q-ANALOGS OF t-WISE BALANCED DESIGNS FROM BOREL SUBGROUPS

MICHAEL BRAUN

Abstract. A \( t-(n, K, \lambda; q) \) design, also called the \( q \)-analog of a \( t \)-wise balanced design, is a set \( B \) of subspaces with dimensions contained in \( K \) of the \( n \)-dimensional vector space \( \mathbb{F}_q^n \) over the finite field with \( q \) elements such that each \( t \)-subspace of \( \mathbb{F}_q^n \) is contained in exactly \( \lambda \) elements of \( B \). In this paper we give a construction of an infinite series of nontrivial \( t-(n, K, \lambda; q) \) designs with \( |K| = 2 \) for all dimensions \( t \geq 1 \) and all prime powers \( q \) admitting the standard Borel subgroup as group of automorphisms. Furthermore, replacing \( q = 1 \) gives an ordinary \( t \)-wise balanced design defined on sets.

1. Introduction

In the following let \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) denote the set of \( k \)-subspaces of the \( n \)-dimensional vector space \( \mathbb{F}_q^n \) over the finite field \( \mathbb{F}_q \) with \( q \) elements. The expression \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^i - q} \) counts the number of \( k \)-subspaces of \( \mathbb{F}_q^n \) and it is called the \( q \)-binomial coefficient.

A \( t-(n, K, \lambda; q) \) design, also called the \( q \)-analog of a \( t \)-wise balanced design or \( t \)-wise balanced design over \( \mathbb{F}_q \), is a set \( B \) of subspaces of \( \mathbb{F}_q^n \) with dimensions contained in \( K \) such that each \( t \)-subspace of \( \mathbb{F}_q^n \) is contained in exactly \( \lambda \) members of the set \( B \).

The set \( B = \cup_{k \in K} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) is already a \( t-(n, K, \lambda_{\text{max}}; q) \) design, the so-called trivial \( t \)-wise balanced design, where \( \lambda_{\text{max}} = \sum_{k \in K} \left[ \begin{array}{c} n-r \\ k-r \end{array} \right] q^r \).

If \( K = \{ k \} \) is a one element set \( B \) is simply called the \( q \)-analog of a \( t \)-design and write \( t-(n, k, \lambda; q) \) instead of \( t-(n, \{ k \}, \lambda; q) \) to indicate the parameters of the design. So far only a few results have been published on \( t-(n, k, \lambda; q) \) designs. Explicit constructions \[1, 5, 6, 10, 11, 12, 13]\.
are known for \( t = 1, 2, 3 \) whereas the existence of these objects for all \( t \geq 1 \) was recently shown [6].

In this paper we concentrate on the case \(|K| = 2\) and present an explicit construction of a series of nontrivial \( t-(n, K, \lambda; q) \) designs for all \( t \geq 1 \) and all prime powers \( q \).

2. Groups of Automorphisms

The general linear group \( \operatorname{GL}(\mathbb{F}_q^n) \) of the vector space \( \mathbb{F}_q^n \), whose elements are represented by \( n \times n \)-matrices, acts on \( \mathbb{F}_q^k \) by left multiplication \( \alpha S := \{ \alpha x \mid x \in S \} \).

An element \( \alpha \in \operatorname{GL}(\mathbb{F}_q^n) \) is called an automorphism of a \( t-(n, K, \lambda; q) \) design \( \mathcal{B} \) if and only if \( \mathcal{B} = \alpha \mathcal{B} := \{ \alpha S \mid S \in \mathcal{B} \} \). The set of all automorphisms of a design forms a group, called the automorphism group of the \( t \)-wise balanced design. Every subgroup of the automorphism group of a \( t \)-wise balanced design is denoted as a group of automorphisms of the \( t \)-wise balanced design.

