A SHARP $L_p$-REGULARITY RESULT FOR SECOND-ORDER
STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH
UNBOUNDED AND FULLY DEGENERATE LEADING
COEFFICIENTS

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Abstract. We present existence, uniqueness, and sharp regularity results of
solution to the stochastic partial differential equation (SPDE)

$$du = (a^{ij}(\omega, t)u_{x^ix^j} + f)dt + (\sigma^{ik}(\omega, t)u_{x^i} + g^k)dw^k_t, \quad u(0, x) = u_0.$$  (0.1)

where $\{w^k_t : k = 1, 2, \cdots\}$ is a sequence of independent Brownian motions.

The coefficients are merely measurable in $(\omega, t)$ and can be unbounded and
fully degenerate, that is, coefficients $a^{ij}, \sigma^{ik}$ merely satisfy

$$\left(\alpha^{ij}(\omega, t)\right)_{d \times d} := \left(a^{ij}(\omega, t) - \frac{1}{2} \sum_{k=1}^{\infty} \sigma^{ik}(\omega, t)\sigma^{jk}(\omega, t)\right) \geq 0.$$  (0.2)

In this article, we prove that there exists a unique solution $u$ to (0.1), and

$$\|u_{xx}\|_{H^\gamma_p(\tau, \delta)} \leq N(d, p) \left(\|u_0\|_{B^{\gamma+2(1-1/p)}_p(\tau, \delta)} + \|f\|_{H^{\gamma_p}_p(\tau, \delta)} + \|g_x\|_{L^{\gamma_p}_p(\tau, \delta)} \right).$$  (0.3)

1. INTRODUCTION

The second-order elliptic and parabolic partial differential equations (PDEs) with
unbounded or degenerate leading coefficients have been widely studied for a long
time (see e.g. [16, 17, 18, 19, 20]). Such equations naturally arise in the modeling
of random phenomenon related to diffusion. For instance, consider the stochastic
process $X_t$ governed by

$$dX_t = b(\omega, t)dt + \sigma(\omega, t)dB_t, \quad X_0 = x,$$

where $b(\omega, t)$ is $R^d$-valued, $\sigma(\omega, t)$ is $d \times d$-matrix-valued, and $B_t$ is $d$-dimensional
Brownian motion. Then, for any smooth function $f(x), u(t, x) := E[f(X_t)]$ satisfies
the parabolic PDE

$$u_t = \frac{1}{2}(\sigma \sigma^*)^{ij} u_{x^ix^j} + b^i u_{x^i}, \quad t > 0; \quad u(0, \cdot) = f(x).$$

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Here $\sigma^*$ is the transpose of $\sigma$. Since $(\sigma\sigma^*)$ is symmetric, it is only guaranteed that
\[
(a^{ij}(\omega, t)) := (\sigma\sigma^*) \geq 0. \tag{1.1}
\]
Such connections between PDEs and stochastic processes illustrate that boundedness and uniform ellipticity conditions of leading coefficients are somewhat restrictive for the study of general PDEs (and SPDEs).

In this article we study a weighted $L_p$-regularity theory ($p \geq 2$) of SPDE
\[
du = (a^{ij}(\omega, t)u_{x_ix_j} + f)dt + (\sigma^{ik}(\omega, t)u_{x_i} + g^k)du^k_t, \quad u(0, x) = u_0, \tag{1.2}
\]
where indices $i, j$ moves from 1 to $d$, $k$ runs through $\{1, 2, 3, \cdots \}$. Einstein’s summation convention with respect to repeated indices $i, j, k$ is assumed. We assume very minimal conditions on the coefficients, that is, the coefficients are merely measurable in $(\omega, t)$ and satisfy
\[
(a^{ij}(\omega, t)) := \left(a^{ij}(\omega, t) - \frac{1}{2}\sum_{k=1}^{\infty} \sigma^{ik}(\omega, t)\sigma^{jk}(\omega, t) \right) \geq 0, \tag{1.3}
\]
together with the local integrability
\[
\int_0^t |a^{ij}(\omega, s)|ds + \int_0^t \sum_{k=1}^{\infty} |\sigma^{ik}(\omega, s)|^2 ds < \infty, \quad (a.s.) \quad \forall t > 0, \ i, j. \tag{1.4}
\]
Actually condition (1.4) is necessary to make sense of equation (1.2).

To the best of our knowledge, the theory of SPDE with degenerate and unbounded leading coefficients was initiated in \cite{8} and \cite{6} respectively, and the result in \cite{8} was extended to the case of system in \cite{2}. Recently, this type of SPDEs have been developed in various directions in $L_2$-spaces. For instance, a regular strong solution to quasilinear degenerate SPDEs is studied in \cite{3} and the existence of an $L_2$-valued continuous solution to SPDEs with space-time dependent random coefficients $a^{ij}(t, x)$ which are allowed to be both unbounded and degenerate is handled in \cite{22}.

