This paper presents an equational theory for the QRAM model of quantum computation, formulated as an embedded language inside of homotopy type theory. The embedded language approach is highly expressive, and reflects the style of state-of-the art quantum languages like Quipper and QWIRE. The embedding takes advantage of features of homotopy type theory to encode unitary transformations as higher inductive paths, simplifying the presentation of an equational theory. We prove that this equational theory is sound and complete with respect to established models of quantum computation.

1 Introduction

One of the most prominent models of quantum computation today is known as the QRAM model, in which a quantum computer works alongside a classical computer to manipulate both quantum and classical data [15]. Programming languages for the QRAM model must provide good access to both quantum data, like qubits, as well as easy-to-use classical abstractions such as booleans, recursion, and functions. In addition, the two sorts of data should interact—it should be possible to measure a qubit and probabilistically obtain a classical boolean value.

Inspired by the QRAM model, several state-of-the-art quantum programming languages are implemented as embedded languages, where domain-specific features for quantum data are added to an existing general-purpose host language. Quipper, embedded in Haskell, utilizes Haskell type classes, monadic programming, and meta-programming [12]. LIQUi⟩, embedded in F#, and Chisel-Q, embedded in Scala, also take advantage of features in their respective host languages [41, 16]. Python boasts a large collection of APIs for quantum computing, including QuTiP [14], QISKit,\(^1\) and pyQuil.\(^2\)

Unfortunately, reasoning about embedded quantum languages has largely been seen as futile; such reasoning would have to account for the quantum behavior of the embedded language as well as the classical behavior of the host language and the interactions between the two. Attempts to formalize Quipper have been restricted to a standalone language in the style of Quipper, not the actual implementation in Haskell [28, 27, 17].

Given a powerful enough host language, however, it is possible to study the meta-theory of an embedded language inside the host language itself. This is the approach is taken by QWIRE, a language implemented in Coq that uses its host language both for programming and for verification [24, 26].

This work extends that technique to study the equational theory of an embedded quantum language. We build on an equational theory proposed by Staton [33], an algebraic account of the relationship between quantum data and classical control. Other works axiomatize particular sets of unitary transformations [5, 19, 20], but, like Staton, we focus on the relationships between quantum and classical

\(^1\)https://github.com/QISKit
\(^2\)http://pyquil.readthedocs.io
HoTT Quantum Equational Theory

features. Staton’s algebraic framework includes measurement-based branching, but it does not contain explicit classical data or features we would expect from a QRAM-style language.

Because our goal is to extend Staton’s equational theory to a richer embedded language, we choose a host language that specializes in equality—homotopy type theory (HoTT). In HoTT, proofs of equality, also called paths, can contain extra computational content. In the past few years, a variety of applications have used paths in HoTT to represent data structures such as containers [1], version control patches [6], and SQL queries [10]. These data structures can all be represented as groupoids, which are generalized by paths in HoTT.

This work builds on the observation that unitary transformations, a core component of quantum computing, form a groupoid. We exploit this structure to encode unitaries in the paths between quantum types. This simplifies Staton’s theory because many of the structural rules in his presentation can be derived in our presentation.

This paper makes the following contributions:

• We present a quantum programming language embedded in homotopy type theory (Sections 3 and 4). The language is both (1) expressive, because users have access to classical higher-order functions, data structures, and more through the host language; and (2) sound, thanks to its use of linear types for quantum data. We have formalized some basic properties of the underlying HoTT data structures in Coq.

• We define an equational theory for the embedded quantum language, consisting of standard rules of \( \alpha \) and \( \beta \) equivalence, as well as variations of \( \eta \) equivalence and commuting conversion rules. In addition, the equational theory contains two quantum-specific axioms, which describe how unitary transformations interact with initialization and measurement, respectively (Section 5).

• We prove the equational theory is sound with respect to a standard semantics in terms of superoperators over density matrices (Section 6).

• Throughout, we use Staton’s algebraic equational theory (Section 3.4) as a specification of the equations we expect to hold in our theory, and we prove that a fragment of our language is sound and complete with respect to Staton’s axioms (Section 7).

A prior version of this work has been presented in Chapters 6 and 7 of the first author’s dissertation [23]. The previous version did not include the formal results in Coq, nor the connection we make in Section 7 with Staton’s algebraic theory.

2 Background and main ideas

2.1 Quantum computing

Quantum computers present a radically different computing environment compared to ordinary classical computers. Instead of bits, quantum computers operate on quibits—superpositions of classical bits of the form \( c_0 |0\rangle + c_1 |1\rangle \) where \( c_0, c_1 : \mathbb{C} \) are amplitudes satisfying \( |c_0|^2 + |c_1|^2 = 1 \). A qubit can be measured, resulting in the bit 0 with probability \( |c_0|^2 \), or the bit 1 with probability \( |c_1|^2 \). Qubits \( e \) can also be manipulated by applying one of a set of unitary matrices \( U \), with application written \( U \# e \). For example, the X (pronounced “not”) unitary swaps the amplitudes of \( |0\rangle \) and \( |1\rangle \), so measuring a qubit of the form

\[ A thorough introduction to quantum computation is beyond the scope of this work; we refer the interested reader to Nielsen and Chuang [21]. \]
X \# e is the same as (results in the same probability distribution as) negating the measurement of e: \text{meas}(X \# e) = \neg(\text{meas } e).

Quantum programs, therefore, do four main things: they initialize qubits, apply unitary gates, measure qubits, and invoke classical programs to process classical measurement results. This means that quantum programs are low-level—they often lack basic abstractions like data structures and parametricity—and effectful—measuring a qubit can non-locally affect a different part of the quantum state.

Researchers realized early on that quantum programming languages would benefit from formal study, including type systems [31, 4], verification [42, 26], and denotational semantics [2]. For example, this work uses linear types to enforce the no-cloning principle of quantum mechanics: unknown quantum states cannot be duplicated. This means that a program that uses quantum data twice, like \lambda x.(x,x), might not correspond to a valid quantum computation. Linear types enforce the fact that quantum variables occur exactly once in a term, so that well-typed programs have a sound denotational semantics [31, 30].

This work also focuses on equational theories, which characterize when two quantum programs are equivalent. Equational theories help programmers understand the meaning of their programs and help compiler writers ensure that their optimizations are sound. Unfortunately, developing sound equational theories for effectful languages in general, and quantum languages in particular, is notoriously difficult.

2.2 Homotopy type theory (HoTT)

Homotopy type theory is, in many ways, a theory of equivalence. In HoTT, proofs of equality \( a = b \), called \textit{paths}, may have computational content. That is, there may be other proofs of equality besides the trivial reflexivity path \( 1_a : a = a \).

In homotopy type theory, we write the type of \textit{propositional} proofs of equality as \( a = b \); that is, \( a = b \) is a type with a single constructor \( 1_a : a = a \). Propositional equality is distinguished from \textit{judgmental} equality \( a \equiv b \), which asserts that \( a \) and \( b \) are equal by definition. The judgment \( a \equiv b \) is not a type; it is only valid in the meta-theory and has no computational content. For more intuition on the difference between propositional and judgmental equality, see the HoTT book [36, Chapter 1].

Homotopy type theory was developed as a type-theoretic alternative to set theory, but it has applications in a wide variety of computational domains [6, 10, 1]. When a domain is difficult to characterize equationally, but uses data in the shape of an equivalence relation or groupoid, HoTT can help.

Consider a type \( A \) that you want to quotient by (the equivalence relation generated by) a relation \( R \). For every element \( a : A \), the equivalence class of \( a \) is written \( [a]_R : A//R \), and whenever \( R(a,b) \), it should be the case that \( [a]_R = [b]_R \). In set theory it is possible to define the equivalence class as a set \( [a]_R = \{ x : A \mid R(a,x) \} \), so that \([a]_R\) contains the same elements as \([b]_R\). However, in programming environments, where the representation of data structures matters, sets are often implemented as lists or arrays, so \([a]_R\) must be carefully constructed so that it has the same representation as \([b]_R\).

Homotopy type theory sidesteps this representation problem with \textit{higher inductive types}: inductive definitions with constructors for both terms and paths.\(^4\)

\textbf{Definition 1} (Informal). \textit{The quotient \( A//R \) of a type \( A \) by an equivalence relation \( R \) on \( A \) is a higher inductive type generated by the following constructors:}

- for \( a : A \), there is a term \( [a]_R : A//R \); and

- for \( a, b : A \) and \( r : R(a,b) \), there is a proof \( [r]_R : [a]_R = [b]_R \).

\(^4\)Definition 1 is an informal, intuitive definition; see Sojakova [32] for a formal definition of set quotients with types and induction principles specified. In that work, Sojakova formalizes higher inductive types as homotopy-initial algebras.
Notice that if \( r_1 \) and \( r_2 \) are two different witnesses (proofs) of \( R(a, b) \), then \([r_1]_R\) is different from \([r_2]_R\)—the structure of the relation \( R \) is preserved in the paths of \( A/R \).

Despite the extra computational content of equality types in HoTT, the usual properties still hold for paths generated by higher inductive types. The principle of path induction states that, given a property \( P : \prod_{b,a} a = b \to \text{Type} \) on paths, if \( P \) holds on the reflexivity path, then \( P \) holds on any path. That is, the induction principle for paths has the following type:

\[
\text{path}\_\text{ind} : (\prod (x : \alpha), P_{x,x}(1_x)) \to \prod (x y : \alpha)(p : x = y), P_{x y}(p).
\]

If \( p : a = b \) for \( a, b : \alpha \) and \( x : Q(a) \) for some property \( Q : \alpha \to \text{Type} \), then it is possible to transport the path \( p \) over \( x \) to obtain a proof \( \text{transport}_Q p : Q(b) \). If \( a \) and \( b \) are types and \( x : a \), we write \( \text{coerce}\, p x \equiv \text{transport}_{\alpha,a} p \, x : b \).

Functions \( f : \alpha \to \beta \) in HoTT are functorial, meaning that paths \( p : a = b \) on \( \alpha \) can be promoted to paths \( a \to f p = fa = fb \). An equivalence \( f : \alpha \cong \beta \) between types \( \alpha \) and \( \beta \) consists of a pair of functions \( f : \alpha \to \beta \) and \( f^{-1} : \beta \to \alpha \), along with proofs \( \eta : \prod a f^{-1}(fa) = a \) and \( \varepsilon : \prod_b f(f^{-1}b) = b \) such that \( \prod a \varepsilon f \eta_a = \varepsilon f a \) [36, Definition 4.2.1]. The univalence axiom states that an equivalence \( f : \alpha \cong \beta \) between types can be treated as a path \( \text{univ}\, f : \alpha = \beta \), such that \( \text{coerce}\, (\text{univ}\, f)\, a = fa \):

\[
\text{univ} : \prod_{\alpha,\beta}(\alpha \cong \beta) \equiv (\alpha = \beta).
\]

### 2.3 Main idea—unitaries as paths

The core idea of this work is to encode the unitary operators used in quantum computing in the higher inductive structure of quantum types.

