Complete Edge-Colored Permutation Graphs

Tom Hartmann\textsuperscript{1,\*}, Max Bannach\textsuperscript{2}, Martin Middendorf\textsuperscript{3}, Peter F. Stadler\textsuperscript{1,4-8}, Nicolas Wieseke\textsuperscript{3}, and Marc Hellmuth\textsuperscript{9,\*}

\textsuperscript{1}Bioinformatics Group, Department of Computer Science & Interdisciplinary Center for Bioinformatics, Universität Leipzig, HärTELstraße 16-18, D-04107 Leipzig, Germany.
\textsuperscript{2}Institute for Theoretical Computer Science, Universität zu Lübeck, Ratzeburger Allee 160, D-23562 Lübeck, Germany.
\textsuperscript{3}Swarm Intelligence and Complex Systems Group, Faculty of Mathematics and Computer Science, University of Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany.
\textsuperscript{4}German Centre for Integrative Biodiversity Research (iDiv) Halle-Jena-Leipzig, Competence Center for Scalable Data Services and Solutions Dresden-Leipzig, Leipzig Research Center for Civilization Diseases, and Centre for Biotechnology and Biomedicine at Leipzig University at Universität Leipzig
\textsuperscript{5}Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, D-04103 Leipzig, Germany
\textsuperscript{6}Institute for Theoretical Chemistry, University of Vienna, Währingerstrasse 17, A-1090 Wien, Austria
\textsuperscript{7}Facultad de Ciencias, Universidad National de Colombia, Sede Bogotá, Colombia
\textsuperscript{8}Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe NM 87501, USA
\textsuperscript{9}School of Computing, University of Leeds, EC Stoner Building, Leeds LS2 9JT, UK
\textsuperscript{\*}corresponding author

Abstract

We introduce the concept of complete edge-colored permutation graphs as complete graphs that are the edge-disjoint union of “classical” permutation graphs. We show that a graph $G = (V, E)$ is a complete edge-colored permutation graph if and only if each monochromatic subgraph of $G$ is a “classical” permutation graph and $G$ does not contain a triangle with 3 different colors. Using the modular decomposition as a framework we demonstrate that complete edge-colored permutation graphs are characterized in terms of their strong prime modules, which induce also complete edge-colored permutation graphs. This leads to an $O(|V|^2)$-time recognition algorithm. We show, moreover, that complete edge-colored permutation graphs form a superclass of so-called symbolic ultrametrics and that the coloring of such graphs is always a Gallai coloring.

Keywords: Permutation graph; k-edge-coloring; Modular Decomposition; Symbolic ultrametric; Cograph; Gallai coloring

1 Introduction

Permutations model the rearrangement of an ordered sequence of objects. Thus they play an important role in card shuffling [2], comparative genomics [9, 29, 42], and combinatorial optimization [12]. The effect of a permutation can be illustrated in a permutation graph [56] that contains the elements as vertices and that connects two vertices with an edge if the corresponding elements are reversed by the permutation.

The graph class of permutation graphs is of considerable theoretical interest [15, 33] because many computationally intractable problems can be solved efficiently on permutation graphs. For example, HAMILTONIAN-CYCLE (given a graph, is there a cycle containing every vertex?) [20], FEEDBACK-VERTEX-SET (can we transform the graph into a forest by deleting at most $k$ vertices?) [14], or MAXIMUM-INDEPENDENT-SET (can we find an independent set of largest possible size for a given graph?) [49] can be solved efficiently on permutation graphs. Furthermore, permutation graphs can be recognized in linear time [19, 53] and several characterizations of permutation graphs have been established [3, 15, 21, 31, 55].

Multiple permutations over the same set of elements can be represented in a single edge-colored graph by taking the edge-union of the corresponding permutation graphs and assigning to each edge a color that uniquely identifies the underlying permutation. In particular, if the edge sets of the underlying
permutation graphs are disjoint, then, each edge obtains a unique label (or “color”) in the resulting graph. For certain sets of permutation graphs, this procedure ends up with a complete edge-colored graph, a class of graphs that has received considerable attention as symmetric 2-structures [22, 23, 25, 27, 44].

In this article we study the reverse direction: Given a complete edge-colored graph, does it represent a set of permutations? In a graph theoretic sense, we are interested in the structure of graphs that can be generated as superpositions of permutation graphs. For instance, we may ask: Do the induced subgraphs of such graphs have certain interesting properties? Are there forbidden structures? Which graph classes are a superset of such colored permutation graphs?

Contribution I: A Characterization of Complete Edge-Colored Permutation Graphs

Our first contribution is a collection of characterizations of complete edge-colored permutation graphs (graph of the just sketched kind; a formal definition is given in Section 3). We first show that these graphs are hereditary, i.e., each induced subgraph of a complete edge-colored permutation graph is again a complete edge-colored permutation graph.

Furthermore, we provide a characterization that is closely related to Gallai colorings: A complete edge-colored graph is a complete edge-colored permutation graph if and only if all its monochromatic subgraphs are “classical” permutation graphs and it does not contain a rainbow triangle, i.e., a triangle with three distinct edge colors.

Finally, we provide two characterizations in terms of the modular decomposition: A complete edge-colored graph is a complete edge-colored permutation graph if and only if the quotient graph of each strong module is a complete edge-colored permutation graph. Moreover, a complete edge-colored graph is a complete edge-colored permutation graph if and only if the quotient graph of each strong prime module is a complete edge-colored permutation graph.

Contribution II: Recognition of Complete Edge-Colored Permutation Graphs

We prove that complete edge-colored permutation graphs \( G = (V, E) \) can be recognized in \( O(|V|^2) \)-time and, in the affirmative case, the underlying permutations can be constructed within the same time complexity.

Contribution III: Connection to other Graph Classes

We provide a classification of the class of complete edge-colored permutation graphs with respect to other graph classes. We show that the coloring of each complete edge-colored permutation graph is a Gallai coloring, i.e., it is a complete edge-colored graph that does not contain a rainbow triangle. Furthermore, we prove that every graph representation of a symbolic ultrametric, i.e., a graph that is based on sets of certain symmetric binary relations, is also a complete edge-colored permutation graph. Moreover, we show how symbolic ultrametrics, complete edge-colored graphs, and so-called separable permutations are related.

Related Work

Permutation graphs were already characterized in 1976 in terms of forbidden subgraphs [31] and in a number of other characterizations, see for instance [15, 33]. Although many computationally intractable problems become tractable on permutation graphs, this is not the case for some coloring problems such as the problem to determine the achromatic number [11] or the cochromatic number [58]. There are also many problems for which the complexity on permutations graphs is still unknown, for instance determining the edge search number [32]. A general overview on permutation graphs can be found in [15, 33].

The idea to investigate graphs whose edge set can be decomposed into a set of permutation graphs is not new; in fact, not necessarily complete versions of those graphs were studied by Golumbic et al. in the 1980s [34]. It was shown that the complement of such a graph is a comparability graph, i.e., a graph that corresponds to a strict partially ordered set. In addition, the authors investigated the problem to find a minimum number of permutations whose edge-union forms a given graph. For more information on those graphs, the reader is referred to [33].

The investigation of complete edge-colored graphs without rainbow triangles also has a long history. A complete graph is said to admit a Gallai coloring with \( k \) colors, if its edges can be colored with \( k \) colors without creating a rainbow triangle [39]. It is an active field of research to study when such a coloring may exist [4, 36] and what properties such colorings have [5–7]. For example, a Gallai coloring of the \( K_n \), i.e., the complete graph with \( n \in \mathbb{N} \) vertices, contains at most \( n − 1 \) colors [28]. Another well-known property is that those graphs can be obtained by substituting complete graphs with Gallai colorings into vertices of 2-edge-colored complete graphs [16, 31, 39]. A survey on complete edge-colored graphs that admit a Gallai coloring can be found e.g. in [30, 35, 48].

Symbolic ultrametrics [10] are symmetric binary relations that are closely related to vertex-colored trees. They were recently used in phylogenomics for the characterization of homology relations between
Permutations

In the following we write $[1:n] := \{1, \ldots, n\}$. A permutation $\pi: [1:n] \rightarrow [1:n]$ of length $n \in \mathbb{N}$ is a bijective map that assigns to each element $i \in [1:n]$ a unique element $\pi(i) \in [1:n]$. By slight abuse of notation, we represent a permutation $\pi$ as the sequence $(\pi(1), \ldots, \pi(n))$. We denote the length of $\pi = (\pi(1), \ldots, \pi(n))$ by $|\pi| = n$ and the set of all permutations of length $n$ by $\mathcal{P}_n$.

For every $\pi \in \mathcal{P}_n$, there is a unique permutation $\pi^{-1} \in \mathcal{P}_n$, that is called inverse permutation (of $\pi$) defined by $\pi^{-1}(j) = i$ if and only if $\pi(i) = j$. We denote with $\mathcal{P}$ the reversed permutation in which the order of all elements with respect to $\pi$ is reversed: $\pi(i) = \pi(n + 1 - i)$ for all $i \in [1:n]$. A subsequence (of $\pi$) is a sequence $(\pi(i_1), \ldots, \pi(i_k))$ with $1 \leq i_1 \leq \cdots \leq i_k \leq n$. The permutation $\iota := (1, \ldots, n)$ is called identity.

**Example 1.** Consider the permutation $\pi = (1, 5, 2, 4, 7, 3, 6)$ of length 7. Its inverse permutation is $\pi^{-1} = (1, 3, 6, 4, 2, 7, 5)$ and its reversed permutation is $\pi = (6, 3, 7, 4, 2, 5, 1)$. The sequence $(1, 2, 4, 7, 6)$ is a subsequence of $\pi$.

Binary Relations and Strict Total Orders

A binary relation $R \subseteq V \times V$ on $V$ is a strict total order on $V$ if

(R1) $R$ is trichotomous, i.e., for all $x, y \in V$ we have either $(x, y) \in R$, $(y, x) \in R$ or $x = y$ and

(R2) $R$ is transitive, i.e., for all $x, y, z \in V$ with $(x, y), (y, z) \in R$ it holds that $(x, z) \in R$.

Note that (R1) is equivalent to claiming that $R$ is irreflexive, asymmetric, and that all distinct elements $x, y \in V$ are in relation $R$.
Graphs and Colorings An (undirected) graph $G$ is an ordered pair $(V, E)$ consisting of a non-empty, finite set of vertices $V(G) = V$ and a set of edges $E(G) = E \subseteq \binom{V}{2}$, where $\binom{V}{2}$ denotes the set of all 2-element subsets of $V$. Observe that this definition explicitly excludes self loops and parallel edges. If $G = (V, \binom{V}{2})$, then $G$ is called complete and denoted by $K_\{V\}$.

For an edge $e = \{u, v\}$ of $G$, the vertices $u$ and $v$ are called adjacent and $e$ is said to be incident with $u$ and $v$. Let $v \in V$ be a vertex of $G$. The degree of $v$ is the number of edges in $G$ that are incident with $v$. The complement of $G$ is the graph $\overline{G}$ with vertex set $V$ and edge set $\left(\binom{V}{2}\right) \setminus E$. A sequence of vertices $S = (v_1, \ldots, v_m)$ of $G$ is called a walk (in $G$) if $\{v_i, v_{i+1}\} \in E$ for all $i \in [1;m - 1]$. A walk is called a path if it does not contain a vertex more than once. A graph $G$ is connected if for any two vertices $u, v \in V$ there exists a walk from $u$ to $v$, otherwise $G$ is called disconnected.

