Domain wall interacting with a black hole:  
A new example of critical phenomena

V.P. Frolov*\(^1\)  A.L. Larsen\(^2\) and M. Christensen\(^3\)

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\(^1\) Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Canada T6G 2J1
\(^2\) Physics Department, University of Odense, Campusvej 55, 5230 Odense M, Denmark

*Electronic address: frolov@phys.ualberta.ca
\(^1\)Electronic address: all@fysik.ou.dk
\(^3\)Electronic address: mc@bose.fys.ou.dk
Abstract

We study a simple system that comprises all main features of critical gravitational collapse, originally discovered by Choptuik and discussed in many subsequent publications. These features include universality of phenomena, mass-scaling relations, self-similarity, symmetry between super-critical and sub-critical solutions, etc.

The system we consider is a stationary membrane (representing a domain wall) in a static gravitational field of a black hole. For a membrane that spreads to infinity, the induced 2+1 dimensional geometry is asymptotically flat. Besides solutions with Minkowski topology there exists also solutions with the induced metric and topology of a 2+1 dimensional black hole. By changing boundary conditions at infinity one finds that there is a transition between these two families. This transition is critical and it possesses all the above-mentioned properties of critical gravitational collapse. It is remarkable that characteristics of this transition can be obtained analytically. In particular, we find exact analytical expressions for scaling exponents and wiggle-periods.

Our results imply that black hole formation as a critical phenomenon is far more general than one might expect.
1 Introduction

In our previous publication [1] we considered stationary axially symmetric membranes (infinitely thin domain walls) embedded in the background of a Schwarzschild black hole. In the approximation when the gravitational field of a membrane can be neglected, the test membrane configuration is an extremal of the Dirac-Nambu-Goto action. Solutions for a membrane which spreads to spatial infinity form a one parameter family. We showed that three different 2+1 dimensional membrane topologies were possible depending on the boundary conditions at infinity: Minkowski topology, wormhole topology and black hole topology. Moreover, we found that the three different membrane topologies are connected via phase transitions of the form first discussed by Choptuik [2], in investigations of scalar field collapse. Besides the first order transition (with mass gap) between black-hole and wormhole topologies, there exists also a second order phase transition (no mass gap) between membranes of Minkowski topology and membranes of black hole topology. The induced metric on the intermediate membrane for the latter transition has a naked singularity. For the membranes of black hole topology we found a mass-scaling relation analogous to that originally found by Choptuik [2] (for a review, see for instance [3] and references given therein). We showed that \( \text{Mass} \propto p^\gamma \) where \( p \) is an external parameter and \( \gamma \approx 0.66 \), and we also found a periodic wiggle in the scaling relation with period \( \omega \approx 3.56 \).

The one-parameter family of stationary axially symmetric membranes in the Schwarzschild background thus essentially contains all the features of critical collapse in Einstein theory of gravity, as have recently been discussed in the literature (see [3] for a review). However, there are two important differences in the physical setup: (1) In the case of membranes we are not actually considering the dynamical formation of black holes; we are merely considering a one-parameter family of stationary membranes of which some have Minkowski topology while others have black hole topology. (2) In the case of membranes the induced 2+1 dimensional metric does not obey Einstein equations but is determined by solving the Dirac-Nambu-Goto equations.

Taking these differences into account, our results [1] indicate that black hole formation as a critical phenomenon is far more general than originally expected.

In the present paper, we continue the study of a test stationary membrane (representing a thin domain wall) interacting with a generic static black hole and demonstrate the universality of critical behavior in this system.
The main observation is that all important features of this critical behavior are determined only by properties of the membrane solution in the close vicinity of the event horizon. The latter allows complete analytical study. We start by considering a one-parameter family of uniformly accelerating axially symmetric membranes in flat Minkowski space, and demonstrate that already this very simple system contains all the features discussed above. We use the results of analytical study of uniformly accelerated membranes to obtain the characteristics of critical behavior in the general case of a membrane interacting with a black hole.

