Microcanonical model for a gas of evaporating black holes and strings, scattering amplitudes and mass spectrum

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We study the system formed by a gas of black holes and strings within a microcanonical formulation. The density of mass levels grows asymptotically as

$$\rho(m) \approx (d m_i + bm_i^2)^{-a} e^{\frac{b}{2c}(d m_i + bm_i^2)},$$

($i = 1, ..., N$). We derive the microcanonical content of the system: entropy, equation of state, number of components $N$, temperature $T$ and specific heat. The pressure and the specific heat are negative reflecting the gravitational unstability and a non-homogeneous configuration. The asymptotic behaviour of the temperature for large masses emerges as the Hawking temperature of the system (classical or semiclassical phase) in which the classical black hole behaviour dominates, while for small masses (quantum black hole or string behaviour) the temperature becomes the string temperature which emerges as the critical temperature of the system. At low masses, a phase transition takes place showing the passage from the classical (black hole) to quantum (string) behaviour. Within a microcanonical field theory formulation, the propagator describing the string-particle-black hole system is derived and from it the interacting four point scattering amplitude of the system is obtained. For high masses it behaves asymptotically as the degeneracy of states $\rho(m)$ of the system (ie duality or crossing symmetry). The microcanonical propagator and partition function are derived from a (Nambu-Goto) formulation of the N-extended objects and the mass spectrum of the black-hole-string system is obtained: for small masses (quantum behaviour) these yield the usual pure string scattering amplitude and string-particle spectrum $M_n \approx \sqrt{n}$; for growing mass it pass for all the intermediate states up to the pure black hole behaviour. The different black hole behaviours according to the different mass ranges: classical, semiclassical and quantum or string behaviours are present in the model.

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I. INTRODUCTION AND RESULTS

We study the system composed by a gaz of black holes and strings. This problem is interesting by its own, it has fundamental and practical physical interest for several reasons. A gaz of primordial black holes and strings could have been formed, existed and decayed in the early universe. In the lack of a microscopic quantum dynamical description of the late stages of black hole evaporation and of a tractable quantum string field theory, the gaz of black holes and strings presents several appealing physical and tractable features. In the early stages of black hole evaporation, most of the emission is in the string massless modes and in the form of Hawking radiation, namely at the Hawking temperature $T_H$, corresponding to the semiclassical evaporation phase [1,2]. As evaporation proceeds, the semiclassical Hawking description breaks down, the quantum string emission is in the most excited massive states, at the string temperature $T_s$, the excited emitted strings undergo a phase transition into a quantum string condensate which then decays as quantum strings do in the usual pure (non mixed) quantum string emission [1,2]. Such phase transition represents the (non perturbative) back reaction effect of the quantum emitted strings on the black hole, (that is when it is not more possible separate the black hole from its emission). The Hawking temperature and the string temperature are the same concept in the two different gravity regimes (semiclassical and quantum regimes) respectively [1-4].

Black holes emit strings, and in their quantum regime, quantum black holes turn to be strings or behave like them, their intrinsic temperature, decay and quantum mass spectrum are similar; on the other hand, strings can, under certain conditions of high excitation and unstability, collapse into black hole states. Thus, a system composed by both, black holes and strings, allow to describe the different phases and black hole behaviours: classical or semiclassical black holes, ie black holes at their early phases of evaporation, emission of strings by the black holes, and quantum black holes at the last stages of their evaporation.

In this paper we study the gaz of black holes and strings within a microcanonical description; the microcanonical approach is the most physical compelling for this system. We first study it within an ideal gaz statistical mechanical ensemble; we also discuss the N-body interacting amplitudes by analogy with the string and dual models. We then study the system within a Quantum Field Theory microcanonical formulation (with interactions) for the propagator, scattering amplitude and partition function. Finally, we derive the microcanonical propagator of the system from the Nambu-Goto action for N- extended objects, which takes into account the non-local effects.

Previously, the statistical properties of the gaz of strings alone [8]-[11] or the gaz of black holes alone [12]-[14] were considered, but the system composed of both extended objects black holes and strings which allow to cover a rich class of situations was never considered before.

We compute the microcanonical partition function $\Omega_n(n, E, V)$ for the gaz of black holes and strings with the density of mass levels having the following asymptotic behaviour

$$\rho(m) \approx (d \ m_i + b \ m_i^2)^{-a} \ e^{\frac{n}{2}(d \ m_i + b m_i^2)} \quad (1.1)$$

$a, b$ and $d$ being constants fixed by the boostrap condition. It appears that the global behaviour of the system depends on two cases: $a = 5/2$ and $a \neq 5/2$, but there are relevant generic features common to the both cases and we will stress them here. We find for the microcanonical partition function

$$\Omega_n(n, E, V) = e^{\frac{8\pi}{2}(d \ E + E^2) b} e^{g(n,E,V)} \quad (1.2)$$

where, by using the Stirling approximation, we find for the density of states $g(n,E,V)$:

$$g(n,E,V) \approx -n \ln n + n \ln [C] + n \ln [H(E)] \quad (1.3)$$

$C$ is a constant which depends on different combinations of the parameters, $a, b, d$ and on $m_0$, the minimal mass in the system ($m_0 << E$), and $H(E)$ is a function which we compute explicitely in the two cases of interest: $a \neq 5/2$ and $a = 5/2$. In this last case $H(E) = n/E$. The first factor in eq.(1.2) is the string-black hole microcanonical partition function and the other factors are deviations from it. The leading part of $\Omega_n(n, E, V)$ is corrected by the $n$ dependent terms.

We study the microcanonical content of the system. The entropy is read from eqs (1.2),(1.3):

$$S \simeq \frac{8\pi}{2}(E d + E^2) + g(n, E, V) \quad , \quad (1.4)$$
and we find for the Temperature:

\[ T = \frac{T_0}{1 + 8\pi E_b T_0 + F(E) T_0} \]

with \( F(E)/E \to 0 \) for \( E \to \infty \) and \( F(E) \to 0 \) for \( E \to 0 \). The function \( F(E) \) is explicitly obtained in the two cases \( a = 5/2 \) and \( a \neq 5/2 \). The asymptotic large energy regime and the small energy regime of the temperature emerge respectively as the Hawking temperature \( T_H \) and the string temperature \( T_0 \) of the system:

\[ T(E \to \infty) = \frac{1}{8\pi E_0} = T_H , \quad T(E \to 0) = T_0 , \quad (T_0 = 1/4\pi d) \]

The pressure is negative, as a function of \( N \):

\[ PV|_{a \neq 5/2} = -\frac{N T}{(5/2 - a)} \ln N \]

while it is positive for \( a = 5/2 \). For \( a \neq 5/2 \) the specific heat \( C_v \) is negative, for low energies it behaves as \( C_v \approx \frac{1}{8\pi E_0} \approx \text{const} \), while for high energies \( C_v \) vanishes, showing up the asymptotic stability point of the system.

The regimes in which the pressure becomes negative reflect the presence of gravothermal unstable behaviour, which is accompanied by the behaviour of the specific heat. The changes in the local energy density given by the microcanonical ensemble allow the negative value for the specific heat \( C_v \). The canonical ensemble (which takes the average of the energy) does not take care of the local fluctuations of the energy density and, for instance, both approaches (canonical and microcanonical) are not equivalent in this case. The behaviour of the microcanonical physical magnitudes indicate that for small energy a phase transition takes place showing the passage from macroscopic (classical black hole) to microscopic (string) behaviour:

We also check for our statistical model the duality (crossing symmetry) of the scattering amplitudes characteristic of string and dual systems: We evaluate the two-body (and \( n \)-body) amplitude \( N_n \) of the string-black hole system and find that \( N_n \approx \rho(m) \approx (d \ m_i + b m^2)^{-a} e^{\frac{V_i}{d}(d \ m_i + b m^2)}, \ (i = 1, ..., N) \) for \( m \to \infty \). That is, the number of open channels does grow in parallel with the degeneracy of states, the number of resonances, as energy is increased.

In order to go further in the understanding of this system, we study it within a microcanonical field theory (or thermodynamic) formulation. We compute (in momentum space) the physical component of the microcanonical propagator \( \Delta^{11}_{\text{phys}}(k) \) and from it the four point scattering amplitude \( A \) of the system, which expresses as the sum of two parts: \( A = A_{\text{str-particle}} + A_{\text{str-bh}} \), the first term corresponds to the usual string-particle spectrum (Veneziano amplitude), the second one describes the new string-black hole component of the amplitude, which we compute explicitly, eq(5.4). For high mass, \( A_{\text{str-bh}} \approx \rho(m) \) and we recover that it increases exponentially as the density of states for the string-black-hole system. We then derive the microcanonical partition function \( \Omega(E, V) \) from the computed field propagators and transition amplitude \( A \), relating \( \ln \Omega(E, V) \) to the trace of the connected part of the imaginary part of the full propagator.

