Asymptotically non-flat Einstein-Born-Infeld-dilaton black holes with Liouville-type potential

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We construct some classes of electrically charged, static and spherically symmetric black hole solutions of the four-dimensional Einstein-Born-Infeld-dilaton gravity in the absence and presence of Liouville-type potential for the dilaton field and investigate their properties. These solutions are neither asymptotically flat nor (anti)-de Sitter. We show that in the presence of the Liouville-type potential, there exist two classes of solutions. We also compute temperature, entropy, charge and mass of the black hole solutions, and find that these quantities satisfy the first law of thermodynamics. We find that in order to fully satisfy all the field equations consistently, there must be a relation between the electric charge and other parameters of the system.

I. INTRODUCTION

It is quite possible that gravity is not given by the Einstein action, at least at sufficiently high energies. In string theory, gravity becomes scalar-tensor in nature. The low energy limit of the string theory leads to the Einstein gravity, coupled non-minimally to a scalar dilaton field [1]. When a dilaton is coupled to Einstein-Maxwell theory, it has profound consequences for the black hole solutions. Some efforts have been done to construct exact solutions of Einstein-Maxwell-dilaton gravity. For example exact charged dilaton black hole solutions of EMd gravity in the absence of a dilaton potential have been constructed by many authors [2, 3]. The dilaton changes the casual structure of the spacetime and leads to curvature singularities at finite radii. These black holes are asymptotically flat. In recent years, non-asymptotically flat black hole spacetimes are attracting much interest in connection with the so called AdS/CFT correspondence. Black hole spacetimes which are neither asymptotically flat nor dS/AdS have been found and investigated by many authors. The uncharged solutions have been found in [4], while the charged solutions have been considered in [5]. In the presence of Liouville-type potential, static charged black hole solutions have also been discovered with a positive constant curvature event horizons and zero or

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negative constant curvature horizons \[6, 7\]. The extension to the dyonic black hole solutions in four-dimensional and higher dimensional EMd gravity with one and two Liouville-type potentials have also been done in \[8\]. These solutions possess both electric and magnetic charge and they are neither asymptotically flat nor dS/AdS.

The idea of the nonlinear electromagnetism was first introduced in 1934 by Born and Infeld in order to obtain a finite value for the self-energy of point-like charges \[9\]. Although it become less popular with the introduction of QED, in recent years, the Born-Infeld action has been occurring repeatedly with the development of superstring theory, where the dynamics of D-branes is governed by the Born-Infeld action \[10, 11\]. For various motivations, extending the Reissner- Nordström black hole solutions in EM theory to the charged black hole solutions in EBI theory has attracted some attention in recent years \[12\]. For example, exact solutions of spherically symmetric Einstein-Born-Infeld black holes in (A)dS spacetime with cosmological horizon in arbitrary dimensions has been constructed \[13\]. The extension to the case where black hole horizon (cosmological horizon) is a positive, zero or negative constant curvature surface have also been studied \[14\]. Unfortunately, exact solutions to the Einstein-Born-Infeld equation coupled to matter fields are too complicated to find except in a limited number of cases. Indeed, exact solutions to the Einstein Born-Infeld dilaton (EBId) gravity are known only in three dimensions \[15\]. Numerical studies of the EBId system in four dimensional static and spherically symmetric spacetime have been done \[16\]. In the absence of a dilaton potential a class of solution to the four-dimensional EBId gravity with magnetic charge has been constructed \[17\]. Our aim in this paper is to generalize these solutions to the case of one and two Liouville type potential and investigate how the properties of the solutions will be changed in the presence of potential for the scalar field. In addition, in each cases, we compute, the mass, electric charge, temperature and entropy of the system. We will consider three special cases: (a) \(V(\phi) = 0\), (b) \(V(\phi) = 2\Lambda e^{2\beta\phi}\) and (c) \(V(\phi) = 2\Lambda_1 e^{2\beta_1\phi} + 2\Lambda_2 e^{2\beta_2\phi}\). The first case corresponds to the action considered in \[18\]. When \(\alpha = 1\), it reduces to the four-dimensional low-energy action obtained from string theory in terms of Einstein metric. Case (b) corresponds to a Liouville-type potential. This kind of potential appear when one applies a conformal transformation on the low energy limit of the string tree level effective action for massless boson sector and write the action in the Einstein frame \[19, 20\]. This potential have been considered previously by a number of authors \[6, 8, 21\]. One may refer to \(\Lambda\) as the cosmological constant, since in the absence of the dilaton field the action reduces to the action of EBI gravity with cosmological constant \[13, 14\]. The potential in case (c) was previously investigated by a number of authors both in the context of \(FRW\) scalar field cosmologies \[22\] and EMd black holes \[6, 8\]. This kind of potential function
can be obtained when a higher-dimensional theory is compactified to four-dimensional spacetime, including various supergravity and string models.