We start by defining unitary matrices in homotopy type theory. Let \( \mathbb{C} \) be a type of complex numbers in HoTT.\(^5\) For any finite types \( \alpha, \beta : \text{FinType} \), let \( \text{Matrix}(\alpha, \beta) \) be the set of complex-valued \( 2^{|\alpha|} \times 2^{|\beta|} \) matrices.\(^6\) We write \( I \) for the identity matrix, \( AB \) for matrix multiplication, \( A^\dagger \) for the conjugate transpose of \( A \), \( A \otimes B \) for the tensor product, and \( A \oplus B \) for the block matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \).

A unitary matrix \( \text{UMatrix}(\alpha, \beta) \) is a matrix \( U : \text{Matrix}(\alpha, \beta) \) such that its conjugate transpose is its own inverse: \( U^\dagger U = UU^\dagger = I \). Unitary transformations \( \text{UMatrix} \) form a groupoid—a category whose objects are finite types, and whose morphisms \( \text{UMatrix}(\alpha, \beta) \) are all invertible.

In this paper we encode unitary matrices in a quantum language by taking the types of our quantum language to be elements of \( \text{QType} \equiv \text{FinType} / \text{UMatrix} \), the groupoid quotient of \( \text{UMatrix} \).\(^7\) The intuition is that qubits are represented as the type \( \text{Bool} \downarrow \text{UMatrix} \) of two-dimensional vector spaces, and unitary transformations \( U : \text{UMatrix}(\alpha, \beta) \) are encoded as paths of type \( [\alpha]_{\text{UMatrix}} = [\beta]_{\text{UMatrix}} \). For quantum types \( \sigma \) and \( \tau \), we write \( \tau / (\sigma, \tau) \) for the unitary path type \( \sigma = \tau \).

Encoding unitaries as paths has two important consequences. First, there is no need for explicit syntax for applying a unitary transformation; unitary application \( U \# e \) is defined to be \( \text{transport}\, U\, e \), the result of transporting the path \( U : \sigma = \tau \) over a term \( \Gamma \vdash e : \sigma \). Second, many of the structural axioms on unitaries can now be proven by path induction. For example, consider the following statement:

---

5The precise formulation of the complex numbers in HoTT is not relevant for this work; for concreteness we can pick a representation based on the Dedekind reals [36, Chapter 11]. But we never need to define functions out of \( \mathbb{C} \), and so all that we require is that basic computational properties, such as arithmetic and the complex conjugate, are valid.

6Here we use \( |\alpha| \) to refer to the size of the finite type \( \alpha \). The fact that \( \text{Matrix}(\alpha, \beta) \) is a set means that for any two paths \( p_1, p_2 : A = B \) between matrices \( A, B : \text{Matrix}(\alpha, \beta) \), it is the case that \( p_1 = p_2 \).

7Section 4.2 extends the set quotient type on equivalence relations \( \alpha / R \) to groupoid quotients \( \alpha / \downarrow G \).
Proposition 2. Suppose \( \Gamma \vdash e : \sigma \). Then, for \( U : \sigma = \tau \) and \( V : \tau = \rho \) we have \( V \# (U \# e) = (V \circ U) \# e \).

Proof. By path induction over \( V \). If \( V \) is the trivial path by reflexivity on \( \sigma \), written \( 1_{\sigma} \), then \( V \circ U = U \).
By the definition of transport, for all \( x \) we have \( 1 \# x = x \). So \( 1 \# (U \# e) = U \# e = (1 \circ U) \# e \).

Crucially, it is not possible to prove the following false statement:

Proposition 3 (False). Let \( \Gamma \vdash e : \sigma \) and \( U : \sigma = \sigma \). Then \( U \# e = e \).

Path induction only applies on proofs \( a = b \) when at least one of \( a \) or \( b \) is a free variable, so it does not apply here. In fact, the statement is false—Proposition 12 will show that \( X \# |0\rangle = |1\rangle \), but it is not the case that \( |0\rangle = |1\rangle \) due to the soundness of the denotational semantics (Theorem 27).

3 A quantum term calculus

In this section we present a specification of an embedded linear quantum language. The embedding, closely related to the linearity monad embedding [25], allows us to use non-linear data in a natural way in the style of linear/non-linear type theory [9]. In Section 4 we will implement this language in HoTT by encoding unitary transformations as paths.

Quantum types \( \sigma, \tau : \text{QType} \) consist of linear pairs \( \sigma \otimes \tau \) and sums \( \sigma \oplus \tau \), as well as host language types \( \text{Lower} \alpha \). Host types \( \alpha : \text{Type} \) are also called classical, in contrast to quantum types. We choose to restrict our semantics to finite-dimensional vector spaces, so we insist that \( \alpha \) be finite for it to be used in a quantum type. That is, for any finite host language type \( \alpha : \text{FinType} \), there is a quantum type \( \text{Lower} \alpha \) containing (superpositions of) host language values of type \( \alpha \).

\[
\begin{array}{cccc}
\alpha : \text{FinType} & \sigma : \text{QType} & \tau : \text{QType} & \sigma : \text{QType} \\
\text{Lower} \alpha : \text{QType} & \sigma \otimes \tau : \text{QType} & \sigma \oplus \tau : \text{QType} &
\end{array}
\]

A typing context \( \Gamma : \text{Ctx} \) is a finite map from linear variables \( \text{Var} \) to \( \text{QTypes} \). For \( \Gamma_1 \) and \( \Gamma_2 : \text{Ctx} \), we write \( \Gamma_1 \sqcup \Gamma_2 \) for the disjoint merge of \( \Gamma_1 \) and \( \Gamma_2 \), which is only defined when \( \Gamma_1 \) and \( \Gamma_2 \) are disjoint. We write \( \emptyset \) for the empty typing context and \( x : \sigma \) for the singleton typing context.

Linear (quantum) expressions are given by the type \( \text{QExp} \) \( \Gamma \), where \( \Gamma \) is a typing context and \( \sigma \) is a quantum type. \( \text{QExp} \) represents a typing judgment; we sometimes write \( \Gamma \vdash e : \sigma \) for \( e : \text{QExp} \Gamma \).

The type \( \text{QExp} \) \( \Gamma \sigma \) is defined inductively by the rules in Figure 1. Most of the constructions are standard for a linear lambda calculus. The exception is the constructors of the non-linear data type \( \text{Lower} \alpha \), which we explain in more detail here.

The introduction rule for \( \text{Lower} \alpha \) says that for any \( a : \alpha \) in the host type theory, there is a linear expression put \( a \) of quantum type \( \text{Lower} \alpha \) that does not use any linear variables. Terms of type \( \text{Lower} \alpha \) correspond to quantum states: vectors in an \( |\alpha| \)-dimensional, complex-valued vector space, where the \( i \)th element \( a_i : \alpha \) corresponds to a vector \((0,\ldots,1,\ldots,0)\) with 1 at index \( i \) and 0 elsewhere.\(^8\)

The elimination rule for \( \text{Lower} \alpha \) has the form \( e >! f \), where \( e \) is a quantum term of type \( \text{Lower} \alpha \) and \( f \) is a host-level function from values of type \( \alpha \) to quantum expressions. The infix constructor \( >! \) is pronounced “let-bang”, in reference to the linear logic operation \( ! \), pronounced “bang”. Intuitively, the expression \( e >! f \) measures\(^9\) the quantum expression \( e \), resulting in a value of type \( \alpha \), and then applies \( f \) to that value. We will sometimes write \( \text{let} \: !x := e \: \text{in} \: e' \) for \( e >! \lambda x.e' \) when it is clear that \( x \) is a host-level variable, as opposed to a linear variable bound by the linear typing context.

\(^8\)This intuition will be formalized in Section 6.
\(^9\)The basis in which this measurement occurs corresponds to the order of the finite type \( \alpha \).
The behavior of this calculus is given as an equational theory, so that we can justify our axioms of quantum computing. We write $\Gamma = x : \sigma \vdash e : \text{QExp} \Gamma \sigma$ for the usual notion of $\beta$ equivalence. Figure 2 shows the $\eta$-expanded version, case $e$ of $(t_1x_1 \to e_1) | t_2x_2 \to e_2) \sim e_1 \sim e_2$.

### 3.1 Equational theory

The behavior of this calculus is given as an equational theory, so that we can justify our axioms of quantum computing. We write $e_1 x_1, \ldots, e_n x_n$ for the simultaneous capture-avoiding substitution of the $e_i$'s for the linear variables $x_i$ in $e$, and we write $e_1 \sim e_2$ for the usual notion of $\alpha$ equivalence. Figure 2 shows the $\beta$ equivalences for the language, which we write as $e_1 \sim e_2$.

Eta equivalence, written $e_1 \sim e_2$, is allowed for product types but not, in general, for sums $\sigma \oplus \tau$ or classical types $\text{Lower} \alpha$. Semantically, case analysis for sums and $>!$ for $\text{Lower} \alpha$ perform quantum measurement (see Section 3.3), so a term $e$ of type $\sigma \oplus \tau$ is not semantically equivalent to its $\eta$-expanded version, case $e$ of $(t_1x_1 \to e_1) | t_2x_2 \to e_2) \sim e_1 \sim e_2$.

However, $\eta$ expansion for the multiplicative product and unit type $\text{Lower} ()$ are admissible, since they do not encode classical information. In fact, for the unit type we have an even stronger property—any two values of unit type are equivalent. This reflects the fact that the unit type is terminal in the category of density matrices in which we define a denotational semantics in Section 6.

$$
\begin{align*}
\Gamma \vdash e : \sigma_1 \otimes \sigma_2 & \quad \text{eta-}\otimes \\
\Gamma \vdash e_1 : \text{Lower} () & \quad \text{eta-}() \\
\Gamma \vdash e_2 : \text{Lower} () & \quad \text{eta-}()
\end{align*}
$$

Recall that we write $\Gamma \vdash e : \tau$ for $e : \text{QExp} \Gamma \tau$. Notice that the usual $\eta$ rule for $\text{Lower} ()$—that $e \sim e_1 \sim e_2$—can be derived from $\eta-()$. 

Figure 1: An embedded linear/non-linear type system.

Figure 2: Linear/non-linear $\beta$ equivalences
A qubit is a quantum type with two basis elements, $|0\rangle$ and $|1\rangle$, so we encode qubits as $\text{Qubit} \equiv \text{Lower Bool}$. Initialization and measurement of qubits is straightforward:

\[
\begin{align*}
\text{init} : \text{Bool} & \to \text{QExp } 0 \text{ Qubit} \\
& \equiv \lambda b. \text{put } b \\
\text{meas} : \text{Qubit} & \to \text{Lower Bool} \\
& \equiv \text{suspend}(\lambda x. \text{let } !b := x \text{ in put } b)
\end{align*}
\]

3.2 Linear functions

The quantum language described so far does not include higher-order functions, as the physical interpretation of higher-order functions in quantum mechanics is not entirely clear [37, 18, 13, 22]. However, we can encode first-order functions in our host language in a data type $\sigma \to \tau$ (pronounced $\sigma$ “lolli” $\tau$) that represents quantum computations with input $\sigma$ and output $\tau$. The type $\sigma \to \tau$ is defined inductively by a single constructor, $\text{suspend}$, that takes an expression of type $\tau$ with a single argument of type $\sigma$:

\[
\text{suspend} : \prod_x \text{QExp } (x : \sigma) \to (\sigma \to \tau).
\]

The eliminator says that for any $k : \prod_{e} \text{QExp } (x : \sigma) \to \beta$ there is a function $\text{rec}_{\text{due}}^{x} : (\sigma \to \tau) \to \beta$ such that $\text{rec}_{\text{due}}^{x}(x e) \equiv k x (f x)$.

\[
\text{rec}_{\text{due}}^{x}(x e) \equiv k x (f x).
\]

To apply a linear function $g : \sigma \to \tau$ to an argument $\Gamma \vdash e : \sigma$, define $\Gamma \vdash \text{force } g e : \tau$ using $\text{rec}_{\text{due}}^{x}$, with $k \equiv \lambda x. e_0.e_0\{e/x\}$, so that $\text{force}(\text{suspend } x e') e \equiv e'\{e/x\}$.

