Some properties of isoclinism in Lie superalgebras

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\textbf{ABSTRACT}

Isoclinism of Lie superalgebras has been defined and studied currently. In this article, it is shown that for finite dimensional Lie superalgebras of same dimension, the notation of isoclinism and isomorphism are equivalent. Furthermore, we show that covers of finite dimensional Lie superalgebras are isomorphic using isoclinism concept.

\textbf{ARTICLE HISTORY}

Received 14 April 2019
Communicated by Prof. K. C. Misra

\textbf{KEYWORDS}
Covers; factor set; isoclinism; Lie superalgebra

\textbf{2010 MATHEMATICS SUBJECT CLASSIFICATION}
Primary 17B30;
Secondary 17B05

\textbf{1. Introduction}

In 1940, P. Hall introduced an equivalence relation on the class of all groups called isoclinism, which is weaker than isomorphism and plays an important role in classification of finite $p$-groups [7]. In 1994, K. Moneyhun [8, 9] gave a Lie algebra analog of the concept of isoclinism. Furthermore, Saeedi and Veisi [13] have defined the same notation for $n$-Lie algebras. Similarly, isoclinism has been defined and studied for Lie superalgebras recently [12].

\textbf{Definition 1.1.} Let $L$ and $K$ be two Lie superalgebras, $\varphi : \frac{L}{Z(L)} \rightarrow \frac{K}{Z(K)}$ and $\theta : L' \rightarrow K'$ be Lie superalgebra homomorphisms such that the following diagram is commutative,

$$
\begin{array}{ccc}
L/Z(L) \times L/Z(L) & \xrightarrow{\mu} & L' \\
\varphi^2 \downarrow & & \downarrow \theta \\
K/Z(K) \times K/Z(K) & \xrightarrow{\rho} & K'
\end{array}
$$

where $\mu((\bar{l}, \bar{m})) := [l, m]$ for $l, m \in L$ and similarly for $\rho((\bar{r}, \bar{s})) := [r, s]$ for $r, s \in K$. Or, equivalently $\varphi$ and $\theta$ are defined in such a way that they are compatible, i.e., $\theta([l, m]) = [k, r]$, where...
In which \( k \in \phi (l + Z(L)) \) and \( r \in \phi (m + Z(L)) \). Then the pair \( (\phi, \theta) \) is called homoclinism and if they are both isomorphisms, then \( (\phi, \theta) \) is called isoclinism.

If \( (\phi, \theta) \) is an isoclinism between \( L \) and \( K \), then \( L \) and \( K \) are said to be isoclinic, which is denoted by \( L \sim K \). Isoclinism is an equivalence relation, and hence we have equivalence classes (or families) of isoclinism of Lie superalgebras.

Below we have listed some results on isoclinism of Lie superalgebras which are useful further (see [12]).

**Lemma 1.2.** If \( L \) is a Lie superalgebra and \( A \) is a abelian Lie superalgebra, then \( L \sim L \oplus A \).

**Lemma 1.3.** Let \( L \) be a Lie superalgebra and \( I \) be a graded ideal, then \( L/I \sim L/(I \cap L') \). In particular, if \( I \cap L' = 0 \) then \( L \sim L/I \). Conversely, if \( L' \) is finite dimensional and \( L \sim L/I \), then \( I \cap L' = 0 \).

**Lemma 1.4.** Let \( L \) and \( M \) be Lie superalgebras and \( f : L \to M \) be an onto homomorphism such that \( \text{Ker}(f) \cap L' = 0 \), then \( L \sim M \).

A Lie superalgebra \( L \) is called a stem Lie superalgebra, whenever \( Z(L) \cong L_0 \).

**Lemma 1.5.** [12] Suppose \( C \) is an isoclinism family of Lie superalgebras. Then

1. \( C \) contains a stem Lie superalgebra.
2. Each finite dimensional Lie superalgebra \( T \in C \) is stem if and only if \( T \) has minimal dimension in \( C \).

The covers and multipliers for Lie algebras are defined and studied by Betten et al. It has been shown that unlike the case of groups the covers of finite dimensional Lie algebras are isomorphic [1, 2]. Likewise covers and multipliers for Lie superalgebras have recently been defined and studied [11].

An extension of a Lie superalgebra \( L \) is a short exact sequence

\[
0 \to M \to K \to L \to 0.
\]

Since \( e : M \to e(M) = \ker(f) \) is an isomorphism we identify \( M \) and \( e(M) \). An extension of \( L \) is then same as an epimorphism \( f : K \to L \). If \( L \) is a Lie superalgebra generated by a \( \mathbb{Z}_2 \)-graded set \( X = X_0 \cup X_1 \) and \( \phi : X \to L \) is a degree zero map, then there exists a free Lie superalgebra \( F \) and \( \psi : F \to L \) extending \( \phi \). Let \( R = \ker(\psi) \). The extension

\[
0 \to R \to F \to L \to 0
\]

is called a free presentation of \( L \) and is denoted by \( (F, \psi) \). With this free presentation of \( L \), we define multiplier of \( L \) as [5]

\[
\mathcal{M}(L) = \frac{[F, F] \cap R}{[F, R]}.
\]

A homomorphism from an extension \( f : K \to L \) to another extension \( f' : K' \to L \) is a Lie superalgebra homomorphism \( g : K \to K' \) satisfying \( f = f' \circ g \); in other words, we have the following commutative diagram,

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{g} & & \downarrow{f'} \\
K' & & 
\end{array}
\]
A central extension of $L$ is an extension (1.1) such that $M \subseteq Z(K)$. The central extension is said to be a stem extension of $L$ if $M \subseteq Z(K) \cap K'$. Finally, we call the stem extension a stem cover if $M \cong M(L)$ and in this case $K$ is said to be a cover of Lie superalgebra $L$.

**Lemma 1.6.** [12] Stem covers exist for each Lie superalgebras.

**Definition 1.7.** Let (1.1) be a stem extension of $L$. The stem extension is maximal if every homomorphism of any other stem extension of $L$ onto $0 \to M \to K \to L \to 0$ is necessarily an isomorphism.

The following is an important result which we use further.

**Lemma 1.8.** [12] The stem extension $0 \to M \to K \to L \to 0$ of finite dimensional Lie superalgebra $L$ is maximal (or, equivalently dimension of $K$ is maximal) if and only if it is a stem cover of $L$.

Finally, below is another useful Lemma.

**Lemma 1.9.** Let the pair $(\varphi, \theta)$ be isoclinism of Lie superalgebras $L$ and $M$. Then

1. $\varphi(x + Z(L)) = \theta(x) + Z(M)$;
2. $\theta([x, y]) = \theta(x)z$, for all $x \in L'$, $y \in L$, $z + Z(M) = \varphi(y + Z(L))$.

