SOBOLEV SPACES WITH NON-ISOTROPIC DILATIONS AND SQUARE FUNCTIONS OF MARCINKIEWICZ TYPE

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ABSTRACT. We consider the weighted Sobolev spaces associated with non-isotropic dilations of Calderón-Torchinsky and characterize the spaces by the square functions of Marcinkiewicz type including those defined with repeated uses of averaging operation.

1. INTRODUCTION

Let $B(x,t)$ be a ball in $\mathbb{R}^n$ with radius $t$ centered at $x$ and for $0 < \alpha < 2$ let

\[(1.1) \quad V_{\alpha}(f)(x) = \left( \int_0^\infty \left| f(x) - \int_{B(x,t)} f(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \]

where $\int_{B(x,t)} f(y) \, dy$ denotes $|B(x,t)|^{-1} \int_{B(x,t)} f(y) \, dy$ and $|B(x,t)|$ the Lebesgue measure. In [1] the operator $V_1$ was used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$ as follows.

**Theorem A.** Let $1 < p < \infty$. Then, $f$ belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $V_1(f) \in L^p(\mathbb{R}^n)$; furthermore,

$$\|V_1(f)\|_p \lesssim \|\nabla f\|_p,$$

which means that there exist positive constants $c_1$, $c_2$ independent of $f$ such that

$$c_1 \|V_1(f)\|_p \lesssim \|\nabla f\|_p \lesssim c_2 \|V_1(f)\|_p.$$

Let $S(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. Define

$$S_0(\mathbb{R}^n) = \{ f \in S(\mathbb{R}^n) : \hat{f} \text{ vanishes near the origin} \},$$

where the Fourier transform $\hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle x,\xi \rangle} \, dx, \quad \langle x,\xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

We also write $\mathcal{F}(f)$ for $\hat{f}$. For $0 < \alpha < n$, $n \geq 2$, let $I_\alpha$ be the Riesz potential operator defined by

\[(1.2) \quad \mathcal{F}(I_\alpha(f))(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi), \quad f \in S_0.
\]
Theorem B. Let $0 < \alpha < 2$ and $1 < p < \infty$. Then
\[
\|S_\alpha(f)\|_p \simeq \|f\|_p.
\]

Theorem A can be derived from this result with $\alpha = 1$ when $n \geq 2$.

The operator $S_\alpha$ is a kind of the Littlewood-Paley operators. Let $\psi \in L^1(\mathbb{R}^n)$ satisfy
\[
(1.4) \quad \int_{\mathbb{R}^n} \psi(x) \, dx = 0.
\]

Put $\psi_t(x) = t^{-n} \psi(t^{-1} x)$. Then the Littlewood-Paley function on $\mathbb{R}^n$ is defined by
\[
(1.5) \quad g_\psi(f)(x) = \left( \int_0^\infty \left| f \ast \psi_t(x) \right|^2 \frac{dt}{t} \right)^{1/2}.
\]

We can see that $S_\alpha(f) = g_{\psi^{(\alpha)}}(f)$, where
\[
(1.6) \quad \psi^{(\alpha)}(x) = L_\alpha(x) - \Phi \ast L_\alpha(x),
\]
with
\[
L_\alpha(x) = \tau(\alpha) |x|^{\alpha-n}, \quad \tau(\alpha) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2} \Gamma(\alpha/2)}
\]
and $\Phi = \chi_0$, $\chi_0 = |B(0,1)|^{-1} \chi_{B(0,1)}$ ($\chi_E$ denotes the characteristic function of a set $E$). We note that $\mathcal{F}(L_\alpha)(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$.

The square function $S_1(f)$ is closely related to the function of Marcinkiewicz on $\mathbb{R}^1$, which is defined by
\[
(1.7) \quad \|\mu(f)\|_p \simeq \|f\|_p,
\]
for $1 < p < \infty$. Also, we consider a variant of $\mu(f)$, which can be regarded as an analogue of $S_1$ in the one-dimensional case:
\[
\nu(f)(x) = \left( \int_0^\infty |F(x + t) + F(x - t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]
where $F(x) = \int_{-\infty}^x f(y) \, dy$, $f \in S(\mathbb{R})$. It is known that

An interesting feature of Theorem A is that it suggests the possibility of defining the Sobolev space analogous to $W^{1,p}(\mathbb{R}^n)$ in metric measure spaces in a reasonable way. In this note, we shall extend Theorem A to the case of the weighted Sobolev spaces with the parabolic metrics of Calderón-Torchinsky [3, 4].
Let $P$ be an $n \times n$ real matrix, $n \geq 2$, such that

\begin{equation}
\langle Px, x \rangle \geq \langle x, x \rangle \quad \text{for all } x \in \mathbb{R}^n.
\end{equation}

A dilation group $\{\delta_t\}_{t>0}$ on $\mathbb{R}^n$ is defined by $\delta_t = tP = \exp((\log t)P)$.

It is known that $|\delta_t x| = |\delta_t x, \delta_t x|^{1/2}$ is strictly increasing as a function of $t$ on $(0, \infty)$ when $x \neq 0$. Let $\rho(x)$, $x \neq 0$, be the unique positive real number $t$ such that $|\delta_{t^{-1}} x| = 1$ and let $\rho(0) = 0$. Then the norm function $\rho$ is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ and satisfies that $\rho(A_t x) = t \rho(x)$, $t > 0$, $x \in \mathbb{R}^n$. We have the following properties of $\rho(x)$ (see [3, 5]):

1. $\rho(-x) = \rho(x)$ for all $x \in \mathbb{R}^n$;
2. $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}^n$;
3. $\rho(x) \leq 1$ if and only if $|x| \leq 1$;
4. $c_1 \rho(x)\tau_1 \leq |x| \leq \rho(x)$ when $|x| \leq 1$ for some $c_1, \tau_1 > 0$;
5. $\rho(x) \leq |x| \leq c_2 \rho(x)^{\tau_2}$ when $|x| \geq 1$ for some $c_2, \tau_2 > 0$.

Also,

(a) $|\delta_t x| \geq t|x|$ for all $x \in \mathbb{R}^n$ and $t \geq 1$;
(b) $|\delta_t x| \leq t|x|$ for all $x \in \mathbb{R}^n$ and $0 < t \leq 1$.

Let $\delta_t^*$ denote the adjoint of $\delta_t$. Then, we can also consider similarly a norm function $\rho^*(x)$ associated with the dilation group $\{\delta_t^*\}_{t>0}$, and we have properties of $\rho^*(x)$ and $\delta_t^*$ analogous to those of $\rho(x)$ and $\delta_t$ above. It is known that a polar coordinates expression for the Lebesgue measure

\begin{equation}
\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \int_{S^{n-1}} f(\delta_t \theta) t^{\gamma - 1} s(\theta) \, d\sigma(\theta) \, dt
\end{equation}

holds, where $\gamma = \text{trace } P$ and $s$ is a strictly positive $C^\infty$ function on $S^{n-1} = \{ |x| = 1 \}$ and $d\sigma$ is the Lebesgue surface measure on $S^{n-1}$ (see [7, 10, 29]). We note that the condition \(1.8\) implies that all eigenvalues of $P$ have real parts greater than or equal to 1 (see [3, pp. 3–4], [13, p. 137]). So we have $\gamma \geq n$.

Let

\begin{equation}
B(x, t) = \{ y \in \mathbb{R}^n : \rho(x - y) < t \}
\end{equation}

be a ball with respect to $\rho$ (a $\rho$-ball) in $\mathbb{R}^n$ with radius $t$ centered at $x$. We say that a weight function $w$ belongs to the Muckenhoupt class $A_p$, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_B \left( |B|^{-1} \int_B w(x) \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all $\rho$-balls $B$ in $\mathbb{R}^n$. The Hardy-Littlewood maximal operator $M$ is defined as

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| \, dy,$$

where the supremum is taken over all $\rho$-balls $B$ in $\mathbb{R}^n$ containing $x$. The class $A_1$ is defined to be the family of weight functions $w$ such that $M(w) \leq C w$ almost everywhere; the infimum of all such $C$ will be denoted by $[w]_{A_1}$. We denote by $L^p_w$ (we also write $L^p(w)$ for $L^p_w$) the weighted $L^p$ space with the norm defined as

$$\|f\|_{L^p_w} = \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}.$$
See [2], [6], [9], [30] for results related to the weight class $A_p$. The following results are known and useful.

**Proposition 1.1.** Let $1 < p < \infty$, $w \in A_p$.

(i) The space $S_0$ is dense in $L^p_w$.

(ii) The maximal operator $M$ is bounded on $L^p_w$.

(iii) If $\varphi \in S$, then $\sup_{t>0} |f * \varphi_t| \leq CM(f)$. Here and in what follows $\varphi_t(x) = t^{-\alpha} \varphi(t^{-1} x)$.

(iv) $\mathcal{F}(g * \varphi_t)(\xi) = \hat{g}(\xi) \hat{\varphi}(\delta_t^* \xi)$ for $g, \varphi \in S$.

