Theta vocabulary I

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Abstract

This paper is an annotated list of transformation properties and identities satisfied by the four theta functions $\theta_1, \theta_2, \theta_3, \theta_4$ of one complex variable, presented in a ready-to-use form. An attempt is made to reveal a pattern behind various identities for the theta-functions. It is shown that all possible 3, 4 and 5-term identities of degree four emerge as algebraic consequences of the six fundamental bilinear 3-term identities connecting the theta-functions with modular parameters $\tau$ and $2\tau$.

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1 Foreword

The theta functions introduced by Jacobi \([1]\) (see also \([B, W, WW, M1]\)) are doubly (quasi)periodic analogues of the basic trigonometric functions \(\sin(\pi u)\) and \(\cos(\pi u)\). Let the two (quasi)periods be 1 and \(\tau \in \mathbb{C}\) with the condition \(\Im \tau > 0\). The basic theta functions are \(\theta_1(u|\tau), \theta_2(u|\tau), \theta_3(u|\tau), \theta_4(u|\tau)\). The theory of theta functions is a sort of “elliptically deformed” trigonometry. In essence the functions \(\sin\) and \(\cos\) are the same because \(\cos x = \sin(x + \pi/2)\), but everybody knows that in practice it is more convenient to work with the two functions rather than one. Likewise, the four theta functions can be obtained from any one of them by simple transformations like shifts of the argument and multiplying by a common factor, but it is more convenient to deal with the set of four instead of one.

The “elliptic deformation” of the trigonometric functions may go in two ways depending on which property of the former one wants to preserve or generalize. One is a deformation in the class of \textit{entire functions} (the north-east arrow in the diagram below). It leads to the quasi-periodic theta functions, which are regular functions in the whole complex plane. The other one is in the class of \textit{doubly periodic functions}. The (infinite) second period of the trigonometric functions becomes finite (equal to \(\tau\)) at the price of breaking the global analyticity, so the elliptic functions \(\text{sn}, \text{cn}\) and \(\text{dn}\), which are doubly periodic analogues of trigonometric \(\sin\) and \(\cos\) are \textit{meromorphic functions} in the complex plane.

\[
\begin{align*}
\{\theta_1, \theta_2, \theta_3, \theta_4\} & \rightarrow \{\sin, \cos\} \\
\{\sin, \cos\} & \rightarrow \{\text{sn}, \text{cn}, \text{dn}\}
\end{align*}
\]

In fact the basic elliptic functions are constructed as ratios of the theta functions and in this sense the latter seem to be more fundamental.

In practical calculations with trigonometric functions (and their hyperbolic cousins), one needs just a few identities for the basic functions \(\sin\) and \(\cos\) like the addition formula \(\sin(x + y) = \sin x \cos y + \sin y \cos x\). It is not difficult to remember them all or derive any forgotten one from scratch using the definitions \(\sin x = -i(e^{ix} - e^{-ix})/2\), \(\cos x = (e^{ix} + e^{-ix})/2\).

For the theta functions, the situation is much more involved. They are connected by a plethora of identities most of which are not obvious, not suitable for memorizing and can not be derived from scratch in any easy way. Here is what Mumford wrote in Chapter 1 of his book “Tata lectures on Theta I” \([M1]\) after presenting a list of ponderous identities for theta functions:

“We have listed these at such length to illustrate a key point in the theory of theta functions: the symmetry of the situation generates rapidly an overwhelming number of formulae, which do not however make a completely elementary pattern. To obtain a clear picture of the algebraic implications of these formulae altogether is then not usually easy.”
All this is aggravated by the fact that there are several different systems of notation for theta functions in use.

In the present paper we make an attempt to bring some order into this conglomeration of formulae. We show that the 3, 4 and 5-term identities of degree four (i.e. with products of four theta functions in each term), referred to as Weierstrass addition formulae, Jacobi relations, and Riemann identities, respectively, can be obtained by purely algebraic manipulations from six basic 3-term theta relations of degree two connecting theta functions with modular parameters \( \tau \) and \( 2\tau \). Starting with the six “elementary bricks”, it is possible to derive 52 fundamental relations of degree four containing four independent variables. Besides, we give the complete list of all important particular identities which are appropriate specifications of the basic bilinear and degree four ones. Recently, Koornwinder has proved [K] that the Weierstrass addition formulae and the Riemann identities are equivalent. We reproduce this result in a very simple way.

In the future we plan to address more specific questions related to the role of theta functions in the theory of integrable systems and lattice models of statistical mechanics.

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2 Theta functions

2.1 Theta functions with characteristics

Fix the modular parameter \( \tau \in \mathbb{C} \) such that \( \Im \tau > 0 \) and consider the infinite series [M1]:

\[
\theta_{a,b}(u|\tau) = \sum_{k \in \mathbb{Z}} \exp \left\{ \pi i \tau (k + a)^2 + 2\pi i (k + a)(u + b) \right\},
\]

(2.1)

where \( i = \sqrt{-1} \) and \( a, b \in \mathbb{R} \). The series is absolutely convergent for any \( u \in \mathbb{C} \) and defines the entire function \( \theta_{a,b}(u|\tau) \). It is called the theta function with characteristics \( a, b \). These functions are connected by the relations

\[
\theta_{a,b}(u + \tau a' + b'|\tau) = e^{-2\pi i a'(u+b+b'+a'\tau/2)}\theta_{a+a',b+b'}(u|\tau),
\]

(2.2a)

\[
\theta_{a+1,b}(u|\tau) = \theta_{a,b}(u|\tau),
\]

(2.2b)

\[
\theta_{a,b+1}(u|\tau) = e^{2\pi i a}\theta_{a,b}(u|\tau).
\]

(2.2c)
In particular, it follows from here that the functions \( \theta_{a,b} \) are quasiperiodic with (quasi)periods 1 and \( \tau \):

\[
\theta_{a,b}(u + 1|\tau) = e^{2\pi i a} \theta_{a,b}(u|\tau), \quad (2.3a)
\]

\[
\theta_{a,b}(u + \tau|\tau) = e^{-\pi i (2u + 2b + \tau)} \theta_{a,b}(u|\tau), \quad (2.3b)
\]

and the shifts by half-periods are given by

\[
\begin{align*}
\theta_{a,b}(u + \frac{1}{2}|\tau) &= \theta_{a,b+\frac{1}{2}}(u|\tau) = e^{2\pi i a} \theta_{a,b-\frac{1}{2}}(u|\tau), \quad (2.4a) \\
\theta_{a,b}(u + \frac{\tau}{2}|\tau) &= e^{-\pi i (u+b+\tau/4)} \theta_{a+\frac{1}{2},b}(u|\tau) = e^{-\pi i (u+b+\tau/4)} \theta_{a-\frac{1}{2},b}(u|\tau). \quad (2.4b)
\end{align*}
\]

It follows from the definition that \( \theta_{a,b}(-u) = \theta_{-a,-b}(u) \). Hence, according to \( (2.2b), (2.2c) \), the functions \( \theta_{a,b}(u) \) have definite evenness properties only for integer or half-integer characteristics. By virtue of \( (2.2b), (2.2c) \), it is sufficient to consider the theta functions with characteristics \( 0 \leq a, b < 1 \).

