The bulk–dislocation correspondence for weak topological insulators on screw–dislocated lattices

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Abstract
A weak topological insulator in dimension three is known to have a topologically protected gapless mode along the screw dislocation. In this paper we formulate and prove this fact with the language of $C^*$-algebra $K$-theory. The proof is based on the coarse index theory of the helical surface.

Keywords: topological insulator, noncommutative geometry, coarse geometry

1. Introduction

The bulk-boundary correspondence is one of the fundamental issues in the theory of topological insulators. This principle states that a topological nature of a quantum system at the bulk ensures the existence of topologically protected boundary states of the same system to which a boundary is inserted. There are several theoretical frameworks demonstrating the bulk-boundary correspondence physically or mathematically. This paper deals with the one based on functional analysis and operator algebra, in which the bulk-boundary correspondence is understood as the boundary map of $C^*$-algebra $K$-theory. For the researches in this direction, we refer the readers to e.g. [BvS94, BKR17, KRS02, Kub17, PS16].

The aim of this paper is to prove the bulk–dislocation correspondence in this functional analytic setup. In short, this principle states that the topology of a three-dimensional Hamiltonian ensures the existence of topologically protected states localized at a screw dislocation. More specifically, a weak topological insulator in the $xy$-direction has a localized state along a screw dislocation at the $z$-axis. This phenomenon is already discovered in the literature of physics such as Ran–Zhang–Vishwanath [RZV09], Teo–Kane [TK10] and Imura–Takane–Tanaka [ITT11]. There is also a mathematical approach to this problem by Hannabuss–Mathai–Thiang...
[HMT16, section 6], in which the $K$-theory of the $C^*$-algebra of Heisenberg groups are studied through the noncommutative $T$-duality. This approach shares a mathematical basis with ours but we deal with a different $C^*$-algebra.

In the functional analytic formulation we are working in, a quantum system is characterized by its Hamiltonian operator. Under the one-particle and tight-binding approximations, it is a bounded self-adjoint operator acting on the Hilbert space of lattices $\ell^2(\mathbb{Z}^d; \mathbb{C}^N)$, where $d$ is the space dimension and $N$ is the internal degree of freedom. We assume that $H$ is insulated, i.e. has a spectral gap at the Fermi energy (which is often assumed to be 0). If we additionally assume that $H$ is of short-range and translation invariant, it is a self-adjoint invertible element in the $C^*$-algebra $C_r^*(\mathbb{Z}^d) \otimes \mathcal{M}_N$, where $C_r^*(\mathbb{Z}^d)$ is the group $C^*$-algebra of the abelian group $\mathbb{Z}^d$. Such an operator is topologically classified by the $K_0$-group of $C_r^*(\mathbb{Z}^d)$. The corresponding boundary Hamiltonian lies in the $C^*$-algebra $\mathcal{T} \otimes C_r^*(\mathbb{Z}^{d-1})$, where $\mathcal{T}$ denotes the Toeplitz algebra.

The bulk-boundary correspondence is understood as the boundary map of the following Toeplitz exact sequence

$$0 \to \mathbb{K}(\ell^2(\mathbb{Z}^2)) \otimes C_r^*(\mathbb{Z}^{d-1}) \to \mathcal{T} \otimes C_r^*(\mathbb{Z}^{d-1}) \to C_r^*(\mathbb{Z}^d) \to 0,$$

where $\mathbb{K}(\ell^2(\mathbb{Z}^2))$ denotes the algebra of compact operators on $\ell^2(\mathbb{Z}^2)$. Indeed, the image of $[H] \in K_0(\mathcal{C}^*_r(\mathbb{Z}^d))$ describes topologically protected boundary states of the corresponding boundary Hamiltonian.

Our formulation of the bulk-dislocation correspondence is given in a similar fashion. We define a $C^*$-algebra exact sequence of the form

$$0 \to \mathbb{K}(\ell^2(\mathbb{Z}^2)) \otimes C_r^*(\mathbb{Z}) \to \mathcal{A}_k \to C_r^*(\mathbb{Z}^2) \to 0,$$

where $\mathcal{A}_k$ is the $C^*$-algebra of observables on the dislocated lattice (a precise definition is given in (2.5)). Then the boundary map of $C^*$-algebra $K$-theory sends a gapped phase without dislocation $[H] \in K_0(\mathcal{C}^*_r(\mathbb{Z}^2))$ to the $K_{d-1}$-element describing the topologically protected dislocation state. The main theorem of this paper, theorem 2.5, determines this homomorphism. Indeed, it turns out to send the two-dimensional Bott generator $\beta_{xy}$ in the $xy$-direction to the generator of $K_1(\mathbb{K}(\ell^2(\mathbb{Z}^2)) \otimes C_r^*(\mathbb{Z}))$ and other Bott generators to zero.

The proof of our main theorem is based on the coarse geometry of the helical surface. As is discussed in subsection 3.1, the bulk-dislocation correspondence is thought of as a bulk-boundary correspondence for Hamiltonians on the helical surface. This identification is given in a ‘large-scale’ way, which matches with the philosophy of coarse geometry. In particular, the key technical lemma for the proof, lemma 3.2, relies on a relatively new technique to lift a finite propagation operator on the covering space, which is originally given in [GWY08]. A remarkable point is that the proof is parallel to the author’s study on the codimension 2 transfer map in higher index theory [KS21, Kub21], which is motivated from differential topology of manifolds. Indeed, the simplest case of this codimension 2 transfer map, a map from the $C^*$-algebra $K$-theory of the group $C^*$-algebra of $\mathbb{Z}^3$ to that of the trivial group, is the same thing as the proof of the bulk-dislocation correspondence given in this paper. This connection is discussed in subsection 4.5.

This paper is organized as follows. In section 2, we give an operator-algebraic setup of the bulk-dislocation correspondence and state the main theorem. In section 3, we give a heuristic discussion introducing the strategy of the proof, summarize the foundation of coarse index theory, and give a proof of the main theorem. In section 4, we list remarks on our main theorem, including several generalizations. In appendix A, we give a proof of the key lemma of this paper, originally given in [Kub21], specified to our setting.
2. A mathematical formulation

We start with formulating the problem with the language of functional analysis and $C^\ast$-algebras. We define the $C^\ast$-algebra of quantum mechanical observables on the lattice possessing the screw dislocation along the Burgers vector $\mathbf{z} := (0, 0, 1) \in \mathbb{Z}^3$. The general case, i.e. the case that the Burgers vector is an arbitrary lattice vector, is discussed later in subsection 4.1.

Let $X := \mathbb{R}^2$ and let $X_0 := X \setminus \{0, 0\}$. We realize the universal covering of $X_0$ as a submanifold of $\mathbb{R}^3$. Let $\tilde{X}_{x,0}$ denote the helical surface of $\mathbb{R}^3$ defined as

$$\tilde{X}_{x,0} := \left\{ (x, y, z + \frac{\arctan(y/x)}{2\pi}) | (x, y, z) \in X_0 \times \mathbb{Z} \right\}.$$

Then the projection $\mathbb{R}^3 \to \mathbb{R}^2$ onto the $xy$-plane restricts to the covering map $\tilde{X}_{x,0} \to X_0$. This is a $\mathbb{Z}$-Galois covering, i.e. $(x, y, z) \mapsto (x, y, z + 1)$ generates a free and proper action of $\mathbb{Z}$ on $\tilde{X}_{x,0}$. We regard $\tilde{X}_{x,0}$ as a metric space by the restriction of the standard metric on $\mathbb{R}^3$. Note that this metric differs from (more specifically, is not even quasi-isometric to) the one induced from the metric as covering manifold of $X_0$, i.e. the metric induced from the Riemannian metric tensor on $\tilde{X}_{x,0}$. We define the dislocated lattice $X_{x,0}$ as a lattice of $\tilde{X}_{x,0}$, that is,

$$X_{x,0} := \tilde{X}_{x,0} \cap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R})$$

$$= \left\{ \left(x, y, z + \frac{\arctan(y/x)}{2\pi}\right) | (x, y, z) \in (\mathbb{Z}^2 \setminus \{0, 0\}) \times \mathbb{Z} \right\}.$$

Set $X_3 := X_{x,0} \cup (\{0\} \times \{0\} \times \mathbb{Z})$. This set is thought of as the configuration of atoms of a screw-dislocated three-dimensional material. The (one-particle tight-binding) Hamiltonian operator is a bounded operator in $C(X_3)$.

