MULTIPLICATIVE STRUCTURES OF THE IMMACULATE BASIS
OF NON-COMMUTATIVE SYMMETRIC FUNCTIONS

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Abstract. We continue our development of a new basis for the non-commutative
symmetric functions. We give non-commutative versions of the Murghahan-Nakayama
and Littlewood-Richardson rules. A surprising new relation among non-commutative
Littlewood-Richardson coefficients is given with implications to the commutative
case. We also provide a geometric interpretation for the structure coefficients as
integer points satisfying certain inequalities.

1. Introduction

The Schur functions appear throughout mathematics: as the representatives for
the Schubert classes in the cohomology of the Grassmannian, as the characters for
the irreducible representations of the general linear group and as an orthonormal
basis for the space of symmetric functions. Their ubiquity makes Schur functions
‘the’ fundamental basis for the ring of symmetric functions.

In [BBSSZ], the authors constructed a basis, called the immaculate basis \( \{ \mathfrak{G}_\alpha \} \),
for the non-commutative symmetric functions \( \text{NSym} \). This basis satisfies many of
the same combinatorial properties of Schur functions, in particular they project onto
Schur functions under the natural map which sends non-commutative symmetric
functions to their commutative counterparts.

An application of the immaculate basis is the development of methods for proving
positivity results in the ring of symmetric functions based on positivity of expressions
in \( \text{NSym} \).

In [BBSSZ2] we gave indecomposable modules for the 0-Hecke algebra whose
characteristics were the dual immaculate basis, the corresponding basis of quasi-
symmetric functions. This provides further evidence to the importance of the im-
maculate basis.

The goal of this paper is to further develop the immaculate basis of \( \text{NSym} \). In
Section 2 we give the symmetric function results which we intend to emulate in
\( \text{NSym} \). Section 3 is a brief introduction to \( \text{NSym} \) and the immaculate basis. Section 4
gives an expansion of the non-commutative Pieri rule (Theorem 4.3) to ribbon shapes

Date: June 4, 2013.
(Theorem 4.4). In Section 5, we prove a non-commutative version of the Murghahan-Nakayama rule (Theorem 5.1) and in Section 6 we give a dual notion on the quasi-symmetric functions (Corollary 6.2). In Section 7 we give our non-commutative analogue of the Littlewood-Richardson rule (Theorem 7.3) and discover a relation of Littlewood-Richardson coefficients (Corollary 7.5) that is not easily deduced from the commutative case. We finish the section with geometric descriptions of our non-commutative Littlewood-Richardson coefficients.

1.1. Acknowledgements. This work is supported in part by NSERC. It is partially the result of a working session at the Algebraic Combinatorics Seminar at the Fields Institute with the active participation of C. Benedetti, C. Ceballos (especially for his help on Corollary 7.30), J. Sánchez-Ortega, O. Yacobi, E. Ens, H. Heglin, D. Mazur and T. MacHenry.

This research was facilitated by computer exploration using the open-source mathematical software Sage [sage] and its algebraic combinatorics features developed by the Sage-Combinat community [sage-combinat].

2. Symmetric Function Background

In this section, we build notation in order to state the classical version of the results we emulate on the immaculate basis of NSym. For more details, we refer the reader to references on symmetric functions ([Sta], [Sagan] or [M]).

2.1. Partitions. A partition of a non-negative integer \( n \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) of non-negative integers satisfying \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) which sum to \( n \); it is denoted \( \lambda \vdash n \). Partitions are of particular importance to algebraic combinatorics; among other things, partitions of \( n \) index a basis for the symmetric functions of degree \( n \).

2.2. Symmetric functions. We let \( \Lambda \) denote the ring of symmetric functions. As an algebra, \( \Lambda \) is the ring over \( \mathbb{Q} \) freely generated by commutative elements \( \{h_1, h_2, \ldots\} \). The algebra \( \Lambda \) has a grading, defined by giving \( h_i \) degree \( i \) and extending multiplicatively. A natural basis for the degree \( n \) component of \( \Lambda \) is the basis of complete homogeneous symmetric functions of degree \( n \), \( \{h_\lambda := h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_m}\}_{n} \). The algebra \( \Lambda \) can be realized as the invariants of the algebra \( \mathbb{Q}[x_1, x_2, \ldots] \) in commuting variables \( \{x_1, x_2, \ldots\} \). Under this identification, \( h_i \) denotes the sum of all monomials in the \( x \) variables of degree \( i \).

2.3. Schur functions. We define the basis of Schur functions via their relationship to the complete homogenous basis \( \{h_\lambda : \lambda \vdash n\} \). For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n \)
we define
\[ s_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_{\ell}-\ell+1} & h_{\lambda_{\ell}-\ell+2} & \cdots & h_{\lambda_{\ell}} \end{bmatrix} = \det |h_{\lambda_i+j-i}|_{1 \leq i, j \leq \ell} \]
where we use the convention that \( h_0 = 1 \) and \( h_{-m} = 0 \) for \( m > 0 \).

2.4. Power sum symmetric functions. The power sum symmetric function \( p_k \) is defined as:
\[ p_k = \sum_i x_i^k. \]
The collection \( \{ p_\lambda := p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_m} \}_{\lambda \vdash n} \) form a basis for the degree \( n \) component of \( \Lambda \).

2.5. The Murghahan-Nakayama rule. The Murghahan-Nakayama rule gives an explicit combinatorial expansion for the product of a Schur function with a power sum generator \( p_k \). For two partitions \( \lambda \) and \( \mu \) with \( \mu \subset \lambda \), we say that \( \lambda/\mu \) is a border strip if the diagram consisting of \( \lambda \) without \( \mu \) contains no \( 2 \times 2 \) square. The height of a border strip is the number of rows it contains.

**Theorem 2.1.** For \( k > 0 \) and a partition \( \mu \),
\[ s_\mu p_k = \sum (-1)^{ht(\lambda/\mu)-1} s_\lambda, \]
the sum over partitions \( \lambda \) for which \( \lambda/\mu \) is a border strip of size \( k \).

**Example 2.2.** For \( \lambda = (2, 2, 2) \) and \( k = 3 \):
\[ s_{222}p_3 = s_{222111} - s_{222121} + s_{333} - s_{432} + s_{522}. \]

2.6. The Littlewood-Richardson rule. The Littlewood-Richardson rule gives an explicit combinatorial expansion for the product of two Schur functions.

For a word \( w \) consisting of letters \( \{1, \ldots, m\} \), we say \( w \) is Yamanouchi if for every positive integer \( j \) and every prefix of \( w \), there are at least as many occurrences of \( j \) as there are of \( j+1 \). The reading word of a skew tableau \( T \) is the word which starts in the top row and reads from right to left, proceeding down the rows.

**Theorem 2.3.** For partitions \( \lambda \) and \( \mu \),
\[ s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu, \]
where \( \nu \) is a partition of \( |\lambda|+|\mu| \) and \( c^\nu_{\lambda,\mu} \) is the number of skew tableaux of shape \( \nu/\lambda \) whose reading word is a Yamanouchi word of content \( \mu \).
Example 2.4. We give an example with \( \mu = \lambda = (2, 1) \).

\[
\begin{align*}
s_{21} & = s_{211} + s_{222} + s_{311} + 2s_{321} + s_{33} + s_{411} + s_{42} \\
\end{align*}
\]

3. The non-commutative symmetric functions

3.1. Compositions. A composition of a non-negative integer \( n \) is a list \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m] \) of positive integers which sum to \( n \), often written \( \alpha \vdash n \). The entries \( \alpha_i \) of the composition are referred to as the parts of the composition. The size of the composition is the sum of the parts and will be denoted \( |\alpha| := n \). The length of the composition is the number of parts and will be denoted \( \ell(\alpha) := m \). We let sort(\( \alpha \)) denote the partition obtained by sorting the terms in \( \alpha \). In this paper we study a Hopf algebra whose bases at level \( n \) are indexed by compositions of \( n \).

Compositions of \( n \) correspond to subsets of \( \{1, 2, \ldots, n - 1\} \). We will follow the convention of identifying \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m] \) with the subset \( D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1}\} \).

If \( \alpha \) and \( \beta \) are both compositions of \( n \), we say that \( \alpha \leq \beta \) in refinement order if \( D(\beta) \subseteq D(\alpha) \). For instance, \([1, 1, 2, 1, 3, 2, 1, 4, 2] \leq [4, 2, 7], \) since \( D([1, 1, 2, 1, 3, 2, 1, 4, 2]) = \{1, 2, 4, 5, 8, 10, 11, 15\} \) and \( D([4, 2, 7]) = \{4, 8, 10\} \).

We introduce a notion which will arise in our Pieri rule (Theorem 4.3); we say that \( \alpha \subset_i \beta \) if:

1. \( |\beta| = |\alpha| + i \),
2. \( \alpha_j \leq \beta_j \) for all \( 1 \leq j \leq \ell(\alpha) \),
3. \( \ell(\beta) \leq \ell(\alpha) + 1 \).

