New types of instability and CP violation in electroweak theory

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It is known that the Schwinger mechanism in vector-like QED theory is afflicted by a logarithmic singularity for a hypothetical massless charged fermion. We extend singularity analysis to a more realistic case of the chiral electroweak theory, to show that the effective lagrangian under background gauge field at zero temperature exhibits a similar instability proportional to \( \ln(1/m_\nu^2) \) with \( m_\nu \) a small neutrino mass. Moreover, the effective lagrangian of chiral fermion loop contains CP violating pieces proportional to background gauge fields in odd powers of \( \vec{E}_Z \cdot \vec{B}_Z \) or \((\vec{E}_{W^+} \cdot \vec{B}_{W^-} + \vec{E}_{W^-} \cdot \vec{B}_{W^+})/2 \). This brings in a new source of CP violation and time-reversal symmetry violation in the standard particle theory independent of the Kobayashi-Maskawa phase of quark mass mixing matrix. The effective action in thermal equilibrium at finite temperature \( T \) is then calculated under background SU(2) \( \times \) U(1) gauge fields in the spontaneously broken phase. An even more singular power-law behavior \( \propto (m_\nu T)^{-5/2} \) is found and it contains CP violating term as well.

The case of Majorana neutrino satisfies almost all necessary conditions to generate a large lepton number asymmetry, though not necessarily convertible to a baryon asymmetry due to lower cosmic temperatures at which this may occur.

I. INTRODUCTION

Instabilities of vector-like gauge theories are well known in QCD (Quantum ChromoDynamics) and in QED (Quantum ElectroDynamics). QCD vacuum instability is understood by instanton effects in Euclidean field theory \[1\]. This led to discovery of the strong CP violation problem, giving its possible resolution by a chiral symmetry known as Peccei-Quinn symmetry \[2\]. Instability in QED is known as Schwinger mechanism \[3\] related to \( e^\pm \) pair production. Non-abelian extension of Schwinger mechanism \[4\], \[5\], \[6\] has deepened our understanding of instability: non-abelian instability is related to the magnetic property of vector gauge boson \[7\] believed to be the origin of asymptotic freedom as well.

Effects we discuss in the present work are different: it works in chiral gauge theories such as the electroweak theory. The proper-time background method used in \[3\], \[4\], \[5\] is powerful enough to be extended to the electroweak gauge field theory, as shown in the present work. Results in chiral theory can be derived from a simple transformation rule from the corresponding vector-like gauge theory.

Results of physical relevance we derive are a surprise: we find a new source of CP violation and time-reversal symmetry violation. This adds to the well-known Kobayashi-Maskawa phase of CP violation, and is more dynamical. The present work is limited to our fundamental findings and many possible applications are relegated to future works.

The rest of this paper is organized as follows.

II. THE PROPER-TIME BACKGROUND FIELD METHOD: A SUMMARY

II. THE PROPER-TIME BACKGROUND FIELD METHOD: A SUMMARY

We recapitulate the proper time formalism of QED under constant electric and magnetic background field denoted by \( \vec{E}, \vec{B} \). Heisenberg-Euler non-linear QED \[8\] along with \( e^\pm \) pair production rate is derived in this formalism.
The starting point in QED is the electron one-loop effective lagrangian (more properly lagrangian density) prior to renormalization,
\[ \mathcal{L}^{(1)}(x) = \frac{i}{2} \int_0^\infty \frac{ds}{s} e^{-m_2^2 s} \text{tr}(x U(s)|x), \]
(1)
\[ U(s) = e^{-i\mathcal{H}s}, \quad \mathcal{H} = (p_\mu - A_\mu)^2 - \frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu}. \]
(2)
\[ \mathcal{H} \] is the quadratic form of Dirac hamiltonian in the proper-time formalism. We include the gauge coupling constant \( \epsilon \) in the definition of \( A_\mu, \bar{E}, \bar{B} \) to make non-abelian generalization easier. This formula encompasses all even orders of background fields \( F^\mu\nu \) in external lines, odd powers being forbidden by the Furry’s theorem. One can extract contribution from a specific powers of fields by a functional derivative of this formula. The attractive feature is that the formula is written in terms of gauge invariant quantity, since one may use c-number \( k_\mu = p_\mu - A_\mu \).

Since there is no field operator involved in the above formula, c-number eigenvalues of matrix \( \sigma_{\mu\nu} F^{\mu\nu} \) can be used. Noting the identity,
\[ \frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu} = 2(-\mathcal{F} + i\gamma_5 \mathcal{G}), \]
(3)
\[ \mathcal{F} = \frac{1}{2} (E^2 - \bar{B}^2), \quad \mathcal{G} = \bar{E} \cdot \bar{B}, \]
(4)
one derives four eigenvalues of \( \frac{1}{2} \sigma_{\mu\nu} F^{\mu\nu} \),
\[ \pm (2(-\mathcal{F} \pm i\mathcal{G}))^{1/2}. \]
(5)
Schwinger uses one of these,
\[ X = (2(-\mathcal{F} + i\mathcal{G}))^{1/2}, \]
(6)
others being related by complex conjugation and sign reversal.

Extension of Schwinger’s U(1) gauge theory to non-abelian gauge theories has been completed by many. We mention works of \[ 4 \sim 10 \] for convenience of our subsequent discussion. In non-abelian extensions U(1) gauge invariants are generalized to
\[ \mathcal{F} = \frac{1}{2} \sum_a t_a (E_a^2 - \bar{B}_a^2), \quad \mathcal{G} = \sum_a t_a E_a \cdot \bar{B}_a, \]
(7)
with the index \( a \) running over independent gauge degrees of freedom. \( t_a \)'s are coefficients giving eigenvalues of non-abelian adjoint matrix diagonalization, as described in Appendix B. In electroweak SU(2) case they are \( \pm 1/2 \), and one can effectively take out \( t_a \) factors in the above definition. For SU(3) the problem is more complicated. See Appendix B for this.

A problem of the Schwinger variable \( X \) is coexistence of CP even and odd terms: \( \mathcal{F} \) is CP-even and \( \mathcal{G} \) is CP-odd. This makes application of this formalism to chiral theories difficult, in particular in clarifying CP properties of its effective action. We shall use other variables used in \[ 3, 6 \]. Including coupling factors and introducing non-abelian indexes, these are two variables \( a, b \) defined by
\[ a^2 (\mathcal{F}, \mathcal{G}) = \mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2}, \]
\[ b^2 (\mathcal{F}, \mathcal{G}) = -\mathcal{F} + \sqrt{\mathcal{F}^2 + \mathcal{G}^2}. \]
(8)
Both of these, \( a^2, b^2 \), are CP-even, but their square-root product \( ab = \mathcal{G} \) is CP-odd, when the square-root cut singularity is analytically continued.

