Excitation of zonal flow by the modulational instability in electron temperature gradient driven turbulence

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Abstract

The generation of large-scale zonal flows by small-scale electrostatic drift waves in electron temperature gradient (ETG) driven turbulence model is considered. The generation mechanism is based on the modulational instability of a finite amplitude monochromatic drift wave. The threshold and growth rate of the instability as well as the optimal spatial scale of zonal flow are obtained.

1 Introduction

It is now an established fact that zonal flows (ZF) (i.e., azimuthally symmetric modes that depend only on the radial coordinate) play a crucial role in regulating the nonlinear evolution of drift-wave instabilities in tokamaks, and consequently, the level of turbulent transport [1, 2]. In particular, it is known that triggering of the L-H transition in tokamaks is related to the emergence of ZFs in the poloidal direction which suppresses the fluctuations and builds up a barrier to the turbulent transport. It is widely thought that zonal flows are generated by the modulational instability of the turbulent spectrum of electrostatic drift-wave perturbations [3, 4]. Under this, two possible regimes can be indicated: (a) when the spectrum is broad, integration over all wavenumbers yields a resonant instability; (b) when the drift wave spectrum is narrow, one can consider the instability of a monochromatic drift wave and the instability is of the modulational type. In the latter case, standard schemes of the four-wave interaction can be applied [5–7].

Ion temperature gradient (ITG) modes and trapped electron modes are generally regarded as the main candidates to explain the anomalous transport, but,
recently, there has been the growth of interest in electron temperature gradient driven turbulence, produced by the electron temperature gradient driven modes [8–11]. This is related to the fact that the study of interaction of ETG modes with large-scale motions like ZFs or streamers (radially elongated vortex-type structures) is important for understanding the electron thermal transport within an internal transport barriers, when ITG turbulence is suppressed by $E \times B$ shear flow [12]. Note, that there is an important difference between the ITG and ETG models: in the ETG model we have the Boltzmann ion response for both waves and zonal flows, while the electron response to zonal flow perturbations is hydrodynamic in the ITG model [1, 13].

In the present paper we consider the excitation of ZFs by a finite amplitude monochromatic drift wave in the framework of ETG turbulence model. The corresponding ETG mode is assumed to be stable, i.e. we consider the region below the marginal stability boundary. We derive a set of coupled equations describing the nonlinear interaction of drift ETG modes and ZFs and show that ZFs can be readily excited by the modulational instability.

This paper is organized as follows. In Sec. 2 a set of normalized fluid equations describing ETG drift modes – ZF interaction is introduced. In Sec. 3 the nonlinear dispersion relation is obtained and the modulational instability growth rate behavior is analyzed. Sec. 4 contains summary and conclusions.

2 Model equations

Assuming a slab two-dimensional geometry, charge quasineutrality and the adiabatic ion response, we consider the following simplified model describing curvature driven ETG turbulence in the inviscid limit (see, e.g. [9]):

\[
\frac{\partial}{\partial t} (1 - \Delta_\perp) \varphi + \frac{\partial}{\partial y} (\varphi + P) - \{\varphi, \Delta_\perp \varphi\} = 0, \tag{1}
\]

\[
\frac{\partial}{\partial t} P - \frac{r}{\rho_s} \frac{\partial \varphi}{\partial y} - \{P, \varphi\} = 0, \tag{2}
\]

where $\varphi$ and $P$ are the normalized electrostatic potential and plasma pressure respectively, and $\{A, B\} = \partial_x A \partial_y B - \partial_y A \partial_x B$ is the Jakobian. In equations (1) and (2), following the notations of Ref. [9], we have rescaled the variables as follows

\[
r = \frac{\epsilon_B \epsilon_{se}}{c_{s1}},
\]

\[
\varphi = \frac{1}{\epsilon_{s1} T_i} \epsilon \varphi, \quad P = \frac{\epsilon_B}{\epsilon_{s1} P_0} P,
\]

