Generalized existential completions and their regular and exact completions

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Abstract

This paper aims to apply the tool of generalized existential completions of conjunctive doctrines, concerning a class $\Lambda$ of morphisms of their base category, to deepen the study of regular and exact completions of existential elementary Lawvere’s doctrines.

After providing a characterization of generalized existential completions, we observe that both the subobjects doctrine $\text{Sub}_C$ and the weak subobjects doctrine $\Psi_C$ of a category $C$ with finite limits are generalized existential completions of the constant true doctrine, the first along the class of all the monomorphisms of $C$ while the latter along all the morphisms of $C$. We then name full existential completion a generalized completion of a conjunctive doctrine along the class of all the morphisms of its base.

From this we immediately deduce that both the regular and the exact completion of a finite limit category are regular and exact completions of full existential doctrines since it is known that both the regular completion $(\mathcal{D})_{\text{reg/lex}}$ and the exact completion $(\mathcal{D})_{\text{lex/lex}}$ of a finite limit category $\mathcal{D}$ are respectively the regular completion $\text{Reg} (\Psi_\mathcal{D})$ and the exact completion $\mathcal{T}_{\Psi_\mathcal{D}}$ (as an instance of the tripos-to-topos construction) of the weak subobjects doctrine $\Psi_\mathcal{D}$ of $\mathcal{D}$.

Here we prove that the condition of being a generic full existential completion is also sufficient to produce a regular/exact completion equivalent to a regular/exact completion of a finite limit category.

Then, we show more specialized characterizations from which we derive known results as well as remarkable examples of exact completions of full existential completions, including all realizability toposes and supercoherent localic toposes.

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1. Introduction

The process of completing a category with quotients by producing an exact or regular category introduced in [1, 2] has been widely studied in the literature of category theory, with applications both to mathematics and computer science, see [3, 4, 5, 6].

In particular, the notion of exact completion of a finite limit (or lex) category, for short ex/lex completion, as well as that of exact completion of a regular category, for short ex/reg completion, have been proved to be an instance of a more general exact completion $T_P$ relative to an elementary existential Lawvere’s doctrine $P$, see [7]. The construction of the exact category $T_P$ is, essentially, provided by the tripos-to-topos construction of J.M.E. Hyland, P.T. Johnstone and A.M. Pitts, see [8, 9]. In particular the ex/lex completion $(\mathcal{D})_{\text{ex/lex}}$ of a category $\mathcal{D}$ is the exact completion $T_{\Psi_D}$ of the weak subobjects doctrine $\Psi_D$ of $\mathcal{D}$ while the ex/reg completion $(\mathcal{C})_{\text{ex/reg}}$ of a regular category $\mathcal{C}$ is the exact completion $T_{\text{Sub}_C}$ of the subobjects doctrine $\text{Sub}_C$ of $\mathcal{C}$.

Similarly, also the regular completion $(\mathcal{D})_{\text{reg/lex}}$ of a lex category $\mathcal{D}$ introduced in [10] can be seen as an instance of the regular completion $\text{Reg}(P)$ of an elementary existential doctrine $P$ introduced in [11], namely the regular completion $\text{Reg}(\Psi_D)$ of the weak subobjects doctrine $\Psi_D$ of $\mathcal{D}$.

Moreover, in [3] it has been shown that the exact completion of an existential doctrine $P$ can be decomposed as the ex/reg completion of the regular completion of $P$. Such a decomposition generalizes the well-known decomposition shown in [2] stating that the ex/lex completion of a lex category $\mathcal{C}$ can be seen as the ex/reg completion of the regular completion of $\mathcal{C}$.

Then, this analysis has been pushed further in [11] by observing that the exact completion of an existential elementary doctrine $P$ can be also decomposed as the regular completion $\text{Reg}(Q_P)$ of the elementary quotient completion $Q_P$ of $P$ introduced in [3, 12].

This paper contributes to the problem of establishing when the exact or regular completion of an elementary existential doctrine happens to coincide respectively with the ex/lex or reg/lex completion of a lex category.
To reach our purposes we employ the construction of the generalized existential completion of a conjunctive doctrine $P$, concerning a suitable class $\Lambda$ of morphisms of their base category, originally introduced in [13].

We start by providing a characterization of generalized existential completions based on an algebraic description of the concept of existential free formulas by means of choice principles. By employing such a characterization, we show that examples of generalized existential completions include: every subobjects doctrine of a lex category, every $\mathcal{M}$-subobjects doctrine relative to a $\mathcal{M}$-category [14, 15], every weak subobjects doctrine of a lex category, every realizability tripos and, among localic triposes, exactly those associated to a supercoherent locale in the sense of [16]. In particular both the subobjects doctrine $\text{Sub}_C$ and the weak subobjects doctrine $\Psi_C$ of a category $C$ with finite limits are generalized existential completions of the constant true doctrine, the first along the class of all the monomorphisms of $C$ while the latter along all the morphisms of $C$. We then name full existential completion a generalized completion of a conjunctive doctrine along the class of all the morphisms of its base.

From this we immediately deduce that both the regular completion $(\mathcal{D})_{\text{reg/lex}}$ and the exact completion $(\mathcal{D})_{\text{ex/lex}}$ of a finite limit category are regular and exact completions of full existential completions thanks to their above mentioned doctrinal presentation through the weak subobjects doctrine $\Psi_\mathcal{D}$.

Here we prove that the condition of being a generic full existential completion is also sufficient to produce a regular/exact completion equivalent to a regular/exact completion of a finite limit category.

For a better exposure of this and related results, we introduce the notion of regular Morita-equivalence and exact Morita-equivalence for elementary existential doctrines, motivated by the notion of Morita equivalence between regular theories in [17]. In more detail, we declare that two elementary and existential doctrines $P$ and $R$ are regular Morita-equivalent/ exact Morita-equivalent when their regular/exact completions are equivalent.

Employing these notions we show that the regular completion of an elementary existential doctrine $P$ is equivalent to the reg/lex completion of a lex category if and only if $P$ is regular Morita-equivalent to a full existential completion, or equivalently $P$ is regular Morita-equivalent to the weak subobjects doctrine $\Psi_\mathcal{D}$ of a lex category $\mathcal{D}$ if and only if $P$ is regular Morita-equivalent to a full existential completion.

Furthermore, from this result and the decomposition of exact completions presented in [7], we also derive that the exact completion $\mathcal{T}_\mathcal{P}$ of an elementary existential doctrine $P$ is equivalent to the ex/lex completion of a lex category if and only if $P$ is exact Morita-equivalent to a full existential completion, or equivalently, $P$ is exact Morita-equivalent to the weak subobjects doctrine $\Psi_\mathcal{D}$ of a lex category $\mathcal{D}$ if and only if $P$ is exact Morita-equivalent to a full existential completion.

More in detail, these characterizations have been derived by two other characterizations identifying regular/exact completions of an elementary existential doctrine which coincide respectively with regular/exact completions of specific categories.

Indeed, we show that the regular completion of an elementary existential doctrine $P$ on a lex category $\mathcal{C}$ coincides with the regular completion $(\mathcal{G}_{P'})_{\text{ex/lex}}$ of the Grothendieck category of a primary subdoctrine $P'$ of $P$ (via an equivalence preserving the embedding of $P'$ in $P$ and in the weak subobjects doctrine of $\mathcal{G}_P$) if and only if $P$ is the full existential completion of $P'$. Combining this with the decomposition of exact completions in [2] we
also derive that the exact completion of an existential doctrine $P$ on a lex category $C$ coincides with the exact completion $(G_{P'})_{ex/lex}$ of the Grothendieck category of a primary subdoctrine $P'$ of $P$ (via an equivalence preserving the embedding of $P'$ in $P$ and in the weak subobjects doctrine of $G_{P'}$) if and only if $P$ is the full existential completion of $P'$.

Relevant applications of the above results to tripos-to-topos constructions include all realizability toposes and localic toposes associated to a supercoherent locale [16].

Furthermore, we show similar characterizations for the case in which $P$ is a pure existential completion of a primary doctrine $P'$, namely when $P$ is a generalized existential completion of $P'$ only with respect to the class of projections of a finite product category, a concept introduced under the name of “existential completion” in [13].

More in detail, we show that the regular completion of an elementary existential doctrine $P$ coincides with the regular completion $(Prd_{P'})_{reg/lex}$ of the category of predicates of a primary subdoctrine $P'$ of $P$ (via an equivalence preserving the embedding of $P'$ in $P$ and in the weak subobjects doctrine of $Prd_{P'}$) if and only if $P$ is a pure existential completion of $P'$. Combining again the decomposition of exact completions in [7], with our result, we deduce that the exact completion of an elementary existential doctrine $P$ coincides with the exact completion $(Prd_{P'})_{ex/lex}$ of the category of predicates of a primary subdoctrine $P'$ of $P$ (via an equivalence preserving the embedding of $P'$ in $P$ and in the weak subobjects doctrine of $Prd_{P'}$) if and only if $P$ is a pure existential completion of $P'$.

Finally we show that the the above characterization generalizes the characterization in [11] asserting that a necessary and sufficient condition for a tripos $P$ to produce a topos $T_P$ coinciding with the exact completion of the lex category $Prd_P$ (via an equivalence given by the canonical exact functor from $(Prd_{P'})_{ex/lex}$ to $T_P$) is to be equipped with Hilbert’s $\epsilon$-operators. The crucial fact linking the two characterizations is that $P$ is a pure existential completion of itself if and only if $P$ is equipped with Hilbert’s $\epsilon$-operators.

We conclude by underlying that a preliminary version of our characterization of generalized existential completion was presented in https://www.youtube.com/watch?v=sAMVU_5RQJQ&t=688s in 2020 and a similar notion was independently presented in [18]. Furthermore, such a characterization together with the notion of existential free object have already been fruitfully employed in recent works [13, 20, 21, 22] to give a categorical version of Gödel’s dialectica interpretation [23] in terms of quantifier-completions. Instead another related study of triposes via different completions can be found in [24, 25].
Definition 2.1. A conjunctive doctrine is a functor \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) from the opposite of the category \( \mathcal{C} \) to the category of inf-semilattices.

Definition 2.2. A conjunctive doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) is a primary doctrine if the category \( \mathcal{C} \) has finite products.

We add the definition of fibred subdoctrine for conjunctive subdoctrines of conjunctive doctrines on the same base category:

Definition 2.3. A conjunctive doctrine \( P': \mathcal{C}^{\text{op}} \to \text{InfSL} \) is said a fibred subdoctrine of a conjunctive doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) if each fibre of \( P'(A) \) is a full subposet of \( P(A) \) for every object \( A \).

Definition 2.4. A primary doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) is elementary if for every \( A \) in \( \mathcal{C} \) there exists an object \( \delta_A \) in \( P(A \times A) \) such that

1. the assignment \( \exists_{(\text{id}_A, \text{id}_A)}(\alpha) := P_{\text{pr}_1}(\alpha) \land \delta_A \) for an element \( \alpha \) of \( P(A) \) determines a left adjoint to \( P_{(\text{id}_A, \text{id}_A)}: P(A \times A) \to PA \);

2. for every morphism \( e \) of the form \( \langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \to X \times A \times A \) in \( \mathcal{C} \), the assignment \( \exists_e(\alpha) := P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\alpha) \land P_{\langle \text{pr}_2, \text{pr}_2 \rangle}(\delta_A) \) for \( \alpha \) in \( P(X \times A) \) determines a left adjoint to \( P_e: P(X \times A \times A) \to P(X \times A) \).

Definition 2.5. A primary doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) is existential if, for every object \( A_1 \) and \( A_2 \) in \( \mathcal{C} \), for any projection \( \text{pr}_i: A_1 \times A_2 \to A_i \), \( i = 1, 2 \), the functor \( P_{\text{pr}_i}: P(A_i) \to P(A_1 \times A_2) \) has a left adjoint \( \exists_{\text{pr}_i} \), and these satisfy:

(BCC) Beck-Chevalley condition: for any pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{pr'} & A' \\
\downarrow f' & & \downarrow f \\
X & \xrightarrow{pr} & A
\end{array}
\]

with \( pr \) and \( pr' \) projections, for any \( \beta \) in \( P(X) \) the canonical arrow

\[ \exists_{pr'} P_{f'}(\beta) \leq P_f \exists_{pr}(\beta) \]

is an isomorphism;
(FR) **Frobenius reciprocity**: for any projection \( pr: X \rightarrow A \), for any object \( \alpha \) in \( P(A) \) and \( \beta \) in \( P(X) \), the canonical arrow

\[
\exists_{pr}(P_{pr}(\alpha) \land \beta) \leq \alpha \land \exists_{pr}(\beta)
\]

in \( P(A) \) is an isomorphism.

**Remark 2.6.** In an existential elementary doctrine, for every map \( f: A \rightarrow B \) in \( C \) the functor \( P_f \) has a left adjoint \( \exists_f \) that can be computed as

\[
\exists_{pr_2}(P_{f \times \text{id}_B}(\delta_B) \land P_{pr_1}(\alpha))
\]

for \( \alpha \) in \( P(A) \), where \( pr_1 \) and \( pr_2 \) are the projections from \( A \times B \). However, observe that such a definition guarantees only the validity of the corresponding Frobenius reciprocity condition for \( \exists_f \) but it does not guarantee that of Beck-Chevalley conditions with respect to pullbacks along \( f \).

**Examples 2.7.** The following examples are discussed in [26].

1. Let \( C \) be a category with finite limits. The functor

\[
\text{Sub}_C: C^{op} \rightarrow \text{InfSL}
\]

assigns to an object \( A \) in \( C \) the poset \( \text{Sub}_C(A) \) of subobjects of \( A \) in \( C \) and, for an arrow \( B \xrightarrow{f} A \) the morphism \( \text{Sub}_C(f): \text{Sub}_C(A) \rightarrow \text{Sub}_C(B) \) is given by pulling a subobject back along \( f \). The fiber equalities are the diagonal arrows. This is an existential elementary doctrine if and only if the category \( C \) is regular. See [29].

2. Consider a category \( D \) with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects (or variations)

\[
\Psi_D: D^{op} \rightarrow \text{InfSL}
\]

where \( \Psi_D(A) \) is the poset reflection of the slice category \( D/A \), whose objects are indicated with \([f]\) for any arrow \( B \xrightarrow{f} A \) in \( D \), and for an arrow \( B \xrightarrow{f} A \), the homomorphism \( \Psi_D([f]): \Psi_D(A) \rightarrow \Psi_D(B) \) is given by the equivalence class of a weak pullback of an arrow \( X \xrightarrow{g} A \) with \( f \). This doctrine is existential, and the existential left adjoint are given by the post-composition.

3. Let \( T \) be a theory in a first order language \( L \). We define a primary doctrine

\[
LT: C_T^{op} \rightarrow \text{InfSL}
\]

where \( C_T \) is the category of lists of variables and term substitutions:

- **objects** of \( C_T \) are finite lists of variables \( \vec{x} := (x_1, \ldots, x_n) \), and we include the empty list (\( () \));
• a morphisms from \((x_1, \ldots, x_n)\) into \((y_1, \ldots, y_m)\) is a substitution \([t_1/y_1, \ldots, t_m/y_m]\)
  where the terms \(t_i\) are built in \(\mathbf{Sg}\) on the variable \(x_1, \ldots, x_n\);

• the composition of two morphisms \(\vec{t}/\vec{y}: \vec{x} \longrightarrow \vec{y}\) and \(\vec{z}/\vec{z}: \vec{y} \longrightarrow \vec{z}\)
  is given by the substitution \([s_1[\vec{t}/\vec{y}]/z_k, \ldots, s_k[\vec{t}/\vec{y}]/z_k]: \vec{x} \longrightarrow \vec{z}\).

The functor \(LT: \mathbf{C^{op}} \longrightarrow \mathbf{InfSL}\) sends a list \((x_1, \ldots, x_n)\) to the partial order
\(LT(x_1, \ldots, x_n)\) of equivalence classes \([\phi]\) of well formed formulas \(\phi\) in the context
\((x_1, \ldots, x_n)\) where \([\psi] \leq [\phi]\) for \(\phi, \psi \in LT(x_1, \ldots, x_n)\) if \(\psi \vdash_T \phi\) and two formulas
are equivalent if they are equiprovable in the theory. Given a morphism of \(\mathbf{C}_T\)
\([t_1/y_1, \ldots, t_m/y_m]: (x_1, \ldots, x_n) \longrightarrow (y_1, \ldots, y_m)\)
the functor \(LT[\vec{t}/\vec{y}]\) acts as the substitution \(LT[\vec{t}/\vec{y}](\psi(y_1, \ldots, y_m)) = \psi[\vec{t}/\vec{y}]\).

The doctrine \(LT: \mathbf{C^{op}} \longrightarrow \mathbf{InfSL}\) is elementary exactly when \(\mathbb{T}\) has an equality
predicate and it is existential. For all the detail we refer to \([6]\), and for the case of
a many sorted first order theory we refer to \([30]\).

4. Let \(\mathbf{Set}_\ast\) be the category of non-empty sets and let \(\xi\) be an ordinal with greatest
element, and \(\mathcal{H}\) its frame. Then the doctrine
\(\mathcal{H}(-): \mathbf{Set}_\ast^{op} \longrightarrow \mathbf{InfSL}\)
is elementary and existential, in particular for every \(\alpha \in \mathcal{H}^{A \times B}\), the left adjoint
\(\exists_{pr\alpha}\) is defined as
\(\exists_{pr\alpha}(\alpha)(a) = \bigvee_{b \in B} \alpha(a, b)\)
and the equality predicate \(\delta(i, j) \in \mathcal{H}^{A \times A}\) is defined as the top element if \(i = j\),
and the bottom otherwise. See \([11]\) for all the details.

The category of primary doctrines \(\mathbf{PD}\) is a 2-category, where:

• a 1-cell is a pair \((F, b)\)

\[\begin{array}{c}
\mathcal{C}^{op} \\
\downarrow F^{op} \\
\mathcal{D}^{op}
\end{array}\]

\[\begin{array}{c}
\text{InfSL} \\
\downarrow R \\
\text{InfSL}
\end{array}\]

such that \(F: \mathcal{C} \longrightarrow \mathcal{D}\) is a functor, and \(b: P \longrightarrow R \circ F^{op}\) is a natural transformation.
a 2-cell is a natural transformation \( \theta: F \rightarrow G \) from \( (F, b) \) to \( (G, c) \), namely from \( b: P \rightarrow R \circ F^{\text{op}} \) to \( c: P \rightarrow R \circ G^{\text{op}} \), such that for every \( A \in \mathcal{C} \) and every \( \alpha \in P(A) \), we have \( b_A(\alpha) \leq R_\theta_A(c_A(\alpha)) \).

We denote by \( \text{ExD} \) the 2-full subcategory of \( \mathcal{PD} \) whose elements are existential doctrines, and whose 1-cells are those 1-cells of \( \mathcal{PD} \) which preserve the existential structure.

We conclude this section by recalling some choice principles from [11].

**Definition 2.8.** Let \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an elementary existential doctrine. We say that \( P \) satisfies the \textbf{Rule of Unique Choice} (RUC) if for every entire functional relation \( \phi \in P(A \times B) \) there exists an arrow \( f: A \rightarrow B \) such that

\[
\top_A \leq P(\langle \text{id}_A, f \rangle)(\phi)
\]

**Examples 2.9.** The subobjects doctrine \( \text{Sub}_C: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) presented in Example 2.7 satisfies the (RUC) as observed in [11].

Now we recall the notion of Extended Rule of Choice and its particular instance called Rule of Choice.

**Definition 2.10.** An existential doctrine \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) satisfies the \textbf{Extended Rule of Choice} (ERC) if for every \( \phi \in P(B) \) and for every \( g: B \rightarrow A \) such that \( \top_A \leq \exists g(\phi) \) there exists an arrow \( f: A \rightarrow B \) in \( \mathcal{C} \) such that \( gf = \text{id}_A \) and

\[
\top_A \leq P_f(\phi).
\]

**Definition 2.11.** For an existential doctrine \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \), we say that \( P \) satisfies the \textbf{Rule of Choice} (RC) if it satisfies the Extended Rule of Choice only for projections, namely for every \( \phi \in P(A \times B) \) such that \( \top_A \leq \exists_{\text{pr}_1}(\phi) \) there exists an arrow \( f: A \rightarrow B \) in \( \mathcal{C} \) such that

\[
\top_A \leq P_{\langle \text{id}_A, f \rangle}(\phi).
\]

**Examples 2.12.** Recall from [11] that the doctrine of weak subobjects \( \Psi_D: \mathcal{D}^{\text{op}} \rightarrow \text{InfSL} \) presented in Example 2.7 (2) satisfies the Extended Rule of Choice.
Definition 2.13. Let $P : \mathcal{C} \text{op} \to \text{InfSL}$ be an existential doctrine. An object $B$ of $\mathcal{C}$ is equipped with Hilbert’s $\epsilon$-operators if, for any object $A$ in $\mathcal{C}$ and any $\alpha$ in $P(A \times B)$ there exists an arrow $\epsilon_{\alpha} : A \to B$ such that

$$\exists_{\text{pr}_1}(\alpha) = P_{(\text{id}_A, \epsilon_{\alpha})}(\alpha)$$

holds in $P(A)$, where $\text{pr}_1 : A \times B \to A$ is the first projection.

Definition 2.14. We say that an elementary existential doctrine $P : \mathcal{C} \text{op} \to \text{InfSL}$ is equipped with Hilbert’s $\epsilon$-operators if every object in $\mathcal{C}$ is equipped with $\epsilon$-operators.

Examples 2.15. We recall from [11] that the doctrine $H(-) : \text{Set}^{\text{op}} \to \text{InfSL}$ presented in Example 2.7 (4) is equipped with $\epsilon$-operators. In particular for every element $\alpha \in H A \times B$, and for every $a \in A$ one can consider the (non empty) set $I_{\alpha}(a) = \{ b \in B \mid \alpha(a, b) = \bigvee_{c \in B} \alpha(a, c) \}$.

Then, by the axiom of choice, there exists a function $\epsilon_{\alpha} : A \to B$ such that $\epsilon_{\alpha}(a) \in I_{\alpha}(a)$. Therefore we have that

$$\alpha(a, \epsilon_{\alpha}(a)) = \bigvee_{b \in B} \alpha(a, b) = \exists_{\text{pr}_1}(\alpha)(a)$$

and this prove that $H(-)$ satisfies the epsilon rule.

We conclude this section recalling from [26, 12, 6] the notion of doctrine with (strong) comprehensions. This notion connects an abstract elementary doctrine with that of the subobjects of the base when this has finite limits and provides an abstract algebraic counterpart of the set-theoretic “comprehension axiom”.