The quantum identity function is $\text{suspend}(x,x)$, and the function that swaps the elements of a pair is $\text{suspend}((x,y),(y,x))$ of type $\sigma \otimes \tau \to \tau \otimes \sigma$.

3.3 Quantum data, and measurement as case analysis

init : Bool → QExp 0 Qubit

meas : Qubit → Lower Bool

= λb. put b

= suspend(λx. let !b := x in put b)

The induction principle says that for any $P : (\sigma \to \tau) \to \text{Type}$, given a proof $p : \prod_{e} \prod_{\alpha, \psi : \lambda e. \text{QExp } (x : \sigma) \to \alpha} \tau \ P (\text{suspend } x e)$, there is a function $\text{ind}_{\text{due}}^{x} : \prod_{\sigma \to \tau} P f$ such that $\text{ind}_{\text{due}}^{x}(\text{suspend } x e) \equiv p x e$. 

Figure 3: Commuting conversion rules

Commuting conversions, written $e \sim_{cc} e'$, describe how elimination forms can move within an expression [11, Chapter 10]. Rules of this form, shown in Figure 3, are common for linear lambda calculi [34].
On first glance, these definitions appear not to be doing anything—force(meas \(e\)) in particular is just the \(\eta\) expansion of \(e\). But this highlights a critical semantic fact of our system: *case analysis performs quantum measurement*. This has a number of consequences for the theory of the language, including the fact that \(\eta\) expansion is not sound in general: a measured qubit is not equivalent to an unmeasured one.

By choosing to encode measurement as case analysis, we open the door to a very expressive quantum theory. For example, the type Lower(Boolean \(\times\) Boolean) is equivalent to the two-qubit system Lower Boolean \(\otimes\) Lower Boolean. Qutrits (a base-3 quantum system) can be encoded as Lower(()) \(\oplus\) Lower(()) \(\oplus\) Lower(()) and finite, length-indexed lists of qubits can be encoded as Lower(Vector n Boolean).

### 3.4 Unitary transformations.

The remaining components of the language are unitary transformations \(U(\sigma, \tau)\). A unitary \(U : U(\sigma, \tau)\) can be applied to an expression of type \(\sigma\) to produce an expression of type \(\tau\):

\[
\begin{align*}
U : U(\sigma, \tau) & \quad e : \text{QExp } \Gamma \sigma \\
\hline
U \# e : \text{QExp } \Gamma \tau
\end{align*}
\]

Section 4 will show how to derive the unitaries in the lambda calculus described so far; this section will spell out a specification of the properties we expect to hold of the quantum fragment.

Following Staton [33], we focus on four main ways to combine unitaries: if \(U : U(q, r)\) and \(V : U(q', r')\), then \(U \otimes V : U(q \otimes q', r \otimes r')\) is the tensor product of \(U\) by \(V\), and \(U \oplus V : U(q \oplus q', r \oplus r')\) is the block matrix. In addition, unitaries form a groupoid: there is an identify unitary, written \(1\); unitaries are subject to composition, written \(V \circ U\); and they are invertible, written \(U^†\). Staton proves that all unitary matrices can be constructed from 1-qubit unitaries with the direct sum and tensor product [33].

The equational theory of unitaries is divided into three classes. First, the “structural” axioms, shown in Figure 4, characterize how unitaries interact with syntactic forms of the language. For example, Equation \(U\)-\(\otimes\)-INTRO describes how the tensor product \(U_1 \otimes U_2\) distributes over pairs.

Second, the axioms in Figure 5 characterize that unitaries form a groupoid.

The third set of axioms describe how certain unitaries interact with initialization and measurement, for instance those shown in Figure 6. Such unitaries are completely defined by isomorphisms on their basis sets, which we call *unitary equivalences* \(\sigma \simeq \tau\), and define formally in Section 5.

Every unitary equivalence \(f\) can be lifted to a unitary transformation \(\widetilde{f} : U(\sigma, \tau)\). For example, for the equivalence \(\text{swap} : \Pi_{\sigma_1, \sigma_2} \sigma_1 \otimes \sigma_2 \simeq \sigma_2 \otimes \sigma_1\), it should be the case that \(\text{swap} \# (e_1, e_2) \approx (e_2, e_1)\) as shown in Figure 6. We call \((e_1, e_2)\) the *partial initialization* of the quantum system \(\Pi_{\sigma_1, \sigma_2} \sigma_1 \otimes \sigma_2\), reflected by the fact that \(\text{swap}\) quantifies over all types \(\sigma_1\) and \(\sigma_2\). Partial initialization and its counterpart, partial measurement, precisely characterize the behavior of unitary equivalences \(f : \sigma \simeq \tau\):

\[
\begin{align*}
\widetilde{f} \# \text{init}_\sigma b & \approx \text{init}_\tau(fb) & \text{(U-INTRO)} \\
\text{match}_\tau(\widetilde{f} \# e) \text{ with } g & \approx \text{match}_\sigma e \text{ with } g \circ f. & \text{(U-ELIM)}
\end{align*}
\]

### 4 Deriving equational rules in HoTT

The goal of this section is to encode unitaries in quantum types, to minimize the number of axioms needed to recover the equational theory described in the previous section. As described in Section 2.3, we do this by encoding unitaries in the groupoid quotient QType \(\equiv\) FinType/\_\_UMatrix.
relations, along with the quantum-specific axiom that will be defined in Section 5.

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Figure 4: Structural axioms. The relation \( e \approx e' \) is the union of \( \alpha, \beta, \eta \), and commuting conversion relations, along with the quantum-specific axiom that will be defined in Section 5.

\[
\begin{align*}
(U_1 \otimes U_2) \# (e_1, e_2) & \approx (U_1 \# e_1, U_2 \# e_2) \quad \text{(U-\otimes-INTRO)} \\
\text{let } (x_1, x_2) & := (U_1 \otimes U_2) \# e \text{ in } e' \\
& \approx \text{let } (y_1, y_2) := e \text{ in } e'\{U_1 \# y_1/x_1, U_2 \# y_2/x_2\} \quad \text{(U-\otimes-ELIM)} \\
U \# (\text{let } (x_1, x_2) := e \text{ in } e') & \approx \text{let } (x_1, x_2) := e \text{ in } U \# e' \quad \text{(U-\otimes-COMM)}
\end{align*}
\]

\[
\begin{align*}
(U_1 \oplus U_2) \# (t_1 e) & \approx U_1 \# e \quad \text{(U-\oplus-INTRO1)} \\
(U_1 \oplus U_2) \# (t_2 e) & \approx U_2 \# e \quad \text{(U-\oplus-INTRO2)} \\
\text{case } (U_1 \oplus U_2) \# e \text{ of } (t_1 x_1 \rightarrow e_1 \mid t_2 x_2 \rightarrow e_2) & \approx \text{case } e \text{ of } (t_1 y_1 \rightarrow e_1\{U_1 \# y_1/x_1\} \mid t_2 y_2 \rightarrow e_2\{U_2 \# y_2/x_2\}) \quad \text{(U-\oplus-ELIM)} \\
U \# (\text{case } e \text{ of } (t_1 x_1 \rightarrow e_1 \mid t_2 x_2 \rightarrow e_2)) & \approx \text{case } e \text{ of } (t_1 x_1 \rightarrow U \# e_1 \mid t_2 x_2 \rightarrow U \# e_2) \quad \text{(U-\oplus-COMM)}
\end{align*}
\]

\[
U \# (e > ! f) \approx e > ! \lambda x \rightarrow U \# (f x) \quad \text{(U-LOWER-COMM)} \\
U \# e > ! \lambda \_e' \approx e > ! \lambda \_e' \quad \text{(U-LOWER-COMM)}
\]

Figure 5: Groupoid axioms

\[
\begin{align*}
U \# (V \# e) & \approx (U \circ V) \# e \quad \text{(U-COMPOSE)} \\
I \# e & \approx e \quad \text{(U-I)} \\
U^\dagger \# U \# e & \approx e \quad \text{(U-\dagger)}
\end{align*}
\]

Figure 6: Examples of axioms describing the behavior of the unitaries \( X : \mathcal{U}(\text{Qubit}, \text{Qubit}) \), \( \text{SWAP} : \mathcal{U}(\sigma_1 \otimes \sigma_2, \sigma_2 \otimes \sigma_1) \), and \( \text{DISTR} : \mathcal{U}(\text{Qubit} \otimes \tau, \tau \otimes \tau) \).
4.1 Groupoid quotient as a higher inductive type

**Definition 4** (Sojakova [32], Section 4.3). If $G$ is a groupoid with objects $\alpha$, then the groupoid quotient of $G$, written $\alpha/\!\!/G$, is a higher inductive type with the following constructors:

- $\text{point} : \alpha 	o \alpha/\!\!/G$
- $\text{cell} : \prod_{a,b} G(a,b) \to \text{point } a = \text{point } b$
- $\text{cell_compose} : \prod_{f,g} \text{cell}(g \circ f) = \text{cell } g \circ \text{cell } f$
- $\text{is-1-type} : 1\text{-type}(\alpha/\!\!/G)$

The fact that $\alpha/\!\!/G$ is a 1-type means that, for $x,y : \alpha/\!\!/G$, if $f,g : x = y$ and $p,q : f = g$, then $p = q$.

The induction principle states: for a predicate $P$ on $\alpha/\!\!/G$, there is a proof $\text{ind}_P$ of $\prod x, P(x)$, provided

- For all $x$, $P(x)$ is a 1-type;
- For all $a : \alpha$, there is a proof $P_{\text{point } a}$ of $P(\text{point } a)$;
- For all $f : G(a,b)$, there is a proof $P_{\text{cell } f}$ that $\text{transport}_P(\text{cell } f)(P_{\text{point } a}) = P_{\text{point } b}$;
- For $f : G(a,b)$ and $g : G(b,c)$, the diagram in Figure 7 commutes.

Furthermore, $\text{ind}_P$ satisfies the following computation laws:

$$\text{ind}_P(\text{point } a) \equiv P_{\text{point } a} \quad \text{and} \quad \text{apd}_P(\text{cell } f) \equiv P_{\text{cell } f}$$

where, for $f : \prod x : \alpha, P(x)$ and $p : a = b$ at type $\alpha$, we have $\text{apd}_P(p) : \text{transport}_P p (fa) = fb$.

4.2 QType as a groupoid quotient

Define QType to be the groupoid quotient of UMatrix: $\text{QType} \equiv \text{FinType}/\!\!/\text{UMatrix}$. Intuitively, for $\sigma, \tau : \text{QType}$, the type $\sigma = \tau$ corresponds to unitary transformations from $\sigma$ to $\tau$. The groupoid quotient ensures that the identity and inverse of paths correspond to the appropriate operations on matrices.