The organization of this paper is as follows: Section II is devoted to a brief review of the field equations and general equations of motion. In Sec. III we consider EBId black holes without potential. In Sec. IV we present two classes of solutions with a Liouville type potential and general dilaton coupling. In Sec. V we extend these solutions to the case of two Liouville potentials. We finish our paper with some concluding remarks.

II. FIELD EQUATIONS

We consider the four-dimensional action in which gravity is coupled to dilaton and Born Infeld field with an action

\[ S = \int d^4x \sqrt{-g} \left( \mathcal{R} - 2(\nabla \phi)^2 - V(\phi) + L(F, \phi) \right) \]  

where \( \mathcal{R} \) is the Ricci scalar curvature, \( \phi \) is the dilaton field and \( V(\phi) \) is a potential for \( \phi \). The Born-Infeld \( L(F, \phi) \) part of the action is given by

\[ L(F, \phi) = 4\gamma e^{-2\alpha \phi} \left( 1 - \sqrt{1 + \frac{F^{\mu \nu} F_{\mu \nu}}{2\gamma}} \right). \]  

Here, \( \alpha \) is the dilaton coupling constant and \( \gamma \) is called the Born-Infeld parameter with dimension of mass. In the limit \( \gamma \to \infty \), \( L(F, \phi) \) reduces to the standard Maxwell field coupled to a dilaton field

\[ L(F, \phi) = -e^{-2\alpha \phi} F^{\mu \nu} F_{\mu \nu}. \]

On the other hand, \( L(F, \phi) \to 0 \) as \( \gamma \to 0 \). It is convenient to set

\[ L(F, \phi) = 4\gamma e^{-2\alpha \phi} \mathcal{L}(Y). \]  

where

\[ \mathcal{L}(Y) = 1 - \sqrt{1 + Y}, \]  

\[ Y = \frac{F^2}{2\gamma}. \]

where \( F^2 = F^{\mu \nu} F_{\mu \nu} \). The equations of motion can be obtained by varying the action (1) with respect to the gravitational field \( g_{\mu \nu} \), the dilaton field \( \phi \) and the gauge field \( A_\mu \) which yields the following field equations

\[ R_{\mu \nu} = 2\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu \nu} V(\phi) - 4e^{-2\alpha \phi} \partial_Y \mathcal{L}(Y) F_{\mu \eta} F_{\nu}^{\eta} + 2\gamma e^{-2\alpha \phi} \left[ 2Y \partial_Y \mathcal{L}(Y) - \mathcal{L}(Y) \right] g_{\mu \nu}. \]
\[ \nabla^2 \phi = \frac{1}{4} \frac{\partial V}{\partial \phi} + 2\gamma \alpha e^{-2\alpha \phi} \mathcal{L}(Y), \quad (8) \]

\[ \nabla_{\mu} \left( e^{-2\alpha \phi} \partial_{\nu} \mathcal{L}(Y) F^{\mu\nu} \right) = 0. \quad (9) \]

We wish to find static and spherically symmetric solutions of the above field equations. The most general such metric can be written in the form

\[ ds^2 = -U(r) dt^2 + \frac{dr^2}{U(r)} + R^2(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (10) \]

The Maxwell equation can be integrated immediately, where all the components of \( F^{\mu\nu} \) are zero except \( F^{rt} \):

\[ F^{rt} = \frac{q e^{2\alpha \phi}}{R^2(r) \sqrt{1 + \frac{q^2 e^{4\alpha \phi}}{\gamma R^4(r)}}} \quad (11) \]

Here \( q \) is the electric charge, defined through the integral

\[ q = \frac{1}{4\pi} \int_{s^2} e^{-2\alpha \phi} * F d\Omega, \quad (12) \]

where \( * \) is the Hodge dual and \( s^2 \) is any two-sphere defined at spatial infinity, which its volume element denoted by \( d\Omega \). We note that the electric field is finite at \( r = 0 \). This is expected in Born-Infeld theories. It is interesting to consider three limits of (11). First, for large \( \gamma \) (where the BI action reduces to Maxwell case) we have \( F^{rt} = \frac{q e^{2\alpha \phi}}{R^2(r)} \) as presented in [6]. On the other hand, if \( \gamma \to 0 \) we get \( F^{rt} = 0 \), finally in the case of \( \alpha \to 0 \) it reduces to the case of Einstein Born Infeld theory without dilaton field [13, 14]. With the metric (10) and Maxwell field (11), the field equations (7)-(9) and (13)-(15) reduce to the following system of coupled ordinary differential equations

