A UNIQUENESS CRITERION FOR UNBOUNDED SOLUTIONS TO THE VLASOV-POISSON SYSTEM

EVELYNE MIOT

Abstract. We prove uniqueness for the Vlasov-Poisson system in two and three dimensions under the condition that the $L^p$ norms of the macroscopic density growth at most linearly with respect to $p$. This allows for solutions with logarithmic singularities. We provide explicit examples of initial data that fulfill the uniqueness condition and that exhibit a logarithmic blow-up. In the gravitational two-dimensional case, such states are intimately related to radially symmetric steady solutions of the system. Our method relies on the Lagrangian formulation for the solutions, exploiting the second-order structure of the corresponding ODEs.

1. Introduction

The purpose of this article is to establish a uniqueness result for the Vlasov-Poisson system in dimension $n = 2$ or $n = 3$

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f &= 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \\
E(t, x) &= \gamma \int_{\mathbb{R}^n} \frac{x - y}{|x - y|^n} \rho(t, y) \, dy \\
\rho(t, x) &= \int_{\mathbb{R}^n} f(t, x, v) \, dv,
\end{align*}
\]

where $\gamma = \pm 1$. The system (1.1) is a physical model for the evolution of a system of particles interacting via a self-induced force field $E$. The interaction is gravitational if $\gamma = -1$ or Coulombian if $\gamma = 1$. The unknown $f = f(t, x, v) \geq 0$ denotes the microscopic density of the particles, and $\rho = \rho(t, x) \geq 0$ their macroscopic density.

A wide literature has been devoted to the Cauchy theory for the Vlasov-Poisson system. Ukai and Okabe [15] established global existence and uniqueness of smooth solutions in two dimensions. In any dimension, global existence of weak solutions with finite energy is a result due to Arsen'ev [2]. In three dimensions, global existence and uniqueness of compactly supported classical solutions where obtained by Pfaffelmoser [18] by Lagrangian techniques. Simultaneously, Lions and Perthame [11] constructed global weak solutions with finite velocity moments. More precisely, they proved that if

\[f_0 \in L^1 \cap L^\infty(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f_0 < \infty \quad \text{for some} \ m > 3,\]

Date: September 25, 2014.

2010 Mathematics Subject Classification. Primary 35Q83; Secondary 35A02, 35A05, 35A24.

Key words and phrases. Vlasov-Poisson system, uniqueness condition, propagation of the moments, logarithmic blow-up, steady states in two dimensions.
then there exists a corresponding solution \( f \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^3)) \) such that
\[
\forall T > 0, \quad \sup_{t \in [0,T]} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(t, x, v) \, dx \, dv < \infty.
\]
If \( m > 6 \) such a solution generates a uniformly bounded force field. We also refer to the works by Gasser, Jabin and Perthame \cite{Gasser}, Salort \cite{Salort} and Pallard \cite{Pallard} for further results concerning global existence and propagation of the moments. Another issue in the setting of weak solutions consists in determining sufficient conditions for uniqueness. Robert \cite{Robert} established uniqueness among weak solutions that are compactly supported. This result was extended by Loeper \cite{Loeper}, who proved uniqueness in the class of weak solutions with bounded macroscopic density
\[
\forall T > 0, \quad \rho \in L^\infty([0, T], L^\infty(\mathbb{R}^n)).
\]
The main result of this paper generalizes Loeper’s uniqueness condition (1.2) as follows:

**Theorem 1.1.** Let \( T > 0 \). There exists at most one weak solution \( f \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)) \) of the Vlasov-Poisson system on \([0, T]\) such that
\[
\sup_{[0,T]} \sup_{p \geq 1} \|\rho(t)\|_{L^p} < +\infty.
\]

Our next task is to determine sufficient conditions on the initial data for which any corresponding weak solution satisfies the uniqueness criterion of Theorem 1.1. We observe that (1.3) is fulfilled if for example
\[
\forall t \in [0, T], \quad \rho(t, x) \leq C(1 + \ln \, |x - \xi(t)|)
\]
for some \( \xi(t) \in \mathbb{R}^n \) (see \cite{Caprino}). Such densities where constructed by Caprino, Marchioro, Miot and Pulvirenti \cite{Caprino} as solutions of a related equation to (1.1). On the other hand, there exist solutions of (1.1) that satisfy (1.4) initially, as will be shown in Theorems 1.3 and 4.2. However, in general, it is not clear whether a logarithmic divergence like (1.4) persists at positive times. In fact, in order to propagate a control on the \( L^p \) norms of the macroscopic density we also need a description of the initial data at the microscopic level. In the above-mentioned previous works \cite{Caprino} \cite{Marchioro} \cite{Miot} \cite{Pulvirenti}, the condition \( L^p \) is met by assuming that the initial data satisfy
\[
\forall R > 0, \quad \forall T > 0, \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{|y - x| \leq RT, |v - w| \leq RT^2} f_0(y + vt, w) \, dv < +\infty.
\]
In the present paper we shall require instead a suitable control on the velocity moments, having in mind the well-known property that velocity moments control the norms of the density, see (3.1):

**Theorem 1.2.** Let \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) be nonnegative and such that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |v|^m f_0(x, v) \, dx \, dv < +\infty.
\]
for some \( m > n^2 - n \). Let \( T > 0 \) and let \( f \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)) \) be a weak solution provided by [11], Theo. 1. If \( f_0 \) satisfies
\[
\forall k \geq 1, \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f_0(x, v) \, dx \, dv \leq (C_0 k)^{\frac{k}{n}},
\]
for some constant \( C_0 \), then \( f \) satisfies the uniqueness condition (1.3).

