ON THE LOGARITHMIC DERIVATIVE OF CHARACTERISTIC POLYNOMIALS
FOR RANDOM UNITARY MATRICES

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ABSTRACT. Let $U \in U(N)$ be a random unitary matrix of size $N$, distributed with respect to the Haar measure on $U(N)$. Let $P(z) = P_U(z)$ be the characteristic polynomial of $U$. We prove that for $z$ close to the unit circle, $\frac{dP}{dz}(z)$ can be approximated using zeros of $P$ very close to $z$, with a typically controllable error term. This is an analogue of a result of Selberg for the Riemann zeta-function. We also prove a mesoscopic central limit theorem for $\frac{dP}{dz}(z)$ away from the unit circle, and this is an analogue of a result of Lester for zeta.

1. INTRODUCTION

Let $\zeta(s)$ be the Riemann zeta-function, and let $\rho = \beta + i\gamma$ denote a generic nontrivial zero of zeta. A beautiful result of Selberg [19] says that for $s = \frac{1}{2} + it$ with $t \in [T, 2T]$ and $s \neq \rho$ we can write

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma| < \frac{1}{\log T}} \frac{1}{s-\rho} + D,$$  \hspace{1cm} (1)

where the error term $D$ is in terms of an explicit Dirichlet polynomial and satisfies

$$\frac{1}{T} \int_T^{2T} |D|^{2K} dt \ll K \log^{2K} T$$

for all positive integers $K$. In other words, $\frac{\zeta'}{\zeta}(s)$ can be approximated using zeros very close to $s$, with a typically controllable error. Radziwill [17] observed that Selberg’s argument with some modification also gives that for every constant $0 < c \leq 1$ and $s$ close to the critical line ($0 \leq \Re(s) – 1/2 \ll 1/\log T$, say),

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma| < \frac{c}{\log T}} \frac{1}{s-\rho} + E,$$ \hspace{1cm} (2)

with

$$\frac{1}{T} \int_T^{2T} |E|^{2K} dt \ll K \left(\frac{\log T}{c}\right)^{2K}.$$  

(Radziwill’s paper assumed the Riemann Hypothesis mainly for other purposes; for (2) alone one can show it holds unconditionally.) For applications of these results, see for example Selberg [19], Radziwill [17], and Ge [11].
It is well known that characteristic polynomials of the circular unitary ensemble (CUE) models the Riemann zeta-function $\zeta(s)$, with the matrix size $N$ about the same as $\log T$. Our first result in this paper is a CUE analogue of (1) and (2). Throughout, let $U(N)$ be the set of unitary matrices of size $N$, equipped with Haar measure. For $U \in U(N)$, write

$$P(z) = P_U(z) = \prod_{j=1}^{N} (z - z_j)$$

for the characteristic polynomial of $U$, where

$$z_j = e^{i\theta_j} \text{ with } -\pi < \theta_j \leq \pi, \text{ for } j = 1, 2, ..., N.$$

**Theorem 1.1.** Let $0 < c \leq 1$ be a constant. For $1 - \frac{1}{N} \leq z \leq 1$ and $z \neq z_j$ we have

$$\frac{P'(z)}{P(z)} = \sum_{|\theta_j| < \frac{2\pi}{N}} \frac{1}{z - z_j} + \mathcal{E},$$

where the error term $\mathcal{E} = \sum_{|\theta_j| \geq \frac{2\pi}{N}} \frac{1}{z - z_j}$ satisfies

$$\mathbb{E} |\mathcal{E}|^{2K} \ll_K \left( \frac{N}{c} \right)^{2K}$$

for every positive integer $K$. Here the expectation $\mathbb{E}$ is over $U(N)$ with respect to the Haar measure.

**Remark 1.** Since the Haar density of a configuration of eigenvalues in CUE is invariant under rotations, Theorem 1.1 holds not just for real $z$ but for all $1 - \frac{1}{N} \leq |z| \leq 1$ with obvious modifications in the statement.

**Remark 2.** Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein and Snaith [3] proved that for $K \in \mathbb{Z}^+$

$$\int_{U(N)} \left| \frac{P'}{P} \left( 1 - \frac{a}{N} \right) \right|^{2K} dU \sim \left( \frac{2K - 2}{K - 1} \right) \frac{N^{2K}}{(2a)^{2K-1}}$$

(3)

where $a \to 0$ as $N \to \infty$. They also conjectured a similar asymptotic for zeta, and this was studied in [11]. In [1] Alvarez and Snaith proved analogous results of (3) for orthogonal and symplectic random matrices. More recently, Alvarez, Bousseyroux and Snaith [2] extended the corresponding result for the odd orthogonal ensemble to non-integer moments (namely, $K \in \mathbb{R}^+$). In [12] we shall apply our Theorem 1.1 to obtain asymptotics for real moments analogues of (3) in unitary, even orthogonal, and symplectic ensembles.

