Universal thermal corrections to symmetry-resolved entanglement entropy and full counting statistics

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ABSTRACT: In the presence of a global symmetry, the entanglement entropy of a quantum system can be decomposed among the individual symmetry sectors, dubbed symmetry-resolved entanglement entropy. For a conformal field theory with Abelian symmetry, it obeys the equipartition theorem in the scaling limit, i.e., at the leading order, the entanglement is distributed equally between different symmetry sectors. In this work, we examine the thermal corrections to a single interval symmetry-resolved Rényi and entanglement entropies for two-dimensional conformal field theories on a circle. Using a low-temperature expansion of the thermal density matrix, we find that in addition to the mass gap and the degeneracy of the first excited state, these universal corrections depend also on the four-point correlation function of the primary fields. We also obtain thermal corrections to the full counting statistics of the ground state and define the probability fluctuations function which scales as $e^{-2\pi\Delta_\psi \beta/L}$, where $\Delta_\psi$ is the scaling dimension of the lowest weight states. As an example, we explicitly evaluate the thermal corrections to the symmetry-resolved entanglement entropy and FCS for the spinless fermions and find a term breaking entanglement equipartition at order $(\log l)^{-2}$.

KEYWORDS: Conformal and W Symmetry, Field Theories in Lower Dimensions, Global Symmetries

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1 Introduction

One of the outstanding features of quantum mechanics is the concept of entanglement [1], which relates to non-local correlations between different subsystems of a given quantum system. It has a wide range of applications in various branches of physics. Its measures, especially the entanglement entropy (EE), can be used as an order parameter for characterizing phase transitions [2–4] and topological phases [5, 6] in many-body quantum systems; providing new tools for quantifying quantum correlations and describing renormalization group flow in quantum field theory [7–11]; entanglement entropy also plays an important role in the connection between quantum states and quantum gravity in the framework of holography [12, 13] and is a key concept in describing the physics of the black hole information loss paradox [14, 15].

The computation of the EE is often based on the replica trick, by introducing $n$ copies of the system. For bipartite quantum system $A \cup B$, the $n$th Rényi entropy (RE) is defined as

$$S_n \equiv \frac{1}{1-n} \log \text{Tr} (\rho_A)^n,$$

(1.1)

where $\rho_A = \text{Tr}_B \rho$ is the reduced density matrix of subsystem $A$. EE is given by

$$S_E = -\text{Tr} \rho_A \ln \rho_A = \lim_{n \to 1} S_n.$$

(1.2)

Based on the path integral language [7, 8], the calculation of $S_n$ reduces to computing the partition function $Z_n$ on a $n$-sheeted Riemann surface $\mathcal{R}_n$ obtained by gluing cyclically $n$ copies of the original geometry along the cut $A$,

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n},$$

(1.3)

where $Z_1$ is a partition function on an original Riemann surface.
Using this approach, one can find the entanglement entropy of the ground state of a \((1 + 1)\)-dimensional conformal field theory with central charge \(c\) [7, 8],
\[
S_E = \frac{c}{3} \log \left[ \frac{L}{\pi} \sin \left( \frac{\pi l}{L} \right) \right] + \cdots
\]
(1.4)

where \(l\) denotes the length of the subsystem \(A\) embedded in the finite line with length \(L\).

It is well known that entanglement entropy is a good entanglement measure for quantum systems in their ground state. But, for mixed states, the entanglement entropy is no longer a proper measure of entanglement. Thermal states are examples of those states. At finite temperatures, in fact, the entanglement entropy of subsystem \(A\) is contaminated by thermal fluctuation and is dominated by thermal entropy in the high-temperature limit. To determine the quantum entanglement of thermal systems, the thermal contribution to the entanglement entropy must be subtracted.

For the systems with the mass gap \(m_{\text{gap}}\), the authors [16] conjectured that, in the limit \(\beta m_{\text{gap}} \gg 1\), these corrections scale as \(e^{-\beta m_{\text{gap}}}\). They provided numerical evidence for their conjecture. The authors [17] started with thermal states \(e^{-\beta H_{\text{tot}}}/e^{-\beta H}\) (where \(\beta\) is the inverse temperature and \(H\) is the Hamiltonian) and calculated the coefficient of the Boltzmann factor by putting the conformal field theory on the cylinder and introducing the mass gap between the ground state and the first excited state through the finite size of the system. They found that these corrections are universal and depend on the size of the mass gap and the degeneracy of the first excited state,

\[
\delta S_n = \frac{ng}{1 - n} \left[ \frac{1}{n^{2\Delta_\psi}} \sin^{2\Delta_\psi} \left( \frac{\pi l}{nL} \right) \right] e^{-2\pi\Delta_\psi \beta/L} + o(e^{-2\pi\Delta_\psi \beta/L})
\]
(1.5)

\[
\delta S_E = 2g\Delta_\psi \left( 1 - \frac{\pi l}{L} \cot \left( \frac{\pi l}{L} \right) \right) e^{-2\pi\Delta_\psi \beta/L} + o(e^{-2\pi\Delta_\psi \beta/L})
\]
(1.6)

where \(g\) is the degeneracy of the first excited state, \(\Delta_\psi\) is the smallest scaling dimension among the set of operators not equal to the identity, \(L\) is the circumference of the cylinder, and \(l\) is the interval length. It is worth noting that in order for these formulae to hold, the conformal field theory on the circle is assumed to have a unique ground state and a mass gap \(m_{\text{gap}} = 2\pi\Delta_\psi/L\) induced by the finite size of the system [17].