A construction approach derives from the Kramer and Mesner construction of ordinary \( t \)-designs [8]:

A subgroup \( G \) of \( \operatorname{GL}(\mathbb{F}_q^n) \) induces an equivalence relation on the set of \( k \)-subspaces of \( \mathbb{F}_q^n \) by defining \( S \simeq_G S' :\iff \exists \alpha \in G : \alpha S = S' \). The corresponding equivalence class of \( S \) is called the \( G \)-orbit on \( S \) and it is denoted by \( G(S) := \{ \alpha S \mid \alpha \in G \} \). The stabilizer of \( S \) is abbreviated by \( G_S := \{ \alpha \in G \mid \alpha S = S \} \).

Now, a \( t-(n, K, \lambda; q) \) design \( \mathcal{B} \) admits a subgroup \( G \) of the general linear \( \operatorname{GL}(\mathbb{F}_q^n) \) as a group of automorphisms if and only if \( \mathcal{B} \) consists of \( G \)-orbits on \( \bigcup_{k \in K} \mathbb{F}_q^k \).

The \( G \)-incidence matrix \( A_{t,k}^G \) is defined to be the matrix whose rows and columns are indexed by the \( G \)-orbits on the set of \( t \)- and \( k \)-subspaces of \( \mathbb{F}_q^n \), respectively. The entry indexed by the orbit \( G(T) \) on \( \mathbb{F}_q^t \) and by the orbit \( G(S) \) on \( \mathbb{F}_q^k \) is defined to be the number \( |\{ S' \in G(S) \mid T \subseteq S' \}| \). Note that each row of \( A_{t,k}^G \) adds up to the constant value \( \binom{n-t}{k-t} q^{t(k-t)} \).
If $A^G_{t,K} := \big|_{k \in K} A^G_{t,k}$ denotes the concatenation of all $G$-incidence matrices $A^G_{t,k}$ for all $k \in K$ we obtain the following result:

**Theorem 1.** A $t$-$(\n, \K, \lambda; q)$ design admitting $G \leq \text{GL}(\mathbb{F}_q^n)$ as a group of automorphisms exists if and only if there is a $0/1$-column vector $x$ satisfying $A^G_{t,K} x = \lambda \mathbf{1}$, where $\mathbf{1}$ denotes the all-one column vector. The vector $x$ represents the corresponding selection of $G$-orbits on the set of subspaces $\bigcup_{k \in K} [\mathbb{F}_q^n]_k$.

For instance, Tables 1 and 2 show a list of $q$-analsogs of $t$-wise balanced designs over finite fields we constructed with the Kramer-Mesner approach using a computer search. All prescribed groups of automorphisms we used are subgroups of the normalizer of a Singer cycle, which is generated by a Singer cycle $\sigma$ of order $q^n - 1$ and the Frobenius automorphism $\phi$ of order $n$. This group and its subgroups have already been used for the successful construction of designs over finite fields (see [4, 5]).

Note, that we were also able to find a large set of $t$-$(\n, \K, \lambda; q)$ designs which is a set of disjoint $t$-$(\n, \K, \lambda; q)$ designs such that their union cover the whole set $\bigcup_{k \in K} [\mathbb{F}_q^n]_k$ for some set of parameters:

- three disjoint 2-$(7, \{3,4\}, 62; 2)$ designs
- two disjoint 2-$(7, \{3,4\}, 93; 2)$ designs
- two disjoint 2-$(8, \{3,4\}, 357; 2)$ designs

### 3. Echelon Equivalence

In the following we consider subspaces as column spaces. Let $S$ be a $k$-subspace of $\mathbb{F}_q^n$. A matrix $\Gamma$ having $n$ rows, $k$ columns, and entries in $\mathbb{F}_q$ is called a generator matrix of $S$ if and only if the columns of $\Gamma$ yield a base of $S$. There are several generator matrices for the same $k$-subspace $S$ but using elementary Gaussian transformations of the columns yields a uniquely determined generator matrix, the canonical generator matrix, $\Gamma_C(S)$ of the subspace $S$, having the structure shown in Table 3.
Table 1. $t$-$(n, K, \lambda; q)$ designs with $|K| = 2$

