$\mathcal{PT}$-Symmetric Generalized Extended Momentum Operator

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We develop further the concept of generalized extended momentum operator (GEMO), which has been introduced very recently in [5], and propose the so called $\mathcal{PT}$-symmetric GEMO. In analogy with GEMO, the $\mathcal{PT}$-symmetric GEMO also satisfies the extended uncertainty principle (EUP) relation. Moreover, the corresponding Hamiltonian that is constructed upon the $\mathcal{PT}$-symmetric GEMO, with a real or $\mathcal{PT}$-symmetric potential, remains non-Hermitian but $\mathcal{PT}$-symmetric and consequently its energy and momentum eigenvalues are real. We apply our formalism to a quasi-free quantum particle and the exact solutions for the energy spectrum are presented.

I. INTRODUCTION

The $\mathcal{PT}$-symmetric quantum mechanics has been introduced by Bender and Boettcher in 1998 [1–4]. In spite of Hermiticity which is a pure mathematical principle, $\mathcal{PT}$-symmetry is a physical axiom. It is a form of non-Hermitian operator representing the parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) symmetry in a quantum system. It is able to define a Hamiltonian to be an observable due to producing real energy eigenvalues. Additionally, the probability is conserved holding the time evolution unitary. In a quantum system, the momentum is not only a Hermitian operator, but also it keeps the $\mathcal{PT}$-symmetry attitudes. To find a $\mathcal{PT}$-symmetric Hamiltonian, one has to examine the potential, $V(x)$, whether it meets the $\mathcal{PT}$-symmetric criteria or not. In this respect, the potential approves the parity and time reversal symmetries i.e., $\mathcal{PT}V(x) \equiv V^*(-x) = V(x)$. There exist several remarkable studies in the sense of non-Hermitian Hamiltonian concerning $\mathcal{PT}$-symmetric potential in the literature [3]. Also, the transition between broken and unbroken $\mathcal{PT}$-symmetry regions has been observed experimentally [7]. Moreover, Mostafazadeh has given a wider view of non-Hermitian quantum theory in a series of works entitled as pseudo-Hermitian [6].

Keeping in mind the value of the short history of $\mathcal{PT}$-symmetric Hamiltonians, now in this study, we introduce an alternative approach for producing $\mathcal{PT}$-symmetric Hamiltonians using $\mathcal{PT}$-symmetric GEMO. The latter contribution of the novel form of momentum is incorporated with the extended uncertainty principle (EUP) which is discussed earlier in [5]. We propose a complexified GEMO (CGEMO) consist of a $\mathcal{PT}$-symmetric auxiliary function $\mu(x)$ such that the CGEMO satisfies the so called extended uncertainty principle (EUP) commutation relation. Furthermore, we construct the corresponding $\mathcal{PT}$-symmetric Hamiltonian, utilizing the CGEMO. On the other hand, since the Hamiltonian and $\mathcal{PT}$ operator commute, i.e., $[\mathcal{H}, \mathcal{PT}] = 0$, the $\mathcal{PT}$-symmetry of such a system remains unbroken.

This paper is organized in the following order. In Sec. II, we propose the concept of CGEMO and prove that it is an observable in the sense that its eigenvalues are real. Afterwards, in Sec. III, one example of complex GEMO is demonstrated with the corresponding eigenfunctions and energy eigenvalues. In Conclusion we summarize our paper.

II. OBSERVABLE GENERALIZED MOMENTUM OPERATOR

Recently we have studied the Generalized Extended Momentum Operator (GEMO) [5] in the context of EUP, given by

$$p = -i\hbar (1 + \mu(x)) \frac{d}{dx} - \frac{i\hbar}{2} \frac{d\mu(x)}{dx},$$

(1)
in which the auxiliary function $\mu(x)$ is a real function of position operator $x$. The GEMO is Hermitian, $p = p^\dagger$ and satisfies the EUP relation, i.e.,

$$[x, p] = i\hbar (1 + \mu(x)).$$

(2)

