SEIDEL’S THEOREM VIA GAUGE THEORY

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Abstract. A new proof is given that Seidel’s generalized Dehn twist is not symplectically isotopic to the identity.

1. Introduction. It has been shown by Seidel [13, 12] that symplectic 4-manifolds may admit symplectic diffeomorphisms which are smoothly isotopic to the identity but not symplectically so. He proved the following theorem (see [13, Cor. 1.6]).

Theorem 1 (Seidel, [13]). Let $X$ be a complete intersection that is neither $\mathbb{CP}^1 \times \mathbb{CP}^1$ nor $\mathbb{CP}^2$. Then there exists a symplectomorphism $\varphi: X \to X$ that is smoothly, yet not symplectically, isotopic to the identity.

The symplectomorphism $\varphi = [\tau]^2$ in Thm.1 is the (square of) generalized Dehn twist described below. Seidel computed the Floer cohomology group $HF^*(\tau)$ of $\tau$ with its module structure over the quantum cohomology ring $QH^*(X)$ and then showed that it differs from that of $HF^*([\tau]^{-1})$. As a result, he proved that $[\tau]$ is not isotopic to $[\tau]^{-1}$, hence not isotopic to the identity. This note aims to give a different proof of this result, though one that so far works only for K3 surfaces (see Thm.2 below). The new proof does not rely on any Floer-theoretic considerations, but instead follows the approach of Kronheimer (see [3]) and uses invariants derived from the Seiberg-Witten equations.

We define now Seidel’s generalized Dehn twist as a Picard-Lefschetz monodromy map of a pencil of surfaces. To be specific, we restrict ourselves to the case of hypersurfaces of $\mathbb{CP}^3$. Inside the space $\mathbb{CP}^{N_d}$ of all hypersurfaces in $\mathbb{CP}^3$ of fixed degree $d$, there is a codimension-1 subvariety $\Sigma$ which parameterizes singular hypersurfaces. Smooth points of $\Sigma$ correspond to surfaces which have a single double-point singularity. Pick a smooth point $p \in \Sigma$ and a small complex disk $\Delta$ meeting $\Sigma$ at the point $p$, transverse to $\Sigma$. Fix a local parameter $t: \Delta \to \mathbb{C}$ such that $t(p) = 0$. Letting $X_t$ denote the hypersurface corresponding to the point $t \in \Delta$, we set

$$X = \{(t, x) \in \Delta \times \mathbb{CP}^3 | x \in X_t\}. \quad (1.1)$$

We let $\{X_t\}_{t \in \Delta \sim \{0\}}$ be the family of non-singular projective surfaces obtained from $X$ by removing the singular fiber $X_0$. The Fubini-Study form of $\mathbb{CP}^3$ gives rise to a family of Kähler forms $\omega_t \in H^{1,1}(X_t; \mathbb{R})$ for each $t \in \Delta \sim \{0\}$. Moser’s trick says that the family of symplectic manifolds $\{(X_t, \omega_t)\}_{t \in \Delta \sim \{0\}}$ is locally-trivial, so there is a representation

$$\pi_1(\Delta \sim \{0\}, t_0) \to \pi_0(\text{Symp}(X_{t_0})), \quad \pi_1(\Delta \sim \{0\}, t_0) \to \pi_0(\text{Symp}(X_{t_0})).$$

where $t_0 \in \Delta \sim \{0\}$ is some fixed base-point. The mapping class corresponding to the generator of $\pi_1(\Delta \sim \{0\}) \cong \mathbb{Z}$ is called Seidel’s generalized Dehn twist and it is denoted by $[\tau] \in \pi_0(\text{Symp}(X_{t_0}))$. It is a classical fact that $[\tau]$ acts as a reflection in $H_2(X_{t_0}; \mathbb{Z})$. Hence,

$$[\tau]^2 \in \text{Ker}[\pi_0(\text{Symp}(X_{t_0})) \to \text{Aut}(H_2(X_{t_0}; \mathbb{Z}))].$$

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In fact, it is also well known (see \[13, 3\]) that
\[
[\tau]^2 \in \text{Ker} \left[ \pi_0(\text{Symp}(X_{t_0})) \to \pi_0(\text{Diff}(X_{t_0})) \right].
\] (1.2)

What we will prove is:

**Theorem 2.** If \(d = 4\), then \([\tau]^2 \in \pi_0(\text{Symp}(X_{t_0}))\) is a non-trivial element.

**Remark 1.** Throughout the paper we work with \(\mathbb{Z}_2\) coefficients. However, if one is willing to work with Seiberg-Witten invariants over \(\mathbb{Z}\), it is fairly easy to show that \([\tau]^2\) is non only non-trivial but also an element of infinite order.

