Infrared Gupta-Bleuler Quantum Electrodynamics: Solvable Models And Perturbative Expansion

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Abstract. We study two Hamiltonian models, based on infrared approximations which render them solvable, in order to obtain an operator formulation of the soft-photon corrections to the scattering of a single electron, as given in Quantum Electrodynamics by the method of Feynman’s diagrams. The first model is based on the same approximations of the Pauli-Fierz Hamiltonian, the second one stems from an expansion in powers of the four-momentum transfer, along the lines of Bloch and Nordsieck. For both models, the dynamics of the charge is accounted for by suitably chosen classical currents, interacting with the quantum e.m. potential.

Möller operators, preserving respectively the Hilbert scalar product, for the Coulomb-gauge formulation of the models, and an indefinite metric, for the formulation of the models in the Feynman-Gupta-Bleuler gauge, are obtained in the presence of an infrared cutoff, with the help of suitable renormalization counterterms.

We show that the soft-photon corrections to the electron scattering under consideration are reproduced by suitable matrix elements of the Möller operators pertaining to the model “of the Bloch-Nordsieck type”, both in the FGB gauge and in the Coulomb gauge.

Further, we prove that if one assumes that the charged particle is non relativistic and employs a dipole approximation, the resulting low-energy radiative corrections admit an operator formulation as well, in terms of the Möller operators of the model “of Pauli-Fierz type”, but lack the invariance property with respect to the gauge employed in their calculation; spurious, yet infrared-relevant, contributions in fact arise, causing in particular a discrepancy between the corrections in the FGB gauge and their Coulomb-gauge expression. The reason why such a discrepancy occurs is finally traced back in full generality, also in connection with the Gupta-Bleuler formulation of non-relativistic models.

Key words: quantum electrodynamics; infrared problem; local and covariant gauge; indefinite metric; solvable models

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Introduction

In Quantum Electrodynamics (QED), the description of states at asymptotic times and the derivation of the scattering matrix are still open issues. At the perturbative level, transition amplitudes between states containing a finite number of photons are ill-defined, since radiative corrections typically exhibit low-energy logarithmic divergences. As a consequence, in contrast with ordinary quantum field theories, Dyson’s $S$-matrix ([Dys49a, Dys51]) is defined only in the presence of an infrared (IR) cutoff and the problem of a proper identification of asymptotic states arises.

As early as 1937, in their pioneering paper on the subject ([BN37]), Bloch and Nordsieck pointed out that infrared singularities arise in perturbation theory because of fundamental physical facts; they in fact argued that, on the basis of the correspondence principle, one has to expect a vanishing probability for the emission of a finite number of photons in any collision process involving electrically charged particles. Exponentiation of the infrared radiative corrections was conjectured by Schwinger ([Schw49]) and proved by Yennie, Frautschi and Suura ([YFS61]) within the framework of the local and covariant Feynman-Dyson formulation ([Dys49b, Feyn49, Feyn50]) of QED. The paper [YFS61] laid the basis of the recipe currently adopted to cope with the soft-photon divergences: The IR cutoff is removed after summing the transition rates over all final photon states with energy below the threshold of the detectors. Finiteness of the resulting inclusive cross-sections is ensured by cancellations, at each perturbative order, between the infrared singularities associated respectively to low-energy radiative corrections and to soft-photon emission.

As emphasized by Steinmann ([Stei]), such a procedure involves an exchange of limits. Moreover, its relation with structural (non-perturbative) properties, such as the spontaneous breaking of the symmetry under Lorentz boosts in the charged superselection sectors ([FM57a, FM57b, Buc86]) and the absence of a sharp eigenvalue for the mass operator of a particle carrying an electric charge ([FM57a, Buc86]), is still unclear. In addition, local and covariant quantizations of abelian gauge theories, within which the perturbation-theoretic formulation of QED is mainly developed, are incompatible with positivity ([S67]) and require a suitable generalization of Wightman’s axioms ([SW74, MS80]).

Quite generally, the problem of filling the gap that separates the perturbative approach from a collision theory of “infraparticles” ([Schr63]), in which the structural features of the IR problem are taken into account, is a relevant issue, not only conceptually but also from a practical point of view: perturbation theory remains in fact the only source of detailed information on Quantum Electrodynamics and its local and covariant version is the best controlled one regarding renormalization procedures.

In the present paper we wish to take a small step forward, by carrying out an analysis of suitable Hamiltonian models, especially focused on a comparison with the perturbative treatment of the infrared divergences.

The aim of such an analysis is twofold. First, we would like to give a mathematical formulation of the structures underlying the diagrammatic treatment of the soft-photon contributions. Secondly, since the electric dipole approximation has been employed in non-relativistic models of infrared QED in order to obtain non-perturbative constructions, both in the Coulomb gauge ([Hanz90, Arm83, Green00]) and within the Gupta-Bleuler formulation ([Hisa09]), we wish to check its reliability with respect to the perturbative low-energy approximations and results.

We shall discuss two Hamiltonian models of a single charged particle, whose dynamics is accounted for by suitably chosen classical currents, interacting with the quantum e.m. potential. Such models are based on infrared approximations which render them solvable and might seem to be equally suited, from a physical point of view, for an investigation of soft-photon effects; the dipole approximation, following the treatment of Pauli and Fierz, and the expansion in powers of the four-momentum transfer around a fixed (asymptotic) charged particle four-momentum, along the lines of Bloch and Nordsieck.

Our main result is that the soft-photon corrections to the scattering of a single charged particle, computed in perturbation theory under the standard infrared approximations ([YFS61]), for a fixed value of a low-energy cutoff, admit an operator formulation in terms of the Möller operators of the model “of the Bloch-Nordsieck type”, both in the FGB gauge ([PhDth]) and in the Coulomb gauge.

Likewise, we show that the infrared radiative corrections to the scattering of a non-relativistic electron, in the presence of a dipole approximation, are reproduced in terms of the Möller operators of the model “of the Pauli-Fierz type”. However, we shall also see that the introduction of a dipole approximation turns out to be too strong an assumption, for its effect on current conservation, also recognized and
studied in [HSu09], leads to a discrepancy of the $FGB$-gauge soft-photon corrections, by a factor $3/2$, with respect to the corresponding Coulomb-gauge expressions and thus to a violation of the invariance property of such corrections with respect to the gauge adopted in their calculation.

The content of the paper is as follows. In Section 1 we first recall the basic features of the Pauli-Fierz model, introduced in [PF38] and later reconsidered by Blanchard ([Blan69]), who studied it in the interaction representation, focusing on the mathematical issues connected with the basic physical fact that an infinite number of photons is emitted in any collision process involving electrically charged particles.

Although our treatment parallels Blanchard’s analysis to a large extent, we find it useful to give a complete presentation; this choice will in fact enable us to mainly devote Section 2 to the operator formulation of infrared \textit{QED}, while omitting a thorough description of similar mathematical procedures and techniques employed therein.

The main differences with respect to the Pauli-Fierz-Blanchard setting stem from the fact that our aim is to reproduce the infrared-regularized Feynman amplitudes associated to a given scattering process, involving a single charged particle; a low-energy cutoff will thus be adopted throughout the discussion and a suitably chosen classical current, accounting for the asymptotic dynamics of the charge, will be introduced. The question concerning the existence of the large-time limits of the evolution operators once the infrared regularization is removed will instead not be addressed, because it is already discussed in [Blan69]. In addition, an adiabatic switching of the interaction will be used as an auxiliary tool, in order to keep contact with the perturbative procedures and calculations.

In order to set up a comparison with the Feynman-Dyson expansion, we then formulate the model in the $FGB$ gauge, define its dynamics and obtain Möller operators, acting as isometries on an indefinite inner-product space, as weak asymptotic limits of the evolution operator (in the interaction representation), for a fixed IR cutoff. A preliminary discussion of the spurious effects induced by the dipole approximation is given at the end of Section 1.

In Section 2 we introduce a model based on an expansion already implicit in [BN37], hereafter referred to as Bloch-Nordsieck (BN) model, and formulate it both in the Coulomb gauge and in the $FGB$ gauge. The definition of the dynamics and the control of the asymptotic limits follow the same pattern of the first Section. We prove that the soft-photon corrections to the scattering of an electron, which result from the diagrammatics of $QED$ in the $FGB$ gauge under the standard low-energy approximations, are given by suitable matrix elements of the Möller operators of the BN model in the same gauge.

In particular, we reproduce the exponentiation of the second-order soft-photon radiative corrections, including the interplay with renormalization procedures (Propositions 5, 6), and the contributions due to the emission of soft photons (Lemma 3, Corollary 1). Moreover, we recover the independence of the expressions associated to soft-photon emission with respect to the low-energy radiative corrections (Proposition 7). We then show that the BN model also allows to reproduce the soft-photon corrections to the electron scattering in the Coulomb gauge (Propositions 8, 9).

The low-energy corrections of $QED$, obtained under the standard infrared approximations, enjoy an invariance property with respect to the gauge adopted in their calculation; in Section 3 we give an operator proof of such a property for the corrections discussed in Section 2 (Proposition 10 and Corollaries 3, 4). Afterwards, we show that the soft-photon radiative corrections to the scattering of a non-relativistic electron, evaluated with the aid of the diagrammatic rules in the presence of a dipole approximation, are not invariant with respect to the choice of a gauge. Such corrections are likewise proved to admit an operator formulation (Corollaries 6, 7), in terms of the Möller operators constructed in Section 1, and the reason at the root of the discrepancy between their expressions in the $FGB$ gauge and in the Coulomb gauge is finally traced back (Lemma 8).

An outlook for future research is finally given. Appendix A is devoted to the proof of the self-adjointness of the Pauli-Fierz Hamiltonian. In Appendix B we describe the construction of a suitable indefinite-metric space, on which the evolution operators of the $FGB$-gauge models studied in this work are shown to be well defined and unique. In Appendix C the main results about the Gupta-Bleuler quantization of the free e.m. field and its relationship with the Coulomb-gauge quantization are given. No pretension of completeness is made about the bibliography.
Notations

The metric \( g^{\mu\nu} = \text{diag} (1, -1, -1, -1) \) is adopted and natural units are used (\( h = c = 1 \)). We shall reserve the symbol \( v \) for a four-vector or, alternatively, denote its components by greek indices, say \( v^\mu = (v^0, v^1) \). A three-vector will be denoted by \( \mathbf{v} \) or by labelling its components with latin indices, say \( \mathbf{v}^I \). The symbol \( \mathbf{v} \) will also indicate the module of the three-vector: when confusion may arise, the notation \( |\mathbf{v}| \) will be instead employed. We use the symbol \( \mathbf{c} \cdot \mathbf{d} \) for the indefinite inner product between four-vectors \( \mathbf{c} \) and \( \mathbf{d} \), and likewise for the scalar product between three-vectors.

The Hilbert scalar product is denoted by \( \langle \ldots \rangle \) and indefinite inner products by \( \langle \ldots, \ldots \rangle \), all products being taken to be linear in the second factor. The norm of \( \phi \in L^2 \) is indicated by \( \|\phi\| \). The adjoint of an operator \( A \) on a Hilbert space is denoted by \( A^\dagger \) and the symbol \( B^\dagger \) will stand for the hermitian conjugate, with respect to the inner product, of an operator \( B \) defined on an indefinite-metric space.

The symmetric Fock space \( \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^{(n)} \) over a Hilbert space \( \mathcal{H} \) will be denoted by \( \mathcal{F} \), the symbols \( \mathcal{H}^{(n)}, S_n \) standing respectively for the \( n \)-fold tensor product \( \bigotimes_n \mathcal{H} \) and for the symmetrization operator, defined in terms of the permutation group of degree \( n \). The norm in \( \mathcal{F} \) will be denoted by \( \|\cdot\| \), the no-particle vector by \( \Psi_F \) and the number operator by \( N \). We let \( \phi^{(n)} \) be the orthogonal projection of \( \phi \in \mathcal{F} \) onto the \( n \)-particle subspace \( S_n \mathcal{H}^{(n)} \).

The symbols \( \delta_{\mathbf{y}}(\mathbf{x}) \), \( \delta(\mathbf{x} - \mathbf{y}) \) will denote the Dirac delta measure at \( \mathbf{y} \). \( G_{\mathbf{y}}(\mathbf{x}) \) will stand for the Green’s function for Poisson’s equation in \( \mathbb{R}^3 \):

\[
-\Delta G_{\mathbf{y}}(\mathbf{x}) = \delta_{\mathbf{y}}(\mathbf{x}).
\] (1)

In the Coulomb gauge, \( a_s(a_s^\dagger) \) will stand for the photon annihilation (creation) operator-valued distribution, fulfilling the canonical commutation relations (CCR)

\[
[ a_s(\mathbf{k}), a_{s'}^\dagger(\mathbf{k'}) ] = \delta_{ss'} \delta(\mathbf{k} - \mathbf{k'}),
\] (2)

with \( s \) and \( s' \) polarization indices.

In the same gauge, we denote the free e.m. Hamiltonian by \( H_0^{c,m} \) and the free vector potential at time \( t = 0 \) by

\[
\mathbf{A}_C^c(\mathbf{x}) \equiv \sum_s \int \frac{d^3k}{(2\pi)^3/2 \sqrt{2k}} \epsilon_s^+(\mathbf{k}) [ a_s(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a_s^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} ]
\equiv \sum_s \int \frac{d^3k}{(2\pi)^3/2 \sqrt{2k}} [ a_C(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a_C^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} ],
\] (3)

with \( \epsilon_s(\mathbf{k}) \), \( s = 1, 2 \), orthonormal polarization vectors satisfying \( \mathbf{k} \cdot \epsilon_s(\mathbf{k}) = 0 \). The annihilation and creation operator-valued distributions in the FGB gauge, denoted respectively by \( a^\mu(\mathbf{k}) \) and \( a^{\mu\dagger}(\mathbf{k}) \), fulfill the CCR

\[
[ a^\mu(\mathbf{k}), a^{\mu\dagger}(\mathbf{k'}) ] = -g^{\mu\nu} \delta(\mathbf{k} - \mathbf{k'}).
\] (4)

In the same gauge, the Hamiltonian of the free e.m. field is denoted by \( H_0^{c,m} \), and the free vector potential at time \( t = 0 \) by

\[
A^\mu(\mathbf{x}) \equiv \sum_s \int \frac{d^3k}{(2\pi)^3/2 \sqrt{2k}} [ a^\mu(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a^{\mu\dagger}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} ] \equiv A_+^\mu(\mathbf{x}) + A_-^\mu(\mathbf{x}).
\] (5)

The convolution with a form factor \( \rho \) is indicated by

\[
A^\mu(\rho, \mathbf{x}) \equiv \int d^3\xi \rho(\xi) A^\mu(\mathbf{x} - \xi),
\] (6)

and likewise for \( A_C \). We shall use the symbol

\[
A_C^c(\mathbf{f}) \equiv \int d^4x \ A_C^c(x) \ F^c(x) = A_C^c(F^c)
\] (7)

for the smearing of \( A \) with a vector test function \( \mathbf{f} \) and the symbol \( A(f) \equiv A^\mu(f_\mu) \) for the smeared four-vector potential in the FGB gauge.
We denote by $\tilde{f}(k)$, or simply by $f(k)$ when no confusion should arise, the Fourier transform of a function $f(x)$ on Minkowski space; we employ the conventions

$$f(x) = \frac{1}{(2\pi)^4} \int d^4k \ e^{-i(k \cdot x)} \ f(k). \tag{8}$$

For brevity we shall also write

$$a(f(t)) \equiv \int d^3k \ a^\mu(k) \ \tilde{f}_\mu(t, k), \tag{9}$$

with

$$f(t, x) = \frac{1}{(2\pi)^3} \int d^3k \ e^{i(k \cdot x)} \ \tilde{f}(t, k), \tag{10}$$

and denote the corresponding sum in the Coulomb gauge by $a_C(f(t))$.

$\mathcal{S}(\mathbb{R}^3)$ stands for the Schwartz space of $C^\infty$ functions of rapid decrease defined on $\mathbb{R}^3$.

Throughout the paper we shall frequently encounter integrals of the type

$$I \equiv \int d^4k \ \tilde{h}(-k) \ \tilde{G}(k) \ h(k), \quad J \equiv \int d^4k \ \tilde{f}(k) \ \tilde{G}(k), \tag{11}$$

with $\tilde{G}(k) = \theta(k_0) \ \delta(k^2) \ \tilde{F}(k)$, $\tilde{F}(k) \in \mathcal{S}(\mathbb{R}^3)$, with the symbol $\theta$ standing for the Heaviside step distribution. As an infrared regularization for such expressions, we shall restrict integrations in momentum space outside a sphere of radius $\lambda$. The regularized $I$ will be denoted by

$$I_\lambda \equiv \int_{k > \lambda} \frac{d^4k}{(2\pi)^4} \ \tilde{h}(-k) \ \tilde{G}(k) \ h(k) = \int_{k > \lambda} \frac{d^3k}{(2\pi)^3} \ 2k \ \tilde{h}(-k) \ \tilde{F}(k) \ h(k) \tag{12}$$

and its expression as a space-time integral by

$$I_\lambda = \int_\lambda \ d^4x \ d^4y \ h(x) \ G(x - y) \ h(y). \tag{13}$$

The same notations will be used for the infrared regularization of $J$.