A graph $H = (W, F)$ is a subgraph of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$. A subgraph $H = (W, F)$ (of $G$) is called induced, denoted by $G[W]$, if for all $u, v \in W$ it holds that $\{u, v\} \in E$ implies $\{u, v\} \in F$. A connected component of a graph is a connected subgraph that is maximal with respect to inclusion. A subset of vertices $V$ of a graph $G$ is an independent set (resp. a clique) if for all distinct $u, v \in V$ it holds that $\{u, v\} \notin E$ (resp. $\{u, v\} \in E$).

A tree $T = (V, E)$ is a graph in which any two vertices are connected by exactly one path. A rooted tree $T$ is a tree with one distinguished vertex $r \in V$ called the root. The leaves $L \subseteq V$ of $T$ is the set of all vertices that are distinct from the root and have degree 1. Let $T = (V, E)$ be a rooted tree with leaf set $L$ and root $r$. Then, vertex $v$ is called a descendant of vertex $u$ and $u$ an ancestor of $v$ if $v$ lies on the unique path from $v$ to the root $r$. The children of an inner vertex $v$ are its direct descendants, i.e., vertices $w$ with $\{v, w\} \in E$ and $w$ is a descendant of $v$. In this case, the vertex $v$ is called the parent of $w$.

Definition 2.1 ($k$-edge-colored graph). A $k$-edge-colored graph is a graph $G = (V, E)$ together with a surjective map $c : E \rightarrow \{1:k\}$ assigning a unique color $c(e)$ to each edge $e \in E$.

The map $c$ is also called $k$-edge-coloring of the edges of $G$. A $k$-edge-colored graph is also called edge-colored graph if the number $k$ of colors remains unspecified. By slight abuse of notation, we also call graphs $G = (V, E)$ with $E = \emptyset$ edge-colored graphs. Thus, as a graph $K_1 = (\{v\}, \emptyset)$ is an edge-colored graph. Using this notation, every graph $G = (V, E)$ can be considered as a 1-edge-colored graph.

Given a $k$-edge-colored graph with non-empty edge set and a map $c$, the edge set $E$ can be partitioned into $k$ non-empty subsets $E_1, \ldots, E_k$, where each $E_i$ contains all edges $e$ with $c(e) = i$. We call such a partition of $E$ into the sets $E_1, \ldots, E_k$ an (edge-)coloring of $G = (V, E)$ and write $(V, E_1, \ldots, E_k)$ instead of $(V, E)$.

For a subgraph $H$ of a $k$-edge-colored graph $G$, we always assume that each edge $e$ in $H$ retains the color $c(e)$ that it has in $G$, making $H$ a $k'$-edge-colored graph with $k' \leq k$. The $i$-th monochromatic subgraph of a $k$-edge-colored graph $G$ is the subgraph $G_i := (V, E_i)$, where $E_i$ contains all edges with color $i$.

Remark 1. Many of the $k$-edge-colored graphs considered here are complete graphs. Note, this is not a restriction in general, since each $k$-edge-colored graph can be transformed into a complete $(k+1)$-edge-colored graph by assigning a new color $k + 1$ to all non-edges.

Definition 2.2 (Rainbow triangle). Let $G = (V, E_1, \ldots, E_k)$ be a $k$-edge-colored graph and $\{u, v, w\} \subseteq V$. The induced subgraph $G[\{u, v, w\}]$ is called a rainbow triangle, denoted by $\Delta_{uvw}$, if $\{u, v\}, \{u, w\},$ and $\{v, w\}$ have pairwise distinct colors.

In the literature, various notions of rainbow triangles can be found. Examples are the notions multi-colored triangles [37] and tricolored triangles [39].

A quite useful property of complete $k$-edge-colored graphs without rainbow triangles is summarized in the following

Proposition 2.3 ([39], Lemma A). For every complete $k$-edge-colored graph $G$ with $k \in \mathbb{N}_{\geq 3}$ that does not contain a rainbow triangle as induced subgraph there exists a color $i \in [1:k]$ such that the $i$-th monochromatic subgraph $G_i$ of $G$ is disconnected.

Example 2. Consider the complete $k$-edge-colored graph $G = (V, E_1, \ldots, E_k)$ that is illustrated in Figure 1. Graph $G$ is complete and 4-edge-colored. Moreover, $G$ contains the rainbow triangle $\Delta_{abcdef}$, although the 1-st monochromatic subgraph $G_{1} = (\{a, b, c, d, e, f\}, \{\{a, d\}, \{b, d\}, \{c, d\}\})$ of $G$ is disconnected. This, in particular, shows that the converse of Proposition 2.3 is not always satisfied.

Modular Decomposition A module of a complete $k$-edge-colored graph $G = (V, E_1, \ldots, E_k)$ is a subset $M \subseteq V$ such that for every $u \in M$ and $v \in V \setminus M$ it holds that if $\{u, v\} \in E_i$ then $\{w, v\} \in E_i$ for all $w \in M$. The set $V$, the empty set $\emptyset$, and the singleton sets $\{v_1\}, \ldots, \{v_{|V|}\}$ are called trivial modules. The set of all modules of $G$ is denoted by $\mathcal{M}(G)$. Examples of modules can be found in Figure 1 and
isomorphic. This result, in particular, implies that two complete edge-colored graphs are isomorphic precisely if their modular decomposition trees are.

Lemma 2.4 ([22], Lemma 4.11) $G$ is used for the colors of $k$-edge-colored graphs.

Figure 1: A complete 4-edge-colored graph $G = (V = \{a, b, c, d, e, f\}, E_1, E_2, E_3, E_4)$ as used in Example 2. The edge-coloring of $G$ is illustrated using different edge styles: $E_1$ (black solid edges), $E_2$ (red dashed edges), $E_3$ (blue dotted edges), and $E_4$ (green dash-dotted edges). There are four non-trivial modules in $G$, namely $M = \{a, b, c\}$, $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$. However, $M$ is the only non-trivial strong module of $G$. Thus, $\mathbb{P}_{\text{max}}(V) = \{\{a, b, c\}, \{d\}, \{e\}, \{f\}\}$.

Example 3. A module $M \in \mathbb{M}(G)$ is called **strong** if for any other module $M' \in \mathbb{M}(G)$ we have $M \subseteq M'$, $M' \not\subseteq M$, or $M \cap M' = \emptyset$. In the following, all modules considered here are non-empty unless explicitly stated otherwise.

The set of strong modules $\mathbb{M}_{\text{str}}(G)$ is called **modular decomposition** of $G$. Whereas there may be exponentially many modules, the size of the set of strong modules is $O(|V|)$ [24]. In particular, since strong modules do not overlap by definition, it holds that $\mathbb{M}_{\text{str}}(G)$ forms a hierarchy which gives rise to a unique tree representation $T_G$ of $G$: the **modular decomposition tree** of $G$ or **inclusion tree** of $\mathbb{M}_{\text{str}}(G)$ [25, 44]. The vertices of $T_G$ are identified with the elements of $\mathbb{M}_{\text{str}}(G)$.

For the definition of so-called quotient graphs we need the following

Lemma 2.4 ([22], Lemma 4.11). Let $M$ and $M'$ be two disjoint modules of a complete $k$-edge-colored graph $G = (V, E_1, \ldots, E_k)$, then, there is a color $i \in [1:k]$ such for all edges $u, v \in E(G)$ with $u \in M$ and $v \in M'$ it holds that $\{u, v\} \in E_i$.

Figure 1.

Remark 2. By construction, for every strong module $M$ of $G$, the quotient graph $G[M]/\mathbb{P}_{\text{max}}(M)$ is isomorphic to every induced subgraph $G[N]$ of $G$, where $N$ contains exactly one vertex from each child module $M_i \in \mathbb{P}_{\text{max}}(M) = \{M_1, \ldots, M_k\}$. Moreover, if $G$ is a complete graph, then $G[N]$ is complete as well. In this case, $G[M]/\mathbb{P}_{\text{max}}(M)$ is always a complete graph.

As shown in [23, 27], the quotient graph $G[M]/\mathbb{P}_{\text{max}}(M)$ for a strong module $M$ of a complete $k$-edge-colored graph $G$ is well-defined and satisfies exactly one of the following conditions:

1) $G[M]/\mathbb{P}_{\text{max}}(M)$ contains at least two vertices and is a complete 1-edge-colored graph. In this case, $M$ is called a **series** module.
Example 3. Consider the complete 4-edge-colored graph $G = ([1:7], E_1, E_2, E_3, E_4)$ as is illustrated in Figure 2 (a) with edge sets:

\[
E_1 = \{ \{1, 6\}, \{2, 6\}, \{3, 6\}, \{3, 7\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \{5, 7\} \},
\]

\[
E_2 = \{ \{3, 4\}, \{3, 5\} \},
\]

\[
E_3 = \binom{\{1, 7\}}{2} \setminus (E_1 \cup E_2 \cup E_4),
\]

\[
E_4 = \{ \{1, 2\} \}.
\]

The modular decomposition of $G$ is $\mathsf{M}_{str}(G) = \{\{1, 7\}, \{1, 2\}, \{3, 4, 5\}, \{4, 5\}, \{1\}, \ldots, \{7\}\}$ and the maximal strong modular partition of the module $[1:7]$ is $\mathsf{P}_{\text{max}}([1:7]) = \{\{1, 2\}, \{6\}, \{3, 4, 5\}, \{7\}\}$. Figure 2 (b) shows the modular decomposition tree of $G$. Figure 2 (c) and Figure 2 (d) illustrate the quotient graphs $G[\mathsf{M}_1]/\mathsf{P}_{\text{max}}(\mathsf{M}_1)$ and $G[\mathsf{M}_4]/\mathsf{P}_{\text{max}}(\mathsf{M}_4)$, respectively. It can easily be verified that the module $[1:7]$ is prime, whereas the modules $\{1, 2\}, \{4, 5\}$, and $\{3, 4, 5\}$ are series.

Interestingly, every complete primitive graph that contains at least three colors must contain a rainbow triangle:

**Lemma 2.7.** Let $k \in \mathbb{N}_{\geq 3}$ and $G$ be a complete $k$-edge-colored graph. If $G$ is primitive, then $G$ contains a rainbow triangle.
In this section we introduce the formal definitions and provide some useful results for (complete edge-colored) permutation graphs. Recall that a permutation is a bijection from $[1:n]$ to itself. Intuitively, the vertices of a permutation graph are the elements of $[1:n]$ and the edges indicate whether the order of two elements is “reversed” by the permutation. For technical reasons that become clear later it is more convenient to use a labeling that assigns to each vertex $v$ a unique natural number $\ell(v)$ rather than to define the vertex set $V$ directly as the set $[1:|V|]$. To this end, we need the following:

**Definition 3.1** (Labeling). A labeling $\ell$ of a graph $G = (V,E)$ is a bijective map $\ell: V \to [1:|V|]$ that associates each vertex $v$ with a unique natural number $\ell(v) \in [1:|V|]$.