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Here, in this study we propose \( \mu(x) \) to be a \( \mathcal{PT} \)-symmetric complex function i.e., \( \mathcal{PT}\mu(x) \equiv \mu^*(-x) = \mu(x) \) which in turn makes the GEMO non-Hermitian but \( \mathcal{PT} \)-symmetric. This means that, \( p \neq p^\dagger \), but \([\mathcal{P}T, p] = 0\). Having GEMO to be \( \mathcal{PT} \)-symmetric, implies that its eigenvalues are real. To prove that we start from the eigenvalue equation of the momentum operator

\[
p \Phi_p(x) = p \Phi_p(x)
\]

in which \( p \) and \( \Phi_p \) are the momentum eigenvalue and corresponding eigenfunction, respectively. Applying \( \mathcal{P}T \) operator from the left side on (3) yields

\[
\mathcal{P}T (p \Phi_p(x)) = p^* (\mathcal{P}T \Phi_p(x))
\]

which after the fact that \( p \) and \( \mathcal{P}T \) commute, one finds

\[
p \Phi_p(x) = p^* \Phi_p(x)
\]

which in turn implies that \( p = p^* \), indicating \( p \) is real and consequently, \( p \) is a physical observable.

Furthermore, the corresponding Hamiltonian of a particle with a \( \mathcal{PT} \)-symmetric GEMO undergoing a one-dimensional potential \( V(x) \) is given by

\[
\mathcal{H} = -\frac{\hbar^2}{2m} \left( (1 + \mu)^2 \frac{d^2}{dx^2} + 2(1 + \mu) \frac{d}{dx} + \frac{1}{2} (1 + \mu) \mu'' + \frac{1}{4} (\mu')^2 \right) + V(x)
\]

in which a prime stands for the derivative with respect to \( x \). Herein, with a \( \mathcal{PT} \)-symmetric potential i.e., \( \mathcal{PT}V(x) = V(x) \), \( \mathcal{H} \) becomes \( \mathcal{PT} \)-symmetric which indicates

\[\left[ \mathcal{P}T, \mathcal{H} \right] = 0.\]

In a similar manner as of GEMO, one can show that the eigenvalues of the Hamiltonian are real, i.e., in

\[
\mathcal{H} \phi(x) = E \phi(x)
\]

\( E \) is real. Next, we apply the so-called Point-Canonical-Transformation (PCT), upon which we introduce

\[
z = z(x) = \int_x^{z_0} \frac{1}{1 + \mu(y)} dy
\]

in which \( z_0 \) is a gauge constant and the wavefunction is decomposed as

\[
\phi(x) = \frac{1}{\sqrt{1 + \mu(x)}} \chi(z(x))
\]

to transformed the Schrodinger equation (8) to the standard form of the Schrodinger equation in \( z \)-space given by

\[
-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \chi(z) + V(x(z)) \chi(z) = E \chi(z).
\]

We observe here that, due to the PCT, although the Schrodinger equation in \( x \)-space is transformed into the standard Schrodinger equation in \( z \)-space, but the energy eigenvalues remained the same. In the other words, the energy is invariant under the PCT transformation.

III. QUASI FREE PARTICLE

Following the first example in Ref. [5], we choose

\[
\mu(x) = \alpha^2 x^2 + i2\beta x
\]

in which \( \alpha \) and \( \beta \) are two real constants such that \( \mu(x) \) remains \( \mathcal{PT} \)-symmetric. After, (12), the CGEMO becomes

\[
p = -i\hbar \left( 1 + \alpha^2 x^2 + i2\beta x \right) \frac{d}{dx} - i\hbar \left( \alpha^2 x + i\beta \right)
\]
which is \( \mathcal{PT} \)-symmetric and satisfies the EUP relation \([x, p] = i\hbar (1 + \alpha^2 x^2 + 2i\beta x)\). Here let’s find the eigenfunctions and eigenvalues of the CGEMO. To do so we start with

\[
p\Phi_p(x) = p\Phi_p(x)
\]

which yields

\[
(1 + \alpha^2 x^2 + i2\beta x) \frac{d\Phi_p(x)}{dx} + \left(\alpha^2 x + i\beta - \frac{ip}{\hbar}\right)\Phi_p(x) = 0.
\]