Typically, Theorem 1.2 allows to consider initial densities with compact support in velocity as well as Maxwell-Boltzmann distributions of the type
\[
f_0(x, v) = e^{-|v|^p} h_0(x, v), \quad p \geq 0, \quad h_0 \in L^1 \cap L^\infty(\mathbb{R}^n).\]
Theorem 1.2 also does include some initial data with unbounded macroscopic density:

**Theorem 1.3.** There exists a nonnegative \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) satisfying the assumptions of Theorem 1.2 and such that
\[
\rho_0(x) = \omega_n \ln |x|, \quad \forall x \in \mathbb{R}^n.
\]

Let us next explain the main idea for proving Theorem 1.1. The argument of Loeper [12] in the context of uniformly bounded macroscopic densities (see also [13], Chapter 2) uses loglipschitz regularity for the force field
\[
|E(t, x) - E(t, y)| \leq (\|\rho(t)\|_{L^1} + \|\rho(t)\|_{L^\infty}) |x - y| (1 + |\ln |x - y||),
\]
which enables to perform a Gronwall estimate involving the distance between the Lagrangian flows associated to the solutions.

The loglipschitz regularity fails in the setting of unbounded densities. However, for \( L^p \) solutions, Sobolev embeddings imply that \( E \) is Hölder continuous with exponent and semi-norm estimated explicitly in terms of \( p \) and \( \|\rho(t)\|_{L^p} \), see Lemma 2.2 below. This estimate turns out to be sufficient to close the Gronwall estimate as \( p \to +\infty \) provided the \( L^p \) norms satisfy the condition in Theorem 1.1.

The Vlasov-Poisson system presents lots of analogies with the Euler equations for two-dimensional incompressible fluids
\[
\begin{aligned}
\partial_t \omega + u \cdot \nabla \omega &= 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2, \\
\omega &= \text{curl} u, \quad \text{div} u = 0,
\end{aligned}
\]
where \( u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \) is the velocity and \( \omega = \text{curl} u \) is the vorticity. Because of their analogous transport structure, both equations (1.1) and (1.5) are often handled similarly, especially for the uniqueness issue. In [12], Loeper extends his uniqueness proof for (1.1) to (1.5). Also the proof of uniqueness in [19] applies to both equations. We emphasize that this is not the case in the present paper, as explained in Remarks 2.4 and 2.5.

This is due to the fact that for the Vlasov-Poisson system the Lagrangian trajectories satisfy a second-order ODE, while for the Euler equations they satisfy a first-order ODE. This crucial observation was already exploited in [1], where it was proved that a logarithmic divergence on the macroscopic

\(^1\)The result of [11] is stated for \( n = 3 \). The case \( n = 2 \) can be obtained by a straightforward adaptation.

\(^2\)Here \( \omega_n \) denotes the volume of the unit ball of \( \mathbb{R}^n \).
density still yields enough regularity for the force field to get well-posedness for the corresponding ODE.

The paper is organized as follows. In the next Section 2 we recall a Hölder estimate for a field that controls the force field. As a consequence we derive a second-order Gronwall estimate on a distance between the Lagrangian flows of two solutions, which leads to the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Finally in Section 4 we prove Theorem 1.3 and we display in Proposition 4.1 a large class of initial densities for which uniqueness holds. We conclude by commenting on the link with radially symmetric steady states in the two-dimensional gravitational case.

**Notation.** In the remainder of the paper, the notation $C$ will denote a constant that can change from one line to another, depending only on $T, n, \|f\|_{L^\infty([0,T],L^1\cap L^\infty(\mathbb{R}^n))}$, and $\int\int |v|^mf_0$ (this latter quantity only for the proof of Theorem 1.2) but independent on $p$ and $k$ as $p, k \to +\infty$.

2. Proof of Theorem 1.1

2.1. Lagrangian formulation for weak solutions. We consider a weak solution $f \in L^\infty([0,T],L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ of (1.1) on $[0,T]$. We assume that $\rho \in L^\infty([0,T],L^1 \cap L^p(\mathbb{R}^n))$ for some $p > n$. By potential estimates we have $E = c(n)\nabla \Delta^{-1}\rho \in L^\infty([0,T] \times \mathbb{R}^n)$, and

$$\|E\|_{L^\infty([0,T],L^\infty)} \leq C_p \|\rho\|_{L^\infty([0,T],L^1 \cap L^p)}.$$ 

Moreover, $\nabla E \in L^\infty([0,T],L^p(\mathbb{R}^n))$ by virtue of the Caldéron-Zygmund inequality, see [7, Theo. 4.12]. Therefore it follows from DiPerna and Lions theory on transport equations [5, Theo. III2] that there exists a map $\Phi = (X,V) \in L^{1}_{loc}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n \times \mathbb{R}^n)$ such that for a.e. $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \mapsto (X,V)(t,x,v)$ is an absolutely continuous integral solution of the ODE

$$\begin{cases}
\dot{X}(t,x,v) = V(t,x,v), & X(0,x,v) = x \\
\dot{V}(t,x,v) = E(t,X(t,x,v)), & V(0,x,v) = v.
\end{cases}$$

Moreover,

$$\forall t \in [0,T], \quad f(t) = \Phi(t)\#f_0$$

which means that $f(t)(B) = f_0 ((\Phi(t,\cdot,\cdot)^{-1}(B))$ for all Borel set $B \subset \mathbb{R}^n$. Such a map is unique and is called Lagrangian flow associated to $E$. We refer also to [11 Theo. 5.7] for a more recent statement and for further developments on the theory.