We also investigate the value distribution of $\frac{P'(z)}{P(z)}$ in the mesoscopic range away from the unit circle, and prove that it obeys a central limit theorem (CLT). To put it in context, Selberg’s central limit theorem states that, roughly speaking, the value distribution of the vector

$$\frac{1}{\sqrt{\frac{1}{2} \log \log T}} (\Re \log \zeta(1/2 + it), \Im \log \zeta(1/2 + it)),$$  \quad t \in [T, 2T]
converges to a normal random vector \((X, Y)\) as \(T\) tends to infinity. Here \(X\) and \(Y\) are independent and both have mean 0 and variance 1, and \(t\) is drawn uniformly from \([T, 2T]\). The result also holds for \(\sigma + it\) in place of \(1/2 + it\) if \(\sigma\) is close to 1/2, but the variance changes accordingly. See Tsang’s thesis [22] for a detailed description and proof. The imaginary part of \(\log \zeta(1/2 + it)\) is related to the counting of zeta zeros via the Riemann-von Mangoldt formula; in this direction Selberg’s CLT for \(\Im \log \zeta(1/2 + it)\) is related to macroscopic and mesoscopic CLTs for value distributions (again \(t\) being drawn uniformly in \([T, 2T]\)) of quantities of the form \(\sum \eta(\Delta \cdot (\rho - t))\), where the sum is over zeta zeros, \(\eta\) is a suitable function, and \(\Delta\) is a scaling factor. Here macroscopic scale refers to the case when \(\Delta\) is of constant size, and mesoscopic scale is the case when \(\Delta \to \infty\) with \(T\) but \(\Delta = o(\log T)\). See Fujii [10], Bourgade and Kuan [5], Rodgers [13], and Maples and Rodgers [16]. Another related result is a mesoscopic CLT of Lester [15] for the logarithmic derivative of zeta away from the critical line, who proved that when \(t\) ranges from \(T\) to \(2T\) the real part and the imaginary part of \(\frac{\zeta'}{\zeta} \left( \frac{1}{2} + \frac{\psi(T)}{\log T} + it \right)\) are close to independent normal with mean 0 and variance

\[
V_{\text{zeta}} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{\Lambda(n)^2}{n^{1 + \frac{\psi(n)}{\log n}}},
\]

provided that \(\psi(T) = o(\log T)\) and \(\psi(T) \to \infty\) with \(T\). A standard calculation shows that

\[
V_{\text{zeta}} \sim \frac{1}{8} \left( \frac{\log T}{\psi(T)} \right)^2. \tag{4}
\]

Analogues of Selberg’s CLT as well as macroscopic and mesoscopic CLTs for sums over zeta zeros are also known in the CUE setting; see Keating and Snaith [14], Hughes, Keating and O’Connell [13], Bourgade [4], Szegö [21], Wieand [23], Diaconis and Evans [8], and Soshnikov [20]. Related to Lester’s result, in the CUE setting there is a mesoscopic CLT away from the real line proved by Chhaibi, Najnudel and Nikeghbali [6] for a limiting object of the characteristic polynomials. Our next result is a CUE analogue of Lester’s CLT.

**Theorem 1.2.** Let \(L = L_N\) be a quantity such that \(L_N = o(N)\) and \(L_N \to \infty\) as \(N \to \infty\). Then with \(U\) drawn from \(U(N)\) with respect to the Haar measure, the random vector

\[
\left( \Re \left[ \frac{L}{N} \cdot \frac{P'}{P} \left( 1 - \frac{L}{N} \right) \right], \Im \left[ \frac{L}{N} \cdot \frac{P'}{P} \left( 1 - \frac{L}{N} \right) \right] \right)
\]

converges in distribution to a normal vector \((G, H)\) as \(N \to \infty\), where \(G\) and \(H\) are independent normals with mean 0 and variance 1/8.

Therefore, for large \(N\) the real part and the imaginary part of \(\frac{P'}{P} \left( 1 - \frac{L}{N} \right)\) are close to independent normal random variables with mean 0 and variance \(V\) with

\[
V = \frac{1}{8} \left( \frac{N}{L} \right)^2. \tag{5}
\]
This variance agrees with Lester’s variance (4) in the zeta case.

2. PROOF OF THEOREM 1.1

Define

\[ z_0 = 1 - \frac{1}{N} \]  

and let

\[ z_0 \leq z \leq 1. \]  

We write

\[ \frac{P'(z)}{P(z)} = \sum_{|\theta_j| < \frac{1}{N}} \frac{1}{z - z_j} + X_1 + X_2 - X_3, \]

where

\[ X_1 = \frac{P'(z_0)}{P(z_0)} = \sum_{j=1}^{N} \frac{1}{z_0 - z_j}, \]

\[ X_2 = \sum_{|\theta_j| \geq \frac{1}{N}} \left( \frac{1}{z - z_j} - \frac{1}{z_0 - z_j} \right), \]

\[ X_3 = \sum_{|\theta_j| < \frac{1}{N}} \frac{1}{z_0 - z_j}. \]

Theorem 1.1 follows from the next three propositions.