The better formula of one-loop fermion contribution, after renormalization, is given by
\[ \Delta \mathcal{L}_f = -\frac{1}{8\pi^2} \int_{0+\alpha+i}^{\infty+\alpha+i} \frac{ds}{s^3} m^2 \mathcal{M}_f(a, b; s), \]
(9)
\[ \mathcal{M}_f(a, b; s) = s^2 ab \cot(sa) \coth(sb) - 1 + \frac{s^2}{3}(a^2 - b^2). \]
(10)
When this formula is restricted to the abelian case, it gives equivalent lagrangian to QED’s. This formula contains both real and imaginary parts. The imaginary part may be derived by taking \( \delta \mathcal{L}_f - (\delta \mathcal{L}_f)^* \). This difference gives an infinite discrete series of contour integrations around poles (arising from \( \cot(sa) \)) on the imaginary axis, at \( s = in\pi/a, n = 1, 2, \ldots \). The total imaginary part is \[ 6 \]
\[ \Im \Delta \mathcal{L}_f = \frac{1}{8\pi^2} ab \sum_{n=1}^{\infty} \frac{1}{n} e^{-m^2 n \pi/|a|} \coth \frac{bn\pi}{a}. \]
(11)

Despite of the explicit appearance \( ab = \mathcal{G} \) as an overall factor \( ab \cot(sa) \coth(sb) \) and apparent evenness in \( \mathcal{G} \) of \( a \) and \( b \), the function \( \mathcal{M}_f(a, b; s) \) is even in the variable \( ab = \mathcal{G} \), viewed as two-variable function, \( ab \) and \( a^2 - b^2 = \mathcal{F} \). This ensures that the effective action is CP conserving. On the other hand, the formula \( 10 \) contains both even and odd power terms of \( a^2 - b^2 \), which becomes important later in application to chiral theories. This property of \( \mathcal{G} \) even function is due to that the analytic structure of this function around the double zero \( \mathcal{G} = 0, \mathcal{F} = 0 \): there are branch cut singularities of the square root type along the imaginary axis at \( 3\mathcal{G} = 0 \sim \sim i\infty, 3\mathcal{F} = 0 \sim \sim i\infty \). We have verified by using the power series expansion of a computer software the property that \( \mathcal{F} \) odd terms appear in total odd powers of \( \mathcal{F} \) and \( \mathcal{G} \). Total even power terms contain summed products of even \( \mathcal{F} \) and even \( \mathcal{G} \) powers.

Expansion of \( \mathcal{M}_f(a, b; s) \) in power series in \( s \) is equivalent to field power expansion. The first few expansion terms are
\[ \mathcal{M}_f(a, b; s) = -\frac{1}{45}(4\mathcal{F}^2 + 7\mathcal{G}^2)s^4 \]
\[ -\frac{2}{945}\mathcal{F}(8\mathcal{F}^2 + 13\mathcal{G}^2)s^6 + O(s^8). \]
(12)
The expansion contains even \( s \) powers, and \( s^{2(2n+1)} \) terms contain odd \( \mathcal{F} \) powers, and all other power terms are even in both variables. \( s \) integration over the first
term gives the well-known Heisenberg-Euler non-linear QED lagrangian [8].

The infrared limit of massless fermion becomes of great interest in chiral theories due to small neutrino masses, and it would be instructive to work out the massless electron limit although this is a fictitious setting. It is easy to explain the logarithmic behavior of the imaginary part given by [12]. Assuming that the interchange of the limit and the sum in this equation is allowed, the large \( n \) behavior of the sum is of order \( 1/n \), which gives a divergence. On the other hand, a finite mass \( m \) gives a well-defined convergent result, implying a logarithmic singular behavior. Indeed, according to calculation in Appendix A, the massless limit gives

\[
\mathfrak{F}_f = \frac{ab}{8\pi^2} \ln \left| \frac{a|t_0}{m^2\pi} \right| ,
\]

with a factor \( t_0 \) of order unity not precisely determined. The real part of effective lagrangian exhibits the logarithmic singularity as well.

The color or flavor matrix diagonalization in the standard theory of particle physics is given in Appendix B. Non-abelian gauge boson and scalar one-loop contributions have been worked out in the literature, and we list in Appendix C functions, \( M_2, M_3 \) that correspond to the fermion \( M_f \).

III. CHIRAL SU(2) \( \times U(1) \) ELECTROWEAK THEORY

We discuss in the present work the gauge invariant effective action under non-abelian gauge field background, which automatically selects an even number of external gauge fields. This circumstance makes following the trick to work.

A. Chiral theory from vector-like theory

Chiral gauge theories are theories in which fermions have gauge coupling asymmetric in the left-handed current proportional to \( \gamma(1 - \gamma_5)/2 \) and the right-handed current \( \propto \gamma(1 + \gamma_5)/2 \). The most important case is the electroweak theory in which the SU(2) triplet gauge boson \( W \) couples to the left-handed chiral fermion and the singlet gauge boson \( B \) couples to fermions of both chirality, with different gauge coupling constants, \( g, g' \) via covariant derivatives,

\[
D_\mu = \partial_\mu - ig' Y B_\mu ,
\]

with the hypercharge \( Y \) assigned to quarks and leptons, separately. These are \( Y = Y^q = 1/6 \) for quark doublets, \( Y_L = -1/2 \) for lepton doublets, and all different numbers for right-handed \( \bar{u}_R, d_R, \bar{e}_R, e_R \) SU(2) singlets. For the moment we assume neutrinos of finite Dirac type masses.

We work out results in the broken phase, thus all fermions and gauge bosons except the photon are massive.

Charge neutral gauge field \( (Z \text{ and } \gamma) \) couplings are thus summarized as an ordinary electromagnetic coupling to photon (with \( e Q_f \) charges of fermion \( f \) and Z-boson coupling of the well-known form,

\[
Z \cdot \gamma \sqrt{g^2 + (g')^2} \left( T_3 L - \sin^2 \theta_W Q_f (L + R) \right) ,
\]

with \( T_3 = \sigma_3/2 \) SU(2) quantum numbers and \( (L, R) = (1 \pm \gamma_5)/2 \). It is important to keep in mind that each chiral component maintains the same chiral property in all orders of couplings. For convenience we call the contribution \( \propto \gamma_5 \) the axial-vector contribution, and the one without it the vector contribution.