The paper is organized as follows. In section 2, we study membranes in Rindler space and describe a one-parameter family of membranes which are stationary in a Rindler frame. We also discuss some general properties of these solutions. In section 3, we give a different mathematical description of the problem in terms of phase-portrait and critical points. This will shed light on some general properties of the membrane solutions. In section 4, we derive analytically the so-called mass-scaling relation, that is, the exact expression of the “2+1 dimensional mass” of the black hole topology membranes in terms of an external “impact” parameter. In section 5, we return to the problem of stationary membranes in the black-hole background. We demonstrate that mass-scaling relation is valid, and obtain exact analytical expressions for the scaling exponent and the wiggle-period. Finally in section 6, we discuss the obtained results.

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2 Stationary Membranes in Rindler Frame

Our starting point is the Dirac-Nambu-Goto action

\[ S = \mu \int d^3\zeta \sqrt{-\det G_{AB}} \]  

(2.1)

describing a test membrane in an external gravitational field \( g_{\mu\nu} \). Here \( \mu \) is the membrane tension, and

\[ G_{AB} = g_{\mu\nu} X^\mu_{,A} X^\nu_{,B} \]  

(2.2)

the induced metric on the world-volume. We denote by \( X^\mu (\mu = 0, 1, 2, 3) \) spacetime coordinates, while \( \zeta^A (A = 0, 1, 2) \) are coordinates on the membrane world-volume.
We consider first axially symmetric membranes in flat Minkowski space which are stationary in a Rindler frame, that is, stationary as seen by a uniformly accelerating observer in Minkowski space. The line-element of Minkowski space in Rindler frame is

\[ ds^2 = -\alpha^2 Z^2 dT^2 + dZ^2 + dR^2 + R^2 d\phi^2 , \]  
\[(2.3)\]

where \( \alpha^{-1} \) is the acceleration in the \( Z \)-direction. The stationary axially symmetric membranes are parametrized as follows

\[ T = \tau , \; \phi = \sigma , \; R = R(Z) . \]  
\[(2.4)\]

For such membranes the equations of motion, corresponding to the action (2.1), reduce to the following second order non-linear ordinary differential equation

\[ ZRR'' + (RR' - Z)(1 + R'^2) = 0 , \]  
\[(2.5)\]

where prime denotes derivative with respect to \( Z \). A general solution of this equation depends on 2 arbitrary constants. Points where \( Z = 0 \) or \( R = 0 \) are singular points of the equation. We shall see later that solutions which are regular at these singular points form a 1-parameter family.

Equation (2.5) is non-integrable. To demonstrate this, we introduce parametrization \( Z = x(s) , \; R = y(s) \), where the parameter \( s \) is defined by

\[ \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 = x^2 y^2 . \]  
\[(2.6)\]

Then eqs. (2.5), (2.6) become

\[ \frac{d^2x}{ds^2} = y^2 x , \]
\[ \frac{d^2y}{ds^2} = x^2 y , \]  
\[(2.7)\]

which are exactly the Hamilton equations corresponding to the Hamiltonian

\[ H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) - \frac{1}{2} x^2 y^2 . \]  
\[(2.8)\]

A Hamiltonian similar to (2.8) but with the opposite sign of the potential, is well-known in the literature in connection with Yang-Mills equations [5, 6, 7].
and the corresponding dynamics has in fact been shown to be chaotic [3, 4, 5]. The system is thus non-integrable, and the change of sign in the potential does not affect this property.

A general solution to eq. (2.3) can therefore not be obtained in explicit form, but a very simple special solution is provided by

$$R = Z.$$  \hspace{1cm} (2.9)

This solution will turn out to play a very special role in our context of membranes, 2+1 dimensional black holes and critical phenomena. Moreover, it is easy to show that the asymptotic ($Z \to \infty$) behavior of a solution of (2.3) is given by

$$R = Z + \frac{\alpha^{-3/2}}{\sqrt{Z}} \left[ a \cos \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) + b \sin \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) \right], \hspace{1cm} (2.10)$$

where $(a, b)$ are dimensionless constants. A membrane is uniquely specified by its asymptotic behavior, that is by the 2-vector

$$\vec{p} = \begin{pmatrix} a \\ b \end{pmatrix}. \hspace{1cm} (2.11)$$

For solutions which are regular at the singular points $Z = 0$ or $R = 0$ of the equation (2.3) the constants $a$ and $b$ are not independent.