Definition 2.16. Let $P : \mathcal{C} \text{op} \to \text{InfSL}$ be a primary doctrine and $\alpha$ be an object of $P(A)$. A comprehension of $\alpha$ is an arrow $\{ \alpha \} : X \to A$ such that $P_{\{ \alpha \}}(\alpha) = \top_X$ and, for every $f : Z \to A$ such that $P_f(\alpha) = \top_Z$, there exists a unique map $g : Z \to X$ such that $f = \{ \alpha \} \circ g$.

Intuitively, the domain of the comprehension morphism of the predicate $\alpha$ represents the set $\{ x \in A \mid \alpha(x) \}$ containing the elements of the object $A$ satisfying $\alpha$. Then, one says that $P$ has comprehensions if every $\alpha$ has a comprehension, and that $P$ has full comprehensions if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{ \alpha \}$ factors through $\{ \beta \}$.

Note that each morphism can be the full comprehension of a unique object:

Lemma 2.17. Let us consider a doctrine

$$P : \mathcal{C} \text{op} \to \text{InfSL}$$

such that $P$ has full comprehensions. If $\{ \alpha_1 \} = \{ \alpha_2 \}$ then $\alpha_1 = \alpha_2$. 

Remark 2.18. For every $f: A' \rightarrow A$ in $C$ the mediating arrow between the comprehensions $\{\alpha\} : X \rightarrow A$ and $\{P_f(\alpha)\} : X' \rightarrow A'$ produces a pullback

Thus comprehensions are stable under pullbacks. Moreover it is straightforward to verify that if $\{\alpha\} : X \rightarrow A$ is a comprehension of $\alpha$, then $\{\alpha\}$ is monic.

Notation: given an $\alpha \in P(A)$, we will denote by $A_\alpha$ the domain of the comprehension $\{\alpha\}$.

Lemma 2.19. Let $P : C^{op} \rightarrow \text{InfSL}$ be an doctrine with full comprehensions, and suppose that $P$ has left adjoints along comprehensions, satisfying (BCC), i.e. for every pullback

we have $P_f \exists_{\{\alpha\}} = \exists_{\{P_f(\alpha)\}} P_f$. Then we have

$$\alpha = \exists_{\{\alpha\}}(\top_{A_\alpha})$$

for every element $\alpha$ in $P(A)$.

Proof. Let $\alpha$ be an element of $P(A)$, and let us consider the comprehension $\{\alpha\} : A_\alpha \rightarrow A$. First, it is direct to see that $\exists_{\{\alpha\}}(\top_{A_\alpha}) \leq \alpha$ since $\exists_{\{\alpha\}}$ is left adjoint to $P_{\{\alpha\}}$ and $P_{\{\alpha\}}(\top_{A_\alpha}) = \top_{A_\alpha}$. Notice that since comprehensions are monomorphisms, and pullbacks along comprehensions always exist by Remark 2.18, we have that $P_{\{\alpha\}} \exists_{\{\alpha\}} = \text{id}$ by BCC. In particular we have $P_{\{\alpha\}}(\exists_{\{\alpha\}}(\top_{A_\alpha})) = \top_{A_\alpha}$. So, by fullness, $\alpha \leq \exists_{\{\alpha\}}(\top_{A_\alpha})$, and then we can conclude that $\alpha = \exists_{\{\alpha\}}(\top_{A_\alpha})$.

We introduce a class of doctrines whose fibre equality turns out to be equivalent to the morphism equality of their base morphisms (see proposition 2.2 in [11] and the original notion called “comprehensive equalizers” in [6]).

Definition 2.20. An elementary doctrine $P : C^{op} \rightarrow \text{InfSL}$ has comprehensive diagonals if the arrow $\Delta_A$ is the comprehension of the element $\delta_A \in P(A)$ for every objects $A$. 

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Recall from [11] that:

**Definition 2.21.** An elementary doctrine is called **m-variational** if it has full comprehensions and comprehensive diagonals.

We summarize some useful properties and results about existential m-variational doctrines.

**Lemma 2.22.** Let \( P : \mathbb{C}^{op} \to \text{InfSL} \) be an existential m-variational doctrine. Then

1. an arrow \( f : A \to B \) is monic if and only if \( P_{f \times f}(\delta_B) = \delta_A \);
2. an element \( \phi \in P(A \times B) \) is functional, i.e. it satisfies

\[
P_{(\text{pr}_1, \text{pr}_2)}(\phi) \land P_{(\text{pr}_1, \text{pr}_3)}(\phi) \leq P_{(\text{pr}_2, \text{pr}_3)}(\delta_B)
\]

in \( P(A \times B \times B) \) if and only if \( \text{pr}_4(\phi) \) is monic, where \( \text{pr}_A : A \times B \to A \) is the first projection.

**Proof.** 1. If \( f \) is monic then \( P_{f \times f}(\delta_B) = \delta_A \) follows by [12, Cor. 4.8], while the other direction follows from [11, Rem. 2.14].

2. Suppose that \( \phi \in P(A \times B) \) is functional, and let us consider its comprehension \( \{ \phi \} : E \to A \times B \), which is an arrow of the form \( \{ \phi \} := (f_1, f_2) \). Then, we have that proving that \( \text{pr}_4(\phi) \) is monic is equivalent to prove that \( P_{f_1 \times f_1}(\delta_A) = \delta_E \) by (1). Recall that every comprehension is monic, in particular we have that

\[
\delta_E = P_{\{\phi\} \times \{\phi\}}(\delta_{A \times B}). \tag{1}
\]

Hence, since by definition we have that \( \delta_{A \times B} = P_{(\text{pr}_1, \text{pr}_3)}(\delta_A) \land P_{(\text{pr}_2, \text{pr}_3)}(\delta_B) \), we have that [11] implies

\[
\delta_E = P_{f_1 \times f_1}(\delta_A) \land P_{f_2 \times f_2}(\delta_B) \tag{2}
\]

Now, since \( \phi \) is functional we have that

\[
P_{(\text{pr}_1, \text{pr}_2)}(\phi) \land P_{(\text{pr}_1, \text{pr}_3)}(\phi) \leq P_{(\text{pr}_2, \text{pr}_3)}(\delta_B)
\]

in \( P(A \times B \times B) \), and also

\[
P_{(\text{pr}_1, \text{pr}_2)}(\phi) \land P_{(\text{pr}_3, \text{pr}_4)}(\phi) \land P_{(\text{pr}_1, \text{pr}_3)}(\delta_A) \leq P_{(\text{pr}_2, \text{pr}_3)}(\delta_B) \land P_{(\text{pr}_1, \text{pr}_2)}(\delta_A)
\]

in \( P(A \times B \times A \times B) \). Finally

\[
P_{\{\phi\} \times \{\phi\}}(P_{(\text{pr}_1, \text{pr}_2)}(\phi) \land P_{(\text{pr}_3, \text{pr}_4)}(\phi) \land P_{(\text{pr}_1, \text{pr}_3)}(\delta_A)) \leq P_{\{\phi\} \times \{\phi\}}(P_{(\text{pr}_2, \text{pr}_3)}(\delta_B) \land P_{(\text{pr}_1, \text{pr}_2)}(\delta_A)) = \delta_E.
\]

Notice that the left side of the inequality is exactly

\[
P_{\{\phi\} \times \{\phi\}}(P_{(\text{pr}_1, \text{pr}_2)}(\phi) \land P_{(\text{pr}_3, \text{pr}_4)}(\phi) \land P_{(\text{pr}_1, \text{pr}_3)}(\delta_A)) = P_{f_1 \times f_1}(\delta_A) \tag{3}
\]

because \( P_{\{\phi\} \times \{\phi\}}(P_{(\text{pr}_1, \text{pr}_2)}(\phi)) = P_{\text{pr}_E}P_{\{\phi\}}(\phi) = \top \) and similarly \( P_{\{\phi\} \times \{\phi\}}(P_{(\text{pr}_3, \text{pr}_4)}(\phi)) = \top. \) Hence we have

\[
P_{f_1 \times f_1}(\delta_A) \leq \delta_E
\]
and then we obtain $P_{f_1 \times f_2}(\delta_A) = \delta_E$ and by (1) we conclude $\text{pr}_A \{ \phi \}$ is monic.

One can prove the other direction using similar arguments. We just sketch the idea. Observe that if $\text{pr}_1 \{ \phi \}$ is monic, then we have that

$$P_{f_1 \times f_2}(\delta_A) \leq P_{f_2 \times f_2}(\delta_B).$$

This follows from the fact that comprehensions are monic, i.e. $\delta_E = P_{\{ \phi \} \times \{ \phi \}}(\delta_A \times B)$, from (2) and that $f_1$ is monic by our assumption, i.e $\delta_E = P_{f_1 \times f_1}(\delta_A)$.

Therefore we have that $\exists f_2 \times f_2 P_{f_1 \times f_1}(\delta_A) \leq \delta_B$. Now we employ the same argument as before: notice that the equality (3) still holds, so we have $P_{f_1 \times f_1}(\delta_A) = P_{\{ \phi \} \times \{ \phi \}}(P_{(pr_1, pr_2)}(\phi) \land P_{(pr_2, pr_3)}(\phi) \land P_{(pr_1, pr_3)}(\delta_A))$. Hence we have that

$$\exists f_2 \times f_2 P_{f_1 \times f_1}(\delta_A) = \exists \{pr_2, pr_4\} \exists \{\phi \} \times \{\phi \} P_{\{pr_1, pr_2\}}(\phi) \land P_{(pr_2, pr_4)}(\phi) \land P_{(pr_1, pr_3)}(\delta_A))$$

and then we can conclude that

$$\exists \{pr_2, pr_4\} (P_{(pr_1, pr_2)}(\phi) \land P_{(pr_2, pr_4)}(\phi) \land P_{(pr_1, pr_3)}(\delta_A)) \leq \delta_B$$

and hence

$$P_{(pr_1, pr_2)}(\phi) \land P_{(pr_2, pr_4)}(\phi) \land P_{(pr_1, pr_3)}(\delta_A) \leq P_{(pr_2, pr_4)}(\delta_B).$$

In particular, applying to both left and right side of the previous inequality the functor $P_{(pr_1, pr_2, pr_3, pr_4)}$, where we consider the projections from $A \times B \times B$, we can conclude that

$$P_{(pr_1, pr_2)}(\phi) \land P_{(pr_2, pr_4)}(\phi) \land P_{(pr_1, pr_3)}(\delta_A) \leq P_{(pr_2, pr_4)}(\delta_B)$$

since $P_{(pr_1, pr_2, pr_3, pr_4)}(P_{(pr_1, pr_3)}(\delta_A)) = P_{(pr_1, pr_3)}(\delta_A) = \top$. 

As pointed out in [11, 12, 31], both full comprehensions and comprehensive diagonals can be freely added to a given doctrine. In particular, recall that the Grothendieck category $\mathcal{G}_P$ of points of the doctrine $P: \mathcal{C}^{op} \rightarrow \mathsf{InfSL}$ provides the free addition of comprehensions. This construction is called comprehension completion.

**Definition 2.23.** Given a primary doctrine $P: \mathcal{C}^{op} \rightarrow \mathsf{InfSL}$ we can define the comprehension completion $P_e: \mathcal{G}_P^{op} \rightarrow \mathsf{InfSL}$ of $P$ as follows:

- an object of $\mathcal{G}_P$ is a pair $(A, \alpha)$ where $A$ is a set and $\alpha \in P(A)$;
- an arrow $f: (A, \alpha) \rightarrow (B, \beta)$ if a arrow $f: A \rightarrow B$ such that $\alpha \leq P(f)(\beta)$.

The fibres $P_e(A, \alpha)$ are given by those elements $\gamma$ of $P(A)$ such that $\gamma \leq \alpha$. Moreover, the action of $P_e$ on morphism $f: (B, \beta) \rightarrow (A, \alpha)$ is defined as $P_e(f)(\gamma) = P(f)(\gamma) \land \beta$ for $\gamma \in P(A)$ such that $\gamma \leq \alpha$.

Similarly, the construction which freely adds comprehensive diagonal is provided by the extensional reflection.
Definition 2.24. Given an elementary doctrine $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ we can define extensional reflection $P_x : \mathcal{X}^{\text{op}}_P \to \text{InfSL}$ of $P$ as follows: the base category $\mathcal{X}_P$ is the quotient category of $\mathcal{C}$ with respect to the equivalence relation where $f \sim g$ when $\top_A \leq P(f,g)(\delta_B)$. The equivalence class of a morphism $f$ of $\mathcal{C}$, i.e. an arrow of $\mathcal{X}_P$, is denoted by $[f]$.

Finally, we denoted by $\text{Prd}_P$ the category of predicates of a doctrine $P$, i.e. the category defined as $\text{Prd}_P : = \mathcal{X}_P$.

Again we refer to [11] for a complete description of these constructions.

3. Generalized existential completion

We recall here from [13] how to complete a conjunctive doctrine to a generalized existential doctrine with respect to a class $\Lambda$ of morphisms of $\mathcal{C}$ closed under composition, pullbacks and containing the identities.

Definition 3.1. A class of morphisms $\Lambda$ of a category $\mathcal{C}$ is called a left class of morphisms if it satisfies the following conditions:

- given an arrow $fh$ of $\mathcal{C}$, if $f \in \Lambda$ and $h \in \Lambda$, then we have $fh \in \Lambda$;
- for every arrow $f \in \Lambda$ and $g$ of $\mathcal{C}$, the pullback of $f$ and $g$ exists and $g^* f \in \Lambda$;
- $\text{id}_A \in \Lambda$ for every object $A$ of $\mathcal{C}$.

Examples 3.2. For any category with finite products the class of product projections is an example of a left class of morphisms.

Actually, the class $\Lambda$ represents generalized projections with respect to which we complete a conjunctive doctrine to a generalized existential ones.

Definition 3.3. A left class doctrine is a pair $(P, \Lambda)$, where $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ is a conjunctive doctrine and $\Lambda$ is a left class of morphisms.

Now, we generalize the notion of “existential doctrine” to a doctrine closed under left adjoint to functors $P_f$ for any morphism $f$ in the left class $\Lambda$:

Definition 3.4. A left class doctrine $(P, \Lambda)$ is called a generalized existential doctrine with respect to a left class of morphisms $\Lambda$ if, for any arrow $f : A \to B$ of $\Lambda$, the functor $P_f : P(B) \to P(A)$ has a left adjoint $\exists_f$, and these satisfy:
(BCC) \textit{Beck-Chevalley condition}: for any pullback

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & A' \\
\downarrow f' & & \downarrow f \\
X & \xrightarrow{g} & A
\end{array}
\]

with \( g \in \Lambda \) (hence also \( g' \in \Lambda \)), for any \( \beta \in \mathcal{P}(X) \) the following equality holds

\[ \exists g' P_{f'}(\beta) = P_f \exists g(\beta). \]

(FR) \textit{Frobenius reciprocity}: for every morphism \( f: X \to A \) of \( \Lambda \), for every element \( \alpha \in \mathcal{P}(A) \) and \( \beta \in \mathcal{P}(X) \), the following equality holds

\[ \exists f(P_f(\alpha) \land \beta) = \alpha \land \exists f(\beta). \]

In order to simplify the notation, sometimes we will simply say that \( P \) is a \textit{\( \Lambda \)-existential doctrine} in \[13\] to indicate a left class doctrine \((P, \Lambda)\) that is a generalized existential doctrine.

Moreover, we adopt the following specific names for \( \Lambda \)-existential doctrines when \( \Lambda \) is the class of all the base morphisms or the class of projections:

\begin{definition}
Let \( \mathcal{C} \) be a finite limit category. A \textit{full existential doctrine} is a generalized existential doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) with respect to the class of all the base morphisms.
\end{definition}

\begin{definition}
Let \( \mathcal{C} \) be a finite product category. A \textit{pure existential doctrine} is a generalized existential doctrine \( P: \mathcal{C}^{\text{op}} \to \text{InfSL} \) with respect to the class of all the projections of its base category.
\end{definition}

Now, we define the 2-category \textbf{Lc-CD} as follows:

- \textbf{objects} are left class doctrines \((P, \Lambda)\);
- \textbf{1-cells} are pairs \((F, b): (P, \Lambda) \to (R, \Lambda')\) where \( F: \mathcal{C} \to \mathcal{D} \) is a functor such that for every \( f \in \Lambda \), we have \( F(f) \in \Lambda' \), \( F \) preserves pullbacks along morphisms of \( \Lambda \) and \( b: P \to R \circ F^{\text{op}} \) is a natural transformation.
- a \textbf{2-cell} is a natural transformation \( \theta: F \to G \) from \((F, b)\) to \((G, c)\) such that for every \( A \) in \( \mathcal{C} \) and every \( \alpha \) in \( \mathcal{P}(A) \), we have

\[ b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)). \]

Similarly, we denote by \textbf{Lc-ExD} the 2-full subcategory of \textbf{Lc-CD} whose objects are objects \( \Lambda \)-existential doctrine \((P, \Lambda)\), and a 1-cell of \textbf{Lc-ExD} is a 1-cell \((F, b)\) of \textbf{Lc-CD} such that the natural transformation \( b \) preserves left adjoints of the opportune left classes,
i.e. given \((P, \Lambda)\) and \((R, \Lambda')\), for every \(f : A \rightarrow B\) arrow of \(\Lambda\), we have \(b_B \exists f \in \exists_{P(f)} b_A\). As in the case of the ordinary existential doctrines, the 2-cell remains the same.

Given this setting, we present the construction of the \textit{generalized existential completion} introduced in [13] and consisting of a \(\Lambda\)-existential doctrine \(\text{Ex}^\Lambda(P) : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}\) freely generated from a left class doctrine \((P, \Lambda)\). Such a free construction provides a left 2-adjoint to the forgetful functor \(U : \text{Lc-ExD} \rightarrow \text{Lc-CD}\).

**Definition 3.7 (Generalized existential completion).** For every object \(A\) of \(\mathcal{C}\) consider the following preorder:

- the objects are pairs \((B \xrightarrow{g \in \Lambda} A, \alpha \in PB)\);
- \((B \xrightarrow{h \in \Lambda} A, \alpha \in PB) \leq (D \xrightarrow{f \in \Lambda} A, \gamma \in PD)\) if there exists \(w : B \rightarrow D\) such that

\[
\begin{array}{c}
B \\
\downarrow^f
\end{array}
\begin{array}{c}
D \\
\downarrow^w
\end{array}
\begin{array}{c}
A
\end{array}
\]

commutes and \(\alpha \leq Pw(\gamma)\).

It is easy to see that the previous data give a preorder. We denote by \(\text{Ex}^\Lambda(P)(A)\) the partial order obtained by identifying two objects when

\((B \xrightarrow{h \in \Lambda} A, \alpha \in PB) \equiv (D \xrightarrow{f \in \Lambda} A, \gamma \in PD)\)

in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.

Given a morphism \(f : A \rightarrow B\) in \(\mathcal{C}\), let \(\text{Ex}^\Lambda(P)_f(C \xrightarrow{g \in \Lambda} B, \beta \in PC)\) be the object

\((D \xrightarrow{g \circ f} A, P_g \circ f(\beta) \in PD)\)

where

\[
\begin{array}{c}
D \\
\downarrow^{g \circ f}
\end{array}
\begin{array}{c}
C
\end{array}
\begin{array}{c}
B
\end{array}
\begin{array}{c}
A
\end{array}
\]

is a pullback because \(g \in \Lambda\).

The assignment \(\text{Ex}^\Lambda(P) : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL}\) is called \textit{the generalized existential completion} of \(P\).

**Theorem 3.8.** For every left class doctrine \((P, \Lambda)\) the doctrine \((\text{Ex}^\Lambda(P), \Lambda)\) is a generalized existential doctrine with respect to the left class \(\Lambda\).
Proof. See [13, Thm 4.3].

Again, we fix the notation for the specific cases in which the left class of morphisms of the base category is the class of projections or of all the morphisms:

**Definition 3.9.** Let \( P : C^{\text{op}} \rightarrow \text{InfSL} \) be a conjunctive doctrine on a finite limit (lex) category \( C \). A **full existential completion** of \( P \), denoted with the symbol \( \text{fEx}(P) \), is the generalized existential completion of \( P \) with respect to the class of all the base morphisms.

**Definition 3.10.** Let \( P : C^{\text{op}} \rightarrow \text{InfSL} \) be a conjunctive doctrine on a category \( C \) with finite products. A **pure existential completion**, denoted with the symbol \( \text{pEx}(P) \), is the generalized existential completion of \( P \) with respect to the class of all the projections in its base category.

Observe that we can define a canonical injection of \( \Lambda \)-doctrines

\[
(id_C, \eta_P) : (P, \Lambda) \rightarrow (\text{Ex}^\Lambda(P), \Lambda)
\]

where \((\eta_P)_A : P(A) \rightarrow \text{Ex}^\Lambda(P)(A)\) acts sending

\[
\alpha \mapsto (A \xrightarrow{id_A} A, \alpha \in P(A)).
\]

Similarly, if \( P \) is \( \Lambda \)-existential, we can define the 1-cell of \( \Lambda \)-existential doctrines

\[
(id_C, \varepsilon_P) : (\text{Ex}^\Lambda(P), \Lambda) \rightarrow (P, \Lambda)
\]

which preserves the left-adjoints along morphisms of \( \Lambda \), where \((\varepsilon_P)_A : \text{pEx}^\Lambda(A) \rightarrow P(A)\) acts sending

\[
( B \xrightarrow{f \in \Lambda} A, \beta \in P(B)) \mapsto \exists f(\alpha).
\]

It is direct to check that, if \((P, \Lambda)\) is \( \Lambda \)-existential, then \( \varepsilon_P \eta_P = \text{id}_P \) and that \( \text{id}_{\text{Ex}^\Lambda(P)} \leq \eta_P \varepsilon_P \).

**Remark 3.11.** Observe that every element \(( B \xrightarrow{f \in \Lambda} A, \beta \in P(B))\) of \( \text{Ex}^\Lambda(P)(A) \) is equal to \( \exists f \eta_B(\beta) \) because \( \exists f \) acts as the post-composition.