Let $\beta \in \mathbb{R}$ and define the Riesz potential operator $\mathcal{I}_\beta$ associated with the dilations $\delta_t^*$ by

$$
\mathcal{I}(\mathcal{I}_\beta(f))(\xi) = \rho^*(\xi)^{-\beta} \hat{f}(\xi)
$$

for $f \in S_0$. Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$. Define the weighted parabolic Sobolev space $W^{\alpha, p}_w$ by

$$
W^{\alpha, p}_w = \{ f \in L^p_w : f = \mathcal{I}_\alpha(g) \text{ for some } g \in L^p_w \},
$$

where $f = \mathcal{I}_\alpha(g)$ means that

$$
\int_{\mathbb{R}^n} f(x) h(x) \, dx = \int_{\mathbb{R}^n} g(x) \mathcal{I}_\alpha(h) \, dx \quad \text{for all } h \in S_0.
$$

We note that the function $g \in L^p_w$ is uniquely determined by $f$, since $\mathcal{I}_\alpha$ is a bijection on $S_0$ and $S_0$ is dense in $L^p(w^{-\rho'/p})$, the dual space of $L^p(w)$, where $1/p + 1/p' = 1$. We write $g = \mathcal{I}_{-\alpha}(f)$.

We have analogues of Theorems A and B in the case of non-isotropic dilations $\delta_t$ with weights. Let $B(x,t)$ be as in (1.11) and

$$
B_\alpha(f)(x) = \left( \int_0^\infty \left| f(x) - \int_{B(x,t)} f(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0.
$$

**Theorem 1.2.** Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < 2$. Then $f \in W^{\alpha, p}_w$ if and only if $f \in L^p_w$ and $B_\alpha(f) \in L^p_w$; also,

$$
\| \mathcal{I}_{-\alpha}(f) \|_{p,w} \approx \| B_\alpha(f) \|_{p,w}.
$$

Let

$$
C_\alpha(f)(x) = \left( \int_0^\infty \left| \mathcal{I}_\alpha(f)(x) - \int_{B(x,t)} \mathcal{I}_\alpha(f)(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}.
$$

Then, Theorem 1.2 can be derived from the following result.

**Theorem 1.3.** Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha < 2$ and let $C_\alpha$ be as in (1.15). Then

$$
\| C_\alpha(f) \|_{p,w} \approx \| f \|_{p,w}, \quad f \in S_0(\mathbb{R}^n).
$$

The range of $\alpha$ in Theorem 1.2 will be extended in Theorem 4.2 in Section 4 by considering square functions with repeated uses of averaging operation $f_p$. We consider square functions generalizing $B_\alpha$ and $C_\alpha$ in (1.14) and (1.15). Let $\Phi$ be a bounded function on $\mathbb{R}^n$ with compact support. We say $\Phi \in \mathcal{M}^\alpha$, $\alpha \geq 0$, if $\Phi$ satisfies
Theorem 1.4. Let $H_\alpha$ be as in (1.18) and $0 < \alpha < \gamma, 1 < p < \infty, w \in A_p$. Then

$$\|H_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in S_0(\mathbb{R}^n).$$

Applying Theorem 1.4 we have the following.

**Theorem 1.5.** Suppose that $1 < p < \infty, w \in A_p$ and $0 < \alpha < \gamma$. Let $G_\alpha$ be as in (1.17). Then $f \in W^\alpha_w$ if and only if $f \in L^p_w$ and $G_\alpha(f) \in L^p_w$; furthermore,

$$\|\mathcal{L}_-\alpha(f)\|_{p,w} \simeq \|G_\alpha(f)\|_{p,w}.$$  

Theorems 1.2 and 1.3 follow from Theorems 1.5 and 1.4, respectively. The proofs of Theorems 1.4 and 1.5 will be given in Section 3. To prove Theorem 1.4 we consider Littlewood-Paley functions

$$g_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where $\psi_t(x) = t^{-\gamma} \psi(t^{-1}x)$ with $\psi \in L^1(\mathbb{R}^n)$ satisfying (1.4). Then we can see that $H_\alpha(f) = g_{\psi(\alpha)}$ for some $\psi(\alpha)$ analogous to the one in (1.6). We shall prove Theorem 1.4 by applying Theorem 2.1 below in Section 2 which is a result for parabolic Littlewood-Paley functions complementing the boundedness result given in [25] and generalizing [22, Corollary 2.6] to the case of non-isotropic dilations.

The proof of Theorem 2.1 will be completed by applying Theorem 2.8 below in Section 2 which provides the estimates

$$\|f\|_{p,w} \leq C\|g_\psi(f)\|_{p,w}$$

under certain conditions. Theorem 2.8 is proved by Corollary 2.7 in Section 2 which is a result on the invertibility of Fourier multipliers homogeneous of degree 0 with respect to $\delta^\alpha$ generalizing [22, Corollary 2.6] to the case of general homogeneity. Corollary 2.7 will follow from a more general result (Theorem 2.3).

Here we review some recent developments of the theory related to the results given in this note after the article [1] (see also the remarks at the end of this note).
Theorem A was generalized to the weighted Sobolev spaces by [10]. Also, Theorems A and B were extended to the weighted Sobolev spaces in [19] by applying a theorem in [17] for the boundedness of Littlewood-Paley functions $g_\psi$ in $L^p$ spaces, which is partly a special case of Theorem 2.1 in Section 2. In [19] it was shown that the theorem of [17] is particularly suitable for handling the square functions in Theorem 1.4 above for the case of the Euclidean structures (with the Euclidean norm and the ordinary dilation). Some results of [19] were generalized in [22] by introducing the function class $M_\alpha$ and by proving the weighted $L^p$ norm equivalence between $g_\psi(f)$ in (1.5) and $f$, part of which was not included in [17]; the estimates in (1.20) in the case of the Euclidean structures for sufficiently large class of $\psi$ and $p \in (1, \infty)$, $w \in A_p$ were absent from [17]. In [20] and [22], discrete parameter versions of Littlewood-Paley functions $g_\psi(f)$ in (1.5) of the form

$$\Delta_\psi(f)(x) = \left( \sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2 \right)^{1/2}$$

are also considered to characterize the Sobolev spaces. See also [10] and [21] for applications of the square function $D_\alpha(f)(x) = \left( \int_0^\infty |t^{-\alpha} \int_{S^{n-1}} (f(x-t\theta) - f(x)) d\sigma(\theta)|^2 \frac{dt}{t} \right)^{1/2}$ in the theory of Sobolev spaces.

In Section 4, we shall establish another characterization of the Sobolev spaces $W^{\alpha,p}_w$ similar to Theorem 1.2 (Theorem 4.2), which is novel even in the case of the Euclidean structures. In Theorem 1.2, the averaging operator $\mathfrak{f}_B f$ is used to define the square function $B_\alpha(f)$ in (1.13) which is applied to characterize $W^{\alpha,p}_w$ for $\alpha \in (0, 2)$. In Theorem 4.2 we shall extend the range of $\alpha$ by introducing square functions which are defined with repeated uses of averaging operation.

Finally, in Section 5 we shall illustrate by example how the Sobolev spaces $W^{\alpha,p}_w$ defined above can be characterized by distributional derivatives in some cases, by the arguments similar to the one in the proof of [28] Theorem 3 of Chap. V.

2. Invertibility of Fourier multipliers homogeneous with respect to $\delta_0^*$ and Littlewood-Paley operators

We consider a majorant of $\psi$ defined by

$$H_\psi(x) = h(\rho(x)) = \sup_{\rho(y) \geq \rho(x)} |\psi(y)|$$

and two seminorms $B_\epsilon$ and $D_u$ defined as

$$B_\epsilon(\psi) = \int_{|x| > 1} |\psi(x)| |x|^\epsilon \, dx \quad \text{for} \quad \epsilon > 0,$$

$$D_u(\psi) = \left( \int_{|x| < 1} |\psi(x)|^u \, dx \right)^{1/u} \quad \text{for} \quad u > 1.$$
Then $g_{\psi}$ defined by (1.19) is bounded on $L^p_w$:

$\|g_{\psi}(f)\|_{p,w} \leq C\|f\|_{p,w}$ for all $p \in (1, \infty)$ and $w \in A_p$,

where the constant $C$ depends only on $p$, $w$, $\epsilon$, $u$ and $C_j$, $1 \leq j \leq 3$ and does not depend on $\psi$ in the other regards. If we further assume the non-degeneracy condition:

$\sup_{t>0} |\hat{\psi}(\delta_t^* \xi)| > 0$ for $\xi \neq 0$,

then we also have the reverse inequality of (2.1) and hence

$\|g_{\psi}(f)\|_{p,w} \simeq \|f\|_{p,w}$ for all $p \in (1, \infty)$ and $w \in A_p$.

By [25, Theorem 1.1], which generalizes a result of [17] to the case of non-isotropic dilations, we have the boundedness (2.1) under the conditions (1), (2), (3) of Theorem 2.1 and the quantitative property of the constant $C$ specified follows by checking the proof given in [25]. The proof of [25, Theorem 1.1] is based on estimates for certain oscillatory integrals in [18].

**Remark 2.2.** If there exist positive numbers $\sigma_1, \sigma_2$ such that

$|\psi(x)| \leq C(1 + \rho(x)^{-1})^{\gamma - \sigma_1}(1 + \rho(x))^{-\gamma - \sigma_2}$ for all $x \in \mathbb{R}^n$,

then the conditions (1), (2), (3) of Theorem 2.1 are satisfied with some $\epsilon$, $u$ and $C_j$, $1 \leq j \leq 3$. To see this the formula (1.19) is useful.