To demonstrate this, let us consider the induced metric on the membrane world-volume

$$d\Sigma^2 = -\alpha^2 Z^2 d\tau^2 + (1 + R'^2) dZ^2 + R^2 d\sigma^2,$$  \hspace{1cm} (2.12)

where $R = R(Z)$. The scalar curvature corresponding to the 2+1 dimensional geometries (2.12) is

$$\mathcal{R} = \frac{-1}{R^2 Z^2} \left( \frac{Z^2 + R^2 R'^2 + (RR' - Z)^2}{1 + R'^2} \right), \hspace{1cm} (2.13)$$

so that $\mathcal{R} < 0$ but asymptotically $\mathcal{R} \to 0_-$ \,(. In other words, the induced geometry is asymptotically flat at infinity $R \to \infty$. The induced curvature $\mathcal{R}$ diverges at points where either $Z = 0$ or $R = 0$ unless the following conditions are satisfied

$$\left( \mathbf{I} \right) \quad R = 0 \quad , \quad \frac{dZ}{dR} = 0; \hspace{1cm} (2.14)$$
These boundary conditions single out regular stationary (for a Rindler observer) axially symmetric membranes in Minkowski space. Imposing these conditions reduces the number of independent parameters characterizing a solution from 2 to 1. Some examples of solutions from the 1-parameter family of regular solutions are shown in figs. 1, 2, and the parameter space is shown in fig. 3.

Membranes obeying boundary condition (2.14) have the topology of 2+1 dimensional Minkowski spacetime, and can be thought of as deformed versions of a planar membrane.

An induced geometry for a membrane obeying boundary condition \( R(0) = R_{ah} \) for some \( R_{ah} > 0, (2.15) \), has an apparent horizon at \( Z = 0 \). These solutions describe a 2+1 dimensional black hole with “area”

\[
A_{ah} = 2\pi R_{ah}.
\]  

An exceptional case is provided by the particular solution (2.9). This membrane, which is the limiting solution separating the Minkowski topology membranes from the black hole topology membranes, has the topology of a cone, and its induced metric has a naked singularity at \((R, Z) = (0, 0)\).

Comparing with the results of the critical collapse in 3+1 dimensional Einstein gravity \([2, 3, 8, 9]\), the results of this section already suggest a close connection. In particular, the “critical” membrane solution (2.9), separating the Minkowski topology membranes from the black hole topology membranes, has a naked singularity and it is continuously self-similar in the sense that

\[
\mathcal{L}_\xi G_{AB} = 2G_{AB},
\]

where \( G_{AB} \) is the induced metric (2.2) for the critical solution (2.9), and the vector \( \xi \) is given by

\[
\xi^A = \begin{pmatrix} 0 \\ Z \\ 0 \end{pmatrix}.
\]

For the membranes with black hole topology, it is convenient to use as the “external parameter”

\[
p \equiv |\vec{p}| = \sqrt{a^2 + b^2}
\]
With this definition, the critical solution (2.9) corresponds to \( p = 0 \), and \( p \) is a monotonically decreasing parameter when approaching the critical solution; see fig. 3.

The goal is now to obtain the so-called “mass-scaling” relation, i.e. the relation between the apparent horizon area (2.16) and the parameter \( p \) for the black hole topology membranes. It should be stressed that, as in the case of scalar field collapse \([2]\), the precise form of the mass-scaling relation obviously depends on the precise definition of the “external parameter” \( p \).

In the present case we find that, due to the asymptotic form (2.10), the definition (2.19) is the most natural. In the case of stationary membranes in the Schwarzschild background \([1]\), a similar definition was used.

### 3 Phase-Portrait and Critical Points

Before deriving mass-scaling relation, we give an alternative mathematical description of the differential equation (2.5).

It is convenient to introduce new coordinates \((x, y)\) (not to be confused with those of eqs. (2.6)-(2.8)) by

\[
\begin{align*}
x &= R', \\
y &= \frac{R}{Z} R',
\end{align*}
\]

as well as a new parameter \( s \)

\[
\frac{dZ}{Z} = y(s) \, ds.
\]

Then eq. (2.5) becomes a first order regular autonomous system

\[
\begin{align*}
\frac{dx}{ds} &= x (1 - y) (1 + x^2), \\
\frac{dy}{ds} &= y \left( 1 - 2y + x^2 (2 - y) \right).
\end{align*}
\]

The corresponding phase-portrait is shown in fig. 4. Notice that there are four critical points;

\[
\begin{align*}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{node} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} & \text{saddle point}
\end{align*}
\]
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
\pm 1 \\
1
\end{pmatrix}
\] focus points

one being a node, another being a saddle point and the last two are focus points.

