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Max-out-in pivot rule with Dantzig’s safeguarding rule for the simplex method

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Abstract. The simplex method is used to solve linear programming problem by improving the current basic feasible solution. It uses a pivot rule to guide the search in the feasible region. The pivot rule is used to select an entering index in simplex method. Nowadays, many pivot rule have been presented, but no pivot rule shows superior performance than other. Therefore, this is still an active research in linear programming. In this research, we present the max-out-in pivot rule with Dantzig’s safeguarding for simplex method. This rule is based on maximum improvement of objective value of the current basic feasible point similar to the Dantzig’s rule. We can illustrate by Klee and Minty problems that our rule outperforms that of Dantzig’s rule by the number of iterations for solving linear programming problems.

Keywords: Linear programming, Simplex method, Pivot rule, Dantzig’s rule

1. Introduction

The simplex method is the solution method for solving a linear programming problem (LP) (Dantzig, 1949). This method starts at a feasible solution of a feasible region and moves to the better solution by increasing value of a nonbasic variable called the entering variable and decreasing value of a basic variable called the leaving variable [2]. The criteria to select the entering and leaving variable is called the pivot rule. Many researchers have attempted to improve the simplex performance, for examples, see Pan (1990) [9], Junior and Lins (2005) [4], Hu (2007) [5] and Arsham (2007) [3]. The pivot rule affects the simplex performance. So a lot of researches have been proposed new pivot rules such as Devex rule [7], Steepest-edge rule [1, 6], and a largest-distance pivot rule [8].

In this paper, we propose the out-in pivot rule called max-out-in pivot rule safeguarding with Dantzig’s rule for the simplex method. This rule is first selecting the leaving variable that has the maximum value from the current basic variable set. Then it chooses the best corresponding entering variable that gives the smallest positive contribution to the binding constraint of the leaving variable. If the selected basic variable and nonbasic variable can not swap or there is no corresponding nonbasic candidate, then we use Dantzig’s rule instead. We can show by Klee and Minty problem [10] that our method improves Dantzig’s rule.

This paper is organized as follows. In section 2, we explain the main idea of our pivot rule and give the max-out-in pivot rule. We give the example to illustrate the implementation our rule in section 3. The application of our rule to Klee and Minty problem is given in section 4. Finally, the conclusion is given in section 5.
2. Max-Out-In Pivot Rule

2.1. The Main Idea

Consider the linear programming problem in the following form:

\[
\begin{align*}
\text{Maximize} & \quad 1^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]  

(7)

where \( b \in \mathbb{R}^m \), \( 1 \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} (m < n) \) and \( \text{rank}(A) = m \). In this form, the value of each decision variable is equal to its contribution to the objective value. Since the leaving variable decreases to zero when it changes to the nonbasic variable, and the increment in value of the entering variable depends on the decrement in value of the leaving variable. Then we should select the leaving variable that has the maximum value among all basic variables and select the nonbasic variable that can improve the objective value and allows the maximum increase.

Rewrite the system (7) into the tableau

\[
\begin{array}{cccc}
 x_B & x_N & \text{RHS} \\
\hline
 0^T & c_B^T B^{-1} N - c_N^T & c_B^T B^{-1} b \\
 I_m & A = B^{-1} N & b \\
\end{array}
\]

where \( I_m \) is an identity matrix of size \( m \), \( A = (\bar{a}_{ij}) \in \mathbb{R}^{m \times (n-m)} \), \( c_B = 1 \in \mathbb{R}^m \), \( c_N = 1 \in \mathbb{R}^{n-m} \), \( \bar{b} = B^{-1} b \in \mathbb{R}^m \), \( \bar{b} \geq 0 \) and \( 0 \in \mathbb{R}^m \). If we select \( x_r \) to be the leaving variable. Consider the binding constraint of \( x_r \):

\[
x_r + \sum_{i \in I_N \setminus J} \bar{a}_{ri} x_i + \sum_{j \in J} \bar{a}_{rj} x_j = \bar{b}_r
\]

(8)

In order to improve objective value, we select the entering variable form the set \( J \) that allows the maximum increase. We select the leaving variable \( x_j \) such that

\[
\tilde{j} = \text{Argmin} \{ \bar{a}_{rj} | \bar{a}_{rj} > 0, j \in J \}.
\]

2.2. Max-Out-In Pivot Rule (with Dantzig’s rule safeguarding)

From our main idea, we state our proposed pivot rule called the max-out-in pivot rule as following.