Finally, we give a Nambu-Goto formulation of the N-extended body system composed by black holes and strings and derive from it the microcanonical propagator

\[ D_E(k, m) = \frac{\delta(E)}{\omega^2 - k^2 - m^2 + i\varepsilon} - D_{E \ (\text{str-bh})}(k, m) \]  

(1.5)

\[ D_{E \ (\text{str-bh})}(k, m) = 8\pi i\alpha \delta(\omega^2 - k^2 - m^2) \sum_{l=1}^{\infty} \frac{K_{-1}(\alpha |l\omega_k - E|)}{E^2} \frac{\Omega(E - l\omega_k)}{\Omega(E)} \theta(E - l\omega_k) \]  

(1.6)

where \( l, \omega_k \) and \( \theta(E - l\omega_k) \) are the mode number, the dispersion relation and the step function respectively, and \( K_{-1} \) is the Mac Donald’s function. The first term in the microcanonical propagator eq.(1.6) is the usual Feynman propagator, the second one is the new microcanonical part \( D_{E \ (\text{str-bh})}(k, m) \) describing the string-black hole component. From this term, we obtain for \( E \to 0 \), being \( E = M \) the total mass of the system, the pure string amplitude (Veneziano amplitude). When \( E \) grows it pass for all intermediate states up to the pure black hole behaviour. This propagator includes all non-local effects from the \( N \)-bodies of the system considered as extended objects and from its derivation as in the string Nambu-Goto formulation. For instance, the propagator eq.(1.6) becomes the (point
particle) Quantum Field Theory propagator \( \Delta^{11}_{11}(k) \) when the string becomes point-particles and the Hamiltonian is quadratic in the momenta (constrained particle Hamiltonian). We analyze the spectrum (singular points of the microcanonical part). In the excited regime \( l \to M/\omega_k \), the spectrum becomes

\[
\left( \frac{8\pi d}{a} \right)^2 M^2 \approx n \ln \left( \frac{8\pi}{a} \right) - \ln 2, \quad n \gg 1 \tag{1.7}
\]

which is like the usual quantum string spectrum, that is the mass of quantum black holes is quantized in the same way as quantum strings.

This study allowed us to derive a wide class of properties and physical magnitudes of the system composed by black holes and strings, covering the different mass ranges and the classical, semiclassical and quantum behaviours of the system.

This paper is organized as follows: In section II we formulate the statistical mechanics of the gas of strings and black holes in the microcanonical ensemble and derive the partition function of the system; in section III we compute from it the physical magnitudes of the system: entropy, temperature, pressure and specific heat, and analyze their properties. In section IV we discuss the duality (crossing) property of the scattering amplitudes of the system in analogy with that of string and dual models. In section V we provide a quantum field theory description of the system (propagator and scattering amplitudes within the operator approach). In section VI we describe the system of \( N \)-black holes and strings as \( N \)-extended objects in the Nambu-Goto formulation, and derive in this context the microcanonical propagator and mass spectrum.

### II. STATEMENT OF THE PROBLEM

In the microcanonical ensemble, we consider the following density of states for a configuration with \( n \) black holes and strings

\[
\Omega_n(n, E, V) = \sum_{n=0}^{\infty} \left[ \frac{V}{(2\pi)^3} \right]^n \frac{1}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} dm_i \int_{-\infty}^{\infty} \rho(m_i) dp_i^3 \delta \left( E - \sum_{i=1}^{n} E_i \right) \delta^3 \left( \sum_{i=1}^{n} \vec{p}_i \right) \tag{2.1}
\]

where we make the following considerations:

(i) the density of mass levels has the following asymptotic behaviour

\[
\rho(m) \approx c (d m_i + m_i^2)^{-a} e^{\frac{8\pi}{a} (d m_i + m_i^2) b} \tag{2.2}
\]

\( a, b, c \) and \( d \) being positive parameters whose meaning will be clear in the sequel (they are fixed by the bootstrap condition eq. (2.21)). This density of mass levels takes into account both the string component and the black hole component of the system.

(ii) a particle-like relation of dispersion

\[
E_i = \sqrt{m_i^2 + |\vec{p}_i|^2} \quad (G = c = 1)
\]

\( E_i \) being the energy of the \( i \)th black hole with linear momentum \( \vec{p}_i \).

(iii) the conditions: \( b^{-1} < m_i^2 \) and \( d^{-1} < m_i \), which are the usual ones in statistical hadronic systems. Here these conditions mean that the constants \( b \) and \( d \) arise naturally from the statistical system considered as delimiters of the kinetic status of the particles.

(iv) the mass \( m_0 \) is the mass of the less massive component (black hole or string) in the gas, namely \( m_0 \ll M \), being \( M \) the total mass of the system (black holes and strings).

(v) the factor \( \delta^3 \left( \sum_{i=1}^{n} \vec{p}_i \right) \) can be neglected (over-all momentum conservation).

Therefore, the expression (2.1) takes the following form

\[
\Omega_n(n, E, V) = \left[ \frac{V}{(2\pi)^3} \right]^n \frac{1}{n!} \prod_{i=1}^{n} \int_{m_0}^{\infty} dm_i \int_{-\infty}^{\infty} \rho(m_i) dp_i^3 \delta \left( E - \sum_{i=1}^{n} E_i \right) \tag{2.3}
\]

We rewrites the energy \( E_i = m_i + Q_i \) in terms of the kinetic energy \( Q_i \) and consider the dominant part of it in the usual form \( Q_i \approx \frac{p_i^2}{2m_i} \). Then,
\[
\prod_{i=1}^{n} e^{\frac{n}{2b}(d m_i + m_i^2 b)} = e^{\frac{n}{2b}(d E + E^2 b)} \prod_{i=1}^{n} e^{-8\pi b\left(\frac{Q_i^2}{2} + m_i Q_i\right)} e^{-\frac{n}{2b}Q_i},
\] (2.4)

and define
\[
I_i(\Lambda_i) \equiv \int_{m_0}^{\Lambda_i} c (d m_i + m_i^2 b)^{-a} \ d m_i \int e^{-8\pi b}\left[\frac{Q}{2} + m_i (1 + \frac{L_i}{Q_i})\right] \ dp_i^3
\]

where the cut-off \(\Lambda_i\) on momentum integration for \(Q_i^2 \approx b^{-1} \approx d^{-2}\) is naturally introduced by the condition (iii). We then have
\[
I_i(\Lambda_i) = \int_{m_0}^{\Lambda_i} c (d m_i + m_i^2 b)^{-a} \ d m_i \int e^{-8\pi b}\left[\frac{Q}{2} + m_i (1 + \frac{L_i}{Q_i})\right] \ dp_i^3
\]

where \(D_{-3/2}(x)\) is the Whittaker function. From eqs. (2.2) and (2.3) we get
\[
\Omega_n(n, E, V) = \frac{e^{\frac{n}{2b}(d E + E^2 b)}}{n!} \left[\frac{V}{(2\pi)^3} c \left(\frac{\pi}{b^3}\right)^2 \right]^n \prod_{i=1}^{n} \int_{m_0}^{\infty} \left(\frac{\hat{m}_i}{2}\right)^{3/2} (d m_i + m_i^2 b)^{-a} \ Y \delta(\sum_{i=1}^{n} m_i - E) \ dm_i \quad (2.5)
\]

where
\[
\hat{m}_i = m_i + \frac{d}{2b}
\] (2.6)

and
\[
\Upsilon \equiv \Gamma(-1/4) \ \Gamma[3/4, 1/2, 4\pi b \hat{m}_i^2] + 2\sqrt{2\pi b} \hat{m}_i \ \Gamma(1/4) \ \Gamma[5/4, 3/2, 4\pi b \hat{m}_i^2]
\] (2.7)

\(1F_1\) being the confluent hypergeometric function. Let us now define
\[
\prod_{i=1}^{n} I_i(\Lambda_i) = \frac{e^{\frac{n}{2b}(d E + E^2 b)}}{n!} \left[\frac{c}{2^{3/2} (\frac{\pi}{b^3})^{3/2}}\right]^n \int_{m_0}^{E} \ d X \prod_{i=1}^{n} \int_{m_0}^{\infty} \ d m_i \left(\frac{\hat{m}_i}{2}\right)^{3/2} (d m_i + m_i^2 b)^{-a} \ Y \delta(\sum_{i=1}^{n} m_i - X)
\] (2.8)

such that
\[
\Omega_n(n, E, V) = \frac{e^{\frac{n}{2b}(d E + E^2 b)}}{n!} \left[\frac{V}{(2\pi)^3}\right]^n \frac{d (\prod_{i=1}^{n} I_i(\Lambda_i))}{d E} = e^{\frac{n}{2b}(d E + E^2 b)} \ e^{\varphi(n>E,V)}
\] (2.9)

where the constraint is obviously
\[
\sum_{i=1}^{n} \Lambda_i = E
\] (2.10)

The maximum contribution to \(\prod_{i=1}^{n} I_i(\Lambda_i)\) is obtained when all the \(\Lambda_i\) are of the order \(E/n\). We have \(I(\Lambda_i)\) with the hypergeometric confluent functions in asymptotic form for \(2\sqrt{2\pi b} \hat{m}_i >> 1\) since the condition (iii). We get in this manner
\[
\prod_{i=1}^{n} I_i(\Lambda_i) = \left[\frac{c}{2^{3/2} (\frac{\pi}{b^3})^{3/2}}\right]^n \prod_{i=1}^{n} \int_{m_0}^{\infty} (1 + \frac{d}{2bm_i})^{-3/2} (d m_i + m_i^2 b)^{-a} \left[1 - \frac{3b}{2.4.(2\pi b)m_i^2} + \ldots\right] \delta(\sum_{i=1}^{n} m_i - E) \ dm_i
\] (2.11)

From the above expression, two different behaviours show up depending on the value of \(a\):

(1) \(a \neq 5/2\)
\[
\prod_{i=1}^{n} I_i(\Lambda_i) = \left[\frac{c}{2^{3/2} (\frac{\pi}{b^3})^{3/2}}\right]^n \left\{\left(\frac{2b}{a}\right)^{3/2} \ \frac{1}{5/2 - a} \ [\Xi(\Lambda_i) + \Xi(m_0)]\right\}^n
\] (2.12)
where

\[ \Xi(x) \equiv x^{3/2-a} \left[ F_1\left(\frac{5}{2} - a, 3/2 + a, 7/2 - a, -\frac{2bx}{d}\right) - \frac{15}{16\pi^2} b F_1\left(\frac{5}{2} - a, 7/2 + a, 7/2 - a, -\frac{2bx}{d}\right) \right] \]