We now derive a general replacement rule that the one-loop contribution proportional to \( \gamma_5 \) is derived from the contribution without \( \gamma_5 \). Consider numerator factor in perturbation expansion of \( 2n \) orders under gauge field background. In perturbation theory background fields are given by vector potentials, and we give input momenta to vector fields, later taken to recover the limiting constant field strength of zero momentum. Decomposition into chiral components shows that mass terms drop out, and the probability amplitude is

\[
\text{tr} A_1 \cdot \gamma \left( 1 \mp \gamma_5 \right) k_1 \gamma A_2 \cdot \gamma \left( 1 \mp \gamma_5 \right) k_2 \gamma \cdots A_{2n} \cdot \gamma \left( 1 \mp \gamma_5 \right) k_{2n} \gamma = \text{tr} A_1 \cdot \gamma k_1 \gamma A_2 \cdot \gamma k_2 \gamma \cdots A_{2n} \cdot \gamma k_{2n} \gamma \left( 1 \mp \gamma_5 \right)^2 / 2 ,
\]

where \( A_i \)'s are external gauge fields, and \( k_i \) are input momenta. This shows the amplitude relation in chiral theories such that the axial-vector contribution is related to the vector contribution,

\[
\text{tr} (x| \mathcal{H}_V^n | x) = \pm \text{tr} (x| \mathcal{H}_V^n \gamma_5 | x) ,
\]

in expansion of \( \gamma_5 \).

In the proper-time formalism the eigenvalue formula may be used to relate the axial-vector contribution to the vector-relation. To extract the axial-vector contribution, note

\[
\left( (p_\mu - A_\mu)^2 - \frac{1}{2} \sigma_{\mu\nu} F^\mu\nu \right)^n \equiv \mathcal{H}_V^n + \gamma_5 \mathcal{H}_A^n ,
\]

The axial-vector and the vector contributions contain numerator factors,

\[
\text{tr} (x| \mathcal{H}_V^n | x) = -F_V \text{tr} (x| \mathcal{H}_V^{2(n-1)} | x) ,
\]

\[
\text{tr} (x| \mathcal{H}_V^n | x) = iG_V \text{tr} (x| \mathcal{H}_V^{2(n-1)} | x) .
\]

Thus, the vector and axial-vector contributions are interchanged by the rule, \( -F_V \leftrightarrow iG_V \) in vector-like theories. Thus, two integrand functions, vector and axial-vector functions given using variables \( F, G \), are related by

\[
\mathcal{M}_{A_L}^{R}(F, G) = \mp \mathcal{M}_{V_L}^{R}(-iG, iF) ,
\]

\[
\mathcal{M}_{L}^{R}(F, G) = \mathcal{M}_{V_L}^{R}(F, G) - \mathcal{M}_{L}^{R}(-iG, iF) ,
\]

\[
\mathcal{M}_{R}^{R}(F, G) = \mathcal{M}_{V_L}^{R}(F, G) + \mathcal{M}_{R}^{R}(-iG, iF) .
\]
It becomes possible for the axial-vector contribution to contain $G$ odd terms, since the vector contribution contains $F$ odd terms. In the example given by (23), $s^6$ power terms have $F$ odd powers, hence the corresponding axial-vector contribution is
\[ 2 \mathcal{M}_A(F, G) = \frac{1}{45}(4G^2 + 7F^2)s^4 \]
\[ -i \frac{2}{945}G(8G^2 + 13F^2)s^6 + O(s^8). \] (26)
The second term $\propto G$ in this equation is CP violating (CPV). When this is integrated over $s$ variable, it gives CPV effective lagrangian of the form,
\[ i \frac{G}{630\pi^2} \frac{(8G^2 + 13F^2)}{m^8} s^2. \] (27)
Formulas here are valid for small $(F/G)/m$, CPV terms are much larger at field regions of $(F/G) = O(m^4)$. Thus, it does not imply emergence of $1/m^8$ singularity. We shall discuss the true massless limit in Subsection D.

B. Flavor dependence of gauge coupling

We shall first recover gauge coupling constants in field strength $F, G$, which was deleted so far for simplicity. These gauge constants differ depending on background gauge fields. Under $W^\pm$ backgrounds alone (namely without mixed ones such as $W^+W^-Z^0$), charged fermions contribute with the single gauge coupling constant $1/2$, multiplied by $T_3$ quantum numbers $\pm 1/2$. Since even number of fields are relevant, all charged fermions contribute with the same factor $(g/2)^2$. Gauge coupling to photon background alone is also simple: $eQ_f$ with $Q_f$ the charge of each fermion $f$ in the unit of proton charge is the relevant one.

We next work out the effective action under $Z$-boson background in the spontaneously broken phase, which is sensitive to details of flavor content. There are six types of fermions with their chiralities specified circulating loops: $\nu_L, e_L, e_R, u_L, u_R, d_L, d_R$. Their coupling factors are
\[ \nu_L : \frac{1}{2} = 0.5, \quad e_L : -\frac{1}{2} + 2\sin^2 \theta_w \sim -0.04, \] (28)
\[ e_R : \sin^2 \theta_w \sim 0.23, \quad u_L : \frac{1}{2} - \frac{2}{3}\sin^2 \theta_w \sim 0.35, \] (29)
\[ u_R : -\frac{2}{3}\sin^2 \theta_w \sim -0.15, \] (30)
\[ d_L : -\frac{1}{2} + \frac{1}{3}\sin^2 \theta_w \sim -0.42, \quad d_R : \frac{1}{3}\sin^2 \theta_w \sim 0.077. \] (31)
where numerical values are obtained with $\sin^2 \theta_w = 0.23$ (close to the experimental value). There is a wide variety of numerical range. There are two more of the second and the third generations, these contributions being characterized by their different masses. We denote these factors by $f_{L,R}$ or $f_{L,R} = g_{L,R} / \sqrt{g^2 + (g')^2}$.

Flavor dependence of CP conserving part is given by
\[ \Re \Delta L_{Z}^{CPC} = -\frac{1}{120\pi^2} \frac{F^2 - G^2}{m^4} \sum_f (g_{L}^f)^4 + (g_{R}^f)^4. \] (32)
This lagrangian is manifestly real.

