Explicit Numerical Methods for High Dimensional Stochastic Nonlinear Schrödinger Equation: Divergence, Regularity and Convergence

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Abstract. This paper focuses on the construction and analysis of explicit numerical methods of high dimensional stochastic nonlinear Schrödinger equations (SNLSEs). We first prove that the classical explicit numerical methods are unstable and suffer from the numerical divergence phenomenon. Then we propose a kind of explicit splitting numerical methods and prove that the structure-preserving splitting strategy is able to enhance the numerical stability. Furthermore, we establish the regularity analysis and strong convergence analysis of the proposed schemes for SNLSEs based on two key ingredients. One ingredient is proving new regularity estimates of SNLSEs by constructing a logarithmic auxiliary functional and exploiting the Bourgain space. Another one is providing a dedicated error decomposition formula and a novel truncated stochastic Gronwall’s lemma, which relies on the tail estimates of underlying stochastic processes. In particular, our result answers the strong convergence problem of numerical methods for 2D SNLSEs emerged from [C. Chen, J. Hong and A. Prohl, Stoch. Partial Differ. Equ. Anal. Comput. 4 (2016), no. 2, 274–318] and [J. Cui and J. Hong, SIAM J. Numer. Anal. 56 (2018), no. 4, 2045–2069].

1. Introduction

Consider the following SNLSEs which model the propagation of nonlinear dispersive waves in an inhomogeneous or random media (see e.g. [27 2 21] and references therein):

\[
\begin{align*}
\frac{du}{dt} &= i\Delta u + f(u)dt - \frac{1}{2}\alpha ud + g(u)dW(t), \\
u(0) &= \Psi,
\end{align*}
\]

where \(f(\xi) = i\lambda|\xi|^{2\sigma}\xi\) with \(\sigma > 0\) and the parameter \(\lambda = 1\) and \(-1\) corresponding to the focusing case and defocusing case in physics, respectively. Here the spatial domain \(\mathcal{O}\) is either a bounded Lipschitz domain in \(\mathbb{R}^d, d \leq 2\) equipped
with a suitable boundary condition (such as the homogeneous Dirichlet or Neumann boundary condition) or a compact Riemannian manifold of $d \geq 2$ without boundary. The operator $g$ is either the Nemytskii operator of the constant function, which is related to the additive noise case, or the Nemytskii operator of $i\xi$ which corresponds to the multiplicative noise case. The diffusion term $g(u)dW(t)$ represents the fluctuation effect of a physical process in the complex media \[18\], where $\{W(t)\}_{t\geq 0}$ is an $L^2(O;\mathbb{C})$-valued Q-Wiener process on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. This implies that there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $L^2(O;\mathbb{C})$ and a sequence of mutually independent Brownian motions $\{\beta_j\}_{j\in\mathbb{N}}$ such that $W(t, x) = \sum_j Q^2 e_j(x) \beta_j(t)$. The real-valued function $\alpha(x), x \in \mathbb{R}$, which measures the damping effect during the propagation of waves over long distance, is set to be smooth enough and may depend on the covariance operator of $W$ \[12\].

SNLSEs have been investigated both theoretically and numerically in the recent decades. For the well-posedness in $L^2(O;\mathbb{C})$ and $H^1(O;\mathbb{C})$ and the global and asymptotic behaviors of SNLSEs, we refer to \[18, 7, 6, 16, 17\] and references therein. Among them, the numerical approximation turns out to be a useful and important tool (see, e.g., \[20, 29\]) since it is impossible to find the analytic solution of Eq. (1.1) in general. For SNLSEs with Lipschitz and smooth coefficients, there exist fruitful numerical results on the stability and strong convergence (see, e.g., \[19, 1\]), and on the structure-preserving properties and long-time dynamic behaviors (see, e.g., \[25, 10\]). However, the basic mathematical mechanism, such as the stability and strong convergence, of numerical methods for SNLSEs with non-monotone coefficients, like (1.1), has not been completely understood. To deal with the strong nonlinearity in SNLSEs, many authors use the stopping time techniques and truncated SNLSEs to consider the convergence rates of numerical methods in probability or in pathwise sense (see, e.g., \[19, 28, 11\]) which is weaker than the strong one. Some progress has been achieved by studying exponential integrability of exact and numerical solutions (see, e.g., \[13, 12, 14, 5\]). For 1D stochastic cubic Schrödinger equation, the authors in \[13, 12, 14\] derive the optimal strong and weak convergence rates of a kind of temporal splitting Crank-Nicolson schemes and their full discretizations. Nevertheless, the convergence problem of numerical methods for SNLSEs with general polynomial nonlinearity in higher dimensions remains open (see, e.g., \[11, 28, 12\]), which is one main motivation of this work.

Another motivation lies on the divergence phenomenon and instability of explicit numerical methods for stochastic differential equations (SDEs) with superlinear coefficients \[24\]. It has been also shown that for stochastic partial differential equations (SPDEs) of parabolic type with polynomial nonlinearities, the moments of exponential and linear-implicit Euler method are divergent \[3\]. Naturally, we are inspired to ask whether the divergence phenomenon of explicit numerical schemes also exists for SPDEs of hyperbolic type. This finding may be used to explain why the structure-preserving strategy, the truncated or tamed technique, as well as the adaptive method, are needed and important in designing numerical schemes for SPDEs of hyperbolic type with superlinear coefficients.

In this paper, we show the divergence of several explicit temporal numerical schemes, which include the classical exponential Euler method, for Eq. (1.1), based on the property that the double-exponent moment of the Wiener process is infinite (see Section 3). To overcome the divergence issue, we use the structure-preserving idea to construct a kind of explicit numerical methods by some suitable spatial
discretizations. Via the Lie–Trotter splitting technique, we propose the following explicit structure-preserving splitting scheme, which is unconditionally stable in $L^p(\Omega; \mathbb{H}), p \in \mathbb{R}$ (see Proposition 4.1),

$$v_{n+1}^M = \Phi_{S,n}^M(\delta t, \Phi_{D,n}^M(\delta t, u_n^M)),$$

where $\Phi_{D,n}^M, \Phi_{S,n}^M, t \in [t_n, t_{n+1}]$ are the phase flows of the following subsystems

$$du_{D,n}^M(t) = i\Delta v_{D,n}^M(t)dt, \quad v_{D,n}^M(t_n) = u_n^M, \quad u_{S,n}(t_n) = v_{D,n}^M(t_n),$$

$$du_{S,n}^M(t) = P^M(\Phi_1|u_{S,n}^M(t_n)|^{2\sigma} u_{S,n}^M(t) - \frac{1}{2} \alpha u_{S,n}^M(t))dt + P^M g(u_{S,n}^M(t))dW(t).$$

Here $u_0 = \Psi$, $\delta t$ is the time stepsize, $\delta_n W := W(t_{n+1}) - W(t_n), n \leq N - 1, N\delta t = T$, $M$ is the parameter of the spectral Galerkin projection operator $P^M$. One can also use the truncated strategy to construct stable explicit numerical schemes, such as the nonlinearity-truncated exponential Euler method,

$$u_{n+1}^M = S(\delta t) P^M u_n^M + \mathbb{1}_{\{u_n^M \geq R\}} S(\delta t) P^M (f(u_n^M) - \frac{1}{2} \alpha u_n^M)\delta t + S(\delta t) P^M g(u_n^M)\delta_n W,$$

where $S(t) = \exp(i\Delta t)$, $R$ is the truncated number and $\kappa \geq 0$ is the Sobolev index in the truncated function. We would like to remark that the spatial discretization can be also chosen as the finite element methods, the finite difference method, as well as other methods with suitable inverse inequalities.

In the second part of this work, we are interested in the strong convergence of the proposed numerical methods for Eq. (1.1). At the outset of the convergence problem, there is no a higher Sobolev regularity estimate, i.e., $\mathbb{H}^s$-estimate with $s > 1$, of both exact and numerical solutions for Eq. (1.1) in $d \geq 2$. (see, e.g., [11], Remarks 2 and 3). Up to now, the existing results on higher regularity estimates for the exact or numerical solution of SNLSEs are only limited to 1D case. To solve the regularity problems, we introduce the logarithmic Sobolev inequality to construct a new Lyapunov functional and establish the higher regularity estimate for 2D SNLSEs. For SNLSEs on a compact Riemannian manifold of $d \geq 2$, by imposing additional assumptions on $W$, we exploit the Bourgain space and its nonlinear estimate (see, e.g., [8]) to show the desirable higher regularity almost surely (see Section 2). With regard to the numerical solution, we prove that there always exists a relationship between $\delta t$ and the eigenvalue $\lambda_M$ of the Laplacian operator such that the splitting numerical method could inherit the regularity of SNLSEs (see Section 4).

Another bottleneck problem in strong convergence analysis for SPDEs with non-monotone coefficients, including Eq. (1.1), is the lack of a systematic way to transform the pointwise nonlinear error estimate into the desirable strong error estimate. Several attempts have been made on this topic in the recent years. For instance, the approach based on the exponential integrability and stochastic Gronwall’s inequalities of the exact and numerical solutions shows the ability to deal with a large class of non-monotone stochastic ordinary differential equations [23]. In contrast, several restrictions on the dimension and nonlinearity have to be imposed for non-monotone SPDEs via this approach. To tackle these issues, we provide a new error decomposition formula which relies on the tail estimates of the underlying stochastic processes (see Section 5). These tail estimates are naturally obtained via the established higher regularity of numerical and exact solutions. As a consequence, we prove the strong convergence of the proposed numerical methods for SNLSEs, including 1D SNLS (Theorem 5.1), 2D stochastic cubic Schrödinger
equation (Theorem 5.2), and higher dimensional cubic Schrödinger equations with random coefficients (Theorem 5.3). In particular, our result gives a positive answer to the open problem on the strong convergence of 2D SNLSEs emerged from [11, 28, 12]. The proposed approach also has a potential to be extended to deal with the convergence of numerical schemes for other non-monotone SPDEs, such as the stochastic nonlinear wave equation, stochastic Korteweg–De Vries equation and stochastic Burgers equation, and this will be investigated in the future.

To conclude, the main contributions of this paper are summarized as follows:

- We show the divergence of explicit numerical methods for SNLSEs, and prove that the splitting strategy is able to enhance the stability and regularity of the numerical solution.
- We provide a generic approach to study the strong convergence of numerical methods for SDEs with non-monotone coefficients based on the tail estimates and a truncated stochastic Gronwall’s lemma.
- We establish the convergence analysis of the proposed explicit splitting numerical methods via proving new $H^2$-regularity of stochastic nonlinear Schrödinger equation in high dimensions.

2. SNLSEs: Higher regularity estimates and tail estimates

In this section, we present the basic notations and definitions, and some new properties for SNLSEs, including the higher Sobolev regularity estimates and tail estimates of 1D SNLSEs with general polynomial nonlinearities, $H^2$-regularity estimates and tail estimates for 2D stochastic cubic Schrödinger equation and defocusing cubic Schrödinger equations with random coefficients on a compact Riemannian manifold. Throughout this paper, we use $C$ to denote a generic constant which is independent of $\delta t, M, R$ and may differ from line to line.

2.1. Preliminaries. In this part, we introduce some frequently used notations and assumptions in the study of SNLSEs. The norm of $\mathbb{H} := L^2(\mathcal{O}; \mathbb{C})$ is denoted by $\| \cdot \|$ and the inner product $\langle \cdot, \cdot \rangle$ is defined by $\langle v, w \rangle := \text{Re} \int_{\mathcal{O}} \bar{v}(x)w(x)dx$ for $v, w \in \mathbb{H}$. We denote $L^p := L^p(\mathcal{O}; \mathbb{C})$ and $W^{k,p} := W^{k,p}(\mathcal{O}; \mathbb{C}), k \in \mathbb{R}, p \geq 1$. When $p = 2$, $W^{k,2}$ is denoted by $H^k$. We denote the interpolation Sobolev space of the corresponding Laplacian operator on $\mathcal{O}$ by $H^\gamma, \gamma \in \mathbb{R}$ and the corresponding eigenvalue by $\{\lambda_i\}_{i \in \mathbb{N}}$. An operator $\Psi \in L^2_{\mathbb{C}}$ with $s \in \mathbb{N}$ if $\|Q_s^e\|_{L^2_{\mathbb{C}}} := \sum_{i} \|Q_s^e e_i\|_{L^2_{\mathbb{C}}} < \infty$, where $\{e_i\}_{i \in \mathbb{N}}$ is any orthonormal basis of $\mathbb{H}$. For simplicity, we assume that $\sup_i \|e_i\|_{L^\infty} < \infty$. The length of the time interval is a positive number $T > 0$.

Let us briefly recall the previous well-posedness result of SNLSEs before we state our main assumptions. The mild solution of Eq. (11) is defined by the following integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds - \int_0^t S(t-s)\frac{1}{2}\alpha u(s)ds$$

$$+ \int_0^t S(t-s)g(u(s))dW(s), \text{ a.s.},$$

where $S(t) := \exp(i\Delta t)$ is the group generated by $i\Delta$. For the global well-posedness of the mild solution of Eq. (11), we refer to, e.g., [18], in the subcritical case, i.e., $\sigma < \frac{2}{(d-2)^\gamma}$ if $\lambda = -1$ and $\sigma < \frac{2}{d}$ if $\lambda = 1$. It is also known that in the additive noise case ($g(\xi) = 1$), the mass and energy conservation laws fail and
that in multiplicative noise \((g(ξ) = ξ)\), the mass conservation law holds if \(W(\cdot)\) is \(L^2(Ω; ℝ)\)-valued and \(α = \sum \{\hat{Q}e_i\}^2\) (see e.g. [18]). We are also interested in the multiplicative focusing critical case \((g(ξ) = iξ, λ = 1, σ = \frac{2}{3})\) where the initial value is required to satisfy \(\|Ψ\| < C_{Thr}\), where \(C_{Thr}\) is the \(L^2\)-norm of the ground state solution of the elliptic equation \(Δv - v + v^{2σ+1} = 0\) (see e.g. [31, 16]). Among these works, the evolution of the energy

\[ H(w) := \frac{1}{2}\|\hat{\nabla}w\|^2 + \frac{1}{2σ + 2}\|w\|^{2σ+2}_{L^{2σ+2}} \]

plays a key role.

We will assume that the initial value \(Ψ\) is deterministic unless it is necessary to avoid confusion. To simplify the presentation, we impose the following conditions on the diffusion term and nonlinearity which will be frequently used in this section and section 4.

