Quantum phase transition in overdoped La$_{2-x}$Sr$_x$CuO$_4$ evinced by the superfluid density

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The superfluid density $\rho_{||sf}(T) = \lambda^{2-}_||^{\perp}(T)$ of overdoped La$_{2-x}$Sr$_x$CuO$_4$ thin films of high quality have been measured with $T_c$ (defined by the onset of the Meissner effect) from 5.1 to 41.6 K by Božović et al. [5]. Given this $T_c$, the superfluid density $\rho_{||sf}(T)$ shows no clear evidence of critical fluctuations and no indication of vortex unbinding as $T \to T_c$. Nevertheless, $\rho_{||sf}(T)$ displays the behavior expected for a quantum phase transition (QPT) in the (3+1)-D-xy dimensional universality class, with $\rho_{||sf}(T) \propto L^{-2}_c$ as $T \to 0$. However, this relation is also a hallmark of dirty superconductors, treated in the mean-field approximation. Here we attempt to clear out the nature of the suppression of $\rho_{||sf}(T)$ as $T_c \to 0$. Noting that for any finite system the continuous transition will be rounded we perform a finite size scaling analysis. It uncovers that the $\rho_{||sf}(T)$ data are consistent with a finite length limited 3D-xy transition. In some films it is their thickness and in others their inhomogeneity that determines the limiting length. Having established the precondition for the occurrence of a QPT mapping on the (3+1)-D-xy model, we explore the consistency with the hallmarks of this transition. In particular with the relations $\rho_{||sf}(T)/\rho_{||sf}(0) = 1 - y/T/T_c$, $T_c \propto 1/\lambda^{\perp}(0)$ as $T_c \to 0$ and $y_\perp = \alpha\lambda^{\perp}(0)/T_c$, where $\alpha$ is the coefficient in $\rho_{||sf}(T) = \rho_{||sf}(0) - \alpha T$. The emerging agreement with these characteristics points clearly to a quantum fluctuations induced suppression, revealing the crossover from the thermal to the quantum critical regime as $T_c \to 0$. In the classical-quantum mapping it corresponds to a 3D to (3+1)D crossover.

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I. INTRODUCTION

An essential unsolved puzzle in cuprate superconductivity is that, despite the evidence of more conventional normal state properties [2,3], the superfluid density $\rho_{||}(T) = \lambda^{2-}_||^{\perp}(T)$ of overdoped La$_{2-x}$Sr$_x$CuO$_4$ seems to be controlled by phase fluctuations [1] [5][7]. Furthermore, the vanishing superfluid density in overdoped La$_{2-x}$Sr$_x$CuO$_4$ films is consistent with $\rho_{||}(T) \propto T^2_c$ as $T \to 0$ [1], expected for a quantum phase transition in the (3+1)-dimensional (D)-xy universality class [8][9]. This implies 3D-xy critical behavior at any $T > 0$, characteristic for uncharged superfluids [10], as observed near optimum doping in YBa$_2$Cu$_3$O$_{6+y}$ single crystals [11] [12] and other cuprates [9] [13][15]. However, the measurements of $\rho_{||}(T)$ on the overdoped La$_{2-x}$Sr$_x$CuO$_4$ films show no clear evidence of 3D-xy critical behavior [1]. Rather the approach to $T_c$ is mean-field-like. Taking this mean-field-like behavior of $\rho_{||s}(T)$ for granted the suppression of $T_c$ is also attributable to pair breaking in a d-wave superconductor. In the dirty limit it leads to the same scaling relation, namely $\rho_{||}(T) \propto T^2_c$ [16][17]. The puzzle, namely what suppresses the superfluid density in overdoped cuprates, is what we attempt to solve.

First we perform a finite size scaling analysis of the superfluid density data of Božović et al. [1]. It turns out that the observed mean-field-like behavior is a finite size effect. It stems from the fact that in an infinite and homogeneous 3D-xy system the transverse correlation length diverges as $\xi^{\perp}_||^{\perp}(T) = \xi^{\perp}_||^{\perp}(0)T^{-\nu}$, where $t = (1 - T/T_c)$ and $\nu \approx 2/3$ [18], while in the films $\xi^{\perp}_||^{\perp}(T_c)$ is set by the limiting length $L_c$. In homogeneous films it is determined by the effective film thickness $L_{sf}$ and in inhomogeneous films by $L_{sf} < L_{sf}$, the size of the homogenous domains. Nevertheless, in the films analyzed here, the 3D-xy critical regime is reached. This analysis provides estimates for the transition temperatures and the critical amplitude $\lambda^{\perp}_||^{\perp}(T_c)$ of the infinite system appearing in $\lambda^{\perp}_||^{\perp}(T) = \lambda^{\perp}_||^{\perp}(0)T^\alpha$ as $T \to T_c$. Because the 3D-xy transition temperatures are lower than those estimated from the onset of the Meissner effect [1], the mean-field-like behavior of $\rho_{||sf}(T) = \lambda^{2-}_||^{\perp}(T)$ is traced back to a finite size rounded transition. In addition the estimates for $T_c$ and $\lambda^{\perp}_||^{\perp}$ uncover the relationship $T_c \propto 1/\lambda^{\perp}_||^{\perp}$ as $T_c \to 0$. Accordingly, the system approaches the extreme type II limit $\lambda^{\perp}_||^{\perp} \to \infty$ as $T_c \to 0$. In this regime it is expected that the static critical properties at zero external field are dominated by phase fluctuations [19][20].