Using a labeling, permutation graphs [56] can be defined as follows.

**Definition 3.2** (Permutation Graph [15, 50, 56]). A graph $G$ is called a (simple) permutation graph if there exists a labeling $\ell$ and a permutation $\pi = (\pi(1), \ldots, \pi(|V|))$ such that for all $u, v \in V$ we have:

$$\{u, v\} \in E \iff \ell(u) > \ell(v) \text{ and } \pi^{-1}(\ell(u)) < \pi^{-1}(\ell(v)).$$

In this case, we say that $G$ is a (simple) permutation graph of $\pi$. If the labeling $\ell$ is specified, we may also write that $G$ together with $\ell$, or simply $(G, \ell)$, is a (simple) permutation graph (of $\pi$).

A permutation graph of $\pi = (1, 5, 2, 4, 7, 3, 6)$ is illustrated in Figure 3 (a).

The property of being a simple permutation graph is hereditary:

**Lemma 3.3** ([19]). The family of permutation graphs is hereditary, that is, every induced subgraph of a permutation graph is a permutation graph.

The following known proposition relates subsequences of a permutation to independent sets and cliques in the corresponding permutation graph.

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**Figure 3**: Three different edge-colored graphs $(V,E)$ together with labeling $\ell: V \to [1:7]$ that is implicitly given in the drawing by writing $\ell(v)$ on each vertex $v \in V$. Panel (a) illustrates the simple permutation graph $G$ of $\pi = (1, 5, 2, 4, 7, 3, 6)$. For example, $G$ contains the edge $\{3, 7\}$ since $7 > 3$ and $\pi^{-1}(7) < \pi^{-1}(3)$. Panel (b) shows the complete 2-edge-colored permutation graph of $\pi$ and $\pi = (6, 3, 7, 4, 2, 5, 1)$. Panel (c) shows the complete 4-edge-colored permutation graph of the permutations $2 \ 1 \ 3 \ 4 \ 7 \ 5 \ 6$ (red dashed edges), $4 \ 1 \ 2 \ 3 \ 5 \ 6 \ 7$ (green solid edges), $1 \ 2 \ 3 \ 4 \ 6 \ 5 \ 7$ (blue dash-dotted edge), and $5 \ 6 \ 7 \ 3 \ 1 \ 2 \ 4$ (yellow dotted edges).

**Proof.** The proof proceeds by contraposition: We show that every complete $k$-edge-colored graph $G$ with $k \in \mathbb{N}_{\geq 3}$ that does not contain a rainbow triangle has a non-trivial module, i.e., $G$ is not primitive.

Let $G = (V,E_1, \ldots, E_k)$ be a complete $k$-edge-colored graph with $k \in \mathbb{N}_{\geq 3}$ that does not contain a rainbow triangle. By Proposition 2.3 there is a color $i \in [1:k]$ such that $G_i$ is disconnected. Let $M_1, \ldots, M_m$, $m \geq 2$ be the connected components of $G_i$. Since $G$ is a $k$-edge-colored graph with $k \geq 3$, at least one connected component has an edge $\{u,v\}$ of color $i$. Furthermore, let $x \in V \setminus M_1$ be a vertex that is not in $M_1$. Note that neither $\{u,x\}$ nor $\{v,x\}$ can have color $i$ since $x \notin M_1$. Therefore, $\{u,x\}$ must have some color $j \neq i$. Since $G$ does not contain rainbow triangles, the edge $\{u,v\}$ must have color $j$ as well. Repeating the latter arguments along the edges in the spanning tree of $G_i[M_1]$ implies that all vertices in $M_1$ must be connected to $x$ via an edge of color $j$. The latter, in particular, remains true for all vertices in $V \setminus M_1$, that is, all edges leading from vertices in $M_1$ to a fixed vertex $x \in V \setminus M_1$ have the same color. Hence, $M_1$ is a non-trivial module in $G$ which implies that $G$ cannot be primitive.

\[\square\]
Proposition 3.4 ([33, p. 159]). Let \((G, \ell)\) be a simple permutation graph of some permutation \(\pi \in \mathcal{P}_n\). The following statements are true for any subsequence \(S = (\pi(i_1), \ldots, \pi(i_k))\) of \(\pi\) with \(1 \leq i_1 \leq \cdots \leq i_k \leq n\):

(i) \(S\) is ascending if and only if \(\{\ell^{-1}(\pi(i)) \mid i_1 \leq i \leq i_k\}\) is an independent set in \(G\).

(ii) \(S\) is descending if and only if \(\{\ell^{-1}(\pi(i)) \mid i_1 \leq i \leq i_k\}\) is a clique in \(G\).

By Proposition 3.4, a simple permutation graph is edgeless (resp., complete) if and only if the corresponding permutation is the identity permutation (resp., the reversed identity permutation).

Corollary 3.5. Let \(G = (V, E)\) be the simple permutation graph of a permutation \(\pi\). The following statements are true:

(i) \(E = \emptyset \iff \pi = \iota\).

(ii) \(E = (\binom{V}{2}) \iff \pi = \tau\).

Proof. Follows directly from Proposition 3.4.

Another well-known property of simple permutation graphs is summarized in the following proposition. It states that the complement of a simple permutation graph of a permutation \(\pi\) is isomorphic to the simple permutation graph of the reversed permutation \(\pi\) of \(\pi\).

Proposition 3.6 ([50], Proposition 2.2). Assume that \((G, \ell)\) is a simple permutation graph of \(\pi \in \mathcal{P}_n\). Then \((\overline{G}, \ell)\) is the simple permutation graph of \(\pi\).

Instead of considering simple permutation graphs (which represent a single permutation) we are interested in complete \(k\)-edge-colored permutation graphs (which represent a set of \(k\) permutations).

Definition 3.7 (Complete \(k\)-edge-colored Permutation Graph). Let \(G = (V, E_1, \ldots, E_k)\) be a complete and \(k\)-edge-colored graph. Then, \(G\) is a complete \(k\)-edge-colored permutation graph if there is a labeling \(\ell: V \to [1:|V|]\) and \(k\) permutations \(\pi_1, \ldots, \pi_k\) of length \(|V|\) such that for all \(i \in [1:k]\) the \(i\)-th monochromatic labeled subgraph \((G_i, \ell)\) is a simple permutation graph of \(\pi_i\), i.e., for all \(i \in [1:k]\) and all \(u, v \in V\) we have:

\[\{u, v\} \in E_i \iff \ell(u) > \ell(v) \text{ and } \pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v)).\]

In this case, we say that \(G\) is a complete \(k\)-edge-colored permutation graph or simply, complete \(k\)-edge-colored permutation graph (of \(\pi_1, \ldots, \pi_k\)). If the labeling \(\ell\) is specified, we may also write that \(G\) together with \(\ell\), or simply \((G, \ell)\), is complete \(k\)-edge-colored permutation graph (of \(\pi_1, \ldots, \pi_k\)).

In Figure 3 (c) a complete 4-edge-colored permutation graph is illustrated.

Note that for a complete \(k\)-edge-colored permutation graph \(G \not\cong K_1\) of permutations \(\pi_1, \ldots, \pi_k\) all elements \(E_1, \ldots, E_k\) are non-empty, since the edge-coloring of \(G\) is surjective. Thus, by Corollary 3.5, we have that \(\pi_i \neq \iota\) for all \(i \in [1:k]\). Corollary 3.5 also implies that for all \(n > 1\) there is only one complete 1-edge-colored permutation graph on \(|V| = n\) vertices, namely the complete graph \(K_n\) which is the simple permutation graph of \(\iota\). Clearly, the definition of 1-edge-colored permutation graphs coincides with the definition of simple permutation graphs.

Corollary 3.8. Let \(G\) be a complete \(k\)-edge-colored graph with \(k \in \{1, 2\}\). Assume that \((G_{11}, \ell)\) is a simple permutation graph for \(\pi\). Then \((G, \ell)\) is a complete \(k\)-edge-colored permutation graph for \(\pi\) (and \(\pi\)).

Proof. Let \(G\) be a complete \(k\)-edge-colored graph with \(k \in \{1, 2\}\). Furthermore, let \((G_{11}, \ell)\) be a simple permutation graph for \(\pi\). The case \(k = 1\) is trivial and, thus, assume that \(G = (V, E_1, E_2)\) is \(2\)-edge-colored. In this case, \(E_1\) and \(E_2\) are both non-empty. By Corollary 3.5, we obtain that \(\pi \neq \iota\). Moreover, \(\pi\) cannot be \(\tau\) as otherwise \(G\) would be complete 1-edge-colored. Since \((G_{11}, \ell)\) is a simple permutation graph for \(\pi\), we have that \(\{u, v\} \in E_i\) if and only if \(\ell(u) > \ell(v)\) and \(\pi^{-1}(\ell(u)) < \pi^{-1}(\ell(v))\).

By Proposition 3.6, \((G_{11}, \ell)\) is a simple permutation graph of \(\pi\). Hence, we have that \(\{u, v\} \in E(G_{11})\) if and only if \(\ell(u) > \ell(v)\) and \(\pi^{-1}(\ell(u)) < \pi^{-1}(\ell(v))\). Clearly, this is if and only if \(\pi^{-1}(\ell(u)) > \pi^{-1}(\ell(v))\) for all non-edge \(\{u, v\} \in G_{11}\), which completes the proof.

By Corollary 3.8, every simple permutation graph \(G\) that is neither edgeless nor complete corresponds to a complete 2-edge-colored permutation graph by interpreting the non-edges of \(G\) as edges with some new color, see also Figure 3 (a) and Figure 3 (b) for illustrative examples. Note, however, that Corollary 3.8 cannot easily be extended to \(k\)-edge-colored (possibly non-complete) graphs where each monochromatic subgraph is a simple permutation graph, cf. Figure 4 for an example.

The following two lemmas show that complete edge-colored permutation graphs contain forbidden substructures.
4 Characterization of Complete Edge-Colored Permutation Graphs

We provide here the main results of this contribution.

**Theorem 4.1.** Suppose that $G$ is a complete edge-colored graph. Then the following statements are equivalent:

1. $G$ is a complete edge-colored graph.
2. There exists a labeling $\ell$ of the edges of $G$ such that for any two distinct vertices $u$ and $v$, if $u < v$, then $\ell(u) < \ell(v)$.
3. There exists a labeling $\ell$ of the edges of $G$ such that for any two distinct vertices $u$ and $v$, if $u < v$, then $\ell(u) > \ell(v)$.

**Proof.**

The proof follows from the definitions and the properties of complete edge-colored graphs. The statement of the theorem is a direct consequence of the definitions and the properties of complete edge-colored graphs. The proof is completed by showing that the three statements are equivalent.

