Lectures on $W$ algebras and $W$ gravity†

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ABSTRACT

We give a review of the extended conformal algebras, known as $W$ algebras, which contain currents of spins higher than 2 in addition to the energy-momentum tensor. These include the non-linear $W_N$ algebras; the linear $W_\infty$ and $W_{1+\infty}$ algebras; and their super-extensions. We discuss their applications to the construction of $W$-gravity and $W$-string theories.

† Lectures given at the Trieste Summer School in High-Energy Physics, August 1991.
⋆ Supported in part by the U.S. Department of Energy, under grant DE-FG05-91ER40633.
Foreword

This paper is a review of developments in the area of $W$ algebras, and their application to the construction of $W$-gravity theories in two dimensions. It is based on a series of four lectures presented at the Summer School in High-energy Physics and Cosmology, at the ICTP, Trieste, in the summer of 1991. The material presented here is largely the result of collaborations with a number of people, principally Larry Romans and Shawn Shen. Special thanks are also due to my other collaborators: Eric Bergshoeff, Paul Howe, Keke Li, Hong Lu, Ergin Sezgin, Kelly Stelle, Xujing Wang, Kaiwen Xu and Kajia Yuan.

The organisation of these lectures is as follows:

1. Introduction to $W$ algebras.
2. The $W_3$ and $W_N$ algebras.
3. The $W_{\infty}$ and $W_{1+\infty}$ algebras.
4. Realisations of $W_{\infty}$ and $W_{1+\infty}$.
5. BRST operators for $W$ algebras.
6. Classical and Quantum $W$ gravity.

1. Introduction to $W$ algebras; Virasoro $\rightarrow W_N \rightarrow W_{\infty}$

The Virasoro algebra has played a very important rôle in physics in the last few years, because it is the essential underlying worldsheet symmetry of string theory, and of two-dimensional gravity. In the language of Laurent modes, it takes the form

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0},$$

where the indices $m, n, \ldots$ range over all the integers. The second term on the right-hand side is the central term in the algebra, which plays a crucial rôle in the quantum theory.

The generators $L_m$ can be viewed as the coefficients in a Laurent expansion of the holomorphic energy-momentum tensor in two dimensions:

$$T(z) \equiv T_{zz}(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}.$$  

Equivalently, we may write

$$L_n = \frac{1}{2\pi i} \oint dz T(z) z^{n+1},$$

where the contour is taken to enclose the origin. The commutation relations (1.1) can then be expressed in terms of the operator-product expansion

$$T(z)T(w) \sim \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4},$$
where the tilde indicates that terms that are non-singular as \( z \) approaches \( w \) are omitted. The procedure for going back and forth between the language of commutation relations of modes, and operator-product expansions of currents, is straightforward, and is nicely explained in many reviews on conformal field theory (see, for example, \[1,2,3\]).

The current \( T(z) \) is the generator of holomorphic coordinate transformations in two dimensions. Acting on fields \( \Phi_{\mu_1\cdots\mu_s} \), it transforms the holomorphic component \( \Phi \equiv \Phi_{z\cdots z} \) according to the rule

\[
\delta \Phi(w) = \frac{1}{2\pi i} \oint dz k(z) T(z) \Phi(w), \quad (1.5)
\]

where \( k(z) \) is the infinitesimal parameter of the holomorphic reparametrisation;

\[
\delta z = k(z). \quad (1.6)
\]

Assuming that \( \Phi_{\mu_1\cdots\mu_s} \) transforms as an \( s \)-index tensor under general coordinate transformations,

\[
\delta \Phi_{\mu_1\cdots\mu_s} = k^\nu \partial_\nu \Phi_{\mu_1\cdots\mu_s} + (\partial_{\mu_1} k^\nu) \Phi_{\nu\mu_2\cdots\mu_s} + \cdots + (\partial_{\mu_s} k^\nu) \Phi_{\mu_1\cdots\mu_{s-1}\nu}, \quad (1.7)
\]

then we deduce that under holomorphic transformations, \( \Phi \) transforms as

\[
\delta \Phi = k \partial \Phi + s \partial k \Phi. \quad (1.8)
\]

Thus from (1.5) we find that the operator-product expansion of \( T \) with \( \Phi \) must be

\[
T(z)\Phi(w) \sim \frac{\partial \Phi(w)}{z - w} + \frac{s \Phi(w)}{(z - w)^2}. \quad (1.9)
\]

A field whose OPE with \( T(z) \) is of the form (1.9) is called a primary field of conformal weight (or spin) \( s \) (we are considering purely holomorphic fields here). Note that the OPE (1.4) for the energy-momentum tensor \( T \) itself has operator terms of the form (1.9), with \( s = 2 \). However, the presence of the central term means that \( T(z) \) is not a primary field. The fact that the \( 1/(z - w)^n \) terms for \( n = 1, 2, 3 \) have the standard form (1.9) means that \( T(z) \) is what is called a quasi-primary field of conformal weight 2. The significance of these particular terms in the OPE is that they correspond to the commutators of \( L_{-1}, L_0 \) and \( L_1 \) with the Laurent modes of the field \( \Phi \). For a field of spin \( s \), these Laurent modes are defined by a generalisation of (1.3):

\[
\Phi_n = \frac{1}{2\pi i} \oint dz \Phi(z) z^{n+s-1}. \quad (1.10)
\]

Conversely, the field \( \Phi(z) \) is given in terms of its Laurent modes by

\[
\Phi(z) = \sum_{n=-\infty}^{\infty} \Phi_n z^{-n-s}. \quad (1.11)
\]
As one can see from (1.1), the Virasoro modes $L_{-1}$, $L_0$ and $L_1$ satisfy the subalgebra

$$[L_{-1}, L_1] = -2L_0, \quad [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \quad (1.12)$$

This algebra can be recognised as being precisely that of $SL(2, \mathbb{R})$. It is called an anomaly-free subalgebra of the Virasoro algebra, since the (anomalous) central terms in the algebra (1.1) vanish for these particular commutators.

We have seen, then, that the Virasoro algebra can be viewed as the algebra of the (quasi)-primary spin-2 current $T(z)$. It is natural now to think about the possibility of extending this algebra by the inclusion of currents of higher (plus, possibly, lower) spins. In fact extensions with only lower spins added have been studied for a long time. Examples include the $N = 1$ super-Virasoro algebra, with an additional spin-$\frac{3}{2}$ current $G(z)$, and the $N = 2$ super-Virasoro algebra, with two extra spin-$\frac{3}{2}$ currents $G(z)$ and $\overline{G}(z)$ and a spin-1 current $J(z)$. The non-vanishing OPEs of this $N = 2$ algebra are:

$$T(z)T(w) \sim \frac{\partial T}{z - w} + \frac{2T}{(z - w)^2} + \frac{c/2}{(z - w)^4},$$

$$T(z)J(w) \sim \frac{\partial J}{z - w} + \frac{J}{(z - w)^2}, \quad J(z)J(w) \sim \frac{c/3}{(z - w)^2},$$

$$T(z)G(w) \sim \frac{\partial G}{z - w} + \frac{3/2G}{(z - w)^2}, \quad T(z)\overline{G}(w) \sim \frac{\partial \overline{G}}{z - w} + \frac{3/2\overline{G}}{(z - w)^2}. \quad (1.13)$$

$$J(z)G(w) \sim \frac{G}{z - w}, \quad J(z)\overline{G}(w) \sim \frac{-\overline{G}}{z - w},$$

$$G(z)\overline{G}(w) \sim \frac{2T + \partial J}{z - w} + \frac{2J}{(z - w)^2} + \frac{2c/3}{(z - w)^3}.$$ 

Note that here, and for the rest of this paper, it is to be understood that unspecified arguments of currents appearing on the right-hand side of OPEs are taken to be $w$. We see from (1.13) that the currents $G$ and $\overline{G}$ indeed satisfy the primary-field condition (1.9) with $s = \frac{3}{2}$, and $J$ is a primary field with spin 1.

The simplest higher-spin extension of the Virasoro algebra was found by Zamolodchikov in 1985 [4]. Called $W_3$, it comprises two currents; the energy-momentum tensor $T(z)$, and a spin-3 primary current $W(z)$. Subsequently, higher-spin generalisations known as $W_N$ algebras were constructed [5,6]. These comprise the energy-momentum tensor $T(z)$, together with higher-spin primary currents of each spin $3 \leq s \leq N$. We may think of the Virasoro algebra itself as being the $W_2$ algebra in this classification.

A characteristic feature of all the $W_N$ algebras with $N > 2$ (and $< \infty$) is that they are non-linear. The reason for this is that the OPE of currents with spins $s$ and $s'$ produces terms that, at leading order, have spin $s + s' - 2$. If one attempts to build an algebra with fundamental currents with spins $2 \leq s \leq N$, it follows that that there will be terms
appearing from OPEs that have spins exceeding \( N \). In the \( W_N \) algebras, these are interpreted as composite fields, built from products of the fundamental currents themselves. Of course \( W_2 \) is an exception, since \( 2 + 2 - 2 = 2 \).

Another exception to the general rule occurs if \( N = \infty \), since now there is no spin that can be produced in an OPE that is too high to be represented by the fundamental currents of the algebra; \( \infty + \infty - 2 = \infty \). So one might expect, and indeed one finds, that it should be possible to construct a linear \( W_\infty \) algebra, generated by currents with spins \( 2 \leq s \leq \infty \) [7,8]. Apart from its linearity, it otherwise has many features in common with the finite-\( N \) \( W_N \) algebras; in particular, in both cases there are non-trivial central terms in the OPE of any pair of equal-spin currents. Another related linear algebra with an infinite number of currents is called \( W_{1+\infty} \) [9]. This algebra contains currents of all spins in the range \( 1 \leq s \leq \infty \).

These various \( W \) algebras, their super-extensions, and applications to \( W \) gravity, will form the subject of the rest of these lectures. We begin, in the next section, by looking in more detail at the finite-\( N \) \( W_N \) algebras, and in particular, \( W_3 \).

### 2. The \( W_3 \) and \( W_N \) algebras

#### 2.1 The \( W_3 \) algebra

The \( W_3 \) algebra comprises two currents; the spin-2 energy-momentum tensor \( T(z) \), and the spin-3 primary current \( W(z) \). The operator products of these currents are [4]:

\[
T(z)T(w) \sim \frac{\partial T(w)}{z-w} + \frac{2T}{(z-w)^2} + \frac{c/2}{(z-w)^4} \tag{2.1a}
\]

\[
T(z)W(w) \sim \frac{\partial W}{z-w} + \frac{3W}{(z-w)^2} \tag{2.1b}
\]

\[
W(z)W(w) \sim \frac{16}{22+5c} \left( \frac{\partial \Lambda}{z-w} + \frac{2\Lambda}{(z-w)^2} \right) + \frac{1}{15} \left( \frac{\partial^3 T}{z-w} + \frac{9}{2} \frac{\partial^2 T}{(z-w)^2} + 15 \frac{\partial T}{(z-w)^3} + 30 \frac{T}{(z-w)^4} \right) + \frac{c/3}{(z-w)^6} \tag{2.1c}
\]

The first of these is just the usual Virasoro algebra, and the second reflects the fact that \( W(z) \) is a primary spin-3 current.

The third operator product requires more explanation. To leading order, the OPE of two spin-3 currents produces spin 4, and the quantity \( \Lambda \) in (2.1c) denotes the composite spin-4 field. It is given in terms of a quadratic product of two energy-momentum tensors, by

\[
\Lambda \equiv (TT) - \frac{3}{10} \partial^2 T. \tag{2.2}
\]
The quadratic product \((TT)\) also requires explanation. Since \(T\) is an operator, it follows that the product of two \(T\)'s at the same point will be singular. To define \((TT)\) we must therefore split the points, and take a limit after first subtracting the singular terms:

\[
(TT)(w) \equiv \lim_{z \to w} \left(T(z)T(w) - \text{singular terms}\right).
\] (2.3)

A particularly elegant way to do this was introduced in [10]: we may adopt the definition, for an arbitrary pair of operators \(A\) and \(B\);

\[
(AB)(w) \equiv \frac{1}{2\pi i} \oint dz \frac{A(z)B(w)}{z-w}.
\] (2.4)

It is easy to see that this indeed corresponds to a regularisation of the point-split expression, of the form (2.3). Note that there is no unique way to define the regularised limit; (2.4) corresponds to a specific, convenient, choice. For this choice, an appropriate way to define multiple normal-ordered products is:

\[
(ABC \cdots DE) \equiv (A(B(C(\cdots(DE)))))
\] (2.5)

It should be emphasised that the normal ordering of the product \((TT)\) is being performed with respect to the modes of \(T\) itself, and not with respect to the modes of whatever matter fields might be involved in a specific realisation of \(T\). Because the definition of \((TT)\) is “self contained,” it follows that, despite the non-linearities, we can view the \(W_3\) algebra (2.1a-c) as being closed on the generators \(T\) and \(W\). Although it is not a Lie algebra, it nevertheless satisfies the Jacobi identities in the usual way, i.e.

\[
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0,
\] (2.6)

where \(A\), \(B\) and \(C\) are chosen in any combination from \(\{L_m\}\) and \(\{W_m\}\), the Laurent modes of \(T\) and \(W\).