An intriguing problem (in this context) concerns understanding the interplay between entanglement and symmetry. It is interesting to understand how entanglement decomposes into the various symmetry sectors in the presence of global symmetry. More generally, in the presence of global symmetry, the density matrix \(\rho\) of the total system commutes with a conserved charge \(\hat{Q}\), \([\rho, \hat{Q}] = 0\). By tracing over the degrees of freedom of the subsystem \(B\), we find that \([\rho_A, \hat{Q}_A] = 0\), where \(\hat{Q}_A\) is the total charge in the \(A\) subsystem. This means that the reduced density matrix \(\rho_A\) can be decomposed into block diagonal form associated with each charge sector,

\[
\rho_A = \bigoplus_Q P(Q_A)\rho_A(Q_A),
\]
(1.7)
$P(Q_A)$ is the probability of finding $Q$ as an outcome of a measurement of $\hat{Q}_A$. Symmetry-resolved entanglement entropy is defined as

$$S(Q) = -\text{Tr} \rho_A(Q) \ln \rho_A(Q).$$

(1.8)

The total entanglement entropy can be split into two terms,

$$S_E = \sum_{Q_A} P(Q_A) S(Q_A) - \sum_{Q_A} P(Q_A) \log(P(Q_A))$$

$$= S^c + S^n. \quad (1.9)$$

$S^c$ is the configurational entropy [18], which measures the average of the entanglement in each charge sector. $S^n$ is called number entropy [18–21], which quantifies the entropy due to the fluctuations of the charge within subsystem $A$.

Using the path integral language, the authors [22] via the insertion of an Aharonov-Bohm flux in the Riemann geometry could decompose entanglement measures into the contribution of individual charge sectors in the presence of global symmetry. Hence, the computation of the path integral in the presence of this flux gives the quantity, charged moments

$$Z_n(\alpha) = \text{Tr} \left( \rho^n_A e^{i \alpha \hat{Q}_A} \right),$$

(1.10)

which enables us to identify and compute the contributions to the entanglement related to each symmetry sector. $\hat{Q}_A$ is the total charge in the $A$ subsystem. They provide a theoretical framework for evaluating the contribution of each symmetry sector by relating the symmetry-resolved entanglement entropy to the Fourier transform of partition function on the $n$-sheeted Riemann surface with generalized Aharonov-Bohm flux:

$$Z_n(\alpha) = \text{tr} \rho^n_A \mathcal{P}_{Q_A} = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} Z_n(\alpha) e^{-i\alpha Q_A}.$$  

(1.11)

$\mathcal{P}_{Q_A}$ denotes the projection into the subspace of states of region $A$ with charge $\hat{Q}_A$. Symmetry-resolved Rényi entropies defined as,

$$S_n(Q_A) = \frac{1}{1-n} \log \left[ \frac{Z_n(Q_A)}{Z_1^n(Q_A)} \right],$$

(1.12)

and symmetry-resolved entanglement entropy is obtained as,

$$S(Q_A) = \lim_{n \to 1} S_n(Q_A).$$ \quad (1.13)

Probability is then recovered as $P(Q_A) = Z_1(Q_A)$. Various works have been done in different kinds of literature, including e.g. 1+1-dimensional conformal field theories [22–28], free and interacting integrable quantum field theories [29–33], spin systems [34–43], and also holographic settings [44–47], topologically ordered systems [48, 49], negativity [50–53], and experimental [18, 54]. Besides these developments, symmetry-resolved entanglement measures were also investigated in out-of-equilibrium situations [55–58]. One of the main features emerging from the literature is that due to conformal invariance, the entanglement entropy is equally distributed among the different sectors. Strictly speaking, at the leading order, the symmetry-resolved entanglement entropy is independent of the charge sector.
Another interesting quantity that can be deduced from charged moments is the full counting statistics (FCS). FCS is a measure of the charge distribution in a given region. This object encodes the information about charge fluctuations as well as all higher-order correlations. Moreover, there is a relation between the entanglement spectrum and the FCS of charge fluctuations between the two subsystems. For example, for non-interacting fermions, the fluctuation of conserved $U(1)$ charge such as particle number $\hat{N}$ encodes some features of quantum systems, such as entanglement properties. In fact, the entanglement entropy is entirely encoded in the even cumulants [59–61]. Furthermore, bipartite charge fluctuations can be shown to be proportional to the (Rényi) entanglement entropy for, e.g., (1+1)-dimensional conformal field theories or Fermi gases [59, 62, 63]. In addition, the FCS is explored at thermal or interacting one-dimensional Bose gas [63, 65], spin chain [64–67].