For $v \in \mathbb{Z}^2 \times \mathbb{R}$, let $[v]$ stand for the vector in $X_3$ which is nearest to $v$ (if there are two such vectors, then $[v]$ stands for the one whose $z$-coordinate is larger). Note that $v$ and $[v]$ share the $(x, y)$-coordinate. We use this notation to define three unitary operators $\tilde{S}_x, \tilde{S}_y, \tilde{S}_z$ on $\ell^2(X_3)$ as

$$\tilde{S}_x \delta_x := \delta_x + \delta_y, \quad \tilde{S}_y \delta_x := \delta_x + \delta_z, \quad \tilde{S}_z \delta_x := \delta_x,$$

where $x := (1, 0, 0)$ and $y := (0, 1, 0)$, and $\delta_x$ are regarded as the dislocated lattice translation in the $x, y, z$-directions respectively.

Lemma 2.1. Let $\tilde{S}_x, \tilde{S}_y, \tilde{S}_z$ and $\Phi_x$ be as above. We write $S_x, S_y, S_z$ for the lattice translation on $\ell^2(\mathbb{Z}^3)$ in the $x, y, z$-directions. Then the following hold.

(a) We have

$$\Phi_x \tilde{S}_x \Phi_x = \tilde{S}_x,$$

$$\Phi_x \tilde{S}_y \Phi_x = \tilde{S}_y(1 - P + PS_x),$$

$$\Phi_x \tilde{S}_z \Phi_x = \tilde{S}_z,$$

where $P \in C(\ell^2(\mathbb{Z}^3))$ denotes the projection onto $\ell^2(\mathbb{Z}_{\geq 0} \times \{-1\} \times \mathbb{Z})$.

(b) We have $[\tilde{S}_x, \tilde{S}_y] = 0, [\tilde{S}_y, \tilde{S}_z] = 0$ and

$$[\tilde{S}_x, \tilde{S}_y] = \Phi_x(S_w S_y P \otimes 1)(1 - S_z^\ast)\Phi_x^\ast.$$


where \( p \in \mathcal{K}(\ell^2(\mathbb{Z}^2)) \) denotes the (rank one) projection onto \( \ell^2(\{-1, -1\}) \).

**Proof.** We first show (a). The third equality is obvious from the definition, and hence we prove the first and the second one. For \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{Z}^3 \), the dislocated lattice point \([\mathbf{v}] \in \mathbb{X}_z\) is located above \( \mathbf{v} \) (i.e. \([\mathbf{v}] = (v_1, v_2, v_3)\) satisfies \( v'_3 \geq v_3 \)) if and only if \( v_2 \geq 0 \), and otherwise below \( \mathbf{v} \) (i.e. \( v'_3 < v_3 \)). This, together with the monotonicity of the arc tangent function, shows that both \( \tilde{S}_z \Phi_Z \delta_x = \delta_{[\mathbf{v}] + x} \) and \( \Phi_Z \tilde{S}_z \delta_x = \delta_{[\mathbf{v}] + x} \) are supported above \( \mathbf{v} + x \), and hence \( \Phi_Z \tilde{S}_z \Phi_Z = \mathcal{S}_z \). By the same reason, the support of \( \tilde{S}_z \Phi_Z \Phi_Z = \delta_{[\mathbf{v}] + y} \) and \( \Phi_Z \tilde{S}_z \Phi_Z = \delta_{[\mathbf{v}] + y} \) may differ only if \( v_2 = -1 \). If \( v_1 \geq 0 \), then both \([\mathbf{v}]_1 + y \) and \( \mathbf{v}_1 + y \) are located near \( \mathbf{v} + y \), and hence \([\mathbf{v}]_1 + y = [\mathbf{v}] + y \). On the other hand, if \( v_1 < 0 \), then \([\mathbf{v}] + y \) is located above \( \mathbf{v} + y \) though \([\mathbf{v}] + y \) is located below \( \mathbf{v} \). This shows \( [\mathbf{v}] + y = [[\mathbf{v}] + y] - z \), and hence \( \Phi_Z \tilde{S}_z \Phi_Z = \mathcal{S}_z(1 - P + PS_z) \).

The claim (b) follows from (a).

**Definition 2.2.** We write \( \mathcal{A}_Z \) for the \( C^* \)-subalgebra of \( \mathbb{B}(\ell^2(\mathbb{X}_z)) \) generated by \( \tilde{S}_z, \tilde{S}_y, \tilde{S}_x \) and the \( C^* \)-subalgebra \( \Phi_Z(\mathcal{K}(\ell^2(\mathbb{Z}^2)) \otimes C_r^*(\mathbb{Z})) \Phi_Z^* \).

There is an abstract characterization of the \( C^* \)-algebra \( \mathcal{A}_Z \). To this aim we introduce two standard terminologies in coarse geometry [Roe03], both of which will be used in section 3 again.

(a) The support of an operator \( T \in \mathbb{B}(\ell^2(\mathbb{X}_z)) \) is defined as the support of the kernel function of \( T \). In other words,

\[
\text{supp}(T) := \{ (\mathbf{v}, \mathbf{w}) \in \mathbb{X}_z^2 | p_T \mathbf{w} \neq 0 \},
\]

where \( p_T \) denotes the rank one projection onto \( \ell^2(\{\mathbf{v}\}) \).

(b) An operator \( T \in \mathbb{B}(\ell^2(\mathbb{X}_z)) \) is of finite propagation if there is \( R > 0 \) such that \( \text{supp}(T) \) is included to the \( R \)-neighborhood of the diagonal, in other words, \( p_T \mathbf{w} = 0 \) for any \( \mathbf{v}, \mathbf{w} \in \mathbb{X}_z \) such that \( d(\mathbf{v}, \mathbf{w}) > R \) (the infimum of such \( R > 0 \) is called the propagation of \( T \)).

We call an operator \( T \in \mathbb{B}(\ell^2(\mathbb{X}_z)) \) xy-translation invariant away from the z-axis if there is \( R > 0 \) such that the commutators \([T, \tilde{S}_x], [T, \tilde{S}_y] \) are both supported in \( U_R \times U_R \), where \( U_R \) denotes the \( R \)-neighborhood of the z-axis in \( \mathbb{X}_z \).

**Lemma 2.3.** The closure of the set of operators satisfying the conditions

(a) Of finite propagation,

(b) xy-translation invariant away from the z-axis, and

(c) Translation invariant in the z-direction, i.e. \([T, \tilde{S}_z] = 0 \), coincides with \( \mathcal{A}_Z \).

Note that the conditions (a), (b) have a similar flavor to the quasi-equivariant Roe algebra introduced in [LT20].

**Proof.** We write \( \mathcal{A}_Z^\prime \) for the closure of the set of operators satisfying (a), (b) and (c). Then we have \( \mathcal{A}_Z \subset \mathcal{A}_Z^\prime \) since \( \tilde{S}_x, \tilde{S}_y, \tilde{S}_z \) and any operator of the form \( \Phi_Z(k \otimes i) \Phi_Z^* \), where \( k \in \mathcal{K}(\ell^2(\mathbb{Z}^2)) \) is of finite rank and \( i \in \mathcal{C}(\mathbb{Z}) \), are in \( \mathcal{A}_Z^\prime \). Now we show the converse.

A finite propagation operator \( T \) is decomposed into a finite sum

\[
T = \sum_{\ell,m,n} r_{m,n}(\mathbf{v}) \tilde{S}_\ell \tilde{S}_m \tilde{S}_n,
\]

where...
where each $f_{I,G}(v)$ is a bounded function on $X_3$. Under this expression of $T$, the condition (c) is equivalent to $f_{I,G}(v) = f_{I,G}(v + z)$ for all $v \in X_3$. Similarly, the condition (b) corresponds to $f_{I,G}(v) = f_{I,G}(v + x)$ for any $v \in X_3 \setminus U_R$. This shows that there is $f_{I,G}(v) = f_{I,G}(v + y)$ for any $v \in X_3 \setminus U_R$. Hence the quotient $\mathbb{C}^*$-algebra $\Phi_2(\mathbb{K} \otimes C_r^*(\mathbb{Z})) \Phi_2^*$ is an ideal of $A_x$. For simplicity of notations, we write this ideal shortly as $\mathbb{K} \otimes C_r^*(\mathbb{Z})$. Moreover, by lemma 2.1 (b), $\tilde{S}_x$ and $\tilde{S}_y$ commutes modulo this ideal. This shows that there is a $*$-homomorphism $\phi : A_x \to C_r^*(\mathbb{Z})$ determined by $\phi(S_x) = S_x$, $\phi(S_y) = S_y$, and $\phi(S_z) = S_z$. That is, there is an extension of $\mathbb{C}^*$-algebras

$$0 \to \mathbb{K} \otimes C_r^*(\mathbb{Z}) \to A_x \to C_r^*(\mathbb{Z}) \to 0.$$  
(2.5)

This induces the boundary map

$$\partial : K_0(C_r^*(\mathbb{Z})) \to K_1(\mathbb{K} \otimes C_r^*(\mathbb{Z})) \cong K_1(C_r^*(\mathbb{Z}))$$

in $\mathbb{C}^*$-algebra $K$-theory.