The focus points (3.6), which in the phase-portrait are approached by self-similar (logarithmic) spirals, correspond to the two critical membranes \( R = \pm Z \). Constraining ourselves to the one half part of spacetime \( Z > 0 \) lying above the Rindler horizon, only the focus with positive \( x \)-component is relevant.

The saddle point (3.5) precisely corresponds to the boundary condition (2.17) for the membranes of 2+1 dimensional black hole topology, that is

\[ R' = 0, \quad Z = 0, \quad R \text{ arbitrary}. \] (3.7)

The important consequence is that, when using these \((x, y)\)-coordinates, all membranes of 2+1 dimensional black hole topology are represented by only one curve, namely the curve connecting the saddle point \((0, 1/2)\) and the focus point \((1, 1)\). So it is necessary to make only one numerical integration of the system (3.3); then all membranes of 2+1 dimensional black hole topology can be reconstructed. In the next section we shall see that this fact is closely related to a certain symmetry of the equation (2.5).

Notice also that the node \((0, 0)\) is represented neither by the critical solution nor by any other solution obtained from the boundary conditions (2.14), (2.15) and hence it has no relevance here.

### 4 Analytical Approach to Mass-Scaling

Now return to the original equation (2.7). There are two symmetries that will allow us to give a complete analytical description of the critical phenomena concerning the transition between membranes of 2+1 dimensional Minkowski topology and membranes of 2+1 dimensional black hole topology.

Notice first that the boundary conditions (2.14), (2.15) are symmetric under interchange of \( R \) and \( Z \)

\[ R \leftrightarrow Z. \] (4.1)

This symmetry actually extends to the solutions themselves as follows from eq. (2.3) (see also eq. (2.7)). In our context it means that for any Minkowski
topology membrane $R = F(Z)$, there is a corresponding black hole topology membrane $Z = F(R)$ with the same function $F$ (see Figure 1).

It is also easy to see that if $R(Z)$ is a solution to eq. (2.3) then, for arbitrary $k > 0$, $R(kZ)/k$ is also a solution

$$R(Z) \leftrightarrow \frac{R(kZ)}{k}. \quad (4.2)$$

Transformation (4.2) preserves boundary condition (2.15), but it shifts the apparent horizon (2.16). Thus, from one black hole topology membrane, we can construct all the others by the transformation (4.2). Using the transformation (4.1), we can then also construct all the Minkowski topology membranes corresponding to the boundary conditions (2.14).

To make these statements more precise, consider some fixed “reference-solution” $\tilde{R}(Z)$, corresponding to a black hole topology membrane with apparent horizon $\tilde{R}_{ah}$ and asymptotic behavior (2.10) with constants ($\tilde{a}, \tilde{b}$), that is

$$\tilde{R}(0) = \tilde{R}_{ah}, \quad \tilde{R}'(0) = 0, \quad (4.3)$$

and for $Z \to \infty$

$$\tilde{R}(Z) = Z + \frac{\alpha^{-3/2}}{\sqrt{Z}} \left[ \tilde{a} \cos \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) + \tilde{b} \sin \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) \right]. \quad (4.4)$$

Any other black-hole-topology membrane $R(Z)$ can then be generated by the transformation (4.2) for some $k > 0$. This new solution is characterized by

$$R(0) = \frac{\tilde{R}(0)}{k} = \frac{\tilde{R}_{ah}}{k} \equiv R_{ah}, \quad R'(0) = 0, \quad (4.5)$$

and for $Z \to \infty$

$$R(Z) = Z + \frac{\alpha^{-3/2}}{k^{3/2}\sqrt{Z}} \left[ a \cos \left( \frac{\sqrt{7}}{2} \ln (\alpha k Z) \right) + b \sin \left( \frac{\sqrt{7}}{2} \ln (\alpha k Z) \right) \right]$$

$$\equiv Z + \frac{\alpha^{-3/2}}{\sqrt{Z}} \left[ a \cos \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) + b \sin \left( \frac{\sqrt{7}}{2} \ln (\alpha Z) \right) \right]. \quad (4.6)$$

