On \( n \)th roots of normal operators

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Abstract

For \( n \)-normal operators \( A \) [2, 4, 5], equivalently \( n \)-th roots \( A \) of normal Hilbert space operators, both \( A \) and \( A^* \) satisfy the Bishop–Eschmeier–Putinar property \((\beta)\), \( A \) is decomposable and the quasi-nilpotent part \( H_0(A - \lambda) \) of \( A \) satisfies \( H_0(A - \lambda)^{-1}(0) = (A - \lambda)^{-1}(0) \) for every non-zero complex \( \lambda \). \( A \) satisfies every Weyl and Browder type theorem, and a sufficient condition for \( A \) to be normal is that either \( A \) is dominant or \( A \) is a class \( A(1, 1) \) operator.

1. Introduction

Let \( B(H) \) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Hilbert space \( H \) into itself. Every normal operator \( A \in B(H) \), i.e., \( A \in B(H) \) such that \([A^*, A] = A^*A - AA^* = 0\), has an \( n \)th root for every positive integer \( n > 1 \). Thus given a normal \( A \in B(H) \), there exists \( B \in B(H) \) such that \( B^n = A \) (and then \( \sigma(B^n) = \sigma(B)^n = \sigma(A) \)). A straightforward application of the Putnam-Fuglede commutativity theorem ([14, Page 103]) applied to \([B, B^n] = 0\) then implies \([B^*, B^n] = 0\). (Conversely, \([B^*, B^n] = 0\) implies \( B^n \) is normal). Operators \( B \in B(H) \) satisfying \([B^*, B^n] = 0\) have been called \( n \)-normal, and a study of the spectral structure of \( n \)-normal operators, with emphasis on the properties which \( B \) inherits from its normal avatar \( B^n \), has been carried out in (2, 4, 5).

Given \( A \in B(H) \), let \( \sigma(A) \subseteq \angle < \frac{2\pi}{n} \) denote that \( \sigma(A) \) is contained in an angle \( \angle \), with vertex at the origin, of width less than \( \frac{2\pi}{n} \). Assuming \( \sigma(B) \subseteq \angle < \frac{2\pi}{n} \) for an \( n \)-normal operator in \( B \in B(H) \), the authors of (2, 4, 5) prove that \( B \) inherits a number of properties from \( B^n \), amongst them that \( B \) satisfies Bishop–Eschmeier–Putinar property \((\beta)\), \( B \) is polaroid (hence also isoloid) and \( \lim_{m \to \infty} \langle x_m, y_m \rangle = 0 \) for sequences \( \{x_m\}, \{y_m\} \subseteq H \) of unit vectors such that \( \lim_{m \to \infty} \| (B - \lambda)x_m \| = 0 = \lim_{m \to \infty} \| (B - \mu)y_m \| \) for distinct scalars \( \lambda, \mu \in \sigma(B) \).

(All our notation is explained in the following section.) That \( B \) inherits a property from \( B^n \) in many a case has little to do with the normality of \( B^n \), but is instead a consequence of the fact that \( B^n \) has the property. Thus, if the approximate point spectrum \( \sigma_a(B^n) = \sigma_a(B)^n \) of \( B^n \) is normal (recall: \( \lambda \in \sigma_a(B^n) \) is normal if \( \lim_{m \to \infty} \| (B^n - \lambda)x_m \| = 0 \) for a sequence \( \{x_m\} \subseteq H \) of unit vectors implies \( \lim_{m \to \infty} \| (B^n - \lambda)^*x_m \| = 0 \); hyponormal operators, indeed dominant operators.

AMS(MOS) subject classification (2010). Primary: Primary47A05, 47A55 Secondary47A80, 47A10.