### 2.2 Basic theta functions

The standard theta functions with half-integer characteristics [J] [B] are defined as follows:

\[
\begin{align*}
\theta_1(u|\tau) &= -\theta_{\frac{1}{2},\frac{1}{2}}(u|\tau) = \sum_{k \in \mathbb{Z}} (-1)^{k} q^{(k+\frac{1}{2})^2} e^{\pi i (2k+1)u}, \\
\theta_2(u|\tau) &= \theta_{0,0}(u|\tau) = \sum_{k \in \mathbb{Z}} q^{(k+\frac{1}{2})^2} e^{\pi i (2k+1)u}, \\
\theta_3(u|\tau) &= \theta_{0,0}(u|\tau) = \sum_{k \in \mathbb{Z}} q^{k^2} e^{2\pi i ku}, \\
\theta_4(u|\tau) &= \theta_{0,0}(u|\tau) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2} e^{2\pi i ku},
\end{align*}
\]

where

\[
q := e^{\pi i \tau}, \quad |q| < 1. \quad (2.6)
\]

In the limit \( \tau \to \infty \) they are: \( \theta_1(u|\tau) = 2q^{\frac{1}{2}} \sin \pi u + O(q^{\frac{3}{2}}), \theta_2(u|\tau) = 2q^{\frac{1}{2}} \cos \pi u + O(q^{\frac{3}{2}}), \theta_3(u|\tau) = 1 + O(q), \theta_4(u|\tau) = 1 + O(q) \).

In what follows we often write \( \theta_r(u|\tau) := \theta_{r}(u), \ r = 1, 2, 3, 4 \) if this does not cause confusion. From \( (2.3) \) it is clear that the function \( \theta_1 \) is odd, \( \theta_1(-u) = -\theta_1(u) \); the other three are even, \( \theta_s(-u) = \theta_s(u), \ s = 2, 3, 4 \).

The values \( \theta'_1(0), \theta_2(0), \theta_3(0), \theta_4(0) \) are called theta constants.

### 2.3 Shifts by periods and half-periods

Here we list the essential transformation properties for the theta functions \( (2.5) \) which follow from \( (2.2) \).
Shifts by periods:

\[
\begin{align*}
\theta_1(u+1) &= -\theta_1(u), & \theta_1(u+\tau) &= -e^{-\pi i (2u+\tau)} \theta_1(u), \\
\theta_2(u+1) &= -\theta_2(u), & \theta_2(u+\tau) &= e^{-\pi i (2u+\tau)} \theta_2(u), \\
\theta_3(u+1) &= \theta_3(u), & \theta_3(u+\tau) &= e^{-\pi i (2u+\tau)} \theta_3(u), \\
\theta_4(u+1) &= \theta_4(u), & \theta_4(u+\tau) &= -e^{-\pi i (2u+\tau)} \theta_4(u).
\end{align*}
\] (2.7)

Shifts by half-periods:

\[
\begin{align*}
\theta_1(u + \frac{1}{2}) &= \theta_2(u), & \theta_1(u + \frac{\tau}{2}) &= ie^{-\pi i (u+\tau/4)} \theta_4(u), \\
\theta_2(u + \frac{1}{2}) &= -\theta_1(u), & \theta_2(u + \frac{\tau}{2}) &= e^{-\pi i (u+\tau/4)} \theta_3(u), \\
\theta_3(u + \frac{1}{2}) &= \theta_4(u), & \theta_3(u + \frac{\tau}{2}) &= e^{-\pi i (u+\tau/4)} \theta_2(u), \\
\theta_4(u + \frac{1}{2}) &= \theta_3(u). & \theta_4(u + \frac{\tau}{2}) &= ie^{-\pi i (u+\tau/4)} \theta_1(u).
\end{align*}
\] (2.8)

2.4 Zeros of theta functions

These relations imply that the (first order) zeros of the theta functions are as follows:

\[
\begin{align*}
\theta_1(u) &= 0 : & u &= n + m\tau, \\
\theta_2(u) &= 0 : & u &= n + \frac{1}{2} + m\tau, \\
\theta_3(u) &= 0 : & u &= n + \frac{1}{2} + (m + \frac{1}{2})\tau, \\
\theta_4(u) &= 0 : & u &= n + (m + \frac{1}{2})\tau,
\end{align*}
\] (2.9)

where \( n, m \in \mathbb{Z}. \)

Indeed, in accordance with (2.7), (2.8), the functions \( \theta_1(u + n + m\tau), \theta_2(u + n + \frac{1}{2} + m\tau), \theta_3(u + n + \frac{1}{2} + (m + \frac{1}{2})\tau), \theta_4(u + n + (m + \frac{1}{2})\tau) \) are proportional to the odd function \( \theta_1(u) \). Hence the corresponding zeros are as in (2.9). To complete the proof, it is sufficient to show that the function \( \theta_1(u) \) has precisely one simple zero in the parallelogram \( \Pi \) with the vertices \( -\frac{1}{2} \pm \frac{\tau}{2}, \frac{1}{2} \pm \frac{\tau}{2} \). The standard argument is to compute the contour integral \( \frac{1}{2\pi i} \oint_{\partial\Pi} d\log \theta_1(u) = 1 \) which means that the zero is simple (see [WW], [M] for details).
2.5 Theta functions as infinite products

One has the following infinite product representations:

\[ \theta_1(u|\tau) = 2q^{\frac{1}{4}} \sin \pi u \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n}e^{2\pi iu})(1 - q^{2n}e^{-2\pi iu}), \]  
\[ (2.10a) \]

\[ \theta_2(u|\tau) = 2q^{\frac{1}{4}} \cos \pi u \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n}e^{2\pi iu})(1 + q^{2n}e^{-2\pi iu}), \]  
\[ (2.10b) \]

\[ \theta_3(u|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2\pi iu})(1 + q^{2n-1}e^{-2\pi iu}), \]  
\[ (2.10c) \]

\[ \theta_4(u|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1}e^{2\pi iu})(1 - q^{2n-1}e^{-2\pi iu}). \]  
\[ (2.10d) \]

To prove (2.10d), we note that the product

\[ p(u|\tau) := \prod_{n=1}^{\infty} (1 - q^{2n-1}e^{2\pi iu})(1 - q^{2n-1}e^{-2\pi iu}) \]

has the same zeros as \( \theta_4(u|\tau) \) and the ratio \( \theta_4(u|\tau)/p(u|\tau) \) is a doubly periodic function with periods 1 and \( \tau \). Hence the ratio is constant and one has \( \theta_4(u|\tau) = Ap(u|\tau) \). To find the constant \( A \), put \( u = 0 \) thus getting

\[ A = \frac{\theta_4(0|\tau)}{p(0|\tau)}. \]  
\[ (2.11) \]

Rewriting \( \prod_{n=1}^{\infty} (1 - q^{2n-1}) = \prod_{n=1}^{\infty} (1 - q^n)/(1 - q^{2n}) \), one gets \( A = \prod_{n=1}^{\infty} (1 - q^n) \) and formula (2.10d) is proved. Equations (2.10a–2.10c) are obtained from (2.10c) by appropriate shifts of \( u \) in accordance with (2.8).