**Proof.** We have the following commutative diagram,

\[
\begin{array}{ccc}
L/Z(L) \times L/Z(L) & \xrightarrow{\mu} & L' \\
\downarrow{\varphi^2} & & \downarrow{\theta} \\
M/Z(M) \times M/Z(M) & \xrightarrow{\rho} & M'
\end{array}
\]

Let $x = [x_1, x_2] \in L'$, then

\[
\theta(x) + Z(M) = \theta(x_1, x_2) + Z(M) \\
= \theta\mu(x_1 + Z(L), x_2 + Z(L)) + Z(M) \\
= \rho\varphi^2(x_1 + Z(L), x_2 + Z(L)) + Z(M) \\
= \rho(\varphi(x_1 + Z(L)), \varphi(x_2 + Z(L))) + Z(M) \\
= [\varphi(x_1 + Z(L)), \varphi(x_2 + Z(L))] + Z(M) \\
= \varphi([x_1 + Z(L), x_2 + Z(L])] = \varphi([x_1, x_2] + Z(L)) = \varphi(x + Z(L))
\]

which proves (1). To prove the second part, consider for $y \in L$, we have $z + Z(M) = \varphi(y + Z(L))$.

\[
\theta([x, y]) = \theta\mu(x + Z(L), y + Z(L)) = \rho\varphi^2(x + Z(L), y + Z(L)) \\
= \rho(\varphi(x + Z(L)), \varphi(y + Z(L))) \\
= [\varphi(x + Z(L)), \varphi(y + Z(L))] \\
= [\theta(x) + Z(M), z + Z(M)] = [\theta(x), z].
\]
In Section 2, we recall some notations of Lie superalgebras. In Section 3, we show that isoclinism and isomorphism are equivalent for finite dimensional Lie superalgebras having the same dimension. Specifically, we have introduced factor sets in Lie superalgebras to do the same. This result was first proved by Moneyhun [8] in 1994 for Lie algebras and then for \( n \)-Lie algebras by Eshrati et al. [4]. Finally in Section 4, as an application to the concept of isoclinism it is shown that covers for finite dimensional Lie superalgebras are isomorphic. The same result hold for Lie algebras (see [1, 2, 8, 9]) and for \( n \)-Lie algebras (see [4]).

2. Preliminaries

Here we fix some terminology on Lie superalgebras (see [6, 10]) and recall some notions. Set \( \mathbb{Z}_2 = \{0, 1\} \) is a field. A \( \mathbb{Z}_2 \)-graded vector space \( V \) is simply a direct sum of vector spaces \( V_0 \) and \( V_1 \), i.e., \( V = V_0 \oplus V_1 \). It is also referred as a superspace. We consider all vector superspaces and superalgebras are over \( \mathbb{F} \) (characteristic of \( \mathbb{F} \neq 2, 3 \)). Elements in \( V_0 \) (resp. \( V_1 \)) are called even (resp. odd) elements. Non-zero elements of \( V_0 \cup V_1 \) are called homogeneous elements. For a homogeneous element \( v \in V_0 \) with \( \sigma \in \mathbb{Z}_2 \) we set \( \deg(v) = \sigma \) is the degree of \( v \). A vector subspace \( U \) of \( V \) is called \( \mathbb{Z}_2 \)-graded vector subspace (or superspace) if \( U = (V_0 \cap U) \oplus (V_1 \cap U) \). We adopt the convention that whenever the degree function appeared in a formula, the corresponding elements are supposed to be homogeneous.

A Lie superalgebra is a superspace \( L = L_0 \oplus L_1 \) with a bilinear mapping \([\cdot, \cdot] : L \times L \to L\) satisfying the following identities:

(1) \([L_x, L_\beta] \subset L_{x+\beta}\), for \( x, \beta \in \mathbb{Z}_2 \) (\( \mathbb{Z}_2 \)-grading),
(2) \([x, y] = -(\text{deg}(x)\text{deg}(y))[y, x]\) (graded skew-symmetry),
(3) \((-1)^{\text{deg}(x)\text{deg}(z)}[x, [y, z]] + (-1)^{\text{deg}(y)\text{deg}(x)}[y, [z, x]] + (-1)^{\text{deg}(z)\text{deg}(y)}[z, [x, y]] = 0\) (graded Jacobi identity),

for all \( x, y, z \in L \).

For a Lie superalgebra \( L = L_0 \oplus L_1 \), the even part \( L_0 \) is a Lie algebra and \( L_1 \) is a \( L_0 \)-module. If \( L_1 = 0 \), then \( L \) is just Lie algebra. But in general a Lie superalgebra is not a Lie algebra. Lie superalgebra without even part, i.e., \( L_0 = 0 \), is an abelian Lie superalgebra, as \([x, y] = 0\) for all \( x, y \in L \). A sub superalgebra of \( L \) is a \( \mathbb{Z}_2 \)-graded vector subspace which is closed under bracket operation. A \( \mathbb{Z}_2 \)-graded subspace \( I \) is a graded ideal of \( L \) if \([I, L] \subseteq I\). The ideal \( Z(L) = \{z \in L : [z, x] = 0\}\) for all \( x \in L \) is a graded ideal and it is called the center of \( L \). Clearly, if \( I \) and \( J \) are graded ideals of \( L \), then so is \([I, J]\). If \( I \) is an ideal of \( L \), the quotient Lie superalgebra \( L/I \) inherits a canonical Lie superalgebra structure such that the natural projection map becomes a homomorphism.

By a homomorphism between superspaces \( f : V \to W \) of degree, \( \text{deg}(f) \in \mathbb{Z}_2 \), we mean a linear map satisfying \( f(V_x) \subseteq W_{x+\text{deg}(f)} \) for \( x \in \mathbb{Z}_2 \). In particular, if \( \text{deg}(f) = 0 \), then the homomorphism \( f \) is called homogeneous linear map of even degree. A Lie superalgebra homomorphism \( f : L \to M \) is a homogeneous linear map of even degree such that \( f[x, y] = [f(x), f(y)] \) holds for all \( x, y \in L \). The notions of epimorphisms, isomorphisms, and automorphisms have the obvious meaning. Throughout for superdimension of Lie superalgebra \( L \) we simply write \( \text{dim}(L) = (m|n) \), where \( \dim L_0 = m \) and \( \dim L_1 = n \).

3. Factor set in Lie superalgebras

The notation for factor sets in Lie algebras was defined by Moneyhun in [8]. In this section, we start with defining factor sets in Lie superalgebras and then study some of its properties.
Definition 3.1. Let \( L = L_0 \oplus L_1 \) be a finite dimensional Lie superalgebra over a field \( \mathbb{F} \). The bilinear map,

\[
    r : L/Z(L) \times L/Z(L) \to Z(L)
\]

is said to be a factor set if the following conditions are satisfied:

1. \( r(\bar{a}, \bar{b}) \subseteq Z(L)_{\alpha + \beta}, \alpha, \beta \in \mathbb{Z}_2 \)
2. \( r(\bar{a}, \bar{b}) = -(-1)^{\deg(\bar{a})\deg(\bar{b})} r(\bar{b}, \bar{a}) \)
3. \( r([\bar{a}, \bar{b}], \bar{c}) = r(\bar{a}, [\bar{b}, \bar{c}]) - (-1)^{\deg(\bar{a})\deg(\bar{b})} r(\bar{b}, [\bar{a}, \bar{c}]) \)

for all \( \bar{a}, \bar{b}, \bar{c} \in L/Z(L) \).

Lemma 3.2. Let \( L \) be a Lie superalgebra and \( r \) be a factor set on \( L \).

1. The set

\[
    R = \{ (x, \bar{a}) : x \in Z(L), \bar{a} \in L/Z(L) \}
\]

is a Lie superalgebra under the component-wise addition and the multiplication

\[
    [(x_1, \bar{a}), (x_2, \bar{b})] = (r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]).
\]

2. \( Z_R = \{ (x, 0) \in R : x \in Z(L) \} \cong Z(L) \).