To prove the reverse inequality of (2.1), we apply a result on the invertibility on the weighted $L^p$ spaces of Fourier multipliers homogeneous with respect to $\delta_t^*$. Let $m \in L^\infty(\mathbb{R}^n)$, $w \in A_p$, $1 < p < \infty$. The Fourier multiplier operator $T_m$ is defined by

$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i (x, \xi)} \, d\xi$.

We say that $m$ is a Fourier multiplier for $L^p_w$ and write $m \in M^p_w$ (we also write $M^p(w)$ for $M^p_w$) if there exists a constant $C > 0$ such that

$\|T_m(f)\|_{p,w} \leq C\|f\|_{p,w}$ for all $f \in S$.

We define $\|m\|_{M^p(w)}$ to be the infimum of the constants $C$ satisfying (2.4). Since $S$ is dense in $L^p_w$, we have a unique extension of $T_m$ to a bounded linear operator on $L^p_w$ if $m \in M^p_w$. We observe that $M^p(w) = M^{p'}(\bar{w}^{-p'}/p)$ by duality, where $\bar{w}(x) = w(-x)$. See [12] for relevant results.

We need the following result generalizing [22, Theorem 2.5] to the case of non-isotropic dilations.

**Theorem 2.3.** Let $m$ be a bounded function on $\mathbb{R}^n$ which is continuous on $\mathbb{R}^n \setminus \{0\}$. Suppose that $m$ is homogeneous of degree 0 with respect to $\delta_t^*$ and that $m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Also, suppose that $m \in M^p_r$ for all $r \in (1, \infty)$ and all $v \in A_v$. Let $1 < p < \infty$, $w \in A_p$ and let $F(z)$ be holomorphic in $D = \mathbb{C} \setminus \{0\}$. Then $F(m(\xi)) \in M^p_w$. 
For $m \in M^p_w$, $1 < p < \infty$, $w \in A_p$, we consider the spectral radius operator

$$\rho_{p,w}(m) = \lim_{k \to \infty} \|m^k\|^{1/k}_{M^p_w}.$$

To prove Theorem 2.3 we need the following.

**Proposition 2.4.** Suppose that $1 < p < \infty$, $w \in A_p$ and $m \in L^\infty(\mathbb{R}^n)$. Let $m$ be homogeneous of degree 0 with respect to the dilations $\delta_1^n$ and continuous on $S^{n-1}$. We assume that $m \in M^p_w$ for all $r \in (1, \infty)$ and all $v \in A_r$. Then, for any $\epsilon > 0$, there exists $\ell \in M^p_w$ which is homogeneous of degree 0 with respect to $\delta_1^n$ and in $C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\|m - \ell\|_\infty < \epsilon$ and $\rho_{p,w}(m - \ell) < \epsilon$.

To prove Proposition 2.4 we apply the following lemmas.

**Lemma 2.5.** Let $\eta \in C^\infty(\mathbb{R})$, supp $\eta \subset [1, 2]$, $\eta \geq 0$ and $\int_{\mathbb{R}} |\eta(t)|^2 dt/t = 1$. Define a real function $p_0(\xi) = (\eta^\prime(\xi))$. Then

$$\|\eta(f)\|_{p,w} \leq C \|f\|_{p,w}$$

for all $p \in (1, \infty)$ and $w \in A_p$.

**Lemma 2.6.** Suppose that $m \in L^\infty(\mathbb{R}^n)$, $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and that $m$ is homogeneous of degree 0 with respect to $\delta_1^n$. Then $m \in M^p_w$ for all $p \in (1, \infty)$ and $w \in A_p$ and

$$\|m\|_{M^p_w} \leq C \sup_{1 \leq \rho^\prime(\xi) \leq 2, |\xi| \leq 1} |(\partial_\xi)^am(\xi)|$$

with a constant $C$ independent of $m$, where $(\partial_\xi)^a = (\partial/\partial_\xi_1)^{a_1} \cdots (\partial/\partial_\xi_n)^{a_n}$ with $a = (a_1, \ldots, a_n)$, $a_j \in \mathbb{Z}$, $a_j \geq 0$, $1 \leq j \leq n$.

Proof of Lemma 2.5. By [25] Theorem 1.1 we see that $\|\eta(f)\|_{p,w} \leq C \|f\|_{p,w}$ for all $p \in (1, \infty)$ and $w \in A_p$. To prove the reverse inequality we note that $\|\eta(f)\|_2 = \|f\|_2$. Thus the polarization implies that for real-valued $f, h \in \mathcal{S}$

$$4 \int_{\mathbb{R}^n} f(x)h(x) dx = \int_{\mathbb{R}^n} (f(x) + h(x))^2 dx - \int_{\mathbb{R}^n} (f(x) - h(x))^2 dx$$

$$= \int_{\mathbb{R}^n} (g_\psi(f + h)(x))^2 dx - \int_{\mathbb{R}^n} (g_\psi(f - h)(x))^2 dx$$

$$= \int_{\mathbb{R}^n} \int_0^\infty f * \psi(x)h * \psi(x) \frac{dt}{t} dx.$$

Therefore, by the inequalities of Schwarz and Hölder we have

$$\left| \int_{\mathbb{R}^n} f(x)h(x) dx \right| \leq \|g_\psi(f)\|_{p,w} \|g_\psi(h)\|_{p^\prime,w-p^\prime/p} \leq C \|g_\psi(f)\|_{p,w} \|h\|_{p^\prime,w-p^\prime/p}.$$

Taking the supremum in $h$ with $\|h\|_{p^\prime,w-p^\prime/p} \leq 1$, we have $\|f\|_{p,w} \leq C \|g_\psi(f)\|_{p,w}$, from which we can derive the desired estimates for complex valued functions. \hfill \Box

Proof of Lemma 2.6. Let $\psi$ be as in Lemma 2.5 and define $\psi_m$ by $\mathcal{F}(\psi_m)(\xi) = \hat{\psi}(\xi)m(\xi)$. Then $g_\psi(T_m f) = g_{\psi_m}(f)$. So, by Lemma 2.5 for $w \in A_p$, $1 < p < \infty$, we have

$$\|T_m f\|_{p,w} \leq C \|g_\psi(T_m f)\|_{p,w} = C \|g_{\psi_m}(f)\|_{p,w}.$$
Since $ψ_m ∈ S_0$, $g_{ψ_m}$ is bounded on $L^p_w$. To specify the operator bounds, we apply the estimates (2.1). It is sufficient to observe the following estimates:

\[(2.6) \quad |ψ_m(x)| = \left| \int_{\mathbb{R}^n} \hat{ψ}(ξ) m(ξ) e^{2πi(x,ξ)} \, dξ \right| \leq C(1 + |x|)^{-|γ|-1} \sup_{1 \leq ρ^*(ξ) ≤ 2, |a| ≤ |γ|+1} |(∂^a_ξ)^{\alpha} m(ξ)|,
\]

which follows by integration by parts, with the constant $C$ independent of $m$. Combining (2.5), (2.6) and the estimates (2.1), we have the conclusion.

\[\square\]

**Proof of Proposition 2.4.** As in [11, 22], we take a sequence of functions $\{φ_j\}_{j=1}^∞$ on the orthogonal group $O(n)$ with the following properties (1) and (2):

1. Functions $φ_j$ are infinitely differentiable, non-negative and satisfy $\int_{O(n)} φ_j(A) \, dA = 1$, where $dA$ is the Haar measure on $O(n)$.
2. For any neighborhood $U$ of the identity of $O(n)$, there exists a positive integer $N$ such that $\text{supp}(φ_j) ⊂ U$ for $j ≥ N$.

For $ξ ∈ S^{n-1}$, let

\[\tilde{m}_j(ξ) = \int_{O(n)} m(Aξ) φ_j(A) \, dA.\]

Then $\tilde{m}_j$ is $C^∞$ on $S^{n-1}$ (see [11] pp. 123–124). For $ξ ∈ \mathbb{R}^n \setminus \{0\}$, let

\[m_j(ξ) = \tilde{m}_j \left( δ^*_j(ξ) \right).\]

Then $m_j$ is homogeneous of degree 0 with respect to $δ^*_j$, $m_j ∈ C^∞(\mathbb{R}^n \setminus \{0\})$ and $m_j = \tilde{m}_j$ on $S^{n-1}$.

We prove

\[(2.7) \quad \rho_{r,v}(m_j) ≤ ||m||_∞, \quad r ∈ (1, ∞), \ v ∈ A_r.\]

For this it suffices to show that

\[||m_j||_{L^r(v)} ≤ C_j ||m||_∞^{k+1},\]

where $C_j$ is independent of $k$. This follows by Lemma 2.6 since

\[\sup_{1 ≤ ρ^*(ξ) ≤ 2, |a| ≤ |γ|+1} |(∂^a_ξ)^{\alpha} m_j(ξ)| ≤ C_j ||m||_∞^{k+1} ||m||_∞^k.\]

To see this, it is helpful to refer to [11] pp. 123–124.