The FCS is characterized by the cumulant generating function whose inverse Fourier transform gives the probability distribution,

$$\chi(\alpha) = \langle e^{i\alpha \hat{Q}_A} \rangle$$

On the other hand, in the presence of global symmetry, as was mentioned, the reduced density matrix $\rho_A$ has a block-diagonal structure, in which each block corresponds to an eigenvalue $Q_A$ of the charge operator $\hat{Q}_A$. The moments of $\rho_A$ restricted to the $Q$ charge sector are the Fourier transform of $Z_n(\alpha) = \text{Tr} (\rho_A^n e^{i\alpha \hat{Q}_A})$. This quantity reduces to cumulant generating function $\chi(\alpha)$ for $n = 1$. This denotes the connections between the FCS and the entanglement entropy in a given charge sector in the presence of a global symmetry.

The interesting questions that motivated this work are: what are the thermal corrections to the contribution of individual system charge sectors? What are the scaling behaviors of these correction terms? If these corrections are universal, what are their physical meaning? In this work, we are addressing these questions. We introduce thermal charged moments and derive its low-temperature expansion based on the [17] approach. We find that these thermal corrections are encoded in the four-point function of the primary fields, the scaling dimension of the lowest weight primary field, and its degeneracy. Consequently, we can find the thermal corrections to the symmetry-resolved Rényi and entanglement entropies. We also obtain thermal corrections to the full counting statistics of the ground state (FCS) and define the fluctuations of probabilities. It scales as $e^{-2\pi \Delta_{\psi} \beta / L}$.

2 Thermal charged moments and universal corrections

The main quantity to calculate the FCS and the symmetry-resolved entanglement entropy is the charged moments $Z_n(\alpha)$. Therefore, to calculate thermal corrections, we introduce thermal charged moments and then find its low-temperature expansion. For this aim, we consider a unitary CFT with central charge $c$ defined on an infinite cylinder of circumference $L$ with coordinate $w = y - it$. The corresponding Hamiltonian is $H = \left( \frac{2\pi}{L} \right) \left( L_0 + \bar{L}_0 - \frac{c}{24} \right)$, where $L_0$ and $\bar{L}_0$ are the zeroth level left- and right-moving Virasoro generators, respectively. According to state-operator correspondence, for any Virasoro primary operator $\psi(w) \neq 1$ there is a state $|\psi\rangle = \lim_{t \to -\infty} \psi(w)|0\rangle$ where $L_0|\psi\rangle = \hbar \psi|\psi\rangle$, $\bar{L}_0|\psi\rangle = \bar{\hbar} \psi|\psi\rangle$ and $\Delta_\psi = \hbar + \bar{\hbar}$, and vice versa. If the $|\psi\rangle$ are the lowest weight states in the CFT, then the smallest
nonzero eigenvalue $E_\psi = \frac{2\pi}{L} (\Delta_\psi - \frac{c}{12})$ must correspond to a primary operator $\psi(w)$. Note that when $\Delta > 2$, two descendants of the identity operator, the stress tensor $T(w)$ and its conjugate $\tilde{T}(\bar{w})$, give the dominant correction [17]. Note that the CFT is assumed to be gapped due to the fine size of the system.

To find low-temperature expansion, we follow [17], and start with the thermal density matrix, which can be written as a Boltzmann sum by introducing a complete set of states,

$$\rho = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}} = \frac{|0\rangle\langle 0| + |\psi\rangle\langle \psi| e^{-2\pi \Delta_\psi \beta / L} + \ldots}{1 + e^{-2\pi \Delta_\psi \beta / L} + \ldots}$$

(2.1)

where $|\psi\rangle$ is the first excited state with scaling dimension $\Delta_\psi$. The $\Delta_\psi$ is the smallest scaling dimension of the primary operator, $\psi(w)$, among the set of non-identity operators including the stress tensor. At the lowest order, it follows

$$\rho \sim \left( |0\rangle\langle 0| + |\psi\rangle\langle \psi| e^{-2\pi \Delta_\psi \beta / L} \right) \left( 1 - e^{-2\pi \Delta_\psi \beta / L} \right)$$

(2.2)

Therefore, the low-temperature expansion of the thermal density matrix becomes as

$$\rho \sim \rho_0 + e^{-2\pi \Delta_\psi \beta / L} (\rho_\psi - \rho_0)$$

(2.3)

where $\rho_0 = |0\rangle\langle 0|$ and $\rho_\psi = |\psi\rangle\langle \psi|$ are density matrices associated with vacuum and first excited states, respectively. By partitioning the spatial circle into regions $A$ and $B$, we can compute the reduced density matrix $\rho_A$ by tracing out the degrees of freedom inside the region $B$. Accordingly, the low-temperature expansion of the reduced density matrix becomes as

$$\rho_A \sim \rho_{0,A} + e^{-2\pi \Delta_\psi \beta / L} (\rho_{\psi,A} - \rho_{0,A})$$

(2.4)

where $\rho_{0,A} = \text{Tr}_B (|0\rangle\langle 0|)$ and $\rho_{\psi,A} = \text{Tr}_B (|\psi\rangle\langle \psi|)$. We define the thermal charged moments as

$$Z_n^{(\text{th})}(\alpha) = \text{Tr} \left( \rho_A^n e^{i\alpha Q_A} \right)$$

$$= Z_n^{(0)}(\alpha) + n Z_n^{(0)}(\alpha) e^{-2\pi \Delta_\psi \beta / L} F_n(\alpha).$$

(2.5)

The first term is nothing but the charged moments at zero temperature that gives symmetry-resolved entanglement entropy for a finite system [24]

$$Z_n^{(0)}(\alpha) = c_{n,\alpha} \left[ \frac{L}{\pi} \sin \left( \frac{\pi \ell}{T} \right) \right]^{1 - n^2 / 6} 2 \frac{\Delta_V}{\pi}$$

(2.6)

where $c_{n,\alpha}$ is non-universal constant, $\ell$ is the length of $A$, and $\Delta_V$ denotes the scaling dimension of the operator $\mathcal{V}$ generating the Aharonov-Bohm flux.