Proof. Here we have \( Z(L) \) is a graded ideal and \( L/Z(L) \) is a quotient Lie superalgebra. Hence there is a natural \( \mathbb{Z}_2 \)-grading on set \( R \). Any \( (x, \bar{a}) \in R \) is an even element when \( x \in Z(L)_0 \), \( \bar{a} \in (L/Z(L))_0 \), and it is an odd element when \( x \in Z(L)_1 \), \( \bar{a} \in (L/Z(L))_1 \). It is easy to check that \( R \) is a superalgebra. Now we intend to show that \( R \) is a Lie superalgebra.

The way \([,]_R\) is defined clearly it is a bilinear map. Let \((x_1, \bar{a}), (x_2, \bar{b})\) be two homogeneous elements in \( R \), then, \( \deg(\bar{a}) = \deg([x_1, \bar{a}]) \) and \( \deg(\bar{b}) = \deg([x_2, \bar{b}]) \).

1. To check graded skew-symmetric property consider

\[
    [(x_1, \bar{a}), (x_2, \bar{b})] = (r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]) = -(-1)^{\deg(\bar{a})\deg(\bar{b})} (r(\bar{b}, \bar{a}), [\bar{b}, \bar{a}])
\]

\[
    = -(-1)^{\deg(\bar{a})\deg(\bar{b})} [(x_2, \bar{b}), (x_1, \bar{a})]
\]

\[
    = -(-1)^{\deg(\bar{a})\deg(\bar{b})} [(x_2, \bar{b}), (x_1, \bar{a})]. \tag{3.1}
\]

2. To check the graded Jacobi identity, consider

\[
    [[(x_1, \bar{a}), (x_2, \bar{b})], (x_3, \bar{c})] = [(r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]), (x_3, \bar{c})]
\]

\[
    = (r([\bar{a}, \bar{b}], \bar{c}), [[\bar{a}, \bar{b}], \bar{c}])
\]

\[
    = \left( (r(\bar{a}, [\bar{b}, \bar{c}]) - (-1)^{\deg(\bar{a})\deg(\bar{b})} r(\bar{b}, [\bar{a}, \bar{c}])) \right)
\]

\[
    \left( [\bar{a}, [\bar{b}, \bar{c}]] - (-1)^{\deg(\bar{a})\deg(\bar{b})} [\bar{b}, [\bar{a}, \bar{c}]] \right)
\]

\[
    = \left( (r(\bar{a}, [\bar{b}, \bar{c}]), [\bar{a}, [\bar{b}, \bar{c}]])) - (-1)^{\deg(\bar{a})\deg(\bar{b})} (r([\bar{b}, [\bar{a}, \bar{c}]])) \right)
\]

\[
    = \left( (x_1, \bar{a}), (r(\bar{b}, \bar{c}), [\bar{b}, \bar{c}]) \right) - (-1)^{\deg(\bar{a})\deg(\bar{b})} \left( (x_2, \bar{b}), (r(\bar{a}, \bar{c}), [\bar{a}, \bar{c}]) \right)
\]

\[
    = \left( (x_1, \bar{a}), (x_2, \bar{b}), (x_3, \bar{c}) \right) - (-1)^{\deg(\bar{a})\deg(\bar{b})} \left( (x_2, \bar{b}), (x_1, \bar{a}), (x_3, \bar{c}) \right). \tag{3.2}
\]

Hence, \( R \) is a Lie superalgebra with the given multiplication. Proof of part(ii) is evident. \( \Box \)
The Lemma below shows that factor sets do exist for any Lie superalgebras.

**Lemma 3.3.** There exists a factor set \( r \), for any Lie superalgebra \( L \) such that \( L \cong (Z(L), L/Z(L), r) \).

**Proof.** Let \( K \) be a vector superspace complement to \( Z(L) \) in \( L \), i.e., \( L = K \oplus Z(L) \). Let us define \( T : L/Z(L) \to L \) by \( T(\bar{a}) = k \), where \( \bar{a} \in L/Z(L) \) with \( a = k + l \), \( k \in K \) and \( l \in Z(L) \). Here, \( T \) is a well-defined map and is also homogeneous linear map of even degree. Clearly, we have \( T(\bar{a}) = \bar{a} \). Now for \( \bar{a}, \bar{b} \in L/Z(L) \), consider \([\bar{a}, \bar{b}] = [k + l + Z(L), k' + l' + Z(L)] = [k, k'] + Z(L)\), then

\[
[T(\bar{a}), T(\bar{b})] - T[\bar{a}, \bar{b}]) + Z(L) = [k, k'] - T([k, k']) + Z(L) = [k, k'] - T([k, k']) = 0 + Z(L)
\]

implies \([T(\bar{a}), T(\bar{b})] - T[\bar{a}, \bar{b}] \in Z(L)\). Define

\[
r : L/Z(L) \times L/Z(L) \to Z(L)
\]

by, \( r(\bar{a}, \bar{b}) = [T(\bar{a}), T(\bar{b})] - T[\bar{a}, \bar{b}] \). We show that \( r \) is a factor set.

Since \( T \) is a homogeneous linear map of even degree and bracket respects gradation, so \( T(\bar{a}, \bar{b}) \in (L/Z(L))_{x+\beta} \) provided \( \bar{a} \in (L/Z(L))_x \) and \( \bar{b} \in (L/Z(L))_\beta \), for \( x, \beta \in \mathbb{Z}_2 \). To check skew-symmetric property of \( r \), take

\[
r(\bar{a}, \bar{b}) = [T(\bar{a}), T(\bar{b})] - T([\bar{a}, \bar{b}])
\]

\[
= -(-1)^{\text{deg}(\bar{a}) \text{deg}(\bar{b})} ([T(\bar{b}), T(\bar{a})] - T([\bar{b}, \bar{a}]))
\]

\[
= -(-1)^{\text{deg}(\bar{a}) \text{deg}(\bar{b})} r(\bar{b}, \bar{a})
\]

for all \( \bar{a}, \bar{b} \in L/Z(L) \). To check Jacobi identity, take

\[
r([\bar{a}, \bar{b}], \bar{c}) = [T([\bar{a}, \bar{b}]), T(\bar{c})] - T([\bar{a}, \bar{b}], \bar{c})
\]

\[
= [[T(\bar{a}), T(\bar{b})] + Z(L), T(\bar{c}) + Z(L)] - T([\bar{a}, \bar{b}], \bar{c})
\]

\[
= [[T(\bar{a}), T(\bar{b})], T(\bar{c})] + Z(L) - T([\bar{a}, \bar{b}], \bar{c})
\]

\[
= \left( [T(\bar{a}), [T(\bar{b}), T(\bar{c})]] - (-1)^{\text{deg}(\bar{a}) \text{deg}(\bar{b})} [T(\bar{b}), [T(\bar{a}), T(\bar{c})]] \right) + Z(L) - T[\bar{a}, [\bar{b}, \bar{c}]]
\]

\[
= \left( [T(\bar{a}), [T(\bar{b}), T(\bar{c})]] + Z(L) - T[\bar{a}, [\bar{b}, \bar{c}]] \right)
\]