3.2. NSym. As an algebra, the non-commutative symmetric functions, NSym, are generated by elements \( H_1, H_2, \ldots \), with no relations. The algebra NSym is graded; the generator \( H_i \) has degree \( i \) and this degree is extended multiplicatively. Therefore a basis for the degree \( n \) component of NSym is the collection of complete homogenous non-commutative functions \( H_\alpha := H_{\alpha_1}H_{\alpha_2}\cdots H_{\alpha_m} \) for \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m] \) a composition of \( n \). There is a projection \( \chi : \text{NSym} \to \Lambda \), which sends the non-commutative
generator $H_i$ to the corresponding commutative generator $h_i$ and is extended multiplicatively (so that $\chi(H_\alpha) = h_{\text{sort}(\alpha)}$).

In this section, we will also remind the reader of several other bases of $\text{NSym}$. The algebra $\text{NSym}$ is isomorphic to the Grothendieck ring of the finitely generated projective representations of the 0-Hecke algebra (see [KrTh, DKLT]). Under this isomorphism, the projective indecomposable modules are identified with the ribbon functions. The ribbon functions $R_\alpha$ can be described in terms of the complete homogeneous non-commutative basis:

$$R_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} H_\beta$$

or equivalently

$$H_\alpha = \sum_{\beta \geq \alpha} R_\beta.$$

Under the canonical projection $\chi$, the ribbon basis of $\text{NSym}$ is mapped onto the ribbon Schur function (see [GKLLRT] for more details).

Ribbons have a simple product formula. For two compositions $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m], \beta = [\beta_1, \beta_2, \ldots, \beta_l]$, let $[\alpha, \beta]$ denote the composition of concatenation of $\alpha$ and $\beta$; in other words, $[\alpha, \beta] = [\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l]$. Let $\alpha \land \beta$ denote $[\alpha_1, \alpha_2, \ldots, \alpha_m + \beta_1, \beta_2, \ldots, \beta_l]$. Then

(1)  

$$R_\alpha R_\beta = R_{[\alpha, \beta]} + R_{\alpha \land \beta}.$$

In [GKLLRT], two bases of $\text{NSym}$ which are analogues of power sum symmetric functions were defined. We focus on the $\Psi$ basis. We will define $\Psi_k$ via its relationship to the ribbon basis:

(2)  

$$\Psi_k = \sum_{i=0}^{k-1} (-1)^i R_{1^i, k-i}$$

and let $\Psi_\alpha = \Psi_{\alpha_1} \cdots \Psi_{\alpha_m}$. It is known that $\chi(\Psi_k) = p_k$.

The algebra $\text{NSym}$ is also a Hopf algebra with coproduct defined on the generators as:

(3)  

$$\Delta(H_\alpha) = \sum_{i=0}^{r} H_i \otimes H_{r-i},$$

and on the basis $H_\alpha$ by $\Delta(H_\alpha) = \Delta(H_{\alpha_1})\Delta(H_{\alpha_2})\cdots \Delta(H_{\alpha_m})$. The elements $\Psi_r$ are primitive with respect to the coproduct.

3.3. $\text{QSym}$. The dual Hopf algebra to $\text{NSym}$ is the Hopf algebra of quasi-symmetric functions, $\text{QSym}$. As a vector space, $\text{QSym}$ is the subspace of $\mathbb{Q}[[x_1, x_2, \ldots]]$ spanned by the monomial quasi-symmetric functions $M_\alpha$ defined by:

(4)  

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_m}^{\alpha_m}.$$

The basis $\{M_\alpha\}_\alpha$ is dual to $\{H_\alpha\}_\alpha$ with respect to a pairing, $(\cdot, \cdot) : \text{NSym} \times \text{QSym} \rightarrow \mathbb{Q}$ such that $(H_\alpha, M_\beta) = \delta_{\alpha, \beta}$. The product and coproduct on $\text{QSym}$ are defined such
that for $F, G \in \text{NSym}$ and $H, K \in \text{QSym}$,

$$\langle F \otimes G, \Delta(H) \rangle = \langle F \cdot G, H \rangle$$

and

$$\langle F, H \cdot K \rangle = \langle \Delta(F), H \otimes K \rangle .$$

3.4. The immaculate basis of $\text{NSym}$. In [BBSSZ], we defined a new basis, called the immaculate basis, which we denoted $\mathcal{G}_\alpha$. The immaculate basis can be defined in terms of the complete basis; its expansion in the complete basis is an analogue of the Jacobi-Trudi formula:

$$\mathcal{G}_\alpha = \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1, \alpha_2+\sigma_2-2, \ldots, \alpha_m+\sigma_m-m},$$

where $H_0 = 1$ and $H_i = 0$ if $i < 0$. It follows from the definition that, for a partition $\lambda$, $\chi(\mathcal{G}_\lambda) = s_\lambda$.

The set of immaculate functions indexed by compositions of $m$ are triangularly related to the basis of complete functions indexed by compositions of $m$ and hence form a basis.

An equivalent formulation of the immaculate basis that has motivated their development has been in terms of a family of creation operators. In analogy with the Bernstein operators [M, Ze] which are creation operators for the Schur functions, we define for $m \in \mathbb{Z}$,

$$\mathbb{B}_m = \sum_{i \geq 0} (-1)^i H_{m+i} M_{1_i}^\perp,$$

where $M_{1_i}^\perp$ denotes the adjoint operator to multiplication by $M_{1_i}$ in $\text{QSym}$. In [BBSSZ], we showed that a composition of the creation operators yielded Equation (7).

**Proposition 3.1.** ([BBSSZ] Theorem 3.23) For $\alpha \in \mathbb{Z}^m$,

$$\mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_m} 1 = \mathcal{G}_\alpha. $$

4. **Ribbon multiplication**

In this section, we will prove that the product of an immaculate function with a ribbon function has a positive expansion in the immaculate basis.
4.1. **Skew immaculate tableaux.** For two compositions $\alpha$ and $\beta$ with $\alpha_i \geq \beta_i$ for all $i$, we view the skew diagram $\alpha / \beta$ as the diagram of $\alpha$ with the cells of $\beta$ removed (and $\beta$ is embedded in the upper left corner of $\alpha$).

**Definition 4.1.** A composition tableau is a map from the cells of $\alpha / \beta$ to $\mathbb{N}$. The shape of the tableau is $\alpha / \beta$ and is denoted $sh(T)$. The content of the tableau is an integer vector where the $i^{th}$ entry is equal to the number of $i$'s which appear in the tableau; this vector is denoted $c(T)$. A tableau is called *immaculate* if the entries in each row are weakly increasing from left to right and the entries in the first column are strictly increasing if they are read from top to bottom. The *reading word* of the tableau (denoted $read(T)$) is the word of entries read starting in the top row from right to left, then proceeding down the rows. An immaculate tableau is *standard* if the reading word is a permutation of the numbers 1 through $|\alpha| - |\beta|$. The *descent set* of a standard immaculate tableau is the set of all positive integers $j$ which appear in a row above $j + 1$. The *descent composition* of a standard immaculate tableau $T$ is the composition associated with the descent set under the map $D$. It will be denoted $D(T)$.

**Example 4.2.** An example of a standard skew immaculate tableau of shape $[2, 3, 2, 1]/[1, 2]$ is

```
2
[ 4 ]
[ 1 3 ]
[ 5 ]
```

The reading word of this tableau is 24135. The descent set is $\{2, 4\}$, so its descent composition is $[2, 2, 1]$. 

4.2. **The product rule.** In [BBSSZ], we established a Pieri rule for the immaculate basis. Specifically, we proved:

**Theorem 4.3.** The $S_\alpha$ satisfy a multiplicity free right-Pieri rule for multiplication by $H_s$:

$$S_\alpha H_s = \sum_{\alpha \subseteq \beta} S_\beta.$$ 

The Pieri rule should be seen as a particular instance of the following more general result (since $R_s = H_s$).

**Theorem 4.4.** For compositions $\alpha$ and $\beta$,

$$S_\alpha R_\beta = \sum_{sh(T) = \gamma/\alpha, D(T) = \beta} S_\gamma,$$

the sum over all standard immaculate tableau of shape $\gamma/\alpha$ and descent composition $\beta$. 
Proof. We prove this by induction on the length of $\beta$. If $\ell(\beta) = 1$ then this is just the Pieri rule. Suppose $\beta$ is of length $m$. The product rule for ribbons, Equation (1), implies that

$$H_{\beta_1}R_{\beta_2,\beta_3,\ldots,\beta_m} = R_{\beta_1}R_{\beta_2,\beta_3,\ldots,\beta_m} = R_{\beta_1+\beta_2,\beta_3,\ldots,\beta_m} + R_{\beta}.$$  

Since

$$S_\alpha H_{\beta_1}R_{\beta_2,\beta_3,\ldots,\beta_m} = \sum_{sh(S)=\eta/\alpha,sh(P)=\gamma/\eta} S_\gamma$$

by the Pieri rule (Theorem 4.3) and induction, and since

$$S_\alpha R_{\beta_1+\beta_2,\beta_3,\ldots,\beta_m} = \sum_{sh(Q)=\delta/\alpha} S_\delta$$

by induction, then

$$S_\alpha R_\beta = S_\alpha (H_{\beta_1}R_{\beta_2,\beta_3,\ldots,\beta_m} - R_{\beta_1+\beta_2,\beta_3,\ldots,\beta_m}) = \sum_{sh(T)=\gamma/\alpha} S_\gamma,$$

since the tableaux in the sum in Equation (10) either have descent set $\beta$ or $[\beta_1+\beta_2,\beta_3,\ldots,\beta_m]$. \[\square\]

Example 4.5. We give an example with $\alpha = [2,1]$ and $\beta = [1,2]$.

$$S_{21}R_{12} = S_{2112} + S_{2121} + S_{2121} + S_{2211} + S_{222} + S_{231} + 2S_{321} + S_{312} + S_{322} + S_{411} + S_{42}$$

5. The Murghahan-Nakayama rule for immaculate functions

We now prove a generalization of the Murghahan-Nakayama rule on the immaculate basis $S_\alpha$.