Keywords: Normal operator, \( n \)-th root, property \((\beta)\), decomposable, quasi-nilpotent part, pole, dominant operator, Weyl and Browder theorems.
satisfy this property), \( \sigma(B) \subseteq \angle < \frac{2\pi}{n} \), and \( \{x_m\}, \{y_m\} \) are sequences of unit vectors in \( \mathcal{H} \) such that \( \lim_{m \to \infty} \|B^n x_m\| = 0 = \lim_{m \to \infty} \|B^n - \mu^n\| y_m \) for some distinct \( \lambda, \mu \in \sigma_a(B) \), then

\[
\lim_{m \to \infty} \lambda^n \langle x_m, y_m \rangle = \lim_{m \to \infty} \langle B^n x_m, y_m \rangle = \lim_{m \to \infty} \langle x_m, B^n y_m \rangle = \mu^n \lim_{m \to \infty} \langle x_m, y_m \rangle
\]

implies

\[
(\lambda - \mu) \lim_{m \to \infty} \langle x_m, y_m \rangle = 0 \iff \lim_{m \to \infty} \langle x_m, y_m \rangle = 0
\]

(cf. [4, Theorem 2.4]). It is well known that \( w \)-hyponormal operators satisfy property \((\beta)_c \) ([3]). If \( B^n \in (\beta)_c \) (i.e., \( B^n \) satisfies property \((\beta)_c \) and \( \sigma(B) \subseteq \angle < \frac{2\pi}{n} \)), then \([7, \text{Theorem 2.9 and Corollary 2.10}]\) imply that \( B + N \in (\beta)_c \) for every nilpotent operator \( N \) which commutes with \( B \) (cf. \([5, \text{Theorem 3.1}]\)). Again, if \( B^n \) is polaroid and \( \sigma(B) \subseteq \angle < \frac{2\pi}{n} \), then \( B \) is polaroid (hence also, isoloid) ([9, Theorem 4.1]). Observe that paranormal operators are polaroid. \( N \)-th roots of normal operators have been studied by a large number of authors (see \([18, 17, 6, 11, 13] \)) and there is a rich body of text available in the literature. Our starting point in this note is that an \( n \)-normal operator \( B \) considered as an \( n \)-th root of a normal operator has a well defined structure ([13, Theorem 3.1]). The problem then is that of determining the "normal like" properties which \( B \) inherits. We prove in the following that the condition \( \sigma(B) \subseteq \angle < \frac{2\pi}{n} \) may be dispensed with in many a case (though not always). Just like normal operators, \( n \)-th roots \( B \) have SVEP (the single-valued extension property) everywhere, \( \sigma(B) = \sigma_a(B) \), \( B \) is polaroid (hence also, isoloid). \( B \in (\beta)_c \) (as also does \( B^n \)) and (the quasinilpotent part) \( H_0(B - \lambda) = (B - \lambda)^{-1}(0) \) at every \( \lambda \in \sigma_p(B) \) except for \( \lambda = 0 \) when we have \( H_0(B) = B^{-n}(0) \). Again, just as for normal operators, \( B \) satisfies various variants of the classical Weyl's theorem \( \sigma(B) \setminus \sigma_w(B) = E_0(B) \) (resp., Browder's theorem \( \sigma(B) \setminus \sigma_w(B) = \Pi_0(B) \)). It is proved that dominant and class \( A(1,1) \) operators \( B \) are normal.

2. Notation and terminology

Given an operator \( S \in B(\mathcal{H}) \), the point spectrum, the approximate point spectrum, the surjectivity spectrum and the spectrum of \( S \) will be denoted by \( \sigma_p(S), \sigma_a(S), \sigma_{su}(S) \) and \( \sigma(S) \), respectively. The isolated points of a subset \( K \) of \( \mathbb{C} \), the set of complex numbers, will be denoted by \( \text{iso}(K) \). An operator \( X \in B(\mathcal{H}) \) is a quasi-affinity if it is injective and has a dense range, and operators \( S, T \in B(\mathcal{H}) \) are quasi-similar if there exist quasi-affinities \( X, Y \in B(\mathcal{H}) \) such that \( SX = XT \) and \( YS = TY \).

