Recently a new approach in constructing the conserved charges in cosmological Einstein’s gravity was given. In this new formulation, instead of using the explicit form of the field equations a covariantly conserved rank four tensor was used. In the resulting charge expression, instead of the first derivative of the metric perturbation, the linearized Riemann tensor appears along with the derivative of the background Killing vector fields. Here we give a detailed analysis of the first order and the second order perturbation theory in a gauge-invariant form in cosmological Einstein’s gravity. The linearized Einstein tensor is gauge-invariant at the first order but it is not so at the second order, which complicates the discussion. This method depends on the assumption that the first order metric perturbation can be decomposed into gauge-variant and gauge-invariant parts and the gauge-variant parts do not contribute to physical quantities.

I. INTRODUCTION

In General Relativity finding an exact solution is often very difficult and therefore one needs to use perturbation theory, by starting from an exact background solution with symmetries, which provides a lot of information about the physical problem at hand. In the absence of a source, any generic gravity field equations in local coordinates read

\[ \mathcal{E}_{\mu\nu}(g(\lambda)) = 0, \]

where \( \lambda \) parametrizes the solution set. We have the exact solution plus the perturbations defined as

\[ g(\lambda = 0) := \bar{g}, \quad h_{\mu\nu} := \frac{dg_{\mu\nu}}{d\lambda} \bigg|_{\lambda=0}, \quad k_{\mu\nu} := \frac{1}{2} \frac{d^2 g_{\mu\nu}}{d\lambda^2} \bigg|_{\lambda=0}, \]

where \( \bar{g} \) is the background solution that we carry out the perturbations around, \( h \) denotes the first order perturbation of the metric tensor and \( k \) denotes the second order perturbation. When we consider the perturbation of the field equations (1) about the background spacetime solution \( \bar{g} \), we obtain expansion of the field equations up to \( \mathcal{O}(\lambda^3) \) as

\[ \mathcal{E}_{\mu\nu}(\bar{g}) + \lambda (\mathcal{E}_{\mu\nu})^{(1)}(h) + \lambda^2 \left( (\mathcal{E}_{\mu\nu})^{(2)}(h, h) + (\mathcal{E}_{\mu\nu})^{(1)}(k) \right) = 0. \]

Here by assumption \( \mathcal{E}_{\mu\nu}(\bar{g}) = 0 \) and \( (\mathcal{E}_{\mu\nu})^{(1)}(h) \) denotes the first order linearized field equations while the combination \( (\mathcal{E}_{\mu\nu})^{(2)}(h, h) + (\mathcal{E}_{\mu\nu})^{(1)}(k) \) denotes the second order perturbations of the field equations. Of course not all background solutions can be integrable to an exact solution,
since once $\bar{g}$ solves the background field equations, the solution of the first order linearized field equations, $h$, must satisfy the given relation \( (2) \). Similarly the second order metric perturbation must satisfy the given definition with the second order field equations

\[
(\mathcal{E}_{\mu\nu}^{(2)})(h,h) + (\mathcal{E}_{\mu\nu}^{(1)})(k) = 0.
\]

It means even if we find the linearized solutions, $h$, to the first order perturbations of the field equations \( (E_{\mu\nu})^{(1)}(h) = 0 \), there exists an additional constraint on it which comes from the second order field equations. To see this situation explicitly let us consider $\bar{\xi}^\mu$, a Killing vector field of the background spacetime. Contraction of \( (4) \) with $\bar{\xi}^\mu$ and integration of the result over a hypersurface $\Sigma$ of the spacetime manifold $\mathcal{M}$ gives

\[
\int d^{n-1}x \sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(1)}_{\mu\nu})(k) = - \int d^{n-1}x \sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(2)}_{\mu\nu})(h,h),
\]

where we have used the background metric and the inverse metric to lower and raise the indices respectively and $\bar{\gamma}$ denotes the metric of the hypersurface. Once the field equations of the theory are given, we can express the left-hand side of \( (5) \) as a pure divergence of an antisymmetric field $F_{\mu\nu}$

\[
\sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(1)}_{\mu\nu})(k) = \partial_\mu \left( \sqrt{\bar{\gamma}} F^{\mu\nu} \right).
\]

When the left-hand side of \( (5) \) is expressed in terms of the metric perturbation, it is known as the Abbott-Deser-Tekin (ADT) current (or charges) \([1, 2]\) and it is an extension of the Abbott-Deser-Misner (ADM) \([3]\) charges of flat spacetime. Substituting the last expression in \( (5) \) we conclude that the right-hand side, which is called the Taub charge \([4]\), must also be expressed as a pure boundary. Then one ends up with the equality of the Taub and ADT charges

\[
Q_{ADT} := \int_{\partial\Sigma} d\Sigma_\mu \sqrt{\bar{\sigma}} \hat{n}_\nu \bar{\xi}_\mu F^{\mu\nu} = - \int d^{n-1}x \sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(2)}_{\mu\nu})(h,h) =: -Q_{Taub},
\]

where $\partial\Sigma$ is the boundary of the hypersurface $\Sigma$, $\bar{\sigma}$ is the pull-back metric on it and $\hat{n}_\nu$ is the outward unit normal vector on $\partial\Sigma$. If the background spacetime has no boundary, one arrives at the integral constraint on the solutions of the linearized equations

\[
\int d^{n-1}x \sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(2)}_{\mu\nu})(h,h) = 0.
\]

When this integral constraint is satisfied, we say $\bar{g}$ is linearization stable and the perturbation $h$ can be integrable to an exact solution, but if this is not the case the background solution has linearization instability and we cannot improve it to get an exact solution, in other words $\bar{g}$ is an isolated solution. This issue was studied for Einstein’s theory in \([5–11]\), summarized in \([12, 13]\); and it was extended to the generic gravity theories recently in \([14, 15]\) and to chiral gravity in \([16]\). For the cosmological Einstein’s theory it was shown that $\sqrt{\gamma} \xi_\mu (\mathcal{E}^{(2)}_{\mu\nu})(h,h)$ cannot be expressed as a pure boundary \([17]\), it has an additional bulk part which becomes a constraint on the linear order perturbation of the metric tensor. The constraint in Einstein’s theory reads

\[
\frac{1}{\Lambda} \int \sigma d^{n-1}x \sqrt{\bar{\gamma}} \bar{\xi}_\mu (\mathcal{E}^{(1)}_{\mu\nu})(k) \gamma^\rho \xi^\sigma (\mathcal{P}^{\mu\nu} \gamma^\rho) = 0.
\]
This paper is organized as follows: in section II we consider the cosmological Einstein’s gravity and give the Abbott-Deser (AD) formula of the conserved charges \[1\] for background Einstein spacetimes and we summarize the new formulation \[18, 19\] to construct the conserved charges. Then we give the linear order perturbation of the new formula and its behavior under gauge transformations for (anti) de Sitter background spacetime. In section III, we discuss the second order perturbations of the new formula and construct the gauge transformation of the result. In section IV we discuss the results in terms of second order gauge-invariant perturbation theory of Nakamura \[20–23\], which is a useful technique to construct the relevant quantities as gauge-variant and invariant parts explicitly. Since the computations are somewhat lengthy we relegate them to the Appendices.

II. FIRST ORDER PERTURBATION IN THE COSMOLOGICAL EINSTEIN THEORY

The linear order expansion of the cosmological Einstein tensor\(^1\) about a generic background is

\[
(G_{\mu\nu})^{(1)} := (R_{\mu\nu})^{(1)} - \frac{1}{2} \bar{g}_{\mu\nu} (R) - \frac{1}{2} h_{\mu\nu} \bar{R} + \Lambda h_{\mu\nu}.
\]

This background tensor can be written as two parts \[2, 24\]

\[
(G_{\mu\nu})^{(1)} = \nabla_{\alpha} \nabla_{\beta} K^{\mu\alpha\nu\beta} + X^{\mu\nu},
\]

with

\[
X^{\mu\nu} = \frac{1}{2} \left( h^{\mu\alpha} \bar{R}_{\alpha\nu} - \bar{R}^{\mu\alpha\nu\beta} h_{\alpha\beta} \right) + \frac{1}{2} \bar{g}^{\mu\nu} h_{\rho\sigma} R_{\rho\sigma} + \Lambda h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \bar{R},
\]

and

\[
K^{\mu\alpha\nu\beta} = \frac{1}{2} \left( g^{\mu\rho} \bar{h}^{\rho\beta} + g^{\mu\beta} \bar{h}^{\rho\nu} - g^{\alpha\beta} \bar{h}^{\mu\nu} - g^{\mu\nu} \bar{h}^{\alpha\beta} \right).
\]