The $F_n(\alpha)$ is defined as:

$$F_n(\alpha) = M_n(\alpha) - 1,$$

$$M_n(\alpha) = \frac{\text{Tr} \left( \rho_{\psi,A}^{n-1} e^{i\alpha Q_A} \right)}{\text{Tr} \left( \rho_{\psi,A}^{n} e^{i\alpha Q_A} \right)}.$$

(2.7)
The denominator is the charged moments at zero temperature. The numerator is a new term. It can be interpreted as a two-point function in the presence of Aharonov-Bohm flux.

In other words, for interpretations of these terms, let us begin with the geometric construction of each term in the equation (2.7). When \( \rho \) is the ground state of a CFT, \( \text{Tr} \rho^n_A \) is a partition function over an \( n \)-sheeted Riemann surface \( \mathcal{R}_n \). Similarly, \( \text{Tr} \left( \rho^n_A e^{i\alpha Q_A} \right) \) can be seen as a partition function over the \( n \)-sheeted Riemann surface \( \mathcal{R}_n \) with an inserted Aharonov-Bohm flux \( \alpha \) [22]. The insertion of a flux corresponds to a twisted boundary condition, which can be implemented by some local fields \( \mathcal{V}_\alpha \) acting on the boundary of subsystem \( A \). More precisely, if subsystem \( A \) is an interval \([u, v]\), then one can identify

\[
e^{i\alpha Q_A} = \mathcal{V}_\alpha(u, 0)\mathcal{V}_\alpha^{-1}(v, 0). \tag{2.8}
\]

Therefore, \( \text{Tr} \left( \rho^n_A e^{i\alpha Q_A} \right) \) can be interpreted as a partition function over the \( n \)-sheeted Riemann surface \( \mathcal{R}_n \) with twisted boundary conditions \( \langle \mathcal{V}_\alpha \rangle \) or, equivalently, as a correlation function over the \( n \)-sheeted Riemann surface \( \mathcal{R}_n \) with periodic boundary conditions [22, 24],

\[
\text{Tr} \left( \rho^n_A e^{i\alpha Q_A} \right) = \left\langle e^{i\alpha Q_A} \right\rangle_{\mathcal{R}_n} = \langle \mathcal{V}_\alpha \mathcal{V}_\alpha^{-1} \rangle_{\mathcal{R}_n}. \tag{2.9}
\]

Therefore, in the above sense, the expression (2.7) then takes the following form,

\[
\mathcal{M}_n = \frac{\langle \mathcal{V}_\alpha \mathcal{V}_\alpha^{-1} \psi \psi \rangle_{\mathcal{R}_n}}{\langle \mathcal{V}_\alpha \mathcal{V}_\alpha^{-1} \rangle_{\mathcal{R}_n} \langle \psi \psi \rangle_{\mathcal{R}_1}}. \tag{2.10}
\]

The numerator term is a four-point function of two \( \mathcal{V}_\alpha \) and two \( \psi \) in the replicated geometry \( \mathcal{R}_n \), which is the \( n \)-sheeted cylinder, for a particular insertion of points. The denominator terms are two two-point functions of the \( \mathcal{V}_\alpha \) in the replicated geometry \( \mathcal{R}_n \) and \( \psi \) in the original geometry \( \mathcal{R}_1 \), cylinder. The latter two-point function is due to the normalization condition, \( \langle \psi | \psi \rangle = 1 \), on the original cylinder.

According to the Riemann-Hurwitz theorem, this \( n \)-sheeted Riemann surface \( \mathcal{R}_n \) has genus zero and can be uniformized through the conformal map [17]

\[
\zeta^{(n)} = \left( \frac{e^{2\pi i w/L} - e^{i\theta_1}}{e^{2\pi i w/L} - e^{i\theta_2}} \right)^{1/n}. \tag{2.11}
\]

This map takes the \( n \)-sheeted cylinder to the plane, we evaluate these correlation functions on the plane as follows. The parameters \( \theta_1 \) and \( \theta_2 \) are selected so that the map is branched over the interval \( A \) with endpoints on the cylinder, such that \( \theta_2 - \theta_1 = \frac{2\pi \ell}{L} \). Subsequently, the insertion points of the primary fields on the \( j \)th cylinder at the points \( t = -\infty \) and \( t = \infty \) are transformed to the points \( \zeta^{(n)}_1 \equiv \zeta^{(n)}_\infty = e^{i(\theta_2 - \theta_1)/n} e^{-2\pi i j/n} \) and \( \zeta^{(n)}_3 \equiv \zeta^{(n)}_\infty = e^{2\pi i j/n} \), respectively, on the \( \zeta^{(n)} \) plane. While the operators on the endpoints of the interval are mapped to the points \( \zeta^{(n)}_1 = \infty \) and \( \zeta^{(n)}_2 = 0 \) on the \( \zeta^{(n)} \) plane.