Second, having established that 3D-xy fluctuations determine the finite temperature critical behavior, limited by the finite size effect, we turn to the analysis of the low temperature data extending down to 0.3 K. Therefore, an extrapolation is necessary to estimate $\lambda^{\perp}_||^{\perp}(0)$. It turns out that the linear extrapolation with $\lambda^{\perp}_||^{\perp}(T) = \lambda^{\perp}_||^{\perp}(0) - \alpha T$ fits the data up to 1 K very well. Given then the estimates of $T_c$, $\lambda^{\perp}_||^{\perp}(0)$, $\alpha$, and the $T_c$ dependence of the latter, their consistency with a QPT, mapped on the (3+1)-D-xy universality class, can now be tested. This mapping requires that the scaling relations $T_c = f_{||}(0)/\lambda^{\perp}_||^{\perp}(0)$, $\alpha = y_\perp T_c / f^2_{||}(0)$ are satisfied and the data plotted as $\lambda^{\perp}_||^{\perp}(0)/\lambda^{2-}_||^{\perp}(T)$ vs. $T_c$ collapse on a single curve. In the limit $T_c / T_c \to 0$ and sufficiently small $T_c$ it should tend to the line $\lambda^{\perp}_||^{\perp}(0)/\lambda^{2-}_||^{\perp}(T) = 1 - y_\perp T_c / T_c$.
and in the $T/T_c \to 1$ limit to $t^{2/3}$. However, the rounded transition leads around $T/T_c \approx 1$ to characteristic deviations. $\lambda^2_0 (0) / \lambda^2_{\parallel} (T)$ exceeds $T/T_c = 1$ up to the temperature set by the onset of the Meissner effect. As $T_c \to 0$ the rounding becomes stronger because the ratio $L_c/\xi^\perp (T) = (L_c/\xi^\perp (0))^{2/3}$, controlling the finite size effect, decreases because $\xi^\perp (0)$ grows like $\xi^\perp (0) \propto 1/T_c$. On the other hand, as $T/T_c$ drops the collapse is no longer affected by the finite size effect. The crossover from the thermal to the quantum fluctuation dominated regime implies that $\lambda^\perp_0 (T) = \lambda^\perp_0 (0)^{2/3}$ approaches the low $T_c$ behavior and thus expands the critical thermal regime $\lambda^\perp_0 (T) = \lambda^\perp_0 (0)^{2/3}$ towards lower temperatures. Noting that the QPT has been mapped on the $(3+1)$D-xy model it corresponds here to a dimensional crossover from 3D to $(3+1)$D. Our analysis reveals that the $\rho_{\parallel}(T) = \lambda^\parallel_0 (T)$ data of the thin and overdoped La$_2-\delta$Sr$_x$CuO$_4$ films are remarkably consistent with the outlined hallmarks of this QPT. In summary, our analysis reveals that the suppression of the superfluid density is driven by quantum phase fluctuations and uncovers the crossover from the thermal to the quantum critical regime. The thermal regime shows finite size limited 3D-xy criticality and the quantum regime is compatible with (3+1)-D-xy criticality.

In Sec. II we sketch the theoretical background including the finite size scaling approach, the scaling relations and scaling forms of the QPT mapped on the classical (3+1)-D-xy model. Sec. III is devoted to the analysis of the superfluid density data of Božović et al. [1]. We close with a brief summary and conclusions.

II. THEORETICAL BACKGROUND

A precondition for the occurrence of 3D-xy critical behavior in homogeneous superconducting films with effective thickness $L_c$ originates from the vortex interaction. Indeed, a Berezinskii-Kosterlitz-Thouless (BKT) transition might occur, provided that the Pearl length $\lambda^\perp_0 (T_{BKT})$ exceeds $L_c$, the lateral extent of the film. In this case, a precondition for a BKT transition, the logarithmic intervortex interaction, is satisfied. Invoking the Nelson-Kosterlitz relation [21],

$$T_{BKT} = \pi \Lambda L_c / \left( 2 \lambda^\parallel (T_{BKT}) \right)$$

we obtain the precondition

$$\lambda_{pearl} = \frac{\pi \Lambda}{T_c} > L_c,$$

where $\Lambda = \Phi^2_0 / (16\pi^3 k_B) \approx 6259 \, \mu m \, K$. For the films considered here $L_c = 1 cm$ and with that $T_{BKT} < \pi \Lambda / L_c \approx 1.97 \, K$, which is considerably below $T_c \approx 4.4 \, K$, the lowest value attained by Božović et al. [1]. In a homogeneous and sufficiently large system 3D-xy critical behavior implies an infinite correlation length below $T_c$ [24]. One can extract an alternative diverging length the so called transverse correlation length defined in terms of the helicity modulus $\Upsilon (T)$ as $\xi^\parallel (T) = \Upsilon (T) / k_B T$, where $\Upsilon (T) = \Phi^2_0 / (16\pi^3)$ and $\Lambda$ denotes the penetration depth [8,9]. It probes the global superconducting phase coherence across the system. In superconductors with uniaxial symmetry ($\lambda_x = \lambda_y = \lambda_\parallel, \lambda_z = \lambda_\perp$) it leads with

$$\xi^\perp_0 (t) = \xi^\perp_0 (0)^{-\nu}, \lambda^\parallel_0 (t) = \lambda^\parallel_0 (0)^{-\nu}, t = 1 - T/T_c, \nu \approx 2/3,$$

to the universal relation [9]

$$T_c = \Lambda^\perp_0 / \lambda^\parallel_0.$$

It controls the $T_c$ dependence of the critical amplitudes $\lambda^\perp_0$ and $\lambda^\parallel_0$.

Other hallmarks of 3D-xy criticality follow from the derivative $d\lambda^\perp_0 / dT$ and the crossing of the curves $\lambda^\parallel (T)^{-2}$ and $\lambda^\perp (T)^{-2}$. From Eq. (2) we obtain

$$d\lambda^\perp_0 / dT = -2/ \left( 3 \lambda^\parallel_0 T_c \right)^{t^{-1/3}}.$$

Contrariwise there is no singularity in the mean-field treatment where $\lambda^\perp_0(t) = \lambda^\perp_0 (0)^{-1}$ yields $d\lambda^\perp_0 / dT = -1/ \lambda^\perp_0 T_c$. Given 3D-xy criticality (Eq. (2)) the curves $\lambda^\parallel (T)^{-2}$ and $\lambda^\perp (T)^{-2}$ cross at

$$T_{cr} = T_c \left( 1 - (\lambda^\parallel_0/a)^3 \right),$$

as long as $\lambda^\parallel_0/a < 1$ with $a = 1$ in units of $\lambda^\parallel_0$. In this case $\lambda^\perp (T)^{-2} < \lambda^\parallel (T)^{-2}$ for $T < T_{cr}$ and $\lambda^\perp (T) > \lambda^\parallel (T)$ for $T > T_{cr}$. Contrariwise, if $\lambda^\parallel_0/a > 1$ there is no crossing point but

$$\lambda^\parallel (T)^{-2} > \lambda^\perp (T).$$

Given then magnetic penetration depth data $\lambda^\perp (T)$ of a homogeneous and infinite system the transition temperature $T_c$ and the critical amplitude $\lambda^\parallel_0$ can be estimated from fits to Eq. (2), its derivative (Eq. (4)) and cross-rechecked with the crossingpoint relations (5) and (6).

In homogeneous films the outlined critical behavior will be modified because the transverse correlation length $\xi^\parallel_0 (T)$ does not diverge at $T_c$ and is analytic for all $t$. However, for a sufficiently large limiting length $L_c$ the finite size scaling theory predicts a scaling behavior of the form [26,27]

$$\xi^\parallel_0 (t) / \xi^\perp_0 (t) = g \left( L_c / \xi^\perp_0 (t) \right).$$

In homogeneous films the limiting length is set by the effective film thickness $L_{cf} = L_c$ and in inhomogeneous films by $L_c = L_{ci} < L_{cf}$, the size of the homogeneous domains. $g (x)$ is the finite size scaling function. The critical 3D-xy critical regime is reached when $L_c / \xi^\perp_0 (t) > 1$ and $g \left( L_c / \xi^\perp_0 (t) \right) \to 1 = \xi_{1f} (t) / \xi^\perp_0 (t)$. In contrast if $L_c / \xi^\perp_0 (t) < 1$ the transition is rounded because $g \left( L_c / \xi^\perp_0 (t) \right) \to L_c / \xi^\perp_0 (t)$ and with that $\xi_{1f} (t) \to L_c$. 