**Definition 3.12 (\( \Lambda \)-weak subobjects doctrine).** Let \( C \) be a category and let \( \Lambda \) be a left class of morphisms of \( C \). We define the \( \Lambda \)-**weak subobjects** doctrine, or the doctrine of \( \Lambda \)-**weak subobjects**, as the functor

\[
\Psi_\Lambda : C^{\text{op}} \rightarrow \text{InfSL}
\]

assigning to every object \( A \) of \( C \) the poset

\[
\Psi_\Lambda(A) := \{ B \xrightarrow{f} A \mid f \in \Lambda \}
\]

with the usual order given by the factorization, i.e. the partial order induced by the usual preorder on morphisms given by \( f \leq g \) if there exists an arrow \( h \) such that \( f = gh \). The re-indexing functors \( \Psi_\Lambda(f) \) act by pulling back the elements of the fibres.
One can directly check that $\Psi_\Lambda$ is a $\Lambda$-existential doctrine, since the left adjoints are given by the post-composition of arrows of $\Lambda$.

**Definition 3.13.** Given a category $\mathcal{C}$, we call the **trivial doctrine** on $\mathcal{C}$ the functor

$$\Upsilon: \mathcal{C}^{\text{op}} \to \text{InfSL}$$

assigning the poset with only an element to every object $A$ of $\mathcal{C}$ denoted as $\Upsilon(A) = \{ \top \}$.

By definition of generalized existential completion, we immediately obtain the following theorem.

**Theorem 3.14.** The $\Lambda$-existential doctrine $\Psi_\Lambda: \mathcal{C}^{\text{op}} \to \text{InfSL}$ is the generalized existential completion of the trivial doctrine $\Upsilon: \mathcal{C}^{\text{op}} \to \text{InfSL}$.

**Proof.** Observe that the top elements cover the weak subobjects doctrine, i.e. for every object $A$ and for every element $f: B \to A$ in the fibre $\Psi_D(A)$ we have $f = \exists f([\top_B])$, since the left adjoints $\exists f$ are given by the post-composition and the top element $\top_B$ is the identity morphism $\text{id}_B$.

**Examples 3.15.** Observe that the weak subobjects doctrine defined in example 2.12 is a $\Lambda$-weak subobjects doctrine where $\Lambda$ is the class $\Lambda_D$ of all the morphisms of a finite limit base category $\mathcal{D}$. Hence from Theorem 3.14 the weak subobjects doctrine is a full existential completion of the trivial doctrine $\Upsilon: \mathcal{D}^{\text{op}} \to \text{InfSL}$.

Notice that the universal properties of existential completion shown in [13], can be generalized for the arbitrary case of the generalized existential completion.

In particular, the proof of [13, Thm. 4.14] can be adapted to this more general setting, just observing that in the proof the fact the class $\Lambda$ is the class of projections, is not concretely used. The proof depends only on the properties of closure under pull-backs, composition and identities of the class of projections, and hence it can be directly generalized as follow.

**Theorem 3.16.** The forgetful 2-functor $U: \text{Lc-ExD} \to \text{Lc-CD}$ has a left 2-adjoint 2-functor $E: \text{Lc-CD} \to \text{Lc-ExD}$, acting on the objects as $(P, \Lambda) \mapsto (\text{Ex}^\Lambda(P), \Lambda)$.

In the following we are going to define those elements of a $\Lambda$-existential doctrine $P$ which are free from the left adjoints $\exists f$ along $\Lambda$ as an algebraic counterpart of the logical notion of existential-free formula.

To this purpose we first define:

**Definition 3.17.** Let $P: \mathcal{C}^{\text{op}} \to \text{InfSL}$ be an $\Lambda$-existential doctrine. An object $\alpha$ of the fibre $P(B)$ is said an $\Lambda$-existential splitting if for every morphism

$$C \xrightarrow{g \in \Lambda} B$$

and for every element $\beta$ of the fibre $P(C)$, whenever $\alpha \leq \exists g(\beta)$ holds then there exists an arrow $h: B \to C$ such that

$$\alpha \leq P_h(\beta)$$

and $gh = \text{id}$.

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Now we are ready to introduce the notion of \textit{\(\Lambda\)-existential-free} object of a doctrine:

\textbf{Definition 3.18.} Let \(P: C^{\text{op}} \to \text{InfSL}\) be an \(\Lambda\)-existential doctrine. An object \(\alpha\) of the fibre \(P(A)\) is said a \textit{\(\Lambda\)-existential-free} if for every morphism \[ B \xymatrix{ \ar[r]^f & A } \]

\(P_f(\alpha)\) is a \(\Lambda\)-existential splitting.

\textbf{Remark 3.19.} The name \textit{\(\Lambda\)-existential-free} object represents an algebraic version of the syntactic notion of \textit{existential-free formula} because if a \textit{\(\Lambda\)-existential-free} object \(\alpha\) is equal to the existential quantification of an object \(\beta\) along an arrow \(f\) in \(\Lambda\), then \(\alpha\) is equal to a reindexing of \(\beta\). Indeed, if \(\alpha = \exists_f(\beta)\) with \(\beta \in P(B)\) and \(f: B \to A\), then from \(\alpha \leq \exists_f(\beta)\) it follows that there exists an arrow \(h: B \to A\) such that \(\alpha \leq P_h(\beta)\). Moreover, since \(\exists_f(\beta) \leq \alpha\), then \(\beta \leq P_f(\alpha)\) and also \(P_h(\beta) \leq P_h(P_f(\alpha)) = \alpha\) because \(fh = \text{id}_A\). Hence we conclude that \(\alpha = P_h(\beta)\).

Now we are going to observe how the notion of \(\Lambda\)-existential-free object is related to well known choice principles. To this purpose we first introduce a generalization of the notion of Rule of Choice by relativizing the existence of a witness to the arrows of the class \(\Lambda\).

\textbf{Definition 3.20.} For a \(\Lambda\)-existential doctrine \(P: C^{\text{op}} \to \text{InfSL}\), we say that \(P\) satisfies the \textit{Generalized Rule of Choice} with respect to the left class \(\Lambda\) of morphisms, also called \(\Lambda\)-(RC), if whenever \[ \top_A \leq \exists_g(\alpha) \]

where \(g: B \to A\) is an arrow of \(\Lambda\), then there exists an arrow \(f: A \to B\) such that \[ \top_A \leq P_f(\alpha) \]

and \(gf = \text{id}\).

\textbf{Remark 3.21.} A \(\Lambda\)-existential doctrine \(P: C^{\text{op}} \to \text{InfSL}\) has \(\Lambda\)-(RC) if and only if for every object \(A\) of \(C\), the top element \(\top_A \in P(A)\) is \(\Lambda\)-existential-free object.

\textbf{Definition 3.22.} Given a \(\Lambda\)-existential doctrine \(P: C^{\text{op}} \to \text{InfSL}\), we also say that an element \(\alpha\) of the fibre \(P(A)\) is \textit{\(\Lambda\)-covered} by an element \(\beta \in P(B)\) if \(\beta\) is a \(\Lambda\)-existential-free object and there exists an arrow \(f: B \to A\) of \(\Lambda\) such that \(\alpha = \exists_f(\beta)\).

\textbf{Definition 3.23.} We say that a \(\Lambda\)-existential doctrine \(P\) has \textit{enough \(\Lambda\)-existential-free objects} if for every object \(A\) of \(C\) and for every element \(\alpha \in P(A)\) there exists an object \(B\), a \(\Lambda\)-existential-free element \(\beta\) and an arrow \(g: B \to A\) in \(\Lambda\) such that \[ \alpha = \exists_g(\beta)\].

Notice that similar notions were introduced independently in [18] (see also the talk of the second author \url{https://www.youtube.com/watch?v=sAMVU_5RQJQ&t=688s}).

Here we fix some notation for the specific cases in which the left class of morphisms of the base category is the class of projections or of all the morphisms. In particular:
• when the case Λ is the class of all the morphisms we will speak of pure-existential splitting, pure-existential-free and enough-pure-existential-free objects;

• when Λ is the class of all the morphisms of the base category we will speak of full-existential splitting, full-existential-free and enough-full-existential-free objects.

Before proving the main result of this section, we present some useful technical lemmas.

Lemma 3.24. Let $P : \mathsf{C}^{\text{op}} \to \text{InfSL}$ be a Λ-existential doctrine, and let us consider a Λ-existential-free element $\alpha \in P(B)$. If $\exists_f(\alpha) \leq \exists_g(\beta)$ where $f : B \to A$ and $g : C \to A$ are arrows of the base category $\mathsf{C}$ and $\beta \in P(C)$, then there exists an arrow $m : B \to D$ where $D$ is the vertex of the pullback of $f$ along $g$ such that

- $(f^*g)m = \text{id}$;
- $\alpha \leq P(g^*f)m(\beta)$.

And hence also $f = g(g^*f)m$.

Proof. If $\exists_f(\alpha) \leq \exists_g(\beta)$, then $\alpha \leq P_f \exists_g(\beta)$, and applying BCC, we have $\alpha \leq \exists_f g P_g f(\beta)$, where

$$
\begin{array}{c}
D \xrightarrow{f^*g} B \\
\downarrow \quad \quad \quad \downarrow f \\
C \xrightarrow{g} A
\end{array}
$$

is a pullback. Thus, since $\alpha$ is ΛC-existential-free, there exists a morphism $m : B \to D$ such that $\alpha \leq P(g^*f)m(\beta)$ and $(f^*g)m = \text{id}$. In particular $f = g(g^*f)m$.

Lemma 3.25. Let $P : \mathsf{C}^{\text{op}} \to \text{InfSL}$ be a Λ-doctrine. If every element of the form $\eta_A(\alpha)$ for any object $\alpha$ of $P(A)$ with $A$ object of $\mathsf{C}$ is Λ-existential splitting, then every element of the form $\eta_A(\alpha)$ is a Λ-existential-free object.

Proof. It follows by the naturality of $\eta_A$.

Proposition 3.26. Let $P : \mathsf{C}^{\text{op}} \to \text{InfSL}$ be a Λ-existential doctrine equipped with a fibred subdoctrine $P' : \mathsf{C}^{\text{op}} \to \text{InfSL}$ such that every element of $P'$ is a Λ-existential-free element in $P$. Then, if every element of $\alpha$ of $P(A)$ is Λ-covered by an element of $P'$, the elements of $P'$ are exactly those elements of $P$ which are Λ-existential-free.

Proof. Let $\alpha \in P(A)$ be a Λ-existential-free object. We have to prove that $\alpha \in P'(A)$. Now, since every element of $P$ is covered by an element of $P'$, we have that $\alpha = \exists_f(\beta)$ where $\beta$ is Λ-existential-free, $\beta \in P'(B)$ and $f : B \to A$ is in Λ. Then, since $\alpha \in P(A)$ is a Λ-existential-free object, from $\alpha \leq \exists_f(\beta)$ it follows that there exists an arrow
Proposition 3.27. Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a \( \Lambda \)-doctrine. Its generalized existential completion \( \text{Ex}^\Lambda(P) \) satisfies the following conditions:

1. the elements \( \eta_A(\alpha) \) for any object \( \alpha \) of \( P(A) \) with \( A \) object of \( \mathcal{C} \) are \( \Lambda \)-existential-free objects of \( \text{Ex}^\Lambda(P) \);

2. \( \text{Ex}^\Lambda(P) \) has enough-\( \Lambda \)-existential-free objects of the form \( \eta_A(\alpha) \) for any object \( \alpha \) of \( P(A) \) with \( A \) object in \( \mathcal{C} \).

3. the \( \Lambda \)-existential-free objects of \( \text{Ex}^\Lambda(P) \) are exactly the elements \( \eta_A(\alpha) \) for any object \( \alpha \) of \( P(A) \) with \( A \) object in \( \mathcal{C} \).

Proof. (1) We first show that every element of the form \( \eta_A(\alpha) \) is a \( \Lambda \)-existential splitting. Thus, let \( \beta := (C \xrightarrow{f} B, \beta) \) be an object of the fibre \( \text{Ex}^\Lambda(P)(B) \), and suppose that

\[
\eta_A(\alpha) \leq \exists^\Lambda_g(\beta).
\]  
(4)

where \( g: B \rightarrow A \). Recall that, by definition of the doctrine \( \text{Ex}^\Lambda(P) \), the inequality \( \exists^\Lambda_g(\beta) \) means that there exists an arrow \( h: A \rightarrow C \) such that the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{\text{id}_A} & & \downarrow{gf} \\
A & & B
\end{array}
\]

and

\[
\alpha \leq P_h(\beta)
\]  
(5)

We claim that

\[
\eta_A(\alpha) \leq \text{Ex}^\Lambda(P)_{fh}(\beta).
\]  
(6)

Thus, let us consider the pullback

\[
\begin{array}{ccc}
D & \xrightarrow{(fh)^*} & C \\
\downarrow{\text{id}_D} & & \downarrow{f} \\
A & \xrightarrow{fh} & B
\end{array}
\]

Moreover, we have by definition

\[
\text{Ex}^\Lambda(P)_{fh}(\beta) = (D \xrightarrow{(fh)^* f} A, P_{f^*(fh)}(\beta)).
\]
Thus, by the universal property of pullbacks, there exists an arrow $w: A \rightarrow D$ such that the following diagram commutes,

\[
\begin{array}{ccc}
A & \xrightarrow{w} & D \\
\downarrow & & \downarrow \\
D & \xrightarrow{f^*(f_h)} & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{f_h} & B.
\end{array}
\]

Hence, combining (5) with the triangle (●●) we have that

$$\alpha \leq P_h(\beta) = P_w(P_f(f_h)(\beta)).$$

From this and the diagram (●) the claim [6] follows. This ends the proof that an element of the form $\eta_A(\alpha)$ is $\Lambda$-existential splitting. By Lemma 3.25 we conclude that all such elements are $\Lambda$-existential-free objects.

(2) It follows by Remark 3.11.

(3) It follows from (1) and (2) and Proposition 3.26.

\[\square\]

**Corollary 3.28.** Let $P: C^{\text{op}} \rightarrow \text{InfSL}$ be a $\Lambda$-doctrine. Then the generalized existential completion $\text{Ex}^\Lambda(P)$ of the doctrine $P$ satisfies the Generalized Rule of Choice with respect to $\Lambda$.

**Proof.** It follows by Theorem 3.27 since in each fibre of $P: C^{\text{op}} \rightarrow \text{InfSL}$ the top element $\top_A$ is existential splitting for each object $A$ in $C$.

\[\square\]

**Examples 3.29.** Recall that if $\Lambda$ is the class of all the base morphisms of a left class doctrine $P: C^{\text{op}} \rightarrow \text{InfSL}$, then $P$ satisfies the Extended Rule of Choice of 2.10. Hence, from the above corollary we have a proof alternative to that in [11] that the doctrine $\Psi_D$ of weak subobjects satisfies the Extended Rule of Choice.

**Remark 3.30.** Note that full and pure existential completions both satisfy the rule of choice (RC) as a consequence of remark 3.24 since they are generalized existential completions with respect to a class $\Lambda$ containing all the projections of the base category.

Now let us consider a left class doctrine $(P, \Lambda)$ and the forgetful functor $U: \mathcal{G}_P \rightarrow C$ from the Grothendieck category $\mathcal{G}_P$ of $P$. We denote $U^{-1}(\Lambda)$ the class of morphisms in $\mathcal{G}_P$ of the form $f: (A, \alpha) \rightarrow (B, \beta)$ where $f \in \Lambda$.

Notice that, since a left class of morphisms contains identities, we have that every arrow of the form $\text{id}_A: (A, \alpha) \rightarrow (A, \top)$ is contained in $\mathcal{G}_P$.

**Lemma 3.31.** Let $(P, \Lambda)$ a left class doctrine with $P: C^{\text{op}} \rightarrow \text{InfSL}$. Then $(P, U^{-1}(\Lambda))$ is a $\Lambda$-doctrine, too.
Proof. Observe that for any morphism $f : (A, \alpha) \to (B, \beta)$ with $f \in \Lambda$ and for any other morphism $h : (C, \xi) \to (B, \beta)$ by using the pullback projections $h^* f : D \to C$ and $f^* h : D \to A$ of $f$ along $h$ claimed to exist in $C$ because $\Lambda$ is a left class of morphisms, then the following diagram is a pullback diagram in $G_P$

$$
\begin{array}{ccc}
(D, P_h \cdot f(\xi) \wedge P_f \cdot h(\beta)) & \xrightarrow{h^* f} & (C, \xi) \\
\downarrow f^* h & & \downarrow h \\
(A, \alpha) & \xrightarrow{f} & (B, \beta)
\end{array}
$$

and $h^* f : (D, P_h \cdot f(\xi) \wedge P_f \cdot h(\beta)) \to (C, \xi)$ is in $U^{-1}(\Lambda)$ since $h^* f$ is in $\Lambda$. \hfill \Box

Then, for every left class doctrine we can define a morphism of left class doctrines as follows:

**Definition 3.32.** Let $P : C^{\text{op}} \to \text{InfSL}$ be a left class doctrine and let us consider the $U^{-1}(\Lambda)$-weak subobjects on the Grothendieck category $G_P$, $\Psi_{U^{-1}(\Lambda)} : G_P^{\text{op}} \to \text{InfSL}$.

We define the following left class doctrine morphism

$$(L, l) : P \to \Psi_{U^{-1}(\Lambda)}$$

as the composition of the embedding of $P$ into its free comprehension completion $P_c$ followed by the comprehension doctrine morphism $\{ - \}$ associating to an object its comprehension arrow:

$$P^c \xrightarrow{\{ - \}} P_c \xrightarrow{\{ - \}} \Psi_{U^{-1}(\Lambda)}.$$

namely

- $L(A) = (A, \top_A)$ for each object $A$ of $C$
- $L(f) = f$ for each morphism $f$ of $C$
- $l(\alpha) = \| \alpha \|$, i.e. $l(\alpha)$ is the **comprehension map** $l(\alpha) := (A, \alpha) \xrightarrow{\text{id}_A} (A, \top)$ for each object $\alpha$ of $P(A)$.

Observe that:

**Lemma 3.33.** For every object $A$ of the base category $C$, and any left class doctrine $P$, the functor $l : P(A) \to \Psi_{U^{-1}(\Lambda)} (A, \top)$ is full and faithful, i.e. its preserves and reflects the order and preserves finite conjunctions.

*Proof.* It follows from [11, Thm 2.15]. \hfill \Box
We now show a key lemma expressing that comprehensions in the image of $t$ are existential free objects:

**Lemma 3.34.** Every element of the doctrine $\Psi_{U^{-1}(\Lambda)}: \mathcal{G}^{\text{op}}_P \to \text{InfSL}$ of the form $\text{id}_A: (A, \alpha) \to (A, \gamma)$ is a $U^{-1}(\Lambda)$-existential-free element. In particular, comprehension maps are $U^{-1}(\Lambda)$-existential-free elements.

**Proof.** It is direct to check that every element of the form $(A, \alpha) \overset{\text{id}_A}{\to} (A, \gamma)$ is a $U^{-1}(\Lambda)$-existential splitting because left adjoints $\exists_g$ in $\Psi_{U^{-1}(\Lambda)}$ with $g \in U^{-1}(\Lambda)$ are given by the post-composition, and the order in the fibres of $\Psi_{U^{-1}(\Lambda)}$ is given by the factorization. Moreover, it is also direct to check that $(A, \alpha) \overset{\text{id}_A}{\to} (A, \gamma)$ is a $U^{-1}(\Lambda)$-existential free element because for every arrow $h: (B, \beta) \to (A, \gamma)$ of $\mathcal{G}_P$ we have that, by definition, $\Psi_{U^{-1}(\Lambda)}(h)((A, \alpha) \overset{\text{id}_A}{\to} (A, \gamma)) = (B, \beta \land P(h)(\alpha)) \overset{\text{id}_B}{\to} (B, \beta)$.

Notice also that one can check that every element of the fibre $\Psi_{U^{-1}(\Lambda)}(A, \top)$ can be presented just using left adjoints along the class $U^{-1}(\Lambda)$ and comprehension maps as follows:

**Lemma 3.35.** For every element $(B, \beta) \overset{f}{\to} (A, \top)$ of the fibre $\Psi_{U^{-1}(\Lambda)}(A, \top)$ we have that $\left( (B, \beta) \overset{f}{\to} (A, \top) \right) = \exists_f((B, \beta) \overset{\text{id}_B}{\to} (B, \top))$ with $f: (B, \top) \to (A, \top)$ arrow of $U^{-1}(\Lambda)$.

**Proof.** It directly follows by the fact that left adjoints act as the post composition.

Finally, observe that full and faithful doctrine morphisms preserve and reflects existential-free objects as follows:

**Lemma 3.36.** Let us consider a $\Lambda$-existential doctrine $P$, a $\Lambda'$-existential doctrine $R$ and a morphism of generalized existential doctrines $(F, f): P \to R$ such that both $F$ and $f$ are full and faithful. Then we have that an element $\alpha \in P(A)$ is $\Lambda$-existential-free if and only if $f(\alpha) \in R(FA)$ is $F(\Lambda)$-existential-free.

**Proof.** Just recall that $F(\Lambda) \subseteq \Lambda'$ by definition of morphisms between generalized existential doctrines.

Now we are ready to prove the main result of this section.

**Theorem 3.37.** Let $P: \mathcal{C}^{\text{op}} \to \text{InfSL}$ be a $\Lambda$-existential doctrine. Then the following are equivalent:

1. $P$ is a $\Lambda$-existential completion.
2. There exists a fibred conjunctive sub-doctrine $P' : C^{\text{op}} \longrightarrow \text{InfSL}$ of $P$ and a morphism $(\mathcal{T}, \mathcal{I})$ of $\Lambda$-existential doctrines such that the diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{(\text{id}, l)} & P \\
\downarrow^{(L, l)} & & \downarrow^{(\mathcal{T}, \mathcal{I})} \\
\Psi_{U^{-1}(\Lambda)} & & \\
\end{array}
\]

commutes, where $\Psi_{U^{-1}(\Lambda)} : G^{\text{op}}_{P'} \longrightarrow \text{InfSL}$ is the $U^{-1}(\Lambda)$-weak subobjects doctrine on the Grothendieck category $G_{P'}$ of $P'$. Moreover for every object $A$ in $C$ the natural transformation $\overline{T}_A : PA \longrightarrow \Psi_{U^{-1}(\Lambda)}(A, \top)$ is an isomorphism.

3. $P$ satisfies the following points:

(a) $P$ satisfies $\Lambda$-(RC), i.e. each top element $\top_A$ of the fibre $P(A)$ is a $\Lambda$-existential-free object.

(b) for every $\Lambda$-existential-free object $\alpha$ and $\beta$ of $P(A)$, then $\alpha \land \beta$ is a $\Lambda$-existential-free object.

(c) $P$ has enough-$\Lambda$-existential-free objects.

