - i(S\psi, T\psi) \text{ for } \psi \in \mathcal{D}(T) \cap \mathcal{D}(S),
\]
and then extend it to a larger space.

Let us provide an example of a commutator formula derived from Lemma 2.2.

Corollary 2.3. Let $T$ be a self-adjoint operator on $\mathcal{H}$, $S$ be a symmetric relatively $T$-bounded operator, and assume that there exists a bounded extension
\[
(|T| + 1)^{-\epsilon/2}(i[T, S])(|T| + 1)^{-\epsilon/2} \in \mathcal{L}(\mathcal{H}) \quad \text{for some } \epsilon \in [0, 2].
\]
Let a real-valued $f \in \mathcal{F}^t$ with $t < 1 - \epsilon$ be given, and let $\mathrm{d}\mu_f$ be given by (2.5a). Then, as a quadratic form on $\mathcal{D}(f(T)) \cap \mathcal{D}(S)$,
\[
i[f(T), S] = - \int_{\mathbb{C}} (T - z)^{-1} (i[T, S]) (T - z)^{-1} \, \mathrm{d}\mu_f(z),
\]
and it extends to a bounded self-adjoint operator on $\mathcal{H}$.

Proof. By (2.5b) the assertion for $t < 0$ is obvious. Then the general case $t < 1 - \epsilon$ follows by approximating $f$ by functions from $C^\infty_c(\mathbb{R})$. We omit the details. \qed

Another example is the commutator $[f(H), r^s]$ treated below. As in [GIS] ‘phase-space localizations’ stated in terms of functions of $H$, $r$ and $B$ will be important. The other two commutators of this triple of operators will be discussed later in Lemmas 2.8 and 2.12. In the proofs of the main theorems we will repeatedly use Lemmas 2.4, 2.8, 2.12 and 2.14.

Lemma 2.4. Suppose Condition 1.1 or Condition 1.2.

(1) For any $f \in \mathcal{F}^t$ with $t < 1/2$ the operator $f(H)$ is of order 0.

(2) Let $f \in \mathcal{F}^t$ with $t < 1/2$, $R \geq 1$ and $s \in \mathbb{R}$. Then $i[f(H), r^s]$ has an expression, as a sesquilinear form on $\mathcal{D}(f(H)) \cap L^2_{\max\{0,s\}}$,
\[
i[f(H), r^s] = -s \int_{\mathbb{C}} (H - z)^{-1} (\text{Re}(r^s\omega \cdot p)) (H - z)^{-1} \, \mathrm{d}\mu_f(z). \tag{2.6}
\]
In particular $i[f(H), r^s]$ is of order $s - 1$.

Proof. (1) Fix any $R \geq 1$, which defines $r = r_R$. By (2.5b) it suffices to show that for each $s \in \mathbb{R}$ there exist $C(s) > 0$ such that
\[
\|(H - z)^{-1}\|_{\mathcal{L}(L^2)} \leq C(s) |\text{Im } z|^{-1} \langle z \rangle / |\text{Im } z|^{s+1} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7a}
\]
By considering the adjoint we may assume \( s \geq 0 \) without loss of generality. We first let \( s \in [0, 1) \). Then as an operator in \( \mathcal{L}(L^2_{\mathbb{R}}, L^2_{\mathbb{R}}) \) we can calculate

\[
i[(H - z)^{-1}, r^s] = s \lim_{t \to 0} t^{-1} \left( (H - z)^{-1} e^{it r^s} - e^{it r^s} (H - z)^{-1} \right) = s \lim_{t \to 0} t^{-1} (H - z)^{-1} (e^{it r^s} H_0 - H_0 e^{it r^s}) (H - z)^{-1}
\]

(2.7b)

\[
= s \lim_{t \to 0} (H - z)^{-1} \left[ -s \frac{d}{d} \left( r^s e^{it r^s} \omega \cdot p + p \cdot \omega e^{it r^s} r^{-s} \right) \right] (H - z)^{-1} = -s (H - z)^{-1} \left( \text{Re} (r^s \omega \cdot p) \right) (H - z)^{-1}.
\]

Here, noting that \( \mathcal{H}^2 \subset \mathcal{H}^1 \) and that \( e^{it r^s} \) preserves \( \mathcal{H}^1 \), we could safely implement integrations by parts for the third equality of (2.7b) without boundary contributions even under Condition 1.1. The last expression of (2.7b) is obviously bounded on \( \mathcal{H} \). Whence by the formula

\[
r^s (H - z)^{-1} r^{-s} = (H - z)^{-1} - s (H - z)^{-1} \left( \text{Re} (r^s \omega \cdot p) \right) (H - z)^{-1} r^{-s},
\]

(2.7c) for \( s \in [0, 1] \) follows; here we use the elementary bound

\[
\|(H - z)^{-1} p\| \leq C (| \text{Im} z|^{-1/2} + |z|^{1/2}) | \text{Im} z|^{-1/2} \leq 2 C (|z|/|z|).
\]

(2.7d)

For \( s \geq 1 \) we write \( s = s' + |s| \) in terms of the integer part \(|s|\) of \( s \), and first use (2.7c) with \( s \) replaced by \( s' \). Next we move each of the \(|s|\) factors of \( r \) through the resolvents to the right using the formula (2.7c) repeatedly. This procedure yields an expansion of \( r^s (H - z)^{-1} r^{-s} \) into a sum of terms having at most \( 2 + |s| \) factors of \( r \) distributed so that (2.7d) applies.

(2) For \( s \leq 0 \) the formula (2.6) follows from Corollary 2.3, and by the proof of (1) we see that indeed the right-hand side of (2.6) is of order \( s - 1 \). For \( s > 0 \) we apply (2.6) to \( r \) replaced by \( r/(1 + \epsilon r) \) for \( \epsilon > 0 \). We let \( \epsilon \to 0 \) and obtain (2) in that case also.

\[
\square
\]

2.3. Self-adjoint realization. Now we provide the self-adjoint realization of the operator \( B \). Recall the definition (2.2) of \( \mathcal{H}^s \).

Lemma 2.5. Let \( R \geq 1 \) be sufficiently large. Then the operator \( B \) defined as (1.15) is essentially self-adjoint on \( C^\infty_c (\Omega) \), and the self-adjoint extension, denoted by \( B \) again, satisfies that for some \( C > 0 \)

\[
\mathcal{D}(B) \supset \mathcal{H}^1, \quad \| B \psi \|_{\mathcal{H}^1} \leq C \| \psi \|_{\mathcal{H}^1} \quad \text{for any } \psi \in \mathcal{H}^1.
\]

(2.8)

In addition, \( e^{it B} \) for each \( t \in \mathbb{R} \) naturally restricts/extends as bounded operators \( e^{it B} : \mathcal{H}^{\pm 1} \to \mathcal{H}^{\pm 1} \), and they satisfy

\[
\sup_{t \in [-1, 1]} \| e^{it B} \|_{\mathcal{L}(\mathcal{H}^{\pm 1})} < \infty,
\]

(2.9)

respectively. Moreover, the restriction \( e^{it B} \subset \mathcal{L}(\mathcal{H}^1) \) is strongly continuous in \( t \in \mathbb{R} \).

Remarks 2.6. (1) The same assertions except for (2.8) hold true also for the operator \( A \) from (1.15), but we do not state it since we do not use it.

(2) For related results in more general geometric settings, see [IS1, Lemma A.8] and [IS3, Lemma 2.8].
By the last expression from (2.12) the generator $B$ rescaled vector field $\omega$

Proof. By the properties (1.13) and (1.14) we can find large flow is preserved under $U$. By using (2.13) we can compute and bound its derivative as

where $J$ guarantees that $H_\infty$ for any $R \geq 1$. Moreover in that case we also have invariance of $H^{l=1}$ and in fact

the proof is similar.

Proof. By the properties (1.13) and (1.14) we can find large $R \geq 1$ such that the rescaled vector field $\omega = \omega_R$ is complete on $\Omega$. Then there exists a globally defined flow

generated by $\omega$. In other words, $y$ is a solution to the equation

We introduce the associated one-parameter group $\{U(t)\}_{t \in \mathbb{R}}$ of unitary operators on $H$ by

where $J(t, \cdot)$ is the Jacobian of the mapping $y(t, \cdot): \Omega \to \Omega$.

Now we define $B$ as the generator of the group $\{U(t)\}_{t \in \mathbb{R}}$:

Since $U(t)$ is unitary, the generator $B$ is self-adjoint on $H$. In addition, since $C^\infty_c(\Omega)$ is preserved under $U(t)$, the space $C^\infty_c(\Omega)$ is a core for $B$ by [RS, Theorem X.49]. By the last expression from (2.12) the generator $B$ takes the form

which actually coincides with (1.15). Then (2.8) follows by extension from the dense subspace $C^\infty_c(\Omega) \subset H^1$.

To prove (2.9) it suffices to discuss the upper case by taking the adjoint. For any $\psi \in C^\infty_c(\Omega) \subset H^1$ consider the quantity

By using (2.13) we can compute and bound its derivative as

where $h = \text{Hess } r$, and $C_1$ is a constant independent of $\psi \in C^\infty_c(\Omega)$, cf. (2.15) and (2.16) below. Then by the Gronwall lemma and a density argument we obtain the uniform boundedness of $U(t) = e^{itB}: H^1 \to H^1$ for $t \in [-1,1]$.

Finally by a density argument, (2.9) and the regularity of the flow (2.11) it is easy to see that $e^{itB} \in L(H^1)$ is strongly continuous in $t \in \mathbb{R}$. \hfill \square
Lemma 2.7. Let \( v \in \mathcal{X}(\Omega) \) be a smooth and complete vector field on \( \Omega \), and assume that there exists \( C > 0 \) such that for any \( x \in \mathcal{X} \)
\[
|v(x)| \leq C, \quad |v'(x)| \leq C, \quad |\text{grad}(\text{div} v)(x)| \leq C.
\]
Then the differential operator
\[
B_v = \text{Re}(v \cdot p) = \frac{1}{2}(v \cdot p + p \cdot v)
\]
is essentially self-adjoint on \( C_c^{\infty}(\Omega) \), and the self-adjoint extension, denoted by \( B_v \)
again, satisfies for some \( C' > 0 \)
\[
\mathcal{D}(B_v) \supset \mathcal{H}^1, \quad \|B_v \psi\| \leq C'\|\psi\|_{\mathcal{H}^1} \quad \text{for any } \psi \in \mathcal{H}^1.
\]
In addition the operators \( e^{itB_v}, t \in \mathbb{R} \), naturally restrict/extend to bounded operators \( e^{itB_v} : \mathcal{H}^{\pm 1} \to \mathcal{H}^{\pm 1} \), and they satisfy
\[
\sup_{t \in [-1,1]} \|e^{itB_v}\|_{\mathcal{L}(\mathcal{H}^{\pm 1})} < \infty,
\]
respectively. Moreover, the restriction \( e^{itB_v} \in L(\mathcal{H}^1) \) is strongly continuous in \( t \in \mathbb{R} \).

Finally in this subsection we present some basic properties of \( B \) (proved in the same manner as we proved Lemma 2.4).

Lemma 2.8. Let \( R \geq 1 \) be sufficiently large.

1. For any \( F \in \mathcal{F}^t \) with \( t < 0 \) the operator \( F(B) \) is of order 0.
2. Let \( F \in \mathcal{F}^t \) with \( t < 1 \), and let \( s \in \mathbb{R} \). Then \( i[F(B), r^s] \) has an expression, as a sesquilinear form on \( \mathcal{D}(F(B)) \cap L^2_{\max\{0,s\}} \):
\[
i[F(B), r^s] = -s \int_\mathcal{C}(B - z)^{-1}(\omega^2 r^{s-1})(B - z)^{-1} \, d\mu_F(z).
\]

In particular \( i[F(B), r^s] \) is of order \( s - 1 \).

2.4. First commutator. Here we are going to compute the commutator \( i[H, B] \), and bound it below. To be rigorous about (form) domains we define \( i[H, B] \) first as a (bounded) quadratic form on \( \mathcal{H}^2 \):
\[
\langle i[H, B] \rangle_\psi = 2\langle \text{Im}(BH) \rangle_\psi = \langle H\psi, B\psi \rangle - \langle B\psi, H\psi \rangle \quad \text{for } \psi \in \mathcal{H}^2. \quad (2.14)
\]
Let us set
\[
\bar{\omega} = \frac{1}{2} \text{grad} r^2, \quad \bar{h} = \frac{1}{2} \text{Hess} r^2, \quad \omega = \text{grad} r, \quad h = \text{Hess} r.
\]
Then formal computations would suggest that
\[
A = i[H, r^2], \quad B = i[H, r], \quad A = r^{1/2}B r^{1/2},
\]
and hence that
\[
i[H, A] = p \cdot \bar{h} - \frac{1}{8}(\Delta^2 r^2) - \bar{\omega} \cdot (\nabla V), \quad (2.15)
i[H, B] = r^{-1/2}(i[H, A] - B^2) r^{-1/2} + \frac{8}{3}r^{-2} \omega \cdot h \omega. \quad (2.16)
\]
Thus we could expect that \( i[H, B] \) extends continuously onto larger spaces, and this is partly justified in the following lemma, which provides a formula for the commutator. We note that in the case where Condition 1.1 is adopted direct integration by parts
Proof. We may consider $i[H, B]$ as the unique extension of the corresponding form on $C_c^\infty(\Omega)$. Instead, we shall compute $i[H, B] \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2})$ by combining the realization (2.21) below with a density argument, cf. (2.7b).