As a corollary of (2.10) one has infinite product representations for the theta constants:

\[ \theta_1'(0) = 2\pi q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})^3, \]  
\[ (2.12a) \]

\[ \theta_2(0) = 2q^{\frac{1}{4}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2, \]  
\[ (2.12b) \]

\[ \theta_3(0) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2, \]  
\[ (2.12c) \]

\[ \theta_4(0) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2. \]  
\[ (2.12d) \]

Since \( \prod_{n \geq 1} (1 + q^{2n})(1 + q^{2n-1})(1 - q^{2n-1}) = 1 \), this implies the famous identity for the theta constants [J p. 517]:

\[ \theta_1'(0) = \pi \theta_2(0)\theta_3(0)\theta_4(0). \]  
\[ (2.13) \]
2.6 Modular transformations

The transformation \( \tau \to \tau + 1 \):

\[
\begin{align*}
\theta_1(u|\tau + 1) &= e^{\frac{\pi i}{2}u} \theta_1(u|\tau), \\
\theta_2(u|\tau + 1) &= e^{\frac{\pi i}{2}u} \theta_2(u|\tau), \\
\theta_3(u|\tau + 1) &= \theta_4(u|\tau), \\
\theta_4(u|\tau + 1) &= \theta_3(u|\tau).
\end{align*}
\]

(2.14)

Since \( \tau \to \tau + 1 \) implies \( q \to -q \), equations (2.14) follow from (2.5) or (2.10).

The transformation \( \tau \to -1/\tau \):

\[
\begin{align*}
\theta_1(u/\tau - 1/\tau) &= -i \sqrt{-i \tau} e^{\pi i u^2/\tau} \theta_1(u|\tau), \\
\theta_2(u/\tau - 1/\tau) &= \sqrt{-i \tau} e^{\pi i u^2/\tau} \theta_2(u|\tau), \\
\theta_3(u/\tau - 1/\tau) &= -\sqrt{i \tau} e^{\pi i u^2/\tau} \theta_3(u|\tau), \\
\theta_4(u/\tau - 1/\tau) &= \sqrt{i \tau} e^{\pi i u^2/\tau} \theta_2(u|\tau).
\end{align*}
\]

(2.15)

The branch of the square root here is such that \( \Re \sqrt{-i \tau} > 0 \).

The proof is well known. Since the ratio \( e^{-\pi i u^2/\tau} \theta_3(u/\tau - 1/\tau) / \theta_3(u|\tau) := C \) is an entire doubly periodic function of \( u \), it is a constant (which may depend only on \( \tau \)). Shifting \( u \) by \( \frac{1}{2}, \frac{\tau}{2}, \frac{\tau + 1}{2} \), one obtains three more relations of the same kind with the same constant \( C \). Then the substitution of these formulas to (2.13), yields \( C^2 = -i \tau \). The sign of the square root is determined by the argument that if \( \tau \in i \mathbb{R}_+ \), then both \( \theta_3(0| -1/\tau) \) and \( \theta_3(0| \tau) \) are real and positive.

2.7 Other notation for the theta functions

The notations for theta functions used in the literature are of a great variety. This can be a source of confusion. Here we briefly comment on the main systems of notation other than the one adopted in this paper. In the theory of elliptic integrals, the theta functions

\[
\Theta_r(u|\tau) = \theta_r \left( \frac{u}{2K} \bigg| \tau \right), \quad K = \frac{\pi}{2} \theta_3^2(0|\tau)
\]

(2.16)

introduced by Riemann are commonly encountered. The number \( K \) is the full elliptic integral (of the first kind). In [A] and some other books our \( \theta_r \) is denoted as \( \vartheta_r \) while \( \Theta_r \) defined in (2.16) is just \( \theta_r \). The antiquated Jacobi notation (still preferred by some authors) are \( H, H_1, \Theta_1, \Theta \) for \( \Theta_1, \Theta_2, \Theta_3, \Theta_4 \) respectively. The “multiplicative notation” \( \theta_r(z|q) \) for \( \theta_r(u|\tau) \), where \( q = e^{\pi i \tau}, z = e^{2\pi i u} \), is widely used in the modern literature on elliptic hypergeometric series and related problems.

Lastly, let us mention a few of the minor differences in notation encountered in the literature. In [W], [HC] the functions \( \Theta_{a,b}(u) \) have been considered which are related with \( \theta_{a,b}(u) \) by

\[
\begin{align*}
\Theta_{a,b}^W(u) &= e^{\pi i a b} \theta_a \left( \frac{u}{2} \bigg| \frac{1}{2} \right), \quad \Theta_{a,b}^{HC}(u) &= e^{-\pi i a b} \theta_a \left( \frac{u}{2} \bigg| \frac{1}{2} \right).
\end{align*}
\]

(2.17)
The set of our theta functions (2.5) is related with the corresponding functions in \([\text{WW}]\) as \(\theta_r(u|\tau) = \theta_r^{\text{WW}}(\pi u|\tau)\), \(r = 1, 2, 3, 4\). Following the original notation \([\text{J}]\), in \([\text{A}]\) and in some other books the notation \(\theta_0\) is used instead of \(\theta_4\).

### 3 Four types of identities between theta functions

#### 3.1 Preliminaries

The number of identities satisfied by the theta functions is enormous. It is still fairly big if we consider identities involving up to four independent variables. They can be split into four types:

- **B.** Three-term bilinear identities involving two independent variables. They relate products of two theta functions with modular parameter \(\tau\) to linear combinations (actually, sums or differences) of similar products of theta functions with modular parameter \(2\tau\).
- **W.** Three-term identities of degree 4 (the Weierstrass addition formulae).
- **J.** Four-term identities of degree 4 (the Jacobi formulae).
- **R.** Five-term identities of degree 4 (the Riemann identities).

The identities of types \(\text{W}, \text{J}, \text{R}\) include theta functions with the same modular parameter \(\tau\) and contain four independent variables. The identities of type \(\text{B}\) are the most fundamental ones: all the others are algebraic consequences of these together with the evenness properties of the theta functions \(\theta_r(-u) = (-1)^{\delta_{r,1}}\theta_r(u), r = 1, 2, 3, 4\). Namely, we shall show how to derive \(\text{W}\) from \(\text{B}\) etc., according to the scheme \(\text{B} \rightarrow \text{W} \rightarrow \text{J} \rightarrow \text{R}\). It also turns out that the Jacobi and Riemann identities are equivalent in a very simple way. At the end of this section, we prove the arrow \(\text{W} \leftarrow \text{J}\) which implies equivalence of the Weierstrass and Jacobi identities.