1. Pauli-Fierz-Blanchard Models

In this Section we introduce a model, based on the approximations of the Pauli-Fierz-Blanchard (PFB) Hamiltonian, and discuss its formulation both in the Coulomb gauge and in the $FGB$ gauge. First, we introduce the infrared-regularized PFB Hamiltonian,

$$H^{(PFB)}_\lambda = \frac{p^2}{2m} + H_\delta^{\lambda, m} + H_{\text{int}, C} \equiv H_0 + H_{\text{int}, C}, \tag{14}$$

$$H_{\text{int}, C} = -\frac{e}{m} \ p \cdot A_{C, \lambda}(\rho, x = 0). \tag{15}$$

We will also call electron the particle of mass $m$, charge $e$ and spherically symmetric distribution of charge $e \rho \in \mathcal{S}(\mathbb{R}^3)$. The subscript $\lambda$ on the left-hand side of (14) denotes the infrared regularization, the $\lambda$-dependence in $H_\delta^{\lambda, m}$ and in $H_{\text{int}, C}$ being understood. The functional form of the interaction Hamiltonian (15) is dictated by the electric dipole approximation and implies that the electron momentum is conserved, while the total one is not.

The Hilbert space of states of the model is $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{H}_C$, with $L^2$ the one-particle space and $\mathcal{H}_C \equiv \mathcal{F}$ the Fock space associated to the Coulomb-gauge canonical photon variables.

The free Hamiltonian $H_0$ is self-adjoint, since it is the sum of two positive, commuting self-adjoint operators; further, it is essentially self-adjoint (e.s.a.) on $D_0 \equiv \mathcal{S}(\mathbb{R}^3) \otimes D_{F_0}$, where $F_0$ is the dense set spanned by the finite-particle vectors of $\mathcal{F}$ and $D_{F_0} \equiv (\psi \in F_0; \psi^{(n)} \in S_n \otimes k = 1 \mathcal{S}(\mathbb{R}^3), \forall n)$.

The essential self-adjointness of the (infrared-regularized) PFB Hamiltonian (14) can be established by methods exploited for instance in [Nels64, Arai81]; for the sake of completeness, the details are given
in Appendix A. Since the PFB Hamiltonian commutes with the electron momentum operator, it is e.s.a. on (almost) any of the subspaces, on which \( p \) takes a constant value, obtained by decomposing \( D_0 \) on the joint spectrum of the components of \( p \).

With the aim of reproducing the soft-photon corrections to the scattering \( p_{in} \rightarrow p_{out} \) of a non-relativistic electron, with initial (final) state of definite three-momentum \( p_{in} (p_{out}) \), in the presence of a dipole approximation, we introduce the Coulomb-gauge Hamiltonian

\[
H^{(\text{Coul})}_{\lambda} (\text{dip}) \equiv H^{e,m}_{\lambda} + H^{(\text{Coul})}_{\text{int}} (\text{dip}),
\]

(16)

and

\[
H^{(\text{Coul})}_{\text{int}} (\text{dip}) \equiv -i \int \lambda \, d^3 x \, j^{(\text{dip})} (x) \cdot A_C (x),
\]

(17)

\[
j^{(\text{dip})} (t, x) \equiv \int d^3 y \, H^{(\text{dip})}_{nl} (x - y) \, j^{(\text{dip})}_{nl} (t, y),
\]

(18)

with

\[
j^{(\text{dip})} (x) \equiv \frac{\rho (x)}{m} \left[ \theta (-t) \, p_{in} + \theta (t) \, p_{out} \right]
\]

(19)

a non-relativistic classical current obeying the dipole approximation and

\[
\delta_{tr}^{lm} (x - y) \equiv \delta^{lm} (x - y) + \partial_y \frac{\partial}{\partial y} G_y (x).
\]

(20)

The Fourier transform of the “transverse Dirac delta” \((20)\) is given by the orthogonal projection onto the transverse components of a momentum-space vector:

\[
\delta_{tr}^{lm} (x - y) = \frac{1}{(2 \pi)^3} \int d^3 k \, e^{i \mathbf{k} \cdot \mathbf{x} - y} \, P^{lm} (k),
\]

(21)

\[
P^{lm} (k) \equiv \delta^{lm} - \frac{k^l k^m}{k^2}.
\]

(22)

The Hamiltonian \((16)\) is e.s.a. since the same property holds for \( H^{(\text{PFB})}_{\lambda} \) at fixed \( p \); existence and uniqueness of the dynamics of the model thus follow by Stone’s theorem \((\text{RSI})\).

Next we discuss the formulation of the dynamics in the interaction picture. In such a representation, the time-evolution operator corresponding to a Hamiltonian \( H = H_0 + H_{\text{int}} \), with \( H_0 \) the free part and \( H_{\text{int}} \) a time-independent interaction, is

\[
U_I (t) \equiv \exp (i \, H_0 \, t) \, \exp (-i \, H \, t)
\]

(23)

and satisfies

\[
i \frac{d U_I (t)}{dt} = H_I (t) \, U_I (t),
\]

(24)

Equation \((24)\) identifies \( U_I (t) \) (a more general result, also applying to time-dependent interactions, is stated below). If the commutator of \( H_I \) evaluated at different times is a multiple of the identity operator, one can solve \((24)\) by making use of the formula \((\text{Wil67})\)

\[
e^{A + B} = e^A e^B e^{-\frac{1}{2} [A, B]}.
\]

(25)

The resulting evolution operator is

\[
U_I (t) = \exp (-i \int_0^t \, dt' \, H_I (t')) \, \exp (-\frac{1}{2} \int_0^t \, dt' \int_0^{t'} \, dt'' \, [H_I (t'), H_I (t'')]),
\]

(26)

whence

\[
U (t) = \exp (-i \, H_0 \, t) \, U_I (t). \quad \text{Since the commutator of}
\]

\[
H^{(\text{Coul})}_{\lambda} (\text{dip}) (t) = \exp (i \, H^{e,m}_{\lambda} \, t) \, H^{(\text{Coul})}_{\text{int}} (\text{dip}) \, \exp (-i \, H^{e,m}_{\lambda} \, t)
\]

(27)

\footnote{The e.s.a. of the PFB Hamiltonian at fixed \( p \) also follows by the Kato-Rellich theorem.}
at different times is a multiple of the identity operator at each definite momentum, the (interaction-picture) time-evolution operator of our model can be computed with the help of the formula (26)

By an explicit calculation, with the aid of the expansion (3) and of the Fourier transform of $\mathbf{j}^{(\text{disp})}_{C, nr}(x)$ with respect to $x$, given by

$$
\mathbf{j}^{(\text{disp})}_{C, nr}(t, \mathbf{k}) = \frac{\rho(\mathbf{k})}{m} P^{\dagger r}(\mathbf{k}) \left[ \theta(-t) \mathbf{p}_{in} + \theta(t) \mathbf{p}_{out} \right],
$$

we obtain

$$
U_{I, C, p}(t) = c_p(t) \exp \left( \frac{i e^2}{3 m^2} d(t) \right),
$$

$$
d(t) = \int_{k > \lambda} \frac{d^3 k}{(2 \pi)^3} \tilde{\rho}^2(\mathbf{k}) \left( t - \frac{\sin k t}{k} \right),
$$

$$
f_{p_s}(t, \mathbf{k}) = \frac{\rho(\mathbf{k})}{(2 \pi)^3 / 2} \frac{\mathbf{p} \cdot \epsilon_x(\mathbf{k})}{m} \frac{e^{ik t} - 1}{i k},
$$

with $\mathbf{p} = \mathbf{p}_{out}$, $t > 0$, and $\mathbf{p} = \mathbf{p}_{in}$, $t < 0$.

Since $U_{I, C, p}(t)$ does not converge for large times, we renormalize (16) by introducing suitable counterterms, in order to get well-defined asymptotic limits for the corresponding time-evolution operator; further, we adopt an adiabatic switching of the interaction, in the form $e^{(ad)}(t) \equiv e^{-\epsilon |t|}$.

Precisely, we consider the Hamiltonian

$$
H^{(\text{Coul})}_{\lambda, \text{ren}, \epsilon}(\epsilon) \equiv H^{e, m}_{0, C} - e \int_{\lambda} d^3 x \mathbf{j}^{(\text{disp})}_{C, nr}(x) : \mathbf{A}_C(x) + e^2 \left[ \theta(-t) \frac{\mathbf{p}_{in}^2}{m^2} + \theta(t) \frac{\mathbf{p}_{out}^2}{m^2} \right] z e^{-\epsilon |t|} \equiv H^{e, m}_{0, C} + H^{(\text{Coul})}_{\text{int, ren}, \epsilon},
$$

with

$$
\mathbf{j}^{(\text{disp})}_{C, nr}(x) \equiv \mathbf{j}^{(\text{disp})}_{C, nr}(x) e^{-\epsilon |t|},
$$

$$
z = \frac{1}{3} \int_{k > \lambda} \frac{d^3 k}{(2 \pi)^3} \tilde{\rho}^2(\mathbf{k}).
$$

The $\lambda$-dependence in $H^{(\text{Coul})}_{\text{int, ren}, \epsilon}$ is again understood. In the sequel, we state the results holding for positive times; the corresponding expressions for $t < 0$ are obtained by replacing $\epsilon$ by $-\epsilon$ and $\mathbf{p}_{out}$ by $\mathbf{p}_{in}$.

Although (33) is time dependent, existence and uniqueness of the time-evolution unitary operators follow from the independence of time of their selfadjointness domain and strong differentiability of $(H(t) - i)(H(0) - i)^{-1}$, through the results of [Kato56]. Insertion of

$$
H^{(\text{Coul})}_{\lambda, \text{ren}, \epsilon}(\epsilon) \equiv \exp \left( i H^{e, m}_{0, C} \right) H^{(\text{Coul})}_{\text{int, ren}, \epsilon} \exp \left( -i H^{e, m}_{0, C} \right)
$$

into the right-hand side (r.h.s.) of (26) yields the time-evolution operator of the model in the interaction representation,

$$
U_I^{(\epsilon)}(t) = c_{p_s}^{(\epsilon)}(t) \exp \left( i e \left[ a_C^* \lambda (\mathbf{f}_{p_s}^{(\epsilon)}(t)) + a_C \lambda (\mathbf{T}_{p_s}^{(\epsilon)}(t)) \right] \right),
$$

where

$$
c_{p_s}^{(\epsilon)}(t) = c_{p_s}^{(\epsilon)}(t) \exp \left( \frac{i e^2}{3 m^2} d^{(\epsilon)}(t) \right),
$$

$$
d^{(\epsilon)}(t) = -\int_{k > \lambda} \frac{d^3 k}{(2 \pi)^3} \tilde{\rho}^2(\mathbf{k}) \left[ e^{-\epsilon t} \sin k t + \frac{k}{2} \frac{e^{-2 \epsilon t} - 1}{e^{-2 \epsilon t} - 1} \right],
$$

$$
f_{p_s}^{(\epsilon)}(t, \mathbf{k}) = \frac{\rho(\mathbf{k})}{(2 \pi)^3 / 2} \frac{\mathbf{p}_{out} \cdot \epsilon_x(\mathbf{k})}{m} \frac{e^{i k t - \epsilon}}{i k - \epsilon}.
$$

The asymptotic limits are controlled by the following
Proposition 1. Both the large-time limits and the adiabatic limit of the evolution operator $\mathcal{V}_\epsilon$, defining the Möller operators, exist in the strong operator topology, for each value of the infrared cutoff $\lambda$:

$$
\Omega^{(\lambda)}_{\pm, C, p} = \text{lim}_{\epsilon \to 0} \lim_{t \to \infty} U^{(\lambda)}_{t, C, p_{\pm}, z}(t) = \text{lim}_{\epsilon \to 0} \lim_{t \to \infty} U^{(\lambda)}_{t, C, p_{\pm}, z}(t)
$$

$$
\Omega^{(\lambda)}_{\pm, C, p_{\pm}} = \exp \left( -i \epsilon \left[ a_{C, \lambda}^{+} (f_{p_{\pm}}) + a_{C, \lambda} (\overline{T}_{p_{\pm}}) \right] \right),
$$

where $f_{p_{\pm}}(s) \equiv L^2 - \text{lim}_{\epsilon \to 0} f^{(\pm)}_{p_{\pm}}(s)(k) \equiv L^2 - \text{lim}_{\epsilon \to 0} \text{lim}_{t \to \infty} f^{(\pm)}(t, k)$.

$$
f_{p_{\pm}}(k) = \frac{\bar{\rho}(k)}{2 \pi^{3/2} \sqrt{2 k}} \frac{p_{\pm} \cdot \epsilon_{s}(k)}{m} i k, p_{+} \equiv p_{\text{out}}, p_{-} \equiv p_{\text{in}}.
$$

Proof. The field operator $\Phi_{\lambda}(f_{p_{\pm}}(t, k)) \equiv a_{C, \lambda}(f_{p_{\pm}}(t, k)) + a_{C, \lambda}^{+}(\overline{T}_{p_{\pm}}(t, k))$ admits $F_0$ as a dense and invariant set of analytic vectors, hence it is e.s.a. therein due to Nelson’s analytic vector theorem (Theorem X.39 in [RSII]). The limits of $f_{p_{\pm}}(t, k)$ exist pointwise, hence, by the dominated convergence theorem, also in the strong $L^2$ topology; by linearity and standard Fock-space estimates, $\Phi_{\lambda}(f_{p_{\pm}}(t, k))$ thus converges strongly to $\Phi_{\lambda}(f_{p_{\pm}}(t, k))$ on $F_0$. Since $\Phi_{\lambda}(f_{p_{\pm}}(k))$ is e.s.a. on $F_0$, convergence is in the strong generalized sense and the existence of the time-limits in follows by Theorem VIII.20 in [RSI]. A similar proof allows to establish the strong convergence of $\Phi_{\lambda}(f_{p_{\pm}}(k))$ for $\epsilon \to 0$.

In order to recover the Coulomb-gauge soft-photon contribution to the process $p_{\text{in}} \to p_{\text{out}}$, we define the scattering matrix

$$
S^{(\text{Coul}), (\text{dp})}_{\lambda, p_{\text{out}}, p_{\text{in}}} = \text{lim}_{\epsilon \to 0} S^{(\text{Coul}), (\text{dp}), (\epsilon)}_{\lambda, p_{\text{out}}, p_{\text{in}}} = \text{lim}_{\epsilon \to 0} \Omega^{(\lambda), (\epsilon)}_{-; C, \lambda} \Omega^{(\lambda), (-\epsilon)}_{+; C, \lambda, p_{\text{in}}},
$$

The comparison of the infrared diagrammatics resulting from the Feynman-Dyson expansion of QED with the expansion of Möller’s operators in powers of the electric charge requires to adopt a formulation in a gauge employing four independent photon degrees of freedom, such as the FGB gauge.

With this aim, we introduce the Hamiltonian

$$
H_{\lambda}^{(\text{FGB}), (\text{dp})} = H_{\lambda}^{\text{e.m.}} + H_{\lambda}^{(\text{FGB}), (\text{dp})},
$$

$$
H_{\lambda}^{(\text{FGB}), (\text{dp})} \equiv \epsilon \int_{\lambda} d^3 x \ j^{(\text{dp})}_{\text{nr}, \mu}(x) A^\mu(x),
$$

with

$$
\int_{\lambda} j^{(\text{dp})}_{\text{nr}, \mu}(x) \equiv (\rho(x), j^{(\text{dp})}_{\text{nr}}(x)).
$$

Since (47) is obtained by imposing a dipole approximation on the non-relativistic four-current $j^\mu_{\text{nr}}(x)$,

$$
j^0_{\text{nr}}(x) \equiv \theta(-t) \rho(|x - \frac{p_{\text{in}}}{m} t|) + \theta(t) \rho(|x - \frac{p_{\text{out}}}{m} t|),
$$

$$
j_{\text{nr}}(x) \equiv \theta(-t) \rho(|x - \frac{p_{\text{in}}}{m} t|) \frac{p_{\text{in}}}{m} + \theta(t) \rho(|x - \frac{p_{\text{out}}}{m} t|) \frac{p_{\text{out}}}{m},
$$

the Hamiltonian (45) shares the same approximations of (14) and (15).

The space $\mathcal{G}$ results from the application to a non-positive linear functional and a $*$-algebra, containing the Weyl exponentials of the photon canonical variables, of a generalization of the Gelfand-Naimark-Segal (GNS) reconstruction theorem [GN43, Theorem 17]. The representation of the Weyl operators defines the corresponding exponentials of the creation and annihilation operators on $\mathcal{G}$; such exponentials are also given by their series, which are weakly converging on $\mathcal{G}$. For the details, we refer to Appendix B.

By the same token, one can establish formula (29) also in the indefinite-metric case, a fact which we will rely upon in the definition of the dynamics of the model.
The domain \( \mathcal{G}_0 \) of the Hamiltonian \( (45) \) is a weakly dense invariant subspace of \( \mathcal{G} \); we again refer to Appendix \( \[A] \) for the details of its construction. The inner product of \( \mathcal{G}_0 \) is also denoted by \( \langle \ldots \rangle \), since this should not be a source of confusion.