\[
x = \frac{x'}{\rho_s \sqrt{\tau}}, \quad y = \frac{y'}{\rho_s \sqrt{\tau}}, \quad t = \epsilon_{s1} \omega_B t',
\]
$x'$, $y'$ and $t'$ being the original physical coordinates (with $x'$ the poloidal and $y'$ the radial coordinate),

$$
\epsilon_{si} = \frac{\rho_s \sqrt{T_e}}{L_n}, \quad \epsilon_B = \frac{\rho_s \sqrt{T_e}}{L_B}, \quad \epsilon_{se} = \frac{\rho_s \sqrt{T_e}}{L_p},
$$

where $L_n$, $L_B$ and $L_p$ are the background gradient scales for the density, magnetic field and pressure respectively, $\rho_s$ is the ion gyroradius calculated at the electron temperature $T_e$, and $\tau = T_i/T_e$.

In the linear limit, equations (1) and (2) give the dispersion relation for ETG modes

$$
\omega_{1,2} = \frac{k_y}{2(k^2 + 1)} \left[ 1 \pm \sqrt{1 - 4r(k^2 + 1)} \right],
$$

where the plus sign describes the drift waves dispersion, while the minus sign corresponds to convective cells. The linear stability condition then reads as

$$
1 - 4r(k^2 + 1) \geq 0.
$$

If $r > 1/4$, the linear stability condition never holds, and ETG mode is always unstable. For $r < 1/4$, the stable ETG modes are confined inside the region $k^2 \leq 1/(4r) - 1$. Below we restrict our analysis to the case of linearly stable ETG drift waves.

### 3 Modulational instability of ETG drift waves and zonal flow generation

Assuming that the zonal flow varies on much larger timescale than ETG drift waves do, the standard decomposition into fast ($\tilde{\varphi}$, ETG drift wave related) and slow ($\hat{\varphi}$, zonal flow related) motions can be performed. The perturbations of electrostatic potential $\varphi$ and plasma pressure $P$ are presented as a sum of fast and slow parts

$$
\varphi = \tilde{\varphi} + \hat{\varphi}, \quad P = \tilde{P} + \hat{P}.
$$

Following the standard averaging procedure (recall that we consider the region below the marginal stability boundary, i.e. the subcritical turbulence) one gets the following set of equations

$$
\frac{\partial}{\partial t} (1 - \Delta_\perp) \tilde{\varphi} = \{\tilde{\varphi}, \Delta_\perp \tilde{\varphi}\},
$$

$$
\frac{\partial}{\partial t} (1 - \Delta_\perp) \hat{\varphi} + \frac{\partial}{\partial y} \left( \hat{\varphi} + \tilde{P} \right) = \{\hat{\varphi}, \Delta_\perp \tilde{\varphi}\} + \{\tilde{\varphi}, \Delta_\perp \hat{\varphi}\},
$$

$$
\frac{\partial}{\partial t} \hat{P} = \{\hat{P}, \tilde{\varphi}\},
$$

$$
\frac{\partial}{\partial t} \tilde{P} - r \frac{\partial}{\partial y} = \{\tilde{P}, \tilde{\varphi}\} + \{\tilde{\varphi}, \tilde{P}\}. 
$$
When obtaining the system (4) - (7), we have assumed that the mean field is described by the same model equations as the fluctuations, but is driven by the Reynolds stress. This follows from the assumption of adiabatic ion response for the flow as well as fluctuations, which is valid for the ETG modes. We describe the interaction between drift ETG modes and ZF in terms of a four-wave coupling scheme, i.e., each fluctuation is taken to be coherent. Then, the ETG drift waves are considered as a superposition of the pump wave \((\vec{k}, \omega_k)\) and two sidebands \((\vec{k} \pm \vec{q}, \omega_{k \pm})\), i.e.,

\[
\tilde{\varphi} = \tilde{\varphi}_0 + \tilde{\varphi}_+ + \tilde{\varphi}_- , \quad \tilde{P} = \tilde{P}_0 + \tilde{P}_+ + \tilde{P}_- ,
\]

where

\[
\tilde{\varphi}_0 = \varphi_0 \exp \left( i \vec{k} \vec{r} - i \omega_k t \right) + \text{c.c.} ,
\]

\[
\tilde{\varphi}_\pm = \varphi_\pm \exp \left( i \vec{k}_\pm \vec{r} - i \omega_{k \pm} t \right) + \text{c.c.} ,
\]

\[
\tilde{P}_0 = P_0 \exp \left( i \vec{k} \vec{r} - i \omega_k t \right) + \text{c.c.} ,
\]

\[
\tilde{P}_\pm = P_\pm \exp \left( i \vec{k}_\pm \vec{r} - i \omega_{k \pm} t \right) + \text{c.c.} .
\]