However, roughly speaking, if $p > 2$ and the coefficients are degenerate then the results in the literature (see e.g. \cite{8}) only say that equation (1.2) has a unique continuous $L_p$-valued solution $u$ and
\[
\mathbb{E}\sup_{t \leq T} \|u\|_{L_p}^p \leq N(d, p, T) \left(\mathbb{E}\|u_0\|_{L_p}^p + \|f\|_{L_p(T)}^p + \|g|_{L^2}^p \|_{L_p(T)}\right), \tag{1.5}
\]
where $\mathbb{L}_p(T) = L_p(\Omega \times [0, T], L_p(\mathbb{R}^d))$. Note that in estimate (1.5) the solution $u$ is not smoother than $u_0, f$ and $g$. Actually (1.5) is the best possible estimation in the extreme degenerate case, i.e. if $a^{ij}(\omega, t) = 0$ and $\sigma^{ik}(\omega, t) = 0$ for all $i, j, k, \omega, t$. Because, in this case, we have
\[
u(t, x) = u_0(x) + \int_0^t f(s, x)ds + \int_0^t g^k(s, x)dw^k_s,
\]
and thus it cannot be expected that the solution $u$ is smoother than data $u_0, f,$ and $g$. In other words, if degeneracy of diffusion is too strong, then there is no smoothing effect enough to make solutions regular than data. However, if the matrix $(a^{ij}(t))_{d \times d}$ in (1.2) is not identically zero then the question whether $u_{xx} \in L_p(\mathbb{R}^d)$ on the set $\{ (\omega, t) : (a^{ij})_{d \times d} > 0 \}$ naturally arises.
It turns out the the answer to the above question is “yes”. In this article we prove that under the conditions \((1.3)\) and \((1.4)\), it holds that
\[
\mathbb{E} \int_0^T \|u_{xx}\|_{H^p}^p \delta \, dt \leq N(d, p) \left( \mathbb{E} \|u_0\|_{B^{1/2}_{p,2}}^p + \mathbb{E} \int_0^T \|f\|_{L^p}^p \delta^{1-p} \, dt \right) + \mathbb{E} \int_0^T \|g_x\|_{H^p}^p (|\sigma|^p \delta^{1-p} + \delta^{1-p/2}) \, dt,
\]
where \(p \geq 2\), \(\gamma \in \mathbb{R}\), \(\tau\) is an arbitrary stopping time, \(\delta = \delta(\omega, t)\) is the smallest eigenvalue of \((\alpha^{ij}(\omega, t))\), \(|\sigma(\omega, t)| = \max_i |\sigma^i|_{L^2}\), \(H^p\) is a Sobolev space, and \(B^{\gamma+2(1-1/p)}_{p,2}\) is a Besov space.

We mention that our weights in estimate \((1.6)\) are not in \(A_p\)-weight class which is a very important function class in the Fourier analysis (see Remark 2.8 below). Thus, even if the coefficients are not random, estimate \((1.6)\) cannot be obtained based on the estimation of the sharp function \((u_{xx})^\#\) or Calderón-Zygmund approach. See e.g. \([14, 7]\) for detail of such approaches.

In summary, we list the novelty of our result:

1. Coefficients \(a^{ij}(\omega, t)\) and \(\delta^{jk}(\omega, t)\) are not necessarily bounded.
2. The matrix \((\alpha^{ij})_{d \times d}(\omega, t) := (a^{ij}(\omega, t) - \frac{1}{2} \sum_{k=1}^d \sigma^{ik}(\omega, t) \sigma^{jk}(\omega, t))\) can be (fully) degenerate.
3. Coefficients \(a^{ij}(\omega, t)\) and \(\delta^{jk}(\omega, t)\) can be random, and merely measurable in \((\omega, t)\).
4. A sharp weighted \(L_p\)-regularity result is obtained for any \(p \geq 2\).

This article is organized as follows. In Section 2, we introduce our main result together with some related function spaces. In Section 3, we prove the solvability and a priori estimate for deterministic equations without boundedness and ellipticity conditions on the leading coefficients. In Section 4, stochastic PDEs with additive noises are treated, and finally the proof of the main theorem is given in Section 5.

We finish the introduction with notation used in the article.

- \(\mathbb{N}\) and \(\mathbb{Z}\) denote the natural number system and the integer number system, respectively. As usual \(\mathbb{R}^d\) stands for the Euclidean space of points \(x = (x^1, \ldots, x^d)\), \(\mathbb{R}^d_+ := \{x = (x^1, \ldots, x^d) \in \mathbb{R}^d : x^1 > 0\}\) and \(B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}\). For \(i = 1, \ldots, d\), multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_d)\), \(\alpha_i \in \{0, 1, 2, \ldots\}\), and functions \(u(x)\) we set
  \[u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad \nabla u = (u_{x^1}, u_{x^2}, \cdots, u_{x^d}).\]

- We also use the notation \(D^m\) for a partial derivative of order \(m\) with respect to \(x\).
- \(C^\infty(\mathbb{R}^d)\) denotes the space of infinitely differentiable functions on \(\mathbb{R}^d\). \(\mathcal{S}(\mathbb{R}^d)\) is the Schwartz space consisting of infinitely differentiable and rapidly decreasing functions on \(\mathbb{R}^d\). By \(C^\infty_c(\mathbb{R}^d)\), we denote the subspace of \(C^\infty(\mathbb{R}^d)\) consisting of functions with compact support.
- \(\mathcal{F}(\mathcal{M})\) denotes the space of all \(\mathcal{F}\)-valued \(\mathcal{M}\)-measurable functions \(u\) so that
  \[\|u\|_{\mathcal{L}_p(\mathcal{M})} := \left( \mathbb{E} \|u(x)\|_F^p \mu(dx) \right)^{1/p} < \infty,\]
where $\mathcal{M}^\mu$ denotes the completion of $\mathcal{M}$ with respect to the measure $\mu$. We write $u \in L_\infty(X, \mathcal{M}, \mu; F)$ iff
\[
\sup_{x} |u(x)| := \|u\|_{L_\infty(X, \mathcal{M}, \mu; F)} := \inf \{ \nu \geq 0 : \mu(\{ x : \|u(x)\|_F > \nu \}) = 0 \} < \infty.
\]

If there is no confusion for the given measure and $\sigma$-algebra, we usually omit the measure and the $\sigma$-algebra. Moreover, if a topology is given on $X$, then the subspace of all continuous functions in $L_\infty(X, \mathcal{M}, \mu; F)$ is denoted by $C(X; F)$.