Here \(\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} h\). Let us assume that the background spacetime has at least one Killing vector field, say \(\bar{\xi}_\nu\). Contraction of the background Killing vector \(\bar{\xi}_\nu\) with \((G_{\mu\nu})^{(1)}\) yields

\[
\bar{\xi}_\nu \left( G^{\mu\nu} \right)^{(1)} = \nabla_{\alpha} \left( \bar{\xi}_\nu \nabla_{\beta} K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha} \nabla_{\beta} \bar{\xi}_\nu \right) + K^{\mu\nu} \bar{R}_{\beta\alpha\rho} \bar{\xi}_\rho + X^{\mu\nu} \bar{\xi}_\nu,
\]

where the last two terms vanish for a background Einstein spacetime and, therefore the current can be written as pure divergence

\[
\bar{\xi}_\nu \left( G^{\mu\nu} \right)^{(1)} = \nabla_{\mu} \nabla_{\alpha} \left( \bar{\xi}_\nu \nabla_{\beta} K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha} \nabla_{\beta} \bar{\xi}_\nu \right) := \nabla_{\mu} F^{\mu\nu}.
\]

One natural question is to ask is how this expression changes when one changes the coordinates on the background spacetime. Under a small diffeomorphism generated by a vector field \(X\), this equation does not change since \(\delta_X \left( G^{\mu\nu} \right)^{(1)} = \mathcal{L}_X G^{\mu\nu}\), which vanishes for the background Einstein spaces. Although the result is gauge-invariant, the antisymmetric tensor \(F^{\mu\nu}\) as defined \([15]\) is gauge-invariant only up to a boundary. The change of \(F^{\mu\nu}\) under gauge transformations is complicated and was given in \([19]\). On the other hand, for (anti) de Sitter background spacetime

\[1\] The details of the computations are given at Appendix A.
it is possible to express the current in a completely gauge-invariant way \cite{18, 19}, starting from the second Bianchi identity on the Riemann tensor

\[ \nabla_\nu R_{\sigma\beta\mu\rho} + \nabla_\sigma R_{\beta\nu\mu\rho} + \nabla_\beta R_{\nu\sigma\mu\rho} = 0. \] (16)

Using the contracted Bianchi identity \( \nabla_\mu G^{\mu\nu} = 0 \), the metric compatibility \( \nabla_\mu g_{\alpha\beta} = 0 \); and carrying out the \( g^{\mu\nu} \) multiplication, one can construct a divergence-free rank four tensor (let us denote it as \( \mathcal{P}^{\mu\nu}_{\beta\sigma} \)) which has additional properties. It has the same symmetries as the Riemann tensor, it vanishes for the background (anti) de Sitter space, \( \bar{\mathcal{P}}^{\mu\nu}_{\beta\sigma} = 0 \), its trace is the cosmological Einstein tensor, \( \mathcal{P}^{\mu\nu}_{\beta\sigma} := \mathcal{P}^{\mu\nu}_{\nu\sigma} = (3 - n) \bar{G}^\mu_{\beta\sigma} \). Explicitly the \( \mathcal{P} \)-tensor reads as

\[ \mathcal{P}^{\mu\nu}_{\beta\sigma} := R^{\mu\nu}_{\beta\sigma} + \delta^\nu_{\beta} G^\mu_{\sigma} - \delta^\sigma_{\beta} G^\mu_{\nu} + \delta^\mu_{\alpha} G^\alpha_{\nu\beta} - \delta^\nu_{\alpha} G^\alpha_{\mu\beta} + \left( \frac{R}{2} - \frac{\Lambda(n + 1)}{n - 1} \right) (\delta^\nu_{\beta} \delta^\mu_{\alpha} - \delta^\nu_{\alpha} \delta^\mu_{\beta}) . \] (17)

This tensor was used to give a new formulation of conserved charges in \cite{18}, also the construction is improved for the extensions of the Einstein’s gravity in \cite{19}. Let us summarize how one can construct the conserved charges by using the \( \mathcal{P} \)-tensor. Consider the exact equation

\[ \nabla_\nu (\mathcal{P}^{\mu\nu}_{\beta\sigma} \nabla^\beta \xi^\sigma) - \mathcal{P}^{\mu\nu}_{\beta\sigma} \nabla_\nu \nabla^\beta \xi^\sigma = 0, \] (18)

which is valid for all smooth metrics without the use of the field equations. Consider the background to be the \( n \)-dimensional (anti) de Sitter spacetime with the following equations

\[ \bar{R}_{\mu\nu\beta\gamma} = \frac{2}{(n - 2)(n - 1)} \Lambda (\bar{g}_{\mu\nu} \bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta} \bar{g}_{\alpha\nu}) , \quad \bar{R}_{\mu\nu} = \frac{2}{n - 2} \Lambda \bar{g}_{\mu\nu} , \quad \bar{R} = \frac{2n \Lambda}{n - 2}. \] (19)

First order expansion of (18) about the background (anti) de Sitter spacetime gives

\[ \bar{\nabla}_{\nu} \left( (\mathcal{P}^{\mu\nu}_{\beta\sigma})^{(1)} \bar{\nabla}^\beta \bar{\xi}^\sigma \right) - (\mathcal{P}^{\mu\nu}_{\beta\sigma})^{(1)} \bar{\nabla}_{\nu} \bar{\nabla}^\beta \bar{\xi}^\sigma = 0 , \] (20)

where the linear order expansion of the \( \mathcal{P} \)-tensor about the (anti) de Sitter spacetime reads

\[ (\mathcal{P}^{\mu\nu}_{\beta\sigma})^{(1)} = (R^{\mu\nu}_{\beta\sigma})^{(1)} + 2 (G^\nu_{[\beta \sigma]} \delta^\mu_{[\beta \sigma]} + 2 (G^\nu_{[\sigma \beta]} \delta^\mu_{[\sigma \beta]} + (R)^{(1)} [\beta \sigma] . \] (21)

Substituting the linearized \( \mathcal{P} \)-tensor, assuming \( \bar{\xi}^\mu \) to be Killing vector and using the identity \( \bar{\nabla}_\nu \bar{\nabla}_\beta \bar{\xi}_\sigma = \bar{R}_{\lambda\nu\beta\sigma} \bar{\xi}^\lambda \), the linearized equation (20) becomes

\[ \bar{\xi}_{\nu} (G^\nu_{[\beta \sigma]} )^{(1)} = c \bar{\nabla}_\nu \left( (\mathcal{P}^{\mu\nu}_{\beta\sigma})^{(1)} \bar{\nabla}^\beta \bar{\xi}^\sigma \right) , \] (22)

where we have defined \( c = \frac{(n - 1)(n - 2)}{4 \Lambda(n - 3)} \). Since \( (G^{\mu\nu})^{(1)} \) and \( (R)^{(1)} \) vanish on the boundary, the conserved charges of the cosmological Einstein’s theory can be written as

\[ Q = \frac{c}{2G \Omega_{n - 2}} \int d^{n - 2} \nu \sqrt{\bar{\sigma}} \bar{n}_\mu \bar{\sigma}_\nu (R^{\mu\nu}_{\beta\sigma})^{(1)} \bar{\nabla}^\beta \bar{\xi}^\sigma , \] (23)

where \( \bar{\sigma}_\nu \) is the unit outward normal vector on the boundary of the hypersurface, \( \partial \bar{\Sigma} \). For a general background spacetime, under a variation generated by the vector field \( X \) the first order linearized Riemann tensor changes as \( \delta_X (R^{\mu\nu}_{\beta\sigma})^{(1)} = \mathcal{L}_X \bar{R}^{\mu\nu}_{\beta\sigma} \), which vanishes for (anti) de Sitter background (for more details see \cite{19}). It turns out, the conserved charges are given with a gauge-invariant expression which involves the linearized Riemann tensor explicitly.
III. SECOND ORDER PERTURBATION THEORY IN THE COSMOLOGICAL EINSTEIN GRAVITY

Here we discuss the second order perturbations of the cosmological Einstein tensor following [17]. After using the linearized equation \( \nabla_\nu (P_{\nu \mu} \beta^\sigma) = 0 \), the second order perturbation of equation (18) about background (anti) de Sitter spacetime reduces to the divergence and non-divergence parts as

\[
\xi^\nu (G^\mu_{\nu \beta \sigma}) = c \left( \nabla_\nu \left( \nabla^\beta \xi^\sigma (T_{\nu \mu \beta \sigma})^{(2)} \right) - 2 (T_{\nu \rho}^{(1)}) \nabla^\rho \xi^\sigma (P_{\nu \mu \beta \sigma})^{(1)} \right),
\]

where we have defined a second order background tensor

\[
(T_{\nu \mu \beta \sigma})^{(2)} := (P_{\nu \mu \beta \sigma})^{(2)} + \frac{h}{2} (P_{\nu \mu \beta \sigma})^{(1)},
\]

and the constant \( c \) was defined below (22). Using the explicit form of the cosmological Einstein gravity field equations, it was shown that the left-hand side of (24) cannot be written as a pure divergence term [18]. It turns out, the non-divergence part can involve some divergence terms, but it cannot be completely written as a divergence term. It is obvious that, for a manifold with a compact hypersurface \( \Sigma \) without a boundary, the non-divergence part of (24) becomes an integral constraint on the solutions to the first order linearized equations. Note that if the spacetime \( \mathcal{M} \) has a compact hypersurface with a boundary, then we obtain the equality (7), which relates the solutions of the first order linearized equations to the solutions of the second order equations. If solutions to the first and the second order perturbed equations, say \( h \) and \( k \) respectively, come from linearization of an exact solution \( g \), then the integral constraint is automatically satisfied for a spacetime manifold which has a compact hypersurface without a boundary. Similarly, if the spacetime \( \mathcal{M} \) has a compact hypersurface with a boundary, the equality of the conserved charges (7) will also be satisfied. Otherwise, we say \( \bar{g} \) is linearization unstable and the perturbation theory about it does not make sense.