**Proposition 5.** Let $I : \text{UMatrix}(\alpha, \alpha)$ be the identity matrix on $\alpha$. Then $\text{cell } I = 1_{\text{point } \alpha}$.  

\[\text{Figure 7: A condition for quotient induction.}\]
Proof. Since \( \mathbf{I} = \mathbf{I} \circ \mathbf{I} \), by the compositionality of \( \text{cell} \) we know that \( \text{cell} \mathbf{I} = \text{cell} \mathbf{I} \circ \text{cell} \mathbf{I} \). But for any path \( p : x = x \), if \( p \circ p = p \) then \( p \) must be \( 1_x \). \( \Box \)

**Proposition 6.** Let \( U : \text{UMatrix}(\alpha, \beta) \). Then \( (\text{cell} \ U)^{-1} = \text{cell} \ U^\dagger \).

**Proof.** By the compositionality of \( \text{cell} \) and Proposition 5, \( \text{cell} \ U \circ \text{cell} \ U^\dagger = \text{cell} (U \circ U^\dagger) = \text{cell} \mathbf{I} = \mathbf{I} \). \( \Box \)

Next we need to define the type formers \( \text{Lower}, \otimes, \) and \( \oplus \) \( \) and \( \otimes \) and \( \oplus \) are defined by quotient induction. To define \( \otimes : \text{QType} \rightarrow \text{QType} \rightarrow \text{QType} \), we apply a variant of the quotient recursion principle on two variables.

**Lemma 7.** Let \( G_1 \) and \( G_2 \) be groupoids with objects \( \alpha_1 \) and \( \alpha_2 \) respectively. Let \( \beta \) be a \( 1 \)-type, and let \( f : \alpha_1 \rightarrow \alpha_2 \rightarrow \beta \) be a function, with

\[
\begin{align*}
f^\text{cell} : \prod_{x_1, y_1 : \alpha_1} \prod_{x_2, y_2 : \alpha_2} G_1(x_1, y_1) & \rightarrow G_2(x_2, y_2) \rightarrow f \ x_1 \ x_2 = f \ y_1 \ y_2
\end{align*}
\]

such that \( f^\text{cell} (h_1 \circ g_1) (h_2 \circ g_2) = f^\text{cell} (h_1 \ h_2 \circ f^\text{cell} g_1 \ g_2 \ \text{for all} \ g_i : G_i(x_i, y_i) \ \text{and} \ h_i : G_i(y_i, z_i)).\)

Then there is a function \( f^{\text{rec}} : \alpha_1/1 \rightarrow \alpha_2/1 \rightarrow \beta \rightarrow \beta \) with

\[
f^{\text{rec}} (\text{point} \ x_1) (\text{point} \ x_2) = f \ x_1 \ x_2 \ \text{and} \ \text{ap}_{f^{\text{rec}}} \ \text{cell} \ \text{cell} \ g_1 \ \text{cell} \ g_2 = f^\text{cell} \ g_1 \ g_2
\]

for all \( x_1, y_1 : \alpha_1, x_2, y_2 : \alpha_2, g_1 : G_1(x_1, y_1), \) and \( g_2 : G_2(x_2, y_2), \) where

\[
\text{ap}_{f^{\text{rec}}} : \prod_{f : A \rightarrow B \rightarrow C} \prod_{a_1 : A} \prod_{a_2 : A} \prod_{b_1 : B} \prod_{b_2 : B} a_1 = a_2 \rightarrow b_1 = b_2 \rightarrow f \ a_1 \ b_1 = f \ a_2 \ b_2.
\]

Thus, to define \( \otimes \), it suffices to define how it acts on points and cells, and then show that it is bilinear. We define point \( \alpha_1 \otimes \text{point} \ \alpha_2 \equiv \text{point} (\alpha_1 \times \alpha_2) \). If \( U : \text{UMatrix}(\alpha, \alpha') \) and \( V : \text{UMatrix}(\beta, \beta') \) then we have \( \text{cell} (U \times V) : \text{point} (\alpha \times \beta) = \text{point} (\alpha' \times \beta') \). The remaining condition is that

\[
\text{cell} (U_2 \otimes V_2) \circ \text{cell} (U_1 \otimes V_1) = \text{cell} ((U_2 \circ U_1) \otimes (V_2 \circ V_1)),
\]

which follows from the fact of linear algebra that \( (U_2 \circ U_1) \otimes (V_2 \circ V_1) = (U_2 \otimes V_2) \circ (U_1 \otimes V_1) \).

For \( U : \sigma = \tau \) and \( U' \ : \sigma' = \tau' \), we lift the tensor product to \( U \otimes U' \equiv \text{ap}_{\otimes} (U, U') : \sigma \otimes \sigma' = \tau \otimes \tau' \). The computation principle states that \( \text{cell} \ U \otimes \text{cell} \ U' = \text{cell} (U \otimes U') \).

A similar argument is used to define \( \oplus \).

### 4.3 Deriving the groupoid axioms

The fact that unitaries are paths means that Figure 5’s groupoid axioms can be derived for free.

**Proposition 8 (U-COMPOSE).** Let \( V : \sigma = \tau \) and \( U : \tau = \rho \). Then \( U \# (V \# e) = (U \circ V) \# e \).

**Proof.** By path induction on \( V \). Since \( 1 \# e \equiv e \) and \( U \circ 1 = U \), the equation reduces to \( U \# e = U \# e \). \( \Box \)

**Proposition 9 (U-I).** If \( \sigma : \text{QExp} \Gamma \sigma \) then \( \text{cell} \ I = e \).

**Proof.** Follows from Proposition 5 which states that \( \text{cell} \mathbf{I} = \mathbf{I} \). \( \Box \)

**Proposition 10 (U-†).** If \( U : \sigma = \tau \) and \( e : \text{QExp} \Gamma \sigma \) then \( U^\dagger \# U \# e = e \).

**Proof.** Follows from Proposition 8 and the fact that, as matrices, \( U^\dagger \circ U = \mathbf{I} \). \( \Box \)
4.4 Deriving the structural axioms

With one exception, the structural axioms from Figure 4 are trivial by path induction, and we omit their proofs here. The exception is the behavior of \( U \text{-LOWER-ELIM} \): when a qubit is measured but the result is not relevant to the rest of the computation.

**Proposition 11.** If \( e : \text{QExp} \Gamma \text{ Qubit and } U : \text{Qubit} = \text{Qubit} \), then

\[
\text{let } ! := \text{meas}(U \# e) \text{ in } e' \approx \text{let } ! := \text{meas } e \text{ in } e'.
\]

**Proof.** It is not possible to do induction on \( U \) here, since its endpoints are both fixed. However, Proposition 11 follows from the \( \eta \) rule for the unit type: for any two terms \( e_1, e_2 : \text{QExp} \Gamma (\text{Lower }()) \), \( e_1 \sim_{\eta} e_2 \):

\[
\text{let } ! := \text{meas}(U \# e) \text{ in } e' \sim_{\eta} \text{let } ! := \text{meas}(U \# e) \text{ in } \text{put } () \text{ in } e'
\]

\[
\sim_{\eta} \text{let } ! := \text{meas } e \text{ in } \text{put } () \text{ in } e'
\]

5 Equivalence of unitaries

This section establishes the soundness of the unitary equivalences shown in Figure 6. First, the “not” unitary \( X \):

**Proposition 12** (X-INTRO and X-ELIM).

\[
\text{cell } X \# \text{put } b = \text{put } (-b) \quad \text{and} \quad (\text{cell } X \# e) > ! f = e > ! \lambda b . f (-b)
\]

The proof of this proposition relies on the following two lemmas, both proved by path induction:

**Lemma 13.** For any \( f : \alpha = \beta \) and \( a : \alpha \):

\[
\text{ap}_{\text{point}} f \# \text{put } a = \text{put } (\text{coerce } f a) \quad \text{and} \quad (\text{ap}_{\text{point}} f \# e) > ! g = e > ! \lambda x . g (\text{coerce } f x).
\]

**Lemma 14.** If \( U : \text{UMatrix}(\alpha_1, \alpha_2) \) and \( H : \alpha_2 = \alpha_3 \), then

\[
\text{cell}(\text{transport } H U) = \text{ap}_{\text{point}} H \circ \text{cell } U.
\]

**Proof of Proposition 12.** Instantiating Lemma 13 with \( (\text{univ }-) \), it suffices to check that \( \text{cell } X = \text{ap}_{\text{point}} (\text{univ }-) \). Notice that, as matrices, \( X = \text{transport}_{\text{UMatrix(Bool, -)}} (\text{univ }-) I \). Then

\[
\text{cell}(\text{transport}_{\text{UMatrix(Bool, -)}} (\text{univ }-) I) = \text{ap}_{\text{point}} (\text{univ }-) \circ \text{cell } I \quad \text{(Lemma 14)}
\]

\[
= \text{ap}_{\text{point}} (\text{univ }-) \quad \text{(Proposition 5)}
\]

This technique does not extend to polymorphic equivalences such as \( \text{swap} : \prod \alpha \beta . \alpha \times \beta = \beta \times \alpha \). Lemma 13 tells us how \( \text{SWAP} \equiv \text{ap}_{\text{point}} \text{swap} \) behaves on classical states: \( \text{SWAP} \# \text{put } (a, b) = \text{put } (b, a) \).

But Equation SWAP-INTRO is an even stronger statement: that \( \text{SWAP} \# (e_1, e_2) \sim_q (e_2, e_1) \) for any \( e_1 \) and \( e_2 \). Similarly, the elimination form of Lemma 13 tells us that measuring both components of \( \text{SWAP} \# e \), where \( e \) is a pair of qubits, is the same as measuring \( e \) and then swapping its arguments. However, Equation SWAP-ELIM doesn’t measure both qubits; it only eliminates the pair:

\[
\text{let } (x, y) := \text{SWAP } e \text{ in } e' \sim_q \text{let } (x, y) := e \text{ in } e'.
\]