- For functions depending on $\omega$, $t$, and $x$, the random parameter $\omega \in \Omega$ is usually omitted.
- By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the d-dimensional Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}[f](\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ and $\mathcal{F}^{-1}[f](x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi$.
- If we write $N = N(a, b, \cdots)$, this means that the constant $N$ depends only on $a, b, \cdots$.

2. Setting and main results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. By $\mathcal{P}$ we denote the predictable $\sigma$-algebra generated by $\{\mathcal{F}_t, t \geq 0\}$ and we assume that on $\Omega$ there exist independent one-dimensional Wiener processes $w^1_t, w^2_t, \ldots$, each of which is a Wiener process relative to $\{\mathcal{F}_t, t \geq 0\}$.

We study the following initial value problem on $\mathbb{R}^d$:
\[
du = (a^{ij}(t)u_{x_i x_j} + f) dt + (\sigma^{ik}(t)u_{x_i} + g^k) dw^k_t, \quad t > 0; \quad u(0, \cdot) = u_0. \tag{2.1}
\]

As mentioned in the introduction, Einstein’s summation convention with respect to indices $i, j, k$ is assumed and the argument $\omega$ is omitted in the above equation for the simplicity of notation.

First, we introduce some deterministic function spaces related to our results. For $p > 1$ and $\gamma \in \mathbb{R}$, let $H^\gamma_p = H^\gamma_p(\mathbb{R}^d)$ denote the class of all (tempered) distributions $u$ on $\mathbb{R}^d$ such that
\[
\|u\|_{H^\gamma_p} := \|(1 - \Delta)^{\gamma/2} u\|_{L_p} < \infty, \tag{2.2}
\]
where
\[
(1 - \Delta)^{\gamma/2} u = \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\gamma/2} \mathcal{F}[u] \right].
\]

It is well-known that if $\gamma = 1, 2, \cdots$, then
\[
H^\gamma_p = W^\gamma_p := \{ u : D^\alpha u \in L_p(\mathbb{R}^d), \ |\alpha| \leq \gamma \}, \quad H^{-\gamma}_p = \left( H^\gamma_{p/(p-1)} \right)^*,
\]
where $\left( H^\gamma_{p/(p-1)} \right)^*$ is the dual space of $H^\gamma_{p/(p-1)}$. For a tempered distribution $u \in H^\gamma_p$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, the action of $u$ on $\phi$ (or the image of $\phi$ under $u$) is defined as
\[
(u, \phi) = \left( (1 - \Delta)^{\gamma/2} u, (1 - \Delta)^{-\gamma/2} \phi \right) = \int_{\mathbb{R}^d} (1 - \Delta)^{\gamma/2} u(x) \cdot (1 - \Delta)^{-\gamma/2} \phi(x) dx.
\]

Let $l_2$ denote the set of all sequences $a = (a^1, a^2, \cdots)$ such that
\[
|a|_{l_2} := \left( \sum_{k=1}^{\infty} |a^k|^2 \right)^{1/2} < \infty.
\]
By $H^\gamma_0(l_2) = H^\gamma_0(\mathbb{R}^d; l_2)$ we denote the class of all $l_2$-valued (tempered) distributions $v = (v^1, v^2, \cdots)$ on $\mathbb{R}^d$ such that

$$\|v\|_{H^\gamma_0(l_2)} := \|(1 - \Delta)^{\gamma/2} v\|_{l_2} < \infty.$$  

In particular, we set

$$L_p := H^0_p \quad \text{and} \quad L_p(l_2) := H^0_1(l_2).$$

To state our assumption for the initial data, we introduce the Besov space $B^\gamma_p(\mathbb{R}^d)$ characterized by the Littlewood-Paley operator. See [1, Chapter 6] or [4, Chapter 6] for more details. Let $\Psi$ be a nonnegative function on $\mathbb{R}^d$, we define

$$\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},$$  

where $B_r(0) := \{ x \in \mathbb{R}^d : |x| \leq r \}$ and $\hat{\Psi}$ is the Fourier transform of $\Psi$. For a tempered distribution $u$, we define

$$\Delta_j u(x) := \Delta_j^\Psi u(x) := \mathcal{F}^{-1} \left[ \hat{\Psi}(2^{-j} \xi) \mathcal{F} u(\xi) \right](x)$$  

and

$$S_0(u)(x) = \sum_{j = -\infty}^{0} \Delta_j u(x),$$

where the convergence is understood in the sense of distributions. Due to (2.3),

$$u(x) = S_0(u)(x) + \sum_{j = 1}^{\infty} \Delta_j u(x).$$  

The Besov space $B^\gamma_p = B^\gamma_p(\mathbb{R}^d)$ with the order $\gamma$ and the exponent $p$ is the space of all tempered distributions $u$ such that

$$\|u\|_{B^\gamma_p} := \|S_0(u)\|_{L_p} + \left( \sum_{j = 1}^{\infty} 2^{j \gamma p} \|\Delta_j u\|^p_{L_p} \right)^{1/p} < \infty.$$  

Similarly, the homogeneous Besov space $\dot{B}^\gamma_p = \dot{B}^\gamma_p(\mathbb{R}^d)$ with the order $\gamma$ and the exponent $p$ is the space of all tempered distributions $u$ such that

$$\|u\|_{\dot{B}^\gamma_p} := \left( \sum_{j = -\infty}^{\infty} 2^{j \gamma p} \|\Delta_j u\|^p_{L_p} \right)^{1/p} < \infty.$$  

Remark 2.1. The followings are well-known (cf. [1, 4]).