Since $m_j → m$ as $j → ∞$ uniformly on $S^{n-1}$, we can take $ℓ = m_j$ for $j$ large enough to get $\|m - ℓ\|_∞ < ε$. Let $p ∈ (1, ∞)$, $w ∈ A_p$. Confirming that a result analogous to [22] Proposition 2.2 holds true in the setting of non-isotropic dilations, we can find $r > 1$, $s > 1$ and $θ ∈ (0, 1)$ such that $w^s ∈ A_r$ and

\[\| (m - m_j)^k \|_{L^r(w)} ≤ \| (m - m_j)^k \|_{∞}^{1-θ} \| (m - m_j)^k \|_{L^r(\theta)}.\]

Thus

\[\rho_{p,w}(m - m_j) ≤ \|m - m_j\|_∞^{1-θ} \rho_{r,w^s}(m - m_j)^θ,\]

Since

\[\rho_{r,w^s}(m - m_j) ≤ \rho_{r,w^s}(m) + \rho_{r,w^s}(m_j),\]

(see Riesz-Nagy [15, p. 426]), it follows that

\[\rho_{p,w}(m - m_j) ≤ \|m - m_j\|_∞^{1-θ} \left( \rho_{r,w^s}(m) + \rho_{r,w^s}(m_j) \right)^θ \leq \|m - m_j\|_∞^{1-θ} \left( \rho_{r,w^s}(m) + \|m\|_∞ \right)^θ,\]

and

\[\rho_{p,w}(m - m_j) ≤ \|m - m_j\|_∞^{1-θ} \left( \rho_{r,w^s}(m) + \|m\|_∞ \right)^θ.\]
where the last inequality follows from (2.7). Since $||m - m_j||_\infty \to 0$ as $j \to \infty$, for a given $\epsilon > 0$, taking $\ell = m_j$ with $j$ large enough, we have $\rho_{p,w}(m - \ell) < \epsilon$ and $||m - \ell||_\infty < \epsilon$.

**Proof of Theorem 2.3.** The proof is similar to that of [22, Theorem 2.5]. Let

$$\epsilon_0 = \frac{1}{4} \min_{\xi \in \mathbb{S}^{n-1}} |m(\xi)|.$$  

Applying Proposition 2.4, we can find $\ell \in M^p_{\delta}$ which is homogeneous of degree 0 with respect to $\delta^*_1$ and belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $||m - \ell||_\infty < \epsilon_0$ and $\rho_{p,w}(m - \ell) < \epsilon_0$. Let $C : \ell(\xi) + 2\epsilon_0 e^{i\theta}, 0 \leq \theta \leq 2\pi$, be a circle in $D$. Apply Cauchy’s formula to get

$$F(m) = \frac{1}{2\pi i} \int_{C} \frac{F(\zeta)}{\zeta - m(\xi)} d\zeta = \frac{\epsilon_0}{\pi} \int_{0}^{2\pi} \frac{F(\ell(\xi) + 2\epsilon_0 e^{i\theta})}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} e^{i\theta} d\theta$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. We expand the integrand in the last integral into a power series by using

$$e^{i\theta} \frac{\epsilon_0}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} = \frac{1}{2\epsilon_0} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right)^k,$$

where the series converges uniformly in $\theta \in [0, 2\pi]$ since

$$\left| \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right| \leq \frac{1}{2}.$$

Substituting (2.9) in (2.8), we have

$$F(m) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right)^k N_k(\xi),$$

where

$$N_k(\xi) = \int_{0}^{2\pi} F(\ell(\xi) + 2\epsilon_0 e^{i\theta}) e^{-ik\theta} d\theta$$

and the series on the right hand side of (2.10) converges uniformly in $\xi \in \mathbb{R}^n \setminus \{0\}$, since

$$\left| \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right| \leq \frac{1}{2}, \quad \epsilon_0 \leq |\ell(\xi) + 2\epsilon_0 e^{i\theta}| \leq ||m||_\infty + 3\epsilon_0.$$

Also, $N_k(\xi)$ is homogeneous of degree 0 with respect to $\delta^*_1$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ and

$$\sup_{1 \leq \rho^*(\xi) \leq 2, |\gamma| \leq |\gamma| + 1} |(\partial_\xi)^\gamma N_k(\xi)| \leq C$$

with $C$ independent of $k$. Therefore, by Lemma 2.6 we have $||N_k||_{M^p_{\delta}} \leq C$ with a constant $C$ independent of $k$. Thus we see that

$$\sum_{k=0}^{\infty} (2\epsilon_0)^{-k} ||(m - \ell)^k||_{M^p_{\delta}} ||N_k||_{M^p_{\delta}} \leq C \sum_{k=0}^{\infty} (2\epsilon_0)^{-k} ||(m - \ell)^k||_{M^p_{\delta}}$$

and that the last series converges since $||m - \ell||_{M^p_{\delta}} \leq \epsilon_0$ if $k$ is sufficiently large. From this and (2.10) we can infer that $F(m) \in M^p_{\delta}$. This completes the proof. \qed

By Theorem 2.3 in particular we have the following.
Corollary 2.7. Let $1 < p < \infty$ and $w \in A_p$. Suppose that $m$ is homogeneous of degree 0 with respect to $\delta_t^*$ and that $m \in M^p_w$ for all $r \in (1, \infty)$ and all $v \in A_r$. We further assume that $m$ is continuous on $S^{n-1}$ and does not vanish there. Then $m^{-1} \in M^p_w$.

Proof. Take $F(z) = 1/z$ in Theorem 2.8. \hfill \square

Applying Corollary 2.7 in the theory of the Littlewood-Paley functions, we can prove the following.

Theorem 2.8. Let $\psi \in L^1(\mathbb{R}^n)$ satisfy (1.4). Suppose that $\|g_\psi(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v}$, $f \in \mathcal{S}$, for all $r \in (1, \infty)$ and all $v \in A_r$ and that $m(\xi) = \int_0^\infty |\psi(\delta_t^*\xi)|^2 \frac{dt}{t}$ is continuous and strictly positive on $S^{n-1}$. Let $f \in \mathcal{S}$. Then we have

$$\|f\|_{p,w} \leq C_{p,w}\|g_\psi(f)\|_{p,w}$$

for all $p \in (1, \infty)$ and all $w \in A_p$.

To prove Theorem 2.8, we also need the following lemma.

Lemma 2.9. Suppose that $\|g_\psi(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v}$, $f \in \mathcal{S}$, for all $r \in (1, \infty)$ and all $v \in A_r$. Then, if $m(\xi)$ is defined as in Theorem 2.8 and if $1 < p < \infty$, $w \in A_p$, we have $m \in M^p_w$.

Proof. For $\epsilon \in (0,1)$, let

$$\Psi^{(\epsilon)}(x) = \int_\epsilon^{r-1} \int_{\mathbb{R}^n} \psi_t(x+y)\overline{\psi}_t(y) \frac{dy}{t},$$

where $\overline{\psi}_t$ denotes the complex conjugate. We note that

$$\mathcal{F}(\Psi^{(\epsilon)})(\xi) = \int_\epsilon^{r-1} \hat{\psi}(\delta_t^*\xi)\overline{\hat{\psi}}(-\delta_t^*\xi) \frac{dt}{t} = \int_\epsilon^{r-1} |\hat{\psi}(\delta_t^*\xi)|^2 \frac{dt}{t} =: m^{(\epsilon)}(\xi).$$

Therefore $\Psi^{(\epsilon)} * f = T_{m^{(\epsilon)}}f$. We observe that

$$\Psi^{(\epsilon)} * f(x) = \int_\epsilon^{r-1} \int_{\mathbb{R}^n} \psi_t * f(y)\overline{\psi}_t(y-x) \frac{dy}{t};$$

$$\int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x)h(x) \, dx = \int_\epsilon^{r-1} \int_{\mathbb{R}^n} \psi_t * f(y)\overline{\psi}_t(y) \frac{dy}{t} \frac{dy}{t}$$

for $f, h \in \mathcal{S}$. Thus by the inequalities of Schwarz and Hölder we have

$$\left|\int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x)h(x) \, dx\right| \leq \int_{\mathbb{R}^n} g_\psi(f)(y)g_\psi(\overline{h})(y) \, dy \leq \|g_\psi(f)\|_{p,w}\|g_\psi(\overline{h})\|_{p',w-p'/p} \leq C\|g_\psi(f)\|_{p,w}\|h\|_{p',w-p'/p}.$$
Proof of Theorem 2.8. Let $m$ be as in Theorem 2.8. Then by Lemma 2.9 $m \in M_w^p$ for all $p \in (1, \infty)$ and $w \in A_p$. So we can apply Corollary 2.7 to $m$ to conclude that $m^{-1} \in M_w^p$ if $1 < p < \infty$, $w \in A_p$ and hence using (2.11), we have

$$\|f\|_{p,w} = \|T_m^{-1}T_{mf}\|_{p,w} \leq C\|T_{mf}\|_{p,w} \leq C\|g_\Psi(f)\|_{p,w}$$

for $f \in C$, which implies the conclusion. \hfill \Box

Proof of Theorem 2.11. It remains to prove the reverse inequality of (2.11). If $m(\xi) = \int_0^{\infty} |\hat{\psi}(\delta_t \xi)|^2 \, dt/t$, then by the non-degeneracy (2.2) we have $m(\xi) \neq 0$ for $\xi \neq 0$. Therefore, by Theorem 2.8 we have only to show that $m$ is continuous on $S^{n-1}$. In [25], it has been shown that