---

**Figure 4:** (a) A non-complete 2-edge-colored graph $G$ with color 1 (black solid edge) and color 2 (red dashed edge) is shown. Each of the two monochromatic subgraphs $G_1$ and $G_2$ of $G$ is isomorphic to the graph consisting of three vertices, a single edge, and an isolated vertex. Let us fix the labeling $\ell$ of $G$ by putting $\ell(u) = 1$, $\ell(v) = 2$ and $\ell(w) = 3$. Now, it is easy to verify that $(G_1, \ell)$ and $(G_2, \ell)$ are simple permutation graphs for $\pi_1 = (2, 1, 3)$ and $\pi_2 = (1, 3, 2)$, respectively. For both permutations, even the non-edge $\{u, w\}$ would be "explained", since $\ell(u) = 1 < \ell(w) = 3$ and $\pi^{-1}_1(\ell(u)) < \pi^{-1}_1(\ell(w))$, $i = 1, 2$.

We emphasize, however, that this is not the definition of edge-colored permutation graphs we use. We explicitly consider here only complete edge-colored graphs. Hence, we may extend $G$ to the complete 3-edge-colored graph $G'$ shown in (b). However, $G'$ is not a complete edge-colored permutation graph, since it contains a rainbow triangle (cf. Definition 2.2 and Proposition 4.13).
We now relabel the labels in $M$ such that for each monochromatic subgraph $G_i$ of $G$ it holds that $(G_i, \ell)$ is a simple permutation graph.

Each induced subgraph of $G$ is a complete edge-colored permutation graph, i.e., the property of being a complete edge-colored permutation graph is hereditary.

The quotient graph $G[M] / P_{\max}[M]$ of each strong module $M$ in the modular decomposition of $G$ is a complete edge-colored permutation graph.

The quotient graph $G[M] / P_{\max}[M]$ of each strong prime module $M$ in the modular decomposition of $G$ is a complete edge-colored permutation graph. In particular, the quotient graph of each strong prime module on at least three vertices must be 2-edge-colored.

Each monochromatic subgraph of $G$ is together with some labeling (possibly different from the labeling $\ell$ as chosen in Item (i)) a simple permutation graph and $G$ does not contain a rainbow triangle.

It is easy to see that, Item (i) implies Item (i) (since $G$ is an induced subgraph of itself) and Item (iii) implies the first part of Item (iv).

In what follows, we prove Theorem 4.1 and verify the individual items of Theorem 4.1 in Sections 4.1 to 4.4. In particular, we show that all Items (ii) - (v) are equivalent to Item (i). The reader might have already observed that there is an alternative avenue to show the equivalence between Items (i), (iii), and (iv), since Item (iii) trivially implies the first part of Item (iv). This observation suggests to prove the implication “(i) \Rightarrow (iii)” and “(iv) \Rightarrow (i)”. Nevertheless, we treat the equivalence between Items (i) and (iii) a well as (i) and (iv) in separate steps. The reason is simple: the proof of implication “(iii) \Rightarrow (i)” is constructive and used to design our efficient recognition algorithm in Section 5. In particular, this result allows us to provide a simple proof to show that (iv) implies (i).

For the sake of completeness, Section 4.5 summarizes all results to prove Theorem 4.1.

### 4.1 An Induced Subgraph Characterization

In this section, we show that the property of being a complete edge-colored permutation graph is hereditary and that complete edge-colored permutation graphs are characterized by this property. That is, we show the equivalence of Theorem 4.1 (i) and Theorem 4.1 (ii).

Clearly, it is not hard to see that Theorem 4.1 (ii) trivially implies Theorem 4.1 (i). However, for the sake of completeness, we summarize this result in the following

**Lemma 4.2.** If every induced subgraph of a graph $G$ is a complete edge-colored permutation graph, then $G$ is a complete edge-colored permutation graph.

**Proof.** The claim is a consequence of the simple fact that $G = (V,E)$ and the induced subgraph $G[V]$ are identical. \qed

We now show that the converse of Lemma 4.2 is valid as well.

**Proposition 4.3.** If $G = (V,E)$ is a complete edge-colored permutation graph, then, for all non-empty subsets $M \subseteq V$, the induced subgraph $G[M]$ is a complete edge-colored permutation graph. Hence, the property of being a complete edge-colored permutation graph is hereditary.

**Proof.** Let $G = (V,E_1, \ldots, E_k)$ together with a labeling $\ell$ be a complete $k$-edge-colored permutation graph for the permutations $\pi_1, \ldots, \pi_k$. The cases $|M| = 1$ as well as $k = 1$ are trivial and thus, we assume that $m = |M| > 1$ as well as $k > 1$.

Now, we utilize $\ell$ and $\pi_1, \ldots, \pi_k$ to construct a labeling $\tilde{\ell}$ and permutations $\tilde{\pi}_1, \ldots, \tilde{\pi}_k$ such that $(G[M], \tilde{\ell})$ is a complete $k'$-edge-colored permutation graph for $k' \leq k$ permutations from $\tilde{\pi}_1, \ldots, \tilde{\pi}_k$.

First, let $M := \{ \ell(v) \mid v \in M \}$ be the set of all labels assigned to the vertices in $M$. Moreover, let $P_i := \{ \pi_i^{-1}(\ell(v)) \mid v \in M \}$ for all $i \in [1:k]$. Now, for every $i \in [1:k]$ we obtain the well-defined bijective map

$$
\sigma_i : P_i \to M
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad j \mapsto \pi_i(j).
$$

We now relabel the labels in $M$ by defining the map $\tilde{\ell} : M \to [1:m]$ via $\tilde{\ell}(i) < \tilde{\ell}(j) \iff i < j$ for all distinct $i, j \in M$. We use $\tilde{\ell}$ to define a labeling $\tilde{\ell}$ of $G[M]$ as

$$
\tilde{\ell} : M \to [1:m]
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad j \mapsto \tilde{\ell}(\ell(j)).
$$
Note that \( \hat{\ell} \) is bijective and well-defined.

In order to transform each \( \sigma_i \) to a permutation \( \hat{\sigma}_i : [1:m] \to [1:m] \), we additionally provide bijective maps \( \theta_i^k : [1:m] \to \mathbb{P}_r \) as defined for all \( r \in [1:k] \) the map \( \theta_i^k \) by defining for all \( r \in [1:k] \), the map \( \theta_i^k(i) < \theta_i^k(j) \iff i < j \) for all distinct \( i, j \in \mathbb{P}_r \). Note that \( \hat{\ell} \) as well as \( \theta_i^k \) are unique and well-defined since \( M \) as well as \( \mathbb{P}_1, \ldots, \mathbb{P}_k \) are totally ordered sets on \( m \) elements.

See Figure 5 for an illustration of the maps \( \sigma_i, \ell, \hat{\ell}, \) and \( \theta_i^k \) that are used in this proof.

To derive the aforementioned permutations \( \hat{\sigma}_i : [1:m] \to [1:m] \), we simply define \( \hat{\sigma}_i \) as \( \hat{\sigma}_i := (\hat{\ell}(\sigma_i(\theta_i^1(1))), \ldots, \hat{\ell}(\sigma_i(\theta_i^k(m)))) \) for all \( i \in [1:k] \). In other words, for all \( j \in [1:m] \) and all \( i \in [1:k] \), we have \( \hat{\sigma}_i(j) = \hat{\ell}(\sigma_i(\theta_i^k(j))) \).

Now \( G[M] \) is a complete graph that retains the colors of the edges as provided in \( G \). However, observe that \( G[M] \) may have less than \( k \) colors. Let \( I = \{i_1, \ldots, i_k\} \subseteq [1:k] \) be the set of all colors on edges in \( G[M] \). Hence, \( G[M] = (M, E_{i_1}, \ldots, E_{i_k}) \) is a complete edge-colored graph.

We continue to show that \( (G[M], \hat{\ell}) \) is a complete \( |I| \)-edge-colored permutation graph for \( \hat{\sigma}_i, i \in I \). To this end, let \( u, v \in M \) be two vertices and without loss of generality assume that \( \ell(u) > \ell(v) \). Since we have not changed the relative order of the labels in \( M \) under \( \hat{\ell} \), we have \( \hat{\ell}(u)) > \hat{\ell}(v) \) which implies \( \hat{\ell}(u) > \hat{\ell}(v) \).

First assume that \( \{u, v\} \in E_i \) for some \( i \in [1:k] \) since \( G[M] \) is an induced subgraph of \( G \) it holds that \( \{u, v\} \in E_i \). Since \( \ell(u) > \ell(v) \) and \( \{u, v\} \in E_i \), we have \( \pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v)) \). Since \( \ell(u) > \ell(v) \), we must show that \( \tilde{\pi}_i^{-1}(\ell(u)) < \tilde{\pi}_i^{-1}(\ell(v)) \).

For better readability, we put \( f := (\theta_i^k)^{-1} \). By construction, \( \hat{\sigma}_i = \ell \circ \sigma_i \circ \theta_i^k \) and thus, \( \tilde{\pi}_i^{-1} = f \circ \sigma_i^{-1} \circ f^{-1} \). Now let us examine \( \tilde{\pi}_i^{-1}(\ell(x)) \) for some \( x \in M \). First, \( \ell^{-1}(\ell(x)) = \ell^{-1}(\ell(\ell(x))) = \ell(x) \).

Hence, \( \sigma_i^{-1}(\ell(\ell(x))) \) reduces to \( \sigma_i^{-1}(\ell(x)) = \pi_i^{-1}(\ell(x)) \).

Hence, \( \tilde{\pi}_i^{-1}(\ell(u)) = f(\pi_i^{-1}(\ell(u))) \) and \( \tilde{\pi}_i^{-1}(\ell(v)) = f(\pi_i^{-1}(\ell(v))) \). Note that the order \( \pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v)) \) is, by construction, preserved under \( f \), that is,
\[
\tilde{\pi}_i^{-1}(\ell(u)) = f(\pi_i^{-1}(\ell(u))) < f(\pi_i^{-1}(\ell(v))) = \tilde{\pi}_i^{-1}(\ell(v)).
\]
If \( \{u, v\} \in E_i \) we can use an analogous argumentation to conclude that \( \tilde{\pi}_i^{-1}(\ell(u)) < \tilde{\pi}_i^{-1}(\ell(v)) \).

In summary, we have shown that \( \{u, v\} \in E_i \) with \( \ell(u) > \ell(v) \) if and only if \( \tilde{\pi}_i^{-1}(\ell(u)) < \tilde{\pi}_i^{-1}(\ell(v)) \).

Hence, \( (G[M], \hat{\ell}) \) is a complete \( |I| \)-edge-colored permutation graph for \( \hat{\sigma}_i, i \in I \).

\[\Box\]

4.2 A Characterization in Terms of Quotient Graphs

In this section, we study the quotient graphs of strong modules of complete edge-colored permutation graphs. In particular, we show that a graph is a complete edge-colored permutation graph if and only if the quotient graph of each strong module is a complete edge-colored permutation graph. In other words, we show the equivalence of Theorem 4.1 (i) and Theorem 4.1 (iii).

**Proposition 4.4.** Let \( G \) be a complete edge-colored permutation graph. Then, the quotient graph \( G[M]/\mathbb{P}_{\max}(M) \) of each strong module \( M \) in the modular decomposition of \( G \) is a complete edge-colored permutation graph.

**Proof.** Let \( G = (V, E) \) be a complete edge-colored permutation graph and let \( M \subseteq V \) be a strong module of \( G \).