We note that (2.1) implies that $t \mapsto \Phi(t,x,v) \in W^{1,\infty}([0,T])$ for a.e. $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$.

As a byproduct of our analysis we shall see in Paragraph 2.4 that under the assumptions of Theorem 1.1 the Lagrangian flow actually corresponds to the classical notion of flow.
2.2. Estimate on the Lagrangian trajectories. We consider two solutions \( f_1 \) and \( f_2 \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)) \) such that \( \rho_1 \) and \( \rho_2 \) belong to \( L^\infty([0,T], L^p(\mathbb{R}^n)) \) for some \( p > n \). Denoting by \( \Phi_1 = (X_1, V_1) \) and \( \Phi_2 = (X_2, V_2) \) the corresponding Lagrangian flows, we introduce the distance

\[
D(t) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |X_1(t, x, v) - X_2(t, x, v)|f_0(x, v) \, dx \, dv.
\]

We infer from (2.2) that

\[
|X_1(t, x, v) - X_2(t, x, v)| \leq \int_0^t \int_0^s |E_1(\tau, X_1(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))| \, d\tau \, ds.
\]

In particular, \( \sup_{(x,v)} |X_1(t, x, v) - X_2(t, x, v)| \leq CT^2(\|E_1\|_{L^\infty} + \|E_2\|_{L^\infty}) \), which shows that \( D \) defines a continuous function on \([0, T]\). The purpose of this paragraph is to establish the estimate

**Proposition 2.1.** For all \( t \in [0,T] \) and for all \( p > n \),

\[
D(t) \leq C_p \max \left( 1 + \|\rho_1\|_{L^\infty([0,T], L^p)}, \|\rho_2\|_{L^\infty([0,T], L^p)} \right) \int_0^t \int_0^s D(\tau) \, d\tau \, ds.
\]

The proof of Proposition 2.1 relies on the following potential estimate, the proof of which is postponed at the end of this paragraph.

**Lemma 2.2.** There exists \( C > 0 \) such that for all \( p > n \) and \( g \in L^1 \cap L^p(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} \frac{|x - z|}{|x - z|^n} - \frac{y - z}{|y - z|^n} |g(z)| \, dz \leq C_p(\|g\|_{L^p} + \|g\|_{L^1})|x - y|^{1 - \frac{n}{p}}.
\]

**Remark 2.3.** Setting \( E[g] = x/|x|^n \ast g = c(n) \nabla \Delta^{-1} g \) we observe that Lemma 2.2 implies the estimate

\[
|E[g](x) - E[g](y)| \leq C_p(\|g\|_{L^p} + \|g\|_{L^1})|x - y|^{1 - n/p}.
\]

This latter inequality can be obtained by combining Morrey’s inequality, which implies that \( |E[g](x) - E[g](y)| \leq C\|\nabla E[g]\|_{L^p}|x - y|^{1 - \frac{n}{p}} \), and Calderón-Zygmund inequality, see [7, Theo. 4.12], which implies that \( \|\nabla E[g]\|_{L^p} \leq C_p\|g\|_{L^p} \).

**Proof of Proposition 2.1.**

By (2.4), we have

\[
D(t) \leq \int_0^t \int_0^s \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} |E_1(\tau, X_1(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))|f_0(x, v) \, dx \, dv \right) \, d\tau \, ds
\]

\[
\leq \int_0^t \int_0^s \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} |E_1(\tau, X_1(\tau, x, v)) - E_1(\tau, X_2(\tau, x, v))|f_0(x, v) \, dx \, dv \right) \, d\tau \, ds
\]

\[
+ \int_0^t \int_0^s \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} |E_1(\tau, X_2(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))|f_0(x, v) \, dx \, dv \right) \, d\tau \, ds
\]