(2) \( a = 5/2 \):

\[
\Pi^a(\Lambda_i) = \left[ \frac{c}{2\sqrt{2} \left( \frac{\pi b}{h} \right)^{\frac{1}{2}}} \right] n \left\{ A \left[ \text{arctanh} \sqrt{h(\Lambda_i)} - \text{arctanh} \sqrt{h(m_0)} \right] + \left( \frac{2b}{d} \right)^{3/2} \left[ \frac{1}{\sqrt{h(\Lambda_i)}} - \frac{1}{\sqrt{h(m_0)}} \right] - \frac{15}{32\pi b} (2b)^{7/2} \left[ p(\Lambda_i) - p(m_0) \right] \right\}^n \tag{2.13}
\]

\[
\text{with } h(x) \equiv 1 + \frac{2bx}{d}; \quad A \equiv \frac{15}{32\pi b} \left( \frac{2b}{d} \right)^{3/2} \left[ \left( \frac{2b}{d} \right)^2 - 1 \right]; \quad p(x) \equiv \frac{1}{d^2/2 h(x)^{5/2}} \left( \frac{1}{5} + \frac{h(x)}{3} + h(x)^2 \right) \tag{2.14}
\]

Then, from eqs (2.11) and (2.12)-(2.14), the microcanonical density \( g(n, E, V) \) in the two cases \((a \neq 5/2 \text{ and } a = 5/2 \text{ respectively})\) is given by :

\[
\begin{align*}
\text{for } a \neq 5/2 & \approx \frac{1}{n!} \left[ \frac{(2b)^{3/2}}{\sqrt{5/2-a}} \right] \left[ \frac{1}{n} \left\{ \eta \left( \frac{E}{n} \right)^{3/2-a} - m_0^{3/2-a} \right\} \right]^{n-1} \\
& \times \left[ \frac{3/2-a}{E} \left( \frac{E}{n} \right)^{3/2-a} - \frac{5/2-a}{7/2-a} \left( \frac{E}{n} \right)^{5/2-a} \frac{b}{Ed} \left( 3 - \frac{7b}{2d^2} \right) \right] \left( \frac{n}{E} \right) \tag{2.15}
\end{align*}
\]

\[
\text{for } a = 5/2 \approx \left[ \frac{\gamma q(1 - \frac{15q^2}{32\pi b})}{n!} \right] \left[ 1 + \frac{f(m_0)}{q(1 - \frac{15q^2}{32\pi b})} \right]^{n-1} \left( \frac{n}{E} \right) \tag{2.16}
\]

where

\[
f(m_0) \equiv \lambda \text{arctanh} \left( \sqrt{h(m_0)} \right) + \left( \frac{2b}{d} \right)^{3/2} \left( 1 - \frac{1}{\sqrt{h(m_0)}} \right) - \frac{1}{(2b)^{7/2}} \frac{15}{32\pi b} \left[ \frac{23}{15d^2/2} - p(m_0) \right]
\]

\[
\gamma \equiv \left[ \frac{V}{(2\pi)^2 c} \left( \frac{\pi b}{h} \right)^{\frac{1}{2}} \right]; \quad \eta \equiv \left( 1 - \frac{15}{16\pi b d^2} \right); \quad q \equiv \left( \frac{2b}{d} \right)^{3/2} \tag{2.17}
\]

Using the Stirling approximation, the density of states \( g(n, E, V) \) eqs.(2.15) and (2.16) can be expressed as

\[
g(n, E, V)_{a \neq 5/2} \approx -n \ln n + n + n \ln \left[ \frac{\gamma (2b)^{3/2}}{5/2-a} \right] + n \ln \left\{ \eta \left( \frac{E}{n} \right)^{3/2-a} - m_0^{3/2-a} \right\} + \left( \frac{3/2-a}{E} \left( \frac{E}{n} \right)^{3/2-a} - \frac{5/2-a}{7/2-a} \left( \frac{E}{n} \right)^{5/2-a} \frac{b}{Ed} \left( 3 - \frac{7b}{2d^2} \right) \right] \tag{2.18}
\]

\[
+ \ln \left[ \frac{3/2-a}{E} \left( \frac{E}{n} \right)^{3/2-a} - \frac{5/2-a}{7/2-a} \left( \frac{E}{n} \right)^{5/2-a} \frac{b}{Ed} \left( 3 - \frac{7b}{2d^2} \right) \right] \tag{2.19}
\]

and

\[
g(n, E, V)_{a = 5/2} \approx -n \ln n + n + n \ln \left[ \gamma q \left( 1 - \frac{15q^2}{32\pi b} \right) \right] + (n-1) \ln \left[ 1 + \frac{f(m_0)}{q(1 - \frac{15q^2}{32\pi b})} \right] + \ln \left( \frac{n}{E} \right) \tag{2.20}
\]
It is interesting to see from eq.(2.20) and eq.(2.16) that when $m_o \approx 0$, then $f (m_o) \approx 1$, and we have

$$g(n, E, V)_{a=5/2} \simeq -n \ln n + n + n \ln \left[ \gamma q \left( 1 - \frac{15q^2}{32\pi b} \right) + \gamma \right] + \ln \left( \frac{n}{E} \right)$$

which express in a more clear form how the leading part of the microcanonical density of states $\Omega_n(n, E, V)$ is corrected by the $n$ dependent terms

$$\Omega_n(n, E, V) \simeq e^{\frac{8\pi}{3}(d E + E^2 b)} \times \frac{\eta n}{E} \left( \frac{const.}{n} \right)^n$$

It is easy to see that the first factor is the string-black hole microcanonical partition function and the other factors are deviations from it.

The extent of the parameters $(a, b, c, d)$ depends on the bootstrap constraint. If we use the strong condition, explicitly

$$\lim_{E \to \infty} \left[ \frac{\Omega(E, V)}{\rho(E)} \right] \to 1 , \quad (2.21)$$

this condition holds for

$$b = 1 = d^2 , \quad c \neq 0 \quad \text{arbitrary}, \quad (2.22)$$

and allowing two cases for the $a$ parameter: (i) $a \neq 5/2$ with $2 \leq a < 5/2$ or $a > 5/2$; and (ii) $a = 5/2$.

With the units restored $b = t^{-2}_{Pl}$, $d = t^{-1}_{s}$, $t_{Pl}$ and $t_{s}$ being the fundamental Planck temperature and fundamental string temperature respectively: $t_{Pl} = \sqrt{\hbar c / G}$, $t_{s} = \sqrt{\hbar c / \alpha'}$.

Let us now discuss the physical microcanonical content of the system.

### III. MICROCANONICAL CONTENT OF THE MODEL

#### A. Entropy and Number of components of the system

In the microcanonical formulation, we have for the entropy of our system

$$S = \ln \Omega_n(E, V) = \frac{8\pi}{2} (Ed + E^2 b) + g(n, E, V) \quad (3.1)$$

From eqs.(2.18)-(2.20) we have

$$S(a \neq 5/2) \simeq \frac{8\pi}{2} (Ed + E^2 b) - n \ln n + n + n \ln \left[ \gamma \left( \frac{2b}{d} \right)^{3/2} \frac{1}{5/2 - a} \right] + n \ln \left\{ \eta \left[ \frac{E}{n} \right]^{\frac{2}{2-a}} - m_0^{3/2-a} \right\} \quad (3.2)$$

$$S(a = 5/2) \simeq \frac{8\pi}{2} (Ed + E^2 b) - n \ln n + \ln \left( \frac{n}{E} \right) + n \ln \left[ \gamma q \left( 1 - \frac{15q^2}{32\pi b} \right) \right] + (n - 1) \ln \left[ 1 + \frac{f (m_o)}{q \left( 1 - \frac{15q^2}{32\pi b} \right)} \right] \quad (3.3)$$

where $f (m_o)$ is given by eq.(2.17), and we can see that the first and second terms in the expressions above correspond to the string entropy and to the black-hole entropy respectively. The other $n$-dependent and $E$-dependent terms are corrections to the entropy of the system.

The most probable number of components $N$ for the system is obtained by maximizing the partition function with respect to $n$. For $a \neq 5/2$ the result is

$$n = N \approx \left[ \frac{\gamma \eta (1 - \Delta^{3/2-a}) E^{3/2-a}}{(5/2 - a - \Delta^{3/2-a}) m_0^{-a}} \right]^{\frac{1}{3/2-a}} \quad (3.4)$$
where
\[
\tilde{\gamma} \equiv \gamma \left( \frac{2b}{d} \right)^{3/2} \frac{1}{5/2-a} ; \quad \Delta \equiv \left( \frac{N m_0}{E} \right)
\]

The units of the constant \( c \) are such that \( \gamma \) is dimensionless. The particle number density \( N/V \) is a function of the energy density \( E/V \) only. The energy as a function of \( N \) is
\[
E = \left[ \frac{N^{5/2-a}(5/2-a - \Delta^{3/2-a})}{\tilde{\gamma} \eta (1 - \Delta^{3/2-a}) m_0^3} \right]^{1/2-a}
\]

We see that \( N \) goes to zero for \( E \rightarrow N m_0 \) (the lowest energy of the system) with \( a < 5/2 \) or \( a \rightarrow 3/2 \), while \( N \rightarrow \infty \) when \( \Delta \rightarrow (5/2-a)^{3/2-a} \) and \( a < 5/2 \).