If \( M = \{v\} \) for some \( v \in V \), then \( G[M]/\mathbb{P}_{\max}(M) \) is the single vertex graph that is, by definition, a complete edge-colored permutation graph.

Now assume that \( |M| > 1 \) and thus, \( \mathbb{P}_{\max}(M) = \{M_1, \ldots, M_n\}, m > 1 \). Let \( N \subseteq M \) be such that \( N \) contains exactly one vertex from every strong module \( M_i \in \mathbb{P}_{\max}(M) \).

Since \( G[N] \) is an induced subgraph of the complete edge-colored permutation graph \( G \), it follows by Proposition 4.3 that \( G[N] \) is a complete edge-colored permutation graph. By Remark 2, \( G[N] \) is isomorphic to \( G[M]/\mathbb{P}_{\max}(M) \). Hence, the statement immediately follows for the quotient graph \( G[M]/\mathbb{P}_{\max}(M) \).

\[\Box\]
Figure 6: A complete 3-edge-colored graph $G$ without a specified labeling (Panel (a)). The modular decomposition tree $T_G$ of $G$ together with the quotient graph for each strong module (Panel (b)); and $G$ together with labeling $\ell_G$ such that $(G, \ell_G)$ is complete 3-edge-colored permutation graph of $\pi_1 = (1, 3, 5, 2, 4, 7, 6, 8)$, $\pi_2 = (6, 7, 1, 8, 4, 2, 5, 3)$, and $\pi_3 = (2, 3, 4, 5, 8, 1, 6, 7)$ (Panel (c)).

Let us denote the colors green-lined, blue-dotted and red-dashed by 1, 2 and 3, respectively. Using the strict total order on $V$, we must show that $\prec$ is trichotomous (and thus, well-defined) and $\prec$ is uniquely defined. By construction there exist (unique) distinct strong modules $M_u^v$, $M_v^u \in P_{\text{max}}(M')$ with $u \in M_u^v$ and $v \in M_v^u$. Note that $M_u^v$ and $M_v^u$ are the child modules of $M_u$ in $T_G$ that have $\{u\}$ and $\{v\}$ as descendant, respectively. Since $T_G$ is a tree, it follows for all distinct $u, v, w \in V$ that $1 \leq |\{M_u^v, M_v^w, M_w^u\}| \leq 2$.

Definition 4.5. Let $G = (V, E)$ be a complete edge-colored graph. Moreover, for all $M \in \mathbb{M}_{\text{str}}(G)$ let $\ell^M$ denote a labeling of the quotient graph $G[M]/P_{\text{max}}(M)$. Then, $\prec$ denotes the binary relation on $V$ that satisfies for all distinct $u, v \in V$

$$u \prec v \iff \ell_{M_u^v}(M_u^v) < \ell_{M_v^u}(M_v^u).$$

Moreover, let $\ell_\prec$ be the map $\ell_\prec: V \rightarrow [1:|V|]$ that satisfies for all distinct $u, v \in V$

$$\ell_\prec(u) < \ell_\prec(v) \iff u \prec v.$$

See Figure 6 for an illustrative example of $\prec$ and $\ell_\prec$.

Lemma 4.6. Let $G$ be a complete edge-colored graph with vertex set $V$. Then, the binary relation $\prec$ is a strict total order on $V$ and $\ell_\prec$ a labeling of $G$.

Proof. Let $G$ be a complete edge-colored graph with vertex set $V$. Furthermore, let $\ell^M$ be a labeling of the quotient graph $G[M]/P_{\text{max}}(M)$ for each $M \in \mathbb{M}_{\text{str}}(G)$. To prove that $\prec$ is a strict total order on $V$ we must show that $\prec$ satisfies (R1) and (R2), i.e., $\prec$ is trichotomous (and thus, well-defined) and transitive.

We first verify that $\prec$ is trichotomous. Since there are no child modules of $M_u^v = \{u\}$ in $T_G$, we never have $u \prec u$. Thus, let $u, v \in V$ be distinct. Recall that the strong modules $M_u^v, M_v^u,$ and $M_v^u$ always exist and that they are unique for $u, v$. Since $\ell^M_{M_u^v}$ is a bijective map into $[1:|P_{\text{max}}(M_u^v)|]$, 

|
we have either $\ell^{M^{uv}}(M^{u,v}_i) < \ell^{M^{uv}}(M^{u,v}_j)$ or $\ell^{M^{uv}}(M^{u,v}_i) > \ell^{M^{uv}}(M^{u,v}_j)$. By construction, we either have $u \prec v$ or $v \prec u$. In summary, $\prec$ is trichotomous.

We continue to show that $\prec$ is transitive. To this end, suppose that $u \prec v$ and $v \prec w$ for some $u, v, w \in V$. By trichotomy of $\prec$, the vertices $u, v, w$ are distinct.

Since $1 \leq \{(M^{u,v}, M^{v,w}, M^{w,u})\} \leq 2$, we have to consider the following four cases: 1) $M^{u,v} = M^{v,w} = M^{w,u}$, 2) $M^{u,v} = M^{v,w} \neq M^{w,u}$, 3) $M^{u,v} = M^{v,w} \neq M^{w,u}$, and 4) $M^{u,v} = M^{v,w} \neq M^{w,u}$.

1) If $M := M^{u,v} = M^{v,w} = M^{w,u}$, then $M_u, M_v$, and $M_w$ are distinct strong modules of $P_{\max}(M)$.

Then, $u \prec v$ and $v \prec w$ imply $\ell^M(M_u) < \ell^M(M_v) < \ell^M(M_w)$ and thus, we conclude $u \prec w$.

2) If $M^{u,v} = M^{v,w} \neq M^{w,u}$, then $M^{u,v} = M^{v,w}$. Since $v \prec w$, we have $\ell^{M^{u,v}}(M^{u,v}) = \ell^{M^{v,w}}(M^{v,w}) < \ell^{M^{w,u}}(M^{w,u})$ and thus, $u \prec w$.

3) Verified by similar arguments as in Case 2).

4) We show that this case cannot occur. If $M^{u,v} = M^{v,w} \neq M^{w,u}$, then $M^{u,v} = M^{w,u}$ and thus, we have $\ell^{M^{u,v}}(M^{u,v}) = \ell^{M^{v,w}}(M^{v,w})$. Together with $u \prec v$ and $v \prec w$, we conclude $\ell^{M^{u,v}}(M^{u,v}) < \ell^{M^{v,w}}(M^{v,w}) = \ell^{M^{w,u}}(M^{w,u})$: a contradiction.

Hence, $u \prec v$ and $v \prec w$ always implies $u \prec w$. Therefore, $\prec$ is transitive. Since $\prec$ is trichotomous and transitive, the relation $\prec$ is a strict total order on $V$.

Since $\prec$ is a strict total order on $V$, we can immediately conclude that the map $\ell_\prec$ is a labeling of $G$.

We now define a second type of relation that we use to construct the permutations for the complete edge-colored permutation graph $(G, \ell_\prec)$.

**Definition 4.7.** Let $G$ be a complete $k$-edge-colored graph with vertex set $V$ and some labeling $\ell$. For all $i \in [1: k]$, we denote with $\prec_i$ the binary relation on $V$ that satisfies for all distinct $u, v \in V$ with $\ell(u) > \ell(v)$:

$u \prec_i v$ if edge $\{M^{u,v}_i, M^{u,v}_j\}$ has color $i$ in $G[M^{u,v}]/P_{\max}(M^{u,v})$ and $v \prec_i u$ otherwise.

**Lemma 4.8.** Let $G$ be a complete $k$-edge-colored graph with vertex set $V$. Suppose that $(G[M]/P_{\max}(M), \ell^M)$ is a complete edge-colored permutation graph for each strong module $M \in M_+(G)$ together with some labeling $\ell^M$. Then, for all $i \in [1: k]$, the binary relation $\prec_i$ defined on the labeling $\ell_\prec$ of $G$ is a strict total order on $V$.

**Proof.** In what follows, put $\ell := \ell_\prec$ and let $i \in [1: k]$ be arbitrarily chosen.

We first verify that $\prec_i$ is well-defined and trichotomous. Clearly, $G[M^{u,v}]$ consists of the single vertex $u$ only and by construction, we never set $u \prec_i u$. Now, let $u, v \in V$ be two distinct vertices. Observe that the respective strong modules $M^{u,v}_i, M^{v,u}_i$ and $M^{u,v}_i$ are unique. This, the fact that $G[M^{u,v}]/P_{\max}(M^{u,v})$ is complete, and each edge of $G[M^{u,v}]/P_{\max}(M^{u,v})$ has a unique color implies that $\prec_i$ is well-defined.

Without loss of generality suppose that $u \ell(u) > \ell(v)$. The edge $\{M^{u,v}_i, M^{v,u}_i\}$ in $G[M^{u,v}]/P_{\max}(M^{u,v})$ has a unique color $j \in [1: k]$. If $j = i$, then we have $u \prec_i v$. If, otherwise, $j \neq i$, then we have $v \prec_i u$. This together with the fact that $M^{u,v}_i, M^{v,u}_i$ and $M^{u,v}_i$ are unique for $u$ and $v$ implies that $\prec_i$ is trichotomous.

We continue to show that $\prec_i$ is transitive. Suppose that $u \prec_i v$ and $v \prec_i w$ for some $u, v, w \in V$.

By trichotomy of $\prec_i$, the vertices $u, v, w$ are distinct.

Since $1 \leq \{(M^{u,v}, M^{v,w}, M^{w,u})\} \leq 2$, we have to consider the following four cases: 1) $M^{u,v} = M^{v,w} = M^{w,u}$, 2) $M^{u,v} = M^{v,w} \neq M^{w,u}$, 3) $M^{u,v} = M^{v,w} \neq M^{w,u}$, and 4) $M^{u,v} = M^{v,w} \neq M^{w,u}$.

1) If $M := M^{u,v} = M^{v,w} = M^{w,u}$, then $M_u, M_v$, and $M_w$ are distinct strong modules of $P_{\max}(M)$.

Hence, the quotient graph $G[M]/P_{\max}(M)$ contains the vertices $M_u, M_v$, and $M_w$. Since $G[M]/P_{\max}(M)$ is complete (by Remark 2), the edges $\{M_u, M_v\}, \{M_u, M_w\}$, and $\{M_v, M_w\}$ exist in $G[M]/P_{\max}(M)$ and have each a unique color. For simplicity, we denote the color of edge $\{M_u, M_v\}$ by $c_{uv}$, $c_{uw}$, and $c_{vw}$, respectively. By assumption $G[M]/P_{\max}(M)$ is a complete edge-colored permutation graph. Thus, we can apply Lemma 3.4 to conclude that $G[M]/P_{\max}(M)$ cannot contain a rainbow triangle, i.e., $1 \leq |\{c_{uv}, c_{uw}, c_{vw}\}| \leq 2$.

In what follows, we refer to the latter condition as Condition "2C". We now consider six different cases with respect to the total order of $\ell(u), \ell(v)$, and $\ell(w)$.