The $K$-theory of the group $C^*$-algebra $C_r^*(\mathbb{Z})^3$ is isomorphic to the topological $K$-theory of the three-dimensional torus, and hence

$$K_n(C_r^*(\mathbb{Z})) \cong K_n \oplus (K_{n-1})^3 \oplus (K_{n-2})^3 \oplus K_{n-3},$$  
(2.6)

where $K_n := K_n(\mathbb{C})$ (which is isomorphic to the topological $K$-group $K^{-1} \ast (pt)$). Since $K_0 \cong \mathbb{Z}$ and $K_1 \cong 0$, it follows that $K_n(C_r^*(\mathbb{Z}))$ is generated by 8 elements $\beta_0, \beta_1, \beta_2, \beta_3, \beta_x, \beta_y, \beta_z, \beta_{x+y}, \beta_{x+z}, \beta_{y+z}, \beta_{x+y+z}$. That is,

$$K_0(C_r^*(\mathbb{Z})) \cong \mathbb{Z} \beta_0 \oplus \mathbb{Z} \beta_1 \oplus \mathbb{Z} \beta_2 \oplus \mathbb{Z} \beta_3,$$

$$K_1(C_r^*(\mathbb{Z})) \cong \mathbb{Z} \beta_x \oplus \mathbb{Z} \beta_y \oplus \mathbb{Z} \beta_z \oplus \mathbb{Z} \beta_{x+y},\beta_{x+z}, \beta_{y+z}.$$  

**Remark 2.4.** In the theory of topological insulators, real $K$-theory plays an important role as well as complex $K$-theory. The Real version of $C^*$-algebra $K$-theory is defined for $C^*$-algebras equipped with a Real structure, i.e. an antilinear $*$-automorphic involution $\bar{\cdot} : A \to A$. We impose the real structure onto $A_x$ induced from the complex conjugation on $\ell^2(X_3)$. Note that this complex conjugation satisfies

$$\bar{S}_x = \bar{S}_y = \bar{S}_z = \bar{x}.$$  

Hence the quotient $*$-homomorphism $A_x \to C_r^*(\mathbb{Z})$ preserves the real structure if we impose the real structure on $C_r^*(\mathbb{Z})$ determined by $\bar{S}_x = S_x$, $\bar{S}_y = S_y$, $\bar{S}_z = S_z$. Through the Gelfand–Naimark duality, this real $C^*$-algebra is isomorphic to the continuous function algebra $C(\mathbb{T}^3)$ on the three-dimensional Brillouin torus, with the real structure $f(k) = \bar{f}(-k)$ for any $k \in \mathbb{T}^3$.

For a unital real $C^*$-algebra $A$, the real $K$-theory $KR_0(A)$ is defined as the group completion of the monoid of homotopy classes of conjugation-invariant projections in $\bigcup_n M_n(A)$. When $A = C_r^*(\mathbb{Z}) \cong C(\mathbb{T}^3)$ as above, the real $K$-group $KR_r(C_r^*(\mathbb{Z}))$ is isomorphic to the $KR$-group $KR^{-1}(\mathbb{T}^3, \tau)$, where $\tau(k) = -k$. Hence it is isomorphic to the direct sum of 8 groups

$$KR_0(C_r^*(\mathbb{Z})) \cong KR_0 \oplus (KR_{n-1})^3 \oplus (KR_{n-2})^3 \oplus KR_{n-3}.$$  


in the same way as (2.6). Each direct summand is generated by a single element $\beta_0^n, \beta_x^{m-1}, \beta_y^{m-1}, \beta_x^{-1}, \beta_y^{-1}, \beta_x^{-2}, \beta_y^{-2}, \beta_x^{-3}$ and $\beta_y^{-3}$ respectively. Here the element $\beta_0^l \in K_R$ is a free generator if $l \equiv 0, 4 \mod 8$, a two-torsion element if $l \equiv 1, 2 \mod 8$, and otherwise 0.

The goal of this paper is to determine the image of these generators by the boundary map $\partial$ associated to the exact sequence (2.5).

**Theorem 2.5.** The boundary map $\partial: K_*(C^*_r(\mathbb{Z}^3)) \to K_{*+1}(\mathbb{K} \otimes C^*_r(\mathbb{Z}))$ associated to the extension (2.5) sends the Bott element $\beta_{xy}$ in the xy-direction to the generator $\beta$ of $K_1(C^*_r(\mathbb{Z}))$ and the other Bott generators to zero.

The essential part of this theorem is to determine $\partial(\beta_{xy})$. Before that, we prove the remainder.

**Lemma 2.6.** The boundary map $\partial$ sends $\beta_x, \beta_y, \beta_z, \beta_{xz}$ and $\beta_{yz}$ to zero.

**Proof.** Let $i_{yz}: C^*_r(\mathbb{Z}^2) \to C^*_r(\mathbb{Z}^3)$ denote the inclusion to the $yz$-component. As is seen in lemma 2.1 (b), the unitaries $S_x$ and $S_y$ commute. Hence the map $i_{yz}(S_x) = S_y$ and $i_{yz}(S_y) = S_z$ gives rise to a $*$-homomorphism

$$\tilde{i}_{yz}: C^*_r(\mathbb{Z}^2) \to A_z$$

such that the diagram

$$
\begin{array}{c}
C^*_r(\mathbb{Z}^2) \\
\bigg\downarrow \tilde{i}_{yz} \\
0 \longrightarrow \mathbb{K} \otimes C^*_r(\mathbb{Z}) \longrightarrow A_z \longrightarrow C^*_r(\mathbb{Z}^3) \longrightarrow 0
\end{array}
$$

commutes. Therefore we get

$$\partial(\beta_{yz}) = \partial(i_{yz}(\beta)) = (\partial \circ \phi_\ast)(\tilde{i}_{yz}(\beta)) = 0,$$

where $\partial \circ \phi_\ast = 0$ comes from the $C^*$-algebra $K$-theory long exact sequence. This discussion also shows that $\partial\beta_0 = 0, \partial\beta_z = 0$ and $\partial\beta_z = 0$, because $\beta_x, \beta_y, \beta_z$ are all in the image of $i_{yz}$.

The claims $\partial(\beta_x) = 0$ and $\partial(\beta_y) = 0$ are proved in the same way. $\square$

The claim $\partial\beta_{xy} = 0$ is reduced to $\partial\beta_{xy} = \beta$ in the following way. Let $i_{xy}: C^*_r(\mathbb{Z}^2) \to C^*_r(\mathbb{Z}^3)$ denote the inclusion to the $xy$-component. Let $B_z$ denote the preimage of $C^*_r(\mathbb{Z}^3)$ in $A_z$ (in other words, $B_z$ denotes the $C^*$-algebra generated by $S_x, S_y$ and $\Phi_u(\mathbb{K}(\ell^2(\mathbb{Z}^2))) \otimes C^*_r(\mathbb{Z}))$. Then there is an exact sequence

$$0 \longrightarrow \mathbb{K} \otimes C^*_r(\mathbb{Z}) \longrightarrow B_z \longrightarrow C^*_r(\mathbb{Z}^3) \longrightarrow 0.$$  \hspace{1cm} (2.10)

**Lemma 2.7.** Suppose that $\partial\beta_{xy} = \beta \in K_{*+1}(\mathbb{K} \otimes C^*_r(\mathbb{Z}))$ is verified. Then $\partial\beta_{xy} = 0$ holds.

**Proof.** The map $u^k \otimes T \mapsto T\widetilde{S}_k^u$ (where $u$ denotes the generator of $C^*_r(\mathbb{Z})$) induces a $*$-homomorphism $m: C^*_r(\mathbb{Z}) \otimes B_z \to A_z$. This extends to a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow C^*_r(\mathbb{Z}) \otimes (\mathbb{K} \otimes C^*_r(\mathbb{Z})) \longrightarrow C^*_r(\mathbb{Z}) \otimes B_z \longrightarrow C^*_r(\mathbb{Z}) \otimes C^*_r(\mathbb{Z}^2) \longrightarrow 0 \\
\bigg\downarrow m \bigg\downarrow m \\
0 \longrightarrow \mathbb{K} \otimes C^*_r(\mathbb{Z}) \longrightarrow A_z \longrightarrow C^*_r(\mathbb{Z}^3) \longrightarrow 0,
\end{array}
$$
where the right vertical map is an isomorphism and the left vertical map $m$ is identified, through the Gelfand–Naimark duality $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$, with the pull-back with respect to the diagonal embedding $\mathbb{T} \to \mathbb{T} \times \mathbb{T}$.