C. Maximum CP violation under gauge field background

It is possible to separate CPV terms by a sort of projection, to derive
\[ \mathcal{M}^{CPV}(F, G; t) = \frac{1}{2} \mathcal{M}_A(F, G; t) - \mathcal{M}_A(F, -G; t), \] (33)
\[ \mathcal{M}^{CPV}(F, G; t) = t^2 F \times \sin \frac{tc}{\sqrt{2}} \cos \frac{td}{\sqrt{2}} \sinh \frac{t d}{\sqrt{2}} - (c \leftrightarrow d) \]
\[ \left( \cos \frac{tc}{\sqrt{2}} \sinh \frac{td}{\sqrt{2}} - i \sin \frac{tc}{\sqrt{2}} \cosh \frac{td}{\sqrt{2}} \right) (c \rightarrow d) \]
\[ + \frac{t^2}{2} (c^2 - d^2), \] (34)
\[ c^2(F, G) = -G + \sqrt{F^2 + G^2}, \] (35)
\[ d^2(F, G) = G + \sqrt{F^2 + G^2}. \] (36)
In these formulas gauge coupling factors are included in $F, G$, and they can be readily recovered. Renormalization subtraction is made so that the renormalized lagrangian contains no CPV term proportional to $G$. When expanded in powers of the integration variable $t$, it gives the result,
\[ \mathcal{M}^{CPV}(F, G; t) = -\frac{i}{12} (e^4 - d^4) t^4 + O(t^6), \] (37)
\[ \Delta L^{CPV}(F, G; t) = -i \frac{G \sqrt{F^2 + G^2}}{24m^2}. \] (38)

The contributions we calculated so far arises from the principal part integral around $t = 0$. A more important contribution to CPV effective action may arise from elsewhere, from summed small half-circle integrals around pole terms. The contour integral parallel to the positive real axis may be deformed to the line along the imaginary axis, $0 \sim -i\infty$. This deformation picks up infinitely many contour integrations around poles at $t = e^{-i\pi/4}n\pi/(c, d), n = 1, 2, \ldots$. This can be seen more clearly by an equivalent integrand formula,
\[ \mathcal{M}^{CPV}(F, G; t) = -t^2 G + \frac{i t^2}{2} F \left( \cot(tc e^{i\pi/4}) \coth(td e^{i\pi/4}) - \cot(td e^{i\pi/4}) \coth(tc e^{i\pi/4}) \right). \] (39)

Summing up all pole terms, one arrives at the CPV ef-
effective action,

$$\Delta \mathcal{L}_{cp} = -\frac{F}{8\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \times \left( \exp[-\frac{m^2n\pi}{\sqrt{2\epsilon}}(1 - i)] \coth \frac{dn\pi}{\epsilon} - \exp[-\frac{m^2n\pi}{\sqrt{2|d|}}(1 - i)] \right).$$

For definiteness we shall assume a positive $\mathcal{G}$, which implies $d(F, \mathcal{G}) > c(F, \mathcal{G})$. One can use in the above formula $\coth x = 1 + 2e^{-2x}/(1 - e^{-2x})$ with $x = n\pi d/c$ or $x = n\pi c/d$. Let us assume $d/c \gg 1$ which leads to

$$\Delta \mathcal{L}_{cp} \approx -\frac{F}{8\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \times \left( \exp[-\frac{m^2n\pi}{\sqrt{2\epsilon}}(1 - i)] - \frac{2c}{n\pi d} \exp[-\frac{m^2n\pi}{\sqrt{2|d|}}(1 - i)] \right).$$

The power series here can be analytically calculated, to give

$$\Delta \mathcal{L}_{cp} \approx -\frac{1}{24\pi} \frac{F^2}{\mathcal{G}} + \frac{F}{8\pi^2} \ln \frac{m^2\pi(1 - i) + \sqrt{\mathcal{G}}}{m^2\pi(1 - i)}.$$ 

Strictly, this contains $\mathcal{G}$ even terms, which should be dropped. In the small mass limit the second term dominates:

$$\Delta \mathcal{L}_{cp} \approx \frac{F}{16\pi^2} \ln \frac{F^2}{m^4\mathcal{G}}.$$ 

In the large mass limit the first term $\sim -\frac{F^2}{24\pi\mathcal{G}}$ dominates.

The presence of inverse $\mathcal{G}$—powers in these results is annoying, and it is better to use the direct integral formula along the real $t$ axis of $\mathcal{G}$, in which we encounter no singularity.

### D. Light neutrino one-loop contribution

From results of the preceding subsections, the neutrino one-loop effective action occurs under $Z$-field background. Its relevant squared gauge coupling is $(g^2 + (g')^2)/4$.

As to the nature of neutrino mass types there are two possibilities, of Dirac type and of Majorana type. We assume that light Majorana neutrinos are generated by the seesaw mechanism [18] due to the mixing of Dirac type Higgs coupling with heavy right-handed neutral Majorana leptons, necessarily SU(2) $\times$ U(1) gauge singlet. Following Appendix D, one can deal with the Majorana type without much difficulty. Calculations are the same for both types of neutrino masses, and the difference appears in the lepton number. As is well known, the Majorana neutrino has an indefinite lepton number, or rather one can assign its lepton number $+1$. The energy shift and the decay rate related to the real and the imaginary parts of effective action depend on the absolute value of neutrino masses and not on its mass type.

We focus on a field value region larger than $m_\nu^2/\epsilon$ with $\epsilon = \sqrt{4\pi\alpha}$. Numerically, this is the region of field strength,

$$\sqrt{|\mathcal{F}|}, \sqrt{|\mathcal{G}|} \gg \frac{m_\nu^2}{\epsilon} \sim 170 \text{ mVcm}^{-1}(\frac{m_\nu}{\text{meV}})^2.$$ (44)

The relevant mathematical limit is then $m_\nu \to 0$, and one derives, following the calculation in Appendix A that led to (78),

$$\exists \mathcal{M}_\nu(F_Z, \mathcal{G}; s) \approx \frac{g^2 + (g')^2}{4} G_Z \frac{a_Z}{8\pi^2} \ln \frac{|a_Z| t_0}{m_\nu^2 \pi},$$ (45)

$$a^2_Z = F_Z + \sqrt{F^2_Z + G^2_Z},$$ (46)

with $t_0$ of order unity dimensionless number. Three neutrino species (mass eigenstates) add with the same sign.