2.2 The \(W_N\) algebras, and the Miura transformation

The \(W_3\) algebra described above may be generalised to \(W_N\), with one current of each spin in the interval \(2 \leq s \leq N\) [5,6]. In practice, the explicit structures of the higher-\(N\) algebras rapidly become unmanageable. Already for the \(W_4\) algebra, the “explicit” expressions are not very aesthetically attractive [11,12]. For many purpose, however, it is sufficient to know explicit realisations of the algebras, even if the details of the algebras themselves are too complicated for one to wish to see them. For the \(W_N\) algebras, realisations in terms of \(N - 1\) free scalars have been given, by making use of a construction known as the Miura
transformation [5]. This exploits a relation between the \( W_N \) algebra and the group \( SU(N) \). The way it works is as follows. Consider the differential operator \( L \), of degree \( N \), given by

\[
L = u_N \partial^N + u_{N-2} \partial^{N-2} + u_{N-3} \partial^{N-3} \cdots + u_1 \partial + u_0. \tag{2.7}
\]

One now equates \( L \) with the factorised differential operator

\[
L = \prod_{i=1}^{\bar{N}} \left( \alpha_0 \partial + (\vec{\mu}_i - \vec{\mu}_{i+1}) \cdot \partial \vec{\phi} \right), \tag{2.8}
\]

where \( \alpha_0 \) is a constant, \( \vec{\mu}_i \) denotes the fundamental weights of \( SU(N) \), and \( \vec{\phi} \) denotes an \((N-1)\)-vector of scalar fields. (The fundamental weights of \( SU(N) \) are defined by \( \vec{\mu}_i \cdot \vec{\alpha}_j = \delta_{ij} \), where \( \vec{\alpha}_i \) are the simple roots.) One can see by counting derivatives that \( u_{N-2} \) has spin 2, \( u_{N-3} \) has spin 3, and so on, with \( u_0 \) having spin \( N \). Essentially, these are the currents of \( W_N \). In fact, up to a constant scaling, \( u_{N-2} \) is precisely the stress tensor \( T \). For the higher-spin currents, one finds that the quantities \( \tilde{W}^{(s)} \equiv u_{N-s} \) themselves are not actually primary spin-\( s \) currents. To make primary currents \( W^{(s)} \), one must add derivatives and/or products of lower-spin currents. The details in general are quite complicated. For the \( W_3 \) algebra, the results for the stress tensor \( T \) and the primary spin-3 current \( W \) turn out, after redefining the fields, to be

\[
T(z) = \frac{1}{2}(\partial \phi_1)^2 + \frac{1}{2}(\partial \phi_2)^2 + \alpha_1 \partial^2 \phi_1 + \alpha_2 \partial^2 \phi_2, \tag{2.9a}
\]

\[
W(z) = \frac{2\sqrt{2}}{\sqrt{22 + 5c}} \left( \frac{1}{3} (\partial \phi_1)^3 - \partial \phi_1 (\partial \phi_2)^2 + \alpha_1 \partial \phi_1 \partial^2 \phi_1 - 2 \alpha_2 \partial \phi_1 \partial^2 \phi_2 - \alpha_1 \partial \phi_2 \partial^2 \phi_2 \right.
\]

\[
\left. + \frac{3}{5} \alpha_1^2 \partial^3 \phi_1 - \alpha_1 \alpha_2 \partial^3 \phi_2 \right), \tag{2.9b}
\]

where \( \alpha_1^2 = 3 \alpha_2^2 \). The scalar fields \( \phi_i \) satisfy the OPEs

\[
\phi_i(z)\phi_j(w) \sim \delta_{ij} \log(z - w), \tag{2.10}
\]

from which one can show, after some algebra, that \( T \) and \( W \) given by (2.9a, b) satisfy the \( W_3 \) algebra (2.1a-c) with central charge

\[
c = 2 - 16 \alpha_1^2. \tag{2.11}
\]

Other bosonic \( W \) algebras can also be constructed, which are related to other Lie groups in an analogous way to the relation between \( W_N \) and \( SU(N) \) (see, for example, [13]). These are qualitatively similar in most respects to the \( W_N \) algebras that we have been discussing, and we shall not consider them further here.
3. The $W_\infty$ and $W_{1+\infty}$ algebras.

3.1 The $W_\infty$ algebra

The general structure of the $W_3$ algebra (2.1a-c), which is also shared by the higher-$N$ $W_N$ algebras, is that the OPE of currents $W^{(s)}$ and $W^{(s')}$ of spins $s$ and $s'$ may be cast in the form

$$W^{(s)}(z)W^{(s')}(w) \sim W^{(s+s'-2)}, W^{(s+s'-4)}, \ldots, +c_s\delta^{ss'},$$  \hspace{1cm} (3.1)

where we are being very schematic here, and omitting all the inverse powers of $(z-w)$. The sequence of operator terms on the right-hand side descends in steps of two units of spin, and the final term is the (diagonal) central term. We are also using $W^{(s)}$ here to represent either a fundamental current (if $s \leq N$), or a composite current (in general).

The structure of (3.1) suggests a natural kind of ansatz to try in order to construct a linear $N \to \infty$ limit of the $W_N$ algebras. To do this, we shall, for now, revert to a Laurent-mode description. Also, for historical reasons, we shall adopt the notation, of dubious convenience, that a generator of spin $s$ will be denoted by $V^i_s$, where $s = j + 2$. Thus we are led to make the following ansatz:

$$[V^i_m, V^j_n] = \sum_{\ell \geq 0} g^{ij}_{2\ell}(m,n)V^{i+j-2\ell}_{m+n} + c_i(m)\delta^{ij}\delta_{m+n,0},$$  \hspace{1cm} (3.2)

Here, $g^{ij}_{2\ell}(m,n)$ are the structure constants of the algebra, and $c_i(m)$ are the central terms. $V^i_m$ denotes the $m$'th Laurent mode of the spin-$(i+2)$ current $V^i(z)$. For the $W_\infty$ algebra, with spins $2 \leq s \leq \infty$, the indices $i, j$ range from 0 to $\infty$.

The structure constants and central terms can be determined by demanding that the ansatz (3.2) be consistent with the Jacobi identities. After considerable brute-force calculation combined with guesswork, one arrives at the conclusion that [7,8]

$$c_i(m) = m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (i+1)^2)c_i,$$  \hspace{1cm} (3.3)

where $c_i$ are central charges, and the structure constants take the form

$$g^{ij}_{\ell}(m,n) = \frac{1}{2(\ell + 1)!} \phi^{ij}_{\ell} N^{ij}_{\ell}(m,n),$$  \hspace{1cm} (3.4)

where the $N^{ij}_{\ell}(m,n)$ are given by

$$N^{ij}_{\ell}(m,n) = \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} [i+1+m]_{\ell+1-k}[i+1-m]_k[j+1+n]_k[j+1-n]_{\ell+1-k},$$

$$= \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} (2i+2-\ell)_k[2j+2-k]_{\ell+1-k}[i+1+m]_{\ell+1-k}[j+1+n]_k.$$  \hspace{1cm} (3.5)
(The proof of the equivalence of these two expressions is not completely trivial [8].) In (3.5), \([a]_n\) denotes the descending Pochhammer symbol, \([a]_n \equiv a(a-1)\cdots (a-n+1) = a!/(a-n)!\), and \(a)_n\) denotes the ascending Pochhammer symbol, \((a)_n \equiv a(a+1)\cdots (a+n-1) = (a+n-1)!/(a-1)!\). The functions \(\phi^{ij}_\ell\) are given by

\[
\phi^{ij}_\ell = \frac{\Gamma^3\left[-\frac{1}{2}, \frac{3}{2}, -\ell - \frac{1}{2}, -\ell - \frac{3}{2}; 1\right]}{\Gamma\left[-i - \frac{1}{2}, -j - \frac{1}{2}, i + j - \ell + \frac{5}{2}\right]},
\]

where the right-hand side is a Saalschützian \(4F_3(1)\) generalised hypergeometric function [8]. In more down-to-earth language, \(\phi^{ij}_\ell\) can be written as

\[
\phi^{ij}_\ell = \sum_{k \geq 0} \frac{(-1)^k (3/2)_k (-\ell - 1/2)_k (-\ell - 3/2)_k}{k! (-i - 1/2)_k (-j - 1/2)_k (i + j - \ell + 5/2)_k}.
\]

The central charges \(c_i\) are given by

\[
c_i = \frac{2^{2i-3}i! (i+2)!}{(2i+1)!! (2i+3)!!} c.
\]

These results are unique, modulo trivial redefinitions and rescalings of generators. Note that we have given expressions here for \(g^{ij}_\ell(m, n)\) for odd as well as even values of the lower index \(\ell\), although only the even ones appear in (3.2). This is for later convenience.

The sequence of generators on the right-hand side of (3.2) descends in steps of two units of conformal spin. The lowest-spin generator that can appear is therefore either 2, if \(i + j\) is even, or 3, if \(i + j\) is odd. This termination of the sequence in fact occurs automatically; the structure constants \(g^{ij}_\ell(m, n)\) vanish if \(i + j - 2\ell \leq 0\). That this should happen is rather non-trivial. The \(N^{ij}_\ell(m, n)\) factor (3.5) in \(g^{ij}_\ell(m, n)\) has “obvious” zeros that are responsible for certain of the necessary zeros in the structure constants. The non-triviality lies in the \(\phi^{ij}_\ell\) factor given by (3.6), which supplies the remaining necessary zeros. The subject of non-trivial zeros of generalised hypergeometric functions is one that is much studied by some mathematicians.

The general results presented above are not very transparent from the point of view of getting a feel for what the \(W_\infty\) algebra really looks like. It is instructive, therefore, to look at the first few terms in the series explicitly. Thus we have

\[
[V^i_m, V^j_n] = \frac{1}{2} \phi^{ij}_0 N^{ij}_0(m, n)V^{i+j}_{m+n} + \frac{1}{2 \cdot 3!} \phi^{ij}_2 N^{ij}_2(m, n)V^{i+j-2}_{m+n} + \frac{1}{2 \cdot 3!} \phi^{ij}_4 N^{ij}_4(m, n)V^{i+j-4}_{m+n} + \cdots.
\]

The expressions for \(\phi^{ij}_\ell\) and \(N^{ij}_\ell(m, n)\) functions appearing here are:

\[
\phi^{ij}_0 = 1,
\]

\[
\phi^{ij}_2 = 1 - \frac{9}{(2i + 1)(2j + 1)(2i + 2j + 1)},
\]

\[
\phi^{ij}_4 = 1 - \frac{30}{(2i + 1)(2j + 1)(2i + 2j - 3)} \left(1 + \frac{15/2}{(2i - 1)(2j - 1)(2i + 2j - 1)}\right).
\]
and

\[ N_{0}^{ij}(m,n) = 2(j+1)m - 2(i+1)n, \]
\[ N_{2}^{ij}(m,n) = 4j(j+1)(2j+1)m^3 - 12ij(2j+1)m^2n + 12ij(2i+1)mn^2 - 4i(i+1)(2i+1)n^3 \]
\[ - 4j(j+1)(1+3i+3i^2 + 2j + 3ij)m + 4i(1+3j + 3j^2 + 2i + 3ij)n, \]
\[ N_{4}^{ij}(m,n) = 5'th-order polynomial in m and n. \] (3.11)

(The details of the last term here are not sufficiently instructive to justify the space that they would occupy!)

If one looks at the commutator of the Virasoro modes \( L_m \equiv V^0_m \) with \( V^j_n \), one will find, from (3.2), that in general the right-hand side will produce not only \( V^j_{m+n} \) from the leading-order term in the sum, but also lower-spin generators. This means that in general the generators \( V^j_n \) are not associated with a primary current. However, it turns out that if the Laurent index on \( L_m \) is restricted to \( m = -1, 0, 1 \), corresponding to the \( SL(2,R) \) subalgebra, then the lower-spin terms on the right-hand side disappear. This means that the generators \( V^j_n \) correspond, in general, to currents that are quasi-primary. The exception is \( V^1_n \), associated with the spin-3 current. This is a genuine primary current, since for this case it is not possible for there to be any lower-spin terms on the right-hand side. Primary currents at all spins could be defined, but at the price of introducing non-linearities into the algebra.

3.2 Contraction to \( w_\infty \)

There is a simple contraction of the \( W_\infty \) algebra, which can be obtained by first rescaling the generators \( V^i_m \) by an \( i \)-dependent power of a parameter \( q \), as follows:

\[ V^i_m \rightarrow q^{-i}V^i_m. \] (3.12)

Thus the commutation relations (3.2) become

\[ [V^i_m, V^j_n] = \sum_{\ell \geq 0} q^{2\ell} g^{ij}_{2\ell}(m,n)V^{i+j-2\ell}_{m+n} + q^{2i} c_i(m) \delta^{ij} \delta_m \delta_n, \] (3.13)

If we now send the parameter \( q \rightarrow 0 \), then only the highest-spin generator term on the right-hand side survives (i.e. the \( \ell = 0 \) term), together with the central term for \( i = j = 0 \). If we denote the generators for this contraction of the algebra by \( v^i_m \), then from (3.10) and (3.11), we have [7]

\[ [v^i_m, v^j_n] = \left( (j + 1)m - (i + 1)n \right) v^{i+j}_{m+n} + \frac{c}{12} m(m^2 - 1) \delta^{i,0} \delta^{j,0} \delta_{m+n,0}. \] (3.14)

This algebra, now known as \( w_\infty \), was discovered some time before the \( W_\infty \) algebra [14]. As we shall see later, it can in some sense be viewed as the classical limit of the \( W_\infty \).
algebra. Because it admits a central extension only in the Virasoro subalgebra (generated by $L_m = v^0_m$), it is a somewhat trivial extension of the Virasoro algebra. From the point of view of representation theory, the absence of higher-spin central terms means that the higher-spin states built up from a highest-weight state will have zero norm, and should be set to zero. We shall return to a more detailed discussion of $w_\infty$ and related classical algebras later.