\[
= \left( -(-1)^{\text{deg}(\bar{a}) \text{deg}(\bar{b})} r(\bar{b}, [\bar{a}, \bar{c}]) \right)
\]

for all \( \bar{a}, \bar{b}, \bar{c} \in L/Z(L) \). Consider \( \theta : (Z(L), L/Z(L), r) \to L \) which is given by, \( \theta(x, \bar{a}) = x + T(\bar{a}) \) for \( x \in Z(L) \) and \( \bar{a} = a + Z(L) = k + Z(L) \), where \( k \in K \). Suppose, \( (x, \bar{a}) = (y, \bar{b}) \iff x + k = y + k' \iff x - y = k - k' \in Z(L) \cap K \iff \theta(x, \bar{a}) = \theta(y, \bar{b}) \). Thus, \( \theta \) is well-defined map and it is injective. Evidently, \( \theta \) is surjective. Also \( \theta \) is a homogeneous linear map of even degree and as

\[
\theta\left( [x, \bar{a}] \right) = \theta(x, \bar{a}) \bar{b} \equiv \theta(r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}])
\]

\[
= r(\bar{a}, \bar{b}) + T([\bar{a}, \bar{b}]) = [T(\bar{a}), T(\bar{b})]
\]

\[
= [x + T(\bar{a}), y + T(\bar{b})] = \left[ \theta(x, \bar{a}), \theta(y, \bar{b}) \right]
\]

so it is a Lie superalgebra homomorphism, therefore \( \theta \) is the required isomorphism. \( \square \)
Lemma 3.4. Let $L$ be a stem Lie superalgebra in an isoclinism family of Lie superalgebras $C$. Then for any stem Lie superalgebra $M$ of $C$, there exists a factor set $r$ over $L$ such that $M \cong (Z(L), L/Z(L), r)$.

Proof. Let $L \sim M$ and let the pair $(\varphi, \theta)$ be isoclinism of Lie superalgebras $L$ and $M$. We have $ZL \subseteq L'$ and $ZM \subseteq M'$. Then, by Lemma 1.9, $\theta(Z(L)) = Z(M)$. According to Lemma 3.3, there exists a factor set $s$ such that $M \cong (Z(M), M/Z(M), s)$.

Now, define the factor set $r : L/Z(L) \times L/Z(L) \to Z(L)$ by,

$$r(\bar{a}, \bar{b}) = \theta^{-1}(s(\varphi(\bar{a}), \varphi(\bar{b}))),$$

where $\bar{a}, \bar{b} \in L/Z(L)$. Further define

$$\psi : (Z(L), L/Z(L), r) \to (Z(M), M/Z(M), s)$$

by,

$$\psi(x, \bar{a}) = (\theta(x), \varphi(\bar{a})).$$

Since, $\theta$ and $\varphi$ are isomorphisms, so $\psi$ is a bijection and is also homogeneous linear map of even degree. Finally, consider

$$\psi\left(\left(\begin{array}{c} x, \bar{a} \\ y, \bar{b} \end{array}\right)\right) = \psi(r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}])$$

$$= (\theta(r(\bar{a}, \bar{b})), \varphi([\bar{a}, \bar{b}])) = \left(s(\varphi(\bar{a}), \varphi(\bar{b})), \varphi([\bar{a}, \bar{b}])\right)$$

(3.4)

$$= \left(\theta(x), \varphi(\bar{a}), \psi(y, \bar{b})\right) = \left(\theta(x), \varphi(\bar{a}), \psi(y, \bar{b})\right),$$

implies $\psi$ is a Lie superalgebra homomorphism. Hence, $\psi$ is an isomorphism, i.e.,

$$(Z(L), L/Z(L), r) \cong (Z(M), M/Z(M), s).$$

It follows that,

$$M \cong (Z(L), L/Z(L), r).$$

Lemma 3.5. Let $L$ be a Lie superalgebra, $r$ and $s$ be two factor sets over $L$. Assume that

$$R = (Z(L), L/Z(L), r), \quad Z_R = \{(x, 0) \in R : x \in Z(L)\} \cong Z(L),$$
$$S = (Z(L), L/Z(L), s), \quad Z_S = \{(x, 0) \in S : x \in Z(L)\} \cong Z(L).$$

Let $\lambda$ is an isomorphism from $R$ to $S$ satisfying $\lambda(Z_R) = Z_S$, then the restriction of $\lambda$ on $L/Z(L)$ and $Z(L)$ define the automorphisms $\mu \in Aut(L/Z(L))$ and $\nu \in Aut(Z(L))$, respectively.

Proof. By assumption, $\lambda(Z_R) = Z_S$, then define the map $\bar{\lambda} : R/Z_R \to S/Z_S$ given by, $\bar{\lambda}((x, \bar{a}) + Z_R) = \lambda(x, \bar{a}) + Z_S$. Here $\bar{\lambda}$ is an isomorphism. Define $\mu$ such that the following diagram is commutative;
where $\phi$ is defined as $\phi(\bar{a}) = (0, \bar{a}) + Z_R$ and $\psi(\bar{a}) = (0, \bar{a}) + Z_S$. In other words $\lambda(0, \bar{a}) + Z_S = (0, \mu(\bar{a})) + Z_S$, for all $\bar{a} \in L/Z(L)$. Certainly $\phi, \psi$ are both homogeneous linear maps of even degree. Moreover for all $\bar{a}, \bar{b} \in L/Z(L)$, we have

$$L/Z(L) \xrightarrow{\mu} L/Z(L)$$

$$\phi \quad \psi$$

$$R/Z_R \xrightarrow{\lambda} S/Z_S$$

where $\phi$ is defined as $\phi(\bar{a}) = (0, \bar{a}) + Z_R$ and $\psi(\bar{a}) = (0, \bar{a}) + Z_S$. In other words $\lambda(0, \bar{a}) + Z_S = (0, \mu(\bar{a})) + Z_S$, for all $\bar{a} \in L/Z(L)$. Certainly $\phi, \psi$ are both homogeneous linear maps of even degree. Moreover for all $\bar{a}, \bar{b} \in L/Z(L)$, we have

$$\phi[\bar{a}, \bar{b}] = (0, [\bar{a}, \bar{b}]) + Z_R$$

$$= (r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]) + Z_R = [(0, \bar{a}), (0, \bar{b})] + Z_R$$

$$= [(0, \bar{a}) + Z_R, (0, \bar{b}) + Z_R] = [\phi(\bar{a}), \phi(\bar{b})].$$

So, $\phi$ is a homomorphism and similarly $\psi$ is also a homomorphism. Since, $\lambda$ is an isomorphism, $\mu$ is a homogeneous linear map of even degree. Now, to check the map $\mu$ is well-defined and injective, let $\bar{a} = \bar{b}$ implies $(0, \bar{a}) = (0, \bar{b})$ in $L/Z(L)$, so in $R/Z_R$ we have $(0, \bar{a}) + Z_R = (0, \bar{b}) + Z_R$,

$$\iff \lambda((0, \bar{a}) + Z_R) = \lambda((0, \bar{b}) + Z_R)$$

$$\iff \lambda(0, \bar{a}) + Z_S = \lambda(0, \bar{b}) + Z_S$$

$$\iff (0, \mu(\bar{a})) + Z_S = (0, \mu(\bar{b})) + Z_S$$

$$\iff (0, \mu(\bar{a}) - \mu(\bar{b})) \in Z_S.$$

But any element in $Z_S$ is of the form $(x, 0)$ for $x \in Z(L)$, which implies $\iff (\mu(\bar{a}) - \mu(\bar{b})) = 0$ in $L/Z(L) \iff \mu(\bar{a}) = \mu(\bar{b})$ in $L/Z(L)$.