For \( a \neq 5/2 \), from eqs. (2.18) and (3.4), \( g(N, E, V) \) is expressed as:
\[
g(N, E, V)_{a \neq 5/2} \approx \left[ \frac{\tilde{\gamma} \eta (1 - \Delta^{3/2-a}) E^{3/2-a}}{(5/2-a - \Delta^{3/2-a}) m_0^3} \right]^{1/2-a} \left\{ 1 + \frac{1}{5/2-a} \ln \left[ \frac{(5/2 + a - \Delta^{3/2-a}) m_0^{-a}}{\tilde{\gamma} \eta (1 - \Delta^{3/2-a}) E^{3/2-a}} \right] \right\}
\]

which has the following behaviours:

| \( a \) \( < 3/2 \) | \( g(N, E, V)_{E \rightarrow 0} \) | \( g(N, E, V)_{E \rightarrow \infty} \) |
|----------------|------------------|------------------|
| \( \frac{\tilde{\gamma} \eta}{m_0^{3/2}} \) \( E^{5/2-a} \rightarrow 0 \) | \( \frac{\tilde{\gamma} \eta}{m_0^{3/2}} \) \( E^{5/2-a} \rightarrow \infty \) |
| \( \frac{\tilde{\gamma} \eta}{(5/2-a) m_0^3} \) \( E^{7/2-a} \rightarrow 0 \) | \( \frac{\tilde{\gamma} \eta}{m_0^{3/2}} \) \( E^{5/2-a} \rightarrow \infty \) |
| \( \frac{\tilde{\gamma} \eta}{(5/2-a) m_0^3} \) \( E^{7/2-a} \rightarrow \infty \) | \( \frac{\tilde{\gamma} \eta}{m_0^{3/2}} \) \( E^{5/2-a} \rightarrow 0 \) |

For \( a = 3/2 \), \( g(N, E, V) \) does not depend on the energy \( E \):
\[
g(N, E, V) \approx \left( \frac{\tilde{\gamma} \eta}{m_0^{3/2}} \right) \left[ 1 + \ln \left( \frac{m_0^{-3/2}}{\tilde{\gamma} \eta} \right) \right]
\]

For \( a = 5/2 \), the most probable number of components of the system is
\[
n = N \approx \gamma \left[ f(m_0) + \lambda \right] , \quad \lambda \equiv \left( \frac{2b}{d} \right)^{3/2} \left[ 1 - \left( \frac{2b}{d} \right)^2 \left( \frac{15}{32\pi b} \right) \right]
\]

where \( f(m_0) \) and \( \gamma \) given by eq. (2.17). For \( a = 5/2 \), \( N \) depends only on the volume \( V \) (linearly through \( \gamma \)) and it does not depend on the energy \( E \), and the function \( g(N, E, V) \) is
\[
g(N, E, V)_{a=5/2} \approx \frac{\gamma^N}{(N-1)!} \left[ f(m_0) + \lambda \right]^{N-1} \frac{\lambda}{E}
\]

**B. Temperature**

From eqs.(2.9) and(2.15) the temperature for the case \( a \neq 5/2 \) is given by
\[
T = \left[ \frac{\partial \ln \Omega_N(E, V)}{\partial E} \right]^{-1}
\]
\[ T^{-1} - T_0^{-1} = 8\pi Eb - \frac{3/2 - a}{(5/2 - a)^2} \left[ \frac{\gamma\eta(1 - \Delta^{3/2-a})^{-1}}{m_0^{-a}(5/2 - a - \Delta^{3/2-a})^{3/2-a}} \right] \frac{3/2-a}{\gamma\eta E^{3/2-a}} \ln N \]  

\( \Delta \equiv \left( \frac{\Delta m}{E} \right) \), and where we can see the emergence of a critical temperature \( T_0^{-1} \equiv 4\pi d \)

such that the temperature satisfies \( T \leq T_0 \). We know that the partition function \( Z(V,T) \) is just the Laplace transform of the density of states \( \Omega_N(E,V) \) and both are, mathematically speaking, equivalent.

\[ Z(V_0, T) = \int_0^\infty \Omega_N(m, V_0) e^{-m/T} dm \]

Since \( \Omega_N(E, V_0) \) is of the form \( e^{\frac{\Delta N}{E+m^2}} e^{g(m, V_0)} \), \( Z(V_0, T) \) reads

\[ Z(V_0, T) = \int_0^\infty e^{-m\tau} e^{\left( \frac{m}{\pi\eta} \right)^2} e^{g(m, V_0)} dm \quad, \quad \tau \equiv \frac{T_0 - T}{TT_0} \]

and all thermodynamical functions (as the energy and pressure) depend on the factor \( \frac{T_0 - T}{TT_0} \) too. Since the exponentially increasing term for high \( m \), this function diverges for all temperature reflecting the fact that the canonical frame does not exist. The function \( g(m, V_0) \) behaves smoothly for large \( m \).

From eq.(3.9) when \( \Delta^{3/2-a} \rightarrow 0 \), we have for the temperature:

\[ T^{-1} - T_0^{-1} \approx 8\pi Eb + \frac{3/2 - a}{5/2 - a} \left[ \frac{\gamma\eta}{m_0^{-a}(5/2 - a)^{3/2-a}} \right] \frac{3/2-a}{\gamma\eta E^{3/2-a}} \ln \left[ \frac{(5/2 - a)m_0^{-a}}{\gamma\eta E^{3/2-a}} \right] \]  

(3.10)

while when \( \Delta^{3/2-a} \rightarrow \infty \), we have:

\[ T^{-1} - T_0^{-1} \approx 8\pi Eb + \frac{3/2 - a}{(5/2 - a)^2} E^\gamma_{\frac{3}{2-a}} \ln \left[ \frac{m_0^{-a}}{\gamma\eta E^{3/2-a}} \right] \]  

(3.11)

The behaviour of the temperature as a function of the energy \( E \) is as follows:

| \( a < 3/2 \) | \( T \rightarrow \infty \) | \( T \rightarrow 0 \) |
|----------------|---------------------|---------------------|
| \( a < 3/2 \) | \( T_0 \) \( 1+8\pi Eb \) | \( 1+8\pi Eb \) | \( 1+8\pi Eb \) |
| \( a > 5/2 \) | \( T_0 \) \( 1+8\pi Eb \) | \( 1+8\pi Eb \) | \( 1+8\pi Eb \) |
| \( 3/2 < a < 5/2 \) | \( T_0 \) \( 1+8\pi Eb \) | \( 1+8\pi Eb \) | \( 1+8\pi Eb \) |
| \( a = 3/2 \) | \( T_0 \) \( 1+8\pi Eb \) | \( 1+8\pi Eb \) | \( 1+8\pi Eb \) |

where

\[ T_0^{-1} \equiv 4\pi d \]

is the string temperature, and \( T_H = \frac{1}{8\pi Eb} \) which emerges as the Hawking temperature of the system.
For \( a = 5/2 \) the temperature is

\[
T^{-1} - T_0^{-1} = 8\pi Eb - \frac{\gamma^N}{(N-1)!} [f (m_0) + \lambda]^{\frac{N-1}{2}} \frac{\lambda}{E^2}
\]

\[
T \approx \frac{T_0}{(1 + 8\pi EbT_0)} - \left\{ T_0 \frac{\gamma^N}{(N-1)!} [f (m_0) + \lambda]^{\frac{N-1}{2}} \frac{\lambda}{E^2} \right\}
\]

The above temperature behaviours show that the asymptotic large energy behaviour is characterized by the Hawking temperature \( T_H \) while the small energy limit is characterized by the string temperature \( T_0 \). Large energies correspond to large masses and so to the classical/semiclassical behaviour of the system while small masses correspond to the quantum string limit. Notice that for \( a = 5/2 \), \( T \to \infty \) when

\[
T_0 \frac{\gamma^N}{(N-1)!} [f (m_0) + \lambda]^{\frac{N-1}{2}} \frac{\lambda}{E^2} \to (1 + 8\pi EbT_0)
\]

that is, superheating points do appear. For low \((E \to m_0)\) and high \((E >> m_0)\) energies, these points are respectively:

\[
E_{low} \approx \sqrt{\left( \frac{\lambda \gamma^N}{(N-1)!} [f (m_0) + \lambda]^{\frac{N-1}{2}} \right) T_0}, \quad E_{high} \approx 2\pi T_0
\]

This is the analogue to the Carlitz situation \([11]\) for the string gas alone where the energy for the case \( a < 5/2 \) is \( E \simeq (a + 3/2) T_0^2 / (T_0 - T) \). This means that this case is obviously an extremely nonuniform configuration, and the thermodynamic properties of the various subsystems of the statistical system under consideration are inequivalent to the properties of the system as a whole. The microcanonical approach is sensible to the local equilibrium state, and as a consequence the equivalence between microcanonical and canonical ensembles is no longer guaranteed in this case. Temperatures greater than \( T_0 \) are allowed in the microcanonical ensemble, and the system as a whole can be unstable, characterized by a negative specific heat. As strong interactions are virtually present everywhere, \( T_0 \) is a universal highest temperature for equilibrium states. A gas of \( N \) particles initially at \( T > T_0 \) will, after some time, create other particles, and then cool down to \( T < T_0 \), which means that initially the gas was not in an equilibrium state.