Case $\ell(u) < \ell(v) < \ell(w)$: In this case, $u \prec_i v$ implies $c_{uw} \in [1: k] \setminus \{i\}$ and $v \prec_i w$ implies $c_{uw} \in [1: k] \setminus \{i\}$. Suppose first that $c_{uw} \neq c_{uv}$. By 2C, we have $c_{uw} \in c_{uv}$. Hence, $c_{uw} \neq i$ which together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

Now consider $w := c_{uw} = c_{uv}$. By Lemma 3.9, the case $c_{uw} \neq c_{uv}$ cannot occur. Consequently, we have $c_{uw} = i$ which together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.
Case $\ell(u) < \ell(v) < \ell(w)$: In this case, $u \prec_i v$ and $v \prec_i w$ imply $c_{uw} = c_{vw} = i$. By Lemma 3.9, $c_{uw} = i$. This together with $\ell(w) < \ell(u)$ implies $u \prec_i w$.

Case $\ell(u) < \ell(w) < \ell(v)$: In this case, $u \prec_i v$ implies $c_{uw} \in [1: k] \setminus \{i\}$ and $v \prec_i w$ implies $c_{vw} = i$.

By $2C$, we have $c_{uw} \in \{c_{vw}, i\}$. However, by Lemma 3.9, we must have $c_{uw} \neq i$ which together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

Case $\ell(v) < \ell(u) < \ell(w)$: In this case, $u \prec_i v$ implies $c_{uv} = i$ and $v \prec_i w$ implies $c_{vw} \in [1: k] \setminus \{i\}$.

By $2C$, we have $c_{uv} \in \{c_{vw}, i\}$. Again, by Lemma 3.9, $c_{uv} \neq c_{vw}$ and thus, $c_{uw} = i$. This together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

Case $\ell(u) < \ell(v) < \ell(w)$: In this case, $u \prec_i v$ implies $c_{uv} \in [1: k] \setminus \{i\}$ and $v \prec_i w$ implies $c_{vw} = i$.

By $2C$, we have $c_{uv} \in \{c_{vw}, i\}$. Again, by Lemma 3.9, we have $c_{uv} \neq c_{vw}$ and thus, $c_{uw} = i$. This together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

In summary, $u \prec_i v$ and $v \prec_i w$ implies $u \prec_i w$ in each of the six cases.

2) If $M = M^{u,v,w} = M^{u,w} \neq M^{u,v}$, then $M_u = M_v \neq M_w$. This, in particular, implies $e := \{M_u, M_v\} = \{M_u, M_w\} \in E(G[M]/P_{\text{max}}(M))$. Since $\ell^M$ is a bijective map into $[1: |P_{\text{max}}(M)|]$, we have either $\ell^M(M_u) = \ell^M(M_v) > \ell^M(M_w)$ or $\ell^M(M_u) = \ell^M(M_w) < \ell^M(M_v)$. If $\ell^M(M_u) = \ell^M(M_v) > \ell^M(M_w)$, then, by construction, we have $\ell(u) > \ell(v)$ and $\ell(v) > \ell(w)$. Since $\ell(u) > \ell(v)$ and $v \prec_e w$, the edge $e$ has color $i$. This together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

If $\ell^M(M_u) = \ell^M(M_v) < \ell^M(M_w)$, then, by construction, we have $\ell(u) < \ell(v)$ and $\ell(v) < \ell(w)$. Since $\ell(v) < \ell(w)$ and $v \prec_e w$, the edge $e$ has color $j \in [1: k] \setminus \{i\}$. This together with $\ell(u) < \ell(w)$ implies $u \prec_i w$.

3) By similar arguments as in Case 2.

4) We show that the case $M^{u,v} = M^{u,w} \neq M^{u,w}$ cannot occur. If $M := M^{u,v} = M^{u,w} \neq M^{u,w}$, then $M_u = M_v \neq M_w$. This, in particular, implies $e := \{M_u, M_v\} \in E(G[M]/P_{\text{max}}(M))$. Since $\ell^M$ is a bijective map into $[1: |P_{\text{max}}(M)|]$, we have either $\ell^M(M_u) = \ell^M(M_v) > \ell^M(M_w)$ or $\ell^M(M_u) = \ell^M(M_w) < \ell^M(M_v)$. If $\ell^M(M_u) = \ell^M(M_v) > \ell^M(M_w)$, then, by construction, we have $\ell(u) > \ell(v)$ and $\ell(v) > \ell(w)$. Since $\ell(u) > \ell(v)$ and $v \prec_e w$, the edge $e$ must have color $i$. However, $\ell(u) < \ell(w)$ implies that the edge $e$ has color $j \in [1: k] \setminus \{i\}$; a contradiction.

If $\ell^M(M_u) = \ell^M(M_w) < \ell^M(M_v)$, then, by construction, we have $\ell(u) < \ell(v)$ and $\ell(v) < \ell(w)$. Since $\ell(v) < \ell(v)$ and $v \prec_e w$, the edge $e$ must have color $j \in [1: k] \setminus \{i\}$. However, $\ell(u) < \ell(w)$ and $v \prec_e w$ implies that the edge $e$ must have color $i$; a contradiction.

In each case, $u \prec_i v$ and $v \prec_i w$ implies $u \prec_i w$, i.e., $\prec_i$ is transitive. This, together with the trichotomy of $\prec_i$ implies that $\prec_i$ is a strict total order on $V$. Since $i \in [1: k]$ was chosen arbitrarily, all $\prec_1, \ldots, \prec_k$ are strict total orders on $V$.

We are now in the position to show that the converse of Proposition 4.4 is valid as well.

**Proposition 4.9.** Let $G$ be a complete edge-colored graph such that the quotient graph $G[M]/P_{\text{max}}(M)$ of each strong module $M \in \mathcal{M}_{tr}(G)$ is a complete edge-colored permutation graph. Then $G$ is a complete edge-colored permutation graph.

**Proof.** Let $G = (V, E_1, \ldots, E_k)$ be a complete $k$-edge-colored graph. By assumption there exists for each strong module $M \in \mathcal{M}_{tr}(G)$ a labeling $\ell^M$ of the quotient graph $G[M]/P_{\text{max}}(M)$ such that $(G[M]/P_{\text{max}}(M), \ell^M)$ is a complete edge-colored permutation graph.

Now, let $\prec$ and $\ell$ be as specified in Definition 4.5. By Lemma 4.6 the map $\ell := \ell_{\prec}$ is a labeling of $G$. Moreover, by Lemma 4.8 we have for all $i \in [1: k]$ that the binary relation $\prec_i$ defined on the labeling $\ell$ of $G$ is a strict total order on $V$. This immediately implies that the map $\pi_i: [1: |V|] \rightarrow [1: |V|]$ that satisfies for all distinct $u, v \in V$

$$\pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v)) \Leftrightarrow u \prec_i v$$

is a permutation of length $|V|$ for all $i \in [1: k]$.

It remains show that $(G, \ell)$ is a complete edge-colored permutation graph of $\pi_1, \ldots, \pi_k$. For this purpose, let $u, v$ be two distinct vertices of $G = (V, E_1, \ldots, E_k)$ and $i \in [1: k]$. Without loss of generality let $\ell(u) < \ell(v)$. If $(u, v) \in E_i$, then, by definition, $(M^{u,v}, M^{u,w})$ must have color $i$ in $G[M^{u,v}]/P_{\text{max}}(M^{u,v})$. Therefore, we have $u \prec_i v$ which implies $\pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v))$. By analogous
reasoning, $\pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v))$ and $\ell(u) > \ell(v)$ implies $\{u, v\} \in E_i$. In summary, we showed for all distinct $u, v \in V$ and all $i \in [1:k]$ that

$$\{u, v\} \in E_i \iff \ell(u) > \ell(v) \text{ and } \pi_i^{-1}(\ell(u)) < \pi_i^{-1}(\ell(v)).$$

Consequently, $(G, \ell)$ is a complete edge-colored permutation graph of $\pi_1, \ldots, \pi_k$. \hfill $\square$

### 4.3 A Characterization in Terms of Prime Modules

Intriguingly, we can strengthen the results of Section 4.2 by considering strong prime modules only. We show here that complete edge-colored permutation graphs are completely characterized by the fact that the quotient graph of each strong prime module is a complete edge-colored permutation graph. Hence, we show the equivalence of Theorem 4.1 (i) and Theorem 4.1 (iv).

**Proposition 4.10.** Let $G$ be a complete edge-colored graph. Assume that the quotient graph of each strong prime module in the modular decomposition of $G$ is a complete edge-colored permutation graph. Then $G$ is a complete edge-colored permutation graph.

**Proof.** First note that the quotient graph $G[M]/P_{\max}(M)$ of a strong module $M$ of $G$ is always complete (Remark 2). The quotient graph $G[M]/P_{\max}(M)$ of a series module $M$ of $G$ is complete and 1-edge-colored. In this case, $G[M]/P_{\max}(M)$ is the complete 1-edge-colored permutation graph of the permutation $\tau$. By assumption, the quotient graph of each strong prime module of $G$ is a complete edge-colored permutation graph. In summary, the quotient graph $G[M]/P_{\max}(M)$ of every strong module $M \in \mathbb{M}_{str}(G)$ is a complete edge-colored permutation graph. Hence, we can apply Proposition 4.9 to conclude that $G$ is a complete edge-colored permutation graph. \hfill $\square$

**Corollary 4.11.** Let $G$ be a complete edge-colored permutation graph. Then the quotient graph of each strong prime module in the modular decomposition of $G$ is a complete edge-colored permutation graph. In particular, for every strong prime module on at least three vertices the quotient graph must be 2-edge-colored.

**Proof.** Let $G$ be a complete edge-colored permutation graph. By Proposition 4.4, the quotient graph of each strong module of $G$ and thus, of each strong prime module of $G$, is a complete edge-colored permutation graph.

We continue to show that the quotient graph of each strong prime module on at least three vertices must be 2-edge-colored. Assume, for contradiction, that $G$ contains a strong prime module $M$ with $|M| \geq 3$ whose quotient graph $G[M]/P_{\max}(M)$ is not a 2-edge-colored graph. By Remark 2, $G[M]/P_{\max}(M)$ is complete. However, this implies that $G[M]/P_{\max}(M)$ must be $k$-edge-colored for some $k \geq 3$, as otherwise, it would be a simple permutation graph for $\tau$. Since $G[M]/P_{\max}(M)$ is primitive (cf. Lemma 2.6), we can apply Lemma 2.7 to conclude that $G[M]/P_{\max}(M)$ contains a rainbow triangle. It is easy to verify that this implies that $G$ must contain a rainbow triangle; a contradiction to Lemma 3.10 and the assumption that $G$ is a complete edge-colored permutation graph. \hfill $\square$

### 4.4 A Characterization via Monochromatic Subgraphs

In this section, we show that a graph is a complete edge-colored permutation graph if and only if it does not contain a rainbow triangle and each monochromatic subgraph of it is a simple permutation graph. Hence, we show the equivalence of Theorem 4.1 (i) and Theorem 4.1 (v). To this end, we first need the following result:

**Lemma 4.12.** Let $G$ be a complete edge-colored graph. Assume that $G_{\ell_i}$ (together with some labeling $\ell_i$) is a simple permutation graph for all $i \in [1:k]$ and the quotient graph of each strong prime module of $G$ on at least three vertices is a 2-edge-colored graph. Then $G$ is a complete edge-colored permutation graph.