**Proposition 2.1.** For \( K \in \mathbb{Z}^+ \) we have \( \mathbb{E}|X_1|^{2K} \ll_K N^{2K}. \)

**Proposition 2.2.** For \( K \in \mathbb{Z}^+ \) we have \( \mathbb{E}|X_2|^{2K} \ll_K \left( \frac{N}{c} \right)^{2K}. \)

**Proposition 2.3.** For \( K \in \mathbb{Z}^+ \) we have \( \mathbb{E}|X_3|^{2K} \ll_K N^{2K}. \)

2.1. Proof of Proposition 2.1. We will need the following ratios formula from Conrey and Snaith [7].

**Theorem 2.4.** (Conrey and Snaith.) If \( \Re \alpha_j > 0 \) and \( \Re \beta_j > 0 \) for \( \alpha_j \in A \) and \( \beta_j \in B \), then \( J(A; B) = J^*(A; B) \) where

\[ J(A; B) := \int_{U(N)} \prod_{\alpha \in A} (-e^{-\alpha}) \frac{P_U^\alpha}{P_U} \prod_{\beta \in B} (-e^{-\beta}) \frac{P_U^\beta}{P_U^\beta} (e^{-\beta}) dU, \]

\[ J^*(A; B) := \sum_{S \subset A, T \subset B} e^{-N(\sum_{\alpha \in S} \hat{\alpha} + \sum_{\beta \in T} \beta)} \frac{Z(S, T)Z(S^-, T^-)}{Z^\dagger(S, S^-)Z^\dagger(T, T^-)} \sum_{(A-S)+(B-T)} r=1 \prod_{u_{r \leq 2}} H_{S,T}(U_r). \]
and

\[
H_{S,T}(W) = \begin{cases} 
\sum_{\alpha \in S} \frac{e^z}{z} (\alpha - \hat{\alpha}) - \sum_{\beta \in T} \frac{e^z}{z} (\beta - \hat{\beta}) & \text{if } W = \{\alpha\} \subset A - S \\
\sum_{\beta \in T} \frac{e^z}{z} (\beta - \hat{\beta}) - \sum_{\alpha \in S} \frac{e^z}{z} (\alpha + \hat{\alpha}) & \text{if } W = \{\beta\} \subset B - T \\
\left(\frac{e^z}{z}\right)' (\alpha + \beta) & \text{if } W = \{\alpha, \beta\} \text{ with } \alpha \in A - S, \beta \in B - T \\
0 & \text{otherwise.}
\end{cases}
\]

Here \( z(x) = (1 - e^{-x})^{-1} \), \( S^- = \{-s : s \in S\} \) (similarly for \( T^- \)) and \( Z(A, B) = \prod_{\alpha \in A} \prod_{\beta \in B} \frac{z(\alpha + \beta)}{z(\alpha - \beta)} \), with the dagger on \( Z^*(S, S^-) \) imposing the additional restriction that a factor \( z(x) \) is omitted if its argument is zero.

Here the \( U^* \) is the conjugate transpose of \( U \), and it is easy to see that

\[
E \left| \frac{P_U'(z)}{P_U(z)} \right|^{2K} = \int_{U(N)} \left( \frac{P_U'(z)}{P_U(z)} \right)^K \left( \frac{P_U^*(z)}{P_U^*(z)} \right)^K dU.
\]

One may try to apply the above theorem by letting all the \( \alpha \)'s in \( A \) equal and similar for \( \beta \)'s in \( B \). However, this will cause complications because many terms in the \( J^* \) will have poles. To get around this, we will apply the ratios formula to a discretized set of points, as follows. Let \( r = \frac{1}{2N}, r_k = \frac{r}{2^k} \), and define

\[
G(x_1, \ldots, x_K) = \int_{U(N)} \frac{P_U'(x_1)}{P_U(x_1)} \cdots \frac{P_U'(x_K)}{P_U(x_K)} dU.
\]

Since \( z_0 \) is real, to prove the proposition it suffices to show that \( G(z_0, \ldots, z_0) \ll \frac{N^{2K}}{K} \). Observe that \( G \) is holomorphic in each variable in a certain domain around \( z_0 \), so applying the maximum modulus principle to \( G(x, z_0, \ldots, z_0) \) as a function of \( x \), we see that there exists a \( w_1 \) on the circle \( C(z_0, r_1) \) centered at \( z_0 \) with radius \( r_1 \) such that

\[
|G(w_1, z_0, \ldots, z_0)| \geq |G(z_0, z_0, \ldots, z_0)|.
\]