\[ \frac{1}{R^2} \frac{d}{dr} \left( U \frac{dR^2}{dr} \right) = \frac{2}{R^2} \left( V(\phi) - 4\gamma e^{-2\alpha \phi} \left[ 2Y \partial_Y \mathcal{L}(Y) - \mathcal{L}(Y) \right] \right), \quad (13) \]

\[ \frac{1}{R^2} \frac{d}{dr} \left( R^2 U \frac{d\phi}{dr} \right) = \frac{1}{4} \frac{dV}{d\phi} + 2\gamma \alpha e^{-2\alpha \phi} \mathcal{L}(Y), \quad (14) \]

\[ \frac{1}{R} \frac{d^2R}{dr^2} + \left( \frac{d\phi}{dr} \right)^2 = 0. \quad (15) \]

In particular, in the case of the linear electrodynamics with \( \mathcal{L}(Y) = -\frac{1}{2} Y \), the system of equations (7)-(9) and (13)-(15) reduce to the well-known equations of EMd gravity [6].

To solve these equations, we make the ansatz

\[ R(r) = e^{\alpha \phi(r)}, \quad (16) \]
By this ansatz, eq. (10) becomes \( Y = -\frac{q^2}{q^2 + \gamma} \). In addition, we introduce the constant \( A \):

\[
A = \gamma \left[ 2Y \partial_r \mathcal{L}(Y) - \mathcal{L}(Y) \right] = \sqrt{\gamma(q^2 + \gamma) - \gamma}. \tag{17}
\]

Using (16) in equation (15), immediately gives

\[
\phi(r) = \alpha \frac{1 + \alpha^2}{\ln(br - c)}, \tag{18}
\]

where \( b \) and \( c \) are integration constants. For later convenience, without loss of generality, we set \( b = 1 \) and \( c = 0 \).

### III. SOLUTIONS WITH \( V(\phi) = 0 \)

Let us begin by looking for the solutions without Liouville potential \( V(\phi) = 0 \).

#### A. String coupling case \( \alpha = 1 \)

We first consider the string coupling case \( \alpha = 1 \) with \( V(\phi) = 0 \). In this case we find the following solution

\[
U(r) = 2r(1 - 2A - \frac{r_0}{2r}), \tag{19}
\]

with \( A \) is a constant related to the electric charge \( q \) as one can see from eq. (17) and \( r_0 > 0 \) is an integration constant related to the mass of the system. In order to fully satisfy the system of equations, there must be a relation between electric charge \( q \) and \( \gamma \) parameter

\[
q^2 = 1 + \frac{\sqrt{16\gamma^2 + 1}}{8\gamma}. \tag{20}
\]

Now we compute the mass of the system. In order to define the mass, we use the so-called quasilocal formalism [24]. The quasilocal mass is given by

\[
M = \frac{1}{2} \frac{dR_s(r)}{dr} U^{1/2}(r) \left[ U_0^{1/2}(r) - U^{1/2}(r) \right], \tag{21}
\]

where \( U_0(r) \) is an arbitrary non-negative function which determines the zero of the energy for a background spacetime and \( r \) is the radius of the spacelike hypersurface boundary. If no cosmological horizon is present, the large \( r \) limit of (21) determines the asymptotic mass \( M \). For the solution under consideration, there is no cosmological horizon and the natural choice for the background is \( U_0(r) = 2(1 - 2A)r \). The large \( r \) limit of (21) gives the mass of the solution

\[
M = \frac{r_0}{4}. \tag{22}
\]
The metric corresponding to (19) and the other metric that we will present in this paper are neither asymptotically flat nor (anti)-de Sitter. The solution has several properties. First, there is an event horizon at \( r_h = 2M/(1 - 2A) \). In order to study the general structure of these solutions, we first look for the curvature singularities in the presence of dilaton gravity. It is easy to show that the Kretschmann scalar \( R_{\mu \nu \lambda \kappa} R^{\mu \nu \lambda \kappa} \) diverges at \( r = 0 \), it is finite for \( r \neq 0 \) and goes to zero as \( r \to \infty \). Also, it is notable to mention that the Ricci scaler is finite everywhere except at \( r = 0 \), and goes to zero as \( r \to \infty \). Therefore \( r = r_h \) is a regular horizon and we have an essential singularity located at \( r = 0 \). This can be seen from the explicit expression of \( K \) and \( R \):

\[
K = \frac{1}{4r^4} \left[ 4r^2 (3 - 4A + 12A^2) + 4rr_0(1 + 2A) + 3r_0^2 \right],
\]

\[
R = \frac{2r(6A - 1) - r_0}{2r^2}.
\]