Zonal flow related electrostatic potential and pressure are taken in the form

\[
\hat{\varphi} = \varphi_q \exp \left( i \vec{q} \vec{r} - i \omega_q t \right) + \text{c.c.} ,
\]

\[
\hat{P} = P_q \exp \left( i \vec{q} \vec{r} - i \omega_q t \right) + \text{c.c.} ,
\]

where \(\vec{q} = (q, 0)\) is the wave vector of the zonal flow, and the resonant conditions \(\vec{k}_\pm = \vec{k} \pm \vec{q}\) and \(\omega_{k \pm} = \omega_k \pm \Omega\) hold, where \(\omega_k\) is the ETG drift mode frequency, given by equation (3) with the ” + ” sign in front of the square root.

Substitution of (8) - (14) into the system (4) - (7) gives:

\[
\Omega \varphi_q = -i \frac{[\vec{k}, \vec{q}]}{(q^2 + 1)} \left\{ (k_+^2 - k_-^2) \varphi_0^* \varphi_+ - (k_+^2 - k_-^2) \varphi_0 \varphi_+^* \right\} ,
\]

\[
\varphi_+ \left\{ \omega_+ (k_+^2 + 1) - k_+ y \right\} - k_+ y P_+ = i \varphi_0 \varphi_q q k_y (k^2 - q^2) ,
\]

\[
\varphi_- \left\{ \omega_- (k_-^2 + 1) - k_- y \right\} - k_- y P_- = i \varphi_0^* \varphi_q q k_y (k^2 - q^2) ,
\]

\[
\Omega P_q = i \left[ \vec{k}, \vec{q} \right]_z \left\{ \varphi_0 P_-^* + \varphi_+ P_0^* - \varphi_-^* P_0 - \varphi_+^* P_- \right\} ,
\]

\[
\omega_+ P_+ + k_+ y r \varphi_+ = i q k_y \left[ \varphi_q P_0 - \varphi_0 P_q \right] ,
\]

\[
\omega_- P_- + k_- y r \varphi_- = i q k_y \left[ \varphi_q P_0^* - \varphi_0^* P_q \right] .
\]

Combining equations (15) - (20), one can calculate the amplitudes of the up-shifted and down-shifted satellites

\[
\varphi_+ = \frac{i q k_y}{S_+} \left\{ \varphi_q \varphi_0 (k^2 - q^2) + \frac{k_+ y}{\omega_+} (\varphi_q P_0 - \varphi_0 P_q) \right\} ,
\]
\[
\varphi_+ = \frac{i q k_y}{\omega_+} \left\{ \varphi_q \varphi_0^* (k^2 - q^2) + \frac{k_y y}{\omega_+} (\varphi_q P_0^* - \varphi_0^* P_q) \right\}, \tag{22}
\]

\[
P_+ = \frac{i q k_y}{\omega_+} \left\{ \varphi_q P_0 - \varphi_0 P_q - \frac{k_y y r}{S_+} \varphi_q \varphi_0 (k^2 - q^2) \right. \\
- \left. \frac{k_y^2 r}{\omega_+ S_+} (\varphi_q P_0 - \varphi_0 P_q) \right\}, \tag{23}
\]

\[
P_- = \frac{i q k_y}{\omega_-} \left\{ \varphi_q P_0^* - \varphi_0^* P_q - \frac{k_y y r}{S_-} \varphi_q \varphi_0^* (k^2 - q^2) \right. \\
- \left. \frac{k_y^2 r}{\omega_- S_-} (\varphi_q P_0^* - \varphi_0^* P_q) \right\}, \tag{24}
\]

where we have introduced the notation

\[
S_{\pm} = \omega_{\pm} \left( k_{\pm}^2 + 1 \right) - k_{\pm} \pm \frac{k_y^2 r y}{\omega_{\pm}}.
\]