Theorem 5.1. For $\alpha$ a composition and $k$ a positive integer,

$$S_\alpha \Psi_k = \sum_{\beta=k} (-1)^{\ell(\beta)} S_{[\alpha,\beta]} + \sum_{j=1}^{\ell(\alpha)} S_{[\alpha_1,\ldots,\alpha_j+k,\ldots,\alpha_{\ell(\alpha)}]}.$$
Proof. We use the expansion of \( \Psi_k \) given in Equation (2) and the product formula coming from Theorem 4.4. Then

\[
\mathcal{S}_\alpha \Psi_k = \sum_{i=0}^{k-1} (-1)^i \mathcal{S}_\alpha R_{1^i, k-i} = \sum_{i=0}^{k-1} \sum_{sh(T) = \gamma/\alpha \atop D(T) = [1^i, k-i]} (-1)^i \mathcal{S}_\gamma.
\]

The coefficient of any given term \( \mathcal{S}_\delta \) is then

\[
\sum_{sh(T) = \delta/\alpha \atop D(T) = [1^i, k-i]} (-1)^i.
\]

If \( \delta \) is of the form \([\alpha, \beta]\) for some \( \beta \), then it is easy to see there is only one skew standard immaculate tableau of this form: for \( i \) in \( \{1, \ldots, \ell(\beta)\} \), we place \( i \) in the \( i^{th} \) row below \( \alpha \). The rest of the entries of the tableau are placed in order, starting from the bottom. This tableau has descent composition \([1^{\ell(\beta)}, k - \ell(\beta)]\), yielding the correct coefficient.

If \( \delta \) is of the form \([\alpha_1, \ldots, \alpha_j + k, \ldots, \alpha_{\ell(\alpha)}]\), then there is again only one skew standard immaculate tableau: place 1, 2, \ldots, \( k \) in order in row \( j \). This has descent composition \([1^0, k]\), so the coefficient is always +1.

For any other skew shape, we will count the number of valid skew immaculate tableaux. Suppose \( \delta/\alpha \) has cells in \( m \) distinct rows. Any skew standard immaculate tableau with descent composition of the form \([1^i, k - i]\) will have the first \( i \) letters put in lower rows, and the remaining \( k - i \) letters can only be placed in one way to create no more descents. Clearly, the bottom-most, left-most box will have to be the start of the \( k - i \) letters with no descent. For the remaining \( m - 1 \) rows, choose \( i \) of them. Then there is a unique skew immaculate tableau which has 1, 2, \ldots, \( i \) in these rows. Therefore the coefficient is \( \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} = 0 \). □

**Example 5.2.** We demonstrate the theorem for \( \alpha = [1, 3, 2] \) and \( k = 3 \).

\[
\mathcal{S}_{132} \Psi_3 = \mathcal{S}_{132111} - \mathcal{S}_{13212} - \mathcal{S}_{13221} + \mathcal{S}_{1323} + \mathcal{S}_{135} + \mathcal{S}_{162} + \mathcal{S}_{432}.
\]

We may reformulate the Murghahan-Nakayama rule to give a positive expansion in terms of immaculate functions indexed by sequences of non-negative integers (these functions do not form a basis of \( \text{NSym} \)).

**Corollary 5.3.** For a composition \( \alpha \) and a positive integer \( k \),

\[
\mathcal{S}_\alpha \Psi_k = \sum_{j=1}^{k} \mathcal{S}_{[\alpha_1, \ldots, \alpha_{\ell(\alpha)}; 0, 0, \ldots, 0, k]} + \sum_{j=1}^{\ell(\alpha)} \mathcal{S}_{[\alpha_1, \alpha_2, \ldots, \alpha_j + k, \ldots, \alpha_{\ell(\alpha)}]}.
\]
Proof. To reduce notation we use $a^b$ to represent $a,a,\ldots,a$. By Theorem 5.1, it is enough to show that

$$
\mathcal{G}_{[\alpha_1,\ldots,\alpha_{\ell(\alpha)},0^{j-1},k]} = (-1)^{j+1} \sum_{\ell(\beta)=j} \mathcal{G}_{[\alpha,\beta]}
$$

where the sum is over compositions $\beta$ of $k$ with length equal to $j$. This result is a consequence of the following Lemma 5.4 since

$$
\mathcal{G}_{[\alpha_1,\ldots,\alpha_{\ell(\alpha)},0^{j-1},k]} = \mathbb{B}_1 \cdots \mathbb{B}_{\alpha_{\ell(\alpha)}} \mathcal{G}_{[0^{j-1},k]}.
$$

\[ \square \]

**Lemma 5.4.** For $k > 0$ and for $1 \leq j \leq k$,

$$
\mathcal{G}_{[0^{j-1},k]} = (-1)^{j+1} R_{[1^{j-1},k-j+1]} = (-1)^{j+1} \sum_{\ell(\beta)=j} \mathcal{G}_{\beta}
$$

where the sum is over $\beta \trianglerighteq k$ such that $\ell(\beta) = j$.

Proof. The second equality follows from Theorem 4.4 since there is exactly one standard immaculate tableau of shape $\beta \trianglerighteq k$ of length $j$ with the descent composition equal to $[1^{j-1},k-j+1]$. That tableau has entries $1,2,\ldots,j$ reading down in the first column and entries $j+1,j+2,\ldots,k$ in the rest of the tableau such that there is no descent.

The first equality follows by induction and applying the operator $\mathbb{B}_0$ on $R_{[1^{j-1},k-j+1]}$. If $j = 1$, the result follows since $R_{[k]} = H_k = \mathcal{G}_{[k]}$. For $j > 1$, recall that

$$
M_{i,d} R_{[1^{j-1},d]} = \begin{cases} 
R_{[1^{j-1},d]} & \text{if } i = 0 \text{ or } d = 1 \\
R_{[1^{j-1},d-1]} + R_{[1^{j-1},d]} & \text{if } i < j \\
R_{[d-1]} & \text{if } i = j \\
0 & \text{if } i > j
\end{cases}
$$

Therefore if $d > 0$,

$$
\mathbb{B}_0(R_{[d]}) = R_0 R_d - R_1 R_{d-1} = -R_{[1,d-1]}.
$$

For $d > 1$ and $j > 0$,

$$
\mathbb{B}_0(R_{[1,d]}) = R_{[1,d]} + \sum_{i=1}^{j} (-1)^i R_{[i]}(R_{[1^{j-i},d]} + R_{[1^{j-i+1},d-1]}) + (-1)^{j+1} R_{[j+1]} R_{[d-1]}
$$

$$
= R_{[1,d]} + \sum_{i=1}^{j-1} (-1)^i (R_{[i,1^{j-i},d]} + R_{[i+1,1^{j-i+1},d]}) + (-1)^j (R_{[j,d]} + R_{[j,d]})
$$

$$
+ \sum_{i=1}^{j} (-1)^i (R_{[i,1^{j-i+1},d-1]} + R_{[i+1,1^{j-i},d-1]}) + (-1)^{j+1} (R_{[j+1,d-1]} + R_{[j,d]})
$$

Therefore $\mathcal{G}_{[0^{j-1},k]} = (-1)^{j+1} R_{[1^{j-1},k-j+1]}$.
All terms of this sum cancel except for $-R_{1^{j-1}, d-1}$.

By induction, $S_{1^{j-1}, k} = B_0(\mathfrak{S}_{1^{j-2}, k}) = (-1)^j B_0(R_{1^{j-2}, k-j+2}) = (-1)^{j+1} R_{1^{j-1}, k-j+1}$.

\[\square\]

Example 5.5. One may check that

\[S_{132} \Psi_3 = S_{432} + S_{162} + S_{132} + S_{13203} + S_{13203}\]

6. The dual Murnaghan-Nakayama rule for immaculate functions

Because the Schur functions are self dual in the ring of the symmetric functions, the Murnaghan-Nakayama rule both dually describes the action of multiplication by $p_k$ on a Schur function and the action of $p^\perp_k$ on a Schur function. We can also give the formula for the action of $M^\perp_r$ on an Immaculate function.

Theorem 6.1. For a sequence of integers $\alpha$ and a composition $\beta$,

\[M^\perp_\beta S_\alpha = \sum \mathfrak{S}_\gamma\]

where the sum is over all sequences of integers $\gamma$ such that removing the zeros from $\alpha - \gamma$ yields the composition $\beta$.