\[
\leq I + J.
\]
On the other hand, since \( f \in E \) By the same arguments as in the proof of Proposition 2.1, it satisfies
\[
(2.7)
\]
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |E_1(\tau, X_1(\tau, x, v)) - E_1(\tau, X_2(\tau, x, v))| f_0(x, v) \, dx \, dv
\]
\[
\leq C p \left( 1 + \|\rho_1\|_{L^\infty(L^p)} \right) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |X_1(\tau, x, v) - X_2(\tau, x, v)|^{1 - \frac{n}{p}} f_0(x, v) \, dx \, dv.
\]
Therefore by Jensen's inequality we find
\[
(2.6)
\]
\[
I \leq C p \left( 1 + \|\rho_1\|_{L^\infty(L^p)} \right) \int_0^t \int_0^s \mathcal{D}(\tau)^{1 - \frac{n}{p}} \, d\tau \, ds.
\]
Next, inserting that \( f_2(\tau) = \Phi_2(\tau)_# f_0 \), we obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |E_1(\tau, X_2(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))| f_0(x, v) \, dx \, dv
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |E_1(\tau, x) - E_2(\tau, x)| f_2(x, v) \, dx \, dv.
\]
On the other hand, since \( f_1(\tau) = \Phi_1(\tau)_# f_0 \) and \( f_2(\tau) = \Phi_2(\tau)_# f_0 \),
\[
(2.7)
\]
\[
E_1(\tau, x) - E_2(\tau, x) = \gamma \int_{\mathbb{R}^n} \left( \frac{x - X_1(\tau, y, w)}{|x - X_1(\tau, y, w)|^n} - \frac{x - X_2(\tau, y, w)}{|x - X_2(\tau, y, w)|^n} \right) f_0(y, w) \, dy \, dw.
\]
Therefore, we obtain by Fubini's theorem
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |E_1(\tau, X_2(\tau, x, v)) - E_2(\tau, X_2(\tau, x, v))| f_0(x, v) \, dx \, dv
\]
\[
\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \frac{x - X_1(\tau, y, w)}{|x - X_1(\tau, y, w)|^n} - \frac{x - X_2(\tau, y, w)}{|x - X_2(\tau, y, w)|^n} \right) f_0(y, w) \, dy \, dw \right) \rho_2(\tau, x) \, dx
\]
\[
\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \frac{x - X_1(\tau, y, w)}{|x - X_1(\tau, y, w)|^n} - \frac{x - X_2(\tau, y, w)}{|x - X_2(\tau, y, w)|^n} \right) \rho_2(\tau, x) \, dx \right) f_0(y, w) \, dy \, dw
\]
\[
\leq \int_{\mathbb{R}^n} C p \left( \|\rho_2(\tau)\|_{L^1} + \|\rho_2(\tau)\|_{L^p} \right) |X_1(\tau, y, w) - X_2(\tau, y, w)|^{1 - \frac{n}{p}} f_0(y, w) \, dy \, dw,
\]
where we have applied Lemma 2.2 in the last inequality. Hence Jensen's inequality yields
\[
(2.8)
\]
\[
J \leq C p \left( 1 + \|\rho_2\|_{L^\infty(L^p)} \right) \int_0^t \int_0^s \mathcal{D}(\tau)^{1 - \frac{n}{p}} \, d\tau \, ds.
\]

The conclusion follows from (2.6) and (2.8).

Remark 2.4. A similar function can be introduced to establish uniqueness for (1.9) with bounded vorticity, see e.g. [13] Theo. 3.1, Chapter 2,
\[
\tilde{\mathcal{D}}(t) = \int_{\mathbb{R}^2} |X_1(t, x) - X_2(t, x)| \omega_0(x) \, dx,
\]
where \( X_1 \) and \( X_2 \) denote the Lagrangian flows
\[
\dot{X}_i(t, x) = u_i(t, X_i(t, x)), \quad X(0, x) = x.
\]
By the same arguments as in the proof of Proposition 2.1 it satisfies
\[
\tilde{\mathcal{D}}(t) \leq C p \max \left( \|\omega_1\|_{L^\infty(L^1 \cap L^p)}, \|\omega_2\|_{L^\infty(L^1 \cap L^p)} \right) \int_0^t \tilde{\mathcal{D}}^{1 - \frac{n}{p}}(s) \, ds
\]
therefore, by conservation of the $L^p$ norms of the vorticity,

$$\tilde{D}(t) \leq C_p \|\omega_0\|_{L^1 \cap L^p} \int_0^t \tilde{D}^{1-\frac{2}{p}}(s) \, ds.$$  

**Proof of Lemma 2.2**

The proof for $p = \infty$ is well-known, see e.g. [14, Chapter 8] for the case $n = 2$. When $p < +\infty$ it is obtained by very similar arguments, but we provide the full details because we are not aware of any reference in the literature. Let $p_0 > n$. By Hölder inequality, we have

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{x - z}{|x - z|^n} |g(z)| \, dz \leq \sup_{z \in \mathbb{R}^n} \left( \int_{|x - z| \leq 1} \frac{|g(z)|}{|x - z|^{n-1}} \, dz + \int_{|x - z| \geq 1} \frac{|g(z)|}{|x - z|^{n-1}} \, dz \right)$$

$$\leq \|g\|_{L^{p_0}(\mathbb{R}^n)} \|z|^{-n+1}\|_{L^{p_0}(B(0,1))} + \|g\|_{L^1}$$

$$\leq C_{p_0} (\|g\|_{L^1} + \|g\|_{L^{p_0}}),$$

with $C_{p_0}$ depending only on $p_0$. Hence it suffices to establish Lemma 2.2 for $|x - y| < 1$. Let us introduce $d = |x - y|$ and $A = (x + y)/2$. We split the integral as

$$\int_{\mathbb{R}^n} \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} |g(z)| \, dz = \int_{\mathbb{R}^n \setminus B(A,1)} \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} |g(z)| \, dz$$

$$+ \int_{B(A,1) \setminus B(A,d)} \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} |g(z)| \, dz + \int_{B(A,d)} \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} |g(z)| \, dz = I + J + K.$$  

For $|z - A| \geq 1$ we have $\min(|x - z|, |y - z|) \geq 1 - d/2 \geq 1/2$, hence

$$I \leq \int_{\mathbb{R}^n \setminus B(A,1)} \frac{|g(z)|}{|x - z|^{n-1}} \, dz + \int_{\mathbb{R}^n \setminus B(A,1)} \frac{|g(z)|}{|y - z|^{n-1}} \, dz \leq C\|g\|_{L^1}.$$  