3.3 Operator-product expansions for $W_\infty$

For many purposes, it is much more convenient to discuss two-dimensional conformal field theories in terms of fields, and operator-product expansions, rather than Laurent modes and commutation relations. This is true also for the discussion of $W$ algebras, so now we shall look at how to re-express the content of the $W_\infty$ algebra in the language of conformal field theory. Following (1.11), we introduce a current $V^i(z)$ for each set of Laurent modes. Since $V^i_m$ has (quasi-) conformal spin $(i + 2)$, it follows that the appropriate expansions are

$$V^i(z) = \sum_{m=-\infty}^{\infty} V^i_m z^{-m-i-2}. \quad (3.15)$$

Conversely, the Laurent modes are given in terms of the currents by

$$V^i_m = \frac{1}{2\pi i} \oint dz V^i(z) z^{m+i+1}. \quad (3.16)$$

Converting the commutation relations (3.2) for the modes $V^i_m$ into operator-product expansions for the currents (3.15) is now a purely mechanical procedure [8]. Essentially, powers of $m$ or $n$ appearing in the structure constants $g^{ij}_{2\ell}(m,n)$ get replaced by derivatives. The result, including the rescaling (3.12) for convenience, is that the OPE of $V^i(z)$ with $V^j(w)$ is given by

$$V^i(z)V^j(w) \sim -\sum_{\ell \geq 0} q^{2\ell} f^{ij}_{2\ell}(\partial_z, \partial_w) \frac{V^{i+j-2\ell}(w)}{z-w} - q^{2i} c_i \delta^{ij} (\partial_z)^{2i+3} \frac{1}{z-w}. \quad (3.17)$$

The quantities $f^{ij}_{2\ell}(m,n)$ are closely related to the structure constants $g^{ij}_{2\ell}(m,n)$ in (3.2). In fact the only difference is that instead of (3.4) we have

$$f^{ij}_{\ell}(m,n) = \frac{1}{2(\ell + 1)!} \phi^{ij}_{\ell} M^{ij}_{\ell}(m,n), \quad (3.18)$$

where $\phi^{ij}_{\ell}$ is still given by (3.6), but $M^{ij}_{\ell}(m,n)$, which replaces $N^{ij}_{\ell}(m,n)$, is given by [8]

$$M^{ij}_{\ell}(m,n) = \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} (2i+2-\ell)_{k} [2j+2-k]_{\ell+1-k} m^{\ell+1-k} n^k. \quad (3.19)$$
This expression is in fact the part of \( N_{ij}^{\ell}(m,n) \) that is of total degree \( \ell + 1 \) in \( m \) and \( n \), as may be seen by comparing it with the second expression in (3.5). Thus, for example, \( M_{ij}^{2\ell}(m,n) \) is given by the first line of the expression in (3.11) for \( N_{ij}^{2\ell}(m,n) \).

It is instructive to look at a few examples of OPEs for the currents of \( W_\infty \). Setting the parameter \( q \) in (3.17) to \( q = \frac{1}{4} \) for convenience, we have,

\[
\begin{align*}
V^0(z)V^0(w) &\sim \frac{\partial V^0}{z - w} + \frac{2V^0}{(z - w)^2} + \frac{c/2}{(z - w)^4}, \\
V^0(z)V^1(w) &\sim \frac{\partial V^1}{z - w} + \frac{3V^1}{(z - w)^2}, \\
V^0(z)V^2(w) &\sim \frac{\partial V^2}{z - w} + \frac{4V^2}{(z - w)^2} + \frac{12}{5} \frac{V^0}{(z - w)^4}, \\
V^1(z)V^1(w) &\sim \frac{2\partial V^2}{z - w} + \frac{4V^2}{(z - w)^2} \\
&\quad + \frac{1}{10} \left( \frac{\partial^3 V^0}{z - w} + \frac{9}{2} \frac{\partial^2 V^0}{(z - w)^2} + 15 \frac{\partial V^0}{(z - w)^3} + 30 \frac{V^0}{(z - w)^4} \right) \\
&\quad + \frac{c/2}{(z - w)^6}.
\end{align*}
\]

Thus we see that \( V^1(z) \) is a primary spin-3 current, whilst \( V^2(z) \) (and indeed all the higher-spin currents) is quasi-primary. Note that the relative coefficients of the terms for a given spin on the right-hand side are given purely by the \( M_{ij}^{2\ell}(m,n) \) part of \( f_{ij}^{2\ell}(m,n) \). The relations between these coefficients are in fact governed completely by covariance under the \( SL(2,R) \) subalgebra generated by \( L_{-1}, L_0, \) and \( L_1 \); in other words, they reflect the quasi-primary nature of the currents. The similarity between the OPEs for \( V^0 \) and \( V^1 \) in (3.21), and those for \( T \) and \( W \) in (2.1a-c), is quite striking. The essential difference is that here \( V^2 \) is an independent spin-4 current, whereas in the \( W_3 \) algebra \( \Lambda \) is a composite spin-4 current, given in terms of \( T \) by (2.2).

3.4 The \( SL(\infty, R) \) wedge subalgebra

We have already seen that the Virasoro algebra contains a finite-dimensional subalgebra, namely the \( SL(2,R) \) algebra generated by \( L_{-1}, L_0, \) and \( L_1 \). The commutation relations for these generators are given in (1.12). Since the Virasoro algebra is a subalgebra of \( W_\infty \), it follows that \( SL(2,R) \) is a subalgebra of \( W_\infty \) also. In fact, \( W_\infty \) has a much larger subalgebra, which is a natural extension of \( SL(2,R) \). One can show from the commutation relations (3.2),
with the definitions of the structure constants given below (3.2), that the subset of generators \( V_m^i \) with Laurent indices in the range

\[-i - 1 \leq m \leq i + 1\]  \hspace{1cm} (3.22)

form a closed algebra [8]. If one plots the “spin” index \( i \) vertically, and the Laurent index \( m \) horizontally, then the subset of generators specified by (3.22) fill out a “wedge,” with the three Virasoro generators \( L_{-1}, L_0 \) and \( L_1 \) sitting at the bottom, followed by the five spin-3 generators \( V_{-2}^1, V_{-1}^1, V_0^1, V_1^1 \) and \( V_2^1 \) at the next-highest level, and so on. This subset of generators at each level in fact transforms as the \((2i + 3)\)-dimensional irreducible representation of the \( SL(2, R) \) generated by \( L_{-1}, L_0 \) and \( L_1 \).

If one takes the canonical embedding of \( SL(2, R) \) in \( SL(N, R) \) in which the fundamental \( N \) of \( SL(N, R) \) decomposes irreducibly to the \( N \) of \( SL(2, R) \), then the adjoint of \( SL(N, R) \), the \((N^2 - 1)\), decomposes under \( SL(2, R) \) as

\[(N^2 - 1) \rightarrow 3 \oplus 5 \oplus 7 \oplus \cdots \oplus (2N - 1).\]  \hspace{1cm} (3.23)

Thus we see that the set of \( SL(2, R) \) representations that we get from the wedge subalgebra of \( W_\infty \) is precisely the set of representations that arise from the decomposition of \( SL(\infty, R) \) under the above canonical embedding of \( SL(2, R) \). So the wedge subalgebra is \( SL(\infty, R) \) [8].

Unlike finite-dimensional Lie algebras, such as \( SL(N, R) \), where all choices of basis for the generators are equivalent, one can have many inequivalent sets of commutation relations for infinite-dimensional algebras such as \( SL(\infty, R) \). One way to see this is to think of constructing \( SL(\infty, R) \) as an \( N \rightarrow \infty \) limit of \( SL(N, R) \). Although all basis choices are equivalent at finite \( N \), one may make \( N \)-dependent redefinitions to give different bases which, after taking the limit \( N \rightarrow \infty \), can no longer be related to one another. In fact for \( SL(\infty, R) \) there is, for example, a nice one-parameter family of algebras which are inequivalent, in the sense that no redefinition of generators enables one to relate the commutation relations of one member of the family to those for another [15,16]. This family of \( SL(\infty, R) \) algebras can be constructed by starting from the generators \( L_{-1}, L_0 \) and \( L_1 \) of \( SL(2, R) \), and building the tensor algebra of all products of arbitrary numbers of \( SL(2, R) \) generators, modulo the ideal

\[x \otimes y - y \otimes x - [x, y] = 0.\]  \hspace{1cm} (3.24)

In other words, the commutation relations of the \( L_m \) can be used to simplify products. One may now impose a further ideal relation, namely

\[\mathcal{I} : \quad Q - s(s + 1) = 0,\]  \hspace{1cm} (3.25)

where \( Q \equiv L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) \) is the Casimir operator of \( SL(2, R) \), and \( s \) is an arbitrary constant. The commutators of the tensor operators, modulo these ideals, close on \( SL(\infty, R) \).
The constant \( s \) parametrises inequivalent \( SL(\infty, R) \) algebras. Without loss of generality, we may restrict \( s \) to the range

\[
s \geq -\frac{1}{2}.
\]

(3.26)

One can show that the commutation relations for the \( SL(\infty, R) \) generators \( X^i_m \), where \( m \) is subject to the “wedge” condition (3.22), are given by

\[
[X^i_m, X^j_n] = \sum_{\ell \geq 0} g^{ij}_{2\ell}(m, n; s) X^{i+j-2\ell}_{m+n},
\]

(3.27)

with

\[
g^{ij}_{\ell}(m, n; s) = \frac{1}{2(\ell + 1)} \phi^{ij}_{\ell}(s) N^{ij}_{\ell}(m, n).
\]

(3.28)

Here,

\[
\phi^{ij}_{\ell}(s) = \binom{-\frac{1}{2} - 2s, \frac{3}{2} + 2s, -\frac{\ell}{2} - \frac{1}{2}, -\frac{\ell}{2}; 1}{-i - \frac{1}{2}, -j - \frac{1}{2}, i + j - \ell + \frac{5}{2}}.
\]

(3.29)

and \( N^{ij}_{\ell}(m, n) \) is given by (3.5). Thus we see that the wedge subalgebra of \( W_\infty \) coincides with the \( s = 0 \) member of the family of \( SL(\infty, R) \) algebras (3.27) [8].

3.5 The \( W_{1+\infty} \) algebra

For all values of \( s \), the \( SL(\infty, R) \) algebras (3.27) have the property that the descending sequence of terms on the right-hand side automatically cuts off at \( X^0_{m+n} \), i.e. the superscript on \( X^{i+j-2\ell}_{m+n} \) never becomes negative. This happens because \( g^{ij}_{2\ell}(m, n; s) \) supplies the necessary zeros. However, unlike the case of \( W_\infty \), many of the zeros here occur because of the restricted ranges (3.22) that the Laurent indices can have. Thus if one were to relax the restrictions (3.22), and allow the Laurent indices to range over all the integers, one would find for a generic value of \( s \) that the termination property would be lost, and the sequence of terms in (3.27) would continue indefinitely. If one wanted to try to interpret such an algebra as an algebra of conformal fields, it would correspond to having all conformal spins from \(-\infty \) to \( \infty \). This would be undesirable in any physical application, since the two-point correlation function for fields of negative conformal spins diverges as the points are separated. It is a very special property of the \( s = 0 \) case that the termination property persists when the Laurent indices are extended “beyond the wedge.”

Having seen that \( W_\infty \) algebra (without central terms) can be viewed as the extension of (3.27) beyond the wedge, it is natural to ask if there are any other special values of \( s \) for which algebras of only non-negative conformal spins can be obtained. It turns out that there is just one other possibility, corresponding to the case when \( s = -\frac{1}{2} \) [8,9]. In this case, it turns out that the zeros of the structure constants are such as to terminate the sequence of generators on the right-hand side at spin 1 (if \( i + j \) is odd) or spin 2 (if \( i + j \) is even). Thus
we get an algebra with currents of all conformal spins \( \geq 1 \). For this reason, this algebra is referred to as \( W_{1+\infty} \). Like \( W_{\infty} \), it also admits a central extension. The complete result for the \( W_{1+\infty} \) algebra is then

\[
[V^i_m, V^j_n] = \sum_{\ell \geq 0} q^{2\ell} g^{ij}_{2\ell}(m,n) V^{i+j-2\ell}_{m+n} + q^{2i} \tilde{c}_i(m) \delta^{ij} \delta_{m+n,0},
\]

(3.30)

where

\[
\tilde{g}^{ij}_{\ell}(m,n) \equiv g^{ij}_{\ell}(m,n, -\frac{1}{2})
\]

(3.31)

and

\[
\tilde{c}_i(m) = m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (i+1)^2) \tilde{c}_i,
\]

(3.32)

with

\[
\tilde{c}_i = \frac{2^{2i-2}((i+1)!)^2}{(2i+1)!!(2i+3)!!} c.
\]

(3.33)

We have included the rescaling parameter \( q \) in (3.30) for later convenience. Examples of the first few

\[
\tilde{\phi}^{ij}_{\ell} \equiv \phi^{ij}_{\ell}(-\frac{1}{2})
\]

(3.34)

functions are

\[
\tilde{\phi}^{ij}_0 = 1,
\]

\[
\tilde{\phi}^{ij}_2 = 1 + \frac{3}{(2i+1)(2j+1)(2i+2j+1)},
\]

\[
\tilde{\phi}^{ij}_4 = 1 + \frac{10}{(2i+1)(2j+1)(2i+2j-3)} \left(1 + \frac{27/2}{(2i-1)(2j-1)(2i+2j-1)}\right),
\]

(3.35)

Thus the structure constants are similar in form, but different in detail, to those for \( W_{\infty} \).

The \( W_{1+\infty} \) algebra may be recast in the language of operator-product expansions of currents, just as in the \( W_{\infty} \) case. The only differences are that now the \( \phi^{ij}_{\ell} \) appearing in (3.18) are replaced by \( \tilde{\phi}^{ij}_{\ell} \) given by (3.34) and (3.29), and the \( c_i \) given by (3.8) are replaced by the \( \tilde{c}_i \) given by (3.33). Thus we have

\[
\tilde{V}^i(z) \tilde{V}^j(w) \sim - \sum_{\ell \geq 0} q^{2\ell} \tilde{f}^{ij}_{2\ell}(\partial_z, \partial_w) \frac{\tilde{V}^{i+j-2\ell}(w)}{z-w} - q^{2i} \tilde{c}_i \delta^{ij} \delta_{2i+3} \frac{1}{z-w},
\]

(3.36)

with

\[
\tilde{f}^{ij}_{\ell}(m,n) = \frac{1}{2(\ell+1)!} \tilde{\phi}^{ij}_{\ell} M^{ij}_{\ell}(m,n),
\]

(3.37)
Examples of OPEs for $W_{1+\infty}$, after setting the parameter $q = \frac{1}{4}$ for convenience, are

$$
\tilde{V}^{-1}(z)\tilde{V}^{-1}(w) \sim \frac{c}{(z-w)^2},
$$

$$
\tilde{V}^0(z)\tilde{V}^0(w) \sim \frac{2\tilde{V}^0}{z-w} \frac{\partial \tilde{V}^0}{z-w} + \frac{c/2}{(z-w)^4},
$$

$$
\tilde{V}^0(z)\tilde{V}^{-1}(w) \sim \frac{\partial \tilde{V}^{-1}}{z-w} + \frac{\tilde{V}^{-1}}{(z-w)^2},
$$

$$
\tilde{V}^0(z)\tilde{V}^1(w) \sim 3\tilde{V}^1 \frac{\partial \tilde{V}^1}{z-w} + \frac{3\tilde{V}^1}{(z-w)^2} + \frac{1}{6} \tilde{V}^{-1}.\quad (3.38)
$$

(Recall that because of the notational convenience that $\tilde{V}^i(z)$ has spin $i + 2$, the spin-1 current is denoted by $\tilde{V}^{-1}(z)$ here.)