We can think of SWAP’s behavior as acting on a state whose structure is only partially known, corresponding to the polymorphism of its underlying function \( \text{swap} \). Our solution is to define a sort of partial initialization and partial measurement that generalizes this notion for \( \text{swap} \) and other polymorphic paths.
\[ X^m \equiv mX \]
\[ [\text{Lower } \alpha]^m \equiv \alpha \]
\[ [\sigma_1 \otimes \sigma_2]^m \equiv [\sigma_1]^m \times [\sigma_2]^m \]
\[ [\sigma_1 \oplus \sigma_2]^m \equiv [\sigma_1]^m + [\sigma_2]^m \]

\[ \gamma_\text{Lower}^m (a : \alpha) \equiv 0 \]
\[ \gamma_\text{Lower}^m \ x \equiv x \]
\[ \text{init}_\text{Lower}^m \ a \equiv \text{put } a \]

\[ \gamma_\sigma^m (b_1, b_2) \equiv \gamma_\sigma^m (b_1) \iff \gamma_\sigma^m (b_2) \]
\[ \text{init}_\sigma^m (\text{init}_{\sigma_1}^m (b_1), \text{init}_{\sigma_2}^m (b_2)) \]
\[ \text{init}_{\sigma_1 \otimes \sigma_2}^m (\text{init}_{\sigma_1}^m (b_1), \text{init}_{\sigma_2}^m (b_2)) \]
\[ \text{init}_{\sigma_1 \otimes \sigma_2}^m (\text{init}_{\sigma_1}^m (b_1), \text{init}_{\sigma_2}^m (b_2)) \]
\[ \text{init}_{\sigma_1 \otimes \sigma_2}^m (\text{init}_{\sigma_1}^m (b_1), \text{init}_{\sigma_2}^m (b_2)) \]

\[ \text{match}_\text{X} e \text{ with } bs \equiv (bs \times \{e/x\}) \quad \text{where } x \text{ is fresh} \]
\[ \text{match}_{\text{Lower } \alpha} e \text{ with } bs \equiv e > ! bs \]
\[ \text{match}_{\sigma_1 \otimes \sigma_2} e \text{ with } bs \equiv \text{let } (x_1, x_2) := e \text{ in } \text{match}_{\sigma_1} x_1 \text{ with } \lambda b_1. \text{match}_{\sigma_2} x_2 \text{ with } \lambda b_2. bs(b_1, b_2) \]
\[ \text{match}_{\sigma_1 \oplus \sigma_2} e \text{ with } bs \equiv \text{case } e \text{ of } \begin{cases} \text{i}_0 x_0 \rightarrow \text{match}_{\sigma_0} x_0 \text{ with } \lambda b_0. bs(\text{init}_{\sigma_0}^m b_0) \\ \text{i}_1 x_1 \rightarrow \text{match}_{\sigma_1} x_1 \text{ with } \lambda b_1. bs(\text{init}_{\sigma_1}^m b_1) \end{cases} \]

Figure 8: Operations on open quantum types

\section{Partial initialization and measurement}

Consider quantum types with the addition of type variables \( X : \text{TVar} \):

\[ \sigma ::= X \mid \text{Lower } \alpha \mid \sigma_1 \otimes \sigma_2 \mid \sigma_1 \oplus \sigma_2. \]

We call these \textit{open quantum types}. Given a map \( m : \text{TVar} \rightarrow \text{Type} \), we can define a basis set corresponding to \( \sigma \), written \([\sigma]^m\), as shown in Figure 8.

Let \( m : \text{TVar} \rightarrow \text{Type} \) and let \( \text{Var} \) be the constant map \( \lambda \_ \text{Var} \). Then every \( b : [\sigma]^\text{Var} \) gives rise to a typing context \( \gamma_\sigma^m(b) \) as well as a term using these variables: if \( \Gamma = \gamma_\sigma^m(b) \) then \( \Gamma \vdash \text{init}_\sigma^m b : \text{point } [\sigma]^m \) is called \textit{partial initialization}, as defined in Figure 8.

Open quantum types also dictate how to eliminate terms of type \text{point } [\sigma]^m \), called \textit{partial measurement}:

\[ \Gamma \vdash e : \text{point } [\sigma]^m \quad \text{bs} : \prod_{b : [\sigma]^{\text{Var}}} \gamma_\sigma^m(b), \Gamma' \vdash - : \tau \]

\[ \Gamma, \Gamma' \vdash \text{match}_\sigma e \text{ with } bs : \tau \]

A unitary equivalence \( \sigma \leftrightarrow \tau \) of open quantum types is a proof that \([\sigma]^m \equiv [\tau]^m \) for every \( m \). For example, the equivalence \( X \otimes Y \leftrightarrow Y \otimes X \) is given by \( \lambda m. \lambda(x,y). (y,x) \).

\[ \sigma \leftrightarrow \tau \equiv \prod_m [\sigma]^m \equiv [\tau]^m. \]

\textbf{Lemma 15.} If \( f : \sigma \leftrightarrow \tau \) then for every \( b : [\sigma]^{\text{Var}} \) there is a path \( \gamma_\tau^m(fb) = \gamma_\sigma^m(b) \).

\section{Proof of Lemma 15}

The proof of Lemma 15 depends on the observation that open type equivalence \( \sigma \leftrightarrow \tau \) is equivalent to the inductively defined relation \( \sigma \equiv \tau \) presented in Figure 9. It is easy to check that every proof \( f : \sigma \equiv \tau \) corresponds to an equivalence \( \hat{f} : \sigma \leftrightarrow \tau \), and it is also easy to check that Lemma 15 follows for inductively-generated equivalences \( f : \sigma \equiv \tau \).
Figure 9: Inductive presentation of open type equivalence. The relation $\sigma \simeq \tau$ is an Abelian rig—a ring without negation—where $\text{Lower}$ is a map from finite types to open quantum types that respects both addition and multiplication.

**Lemma 16.** If $f : \sigma \simeq \tau$ and $b : [\sigma]^{\text{Var}}$, then $\gamma^\sigma(fb) = \gamma^\tau(b)$.

**Proof.** By induction on $f$. □

To complete the proof of Lemma 15 we need to show that $\sigma \iff \tau$ implies $\sigma \simeq \tau$. We show:

1. Every open quantum type $\sigma$ corresponds to one in a normal form $N_\sigma$ such that $\sigma \simeq N_\sigma$.
2. If $N_\sigma \iff N_\tau$ then $N_\sigma \simeq N_\tau$.

So, if $\sigma \iff \tau$ then by (1) it is the case that $\sigma \simeq N_\sigma$ and $\tau \simeq N_\tau$. This implies $\sigma \iff N_\sigma$ and $\tau \iff N_\tau$, and so $N_\sigma \iff \sigma \iff \tau \iff N_\tau$. By (2) we can conclude $\sigma \simeq N_\sigma \simeq N_\tau \simeq \tau$.

Normal quantum types $N$ have the following structure:

$$(\text{Lower } \alpha_1 \otimes X^i_1 \otimes \cdots \otimes X^i_n) \oplus \cdots \oplus (\text{Lower } \alpha_m \otimes X^m_1 \otimes \cdots \otimes X^m_m)$$

**Proposition 17.** For every $\sigma$, there is a normal quantum type $N_\sigma$ such that $\sigma \simeq N_\sigma$.

**Proof.** First we define $N_\sigma$ by induction on $\sigma$.

$$
N_{\sigma \otimes \tau} \equiv \bigoplus_{1 \leq i \leq n, 1 \leq j \leq m} \text{Lower}(\alpha_i \times \beta_j) \otimes \tilde{X}^i \otimes \tilde{Y}^j
$$

where $N_\sigma \equiv \bigoplus_{1 \leq i \leq n} \text{Lower } \alpha_i \otimes \tilde{X}^i$

and $N_\tau \equiv \bigoplus_{1 \leq j \leq m} \text{Lower } \beta_j \otimes \tilde{Y}^j$

To complete the proof we check that $N_{\sigma \otimes \tau} \simeq N_\sigma \otimes N_\tau$, which follows from distributivity of $\otimes$ over $\oplus$. □
Now, let \( f : N \cong N' \) where \( N \equiv \bigoplus_{1 \leq i \leq n} (\text{Lower } \alpha_i \otimes \bar{X}_i) \) and \( N' \equiv \bigoplus_{1 \leq j \leq n'} (\text{Lower } \beta_j \otimes \bar{Y}_j) \), where each \( \bar{X}_i \) and \( \bar{Y}_j \) are \( \otimes \)-separated sequences of type variables. That means \( f \) has the form

\[
f : \Pi_{m \cdot \text{TVar} \rightarrow \text{Type}} \Sigma_{i : \mathbb{N}_n} \alpha_i \times m(\bar{X}_i) \cong \Sigma_{j : \mathbb{N}_{n'}} \beta_j \times m(\bar{Y}_j).
\]

Let \( \mathcal{R}_f \subseteq \mathcal{P}(\mathbb{N}_n \times \mathbb{N}_{n'}) \) be a relation defined as follows:

\[
(i, j) \in \mathcal{R}_f \leftrightarrow \sum_{a : \alpha_i, b : \beta_j} f((\lambda (i, j)) = (j, b).
\]

That is, \( f_{\lambda (i, j)} \) has type \( \Sigma i, \alpha_i \equiv \Sigma j, \beta_j \) and \( (i, j) \in \mathcal{R}_f \) says there is some \( a : \alpha_i \) that \( f \) maps to some \( b : \beta_j \).

Importantly, this implies a broader property by a parametricity argument:

**Proposition 18.** For any \( m_1 \) and \( m_2 \) of type \( \text{TVar} \rightarrow \text{Type} \), and for \( a : \alpha_i, x_1 : m_1(\bar{X}_i), \) and \( x_2 : m_2(\bar{X}_j), \)

\[
\pi_1(f_{m_1}(i, a, x_1)) = \pi_1(f_{m_2}(i, a, x_2)) \quad \text{and} \quad \pi_2(f_{m_1}(i, a, x_1)) = \pi_2(f_{m_2}(i, a, x_2)).
\]

**Proof.** Follows from the abstraction theorem by Uemura [35]. \( \square \)

**Lemma 19.** If \( (i, j) \in \mathcal{R}_f \) then \( \bar{X}_i \cong \bar{Y}_j \).

**Proof.** First, observe that \( \bar{X}_i \cong \bar{Y}_j \). For a fixed \( m \), let \( x : m(\bar{X}_i) \). Now, take \( a \) to be the element of \( \alpha_i \) witnessed by \( (i, j) \in \mathcal{R}_f \). Then by Proposition 18 we know that there exists some (unique) \( b : \beta_j \) and \( y : m(\bar{Y}_j) \) such that \( f_m(i, a, x) = (j, b, y) \). The map \( x \mapsto y \) is in fact an equivalence.

It is easy to see, then, that \( \bar{X}_i \cong \bar{Y}_j \), by induction on the sizes of \( \bar{X}_i \) and \( \bar{Y}_j \). \( \square \)

Finally, we can prove the main property of this section.

**Proof of Lemma 15.** The proof is by induction on \( n + n' \). We consider five cases: either \( \mathcal{R}_f \) is an isomorphism, or it is either not functional, not well-defined on all input, not injective, or not surjective. Since \( \mathcal{R}_f \) is finite, this property is decidable.