#### 3.2 Three-term bilinear identities connecting theta functions with \(\tau\) and \(2\tau\)

**B.I.** There are six basis bilinear identities:

\[
\begin{align*}
\theta_1(u|\tau)\theta_1(v|\tau) &= \theta_3(u + v|2\tau)\theta_2(u - v|2\tau) - \theta_2(u + v|2\tau)\theta_3(u - v|2\tau), \\
\theta_1(u|\tau)\theta_2(v|\tau) &= \theta_1(u + v|2\tau)\theta_4(u - v|2\tau) + \theta_4(u + v|2\tau)\theta_1(u - v|2\tau), \\
\theta_2(u|\tau)\theta_2(v|\tau) &= \theta_2(u + v|2\tau)\theta_3(u - v|2\tau) + \theta_3(u + v|2\tau)\theta_2(u - v|2\tau), \\
\theta_3(u|\tau)\theta_3(v|\tau) &= \theta_3(u + v|2\tau)\theta_3(u - v|2\tau) + \theta_2(u + v|2\tau)\theta_2(u - v|2\tau), \\
\theta_3(u|\tau)\theta_4(v|\tau) &= \theta_4(u + v|2\tau)\theta_4(u - v|2\tau) - \theta_1(u + v|2\tau)\theta_1(u - v|2\tau), \\
\theta_4(u|\tau)\theta_4(v|\tau) &= \theta_3(u + v|2\tau)\theta_3(u - v|2\tau) - \theta_2(u + v|2\tau)\theta_2(u - v|2\tau).
\end{align*}
\]
B.II. A system equivalent to (3.1):

\[ 2\theta_1(u + v|2\tau)\theta_1(u - v|2\tau) = \theta_1(u|\tau)\theta_3(v|\tau) - \theta_3(u|\tau)\theta_4(v|\tau), \]  
\[ 2\theta_1(u + v|2\tau)\theta_4(u - v|2\tau) = \theta_1(u|\tau)\theta_2(v|\tau) + \theta_2(u|\tau)\theta_1(v|\tau), \]  
\[ 2\theta_2(u + v|2\tau)\theta_2(u - v|2\tau) = \theta_3(u|\tau)\theta_3(v|\tau) - \theta_4(u|\tau)\theta_4(v|\tau), \]  
\[ 2\theta_2(u + v|2\tau)\theta_3(u - v|2\tau) = \theta_2(u|\tau)\theta_3(v|\tau) - \theta_1(u|\tau)\theta_1(v|\tau), \]  
\[ 2\theta_3(u + v|2\tau)\theta_3(u - v|2\tau) = \theta_3(u|\tau)\theta_3(v|\tau) + \theta_4(u|\tau)\theta_4(v|\tau), \]  
\[ 2\theta_4(u + v|2\tau)\theta_4(u - v|2\tau) = \theta_3(u|\tau)\theta_4(v|\tau) + \theta_4(u|\tau)\theta_3(v|\tau). \]  

\textbf{Remark 3.1} Starting with any identity in (3.1), one can derive all the other ones by appropriate shifts of the variables \(u, v\).

The proof is standard. Let us prove, for example, (3.1b). Consider the function

\[ F(v) := \theta_1(u + v|2\tau)\theta_4(u - v|2\tau) + \theta_4(u + v|2\tau)\theta_1(u - v|2\tau). \]

By virtue of (2.7) and (2.8), \( F(v + 1) = -F(v) \), \( F(v + \tau) = e^{-\pi i (2v + \tau)} F(v) \) and \( F(\frac{1}{2}) = 0 \). Hence zeros of \( F(v) \) are \( v_{n,m} = n + \frac{1}{2} + m\tau \), \( n, m \in \mathbb{Z} \) and the ratio \( F(v)/(\theta_1(u|\tau)\theta_2(v|\tau)) \) is an entire function doubly periodic in \( v \) with periods 1, \( \tau \). Therefore, this ratio does not depend on \( v \): \( F(v)/\theta_1(u|\tau)\theta_2(v|\tau) = C(u) \). Setting \( v = u \), one has:

\[ C(u) = \frac{\theta_1(2u|2\tau)\theta_4(0|2\tau)}{\theta_1(u|\tau)\theta_2(u|\tau)}. \]

By virtue of (2.10a), (2.10b) and (2.11), \( \theta_1(u|\tau)\theta_2(u|\tau) = \theta_1(2u|2\tau)\theta_4(0|2\tau) \) and thus \( C(u) = 1 \).

### 3.3 Three-term Weierstrass addition identities

There are twelve addition formulae (see below). We start with the identity

\[ \theta_1(u + x)\theta_1(u - x)\theta_1(v + y)\theta_1(v - y) - \theta_1(u + y)\theta_1(u - y)\theta_1(v + x)\theta_1(v - x) \]
\[ = \theta_1(u + v)\theta_1(u - v)\theta_1(x + y)\theta_1(x - y) \]  

which was originally discovered and proved by Weierstrass [We] p. 155. All the identities listed below in this section can be derived from it by appropriate shifts of the variables in accordance with relations (2.3). Our approach is different. We show that all the identities of Weierstrass’ type are simple algebraic consequences of the bilinear system (3.1) together with the evenness conditions \( \theta_r(-u) = (-1)^{r-1}\theta_r(u) \), \( r = 1, 2, 3, 4 \). This argument is independent of (2.3).

To prove (3.3), one should rewrite (3.1a) as

\[ \theta_1(u + x|\tau)\theta_1(u - x|\tau) = \theta_3(2u|2\tau)\theta_2(2x|2\tau) - \theta_2(2u|2\tau)\theta_3(2x|2\tau). \]
Multiply this by the similar expression for \( \theta_1(v + y|\tau)\theta_1(v - y|\tau) \) and subtract the same with the change \( x \leftrightarrow y \). Using (3.1a) once again, we arrive at (3.3). All the equations below in this section can be obtained from system (3.1) in a similar way.

**W.I.** Symmetric system:

\[
\theta_1(u + x)\theta_1(u - x)\theta_r(v + y)\theta_r(v - y) - \theta_1(v + x)\theta_1(v - x)\theta_r(u + y)\theta_r(u - y) = \theta_1(u + v)\theta_1(u - v)\theta_r(x + y)\theta_r(x - y),
\]

\( r = 1, 2, 3, 4. \)

**W.II.** Complimentary system:

\[
\theta_2(u + x)\theta_2(u - x)\theta_3(v + y)\theta_3(v - y) - \theta_2(v + x)\theta_2(v - x)\theta_3(u + y)\theta_3(u - y) = -\theta_1(u + v)\theta_1(u - v)\theta_4(x + y)\theta_4(x - y),
\]

\( 3.5a \)

\[
\theta_2(u + x)\theta_2(u - x)\theta_4(v + y)\theta_4(v - y) - \theta_2(v + x)\theta_2(v - x)\theta_4(u + y)\theta_4(u - y) = -\theta_1(u + v)\theta_1(u - v)\theta_3(x + y)\theta_3(x - y),
\]

\( 3.5b \)

\[
\theta_3(u + x)\theta_3(u - x)\theta_4(v + y)\theta_4(v - y) - \theta_3(v + x)\theta_3(v - x)\theta_4(u + y)\theta_4(u - y) = -\theta_1(u + v)\theta_1(u - v)\theta_2(x + y)\theta_2(x - y).
\]

\( 3.5c \)

**W.III.** Asymmetric system:

\[
\theta_r(u + x)\theta_r(u - x)\theta_r(v + y)\theta_r(v - y) - \theta_r(u + y)\theta_r(u - y)\theta_r(v + x)\theta_r(v - x) = (-1)^{r-1}\theta_1(u + v)\theta_1(u - v)\theta_1(x + y)\theta_1(x - y),
\]

\( r = 1, 2, 3, 4. \)