Second, even though \( \phi(r) \) diverges at \( r \to \infty \), but since the mass, charge and curvature all remain finite, the solution is well behaved at infinity. Note that the dilaton field is regular on the horizon, too. The spacial infinity is conformally null and the solution describes a black hole with the same causal structure as the Schwarzschild spacetime. Black hole entropy typically satisfies the so called area law of the entropy [25], which states that the entropy is a quarter of the event horizon area. It is easy to see that the temperature and entropy of the black hole can be written as

\[
T = \frac{1}{4\pi} \frac{dU}{dr}(r_h) = \frac{1 - 2A}{2\pi},
\]

\[
S = \pi r_h = \frac{2\pi M}{1 - 2A}.
\]

Note that the temperature depends on \( A \) and is independent of \( M \). In the limit \( A \to 1/2 \), the temperature goes to zero, while the entropy \( S \) becomes infinite. Since the temperature is always non negative quantity thus \( A \leq 1/2 \).

Finally we investigate the first law of thermodynamics. The black hole solution we found here have mass and charge, thus in general all thermodynamic quantities are functions of \( M \) and \( q \). Since the electric charge is fixed, thus the first law of thermodynamics may be written as

\[
dM = TdS.
\]

Then it is easy to see that thermodynamics quantities obtained above satisfy the first law [27]. The solution with zero mass \( (M = 0) \) is singular with a null singularity at \( r = 0 \). The case \( M < 0 \) corresponds to naked timelike singularity located at \( r = 0 \).
B. General dilaton coupling \( \alpha \)

It is straightforward to generalize the solution (19) to arbitrary dilaton coupling constant \( \alpha \). In this case, we find the following solution

\[
U(r) = r^{2-2N} \left( (1 - 2A)/N - \frac{r_0}{r} \right),
\]

with \( N = \alpha^2/(1 + \alpha^2) \). Here \( r_0 > 0 \) is again an integration constant related to the mass of the black hole. The consistency of all field equations force that, the electric charge \( q \) satisfy in the following equation:

\[
\sqrt{\gamma + q^2} \left[ 2\gamma (\alpha^2 - 1) - 1 \right] + 2\sqrt{\gamma} \left( q^2 - \gamma (\alpha^2 - 1) \right) = 0.
\]

Note that the solution is ill defined for \( \alpha = 0 \). In the particular case \( \alpha = 1 \), the solution reduces to the (19). There is no cosmological horizon and the mass can be computed from (21). For the background function \( U_0(r) = r^{2-2N}(1 - 2A)/N \), the mass is found to be

\[
M = \frac{Nr_0}{2}.
\]

On the other hand, if \( \alpha \) and \( \gamma \) go to infinity, we will have \( N = 1 \) and \( A = 0 \), respectively. Thus the metric reduces to the Schwarzschild black hole. There is an event horizon at \( r_h = 2M/(1 - 2A) \) which is regular only for \( A < 1/2 \). The Kretschmann invariant and Ricci scalar are finite at \( r = r_h \), diverge at \( r = 0 \), and both of them vanish at \( r \to \infty \), therefore \( r = r_h \) is a regular horizon and we have an essential singularity located at \( r = 0 \). Note that the dilaton field is regular on the horizon, too. The temperature and the entropy of the black hole on the event horizon are

\[
T = \frac{1}{4\pi} \frac{dU}{dr}(r_h) = \frac{1 - 2A}{4\pi N r_h^{1-2N}},
\]

\[
S = \pi r_h^{2N}.
\]

Finally we investigate the first law of thermodynamics. Again, since the electric charge is fixed, thus the first law of thermodynamics can be written as

\[
dM = TdS.
\]

The solution with zero mass \( (M = 0) \) is singular with a null singularity at \( r = 0 \).

IV. SOLUTION WITH A LIOUVILLE TYPE POTENTIAL

In this section, we consider the action (1) with a Liouville type potential,

\[
V(\phi) = 2\Lambda e^{2\beta\phi},
\]
where $\Lambda$ and $\beta$ are constants. One may refer to $\Lambda$ as the cosmological constant, since in the absence of the dilaton field the action reduces to the action of EBI gravity with cosmological constant \[13, 14\].