We now sketch an argument (due to Kronheimer) that establishes (1.2). If \(sq: \Delta \to \Delta\) is the map \(sq(t) = t^2\), then define a complex-analytic fiber space \(\mathcal{X}'\) as the base change
\[
\mathcal{X}' \longrightarrow \mathcal{X} \\
\downarrow \quad \downarrow \\
\Delta \quad \Delta
\] (1.3)

which is equivalent to setting
\[
\mathcal{X}' = \left\{ (t, x) \in \Delta \times \mathbb{CP}^3 \mid x \in X_{\text{sq}(t)} \right\}. \quad (1.4)
\]

The space \(\mathcal{X}'\) is a non-smooth complex 3-fold with a single double-point in the fiber over 0. We proceed by recalling a result of Atiyah which allows us to get rid of the double-point in the central fiber.

**Theorem 3** (Atiyah, \[1\]). There exists a complex-analytic family of non-singular surfaces \(\mathcal{Y} \to \Delta\) and a morphism of families \(h: \mathcal{Y} \to \mathcal{X}'\),
\[
\mathcal{Y} \quad \leftarrow \quad Y_t \\
\downarrow \quad \downarrow \\
\mathcal{X}' \quad \leftarrow \quad X'_t
\] (1.5)

such that for each \(t \in \Delta - \{0\}\), \(h_t: Y_t \to X'_t\) is an isomorphism, whereas \(h_0: Y_0 \to X_0\) is the minimal resolution. The exceptional divisor of \(h_0\), which is a smooth rational curve \(C \subset Y_0\) of self-intersection number \((-2)\), is embedded in \(\mathcal{Y}\) as a \((-1, -1)\)-curve, that is, a curve whose normal bundle is isomorphic to \(\mathcal{O}(-1) \oplus \mathcal{O}(-1)\).

This is the statement of Thm. 2 in \[1\] except for the assertion about \(C\), which is explained in \[1\] §3. The family morphism \(h\) is an isomorphism away from the central fibers, so the monodromy of \(\{Y_t\}_{t \in \Delta - \{0\}}\) is equal to that of \(\{X'_t\}_{t \in \Delta - \{0\}}\). However, as the only singular fiber of \(\mathcal{X}'\) has been replaced by a smooth surface, the fibers of \(\mathcal{Y}\) are all smooth. Hence, the monodromy is smoothly isotopic to the identity and (1.2) follows.

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2. Family Seiberg-Witten invariants. Here, we briefly recall the definition of the Seiberg-Witten invariants in both classical and family settings. The given exposition is extremely brief, meant mainly to fix the notations and also to show just how little of Seiberg-Witten theory we need to prove Thm.2. We refer the reader to [8] for a comprehensive introduction to four-dimensional gauge theory (see also [11] for a quick survey), whereas the theory of family invariants can be read from the seminal paper [5].

Let $X$ be a closed oriented simply-connected 4-manifold endowed with a Riemannian metric $g$, a self-dual form $\eta$, and a spin$^C$-structure $s$. Associated to a spin$^C$-structure are spinor bundles $W^\pm$ and a determinant line bundle $L = \text{det} W^+$ over $X$. Define the monopole map

$$\mu: \Gamma(W^+) \times A \to \Gamma(W^-) \times i\Gamma(\Lambda_+^2) \quad \text{by} \quad \mu(\varphi, A) := (D^A \varphi, F_A^+ - \sigma(\varphi) - i\eta),$$

where $\varphi \in \Gamma(W^+)$ is a self-dual spinor field, $A \in A$ is a $U(1)$-connection on $L$, and $F_A^+$ is the self-dual part of the curvature $F_A$. Finally, $\sigma: W^+ \to i\Lambda_+^2$ stands for the squaring map.

The monopole space (or, the Seiberg-Witten solution space) is, by definition, the zero set of $\mu$, and the moduli space of monopoles, denoted by $\mathcal{M}_{g,\eta}$, is defined to be the quotient of $\mu^{-1}(0)$ by the gauge group

$$\mathcal{G} = \{g: X \to S^1\}.$$

If now $g$ is an element of $\mathcal{G}$, the corresponding gauge transformation is given by

$$g \cdot (\varphi, A) := (e^{-i\beta} \varphi, A + 2i \beta f), \quad \text{where} \quad g = e^{i\beta}.$$

Such $f$ exists because $X$ is simply-connected. Gauge transformations preserve the property of being a monopole.