Under conformal transformations, the primary field transforms as [68]

\[
\psi(w, \bar{w}) = \left( \frac{d \zeta^{(n)}_1}{dw} \right)^{h_\psi} \left( \frac{d \bar{\zeta}^{(n)}_1}{d\bar{w}} \right)^{\bar{h}_\psi} \psi(\zeta^{(n)}_1, \bar{\zeta}^{(n)}_1), \tag{2.12}
\]
where \((h_\psi, \tilde h_\psi)\) are conformal weights of primary field \(\psi\). Using the following identity
\[
\frac{d\zeta^{(n)}/dw}{d\zeta^{(1)}/dw} = \frac{1}{n} \zeta^{(1)},
\]
and applying the conformal map (2.11), one can easily express the equation (2.10) in terms of correlation functions on the plane
\[
\mathcal{M}_n(\alpha) = \frac{1}{n^{2\Delta_\psi}} \left(\frac{\zeta^{(n)}_3 \zeta^{(n)}_4}{\zeta^{(1)}_3 \zeta^{(1)}_4}\right)^{h_\psi} \left(\frac{\zeta^{(n)}_3 \zeta^{(n)}_4}{\zeta^{(1)}_3 \zeta^{(1)}_4}\right)^{\tilde h_\psi} \left\langle \mathcal{V}_\alpha \left(\zeta^{(n)}_1, \zeta^{(n)}_1\right) \mathcal{V}_{-\alpha} \left(\zeta^{(n)}_2, \zeta^{(n)}_2\right) \psi \left(\zeta^{(n)}_3, \zeta^{(n)}_3\right) \psi \left(\zeta^{(n)}_4, \zeta^{(n)}_4\right) \right\rangle.
\]
(2.14)
The correlation functions on the plane are as follows (2.10). Two-point functions are
\[
\left\langle \psi \left(\zeta^{(1)}_1, \zeta^{(1)}_3\right) \psi \left(\zeta^{(1)}_4, \zeta^{(1)}_4\right) \right\rangle = \frac{1}{\left(\zeta^{(1)}_3 \zeta^{(1)}_4\right)^{h_\psi} \left(\zeta^{(1)}_1 \zeta^{(1)}_2\right)^{2h_\psi}},
\]
\[
\left\langle \mathcal{V}_\alpha \left(\zeta^{(n)}_1, \zeta^{(n)}_1\right) \mathcal{V}_{-\alpha} \left(\zeta^{(n)}_2, \zeta^{(n)}_2\right) \right\rangle = \frac{1}{\left(\zeta^{(n)}_3 \zeta^{(n)}_4\right)^{2h_\psi} \left(\zeta^{(n)}_1 \zeta^{(n)}_2\right)^{2h_\psi}}.
\]
(2.15)
We used the standard CFT normalization for the two-point function. And the four-point function is
\[
\left\langle \mathcal{V}_\alpha \left(\zeta^{(n)}_1, \zeta^{(n)}_1\right) \mathcal{V}_{-\alpha} \left(\zeta^{(n)}_2, \zeta^{(n)}_2\right) \psi \left(\zeta^{(n)}_3, \zeta^{(n)}_3\right) \psi \left(\zeta^{(n)}_4, \zeta^{(n)}_4\right) \right\rangle = \frac{1}{\left(\zeta^{(n)}_1 \zeta^{(n)}_2 \zeta^{(n)}_3 \zeta^{(n)}_4\right)^{2h_\psi} \left(\zeta^{(n)}_1 \zeta^{(n)}_2 \zeta^{(n)}_3 \zeta^{(n)}_4\right)^{2h_\psi}} G_{n,\alpha}(z, \bar z)
\]
(2.16)
Using the correlation functions on the plane (2.15) and (2.16), the expression (2.14) can be simplified to the following expression,
\[
\mathcal{M}_n(\alpha) = \frac{g}{n^{2\Delta_\psi} \sin^{2\Delta_\psi}(\pi x/n)} \sin^{2\Delta_\psi}(\pi x/n) G_{n,\alpha}(z, \bar z),
\]
(2.17)
where \(x = \frac{\zeta}{\ell}\). \(g\) denotes the degeneracy of the first excited state. In fact, if we have several states \(\ket{\psi_i}, (i = 1, 2, \ldots, g)\) with the same dimension \(\Delta_\psi\), each \(\ket{\psi_i}\) contributes equally, then the factor of \(g\) is included in the equation (2.17). The \(G_{n,\alpha}(z, \bar z)\) is a function of cross ratios
\[
z = \frac{\zeta^{(n)}_1 \zeta^{(n)}_2 \zeta^{(n)}_3}{\zeta^{(n)}_4 \zeta^{(n)}_2}, \quad \bar z = \frac{\zeta^{(n)}_1 \zeta^{(n)}_2 \zeta^{(n)}_3}{\zeta^{(n)}_4 \zeta^{(n)}_2},
\]
(2.18)
which are invariants of the global conformal group \(\text{SL}(2, \mathbb{C})\). The equation (2.17) is one of the main results of this work. It can be regarded as the building block of our computations. All information about the thermal corrections to the charged moments and the symmetry-resolved entanglement entropy is encoded in the \(G_{n,\alpha}(z, \bar z)\), which has a conformal block expansion as
\[
G_{n,\alpha}(z, \bar z) = \sum_a C_{\mathcal{V}_\alpha \mathcal{V}_{-\alpha} \mathcal{O}_a} C_{\psi \bar \psi \bar \psi} z^{2h_{\mathcal{V}_\alpha}} \bar z^{2h_{\mathcal{V}_{-\alpha}}} \mathcal{F}(h_{\mathcal{V}_\alpha}, h_\psi, h_{\bar \psi}; z) \mathcal{F}(h_{\mathcal{V}_{-\alpha}}, \bar h_\psi, \bar h_{\bar \psi}; \bar z),
\]
(2.19)
where the sum runs over all the primary operators $O_a$ that appear in both the $\mathcal{V}_\alpha \times \mathcal{V}_{-\alpha}$ and $\psi \times \psi$ OPEs, and $(h_a, \bar{h}_a)$ are the conformal weights of the primaries $O_a$. The $\mathcal{F}(h_{\mathcal{V}_\alpha}, h_\psi, h_a; z)$ is called the Virasoro block, which, in the small limit of $z$, has an expansion as