Proof. $(1 \Rightarrow 2)$ This follows from the universal property of the generalized existential completion. In particular, if $P$ is the $\Lambda$-existential completion of a doctrine $P'$, then the arrow $(\overline{T}, \overline{I}) : P \longrightarrow \Psi_{U^{-1}(\Lambda)}$ in definition 3.32 exists by the universal properties of the generalized existential completion. Furthermore, $\overline{T} : P(A) \longrightarrow \Psi_{U^{-1}(\Lambda)}(A, \top)$ is surjective since as observed in Lemma 3.35, every element $g : (B, \beta) \longrightarrow (A, \top)$ can be written as

$$\exists_g(\{\beta\}) = \exists_g(l_B(\beta)) = \exists_g(\overline{T}_B(l_B(\beta))) = \overline{T}_A \exists_g(l_B(\beta)).$$

It remains to prove that $\overline{T}_A$ reflects the order. Now suppose that $\overline{T}_A(\exists f \iota_B(\beta)) \leq \overline{T}_A(\exists \gamma \iota_C(\xi))$. Then this means that

$$(B, \iota_B(\beta)) \xrightarrow{f} (A, \top) \leq (C, \iota_C(\xi)) \xrightarrow{g} (A, \top)$$

in the fibre $\Psi_A(A, \top)$, i.e. that there exists an arrow $h : (B, \iota_B(\beta)) \longrightarrow (C, \iota_C(\xi))$ such that $f = gh$. Moreover, we have that $\iota_B(\beta) \leq P_h(\iota_C(\xi))$ by definition of morphisms in $G_{P'}$. Therefore we have that

$$\exists f \iota_B(\beta) \leq \exists f P_h(\iota_C(\gamma)) = \exists_g \exists h P_h(\iota_C(\gamma)) \leq \exists_g(\iota_C(\gamma)).$$

This ends the proof that for every object $A$, $\overline{T}_A : P(A) \longrightarrow \Psi_{U^{-1}(\Lambda)}(A, \top)$ is an isomorphism of inf-semilattices.
(2 ⇒ 3) By Lemma 3.34 and Lemma 3.36 we have that every element of the form \( \iota_A(\alpha) \) where \( \alpha \in P'(A) \) is \( \Lambda \)-existential-free in \( P(A) \), and by Lemma 3.35 and the fact that \( \iota \) is an isomorphism we conclude that the doctrine \( P \) has enough-\( \Lambda \)-existential-free elements. Finally, observe that the top element and the binary conjunction of \( \Lambda \)-existential free elements are \( \Lambda \)-existential-free elements because \( P' \) is a full conjunctive sub-doctrine of \( P \), i.e. \( \iota \) preserves and reflects the order.

(3 ⇒ 1) Let \( (P,A) \) be a \( \Lambda \)-existential doctrine satisfying (a), (b) and (c). The we can define the conjunctive doctrine

\[
\overline{P} : \mathbb{C}^{\text{op}} \longrightarrow \text{InfSL}
\]

as the functor which sends an object \( A \) to the poset \( \overline{P}(A) \) whose elements are the \( \Lambda \)-existential-free objects of \( P(A) \) with the order induced from that of \( P(A) \), and such that \( \overline{P}f = Pf \) for every arrow \( f \) of \( \mathbb{C} \). Notice that this functor is a conjunctive doctrine because \( \Lambda \)-existential-free objects are stable under re-indexing by definition, and they are closed under the top element and binary conjunctions by the assumptions (a) and (c). Therefore, by the universal property of the generalized existential completion, there exists a 1-cell of \( \text{Le-ExD} \)

\[
\begin{array}{ccc}
\mathbb{C}^{\text{op}} & \xrightarrow{\overline{P}} & \text{InfSL} \\
\downarrow{\text{id}} & & \downarrow{\varphi} \\
\mathbb{C}^{\text{op}} & & \text{InfSL}
\end{array}
\]

where the map \( \varrho_A : \text{Ex}^\Lambda(\overline{P})(A) \longrightarrow P(A) \) sends

\[
( B \xrightarrow{f} A, \alpha ) \mapsto \exists f(\alpha).
\]

In particular, \( \varrho \) is a morphism of inf-semilattices and it is natural with respects to left adjoints along the class \( \Lambda \). Moreover, notice that for every object \( A \), we have that \( \varrho_A : \text{Ex}^\Lambda(\overline{P})(A) \longrightarrow P(A) \) is surjective on the objects since \( P \) has enough-\( \Lambda \)-existential-free objects, i.e. every object \( \alpha \) of \( P(A) \) is of the form \( \exists g(\beta) \) for some \( g : B \longrightarrow A \) and \( \beta \in P(B) \). Now we want to show that \( \varrho \) reflects the order, and hence that it is an isomorphism. Suppose that

\[
\exists f(\alpha) \leq \exists g(\beta)
\]

where \( f : B \longrightarrow A \) and \( g : C \longrightarrow A \) are arrows of \( \mathbb{C} \), and \( \alpha \in P(B) \) and \( \beta \in P(C) \) are \( \Lambda \)-existential-free objects. Then we have to prove that

\[
( B \xrightarrow{f} A, \alpha ) \leq ( C \xrightarrow{g} A, \beta ).
\]
By (7) we have that $\alpha \leq P_f \exists g(\beta)$. Now we can consider the pullback in $C$

\[
\begin{array}{ccc}
D & \xrightarrow{f^*g} & B \\
\downarrow & & \downarrow \\
g^*f & \xrightarrow{f} & f \\
\downarrow & & \downarrow g \\
C & \xrightarrow{g} & A
\end{array}
\]

and, after applying BCC, we obtain

$$\alpha \leq \exists f^*g P g^*f(\beta).$$

Therefore, since $\alpha$ is a $\Lambda$-existential-free object, there exists an arrow $h: B \to D$ such that

\[
\begin{array}{ccc}
B & \xrightarrow{h} & D \\
\downarrow & & \downarrow \\
n_B & \xrightarrow{(f^*g)h} & (f^*g) \\
\downarrow & & \downarrow \text{id}_B \\
B & & B
\end{array}
\]

and

$$\alpha \leq P_h P g^*f(\beta)$$

From this, it follows that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{(g^*f)h} & C \\
\downarrow & & \downarrow g \\
f & \xrightarrow{g} & A
\end{array}
\]

commutes because $g(g^*f)h = f(f^*g)h = f$, and $\alpha \leq P(g^*f)h(\beta)$. Thus, we have proved that $(B \xrightarrow{f} A, \alpha) \leq (C \xrightarrow{g} A, \beta)$. Therefore, we can conclude that $(\text{id}, g)$ is an invertible 1-cell of $\text{Le-ExD}$, and then $\text{Ex}^\Lambda(\mathcal{P}) \cong P$. 

Observe that when the existential completion of a conjunctive doctrine $P: \mathcal{C}^\text{op} \to \text{InfSL}$ is full, i.e. the existential completion of $P$ is performed with respect to the whole class of morphisms of $\mathcal{C}$, then such an existential completion comes equipped with comprehensive diagonals. From a logical perspective this does not surprise, since the full existential completion freely adds left adjoints, and in particular equality predicates as left adjoint along diagonals.

**Corollary 3.38.** Let $P: \mathcal{C}^\text{op} \to \text{InfSL}$ be the full existential completion of a conjunctive subdoctrine $P'$. Then $P$ has comprehensive diagonals.
Proof. By Theorem 3.37 we have that every top element is full-existential-free. Now, if we consider two arrows \( f, g: A \rightarrow B \), if \( \top_A \leq P(f,g)(\top_B) \) then we have that \( \exists(f,g)(\top_A) \leq \delta_B = \exists \Delta_B(\top_B) \). The by Lemma 3.24 we can conclude that there exists an arrow \( m \) such that \( \langle f, g \rangle = \Delta_B m \), and this means that \( f = g \). So by proposition 2.2 in [11] we conclude that \( P \) has comprehensive diagonals.

Finally we conclude this section by showing another equivalent characterization of \( \Lambda \)-existential completions via the comprehension completion construction.

**Proposition 3.39.** Let \( P: \text{C}^{\text{op}} \rightarrow \text{InfSL} \) be a \( \Lambda \)-existential doctrine, and let us consider a fibred conjunctive subdoctrine \( P': \text{C}^{\text{op}} \rightarrow \text{InfSL} \) of \( P \). Then the following conditions are equivalent:

1. \( P = \text{Ex}^\Lambda(P') \) is the \( \Lambda \)-existential completion of \( P' \).
2. There exists an isomorphism \((\hat{I}, \hat{\iota}): P_c I^{\text{op}} \rightarrow \Psi_{U^{-1}(\Lambda)} \) of \( \Lambda \)-existential doctrines such that the diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{(\hat{I}, \hat{\iota})} & P_c I^{\text{op}} \\
\downarrow & & \downarrow \text{id}_{P_c I^{\text{op}}} \\
\Psi_{U^{-1}(\Lambda)} & \xrightarrow{(L, I)} & \Psi_{U^{-1}(\Lambda)}
\end{array}
\]

commutes.

Proof. Notice that given a \( \Lambda \)-existential doctrine \( P: \text{C}^{\text{op}} \rightarrow \text{InfSL} \) and a fibred subdoctrine \( P': \text{C}^{\text{op}} \rightarrow \text{InfSL} \) we always have a full and faithful injection of categories

\[
I: \mathcal{G}_P \hookrightarrow \mathcal{G}_{P'}
\]

given by \( I(A, \alpha) = (A, \alpha) \) and \( I(f) = f \). Thus, we can consider the \( \Lambda \)-existential doctrine given by the functor \( P_c I^{\text{op}} \)

\[
\mathcal{G}_P^{\text{op}} \xrightarrow{I^{\text{op}}} \mathcal{G}_{P'}^{\text{op}} \xrightarrow{P_c} \text{InfSL}
\]

which is essentially the restriction of the comprehension completion doctrine \( P_c \) to \( \mathcal{G}_{P'} \).

Moreover, notice that we have an injection of doctrines \((\hat{I}, \hat{\iota}): P_c I^{\text{op}} \rightarrow P_c I^{\text{op}} \)

\[
\hat{I}(A) = (A, \top) \text{ and } \hat{\iota}(\alpha) = (A, \alpha) \xrightarrow{\text{id}_A} (A, \top).
\]

Therefore, (2) \( \Rightarrow \) (1) follows because the commutativity of the diagram implies the second point (2) of Theorem 3.37 as a particular case.

Instead (1) \( \Rightarrow \) (2) follows since the fibres \( P_c(A, \alpha) \) are contained in \( P(A) \), which is isomorphic to \( \Psi_{U^{-1}(\Lambda)}(A, \top) \) by second point (2) of Theorem 3.34.

In the particular case of the full existential completion we obtain the following corollary relating full existential completions and weak subobjects doctrines.

27
Corollary 3.40. Let \( P': \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a fibred conjunctive subdoctrine of a full existential doctrine \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \). Then the following conditions are equivalent:

1. \( P = \text{fEx}(P') \) is the full existential completion of \( P' \).

2. There exists an isomorphism \((\text{id}_{\mathcal{G}_P}, \hat{1}): P_{c} \text{I} \mathcal{G}_P' \rightarrow \Psi \mathcal{G}_P'\) commuting with the canonical injections of \( P' \) in \( P \) and of \( P' \) in the weak subobjects doctrine \( \Psi \mathcal{G}_P' \).

3.1. A characterization of pure existential completions

Here we focus our attention on pure existential completions \( \text{pEx}(P) \), i.e. to existential completions with respect to the class of product projections in the base category \( \mathcal{C} \) of a primary doctrine \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) as originally introduced in [13]. Contrary to other existential completions, pure existential ones \( \text{pEx}(P) \) inherit the elementary structure from any primary doctrine \( P \) generating them and they also induce it on \( P \) when they are elementary.

Our aim is to produce a specific characterization for elementary pure existential completions \( \text{pEx}(P') \) of a primary doctrine \( P' \) analogous to Proposition 3.39, but considering the weak subobjects of the category of predicates \( \mathcal{P}_{\mathcal{D}} \) instead of the Grothendieck category \( \mathcal{G}_P' \).

In [13] it is shown that the assignment \( P \mapsto \text{pEx}(P) \) of a primary doctrine to its pure existential completion extends to a lax-idempotent 2-monads \( T^{\text{ex}}: \mathcal{PD} \rightarrow \mathcal{PD} \) on the 2-category of primary doctrines, and that the 2-category \( T^{\text{ex}}\text{-Alg} \) of algebras is isomorphic to the 2-category \( \text{ExD} \) of existential doctrines. Moreover, the 2-monad \( T^{\text{ex}} \) preserves the elementary structure, i.e. it can be restricted to the 2-category of elementary doctrines.

**Examples 3.41 (Regular fragment of Intuitionistic Logic).** Let \( \mathcal{L}_{=,3} \) be the Regular fragment of first order intuitionistic logic [17], i.e. the fragment with equality, conjunction and existential quantification of first order intuitionistic logic. Then the elementary existential doctrine

\[
\text{LT}_{=,3}: \mathcal{C}_{\mathcal{L}_{=,3}}^{\text{op}} \rightarrow \text{InfSL}
\]

is the existential completion of the syntactic elementary doctrine

\[
\text{LT}_{=}: \mathcal{C}_{\mathcal{L}_=}^{\text{op}} \rightarrow \text{InfSL}
\]

associated to the Horn fragment \( \mathcal{L}_= \), i.e. the fragment of first order intuitionistic logic with conjunctions and equality. This result is a consequence of the fact that extending the language \( \mathcal{L}_= \) with existential quantifications is a free operation, so by the known equivalence between doctrines and logic, the elementary existential doctrine \( \text{pEx}(\text{LT}_{\mathcal{L}_=}) \) must coincide with the syntactic doctrine \( \text{LT}_{=,3} \), since both completions are free.

From theorem 3.39 we immediately deduce:

**Corollary 3.42.** Let \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a pure existential doctrine. Then \( P \) is an instance of the pure existential completion construction if and only if the following conditions hold:
1. $P$ satisfies (RC);

2. $P$ has enough-pure-existential-free objects;

3. for every pure-existential-free object $\alpha$ and $\beta$ of $P(A)$, $\alpha \land \beta$ is pure-existential-free object.

From an algebraic point of view, since the existential doctrines are exactly the $T^\text{ex}$-algebras [13], corollary 3.42 characterizes free algebras of the monad $T^\text{ex}$.

Moreover, in [13] it was shown that the pure existential completion preserves the elementary structure. Here we also prove that an elementary pure existential completion must be the pure existential completion of an elementary doctrine:

**Theorem 3.43.** Let $P^\prime: C^{op} \longrightarrow \text{InfSL}$ a primary doctrine and $P = \text{pEx}(P^\prime)$ its pure existential completion. The following conditions are equivalent:

1. $P^\prime$ is elementary

2. $\text{pEx}(P^\prime)$ is elementary.

*Proof.* For $(1) \Rightarrow (2)$ see [13, Prop. 6.1].

For $(2) \Rightarrow (1)$ we proceed as follows. First of all recall that left adjoints along the functors $P_{\Delta \times \text{id}}$ are of the form

$$\exists_{\Delta \times \text{id}}(\alpha) = P_{(\text{pr}_1, \text{pr}_2)}(\alpha) \land P_{(\text{pr}_2, \text{pr}_2)}(\delta_A)$$

Since $P^\prime$ is the subdoctrine of pure-existential-free objects of $P$, in order to show that $P^\prime$ is elementary it is enough to show that the equality predicates $\delta_A$ are pure-existential-free objects so that $\exists_{\Delta \times \text{id}}$ restricts to $P^\prime$.

We start by showing that $\delta_A$ is a pure-existential-splitting object. Suppose $\delta_A = \exists_{\Delta_A} (\top_A) \leq \exists_{\text{pr}_A \times A}(\beta)$. Then by the equality adjunction

$$\top_A \leq P_{\Delta_A} \exists_{\text{pr}_A \times A}(\beta).$$

Hence, by BCC, we have that

$$\top_A \leq \exists_{\text{pr}_A} P_{\Delta_A \times \text{id}_B}(\beta).$$

Since $P$ satisfies the rule of choice by corollary 3.42, then there exists an arrow $f: A \longrightarrow B$ such that $\top_A \leq P_{(\text{id}_A, f)} P_{\Delta_A \times \text{id}_B}(\beta)$. Now since $(\Delta_A \times \text{id}_B)(\text{id}_A, f) = (\Delta_A \times f)\Delta_A$ we obtain $\top_A \leq P_{\Delta_A} P_{\Delta_A \times f}(\alpha)$ and by the equality adjunction we conclude

$$\delta_A = \exists_{\Delta_A}(\top) \leq P_{\Delta_A \times f}(\beta).$$

This ends the proof that $\delta_A$ is pure-existential-splitting object.

Moreover from $\exists_{\text{pr}_A \times A}(\alpha) \leq \delta_A$ we obtain $\alpha \leq P_{\text{pr}_A \times A}(\delta_A)$ and hence we have $P_{(\text{id}_A, \ast, \varnothing)}(\alpha) \leq P_{(\text{id}_A, \ast, \varnothing)}(P_{\text{pr}_A \times A}(\delta_A)) = \delta_A$. We then conclude that $P_{(\text{id}_A, \ast, \varnothing)}(\alpha) = \delta_A$, namely that $\delta_A$ is a pure-existential-free object.

□
Remark 3.44. Theorem 3.43 does not hold for all generalized existential completions. In sections 3.4 and 3.5 we provide examples of full existential completions which are elementary but their generating primary doctrines are not.

Moreover, a Λ-weak subobjects doctrine Ψ: C^{op} → InfSL on a finite product category C is not necessarily elementary if Λ does not contain the diagonal arrow, whilst its generating trivial primary doctrine Υ: C^{op} → InfSL is elementary, instead.

Remark 3.45. Recall from the Remark 2.6 that any elementary doctrine P: C^{op} → InfSL has left adjoint of every re-indexing functor P_g given by the assignment
\[ \exists_g(\alpha) = \exists_{pr_A}(P_{pr_B}(\alpha) \land P_{(pr_A,g pr_B)}(\delta_A)) \]

However, as observed in [11], these left adjoints do not necessarily satisfies BCC conditions unless P has full comprehensions and comprehensive diagonals. In this case P does not only satisfies the Rule of Choice by 3.42 but also the Extended Rule of Choice as stated in lemma 5.8 of [11].

Generalized existential completions with an elementary structure inherited by their generating subdoctrine are essentially pure existential completions:

**Proposition 3.46.** Let P: C^{op} → InfSL be an elementary existential doctrine and Λ is a left class of morphisms in C containing the projections. Assume also that P = Ex^\Lambda(P') is the generalized existential completion of an elementary doctrine P' and that the canonical injection of P' in P preserves the elementary structure. Then P = pEx(P) is the pure existential completion of P'.

*Proof.* Clearly the pure existential completion, which call Q, of P' is a subdoctrine of P since Λ contains projections. The statement of the proposition amounts to show that Q is actually isomorphic to P.

To meet our purpose it is enough to show that every element α ∈ P(A) is of the form α = \exists_{pr_A}(γ) with γ pure-existential-free element and hence it is a fibre object of Q. Now, since P is the generalized existential completion of P', there exists an arrow g: B → A and a Λ-existential-free element β such that α = \exists_g(β). Moreover, since P is elementary and existential we have that
\[ \alpha = \exists_g(\beta) = \exists_{pr_A}(P_{pr_B}(\beta) \land P_{(pr_A,g pr_B)}(\delta_B)) \]

which actually gives the claimed representation because we can show that P_{pr_B}(β) \land P_{(pr_A,g pr_B)}(\delta_B) is a pure-existential-free element of P. Indeed, P_{(pr_A,g pr_B)}(\delta_B) is Λ-existential-free because P' is elementary and the canonical injection of P' in P is elementary, i.e. δ_B is in P' and hence Λ-existential-free. Moreover since β is also Λ-existential-free, by closure of Λ-existential-free objects under reindexing and conjunctions we conclude that P_{pr_B}(β) \land P_{(pr_A,g pr_B)}(\delta_B)) is also Λ-existential-free and finally also pure-existential-free since Λ contains the projections. This concludes the proof that P is isomorphic to Q, i.e. P is the pure existential completion of P'.

Remark 3.47. Observe that by Theorem 3.44 and (2) of Theorem 3.37 under the hypothesis of Theorem 3.43 we obtain an arrow (\tilde{\mathcal{L}}, \tilde{\xi}): P → Ψ_{U^{-1}(A)} which is both a morphism of elementary and existential doctrines. This is because the generalized weak subobjects relative to the class of projections is elementary and existential, since P' is elementary.

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We can prove a stronger result, which will be fundamental in the sequel of our work and shows the analogous of Proposition \[3.39\] but considering the weak subobjects of the category of predicates \( \mathcal{P}_{dP} \) of \( P' \) instead of the Grothendieck category \( \mathcal{G}_{P} \).

To this purpose we first need to show a lemma. From here on we denote the injection of \( \mathcal{P}_{dP} \) into \( \mathcal{P}_{dP} \) by \( I \):

\[ \mathcal{P}_{dP} \rightarrow \mathcal{P}_{dP} \]

**Lemma 3.48.** Let \( P = p\text{Ex}(P') \) be the pure existential completion of an elementary doctrine \( P' \). Then, the doctrine \( P_{cx}I_{op}: \mathcal{P}_{dP,op} \rightarrow \text{InfSL} \) is elementary and existential with comprehensive diagonals and full weak comprehensions. Furthermore, it satisfies the rule of choice (RC).

**Proof.** First, \( P_{cx}I_{op} \) is elementary and pure existential because both the comprehension and the diagonal comprehensive completion preserve the elementary and existential structures, and \( I_{op} \) preserves projections. Now we show that \( P_{cx}I_{op} \) has weak comprehensions. Let us consider an element \( \beta \in P_{cx}(A, \alpha) \), where \( \alpha \in P'(A) \). Now, since \( P \) has enough-pure-existential-free elements, we have that \( \beta = \exists_{pr_A}(\gamma) \) where \( \gamma \in P'(B \times A) \) and \( pr_A: B \times A \rightarrow A \). Then we define \( \{ \beta \} := (B \times A, \gamma) \xrightarrow{pr_A} (A, \alpha) \), that is a morphism of \( \mathcal{P}_{dP} \). Notice that \( \top(B, \gamma) \leq (P_{cx})(\beta) \) since \( \top(B, \gamma) = f \leq (P_{cx})(\exists_{pr_A}(\gamma)) = (P_{cx})(\beta) \). Moreover, if we consider an arrow \( g: (C, \sigma) \rightarrow (A, \alpha) \) of \( \mathcal{P}_{dP} \) such that \( \top(C, \sigma) = \sigma \leq (P_{cx})(\beta) \), i.e. \( \sigma \leq P_{op}(\beta) \), in particular we have that \( \sigma \leq P_{op}(\exists_{pr_A}(\gamma)) \) then, by BCC we have \( \sigma \leq \exists_{pr_C}P_{dB \times g} \) where \( pr_C: B \times C \rightarrow C \). Now, since \( \sigma \) is an element of \( P'(C) \), it is a pure-existential-free element of \( P \), hence there exists an arrow \( f: C \rightarrow B \) such that \( \sigma \leq P(f, g)(\gamma) \) and \( \langle f, g \rangle: (C, \sigma) \xrightarrow{\top} (B \times A, \gamma) \) is such that \( g = \{ \beta \} \langle f, g \rangle \).