For each $\bar{b} \in L/Z(L)$, there exist $\bar{a} \in L/Z(L)$ such that $\lambda((0, \bar{a}) + Z_R) = (0, \bar{b}) + Z_S$ and on the other hand, $\lambda((0, \bar{a}) + Z_R) = \lambda(0, \bar{a}) + Z_S = (0, \mu(\bar{a})) + Z_S$. Hence, $\mu(\bar{a}) = \bar{b}$, i.e., $\mu$ is surjective. Now, for

$$\lambda((0, \mu([\bar{a}, \bar{b}])) + Z_S = \lambda(0, [\bar{a}, \bar{b}]) + Z_S$$

$$= \lambda(0, [\bar{a}, \bar{b}] + Z_R) = \lambda([\bar{a}, \bar{b}])$$

$$= \lambda(0, [\bar{a}, \bar{b}]) = \lambda([\bar{a}, \bar{b}])$$

$$= [(\lambda(0, \bar{a}) + Z_S) = (0, \mu(\bar{a})) + Z_S].$$

Hence, $\mu(\bar{a}) = \bar{b}$, i.e., $\mu$ is an automorphism. Consider the map $\bar{\lambda} : Z_R \to Z_S$ defined as $\bar{\lambda}(x, 0) = \lambda(x, 0)$ for all $x \in Z(L)$, is an isomorphism. Define $\nu$ in such a way that the following diagram is commutative;
i.e., \( \lambda(x, 0) = (\nu(x), 0) \), for all \( x \in Z(L) \). It is easy to check that \( \nu \) is an automorphism.

\( \square \)

**Lemma 3.6.** Let \( L \) be a Lie superalgebra and \( R, S, Z_R \) and \( Z_S \) be as in the Lemma 3.5.

1. Consider \( \lambda : R \to S \) is a Lie superalgebra isomorphism such that \( \lambda(Z_R) = Z_S \). Let \( \mu \in \text{Aut}(L/Z(L)) \) and \( \nu \in \text{Aut}(Z(L)) \) be the automorphisms induced by \( \lambda \). Then there exists a homogeneous linear map of even degree, \( \gamma : L/Z(L) \to Z(L) \) such that,

\[
\nu(r(\tilde{a}, \tilde{b})) + \gamma[\tilde{a}, \tilde{b}] = s(\mu(\tilde{a}), \mu(\tilde{b})).
\]

2. If \( \mu \in \text{Aut}(L/Z(L)), \nu \in \text{Aut}(Z(L)) \) and \( \delta : L/Z(L) \to Z(L) \) is a homogeneous linear map of even degree such that

\[
\nu(r(\tilde{a}, \tilde{b})) + \delta[\tilde{a}, \tilde{b}] = s(\mu(\tilde{a}), \mu(\tilde{b})),
\]

holds, then there exists an isomorphism \( \lambda : R \to S \) which is induced by \( \mu \) and \( \nu \) satisfying \( \lambda(Z_R) = Z_S \).

**Proof.** We have \( \lambda(0, \tilde{a}) + Z_S = (0, \mu(\tilde{a})) + Z_S \) which means \( \lambda(0, \tilde{a}) - (0, \mu(\tilde{a})) \in Z_S \). Say, \( \lambda(0, \tilde{a}) - (0, \mu(\tilde{a})) = (x_a, 0) \), for some \( x_a \in Z(L) \). Define the map \( \gamma : L/Z(L) \to Z(L) \) by \( \gamma(\tilde{a}) = x_a \), for all \( \tilde{a} \in L/Z(L) \). It is easy to see that the map \( \gamma \) is well defined. For \( \tilde{a}, \tilde{b} \in \frac{L+Z(L)}{Z(L)} \) with \( \tau \in Z_2 \), we have

\[
\left( \gamma(\tilde{a} + \tilde{b}), 0 \right) = \left( \frac{x_a + x_b}{\tau}, 0 \right) = \lambda(0, \tilde{a} + \tilde{b}) - (0, \mu(\tilde{a} + \tilde{b}))
= \lambda(0, \tilde{a}) - (0, \mu(\tilde{a})) + \lambda(0, \tilde{b}) - (0, \mu(\tilde{b}))
= (x_a, 0) + (y_b, 0) = \left( \gamma(\tilde{a}), \gamma(\tilde{b}), 0 \right).
\]

Hence, we get \( \gamma(\tilde{a} + \tilde{b}) = \gamma(\tilde{a}) + \gamma(\tilde{b}) \) implies \( \gamma \) is a linear map. Further as \( \lambda \) and \( \mu \) are Lie superalgebra isomorphisms, that immediately implies \( \gamma \) is a homogeneous linear map of even degree. We have for \( x \in Z(L), \lambda(\tilde{x}, 0) = (\nu(x), 0) \).

\[
\lambda(\tilde{x}, 0) = \nu(x, 0) + (0, \mu(\tilde{a})) + (\gamma(\tilde{a}), 0)
= (\nu(x) + \gamma(\tilde{a}), \mu(\tilde{a})).
\]

On one hand,

\[
\lambda[(x, \tilde{a}), (y, \tilde{b})] = \lambda(0, \tilde{a}) + \lambda(0, \tilde{b})
= \left[ (0, \mu(\tilde{a})), (x, 0), (0, \mu(\tilde{b})), (y, 0) \right]
= \left[ (\gamma(\tilde{a}), \mu(\tilde{a})), (\gamma(\tilde{b}), \mu(\tilde{b})) \right] = \left( s(\mu(\tilde{a}), \mu(\tilde{b})), [\mu(\tilde{a}), \mu(\tilde{b})] \right).
\]

On the other hand, \( \lambda[(x, \tilde{a}), (y, \tilde{b})] = \lambda(r(\tilde{a}, \tilde{b}), [\tilde{a}, \tilde{b}]) = (\nu(r(\tilde{a}, \tilde{b})) + \gamma[\tilde{a}, \tilde{b}], [\mu(\tilde{a}), \mu(\tilde{b})]) \). Hence,

\[
\nu(r(\tilde{a}, \tilde{b})) + \gamma[\tilde{a}, \tilde{b}] = s(\mu(\tilde{a}), \mu(\tilde{b})),
\]

which proves our first assertion.