$$\mathcal{F}(h_{\mathcal{V}_\alpha}, h_\psi, h_a; z) = z^{h_a - 2h_{\mathcal{V}_\alpha}} (1 + \cdots),$$ (2.20)

where the leading contribution comes from the primary operator $O_a$.

### 3 Full counting statistics

Thermal charged moments can be defined as a generating function of FCS at finite temperatures. FCS defines the distribution probability of conserved charge in the subsystem $A$ with length $l$. It can be defined via generating function

$$\chi(\alpha) = \sum_{Q_A=-\infty}^{\infty} P(Q_A) e^{i\alpha Q_A} = \langle e^{i\alpha \hat{Q}_A} \rangle$$ (3.1)

which encodes all the cumulants $C_m$,

$$\ln \chi(\alpha) = \sum_{Q_A=1}^{\infty} \frac{(i\alpha)^m C_m}{m!}$$ (3.2)

where $C_m = (-i\partial_{\alpha})^m \ln \chi(\alpha)$. $C_m$ describe properties of the distribution probability $P(Q_A)$. For example, the mean $C_1 = \langle \hat{Q}_A \rangle$, the fluctuations $C_2 = \langle (\hat{Q}_A - \langle \hat{Q}_A \rangle)^2 \rangle$, and so on.

In this section, we derive the thermal corrections to the FCS.

Let us first define the quantity:

$$f_n(\alpha, T) = \frac{Z_n^{(th)}(\alpha)}{Z_n^{(0)}(\alpha)} = 1 + n F_n(\alpha) e^{-2\pi \Delta \psi \beta / L}. $$ (3.3)

If we take a logarithm of the above expression, we reach the universal quantity

$$g_n(\alpha, T) = \log f_n(\alpha, T)$$ (3.4)

that can be used to define the excess-cumulant generating function, such that, its different derivative in $\alpha = 0$ gives the excess of various moments of $\hat{Q}_A$,

$$\Delta C_{n,m} = (-i\partial_{\alpha})^m g_n(\alpha, T)|_{\alpha=0}$$

$$= g e^{-2\pi \Delta \psi \beta / L} \frac{\sin^{2\Delta \psi} (\pi x)}{\sin^{2\Delta \psi} (\pi \bar{x})} (-i\partial_{\alpha})^m G_{n,\alpha}(z, \bar{z})|_{\alpha=0}.$$ (3.5)

The relation (3.5), for $\Delta C_{1,m} = \Delta(\Delta Q^m_A)$ with $\Delta Q_A = Q_A - \langle \hat{Q}_A \rangle$, denotes that the exceed-FCS depend on the mass gap, degeneracy $g$, and the field content of the theory. FCS was previously calculated for the ground state [69, 70]. It was also calculated for the excited state in free compact boson [24]. Here, we derived thermal corrections for the FCS for any two-dimensional conformal field theory.
The symmetry-resolved thermal partition function becomes

\[ Z_n^{(th)}(Q_A) = Z_n^{(0)}(Q_A) \left[ 1 + gnF_n(Q_A)e^{-2\pi \Delta \phi / L} \right], \]

\[ F_n(Q_A) = \frac{1}{n^2 \Delta \phi} \sin^2 \frac{\Delta \phi \pi x}{n} \frac{X_n(Q_A)}{Z_n^{(0)}(Q_A)} - 1, \]

\[ X_n(Q_A) = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} Z_n^{(0)}(\alpha) G_n,\alpha(z, \bar{z}) e^{-i\alpha Q_A}. \] (3.6)