Thus, we have proved that \( \{ \beta \} := (B \times A, \gamma) \xrightarrow{pr_A} (A, \alpha) \) is a weak comprehension of \( \beta \). It is direct to check weak comprehensions are full and that \( P_{cx}I_{op} \) has comprehensive diagonals. Finally, \( P_{cx}I_{op} \) satisfies (RC) because the base category of this doctrine is \( \mathcal{P}_{dP} \), and hence every object of this category is of the form \( (A, \alpha) \) with \( \alpha \) an element of \( P'(A) \) which is in particular, a pure-existential-free elements of \( P \). Hence, if \( \top(A, \alpha) \leq \exists_{pr_A}(\beta) \) then, since \( \top(A, \alpha) = \alpha \), we can use the universal property of pure-existential-free elements to construct the witness, and concluding that \( P_{cx}I_{op} \) satisfies the rule of choice. \qed

**Remark 3.49.** Observe that proposition \[3.48\] holds also for elementary generalized existential completions \( P \) of a primary elementary sub doctrine \( P' \) with respect to a class \( \Lambda \) of morphisms containing the projections and whose embedding of \( P' \) in \( P \) preserves the elementary structure. However, by proposition \[3.48\] such existential completions are actually pure existential completions. So the above statement is the most general one.

Therefore, we conclude:

**Theorem 3.50.** Let \( P': C_{op} \rightarrow \text{InfSL} \) be an elementary fibred subdoctrine of an existential elementary doctrine \( P: C_{op} \rightarrow \text{InfSL} \). The following conditions are equivalent:
1. $P = \text{pEx}(P')$ is the pure existential completion of $P'$.

2. There exists an isomorphism $(\text{id}_{\mathcal{P}'}, \hat{l}): P_{\text{cx}} \mathcal{I}^{\text{op}} \to \Psi_{\mathcal{P}'\mathcal{L}}$ of existential elementary doctrines such that the diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{(\hat{f}, \hat{i})} & P_{\text{cx}} \\
\downarrow & & \downarrow \\
(L, \hat{l}) & \xrightarrow{(\text{id}_{\mathcal{P}'}, \hat{l})} & \Psi_{\mathcal{P}'\mathcal{L}}
\end{array}
\]

commutes.

Proof. $(1) \Rightarrow (2)$ It follows from the fact that, if $P = \text{pEx}(P')$ by Lemma 3.48 we have that the doctrine $P_{\text{cx}} \mathcal{I}^{\text{op}}: \mathcal{P}' \mathcal{L}^{\text{op}} \to \text{InfSL}$ is an existential variational doctrine satisfying the rule of choice, and then, by [11, Thm. 5.9], $P_{\text{cx}} \mathcal{I}^{\text{op}}: \mathcal{P}' \mathcal{L}^{\text{op}} \to \text{InfSL}$ has to be isomorphic to the weak subobjects doctrine on $\mathcal{P}' \mathcal{L}^{\text{op}}$.

$(2) \Rightarrow (1)$ It follows by employing the same arguments in Theorem 3.37 and Theorem 3.50. Observe that while in general $[f][g] = [\text{id}]$ in $\mathcal{P}' \mathcal{L}$ does not imply that $fg = \text{id}$ in $\mathcal{C}$ (recall the notation $[f]$ is introduced in Definition 2.24), for the specific case of projections if $[\text{pr}][g] = \text{id}$ in $\mathcal{P}' \mathcal{L}$ then we have that there exists an arrow $g'$ in $\mathcal{C}$ such that $\text{pr} g' = \text{id}$. Using this, the hypothesis that $P'$ is an elementary subdoctrine of $P$ and the fact that left adjoint can be written as $\exists \gamma(\alpha) = \exists \text{pr}_1 (P_{\text{pr}_1}(\alpha) \wedge (\text{pr}_1, g_\mathcal{L})(\delta))$, we can employ the same arguments used for the general case and conclude.

Combining the previous result with Proposition 3.46 we conclude:

**Corollary 3.51.** Let $P: \mathcal{C}^{\text{op}} \to \text{InfSL}$ be an elementary $\Lambda$-existential doctrine, where $\Lambda$ is left class of morphisms containing the projections. Assume also that $P = \text{Ex}(P')$ is the generalized existential completion of an elementary doctrine $P'$ with respect to $\Lambda$ and that the canonical injection of $P'$ in $P$ preserves the elementary structure. Then, there exists an isomorphism $(\text{id}_{\mathcal{P}'}, \hat{l}): P_{\text{cx}} \mathcal{I}^{\text{op}} \to \Psi_{\mathcal{P}'\mathcal{L}}$ of pure existential, elementary doctrines such that the diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{(\hat{f}, \hat{i})} & P_{\text{cx}} \\
\downarrow & & \downarrow \\
(L, \hat{l}) & \xrightarrow{(\text{id}_{\mathcal{P}'}, \hat{l})} & \Psi_{\mathcal{P}'\mathcal{L}}
\end{array}
\]

commutes.
Examples 3.52 (Regular fragment of Intuitionistic Logic). Observe that the pure-existential-free objects of the existential doctrine

\[ \text{LT}_{=, 3} : \mathcal{C}_{\text{Ex}}^{\text{op}} \longrightarrow \text{InfSL} \]

are exactly the formulae which are free from the existential quantifier, and this doctrine has enough-pure-existential-free objects since every formula in the Regular fragment of intuitionistic logic is provably equivalent to a formula in prenex normal form.

Then, we have that the Regular fragment of Intuitionistic logic \( \text{LT}_{=, 3} \) satisfies the Rule of Choice by Theorem 3.42, and more generally, for every formula \([a : A] \mid \alpha(a)\) which is quantifier-free, we have that

\[ [a : A] \mid \alpha(a) \vdash \exists b : B \beta(a, b) \]

then there exists a term \([a : A] \mid t(a) : B\) such that

\[ [a : A] \mid \alpha(a) \vdash \beta(a, t(a)). \]

This property is called Existence Property in [32].

3.2. A characterization of idempotent \( \text{Tex} \)-algebras

Now we turn our attention to the idempotent \( \text{Tex} \)-algebras, and we show that they coincide with those doctrines which are equipped with Hilbert’s epsilon operator.

The logical relevance of elementary existential doctrines with Hilbert’s epsilon operator is provided in [11], where the authors show that these doctrines are exactly those elementary existential doctrines \( \mathcal{P} \) such that their exact completions \( T_\mathcal{P} \) is equivalent to the exact completion \((\text{Pd}_\mathcal{P})_{\text{ex/lex}}\) of the category of predicates \( \text{Pd}_\mathcal{P} \) via an equivalence induced by the canonical embedding of \( \text{Pd}_\mathcal{P} \) in \( T_\mathcal{P} \).

Actually we will derive such a result as a consequence of a more general characterization.

Theorem 3.53. Let \( \mathcal{P} : \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL} \) be an existential doctrine. Then \( \mathcal{P} \) is isomorphic to \( \text{pEx}(\mathcal{P}) \) if and only if \( \mathcal{P} \) is equipped with an \( \epsilon \)-operators.

Proof. We need to show that \( \mathcal{P} \) is isomorphic to \( \text{pEx}(\mathcal{P}) \) if and only if for every \( A \) and \( B \) of \( \mathcal{C} \) and every \( \alpha \in \mathcal{P}(A \times B) \) there exists arrow \( f : A \longrightarrow B \) such that

\[ \exists \text{pr}_1(\alpha) \leq P_{(\text{id}_A, f)}(\alpha) \quad (11) \]

where \( \text{pr}_1 : A \times B \longrightarrow A \) is the first projection. Recall that by [13] we have the fol-
following adjunction

\[
\begin{array}{c}
\text{C} \text{op} \\
\downarrow \\
\text{P} \rightarrow \\
\downarrow \\
\text{InfSL} \\
\end{array}
\]

with

\[\text{id}_P = \epsilon_P \eta_P \quad \text{and} \quad \text{id}_{P^{ex}} \leq \eta_P \epsilon_P.\]

In particular the doctrine \(P\) is isomorphic to \(P^{ex}\) if and only if \(\eta_P \epsilon_P \leq \text{id}_{P^{ex}}\). By definition of the pre-order in the doctrine \(P^{ex}\), this inequality holds if and only if for every object \(A\) of \(C\) and every \((A \times B \xrightarrow{pr_1} A, \alpha \in P(A \times B)) \in P^{ex}(A)\), there exists an arrow \(f: A \to B\) such that \(\exists_{pr_1}(\alpha) \leq P\langle \text{id}_A, f \rangle(\alpha)\).

**Examples 3.54.** A notable example of doctrine equipped with Hilbert’s \(\epsilon\)-operator, and hence of \(T^{ex}\)-idempotent algebra thanks to Theorem 3.53, is provided by the well-known hyperdoctrine of subsets \(P: \text{Set} \text{op} \to \text{InfSL}\).

In particular, thanks to the Axiom of Choice, one can easily check the doctrine \(P: \text{Set} \text{op} \to \text{InfSL}\) is equipped with \(P: \text{Set} \text{op} \to \text{InfSL}\).

A second example of \(T^{ex}\)-idempotent algebra is provided by the localic hyperdoctrine presented in Example 2.15.

### 3.3. Examples of generalized existential completions with full comprehensions

In this section we show that relevant examples of generalized existential doctrines are given by doctrines with full comprehensions closed under compositions.

**Definition 3.55.** Given a conjunctive doctrine

\[
P: \text{C} \text{op} \to \text{InfSL}
\]

with full comprehensions, we say that \(P\) has **composable comprehensions** if its comprehensions are closed under compositions, namely if for every objects \(\alpha\) in \(P(A)\) and \(\beta\) in \(P(B)\) with \(\|\alpha\|: B \to A\) then we have \(\|\alpha\|\|\beta\| = \|\gamma\|: C \to A\) for \(\gamma\) in \(P(C)\).

Note that \(\gamma\) is unique by Lemma 2.17.

Our aim now is to show that a doctrine \(P: \text{C} \text{op} \to \text{InfSL}\) with full composable comprehensions is an instance of the generalized existential completion construction with respect to the class \(\Lambda_{\text{comp}}\) of comprehensions.

To this purpose, we first show that such a \(P\) is closed under left adjoints along comprehensions:

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Proposition 3.56. Let $P: C^{\text{op}} \to \text{InfSL}$ be a conjunctive doctrine with full comprehensions. Then the following conditions are equivalent:

1. $P$ has composable comprehensions.

2. $P$ has left adjoints along the comprehensions satisfying BCC and FR.

Proof. We start by showing $(2) \Rightarrow (1)$.

Suppose that the doctrine $P$ has left adjoints along comprehensions, and consider two comprehensions $\{\beta\}: C \to B$ and $\{\alpha\}: B \to A$. We claim that the comprehension

$\{\exists_{\{\alpha\}}(\beta)\}: D \to A$

is the composition $\{\alpha\} \{\beta\}$. First of all from the unit of the adjunction $\beta \leq P\{\alpha\} \exists_{\{\alpha\}}(\beta)$ it is direct to verify that

$\top_{C} \leq P\{\beta\} P\{\alpha\} \exists_{\{\alpha\}}(\beta) = P\{\alpha\} \exists_{\{\alpha\}}(\beta)$

Therefore, by comprehension, there exists a unique $g: C \to D$ such that the following diagram commutes

\[
\begin{array}{c}
C \\
\downarrow \{\beta\} \\
B \\
\downarrow \{\alpha\} \\
A
\end{array} \quad \quad \begin{array}{c}
\xrightarrow{g} \\
\exists_{\{\alpha\}}(\beta) \\
\xleftarrow{\exists_{\{\alpha\}}(\beta)} \\
\end{array} \quad \quad \begin{array}{c}
D \\
\downarrow \\
B \\
\downarrow \\
A
\end{array}
\quad \quad \text{(12)}
\]

Now we are going to prove that $g$ is invertible. Observe that by Lemma 2.19 (or see [11]), we have $\alpha = \exists_{\{\alpha\}}(\top_{B})$ from which

$\exists_{\{\alpha\}}(\beta) \leq \exists_{\{\alpha\}}(\top_{B}) \leq \alpha$. Then

$\top_{D} \leq P[\exists_{\{\alpha\}}(\beta)](\exists_{\{\alpha\}}(\beta)) \leq P[\exists_{\{\alpha\}}(\beta)](\alpha)$.

Hence by comprehension there exists a unique $h: D \to B$ such that the following diagram commutes

\[
\begin{array}{c}
D \\
\downarrow \{\exists_{\{\alpha\}}(\beta)\} \\
\xrightarrow{h} \\
B \\
\downarrow \{\alpha\} \\
A
\end{array}
\]

Hence

$\top_{D} \leq P[\exists_{\{\alpha\}}(\beta)](\exists_{\{\alpha\}}(\beta)) = P_{h}(P\{\alpha\} \exists_{\{\alpha\}}(\beta)) \leq P_{h}\beta$ since $P\{\alpha\} \exists_{\{\alpha\}}(\beta) = \beta$ by BCC and the fact that $\{\alpha\}$ is monic.
Therefore by comprehension from \( \top \leq P_h(\beta) \) there is a unique \( l : D \rightarrow C \) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{[\beta]} & B & \xrightarrow{[\alpha]} & A \\
\downarrow & & \downarrow & & \downarrow \\
D & \xrightarrow{l} & \{ \exists_{[\alpha]}(\beta) \} & \rightarrow & A
\end{array}
\]

commutes. Then, since \( \{ \exists_{[\alpha]}(\beta) \} g = \{ \alpha \} \{ \beta \} \) by (12), putting together the known diagrams

\[
\begin{array}{ccc}
D & \xrightarrow{l} & C & \xrightarrow{g} & D \\
\{ \exists_{[\alpha]} \} & \rightarrow & \{ \alpha \} \{ \beta \} & \rightarrow & \{ \exists_{[\alpha]} \}
\end{array}
\]

we get \( \{ \exists_{[\alpha]}(\beta) \} g l = \{ \exists_{[\alpha]}(\beta) \} \) and since \( \{ \exists_{[\alpha]}(\beta) \} \) is monic we conclude \( g l = \text{id}_D \). Analogously, we get

\[
\{ \alpha \} \{ \beta \} l g = \{ \alpha \} \{ \beta \}
\]

and hence \( lg = \text{id}_C \) since \( \{ \alpha \} \{ \beta \} \) is monic. Therefore \( \{ \exists_{[\alpha]}(\beta) \} = \{ \alpha \} \{ \beta \} \), namely we have shown that comprehensions of \( P \) compose.

Now we show that (1) \( \Rightarrow \) (2), namely that if comprehensions compose then there exists the left adjoint along \( \{ \alpha \} \) defined as

\[
\exists_{[\alpha]}(\beta) := \gamma
\]

where \( \gamma \) is the unique element of \( P(A) \) such that \( \{ \alpha \} \{ \beta \} = \{ \gamma \} \).

First, observe that, since comprehensions are full, we have that \( \gamma \leq \sigma \), where \( \sigma \in P(A) \), if and only if

\[
\top \leq P_{[\gamma]}(\sigma)
\]

and then, by definition of \( \gamma \), this holds if and only if

\[
\top \leq P_{[\alpha]}(\alpha)(\sigma).
\]

Then, again by fullness, the previous inequality holds if and only if

\[
\beta \leq P_{[\alpha]}(\sigma).
\]

Therefore, we have that \( \exists_{[\alpha]} \) is left adjoint to \( P_{[\alpha]} \), and it is a direct to verify that these left adjoints satisfy (BCC) and (FR).

\( \blacksquare \)

Examples 3.57. Given an \( M \)-category \( (\mathcal{C}, M) \) one can define the doctrine of \( M \)-subobjects

\[
\begin{array}{ccc}
\text{Sub}_M : C^\text{op} & \longrightarrow & \text{InfSL}
\end{array}
\]

where \( \text{Sub}_M(X) \) if the inf-semilattice of \( M \)-subobjects of \( M \), and the action of \( \text{Sub}_M \) on a morphism \( f : X \rightarrow Y \) of \( \mathcal{C} \) is given by pulling back the \( M \)-subobjects of \( Y \) along
Recall that a \( \mathcal{M} \)-category is a pair \((\mathcal{C}, \mathcal{M})\) where \(\mathcal{C}\) is a category and \(\mathcal{M}\) is a stable system of monics, i.e. \(\mathcal{M}\) is a collection of monics which includes all isomorphisms and is closed under composition and pullbacks. Observe that a stable system of monics is essentially what was called a dominion in [33], an admissible system of subobjects in [34], a notion of partial maps in [15] and a domain structure in [35].

It is direct to prove that given an \(\mathcal{M}\)-category \((\mathcal{C}, \mathcal{M})\), the doctrine of \(\mathcal{M}\)-subobjects \(\text{Sub}_\mathcal{M} : \mathcal{C}^{\text{op}} \to \text{InfSL}\) has full composable comprehensions.

**Proposition 3.58.** Let \(P : \mathcal{C}^{\text{op}} \to \text{InfSL}\) be a conjunctive doctrine with full composable comprehensions, and let \(\Lambda_{\text{comp}}\) be the class of the comprehensions. Then:

1. \(\Lambda_{\text{comp}}\) is a left class of morphisms of \(\mathcal{C}\);
2. for every \(\alpha \in P(A)\), \(\alpha = \exists_{\{\alpha\}}(\top_A)\);
3. an element \(\alpha \in P(A)\) is a \(\Lambda_{\text{comp}}\)-existential-free object if and only if is the top element. In particular the doctrine \(P\) satisfies the \(\Lambda_{\text{comp}}\)-(RC).

**Proof.**

1. The class of comprehensions contains identities, and comprehensions are stable under pullbacks by Remark 2.18 while they compose by Proposition 3.56.

2. The first point follows from Lemma 2.19.

3. First we show that every top element is \(\Lambda_{\text{comp}}\)-existential-free. If \(\top_A \leq \exists_{\{\alpha\}}(\beta)\), then we have that \(\exists_{\{\alpha\}}(\top_A)\) factors on \(\exists_{\{\alpha\}}(\beta)\), which is equal to the arrow \(\{\alpha\} \{\beta\}\) by Proposition 3.50. Then there exists an arrow \(h\) such that the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow{\text{id}_A} & & \downarrow{\{\beta\}} \\
A. & \downarrow{\{\alpha\}} & \\
A.
\end{array}
\]

Now we define \(f := \{\beta\}h\). Thus, we have that \(\{\alpha\}f = \text{id}_A\) and

\[P_f(\beta) = P_f P_{\{\beta\}}(\beta) = \top_A.\]

Now we prove the converse. By point 2. we have that \(\alpha = \exists_{\{\alpha\}}(\top_A)\), and then if \(\alpha\) is a \(\Lambda_{\text{comp}}\)-existential-free object, \(\alpha \leq \exists_{\{\alpha\}}(\top_A)\) implies that there exists an arrow \(f : A \to A_{\{\alpha\}}\) such that \(\alpha \leq P_f(\top_A) = \top_A\) and \(\{\alpha\}f = \text{id}_A\). Since \(\text{id}_A = \{\top_A\}\) by fullness of comprehensions we conclude \(\top_A \leq \alpha\) and hence \(\top_A = \alpha\).

From this theorem we obtain the following one:

**Theorem 3.59.** Every conjunctive doctrine \(P : \mathcal{C}^{\text{op}} \to \text{InfSL}\) with full and composable comprehensions is an instance of the generalized existential completion construction.
with respect to the class \( \Lambda_{\text{comp}} \) of comprehensions. In particular \( P \) is the generalized existential completion of the conjunctive doctrine

\[
\Upsilon : \mathcal{C}^{\text{op}} \to \text{InfSL}
\]

where for every object \( A \) the poset \( \Upsilon(A) \) contains only the top element.

**Proof.** By point (3) of Theorem 3.37 and Proposition 3.58.\( \square \)

**Remark 3.60.** When \( P : \mathcal{C}^{\text{op}} \to \text{InfSL} \) is a doctrine with full composable comprehensions, then the generalized existential completion of the trivial doctrine \( \Upsilon : \mathcal{C}^{\text{op}} \to \text{InfSL} \) is exactly the doctrine \( \text{Sub}_{\Lambda_{\text{comp}}} : \mathcal{C}^{\text{op}} \to \text{InfSL} \) where \( \text{Sub}_{\Lambda_{\text{comp}}}(A) \) is the class of \( \Lambda_{\text{comp}} \)-subobjects of \( A \).

In particular, by Theorem 3.59 we have an isomorphism given by the 1-cell

\[
\begin{array}{ccc}
\mathcal{C}^{\text{op}} & \xrightarrow{P} & \text{InfSL} \\
\downarrow \text{id}_{\mathcal{C}^{\text{op}}} & & \downarrow \text{id}_{\text{InfSL}} \\
\mathcal{C}^{\text{op}} & \xrightarrow{\text{Sub}_{\Lambda_{\text{comp}}}} & \text{InfSL}
\end{array}
\]

where \( \llbracket - \rrbracket_A : P(A) \to \text{Sub}_{\Lambda_{\text{comp}}}(A) \) sends \( \alpha \) to \( \llbracket \alpha \rrbracket \).

In particular, one can directly check that the previous result, together with Example 3.57, extends to an isomorphism of 2-categories.

**Theorem 3.61.** We have an isomorphism of 2-categories

\[
\mathcal{M}\text{-Cat} \cong \text{CE}_c
\]

where \( \mathcal{M}\text{-Cat} \) is the 2-category of \( \mathcal{M} \)-categories and the 2-category \( \text{CE}_c \) of doctrines with full composable comprehensions.

**Corollary 3.62.** Every \( m \)-variational existential doctrine is an instance of generalized existential completion construction with respect to the class of comprehensions.

**Remark 3.63.** Recall that in the case of existential \( m \)-variational doctrines, the equivalence of Theorem 3.61 restricts to an equivalence between the 2-category of existential \( m \)-variational doctrines and the 2-category of proper stable factorization systems \( \text{CE}_c \). We refer to \( [7] \) and \( [24] \) the proof of this equivalence.