Here the second line defines the constants $(a, b)$ according to eq. (2.10). It follows that

$$a = k^{-3/2} \left[ \tilde{a} \cos \left( \frac{\sqrt{7}}{2} \ln (k) \right) + \tilde{b} \sin \left( \frac{\sqrt{7}}{2} \ln (k) \right) \right],$$

$$b = k^{-3/2} \left[ \tilde{b} \cos \left( \frac{\sqrt{7}}{2} \ln (k) \right) - \tilde{a} \sin \left( \frac{\sqrt{7}}{2} \ln (k) \right) \right]. \quad (4.7)$$
This shows that we can generate the complete parameter space, fig. 3, from the “reference point” \((\tilde{a}, \tilde{b})\), and that the parameter space consists of two logarithmic spirals. Notice that the points corresponding to Minkowski topology membranes are obtained from the points corresponding to black hole topology membranes by inversion \((a, b) \leftrightarrow (-a, -b)\), as follows from eq. (4.1).

However, we can go one step further. Consider again the relations (4.5), (4.7) for the black hole topology membranes. The transformation (4.7) is a combined scaling and rotation in parameter space. Since

\[ k = \frac{\tilde{R}_{\text{ah}}}{R_{\text{ah}}} = \frac{\tilde{A}_{\text{ah}}}{A_{\text{ah}}} , \]  

(4.8)

the rotation matrix can be written as a product of two rotation matrices; one involving only \(\tilde{A}_{\text{ah}}\) and the other involving only \(A_{\text{ah}}\). That is, relations (4.7) are equivalent to

\[ A^{-3/2}_{\text{ah}} \begin{pmatrix} \cos\left(\frac{\sqrt{7}}{2} \ln \left(\tilde{A}_{\text{ah}}\right)\right) & \sin\left(\frac{\sqrt{7}}{2} \ln \left(\tilde{A}_{\text{ah}}\right)\right) \\ -\sin\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) & \cos\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \tilde{A}^{-3/2}_{\text{ah}} \begin{pmatrix} \cos\left(\frac{\sqrt{7}}{2} \ln \left(\tilde{A}_{\text{ah}}\right)\right) & \sin\left(\frac{\sqrt{7}}{2} \ln \left(\tilde{A}_{\text{ah}}\right)\right) \\ -\sin\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) & \cos\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}, \]  

(4.9)

where the horizon area is measured in units of \(\alpha^{-1}\). It follows that the combination:

\[ A^{-3/2}_{\text{ah}} \begin{pmatrix} \cos\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) & \sin\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) \\ -\sin\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) & \cos\left(\frac{\sqrt{7}}{2} \ln \left(A_{\text{ah}}\right)\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \]  

(4.10)

is a constant vector, i.e. the same for all membranes of black hole topology.

Taking the norm and using definition (2.19) finally leads to:

\[ \frac{p^2}{A^3_{\text{ah}}} = \text{const}. \]  

(4.11)

It follows that we get the exact mass-scaling relation

\[ \ln \left(A_{\text{ah}}\right) = \frac{2}{3} \ln \left(p\right) + c \]  

(4.12)

corresponding to a critical exponent \(\gamma = 2/3\), and \(c\) is an unimportant constant. This exactly linear mass-scaling relation without any wiggle was to
be expected due to the continuous self-similarity (2.17), (2.18), if we believe in the analogy with the case of scalar field collapse [2, 10, 11]. The relation (4.11) also implies that the "phase-transition" between membranes of Minkowski topology and membranes of black hole topology is of second order (no mass gap).

In this section we have obtained the mass-scaling relation for the black hole topology membranes, that is, the super-critical solutions. However, since we have the exact symmetry (4.1), we could as well have obtained the scaling relation for the Minkowski topology membranes, that is, the sub-critical solutions. In that case one would use instead of the apparent horizon area (2.16), the minimal proper distance between the Minkowski topology membrane and the Rindler horizon. The scaling relations in the two cases would then obviously be identical. It is interesting to mention that such "symmetry" between sub-critical and super-critical scaling relations have been found numerically in the case of scalar field collapse also [12]. In our case, this symmetry is completely trivial.