In principle, thick films are required for observing the signatures of criticality such as Eq. (4), and for accurate determination of $T_c$ and the critical amplitudes $\xi_{\perp 0}$ and $\lambda_{\parallel 0}$. Noting that the divergence of $\xi_{\perp}(t)$ is cut off at $\xi_{\perp}(t_{L_c}) = \xi_{\perp 0}(1 - T_{L_c}/T_c)^{-2/3} = L_c$ we obtain from $T_{L_c}/T_c = 1 - (\xi_{\perp 0}/L_c)^{3/2}$ the precondition
$$\xi_{\perp 0} \ll L_c, \quad (8)$$
for a reliable finite size scaling analysis. Given the penetration depth data and the universal relation (3) it is convenient to transform the finite size scaling form (7) to
$$\lambda_{\parallel}^2(t)/\lambda_{\perp}^2(t) = g\left(L_{\parallel}/\lambda_{\perp}^2(t)\right), \quad (9)$$
with
$$L_c = L_{\parallel}^2 T_c/\Lambda. \quad (10)$$
The critical 3D-xy regime is reached when $L_{\parallel}^2/\lambda_{\parallel}^2(t) < 1$ and $g\left(L_{\parallel}/\lambda_{\parallel}^2(t)\right) \to 1 = \lambda_{\parallel}^2(t)/\lambda_{\parallel}^2(t)$. In contrast if $L_{\parallel}^2/\lambda_{\parallel}^2(t) > 1$ the transition is rounded because $g\left(L_{\parallel}/\lambda_{\parallel}^2(t)\right) \to L_{\parallel}/\lambda_{\parallel}^2(t)$ and $\lambda_{\parallel}^2(t) \to L_{\parallel}$ by which $L_c$ can be determined with Eq. (10). Using the estimates for $T_c$, $1/\lambda_{\parallel}^0$, and $L_{\parallel}^0$ the plot of $\lambda_{\parallel}^2(t)/\lambda_{\parallel}^2(t)$ vs. $L_{\parallel}^2/\lambda_{\parallel}^2(t)$ should let the experimental data collapse onto a single line. For this to happen, the estimates for $T_c$, $1/\lambda_{\parallel}^0$, and $L_{\parallel}^0$ must be correct. On the one hand, this plot enables the reliability of the estimated values $T_c$, $\lambda_{\parallel}^0$, and $L_c$ to be checked, and on the other hand it uncovers the nature of the limiting length. If $L_c < L_{cf}$ it is traced back to the size of the homogenous domains $L_{ci}$ and if $L_c \approx L_{cf}$ to the effective film thickness $L_{cf}$.

In summary, the finite scaling analysis provides cross-check estimates of $T_c$, the critical amplitudes $\xi_{\perp 0}$ and $\lambda_{\parallel 0}$ of the infinite and homogeneous system and the $T_c$ dependence of the latter. Of particular interest is the resulting phase transition line $T_c(1/\lambda_{\parallel}^0)$ which shows how the finite temperature superfluid density depends on $T_c$. Supposing that $T_c$ vanishes as
$$T_c = f_{\parallel 0}/\lambda_{\parallel 0}, \quad (11)$$
the universal relation (3) points to a QPT because the transverse correlation length
$$\xi_{\perp 0} = f_{\parallel 0}/\Lambda T_c. \quad (12)$$
diverges as $T_c \to 0$, while the temperature dependence of $\lambda_{\parallel}^2(t) = \lambda_{\parallel 0}^{-2(T-\lambda_{\parallel 0}/2)}$ and $\xi_{\perp}(t)$ is not affected. In this case the system approaches the extreme Type II limit where $\lambda_{\parallel} \to \infty$ as $T_c \to 0$.

Approaching the corresponding quantum critical point at $T = 0$ we can invoke the scaling theory of quantum critical phenomena [8, 9]. A hallmark of a QPT mapped on the classical $(3+1)$ D-xy model is
$$T_c = f_{\parallel 0}/\lambda_{\parallel 0}, \quad (13)$$
the quantum counterpart of Eq. (11). Since $D = 4$ is the upper critical dimension of the XY model, mean-field critical behavior follows, supplemented by logarithmic corrections [28, 29]. The scaling relations of this transition include the quantum analog of Eq. (3)
$$T_c = RA\xi_{\perp}^2(0)/\lambda_{\parallel}^2(0), \quad (14)$$
where $R$ is a nonuniversal coefficient. It yields with Eq. (13) the important relations
$$T_c = f_{\parallel 0}/\lambda_{\parallel 0} = f_{\parallel 2}^2(0)/(RA\xi_{\perp}^2(0)), \quad (15)$$
In addition there is the scaling form of
$$\lambda_{\parallel}^2(0)/\lambda_{\parallel}^2(T) = F(y_{\parallel}T/T_c), \quad F(0) = 1 \quad (16)$$
valid in the limit $T \to 0$ and low $T_c$. $y_{\parallel} = ab$ denotes the universal critical value of the scaling variable $y = aT\delta^{-1/2}$ at which the scaling function $F(T = 0, \delta)$ exhibits a singularity at
$$T_c = bd^{3/2} \propto 1/\lambda_{\parallel}^2(0), \quad (17)$$
where $d$ denotes the tuning parameter. A crossover to 3D-xy criticality (Eq. (2)) then takes place as the temperature increases. In 3D it is the crossover from the quantum to the thermal critical regime and in the quantum 3D-xy model mapped onto the $(3+1)$D-xy model a dimensional crossover from $(3+1)$ to 3D. To check the compatibility with scaling relations (13), (14) and the scaling form (16) estimates for $\lambda_{\parallel}^0$ are required. Given the data extending down to 0.3 K we have to rely on an extrapolation. A linear temperature dependence $\lambda_{\parallel}^2(T) = \lambda_{\parallel}^2(0) - aT$ leads with Eq. (16) to the scaling form
$$\lambda_{\parallel}^2(0)/\lambda_{\parallel}^2(T) = F(y_{\parallel}T/T_c) = 1 - y_{\parallel}T/T_c$$
$$= 1 - y_{\parallel}Tc\lambda_{\parallel}^0(0)/f(0), \quad (18)$$
and the scaling relation
$$y_{\parallel} = \alpha\lambda_{\parallel}^2(0) T_c = \alpha f_{\parallel 0}^2(0) \lambda_{\parallel}^0(0) / \alpha^2 f_{\parallel 2}^2(0)/T_c, \quad (19)$$
setting a stringent constraint on the coefficient $\alpha$. Given then estimates for $\alpha$ and $1/\lambda_{\parallel}^2(0)$, consistency with the $(3 + 1)$D-xy QPT requires that the data, plotted as $\lambda_{\parallel}^2(0)/\lambda_{\parallel}^2(T)$ vs. $T\lambda_{\parallel}^0(0)$ or vs. $T/Tc$, collapse for sufficiently low $T_c$ as $T \to 0$ on the line given by Eq. (18), provided that the estimates for $\lambda_{\parallel}^0(0)$, $\alpha$ and $T_c$ are correct and $\alpha$ scales according to Eq. (18) as
$$\alpha = y_{\parallel}Tc/f_{\parallel 2}^2(0). \quad (20)$$
This linear temperature dependence $d\lambda_{\parallel}/dT|_{T=0} = \alpha\lambda_{\parallel}^2(0)/2$ was proposed by several authors in terms of phase fluctuations [20, 22].