IV. GAUGE INVARIANT PERTURBATION THEORY

The second order gauge-invariant perturbation theory was studied in detail in [21–23] and the existence of the two perturbation parameters are included in [20]. Gauge-invariant perturbation theory is a technique that allows one to express the tensor fields in terms of gauge-variant and invariant terms. Of course, one cannot use this method on any arbitrary background spacetime since the main assumption of the theory is decomposing the first order metric perturbation as

\[
h_{\mu \nu} := \tilde{h}_{\mu \nu} + \mathcal{L}_X \bar{g}_{\mu \nu},
\]

here \( \tilde{h}_{\mu \nu} \) denotes the gauge-invariant part, and the gauge-variant term \( \mathcal{L}_X \bar{g}_{\mu \nu} \) denotes the Lie derivative of the background metric with respect to vector field \( X \) which is the generator of the gauge transformation. In the following discussion, we denote the gauge-variant quantities with a tilde and the background quantities with a bar. If such a decomposition exists, one can express the linear order perturbation of any tensor field \( T \) as

\[
(T)^{(1)} = (\bar{T})^{(1)} + \mathcal{L}_X \bar{T}.
\]

The second order perturbation of the metric tensor can be expressed as

\[
k_{\mu \nu} := \frac{1}{2} \tilde{k}_{\mu \nu} + \mathcal{L}_X h_{\mu \nu} + \frac{1}{2} \left( \mathcal{L}_Y - \mathcal{L}_X \right) \bar{g}_{\mu \nu},
\]
where $Y$, just like $X$ generates the gauge transformations. Using (26, 28) the second order perturbation of any generic tensor field $T$ can be written as

$$(T)^{(2)} = (\tilde{T})^{(2)} + \mathcal{L}_X (T)^{(1)} + \frac{1}{2} \left(\mathcal{L}_X - \mathcal{L}_Y^2\right) \mathcal{T}. \tag{29}$$

Note that since the metric tensor involves irreducible gauge-invariant terms at the first and the second orders, the gauge-invariant part of any generic tensor field has the same form. Of course, the irreducible gauge-invariant part of the tensor field only includes $\tilde{h}_{\mu\nu}$ and $\tilde{k}_{\mu\nu}$. Details of the calculations are given in Appendix C. Here we discuss the conserved charges, which are constructed by using the $\mathcal{P}$-tensor, in terms of the gauge-invariant perturbation theory. Let us start with the first order linearized equation (22), which we can use to construct the conserved charges. In terms of the gauge-invariant perturbation theory, the left-hand side of the equation (22) is gauge-invariant

$$\bar{\xi}_\nu \left( (\tilde{G}^{\mu\nu})^{(1)} + \mathcal{L}_X \tilde{G}^{\mu\nu} \right) = \bar{\xi}_\nu \left( \left( \tilde{P}^{\nu\mu}_{\beta\sigma} \right)^{(1)} - \mathcal{L}_X \tilde{P}^{\nu\mu}_{\beta\sigma} \right). \tag{30}$$

This reduces to

$$\bar{\xi}_\nu \left( (\tilde{P}^{\nu\mu}_{\beta\sigma})^{(1)} \right) = \bar{\xi}_\nu \left( \left( \tilde{P}^{\nu\mu}_{\beta\sigma} \right)^{(1)} \right). \tag{31}$$

by using the vanishing of the $\mathcal{P}$-tensor for the (anti) de Sitter background spacetime, $\tilde{P}^{\nu\mu}_{\beta\sigma} = 0$. So, as in the case of the usual perturbation theory the current is gauge-invariant. At the second order, the left-hand side of the equation (24) is gauge-invariant, since we have

$$(G^{\mu\nu})^{(2)} = \mathcal{L}_Y (G^{\mu\nu})^{(1)} + \frac{1}{2} \left(\mathcal{L}_X - \mathcal{L}_Y^2\right) \tilde{G}^{\mu\nu}, \tag{32}$$

which becomes

$$(G^{\mu\nu})^{(2)} = \left( \tilde{G}^{\mu\nu} \right)^{(2)}, \tag{33}$$

where we used $(G^{\mu\nu})^{(1)} = 0 = \tilde{G}^{\mu\nu}$ in (anti) de Sitter background spacetime. Now let us compute the right-hand side of (24). For the second order perturbation of the $\mathcal{P}$-tensor, we get

$$(\mathcal{P}^{\nu\mu}_{\beta\sigma})^{(2)} = \left( \tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma} \right)^{(2)} \mathcal{L}_X (\mathcal{P}^{\nu\mu}_{\beta\sigma})^{(1)} + \frac{1}{2} \left(\mathcal{L}_Y - \mathcal{L}_X^2\right) \tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma}, \tag{34}$$

where the last term vanishes at the (anti) de Sitter background spacetime and so we obtain

$$(\mathcal{P}^{\nu\mu}_{\beta\sigma})^{(2)} = \left( \tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma} \right)^{(2)} + \mathcal{L}_X (\tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma})^{(1)}. \tag{35}$$

Inserting the results in (24) we can write

$$\tilde{\xi}^{\nu} (\tilde{G}^{\mu\nu})^{(2)} = c \bar{\xi}^{\nu} \left( \nabla^\beta \tilde{\xi}^\sigma \left( \tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma} \right)^{(2)} + \nabla^\beta \tilde{\xi}^\sigma \mathcal{L}_X (\tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma})^{(1)} + \frac{h}{2} \nabla^\beta \tilde{\xi}^\sigma (\tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma})^{(1)} \right)$$

$$- 2c (\Gamma^{\nu\mu\rho})^{(1)} \nabla^\rho \tilde{\xi}^\sigma (\tilde{\mathcal{P}}^{\nu\mu}_{\beta\sigma})^{(1)}, \tag{36}$$
where the left-hand side and the first term on the right-hand side are already in a gauge-invariant form. Then, let us concentrate on the gauge-variant terms. The second term reads

$$\nabla_\nu \left( \bar{\nabla}^\beta \bar{\xi}_\sigma \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} \right) = \left( \nabla_\nu \nabla^\beta \bar{\xi}_\sigma \right) \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} + \bar{\nabla}^\beta \bar{\xi}_\sigma \nabla_\nu \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)},$$

(38)

where the first term vanishes after using the identity \(\nabla_\nu \nabla^\beta \bar{\xi}_\sigma = \bar{R}_{\lambda\nu}^{\beta\sigma} \bar{\xi}_\lambda\), and then we obtain

$$\nabla_\nu \left( \bar{\nabla}^\beta \bar{\xi}_\sigma \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} \right) = \bar{\nabla}^\beta \bar{\xi}_\sigma \nabla_\nu \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)}.$$  

(39)

Using the identity (74) in Appendix B, we get

$$\bar{\nabla}^\beta \bar{\xi}_\sigma \nabla_\nu \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} = \bar{\nabla}^\beta \bar{\xi}_\sigma \left( \mathcal{L}_X \nabla_\nu (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} - \delta_X (\Gamma^\nu_{\nu\lambda})^{(1)} (\bar{\mathcal{P}}^\lambda_{\nu\beta})^{(1)} + 2\delta_X (\Gamma^\lambda_{\nu\beta})^{(1)} (\bar{\mathcal{P}}^\nu_{\nu\lambda})^{(1)} \right).$$

(40)

So one has

$$\bar{\nabla}^\beta \bar{\xi}_\sigma \nabla_\nu \mathcal{L}_X(\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} = \bar{\nabla}^\beta \bar{\xi}_\sigma \left( \delta_X (\Gamma^\nu_{\nu\lambda})^{(1)} (\bar{\mathcal{P}}^\lambda_{\nu\beta})^{(1)} + 2\delta_X (\Gamma^\lambda_{\nu\beta})^{(1)} (\bar{\mathcal{P}}^\nu_{\nu\lambda})^{(1)} \right),$$

(41)

where we have used the first order linearization of \(\nabla_\nu \bar{\mathcal{P}}^\nu_{\beta\sigma} = 0\) about the (anti) de Sitter background metric. Substituting the results in (37) and using the decomposition of the linear order perturbation of the metric tensor (26), we arrive at

$$\bar{\xi}_\nu (\bar{G}_\nu^\mu)^{(2)} = c \nabla_\nu \left( \nabla^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(2)} + \frac{z}{2} \bar{\nabla}^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} + \nabla_\rho X^\rho \nabla^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} \right)$$

$$- c \bar{\nabla}^\beta \bar{\xi}_\sigma \delta_X (\Gamma^\nu_{\nu\lambda})^{(1)} (\bar{\mathcal{P}}^\lambda_{\nu\beta})^{(1)} + 2 c (\bar{\mathcal{P}}^\nu_{\nu\lambda})^{(1)} \bar{\nabla}^\beta \bar{\xi}_\sigma \left( \delta_X (\Gamma^\lambda_{\nu\beta})^{(1)} - (\Gamma^\lambda_{\nu\beta})^{(1)} \right),$$