The real part of massive fermion loop is given by

$$\Re \Delta \mathcal{L}_f = \frac{ab}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n} \left( \coth \frac{a n\pi}{|b|} I_1 \left( \frac{m^2n\pi x}{|b|} \right) - \coth \frac{b n\pi}{a} I_2 \left( \frac{m^2n\pi x}{|a|} \right) \right).$$ (47)

$$I_1(\mu_n) = \int_0^\infty dx \frac{x}{x^2 + 1} e^{-\mu_n x}, \quad \mu_n = \frac{m^2n\pi x}{|b|},$$ (48)

$$I_2(\mu_n') = \int_0^\infty dx \frac{x}{x^2 + 1} \cos(\mu_n' x), \quad \mu_n' = \frac{m^2n\pi x}{|a|}.$$ (49)

The dominant term in the massless limit is

$$\Re \Delta \mathcal{L}_f \approx \frac{ab}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n} \left( I_1 \left( \frac{m^2n\pi x}{|b|} \right) - I_2 \left( \frac{m^2n\pi x}{|a|} \right) \right).$$ (50)

Hence, discussion of the massless limit $\mu \to 0$ depends on the limiting behaver of two integrals, $I_1(\mu)$.

We attempted to derive the massless limit of the real part of effective lagrangian using analytic method similar to the one in Appendix A, but failed, presumably under a bad control of interchangeable operations. Numerical simulations, as illustrated in Fig[1], however, show that the limiting behavior follows the logarithmic singularity $\sim \ln(1/m^2)$ as $m \to 0$.

### IV. ELECTROWEAK THEORY UNDER FIELD BACKGROUND AT FINITE TEMPERATURE

#### A. Effective action under background gauge fields at finite temperature

The usual procedure from the real time quantum field theory to quantum field theory in thermal equilibrium is to use the Euclidean field theory in a finite Euclidean time, $0 \sim \eta / 2 = \eta / T$ [9] with $T$ the temperature in the unit of Boltzmann constant taken to be unity. It is
most unambiguous to go back to the starting equation of Schwinger in the real-time formalism,
\[ \mathcal{L}^{(1)}(x) = \frac{i}{2} \int \frac{ds}{s} e^{-im^{2}s} \text{tr}(x) \exp[-i(\Pi^{2} - \frac{1}{2} \sigma \cdot F)]|x|. \] (51)
The proper-time \( s \) here has \(-2\) mass dimension, while it is \(-1\) in the usual approach. It is legitimate to replace \( ms \) by \(-is\beta = -is/T\) for finite-temperature quantum field theory \([9]\). Hence the finite Euclidean interval should be \(0 \sim -i/(mT)\) for transformation from the Schwinger proper-time approach. Our ansatz of background field theory under constant gauge fields is thus to use the integration path,
\[ \frac{i}{2} \int_{0}^{-i\beta/m} \frac{ds}{s} e^{-im^{2}s} \text{tr}(x) \exp[-i(\Pi^{2} - \frac{1}{2} \sigma \cdot F)]|x|, \] (52)
for \(\mathcal{L}^{(1)}(\bar{x}, T)\).

It is useful to rescale the Euclidean proper-time by introducing dimensionless Euclidean time and making Wick rotation. We shall present the case of fermion one-loop. Results for the effective action per unit space-volume \(\Delta W^{(T)}_{\text{eff}}/V\) are for the vector contribution,
\[
\frac{(mT)^{3/2}}{16\pi^{2}} \int_{0}^{1} \frac{dt}{t^{3}} e^{im^{2}t^{2}/m^{2}T} \mathcal{M}_{V}(\mathcal{F}, \mathcal{G}; \tau/mT), \] (53)
and for the axial-vector contribution,
\[
\frac{(mT)^{3/2}}{16\pi^{2}} \int_{0}^{1} \frac{dt}{t^{3}} e^{im^{2}t^{2}/m^{2}T} \mathcal{M}_{A}(\mathcal{F}, \mathcal{G}; \tau/mT). \] (54)
Relation between the axial-vector and the vector cases is the same as before: \(-\mathcal{F} \leftrightarrow i\mathcal{G}\) rule holds. For example, the same equations for \(\mathcal{M}\) such as \(\mathcal{M}_{V} \sim \mathcal{M}_{A}\) may be used provided the integration variable is replaced by \(t \rightarrow -i/m^{2}t\).

Extension to scalar and gauge boson loops at finite temperature should be evident.

**B. Massless limit at finite temperature**

The massless limit of fermion one-loop contribution to the effective lagrangian is mathematically equivalent to zero temperature limit, since various \(\mathcal{M}\) functions at finite temperature depends on of dimensionless variable in the combinations, \(\mathcal{F}/(mT)^{2}, \mathcal{G}/(mT)^{2}\). General theory of finite temperature perturbation theory \([9]\) suggests that the zero temperature limit should coincide with the usual effective action without temperature, which is shown above to exhibit the logarithmic singularity \(\sim \ln(1/m^{2})\).

Whether this argument is correct or not is not evidently clear under the presence of background gauge fields, however. Let us look into one of typical finite temperature formula in our setting:
\[
I = \frac{(mT)^{3/2}}{16\pi^{2}} \int_{0}^{1} \frac{dt}{t^{3}} e^{im^{2}t^{2}/m^{2}T} \mathcal{M}_{V}(\mathcal{F}, \mathcal{G}; \tau/mT), \] (55)
\[
\mathcal{M}(x, y) = ixy \frac{(1 + i \tanh \frac{\sqrt{2}}{2} \tan \frac{\sqrt{2}}{2})(1 - i \tanh \frac{\sqrt{2}}{2} \tan \frac{\sqrt{2}}{2})}{(\tanh \frac{\sqrt{2}}{2} + i \tan \frac{\sqrt{2}}{2})(i \tanh \frac{\sqrt{2}}{2} + \tan \frac{\sqrt{2}}{2})}, \] (56)
where \(c(\mathcal{F}, \mathcal{G}), d(\mathcal{F}, \mathcal{G})\) are given by \([55]\) and \([56]\). One can separate \(\tau\)–integration range into two parts, one near \(\tau = 0\) and the rest. The former contribution is relevant only for large values of the mass, and is irrelevant to discussion here. In the latter contribution the large \((x, y)\) variable limit of \(\mathcal{M}(x, y)\) can be taken, to use an approximate formula,
\[
\mathcal{M}(x, y) \approx xy - \frac{i}{3}(x^{2} - y^{2}). \] (57)
Integration near the upper boundary $\tau = 1$ gives
\[
I \to \frac{(mT)^{-5/2}}{16\pi^2} \int_0^1 d\tau \, e^{i\tau T} \times \left( c(F, G)d(F, G) - \frac{i}{3}(c^2(F, G) - d^2(F, G)) \right).
\]
(58)
Thus, the limiting behavior as $m \to 0$ is
\[
I \to O((mT)^{-5/2}) \frac{1}{32\pi^2} \left( F^2 + \frac{2i}{3}G\sqrt{F^2 + G^2} \right).
\]
(59)
The second term of this equation is CP violating, since
\[
e^2 - d^2 = 4GZ\sqrt{F_Z^2 + G_Z^2},
\]
(60)
is $G_Z$-odd. Relevant gauge coupling is $g^2 + (g')^2 = e^2/(\sin^2 \theta_w \cos^2 \theta_w)$, and the entire limit function $I_0$ has the form, recovering gauge coupling constants,
\[
I_0 = \frac{(g^2 + (g')^2)^2}{32\pi^2} \left( F_Z^2 - \frac{i}{4}GZ\sqrt{F_Z^2 + G_Z^2} \right)
\times \sum f(m_\nu T)^{-5/2} f(T/m_\nu),
\]
(61)
\[
f(x) = \int_0^1 d\tau \, \tau e^{i\tau T} = -1 + e^{ix}(1 - ix)\frac{1}{x^2}.
\]
(62)
The function $f(x)$ here has both the real and the imaginary parts.