\( S \in B(\mathcal{H}) \) has SVEP, the single-valued extension property, at a point \( \lambda_0 \in \mathbb{C} \) if for every open disc \( \mathcal{D} \) centered at \( \lambda_0 \) the only analytic function \( f : \mathcal{D} \to \mathcal{H} \) satisfying \((S - \lambda)f(\lambda) = 0 \) is the function \( f \equiv 0 ; \) \( S \) has SVEP if it has SVEP everywhere in \( \mathbb{C} \). (Here and in the sequel, we write \( S - \lambda \) for \( S - \lambda I \).) Let, for an open subset \( \mathcal{U} \) of \( \mathbb{C} \), \( \mathcal{E}(\mathcal{U}, \mathcal{H}) \) (resp., \( \mathcal{O}(\mathcal{U}, \mathcal{H}) \)) denote the Fréchet space of all infinitely differentiable (resp., analytic) \( H \)-valued functions on \( \mathcal{U} \) endowed with the topology of uniform convergence of all derivatives (resp., topology of uniform convergence) on compact subsets of \( \mathcal{U} \). \( S \in B(\mathcal{H}) \) satisfies property \((\beta)_c \), \( S \in (\beta)_c \), at \( \lambda \in \mathbb{C} \) if there exists a neighborhood \( \mathcal{N} \) of \( \lambda \) such that for each subset \( \mathcal{U} \) of \( \mathcal{N} \) and sequence \( \{f_n\} \) of \( H \)-valued functions in \( \mathcal{E}(\mathcal{U}, \mathcal{H}) \),

\[(S - z)f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}) \implies f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H})\]
(resp., $S$ satisfies property $(\beta)$, $S \in (\beta)$, at $\lambda \in \mathbb{C}$ if there exists an $r > 0$ such that, for every open subset $\mathcal{U}$ of the open disc $\mathcal{O}(\lambda; r)$ of radius $r$ centered at $\lambda$ and sequence $\{f_n\}$ of $\mathcal{H}$-valued functions in $\mathcal{O}(\mathcal{U}, \mathcal{H})$, 

$$(S - z)f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{H}) \implies f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U}, \mathcal{H}).$$

The following implications are well known (H2, H3):

$$S \in (\beta)_e \implies S \in (\beta) \implies S \text{ has SVEP}; S, S^* \in (\beta) \implies S \text{ decomposable}.$$ 

The ascent $\text{asc}(S - \lambda)$ (resp., descent $\text{dsc}(S - \lambda)$) of $S$ at $\lambda \in \mathbb{C}$ is the least non-negative integer $p$ such that $(S - \lambda)^{-p}(0) = (S - \lambda)^{-(p+1)}(0)$ (resp., $(S - \lambda)^p(\mathcal{H}) = (S - \lambda)^{(p+1)}(\mathcal{H})$). A point $\lambda \in \text{iso}(S)$ (resp., $\lambda \in \text{iso}_a(S)$) is a pole (resp., left pole) of the resolvent of $S$ if $0 < \text{asc}(S - \lambda) = \text{dsc}(S - \lambda) < \infty$ (resp., there exists a positive integer $p$ such that $\text{asc}(S - \lambda) = p$ and $(S - \lambda)^{p+1}(\mathcal{H})$ is closed). Let

$$\Pi(S) = \{ \lambda \in \text{iso}(S) : \lambda \text{ is a pole (of the resolvent) of } S \}$$

$$\Pi^a(S) = \{ \lambda \in \text{iso}_a(S) : \lambda \text{ is a left pole (of the resolvent) of } S \}.$$ 

Then $\Pi(S) \subseteq \Pi^a(S)$, and $\Pi^a(S) = \Pi(S)$ if (and only if) $S^*$ has SVEP at points $\lambda \in \Pi^a(S)$. We say in the following that the operator $S$ is polaroid if $\{ \lambda \in \mathbb{C} : \lambda \in \text{iso}(S) \} \subseteq \Pi(S)$. Polaroid operators are isoloid (where $S$ is isoloid if $\{ \lambda \in \mathbb{C} : \lambda \in \text{iso}(S) \} \subseteq \sigma_p(S)$). Let $\sigma_x = \sigma$ or $\sigma_a$. The sets $E^x(S) = E(S)$ or $E^a(S)$ and $E^x_0(S) = E_0(S)$ or $E^a_0(S)$ are then defined by

$$E^x(S) = \{ \lambda \in \text{iso}_x(S) : \lambda \in \sigma_p(S) \}, \quad \text{and}$$

$$E^x_0(S) = \{ \lambda \in \text{iso}_x(S) : \lambda \in \sigma_p(S), \dim(S - \lambda)^{-1}(0) < \infty \}.$$ 

It is clear that

$$\Pi^x(S) \subseteq E^x(S) \quad \text{and} \quad \Pi^a(S) \subseteq E^a_0(S)$$

(where $\Pi^a_0(S) = \{ \lambda \in \Pi^a(S) : \dim(S - \lambda)^{-p}(0) < \infty \}$).