At energies higher than those of the superheating points, the behaviour of the temperature in the system is of the form

\[
T \big|_{E>E_{high}} \approx \frac{T_0}{(1 + 8\pi EbT_0)} \simeq \frac{1}{8\pi Eb} = T_H
\]

which shows that the temperature after reaching the superheating point, begin to have a behaviour inverse to the energy stabilizing the system. This behaviour shows the Hawking temperature \( T_H \) of the system. This phase corresponds to the large masses and very low temperature: that is, to the semiclassical behaviour characterized by the Hawking temperature \( T_H = \frac{1}{8\pi Eb} \). In this phase, the temperature goes to zero stabilizing the system.

On the other hand, for low energies, namely \( E \) lower than \( E_{low} \), the behaviour of the temperature is quite different

\[
T \big|_{E<E_{low}} \approx \frac{1}{\gamma^N (N-1)!} [f (m_0) + \lambda]^{\frac{N-1}{2}} \frac{\lambda}{E^2}
\]

which shows that before reaching the superheating point, the temperature is proportional to the square of the energy of the system.

C. Pressure and Specific Heat

In the microcanonical ensemble, the pressure is defined by

\[
P = T \frac{\partial [\ln \Omega_N(E,V)]}{\partial V}
\]
Explicitly, for \( a \neq 5/2 \) we have:

\[
PV \approx \frac{T}{(5/2 - a)^2} \left[ \frac{\hat{\gamma} \eta (1 - \Delta^{3/2-a}) E^{3/2-a} m_0^{-a}}{(5/2 - a - \Delta^{3/2-a}) m_0^a} \right]^{1/2} \ln \left[ \frac{(5/2 - a - \Delta^{3/2-a}) m_0^a}{\hat{\gamma} \eta (1 - \Delta^{3/2-a}) E^{3/2-a}} \right]
\]

Or, as a function of \( N \):

\[
PV|_{a \neq 5/2} \approx \frac{NT}{(5/2 - a)^2} \ln \left[ N^{-{(5/2-a)}} \right] = -\frac{NT}{(5/2 - a)} \ln N
\]

The behaviour of the pressure as a function of the energy \( E \) is as follows:

| \( E \rightarrow 0 \) | \( E \rightarrow \infty \) |
|------------------|------------------|
| \( a < 3/2 \)    | \( PV \rightarrow 0 \) | \( PV \rightarrow \infty \) |
| \( a > 5/2 \)    | \( PV \rightarrow 0 \) | \( PV \rightarrow \infty \) |
| \( 3/2 < a < 5/2 \) | \( PV \rightarrow \infty \) | \( PV \rightarrow 0 \) |

and the behaviour for \( a = 3/2 \) is:

\[
PV \approx T \left[ \frac{\hat{\gamma} \eta}{m_0^{-3/2}} \right] \ln \left[ \frac{m_0^{-3/2}}{\hat{\gamma} \eta} \right] = -NT \ln N
\]

Notice that in this case \( PV \) is independent of \( E \).

For \( a = 5/2 \), the expression for the pressure takes the form

\[
P|_{a=5/2} \approx \frac{TN}{VE \ (N-1)} \gamma^N \left[ f(m_0) + \lambda \right]^{N-1} \lambda
\]

where \( f(m_0) \) and \( \gamma \) are given by eq.(2.17) and the constant \( \lambda \) is given by eq.(3.7).

We see the different behaviours of the system depending on the parameter \( a \). The regimes in which the pressure becomes negative reflect the presence of gravothermal unstable behaviour, which is accompanied, as we see below, by the behaviour of the specific heat.

Let us discuss the specific heat \( C_v = \frac{\partial E}{\partial T} \big|_{V=\text{fixed}} \). From eq.(3.9) it takes for our system the explicit form

\[
C_v \approx -\frac{1}{T^2} \left\{ \frac{1}{8\pi b - X \left( \frac{1}{N} + \ln N \ E^{7/2-a} Y(\Delta) \right)} \right\}
\]

where

\[
X \equiv \frac{(T_0^{-1} + 8\pi b E) - T^{-1}}{\ln N} \ (5/2 - a) \ ; \quad \Delta \equiv \left( \frac{N \ m_0}{E} \right)
\]

\[
Y(\Delta) \equiv \frac{(3/2 - a) \Delta^{3/2-a} \left( 1 - \Delta^{3/2-a} \right)}{[3/2 - a + (1 - \Delta^{3/2-a})^2]} - \left[ 1 + \frac{(3/2 - a) \Delta^{3/2-a}}{5/2 - a} \left( \frac{3/2 - a}{(1 - \Delta^{3/2-a})} + \frac{7/2 - a}{(5/2 - a - \Delta^{3/2+a})} \right)^{3/2-a} \ \right]
\]

The behaviour of the specific heat as a function of the energy \( E \) is as follows:
\[
\begin{array}{|c|c|c|}
\hline
   & E \to 0 & E \to \infty \\
\hline
a < 3/2 & C_v \to 0_- & C_v \to \frac{-1}{8\pi bT^2} \approx \text{const} \\
\hline
a > 5/2 & C_v \to \frac{-1}{8\pi bT^2} \approx \text{const} & C_v \to 0_- \\
\hline
3/2 < a < 5/2 & C_v \to 0_- & C_v \to \frac{-1}{8\pi bT^2} \approx \text{const} \\
\hline
\end{array}
\]

For \( a = 5/2 \), \( C_v \) takes the form

\[
C_v \approx -\frac{E^3}{T^2} \left[ 8\pi E^3 b + 2\frac{\gamma^N}{(N-1)!} \left[ f(m_0) + \lambda \right]^{N-1} \lambda \right]
\]

with the following behaviours

\[
C_v \mid_{E \to \infty} \to \frac{-1}{8\pi bT^2} \approx \text{const} \quad ; \quad C_v \mid_{E \to 0} \to 0_- 
\]

Notice that the changes in the local energy density given by the microcanonical ensemble generate an instability that is here directly translated into the negative value for the specific heat \( C_v \). The canonical ensemble (which takes the average of the energy) does not take care of the local fluctuations of the energy density and, for instance, both approaches (canonical and microcanonical) are not equivalent in this case.

**IV. DUALITY (CROSSING) OF THE SCATTERING AMPLITUDES**

A characteristic property common to string systems and dual models is the duality of the scattering amplitudes (crossing symmetry). This means that the four point amplitude can be expressed as a sum over resonances either in the \( s \) or \( t \) channel, even at very high energies, \((s,t\) being the center of mass energy and the transverse momentum of the components). As was pointed out in ref.[10], in order for duality to be valid, the number of \( n \)-body channels open in the statistical model, and so the total number of open channels, must rise in parallel with the number of resonances as the center of mass energy is increased

\[
N_n(m) \sim \rho_{\text{string}}(m) \sim d m^{-a} e^{\frac{b d m}{a}} , \quad m \to \infty
\]

where \( N_n(m) \) is the number of open \( n \)-body channels at the center of mass energy \( m \). In our system of black-holes and strings, we expect \( N_n(m) \sim \rho(m) \), with \( \rho(m) \) as given by eq.(2.2). An explicit expression for the two-body amplitude is

\[
N_2(m) = \frac{1}{2!} \int_{m_0}^{m-m_0} \rho(m_2) dm_2 \int_{m_0}^{m-m_2} \rho(m_1) dm_1 \tag{4.1}
\]

If duality of the scattering amplitude can be argued to be a symmetry of the string-black-hole system, one should support it with a direct computation of \( N_2(m) \) eq. (4.1) by using \( \rho(m) \) given by eq. (2.2). If we assume the lowest mass of the system \( m_0 = 0 \), we obtain

\[
N_2(m) = \frac{c^2}{2!} \int_0^m dm_2 \int_0^{m-m_2} dm_1 e^{r_1+r_2}
\]

where

\[
r_i = -a \log \left[ m_i(d + m_i b) \right] + 4\pi m_i(d + m_i b)
\]

and \( b, d \) as fixed by eq.(2.22). We can easily see, that the dominant contribution is obtained when \( m \gg m_2 \) and \( m \approx m_1 \). This is similar to the evaluation of the density of states with \( n \)-strings, in which most of the energy is carried by one body of the system and the \( n-1 \) others share the light remnants. We obtain in this approximation,

\[
N_2(m) \approx \frac{c^2}{2} \left( d m + b m^2 \right)^{a} e^{\frac{b \pi}{2}(d m^2)} , \quad m \to \infty \tag{4.2}
\]
which is precisely the degeneracy of states $\rho (m)$ of the string-black-hole system in which the exponential part dominates. That means that this argument can be extended to any $n$, and we find for our string-black hole system, the same result as in the string and dual models: the number of open channels does grow in parallel with the degeneracy of states, here given by eq.(4.2), as energy is increased.