Max-out-in pivot rule: If \( J \neq \phi \).

Select \( x_r \) to leave the basic by \( x_r = \max \{ x_i | i \in I_B \} \).

Select \( x_j \) to enter the basic by \( \tilde{j} = \text{Argmin} \{ \bar{a}_{rj} | \bar{a}_{rj} > 0, j \in J \} \).

When it performs the max-out-in pivot rule, there are the possible two cases.

Case 1. One basic variable \( x_r \) and one nonbasic variable \( x_j \) can be swapped.

Case 2. The max-out-in pivot rule can not be used.

- If the selected basic variable violates the minimum ratio test.
- If no corresponding nonbasic variable exists.

In case 1, we can use the max-out-in pivot rule to perform the pivot step. However, we cannot perform the pivot step if case 2 occurs. We apply Dantzig’s rule as the safeguarding rule instead.
3. Illustrative Examples
In this section, we give the example to illustrate the implementation of the proposed method.

Example 1. Consider the following problem:

$$\text{Maximize } -86x_1 - 6x_2 - 28x_3 + 2x_4 + 5x_5$$
subject to: $$54x_1 + 84x_2 + 46x_3 + 16x_4 + 90x_5 \leq 14000$$
$$51x_1 + 59x_2 + 34x_3 + 3x_4 + 94x_5 \leq 12000$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$ 

Replacing $$x_j$$ by $$|c(j)|\tilde{x}_j$$ for $$j = 1, 2, \ldots, 5,$$ and let $$\tilde{x}_6$$ and $$\tilde{x}_7$$ be slack variables associated with the first and the second constraint, respectively. Then the initial tableau for above problem is:

|   | $$\tilde{x}_1$$ | $$\tilde{x}_2$$ | $$\tilde{x}_3$$ | $$\tilde{x}_4$$ | $$\tilde{x}_5$$ | $$\tilde{x}_6$$ | $$\tilde{x}_7$$ | RHS |
|---|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-----|
| z | 1              | 1              | 1              | -1             | -1             | 0              | 0              | 0   |
| $$x_6$$ | 27/43        | 14/14          | 23             | 8              | 18             | 1              | 0              | 14000 |
| $$x_7$$ | 51/86        | 59/14          | 17             | 3              | 94             | 0              | 1              | 12000 |

Since the maximum value among all basic variables is at $$x_6 = 14000$$ corresponding to $$r = 1.$$ Then $$\{\tilde{a}_{ij} \mid j \in J \text{ and } \tilde{a}_{ij} > 0\} = \{\tilde{a}_{14}, \tilde{a}_{15}\} = \{8, 18\}.$$ Then $$4 = \text{Argmin}\{\tilde{a}_{14}, \tilde{a}_{15}\}.$$ Since $$\{\frac{b_k}{\tilde{a}_{k4}} > 0, \text{for } k = 1, 2\} = \{\frac{b_1}{\tilde{a}_{14}}, \frac{b_2}{\tilde{a}_{24}}\} = \{1750, 8000\}.$$ From case 1, we select $$x_6$$ to be the leaving variable and select $$x_4$$ to be the entering variable. After pivoting, the simplex tableau is:

|   | $$\tilde{x}_1$$ | $$\tilde{x}_2$$ | $$\tilde{x}_3$$ | $$\tilde{x}_4$$ | $$\tilde{x}_5$$ | $$\tilde{x}_6$$ | $$\tilde{x}_7$$ | RHS |
|---|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-----|
| z | 371/344        | 11/4           | 135/112        | 0              | 5              | 1              | 0              | 1750 |
| $$x_4$$ | 27/344        | 7/4            | 23/112         | 1              | 9              | 1750           | 0              | 1750 |
| $$x_6$$ | 327/688       | 173/24         | 29/32          | 617/40         | -3             | 1              | 4              | 9375 |