As a consequence this computation, the dual Murnaghan-Nakayama rule is a special case since $p_r = M_r$.

Corollary 6.2. If $\alpha$ is a sequence of integers which sums to $d$ and $r > 0$, then

\[M^\perp_r S_\alpha = \sum_{i=1}^{\ell(\alpha)} \mathfrak{S}_{\alpha-(0^{i-1},0^{\ell(\alpha)-i})}\]

where the expression $\alpha-(0^{i-1},0^{\ell(\alpha)-i})$ represents the result subtracting $r$ from the $i^{th}$ entry of $\alpha$.

Proof. Recall that in [BBSSZ, Lemma 2.4] we give a Hopf algebra computation that can be used to compute the commutation relation between $G^\perp$ for $G \in \mathbb{QSym}$ and multiplication by $f$ in $\mathbb{NSym}$. In particular, since $\Delta(M_\beta) = \sum_{(\gamma, \tau) \in \beta} M_\gamma \otimes M_\tau$ then $M^\perp_r(H_m) = H_{m-r}$ and $M^\perp_\beta(H_m) = 0$ when $\ell(\beta) > 1$, therefore $M^\perp_\beta \circ H_m = H_{m-\beta_1} \circ M^\perp_{(\beta_2, \ldots, \beta_\ell)} + H_m \circ M^\perp_\beta$. Now we can use this relation to compute the commutation relation with creation operator for the immaculate basis and we have

\[M^\perp_\beta \circ B_m = \sum_{d \geq 0} (-1)^d M^\perp_\beta \circ H_{m+d} M^\perp_1\]

\[= \sum_{d \geq 0} (-1)^d (H_{m+d-\beta_1} \circ M^\perp_{(\beta_2, \ldots, \beta_\ell)} + H_{m+d} \circ M^\perp_\beta) M^\perp_1\]

\[= B_{m-\beta_1} \circ M^\perp_{(\beta_2, \ldots, \beta_\ell)} + B_m \circ M^\perp_\beta.\]
The result follows by induction since $S_{\alpha} = B_{\alpha_1}B_{\alpha_2} \cdots B_{\alpha_{\ell(\alpha)}}(1)$ and since $M_r^1(1) = 0$.

As in the last section we need to potentially consider functions indexed by sequence of integers in this formula if $\alpha_i < r$. If $r < \alpha_i$ for all $1 \leq i \leq \ell(\alpha)$ then this result expression expands positively in the immaculate basis.

7. The product of immaculate functions

In [BBSSZ], we conjectured that the product of two immaculate functions expand positively in the immaculate basis in certain cases. The main goal of this section is to prove this conjecture by giving an explicit combinatorial formula. Our proof follows some techniques developed in [BG90] and [BS02].

7.1. The combinatorial formula for the structure coefficients.

Definition 7.1. A skew immaculate tableau $T$ will be called Yamanouchi if the reading word read$(T)$ of $T$ is Yamanouchi. For compositions $\alpha$, $\beta$ and a partition $\lambda$ such that $|\alpha| + |\lambda| = |\beta|$ we let $C^\beta_{\alpha,\lambda}$ denote the number of Yamanouchi immaculate tableaux of shape $\beta/\alpha$ and content $\lambda$.

Example 7.2. Let $\alpha = (1, 2)$, $\lambda = (3, 2, 1)$ and $\beta = (3, 4, 2)$. The tableaux drawn below are the only Yamanouchi immaculate tableaux, so $C^\beta_{\alpha,\lambda} = 2$.

$$
\begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 1 \\
& & 2 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 2 \\
& 2 & 3 \\
& & 2 & 2 \\
\end{array}
$$

Theorem 7.3. For a composition $\alpha$ and a partition $\lambda$,

$$
\mathcal{G}_\alpha \mathcal{G}_\lambda = \sum_{\beta \vdash |\alpha| + |\lambda|} C^\beta_{\alpha,\lambda} \mathcal{G}_\beta.
$$

Proof. Using (7), we expand $\mathcal{G}_\lambda$ in the product $\mathcal{G}_\alpha \mathcal{G}_\lambda$:

$$
\mathcal{G}_\alpha \mathcal{G}_\lambda = \sum_{\sigma \in S_m} (-1)^\sigma \mathcal{G}_\alpha H_{\lambda_1+\sigma_1-1,\lambda_2+\sigma_2-2,\ldots,\lambda_m+\sigma_m-m},
$$

where $H_0 = 1$ and $H_i = 0$ if $i < 0$. An iterative use the Pieri rule [Theorem 4.3] gives

$$
\mathcal{G}_\alpha H_\tau = \sum_{\gamma \vdash |\tau|} \mathcal{G}_\gamma,
$$

where the sum is over skew immaculate tableaux $T$ and $c(T)$ denotes the content of $T$. Substituting this back in (11) we get

$$
\mathcal{G}_\alpha \mathcal{G}_\lambda = \sum_{\sigma \in S_m} \sum_{\gamma \vdash |c(T)|=|\lambda|+|\sigma|} (-1)^\sigma \mathcal{G}_\gamma.
$$
The double sum on the right is over pairs \((\sigma, T)\) where \(\sigma \in S_m\) and \(T\) is a skew immaculate tableau of shape \(\gamma/\alpha\) and content \(C(T) = \lambda + \sigma - Id\). Note that \(\sigma\) depends on \(T\), since \(\sigma = c(T) - \lambda + Id\). Let

\[
\mathcal{T}_\alpha^\lambda = \left\{ T : \begin{array}{l}
T \text{ is a skew immaculate tableau of inner skew shape } \alpha \\
\text{for which } c(T) - \lambda + Id \text{ is a permutation}
\end{array} \right\}
\]

and for \(T\) in \(\mathcal{T}_\alpha^\lambda\), we denote the corresponding permutation by

\[
\sigma(T) = c(T) - \lambda + Id.
\]

Thus, (13) can be rewritten as

\[
\mathcal{S}_\alpha \mathcal{S}_\lambda = \sum_{T \in \mathcal{T}_\alpha^\lambda} (-1)^{\sigma(T)} \mathcal{S}_{sh(T)}.
\]

The theorem will follow from a sign reversing involution on \(\mathcal{T}_\alpha^\lambda\). More specifically, in Definition 7.25 we construct a map \(\Phi\) on \(\mathcal{T}_\alpha^\lambda\) with the following properties:

1. \(\Phi(T) \in \mathcal{T}_\alpha^\lambda\) for \(T \in \mathcal{T}_\alpha^\lambda\).
2. For all \(T \in \mathcal{T}_\alpha^\lambda\), \(\Phi^2(T) = T\).
3. If \(T \in \mathcal{T}_\alpha^\lambda\) is Yamanouchi and \(\sigma(T) = Id\), then \(T\) is a fixed point of \(\Phi\).
4. If \(T \in \mathcal{T}_\alpha^\lambda\) is not Yamanouchi or \(\sigma(T) \neq Id\), then \(T' = \Phi(T) \in \mathcal{T}_\alpha^\lambda\) is such that:
   a. The shape of \(T\) and \(T'\) are equal.
   b. \(\sigma(T') = t_r \sigma(T)\) for an integer \(r\) (where \(t_r\) is the transposition which interchanges \(r\) and \(r + 1\)).

Properties (1), (2), (3), and (4) are proven in Proposition 7.26. The result follows from the existence of such a map.

Example 7.4. We give an example with \(\alpha = [1, 2]\) and \(\lambda = [2, 1]\).

\[
\begin{align*}
\mathcal{S}_{12} \times \mathcal{S}_{21} & = \mathcal{S}_{1221} + \mathcal{S}_{1311} + \mathcal{S}_{132} + \mathcal{S}_{2211} + \mathcal{S}_{222} \\
& + 2\mathcal{S}_{231} + \mathcal{S}_{141} + \mathcal{S}_{24} + \mathcal{S}_{33} + \mathcal{S}_{321}
\end{align*}
\]

Corollary 7.5. For compositions \(\alpha, \beta, \nu\) and a partition \(\lambda\) such that \(\ell(\nu) \leq \ell(\alpha)\),

\[
C_{\alpha, \lambda}^\beta = C_{\alpha+\nu, \lambda}^\beta + C_{\alpha+\nu, \lambda}^\beta
\]

where \(\alpha + \nu\) and \(\beta + \nu\) are the compositions obtained by component-wise addition.
Applying the projection to symmetric functions tells us that we can compute the product of any Schur function with two parts by the Schur function and content in their row words.

Example 7.8. Let \( \lambda = (2,1) \). Then \( \mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_1 = \mathcal{S}_3 \mathcal{S}_2 \mathcal{S}_2 \).

We can compute the product of any Schur function with two parts by the Schur function \( s_2 \). For instance, \( \mathcal{S}_2 \mathcal{S}_2 = \mathcal{S}_3 \mathcal{S}_3 + \mathcal{S}_3 \mathcal{S}_2 + \mathcal{S}_2 \mathcal{S}_2 + 2 \mathcal{S}_1 \mathcal{S}_2 + \mathcal{S}_0 \mathcal{S}_3 + \mathcal{S}_0 \mathcal{S}_2 + \mathcal{S}_0 \mathcal{S}_1 + \mathcal{S}_0 \).