Let us define \( \lambda \) by

\[
\lambda : R \to S
\]

\[
\lambda(x, \tilde{a}) = (\nu(x) + \delta(\tilde{a}), \mu(\tilde{a})),
\]

for all \( \tilde{a} \in \frac{L+Z(L)}{Z(L)} \) and for all \( x \in L_L \cap Z(L) \) with \( \tau \in Z_2 \). Here \( \lambda \) is a well defined map and also a bijection. Further as \( \nu, \delta, \mu \) are all homogeneous linear maps of even degree hence so is \( \lambda \). Now,
\[
\lambda[(x, \bar{a}), (y, \bar{b})] = \lambda(r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]) \\
= (\nu(r(\bar{a}, \bar{b}) + [\bar{a}, \bar{b}] + \delta[\bar{a}, \bar{b}], \mu(\bar{a}, \bar{b})) = (s(\mu(\bar{a}), \mu(\bar{b})), \mu(\bar{a}, \bar{b})).
\]

Also
\[
\lambda((x, \bar{a}), (y, \bar{b})) = \left(\nu(x) + \delta(\bar{a}), \mu(\bar{a})\right), \left(\nu(y) + \delta(\bar{b}), \mu(\bar{b})\right)] \\
= (s(\mu(\bar{a}), \mu(\bar{b})), [\mu(\bar{a}), \mu(\bar{b})]).
\]

Finally, we get \(\lambda[(x, \bar{a}), (y, \bar{b})] = [\lambda(x, \bar{a}), \lambda(y, \bar{b})]\), so \(\lambda\) is a Lie superalgebra isomorphism. Evidently \(\lambda(x, 0) = (\nu(x), 0)\) and \(\lambda(0, \bar{a}) = (\delta(\bar{a}), \mu(\bar{a}))\). Now,
\[
\lambda(0, \bar{a}) + Z_S = (\nu(0) + \delta(\bar{a}), \mu(\bar{a})) + Z_S \\
= (0, \mu(\bar{a})) + (\delta(\bar{a}), 0) + Z_S \\
= (0, \mu(\bar{a})) + Z_S,
\]
where the equality holds if \((0, \mu(\bar{a})) \in S\) satisfying \(\lambda(Z_R) = Z_S\) which proves our second assertion. □

The following Theorem plays an important role in proving the main result of this section. This result was first proved by Moneyhun for Lie algebras [8].

**Theorem 3.7.** Let \(L\) and \(M\) be two finite dimensional stem Lie superalgebras. Then \(L \sim M\) if and only if \(L \cong M\).

**Proof.** If \(L\) and \(M\) are finite dimensional stem Lie superalgebras such that \(L \cong M\), then clearly \(L \sim M\). Conversely, let \(L \sim M\). By Lemma 3.3 we have, \(L \cong (Z(L), \frac{L}{Z(L)}, r) = R\) and \(M \cong (Z(L), \frac{Z(L)}{Z(L)}, s) = S\) for factor sets \(r, s\) in \(L, M\) respectively. Let \((\alpha, \beta)\) be the isoclism between the Lie superalgebras \(R\) and \(S\). Certainly \(Z(L) \cong Z_R\) and also \(Z(R) \cong Z_L\), hence \(Z(R) \cong Z_R\). Similarly \(Z(S) \cong Z_S\). As \(Z_R \subseteq Z(R)\), we get \(Z_R = Z(R)\). Let the map \(\mu \in Aut(L/Z(L))\), be defined by \(\mu((0, \bar{a}) + Z_R) = (0, \mu(\bar{a})) + Z_S\), for all \(\bar{a} \in L/Z(L)\). Let us consider the following commutative diagram;

\[
\begin{array}{ccc}
\frac{L}{Z(L)} \times \frac{L}{Z(L)} & \xrightarrow{\rho} & \frac{R}{Z_R} \times \frac{R}{Z_R} \\
\downarrow \mu^2 & & \downarrow \alpha^2 \\
\frac{L}{Z(L)} \times \frac{L}{Z(L)} & \xrightarrow{\sigma} & \frac{S}{Z_S} \times \frac{S}{Z_S} \\
\downarrow \xi & & \downarrow \beta \\
\end{array}
\]

in which
\[
\rho(\bar{a}, \bar{b}) = \left((0, \bar{a}) + Z_R, (0, \bar{b}) + Z_R\right),
\sigma(\bar{a}, \bar{b}) = \left((0, \bar{a}) + Z_S, (0, \bar{b}) + Z_S\right),
\xi(\lambda(x, \bar{a}) + Z_S, (y, \bar{b}) + Z_S) = \left(s(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]\right),
\theta(\lambda(x, \bar{a}) + Z_R, (y, \bar{b}) + Z_R) = (r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]).
\]

Again let \(\nu \in Aut(Z(L))\) be defined by \(\beta(x, 0) = (\nu(x), 0)\), for all \(x \in Z(L)\). Now for \(\bar{a}, \bar{b} \in L/Z(L)\), consider
\[ \beta \theta \left( (0, \bar{a}) + Z_R, (0, \bar{b}) + Z_R \right) = \beta \left[ (0, \bar{a}), (0, \bar{b}) \right] \]

and further
\[ \xi z \left( (0, \bar{a}) + Z_R, (0, \bar{b}) + Z_R \right) = \xi \left( (0, \mu(\bar{a})), (0, \mu(\bar{b})) + Z_S \right) \]
\[ = \left[ (0, \mu(\bar{a})), (0, \mu(\bar{b})) \right] \]
\[ = (s((\mu(\bar{a})), (\mu(\bar{b}))) \right). \]

Hence, we have \[ \beta[(0, \bar{a}), (0, \bar{b})] = (s((\mu(\bar{a})), (\mu(\bar{b})))) \]. Let us define the map \( \delta : L'/Z(L) \to Z(L) \) as follows,
\[ \beta \left[ (0, \bar{a}), (0, \bar{b}) \right] = \beta(r(\bar{a}, \bar{b}), [\bar{a}, \bar{b}]) \]
\[ = \beta(r(\bar{a}, \bar{b}), 0) + \beta(0, [\bar{a}, \bar{b}]) \]
\[ = (\nu(r(\bar{a}, \bar{b})), 0) + (\delta([\bar{a}, \bar{b}]), t) \]
\[ = (\nu(r(\bar{a}, \bar{b}))) + \delta([\bar{a}, \bar{b}]), t), \]
where \( t \in L/Z(L) \) and hence, we get
\[ \nu(r(\bar{a}, \bar{b}))) + \delta([\bar{a}, \bar{b}] = s((\mu(\bar{a})), (\mu(\bar{b}))). \]

To apply Lemma 3.6, we may extend \( \delta \) to \( L'/Z(L) \) by defining \( 0 \) on the complement of \( L'/Z(L) \) in \( L/Z(L) \). Then, we get \( R \cong S \). \( \square \)

**Theorem 3.8.** Let \( C \) be an isoclinism family of finite dimensional Lie superalgebras. Then any \( L \in C \) can be expressed as \( L = T \oplus A \) where \( T \) is a stem Lie superalgebra and \( A \) is some finite dimensional abelian Lie superalgebra.