(42)

where the last two terms together form a gauge-invariant combination from the decomposition of the Christoffel connection

$$(\Gamma^\lambda_{\nu\beta})^{(1)} - \delta_X (\Gamma^\lambda_{\nu\beta})^{(1)} = (\bar{\Gamma}^\lambda_{\nu\beta})^{(1)}.$$  

(43)

Also, after a straightforward calculation one has

$$\nabla_\nu \left( \nabla_\rho X^\rho \bar{\nabla}^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} \right) - \bar{\nabla}^\beta \bar{\xi}_\sigma \delta_X (\Gamma^\nu_{\nu\lambda})^{(1)} (\bar{\mathcal{P}}^\lambda_{\nu\beta})^{(1)} = 0,$$

(44)

which proves the vanishing of the gauge-variant terms. Collecting the pieces together, one ends up with

$$\bar{\xi}_\nu (\bar{G}_\nu^\mu)^{(2)} = c \nabla_\nu \left( \bar{\nabla}^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(2)} + \frac{\bar{h}}{2} \bar{\nabla}^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\beta\sigma})^{(1)} \right) - 2 c \bar{\nabla}^\beta \bar{\xi}_\sigma (\bar{\mathcal{P}}^\nu_{\nu\lambda})^{(1)} (\bar{\Gamma}^\lambda_{\nu\beta})^{(1)},$$

(45)

where the result involves divergence and non-divergence terms; \(\bar{h}\) refers to the gauge-variant trace of the metric perturbation. Unlike the case of usual perturbation theory, the second order cosmological Einstein tensor is gauge-invariant in this formulation, so are the conserved charges. For the compact hypersurfaces without a boundary, vanishing of the last term becomes an integral constraint on solutions of the first order linearized equations.
V. CONCLUSIONS

The general covariance principle introduces a large gauge degree of freedom since there is no preferred coordinate system in General Relativity. In perturbation theory, computing gauge-invariant results plays an important role since the gauge-variant results can include some unphysical parts which depend on our choice of the coordinate system. On the other hand, the second order gauge-invariant perturbation theory allows a consistent formulation to compute the gauge-invariant parts of the relevant expressions. In this technique one can construct the relevant quantities as gauge-variant and invariant parts. So there is no further need to discuss for the gauge invariance, since the quantities involve all information that we need.

In cosmological Einstein’s theory, construction of the gauge-invariant conserved charges is generally done by using the explicit form of the field equations. The current does not have to be a gauge-invariant quantity. Of course finding a gauge-invariant current is more valuable since one only has the physical terms in this case. At the first order, starting with the second Bianchi identity, one can compute a gauge-invariant current that involves the Riemann tensor explicitly. At the second order neither the cosmological Einstein tensor nor the conserved charges are gauge-invariant. They are only gauge-invariant up to a boundary term.

In gauge-invariant perturbation theory, at the first order one has gauge-invariant current and conserved charges as expected. At the second order, one has a gauge-invariant cosmological Einstein tensor which is different from the usual perturbation theory case. So, the conserved charges and the current are all gauge-invariant in this theory.

APPENDIX A: SECOND ORDER PERTURBATION THEORY

Here we give the explicit expressions of the perturbation theory about the background spacetime $\bar{g}$, up to and including the second order terms by considering the following metric tensor decomposition

$$g_{ab} := \bar{g}_{ab} + \lambda h_{ab} + \lambda^2 k_{ab}, \quad (46)$$

where $\lambda$ is a small parameter, $h_{ab}$ and $k_{ab}$ are the linear and the second order metric tensor perturbations respectively. Using $g_{ab} g^{bc} = \delta^b_a$, we can compute the expansion of the inverse metric as

$$\bar{g}^{ab} = g^{ab} - \lambda h^{ab} + \lambda^2 \left( h^{a}_c h^{cb} - k^{ab} \right) \quad (47)$$

Let $T$ be a generic tensor, it can be perturbed about the background spacetime $\bar{g}$ as follows

$$T = \bar{T} + \lambda (T)(1) + \lambda^2 (T)(2). \quad (48)$$

The Christoffel symbol $\Gamma^c_{ab}$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left( \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right), \quad (49)$$

is not a tensor quantity but it can be decomposed in the same way
\[ \Gamma^c_{ab} = \bar{\Gamma}^c_{ab} + \lambda (\Gamma^c_{ab})^{(1)} + \lambda^2 (\Gamma^c_{ab})^{(2)}. \] (50)

Inserting the given expressions for the metric and its inverse, we obtain the linear order perturbation of the Christoffel symbol as

\[ (\Gamma^c_{ab})^{(1)} = \frac{1}{2} \left( \nabla_a h^c_b + \nabla_b h^c_a - \nabla^c h_{ab} \right), \] (51)

and the second order perturbation as

\[ (\Gamma^c_{ab})^{(2)} = K^c_{ab} - h^c_d (\Gamma^d_{ab})^{(1)}, \] (52)

where we have defined

\[ K^c_{ab} = \frac{1}{2} \left( \nabla_a k^c_b + \nabla_b k^c_a - \nabla^c k_{ab} \right). \] (53)

We can write the linear order perturbation of the Riemann tensor as

\[ (R^a_{bcd})^{(1)} = \nabla_c (\Gamma^a_{bd})^{(1)} - \nabla_d (\Gamma^a_{bc})^{(1)}, \] (54)

and the second order Riemann tensor as

\[ (R^a_{bcd})^{(2)} = 2 \nabla_c (\Gamma^a_{bd})^{(2)} - \nabla_d (\Gamma^a_{bc})^{(2)} + (\Gamma^e_{bd})^{(1)} (\Gamma^a_{ce})^{(1)} - (\Gamma^e_{cb})^{(1)} (\Gamma^a_{de})^{(1)}, \] (55)

which reduces to

\[ (R^a_{bcd})^{(2)} = 2 \nabla_c K^a_{db} - \nabla_c \left( h^a_c (\Gamma^e_{bd})^{(1)} \right) + \nabla_d \left( h^a_c (\Gamma^e_{bc})^{(1)} \right) + (\Gamma^e_{bd})^{(1)} (\Gamma^a_{ce})^{(1)} - (\Gamma^e_{cb})^{(1)} (\Gamma^a_{de})^{(1)}, \] (56)

after using the second order Christoffel connection given in (52). The first and the second order Ricci tensors are obtained from the contraction, \( R_{ab} := R^c_{acb} \), and we get the linear order perturbation of the Ricci tensor

\[ (R_{ab})^{(1)} = \nabla_c (\Gamma^c_{ab})^{(1)} - \nabla_a (\Gamma^c_{cb})^{(1)}, \] (57)

and the second order Ricci tensor

\[ (R_{ab})^{(2)} = 2 \nabla_c K^c_{ab} - \nabla_c \left( h^c_c (\Gamma^e_{ab})^{(1)} \right) + \nabla_a \left( h^c_c (\Gamma^e_{cb})^{(1)} \right) + (\Gamma^e_{ab})^{(1)} (\Gamma^c_{ce})^{(1)} - (\Gamma^e_{ac})^{(1)} (\Gamma^c_{be})^{(1)}. \] (58)

The first order linearization of the scalar curvature becomes

\[ (R)^{(1)} = \bar{g}^{ab} (R_{ab})^{(1)} - \bar{R}_{ab} h^{ab}, \] (59)

and the second order Ricci scalar is
\[(R)^{(2)} = \bar{R}_{ab} \left( \bar{h}^{cd} \bar{h}_{cd} - k^{ab} \right) - \left( R_{ab} \right)^{(1)} h^{ab} + \bar{g}_{ab} \left( R_{ab} \right)^{(2)}. \]  

The cosmological Einstein tensor

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab}, \]

at first order yields

\[ (G_{ab})^{(1)} = \left( R_{ab} \right)^{(1)} - \frac{1}{2} \bar{g}_{ab} \left( R \right)^{(1)} - \frac{1}{2} \bar{R} h_{ab} + \Lambda h_{ab}, \]

and at the second order becomes

\[ (G_{ab})^{(2)} = \left( R_{ab} \right)^{(2)} - \frac{1}{2} \left( \bar{g}_{ab} \left( R \right)^{(2)} + h_{ab} \left( R \right)^{(1)} + k_{ab} \bar{R} + 2\Lambda k_{ab} \right). \]