(i) Let $1 < p < \infty$ and $\gamma > 0$. Then, two norms $\| \cdot \|_{B^\gamma_p}$ and $\| \cdot \|_{\dot{B}^\gamma_p}$ are equivalent, and for any $c > 0$, we have $\|u(cx)\|_{\dot{B}^\gamma_p} = c^{-d/p} \|u\|_{\dot{B}^\gamma_p}$.

(ii) Let $p \geq 2$. Then $B^\gamma_p \supset H^\gamma_p$ and $B^\gamma_p \subset H^\gamma_p$ for any $\gamma' < \gamma$.

Next, we introduce stochastic Banach spaces. Denote

$$\mathbb{H}^\gamma_p := L_p(\Omega, \mathcal{F}_0; B^\gamma_p), \quad \dot{\mathbb{H}}^\gamma_p := L_p(\Omega, \mathcal{F}_0; \dot{B}^\gamma_p), \quad \mathbb{H}^\gamma_p := L_p(\Omega, \mathcal{F}_0; H^\gamma_p),$$

and for a stopping time $\tau$ and weight function $\delta = \delta(\omega, t) \geq 0$, denote

$$\mathbb{H}^\gamma_p(\tau, \delta) = L_p(\Omega \times [0, \tau], dP \times \delta(t) dt, \mathcal{P}; H^\gamma_p), \quad \mathbb{H}^\gamma_p(\tau) := \mathbb{H}^\gamma_p(\tau, 1).$$
and
\[ H_p^\gamma(\tau, \delta, l_2) = L_p(\Omega \times [0, \tau], dP \times \delta(t)dt, P; H_p^\gamma(l_2)), \quad H_p^\gamma(\tau, l_2) := H_p^\gamma(\tau, 1, l_2). \]

For the notational convenience, we use \( L_p \) instead of \( H_p^0 \). We write
\[ u \in L_p(\Omega, \mathcal{F}; C([0, \tau]; H_p^\gamma)) \]
if \( u \) is an \( H_p^\gamma \)-valued predictable process such that \( u(\omega) \in C([0, \tau(\omega)]; H_p^\gamma) \) \((\text{a.s.})\), and
\[ \mathbb{E}\sup_{t \leq \tau} \|u\|_{H_p^\gamma} < \infty. \]

**Remark 2.2.** It is easy to check that if \( \tau \) is bounded, then \( L_p(\Omega, \mathcal{F}; C([0, \tau]; H_p^\gamma)) \) is a Banach space.

Let \( \mathcal{D} \) be the space of distributions (generalized functions) on \( C^{\infty}_c(\mathbb{R}^d) \), and let \( \mathcal{D}(l_2) \) denote the space of \( l_2 \)-valued distributions (generalized functions) on \( C^{\infty}_c(\mathbb{R}^d) \).

**Definition 2.3.** Let \( u_0 \) be \( \mathcal{D} \)-valued random variable, \( u \) and \( f \) be \( \mathcal{D} \)-valued predictable stochastic processes, and \( g \) be \( \mathcal{D}(l_2) \)-valued predictable stochastic process. We say that \( u \) satisfies (or is a solution to) the equation
\[ du(t,x) = f(t,x)dt + g(t,x)dw^d_t, \quad (t,x) \in [0,\tau] \times \mathbb{R}^d \]
\[ u(0,\cdot) = u_0 \quad (2.7) \]
in the sense of distributions if for any \( \phi \in C^{\infty}_c(\mathbb{R}^d) \), the equality
\[ (u(t,\cdot), \phi) = (u_0, \phi) + \int_0^t (f(s,\cdot), \phi)ds + \sum_k \int_0^t (g^k(s,\cdot), \phi)dw^d_t \quad (2.8) \]
holds for all \( t \leq \tau \) \((\text{a.s.})\).

In particular, if \( u_0 \in \mathbb{L}_p \), \( u, f \in \mathbb{L}_p(\tau), \) and \( g \in \mathbb{L}_p(\tau, l_2) \), then \( u \) is a solution to \( (2.7) \) if
\[ (u(t,\cdot), \phi) = (u_0, \phi) + \int_0^t \left[ \left( a^{ij}(t)u(s,\cdot), \phi_{x^i x^j} \right) + \left( f(s,\cdot), s, \cdot, \phi \right) \right] ds \]
\[ + \sum_k \int_0^t \left[ -\left( \sigma^{ik}(t)u(s,\cdot), \phi_{x^i} \right) + \left( g^k(u(s,\cdot), s, \cdot, \phi) \right) \right] dw^d_t \]
for all \( t \in [0,\tau] \) \((\text{a.s.})\).

**Remark 2.4.** Suppose that \( u_0 \in \mathbb{L}_p \), \( u, f \in \mathbb{L}_p(\tau) \), and \( g \in \mathbb{L}_p(\tau, l_2) \). In Definition \( 2.3 \) the subset \( \Omega' \subset \Omega \) such that \( P(\Omega') = 1 \) and \( (2.8) \) holds for all \( (\omega', t) \in \Omega' \times [0,\tau(\omega')] \) depends on the test function \( \phi \). However, taking the countable dense subset of \( C^{\infty}_c(\mathbb{R}^d) \) in \( L_q(\mathbb{R}^d) \) with \( q = \frac{p}{p-1} \), one can find a \( \Omega' \subset \Omega \) such that \( P(\Omega') = 1 \) and \( (2.8) \) holds for all \( (\omega', t) \in \Omega' \times [0,\tau(\omega')] \) and \( \phi \in C^{\infty}_c(\mathbb{R}^d) \).