Let \( \mathcal{H} \) be the Hilbert space arising from the standard positive scalar product on \( \mathcal{G} \):

\[
(f, g) \equiv (f^0, g^0) + \sum_i (f^i, g^i).
\]  

(49)

The free and full dynamics of the model are determined by isometries of \( \mathcal{G} \) which leave \( \mathcal{G}_0 \) invariant and are differentiable on \( \mathcal{G}_0 \) in the strong topology of \( \mathcal{H} \), with time-derivatives given respectively by \( H^\text{free} \) and \( H^{(FGB)} \). The formulation and the proof of uniqueness of such evolution operators are given in Appendix \( \[C] \) (Lemma \( \[10] \)).

The dynamics of the model in the interaction representation is governed by a \( (\ldots) \)-isometric operator \( U^{(\lambda)}_I, p(t) = \tilde{c}_\tilde{\varphi}(t) \exp (\cdot i \epsilon \int a^\dagger_\lambda (f_\varphi(t)) + a_\lambda (\overline{f}_\varphi(t)))\),

(51)

\[
\tilde{c}_\tilde{\varphi}(t) \equiv \exp (- \frac{i \epsilon^2 \tilde{\varphi}^2}{2} d(t)),
\]

(52)

\[
f^\mu_\varphi(t, k) = \frac{\tilde{\rho}(k) \tilde{\varphi}^\mu}{(2\pi)^{3/2} \sqrt{2k}} \exp \left( - \frac{ik \tau}{\epsilon} - 1 \right).
\]

(53)

It can be immediately checked that the operator \( \overline{51} \) fulfills equation \( \overline{24} \) in the strong topology of \( \mathcal{H} \), on the invariant subspace \( \mathcal{G}_0 \).

Proceeding as for the Coulomb-gauge formulation of the model, we consider the Hamiltonian resulting from the introduction of suitable renormalization counterterms and of an adiabatic switching in \( \overline{45} \):

\[
H^{(FGB)} \equiv \overline{H}_0^\text{free} + e \int \overline{d}^3 x J^{(\text{free})}(x) \overline{\mathcal{A}}(x) - e^2 \overline{\theta}(-t) \tilde{\varphi}^2\overline{\mathcal{A}}_\varphi(x) + \overline{\mathcal{A}}_\varphi(x) - e^2 \overline{\theta}(-t) \tilde{\varphi}^2\overline{\mathcal{A}}_\varphi(x) + \overline{\mathcal{A}}_\varphi(x),
\]

(54)

with

\[
J^{(\text{free})}(x) = \overline{J}^{(\text{free})}(x) e^{-\epsilon |t|},
\]

(55)

\[
\tilde{z} = \frac{3}{2} \overline{z} = \frac{1}{2} \int_{k > \lambda} d^3 k \frac{\overline{d}^3 k}{(2\pi)^{3/2} k^2} \overline{\tilde{\rho}}^2(k).
\]

(56)

For positive times, the corresponding evolution operator in the interaction picture is

\[
U^{(\lambda), \varphi}_I, t < 0 = \tilde{c}^{(\varphi)}_{\tilde{\varphi}} (t, \tilde{\varphi}(t) \exp (\cdot i \epsilon \int a^\dagger_\lambda (f^{(\varphi)}_{\tilde{\varphi}}(t)) + a_\lambda (\overline{f}^{(\varphi)}_{\tilde{\varphi}}(t)))],
\]

(57)

with

\[
\tilde{c}^{(\varphi)}_{\tilde{\varphi}}(t) = \exp \left( - \frac{i \epsilon^2 \tilde{\varphi}^2}{2} d^{(\varphi)}(t) \right),
\]

(58)

\[
\tilde{f}^{(\varphi)}_{\tilde{\varphi}}(t, k) = \frac{\tilde{\rho}(k) \tilde{\varphi}^\mu}{(2\pi)^{3/2} \sqrt{2k}} \exp \left( - \frac{ik \tau}{\epsilon} - 1 \right).
\]

(59)

For \( t < 0 \), one has to replace \( \epsilon \) by \( -\epsilon \) and \( \tilde{\varphi} \) by \( \tilde{\varphi} \) in the expressions above.

The asymptotic limits are given by the following

9
Proposition 2. The large-time limits and the adiabatic limit of the time-evolution operator \( \Omega^{(\lambda)}_{\pm}, \tilde{v}_{\mp} \) are given by

\[
\Omega^{(\lambda)}_{\pm}, \tilde{v}_{\mp} = \tau_{w} - \lim_{\epsilon \to 0} \Omega^{(\lambda)}_{\pm}, (\mp \epsilon) = \exp \left( i e \left[ a_{\lambda} \left( f_{\pm} \right) + a_{\lambda} \left( \mathcal{F}_{\pm} \right) \right] \right),
\]

(61)

\[
\Omega^{(\lambda)}_{\mp}, \tilde{v}_{\pm} = \tau_{w} - \lim_{t \to \pm \infty} U^{(\lambda)}_{\pm}, (\mp \epsilon, t) = \tilde{c}_{\pm}(\pm \epsilon) \times \exp \left( i e \left[ a_{\lambda} \left( f_{\pm} \right) + a_{\lambda} \left( \mathcal{F}_{\pm} \right) \right] \right),
\]

(62)

\[
f_{\pm}^{\mu}(k) = \frac{\tilde{\rho}(k)}{(2 \pi)^{3/2} \sqrt{2k}} \frac{i}{\tilde{v}_{\mp}} \equiv \tilde{v}_{\mpout} \equiv \tilde{v}_{\mpin},
\]

(64)

exist on \( \mathcal{G} \), for a fixed value of the infrared cutoff \( \lambda \), and define Möller operators as isometries of \( \mathcal{G} \).

Proof. By (199)–(203) in Appendix B the convergence of the coherence functions (60) in \( L^2 \) implies the weak convergence of the corresponding expectations on \( \mathcal{G} \) of polynomials and exponentials of the smeared photon variables. The Möller operators (61) are invertible and preserve the inner product \( \langle \ldots, \ldots \rangle \) as a consequence of the GNS representation of relations (198), (199); they define therefore isometries of \( \mathcal{G} \), uniquely determined by their restrictions to \( \mathcal{G} \).

As in the Coulomb-gauge formulation of the model, we define a scattering operator

\[
S^{(\text{FGB})}_{\lambda, \tilde{v}_{\mpout}, \tilde{v}_{\mpin}}(\text{dip}) = \tau_{w} - \lim_{\epsilon \to 0} S^{(\text{FGB})}_{\lambda, \tilde{v}_{\mpout}, \tilde{v}_{\mpin}}(\text{dip}), (\epsilon),
\]

(65)

\[
S^{(\text{FGB})}_{\lambda, \tilde{v}_{\mpout}, \tilde{v}_{\mpin}}(\text{dip}), (\epsilon) = \Omega^{(\lambda)}_{\pm}, (\epsilon) \mp \tilde{v}_{\mpout}, \tilde{v}_{\mpin}, \Omega^{(\lambda)}_{\pm}, \tilde{v}_{\mpin}, \tilde{v}_{\mpout}.
\]

(66)

We give here a preliminary discussion of the extent and limitations of the comparison of this model with the perturbative (infrared) structures and results.

In Section 3 we prove that the perturbative expressions for the overall soft-photon radiative corrections to the process \( p_{\text{in}} \to p_{\text{out}} \), in the presence of a dipole approximation, are indeed reproduced in terms of the Möller operators of the PFB model, both in the Coulomb gauge and in the FGB gauge. In particular, the Coulomb-gauge expression for such corrections is given by the vacuum expectation

\[
\langle \Psi_{F}, S_{\lambda, \tilde{v}_{\mpout}, \tilde{v}_{\mpin}}^{(\text{Coul})}, (\text{dip}) \rangle_{\Psi_{F}} = \exp \left( -\frac{e^{2}}{2m^2} \left| p_{\text{out}} - p_{\text{in}} \right|^2 \int_{k > \lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} \tilde{\rho}^2 (k) \right),
\]

(67)

for each value of the infrared cutoff \( \lambda \). Likewise, the soft-photon radiative corrections to the same process in the FGB gauge are recovered in terms of the vacuum expectation

\[
\langle \Psi_{0}, S_{\lambda, \tilde{v}_{\mpout}, \tilde{v}_{\mpin}}^{(\text{FGB})}, (\text{dip}) \rangle_{\Psi_{0}} = \exp \left( -\frac{e^{2}}{2m^2} \left| p_{\text{out}} - p_{\text{in}} \right|^2 \int_{k > \lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} \tilde{\rho}^2 (k) \right). \]

(68)

However, the expression on the r.h.s. of (68) is not equal to its Coulomb-gauge counterpart (67), while radiative corrections are not expected to depend upon the gauge employed in their calculations.

In order to clarify this apparent paradox, we recall that an invariance property with respect to gauge employed in their calculation holds indeed for the expression of the soft-photon corrections, as given in the FGB gauge (for a fixed infrared cutoff) by the standard low-energy Feynman rules, introduced and employed for instance in [Wein], Chap. 13. If additional infrared approximations are adopted, such an invariance property is however no more guaranteed.

In Section 3 we will see that for the FGB-gauge soft-photon corrections to the scattering of a single charged particle to be invariant in the above sense, it is crucial that the classical four-current of the associated Hamiltonian model fulfills the continuity equation. Furthermore, we shall also show that the discrepancy between the expression (65) and the Coulomb-gauge result (67) is quantitatively related to the non conservation of the four-current (17).

\footnote{This result follows from Lemma 3 in Appendix B.}

\footnote{In the literature of theoretical physics, this property is generally referred to as the gauge-invariance of Feynman amplitudes. However, since gauge-invariance is actually a concept with much broader scope ([SW74]), we prefer to avoid the use of such a terminology.}

10
2 Bloch-Nordsieck Models And Feynman-Dyson Expansion

In the present Section we introduce a Hamiltonian model based on an approximation first devised by Bloch and Nordsieck (BN37). As a useful starting point we recall the basic features of the original treatment, which in a nutshell amounts to a first-order expansion around a fixed four-momentum of each charged particle with respect to the energy-momentum transfer. Consider the one-particle Dirac Hamiltonian with minimal coupling,

\[ H = \alpha \cdot (p - eA) + \beta m + eA^0 \equiv H_D - e \alpha \cdot A + eA^0, \]  

(69)

and an eigenstate \( \psi_{+\mu} (x) = e^{-ip \cdot x} \mathcal{U}_r (p) \) of \( H_D \) with momentum \( p \) and (positive) energy \( E_p \), \( \mathcal{U}_r (p) \) being the corresponding momentum-space spinor with helicity \( r \). The Dirac equation and the algebraic relations for Dirac’s matrices yield

\[ H_D \psi_{0\mu} (x) = \left[ u \cdot p + \sqrt{1 - u^2} m \right] \psi_{0\mu} (x) + O(p - p_0), \] 

(70)

with \( \psi_{0\mu} (x) \equiv e^{-ip \cdot x} \mathcal{U}_r (p_0), u \equiv p_0/E_p, p_0 = (E_{p_0}, p_0) \) being a fixed four-momentum.\(^4\)

More formally, the \( u \)-dependent terms on the r.h.s. of (70) may be obtained as a result of replacing \( \alpha \) and \( \beta \) in (69) respectively by the (diagonal in the spinor indices) matrices \( u \) and \( \sqrt{1 - u^2} \). Although the outcome of such an approximation may seem to depend upon the linearity of the \( \alpha \) matrices, it is indeed more general; for instance, the same expression would be obtained starting from the eigenvalue equation for the Klein-Gordon Hamiltonian.

The above discussion leads to a model defined, respectively in the Coulomb-gauge and in the FGB gauge, by the Hamiltonians

\[ H_{u}^{(Coul)} = p \cdot u + H_{0}^{e.m.} - e \ u \cdot A_{C} (\rho, x), \]  

(71)

\[ H_{u}^{(FGB)} = p \cdot u + H_{0}^{e.m.} + e \ u \cdot A (\rho, x), \]  

(72)

with \( u \equiv (1, u) \), \( u \) being a triple of self-adjoint operators, to be identified as the observable associated to the asymptotic velocity of the particle, which commute with the Weyl algebra \( \mathcal{A}_{\text{ch}} \) generated by \( x \) and \( p \) and with the polynomial algebras generated by the photon canonical variables.

In the sequel we let \( \alpha \to \beta \) denote the scattering of an electron by a potential, with the initial (final) particle-state being of definite momentum and four-velocity \( u_{\text{in}} (u_{\text{out}}) \). In order to establish an operator formulation of the soft-photon corrections to this process, the full structure of (71), (72) will not be needed; in fact it will suffice to suppose that the dynamics of the charge is governed by classical mechanics and that its time evolution is given by \( x(t) = \theta (-t) \ u_{\text{in}} t + \theta (t) \ u_{\text{out}} t \).

Precisely, starting from the Hamiltonians

\[ H_{u}^{(Coul)} = H_{0}^{e.m.} - e \ u \cdot A_{C} (\rho, x), \]  

(73)

\[ H_{u}^{(FGB)} = H_{0}^{e.m.} + e \ u \cdot A (\rho, x), \]  

(74)

indexed by the value of the (classical) velocity \( u \), we introduce the infrared-regularized Hamiltonians

\[ H_{\lambda}^{(Coul)} = H_{0}^{e.m.} - e \int_{\lambda} d^3 x \ j_{C} (x) \cdot A_{C} (x) \equiv H_{0}^{e.m.} + H_{\text{int}}^{(Coul)}, \]  

(75)

\[ H_{\lambda}^{(FGB)} = H_{0}^{e.m.} + e \int_{\lambda} d^3 x \ j_{\mu} (x) A_{\mu} (x) \equiv H_{0}^{e.m.} + H_{\text{int}}^{(FGB)}. \]  

(76)

Again, the \( \lambda \)-dependence in \( H_{0}^{e.m.}, H_{\text{int}}^{(Coul)}, H_{\text{int}}^{(FGB)} \) is understood. The classical four-current

\[ j_{\mu} (x) = \theta (-t) \rho (|x - u_{\text{in}} t|) u_{\text{in}}^{\mu} + \theta (t) \rho (|x - u_{\text{out}} t|) u_{\text{out}}^{\mu} = \left( j_{0} (x), j (x) \right) = j_{\text{in}}^{\mu} (x) + j_{\text{out}}^{\mu} (x) \]  

(77)

\(^4\)We denote the four-velocity by the symbol \( u \) in order to avoid possible ambiguities with the notations of Section\[11\].
and the classical current
\[ j^e_C(t, x) = \int d^3y \, \delta_{1\nu}^m (x - y) \, j^m(t, y) \] (78)
fulfill the continuity equation \( \partial_\mu j^\mu(x) = 0 \) and the Coulomb-gauge condition \( \partial_t j^e_C(x) = 0 \), respectively. The quantum e.m. potentials occurring in the interaction Hamiltonians of (75), (76) will be interpreted as describing the soft-photon degrees of freedom.

We first consider the model with (Coulomb-gauge) Hamiltonian (75). The Hilbert space of photon states of this model is \( \mathcal{H}_C \equiv \mathcal{F} \), with \( \mathcal{F} \) the boson Fock space already considered in Section 1. Proceeding in a similar way as in the proof of self-adjointness of the Pauli-Fierz Hamiltonian, one finds that \( H^{(Coul)}_\lambda \) is e.s.a. on \( D_{F_o} \subset \mathcal{H}_C \); Stone’s theorem then implies the existence and uniqueness of the dynamics.

With the aid of (25) and employing the expansion (3) and the Fourier transform of \( j^e_C(x) \) with respect to \( x \), given by
\[ \hat{j}^r_C(t, k) = \hat{\rho}(k) \, D^{rs}(k) \left[ \theta(-t) \, u^s_{in} e^{-i u_{in} \cdot k t} + \theta(t) \, u^s_{out} e^{-i u_{out} \cdot k t} \right] , \] (79)
we obtain a solution of the equations of motion in the interaction representation, in the form
\[ \mathcal{W}^{(\lambda)}_{I, C, u}(t) = c_u(t) \exp \left( i e \left[ a^*_C, \lambda \left( f_u(t) \right) + a_C, \lambda \left( T_u(t) \right) \right] \right) , \] (80)
\[ c_u(t) = \exp \left( -\frac{i e^2 u^2}{3} \, d_u(t) \right) , \] (81)
\[ d_u(t) = \int_{k > \lambda} \left( \frac{2}{3} \, \frac{1}{k} \, u \cdot k \right) \hat{\rho}^2(k) \left( t - \sin \frac{u \cdot k}{u \cdot k} \right) , \] (82)
where \( k_0 = k \) and \( u = u_{out}, \, t > 0 \), \( u = u_{in}, \, t < 0 \).