**ASSUMPTION 2.1.** Let \(g(ξ) = 1, α = 0, σ ∈ ℕ^+, Ψ ∈ ℋ^s\) and \(Q^\dagger ∈ L^s_2\) for some \(s ∈ ℕ\). Suppose that one of the following condition holds,

(i) when \(d = 1\), it holds that \(σ = 1\) if \(λ = 1\) and \(σ ∈ ℕ^+\) if \(λ = -1\).
(ii) when \(d = 2\), it holds that \(σ = 1\) and \(λ = -1\).

**ASSUMPTION 2.2.** Let \(g(ξ) = iξ, α = \sum_i |Q^\dagger e_i|^2, σ ∈ ℕ^+, Ψ ∈ ℋ^s, W(\cdot)\) be \(L^2(Ω; ℝ)\)-valued and \(Q^\dagger ∈ L^s_2\) satisfy \(\sum_i \|Q^\dagger e_i\|_{H^{s+∞}} < ∞\) for some \(s ∈ ℕ\). Suppose that one of the following condition holds,

(i) when \(d = 1\), it holds that \(σ < 2\),
(ii) when \(d = 2\), it holds that \(σ = 1\).

Furthermore, if \(σ = \frac{2}{3}, Ψ\) is \(1\), we in addition assume that \(\|Ψ\| < C_{Thr}\).

With a slight modification, one could follow our approach and investigate the additive noise case with linear damping effect \(α ≠ 0\) and the multiplicative noise case with the complex-valued Wiener process. Our assumptions \(σ ∈ ℕ^+\) could be also extended to general positive real number, i.e., \(σ > 0\) if \(d = 1\) and \(σ > \frac{2}{3}\) if \(d = 2\), but there still exist some limitations on the upper bound of \(σ\) since the global \(H^2\)-regularity estimate of SNLSEs is still unclear when \(σ\) is large. In the following, we collect the higher regularity results for 1D SNLSEs and present some new results for higher dimensional SNLSEs.

### 2.2. 1D stochastic nonlinear Schrödinger equation

In this part, we show the regularity results of 1D SNLSEs with general polynomial nonlinearities. This higher regularity result could be obtained by studying the auxiliary functional \(V(v) = \|(-Δ)^{\frac{s}{2}}v\|^2 - λ\|(-Δ)^{s-1}v, |v|^{2σ}v\|^2\) (see, e.g., [13] Section 2), and thus we omitted its proof. Part of them has been reported in [13].

**PROPOSITION 2.1.** Let \(T > 0, d = 1, s ∈ ℕ^+, Ψ ∈ ℋ^s, σ ∈ ℕ^+, Q^\dagger ∈ L^s_2\). Suppose that \(σ \in (0, 2), λ = 1 or σ \in (0, ∞), λ = -1\) if \(g(ξ) = 1\) and \(α = 0\), and that \(σ \in (0, 2), λ = 1 or σ \in (0, ∞), λ = -1\) if \(g(ξ) = iξ, α = \sum_i |Q^\dagger e_i|^2\) and \(W(\cdot)\) is \(L^2(Ω; ℝ)\)-valued. Furthermore, if \(g(ξ) = iξ, λ = 1, σ = 2\), in addition assume that \(\|Ψ\| < C_{Thr}\). There exists a unique mild solution of (1.1) satisfying

\[
\mathbb{E}\left[\sup_{s ∈ [0, T]} \|u(s)\|_{L^p}\right] ≤ C(T, Q, Ψ, λ, σ, p).
\]


Similar to the deterministic case [9], it is not hard to check that to obtain $\mathbb{H}^2$-regularity, one needs to take $\sigma \geq \frac{1}{2}, \sigma \in \mathbb{R}^+$. Proposition 2.1 implies the following useful tail estimates of the mild solution under different norms.

**Corollary 2.1.** Under the condition of Proposition 2.1, it holds that for a large $R > 0$,

\begin{equation}
(2.1) \quad \mathbb{P}(\sup_{s \in [0,T]} \|u(s)\| \geq R) \leq C(T, Q, \Psi, \lambda, \sigma) \exp(-\eta R^2)
\end{equation}

with some $\eta = \eta(T, Q, \Psi, \lambda, \sigma) > 0$, and

\begin{equation}
(2.2) \quad \mathbb{P}(\sup_{s \in [0,T]} \|u(s)\|_{\mathbb{H}^\sigma} \geq R_1) \leq C(T, Q, \Psi, \lambda, \sigma, p_1)R_1^{-2p_1}, \quad \forall p_1 \in \mathbb{N}^+.
\end{equation}

Furthermore, if $g(\xi) = i\xi$, then for large $R_2, R_3 > 0$, there exist $\eta_1 = \eta_1(T, Q, \Psi, \lambda, \sigma)$, $\eta_2 = \eta_2(T, Q, \Psi, \lambda, \sigma) > 0$ such that

\begin{equation}
(2.3) \quad \mathbb{P}(\sup_{s \in [0,T]} \|u(s)\|_{\mathbb{H}^1} \geq R_2) \leq C(T, Q, \Psi, \lambda, \sigma) \exp(-\eta_1 R_2^4),
\end{equation}

and that

\begin{equation}
(2.4) \quad \mathbb{P}(\sup_{s \in [0,T]} \|u(s)\|_{L^{\infty}} \geq R_3) \leq C(T, Q, \Psi, \lambda, \sigma) \exp(-\eta_2 R_3^4).
\end{equation}

**Proof.** The tail estimate (2.1) in the additive noise case could be obtain by applying [13] Lemma 3.1 to the mass functional $\|u(t)\|^2$. In the multiplicative noise case, the mass conservation law immediately implies (2.1). The Chebyshev inequality, together with Proposition 2.1 leads to (2.2). Applying [13] Lemma 3.1 to the energy functional $H(u(t))$ and using the mass conservation law, one can obtain the exponential integrability $\mathbb{E}\left[\sup_{t \in [0,T]} \exp(e^{-\alpha_1 t} \|u(t)\|^2)\right] < \infty$ for some $\alpha_1 = \alpha_1(\Psi, Q, \lambda, \sigma, T)$. Using the arguments in the proof of [13] Corollary 4.1, we obtain (2.3) where $\eta_1 = e^{-\alpha_1 T}$. By using the Gagliardo–Nirenberg interpolation inequality, there exists $C' > 0$ such that $\|u\|_{L^{\infty}} \leq C' \|\nabla u\|^{\frac{1}{2}} \|u\|^{\frac{1}{2}}$. It follows that

\begin{align*}
\mathbb{P}\left(\sup_{s \in [0,T]} \|u(s)\|_{L^{\infty}} \geq R_3\right) &\leq \mathbb{P}\left(\sup_{s \in [0,T]} \|\nabla u(s)\| \geq \frac{R_3^2}{(C')^2\|u(0)\|}\right) \\
&\leq C(T, Q, u_0, \lambda) \exp\left(-\frac{\eta_1}{(C')^4\|u(0)\|^2} R_3^4\right),
\end{align*}

which completes the proof of (2.4).\hfill \square

**2.3. 2D stochastic cubic Schrödinger equation.** In this part, we present the higher regularity estimate and tail estimate for 2D stochastic cubic Schrödinger equation, which has not been reported in the literature. To study the higher regularity, our key tool is the following critical Sobolev interpolation inequality whose proof is in the appendix.

**Lemma 2.1.** Let $v \in \mathbb{H}^2$. It holds that for some $C_0 > 0$,

\[ \|v\|_{L^{\infty}} \leq C_0\|v\|_{\mathbb{H}^1}(1 + \sqrt{\log(1 + \|v\|^2_{\mathbb{H}^2})}). \]

For simplicity, let us assume that $\mathcal{O}$ is a bounded domain equipped with homogeneous Dirichlet boundary condition. We introduce a new auxiliary functional $\tilde{U}$ defined by

\[ \tilde{U}(w) = \log(1 + \log(1 + \|\Delta w\|^2)). \]
We would like to remark when considering the Cauchy problem posed on a bounded domain $\Omega$ equipped with homogeneous Neumann boundary condition or periodic boundary condition, one need to consider 

$$\tilde{O}_s$$

holds with $t = 18, 13$

tions, by using the energy evolution (see, e.g., \[ proof of the additive noise case is similar and simpler. According to our assump-

boundedness of any finite $p$-th moment, 

$$= \tilde{U}^p(u(t))$$

Applying Itô’s formula to $\tilde{U}^p(u(t))$, we obtain that 

\[ E\left[ \sup_{t \in [0,T]} \tilde{U}^p(u(t)) \right] \leq C(T, Q, \Psi, \lambda, p). \]

Proof. We only present the proof of the multiplicative noise case since the proof of the additive noise case is similar and simpler. According to our assumptions, by using the energy evolution (see, e.g., \[18, [13]), one can obtain the following boundedness of any finite $p$-th moment, 

\[ E\left[ \sup_{t \in [0,T]} \|u(t)\|_{H^p}^2 \right] \leq C(T, Q, \Psi, \lambda, p). \]

Applying Itô’s formula to $\tilde{U}^p(u(t))$, we obtain that 

\[ \tilde{U}^p(u(t)) = \tilde{U}^p(u(0)) + \int_0^t p\tilde{U}^{p-1}(u) \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \frac{1}{1 + \|\Delta u\|^2} \langle \Delta u, i\lambda 2 Re(\bar{u}\Delta u) u + i\lambda 4 Re(\bar{u}\nabla u)\nabla u + i\lambda 2 |\nabla u|^2 u \rangle \right] + \int_0^t 2\nabla u \nabla Q^\frac{1}{2} e_i, + \Delta u |Q^\frac{1}{2} e_i|^2 + 2u \Delta Q^\frac{1}{2} e_i, Q^\frac{1}{2} e_i \rangle ds \]

\[ + \int_0^t p\tilde{U}^{p-1}(u) \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \frac{1}{1 + \|\Delta u\|^2} \langle \Delta u, i\Delta(u, i\Delta(u dW(s))) \rangle + \int_0^t 2p\tilde{U}^{p-1}(u) \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \frac{1}{1 + \|\Delta u\|^2} \langle \|\nabla u \nabla Q^\frac{1}{2} e_i\|^2 + \|\Delta u Q^\frac{1}{2} e_i\|^2 \rangle \]

\[ + \int_0^t -2p\tilde{U}^{p-1}(u) \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \frac{1}{1 + \|\Delta u\|^2} \left(1 + \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \right) \sum_{i \in \mathbb{N}^+} \left( \langle \Delta u, i\nabla u \nabla Q^\frac{1}{2} e_i \rangle + \langle \Delta u, i\nabla u \nabla Q^\frac{1}{2} e_i \rangle \right)^2 ds. \]

Taking supreme over $t \in [0, t_1]$ and taking expectation, applying Hölder’s and Young’s inequalities, using the Gagliardo–Nirenberg interpolation inequality and Lemma\[2\] as well as Burkerholder’s inequality, we obtain that for a small $\epsilon \in (0, 1)$,

\[ E\left[ \sup_{t \in [0,t_1]} \tilde{U}^p(u(t)) \right] \leq E[\tilde{U}^p(u(0))] + C \int_0^{t_1} E\left[ \tilde{U}^{p-1}(u) \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \right] \left(1 + \|u\|^2_{H^\infty} \right) ds. \]
\[+\|u\|_{L^\infty}\|\nabla u\| + \|u\|^2 + \|\nabla u\|^2\] \, ds
\]
\[+ \mathbb{E}\left[\sup_{t \in [0,t_1]} \int_0^{t_1} \frac{1}{1 + \log(1 + \|\Delta u\|^2)} \frac{1}{1 + \|\Delta u\|^2} (\Delta u, i\Delta (udW(s))) \right] \]
\[\leq \mathbb{E}[\tilde{U}^p(u(0))] + C \int_0^{t_1} \mathbb{E}\left[\tilde{U}^{p-1}(u) \left(1 + \|u\|^2 + \|\nabla u\|^2\right)\right] ds \]
\[+ C\mathbb{E}\left[\left(\int_0^{t_1} \tilde{U}^{2p-2}(u)(1 + \|u\|^2 + \|\nabla u\|^2)ds\right)^2\right] \]
\[\leq \mathbb{E}[\tilde{U}^p(u(0))] + \mathbb{E}\left[\sup_{t \in [0,t_1]} \tilde{U}^p(u(t))\right] + C(\epsilon) \int_0^{t_1} \mathbb{E}[\tilde{U}^p(u)] ds \]
\[+ C(\epsilon) \int_0^t \mathbb{E}\left[1 + \|u\|^2 + \|\nabla u\|^2\right] ds. \]

The Gronwall’s inequality and (2.5) yield that
\[\mathbb{E}\left[\sup_{t \in [0,T]} \tilde{U}^p(u(t))\right] \leq C(T, Q, \Psi, \lambda, p),\]
which completes the proof. \(\square\)

Thanks to the above regularity estimate, we are able to present the tail estimate for (1.2) with \(d = 2\).

**Corollary 2.2.** Under the condition of Proposition 2.4, for large \(R, R_1 > 0\), (2.1) and (2.2) hold with \(s = 1\). Furthermore, for a large \(R_2 > 0\), it holds that for any \(p_1 \geq 1\),

\[\mathbb{P}\left(\sup_{s \in [0,T]} \|u(s)\|_{L^\infty} \geq R_2\right) \leq C(T, Q, \Psi, \lambda, p_1) [\log(R_2)]^{-p_1}, \]

\[\mathbb{P}\left(\sup_{s \in [0,T]} \|u(s)\|_{L^2} \geq R_2\right) \leq C(T, Q, \Psi, \lambda, p_1) [\log(1 + \log(1 + R_2^2))]^{-p_1}. \]

Furthermore, if \(g(\xi) = i\xi\), then (2.6) holds.

**Proof.** The proofs of (2.1) and (2.2) are similar to those in Proposition 2.1. We only prove (2.6). Applying Lemma 2.1, 2.2, the Chebyshev inequality and Proposition 2.2, we obtain that for any \(p_1 \geq 1\),

\[\mathbb{P}\left(\sup_{s \in [0,T]} \|u(s)\|_{L^\infty} \geq R_2\right) \]
\[\leq \mathbb{P}\left(\sup_{s \in [0,T]} \left[\log(1 + \|u(s)\|_{H^1}) + \log(1 + \log^2(1 + \|\Delta u(s)\|^2))\right] \geq \log(R_2) - \log(C)\right) \]
\[\leq \mathbb{P}\left(\sup_{s \in [0,T]} \log(1 + \|u(s)\|_{H^1}) \geq \frac{\log(R_2) - \log(C)}{2}\right) \]
\[+ \mathbb{P}\left(\sup_{s \in [0,T]} \tilde{U}(u(s)) \geq \left(\frac{\log(R_2) - \log(C)}{2}\right)^2\right) \]
\[\leq C(T, Q, \Psi, \lambda, p_1) [\log(R_2)]^{-p_1}. \]

The estimate of \(\mathbb{P}\left(\sup_{s \in [0,T]} \|u(s)\|_{L^2} \geq R_2\right)\) is similar. \(\square\)
2.4. Higher dimensional stochastic cubic Schrödinger equation. In this part, we suppose that $\mathcal{O}$ is any compact Riemannian manifold of $d \geq 2$ without boundary. For simplicity, we will only consider the cubic Schrödinger equation with random coefficient, namely,

$$(2.7) \quad du = i\Delta u dt + f(u)dt + g(u)B(t)dt, \quad u(0) = u_0,$$

where $\sigma = 1$, the random coefficient $B(t, x) = \mathcal{V}(x)\beta(t), x \in \mathbb{R}$ with a smooth real-valued potential $\mathcal{V}$ and a standard Brownian motion $\beta(t)$. Formally speaking, one may think that the driving process is $W(t) = \int_0^t B(r)dr$ in (1.1). However, one may follow our approach to deal with (1.1) directly since its temporal regularity is not enough to introduce the Bourgain space. In particular, it has been proved that (1.1) is well-posed in $X^{s, b}$ directly since its temporal regularity is not enough to introduce the Bourgain space.