**W.IV.** Complimentary identity:

\[
\theta_3(u + x)\theta_3(u - x)\theta_3(v + y)\theta_3(v - y) - \theta_4(v + x)\theta_4(v - x)\theta_4(u + y)\theta_4(u - y) = \theta_2(u + v)\theta_2(u - v)\theta_2(x + y)\theta_2(x - y).
\]

\( 3.7 \)

**W.V.** Mixed identity:

\[
\theta_1(u + x)\theta_2(u - x)\theta_3(v + y)\theta_4(v - y) - \theta_1(u - y)\theta_2(u + y)\theta_3(v - x)\theta_4(v + x) = \theta_1(x + y)\theta_2(x - y)\theta_3(u + v)\theta_4(u - v).
\]

\( 3.8 \)

**Remark 3.2** Sometimes the Weierstrass addition formula (3.3) is referred to as Fay identity. In fact, it is a generalization of Jacobi’s results (see Section 4.2 below).
3.4 Four-term Jacobi identities

In Sect. 3.3 we have presented twelve three-term identities of degree four depending on four variables \( u, v, x, y \). Here we introduce another set of variables \( W, X, Y, Z \) and their “dual” counterparts \([WW]\):

\[
W' = \frac{1}{2}(-W + X + Y + Z),
X' = \frac{1}{2}(W - X + Y + Z),
Y' = \frac{1}{2}(W + X - Y + Z),
Z' = \frac{1}{2}(W + X + Y - Z).
\]

One can easily verify that \( W, X, Y, Z \) are expressed via the “dual” variables \( W', X', Y', Z' \) by the same formulae, i.e., the “prime procedure” applied to (3.9) yields \((W')' = W = \frac{1}{2}(-W' + X' + Y' + Z')\) etc. We employ the short-hand notation

\[
[pqrs] := \theta_r(W)\theta_q(X)\theta_r(Y)\theta_s(Z), \quad [pqrs]' := \theta_r(W')\theta_q(X')\theta_r(Y')\theta_s(Z')
\]

which is widely used in \([WW]\). If all the indices of the theta functions coincide, this is further abbreviated to

\[
[r] := \theta_r(W)\theta_r(X)\theta_r(Y)\theta_r(Z), \quad [r]' := \theta_r(W')\theta_r(X')\theta_r(Y')\theta_r(Z').
\]

Below we list all four-term basic identities of degree four which were essentially obtained by Jacobi \([J]\) p. 507. Here we present these in a more symmetric and comprehensive form.

The simplest (and most important) ones are:

\[
[1] + [2] = [1]' + [2]', \quad (3.10a)
\]

\[
[1] - [2] = [4]' - [3]', \quad (3.10b)
\]

\[
[3] + [4] = [3]' + [4]', \quad (3.10c)
\]

\[
[3] - [4] = [2]' - [1]', \quad (3.10d)
\]

The system (3.10) is a direct algebraic corollary of appropriate addition formulae given in Section 3.3. To see this, we relate the variables \( u, v, x, y \) with the variables of the present section as follows:

\[
\begin{align*}
& W = u + x, \\
& X = u - x, \\
& Y = v + y, \\
& Z = v - y.
\end{align*} \quad \iff \quad \begin{align*}
& W' = v - x, \\
& X' = v + x, \\
& Y' = u - y, \\
& Z' = u + y.
\end{align*}
\]

Further, the products of theta functions containing “inappropriate” combinations \( u \pm v, x \pm y \) can be excluded from the addition formulae. Then identities (3.10a), (3.10c) emerge as particular cases of (3.6). Changing \( v \leftrightarrow x \) in (3.6) (with \( r = 3 \)) and (3.7), one obtains (3.10b). Finally, (3.10d) is a “dual” version of (3.10b).
Remark 3.3 Equations (3.10) are a part of the system of twelve identities written in [WW pp. 468, 488]. It is easy to see that all additional relations are appropriate linear combinations of the basic ones, (3.10). For completeness, we give here the full list:

\begin{align}
[1] + [2] &= [1]' + [2]', & [1] + [3] &= [2]' + [4]', & [1] + [4] &= [1]' + [4]', \\
[2] + [3] &= [2]' + [3]', & [2] + [4] &= [1]' + [3]', & [3] + [4] &= [3]' + [4]', \\
[1] - [2] &= [4]' - [3]', & [1] - [3] &= [1]' - [3]', & [1] - [4] &= [2]' - [3]', \\
[2] - [3] &= [1]' - [4]', & [2] - [4] &= [2]' - [4]', & [3] - [4] &= [2]' - [1]' .
\end{align}

Now we list symmetric self-dual identities for products of type $[rrss]$ which can also be derived from the addition formulae by algebraic manipulations:

\begin{align}
[1122] + [2211] &= [1122]' + [2211]', \\
[1133] + [3311] &= [1133]' + [3311]', \\
[1144] + [4411] &= [1144]' + [4411]', \\
[2233] + [3322] &= [2233]' + [3322]', \\
[2244] + [4422] &= [2244]' + [4422]', \\
[3344] + [4433] &= [3344]' + [4433]' .
\end{align}

Further, there are simple complimentary relations:

\begin{align}
[1122] - [2211] &= [3344]' - [4433]', \\
[1133] - [3311] &= [2244]' - [4422]', \\
[1144] - [4411] &= [2233]' - [3322]', \\
[2233] - [3322] &= [1144]' - [4411]', \\
[2244] - [4422] &= [1133]' - [3311]', \\
[3344] - [4433] &= [1122]' - [2211]' .
\end{align}

The subsystems (3.13a)–(3.13c) and (3.14d)–(3.14f) are dual to each other. Thus, the systems (3.13), (3.14) can be represented in very compact form:

\begin{align}
[rrss] + [ssrr] &= [rrss]' + [ssrr]', \\
[rrss] - [ssrr] &= [\tilde{r}\tilde{r}\tilde{s}\tilde{s}]' - [\tilde{s}\tilde{s}\tilde{r}\tilde{r}]' ,
\end{align}

where $r, s \in \{1, 2, 3, 4\}, r < s$ and $\tilde{r}, \tilde{s} \in \{1, 2, 3, 4\}\setminus\{r, s\}$, $\tilde{r} < \tilde{s}$.

Finally, there are four “fully mixed” identities:

\begin{align}
[1234] + [2143] &= [3412]' + [4321]', \\
[1234] - [2143] &= [2143]' - [1234]', \\
[3412] + [4321] &= [1234]' + [2143]', \\
[3412] - [4321] &= [4321]' - [3412]' .
\end{align}
Identities (3.13a)-(3.13c) follow from (3.4). Indeed, one can write (3.4) as

\[ [11rr] - [11rr]' = \theta_1(u + v)\theta_1(u - v)\theta_r(x + y)\theta_r(x - y), \quad r = 1, 2, 3, 4. \]

Changing here \(x \leftrightarrow y\), one gets \([rr11]' - [rr11] = \theta_1(u + v)\theta_1(u - v)\theta_r(x + y)\theta_r(x - y) = [11rr] - [11rr]'\). Similarly, (3.13d)-(3.13f) follow from (3.5a)-(3.5c), respectively. To prove (3.14a), we write (3.13a) in terms of the variables \(u, v, x, y\) and exchange \(u \leftrightarrow x\). Then

\[ [2211] - [1122] = -\theta_1(u + v)\theta_1(u - v)\theta_2(x + y)\theta_2(x - y) + \theta_2(u + v)\theta_2(u - v)\theta_1(x + y)\theta_1(x - y). \]

Now (3.14a) holds by virtue of (3.5c). Identities (3.14b)-(3.14f) can be proved in a similar way. Finally, it is easy to see that (3.16) follows from (3.8). Indeed, subtracting (3.8) from the same identity with the exchange \(x \leftrightarrow y\) yields (3.16a). All other identities in (3.16) are proved in a similar way.