Both the real and the imaginary parts of effective lagrangian $I_0$ contain sinusoidal behaviors in the temperature-related variable $m_\nu T$. This exhibits slowly varying oscillatory components with small neutrino masses. Their oscillation time period $t_p$ at time $t$ is estimated by
\[
t_p = 4\pi \frac{T}{m_\nu}.
\]
(63)
We speculate that a large time-period implied by a small neutrino mass may have observational impact at the relevant epoch of cosmological evaluation of $t_p/t = O(1)$.

It is useful here to explain how CP violating observables may emerge from CP conserving initial states. Take as an example the formula given by (61) and focus on the real part of the second contribution. The combination $G_Z\sqrt{F_Z^2 + G_Z^2}$ involves four external gauge fields, and one may take two of them as initial state and the two other as final state. When one takes $\sqrt{F_Z^2 + G_Z^2}$ for the initial state, it is CP conserving, while the final one $G$ is CP violating. This means that the final CPV state is created from an initial CPC state. The real part of this effective lagrangian gives the transition amplitude for this CPV process. What is surprising here is that this CPV transition rate is enhanced by a large factor $\propto (m_\nu T)^{-5}$ due to the small neutrino mass.

It is conceivable to think of mixed background made of electromagnetic fields and $Z$-fields, considering electron one-loop. In this case CP-even electromagnetic field can create CPV transition. For example, high intensity laser collision can create CPV effect, although rate of this mixed case is suppressed by inverse powers of electron mass.

C. High temperature limit and a mechanism of generating lepton number asymmetry

High temperature expansion gives the leading contribution to CP conserving (CPC) and CP violating (CPV) effective action of the form,
\[
\frac{\Delta W_{CPV}}{V} \approx \frac{g^6_A}{2520\pi^2} \frac{G}{m^8 T}\left(13F^2 + 8G^2\right)\left(1 - e^{imT/T}\right),
\]
(64)
\[
\frac{\Delta W_{CPC}}{V} \approx -\frac{1}{1440\pi^2} \frac{S}{m^4 T}\left(1 - e^{imT/T}\right),
\]
(65)
\[
S = (7g_A^4 - 4g_A^8 F^2 + (4g_A^4 - 7g_A^8)G^2),
\]
(66)
recovering gauge coupling constants of vector and axial-vector parts, $g_V, g_A$. High temperature expansion is equivalent to small field expansion, since these dependence appears via the combination, $F/(mT)^2, G/(mT)^2$ in various $M$ functions.

Flavor dependence of the effective lagrangian is as follows:
\[
\Delta L_{CPV} = \frac{1}{2520\pi^2} \times \sum \left( \frac{g_A^6}{m_\nu F} \sqrt{m_\nu T} \left(13F^2 + 8G^2\right) \frac{1}{m^8 T}\left(1 - e^{imT/T}\right) \right),
\]
(67)
\[
\Delta L_{CPC} = -\frac{1}{1440\pi^2} \times \sum \left( \frac{3g_A^8}{m_\nu T} \sqrt{m_\nu T} \left(13F^2 + 8G^2\right) \frac{1}{m^4 T}\left(1 - e^{imT/T}\right) \right).
\]
(68)
We neglected electromagnetic field background contributions. $f_L, f_R$ are values listed in $[25] \sim [31]$.

If the neutrino mass is of Majorana type, the lepton number is violated by $Z$-field fluctuation. Along with CP violation discussed here, the combined symmetry violation may lead to a mechanism of generating lepton number asymmetry. A missing out-of-equilibrium condition $[10]$ can be replaced by a generation of the lepton number chemical potential $[11]$ due to the lepton number
violating scattering involving right-handed heavy Majorana $N_R$ [12]. The lepton asymmetry is not converted to a baryon number since the generation occurs at epochs much later than sphaleron formation [13] unlike the usual lepto-genesis scenario [14]. The lepton asymmetry thus generated may however lead to a large neutrino degeneracy left to the present cosmic epoch.

D. Speculation on a possible phase transition

Skeptics might have detected a peculiar behavior of approach to zero temperature. The behavior $\propto (mT)^{-5/2}$ is inconsistent with a smooth approach to zero temperature having a milder logarithmic singularity on the neutrino mass. Namely, a discontinuity at zero temperature is present. The only way we can think of for a proper behavior is to assume a phase transition at some critical temperature $T = T_c$ or $\beta = \beta_c$, and our result so far is applicable only in the higher temperature region $T > T_c$.

The effective potential $(-\text{the effective lagrangian})$ is identified to $-\text{the free energy in many-body system in thermal equilibrium}$. We postulate that the free energy behaves as

$$m_\nu^{-5/2}(T - T_c)^{-5/2}\alpha^2(F, G)^2\theta(T - T_c),$$

with $\alpha$ the fine structure constant near and above the critical temperature. We neglected all complications of order unity. We leave the appropriate combination of field strength unspecified. There are two possibilities on how to equate the discontinuous quantity,

$$(m_\nu T_c)^{-5/2}\alpha^2(F, G)^2,$$

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$$(m_\nu T_c)^{-5/2}\alpha^2(F, G)^2,$$

to: the first is to take $T_c^3$ and the second is to choose $(m_\nu T_c)^{3/2}$. We believe the first possibility more likely, and present the result in this case. The estimated critical temperature is

$$T_c = m_\nu^{-5/11}(\alpha|F, G||)^{4/11}.$$  

For field strength $\geq m_\nu^2$, the critical temperature $T_c \geq \alpha^{4/11} m_\nu$.