The quasi-nilpotent part $H_0(S)$ and the analytic core $K(S)$ of $S \in B(\mathcal{H})$ are the sets

$$H_0(S) = \left\{ x \in \mathcal{H} : \lim_{n \to \infty} \|S^n x\|^\frac{1}{n} = 0 \right\}, \quad \text{and}$$

$$K(S) = \{ x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subset \mathcal{H} \text{ and } \delta > 0 \text{ for which } x = x_0, Sx_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n\|x\| \text{ for all } n = 1, 2, \ldots \}$$

(\Pi). If $\lambda \in \text{iso}(S)$, then $\mathcal{H}$ has a direct sum decomposition $\mathcal{H} = H_0(S - \lambda) \oplus K(S - \lambda)$, $S - \lambda|_{H_0(S - \lambda)}$ is quasipoloid and $S - \lambda|_{K(S - \lambda)}$ is invertible. A necessary and sufficient condition for a point $\lambda \in \text{iso}(S)$ to be a pole of $S$ is that there exist a positive integer $p$ such that $H_0(S - \lambda) = (S - \lambda)^{-p}(0)$.

In the following we shall denote the upper semi-Fredholm, the lower semi-Fredholm and the Fredholm spectrum of $S$ by $\sigma_{usf}(S), \sigma_{lsf}(S)$ and $\sigma_f(S)$; $\sigma_{uaw}(S), \sigma_{lw}(S)$ and $\sigma_a(S)$ (resp., $\sigma_{ub}(S), \sigma_{lb}(S)$ and $\sigma_b(S)$) shall denote the upper Weyl, the lower Weyl and the Weyl (resp., the upper Browder, the lower Browder and the Browder) spectrum of $S$. Additionally, we shall denote the upper $B$-Weyl, the lower $B$-Weyl and the $B$-Weyl (resp., the upper $B$-Browder, the lower $B$-Browder and the $B$-Browder) spectrum of $S$ by $\sigma_{ubw}(S), \sigma_{lbw}(S)$ and $\sigma_{bw}(S)$ (resp., $\sigma_{ubb}(S), \sigma_{lbb}(S)$ and $\sigma_{bb}(S)$). We refer the interested reader to the monograph (H1) for definition, and other relevant information, on these distinguished parts of the spectrum; our interest here in these spectra is at best peripheral.
3. Results.

Throughout the following, \( A \in B(\mathcal{H}) \) shall denote an \( n \)-normal operator. Considered as an \( n \)th root of the normal operator \( A^n \), \( A \) has a direct sum representation

\[
A = \bigoplus_{i=0}^{\infty} A |_{\mathcal{H}_i} = \bigoplus_{i=0}^{\infty} A_i, \quad \mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i,
\]

where \( A_0 \) is \( n \)-nilpotent and \( A_i \), for all \( i = 1, 2, \ldots \), is similar to a normal operator \( N_i \in B(\mathcal{H}_i) \). Equivalently,

\[
A = B_1 \oplus B_0, \quad B_0 = A_0 \quad \text{and} \quad B_1 = \bigoplus_{i=1}^{\infty} A_i,
\]

where \( B_0^n = 0 \) and \( B_1 \) is quasi-similar to a normal operator \( N = \bigoplus_{i=1}^{\infty} N_i \in B \left( \bigoplus_{i=1}^{\infty} \mathcal{H}_i \right) \). Quasi-similar operators preserve SVEP; hence, since the direct sum of operators has SVEP at a point if and only if the summands have SVEP at the point, \( A \) and \( A^* \) have SVEP (everywhere). Consequently (\([\Pi]\)):

\[
\sigma(A) = \sigma(B_1) \cup \{0\} = \sigma(N) \cup \{0\} = \sigma_{sa}(A) = \sigma_{su}(A),
\]

\[
E^a(A) = E(A), \quad E^a_0(A) = E_0(A), \quad \Pi^a(A) = \Pi(A), \quad \Pi^a_0(A) = \Pi_0(A);
\]

furthermore:

\[
\sigma_f(A) = \sigma_{usf}(A) = \sigma_{isf}(A) = \sigma_{w}(A) = \sigma_{ww}(A) = \sigma_{lw}(A) = \sigma_{lb}(A) = \sigma_{ub}(A) = \sigma_{bb}(A),
\]

\[
\sigma_b(A) = \sigma_{bw}(A) = \sigma_{ubw}(A) = \sigma_{lbw}(A) = \sigma_{bb}(A) = \sigma_{ubb}(A) = \sigma_{lb}(A).
\]