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(\delta_t \xi)|^2 \frac{dt}{t} \leq C \min \left( \delta_k^2 \xi^1, \delta_k^2 \xi^2 \right)$$

for $\xi \in S^{n-1}$ and $k \in \mathbb{Z}$ with some $\epsilon > 0$ (see [25] Lemmas 3.1 and 3.3). By analogues for $\delta_t$ of (a), (b) for $\delta_k$ in Section 1 it follows that

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(\delta_t \xi)|^2 \frac{dt}{t} \leq C \min \left( 2^{2k-1}, 2^{-2k} \right).$$

This implies that

$$\int_{\epsilon}^{1} |\hat{\psi}(\delta_t \xi)|^2 \frac{dt}{t} \to \int_0^{\infty} |\hat{\psi}(\delta_t \xi)|^2 \frac{dt}{t} \quad \text{as } \epsilon \to 0$$

uniformly in $\xi \in S^{n-1}$. We note that $\int_{\epsilon}^{1} |\hat{\psi}(\delta_t \xi)|^2 \frac{dt}{t}$ is continuous on $S^{n-1}$ for each fixed $\epsilon > 0$. Thus the continuity of $m$ on $S^{n-1}$ follows by the uniform convergence. \hfill \Box

Remark 2.10. Let $\psi^{(j)} \in L^1(\mathbb{R}^n)$ for $j = 1, \ldots, \ell$. Suppose that $\psi^{(j)}$ satisfies (1.4) and (1), (2) and (3) of Theorem 2.1 for every $j$, $1 \leq j \leq \ell$. Let

$$\Psi_t(x) = \left( \psi^{(1)}_t(x), \ldots, \psi^{(\ell)}_t(x) \right),$$

$$\Psi_t(x) = \left( \psi^{(1)}_t(x), \ldots, \psi^{(\ell)}_t(x) \right), \quad \mathcal{F}(\Psi_t)(\xi) = \left( \mathcal{F}\left( \psi^{(1)}_t \right)(\xi), \ldots, \mathcal{F}\left( \psi^{(\ell)}_t \right)(\xi) \right).$$

We further assume that

$$\sup_{t \geq 0} \left| \mathcal{F}(\Psi_t) (\xi) \right| = \sup_{t \geq 0} \left( \sum_{j=1}^{\ell} \left| \mathcal{F}\left( \psi^{(j)}_t \right) (\delta_t \xi) \right|^2 \right)^{1/2} > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \quad (2.12)$$

Let

$$f \ast \Psi_t(x) = \left( f \ast \psi^{(1)}_t(x), \ldots, f \ast \psi^{(\ell)}_t(x) \right)$$

and

$$g_\Psi(f)(x) = \left( \int_0^{\infty} |f \ast \Psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad \left| f \ast \Psi_t(x) \right| = \left( \sum_{j=1}^{\ell} \left| f \ast \psi^{(j)}_t(x) \right|^2 \right)^{1/2}.$$

Then by Theorem 2.1 we have $\|g_\Psi(f)\|_{p,w} \leq C\|f\|_{p,w}$. We can also prove the reverse inequality by adapting the arguments given above when $\ell = 1$ for the present situation, applying the non-degeneracy (2.12). Thus we have

$$\|g_\Psi(f)\|_{p,w} \approx \|f\|_{p,w}. \quad (2.13)$$
Example. We give an example in the case of the Euclidean structures \( (\rho(x) = |x|, \delta t(x) = tx) \) for which we can apply Remark 2.10 to get the norm equivalence in (2.13). Let \( P_t(x) \) be the Poisson kernel on the upper half space \( \mathbb{R}^n \times (0, \infty) \) defined by

\[
P_t(x) = c_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}} = \int_{\mathbb{R}^n} e^{-2\pi t |\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi.
\]

Let \( \psi^{(j)}(x) = (\partial/\partial x_j) P_t(x), 1 \leq j \leq n. \) Then

\[
\mathcal{G} \left( \psi^{(j)} \right)(\xi) = 2\pi i \xi e^{-2\pi |\xi|}.
\]

We can see that all the requirements in Remark 2.10 are fulfilled; in particular, (2.12) follows from

\[
|\mathcal{G} (\Psi_t)(\xi)| = 2\pi t |\xi| e^{-2\pi t |\xi|}.
\]

Thus we have (2.13) for \( \Psi = ((\partial/\partial x_1) P_1, \ldots, (\partial/\partial x_n) P_1). \)

3. PROOFS OF THEOREMS 1.4 AND 1.5

We apply the following estimates in proving Theorem 1.4.

Lemma 3.1. Let \( F \) be a function in \( C^\infty(\mathbb{R}^n \setminus \{0\}) \) which is homogeneous of degree \( \alpha \) with respect to \( \delta_t \). Then, for \( \rho(x) \geq 1 \) we have

\[
|\langle \partial^a \rangle F(x)| \leq C_\alpha \rho(x)^{d - |a|}
\]

for all multi-indices \( a \) with a positive constant \( C_\alpha \) independent of \( x. \)

Proof. We write \( \delta_i = (\delta_{ij}(t)), 1 \leq i, j \leq n. \) We have \( t^d F(x) = F(\delta_t x) \). Differentiating both sides by using the chain rule on the right hand side, we have

\[
t^d \langle \partial^a \rangle F(x) = \left( \prod_{j=1}^n \left( \sum_{i=1}^n \delta_{ij}(t) \partial / \partial x_i \right)^{a_j} \right) F(\delta_t x).
\]

Substituting \( t = \rho(x)^{-1} \) in this equation, we have

\[
|\langle \partial^a \rangle F(x)| \leq C \left( \sup_{|\alpha| = 1} |\langle \partial^b \rangle F(x)| \right) \left( \sup_{|\alpha| = 1} \delta_{ij}(\rho(x)^{-1}) \right)^{|a|} \rho(x)^d.
\]

This implies what we need, since \( |\delta_{ij}(t)| \leq Ct \) for \( 0 < t \leq 1 \) by (b) of Section 1. \( \square \)

Proof of Theorem 1.4. Let \( 0 < \alpha < \gamma \) and \( \mathcal{L}_\alpha = \mathcal{G}^{-1}(\rho^\alpha(\xi)^{-\alpha}) \). Then \( \mathcal{L}_\alpha \) is homogeneous of degree \( \alpha - \gamma \) with respect to \( \delta_t \) and belongs to \( C^\infty(\mathbb{R}^n \setminus \{0\}) \) (see [4, pp. 162–165]). Let \( \psi^{(\alpha)} = \mathcal{L}_\alpha - \mathcal{L}_\alpha \ast \Phi. \) Then \( H_\alpha(f) = g_{\psi^{(\alpha)}}(f). \)

We easily see that

\[
|\psi^{(\alpha)}(x)| \leq C \rho(x)^{\alpha - \gamma} \quad \text{for } \rho(x) \leq 2.
\]

Since

\[
\psi^{(\alpha)}(x) = \int_{\mathbb{R}^n} (\mathcal{L}_\alpha(x) - \mathcal{L}_\alpha(x-y)) \Phi(y) dy,
\]

and

\[
|\langle \partial^a \rangle \mathcal{L}_\alpha(x)| \leq C_\alpha \rho(x)^{\alpha - \gamma - |a|} \quad \text{for } \rho(x) \geq 2
\]

for all multi-indices \( a \) by Lemma 3.1 using Taylor’s formula with (1.16) and noting that \( \Phi \) is compactly supported, we see that

\[
|\psi^{(\alpha)}(x)| \leq C \rho(x)^{\alpha - \gamma - |a| - 1} \quad \text{for } \rho(x) \geq 2,
\]
where $\alpha - \gamma - [\alpha] - 1 < -\gamma$. By (3.1), (3.2) and (1.9) it follows that $\psi(\alpha) \in L^1$ (see Remark 2.2). Also, we have
\[
|F(\psi(\alpha))(\xi)| = |\rho^*(\xi)^{-\alpha}(1 - \hat{\Phi}(\xi))| \leq C\rho^*(\xi)^{-\alpha}[\xi]^{|\alpha|+1} \leq C\rho^*(\xi)^{-\alpha+|\alpha|+1}
\]
for $\rho^*(\xi) \leq 1$ by the analogue for $\rho^*$ of (4) for $\rho$ of Section 1. So we have $F(\psi(\alpha))(0) = 0$, i.e., $\int \psi(\alpha) = 0$, by which combined with (3.1), (3.2) and (5) for $\rho$ of Section 1 we see that the conditions (1), (2), (3) of Theorem 2.1 are satisfied for $\psi(\alpha)$. Further, it is easy to see that
\[
\sup_{t>0} |F(\psi(\alpha))(\delta_t^*\xi)| > 0
\]
for $\xi \not= 0$. Thus all the assumptions of Theorem 2.1 are fulfilled for $\psi(\alpha)$ and the conclusion of Theorem 1.4 follows by applying Theorem 2.1 to $g(\psi(\alpha))$.