The leading correction term to the probability distribution of charge in each sector can be obtained by \( P^{(th)}(Q_A) = Z_n^{(th)}(Q_A) \). We define the quantity:

\[ g_n(Q_A, T) = \log f_n(Q_A, T) = ngF_n(Q_A)e^{-2\pi \Delta \phi / L}, \] (3.7)

where \( f_n(Q_A, T) = \frac{Z_n^{(th)}(Q_A)}{Z_n^{(0)}(Q_A)} \). \( Z_n^{(th)}(Q_A) \) denotes the probability of finding the charge \( Q_A \) at the temperature \( \beta^{-1} \). By choosing \( n = 1 \), we find

\[ g_1(Q_A, T) = \left( \frac{X_1(Q_A)}{Z_1^{(0)}(Q_A)} - 1 \right) g e^{-2\pi \Delta \phi / L}. \] (3.8)

This quantity represents the probability fluctuations. It states that, at low temperatures, the ratio of the probability of finding a charge \( Q_A \) at temperature \( \beta^{-1} \) to its value at zero temperature scales as \( e^{-2\pi \Delta \phi / L} \). Its coefficient depends on the degeneracy of the first excited state, the charge of the sector, and the field content of the theory.

It is worth noting that the starting and common point of the FCS scenario and the symmetry-resolved entanglement entropy is the thermal charged moments \( Z_n^{(th)}(\alpha) \). Its low-temperature expansion provides a source of thermal correction to both quantities. Then we expect the scaling of these corrections to be the same for both and scales as \( e^{-2\pi \Delta \phi / L} \). In the next section, the thermal corrections of symmetry-resolved entanglement entropy will be obtained.

### 4 Entanglement measures

Using the relations (3.6), in general, the thermal correction to the symmetry-resolved Rényi and entanglement entropies take the following forms:

\[ \delta S_n(Q_A) = \frac{ng}{1-n} \left[ \frac{1}{n^2 \Delta \phi} \sin^2 \frac{\Delta \phi \pi x}{n} \frac{X_n(Q_A)}{Z_n^{(0)}(Q_A)} - \frac{X_1(Q_A)}{Z_1^{(0)}(Q_A)} \right] e^{-2\pi \Delta \phi / L} \]

\[ + o(e^{-2\pi \Delta \phi / L}), \] (4.1)

\[ \delta S_E(Q_A) = g \left[ 2\Delta \phi \left( 1 - \pi x \cot(\pi x) \right) \frac{X_1(Q_A)}{Z_1^{(0)}(Q_A)} + \partial_n \left( \frac{X_n(Q_A)}{Z_n^{(0)}(Q_A)} \right) \right] |_{n=1} e^{-2\pi \Delta \phi / L} \]

\[ + o(e^{-2\pi \Delta \phi / L}). \] (4.2)

Note that in the above expression, we assume that the finite size of the system induces a mass gap that separates the ground state from the first excited state. As we see, these
symmetry sector corrections scale as $e^{-2\pi \Delta \psi \beta / L}$. Their coefficients, besides the size of the mass gap and degeneracy of the first excited state, depend on the four-point correlation function of primary fields. Compared to the total entanglement entropy, the scaling of these sectors is similar to the scaling of the total entanglement entropy, except that the coefficients in the latter depend only on the mass gap and the degeneracy of the excited state and are independent of the charge of the sector \[17\].

Before proceeding, we recall that a similar calculation was performed in refs. \[24, 27\], where the authors considered symmetry-resolved entanglement and relative entropy in excited states of the CFT, by inserting two operators $\psi(z, \bar{z})$ on all of the $n$-sheets of the cylinder. The authors \[24\] derived the cumulants and the symmetry-solved entanglement entropy for compact boson CFT (aka Luttinger liquid) for vertex and derivative excitations using the $(2n + 2)$-correlation functions. They found a term breaking entanglement equipartition at order $(\log l)^{-2}$. By contrast, the leading correction term in our calculation comes from the 4-point function. As an explicit example, in the next section, we will obtain the exact result for the derivative operator, which explicitly represents the breaking entanglement equipartition at order $(\log l)^{-2}$. The authors \[27\], derived the symmetry-resolved subsystem trace distance and relative entropy for reduced density matrices associated to excited states in the massless compact boson. They also computed the $(2n + 2)$-correlation functions.

5 Examples

As an example, we specialize in compactified massless bosons with $c = 1$ \[68, 71, 72\]. In this CFT, there are two holomorphic primary fields, vertex operator $V_\beta = e^{i\beta \phi_j}$ and derivative operator $i \partial \phi$, with scaling dimensions $h_V = K \frac{\beta^2}{2}$ and $h_{i \partial \phi} = 1$, respectively. The Luttinger parameter $K$ is related to the compactification radius via the bosonization relation. The lowest scaling primary field depends on the range $\beta$ that is determined via the Luttinger parameter $K$. In the following, we will consider both cases.