Due to the above fact, one can use Sobolev’s mollifier to approximate \( u \) with smooth functions as in the deterministic case. Indeed, let \( \phi \in C^{\infty}_c(\mathbb{R}^d) \) have a unit integral, and denote \( \phi^\varepsilon(x) = \frac{1}{\varepsilon^d} \phi(x/\varepsilon) \). Plugging in \( \phi^\varepsilon(x-\cdot) \) in \( (2.8) \) in place of \( \phi \), we get
\[ u^\varepsilon(t,x) = u_0^\varepsilon + \int_0^t f^\varepsilon(s,x)ds + \sum_k \int_0^t g^{k\varepsilon}(s,x)dw^d_t. \]
for all $t \leq \tau$, $x \in \mathbb{R}^d$, (a.s.), where

$$u^\varepsilon(t, x) = \int_{\mathbb{R}^d} u(t, y) \phi^\varepsilon(x - y) dy, \quad u^\varepsilon_0(x) = \int_{\mathbb{R}^d} u_0(y) \phi^\varepsilon(x - y) dy$$

and

$$f^\varepsilon(t, x) = \int_{\mathbb{R}^d} f(t, y) \phi^\varepsilon(x - y) dy, \quad g^{k, \varepsilon}(t, x) = \int_{\mathbb{R}^d} g^k(t, y) \phi^\varepsilon(x - y) dy.$$ 

Now we introduce our assumptions on the coefficients $a^{ij}(t)$ and $\sigma^{ik}(t)$. Set

$$\alpha^{ij}(t) := a^{ij}(t) - \frac{1}{2}(\sigma^i(t), \sigma^j(t))_{l_2}$$

and

$$|\sigma(t)| = \max_{i=1, \ldots, d} |\sigma^i(t)|_{l_2}.$$ 

**Assumption 2.5.** (i) The coefficients $a^{ij}(t)$, $\sigma^{ik}(t)$ are predictable for all $i$, $j$, $k$, and

$$a^{ij}(t)\xi^i\xi^j \geq 0, \quad \forall (\omega, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^d.$$ 

(ii) The coefficients $a^{ij}(t)$, $|\sigma^{ik}(t)|^2$ are locally integrable, i.e.

$$\int_0^t |a^{ij}(s)|ds + \int_0^t \sum_{k=1}^\infty |\sigma^{ik}(s)|^2 ds < \infty \quad \forall t > 0, i, j$$ 

(2.9)

**Remark 2.6.** (i) Obviously, Assumption 2.5 allows the coefficients to be unbounded or degenerate.

(ii) Without loss of generality, we may assume that the coefficients $a^{ij}(t)$ and $\alpha^{ij}(t)$ are symmetric, i.e.

$$a^{ij}(t) = a^{ji}(t) \quad \text{and} \quad \alpha^{ij}(t) = \alpha^{ji}(t) \quad \forall i, j.$$ 

Thus if we denote by $\delta(t)$ the smallest eigenvalue of the matrix $(a^{ij}(t))$, then Assumption 2.5 implies

$$a^{ij}(t)\xi^i\xi^j \geq \delta(t)|\xi|^2 \geq 0 \quad \forall (\omega, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^d.$$ 

(2.10)

Here is the main result of this article.

**Theorem 2.7.** Let $p \in [2, \infty)$, $T \in [0, \infty)$, $\delta(t)$ be the smallest eigenvalue of $a^{ij}(t)$, $\tau \leq T$ be a stopping time, $\gamma \in \mathbb{R}$, $u_0 \in H_{\gamma}^p(\tau)$, $f \in H_{\gamma}^p(\tau, \delta^1 - p)$, and $g \in H_{\gamma}^p(\tau, l_2) \cap H_{\gamma}^{p+1}(\tau, \delta^1 - p/2, l_2)$. Suppose that Assumption 2.5 holds and $g_x \in H_{\gamma}^p(\tau, |\sigma|^p, l_2) \cap H_{\gamma}^{p}(\tau, |\sigma|^p \delta^1 - p, l_2)$.

Then there exists a unique solution $u \in L_p(\Omega, \mathcal{F}; C([0, \tau]; H_p^\gamma))$ to (2.1), and for this solution we have

$$\mathbb{E} \sup_{t \in [0, \tau]} ||u(t, \cdot)||_{H_p^\gamma}^p \leq N_1 \left(||u_0||_{H_p^\gamma}^p + ||f||_{H_p^\gamma(\tau)}^p + ||g||_{H_p^\gamma(\tau, l_2)}^p + ||g_x||_{H_p^\gamma(\tau, l_2)}^p \right);$$

(2.11)

and

$$||u_{xx}||_{H_p^\gamma(\tau, \delta)} \leq N_2 \left(||u_0||_{H_p^{\gamma+2(1-1/p)}}^p + ||f||_{H_p^\gamma(\tau, \delta^1 - p)}^p + ||g||_{H_p^\gamma(\tau, \delta^1 - p/2, l_2)}^p + ||g_x||_{H_p^\gamma(\tau, \delta^1 - p/2, l_2)}^p \right),$$

(2.12)
where $N_1 = N_1(p, T)$ and $N_2 = N_2(d, p)$.