With the same motivations and following the same treatment as in the first Section, we introduce the renormalized adiabatic Hamiltonian
\[ H^{(Coul), \text{ren}}_{\lambda, u}(t) \equiv H_{0, C} e_m \quad \text{and} \quad H^{(Coul), \text{ren}}_{\lambda, u}(t) = \int_{k > \lambda} \left( \frac{2}{3} \, \frac{1}{k} \, u \cdot k \right) \hat{\rho}^2(k) \, t , \] (84)

Kato’s result on time-dependent Hamiltonians applies as in Section 1 and, using (25), one obtains the corresponding evolution operator in the interaction representation; for positive times
\[ \mathcal{W}^{(\lambda), \text{ren}}_{I, C, u_{out}, z_1}(t) = c_{u_{out}, z_1}(t) \exp \left( i e \left[ a^*_C, \lambda \left( f^{(\varepsilon)}_{u_{out}}(t) \right) + a_C, \lambda \left( T^{(\varepsilon)}_{u_{out}}(t) \right) \right] \right) , \] (85)
\[ c_{u_{out}, z_1}(t) \equiv c_{u_{out}}(t) \exp \left( i e^2 \left( u_{out} \right) 2 \, u_{out} - \frac{2 e - 1}{2 e} \right) , \] (86)
\[ d^{(\varepsilon)}_{u_{out}}(t) = -\int_{k > \lambda} \left( \frac{2}{3} \, \frac{1}{k} \, u \cdot k \right) \hat{\rho}^2(k) \left[ e^{-e t} \sin u_{out} \cdot k t + \frac{u_{out} \cdot k}{2 e} \left( e^{-2 e t} - 1 \right) \right] , \] (87)
\[ f^{(\varepsilon)}_{u_{out}^{(s)}}(t, k) = \frac{\hat{\rho}(k)}{\left( 2 \pi \right)^{3/2} \sqrt{2 k}} \, u_{out} \cdot \epsilon_s(k) \, e^{i u_{out} \cdot k t - e t - 1} . \] (88)

The results holding for \( t < 0 \) are obtained by replacing \( u_{out} \) and \( e \) respectively by \( u_{in} \) and \( -e \) in the expressions above. One can then prove the following
Proposition 3. The evolution operator \([59]\) in the interaction representation corresponding to the renormalized Hamiltonian \([53]\) admits asymptotic limits, yielding the Möller operators in the Coulomb gauge, for each value of the infrared cutoff \(\lambda\):

\[
\Omega^{(\lambda)}_{\pm, C, u_z} = s - \lim_{\epsilon \to 0} \lim_{t \to \pm \infty} \Psi^{(\lambda)}_{I, C, u_z, z_1}(t) = s - \lim_{\epsilon \to 0} \Omega^{(\lambda)}_{\pm, C, u_z} = \exp(-i \epsilon [a^{\dagger}_{C, \lambda}(f_{u_z}) + a_{C, \lambda}(T_{u_z})]),
\]

\[
f^{u_z, s}_{-}\left(k\right) = \frac{\tilde{\rho}\left(k\right)}{(2\pi)^{3/2}} \frac{i \mu_{u_z, \epsilon}(k)}{u_{\pm} \cdot k}, u_{\pm} \equiv u_{out}, u_{-} \equiv u_{in}.
\]

The \(t \to \pm \infty\) and \(\epsilon \to 0\) limits in \([59]\) exists in the strong topology.

**Proof.** The control of the limits is very similar to the case discussed in Proposition 1.

---

We introduce the Coulomb-gauge photon scattering operator of the model,

\[
S^{(Coul)}_{\lambda, u_{out}, u_{in}} = s - \lim_{\epsilon \to 0} S^{(Coul)}_{\lambda, u_{out}, u_{in}},
\]

\[
S^{(Coul)}_{\lambda, u_{out}, u_{in}} = \Omega^{(\lambda)}_{\lambda, (-\epsilon)} \Omega^{(\lambda)}_{\epsilon, C, u_{out}} \Omega^{(\lambda)}_{\epsilon, C, u_{in}}.
\]

We shall now turn to the formulation of the model in the FGB gauge. Concerning the problems posed by the absence of a positive scalar product on the state space of the model, we adopt the same choices as in Section 1 following the same steps with identical results; in particular, as discussed in detail in Appendix \(\mathcal{A}\) a (generalized) GNS construction provides an indefinite space \(\mathcal{G}\) and a weakly dense domain \(\mathcal{G}_0\). Isometric evolution operators on \(\mathcal{G}_0\) are constructed as in Section 1 and are unique in the same sense.

With the aid of the expansion \([5]\) and of the Fourier transform of \(j^\mu(x)\) with respect to \(x\), whose explicit expression is

\[
\tilde{j}^\mu(t, k) = \tilde{\rho}(k) [\theta(-t) u^{in}_\mu e^{-i u_{in} \cdot k t} + \theta(t) u^{out}_\mu e^{-i u_{out} \cdot k t}],
\]

we obtain the evolution operator in the interaction representation, in the form

\[
\Psi^{(\lambda)}_{I, u}(t) = \mathcal{C}_u(t) \exp(-i \epsilon [a^{\dagger}_{\lambda}(f_{u}(t)) + a_{\lambda}(T_{u}(t))]),
\]

\[
\mathcal{C}_u(t) = \exp(-i \epsilon^2 u_\mu^2 / 2) d_u(t),
\]

\[
f^\mu(t, k) = \frac{\tilde{\rho}(k) u^{\mu}_{u}}{(2\pi)^{3/2}} \frac{e^{i u \cdot k t} - 1}{i u \cdot k},
\]

where \(k_0 = k\) and \(u = u_{out}, t > 0, u = u_{in}, t < 0\).

In order to construct Möller operators, we introduce the adiabatic Hamiltonian

\[
H^{(FGB)}_{\lambda, \epsilon_0} = H_{\lambda, \epsilon_0} + \epsilon \int_{\lambda} d^3x \tilde{j}^{(\epsilon)}(x) A^\mu(x),
\]

\[
j^{(\epsilon)}(x) \equiv j^\mu(x) e^{-\epsilon|x|} = j^{(\epsilon)}_{in, \mu}(x) + j^{(\epsilon)}_{out, \mu}(x),
\]

and renormalize it via suitable counterterms; the resulting expression is

\[
H^{(FGB)}_{\lambda, \epsilon_0, \epsilon_0} = H^{(FGB)}_{\lambda, \epsilon_0} - \epsilon^2 [\theta(-t) z_2(u_{in}) u^{2}_{in} + \theta(t) z_2(u_{out}) u^{2}_{out}] e^{-2 \epsilon |t|},
\]

with

\[
z_2(u) = \frac{3}{2} z_1(u) = \frac{1}{2} \int_{k > \lambda} \frac{d^3k}{(2\pi)^3} \frac{\tilde{\rho}^2(k)}{u \cdot k}.
\]

For \(t > 0\), the corresponding evolution operator in the interaction representation is

\[
\Psi^{(\lambda)}_{I, u_{out}, z_2}(t) = \mathcal{C}^{(\epsilon)}_{u_{out}, z_2}(t) \exp(i \epsilon [a^{\dagger}_{\lambda}(f^{(\epsilon)}_{u_{out}, z_2}(t)) + a_{\lambda}(T^{(\epsilon)}_{u_{out}, z_2}(t))]),
\]
\[ \mathcal{G}^{(\epsilon)}_{u_{\text{out}}, z_2}(t) \equiv \mathcal{G}^{(\epsilon)}_{u_{\text{out}}}(t) \exp \left( -i e^2 z_2 \left( u_{\text{out}} \right) u_{\text{out}}^2 \frac{e^{-2 \epsilon t} - 1}{2 \epsilon} \right), \tag{100} \]

\[ f^{(\epsilon), \mu}_{u_{\text{out}}}(t, k) = \frac{\hat{\rho}(k) u_{\text{out}}^\mu}{(2 \pi)^{3/2} \sqrt{2 k}} \frac{e^{i u_{\text{out}} \cdot k - \epsilon t - 1}}{i u_{\text{out}} \cdot k - \epsilon}. \tag{101} \]

The results holding for \( t < 0 \) are obtained by replacing \( \epsilon \) by \( -\epsilon \) and \( u_{\text{out}} \) by \( u_{\text{in}} \) in the expressions above. The asymptotic limits are controlled through the following

**Proposition 4.** The large-time limits and the adiabatic limit of the evolution operator \([77]\), corresponding to the renormalized adiabatic Hamiltonian \([77]\),

\[ \Omega^{(\lambda)}_{\pm, u_{\pm}} \equiv \tau_w - \lim_{\epsilon \to 0} \Omega^{(\lambda), (\pm \epsilon)}_{\pm, u_{\pm}} = \exp \left( i e \left[ a_\lambda^f (f_{u_\mp}) + a_\lambda (\mathcal{F}_{u_\mp}) \right] \right), \tag{102} \]

\[ \Omega^{(\lambda), (\pm \epsilon)}_{\pm, u_{\pm}} \equiv \tau_w - \lim_{t \to \pm \infty} \mathcal{G}^{(\lambda), (\pm \epsilon)}_{\pm, u_{\pm}}(t) = \mathcal{G}^{(\epsilon)}_{u_{\pm}, z_2}(t), \tag{103} \]

\[ \mathcal{G}^{(\pm \epsilon)}_{u_{\pm}, z_2}(t), \tag{104} \]

\[ f^{\mu}_{u_{\pm}, z_2}(k) = \frac{\hat{\rho}(k) u_{\pm}^\mu}{(2 \pi)^{3/2} \sqrt{2 k}} \frac{i}{u_{\pm} \cdot k}, u_{\pm} \equiv u_{\text{out}}, u_{\mp} \equiv u_{\text{in}}, \tag{105} \]

exist on \( \mathcal{G}_0 \) and define the Møller operators of the four-vector BN model as invariants in \( \mathcal{G} \), for each value of the infrared cutoff \( \lambda \).

**Proof.** The proof is the same as that of Proposition 2.

We turn now to the comparison of our model with the soft-photon contributions of QED. As regards infrared approximations and results in perturbation theory, we refer to the streamlined approach of [JR], Chap. 16, [Wein], Chap. 13, while a more detailed analysis can be found in the classic paper of Yennie, Frautschi and Suura ([YFS61]; for the convenience of the reader, the relevant results of such treatments are recalled below.

The effect to all orders of radiative corrections due to “virtual soft-photons” is to multiply the \( S \)-matrix element for a process \( \gamma \to \delta \), involving a fixed number of charged particle and photons, by an exponential factor. Precisely, in presence of an infrared cutoff \( \lambda \) one has

\[ S_{\lambda, \delta \gamma} = \exp \left( e^2 M_{\lambda, \delta \gamma} \right) S_{\lambda, \delta \gamma}^{(\text{hard})}, \tag{106} \]

with \( S_{\lambda, \delta \gamma}^{(\text{hard})} \) yielding the overall infrared-finite contributions to the transition amplitude \( \gamma \to \delta \). From a diagrammatic point of view, it is given by the sum of a family of connected Feynman diagrams, sharing the same number of external lines and possibly different internal structure; it is customary to represent such a sum by a single diagram, in which the external lines common to the diagrams of the family are recalled below.

The exponential on the r.h.s. of (106) gives those radiative corrections which are associated to the insertion of soft-photon propagators on the external lines of the bubble; such corrections are singular as the low-energy cutoff is removed. The FGB-gauge expression of the corresponding exponent is

\[ M_{\lambda, \delta \gamma}^{(\text{FGB})} = \frac{1}{2 (2 \pi)^3} \sum_{m, n} \eta_m \eta_n \rho_m \cdot \rho_n \int_{\Lambda} \frac{d^3 k}{2 k} \frac{\rho_{m \cdot k} \rho_{n \cdot k}}{\rho_{m \cdot k} \rho_{n \cdot k}}, \tag{107} \]

with the sum running over the external four-momenta of the charged particles involved in the process \( \gamma \to \delta \); in (107), \( \eta = 1 \) for the outgoing electronic lines, \( \eta = -1 \) for the incoming ones and \( \Lambda \) is an energy scale, chosen in such a way that the approximations made in the analysis of the soft-photon contributions to the perturbative series of QED are justified for photons with energy below \( \Lambda \).

Concerning the corrections due to soft-photon emission, one has to take into account the contributions of additional vertices and electron propagators, which arise by virtue of the insertion of photon legs,
representing the emitted photons, on the external lines of the bubble. Since such contributions give rise to infrared divergences in the calculations of inclusive cross-sections, a low-energy cutoff on the photon momenta must be introduced. As explained in [Wein], § 13.3, one has to adopt the same infrared cutoff both for the momentum-space integrals which yield the low-energy radiative corrections and for the momenta of the emitted photons, in order to preserve unitarity at each stage of the calculations.

The Feynman rules which result from the standard low-energy approximations are as follows. A vertex on an external charged line carrying four-velocity \( u = p / E_p \) contributes a factor \(-i e u^\mu\). An electron propagator carrying four-momentum \( p + k \) is associated to a term \( i / (u \cdot k + i \epsilon) \), where \( k \) is the overall photon contribution to the four-momentum carried by the propagator and \( i \epsilon \) is the Feynman prescription for Green’s functions (for its definition we refer for instance to [MaSh], § 3.4). The rule for the photon propagator is the standard one: Letting

\[
\Delta_F(x) = \theta(x_0) \Delta^+(x) + \theta(-x_0) \Delta^+(x),
\]

\[
g^{\mu \nu} \Delta^+(x-y) \equiv i[A_\mu(x), A_\nu(y)],
\]

\[
\Delta^+(x) = \frac{1}{(2\pi)^4} \int d^4k \ e^{-i k \cdot x} \Delta^+(k),
\]

\[
\Delta^+(k) = -2\pi i \theta(k^0) \delta(k^2),
\]

the space-time expression of the photon propagator is

\[
i \Delta^{\mu \nu}_F(x) \equiv \frac{i}{(2\pi)^4} \int d^4k \ e^{-i k \cdot x} \Delta^{\mu \nu}_F(k) = -i g^{\mu \nu} \Delta_F(x)
\]

and the term \( i \Delta^{\mu \nu}_F(k) \) is associated to a (photon) line carrying four-momentum \( k \) and joining two vertices with indices \( \mu \) and \( \nu \). In (109), the positive-frequency part of the Pauli-Jordan function \( \Delta(x) \) of a free massless scalar field has been introduced. We shall also make use of the well-known relation

\[
[A^\mu(x), A^\nu(y)] = -i g^{\mu \nu} \Delta(x-y).
\]

Let

\[
S^{(\text{FGB})}_{\lambda, u_{\text{out}}, u_{\text{in}}} = \tau \lim_{\epsilon \to 0} S^{(\text{FGB})}_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\epsilon)},
\]

\[
S^{(\text{FGB})}_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\epsilon)} = \Omega^{(\lambda)}(\epsilon)\Omega^{(\lambda)}(\epsilon)^\dagger \Omega^{(\lambda)}(\epsilon)^\dagger \Omega^{(\lambda)}(\epsilon),
\]

be the S-matrix for the BN model in the FGB gauge. We wish to show that suitable matrix elements of (114) reproduce the soft-photon contributions to the process \( \alpha \to \beta \), as given by the diagrammatic rules discussed above.

**Remark 1.** We shall recover the perturbative expressions corresponding to mass-shell (in the sequel referred to as on-shell) renormalization prescriptions, in accordance with the analysis carried out in [JR, Wein], and to the choice of a form factor \( \tilde{\rho}(k) \) as an ultraviolet cutoff in momentum-space integrals. This leads to the following minor change for the vertex rule: A contribution \(-i e \tilde{\rho}(k) u^\mu\) will be assigned to a vertex with index \( \mu \) on a fermion line carrying four-velocity \( u \).

In the following, we let \( \mathcal{D} \) and \( \mathcal{M}^{(\text{FGB})}_{\beta \alpha} \) denote respectively the bubble diagram associated to the basic process \( \alpha \to \beta \) and the overall soft-photon radiative corrections at second order in the FGB gauge. Further, we let \( \mathcal{D}_1 \) be the diagram obtained by attaching a photon line to the outgoing fermion leg of \( \mathcal{D} \). As a result of the extra photon line, \( \mathcal{D}_1 \) contains one more vertex and an additional electron propagator with respect to \( \mathcal{D} \).

We begin our analysis by writing down the amplitudes which contribute to \( \mathcal{M}^{(\text{FGB})}_{\beta \alpha} \). By virtue of the (infrared) diagrammatic rules discussed above, the correction associated to a soft photon, emitted from the incoming fermion leg of \( \mathcal{D} \) and absorbed from the outgoing one, is given by

\[
e^2 \Gamma_{u_{\text{out}}, u_{\text{in}}} = \lim_{\epsilon \to 0} (i e)^2 \int_{k > \lambda} \frac{d^4k \tilde{\rho}^2(k)}{(2\pi)^4} \frac{i u_{\text{out}, \mu}}{-u_{\text{out}} \cdot k + i \epsilon} \frac{i u_{\text{in}, \nu}}{-u_{\text{in}} \cdot k + i \epsilon} \Delta^{\mu \nu}_F(k) = e^2 u_{\text{out}} \cdot u_{\text{in}} \int_{k > \lambda} \frac{d^4k \tilde{\rho}^2(k)}{(2\pi)^4} \frac{i}{-u_{\text{out}} \cdot k} \frac{i}{-u_{\text{in}} \cdot k} \Delta^+(k),
\]
where the method of residues has also been employed.