The monopole map depends on the choices of Riemannian metric, perturbation form, and spin$^C$-structure. Moreover, a spin$^C$ structure itself is defined with respect to a metric on $X$. However, it is explained in [9] § 2.2 that a spin$^C$ structure for one metric can be extended to all of them. Therefore, we can consider a family of monopole maps parametrized by $(g, \eta)$. Let $\Pi$ denote the space of pairs $(g, \eta)$, where $g$ is a metric on $X$ and $\eta$ is a $g$-self-dual form. Note that $\Pi$ is naturally a vector bundle over the space of Riemannian metrics $\mathcal{R}$ on $X$. Using the parameterized monopole map, one defines the universal moduli space as follows

$$\mathfrak{M}^s := \bigcup_{(g, \eta) \in \Pi} \mathcal{M}_{g, \eta}^s.$$

Let $\Pi_{\text{red}} \subset \Pi$ be the subset of pairs $(g, \eta)$ for which $\mu^{-1}(0)$ contains pairs of the form $(0, A)$, the so-called reducible monopoles. Unless $\varphi = 0$, the stabilizer of $(\varphi, A)$ w.r.t. $\mathcal{G}$ is trivial. However, the stabilizer of $(0, A)$ is $U(1)$. Therefore, reducible monopoles obstruct $\mathfrak{M}$ to be a manifold. In case $\varphi = 0$, the equation $\mu(\varphi, A) = 0$ takes the simple form

$$F_A^+ - i\eta = 0. \quad (2.1)$$

A solution to (2.1) exists iff

$$\langle F_A^+ \rangle_g = i\langle \eta \rangle_g, \quad (2.2)$$

where the brackets in both sides denote the self-dual harmonic part of the 2-form in question. Recall here that the space $\Gamma(\Lambda_+^2)$ of self-dual 2-forms splits as $H_g \oplus \text{Im} d^+$, where $H_g$ stands for the space of harmonic self-dual forms on $X$, and $\text{Im} d^+$ stands for the image of $d^+: \Gamma(\Lambda^1) \to \Gamma(\Lambda_+^2)$. For abbreviation, we will drop the subscript and write $\langle \cdot \rangle$ instead of $\langle \cdot \rangle_g$ when no confusion can arise. Since the (self-dual) harmonic part of a closed form depends on its cohomology class but not on the specific representative, we restate (2.2) as

$$\langle \eta + 2\pi c_1(L) \rangle = 0.$$
Now, we set
\[ \Pi^* = \{(g, \eta) \in \Pi | \langle \eta + 2\pi c_1(L) \rangle_\rho \neq 0 \}. \]
Let us describe the homotopy type of this space. Denote by \( \mathcal{H} \) the vector bundle over \( \mathcal{R} \) whose fiber over \( g \in \mathcal{R} \) is the space \( H_g \) of all \( g \)-self-dual harmonic forms, and denote by \( \mathcal{H}^* \) the complement, in \( \mathcal{H} \), of the section given by
\[ -2\pi \langle c_1(L) \rangle. \]
As \( \mathcal{H} \) is a vector bundle of rank \( b^+(X) \) on a contractible space, it follows that \( \mathcal{H}^* \) has the homotopy type of \( S^{b^+(X)-1} \). Now, observe that the bundle map
\[ \Pi \rightarrow \mathcal{H}, \quad (g, \eta) \rightarrow \langle \eta \rangle \]
sits in the diagram
\[ \xymatrix{ \Pi \ar[r]^{(2.3)} & \mathcal{H} \ar[d]^\uparrow \ar[u]^\uparrow \\
\Pi^* \ar[r]_{(2.3)} & \mathcal{H}^* } \]
and that is has contractible fibers. Thus \( \Pi^* \) has the same homotopy type as \( \mathcal{H}^* \).
Let us now consider the piece of \( \mathcal{M} \), denoted by \( \mathcal{M}^* \), that lies over \( \Pi^* \). A classical result (see [4, Lem. 5]) is that the projection
\[ \text{pr}_s: \mathcal{M}^* \rightarrow \Pi^* \]
is a proper Fredholm map of index
\[ d(s) = \frac{1}{4}(c_1^2(s) - 3\sigma(X) - 2\chi(X)), \]
where \( c_1(s) \), the Chern class of the spin\(^C\) structure \( s \), is simply \( c_1(L) \). The Sard-Smale theorem then asserts that for a generic \( (g, \eta) \in \Pi^* \) the moduli space \( \mathcal{M}^s_{(g, \eta)} = \text{pr}_s^{-1}(g, \eta) \) is either empty or a compact manifold of dimension \( d(s) \). If \( d(s) = 0 \), then \( \mathcal{M}^s_{(g, \eta)} \) is zero-dimensional, and thus consists of finitely-many points. We call
\[ \text{SW}_{(g, \eta)}(s) := \# \{ \text{points of } \mathcal{M}^s_{(g, \eta)} \} \mod 2 \]
the \((\mathbb{Z}_2)\)-Seiberg-Witten invariant for the spin\(^C\) structure \( s \) w.r.t. \( (g, \eta) \). If \( b_2^+(X) > 1 \), then \( \Pi^* \) is connected, and then (again, by the Sard-Smale theorem), for every two pairs \( (g_1, \eta_1) \) and \( (g_2, \eta_2) \) and every generic path \( (g_t, \eta_t) \) connecting them, the corresponding moduli space
\[ \bigcup_t \mathcal{M}^s_{(g_t, \eta_t)} \]
is a smooth one-dimensional manifold which draws a cobordism between \( \mathcal{M}^s_{(g_0, \eta_0)} \) and \( \mathcal{M}^s_{(g_1, \eta_1)} \). Hence,
\[ \text{SW}_{(g_0, \eta_0)}(s) = \text{SW}_{(g_1, \eta_1)}(s). \]
Another classical fact (see e.g. [8, Prop. 2.2.22]) is that there is a charge conjugation involution \( s \rightarrow -s \) on the set of spin\(^C\) structures, which changes the sign of \( c_1(s) \), and there is a canonical isomorphism between
\[ \mathcal{M}^s_{(g, \eta)} \quad \text{and} \quad \mathcal{M}^{s}_{(g, -\eta)}. \]
Hence,
\[ \text{SW}_{(g, \eta)}(s) = \text{SW}_{(g, -\eta)}(-s). \]
The corresponding \( \mathbb{Z}_2 \)-valued Seiberg-Witten invariants are also equal to each other, but only up to sign. See [8, Prop. 2.2.26] for the precise statement.
We now recall a version of the family Seiberg-Witten invariants that is adequate for our purpose. This simplest version, along with other kinds of the family invariants, were systematically studied
by Li-Liu in [3]. Let $B$ be a compact smooth manifold and suppose we have a fiber bundle $\mathcal{X} \to B$ whose fibers are diffeomorphic to $X$. Pick a spin$^c$ structure $s$ on the vertical tangent bundle $T_{\mathcal{X}/B}$ of $\mathcal{X}$. For a family of fiberwise metrics $\{g_b\}_{b \in B}$ and a family of $g_b$-self-dual forms $\{\eta_b\}_{b \in B}$, we consider the parameterized moduli space