V. MICROCANONICAL FIELD FORMULATION: SCATTERING AMPLITUDES AND THE STRING/PARTICLE-BLACK HOLE SYSTEM

Is clear from the previous sections that the correct thermodynamical interpretation of the gas of black-holes and strings is in the microcanonical formulation. Let us go further in the understanding of this system and study a field theory description within the microcanonical field (or thermo field) formulation for this system. The physical component of the propagator in the microcanonical field formulation [15] is given by

$$\Delta_{E}^{11}(k) = \frac{1}{k^2 - m^2 + i\epsilon} - 2\pi i\delta (k^2 - m^2) n_{E}(m,k)$$  \hspace{1cm} (5.1)

where the second term corresponds to the statistical (microcanonical) part and $n_{E}(m,k)$ is the microcanonical number density:

$$n_{E}(m,k) = \sum_{l=1}^{\infty} \frac{\Omega(E-l\omega_{k}(m))}{\Omega(E)} \theta(E-l\omega_{k}),$$

$l, \omega_{k}$ and $\theta(E-l\omega_{k})$ being the mode number, the dispersion relation and the step function respectively. In our case, $n_{E}(m,k)$ is explicitly given by

$$n_{E}(m,k) = \sum_{l=1}^{\infty} c \left[ d^2 (E-l\omega_{k}(m))^2 + d(E-l\omega_{k}(m)) \right]^{-\alpha} e^{4\pi d(E-l\omega_{k})} e^{4\pi d^2(E-l\omega_{k})^2} \frac{\theta(E-l\omega_{k})}{c (dE + d^2E^2)^{\alpha} e^{4\pi d E^2}}$$

or (with $E = M$, being $M$ the total mass of the system):

$$n_{M} = \sum_{l=1}^{M/\omega_{k}} \left( 1 - \frac{l\omega_{k} d}{(M d + 1)} \right)^{-\alpha} \left( 1 - \frac{l\omega_{k}}{M} \right)^{\alpha} e^{-4\pi d l\omega_{k}} e^{-8\pi d^2 M l\omega_{k}} e^{4\pi (dl\omega_{k})^2}$$

Let us customize the above expression for the number density as

$$n_{M} = \sum_{l=1}^{M/\omega_{k}} \left( 1 - \frac{l\omega_{k}}{M} \right)^{\alpha} e^{-4\pi d l\omega_{k}} e^{-8\pi d^2 M l\omega_{k}} e^{4\pi (dl\omega_{k})^2}$$

where

$$\zeta = \frac{8\pi d^2 l\omega_{k}}{\alpha}$$

In order to obtain a complete picture of the string-black-hole system, it is instructive to compute explicitly from the microcanonical statistical propagator eq.(5.1) the four point transition amplitude $A$. To do this, it is necessary to insert in eq.(5.1) the expression eq.(5.2) for $n_{M}$ and use the relation between the propagator and the four point transition amplitude process given explicitly in the time ordered perturbation theory. Thus, we find for the total amplitude:

$$A_{total} = A_{0} + A_{str-bh},$$

$$A_{0} = \frac{\lambda^2}{4\pi^2 i} m^2 \delta (p_{1}' + p_{2}' - p_{1} - p_{2}) \left[ \frac{1}{(p_{1}' - p_{1})^2 + m^2 + i\epsilon} + \frac{1}{(p_{2}' - p_{1})^2 + m^2 + i\epsilon} \right],$$

$$A_{str-bh} = -2\pi i \sum_{l=1}^{M/\omega_{str}} \left[ -\frac{l\omega_{str}}{M} \right]^{\alpha} \left( 1 - \frac{l\omega_{str}}{M} \right)^{\alpha} e^{-\zeta}$$

$$\sum_{n=0}^{\infty} \frac{(M \zeta)^{n}}{n!} \left( 1 - \frac{l\omega_{str}}{M} \right)^{\alpha} e^{-\zeta}$$

$$\left( 1 - \frac{lw_{str}d}{(Md + 1)} \right)^{-\alpha} e^{\zeta}$$

(5.4)
\( \chi \equiv 4\pi \left[ (dl\omega(m))^2 - l\omega(m)d \right] \)

\( \lambda_0 \) being the coupling constant of the scattering amplitude and the energy condition is on mass-shell

\[ \varepsilon'_1 \equiv p_0 = \sqrt{m^2 + \vec{p}^2} \]

As is known, the transition amplitude \( A \) (in the momentum representation) between an initial state \(|i\rangle\) and final state \(|f\rangle\) is defined from the \( S \) matrix \((S = 1 + iT)\) as \( \langle f | T | i \rangle = (2\pi)^4 \delta(p' - P) \langle f | A | i \rangle \), \( P', P \) being the initial and final four-momenta. In order to establish and show the kind of relation between the string-particle part and the black-hole part of the spectrum, let us use the formal correspondence \([18]\) between the Feynman propagator \( \Delta_{ij} \) and the expression for dual string amplitudes as \([19]\):

\[ \Delta_{ij} = [s_{ij} + \alpha' M^2 + \alpha(0)]^{-1} \]

where \( s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2 \); \( \alpha(0) \) is the Regge trajectory at zero momentum; \( \alpha(s_{ij}) = \alpha(0) + \alpha's_{ij} \) and \( \alpha'M^2 = \sum_{n=1}^{\infty} \alpha_n^+ \alpha_n^- \) is the usual mass operator. With these definitions, we can see that the expression eq.(5.4) contains the full spectrum of the system: the known string-particle spectrum and on the other hand, the new part \( A_{str-bh} \) describing the string-black hole component of the spectrum:

\[ A \approx A_{str-particle} + A_{str-bh} \]

\[ A_{str-particle} = \frac{\lambda^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \alpha(t))}{n!\Gamma(1 + \alpha(t))} \frac{1}{n + \alpha(s)} \]

where \( s, t \) are the channels of the process in the Mandelstam formulation and we used the well known relation between the Euler Gamma function and the integral representation of the Veneziano amplitude

\[ A_{str-particle} = \int dx x^{p_1 \cdot p_2} (1 - x)^{p_1' \cdot p_2} = \frac{\lambda^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \alpha(t))}{n!\Gamma(1 + \alpha(t))} \frac{1}{n + \alpha(s)} \]

On the other hand, the microcanonical statistical part \( A_{str-bh} \) given explicitly by eq.(5.4) describes the string-black hole component, and we can easily see that in the limit \( l \to M/\omega(m) \) the exponentially increasing mass density of states for the string-black-hole system is recovered:

\[ A_{str-bh} \approx [(Md)^2 + Md]^{-\alpha} e^{4\pi [(Md)^2 + Md]} \approx \rho(m) \]

We analize the mass spectrum of the system in section VI below.

**A. Microcanonical Partition function from the propagators and scattering amplitudes**

Now, we can derive the partition function in the microcanonical field approach from the propagators and the scattering amplitudes computed above. The relation between the \( S \) matrix formulation in Quantum Field Theory and the statistical operator in the canonical ensemble \([16,20]\) can be easily extended to the microcanonical description as follows. The total propagator operator (with its statistical microcanonical part \( G(E) \))

\[ \hat{G}(E) = \frac{1}{E - H - i\epsilon} + G(E), \]

and the \( S \)-matrix are related as

\[ \Im \hat{G}(E) = \Im \hat{G}_0(E) + \frac{1}{4i} \hat{S}^{-1}(E) \frac{\partial}{\partial E} \hat{S}(E) \]

where \( \Im \) stands for the imaginary part, \( \hat{S}(E) \) is the scattering operator at the energy \( E \) and \( \hat{G}_0(E) \) is the free part of the full propagator (which yields non-connected graphs in the cluster expansion). As is known, the relation to the
physical $\hat{T} (E)$ scattering matrix is $\hat{S} (E) = 1 + i \delta \left( E - \hat{H}_0 \right) \hat{T} (E)$, $\hat{H}_0$ being the free Hamiltonian. The logarithm of the trace of the canonical partition function $Z (T, V)$ can be written as [16]

$$\ln Z (T, V) = \ln Z_0 (T, V) + \text{Tr} \int d^4P e^{-\frac{E}{\hbar} \mu \sigma} \frac{-1}{\pi} \delta^3 \left( \mathbf{P} - \mathbf{P}_0 \right) I_m \left[ \hat{G} (E) - \hat{G}_0 (E) \right]_{\text{connected}}$$

where the four-vector temperature $\mathbf{T}^\mu$ is defined by the identity $\mathbf{T}^\mu = \frac{1}{T} u^\mu$, $T$ being the temperature in the rest frame of the box enclosing the thermodynamical system, $u^\mu u_\mu = 1$ being its four-velocity. The index $\text{connected}$ means that only the connected part of the full propagator $\hat{G} (E)$ must be taken, which is simply done by subtracting from the full propagator the free part $\hat{G}_0 (E)$ corresponding to an ideal gas configuration. Recalling that the operation of taking the connected part leaves the operators invariant, we can write

$$\ln Z (T, V) = \ln Z_0 (T, V) + \int e^{-\frac{E}{\hbar} \mu \sigma} d^4P \rho_I (P^2, \mathbf{V}, P, \mathbf{V}^2) , \; \mathbf{V}^\mu = \frac{2V}{(2\pi)^3} u^\mu \; \text{(four-vector volume)}$$

where, by analogy with the interaction level density, the function which we call mass spectrum, or cluster level density, is given by

$$\rho_I (P^2, \mathbf{V}, P, \mathbf{V}^2) = \left\{ \frac{-1}{\pi} \delta \left( \mathbf{P} - \mathbf{P}_0 \right) I_m \left[ \hat{G} (E) - \hat{G}_0 (E) \right] \right\}_{\text{connected}} (5.8)$$

On the other hand, from the pure statistical point of view:

$$\rho_I (P^2, \mathbf{V}, P, \mathbf{V}^2) = \rho - \rho_0$$

$\rho$ being the full density of states of the system under consideration and $\rho_0$ the density of states in the free configuration,

$$\rho_0 = \sum_{k=1}^\infty \mathbf{V}^\mu P_\mu \delta \left( P^2 - k^2 m^2 \right) . \; \; \; \; \; \; (5.9)$$

$\rho$ satisfies the following relation

$$\rho (P^2, \mathbf{V}, P, \mathbf{V}^2) = \mathbf{V}^\mu P_\mu \rho \left( \sqrt{P^2} \right)$$

(5.11)

This condition means that the cluster counting $\rho (P^2, \mathbf{V}, P, \mathbf{V}^2)$ in the rest frame of the system can be reexpressed as the counting of states for a single particle of the mass degeneracy $\rho (m)$ moving in the volume $V$. Now, since the mathematical mapping between the microcanonical and canonical ensembles, is easy to see that the partition function in the microcanonical ensemble we are looking for is given by

$$\ln \Omega (E, V) = \ln \Omega_0 (E, V) + \text{Tr} \int d^4P \left\{ \frac{-1}{\pi} \delta^3 \left( \mathbf{P} - \mathbf{P}_0 \right) I_m \left[ \hat{G} (E) - \hat{G}_0 (E) \right] \right\}_{\text{connected}} (5.12)$$

Considering that the imaginary part $I_m \left[ \hat{G} (E) - \hat{G}_0 (E) \right]$ is the connected part of the full propagator $\hat{G} (E)$, precisely corresponds to the physical component $\Delta_E^{\text{phys}} (k)$ eq.(5.1) of the microcanonical field formulation which leads to the scattering amplitude expression eq.(5.5). From the comparison between the propagator eq.(5.1), (5.2) with the expressions eq.(5.7),(5.8) yielding the level density of states $\rho_I$, we can easily see that the dynamical information from the connected part of the propagator eq.(5.8) is automatically translated into a variation of the energy (mass) levels in the statistical ensemble (given by $\rho (m)$) and the statistical information of the propagator eq.(5.1).