5 Black Hole Background Case

In a previous publication [1], we considered stationary axially symmetric membranes in the Schwarzschild spacetime

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right). \]  

(5.1)

Such membranes are parametrized by

\[ t = \tau, \quad \phi = \sigma, \quad \theta = \theta(r), \]  

(5.2)

and the equation of motion corresponding to the action (2.1) is

\[ \theta'' + (2r - 3M)\theta^3 - \frac{\theta'^2}{\tan(\theta)} + \frac{3r - 4M}{r(r - 2M)}\theta' - \frac{1}{r(r - 2M)\tan(\theta)} = 0. \]  

(5.3)

where the prime now denotes derivative with respect to \( r \).

In reference [1] we showed that three different membrane topologies were possible depending on the boundary conditions at infinity: 2+1 dimensional Minkowski topology, 2+1 dimensional wormhole topology and 2+1 dimensional black hole topology. We also found that the different membrane
topologies are connected via phase transitions of the form first discussed by Choptuik \cite{2} in investigations of scalar field collapse. In particular, we found a second order phase transition (no mass gap) between membranes of Minkowski topology and membranes of black hole topology. The corresponding mass-scaling relation for the black hole topology membranes was numerically found to have the approximate form

$$\ln(Mass) \approx \gamma \ln(p) + f(\ln(p)),$$

where $p$ is the external parameter and $f$ is a periodic function with period $\omega$. The numerical computations gave \cite{1}

$$\gamma \approx 0.66, \quad \omega \approx 3.56.$$  \hspace{1cm} (5.5)

We shall now show analytically that $\gamma = 2/3$ and $\omega = 3\pi/\sqrt{7}$, and that these numbers are universal, in the sense that they hold true for stationary axially symmetric membranes in a variety of background spacetimes, including Schwarzschild, Reissner-Nordström and Rindler spacetimes (in the latter case, the periodic function has zero amplitude, as will be discussed later).

Our main observation is that characteristics of critical behavior are determined only by properties of a membrane in the close vicinity of the event horizon of the background geometry. To illustrate this, we consider black-hole-topology membranes in the Schwarzschild background, although the arguments are more general. It is well-known that the near-horizon region of a Schwarzschild (as of any other static) black hole is Rindler space. To be more precise, consider the region near the $\theta = \pi$-pole: $r = 2M + \xi$, $\theta = \pi - \eta$, of the Schwarzschild black hole, and define

$$Z = \sqrt{8M\xi}, \quad R = 2M\eta.$$ \hspace{1cm} (5.6)

To the lowest order the line element (5.1) then becomes

$$ds^2 = -\frac{Z^2}{16M^2}dt^2 + dZ^2 + dR^2 + R^2d\phi^2,$$ \hspace{1cm} (5.7)

which is precisely the Rindler line element (2.3) for $\alpha = \frac{1}{4M}$. Moreover, the membrane-embedding equation (5.3) becomes:

$$ZRR'' + (RR' - Z) \left( 1 + R^2 \right) = -\frac{Z^2RR'^3}{4M^2} - \frac{Z^2RR'}{4M^2} \left[ \frac{1}{1 + \left( \frac{Z}{4M} \right)^2} \right]$$

$$- Z \left( 1 - \frac{R}{2M \tan \left( \frac{R}{2M} \right)} \right) \left[ \frac{1}{1 + \left( \frac{Z}{4M} \right)^2} \right] = ZR'^2 \left( 1 - \frac{R}{2M \tan \left( \frac{R}{2M} \right)} \right),$$ \hspace{1cm} (5.8)
which is precisely eq. (2.5) near the horizon, since the terms on the right hand side are negligible there.

As in Rindler space, a membrane in the Schwarzschild spacetime is specified by a 2-vector $\vec{p} = (a, b)$ which parametrizes the solutions of the embedding equation at spatial infinity. Near the critical membrane (which separates the Minkowski topology membranes from the black hole topology membranes), it is convenient to choose the parametrization such that the critical membrane corresponds to $(a, b) = (0, 0)$ and such that $p \equiv \sqrt{a^2 + b^2}$ is a decreasing function when approaching the critical solution (see ref. [1] for more details). In the Schwarzschild background, the critical membrane enters the horizon at $r = 2M$, $\theta = \pi$ where it has a naked singularity. Near-critical membranes are thus described by equation (5.8), with vanishing right hand side near the horizon.