$$\mathcal{M}_s^{g_b, \eta_b} := \bigcup_{b \in B} \mathcal{M}_s^{g_b, \eta_b}.$$ 

Assume that

$$\langle \eta_b + 2\pi c_1(L) \rangle \neq 0 \quad \text{for each } b \in B. \quad (2.7)$$

Under this assumption and, perhaps after a small perturbation of $\{(g_b, \eta_b)\}_{b \in B}$, the moduli space $\mathcal{M}_s^{g_b, \eta_b}$, if not empty, is a manifold of dimension

$$d(s, B) = \frac{1}{4} (c_1^2(s_X) - 3\sigma(X) - 2\chi(X)) + \dim B,$$

where $s_X$ is the restriction of $s$ to any fiber $X$. Suppose that $d(s, B) = 0$. Then, as in (2.4), we define the family Seiberg-Witten invariant $\text{FSW}_{(g_b, \eta_b)}(s)$ by counting the points of $\mathcal{M}_s^{g_b, \eta_b}$. Just like the ordinary Seiberg-Witten invariants (see (2.5)), the family invariants are unchanged under the homotopies of $(g_b, \eta_b)$ that satisfy (2.7), and, just like the ordinary invariants (see (2.6)), they share the conjugation symmetry

$$\text{FSW}_{(g_b, \eta_b)}(s) = \text{FSW}_{(g_b, -\eta_b)}(-s). \quad (2.8)$$

Already this version of the family invariants is capable to detect non-trivial families of cohomologous symplectic forms (see [3]) as well as to distinguish between different connected components of positive scalar curvature metrics (see [10]).

Let us now turn to the special case of $b_2^+(X) = 3$.

3. Unwinding families. As before, let $\mathcal{X}$ be a smooth fiber bundle over $B$ with fiber $X$. From now on, we assume that $B$ is the 2-sphere $S^2$ and $X$ is the K3 surface. Pick a family $\{g_b\}_{b \in B}$ of fiberwise metrics on the fibers of $\mathcal{X}$. Suppose now that a fiber $X_b$ is given a spin$^c$ structure $s_X$. Then it is easy to show (see e.g. [4, Prop. 2.1]) that, under the topological assumptions that we imposed on $\mathcal{X}$, there exists a spin$^c$ on $T_{\mathcal{X}/B}$ whose restriction to $X$ is $s_X$.