1. Suppose \( \mathcal{R}_f \) is an isomorphism. Then, observe that whenever \( (i, j) \in \mathcal{R}_f \), we have \( \alpha_i \equiv \beta_j \). This isomorphism is witnessed by the map \( a \mapsto \pi_2(f_{\lambda (i, j)}) \); since \( \mathcal{R}_f \) is injective, we can be sure that this value is in \( \beta_j \). Then, applying this fact as well as Lemma 19 we have that

\[
N = \bigoplus_{1 \leq i \leq n} (\text{Lower } \alpha_i \otimes \bar{X}_i) \equiv \bigoplus_{1 \leq i \leq n} (\text{Lower } \beta_{f(i)} \otimes \bar{X}_i)
\]

\[
\cong \bigoplus_{1 \leq i \leq n} (\text{Lower } \beta_{f(i)} \otimes \bar{Y}_{f(i)}) \equiv N'.
\]

2. Suppose \( \mathcal{R}_f \) is not functional, meaning that there exits some \( (i, j_1) \in \mathcal{R}_f \) and \( (i, j_2) \in \mathcal{R}_f \) with \( j_1 \neq j_2 \). We know \( \bar{Y}_{j_1} \cong \bar{X}_i \cong \bar{Y}_{j_2} \) by Lemma 19, so we have that

\[
N' \equiv (\text{Lower } \beta_{j_1} \otimes \bar{Y}_{j_1}) \oplus (\text{Lower } \beta_{j_2} \otimes \bar{Y}_{j_2}) \oplus \bigoplus_{j \neq j_1, j_2} (\text{Lower } \beta_j \otimes \bar{Y}_j)
\]

\[
\cong (\text{Lower } \beta_{j_1} \otimes \bar{Y}_{j_1}) \oplus \bigoplus_{j \neq j_1, j_2} (\text{Lower } \beta_j \otimes \bar{Y}_j).
\]

Call this new normal type \( N'' \). We still have \( N'' \cong N \), but the number of clauses of \( N'' \) is smaller than that of \( N \), so we can invoke the induction hypothesis to show \( N'' \equiv N \). By transitivity, \( N \equiv N' \).
3. If $R_f$ is not injective, we invoke a similar argument to the case that $R_f$ is not functional by reducing the number of clauses of $N$ instead of $N'$.

4. Suppose $R_f$ is not well-defined on its domain, meaning that there is some $i_0$ not in the domain of $R_f$. Observe first that $\alpha_{i_0}$ must be equal to the empty type, Void. If not, then there is some $a : \alpha_{i_0}$, and let $f = \pi_1(f(i_0,a))$; we have $(i_0, j) \in R_f$, a contradiction. Thus

$$N \simeq (\text{Lower } \alpha_{i_0} \otimes \bar{X}_{i_0}) \oplus \bigoplus_{i \neq i_0} (\text{Lower } \alpha_i \otimes \bar{X}_i) \simeq (\text{Lower Void } \otimes \bar{X}_{i_0}) \oplus \bigoplus_{i \neq i_0} (\text{Lower } \alpha_i \otimes \bar{X}_i)$$

Again, call this new type $N''$. By the induction hypothesis, $N'' \simeq N'$, and by transitivity $N \simeq N'$.

5. If $R_f$ is not surjective, the proof follows parallel to the case that $R_f$ is not well-defined.

\[ \square \]

5.3 Axioms of partial initialization and measurement.

Having established Lemma 15, we can finally complete the equational theory for our quantum language by defining two axioms, written $\sim_q$, about the behavior of partial initialization and partial measurement.

For $f : \sigma \leftrightarrow \tau$ and $m : \text{TVar} \rightarrow \text{Type}$, let us write $[f]^m : \text{point } [\sigma]^m = \text{point } [\tau]^m$ for $\text{ap}_{\text{point}}(\text{univ } f_m)$.

**Axiom 20.** Let $f : \sigma \leftrightarrow \tau$ and $b : [\sigma]^\text{Var}$, and let $e : \text{QExp } \Gamma (\text{point } [\sigma]^m)$ and $bs : \prod_{y' \in \text{Var}} \text{QExp } (\Gamma', y'^m(b'))(\tau)$. Then

$$[f]^m \# \text{init}^m_\sigma b \sim_q \text{init}^m_\tau (f_{\text{Var}} b) \quad \text{(U-INTRO)}$$

$$\text{match}_\tau [f]^m \# e \text{ with } bs \sim_q \text{match}_\sigma e \text{ with } (bs \circ f_{\text{Var}}) \quad \text{(U-ELIM)}$$

**Definition 21.** Define the relation $e_1 \equiv_q e_2$ on expressions as $\equiv_q \equiv \equiv_{\alpha} \cup \equiv_{\beta} \cup \equiv_\eta \cup \equiv_{\text{cc}} \cup \equiv_q$. We write $e_1 \equiv e_2$ for equality modulo $\equiv_q$, i.e., the type $[e_1]_{\equiv_q} = [e_2]_{\equiv_q}$.

5.4 Instances of equational axioms

**Proposition 22** (SWAP-INTRO and SWAP-ELIM). Let SWAP be the unitary $[\text{swap}]^m$, where swap is the equivalence $\lambda(x,y) . (y,x)$ of type $X \otimes Y \leftrightarrow Y \otimes X$. Then

$$\text{SWAP } # (e_1, e_2) \equiv (e_2, e_1) \quad \text{(SWAP-INTRO)}$$

$$\text{let } (y,x) := \text{SWAP } # e \text{ in } e' \equiv \text{let } (x,y) := e \text{ in } e' \quad \text{(SWAP-ELIM)}$$

**Proof.** For the introduction rule, it suffices to show that $\text{SWAP } # (x,y) \equiv (y,x)$ for free variables $x$ and $y$.

$$\text{SWAP } # (x,y) \equiv \text{SWAP } # \text{init}_{X \otimes Y} (x,y) \sim_q \text{init}_{Y \otimes X} (\text{swap}(x,y)) \quad \text{(U-INTRO)}$$

Elimination is similarly straightforward from Axiom U-ELIM.

$$\text{let } (y,x) := \text{SWAP } # e \text{ in } e' \equiv \text{match}_{X \otimes Y} (\text{SWAP } # e) \text{ with } \lambda(y,x) . e'$$

$$\sim_q \text{match}_{Y \otimes X} e \text{ with } \left( \lambda(y,x) . e' \circ \text{swap} \right)$$

$$\equiv \text{match}_{X \otimes Y} e \text{ with } \lambda(x,y) . e' \quad \equiv \text{let } (x,y) := e \text{ in } e' \quad \square$$
Proposition 23. Let $\text{CNOT}$ be the unitary $\rho_{\text{point}} \text{cnot}$, where $\text{cnot}$ is the equivalence

$$\lambda(b, b'). (b, \text{if } b \text{ then } \neg b' \text{ else } b')$$

of type $\text{Bool} \times \text{Bool} \cong \text{Bool} \times \text{Bool}$. Then:

$$\text{CNOT}(\text{put } b, e) \approx (\text{put } b, \text{if } b \text{ then } X \# e \text{ else } e) \quad \text{(CNOT-INTRO)}$$

let $(!_y, _y) := \text{CNOT}(e)$ in $e' \approx \text{let } (!b, y') := e \text{ in } b \text{ if } b \text{ then } e'\{X \# y'/y\} \text{ else } e'\{y'/y\} \quad \text{(CNOT-ELIM)}$

Proof. Let $\text{DISTR}$ be the unitary $[\text{distr}]^m$ where $\text{distr} : \text{Lower Bool} \otimes X \leftrightarrow X \oplus X$ is defined by

$$\lambda(b, x). \text{if } b \text{ then } \text{inr } x \text{ else } \text{inl } x.$$

From Equation U-INTRO we can derive that for booleans $b$ and expressions $e$, we have

$$\text{DISTR}(\text{put } b, e) \approx_q \text{if } b \text{ then } t_1 e \text{ else } t_0 e \quad \text{and} \quad \text{DISTR}^{-1}(t_0 e) \approx_q (\text{put } \text{false}, e)$$

$$\text{DISTR}^{-1}(t_1 e) \approx_q (\text{put } \text{true}, e)$$

As a matrix, $\text{CNOT}$ is equal to $\text{DISTR}^{-1} \circ (I \oplus X) \circ \text{DISTR}$. Thus,

$$\text{CNOT}(\text{put } b, e) = \text{DISTR}^{-1}(I \oplus X) \text{DISTR}(\text{put } b, e) \quad \text{(U-COMPOSE)}$$

$$\approx \text{DISTR}^{-1}(I \oplus X) \text{if } b \text{ then } t_1 e \text{ else } t_0 e \quad \text{(U-INTRO)}$$

$$= \text{if } b \text{ then } (\text{DISTR}^{-1}(I \oplus X) t_1 e) \text{ else } (\text{DISTR}^{-1}(I \oplus X) t_0 e) \quad \text{(U-\oplus-INTRO)}$$

$$\approx \text{if } b \text{ then } (\text{DISTR}^{-1}(I \oplus X) (\text{DISTR}^{-1} t_0(I \# e))) \quad \text{(U-\oplus-INTRO)}$$

$$= (\text{put } b, \text{if } b \text{ then } X \# e \text{ else } e) \quad \text{(U-\oplus-INTRO)}$$

There is a similar argument for the elimination form. From Equation U-ELIM we can derive that

$$\text{case } \text{DISTR}(e) \text{ of } (t_0y \rightarrow e_0 | t_1y \rightarrow e_1) \approx_q \text{let } (!b, y) := e \text{ in } b \text{ if } b \text{ then } e_1 \text{ else } e_0$$

$$\text{let } (x, y) := \text{DISTR}^{-1}(e) \text{ in } x >! f \approx_q \text{case } e \text{ of } (t_0y \rightarrow f(\text{false}) | t_1y \rightarrow f(\text{true}))$$

Then:

$$\text{let } (!_y, _y) := \text{CNOT}(e) \text{ in } e' = \text{let } (!_y, _y) := \text{DISTR}^{-1}(I \oplus X) \text{DISTR}(e) \text{ in } e' \quad \text{(U-COMPOSE)}$$

$$\approx \text{case}(I \oplus X \text{DISTR}(e) \text{ of } (t_0y \rightarrow e' | t_1y \rightarrow e')) \quad \text{(U-ELIM)}$$

$$= \text{case}(\text{DISTR}(e) \text{ of } (t_0y' \rightarrow e'\{I \# y'/y\} | t_1y' \rightarrow e'\{X \# y'/y\})) \quad \text{(U-\oplus-ELIM)}$$

$$\approx \text{let } (!b, y) := e \text{ in } b \text{ if } b \text{ then } e'\{X \# y'/y\} \text{ else } e'\{y'/y\} \quad \text{(U-ELIM)} \quad \square$$

6 Soundness

In this section we give a denotational semantics for the quantum term calculus with respect to superoperators over density matrices [21, Chapter 2]. A density matrix is a real-valued square matrix whose trace sums to one; every density matrix has the form $\rho \equiv \sum_j p_j |\varphi_j\rangle \langle \varphi_j|$, where each $|\varphi_j\rangle$ is a pure state vector.
and the coefficients $p_j$ sum to one. A superoperator is a completely positive map over density matrices that does not increase the trace of its input.

For a quantum type $\sigma : \text{QType}$, we define the type $\text{Density} \sigma$ of density matrices of type $\sigma$ by quotient induction. First, define $\text{Density}(\text{point} \; \alpha)$ to be the collection of density matrices of type $\text{Matrix}(\alpha, \alpha)$. Next, we must show that for any unitary $U : \text{UMatrix}(\alpha, \beta)$, we have $\text{Density}(\text{point} \; \alpha) = \text{Density}(\text{point} \; \beta)$. This path can be obtained through univalence from the equivalence $U^* \equiv \lambda \rho. U \rho U^\dagger$. The function $U^*$ is a superoperator when $U$ is unitary, and it is invertible via the function $(U^\dagger)^*$.

A reader might be concerned that because we defined $\text{Density} \tau$ by quotient induction, we have inadvertently collapsed all two density matrices $\rho_1, \rho_2$ such that $\rho_2 = U^* \rho_1$. This is not true. Just because $\rho : \text{Density} \sigma$ and $\text{Density} \sigma = \text{Density} \sigma$ does not mean that there is a path $\rho = U \rho U^\dagger$. In particular, notice that $\text{Density} \text{Qubit}$ is just a set of $2 \times 2$ matrices.