**APPENDIX B: IDENTITIES ON LIE AND COVARIANT DERIVATIVES**

Lie derivative plays an important role in the second order gauge-invariant perturbation theory and also in the usual gauge transformations generated by a vector field. Here we derive some useful identities which heavily used in the computations. Since Lie and covariant derivatives do not commute, we need to introduce the expressions in a compact way, that appears when we change the order of these differentiations. In order to obtain the desired expressions, let us start with Lie derivative of a rank two tensor \( T \)

\[ L_X T_{ab} = X^c \bar{\nabla} c T_{ab} = X^f \bar{\nabla} f T_{ab} + T_{fa} \bar{\nabla} a X^f + T_{fb} \bar{\nabla} b X^f. \]  

Covariant derivative of this expression yields

\[ \bar{\nabla} c L_X T_{ab} = \bar{\nabla} c X^f \bar{\nabla} f T_{ab} + X^f \bar{\nabla} c \bar{\nabla} f T_{ab} + \bar{\nabla} a X^f \bar{\nabla} a T_{ab} + \bar{\nabla} b X^f \bar{\nabla} b T_{ab}, \]

When we change the order of the derivatives we get

\[ \mathcal{L}_X \bar{\nabla} c T_{ab} = X^f \bar{\nabla} f \bar{\nabla} c T_{ab} + \left( \bar{\nabla} c X^f \bar{\nabla} f + \bar{\nabla} a X^f \bar{\nabla} a + \bar{\nabla} b X^f \bar{\nabla} b \right) T_{af}, \]

and subtraction of the results yields

\[ \bar{\nabla} c \mathcal{L}_X T_{ab} - \mathcal{L}_X \bar{\nabla} c T_{ab} = X^f \left[ \bar{\nabla} c, \bar{\nabla} f \right] T_{ab} + \left( \bar{\nabla} c \bar{\nabla} a X^f \right) T_{fb} + \left( \bar{\nabla} c \bar{\nabla} b X^f \right) T_{af}. \]

Using

\[ \left[ \bar{\nabla} c, \bar{\nabla} f \right] T_{ab} = \bar{R}_{cfa} e^c T_{eb} + \bar{R}_{cfa} e^c T_{ae}, \]

one can rewrite as
\[ \nabla_c \mathcal{L}_X T_{ab} = \mathcal{L}_X \nabla_c T_{ab} + \left( \nabla_c \nabla_a X^e + \mathcal{T}_{cfa}^e X^f \right) T_{eb} + \left( \mathcal{T}_{cfb}^e X^f + \nabla_c \nabla_b X^e \right) T_{ae}. \]  

We can relate the last expression with the gauge transformation of the linearized Christoffel connection as follows. Recall that under the gauge transformations generated by the vector field \( X \), the linear order metric perturbation transforms as \( \delta_X h_{ab} = \nabla_a X_b + \nabla_b X_a = \mathcal{L}_X \bar{g}_{ab} \), then the gauge transformation of the linearized Christoffel symbol becomes

\[ \delta_X (\Gamma^{(1)}_{ab}) = \frac{1}{2} \left( \nabla_a \delta_X h_b^e + \nabla_b \delta_X h_a^e - \nabla^c \delta_X h_{ab} \right), \]

which can be rewritten as

\[ \delta_X (\Gamma^{(1)}_{ab}) = \nabla_a \nabla_b X^c + \bar{R}^{e}_{bda} X^d. \]

Using the last expression, (69) can be expressed as

\[ \nabla_c \mathcal{L}_X T_{ab} = \mathcal{L}_X \nabla_c T_{ab} + \delta_X (\Gamma^{(1)}_{ca}) T_{eb} + \delta_X (\Gamma^{(1)}_{cb}) T_{ae}. \]

Similar computation for a \((1,1)\) tensor ends up with

\[ \nabla_c \mathcal{L}_X T^{a}{}_{b} = \mathcal{L}_X \nabla_c T^{a}{}_{b} + T^e{}_{a} \delta_X (\Gamma^{(1)}_{ea}) - T^e{}_{b} \delta_X (\Gamma^{(1)}_{ce}) \]

We can extend the computation for a general \((m,n)\) tensor as

\[ \nabla_c \mathcal{L}_X T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_n} = \mathcal{L}_X \nabla_c T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_n} \]

\[ + \delta_X (\Gamma^{d}_{cb_1}) T^{a_1 a_2 \ldots a_m}{}_{db_2 \ldots b_n} + \delta_X (\Gamma^{d}_{cb_2}) T^{a_1 a_2 \ldots a_m}{}_{b_1 db_3 \ldots b_n} + \ldots + \delta_X (\Gamma^{d}_{cb_n}) T^{a_1 a_2 \ldots a_m}{}_{b_1 b_2 \ldots b_{n-1} d} \]

\[ - \delta_X (\Gamma^{a_1}_{cd}) T^{da_2 \ldots a_m}{}_{b_1 b_2 b_3 \ldots b_n} - \delta_X (\Gamma^{a_2}_{cd}) T^{a_1 d \ldots a_m}{}_{b_1 b_2 \ldots b_n} - \ldots - \delta_X (\Gamma^{a_m}_{cd}) T^{a_1 a_2 \ldots d}{}_{b_1 b_2 \ldots b_n}, \]

which simplifies the computations.

**APPENDIX C: SECOND ORDER GAUGE INVARIANT PERTURBATION THEORY**

Here we summarize the results of the second order gauge-invariant perturbation theory following 22. The gauge transformation of a physical quantity \( T \) reads

\[ T(p) = \bar{T}(\bar{p}) + \delta T(p) \]

where \( T(p) \) denotes the physical quantity on spacetime \( \mathcal{M} \) at point \( p \), \( \bar{T}(\bar{p}) \) denotes the same quantity on the background spacetime \( \mathcal{M}_0 \) at point \( \bar{p} \) and \( \delta T(p) \) denotes the deviation of \( T(p) \) from its background value \( \bar{T}(\bar{p}) \). We show the metric on \( \mathcal{M} \) with \( g \) and the metric on the background spacetime \( \mathcal{M}_0 \) with \( \bar{g} \). Let \( X \) and \( Y \) denote two different gauge choices and let \( \xi_1 \) and \( \xi_2 \) denote the generators of the gauge transformations. One can compute the following difference

\[ (T)^{(1)}_Y - (T)^{(1)}_X = \mathcal{L}_{\xi_1} \bar{T}, \]
where \( (T)_Y^{(1)} \) is the linear order perturbation of the physical quantity \( T(p) \) in the gauge \( Y \) and \( (T)_X^{(1)} \) denotes the same quantity in the gauge \( X \). For the second order perturbation of the physical quantity \( T(p) \) we have a similar expression

\[
(T)_Y^{(2)} - (T)_X^{(2)} = \mathcal{L}_{\xi_1} (T)_X^{(1)} + (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2) T,
\]

which shows the difference of the perturbations under the change of the coordinate system. The generators \( \xi_1 \) and \( \xi_2 \) can be expressed as follows

\[
\xi_1 := Y - X
\]

and

\[
\xi_2 := [Y, X],
\]

note that \( \xi_1 \) and \( \xi_2 \) may be different. Following Nakamura [22], we assume that the linear order metric perturbation can be decomposed to gauge-variant and invariant parts as

\[
h_{ab} := \tilde{h}_{ab} + \nabla_a X_b + \nabla_b X_a = \tilde{h}_{ab} + \mathcal{L}_X g_{ab},
\]

where \( \tilde{h}_{ab} \) is gauge-invariant term and the \( \mathcal{L}_X g_{ab} \) denotes the gauge-variant part. From the gauge transformation (76), we can write

\[
\delta_Y \tilde{h}_{ab} - \delta_X \tilde{h}_{ab} = 0,
\]

which shows the gauge invariance of the \( \tilde{h}_{ab} \). Note that this assumption depends on the properties of the background spacetime. If we accept this decomposition, the second order metric perturbation can be expressed as

\[
2k_{ab} := \tilde{k}_{ab} + 2\mathcal{L}_X h_{ab} + \mathcal{L}_Y - \mathcal{L}_X^2 \bar{g}_{ab},
\]

where \( \tilde{k}_{ab} \) is the gauge-invariant part and the additional terms are all gauge-variant. Using the given decompositions of the first and the second order metric perturbations, the linear order perturbation of a generic tensor field reads

\[
(T)_Y^{(1)} = (\tilde{T})_Y^{(1)} + \mathcal{L}_X T,
\]

which means gauge-variant part of the tensor field is equivalent to the Lie derivative of this tensor field evaluated at the background spacetime. For the second order perturbations, we obtain a similar expression as

\[
(T)_Y^{(2)} = (\tilde{T})_Y^{(2)} + \mathcal{L}_X (T)_Y^{(1)} + \frac{1}{2} \left( \mathcal{L}_X - \mathcal{L}_Y^2 \right) T.
\]

Here \( (\tilde{T})^{(2)} \) is the gauge-variant part of the second order tensor \( (T)^{(2)} \) and the remaining terms are gauge-variant. Using (80), the linear order perturbation of the Christoffel symbol (51), can be
written as

\[ (\Gamma^c_{ab})^{(1)} = \frac{1}{2} \left( \nabla_a (\tilde{h}_b^c + \nabla_b X^c) + \nabla_b (\tilde{h}_a^c + \nabla_a X^c + \nabla^c X_a) - \nabla^c (\tilde{h}_{ab} + \nabla_a X_b + \nabla_b X_a) \right). \]  