Now consider such a near-critical membrane of black hole topology; we think of this solution as being obtained by linearization around the critical membrane. The idea is then to first read off Rindler-data $(a_R, b_R)$ in the Rindler region, and then to relate this to the Schwarzschild-data $(a, b)$ which we read off at spatial infinity. This makes sense since a membrane sufficiently close to the critical membrane will make a sufficient number of oscillations (see fig. 5) in the Rindler region (this statement actually follows from the symmetry (4.2)) to allow determination of the coefficients $(a_R, b_R)$, as defined by eq. (2.10). Moreover, since this near-critical membrane is obtained by linearization around the critical membrane, there will to lowest order be a linear relationship between the Rindler-data and the Schwarzschild-data:

$$
\begin{pmatrix}
a \\
b 
\end{pmatrix}
= \mathcal{A}
\begin{pmatrix}
a_R \\
b_R 
\end{pmatrix}
$$

(5.9)

for some constant $2\times2$ matrix $\mathcal{A}$, which somehow have encoded the spacetime curvature (for “pure” Rindler space, the matrix $\mathcal{A}$ is just the identity). By repeating the argument of section 4 it is now possible to obtain the exact mass-scaling relation for black hole topology membranes in the Schwarzschild background.

Consider again some fixed “reference-solution” corresponding to a black hole topology membrane with apparent horizon $\bar{R}_{ah}$, as well as another solution corresponding to the apparent horizon $R_{ah}$. Both solutions are very close to the critical membrane solution and thus have respective Rindler-data $(\bar{a}_R, \bar{b}_R)$ and $(a_R, b_R)$, as well as Schwarzschild-data $(\bar{a}, \bar{b})$ and $(a, b)$.
Now using relations \((5.3)\) and \((4.7)\) we get

\[
\begin{pmatrix} a \\ b \end{pmatrix} = k^{-3/2} \mathcal{A} \begin{pmatrix} \cos \left( \frac{\sqrt{7}}{2} \ln(k) \right) & \sin \left( \frac{\sqrt{7}}{2} \ln(k) \right) \\ -\sin \left( \frac{\sqrt{7}}{2} \ln(k) \right) & \cos \left( \frac{\sqrt{7}}{2} \ln(k) \right) \end{pmatrix} \mathcal{A}^{-1} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix},
\]

where \(k = \tilde{A}_{ah}/A_{ah}\). That is, by repeating the analysis of eqs. \((4.9)-(4.10)\), a vector being constant for all near-critical membranes of black hole topology exists

\[
A_{ah}^{-3/2} \begin{pmatrix} \cos \left( \frac{\sqrt{7}}{2} \ln(A_{ah}) \right) & \sin \left( \frac{\sqrt{7}}{2} \ln(A_{ah}) \right) \\ -\sin \left( \frac{\sqrt{7}}{2} \ln(A_{ah}) \right) & \cos \left( \frac{\sqrt{7}}{2} \ln(A_{ah}) \right) \end{pmatrix} \mathcal{A}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}.
\]

(5.11)

Taking the norm leads to

\[
p^2 \equiv a^2 + b^2 = A_{ah}^3 C_1 \left[ 1 + C_2 \cos \left( \sqrt{7} \ln(A_{ah}) - \varphi \right) \right],
\]

(5.12)

where \((C_1, C_2, \varphi)\) are constants depending only on the background spacetime! Furthermore, from eq. \((5.11)\) follows directly

\[
\ln(A_{ah}) = \frac{2}{3} \ln(p) - \frac{1}{3} \ln(C_1)
- \frac{1}{3} \ln \left[ 1 + C_2 \cos \left( \sqrt{7} \ln(A_{ah}) - \varphi \right) \right].
\]

(5.13)

Now let us compare with the numerical results obtained in ref. \[1\]. The numerically obtained mass-scaling relation is shown in fig. \[3\], together with a fit for the parameters \((C_1, C_2, \varphi)\) obtained using eq. \((5.13)\). It follows that there is indeed complete agreement.