Finally, the relationship between the zero temperature penetration depth $\lambda_{\parallel}^0(0)$, the correlation length $\xi_{\perp}(0)$ and the respective critical amplitudes, $\lambda_{\parallel 0}$ and $\xi_{\perp 0}$, need
to be clarified. The crossover from quantum to classical critical behavior implies \( \lambda_{\parallel}^{2}(0)/\lambda_{\perp}^{2}(0) = F(y_{T}/T_{c}) \rightarrow \lambda_{\parallel}^{2}/\lambda_{\perp}^{2}(T) t^{2/3} \) as \( T/T_{c} \) increases and with that the crossover from \( \lambda_{\parallel}(0) \) to the finite temperature amplitude \( \lambda_{\parallel}(0) \). This crossover does not modify the characteristic temperature dependence of 3D-xy criticality \( (t^{2/3}) \) but reflects the quantum fluctuation induced change of the critical amplitude \( \lambda_{\parallel}(0) \). With \( \lambda_{\parallel}(0) \rightarrow \lambda_{\parallel}(0) \) as \( T_{c} \rightarrow 0 \) the scaling relations (11) and (13) reduce then to

\[
T_{c} = f_{\parallel}(0)/\lambda_{\parallel}(0) = f_{\parallel}(0)/\lambda_{\parallel}(0).
\]

Combining this result with the scaling relations (3), (15) we obtain the desired relationship

\[
T_{c} = \frac{f_{\parallel}(0)}{\lambda_{\parallel}(0)} = \frac{f_{\parallel}(0)}{\lambda_{\parallel}(0)} = \frac{f_{\parallel}(0)}{\lambda_{\parallel}(0)} = \frac{f_{\parallel}(0)}{\lambda_{\parallel}(0)}.
\]

An essential implication of this crossover is the growth of the amplitude \( \xi_{1,0}^{\|} \) as \( T_{c} \rightarrow 0 \). It crosses over to \( \xi_{1,0}^{\|}(t) = \xi_{1,0}^{\|}(0) = \left( f_{\parallel}^{2}(0)/\lambda_{\parallel}\right) t^{2/3} \) which leads to an inevitable enhancement of the finite size effect controlled by the ratio \( L_{c}/\xi_{1,0}^{\|}(0) = (L_{c}/\xi_{1,0}^{\|}) t^{2/3} \) (Eq. (7)) because \( L_{c}/\xi_{1,0}^{\|}(0) = (L_{c}/\xi_{1,0}^{\|}) t^{2/3} \approx L_{c} t^{2/3} \) as \( T_{c} \rightarrow 0 \). In contrast, the zero temperature counterpart \( L_{c}/\xi_{1,0}^{\|}(0) \) remains large in spite of \( \xi_{1,0}^{\|}(0) \propto 1/T_{c} \) for any reasonable \( L_{c} \) because \( t^{2/3} \) does not enter.

Another manifestation of the quantum fluctuations comes from their effect on the condensate and Cooper-pair density. In the Ginzburg-Landau version of the D-xy-model, where amplitude |Ψ(\( R \))| and phase \( \varphi(\( R \)) \) are taken into account, these densities can be expressed in terms of the complex order parameter \( \Psi(\( R \)) \exp(\( i\varphi(\( R \)) \)). The superfluid density \( \rho_{s}(0) = \lambda_{\parallel}^{-2}(0) \), the helicity modulus \( \Upsilon_{\parallel}(0) \), the condensate density \( \langle |\Psi|^{2} \rangle \) and the Cooper-pair counterpart \( \langle |\Psi|^{2} \rangle \) are found to scale as [20]

\[
\rho_{s}(0) = \lambda_{\parallel}^{-2}(0) = (16\pi^{3}/\Phi_{0}^{2}) \Upsilon_{\parallel}^{2} \varphi(0) = (h^{2}/2m_{\|}) \langle |\Psi|^{2} \rangle,
\]

\[
\langle |\Psi|^{2} \rangle \propto t^{2/3}, \quad \langle |\Psi|^{2} \rangle > \langle |\Psi|^{2} \rangle, \quad \langle |\delta\Psi|^{2} \rangle > 0.
\]

Accordingly, the superfluid and condensate densities vanish as \( T \rightarrow T_{c} \) while the Cooper pair density remains finite at and above \( T_{c} \). Indeed, at \( T_{c} \) it scales as

\[
\langle |\delta\Psi|^{2} \rangle_{T=T_{c}} \propto T_{c}
\]

The difference between the superfluid and condensate density stems from the fact that in the 3D-xy model \( \nu = 2\beta - \eta \) with \( \eta \approx 0.0381 \). The appearance of 3D-xy critical behavior stems from the fact that in extreme type II superconductors the amplitude fluctuations are dominated by phase fluctuations over a sizeable temperature regime [20]. Note that in the QPT considered here, where \( T_{c} \approx 1/\lambda_{\|}(0) \approx 1/\lambda_{\parallel}(0) \) (Eq. (21)), the extreme type II limit \( (\lambda_{\parallel} \rightarrow \infty) \) is attained as \( T_{c} \rightarrow 0 \).