(85)

For simplicity, let us define a new gauge-invariant background tensor

\[ (\tilde{\Gamma}^c_{ab})^{(1)} = \frac{1}{2} \left( \nabla_a \tilde{h}_b^c + \nabla_b \tilde{h}_a^c - \nabla^c \tilde{h}_{ab} \right). \]  

(86)

Then we have

\[ (\Gamma^c_{ab})^{(1)} = (\tilde{\Gamma}^c_{ab})^{(1)} + \frac{1}{2} \left( 2\nabla_a \nabla_b X^c + [\nabla_a, \nabla^c] X_b + [\nabla_b, \nabla^c] X_a \right), \]  

(87)

which reduces to

\[ (\Gamma^c_{ab})^{(1)} = (\tilde{\Gamma}^c_{ab})^{(1)} + \nabla_a \nabla_b X^c + \tilde{R}^c_{bda} X^d, \]  

(88)

where we used the identity \( [\nabla_a, \nabla_b] X^c = \tilde{R}_{ab} \gamma^c X_d \), and the first Bianchi identity \( \tilde{R}_{abcd} + \tilde{R}_{bcda} + \tilde{R}_{cadb} = 0 \). Furthermore, from \( (91) \) we get

\[ (\Gamma_{ab})^{(1)c} = (\tilde{\Gamma}_{ab})^{(1)c} + \delta_X (\Gamma_{ab})^{(1)c}, \]  

(89)

which relates the linearized Christoffel connection with the usual gauge transformation of the linearized Christoffel symbol generated by the vector field \( X \). Similarly the first order expansion of the Riemann tensor \([91]\) can be expressed as

\[ (R^a_{\ bcd})^{(1)} = 2 \nabla_{\ [c} (\tilde{R}_{\ d]b})^{(1)} + [\nabla_{c}, \nabla_{d}] \nabla_b X^a + \tilde{R}^a_{\ bed} \nabla_c X^e - \tilde{R}^e_{\ bec} \nabla_d X^e + X^e (\nabla_c \tilde{R}^a_{\ bed} - \nabla_d \tilde{R}^a_{\ bed}) \]  

(90)

and reduces to

\[ (R^a_{\ bcd})^{(1)} = 2 \nabla_{\ [c} (\tilde{R}_{\ d]b})^{(1)} + X^e \nabla_c \tilde{R}^a_{\ bed} + \tilde{R}^e_{\ bed} \nabla_c X^e - \tilde{R}^e_{\ bec} \nabla_d X^e + \tilde{R}^e_{\ ced} \nabla_b X^e - \tilde{R}^e_{\ bde} \tilde{R}^e_{\ cda}, \]  

(91)

after using the second Bianchi identity \( \nabla_a \tilde{R}_{bde} + \nabla_b \tilde{R}_{cde} + \nabla_c \tilde{R}_{abe} = 0 \). Note that the gauge-invariant part is obviously given as the Lie derivative of the Riemann tensor evaluated at the background spacetime. Then the final expression becomes

\[ (R^a_{\ bcd})^{(1)} = 2 \nabla_{\ [c} (\tilde{R}_{\ d]b})^{(1)} + \mathcal{L}_X \tilde{R}^a_{\ bcd}, \]  

(92)

which is consistent with the aim of the gauge-invariant perturbation theory. The first order linearized Ricci tensor can be found from the contraction of the first and the third indices, \( (R_{ab})^{(1)} := (R^c_{\ abc})^{(1)} \), so we have

\[ (R_{ab})^{(1)} = 2 \nabla_{\ [c} (\tilde{R}_{\ d]b})^{(1)} + \mathcal{L}_X \tilde{R}_{ab}. \]  

(93)
Since the first order linearized Christoffel connection is a background tensor, we can lower and raise the indices with the background metric and the inverse metric respectively. For an example we use \( (\Gamma_{a}^{cd})^{(1)} := g_{bd}(\Gamma_{a}^{bd})^{(1)} \), where the up index is lowered as the last down index. The first order linearized scalar curvature, by using (59) and the previous results, becomes

\[
(R)^{(1)} = 2\tilde{\nabla}_{[b}(\tilde{\Gamma}_{a}]^{ab})^{(1)} + \tilde{g}^{ab}\mathcal{L}_{X}\tilde{R}_{ab} - \tilde{R}_{ab}(\tilde{h}^{ab} - \mathcal{L}_{X}\tilde{g}^{ab}).
\]  

(94)

Equivalently, it can be written as

\[
(R)^{(1)} = 2\tilde{\nabla}_{[b}(\tilde{\Gamma}_{a}]^{ab})^{(1)} - \tilde{R}_{ab}\tilde{h}^{ab} + \mathcal{L}_{X}(\tilde{R}).
\]

(95)

Inserting the corresponding expressions in the first order linearized cosmological Einstein tensor \((\mathcal{G}_{ab})^{(1)}\), we get

\[
(\mathcal{G}_{ab})^{(1)} = 2\tilde{\nabla}_{[c}(\tilde{\Gamma}_{a]c)}^{ab})^{(1)} + \tilde{g}_{ab}\tilde{\nabla}_{[c}(\tilde{\Gamma}_{d]}^{cd})^{(1)} + \frac{1}{2}\tilde{g}_{ab}\tilde{R}_{cd}\tilde{h}^{cd} + \tilde{h}_{ab}\left(\Lambda - \frac{1}{2}\tilde{R}\right) + \mathcal{L}_{X}\tilde{G}_{ab},
\]

(96)

where only the last term is gauge-variant and it vanishes if \( \tilde{g} \) is a background solution, if this is the case \((\mathcal{G}_{ab})^{(1)}\) becomes gauge-invariant.

Now, we compute the decompositions of the second order tensors in terms of gauge-variant and invariant parts. We can compute (53) by using (52) as

\[
K^{c}_{ab} = \frac{1}{4}(\tilde{\nabla}_{a}\tilde{k}^{c}_{b} + \tilde{\nabla}_{b}\tilde{k}^{c}_{a} - \tilde{\nabla}^{c}\tilde{k}_{ab}) + \frac{1}{4}\tilde{g}^{cd}\left(\tilde{\nabla}_{a}\mathcal{L}_{X}\left(h_{bd} + \tilde{h}_{bd}\right) + \tilde{\nabla}_{b}\mathcal{L}_{X}\left(h_{ad} + \tilde{h}_{ad}\right) - \tilde{\nabla}_{d}\mathcal{L}_{X}\left(h_{ab} + \tilde{h}_{ab}\right)\right)
\]

(97)

After defining a new gauge-invariant second order background tensor

\[
\tilde{K}^{c}_{ab} = \frac{1}{2}\left(\tilde{\nabla}_{a}\tilde{k}^{c}_{b} + \tilde{\nabla}_{b}\tilde{k}^{c}_{a} - \tilde{\nabla}^{c}\tilde{k}_{ab}\right),
\]

(98)

we obtain

\[
2K^{c}_{ab} = \tilde{K}^{c}_{ab} + \frac{1}{2}\tilde{g}^{cd}\mathcal{L}_{X}\left(\tilde{\nabla}_{a}\left(h_{bd} + \tilde{h}_{bd}\right) + \tilde{\nabla}_{b}\left(h_{ad} + \tilde{h}_{ad}\right) - \tilde{\nabla}_{d}\left(h_{ab} + \tilde{h}_{ab}\right)\right)
\]

(99)

Note that we have used the identity (72) given in Appendix B to get the last expression. After a straightforward calculation the result reduces to

\[
2K^{c}_{ab} = \tilde{K}^{c}_{ab} + \mathcal{L}_{X}\left((\Gamma^{c}_{ab})^{(1)} + (\tilde{\Gamma}^{c}_{ab})^{(1)}\right) - \mathcal{L}_{X}\tilde{g}^{cd}\left((\Gamma^{c}_{ad})^{(1)} + (\tilde{\Gamma}^{c}_{ad})^{(1)}\right)
\]

\[
+ \left(h^{c}_{e} + \tilde{h}^{c}_{e}\right)\delta_{X}(\Gamma^{e}_{ab})^{(1)} + \delta_{Y}(\Gamma^{e}_{ab})^{(1)}.
\]

(100)

We can construct the following tensor
the second order perturbation of the Riemann tensor \(56\). Using \(74\), it can be written as

\[
4\nabla_c K^a_{db} = 2\nabla_c \bar{K}^a_{db} + \nabla_c \left( \mathcal{L} X \left( (\Gamma^a_{bd})^{(1)} + (\tilde{\Gamma}^a_{bd})^{(1)} \right) \right) - \nabla_c \left( \mathcal{L} X g^a \left( (\Gamma_{bcd})^{(1)} + (\tilde{\Gamma}_{bcd})^{(1)} \right) \right)
\]

\[
+ \nabla_c \left( \left( h^a_c + \tilde{h}^a_c \right) \delta_X \left( (\Gamma^e_{bd})^{(1)} \right) + \nabla_c \delta_Y \left( (\Gamma^e_{bd})^{(1)} \right) \right)
\]

\[
- \nabla_d \left( \mathcal{L} X \left( (\Gamma^a_{bc})^{(1)} + (\tilde{\Gamma}^a_{bc})^{(1)} \right) \right) + \nabla_d \left( \mathcal{L} X g^a \left( (\Gamma_{bce})^{(1)} + (\tilde{\Gamma}_{bce})^{(1)} \right) \right)
\]

\[
- \nabla_d \left( \left( h^a_d + \tilde{h}^a_d \right) \delta_X \left( (\Gamma^e_{bc})^{(1)} \right) - \nabla_d \delta_Y \left( (\Gamma^e_{bc})^{(1)} \right) \right),
\]  