Lemma 2.9. Suppose Condition 1.1 or Condition 1.2, and let $R \geq 1$ be sufficiently large. Denoting the extension of the quadratic form $DB := i[H, B]$ given in (2.14) by the same notation, it has expressions

$$DB = s\lim_{t \to 0} t^{-1} (He^{itB} - e^{itB}H) \text{ in } \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2})$$

(2.17)

and, more explicitly,

$$DB = r^{-1/2}(L - B^2)r^{-1/2},$$

(2.18)

where $L \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2})$ is defined as

$$L = p \cdot \Delta - \frac{1}{8}(\Delta^2 r^2) + \frac{1}{4}r^{-1}\omega \cdot h\omega$$

$$+ \sum_{a \in A} \left( -\tilde{\omega}^a \cdot (\nabla^a V^r) + (V^r \cdot \nabla^a)V^s + V^r \cdot \nabla^a V^s + V^s \cdot \nabla^a V^s \right).$$

(2.19)

Here $\tilde{\omega}^a$ and $\nabla^a$ denote the projection onto the internal components of $\tilde{\omega}$ and $\nabla$, respectively, for any cluster decomposition $a \in A$.

Remarks. (1) Under Condition 1.1 the quantities $DB$ and $L$ actually belong to $\mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1})$, and in this case the limit (2.17) may be taken in $\mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1})$, cf. (2.23).

(2) If we consider $DA = i[H, A]$ in some extended sense, we can also write

$$L = DA + \frac{1}{4}r^{-1}\omega \cdot h\omega,$$

(2.20)

cf. (2.15) and (2.19). However, since we will not ‘undo’ the commutator $i[H, A]$ or in other ways use the operator $A$ itself, we have suppressed $A$ from (2.19). We emphasize that our theory does not depend on $A$ but on $B$.

Proof. We may consider $i[H, B] \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2})$ since $\mathcal{H}^1 \subset \mathcal{D}(B)$. By Lemma 2.5 it follows that

$$i[H, B] = s\lim_{t \to 0} t^{-1} (He^{itB} - e^{itB}H) \text{ in } \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2}).$$

(2.21)

Let us write

$$H = H_0 + V^r + V^s + V^s, \quad V^s = \sum_{a \in A} V^s_a \text{ for } * = lr, sr, si.$$

We first consider Condition 1.1. Then by Lemma 2.5 in fact $He^{itB} - e^{itB}H \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1})$ for each $t \in \mathbb{R}$. For sufficiently large $R \geq 1$ we compute as a quadratic form on $C_c^\infty(\Omega) \subset \mathcal{H}^1$

$$He^{itB} - e^{itB}H = \int_0^t \frac{d}{ds}e^{i(t-s)B}He^{isB}ds$$

$$= \int_0^t e^{i(t-s)B}(i[H_0 + V^r + V^s, B])e^{isB}ds$$

$$= \int_0^t e^{i(t-s)B}r^{-1/2}(L - B^2)r^{-1/2}e^{isB}ds.$$
Hence for each $t \in \mathbb{R} \setminus \{0\}$ we obtain
\[
t^{-1}(He^{itB} - e^{itB}H) = t^{-1} \int_0^t e^{i(t-s)B} r^{-1/2}(L - B^2) r^{-1/2} e^{isB} \, ds \tag{2.22}
\]
as a quadratic form on $C_c^\infty(\Omega)$, but both sides of (2.22) obviously extend continuously to $\mathcal{H}^1$. The right-hand side of (2.22) has a strong limit $r^{-1/2}(L - B^2)r^{-1/2}$ for $t \to 0$, and consequently it follows that
\[
s\lim_{t \to 0} t^{-1}(He^{itB} - e^{itB}H) = r^{-1/2}(L - B^2)r^{-1/2} \text{ in } \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1}). \tag{2.23}
\]
Hence the lemma is proven under Condition 1.1.

If Condition 1.2 holds we compute as a quadratic form on $C_c^\infty(X)$
\[
i[H_0 + V^r + V^{sr}, B] = r^{-1/2}(L - B^2)r^{-1/2},
\]
leading to (2.22) as a quadratic form on $C_c^\infty(X)$. Due to Remark 2.6 (3) we can extend (2.22) (uniquely) to $\mathcal{H}^2$. The right-hand side has the strong limit $r^{-1/2}(L - B^2)r^{-1/2}$ in $\mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-1}) \cap \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-2})$. The proof of the lemma is complete under Condition 1.2 also.

\[\square\]

Remark. The completeness of the vector field $\omega$ comes in handy in giving an interpretation of the formal commutator $i[H, B]$. If $\omega$ is incomplete in $\Omega$, we can not freely ‘do and undo’ the commutator $i[H, B]$ due to the boundary contribution coming from integration by parts. See [BGS, Proposition 6.2] for an explicit formula for this boundary contribution under regularity conditions on the boundary of $\Omega$. We also note that for $N = 1$ only the forward completeness suffices, see [IS3].

We quote the following result without proof. It is the so-called Mourre estimate for the $N$-body Schrödinger operator. Here again we suppress the usual conjugate operator $A$, since in this paper it suffices to have this estimate for the quadratic form $L$ defined by (2.19); the expression in terms of $A$ is not needed.

Lemma 2.10. Suppose Condition 1.1 or Condition 1.2. Let $I \subset \mathbb{R} \setminus \mathcal{T}(H)$ be a compact interval, let $\epsilon > 0$, and take $R \geq 1$ large enough. Then for any $\lambda \in I$ there exist a neighbourhood $U$ of $\lambda$ and a compact operator $K$ on $\mathcal{H}$ such that for all real-valued $f \in C_c^\infty(U)$
\[
f(H)^*Lf(H) \geq f(H)^*(2d(\lambda) - \epsilon - K)f(H).
\]

Proof. We proved this version of the Mourre estimate in [IS1] for a class of more regular pair-potentials using the properties of the rescaled Graf function stated in Subsection 1.2.3. Note that although there is an extra term $\frac{1}{4}r^{-1} \omega \cdot h\omega$ in (2.20), it is obviously harmless. We omit the details. \[\square\]

We will always implement Lemma 2.10 in combination with Lemma 2.9 in the following form.

Corollary 2.11. Suppose Conditions 1.1 or 1.2. Let $\lambda \in \mathbb{R} \setminus \mathcal{T}(H)$ and $\sigma \in (0, \gamma(\lambda))$, and take $R \geq 1$ large enough and a neighborhood $U \subset \mathbb{R}$ of $\lambda$ small enough. Then for any real-valued function $f \in C_c^\infty(U)$ there exists $C > 0$ such that, as quadratic forms on $\mathcal{H}$,
\[
f(H)(DB)f(H) \geq f(H)r^{-1/2}(\sigma^2 - B^2)r^{-1/2}f(H) - Cr^{-2}.
\]
Proof. In order to apply Lemma 2.10 we fix variables in the following order: First fix any \( \lambda \in \mathbb{R} \setminus \mathcal{T}(H) \) and \( \sigma \in (0, \gamma(\lambda)) \) as in the assertion. We then let \( I = \{ \lambda \} \) and take any \( \epsilon \in (0, 2d(\lambda) - \sigma^2) \). With these quantities \( I \) and \( \epsilon \) fixed we consider any large \( R \geq 1 \) in agreement with both Lemmas 2.9 and 2.10. We fix a neighbourhood \( \mathcal{U} \) of \( \lambda \) and a compact operator \( K \) on \( \mathcal{H} \) as in Lemma 2.10, and let \( f \in C_c^\infty(\mathcal{U}) \) be any real-valued function. We need to show the quadratic form bound.

First, by Lemma 2.9 we have
\[
f(H)(DB)f(H) = f(H)r^{-1/2}(L - B^2)r^{-1/2}f(H)
\geq r^{-1/2}f(H)Lf(H)r^{-1/2} - f(H)r^{-1/2}B^2r^{-1/2}f(H)
+ [f(H), r^{-1/2}]L[r^{-1/2}, f(H)]
+ 2\text{Re}(r^{-1/2}f(H)L[r^{-1/2}, f(H)]).
\]
(2.24)

By Lemma 2.10 we can bound the first term on the right-hand side of (2.24) as
\[
r^{-1/2}f(H)Lf(H)r^{-1/2} \geq r^{-1/2}f(H)(2d(\lambda) - \epsilon - K)f(H)r^{-1/2}.
\]
(2.25)

Since \( K \) is compact on \( \mathcal{H} \), we can choose \( m \in \mathbb{N}_0 \) large enough that
\[
2d(\lambda) - \epsilon - \|K - \chi_mK\chi_m\| \geq \sigma^2,
\]
(2.26)

where \( \chi_m \) is the smooth cut-off function from (2.4). The bounds (2.25) and (2.26) imply that
\[
r^{-1/2}f(H)Lf(H)r^{-1/2} \geq \sigma^2 r^{-1/2}f(H)^2 r^{-1/2} - r^{-1/2}f(H)\chi_mK\chi_m f(H)r^{-1/2}
= \sigma^2 f(H)r^{-1}f(H) - \sigma^2[r^{-1/2}, f(H)] [f(H), r^{-1/2}]
+ 2\sigma^2 \text{Re}(r^{-1/2}f(H)[f(H), r^{-1/2}])
- r^{-1/2}f(H)\chi_mK\chi_m f(H)r^{-1/2}.
\]
(2.27)

By (2.24), (2.27) it remains to bound the third and fourth terms of (2.24) and the second to fourth terms of (2.27), but all of them can be treated by using Lemma 2.4. In fact one easily checks using the explicit representations (2.6) and (2.19) that all of these terms are of order \(-2\) in the sense of (2.1).

Finally we compute and bound commutators of functions of \( H \) and \( B \).

Lemma 2.12. Suppose Condition 1.1 or Condition 1.2, and let \( R \geq 1 \) be sufficiently large. For any \( f \in \mathcal{F}^t \) and \( F \in \mathcal{F}^t \) with \( t < -1/2 \) and \( t' < 1 \) the commutators \( i[f(H), B], i[f(H), F(B)] \) extend to bounded sesquilinear forms on \( \mathcal{H} \) from \( \mathcal{D}(B), \mathcal{D}(F(B)) \), and they have expressions
\[
i[f(H), B] = - \int_C (H - z)^{-1} (DB)(H - z)^{-1}d\mu_f(z),
i[f(H), F(B)] = - \int_C (B - z)^{-1}(i[f(H), B])(B - z)^{-1}d\mu_F(z),
\]
respectively. Moreover, with the notation (2.1)
\[
i[f(H), B] = O(r^{-1}), \quad i[f(H), F(B)] = O(r^{-1}).
\]

Proof. By (2.17) and interpolation \( i[H, B] \in \mathcal{L}(\mathcal{H}^{3/2}, \mathcal{H}^{-3/2}) \). This yields the first formula by Corollary 2.3. The second follows from the first and Corollary 2.3. Using the expression for \( DB \) from Lemma 2.9 one easily checks the last assertions, cf. the proofs of Lemmas 2.4 and 2.8.  \( \square \)
2.5. **Second commutator.** Here we provide a realization of the second commutator $i[DB, B]$, and bound it in some operator space. Under Condition 1.2 one can naturally define and interpret this second commutator. On the other hand, under Condition 1.1 there is a ‘domain problem’, and this prevents us from directly defining $i[DB, B]$ as a quadratic form even on $\mathcal{H}^2$, like we first did for $DB = i[H, B]$. However, for our application it suffices to consider an alternative strong limit of the form (2.28) below, cf. (2.7b) and (2.17). We note that only the first commutator does not suffice for the LAP bounds either in the abstract Mourre theory, see [ABG2].

Note that by Lemmas 2.5 and 2.9 we may consider

$$(DB)e^{itB} - e^{itB}(DB) \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2}).$$

**Lemma 2.13.** Suppose Condition 1.1 or Condition 1.2, and let $R \geq 1$ be sufficiently large. Then there exists the strong limit

$$i[DB, B] := s\text{-}\lim_{t \to 0} t^{-1}(i(DB)e^{itB} - e^{itB}(DB)) \text{ in } \mathcal{L}(\mathcal{H}^2, \mathcal{H}^{-2}).$$

Moreover, with the notation (2.1)

$$(H - i)^{-1}(i[DB, B])(H + i)^{-1} = O(r^{-1-2\kappa}),$$

where $\kappa = \delta/(1 + 2\delta)$ as in (1.17).