The mapping of the QPT on the \((3+1)\) D-xy model implies mean-field critical behavior. Therefore, taking the quantum fluctuations in the 3D-xy model into account, the superfluid density \( \rho_{s}(0) = \lambda_{\|}^{-2}(0) \), the helicity modulus \( \Upsilon_{\|}(0) \), the condensate density \( \langle |\Psi|^{2} \rangle \) and the Cooper-pair counterpart \( \langle |\Psi|^{2} \rangle \) scale as

\[
\rho_{s}(0) = \lambda_{\|}^{-2}(0) = (16\pi^{3}/\Phi_{0}^{2}) \Upsilon_{\parallel}(0) = (h^{2}/2m_{\|}) \langle |\Psi|^{2} \rangle_{T=0},
\]

\[
\langle |\Psi|^{2} \rangle_{T=0} = \langle |\Psi|^{2} \rangle_{T=0} \propto \delta \propto T_{c}^{2},
\]

where we used Eq. (17). \( \delta \) denotes the tuning parameter of the QPT. Remarkably, not only the superfluid density is suppressed by quantum fluctuations as \( T_{c} \rightarrow 0 \) but also the condensation and Cooper-pair density.

### III. Data Analysis

We are now prepared to analyze the penetration depth data. In the nearly optimally doped film S1 evidence for the relevance of 3D-xy fluctuations emerges from Fig. 1 revealing a crossing point in the plots \( \lambda_{f}(T), \lambda_{f}(T) \) vs. \( T \) (Eq. (6)). Indeed, \( \lambda_{f}^{-2}(t) = \lambda_{f}^{-2}(0) \) \((1 - T_{c}/T_{c})^{2/3} \) is a characteristic of 3D-xy criticality. The experimental data is seen to cross at \( T_{c} \) \( \simeq 40.2 \) K, a little higher than \( T_{c} \approx 40 \) K, the value that corresponds to the estimated 3D-xy critical behavior \( T_{c} = 40.4 \) K (Eq. (15) Table 1). This and the fact that \( \lambda_{f}^{-2}(T) \) does not vanish at and extends beyond \( T_{c} \) points to a finite size effect that rounds the sharp 3D-xy-transition, marked by the black line. It implies that the 3D-xy transition temperature \( T_{c} = 40.4 \) K is considerably lower than the \( T_{c} \approx 41.6 \) K, defined by the onset of the Meissner effect [1]. Additional evidence for a finite size induced rounding of the transition comes from \( d\lambda_{f}(T)/dT \) vs. \( T \) , shown in the inset of Fig. 1. As the estimated \( T_{c} = 40.4 \) K is approached from below the data tends to the characteristic 3D-xy singularity, \( d\lambda_{f}^{-2}/dT = -2/(3\lambda_{f}^{2}T_{c}) t^{1/3} \) (Eq. (11)), marked by the solid line. However, close to \( T_{c} \) the singularity is cut off and \( \lambda_{f} \) approaches \( \lambda_{f}(T) = L_{c}(\Upsilon_{\|}(\text{Eqs. (9), (10)})) \).

To unravel the nature of the weakly rounded transition we performed the finite size scaling analysis of the data. Fig. 2 shows \( \lambda_{f}^{2}(t)/\lambda_{f}(T) \) vs. \( 1/\lambda_{f}(t) \) for the films S1, S62, and S97. From the limiting behavior \( \lambda_{f}^{2}(t)/\lambda_{f}(t) \rightarrow L_{c}^{2}/\lambda_{f}^{2}(t) \) (Eq. (9)) we derive \( L_{c}^{2} \simeq 1.2 \mu m^{2} \) for the film S1, yielding with Eq. (14) \( L_{c} \approx 8 \) nm for the limiting length, which is slightly larger than the quoted film thickness \( L_{c} \approx 6.6 \text{ nm} \) [1]. Nevertheless, the 3D-xy transition is only slightly rounded in this thin film because the amplitude of the transverse correlation length \( \xi_{0} = 0.3 \) nm (Table 1) is small compared to the film thickness (Eq. (8)). Therefore, the analysis compiled in Figs. 1 and 2 shows clearly that this film is homogeneous and exhibits due to its thickness a...
FIG. 1. $\lambda_{tf}^{-2} (T)$, $\lambda_{tf}^{-3} (T)$ vs. $T$ of film S1 marked by the red and blue lines. The solid black and the dashed-dot lines mark the 3D-xy critical behavior in terms of $\lambda_{t0}^{-2} (T) = \lambda_{t0}^{-2} t^{-2/3}$, $\lambda_{t0}^{-3} (T) = \lambda_{t0}^{-3} t$ with $\lambda_{t0}$ and $T_c$ values listed in Table I. The crossing point of the experimental data occurs at $T_{crf} \simeq 40.2$ K and the 3D-xy counterpart at $T_{crf} \simeq 40$ K. The inset shows $d\lambda_{tf}^{-2} (T)/dT$ vs. $T$ and the solid line marks the 3D-xy critical behavior (Eq. [4]) with the parameters listed in Table I. The black solid, dashed, and dashed-dot lines are $\lambda_{tf}^{-2} (T)/\lambda_{t0}^{-2} (T) \to \tanh(L_{t0}^{-1}/\lambda_{t0}^{-2} (t))$ which describe the crossover from the finite size $(\lambda_{tf}^{-2} /\lambda_{t0}^{-2} (t))^2 < 1$ to the 3D-xy critical regime $(\lambda_{tf}^{-2} /\lambda_{t0}^{-2} (t))^2 \to 1$ remarkably well.

weakly rounded 3D-xy-transition. Moreover, the approxi-
mately linear temperature dependence of $1/\lambda_{tf}^{2}$, over a
rather wide temperature range, is then traced back to the
film thickness induced finite size effect and is not the
fingerprint of a d-wave superconductor in the dirty limit
[4, 16, 17]. Nevertheless, associating the drop of $T_c$ with
increased dopant concentration, it seems plausible that in
the limit $T_c \to 0$ the size of the homogenous domains
instead of the film thickness set the limiting length.

To clarify this issue we turn to the analysis of the
films S62 and S97 with $T_c \approx 9.7$ K and $T_c \approx 4.45$ K,
respectively. Fig. 3 shows $\lambda_{tf}^{-2} (T)$, $\lambda_{tf}^{-3} (T)$ vs. $T$ of
the film S62. The crossing point at $T_{crf} \simeq 7.5$ K uncov-
ers the presence of 3D-xy fluctuations. How-
ever, the deviations from the estimated 3D-xy critical
behavior near and above $T_c$ reveal a strongly rounded
transition. This behavior is also visible in $d\lambda_{t}^{-2} (T)/dT$
shown in the inset of Fig. 3. The critical behavior $d\lambda_{t}^{-2} (T)/dT = -2/(3\lambda_{t0}^{2} T_c t^{-1/3}$ (Eq. [4]) is attained at rather low temperatures. Both phenomena
are consistent with the crossover from the thermal to the
quantum critical regime where the transverse correlation length scales as $\xi_{t}^{-1} (t) \propto t^{-2/3}$ (Eq. [22]). Indeed, the crossover $\xi_{t}^{-1} (t) = \xi_{t0}^{-1} t^{-2/3} \to \xi_{t0}^{-1} (0) t^{-2/3}$
extends the 3D-xy critical regime to lower temperatures
and enhances the finite size effect for fixed $L_c$, because
$L_c/\xi_{t}^{-1} (t) = L_c/\xi_{t0}^{-1} t^{-2/3} \to L_c T_c t^{-2/3}$ (Eq. [22]).