The point spectrum of a normal operator consists of normal eigenvalues (i.e., the corresponding eigenspaces are reducing): This fails for the operator \( A \) ([\([\Pi]\) Remark 2.17]), and a sufficient condition is that \( \sigma(A) \subseteq \angle \left< \frac{2\pi}{n} \right> \) (for then \( (A - \lambda)x = 0 \iff (A^n - \lambda^n)x = 0 \iff (A^n - \lambda^n)x = 0 \iff (A^* - \bar{\lambda})x = 0 \)).

The polaroid property travels from \( A^n \) to \( A \), no restriction on \( \sigma(A) \). (This would then imply that \( E^a(A) = E(A) = \Pi(A) = \Pi^a(A) \) and \( E^a_0(A) = E_0(A) = \Pi_0(A) = \Pi^a_0(A) \).) We start by proving that the quasi-similarity of \( B_1 \) and \( N \) transfers to the Riesz projections \( P_{B_1}(\lambda) \) and \( P_N(\lambda) \) corresponding to points \( \lambda \in \text{iso}\sigma(B_1) = \text{iso}\sigma(N) \). Let \( \Gamma \) be a positively oriented path separating \( \lambda \) from \( \sigma(B_1) \) and let \( X, Y \) be quasi-affinities such that \( B_1X = XN \) and \( YB_1 = NY \). Then, for all \( \mu \not\in \sigma(B_1) \),

\[
P_{B_1}(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} (\mu - B_1)^{-1} d\mu \Longleftrightarrow YP_{B_1}(\lambda) = Y \left\{ \frac{1}{2\pi i} \int_{\Gamma} (\mu - B_1)^{-1} d\mu \right\}
\]

\[
\Longleftrightarrow YP_{B_1}(\lambda) = \left\{ \frac{1}{2\pi i} \int_{\Gamma} (\mu - N)^{-1} d\mu \right\} Y = P_N(\lambda)Y.
\]

A similar argument proves

\[
P_{B_1}(\lambda)X = XP_N(\lambda).
\]

**Theorem 3.1** \( A \) is polaroid.
Proof. Continuing with the argument above, the normality of $N$ implies that the range $H_0(N - \lambda)$ of $P_N(\lambda)$ coincides with $(N - \lambda)^{-1}(0)$. Hence $(N - \lambda)P_N(\lambda) = 0$, and

$$Y(B_1 - \lambda)P_{B_1}(\lambda) = (N - \lambda)YP_{B_1}(\lambda) = (N - \lambda)P_N(\lambda)Y = 0$$

$$\implies (B_1 - \lambda)P_{B_1}(\lambda) = 0 \iff H_0(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0).$$

Since $\lambda \in \text{iso}\sigma(B_1)$,

$$\bigoplus_{i=1}^{\infty} H_i = H_0(B_1 - \lambda) \oplus K(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0) \oplus K(B_1 - \lambda)$$

$$\implies \bigoplus_{i=1}^{\infty} H_i = (B_1 - \lambda)^{-1}(0) \oplus (B_1 - \lambda) \bigoplus_{i=1}^{\infty} H_i,$$

i.e., $\lambda$ is a (simple) pole. The $n$-nilpotent operator $B_0$ being polaroid, the direct sum $B_0 \oplus B_1$ is polaroid (since $\text{asc}(A - \lambda) \leq \text{asc}(B_0 - \lambda) \ominus \text{asc}(B_1 - \lambda)$ and $\text{dsc}(A - \lambda) \leq \text{dsc}(B_0 - \lambda) \oplus \text{dsc}(B_1 - \lambda)$ for all $\lambda$ (\cite[Exercise 7, Page 293]{20})).