**Proof.** First we consider Condition 1.1. By (2.18) and (2.19)

$$DB = \mathcal{L} + \sum_{a \in A} \mathcal{V}_a;$$

$$\mathcal{L} = r^{-1/2}(p \cdot \hbar p - \frac{1}{8}(\Delta^2 r^2) + \frac{1}{4}r^{-1}\omega \cdot \hbar \omega - B^2)r^{-1/2},$$

$$\mathcal{V}_a = -\omega^a \cdot (\nabla^a V^i_a) + V^{sr}_a \text{div } \omega^a - 2 \text{Im}((V^a_a \omega^a) \cdot p^a).$$

We consider the contribution from each of these terms.

The contribution from $\mathcal{L}$ is straightforward. Similarly to the proof of Lemma 2.9 we can prove the existence of the strong limit

$$i[\mathcal{L}, B] := s\text{-}\lim_{t \to 0} t^{-1}(\mathcal{L}e^{itB} - e^{itB}\mathcal{L}) \text{ in } \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1})$$

by extension from $C^\infty_c(\Omega)$, cf. (2.22) and (2.23). In fact the expression $\mathcal{L}$ simplifies as $\mathcal{L} = p \cdot \hbar p - \frac{1}{4}(\Delta^2 r)$, cf. the familiar formula (for any $v \in \mathcal{X}(\Omega)$)

$$i[p^2, v \cdot p + p \cdot v] = 2p(v' + v^t)p - (\Delta \text{div } v).$$

Using this representation we compute

$$i[\mathcal{L}, B] = p \cdot \left(2\hbar^2 - ((\omega \cdot \nabla)h)\right)p - \frac{1}{2}(\nabla \cdot \hbar \nabla \Delta r) + \frac{1}{4}(\omega \cdot \nabla(\Delta^2 r)).$$

The contribution from the right-hand side is easily checked using (1.14) to agree with (2.29) (the contribution is an operator of order $-2$).

Next we consider the contribution from $\mathcal{V}_a$. Since we can write

$$(H - i)^{-1}(\mathcal{V}_ae^{itB} - e^{itB}\mathcal{V}_a)(H + i)^{-1}$$

$$= (H - i)^{-1}\mathcal{V}_a(H + i)^{-1}e^{itB} - e^{itB}(H - i)^{-1}\mathcal{V}_a(H + i)^{-1}$$

$$+ (H - i)^{-1}\mathcal{V}_a(H + i)^{-1}(He^{itB} - e^{itB}H)(H + i)^{-1}$$

$$+ (H - i)^{-1}(He^{itB} - e^{itB}H)(H - i)^{-1}\mathcal{V}_a(H + i)^{-1},$$

(2.32)
there exists the strong limit
\[ i[V_a, B] := \operatorname{s-lim}_{t \to 0} \left( V_a e^{itB} - e^{itB} V_a \right) \text{ in } L(H^2, H^{-2}), \]
which has the expression, with appropriate weights from both sides,
\[
(H - i)^{-1} i[V_a, B](H + i)^{-1} \\
= -2 \operatorname{Im}((H - i)^{-1} V_a (H + i)^{-1} B) \\
+ 2 \operatorname{Re}((H - i)^{-1} V_a (H + i)^{-1} (\mathcal{D}B)(H + i)^{-1}).
\]  
(2.33)

Using the expression (2.30) it follows that the last term on the right-hand side of (2.33) agrees with (2.29), so it only remains to examine the first term. We set
\[
\bar{\eta}_b(x) = \eta_b \left( x/r^{1/(1+2\delta)} \right) = \eta_{1,b} \left( x/(R r^{1/(1+2\delta)}) \right),
\]
and decompose
\[
B = \sum_{b \in A} \tilde{B}_b; \quad \tilde{B}_b = \frac{1}{2} \left( (\bar{\eta}_b \omega) \cdot p + p \cdot (\bar{\eta}_b \omega) \right).
\]

Then the investigation of the first term of (2.33) reduces to that of
\[
2 \operatorname{Im}((H - i)^{-1} V_a (H + i)^{-1} \tilde{B}_b) \text{ for } b \in A.
\]  
(2.34)

We first consider the case \( a \nsubseteq b \). Then by (1.11) we have
\[
|x^a| \geq c_1 R r^{1/(1+2\delta)} \text{ on supp } \bar{\eta}_b.
\]
This combined with (1.4) implies that for \( a \nsubseteq b \) the contribution (2.34) agrees with (2.29).

It remains to consider the case \( a \subset b \). We further decompose
\[
\tilde{B}_b = (\tilde{B}_b)^b + (\tilde{B}_b)_b
\]
with
\[
(\tilde{B}_b)^b = \frac{1}{2} \left( (\bar{\eta}_b \omega)^b \cdot p^b + p^b \cdot (\bar{\eta}_b \omega)^b \right), \quad (\tilde{B}_b)_b = \frac{1}{2} \left( (\bar{\eta}_b \omega)_b \cdot p_b + p_b \cdot (\bar{\eta}_b \omega)_b \right).
\]

Accordingly (2.34) decomposes as
\[
2 \operatorname{Im}((H - i)^{-1} V_a (H + i)^{-1} \tilde{B}_b) = 2 \operatorname{Im}((H - i)^{-1} V_a (H + i)^{-1} (\tilde{B}_b)^b) \\
+ 2 \operatorname{Im}((H - i)^{-1} V_a (H + i)^{-1} (\tilde{B}_b)_b).
\]  
(2.35)

To bound the first term of (2.35) note that (1.11) implies that
\[
|x^b| \leq C_1 R r^{1/(1+2\delta)} \text{ on supp } \bar{\eta}_b;
\]
so that, combined with (1.14),
\[
|\omega^b| \leq C_2 R r^{-2\delta/(1+2\delta)} \text{ on supp } \bar{\eta}_b.
\]  
(2.36)

Hence the first term of (2.35) certainly agrees with (2.29). As for the second term of (2.35), we first note that the vector field \( v_b = \bar{\eta}_b \omega_b \in \mathfrak{X}(\Omega) \) is complete on \( \Omega \). To see this it suffices (since it is bounded) to show that \( v_b \) is tangent to \( \partial (\Omega_c + X_c) \) for all \( c \in A \): If \( c \nsubseteq b \), we have
\[
|x^c| \geq c_2 R r^{1/(1+2\delta)} \text{ on supp } \bar{\eta}_b,
\]
and hence by boundedness of \( \partial \Omega_c \subset X_c \)
\[
v_b = 0 \text{ on } \partial (\Omega_c + X_c)
\]
for large $R \geq 1$. If $c \subset b$

$v_b \in X_b \subset X_c$, also implying that $v_b$ is tangent to $\partial(\Omega_c + X_c)$. Now $v = v_b$ is complete on $\Omega$, and indeed it satisfies the assumptions of Lemma 2.7. By using the lemma we can move the operator $(\tilde{B}_b)_b$ to the center in the second term of (2.35). We calculate similarly to (2.32)

$$2 \text{Im}((H - i)^{-1}V_a(H + i)^{-1}(\tilde{B}_b)_b) = s\lim_{t \to 0} t^{-1}(H - i)^{-1} \left( - (V_a e^{it(\tilde{B}_b)_b} - e^{it(\tilde{B}_b)_b}V_a) \right. \right.$$  

$$\left. + (He^{it(\tilde{B}_b)_b} - e^{it(\tilde{B}_b)_b}H)(H - i)^{-1}V_a \right) \left. + V_a(H + i)^{-1}(He^{it(\tilde{B}_b)_b} - e^{it(\tilde{B}_b)_b}H) \right) (H + i)^{-1}.$$  

(2.37)

Noting that $(\nabla_b V^{sr}_a) = 0$ due to the property $a \subset b$, we can mimic the proof of Lemma 2.9 and calculate the strong limit

$$i[\mathcal{V}_a, (\tilde{B}_b)_b] := s\lim_{t \to 0} t^{-1}(V_a e^{it(\tilde{B}_b)_b} - e^{it(\tilde{B}_b)_b}V_a) \quad \text{in } \mathcal{L}(H^1, H^{-1})$$

as the explicit expression (with $v_b = \tilde{\eta}_b \omega_b$)

$$i[\mathcal{V}_a, (\tilde{B}_b)_b] = ((v_b \cdot \nabla_b) \omega^a) \cdot (\nabla^a V^{sr}_a) - (v_b \cdot \nabla_b \text{div} \omega^a) V^{sr}_a$$

$$+ 2 \text{Im}((V^{sr}_a (v_b \cdot \nabla_b) \omega^a) \cdot p^a) - 2 \text{Im}((V^{sr}_a \omega^a \cdot \nabla) v_b) \cdot p_b + (V^{sr}_a \omega^a \cdot \nabla) \text{div} v_b$$

$$=: T_1 + \cdots + T_5.$$  

To see that the contribution from the first term in the big brackets of (2.37) agrees with (2.29) we plug in (2.29) and bound separately. To treat $T_1, T_2$ and $T_3$ we use

$$\omega_b \cdot \nabla_b = r^{-1} x \cdot \nabla + r^{-1} (\tilde{\omega} - x) \cdot \nabla - \omega^b \cdot \nabla^b,$$  

(2.38)

which together with (2.36) to obtain the $r^{-1-2\kappa}$ decay. The terms $T_4$ and $T_5$ contribute by terms with this decay too thanks to the bounds

$$|\partial^\beta v_b| = O(r^{-1/(1+2\beta)}) = O(r^{-2\kappa}); \quad |\beta| \geq 1.$$  

(2.39)

We can compute the expressions for the second and third terms in the big brackets of (2.37) as in (2.32) and (2.33). Using Lemma 2.7 with $v = \tilde{\eta}_b \omega_b$, (2.31), (2.39) and the proof of Lemma 2.9 we easily check that these contributions also agree with (2.29). Hence we are done with the proof under Condition 1.1.

The proof under Condition 1.2 is simpler. In fact we may then consider $i[D, B]$ naturally extended from $C^\infty_c(X)$, and the extension coincides with (2.28), cf. Remarks 2.6 (3) and the proof of Lemma 2.9. Again we use the decomposition (2.30). However now we can treat the contribution from $\mathcal{V}_a$ more directly by using the more freedom of distributing factors of momenta (avoiding the previous somewhat technical ‘commuting back and forth argument’). Noting that

$$B = B_a + B; \quad B_a = \frac{1}{2}(\omega^a \cdot p^a + p^a \cdot \omega^a), \quad B = \frac{1}{2}(\omega_a \cdot p_a + p_a \cdot \omega_a),$$  

(2.40)

we decompose

$$i[\mathcal{V}_a, B] = i[\mathcal{V}_a, B_a] + i[\mathcal{V}_a, B_a].$$
By not doing the commutation the contribution from the first term is seen to be of the form $O(r^{-1-2\varepsilon})$. Next, for the second term, we compute as above
\[
i [V_a, B_a] = \left((\omega_a \cdot \nabla_a)\omega^a\right) \cdot \left(\nabla^a V_a^r\right) - \left(\omega^a \cdot \nabla_a \operatorname{div} \omega^a\right)V_a^{sr} + 2\operatorname{Im}\left((V^s_{a\omega^a} \cdot \nabla)\omega^a\right) \cdot p^a) + (V^s_{a\omega^a} \cdot \nabla) \operatorname{div} \omega_a.
\]
For the first three terms (containing $\omega_a \cdot \nabla_a$) we substitute (2.38) (now with $b = a$), bound separately and then conclude that the contribution from $i[V_a, B_a]$ is of the form $O(r^{-2})$. In conclusion we obtain that the contribution from $i[V_a, B]$ to (2.29) under Condition 1.2 is of the form $O(r^{-1-2\varepsilon}) = O(r^{-1-2\varepsilon})$.

Finally we consider as a continuation of Lemma 2.12 the second commutator of a function of $H$ and $B$.

**Lemma 2.14.** Suppose Condition 1.1 or Condition 1.2, and let $R \geq 1$ be sufficiently large. For any $f \in \mathcal{F}$ with $t < -1$ the second commutator $i[i[f(H), B], B]$ extends to a bounded sesquilinear form on $H$ from $\mathcal{D}(B)$, and it has the expression
\[
i[i[f(H), B], B] = -\int_C (H - z)^{-1} (i[DB, B]) (H - z)^{-1} d\mu_f(z) + 2\int_C (H - z)^{-1} (DB)(H - z)^{-1} (DB)(H - z)^{-1} d\mu_f(z).
\]
Moreover, with the notation (2.1)
\[
i[i[f(H), B], B] = O(r^{-1-2\varepsilon}).
\]

**Proof.** By Lemmas 2.9, 2.12 and 2.13 we calculate, as a sesquilinear form on $\mathcal{D}(B)$,
\[
i[i[f(H), B], B] = \lim_{t \to 0} t^{-1} (i[f(H), B]e^{itB} - e^{itB}i[f(H), B])
\]
\[
= -\int_C (H - z)^{-1} (i[DB, B]) (H - z)^{-1} d\mu_f(z) + 2\int_C (H - z)^{-1} (DB)(H - z)^{-1} (DB)(H - z)^{-1} d\mu_f(z).
\]
By using the expression for $DB$ from Lemma 2.9 and (2.29) we obtain the boundedness and (2.41) from the above representation.

