ON CYCLIC HOMOLOGY OF $A_\infty$-ALGEBRAS

Masoud Khalkhali

The category of $A_\infty$-algebras extend the category of differential graded (DG) algebras. The main result of the present paper asserts that the periodic cyclic homology $HP_\ast(A,m)$ of an $A_\infty$-algebra $(A,m)$ is equal to ordinary periodic cyclic homology $HP_\ast(H_0(A))$ of the homology of $(A,m_1)$ in degree zero. This result extends a well known result of T. Goodwillie [Go] for the periodic cyclic homology of DG algebras. We notice, however, that while the study of, at least certain aspects of, cyclic homology of DG algebras can be “reduced” to the study of cyclic homology of algebras by carefully adapting the algebra case to cyclic objects in the category of chain complexes as in [Go], no such technique is available for $A_\infty$-algebras. In fact, although this is perhaps possible, and certainly interesting to try, we are not aware of any generalization of Connes’ cyclic category [C3] to a hypothetical “$\infty$-cyclic category”, so that an $A_\infty$-algebra yields a “$\infty$-cyclic” object.

Hochschild and cyclic homology of $A_\infty$-algebras were first defined by Getzler and Jones in [GJ], where a (b,B) type bicomplex was defined for an $A_\infty$-algebra. Our approach is however different and is based on ideas of Cuntz and Quillen [CQ1, CQ2, Q] and especially their X-complex approach to cyclic homology. In fact this is crucial for the Cartan homotopy formula that we need. As is evident in the present paper, this approach allows a unified treatment of cyclic homology type theories for various algebraic structures. A key element in the proof of our main theorem is a Cartan homotopy formula for quasifree DG (co)algebras from [Kh], which in turn is a generalization of the Cartan homotopy formula of Cuntz and Quillen [CQ1].

The concepts of $A_\infty$-spaces and $A_\infty$-algebras are due to Stasheff in [S], where it is shown that a topological space has the homotopy type of a (based) loop space if and only if it is an $A_\infty$-space. The Moore model of a loop space has the extra property that it has a strictly associative product and hence its singular chains is a DG algebra. This is used in [Go] to identify the $SO(2)$-equivariant homology of the free loop space with the cyclic homology of based loop space. It follows from Proposition 2.3 that this is indeed true for any model of the loop space.

Apart from applications in [GJ] to the $SO(2)$-equivariant version of Chen’s iterated integrals, the new surge of interest in $A_\infty$-algebras goes back in part to a paper of M. Kontsevich [K], where it is shown that cyclic cohomology classes in degree zero of any $A_\infty$-algebra can be used to construct homology classes on the moduli spaces of algebraic curves (see also [PS] where this construction is further explained).

* The author is grateful to J. Williams for typing the manuscript. The author is supported by NSERC of Canada.
1 The bar construction for $A_\infty$-algebras

The relevance of bar construction to the study of cyclic homology of associative algebras is explained in [Q]. Since a similar construction is available for $A_\infty$-algebras [S,GJ], it is natural to expect that it would play a similar role for Hochschild and cyclic homology of $A_\infty$-algebras. In general, the bar construction makes the definitions and the identities to be satisfied by various operators in an $A_\infty$-algebra setting completely natural.

Let us fix our notations and recall some elementary concepts from “graded mathematics”. Throughout this paper we work over a field $K$ of characteristic zero and all homomorphisms and tensor products are over $K$. If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded vector space, we denote by $A[n]$ the shifted graded vector space defined by $A[n]_i = A_{i-n}$, $i \in \mathbb{Z}$. We have $A[n][m] = A[n+m]$. The degree of a homogeneous element $a$ in a graded space is denoted by $|a|$. In case of several gradings, we will make the required grading explicit.

A homogeneous map $f : A \to B$ between graded spaces has degree $k \in \mathbb{Z}$, if $f(A_i) \subset B_{i+k}$ $\forall \ i \in \mathbb{Z}$. We have

$$\text{Hom}^k(A,B) = \prod_{i \in \mathbb{Z}} \text{Hom}(A_i,B_{i+k}) .$$

The tensor product of graded spaces $A$ and $B$ is defined by $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$. If $f : A \to A'$, $g : B \to B'$ are graded maps, then their graded tensor product $f \otimes g : A \otimes B \to A' \otimes B'$ is defined by

$$(f \otimes g)(a \otimes b) = (-1)^{|a||g|}f(a) \otimes g(b) .$$

Note that $|f \otimes g| = |f| + |g|$. The graded twist $\sigma : A \otimes B \to B \otimes A$ is defined by $\sigma(a \otimes b) = (-1)^{|a||b|}b \otimes a$. We will denote a tensor $a_1 \otimes \cdots \otimes a_n$ in $A^\otimes n$ by $(a_1, \ldots, a_n)$. Note that with our conventions, $|(a_1, \ldots, a_n)| = \sum_{i=1}^n |a_i|$ and hence the r-th components of $A^\otimes n$ is given by

$$(A^\otimes n)_r = \{(a_1, \ldots, a_s) \mid \sum |a_i| = r\} = \bigoplus_{s \geq 0} \bigoplus_{i_1 + \cdots + i_s = r} A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_s} .$$

Let $T^c A$ denote the cofree coaugmented graded coalgebra generated by the positively graded vector space $A = \bigoplus_{i \geq 0} A_i$. We have $(T^c A)_n = A^\otimes n$. Explicitly,

$$(T^c A)_0 = K \oplus A_0 \oplus A_0^\otimes 2 \oplus A_0^\otimes 3 \oplus \ldots$$

$$(T^c A)_1 = A_1 \oplus A_0 \otimes A_1 \oplus A_1 \otimes A_0 \oplus \ldots$$

$$(T^c A)_2 = A_2 \oplus A_1 \otimes A_1 \oplus A_0 \otimes A_1 \oplus A_1 \otimes A_0 \oplus \ldots$$