In order to study the $H^\frac{d}{2}$-regularity, we introduce the following definition ($\mathcal{P}_{s_0}$) which gives a kind of bilinear Strichartz estimate (see e.g., [4]).

**Definition 2.1.** Let $s_0 \in [0, 1)$. We say $S(t) = \exp(it\mathcal{A})$ satisfies the property ($\mathcal{P}_{s_0}$) if for all the dyadic numbers $M, L$, and $v_0, w_0 \in H$ localized on the dyadic intervals of order $M, L$, i.e.,

$$\|_{M \leq \sqrt{-\Delta} \leq 2M}(v_0) = v_0, \quad \|_{L \leq \sqrt{-\Delta} \leq 2L}(w_0) = w_0,$$

it holds that

$$\|S(t)v_0S(t)w_0\|_{L^2([0, 1] \times \mathcal{O})} \leq C(\min(M, L))^{s_0}\|v_0\|\|w_0\|.$$  

Such definition was used by several authors in the context of the wave equations and the Schrödinger equations in deterministic case (see e.g., [32]). For instance, it has been proved that ($\mathcal{P}_{s_0}$) holds with $s_0 = (\frac{1}{4})^+$ for $S^3$ and $s_0 = (\frac{3}{4})^+$ for $S^2 \times S^1$. When $\mathcal{O} = T^2$ and $T^3$, $s_0 = 0^+$ and $(\frac{1}{2})^+$ respectively. For general manifolds with boundary, $s_0$ is shown to be $(\frac{1}{4})^+$.

Beyond the bilinear Strichartz estimate, we also need to introduce the Bourgain spaces:

**Definition 2.2.** The Bourgain space $X^{s, b}(\mathbb{R} \times M)$ is the completion of $C_0^\infty(\mathbb{R}; H^s)$ under the norm

$$\|u\|_{X^{s, b}(\mathbb{R} \times M)} = \sum_{\tau} \left\| \left( \tau + \lambda_k \right)^{\frac{1}{2}}(1 + \lambda_k)^{\frac{1}{2}}(\widehat{u_k}C_k)(\tau) e^{-i\Delta u(t, \cdot)} \right\|_{L^2(\mathbb{R}; H^s)}$$

where $\widehat{\cdot}(\tau)$ denotes the Fourier transform with respect to time variable. For any $T > 0$, we denote by $X^{s, b}_T$ the space of restrictions of the elements of $X^{s, b}(\mathbb{R} \times M)$ endowed with the norm

$$\|u\|_{X^{s, b}_T} = \inf\{\|u\|_{X^{s, b}(\mathbb{R} \times M)} : u|_{[0, T] \times \mathcal{O}} = u\}.$$

The basic properties of the Bourgain space (see e.g., [8]), which will be used in the investigation $H^s$-regularity, is summarized as follows. The Sobolev embedding theorem holds, i.e., $X^{s, b}_T \hookrightarrow C([0, T]; H^s)$ and $X^{s, b}_T \hookrightarrow L^4([0, T]; H)$, as well as $X^{s_1, b_2}(\mathbb{R} \times M) \hookrightarrow X^{s_1, b_1}_T(\mathbb{R} \times M)$ for $s_1 \leq s_2$ and $b_1 \leq b_2$. In the following, we state the nonlinear estimate which is taken from [32] Proposition 3.3 and Remark 3.4].
LEMMA 2.2. Under the condition of $(P_{s_0})$, let $s > s_0$. There exists $(b, b')$ satisfying $0 < b' < \frac{1}{2} < b, b + b' < 1$, and $\lambda > 0$ such that for any $u, v, w \in X^{s,b}(\mathbb{R} \times \mathcal{O})$,

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})} \leq C\|v\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})}\|w\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})}\|y\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})}.$$ 

In particular, when $v = w = y$, $s \geq 1$, there exists $(b, b') \in \mathbb{R}^2$ satisfying $0 < b' < \frac{1}{2} < b, b + b' < 1$ and $\lambda > 0$ such that

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})} \leq C\|v\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})}\|w\|_{X^{s,b}(\mathbb{R} \times \mathcal{O})}^2.$$ 

Now we are in a position to prove the $\mathbb{H}^s$-regularity, $s \geq 2$, in the pathwise sense.

PROPOSITION 2.3. Let $T > 0$, $\sigma = 1$, $d \geq 2$, $\lambda = -1$, the condition $(P_{s_0})$ hold and $u(0) \in \mathbb{H}^s$, $V \in \mathbb{W}^{s,\infty}$, $s > s_0$. The solution of the Cauchy problem $(1.1)$ is locally well-posed in $\mathbb{H}^s$, a.s. Moreover, if $s \geq 1$, then the solution exists globally in $C([0, T]; \mathbb{H}^s)$, a.s.

PROOF. The proof of the well-posedness of the additive noise case is similar to that of [18] Theorem 3.4] and thus is omitted. Below, we only present the proof of the multiplicative noise case. Let $T_1$ be a sufficient small number which will be determined later. According to Duhamel’s principle, we define a map $\Gamma$ from $X^{s,b}_{T_1}$ to itself by

$$\Gamma u = S(t)u(0) + \int_0^t S(t - s)f(v(s))ds + \int_0^t S(t - s)g(v(s))B(s)ds.$$ 

By using [8 Proposition 2.11], Lemma 2.2 and $X^{s,b}_{T} \hookrightarrow L^4([0, T]; \mathbb{H}^s)$, we achieve that

$$\|S(t)u(0)\|_{X^{s,b}_{T_1}} \leq C\|u(0)\|_{\mathbb{H}^s},$$

$$\left\| \int_0^t S(t - r)f(v(r))dr \right\|_{X^{s,b}_{T_1}} \leq C T_1^{1 - b - b'}\|v\|_{X^{s,b}_{T_1}}^3,$$

$$\left\| \int_0^t S(t - r)g(v(r))B(r)dr \right\|_{X^{s,b}_{T_1}} \leq C T_1^{1 - b - b'}\|v\|_{X^{s,b}_{T_1}}\|B\|_{X^{\frac{1}{2} - \frac{\epsilon}{2}}_{T_1}}.$$ 

Since $\beta(t)$ is $(\frac{1}{2} - \epsilon)$-Hölder continuous, $\epsilon \in (0, \frac{1}{2})$, and $V \in \mathbb{W}^{s,\infty}$, we have that $\|B\|_{X^{\frac{1}{2} - \epsilon}_{T_1}} < \infty$, a.s. Thus, by taking $T_1 \sim \min(\|u(0)\|_{\mathbb{H}^s}, \|B\|_{X^{\frac{1}{2} - \epsilon}_{T_1}})$ and applying the fixed point theorem, it follows that the unique fixed point $u \in X^{s,b}_{T_1}$ satisfies $\|u(t)\|_{X^{s,b}_{T_1}} \leq C\|u(0)\|_{\mathbb{H}^s}.$

Next, we show the global existence of the solution when $s = 1$. By the above arguments, it follows that there exists $T_1(\|u(0)\|_{\mathbb{H}^1}, \|B\|_{X^{\frac{1}{2} - \epsilon}_{T_1}})$ such that $\|u(t)\|_{X^{1,b}_{T_1}} \leq C\|u(0)\|_{\mathbb{H}^1}$, a.s. By applying the chain rule, Hölder’s and Young’s inequalities, one can obtain the finiteness of mass and energy in any $p$-moment sense. This, together with $\lambda = -1$, implies that

$$\sup_{t \in [0, T]} \|u(t)\|_{\mathbb{H}^1} \leq C_2(T, \|u(0)\|_{\mathbb{H}^1}), \text{ a.s.,}$$

where $C_2(T, \|u(0)\|_{\mathbb{H}^1})$ is a random variable independent of $\tau \in [0, T]$. As a consequence, we could extend the local solution to the global solution almost surely.
When $s \geq 1$, applying Lemma 2.2, it follows that there exists
\begin{equation}
T_1 \sim \min(C_2(T, \|u(0)\|_H), \|B\|_{\mathcal{L}^1(X_T^s)}, \|\cdot\|_{\mathcal{L}^s(X_T^s)})
\end{equation}
such that the solution exists and satisfies $\|u\|_{X_T^s} \leq C\|u(0)\|_H$. As $T_1$ depends only on $\|u(0)\|_H$ and $\|B\|_{\mathcal{L}^1(X_T^s)}$, by elementary iterating process, the solution is global.

Due to loss of the moment estimate of $\|u(\cdot)\|_H$, $s > 1$, or $\|u(\cdot)\|_{L^\infty}$, we only have the following result without any explicit decay rate on the tail estimate of (1.1).

**Corollary 2.3.** Under the condition of Proposition 2.5, it holds that
\begin{equation}
\lim_{R_1 \to \infty} \mathbb{P}( \sup_{r \in [0, T]} \|u(r)\|_H \geq R_1) = 0.
\end{equation}
Furthermore, when $s > \frac{d}{2}$, it holds that
\begin{equation}
\lim_{R_1 \to \infty} \mathbb{P}( \sup_{r \in [0, T]} \|u(r)\|_{L^\infty} \geq R_1) = 0.
\end{equation}

3. Divergence phenomenon of explicit numerical methods for SNLSEs

Since the solution of SNLSE usually does not have an analytical expression, many researchers have been devoted to approximating it numerically and the structure-preserving numerical method has become one popular choice to discretize SNLSEs. In this section, we illustrate why the structure-preserving, truncated and tamed strategies are needed to approximate SNLSEs via several explicit schemes. We would like to mention that the divergence phenomenon of explicit methods for SDEs with polynomial coefficients has been reported in [24]. But less result is known in the infinite-dimensional case.

Below we prove the the divergence of a class of explicit schemes for Eq. (1.1), including the exponential Euler scheme
\begin{equation}
u^{(1)}_{n+1} = S(\delta t)\left(u^{(1)}_n + \delta tf(u^{(1)}_n) + g(u^{(1)}_n)\right) \delta_n W - \delta t\alpha u^{(1)}_n,
\end{equation}
the linear-implicit modified mid-point scheme
\begin{equation}
u^{(2)}_{n+1} = S_{\delta t}u^{(2)}_n + S_{\delta t}\delta tf(u^{(2)}_n) + S_{\delta t}g(u^{(2)}_n)\delta_n W - \delta tT_{\delta t}\alpha u^{(2)}_n,
\end{equation}
where $S_{\delta t} := \frac{1 + \frac{1}{2}\Delta t}{1 - \frac{1}{2}\Delta t}$ and $T_{\delta t} := \frac{1}{1 - \frac{1}{2}\Delta t}$, and the linear-implicit modified Euler scheme
\begin{equation}
u^{(3)}_{n+1} = \tilde{T}_{\delta t}u^{(3)}_n + S_{\delta t}\delta tf(u^{(3)}_n) + S_{\delta t}g(u^{(3)}_n)\delta_n W - \delta t\tilde{T}_{\delta t}\alpha u^{(3)}_n,
\end{equation}
where $\tilde{T}_{\delta t} = \frac{1}{1 - \frac{1}{2}\Delta t}$. Here $n \leq N, n \in \mathbb{N}, T = N\delta t$. The following properties of these semi-groups
\begin{equation}||S(\delta t)||_{\mathcal{L}(H,H)} = ||S_{\delta t}||_{\mathcal{L}(H,H)} = 1, \quad ||\tilde{T}_{\delta t}||_{\mathcal{L}(H,H)} \leq C_0, \quad ||T_{\delta t}||_{\mathcal{L}(H,H)} \leq C_0
\end{equation}
for some $C_0 > 0$ will be frequently used in this section.
Lemma 3.1. Let $T > 0$, $d \geq 1$, $\Psi$ be an $L^{4\sigma+2}$-valued random variable with $\sigma > 0$ and $\omega \in \Omega$. By induction, assume that $\|u^n(\omega)\| \geq K$, $n \geq N_0$, with $\mathbb{P}[\Omega_N] \geq c^{1-N/\epsilon}$ and $u^{(l)}_N(\omega) \geq 2^{(2\sigma+1-\epsilon)(N-1)}$ for $\epsilon > 0$ small enough and $\omega \in \Omega_N$, $N \in \mathbb{N}^+$. Moreover, the numerical approximations satisfy $\lim_{N \to \infty} \mathbb{E}[\|u^{(l)}_N\|^p] = \infty$, $l = 1, 2, 3$, for $p \in [1, \infty)$.

Proof. For simplicity, we only present the result of $u^{(1)}_N$ since the proof of the other cases is similar. We first show the divergence in the multiplicative noise case. Denote $m(O)$ the Lebesgue measure of $O$. Notice that for the polynomial $\delta t m(O)^{-\sigma} \xi^{2\sigma} - \delta t(\frac{1}{2}\|\alpha\|_{L^\infty}+1)-1$, we can always find $C' \geq \frac{2m(O)^{-\frac{1}{2}}}{{\delta t \|\alpha\|_{L^\infty}+1}^2} m(O)^{\sigma}$ and $\epsilon \in (0, 2\sigma)$ such that for $\xi \geq \max(C', (\frac{2m(O)^{\sigma}}{\delta t})^\epsilon)$, it holds that

$$\delta t m(O)^{-\sigma} \xi^{2\sigma} - \delta t(\|\alpha\|_{L^\infty}+1)-1 \geq \frac{1}{2}\delta t m(O)^{-\sigma} \xi^{2\sigma} \geq \xi^{2\sigma-\epsilon}.$$ 

For $C_0' = \frac{1}{2}\|\alpha\|_{L^\infty}+1$, we define

$$r_N := \max \left( 2, C_0', \left( \frac{2}{\delta t} + C_0' \right)^{\frac{1}{\sigma}}, C', \left( \frac{2m(O)^{\sigma}}{\delta t} \right)^\epsilon \right)$$

and the events $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}^+$ by

$$\Omega_N := \left\{ \omega \in \Omega| \|\delta t_n W\|_{H^r} \geq \frac{1}{2}\|\alpha\|_{L^\infty}, \|\delta t W_0\| \geq K(r_N + K), \min_{x \in \Omega} |\Psi(x)| \geq \frac{1}{K}, \|u_0\| + T\|f(u_0) - \alpha u_0\| \leq K \right\},$$

where $K > 1$, $s = 0$ for additive noise and $s > \frac{d}{2}$ for the multiplicative noise.