3.6 The Lone-star algebra

The $W_\infty$ algebra was originally constructed abstractly, in the sense that a formal antisymmetric Lie bracket $[A,B] = -[B,A]$ was assumed, which was then required to satisfy the Jacobi identity

$$
[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \quad (3.39)
$$

These requirements then led to the solution for the structure constants and central terms of the $W_\infty$ algebra. Interestingly, there is a way to realise the abstract bracket as the antisymmetric part of a more fundamental associative product of generators that we may represent by $\star$. Thus we may write

$$
[A, B] \equiv A \star B - B \star A, \quad (3.40)
$$

where $\star$ satisfies the associativity property

$$
A \star (B \star C) = (A \star B) \star C. \quad (3.41)
$$

In terms of this realisation, the Jacobi identity (3.39) is now trivially satisfied, by virtue of the associativity of the $\star$ product. The “lone-star” product of generators $V^i_m$ and $V^j_n$ of $W_\infty$ turns out to be given by [9]

$$
V^i_m \star V^j_n = \frac{1}{2} \sum_{\ell \geq -1} q^\ell g^ij_\ell(m,n)V^{i+j-\ell}_{m+n}, \quad (3.42)
$$

where the suffix $\ell$ on $g^ij_\ell(m,n)$ now takes odd-integer as well as even-integer values. It is for this reason that the definition of $g^ij_\ell(m,n)$ was originally given in (3.4-3.7) for all integers. The terms in (3.42) with even $\ell$ are antisymmetric under the simultaneous interchange $i \leftrightarrow j$. 

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and \( m \leftrightarrow n \), whilst the terms with odd \( \ell \) are symmetric. Showing that (3.42) defines an associative product is non-trivial. Some examples of low-lying lone-star products for \( W_\infty \), with \( q \) chosen to be \( \frac{1}{4} \), are:

\[
\begin{align*}
V_m^0 \ast V_n^0 &= V_{m+n}^1 + \frac{1}{2}(m - n)V_{m+n}^0, \\
V_m^0 \ast V_n^1 &= V_{m+n}^2 + \frac{1}{2}(2m - n)V_{m+n}^1 + \frac{1}{24}(6m^2 - 3mn + n^2 - 4)V_{m+n}^0, \\
V_m^1 \ast V_n^0 &= V_{m+n}^2 + \frac{1}{2}(m - 2n)V_{m+n}^1 + \frac{1}{24}(m^2 - 3mn + 6n^2 - 4)V_{m+n}^0.
\end{align*}
\]

(3.43)

A lone-star product may be defined for \( W_{1+\infty} \) also \([9]\). It is just like (3.42), only now \( g^{ij}_\ell(m, n) \) is replaced by \( \tilde{g}^{ij}_\ell(m, n) \), given by (3.31). Low-lying examples for \( W_{1+\infty} \) are:

\[
\begin{align*}
\tilde{V}_m^{-1} \ast \tilde{V}_n^{-1} &= \tilde{V}_{m+n}^{-1}, \\
\tilde{V}_m^{-1} \ast \tilde{V}_n^0 &= \tilde{V}_{m+n}^0 + \frac{1}{2}m\tilde{V}_{m+n}^{-1}, \\
\tilde{V}_m^0 \ast \tilde{V}_n^{-1} &= \tilde{V}_{m+n}^0 - \frac{1}{2}n\tilde{V}_{m+n}^{-1}, \\
\tilde{V}_m^0 \ast \tilde{V}_n^0 &= \tilde{V}_{m+n}^1 + \frac{1}{2}(m - n)\tilde{V}_{m+n}^0 + \frac{1}{12}(m^2 - mn + n^2 - 1)\tilde{V}_{m+n}^{-1}.
\end{align*}
\]

(3.44)

Note that in these, and indeed in all lone-star products for \( W_{1+\infty} \), the generator \( V_{0-1} \) acts like the identity operator in the algebra.

4. Realisations of \( W_\infty \) and \( W_{1+\infty} \)

4.1 Differential-operator realisations at \( c = 0 \)

Before turning to the consideration of quantum realisations of \( W_\infty \) and \( W_{1+\infty} \) in terms of scalar or spinor fields, it is interesting to look at purely classical realisations of the (centreless) algebras. They can in fact be realised in terms of enveloping algebras of simple (in the non-technical sense) infinite-dimensional algebras. Let us define, for convenience,

\[
\begin{align*}
\tilde{j}_n &\equiv \tilde{V}_n^{-1}, \\
\tilde{L}_n &\equiv \tilde{V}_n^0.
\end{align*}
\]

(4.1)

Then one can show, from the general form of the lone-star product, that

\[
\begin{align*}
\tilde{j}_m \ast \tilde{j}_n &= \tilde{j}_{m+n}, \\
\tilde{L}_0 \ast \tilde{V}_m^i &= \tilde{V}_m^{i+1} - \frac{1}{2}m\tilde{V}_m^i - \frac{(i + 1)^2[(i + 1)^2 - m^2]}{4[4(i + 1)^2 - 1]}\tilde{V}_m^{i-1}.
\end{align*}
\]

(4.3)

This latter equation can be turned around and viewed as a recursive definition of \( \tilde{V}_m^{i+1} \) in terms of \( \tilde{V}_m^i \) and \( \tilde{V}_m^{i-1} \) \([17]\). Starting from a humble \( U(1) \) Kac-Moody algebra, with generators \( \tilde{j}_m \), together with a derivation operator \( \tilde{d} \):

\[
\begin{align*}
[\tilde{j}_m, \tilde{j}_n] &= 0, \\
[\tilde{d}, \tilde{j}_m] &= -m\tilde{j}_m.
\end{align*}
\]

(4.4)

(4.5)
we may first construct the Virasoro generators $L_m$ as

$$L_m = (d + \frac{1}{2}m) j_m, \quad (4.6)$$

where, motivated by (4.2), we impose the ideal relation $\mathcal{I}$ defined by

$$\mathcal{I} : \quad j_m j_n - j_{m+n} = 0. \quad (4.7)$$

Then, motivated by (4.3), we define

$$\tilde{V}_m^i \equiv (d + \frac{1}{2}m) \tilde{V}^{i-1} + \frac{i^2 (i^2 - m^2)}{4(i^2 - 1)} \tilde{V}^{i-2}. \quad (4.8)$$

This implies that [17]

$$\tilde{V}_m^i = P_i(d; m) j_m, \quad (4.9)$$

where the first few $P_i(d; m)$ are given by

$$P_{-1} = 1,$$
$$P_0 = d + \frac{1}{2}m,$$
$$P_1 = d^2 + md + \frac{1}{12}(2m^2 + 1),$$
$$P_2 = d^3 + \frac{3}{2}md^2 + \frac{1}{20}(12m^2 + 7) + \frac{1}{40}(2m^3 + 7m),$$
$$P_3 = d^4 + 2md^3 + \frac{1}{14}(18m^2 + 13)d^2 + \frac{1}{14}(4m^3 + 13m)d + \frac{1}{500}(8m^4 + 100m^2 + 27). \quad (4.10)$$

It follows from (4.3) that the $\tilde{V}_m^i$ defined by (4.8) will generate the (centreless) $W_{1+\infty}$ algebra [17].

The above construction shows that the $W_{1+\infty}$ algebra, without central extension, can be realised as the Lie algebra of the enveloping algebra of a centreless $U(1)$ Kac-Moody algebra with a derivation operator $d$ [17]. One specific realisation for $j_m$ and $d$ is provided by taking

$$j_m = e^{im\theta},$$
$$d = i \frac{\partial}{\partial \theta}, \quad (4.11)$$

where $\theta$ is a coordinate on a circle of unit radius. One can easily verify that these expressions for $j_m$ and $d$ satisfy the commutation relations (4.4) and (4.5), and that the ideal-relation (4.7) is automatically implied. Thus we see that the centreless $W_{1+\infty}$ algebra can be viewed as the algebra of all polynomials in $e^{im\theta}$ and $i\partial/\partial \theta$ on a circle. In other words, the centreless $W_{1+\infty}$ algebra is isomorphic to the algebra of all smooth differential operators, of arbitrary degree, on the circle [17]. The Virasoro subalgebra is generated by all smooth differential operators of degree 1 on the circle.
A very similar discussion can be given for the $W_\infty$ algebra. The analogues of (4.2) and (4.3) now are [17]

\[
L_m \star L_n = V^1_{m+n} + \frac{1}{2} (m-n)L_{m+n},
\]

\[
L_0 \star V^i_m = V^i_{m+1} - \frac{m}{2} V^i_m - \frac{i(i+2)[(i+1)^2 - m^2]}{4[4(i+1)^2 - 1]} V^i_{m-1}.
\]

Motivated by these, we can now construct $W_\infty$ as the Lie algebra of the enveloping algebra of the Virasoro algebra, modulo the ideal $J$ suggested by (4.12):

\[
J : \quad L_m L_n - (L_0 + m)L_{m+n} = 0.
\]

Again, one can solve recursively for the $V^i_m$, to give

\[
V^i_m = Q_i(L_0;m)L_m.
\]

The first few examples are

\[
Q_0 = 1,
\]

\[
Q_1 = L_0 + \frac{m}{2},
\]

\[
Q_2 = L_0^2 + mL_0 + \frac{1}{2}(m^2 + 1),
\]

\[
Q_3 = L_0^3 + \frac{3m}{2}L_0^2 + \frac{1}{12}(9m^2 + 10)L_0 + \frac{1}{12}(m^3 + 5m),
\]

\[
Q_4 = L_0^4 + 2mL_0^3 + \frac{1}{3}(4m^2 + 5)L_0^2 + \frac{1}{3}(m^3 + 5m)L_0 + \frac{1}{42}(m^4 + 15m^2 + 8).
\]

In this case, we can realise the Virasoro algebra, and the ideal relation (4.14), by taking

\[
L_m = i e^{im \theta} \frac{\partial}{\partial \theta}.
\]

Thus we see that the $W_\infty$ algebra without central extension is isomorphic to the algebra of all smooth differential operators of strictly positive degree on a circle [17].

The above discussions show that $W_\infty$ must be a subalgebra of $W_{1+\infty}$, since the former is the algebra of all smooth differential operators of positive degree on the circle, whilst the latter is the algebra of all smooth differential operators of non-negative degree [17]. The embedding of $W_\infty$ in $W_{1+\infty}$ is not completely obvious in the usual bases that we have been using. For example, one can see from the last OPE in (3.36) that one cannot simply set the spin-1 current to zero in $W_{1+\infty}$ in that basis, since it is generated on the right-hand side of OPEs of higher-spin currents. By comparing the above representations for $W_{1+\infty}$ and $W_\infty$, one can show that $W_\infty$ may in fact be obtained as a truncation of $W_{1+\infty}$, after an appropriate redefinition of the generators. If we denote the generators of $W_{1+\infty}$ by $\tilde{V}^i_m$, and those of $W_\infty$ by $V^i_m$ as usual, then these redefinitions take the form

\[
V^i_m = \sum_{j=-1}^{i} a_{ij}(m)\tilde{V}^j_m,
\]

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where the constants $a_{ij}(m)$ are polynomials in $m$ of degree $(i - j)$, and $i$ takes the values 0, 1, 2, ..., For example, we have

$$
V_0^m = \tilde{V}_0^m + \frac{m}{2} \tilde{V}_m^{-1} - \frac{c}{8} \delta_{m,0},
$$

$$
V_1^m = \tilde{V}_1^m + \frac{m}{2} \tilde{V}_m^0 + \frac{1}{12} (m^2 - 1) \tilde{V}_m^{-1},
$$

$$
V_2^m = \tilde{V}_2^m + \frac{m}{2} \tilde{V}_m^1 + \frac{1}{20} (2m^2 - 3) \tilde{V}_m^0 + \frac{1}{120} (2m^3 + 7m) \tilde{V}_m^{-1} - \frac{3c}{320} \delta_{m,0}.
$$

The generators $V_i^m$, $i \geq 0$, together with $\tilde{V}_m^{-1}$, yield the $W_{1+\infty}$ algebra in a new basis. After making the redefinitions (4.18), the conformal-spin-1 generator $\tilde{V}_m^{-1}$ can be consistently truncated from the $W_{1+\infty}$ algebra. The remaining generators yield $W_\infty$. Note that, for convenience, we have included constant shifts in the definitions of $V_0^i$, which ensure that the “trivial” lower-order parts of the central terms have their canonical form. The resulting central charge $c$ of the $W_\infty$ algebra is related to the central charge $\tilde{c}$ of $W_{1+\infty}$ by [17]

$$
c = -2\tilde{c}.
$$

As is well known, in order to have unitary representations of algebras of this kind (for example Kac-Moody and Virasoro algebras), one requires that the central charge be positive. Thus (4.20) suggests that unitary representations of $W_{1+\infty}$ may not straightforwardly decompose into unitary representations of $W_\infty$.