The definition of density matrices can be extended to typing contexts by interpreting a context $\Gamma \equiv x_1 : \sigma_1, \ldots, x_n : \sigma_n$ as a quantum type $\sigma_1 \otimes \cdots \otimes \sigma_n$. That is, we write $\text{Density} \Gamma$ for $\text{Density}(\sigma_1 \otimes \cdots \otimes \sigma_n)$.

The category of superoperators over density matrices is dagger compact closed [29], has sums (given by the direct product), and has a terminal object $(\text{Density}(\text{Lower}()))$. We sketch some of its properties:

- For superoperators $f, g : \text{Density} \sigma \rightarrow \text{Density} \tau$, pointwise addition $f + g$ is $(f + g) \rho \equiv f \rho + g \rho$.
- Superoperators are strict symmetric monoidal, which means that for $f : \text{Density} \sigma \rightarrow \text{Density} \tau$ and $g : \text{Density} \sigma' \rightarrow \text{Density} \tau'$, we can define $f \otimes f' : \text{Density}(\sigma \otimes \sigma') \rightarrow \text{Density}(\tau \otimes \tau')$. The unit of $\otimes$ is $\text{Density} \text{Lower}()$. The strictness condition means that $\text{Density}(\sigma_1 \otimes (\sigma_2 \otimes \sigma_3))$ is equal to $\text{Density}((\sigma_1 \otimes \sigma_2) \otimes \sigma_3)$, and that $\text{Density} \text{Lower}() \otimes \sigma$ is equal to $\text{Density} \sigma$.
- $\text{Density}(\text{Lower}())$ is a terminal element, meaning that for every $\sigma$ there is a unique map $! : \text{Density} \sigma \rightarrow \text{Density}(\text{Lower}())$ that takes the trace of its input matrix.
- For every $\sigma_1$ and $\sigma_2$ there are maps $t_i^\sigma : \text{Density} \sigma_i \rightarrow \text{Density}(\sigma_1 \oplus \sigma_2)$ defined as $\lambda \rho. t_i \rho t_i^\dagger$, where $t_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $t_2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ respectively. Given $f_1 : \text{Density} \sigma_1 \rightarrow \text{Density} \tau$ and $f_2 : \text{Density} \sigma_2 \rightarrow \text{Density} \tau$, there is a unique function $[f_1, f_2] : \text{Density}(\sigma_1 \oplus \sigma_2) \rightarrow \text{Density} \tau$ that commutes with $t_i^\sigma$ in the usual way, defined as $[f_1, f_2] \rho \equiv f_1^\sigma t_i^\sigma \rho + f_2^\sigma t_i^\sigma \rho$.
- Generalizing the notion of sum, if $a_i$ is the $i$th element of a finite type $\alpha$, there is a matrix $\delta_{i\alpha} : \text{Density}(\text{point} \alpha)$ characterized by a value 1 at index $(i, i)$ and 0 elsewhere. For any function $f : \alpha \rightarrow \text{Density} \tau$, there is a map $[f] : \text{Density}(\text{point} \alpha) \rightarrow \text{Density} \tau$ defined by $[f] \rho \equiv \sum_{i\alpha} \delta_{i\alpha} \rho$.

### 6.1 Semantics of quantum expressions

The denotational semantics of the quantum term language maps expressions $e : \text{QExp} \Gamma \sigma$ to a superoperator $[e] : \text{Density} \Gamma \rightarrow \text{Density} \varrho$ between density matrices. We define $[e]$ by induction on $e$.

- For a variable $x : \sigma \vdash x : \sigma$, $[x]$ is the identity function.
- For a let binding $\Gamma, \Gamma' \vdash \text{let} \; x := e \; \text{in} \; e' : \tau$ where $\Gamma \vdash e : \sigma$ and $x : \sigma, \Gamma' \vdash e' : \tau$, define $[\text{let} \; x := e \; \text{in} \; e']$ as $[e'] \circ ([e] \otimes \text{id})$.
- The category of density matrices is symmetric monoidal, which tells us how to interpret the multiplicative unit and product. That is, given $\Gamma_1 \vdash e_1 : \sigma_1$ and $\Gamma_2 \vdash e_2 : \sigma_2$, define $[\Gamma_1, \Gamma_2 \vdash (e_1, e_2) : \sigma_1 \otimes \sigma_2]$ as $[e_1] \otimes [e_2]$.
- Given $\Gamma, \Gamma' \vdash \text{let} \; (x_1, x_2) := e \; \text{in} \; e' : \tau$ where $\Gamma \vdash e : \sigma_1 \otimes \sigma_2$ and $x_1 : \sigma_1, x_2 : \sigma_2, \Gamma' \vdash e' : \tau$, define $[\text{let} \; (x_1, x_2) := e \; \text{in} \; e']$ as $[e'] \circ ([e] \otimes \text{id})$. 

- Given $\Gamma \vdash e : \sigma$, define $[\Gamma \vdash t_e : \sigma_1 \oplus \sigma_2]$ as $[\Gamma, e] = [\Gamma, t_1, e_1] = [\Gamma, t_2, e_2]$.
- Given $\Gamma \vdash e : \sigma_1 \oplus \sigma_2$ and $x : \sigma, \Gamma, e : \tau$, define $[\Gamma, \Gamma' \vdash e \triangleq (t_1x_1 \rightarrow e_1 \mid t_2x_2 \rightarrow e_2) : \tau]$ as $[[[\Gamma, e_1]], [[\Gamma, e_2]]] \odot \text{DISTR} \circ ([\Gamma] \otimes \text{id})$, where $\text{DISTR} : \text{UMatrix}((\alpha_1 + \alpha_2) \times \beta, (\alpha_1 \times \beta) + (\alpha_2 \times \beta))$.
- Define $\emptyset \vdash \text{put} a : \text{Lower } \alpha$ as $\lambda_\alpha \delta_a$.
- Given $\Gamma \vdash e : \text{Lower } \alpha$ and $f : \alpha \rightarrow \Gamma' \vdash - : \tau$, define $[\Gamma, \Gamma' \vdash e \triangleright f : \tau]$ as $\sum_{a : \alpha} [\Gamma, f a] \circ ([\Gamma] \otimes \text{id})$.

### 6.2 Soundness of the equational theory

The soundness of $\beta, \eta$, and commuting conversion equivalences comes down to the fact that the category of density matrices is symmetric monoidal, has sums, and has a terminal object.

**Theorem 24.** If $e_1 \sim_o e_2$ for $o \in \{\beta, \eta, \text{cc}\}$, then $[\Gamma, e_1] = [\Gamma, e_2]$.

For any $U : \sigma = \tau$, define $[U] : \text{Density } \sigma \rightarrow \text{Density } \tau$ trivially by path induction such that $[1] = \lambda_\lambda x.x$.

**Lemma 25.** If $U : \sigma = \tau$ and $\Gamma \vdash e : \sigma$, then $[\Gamma, U \# e] = [U] \circ [\Gamma, e]$.

We can now prove the soundness of the axioms regarding the behavior of unitary equivalences.

**Theorem 26** (Soundness of Axiom 20). Let $f : \sigma \leftrightarrow \tau$ and $b : [\sigma]_{\text{Var}}$. Then

$$[[\text{init}_\sigma f \# | b]] = [[\text{init}_\sigma f]_{\text{Var}}]$$

If $e : \text{QExp } \Delta$ (point $[\sigma]^m$) and $b \times \prod_{b : \tau} \text{QExp } (f^m_{\tau}(b), \Delta') q$, then

$$[[\text{match}_\tau (f^m \# e) \times b]] = [[\text{match}_\sigma e \times b]_{\text{Var}}]$$

**Proof.** Section 5.2 introduces an inductively-defined relation $\sigma \simeq \tau$ that holds exactly when $\sigma \sim \tau$. Thus, it suffices to prove this property with respect to $f : \sigma \simeq \tau$. First we check the properties with respect to reflexivity, symmetry, transitivity, and congruence.

Reflexivity and symmetry follow directly from Proposition 9 and Proposition 8 respectively, and congruence follows from the congruence of density matrices.

Next we check the behavior of the ten specific unitaries. Figure 10 lists the equations we expect to hold; these equalities follow from the properties of density matrices and by unfolding definitions.

**Theorem 27** (Soundness). If $e_1 \approx e_2$, then $[[\Gamma, e_1]] = [[\Gamma, e_2]]$.

### 7 Completeness

This section will draw a formal connection between the HoTT calculus defined in this paper and Staton’s algebraic presentation, which is shown in Figure 11.

The algebraic calculus is presented in continuation-passing style, with judgment $\Gamma \vdash \Delta \vdash t$ where $\Gamma$ and $\Delta$ are linear contexts in the sense of Section 3. The intuition is that $\Gamma$ contains continuation variables, to which quantum variables in $\Delta$ can be passed. In Staton’s original presentation, variables could hold only qubits, but here we allow variables to hold arbitrary tuples. However, we do restrict types that occur in
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\[
\begin{align*}
\text{SWAP} \# (e_1, e_2) & = [(e_2, e_1)] \\
\text{SWAP} \# t_1 e & = [t_1 e] \\
\text{ASSOC} \# (e_1, (e_2, e_3)) & = [[[e_1, e_2], e_3]] \\
\text{ASSOC} \# t_0 e & = [t_0 (t_0 e)] \\
\text{ASSOC} \# t_1 (t_0 e) & = [t_0 (t_1 e)] \\
\text{ASSOC} \# t_1 (t_1 e) & = [t_1 e] \\
\text{DISTR} \# (e_1, t_2) & = [t_1 (e_1, e_2)]
\end{align*}
\]

[let \((x_2, x_3) := \text{SWAP} \# e \text{ in } e'\] = [let \((x_1, x_2) := e \text{ in } e'\]

[case(SWAP \# e) of \((t_0 x_0 \rightarrow e_0 \mid t_1 x_1 \rightarrow e_1)\] = [case e of \((t_0 x_0 \rightarrow e_1 \mid t_1 x_0 \rightarrow e_0)\]

[let \((x_1, x_2, x_3) := \text{ASSOC} \# e \text{ in } e'\] = [let \((x_1, x_2, x_3) := e \text{ in } e'\]

[match\(x_1, (x_2, x_3) \text{ with } e\] = [match\(x_1, x_2, x_3 \text{ with } e\]

[let \(! (a, b) := \text{Lower} \# e \text{ in } e'\] = [let \(! (a, b) := e \text{ in } e'\]

[case(DISTR \# e) of \((t_0 x_0 \rightarrow e_0 \mid t_1 x_1 \rightarrow e_1)\] = [let \((x, y) := e \text{ in case } y \text{ of } \(t_0 y_0 \rightarrow e_0 \mid t_1 y_1 \rightarrow e_1)\]

[Lower \# e ! f ] = [case e of \((t_0 x_0 \rightarrow x_0 \Rightarrow f \circ \text{inl} \mid t_1 x_1 \rightarrow x_1 \Rightarrow f \circ \text{inr})\]

[let \(!(), x) := \text{lunit} \# e \text{ in } e'\] = [let \(x := e \text{ in } e'\]

[LZERO \# e ! f ] = [let \(! a := e \text{ in } f a\]

Figure 10: Proof that Axiom 20 is sound with respect to initialization and measurement. Note that the equations containing values \(a : \text{Void}\) are vacuously true.

\[
\begin{align*}
\Gamma, x : \sigma_1 \otimes \cdots \otimes \sigma_n, \Gamma' & \vdash a_1 : \sigma_1, \ldots, a_n : \sigma_n \vdash x(a_1, \ldots, a_n) \quad \text{VAR} \\
\Gamma & \vdash \Delta, a : \sigma_1 \otimes \sigma_2 \vdash t \quad \otimes
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \Delta, a : \text{Qubit} \vdash t \\
\Gamma & \vdash \Delta \vdash \text{new}(a,t) \quad \text{NEW}
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \Delta \vdash u \\
\Gamma & \vdash \Delta, a : \text{Qubit} \vdash \text{meas}(a,t,u) \quad \text{MEAS}
\end{align*}
\]

\[
\begin{align*}
U : \mathcal{H}(\sigma, \tau) & \quad \Gamma & \vdash \Delta, b : \tau \vdash t \\
\Gamma & \vdash \Delta, a : \sigma \vdash U(a,b,t) \quad U
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \Theta, a : \sigma \vdash u \\
\Gamma & \vdash \Delta, \Theta \vdash t[x(a) \mapsto u] \quad \text{SUBST}
\end{align*}
\]

Figure 11: Algebraic structure of quantum computation.
the following shorthand: that measuring a qubit set of "interesting" axioms correspond closely to those shown in Figure 6. For example, Equation A says

\[ \text{Equation A} \]

The second set of "administrative" axioms correspond to Figures 4 and 5.