The second cohomology group of a K3 surface is a free $\mathbb{Z}$-module of rank 22 which, when endowed with the bilinear form coming from the cup product, becomes a unimodular lattice of signature $(3, 19)$. Let us fix (once and for all) an abstract lattice $L$ which is isometric to $H^2(X; \mathbb{Z})$ and an isometry $L \to H^2(X_b; \mathbb{Z})$, where $b \in B$ is some fixed base-point. The dependence on the choice of base-point will be inessential: as $B$ is simply-connected, the groups $\{H^2(X_b; \mathbb{R})\}_{b \in B}$ are all canonically isomorphic. We also need to introduce the (open) positive cone

$$K = \{ \kappa \in L | \kappa^2 > 0 \},$$

which is homotopy-equivalent to $S^2$.

As before, we let $\mathcal{H}$ denote the bundle on $B$ whose fiber over $b \in B$ is the space of harmonic $g_b$-self-dual forms. Pick a family $\{\eta_b\}_{b \in B}$ of $g_b$-self-dual forms. Suppose that $(g_b, \eta_b)$ satisfies

$$\langle \eta_b \rangle \neq 0 \quad \text{for each } b \in B,$$

so that the correspondence $b \to \langle \eta_b \rangle$ yields a nowhere vanishing section of $\mathcal{H}$. Then, associated to this section, there is a map:

$$B \to K, \quad b \to [\langle \eta_b \rangle] \in L,$$

where the brackets $[\cdot]$ signify the cohomology class of $\langle \eta_b \rangle$. Since both $B$ and $K$ are homotopy $S^2$, this map has a degree, called the winding number of the family $(g_b, \eta_b)$.
Lemma 1. Suppose that the winding number of \((g_b, \eta_b)\) vanishes. Then

\[
\text{FSW}_{(g_b, \lambda \eta_b)}(\mathfrak{s}) = \text{FSW}_{(g_b, -\lambda \eta_b)}(\mathfrak{s})
\]

for \(\lambda\) sufficiently large.

Proof. By choosing \(\lambda\) large enough, we can make

\[
\lambda^2 \min_{b \in B} \int_{X_b} \langle \eta_b \rangle^2 > 4\pi^2 \max_{b \in B} \int_{X_b} \langle c_1(L) \rangle^2,
\]

so that both \((g_b, \lambda \eta_b)\) and \((g_b, -\lambda \eta_b)\) satisfies (2.7) for \(\lambda\) large enough. Thus, both sides of (3.1) are well defined. Let us show that there exists a homotopy between \(\lambda \eta_b\) and \(-\lambda \eta_b\) that satisfies (2.7). To begin with, we can assume that \(\eta_b\) satisfies (2.7), then so does \((1 - t)\eta_b + t\langle \eta_b \rangle\).

If (3.2) holds, then the range of both maps

\[
b \mapsto \lambda[\eta_b], \quad b \mapsto -\lambda[\eta_b]
\]

lies in the complement of the ball \(O \subset K\),

\[
O = \{ \kappa \in K | \kappa^2 < 4\pi^2 \max_{b \in B} \langle c_1(L) \rangle \}.
\]

Observe that, for every map \(f : B \to K\), there exists a unique section \(\tilde{f} : B \to K\) such that the diagram

\[
\begin{array}{ccc}
\mathcal{H} & \longrightarrow & K \\
\tilde{f} \uparrow & & \downarrow \quad [f] \\
B & \longrightarrow & K
\end{array}
\]

commutes. Also, if \(\text{Range} f \subset K - O\), then \(\tilde{f}(b)\) satisfies (2.7) for each \(b \in B\). To conclude the proof, it suffices to show that the maps (3.3) are homotopic as maps from \(B\) to \(K - O\). Since \(K - O\) is a homotopy \(S^2\), the maps (3.3) are homotopic iff their degrees are equal to each other. This is the case, as the winding number of \((g_b, \pm \lambda \eta_b)\) is equal to that of \((g_b, \pm \eta_b)\), and the latter is zero. \(\Box\)

Combining (3.1) and (2.8), we obtain

\[
\text{FSW}_{(g_b, \lambda \eta_b)}(-\mathfrak{s}) = \text{FSW}_{(g_b, \lambda \eta_b)}(\mathfrak{s}) \quad \text{for} \quad \lambda \text{ sufficiently large.}
\]

Example 1 (Constant families). Suppose that \(X \to B\) is a trivial bundle. Then there is a canonical family \((g_b, \eta_b)\), corresponding to some constant metric \(g_b = g\) and self-dual form \(\eta_b = \eta\). Suppose that \(\langle \eta \rangle \neq 0\). Then such a family has vanishing winding number, and hence it satisfies (3.4). In fact, in this case, both sides of (3.4) must vanish.