3. **Proof of Rellich type theorems**

In this section we prove Theorem 1.4 under Condition 1.1 or Condition 1.2, and we prove Theorem 1.5 under Condition 1.1. The proofs are given in Sections 3.1 and 3.2, respectively. The idea of proof comes from a combination of [IS1], [IS2], [IS3] and [FH]. Note that in [IS1] the second commutator $i[i[H, B], B]$ was not needed, but for Theorem 1.4 it is. We also note that our arguments get simpler if one considers only polynomial decay estimates at non-threshold energies $E$, see [IS2].

3.1. **Exponential decay estimates.** Throughout this subsection we impose Conditions 1.1 or 1.2. We introduce the regularized weights
\[
\Theta = \Theta_{a, b, R, \nu} = \chi_{m, n} e^{\theta}, \quad n, m \in \mathbb{N}_0, \ n > m, \ R \geq 1
\]
with exponents $\theta$ given by
\[
\theta = \theta^{a, b, \nu} = \alpha r + \beta \int_0^r \frac{1}{s^{1/2}} ds; \quad \alpha, \beta \geq 0, \ n \in \mathbb{N}_0.
\]
Here $r = r_R$ indeed depends on $R \geq 1$, and $\kappa \in (0, 1/4]$ is from (1.17). We are going to investigate the Heisenberg derivative of the ‘propagation observable’ $P$ defined as
\[ P = P_{m,n,R,\nu,\epsilon}^\alpha = \Theta f(H)\zeta(B)f(H)\Theta \in \mathcal{L}(\mathcal{H}); \quad f \in C_c^\infty(\mathbb{R}), \quad \epsilon \in (0, 1), \tag{3.2} \]
where $\zeta = \zeta_c \in C_c^\infty(\mathbb{R})$ is the smooth sign function from Section 2.1.

In the following we denote the derivatives of $\Theta$ and $\theta$ in $r$ by primes, e.g.,
\[ \theta' = \alpha + \beta \theta_0^{-1-2\kappa}, \quad \theta'' = -\beta (1 + 2\kappa) 2^{-\nu} \theta_0^{-2-2\kappa}; \quad \theta_0 = 1 + r/2\nu. \tag{3.3} \]
In particular, noting that $2^{-\nu} \theta_0^{-1} \leq r^{-1}$, we have
\[ |\theta^{(k)}| \leq C_k r^{1-k} \theta_0^{-1-2\kappa} \quad \text{for} \quad k = 2, 3, \ldots. \]

**Lemma 3.1.** Suppose Condition 1.1 or Condition 1.2. Let $E \in \mathbb{R}$ and $\alpha_0 \geq 0$ satisfy $\lambda := E + \alpha_0^2/2 \notin \mathcal{T}(H)$. Then there exist $c, C > 0$, $n_0 \in \mathbb{N}$, $R \geq 1$, $\alpha_1 \in \{0\} \cup (0, \alpha_0)$, $\beta, \epsilon \in (0, 1)$ and real-valued $f \in C_c(\mathbb{R})$, such that for all $n > m \geq 2n_0$, $\nu \geq 2n_0$ and $\alpha \in [\alpha_1, \alpha_0]$
\[ 2 \text{Im} \left( P(H - E) \right) \geq cr^{-1}\theta_0^{-2\kappa} \Theta^2 - C(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)r^{-1} e^{2\nu} \]
\[ - \text{Re} \left( \Theta Q_1 \Theta(H - E) \right) - \text{Re} \left( \Theta\theta_0^{-\kappa} Q_2 \theta_0^{-\kappa} \Theta(H - E) \right); \tag{3.4} \]
here $Q_1, Q_2 \in \mathcal{L}(L^2_{-1/2}, L^2_{1/2})$ are symmetric, (possibly) depending on $\alpha$ and the other parameters except though for $n, m$ and $\nu$, and the estimate (3.4) is understood as a quadratic form on $H^2$.

**Remark.** We have stated more properties of $Q_1$ and $Q_2$ than needed, for example boundedness on $L^2$ suffices and the independence of $n, m$ and $\nu$ is irrelevant.

**Remark 3.2.** We note that the constants, in particular $c, C > 0$, can be chosen locally uniformly in $E \in \mathbb{R}$ and $\alpha_0 > 0$ with $E + \alpha_0^2/2 \notin \mathcal{T}(H)$. This in fact enables us to apply the arguments of [Pe] to conclude that the set of non-threshold eigenvalues of $H$ can accumulate only at points in $\mathcal{T}(H)$ from below.

**Proof. Step I.** We intend to apply Corollary 2.11, and for that purpose we fix some variables in the following order: Let $E \in \mathbb{R}$ and $\alpha_0 \geq 0$ be given and define $\lambda$ correspondingly. Fix then any $\sigma \in (0, \gamma(\lambda))$. Choose a big $R \geq 1$ and a neighborhood $U \subset \mathbb{R}$ of $\lambda$ as in Corollary 2.11, and let $f \in C_c(\mathbb{R})$ be a function such that $0 \leq f \leq 1$ in $U$ and $f = 1$ in a neighborhood $I$ of $\lambda$. Then Corollary 2.11 asserts that
\[ f(H)(DB)f(H) \geq f(H)r^{-1/2}(\sigma^2 - B^2)r^{-1/2} f(H) - C_4 r^{-2}, \tag{3.5} \]
which we will implement in Step IV below. We fix $\alpha_1 \in \{0\} \cup (0, \alpha_0)$ such that
\[ \inf_{\alpha \in [\alpha_1, \alpha_0]} d(E + \frac{1}{2} \alpha^2, \mathbb{R} \setminus I) > 0. \]

With these variables we consider the operator $P$ given in (3.2). Note though that we have not yet fixed $c, C > 0$, $n_0 \geq 1$, $\beta, \epsilon \in (0, 1)$, $Q_1$ and $Q_2$. We will need $\epsilon^2 < \sigma^2/2$. These quantities will be chosen in Step II. In the following estimates the dependence on $\beta, \epsilon \in (0, 1)$, $\epsilon^2 < \sigma^2/2$, and $n_0 \geq 1$ will always be emphasized, and the estimates will be uniform in $n, m, \nu$ and $\alpha$ fulfilling $n > m \geq 2n_0$, $\nu \geq 2n_0$ and $\alpha \in [\alpha_1, \alpha_0]$ (as required for (3.4)).

We will repeatedly use (small variations of) Lemmas 2.4, 2.8 and 2.12 to bound commutators of functions of $H$, $r$ and $B$, mostly without reference. It is assumed
that $R \geq 1$ is chosen so large that not only (3.5) is valid, but also that these lemmas apply for this (fixed) $R$.

**Step II.** We are calculate and bound the left-hand side of (3.4). By the definition (3.2) we compute

$$2 \text{Im}(P(H - E)) = \mathbf{D}P = 2 \text{Re}[(D\Theta)f(H)\zeta(B)f(H)\Theta] + \Theta f(H)(D\zeta(B))f(H)\Theta.$$ (3.6)

We claim that the two terms on the right-hand side of (3.6) are bounded from below as

$$2 \text{Re}[(D\Theta)f(H)\zeta(B)f(H)\Theta] \geq 2^{n_0+1}\alpha r^{-1/2}\Theta f(H)B\zeta(B)f(H)\Theta r^{-1/2} - (C_2 + C_3(\epsilon)2^{-n_0})\alpha r^{-1}\Theta^2$$

$$+ 2^n\beta r^{-1/2}\Theta f(H)B\zeta(B)f(H)\Theta r^{-1/2} - C_3(\epsilon)\beta r^{-1}\Theta^2$$

$$- C_3(\epsilon)r^{-2}\Theta^2 - C_3(\epsilon)\left(\lambda^2_{m-1,n+1} + \lambda^2_{n-1,n+1}\right)r^{-1}e^{2\theta},$$ (3.7)

and

$$\Theta f(H)(D\zeta(B))f(H)\Theta \geq \left(\frac{\sigma^2}{\epsilon^2} - \epsilon^2\right)r^{-1/2}\Theta f(H)'\zeta'(B)f(H)\Theta r^{-1/2}$$

$$+ \frac{1}{2}\sigma^2 r^{-1/2}\Theta f(H)'\zeta'(B)f(H)\Theta \Theta r^{-1/2} - C_3(\epsilon)r^{-1}2^\kappa\Theta^2,$$ (3.8)

both of which are uniform in $\beta, \epsilon \in (0, 1)$, $\epsilon^2 < \sigma^2/2$ and $n_0 \geq 1$, and also in $\alpha \in [\alpha_1, \alpha_0]$, $n > m \geq 2n_0$ and $\nu \geq 2n_0$. We will prove these bounds (3.7) and (3.8) later in Steps III and IV, respectively. For the moment let us assume them. Then by (3.6)–(3.8) and Lemma 2.1 we obtain

$$2 \text{Im}(P(H - E)) \geq \min\{2^{n_0-2}\alpha\epsilon, (\sigma^2 - 2\epsilon^2)/\epsilon\}r^{-1/2}\Theta f(H)^2\Theta r^{-1/2}$$

$$- (C_2 + C_3(\epsilon)2^{-n_0})\alpha r^{-1}\Theta^2$$

$$+ \min\{2^{n_0-3}\beta\epsilon, \sigma^2/\epsilon\}r^{-1/2}\Theta f(H)^2\Theta r^{-1/2} - C_3(\epsilon)\beta r^{-1}\Theta^2$$

$$- C_3(\epsilon)\left(\lambda^2_{m-1,n+1} + \lambda^2_{n-1,n+1}\right)r^{-1}e^{2\theta}.$$ (3.9)

Now we are going to remove $f(H)^2$ from the first and third terms of (3.9) with some controllable errors, and for that end we introduce $f_1 \in \mathcal{F}^{-1}$ by

$$f_1(t) = \left(1 - f(t)^2\right)(t - \lambda)^{-1},$$ (3.10)

so that (uniformly in $\alpha \in [\alpha_1, \alpha_0]$)

$$1 - f(H)^2 \leq C_4 f_1(H)(H - E - \frac{1}{2}\alpha^2).$$

Then we estimate

$$r^{-1/2}\Theta(1 - f(H)^2)\Theta r^{-1/2}$$

$$\leq C_4 \text{Re}\left[r^{-1/2}\Theta f_1(H)\Theta r^{-1/2}(H - E - \frac{1}{2}\alpha^2)\right]$$

$$+ \frac{1}{2}C_4 \text{Re}\left[r^{-1/2}\Theta f_1(H)\omega^2(\Theta r^{-1/2})^n\right] + C_4 \text{Im}\left[r^{-1/2}\Theta f_1(H)B(\Theta r^{-1/2})\right]$$

$$\leq C_4 \text{Re}\left[r^{-1/2}\Theta f_1(H)\Theta r^{-1/2}(H - E)\right]$$

$$+ C_5 r^{-3/2}\Theta^2 + C_5\beta r^{-1}\Theta^2 + C_5\left(\lambda^2_{m-1,n+1} + \lambda^2_{n-1,n+1}\right)r^{-2}e^{2\theta}. $$ (3.11)
Here we have used the inequality
\[ |ω^2 - 1| \leq C_7 r^{-1/2}, \]
which is a consequence of (1.14) (a stronger bound holds, but this is not needed). Hence by (3.11) it follows that uniformly in large \( n_0 \geq 1 \) and small \( β \in (0, 1) \)
\[
r^{-1/2}θf(H)^2θr^{-1/2} \geq c_1r^{-1}θ^2 - \text{Re}(ΘQ_1Θ(H - E))
- C_5(χ_{m-1,m+1}^2 + χ_{n-1,n+1}^2)r^{-2}e^{2θ},
\]
where \( Q_1 = C_4r^{-1/2}f_1(H)r^{-1/2} \). Similarly, we can show that uniformly in large \( n_0 \geq 1 \) and small \( β \in (0, 1) \)
\[
r^{-1/2}θ_0^\alphaθf(H)^2θ_0^\alpha r^{-1/2} \geq c_2r^{-1}θ_0^{-2α}θ^2 - \text{Re}(Θθ_0^αQ_2θ_0^αΘ(H - E))
- C_6(χ_{m-1,m+1}^2 + χ_{n-1,n+1}^2)r^{-2}e^{2θ},
\]
where \( Q_2 = C_4r^{-1/2}f_1(H)r^{-1/2} \).