Theorem 3.1 implies:

Corollary 3.2 A is isoloid (i.e., points $\lambda \in \text{iso}\sigma(A)$ are eigenvalues of $A$).

More is true, and, indeed, Theorem 3.1 is a consequence of the following result which shows that $H_0(A - \lambda) = (A - \lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$.

Theorem 3.3 $H_0(A - \lambda) = (A - \lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$ and $H_0(A) = A^{-n}(0)$. In particular, $A$ is polaroid.

Proof. Following the same notation as above, the normality of $N$ implies $H_0(N - \lambda) = (N - \lambda)^{-1}(0)$ for all $\lambda \in \sigma(N)$ ($= \sigma(B_1)$). Since

$$NY = YB_1 \iff (N - \lambda)Y = Y(B_1 - \lambda), \text{ all } \lambda,$$

it follows that

$$\|(N - \lambda)^nYx\|^{\frac{1}{n}} = \|Y(B_1 - \lambda)^nx\|^{\frac{1}{n}} \leq \|Y\|^{\frac{1}{n}} \|(B_1 - \lambda)^nx\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty$$

for all $x \in H_0(B_1 - \lambda)$. Consequently,

$$Yx \in H_0(N - \lambda) = (N - \lambda)^{-1}(0) \implies Y(B_1 - \lambda)x = (N - \lambda)Yx = 0 \iff x \in (B_1 - \lambda)^{-1}(0),$$

and hence

$$H_0(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0)$$

for all $\lambda \in \sigma(B_1)$. Evidently,

$$H_0(A) = H_0(B_1 \ominus B_0) = B_1^{-1}(0) \oplus B_0^{-n}(0) \subseteq A^{-n}(0).$$

Argue now as in the proof of Theorem 3.1 to prove that $A$ is polaroid.

The Riesz projection $P_A(\lambda)$ corresponding to points $(0 \neq) \lambda \in \text{iso}\sigma(A)$ are, in general, not self-adjoint. Since $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ ensures $(A - \lambda)^{-1}(0) \subseteq (A^* - \lambda)^{-1}(0)$ for all $0 \neq \lambda \in \sigma_p(A)$, $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ forces $P_A(\lambda) = P_A(\lambda)^*$ for all $\lambda \neq 0$. 

Corollary 3.4  If \( \sigma(A) \subseteq \mathcal{L} \subsetneq \frac{2\pi}{n} \), then the Riesz projection corresponding to non-zero \( \lambda \in \text{iso}(A) \) is self-adjoint.

Remark 3.5  Theorem 3.1 and 3.3 generalize corresponding results from [2], [4], [5] by removing the hypothesis that \( \sigma(A) \subseteq \mathcal{L} \subsetneq \frac{2\pi}{n} \), and, in the case of Theorem 3.3, the hypothesis on the points \( \lambda \) being isolated in \( \sigma(A) \). Recall from [1, Page 336] that an operator \( S \in B(\mathcal{H}) \) is said to have property \( Q \) if \( H_0(S_\lambda) \) is closed for all \( \lambda \).

Theorem 3.3 says that the \( n \)th roots \( A \) have property \( Q \). Another proof of Theorem 3.3, hence also of the fact that the operators \( A \) satisfy property \( Q \), follows from the argument below proving the subscalarity of \( A \).

Property \((\beta)_\epsilon\) (similarly \((\beta)\)) does not travel well under quasi-affinities. Thus \( CX = XB \) and \( B \in (\beta)_\epsilon \) does not imply \( C \in (\beta)_\epsilon \) (see [7, Remark 2.7] for an example). However, \( C \in (\beta)_\epsilon \) implies \( B \in (\beta)_\epsilon \). The operator \( A \) being the direct sum \( B_1 \oplus B_0 \), where \( B_0, B_0^* \) being nilpotent satisfy \((\beta)_\epsilon\), to prove the theorem it will suffice to prove \( B_1, B_1^* \in (\beta)_\epsilon \). But this is immediate from the argument above, since normal operators \( N \) satisfy \( N, N^* \in (\beta)_\epsilon \) and since there exist quasi-affinities \( X \) and \( Y \) in \( B(\bigoplus_{i=1}^{\infty} \mathcal{H}_i) \) such that \( N^*X^* = X^*B_1^* \) and \( NY = YB_1 \).