In the case of Dirac neutrino one could add another order parameter, the chemical potential $\mu_\nu$, to make the phase diagram in the three parameter space $(T, \mu_\nu)$ and field strength. We need a separate study to analyze the phase space diagram in a more rigorous way.

V. SUMMARY AND OUTLOOK

We extended the gauge invariant proper-time background field method to chiral gauge theories which makes possible to derive the effective action for the standard electroweak theory. Result in a chiral gauge theory is obtained by using a simple transformation rule from the corresponding vector-like gauge theory. The rule for the integrand function $\mathcal{M}(F, G)$ is given by (23) where the right-hand side is from the corresponding vector-like theory, (10), (11). We applied this method both at zero and finite temperature.

We found interesting results on the infrared singularity in the massless neutrino limit, and identified a new source of CP violation in addition to the well-known and established Kobayashi-Maskawa CP violating phase of quark mass mixing matrix. CPV contribution at finite temperature is very strong behaving $\propto (m_\nu T)^{-5/2}$ in the small neutrino mass limit. The limit formula of the effective action is given by (61).

In the last section we speculated on the phase transition that suggests existence of a shifted critical temperature under gauge field background. A more elaborate analysis is required to determine whether this possibility is real.

Search for physical consequences within a observational reach is yet to come. We can list an important unsolved problem ahead of us: lack of a reliable estimation to calculate gauge field backgrounds that may exist in problems at our hand. The problem may be solved in cosmology at high temperatures by some elaborate technique. Cases at zero temperature do depend on special features of the problem, and we need to work out individual cases. The possibility of coherence generation is necessary to discuss whether condensate formation of gauge fields becomes possible. The condensation may be realized deep interior of neutron stars.

Despite of missing concrete application we hope that this work laid down some fundamentals to a new area of research activity.

VI. APPENDIX: ADDITIONAL NOTES

A. Logarithmic singularity in the massless fermion limit

We prove that the imaginary part given by eq.(12) leads to the logarithmic singularity of the type $\propto \ln(1/m^2)$.

The first step is to use the identity,

$$\coth A_n = 1 + 2\frac{e^{-2A_n}}{1 - e^{-2A_n}}, \quad A_n = \frac{b n \pi}{a},$$

and observe the power series in $[12]$ from the second term to converge after the interchange of the limit $m \to 0$ and the sum. This means that one can replace $\coth A_n$ by the unity 1 for the discussion of the infrared limit,

$$\lim_{m \to 0} \Im \Delta L_f = \frac{ab}{8\pi^2} \lim_{m \to 0} \sum_{n=1}^{\infty} e^{-m^2 n \pi/|a|} \frac{n}{n}.$$  

The problem thus reduces to the limit of a special case $\phi(1, e^{-m^2 n \pi/|a|})$ called Appell function,

$$\phi(z, s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$  

According to [13], this function has an integral representation,
\[ \phi(1, s) = s \int_0^\infty \frac{dt}{e^t - s}, \quad s = \exp[-\frac{m^2 \pi}{|a|}]. \] (75)

In the massless limit one can approximate \( s \) by
\[ s \approx 1 - \epsilon, \quad \epsilon = \frac{m^2 \pi}{|a|}, \] (76)

to give
\[ \phi(1, s) \approx \int_0^\infty \frac{dt}{e^t - 1 + \epsilon}. \] (77)
The integrand has a pole outside the integration region at \( t = \ln(1 - \epsilon) \sim -\epsilon < 0 \).

A standard estimation of this integral in the small \( \epsilon \) limit would be to separate the integral into two parts, the one \([0, t_0]\) and the other \([t_0, \infty]\) taking \( t_0 \) of order unity \( \gg \epsilon \). The second contribution is \( \epsilon \)-independent and may be dropped in the limit. The integration of the first contribution near the end point 0 is crucial, and one may approximate this integral to derive
\[ \lim_{m \to 0} \phi(1, s) \approx \int_0^{t_0} \frac{dt}{t + \epsilon} = \ln \frac{t_0 + \epsilon}{\epsilon} \to \ln \frac{|a| t_0}{m^2 \pi}. \] (78)

One cannot determine \( t_0 \) precisely, but this is irrelevant to the logarithmic behavior.

We checked this derivation numerically as well.

### B. Color matrix eigenvalues

The problem is to derive \( z \) eigenvalues of the algebraic equation for SU(3) color,
\[ \det \left( z \delta_{ab} - \sum_{i=1}^8 \frac{\lambda_i}{2} t_i \right) = 0, \] (79)
where \( \lambda_i \) is 3 \times 3 hermitian Gell-Mann matrices, and \( t_i \)'s are real numbers.

To simplify the equation, we first use the unitary transformation such that the matrix part is diagonal, taking only \( t_3, t_8 \) non-vanishing. The third order algebraic eigenvalue equation is then
\[ f_3(z) = z^3 - \frac{1}{4} (t_3^2 + t_8^2) + \frac{1}{12 \sqrt{3}} t_8 (t_3^2 - 3 t_8^2) \]
\[ = \left( \frac{t_3^2 + t_8^2}{4} \right)^{3/2} (x^3 - x + c), \] (80)
\[ x = \frac{2z}{(t_3^2 + t_8^2)^{1/2}}; \quad c = \frac{1}{6 \sqrt{3}} \frac{t_8 (t_3^2 - 3 t_8^2)}{(t_3^2 + t_8^2)^{3/2}}. \] (81)
c is an odd function of \( t_8 \), and ranges between \(-0.0962 \sim 0.0962 \) with extrema given at \( t_8/t_3 = \pm 0.57735 \). At \( t_8/t_3 = \infty \) it goes to the limiting value \( 1/(6 \sqrt{3}) = 0.0962 \).