$\vdots$

The coproduct $\Delta : T^c A \to T^c A \otimes T^c A$, which is a degree zero map, is defined by $\Delta(1) = 1 \otimes 1$ and

$$\Delta(a_1, \ldots, a_n) = 1 \otimes (a_1, \ldots, a_n) + \sum_{i=1}^n (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n) + (a_1, \ldots, a_n) \otimes 1 .$$
Let $C$ be a graded counital coalgebra. By a graded $C$-bicomodule we mean a graded vector space $M$ such that the left and right coactions $\Delta_l : C \to C \otimes M$ and $\Delta_r : C \to M \otimes C$ are of degree zero. We further assume that our bicomuldes are counitary. A graded coderivation of degree $k \in \mathbb{Z}$ is a degree $k$ map $\delta : M \to C$ such that

$$(1 \otimes \delta)\Delta + (\delta \otimes 1)\Delta_r = \Delta \delta.$$

A universal coderivation of degree $k$ consists of the following data: A $C$-bicomodule $\Omega^1_k C$ and a graded coderivation of degree $k$, $d_k : \Omega^1_k C \to C$ which is universal. That is, for any degree $k$ coderivation $\delta : M \to C$, there exists a unique degree zero $C$-bicomodule map $m : M \to \Omega^1_k C$ such that $\delta = d_k \circ m$. We refer to $\Omega^1_k C$ as the comodule of universal codifferentials over $C$.

It is not difficult to see that universal graded coderivations of any degree exist. One can simply define $\Omega^1_k C = \text{coker}\{\Delta_k : C[-k] \to C \otimes C\}$ with its $C$-bicomodule structure induced from $C \otimes C$. Here $\Delta_k$ is the composition $C[-k] \to C \xrightarrow{\Delta} C \otimes C$ and $d_k = \eta \otimes 1 - 1 \otimes \eta$, where $\eta : C \to K$ is the counit.

In the cofree case $C = T^c A$, this universal coderivation can be identified as follows. Let $\Omega^1_k T^c A = T^c A \otimes A[-k] \otimes T^c A$ be the free bicomodule over $T^c A$ generated by $A[-k]$. Define $d_k : \Omega^1_k T^c A \to T^c A$ by

$$d_k(\alpha \otimes a \otimes \beta) = (-1)^{|\alpha|}(\alpha, a, \beta),$$

where $\alpha, \beta \in T^c A$ and $a \in A[-k]$. It is not difficult to see that $d_k$ is a degree $k$ coderivation and moreover it is universal.

If in the adjunction formula

$$\text{Coder}^k(M, T^c A) \simeq \text{Hom}(M, \Omega^1_k T^c A),$$

we take $M = C$, with its natural bocomodule structure, we obtain, for $k \in \mathbb{Z}$,

$$\text{Coder}^k(T^c A, T^c A) \simeq \text{Hom}^k(T^c A, \Omega^1_k T^c A).$$

However, $\Omega^1_k T^c A = T^c A \otimes A[-k] \otimes T^c A$, is a cofree $T^c A$-bicomodule, and hence

$$\text{Hom}^k(T^c A, \Omega^1_k T^c A) \simeq \text{Hom}^0(T^c A, A[-k]) \simeq \text{Hom}^k(T^c A, A),$$

so, that we have an isomorphism of graded vectors spaces

$$\text{Coder}(T^c A, T^c A) \simeq \text{Hom}(T^c A, A). \tag{1}$$

This isomorphism works as follows. A graded coderivation $b' : T^c A \to T^c A$ of degree $k$ defines a degree $k$ map $m_{b'} : T^c A \to A$ as the composition

$$T^c A \xrightarrow{b'} T^c A \longrightarrow A,$$
where the last map is the projection. Conversely, given a degree $k$ map $m : T^c A \to A$, define a degree $k$ coderivation $b'_m : T^c A \to T^c A$ by

$$b'_m = d_k(1 \otimes m \otimes 1) (1 \otimes \Delta) \Delta.$$  

Chasing the formulas for $\Delta$ and $d_k$, we obtain the formula for $b'_m$. We have $b'_m(1) = 0$ and

$$b'_m(a_1, \ldots, a_n) = \sum_{i=1}^n b'_{m_i}(a_1, \ldots, a_n)$$

$$= \sum_{i=1}^n \left( \sum_{j=1}^{n-i} (-1)^{\varepsilon_{ij}} (a_1, \ldots, m_i(a_j, \ldots, a_{j+i}), \ldots, a_n) \right).$$

Finally, note that $T^c A$ has also a universal property with respect to morphisms of graded coalgebras. Namely, for any graded coaugmented counital coalgebra $C$, we have an isomorphism of graded vector spaces

$$\text{Hom}^{GC}(C, T^c A) \simeq \text{Hom}(C, A),$$

where on the left hand side $\text{Hom}^{GC}$ means coaugmented graded coalgebra morphisms.

Under this isomorphism, a graded linear map $f : C \to A$ of degree $k$ defines a coalgebra map $\hat{f} : C \to T^c A$, where its degree $n$ component is given by $\hat{f}_n = f^\otimes n \circ \Delta^{(n)}$. Here $\Delta^{(n)}$ denotes the $n^{th}$ iteration of the comultiplication $\Delta$.

Next we turn to the cofree coalgebra $C = T^c A[1]$, generated by the suspension $A[1]$, where, as before $A$ is positively graded. The reason for this is twofold. First of all, similar to algebra case [Q], various cyclic bicomplexes for an $A_\infty$-algebra are obtained from $T^c A[1]$. Secondly, and independently, the all important Gerstenhaber product that we are going to define preserves only the total grading (= length + degree), which is the same as the grading in $A[1]$. We have

$$C_0 = K$$
$$C_1 = A_0,$$
$$C_2 = A_1 \oplus A_0 \otimes A_0,$$
$$C_3 = A_2 \oplus A_0 \otimes A_1 \oplus A_1 \otimes A_0,$$
$$\vdots$$

In general

$$(T^c A[1])_n = \bigoplus_{r=0}^n \bigoplus_{i_1+\cdots+i_r=r} A_{i_1} \otimes \cdots \otimes A_{i_r}.$$  

A linear map $m : T^c A[1] \to A[1]$ of degree $k$ such that $m(1) = 0$ is given by a sequence of linear maps

$$m_n : A^\otimes n \to A \quad n = 1, 2, \ldots$$
such that $|m_n| = n - 1 + k$. We refer to $k$ as the suspended degree of $m$.