FIG. 2. Finite size scaling plots $1/\lambda_{tf}^{-2} (t)$, $1/\lambda_{tf}^{-3} (t)$ vs. $T$ for the films S1, S62, and S97. The limiting behavior $(\lambda_{tf}^{-2} /\lambda_{t0}^{-2} (t))^2 \to (L_{t0}^{-1}/\lambda_{t0}^{-2} (t))^2$ yields $L_{t0}^{-1}$ and with
Eq. [4] the estimates for the limiting length $L_c$ listed in Table I. The black solid, dashed, and dashed-dot lines are $\lambda_{tf}^{-2} (t)/\lambda_{t0}^{-2} (t) \to \tanh(L_{t0}^{-1}/\lambda_{t0}^{-2} (t))$ which describe the
crossover from the finite size $(\lambda_{tf}^{-2} /\lambda_{t0}^{-2} (t))^2 < 1$ to the 3D-xy critical regime $(\lambda_{tf}^{-2} /\lambda_{t0}^{-2} (t))^2 \to 1$ remarkably well.

To clarify this issue we turn to the analysis of the
films S62 and S97 with $T_c \approx 9.7$ K and $T_c \approx 4.45$ K,
respectively. Fig. 3 shows $\lambda_{tf}^{-2} (T)$, $\lambda_{tf}^{-3} (T)$ vs. $T$ of
the film S62. The crossing point at $T_{crf} \simeq 7.5$ K uncov-
ners the presence of 3D-xy fluctuations. How-

Table I. Estimates for $T_c$, the 3D-xy critical amplitudes
$\lambda_{t0}$, $\xi_{t0}$ (Eq. [2]), the limiting length $L_{t0}$ of the
penetration depth $\lambda_{t0}$, the resulting limiting length $L_c$, 
the zero temperature penetration depth $\lambda_{t0} (0)$, and the
coefficient $\alpha$ entering the linear temperature dependence of $\lambda_{t0}^{-2} (T)$ in the limit $T \to 0$ (Eq. [18]). The quoted film thickness is $L_{cf} = 6.6$ nm.

| $T_c$ (K) | $\lambda_{t0}$ (μm) | $\xi_{t0}$ (μm) | $L_{t0}$ (μm) | $L_c$ (μm) | $\lambda_{t0} (0)$ (μm) | $\alpha$ |
|----------|-------------------|----------------|--------------|-------------|----------------------|--------|
|          | $\lambda_{t0}^{-2}$ (μm) | $\lambda_{t0}^{-3}$ (μm) | $\lambda_{t0}^{-4}$ (μm) | $\alpha$ (μm K) |
| S1       | 40.4              | 21.70          | 2.0          | 1.7         | 0.30                 | 6.37   |
| S62      | 9.7               | 2.62           | 1.9          | 1.5         | 0.59                 | 2.54   |
| S73      | 6.73              | 1.10           | 1.2          | 1.1         | 0.98                 | 1.07   |
| S75      | 5.64              | 0.76           | 1.1          | 0.8        | 1.19                 | 3.00   |
| S97      | 4.45              | 0.49           | 1.4          | 0.68        | 1.45                 | 6.84   |
| S98      | 4.27              | 0.47           | 1.0          | 0.46        | 1.45                 | 6.84   |

The solid black line and the dash-dot line indicate the es-
teimated 3D-xy critical behavior of $\lambda_{tf}^{-2} (T)$ and $\lambda_{tf}^{-3} (T)$.
Essential features include the absence of the crossing point, the strongly rounded transition and the associated shift of the attainable 3D-xy critical regime to lower temperatures. The absence of the crossing point is according to Eq. (5) a consequence of 3D-xy criticality because the critical amplitude \(\lambda_{||0}^3\) of this film satisfies the condition \(\lambda_{||0}^{-2} \approx 0.49 \mu m^{-2} < 1 \mu m^{-2}\) whereupon \(\lambda_{||}(T)^{-2} > \lambda_{||}^{-3}(T)\) follows (Eq. (6)). To clarify the origin of the strongly rounded transition, also visible in \(d\lambda_{||}^{-2}(T)/dT\) vs. \(T\) depicted in the inset of Fig. 2 we turn to the results of the finite size scaling analysis shown in Fig. 2. The essential result is that the finite size limited 3D-xy criticality mimics the heavily smeared transition well. But in this case with a limiting length \(L_c = 7.1 \text{ nm} \) (Table I), compatible with the quoted film thickness. Therefore, in spite of the pronounced overdoping, the finite size scaling analysis uncovers the homogeneity of this film. We have carried out the same analysis for the films S73, S87 and S98. The result is that all show agreement with a film thickness limited 3D-xy criticality. Comparing the limiting lengths \(L_c\) listed in Table I with the film thickness, \(L_{cf} = 6.6 \text{ nm}\) it follows that the limiting length responsible for the rounded 3D-xy transitions in S87 and S98 stems from the film thickness, because \(L_c \approx L_{cf}\), while in S73, where \(L_c\) is considerably smaller than \(L_{cf}\), inhomogeneities set the limiting length. However, the missing systematic correlation between \(T_c\) and \(L_c\) suggests that \(L_c < L_{cf}\) simply points to worse film quality and not to an unavoidable doping-induced phase separation, according to which the size of the homogeneous domains should shrink with reduced \(T_c\).

Finally we crosscheck the finite size scaling results. In Fig. 3 we depicted the finite size scaling plot \(\lambda_{||}^{-2}(T), \lambda_{||}^{-3}(T)\) vs. \(T\) of film S62 marked by the red and blue lines. The solid black and the dash-dot lines mark the 3D-xy critical behavior in terms of \(\lambda_{||}^{-2}(T) = \lambda_{||0}^{-2} T^{-2/3}\), \(\lambda_{||}^{-3}(T) = \lambda_{||0}^{-3} T\) with \(\lambda_{||0}^{-2}\) and \(T_c\) values listed in Table I. The crossing point of the experimental data occurs at \(T_{cr} \approx 7.5 \text{ K}\) and the 3D-xy counterpart at \(T_{c3d} \approx 7.4 \text{ K}\).