Thus \( B \in (\beta)_\epsilon \).

Theorem 3.6  \( A \) and \( A^* \) satisfy property \((\beta)_\epsilon\).

Proof. Recall from [7, Lemma 2.2] that a direct sum of operators satisfies \((\beta)_\epsilon\) if and only if the individual operators satisfy \((\beta)_\epsilon\). The operator \( A \) being the direct sum \( B_1 \oplus B_0 \), where \( B_0, B_0^* \) being nilpotent satisfy \((\beta)_\epsilon\), to prove the theorem it will suffice to prove \( B_1, B_1^* \in (\beta)_\epsilon \). But this is immediate from the argument above, since normal operators \( N \) satisfy \( N, N^* \in (\beta)_\epsilon \) and since there exist quasi-affinities \( X \) and \( Y \) in \( B(\bigoplus_{i=1}^{\infty} \mathcal{H}_i) \) such that \( N^*X^* = X^*B_1^* \) and \( NY = YB_1 \).

\( A \in (\beta)_\epsilon \) implies \( A \in (\beta) \), and \( A, A^* \in (\beta) \) implies \( A \) is decomposable ([16]). Hence:

Corollary 3.7  \( A \) is decomposable.
(see [1] Definitions 6.59, 6.81]). Let $S \in \mathrm{Wt}$, $S \in a-Wt$, $S \in gBt$, $S \in a\!-\!gBt$, $S \in Bt$ and $S \in a-Bt$ denote, respectively, that

$S$ satisfies Weyl’s theorem : $\sigma(S) \setminus \sigma_w(S) = E_0(S)$,

$S$ satisfies a – Weyl’s theorem : $\sigma_a(S) \setminus \sigma_{aw}(S) = E_0^a(S)$,

$S$ satisfies generalized Browder’s theorem : $\sigma(S) \setminus \sigma_{Bw}(S) = \Pi(S)$,

$S$ satisfies generalized a – Browder’s theorem : $\sigma_a(S) \setminus \sigma_{aw}(S) = \Pi^a(S)$,

$S$ satisfies Browder’s theorem : $\sigma(S) \setminus \sigma_w(S) = \Pi_0(S)$,

$S$ satisfies a – Browder’s theorem : $\sigma_a(S) \setminus \sigma_{aw}(S) = \Pi^a_0(S)$.

(see [1] Chapter 6]). The following implications are well known ([1] Chapters 5, 6]):

\[
S \in a - gWt \implies \begin{cases} S \in a - Wt \implies S \in Wt \implies S \in Bt, \\
S \in gWt \implies S \in Bt\end{cases}
\]

\[
S \in a - gWt \implies \begin{cases} S \in a - Wt \implies S \in a - Bt \implies S \in Bt, \\
S \in a - gBt \implies S \in a - Bt \iff S \in Bt\end{cases}
\]

$A$ has SVEP (guarantees $A \in a - gBt$ ([1] Theorem 5.37)) and $\sigma(A) = \sigma_a(A)$ guarantee the equivalence of a-gBt and gBt (hence also of $a$-gBt with $a$-Bt and Bt) for $A$. The fact that $A$ is polaroid and $\sigma(A) = \sigma_a(A)$ guarantees also that $E(A) = E^a(A) = \Pi^a(A) = \Pi(a)$ (and $E_0(A) = E_0^a(A) = \Pi_0^a(A) = \Pi_0(a)$). Hence all Weyl’s theorems (listed above) are equivalent for $A$ and :

**Theorem 3.8** $A \in a - gWt$

**Normal $A$**. For the operator $A = B_1 \oplus B_0$ to have any chance of being a normal operator, it is necessary that (either $B_0$ is missing, or) $B_0 = 0$. The hypothesis ($B_0$ is missing, or) $B_0 = 0$ is, however, in no way sufficient to ensure the normality of $A$. Additional hypotheses are required. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be dominant (resp., class $A(1,1)$) if to every complex $\lambda$ there corresponds a real number $M_\lambda > 0$ such that $\| (S - \lambda)^* x \| \leq M_\lambda \| (S - \lambda) x \|$ for all $x \in \mathcal{H}$ (resp., $|S|^2 \leq |S|^2$) ([19], [15]). Recall from [10] Lemma 2.1 that if a dominant or class $A(1,1)$ operator $A \in \mathcal{B}(\mathcal{H})$ is a square root of a normal operator, then $A$ is normal. The following theorem, which uses an argument different from that used in [10], proves that this result extends to $n$th roots $A$.