Now, we use induction arguments to show that $\|u_n(\omega)\| \geq (r_N)^{(2\sigma+1-\epsilon)n-1}$, $n \in \mathbb{N}/\{0\}$ for given $\omega \in \Omega_N$. For $n = 1$, the definition of $\Omega_N$ and (3.4) yields that

$$\|u^{(1)}_n(\omega)\| \geq \|g(\Psi)\delta t W_0(\omega)\| - \|u_0(\omega)\| - \|f(u_0(\omega)) - \alpha u_0(\omega)\| \geq \frac{1}{K}\|\delta t W_0(\omega)\| - K \geq r_N \geq C \geq 1.$$ 

By inductions, assume that $\|u^{(1)}_n(\omega)\| \geq (r_N)^{(2\sigma+1-\epsilon)n-1}$ holds for the first $n$ steps. The unitarity of $S(t)$ and Hölder’s inequality yield that

$$\|S(\delta t)(f(u) - \frac{1}{2}\alpha)\| \geq \|u\|^{2\sigma} - \frac{1}{2}\|\alpha\|_{L^\infty} \|u\| = \|u\|^{2\sigma+1} - \frac{1}{2}\|\alpha\|_{L^\infty} \|u\| \geq m(O)^{-\sigma} \|u\|^{2\sigma+1} - \frac{1}{2}\|\alpha\|_{L^\infty} \|u\|.$$
To prove the estimate in the \((n+1)\)-th step, by Hölder’s inequality and the Sobolev embedding theorem, we have

\[
\|u_{n+1}^{(l)}(\omega)\| \geq \delta t\|f(u_n^{(l)}(\omega)) - \alpha u_n(\omega)\| - \|g(u_n^{(l)}(\omega))\| \delta_n W(\omega)\| - \|u_n^{(l)}(\omega)\| \\
\geq m(\mathcal{O})^{-\sigma}\|u_n^{(l)}(\omega)\|^{2\sigma+1}\delta t - \frac{1}{2}\|\alpha\| L_\infty\|u_n^{(l)}(\omega)\|\delta t - \|u_n^{(l)}(\omega)\| \\
\geq \|u_n^{(l)}(\omega)\|\left(m(\mathcal{O})^{-\sigma}\delta t\|u_n^{(l)}(\omega)\|^{2\sigma} - \delta t\left(\frac{1}{2}\|\alpha\| L_\infty + 1\right)\right).
\]

Therefore, by the definition of \(r_N\), we get

\[
\|u_{n+1}^{(l)}(\omega)\| \geq \|u_n^{(l)}(\omega)\|\left(m(\mathcal{O})^{-\sigma}\delta t\|u_n^{(l)}(\omega)\|^{2\sigma} - \delta t\left(\frac{1}{2}\|\alpha\| L_\infty + 1\right)\right) \\
\geq \|u_n^{(l)}(\omega)\|^{2\sigma+1-\epsilon}.
\]

Thus we can obtain

\[
\|u_N^{(l)}(\omega)\| \geq \|u_{N-1}^{(l)}(\omega)\|^{2\sigma+1-\epsilon} \geq (r_N)^{(2\sigma+1-\epsilon)N} \geq 2^{(2\sigma+1-\epsilon)N}
\]

for \(\omega \in \Omega_N\).

Next we give the lower bound of \(\mathbb{P}(\Omega_N)\). Denote \(\mathcal{V}_1 := \mathbb{P}\left[\min_{x \in \mathcal{O}} |\Psi(x)| \geq \frac{x}{4}, \|\Psi\| + T\|f(\Psi) - \frac{1}{2}\alpha\Psi\| \leq K\right] > 0\). Using the lower bound of the tail estimate of normal distribution \(Z\), i.e., for \(x \in [1, \infty)\), \(\mathbb{P}(\|Z\| \geq x) \geq \frac{1}{4}xe^{-x^2}\), it holds that for \(k \in \mathbb{N}^+\) and \(Q^\frac{1}{2}\mathcal{E}_k \neq 0\),

\[
(3.5)\quad \mathbb{P}\left[T^{-\frac{1}{2}}\|W(t)\| \geq x\right] \geq \mathbb{P}\left[T^{-\frac{1}{2}}|\beta_k(T)| \geq \|Q^\frac{1}{2}\mathcal{E}_k\|^{-1}x\right] \geq \frac{x}{4\|Q^\frac{1}{2}\mathcal{E}_k\|}\exp\left(-\frac{\|Q^\frac{1}{2}\mathcal{E}_k\|^2x^2}{2}\right).
\]

Applying (3.4), (3.5), the independent increments of the Wiener process and the property that \(\frac{W(t)}{N}\) has the same distribution of \(\frac{\delta W_0}{\sqrt{N}}\), as well as \(\mathbb{P}(\|Z\| \geq \frac{1}{4}x, x) \geq \frac{x^2e^{-\frac{x^2}{2}}}{4}\), we achieve that for some \(k \in \mathbb{N}^+\) and \(Q^\frac{1}{2}\mathcal{E}_k \neq 0\),

\[
\mathbb{P}(\Omega_N) \geq \mathcal{V}_1\mathbb{P}\left[\|\delta W_0\| \geq K(r_N + K)\right]\left(\mathbb{P}\left[\|\delta W_0\|_{\mathcal{H}^s} \in \left[\frac{1}{2}\delta t, \delta t\right]\right]\right)^{N-1} \\
\geq \mathcal{V}_1\left[\frac{K(kr_N + K)}{4\|Q^\frac{1}{2}\mathcal{E}_k\|\delta t^\frac{1}{2}}\exp\left(-\frac{K^2(r_N + K)^2\|Q^\frac{1}{2}\mathcal{E}_k\|_{\mathcal{H}^s}^2}{\delta t}\right) - T\|Q^\frac{1}{2}\mathcal{E}_k\|_{\mathcal{H}^s}^2\right] \\
\times \left(\frac{\delta t^\frac{1}{2}}{4\|Q^\frac{1}{2}\mathcal{E}_k\|_{\mathcal{H}^s}}\right)^{N} \\
\geq \mathcal{V}_1\left[\frac{N^{-1}}{(4N+1)^N\|Q^\frac{1}{2}\mathcal{E}_k\|_{\mathcal{H}^s}^N}\exp\left(-\frac{K^2(r_N + K)^2\|Q^\frac{1}{2}\mathcal{E}_k\|_{\mathcal{H}^s}^2}{\delta t}\right)\right].
\]

Since \(N \log N \leq N^c\), for \(c \in (1, \infty)\), there exists a constant \(c \in (1, \infty)\) such that

\[
\mathbb{P}(\Omega_N) \geq e^{-cN^r}
\]

for \(N \in \mathbb{N}^+\). The lower bound of \(u_N^{(l)}\) on \(\Omega_N\) and the lower bound of \(\mathbb{P}(\Omega_N)\) yield that for any \(p \geq 1\),

\[
\lim_{N \to \infty} \mathbb{E}(\|u_N^{(l)}\|^p) \geq \lim_{N \to \infty} \left(\mathbb{P}(\Omega_N)(r_N)^{p(2\sigma+1-\epsilon)N^{-1}}\right) \\
\geq \lim_{N \to \infty} \left(\mathbb{P}(\Omega_N)\right)^{2p(2\sigma+1-\epsilon)N^{-1}}
\]
For the additive noise case, the proof is similar. The main difference is that the definition of $\gamma_N$ will rely on the polynomial $\delta t m(\mathcal{O})^{-\sigma} \xi^{2\sigma+1} - \delta t \frac{1}{2} \|\alpha\|_{L^\infty} \xi - \delta t - \xi$. We omit the tedious and analogous procedures for simplicity.

Remark 3.1. The proposed approach can also be extended to prove the divergence of explicit schemes for general stochastic Schrödinger nonlinear equations whose drift and diffusion coefficients have different growing speeds, i.e., there exist $C_1 > 0, C_2 \geq 1, \beta > 1, \beta > \alpha$, such that

\[
\max(\|f(v)\|, \|g(v)\|) \geq C_1 \|v\|^\beta, \quad \text{and} \quad \min(\|f(v)\|, \|g(v)\|) \leq C_2 \|v\|^\alpha,
\]

for all $\|x\| \geq C$ with a large enough number $C > 0$.

4. Explicit numerical methods

Roughly speaking, there exist two popular approaches to treat the divergence problem of numerical methods for SDEs. One can use some stable implicit discretizations such that the $p$-moment of numerical solutions is finite (see, e.g., [22]). This approach often requires the solvability of underlying implicit schemes. Besides, the overall error estimate of the implementable algorithm involved with Newton’s iteration or Picard’s iteration is not clear for non-monotone SPDEs in general. Another approach to overcome the divergence issue is constructing different variants of the explicit schemes, such as the truncated/tamed strategy and the adaptive mesh method (see, e.g., [30, 26]). In this section, we focus on the second approach and will construct and analyze strongly convergent explicit schemes for SNLSEs.

4.1. Construction. Via the Lie–Trotter splitting (see e.g., [15, 14]), (1.1) can be split into several stochastic Hamiltonian systems (see e.g., [28, 14]). Following this structure-preserving strategy, we first propose an explicit splitting scheme which reads:

\[
\psi^M_{n+1} = \Phi^M_{S,n}(\delta t, \Phi^M_{D,n}(\delta t, \psi^M_n)),
\]

where $\Phi^M_{D,n}, \Phi^M_{S,n}, t \in [t_n, t_{n+1}]$ are the phase flows of the following subsystems

\[
\begin{align*}
{v^M_{D,n}(t)} & = i\Delta v^M_{D,n}(t)dt, \quad v^M_{D,n}(t_n) = u^M_n, u_{S,n}(t_n) = v^M_{D,n}(t_n), \\
{v^M_{S,n}(t)} & = P^M \left(1|u^M_{S,n}(t_n)|^2 + u^M_{S,n}(t) - \frac{1}{2} \alpha u^M_{S,n}(t)\right)dt + P^M g(u^M_{S,n}(t))dW(t),
\end{align*}
\]

respectively. Here $n \leq N, N\delta t = T$, $u^0_n = P^M \Psi$ and $P^M$ is the spectral Galerkin projection operator with $M \in \mathbb{N}^+$. We would like to remark that (1.3) is a linear system and thus it is analytically solvable. One can easily construct different explicit splitting numerical schemes via this splitting idea. For instance, splitting (1.1) into three parts yields the following scheme

\[
u^M_{n+1} = \tilde{\Phi}^M_{S,n}(\delta t, \Phi^M_{f,n}(\delta t, \Phi^M_{D,n}(\delta t, \psi^M_n))),
\]

where $\tilde{\Phi}^M_{S,n}, \Phi^M_{f,n}, \Phi^M_{D,n}$ correspond to the subsystems

\[
dx(t) = iP^M \Delta x(t)dt,
\]
\[ dy(t) = iP^M (\lambda |y(t_n)|^{2\sigma} y(t) - \frac{1}{2} \alpha y(t))dt, \]
\[ dz(t) = iP^M g(z(t))dW(t). \]

Another popular way to construct stable explicit scheme is using truncated/tamed method. For example, applying the truncated strategy to the standard exponential Euler schemes leads to the nonlinearity-truncated fully discrete exponential Euler scheme,
\[
u^{M}_{n+1} = S(\delta t)P^M u^{M}_{n} + \mathbb{1}_{\{\|u^{M}_{n}\|_{\mathbb{H}^s} \leq R\}} S(\delta t)P^M (f(u^{M}_{n}) - \frac{1}{2} \alpha u^{M}_{n})\delta t + S(\delta t)P^M g(u^{M}_{n})\delta n W,
\]

where the indicator function is defined on \(\Omega\). One may change the truncated norm \(\mathbb{H}^p, \kappa \in \mathbb{N}^+\), and set the scale of \(R\) to find the suitable relationship of \(R, \delta t, M\) such that this scheme enjoys the desirable higher regularity.

When \(g(\xi) = i\xi\), the proposed splitting method (4.1) is a kind of the nonlinearity-truncated scheme due to the property of the spectral Galerkin method (see e.g. [14, Subsection 4.1]) that
\[
\|u^n_M\| \leq \|u^0_M\|, \text{ a.s. (4.4)}
\]

In the following, we will focus on (4.1) and present the detailed steps for proving the strong convergence of explicit numerical methods of SNLSEs. Following similar steps, the convergence analysis of other explicit splitting numerical methods and the nonlinearity-truncated fully discrete exponential Euler scheme could be established. For simplicity, we omitted the tedious calculations for these schemes.

### 4.2. Stability and Higher regularity estimate.

The first step to prove the strong convergence of (4.1) lies on the stability and higher regularity estimate. Below, we show that there always exists a relationship between \(M\) and \(\delta t\) such that (4.1) inherits the higher regularity estimate of the exact solution.