**Proof.** By Lemma 1.5, \( C \) contains a stem Lie superalgebra say \( T \). Further \( T \sim T \oplus A \) for some abelian Lie superalgebra \( A \) (Lemma 1.2). Consider \( L \in C \) be any arbitrary Lie superalgebra and we have \( Z(L) \cap L' \) is a graded ideal of \( L \). Let \( M = L_0 \cap M \oplus L_1 \cap M \) be a complementary \( \mathbb{Z}_2 \)-graded vector subspace to \( Z(L) \cap L' \) in \( Z(L) \). So,
\[ Z(L) = M \oplus Z(L) \cap L' \]

and clearly \( [M, L] \subseteq M \) implies \( M \) is an ideal of \( L \). Assume \( L/M = T \) and as \( Z(L) \) is direct sum of \( Z(L) \cap L' \) and \( M \) we have \( M \cap L' = 0 \). By Lemma 1.3, we get \( L \sim L/M \). Now,
\[ Z(T) = Z\left( \frac{L}{M} \right) \]
\[ \subseteq \frac{L' + M}{M} \]
\[ \cong \left( \frac{L}{M} \right) ' = T'. \]

Hence \( T \) is a stem Lie superalgebra. As \( M \) is an ideal of \( L \) and \( M \cap L' = 0 \), we get a \( \mathbb{Z}_2 \)-graded vector subspace \( K \) of \( L \) containing \( L' \) complementary to \( M \). Now,
\[ [K, L] \subseteq [L, L] = L' \subseteq K \]
implies \( K \) is an ideal. Further,
\[ L \sim \frac{L}{M} \cong \frac{K \oplus M}{M} \cong K. \]

It is easy to check \( K \) is a stem Lie superalgebra. Finally \( T \sim L \) and \( L \sim K \) implies \( T \sim K \). Using Theorem 3.7, \( T \cong K \). Specifically \( L = K \oplus M \cong T \oplus M \) as required. \( \square \)
Theorem 3.9. If \(L\) and \(K\) be two Lie superalgebra with same dimension. Then \(L \sim K\) if and only if \(L \cong K\).

**Proof.** Consider Lie superalgebras \(L\) and \(K\) with \(L \sim K\). By Theorem 3.8, write \(L = T \oplus M\) and \(K = T_1 \oplus M_1\) for stem Lie superalgebras \(T, T_1\) and abelian Lie superalgebras \(M, M_1\), respectively. As \(T \sim T_1\) by Theorem 3.7 \(T \cong T_1\). Since, \(L\) and \(K\) are of same dimension, so \(\dim M = \dim M_1\) and \(M \cong M_1\), which implies \(T \oplus M \cong T_1 \oplus M_1\). Therefore \(L \cong K\). Converse is obvious. \(

Below is an example which shows that, two isoclinic Lie superalgebras of different dimension may not be isomorphic.

**Example 3.10.** Let \(L = L_0 + L_1\) be a \((2|1)\) dimensional Lie superalgebra with the basis \(\{e_1, e_2, e_3\}\) and the commutator relations are defined by:

\[ [e_1, e_2] = e_1, \quad [e_3, e_3] = e_2, \]

and all other commutator relations are zero. Then \(L' = \langle e_1, e_2 \rangle\) and \(Z(L) = \{0\}\) and hence, \(L/Z(L) \cong L\).

Now, let \(M = M_0 + M_1\) be a \((3|1)\) dimensional Lie superalgebra with the basis \(\{e'_1, e'_2, e'_3, e'_4\}\) and the commutator relations are defined by:

\[ [e'_1, e'_2] = e'_1, \quad [e'_3, e'_4] = e'_2, \]

and all other commutator relations are zero. Then \(M' = \langle e'_1, e'_2 \rangle\) and \(Z(M) = \{e'_3\}\) and hence, \(M/Z(M) = \{e'_1, e'_2, e'_4\}\), where \(e'_i = e'_i + Z(M)\), for \(i = 1, 2, 4\).

It is easy to verify that \(L' \ncong M'\) and \(L/Z(L) \ncong M/Z(M)\). Hence, one can deduce \(L \ncong M\) while \(\dim(L) \neq \dim(M)\).

### 4. Covers of finite dimensional Lie superalgebras

Here we show that for finite dimensional Lie superalgebras covers are isomorphic, using isoclinism. At first we define the notation of universal element.

**Definition 4.1.** Let \(L\) be a Lie superalgebra. Let

\[ C(L) = \{(K, \lambda) | \lambda \in \text{Hom}(K, L), \lambda \text{ is onto and } \text{Ker}(\lambda) \subseteq K' \cap Z(K)\}, \]

for any Lie superalgebra \(K\). The element \((T, \sigma) \in C(L)\) is called a universal element if there exists a map \(\tau \in \text{Hom}(T, K)\) satisfying \(\lambda \circ \tau = \sigma\), for all \((K, \lambda) \in C(L)\).

**Lemma 4.2.** Let \(K = K_0 \oplus K_1\) be a Lie superalgebra of \(\dim K = (m|n)\), then \(Z(K) \cap K'\) is contained in any maximal Lie superalgebra of \(K\).

**Proof.** Let \(W = K_0 \cap W \oplus K_1 \cap W\) be any maximal subalgebra of \(K\). There is a natural gradation for \(Z(K) \cap K' + W\), given by \(Z(K) \cap K' + W = K_0 \cap (Z(K) \cap K' + W) \oplus K_1 \cap (Z(K) \cap K' + W)\). It is a subalgebra of \(K\). Hence either \(Z(K) \cap K' + W = W\) or \(Z(K) \cap K' + W = K\). If \(Z(K) \cap K' + W = K\), then \(K' = W' \subseteq W\), so \(Z(K) \cap K' \subseteq W\) which is a contradiction, i.e., \(Z(K) \cap K' + W = W\). \(\square\)

**Lemma 4.3.** Let \(L\) be a Lie superalgebra of dimension \((m|n)\). Let \((K, \lambda) \in C(L)\) and \(M\) be any Lie superalgebra and \(\sigma \in \text{Hom}(M, L)\) is onto. If \(\tau \in \text{Hom}(M, K)\) such that \(\lambda \circ \tau = \sigma\), then \(\tau\) is onto.

**Proof.** Consider \((K, \lambda) \in C(L)\) and for \(l \in K\), as \(\sigma\) is onto, \(\sigma(m) = \lambda(l)\) for some \(m \in M\). Now \(\sigma(m) = \lambda(\tau(m)) = \lambda(l)\) which implies \(\tau(m) - l \in \text{Ker}(\lambda)\) and moreover \(K = \text{Ker}(\lambda) + \text{Img}(\tau)\). By assumption \(\text{Ker}(\lambda) \subseteq Z(K) \cap K'\) and by **Lemma 4.2**, \(Z(K) \cap K'\) is contained in every maximal
subalgebra of \( K \). Suppose \( \text{Img}(\tau) \neq K \). Now, \( \text{Img}(\tau) \) is a subalgebra of \( K \) and say \( \text{Img}(\tau) \subset N \) for some maximal subalgebra \( N \neq K \). But
\[
K \subseteq \text{Img}(\tau) + (Z(K) \cap K') \subset N,
\]
i.e., \( K = N \) leads to a contradiction. Hence, \( \text{Img}(\tau) = K \) as required. \( \square \)

If \((K, \lambda) \in C(L)\) is universal element, then for each \((M, \tau) \in C(L)\), there exists homomorphism \( \rho \in \text{Hom}(K, M) \) such that \( \tau \circ \rho = \lambda \). By Lemma 4.3, we have \( \rho \) is onto, hence \( \dim M \leq \dim K \). This implies \( K \) is a cover (using Lemma 1.8). So, to show all covers for a finite dimensional Lie superalgebra \( L \) are isomorphic it is enough to show universal elements for \( L \) are isomorphic.