**Theorem 3.9** Dominant or $A(1,1)$ $n$th roots of a normal operator in $\mathcal{B}(\mathcal{H})$ are normal.

**Proof**. Recall that the eigenvalues of a dominant operator are normal (i.e., they are simple and the corresponding eigenspace is reducing). Hence if our $n$th root of $A = B_1 \oplus B_0$ is dominant, then $A = B_1 \oplus 0$ is a dominant operator which satisfies

\[
A (Y \oplus I |_{\mathcal{H}_0}) = (Y \oplus I |_{\mathcal{H}_0}) (N \oplus 0).
\]

The operator $N \oplus 0$ being normal and the operator $Y \oplus I |_{\mathcal{H}_0}$, being a quasi-affinity it follows from [19], [8] that $A$ is normal (and unitarily equivalent to $N \oplus 0$). We consider next $A \in A(1,1)$.
It is well known that $\mathcal{A}(1,1)$ operators have ascent less than or equal to one. (Indeed, operators $S \in \mathcal{A}(1,1)$ are parannormal: $\|Sx\|^2 \leq \|S^2x\| \|x\|$ for all $x \in \mathcal{H}$, hence $\text{asc}(S) \leq 1$). Hence if $A = B_1 \oplus B_0 \in \mathcal{A}(1,1)$, then $B_0 = 0$ and $A \in B \left( A^{-1}(0) \oplus A^{-1}(0)^\perp \right)$ has an upper triangular matrix representation

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$ 

Let $N_1 = N \oplus 0 \mid_{\mathcal{H}_0}$ have the representation

$$N_1 = 0 \oplus N_{22} \in B \left( N_1^{-1}(0) \oplus N_1^{-1}(0)^\perp \right),$$

and let $Y_1 = Y \oplus I \mid_{\mathcal{H}_0} \in B \left( N_1^{-1}(0) \oplus N_1^{-1}(0)^\perp, A^{-1}(0) \oplus A^{-1}(0)^\perp \right)$ have the corresponding matrix representation

$$Y_1 = [Y_{ij}]_{i,j=1}^2.$$ 

Then, given that $Y$ is a quasi-affinity satisfying $B_1 Y = YN$, $Y_1$ is a quasi-affinity such that $AY_1 = Y_1 N_1$. Consequently, $A_{22} Y_2 = 0$. The operator $A_{22}$ being injective, we must have $Y_{21} = 0$ (and then $Y_{11}$ is injective and $Y_{22}$ has a dense range). The operator $A$ being an $n$th root of a normal operator, $A^n$ is normal. Applying the Putnam-Fuglede commutativity theorem to $(AY_1 = Y_1 N_1 \implies) A^n Y_1 = Y_1 N_1^n$, it follows that $A^n Y_1 = Y_1 N_1^n$, and hence $Y_{12} N_{22}^n = 0$. Since the normal operator $N_{22}^n$ has a dense range, $Y_{12} = 0$ (which than implies that $Y_{11}$ and $Y_{22}$ are quasi-affinities). But then $A_{22} Y_{22} = Y_{22} N_{22}^*$ and $A_{22} Y_{22} = Y_{22} N_{22}$ imply that $A_{22}$ is quasi-affinity. Hence, since $(A^n Y_1 = Y_1 N_1^n)$ implies also that $Y_{12} N_{22}^{-1} Y_{11} = 0$, $A_{12} = 0$. Thus $A = 0 \oplus A_{22}$, where $A_{22} \in \mathcal{A}(1,1)$, $A_{22}^{-1}(0) = \{0\}$ and $A_{22} Y_{22} = Y_{22} N_{22}$. Applying Proposition 2.5 and Lemma 2.2 of [10], it follows that $A_{22}$ and $N_{22}$ are (unitarily equivalent) normal operators. Conclusion: $A = 0 \oplus A_{22}$ is a normal $n$th root. □

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