The explicit normalized solution for this equation is found to be

\[
\Phi_p(x) = \sqrt{\frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{1 + \alpha^2 x^2 + i2\beta x}}} \exp\left[-\frac{ip \arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2 + \beta^2}}\right)}{\hbar \sqrt{\alpha^2 + \beta^2}}\right],
\]

in which \( p \) is the continuous real eigenvalue of the momentum operator \( p \). To normalize the momentum eigenfunction (16) we introduce \( x + \frac{i\beta}{\alpha^2} = \xi \) which yields

\[
\Phi_p(\xi) = \sqrt{\frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{1 + \beta^2 + \alpha^2 \xi^2}}} \exp\left[-\frac{ip \arctan\left(\frac{\alpha^2 \xi}{\sqrt{\alpha^2 + \beta^2}}\right)}{\hbar \sqrt{\alpha^2 + \beta^2}}\right].
\]

On \( \xi - axis \) (17) is normalizable in the usual manner, i.e., by assuming \( \xi \) to be a real variable it satisfies

\[
\int_{-\infty}^{\infty} |\Phi_p(\xi)|^2 d\xi = 1.
\]

Next, the corresponding \( \mathcal{PT} \)-symmetric Hamiltonian is obtained to be as given in Eq. (6). Upon applying the PCT, introduced in Eqs. (9) and (10), the corresponding Schrödinger equation (8) in \( x \)-space is transformed into \( z \)-space given in Eq. (11) in which the transformed coordinate is found to be

\[
z(x) = 1 \sqrt{\frac{\alpha^2 + \beta^2}{\omega^2 + \beta^2}} \arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2 + \beta^2}}\right).
\]

We note that, \( z(x) = \zeta + i\eta \) is a complex variable such that

\[
\tan(\omega(\zeta + i\eta)) = \frac{\alpha^2 x + i\beta}{\omega}
\]

in which \( \omega = \sqrt{\alpha^2 + \beta^2} \). Eq. (15) implies

\[
\frac{\alpha^2 x}{\omega} = \frac{\sin(\omega\zeta) \cos(\omega\zeta)}{\cos^2(\omega\zeta) \cosh^2(\omega\eta) + \sin^2(\omega\zeta) \sinh^2(\omega\eta)}
\]

and

\[
\frac{\beta}{\omega} = \frac{\sinh(\omega\eta) \cosh(\omega\eta)}{\cos^2(\omega\zeta) \cosh^2(\omega\eta) + \sin^2(\omega\zeta) \sinh^2(\omega\eta)}.
\]

The domain of \( x \) is \( \mathbb{R} \), such that (16) and (17) are a map of the form \( \mathbb{R} \rightarrow \mathbb{C} \). For a finite nonzero \( \beta \), at the limits \( x \rightarrow \pm \infty \), we find \( \omega\zeta \rightarrow \pm \frac{\pi}{2} \) and \( \omega\eta \rightarrow 0 \).

In Eq. (11) we set the external potential to be zero, i.e., \( V(x) = 0 \) for the entire domain of \( x \) i.e., \( x \in \mathbb{R} \). Imposing \( V(x) = 0 \) in (11), one finds

\[
-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} \chi(z) = E \chi(z)
\]
FIG. 1: Plots of $\frac{1}{\alpha} |\phi_n(\xi)|^2$ in terms of shifted coordinate $\alpha \xi$ in which $\xi = x + i \frac{\beta}{\alpha}$ for $\frac{\beta}{\alpha} = 0.0$ (blue, solid), 0.25 (black, dash), 0.50 (green, dash-dot) and 1.0 (red, long-dash) respectively. With a given value for $\alpha$, the effect of the parameter $\frac{\beta}{\alpha}$ is to decrease the confinement of the particle on $\xi$-axis. The quantum number $n = 1$ and 2 for the left and right panels respectively.