**Lemma 4.4.** Let \( L \) be a Lie superalgebra of dimension \((m|n)\) and \( 0 \to R \to F \to L \to 0 \) be a free presentation of \( L \). Let
\[
B = \frac{R}{[F, R]}, \quad C = \frac{F}{[F, R]}, \quad D = \frac{R \cap F'}{[F, R]},
\]
be quotient Lie superalgebras then

1. \( \dim D \leq \dim(\text{Ker}(\lambda)), \) where \((T, \lambda) \in C(L)\) is the universal element.
2. \( \text{Ker}(\psi) \) is a homomorphic image of \( D \), for every element \((K, \psi) \in C(L)\).

**Proof.** Clearly \([B, C] = 0\), so \( B \subseteq Z(C) \) is an graded ideal of \( C \). Let \( E \) be complementary super-space to \( D \) in \( B \), i.e., \( B = D \oplus E \). As, \( E \subseteq B \subseteq Z(C) \), so \( E \) is a graded ideal of \( C \). We intend to show \((C/E, \tilde{\sigma}) \in C(L)\) where \( \sigma : C \to L \) is given by \( \sigma(a + [F, R]) = a + R \), for all \( a \in F \). Evidently, \( \sigma \) is well defined onto map and also is a homogeneous linear map of even degree. Further for homogeneous elements \( a + [F, R], b + [F, R] \), where \( a, b \in F \), consider \( \sigma([a + [F, R], b + [F, R]]) = [a, b] + R = [a + R, b + R] \). It shows that \( \sigma \) is a homomorphism. Now, \( \sigma \) induces the map
\[
\tilde{\sigma} : C/E \to L
\]
given by
\[
\tilde{\sigma}(x + E) = \sigma(x),
\]
as \( \sigma(B) = R \). So, \( \tilde{\sigma} \) is a surjective Lie superalgebra homomorphism. Now, \( \text{Ker}(\tilde{\sigma}) = \{ x + E \mid \tilde{\sigma}(x + E) = R \} \). Here \( x \in C \), say \( x = b + [F, R] \) for \( b \in F \). Finally, we get \( \text{Ker}(\tilde{\sigma}) = \{ b + [F, R] + E \mid b \in R \} = B/E \). Moreover,
\[
\text{Ker}(\tilde{\sigma}) = B/E \subseteq \frac{Z(C) + E}{E} \subseteq \frac{Z(C)}{E}
\]
and
\[
\text{Ker}(\tilde{\sigma}) = B/E = \frac{D \oplus E}{E} \subseteq \frac{C' + E}{E} = \left( \frac{C'}{E} \right).
\]
Hence \( C/E \in C(L) \). We have \((T, \lambda)\) is the universal element, by Lemma 4.3 \( \dim(C/E) \leq \dim T \). So,
\[
\dim(D) = \dim\left( \frac{D \oplus E}{E} \right) = \dim\left( \frac{B}{E} \right)
\]
\[
= \dim\left( \frac{C'}{E} \right) - \dim(L)
\]
\[
\leq \dim T - \dim L
\]
\[
= \dim L + \dim \text{Ker}(\lambda) - \dim L = \dim \text{Ker}(\lambda),
\]
which proves our first assertion.
Let \((K, \psi)\) be an element of \(C(L)\). Now consider the canonical homomorphism \(\pi\) from \(F\) onto \(L\), so \(\text{Ker}(\pi) = R\). Since \(F\) is free Lie superalgebra, there exists a homomorphism \(\rho : F \to K\) such that \(\psi \circ \rho = \pi\) and we claim, \(\rho\) is onto. Let \(X\) be a \(\mathbb{Z}_2\)-graded set and the free Lie superalgebra \(F\) is generated by \(X\). Let us denote \(W = \langle \{\rho(x) \mid x \in X\} \rangle\), then \(K = \langle \{W + \text{Ker}(\psi)\} \rangle\). Consider the Frattini subalgebra \(\mathfrak{F}(K)\) \([3]\) of \(K\) and let \(\text{Ker}(\psi) \not\subseteq \mathfrak{F}(K)\). Then there exists a maximal subalgebra \(M\) of \(K\) and a non-zero homogeneous element \(y \in \text{Ker}(\psi) \setminus M\). Hence \(M + \langle y \rangle = K\) and as \(\text{Ker}(\psi) \subseteq Z(K) \cap K'\), so \(K' = M'\). On the other hand, \(y \in \text{Ker}(\psi) \subseteq M' \subseteq M\) which is a contradiction. Hence \(\text{Ker}(\psi) \subseteq \mathfrak{F}(K)\) which implies \(K = \langle W \rangle\), i.e., \(\rho\) is onto and \(\rho(R) = \text{Ker}(\psi)\). So, \(\rho[F, R] = [\rho(F), \rho(R)] = [K, \text{Ker}(\psi)] = 0\). Now, let
\[
\rho^* : \frac{F}{[F, R]} \to K
\]
be the map given by,
\[
\rho^*(a + [F, R]) = \rho(K).
\]
Hence we have
\[
\rho^*(D) = \rho^* \left( \frac{F' \cap R}{[F, R]} \right) = \rho(F' \cap R) = \rho(F') \cap \rho(R) = K' \cap \text{Ker}(\psi) = \text{Ker}(\psi).
\]
\[
\square
\]

**Corollary 4.5.** Let \(L\) be finite dimensional Lie superalgebra and \((T, \lambda)\) be the universal element, then
\[
\text{Ker}(\lambda) \cong \frac{F' \cap R}{[F, R]} = D.
\]

**Theorem 4.6.** If \((T, \lambda)\) is a universal element of the Lie superalgebra \(L\), then \(T \sim \frac{F}{[F, R]}\).

**Proof.** Consider the map \(\rho^* : \frac{F}{[F, R]} \to T\) and by Lemma 4.4, \(\rho^*\) is surjective. Let \(x + [R, F] \in \text{Ker}(\rho^*)\), for \(x \in F\). Then \(\rho^*(x + [R, F]) = \rho(x) = 0\), which implies \(\psi(\rho(x)) = \pi(x) = 0\), i.e., \(x \in \text{Ker}(\pi) = R\). We have \(\text{Ker}(\rho^*) \subseteq \frac{R}{[F, R]}\), then
\[
\text{Ker}(\rho^*) \cap \left( \frac{F}{[F, R]} \right)' = \text{Ker}(\rho^*) \cap \left( \frac{R}{[F, R]} \right) \cap \left( \frac{F}{[F, R]} \right)'
\]
and hence
\[
\text{Ker}(\rho^*) \cap \left( \frac{F' \cap R}{[F, R]} \right) = \text{Ker}(\rho^*) \cap D = 0.
\]
By Lemma 1.4 we get \(T \sim \frac{F}{[F, R]}\). \[
\square
\]

Finally, below is the main result of this section.

**Theorem 4.7.** Let \((T_1, \lambda_1)\) and \((T_2, \lambda_2)\) be two universal elements of a finite dimensional Lie superalgebra, then \(T_1 \cong T_2\).

**Proof.** By Theorem 4.6, we have \(T_1 \sim \frac{F}{[F, R]}\) and also \(T_2 \sim \frac{F}{[F, R]}\), which implies \(T_1 \sim T_2\). Further \(\dim T_1 = \dim T_2\), So, using Theorem 3.9 \(T_1 \cong T_2\). \[
\square
\]
Acknowledgement

The authors would like to thank the anonymous referee to go through the paper.

Funding

First and Third authors research are supported by NBHM Postdoctoral Fellowship and NBHM Grant respectively.

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