Applying the projection to symmetric functions tells us that \( s_3 s_2 = s_3 s_3 + s_3 s_2 + s_2 s_3 + s_3 + s_2 + 2 s_1 s_2 + s_0 s_3 + s_0 s_2 = s_3 s_2 + s_3 s_3 + s_3 s_2 + 0 + s_4 s_2 + s_2 s_4 + 2 s_4 + s_1 s_4 + s_2 + s_3 = 0 \).

We conjecture the following generalization to Corollary 7.5.

Conjecture 7.9. For compositions \( \alpha, \beta, \gamma, \) and \( \delta \) such that \( \ell(\gamma) \leq \ell(\alpha) \),

\[
C_{\alpha, \beta}^{\gamma} = C_{\alpha+\gamma, \delta}^{\beta+\gamma}.
\]

Applying the projection map to Theorem 7.3 yields the following result about Schur functions.

Corollary 7.10. For two partitions \( \mu, \lambda \),

\[
s_{\mu} s_{\lambda} = \sum_{\beta = |\mu|+|\lambda|} C_{\mu, \lambda}^{\beta} s_{\beta}.
\]

We have the following relation amongst Schur functions indexed by compositions:

\[
s_\zeta = -s_\zeta \star \tau_r
\]

where

\[
\zeta \star \tau_r = (\zeta_1, \ldots, \zeta_{r-1}, \zeta_{r+1} - 1, \zeta_r + 1, \zeta_{r+2}, \ldots, \zeta_m).
\]

The classical Littlewood-Richardson coefficients \( c_{\mu, \lambda}^{\nu} \) in Theorem 2.3 are thus obtained as follows.

Proof. This follows by a direct bijection of the two sets counted by \( C_{\alpha, \lambda}^{\beta} \) and \( C_{\alpha+\nu, \lambda}^{\beta+\nu} \). If we have an immaculate tableau \( T \) of shape \( \beta/\alpha \) and content \( \lambda \) in the first set that is Yamanouchi, we simply map it to the immaculate tableau of shape \( (\beta+\nu)/(\alpha+\nu) \) (which has the same row distributions, and same first column as \( \beta/\alpha \) and content \( \lambda \) preserving the rows of \( T \). The result is clearly also Yamanouchi since the row word is preserved. The inverse map is similar. 

Remark 7.6. This implies that the coefficient \( C_{\alpha, \lambda}^{\beta} \) can be deduced just from knowing \( C_{\alpha+\nu, \lambda}^{\beta+\nu} \).

Definition 7.7. For a composition \( \beta = (\beta_1, \beta_2, \ldots, \beta_m) \) we define the Schur function indexed by \( \beta \) as

\[
s_\beta = \det |h_{\beta_i+j-i}|_{1 \leq i, j \leq m}.
\]

"
Corollary 7.11. For partitions $\lambda, \mu, \nu$ such that $|\lambda| + |\mu| = |\nu|$, 
\[ c_{\mu,\lambda}^\nu = \sum_{\sigma \in S_{\ell(\nu)}} (-1)^\sigma C_{\mu,\lambda}^{\nu \ast \sigma} \]
with the convention that $C_{\mu,\lambda}^{\nu \ast \sigma} = 0$ if $\nu \ast \sigma$ is not a composition of length $m = \ell(\nu)$ that contains $\mu$.

Remark 7.12. We can construct a sign reversing involution to resolve the cancellations in Corollary 7.11 and deduce the classical Littlewood-Richardson rule. We leave this to the interested reader.

7.2. The details. As mentioned above, we will define a sign reversing involution $\Phi$ on the set of tableaux $\mathfrak{T}_\alpha^\lambda$ defined in (14). This subsection is devoted to constructing $\Phi$ and proving the various properties of $\Phi$ which were claimed in the proof of Theorem 7.3.

Remark 7.13. We will make use of both a left and a right action of a permutation $\sigma$. The symbol $\sigma(r)$ denotes the action of a permutation on a single value. A permutation $\sigma \in S_n$ acts on a list of entries in $\{1, \ldots, n\}$ by $\sigma \cdot (a_1, a_2, \ldots, a_d) = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_d))$. A permutation acts on a list of $n$ entries on the right by $(b_1, b_2, \ldots, b_n) \cdot \sigma = (b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(n)})$.

Definition 7.14. Given $T \in \mathfrak{T}_\alpha^\lambda$, we construct $Y(T)$ as follows. For every cell of content $r$ in the $i$-th row of $T$ we put a cell of content $i$ in the $\sigma(T)(r)$-th row of $Y(T)$. We sort the entries of the row of $Y(T)$ in increasing order.

Example 7.15. Let $\lambda = (2, 2, 2)$ and $\alpha = (1, 2)$. Let 
\[ T_1 = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 \\
3 & 2 
\end{array} \]

Note that $\sigma(T_1) = c(T_1) - \lambda + Id = (2, 3, 1) - (2, 2, 2) + (1, 2, 3) = (1, 3, 2)$. Since $\sigma(T_1)$ is a permutation, $T_1 \in \mathfrak{T}_\alpha^\lambda$. By the construction, 
\[ Y(T_1) = \begin{array}{ccc}
1 & 1 \\
3 \\
1 & 2 & 3 
\end{array} \]

As a second example, again let $\lambda = (2, 2, 2)$ and $\alpha = (1, 2)$. Let 
\[ T_2 = \begin{array}{ccc}
1 & 1 \\
1 & 2 \\
2 & 2 
\end{array} \]
Note that $\sigma(T_2) = c(T_2) - \lambda + Id = (3, 3, 0) - (2, 2, 2) + (1, 2, 3) = (2, 3, 1)$. Again since $\sigma(T_2)$ is a permutation, $T_2 \in \Sigma^\lambda_\alpha$. By the construction,

\[
Y(T_2) = \begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3
\end{array},
\]

where the shaded empty square indicates that there are no cells in the first row of the diagram.

**Lemma 7.16.** Let $\lambda$ be a partition and $\alpha$ a composition. For $T \in \Sigma^\lambda_\alpha$, let $\tau = sh(Y(T))$.

1. $(\tau - Id) = (\lambda - Id) \cdot \sigma(T)^{-1}$.
2. We can recover $T$ uniquely from $Y(T)$ (that is, $Y$ is injective).
3. $\text{Des}(\sigma(T)^{-1}) = \{r : \tau_r < \tau_{r+1} - 1\}$.
4. If $\tau_r < \tau_{r+1}$, then $\tau_r < \tau_{r+1} - 1$.
5. $\tau$ is a partition if and only if $\sigma(T) = Id$.
6. If $\sigma(T) = Id$, then $Y(T)$ is a semi-standard Young tableau (of partition shape and strictly increasing down columns) if and only if $\text{read}(T)$ is Yamanouchi.

**Proof.** (1) By definition $\sigma(T) = c(T) - \lambda + Id$ and so

\[
\tau = c(T) \cdot \sigma(T)^{-1} = \lambda \cdot \sigma(T)^{-1} + Id - \sigma(T)^{-1} = (\lambda - Id) \cdot \sigma(T)^{-1} + Id.
\]

(2) follows from the fact that one can reverse the construction of $Y$. Indeed, the permutation $\sigma(T)$ can be computed from the identity $(\tau - Id) = (\lambda - Id) \cdot \sigma(T)^{-1}$ since $\tau - Id$ and $\lambda - Id$ have distinct parts. Then $T$ is the skew tableau of inner skew shape $\alpha$ whose $i$-th row contains $\sigma(T)^{-1}(u_1), \sigma(T)^{-1}(u_2), \ldots$, where $u_1, u_2, \ldots$, are the rows of $Y(T)$ that contain $i$ (listed according to multiplicity).

(3) From (17), if $d = \sigma(T)^{-1}(r)$ and $c = \sigma(T)^{-1}(r + 1)$, then

\[
\tau_r = \lambda_d - d + r \quad \text{and} \quad \tau_{r+1} = \lambda_c - c + (r + 1).
\]

If $r$ is a descent of $\sigma(T)^{-1}$, then $d > c$ and so $\lambda_d \geq \lambda_c$. This implies that

\[
\tau_r = \lambda_d - d + r < \lambda_c - c + r = \tau_{r+1} - 1.
\]

Conversely, if $r$ is an ascent of $\sigma(T)^{-1}$, then $d < c$ and so $\lambda_d \geq \lambda_c$. This implies that

\[
\tau_r = \lambda_d - d + r > \lambda_c - c + r = \tau_{r+1} - 1,
\]

and so $\tau_r \geq \tau_{r+1}$.

(4) If $\tau_r < \tau_{r+1}$, then by the previous sentence, $r$ cannot be an ascent of $\sigma^{-1}(T)$. Hence, $r$ is a descent of $\sigma^{-1}(T)$, which by (3) implies that $\tau_r < \tau_{r+1} - 1$.