Using the relation \(P_0 = -(r k_y/\omega_k) \varphi_0\) and eliminating \(P_0\), from equations (15), (18) and (21) – (24) one can get the nonlinear dispersion relation in the form

\[
a_{11} a_{22} - a_{12} a_{21} = 0, \tag{25}
\]

where

\[
a_{11} = \Omega + \frac{[\vec{k}, \vec{q}]^2}{q^2 + 1} \varphi_0^2 \left\{ (k^2 - q^2) \left( \frac{k^2 - k^2}{S_+} - \frac{k^2 - k^2}{S_-} \right) \\
- \frac{r k_y^2}{\omega_k} \left( \frac{k^2 - k^2}{S_+} - \frac{k^2 - k^2}{S_-} \right) \right\},
\]

\[
a_{12} = -\frac{[\vec{k}, \vec{q}]^2}{q^2 + 1} k_y \varphi_0^2 \left\{ \frac{k^2 - k^2}{\omega_+ S_+} - \frac{k^2 - k^2}{\omega_- S_-} \right\},
\]

\[
a_{21} = \frac{r k_y}{\omega_k} \left[ \vec{k}, \vec{q} \right]^2 \varphi_0^2 \left\{ \left( \omega_k (k^2 - q^2) + \frac{r k_y^2}{\omega_k} \right) \left( \frac{1}{\omega_+ S_+} - \frac{1}{\omega_- S_-} \right) \\
- \left( k^2 - q^2 \right) \left( \frac{1}{S_+} - \frac{1}{S_-} \right) + \frac{1}{\omega_+} \left( 1 - \frac{r k_y^2}{\omega_+ S_+} \right) - \frac{1}{\omega_-} \left( 1 - \frac{r k_y^2}{\omega_- S_-} \right) \right\},
\]

\[
a_{22} = -\Omega + \frac{[\vec{k}, \vec{q}]^2}{q^2 + 1} \varphi_0^2 \left\{ \frac{r k_y^2}{\omega_k} \left( \frac{1}{\omega_+ S_+} - \frac{1}{\omega_- S_-} \right) \\
+ \frac{1}{\omega_+} \left( 1 - \frac{r k_y^2}{\omega_+ S_+} \right) - \frac{1}{\omega_-} \left( 1 - \frac{r k_y^2}{\omega_- S_-} \right) \right\}.
\]
Generally, ZF dispersion which follows from the equation (25) is a fourth-order in $\Omega$, which, in principle, allow one to treat it analytically. We present below an analysis of nonlinear dispersion relation for the case when the ETG drift mode does not have the $k$-component in the direction of inhomogeneity, i.e. for $k_x = 0$. We also treated the case $k_x \neq 0$ and found that inclusion of nonzero $k_x$ does not introduce any qualitative change to the growth rate behavior except for the decrease of it’s magnitude. Besides that, it turns out that in all studied cases the pump wave with $k_x = 0$ ensures the largest growth rate in the explored region of linear stability.

For the case $k_x = 0$, the nonlinear dispersion relation is reduced to the biquadratic equation

$$c_4 \Omega^4 + c_2 \Omega^2 + c_0 = 0,$$

where

$$
c_0 = 4k^2 q^6 (k^2 - q^2) (k^2 + q^2 + 1) \phi_0^4
+ 2k^2 q^4 [k^2 - 2k (1 + 2k^2) \omega_k + (1 + 5k^2 + 4k^4 - 2 (q^2 + q^4) \omega_k^2)] \phi_0^2
-q^4 (1 + q^2) \omega_k^4,
c_2 = 2k^2 q^2 (k^2 + q^2 + 1) (k^2 + 2 (q^2 + q^4) + 1) \phi_0^2
+ (q^2 + 1) [k^2 - 4k \omega_k (k^2 + q^2 + 1)]
+ 2 (k^2 + q^2 + 1) (2 (k^2 + 1) + q^2) \omega_k^2],
c_4 = -(q^2 + 1) (k^2 + q^2 + 1)^2,
$$

which yields the modulational instability growth rate

$$\gamma = \max \text{ Im} \left( \pm \sqrt{\frac{-c_2 \pm \sqrt{c_2^2 - 4c_0 c_4}}{2c_4}} \right).$$