In the case of the subobjects doctrine, we have that every that the class of comprehensions is exactly those of all the monomorphisms.

**Corollary 3.64.** The subobjects doctrine \( \text{Sub}_C : \mathcal{C}^{\text{op}} \to \text{InfSL} \) of a category \( C \) with finite limits is the generalized existential completion of the trivial doctrine \( \Upsilon : \mathcal{C}^{\text{op}} \to \text{InfSL} \), with respect to the class of all the monomorphisms \( \Lambda_{\text{mono}} \).
The previous characterizations, in particular of Lemma 3.58, provide also another presentation of existential m-variational doctrines satisfying the Rule of Unique Choice.

**Proposition 3.65.** Let \( P: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a primary doctrine. Then the following are equivalent:

1. \( P \) is existential, m-variational, and it satisfies the Rule of Unique Choice;
2. \( \mathcal{C} \) is regular and \( P = \text{Sub}_{\mathcal{C}} \);
3. \( P \) is the \( M \)-existential completion of the trivial doctrine \( \Upsilon: \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \), where \( M \) is a class of morphisms of \( \mathcal{C} \) such that
   - there exists a class \( E \) of morphisms of \( \mathcal{C} \) such that \((E, M)\) is a proper, stable, factorization system on \( \mathcal{C} \);
   - for every arrow \( \text{pr}_A: A \times B \rightarrow A \) of \( \mathcal{C} \), if \( \text{pr}_A f \) is a monomorphism and \( f \in M \) then \( \text{pr}_A f \in M \).

**Proof.** (1 \( \Rightarrow \) 2) It follows from [11, Prop. 5.3] and [31, Thm. 4.4.4 and Thm. 4.9.4].

(2 \( \Rightarrow \) 3) It follows from Corollary 3.3.

(3 \( \Rightarrow \) 1) Under these assumptions, we have that \( P \) is an existential m-variational doctrine by Remark 3.63 and Corollary 3.62 and the arrows of \( M \) are exactly the comprehensions of \( P \). Now, let us consider a functional, entire relation \( \rho \in P(A \times B) \). First, notice that \( \rho \) functional implies \( \text{pr}_A \{ \rho \} \) monic by Lemma 2.22. Hence, by our assumption, \( \text{pr}_A \{ \rho \} \) is a comprehension. Moreover, we have that

\[
\top_A \leq \exists_{\text{pr}_A}(\rho) = \exists_{\text{pr}_A \{ \rho \}}(\top)
\]  

because \( \rho \) is entire and \( \rho = \exists_{\{\rho\}}(\top) \) by Lemma 2.19. Therefore we can apply Lemma 3.58 because \( \text{pr}_A \{ \rho \} \) is a comprehension. Hence, there exists an arrow \( f: A \rightarrow (A \times B)_\rho \) such that

\[
\top_A \leq P(f(\top))
\]  

and \( \text{pr}_A \{ \rho \} f = \text{id}_A \), namely \( \{ \rho \} f = (\text{id}_A, \text{pr}_B \{ \rho \} f) \). In particular, we have that

\[
\top_A \leq P(f(\top)) = P_{\text{id}_A, \text{pr}_B \{ \rho \} f}(\rho) = P_{\text{id}_A, \text{pr}_B \{ \rho \} f}(\rho).
\]

Therefore, \( P \) satisfies the (RUC).

### 3.4. The realizability hyperdoctrine

In this section we are going to prove that all the realizability triposes \( \mathbf{8}, \mathbf{9}, \mathbf{37} \) are full generalized existential completions as shown independently in \( \mathbf{18} \) and presented in the talk by the second author https://www.youtube.com/watch?v=sAHVU_5RQJQ&t=688s in 2020. A different presentation of realizability triposes via free constructions (essentially acting on the representing object of a given representable indexed preorder) can be found in \( \mathbf{27} \).

We start by recalling some related fundamental notions.
A partial combinatory algebra (pca) is specified by a set $A$ together with a partial binary operation $(-) \cdot (-) : A \times A \rightarrow A$ for which there exist elements $k, s \in A$ satisfying for all $a, a', a'' \in A$ that
\[
k \cdot a \uparrow \text{ and } (k \cdot a) \cdot a' \equiv a
\]
and
\[
s \cdot a \uparrow, (s \cdot a) \cdot a' \downarrow, \text{ and } ((s \cdot a) \cdot a') \cdot a'' \equiv (a \cdot a'') \cdot (a' \cdot a'')
\]
where $e \downarrow$ means "$e$ is defined" and $e \equiv e'$ means "$e$ is defined if and only if $e'$ is, and in that case they are equal". For more details we refer to [37].

Given a pca $A$, we can consider the realizability hyperdoctrine $\mathcal{P} : \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ over $\text{Set}$ introduced in [8, 9]. For each set $X$, the partial ordered set $(\mathcal{P}(X), \leq)$ is defined as follow:

- $\mathcal{P}(X)$ is the set of functions $P(A)^X$ from $X$ to the powerset $P(A)$ of $A$;
- given two elements $\alpha$ and $\beta$ of $\mathcal{P}(X)$, we say that $\alpha \leq \beta$ if there exists an element $\pi \in A$ such that for all $x \in X$ and all $a \in \alpha(x)$, $\pi \cdot a$ is defined and it is an element of $\beta(x)$. By standard properties of pca this relation is reflexive and trans itive, i.e. it is a preorder. Then $\mathcal{P}(X)$ is defined as the quotient of $P(A)^X$ by the equivalence relation generated by $\leq$. The partial order on the equivalence classes $[\alpha]$ is that induced by $\leq$.

Given a function $f : X \rightarrow Y$ of $\text{Set}$, the functor $\mathcal{P}f : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ sends an element $[\alpha] \in \mathcal{P}(Y)$ to the element $[\alpha \circ f] \in \mathcal{P}(X)$ given by the composition of the two functions. With the these assignments $\mathcal{P} : \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ is an hyperdoctrine, see [3, Example 2.3] or [37].

In particular we recall from some Heyting operations which give the structure of Heyting algebra to the fibres $\mathcal{P}(X)$. First, we recall from [37] that from the elements $k$ and $s$ one can construct elements $p, p_1$ and $p_2$ which are called pairing and projections operators. Hence, we can consider the following assignments:

- $\top_X := [\lambda x \in X. A]$;
- $[\alpha] \wedge [\beta] := [\lambda x \in X. \{(p \cdot a) \cdot b \mid a \in \alpha(x) \text{ and } b \in \beta(x)\}]$
- $\delta_X := [\lambda(x_1, x_2) \in X \times X. A \text{ if } x_1 = x_2, \emptyset \text{ otherwise}]$;
- for every projection $\text{pr}_X : X \times Y \rightarrow X$, the functor $\exists_{\text{pr}_X}$ sends an element $[\gamma] \in \mathcal{P}(X \times Y)$ to the following:
  \[
  \exists_{\text{pr}_X} ([\gamma]) = [\lambda x \in X. \bigcup_{y \in Y} \gamma(x, y)].
  \]

In particular we have that $\mathcal{P}$ is elementary and existential and then, in particular, it has left adjoints along arbitrary functions, i.e. it is a full existential doctrine.
Notice also these left adjoints along any function satisfy BCC and FR (see [38, Lem. 5.2] or [27]).

Now we show that the realizability hyperdoctrine is an instance of generalized existential completion. Thus, we fix the class \( \mathcal{A}_{\text{Set}} \) to be the class of all functions of Set. The realizability hyperdoctrine is, in particular, existential and elementary, and then it has left adjoints along all the morphisms of Set, so it is a full existential doctrine. The rest of this section is devoted to prove that \( \mathcal{P} \) is a full existential completion.

Hence, we need to understand what are the full-existential-free objects of the realizability hyperdoctrine.

**Definition 3.66.** Let \( \mathcal{A} \) be a pca, and let \( \mathcal{P} : \text{Set}^{\text{op}} \longrightarrow \text{InfSL} \) be the realizability tripos associated to the pca \( \mathcal{A} \). We call an element \( \gamma : X \longrightarrow \mathcal{P}(\mathcal{A}) \) of the fibre \( \mathcal{P}(X) \) a **singleton predicate** if there exists a singleton function \( \alpha : X \longrightarrow \mathcal{P}(\mathcal{A}) \), i.e. \( \alpha(x) = \{a\} \) for some \( a \in \mathcal{A} \), such that \( \gamma \sim \alpha \).

We claim that the singleton predicates are exactly the full-existential-free objects provided that we assume the axiom of choice in our meta-theory as we do.

**Lemma 3.67.** Every singleton predicate is a full-existential-free object.

**Proof.** Assume that \( \gamma : X \longrightarrow \mathcal{P}(\mathcal{A}) \) is a singleton predicate, i.e. \( \gamma \) assigns to every element \( x \) of \( X \) a singleton \( \gamma(x) = \{a\} \) for some \( a \in \mathcal{A} \). In order to show that this is a full-existential-free object is enough to show that it is a full-existential splitting since the action of \( \mathcal{P} \) preserves singletons, i.e. for every function \( m : Z \longrightarrow X \) of Set, we have that \( \mathcal{P}(m)([\gamma]) = [\gamma \circ m] \) is again a singleton predicates. To this purpose, observe that if \( \gamma \leq \exists_{g}([\beta]) \) for some \( \beta \in \mathcal{P}(Y) \), then there exists an element \( \bar{b} \in \mathcal{A} \) such that for every \( x \in X \) and every \( a \in \gamma(x), \bar{b} \cdot a \in \exists_{g}\beta(x) \). By Remark 2.6 we have

\[
\exists_{g}(\beta) = \exists_{\mathcal{P}_{\text{pr}_{X}}}(\mathcal{P}_{\mathcal{P}_{\text{pr}_{Y}}}(\beta) \land \mathcal{P}_{x \times \mathcal{A}}(\delta_{X}))
\]

and since \( \gamma(x) \) is a singleton \( \{a\} \) for every \( x \in X \), we have that \( \bar{b} \cdot a \in \bigcup_{y \in Y} (\mathcal{P}_{\mathcal{P}_{\text{pr}_{Y}}}(\beta) \land \mathcal{P}_{y \times \mathcal{A}}(\delta_{Y}))(x,y) \). Hence we have that \( \bar{b} \cdot a = (p \cdot c_{1}) \cdot c_{2} \) for some \( c_{1} \in \beta(x,y_a), c_{2} \in \mathcal{A} \), and for some \( y_a \in Y \) such that \( g(y_a) = x \). By the Axiom of Choice, we can define a function \( f : X \longrightarrow Y \) such that \( f(x) = y_a \). In particular, we have that \( \lambda y.\mathcal{P}_{f}(\beta) \) realizes \( \gamma \leq \mathcal{P}_{f}(\beta) \), and we have that \( g_{f} = \text{id} \). This concludes the proof that singletons predicates are full-existential splitting. \( \square \)

**Corollary 3.68.** For every set \( X \), \( [\top_{X}] \) is a full-existential-free object.

**Proof.** Let us consider an element \( c \in \mathcal{A} \) and the singleton function \( \iota_{c} : X \longrightarrow \mathcal{P}(\mathcal{A}) \) given by the constant assignment \( x \mapsto \{c\} \). It is straightforward to check that \( \top_{X} \sim \iota_{c} \), i.e. that \( [\top_{X}] \) is a singleton predicate, and then a full-existential-free object by Lemma 3.67. \( \square \)

**Remark 3.69.** Notice that from Lemma 3.68 it follows that the realizability hyperdoctrine \( \mathcal{P} : \text{Set}^{\text{op}} \longrightarrow \text{InfSL} \) satisfies the Extended Rule of Choice.

Now we ready to show the main result of this section.
Theorem 3.70. The realizability hyperdoctrine \( P : \text{Set}^{\text{op}} \rightarrow \text{InfSL} \) is the full existential completion of the primary doctrine \( P^{\text{sing}} : \text{Set}^{\text{op}} \rightarrow \text{InfSL} \) whose elements of the fibre \( P^{\text{sing}}(X) \) are only singleton predicates of \( P(X) \).

Proof. By Lemma 3.67 we have that every singleton predicate is a full-existential-free element, and in particular, by corollary 3.68 we have that also every top element is a full-existential-free element. Notice that singleton predicates are closed under binary meet since if \( [\alpha] \) and \( [\beta] \) are singletons of \( P(X) \), by definition, we have that
\[
([\alpha] \land [\beta]) = [\lambda x \in X. \{(p \cdot a) \cdot b \mid a \in \alpha(x) \text{ and } b \in \beta(x)\}].
\]
is again a singleton function. Moreover, one can directly check that the singleton predicates cover the realizability hyperdoctrine, i.e. for every \( [\beta] \in P(Y) \) there exists a singleton predicate \( [\alpha] \in P(Y) \) and a function \( g : Y \rightarrow X \) such that \( [\beta] = \exists g([\alpha]) \). Then, by Proposition 3.26 we have that singleton predicates are exactly the full-existential-free elements of \( P \). Therefore the realizability hyperdoctrine satisfies all the conditions of Theorem 3.37 and then we can conclude that it is the full existential completion of the primary doctrine \( P^{\text{sing}} \) of singletons.

Remark 3.71. Observe that the doctrine \( P^{\text{sing}} : \text{Set}^{\text{op}} \rightarrow \text{InfSL} \) of singletons is closed under universal quantifiers. Then, by combining Theorem 3.70 with the results presented in [20], we can conclude that the realizability doctrine satisfies the so-called principle of Skolemization:
\[
\forall u \exists x \alpha(u, x, i) = \exists f \forall u \alpha(u, fu, i)
\]
being the existential completion of a universal doctrine whose base is cartesian closed. This form of choice principle plays a central role in the dialectica interpretation [23] and its categorical presentation, see [19, 20] for more details.

3.5. Locadic doctrines

Let us consider a locale \( \mathcal{A} \), i.e. \( \mathcal{A} \) is a poset with finite meets and arbitrary joins, satisfying the infinite distributive law \( x \land (\bigvee_i y_i) = \bigvee_i (x \land y_i) \). Recall from [8, 9] that given a locale \( \mathcal{A} \) we can define the canonical localic doctrine of \( \mathcal{A} \):
\[
\mathcal{A}^{-} : \text{Set}^{\text{op}} \rightarrow \text{InfSL}
\]
by assign \( I \mapsto \mathcal{A}^I \), and the partial order is provided by the pointwise partial order on functions \( f : I \rightarrow \mathcal{A} \). Propositional connectives are defined pointwise. The existential quantifier along a given function \( f : I \rightarrow J \) maps a function \( \phi \in \mathcal{A}^I \) to \( \exists_f \phi \) given by \( j \mapsto \bigvee_{i \in I \mid f(i) = j} \phi(i) \) and these are called existential since they satisfy the FR and BCC conditions.

Now we are going to consider localic triposes [8, 9] whose locale is supercoherent as defined in [16]. For the reader convenience we just recall few related basic notions from [17].

Definition 3.72. An element \( c \) of a locale \( \mathcal{A} \) is said supercompact if whenever \( c \leq \bigvee_{k \in K} b_k \), there exists \( k \in K \) such that \( c \leq b_k \).
Remark 3.73. Note that a supercompact element of a non trivial locale must be different from the bottom of the locale.

**Definition 3.74.** A local $A$ is called **supercoherent** if:

- each element $d \in A$ is a join $d = \bigvee c_i$ of supercompact elements $c_i$;
- supercompact elements are closed under finite meets.

**Remark 3.75.** The category $\text{SCohFrm}$ of supercoherent frames is a coreflexive subcategory of the category of frames. This result together with the general notion of supercoherent frame was introduced in [16] where the authors show that every regular projective complete lattice is exactly a retract of a supercoherent frame [16, Cor. 6].

Let $M$ be the category of meet-semilattices, and let $\text{Frm}$ be the category of frames. Recall from [39] that we can define a functor

$$D: M \rightarrow \text{Frm}$$

sending a meet-semilattice $M$ to the down-set lattice $D(M)$, i.e. the lattice of all the $X \subseteq M$ such that $a \in X$ implies that for all $b \leq a$ we have $b \in X$ and the order is provided by the set-theoretical inclusion. This functor is left adjoint to the inclusion functor, see [16, Lem. 1] or [39, Thm. 1.2]. In particular, we have a natural injection $\eta: M \rightarrow D(M)$ sending $a \mapsto \down{a}$. It is direct to see that sets of the form $\down{a}$ are supercompact elements in $D(M)$.

**Remark 3.76.** The functor $D: M \rightarrow \text{Frm}$ induces an equivalence $M \equiv \text{SCohFrm}$. Essentially this means that supercoherent frames are the frame completion of inf-semilattices.

**Definition 3.77.** Let $A$ be an arbitrary locale, and let $A(-): \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ be the localic doctrine. We call **supercompact predicate** an element $\phi \in A^J$ such that $\phi(j)$ is a supercompact element for every $j \in J$.

**Lemma 3.78.** Every supercompact predicate of the localic doctrine $A(-): \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ is a full-existential-free element.

**Proof.** Let $\phi \in A^J$ be a supercompact predicate. If we have $\phi \leq \exists_f(\psi)$ for some $f: I \rightarrow J$ and $\psi \in A^I$, then in particular

$$\phi(j) \leq \bigvee_{\{i \in I | f(i) = j\}} \psi(i)$$

and, since $\phi(j)$ is supercompact, there exists $\tilde{\tau} \in \{i \in I | f(i) = j\}$ such that $\phi(j) \leq \psi(\tilde{\tau})$. Hence, we can define a function $g: J \rightarrow I$ sending $j \mapsto \tilde{\tau}$. Hence, by definition, we have that $fg = \text{id}$ and $\phi \leq A_g^{-1}(\psi)$, i.e. $\phi$ is full-existential-splitting. Moreover it is direct to see that supercompact predicate are stable under re-indexing, and hence supercompact predicates are full-existential-free. \qed
Theorem 3.79. Let \( \mathcal{A} \) be a locale.

1. If \( \mathcal{A} \) is a supercoherent locale then the localic doctrine \( \mathcal{A}^{(-)} : \text{Set}^{op} \rightarrow \text{InfSL} \)
   is a full existential completion of the primary doctrine \( \mathcal{N}^{(-)} : \text{Set}^{op} \rightarrow \text{InfSL} \)
   of supercompact predicates of \( \mathcal{A} \).

2. If the localic doctrine \( \mathcal{A}^{(-)} : \text{Set}^{op} \rightarrow \text{InfSL} \) is a full existential completion then its locale \( \mathcal{A} \) is supercoherent.

Proof. (1) Let \( \mathcal{A} \) be a supercoherent locale. By Lemma 3.78 supercompact predicates are full-existential-free. Moreover since \( \mathcal{A} \) is supercoherent, the supercompact predicates are closed under finite meets, and since every element of \( a \in \mathcal{A} \) is a join of supercompact elements, then every element of every fibre of \( \mathcal{A}^{(-)} \) is covered by a supercompact predicates. In particular, if we consider an element \( \phi : I \rightarrow \mathcal{A} \), we have that for every \( i \in I \), \( \phi(i) = \bigvee_{j \in J} c^j_i \) with \( c^j_i \) supercompact elements. So, we can define the disjoint sum \( J = \bigoplus_{i \in I} J_i \) and a supercompact predicate \( \psi : J \rightarrow \mathcal{A} \) given by \( \psi(i, j) = c^j_i \). Then it is direct to see that \( \phi = \exists f(\psi) \) where \( f : J \rightarrow I \) is the function mapping \( f(i, j) = i \).

Hence, by Proposition 3.26 we have that supercompact elements are exactly the full-existential-free elements of \( \mathcal{A}^{(-)} \) and by Theorem 3.37 we can conclude that \( \mathcal{A}^{(-)} \) is the full existential completion of the primary doctrine \( \mathcal{N}^{(-)} : \text{Set}^{op} \rightarrow \text{InfSL} \) such that \( \mathcal{N}(I) \) is the inf-semilattice whose objects are the supercompact predicates of \( \mathcal{A}^{(-)} \).

(2) Suppose that the localic doctrine \( \mathcal{A}^{(-)} : \text{Set}^{op} \rightarrow \text{InfSL} \) is a full existential completion. By Lemma 3.78 we know that supercompact elements of \( \mathcal{A} \) are values of full-existential-free elements, hence supercompact elements are closed under finite meets because full-existential-free elements are closed under them. Now suppose that \( \phi \) is a full-existential-free object. Let \( j \) any index in \( J \) and suppose that \( \phi(j) \leq \bigvee_{i \in I} b_i \). Then we can define an element \( \psi : J \times I \rightarrow \mathcal{A} \) such that \( \psi(j, i) = b_i \) and \( \psi(j, i) = \top \) for every \( j \neq j \). Hence it follows that \( \phi(j) \leq \bigvee_{i \in I} \psi(j, i) \) for each \( j \) in \( J \) which means \( \phi \leq \exists f(\psi) \).

Then, since \( \phi \) is a full-existential-free object, there exists a function \( g : J \rightarrow I \) such \( \phi \leq \mathcal{A}^{(-)}(\psi, g) \), and then, in particular \( \phi(g(j)) \leq \psi(g(j)) = b_i \). Hence every element \( \phi(g) \) is supercompact. This concludes the proof that the full-existential-free elements of \( \mathcal{A}^{(-)} \) are exactly the supercompact predicates.

Finally, every element \( a \in \mathcal{A} \) is the join of supercompact elements because for such an element we can define the function \( \alpha : \{\ast\} \rightarrow \mathcal{A} \) as \( \ast \mapsto a \), and since the doctrine has enough-full-existential-free elements, we have \( \alpha = \exists f(\phi) \) for a full-existential-free object \( \phi \), i.e. \( a \) is the join of supercompact elements.

4. Regular completion via generalized existential completion

The notion of elementary existential doctrine contains the logical data which allow to describe relational composition as well as functionality and entirety. Given an elementary and existential doctrine \( P : \mathcal{C}^{op} \rightarrow \text{InfSL} \), an element \( \alpha \in P(A \times B) \) is said entire from \( A \) to \( B \) if

\[
\exists_{pr,A}(\alpha) \\
\]
Moreover it is said \textit{functional} if

\[ P_{(pr_1, pr_2)}(\alpha) \land P_{(pr_1, pr_2)}(\alpha) \leq P_{(pr_2, pr_3)}(\delta_B) \]

in \( P(A \times B \times B) \). Notice that for every relation \( \alpha \in P(A \times B) \) and \( \beta \in P(B \times C) \), the \textit{relational composition of} \( \alpha \) and \( \beta \) is given by the relation

\[ \exists_{(pr_1, pr_3)}(P_{(pr_1, pr_2)}(\alpha) \land P_{(pr_2, pr_3)}(\beta)) \]

in \( P(A \times B) \), where \( pr_i \) are the projections from \( A \times B \times C \).