The Gerstenhaber product (see [G] for the non-graded version) is a degree zero non-
associative product

$$Hom(T^c A[1], A[1]) \otimes Hom(T^c A[1], A[1]) \to Hom(T^c A[1], A[1])$$

defined as follows. First, for $m : A^{\otimes k+1} \to A$, $m' : A^{\otimes \ell+1} \to A$, with suspended degrees $|m|$ and $|m'|$, define

$$m \circ m' : A^{\otimes k+\ell+1} \to A,$$

by

$$(m \circ m')(a_1, \ldots, a_{k+\ell+1}) = \sum_{i=1}^k (-1)^{\varepsilon_i} m(a_1, \ldots, a_{i-1}, m'(a_i, \ldots, a_{i+\ell}), \ldots, a_{k+\ell+1}),$$

where $\varepsilon_i = |m'| \left( \sum_{j=1}^{i-1} |a_j| + i - 1 \right)$. Note that we have the equality of suspended degrees

$$|m \circ m'| = |m| + |m'|.$$ We extend this product to arbitrary cochains $m = \sum_{i=1}^\infty m_i$ and

$$m' = \sum_{i=1}^\infty m'_i \in Hom(T^c A[1], A[1])$$

by

$$m \circ m' = \sum_{n=2}^\infty \left( \sum_{i+j=n} m_i \circ m_j \right).$$

In [G], where the non-graded case is treated, Gerstenhaber proves that $[m, m'] := m \circ m' - (-1)^{|m|} m' \circ m$ is a graded Lie bracket on $Hom(T^c A[1], A[1])$. This is a surprise, given the fact that $\circ$ is not associative. This led to investigating the full structure of higher homotopies that is hidden here [GV] (see also section 3 of the present article). The Lie algebra structure itself, however, can be understood using isomorphism (1). In fact, since the bracket of two (graded) (co)derivation is again a (graded)(co)derivation, the Lie structure on the left hand side of (1) is obvious and hence suffices to show that (1) preserves the brackets.

Now a map $m : T^c A[1] \to A[1]$ of degree $-1$ defines a degree $-1$ coderivation

$$b'_m : T^c A[1] \to T^c A[1].$$

We have $b'^2_m = \frac{1}{2}[b'_m, b'_m] = \frac{1}{2} b''_{[m,m]} = b'_{[m,m]}$, so that $b'_m$ is a codifferential iff $m \circ m = 0$. Writing $m = \sum_{i=1}^\infty m_i$, we have $m \circ m = 0$ iff

$$\sum_{i+j=n} m_i \circ m_j = 0 \quad n = 2, 3, \ldots.$$
An $A_\infty$-algebra structure (also called strongly homotopy associative algebra structure) on a graded vector space $A = \bigoplus_{i=0}^{\infty} A_i$, is a degree $-1$ map $m : T^c A[1] \to A[1]$ such that $m \circ m = 0$. Equivalently, it is defined by a coderivation $b'_m : T^c A[1] \to T^c A[1]$ of degree $-1$, such that $b^2_m = 0$. This concept is due to Stasheff [S].

Note that if $(A, m)$ is an $A_\infty$-algebra, the homotopy associative product induces a strictly associative product on $H_\bullet(A, m) := H_\bullet(A, m_1) = \bigoplus_{i \geq 0} H_i(A, m_1)$ and turns it into an associative graded algebra. In particular $H_0(A, m_1)$ is an associative algebra.

A morphism $(A, m) \to (B, m)$ of $A_\infty$-algebras is, by definition [GJ], a morphism $(T^c A[1], b'_m) \to (T^c B[1], b'_m)$ of the corresponding coaugmented DG coalgebras. The universal property (2) has an obvious extension from graded vector spaces to complexes. Using this, we see that there is a 1-1 correspondence between morphisms of $A_\infty$-algebras $(A, m) \to (B, m)$ and morphisms of complexes $f : (T^c A[1], b'_m) \to (B[1], m_1)$.

This notion of morphism between $A_\infty$-algebras may seem too general, but, as we will see, all homological invariants that we construct are in fact functorial with respect to these morphisms. A strict morphism $(A, m) \to (B, m)$ of $A_\infty$-algebras [GJ] is a graded linear map $f : A \to B$ commuting with defining cochains of $A$ and $B$, that is

$$m_n(f(a_1), \ldots, f(a_n)) = f(m_n(a_1, \ldots, a_n))$$

for all $n$ and all $a_i \in A$. Note that while associative algebras form a full subcategory of the category of $A_\infty$-algebras, the inclusion of DG algebras into the category of $A_\infty$-algebras is not full.

2 The $X$-machine

In this section we derive a bicomplex for $A_\infty$-algebras from which every other (bi)complex to calculate various kinds of Hochschild and cyclic homology theories for $A_\infty$-algebras can be defined. This bicomplex is the exact analogue of Connes-Tsygan bicomplex $[C_1, T]$, originally defined for associative algebras. The key idea here is to extend Quillen’s approach [Q] for associative algebras, to $A_\infty$-algebras. Once this bicomplex is defined, the rest of the homological algebra of $A_\infty$-algebras “follows the book”. In particular, the $(b, B)$-bicomplex, the $S$-coperation and Connes’ long exact sequence follow the same pattern as in cyclic homology $[C_1, C_2]$.

Our main tool to define the cyclic homology of an $A_\infty$-algebra is the $X$-complex. The $X$-complex of a (DG)(co) algebra $C$ is only a first approximation to its various cyclic homology theories. However, once $X$ is applied to certain universal functors on $C$ one obtains complexes
which are (quasi-) isomorphic to the standard complexes. One advantage of this approach is that these functors, like the bar construction, are defined for algebraic structures which are more flexible than (DG)(co) algebras.