A relevant feature of the perturbation-theoretic treatment of infrared QED is that if the renormalization procedure is performed by means of on-shell prescriptions, it yields an additional low-energy divergent contribution for each external (charged) line ([JR], § 16.1). The order-$e^2$ expression is obtained as follows. Letting $\Sigma (u)$ be the self-energy function associated to the insertion of a second-order self-energy subgraph on an external fermion line carrying four-velocity $u$, the coefficient $B$ of the linear term in the expansion of $\Sigma (u)$ around the electron mass-shell ([MaSh], § 9.3) depends logarithmically upon the low-energy cutoff; the resulting correction takes the form $-e^2 B_{IR}/2$, with $B_{IR}$ the infrared part of $B$.

Considering for definiteness the expression associated to the outgoing line of $D$, the perturbation-theoretic result that we wish to reproduce (as can be inferred by taking $m = n$ in [107] and recalling Remark [4] is

\[
-\frac{e^2}{2} B_{IR}^{out} = \frac{e^2 u_{out}^2}{2} \int_{k > \lambda} \frac{d^3k}{(2\pi)^3} \frac{d^3k}{2k} \tilde{\rho}^2 (k) \frac{1}{(u_{out} \cdot k)^2}. \tag{117}
\]

Next Lemma shows that the infrared radiative corrections at second order can be expressed as a functional of the four-current (77).

**Lemma 1.** The order-$e^2$ infrared radiative corrections to the process $\alpha \to \beta$, obtained in the FGB gauge with the aid of the low-energy Feynman rules, can be written as

\[
e^2 \mathcal{M}^{(FGB)}_{\alpha \beta} = \frac{i e^2}{2} \int_{\lambda} d^4x \, d^4y \, j_\mu^\alpha (x) \Delta^+(x-y) \tilde{j}_\mu^\beta (y). \tag{118}
\]

**Proof.** We have to evaluate

\[
\mathcal{M}^{(FGB)}_{\alpha \beta} = \frac{1}{2} (\Gamma_{u_{out} u_{in}} + \Gamma_{u_{in} u_{out}}) - \frac{1}{2} (B_{IR}^{in} + B_{IR}^{out}). \tag{119}
\]

It suffices to prove that

\[
e^2 \mathcal{M}^{(FGB)}_{\alpha \beta} = \frac{i e^2}{2} \int_{k > \lambda} \frac{d^4k}{(2\pi)^4} \tilde{j}_\mu \Delta^+ (k) \tilde{j}_\mu (k). \tag{120}
\]

with

\[
\tilde{j}_\mu (k) = i \tilde{\rho} (k) \left( \frac{u_{in}^\mu}{u_{in} \cdot k} + \frac{u_{out}^\mu}{u_{out} \cdot k} \right) \equiv \tilde{j}_{in}^\mu (k) + \tilde{j}_{out}^\mu (k) \tag{121}
\]

the Fourier transform of the four-current (77). We can express (117) as

\[
-\frac{e^2}{2} B_{IR}^{out} = \frac{e^2}{2} \int_{k > \lambda} \frac{d^4k}{(2\pi)^4} \tilde{j}_{out, \mu} \Delta^+(k) \tilde{j}_{out}^\mu (k), \tag{122}
\]

recalling (111), and write (116) as

\[
e^2 \Gamma_{u_{out} u_{in}} = i e^2 \int_{k > \lambda} \frac{d^4k}{(2\pi)^4} \tilde{j}_{out, \mu} \Delta^+(k) \tilde{j}_{in}^\mu (k); \tag{123}
\]

hence (120) immediately follows. \qed

**Remark 2.** The contributions of the infrared diagrammatic rules to the r.h.s. of (137) which are specific to the FGB gauge are those given by the vertex, by the electron propagator and by the coefficient $-\frac{g_{\mu\nu}}{2}$ arising from the tensor structure of the photon propagator. Therefore, in order to compute the amplitude associated to the same Feynman diagrams in a different gauge it suffices to replace the above contributions by those pertaining to that gauge.

Consider the insertion of an order-$e^2$ unrenormalized self-energy loop on the outgoing line of $D$ and let $e^2 \Sigma^{(\epsilon)}_{\mu\nu}(u_{out})$ be the associated amplitude, obtained by adopting the expression $i / (u \cdot k + i \epsilon)$ for the fermion propagator of the loop; the infrared contribution (122) is related to the renormalization counterterms introduced in (107) through the following.
Lemma 2. The FGB-gauge second-order infrared radiative corrections to the process $\alpha \rightarrow \beta$, which arise from on-shell renormalization prescriptions and are associated to the outgoing line of $\mathcal{D}$, can be obtained as the limit
\[
\lim_{\epsilon \to 0} \frac{i e^2}{2 \epsilon} \Sigma^{(\epsilon)}(u_{\text{out}}) = -\frac{e^2}{2} \frac{\partial}{\partial \left( \frac{i \epsilon}{2} \right)} \sum_{\text{unr}}^{(\epsilon)}(u_{\text{out}}) \bigg|_{\epsilon = 0} = -\frac{e^2}{2} B_{\text{IR}}^\text{out},
\]
(124)
\[
\Sigma^{(\epsilon)}(u_{\text{out}}) = \sum_{\text{unr}}^{(\epsilon)}(u_{\text{out}}) + u_{\text{out}}^2 z_2(u_{\text{out}}).
\]
(125)

Proof. The FGB-gauge infrared diagrammatic rules and the method of residues yield\(^5\)
\[
i e^2 \sum_{\text{unr}}^{(\epsilon)}(u_{\text{out}}) \equiv e^2 u_{\text{out}}^2 \int_{k > \lambda} \frac{d^4k}{(2\pi)^4} \rho^2(k) \frac{i}{-u_{\text{out}} \cdot k + i \epsilon} i \Delta^+(k).
\]
(126)

It then suffices to compute the sum \([125]\) and to take the limit \([124]\). \(\square\)

The role of the renormalization counterterms in reproducing the infrared radiative corrections of QED to all orders is conveniently displayed by explicitly writing down their contributions to the S-matrix \([114]\). Notice first that, recalling \([99]\) and \([100]\), we can write
\[
\Psi_{\lambda, u_{\text{out}}, \text{unr}}(t) = \exp\left(-i e^2 z_2(u_{\text{out}}) u_{\text{out}}^2 \frac{e^{-2 \epsilon \epsilon} - 1}{2 \epsilon}\right) \Psi_{\lambda, u_{\text{out}}, \text{unr}}(0).
\]
where $\Psi_{\lambda, u_{\text{out}}, \text{unr}}(t)$ is the evolution operator for positive times, associated to the adiabatic (unrenormalized) Hamiltonian \([135]\). In order to obtain the corresponding result for $t < 0$, it suffices to replace $u_{\text{out}}$ by $u_{\text{in}}$ and $\epsilon$ by $-\epsilon$ in the above expressions.

With the same notations employed in Proposition \([4]\), we introduce the unrenormalized Möller operators, for a fixed value of $\epsilon$,
\[
\Omega^{(\lambda), (\pm \epsilon)}_{\mp, u_{\pm}, \text{unr}} = \tau_\omega - \lim_{t \to \pm \infty} \Psi_{\lambda, u_{\text{out}}, \text{unr}}(t).
\]
(128)

The renormalized Möller operators \([110]\) can then be written as
\[
\Omega^{(\lambda), (\pm \epsilon)}_{\mp, u_{\pm}, \text{unr}} = \exp\left(\pm i e^2 u_{\pm}^2 \frac{z_2(u_{\pm})}{2 \epsilon}\right) \Omega^{(\lambda), (\pm \epsilon)}_{\mp, u_{\pm}, \text{unr}},
\]
(129)
whence
\[
S^{(\text{FGB}), (\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}} = \exp\left(-i e^2 u_{\text{out}}^2 \frac{z_2(u_{\text{out}})}{2 \epsilon}\right) S^{(\text{FGB}), (\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}, \text{unr}} \exp\left(-i e^2 u_{\text{in}}^2 \frac{z_2(u_{\text{in}})}{2 \epsilon}\right),
\]
(130)
with
\[
S^{(\text{FGB}), (\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}, \text{unr}} \equiv \Omega^{(\lambda), (\epsilon)}_{\mp, u_{\text{out}}, \text{unr}} \Omega^{(\lambda), (-\epsilon)}_{\pm, u_{\text{in}}, \text{unr}}
\]
(131)
the corresponding unrenormalized scattering matrix.

Lemma \([2]\) suggests that it should be possible to recover the exponentiation of the correction \([117]\) within a Hamiltonian framework, by taking into account the effect to all orders of the counterterm with coefficient $z_2(u_{\text{out}})$. The following Proposition shows that this is indeed the case.

Proposition 5. One has
\[
\langle \Psi_0, \Omega^{(\lambda), (\epsilon)}_{-\epsilon, u_{\text{out}}, \text{unr}} \Psi_0 \rangle = \exp\left(\frac{i e^2}{2} \int \lambda d^4x d^4y j^{(\epsilon)}_{\text{out}, \mu}(x) \Delta_{\epsilon^2}(x-y) j^{(\epsilon)}_{\text{out}, \mu}(y)\right).
\]
(132)

Upon including the contribution of the renormalization counterterm, as given in \([129]\), evaluating the vacuum expectation value of the resulting operator and taking the adiabatic limit, one obtains the overall contribution to the infrared radiative corrections to the process $\alpha \rightarrow \beta$, which arises from on-shell renormalization prescriptions and is associated to the outgoing line of $\mathcal{D}$:
\[
\lim_{\epsilon \to 0} \langle \Psi_0, \Omega^{(\lambda), (\epsilon)}_{-\epsilon, u_{\text{out}}} \Psi_0 \rangle = \exp\left(-\frac{e^2}{2} B_{\text{IR}}^\text{out}\right).
\]
(133)

\(^5\)Regarding the definition of the $\Sigma$-function, we adopt the convention of \([139]\).
Proof. By evaluating the evolution operator \( \mathcal{U}_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda)} (t) \) for positive times, in terms of the interaction Hamiltonian of the model as given in (95), and taking the asymptotic limit, we obtain

\[
\Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda), (\epsilon)} = \mathcal{G}_{u_{\text{out}}}^{(\epsilon)} \exp \left( i e \int_{\lambda} d^4 x \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, A^\mu (x) \right),
\]

whence

\[
\mathcal{G}_{u_{\text{out}}}^{(\epsilon)} \equiv \lim_{t \to +\infty} \mathcal{G}_{u_{\text{out}}}^{(\epsilon)} (t) = \exp \left( - \frac{i e^2}{2} \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right),
\]

where

\[
\langle \Psi_0, \Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda), (\epsilon)} \Psi_0 \rangle = \exp \left( - \frac{\epsilon^2}{2} \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta^+ (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right).
\]

Since

\[
\theta (x_0 - y_0) \Delta^+ (x - y) = \theta (x_0 - y_0) \Delta^+ (x - y) + \left[ \theta (y_0 - x_0) - 1 \right] \Delta^+ (y - x),
\]

one gets (133).

By virtue of (102), in order to prove (133) it is enough to show that

\[
\langle \Psi_0, \Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda)} \Psi_0 \rangle = \exp \left( - \frac{\epsilon^2}{2} \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta^+ (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right).
\]

Since

\[
\theta (x_0 - y_0) \Delta^+ (x - y) = \theta (x_0 - y_0) \Delta^+ (x - y) + \left[ \theta (y_0 - x_0) - 1 \right] \Delta^+ (y - x),
\]

one gets (133).

By virtue of (102), in order to prove (133) it is enough to show that

\[
\langle \Psi_0, \Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda)} \Psi_0 \rangle = \exp \left( - \frac{\epsilon^2}{2} \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta^+ (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right).
\]

Recalling (129), we can write

\[
\Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda), (\epsilon)} = \mathcal{G}_{u_{\text{out}}}^{(\epsilon)} \exp \left( i \epsilon^2 \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta^+ (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right)
\]

whence

\[
\langle \Psi_0, \Omega_{-\lambda, u_{\text{out}}, u_{\text{in}}}^{(\lambda)} \Psi_0 \rangle = \exp \left( - \frac{\epsilon^2}{2} \int_{\lambda} d^4 x \, d^4 y \, j_{\text{out}, \mu}^{(\epsilon)} (x) \, \Delta^+ (x - y) \, j_{\text{out}, \mu}^{(\epsilon)} (y) \right).
\]

Taking the Fourier transform of the r.h.s. of (140), we obtain (133).

Remark 3. Proposition 5 and Lemma 2 make it clear that the contribution of the renormalization counterterm is instrumental in order to obtain an operator formulation of the infrared corrections related to the outgoing line. The same reasoning applies to the incoming-line expressions.

Employing (144) and the corresponding expression for \( \Omega_{+\lambda, u_{\text{in}}, u_{\text{out}}}^{(\lambda), (\epsilon)} \), the scattering matrix (115) can be written as

\[
S_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\text{FGB}), (\epsilon)} = \mathcal{G}_{u_{\text{in}}}^{-1} \mathcal{G}_{u_{\text{out}}} \exp \left( - i \epsilon^2 \int_{\lambda} d^4 x \, j_{\text{in}, \mu}^{(\epsilon)} (x) \, A^\mu (x) \right)
\]

whence

\[
S_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\text{FGB})} = S_{\lambda} [ j^{(\epsilon)} ].
\]
Proposition 6. Taking the vacuum expectation value of the unrenormalized scattering operator \( \mathcal{F}_\Lambda \), for fixed \( \epsilon \) and \( \lambda \), one obtains

\[
\langle \Psi_0, S^{\{FGB\}}_\lambda (\epsilon) \Psi_0 \rangle = \exp \left( \frac{ie^2}{2} \int d^4 x \, d^4 y \, j^{(\epsilon)}_\mu (x) \, \Delta_F (x - y) \, j^{(\epsilon)\mu} (y) \right).
\]

The overall soft-photon radiative corrections to the process \( \alpha \to \beta \) in the FGB gauge are given by

\[
\lim_{\epsilon \to 0} \langle \Psi_0, S^{\{FGB\}}_\lambda (\epsilon) \Psi_0 \rangle = \exp \left( e^2 \mathcal{M}^{\{FGB\}}_\lambda, \beta \alpha \right),
\]

for each value of the low-energy cutoff \( \lambda \).

Proof. The proof of (144) requires to repeatedly apply the indefinite-metric generalization of (25) and to recall (132), (96) and (108). By virtue of (114) and (143), in order to establish (145) it suffices to prove

\[
\langle \Psi_0, S_\lambda [j] \Psi_0 \rangle = \exp \left( e^2 \mathcal{M}^{\{FGB\}}_\lambda, \beta \alpha \right).
\]

Proceeding as for the evaluation of (140), we obtain

\[
\langle \Psi_0, S_\lambda [j] \Psi_0 \rangle = \exp \left( \frac{ie^2}{2} \int d^4 x \, d^4 y \, j_{\text{out}, \mu} (x) \, \Delta_\mu^+ (x - y) \, j_{\text{in} \mu} (y) \right).
\]

Recalling Lemma \( \text{L} \), the result is proved.

\[ \square \]

Remark 4. As it might already be clear from Proposition 6, the renormalization counterterms do not play a role in reproducing the exponentiation of the \( \Gamma \)-amplitudes which contribute to (144). We may also directly check this fact by showing that the term on the r.h.s. of (145) which is associated to the emitted photon and carrying out the momentum-space integration, one obtains the perturbative result. We turn now to the analysis of the infrared corrections to the process \( \alpha \to \beta \) which are associated to the emission of soft photons. Let \( \mathcal{H}_{\text{phys}} \) be the Hilbert space of (physical) free-photon states\(^6\) and let \( \Psi_f \equiv a_\epsilon (f) \Psi_0 \in \mathcal{H}_{\text{phys}} \) describe a “soft-photon state”; namely, \( f^\mu (k) \) is selected in such a way that the approximations leading to the infrared Feynman rules are justified; for instance, we may choose \( f^\mu \) with support contained in a sphere of radius \( r \leq \Lambda \), \( \Lambda \) being the energy scale introduced in (107).

We first prove a Lemma concerning the operator formulation of the correction due to the emission of a single soft photon.