The proof of this theorem is given in Section 5.

Remark 2.8. (i) A nonnegative function $w(x)$ is said to be of class $A_p$ if

$$\sup \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the sup is taken over all cubes on $\mathbb{R}^d$ (cf. Section 7.1). Note that, due to the term $\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx$, it is required that $w(x) > 0$ (a.e.).

However since our coefficients $a^{ij}(t)$ and $|\sigma|^p l_2$ can be degenerate on sets with positive measures, our weights are generally not in $A_p$-class, which makes us unable to use $A_p$-weight theories from the Fourier analysis.

(ii) Suppose that $a^{ij}$ and $|\sigma|^p l_2$ are bounded. Then, since $\delta$ is also bounded, the conditions for $f$ and $g$ in Theorem 2.7 are as follows:

$$f \in H_p^\gamma(\tau) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau, \delta^{1-p}),$$

$$g \in H_p^\gamma(\tau) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau, \delta^{1-p/2}, l_2),$$

and

$$g_x \in H_p^\gamma(\tau, |\sigma|^p, l_2) \cap H_p^{\gamma+1}(\tau, |\sigma|^p \delta^{-1/p}, l_2) = H_p^\gamma(\tau, |\sigma|^p, l_2).$$

(iii) If the matrix $(a^{ij}(t))$ is uniformly elliptic, that is, there exists a positive constant $\varepsilon > 0$ such that $\delta(t) \geq \varepsilon$, then in Theorem 2.7 it is assumed that

$$f \in H_p^\gamma(\tau) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau),$$

$$g \in H_p^\gamma(\tau, l_2) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau, l_2),$$

and

$$g_x \in H_p^\gamma(\tau, l_2) \cap H_p^{\gamma+1}(\tau, |\sigma|^p \delta^{-1/p}, l_2) = H_p^\gamma(\tau, l_2).$$

Furthermore, if $\delta(t) \geq \varepsilon$ and $|\sigma|_{l_2}$ is bounded, then our data spaces are given by

$$f \in H_p^\gamma(\tau) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau),$$

$$g \in H_p^\gamma(\tau, l_2) \cap H_p^{\gamma+1}(\tau, \delta^{-1/p}, l_2) = H_p^\gamma(\tau, l_2),$$

$$g_x \in H_p^\gamma(\tau, |\sigma|^p, l_2) \cap H_p^{\gamma+1}(\tau, |\sigma|^p \delta^{-1/p}, l_2) = H_p^\gamma(\tau, |\sigma|^p, l_2).$$

Therefore our data spaces for $f, g$ obviously include the classical data spaces (cf. [11]).

(iv) If $p = 2$, then $1 - p/2 = 0$. Thus $\delta(t)^{1-p/2}$ is not well-defined if $\delta(t) = 0$. In this case, we define $\delta(t)^{1-p/2} = 1$.

(v) We chose the smallest eigenvalue $\delta(t)$ of $(a^{ij}(t))$ as the weight in our results. However, it is possible that Theorem 2.7 holds with any function $\delta(t)$ satisfying (2.10) in place of the smallest eigenvalues.
3. Deterministic linear equations

In this section, we consider the following deterministic equation on $\mathbb{R}^d$:

$$\frac{du}{dt} = a^{ij}(t)u_{x_ix_j} + f, \quad t \in (0,T]; \quad u(0, \cdot) = u_0. \quad \text{(3.1)}$$

The coefficients $a^{ij}$ depend only on $t$. We say that $u$ is a (weak) solution to [3.1] if it holds in the sense of distributions, that is, for any $\phi \in C_c^\infty(\mathbb{R}^d)$ the equality

$$\int_{\mathbb{R}^d} u(t, x)\phi(x)dx$$

$$= \int_{\mathbb{R}^d} u_0(x)\phi(x)dx + \int_0^t \int_{\mathbb{R}^d} (a^{ij}(s)u(s, x)\phi_{x_ix_j}(x) + f(t, x)\phi(x)) \, dx \, ds \quad \text{(3.2)}$$

holds for all $t \leq T$.

Here we assume

$$a^{ij}(t)\xi^i\xi^j \geq 0, \quad \forall (t, \xi) \in (0, \infty) \times \mathbb{R}^d \quad \text{(3.3)}$$

and set

$$|a(t)| = \max_{i,j} |a^{ij}(t)|. \quad \text{(3.4)}$$

We emphasize that there is no bounded assumption on $a^{ij}(t)$. However, to make sense of equality [3.2], it is at least required that

$$\left| \int_0^t \int_{\mathbb{R}^d} a^{ij}(s)u(s, x)\phi_{x_ix_j}(x) \, dx \, ds \right| < \infty \quad \forall t \in [0, T],$$

which holds if

$$u \in L_1 ((0, T), |a(t)|dt; L_p), \quad p > 1.$$

Indeed, if $u \in L_1 ((0, T), |a(t)|dt; L_p)$, then by Hölder’s inequality, with $q = \frac{p}{p-1}$,

$$\left| \int_0^t \int_{\mathbb{R}^d} a^{ij}(s)u(s, x)\phi_{x_ix_j}(x) \, dx \, ds \right| \leq \|\phi_{xixj}\|_{L_q} \int_0^t \|u(s, \cdot)\|_{L_p} |a(s)| \, ds < \infty. \quad \text{(3.5)}$$