(5) follows from (3) by remarking that the identity permutation is the only permutation with no descents.
To prove (6), note that in \( Y(T) \), the \((i, j)\) entry is a \( k \) if in \( T \), the \( j\)-th appearance of the letter \( i \) in \( \text{read}(T) \) comes from a cell in row \( k \). The result follows. □

We now define a mapping on \( Y(\mathfrak{S}^\lambda_\alpha) \), which we will prove is an involution. We say that a cell \( x \) not in the first row of a tableau and containing the value \( a \) is nefarious if the cell above \( x \) is either empty or it contains \( b \) with \( b \geq a \):

\[
\begin{array}{c|c}
\hline
a & \text{or} & b \\
\hline
\end{array}
\]

The most nefarious cell of a tableau is the bottom-most nefarious cell contained in the left-most column of the tableau that contains a nefarious cell.

**Definition 7.17.** For a partition \( \lambda \) and a composition \( \alpha \), we define a map \( \Theta \) on \( Y(\mathfrak{S}^\lambda_\alpha) \) as follows (cf. Figure 1). Let \( Y(T) \in Y(\mathfrak{S}^\lambda_\alpha) \).

1. If \( Y(T) \) contains no nefarious cells, then \( Y(T) \) is a fixed point of \( \Theta \).
2. Otherwise, let \( x \) be the most nefarious cell of \( Y(T) \).
   - (a) If the cell \( y \) above \( x \) is not empty, then define \( \Theta(Y(T)) \) to be the tableau obtained from \( Y(T) \) by moving:
     - all the cells strictly to the right of \( x \) into the row above \( x \); and
     - all the cells weakly to the right of \( y \) (including \( y \)) into the row containing \( x \).
   - (b) Otherwise, define \( \Theta(Y(T)) \) to be the tableau obtained from \( Y(T) \) by moving all the cells strictly to the right of \( x \) into the row above \( x \).

**Example 7.18.** Continuing Example 7.15, the most nefarious cell of \( Y(T_1) \) is in the third row of the first column. Applying \( \Theta \) yields:

\[
\begin{pmatrix}
1 & 1 \\
3 & 1 \\
1 & 2 & 3
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
2 & 3 \\
1 & 3
\end{pmatrix}
\]

Figure 1. The effect of \( \Theta \) on the cells to the right of the two types of a nefarious cell \( x \).
In $Y(T_2)$, the most nefarious cell is in the second row of the first column and so

$$\Theta \left( \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 3 \end{array} \right) = \begin{array}{ccc} 1 & 3 \\ 1 & 2 & 3 \end{array}. $$

**Lemma 7.19.** If $Y(T) \in Y(\Sigma^\lambda_\alpha)$ contains a nefarious cell, then $\Theta(Y(T)) \neq Y(T)$.

**Proof.** Suppose the most nefarious cell $x$ of $Y(T)$ lies in row $r+1$. Let $\tau = sh(Y(T))$.

If $\tau_r \geq \tau_{r+1}$, then the cell $y$ above $x$ is not empty and so we find ourselves in case (2a) of Definition 7.17. Hence, the number of cells that move from row $r$ into row $r+1$ is greater than the number of cells that move from row $r+1$ into row $r$. In particular, $\Theta(Y(T)) \neq Y(T)$.

Suppose instead that $\tau_r < \tau_{r+1}$. Then we have that $\tau_r < \tau_{r+1} - 1$ by 7.16(4). Hence, the number of cells that move from row $r+1$ into row $r$ is greater than the number of cells that move from row $r$ into row $r+1$. This insures that $\Theta(Y(T)) \neq Y(T)$ in this case as well. \qed

**Lemma 7.20.** For a composition $\alpha$, a partition $\lambda$ and $T \in \Sigma^\lambda_\alpha$, if there exists a $T' \in \Sigma^\lambda_\alpha$ for which $Y(T') = \Theta(Y(T))$ and $T' \neq T$, then there exists an $1 \leq r < \ell(sh(Y(T)))$ such that $\sigma(T') = t_r \circ \sigma(T)$.

**Proof.** Suppose that $\Theta(Y(T)) = Y(T')$ and that $T' \neq T$. Then $Y(T)$ is not a fixed point of $\Theta$ and so it contains a nefarious cell. Suppose the most nefarious cell $x$ of $Y(T)$ lies in row $r+1$. Then the shapes of $\Theta(Y(T))$ and $Y(T)$ differ only in rows $r$ and $r+1$. More explicitly,

$$sh(\Theta(Y(T))) = (sh(Y(T)) + t_r - Id) \cdot t_r$$

since $t_r - Id = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ is the vector whose nonzero entries occur in positions $r$ and $r+1$. Using Equation (17), we rewrite $sh(Y(T))$ to obtain

$$sh(\Theta(Y(T))) = (\lambda \cdot \sigma(T)^{-1} + t_r - \sigma(T)^{-1}) \cdot t_r$$

$$= \lambda \cdot (\sigma(T)^{-1} \circ t_r) + Id - \sigma(T)^{-1} \circ t_r.$$ 

But, also from Equation (17), we have $sh(Y(T')) = (\lambda - Id) \cdot \sigma(T')^{-1} + Id$ so that

$$(\lambda - Id) \cdot \sigma(T')^{-1} = sh(Y(T')) - Id$$

$$= (\lambda \cdot (\sigma(T)^{-1} \circ t_r) + Id - \sigma(T)^{-1} \circ t_r) - Id$$

$$= (\lambda - Id) \cdot (\sigma(T)^{-1} \circ t_r).$$

Since all the parts of $\lambda - Id$ are distinct, we conclude that $\sigma(T')^{-1} = \sigma(T)^{-1} \circ t_r$. \qed

**Lemma 7.21.** For every immaculate tableau $T$, there exists an immaculate tableau $T'$ such that $Y(T') = \Theta(Y(T))$. 
Proof. The proof of Lemma 7.16(2) gives a procedure to compute the preimage of the map $Y$. We will apply this procedure to $\Theta(Y(T))$ and show that the resulting skew tableau $T'$ belongs to $\mathcal{T}_\alpha^\lambda$. To simplify the notation, let $\sigma = \sigma(T)$.

Suppose that the most nefarious cell $x$ of $Y(T)$ occurs in row $r+1$. Let $T'$ denote the skew tableau of inner skew shape $\alpha$ whose $j$-th row contains $(t_r \circ \sigma)^{-1}(v_1)$, \ldots, $(t_r \circ \sigma)^{-1}(v_a)$ (listed in weakly increasing order), where $v_1, \ldots, v_a$ are the rows of $\Theta(Y(T))$ that contain $j$. For examples of this, see Examples 7.22 and 7.23. By construction, the entries in each row of $T'$ are weakly increasing, so it remains to show that the entries in the first column of $T'$ are strictly increasing. We will show that the first column of $T'$ is equal to the first column of $T$.

Claim 1. The tableau $T'$ is obtained from $T$ by changing some $\sigma^{-1}(r)$ to $\sigma^{-1}(r+1)$ and some $\sigma^{-1}(r+1)$ to $\sigma^{-1}(r)$.

An entry $e$ in $T$ corresponds to an entry in row $\sigma(e)$ of $Y(T)$; and an entry in row $s$ of $\Theta(Y(T))$ corresponds to an entry $(t_r \circ \sigma)^{-1}(s)$ in $T'$. Therefore, an entry $e$ in $T$ corresponds to an entry $(t_r \circ \sigma)^{-1}(\sigma(e))$ in $T'$, which is equal to $\sigma^{-1}(r+1)$ if $e = \sigma^{-1}(r)$; to $\sigma^{-1}(r)$ if $e = \sigma^{-1}(r+1)$; and to $e$ otherwise.

Claim 2. The first entry in the $j$-th row of $T$ is $f_j(T) = \min \{\sigma^{-1}(s) : \text{row } s \text{ of } Y(T) \text{ contains } j\}$;

and the first entry in the $j$-th row of $T'$ is $f_j(T') = \min \{(t_r \circ \sigma)^{-1}(s) : \text{row } s \text{ of } \Theta(Y(T)) \text{ contains } j\}$.

Claim 3. The maximal entry in row $\sigma(f_j(T))$ of $Y(T)$ is $j$.

Since $T$ is a skew immaculate tableau and $f_j(T)$ is the first entry in row $j$ of $T$, it follows that $f_j(T)$ does not appear in rows $j+1, j+2, \ldots$ of $T$. Hence, none of $j+1, j+2, \ldots$ appear in row $\sigma(f_j(T))$ of $Y(T)$.

Now suppose that the first column of $T$ is not equal to the first column of $T'$ and let $j > l(\alpha)$ be a row in which the first entries of $T$ and $T'$ differ: $f_j(T) \neq f_j(T')$. As shown above, these entries are necessarily $\sigma^{-1}(r)$ and $\sigma^{-1}(r+1)$.

Below we will make reference to the following diagram illustrating the effect of $\Theta$ on the rows $r$ and $r+1$ of $Y(T)$.

\[
\begin{array}{clll}
\text{row } r & \ldots & c & b & v \\
\text{row } r+1 & \ldots & d & a & u \\
\end{array}
\]

\[
\Theta \quad \rightarrow \quad \begin{array}{clll}
\ldots & c & u \\
\ldots & d & a & b & v \\
\end{array}
\]

Case 1. $f_j(T) = \sigma^{-1}(r)$ and $f_j(T') = \sigma^{-1}(r+1) = (t_r \circ \sigma)^{-1}(r)$.