By the Künneth theorem [RS86], there is an isomorphism $K_* (A) \otimes K_* (C_r^*(\mathbb{Z})) \to K_* (A \otimes C_r^*(\mathbb{Z}))$ which is factorial and is compatible with the $K$-theory long exact sequence. Hence we get

$$\partial (\beta_{xy} \otimes \beta) = \partial \beta_{xy} \otimes \beta = \beta \otimes \beta.$$

This completes the proof as

$$\partial (\beta_{xy}) = m_* \circ \partial (\beta_{xy} \otimes \beta) = m_* (\beta \otimes \beta) = 0.$$

Here $m_* (\beta \otimes \beta) = 0$ follows from $\beta \otimes \beta \in K_* (C_0(\mathbb{R}^2)) \subset K_0 (C(\mathbb{T}^2))$, since the composition $C_0(\mathbb{R}^2) \to C(\mathbb{T}^2)$ is null-homotopic.

\section{3. Proof of the main theorem}

In this section we prove the essential part of our main theorem. Here we focus on complex $K$-theory, and show that the boundary map $\partial$ of the $K$-theory long exact sequence of (2.5) sends the Bott generator $\beta_{xy}$ of the $xy$-component of $K_0 (C_r^*(\mathbb{Z}^3))$ to the generator $\beta$ of $K_1 (\mathbb{K} \otimes C_r^*(\mathbb{Z}))$. We remark that the proof given in this section also works for the corresponding result in Real $K$-theory. This point is discussed in subsection 4.3.

\subsection{3.1. Discussion}

Before going to the formal proof, we shortly illustrate how the weak topology of a bulk Hamiltonian in the $xy$-direction induces a dislocation-localized current along the $z$-axis. Let $H$ be a Hamiltonian on the two-dimensional standard lattice, which is of the form

$$H := \sum_{(a, m) \in \mathbb{Z}^2} A_{nm} S_a^x S_m^y \in \mathcal{M}_N \otimes C_r^*(\mathbb{Z}^3).$$

Here, each $A_{nm}$ is an $N \times N$ matrix satisfying $A_{nm}^* = A_{nm}$. It is usually assumed to be of short-range, i.e. the coefficient decays exponentially as $||A_{nm}|| \leq C_1 e^{-C_2 \sqrt{|n^2 + m^2|}}$ for some $C_1, C_2 > 0$. To simplify the discussion, we impose a stronger assumption; the right-hand side is a finite sum. This corresponds to the finite propagation condition given on page 4. Moreover, we assume that $H$ has a spectral gap at the Fermi level $\mu = 0$. This $H$, regarded as a three-dimensional quantum observable with the same presentation through the inclusion of $C^*$-algebras $C_r^*(\mathbb{Z}^3) \subset C_r^*(\mathbb{Z}^3)$, gives a model of an $xy$-weak topological insulator of type A. As an actual operator acting on the Hilbert space $\mathcal{F} (\mathbb{Z}^3; \mathbb{C}^N)$, this $H$ is the superposition of infinitely many layers of two-dimensional Hamiltonians in the $z$-direction.

The corresponding Hamiltonian on the screw-dislocated lattice $X_\phi$ is

$$H_{\phi} := \sum_{(a, m) \in \mathbb{Z}^2} A_{nm} \tilde{S}_a^x \tilde{S}_m^y.$$

This is a lift of $H \in C^*(\mathbb{Z}^3) \otimes \mathcal{M}_N$ to $A_\phi \otimes \mathcal{M}_N$ with respect to the quotient $\phi$ in (2.10). Therefore, by definition of the boundary map in $K$-theory, the image of the boundary map $\partial [H] \in K_* (\mathbb{K} \otimes C_r^*(\mathbb{Z}))$ is represented by the unitary

$$U_{\phi} := - \exp [ - \pi i \chi (H_{\phi})] \in (\mathbb{K} \otimes C_r^*(\mathbb{Z}))^+,$$
where \( \chi : \mathbb{R} \to [-1, 1] \) is a continuous function such that \( \chi|_{(-\infty,-\epsilon)} \equiv -1 \) and \( \chi|_{(\epsilon,\infty)} \equiv 1 \). This unitary extracts the spectrum of the dislocated Hamiltonian \( H_z \) inside the bulk spectral gap.

We consider replacing this \( H_z \) with another lift of \( H \) in \( \mathcal{A}_z \). Let \( \tilde{X}, R \) and \( X, R \) denote the complement of the \( R \)-neighborhood of the \( z \)-axis in \( \tilde{X}, 0 \) and \( X, R \) respectively. Let \( \Pi_R \) denote the projection onto \( \ell^2(X, R; \mathbb{C}^N) \subset \ell^2(X; \mathbb{C}^N) \). The operator \( H_{z,R} \) is also a lift of \( H \) since \( H_{z,R} - H_z \in \mathbb{K} \otimes C_c^*(\mathbb{Z}) \). If \( R > 0 \) is sufficiently larger than the range of the Hamiltonian \( H \), then the above \( H_{z,R} \) is locally separated into layers, and hence is viewed as a difference operator on (the lattice \( \mathbb{X}_{z,R} \) of) the helical surface \( \tilde{X}, R \), which is a noncompact two-dimensional manifold with boundary. The boundary \( \partial \tilde{X}, R \) is a helix, and hence is \( \mathbb{Z} \)-equivariantly and quasi-isometrically homeomorphic to the real line \( \mathbb{R} \) (on which \( \mathbb{Z} \) acts as the shift by 1). Then the bulk-boundary correspondence for \( \tilde{X}, R \) will show that a non-trivial topology of \( H \) implies an edge current along the boundary helix (figure 1).

In order to shape this heuristic discussion into a rigorous proof of the bulk-dislocation correspondence, there are two difficulties. One is that the helical surface no longer has the translation symmetry in the \( xy \)-direction. Another is that the radius \( R > 0 \) of the boundary helix is chosen after the range of \( H \) is fixed, and hence there is no simultaneous construction which works for all finite range Hamiltonians. Both of these two problems are resolved by working in the framework of coarse index theory. In particular, the second problem is resolved by considering coarse geometry on the helical surface ‘relative to the boundary helix’. Indeed, the concept of relative coarse geometry is already encoded to the definition of the \( \mathbb{C}^* \)-algebra extension (2.10).

3.2. Coarse index theory

Here we discuss the results in coarse geometry and coarse index theory used in the paper. For more details, we refer the reader to [Roe96, Roe03, HR00, WY20].

Let \( W \) be a locally compact metric space equipped with a free proper action of a discrete group \( \Gamma \) (the translation action of \( \mathbb{Z}^2 \) in the \( xy \)-direction on \( X = \mathbb{R}^2 \) and the translation action of \( \mathbb{Z} \) in the \( z \)-direction on \( \tilde{X}, R \) are our examples of interest). Let \( \pi : C_0(W) \to \mathbb{B}(\mathcal{H}) \) be a \( \Gamma \)-equivariant \(*\)-homomorphism which is ample, i.e. any non-zero function \( f \in C_0(W) \) acts as a compact operator. When \( W \) is a Riemannian manifold, we choose as \( (\pi, \mathcal{H}) \) the multiplication representation of \( C_0(W) \) on the \( L^2 \)-space \( L^2(W) \) with respect to the \( \Gamma \)-invariant volume form on \( W \). When \( W \) is a discrete metric space, we choose as \( \mathcal{H} \) the infinite direct sum \( \ell^2(W)^{\oplus \infty} \) on which \( C_0(W) \) acts by multiplication. We define the notion of support and propagation for an operator \( T \in \mathbb{B}(L^2(W)) \) in the same way as on page 6 (for the precise definition, see e.g. [HR00, definition 6.3.3]).
The invariant Roe algebra $C^*(W)^\Gamma$ is the closure of the $*$-algebra $\mathbb{C}[W]^\Gamma$ of $\Gamma$-invariant operators on $H$ which are of finite propagation and locally compact, i.e. $Tf, fT \in \mathbb{K}(H)$ for any $f \in C_c(W)$. When $\Gamma$ is trivial, this $C^*$-algebra is called the Roe algebra and written as $C^*(W)$. Moreover, for a $\Gamma$-invariant subspace $V \subset W$, the ideal $C^*(V \subset W)$ is defined as the closure of $\Gamma$-invariant, finite propagation, locally compact operators whose support is contained in an $R$-neighborhood of $V \times V$ for some $R > 0$.

Let $D^*(W)^\Gamma$ denote the closure of the $*$-algebra $D_{alg}^*(W)^\Gamma$ of $\Gamma$-invariant bounded operators on $L^2(W)$ which are of finite propagation and quasi-local, i.e. the commutator $[T, f]$ is a compact operator for any $f \in C_c(W)$. We also define the ideal $D^*(V \subset W)^\Gamma$ of $D^*(W)^\Gamma$ as the closure of the set of finite propagation quasi-local operators $T$ such that $Tf, fT \in \mathbb{K}(H)$ for any $f \in C_0(W)$ such that $f|_V \equiv 0$.