Next, applying first Hölder inequality, then the mean-value theorem, we obtain

$$J \leq \|g\|_{L^p} \left( \int_{B(A,1) \setminus B(A,d)} \frac{1}{|x - z|^{n-1}} \left. \frac{x - z}{|x - z|^n} - \frac{y - z}{|y - z|^n} \right|^{p'} \, dz \right)^{1/p'}$$

$$\leq \|g\|_{L^p} d \left( \int_{B(A,1) \setminus B(A,d)} \sup_{u \in [x,y]} \frac{1}{|u - z|^{np'}} \, dz \right)^{1/p'}.$$  

Now, for $|z - A| \geq d$ we have $|u - z| \geq |z - A| - |u - A| \geq |z - A|/2$ for any $u \in [x, y]$. Therefore

$$J \leq C d \|g\|_{L^p} \left( \int_{B(A,1) \setminus B(A,d)} \frac{1}{|z - A|^{np'}} \, dz \right)^{1/p'} \leq C d \|g\|_{L^p} d^{n(1 - \frac{1}{p'})} (p' - 1)^{-\frac{1}{p'}}$$

hence

$$J \leq C_p \|g\|_{L^p} d^{1-\frac{n}{p}}.$$
Applying again Hölder inequality, we obtain
\[ K \leq \|g\|_{L^p} \left( \int_{B(A,d)} \frac{1}{|x-z|^{p'(n-1)}} \, dz \right)^{1/p'} + \|g\|_{L^p} \left( \int_{B(A,d)} \frac{1}{|y-z|^{p'(n-1)}} \, dz \right)^{1/p'} . \]

Since for $|z-A| \leq d$ we have $\max(|x-z|, |y-z|) \leq 3d/2$, we finally obtain
\[ K \leq 2\|g\|_{L^p} \left( \int_{B(0,3d/2)} \frac{1}{|u|^{p'(n-1)}} \, du \right)^{1/p'} \leq C\|g\|_{L^p} d^{1-\frac{n}{p}} . \]

2.3. Proof of Theorem 1.1. Given two solutions $f_1$ and $f_2$ of (1.1) satisfying the assumptions of Theorem 1.1 let $\mathcal{D}$ be the corresponding distance function. Since $\max(\|\rho_1\|_{L^\infty(L^p)}, \|\rho_2\|_{L^\infty(L^p)}) \leq Cp$ by assumption, Proposition 2.1 implies that
\[ \mathcal{D}(t) \leq Cp^2 \int_0^t \int_0^s \mathcal{D}^{1-\frac{n}{p}}(\tau) \, d\tau \, ds . \]

Let $F(t) = \int_0^t \int_0^s \mathcal{D}^{1-\frac{n}{p}}(\tau) \, d\tau \, ds$. Since $\mathcal{D} \in C([0,T])$ we have $F \in C^2([0,T])$, with
\[ \forall t \in [0,T], \quad F''(t) \leq Cp^2 F^{1-\frac{n}{p}}(t) . \]

We next argue similarly as in the proof of Lemma 4 in [4]. We multiply the previous inequality by $F'(t) \geq 0$ and integrate on $[0,t]$. We obtain
\[ \forall t \in [0,T], \quad (F'(t))^2 \leq Cp^2 F(t)^{2-\frac{n}{p}} \]
therefore
\[ \forall t \in [0,T], \quad F'(t) \leq CpF(t)^{1-\frac{n}{2p}} . \]

We now conclude as in the proof of the uniqueness of bounded solutions of the 2D Euler equations, see e.g. [23, 14]: integrating the above inequality yields
\[ \forall p > n, \quad \forall t \in [0,T], \quad F(t) \leq (Ct)^{\frac{2p}{p-n}} . \]

Letting $p \to +\infty$ we obtain that $F(t) = 0$ for $t \in [0,1/C]$. Repeating the argument of intervals of length $1/C$ we finally prove that $F$, therefore also $\mathcal{D}$, vanishes on $[0,T]$. This implies that for all $t \in [0,T]$ we have $X_1(t,\cdot,\cdot) = X_2(t,\cdot,\cdot) \int_0^t f_0 \, dx \, dv$ a.e. We infer from (2.17) that for all $t \in [0,T]$, $E_1(t,\cdot) = E_2(t,\cdot)$ on $\mathbb{R}^n$. By (2.21), it follows that $V_1(t,\cdot,\cdot) = V_2(t,\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$. We conclude that for all $t \in [0,T]$ we have $f_1(t,\cdot,\cdot) = f_2(t,\cdot,\cdot)$ a.e. on $\mathbb{R}^n \times \mathbb{R}^n$.

Remark 2.5. In the setting of (1.5), the estimate obtained for $\mathcal{D}$ in Remark 2.4 yields
\[ \forall p > 2, \quad \mathcal{D}(t) \leq (C\|\omega_0\|_{L^p} t)^p , \]
which does not enable to conclude that $\mathcal{D} = 0$ as above unless $\omega_0 \in L^\infty$. 
2.4. The Lagrangian flow is the classical flow. We conclude this section with the following remark: let $f$ be a weak solution of (1.1) satisfying the assumptions of Theorem 1.1. In view of Remark 2.3 we have

$$\forall p > n, \sup_{t \in [0, T]} |E(t, x) - E(t, y)| \leq C p^2 |x - y|^{1 - \frac{n}{p}}.$$  

By space continuity of $E$, Ascoli-Arzela’s theorem implies that for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists a curve $\gamma \in W^{1, \infty}([0, T]; \mathbb{R}^n \times \mathbb{R}^n)$ which is a solution to the ODE (2.2). Moreover, if $\gamma_1$ and $\gamma_2$ are two such integral curves then $d(t) = \int_0^t \int_0^s |\gamma_1 - \gamma_2|(|\tau|) \, d\tau \, ds$ satisfies $d'' \leq C p^2 d^{1-n/p}$. So by exactly the same arguments as in the proof of Theorem 1.1 above, $d = 0$ on $[0, T]$. This means that the ODE (2.2) is well-posed for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ and that the Lagrangian flow is actually a classical flow.