It is sometimes convenient to write eq. \((5.13)\) in a different way. The expression can not be inverted analytically, but since we are near the critical solution, the first term on the right hand side is dominant (this is a reasonably good approximation provided \(C_2\) is not too close to 1). Therefore

\[
\ln(A_{ah}) \approx \frac{2}{3} \ln(p) - \frac{1}{3} \ln(C_1)
- \frac{1}{3} \ln \left[ 1 + C_2 \cos \left( \frac{2}{3} \sqrt{7} \ln(p) - \varphi \right) \right].
\]

(5.14)

This relation is of the standard form \[3\]

\[
\ln(Mass) \approx \gamma \ln(p) + C + f(\ln(p)),
\]

(5.15)
where \( f \) is a periodic function; \( f(z) = f(z + \omega) \). It must be stressed, however, that eq. (5.13) gives a more precise expression for the mass-scaling relation. But following eqs. (5.14)-(5.15), we find in our case

\[
\gamma = \frac{2}{3}, \quad \omega = \frac{3\pi}{\sqrt{7}},
\]

in agreement with eq. (5.5), while

\[
C = -\frac{1}{3} \ln(C_1), \\
f(z) = -\frac{1}{3} \ln \left[1 + C_2 \cos \left(\frac{2\sqrt{7}}{3}z - \varphi\right)\right].
\]

The constants \((C_1, C_2, \varphi)\) determining the constant \( C \) and the function \( f(z) \) can be obtained numerically (c.f. fig. 5), but they depend on the embedding spacetime. The important point is that the constants \( \gamma \) and \( \omega \) are universal, that is they have the same values (5.16) for an arbitrary black-hole background geometry. In this respect, membranes in “pure” Rindler space as discussed in Sections 1-4, represents a “degenerate” case. Indeed, in Rindler space the matrix \( \mathcal{A} \), as introduced in eq. (5.9), is just the identity matrix, and therefore the constant \( C_2 \) in eq. (5.17) equals 0. Thus in that case, there is no periodic wiggle in the scaling relation (more precisely, its amplitude vanishes) and the result reduces to eq. (4.12).

6 Conclusion

In this paper we have considered stationary cosmic membranes in Rindler space as well as in black hole spacetimes, thus generalizing and completing the previously obtained results [1]. Using exact analytical methods, we have analyzed in detail the transition between the different membrane topologies. We have shown that this simple system comprises all features of gravitational collapse of scalar and Yang Mills fields, as originally discovered and discussed by Choptuik [2]. These features include mass-scaling relations (with or without wiggles), universality of phenomena and certain dimensionless quantities (critical exponent and wiggle-period), mass gap (or no mass gap), self-similarity (continuous or discrete), symmetry between super-critical and sub-critical solutions, etc.
Using our simple system, we have thus been able to account for all these features in a completely analytical way. The obtained results are in complete agreement with previous numerical results and explain them. For instance, we have obtained exact analytical expressions for scaling exponents and wiggle-periods.

Our results indicate that black hole formation as a critical phenomenon might be far more general than expected. In particular, the critical phenomena of black hole formation are not restricted to time-dependent solutions of any particular physical equation, like the Einstein equation.

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Figure 1: Stationary and axially symmetric cosmic membranes in Rindler space fall into two families: Those of 2+1 dimensional Minkowski spacetime topology and those of 2+1 dimensional black hole spacetime topology. The limiting membrane $R = Z$ has a curvature singularity at $(0, 0)$. To obtain the full spatial structure of the membranes, the curves must be rotated around the $Z$-axis.
Figure 2: The Minkowski topology and black hole topology solutions oscillate around the critical solution with a fixed period in a logarithmic plot. Notice the symmetry relating the two topologies.
Figure 3: Conformal magnification of the parameter space near the critical solution. This illustrates how the membranes of the two topologies approach the critical solution along logarithmic spiral arms.
Figure 4: Phase-portrait of the system of equations (3.3). It has a saddle point at $(0, 1/2)$ and a focus point at $(1, 1)$. The curve connecting these two points represents all the black hole topology membranes.
Figure 5: A simple fit of the non-inverted mass-scaling relation (5.13) for the black hole topology membranes in the Schwarzschild background indicates that the constant $C_2$ is indeed quite large. Here the values $C_1 \approx 0.0188$, $C_2 = 0.858$ and $\varphi = 4.69$ have been used.