We derive a contradiction by showing $\sigma^{-1}(r) > \sigma^{-1}(r+1)$ and $\sigma^{-1}(r) < \sigma^{-1}(r+1)$.

The equality $f_j(T) = \sigma^{-1}(r)$ implies that $j$ is the maximal entry in row $r$ of $Y(T)$, by Claim 3, which means that $j$ is contained in row $r+1$ of $\Theta(Y(T))$. Since $f_j(T')$
is such that \( f_j(T') \leq (t_r \circ \sigma)^{-1}(s) \) for all rows \( s \neq r \) of \( \Theta(Y(T)) \) that contain \( j \), it follows that
\[
\sigma^{-1}(r) = (t_r \circ \sigma)^{-1}(r + 1) > f_j(T') = (t_r \circ \sigma)^{-1}(r) = \sigma^{-1}(r + 1).
\]

The equality \( f_j(T') = (t_r \circ \sigma)^{-1}(r) \) implies that row \( r \) of \( \Theta(Y(T)) \) contains \( j \). We argue that \( j \) occurs in \( u \). If \( j \leq c \), then \( j \leq c < d \leq a \leq b \) since the cell containing \( a \) is the most nefarious cell of \( Y(T) \). This inequality contradicts the fact that \( j \) is the maximal entry in row \( r \) of \( Y(T) \). Hence, \( j \) occurs in \( u \), which means that \( j \) also occurs in row \( r + 1 \) of \( Y(T) \). This implies that \( \sigma^{-1}(r) = f_j(T) < \sigma^{-1}(r + 1) \).

Case 2. \( f_j(T) = \sigma^{-1}(r + 1) \) and \( f_j(T') = (t_r \circ \sigma)^{-1}(r + 1) \).

Arguing as in the first part of the previous case, we conclude that
\[
\sigma^{-1}(r) < \sigma^{-1}(r + 1).
\]

The equality \( f_j(T') = (t_r \circ \sigma)^{-1}(r + 1) \) implies that \( j \) occurs in row \( r + 1 \) of \( \Theta(Y(T)) \).

Suppose first that \( j \geq b \). Then \( b = j \) or \( j \) occurs in \( v \). In both cases, we have that \( j \) appears in row \( r \) of \( Y(T) \). Hence, \( \sigma^{-1}(r) > f_j(T) = \sigma^{-1}(r + 1) \), a contradiction.

Suppose instead that \( j < b \). Then \( j \leq a \). Since \( j \) is the maximal entry of row \( r + 1 \) of \( Y(T) \), we have that \( a = j \) and all the entries of \( u \) are equal to \( j \).

Therefore, row \( b \) of \( T \) contains the entry \( \sigma^{-1}(r) \). Since \( b > j \) and the first entry of row \( j \) is \( f_j(T) = \sigma^{-1}(r + 1) \), it follows that \( \sigma^{-1}(r) > \sigma^{-1}(r + 1) \), a contradiction.

Example 7.22. Continuing Example 7.15 and Example 7.18 we see that there exists a tableau \( T'_1 \in \tilde{\Sigma}_\alpha^\lambda \) for which \( Y(T'_1) = \Theta(Y(T_1)) \):

\[
T_1 = \begin{bmatrix}
1 & 2 & 1 & 3 \\
2 & 3 & 2 & 1 \\
\end{bmatrix} \quad Y \quad \begin{bmatrix}
1 & 1 \\
3 & 1 \\
\end{bmatrix} \quad \Theta \quad \begin{bmatrix}
1 & 1 & 3 \\
2 & 3 & 2 \\
\end{bmatrix} = T'_1.
\]

Furthermore, \( \sigma(T'_1) = (2, 2, 2) - (2, 2, 2) + (1, 2, 3) = (1, 2, 3) = t_2 \circ (1, 3, 2) = t_2 \circ \sigma(T_1) \).

Similarly, there exists a tableau \( T'_2 \in \tilde{\Sigma}_\alpha^\lambda \) for which \( Y(T'_2) = \Theta(Y(T_2)) \):

\[
T_2 = \begin{bmatrix}
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 2 \\
\end{bmatrix} \quad Y \quad \begin{bmatrix}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{bmatrix} \quad \Theta \quad \begin{bmatrix}
1 & 3 \\
1 & 2 \\
\end{bmatrix} = T'_2.
\]

Furthermore, \( \sigma(T'_2) = (2, 3, 1) - (2, 2, 2) + (1, 2, 3) = (1, 3, 2) = t_1 \circ (2, 3, 1) = t_1 \circ \sigma(T_2) \).

Example 7.23. Let \( \alpha = (1, 2) \), \( \lambda = (3, 3, 0) \) and let \( T \) be the tableau:

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 2 \\
\end{bmatrix}.
\]
Then \( \sigma(T) = c(T) - \lambda + Id = (4, 2, 0) - (3, 3, 0) + (1, 2, 3) = (2, 1, 3) \), so \( T \in \mathfrak{T}_\lambda^\alpha \).

**Proposition 7.24.** \( \Theta \) is an involution on \( Y(\mathfrak{T}_\lambda^\alpha) \).

*Proof.* Suppose \( Y(T) \in Y(\mathfrak{T}_\lambda^\alpha) \). By construction, if \( x \) is the most nefarious cell of \( Y(T) \), then also \( x \) is a nefarious cell of \( \Theta(Y(T)) \). Moreover, it is the most nefarious cell of \( \Theta(Y(T)) \) since the only cells changed by \( \Theta \) occur to the right of and above \( x \). By Lemma 7.21, \( \Theta(Y(T)) \in Y(\mathfrak{T}_\lambda^\alpha) \). Thus, part (2) of the definition of \( \Theta \) applies to the same nefarious cell \( x \) and it will undo the modifications effected by \( \Theta \). \( \square \)

**Definition 7.25.** For a composition \( \alpha \), a partition \( \lambda \), and a tableau \( T \in \mathfrak{T}_\lambda^\alpha \), we define the map \( \Phi \) on \( \mathfrak{T}_\lambda^\alpha \) by \( \Phi(T) = Y^{-1}(\Theta(Y(T))) \).

**Proposition 7.26.** For a composition \( \alpha \) and a partition \( \lambda \),

1. \( \Phi(T) \in \mathfrak{T}_\lambda^\alpha \) for \( T \in \mathfrak{T}_\lambda^\alpha \).
2. For all \( T \in \mathfrak{T}_\lambda^\alpha \), \( \Phi^2(T) = T \).
3. If \( T \in \mathfrak{T}_\lambda^\alpha \) is Yamanouchi and \( \sigma(T) = Id \), then \( T \) is a fixed point of \( \Phi \).
4. If \( T \in \mathfrak{T}_\lambda^\alpha \) is not Yamanouchi or \( \sigma(T) \neq Id \), then \( T' = \Phi(T) \in \mathfrak{T}_\lambda^\alpha \) is such that:
   a. The shape of \( T \) and \( T' \) are equal.
   b. \( \sigma(T') = t_r \sigma(T) \) for an integer \( r \) (where \( t_r \) is the transposition which interchanges \( r \) and \( r + 1 \)).

*Proof.* Condition (1) is a direct consequence of Lemma 7.21. Condition (2) is a direct consequence of Proposition 7.24. Condition (3) follows from the definition of \( \Theta \) and Lemma 7.16 (6). Condition (4a) follows from the fact that \( \Theta \) preserves contents and \( Y \) interchanges shape with content and Condition (4b) follows from Lemma 7.20. \( \square \)

### 7.3. A geometric interpretation of the coefficients

There are several geometric constructions of the Littlewood-Richardson coefficients \( c_{\mu,\lambda}^{\alpha} \) counting integral points inside a certain polytope (for instance [BZ] and [GZ]). Given that the coefficients \( C_{\alpha,\nu}^{\beta} \) are non-negative for \( \alpha \) and \( \beta \) compositions and \( \nu \) a partition such that \( |\beta| = |\alpha| + |\nu| \), we decided to investigate if it was possible to also construct a polytope such that \( C_{\alpha,\nu}^{\beta} \) is the number of integral points inside. Following the exposition of [PV] we present here a set of equations which allow us to give a geometric interpretation to immaculate structure coefficients and make a comparison to the Littlewood-Richardson coefficients.