3. Proof of Theorem 1.2

We start by recalling an elementary inequality, which can be found in [11] (14) for the case $n = 3$, and which can be easily adapted to the case $n = 2$. Let $f \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be nonnegative and $\rho_f(x) = \int f(x, v) \, dv$. Then

$$\forall k \geq 1, \|\rho_f\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_{L^\infty} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f(x, v) \, dx \, dv\right)^{\frac{1}{k}},$$

where $C$ is a constant independent on $k$.

Now, let $f_0$ satisfy the assumptions of Theorem 1.2 and let $f$ be any weak solution on $[0, T]$ with this initial data given by [11] Theo. 1. By construction we have

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^n \times \mathbb{R}^n} |v|^m f(t, x, v) \, dx \, dv < +\infty.$$  

In view of (3.1), in order to control the norms $\|\rho(t)\|_{L^p}$ for large $p$ it suffices to prove that

$$\forall k > 0, \sup_{t \in [0, T]} \|f(t)\|_{L^\infty} M_k(t)^{\frac{n}{k-n}} \leq C k,$$

where

$$M_k(t) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f(t, x, v) \, dx \, dv = \int_{\mathbb{R}^n \times \mathbb{R}^n} |V(t, x, v)|^k f_0(x, v) \, dx \, dv.$$  

Since $f \in L^\infty([0, T], L^\infty(\mathbb{R}^n))$ this amounts to showing that

$$\forall k > 0, \sup_{t \in [0, T]} M_k(t)^{\frac{n}{k-n}} \leq C k.$$  

At this stage it is not known whether all the $M_k(t)$ remain finite for $t > 0$. We prove next that this is indeed the case and that (3.3) can be achieved thanks to (3.2) in a much easier way as for the propagation (3.2) itself, which is the heart of the matter of [11]. As a matter of fact, since $m > n^2 - n$ we infer from (3.1) and (3.2) that $\rho \in L^\infty([0, T], L^{p_0}(\mathbb{R}^n))$ with $p_0 = (m + n)/n > n$ depending only on $n$ and $m$. It follows that $E \in L^\infty([0, T], L^\infty(\mathbb{R}^n))$ by 2.1.
For \( k > m \), we have by (2.2)
\[
|V(t, x, v)|^k \leq |v|^k + k \int_0^t |V(s, x, v)|^{k-1} |E(s, X(s, x, v))| \, ds
\]
\[
\leq |v|^k + k \|E\|_{L^\infty([0,T] \times \mathbb{R}^n)} \int_0^t |V(s, x, v)|^{k-1} \, ds.
\]
Integrating with respect to \( f_0(x, v) \, dx \, dv \) we get
\[
M_k(t) \leq M_k(0) + k \|E\|_{L^\infty([0,T] \times \mathbb{R}^n)} \int_0^t M_{k-1}(s) \, ds.
\]
By induction, we first infer that \( \sup_{t \in [0,T]} M_k(t) \) is finite for any \( k > m \). On the other hand, we obtain by Hölder inequality
\[
M_{k-1}(s) \leq \|f(s)\|_{L^1}^\frac{1}{k} M_k(s)^{1-\frac{k}{k}},
\]
therefore, since \( \|f(s)\|_{L^1} = \|f_0\|_{L^1} \) by (2.3) we get
\[
M_k(t) \leq M_k(0) + Ck \int_0^t M_k(s)^{1-\frac{1}{k}} \, ds.
\]
Integrating this Gronwall inequality leads to
\[
\sup_{t \in [0,T]} M_k(t)^\frac{1}{k} \leq M_k(0)^{\frac{1}{k}} + C.
\]
By assumption on \( M_k(0) \) we find
\[
\sup_{t \in [0,T]} M_k(t)^\frac{1}{k} \leq (C_0k)^{\frac{1}{k}} + C \leq (Ck)^{\frac{1}{k}}
\]
therefore, finally,
\[
\sup_{t \in [0,T]} M_k(t)^\frac{1}{kn} \leq Ck,
\]
and the conclusion follows.

4. Proof of Theorem 1.3

4.1. Seeking for initial data. In this section we construct a collection of initial densities that satisfy the assumptions of Theorem 1.2 and that do not necessarily enter in the framework of Loeper’s uniqueness condition. We will consider nonnegative measurable functions \( \varphi \) on \( \mathbb{R} \) such that
\[
\sup\{\varphi \} \subset ]-\infty, M]\] for some \( M \in \mathbb{R} \).