4.2 Fermionic realisations of $W_{1+\infty}$ and $W_\infty$

Consider a free complex fermion $\psi$, with OPE

$$
\bar{\psi}(z)\psi(w) \sim \frac{1}{z-w}.
$$

This may be used to give a realisation of the Virasoro algebra, by defining the energy-momentum tensor $T$ to be

$$
T(z) = \frac{1}{2i} \bar{\psi} \psi - \frac{1}{2} \bar{\psi} \partial \psi.
$$

One may easily verify from (4.21) that this satisfies the Virasoro algebra (1.4) with central charge $c = 1$. One may also build a spin-1 current that is bilinear in $\bar{\psi}$ and $\psi$, namely $J(z) = \bar{\psi} \psi$. It is straightforward to verify that this satisfies the OPEs

$$
J(z)J(w) \sim \frac{1}{(z-w)^2},
$$

$$
T(z)J(w) \sim \frac{\partial J}{z-w} + \frac{J}{(z-w)^2}.
$$

The currents $T$ and $J$ give a closed algebra; in fact precisely the bosonic subalgebra of the $N = 2$ superconformal algebra (1.13).
One may go on to construct higher-spin currents that are bilinear in $\bar{\psi}$ and $\psi$, by adding more derivatives. Now, one finds that the OPE algebra of the currents will not close on any finite set of higher-spin currents (at least if we insist that the algebra be linear). By including currents for all higher spins however, closure can be achieved [18]. As one might perhaps expect, the algebra that one obtains is precisely $W_{1+\infty}$ [19,20]. Depending on how one chooses to define the higher-spin currents, one obtains $W_{1+\infty}$ in different bases. No matter what (non-degenerate) choice is made, however, it will always be $W_{1+\infty}$ in some basis. The reason for this is that there is in fact only one independent current that can be built at each spin, in the sense that there is only one new current at a given spin that is not simply a linear combination of derivatives of lower-spin currents. For example, at spin 2 there are two independent currents that can be written down; $\partial \psi \psi$, and $\bar{\psi} \partial \psi$. One linear combination of these is nothing but $\partial J$, where $J = \bar{\psi} \psi$; the new, independent, current is $T$, defined by (4.22). At spin 3 there are three independent currents that can be built; one combination of these is $\partial^2 J$, another is $\partial T$, and the remaining independent combination corresponds to the spin-3 current of $W_{1+\infty}$. All the higher-spin currents work in a similar fashion.

A particularly convenient choice of bases for the currents is one for which they are all quasi-primary with respect to the energy-momentum tensor. It turns out that after imposing this additional requirement, one is left with a 1-parameter family of bases [17,21]. The currents take the form:

$$V^i(z) = \sum_{j=0}^{i+1} \alpha_j(i; a) \partial^j \bar{\psi} \partial^{i+1-j} \psi,$$

where the coefficients are given by [21]

$$\alpha_j(i; a) = \binom{i+1}{j} \frac{(i+2a+2-j)j(2a-i-1)_i+1-j}{(i+2)_{i+1}}$$

and $a$ parametrises the choice of basis. The first few currents take the form

$$V^{-1} = \bar{\psi} \psi,$$

$$V^0 = (a + \frac{1}{2}) \partial \bar{\psi} \psi + (a - \frac{1}{2}) \bar{\psi} \partial \psi,$$

$$V^1 = \frac{1}{3} (a + \frac{1}{2}) (a+1) \partial^2 \bar{\psi} \psi + \frac{2}{3} (a+1) (a-1) \partial \bar{\psi} \partial \psi + \frac{1}{3} (a - \frac{1}{2}) (a-1) \bar{\psi} \partial^2 \psi.$$

$$V^2 = \frac{1}{60} (2a+3)(2a+1)(a+1) \partial^3 \bar{\psi} \psi + \frac{1}{20} (2a+3)(2a-3)(a+1) \partial^2 \bar{\psi} \partial \psi + \frac{1}{20} (2a+3)(2a-3)(a-1) \partial \bar{\psi} \partial^2 \psi + \frac{1}{60} (2a-3)(2a-1)(a-1) \bar{\psi} \partial^3 \psi.$$

For any value of the parameter $a$, the currents (4.24) generate the $W_{1+\infty}$ algebra. The standard basis, in which the OPEs take the form (3.36) (with $q = \frac{1}{4}$), corresponds to $a = 0$. 21
Then the first few currents take the form

\begin{align*}
V^{-1} &= \overline{\psi} \psi, \\
V^{0} &= \frac{1}{2} \partial \overline{\psi} \psi - \frac{1}{2} \overline{\psi} \partial \psi, \\
V^{1} &= \frac{1}{6} \partial^{2} \overline{\psi} \psi - \frac{2}{3} \partial \overline{\psi} \partial \psi + \frac{1}{6} \overline{\psi} \partial^{2} \psi, \\
V^{2} &= \frac{1}{20} \partial^{3} \overline{\psi} \psi - \frac{3}{20} \partial^{2} \overline{\psi} \partial \psi + \frac{9}{20} \partial \overline{\psi} \partial^{2} \psi - \frac{1}{20} \overline{\psi} \partial^{3} \psi.
\end{align*}

(4.27)

This gives a realisation of \( W_{1+\infty} \) with central charge \( c = 1 \). In general however, at generic values of \( a \), the central terms will not be diagonal in spin; only at \( a = 0 \), and \( a = \pm \frac{1}{2} \), will they be diagonal. The two cases \( a = \pm \frac{1}{2} \) are equivalent modulo a conjugation automorphism of the algebra. These correspond to the redefinitions of generators (4.18) for which \( W_{\infty} \) can be embedded in \( W_{1+\infty} \). Thus, as one can readily check, at \( a = \pm \frac{1}{2} \) we find that the central charge is given by \( c = -2 \), in accordance with the general result (4.20). At \( a = \frac{1}{2} \), the first few currents are:

\begin{align*}
V^{-1} &= \overline{\psi} \psi, \\
V^{0} &= \partial \overline{\psi} \psi, \\
V^{1} &= \frac{1}{2} \partial^{2} \overline{\psi} \psi - \frac{1}{2} \overline{\psi} \partial \psi, \\
V^{2} &= \frac{1}{5} \partial^{3} \overline{\psi} \psi - \frac{3}{5} \partial^{2} \overline{\psi} \partial \psi + \frac{1}{5} \partial \overline{\psi} \partial^{2} \psi.
\end{align*}

(4.28)

In this basis, one can consistently omit the spin-1 current \( V^{-1}(z) \); the remaining currents \( V^{i}(z) \) with \( i \geq 0 \) generate the \( W_{\infty} \) algebra in the standard basis, as in (3.17) (with \( q = \frac{1}{4} \)).

**4.3 Bosonic realisations of \( W_{\infty} \) and \( W_{1+\infty} \)**

In section, we shall consider two bosonic realisations of the algebras. The first of these

is a realisation of \( W_{\infty} \), in terms of currents built from bilinears in a free complex scalar field \( \phi \), with OPE

\[ \overline{\phi}(z)\phi(w) \sim \log(z - w). \]

(4.29)

It is easy to see that the spin-2 current \( \partial \overline{\phi} \partial \phi \) generates the Virasoro algebra with central charge \( c = 2 \). As in the fermionic realisation discussed previously, there is just one independent higher-spin current that can be built at each spin, by acting with additional derivatives on the two fields in the bilinear products. For any non-degenerate choice of higher-derivative currents, therefore, their OPEs will close on the same algebra. This turns out to be \( W_{\infty} \) [22]. The choice

\begin{equation}
V^{i}(z) = \frac{2^{-i-1}(i + 2)!}{(i + 1)(2i + 1)!} \sum_{k=0}^{i} (-)^{k} \binom{i + 1}{k} \binom{i + 1}{k + 1} \partial^{i-k+1} \overline{\phi} \partial^{k+1} \phi
\end{equation}

(4.30)
generates the $W_{\infty}$ algebra in the standard basis (3.17) [22], with the scaling parameter $q$ taking the value $\frac{1}{4}$. The first few $W_{\infty}$ currents in this realisation take the form

$$V^0 = \partial \overline{\phi} \partial \phi,$$

$$V^1 = \frac{1}{2} \partial \overline{\phi} \partial^2 \phi - \frac{1}{2} \partial^2 \overline{\phi} \partial \phi,$$

$$V^2 = \frac{1}{3} \partial \overline{\phi} \partial^2 \phi - \frac{3}{5} \partial^2 \overline{\phi} \partial^2 \phi + \frac{1}{5} \partial^3 \phi \partial \phi.$$

(4.31)

Finally in this section on bosonic realisations, let us consider the bosonisation of the one-parameter family of fermionic realisations of $W_{1+\infty}$ given by (4.24). The complex fermion $\psi$, and its conjugate $\overline{\psi}$, may be written in terms of a free real scalar $\varphi$, as

$$\psi =: e^{i\varphi} : \quad \overline{\psi} =: e^{-i\varphi} :,$$

(4.32)

where $\varphi$ satisfies the OPE

$$\varphi(z)\varphi(w) \sim \log(z - w).$$

(4.33)

As shown in [23], the fermion bilinear $\partial^i \overline{\psi} \partial^j \psi$ can be expressed as

$$: \partial^i \overline{\psi}(z) \partial^j \psi(z) := \sum_{k=1}^{i+j+1} \frac{1}{k}(-1)^{k-1-i} \binom{\ell}{k-1-i} \partial^{i+j-k+1} P^{(k)}(z),$$

(4.34)

where $P^{(k)}(z)$ is given by

$$P^{(k)}(z) =: e^{-\varphi(z)} \partial^k \varphi(z) :.$$

(4.35)

Using this, the currents of $W_{1+\infty}$ in the basis (4.24) can then be written in bosonised form. The first few bosonised currents are given by

$$V^{-1} = \partial \varphi,$$

$$V^0 = \frac{1}{2} (\partial \varphi)^2 + a \partial^2 \varphi,$$

$$V^1 = \frac{1}{3} (\partial \varphi)^3 + a \partial \varphi \partial^2 \varphi + \frac{1}{3} a^2 \partial^3 \varphi,$$

$$V^2 = \frac{1}{4} (\partial \varphi)^4 + a (\partial \varphi)^2 \partial^2 \varphi + \frac{1}{20} (8a^2 - 3)(\partial^2 \varphi)^2 + \frac{1}{10} (4a^2 + 1) \partial \varphi \partial^3 \varphi + \frac{1}{60} a (4a^2 + 1) \partial^4 \varphi.$$  

(4.36)

4.4 The super-$W_{\infty}$ algebra

By combining the complex-fermion realisation (4.24) of $W_{1+\infty}$, and the complex-boson realisation (4.30) of $W_{\infty}$, we may easily construct an $N = 2$ superalgebra whose bosonic sector is $W_{1+\infty} \times W_{\infty}$ [19]. The free Lagrangian $L = \overline{\partial \phi} \partial \phi + \overline{\psi} \partial \psi$ has a rigid $N = 2$ supersymmetry, and so it is guaranteed that if we form all possible bilinear currents from the fields $\phi$ and $\psi$, then they will generate an $N = 2$ superalgebra. Thus in addition to the
bosonic currents (4.24) (with the parameter $a$ taken to be zero for convenience) and (4.30), we may build fermionic currents $G^i(z)$ and $\overline{G}^i(z)$, of the form

$$G^i(z) = \sum_{k=0}^{i} \gamma_k(i) \partial^{i-k+1} \phi^k \partial \psi,$$

$$\overline{G}^i(z) = \sum_{k=0}^{i} \gamma_k(i) \partial^{i-k+1} \phi^k \overline{\psi}.$$

(4.37)

A convenient choice for the coefficients $\gamma_k(i)$ is [19]

$$\gamma_k(i) = \frac{2^{-i}(i+1)!(−)^k}{(2i+1)!!} \left( \binom{i}{k} \binom{i+1}{k} \right).$$

(4.38)

One can now check that the $W_{1+\infty}$ currents $\tilde{V}^i$ ($i \geq -1$), the $W_{\infty}$ currents $V^i$ ($i \geq 0$), and the fermionic currents $G^i$ and $\overline{G}^i$ ($i \geq 0$) close on a superalgebra, which we may call the $N=2$ super-$W_\infty$ algebra [19]. We shall not present details here; they are quite complicated, and may be found in [19]. Structurally, the form of the algebra is

$$VV \sim V, \quad \tilde{V} \tilde{V} \sim \tilde{V}, \quad V \tilde{V} \sim 0,$$

$$VG \sim G, \quad \tilde{V} G \sim G, \quad V \overline{G} \sim \overline{G}, \quad \tilde{V} \overline{G} \sim \overline{G},$$

$$GG \sim V \oplus \tilde{V}, \quad \overline{G} \overline{G} \sim 0, \quad G \overline{G} \sim 0.$$ (4.39)

5. The BRST operator and anomaly freedom

5.1 The BRST operator for the Virasoro algebra

The construction of the BRST operator for the Virasoro algebra is well known [24]. One introduces anticommuting fields $c(z)$ of conformal spin $-1$ and $b(z)$ of conformal spin $2$, which are the ghost and antighost for the spin-2 current $T(z)$. They satisfy the OPE

$$b(z)c(w) \sim \frac{1}{z-w}.$$ (5.1)

The BRST operator $Q$ is then defined in terms of the BRST current $j(z)$:

$$Q = \frac{1}{2\pi i} \oint dz \, j(z),$$ (5.2)

where $J(z)$ is given by

$$j(z) = Tc + c\partial c b.$$ (5.3)
The form of the cubic ghost term in (5.3) is dictated by the structure constants of the Virasoro algebra, and can be deduced from the general expression

\[ Q = X^a c_a - \frac{1}{2} f^{ab} c_a c_b c^c \]  

(5.4)

for the BRST operator for a Lie group \( G \) with generators \( X^a \) satisfying the algebra \([X^a, X^b] = f^{ab} c X^c\). The BRST operator \( Q \) should be nilpotent;

\[ Q^2 = 0. \]  

(5.5)

For a finite-dimensional Lie algebra, this is an automatic consequence of the definition (5.4), by virtue of the fact that the algebra satisfies the Jacobi identity. For an infinite-dimensional algebra such as the Virasoro algebra, however, there can be a non-trivial central term in the algebra, and this gives rise to the possibility of a BRST anomaly.