**Proposition 4.1.** Let \(p \in \mathbb{N}^+\) and \(T > 0\). Suppose that either Assumption 2.1 or Assumption 2.2 holds with \(s = 1\). The scheme (4.1) is unconditionally stable, i.e.,
\[
\mathbb{E}\left[ \sup_{n \leq N} \|u^n_M\|_{\mathbb{H}^1}^{2p} \right] < \infty,
\]
and enjoys \(\mathbb{H}^1\)-regularity, i.e.,
\[
\mathbb{E}\left[ \sup_{n \leq N} \|u^n_M\|_{\mathbb{H}^1}^{2p} \right] < \infty
\]
under the condition that
\[
\frac{\lambda^{2+2\kappa_1} \sigma}{M} \delta t \sim O(1), \text{ for some } \kappa_1 > \frac{d}{2}.
\]

**Proof.** We first present the details of the proof for the multiplicative noise case. When \(g(\xi) = i\xi\), since (4.2) is a Hamiltonian system, and the mass of (4.3) decays, (4.3) holds, which implies the unconditional stability. In order to show the \(\mathbb{H}^1\)-estimate, it suffices to prove the a priori estimate of the energy functional. This will be done via a bootstrap argument as follows. For simplicity, we only present the details for \(p = 1\). Applying the Itô formula to \(H(u^n_{S,n}(t_{n+1}))\) and using the integration by parts, we get
\[
H(u^n_{S,n}(t_{n+1}))
\]
\[ H(u^M_{S,n}(t_n)) + \int_{t_n}^{t_{n+1}} \langle i\Delta v^M_{D,n}(s), i\lambda u^M_{S,n}(t_n) \rangle ds + \int_{t_n}^{t_{n+1}} \langle -\Delta u^M_{S,n}(s), i\lambda u^M_{S,n}(t_n) \rangle ds + \int_{t_n}^{t_{n+1}} \langle -\Delta u^M_{S,n}(s), g(u^M_{S,n}(s)) dW(s) \rangle \\
+ \int_{t_n}^{t_{n+1}} \frac{1}{2} \sum_i \langle \nabla g(u^M_{S,n}(s)) Q^* e_i, \nabla g(u^M_{S,n}(s)) Q^* e_i \rangle ds \\
+ \int_{t_n}^{t_{n+1}} \langle \lambda f(u^M_{S,n}(s)), P^M \left( i\lambda u^M_{S,n}(t_n) \right)^{2\sigma} u^M_{S,n}(s) - \frac{1}{2} \alpha u^M_{S,n}(s) \rangle \\
+ \int_{t_n}^{t_{n+1}} \langle \lambda f(u^M_{S,n}(s)), P^M g(u^M_{S,n}(s)) dW(s) \rangle \\
+ \int_{t_n}^{t_{n+1}} \frac{1}{2} \sum_i \langle \lambda |u^M_{S,n}(s)|^{2\sigma} P^M g(u^M_{S,n}(s)) Q^* e_i, P^M g(u^M_{S,n}(s)) Q^* e_i \rangle ds \\
+ \int_{t_n}^{t_{n+1}} \frac{1}{2} \sum_i \langle \lambda |u^M_{S,n}(s)|^{2\sigma-2} 2 \Re \left( u^M_{S,n}(s) P^M g(u^M_{S,n}(s)) Q^* e_i u^M_{S,n}(s), P^M g(u^M_{S,n}(s)) Q^* e_i \right) u^M_{S,n}(s), \\
P^M g(u^M_{S,n}(s)) Q^* e_i \rangle ds. \\
\]

Using the fact that
\[(4.6) \quad \|(S(\delta t) - I)w\| \leq C\delta t^{\min(\frac{2\sigma}{d} + 1)} \|w\|_{L_{\infty}^p}, \quad s' \in \mathbb{N}^+,\]
and applying the Gagliardo–Nirenberg interpolation inequality,
\[\|w\|_{L^{d+2}} \leq C \|\nabla w\|^{\frac{d}{d+2}} \|w\|^{\frac{2}{d+2}}, \quad d \leq 2,\]
as well as the inverse inequality \[\|w\|_{L^{p_1}} \leq C \lambda_M \|w\|, \quad \forall \ w \in P^M \mathbb{H}, \quad s_1 \in \mathbb{N}^+,\]
one can show that for \(s \in [t_n, t_{n+1}],\)
\[(4.7) \quad \|v^M_{D,n}(s) - u^M_{n}(s)\| \leq C\delta t \lambda_M \|u^M_{n}(s)\| \\
(4.8) \quad \|u^M_{S,n}(s) - u^M_{n}(s)\| \leq C\delta t^{\max(\frac{2\sigma}{d} + 1)} \lambda_M^\frac{d}{d+2} \|u^M_{n}(s)\| + C \left\| \int_{t_n}^{s} u^M_{S,n}(r) dW(r) \right\| \\
+ \lambda_M^\frac{d}{d+2} \left( 1 + \|u^M_{S,n}(r)\|^{2\sigma + 1} + \|u^M_{S,n}(t_n)\|^{2\sigma + 1} \right) dr.\]

According to (4.7) and (4.8), using the Gagliardo–Nirenberg interpolation inequality, Hölder’s and Young’s inequalities, and the inverse inequality, we have that
\[\left| \int_{t_n}^{t_{n+1}} \langle i\Delta v^M_{D,n}(s), i\lambda u^M_{S,n}(t_n) \rangle ds + \int_{t_n}^{t_{n+1}} \langle -\Delta u^M_{S,n}(s), i\lambda u^M_{S,n}(t_n) \rangle ds \right| \]
\[\leq C \left[ \int_{t_n}^{t_{n+1}} \lambda_M \left( 1 + \|v^M_{D,n}(s)\|^{2\sigma + 1} + \|u^M_{S,n}(t_n)\|^{2\sigma + 1} \right) \|v^M_{D,n}(s) - u^M_{S,n}(s)\| ds \right. \\
+ C \left. \int_{t_n}^{t_{n+1}} \lambda_M^{1 + \gamma \sigma} \left( 1 + \|v^M_{D,n}(s)\|^{2\sigma + 1} + \|u^M_{S,n}(t_n)\|^{2\sigma + 1} \right) \|v^M_{D,n}(s) - u^M_{S,n}(s)\| ds \right].
\]
Taking expectation on the equation of \(H(u^M_{S,n}(t_{n+1})),\) and using Hölder’s and Young’s inequalities, as well as the Gagliardo–Nirenberg interpolation inequality
and the property of Galerkin projection operator,

\[
\|(I - P^M)w\| \leq C\lambda_M^{-\frac{1}{2}}\|w\|_{H^r},
\]

we obtain that for \(\lambda_M^{2+2\kappa_1}\sigma \delta t \sim \mathcal{O}(1), \kappa_1 > \frac{\delta}{2}\),

\[
E\left[H(u_{S,n}^M(t_{n+1}))\right] \\
\leq E\left[H(u_{n}^M)\right] + C\delta t + C \int_{t_n}^{t_{n+1}} H(u_{S,n}^M(s))ds \\
+ C \int_{t_n}^{t_{n+1}} \|(I - P^M)u_{S,n}^M(s)\alpha\|_{L^2}^{2\sigma+2}ds \\
+ C \int_{t_n}^{t_{n+1}} \sum_i \|(I - P^M)u_{S,n}^M(s)\sigma_i\|_{L^2}^{2\sigma+2}ds \\
+ C \int_{t_n}^{t_{n+1}} \mathcal{Q}\mathcal{P}(\mathcal{S},\mathcal{M})^2 e_i \|\sigma\|_{L^2}^{2\sigma+2}ds \\
+ C \int_{t_n}^{t_{n+1}} \mathcal{Q}\mathcal{P}(\mathcal{S},\mathcal{M})^2 e_i \|\sigma\|_{L^2}^{2\sigma+2}ds \\
+ C \int_{t_n}^{t_{n+1}} \mathcal{Q}\mathcal{P}(\mathcal{S},\mathcal{M})^2 e_i \|\sigma\|_{L^2}^{2\sigma+2}ds \\
\leq E\left[H(u_{n}^M)\right] + C\delta t + C \int_{t_n}^{t_{n+1}} H(u_{S,n}^M(t_n))ds \\
+ C \int_{t_n}^{t_{n+1}} \mathcal{Q}\mathcal{P}(\mathcal{S},\mathcal{M})^2 e_i \|\sigma\|_{L^2}^{2\sigma+2}ds \\
\leq E\left[H(u_{n}^M)\right] + C\delta t + C \int_{t_n}^{t_{n+1}} \mathcal{Q}\mathcal{P}(\mathcal{S},\mathcal{M})^2 e_i \|\sigma\|_{L^2}^{2\sigma+2}ds,
\]

where we use the condition \(\sigma d \leq 2\) in Assumption 2.2 in the last step. By Gronwall’s inequality, we have \(\sup_{n \leq N} E\left[H(u_{n}^M)\right] < \infty\). Similarly, applying Itô formula to \(H(u_{S,n}^M(t_{n+1}))\) and repeating the above procedures, we have that

\[
\sup_{n \leq N} \sup_{t \in [t_n, t_{n+1}]} E\left[H(u_{S,n}^M(t))\right] < \infty.
\]

One can introduce the global auxiliary processes \(v_{D,M}^M\) and \(u_{S}^M\) defined by

\[
v_{D,M}^M(t) = v_{D,M,n}(t), \quad \text{if } t \in [t_n, t_{n+1}]; \quad v_{D,M}^M(t_{n+1}) = u_{n+1},
\]

\[
u_{S}^M(t) = u_{S,n}(t), \quad \text{if } t \in (t_n, t_{n+1}], \quad \text{and } u_{S,n-1}(t_n).
\]

It can be seen that \(v_{D,M}^M\) and \(u_{S}^M\) are \(\mathcal{F}_t\)-adapted and continuous piece-wisely. Following the arguments in the proof of 14, Corollary 2.1 and Lemma 3.3, it is not hard to show that

\[
E\left[\sup_{s \in [0,T]} \|v_{D,M}(s)\|_{L^2}^{2p}\right] + E\left[\sup_{s \in [0,T]} \|u_{S}(s)\|_{L^2}^{2p}\right] < \infty.
\]

As a consequence, it follows that \(E\left[\sup_{n \leq N} \|u_{n,\mathcal{S}}\|_{L^2}^{2p}\right] < \infty\). For the additive noise case, the proof is similar and simpler since when applying the Itô formula, the second derivative of \(H(u_{S,n}^M(t))\) can be bounded by the energy itself.

**Remark 4.1.** It can be seen that the condition \(\lambda_M^{2+2\kappa_1}\sigma \delta t \sim \mathcal{O}(1)\), is just for simplicity and could be improved. One may add a tamed term or truncated term into the stochastic nonlinear subsystem to improve this restrictive condition. Our numerical analysis of the explicit numerical methods is also applicable to other
spatial discretizations, like the finite difference methods and finite element methods, as long as the corresponding inverse holds.

Next, we show that the scheme (4.1) could inherit the higher regularity estimate in $H^s$, $s \geq 2$, of the exact solution, whose proof is given in the appendix.

**Proposition 4.2.** Let $p \in \mathbb{N}^+$, $T > 0$ and (4.5) hold. Suppose that either Assumption 2.1 or Assumption 2.2 holds with $s \geq 2$. Then the numerical solution of (4.1) satisfies that for $d = 1$,

$$E \left[ \sup_{n \leq N} \| u^M_n \|_{H^s}^{2p} \right] < \infty,$$

and for $d = 2$,

$$E \left[ \sup_{n \leq N} \tilde{U}(u^M_n) \right] < \infty.$$

One could also derive the regularity of the numerical scheme when $\sigma$ is not an integer and show that under the condition of Corollary 4.2 with $\sigma \geq \frac{1}{2}$, (4.14) holds. As a consequence, the following tail estimates hold.

**Corollary 4.1.** Let the condition of Proposition 4.2. For a large enough $R_1 \geq 1$ and $p_1 \in \mathbb{N}^+$, it holds that for $d = 1$,

$$\mathbb{P} \left( \sup_{n \leq N} \| u^M_n \|_{H^s} \geq R_1 \right) \leq C_1 R_1^{-p_1},$$

and that for $d = 2$,

$$\mathbb{P} \left( \sup_{n \leq N} \| u^M_n \|_{H^1} \geq R_1 \right) \leq C_1 R_1^{-p_1},$$

$$\mathbb{P} \left( \sup_{n \leq N} \| u^M_n \|_{L^\infty} \geq R_1 \right) \leq C_1 \log^{-p_1}(R_1).$$

To end this section, we present the exponential integrability of the energy functional for 1D stochastic cubic Schrödinger equation whose proof is based on [13], Lemma 3.1], (4.4) and the Gagliardo–Nirenberg interpolation inequality $\| w \|_{L^\infty} \leq C_0 \| \nabla w \|_{1/2} \| w \|_{1/2} + C'_0 \| w \|$. We omit the details of this proof for simplicity.

**Proposition 4.3.** Let $T > 0$, $d = 1$, (4.5) and Assumption 2.2 hold with $s = 1$ and $\sigma = 1$. Then for a large enough $R_1 \geq 1$ and $p_1 \in \mathbb{N}^+$, the scheme (4.1) satisfies

$$\mathbb{P} \left( \sup_{n \leq N} \| u^M_n \|_{H^1} \geq R_1 \right) \leq C_1 \exp(-\eta_1 R_1^2),$$

and

$$\mathbb{P} \left( \sup_{n \leq N} \| u^M_n \|_{L^\infty} \geq R_1 \right) \leq C_1 \exp(-\eta_1 R_1^4).$$

### 5. Error decomposition and strong convergence

In this section, we are in a position to present our methodology to analyze the convergence of explicit numerical methods and then to establish strong convergence analysis of (4.1).
5.1. Error decomposition. By iteratively rewriting (4.1) into its integral form, we have that
\[
\begin{align*}
\sigma_n & = S(t_{n+1}) P^M u_0^M + \int_0^{t_{n+1}} S_{n,n+1}^M(s) \lambda |u_S^M(t_n)|^{2\sigma} u_S^M(s)ds \\
& \quad - \int_0^{t_{n+1}} S_{n,n+1}^M(s) \frac{1}{2} \alpha u_S^M(s)ds + \int_0^{t_{n+1}} S_{n,n+1}^M(s) g(u_S^M(s))dW(s),
\end{align*}
\] where $S_{n,n+1}^M(s)$ is defined by
\[
S_{n,n+1}^M(s) = \sum_{j=0}^{n-1} I_{[t_j,t_{j+1})}(s) S(t_n - t_j) P^M + I_{[t_{n+1},T]}(s) P^M.
\]

Thanks to $H^s$-regularity of $u(t)$ in section 2, it follows that
\[
\|u(t) - P^M u(t)\|^{2p} \leq C\lambda M^{2p} \|u(t)\|^{2p}.
\]

Therefore, in order to show the strong convergence of (4.1), it suffices to estimate the error $P^M u(t_n) - u_n^M$. To simplify the presentation, we only present the detailed steps of the strong convergence analysis of (4.1) in the multiplicative noise case. Define a sequence of subsets of $\Omega$,
\[
\begin{align*}
\Omega_{R_1}^{n+1} & := \left\{ \sup_{s \in [0,(n+1)\delta t]} \left( \|u(s)\|_{H^1} + \|u_M^M\|_{H^1} \right) \leq R_1 \right\}, \text{ if } d = 1, \text{ and} \\
\Omega_{R_1}^{n+1} & := \left\{ \sup_{s \in [0,(n+1)\delta t]} \left( (1 + \|u(s)\|_{H^1}) (1 + \log(1 + \|u(s)\|_{H^2})) \\
& \quad + (1 + \|u_M^M\|_{H^2}) (1 + \log(1 + \|u_M^M\|_{H^2})) \right) \leq R_1 \right\}, \text{ if } d = 2.
\end{align*}
\]

It can be seen that $\Omega_{R_1}^n$ is increasing w.r.t. $R_1$ and decreasing w.r.t. $n$. We omit the details in the additive noise case and only present its error representation formula here.