Moreover, as pointed out in \[36,11\], for elementary existential m-variational doctrines, i.e. with full comprehensions and comprehensive diagonals, we can define their \textit{regular completion} as follows:

\textbf{Definition 4.1.} Let \( P \) be an elementary and m-variational doctrine. Then we define \textit{its m-variational regular completion} as the category \( Ef_P \), whose objects are those of \( C \), and whose morphisms are entire functional relations.

Recall from Section \[2\] that the doctrine \( P_c \) denotes the free-completion with full comprehensions of the doctrine \( P \).

\textbf{Definition 4.2.} Let \( P \) be a an elementary existential doctrine. We call the category \( \text{Reg}(P) := Ef_P \) the \textit{the regular completion} of \( P \).

Then, we recall the following result from \[11\], Thm. 3.3:

\textbf{Theorem 4.3.} The assignment \( P \mapsto \text{Reg}(P) \) extends to a 2-functor

\[ \text{Reg}(-) : \text{EED} \longrightarrow \text{Reg} \]

which is left biadjoint to the inclusion of the 2-category \( \text{Reg} \) of regular categories in the 2-category \( \text{EED} \) of elementary and existential doctrines.

Notice that the regular completion of an elementary existential doctrine depends only on the objects of the base category and on the fibres. In particular we have the following lemma.

\textbf{Lemma 4.4.} Let \( P \) and \( R \) be two elementary existential doctrines. If \( (F, f) : P \longrightarrow R \) is a morphism of elementary existential doctrine such that \( f_A : P(A) \longrightarrow R(FA) \) is an isomorphism for every object \( A \) of the base, then \( \text{Reg}(F, f) : \text{Reg}(P) \longrightarrow \text{Reg}(R) \) is full and faithful functor.

\textit{Proof.} By hypothesis we have that \( f_A : P(A) \longrightarrow R(FA) \) is an isomorphism, and then when we consider the comprehension completion we have again that each component \((f_{c})(A, \alpha) : P_c(A, \alpha) \longrightarrow R_c(FA, f_A(\alpha)) \) an isomorphism. Since \((F, f)\) is a morphism of elementary and existential doctrines and \( f \) is a natural transformation whose components are isomorphisms we have that the \( \phi \) is an entire functional relation of \( P_c(A, \alpha) \) if and only if \( f_{c}(\phi) \) is an entire functional relation of \( R_c(FA, f_A(\alpha)) \). Therefore, by definition of regular completion, we have that the functor \( \text{Reg}(F, f) : \text{Reg}(P) \longrightarrow \text{Reg}(R) \) is a full and faithful.

\[ \square \]
The regular completion construction $\text{Reg}(P)$ of an elementary existential doctrine $P$ generalizes in the setting of doctrines the regular completion construction $(\mathcal{D})_{\text{reg/lex}}$ of a category $\mathcal{D}$ with finite limits introduced in [2, 1]. In particular, such a regular completion is an instance of the $(-)_{\text{reg/lex}}$ completion.

We briefly recall from [2] the construction of the category $(\mathcal{D})_{\text{reg/lex}}$ associate to a category $\mathcal{D}$ with finite limits.

**Definition 4.5.** Let $\mathcal{D}$ be a category with finite limits. The *regular completion* of $\mathcal{D}$ is the category $(\mathcal{D})_{\text{reg/lex}}$ defined as follows:

- its objects are arrows $X \xrightarrow{g} Y$ of $\mathcal{D}$;
- an arrow $X \xrightarrow{g} Y$ in $(\mathcal{D})_{\text{reg/lex}}$ is an equivalence class of arrows $m: X \rightarrow U$ such that $fmg_1 = fmg_2$ where $g_1$ and $g_2$ is a kernel pair of $g$. Two such arrows $m$ and $u$ are considered equivalent if $fn = fu$.

**Corollary 4.6.** The regular completion $(\mathcal{D})_{\text{reg/lex}}$ of category with finite limits $\mathcal{D}$ is equivalent to the regular completion $\text{Reg}(\Psi\mathcal{D})$ of the doctrine $\Psi\mathcal{D}: \mathcal{D}^{\text{op}} \rightarrow \text{InfSL}$ of weak subobjects of $\mathcal{D}$.

**Proof.** This follows from Theorem 4.3 by assigning to a lex category $\mathcal{D}$ the regular completion $\text{Reg}(\Psi\mathcal{D})$ of its weak subobjects doctrine after observing that the embedding of $\mathcal{D}$ into $\text{Reg}(\Psi\mathcal{D})$ is a finite limit functor. Moreover, any finite limit functor $L$ from $\mathcal{D}$ into a regular category $\mathcal{R}$ can be completed uniquely to an elementary existential doctrine morphism $(L, l)$ from $\Psi\mathcal{D}$ to the subobjects doctrine of $\mathcal{R}$ by defining $l(f) = \exists_{L(f)}(\top)$, namely the image of $L(f)$ for any $f$ in the fibre of $\Psi\mathcal{D}$, since $f = \exists f(\top)$ and $l$ must preserve existential quantifiers.

**Lemma 4.7.** Let $P: \mathcal{D}^{\text{op}} \rightarrow \text{InfSL}$ be a conjunctive doctrine such that the base category has finite limits. The category $\mathcal{G}_P$ has finite limits, and every object of $(\mathcal{G}_P)_{\text{reg/lex}}$ is isomorphic to one of the form $(A, \alpha) \xrightarrow{f} (B, \top_B)$.

**Proof.** The fact that $\mathcal{G}_P$ has finite limits follows because the base category $\mathcal{D}$ has finite limits and every fibre of $P$ is an inf-semilattice. Moreover, if we consider an object $(A, \alpha) \xrightarrow{f} (B, \beta)$ of the category $(\mathcal{G}_P)_{\text{reg/lex}}$, then it is direct to check that

$$(A, \alpha) \xrightarrow{f} (B, \beta)$$

is an arrow of $(\mathcal{G}_P)_{\text{reg/lex}}$ and that is an isomorphism.
Remark 4.8. Let us consider the weak subobjects doctrine $\Psi_{G_P}: \mathbb{C}^{\text{op}} \longrightarrow \text{InfSL}$, where $P$ is a conjunctive doctrine. Then every object of $\text{Reg}(\Psi_{G_P})$ is of the form $\langle (B, \beta), \exists_f(\top) \rangle$, with $f: (A, \alpha) \longrightarrow (B, \beta)$, since $\Psi_{G_P}$ is the full existential completion of the trivial doctrine $\Upsilon_{G_P}$. Notice that it is direct to check that $\langle (B, \beta), \exists_f(\top) \rangle \cong \langle (B, \top), \exists_f(\top) \rangle$ via the isomorphism $\exists \langle f, f \rangle(\top)$. This is the correspondent result of Lemma 4.7, since $(G_P)_{\text{reg}/\text{lex}} \cong \text{Reg}(\Psi_{G_P})$.

From what said in example 3.15, we conclude:

**Corollary 4.9.** The regular completion of $\Psi_{D}: \mathbb{D}^{\text{op}} \longrightarrow \text{InfSL}$ of a category with finite limits is the regular completion of a full generalized existential completion.

Notice that for any regular theory $T$ according to Johnstone’s definition \[1\] the construction of its syntactic category $\mathbb{C}_T$ in the sense of \[1\] is equivalent to the regular completion $\text{Reg}(LT_T)$ of the syntactic elementary existential doctrine $LT: \mathbb{C}_T^{\text{op}} \longrightarrow \text{InfSL}$ associated to the theory $T$, defined in Example 2.7. Therefore it is natural to present and consider the notion of regular Morita-equivalent for elementary existential doctrines defined as follows:

**Definition 4.10.** Two elementary existential doctrines $P$ and $P'$ are said regular Morita-equivalent if their regular completions are equivalent, i.e. when $\text{Reg}(P) \equiv \text{Reg}(P')$.

Hence, by using Definition 4.10 we can state:

**Theorem 4.11.** Let $P: \mathbb{C}^{\text{op}} \longrightarrow \text{InfSL}$ be a elementary and existential doctrine. If $\text{Reg}(P) \equiv (D)_{\text{reg}/\text{lex}}$, then $P$ is regular-Morita equivalent to a full existential completion whose base is $D$.

*Proof.* It follows from Corollaries 4.9 and 4.6.

Moreover, the notion of regular Morita-equivalence among weak subobjects doctrines coincides with their equivalence as doctrines:

**Theorem 4.12.** Let $\Psi_{C}: \mathbb{C}^{\text{op}} \longrightarrow \text{InfSL}$ and $\Psi_{D}: \mathbb{D}^{\text{op}} \longrightarrow \text{InfSL}$ be two weak subobjects doctrines whose base categories $\mathbb{C}$ and $\mathbb{D}$ have finite limits. Then $\Psi_{C}$ and $\Psi_{D}$ are regular Morita-equivalent if and only if $\Psi_{C} \cong \Psi_{D}$.

*Proof.* It follows by the notion of regular Morita-equivalence, Corollary 4.9 and the characterization of regular completions in terms of regular projectives due to Carboni \[1, \text{Lem. 5.1},\]

The main purpose of this section it to show that, for a given elementary existential doctrine $P$, the condition of being regular-Morita equivalent to a full existential completion is not only necessary but also sufficient for obtaining $\text{Reg}(P)$ equivalent to the regular completion of a finite limit category.

---

1. Such a notion of regular logic is indeed the internal language of existential elementary doctrines in the sense of \[11\]. In \[11\] the internal language of regular categories is presented as a dependent type theory.

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In particular, we are going to show that, if we consider a conjunctive doctrine $P: \mathcal{C} \to \text{InfSL}$ where $\mathcal{C}$ is a category with finite limits and we consider the left class of morphisms of all the morphisms of $\mathcal{C}$, the regular completion of the full existential completion $f\text{Ex}(P)$ of $P$ is equivalent to the regular category associated to the Grothendieck category $\mathcal{G}_P$ of the doctrine $P$

$$\text{Reg}(f\text{Ex}(P)) \equiv (\mathcal{G}_P)_{\text{reg/lex}}.$$

To this purpose we define the following canonical morphism:

**Definition 4.13.** Given a full existential doctrine $P: \mathcal{C} \to \text{InfSL}$ on a category $\mathcal{C}$ with finite limits and a fibred subdoctrine $P'$ of $P$, we define a morphism of full existential doctrines $(N,n): (\Psi \mathcal{G}_{P'}) \to P$ given by

- $N(A,\alpha) = A$ for every object $(A,\alpha)$ of $\mathcal{G}_{P'}$;
- $N(h) = h$ for every arrow $h: (B,\beta) \to (A,\alpha)$;
- $n(g) = \exists g(\gamma)$ for every element $g: (C,\gamma) \to (A,\alpha)$ of $(\Psi \mathcal{G}_{P'})((A,\alpha))$.

This induces a morphism $(N,n)_c: (\Psi \mathcal{G}_{P'})_c \to P_c$ defined as follows:

- $N_c((A,\alpha), f) = (A, \exists f(\beta))$, where $f: (B,\beta) \to (A,\alpha)$;
- $N_c(h) = h: (A, \exists f(\beta)) \to (C, \exists l(\sigma))$ for any arrow $h: ((A,\alpha), f) \to ((C,\gamma), l)$;
- $n_c(g) = \exists g(\gamma)$ for any given element $g: (C,\gamma) \to (A,\alpha)$ of $(\Psi \mathcal{G}_{P'})_c((A,\alpha), f)$.

We also recall that the action of the functor $\text{Reg}(N,n): \text{Reg}(\Psi \mathcal{G}_{P'}) \to \text{Reg}(f\text{Ex}(P))$ on the objects of $\text{Reg}(\Psi \mathcal{G}_{P'})$ is given essentially by the action of the functor $N_c$, i.e. we have that $\text{Reg}(N,n)((A,\alpha), f) = N_c((A,\alpha), f)$, while the action of $\text{Reg}(N,n)$ on morphisms is given by the action of $n_c$, i.e. $\text{Reg}(N,n)(g) = n_c(g)$.

Combining the previous results, we obtain the following theorem.

**Theorem 4.14.** Let $P: \mathcal{C} \to \text{InfSL}$ be a conjunctive doctrine, whose base category $\mathcal{C}$ has finite limits. Then the regular functor $\text{Reg}(N,n): \text{Reg}(\Psi \mathcal{G}_{P'}) \to \text{Reg}(f\text{Ex}(P))$ is an equivalence, so we have

$$\text{Reg}(f\text{Ex}(P)) \equiv \text{Reg}(\Psi \mathcal{G}_{P'}) \equiv (\mathcal{G}_P)_{\text{reg/lex}}.$$
Proof. By Theorem 3.37 there exists a morphism \((\mathcal{T}, \mathcal{G}) : \text{Ex}(P) \to \Psi_{\mathcal{G}}\) of full existential doctrines where \(\mathcal{T} : \mathcal{C} \to \mathcal{G}\) is full and faithful and for every object \(A\) of \(\mathcal{C}\), \(\mathcal{T}_A\) is an isomorphism between the fibres \(\text{Ex}(P)(A) \cong \Psi_{\mathcal{G}}(A, \top)\). Therefore, by Lemma 4.13 we have that \(\text{Reg}(\mathcal{T}, \mathcal{G}) : \text{Ex}(P) \to \text{Ex}(\mathcal{G})\) is an inclusion of categories, i.e. the functor \(\text{Reg}(\mathcal{T}, \mathcal{G})\) is full and faithful. Now, since by Remark 4.8 every object of \(\text{Reg}(\mathcal{G})\) is isomorphic to one of the form \(((B, \top), \exists f(\top)) = \text{Reg}(\mathcal{T}, \mathcal{G})(B, \exists f(\alpha))\) we can conclude that \(\text{Reg}(\mathcal{T}, \mathcal{G}) = \text{Reg}(\mathcal{G})\) is full and faithful. Now, since comprehensions are full, we have that \(\text{Reg}(\mathcal{N}, n) \text{Reg}(\mathcal{T}, \mathcal{G}) = \text{id}\) and also \(\text{Reg}(\mathcal{T}, \mathcal{G}) \text{Reg}(\mathcal{N}, n) = \text{id}\).

Remark 4.15. Observe that for any conjunctive doctrine \(P : \mathcal{C} \to \text{InfSL}\), whose base category \(\mathcal{C}\) has finite limits, we have that \((\mathcal{N}, n) : \text{Reg}(\mathcal{T}, \mathcal{G}) = \text{id}\) by fullness of comprehensions but not in general that \((\mathcal{T}, \mathcal{G}) : \text{Reg}(\mathcal{G}) = \text{id}\). The composition \((\mathcal{T}, \mathcal{G}) : \text{Reg}(\mathcal{G})\) becomes an identity after applying the regular completion construction for what observed in Remark 4.8.

Theorem 4.16. Let \(P : \mathcal{C} \to \text{InfSL}\) be a full existential doctrine, whose base category \(\mathcal{C}\) has finite limits. Let \(P' : \mathcal{C} \to \text{InfSL}\) be a conjunctive, fibred subdoctrine of \(P\). If the canonical arrow \(\text{Reg}(\mathcal{N}, n) : \text{Reg}(\mathcal{G}) = \text{id}\) is an equivalence preserving the canonical embeddings of \(P'\) into \(P\) and \(\Psi_{\mathcal{G}}\), namely it forms an equivalence with an arrow \((H, h) : \text{Reg}(P) \to \text{Reg}(\mathcal{G})\) making the diagram

\[
P' \xrightarrow{(H, h)} \text{Reg}(P) \xrightarrow{\text{Reg}(\mathcal{G})} \text{Reg}(\mathcal{G})
\]

commute, then \(P\) is the full existential completion of \(P'\), i.e. \(P = \text{Ex}(P')\).

Proof. To prove the result we employ the second point of Theorem 3.37. Hence, we first show that \(P(A) \cong \Psi_{\mathcal{G}}(A, \top)\) for every object \(A\) of \(\mathcal{C}\). Therefore, we have that for every elementary and existential doctrine the following isomorphisms hold

\[
P(A) \cong P(A, \top) \cong \text{Sub}_{\text{Reg}(P)}(A, \top)
\]

and, in particular we have that

\[
\Psi_{\mathcal{G}}(A, \top) \cong (\Psi_{\mathcal{G}}, \text{id}_{(A, \top)}) \cong \text{Sub}_{\text{Reg}(\mathcal{G})}((A, \top), \text{id}_{(A, \alpha)}).
\]

Now, since \(\text{Reg}(\mathcal{N}, n) : \text{Reg}(\mathcal{G}) \to \text{Reg}(P)\) is an equivalence, and by definition of \(N\), we have that
we conclude that
\[ P(A) \cong \Psi_{\mathcal{G}_{P'}}(A, \top). \] (18)

Observe that all the previous mentioned isomorphisms preserve elements of \( P' \) because they are components of canonical morphisms and \( \text{Reg}(N, n) \) preserves the elements of \( P' \) by hypothesis. Hence we conclude that the isomorphism (18) sends every element \( \alpha \in P'(A) \) to the comprehension map \( (A, \alpha) \xrightarrow{\text{id}_A} (A, \top) \).

The naturality of this family of isomorphisms follows because the canonical injections of a weak subobjects doctrine into the completions involved, is full and faithful on the base category. In particular the crucial point is that the functor between the base categories of the canonical injection
\[
\Psi_{\mathcal{G}_{P'}} \longrightarrow \text{Sub}_{\text{Reg}(\Psi_{\mathcal{G}_{P'}})}
\]
is full and faithful. This holds because the weak subobjects doctrines has comprehensive diagonals. Therefore, we can apply the second point of Theorem 3.37 and conclude that \( P \) is the full existential completion of \( P' \).

Remark 4.17. A first obvious example of application of theorem 4.14 is the construction of the \( \text{reg/lex} \) completion of a finite limit category \( D \) itself. Indeed, after recalling that \((D)_{\text{reg/lex}} \equiv \text{Reg}(\Psi_D)\) from corollary 4.6 and that the weak subobjects doctrine is a full existential completions of the trivial doctrine \( \Upsilon: D^{\text{op}} \longrightarrow \text{InfSL} \) from 3.15, we can obviously apply Theorem 4.14 to \( \Psi_D \) by getting \((D)_{\text{reg/lex}} \equiv \text{Reg}(\Psi_D) \equiv \text{Reg}(\text{fEx}(\Upsilon)) \equiv (G_{\Upsilon})_{\text{reg/lex}}\). This chain of equivalences does not yield more information since by Theorem 4.12 we conclude that \( D \equiv G_{\Upsilon} \) as one can immediately check independently.

Finally, we can apply Theorem 4.14 together with Theorem 4.11 obtaining our main result:

**Theorem 4.18.** Let \( P: C^{\text{op}} \longrightarrow \text{InfSL} \) be a full existential doctrine, whose base category has finite limits. Then the following are equivalent:

1. the regular completion \( \text{Reg}(P) \) is equivalent to regular completion \((D)_{\text{reg/lex}}\) of a lex category \( D \);
2. \( P \) is regular Morita-equivalent to the weak-subobjects doctrine \( \Psi_D: D^{\text{op}} \longrightarrow \text{InfSL} \) on a lex category \( D \);
3. \( P \) is regular Morita-equivalent to a full existential completion \( P' \).

**Proof.** \((1 \Rightarrow 2)\) It follows by Corollary 4.6 and by definition of regular Morita-equivalence, Definition 4.10.

\((2 \Rightarrow 3)\) It follows again by definition of regular Morita-equivalence and by the fact that every weak subobjects doctrine is a full existential completion, see Theorem 3.14 and Example 3.15.

\((3 \Rightarrow 1)\) It follows by Theorem 4.14. \( \square \)
Employing the same arguments used in the previous proofs, together with Theorem 3.50, we can prove the analogous results for the case of the pure existential completion of elementary doctrines.

**Theorem 4.19.** Let \( P : \mathcal{C} \to \text{InfSL} \) be a pure existential, elementary doctrine. Then we have the equivalence

\[
\text{Reg}(\text{pEx}(P)) \equiv \text{Reg}(\Psi_{\text{prd}}) \equiv (\text{PrdP})_{\text{reg/lex}}.
\]

**Proof.** Notice that in this case we have to employ Theorem 3.50 to obtain the equivalence between the fibres \( \text{pEx}(P)(A) \sim \Psi_{\text{prd}}(A, \top) \). After this observation, we can use exactly the same argument used in the proof of Theorem 4.14 to conclude.

Similarly we have the following theorem.

**Theorem 4.20.** Let \( P : \mathcal{C} \to \text{InfSL} \) be a pure existential, elementary doctrine, and let \( P' : \mathcal{C} \to \text{InfSL} \) be an elementary fibred subdoctrine of \( P \). If the canonical morphism given by the pair \( \text{Reg}(N, n) : \text{Reg}(\Psi_{\text{prd}}) \to \text{Reg}(P) \) is an equivalence then \( P \) is a pure existential completion of \( P' \).

**Proof.** Notice that by Theorem 3.50 we obtain the equivalence between the fibres \( \text{pEx}(P)(A) \sim \Psi_{\text{prd}}(A, \top) \). Then, this result can be proved exactly using the same argument of Theorem 4.16.

**Examples 4.21.** As application of Theorem 4.19 together with Theorem 3.53 we obtain that if an elementary existential doctrine \( P : \mathcal{C} \to \text{InfSL} \) is equipped with Hilbert’s \( \epsilon \)-operators, then we have the equivalence \( \text{Reg}(P) \equiv (\text{PrdP})_{\text{reg/lex}} \). Therefore, as a particular case of our result, we obtain the result proved in [11, Thm. 6.2(ii)].

### 4.1. The example of the regular category of assemblies of a partial combinatory algebra

From [41, 42, 1] we know that the category of assemblies is the reg/lex completion of the category of partitioned assemblies. We show here that such a result is a consequence of Theorem 4.14 and the fact that the realizability hyperdoctrines on a pca are full existential completions as proved in theorem 3.70.

Recall that an **assembly** on a pca \( \mathcal{A} \) is a pair \((X, E)\) where \( X \) is a set and \( E \) is a function \( E : X \to P^*(\mathcal{A}) \) assigning to every element \( x \in X \) a non-empty subset \( E(x) \subseteq \mathcal{A} \).