Next, apply the maximum modulus principle to \( G(w_1, x, z_0, \ldots, z_0) \) as a function of \( x \), and we see that there exists a \( w_2 \) on the circle \( C(z_0, r_2) \) such that

\[
|G(w_1, w_2, z_0, \ldots, z_0)| \geq |G(w_1, z_0, \ldots, z_0)|.
\]

Repeat this and we conclude that there are points \( w_1, \ldots, w_k \) with \( w_i \) on the circle \( C(z_0, r_i) \) for each \( i \), such that

\[
|G(w_1, \ldots, w_k)| \geq |G(z_0, \ldots, z_0)|.
\]

By the definition \( (5) \) of \( z_0 \), it is easy to see that \( \min_{i,j} |w_i - w_j| \gg r \). Since the function \( \log(1 - x) \) is close to \( -x \) around \( x = 0 \), we conclude that the points \( \log(w_k) \)'s are also well-spaced by a distance \( \gg r \gg \frac{1}{N} \). We can now apply the ratios formula to the \( w_k \)'s. In the formula for \( J^* \), from the above observation that these points are well-spaced, it is easy to see that all the \( Z(S, T) \) and \( Z(S^-, T^-) \) factors on the numerator are each bounded by \( \frac{N^{1|S|^2}}{K} \), and all the \( Z^*(S, S^-) \) and \( Z^*(T, T^-) \) factors on the denominator are each bounded from below by \( \frac{N^{1|S|^2 - |S|}}{K} \). (Indeed, for \( x \) close to \( 0 \) the function \( z(x) \) is close to \( 1/x \), and thus, the \( z \) factors
in the $Z$ and $Z^\dagger$ functions are $\gg$ and $\ll N$ by our choice of the $w_k$’s. The claimed bounds for $Z$ and $Z^\dagger$ follow by noticing that there are $|S|^2$ factors in $Z(S, T)$ and $|S|^2 - |S|$ factors in $Z^\dagger(S, S^-)$. Similarly, all the $H_{S,T}$ factors are bounded by $N^{2K-2|S|}$. Collecting these estimates, we conclude that $J^* \ll K N^{2K}$, and this finishes the proof of Proposition 2.1.

2.2. Proof of Proposition 2.2. Recall that $z_0 = 1 - \frac{1}{N}$ and $z_0 \leq z \leq 1$. Observe that $|z_0 - z_j| \leq |z_0 - z| + |z - z_j| \ll 1/N + |z - z_j|$ and that for $|\theta_j| \geq c/N$,

$$|z - z_j| \gg c/N.$$  

It follows that $c|z_0 - z_j| \ll c/N + c|z - z_j| \ll |z - z_j| + c|z - z_j|$, and thus, for $0 < c \leq 1$ we have $c|z_0 - z_j| \ll |z - z_j|$. Using this we see that

$$X_2 = \sum_{|\theta_j| \geq c/N} \left( \frac{1}{z - z_j} - \frac{1}{z_0 - z_j} \right)$$

$$= \sum_{|\theta_j| \geq c/N} \frac{z_0 - z}{(z - z_j)(z_0 - z_j)}$$

$$\ll \frac{1}{cN} \sum_{|\theta_j| \geq c/N} \frac{1}{|z_0 - z_j|^2}$$

$$\ll \frac{1}{cN} \sum_{j=1}^{N} \frac{1}{|z_0 - z_j|^2},$$

We claim that

$$\sum_{j=1}^{N} \frac{1}{|z_0 - z_j|^2} \ll N(N + |X_1|). \quad (8)$$

From this together with Proposition 2.1 we will obtain Proposition 2.2.

To prove the claim (8), we would like to connect its left-hand side with

$$X_1 = \frac{P'}{P}(z_0) = \sum_{j=1}^{N} \frac{1}{z_0 - z_j},$$

and a natural way is to take the real or imaginary part of $X_1$. However, unlike in the zeta case, after taking real or imaginary parts the summands may have different signs or even be 0, and
this prevents us from controlling the size of the left-hand side of (8). To get around this, we introduce a change of variable

$$\quad P(z) = P(e^s) = Q(s) = \prod_{j=1}^{N} (e^s - e^{i\theta_j}).$$

We then consider a variant of the Weierstrass-Hadamard factorization for each $e^s - e^{i\theta_j}$ and first write for purely imaginary $s$ that

$$e^s - e^{i\theta_j} = e^{\frac{s+i\theta_j}{2}} \cdot (2i) \cdot \sin \left( \frac{-is - \theta_j}{2} \right) = e^{\frac{s+i\theta_j}{2}} \cdot (s - i\theta_j) \cdot \prod_{n \neq 0} \left( 1 + \frac{is + \theta_j}{2n\pi} \right).$$