**Proposition 5.1.** Let $T > 0, p \in \mathbb{N}^+$ and Assumption 2.1 hold with $s > \frac{4}{2}$. Then it holds that
\[
\|u_{n+1}^M - P^M u(t_{n+1})\|_{L^p(\Omega_{R_1}^{n+1};H^s)} \leq C(\delta t + \lambda^{\frac{s}{2}}) \sup_{s \in [0,(n+1)\delta t]} \left( 1 + \|u(s)\|^{2\sigma+1}_{L^{(2\sigma+1)p}(\Omega_{R_1}^{n+1};H^s)} + \|u_M^M(s)\|^{2\sigma+1}_{L^{(2\sigma+1)p}(\Omega_{R_1}^{n+1};H^s)} \right)
\]
\[
+ C \int_0^{t_{n+1}} R_1^{2\sigma'} \|u(r) - P^M u(r)\|_{L^p(\Omega_{R_1}^{n+1};H^s)} ds,
\]
where $\sigma' = \frac{s}{2}$ if $d = 1$, and $\sigma' = \sigma$ if $d = 2$. 

5.1.2. Error decomposition in multiplicative noise case. To eliminate the interaction effects of the unbounded operator and the multiplicative noise, we introduce the exponential transform \( v_n(t) = S(-(t - t_n))u(t), t \in [t_n, t_{n+1}] \) which satisfies the following SODE,

\[
dv_n(t) = S(-(t - t_n))f(S(t - t_n)v_n(t))dt - \frac{1}{2}S(-(t - t_n))\alpha(S(t - t_n)v_n(t))dt + S(-(t - t_n))g(S(t - t_n)v_n(t))dW(t),
\]

Via the above equation and \([33]\), we can transform the error estimate of SPDEs into that of infinite-dimensional SDEs. It can be also seen that to propose the error decomposition, we also need the sequence of subsets \( \Omega_i \) defined in (5.2) which is increasing w.r.t. \( s \) holds that for \( n \in \mathbb{N}^+ \),

\[
dP_M v_n(t) = S(-(t - t_n))P_M f(S(t - t_n)P_M v_n(t) + (I - P_M)u(t))dt - \frac{1}{2}S(-(t - t_n))P_M \alpha(S(t - t_n)v_n(t) + (I - P_M)u(t))dt + S(-(t - t_n))P_M g(S(t - t_n)v_n(t) + (I - P_M)u(t))dW(t).
\]

To propose the error decomposition, we also need the sequence of subsets \( \Omega_i \) defined in (5.2) which is increasing w.r.t. \( R_1 \) and decreasing w.r.t. \( n \).

**PROPOSITION 5.2.** Let \( T > 0, p \in \mathbb{N}^+ \) and Assumption (2.1) hold with \( s > \frac{d}{2} \). It holds that for \( s_1 \in \mathbb{N}^+ \),

\[
\left\| P_M u(t_{n+1}) - u_{n+1}^M \right\|_{L^{2p}(\Omega; \mathbb{H})}^2 \\
\leq C(1 + R_1^{2\sigma'}) \delta t + C \int_0^{t_{n+1}} (1 + R_1^{2\sigma'}) \| P_M u([r]\delta t) - u_r^M \|^2_{L^{2p}(\Omega; \mathbb{H})} dr \\
+ C(1 + R_1^{2\sigma'}) \lambda_M^{-s} \int_0^{t_{n+1}} \left( 1 + \| u(r) \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})} \right) ds \\
+ C(1 + R_1^{2\sigma'}) \lambda_M^{-s} + \delta_{\text{min}(2, s_1)} \sup_{r \in [0, t_{n+1}]} (\| u(r) \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})} + \| u_r^M(r) \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})}) \\
+ C(1 + R_1^{2\sigma'}) \lambda_M^{-s} + \delta_{\text{min}(1, s_1)} \sup_{r \in [0, t_{n+1}]} (\| u(r) \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})} + \| u_r^M(r) \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})}) \\
\times \left( \int_0^{t_{n+1}} \| P_M u([r]\delta t) - u_r^M \|^2_{L^{2p}(\Omega; \mathbb{H}^{s+1})} dr \right)^{\frac{1}{2}},
\]

where \( \sigma' = \frac{d}{2} \) if \( d = 1 \), and \( \sigma' = \sigma \) if \( d = 2 \).

**PROOF.** For convenience, let us define a global auxiliary process \( v(t) \) by \( v(t) = v_n(t) \) for \( t \in [t_n, t_{n+1}] \) and \( v(t_{n+1}) = S(-\delta t)u(t_{n+1}) \). For \( t \in [t_n, t_{n+1}] \), applying the Itô formula to \( \| P_M v(t) - S(-\delta t)u_{n}^S(t) \|^2 \) and using the fact \( \langle w, \bar{w} \rangle = 0, w \in \mathbb{H} \), we obtain

\[
\left\| P_M v(t) - P_M S(-\delta t)u_{n}^S(t) \right\|^2 \\
= \left\| P_M u(t_n) - \lim_{t \to t_n} S(-\delta t)u_{n}^S(t) \right\|^2 \\
+ 2 \int_{t_n}^{t} (P_M v(r) - S(-\delta t)u_{n}^S(r), S(t_n - r) f(S(r - t_n)v(r)) - S(-\delta t) \alpha u_{n}^S(t_n) u_{n}^M(r)) \delta r \\
+ 2 \int_{t_n}^{t} (P_M v(r) - S(-\delta t)u_{n}^S(r), (S(t_n - r) g(S(r - t_n)v(r)) - S(-\delta t) g(u_{n}^S(r))) dW(r)) \\
= \left\| P_M u(t_n) - u_{n}^M \right\|^2
\]
and 2.1, 2.2 and 4.1, it follows that

\[ H_{5.6} \]

\[
+ 2 \int_{t_n}^t (P^M v(r) - S(-\delta t)u^M_S(r), S(t_n - r)f(S(r - t_n)v(r)) - S(-\delta t)f(S(\delta t)v(r)))dr
\]

\[
+ 2 \int_{t_n}^t (P^M v(r) - S(-\delta t)u^M_S(r), S(-\delta t)f(S(\delta t)v(r)) - S(-\delta t)i\lambda|u^M_S(t_n)|^{2\sigma} u^M_S(r))dr
\]

\[
+ 2 \int_{t_n}^t (P^M v(r) - S(-\delta t)u^M_S(r), (S(t_n - r)g(S(r - t_n)v(r)) - S(-\delta t)g(S(\delta t)v(r)))dW(r)
\]

\[
+ 2 \int_{t_n}^t (P^M v(r) - S(-\delta t)u^M_S(r), (S(-\delta t)g(S(\delta t)(I - P^M)v(r)))dW(r)
\]

\[
=: \|P^M[u(t_n) - u^M_S(t_n)]^2 + Err_1^n(t) + Err_2^n(t) + Err_3^n(t) + Err_4^n(t).
\]

Notice that for \( t \in [t_n, t_{n+1}] \),

\[ (5.5) \]

\[
\|v(t) - u(t)\| = \|(S(\xi - t_n) - I)u(t)\| \leq C\delta t^{\min(\frac{1}{2}, 1)}\|u(t)\|_{\mathbb{H}^{s}},
\]

\[
v(t) - v(t_n) = \int_{t_n}^t \left( S(t_n - r)f(S(r - t_n)v(r)) - \frac{1}{2}S(t_n - r)\alpha S(r - t_n)v(r) \right)dr
\]

\[
+ \int_{t_n}^t S(t_n - r)g(S(r - t_n)v(r))dW(r),
\]

and

\[ (5.6) \]

\[
u^M_S(t_n) - u^M_S(t_n) = \int_{t_n}^t P^M \left( i\lambda|u^M_S(t_n)|^{2\sigma} u^M_S(r) - \frac{1}{2}\alpha u^M_S(r) \right)dr
\]

\[
+ \int_{t_n}^t P^M g(u^M_S(r))dW(r).
\]

According to (5.6) and the fact that \( \mathbb{H}^s, s > \frac{d}{2} \) forms an algebra, using the unitarity of \( S(\cdot) \), (4.6), and the Sobolev embedding theorem, we have that

\[
Err_1^n(t) \leq C \int_{t_n}^t \|P^M v(r) - S(-\delta t)u^M_S(r)\|\delta t^{\min(1, \frac{1}{2})}(1 + \|u(r)\|_{\mathbb{H}^{2\sigma+1}}^{2\sigma+1})dr.
\]

Similarly, using (5.6), (4.9), Holder’s and Young’s inequalities, as well as Propositions 2.1, 2.2 and 4.1 it follows that

\[
Err_2^n(t)
\]

\[
\leq C \int_{t_n}^t \|P^M v(r) - S(-\delta t)u^M_S(r)\|^2 \left( 1 + \|P^M v(r)\|_{L_\infty}^{2\sigma} + \|u^M_S(t_n)\|_{L_\infty}^{2\sigma} \right)dr
\]

\[
+ C \int_{t_n}^t \|P^M v(r) - S(-\delta t)u^M_S(r)\| \left( \|u^M_S(t_n)\|_{L_\infty}^{2\sigma} + \|v(r)\|_{L_\infty}^{2\sigma} \right)\|\delta t\|\delta t^{\min(1, \frac{1}{2})}(1 + \|u(r)\|_{\mathbb{H}^{2\sigma+1}}^{2\sigma+1})dr
\]

\[
+ C \int_{t_n}^t \|P^M v(r) - S(-\delta t)u^M_S(r)\| \left( \|u^M_S(t_n)\|_{L_\infty}^{2\sigma} + \|v(r)\|_{L_\infty}^{2\sigma} \right)\|u^M_S(r) - u^M_S(t_n)\|dr
\]

\[
\leq C \left( 1 + R_1^{2\sigma} \right) \left[ \int_{t_n}^t \|P^M v(t_n) - S(-\delta t)u^M_S(t_n)\|^2 dr + \int_{t_n}^t \|v(r)\|_{\mathbb{H}^{2\sigma}}^{2\sigma} \lambda^{n+\frac{d}{2}} \right]
\]

\[
+ \delta t\|u^M_S(t_n)\|_{L_1}^{2\sigma} + \int_{t_n}^t \|u^M_S(r)\|_{\mathbb{H}^{2\sigma}}^{2\sigma+2} ds + \|g(u^M_S(s))dW(s)\|^2 dr
\]

\[
+ C \int_{t_n}^t \left( 1 + \|v(r)\|_{L_4}^{2\sigma+1} + \|u^M_S(r)\|_{L_4}^{2\sigma+1} \right)dr \right|^2 dr
\]
Thus its estimate is omitted.

Using (5.3) and (5.4), as well as Burkholder’s inequality, it follows that for $s_1 \in \mathbb{N}^+$,

$$
\begin{align*}
\| & \sum_{j=0}^{n} \mathcal{E} \mathcal{R}_j(t_{j+1}) + \mathcal{E} \mathcal{R}_j(t_{j+1}) \mathcal{L} \mathcal{P}(\Omega_{t_{j+1}}, \mathbb{R}) \\
& \leq C \left( \int_{0}^{t_{n+1}} \left( \mathcal{E} \left[ \left. \left\| P^M u(r) \right\| \mathcal{L}^2 \mathcal{P}(\Omega_{t_{j+1}}) \right| \delta t \right] \delta t \right)^{\frac{1}{2}} \left( \lambda_M^{\frac{1}{2}} + \delta t \min \left( \frac{1}{2}, 1 \right) \right) \sup_{r \in [0, t_{j+1}]} \| u(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega_{r+1})} \\
& \times \left( \lambda_M^{\frac{1}{2}} + \delta t \min \left( \frac{1}{2}, 1 \right) \right) \sup_{r \in [0, t_{j+1}]} \| u(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega_{r+1})} \right) + \left( \int_{0}^{t_{n+1}} \left( P^M v(r) - \mathcal{P}^M v[r] \right) + \left( S(-\delta t)(u_M S((r)) - u_M S(r)) \right) \mathcal{L}^2 \mathcal{P}(\Omega_{t_{j+1}}) \delta t \right)^{\frac{1}{2}} \\
& \leq C \left( \int_{0}^{t_{n+1}} \left( \mathcal{E} \left[ \left. \left\| P^M u(r) \right\| \mathcal{L}^2 \mathcal{P}(\Omega_{t_{j+1}}) \right| \delta t \right] \delta t \right)^{\frac{1}{2}} \left( \lambda_M^{\frac{1}{2}} + \delta t \min \left( \frac{1}{2}, 1 \right) \right) \sup_{r \in [0, t_{j+1}]} \| u(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega_{r+1})} \\
& + \left( \int_{0}^{t_{n+1}} P^M v(r) + \mathcal{P}^M v[r] + S(-\delta t)(u_M S((r)) - u_M S(r)) \mathcal{L}^2 \mathcal{P}(\Omega_{t_{j+1}}) \delta t \right)^{\frac{1}{2}} \right) \left( \lambda_M^{\frac{1}{2}} + \delta t \min \left( \frac{1}{2}, 1 \right) \right) \sup_{r \in [0, t_{j+1}]} \| u(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega_{r+1})} \right).
\end{align*}
$$

By applying (5.3), (5.4), the Sobolev embedding theorem, Propositions 2.1, 2.2 and 4.1, it follows that for $r \in [t_j, t_{j+1}]$,

$$
\begin{align*}
\| P^M v(r) - \mathcal{P}^M v[r] + S(-\delta t)(u_M S((r)) - u_M S(r)) \|_{\mathcal{L}^2 \mathcal{P}(\Omega; \mathbb{H})} \\
& \leq \left\| \int_{t_j}^{r} \left( S(t_n - s)f(S(s - t_n))v(s) - \frac{1}{2} S(t_n - s)C S(s - t_n)v(s) \right) ds \right\|_{\mathcal{L}^2 \mathcal{P}(\Omega; \mathbb{H})} \\
& + \left\| \int_{t_j}^{r} \left( 1\lambda |u_M S(t_n)|^2 \mathcal{P}^M u_M S(s) - \frac{1}{2} \alpha u_M S(s) \right) ds \right\|_{\mathcal{L}^2 \mathcal{P}(\Omega; \mathbb{H})} \\
& + \left\| \int_{t_j}^{r} P^M S(t_n - s)g(S(s - t_n))v(s) - g(u_M S(s))dW(s) \right\|_{\mathcal{L}^2 \mathcal{P}(\Omega; \mathbb{H})} \\
& \leq C \delta t + C \delta t \left( \lambda_M^{\frac{1}{2}} + \delta t \min \left( \frac{1}{2}, 1 \right) \right) \sup_{r \in [t_j, t_{j+1}]} \left( \| u(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega; H^{r+1})} + \| u_M S(r) \|_{\mathcal{L}^2 \mathcal{P}(\Omega; H^{r+1})} \right). \end{align*}
$$

By expanding $\| P^M v(t) - S(-\delta t)u_M S(t) \|^2$ at the initial error, then taking $\mathcal{L}^p(\Omega_{t_{j+1}}; \mathbb{H})$ and using Propositions 2.1, 2.2 and 4.1, we have that

$$
\begin{align*}
\| P^M v(t_{n+1}) - S(-\delta t)u_M S(t_{n+1}) \|^2_{\mathcal{L}^2 \mathcal{P}(\Omega_{t_{j+1}}; \mathbb{H})} \\
& \leq \sum_{j \leq n} \left( \| \mathcal{E} \mathcal{R}_j(t_{j+1}) \|_{\mathcal{L}^p(\Omega_{t_{j+1}}; \mathbb{H})} + \| \mathcal{E} \mathcal{R}_j(t_{j+1}) \|_{\mathcal{L}^p(\Omega_{t_{j+1}}; \mathbb{H})} \right).
\end{align*}
$$
with respect to $n$ and that for some $C$ that a random positive sequence 

\[ \| Err_{i}^{n}(t_{j+1}) + Err_{4}^{n}(t_{j+1}) \|_{L^{p}(Ω^{n+1},\mathbb{E})}. \]

Summarizing up the estimates of $Err_{i}^{n} - Err_{4}^{n}$, and using Young’s inequality, we obtain (5.4). □

5.2. Strong convergence. In this part, we provide a new type of stochastic Gronwall’s inequality and combine it with Proposition 5.1 and Proposition 5.2 to show the strong convergence of the proposed scheme.