We follow [Q] to define the $X$-complex of a DG coalgebra. Let $C$ be a counital coalgebra over a field $K$ of characteristic zero. Let $\Omega^1 C = \Omega^1_0 C$ denote the $C$-bicomodule of universal differential forms on $C$. Let

$$\Omega^1 C_x = \ker\{\Delta_x - \sigma\Delta : \Omega^1 C \to C \otimes \Omega^1 C\}$$

be the subspace of $C$-cocommutators in $\Omega^1 C$. Let $\partial_0 : \Omega^1 C_x \to C$ be the restriction of $d : \Omega^1 C \to C$ to the cocommutator subspace. Let $\partial_1 : C \to \Omega^1 C_x$ be the analogue of Hochschild boundary for coalgebras. The $X$-complex of $C$, denoted $X(C)$, is the following $\mathbb{Z}$-graded complex which is 2-periodic:

$$\ldots \xrightarrow{\partial_1} C \xrightarrow{\partial_0} \Omega^1 C_x \xrightarrow{\partial_1} C \xrightarrow{\partial_0} \ldots .$$

Next let $(C, b')$ be a counital DG coalgebra with $|b'| = -1$ and $C = \bigoplus_{i \geq 0} C_i$. Then one can repeat the above construction of the $X$-complex to define $X(C)$. This is a complex in the category of complexes, i.e. it is a bicomplex in the usual sense. In fact, more generally, if $\mathcal{C}$ is an abelian tensor category and $C$ is an algebra or coalgebra object in $\mathcal{C}$, then $X(C)$ is defined as a complex in $\mathcal{C}$.

For a DG (co) algebra $A$, we define three homologies $XH_\bullet(A)$, $XC_\bullet(A)$ and $XP_\bullet(A)$ by $XH_\bullet(A) = H_\bullet(\Omega^1 A^e)$, $XC_\bullet(A) = H_\bullet(Tot X^+(A))$ and $XP_\bullet(A) = H_\bullet(Tot X(A))$.

Because of the 2-periodicity in $X(A)$, we obtain a degree 2 map $S : XC_\bullet(A) \to XC_{\bullet-2}(A)$ and a long exact sequence

$$\to XC_n(A) \to XC_{n-2}(A) \to XH_{n-2}(A) \to XC_{n-1}(A) \to ,$$

similar to Connes’ long exact sequence.

A morphism $f : A \to B$ of DG (co) algebras is called an equivalence if the induced map $H_\bullet(A, d) \to H_\bullet(B, d)$ is an isomorphism.

2.1. Proposition. An equivalence $f : A \to B$ induces isomorphisms on $XH_\bullet$, $XC_\bullet$ and $XP_\bullet$.

In particular we can apply this construction to the DG coalgebra $C = (T^e A[1], b'_m)$, where $(A, m)$ is an $A_\infty$-algebra. The corresponding double complex, when $A$ is an associative algebra, is identified in [Q] and shown to be isomorphic to Connes-Tsygan bicomplex. In general, one obtains nothing new for the horizontal differentials $\partial_0$ and $\partial_1$ except extra signs since we are working with graded objects. They are given by $\partial_0 = 1 - \lambda$ and $\partial_1 = N = 1 + \lambda + \cdots + \lambda^n$, where $\lambda : A^{\otimes n+1} \to A^{\otimes n+1}$ is the cyclic shift

$$\lambda(a_0, \ldots, a_n) = (-1)^{\varepsilon}(a_n, a_0, \ldots, a_{n-1}) ,$$

where $\varepsilon = (|a_n| + 1) \left( \sum_{i=0}^{n-1} |a_i| + n \right)$.
Let the induced operator on $\Omega^1T^cA[1]$ be denoted by $B_m$. One obtains the following formula for $b_m$:

$$b_m(a_0, \ldots, a_n) = \sum_{i=0}^{n+1} b'_{m_i}(a_0, \ldots, a_n),$$

where

$$b_{m_i}(a_0, \ldots, a_n) = b'_m(a_0, \ldots, a_n) + \sum_{j=n-i+1}^n (-1)^j (m_i(a_j, \ldots, a_{i+j-n-2}), \ldots, a_{j-1}).$$

Let us denote this double complex by $CC(A,m)$, the part that is in the first quadrant by $CC^+(A,m)$ and the zeroth column by $C(A,m)$. We define the Hochschild homology of the $A_\infty$-algebra $(A,m)$ (with coefficients in $A$) as the homology of $C(A,m)$ and denote it by $HH_*(A,m)$. The cyclic homology of $(A,m)$ is defined as the homology of the total complex $Tot\ CC(A,m)$ and will be denoted by $HC_*(A,m)$. We define the periodic cyclic homology of $(A,m)$, denoted $HP_*(A,m)$, as the homology of $Tot\ CC(A,m)$, where $\hat{\lambda}$ means we take direct product instead of direct sum in the total complex. (The corresponding homology with direct sums is trivial in all degrees.)

Due to its periodicity, the complex $Tot\ CC(A,m)$ has a degree 2 morphism which induces a map $S : HC_*(A,m) \rightarrow HC_{*-2}(A,m)$. One has the analogue of Connes’ long exact sequence

$$\rightarrow HH_*(A,m) \overset{\lambda}{\rightarrow} HC_*(A,m) \overset{S}{\rightarrow} HC_{*-2}(A,m) \overset{B}{\rightarrow} HH_{*-1}(A,m) \rightarrow .$$

Let $C^n_\lambda(A,m) = \text{Coker}\{1 - \lambda : C_n(A) \rightarrow C_n(A)\}$. From the bicomplex relation $b_m(1 - \lambda) = (1 - \lambda)b'_m$ it is clear that $(C^n_\lambda(A,m), b_m)$ is a complex. Moreover, the natural projection

$$Tot\ CC^+(A,m) \rightarrow C^\lambda(A,m)$$

is a quasi-isomorphism.

Defining cyclic cohomology of $A_\infty$-algebras is straightforward. One should simply dualize the bicomplex $CC(A,m)$ by replacing tensor products by multilinear functionals. Let us identify $HC^0(A,m)$. A cocycle in $HC^0(A,m)$ is defined by a closed graded trace, that is a linear map $f : A \rightarrow k$ such that $f(ab - (-1)^{|a||b|}ba) = 0$ and $f(m_1a) = 0$ for all $a,b$ in $A$. Thus $HC^0(A,m)$ is isomorphic to the space of closed graded traces on $(A,m)$.