Finally by (3.9), (3.12) and (3.13) it follows that
\[
2\text{Im}(P(H - E)) \geq \left[c_1 \min\{2n_0^{-2}\alpha ε, (σ^2 - 2ε^2)/ε\} - (C_2 + C_3(ε)2^{-n_0})\right]r^{-1}θ^2
+ \left[c_2 \min\{2n_0^{-3}\beta ε, σ^2/ε\} - C_3(ε)β - 2C_3(ε)θ_0^{-2α}r^{-2α}θ^2
- C_7(n_0, ε)(χ_{m-1,m+1}^2 + χ_{n-1,n+1}^2)r^{-1}e^{2θ}
- \min\{2n_0^{-2}\alpha ε, (σ^2 - 2ε^2)/ε\} \text{Re}(ΘQ_1Θ(H - E))
- \min\{2n_0^{-3}\beta ε, σ^2/ε\} \text{Re}(Θθ_0^αQ_2θ_0^αΘ(H - E))\].

We can bound \( θ_0^αr^{-2α} ≤ 2^{2α}2^{-4αn_0} \) on the support of \( Θ^2 \) for \( ν ≥ 2n_0 \). Now, first let \( ε > 0 \) be small enough, then choose \( β \in (0, 1) \) small enough, and finally let \( n_0 ≥ 1 \) be large enough. We conclude the desired bound (3.4) since the first square bracket expression is then non-negative while the second is bounded from below by some constant \( c > 0 \). This \( c \) and \( C = C_7(n_0, ε) \) work.

**Step III.** We prove (3.7). Substitute the expression
\[
DΘ = \text{Re}(Θ′ω · p) = θ′ΘB + χ′_m,eθB - \frac{1}{2}ω^2(θ″Θ + θ′^2Θ + χ″_m,eθ + 2χ′_m,θ′eθ)
\]
and then we can first write
\[
2\text{Re}[(DΘ)f(H)ζ(B)f(H)Θ]
= 2\text{Re}[θ′ΘBf(H)ζ(B)f(H)Θ] + 2\text{Re}[χ′_m,eθBf(H)ζ(B)f(H)Θ]
+ \text{Im}[ω^2(θ″Θ + θ′^2Θ + χ″_m,eθ + 2χ′_m,θ′eθ)f(H)ζ(B)f(H)Θ].
\]
The first term of (3.14) can be calculated by (3.3) as
\[
2\text{Re}[θ′ΘBf(H)ζ(B)f(H)Θ]
= 2α\text{Re}[θBf(H)ζ(B)f(H)Θ] + 2β\text{Re}[θ_0^{-1-2α}ΘBf(H)ζ(B)f(H)Θ]
\geq 2α\text{Re}[θf(H)Bζ(B)f(H)Θ - (C′_1 + C′_2(ε)r^{-1})αr^{-1}θ^2]
+ 2β\text{Re}[θ_0^{-1-2α}θf(H)Bζ(B)f(H)Θ]
- (C′_1 + C′_2(ε)r^{-1})βr^{-1}θ_0^{-1-2α}θ^2.
\]
We define $Z_1 \in \mathcal{F}^{1/2}$ as

$$ Z_1(b) = b \sqrt{b^{-1} \zeta(b)}. $$

Then we have $Z_1(b)^2 = b \zeta(b)$, so that for the first term of (3.15)

$$ 2\alpha \Theta f(H)B\zeta(B)f(H)\Theta $$

$$ \geq 2\alpha \Theta f(H)Z_1(B)(2^{n_0}r^{-1}\chi_{n_0})Z_1(B)f(H)\Theta $$

$$ \geq 2^{n_0+1}\alpha r^{-2}\Theta f(H)B\zeta(B)f(H)\Theta r^{-1/2} - \left(C'_3 + C'_4(\epsilon)r^{-1}\right)2^{n_0}\alpha r^{-2}\Theta^2; $$

(3.16)

here we used Lemma 2.8 repeatedly.

Similarly, for the third term of (3.15)

$$ 2\beta \text{Re}\left[\theta_0^{1-2\kappa}\Theta f(H)B\zeta(B)f(H)\Theta\right] $$

$$ \geq 2\beta \theta_0^{-\kappa}f(H)Z_1(B)\theta_0^{-1}Z_1(B)f(H)\Theta \theta_0^{-\kappa} - C'_5(\epsilon)\beta r^{-1}\theta_0^{-1-2\kappa}\Theta^2 $$

$$ \geq 2\beta \theta_0^{-\kappa}f(H)Z_1(B)(2^{n_0}r^{-1}\chi_{n_0})Z_1(B)f(H)\Theta \theta_0^{-\kappa} $$

$$ - C'_5(\epsilon)\beta r^{-1}\theta_0^{-1-2\kappa}\Theta^2 $$

(3.17)

By (3.15), (3.16) and (3.17) we obtain (3.7).

By (3.15), (3.16) and (3.17) we obtain for the first term of (3.14)

$$ 2\text{Re}\left[\theta^\prime \Theta B f(H)\zeta(B)f(H)\Theta\right] $$

$$ \geq 2^{n_0+1}\alpha r^{-1/2}\Theta f(H)B\zeta(B)f(H)r^{-1/2}\Theta - \left(C'_7 + C'_8(\epsilon)2^{-n_0}\right)\alpha r^{-1}\Theta^2 $$

(3.18)

$$ + 2^{n_0}\beta r^{-1/2}\theta_0^{-\kappa}\Theta f(H)B\zeta(B)f(H)r^{-1/2}\theta_0^{-\kappa}\Theta - C'_9(\epsilon)\beta r^{-1}\theta_0^{-1-2\kappa}\Theta^2. $$

On the other hand, note that by (1.14) we have

$$ |\omega \cdot \nabla \omega^2| \leq C_0r^{-2}. $$

Then the second and third terms of (3.14) are bounded as

$$ 2\text{Re}\left[\chi'_{m,n}B f(H)\zeta(B)f(H)\Theta\right] $$

$$ + \text{Im}\left[\omega^2(\theta^\prime\Theta + \theta^\prime\Theta + \chi''_{m,n}e^\theta + 2\chi'_{m,n}\theta e^\theta) f(H)\zeta(B)f(H)\Theta\right] $$

$$ \geq -C'_{10}\alpha r^{-1}\Theta^2 - C'_{11}(\epsilon)\beta r^{-1}\theta_0^{-1-2\kappa}\Theta^2 - C'_{11}(\epsilon)r^{-2}\Theta^2 $$

(3.19)

$$ - C'_{11}(\epsilon)\left(\chi_{m-1,m+1} + \chi_{n-1,n+1}\right)r^{-1}e^{2\Theta}. $$

By (3.14), (3.18) and (3.19) we obtain (3.7).

**Step IV.** Here we prove (3.8). Take a real-valued function $\tilde{f} \in C^\infty_c(\mathcal{U})$ such that $\tilde{f} = 1$ on supp $f$, and set

$$ g(\lambda) = \lambda \tilde{f}(\lambda). $$

Then we can write

$$ f(H)(\mathbf{D}\zeta(B))f(H) = f(H)(i[g(H), \zeta(B)])f(H). $$

(3.20)

By Lemmas 2.12, 2.2 and 2.14 we have

$$ i[g(H), \zeta(B)] = \text{Re}[C'(B)(i[g(H), B])] $$

$$ + \text{Re}\int_{\mathbb{C}}(B - z)^{-2}[i[g(H), B], B](B - z)^{-1}d\mu_\zeta(z) $$

(3.21)

$$ \geq Z_2(B)(i[g(H), B])Z_2(B) - C'_1(\epsilon)r^{-1-2\kappa}, $$
where we have set $Z_2 = \sqrt{\zeta} \in C^\infty_c(\mathbb{R})$. Using (3.5) the contribution from the first term of (3.21) is bounded as
\[
 f(H)Z_2(B)(i[g(H), B])Z_2(B)f(H) \\
 \geq Z_2(B)f(H)(i[H, B])f(H)Z_2(B) = C'_2(\epsilon)r^{-2} \\
 \geq Z_2(B)f(H)r^{-1/2}(\sigma^2 - B^2)r^{-1/2}f(H)Z_2(B) - C'_4(\epsilon)r^{-2} \\
 \geq \left(\frac{1}{2}\sigma^2 - \epsilon^2\right)r^{-1/2}f(H)\zeta(B)f(H)r^{-1/2} \\
 + \frac{1}{2}\sigma^2r^{-1/2}\theta_0^{-\kappa}f(H)\zeta(B)f(H)\theta_0^{-\kappa}r^{-1/2} - C'_4(\epsilon)r^{-2}.
\]

We obtain (3.8) by combining (3.20)–(3.22). □

Proof of Theorem 1.4. Let $\phi \in B_0^* \cap H_{1,\text{loc}}^1(\Omega)$, $E \in \mathbb{R}$, $\rho \geq 0$, and $\alpha_0 \in [0, \infty]$ be as in the assumptions of Theorem 1.4. We assume
\[
 E + \frac{1}{2}\alpha_0^2 \notin T(H) \cup \{\infty\}
\]
and deduce a contradiction. For the above $E$ and $\alpha_0$ we choose $c$, $C$, $n_0$, $R$, $\alpha_1$, $\beta$, $\epsilon$, $f$ and $Q$ in agreement with Lemma 3.1. Note that we may take $n_0$ larger if necessary so that $2^{n_0-\beta} > \rho$. Note that for all $n > m \geq 2n_0$
\[
 \chi_{m-2,n+2}\phi \in D(H).
\]

We can also choose $\alpha \in [\alpha_1, \alpha_0]$ such that $\alpha + \beta > \alpha_0$. With these variables we evaluate the inequality (3.4) in the state $\chi_{m-2,n+2}\phi \in D(H)$ and then obtain for all $n > m \geq 2n_0$ and $\nu \geq 2n_0$
\[
 \|r^{-1/2}\theta_0^{-\kappa}\Theta\phi\|^2 \leq C_1(m)\|\chi_{m-1,m+1}\phi\|^2 + C_2(\nu)2^{-n}\|\chi_{n-1,n+1}e^{\epsilon r}\phi\|^2.
\]

The second term on the right-hand side of (3.23) vanishes when $n \to \infty$ since $e^{\epsilon r}\phi \in B_0^*$, and consequently by Lebesgue’s monotone convergence theorem
\[
 \|\tilde{\chi}_m r^{-1/2}\theta_0^{-\kappa}e^{\epsilon r}\phi\|^2 \leq C_1(m)\|\chi_{m-1,m+1}\phi\|^2.
\]

Next we let $\nu \to \infty$ in (3.24) invoking again Lebesgue’s monotone convergence theorem, and then it follows that
\[
 \tilde{\chi}_m r^{-1/2}e^{(\alpha + \beta)r}\phi \in H.
\]

Consequently $e^{(\alpha + \beta)r}\phi \in B_0^*$, but this is a contradiction that $\alpha + \beta > \alpha_0$. □

3.2. Super-exponentially decaying eigenfunctions. Throughout this subsection we impose Condition 1.1. We shall use a function introduced in [Ya2]. Although we do not present its construction, we list the properties required in the arguments of the paper.

Lemma 3.3. There exists a real-valued function $Y \in C^\infty(X \setminus \{0\})$ such that

1. $Y$ is homogeneous of degree one;
2. $Y(x) \geq 1$ for $|x| = 1$;
3. $Y$ is convex;
4. There exists $c \in (0, 1)$ such that for all $a \in A$
\[
 Y(x) = Y(x_a) \text{ for } |x_a| \geq (1 - c)|x|.
\]

Proof. We omit the proof. See [Ya2] or [HuS, Theorem 7.4]. □
Define
\[ q(x) = q_R(x) = \chi(|x|/R) + (1 - \chi(|x|/R))Y(x); \quad R \geq 1. \]

We set
\[ \omega_q = \text{grad } q, \quad h_q = \text{Hess } q, \]
and
\[ B_q = \text{Re}(\omega_q \cdot p) = \frac{1}{2}(\omega_q \cdot p + p \cdot \omega_q). \]

Lemma 3.4. Let \( R \geq 1 \) be sufficiently large. Then the operator \( B_q \) defined by (3.26) is essentially self-adjoint on \( C^\infty_c(\Omega) \), and the self-adjoint extension, denoted by \( B_q \) again, satisfies that for some \( C > 0 \)
\[ D(B_q) \supset H^1, \quad \| B_q \psi \|_{H^1} \leq C \| \psi \|_{H^1} \quad \text{for any } \psi \in H^1. \]

In addition the operators \( e^{itB_q}, \ t \in \mathbb{R}, \) naturally restrict/extend as bounded operators \( e^{itB_q} : H^{\pm 1} \to H^{\pm 1}, \) and they satisfy
\[ \sup_{t \in [-1,1]} \| e^{itB_q} \|_{\mathcal{L}(H^{\pm 1})} < \infty, \]
respectively.