(101)

Since the last equation is complicated we use the results given below to get a compact form. We have

\[
\nabla_c \delta_Y \left( (\Gamma^a_{bd})^{(1)} \right) - \nabla_d \delta_Y \left( (\Gamma^a_{bc})^{(1)} \right) = \mathcal{L} Y \bar{R}^a_{bcd}
\]

(103)

and from \(92\)

\[
\nabla_c \delta_X \left( (\Gamma^e_{bd})^{(1)} \right) - \nabla_d \delta_X \left( (\Gamma^e_{bc})^{(1)} \right) = \mathcal{L} X \bar{R}^e_{bcd} = (R^e_{bcd})^{(1)} - 2\nabla_c \left( \tilde{\Gamma}^a_{db} \right)^{(1)} ,
\]

(104)

and

\[
\mathcal{L} X \left( (\Gamma^a_{bd})^{(1)} + (\tilde{\Gamma}^a_{bd})^{(1)} \right) - \nabla_d \left( (\Gamma^a_{bc})^{(1)} + (\tilde{\Gamma}^a_{bc})^{(1)} \right) = \mathcal{L} X \left( 2(R^a_{bcd})^{(1)} - \mathcal{L} X \bar{R}^a_{bcd} \right),
\]

(105)

and also

\[
\mathcal{L} X g^a \left( (\Gamma^e_{bde})^{(1)} + (\tilde{\Gamma}_{bde})^{(1)} \right) - \nabla_d ((\Gamma^e_{bce})^{(1)} + (\tilde{\Gamma}_{bce})^{(1)} \right) = - \left( \nabla^a X_e + \nabla_c X^a \right) \left( 2(R^e_{bcd})^{(1)} - \mathcal{L} X \bar{R}^e_{bcd} \right),
\]

(106)

and

\[
(\Gamma^a_{ed})^{(1)} + (\tilde{\Gamma}^a_{ed})^{(1)} - \nabla_d (h^a_e + \tilde{h}^a_e) = - \left( (\Gamma^a_{d e})^{(1)} + (\tilde{\Gamma}^a_{d e})^{(1)} \right).
\]

(107)
Similarly we have

\[(\Gamma^a_{ce})^{(1)} + (\tilde{\Gamma}^a_{ce})^{(1)} - \nabla_c (\Delta^a_e + \tilde{\Delta}^a_e) = - \left( (\Gamma^a_{ce})^{(1)} + (\tilde{\Gamma}^a_{ce})^{(1)} \right) \]  

(108)

and

\[\delta_X (\Gamma^a_{ce})^{(1)} - \nabla_c (\Delta^a_e + \tilde{\Delta}^a_e) = - \delta_X (\Gamma^a_{de})^{(1)} \]  

(109)

and also

\[\delta_X (\Gamma^a_{de})^{(1)} - \nabla_d (\Delta^a_e + \tilde{\Delta}^a_e) = - \delta_X (\Gamma^a_{de})^{(1)} . \]  

(110)

Inserting the above results we obtain

\[ \begin{align*}
4\nabla_{c}K^a_{db} & = 2\nabla_{c}\tilde{K}^a_{db} + 2\mathcal{L} X (R^{a}{}_{bcd})^{(1)} - \mathcal{L}^2 \tilde{R}^{a}{}_{bcd} \\
& + (h^a_{\alpha} + \tilde{h}^a_{\alpha}) \mathcal{L} X \tilde{R}^a_{bcd} + \mathcal{L} Y \tilde{R}^a_{bcd} + (\nabla^a X_e + \tilde{\nabla}_e X^a) \left( 2(R^e_{bcd})^{(1)} - \mathcal{L} X \tilde{R}^e_{bcd} \right) \\
& - \left( (\Gamma^a_{de})^{(1)} + (\tilde{\Gamma}^a_{de})^{(1)} \right) \delta_X (\Gamma^b_{eb})^{(1)} + \left( (\Gamma^e_{de})^{(1)} + (\tilde{\Gamma}^e_{de})^{(1)} \right) \delta_X (\Gamma^b_{eb})^{(1)} \\
& + \left( (\Gamma^a_{de})^{(1)} + (\tilde{\Gamma}^a_{de})^{(1)} \right) \delta_X (\Gamma^b_{eb})^{(1)} - \left( (\Gamma^a_{de})^{(1)} + (\tilde{\Gamma}^a_{de})^{(1)} \right) \delta_X (\Gamma^b_{eb})^{(1)},
\end{align*} \]  

(111)

which can be rewritten as

\[ \begin{align*}
4\nabla_{c}K^a_{db} & = 2\nabla_{c}\tilde{K}^a_{db} - 4\tilde{h}^a_{\alpha} \nabla_{c}(\tilde{\Gamma}^a_{db})^{(1)} + 2(\tilde{\Gamma}^a_{de})^{(1)}(\tilde{\Gamma}^e_{db})^{(1)} - 2(\tilde{\Gamma}^a_{de})^{(1)}(\tilde{\Gamma}^e_{db})^{(1)} \\
& + 2\mathcal{L} X (R^{a}{}_{bcd})^{(1)} + \left( \mathcal{L} Y - \mathcal{L}^2 \tilde{X} \right) \tilde{R}^{a}{}_{bcd} + 2\tilde{h}^a_{\alpha} (R^{e}{}_{bcd})^{(1)} \\
& + 2(\tilde{\Gamma}^a_{de})^{(1)}(\Gamma^b_{eb})^{(1)} - 2(\Gamma^a_{de})^{(1)}(\Gamma^b_{eb})^{(1)}.
\end{align*} \]  

(112)

Using the last expression we can construct the second order perturbation of the Riemann tensor \([56]\) in terms of gauge-invariant and variant quantities

\[ \begin{align*}
(R^{a}{}_{bcd})^{(2)} & = \nabla_{c}\tilde{K}^a_{db} - 2\tilde{h}^a_{\alpha} \nabla_{c}(\tilde{\Gamma}^a_{db})^{(1)} + (\tilde{\Gamma}^a_{de})^{(1)}(\tilde{\Gamma}^e_{db})^{(1)} - (\tilde{\Gamma}^a_{de})^{(1)}(\tilde{\Gamma}^e_{db})^{(1)} \\
& + \mathcal{L} X (R^{a}{}_{bcd})^{(1)} + \left( \mathcal{L} Y - \mathcal{L}^2 \tilde{X} \right) \tilde{R}^{a}{}_{bcd},
\end{align*} \]  

(113)

where the second line shows the gauge-variant terms and this result is consistent with the aim of the gauge-invariant perturbation theory. Contraction of the indices yields the decomposition of the second order Ricci tensor

\[ \begin{align*}
(R_{ab})^{(2)} & = \nabla_{c}\tilde{K}^c_{ab} - 2\tilde{h}^c_{\alpha} \nabla_{c}(\tilde{\Gamma}^c_{ab})^{(1)} + (\tilde{\Gamma}^c_{de})^{(1)}(\tilde{\Gamma}^e_{ab})^{(1)} - (\tilde{\Gamma}^c_{de})^{(1)}(\tilde{\Gamma}^e_{ab})^{(1)} \\
& + \mathcal{L} X (R_{ab})^{(1)} + \left( \mathcal{L} Y - \mathcal{L}^2 \tilde{X} \right) \tilde{R}_{ab},
\end{align*} \]  

(114)

The second order Ricci scalar \([60]\) becomes

\[ \begin{align*}
(R)^{(2)} & = \nabla_{c}\tilde{K}^c_{a} - 2\tilde{h}^c_{\alpha} \nabla_{c}(\tilde{\Gamma}^c_{a})^{(1)} + 2(\tilde{\Gamma}^c_{de})^{(1)}(\tilde{\Gamma}^e_{a})^{(1)} + \tilde{\Gamma}^{cb} \mathcal{L} X (R_{ab})^{(1)} \\
& + \frac{1}{2} \tilde{\Gamma}^{ab} (\mathcal{L} Y - \mathcal{L}^2 \tilde{X}) R_{ab} - \tilde{h}^{ab} (\mathcal{L} Y - \mathcal{L}^2 \tilde{X}) (2 \nabla_{c}\tilde{\Gamma}^c_{ab}) + \mathcal{L} X (R_{ab})^{(1)} \\
& + (\tilde{h}^{ac} - \mathcal{L} X \tilde{\Gamma}^{ac}) (\tilde{h}_{ab} + \mathcal{L} X \tilde{g}_{ab}) R_{ab} - \frac{1}{2} (\tilde{k}_{ab} + 2 \mathcal{L} X \tilde{h}_{ab} + (\mathcal{L} Y - \mathcal{L}^2 \tilde{X}) \tilde{g}_{ab}) R_{ab},
\end{align*} \]  