The inclusions $C^*(W)^\Gamma \subset D^*(W)^\Gamma$ and $C^*(V \subset W)^\Gamma \subset D^*(V \subset W)^\Gamma$ are ideals. We write $Q^*(W)^\Gamma$ and $Q^*(V \subset W)^\Gamma$ for the quotients $D^*(W)^\Gamma/C^*(W)^\Gamma$ and $D^*(V \subset W)^\Gamma/C^*(V \subset W)^\Gamma$ respectively.

**Remark 3.1.** Here we list some basic facts on the $K$-theory of these coarse $C^*$-algebras, which will be used in the proof of our main theorem.

(a) Let $\Gamma$ act on $W$ cocompactly. By choosing a Borel subset $U \subset W$ such that $W = \bigsqcup_{g \in \Gamma} g \cdot U$, the Hilbert space $L^2(W)$ is identified with $l^2(\Gamma) \otimes L^2(U)$. This induces a $*$-isomorphism $C^*(W)^\Gamma \cong C^*(\Gamma) \otimes \mathbb{K}(L^2(U))$ ([Roe96, lemma 5.14]).

(b) For any $\Gamma$-invariant subspace $V \subset W$, the Roe algebras $C^*(V \subset W)^\Gamma$ and $C^*(V)^\Gamma$ have the same $K$-theory. Also, $Q^*(V \subset W)^\Gamma$ and $Q^*(V)^\Gamma$ have the same $K$-theory ([2012, proposition 4.3.34]).

(c) When $W$ is an even dimensional complete Riemannian manifold with a spin structure, the Dirac operator $D$ determines a $K$-theory class $[W] \in K_0(Q^*(W)^\Gamma)$ called the Dirac fundamental class. In the same way, if $W$ is a free proper $\Gamma$-manifold with $\Gamma$-invariant boundary, the relative fundamental class $[W, \partial W] \in K_0(Q^*(W)^\Gamma/Q^*(\partial W \subset W)^\Gamma)$ is defined.

(d) The boundary map

$$
\partial : K_*(Q^*(W)^\Gamma/Q^*(\partial W \subset W)^\Gamma) \to K_{*-1}(Q^*(\partial W \subset W)^\Gamma)
$$

sends $[W, \partial W]$ to the fundamental class $[\partial W]$ of the boundary under the isomorphism $K_*(Q^*(\partial W \subset W)^\Gamma) \cong K_*(Q^*(\partial W)^\Gamma)$ mentioned in (b) (this fact is known as the ‘boundary of Dirac is Dirac’ principle, see e.g. [HR00, proposition 11.2.15]).

(e) The boundary map

$$
\partial : K_0(Q^*(W)^\Gamma) \to K_0(C^*(W)^\Gamma)
$$

is called the equivariant coarse index map and denoted by $\text{Ind}$. When $W = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ acting on $W$ by the translation, then the equivariant coarse index is the same as the family index ([BCH94, example 3.11]), and in particular sends the fundamental class $[\mathbb{R}^n] \in K_0(Q^*(\mathbb{R}^n)^{\mathbb{Z}^n})$ to the Bott generator of the top degree in $K_0(C^*(\mathbb{R}^n)^{\mathbb{Z}^n}) \cong K_0(C^*(\mathbb{Z}^n)) \cong K^0(\mathbb{T}^n)$.
3.3. Equivariant coarse geometry of helical surfaces

Here we apply the facts listed in the previous subsection to the $\mathbb{Z}$-equivariant coarse geometry of the helical surface. The following lemma, which lifts a finite propagation operator on $X = \mathbb{R}^2$ to the covering space $\tilde{X}_{x,R}$, is a key ingredient of the proof of theorem 2.5.

**Lemma 3.2.** There is a $*$-homomorphism

$$s : C^*(X) \to C^*(\tilde{X}_{x,R})^\mathbb{Z}/C^*(\partial\tilde{X}_{x,1} \subset \tilde{X}_{x,1})^\mathbb{Z},$$

which extends to

$$s : D^*(X) \to D^*(\tilde{X}_{x,R})^\mathbb{Z}/D^*(\partial\tilde{X}_{x,1} \subset \tilde{X}_{x,1})^\mathbb{Z}.$$

They induce a corresponding $*$-homomorphism between $Q^*$ coarse $C^*$-algebras. Moreover, the image $s_*[X]$ of the fundamental class of $X$ is $[\tilde{X}_{x,1}, \partial\tilde{X}_{x,1}]$.

A proof of this lemma is given in a more general geometric setting in [Kub21]. In appendix A, we give a full proof which is specific to our setting. Here we only sketch the construction of $s$.

**Lemma 3.3.** Let $K$ be a bounded operator on $L^2(X)$ with propagation less than $R > 0$. Moreover, we assume that $K$ is represented by convolution with a kernel function $k : X \times X \to \mathbb{C}$ as $K\xi(x) = \int Xk(x,y)\xi(y)dy$. The set of such operators is dense in $C^*(X)$. We define the function $\tilde{k}_R : \mathbb{X}_{x,R} \times \mathbb{X}_{x,R} \to \mathbb{C}$ as

$$\tilde{k}_R(x,y) = \begin{cases} k(\pi(x), \pi(y)) & \text{if } d(\tilde{x}, \tilde{y}) < R, \\ 0 & \text{otherwise,} \end{cases}$$

where $d$ denotes the metric on $\mathbb{X}_{x,R}$ induced from its Riemannian metric, and set

$$\tilde{K}_R \xi(x) := \int_{y \in \mathbb{X}_{x,R}} \tilde{k}_R(x,y)\xi(y)dy.$$ \hspace{1cm} (3.5)

This $\tilde{K}_R$ determines a bounded operator (this is a non-trivial part of the proof), which is locally compact and has propagation less than $R$. For $0 < R < S$, the difference $\tilde{K}_S - \tilde{K}_R$ is supported in the $(S + R)$-neighborhood of $\partial\tilde{X}_{x,1}$, and hence lies in the ideal $C^*(\partial\tilde{X}_{x,1} \subset \tilde{X}_{x,1})$. Therefore, $s(K) := \tilde{K}_R$ gives rise to a well-defined linear map from a dense subalgebra of $C^*(X)$ to $C^*(\tilde{X}_{x,R})^\mathbb{Z}/C^*(\partial\tilde{X}_{x,1} \subset \tilde{X}_{x,1})^\mathbb{Z}$. Note that it remains to prove that this $s$ extends to $C^*(\tilde{X}_{x,1})^\mathbb{Z}$, in other words, $s$ is a bounded linear map.

Next we relate this $s$ with the extension (2.10). Let us fix $0 < \epsilon < 1/2$. Let $\psi_0$ be an $L^2$-function on $L^2(X)$ supported in the $\epsilon$-neighborhood of $0$. For $v \in \mathbb{Z}^2$, let $\psi_v(x) := \psi_0(x - v)$. We define the $\mathbb{Z}^2$-equivariant isometry of Hilbert spaces

$$V : \ell^2(\mathbb{Z}^2) \to L^2(X), \quad V(\delta_v) = \psi_v.$$

Moreover, for each $\hat{v} \in \mathbb{Z}^2$, let $\psi_\hat{v}$ denote the restriction of the pull-back $\pi^*\psi_v$ to the $\epsilon$-neighborhood of $\hat{v}$. This gives rise to a $\mathbb{Z}$-equivariant isometry

$$\hat{V} : \ell^2(\mathbb{Z}_{x,1}) \to L^2(\tilde{X}_{x,1}), \quad \hat{V}(\delta_v) := \psi_\hat{v}.$$
Now, by the definition of $s$ sketched in lemma 3.3, we have

$$s(VS_x V^*) = s \left( \sum_{\psi} |\psi + x\rangle \langle \psi| \right) = \sum_{\psi} |\psi + x\rangle \langle \psi| = \bar{V}_x V^*$$

modulo $C^*(\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})$. Similarly, we have $s(VS_y V^*) = \bar{V}_y V^*$ modulo $C^*(\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})$. That means that $\text{Ad}(\bar{V})$ sends $B_x$ to $C^*(\bar{X}_{\mathbf{z},1})^2$ and the diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{K} \otimes C^*_\mathbb{Z}(\mathbb{Z}) & \rightarrow & B_x & \rightarrow & C^*_\mathbb{Z}(\mathbb{Z}^2) & \rightarrow & 0 \\
& & \downarrow \text{Ad}(\bar{V}) & & \downarrow \text{Ad}(\bar{V}) & & \downarrow \circ F \circ \text{Ad}(V) & \\
0 & \rightarrow & C^*(\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})^2 & \rightarrow & C^*(\bar{X}_{\mathbf{z},1})^2 & \rightarrow & \frac{C^*(\bar{X}_{\mathbf{z},1})^2}{\mathbb{C}^* (\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})^2} & \rightarrow & 0
\end{array}
\]

commutes, where $F : C^*(X)^2 \rightarrow C^*(X)$ denotes the inclusion. This induces the commutative diagram of $K$-theory

\[
\begin{array}{cccccc}
K_0(C^*_\mathbb{Z}(\mathbb{Z}^2)) & \xrightarrow{\partial} & K_0(C^*_\mathbb{Z}(\mathbb{Z})) & \xrightarrow{\text{Ad}(\bar{V})} & K_{-1}(\mathbb{K} \otimes C^*_\mathbb{Z}(\mathbb{Z})) \\
& & \downarrow \circ s \circ F \circ \text{Ad}(V) & & \downarrow \text{Ad}(\bar{V}) \\
K_0(C^*(X)^2) & \xrightarrow{\partial} & K_0(C^*(X)) & \xrightarrow{\text{Ind}} & K_{-1}(C^*(\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})^2).
\end{array}
\]

Moreover, by remark 3.1 (a), the left and the right vertical maps in (3.6) are isomorphisms.