### A. Solution with $\alpha = 1$

At first, we consider the case $\alpha = 1$. We have found the following solution

$$U(r) = 2r\left(1 - 2A - \Lambda - \frac{r_0}{2r}\right),$$

with $\beta = -1$ and $r_0 = 4M$, where $M$ is the mass of the system defined via the eq. (21). The electric charge is given by eq. (20). For the background function $U_0(r) = (1 - 2A - \Lambda)r$. There is an event horizon at $r_h = \frac{2M}{1 - 2A - \Lambda}$ which is regular only for $\Lambda < 1 - 2A$. The Kretschmann invariant and Ricci scalar are regular except for $r = 0$, where they diverge. Note that the dilaton field is regular on the horizon, too. As an illustration we present the Ricci scalar $\mathcal{R}$:

$$\mathcal{R} = \frac{2r(6A - 1 + 3\Lambda) - r_0}{2r^2},$$

The temperature and the entropy of the black hole on the event horizon are

$$T = \frac{1 - 2A - \Lambda}{2\pi},$$

$$S = \frac{2\pi M}{1 - 2A - \Lambda},$$

which satisfy the first law. Note that the temperature depends on $A$ and $\Lambda$ and is independent of $M$. In the limit $\Lambda \to (1 - 2A)$, the temperature goes to zero, while the entropy $S$ become infinite. It is interesting to see that our solutions are well behaved in the limit $\Lambda \to 0$. In other words all of our results presented in this section reduce to the ones presented in section (III A), in this limit.

### B. Solutions with general coupling $\alpha$

In this section, we present exact black hole solutions of EBI d with an arbitrary dilaton coupling $\alpha$ and Liouville potential $V(\phi) = 2\Lambda e^{2\beta \phi}$. In this case we can distinguish two classes of solutions which satisfy all the field equations depending of the suitable choice of the $\beta$ parameter.

I. $\beta = -\alpha$. In this case, using (16) and (18), one can easily show that eqs. (13) and (14) have solution of the form

$$U(r) = \frac{r^{2 - 2N}}{N} \left(1 - 2A - \Lambda - \frac{2M}{r}\right),$$

(39)
where $M$ is the mass of the system define via the eq. (21). Again, the consistency of all field equations force that, the electric charge $q$ satisfy in the following equation:

$$\sqrt{\gamma + q^2} \left[ (2\gamma - \Lambda)(\alpha^2 - 1) - 1 \right] + 2\sqrt{\gamma} \left( q^2 - \gamma(\alpha^2 - 1) \right) = 0. \quad (40)$$

For the background function $U_0(r) = r^{2-2N}(1 - 2A - \Lambda)/N$. There is an event horizon at $r_h = \frac{2M}{1 - \Lambda - 2A}$ which is regular only for $\Lambda < (1 - 2A)$. The Kretschmann invariant and Ricci scalar, are regular except for $r = 0$, where they diverge. Again the dilaton field is regular on the horizon.

The temperature and the entropy of the black hole on the event horizon are

$$T = \frac{1}{4\pi} \frac{dU}{dr}(r_h) = \frac{1 - \Lambda - 2A}{4\pi N} r_h^{1-2N}, \quad (41)$$

$$S = \pi r_h^{2N} = \frac{2\pi M}{1 - \Lambda - 2A} r_h^{2N-1}. \quad (42)$$

which satisfy in the first law (27). In the limit $\Lambda \to (1 - 2A)$, the temperature goes to zero, while the entropy $S$ become infinite. One may note that the solution is ill defined for $\alpha = 0$. In the particular case $\alpha = 1$, the solution reduces to the (35), while in the absence of Liouville potential ($\Lambda = 0$), the above solutions reduce to (28).

II. $\beta = -1/\alpha$. In this case, using (16) and (18), one can easily show that eqs. (13) and (14) have solution of the form

$$U(r) = \frac{r^{2-2N}}{N} \left( 1 - 2A - \frac{2M}{r} + \frac{\Lambda(1 + \alpha^2)}{1 - 3\alpha^2} r^{2(2N-1)} \right), \quad (43)$$

In order to have consistency of all the field equations, the electric charge satisfy in eq.(29). The Kretschmann invariant and Ricci scalar, diverge at $r = 0$, and both of them vanish as $r$ goes to infinity, so there is a singularity located at $r = 0$. Note that the solution is ill defined for $\alpha^2 = 1/3$.

In the limit $\alpha^2 \to 1$ it reduces to the solution of section (IV A), and in the limit $\Lambda \to 0$ the solution reduce to that with $V(\phi) = 0$.