Next we turn to the analogue of Connes’ operator $B$ and in particular a $(b, B)$ bicomplex for $A_\infty$-algebras. This is already achieved in [GJ] and the relations

$$B^2 = b_mB + Bb_m = 0$$

verified. In our approach this comes about as follows. Let $(A,m)$ be a unital $A_\infty$-algebra. This means there exist an element $1 \in A_0$ such that $m_2(a,1) = m_2(1,a) = a$ for all $a \in A$ and $m_n(a_1, \ldots, a_{i-1}, 1, a_{i+1} \ldots a_n) = 0$ for $n \neq 2$ and all $a_i \in A$. Let $s : A^\otimes n \rightarrow A^\otimes n+1$ be the standard map $s(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$. Let $B = (1 - \lambda^{-1})sN : A^\otimes n \rightarrow A^\otimes n+1$. 

8
The relations $B^2 = b_mB + Bb_m = 0$ are consequences of bicomplex relations $N(1-\lambda) = (1-\lambda)N = 0$, $b_m(1-\lambda) = (1-\lambda)b'_m$, $b_m^2 = b'_m = 0$ and the relation $sb'_m + b'_ms = 0$.

The well-known homotopy equivalence between the cyclic and $(b, B)$-bicomplex carries over to the $A_\infty$-case verbatim. One can also consider a normalized $(b, B)$-bicomplex.

We need the following concept and the next proposition for the proof of theorem 4.4 in Sect. 4. A morphism $(A, m) \to (B, m)$ of $A_\infty$-algebras is said to be an equivalence if the corresponding map $(T^cA[1], b'_m) \to (T^cB[1], b'_m)$ is a quasi-isomorphism, i.e. induces an isomorphism $H_*(T^cA[1], b'_m) \xrightarrow{\sim} H_*(T^cB[1], b'_m)$.

2.2. Lemma. A strict morphism $(A, m) \to (B, m)$ of $A_\infty$-algebras is an equivalence iff the induced map $H_*(A, m_1) \to H_*(B, m_1)$ is an isomorphism.

2.3. Proposition. An equivalence $(A, m) \to (B, m)$ of $A_\infty$-algebras induces isomorphisms of $HH_*, HC_*$, and $HP_*$. 

Proof. This is a special case of prop. 2.1 for the DG coalgebra $T^cA[1]$ and $T^cB[1]$. 

3 Deformation theory of $A_\infty$-algebras

The link between deformations of an (associative, commutative, Lie, etc.) algebra $A$ and the (Hochschild, Harrison, Chevally-Eilenberg, etc.) cohomology of $A$ with coefficients in $A$ is well known. One knows that obstructions for extending a deformation in each order live in $H^3(A, A)$ and isomorphism classes of deformations are classified by $H^2(A, A)$. Moreover, in all of the above cases the Hochschild cohomology is a Gerstenhaber algebra, i.e., a graded Poisson algebra.

The link between cyclic cohomology of an associative algebra $A$ and its deformation theory was first elucidated in [CFS], where it is shown that if we restrict to deformations that preserve a trace (closed deformations), then the corresponding obstructions are in $HC^2(A)$.

In this section we define and study the deformation complex of an $A_\infty$-algebra much in the spirit of the rest of this paper. We also establish the link between cyclic cohomology of $A$ and its deformation complex. Apparently this fact is of importance in the cohomology of graph complexes [K,PS].

Let $(A, m)$ be an $A_\infty$-algebra. We can take

\[ C^\bullet(A, A) = \text{Hom}(T^cA[1], A[1]) \simeq \text{Coder}(T^cA[1], T^cA[1]) \]

as the underlying graded vector space to define the Hochschild cohomology $H^\bullet(A, A)$. Note that this is $\mathbb{Z}$-graded, although $A$ is only positively graded. Also a Lie bracket is defined on $C^\bullet(A, A)$. Define a differential

\[ \delta : C^\bullet(A, A) \to C^{\bullet-1}(A, A) \]

by $\delta x = [x, b'_m]$, where we are interpreting $x$ as a coderivation. From $b'_m^2 = 0$ and the Jacobi identity, it easily follows that $\delta^2 = 0$ and that $\delta$ is a graded derivation:

\[ \delta[x, y] = [\delta x, y] + (-1)^{|x|}[x, \delta y] , \]
i.e. \((C^\bullet(A, A), \delta, [\cdot, \cdot])\) is a differential \(\mathbb{Z}\)-graded Lie algebra. The formulas for the differential and brackets are as follows. Let \(f = \sum_{i=0}^{\infty} f_i\) be a degree \(k\) cochain. We have

\[
\delta f = \sum_{n=1}^{\infty} \sum_{i+j=n} (f_i \circ m_j - (-1)^j m_j \circ f_i)
\]

\[
[f_1, f_2] = \sum_{n=0}^{\infty} \sum_{i+j=n} (f_i \circ f_j - (-1)^j f_j \circ f_i)
\]

There is, however, more structure hidden in \(C^\bullet(A, A)\). We need the following simple lemma.

3.1. Lemma. Let \(C\) be a graded coalgebra and \((A, m)\) an \(A_\infty\)-algebra. Then there is a natural \(A_\infty\)-algebra structure on \(\text{Hom}(C, A)\).

Proof. Use “multiplications” on \(A\) and comultiplication on \(C\) to define cochains

\[
\tilde{m}_n : \text{Hom}(C, A)^{\otimes n} \to \text{Hom}(C, A), \ n \geq 1
\]

Let \(\Delta^n : C \to C^{\otimes n}\) be the \(n\)-th iteration of the comultiplication \(\Delta : C \to C \otimes C\) of \(C\). Let \(\tilde{m}_n(f_1, \ldots, f_n) = m_n \circ (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{(n)}\). Checking the \(A_\infty\) condition is straightforward.