where $\chi(z) \to 0$ when $\zeta \to \pm \pi$ and $\eta \to 0$. Hence, one writes the general solution for (18) as

$$\chi(z) = C_1 \sin(Kz) + C_2 \cos(Kz)$$

in which $K^2 = \frac{2mE}{\hbar^2}$. Applying the boundary conditions, one obtains the eigenfunctions and eigenvalues given by

$$\chi_n(z) = \begin{cases} C_2 \cos(n \sqrt{\alpha^2 + \beta^2} z), & \text{odd } n \\ C_1 \sin(n \sqrt{\alpha^2 + \beta^2} z), & \text{even } n \end{cases}$$

(24)

and

$$E_n = \frac{n^2 \hbar^2 (\alpha^2 + \beta^2)}{2m},$$

(25)

respectively.

Please note that, $C_1$ and $C_2$ are the normalization constants to be found. To find the corresponding wavefunction in $x$-space, we write

$$\phi_n(x) = \frac{1}{\sqrt{1 + \alpha^2 x^2 + 2i\beta x}} \chi_n(z)$$

(26)

which yields

$$\phi_n(x) = \begin{cases} C_2 \cos \left( n \arctan \left( \frac{\alpha^2 x + \beta}{\sqrt{\alpha^2 + \beta^2 x^2 + 2i\beta x}} \right) \right), & \text{odd } n \\ C_1 \sin \left( n \arctan \left( \frac{\alpha^2 x + \beta}{\sqrt{\alpha^2 + \beta^2 x^2 + 2i\beta x}} \right) \right), & \text{even } n \end{cases}$$

(27)

with real eigenvalues given in (25). Unlike $\chi_n(z)$, $\phi_n(x)$ can be normalized on the specific contour $\mathcal{C}$ such that

$$\int_{\mathcal{C}} (\mathcal{CPT} \phi_n(x)) \phi_n(x) \, dx = \int_{\mathcal{C}} (\phi_n(x))^2 \, dx$$

(28)

in which $\mathcal{C}$ operator is known to be the charge operator having eigenvalues $\pm 1$ and commutes with $\mathcal{PT}$ operator [2]. As one can see in (27), $(\phi_n(x))^2$ is complex function implying that it can not be the probability density. However,
on the specific contour $C$ on the complex $x$-plane where $C$ satisfies the certain conditions, i) $Im \left[ (\phi_n(x))^2 \right] dx = 0$, ii) $Re \left[ (\phi_n(x))^2 \right] dx \geq 0$ and iii) $\int_C (\phi_n(x))^2 dx = 1$, it can be considered as the probability density \[4\]. Here, the contour $C$ is a line parallel to the real axis defined by $Im(x) = \beta$ and the normalization constants are found to be

$$C_1 = C_2 = \sqrt{\frac{2\sqrt{\alpha^2 + \beta^2}}{\pi}}.$$ \[(29)\]

In Fig. 1 we plot the ground state and the first excited state corresponding to $n = 1$ and $n = 2$ in the solution (27). Increasing the value of $\beta$ for a given $\alpha$, the probability density admits a larger uncertainty for $\xi = x + \frac{\beta}{\alpha^2}$.

IV. CONCLUSION

A $\mathcal{PT}$-symmetric quantum Hamiltonian admits a real energy spectrum. For such a Hamiltonian, a complex potential is usually responsible for the non-Hermiticity. Here in this research, we have shown that, in the context of the GEMO, one may introduce a non-Hermitian Hamiltonian through a non-Hermitian GEMO. In doing so, we have introduced a $\mathcal{PT}$-symmetric GEMO in virtue of a $\mathcal{PT}$-symmetric auxiliary function $\mu(x)$. The momentum eigenvalues are real and the Hamiltonian is non-Hermitian but $\mathcal{PT}$-symmetric. Hence, the energy spectrum is real. We have also presented an explicit example with a specific choice for $\mu(x)$ for a quasi-free particle.

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