On the other hand, if $\alpha$ and $\gamma$ go to infinity, the solution become

$$U(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2, \quad (44)$$

which is the Schwarzschild ds/Ads black hole, depending on the sign of $\Lambda$. In order to investigate the causal structure of the solution, we must investigate the zeros of the metric function $U(r)$. In fact, for $0 < r < \infty$ the zeros of $U(r)$ are governed by the function

$$f(r) = 1 - 2A - \frac{2M}{r} + \frac{\Lambda(1 + \alpha^2)}{(1 - 3\alpha^2)} r^{2(2N-1)}. \quad (45)$$
We investigate the function $g(r) = rf(r)$, for simplicity. The cases with $\alpha^2 > 1/3$ and $\alpha^2 < 1/3$ should be considered separately. We should also consider the sign of the parameter $\Lambda$ in each case.

In the first case where $\alpha^2 < 1/3$ and $\Lambda < 0$ we may have one horizon since $\frac{dg}{dr} > 0$. But the more interesting case happens for $\Lambda > 0$ where we only obtain one local minimum at $r = r_{\text{min}}$ where

$$r_{\text{min}} = \left(1 - 2A\right)\frac{\alpha^2 + 1}{\frac{1}{2} + \frac{3}{2} \alpha^2 - 2}.$$  \hspace{1cm} (46)

The function $g(r)$ possesses horizon if $g(r_{\text{min}}) < 0$. There are two zeros for $g(r_{\text{min}}) < 0$ and one degenerate zero for $g(r_{\text{min}}) = 0$ which corresponds to an extremal black hole. The condition $g(r_{\text{min}}) < 0$ gives

$$M > \frac{(1 - 2A)(1 - \alpha^2)}{1 - 3\alpha^2} \left(\frac{1 - 2A}{\Lambda}\right)^{\frac{\alpha^2 + 1}{\frac{1}{2} + \frac{3}{2} \alpha^2 - 2}}.$$ \hspace{1cm} (47)

In the second case for $\alpha^2 > 1/3$ and $\Lambda < 0$, the function $g(r)$ increases monotonically, thus we can conclude that there is one point where $g(r) = 0$ which is the black hole horizon. For $\Lambda > 0$ we find local extremum for the function. The sign of $\frac{d^2g(r)}{dr^2}$ determines whether we have local maximum or minimum. For $\alpha^2 > 1$ we have local maximum and $g(r_{\text{max}})$ should be positive in order to have any horizon. The latter condition gives

$$M < \frac{(1 - 2A)(1 - \alpha^2)}{1 - 3\alpha^2} \left(\frac{1 - 2A}{\Lambda}\right)^{\frac{\alpha^2 + 1}{\frac{1}{2} + \frac{3}{2} \alpha^2 - 2}}.$$ \hspace{1cm} (48)

If we have $\frac{1}{3} < \alpha^2 < 1$, then we would have local minimum and in case of any horizon existing $g(r_{\text{min}})$ eq. (46) should be negative which implies eq. (47).

The above considerations show that the solutions describe black holes with two horizons or an extremal black hole hiding a singularity at the origin $r = 0$, when the mass satisfies (47), (48). The radius of inner and outer horizons can not be expressed in a closed analytical form except for the extremal case. The radius of the extremal solution $r_{\text{ext}}$, coincides with $r_{\text{min}}$:

$$r_{\text{ext}} = \left(1 - 2A\right)\frac{\alpha^2 + 1}{\frac{1}{2} + \frac{3}{2} \alpha^2 - 2} = \frac{(1 - 3\alpha^2)M_{\text{ext}}}{(1 - 2A)(1 - \alpha^2)}.$$ \hspace{1cm} (49)

Unfortunately, because of the nature of the exponents of $r$ in (45), the event horizon determined by $f(r) = 0$ can not be expressed in a closed analytical form for arbitrary $\alpha$. As an example, we consider the special case $\alpha = \sqrt{3}$. For this value of $\alpha$, and large $\gamma$ limit, the action (1) is simply the Kaluza-Klein action which is obtained by dimensionally reducing the five dimensional vacuum Einstein action. For details, see [26, 27, 28].
In this case, there are two zeros of \( f(r) \) at \( r_{\pm} \):

\[
r_{\pm} = \frac{1}{\Lambda} \left( 1 - 2A \pm \sqrt{(1 - 2A)^2 - 4\Lambda M} \right).
\] (50)

The extremal solution corresponds to \( M_{\text{ext}} = \frac{(1 - 2A)^2}{4\Lambda} \). In this case \( f(r) \) has only one root at \( r_{\text{ext}} = \frac{1 - 2A}{\Lambda} \). When \( \Lambda < \frac{(1 - 2A)^2}{4M} \) we have two horizons located at \( r = r_{\pm} \). For \( \Lambda > \frac{(1 - 2A)^2}{4M} \), there is a naked singularity at \( r = 0 \).