The eigenvalue equation thus reduces to solving
\[ g_3(x) = 0, \quad g_3(x) = x^3 - x + c, \] (82)
where \( c \) is a function of single variable combination, \( t_8/\sqrt{t_3^2 + t_8^2} \). The function \( x^3 - x \) has two local extrema at \( x = \pm 1/\sqrt{3} \sim 0.5774 \): local maximum \( 2/(3 \sqrt{3}) \sim 0.3849 \) at \( x = -1/\sqrt{3} \) and local minimum \( -2/(3 \sqrt{3}) \sim -0.3849 \) at \( x = 1/\sqrt{3} \). Hence it is easy to see that three real zeros of \( x \) exist: two positive and one negative for \( t_8 < 0 \) and one positive and two negative for \( t_8 > 0 \). At \( t_8 = 0 \) three eigenvalues are \( x = 0, \pm 1 \), hence \( z = 0, \pm \frac{t_8}{2} \). In general eigenvalues are functions of \( \sqrt{t_3^2 + t_8^2}/2 \times (-1 \sim 1) \). Prior to the diagonalization they are the Casimir invariant \( \sum \sqrt{t_i^2/2} \) numbers of absolute value less than or equal to unity.

The eigenvalue problem of SU(2) group is simpler and its solution is trivially derived.

### C. Scalar and gauge boson loops

For completeness we record known relevant functions, \( \mathcal{M}_s \) for scalar and \( \mathcal{M}_g \) for gauge boson [4, 6], that replace the fermion \( \mathcal{M}_f \) in the same integral formula of \( \Delta \mathcal{L} \), eq. (10) with appropriate choice of mass in \( e^{-m^2 s} \).

\[ \mathcal{M}_s = -\frac{1}{4} \left( \frac{s^2 a b}{\sin(s a) \sinh(s b)} - 1 - \frac{s^2}{6} (a^2 - b^2) \right), \] (83)
\[ \mathcal{M}_g = \frac{s^2 a b}{\sinh(s a) \sin(s b)} (\sinh^2(s a) - \sin^2(s b)) \]
\[ - s^2 (a^2 - b^2). \] (84)

There is no massless singularity in the effective action calculated from \( \mathcal{M}_s \) and \( \mathcal{M}_g \).

### D. General four-component neutral lepton system

It is best to introduce four-component spinor field incorporating two 2-component \( \nu_L, N_R \), and discuss the seesaw mechanism in general by introducing three independent mass mixing terms. Some elementary explanation of Majorana fermion is given in the textbook of Aitchinson-Hey [10]. Quantization based on two-component field is explained in [17].

One introduces creation and annihilation operators, both for two-component fields, denoted by \( c_{\nu}^\dagger, c_{\nu}, d_{\nu}^\dagger, d_{\nu} \). Quantum field projected on plane waves is described by
\[ \psi(x; p) = e^{-i p \cdot x} \left( u(p)c_{\nu} + i \sigma_2 \nu(p) d_{\nu} \right) + e^{i p \cdot x} \left( v(p)c_{\nu}^\dagger + i \sigma_2 u(p) d_{\nu}^\dagger \right), \] (85)
with \( u, v \) two-spinors. General \( 4 \times 4 \) mass matrix is introduced;
\[ \mathcal{M} = \begin{pmatrix} m_L & m_D \noalign{\medskip} m_D & m_R \end{pmatrix}. \] (86)
For simplicity we assume real $m_i$. In the usual $SU(2) \times U(1)$ theory $m_L = 0$. Note that our $\gamma$ matrices are not conventional in order to make $\gamma_5$ diagonal, in particular, $\gamma^0$ is off-diagonal, and

$$
\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

(87)

$$
\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(88)

The mass and the kinetic terms are defined as usual, giving

$$
\bar{\psi} \mathcal{M} \psi = m_L c^\dagger_p c_p (u(p)^\dagger (i\sigma_2) u(p)^* - u(p)^T (i\sigma_2) u(p))
+ m_R d^\dagger_p d_p (v(p)^\dagger (i\sigma_2) v(p)^* - v(p)^T (i\sigma_2) v(p))
+ m_D (c^\dagger_p c_p u(p)^\dagger v(p) + d^\dagger_p d_p v(p)^\dagger u(p))
+ \bar{\psi} (-i\gamma^5 \cdot \gamma) \psi = 2c^\dagger_p c_p u(p)^\dagger (p_\mu - \vec{p} \cdot \vec{\sigma}) u(p)
+ 2 d^\dagger_p d_p v(p)^\dagger (p_\mu - \vec{p} \cdot \vec{\sigma}) v(p).
$$

(89)

We neglected highly oscillating terms $\propto e^{\pm 2ip \cdot x}$ and irrelevant c-number terms. The first two mass terms $\propto m_L$, $m_D$ give Majorana type masses, and the last $\propto m_D$ is of Dirac type. In the kinetic term $\gamma^0 \gamma \cdot \partial = \partial_0 + \gamma_5 \vec{\nabla} \cdot \vec{\Sigma}$, a $2 \times 2$ block-diagonal form. Two kinetic terms correspond to chirally projected kinetic terms, $(1 \pm \gamma_5)/2$.

In the chiral basis of $(1 \mp \gamma_5)/2$ Majorana mass terms are diagonal, while Dirac terms are off-diagonal, causing a mixing between two different chirality states. This gives rise to the seesaw mechanism \[13\]. Suppose a hierarchical parameter pattern, $m_R \gg m_D \gg m_L$ in the limit $m_L \to 0$. Diagonalization of the mass matrix $\mathcal{M}$ leads to eigenvalues and their eigenvectors of the form,

$$
m \approx -\frac{m_D}{m_R}, \text{ state } |m\rangle \sim |\nu_L\rangle \mp \frac{m_D}{m_R} |N_R\rangle,
$$

(91)

$$
M \approx m_R, \text{ state } |M\rangle \sim |N_R\rangle \pm \frac{m_D}{m_R} |\nu_L\rangle.
$$

(92)

By a phase change one can take $m$ positive.

Thus, the four-component chiral formalism can be constructed in terms of the chiral basis by assuming a small Majorana mass $m$ for chiral $|\nu_L\rangle$ state and a large Majorana mass $M$ for $|\nu_R\rangle$ state. The chiral projection of $(1 - \gamma_5)/2$ makes calculation in the text for Dirac neutrinos equally applicable to light Majorana neutrinos generated by the seesaw mechanism, giving the same energy shift and the same decay rate.

There is, however, an important difference with regard to lepton number violation: the real neutrino-pair production process of its rate given by the imaginary part of effective lagrangian is lepton-number violating, and CP-violating.

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