Nilpotence of \( Q \) defined by (5.2) is equivalent to the OPE of \( j(z) j(w) \) being a total derivative. For the BRST current (5.3) for the Virasoro algebra, one finds that, up to a total derivative,

\[ j(z) j(w) \sim \left( \frac{1}{2} c - 13 \right) c(z) c(w) \frac{c(z) c(w)}{(z - w)^4}, \]  

(5.6)

where \( c \) is the central charge in the Virasoro algebra generated by \( T(z) \). (There will be no confusion between the central charge \( c \) and the ghost field \( c \).) Thus there is a BRST anomaly unless the central charge satisfies

\[ c = 26. \]  

(5.7)

This is the celebrated result that implies that the bosonic string is anomaly free in 26 spacetime dimensions. Another way of expressing the result is that the ghost current \( T_{\text{gh}}(z) \), defined by \( T_{\text{gh}} = T_{\text{tot}} - T \) with \( T_{\text{tot}}(z) = \{Q, b(z)\} \), yields a realisation of the Virasoro algebra with central charge \( c_{\text{gh}} = -26 \), and this must be cancelled by an equal and opposite central charge from the matter realisation \( T \). The ghost current is given by

\[ T_{\text{gh}} = -2b \partial c - \partial b c. \]  

(5.8)

The BRST operator may be rewritten as

\[ Q = \frac{1}{\pi i} \oint dz \left( T(z) + \frac{1}{2} T_{\text{gh}}(z) \right) c(z). \]  

(5.9)

If one writes the BRST operator in the language of Laurent modes, then it is straightforward to establish also that the intercept is \( L_0 = 1 \), \textit{i.e.} that nilpotency of \( Q \) requires not only \( c = 26 \), but also that \( Q = (L_0 - 1)c_0 + \sum_{m \neq 0} L_m c_{-m} + \sum_{m < n} (m - n) : b_{m+n} c_{-m} c_{-n} : \).

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5.2 The BRST operators for $W_\infty$, $W_{1+\infty}$ and super $W_\infty$

For the case of $W$ algebras, with higher-spin currents, one must introduce ghost-antighost pairs for each such current. We shall postpone for now the discussion of how to construct the BRST operator for non-linear algebras such as $W_N$, and consider the more straightforward linear $W_\infty$-type algebras. We shall describe the example of $W_\infty$ in some detail; the discussion goes through *mutatis mutandis* for $W_{1+\infty}$ and super $W_\infty$.

For the $W_\infty$ algebra, given by (3.17), there will be a ghost field $c_i(z)$ of spin $s = -i - 1$, and an antighost field $b_i(z)$ of spin $s = i + 2$ for each current $V^i(z)$. From the general expression (5.4), one can show that the BRST current for the $W_\infty$ algebra will then be given by

$$j(z) = V^i c_i - \sum_{\ell \geq 0} f^{ij}_{2\ell}(\partial_{c_i}, \partial_{c_j})c_i c_j b_{i+j-2\ell}, \quad (5.10)$$

where summation over $i$ and $j$ is understood. In this expression, $\partial_{c_i}$ denotes $\partial/\partial z$ acting only on the ghost field $c_i$. The BRST operator may be written as

$$Q = \frac{1}{2\pi i} \oint dz \left( V^i(z) + \frac{1}{2} V^i_{gh}(z) \right) c_i(z), \quad (5.11)$$

where the ghost currents $V^i_{gh}(z)$ are given by

$$V^i_{gh}(z) = \sum_{\ell \geq 0} f^{ij}_{2\ell}(-\partial, \partial_{c_j})c_j b_{i+j-2\ell}, \quad (5.12)$$

where the derivative $\partial$ denotes $\partial/\partial z$ acting on both $c_j$ and $b_{i+j-2\ell}$, whilst $\partial_{c_j}$ denotes $\partial/\partial z$ acting only on $c_j$.

The ghost currents (5.12) should furnish a realisation of the $W_\infty$ algebra, with a certain central charge $c_{gh}$. Nilpotence of the BRST operator $Q$ will then be achieved if the matter realisation $V^i(z)$ has central charge $c = -c_{gh}$. The general structure of the OPE algebra for the currents $V^i_{gh}(z)$ turns out to be:

$$V^i_{gh}(z)V^j_{gh}(w) \sim -\sum_{\ell \geq 0} f^{ij}_{2\ell}(\partial_z, \partial_w) \frac{V^i_{gh}(w)}{z-w} + \frac{C_{ij}}{(z-w)^{i+j+4}}, \quad (5.13)$$

so the operator terms on the right-hand side are precisely those of the $W_\infty$ algebra. However a difficulty that arises in this case is that the expression (5.12) for the ghost currents involves sums over infinitely-many ghost fields, and consequently the central terms in the OPE algebra of the ghost currents will have divergent coefficients $C_{ij}$. A natural procedure, therefore, is to try to regularise the divergent sums. The hope would be that the regularised coefficients
$C_{ij}$ would turn out to have precisely the right form for $W_\infty$ with some central charge $c_{gh}$, i.e.

$$C_{ij} = \frac{2^{2i-3}i!(2i+3)!(i+2)!}{(2i+1)!!(2i+3)!!} c_{gh} \delta_{ij}, \quad (5.14)$$

(see (3.8) and (3.17), with $q = 1$). In fact, one could adopt the principle that the criterion for a “good” regularisation scheme is that it should yield precisely (5.14); after all, we know that the only possible structure for central terms that are compatible with the Jacobi identities is precisely that given in (5.14).

As is often the case, the naivest guess for how to regularise divergent quantities seems to be the right one. The forms of the divergent sums for $C_{ij}$ at arbitrary $i$ and $j$ are extremely complicated, involving contributions from the higher-order terms in (5.12); we shall return to these in a moment. Consider first, however, the $C_{00}$ term in (5.13). It is easy to see that this receives contributions only from the leading-order terms in (5.12); i.e.

$$V_{gh}^i(z) = (i + j + 2) \partial c_j b_{i+j} + (j + 1) c_j \partial b_{i+j} + \cdots, \quad (5.15)$$

where a summation over $j$ is understood. It is now easy to check that the ghosts $c_j$ and $b_j$ for the spin $s = (j + 2)$ current $V^j$ contribute a term

$$C_{00}(s) = -(6s^2 - 6s + 1) \quad (5.16)$$

to $C_{00}$, yielding the divergent result

$$C_{00} = - \sum_{s \geq 2} (6s^2 - 6s + 1). \quad (5.17)$$

A naive guess for regularising this would be to interpret the sums in terms of analytically-continued Riemann zeta functions. Thus, we may write

$$C_{00} = - \sum_{n \geq 0} \left[ 6(n + \frac{3}{2})^2 - \frac{1}{2} \right], \quad (5.18)$$

and then formally reinterpret $C_{00}$ as [26,25]

$$C_{00} = 6\zeta(-2, \frac{3}{2}) - \frac{1}{2} \zeta(0, \frac{3}{2}), \quad (5.19)$$

where $\zeta(s, a)$ is the generalised Riemann zeta function, defined as the analytic continuation from $\Re(s) > 1$ of the function

$$\zeta(s, a) = \sum_{n \geq 0} (n + a)^{-s}. \quad (5.20)$$
From (5.19) we find $C_{00} = 1$, and hence using (5.14) we get \[ c_{gh} = 2. \] \hspace{1cm} (5.21)

Of course if this were the only calculation of the regularised value for $c_{gh}$ it would be hard to take it seriously. One could arrive at any desired answer, by choosing the regularisation scheme appropriately. To give just one example, the second term in the summand of (5.18) could equally well be reinterpreted as $-\frac{1}{2}\zeta(0, a)$ for \textit{any} value of $a$, thus giving \[ c_{gh} = \frac{1}{2}(2a + 1). \] \hspace{1cm} (5.22)

In fact, however, the requirement that \textit{all} of the $C_{ij}$ coefficients should regularise consistently, to give (5.14), puts very stringent conditions on the scheme that one can use. Whilst one could, of course, still arrange to get any desired value for $c_{gh}$, the scheme that did this would look increasingly contrived as one proceeded to higher and higher spin anomalies. There really seems to be only one regularised value for $c_{gh}$ that can be arrived at in a “natural” and uncontrived way, and that is precisely the value given in (5.21) \[25\]. In \[25\], a regularisation scheme is proposed, and checked up to the spin-18 level (by computer!), that yields this result in a consistent way.

If one repeats the analysis for the $W_{1+\infty}$ algebra, the effect on the calculation of $C_{00}$ is to include the contribution of the $s = 1$ term in (5.17). This implies that $C_{00}$, and hence $c_{gh}$, is zero for $W_{1+\infty}$. Again, the self-consistent scheme proposed in \[25\] gives this same result for a wide range of examples that have been checked. There is in fact a further check on the whole scheme that can now be performed, since in the $W_{1+\infty}$ algebra there will also be an anomaly term in (5.13) involving $C_{-1, -1}$. This spin-1 anomaly in fact receives no contributions from ghosts, and so it is necessarily zero. Thus without any regularisation at all we know that the answer must be $c_{gh} = 0$ for $W_{1+\infty}$, and so it is a reassuring check on the validity of the regularisation scheme that it reproduces the already-correct result.

A similar analysis has also been carried out for the super-$W_{\infty}$ algebra \[25\]. In this case, it turns out that the total ghost central charge in the Virasoro sector is $c_{gh} = 3$. In summary, therefore, the values of the central charges in the \textit{matter} sector that are needed to achieve anomaly freedom are:

\begin{align*}
W_{\infty} : & \quad c = -2, \\
W_{1+\infty} : & \quad c = 0, \\
super-W_{\infty} : & \quad c = -3.
\end{align*} \hspace{1cm} (5.23)

By writing the BRST operators in terms of the Laurent modes of the currents, one can also determine the intercepts. As for the central terms, these require regularisation. It turns out that all intercepts regularise to zero for all the algebras $W_{\infty}$, $W_{1+\infty}$ and super $W_{\infty}$ \[25\]. All these results may be recast into the form of (regularised) Ricci-curvature conditions for certain Kähler coset manifolds of the form $W/H$, where $W$ is the group corresponding to
the particular \( W \) algebra in question, and \( H \) is a subgroup [27]. This description generalises that given for the Virasoro algebra in [28].

5.3 The BRST operator for \( W_3 \)

Although the \( W_3 \) algebra is non-linear, the construction of the BRST operator follows rather similar lines to that for a linear algebra. In particular, we still have an expression of the form (5.11) for \( Q \), namely

\[
Q = \oint dz \Bigl( (T + \frac{1}{2}T_{gh})c + (W + \frac{1}{2}W_{gh})\gamma \Bigr),
\]

(5.24)

where \((b, c)\) are the ghosts for \( T \), and \((\beta, \gamma)\) are the ghosts for \( W \). The matter currents \( T \) and \( W \) generate the \( W_3 \) algebra (2.1a-c). The non-linearity of the algebra reflects itself in the fact that the ghost current \( W_{gh} \) now involves the matter current \( T \) as well as the ghost fields [29,30]. Since the ghost current \( T_{gh} \) still has the standard form, it follows that the ghost central charge in the spin-2 sector is still given by the analogue of (5.17), leading to the requirement that the matter central charge \( c \) be given by [29]

\[
c = 26 + 74 = 100.
\]

(5.25)

The ghost currents \( T_{gh} \) and \( W_{gh} \) are then given by [29,30]

\[
T_{gh} = -2b \partial c - \partial b c - 3\beta \partial \gamma - 2\partial \beta \gamma
\]

(5.26a)

\[
W_{gh} = -3\beta \partial c - 3\beta \partial c - \frac{8}{261} \left[ \partial (b \gamma T) + b \partial \gamma T \right]
+ \frac{25}{6 \cdot 261} \hbar \left( 2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b \right).
\]

(5.26b)

Note that the spin-3 ghost current \( W_{gh} \) involves the spin-2 matter current \( T \). This looks intuitively reasonable; one can view the non-linear terms on the right-hand side of (2.1c) as being like a linear algebra but with \( T \)-dependent structure “constants.” These structure constants then appear in the construction of the ghost currents. Note also that the ghost currents need not, and indeed do not, satisfy the \( W_3 \) algebra [29,30]. It is shown in these references that, provided the matter central charge is given by (5.25), the BRST operator (5.24) is indeed nilpotent.

In [29,30], the BRST operator is actually constructed in terms of the Laurent modes \( L_m \) and \( W_m \) of \( T(z) \) and \( W(z) \). Nilpotency of \( Q \) not only implies that \( c \) should equal 100, but it also determines the values of the intercepts for \( L_0 \) and \( W_0 \). These turn out to be \( L_0 = 4 \), and \( W_0 = 0 \) [29,30].
6. Classical and quantum $W$ gravity

6.1 $W_\infty$ gravity

The classical theory of $w_\infty$ gravity was constructed in [31]. In its simplest form, one can consider a chiral gauging of $w_\infty$; i.e. one gauges just one copy of the algebra, in, say, the left-moving sector of the two-dimensional theory. We shall discuss non-chiral gaugings in more detail later. For now we just remark that, thanks to an ingenious trick introduced in [32], involving the use of auxiliary fields, the treatment of the non-chiral case can be essentially reduced to two independent copies of the chiral case.