The temperature and the entropy of the black hole on the horizons are

\[
T_{\pm} = \frac{r_{\pm}^{-3/2}}{12\pi} \left( 4M + 2(1 - 2A)r_{\pm} - 3\Lambda r_{\pm}^2 \right),
\] (51)

\[
S_{\pm} = \pi r_{\pm}^{3/2}.
\] (52)

which satisfy in the first law of thermodynamics.

V. SOLUTIONS WITH A GENERAL COUPLING PARAMETER AND TWO LIOUVILLE POTENTIALS

In this section, we present exact solutions to the EBId gravity equations with an arbitrary dilaton coupling parameter \( \alpha \) and dilaton potential

\[
V(\phi) = 2\Lambda_1 e^{2\beta_1 \phi} + 2\Lambda_2 e^{2\beta_2 \phi}.
\] (53)

Where \( \Lambda_1 \), and \( \Lambda_2 \), \( \beta_1 \) and \( \beta_2 \) are constants. This kind of potential was previously investigated by a number of authors both in the context of FRW scalar field cosmologies and EMd black holes. This generalizes further the potential (34). If \( \beta_1 = \beta_2 \), then (53) reduces to (34), so we will not repeat these solutions. Requiring \( \beta_1 \neq \beta_2 \), one obtains

\[
U(r) = \frac{r^{2-2N}}{N} \left( 1 - 2A - \Lambda_1 - \frac{2M}{r} + \frac{\Lambda_2(1 + \alpha^2)}{1 - 3\alpha^2} r^{2(2N - 1)} \right).
\] (54)

In order to fully satisfy the system of equations, the \( \beta_1 \) and \( \beta_2 \) parameters must satisfy \( \beta_1 = 1/\beta_2 = -\alpha \), and \( q \) parameter should be satisfy in eq. (40), with replacing \( \Lambda \rightarrow \Lambda_1 \). Note that the solution is ill defined for \( \alpha^2 = 1/3 \). In the particular case \( \Lambda_2 = 0 \), this solution reduces to (39) and when \( \Lambda_1 = 0 \), it reduces to (43). The Kretschmann invariant and Ricci scalar, diverge at \( r = 0 \), and both of them vanish as \( r \) goes to infinity, so there is a singularity located at \( r = 0 \). Another solution with the same spacetime metric is generated via the discrete transformation \( \beta_1 \leftrightarrow \beta_2 \) and \( \Lambda_1 \leftrightarrow \Lambda_2 \).
In order to investigate the causal structure of the solution and subsequently find the horizons (similar to what was done in the previous section) we find the zeros of the function

\[ f(r) = (1 - 2A - \Lambda_1) - \frac{2M}{r} + \frac{\Lambda_2(1 + \alpha^2)}{1 - 3\alpha^2} r^{4N - 2}, \]  

(55)

Again, we investigate the function \( g(r) = rf(r) \), for simplicity. The cases with \( \alpha^2 > 1/3 \) and \( \alpha^2 < 1/3 \) should be considered separately. We should also consider the sign of the parameter \( \Lambda_1 \) and \( \Lambda_2 \).

For the first case, where \( \alpha^2 > 1/3 \), we certainly govern extremum if \( \Lambda_1 > 1/2(\Lambda_1 < 1/2) \) and \( \Lambda_2 < 0(\Lambda_2 > 0) \). The sign of second derivative will show whether we have local minimum or maximum. Here, for \( 1/3 < \alpha^2 < 1(\alpha^2 > 1) \) and \( \Lambda_2 > 0(\Lambda_2 < 0) \) the function \( f(r) \) would have local minimum and in opposite, for \( 1/3 < \alpha^2 < 1(\alpha^2 > 1) \) and \( \Lambda_2 < 0(\Lambda_2 > 0) \) the function \( g(r) \) will have local maximum at

\[ r_{\text{min(max)}} = \left( \frac{1 - 2A - \Lambda_1}{\Lambda_2} \right)^{\frac{\alpha^2 + 1}{2(\alpha^2 - 1)}}. \]  

(56)