Moreover, if

$$\int_0^T |a(t)| \, dt < \infty \quad \text{(3.6)}$$

and $\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)} < \infty$, then

$$\int_0^t |a(s)|\|u(s, \cdot)\|_{L_p} \, ds \leq \sup_{s \leq T} \|u(s, \cdot)\|_{L_p} \int_0^t |a(s)| \, ds < \infty. \quad \text{(3.7)}$$

**Lemma 3.1** (A priori estimate). Let $p \in (1, \infty)$, $T \in (0, \infty)$, $f \in L_p((0, T); L_p)$, $u_0 \in L_p$, and $\text{3.1}$ holds. Suppose that $u$ is a solution to equation [3.1] and

$$u \in C ((0, T]; L_p) \cap L_1 ((0, T), |a(t)|dt; L_p).$$

Then

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L_p}^p \leq N \left( \|u_0\|_{L_p}^p + \int_0^T \|f(t, \cdot)\|_{L_p}^p \, dt \right), \quad \text{(3.7)}$$

where $N = N(p, T)$. 

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**SPDE WITH DEGENERATE AND UNBOUNDED COEFFICIENTS**

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Due to (3.3), we give a detailed proof for the sake of the completeness. Nonetheless, if the coefficients are bounded, then the lemma is a classical result and can be found, for instance, in [8, 2]. The proof for general case is similar.

We use Sobolev mollifiers. Fix a nonnegative \( \phi \in C^\infty_c(\mathbb{R}^d) \) such that \( 0 \leq \phi \leq 1 \), \( \int_{\mathbb{R}^d} \phi \, dx = 1 \), and \( \phi(x) = 1 \) near \( x = 0 \). Denote \( \phi^\varepsilon(x) = \varepsilon^{-d} \phi(x/\varepsilon) \), \( u_0^\varepsilon(x) = u_0 * \phi^\varepsilon(x) \), and \( u^\varepsilon(t, x) = u(t, \cdot) * \phi^\varepsilon(x) \). Putting \( \phi^\varepsilon(x - \cdot) \) in (3.2), for all \( (t, x) \in (0, T) \times \mathbb{R}^d \), we have

\[
  u^\varepsilon(t, x) = u_0^\varepsilon(x) + \int_0^t a^{ij}(s)u_x^{\varepsilon}(s, x)ds + \int_0^t f^\varepsilon(s, x)ds.
\]

(3.8)

Note that (3.2) and (3.8) make sense due to (3.4). By the chain rule, for any \( p > 1 \),

\[
  \frac{d}{dt}(|u^\varepsilon|^p) = p|u^\varepsilon|^{p-2}u^\varepsilon u_t^\varepsilon,
\]

and thus by the Fundamental theorem of calculus

\[
  |u^\varepsilon(t)|^p = |u_0^\varepsilon(x)|^p + \int_0^t p|u^\varepsilon|^{p-2}(s, x)u^\varepsilon(s, x)a^{ij}(s)u_{x,i}^\varepsilon(s, x)ds \\
  + \int_0^t p|u^\varepsilon(s, x)|^{p-2}u^\varepsilon(s, x)f^\varepsilon(s, x)ds.
\]

(3.9)

To apply Fubini’s theorem we first note that, since \( \|u^\varepsilon(t)\|_p + \|u_{xx}^\varepsilon(t)\|_p \leq N(\varepsilon)\|u(t)\|_{L^p} \), by Hölder’s inequality

\[
  \int_0^t \int_{\mathbb{R}^d} |u^\varepsilon|^{p-1}|a^{ij}||u_{x,i}^\varepsilon|\,dxds \\
  \leq \int_0^t \|u^\varepsilon(s)\|_{p}^{p-1}\|u_{xx}^\varepsilon(s)\|_p|a(s)|ds \\
  \leq N \sup_{r \leq T} \|u(r)\|_{p}^{p-1}\int_0^t \|u(s)\|_p|a(s)|ds < \infty.
\]

Thus, integrating both sides of (3.9) with respect to \( x \), and applying Fubini’s theorem and the integration by parts, we have

\[
  \int_{\mathbb{R}^d} |u^\varepsilon(t, x)|^p \, dx \\
  = \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^p \, dx + \int_0^t \int_{\mathbb{R}^d} p|u^\varepsilon|^{p-2}(s, x)u^\varepsilon(s, x)a^{ij}(s)u_{x,i}^\varepsilon(s, x)dsdx \\
  + \int_0^t \int_{\mathbb{R}^d} p|u^\varepsilon(s, x)|^{p-2}u^\varepsilon(s, x)f^\varepsilon(s, x)\,dxds \\
  = \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^p \, dx - \int_0^t \int_{\mathbb{R}^d} p(p-1)|u^\varepsilon|^{p-2}(s, x)u_{x,i}^\varepsilon(s, x)a^{ij}(s)u_x^\varepsilon(s, x)dsdx \\
  + \int_0^t \int_{\mathbb{R}^d} p|u^\varepsilon|^{p-2}(s, x)u^\varepsilon(s, x)f^\varepsilon(s, x)\,dxds.
\]

Due to (3.3),

\[
  \int_0^t \int_{\mathbb{R}^d} p(p-1)|u^\varepsilon|^{p-2}(s, x)u_{x,i}^\varepsilon(s, x)a^{ij}(s)u_x^\varepsilon(s, x)\,dsdx \geq 0.
\]
Thus

\[ \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |u^\varepsilon(t,x)|^p dx \leq \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^p dx + \int_0^T \int_{\mathbb{R}^d} p|u^\varepsilon|^{p-2}(s,x) |u^\varepsilon(s,x)| |f^\varepsilon(s,x)| dx ds. \]