Lemma 3. The FGB-gauge correction to the process \( \alpha \to \beta \) associated to the emission of one photon in the state \( \Psi_f \) is reproduced by the matrix element

\[
\langle \Psi_f, \exp \left( -ie \int d^4 x \, j_\mu (x) \, A^\mu (x) \right) \Psi_0 \rangle.
\]

Proof. With the aid of the commutation relations (4), we obtain

\[
\langle \Psi_f, \exp \left( -ie \int d^4 x \, j_\mu (x) \, A^\mu (x) \right) \Psi_0 \rangle = -ie \int_{k > \lambda} d^3 k \, k_\mu \, \Gamma_\mu (k) \frac{f^\mu (k)}{(2\pi)^{3/2} \sqrt{2} k}. \tag*{(149)}
\]

The expression

\[
\Gamma_\mu (k) \equiv -ie \, j_\mu (k) \big|_{k^0 = k}.
\]

gives the infrared contribution due to the new vertex and to the extra electron propagator of \( \mathcal{D} \) (as evaluated for instance in [10], § 16.1). Taking into account the function \( f^\mu (k)/\left((2\pi)^{3/2} \sqrt{2} k\right) \) associated to the emitted photon and carrying out the momentum-space integration, one obtains the perturbative result.\[ \square \]
In the same sense as in the one-photon case, let \( \Psi_{f_1 \ldots f_n} \equiv \alpha^\dagger(f_1) \ldots \alpha^\dagger(f_n) \Psi_0 \in \mathcal{H}_{\text{phys}} \) be a state describing \( n \) soft photons. The contribution associated to emission in such a state is recovered, in the presence of an infrared cutoff \( \lambda \) on the momentum of the photon, through the following

**Corollary 1.** The FGB-gauge correction to the process \( \alpha \rightarrow \beta \) due to the emission of \( n \) photons in the state \( \Psi_{f_1 \ldots f_n} \) is reproduced by the matrix element

\[
\langle \Psi_{f_1 \ldots f_n}, \exp \left( -i e \int_\lambda d^4x \ j_\mu (x) \ A_\mu^\alpha (x) \right) \Psi_0 \rangle.
\]

**Proof.** Taking into account (139), we find

\[
\langle \Psi_{f_1 \ldots f_n}, \exp \left( -i e \int_\lambda d^4x \ j_\mu (x) \ A_\mu^\alpha (x) \right) \Psi_0 \rangle = (-1)^n \Pi_{j=1}^{\eta_2 = 1} \int_{k_j > \lambda} \frac{d^3k_j}{(2 \pi)^{3/2} \sqrt{2k_j^0}} \Psi_0 \approx \Psi_1 \approx \Psi_{f_1 \ldots f_n}.
\]

**Proposition 7.** In the FGB gauge, the overall contribution given by the soft-photon radiative corrections to the process \( \alpha \rightarrow \beta \) and by the corrections associated to the emission of \( n \) photons in the state \( \Psi_{f_1 \ldots f_n} \) is reproduced by the matrix element

\[
\langle \Psi_{f_1 \ldots f_n}, S_{\lambda, \mu_{\text{out}}, \mu_{\text{in}}}^{(\text{FGB})} \Psi_0 \rangle.
\]

**Proof.** With the aid of formula (25), we can write

\[
S_{\lambda, \mu_{\text{out}}, \mu_{\text{in}}}^{(\text{FGB})} = \exp \left( -\frac{e^2}{2} \int_\lambda d^4x d^4y \ j_\mu (x) \ [A_\mu^\alpha (x), A_\nu^\beta (y)] \ j_\nu (y) \right)
\]

\[
\times \exp \left( -i e \int_\lambda d^4x \ j_\mu (x) \ A_\mu^\alpha (x) \right) \exp \left( -i e \int_\lambda d^4x \ j_\mu (x) \ A_\mu^\alpha (x) \right)
\]

on \( \mathcal{H}_{\text{phys}} \), hence

\[
\langle \Psi_{f_1 \ldots f_n}, S_{\lambda, \mu_{\text{out}}, \mu_{\text{in}}}^{(\text{FGB})} \Psi_0 \rangle = \exp \left( \frac{i e^2}{2} \int_\lambda d^4x d^4y \ j_\mu (x) \ \Delta^+ (x - y) \ j_\mu (y) \right)
\]

\[
\times \langle \Psi_{f_1 \ldots f_n}, \exp \left( -i e \int_\lambda d^4x \ j_\mu (x) \ A_\mu^\alpha (x) \right) \Psi_0 \rangle.
\]

By Lemma II and Corollary II,

\[
\langle \Psi_{f_1 \ldots f_n}, S_{\lambda, \mu_{\text{out}}, \mu_{\text{in}}}^{(\text{FGB})} \Psi_0 \rangle = \exp \left( e^2 \mathcal{H}_{\lambda, \beta, \alpha}^{(\text{FGB})} \right) (-1)^n \Pi_{j=1}^{\eta_2 = 1} \int_{k_j > \lambda} \frac{d^3k_j}{(2 \pi)^{3/2} \sqrt{2k_j^0}}
\]

\[
\times \sum_{r=1}^{\eta_2 = 1} \eta_r e \tilde{\rho} (k_j) \frac{u_{r, \mu_j} A_\mu^\beta (k_j)}{u_r \cdot k_j}.
\]

The factorization property discussed above is thus recovered; moreover, the result (155) agrees with the expression obtained in perturbation theory (equation (13.3.1) in [Wein]).

The infrared phases occurring in the transition amplitude for a process involving at least two charged particles, in either the initial or the final state, can also be recovered by explicit calculations, in terms of Möller operators corresponding to the sum of one-particle Hamiltonians. For the sake of conciseness, we do not report the details.
We conclude the comparison of the BN model with the perturbative treatment of infrared QED in the FGB gauge with a number of observations. First, the proof of the exponentiation of the low-energy corrections arising from on-shell renormalization prescriptions is quite streamlined within the Hamiltonian formalism, while it is more cumbersome within the diagrams approach, requiring in particular the identification of relevant terms and the application of Ward’s identity ([JR, §16.1]). Likewise, the proof of the factorization property of the contributions due to the emission of low-energy photons with respect to the soft-photon radiative corrections, while straightforward within an operator formulation, requires a careful treatment in perturbation theory; as pointed out in [Wein, §13.1], one needs in fact to show that certain on-shell amplitudes are not affected by radiative corrections.

The rest of this Section is devoted to establish an operator formulation of infrared QED in the Coulomb gauge. As in [150], let us factor out the contributions of the renormalization counterterms to the electron propagator and the projection of the scattering operator (92),

\[ S^{(\text{Coul}), (\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}} = \exp \left( i \epsilon^2 \frac{u_{\text{out}}^2}{2 \epsilon} \right) S^{(\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}, \text{unr}} \exp \left( i \epsilon^2 \frac{u_{\text{in}}^2}{2 \epsilon} \right), \quad (156) \]

with \( S^{(\text{Coul}), (\epsilon)}_{\lambda, u_{\text{out}}, u_{\text{in}}, \text{unr}} \) the corresponding unrenormalized scattering matrix. Proceeding as for the derivation of (143), the \( S \)-matrix ([12]) can be written as

\[ S^{(\text{Coul})}_{\lambda, u_{\text{out}}, u_{\text{in}}} = \exp \left( i \int_{\lambda} d^4 x \ j_C(x) \cdot A_C(x) \right) \equiv S_{\lambda} [ j_C ] = s \lim_{\epsilon \to 0} S_{\lambda} [ j_C^{(\epsilon)} ] . \quad (157) \]

The Coulomb-gauge Feynman rules which are relevant for the computation of the soft-photon corrections to the process \( \alpha \to \beta \) are the following: A factor \( i e \bar{\rho}(k) u^\gamma \) has to be associated to a vertex with space index \( r \), occurring on an external charged line carrying four-velocity \( u \), and a contribution \( i \Delta_F(k) P^{rs}(k) \) must be supplied for a photon line carrying four-momentum \( k \) and joining two vertices with indices \( r \) and \( s \). We use the symbol

\[ i \nu \Delta_F^{rs}(x) \equiv \frac{i}{(2 \pi)^4} \int d^4 k \ e^{-i k \cdot x} \Delta_F(k) P^{rs}(k) \]

(158)

for the space-time expression of the photon propagator. The diagrammatic rule pertaining to the electron propagator is the same as in the FGB gauge. In the sequel, we state and prove the main results.

**Lemma 4.** The order-\( \epsilon^2 \) soft-photon radiative corrections to the process \( \alpha \to \beta \), obtained in the Coulomb gauge with the aid of the infrared Feynman rules, can be written as

\[ e^2 \mathcal{M}^{(\text{Coul})}_{\lambda, \beta \alpha} = \frac{-i \epsilon^2}{2} \int_{\lambda} d^4 x \ d^4 y \ \Delta^+(x-y) \ j_C(x) \cdot j_C(y) . \quad (159) \]

**Proof.** It suffices to prove that

\[ e^2 \mathcal{M}^{(\text{Coul})}_{\lambda, \beta \alpha} = \frac{-i \epsilon^2}{2} \int_{\lambda} \frac{d^4 k}{(2 \pi)^4} \ \Delta^+(k) \ \tilde{j}_C(-k) \cdot \tilde{j}_C(k) , \quad (160) \]

with \( \tilde{j}_C(k) \) the Fourier transform of the current (125). In the evaluation of the Feynman amplitude \( \mathcal{M}^{(\text{Coul})}_{\lambda, \beta \alpha} \), the infrared (Coulomb-gauge) diagrammatic rules corresponding to the vertex and to the electron propagator and the projection \( P^{rs}(k) \), associated to the tensor structure of the photon propagator, contribute a term \( -e^2 \tilde{j}_C^{(r)}(-k) \cdot \tilde{j}_C^{(s)}(k) \), where \( \tilde{j}_C^{(r)}(k) \equiv \tilde{j}_C^{(r)}(k_0 = k, k) \),

\[ \tilde{j}_C^{(r)}(k) = i \bar{\rho}(k) P^{rs}(k) \left( \frac{u_{\text{in}}}{-u_{\text{in}} \cdot k + i \epsilon} + \frac{u_{\text{out}}}{u_{\text{out}} \cdot k + i \epsilon} \right) \]

(161)

being the Fourier transform of the current \( j_C^{(r)}(x) \). Recalling Remark [2] and ([11]) and taking the adiabatic limit, we obtain (160). \( \Box \)

---

\(^7\)In the case of a process involving a single charged particle, to be considered below, the Coulomb interaction merely provides an ultraviolet divergent self-energy contribution, which can be removed with the aid of a suitable renormalization counterterm and is in any case infrared finite; for this reason we have not introduced the associated term in the Coulomb-gauge Hamiltonian and shall not need the corresponding diagrammatic rule.
The proof of (162) follows the same steps as that of (144). In order to establish (163), it suffices to recall (157) and to proceed as for the proof of (146).

**Proposition 8.** The vacuum expectation value of the unrenormalized Coulomb-gauge $S$-matrix for fixed $\epsilon$ and $\lambda$ is given by

$$ (\Psi_F, S_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\text{Coul})}(\epsilon) \Psi_F) = \exp \left( -\frac{i e^2}{2} \int_\lambda d^4 x \: d^4 y \: \mathbf{J}_C^{(\epsilon)}(x) \: \mathbf{J}_C^{(\epsilon)^*}(y) \right). $$

For each value of the cutoff $\lambda$, the overall soft-photon radiative corrections to the process $\alpha \rightarrow \beta$ in the same gauge are reproduced by the adiabatic limit of the vacuum expectation value of the renormalized $S$-matrix (156):

$$ \lim_{\epsilon \rightarrow 0} (\Psi_F, S_{\lambda, u_{\text{out}}, u_{\text{in}}}^{(\text{Coul})}(\epsilon) \Psi_F) = \exp \left( e^2 \mathcal{M}_{\lambda, \beta \alpha}^{(\text{Coul})} \right). \tag{163} $$

**Proof.** The proof of (162) follows the same steps as that of (144). In order to establish (163), it suffices to recall (157) and to proceed as for the proof of (146).

We turn now to the corrections due to soft-photon emission. Let $\Phi_f \equiv a^*(f) \Psi_F$ be the vector belonging to the Coulomb-gauge space of free photon states $\mathcal{H}_C$ and indexed by the test function $f(k) \in L^2(\mathbb{R}^3)$, $k \cdot f(k) = 0$. As in the analysis carried out in the $FGB$ gauge, we suppose that $\Phi_f$ describes a (free) soft-photon state. The contribution associated to the emission of one photon in such a state is recovered, in the presence of a low-energy cutoff $\lambda$ on the photon momentum, through the following

**Lemma 5.** The Coulomb-gauge correction to the process $\alpha \rightarrow \beta$ due to the emission of one photon in the state $\Phi_f$ is reproduced by the matrix element

$$ (\Phi_f, \exp( i e \int_\lambda d^4 x \: \mathbf{J}_C(x) \cdot \mathbf{A}_{C,-}(x) ) \Psi_F). \tag{164} $$

**Proof.** By means of the CCR, we obtain

$$ (\Phi_f, \exp( i e \int_\lambda d^4 x \: \mathbf{J}_C(x) \cdot \mathbf{A}_{C,-}(x) ) \Psi_F) = \int_{k > \lambda} d^3k \: \Gamma_C^f(k) \frac{\tilde{f}^r(k)}{(2\pi)^{3/2} \sqrt{2k}}. \tag{165} $$

The coefficient function

$$ \Gamma_C^f(k) \equiv i e \mathbf{J}_C^f(k) \big|_{k=\tilde{k}} $$

reproduces the infrared contribution due to the new vertex and to the extra electron propagator of $D_1$, according to the Coulomb-gauge diagrammatic rules discussed above. Taking into account the function $\tilde{f}^r(k)/[(2\pi)^{3/2} \sqrt{2k}]$ associated to the emitted photon and carrying out the momentum-space integration, we recover the perturbative expression.

Letting $\Phi_{f_1 \ldots f_n} \equiv a^*(f_1) \ldots a^*(f_n) \Psi_F \in \mathcal{H}_C$ be a state describing $n$ soft photons, the correction due to the emission of photons in such a state is reproduced by the following

**Corollary 2.** The Coulomb-gauge correction to the process $\alpha \rightarrow \beta$ associated to the emission of $n$ photons in the state $\Phi_{f_1 \ldots f_n}$ is given by the matrix element

$$ (\Phi_{f_1 \ldots f_n}, \exp( i e \int_\lambda d^4 x \: \mathbf{J}_C(x) \cdot \mathbf{A}_{C,-}(x) ) \Psi_F). \tag{166} $$

**Proof.** Taking into account (165), we find

$$ (\Phi_{f_1 \ldots f_n}, \exp( i e \int_\lambda d^4 x \: \mathbf{J}_C(x) \cdot \mathbf{A}_{C,-}(x) ) \Psi_F) = \Pi_{j=1}^n \int_{k_j > \lambda} \frac{d^3k_j}{(2\pi)^{3/2} \sqrt{2k_j}} \sum_{r=1}^2 \eta_r e^\rho(k_j) \frac{u_r \cdot f_j(k_j)}{u_r \cdot k_j}, \tag{167} $$

with $u_1 \equiv u_{\text{in}}$, $u_2 \equiv u_{\text{out}}$, $\eta_2 = 1 = -\eta_1$. 

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Finally, we state the Coulomb-gauge counterpart of Proposition 7.

Proposition 9. In the Coulomb gauge, the overall contribution given by the soft-photon radiative corrections to the process $\alpha \to \beta$ and by the corrections associated to the emission of $n$ photons in the state $\Phi f_1 \cdots f_n$ is reproduced by the matrix element

$$ \langle \Phi f_1 \cdots f_n , S_{\lambda}^{(Coul)} u_{\text{out}} u_{\text{in}} \Psi_F \rangle . $$

(168)

Proof. The proof is the same as that of Proposition 7. \qed

It is straightforward to check that if we choose the vector $\Psi f_1 \cdots f_n \in H_{\text{phys}}$ in Corollary 1 to be indexed by (square-integrable) smearing functions $f_{\mu}^{j} (k_{j}) = (0, f_{j} (k_{j}))$, $j = 1, \ldots, n$, the correction (151) equals the Coulomb-gauge expression (167). A more intrinsic formulation of the invariance properties of the soft-photon corrections to the process $\alpha \to \beta$ with respect to the gauge employed in their calculations, not relying in particular upon a specific choice of test functions, is indeed possible and will be given in the forthcoming Section.

3 Infrared Approximations And Invariance Properties

This Section is devoted to the discussion of the invariance properties of the soft-photon corrections with respect to the gauge chosen for their evaluation. For definiteness, we shall compare the expressions obtained in the $FGB$ gauge with those calculated in the Coulomb gauge; by a similar treatment it is possible to include other local and covariant formulations, such as those employing $\xi$-gauges (discussed for example in BLOT, § 10.2). Next Proposition contains the main result of this Section; relying on the existence of an (isometric) isomorphism $T$ between $H_{\text{phys}}$ and $H_{C}$ (Lemma 11, Appendix C), we prove that the restriction of the photon scattering operator (114) to $H_{\text{phys}}$ is unitarily equivalent to the Coulomb-gauge $S$-matrix (91).

Proposition 10. For each value of the infrared cutoff $\lambda$, the equality

$$ S_{\lambda}^{(FGB)} \bigg|_{H_{\text{phys}}} = T^{-1} S_{\lambda}^{(Coul)} T $$

(169)

holds.