| $t$-$(n, K, \lambda; q)$ | $\lambda_{\text{max}}$; group; values for $\lambda$ |
|--------------------------|-----------------------------------|
| $2$-(6, \{3, 4\}, $\lambda$; 2) | 50; $\langle \sigma \rangle$; 5, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24 |
| $2$-(7, \{3, 4\}, $\lambda$; 2) | 186; $\langle \sigma \rangle$; 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94 |
| $2$-(8, \{3, 4\}, $\lambda$; 2) | 714; $\langle \sigma, \phi \rangle$; 7, 21, 28, 35, 42, 47, 56, 63, 70, 77, 84, 91, 98, 105, 112, 119, 126, 133, 140, 147, 154, 161, 168, 175, 182, 189, 196, 203, 210, 217, 224, 231, 238, 245, 252, 259, 266, 273, 280, 287, 294, 301, 308, 315, 322, 329, 336, 343, 350, 357 |
| $2$-(9, \{3, 4\}, $\lambda$; 2) | 2794; $\langle \sigma, \phi \rangle$; 21, 22, 42, 43, 63, 64, 84, 85, 105, 106, 126, 127, 147, 148, 168, 169, 189, 190, 210, 211, 231, 232, 252, 253, 273, 274, 294, 295, 315, 316, 336, 337, 357, 358, 378, 379, 399, 400, 420, 421, 441, 442, 462, 463, 483, 484, 504, 505, 525, 526, 546, 547, 567, 568, 588, 589, 609, 610, 630, 631, 651, 652, 672, 673, 693, 694, 714, 715, 735, 736, 756, 757, 777, 778, 798, 799, 819, 820, 840, 841, 861, 862, 882, 883, 903, 904, 924, 925, 945, 946, 966, 967, 987, 988, 1008, 1009, 1029, 1030, 1050, 1051, 1071, 1072, 1092, 1093, 1113, 1114, 1134, 1135, 1155, 1156, 1176, 1177, 1197, 1198, 1218, 1219, 1239, 1240, 1260, 1261, 1281, 1282, 1302, 1303, 1323, 1324, 1344, 1345, 1365, 1366, 1387 |
Table 2. $t$-$\left(n, K, \lambda; q\right)$ designs with $|K| = 3$

| $t$-$\left(n, K, \lambda; q\right)$ | $\lambda_{\max}$; group; values for $\lambda$ |
|----------------------------------|-----------------------------------------------|
| $2$-$\left(6, \{3, 4, 5\}, \lambda; 2\right)$ | $65$; $\langle \sigma \rangle$; $23, 30$ |
| $2$-$\left(7, \{3, 4, 5\}, \lambda; 2\right)$ | $341$; $\langle \sigma, \phi \rangle$; $71, 78, 82, 85, 86, 89, 92, 93, 96, 99, 103, 106, 107, 113, 115, 119, 120, 122, 124, 126, 127, 128, 129, 130, 131, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170 |
| $3$-$\left(8, \{4, 5, 6\}, \lambda; 2\right)$ | $341$; $\langle \sigma, \phi \rangle$; $156, 166$ |

Hereby the stars in this matrix represent elements in $\mathbb{F}_q$ and the row numbers where new steps commence are called *base rows indices*.

Now, we introduce an equivalence relation on the set $[\mathbb{F}_q^n]_k$ which we call *Echelon equivalence*:

Two $k$-subspaces $S$ and $S'$ are defined to be *Echelon equivalent*, abbreviated by $S \simeq_E S'$, if and only if the base row indices of the canonical generator matrices $\Gamma_C(S)$ and $\Gamma_C(S')$ are the same.

For instance, the two 3-subspaces of $\mathbb{F}_5^6$ generated by the following generator matrices are Echelon equivalent:

\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 4
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
4 & 0 & 3 \\
1 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

Table 4 shows all Echelon equivalence classes of $\mathbb{F}_5^6$. 
Table 3. The structure of an Echelon matrix

If \( i \) is the number of stars in the matrix then \( q^i \) is the number of \( k \)-subspaces in the corresponding equivalence class. The maximum number of stars is \((n - k)k\).