**Proof of Theorem 2.5.** By lemma 3.2, we have the commutative diagram

\[
\begin{array}{cccccc}
K_1(Q^*(X)^2) & \xrightarrow{\partial} & K_1(Q^*(X)) & \xrightarrow{\text{Ind}} & K_0(Q^*(\partial \bar{X}_{\mathbf{z},1})^2) \\
& & \downarrow \text{Ind} & & \downarrow \text{Ind} \\
K_0(C^*(X)^2) & \xrightarrow{\partial} & K_0(C^*(X)) & \xrightarrow{\text{Ind}} & K_{-1}(C^*(\partial \bar{X}_{\mathbf{z},1} \subset \bar{X}_{\mathbf{z},1})^2).
\end{array}
\]

By (3.6), it suffices to show the composition $\partial \circ s \circ F_x$ in the second row maps the Bott generator $\beta_{xy} \in K_0(C^*(X)^2)$ to $\beta \in K_{-1}(C^*(\partial \bar{X}_{\mathbf{z},1})^2)$. This is checked as

\[
(\partial \circ s \circ F_x)(\beta_{xy}) = (\partial \circ s_x \otimes F_x \circ \text{Ind})([X])
\]

\[
= (\text{Ind} \circ \partial \circ s \circ F_x)([X])
\]

\[
= (\text{Ind} \circ \partial)[\bar{X}_{\mathbf{z},1}, \partial \bar{X}_{\mathbf{z},1}]
\]

\[
= \text{Ind} [\partial \bar{X}_{\mathbf{z},1}] = \beta.
\]

Here the first and the last equalities are due to remark 3.1 (e), the third equality follows from lemma 3.2 and the forth equality is the ‘boundary of Dirac is Dirac’ principle exposed in remark 3.1 (d). □
4. Miscellaneous remarks

We finish the paper by a list of remarks.

4.1. General Burgers vector

Theorem 2.5 generalizes to a dislocated lattice with a general Burgers vector. Let us consider the Burgers vector $b := (b_x, b_y, b_z) \in \mathbb{Z}^3$. We assume that $b$ extends to a $\mathbb{Z}$-basis $\{a, c, b\}$ of $\mathbb{Z}^3$, i.e. there are no $b' \in \mathbb{Z}^3$ and $k \in \mathbb{Z} \setminus \{1, 0, -1\}$ such that $kb' = b$. Let $T_b := (abc) \in GL_3(\mathbb{Z})$, i.e. $T_b x = a$, $T_b y = c$ and $T_b z = b$. Then the dislocated lattice is defined as the discrete subset $T_b \cdot \mathbb{Z}^3 \subset \mathbb{R}^3$. We define the unitary $\Psi_b : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ as $\Psi_b(\delta_k) = \delta_{T_b k}$. We define the $C^*$-algebra $A_b$ as $\text{Ad}(\psi_b)(A_{\mathbb{Z}})$. It is generated by three unitaries $S_a := \psi_b \psi_a \psi_b^*$, $S_c := \psi_b \psi_y \psi_b^*$, $S_b := \psi_b \psi_z \psi_b^*$ and $\text{Ad}(\psi_b)(k \otimes C^*_r(\mathbb{Z}))$.

Then we have a commutative diagram of exact sequences

\[
\begin{array}{cccc}
0 & \rightarrow & K \otimes C^*_r(\mathbb{Z}) & \rightarrow & A_{\mathbb{Z}} & \rightarrow & C^*_r(\mathbb{Z}^3) & \rightarrow & 0 \\
& & \downarrow \text{Ad}(\psi_b) & & \downarrow \text{Ad}(\psi_b) & & \downarrow T_b & & \\
0 & \rightarrow & K \otimes C^*_r(\mathbb{Z}) & \rightarrow & A_b & \rightarrow & C^*_r(\mathbb{Z}^3) & \rightarrow & 0
\end{array}
\]

which shows that the boundary map in $K$-theory with respect to the second row is identified with the one given in theorem 2.5 through $T_b$.

4.2. Translation invariance in the $xy$-direction

In sections 2 and 3, we assume that the Hamiltonian $H$ is translation invariant away from the line defect. Indeed, the proof of theorem 2.5 shows that this assumption is not needed. Let us only assume that the Hamiltonian $H \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is translation invariant in the $z$-direction. Let $C^*_r([Z])^\mathbb{Z}$ denote the closure of the set of finite propagation operators on $\ell^2(\mathbb{Z})$ which is translation invariant in the $z$-direction (this $C^*$-algebra is called the invariant uniform Roe algebra). Here $[Z]$ stands for the lattice $\mathbb{Z}^3$ regarded as a metric space. This is a subalgebra of the Roe algebra $C^*((\mathbb{Z}))$. In [Kub17], the (possibly not translation invariant) topological phases are classified by the $K$-theory of the Roe algebra. (More precisely, in [Kub17] two kinds of classifications are considered; $K$-theory of the uniform Roe algebra and that of the Roe algebra. The latter classification is more rough.) The invariant Roe algebra $C^*((\mathbb{Z}))^{\mathbb{Z}}$ is isomorphic to $C^*((\mathbb{Z}^2)) \otimes C^*_r(\mathbb{Z})$, where the right tensor component is generated by the unitary $S_z$. By the Künneth theorem, its $K$-theory is isomorphic to

\[K(C^*((\mathbb{Z}))) \cong K_{-2} \oplus K_{-3} = Z \beta_{xy} \oplus Z \beta_{yz},\]

where $\beta_{xy}$ and $\beta_{yz}$ are the image of the corresponding element with respect to the inclusion $C^*_r([Z]) \subset C^*((\mathbb{Z}))$. Now the $s$-homomorphism $s$ defined in lemma 3.2 extends to

\[s : C^*((\mathbb{Z}^2)) \cong C^*((\mathbb{Z}^2)) \otimes C^*_r(\mathbb{Z}) \rightarrow C^*(\tilde{X}_{z,1}) \otimes C^*(\partial \tilde{X}_{z,1} \subset \tilde{X}_{z,1})\]

as $s(T \otimes S_{\mathbb{Z}}) := s(T) \otimes S_{\mathbb{Z}}$. The induced map in $K$-theory

\[K_*(C^*((\mathbb{Z}))) \rightarrow K_*\left(\frac{C^*(\tilde{X}_{z,1})}{C^*(\partial \tilde{X}_{z,1} \subset \tilde{X}_{z,1})}\right) \rightarrow K_*(C^*(\partial \tilde{X}_{z,1} \subset \tilde{X}_{z,1}))\]

sends $\beta_{xy}$ to $\beta$ and $\beta_{yz}$ to zero.
Table 1. Strong invariants in dimension 2.

| Type     | A | AIII | AI | AIII | AI | BDI | D | DIII | AI | CH | C | CI |
|-----------|---|------|----|------|----|-----|---|------|----|----|---|----|
| K-group   | Z | 0    | Z  | 0    | 0  | Z   | Z | Z    | Z2 | Z2 | 0 | Z  |

4.3. Bulk-dislocation correspondence of topological insulators

Our proof of theorem 2.5 uses no special features of complex K-theory (indeed it relies only on a formal diagram chasing argument and the fundamental facts of coarse index theory listed in remark 3.1). Therefore the same proof also works in context of real K-theory of C*-algebras. Namely, it is shown that the boundary map $\partial$ of the exact sequence (2.5) sends $\beta_{\text{xy}}^{-2} \in KR_n(C_1(\mathbb{Z}^2))$ (cf remark 2.4) to $\beta \in KR_{n-1}(\mathbb{K} \otimes C_1(\mathbb{Z}))$, and other generators to zero. This enables us to generalize the bulk-dislocation correspondence for topological insulators with one of the symmetries of the Altland–Zirnbauer 10-fold way [AZ97]. A symmetry in this class is generated by some of time-reversal symmetry $T$ (antilinear unitary commuting with $H$), the particle–hole symmetry $C$ (antilinear unitary anticommuting with $H$) and sublattice symmetry $S$ (linear unitary anticommuting with $H$) with the relations $T^2 = \pm 1$, $C^2 = \pm 1$, $S^2 = 1$, $S = CT$. The set of self-adjoint invertible operators that preserve the symmetry is classified by one of two complex and 8 Real K-groups of $C_1(\mathbb{Z}^2)$ (for this, we use Van Daele’s description of Real $\mathbb{Z}_2$-graded C*-algebra K-theory [Kel17] or, equivalently, a twisted equivariant K-theory [Kub16]).