The generation of the Aharonov-Bohm flux is implemented by inserting the vertex operator $V_\alpha = e^{i \alpha \phi}$, with the scaling dimension $h_V = h_{\alpha \phi} = \frac{1}{2} \left( \alpha \frac{\pi}{n} \right)^2 K$, such that generating the total phase $\alpha$ for the field upon going through the entire $n$-sheeted Riemann surface $\mathcal{R}_n$ \[22\]. If the excited state is generated by the vertex operator $V_\beta = e^{i \beta \phi_j}$, we find that $G_{n,\alpha}(z, \bar{z}) = e^{-iK \frac{\alpha \beta x}{n}}$. It follows that, the exceed-cumulant generating function is

$$g_n(\alpha, T, x) = \frac{1}{2} \left( \alpha \frac{\pi}{n} \right)^2 e^{-K \pi \beta^2 \beta / L}.$$  

For example, eq. (5.1) with $n = 1, m = 2$ and $\Delta Q_A = Q_A - \langle \hat{Q}_A \rangle$, implies that the exceed in the variance in the charge (number of particles) is $(\Delta Q_A)^2 = g(K \beta x)^2 e^{-K \pi \beta^2 \beta / L}$.

If the excited state is induced by the derivative primary operator $i \partial \phi$, as a generator of a low-dimensional primary state, we find the conformal block:

$$G_{n,\alpha}(z, \bar{z}) = 1 - K \left( \frac{\alpha}{\pi} \right)^2 \sin^2 \left( \frac{\pi x}{n} \right).$$  

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Using the equation (3.5), we find that the only non-zero exceed-cumulant is second cumulants, which is universal for any \( n \),

\[
\Delta C_{n,2} = \frac{2gK\sin^2(\pi x)}{\sigma^2} e^{-2\pi \beta/L}.
\] (5.3)

The physical meaning for \( n = 1 \) is the excess of the fluctuations \( \Delta C_{1,2} = 2gK\frac{\sin^2(\pi x)}{\pi^2} e^{-2\pi \beta/L} \).

In addition, we have

\[
\frac{\mathcal{X}_n(Q_A)}{Z_n^{(0)}(Q_A)} = 1 - \frac{K\sin^2(\frac{\pi x}{n})}{\pi^2 \sigma^2_n} \left(1 - \frac{(Q_A)}{\sigma_n} \right)^2,
\] (5.4)

where \( \sigma^2_n = \frac{K\log(l)}{n \pi^2} \). Using the equations (3.8) and (5.4), we find the probability fluctuations as

\[
g_1(Q_A) = \frac{K\sin^2(\pi x)}{\pi^2 \sigma^1_1} \left(\left(\frac{Q_A}{\sigma_1}\right)^2 - 1\right) e^{-2\pi \beta/L}.
\] (5.5)

The resolved-symmetry entanglement entropy takes the following form:

\[
\delta S_E(Q_A) = \delta S_E + B(Q_A) e^{-2\pi \beta/L},
\]

\[
B(Q_A) = \left(3\sin^2(\pi x) - \pi(1 + x)\sin(2\pi x)\right) \frac{1}{\ln(l)} + \left(-4\pi\sin^2(\pi x) + \pi(1 + x)\sin(2\pi x)\right) \frac{\pi Q_A^2}{K\ln(l)^2}.
\] (5.6)

where \( \delta S_E = 2\left(1 - \pi x \cot(\pi x)\right) e^{-2\pi \beta/L} \) [17]. We see that the thermal corrections of the charge-sector contributions, at the order \( \ln(l)^{-2} \), are charge-dependent. The same result for the excited state associated with the derivative operator is derived in [24], the scaling of the terms is the same and the equipartition of the entanglement breaks at the order \( (\log l)^{-2} \).

It is worth mentioning that the authors of [24] derived the cumulants and the symmetry-resolved entanglement entropy for derivative excitations of the compact boson CFT (aka Luttinger liquid) using the \((2n + 2)\)-correlation functions. Those \((2n + 2)\)-correlation functions, with some manipulations, can be expressed as a characteristic polynomial of degree \( 2n \), which can then be analytically continued to arbitrary non-integer values of \( n \).

The final results are expressed in terms of the digamma function (the logarithmic derivative of the gamma function). However, as also suggested by the authors of [24], the exact form of coefficients in the symmetry-resolved entanglement entropy is not very illuminating. By contrast, the leading correction term in our calculations, which denotes the contribution of the excited state associated with the derivative operator, comes from the four-point function. However, we need to confirm that the main result is the breaking of the equipartition of entanglement, which occurs at the order \( (\log l)^{-2} \), where both results are the same.

6 Discussion

In this work, we have derived a formula for thermal corrections to the symmetry-resolved Rényi and entanglement entropies for general two-dimensional conformal field theories on a
circle. Besides the size of the mass gap and the degeneracy of the first excited state, these terms depend only on the four-point function of primary fields. It is worth noting that, until now, these thermal corrections have only been studied in the free case, whereas we found it for any two-dimensional CFT. Specially, we have derived the thermal corrections to full counting statistics, and the excess-cumulant generating function.

We also have defined the probability fluctuations functions. It states that, at low temperatures, the ratio of the probability of finding a charge $Q_A$ at temperature $\beta^{-1}$ to its value at zero temperature scales as $e^{-2\pi \Delta \psi / \beta / L}$. Its coefficient depends on the degeneracy of the first excited state, the charge of the sector, and the field content of the theory.

We have explicitly evaluated thermal corrections for the entanglement entropy and FCS in the free compact boson theory for both derivative and vertex operators. We have found that in the first case, the entanglement equipartition breaks at the order of $\ln(l)$.

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