Replacing the stars by arbitrary elements of \( \mathbb{F}_q \) we get a canonical transversal of the Echelon equivalence classes. In order to determine the number of different Echelon equivalence classes we have to calculate the number of possibilities to choose \( k \) base row indices in a matrix with
$q$-analog of $t$-wise balanced designs from borel subgroups

Table 4. The Echelon forms of $\mathbb{F}_q^6$

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | * | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | * | * | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$n$ rows, which is $\binom{n}{k}$. Hence, each Echelon equivalence class corresponds to a unique $k$-subset of the set of possible row indices $\{1, \ldots, n\}$.

More formally, let $e_j$ denote the unit vector having exactly one entry 1 in the $j$th position and 0 in the remaining positions. Representatives of the Echelon equivalence classes are given by generator matrices consisting of unit vectors $\Gamma(\pi) := [e_{\pi_1} \ldots e_{\pi_k}]$ where $\pi = \{\pi_1, \ldots, \pi_k\}$ is
a $k$-subset of $\{1, \ldots, n\}$. The corresponding subspace is denoted by $E(\pi) := \langle e_{\pi_1}, \ldots, e_{\pi_k} \rangle$.

For example, the Echelon equivalence class containing $\Gamma(\pi) = [e_2 | e_3 | e_6] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to the subset $\pi = \{2, 3, 6\}$.

**4. Parabolic and Borel Subgroups**

A flag $[S_1, \ldots, S_r]$ of $\mathbb{F}_q^n$ of length $r$ is a sequence of subspaces of $\mathbb{F}_q^n$ satisfying $\{0\} \subsetneq S_1 \subsetneq \ldots \subsetneq S_r \subseteq \mathbb{F}_q^n$. The maximum length $r$ of a flag is $n$. In this case the sequence of subspaces satisfies $\dim(S_i) = i$ and $S_n = \mathbb{F}_q^n$. Flags of length $n$ are called complete. The complete flag $[E_1, \ldots, E_n]$ consisting of all standard subspaces $E_i := \langle e_1, \ldots, e_i \rangle$ is called the standard flag.

We can also define a group action of $\text{GL}(\mathbb{F}_q^n)$ on the set of flags of $\mathbb{F}_q^n$ by setting $\alpha[S_1, \ldots, S_r] := [\alpha S_1, \ldots, \alpha S_r]$ for all $\alpha \in \text{GL}(\mathbb{F}_q^n)$ and all flags $[S_1, \ldots, S_r]$.

The stabilizer of a flag $[S_1, \ldots, S_r]$ is also called parabolic subgroup $[\mathbb{I}]$ and it is the intersection of all single stabilizers of $S_i$:

$$
\text{GL}(\mathbb{F}_q^n)|_{S_1, \ldots, S_r} = \bigcap_{i=1}^r \text{GL}(\mathbb{F}_q^n)_{S_i}
$$

In order to determine the stabilizer of the standard flag we first have to consider the stabilizer of the standard subspace $E_1$ which consists of all matrices of the form

$$
\begin{bmatrix}
A & B \\
0 & C
\end{bmatrix}
$$

where $A \in \text{GL}(\mathbb{F}_q^i)$, $B \in \mathbb{F}_q^{i \times n-i}$ and $C \in \text{GL}(\mathbb{F}_q^{n-i})$. The intersection of the stabilizers of all $E_i$ yields the stabilizer of the standard flag which
is the set of all nonsingular upper triangular matrices—the so called standard Borel subgroup \[2\]:

\[ B(F_n^q) := \text{GL}(F_n^q)[E_1, ..., E_n] = \bigcap_{i=1}^{n} \text{GL}(F_n^q)E_i \]

Now we establish the following connection between Echelon equivalence and the standard Borel subgroup \[3\]:

The multiplication of \( \alpha \in B(F_n^q) \) to a \( k \times n \) matrix \( \Gamma \) from the left, is equivalent to a series of elementary Gaussian transformations of the rows of \( \Gamma \) such that rows will be multiplied by a nonzero finite field element or such that a multiple of a first row will be added to a second row above to the first row. Applying an arbitrary element \( \alpha \in B(F_n^q) \) to a canonic generator matrix \( \Gamma_C(S) \) from the left, \( \alpha \Gamma_C(S) = \Gamma \), yields a generator matrix \( \Gamma \) of a subspace \( S' \) whose base row indices are the same as in \( \Gamma_C(S) \). Moreover, the Echelon equivalence and the equivalence induced by the group action of the Borel subgroup are the same:

\[ S \cong E \ S' \iff S \cong_{B(F_n^q)} S' \]

This also yields that representatives of the \( B(F_n^q) \)-orbits on the set of \( k \)-subspaces of \( F_n^q \) can be obtained from \( k \)-subsets of \( \{1, \ldots, n\} \):

**Corollary 1.** Representatives of the \( B(F_n^q) \)-orbits on the set \( [F_n^q]_k \) are given by the subspaces \( E(\pi) \) and their generator matrices \( \Gamma(\pi) \) where \( \pi \) is a \( k \)-subset of \( \{1, \ldots, n\} \).

In the following we consider some properties of the \( B(F_n^q) \)-incidence matrices. The following property is immediate from the Echelon equivalence classes:

**Lemma 1.** The entry \( a_{\tau,\pi} \) of the incidence matrix \( A_{t,k}^{B(F_n^q)} \) whose row is indexed by the \( B(F_n^q) \)-orbit on \( E(\tau) \) and whose column is indexed by the \( B(F_n^q) \)-orbit on \( E(\pi) \) is nonzero if and only if \( \tau \subseteq \pi \). In this case, if \( a_{\tau,\pi} \) is nonzero it is a power of \( q \).

**Corollary 2.** If we substitute \( q = 1 \) in \( A_{t,k}^{B(F_n^q)} \) we obtain the incidence matrix between all \( t \)-subsets and \( k \)-subsets of \( \{1, \ldots, n\} \).
Considering the orbits of the standard Borel subgroup on subspaces or the Echelon equivalence classes, respectively, might be understood as the proper form of “\(q\)-analogization”: If the number of stars in an Echelon equivalence class representative—as depicted in Table I—is \(i\) each star can be substituted by a finite field element which yields the cardinality \(q^i\) of this particular class. Setting \(q = 1\) means, that we replace all stars by 0’s and each Echelon class contains only one element. Hence, the Echelon equivalence classes themselves can be considered as subsets. Furthermore, the incidence matrix \(A_{t,k}^{B(F_q^n)}\) becomes the incidence matrix between subsets for \(q = 1\).

**Lemma 2.** The matrix \(A_{t,\{t+1\}+t+2}^{B(F_q^n)} = A_{t,t+1}^{B(F_q^n)} \mid A_{t,t+2}^{B(F_q^n)}\) has the following form:

\[
A_{t,\{t+1\}+t+2}^{B(F_q^n)} = \begin{bmatrix}
q^{n-t-1} & \cdots & \cdots & * & \cdots & * \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & * & \cdots & * \\
A_{t,\{t+1\}+t+2}^{B(F_q^n-1)} & \cdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & * & \cdots & * \\
A_{t-1,t}^{B(F_q^n-1)} & \cdots & \cdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

The first block of columns is indexed by all \((t+1)\)-subsets of \(\{1, \ldots, n\}\) containing 6, the second block of columns is indexed by all \((t+2)\)-subsets of \(\{1, \ldots, n-1\}\), all remaining \((t+1)\)- and \((t+2)\)-subsets of \(\{1, \ldots, n\}\) occur in the third block of columns. The first block of rows is indexed by all \(t\)-subsets of \(\{1, \ldots, n-1\}\) and the second block of rows is indexed by all \(t\)-subsets of \(\{1, \ldots, n\}\) containing 6.