Proof. The assertion is obvious by Lemma 2.7. \qed

Lemma 3.5. Let \( R \geq 1 \) be sufficiently large. Then the quadratic form \( \mathbf{D}B_q := i[H, B_q] \) defined on \( H^2 \) is given by
\[ \mathbf{D}B_q = p \cdot h_q p - \frac{1}{4}(\Delta^2 q) \]
\[ + \sum_{a \in A} \left( -\omega^a_q \cdot (\nabla^a V^l_q) + (V^r_q \omega^a_q) \cdot \nabla - \nabla^a \cdot (V^r_q \omega^a_q) + V^r_q \text{div } \omega_q^a \right). \]

Moreover there exist \( C, C' > 0 \) such that, as quadratic forms on \( H^2 \),
\[ \mathbf{D}B_q \geq -r^{-1/2-\delta}(CH + C')r^{-1/2-\delta}. \]

Proof. We can argue as in the proof of Lemma 2.9. The bound (3.27) follows by using the computed expression for \( \mathbf{D}B_q \) and the fact that \( h_q(x) \geq 0 \) for \( |x| > 2R \). \qed

Now we proceed somewhat as in the previous subsection. We introduce the regularized weights slightly different from (3.1):
\[ \Theta = \Theta^{m,R}_{m,n,R} = \eta_{m,n} e^\theta; \quad n > m \geq 0, \quad R \geq 1. \]

Here we set, as in (2.3) and (2.4),
\[ \eta_m = \chi(q/2^m); \quad \bar{\eta}_m = 1 - \eta_m, \quad \eta_{m,n} = \bar{\eta}_m \eta_n, \]
and
\[ \theta = \theta^{m,R}_{R'} = \alpha(q - q^{1-2R}); \quad \alpha \geq 0, \quad \delta' \in (0, \delta). \]

We are going to investigate the Heisenberg derivative of the ‘propagation observable’ \( P \) defined here as
\[ P = P^{m,R}_{m,n} = \Theta B_q \Theta. \]

In the following we denote the derivatives of \( \Theta \) and \( \theta \) in \( q \) by primes.
Lemma 3.6. Let $E \in \mathbb{R}$ and $\delta^\prime \in (0, \delta)$. Then there exist $c, C, C' > 0$, $\beta_0 \geq 1$, $n_0 \geq 1$ and $R \geq 1$ such that uniformly in $\alpha \geq \beta_0$ and $n > m \geq n_0$, as quadratic forms on $H^2$,  

$$2 \text{Im}(P(H - E)) \geq c \alpha^2 q^{-1/2}\Theta^2 - C \alpha (\eta_{m-1,m+1}^2 + \eta_{n-1,n+1}^2) r^{-2} e^{2\beta} - \text{Re}(Q(H - E)), \tag{3.30}$$

where $Q = C'' q^{-1/2}\Theta^2 + C'' (\eta_{m,n}^2) e^{2\beta}$.

Proof. We proceed in parallel with the proof of Lemma 3.1. We first have

$$2 \text{Im}(P(H - E)) = \mathbf{D} P = 2 \text{Re}((\mathbf{D} \Theta) B_q \Theta) + \Theta(\mathbf{D} B_q) \Theta, \tag{3.31}$$

and further calculate each term on the right-hand side of (3.31). We note that

$$\omega_q \cdot \nabla \omega_q^2 = 2 \omega_q \cdot h_q \omega_q \geq 0 \text{ for } |x| > 2R.$$

Then for the first term of (3.31) by letting $\beta_0 \geq 1$ and $n_0 \geq 1$ be sufficiently large

$$2 \text{Re}((\mathbf{D} \Theta) B_q \Theta)$$

$$= 2 \text{Re}(\theta' B_q^2 \Theta) + 2 \text{Re}(\eta_{m,n}' \eta \Theta) + \text{Im}(\omega_q^2 (\Theta^2 + \Theta'') B_q \Theta) + \text{Im}(\omega_q^2 \eta_{m,n}' \eta' \Theta) + 2 \text{Re}(\omega_q^2 \eta_{m,n}' \eta' \Theta)$$

$$\geq c_1 \alpha \Theta B_q^2 \Theta - C_1 \eta_{m,n}' \eta \Theta$$

$$+ C_2 \alpha \Theta B_q^2 \Theta - C_3 q^{-1/2} \Theta H \Theta q^{-1/2} \delta. \tag{3.32}$$

On the other hand, as for the second term of (3.31), we use Lemma 3.5, and obtain

$$\Theta(\mathbf{D} B_q) \Theta \geq -C_2 q^{-1/2} \Theta^2 - C_3 q^{-1/2} \Theta H \Theta q^{-1/2}.$$ \tag{3.33}

By (3.31), (3.32) and (3.33) it follows that for sufficiently large $\beta_0 \geq 1$ and $n_0 \geq 1$

$$2 \text{Im}(P(H - E)) \geq c_2 \alpha^2 q^{-1/2}\Theta^2 - C_1 \alpha (\eta_{m-1,m+1}^2 + \eta_{n-1,n+1}^2) q^{-2} e^{2\beta}$$

$$- C_3 q^{-1/2} \Theta(H - E)\Theta q^{-1/2} - C_4 \eta_{m,n}' \eta \Theta e^{\Theta}$$

$$\geq c_3 \alpha^2 q^{-1/2}\Theta^2 - C_3 q^{-1/2} \Theta^2 - C_4 \eta_{m,n}' \eta \Theta q^{-2} e^{2\beta}$$

$$- C_3 \frac{1}{2} \Theta(H - E) - C_1 \frac{1}{2} \Theta(\eta_{m,n}' \eta e^{2\beta} (H - E)).$$

This implies the assertion. \hfill \Box

Proof of Theorem 1.5. Let $\phi \in \mathcal{B}_0 \cap H_{0, \text{loc}}^1(\Omega)$, $E \in \mathbb{R}$ and $\rho > 0$ be as in the statement of Theorem 1.5. Fix any $\delta' \in (0, \delta)$, and choose $\beta_0 \geq 1$, $n_0 \in \mathbb{N}$ and $R \geq 1$ in agreement with Lemma 3.6. We may assume that $2^{n_0/3} \geq 1$, so that for all $n > m \geq n_0$

$$\eta_{m-2,n+2} \phi \in \mathcal{D}(H).$$

Let us evaluate the inequality (3.30) in the state $\eta_{m-2,n+2} \phi \in \mathcal{D}(H)$. Then it follows that for any $\alpha \geq \beta_0$ and $n > m \geq n_0$

$$\|q^{-1/2-\delta'} \Theta \phi\|^2 \leq C_1 \alpha^{-1} \left(2^{-2m} \|\eta_{m-1,m+1} e^{a\phi}\|^2 + 2^{-2n} \|\eta_{n-1,n+1} e^{a\phi}\|^2 \right). \tag{3.34}$$

The second term in the parentheses on the right of (3.34) vanishes under the limit $n \to \infty$, and hence by Lebesgue’s monotone convergence theorem we obtain

$$\|\eta_{m} q^{-1/2-\delta'} e^{\Theta} \phi\|^2 \leq C_1 2^{-2m} \alpha^{-1} \|\eta_{m-1,m+1} e^{a\phi}\|^2 ;$$
or
\[ \| \tilde{\eta}_m q^{-1/2 - \delta} [\exp(\theta - 2^m + 2\alpha)] \phi \|^2 \leq C_2(m) \| \eta_{m-1,m+1}\phi \|^2. \] (3.35)

Now assume \( \tilde{\eta}_{m+2}\phi \neq 0 \). Then the left-hand side of (3.35) grows exponentially as \( \alpha \to \infty \) whereas the right-hand side remains bounded. This is a contradiction. Thus \( \tilde{\eta}_{m+2}\phi \equiv 0 \), and hence we are done. \( \square \)

4. PROOF OF LAP BOUNDS

In this section we prove Theorem 1.7. The proof depends on a propagation estimate of commutator type similarly to Section 3, but with different weight functions.

We shall here use the weight functions \( \Theta \) defined as
\[ \Theta = \Theta_{\nu,R} = 1 - (1 + r/2^\nu)^{-1}; \quad r = r_R, \nu \in \mathbb{N}_0, \]
and consider the ‘propagation observable’
\[ P = P_{R,\nu,R} = \Theta^{1/2} f(H)\zeta(B) f(H) \Theta^{1/2}; \quad f \in C_c^\infty(\mathbb{R}), \quad \epsilon \in (0,1), \] (4.1)
where \( \zeta = \zeta_\epsilon \in C^\infty(\mathbb{R}) \) is the smooth sign function from Section 2.1.

As in Section 3 we denote the derivatives of \( \Theta \) in \( r \) by primes and compute
\[ \Theta' = 2^{-\nu}(1 + r/2^\nu)^{-2}, \quad \Theta'' = -2^{-2\nu}(1 + r/2^\nu)^{-3}, \]
and in general
\[ \Theta^{(k)} = (-1)^{k-1}k!2^{-k\nu}(1 + r/2^\nu)^{-1-k} \quad \text{for } k = 1,2,\ldots. \]

From the above expression it follows that for any \( k,l \in \mathbb{N}_0 \) with \( k \geq l \)
\[ 0 < (-1)^{k-l}(k!)^{-1}l!\Theta^{(k)} \leq (-1)^{l-1}(l!)^{-1}l!\Theta^{(l)} \leq \min\{1,r/2^\nu\}. \] (4.2)

Let \( \kappa = \delta/(1 + 2\delta) \) as in (1.17).

Lemma 4.1. Suppose Condition 1.1 or Condition 1.2, and let \( E \in \mathbb{R} \setminus \mathcal{T}(H) \). There exist \( c,C > 0, R \geq 1, \epsilon \in (0,1) \), real-valued \( f \in C_c^\infty(\mathbb{R}) \) and a neighbourhood \( I \subset \mathbb{R} \) of \( E \) such that for all \( \nu \in \mathbb{N}_0 \) and \( z \in I_\pm \)
\[ 2 \Im\{P(H - z)\} \geq c\Theta' - C\nu^{-1}2\epsilon\Theta - \Re\{Q(H - z)\}; \] (4.3)
here \( Q = Q_\nu \in \mathcal{L}(\mathcal{B}) \cap \mathcal{L}(\mathcal{B}^*) \) is bounded uniformly in \( \nu \in \mathbb{N}_0 \), and the estimate (4.3) is understood as a quadratic form on \( H^2 \).

Proof. The proof is very similar to that of Lemma 3.1, and we skip some of the details here. To bound commutators of functions of \( H \), \( r \) and \( B \) we are going to repeatedly use (small variations of) Lemmas 2.4, 2.8 and 2.12 without references again.

Fix the variables in the following order: For any \( \lambda = E \in \mathbb{R} \setminus \mathcal{T}(H) \) and \( \sigma \in (0,\gamma(\lambda)) \) choose \( R \geq 1 \) and a neighborhood \( \mathcal{U} \subset \mathbb{R} \) of \( \lambda \) in accordance with Corollary 2.11. Let \( f \in C_c^\infty(\mathcal{U}) \) be a real-valued function such that \( 0 \leq f \leq 1 \) in \( \mathcal{U} \) and \( f = 1 \) in an open neighborhood \( I \subset \mathbb{R} \) of \( \lambda \). Finally we fix any \( \epsilon \in (0,\sigma) \). With these variables we consider the observable \( P \) defined as (4.1). Note that Corollary 2.11 asserts that
\[ f(H)(DB)f(H) \geq f(H)r^{-1/2}(\sigma^2 - B^2)r^{-1/2}f(H) - C_1r^{-2}, \] (4.4)
and this bound will be implemented in the same manner as in the proof of Lemma 3.1. In the present proof all the estimates are uniform in \( \nu \in \mathbb{N}_0 \) and \( z \in I_\pm \).
Let $I \subset \tilde{I}$ be a compact neighbourhood of $\lambda = E$. We calculate for $z \in I_\pm$

$$2 \text{Im}(P(H - z)) = 2 \text{Re}((D\Theta^{1/2})f(H)\zeta(B)f(H)\Theta^{1/2}) + \Theta^{1/2}f(H)(D\zeta(B))f(H)\Theta^{1/2} - 2(\text{Im } z)P,$$

and bound each term on the right-hand side of (4.5). As for the first term, we substitute

$$D\Theta^{1/2} = \frac{i}{2}\text{Re}(\Theta'\Theta^{-1/2} \omega \cdot p) = \frac{i}{2}\Theta'\Theta^{-1/2} B - \frac{i}{2}\omega^2(\Theta'\Theta^{-1/2})',$$

and then by commuting operators and noting (4.2) we obtain

$$2 \text{Re}[(D\Theta^{1/2})f(H)\zeta(B)f(H)\Theta^{1/2}] \geq f(H)\Theta^{1/2}B\zeta(B)\Theta^{1/2}f(H) - C_2r^{-2}\Theta. \quad (4.6)$$

This part corresponds to Step III of the proof of Lemma 3.1, however the arguments are simpler since we do not need to consider the square root of $B\zeta(B)$ as before. Hence we omit the details verifying (4.6). As for the second term of (4.5), we proceed as in Step IV of the proof of Lemma 3.1. We omit the details again, but with slightly simpler arguments we can actually show that

$$\Theta^{1/2}f(H)(D\zeta(B))f(H)\Theta^{1/2} \geq c_1\Theta^{1/2}f(H)\zeta'(B)f(H)\Theta^{1/2} - C_3r^{-1-2\kappa}\Theta. \quad (4.7)$$