In Fig. 4 we depicted the finite size scaling plot \(\lambda_{||}^{-2}(T), \lambda_{||}^{-3}(T)\) vs. \(T\) of film S97 marked by the red and blue lines. The solid black line and the dash-dot line indicate the estimated 3D-xy critical behavior, \(\lambda_{||}^{-2}(T) = \lambda_{||0}^{-2} T^{-2/3}\), \(\lambda_{||}^{-3}(T) = \lambda_{||0}^{-3} T\) with \(\lambda_{||0}^{-2}\) and \(T_c\) listed in Table I. There is no crossing point because \(\lambda_{||0}^{-2} \approx 0.49 \mu m^{-2} < 1 \mu m^{-2}\). Nevertheless there is still evidence for 3D-xy critical behavior emerging from \(d\lambda_{||}^{-2}(T)/dT\) vs. \(T\) shown in the inset. The solid line represents the corresponding 3D-xy critical behavior (Eq. (4)) with the parameters listed in Table I.

Next we turn to the analysis of the low-temperature penetration depth data extending down to 0.3 K. Therefore, an extrapolation is necessary to estimate \(\lambda_{||}^{-2}(0)\). Fig. 6 shows that the linear temperature dependence \(\lambda_{||}^{-2}(T) = \lambda_{||}^{-2}(0) - \alpha T\) using the data up to 1 K, fits the data remarkably well. In \(\lambda_{||}(T)\) it corresponds to \(d\lambda_{||}^{-2}(T)/dT\bigg|_{T=0}^{T=0} = \alpha \lambda_{||}^{-2}(0)/2\) attributed by several authors to phase fluctuations. In addition, the mapping on the \((3+1)\)-D-xy model implies a linear temperature dependence because \(\Upsilon_{||}(T) \propto \langle |\Psi|^2 \rangle \propto (1 - T/T_c)\). The resulting \(\lambda_{||}^{-2}(0)\) and \(\alpha\) are listed in Table I.
the estimates for $T_c, \lambda_{||}^{-2}(0), \lambda_{\perp}^{-2}(0)$ and $\alpha$ it is now possible to clarify their compatibility with a quantum transition belonging to the (3+1)D-xy universality class. Indeed, consistency with this transition requires that the relationships $T_c = f_{||}(0)/\lambda_{||}(0) = f_{\perp}(0)/\lambda_{\perp}(0)$ (Eq. (21)), $\alpha = y_c T_c / f_{||}(0)$ (Eq. (22)), and $\lambda_{||}^2(0)/\lambda_{\perp}^2(T) = 1 - y_c T_c / T_c$ (Eq. (23)) are met. In Fig. 8 depicting $T_c$ vs. $1/\lambda_{||}(0)$ and $1/\lambda_{\perp}(0)$, we observe that the first relation is confirmed with

$$f_{||}(0) \simeq 6.45 \text{ K\mu m}. \quad (25)$$

Consequently, as $T_c$ drops quantum fluctuations reduce $1/\lambda_{||}(0)$ and $1/\lambda_{\perp}(0)$ in the same manner. Further consistency emerges from the $T_c$ dependence of the coefficient $\alpha$ shown in the inset of Fig. 8 yielding with $\alpha = (y_c / f_{||}^2(0)) T_c$ the estimates

$$y_c / f_{||}^2(0) \simeq 0.014 (\mu m^{-2}K^{-1}), \quad (26)$$

and with $f(0) \simeq 6.45 \text{ K\mu m}$

$$y_c \simeq 0.58, \quad (27)$$

for the universal coefficient. Noting that $\lambda_{||}^{-2}(T) = \lambda_{||}^{-2}(0) - \alpha T$ can be rewritten in the form $\lambda_{||}^{-2}(T) = (T_c / f_{||}(0))^2 - \alpha T_c / T_c$ it is evident that $\alpha$ scales in the limit $T/T_c \to 0$ as $\alpha \propto T_c$.

To complete the consistency checks we show in Fig. 9 the plot $\lambda_{||}^2(0)/\lambda_{||}^2(T)$ vs. $T/T_c$. The outlined linear temperature dependence of $\lambda_{||}^2(T)$ implies that the data should fall on the line $\lambda_{||}^2(0)/\lambda_{||}^2(T) = 1 - u T/T_c$ with $y_c \simeq 0.58$ (Eq. (27)). The approach to this line is clearly seen as $T_c \to 0$. A crosscheck comes from the relation $T_c = f(0)/\lambda_{||}(0)$ (Eq. (15)) after which $T/T_c$ can be replaced by $T\lambda_{||}(0)/f_{||}(0)$. This plot is shown in Fig. 9. Here the data should collapse on the line $\lambda_{||}^2(0)/\lambda_{||}^2(T) = 1 - u T\lambda_{||}(0)$ with

$$u = y_c / f_{||}(0) \simeq 0.09, \quad (28)$$

using the estimates given in Eqs. (23) and (27). Again, as $T_c$ drops the data are seen to approach this line.

To complete the analysis of the selected overdoped La$_{2-x}$Sr$_x$CuO$_4$ thin films we show in Fig. 10 the plot $\lambda_{||}^2(0)/\lambda_{||}^2(T)$ vs. $T/T_c$. It contains the data from 0.3 K to 3 K.
FIG. 8. $T_c$ vs. $1/\lambda_\parallel(0)$ and $1/\lambda_{||0}$ using the estimates listed in Table 1. The solid black line is $T_c = f_1(0)/\lambda_\parallel(0)$ with $f_1(0) = 6.45$ Kµm. The inset shows $\alpha$ vs. $T_c$. The straight line is $\alpha = (y_c/f_2(0))/T_c$ (Eq. (20)) with $y_c/f_2(0) = 0.014\mu m^{-2} K^{-1}$.

FIG. 9. Scaling plot $\lambda_\parallel^2(0)/\lambda_{||f}^2(T) \times T/T_c$ in the limit $T/T_c \to 0$. The line is $\lambda_\parallel^2(0)/\lambda_{||}(T) = 1 - y_cT/T_c$ with the universal coefficient $y_c = 0.58$ (Eq. (27)).

FIG. 10. Scaling plot $\lambda_\parallel^2(0)/\lambda_{||f}^2(T)$ vs. $T\lambda_\parallel(0)$ in the limit $T\lambda_\parallel(0) \to 0$. The line is $\lambda_\parallel^2(0)/\lambda_{||}(T) = 1 - uT/T_c$ with $u = y_c/f(0) \approx 0.09$ (Eq. (28)).

up to the temperature defined by the onset of the Meissner effect. Because the limit $T/T_c \approx 1$ has already been dealt with (see Fig. [1]), we are concentrating on the limit $T/T_c \to 0$. It can be reached along the phase transition line $T_c(1/\lambda_\parallel(0))$ where $\lambda_\parallel^2(0)/\lambda_{||}^2(T) = t^{2/3}$ and $\lambda_{||0} \to \lambda_\parallel(0)$, or along the line $1/\lambda_\parallel(0) \propto \delta^{1/2} \propto T_c$ at $T = 0$ where $\delta$ is the tuning parameter (Eq. (17)). The dashed line in Fig. [11] marks this limit in terms of $\lambda_\parallel^2(0)/\lambda_{||}^2(T) = 1 - y_cT/T_c$ (Eq. (18)). Taking the Sr concentration $x$ in La$_{2-x}$Sr$_x$CuO$_4$ as tuning parameter it implies that the empirical relation $T_c \propto \delta \propto x_o - x$ of Presland at al. [26] is violated as $x \to x_o$. Approaching the QPT along the phase transition line $T_c(1/\lambda_\parallel(0))$ the data is seen to collapse to rather large $T/T_c$ values on the curve $\lambda_\parallel^2(0)/\lambda_{||}^2(T) = t^{2/3}$ because $\lambda_{||0} \to \lambda_\parallel(0)$.