**Proposition 4.1.** Let \( \varphi \in L^\infty(\mathbb{R}, \mathbb{R}_+) \) satisfy (4.1). Let \( \Phi : \mathbb{R}^n \to \mathbb{R} \) and \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \) be two measurable functions. We set
\[
f_0(x, v) = \varphi \left( |v|^2 + \Phi(x) + a(x, v) \right).
\]
We assume that \( \rho_0 = \int f_0 \, dv \) has compact support in \( B \subset \mathbb{R}^n \), and that
\[
\forall p \geq 1, \quad \int_B (M - \Phi(x))^p_+ \, dx \leq (C_0p)^{\frac{2p}{n}},
\]
for some constant \( C_0 \). Then any initial density given by
\[
f_0 h_0, \quad \text{where } h_0 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n),
\]
satisfies the assumptions of Theorem 1.2.
Proof. Since for \((x, v) \in \text{supp} f_0\) we have \(|v|^2 \leq M - \Phi(x)\), we obtain
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f_0(x, v) h_0(x, v) \, dx \, dv \\
\leq \omega_n \|h_0\|_{L^\infty} \int_B (M - \Phi(x))^{\frac{k}{n}} \rho_0(x) \, dx \\
\leq \omega_n \|h_0\|_{L^\infty} \|\varphi\|_{L^\infty} \int_B (M - \Phi(x))^{\frac{k}{n}} \, dx.
\]
Finally, the condition of Theorem 1.2 is fulfilled provided
\[
\forall k \geq 1, \quad \int_B (M - \Phi(x))^{\frac{k}{n}} \, dx \leq (C_0 k)^{\frac{k}{n}},
\]
and this concludes the proof. \(\square\)

4.2. Proof of Theorem 1.3. We consider an initial density given by (4.2) with the choice
\[
\varphi = 1_{\mathbb{R}^n}, \quad \Phi(x) = - (\ln |x|)^{\frac{2}{n}}, \quad a = 0,
\]
so that
\[
\rho_0(x) = \left\{ v : |v|^2 - (\ln |x|)^{\frac{2}{n}} \leq 0 \right\} = \omega_n \ln |x|, \quad \forall x \in \mathbb{R}^n.
\]
Besides, a straightforward computation yields
\[
\forall p \geq 1, \quad \int_{\mathbb{R}^n} (\ln |x|)^p \, dx = \sigma_n n^{-p} p !
\]
where \(\sigma_n\) denotes the surface of \(\partial B(0, 1)\), so that by Stirling’s formula we get
\[
\forall p \geq 1, \quad \int_{\mathbb{R}^n} (\ln |x|)^{\frac{2p}{n}} \, dx \leq (C p)^{\frac{2p}{n}}.
\]
The conclusion follows by invoking Proposition 4.1.

4.3. Steady states in the two-dimensional gravitational case. In this last paragraph we focus on the Vlasov-Poisson equation (1.1) in the gravitational case for \(n = 2\), which can be rewritten as
\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f - \nabla U \cdot \nabla_v f &= 0 \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^2 \\
U(t, x) &= \int_{\mathbb{R}^2} \ln |x - y| \rho(t, y) \, dy \\
\rho(t, x) &= \int_{\mathbb{R}^n} f(t, x, v) \, dv.
\end{aligned}
\]
Every function of the form
\[
\overline{f}(x, v) = \varphi \left( \frac{|v|^2}{2} + U(x) \right),
\]
with \(\varphi \in C^1(\mathbb{R}, \mathbb{R})\), is a stationary solution of (4.3). Existence of steady states of the form (4.6) and their stability properties, especially in three dimensions, have been studied intensively (see [3, 6, 9, 10] and references therein). By variational methods, Dolbeault, Fernández and Sánchez [6, Theo. 1, Theo. 22] obtained the existence of a steady solution \(\overline{f} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)\) of the form (4.6), where \(\varphi\) is continuous, nonincreasing and satisfies
Moreover $μ = \int f dv$ is radially symmetric, compactly supported in $B(0,1)$, and $U$ is of class $C^1$ on $\mathbb{R}^2 \setminus \{0\}$. In particular, $U$ has the simple expression, see [6, Lemma 12]:

$$U(x) = \ln |x| \int_{|y| \leq |x|} \overline{μ}(|y|) \, dy + \int_{|y| > |x|} \ln |y| \rho(|y|) \, dy$$

$$= \ln |x| \left( \int_{\mathbb{R}^2} \overline{μ}(|y|) \, dy \right) + \int_{|y| > |x|} \ln \left( \frac{|y|}{|x|} \right) \rho(|y|) \, dy.$$

Note that $U$ is well defined for all $x \neq 0$ in view of the assumption on the support of $μ$. We remark that $f$ does not have to belong to $L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$.

**Theorem 4.2.** Let $\overline{f}$ be as above. Then for any $K > 0$, any initial density given by $f_1 \{ f \leq K \} h_0$, where $h_0 \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, satisfies the assumptions of Theorem 1.2.