We will consider now the system of black holes and strings as N-body extended objects in a Nambu-Goto formulation and derive the microcanonical propagator and partition function from the Nambu-Goto action of N-body extended objects and from them we will obtain the mass spectrum of the system.

VI. THE NAMBU-GOTO ACTION AND THE STRING-PARTICLE-BLACK HOLE SPECTRUM

It is difficult to study this system in the Hamiltonian framework because of the constraints and the vanishing of the Hamiltonian. As is known, the Nambu-Goto action is invariant under the reparametrizations

$$\tau \to \tilde{\tau} = f_1 (\tau, \sigma) \; \; \; \; \; \; \sigma \to \tilde{\sigma} = f_2 (\tau, \sigma)$$
then, we can make the following choice for the dynamical variable \( x_0 \) and the space variable \( x_1 \), as first proposed in ref. [21] which does not restrict the essential physics and simplifies considerably the dynamics of the system

\[
x_0(\tau, \sigma) \equiv x_0(\tau) ; \quad x_1(\tau, \sigma) \equiv \kappa \sigma \quad (\kappa = \text{const})
\]

For this, it is sufficient to make the chain derivatives and to write the action in the form

\[
S = -\frac{\kappa}{\alpha'} \int_{\tau_1}^{\tau_2} \dot{x}_0 \, d\sigma \, d\tau \sqrt{\left[1 - (\partial_0 x_b)^2\right] \left[1 + (\partial_1 x_a)^2\right]},
\]

where \( a, b = 2, 3; \partial_1 x_a = \varepsilon_{1a} \partial_0 x_b \) and in order to simplify at maximum this action we choose an orthonormal frame; (thus passing from the Nambu-Goto action to the Born-Infeld representation). Therefore, the invariance with respect to the choice of the coordinate evolution parameter means that one of the dynamical variables of the theory \((x_0(\tau)\) in this case) becomes the observed time with the corresponding non-zero Hamiltonian

\[
H_{BI} = \Pi_a \dot{x}^a - L = \sqrt{\alpha^2 - \Pi_a \Pi^a} , \quad \Pi^a = \frac{\partial L}{\partial (\partial_0 x_a)} , \quad \alpha \equiv \frac{\kappa}{\alpha'} \sqrt{1 + (\partial_1 x_a)^2} \quad (6.1)
\]

Now, in order to find the microcanonical partition function of the system from the Nambu-Goto Hamiltonian we proceed as follows: From the most simple quantum path-integral formalism, we have

\[
K(q', t, q, 0) \equiv \left\langle q' \left| (e^{iH})^N \right| q \right\rangle = \left\langle q' \left| \Psi (r, s, t...) \right\rangle \right.
\]

where \( K(q', t, q, 0) \) is the propagator, \( H \) is the Hamiltonian, \( t \) is the time that was fractionated in small lapses \( t = N \varepsilon \) and \( q, q' \) and \( \Psi (r, s, ...) \) are the physical states with \( r, s, ... \) quantum numbers. With the usual path integral operations and introducing the integral representation for a pseudodifferential operator [17]

\[
\int (t^2 + u^2)^{-\lambda} e^{iux} \, dt = \frac{2\pi^{1/2}}{\Gamma(\lambda)} \left[ \frac{|x|}{2u} \right]^{\lambda-1/2} K_{\lambda-1/2} (u |x|)
\]

where \( K_\nu (x) \) is the Mac Donald’s function, the propagator for a sub-interval takes the form

\[
K_{q_j, q_{j+1}} = \delta_{q_j, q_{j+1}} - i \varepsilon \left\langle 4\alpha K_{-1} (\alpha |q_j - q_{j+1}|) \right\rangle
\]

Putting all the subinterval propagators together yields the full propagator

\[
K = \delta_{q_N, q_0} - i N \varepsilon \left\langle 4\alpha K_{-1} (\alpha |q_N - q_0|) \right\rangle
\]

Making, without lost of generality, the transformation \(-it \rightarrow -\beta\), integrating and Fourier transforming to momentum space, yields the canonical partition function

\[
Z = \sum_N \left[ 1 - \beta 4\alpha K_{-1} (\alpha |q_N - q_0|) \right] = \sum_N \exp -\beta \left[ 4\alpha K_{-1} (\alpha |q_N - q_0|) \right] \quad (6.2)
\]

The microcanonical partition function \( \Omega_m \) is obtained as the inverse Laplace transform of the last expression :

\[
\Omega_m = \delta \left\langle E \right\rangle - i \sum_{N=1}^{\infty} \sum_{p_1}^{\infty} \sum_{n_1=1}^{\infty} \ldots \sum_{p_N}^{\infty} \sum_{n_n=1}^{\infty} \left[ 4\alpha K_{-1} (\alpha |\sum_j n_j \varepsilon_j - E|) \right] \frac{1}{n_1 n_2 \ldots n_N}
\]

where the factor \( \frac{1}{n_1 n_2 \ldots n_N} \) allows eliminate the overcounting, and \( \sum_j n_j \varepsilon_j = E_N \).

The microcanonical propagator for the string-particle-black hole system can be consistently formulated using the relation between time ordered products and normal products

\[
-i T \left[ \varphi (x) \varphi (0) \right] = D_F (x) - i : \varphi (x) \varphi (0) :.
\]
where \( D_F(x) = -i \langle J | \varphi(x) \varphi(0) | J \rangle \) is the ordinary Feynman propagator with the expectation value evaluated in the basic states of our system (i.e., for zero temperature)

\[
|J\rangle = \sum_{k, m} \prod_{n_k} \sum_{m} |k, m \rangle \otimes |\bar{n}_{k,m}\rangle
\]

Since the relation between microcanonical and canonical formulations is through a Laplace transform, it is reasonable to perform the following mapping

\[
\int_0^\infty dE' \Omega_{E-E'} D_{E'} = \frac{1}{\Omega_E} \sum_{N=1}^\infty \sum_{n_1=1}^\infty \cdots \sum_{n_N=1}^\infty \left[ 4\alpha \frac{K_{-1} (\alpha \sum n_j \varepsilon_j - E)}{E^2} \right] \langle J : \varphi(x) \varphi(0) : E \rangle
\]

(6.3)

where \( D_{E'} \) is the microcanonical propagator and we defined the microcanonical string/particle-black hole state as

\[
|E\rangle = \frac{1}{\Omega_E} \int_0^\infty dE' \Omega_{E-E'} L^{-1} e^{\beta |\beta|}
\]

\( L^{-1} \) being the inverse Laplace transform. The matrix element for the most general states in our system is

\[
\langle J : \varphi(x) \varphi(0) : E \rangle = \frac{1}{\Omega_E} \sum_{\mathcal{F}} \left[ \frac{n}{\varepsilon_p V} \cos (p.x - \varepsilon pt) \right]
\]

By inserting this expression into the definition of the microcanonical propagator \( D_{E'} \) eq. (6.3) given above and converting the momentum sum into an integral, we have

\[
D_E(t, \vec{p}) = \delta (E) D_F - 4i\alpha \int \frac{d^3 p}{(2\pi)^3} \sum_{\varepsilon_j=1}^\infty \frac{K_{-1} (\alpha |n_j \varepsilon_j - E|)}{E^2} \Omega (E-n_j \varepsilon_j) \cos (p.x - \varepsilon_j t)
\]

Finally, Fourier transform to momentum representation gives the microcanonical propagator of the system:

\[
D_E(k, m) = \frac{\delta (E)}{\omega^2 - k^2 - m^2 + i\varepsilon} - D_{E (str-bh)}(k, m)
\]

(6.4)

\[
D_{E (str-bh)}(k, m) = 8\pi i\alpha \theta (\omega^2 - k^2 - m^2) \sum_{l=1}^\infty \frac{K_{-1} (\alpha |l\omega_k - E|)}{E^2} \Omega (E-l\omega_k) \theta (E-l\omega_k)
\]