**Remark 3.2.** If $\psi(\alpha)$ is as in (1.6), in the case of the Euclidean norm and the ordinary dilation, to prove $\|f\|_{p,w} \leq C\|g(\psi(\alpha))(f)\|_{p,w}$, $0 < \alpha < 2$, $1 < p < \infty$, $w \in A_p$, we can also apply the polarization technique as in the proof of Lemma 2.3 (see also [1]) instead of using Theorem 2.1 with the non-degeneracy condition (2.2), which is applicable in a more general situation of Theorem 1.4. This is the case because $F(\psi(\alpha))$ is a radial function.

To prove Theorem 1.5 we prepare the following lemmas.

**Lemma 3.3.** Let $1 < p < \infty$, $w \in A_p$ and $f \in L^p_w$. For a positive integer $m$, let $f_m = f(\epsilon E_m)$, where
\[
E_m = \{x \in \mathbb{R}^n : |x| \leq m, |f(x)| \leq m\}.
\]
The then $f_m \rightarrow f$ almost everywhere and in $L^p_w$ as $m \rightarrow \infty$.

**Lemma 3.4.** Let $p$, $w$ and $f$ be as in Lemma 3.3. Let $\phi$ be an infinitely differentiable, non-negative function on $\mathbb{R}^n$ such that $\phi(\xi) = 1$ for $\rho^*(\xi) \leq 1$, supp(\phi) \subset \{\rho^*(\xi) \leq 2\}$ and $\phi(\xi) = \phi_0(\rho^*(\xi))$ for some $\phi_0$ on $\mathbb{R}$. Define $\zeta^{(\epsilon)}(\xi) = \phi(\delta_{\epsilon}^*\xi) - \phi(\delta_{\epsilon}^{*,-}\xi)$, $\epsilon \in (0,1/2)$.

We note that $\zeta(\epsilon)(\xi) = \zeta^{(\epsilon/2)}(\xi)\zeta^{(\epsilon)}(\xi)$. Let $f^{(\epsilon)} = f * F^{-1}(\zeta^{(\epsilon)})$. Then $f^{(\epsilon)} \rightarrow f$ almost everywhere and in $L^p_w$ as $\epsilon \rightarrow 0$.

**Proof of Lemma 3.3.** The pointwise convergence is obvious and the norm convergence follows from the dominated convergence theorem of Lebesgue since $|f_m| \leq |f|$.

**Proof of Lemma 3.4.** If $f \in S$, we easily see that $f^{(\epsilon)} \rightarrow f$ pointwise as $\epsilon \rightarrow 0$.

Therefore, for $f \in L^p_w$, we have
\[
\limsup_{\epsilon \rightarrow 0} \left\|f^{(\epsilon)} - f\right\|_{p,w} \leq \limsup_{\epsilon \rightarrow 0} \left\|(f - h)^{(\epsilon)} - (f - h)\right\|_{p,w} \leq C\left\|M(f - h)\right\|_{p,w} \leq C\left\|f - h\right\|_{p,w}
\]
for any $h \in S$. Since $S$ is dense in $L^p_w$, it follows that $\limsup_{\epsilon \rightarrow 0} |f^{(\epsilon)}(x) - f(x)| = 0$ a. e., which implies the pointwise convergence. The norm convergence follows from the pointwise convergence and the dominated convergence theorem of Lebesgue since $|f^{(\epsilon)}| \leq CM(f) \in L^p_w$. \qed
Proof of Theorem 1.3: Define $f_{m,\epsilon} = (f_{(m)})^{(\epsilon)}$ for $f \in L^p_w$. Then $f_{m,\epsilon} \in S_0$. By Theorem 1.4 we see that

$$
\|G_\alpha(f_{m,\epsilon})\|_{p,w} = \|H_\alpha(I_{-\alpha}f_{m,\epsilon})\|_{p,w} \leq \|I_{-\alpha}^{(\epsilon/2)}f_{m,\epsilon}\|_{p,w},
$$

where $I_{\beta}^{(\epsilon/2)}(f) = \mathcal{F}^{-1}(\zeta^{(\epsilon/2)}(\rho^\beta)^{-\epsilon}) * f$, $\beta \in \mathbb{R}$, for $f \in L^p_w$ and we have used the equality $I_{-\alpha}f_{m,\epsilon} = I_{-\alpha}^{(\epsilon/2)}f_{m,\epsilon}$. Using Lemma 3.3 we see that $f_{m,\epsilon} \to f^{(\epsilon)}$ in $L^p_w$, since

$$
\|f_{m,\epsilon} - f^{(\epsilon)}\|_{p,w} \leq C\|M(f_{(m)} - f)\|_{p,w} \leq C\|f_{(m)} - f\|_{p,w}
$$

and also $f_{m,\epsilon} \to f^{(\epsilon)}$ pointwise, since

$$
|f_{m,\epsilon}(x) - f^{(\epsilon)}(x)| \leq \int (f_{(m)}(y) - f(y))\mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y)\,dy \leq \|f_{(m)} - f\|_{p,w} \left(\int |\mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y)|^{p'/p} w(y)^{-\nu'/p}\,dy\right).
$$

Thus $f_{m,\epsilon} - \Phi_t \ast f_{m,\epsilon} \to f^{(\epsilon)} - \Phi_t \ast f^{(\epsilon)}$ a.e. as $m \to \infty$ and by 3.3 we have, via Fatou’s lemma,

$$
\|G_\alpha(f^{(\epsilon)})\|_{p,w} \leq \liminf_{m \to \infty} \|G_\alpha(f_{m,\epsilon})\|_{p,w} \leq C\liminf_{m \to \infty} \|I_{-\alpha}^{(\epsilon/2)}f_{m,\epsilon}\|_{p,w} = C\|I_{-\alpha}^{(\epsilon/2)}f^{(\epsilon)}\|_{p,w},
$$

where the last equality follows since $I_{-\alpha}^{(\epsilon/2)}$ is bounded on $L^p_w$. Thus we see that $G_\alpha(f^{(\epsilon)}) \in L^p_w$. In fact, we also have the reverse inequality. To see this we first note that

$$
\|G_\alpha(f^{(\epsilon)}) - G_\alpha(f_{m,\epsilon})\|_{p,w} \leq \|G_\alpha(f^{(\epsilon)} - f_{m,\epsilon})\|_{p,w} = \|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w}
$$

Since

$$
(f_{(k)} - f_{(m)})^{(\epsilon)} - \Phi_t \ast (f_{(k)} - f_{(m)})^{(\epsilon)} \to (f - f_{(m)})^{(\epsilon)} - \Phi_t \ast (f - f_{(m)})^{(\epsilon)} \quad \text{a.e. as } k \to \infty,
$$

by Fatou’s lemma we have

$$
\|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w} \leq \liminf_{k \to \infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}.
$$

Since $(f_{(k)} - f_{(m)})^{(\epsilon)} \in S_0$, by Theorem 1.4 we have

$$
\|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} = \|I_{-\alpha}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} \leq \|I_{-\alpha}^{(\epsilon/2)}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}.
$$

Since $f_{(m)} \to f$ in $L^p_w$, this implies that

$$
\lim_{k,m \to \infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} = 0.
$$

Thus by 3.3 and 3.5, it follows that $G_\alpha(f_{m,\epsilon}) \to G_\alpha(f^{(\epsilon)})$ in $L^p_w$ as $m \to \infty$. Therefore, letting $m \to \infty$ in 3.3, we have

$$
\|G_\alpha(f^{(\epsilon)})\|_{p,w} \leq \|I_{-\alpha}^{(\epsilon/2)}f^{(\epsilon)}\|_{p,w}.
$$

