Restrictions on Gauge Groups in Noncommutative Gauge Theory

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We show that the gauge groups SU(N), SO(N) and Sp(N) cannot be realized on a flat noncommutative manifold, while it is possible for U(N).

I. INTRODUCTION

The Yang-Mills theories naturally arise as low energy limits of the theory of open strings. One can obtain Yang-Mills theories with different gauge groups by studying different D-brane configurations (see e.g. [1]). For instance, if we place N D-branes on top of each other in the flat space, the corresponding open string theory gives rise to the Yang-Mills theory with the gauge group U(N). One can also obtain gauge theories with other gauge groups such as SO(N) and Sp(N) by using the orientifold construction. In more detail, one combines the spatial reflection σ → π − σ on the string world-sheet with the target space reflection, Xμ → −Xμ, μ = 1, . . . , k and Xμ → Xμ, μ = (k + 1), . . . , 10. It is the goal of this note to study which gauge groups can be realized in the presence of the background B-field when the brane world-volume turns into a noncommutative space [2,3,4].

Interaction with the B-field introduces an extra term into the Polyakov action of the string [4],

\[\Delta S = -i \frac{1}{2} \int \Sigma B_{\mu\nu} e^{ab} \partial_a X^\mu \partial_b X^\nu. \] (1)

Here a, b = 1, 2 are world-sheet indices, Σ is the world-sheet and Bμν = −Bνμ is the B-field on the target space. One requires the expression (1) to be invariant with respect to the orientifold reflection. This implies the following transformation rules for components of the B-field,

\[B_{\|\|} \rightarrow -B_{\|\|} \quad B_{\perp\perp} \rightarrow -B_{\perp\perp} \]
\[B_{\|\perp} \rightarrow B_{\|\perp} \quad B_{\perp\|} \rightarrow B_{\perp\|}. \] (2)

Here the symbols \(\|\) and \(\perp\) stand for the target space indices μ = 1, . . . , k and μ = (k + 1), . . . , 10, respectively. In the orientifold construction we finally let the branes lie on the orientifold. The continuity of the B-field implies that the B-field on the brane, \(B_{\|\|}\), vanishes. Hence, the brane world-volumes are commutative since it is \(B_{\|\|}\) which is responsible for the noncommutativity [2,3,4]. This consideration indicates that one should encounter difficulties in the construction of the gauge theories with gauge groups SO(N) and Sp(N) on noncommutative spaces. Somewhat surprisingly, we find that the reduction from U(N) to SU(N) in the framework of noncommutative geometry also fails.

II. THE CLOSURE OF CLASSICAL LIE ALGEBRAS UNDER THE MOYAL COMMUTATOR

In the flat case the presence of a constant B-field turns the D-branes into noncommutative spaces, with the ordinary pointwise multiplication of functions replaced by the Moyal product,

\[ (X \ast Y)(x) = \exp\left(\frac{i}{2} \theta^{ij} \partial_i x \partial_j y \right) X(x) Y(y) \big|_{x=y} = X Y + \frac{i}{2} \theta^{ij} \partial_i X \partial_j Y + ... \] (3)

Here X and Y are functions on the D-brane world-volume, and \(\theta^{ij}\) is a real-valued constant antisymmetric tensor constructed of the metric and B-field [3]. The Moyal product naturally extends to N by N matrices, formula (3) still applies. One can also introduce the Moyal commutator by the formula,

\[ [X, Y]_\ast = X \ast Y - Y \ast X. \] (4)

In what follows we check whether the matrix Lie algebras of the classical Lie groups SO(N), U(N), SU(N) and Sp(N) are closed under the Moyal commutator. We choose to work in the fundamental representation of these Lie algebras.

The Lie algebra of U(N) consists of anti-Hermitian matrices, \(\bar{X}^T = -X\), where the bar stands for complex conjugation. We first show that this algebra is closed under the Moyal commutator. The key observation is the following property of the Moyal product,

\[ \overline{(X \ast Y)^T} = \overline{Y}^T \ast \overline{X}^T. \] (5)

By using the ordinary rules for the transpose of matrices we get,

\[ (X \ast Y)^T = Y^T X^T + \frac{i}{2} \theta^{ij} \partial_i Y^T \partial_j X^T - \frac{1}{8} \theta^{ijk} \partial_i Y^T \partial_j \partial_k X^T + ... \] (6)
The construction for higher order terms is obvious. Now we apply the complex conjugation and rename the indices of $\theta$ to obtain,

$$
(X \ast Y)^t = Y^t X^t + \frac{i}{2} \theta^{ij} \partial_i Y^t \partial_j X^t - \frac{1}{8} \theta^{ij} \theta^{kl} \partial_i \partial_k Y^t \partial_j \partial_l X^t + \ldots = Y^t \ast X^t. \tag{7}
$$

Taking into account $X^t = -X$ and $Y^t = -Y$ yields,

$$
[X, Y]^t_\ast = (X \ast Y)^t - Y^t \ast X^t = Y^t \ast X^t - X^t \ast Y^t = Y \ast X - X \ast Y = -[X, Y]^t_\ast \tag{8}
$$

which shows that the algebra $U(N)$ is closed under the Moyal commutator.

We now turn to the algebras of $SO(N)$, $SU(N)$, and $Sp(N)$. We first show that for $N = 2$ these algebras are not closed with respect to the Moyal commutator.

The counterexamples for both $SO(2)$ and $Sp(2)$ are given by formulas,

$$
X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}. \tag{9}
$$

and the counterexample for $SU(2)$ is

$$
X = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix}, \quad Y = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix}. \tag{10}
$$

Here $\alpha$ and $\beta$ are coordinates on the manifold chosen so that $\theta^{\alpha\beta} \neq 0$. This can always be done unless $\theta = 0$ and the Moyal product coincides with the ordinary multiplication of matrix-valued functions. With $X$ and $Y$ as given above one can easily compute the Moyal commutator since all derivatives of order higher than one vanish. The result for both counter examples is

$$
[X, Y]^t_\ast = i\theta^{\alpha\beta} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}
$$

Note that this matrix has a nonvanishing trace. Since the Lie algebras of $SO(2)$, $SU(2)$ and $Sp(2)$ consist of traceless matrices, we conclude that they are not closed under the Moyal commutator. This also applies to $SO(N)$, $SU(N)$ and $Sp(N)$ for arbitrary $N$ because they contain $SO(2)$, $SU(2)$ and $Sp(2)$ as subgroups.

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