It follows that

$$\frac{Q'}{Q}(s) = \sum_{j=1}^{N} \frac{(e^s - e^{i\theta_j})'}{e^s - e^{i\theta_j}} = \sum_{j=1}^{N} \left( \frac{1}{2} + \frac{1}{s - i\theta_j} + \sum_{n \neq 0} \frac{1}{-i2n\pi + s - i\theta_j} \right) = \sum_{j=1}^{N} \left( \frac{1}{2} + \sum_{n \in \mathbb{Z}} \frac{1}{-i2n\pi + s - i\theta_j} \right) = \frac{N}{2} + \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \frac{1}{s - i(\theta_j + 2n\pi)}. \quad (9)$$

Remark: One may view this formula as an analogue of the Hadamard fraction formula for zeta, where the $N/2$ in (9) corresponds to the contribution of trivial zeta zeros which is about $-\frac{\log T}{2}$. The difference of the sign comes from the fact that $s$ is to the left of the ‘critical line’ in the CUE case.

These equations extend to all $s \in \mathbb{C}$ (except at poles), and thus,

$$\Re \left( \frac{Q'}{Q}(s) \right) = \frac{N}{2} + \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \frac{\Re(s)}{|s - i(\theta_j + 2n\pi)|^2}.$$

Set $s = s_0$ in the above equation, where $e^{s_0} = z_0$, so that $s_0 = \log(1 - \frac{1}{N})$ is about $-1/N$. We have

$$0 < \sum_{j=1}^{N} \frac{-s_0}{|s_0 - i\theta_j|^2} \leq \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} \frac{-s_0}{|s_0 - i(\theta_j + 2n\pi)|^2} = \frac{N}{2} - \Re \left( \frac{Q'}{Q}(s_0) \right)$$
\[ \frac{N}{2} + \left| \frac{Q'(s_0)}{Q(s_0)} \right| \leq \frac{N}{2} + |Q'(s_0)|. \]

Now observe that
\[ |z_0 - z_j| \gg |s_0 - i\theta_j| \]

and
\[ \left| \frac{Q'(s_0)}{Q(s_0)} \right| = \left| \frac{P'(e^{s_0}) \cdot e^{s_0}}{P(e^{s_0})} \right| \ll \left| \frac{P'(z_0)}{P(z_0)} \right| = |X_1|. \]

Combine these with the above and we see that
\[ \frac{1}{N} \sum_{j=1}^{N} \frac{1}{|z_0 - z_j|^2} \ll \frac{1}{N} \sum_{j=1}^{N} \frac{1}{|s_0 - i\theta_j|^2} \]
\[ \ll \sum_{j=1}^{N} \frac{-s_0}{|s_0 - i\theta_j|^2} \]
\[ \ll N + \left| \frac{Q'(s_0)}{Q(s_0)} \right| \]
\[ \ll N + |X_1|, \]
proving (8).

2.3. **Proof of Proposition 2.3**

We have
\[ X_3 = \sum_{|\theta_j| < \frac{c}{N}} \frac{1}{z_0 - z_j} \ll \left( \sum_{|\theta_j| < \frac{c}{N}} 1 \right) \cdot N \leq \frac{1}{N} \cdot \left( \sum_{|\theta_j| < \frac{c}{N}} 1 \right) \cdot N^2. \]

To bound the number of \(|\theta_j| < 1/N\), we observe that
\[ \sum_{j=1}^{N} \frac{1}{|z_0 - z_j|^2} \gg \sum_{|\theta_j| < 1/N} \frac{1}{|z_0 - z_j|^2} \gg N^2 \cdot \left( \sum_{|\theta_j| < \frac{1}{N}} 1 \right). \]

It follows that
\[ X_3 \ll \frac{1}{N} \cdot \sum_{j=1}^{N} \frac{1}{|z_0 - z_j|^2} \ll N + |X_1|, \]
where the last inequality is by (8). Proposition 2.3 now follows from Proposition 2.1.