5.2.1. Truncated stochastic Gronwall’s inequality. The following lemma is based on a series of truncated subsets of $Ω$ which has its own interests and could be used to study the strong convergence problem for general non-monotone SDEs and SPDEs.

**Lemma 5.1.** Let $\{Ω_{R_{k}}^{n}\}_{n≤N}$ be a sequence of subsets of $Ω$ which is decreasing with respect to $n$ and increasing with respect to $R_{1} ≥ 1, σ_{1} ≥ 0$ and $ε_{N} ≥ 0$. Assume that a random positive sequence $\{a_{n}\}_{n≤N}$ in a Banach space $\mathbb{E}$ satisfies

\[ \sup_{n≤N} \| a_{n} \|_{L^{p}(Ω; \mathbb{E})} ≤ C(q_{0}), \quad ∀ q_{0} ∈ \mathbb{N}^{+}, \]

and that for some $C’ ≥ 0$,

\[ \| a_{n+1} \|_{L^{p}(Ω^{n+1},\mathbb{E})} ≤ C(1 + R_{1}^{2σ_{1}}) \sum_{k=0}^{n} \| a_{k} \|_{L^{p}(Ω^{k},\mathbb{E})} \delta t + ε_{N} + C’(\sum_{k=0}^{n} \| a_{k} \|_{L^{p}(Ω^{k},\mathbb{E})} \delta t)^{\frac{1}{2}} ε_{N}. \]

Then it holds that for some $l > 1$,

\[ \| a_{n} \|_{L^{p}(Ω; \mathbb{E})} ≤ \exp(C(T)R_{1}^{2σ_{1}}) \left( ε_{N} + Cε_{N}^{\frac{1}{2}} \sup_{k≤N} (\mathbb{P}(Ω^{k})^{\frac{1}{2}}) \right) + C \sup_{k≤N} (\mathbb{P}(Ω^{k})^{\frac{1}{2}}). \]

**Proof.** By using the uniform boundedness of $a_{n}$ in $L^{p}(Ω; \mathbb{E})$, Hölder’s inequality and the Chebyshev inequality, we have that for $l > 1$,

\[ \| a_{n+1} \|_{L^{p}(Ω^{n+1},\mathbb{E})} ≤ C(1 + R_{1}^{2σ_{1}}) \sum_{k=0}^{n} \| a_{k} \|_{L^{p}(Ω^{k},\mathbb{E})} \delta t + Cε_{N}^{\frac{1}{2}} \sup_{k≤N} (\mathbb{P}(Ω^{k})^{\frac{1}{2}}) + ε_{N}. \]

By using the discrete Gronwall’s inequality, we achieve that

\[ \| a_{n} \|_{L^{p}(Ω^{n},\mathbb{E})} ≤ \exp(C(T)R_{1}^{2σ_{1}}) \left( ε_{N} + Cε_{N}^{\frac{1}{2}} \sup_{k≤N} (\mathbb{P}(Ω^{k})^{\frac{1}{2}}) \right). \]

Using the Chebyshev inequality and the boundedness of $a_{n}$ in $L^{p}(Ω; \mathbb{E})$ yield that

\[ \| a_{n} \|_{L^{p}(Ω; \mathbb{E})} ≤ \exp(C(T)R_{1}^{2σ_{1}}) \left( ε_{N} + Cε_{N}^{\frac{1}{2}} (\mathbb{P}(Ω_{R_{k}}^{n})^{\frac{1}{2}}) \right) + C(\mathbb{P}(Ω_{R_{k}}^{n})^{\frac{1}{2}}). \]

□

**Corollary 5.1.** Under the assumption of Lemma 5.1, if $\lim_{R_{k}→∞} N→∞ \mathbb{P}(Ω_{R_{k}}^{N})^{\frac{1}{2}} = 0$, and $\lim_{N→∞} \epsilon_{N} = 0$. Then $\sup_{n≤N} \| a_{n} \|_{L^{p}(Ω; \mathbb{E})}$ is convergent to 0.

**Proof.** By applying Lemma 5.1 and letting $ε_{N} → 0$ firstly, we have that

\[ \| a_{n} \|_{L^{p}(Ω; \mathbb{E})} ≤ C \lim_{N→∞} \sup_{k≤N} (\mathbb{P}(Ω_{R_{k}}^{n})^{\frac{1}{2}}) ≤ \lim_{R_{k}→∞} \lim_{N→∞} C(\mathbb{P}(Ω_{R_{k}}^{N})^{\frac{1}{2}}). \]

Taking $R_{1}$ goes to $∞$, we complete the proof. □

When one is interested in the convergence rate of the numerical schemes, the following bootstrap type estimates will play a key role.
PROPOSITION 5.3. Under the assumption of Corollary 5.2, suppose that $P((\Omega_1^{N})^c) \leq C_1 R_1^{-p_1}$ for a large enough $p_1 \in \mathbb{N}^+$. Then it holds that for $\kappa > 1$,

$$\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} \left( \frac{\log((\epsilon_N + \frac{1}{N})^{-1})}{\kappa C(T)} \right)^{-\frac{p_1}{2M}}.$$

Proof. For convenience, we assume that $C(\mathbb{P}((\Omega_1^{N}))^{\frac{1}{\beta}} \leq 1$. According to Lemma 5.1, letting $\exp(R_1^{2\sigma_1} C(T)(\epsilon_N + \frac{1}{N}) \leq CC_1^{\frac{1}{R_1}} R_1^{-\frac{\eta}{pl}}$, we have that for $\kappa > 0$

$$\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} R_1^{-\frac{\eta}{pl}}.$$

Taking $R_1 = \left( \frac{\log((\epsilon_N + \frac{1}{N})^{-1})}{\kappa C(T)} \right)^{\frac{1}{\beta}}$ leads to the desired result. \hfill $\Box$

PROPOSITION 5.4. Under the assumption of Corollary 5.2, suppose that $P((\Omega_1^{N})^c) \leq C_1 \exp(-\eta R_1^{2\sigma_2})$ with $\eta > 0, \sigma_2 > 0$. Then it holds that

$$\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} \exp \left( -\frac{\eta}{pl} \left( \frac{\log((\epsilon_N + \frac{1}{N})^{-1})}{\kappa C(T)} \right) \right).$$

In particular, when $\sigma_1 < \sigma_2$, it holds that for any $\gamma_1 \in (0,1)$,

$$\|a_n\|_{L^p(\Omega;E)} \leq CC_1^{\frac{1}{\gamma_1}}.$$

Proof. For convenience, we assume that $C(\mathbb{P}((\Omega_1^{N}))^{\frac{1}{\beta}} \leq 1$. Applying Lemma 5.1 and letting $\exp(R_1^{2\sigma_1} C(T)(\epsilon_N + \frac{1}{N}) \leq CC_1^{\frac{1}{R_1}} \exp(-\frac{\eta}{pl} R_1^{2\sigma_2})$, we have that

$$\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} \exp(-\frac{\eta}{pl} R_1^{2\sigma_2}).$$

Taking $R_1 = \left( \frac{\log((\epsilon_N + \frac{1}{N})^{-1})}{\kappa C(T)} \right)^{\frac{1}{\beta}}$ yields that

$$\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} \exp \left( -\frac{\eta}{pl} \left( \frac{\log((\epsilon_N + \frac{1}{N})^{-1})}{\kappa C(T)} \right) \right).$$

It can be seen that for $\sigma_1 = \sigma_2$, $\|a_n\|_{L^p(\Omega;E)} \leq 2CC_1^{\frac{1}{R_1}} (\epsilon_N + \frac{1}{N})^{-\frac{\eta}{pl} R_1^{2\sigma_2}}$. When $\sigma_1 < \sigma_2$, we use the following bootstrap arguments to improve the convergence rate. Applying Lemma 5.1 yields that for a large enough $R_1 > 0$ such that

$$\|a_n\|_{L^p(\Omega;E)} \leq \exp(R_1^{2\sigma_1} C(T)) \left( \epsilon_N + C\epsilon_N^{\frac{1}{R_1}}(\mathbb{P}(\Omega_1^{N}))^{\frac{1}{\beta}} \right) + C(\mathbb{P}(\Omega_1^{N}))^{\frac{1}{\beta}}$$

$$\leq \exp(R_1^{2\sigma_1} C(T)) \left( \epsilon_N + \frac{1}{N} \right) + CC_1^{\frac{1}{R_1}} \exp \left( -\frac{\eta}{pl} R_1^{2\sigma_2} \right).$$

Taking $\frac{\eta}{pl} R_1^{2\sigma_2} = \log \left( (\epsilon_N + \frac{1}{N})^{-1} \right)$, we have that

$$\|a_n\|_{L^p(\Omega;E)} \leq (1 + \exp(R_1^{2\sigma_1} C(T))) \epsilon_N^{\frac{1}{R_1}}$$

$$\leq \left( 1 + \exp \left( \frac{pl}{\eta} \log \left( \frac{pl}{\eta} \left( \epsilon_N + \frac{1}{N} \right)^{-1} \right) C(T) \right) \right) \epsilon_N + \frac{1}{N}.$$
Noticing that 
\[ \exp \left( \left( \frac{M}{T} \right)^{\alpha} \log^{\frac{\alpha}{\sigma}} \left( \left( \epsilon_N + \frac{\epsilon}{N} \right)^{-1} \right) \left( C(T) \right) \right) \leq \left( \epsilon_N + \frac{\epsilon}{N} \right)^{-\gamma}, \forall \gamma \in (0, 1), \]

it follows that
\[ \|a_n\|_{L^p(\Omega; \mathbb{E})} \leq \left( \epsilon_N + \frac{\epsilon}{N} \right)^{1-\gamma}. \]

Now applying again Lemma 5.1 and Young's inequality, we get for sufficient small 
\( \gamma > 0 \),
\[ \|a_n\|_{L^p(\Omega; \mathbb{E})} \leq \exp(R^{2\gamma_1}(C(T)) \left( \epsilon_N + \frac{\epsilon}{N} \left( \epsilon_N + \frac{\epsilon}{N} \right)^{1-\gamma} \right) + C(\mathbb{P}(\Omega_N^{R_2}))^{\frac{1}{p}} \]
\[ \leq C \exp(R^{2\gamma_1}(C(T)) \left( \epsilon_N + \epsilon_1^{-\gamma} \right) + C(\mathbb{P}(\Omega_N^{R_2}))^{\frac{1}{p}}. \]

By repeating the above procedures and taking \( \frac{\eta}{p} R^{2\gamma_2} = \log \left( \left( \epsilon_N + \epsilon_1^{-\gamma} \right)^{-1} \right) \), we complete the proof. \( \square \)

At the end of this part, we present one more estimate which will be used in studying 2D SNLSEs.

**Proposition 5.5.** Under the assumption of Corollary 5.1, suppose that \( \mathbb{P}(\Omega_N^{R_1}) \leq C_1 \log^{-p_1}(R_1) \) for a large enough \( p_1 \geq 1 \). Then it holds that
\[ \|a_n\|_{L^p(\Omega; \mathbb{E})} \leq 2CC_1^{\frac{1}{p}} \left( (2\gamma_1) \frac{1}{p} \right) \log^{-p_1} \left( \left( \epsilon_N + \epsilon_1^{-\gamma} \right)^{-1} \right) \log(\kappa C(T)). \]

**Proof.** Without loss of generality, we assume that \( C(\mathbb{P}(\Omega_N^{R_1}))^{\frac{1}{p}} \leq 1 \). Applying Lemma 5.1 and letting \( \exp(R^{2\gamma_1}(C(T)) \left( \epsilon_N + \epsilon_1^{-\gamma} \right) \leq CC_1^{\frac{1}{p}} \log^{-p_1}(R_1) \), we have that
\[ \|a_n\|_{L^p(\Omega; \mathbb{E})} \leq 2CC_1^{\frac{1}{p}} \log^{-p_1}(R_1). \]

Taking \( R_1 = \left( \frac{\log(\left( \epsilon_N + \frac{\epsilon}{N} \right)^{-1})}{\kappa C(T)} \right)^{\frac{1}{2}} \) yields that
\[ \|a_n\|_{L^p(\Omega; \mathbb{E})} \leq 2CC_1^{\frac{1}{p}} \log^{-p_1} \left( \left( \log \left( \left( \epsilon_N + \frac{\epsilon}{N} \right)^{-1} \right) \right) \kappa C(T) \right) \]
\[ \leq 2CC_1^{\frac{1}{p}} \left( (2\gamma_1) \frac{1}{p} \right) \log^{-p_1} \left( \left( \epsilon_N + \epsilon_1^{-\gamma} \right)^{-1} \right) \log(\kappa C(T)). \]

\( \square \)

5.2.2. **Strong convergence.** By combining the truncated stochastic Gronwall’s inequality (Lemma 5.1, Propositions 5.3, 5.4 and Corollary 5.6), and the regularity estimates and tail estimates of the exact and numerical solutions in Section 2 and Section 4, we are in a position to show several convergence properties of the proposed schemes for SNLSEs.