Example 2 (Symplectic families). Now, let us not assume that \(X \to B\) is trivial, or that the family \(\{g_b\}_{b \in B}\) is any special. But let us keep the assumption

\[
\langle \eta_b \rangle = \text{const} \in K.
\]

In this case, the quantities of equality (3.4) may not vanish, but the equality itself holds. Suppose now that the fibers of \(X \to B\) are furnished with a family of symplectic forms \(\{\omega_b\}_{b \in B}\) of some constant cohomology class. Pick a family \(\{J_b\}_{b \in B}\) of \(\omega_b\)-compatible almost-complex structures, so that

\[
g_b(\cdot, \cdot) := \omega_b(\cdot, J_b \cdot)
\]

(3.6)
gives rise to a family of fiberwise metrics. Recall here that the space of compatible almost-complex structures is non-empty and contractible (see e.g. [6, Prop. 4.1.1]). Since

$$\left[\omega_b\right] = \text{const},$$

the winding number of \((g_b, \omega_b)\) must vanish. More generally, we may assume that the cohomology class of \(\omega_b\) is not constant but varies over a small range

$$\left[\omega_b\right] \sim \text{const},$$

so that the mapping

$$B \to K, \quad b \to \left[\omega_b\right]$$

is homotopic to a constant map. Then the winding number of \((g_b, \omega_b)\) would still have to vanish. There is a canonical way to perturb the Seiberg-Witten equation on symplectic 4-manifolds. This is by setting

$$\eta_b = -\rho^2 \omega_b + \text{constant term in } \rho.$$  \hspace{1cm} (3.7)

As \(\rho\) grows, the contribution of the second term gets small. Hence, the winding number of \((g_b, \eta_b)\) is equal to that of \((g_b, \omega_b)\), which is zero. Then (3.4) becomes

$$\text{FSW}_{(g_b, \eta_b)}(s) = \text{FSW}_{(g_b, \omega_b)}(-s) \quad \text{for } \eta_b \text{ as in (3.7) and } \rho \text{ large.}$$  \hspace{1cm} (3.8)

4. Proof of Theorem 2. Let \(\mathcal{X}' \to \Delta \) be the family of quartic K3’s given by the complex-analytic fiber space (1.4), and let \(\mathcal{Y}\) be as in Thm. 3. From (1.4), we have the mapping \(\mathcal{X}' \to \mathbb{C}P^3\) given by \((t, x) \to x\). On the other hand, we also have the resolution map \(h: \mathcal{Y} \to \mathcal{X}'\) suggested by Thm. 3. Let \(\Omega\) be the Funini-Study form on \(\mathbb{C}P^3\) and let \(\Omega_{\mathcal{Y}}\) be the pull-back of \(\Omega\) under the mapping:

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{h} & \mathcal{X}' \\
(t, x) & \xrightarrow{\omega} & \mathbb{C}P^3 \\
\end{array}$$

Note that, for each \(t \in \Delta - \{0\}\), the restriction of \(\Omega_{\mathcal{Y}}\) to \(Y_t\) is Kähler. But the restriction of \(\Omega_{\mathcal{Y}}\) to the central fiber \(Y_0\) is degenerate. We now will construct a family of Kähler forms on the fibers of \(\mathcal{Y}\) by perturbing \(\Omega\) in a neighbourhood of \(Y_0\). To this end, recall that every K3 surface is Kähler. Hence, there exists some Kähler form \(\vartheta_0\) on \(Y_0\). From the theory of complex-analytic families (see [2, Thm. 15]), we recall:

**Theorem 4** (Kodaira-Spencer, [2]). Let \(\mathcal{Y} \to \Delta\) be a complex-analytic family of non-singular varieties. If \(Y_{t_0}\) carries a Kähler form, then, any fiber \(Y_t\), sufficiently close to \(Y_{t_0}\), also admits a Kähler form. Moreover, given any Kähler form on \(Y_{t_0}\), we can choose a Kähler form on each \(Y_t\), which depends differentiably on \(t\) and which coincides for \(t = t_0\) with the given Kähler form on \(Y_{t_0}\).