The relation between the “cup product” and the Lie bracket, even in the case of associative algebras is quite subtle. Nevertheless, in \([G]\) it is shown that \(H^\bullet(A, A)\), for \(A\) an associative algebra, is a Gerstenhaber algebra. That is, the induced cup product in \(H^\bullet\) is graded commutative and is compatible with the induced Lie bracket in the sense that for any \(x \in H^\bullet\), the operator \(a \mapsto [a, x]\) is a graded derivation of the cup product.

To prove a similar result for \(A_\infty\)-algebras, we need the notion of homotopy Gerstenhaber algebra or \(G_\infty\)-algebra due to Gerstenhaber and Voronov \([GV]\). Let \((B, m)\) be an \(A_\infty\)-algebra. A \(G_\infty\)-structure on \((B, m)\) is an associative product \(T^e B[1] \times T^e B[1] \to T^e B[1]\) on the bar construction of \(A\) such that the codifferential \(b'_m\) is a graded derivation of this cup product.

3.2. Lemma. Let \((B, m)\) be a \(G_\infty\)-algebra. Then \(H_\bullet(B, m_1)\) is a Gerstenhaber algebra.

3.3. Corollary. Let \((A, m)\) be an \(A_\infty\)-algebra. Then the Hochschild cohomology \(H^\bullet(A, A)\) is a Gerstenhaber algebra.

At this stage we notice that it is straightforward to define formal deformations of \(A_\infty\)-algebras and link it with \(H^2(A, A)\) and \(H^3(A, A)\). Instead we link \(HC^\bullet(A)\) to deformations that preserve a trace, or, equivalently, an invariant bilinear form. (We are assuming \(A\) is unital.)

To illustrate, let us first consider the case where \(A\) is an associative algebra. We then have the pairings

\[
H^p(A, A) \otimes HH^q(A) \to HH^{p+q}(A)
\]

\[
H^k(A, A) \otimes HC^0(A) \to HC^m(A)
\]
The first map is induced by a morphism of complexes of degree zero
\[ C^\bullet(A, A) \otimes C^\bullet(A) \to C^\bullet(A) , \]
defined as follows. For \( \varphi : A^{\otimes p} \to A \) in \( C^p(A, A) \) and \( \tau : A^{\otimes q+1} \to K \) in \( C^q(A) \), define \( \tilde{\varphi} : A^{\otimes p+q+1} \to K \) by
\[ \tilde{\varphi}(a_0, a_1, \ldots, a_{p+q}) = \tau(a_0 \varphi(a_1, \ldots, a_p), a_{p+1}, \ldots, a_{p+q}) . \]
It is straightforward to check that \( \varphi \times \tau \mapsto \tilde{\varphi} \) is a morphism of complexes and hence the first pairing is defined.

To define the second pairing, we can interpret \( HC^\bullet(A) \) as the cohomology of \( \text{Tot} CC^+ (A) \) and define a morphism of complexes of degree zero
\[ C^\bullet(A, A) \otimes HC^0(A) \to \text{Tot} CC^+ (A) \]
as follows. For \( \tau : A \to K \) a trace on \( A \) and \( \varphi : A^{\otimes n} \to A \) in \( C^n(A, A) \) define in \( \text{Tot} CC^+ (A) \) by
\[ \psi = (\psi_p, \psi_{p-1}, \ldots) \]
\[ \psi_p(a_0, a_1, \ldots, a_p) = \tau(0 \varphi(a_1, \ldots, a_p)), \]
\[ \psi_{k-1}(a_0, \ldots, a_{k-1}) = \tau(\varphi(a_0, \ldots, a_{k-1})) \]
\[ \psi_i = 0, \quad \text{for}\quad i < k - 1 . \]
Then it is not difficult to check that the above map is a morphism of complexes. This means \( b\tilde{\varphi} = \delta \varphi \) which has already appeared in the first pairing and \( (1 - \lambda) \psi_p - \epsilon_p \psi_{p-1} = (\delta \varphi)_p \), and this is easy to verify. This construction has an obvious extension to \( A_\infty \) algebras. Using the second pairing we can transfer cohomological relations in \( H^\bullet(A, A) \) to ones in \( HC^\bullet(A) \). The point is that of course \( HC^\bullet \) is, in general, a “smaller” group than \( H^\bullet \). Closed deformations is an instant where this map can be used.

Let \( A \) be an associative algebra and \( \tau : A \to K \) a trace on \( A \). A formal deformation \((A[[t]], *)\) defined by
\[ a * b = \sum_{i=0}^{\infty} m_i(a, b) t^i , \]
with \( m_0(a, b) = ab \), is called closed (with respect to \( \tau \) [CFS]) if the functional
\[ \tilde{\tau} : A[[t]] \to K[[t]], \]
\[ \tilde{\tau} \left( \sum_{i=0}^{\infty} a_i t^i \right) = \sum_{i=0}^{\infty} \tau(a_i) t^i \]
is a trace on \((A[[t]], *)\). It is easy to see that this is equivalent to
\[ \tau(m_i(a, b)) = \tau(m_i(b, a)) \]
for all $a, b \in A$ and $i \geq 0$.

The question of extending a closed deformation of order $n$ to one of order $n + 1$ amounts to solving the equation

$$\delta m_{n+1} = m_0 \circ m_{n+1} + m_{n+1} \circ m_0 = - \sum_{i+j=n} m_i \circ m_j ,$$

for $m_{n+1}$ such that $\tau m_{n+1}$ be symmetric, given that $\tau m_i$, $0 \leq i \leq n$, are symmetric. This means the right hand side should represent zero in $H^3(A, A)$. Using the above pairing, this is transferred to $HC^3(A)$. There is a similar argument for equivalence of closed $*$-products.

### 3.4. Proposition

Let $(A, m)$ be an $A_{\infty}$-algebra. Let $\tau \in HC^0(A, m)$ be a “trace” on $(A, m)$. Then the obstructions to extending a closed $*$-deformations of $(A, m)$ within the category of $A_{\infty}$-algebras, at any order lie in $HC^3(A, m)$. Similarly, extending an equivalence with closed $*$-products at any order lies in $HC^2(A, m)$.