3. **Proof of Theorem 1.2**

In view of Lester’s result, we expect that the variance of the complex random variable
\[ \frac{P'}{P} \left( 1 - \frac{L}{N} \right) = \sum_{j=1}^{N} \frac{1}{(1 - L/N) - e^{i\theta_j}}. \]
is a constant times $(N/L)^2$. Therefore, we introduce the following rescaling
\[
\begin{align*}
    f(\theta) &= f_{N,L}(\theta) = \frac{L}{N} \cdot \frac{1}{(1 - L/N) - e^{i\theta}}, \\
    g(\theta) &= g_{N,L}(\theta) = \Re f(\theta), \\
    h(\theta) &= h_{N,L}(\theta) = \Im f(\theta),
\end{align*}
\]
and denote
\[
\begin{align*}
    S_N(f) &= \sum_{j=1}^{N} f(\theta_j) = \frac{L}{N} \cdot \frac{P'}{P} \left( 1 - \frac{L}{N} \right), \\
    S_N(g) &= \sum_{j=1}^{N} g(\theta_j) = \Re \left[ \frac{L}{N} \cdot \frac{P'}{P} \left( 1 - \frac{L}{N} \right) \right], \\
    S_N(h) &= \sum_{j=1}^{N} h(\theta_j) = \Im \left[ \frac{L}{N} \cdot \frac{P'}{P} \left( 1 - \frac{L}{N} \right) \right].
\end{align*}
\]

We compute the characteristic function (ch.f.)
\[
\phi_N(u, v) = \mathbb{E} e^{i(uS_N(g) + vS_N(h))}
\]
of the random vector $(S_N(g), S_N(h))$. We will show that
\[
\phi_N(u, v) \to e^{-\frac{1}{2} \cdot \frac{u^2 + v^2}{8}} \quad \text{as } N \to \infty,
\]
for every $(u, v) \in \mathbb{R}^2$, and this will prove Theorem 1.2 according to the convergence theorem
for random vectors (see for example Theorem 3.10.5 in Durrett [9]). To prove (10), it suffices

to show that for every $(u, v) \in \mathbb{R}^2$, the real random variable
\[
uS_N(g) + vS_N(h) \to \text{Normal} \left( 0, \frac{u^2 + v^2}{8} \right)
\]
in distribution, (11)

for this will imply pointwise convergence of the ch.f. of $uS_N(g) + vS_N(h)$, thus in particular
its ch.f. evaluated at 1, which gives (10).

The main tool we use to prove (11) is the following result of Soshnikov [20], which is a
combination of Lemma 1 and the main combinatorial lemma in that paper.

**Proposition 3.1.** (Soshnikov.) Let $F(\theta)$ be a real-valued function on the unit circle with
continuous derivative and satisfy
\[
\sum_{k \in \mathbb{Z}} |k||\hat{F}(k)|^2 < \infty,
\]
where
\[
\hat{F}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} F(\theta)e^{-ik\theta} d\theta
\]
are the Fourier coefficients of $F$. Let $C_\ell(F)$ be the $\ell$-th cumulant of $S_N(F) = \sum_{j=1}^{N} F(\theta_j)$. Then we have

$$C_1(F) = \hat{F}(0) \cdot N,$$

(12)

$$\left| C_2(F) - \sum_{k \in \mathbb{Z}} |k| |\hat{F}(k)|^2 \right| \leq \sum_{|k| > N/2} |k| |\hat{F}(k)|^2,$$

(13)

and for $\ell \geq 3$

$$|C_\ell(F)| \ll \sum_{k_1 + \ldots + k_\ell = 0, |k_1| + \ldots + |k_\ell| > N} |k_1||\hat{F}(k_1)\cdots\hat{F}(k_\ell)|.$$

(14)

We shall apply Proposition 3.1 to

$$F(\theta) = F_{N,L,u,v}(\theta) = ug(\theta) + vh(\theta)$$

for every $(u,v) \in \mathbb{R}^2$, and thus,

$$S_N(F) = uS_N(g) + vS_N(h).$$

Since the normal distribution is determined by cumulants, to prove (11) it suffices to prove that

$$C_1(F) \to 0, \quad C_2(F) \to \frac{u^2 + v^2}{8}, \quad \text{and} \quad C_\ell(F) \to 0 \text{ for } \ell \geq 3$$

(15)

as $N \to \infty$.

We start by computing Fourier coefficients of $f$. Recall that

$$f(\theta) = \frac{L}{N} \cdot \frac{1}{(1 - L/N) - e^{i\theta}}.$$

It follows easily that

$$\hat{f}(k) = \begin{cases} 0, & \text{if } k \geq 0, \\ \frac{i}{N} \cdot (1 - \frac{L}{N})^{-k-1}, & \text{if } k < 0. \end{cases}$$