This is the optimal tableau. The optimal solution is $$\tilde{x}_4 = 1750,$$ $$\tilde{x}_j = 0$$ for $$j = 1, \ldots, 5$$ and $$j \neq 4.$$ Then we get $$x_4 = 875,$$ $$x_j = 0$$ for $$j = 1, \ldots, 5$$ and $$j \neq 4$$ with the optimal value 1750 and the number of iteration is 1. By using the simplex method with Dantzig’s pivot rule, the number of iterations is 3.

4. Application of the max-out-in pivot rule to Klee and Minty problem
Klee and Minty proposed some linear programming problems that the simplex method has exponential worst-case running time in 1972[10]. This problem is called the Klee and Minty problem, which is stated as below.

Klee and Minty problem:

$$\text{Maximize } \sum_{j=1}^{n} 10^{n-j}x_j$$

subject to: $$2\sum_{j=1}^{n} 10^{i-j}x_j + x_i \leq 100^{i-1}, i = 1, \ldots, n$$ (9)
$$x_i \geq 0, i = 1, \ldots, n$$
The simplex method with Dantzig’s pivot rule requires $2^n - 1$ iterations to solve Klee and Minty problem [10]. Next, we show that our rule obtains the optimal solution in 1 iteration.

We first normalize the objective coefficients to be $+1$ or $0$ or $-1$ and we let $x_{n+1}, \ldots, x_{2n}$ be slack variables associate with the first to the $n^{th}$ constraint, respectively. Rewrite this problem into the tableau

|       | $x_1$ | $\cdots$ | $x_n$ | $x_{n+1}$ | $\cdots$ | $x_{2n}$ | RHS  |
|-------|-------|----------|-------|-----------|----------|----------|------|
| $z$   | $-1$  | $\cdots$| $-1$  | $0$       | $\cdots$| $0$       | $0$  |
| $x_{n+1}$ | \[10^{n-r} \] | $\cdots$| $0$  | $1$       | $\cdots$| $0$       | $1$  |
| $\vdots$ |       |         |      |           | $\vdots$|          |      |
| $x_{2n}$ | $2$  | $2$   | $1$  | $0$       | $\cdots$| $1$       | $10^{n-1}$  |

By using the max-out-in pivot rule, $x_{2n}$ is selected to be the leaving variable and $x_n$ is selected to be the entering variable.

|       | $x_1$ | $\cdots$ | $x_n$ | $x_{n+1}$ | $\cdots$ | $x_{2n}$ | RHS  |
|-------|-------|----------|-------|-----------|----------|----------|------|
| $z$   | $1$  | $1$   | $0$  | $0$       | $\cdots$| $0$       | $10^{n-1}$  |
| $x_{n+1}$ | \[10^{n-r} \] | $\cdots$| $0$  | $1$       | $\cdots$| $0$       | $1$  |
| $\vdots$ |       |         |      |           | $\vdots$|          |      |
| $x_{2n}$ | $2$  | $2$   | $1$  | $0$       | $\cdots$| $1$       | $10^{n-1}$  |

This is the optimal tableau. Then the optimal solution is obtained in 1 iteration for Klee and Minty problem of any size $n$.

5. Conclusions

The objective of this paper is to propose a new pivot rule called the max-out-in pivot rule safeguarding with Dantzig’s rule for the simplex method. This rule first normalizes the objective coefficients to be $+1$ or $0$ or $-1$. The keys of this rule are that the maximum basic variable is selected to leave basis, and the corresponding nonbasic variable which allowed the maximum increase is selected to enter basis. We can illustrate by the examples that the max-out-in pivot rule can achieve good results. In addition, our rule is better than Dantzig’s rule over the Klee and Minty problems [10]. Further work will be done for the improvement of the max-out-in pivot rule.

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