Proof. By virtue of (157) and of the transversality of the (free) Coulomb-gauge vector potential (3), the r.h.s. of (169) can be expressed as

$$ T^{-1} S^{(Coul)}_{\lambda, u_{\text{out}} u_{\text{in}}} T = \exp \left( i e \int_{\lambda} d^{4} x \ j (x) \cdot A_T (x) \right) , $$

(170)

with $A_T (x) \equiv T^{-1} A_C (x) T$. Lemma 12 in Appendix C (choosing the components of $j^{\mu}$, defined in (77), as smearing functions in (215)) and Stone’s theorem\(^8\) yield

$$ T^{-1} S_{\lambda, u_{\text{out}} u_{\text{in}}}^{(Coul)} T = \exp \left( -i e \int_{\lambda} d^{4} x j_{\mu}^{(c)} (x) \left[ A^{\mu} (x) - \partial^{\mu} \int d^{3} y \ G_{y} (x) \times ( \partial_{m} A^{m} ) (x_{0}, y) \right] \right) |_{H_{\text{phys}}} . $$

(171)

Making use of the continuity equation for $j^{\mu} (x)$ the Proposition is proved. \qed

As a straightforward consequence of Propositions 6, 8, 10, we obtain the following

Corollary 3. The overall soft-photon radiative corrections to the process $\alpha \to \beta$ take the same value in the $FGB$ gauge and in the Coulomb gauge, for each value of the infrared cutoff $\lambda$.

In order to give an unambiguous formulation of the invariance properties of low-energy corrections including the effect of soft-photon emission, it is convenient to introduce a notion of corresponding states.

\(^8\)Stone’s theorem can be applied since $H_{\text{phys}}$ is a Hilbert space.
Definition. \( \Phi \in \mathcal{H}_C \) is the state corresponding to \( \Psi \in \mathcal{H}_{\text{phys}} \) if \( \Phi = T \Psi \).

Propositions 7, 9, 10 lead then to the following

**Corollary 4.** The FGB-gauge contribution associated to the overall soft-photon radiative corrections to the process \( \alpha \rightarrow \beta \) and to the corrections due to the emission of \( n \) photons in the state \( \Psi_{f_1 \ldots f_n} \in \mathcal{H}_{\text{phys}} \) equals the Coulomb-gauge contribution given by the overall soft-photon radiative corrections to the same process and by the corrections associated to photon emission in the state corresponding to \( \Psi_{f_1 \ldots f_n} \).

We are in a position to discuss the lack of invariance of the soft-phot on corrections, in the presence of the approximations of the model studied in Section 1. As a consequence of Proposition 10 we obtain the following result, which will be relevant in the subsequent analysis.

**Corollary 5.** Let \( f^\mu = (0^0, f) \) be chosen in such a way that the exponential operators

\[
O_\lambda [ f ] \equiv \exp \left( -ie \int_\lambda d^4 x \, f^\mu (x) \, A^\mu (x) \right),
\]

\[
O_\lambda [ f_C ] \equiv \exp \left( ie \int_\lambda d^4 x \, f_C (x) \cdot A (x) \right),
\]

with

\[
f_C (t, x) \equiv \int d^3 y \, \delta (x - y) \, f^m (t, y),
\]

are well-defined, respectively on \( \mathcal{H} \) and on \( \mathcal{H}_C \), for each value of the infrared cutoff \( \lambda \). The equality

\[
O_\lambda [ f ] = T^{-1} O_\lambda [ f_C ] T \exp \left( -ie \int_\lambda d^4 x \, f^\mu (x) \, A^\mu (x) \right) \times \left( \partial_m A^m \right) (x_0, y)
\]

holds on \( \mathcal{H}_{\text{phys}} \).

**Proof.** By the same reasoning leading to equation (171) and taking into account the commutation relations

\[
[ A^\mu (x), \partial^\nu (\partial_m A^m) (x_0, y) ] = 0,
\]

which result from the CCR, we obtain

\[
T^{-1} O_\lambda [ f_C ] T = \exp \left( -ie \int_\lambda d^4 x \, f^\mu (x) \, A^\mu (x) \right) \bigg|_{\mathcal{H}_{\text{phys}}} \exp \left( ie \int_\lambda d^4 x \, f^\mu (x) \right) \times \left( \partial_m A^m \right) (x_0, y).
\]

In the sequel, besides the standard low-energy assumptions employed in the diagrammatic treatment of infrared \( \text{QED} \), we suppose that the electron is non relativistic and introduce a dipole approximation. The resulting (infrared) Feynman rules are as follows. In the FGB gauge, a term \( -ie \tilde{\rho} (k) \tilde{v}^\mu \) has to be associated to a vertex with index \( \mu \) on a charged-particle line carrying energy-momentum \( p \); the Coulomb-gauge expression to be supplied for a vertex with index \( l \) on the same line is \( ie \tilde{\rho} (k) p^l / m \). In both gauges, an electron propagator carrying four-momentum \( p + k \) contributes a term \( i / (k + ie) \), with \( k \) the overall photon contribution to the energy carried by the propagator line.\(^9\)

First, we compute the resulting expressions for the radiative corrections at order \( e^2 \).

\(^9\)The peculiar functional form of the electron propagator is due to the fact that, in the presence of a dipole approximation, the electron momentum is conserved while the total one is not.
Lemma 6. In the presence of a dipole approximation, the Coulomb-gauge soft-photon radiative corrections to the process $\mathbf{p}_{\text{in}} \to \mathbf{p}_{\text{out}}$ at second order can be written as

$$e^2 \mathcal{M}^{(\text{Coul}), (\text{disp})}_{\lambda, \mathbf{p}_{\text{out}} \mathbf{p}_{\text{in}}} = - \frac{i e^2}{2} \int_{\lambda} d^4 x \ d^4 y \ \Delta^+(x-y) \ j_{C, \text{nr}}^{(\text{disp})}(x) \cdot j_{C, \text{nr}}^{(\text{disp})}(y),$$  \hfill (178) 

where $j_{C, \text{nr}}^{(\text{disp})}(x)$ is the current [155]. The FGB-gauge expression for the same corrections is

$$e^2 \mathcal{M}^{(\text{FGB}), (\text{disp})}_{\lambda, \mathbf{p}_{\text{out}} \mathbf{p}_{\text{in}}} = \frac{i e^2}{2} \int_{\lambda} d^4 x \ d^4 y \ \Delta^+(x-y) \ j_{\text{nr}}^{(\text{disp})}(x) \cdot j_{\text{nr}}^{(\text{disp})}(y),$$  \hfill (179) 

with $j_{\text{nr}}^{(\text{disp})}(x)$ the current [155].

Proof. We first prove that

$$e^2 \mathcal{M}^{(\text{Coul}), (\text{disp})}_{\lambda, \beta \alpha} = - \frac{i e^2}{2} \int_{k > \lambda} \frac{d^4 k}{(2\pi)^4} \ \Delta^+(k) \ j_{C, \text{nr}}^{(\text{disp})}(-k) \cdot \tilde{j}_{C, \text{nr}}^{(\text{disp})}(k),$$  \hfill (180) 

with $\tilde{j}_{C, \text{nr}}^{(\text{disp})}(k)$ the Fourier transform of the current [155]. In the evaluation of the Feynman amplitude $e^2 \mathcal{M}^{(\text{Coul}), (\text{disp})}_{\lambda, \beta \alpha}$, the infrared diagrammatic rules pertaining to the vertex and to the electron propagator and the projection $P^{tr}(k)$ associated to the Coulomb-gauge photon propagator provide a contribution $- e^2 \tilde{j}_{C, \text{nr}}^{(\text{disp})}(x) \cdot \tilde{j}_{C, \text{nr}}^{(\text{disp})}(y)(k)$, with $\tilde{j}_{C, \text{nr}}^{(\text{disp})}(x) \equiv \tilde{j}_{C, \text{nr}}^{(\text{disp})}(-x)(k_0 = |k|, k)$,

$$\tilde{j}_{C, \text{nr}}^{(\text{disp})}(x) = \frac{\tilde{\rho}(k)}{m(k_0 + i\epsilon)} P^{tr}(k)(-\mathbf{p}_{\text{in}} + \mathbf{p}_{\text{out}})$$  \hfill (181) 

being the Fourier transform of the four-current [55]. Proceeding as in the proof of Lemma 4 and recalling [111], we obtain [130].

In order to establish [179], we show that

$$e^2 \mathcal{M}^{(\text{FGB}), (\text{disp})}_{\lambda, \beta \alpha} = - \frac{i e^2}{2} \int_{k > \lambda} \frac{d^4 k}{(2\pi)^4} \ \Delta^+(k) \ j_{\text{nr}}^{(\text{disp})}(-k) \cdot \tilde{j}_{\text{nr}}^{(\text{disp})}(k),$$  \hfill (182) 

with $\tilde{j}_{\text{nr}}^{(\text{disp})}(k)$ the Fourier transform of the current [199]. In the calculation of $e^2 \mathcal{M}^{(\text{FGB}), (\text{disp})}_{\lambda, \beta \alpha}$, the (infrared) diagrammatic rules pertaining to the vertex and to the electron propagator and the coefficient $-g^{\mu\nu}$ associated to the photon propagator yield a term $e^2 \tilde{j}_{\text{nr}}^{(\text{disp})}(-k) \ j_{\text{nr}}^{(\text{disp})}(\mu)(k)$, with $\tilde{j}_{\text{nr}}^{(\text{disp})}(x) \equiv \tilde{j}_{\text{nr}}^{(\text{disp})}(\epsilon)(\mu)(k_0 = |k|, k)$,

$$\tilde{j}_{\text{nr}}^{(\text{disp})}(x) = \frac{\tilde{\rho}(k)}{m} \left( \frac{\tilde{\nu}^\mu_{\text{in}}}{-k_0 + i\epsilon} + \frac{\tilde{\nu}^\mu_{\text{out}}}{k_0 + i\epsilon} \right)$$  \hfill (183) 

being the Fourier transform of the four-current [55]. Recalling [111] and taking the adiabatic limit we obtain [132].

Remark 5. Notice that [155], [178] have the same functional dependence upon the respective Coulomb-gauge currents. In contrast, while [118] depends upon the four-current [77], the amplitude [179] only involves the space components of [47].

With the aid of the Feynman rules discussed above, accounting for the non-relativistic electronic motion and for the introduction of a dipole approximation, and proceeding as in the analysis carried out in [YFS61], one obtains the overall soft-photon radiative corrections to the process $\mathbf{p}_{\text{in}} \to \mathbf{p}_{\text{out}}$ in the form of exponentiation of the second-order results [178], [179]. In order to show that such corrections admit an operator formulation, we need the following preliminar result:

Lemma 7. The scattering matrices [144], [229] can be expressed respectively as

$$S^{(\text{Coul}), (\text{disp})}_{\lambda, \mathbf{p}_{\text{out}} \mathbf{p}_{\text{in}}} = \exp \left( i e \int_{\lambda} d^4 x \ j_{C, \text{nr}}^{(\text{disp})}(x) \cdot \mathbf{A}_C(x) \right),$$  \hfill (184) 

$$S^{(\text{FGB}), (\text{disp})}_{\lambda, \mathbf{p}_{\text{out}} \mathbf{p}_{\text{in}}} = \exp \left( i e \int_{\lambda} d^4 x \ j_{\text{nr}}^{(\text{disp})}(x) \cdot \mathbf{A}(x) \right).$$  \hfill (185) 

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Proof. In order to get (185) it suffices to follow the same steps leading to (184).

Concerning the proof of (185), the explicit expressions (184), (181) yield

$$S_{\lambda, \tilde{v}_{out} \tilde{v}_{in}}^{(FBG), (dip)} = \exp \left( i e \left\{ a \left( \frac{r\tilde{v}_{out}}{r\tilde{v}_{in}} - f \right) + a \left( \tilde{T}_{\tilde{v}_{out}} - \tilde{T}_{\tilde{v}_{in}} \right) \right\} \right),$$

whence we obtain (185), again by inverse Fourier transform.

**Remark 6.** A proof of (185) in which the role of functional form of the four-current (47) is more clearly displayed is as follows. Proceeding as for the derivation of formula (147), we get

$$S_{\lambda, \tilde{v}_{out} \tilde{v}_{in}}^{(FBG), (dip), (\epsilon)} = \tilde{c} \tilde{c}^{-1} \exp \left( - i e^2 \tilde{v}_{out} \frac{z}{2\epsilon} \right) \exp \left( - i e^2 \tilde{v}_{in} \frac{z}{2\epsilon} \right) S_{\lambda} \left[ j^{(dip), (\epsilon)} \right].$$

Let

$$I_{\lambda}^{(\epsilon)} = \int_{\lambda} d^4 x \ j_{nr}^{(dip), (\epsilon)}(x) A_0^0(x).$$

It is straightforward to show that

$$\tau_w = \lim_{\epsilon \to 0} \int d t \ e^{-\epsilon |t|} A_0^0(x, t) = 0.$$

Since $j_{nr}^{(dip), \mu=0}(x) = \rho(x) j_{nr}^{(dip), \mu=0}(x)$, (185) implies $\tau_w = \lim_{\epsilon \to 0} I_{\lambda}^{(\epsilon)} = 0$, hence (185) is proven.

By means of the same calculations leading to the proof of (185) and by Lemma 7 we immediately obtain the following

**Corollary 6.** In the presence of a dipole approximation, the overall soft-photon radiative corrections to the process $p_{in} \to p_{out}$ in the Coulomb gauge are reproduced by the vacuum expectation

$$\langle \Psi_F, S_{\lambda, p_{out} p_{in}}^{(Coul), (dip)} \Psi_F \rangle = \lim_{\epsilon \to 0} \langle \Psi_F, S_{\lambda, p_{out} p_{in}}^{(Coul), (dip), (\epsilon)} \Psi_F \rangle = \exp \left( - i e^2 A_{\lambda, p_{out} p_{in}}^{(Coul), (dip)} \right).$$

for each value of the low-energy cutoff $\lambda$.

Likewise, proceeding as for the proof of (165) and taking again into account Lemma 7, we obtain the corresponding result in the FGB gauge:

**Corollary 7.** In the presence of a dipole approximation, the overall FGB-gauge soft-photon radiative corrections to the process $p_{in} \to p_{out}$ are given by the vacuum expectation

$$\langle \Psi_0, S_{\lambda, \tilde{v}_{out} \tilde{v}_{in}}^{(FBG), (dip)} \Psi_0 \rangle = \lim_{\epsilon \to 0} \langle \Psi_0, S_{\lambda, \tilde{v}_{out} \tilde{v}_{in}}^{(FBG), (dip), (\epsilon)} \Psi_0 \rangle = \exp \left( - i e^2 A_{\lambda, p_{out} p_{in}}^{(FBG), (dip)} \right),$$

for each value of the infrared cutoff $\lambda$.

Finally, we state a Lemma concerning the discrepancy between the vacuum expectations (190), (191).

**Lemma 8.** The following relation holds:

$$\langle \Psi_0, S_{\lambda, \tilde{v}_{out} \tilde{v}_{in}}^{(FBG), (dip)} \Psi_0 \rangle = \langle \Psi_F, S_{\lambda, p_{out} p_{in}}^{(Coul), (dip)} \Psi_F \rangle \exp \left( i e \int_{\lambda} d^4 x \xi(x) \right) \times \int d^3 y \ G_y(x) \left( \partial_m A^m(x, y) \right).$$

$$\xi(x) = \partial_t j_{nr}^{(dip), \mu=0}(x) = \partial_{\mu} j_{nr}^{(dip), \mu=0}(x).$$
Proof. Taking into account Lemma 7 and the transversality of \( \mathbf{j}_{\mathbf{C}, \, \mathbf{nr}}^{(\text{dip})} \), the \( S \)-matrix (44) can be written as

\[
S_{\lambda, \, \mathbf{P}_{\text{out}}, \, \mathbf{P}_{\text{in}}}^{(\text{Coul}), \, (\text{dip})} = \exp \left( i e \int_{\lambda} d^4 x \; \mathbf{j}_{\mathbf{C}, \, \mathbf{nr}}^{(\text{dip})} (x) \cdot \mathbf{A} (x) \right).
\]

(194)

Corollary 5 can thus be applied to the scattering matrices (44), (65); as a result, we obtain

\[
S_{\lambda, \, \mathbf{v}_{\text{out}}, \, \mathbf{v}_{\text{in}}}^{(\text{FGB}), \, (\text{dip})} \big|_{\mathcal{M}_{\text{phys}}} = T^{-1} S_{\lambda, \, \mathbf{P}_{\text{out}}, \, \mathbf{P}_{\text{in}}}^{(\text{Coul}), \, (\text{dip})} T \exp \left( i e \int_{\lambda} d^4 x \; \xi (x) \right)
\times \int d^3 y \; G_y (x) (\partial_m A^m) (x_0, \, y),
\]

(195)

hence (192) follows.

We wish to give a few comments about the above results, already stated without proof and briefly discussed at the end of Section 1.

The first statement in Remark 5 and the exponentiation of the order-\( e^2 \) corrections (178) show that the dipole approximation is compatible with the low-energy assumptions made in the perturbation-theoretic treatment of Coulomb-gauge infrared QED; further, the exponentiation of (178) can be recovered within a Hamiltonian framework (Corollary 6). The overall FGB-gauge soft-photon radiative corrections also admit an operator formulation (Corollary 7); however, a discrepancy arises between the explicit expression of such corrections, equation (191), and the Coulomb-gauge result (190).

The second statement in Remark 5 suggests that such a discrepancy should be due to issues arising in the FGB gauge formulation. By Remark 3 it is in fact related to the functional form of the four-vector current (47), specifically, by virtue of Lemma 8, to its non-conservation.