**Remark 3.4** Identities (3.13), (3.14) differ slightly from those written by Jacobi. For example, in [1], p. 507 one can find the relations \(1122 + [4433] = [2211]' + [3344]', \), \(1122 - [4433] = [1122]' - [4433]'\) which are appropriate linear combinations of (3.13a) and (3.14a), (3.14b). We should also stress that these Jacobi identities are direct corollaries of (3.4) at \(r = 2\) and (3.5c). This is in complete agreement with the derivation of (3.13), (3.14) from the Weierstrass addition formulae.

### 3.5 Five term Riemann identities

The Riemann identities (the term is due to Mumford [M1 page 20]) are simple corollaries of the Jacobi relations (3.10), (3.13)-(3.16). They each express a “primed” quantity as a linear combination of some appropriate four “unprimed” ones.

Hence from (3.10) we have the four simplest Riemann identities:

\[ 2[1]' = [1] + [2] - [3] + [4], \]  \hspace{1cm} (3.17a)

\[ 2[2]' = [1] + [2] + [3] - [4], \]  \hspace{1cm} (3.17b)

\[ R.I \]

\[ 2[3]' = -[1] + [2] + [3] + [4], \]  \hspace{1cm} (3.17c)

\[ 2[4]' = [1] - [2] + [3] + [4]. \]  \hspace{1cm} (3.17d)

Let us emphasize that (3.17) is equivalent to (3.10).

Next we list all possible (twelve) identities that are obtained from (3.13), (3.14):

\[ 2[1122]' = [1122] + [2211] + [3344] - [4433], \]  \hspace{1cm} (3.18a)

\[ 2[1133]' = [1133] + [3311] + [2244] - [4422], \]  \hspace{1cm} (3.18b)

\[ 2[1144]' = [1144] + [4411] + [2233] - [3322]. \]  \hspace{1cm} (3.18c)
\[ \begin{align*}
2[2211]' &= [2211] + [1122] + [4433] - [3344], \\
2[2233]' &= [2233] + [3322] + [1144] - [4411], \\
R.II & \quad 2[2244]' = [2244] + [4422] + [1133] - [3311],
\end{align*} \]

\[ \begin{align*}
2[3311]' &= [3311] + [1133] + [4422] - [2244], \\
2[3322]' &= [3322] + [2233] + [4411] - [1144], \\
2[3344]' &= [3344] + [4433] + [1122] - [2211],
\end{align*} \]

\[ \begin{align*}
2[4411]' &= [4411] + [1144] + [3322] - [2233], \\
2[4422]' &= [4422] + [2244] + [3311] - [1133], \\
2[4433]' &= [4433] + [3344] + [2211] - [1122].
\end{align*} \]

Clearly, the system \((3.18)-(3.21)\) is equivalent to \((3.13), (3.14)\). Finally, identities \((3.16)\) are equivalent to

\[ \begin{align*}
2[1234]' &= -[1234] + [2143] + [3421] + [4312], \\
2[2143]' &= -[2143] + [1234] + [3412] + [4321], \\
R.III & \quad 2[3412]' = -[3412] + [4321] + [1234] + [2143], \\
2[4321]' &= -[4321] + [3412] + [1234] + [2143].
\end{align*} \]

**Remark 3.5** The identities presented here essentially coincide with the ones given by Mumford [M] p. 20. See also [WW].

### 3.6 Equivalence of addition formulae and Jacobi identities

In section 3.4, we have obtained the Jacobi identities from the addition formulae. In its turn, one can show that the system \((3.10), (3.13), (3.14)\) implies the addition formulae \((3.5)-(3.8)\). The proof is similar to the one given by Koornwinder for the Riemann identities [K].

In accordance with \((3.11)\), the relation \((3.10a)\) acquires the form

\[ \begin{align*}
\theta_1(u + x)\theta_1(u - x)\theta_1(v + y)\theta_1(v - y) + \theta_2(u + x)\theta_2(u - x)\theta_2(v + y)\theta_2(v - y) \\
= \theta_1(v + x)\theta_1(v - x)\theta_1(u + y)\theta_1(u - y) + \theta_2(v + x)\theta_2(v - x)\theta_2(u + y)\theta_2(u - y).
\end{align*} \]

Changing here \(u \leftrightarrow x\) and \(v \leftrightarrow y\), one obtains two additional relations:

\[ \begin{align*}
-\theta_1(u + x)\theta_1(u - x)\theta_1(v + y)\theta_1(v - y) + \theta_2(u + x)\theta_2(u - x)\theta_2(v + y)\theta_2(v - y) \\
= -\theta_1(u + v)\theta_1(u - v)\theta_1(x + y)\theta_1(x - y) + \theta_2(u + v)\theta_2(u - v)\theta_2(x + y)\theta_2(x - y),
\end{align*} \]
\[
\theta_1(u + v)\theta_1(u - v)\theta_1(x + y)\theta_1(x - y) + \theta_2(u + v)\theta_2(u - v)\theta_2(x + y)\theta_2(x - y) \\
= -\theta_1(v + x)\theta_1(v - x)\theta_1(u + y)\theta_1(u - y) + \theta_2(v + x)\theta_2(v - x)\theta_2(u + y)\theta_2(u - y).
\]

(3.25)

Introduce the notation:

\[
A_j := \theta_j(u + x)\theta_j(u - x)\theta_j(v + y)\theta_j(v - y), \\
B_j := \theta_j(u + y)\theta_j(u - y)\theta_j(v + x)\theta_j(v - x), \\
C_j := \theta_j(u + v)\theta_j(u - v)\theta_j(x + y)\theta_j(x - y).
\]

(3.26)

Then relations (3.23)–(3.25) acquire the form \(A_1 - B_1 = B_2 - A_2, A_1 - C_1 = A_2 - C_2, B_1 + C_1 = B_2 - C_2\) which is a system of linear equations for the unknowns \(A_2, B_2, C_2\). The system is degenerate with compatibility condition \(A_1 - B_1 = C_1\). In terms of the theta functions, this condition is nothing but equation (3.4) with \(r = 1\). Equivalently, one can treat the above equations for \(A_j, B_j, C_j\) as a linear system for the unknowns \(A_1, B_1, C_1\). Then, for example, \(C_1 = B_2 - A_2\) which is (3.6) with \(r = 2\). The other addition formulae can be obtained in a similar way.