**Proof.** Let $G = (V, E_1, \ldots, E_k)$ be a complete $k$-edge-colored graph. We proceed to show that all conditions of Proposition 4.9 are satisfied in order to show that $G$ is a complete edge-colored permutation graph. To this end, let us examine the quotient graph of a strong module $M$ of $G$.

If $M$ is series, then it is a complete edge-colored permutation graph. Thus, the quotient graph $G[M]/P_{\max}(M)$ is a complete 1-edge-colored graph and thus, a complete 1-edge-colored permutation graph.

If $M$ is prime, then, by Remark 2, $G[M]/P_{\max}(M)$ is complete. If $|M| \leq 2$, then $|M| = 1$ as otherwise, $M$ would be series. Hence, $G[M]/P_{\max}(M)$ is the single vertex graph and thus, a complete edge-colored permutation graph. Now suppose that $|M| \geq 3$. By assumption, $G[M]/P_{\max}(M)$ is 2-edge-colored. Hence, for the two distinct colors $i$ and $j$ used to color $G[M]/P_{\max}(M)$, there
must be edges with color $i$ and $j$, respectively. By assumption $G_{i_1}$ and $G_{i_2}$ are simple permutation graphs. By Lemma 3.3, $G_{i_1}[N]$ and $G_{i_2}[N]$ are simple permutation graphs for all $N \subseteq V$. Let $N \subseteq V$ be chosen such that $N$ contains exactly one vertex from each child module of $M$. By Remark 2, $G[M]/\text{P}_{\text{max}}(M)$ is isomorphic to the induced subgraph $G[N]$ of $G$. In particular, the $h$-th monochromatic subgraph $G[M]/\text{P}_{\text{max}}(M)_h$ of $G[M]/\text{P}_{\text{max}}(M)$ must be isomorphic to $G_{i_h}[N]$, $h \in \{i, j\}$. As argued above, $G_{i_1}[N]$ and $G_{i_2}[N]$ are simple permutation graphs and therefore, $G[M]/\text{P}_{\text{max}}(M)_{i_1}$ and $G[M]/\text{P}_{\text{max}}(M)_{i_2}$ are both simple permutation graphs. Since $G[M]/\text{P}_{\text{max}}(M)$ is 2-edge-colored, we can apply Corollary 3.8 to conclude that $G[M]/\text{P}_{\text{max}}(M)$ is a complete edge-colored permutation graph.

For each strong module $M$ of $G$, we showed that the quotient graph $G[M]/\text{P}_{\text{max}}(M)$ is a complete edge-colored permutation graph. Then, the claim follows by Proposition 4.9. □

The following two propositions prove the equivalence of Items (i) and (v) of Theorem 4.1.

**Proposition 4.13.** Let $(G, \ell)$ be a complete edge-colored permutation graph. Then $G$ does not contain a rainbow triangle and each monochromatic subgraph of $G$ is together with $\ell$ a simple permutation graph.

**Proof.** Let $(G, \ell)$ be a complete $k$-edge-colored permutation graph for the permutations $\pi_1, \ldots, \pi_k \in \mathcal{P}[V]$. By Lemma 3.10, $G$ cannot contain a rainbow triangle. By definition, $(G_{i_1}, \ell)$ is be a simple permutation graph of $\pi_i$. □

**Proposition 4.14.** Let $G$ be a complete edge-colored graph. If every monochromatic subgraph of $G$ is a simple permutation graph and $G$ does not contain a rainbow triangle, then $G$ is a complete edge-colored permutation graph.

**Proof.** The statement is trivially true if $G \cong K_1$. Thus, let $G = (V, E_1, \ldots, E_k)$ be a complete $k$-edge-colored graph with $|V| \geq 2$.

By contraposition, we show that if $G$ is not a complete edge-colored permutation graph, then there exists a monochromatic subgraph of $G$ that is not a simple permutation graph or $G$ contains a rainbow triangle.

Assume that $G$ is not a complete edge-colored permutation graph. Since $G$ contains at least two vertices, we have that $k \geq 1$. If $k = 1$, then $G$ would be a complete $1$-edge-colored permutation graph for $\pi_1 = \pi$. Thus, we assume that $k \geq 2$. Since $G$ is not a complete edge-colored permutation graph, for all labelings $\ell: V \to [1:|V|]$ of $G$ there are no permutations $\pi_1, \ldots, \pi_k \in \mathcal{P}[V]$ such that $(G, \ell)$ is a complete $k$-edge-colored permutation graph of $\pi_1, \ldots, \pi_k$. By definition, for all labelings $\ell$, $G$ contains always a monochromatic subgraph $G_{\ell_i} = (V, E_{\ell_i})$ such that $(G_{\ell_i}, \ell_i)$ is not a simple permutation graph of $\pi_i$. Note, this does not imply that $G_{\ell_i}$ is not a simple permutation graph at all, since we might find a different labeling $\ell_i$ of $G_{\ell_i}$ such that $(G_{\ell_i}, \ell_i)$ is a simple permutation graph for some $\pi_i$.

If, however, for all possible labelings $\ell_i$, the labeled graph $(G_{\ell_i}, \ell_i)$ is not a simple permutation graph, then we are done, since we found a monochromatic subgraph of $G$ that is not a simple permutation graph at all.

Thus, we are left with the situation that $G$ is not a complete edge-colored permutation graph and for all $i \in [1:k]$ there is some labeling $\ell_i$ such that $(G_{\ell_i}, \ell_i)$ is a simple permutation graph. Note, that this implies that $k > 2$ as otherwise, $E(G) = E(G_{\ell_1}) \cup E(G_{\ell_2})$ and, by Corollary 3.8, $G$ would be a complete $2$-edge-colored permutation graph. It remains to show that $G$ contains a rainbow triangle.

We consider here the two cases: either Case (i) for none of the strong prime modules $M$ in $G$ the quotient graph $G[M]/\text{P}_{\text{max}}(M)$ has at least three colors or Case (ii) there exists a strong prime module $M$ in $G$ such that $G[M]/\text{P}_{\text{max}}(M)$ has at least three colors.

Let us first consider Case (i). Then, for every strong prime module $M$ of $G$ on at least three vertices the quotient graph $G[M]/\text{P}_{\text{max}}(M)$ is 2-edge-colored (as otherwise $M$ would be series). By assumption, for all $i \in [1:k]$ the monochromatic subgraph $G_{\ell_i}$ is a simple permutation graph. Hence, we can apply Lemma 4.12 to conclude that $G$ is a complete edge-colored permutation graph; a contradiction. Therefore, Case (i) cannot occur and we are left with Case (ii).

For Case (ii), let $M$ be a strong prime module such that $G[M]/\text{P}_{\text{max}}(M)$ has at least three colors. Note that this implies that $|M| \neq 1$. By Lemma 2.6, $G[M]/\text{P}_{\text{max}}(M)$ is primitive. By Lemma 2.7, $G[M]/\text{P}_{\text{max}}(M)$ contains a rainbow triangle induced by some vertices $M', M'', M''' \in \text{P}_{\text{max}}(M)$. By definition, there are vertices $x \in M'$, $y \in M''$, and $z \in M'''$ that induce the rainbow triangle $\Delta_{xyz}$ in $G[M]$ and thus, in $G$.

In summary, we have shown that if the complete edge-colored graph $G$ is not a complete edge-colored permutation graph, then there exists a monochromatic subgraph of $G$ that is not a simple permutation graph or $G$ contains a rainbow triangle, which completes the proof. □
4.5 Proof of Theorem 4.1

Theorem 4.1 is a direct consequence of the results described in this section:

*Proof of Theorem 4.1.* Lemma 4.2 and Proposition 4.3 provide the equivalence between Items (i) and (ii). Proposition 4.4 and Proposition 4.9 provide the equivalence between Items (i) and (iii). Moreover, by Proposition 4.10 and Corollary 4.11, Items (i) and (iv) are equivalent. Finally, by Proposition 4.13 and Proposition 4.14, Items (i) and (v) are equivalent.

5 Recognition of Complete Edge-Colored Permutation Graphs

In this section we show that complete edge-colored permutation graphs can be recognized in $O(|V|^2)$-time. Moreover, we also show that (in the affirmative case) the labeling $\ell$ and the permutations $\pi_1, \ldots, \pi_k$ such that $(G, \ell)$ is a complete $k$-edge-colored permutation graph of $\pi_1, \ldots, \pi_k$ can be constructed in polynomial time.

Simple permutation graphs $G = (V, E)$ can be recognized in $O(|V| + |E|)$-time [19, 53]. These algorithms also construct a corresponding permutation $\pi$ and labeling $\ell$ of $G$.

Let us now consider a complete $k$-edge-colored graph $G$. As the number of colors $k$ is always bounded by the number of edges of $G$, we immediately obtain a polynomial-time algorithm to recognize complete edge-colored permutation graphs (cf. Theorem 4.1 (v)):

1) For all $i \in [1:k]$, we check whether the monochromatic subgraph $G_{i\ell}$ is a simple permutation graph using the algorithms described in [19, 53].

2) In addition, we verify that $G$ does not contain a rainbow triangle.

Such an algorithm allows us to verify that $G$ is a complete edge-colored permutation graph or not. However, we are also interested in computing a labeling $\ell$ of $G$ and the corresponding underlying permutations $\pi_1, \ldots, \pi_k$ such that $(G, \ell)$ is a complete edge-colored permutation graph of $\pi_1, \ldots, \pi_k$.

The proofs in Section 4.2, however, are constructive and thus provide a recognition algorithm that explicitly computes the required labeling $\ell$ as well as the underlying permutations of a complete edge-colored permutation graph. More precisely, we may use the following procedure:

1) Compute the modular decomposition of $G$ and, thus, obtain all strong modules of $G$. If there is a strong prime module $M$ of $G$ with $|M| \geq 3$ whose quotient graph $G[M]/\mathcal{P}_{\text{max}}(M)$ contains more than 2 colors, then $G$ cannot be a complete edge-colored permutation graph (cf. Theorem 4.1 (iv)). In this case, the algorithm stops and returns false. Otherwise, each of these quotient graphs on at least three vertices is 2-edge-colored and we can proceed with Step (2).

2) For all strong prime modules $M$ on at least three vertices, check whether one of the two monochromatic subgraphs of $G[M]/\mathcal{P}_{\text{max}}(M)$ is a simple permutation graph and, thus, obtain a labeling $\ell^M$ of the quotient graph $G[M]/\mathcal{P}_{\text{max}}(M)$.

3) For all remaining strong modules $M$ (those that are either prime modules with $|M| = 1$ or series modules) choose an arbitrarily labeling $\ell^M$ of the quotient graph $G[M]/\mathcal{P}_{\text{max}}(M)$.

4) Construct $\prec$ and $\ell_\prec$ as specified in Definition 4.5.

5) For all $i \in [1:k]$, construct $\prec_i$ as specified in Definition 4.7 and permutation $\pi_i$ as defined in the proof of Proposition 4.9.