Suppose that $f \in W^{\alpha,p}_w$ and let $g = I_{-\alpha}(f)$. We show that

$$
I_{-\alpha}^{(\epsilon/2)}f^{(\epsilon)} = g^{(\epsilon)}
$$
as follows. We have for $h \in S_0$
\[
\int g^{(e)}I_\alpha(h) \, dx = \lim_{m \to \infty} \int g_mI_\alpha(h) \, dx = \lim_{m \to \infty} \int I^{(\varepsilon/2)}_{\alpha}(g_m,\varepsilon)h \, dx \\
= \int I^{(\varepsilon/2)}_{\alpha}(g^{(e)})h \, dx.
\]
Also,
\[
\int g^{(e)}I_\alpha(h) \, dx = \lim_{m \to \infty} \int g_mI_\alpha(h) \, dx = \lim_{m \to \infty} \int g^{(e)}I_\alpha(h^{(e)}) \, dx = \int gI_\alpha(h^{(e)}) \, dx.
\]
By the definition of $g = I_{-\alpha}(f)$, $\int gI_\alpha(h^{(e)}) \, dx = \int fh^{(e)} \, dx$. Thus
\[
\int g^{(e)}I_\alpha(h) \, dx = \int fh^{(e)} \, dx = \lim_{m \to \infty} \int f_mh^{(e)} \, dx = \lim_{m \to \infty} f_mh \, dx = \int f^{(e)}h \, dx.
\]
Therefore
\[
\int I^{(\varepsilon/2)}_{\alpha}(g^{(e)})h \, dx = \int f^{(e)}h \, dx \quad \text{for all } h \in S_0,
\]
which implies that $I^{(\varepsilon/2)}_{\alpha}(g^{(e)}) = f^{(e)}$. Since $I^{(\varepsilon/2)}_{\alpha}$ and $I^{(\varepsilon/2)}_{-\alpha}$ are bounded on $L^p_w$ and the mapping $f \to f^{(e)}$ is also bounded on $L^p_w$, by Lemma 3.3 we see that
\[
I^{(\varepsilon/2)}_{\alpha}(f^{(e)}) = I^{(\varepsilon/2)}_{-\alpha}I^{(\varepsilon/2)}_{\alpha}(g^{(e)}) = \lim_{m \to \infty} I^{(\varepsilon/2)}_{-\alpha}I^{(\varepsilon/2)}_{\alpha}(g_m,\varepsilon) \\
= \lim_{m \to \infty} g_m,\varepsilon = g^{(e)},
\]
which proves (3.7).
By (3.6) and (3.7), we have
\[
\|G_\alpha(f^{(e)})\|_{p,w} \leq C\|g^{(e)}\|_{p,w} \leq C\|M(g)\|_{p,w} \leq C\|g\|_{p,w}.
\]
Letting $\varepsilon \to 0$ and applying Lemma 3.4 and Fatou’s lemma, we have
\[
(3.8)
\|G_\alpha(f)\|_{p,w} \leq C\|I_{-\alpha}(f)\|_{p,w}.
\]
Conversely, let us assume that $f \in L^p_w$ and $G_\alpha(f) \in L^p_w$. By Minkowski’s inequality we see that
\[
(3.9) \quad \|G_\alpha(f^{(e)})\|_{p,w} \leq C\|M(G_\alpha(f))\|_{p,w} \leq C\|G_\alpha(f)\|_{p,w}.
\]
Applying (3.6) and (3.9), we see that
\[
\sup_{\varepsilon \in (0,1/2)} \|I^{(\varepsilon/2)}_{\alpha}f^{(e)}\|_{p,w} \leq C \sup_{\varepsilon \in (0,1/2)} \|G_\alpha(f^{(e)})\|_{p,w} \leq C\|G_\alpha(f)\|_{p,w}.
\]
Therefore, there exist a sequence $\{\varepsilon_k\}$, $0 < \varepsilon_k < 1/2$, and a function $g \in L^p_w$ such that $\varepsilon_k \to 0$ and $I^{(\varepsilon_k/2)}_{-\alpha}f^{(e_k)} \to g$ weakly in $L^p_w$ as $k \to \infty$ and
\[
(3.10) \quad \|g\|_{p,w} \leq C\|G_\alpha(f)\|_{p,w}.
\]
This implies that which combined with (3.8), completes the proof of Theorem 1.5. □

If \( \Phi = \chi \)

We also write \( \Lambda_t \) (4.3)

and we have used the equation (4.4)

Define \( K \) (4.2)

for appropriate functions \( f \).

We show that \( f = \mathcal{I}_\alpha g \) by definition. By (3.10) we have

\[
\|\mathcal{I}_{-\alpha} f\|_{p,w} = \|g\|_{p,w} \leq C\|G_\alpha(f)\|_{p,w},
\]

which combined with (5.8), completes the proof of Theorem 1.5.

4. Characterization of the Sobolev spaces \( W^{\alpha,p}_w \) by square functions defined with repeated uses of averaging operation

Let \( \Phi \in \mathcal{M}^1 \). Define \( \Lambda^j f(x) \), \( j \geq 1 \), by \( \Lambda^j f(x) = f * \Phi^{(j)}(x) \), where

\[
\Phi^{(1)}(x) = \Phi(x), \quad \Phi^{(j)}(x) = \Phi * \cdots * \Phi(x), \quad j \geq 2.
\]

We also write \( \Lambda f(x) \) for \( \Lambda^1 f(x) \). Let \( I \) be the identity operator and \( k \) a positive integer. We consider

\[
(I - \Lambda)^k f(x) = f(x) + \sum_{j=0}^{k} (-1)^j \binom{k}{j} \Lambda^j f(x)
\]

\[
= f(x) - K^{(k)} * f(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y))K^{(k)}(y) \, dy,
\]

for appropriate functions \( f \), where

\[
K^{(k)}(x) = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} \Phi^{(j)}(x),
\]

and we have used the equation

\[
\int_{\mathbb{R}^n} K^{(k)}(x) \, dx = -\sum_{j=1}^{k} (-1)^j \binom{k}{j} = 1.
\]

Define

\[
E^{(k)}_{\alpha}(f)(x) = \left( \int_0^\infty \left| (I - \Lambda)^k f(x) \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0.
\]

If \( \Phi = \chi_0 = |B(0,1)|^{-1} \chi_{B(0,1)} \) and \( k = 2 \) in (4.4), we have

\[
E^{(2)}_{\alpha}(f)(x) = \left( \int_0^\infty \left( f(x) - 2 \int_{B(x,t)} f(y) \, dy + \int_{B(x,t)} f(y) B(y,t) \, dy \right)^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},
\]
where \((f)_{B(y,t)} = \int_{B(y,t)} f\). Also, let

\[
U^{(k)}_{\alpha}(f)(x) = \left( \int_{0}^{\infty} \left| (I - \Lambda t)^{k} \mathcal{I}_{\alpha}(f)(x) \right|^{2} \frac{dt}{t^{1+2\alpha}} \right)^{1/2},
\]

where \(0 < \alpha < \gamma\), \(f \in S_{0}\). Using (4.1), we can rewrite \(E^{(k)}_{\alpha}(f)\) in (4.4) and \(U^{(k)}_{\alpha}(f)\) in (4.5) as follows:

\[
E^{(k)}_{\alpha}(f)(x) = \left( \int_{0}^{\infty} \left| f(x) - K^{(k)}_{t} * f(x) \right|^{2} \frac{dt}{t^{1+2\alpha}} \right)^{1/2},
\]

\[
U^{(k)}_{\alpha}(f)(x) = \left( \int_{0}^{\infty} \left| \mathcal{I}_{\alpha}(f)(x) - K^{(k)}_{t} * \mathcal{I}_{\alpha}(f)(x) \right|^{2} \frac{dt}{t^{1+2\alpha}} \right)^{1/2},
\]

where \(K^{(k)}\) is as in (2.2).

As applications of Theorems 1.4 and 1.5 we have the following theorems.

**Theorem 4.1.** Let \(0 < \alpha < \min(2k, \gamma)\), \(1 < p < \infty\), \(w \in A_{p}\) and let \(U^{(k)}_{\alpha}\) be as in (4.5). Then

\[
\| U^{(k)}_{\alpha}(f) \|_{p,w} \simeq \| f \|_{p,w}, \quad f \in S_{0}(\mathbb{R}^{n}).
\]

**Theorem 4.2.** Let \(1 < p < \infty\), \(w \in A_{p}\) and \(0 < \alpha < \min(2k, \gamma)\). Let \(E^{(k)}_{\alpha}\) be as in (4.4). Then \(f \in W^{\alpha,p}_{w}\) if and only if \(f \in L^{p}_{w} \) and \(E^{(k)}_{\alpha}(f) \in L^{p}_{w}\); also, we have

\[
\| I_{-\alpha}(f) \|_{p,w} \simeq \| E^{(k)}_{\alpha}(f) \|_{p,w}.
\]

**Proofs of Theorems 4.1 and 4.2.** Using the expressions of \(E^{(k)}_{\alpha}(f)\) and \(U^{(k)}_{\alpha}(f)\) in (4.6) and (4.7), respectively, if \(K^{(k)} \in \mathcal{M}^{2k-1}\), since then \(K^{(k)} \in \mathcal{M}^{\alpha}\) for \(\alpha \in (0, \min(2k, \gamma))\).

To show that \(K^{(k)} \in \mathcal{M}^{2k-1}\), first we easily see that \(K^{(k)}\) is bounded and compactly supported. Since we have already noted (4.3), it remains to show that

\[
\int_{\mathbb{R}^{n}} y^{a} K^{(k)}(y) dy = 0 \quad \text{if} \ 1 \leq |a| < 2k.
\]

This can be shown as follows. Since \(\Phi \in \mathcal{M}^{1}\), we have \(\int y^{a} \Phi(y) dy = 0\) for \(|a| = 1\), which implies that \(\partial^{a}_{y} \Phi(0) = 0\) for \(|a| = 1\). Therefore we have, near \(\xi = 0\),

\[
1 - \mathcal{F}(K^{(k)})(\xi) = 1 + \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \Phi(\xi)^{j} = \left( 1 - \hat{\Phi}(\xi) \right)^{k} = O(|\xi|^{2k}).
\]

Also, by Taylor’s formula we see that

\[
\mathcal{F}(K^{(k)})(\xi) = 1 + \sum_{1 \leq |a| < 2k} C_{a} \xi^{a} \partial^{a}_{\xi} \mathcal{F}(K^{(k)})(0) + O(|\xi|^{2k}).
\]

From (4.9) and (4.10) it follows that

\[
\sum_{1 \leq |a| < 2k} C_{a} \xi^{a} \partial^{a}_{\xi} \mathcal{F}(K^{(k)})(0) = O(|\xi|^{2k}).
\]

This implies that \(\partial^{a}_{\xi} \mathcal{F}(K^{(k)})(0) = 0\) for \(1 \leq |a| < 2k\), and hence we have (4.8). \(\square\)

**Remark 4.3.** In the definitions of \(E^{(k)}_{\alpha}\) and \(U^{(k)}_{\alpha}\) in (4.4) and (4.5), if we assume only that \(\Phi\) belongs to \(\mathcal{M}^{0}\), then we have analogues of Theorems 4.1 and 4.2 for the range \((0, \min(k, \gamma))\) of \(\alpha\).
5. The Sobolev spaces $W^{\alpha,p}_w$ and distributional derivatives

In $\mathbb{R}^2$, we consider $P = \text{diag}(1, 2)$, $\delta_k = \text{diag}(t, t^2)$. Then, $\gamma = 3$ and

$$\rho(x_1, x_2) = \frac{1}{\sqrt{2}} \sqrt{x_1^2 + x_1^4 + 4x_2^2},$$

$\rho^* = \rho$, $\delta^*_t = \delta_t$. Under this setting, let $W^{\alpha,p}_w$ be the weighted Sobolev space on $\mathbb{R}^2$ defined in Section 3 with $0 < \alpha < 3$, $1 < p < \infty$, $w \in A_p$. Then $W^{2,p}_w$ can be characterized by using distributional derivatives as follows.