(6.5)

where \( \theta (x) \) is the usual step function and \( K_{-1} \) is the Mac Donald’s function. The first term in the microcanonical propagator is the usual Feynman propagator, the second one is the new microcanonical part \( D_{E (str-bh)}(k, m) \). This part is crucial for the correct description of the full string-particle-black hole system, as we can see by explicitly expanding the Mac Donald’s function \( K_{-1} \) in the second term of the microcanonical propagator eq. (6.4) and expressing it as a function of the mass, being \( E = M \) the total mass of the system:

\[
D_{E (str-bh)}(k, m) = 8\pi i\alpha \theta (\omega^2 - k^2 - m^2) \sum_{l=1}^{M/\omega_k} \ln \left( \frac{\gamma_s \alpha |l\omega_k - M|}{2} \right) \frac{\alpha |l\omega_k - M|}{2} \sum_{s=0}^\infty \frac{\alpha |l\omega_k - M|/2^{s+1}}{s! \Gamma(s+2)} -
\]

\[
- \frac{1}{2} \sum_{s=0}^\infty \frac{\alpha |l\omega_k - M|/2^{s+1}}{s! \Gamma(s)} \left( \sum_{h=1}^s \frac{1}{h} + \sum_{h=1}^{s+1} \frac{1}{h} \right) + \frac{\alpha |l\omega_k - M| \theta (M-l\omega_k) \Omega (M-l\omega_k)}{4 \Omega (M)}
\]

(6.6)

We see that when \( M \to 0 \) this expression yields the pure string-like behaviour (Gamma type string-amplitude). When \( M \) grows it pass for all the intermediate states up to the pure black-hole behaviour. Notice that, previously [18],[19], the relation between the Feynman propagator and the Veneziano amplitude was putted "by hand". We see that this type of structure coming from the statistical microcanonical part is contained in our microcanonical propagator eq. (6.5). Notice that the relation between temporal and normal ordering of the field operators contributes to the statistical part of this full propagator. Explicitly, for \( a > 0 \) the spectrum (singular points of the microcanonical
part) is given by the following expression

\[
\sum_{l=1}^{M/\omega_k} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{8\pi l\omega_k d^2 M}{a} \left( 1 - \frac{8\pi (l\omega_k d)^2}{a(n+1)} \right) \right]^{n} = \sum_{l=1}^{M/\omega_k} \frac{l\omega_k}{M}
\]

(6.7)

It is interesting to note that the main difference between our microcanonical propagator eq.(6.4) and the propagator \(\Delta_{11}\) of the previous section is in that the propagator eq.(6.4) includes all non-local effects: from the \(N\)-bodies of the system as extended objects (i.e. strings) and from the derivation of this propagator from a theory with a Hamiltonian not quadratic in the momenta in the as Nambu-Goto formulation of string theory. For instance, the propagator eq.(6.4) becomes the propagator eq.(5.1) when the extended bodies became point-particles and the Hamiltonian is quadratic in the momenta (constrained particle Hamiltonian. Let us analyze the excited regime \(l \to M/\omega_k\).

The microcanonical part of the propagator eq.(6.4) in the limit \(l \to M/\omega_k\) takes the form

\[
D_E(\text{str-bh})(k,m) = 8\pi i\alpha \delta(\omega^2 - k^2 - m^2) \frac{1}{2M^2} \left\{ 2e^{\frac{8\pi kdM}{a}} \left[ \sum_{n=0}^{\infty} \left( \frac{a}{8\pi} \right)^n \left( 1 - \frac{8\pi (Md)^2}{a(n+1)} \right) \right] - 1 \right\}^{-a} \times \frac{1}{Md+1} 
\]

\[\times e^{4\pi M d(Md-1)} \lim_{l\to M/\omega_k} \frac{1}{|l\omega_k - M|} \]

(6.8)

and the spectrum is given by

\[
e^{-\left(\frac{8\pi kdM}{a}\right)^2} = 2 \sum_{n=0}^{\infty} \left( \frac{a}{8\pi} \right)^n \left( 1 - \frac{8\pi (Md)^2}{a(n+1)} \right)
\]

For \(n >> 1\) the spectrum becomes

\[
\left( \frac{8\pi d}{a} M \right)^2 \simeq n \ln \left( \frac{8\pi}{a} \right) - 2
\]

(6.9)

This is the spectrum of the system in the very quantum regime, and we see that this is like the pure quantum string spectrum. That is, quantum black holes have their mass quantized as quantum strings.

VII. CONCLUSIONS

We have first considered the ideal gase of black holes and strings in a microcanonical formulation and analyzed the microcanonical content of this system (sections II and III). We then considered the system with interactions, first in a quantum field theory microcanonical formulation (sections IV,V), second as N-body extended objects in a Nambu Goto formulation, from which we found the propagator, scattering amplitudes and mass spectrum. This study allowed us to describe a wide class of properties and physical magnitudes of the system, covering the different mass ranges and the classical, semiclassical and quantum behaviours of the system.

In the gase of strings and black-holes, the global behaviour of the string-particle-black hole system turns to be divided in two main cases: \(a = 5/2\) and \(a \neq 5/2\), but several relevant features are generic, common to both cases, as the fact that the Hawking temperature and the string temperature emerge in the asymptotic large energy regime and in the small energy regime respectively. We found that the temperature of the system is given by

\[
T = \frac{T_0}{1 + 8\pi EbT_0 + F(E)T_0}
\]

with \(F(E)/E \to 0\) for \(E \to \infty\) and \(F(E) \to 0\) for \(E \to 0\). The function \(F(E)\) has been explicitly obtained in the two different cases of interest: \(a = 5/2\) and \(a \neq 5/2\).

For \(a = 5/2\), the number of components \(N\) does not depend on the energy \(E\), but depends on the minimum mass of the system \(m_0\). The microcanonical density of states \(\Omega_N(E)\) presents a maximum near the point \(E \to 0\). The pressure \(P\) is inversely proportional to the energy \(E\). The temperature presents a clear critical point \(T_0 = (4\pi d)^{-1}\).
Superheating points for which $T \to \infty$, do appear: at $E_{\text{low}} \approx \sqrt{T_0}$ for low energies and at $E_{\text{high}} \approx T_0$ for high energies. For high energies (large masses) the temperature of the system is essentially the Hawking temperature $T_H$ plus a (non logarithmic) correction, while at low energies (small masses) the temperature is practically the string temperature $T_0$. For $N >> 1$ the specific heat behaves as $C_v \approx -1/(8\pi b T^2)$.

For $a \neq 5/2$, the particle density number $N/V$ is a function of the energy density $E/V$. The density of states $\Omega_N(E)$ presents a maximum near the point $E \approx N m_0$. The temperature presents the critical point $T_0 = (4\pi d)^{-1}$ and there are no superheating points. For $E \to n m_0$ (lowest energy of the system) as well as for $E \to \infty$, the temperature vanishes, while for $E \to 0$, $T$ goes to the critical point $T \to T_0$. For $N >> 1$, the temperature goes to the Hawking temperature $T_H$ plus a (logarithmic) correction. The pressure is negative with the form $P = -NT \ln N/(\frac{5}{2} - a)$. The specific heat $C_v$ is negative, for low energies it behaves as $C_v \approx -1/8\pi b T_0^2 \approx \text{const.}$, while for high energies $C_v$ vanishes, putting in evidence the asymptotic stability point of the system.

For strings, the parameter $a$ is related to the number of space-time dimensions $D$: $a = D$ (closed strings) or $a = (D - 1)/2$ (open strings). Thus, the relevant cases for the number of dimensions of interest in string theory ($D = 4, 10$ or $26$), are the cases $a \neq 5/2$, although it is useful for the sake of completeness, comparison and universality properties of the system to analyze the case $a = 5/2$ too, as both cases appear together.

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**A. References**

[1] N.G. Sanchez, IJMP A19, 4173 (2004).
[2] M. Ramon Medrano and N. Sanchez, Phys. Rev. D61, 084030 (2000).
[3] A. Bouchareb, M. Ramon Medrano and N. Sanchez, IJMP A22, 1627 (2007).
[4] M. Ramon Medrano and N. Sanchez, Phys. Rev. D60, 125014 (1999).
A. Bouchareb, M. Ramon Medrano and N. Sanchez, IJMP D6, 1053 (2007).
[5] M. Ramon Medrano and N. Sanchez, MPLA18, 2537 (2003).
M. Ramon Medrano and N. Sanchez, MPLA22, 1133 (2007).
[6] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[7] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[8] R. Hagedorn, Nuovo Cimento 52 A, 1336 (1967).
[9] R. Hagedorn, Suppl. Nuovo Cimento 3, 147 (1965).
[10] S. Frautschi, Phys. Rev. D3, 2821 (1971).
[11] S. Carlitz, Phys. Rev. D5, 3231 (1972).
[12] B. Harms and Y. Leblanc, Phys. Rev. D46, 2334 (1992).
[13] B. Harms and Y. Leblanc, Phys. Rev. D47, 2438 (1993).
[14] R. Casadio, B. Harms and Y. Leblanc, arXiv:gr-qc9706005.
[15] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States*, North Holland Publishing Co., Amsterdam (1982).

[16] R. Dashen, S. Ma, H.J. Bernstein, Phys. Rev. **187**, 345 (1969).

[17] Yu. A. Brichkov and A. P. Prudnikov, *Integral transform of General Functions*, Nauka, Moscow (1977). (In Russian).

[18] J. Scherk, Rev. Mod. Phys. **47**, 123 (1975).

[19] G. Veneziano, Phys. Rept. **9**, 199 (1974).

[20] L. Sertorio and M. Toller, N.C., **14 A**, 21 (1973).

[21] B.M. Barbashov, N.A. Chernikov, Zh. Eksp. Theor. Fiz. **50**, 1296 (1966). Commun. Math. Phys. **5**, 313 (1966).