Let \( m \) be the maximum of the three integers \( \ell(\alpha), \ell(\beta) \) and \( \ell(\nu) \). We extend the lists \( \alpha, \beta \) and \( \nu \) with 0’s at the end such that they all have length \( m \). The polytope...
will live in a \( \binom{m+2}{2} \) dimensional vector space with coordinates \( a_{ij} \) where \( 0 \leq i \leq j \leq m \) and \( a_{00} = 0 \). We arrange the coordinates into a triangular array as in the diagram below.

\[
\begin{array}{ccccccc}
   a_{00} & a_{01} & a_{11} & a_{12} & a_{22} \\
   a_{02} & a_{12} & \ddots & \ddots & \\
   a_{0m} & a_{1m} & \ddots & a_{m-1m} & a_{mm} \\
\end{array}
\]

We will determine our polytope as the following intersection of the set of planes and half planes below. Fix \( \alpha \) and \( \beta \) compositions and \( \nu \) a partition (possibly padded with parts of size 0) so that they are of length \( m \). Let \( N = |\nu| \), then

1. \( a_{ij} \geq 0 \) for \( 1 \leq i \leq j \leq m \)
2. \( a_{0j} = \alpha_j \) for \( 1 \leq i \leq m \)
3. \( \sum_{p=0}^m a_{pj} = \beta_j \) for \( 1 \leq j \leq m \)
4. \( \sum_{q=i}^m a_{iq} = \nu_i \) for \( 1 \leq i \leq m \)
5. \( \sum_{q=i+1}^j a_{iq} \geq 0 \) for \( 1 \leq i \leq j \leq m \)
6. \( \sum_{p=0}^{i-1} Na_{pj} - \sum_{p=0}^i a_{p,j+1} \geq 0 \) for \( 1 \leq i \leq j < m \)

These equations define a polytope because they are the intersection of planes and half planes. By comparison with the Littlewood-Richardson coefficients, if \( \alpha = \mu, \beta = \lambda \) where \( \mu, \lambda \) and \( \nu \) are partitions such that \( |\lambda| = |\mu| + |\nu| \), then \([PV, \text{Lemma 3.1}]\) states that \( c_{\mu,\nu}^\lambda \) is equal to the number of integer points satisfying conditions (1) through (5) and the inequality

\[
(CS) \quad \sum_{p=0}^{i-1} a_{pj} - \sum_{p=0}^i a_{p,j+1} \geq 0 \text{ for all } 1 \leq i \leq j < m
\]

in place of condition (6).

**Theorem 7.27.** For \( \alpha, \beta \) compositions and \( \nu \) a partition all of length less than or equal to \( m \) such that \( |\beta| = |\alpha| + |\nu| \), \( C_{\alpha,\nu}^\beta \) is equal to the number of integer points satisfying conditions (1) through (6).

**Proof.** We want to show that there is a bijection between the integral points inside the region defined by conditions (1) through (6) and Yamanouchi immaculate tableaux of shape \( \beta/\alpha \) and content \( \nu \). Fix a Yamanouchi immaculate tableau \( T \) of shape \( \beta/\alpha \) and content \( \nu \). Let \( A(T) = (a_{ij})_{0 \leq i \leq j \leq m} \) be the point defined by

1. \( A(T)_{00} = 0 \)
2. \( A(T)_{0j} = \alpha_j \) for \( 1 \leq j \leq m \)
3. \( A(T)_{ij} \) is the number of entries \( i \) in row \( j \) of \( T \), for \( 1 \leq i \leq j \leq m \).
We can check that \( A(T) \) is in the region defined by (1)–(6). Condition (iii) implies that condition (1) will hold and condition (ii) is a statement that (2) will be true. The sum of the entries \( A(T)_{pj} \) for \( 0 \leq p \leq j \) is equal to the number of entries in row \( j \) and hence will sum to \( \beta_j \) (hence (3) will hold). The sum of the entries \( A(T)_{iq} \) for \( i \leq q \leq m \) is the number of entries of \( T \) equal to \( i \) and this is equal to \( \nu_i \) which implies (4).

The sum \( \sum_{q=i}^{j} A(T)_{iq} \) represents the number of entries \( i \) in rows \( i \) through \( j \) in \( T \). If \( T \) is Yamanouchi, then the number of labels \( i \) in rows \( i \) through \( j \) is larger than or equal to the number of labels \( i+1 \) in rows \( i+1 \) through \( j \). This implies the inequality \( \sum_{q=i}^{j} A(T)_{iq} \geq \sum_{q=i+1}^{j} A(T)_{i+1q} \) and (5) holds.

Assume that \( T \) is not immaculate column strict, then there will be a row \( j \) where \( \alpha_j = \alpha_{j+1} = 0 \) and the first non-zero entry in row \( j+1 \) is smaller than or equal to the first non-zero entry in row \( j \). If the first non-zero entry in row \( j+1 \) is \( i \), then \( \sum_{p=0}^{i} A(T)_{pj+1} \) will be non-zero and \( \sum_{p=0}^{i-1} A(T)_{pj} \) will be 0, hence (6) will not hold.

Now assume that \( T \) is immaculate column strict, then every row either has \( A(T)_{0j} = \alpha_j > 0 \) or \( \alpha_j = \alpha_{j+1} = 0 \) and the smallest entry in row \( j \) is smaller than the smallest entry in row \( j+1 \). If \( \alpha_j > 0 \), then because \( N \) is chosen to be large, \( \sum_{p=0}^{i-1} NA(T)_{pj} \gg \sum_{p=0}^{i} A(T)_{pj+1} \) for all \( i \) such that \( 1 \leq i \leq j \). If \( \alpha_j = \alpha_{j+1} = 0 \) and \( \beta_j = \beta_{j+1} = 0 \), then \( \sum_{p=0}^{i-1} NA(T)_{pj} - \sum_{p=0}^{i} A(T)_{pj+1} = 0 \) if \( i < i' \) and \( \sum_{p=0}^{i-1} NA(T)_{pj} \gg \sum_{p=0}^{i} A(T)_{pj+1} \) if \( i \geq i' \). In all of these cases (6) holds.

Also, given a point \((a_{ij})\) which satisfies conditions (1)–(6), let \( T \) be the skew tableau of with inner shape \((a_{01}, a_{02}, \ldots, a_{0m}) = \alpha \) and with \( a_{ij} \) entries \( i \) in row \( j \) for \( 1 \leq i < j \leq m \) such that these entries are arranged so they are weakly increasing in the rows. The \( j^{th} \) row of \( T \) has a total of \( \sum_{p=0}^{i} a_{pj} = \beta_j \) cells. The number of entries \( i \) will be \( \sum_{q=i}^{m} a_{iq} = \lambda_i \). The argument above shows that Condition (5) is equivalent to the tableau is Yamanouchi and Condition (6) implies that \( T \) is strictly increasing in the first column. \( \Box \)

**Example 7.28.** The two Yamanouchi immaculate tableaux from Example 7.2 have arrays \( A(T) \) represented by the following triangles.

```
0    0
1  2  1  2
2 1  1  2  0  2
0  0  1  1  0  1  0  1
```
Example 7.29. A larger example of a skew immaculate Yamanouchi tableau with \( \alpha = (3, 2, 3, 1), \beta = (3, 6, 5, 5, 2) \) and \( \lambda = (5, 3, 3, 1) \) is represented by the following

\[
T = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 \\
2 & 2 & 3 & 3 \\
3 & 4
\end{bmatrix}.
\]

The array for this tableau is

\[
A(T) = \begin{bmatrix}
0 & 3 & 0 \\
2 & 3 & 0 \\
3 & 1 & 1 & 0 \\
1 & 0 & 2 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

We now consider the linear transformation where

\[
h_{ij} = \sum_{p=0}^{i} \sum_{q=p}^{j} a_{pq}.
\]

We will refer the point resulting of this transformation the hive array corresponding to the point or tableau. This transformation sends the region in Theorem 7.27 into the following region cut out by the intersection of the inequalities:

\[
(1') h_{0j} - h_{0j-1} = \alpha_j \text{ for } 1 \leq j \leq m \\
(2') h_{jj} - h_{j-1,j-1} = \beta_j \text{ for } 1 \leq j \leq m \\
(3') h_{im} - h_{i-1m} = \nu_i \text{ for } 1 \leq i \leq m \\
(4') h_{ij} - h_{i,j-1} \geq h_{i-1,j} - h_{i-1,j-1} \text{ for } 1 \leq i < j \leq m \\
(5') h_{ij} - h_{i-1,j} \geq h_{i+1,j+1} - h_{i,j+1} \text{ for } 1 \leq i \leq j < m \\
(6') N(h_{i-1,j} - h_{i-1,j-1}) \geq h_{i,j+1} - h_{ij} \text{ for } 1 \leq i \leq j < m
\]

Corollary 7.30. Equations (1')–(6') determine a polytope such that the number of integral points inside the region is equal to \( C_{\alpha,\lambda}^\beta \).

The polytope defined by the equations (1')–(6') is an analogue of the hive polytope of Knutson and Tao [KnTao].

Example 7.31. The two Yamanouchi immaculate tableaux from Example 7.2 and 7.28 have the following hive arrays.

\[
\begin{bmatrix}
0 & 0 \\
1 & 3 & 1 & 3 \\
3 & 6 & 7 & 3 & 5 & 7 \\
3 & 6 & 8 & 9 & 3 & 6 & 8 & 9
\end{bmatrix}
\]
Example 7.32. The tableau $T$ from Example 7.29 has the following hive array.

\[
\begin{array}{ccccccc}
0 & & & & & & \\
3 & 3 & & & & & \\
5 & 8 & 8 & & & & \\
8 & 12 & 13 & 13 & & & \\
9 & 10 & 16 & 18 & 18 & & \\
9 & 10 & 16 & 19 & 20 & 20 & \\
\end{array}
\]

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