**Theorem 5.1.** Let $f \in L^p_w$ with $1 < p < \infty$, $w \in A_p$. Let $(\partial/\partial x_1)^2 f$, $\partial/\partial x_2 f$ be the distributional derivatives in $\mathcal{S}'$ (the space of tempered distributions). Then, $f \in W^{2,p}_w$ if and only if $(\partial/\partial x_1)^2 f \in L^p_w$ and $\partial/\partial x_2 f \in L^p_w$; further

$$\|T\_\alpha(f)\|_{2,w} \simeq \|(\partial/\partial x_1)^2 f\|_{p,w} + \|\partial/\partial x_2 f\|_{p,w}.$$

**Proof.** Suppose that $f \in W^{2,p}_w$. Let $g = T\_\alpha(f) \in L^p_w$. Then we have

$$\int fh dx = \int gI_2(h) dx \quad \text{for all } h \in \mathcal{S}_0.$$

Let $k(\xi) = -4\pi^2\xi_1^2$. Let $g_{m,\epsilon} = g(m) \ast \mathcal{F}^{-1}(\zeta(\xi))$ be as in Section 3. Then by (5.1) we see that for $h \in \mathcal{S}_0$

$$\int f(\partial/\partial x_1)^2 h dx = \int gI_2((\partial/\partial x_1)^2 h) dx = \int gI_2(Tk h) dx = \lim_{\epsilon \to 0} \lim_{m \to \infty} \int g_{m,\epsilon}I_2(Tk h) dx = \lim_{\epsilon \to 0} \lim_{m \to \infty} \int T_{k(\rho^*)^{-2}}(g_{m,\epsilon}) h dx.
$$

Since $k(\rho^*)^{-2}$ is homogeneous of degree 0 with respect to $\delta^*_t$ and infinitely differentiable in $\mathbb{R}^2 \setminus \{0\}$, by Lemma 2.6 the multiplier operator $T_{k(\rho^*)^{-2}}$ is bounded on $L^p_w$. Thus $T_{k(\rho^*)^{-2}}(g_{m,\epsilon}) \to T_{k(\rho^*)^{-2}}(g)$ in $L^p_w$ as $m \to \infty$, $\epsilon \to 0$ since $g_{m,\epsilon} \to g$ in $L^p_w$ as $m \to \infty$, $\epsilon \to 0$. Therefore, by (5.2) we have

$$\int f(\partial/\partial x_1)^2 h dx = \int T_{k(\rho^*)^{-2}}(g) h dx \quad \text{for all } h \in \mathcal{S}_0,
$$

which implies that

$$\int f(\partial/\partial x_1)^2 \psi dx = \int T_{k(\rho^*)^{-2}}(g) \psi dx \quad \text{for all } \psi \in \mathcal{S}.
$$

It follows that

$$\partial/\partial x_1 f = T_{k(\rho^*)^{-2}}(g) \quad \text{in } \mathcal{S}',
$$

To see this, substitute $\psi - \mathcal{F}^{-1}(\varphi(\delta^{-1}_\epsilon \psi(\xi)))$ for $h$ in (5.3), where $\varphi$ is as in Lemma 3.4 and let $\epsilon \to 0$.

Let $\ell(\xi) = 2\pi^2 \xi_2$. Then, arguing similarly as above and noting that $\ell(\rho^*)^{-2}$ is homogeneous of degree 0 with respect to $\delta^*_t$ and infinitely differentiable in $\mathbb{R}^2 \setminus \{0\}$, we see that $T_{\ell(\rho^*)^{-2}}(g) \in L^p_w$ and

$$- \int f\partial/\partial x_2 \psi dx = \int T_{\ell(\rho^*)^{-2}}(g) \psi dx \quad \text{for all } \psi \in \mathcal{S},
$$

which implies that

$$\partial/\partial x_2 f = T_{\ell(\rho^*)^{-2}}(g) \quad \text{in } \mathcal{S}'.
$$
Combining (5.5) and (5.6), we have
\[(5.7)\quad \|(\partial/\partial x_1)^2 f\|_{p,w} + \|(\partial/\partial x_2 f\|_{p,w} \leq C \|g\|_{p,w} = C \|I_{-2}(f)\|_{p,w}.
\]

Conversely, suppose that \((\partial/\partial x_1)^2 f =: \Theta \in L^p_w\) and \(\partial/\partial x_2 f =: \Xi \in L^p_w\). Then, for \(h \in S_0\) we have
\[
\int f(\partial/\partial x_1)^2 h \, dx = \int \Theta h \, dx, \quad -\int f \partial/\partial x_2 h \, dx = \int \Xi h \, dx,
\]
and hence
\[(5.8)\quad \int f(T_h - T_h) \, dx = \int f((\partial/\partial x_1)^2 h - \partial/\partial x_2 h) \, dx = \int (\Theta + \Xi) h \, dx,
\]
where \(k(\xi)\) and \(\ell(\xi)\) are as above. Let
\[N(\xi) = \frac{k(\xi) - \ell(\xi)}{\rho^*(\xi)^2} = \frac{-4\pi^2 \xi_1^2 - 2\pi i \xi_2}{\rho^*(\xi)^2}.
\]
Then, substituting \(I_2(h)\) for \(h\) in (5.8), we have
\[(5.9)\quad \int fT_nh \, dx = \int (\Theta + \Xi)I_2(h) \, dx.
\]

We note that the functions \(N\) and \(\tilde{N}^{-1}\) are homogeneous of degree 0 with respect to \(\delta_t^*\) and infinitely differentiable in \(\mathbb{R}^2 \setminus \{0\}\), where \(\tilde{N}(\xi) = N(-\xi)\). So, \(T_{\tilde{N}^{-1}}\) is bounded on \(L^p_w\) by Lemma (2.4). Substituting \(T_{\tilde{N}^{-1}}h\) for \(h\) in (5.9), we have
\[(5.10)\quad \int f \, dx = \int (\Theta + \Xi)T_{\tilde{N}^{-1}}(I_2(h)) \, dx = \int T_{\tilde{N}^{-1}}(\Theta + \Xi)I_2(h) \, dx,
\]
where the last equality follows as (5.3), since \(T_{\tilde{N}^{-1}}\) is bounded on \(L^p_w\). By (5.10) we see that \(f \in W^{2,p}_w\) and
\[
I_{-2}(f) = T_{\tilde{N}^{-1}}(\Theta + \Xi)
\]
and
\[
\|I_{-2}(f)\|_{p,w} \leq C \|\Theta\|_{p,w} + C ||\Xi||_{p,w} = C \|\partial/\partial x_1)^2 f\|_{p,w} + C \|\partial/\partial x_2 f\|_{p,w},
\]
which combined with (5.7) completes the proof of the theorem. \(\square\)

We conclude this note with two remarks.

**Remark 5.2.** To characterize the Sobolev spaces \(W^{\alpha,p}\) (unweighted spaces) we can also apply the square functions of the Lusin area integral type instead of the Littlewood-Paley function type (see [26]). In [29], certain Sobolev spaces \((H^1)\) Sobolev spaces) were characterized by using certain square functions of the Lusin area integral type. The characterization of those Sobolev spaces by square functions of the Littlewood-Paley function type analogous to Theorem 1.5 is yet to be proved.

**Remark 5.3.** Let us consider another square function of Marcinkiewicz type:
\[
D_{\alpha}(f)(x) = \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x + y) - I_\alpha(f)(x)|^2 |y|^{-n-2\alpha} \, dy \right)^{1/2},
\]
where \(I_\alpha\) as is in (1.2). Let \(0 < \alpha < 1\) and \(p_0 = 2n/(n + 2\alpha) > 1\). Then it is known that the operator \(D_{\alpha}\) is bounded on \(L^p(\mathbb{R}^n)\) if \(p_0 < p < \infty\) ([27]) and that \(D_{\alpha}\) is of weak type \((p_0, p_0)\) ([8]). In [24] analogues of these results were established in the case of dilations \(\delta_t = t^P\) when \(P\) is diagonal.
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