**Proof.** Note that $f_1 \{ f \leq K \} h_0 \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Since $μ$ is supported in $B(0,1)$ we have $U(x) = \ln |x| (\int \overline{μ})$ for $|x| \geq 1$, from which we infer that $f(x,v) = 0$ whenever $|x| \geq N = \exp(M/(\int \overline{μ}))$. In addition, we observe that $f$ takes the form (4.2), where we have set

$$\Phi(x) = \ln |x| \left( \int_{\mathbb{R}^2} \overline{μ}(|y|) \, dy \right), \quad \alpha(x,v) = \int_{|y| > |x|} \ln \left( \frac{|y|}{|x|} \right) \rho(|y|) \, dy \geq \frac{1}{2}.$$ 

the only difference with the setting of Proposition 4.1 is that $φ$ can be possibly unbounded on $\mathbb{R}$. Mimicking the proof of Proposition 4.1 we still obtain

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^k \left( \int_{\mathbb{R}^2} \overline{f} \, dy \right)(x,v) \, h_0(x,v) \, dx \, dv \leq \| h_0 \|_{L^\infty} K \int_{B(0,N)} \left( \int_{B(0,C(\int \overline{μ}|\ln |x|)|^{1/2})} |v|^k \, dv \right) \, dx \\
\leq C \int_{B(0,N)} \left( M + \left( \int \overline{μ} \right) \ln |x| \right)^{\frac{k+1}{2}} \, dx \\
\leq (Ck)^{\frac{k}{2}},$$

where we have used (4.4) in the last inequality.

**Acknowledgments** The author thanks Daniel Han-Kwan for interesting discussions. She is partially supported by the French ANR projects SchEq ANR-12-JS-0005-01 and GEODISP ANR-12-BS01-0015-01.

**References**

[1] L. Ambrosio and G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields, Transport equations and multi-D hyperbolic conservation laws, 357, Lect. Notes Unione Mat. Ital., 5, Springer, Berlin, 2008.

[2] A. A. Arsenev, Global existence of a weak solution of Vlasov’s system of equations, U. S. R. Comput. Math. Math. Phys. 15 (1975), 131–143.
[3] J. Batt, P. Morrison and G. Rein, Linear stability of stationary solutions of the Vlasov-Poisson system in three dimensions, Arch. Rational Mech. Anal. 130 (1995), no. 2, 163–182.

[4] S. Caprino, C. Marchioro, E. Miot and M. Pulvirenti, On the attractive plasma-charge model in 2-D, Comm. Partial Differential Equations 37 (2012), no. 7, 1237–1272.

[5] R. J. DiPerna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989), 511–547.

[6] J. Dolbeault, J. Fernández and O. Sánchez, Stability for the gravitational Vlasov-Poisson system in dimension two, Comm. Partial Differential Equations 31 (2006), no. 10-12, 1425–1449.

[7] J. Duoandikoetxea, Fourier Analysis, GSM29, Amer. Math. Soc., Providence RI, 2001.

[8] I. Gasser, P. E. Jabin and B. Perthame, Regularity and propagation of moments in some nonlinear Vlasov systems, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000), 1259–1273.

[9] Y. Guo and G. Rein, A non-variational approach to nonlinear stability in stellar dynamics applied to the King model, Comm. Math. Phys. 271 (2007), no. 2, 489–509.

[10] M. Lemou, F. Méhats and P. Raphaël, Orbital stability of spherical systems, Invent Math 187 (2012), no. 1, 145–194.

[11] P. L. Lions and B. Perthame, Propagation of moments and regularity for the 3-dimensional Vlasov-Poission system, Invent. Math. 105 (1991), 415–430.

[12] G. Loeper, Uniqueness of the solution to the Vlasov-Poisson system with bounded density, J. Math. Pures Appl. (9) 86 (2006), no. 1, 68–79.

[13] C. Marchioro and M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids, Springer-Verlag, New York, 1994.

[14] A. J. Majda and A. L. Bertozzi, Vorticity and incompressible flow, Cambridge Texts in Applied Mathematics 27. Cambridge University Press, Cambridge, 2002.

[15] S. Okabe, T. Ukai, On classical solutions in the large in time of the two-dimensional Vlasov equation, Osaka J. Math. 15 (1978), 245–261.

[16] C. Pallard, Moment propagation for weak solutions to the Vlasov-Poisson system, Commun. Partial Differ. Equations 37 (7) (2012), 1273–1285.

[17] C. Pallard, Space moments of the Vlasov-Poisson system: propagation and regularity, SIAM J. Math. Anal. 46 (2014), no. 3, 1754–1770.

[18] K. Pfaffelmoser, Global existence of the Vlasov-Poisson system in three dimensions for general initial data, J. Differ. Equ. 95 (1992), 281–303.

[19] R. Robert, Unicité de la solution faible à support compact de l’équation de Vlasov-Poisson, C. R. Acad. Sci. Paris Sér. I Math. 324, no. 8 (1994), 873–877.

[20] D. Salort, Transport equations with unbounded force fields and application to the Vlasov-Poisson equation, Math. Models Methods Appl. Sci. 19 (2) (2009), 199-228.

[21] J. Schaeffer, Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions, Commun. Partial Differ. Equations 16 (8–9) (1991), 1313–1335.

[22] S. Wollman, Global in time solution to the three-dimensional Vlasov-Poission system, J. Math. Anal. Appl. 176 (1) (1996), 76–91.

[23] Yudovich V. I., Non-stationary flows of an ideal incompressible fluid, Z. Vycist. Mat. i Mat. Fiz. 3 (1963), pp. 1032-1066 (in Russian). English translation in USSR Comput. Math. & Math. Physics 3 (1963), pp. 1407–1456.

(E. Miot) CENTRE DE MATHEMATIQUES LAURENT SCHWARTZ, ECOLE POLYTECHNIQUE, 91128 PALAISEAU, FRANCE
E-mail address: evelyne.miot@math.polytechnique.fr