4 Particular identities

4.1 Consequences of the bilinear identities

One can obtain twelve particular identities from the general system (3.2) putting \(v = 0\) or \(v = \pm u\) (actually, the restriction \(v = -u\) can be applied only for identity (3.2d) which leads to (4.3e) below):

\[
2\theta_1^2(u|2\tau) = \theta_4(u|\tau)\theta_3(0|\tau) - \theta_3(u|\tau)\theta_4(0|\tau),
\]

(4.1a)

\[
2\theta_2^2(u|2\tau) = \theta_3(u|\tau)\theta_3(0|\tau) - \theta_4(u|\tau)\theta_4(0|\tau),
\]

(4.1b)

\[
2\theta_3^2(u|2\tau) = \theta_3(u|\tau)\theta_3(0|\tau) + \theta_4(u|\tau)\theta_4(0|\tau),
\]

(4.1c)

\[
2\theta_4^2(u|2\tau) = \theta_3(u|\tau)\theta_4(0|\tau) + \theta_4(u|\tau)\theta_3(0|\tau),
\]

(4.1d)

\[
2\theta_1(u|2\tau)\theta_4(u|2\tau) = \theta_1(u|\tau)\theta_2(0|\tau),
\]

(4.2a)

\[
2\theta_2(u|2\tau)\theta_3(u|2\tau) = \theta_2(u|\tau)\theta_2(0|\tau),
\]

(4.2b)

\[
2\theta_3(2u|2\tau)\theta_2(0|2\tau) = \theta_3^2(u|\tau) - \theta_4^2(u|\tau),
\]

(4.3a)

\[
2\theta_4(2u|2\tau)\theta_3(0|2\tau) = \theta_2^2(u|\tau) - \theta_4^2(u|\tau),
\]

(4.3b)

\[
2\theta_3(2u|2\tau)\theta_4(0|2\tau) = \theta_2^2(u|\tau) + \theta_4^2(u|\tau),
\]

(4.3c)

\[
2\theta_4(2u|2\tau)\theta_4(0|2\tau) = \theta_3^2(u|\tau) + \theta_4^2(u|\tau),
\]

(4.3d)

\[
\theta_1(2u|2\tau)\theta_4(0|2\tau) = \theta_1(u|\tau)\theta_2(u|\tau),
\]

(4.4a)

\[
\theta_4(2u|2\tau)\theta_4(0|2\tau) = \theta_3(u|\tau)\theta_4(u|\tau).
\]

(4.4b)
In [WW] Section 21.52 there are two particular equations relating theta functions with modular parameters \( \tau \) and \( 2\tau \) which are called the transformations of Landen’s type:

\[
\frac{\theta_4(2u|2\tau)}{\theta_4(0|2\tau)} = \frac{\theta_3(u|\tau)\theta_4(u|\tau)}{\theta_3(0|\tau)\theta_4(0|\tau)}, \tag{4.5a}
\]

\[
\frac{\theta_1(2u|2\tau)}{\theta_4(0|2\tau)} = \frac{\theta_1(u|\tau)\theta_2(u|\tau)}{\theta_3(0|\tau)\theta_4(0|\tau)} \tag{4.5b}.
\]

The first identity is derived from (4.4b) and from the relation

\[
\theta_4^2(0|2\tau) = \theta_3(0|\tau)\theta_4(0|\tau) \tag{4.6}
\]

which is also a corollary of (4.4b). The identity (4.5b) is a ratio of (4.4a) and (4.6). Note also that (4.5b) can be obtained from (4.5a) by the shift \( u \to u + \frac{\tau}{2} \).

### 4.2 Particular addition formulae

One can obtain important particular cases of the Weierstrass addition formulae which include two variables. Here we present the complete list of eighteen identities which easily follow from (3.4)–(3.8):

\[
\theta_1(u + v)\theta_1(u - v)\theta_2^2(0) = \theta_1^2(u)\theta_2^2(v) - \theta_1^2(u)\theta_1^2(v) = \theta_4^2(u)\theta_3^2(v) - \theta_3^2(u)\theta_4^2(v), \tag{4.7a}
\]

\[
\theta_1(u + v)\theta_1(u - v)\theta_3^2(0) = \theta_1^2(u)\theta_3^2(v) - \theta_1^2(u)\theta_1^2(v) = \theta_4^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_4^2(v), \tag{4.7b}
\]

\[
\theta_1(u + v)\theta_1(u - v)\theta_4^2(0) = \theta_1^2(u)\theta_4^2(v) - \theta_1^2(u)\theta_1^2(v) = \theta_3^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_3^2(v), \tag{4.7c}
\]

\[
\theta_2(u + v)\theta_2(u - v)\theta_2^2(0) = \theta_2^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_1^2(v) = \theta_3^2(u)\theta_3^2(v) - \theta_4^2(u)\theta_4^2(v), \tag{4.8a}
\]

\[
\theta_2(u + v)\theta_2(u - v)\theta_3^2(0) = \theta_3^2(u)\theta_2^2(v) - \theta_1^2(u)\theta_1^2(v) = \theta_2^2(u)\theta_3^2(v) - \theta_4^2(u)\theta_1^2(v), \tag{4.8b}
\]

\[
\theta_2(u + v)\theta_2(u - v)\theta_4^2(0) = \theta_3^2(u)\theta_4^2(v) - \theta_1^2(u)\theta_3^2(v) = \theta_2^2(u)\theta_3^2(v) - \theta_4^2(u)\theta_4^2(v), \tag{4.8c}
\]
\[
\theta_3(u + v)\theta_3(u - v)\theta_3^2(0) = \theta_2^2(u)\theta_3^2(v) + \theta_1^2(u)\theta_4^2(v) = \theta_3^2(u)\theta_2^2(v) + \theta_2^2(u)\theta_1^2(v),
\]

(4.9a)

\[
\theta_4(u + v)\theta_3(u - v)\theta_3^2(0) = \theta_1^2(u)\theta_3^2(v) + \theta_2^2(u)\theta_3^2(v) = \theta_3^2(u)\theta_2^2(v) + \theta_2^2(u)\theta_1^2(v),
\]

(4.9b)

\[
\theta_4(u + v)\theta_4(u - v)\theta_2^2(0) = \theta_1^2(u)\theta_2^2(v) + \theta_3^2(u)\theta_2^2(v) = \theta_2^2(u)\theta_1^2(v) + \theta_2^2(u)\theta_3^2(v),
\]

(4.9c)

\[
\theta_4(u + v)\theta_4(u - v)\theta_3^2(0) = \theta_1^2(u)\theta_3^2(v) - \theta_1^2(u)\theta_3^2(v) = \theta_3^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_3^2(v),
\]

(4.10a)

\[
\theta_4(u + v)\theta_4(u - v)\theta_3^2(0) = \theta_1^2(u)\theta_2^2(v) + \theta_3^2(u)\theta_2^2(v) = \theta_2^2(u)\theta_1^2(v) + \theta_2^2(u)\theta_3^2(v),
\]

(4.10b)

\[
\theta_1(u + v)\theta_2(u - v)\theta_3(0)\theta_4(0) = \theta_1(u)\theta_2(u)\theta_3(v)\theta_4(v) + \theta_3(u)\theta_4(u)\theta_1(v)\theta_2(v),
\]

(4.11a)

\[
\theta_1(u + v)\theta_3(u - v)\theta_2(0)\theta_4(0) = \theta_1(u)\theta_3(u)\theta_2(v)\theta_4(v) + \theta_2(u)\theta_4(u)\theta_1(v)\theta_3(v),
\]