We list in table 1 the topological classification of three-dimensional weak insulators in the $xy$-direction, which is the same as the classification of two-dimensional strong topological insulators. There is a non-trivial bulk-dislocation correspondence for type A, AI, D, DIII, AI, C topological insulators. In the literature of physics, this is already discovered by Teo–Kane [TK10].

4.4. Quantum Hall effect along the dislocation

Here we discuss a consequence of theorem 2.5 in the case of type A topological insulators. Let $H$ be a three-dimensional type A topological insulator. Parallel to the case of the bulk-edge correspondence of the integer quantum Hall effect, the topology of the bulk Hamiltonian $H$ induces the Hall conductivity along the line defect. Following [KRS02, equation (24)], the Hall conductance along the dislocation is calculated as

$$
\sigma_{\text{screw}} = \frac{e^2}{h} \left( -\lim_{\Delta \to \{0\}} \frac{1}{|\Delta|} \int_{\mathbb{T}^2} \text{Tr}(\hat{P}_{\Delta}(k)(\partial_{k_x} \hat{H})(k)) dk \right),
$$

where Tr is the (unbounded) trace on the compact operator algebra $\mathbb{K}$, $\hat{P}_{\Delta}$ is the spectral projection of $\hat{H}_x$ with respect to the interval $\Delta$, and the integral is taken through the identification $C_1(\mathbb{Z}) \cong C(\mathbb{T})$. Note that, since the operators $P_{\Delta}$ and $\hat{H}$ are $\mathbb{Z}$-invariant in the $z$-direction, the Fourier transform in the $z$-direction identifies them with the corresponding operator-valued functions on $\mathbb{T}$. As is shown in [KRS02, theorem 1], this value coincides with $\partial[H_x]$ through the identification $K_1(\mathbb{K} \otimes C_1(\mathbb{Z})) \cong \mathbb{Z}$ given by a cyclic one-cocycle on $C_1(\mathbb{Z})$.

Theorem 2.5 shows that $\sigma_{\text{screw}}$ coincides with the weak Chern number in the $xy$-direction of the bulk Hamiltonian $H$, which is calculated as by the Chern–Weil theory as

$$
\sigma_{\text{xy}}^{\text{bulk}} = \frac{e^2}{h} \cdot (2\pi i) \int_{\mathbb{T}^2 \times \{0\}} \text{tr}(P(\partial_{k_x} P, \partial_{k_y} P)) dk_x dk_y,
$$

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where tr denotes the trace on $M_\mathbb{Q}$ and $P$ denotes the spectral projection of $H$ (regarded as a matrix-valued function on $\mathbb{T}^3$ through the isomorphism $C_r^\ast(\mathbb{Z}^3) \cong C(\mathbb{T}^3)$) corresponding to the negative eigenvalues. The right-hand side is analogous to the TKNN formula in dimension two.

4.5. Relation to the codimension two transfer map

The proof of theorem 2.5 given in this paper comes from the $C^\ast$-algebraic codimension two transfer map in higher index theory introduced in [KS21, Kub21]. Let $M$ be a manifold and let $N$ be a codimension two submanifold such that $\pi_1(N) \to \pi_1(M)$ is injective, $\pi_2(N) \to \pi_2(M)$ is subjective, and the normal bundle of $N$ is trivial. Set $\Gamma := \pi_1(M)$ and $\pi := \pi_1(N)$. In [KS21, theorem 1.1], a group homomorphism

$$\tau_\sigma : K_r(C^\ast \Gamma) \to K_{r-2}(C^\ast \pi)$$

is constructed (the notation $\tau_\sigma$ is introduced in [Kub21]). This homomorphism $\tau_\sigma$ satisfies $\tau_\sigma(\alpha_\Gamma(M)) = \alpha_\pi(N)$. Here, for a closed spin manifold $M$, $\alpha_\Gamma(M)$ denotes the higher index of the Dirac operator on the universal covering $\tilde{M}$. In [2021], a $C^\ast$-algebra extension

$$0 \to \mathbb{K} \to C^\ast(C \times \mathbb{Z}) \to A \to C^\ast \Gamma \to 0$$

is constructed, and $\tau_\sigma$ is defined to be the boundary map in $K$-theory.

In the simplest case, when $M = \mathbb{T}^3$ and $N = pt$, this extension is the same thing as (2.10). The fact $\tau_\sigma(\alpha_\Gamma(M)) = \alpha_\pi(N)$ corresponds to theorem 2.5. The proof in this paper, particularly the construction of a lifting map $s$ in lemma 3.2, is a special case of the one constructed in lemma 3.14, proposition 4.3, and 4.7 of [Kub21].

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Proof of the lifting lemma

Lemma 3.2 is an essential ingredient of the proof of theorem 2.5. Its proof is given in [Kub21] in a more general setting concerned with codimension two inclusions of manifolds (cf subsection 4.5). In this appendix, we restate the proof given in [Kub21] in a way that is specific to the setting we need, i.e. the coarse index theory of the helical surface.

For a bounded operator $K$ with propagation less than $R > 0$, we define its lift $\tilde{K}_R$ as (3.5). If $K$ is a locally Hilbert–Schmidt operator (i.e. $Kf$ and $fK$ are Hilbert–Schmidt for any $f \in C_c(X)$), then the kernel function $k$ exists and is a Borel function on $X \times X$. Even if $K$ is not locally Hilbert–Schmidt, the kernel function $k$ makes sense as a distribution on $X \times X$ supported near the diagonal. Its lift (3.4) is also defined, and hence $\tilde{K}_R$ is well-defined as a linear map $\mathcal{D}(X) \to \mathcal{D}'(X)$, where $\mathcal{D}(X)$ denotes the Fréchet space of compactly supported test functions. Note that the image of $\tilde{K}_R$ is included to $L^2(X)$. 

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Step 1. We show that \( \tilde{K}_R \) is a bounded operator with respect to the \( L^2 \)-norms on the domain and the range. For \( v \in \mathbb{Z}^2 \), let \( P_v \) denote the projection onto the \( L^2 \)-space of \( (v + [0,1) \times [0,1)) \). Similarly, for \( \tilde{v} \in X_s \), let \( P_{\tilde{v}} \) denote the projection onto the \( L^2 \)-space of the connected component of \( \pi^*([\tilde{v}(\pi) + [0,1) \times [0,1)] \) containing \( \tilde{v} \). Now \( \tilde{K}_R \) is decomposed into a sum

\[
\tilde{K}_R = \sum_{r \in \mathbb{Z}} \left( \sum_{v \in X_s} P_{[v+r]} \tilde{K}_R P_v \right),
\]

which is finite with respect to \( r \), and the norm of each summand is bounded as

\[
\left\| \sum_{v \in X_s} P_{[v+r]} \tilde{K}_R P_v \right\| \leq \sup_{\tilde{v} \in X_s} \left\| P_{[\tilde{v}]} \tilde{K}_R \right\| = \sup_{\tilde{v} \in X_s} \left\| P_{\tilde{v}} \tilde{K} \right\| \leq \|K\|.
\]

Step 2. We observe that, if \( K \) is locally compact (resp. pseudo-local), then \( \tilde{K}_R \) is also locally compact (resp. pseudo-local). We firstly notice that local compactness and pseudo-locality of an operator is a local condition, i.e. it is enough to check that any \( f \in C_c(X) \) supported in an \( \varepsilon \)-ball satisfies \( \tilde{K}_R f \in \mathbb{K} \) (resp. \([\tilde{K}_R, f] \in \mathbb{K}\)). To see this, notice that \( \tilde{K}_R f \) and \( \tilde{K}_R \) are supported in an \( (\varepsilon + R) \)-ball in \( X_{s,1} \), which is identified with an \( (\varepsilon + R) \)-ball in \( X \) through the covering map. This identification gives rise to a partial isometry of \( L^2 \)-spaces, which identifies \( \tilde{K}_R f \) and \( \tilde{K}_R \) with \( K f \) and \( K \) respectively.