Here we use that $\epsilon \in (0, \sqrt{\sigma})$, (4.4) and (4.2). By using $\|P\|_{\mathcal{H}} \leq 1$ we can bound the last term of (4.5) as

$$-2(\text{Im } z)P \geq -2|\text{Im } z| = \pm 2 \text{Im}(H - z). \quad (4.8)$$

Now by (4.5), (4.6), (4.7), (4.8) and Lemma 2.1 we obtain the lower bound

$$2 \text{Im}(P(H - z)) \geq c_2\Theta^{1/2}f(H)^2\Theta^{1/2} - C_4r^{-1-2\kappa}\Theta \pm 2 \text{Im}(H - z). \quad (4.9)$$

Finally it suffices to remove $f(H)^2$ from the first term of (4.9) with a controllable error. This corresponds to the middle part of Step II of the proof of Lemma 3.1, and again the arguments are simpler. Introducing $f_1 \in \mathcal{F}^{-1}$ by (3.10) we obtain after commutation

$$\Theta^{1/2}(1 - f(H)^2)\Theta^{1/2} \leq C_5 \text{Re}[\Theta^{1/2}f_1(H)\Theta^{1/2}(H - z)] + C_6r^{-2}\Theta. \quad (4.10)$$

The lemma follows from (4.9) and (4.10). \hfill \Box

Proof of Theorem 1.7. Letting $E \in \mathbb{R} \setminus (\sigma_{pp}(H) \cup T(H))$ we prove the assertion for a compact neighbourhood $I \subset \mathbb{R}$ of $E$. This suffices due to compactness. We choose such $I$ and other variables in agreement with Lemma 4.1. We may assume $I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup T(H))$. Then Lemma 4.1 and the Cauchy–Schwarz inequality imply that uniformly in $\nu \in \mathbb{N}_0$ and $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in \mathcal{B}$

$$\|\Theta^{1/2}\phi\|_{\mathcal{H}}^2 \leq C_1(\|\phi\|_{\mathcal{B}} \|\psi\|_{\mathcal{B}} + \|r^{-1/2-\kappa}\Theta^{1/2}\phi\|_{\mathcal{H}}^2). \quad (4.11)$$

Thanks to (4.2) we can bound $\Theta \leq \Theta^\kappa \leq r^{\kappa/2}\kappa\rho$, and by implementing these estimates in (4.11) it follows that

$$\|\Theta^{1/2}\phi\|_{\mathcal{H}}^2 \leq C_2(\|\phi\|_{\mathcal{B}} \|\psi\|_{\mathcal{B}} + 2^{-\kappa\rho}\|\phi\|_{\mathcal{B}}^2). \quad (4.12)$$

On the other hand by taking supremum over $\nu \in \mathbb{N}_0$ it also follows that

$$\|\phi\|_{\mathcal{B}}^2 \leq C_3(\|\psi\|_{\mathcal{B}}^2 + \|r^{-1/2-\kappa}\phi\|_{\mathcal{H}}^2). \quad (4.13)$$
Now suppose by contradiction that there exist \( \phi_n = R(z_n)\psi_n \) with \( z_n \in I_\pm \) and \( \psi_n \in B \) such that
\[
z_n \to E' \in I, \quad \|\psi_n\|_B \to 0, \quad \|\phi_n\|_{B^*} = 1.
\] (4.14)
By using a subsequence we can assume that there exists
\[
\phi := \text{w}^\ast\text{-lim}_{n \to \infty} \phi_n \quad \text{in} \quad L^2_{-(1+\kappa)/2}.
\] (4.15)
By local compactness and an energy estimate we easily see that for any \( \chi \in C_0^\infty(X) \)
\[
\lim_{n \to \infty} \chi \phi_n = \chi \phi \quad \text{in} \quad H^1_0(\Omega),
\]
which implies that \( \phi \in H^1_{0,\text{loc}}(\Omega) \). In addition, we have by (4.12) that
\[
\|\Theta^{1/2}\phi\|_H^2 = \lim_{n \to \infty} \|\Theta^{1/2}\phi_n\|_H^2 \leq C_2 2^{-n},
\]
which implies \( \phi \in B_0^* \). By taking limit in \( (H - z_n)\phi_n = \psi_n \) we have
\[
(H - E')\phi = 0 \quad \text{in the distributional sense.}
\]
Since \( E' \notin \sigma_{\text{pp}}(H) \cup T(H) \), we obtain from Theorem 1.4 that \( \phi = 0 \). By local compactness and (4.15) it then follows that \( \phi_n \to 0 \) in \( L^2_{1/2-k} \). In turn this implies by (4.13) that \( \phi_n \to 0 \) in \( B^* \), contradicting the assumption (4.14). Hence we obtain that for all \( z \in I_\pm \) and \( \psi \in B \)
\[
\|R(z)\psi\|_{B^*} \leq C_4 \|\psi\|_B.
\] (4.16)
To bound the second term on the left-hand side of (1.10) it suffices to note that
\[
p^*\Theta'p \leq C_5 \text{Re}(\Theta'(H - z)) + C_6 \Theta'.
\] (4.17)
By (4.16), (4.17) and the Cauchy–Schwarz inequality we obtain the assertion. \( \square \)

5. Proof of microlocal resolvent bounds and applications

We prove Theorem 1.8 in Section 5.1, and Corollaries 1.9 and 1.10 in Section 5.2.

5.1. Microlocal resolvent bounds. Here we prove Theorem 1.8 partly by using the scheme of [GIS]. The proof consists of two parts, according to spectral components of the radial momentum \( B \).

We first consider the components of high radial momentum. Using the functions \( \bar{\chi}_m \in C^\infty(\mathbb{R}) \) from (2.4), \( m \in \mathbb{N} \) large, we first introduce \( F_m \in \mathcal{F}^0 \) by
\[
F_m(b) = \bar{\chi}_m(|b|); \quad b \in \mathbb{R}.
\] (5.1)

**Lemma 5.1.** Suppose Condition 1.1 or Condition 1.2 and let \( R \geq 1 \) be sufficiently large. Let \( I \subset \mathbb{R} \) be a compact interval and \( s \in \mathbb{R} \). Then for all \( m \) large enough there exists \( C > 0 \) such that for all \( z \in I_\pm \), as a quadratic form on \( (H - i)^{-1}L^2_s \),
\[
F_m(B)r^{2s}F_m(B) \leq Cr^{2s-1} + \text{Re}(F_m(B)Q(H - z)),
\]
where \( Q = Q_z \in \mathcal{L}(L^2_s, L^2_{-s}) \) is bounded uniformly in \( z \in I_\pm \).

**Proof.** Fix real \( f \in C_0^\infty(\mathbb{R}) \) such that \( f = 1 \) in a neighbourhood of \( I \) and decompose with \( T = F_m(B)r^{2s}F_m(B) \)
\[
T = f(H)Tf(H) + \text{Re}((1 + f(H))T(1 - f(H))).
\] (5.2)
We bound each term on the right-hand side of (5.2). The first term is estimated as
\[
f(H)Tf(H) \leq C_1 2^{-2m}f(H)r^sF_m(B)B^2F_m(B)r^s f(H) + C_2(m)r^{2s-1}
\]
\begin{align*}
&\leq C_1 2^{-2m} r^s F_m(B) f(H) B^2 f(H) F_m(B) r^s + C_3(m) r^{2s-1} \\
&\leq C_4 2^{-2m} \text{Re}(r^s F_m(B) f(H)(H - z) f(H) F_m(B) r^s) \\
&\quad + C_5 2^{-2m} r^s F_m(B) f(H) V F_m(B) r^s + C_3(m) r^{2s-1} \\
&\leq C_4 2^{-2m} \text{Re}(F_m(B)r^s f(H) F_m(B) r^s f(H)(H - z)) \\
&\quad + C_5 2^{-2m} f(H) T f(H) + C_6(m) r^{2s-1}.
\end{align*}

Therefore if we \( m \) is large enough we obtain
\[
f(H) T f(H) \leq C_7 r^{2s-1} + \text{Re}(F_m(B) Q_1(H - z)) \tag{5.3a}
\]
with
\[
Q_1 = C_8 2^{-2m} r^s f(H) F_m(B) r^s f(H).
\]
As for the second term of (5.2) we can estimate
\[
\text{Re}((1 + f(H)) T (1 - f(H))) \leq C_9 r^{2s-1} + \text{Re}(F_m(B) Q_2(H - z)), \tag{5.3b}
\]
where
\[
Q_2 = (1 + f(H)) r^{2s} F_m(B) (1 - f(H)) R(z).
\]

The lemma follows from (5.2), (5.3a), and (5.3b).

\( \square \)

**Corollary 5.2.** Suppose Condition 1.1 or Condition 1.2. Let \( R \geq 1 \) be sufficiently large. Let \( I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup T(H)) \) be a compact interval and \( s \in [-1/2, 0) \). Then for all \( m \) large enough there exists \( C > 0 \) such for all \( z \in I_\pm \) and \( \psi \in L^2_s \)
\[
\|F_m(B) R(z) \psi\|_{L^2_s} \leq C \|\psi\|_B.
\]

**Proof.** The assertion is a consequence of Lemma 5.1, Theorem 1.7 and the Cauchy–Schwarz inequality.

\( \square \)

We next study middle components of the radial momentum for the outgoing and the incoming resolvents using a modification of an ‘induction start’ given in [GIS]. We use the function \( \chi \) of (2.3) and define \( \chi_\epsilon(t) = \chi(t/\epsilon) \) for \( \epsilon \in (0, 1) \).

**Lemma 5.3.** Suppose Condition 1.1 or Condition 1.2. Let \( E \in \mathbb{R} \setminus T(H) \) and \( 0 < \sigma' < \sigma < \gamma(E) \). Take \( R \geq 1 \) large and \( \epsilon > 0 \) small. Consider for real \( f \in C^\infty_c(\mathbb{R}) \) and \( \kappa' \in (0, \kappa) \)
\[
P_\pm = \mp f(H) r^{\kappa'} F_\pm(\pm B) r^{\kappa'} f(H); \quad F_\pm(b) = (\sigma' + \kappa 2) 2^{\kappa} \chi_\epsilon^2(b - \sigma'). \tag{5.4}
\]
There exist \( C, > 0 \), real \( f \in C^\infty_c(\mathbb{R}) \) and a neighbourhood \( I \subset \mathbb{R} \) of \( E \) (possibly depending on \( \kappa' \in (0, \kappa) \)) such that for all \( z \in I_\pm \), as quadratic forms on \( (H - i)^{-1} L^2_{1/2 + \kappa'} \),
\[
2 \text{Im}(P_\pm(H - z)) \geq C \chi_\epsilon(\pm B - \sigma') r^{-1+2\kappa'} \chi_\epsilon(\pm B - \sigma') \\
- C r^{-1-2(\kappa - \kappa')} - \text{Re}(\chi_\epsilon(\pm B - \sigma') Q(H - z)),
\]
where \( Q = Q_z \in \mathcal{L}(L^2_{1/2 + \kappa'}, L^2_{3/2 - \kappa'}) \) is bounded uniformly in \( z \in I_\pm \).