(4.11b)

\[
\theta_1(u + v)\theta_4(u - v)\theta_2(0)\theta_3(0) = \theta_1(u)\theta_4(u)\theta_2(v)\theta_3(v) + \theta_2(u)\theta_3(u)\theta_1(v)\theta_4(v),
\]

(4.11c)

\[
\theta_2(u + v)\theta_3(u - v)\theta_2(0)\theta_3(0) = \theta_2(u)\theta_3(u)\theta_2(v)\theta_3(v) - \theta_1(u)\theta_4(u)\theta_1(v)\theta_4(v),
\]

(4.11d)

\[
\theta_2(u + v)\theta_4(u - v)\theta_2(0)\theta_4(0) = \theta_2(u)\theta_4(u)\theta_2(v)\theta_4(v) - \theta_1(u)\theta_3(u)\theta_1(v)\theta_3(v),
\]

(4.11e)

\[
\theta_3(u + v)\theta_4(u - v)\theta_3(0)\theta_4(0) = \theta_3(u)\theta_4(u)\theta_3(v)\theta_4(v) - \theta_1(u)\theta_2(u)\theta_1(v)\theta_2(v).
\]

(4.11f)

Remark 4.1 The complete list of identities (4.7)–(4.11) was originally obtained by Jacobi [1], p. 510 as a particular specification of identities (3.11), (3.13)–(3.16), see also [W] pp. 76–78, [WW] 487–488. Mumford [MI], p. 22 has obtained a part of relations (4.7)–(4.11) as specific cases of the Riemann identities (3.17)–(3.22).
As a byproduct of (4.7)–(4.11), one gets some extra identities:

\[ \theta_1^2(u)\theta_1^2(v) - \theta_2^2(u)\theta_2^2(v) = \theta_1^2(u)\theta_1^2(v) - \theta_3^2(u)\theta_3^2(v), \quad (4.12a) \]
\[ \theta_1^2(u)\theta_1^2(v) - \theta_2^2(u)\theta_2^2(v) = \theta_1^2(u)\theta_1^2(v) - \theta_3^2(u)\theta_3^2(v), \quad (4.12b) \]
\[ \theta_1^2(u)\theta_3^2(v) - \theta_2^3(u)\theta_1^2(v) = \theta_1^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_3^2(v), \quad (4.12c) \]
\[ \theta_1^2(u)\theta_4^2(v) - \theta_2^3(u)\theta_1^2(v) = \theta_3^2(u)\theta_2^2(v) - \theta_2^2(u)\theta_3^2(v). \quad (4.12d) \]

In particular, the following identity holds:

\[ \theta_1^4(u) + \theta_3^4(u) = \theta_1^2(u) + \theta_3^2(u). \quad (4.13) \]

Certainly, the last relation is a corollary of (4.3). Thus, in addition to (2.13), one gets another famous identity for theta constants:

\[ \theta_3^4(0) = \theta_2^4(0) + \theta_4^4(0). \quad (4.14) \]

### 4.3 Duplication formulae

The duplication formulae relate the functions \( \theta_a(2u|\tau) \), \( a = 1, 2, 3, 4 \) with appropriate combinations of the functions \( \theta_a(u|\tau) \). All these identities emerge as further degenerations of addition formulae (1.7)–(1.11). Here is the complete list:

\[ \theta_1(2u)\theta_2(0)\theta_3(0)\theta_4(0) = 2 \theta_1(u)\theta_2(u)\theta_3(u)\theta_4(u), \quad (4.15) \]
\[ \theta_2(2u)\theta_2(0)\theta_3^2(0) = \theta_2^2(u)\theta_3^2(u) - \theta_1^2(u)\theta_3^2(u), \quad (4.16a) \]
\[ \theta_2(2u)\theta_2(0)\theta_3^2(0) = \theta_2^2(u)\theta_2^2(u) - \theta_1^2(u)\theta_2^2(u), \quad (4.16b) \]
\[ \theta_2(2u)\theta_3^2(0) = \theta_1^2(u) - \theta_3^2(u), \quad (4.16c) \]
\[ \theta_2(2u)\theta_3^2(0) = \theta_3^2(u) - \theta_1^2(u). \quad (4.16d) \]
\[ \theta_3(2u)\theta_3(0)\theta_2^2(0) = \theta_2^2(u)\theta_3^2(u) + \theta_1^2(u)\theta_2^2(u), \quad (4.17a) \]
\[ \theta_3(2u)\theta_3(0)\theta_3^2(0) = \theta_3^2(u)\theta_3^2(u) - \theta_1^2(u)\theta_3^2(u), \quad (4.17b) \]
\[ \theta_3(2u)\theta_3^2(0) = \theta_1^2(u) + \theta_3^2(u), \quad (4.17c) \]
\[ \theta_3(2u)\theta_3^2(0) = \theta_3^2(u) + \theta_1^2(u). \quad (4.17d) \]
\[ \theta_4(2u)\theta_4(0)\theta_2^2(0) = \theta_2^2(u)\theta_4^2(u) + \theta_1^2(u)\theta_2^2(u), \quad (4.18a) \]
\[ \theta_4(2u)\theta_4(0)\theta_3^2(0) = \theta_1^2(u)\theta_3^2(u) + \theta_3^2(u)\theta_2^2(u), \quad (4.18b) \]
\[ \theta_4(2u)\theta_4^2(0) = \theta_1^2(u) - \theta_4^2(u), \quad (4.18c) \]
\[ \theta_4(2u)\theta_4^2(0) = \theta_3^2(u) - \theta_2^2(u). \quad (4.18d) \]
One can unify the sets of equations \{ (4.16a), (4.17b), (4.18a) \}; \{ (4.16b), (4.17a), (4.18b) \}; \{ (4.16c), (4.17c), (4.18c) \} and \{ (4.16d), (4.17d), (4.18d) \} by writing down all the 12 identities (4.16)–(4.18) in the compressed form

\[
(-1)^{\beta + \gamma - 1} \theta^2_{\alpha+1}(u) \theta^2_{\gamma+1}(u) = \theta_{\beta+1}(2u) \theta_{\gamma+1}(0) \theta^2_{\gamma+1}(0), \tag{4.19a}
\]

\[
(-1)^{\beta + \gamma - 1} \theta^2_{\alpha+1}(u) - \theta^2_{\beta+1}(u) \theta^2_{\gamma+1}(u) = - \theta_{\gamma+1}(2u) \theta_{\gamma+1}(0) \theta^2_{\gamma+1}(0), \tag{4.19b}
\]

\[
\theta_{\alpha+1}(2u) \theta^2_{\alpha+1}(0) = \theta^4_{\alpha+1}(u) + (-1)^{\alpha} \theta^4_{\alpha+1}(u), \tag{4.19c}
\]

\[
\theta_{\alpha+1}(2u) \theta^2_{\beta+1}(0) = (-1)^{\gamma+1} \theta^4_{\beta+1}(u) + (-1)^{\beta+1} \theta^4_{\gamma+1}(u). \tag{4.19d}
\]

where in (4.19a), (4.19b), and (4.19d) the indices \( \alpha, \beta, \gamma \) are assumed to be any cyclic permutation of \( \{1, 2, 3\} \) and in (4.19c) \( \alpha = 1, 2, 3 \). Together with (4.15), the system (4.19) yields the complete set of duplication formulae.

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