The violation of the continuity equation in the presence of a dipole approximation was already pointed out in [HiSu09] and shown to cause difficulties with the Gupta-Bleuler condition in a specific non-relativistic model; to the best of our knowledge, its implications for the lack of invariance of the soft-photon corrections have however not been stressed before.

Outlook

By exploiting the Hamiltonian control of the soft-photon contributions to the Feynman-Dyson expansion of QED, achieved through the four-vector Bloch-Nordsieck model, the problem of the removal of the infrared cutoff in the perturbative expressions can also be addressed. In particular, it should be possible to outline the extent and limitations of the recipe leading to infrared-finite cross-sections, by exploiting the fact that the approach here developed allows to reproduce the results of the order-by-order diagrammatic treatment in a compact way. We plan to report on these problems in a future work.

Acknowledgments

The content of this work is a development of part of my Ph.D. thesis at the department of Physics of the University of Pisa. I am grateful to Giovanni Morchio and Franco Strocchi for having suggested to me the problem and for having shared with me ideas stemming from their preliminary analysis of Hamiltonian models, devoted to a better understanding of the role of local and covariant formulations in the treatment of the infrared problem in Quantum Electrodynamics. I am indebted to G. Morchio for extensive discussions on these topics, for his helpful advice and for a number of comments on the manuscript. The suggestions of an anonymous referee are also gratefully acknowledged.
Appendices

A Essential Self-Adjointness Of The Pauli-Fierz Hamiltonian

Let \( b_s (f), f \in L^2 \), stand for either \( a_s (f) \) or \( a^*_s (f) \); by virtue of the standard Fock-space estimate \( \| b_s (f) \Psi \| \leq \| f \|_2 \| (N + \mathbb{1})^{1/2} \Psi \|, \forall \Psi \in \text{Dom} (N^{1/2}) \), one obtains
\[
\| A_{\omega, \lambda} (\rho, x = 0) \Psi \| \leq c (\rho) \| (N + \mathbb{1})^{1/2} \Psi \|, \forall \Psi \in D_{F_0},
\]
with \( c (\rho) \) a (positive) constant, for a given form factor \( \rho \).

The relation \( 2 \| (A \otimes B) \Phi \| \leq \| (A^2 \otimes \mathbb{1} + \mathbb{1} \otimes B^2) \Phi \|, \forall \Phi \in D_0 \), applied to the operators \( A = p, B = (N + \mathbb{1})^{1/2} \), and the estimate \( \| (N + \mathbb{1}) \Psi \| \leq \| (\lambda^{-1} H_0 + \mathbb{1}) \Psi \|, \forall \Psi \in D_{F_0} \), yield the bound
\[
\| H^{(PF)}_\Lambda (\Phi) \| \leq d (e, \lambda; \rho) \| (H_0 + \mathbb{1}) \Phi \|, \forall \Phi \in D_0,
\]
for a suitable \( d (e, \lambda; \rho) \). Further, with the aid of (197) and of the CCR, one finds \( g(e, \lambda; \rho) \) such that \( |(\Phi, [H^{(PF)}_\Lambda, H_0] \Phi)| \leq g(e, \lambda; \rho) \| (H_0 + \mathbb{1})^{1/2} \Phi \|, \forall \Phi \in D_0; H^{(PF)}_\Lambda \) is thus e.s.a. on \( D_0 \), for all \( |e|, \lambda > 0 \), by Nelson’s commutator theorem in the formulation given in [FL74].

B Dynamics Of Solvable Models On Indefinite-Metric Spaces

For models employing four-vector potentials, lack of positivity of the scalar product raises substantial questions on selfadjointness and existence and uniqueness of time evolution. In [HiSu09] such problems have been treated on a slightly different version of the model, and a general framework has been provided for existence and uniqueness of the Heisenberg time-evolution within a Hilbert-space formulation.

Since, however, the formal evolution operators defined by the Hamiltonians \( \{A_s \} \) are exponentials of the canonical variables of the photon field, we adopt a more pragmatic approach; in particular, the evolution operators will be defined in terms of Weyl exponentials of fields, introduced starting from their algebraic relations, on a suitable invariant vector space.

Given a linear non-positive and normalized functional \( \omega \) on a unital \(*\)-algebra \( \mathcal{A} \), fulfilling the hermiticity property \( \omega (A^*) = \omega (A), \forall A \in \mathcal{A} \), it is possible to apply a generalized GNS reconstruction procedure (MS00); as a result, one obtains a non-degenerate indefinite vector space carrying a \(*\) representation of \( \mathcal{A} \), with the involution represented by the indefinite-space adjoint \( \dagger \), and expectations over a cyclic vector representing \( \omega \).

Let now \( \mathcal{A}^{e.m.}_{ext} \) be the unital \(*\)-algebra generated by the photon canonical variables and by variables (Weyl operators in momentum space) \( W(g, h), \) indexed by four-vector real-valued functions in \( L^2 (\mathbb{R}^3) \), fulfilling
\[
W(g, h)^* = W(-g, -h), W(g, h)^* W(g, h) = 1, \quad (198)
\]
\[
W(g, h) W(l, m) = \exp \left( i \left[ \langle g, m \rangle - \langle h, l \rangle \right] \right) W(l, m) W(g, h), \quad (199)
\]
\[
[a(f), W(g, h)] = \frac{i}{\sqrt{2}} \langle f, n \rangle W(g, h), \quad n \equiv g + i h, \quad (200)
\]
with the symbol \( ^* \) standing for the algebra involution and with
\[
\langle f, g \rangle \equiv \langle f^0, g^0 \rangle - \sum_i (f^i, g^i). \quad (201)
\]
Let also \( \mathcal{A}^{e.m.}_{ext, \omega} \) be the algebra generated by the photon variables introduced above, smeared with functions in \( \mathcal{S} (\mathbb{R}^3) \), and denote by \( \omega_F \) a hermitian linear (non-positive) functional on \( \mathcal{A}^{e.m.}_{ext}, \mathcal{A}^{e.m.}_{ext, \omega} \), with Fock-type expectations
\[
\omega_F (a(f_1) a^* (f_2)) = - \langle f_1, f_2 \rangle, \quad (202)
\]
\[
\omega_F (W(g, h)) = \exp \left( \frac{1}{4} \left( \langle g, g \rangle + \langle h, h \rangle \right) \right). \quad (203)
\]
The functional $\omega_F$ is identified by (202), (203), since expectations of monomials of $a$ and $a^*$ can be expressed in terms of (202) with the aid of Wick’s theorem ([Wick50]), while those of monomials of $W$ are fixed by (203) up to a phase factor, given by (199), and the other ones follow from (200).

The spaces $\mathcal{G}$ and $\mathcal{G}_0$ are obtained via generalized GNS constructions, applied respectively to $\omega_F(\mathcal{A} e.m.)$ and $\omega_F(\mathcal{A} e.m., S)$. The space $\mathcal{G}_0$ is weakly dense in $\mathcal{G}$ by density of the Schwartz space in $L^2$ and by the Cauchy-Schwartz inequality, applied component-wise to explicit expression of the inner product.

Alternatively, starting from representation spaces $\mathcal{D}$ obtained by applying the generalized GNS theorem to polynomial $*$-algebras, we might have tried to identify completions of $\mathcal{D}$ containing Weyl exponentials, with the help of a suitable topology. In this respect, we recall that an indefinite-metric space resulting from a (generalized) GNS construction does not admit a unique completion, besides not being complete. In some generality, complete spaces can be constructed as Hilbert-space completions of vector spaces obtained via a GNS procedure and Hilbert-space structures can be relevant for the existence and control of limits (the role of such structures in models of indefinite-metric quantum field theories has been discussed in [MS80]).

A different choice, which is the one adopted in this paper, is to employ a vector space. In the case of models only involving polynomials and exponentials of fields, such a choice looks simpler and even more intrinsic; in particular, a strong topology is solely required in order to formulate and settle uniqueness of time-evolution operators.

In order to prove uniqueness of the evolution operators, we need a preliminary result.

**Lemma 9.** Isometries $U$ of a non-degenerate indefinite space $\mathcal{Z}$ are identified by their restriction to a weakly dense subspace $\mathcal{Y}$.

**Proof.** One has in fact
\[
\langle y, Ux \rangle = \langle U^{-1}y, x \rangle = \lim_n \langle U^{-1}y, x_n \rangle = \lim_n \langle y, Ux_n \rangle,
\]
with $y \in \mathcal{Z}$ and $x_n$ a sequence of elements of $\mathcal{Y}$ (weakly) converging to $x$.

Uniqueness is then a consequence of the following observation, only requiring the existence of a positive scalar product $(\cdot, \cdot)$ majorizing the indefinite product:

**Lemma 10.** Two one-parameter families of isometries $U(a), V(a)$ of a vector space $V_0$, endowed with a non-degenerate indefinite inner product $(\cdot, \cdot)$ admitting a majorizing positive scalar product, coincide if they are differentiable on $V_0$ in the corresponding strong topology, with the same derivative $-i H(a)$.

**Proof.** One has, $\forall x, y \in V_0$,
\[
\frac{d}{da} \langle x, V(-a)U(a)y \rangle = \frac{d}{da} \langle V(a)x, U(a)y \rangle = -i \langle V(a)x, H(a)U(a)y \rangle + i \langle H(a)V(a)x, U(a)y \rangle = 0,
\]
since strong differentiability implies strong continuity and hermiticity of $H(a)$ on $V_0$, which by assumption is invariant under $U(a), V(a)$.

### C  Gupta-Bleuler Quantization Of The Free Electromagnetic Field

Within the Gupta-Bleuler quantization of Electrodynamics ([Gup50, Ble50]), physical states are selected by the linear subsidiary condition\(^{10}\)

\[
(\partial \cdot A)(x) \Psi = 0.
\]

\(^{10}\)The decomposition $(\partial \cdot A)(x) = (\partial \cdot A)^+(x) + (\partial \cdot A)^-(x)$ is well defined, since $(\partial \cdot A)(x)$ is a free field in the $FGB$ gauge.
In the free-field case, the solutions of the subsidiary condition (204) in $\mathcal{G}$ define a subspace

$$\mathcal{H}' = \mathcal{A}_{\text{ext}, \text{tr}}^c \Psi_0,$$

(205)

with $\mathcal{A}_{\text{ext}, \text{tr}}^c$ the algebra generated by the canonical photon variables and their Weyl exponentials, smeared with four-vector test functions $f^\mu(k) \in L^2(\mathbb{R}^3)$ obeying the transversality condition

$$\overline{\mathcal{E}}^\mu f_\mu(k) = 0, \overline{\mathcal{E}}^\mu \equiv (|k|, k).$$

(206)

$\mathcal{H}'$ contains a null space

$$\mathcal{H}'' = (\Psi \in \mathcal{H}', \langle \Psi, \Psi \rangle = 0),$$

(207)

whose vectors result from the application to $\Psi_0$ of those elements of $\mathcal{A}_{\text{ext}, \text{tr}}^c$ which are indexed by test functions $f^\mu(k) = \overline{\mathcal{E}}^\mu h(k).$

Since $\mathcal{H}'$ is endowed with a positive-semidefinite product $\langle \ldots \rangle_\ast \equiv - \langle \ldots \rangle$, the space $\mathcal{H}'/\mathcal{H}''$, obtained by introducing equivalence classes in $\mathcal{H}'$, is a pre-Hilbert space with scalar product $\langle \ldots \rangle_\ast$.

We use the symbol

$$\mathcal{H}_{\text{phys}} \equiv \mathcal{H}'/\mathcal{H}''$$

(208)

for the Hilbert space of photon states obtained by completion in the topology of $\langle \ldots \rangle_\ast$. The states of $\mathcal{H}_{\text{phys}}$ have a simple characterization.

**Lemma 11.** $\mathcal{H}_{\text{phys}}$ is isomorphic as a Hilbert space to $\mathcal{H}_C$, the Coulomb-gauge space of free photon states.

**Proof.** Let $g^\mu(k) \in L^2(\mathbb{R}^3), \overline{\mathcal{E}}_\mu g^\mu(k) = 0, g^\mu_C(k) \equiv g^\mu(k) - (\overline{\mathcal{E}}^\mu/\overline{\mathcal{E}}^0) g^0(k),$ denote the equivalence class of $\Psi_g = a^\dagger(g) \Psi_0$ by $[\Psi_g]$ and let $(\ldots)$ be the scalar product of $\mathcal{H}_C$. It is straightforward to check that

$$\langle \Phi_{g_C}, \Phi_{h_C} \rangle = \langle \tilde{\Psi}_g, \tilde{\Psi}_h \rangle_\ast,$$

(209)

where $\tilde{\Psi}_g \equiv a^\dagger(g_C) \Psi_0$ is a representative of $[\Psi_g]$ and $\Phi_{g_C} \equiv a^\ast(g_C) \Psi_F$ is the vector of $\mathcal{H}_C$ corresponding to the (transverse) vector test function $g_C$; hence

$$\langle \Phi_{g_C}, \Phi_{h_C} \rangle = \langle [\Psi_g], [\Psi_h] \rangle_\ast.$$

(210)

With the same notations as for the one-photon states, let

$$\Psi_{g_1 \ldots g_n} \equiv a^\dagger(g_1) \ldots a^\dagger(g_n) \Psi_0, \Phi_{g_C, 1 \ldots g_C, n} \equiv a^\ast(g_C, 1) \ldots a^\ast(g_C, n) \Psi_F$$

and define the linear map

$$T_0 [\Psi_{g_1 \ldots g_n}] = \Phi_{g_C, 1 \ldots g_C, n},$$

(211)

$$T_0 [\Psi_0] = \Psi_F.$$  

(212)

In order to show that the action of $T_0$ does not depend upon the choice of a representative in each equivalence class, it is enough to notice that $g_C(k) = 0$ for square-integrable four-vector functions of the form $g^\mu(k) = \overline{\mathcal{E}}^\mu h(k),$ which index the state vectors belonging to the null space of $\mathcal{G}$.

The equality (210) for one-photon states implies

$$\langle T_0 [\Psi_{g_1 \ldots g_n}], T_0 [\Psi_{h_1 \ldots h_n}] \rangle = \langle [\Psi_{g_1 \ldots g_n}], [\Psi_{h_1 \ldots h_n}] \rangle_\ast.$$

(213)

By linearity, $T_0$ can be extended to $B_0$, the dense set of $\mathcal{H}_{\text{phys}}$ spanned by the (equivalence classes of) finite-particle vectors of $\mathcal{G}$ which describe transverse photons (namely, indexed by test functions fulfilling (206)); further, by (213) such an extension is isometric.

Finally, by virtue of the B. L. T. theorem ([RSI], Theorem I.7), $T_0$ can be uniquely extended to an unitary operator $T$ from $\mathcal{H}_{\text{phys}}$ to $\mathcal{H}_C$. 

$\blacksquare$
The smeared field $A_T(f) = T^{-1} A_C(f) T : T^{-1} \operatorname{Dom}(A_C(f)) \to \mathcal{H}_{\text{phys}}$ fulfills the Coulomb-gauge (transversality) condition
\begin{equation}
(\partial_l A_T^l)(h) = 0.
\end{equation}

Next Lemma shows how to express $A_T(f)$ in terms of the smeared (free) $FGB$-gauge four-vector potential.

**Lemma 12.** On $B_0$, $A_T(f)$ is essentially self-adjoint and fulfills the equality
\begin{equation}
A_T(f) = -A_g(f),
\end{equation}
where, following [Sym71],
\begin{equation}
A_\mu^a(x) \equiv A_\mu(x) - \partial^\mu \int d^3 y \, G_\gamma(x) \left( \partial_m A^m(x_0, y) \right).
\end{equation}

**Proof.** Since $A_C(f)$ is e.s.a. on $F_0$ and, by Lemma 11, $B_0 = T^{-1} F_0$, $A_T(f)$ is e.s.a. on $B_0$. A necessary condition for (215) to make sense is $A_g(f)(B_0) \subset \mathcal{H}_{\text{phys}}$; such a requirement is indeed fulfilled, as $B_0 \subset \operatorname{Dom}(A_g(f))$ and $A_\mu^a$ commutes with $\partial A$ by virtue of the canonical commutation relations ([Sym71]). Maxwell’s equations and (1) yield
\begin{equation}
A_0^a(f_0) = A^0(f_0) - \int d^4 x \, d^3 y \, G_\gamma(x) \left( \partial_m A^m(x_0, y) \right) f_0(x) = 0,
\end{equation}
whence $A_g(f) = -A_g(f)$; it is therefore enough to prove that $A_g(f) = A_T(f)$. We first show that $A_g(f_{tr}) = A_T(f_{tr})$ on $B_0$, for test functions $f_{tr}(x)$ fulfilling $\partial_l f_{tr}^l(x) = 0$; on $B_0$ one in fact has $A_g(f_{tr}) = A(f_{tr})$, since $f_{tr}$ is transverse, and $A(f_{tr}) = A_T(f_{tr})$, by Lemma 11.

Finally, it follows from (11) that
\begin{equation}
(\partial_l A_T^l)(h) = (\partial_l A^l)(h) + \int d^4 x \, d^3 y \, \Delta G_\gamma(x) \left( \partial_m A^m(x_0, y) \right) h(x) = 0.
\end{equation}

The Lemma is thus proved. □

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