The value of the function \( g(r) \) at its extremum is

\[ g(r_{\text{ext}}) = -2M + 2\left( \frac{1 - 2A - \Lambda_1}{1 - 3\alpha^2} \right) \left( 1 - \frac{2A - \Lambda_1}{\Lambda_2} \right)^{\frac{\alpha^2 + 1}{2(\alpha^2 - 1)}}. \]  

(57)

In order to have any horizon, \( g(r_{\text{min}})[g(r_{\text{max}})] \) should be larger(less) than or equal to zero in order to possess any local extremum and subsequently to have any horizon for the black hole. The case \( g(r_{\text{min}})[g(r_{\text{max}})] = 0 \) corresponds to an extremal black hole. The condition \( g(r_{\text{min}}) < 0 \) gives

\[ M > \frac{(1 - 2A - \Lambda_1)(1 - \alpha^2)}{1 - 3\alpha^2} \left( 1 - \frac{2A - \Lambda_1}{\Lambda_2} \right)^{\frac{\alpha^2 + 1}{2(\alpha^2 - 1)}}. \]  

(58)

and we obtain the following inequality for the condition \( g(r_{\text{max}}) > 0 \)

\[ M < \frac{(1 - 2A - \Lambda_1)(1 - \alpha^2)}{1 - 3\alpha^2} \left( 1 - \frac{2A - \Lambda_1}{\Lambda_2} \right)^{\frac{\alpha^2 + 1}{2(\alpha^2 - 1)}}. \]  

(59)

For the second case where \( \alpha^2 < 1/3 \) the function \( g(r) \) possess local minimum for \( \Lambda_2 < 0 \) and local maximum in case of \( \Lambda_2 > 0 \). In this case, the function diverges both at \( r = 0 \) and at infinity. The local minimum(maximum) happens at (56) and the value of the function \( g(r_{\text{min}}) \) is given by (57). Since in this case we have both a local minimum and maximum, condition (59), should hold for both cases.

We see that in both cases we obtain horizons for any given value of the parameter \( \alpha \). Here we express the the radius of the extremal solution like the preceding section

\[ r_{\text{ext}} = \left( \frac{1 - 2A - \Lambda_1}{\Lambda_2} \right)^{\frac{\alpha^2 + 1}{2(\alpha^2 - 1)}} \left( 1 - \frac{3\alpha^2}{1 - 2A - \Lambda_1} \right) M_{\text{ext}} \]  

(60)
Unfortunately, because of the nature of the exponents of $r$ in (45), the event horizon determined by $g(r) = 0$ can not be expressed in a closed analytical form for arbitrary $\alpha$.

VI. CONCLUSION

Born-Infeld theory and dilaton gravity are well-motivated and extensively studied theories, not only separately, and also coupled to each other. In this paper, we derived some classes of exact, electrically charged, static and spherically symmetric black hole solutions to four dimensional Einstein-Born-Infeld-dilaton gravity without potential or with one or two Liouville type potentials. The black hole solutions have unusual asymptotics. They are neither asymptotically flat nor asymptotically (anti-) de Sitter. In particular, in the case of the linear electrodynamics with $\mathcal{L}(Y) = -\frac{1}{2}Y$ the method presented here gives the well-known asymptotically non-flat and non-(A)dS black hole solutions of the EMd gravity \cite{6}. We showed that in the presence of dilaton field, both Kretschmann invariant and Ricci scalar diverge at $r = 0$, they remain finite for $r \neq 0$ and tend to zero as $r \to \infty$. Thus we have an essential singularity located at $r = 0$. We found that in the presence of Liouville type potentials for dilaton field, there exist two classes of solutions which satisfy all the field equations depending on suitable choice of the $\beta$ parameter. We also computed -for each case- temperature, entropy, charge and mass of the black hole solutions, and find that these quantities satisfy the first law of thermodynamics. We found that in order to fully satisfy all the field equations consistently, there must be a relation between the electric charge and other parameters of the system. In general, the electric charge depends on the three parameters $\gamma$, $\alpha$ and $\Lambda$. We found that in the large $\gamma$ and $\alpha$ limit, our solutions reduce to Schwarzschild, and Schwarzschild ds/Ads black holes, depending on the sign of $\Lambda$.

Finally, it should be noted that the solutions were based on an ansatz and consistency checks demanded a relationship between the parameters of the theory. An attempt for finding exact solutions of EBId gravity by relaxing the ansatz \cite{16}, is under investigation. Note that the four dimensional EBId black hole solutions obtained here are static. Therefore, it would be interesting if one could construct charged rotating solutions of EBId gravity in four dimensions. One can also attempt to construct static and rotating solutions of the EBId gravity with both flat and curved horizons in various dimensions.
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