Note that here all $\hat{f}(k)$ are real. From this we deduce the Fourier coefficients for $g$ and $h$:

$$\hat{g}(k) = \frac{1}{2} \left( \hat{f}(k) + \hat{f}(-k) \right) = \begin{cases} 0, & \text{if } k = 0, \\ \frac{-L}{2N} \cdot (1 - \frac{L}{N})^{k-1}, & \text{if } k \neq 0, \end{cases}$$

and

$$\hat{h}(k) = \frac{1}{2i} \left( \hat{f}(k) - \hat{f}(-k) \right) = \begin{cases} 0, & \text{if } k = 0, \\ \frac{L}{2iN} \cdot (1 - \frac{L}{N})^{k-1}, & \text{if } k < 0, \\ \frac{-L}{2iN} \cdot (1 - \frac{L}{N})^{k-1}, & \text{if } k > 0. \end{cases}$$
Since \( F = u g + v h \), we have
\[
\hat{F}(k) = u \hat{g}(k) + v \hat{h}(k) = \begin{cases} 
0, & \text{if } k = 0, \\
\frac{L}{2N} \cdot (1 - \frac{k}{N})^{k-1} \cdot (u - iv), & \text{if } k < 0, \\
\frac{L}{2N} \cdot (1 - \frac{k}{N})^{k-1} \cdot (u + iv), & \text{if } k > 0.
\end{cases} \tag{16}
\]

We first estimate the two sums in (13). A straightforward computation shows
\[
\sum_{k \in \mathbb{Z}} |k| |\hat{F}(k)|^2 = \frac{u^2 + v^2}{2} \left( \frac{L}{N} \right)^2 \left( 1 - \frac{L}{N} \right)^{-2} \sum_{k \geq 1} k \left( 1 - \frac{L}{N} \right)^{2k}.
\]

Denote temporarily \( A(x) = \sum_{k \geq 1} k x^{2k} \) and \( B(x) = \sum_{k \geq 1} x^{2k} \). For \( 0 < x < 1 \) we have \( B(x) = (1 - x^2)^{-1} - 1 \). Differentiating \( B(x) \) yields \( A(x) = x^2 (1 - x^2)^{-2} \) for \( 0 < x < 1 \). Therefore, letting \( x = 1 - L/N \) in the above equation we obtain
\[
\sum_{k \in \mathbb{Z}} |k| |\hat{F}(k)|^2 = \frac{u^2 + v^2}{2} \frac{1}{(2 - \frac{L}{N})^2},
\]
which is finite for fixed \( N, L, u \) and \( v \). Therefore, Proposition 3.1 applies to our function \( F \).

From the above equation we also see
\[
\lim_{N \to \infty} \sum_{k \in \mathbb{Z}} |k| |\hat{F}(k)|^2 = \frac{u^2 + v^2}{8}. \tag{17}
\]

A similar treatment for the second sum in (13) gives
\[
\sum_{|k| > N/2} |k| |\hat{F}(k)|^2 = \frac{u^2 + v^2}{2} \left( \frac{L}{N} \right)^2 \left( 1 - \frac{L}{N} \right)^{-2} \sum_{k \geq N/2} k \left( 1 - \frac{L}{N} \right)^{2k} = \frac{u^2 + v^2}{2} \left( \frac{L}{N} \right)^2 \left( 1 - \frac{L}{N} \right)^{-2} \frac{\left( 1 - \frac{L}{N} \right)^{2M} \left( 1 - \frac{L}{N} \right)^2}{(2 - \frac{L}{N})^2} \left( M + (1 - M) \left( 1 - \frac{L}{N} \right)^2 \right),
\]
where \( M \) is the least integer greater than \( N/2 \). Since we assume \( L = o(N) \), there is no harm to assume \( L/N < 1/2 \), say. Thus, it is not difficult to see that the above is
\[
\ll_{u,v} \left( 1 - \frac{L}{N} \right)^{2M+2} + \left( 1 - \frac{L}{N} \right)^{2M} \cdot M \cdot \frac{L}{N}.
\]

Therefore, letting \( x = 1 - L/N \) in the above equation we obtain
\[
\lim_{N \to \infty} \sum_{|k| > N/2} |k| |\hat{F}(k)|^2 = 0. \tag{18}
\]
Combining (17) and (18) we conclude that
\[ C_2(F) \to \frac{v^2 + u^2}{8} \] (19)
as \( N \to \infty \).

Moreover, from (16) and Proposition 3.1 it follows immediately that
\[ C_1(F) = 0. \] (20)

In view of (15), (20), (19) and (14), it only remains to prove
\[ \sum_{k_1 + \ldots + k_\ell = 0 \atop |k_1| + \ldots + |k_\ell| > N} |k_1||\hat{F}(k_1) \ldots \hat{F}(k_\ell)| \to 0 \]
as \( N \to \infty \), for each \( \ell \geq 3 \). From (16) we have
\[ |\hat{F}(k)| \begin{cases} = 0, & \text{if } k = 0, \\ \ll_{u,v} \frac{1}{N} \cdot \left(1 - \frac{k}{N}\right)^{|k|}, & \text{if } k \neq 0. \end{cases} \]