**Proof.** The proof is straightforward and uses Lemma II. We only look at the block in the upper left corner. The rows correspond to the \(t\)-subsets of \(\{1, \ldots, n-1\}\) and the columns correspond to \((t+1)\)-subsets of \(\{1, \ldots, n\}\) containing the element 6 which arise by all \(t\)-subsets of \(\{1, \ldots, n-1\}\) by just adding the element 6. In order to determine the matrix entry whose row is indexed by the \(t\)-subset \(\{\pi_1, \ldots, \pi_t\}\) and whose column is indexed by the \((t+1)\)-subset \(\{\pi_1, \ldots, \pi_t, 6\}\) we
have to count the number of canonical generator matrices of the form \([e_{\pi_1} \cdots e_{\pi_t}]|v]\) where \(v\) must have the entry 1 in the last position. Since the last entry of \(v\) is 1 and since \(t\) entries are 0 due to the remaining \(t\) base row indices we have \(n - t + 1\) positions in \(v\) for which we can choose any finite field element. Finally, we get \(q^{n-t-1}\) as corresponding incidence matrix entry. \(\Box\)

5. Construction

In this section we finally describe the construction of an infinite family of

\[t-(t+4, \{t+1, t+2\}, q^3 + q^2 + q + 1; q)\]

designs as a union of certain \(B(\mathbb{F}_q^n)\)-orbits on \((t+1)\)- and \((t+2)\)-subspaces of \(\mathbb{F}_q^n\).

As selection of \(B(\mathbb{F}_q^n)\)-orbits we choose the representatives \(T\) of \(B(\mathbb{F}_q^n)\)-orbits belonging to the first two blocks of columns of \(A_{t,(t+1,t+2)}^{B(\mathbb{F}_q^n)}\) as they are given in Lemma 2, i.e. we get the following set of subspaces:

\[T := \{E(\pi \cup \{6\}) | \pi \in \left(\{1, \ldots, n-1\}\right)_{t+1} \} \cup \{E(\pi) | \pi \in \left(\{1, \ldots, n-1\}\right)_{t+2}\}\]

and the union of orbits:

\[B := \bigcup_{S \in T} B(\mathbb{F}_q^n)(S)\]

By this selection the following columns of \(A_{t,(t+1,t+2)}^{B(\mathbb{F}_q^n)}\) are chosen:

\[
M = \begin{bmatrix}
q^{n-t-1} & \cdots & \cdots & A_{t,t+2}^{B(\mathbb{F}_q^n)} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
A_{t-1,t}^{B(\mathbb{F}_q^n)} & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}
\]

The final issue is now to investigate under which conditions the set \(B\) becomes a \(t\)-wise balanced \(t-(n, \{t+1, t+2\}, \lambda; q)\) design. To answer
this question we determine the row sum of the selected matrix. For the first block of rows we get the row sum:

\[
\alpha = q^{n-t-1} + \left[ \frac{(n - 1) - t}{(t + 2) - t} \right] = q^{n-t-1} + \left[ \frac{n - t - 1}{2} \right]_q
\]

and for the second block of rows the rows add up to:

\[
\beta = \left[ \frac{(n - 1) - (t - 1)}{t - (t - 1)} \right] = \left[ \frac{n - t}{1} \right]_q
\]

It is clear that the corresponding selection of \( B(\mathbb{F}_q^n) \)-orbits and columns, respectively, becomes a \( t \)-wise balanced \( t-(n, \{t+1, t+2\}, \lambda; q) \) design if and only if both values are equal \( \alpha = \beta \). In this case we get the index:

\[
\lambda = \alpha = \beta = \left[ \frac{n - t}{1} \right]_q = q^{n-t-1} + \ldots + q^2 + q + 1
\]

It is easy to see that \( \alpha = \beta \) if and only if \( \left[ \frac{n-t-1}{1} \right]_q = \left[ \frac{n-t-1}{2} \right]_q \). From the symmetry of the \( q \)-binomial coefficient we get \( n - t - 1 = 3 \) and hence:

\[
n = t + 4
\]

This shows that \( B \) really defines a \( t-(t+4, \{t+1, t+2\}, q^3+q^2+q+1; q) \) design.