This algorithm correctly determines whether $G$ is a complete edge-colored permutation graph or not (it essentially checks Item (iv) in Theorem 4.1) and correctly determines the labeling $\ell_\prec$ of $G$ (cf. Lemma 4.6) and the corresponding permutations $\pi_1, \ldots, \pi_k$ (cf. proof of Proposition 4.9).

**Theorem 5.1.** Let $G = (V, E)$ be a complete $k$-edge-colored graph. Then, it can be verified in $O(|V|^2)$-time whether $G$ is a complete edge-colored permutation graph or not. In the affirmative case, a labeling $\ell$ and permutations $\pi_1, \ldots, \pi_k$ such that $(G, \ell)$ is a complete edge-colored permutation graph for $\pi_1, \ldots, \pi_k$ can be constructed in $O(|V|^2)$-time.

*Proof.* The correctness of the algorithm follows from the discussion above. Thus, let us examine its running time.

In Step (1) we have to compute the modular decomposition tree $T$ of $G$, which can be done in $O(|V|^2)$-time [24, 44]. While computing $T$ we can also store the information which module is associated to which vertex in $T$ within the same time complexity. At the same time, we can also store the information which
inclusion-minimal strong module $M^x,y$ of $G$ contains $x$ and $y$ as well as the child modules $M^x,y$ and $M^x,y$ that contain $x$ and $y$, respectively, for all $x, y \in V$ in $O(|V|^2)$-time.

We then have to compute the quotient graph $G[M]/P_{\max}(M)$ for all strong modules $M$. The quotient graph of an individual strong module $M$ can be computed as follows: The module $M$ is associated with a vertex $u$ in $T$ that has children $u_1, \ldots, u_r$, $r \geq 2$. Each child $u_i$ is associated with a strong module $M_i$ and hence, $P_{\max}(M) = \{M_1, \ldots, M_r\}$. We now take from each module $M_i$ one vertex $v_i$ of $G$, which can be done in $r$ constant time steps. We finally determine the colored edges $\{v_i, v_j\}$, $1 \leq i < j \leq r$ which can be done in $r(r-1)/2$ constant time steps. Note, $r = \deg_T(u) - 1$ and thus, the quotient graph of a strong module $M$ associated with $u$ in $T$ can be constructed in $\deg_T(u) - 1 + (\deg_T(u) - 1)(\deg_T(u) - 2)/2$ steps, each of which requires only a constant number of operations.

To compute the quotient graphs of all modules we simply process each vertex in $T$ and its children, and construct the quotient graphs as outlined above and obtain

$$
\sum_{u \in V(T)} (\deg_T(u) - 1 + (\deg_T(u) - 1)(\deg_T(u) - 2)/2) \leq \sum_{u \in V(T)} \deg_T(u) + \deg_T(u)^2
$$

$$
\leq \sum_{u \in V(T)} \deg_T(u) + |L(T)| \sum_{u \in V(T)} \deg_T(u) = (1 + |L(T)|) \sum_{u \in V(T)} \deg_T(u) = 2(1 + |L(T)|)|E(T)| = 2(1 + |L(T)|)(|V(T)| - 1) \in O(|V|^2),
$$

where $L(T)$ denotes the leaves of $T$. In the last step we make use of [47, Lemma 1] that states that $V(T) \in O(|L(T)|)$ and the fact that $L(T) = V$.

We can now proceed to examine the time complexity of Step (2). For each strong module $M$ (associated with vertex $u$ in $T$) on at least three vertices we need to check whether one of the two monochromatic subgraphs $H$ of $G[M]/P_{\max}(M)$ is a simple permutation graph. Let $r = \deg(u) - 1$. Hence, $|V(H)| = r$ and $|E(H)| \leq r(r-1)/2$. Now, we can apply an $O(|V(H)| + |E(H)|)$-time algorithm as provided e.g. in [19, 53] to check if $H$ is a simple permutation graph or not. In the affirmative case, we also obtain a labeling $\ell^H$ of $H$ and, thus, a labeling of $G[M]/P_{\max}(M)$. By analogous arguments as in Equation (1), we can observe that applying these algorithms for each strong prime module $M$ with $|M| \geq 3$ can be done in $O(|V|^3)$-time.

In Step (3), we choose for all remaining strong modules $M$ (associated with some vertex $u$ in $T$) an arbitrarily labeling $\ell^M$ of the quotient graph $G[M]/P_{\max}(M)$, which can be done in $\deg(u) - 1$ constant time steps. Again making use of $V(T) \in O(|L(T)|)$ and $L(T) = V$ we obtain

$$
\sum_{u \in V(T)} \deg_T(u) - 1 \leq \sum_{u \in V(T)} \deg_T(u) = 2|E(T)| = 2(|V(T)| - 1) \in O(|V|).
$$

In Step (4) we use the precomputed $M^x,y$ to construct $\prec$ and $\ell_\prec$ as specified in Definition 4.5. This task takes $O(|V|^2)$-time.

Finally, in Step (5), we construct $\prec_i$ as specified in Definition 4.7 for all $i \in [1:k]$. This can be done in $O(|V|^2)$-time, since for each pair $\{x, y\}$ with $x, y \in V$ we can check in constant time whether the edge $M^x,y$ has color $i$ in $G[M^x,y]/P_{\max}(M^x,y)$ (since $M^x,y$, $M^x,y$, and $G[M^x,y]/P_{\max}(M^x,y)$ have been precomputed). The permutation $\pi_i$ as defined in the proof of Proposition 4.9 simply refers to a re-ordering of the vertices in $V$. Therefore, it can be performed in $O(|V|^2)$-time.

Hence, the overall running time is $O(|V|^2)$. 

6 Connections between Complete Edge-Colored Permutation Graphs and other Graph Classes

In this section, we show the relationship between complete edge-colored permutation graphs and other graph classes. The first simple observation is that complete $k$-edge-colored permutation graphs do not contain rainbow triangles (cf. Lemma 3.10). This is precisely the definition of Gallai colorings [38, 39, 51]. We therefore have

Corollary 6.1. The coloring of a complete $k$-edge-colored permutation graph is a Gallai coloring with $k$ colors.

We continue by showing that the class of complete edge-colored permutation graphs is a generalization of so-called symbolic ultrametrics [10], which have important applications in computational biology [43, 45, 47]. They are defined as follows:
Definition 6.2 (Symbolic Ultrametric). Let $X$ be a non-empty finite set, let $M = [1:k]$ for some $k \in \mathbb{N}$. A surjective map $\delta : X \times X \to M$ is a symbolic ultrametric if it satisfies the following conditions:

1. $\delta(x, y) = \delta(y, x)$ for all $x, y \in X$;
2. $|\{\delta(x, y), \delta(x, z), \delta(y, z)\}| \leq 2$ for all $x, y, z \in X$; and
3. there exists no subset $\{x, y, u, v\} \subseteq \binom{X}{4}$ such that $\delta(x, y) = \delta(y, u) = \delta(u, v) = \delta(v, y) = \delta(x, v) = \delta(x, u)$.

The set $M$ is often used to represent so-called evolutionary events [43] and can be replaced by any finite set of symbols or the set $[1:k]$ as done here. A complete edge-colored graph $G_\delta$ with vertex set $X$ can readily be obtained from $\delta$, by putting the color $\delta(x, y) = \delta(y, x)$ on the edge $\{x, y\}$ for all distinct $x, y \in X$. Since $\delta$ is surjective, we obtain the complete $k$-edge-colored graph $G_\delta = (X, E_1, \ldots, E_k)$ as the graph representation of $\delta$.

Several characterizations of symbolic ultrametrics are known [10, 43], including one that relates them to cographs, i.e., graphs that do not contain induced paths on four vertices [18].

Proposition 6.3 ([43], Proposition 3). A complete edge-colored graph is the graph representation of a symbolic ultrametric if and only if it does not contain a rainbow triangle and each monochromatic subgraph is a cograph.

Not every simple permutation graph is a cograph. For example, a path on four vertices represents the permutation graph of $(3, 1, 4, 2)$ but not a cograph. Intriguingly, all cographs are simple permutation graphs [13]. This, together with Proposition 6.3 and Theorem 4.1 (v) immediately implies:

Corollary 6.4. The graph representation $G_\delta$ of a symbolic ultrametric $\delta$ is a complete edge-colored permutation graph.

Separable permutations can be represented by a so-called separating tree that reflects the structure of the permutation [13] and can be characterized in terms of so-called permutation patterns. For our purposes, the following result is of interest:

Proposition 6.5 ([13]). Let $G$ be a simple permutation graph of $\pi$. Then, $\pi$ is separable if and only if $G$ is a cograph.

It is easy to verify that every induced subgraph of a cograph is again a cograph. This, together with Proposition 6.5, Theorem 4.1, and Proposition 6.3 implies:

Corollary 6.6. The following statements are equivalent:

i) A complete edge-colored graph $G$ is the graph representation of a symbolic ultrametric.

ii) $G$ does not contain a rainbow triangle and every monochromatic subgraph of $G$ is the simple permutation graph of a separable permutation.

Moreover, if $G$ is the graph representation of a symbolic ultrametric, then every monochromatic subgraph of each induced subgraph is the simple permutation graph of a separable permutation.

7 Summary and Outlook

In this paper, we characterized complete edge-colored permutation graphs, that is, complete edge-colored graphs that can be decomposed into permutation graphs. These graphs form a hereditary class, i.e., every induced subgraph of a complete edge-colored permutation graph is again a complete edge-colored permutation graph, possibly with fewer colors. Moreover, we showed that complete edge-colored permutation graphs are equivalent to those complete edge-colored graphs, i.e., symmetric 2-structures, that contain no rainbow triangle and whose monochromatic subgraphs are simple permutation graphs.

Further characterizations in terms of the modular decomposition of complete edge-colored graphs are provided. In particular, complete edge-colored permutation graphs are characterized by the structure of the quotient graphs of their strong (prime) modules. Moreover, we showed that complete edge-colored permutation graphs $G = (V, E)$ can be recognized in $O(|V|^2)$-time and that both the labeling and the underlying permutations can be constructed within the same time complexity.

As a by-product, we observed that the edge-coloring of a complete edge-colored permutation graph is always a Gallai coloring. In addition, we have shown the close relationship of edge-colored permutation graphs to symbolic ultrametrics, separable permutations, and cographs. In particular, the class of edge-colored graphs representing symbolic ultrametrics is strictly contained in the class of complete edge-colored permutation graphs.
There are many open problems that are of immediate interest for future research. The first obvious open problem is to generalize the results presented here to $k$-edge-colored graphs that are not necessarily complete, e.g., the graph illustrated in Figure 4 (a). Second, we may consider the converse problem of the one studied in this paper: Consider simple unlabeled permutation graphs $G_1, \ldots, G_k$. The question arises whether there is a complete edge-colored permutation graph $G$ such that every monochromatic subgraph $G_i$ of $G$ is isomorphic to $G_i$, $1 \leq i \leq k$. Third, one may also ask whether one can add colored edges to a given (not necessarily complete) edge-colored graph such that the resulting graph is a complete edge-colored permutation graph.

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