### 4 Derivations, homotopy invariance and a Goodwillie type theorem

To prove our main theorem (Theorem 4.5 below), we first extend the language of derivations and the corresponding Cartan homotopy formula to $A_{\infty}$-algebras. Note that Cartan homotopy formula is the infinitesimal form of homotopy invariance, from which homotopy invariance and other results, like Goodwillie’s theorem on nilpotent extensions follows.

Let $(A, m)$ be an $A_{\infty}$-algebra. By a graded derivation of degree $|D| \in \mathbb{Z}$ of $(A, m)$, we mean a degree $|D|$ map

$$D : T^c A[1] \to A[1]$$

such that

$$\delta(D) = [m, D] = m \circ D - (-1)^{|D|} D \circ m = 0 .$$

Equivalently, $D$ is a cocycle of dimension $|D|$ in the deformation complex of $A$, introduced in Section 3. Explicitly, $D$ is defined by a sequence of maps $D_n : A^\otimes n \to H$, $n = 1, 2, \ldots$ such that $|D_n| = \infty$ and

$$[m, D] = \sum_{n=2}^{\infty} \sum_{n} (m_i \circ D_j - (-1)^{|D|} D_j \circ m_i) = 0 .$$

In particular $m$ is itself a derivation of degree -1. It is, however, trivial as it can be checked that $m = \delta(1)$.

### 4.1. Example

Let $A$ be a (non-graded) associative algebra considered as an $A_{\infty}$-algebra with $A_0 = A$, $A_i = \{0\}$, $i \geq 1$, $m_i = 0$ for $i \neq 2$ and $m_2 = \text{multiplication of } A$. A linear map $D : A^\otimes k+1 \to A$ defines a map $D : T^c A[1] \to A[1]$ of degree $-k$ by extending it by zero. Now $D$ is an $A_{\infty}$-derivation iff the original $D$ is a cocycle for the (standard) Hochschild cohomology $H^{k+1}(A, A)$. In particular a derivation of algebras in the usual sense is a derivation in the above sense.
Now a derivation $D : (A,m) \to (A,m)$ of degree $|D|$ induces a coderivation $b'_D : T^c A[1] \to T^c A[1]$ which is moreover compatible with the original codifferential $b'_m$:

$$[b'_D, b'_m] = b'_{[D,m]} = 0.$$ 

So now we have a DG coalgebra $(T^c A[1], b'_m)$ with a compatible coderivation $b'_D$.

It is well known that derivations of an associative algebra act on various Hochschild and cyclic complexes of $A$ via the so-called Lie derivative and one has a Cartan homotopy formula which implies that the induced action on deRham cohomology and periodic cyclic homology of $A$ is trivial $[C_1, Go]$. Same formula is also crucial for the proof of Goodwillie’s theorem on periodic cyclic homology of nilpotent extensions. In $[Go]$, these results are extended to DG algebras and applied to singular chains on free loop spaces.

A universal point of view on these matters is as follows. This will be very useful in extending these results to $A_\infty$-algebras. Assume we have an abelian tensor category $\mathcal{C}$. One can first extend the Cuntz-Quillen definition of quasi-free algebras $[CQ_1, CQ_2]$, to define a quasi-free algebra or coalgebra object in $\mathcal{C}$. There are several equivalent definitions, but the one based on existence of a connection $\nabla : \Omega^1 A \to \Omega^2 A$ for algebra objects, or a coconnection $\nabla : \Omega^2 C \to \Omega^1 C$ for coalgebra objects is most useful for us. Here $\Omega^* C$ is the DG algebra of noncommutative differential forms which can be defined in $\mathcal{C}$. A connection for algebras is a linear map which is left $A$-linear and has right Leibniz property $\nabla(\omega a) = \nabla \omega \cdot a + \omega da$.

We call a (co) algebra object quasi-free if it admits a connection in the above sense.

Now in the above general setting, a derivation $D : C \to C$ induces a Lie derivative map $L_D : X(C) \to X(C)$ of degree zero as follows. In odd degrees it is simply $D$ itself while in even degrees it is the map induced by $D$ on $\Omega^1 C$.

The Cartan-homotopy formula of Cuntz and Quillen $[CQ_1]$ extends verbatim to show that if $C$ is quasi-free, then there exists an operator $I_D : X(C) \to X(C)$ of odd degree such that $L_D = \partial_1 I_D + I_D \partial_2$.

In extending all this to our setup, the category of DG coalgebras, we have to face the fact that the bar construction is not quasi-free as a DG coalgebra—if it was, there would be no cyclic homology. It is, however, quasi-free, and in fact free, as a graded coalgebra only. In $[Kh]$, this problem is addressed and solved by noticing that the vertical differential in the cyclic bicomplex is given, in even and odd dimension, by $L_m$, and one obtains a cartan homotopy formula of the form

$$L_D = [\partial_1 + \partial_2, I_D] + I_{\delta_D}$$

so that, if $C$ is only quasi-free as a coalgebra and $D$ is compatible with its differential, then $L_D$ still acts trivially. The explicit form of the operator $I_D$ is irrelevant here.

If we specialize the above general result to the case where $C = T^c A[1]$ is the bar construction of an $A_\infty$-algebra $(A,m)$ and $\delta = b'_D$ is the closed coderivation associated to a derivation $D$ of $(A,m)$, we obtain

4.2. Corollary. The induced map $L_D$ on $HP^* (A,m)$ is zero.
4.3. Remark. Similar to \((DG)\) algebras, it is trivial to see that \(L_D\) induces the zero map on \(H^{dr}(A, m)\) and also \(L_D \circ S\) is zero on \(HC_\bullet(A, m)\).

Let \((A, m)\) be an \(A_\infty\)-algebra. By an \(A_\infty\)-ideal we mean a graded subspace \(I \subset A\) such that for all \(n \geq 1\), \(m_n(a_1, \ldots, a_n) \in I\) if for some \(i, a_i \in I\). Note that in particular \(m_1 I \subset I\).

The graded quotient space \(A/I = \bigoplus_{i=0}^{\infty} A_i/I_i\) is an \(A_\infty\)-algebra in a natural way and the quotient map \(A \to A/I\) is a strict morphism of \(A_\infty\)-algebras.