Observe that
\[ \sum_{k_1 + \ldots + k_\ell = 0 \atop |k_1| + \ldots + |k_\ell| > N} |k_1||\hat{F}(k_1) \ldots \hat{F}(k_\ell)| \ll_{\ell} \sum_{k_1 + \ldots + k_\ell = 0 \atop |k_1| + \ldots + |k_\ell| > N} |k_1||\hat{F}(k_1) \ldots \hat{F}(k_\ell)|, \]

and the conditions in the last sum imply that \( |k_1| > N/\ell \). Thus, the above sum is
\[ \ll_{\ell} \sum_{|k| > N/\ell} |k| |\hat{F}(k)| \sum_{k_2, \ldots, k_\ell \leq |k| \atop k + k_2 + \ldots + k_\ell = 0} |\hat{F}(k_2) \ldots \hat{F}(k_\ell)|. \]

Plug in the bounds for \( |\hat{F}(k)| \), and note that the inner sum condition \( k + k_2 + \ldots + k_\ell = 0 \) implies \( |k_2| + \ldots + |k_\ell| \geq |k| \). Thus, the above is
\[ \ll_{\ell,u,v} \left(\frac{L}{N}\right)^\ell \sum_{|k| > N/\ell} |k| \left(1 - \frac{L}{N}\right)^{|k|} \sum_{|k_2| \leq |k| \atop k + k_2 + \ldots + k_\ell = 0} \left(1 - \frac{L}{N}\right)^{|k_2| + \ldots + |k_\ell|} \]
\[ \ll_{\ell,u,v} \left(\frac{L}{N}\right)^\ell \sum_{|k| > N/\ell} |k| \left(1 - \frac{L}{N}\right)^{|k|} \sum_{|k_2| \leq |k| \atop k + k_2 + \ldots + k_\ell = 0} \left(1 - \frac{L}{N}\right)^{|k_2| + \ldots + |k_\ell|} \]
\[ \ll_{\ell,u,v} \left(\frac{L}{N}\right)^\ell \sum_{|k| > N/\ell} |k|^2 |\hat{F}(k)| \sum_{|k_2| \leq |k| \atop k + k_2 + \ldots + k_\ell = 0} 1 \]
\[ \ll_{\ell,u,v} \left(\frac{L}{N}\right)^\ell \sum_{|k| > N/\ell} |k|^\ell - 1 \left(1 - \frac{L}{N}\right)^{2|k|} \]
\[ \ll_{\ell,u,v} \left(\frac{L}{N}\right)^\ell \sum_{k > N/\ell} k^{\ell - 1} \left(1 - \frac{L}{N}\right)^{2k}. \] (21)
Let $D_n(y) = \sum_{k \geq n} y^k = y^n (1 - y)^{-1}$ for $0 < y < 1$. Differentiating $\ell - 1$ times with respect to $y$, we have, for fixed $\ell$ and for $n > 2\ell$, that

$$\sum_{k \geq n} k^{\ell-1} y^k \leq \sum_{k \geq n} k^{\ell-1} y^{\ell-1+1} \ll_{\ell} \left( \frac{d}{dy} \right)^{\ell-1} D_n(y)$$

$$= \ell \sum_{j=0}^{\ell-1} \left( \frac{\ell-1}{j} \right) \cdot \left( \frac{d}{dy} \right)^{j} y^n \cdot \left( \frac{d}{dy} \right)^{\ell-1-j} \frac{1}{1-y}$$

$$\ll_{\ell} \sum_{j=0}^{\ell-1} n^j y^{n-j} \frac{1}{(1-y)^{\ell-j}}.$$

Plug in $y = (1 - L/N)^2$ and $n = \text{the least integer} > N/\ell$, and let $N$ be sufficiently large. We obtain

$$\sum_{k > N/\ell} k^{\ell-1} \left( 1 - \frac{L}{N} \right)^{2k} \ll_{\ell} \sum_{j=0}^{\ell-1} \left( \frac{N}{\ell} \right)^{j} \left( 1 - \frac{L}{N} \right)^{2(n-j)} \frac{1}{(2 - \frac{L}{N})^{\ell-j} \left( \frac{N}{\ell} \right)^{\ell-j}}$$

$$\ll_{\ell} \sum_{j=0}^{\ell-1} N^j \left( 1 - \frac{L}{N} \right)^{2n} \left( \frac{L}{N} \right)^{j-\ell}$$

$$\ll_{\ell} N^{\ell} \left( 1 - \frac{L}{N} \right)^{2N/\ell} L^{-1}.$$

From this and (21) it follows that

$$\sum_{|k_1| + \cdots + |k_\ell| > N} |k_1| \cdots |k_\ell| \ll_{L,u,v} \left( 1 - \frac{L}{N} \right)^{2N/\ell} L^{\ell-1}$$

$$\ll_{L,u,v} e^{-2L/\ell} L^{\ell-1}$$

which tends to 0 as $L \to \infty$ (or $N \to \infty$). This finishes the proof of Theorem 1.2.

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