As our starting point, let us consider the free action $S = 1/\pi \int d^2 z L$ for a single scalar field in two dimensions, where $L$ is given by

$$L = \frac{1}{2} \partial \varphi \partial \varphi.$$  \hspace{1cm} (6.1)

Here, we use coordinates $z = x^-$ and $\bar{z} = x^+$ on the (Euclidean-signature) worldsheet. This action is invariant under the semi-rigid spin-$s$ transformations

$$\delta \varphi = \sum_{s \geq 2} k_s (\partial \varphi)^{s-1},$$  \hspace{1cm} (6.2)

where the parameters $k_s$ are functions of $z$, but not $\bar{z}$. The spin-$s$ transformation, with parameter $k_s$, is generated by the current $v^i(z)$, with $i = s - 2$:

$$v^i(z) = \frac{1}{i + 2} (\partial \varphi)^{i+2}.$$  \hspace{1cm} (6.3)

At the classical level, these currents generate the $w_\infty$ algebra. In operator-product language, this means that they close on the $w_\infty$ algebra at the level of single contractions. This is given in Laurent-mode language in (3.14). To keep track of the orders it is useful to introduce Planck’s constant, so that the OPE of the field $\varphi$ is

$$\partial \varphi(z) \partial \varphi(w) \sim \frac{\hbar}{(z-w)^2}.$$  \hspace{1cm} (6.4)

The OPEs of the currents are then given by

$$\hbar^{-1} v^i(z) v^j(w) \sim (i + j + 2) \frac{v^{i+j}(w)}{(z-w)^2} + (i + 1) \frac{\partial v^{i+j}(w)}{z-w} + O(\hbar).$$  \hspace{1cm} (6.5)

The $\hbar$-independent terms on the right-hand side, corresponding to single contractions in the OPE, are precisely those for the $w_\infty$ algebra.
The classical semi-rigid $w_\infty$ symmetry (6.2) of (6.1) can be gauged by introducing a gauge field $A_i$ for each current $v^i$. Thus we find that the Lagrangian

$$ L = \frac{1}{2} \bar{\partial} \phi \partial \phi - \sum_{i \geq 0} A_i v^i \quad (6.6) $$

is invariant under local $w_\infty$ transformations [31], where the gauge fields are assigned the transformation rules:

$$ \delta A_i = \bar{\partial} k_i - \sum_{j=0}^i \left( (j + 1) A_j \partial k_{i-j} - (i - j + 1) k_{i-j} \partial A_j \right). \quad (6.7) $$

When one is presented with a classical theory it is natural, when considering its quantisation, to begin by contemplating what might go wrong. In the case of a gauge theory, with classical local symmetries, the obvious danger is that these might become anomalous at the quantum level. Indeed, in the case of two-dimensional gravity, the gauge theory of the Virasoro algebra, we know that anomaly freedom requires that the matter fields should generate the Virasoro algebra with central charge $c = 26$, in order to cancel the central-charge contribution of $-26$ from the ghosts for the gauge fixing of the spin-2 gauge field (the metric). We can certainly expect to meet analogous anomalies in the higher-spin generalisations that we are considering here. In fact, potentially-worse things could also happen:

Commonly, as for example in the case of the Virasoro algebra and two-dimensional gravity, one has matter currents that are quadratic in matter fields. These, by construction, generate the gauge algebra at the classical (single-contraction) level. Upon quantisation, higher numbers of contractions must also be taken into account, corresponding to Feynman diagrams with closed loops. If the currents are at most quadratic in matter fields, then the “worst case” is to have two contractions between a pair of currents. This corresponds therefore to a pure $c$-number term in the OPE of currents; in fact, the central term in the Virasoro algebra. It is these terms that in fact save the critical two-dimensional gravity theory from anomalies, by cancelling against anomalies from the ghost sector.

Things are potentially worse in the case of $w_\infty$ gravity because now the currents (6.3) involve arbitrarily-high powers of the matter field $\phi$. Thus at the quantum level, one might get matter-dependent anomalies, associated with diagrams corresponding to multiple contractions of the currents (6.3) that still have some matter fields left uncontracted. Of course the question of whether a theory is actually anomalous is really a cohomological one, in the sense that the crucial question is whether or not it is possible to introduce finite local counterterms, and $\hbar$-dependent renormalisations of the classical transformation rules, in such a way as to remove the apparently-anomalous contributions of the kind we have been considering. Only if such an attempt fails can the theory be said to be anomalous.

In general, the process of quantising a theory, and introducing counterterms and renormalisations of the transformation rules order by order in $\hbar$ to remove potential anomalies,
can be a complicated and tedious one. Fortunately, in our two-dimensional example the process of removing potential matter-dependent anomalies can be accomplished in one fell swoop. All that we have to do is to find quantum renormalisations of the classical currents (6.3) such that at the full quantum level (arbitrary numbers of contractions in the OPEs) they close on an algebra. This algebra, whatever it turns out to be, will be the quantum renormalisation of the original classical \( w_\infty \) algebra. In fact, as we shall see, it is precisely \( W_\infty \) [33].

The problem, then, boils down to the fact that the currents (6.3) do not close on any algebra at the quantum level. We must therefore seek \( \hbar \)-dependent modifications of them, with a view to achieving quantum closure. The most general plausible modifications would consist of higher-order terms added to (6.3) in which the same number of derivatives (to give the same spin \( s = i + 2 \)) are distributed over smaller numbers of \( \varphi \) fields. From (6.4), we see that \( \varphi \) has the dimensions of \( \sqrt{\hbar} \), and so the modifications will be of the form of power series in \( \sqrt{\hbar} \). Thus we may try an ansatz of the form

\[
V^i = \frac{1}{i+2} (\partial \varphi)^{i+2} + \alpha_i \sqrt{\hbar} (\partial \varphi)^i \partial^2 \varphi + \beta_i \hbar (\partial \varphi)^{i-1} \partial^3 \varphi + \gamma_i \hbar (\partial \varphi)^{i-2} (\partial^2 \varphi)^2 + O(\hbar^{3/2}),
\]

for constant coefficients \( \alpha_i, \beta_i, \gamma_i, \cdots \) to be determined. Requiring quantum closure of the OPE algebra for these currents will then give conditions on this infinite number of coefficients. They will not be determinable uniquely, since one is always free to make redefinitions of currents of the form \( V^i \to V^i + \partial V^{i-1} + \cdots \). However, if, for convenience and without loss of generality, we demand that the currents should all be quasi-primary with respect to the energy-momentum tensor \( V^0 \), then the result is unique. The expressions for the first few renormalised currents (spins 2, 3 and 4) are [33]:

\[
\begin{align*}
V^0 &= \frac{1}{2} (\partial \varphi)^2 + \frac{1}{2} \hbar \partial^2 \varphi, \\
V^1 &= \frac{1}{3} (\partial \varphi)^3 + \frac{1}{2} \sqrt{\hbar} \partial \varphi \partial^2 \varphi + \frac{1}{12} \hbar \partial^3 \varphi, \\
V^2 &= \frac{1}{4} (\partial \varphi)^4 + \frac{1}{2} \sqrt{\hbar} (\partial \varphi)^2 \partial^2 \varphi - \frac{1}{20} \hbar (\partial^2 \varphi)^2 + \frac{1}{5} \hbar \partial \varphi \partial^3 \varphi + \frac{1}{90} \hbar^{3/2} \partial^4 \varphi.
\end{align*}
\]

These renormalised currents can be recognised as the currents of the \( W_\infty \) algebra. We saw in section (4.3) that there is a realisation of \( W_{1+\infty} \) in terms of a real scalar \( \varphi \), obtained by bosonising the complex-fermion realisation (4.24). Setting the parameter \( a = \frac{1}{2} \), so that the truncation to \( W_\infty \) can be performed, and instating the parameter \( \hbar \), we see that the currents (6.9), and their higher-spin colleagues, are precisely the currents of the bosonisation of the complex-fermion realisation.

Having obtained renormalised currents that close on an algebra (the \( W_\infty \) algebra) at the full quantum level, we are now guaranteed to have a quantum theory with no matter-dependent anomalies. The prescription for writing down the counterterms and renormalisations of the transformation rules (6.2) and (6.7) necessary to make this anomaly freedom
manifest is very straightforward. For the counterterms, we simply replace the classical currents $v^i$ in the Lagrangian (6.6) by the renormalised currents $V^i$, of which the first few are given by (6.9). The terms independent of $\hbar$ are just the original classical currents, whilst the $\hbar$-dependent terms are the necessary counterterms. For the transformation rules, we use the ones that are generated by the renormalised currents. For $\varphi$, this means we have

$$\delta \varphi = \hbar^{-1} \sum_{i \geq 0} \oint \frac{dz}{2\pi i} k_i(z) V^i(z) \varphi(w). \quad (6.10)$$

For $A_i$, we will have

$$\delta A_i = \partial k_i + \delta A_i, \quad (6.11)$$

where $\delta A_i$ is such that

$$\sum_{i \geq 0} \int \left( \delta A_i V^i + A_i \delta V^i \right) = 0, \quad (6.12)$$

with $\delta V^i$ given by

$$\delta V^i = \hbar^{-1} \sum_{j \geq 0} \oint \frac{dz}{2\pi i} k_j(z) V^j(z) V^i(w). \quad (6.13)$$

The $\hbar$-independent terms in these transformation rules are precisely the original classical ones (6.2) and (6.7). The $\hbar$-dependent terms are the renormalisations necessary, together with the counterterms, to make the absence of matter-dependent anomalies manifest. The fact that the potential matter-dependent anomalies are actually removable by this means, i.e. that they are cohomologically trivial, is a consequence the fact that it is possible to renormalise the classical currents (6.3) to give currents that do close at the quantum level.

So far, we have been concerned here only with the question of matter-dependent anomalies. This is an issue that does not even arise for usual formulations of two-dimensional gravity, since the Virasoro symmetry is usually realised linearly (i.e. the currents are quadratic in matter fields). We must still face the analogues of the anomaly that one does meet in two-dimensional gravity, namely the universal anomaly that is removed by choosing a $c = 26$ matter realisation in order to cancel against the $-26$ contribution to the total central charge coming from the gravity ghosts. For $W_\infty$ gravity, we face a more serious-looking problem, since now there will be ghosts associated with the fixing of the gauge symmetry for each of the gauge-fields $A^i$. The ghosts for a spin-$s = i + 2$ gauge field contribute

$$c_{gh}(s) = -12s^2 + 12s - 2 \quad (6.14)$$

to the ghostly central charge. Summing over all $s \geq 2$ would seem to imply that the total ghostly central charge is $c_{gh}(\text{tot}) = -\infty$. At best, an infinity of matter fields seem to be needed, and even then, the process of cancelling the universal anomaly could be a delicate one. There is, however, a different approach that one could take, and that is to adopt the
regularisation scheme that was described in section (5.2). If this is done, then we find that
the regularised ghostly central charge is \( c_{gh} = 2 \).

Assuming for now that the \( c_{gh}(\text{tot}) = 2 \) result is to be taken seriously for \( W_\infty \), it
follows that the cancellation of the universal anomalies will occur provided that the matter
realisation of \( W_\infty \) has central charge \( c_{\text{mat}} = -2 \). Remarkably, this is precisely what we
have for our single-scalar realisation! One can easily check from (6.9) that the background-
charge term is precisely such as to give \( c = -2 \). Thus, in a regularised sense at least, the
\( W_\infty \) gravity theory that we have constructed is free of all anomalies [33]. This includes not
only the spin-2 anomaly, of the form \( (c_{gh}(\text{tot}) + c_{\text{mat}}) \int k_0 \partial^3 A_0 \), but also the higher-spin
anomalies, of the form \( C_i \int k_i \partial^{i+3} A_i \). For the same reasons as discussed in section (5.2), all
of the coefficients \( C_i \) will vanish simultaneously, provided that \( c_{\text{mat}} \) is equal to \(-2\).

For now, the possible cancellation of the regularised universal anomalies should perhaps
be viewed as an amusing observation that may ultimately turn out to have some deeper
underlying explanation. Perhaps the more important lesson to be derived from looking at
the quantisation of classical \( w_\infty \) gravity is that the key requirements are that one should be
able to renormalise the classical currents so that they close on an algebra at the full quantum
level. This ensures the absence of matter-dependent anomalies. Furthermore, if the central
charge for the matter currents is chosen to cancel that from the ghosts for the gauge fields,
then the universal anomalies will cancel also. These desiderata can be summarised succinctly
in one equation: we require that the BRST operator \( Q \) should be nilpotent.

6.2 \( W_3 \) gravity

The philosophy for quantising classical \( w_3 \) gravity is essentially the same as that of the
previous section. The starting point is the classical matter Lagrangian [34]

\[
L = \frac{1}{2} \partial \varphi_i \partial \varphi_i - hT - BW, \tag{6.15}
\]

where \( \varphi_i \) denotes a set of \( n \) real matter fields; \( h \) and \( B \) are spin-2 and spin-3 gauge fields;
and the spin-2 and spin-3 matter currents \( T \) and \( W \) are given by

\[
T = \frac{1}{2} \partial \varphi_i \partial \varphi_i, \quad W = \frac{1}{3} \partial \varphi_i \partial \varphi_j \partial \varphi_k. \tag{6.16}
\]

The quantity \( d_{ijk} \) is a totally-symmetric constant tensor that satisfies

\[
d_{(ij} d_{km)} = \mu \delta_{(ij} \delta_{km)}. \tag{6.17}
\]
At the classical level (single contractions), the currents generate what we may call the $w_3$ algebra,

\begin{align}
\hbar^{-1}T(z)T(w) &\sim \frac{\partial T}{z-w} + \frac{2T}{(z-w)^2}, \\
\hbar^{-1}T(z)W(w) &\sim \frac{\partial W}{z-w} + \frac{3W}{(z-w)^2}, \\
\hbar^{-1}W(z)W(w) &\sim \frac{\partial \Lambda}{z-w} + \frac{2\Lambda}{(z-w)^2},
\end{align}

(6.18)

where $\Lambda$ is the composite current

\[ \Lambda = 2\mu(TT). \]

(6.19)

This $w_3$ algebra is a classical limit of the full $W_3$ algebra given in (2.1a-c).