Having fixed \(\vartheta_0\) on \(Y_0\), we construct a family of Kähler forms \(\{\vartheta_t\}_{t \in U}\) for a sufficiently small neighbourhood \(U\) of 0 in \(\Delta\). Choose a bump function \(\chi: \Delta \to \mathbb{R}\) which equals 1 at the center of \(\Delta\) and equals 0 outside of the neighbourhood \(U\). Set:

$$\omega_t = \Omega_{\mathcal{Y}}|_{Y_t} + \varepsilon \chi \vartheta_t \quad \text{for } \varepsilon \text{ positive arbitrary small.}$$  \hspace{1cm} (4.1)

Both forms in the right-hand side of (4.1) are of type (1,1). Furthermore, \(\Omega_{\mathcal{Y}}|_{Y_t}\) is positive for each \(t \in \Delta - \{0\}\) and semi-positive for \(t = 0\), while \(\varepsilon \chi \vartheta_t\) is positive for each \(t \in U\). Thus, for every \(Y_t\), the form \(\omega_t\) is a positive (1,1)-form, hence is Kähler. Also, we have that:

$$\left[\omega_t\right] = \text{const} + O(\varepsilon).$$  \hspace{1cm} (4.2)
Since \( \vartheta_t \) is zero for each \( t \in \Delta - U \), it follows that:

\[
[\omega_t] = \text{const} \quad \text{for each } t \in \Delta - U.
\]

We now fix an abstract symplectic K3 surface \((Y,\omega)\) together with a symplectomorphism between \((Y_t,\omega_t)\) and \((Y_{t_0},\omega_{t_0})\) for some base-point \( t_0 \in \Delta - U \). We assume now that the monodromy homomorphism

\[
\pi_1(\Delta - U, t_0) \to \text{Symp}(Y,\omega)
\]

is the zero homomorphism. (4.3)

In other words, we assume that there is a family of symplectomorphisms

\[
f_t: (Y_t,\omega_t) \to (Y,\omega) \quad \text{for every } t \in \partial \Delta .
\]

Via the clutching construction, the family \( \{f_t\}_{t \in \partial \Delta} \) corresponds to the quotient space:

\[
S = Y \cup Y/\sim, \quad \text{where } (t,y) \sim f_t(y) \text{ for each } t \in \partial \Delta \text{ and } y \in Y_t,
\]

which is a fiber bundle over the 2-sphere

\[
B = \Delta/\partial \Delta.
\]

Since, for each \( t \in \partial \Delta \), the mapping \( f_t \) is a symplectomorphism, it follows that \( S \to B \) is a bundle of symplectic manifolds. This bundle is not Hamiltonian: the symplectic forms on the fibers are not cohomologous. However, their cohomology classes must obey (4.2) and so differ from each other very little. This latter property is interesting: it confers an extra symmetry to the family Seiberg-Witten invariants of this bundle; see Ex. 2 in §3. On the contrary, an independent computation will show that this symmetry fails for \( S \), which contradicts (4.3).

First we will need to analyze the Seiberg-Witten equations on the family \( Y \), concerning which we recall:

(i) There is a smooth rational \((-2)\)-curve \( C \subset Y_0 \) which is embedded in \( Y \) as a \((-1,-1)\)-curve.

(ii) There is closed \((1,1)\)-form \( \Omega \) on \( Y \) which is degenerate along \( C \). Hence,

\[
\int_C \Omega = 0.
\]

As \( Y \) is trivial as a differentiable family, we can find a fiber-preserving diffeomorphism

\[
\begin{array}{ccc}
\Delta \times Y_0 & \xrightarrow{\Phi} & Y \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{id} & \Delta \\
\end{array}
\]

(4.4)

Letting \( C_t \) denote \( \Phi(\{t\} \times C) \), we have:

\[
\int_{C_t} \omega_t \geq 0 \text{ for each } t \in \Delta \quad \text{and} \quad \int_{C_t} \omega_t = 0 \text{ for each } t \in \Delta - U.
\]

Letting \( \{g_t\}_{t \in \Delta} \) be the family of Kähler metrics associated to \( \{\omega_t\}_{t \in \Delta} \), we pick a spin\(^C\) structure \( s_C \) on \( T_{Y/\Delta} \) which, when restricted to \( Y_0 \), satisfies:

\[
c_1(s_C) = c_1(Y_0)(= 0) + 2[C],
\]

(4.5)

where \([C] \in H^2(Y_0;\mathbb{Z})\) is the class dual to \( C \). We remark that (4.5) specifies \( s_C \) uniquely. Now, we recall that, for each \( t \in \Delta \), the Levi-Civita connection of the Kähler metric \( g_t \) induces a canonical \( U(1) \)-connection \( A_t \) on \( K^*_Y = \text{det}_C T^*_Y \). Set:

\[
i \eta_t = F^+_A - i \rho^2 \omega_t.
\]

(4.6)
In [3], Kronheimer has proved the following statement:

**Theorem 5** (Kronheimer, [3]). Let \( \{Y_t\}_{t \in \Delta} \) be a family of non-singular surfaces (not necessarily K3 surfaces) given as a complex-analytic fiber bundle \( Y \to \Delta \), and let \( C \subset Y_0 \) be a smooth rational \((-1, -1)\)-curve. Suppose that, for each \( t \in \Delta - \{0\} \), the fiber \( Y_t \) has no effective divisors that are homologous to \( C_t \). Then, for the spin\(^C\) structure \( s_C \) above, \( \eta_t \) as in (4.6), and \( \rho \) large enough, the parameterized moduli space

\[
\mathcal{M}_{s_C}^{(g_t, \eta_t)} = \bigcup_{t \in \Delta} \mathcal{M}_{s_C}^{(g_t, \eta_t)}
\]

consists of a single point which lies above the fiber \( Y_0 \). This moduli space is transverse of the correct dimension.

