ON CONTRACTIBLE ORBIFOLDS

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Abstract. We prove that a contractible orbifold is a manifold.

1. Introduction

Following [Dav10], we call an orbifold $X$ contractible if all of its orbifold homotopy groups $\pi_i^{orb}(X), i \geq 1,$ vanish. We refer the reader to [Dav10] and the literature therein for basics about orbifolds. Davis has asked in [Dav10] whether any contractible orbifold $X$ must be developable. In this note we answer this question affirmatively.

Theorem 1.1. Let $X$ be a smooth contractible orbifold. Then it is a manifold.

Proof. Since $X$ is contractible, it is orientable. Let $n$ be the dimension of $X$. Define on $X$ a Riemannian metric and let the smooth manifold $M$ be the bundle of oriented orthonormal frames on $X$ (cf. [Hae84]). Then $G = SO(n)$ acts effectively and almost freely on $M$ with $X = M/G$.

Let $E$ denote a contractible CW complex on which $G$ act freely, with quotient $E/G = BG$, the classifying space of $G$. Then $\tilde{X} = (M \times E)/G$ is a model for the classifying space of $X$ (cf. [Hae84]). By definition, the orbifold homotopy groups of $X$ are the usual homotopy groups of $\tilde{X}$. Thus, by our assumption, the topological space $\tilde{X}$ is contractible.

The projection $M \times E \to \tilde{X}$ is a homotopy fibration. Thus the contractibility of $\tilde{X}$ implies that the embedding of any orbit of $G$ into $M \times E$ is a homotopy equivalence between $G$ and $M \times E$. Since $E$ is contractible, the projection $M \times E \to M$ is a homotopy equivalence as well. Therefore, for any $p \in M$, the composition $o_p : G \to G \cdot p \to M$ given by orbit map $o_p(g) := g \cdot p$ is a homotopy equivalence.

Assume now that $X$ is not a manifold. Then $G$ does not act freely on $M$. Thus, for some $p \in M$, the stabilizer $G_p$ of $p$ is a finite non-trivial group. Then the orbit map $o_p : G \to M$ factors through the quotient map $\pi_p : G \to G/G_p$. Since the orbit map is a homotopy equivalence, there must exist some map $i : G/G_p \to G$ such that $i \circ \pi : G \to G$ is a homotopy equivalence. However, the manifolds $G$ and $G/G_p$ are orientable and the map $G \to G/G_p$ is a covering of degree $|G_p|$. Thus, for $m = \dim(G) = n(n-1)/2$, the image of $\pi_p^* \in H^m(G,\mathbb{Z}) = \mathbb{Z}$ is a subgroup of $H^m(G,\mathbb{Z})$ of index $|G_p|$. In particular, $(i \circ \pi)^* \pi_p^*$ cannot be surjective, a contradiction. □

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A small observation on the proof above: If we do not assume $X$ to be contractible but merely $k$-connected, then the orbit map $o_p : G \to M$ is $k$-connected as well. Thus, for any $l < k$, the map $H^l(M, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ is surjective. Since the orbit map $o_p : G \to M$ factorizes through $\pi_p : G \to G/G_p$, we deduce as above that $H^l(G/G_p, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ is surjective, for all $l < k$. If $X$ is not a manifold, i.e., if some $G_p$ is non-trivial, the above contradiction shows that $k \leq n(n-1)/2$. However, recall that the free part of $H^*(G)$ is generated by elements of degree at most $2n - 3$ ([Hat02], p. 300). Hence, if $G_p$ is non-trivial, the map $H^l(G/G_p, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ cannot be surjective for all $l \leq 2n - 3$. We deduce that $X$ is a manifold if it is $(2n - 2)$-connected.

We believe that this observation is far from optimal in high dimensions. In fact, we do not know of a single example of a 4-connected non-developable orbifold. We would like to finish the note by formulating two problems.

**Problem 1.2.** Do highly connected non-developable orbifolds exist?

**Problem 1.3.** Does an analogue of Theorem 1.1 hold true for non-smooth orbifolds? Does it hold true for etale groupoids of isometries?

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