**Theorem 5.1.** Let \( T > 0, p \in \mathbb{N}^+, d = 1 \) and (4.5) hold. Suppose that Assumption (2.7) or (2.8) holds with \( s \geq 1 \). Then the scheme (4.11) satisfies
\[ \sup_{n \leq N} \|u(t_n) - u_n\|_{L^{2^p}(\Omega; \mathbb{H})} \leq C(T, Q, \Psi, \lambda, \sigma, p) \log^{-\kappa_1} \left( (\delta t^s + \lambda^{-s})^{-1} \right), \forall \kappa_1 \geq 1. \]

Furthermore, for \( g(\xi) = i\xi \) and \( \sigma = 1 \), it holds that
\[ \sup_{n \leq N} \|u(t_n) - u_n\|_{L^{2^p}(\Omega; \mathbb{H})} \leq C(T, Q, \Psi, \lambda, p) \left( (\delta t^s + \lambda^{-s})^{-1} \right)^{1-\gamma}, \forall \gamma \in (0, 1). \]
Propositions 2.1, 4.2 and 4.3, yield (5.11). In the multiplicative noise case, applying Lemma 5.1 and Proposition 5.3 to (5.4), taking tail estimates in Corollary 2.1 yield (5.10). In the multiplicative noise case, applying Lemma 5.1 and Proposition 5.3 to (5.5), taking a lead to (5.10).

When $g(\xi) = i\xi$ and $\sigma = 1$, applying Lemma 5.1 and (5.8) with $a_n = \|u(t_n) - u_n\|^2, s = s_1$ and $\epsilon_N \sim O(\delta t^\frac{\lambda}{2} + \lambda_M^{-\frac{\sigma}{2}})$, using Corollaries 2.1 and 4.1, Propositions 2.1 and 4.2, lead to (5.10).

Compared with the existing strongly convergent results for 1D SNLSEs [12, 14], our strong analysis only requires $H^1$-regularity to achieve the strong convergence order $\frac{1}{2}$ in time. The advantage of the implicit splitting Crank–Nicolson scheme in [12, 14] is its unconditional stability in energy space. By applying the truncated stochastic Gronwall’s inequality and Proposition 5.3 to its error decomposition one can obtain (5.11) if $d = 1, \sigma < 2$ and the convergence rate $O(\lambda_M^{\frac{\sigma}{2}} + \delta t) = \log \frac{1}{\lambda_M}$ if $d = 1, \sigma = 2$. When $T$, the scale of $W$, or $\|\Psi\|$ is sufficiently small, via the large deviation principle of SNLSEs and our current approach, it seems possible to prove the strong convergence estimate (5.11) for general $\sigma$. This will be investigated in the future.

**Theorem 5.2.** Let $T > 0, p \in \mathbb{N}^+$, $d = 2$ and (4.5) hold. Suppose that Assumption 2.7 or 2.2 holds with $s \geq 2$. The scheme (4.1) satisfies that for $\kappa_1 \geq 1$,

\[
(5.12) \quad \sup_{n \leq N} \|u(t_n) - u_n\|_{L^2(\Omega; H)} \leq C(T, Q, \Psi, \lambda, p) \log^{-\kappa_1} \left( \log (\delta t^\frac{\lambda}{2} + \lambda_M^{-\frac{\sigma}{2}}) \right)^{-1}.
\]

**Proof.** In the additive noise case, applying Lemma 5.1 and Proposition 5.3 to (5.3) with $a_n = \|u(t_n) - u_n\|, s = 1, C' = 0, \sigma_1 = \sigma', \epsilon_N \sim R^2_0 O(\delta t^\frac{\lambda}{2} + \lambda_M^{-\frac{\sigma}{2}})$, and using the tail estimates (2.6) and (4.16) yield (5.12). In the multiplicative noise case, applying Lemma 5.1 and Proposition 5.3 to (5.3), taking $a_n = \|u(t_n) - u_n\|^2, s = 1$, and $\epsilon_N \sim R_0^2 O(\delta t^\frac{\lambda}{2} + \lambda_M^{-\frac{\sigma}{2}}), \sigma_1 = \sigma'$, using (2.6) and (4.16), Propositions 2.2 and 4.2 lead to (5.10).

At last, we present the convergence of (4.1) for Eq. (2.7) with random coefficients on a compact Riemannian manifold in $d \geq 2$ without boundary.

**Theorem 5.3.** Let $d \geq 2, \lambda = -1, \sigma = 1, p \in \mathbb{N}^+, s > \frac{d}{2}, s \in \mathbb{N}^+, \Psi \in H^s, \mathcal{V}(\cdot) \in W^{s, \infty}$, and (4.5) hold. Suppose that the scheme (4.1) satisfies

\[
\lim_{R_2 \to \infty} \lim_{N \to \infty} \mathbb{P}((\tilde{\Omega}_R^N)^c) = 0,
\]

where

\[
\tilde{\Omega}_R^N := \left\{ \sup_{s \in [0, t_n]} \|u_{[s]}^M\|_{H^s} \leq R_2, R_2 \in \mathbb{R}^+, n \leq N \right\}.
\]

Then the numerical solution is strongly convergent to the solution of (2.7), i.e.,

\[
\lim_{N \to \infty} \sup_{n \leq N} \|u(t_n) - u_n\|_{L^2(\Omega; H)} = 0.
\]

**Proof.** The proof is similar to that of Theorem 5.1. For the sake of simplicity, we only present a sketch of this proof since the solution of Eq. (2.7) with random coefficient is more regular than that of Eq. (1.1). At first step, one could follow
the steps in Section 4 to show that $\mathbb{E} \left[ \sup_{t \in [0, T]} \| u(t) \|_{L^p}^{2p} \right] + \mathbb{E} \left[ \sup_{n \leq N} \| u_n^M \|_{L^p}^{2p} \right] < \infty$.

Second, similar to Proposition [5.1] by using the mild formulation of the numerical solution, one obtain the following error decomposition,

$$
\| P^M u(t_{n+1}) - u^M_{n+1} \|_{L^{2p}(\Omega_{t_1}^{t_{n+1}}; \mathbb{H})} 
\leq C(1 + R_1^2) \int_0^{t_{n+1}} \| P^M u([s], s) - \tilde{u}(s) \|_{L^{2p}(\Omega_s^{[c]}; \mathbb{H})} ds + C(1 + R_1^2) (\delta t^{\min(1, \frac{5}{2})} + \lambda_M^{\frac{s}{2}}),
$$

where $\Omega_{R_1}^n := \Omega_{R_1}^{[c]} \cap \{ \sup_{s \in [0, t_n]} \| u(s) \|_{\mathbb{H}} \leq R_1, \sup_{s \in [0, t_n]} \| B(s) \| \leq R_1 \}$. Then applying the assumption $\lim_{R_1 \to \infty} \lim_{N \to \infty} \mathbb{P}(\Omega_{R_1}^N)^c = 0$, Proposition 2.3 Lemma 5.1 and Corollary [5.7] lead to the desired result.

**Remark 5.1.** Based on our analysis, for a given numerical scheme for SNLSE in $d \geq 1$, the logarithmic tail estimate $\mathbb{P}(\Omega_{R_1}^N)^c \leq C_1 \log^{-p} (R_1)$ implies the double logarithmic convergence rate; the polynomial tail estimate $\mathbb{P}(\Omega_{R_1}^N)^c \leq C_1 R_1^{-p/2}$ implies the logarithmic convergence rate; and the exponential tail estimate implies the algebraic convergence rate. If there is no explicit decay rate for the tail estimate, we could use Corollary 5.1 to show the strong convergence without an explicit rate. One can follow this approach to get the strong convergence result for a large class of SODEs and SPDEs with non-monotone coefficients.

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6. Appendix

**Proof of Lemma 2.1.** Consider the Fourier series expansion $v = \sum_i \langle v, e_i \rangle e_i$. The condition that $u \in \mathbb{H}^2$ leads to $\sum_i (u, e_i)^2 (1 + \lambda_i^2) < \infty$. Notice that

$$\|u\|_{L^\infty} \leq \| \sum_i \langle u, e_i \rangle e_i \|_{L^\infty} \leq \sum_i |\langle u, e_i \rangle|.$$
For $\kappa > 0$, according to the Weyl’s law, the eigenvalue $\lambda_m \sim m^2$ with $d = 2$, it holds that
\[
\sum_i |\langle u, e_i \rangle| = \sum_{\sqrt{|\lambda_i|} < \kappa} |\langle u, e_i \rangle| + \sum_{\sqrt{|\lambda_i|} \geq \kappa} |\langle u, e_i \rangle| \\
\leq \sum_{\sqrt{|\lambda_i|} < \kappa} (1 + \sqrt{|\lambda_i|})|\langle u, e_i \rangle| \frac{1}{1 + \sqrt{|\lambda_i|}} + \sum_{\sqrt{|\lambda_i|} \geq \kappa} (1 + |\lambda_i|)|\langle u, e_i \rangle| \frac{1}{1 + |\lambda_i|} \\
\leq C\|u\|_{\mathcal{H}^2}^2 \left( \sum_{\sqrt{|\lambda_i|} < \kappa} \frac{1}{1 + \sqrt{|\lambda_i|}} \right)^{\frac{1}{2}} + C\|u\|_{\mathcal{H}^2}^2 \left( \sum_{\sqrt{|\lambda_i|} \geq \kappa} \frac{1}{1 + |\lambda_i|} \right)^{\frac{1}{2}} \\
\leq C\|u\|_{\mathcal{H}^2}^2 \left( \sum_{\sqrt{|\lambda_i|} < \kappa} \frac{1}{1 + \sqrt{|\lambda_i|}} \right)^{\frac{1}{2}} + C\|u\|_{\mathcal{H}^2}^2 (1 + \kappa)^{-1}.
\]
Taking $\kappa = \|u\|_{\mathcal{H}^2}$, we complete the proof.

PROOF OF PROPOSITION 4.2 When $d = 1$, one can use the auxiliary functional $V(w) = \|(-\Delta)^{\frac{d}{2}}w\|^2 - \lambda \langle (-\Delta)^{\frac{d-2}{2}}w, |w|^{2\sigma}w \rangle$ and follow the arguments in the proof of [13 Theorem 2.1] to show that
\[
\mathbb{E} \left[ \sup_{n \leq N} \|u_n^M\|_{\mathbb{H}^2}^{2p} \right] < \infty.
\]
Next, we focus on the case that $d = 2$, $\sigma = 1$. Since $u^M_S$ preserving $\mathbb{H}^s$-norm for any $s \in \mathbb{N}$, applying the Itô formula to $\widetilde{U}^p(u^M_S(t))$ and using the integration by parts, we obtain
\[
\widetilde{U}^p(u^M_S(t)) \\
= \widetilde{U}^p(u^M_S(0)) + \int_0^t p\widetilde{U}^{p-1}(u^M_S(s)) \frac{1}{1 + \log(1 + \|\Delta u^M_S(s)\|_2) + \|\Delta u^M_S(s)\|_2} \left( \langle \Delta u^M_S(s), 1 \rangle \Re(\langle u^M_S(s) \Delta u^M_S(s) \rangle) u^M_S(s) + 1 \lambda^2 \Re(\langle u^M_S(s) \nabla u^M_S(s) \rangle) \nabla u^M_S(s) + i \lambda^2 |\nabla u^M_S(s)|^2 u^M_S(s) \rangle \right) ds \\
- \int_0^t \frac{1}{1 + \log(1 + \|\Delta u^M_S(s)\|_2) + \|\Delta u^M_S(s)\|_2} \left( \langle \Delta u^M_S(s), 2 \nabla Q^\frac{1}{2} e_i \rangle u^M_S(s) + 2 \nabla u^M_S(s) \nabla Q^\frac{1}{2} e_i + \Delta u^M_S(s) |Q^\frac{1}{2} e_i|^2 \\
+ 2 u^M_S(s) \Delta Q^\frac{1}{2} e_i, Q^\frac{1}{2} e_i \rangle \right) ds \\
+ \int_0^t 2p\widetilde{U}^{p-1}(u^M_S(s)) \frac{1}{1 + \log(1 + \|\Delta u^M_S(s)\|_2) + \|\Delta u^M_S(s)\|_2} \left( \langle \Delta u^M_S(s), i \Delta u^M_S(s) dW(s) \rangle \right) \\
+ \int_0^t \frac{1}{1 + \log(1 + \|\Delta u^M_S(s)\|_2) + \|\Delta u^M_S(s)\|_2} \left( \langle P^M \nabla u^M_S(s) \nabla Q^\frac{1}{2} e_i \rangle + \|P^M \Delta u^M_S(s) Q^\frac{1}{2} e_i \rangle + \|P^M u^M_S(s) \Delta Q^\frac{1}{2} e_i \rangle \\
+ P^M u^M_S(s) \Delta Q^\frac{1}{2} e_i, \nabla u^M_S(s) \nabla Q^\frac{1}{2} e_i \rangle + 2 \langle P^M \nabla u^M_S(s) \nabla Q^\frac{1}{2} e_i, \Delta u^M_S(s) Q^\frac{1}{2} e_i \rangle \right) ds.
\]
\[ \begin{aligned}
+ 2 \langle P^M u_S^M(s) \Delta Q^e e_1, \Delta u_S^M(s) Q^e e_1 \rangle ds \\
+ \int_0^t -2p\tilde{U}_p^{-1}(u_S^M(s)) \frac{1}{1 + \log(1 + \|\Delta u_S^M(s)\|^2)} \frac{1}{(1 + \log(1 + \|\Delta u_S^M(s)\|^2)) - (p - 1)} \sum_{i \in \mathbb{N}^+} \left( \langle \Delta u_S^M(s), i\nabla u_S^M(s) \nabla Q^e e_1 \rangle + \langle \Delta u_S^M(s), iu_S^M(s) \Delta Q^e e_1 \rangle \right)^2 ds.
\end{aligned} \]

Similar arguments as in the proof of Proposition 2.2 yield that for a small \( \epsilon \in (0, 1) \),
\[ \mathbb{E} \left[ \sup_{t \in [0, t_1]} \tilde{U}_p(u_S^M(t)) \right] \leq \mathbb{E}[\tilde{U}_p(u_S^M(0))] + c\mathbb{E}[\sup_{t \in [0, t_1]} \tilde{U}_p(u_S^M(t))] + C(\epsilon) \int_0^{t_1} \mathbb{E}[\tilde{U}_p(u_S^M(s))] ds \\
+ C(\epsilon) \int_0^t \mathbb{E} \left[ 1 + \|u_S^M(s)\|^{2p} + \|\nabla u_S^M(s)\|^{2p} \right] ds. \]

The Gronwall’s inequality and Proposition 4.1 yield that
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \tilde{U}_p(u_S^M(t)) \right] \leq C(T, Q, \Psi, \lambda, \sigma, p), \]
which completes the proof. \( \Box \)

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