The analysis of the growth rate (27) shows that the modulational instability can be excited ($\gamma > 0$) only above some threshold in the input power $\phi_0^2$. This threshold is a function of the scales $k$ and $q$ of ETG drift wave and zonal flow. The minimum threshold corresponds to the case $q \to 0$ and is given by

$$\phi_{th}^2 = \frac{v_{ph}^4}{2 \left(1 - 2(1 + 2k^2)v_{ph} + (1 + k^2)(1 + 4k^2)v_{ph}^2\right)},$$

where $v_{ph} = \omega_k / k$ is the phase velocity of ETG drift wave. As the zonal flow wavenumber $q$ approaches $k$, the instability threshold goes to infinity, as it is illustrated in Figure 1.

Present analysis is restricted to the case when the pump ETG drift wave, as well as both sidebands are linearly stable, i.e. for each value of $r$ parameter, all considered scales $k$ and $q$ are confined in the $(k, q)$ - plane inside the circle

$$k^2 + q^2 = \frac{1}{4r} - 1.$$
Thus, the modulational instability parameter range under consideration is limited. Instability domain in \((k, q)\)-plane is outlined in Figure 2. It is clear that circle shrinks to the origin when \(r\) approaches \(1/4\).

In the Figure 3 modulational instability growth rates are plotted as a functions of \(k\) and \(q\) scales of ETG drift wave and zonal flow respectively for different values of the parameter \(r\) and pump wave amplitude \(\varphi^2_0\). Dashed lines in the contour planes confine the considered instability domain (schematically outlined in Figure 2). Outside the circle given by the equation (29), the growth rate was artificially forced to zero due to linear stability requirements. It is seen that as the parameter \(r\) grows, the region of instability contracts and the growth rate magnitude goes down. In all studied cases, the maximum growth rate always corresponded to the \((k, q)\) - pair from the circle (29) so that for the fixed \(k\) there is an optimal (i.e. corresponding to the maximum growth rate) scale \(q\) of the generated ZFs. With the increase of a pump wave amplitude, the growth rate as well as the instability domain increases, however in the region of physically appropriate values of \(\varphi^2_0\), no qualitative changes to the growth rate behavior were found.

4 Summary and conclusions

In the present paper, we have considered the nonlinear interaction between drift waves and ZFs in subcritical ETG turbulence. We have derived a set of equations describing the dynamics of nonlinearly coupled ETG modes and ZFs. Assuming that the spectrum of fluctuations is sufficiently narrow, we analyzed the obtained set of equations in terms of a four-wave coupling scheme and obtained the nonlinear dispersion relation. We have shown that this dispersion relation predicts the modulational instability of a finite amplitude monochromatic drift wave. Thus, ETG drift fluctuations can be destabilized by the four-wave in-
Interaction mechanism with simultaneous generation of ZFs. We have found the threshold of the modulational instability and the dependence of the instability growth rate on spatial scales of ETG drift waves and excited ZFs. For the fixed wavenumber of the ETG mode $k$, the growth rate always has a maximum which is achieved for some intermediate value of ZF wavenumber $q$. When the amplitude of the ETG pump wave increases (as well as when the parameter $r$ decreases), the region of modulational instability widens towards small $k$-scales. The present results thus demonstrate that ZFs in subcritical ETG turbulence can be excited by the modulational instability.

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Figure 3: Modulational instability growth rate $\gamma$ given by (27) as a function of $(k, q)$ for different $\varphi_2^2$ and $r$. Dashed lines in the contour plane indicate the instability domain.
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