Step 3. We show that the map \( K \mapsto \tilde{K}_R \) is multiplicative modulo the ideal \( C^*(\partial \tilde{X}_{s,1} \subset \tilde{X}_{s,1})^2 \).

For locally Hilbert–Schmidt operator \( K, L \in \mathcal{B}(L^2(X)) \) with \( \text{Prop}(K) < R \) and \( \text{Prop}(L) < S \), the composition \( \tilde{K}_R L_S \) is given by convolution with

\[
t(\tilde{x}, \tilde{z}) = \int_{x \in X_{s,1}} \tilde{k}_R(\tilde{x}, \tilde{y}) L_S(\tilde{y}, \tilde{z})d\tilde{y}.
\]

If \( \tilde{x}, \tilde{z} \) and the boundary \( \partial \tilde{X}_{s,1} \) are separated by a distance \( S + R \), then

\[
t(\tilde{x}, \tilde{z}) = \int_{y \in X_{s,1}} k(\pi(\tilde{x}), y) L_S(y, \pi(\tilde{z}))dy \quad \text{if} \quad d(\tilde{x}, \tilde{z}) < S + R,
\]

\[
t(\tilde{x}, \tilde{z}) = 0 \quad \text{otherwise}.
\]

This shows that \( \tilde{K}_R L_S \) and \( \tilde{K}_R L_{R+S} \) coincides modulo \( C^*(\partial \tilde{X}_{s,1} \subset \tilde{X}_{s,1})^2 \).

Now we get a \(*\)-homomorphism

\[
s : \mathcal{C}_{HS}[X] \to C^*(\tilde{X}_{s,1})^2 / C^*(\partial \tilde{X}_{s,1} \subset \tilde{X}_{s,1})^2,
\]

where \( \mathcal{C}_{HS}[X] \) denotes the set of locally Hilbert–Schmidt operators with finite propagation. This extends to a \(*\)-homomorphism from \( C^*(X) \) since \( s \) is contractive, i.e. \( \|s(T)\| \leq \|T\| \) for any \( T \in \mathcal{C}_{HS}[X] \). This is because the operator norm on \( \mathcal{C}_{HS}[X] \) is the largest norm satisfying the \( C^* \)-condition, which follows from the amenability of the group \( \mathbb{Z}^2 \).

Step 4. We extend \( s \) to a \(*\)-homomorphism between pseudo-local coarse \( C^* \)-algebras. This part is completely the same as [Kub21, proposition 4.3]. We repeat the proof just for
self-consistency of this appendix. As is shown in [Roe96, lemma 5.8], any operator $T \in D^s(X)$ is decomposed as $T = T_0 + T_1$, where $\text{Prop}(T_0) < 1/2$ and $T_1 \in C^s(X)$. Let $\Pi$ denote the projection onto $L^2(\tilde{X}_{z,1}) \subset L^2(\tilde{X}_{z,1/2})$. Let us take the lift $(\tilde{T}_0)_{1/2}$ of $T_0$ on $\tilde{X}_{z,1/2}$ in the sense of (3.5), and set

$$s(T) := \Pi(\tilde{T}_0)_{1/2}\Pi + s(T_1) \in D^s(\tilde{X}_{z,1})^\mathbb{Z}/D^s(\partial\tilde{X}_{z,1} \subset \tilde{X}_{z,1}).$$

This is well-defined independent of the choice of the decomposition $T = T_0 + T_1$. Indeed, if we have another decomposition $T = T'_0 + T'_1$, then

$$(\Pi(\tilde{T}_0)_{1/2}\Pi + s(T_1)) - (\Pi(\tilde{T}'_0)_{1/2}\Pi + s(T'_1)) = \Pi((\tilde{T}_0)_{1/2} - (\tilde{T}'_0)_{1/2}\Pi + s(T_1 - T'_1).$$

Since $T_0 - T'_0 = -(T_1 - T'_1)$ is contained in $C^s(\tilde{X}_{z,1})$ and has propagation less than 1/2, the right-hand side lies in the ideal $C^s(\partial\tilde{X}_{z,1} \subset \tilde{X}_{z,1}).$

We also show that this $s$ is multiplicative. For $T, S \in D^s(X)$, we choose decompositions $T = T_0 + T_1$ and $S = S_0 + S_1$ as $\text{Prop}(T_0)$ and $\text{Prop}(S_0)$ are less than 1/4. Then we have

$$s(T)s(S) - s(TS) = (\Pi(\tilde{T}_0)_{1/2}\Pi(\tilde{S}_0)_{1/2}\Pi - \Pi(\tilde{T}_0)_{1/2}(\tilde{S}_0)_{1/2}\Pi)$$

$$+ (\Pi(\tilde{T}_0)_{1/2}\Pi(\tilde{S}_0)_{1/2}\Pi - s(T_0S_0))$$

$$+ (s(T_1)s(S_1) - s(T_1S_1)).$$

By Step 3, the second, third and fourth components are locally compact and supported in a neighborhood of $\partial\tilde{X}_{z,1}$, i.e. are contained in $C^s(\partial\tilde{X}_{z,1} \subset \tilde{X}_{z,1})$. Moreover, the first term is contained in $D^s(\partial\tilde{X}_{z,1} \subset \tilde{X}_{z,1})$. Indeed, for any $f \in C_c(\tilde{X}_{z,1})$ vanishing at the boundary, the composition

$$(\Pi(\tilde{T}_0)_{1/2}\Pi(\tilde{S}_0)_{1/2}\Pi - \Pi(\tilde{T}_0)_{1/2}(\tilde{S}_0)_{1/2}\Pi)f = \Pi(\tilde{T}_0)_{1/2}(1 - \Pi)(\tilde{S}_0)_{1/2}f$$

$$= \Pi(\tilde{T}_0)_{1/2}(1 - \Pi)[(\tilde{S}_0)_{1/2}, f]$$

is in $\mathbb{K}(L^2(\tilde{X}_{z,1}))$ (here we use $f\Pi = f$). This shows the multiplicativity of $s$. Note that, our choice of the first term of $s(T)$, the operator $\Pi(\tilde{T}_0)_{1/2}\Pi$ obtained by cutting a pseudo-local operator $(\tilde{T}_0)_{1/2}$ on $\tilde{X}_{z,1/2}$ by the projection $\Pi$, is advantageous for proving this equality.

**Step 5.** Finally we show that $s$ sends the fundamental class $[X]$ to $[\tilde{X}_{z,1}, \partial\tilde{X}_{z,1}]$. We start with a definition of these fundamental classes. Let $Z$ be a complete two-dimensional Riemannian manifold obtained by attaching to $X_{z,1}$ an infinite cylinder of the boundary. We choose 0th order pseudo-differential operators $F \in \mathbb{B}(L^2(X, C^2))$ and $F_2 \in \mathbb{B}(L^2(Z, C^2))$ whose principal symbols are the same as that of Dirac operators on $X$ and $Z$ on the unit cosphere bundle (note that the spinor bundle of $X = \mathbb{R}^2$ is $C^2$). We may choose them as $\text{Prop}(F) < 1/2$ and $\text{Prop}(F_2) < 1/2$ ([LM89, corollary III.3.7]). Then both $F \in Q^s(X)$ and $\Pi F \Pi \in Q^s(\tilde{X}_{z,1})/Q^s(\partial\tilde{X}_{z,1} \subset \tilde{X}_{z,1})$ are unitaries. The $K_1$-classes determined by them are denoted by $[X]$ and $[\tilde{X}_{z,1}, \partial\tilde{X}_{z,1}]$ respectively.

Now, the lift $\tilde{F}_{1/2}$ restricted to $\tilde{X}_{z,1/2}$ is a 0th order pseudo-differential operator whose principal symbol is $\sigma(\tilde{F}_{1/2}) = \sigma(F_Z)|_{X_{z,1}}$. Hence, for any $f \in C_c(\tilde{X}_{z,1})$ vanishing at the boundary, we have
\[(\Pi \tilde{F}_{1/2} - \Pi F_2) f = [\tilde{F}_{1/2} - F_2, f] + f(\tilde{F}_{1/2} - F_2) \in K(L^2(\tilde{X}_{z,1})).\]

This shows that \(\Pi \tilde{F}_{1/2} = \Pi F_2\) modulo \(D^* (\partial \tilde{X}_{z,1} \subset \tilde{X}_{z,1}) \oplus C^* (\tilde{X}_{z,1})\), in other words, their images coincide in \(Q^* (\tilde{X}_{z,1}) / Q^* (\partial \tilde{X}_{z,1} \subset \tilde{X}_{z,1})\).

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