Finally, we add three commuting conversion rules to our calculus that define the behavior of the \( \otimes \) rule, which were not explicit in Staton’s original presentation.

\[ \text{Equation M} \]

\[ \text{Equation N} \]

\[ \text{Equation O} \]
\[
x : \text{BExp}(x : \sigma) \quad \Rightarrow \quad y : \sigma \mid x : \sigma \vdash y(x)
\]

\[
e : \text{BExp} \Gamma \sigma \quad e' : \text{BExp}(\Gamma', x : \sigma) \tau
\]

\[
\text{let } x := e \text{ in } e' : \text{BExp}(\Gamma, \Gamma') \tau \quad \Rightarrow \quad y : \sigma \mid \Gamma \vdash (e)^y;
\]

\[
y : \sigma \mid \Gamma \vdash (e)^y \quad z : \tau \mid \Gamma', x : \sigma \vdash (e')^z;
\]

\[
e_1 : \text{BExp} \Gamma_1 \sigma_1 \quad e_2 : \text{BExp} \Gamma_2 \sigma_2
\]

\[
(\text{e}_1, \text{e}_2) : \text{BExp}(\Gamma_1, \Gamma_2)(\sigma_1 \otimes \sigma_2) \quad \Rightarrow \quad y_1 : \sigma_1 \mid \Gamma_1 \vdash (e_1)^y \quad y_2 : \sigma_2 \mid \Gamma_2 \vdash (e_2)^y
\]

\[
e : \text{BExp} \Gamma(\sigma_1 \otimes \sigma_2) \quad e' : \text{BExp}(\Gamma', x_1 : \sigma_1, x_2 : \sigma_2) \tau
\]

\[
\text{let } (x_1, x_2) := e \text{ in } e' : \text{BExp}(\Gamma, \Gamma') \tau \quad \Rightarrow \quad y : \sigma_1 \otimes \sigma_2 \mid \Gamma \vdash (e)^y \quad z : \tau \mid \Gamma', x_1 : \sigma_1, x_2 : \sigma_2 \vdash (e')^z
\]

\[
\text{put } b : \text{BExp } 0 (\text{Qubit}) \quad \Rightarrow \quad x : \text{Qubit} \mid 0 \vdash \text{new}(a.x(a)) \quad , 
\]

\[
\text{b} : \text{Bool} \quad b = \text{false} \quad x : \text{Qubit} \mid 0 \vdash \text{new}(a.X(a.x(a)))
\]

\[
e : \text{BExp} \Gamma(\text{Lower Bool}) \quad f : \text{Bool} \rightarrow \text{BExp} \Gamma' \tau
\]

\[
\quad x : \text{Lower Bool} \mid \Gamma \vdash (e) \quad \prod_b y : \tau \mid \Gamma \vdash (fb)^y
\]

\[
\Rightarrow \quad y : \tau \mid \Gamma, \Gamma' \vdash (e)(x(q) \rightarrow \text{meas}(q, f(false), f(true)))
\]

Figure 14: Encoding of the HoTT quantum calculus defined in this paper into Staton’s algebraic calculus.

7.1 Algebraic to HoTT calculus translation

Let \( x : \sigma \mid \Delta \vdash t \) be a term with exactly one continuation. Then we can define \( \langle t \rangle : \text{QExp} \Delta \sigma \) as follows:

\[
\langle x(a_1, \ldots, a_n) \rangle \equiv (a_1, \ldots, a_n)
\]

\[
\langle \text{new}(a.t) \rangle \equiv \text{let } a := \text{init} 0 \text{ in } (t)
\]

\[
\langle \text{meas}(a, t, u) \rangle \equiv \text{let } !x := a \text{ if } x \text{ then } \langle u \rangle \text{ else } \langle v \rangle
\]

Lemma 28. If \( U : \forall (\sigma, \tau) \text{ then } \langle U(a, b.t) \rangle \equiv \text{let } b := U \# a \text{ in } \langle t \rangle. \)

Proof. By path induction on \( U : \sigma = \tau. \)

Theorem 29. If \( t \approx u \text{ then } \langle t \rangle \approx \langle u \rangle. \)

Proof. Straightforward from the proofs in Sections 4 and 5.

7.2 HoTT to algebraic calculus translation

Since the algebraic calculus can only represent binary types, we omit the term constructors for sum types \( \oplus, \) and we write \( \text{BExp} \Gamma \tau \) for this restricted class of binary expressions. Figure 14 defines a translation from a binary expression \( e : \text{BExp}(\Gamma) \tau \) to \( y : \tau \mid \Gamma \vdash (e)^y. \)

Lemma 30. If \( e : \text{BExp} \Gamma \sigma \) and \( U : \sigma = \tau \text{ then } \langle U \# e \rangle^y \approx \langle e \rangle^x(x(a) \rightarrow U(a, y(a))). \)

Proof. By path induction on \( e, \) it suffices to observe that \( \langle e \rangle^x = \langle e \rangle^x[x(a) \rightarrow y(a)]. \)

Theorem 31. If \( e_1, e_2 : \text{BExp} \Gamma \sigma \) and \( e_1 \approx e_2, \) then \( \langle e_1 \rangle \approx \langle e_2 \rangle. \)
Lemma 32. For all $e : \text{BExp } \Gamma \sigma$ and $x : \tau \vdash t : \langle \langle e \rangle \rangle \approx e$ and $\langle \langle t \rangle \rangle^x \approx t$.

Theorem 33. The $\text{BExp}$ HoTT calculus is sound and complete with respect to the algebraic calculus.

Proof. If $e_1 \approx e_2$ then $\langle \langle e_1 \rangle \rangle \approx \langle \langle e_2 \rangle \rangle$ by Theorem 27, and if $\langle \langle e_1 \rangle \rangle \approx \langle \langle e_2 \rangle \rangle$ then, by Theorem 29 and Lemma 32,

$$e_1 \approx \langle \langle e_1 \rangle \rangle \approx \langle \langle e_2 \rangle \rangle \approx e_2.$$  

Staton proves that the algebraic calculus is sound and fully complete with respect to $C_{CPU}^*$, the category of $C^*$ algebras of dimension $2^n$ with completely positive and unitary transformations [33]. As a consequence, the completeness of the $\text{BExp}$ fragment of the HoTT calculus extends to $C_{CPU}^*$. We speculate but have yet to prove that the unrestricted HoTT calculus is sound and complete with respect to $C_{CPU}^*$ algebras of arbitrary dimension.

8 Discussion

In this section we discuss some of the design decisions made in this work.

Axiom schemes. Our equational theory prioritizes equations based on the structure of the language, such as $\beta$, $\eta$, and commuting conversion rules. Such rules do not depend on any quantum-specific principles, and their meta-theories are well-understood. In addition, we prioritize collecting many axioms into a single axiom scheme, as we do for the equational axioms $\text{U-INTRO}$ and $\text{U-ELIM}$. This approach gives concise axioms that highlight the important structure, but requires more overhead to express.

Our axiom schemes are also somewhat redundant—for example, we proved the equations for the not unitary $X$ in Proposition 12, but they are also a consequence of the $\text{U-INTRO}$ and $\text{U-ELIM}$ axioms.

Unitaries. This work does not axiomatize unitary transformations, in order to focus on the relationship between quantum and non-quantum data. However, axiomatizations based on universal (or even non-universal) sets of unitaries, such as those by Matsumoto and Amano [19] or Amy et al [5], could be incorporated with a higher-inductive type (HIT). As a first approximation, we could define $\text{QType}$ as a HIT that axiomatizes only the behavior of the Hadamard gate $H$, with the following constructors: a type $\text{Qubit} : \text{QType}$; a path $H : (\text{Qubit} = \text{Qubit})$, and a higher path expressing that $H^\dagger = H$. Since unitaries are still encoded in the path type of quantum types, the aspects of the equational theory we derived by path induction would still hold, and it would allow finer control over the ways by which unitaries approximate each other. On the other hand, working with higher inductive types with many constructors can quickly become unwieldy.

Quantum types as host types. It may seem odd to embed arbitrary host language values as quantum data, via the $\text{Lower}$ type. In contrast, Quipper has both host-language booleans and embedded bits, which are different from qubits. We could have designed a similarly restricted system by removing the $\text{put}$ constructor, so that the only way to construct the $\text{Lower}$ type is via explicit measurement, $\text{meas} : \text{Qubit} \rightarrow \text{Lower Bool}$. However, because we have restricted ourselves to finite types, all such data is sound with respect to finitary quantum computing.

Other equational theories of quantum computing. Departing from the embedded QRAM model, other models of quantum computation have elegant equational theories, including algebraic presentations such as the arrow calculus [38] and graphical presentations such as the ZX calculus [7]. The ZX calculus is not a programming language in the usual sense, but is a graphical calculus that makes associativity and symmetry irrelevant. It’s equational theory has been proven sound and complete, but it lacks programming abstractions such as modularity and polymorphism.
On the other hand, the arrow calculus is a λ-calculus that builds on a body of work allowing programmers to construct density matrices and superoperators directly in a programmatic way, for example by adding matrices together directly [3, 39, 40]. This differs from the style of language presented in this paper, where the only way that users can obtain non-unitary results is explicitly through measurement, which corresponds more closely to the QRAM style of interaction with a quantum computer.

**Conclusion.** This paper presents an equational theory for a linear quantum term calculus in a compact and elegant style using homotopy type theory. We justify these claims by deriving an equational theory known to be complete for a less expressive language, and by proving the semantics is sound with respect to a standard model of quantum computation. In doing so, we have both introduced a new tool to the study of quantum equational theory, and also demonstrated the application of homotopy type theory in a programming environment.

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