4.4. Theorem. Let \((A, m)\) be an \(A_\infty\)-algebra over a field of characteristic zero and let \(I \subset A\) be an \(A_\infty\)-ideal such that \(I_0 = 0\). Then the quotient map \((A, m) \to (A/I, m)\) induces an isomorphism

\[
HP_\bullet(A, m) \longrightarrow HP_\bullet(A/I, m).
\]

Proof. As in [Go], let

\[
gr(A, I, m) = \bigoplus_{k \geq 0} I^k/I^{k+1}.
\]

This is an \(A_\infty\)-algebra. There is a derivation, the so-called number operator, acting on \(gr(A, I, m)\) by multiplying a homogeneous element \(a\) by its degree \(|a|\). The cyclic complex of \(gr(A, I, m)\) can be identified as follows. Filter the cyclic complex \(CC_\bullet(A, m) = Tot X^+(T^C A[1])\) by subcomplexes \(F^k, k \geq 0\), where for each \(n\), the \(n\)-tensor components of \(F^k\) are tensors of the form

\[
\bigoplus_{k_0 + \cdots + k_n = k} I^{k_0} \otimes \cdots \otimes I^{k_n}.
\]

Then the cyclic complex of \(gr(A, I, m)\) is given by

\[
CC_\bullet(gr(A, I, m)) \simeq \bigoplus_{k \geq 0} F^k/F^{k+1}.
\]

The operator \(S\) descends to subcomplexes \(F^k\). It suffices to show that the map

\[
H_{n+2k}(F') \xrightarrow{S^k} H_n(F')
\]

is zero for \(n \leq k\). As in [Go], this follows from the following two observations:

1. \(S^k : H_n(F'/F^{k+1}) \to H_n(F'/F^{k+1})\) is zero for all \(n\).
2. \(H_n(F^k) = 0\) for \(n < k\). This is simply true because \(F^k\) has non chains in dimensions less than \(k\). 

\(\square\)
Finally, we prove our main result:

4.5. Theorem. Let \((A, m)\) be an \(A_\infty\)-algebra over a field of characteristic zero. Let \(H_\bullet(A) = H_\bullet(A, m_1)\) be the homology of \(A\) in degree zero with its induced associative product. Then \(HC^\per_\bullet(A, m) \simeq HC^\per_\bullet(H_0(A))\). [Note that in the right hand side we have the ordinary cyclic homology functor.]

Proof. This is a consequence of Lem. 2.2, Prop. 2.3 and Prop. 4.4. Define a new \(A_\infty\)-algebra \(B = \bigoplus_{i \geq 0} B_i\) by \(B_0 = A_0/Im \ m_1\) and \(B_i = A_i, i \geq 1\). Note that \(B_0 = H_0(A, m_1)\). We have a strict morphism of \(A_\infty\)-algebras \((A, m) \to (B, m)\). By Lem. 2.2, this is an equivalence and hence induces an isomorphism on \(HP_\bullet\). Let \(I = \bigoplus_{i \geq 1} B_i\). Then \((B, I)\) satisfies conditions of Prop. 2.3 and hence we have an isomorphism \(HP_\bullet(B, m) \to HP_\bullet(B_0)\). It follows that the map \(HP_\bullet(A, m) \to HP_\bullet(H_0(A, m_1))\), being a composition of two isomorphisms, is an isomorphism.

4.6. Remark. Although we have not checked it, but one can perhaps prove a stronger result. Let us call a morphism of \(A_\infty\)-algebras \((A, m) \to (B, m)\) to be 1-connected if it induces an isomorphism on \(H_0\) and is surjective on \(H_1\). Then a 1-connected map of \((A, m) \to (B, m)\) of \(A_\infty\)-algebras induces an isomorphism

\[HP_\bullet(A, m) \xrightarrow{\sim} HP_\bullet(B, m).\]

If \(A\) and \(B\) are DG algebras, this is Theorem IV.2.1 of [Go].

References

[C1] Connes, A., Non-commutative differential geometry, pub. Math. IHES 62 (1985), 41–144.
[C2] Connes, A., Noncommutative Geometry, Academic Press (1994).
[C3] Connes, A., Cohomologie cyclique et foncteurs Ext\(^n\), C.R. Acad. Sci. Paris, Sér A-B, 296 (1983), 953–958.
[CFS] Connes, A., Flato, M. and Sternheimer D., Closed star products and cyclic cohomology, Lett. Math. Phys. 24 (1992), 1–12.
[CQ1] Cuntz, J. and Quillen, D., Cyclic homology and nonsingularity, J. Amer. Math. Soc. 8 (1995), 373–442.
[CQ2] Cuntz, J. and Quillen, D., Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995), no. 2, 251–289.
[G] Gerstenhaber, M., *The cohomology structure of an associative ring*, Ann. Math. 78 (1963), 267–289.

[GV] Gerstenhaber, M. and Voronov, A., *Homotopy C-algebras and moduli space*, Operad, preprint (1994).

[GJ] Getzler, E. and Jones, J.D.S., *A\textsubscript{\infty}-algebras and the cyclic bar complex*, Illinois Journal of Mathematics 34 (1990), 256–283.

[Go] Goodwillie, T.G., *Cyclic homology, derivations and the free loop space*, Topology 24 (1985), 187–215.

[Kh] Khalkhali, M., *On Cartan homotopy formulas in cyclic homology*, Manuscripta Math. 94 (1997), 111–132.

[K] Kontsevich, M., *Feynman diagrams and low dimensional topology*, Progr. Math., 120 (1994), Birkhauser, 97–121.

[PS] Penkava, M. and Schwarz, A., *A\infty-algebras and the cohomology of moduli spaces*, Amer. Math. Soc. Transl. Ser. 2, 169 (1995), 91–107.

[Q] Quillen, D., *Algebra cochains and cyclic cohomology*, publ. Math. IHES 68 (1989), 139–174.

[S] Stasheff, J., *Homotopy associativity of H-spaces*, II Trans. Amer. Math. Soc., Vol. 108 (1963), 293–312.

Masoud Khalkhali
University of Western Ontario
London, Canada
N6A 5B7
masoud@julian.uwo.ca