(115)
which reduces to

\[(R)^{(2)} = \nabla_{[c} \tilde{K}_{a]}^{ac} - 2\tilde{h}_c^{e} \nabla_{[c} \tilde{\Gamma}_{a]}^{ae} + 2\tilde{\Gamma}_{[c}^{ae} \tilde{\Gamma}_{a]}^e c - 2\tilde{h}_c^{ab} \nabla_{[c} \tilde{\Gamma}_{a]}^{ab} - \frac{1}{2} \tilde{k}_{ab} \tilde{R}^{ab} + \tilde{h}^{ac} \tilde{h}_{bc} \tilde{R}_c \]  

(116)  

Let us concentrate on the gauge variant terms: we can write

\[\tilde{g}^{ab} \mathcal{L}_X (R_{ab})^{(1)} + (R_{ab})^{(1)} \mathcal{L}_X \tilde{g}^{ab} - \tilde{R}^{ab} \mathcal{L}_X h_{ab} = \mathcal{L}_X (R)^{(1)} + h_{ab} \mathcal{L}_X \tilde{R}^{ab}, \]  

(117)  

and

\[\frac{1}{2} \tilde{g}^{ab} (\mathcal{L}_Y - \mathcal{L}_X^2) \tilde{R}_{ab} - \tilde{h}^{ab} \mathcal{L}_X \tilde{R}_{ab} - \frac{1}{2} \tilde{R}^{ab} (\mathcal{L}_Y - \mathcal{L}_X^2) \tilde{g}_{ab} = \frac{1}{2} (\mathcal{L}_Y - \mathcal{L}_X^2) \tilde{R} - h_{ab} \mathcal{L}_X \tilde{R}^{ab} - 2h^{ab} \tilde{R}_a \mathcal{L}_X \tilde{g}_{db} - \tilde{R}_a \mathcal{L}_X \tilde{g}_{ca} \mathcal{L}_X \tilde{g}_{cb}, \]  

(118)  

and also

\[\tilde{h}^{ac} \tilde{R}_a \mathcal{L}_X \tilde{g}_{cb} - \tilde{h}_{cb} \tilde{R}_a \mathcal{L}_X \tilde{g}^{ac} - \tilde{R}_a \mathcal{L}_X \tilde{g}^{ac} \mathcal{L}_X \tilde{g}_{cb} = -\tilde{h}_{cb} \tilde{R}_a \mathcal{L}_X \tilde{g}^{ac} + h^{ac} \tilde{R}_a \mathcal{L}_X \tilde{g}_{cb}. \]  

(119)  

Finally the second order scalar curvature yields

\[(R)^{(2)} = \nabla_{[c} \tilde{K}_{a]}^{ac} - 2\tilde{h}_c^{e} \nabla_{[c} \tilde{\Gamma}_{a]}^{ae} + \tilde{\Gamma}_{[c}^{ae} \tilde{\Gamma}_{a]}^e c - 2\tilde{h}_c^{ab} \nabla_{[c} \tilde{\Gamma}_{a]}^{ab} - \frac{1}{2} \tilde{k}_{ab} (\tilde{k}_{ab} - \tilde{h}_a \tilde{h}_b). \]  

(120)  

Now we can compute the second order perturbation of the cosmological Einstein tensor 63 in terms of gauge-variant and invariant quantities. From the previous results we get

\[(G_{ab})^{(2)} = \nabla_{[c} \tilde{K}_{a]}^{c} - 2\tilde{h}_c^{e} \nabla_{[c} \tilde{\Gamma}_{a]}^{ce} + 2\tilde{\Gamma}_{[c}^{ce} \tilde{\Gamma}_{a]}^e c + \mathcal{L}_X (R_{ab})^{(1)} + \frac{1}{2} (\mathcal{L}_Y - \mathcal{L}_X^2) \tilde{R}_{ab} \]  

(121)
which reduces to

\[
(G_{ab})^{(2)} = \tilde{\nabla}_{[c} \tilde{K}_{a]}^{b} - 2\tilde{h}^{b}_{c} \tilde{\nabla}_{[c} \tilde{\Gamma}_{a]}^{d} + 2\tilde{\Gamma}^{e}_{[c} \tilde{\Gamma}_{a]}^{d} + \tilde{k}_{ab}(\frac{\Lambda}{2} - \frac{\tilde{R}}{4}) - \frac{1}{2}\tilde{h}_{ab}(2\tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{de} - \tilde{R}_{dc}\tilde{h}^{de})
\]

\[
+ \frac{1}{2}\tilde{g}_{ab}\left(\tilde{\nabla}_{[c} \tilde{K}_{d]}^{de} - 2\tilde{h}^{c}_{e} \tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{de} + \tilde{R}^{cd}(\tilde{h}^{e}_{d}\tilde{h}_{ce} - \frac{1}{2}\tilde{k}_{cd}) + 2\tilde{\Gamma}^{e}_{[c} \tilde{\Gamma}_{d]}^{d} + \tilde{R}^{cd}[c \tilde{\Gamma}_{d]}^{e} c - 2\tilde{h}^{cd}\tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{e}\right)
\]

\[
+ \mathcal{L}_{X}(R_{ab})^{(1)} - \frac{1}{2}(R)^{(1)}\mathcal{L}_{X}\tilde{g}_{ab} + (\Lambda - \frac{\tilde{R}}{2})\mathcal{L}_{X}h_{ab} - \frac{1}{2}\tilde{h}_{ab}\mathcal{L}_{X}\tilde{R} - \frac{1}{2}\tilde{g}_{ab}\mathcal{L}_{X}(R)^{(1)}
\]

\[
- \frac{1}{4}\tilde{g}_{ab}(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\tilde{R} + \frac{1}{2}(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\mathcal{R}_{ab} + (\Lambda - \frac{\tilde{R}}{2})(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\tilde{g}_{ab}
\]

where the first two lines denote the gauge-invariant part. Let us consider the gauge-variant terms. We can collect the third line as

\[
\mathcal{L}_{X}(R_{ab})^{(1)} = \mathcal{L}_{X}(G_{ab})^{(1)} + \frac{1}{2}\mathcal{L}_{X}\tilde{g}_{ab}\mathcal{L}_{X}\tilde{R}
\]

and the terms on the last line yield

\[
- \frac{1}{4}\tilde{g}_{ab}\left(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2}\right)\tilde{R} + \frac{1}{2}(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\mathcal{R}_{ab} + (\Lambda - \frac{\tilde{R}}{2})(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\tilde{g}_{ab}
\]

\[
= \frac{1}{2}(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\tilde{g}_{ab} - \frac{1}{2}\mathcal{L}_{X}\tilde{R}\mathcal{L}_{X}\tilde{g}_{ab}
\]

Finally we obtain the second order cosmological Einstein tensor

\[
(G_{ab})^{(2)} = \tilde{\nabla}_{[c} \tilde{K}_{a]}^{b} - 2\tilde{h}^{b}_{c} \tilde{\nabla}_{[c} \tilde{\Gamma}_{a]}^{d} + 2\tilde{\Gamma}^{e}_{[c} \tilde{\Gamma}_{a]}^{d} + \tilde{k}_{ab}(\frac{\Lambda}{2} - \frac{\tilde{R}}{4}) - \frac{1}{2}\tilde{h}_{ab}(2\tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{de} - \tilde{R}_{dc}\tilde{h}^{de})
\]

\[
- \frac{1}{2}\tilde{g}_{ab}\left(\tilde{\nabla}_{[c} \tilde{K}_{d]}^{de} - 2\tilde{h}^{c}_{e} \tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{de} + \tilde{R}^{cd}(\tilde{h}^{e}_{d}\tilde{h}_{ce} - \frac{1}{2}\tilde{k}_{cd}) + 2\tilde{\Gamma}^{e}_{[c} \tilde{\Gamma}_{d]}^{d} + \tilde{R}^{cd}[c \tilde{\Gamma}_{d]}^{e} c - 2\tilde{h}^{cd}\tilde{\nabla}_{[c} \tilde{\Gamma}_{d]}^{e}\right)
\]

\[
+ \mathcal{L}_{X}(G_{ab})^{(1)} + \frac{1}{2}(\mathcal{L}_{Y} - \mathcal{L}_{X}^{2})\tilde{g}_{ab}
\]

where the gauge-variant terms vanish when $\tilde{g}$ is solution to the background equations and $\tilde{h}$ is a solution of the first order linearized equations. In this case we arrive at a pure gauge-invariant second order cosmological Einstein tensor.

Acknowledgments

This work was done in the Physics Department of the Middle East Technical University. The Author would like to thank Prof. Dr. Bayram Tekin for his comments and extended discussions on conserved charges in cosmological Einstein gravity.

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