However, as $T/T_c \to 1$ the finite size induced rounding of the transition sets in.

What remains is the question of which material parameters control the 3D-xy transition. Fundamental parameters are the anisotropy $\gamma_0 = \xi_{||0}(0)/\xi_{||0}$ and the ratio $\kappa_{||0} = \lambda_0/\lambda_{||0}$. Noting that the universal relation [3] transforms with these parameters to

$$\gamma_0\kappa_{||0} = \Lambda/T_c\lambda_{||0}$$

(29)

it is evident that $T_c\lambda_{||0}$ is fixed by $\gamma_0\kappa_{||0}$. In the films considered here we obtain with $f_1(0) \simeq f_{||0} \simeq 6.45$ Kµm (Eq. (25))

$$\gamma_0\kappa_{||0} = \Lambda/T_c\lambda_{||0} = \lambda_{||0}/f_{||0} \simeq 91.5$$

(30)
and with $\gamma_0 \approx 10.6$ [37], $\kappa_{||0} \approx 91.5$ as $T_c \to 0$. This differs drastically from the behavior of highly underdoped YBa$_2$Cu$_3$O$_{6+y}$, where $f_{||0} \approx 6.5$ Kµm [38] but $\gamma_0 \approx 100$ as $T_c \to 0$ [39] and with that $\kappa_{||0} \approx 10$ as $T_c \to 0$.

IV. SUMMARY AND CONCLUSIONS

In summary, our analysis of the penetration depth data from Božović et al. [1] for thin overdoped La$_{2-x}$Sr$_x$CuO$_4$ films revealed: First, the observed mean-field-like behavior of $\lambda_{||f}(T)$ is attributable to a finite size effect. It stems from the fact that in an infinite and homogeneous 3D-xy system the transverse correlation length diverges as $\xi_{||}^4(T) = \xi_{||0}^4 T^{-\nu}$, where $t = (1 - T/T_c)$ and $\nu \approx 2/3$, while in the films $\xi_{||f}(T_c)$ the divergence is cut off. In homogenous films the limiting length is set by the effective film thickness $L_{cf}$ and in the inhomogeneous ones by $L_{c1} < L_{cf}$, the size of the homogenous domains. Nevertheless, in the films analyzed here, the 3D-xy critical regime is reached. The finite size scaling analysis of the rounded transition provided estimates for the $T_c$’s and the critical amplitudes $\lambda_{||0}(T_c)$ of the infinate and homogenous counterpart. Therefore, the $T_c$’s are lower than those estimated from the the onset of the Meissner effect [4], the mean-field-like behavior of $\rho_{||f}(T) = \lambda_{||0}^{-2}(T)$ was traced back to a finite size effect. In addition the estimates for $T_c$ and $\lambda_{||0}$ uncovered the relationship $T_c = f_{||0}/\lambda_{||0}$. Second, having established that the films exhibit 3D-xy critical behavior, rounded by the finite size effect, we performed the analysis of the low temperature data extending down to 0.3 K. Therefore, an extrapolation was necessary to estimate $\lambda_{||0}^{-2}(0)$. It was shown that the linear extrapolation with $\lambda_{||0}^{-2}(T) = \lambda_{||0}^{-2}(0) - \alpha T$ fits the data up to 1K very well. Given the estimates of $T_c$, $\lambda_{||0}^{-2}(0)$, $\alpha$, and the $T_c$ dependence of the latter, their consistency with a QPT mapping on the (3+1)D-xy universality class was tested. This mapping requires that the scaling relations $T_c = f_{||}(0)/\lambda_{||0}(0)$, $\alpha = y_c T_c/\lambda_{||0}(0)$ are satisfied and the data plotted as $\lambda_{||0}^{-2}(0)/\lambda_{||f}^2(T)$ vs. $T/T_c$ collapses on a single curve. In the limit $T/T_c \to 0$ and sufficiently small $T_c$ it should tend to the line $\lambda_{||0}^2(0)/\lambda_{||f}^2(T) = 1 - y_c T/T_c$ and in the $T/T_c \to 1$ limit to $t^{2/3}$. However, due to the rounded transition is this limit not fully attainable and $\lambda_{||0}^2(0)/\lambda_{||f}^2(T)$ exceeds $T/T_c = 1$. Nevertheless, as $T_c$ drops the collapse improved because $\lambda_{||0}^{-2}(T) = \lambda_{||0}^{-2}(0) t^{2/3}$, due to the crossover from the thermal to the quantum fluctuation dominated regime. The consistency shown with these characteristics of 3D-xy- and (3+1)D-xy criticality clearly revealed that the suppression of the superfluid density in the thin and overdoped La$_{2-x}$Sr$_x$CuO$_4$ films of Božović et al. [1] is a quantum effect. It uncovers the crossover from the thermal to the quantum fluctuation dominated regime as $T_c$ drops. In the classical-quantum mapping it is the crossover from 3D to (3+1)D. The thermal regime shows finite size limited 3D-xy criticality and the quantum regime is compatible with (3+1)D-xy criticality. Another manifestation of the relevance of quantum fluctuations comes according to Eq. [24] from the observation that in the overdoped La$_{2-x}$Sr$_x$CuO$_4$ films conductivity measurements show that below $T_c$ a large fraction of the Drude weight remains uncondensed [7]. Moreover, consistent with the importance of thermal fluctuations (Eq. [23]) in the overdoped regime are Nernst, torque, magnetization [10], recent specific heat and ARPES measurements [41], revealing in overdoped Bi-2212 single crystals Cooper pairs which are already formed above $T_c$. However, it should be kept in mind that the observed strong diamagnetism and Nernst signal in a wide temperature window above $T_c$ exceeds the fluctuation dominated regime drastically. Indeed, taking the inhomogeneity of single crystals into account, the effective dimensionality is reduced to zero and reduced dimensionality enhances the thermal fluctuations [42]. Nonetheless, the proven existence of copper pairs above $T_c$ implies that they contribute to the transport properties and with that modify the standard Fermi liquid picture.

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