**Proof.** The proof is similar to those of Lemmas 3.1 and 4.1, and we skip some of the details. In particular we use Lemmas 2.4, 2.8 and 2.12 without references again. Fix \( E = \lambda \in \mathbb{R} \setminus T(H) \) and \( \sigma < \gamma(E) \) and choose then \( R \geq 1 \) and a neighbourhood \( U \subset \mathbb{R} \) of \( \lambda \) in accordance with Corollary 2.11. Let \( f \in C^\infty_c(U) \) be a real-valued function such that \( f = 1 \) in an open neighborhood \( I \subset \mathbb{R} \) of \( \lambda \). Let \( I \subset \tilde{I} \) be a compact neighbourhood of \( \lambda \). Let \( \kappa' \in (0, \kappa) \) and \( \sigma' \in (0, \sigma) \). With these quantities
We use the operator $\tilde{H} = g(H) = H \tilde{f}(H)$ from Step IV of the proof of Lemma 3.1 and let $\tilde{D}$ denote the corresponding Heisenberg derivative. We compute with $\theta_\epsilon = \sqrt{-F_\epsilon'}$ (and using the notation (2.1))

$$f(H) r^{\kappa'} (\tilde{D} F_\epsilon(B)) r^{\kappa'} f(H)$$

$$= -r^{\kappa'} \theta_\epsilon(B) f(H)(\tilde{D} B) f(H) \theta_\epsilon(B) r^{\kappa'} + O(r^{2\kappa'-2\kappa-1}).$$

Note that

$$-\frac{1}{2}(\sigma' + 3\epsilon - b)^{1-2\kappa'} F'_\epsilon(b) = \kappa' \chi_\epsilon^2(b - \sigma') - (\sigma' + 3\epsilon - b)(\chi_\epsilon \chi'_\epsilon)(b - \sigma') \geq 0.$$ 

We also compute

$$\tilde{D} r^{\kappa'} = \kappa' g'(H) r^{\kappa'-1} B + O(r^{\kappa'-2}).$$

Introducing $T = (\sigma' + 3\epsilon - B)^{\kappa'-1/2} \chi_\epsilon(B - \sigma') r^{\kappa'-1/2} f(H)$ this leads to the lower bound

$$2 \Im \left( P_+(H - z) \right) \geq \mathbf{D} P_+$$

$$\geq 2\kappa' T^* (\sigma^2 - B^2 - B(\sigma' + 3\epsilon - B)) T + O(r^{2\kappa'-2\kappa-1}) \quad \text{(5.5)}$$

$$\geq 2\kappa' T^* (\sigma^2 - (\sigma' + 2\epsilon)(\sigma' + 3\epsilon)) T + O(r^{2\kappa'-2\kappa-1})$$

$$= c_1 T^* + O(r^{2\kappa'-2\kappa-1}); \quad c_1 = 2\kappa' (\sigma^2 - (\sigma' + 2\epsilon)(\sigma' + 3\epsilon)) > 0.$$ 

Introducing $S = \chi_\epsilon(B - \sigma') r^{\kappa'-1/2}$ we claim that for some $c_2, C_2 > 0$

$$T^* T \geq c_2 S^* S - C_2 r^{2\kappa'-2} - \Re \left( \chi_\epsilon(B - \sigma') Q(H - z) \right), \quad \text{(5.6)}$$

where $Q = Q_z \in \mathcal{L}(L^2_{1/2+\kappa'}, L^2_{3/2-\kappa'})$ is bounded uniformly in $z \in I_+$. The combination of (5.5) and (5.6) completes the proof of the lemma.

To show (5.6) we first remove the factor $(\sigma' + 3\epsilon - B)^{\kappa'-1/2}$ of $T$. We write $S f(H) = S f(H) \tilde{f}(H)$ and note that

$$[S f(H), \tilde{f}(H)] = O(r^{\kappa'-3/2}).$$

Using the notation $\| \cdot \|_s = \| \cdot \|_{L^2_s}$ this leads to

$$\|S f(H) \phi\| \leq \| \tilde{f}(H) S f(H) \phi\| + C_1 \|\phi\|_{\kappa'-3/2}$$

$$\leq C_1' \|T \phi\| + C_1 \|\phi\|_{\kappa'-3/2}, \quad \phi \in L^2_{\kappa'-1/2},$$

and therefore

$$T^* T \geq c_3 f(H) S^* S f(H) - C_3 r^{2\kappa'-3}.$$ 

Next we remove the factors $f(H)$ of (5.7) writing as in (5.2) and using the notation $P_z = S^* S (1 - f(H)) R(z)$

$$f(H) S^* S f(H)$$

$$= S^* S - \Re \left( (1 + f(H)) S^* S (1 - f(H)) \right)$$

$$= S^* S - \Re \left( (1 + f(H)) P_z (H - z) \right)$$

$$\geq S^* S - \Re \left( \chi_\epsilon(B - \sigma') Q_z (H - z) \right) - C_4 r^{2\kappa'-2},$$

where $Q = Q_z \in \mathcal{L}(L^2_{1/2+\kappa'}, L^2_{3/2-\kappa'})$ is bounded uniformly in $z \in I_+$. 

fixed we now consider for small $\epsilon > 0$ the operator $P_+$ defined by (5.4) (treating only the upper sign).
We obtain (5.6) from (5.7) and (5.8).

**Corollary 5.4.** Suppose Condition 1.1 or Condition 1.2. Let \( E \in \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H)) \), \( \sigma' \in (0, \gamma(E)) \), and take \( R \geq 1 \) sufficiently large and \( \epsilon > 0 \) sufficiently small. There exist for all \( \kappa' \in (0, \kappa) \) a constant \( C > 0 \) and a neighbourhood \( I \subset \mathbb{R} \) of \( E \) such that for all \( z \in I \) and \( \psi \in L^2_{1/2+\kappa'} \),

\[
\|\chi(\pm B - \sigma')R(z)\psi\|_{L^2_{-1/2+\kappa'}} \leq C\|\psi\|_{L^2_{1/2+\kappa'}},
\]

respectively.

**Proof.** The assertion follows from Lemma 5.3, Theorem 1.7 and the Cauchy–Schwarz inequality.

**Proof of Theorem 1.8.** The assertion follows by a covering argument using Corollaries 5.2 and 5.4.

### 5.2. Applications

Here we prove Corollaries 1.9 and 1.10.

**Proof of Corollary 1.9.** We discuss only the upper sign. Let \( I \subset \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H)) \) be a compact interval, and let \( s > 1/2 \) and \( \beta \in (0, \min\{\kappa, s - 1/2\}) \). Decompose for any \( z, z' \in I \) and \( m \in \mathbb{N}_0 \)

\[
R(z) - R(z') = \chi_m R(z)\chi_m - \chi_m R(z')\chi_m + (R(z) - \chi_m R(z)\chi_m) - (R(z') - \chi_m R(z')\chi_m).
\]

(5.9)

We estimate the last two terms of (5.9) as follows. Take any \( s' \in (1/2, s - \beta) \). Then by Theorem 1.7 we have uniformly in \( z \in I \) and \( m \in \mathbb{N}_0 \)

\[
\|R(z) - \chi_m R(z)\chi_m\|_{L^q(I)} \leq C_1\|r^{-s}R(z)(1 - \chi_m)r^{-s}\|_{L^2} + C_1\|r^{-s}(1 - \chi_m)R(z)\chi_m r^{-s}\|_{L^2}
\]

\[
\leq C_22^{-\beta m}.
\]

The same holds true uniformly in \( z' \in I \) and \( m \in \mathbb{N}_0 \)

\[
\|R(z') - \chi_m R(z')\chi_m\|_{L^q(I)} \leq C_22^{-\beta m}.
\]

(5.11)

As for the first and second terms of (5.9) we write

\[
\chi_m R(z)\chi_m - \chi_m R(z')\chi_m = \chi_m R(z)\left(\chi_{m+1} (H - z') - (H - z)\chi_{m+1}\right)R(z')\chi_m
\]

\[
= \chi_m R(z)\left(\left(I - z - z'\right)\chi_{m+1} - [H, \chi_{m+1}]\right)R(z')\chi_m.
\]

(5.12)

Now let us choose \( F_\pm \in \mathcal{F}^0 \) such that

\[
F_- + F_+ = 1, \quad \text{supp} \ F_- \subset (-\infty, -\gamma(I)), \quad \text{supp} \ F_+ \subset (-\gamma(I), \infty).
\]

We write the first term in the parentheses on the right-hand side of (5.12) as

\[
\left(\chi_{m+1} F_-(B) + F_+(B)\chi_{m+1} + [\chi_{m+1}, F_+(B)]\right)
\]

and the second term as

\[
i[H, \chi_{m+1}] = \left(\text{Re}(\chi_{m+1} B)\right)F_-(B) + F_+(B)\text{Re}(\chi_{m+1} B)
\]

\[
+ \left[\text{Re}(\chi_{m+1} B), F_+(B)\right].
\]
Then by (5.12) and Theorem 1.8 it follows that uniformly in $z, z' \in I_+$ and $m \in \mathbb{N}_0$
\[\|\chi_m R(z)\chi_m - \chi_m R(z')\chi_m\|_{L^2(L^2, L^2)} \leq |z-z'|\left|\left|\frac{r^{-s}}{r^{-s}}R(z)\chi_m R(z')r^{-s}\right|\right|_{L(H)} + \|r^{-s}R(z)[H, \chi_m]R(z')r^{-s}\|_{L(H)} \tag{5.13}\]
\[\leq C_3 2^{(1-\beta)m}|z - z'| + C_5 2^{-\beta m}.
\]

Summing up (5.10), (5.11) and (5.13), we obtain uniformly in $z, z' \in I_+$ and $m \in \mathbb{N}_0$
\[\|R(z) - R(z')\|_{L^2(L^2, L^2)} \leq C_3 2^{(1-\beta)m}|z - z'| + C_5 2^{-\beta m}.
\]

For $|z - z'| \leq 1$ we choose $m \in \mathbb{N}_0$ such that $2^m \leq |z - z'|^{-1} < 2^{m+1}$, yielding
\[\|R(z) - R(z')\|_{L^2(L^2, L^2)} \leq C_5 |z - z'|^{\beta}.
\]

This bound is trivial for $|z - z'| \geq 1$. Therefore we obtain (1.18) for $k = 0$. For $k = 1$ we may use the bound with $k = 0$ and the first resolvent equation. The rest of the assertions follow from Theorem 1.7 and (1.18).

Proof of Corollary 1.10. We discuss only the upper sign. Let $\lambda \in \mathbb{R} \setminus (\sigma_{pp}(H) \cup \mathcal{T}(H))$, and take $R \geq 1$ and $\tilde{\gamma} > 0$ sufficiently large as in Theorem 1.8 with $I = \{\lambda\}$. We let $\psi \in r^{-\beta}B$ with $\beta \in [0, \kappa)$, and set $\phi = R(\lambda + i0)\psi$. Then (1) and (2) follows by Theorems 1.7, 1.8 and Corollary 1.9.

Conversely, let $\phi' \in L^2_{-\infty} \cap H^1_{0,loc}(\Omega)$ satisfy (1') and (2'). Set
\[\phi'' = \phi' - \phi; \quad \phi = R(\lambda + i0)\psi.
\]
Since we proved that $\phi$ satisfies (1') and (2') it follows that $\phi''$ satisfies (1') and (2') with $\psi = 0$. Due to Theorem 1.4 it suffices to show that $\phi'' \in B^*_0$.

We first claim that $\phi'' \in L^2_{-1}$. Choose $s < -1$ such that $\phi'' \in L^2_s$ and choose $F_\pm \in \mathcal{F}^0$ such that
\[F_- + F_+ = 1, \quad \text{supp} F_- \subset (-\infty, \gamma'), \quad \text{supp} F_+ \subset (\gamma'/2, \infty).
\]
with $\gamma' > 0$ sufficiently small. For any $t \leq 0$ we estimate uniformly in $m \in \mathbb{N}_0$
\[2\text{Im}(\chi_m r^t(H - \lambda)) = -|\chi_m r^t|^{1/2}B(F_-(B) + F_+(B))|\chi_m r^t|^{1/2}
\]
\[\leq |\chi_m r^t|^{1/2}(c_1 - B)F_-(B)|\chi_m r^t|^{1/2} - c_1|\chi_m r^t|^{1/2} ,
\]
so that
\[|\chi_m r^t|^{1/2} \leq |\chi_m r^t|^{1/2}(1 - C_2B)F_-(B)|\chi_m r^t|^{1/2} - C_3\text{Im}(\chi_m r^t(H - \lambda)) \tag{5.14}\]

We take $t = 2s + 2(<0$ and apply (5.14) to $\phi'' = f(H)\phi''$; here $f \in C^\infty_c(\mathbb{R})$ satisfies $f(\lambda) = 1$. Then we obtain
\[\|\chi_m r^{s+2} \|^{1/2} \phi'' \|_{\mathcal{H}}^2 \leq C_4\|\chi_m r^{s+2} \|^{1/2} \phi'' \|_{\mathcal{H}}\|\chi_m r^{s+2} \|^{1/2} F_-(B)\phi'' \|_{\mathcal{H}} + C_4\|r^s \phi'' \|_{\mathcal{H}},
\]
so that by the Cauchy–Schwarz inequality
\[\|\chi_m r^{s+1/2} \phi'' \|_{\mathcal{H}}^2 \leq C_5\|F_-(B)\phi'' \|_{\mathcal{H}}^2 + C_5\|\phi'' \|_{s}.
\]
This implies that $\phi'' \in L^2_{s+1/2}$, and hence inductively that indeed $\phi'' \in L^2_{-1}$.
Finally we prove $\phi'' \in B^*_0$. We take $t = 0$ and apply again (5.14) to $\phi'' = f(H)\phi''$. Then we obtain

$$
\left\| \left| \chi_m' \right|^{1/2} \phi'' \right\|_H^2 \leq C_6 \left\| \left| \chi_m' \right|^{1/2} \phi'' \right\|_H \left\| \chi_m' \right|^{1/2} F_1(B)\phi'' \right\|_H
+ C_6 \left\| \left| \chi_m' \right|^{1/2} r^{-1/2} \phi'' \right\|_H \left\| r^{-1} \phi'' \right\|_H,
$$

which implies that

$$
\lim_{m \to \infty} \langle |\chi_m'\rangle |\phi'' \rangle = 0,
$$

or equivalently that $\phi'' \in B^*_0$. By Theorem 1.4 it follows that $\phi'' = 0$, and we are done. 

\[\square\]

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