A **morphism of assemblies** \( f : (X, E) \to (Y, F) \) is a function \( F : X \to Y \) with the property that there exists an element \( a \in \mathcal{A} \) such that for every \( x \in X \), for every \( b \in E(x) \), \( a \cdot b \downarrow \) and \( a \cdot b \in F(f(x)) \). This element \( a \) is said to **track** the function \( f \). Assemblies and morphisms of assemblies form a category denoted by \( \text{Asm} (\mathcal{A}) \).

An assembly \((X, E)\) is said **partitioned** if \( E \) is single-valued, i.e. \( E \) is a function from \( X \) to \( \mathcal{A} \). The full subcategory of \( \text{Asm}(\mathcal{A}) \) on partitioned assemblies is written \( \text{PAsm}(\mathcal{A}) \).

We refer to [37, 41] for an extensive analysis of these categories.

First of all observe that:

**Theorem 4.22.** The category of assemblies \( \text{Asm}(\mathcal{A}) \) on a pca \( \mathcal{A} \) is equivalent to the regular completion \( \text{Reg}(P) \) of the realizability doctrine \( P : \text{Set}^\text{op} \to \text{InfSL} \) associated to \( \mathcal{A} \).
Proof. The equivalence $\mathcal{A}sm(\mathbb{A}) \equiv \mathcal{R}eg(\mathcal{P})$ can easily established after observing the following two facts whose proofs are straightforward. First, given an object $(A, \alpha)$ of $\mathcal{R}eg(\mathcal{P})$, then if $A$ is not empty, then $(A, \alpha)$ is isomorphic to an object $(A', \alpha')$ such that $\alpha'(a) \neq \emptyset$ for every $a \in A'$; otherwise it is isomorphic to $(\emptyset, !\mathcal{P}(A))$, where $!\mathcal{P}(A): \emptyset \rightarrow \mathcal{P}(A)$ is the empty function.

Moreover, if $R: (A, \alpha) \rightarrow (B, \beta)$ is an arrow of $\mathcal{R}eg(\mathcal{P})$ where $\alpha$ and $\beta$ are always non-empty, then there exists a unique function $f: A \rightarrow B$ such that

$$R(a, b) = \begin{cases} \alpha(a) & \text{if } b = f(a) \\ \emptyset & \text{otherwise.} \end{cases}$$

Second, notice that $f$ has the following property: there exists an element $n \in A$ such that for every $a \in A$, for every $x \in \alpha(a)$, $n \cdot x \downarrow$ and $n \cdot x \in \beta(f(a))$. \hfill \square

It is immediate to observe that:

Lemma 4.23. The Grothendieck category of $\mathcal{P}^{\text{sing}}$ defined in section 3.4 is equivalent to that of partitioned assemblies

$$\mathcal{G}_{\mathcal{P}^{\text{sing}}} \equiv \mathcal{P}\mathcal{A}sm(\mathbb{A}).$$

Hence, from these facts and our previous results we conclude in an alternative way Robinson and Rosolini’s result:

Corollary 4.24. The category of assemblies is equivalent to $\text{reg/lex completion of the category of partitioned assemblies}$.\hfill \square

Proof. From Theorem 3.70 we know that the realizability hyperdoctrine $\mathcal{P}: \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ associated to a partial combinatory algebra $\mathbb{A}$ is the full existential completion of the primary doctrine $\mathcal{P}^{\text{sing}}: \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ whose elements of the fibre $\mathcal{P}^{\text{sing}}(X)$ are only singleton predicates of $\mathcal{P}(X)$. Therefore, by Theorem 4.14 we have that

$$\mathcal{R}eg(\mathcal{P}) \equiv (\mathcal{G}_{\mathcal{P}^{\text{sing}}})_{\text{reg/lex}}.$$

and by theorem 4.22 and by lemma 4.23 we conclude

$$\mathcal{A}sm(\mathbb{A}) \equiv (\mathcal{P}\mathcal{A}sm(\mathbb{A}))_{\text{reg/lex}}$$

\hfill \square

Remark 4.25. Observe also that in [43] the category of assemblies can be obtained also as a different kind of completion which instead is not an instance of a full existential completion because the generating doctrine does not satisfy the choice principles (RC) in Remark 3.21.

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5. Exact completions via generalized existential completion

The tripos-to-topos construction origins in \[\mathcal{L}\mathcal{S}\mathcal{L}\] is a fundamental construction of categorical logic that relates the notion of tripos with that of topos. This construction is an instance of the exact completion \(T_P\) of an elementary existential doctrine \(P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}\) studied in \[\mathcal{L}\mathcal{S}\mathcal{L}\].

Given an elementary existential doctrine \(P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}\), the category \(T_P\) consists of:

- **objects**: pairs \((A, \rho)\) such that \(\rho \in \mathcal{P}(A \times A)\) satisfies
  
  - symmetry: \(\rho \leq P_{(pr_2, pr_1)}(\rho)\);
  
  - transitivity: \(P_{(pr_1, pr_2)}(\rho) \wedge P_{(pr_2, pr_3)}(\rho) \leq P_{(pr_1, pr_3)}(\rho)\) where \(pr_i\) are the projections from \(A \times A \times A\);

- **arrows**: \(\phi: (A, \rho) \rightarrow (B, \sigma)\) are objects \(\phi \in \mathcal{P}(A \times B)\) such that

  - (i) \(\phi \leq P_{(pr_1, pr_1)}(\rho) \wedge P_{(pr_2, pr_3)}(\sigma)\);
  
  - (ii) \(P_{(pr_1, pr_2)}(\rho) \wedge P_{(pr_2, pr_3)}(\phi) \leq P_{(pr_2, pr_3)}(\rho)\) where \(pr_i\) are projections from \(A \times A \times B\);
  
  - (iii) \(P_{(pr_2, pr_3)}(\sigma) \wedge P_{(pr_1, pr_2)}(\phi) \leq P_{(pr_2, pr_3)}(\phi)\) where \(pr_i\) are projections from \(A \times B \times B\);
  
  - (iv) \(P_{(pr_1, pr_2)}(\phi) \wedge P_{(pr_1, pr_3)}(\rho) \leq P_{(pr_2, pr_3)}(\sigma)\) where \(pr_i\) are projections from \(A \times B \times B\);
  
  - (v) \(P_{(pr_1, pr_1)}(\rho) \leq \exists_{pr_3}(\phi)\) where \(pr_i\) are projections from \(A \times B\).

The construction \(T_P\) is called the **exact completion** of the elementary existential doctrine \(P\). In particular we recall the following result from \[\mathcal{L}\mathcal{S}\mathcal{L}\] Cor. 3.4:

**Theorem 5.1.** The assignment \(P \mapsto T_P\) extends to a 2-functor

\[
\text{EED} \longrightarrow \text{Xct}
\]

from the 2-category \text{EED} of elementary, existential doctrines to the 2-category \text{Xct} of exact categories, and it is left adjoint to the functor sending an exact category \(\mathcal{C}\) to doctrine \(\text{Sub}_C\) of its subobjects.

As done for regular completions, we introduce the following definition of **exact Morita-equivalence** between elementary existential doctrines:

**Definition 5.2.** Two elementary existential doctrines \(P\) and \(P'\) are said **exact Morita-equivalent** if their exact completions are equivalent, i.e. when \(T_P \cong T_{P'}\).

**Remark 5.3.** Recall from \[\mathcal{L}\mathcal{S}\mathcal{L}\] Ex. 3.2 and \[\mathcal{L}\mathcal{S}\mathcal{S}\] Ex. 4.8 that if we consider a regular category \(\mathcal{A}\), the exact completion \(T_{\text{Sub}_{\mathcal{A}}}\) of the subobjects doctrine \(\text{Sub}_{\mathcal{A}}: \mathcal{A}^{\text{op}} \rightarrow \mathbf{InfSL}\) coincides with the exact completion \((\mathcal{A})_{\text{ex}/\text{reg}}\), i.e. with the exact completion \((\mathcal{A})_{\text{ex}/\text{reg}}\) of the regular category \(\mathcal{A}\) introduced by Freyd in \[\mathcal{L}\mathcal{S}\mathcal{S}\]. Therefore, combining Proposition \[\mathcal{L}\mathcal{S}\mathcal{S}\] with \[\mathcal{L}\mathcal{S}\mathcal{S}\] Ex. 4.8 we can provide a description of the exact completion of a regular category in terms of existential completions.

In particular, if a doctrine \(P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}\) is the \(\mathcal{M}\)-existential completion of the trivial doctrine \(\Upsilon: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}\), where \(\mathcal{M}\) is a class of morphisms of \(\mathcal{C}\) such that
• there exists a class $E$ of morphisms of $C$ such that $(E, M)$ is a proper, stable, factorization system on $C$;
• for every arrow $pr_A: A \times B \to A$ of $C$, if $pr_A f$ is a monomorphism and $f \in M$ then $pr_A f \in M$;

then $\mathcal{T}_p \equiv (C)_{ex/\text{reg}}$.

Then, from the results shown in the previous sections we also obtain:

**Corollary 5.4.** The exact completion $C_{ex/\text{reg}}$ of a regular category $C$ is the exact completion of a full existential completion.

**Proof.** As observed in remark 5.3 the $ex/\text{reg}$ completion $C_{ex/\text{reg}}$ of a regular category $C$ coincides with the exact completion of the elementary existential doctrine of the superobjects doctrine $\text{Sub}_C$ and this is the full existential completion with respect to all the morphisms of its base category as shown in Proposition 3.65.

Thanks to the result presented in [7, Sec. 2] and 5.1 we have that:

**Theorem 5.5.** For any elementary existential doctrine $P: C^{op} \to \text{InfSL}$ the following equivalence holds

$$\mathcal{T}_P \equiv (\text{Reg}(P))_{ex/\text{reg}}.$$  

This is indeed a generalization of the well-known result from [1, 2, 10] that the exact completion of a category $D$ with finite limits is equivalent to the construction $(D_{\text{reg/lex}})_{ex/\text{reg}}$.

Furthermore, as first observed in [2] and analogously to what proved for the $\text{reg/lex}$ completion in Corollary 4.6, the following holds:

**Corollary 5.6.** The exact completion $(D)_{ex/\text{lex}}$ of a category $D$ with finite limits happens to be equivalent to the exact completion $\mathcal{T}_{\Psi D}$ of the doctrine $\Psi D: D^{op} \to \text{InfSL}$ of weak subobjects of $D$.

Now, by combining these results we obtain the following analogous result to that of Theorem 4.11.

**Theorem 5.7.** Let $P: C^{op} \to \text{InfSL}$ be a elementary and existential doctrine. If $\mathcal{T}_P \equiv (D)_{ex/\text{lex}}$, then $P$ is exact-Morita equivalent to a full existential completion whose base is $D$.

**Proof.** It follows from Example 5.13 and Corollary 5.16.

As in the case of the regular completion in Theorem 4.12, exact-Morita equivalence between weak subobjects doctrines coincides with their equivalence:

**Theorem 5.8.** Let $\Psi_C: C^{op} \to \text{InfSL}$ and $\Psi_D: D^{op} \to \text{InfSL}$ be two weak subobjects doctrines whose base categories $C$ and $D$ have finite limits. Then $\Psi_C$ and $\Psi_D$ are exact Morita-equivalent if and only if $\Psi_C \cong \Psi_D$.

**Proof.** It follows by the notion of exact Morita-equivalence, Corollary 5.6 and the characterization of exact completions in terms of regular projectives due to Carboni [1, Lem. 2.1].
Furthermore, from these facts and Theorem 4.14 we directly conclude the following:

**Theorem 5.9.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an elementary and existential doctrine, whose base category \( \mathcal{C} \) has finite limits. Let \( P' : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a conjunctive, fibred subdoctrine of \( P \). Then the arrow \( \text{Ex}(N,n) : T_{\Psi_{P'}} \rightarrow T_P \) is an equivalence, preserving the canonical embeddings of \( P' \) into \( P \) and \( \Psi_{P'} \), if and only if \( P \) is the full existential completion of \( P' \).

**Proof.** It follows from Theorem 5.5, Theorem 4.14 and Theorem 4.16 together with the observation that the only if part has a straightforward proof following the same line and technique used in Theorem 4.16 for the case of the regular completion. \( \square \)

**Corollary 5.10.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be a conjunctive doctrine whose base category \( \mathcal{C} \) has finite limits. Then we have the equivalence
\[
T_{\text{Ex}(P)} \equiv (G_P)_{\text{ex}/\text{lex}}.
\]

**Theorem 5.11.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an full existential doctrine, whose base category has finite limits. Then the following are equivalent:

1. the exact completion \( T_P \) is equivalent to regular completion \((\mathcal{D})_{\text{ex}/\text{lex}}\) of a lex category \( \mathcal{D} \);
2. \( P \) is exact Morita-equivalent to the weak-subobjects doctrine \( \Psi_{\mathcal{D}} : \mathcal{D}^{\text{op}} \rightarrow \text{InfSL} \) on a lex category \( \mathcal{D} \);
3. \( P \) is exact Morita-equivalent to a full existential completion \( P' \).

**Proof.** (1 \( \Rightarrow \) 2) It follows by Corollary 5.6 and by definition of exact Morita-equivalence.
(2 \( \Rightarrow \) 3) It follows again by definition of exact Morita-equivalence and by the fact that every weak subobjects doctrine is a full existential completion, see Theorem 3.14 and Example 3.15.
(3 \( \Rightarrow \) 1) It follows by Corollary 5.10. \( \square \)

Furthermore, we obtain analogous results for the notion of pure existential completion of an elementary doctrine.

**Theorem 5.12.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an pure existential, elementary doctrine, and let \( P' : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) be an elementary fibred subdoctrine of \( P \). Then, the canonical morphism given by the pair \( \text{Ex}(N,n) : T_{\Psi_{P'}} \rightarrow T_P \) is an equivalence, preserving the canonical embeddings of \( P' \) into \( P \) and \( \Psi_{P'} \), if and only if \( P \) is a pure existential completion of \( P' \).

**Proof.** It follows from Theorem 5.5, Theorem 4.14 and Theorem 4.16. \( \square \)

**Corollary 5.13.** If \( P : \mathcal{C}^{\text{op}} \rightarrow \text{InfSL} \) is an elementary doctrine then we have the equivalence
\[
T_{\text{Ex}(P)} \equiv (\Psi_{\mathcal{D}})_{\text{ex}/\text{lex}}.
\]
As application of the previous result we obtain as corollary the result regarding doctrines equipped with $\epsilon$-operators and exact completion presented in [11, Thm. 6.2(iii)].

**Corollary 5.14.** An elementary existential doctrine $P: C^{\text{op}} \rightarrow \text{InfSL}$ is equipped with Hilbert's $\epsilon$-operators if and only if the arrow $\text{Ex}(N, n): T_{\Psi_{\text{ex}}^{\text{op}}} \rightarrow T_P$ is an equivalence.

**Proof.** By Theorem 3.53 we have that a doctrine is equipped with Hilbert's Theorem $\epsilon$-operators if and only if it is the pure existential completion of itself. Then, applying Theorem 5.12 we can conclude. □

**Remark 5.15.** A first obvious example of application of theorem 5.10 is the construction of the ex/lex completion of a finite limit category $D$ itself. Indeed, after recalling that $(D)^{\text{ex}} / \text{reg} \equiv T_{\Psi} D$ from corollary 4.6 and that the weak subobjects doctrine is a full existential completions of the trivial doctrine $\delta: D^{\text{op}} \rightarrow \text{InfSL}$ from 3.15, we can obviously apply Theorem 5.10 to $\Psi D$ by getting $(D)^{\text{ex}} / \text{lex} \equiv T_{\Psi} D \equiv T_{\text{Ex}(\delta)} \equiv (\mathcal{G}\delta)^{\text{ex}} / \text{lex}$. This chain of equivalences does not yield more information, since by Theorem 5.8 we conclude that $D \equiv \mathcal{G}\delta$ as one can immediately check independently.

**Examples 5.16.** A relevant exact category having the presentation of Theorem 5.10 is provided by the realizability topos $RT(\mathfrak{A})$ associated to a pca $\mathfrak{A}$. In particular, given a realizability doctrine $P: \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ associated to a pca $\mathfrak{A}$, we have that combining Theorem 4.22, Lemma 4.23, and Theorem 5.10 we can conclude that

$$T_P = T_{\text{Ex}(\mathfrak{A})} \equiv (\mathcal{G}\mathfrak{A})^{\text{ex}} / \text{lex} \equiv (\mathcal{P}\mathfrak{A})^{\text{ex}} / \text{lex} \equiv \text{RT}(\mathfrak{A}).$$

**Examples 5.17.** Notice that the result presented in Theorem 5.10 provides also a useful instrument to study subcategories of $\text{Eff}$ which are instances of $(\cdot)^{\text{ex}} / \text{lex}$ completion. In particular, let us consider the doctrine $\Upsilon: \text{Rec}^{\text{op}} \rightarrow \text{InfSL}$, where $\text{Rec}$ denotes the category (with finite limits) whose objects are subsets of $\mathbb{N}$, and whose arrows are total, recursive functions. Following the notation used in Theorem 3.14, the functor $\Upsilon$ denotes the trivial doctrine, i.e. $\Upsilon(A) = \{\top\}$ for every object $A$ of $\text{Rec}$. It is easy to see that the doctrine $\Upsilon: \text{Rec}^{\text{op}} \rightarrow \text{InfSL}$ is a sub-doctrine of the doctrine $\mathfrak{P}^{\text{sing}}: \text{Set}^{\text{op}} \rightarrow \text{InfSL}$ of singletons of the realizability tripos, and then we have that $(\text{Rec})^{\text{ex}} / \text{lex}$ is a sub-category of $\text{Eff}$. Hence, for every supercoherent local $\mathcal{A}$, we have

$$\mathcal{A} \equiv \text{Sh}(\mathfrak{A}) \equiv (\mathcal{G}\mathfrak{A})^{\text{ex}} / \text{lex}$$

where $\mathcal{N}$ is the inf-semilattice of the supercompact elements of $\mathcal{A}$. 56
Recall from [45] that given a local $\mathcal{A}$ the category of presheaves $\text{PreSh}(\mathcal{A})$ over $\mathcal{A}$ is defined as $(\mathcal{A}_+)^{\text{ex/lex}}$, where $\mathcal{A}_+$ is the coproduct completion of $\mathcal{A}$. Employing our main results we can prove the following theorem, showing that every category of presheaves on a given locale is equivalent to the category of sheaves on the locale provided by the free completion which adds arbitrary sups to a locale.

**Theorem 5.19.** Let $\mathcal{A}$ be a locale. Then we have the following equivalence

$$\text{Sh}(D(\mathcal{A})) \equiv \text{PreSh}(\mathcal{A})$$

where $D(\mathcal{A})$ is the supercoherent locale constructed from $\mathcal{A}$.

**Proof.** Observe that the category called $\mathcal{A}_+$ is exactly the Grothendieck category $G_{\mathcal{A}(-)}$, where $\mathcal{A}(-) : \text{Set}^{\text{op}} \to \text{InfSL}$ is the localic tripos. Therefore, by Theorem 5.10 and Theorem 3.79, we can conclude that

$$\text{Sh}(D(\mathcal{A})) \equiv (G_{\mathcal{A}(-)})^{\text{ex/lex}} \equiv \text{PreSh}(\mathcal{A})$$

where $D(\mathcal{A})$ is the supercoherent locale constructed from $\mathcal{A}$. \hfill $\Box$

6. Conclusions

We have employed the tool of generalized existential completions to study properties of regular and exact completions. We conclude providing pictures summarizing our principal results and examples.

Let $P : C^{\text{op}} \to \text{InfSL}$ be a tripos, which is the full existential completion of a fibred subdoctrine $P'$, and let $\Psi_{G_{P'}} : G_{P'}^{\text{op}} \to \text{InfSL}$ be the weak subobjects doctrine on the Grothendieck category $G_{P'}$. Then we have the following morphisms of doctrines and categories

$$
\begin{array}{cccccc}
P & \cong & f\text{Ex}(P') & \to & P_c & \to & \text{Reg}(P) & \to & T_P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Psi_{G_{P'}} & \to & (\Psi_{G_{P'}})_c & \to & (G_{P'})_{\text{reg/lex}} & \equiv & (G_{P'})^{\text{ex/lex}}.
\end{array}
$$

The previous diagram can be then specialized for realizability triposes and supercoherent localic triposes as follows.

Let $\mathbb{A}$ be a pca and let us consider the realizability tripos associated to $\mathbb{A}$, which we know it is the full existential completion $f\text{Ex}(\mathcal{P}^{\text{sing}})$ of the conjunctive doctrine of singletons. Then let us consider its realizability topos $\text{RT}(\mathbb{A})$ and also the weak subobjects doctrine $\Psi_{\text{PAsm}(\mathbb{A})} : \mathcal{P}\text{Asm}(\mathbb{A})^{\text{op}} \to \text{InfSL}$ on the category of partitioned assemblies over $\mathbb{A}$. In this case, we obtain the following diagram

$$
\begin{array}{cccccc}
P & \cong & f\text{Ex}(\mathcal{P}^{\text{sing}}) & \to & \mathcal{P}_c & \to & \text{Reg}(\mathcal{P}) & \to & \text{RT}(\mathbb{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Psi_{\text{PAsm}(\mathbb{A})} & \to & (\Psi_{\text{PAsm}(\mathbb{A})})_c & \equiv & (\mathcal{P}\text{Asm}(\mathbb{A}))_{\text{reg/lex}} & \equiv & \text{Asm}(\mathbb{A}) & \to & (\mathcal{P}\text{Asm}(\mathbb{A}))^{\text{ex/lex}}.
\end{array}
$$
For any supercoherent locale $A$, where $N$ the inf-semilattice of its supercompact elements, and the localic tripos $A(-): \text{Set}^{op} \rightarrow \text{InfSL}$ associated to $A$ we obtain:

$$A(-) \cong f \text{Ex}(N(-)) \rightarrow A(-) \rightarrow \text{Reg}(A(-)) \rightarrow \mathcal{T}_{A(-)} \equiv \text{Sh}(A)$$

$\Psi \mathcal{G}_{N(-)} \rightarrow \mathcal{A}_{N(-)} \rightarrow (\mathcal{A}_{N(-)})_{\text{reg/lex}} \rightarrow (\mathcal{G}_{N(-)})_{\text{ex/lex}}$

7. Future work

A preliminary version of the characterization of generalized existential completion presented here has already been fruitfully employed in recent works [19, 20, 21, 22] to give a categorical version of Gödel’s dialectica interpretation [23] in terms of quantifier-completions. As future work, we intend to further apply the tools and results developed here to the categorification of Gödel dialectica interpretation and study their connections with dialectica triposes in [46] and modified realizability triposes [25, 27].

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