Furthermore, in \( M \) each row sum is equal to \( q^3 + q^2 + q + 1 \) which means that exactly four entries are nonzero. Hence, substituting \( q = 1 \) in the matrix \( M \) we obtain a constant row sum of 4. The subsets corresponding to the columns of \( M \) finally defines an ordinary nontrivial \( t \)-wise balanced design with parameters

\[
t-(t + 4, \{t + 1, t + 2\}, 4)
\]

for all positive integers \( t \geq 1 \).

Finally, we show an example:

We construct a 2-(6, \{3, 4\}, 15; 2) design. The set of representatives \( \mathcal{T} \) of the chosen \( B(\mathbb{F}_2^6) \)-orbits on 3- and 4-subspaces is the following one:
If the list

\[
E(\{1, 2\}), E(\{1, 3\}), E(\{1, 4\}), E(\{1, 5\}), E(\{2, 3\}), E(\{2, 4\}),
E(\{2, 5\}), E(\{3, 4\}), E(\{3, 5\}), E(\{4, 5\}), E(\{1, 6\}), E(\{2, 6\}),
E(\{3, 6\}), E(\{4, 6\}), E(\{5, 6\}),
\]

denotes the representatives of $B(\mathbb{F}_2)$-orbits on the set of 2-subspaces, we get the following incidence matrix $M$ between the orbits on 2-subspaces and the selected orbits on 3- and 4-subspaces:

\[
M = \begin{bmatrix}
8 & 1 & 2 & 4 & 8 \\
& 8 & 1 & 2 & 4 \\
& & 8 & 1 & 2 \\
& & & 8 & 1 \\
1 & 2 & 4 & 8 \\
1 & 2 & 4 & 8 \\
1 & 2 & 4 & 8 \\
1 & 2 & 4 & 8 \\
\end{bmatrix}
\]

The row sum in each row is exactly 15 which shows that the selected set of orbit representatives on 3- and 4-subspaces yields a 2-(6, \{3, 4\}, 15; 2) design.

Moreover, the given set of 3- and 4-subsets defines a 2-(6, \{3, 4\}, 4) design.
REFERENCES

[1] J. L. Alperin and R. B. Bell. Groups and Representations. Springer-Verlag, 1995.
[2] A. Borel. Groupes Linéaires Algébriques. Annals of Mathematics, 64(1):20–82, 1956.
[3] M. Braun. An Algebraic Interpretation of the $q$-Binomial Coefficients. International Electronic Journal of Algebra, 6:23–30, 2009.
[4] M. Braun, T. Etzion, P. R. J. Östergård, A. Vardy, and A. Wassermann. On the Existence of $q$-Analogs of Steiner Systems. submitted for publication.
[5] M. Braun, A. Kerber, and R. Laue. Systematic Construction of $q$-Analogs of Designs. Designs, Codes and Cryptography, 34(1):55–70, 2005.
[6] A. Fazeli, S. Lovett, and A. Vardy. Nontrivial $t$-Designs over Finite Fields Exist for All $t$. arXiv:1306.2088, 2013.
[7] T. Itoh. A New Family of 2-Designs over $GF(q)$ Admitting $SL_m(q^l)$. Geometriae Dedicata, 69:261–286, 1998.
[8] E. Kramer and D. Mesner. $t$-Designs on Hypergraphs. Discrete Mathematics, 15(3):263–296, 1976.
[9] M. Miyakawa, A. Munemasa, and S. Yoshiara. On a Class of Small 2-Designs over $GF(q)$. Journal of Combinatorial Designs, 3:61–77, 1995.
[10] H. Suzuki. 2-Designs over $GF(2^m)$. Graphs and Combinatorics, 6:293–296, 1990.
[11] H. Suzuki. 2-Designs over $GF(q)$. Graphs and Combinatorics, 8:381–389, 1992.
[12] S. Thomas. Designs over Finite Fields. Geometriae Dedicata, 24:237–242, 1987.
[13] S. Thomas. Designs and Partial Geometries over Finite Fields. Geometriae Dedicata, 63:247–253, 1996.