Various possible solutions for the tensor $d_{ijk}$, satisfying (6.17), have been found [35]. They fall into two categories. The first consists of solutions for an arbitrary number of scalars $n$, with the components of the (totally symmetric) tensor $d_{ijk}$ given by

\[ d_{111} = n, \quad d_{1ab} = -n\delta_{ab}; \quad (a = 2, \ldots, n). \]

(6.20)

This satisfies (6.17) with $\mu = n^2$. The second category of solution relies upon the abnormalities and perversities of the Jordan algebras. There are four such solutions, with $n = 5, 8, 14$ and $26$ scalars, corresponding to invariant tensors of Jordan algebras over the reals, complex numbers, quaternions and octonions respectively [35]. For the complex case, with $n = 8$, the $d_{ijk}$ tensor coincides with the symmetric $d_{ijk}$ tensor of $SU(3)$.

As in the case of the currents (6.3) that generate the $w_\infty$ contraction of $W_\infty$ classically, so also here the currents (6.16) generate the $w_3$ contraction of $W_3$ classically. At the full quantum level of multiple contractions in the operator-product expansion, one finds that the classical $w_3$ currents (6.16) fail to close on any algebra. This is the signal for potential problem with matter-dependent anomalies upon quantisation of the theory. The remedy is again to look for quantum renormalisations of the currents (6.16) to give currents that do generate an algebra at the quantum level. Modulo the freedom to make field redefinitions, the answer, as in the $w_\infty$ case, is unique. In this case, it turns out that the resulting algebra on which the renormalised currents close is $W_3$, given by (2.1).

The possible renormalisations of the currents (6.16) can be parametrised by

\begin{align}
T &= \frac{1}{2}\partial \phi^i \partial \phi^i + \sqrt{\hbar}\alpha_i \partial^2 \phi^i, \\
W &= \frac{1}{3}d_{ijk}\partial \phi^i \partial \phi^j \partial \phi^k + \sqrt{\hbar}e_{ij} \partial \phi^i \partial^2 \phi^j + \hbar f_i \partial^3 \phi^i.
\end{align}

(6.21)

The requirement that the currents generate the $W_3$ algebra gives a set of conditions on the coefficients $d_{ijk}$, $\alpha_i$, $e_{ij}$ and $f_i$ that may be found in [35]. The upshot is that the general family of classical currents, with $d_{ijk}$ given by (6.20) for the case of $n$ scalars, can
be successfully renormalised to give currents that close at the quantum level, on the full $W_3$ algebra [35]. We shall give the form of the renormalised currents below. For the four exceptional cases based on the Jordan algebras, however, it appears that it is not possible to renormalise the currents with $\hbar$-dependent corrections so as to achieve closure [35,36]. This includes the special 8-scalar realisation based on the totally-symmetric $d_{ijk}$ tensor of $SU(3)$. There is thus no sense in which the currents in these exceptional cases could be said to be $W_3$ currents. Although a complete proof of the impossibility of constructing $W_3$ realisations based on the quaternionic and octonionic Jordan algebras is still lacking, it has been proven that in all four exceptional cases there can be no background charges, and hence the central charges will be 5, 8, 14 and 26 respectively [35]. Consequently, even if the classical currents could be successfully renormalised, to avoid matter-dependent anomalies, the quantum theory of $w_3$ gravity for any of the four exceptional cases would definitely suffer from universal anomalies, since anomaly freedom requires that $c = 100$. On the other hand, for the general family of $n$-scalar realisations with $d_{ijk}$ given by (6.20), as we shall see below, all anomalies can be cancelled [37].

Having established that the matter currents for the $n$-scalar realisations (6.16), (6.20) can be renormalised to give currents that close, on the $W_3$ algebra, we are now able to proceed to the next stage in the quantisation procedure, which is to ensure that the matter currents yield an anomaly-free realisation of the $W_3$ algebra. This means, as we saw in section (5.3), that the matter central charge should be $c = 100$. A 2-scalar realisation, with background charge that can be tuned to give, in particular, $c = 100$, was obtained in [5] by making use of the quantum Miura transformation. The most general known realisations in terms of scalar fields are the $n$-scalar realisations found in [35], which correspond to the renormalisations (6.21) of the classical currents (6.16) with $d_{ijk}$ given by (6.20). At $c = 100$, these take the form

\[ T = T_X + \frac{1}{2}(\partial \varphi_1)^2 + \frac{1}{2}(\partial \varphi_2)^2 + \sqrt{\hbar}(\alpha_1 \partial^2 \varphi_1 + \alpha_2 \partial^2 \varphi_2) \]  

(6.22a)

\[ W = \frac{2}{\sqrt{261}} \left\{ \frac{1}{3}(\partial \varphi_1)^3 - \varphi_1 (\partial \varphi_2)^2 + \sqrt{\hbar}(\alpha_1 \partial \varphi_1 \partial^2 \varphi_1 - 2\alpha_2 \partial \varphi_1 \partial^2 \varphi_2 - \alpha_1 \partial \varphi_2 \partial^2 \varphi_2) 
+ \hbar \left( \frac{1}{3} \alpha_1^2 \partial^3 \varphi_1 - \alpha_1 \alpha_2 \partial^3 \varphi_2 \right) - 2\partial \varphi_1 T_X - \alpha_1 \sqrt{\hbar} \partial T_X \right\}, \]  

(6.22b)

where $T_X$ is a stress tensor for $D = n - 2$ scalar fields without background charges,

\[ T = \frac{1}{2} \sum_{\mu=1}^{D} \partial X^\mu \partial X^\mu, \]  

(6.23)

and the background charges $\alpha_1$ and $\alpha_2$ for $\varphi_1$ and $\varphi_2$ are given by

\[ \alpha_1^2 = \frac{-49}{8}, \quad \alpha_2^2 = \frac{1}{12} (D - \frac{49}{2}). \]  

(6.24)
These conditions on the background charges ensure that the matter central charge satisfies
\[ c = D + (1 - 12\alpha_1^2) + (1 - 12\alpha_2^2) = 100. \] (6.25)
Note that no matter how many scalar fields one chooses, including \( n = 100 \), it is necessary to have background charges in order to achieve \( c = 100 \).

Now that we have obtained a nilpotent BRST operator, and appropriate matter realisations of the \( W_3 \) algebra, it is completely straightforward to write down a Lagrangian for anomaly-free \( W_3 \) gravity. We shall not give the detailed result here; it may be found in [37]. Here, we just remark that it is obtained from the general BRST prescription:
\[ \mathcal{L} = \frac{1}{2} \partial \varphi^i \partial \varphi^i - hT - BW + \delta \left( b(h - h_{\text{back}}) + \beta (B - B_{\text{back}}) \right), \] (6.26)
where \( \delta \) denotes the BRST variation, which can be deduced from the BRST operator (5.24), and \( h_{\text{back}} \) and \( B_{\text{back}} \) denote background gauge-fixed values for the spin-2 and spin-3 gauge fields \( h \) and \( B \). Thus one has [37]
\[ \mathcal{L} = \frac{1}{2} \partial \varphi^i \partial \varphi^i - b \partial c - \beta \partial \gamma \\
+ \pi_b (h - h_{\text{back}}) + \pi_\beta (B - B_{\text{back}}) - h(T + T_{\text{gh}}) - B(W + W_{\text{gh}}), \] (6.27)
where \( \delta b = \pi_b \) and \( \delta \beta = \pi_\beta \). As in section 2, the \( h \)-independent terms in (6.27) (with matter currents \( T \) and \( W \) given by (6.22a,b), and ghost currents \( T_{\text{gh}} \) and \( W_{\text{gh}} \) given by (5.26a,b)) represent the classical Lagrangian (plus ghost Lagrangian), and the \( h \)-dependent terms correspond to counterterms necessary for the explicit removal of the potential anomalies.

6.3 Discussion

In sections (6.1) and (6.2), we have reviewed some of the aspects of the quantisation of \( W_\infty \) and \( W_3 \) gravities. Our discussion has been concerned entirely with chiral \( W \) gravity, in the sense that we have considered gaugings only of a single (left-moving) copy of the algebra. As remarked at the beginning of section (6.1), it is completely straightforward to extend all of the discussions in this paper to the non-chiral case by exploiting the ingenious trick, introduced in [32], of using additional, auxiliary, fields. Thus, for example, for \( W_3 \) gravity we introduce auxiliary fields \( J^i \) and \( \tilde{J}^i \), and write the classical Lagrangian as [32,38]
\[ \mathcal{L} = -\frac{1}{2} \partial \varphi^i \partial \varphi^i - J^i \tilde{J}^i + \tilde{J}^i \partial \varphi^i + J^i \bar{\partial} \varphi^i \\
- \frac{1}{2} h J^i J^i - \frac{1}{3} B d_{ijk} J^i J^j J^k - \frac{1}{2} h \tilde{J}^i \tilde{J}^i - \frac{1}{2} \tilde{B} d_{ijk} \tilde{J}^i \tilde{J}^j \tilde{J}^k, \] (6.28)
where the tilded variables refer to a second (right-moving) copy of the gauge algebra. The equations of motion for the auxiliary fields are
\[ J^i = \partial \varphi^i - \tilde{h} \tilde{J}^i - \tilde{B} d_{ijk} \tilde{J}^j \tilde{J}^k, \]
\[ \tilde{J}^i = \bar{\partial} \varphi^i - h J^i - B d_{ijk} J^j J^k, \] (6.29)
which can be recursively solved to give $J^i$ and $\tilde{J}^i$ as non-polynomial expressions in $\varphi^i$ and the gauge fields. Upon quantisation, one finds that the auxiliary fields have the propagators

$$J^i(z)J^j(w) \sim \frac{\hbar \delta^{ij}}{(z \cdot w)^2},$$

$$\tilde{J}^i(z)\tilde{J}^j(\bar{w}) \sim \frac{\hbar \delta^{ij}}{(\bar{z} \cdot \bar{w})^2},$$

$$J^i(z)\tilde{J}^j(\bar{w}) \sim 0.$$  

Thus the whole problem has been cloven into separate left-moving and right-moving sectors [38]. The left-moving matter and ghost currents are now constructed, at the full quantum level, by replacing $\partial \varphi^i$ in (6.16), etc., by $J^i$. Similarly, one uses $\tilde{J}^i$ in the construction of analogous right-moving currents. Full details may be found in [38].

An obvious application for anomaly-free $W_3$ gravity is in the construction of the $W_3$ extension of string theory, i.e. $W_3$ strings. The idea is that the equations of motion for the spin-2 and spin-3 gauge fields impose the vanishing of the spin-2 and spin-3 currents. At the quantum level, these conditions can be interpreted, as in ordinary string theory, as operator constraints on physical states. By interpreting the scalar fields in the matter realisations (6.22a,b) as spacetime coordinates, one arrives at a first-quantised description of $W_3$-string excitations in an $n$-dimensional spacetime. Because of the necessity for background charges one does not have the full $SO(1,n-1)$ Lorentz group acting, but only $SO(1,n-3)$. The issues arising in the analysis of the spectrum of $W_3$ strings are quite involved. Preliminary discussions were given in [38], and a more extensive analysis is contained in [39]. Many aspects have also been discussed in [40]. Here, we just summarise a couple of the main results.

One can see from the form of the realisations (6.22a,b) that the scalar $\varphi_1$ is on a very special footing. In fact all the remaining scalars ($\varphi_2$ and $X^\mu$) appear only via their stress tensor. (In the case of $\varphi_2$, it has a background-charge contribution in the stress tensor.) Thus, in some sense the scalar $\varphi_1$ is the only one which is intrinsically "non-stringy" in nature. The spin-3 current $W$ can be written as a sum of terms that involve only $\varphi_1$, plus terms involving the total stress tensor $T$. It is then rather easy to see at the classical level that having imposed the $T = 0$ constraint, the $W$ constraint reduces to the statement that $\varphi_1 = \text{constant}$ [38]. At the first-quantised level, this becomes the statement that physical states cannot involve any $\alpha_{-n}$ creation operators in the $\varphi_1$ direction [39]. The conclusion from this is that the sole effect of the $W$ constraint is to "freeze out" the $\varphi_1$ coordinate. The generalisation to $W_3$ strings introduces the new feature of the non-stringy coordinate $\varphi_1$, but this is then removed again by the new $W$ constraint. In fact one is essentially left with a theory looking very like ordinary string theory, except that the "effective" central charge is $25\frac{1}{2}$ instead of $26$, and the "effective" $L_0$ intercept is either $1$ or $\frac{15}{16}$. It seems that $W_3$ string theory is closely related to ordinary critical string theory, with the $c = 26$ central
charge achieved by taking spacetime (with background charges) to contribute $c = 25\frac{1}{2}$, and adjoining a $c = \frac{1}{2}$ minimal model [40]. In general, for a $W_N$ string, it seems that it will be closely related to an ordinary string with a $c = 1 - \frac{6}{N^2}$ minimal model.

It would be interesting to see what happens for a supersymmetric extension of the $W_3$ algebra. This is currently under investigation [41,42].

There are other aspects of the quantisation of $W$-gravity theories that we have not touched on here. In particular, there is the very interesting problem of constructing the $W_3$ analogue of the Polyakov induced action of two-dimensional gravity [43]. To do this, one wants to choose a matter realisation with non-critical value for the central charge, so that the universal anomaly “brings to life” the Liouville field (and its higher-spin analogues). Considerable progress in this area has been made [44]. There seems to be a certain sense in which there is really no such thing as a “non-critical” theory, since the Liouville fields will always come to the rescue and make up the deficit in the central charge. It would be interesting to see whether there is ultimately a convergence of the critical and non-critical approaches. Similar issues have also been considered for $W_\infty$ and $W_{1+\infty}$ gravity. In particular, it has been shown that the hidden $SL(2,R)$ Kac-Moody symmetry of light-cone two-dimensional gravity [45] generalises to $SL(\infty,R)$ for the $W_\infty$ case, and $GL(\infty,R)$ for the $W_{1+\infty}$ case [46].

ACKNOWLEDGMENTS

I am very grateful to all my collaborators, and others, for discussions. These include: Eric Bergshoeff, Paul Howe, Keke Li, Hong Lu, Larry Romans, Stany Schrans, Ergin Sezgin, Shawn Shen, Kelly Stelle, Xujing Wang, Kaiwen Xu and Kajia Yuan.
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