To analyze the Seiberg-Witten equations on \( S \) we will use:

**Lemma 2.** Let \( \{J_t\}_{t \in \Delta} \) be the family of complex-structures on the fibers of \( Y \). There exists another family of \( \omega_t \)-compatible almost-complex structures \( \{\hat{J}_t\}_{t \in \Delta} \), which is homotopic to \( \{J_t\}_{t \in \Delta} \), and which satisfies the following:

(i) \( J_t = \hat{J}_t \) for each \( t \in U \);

(ii) \( \hat{J}_t = (f_t^{-1})_* \circ J \circ (f_t)_* \) for all \( t \in \partial \Delta \) and some \( \omega \)-compatible almost-complex structure \( J \) on \( (X, \omega) \).

This condition implies that \( \{\hat{J}_t\}_{t \in \Delta} \) gives a family of almost-complex structures on the fibers of \( S \).

**Proof.** Follows from the well-known fact that, for each \( t \in \Delta \), the space of \( \omega_t \)-compatible almost-complex structures is contractible. \( \square \)

If now \( \{g_t\}_{t \in \Delta} \) be the family of Hermitian metrics defined as \( g_b(\cdot, \cdot) := \omega_b(\cdot, \hat{J}_b \cdot) \), then, for each \( t \in U \), it satisfies \( g_t = g_t \). Hence, given any family of perturbations \( \{\eta_t\}_{t \in \Delta} \), we have:

\[
\mathcal{M}_{s_C}^{(g_t, \eta_t)} = \bigcup_{t \in U} \mathcal{M}_{s_C}^{(g_t, \eta_t)} \quad \text{where} \quad \mathcal{M}_{s_C}^{(g_t, \eta_t)} = \bigcup_{t \in \Delta - U} \mathcal{M}_{s_C}^{(g_t, \eta_t)}.
\]  

(4.7)

From Taubes’ theory of Gromov invariants (see [14, 15]), we recall:

**Theorem 6** (Taubes, [14, 15]). Let \( (X, \omega) \) be a closed symplectic 4-manifold, \( J \) any \( \omega \)-compatible almost-complex structure, and \( g \) the associated Hermitian metric. Choose a cohomology class \( \varepsilon \in H^2(X; \mathbb{Z}) \) such that \( \varepsilon \cdot [\omega] \leq 0 \) and let \( s_\varepsilon \) be the spin\(^C\) structure such that:

\[
c_1(s_\varepsilon) = c_1(X) + 2\varepsilon.
\]

Then there exists a special \( U(1) \)-connection \( A \) on \( K^*_X \) such that for the family of perturbation

\[
i\eta = F^+_A - i\rho^2 \omega \quad \text{and} \quad \rho \text{ large enough},
\]

(4.8)

the moduli space \( \mathcal{M}_{s_\varepsilon}^{(g, \eta)} \) is empty. If \( (X, \omega) \) is Kähler, then the special connection \( A \) is the one induced by the Levi-Civitá connection.

This theorem is also explained in the notes for Ch. 10 in [11].

Thus, if \( \eta_t \) is as in (4.8), then \( \mathcal{M}_{s_C}^{(1)} \) will be empty, whereas \( \mathcal{M}_{s_C}^{(2)} \) will consist of a single point. On the other hand, restricting the spin\(^C\) structure \(-s_{[C]}\) on \( Y_t \), we get:

\[
c_1(-s_{[C]}) = -2[C],
\]
since \( c_1(Y_t) = 0 \). Combining this with Thm.6, we see that \( \mathcal{M}^c_{(g_t, \eta_t)} \) is empty for \( \rho \) sufficiently large. Thus, for the Seiberg-Witten invariant of \( S \to B \), we see that:

\[
\text{FSW}_{(g_b, \eta_b)}(\mathfrak{s}_C) = 1 \quad \text{and} \quad \text{FSW}_{(g_b, \eta_b)}(-\mathfrak{s}_C) = 0 \quad \text{for} \ \eta_b \ \text{as in (3.7)}.
\]

But, by (3.8), these invariants must be equal. This contradiction finishes the proof.

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