By Hölder’s inequality and Young’s inequality, for any constant \( c > 0 \)

\[ \int_0^T \int_{\mathbb{R}^d} |u^\varepsilon|^{p-2}(s,x) u^\varepsilon(s,x) f^\varepsilon(s,x) dx ds \leq \int_0^T \| c^{p-1} u^\varepsilon(s,\cdot) \|_{L^p(\mathbb{R}^d)} \| f^\varepsilon(s,\cdot) \|_{L^p(\mathbb{R}^d)} ds \]

\[ \leq \frac{c^q}{q} \int_0^T \| u(s,\cdot) \|_{L^p(\mathbb{R}^d)} ds + c^{-p} \frac{1}{p} \int_0^T \| f(s,\cdot) \|_{L^p(\mathbb{R}^d)} ds \]

\[ \leq \frac{c^q T}{q} \sup_{s \leq T} \| u(s,\cdot) \|_{L^p(\mathbb{R}^d)} + c^{-p} \frac{1}{p} \int_T^T \| f(s,\cdot) \|_{L^p(\mathbb{R}^d)} ds, \]

where \( q = \frac{p}{p-1} \). Therefore taking \( c > 0 \) small so that \( \frac{c^q T}{q} < 1 \), we obtain

\[ \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |u^\varepsilon(t,x)|^p dx \leq N \left( \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^p dx + \int_0^T \int_{\mathbb{R}^d} |f^\varepsilon(s,x)|^p dx ds \right), \]

where \( N \) depends only on \( p \) and \( T \). Observing

\[ \left( \int_{\mathbb{R}^d} |u_0^\varepsilon(x)|^p dx + \int_0^T \int_{\mathbb{R}^d} |f^\varepsilon(s,x)|^p dx ds \right) \]

\[ \leq \left( \int_{\mathbb{R}^d} |u_0(x)|^p dx + \int_0^T \int_{\mathbb{R}^d} |f(s,x)|^p dx ds \right), \]

and using \( u^\varepsilon \to u \) in \( C([0,T];L_p) \), we finally get (3.7).

\[ \Box \]

**Remark 3.2.** If (3.5) holds, then by (3.4) and (3.6),

\[ C \left( [0,T]; L_p \right) \cap L_1 \left( [0,T], |a(t)| dt; L_p \right) = C \left( [0,T]; L_p \right). \]

In Lemma 3.1 local integrability of the coefficients \( a^{ij}(t) \) is not assumed. However, (3.3) is needed for the proof of the existence as follows.

**Theorem 3.3** (Well-posedness). Let \( p \in (1, \infty) \), \( T \in (0, \infty) \), \( f \in L_p(0,T; L_p) \), and \( u_0 \in L_p \). Suppose that (3.3) and (3.5) hold. Then there exists a unique solution \( u \in C \left( [0,T]; L_p \right) \) to equation (3.1) such that

\[ \sup_{t \in [0,T]} \| u(t,\cdot) \|_{L_p}^p \leq N \left( \| u_0 \|_{L_p}^p + \int_0^T \| f(t,\cdot) \|_{L_p}^p dt \right), \]

where \( N \) depends only on \( p \) and \( T \).

**Proof.** We remark that the theorem is a classical result if the coefficients are bounded, and we give a proof for the general case for the sake of the completeness.

**Part I.** (Estimate and Uniqueness) Due to (3.5),

\[ C \left( [0,T]; L_p \right) \cap L_1 \left( [0,T], |a(t)| dt; L_p \right) = C \left( [0,T]; L_p \right). \]

By this and Lemma 3.1, (3.10) holds if \( u \in C \left( [0,T]; L_p \right) \) is a solution to equation (3.1), and the uniqueness also follows.
Part II. (Existence)

Let $W_t' = (W_t'^1, \cdots, W_t'^d)$ be a $d$-dimensional Wiener process on a probability space $(\Omega', \mathcal{F}', P')$. Since $A(t) := (a_{ij}(t))$ is a nonnegative symmetric matrix, there exists a nonnegative symmetric (non-random) matrix $\sigma'(t) = (\sigma'_{ij}(t))$ such that

$$2A(t) = (\sigma'(t))^2(t).$$

Due to (3.5), $\sigma'(t)$ is Itô integrable (cf. [10, Chapter 6.3]), i.e.

$$\int_0^t |\sigma'(s)|^2 1_{s \leq T} ds < \infty, \quad \forall t.$$

We define

$$X'_t := \int_0^t \sigma'(s) dW'_s, \quad \text{(i.e. } X'_t = \sum_{k=1}^d \int_0^t \sigma'_{ik}(s) dW'_s, \quad (i = 1, 2, \cdots, d)). \quad (3.11)$$

We will first show that $u(t,x)$ defined as

$$u(t,x) := \mathbb{E}'[u_0(x + X'_t)] + \int_0^t \mathbb{E}'[f(s,x + X'_t - X'_s)] ds \quad (3.12)$$

is a solution to equation (3.1) if $u_0$ and $f$ are sufficiently smooth, where $\mathbb{E}'$ is the expectation in the probability space $(\Omega', \mathcal{F}', P')$. Then by using an approximation, we finally prove the existence of a solution for general $u_0$ and $f$.

We divide the details into several steps.

(i) Let $u_0 \in C^2 \cap H^2_p$ and $f = 0$. Then by Itô’s formula, for all $t \in [0,T]$,

$$u(t,x) := \mathbb{E}'[u_0(x + X'_t)] = \mathbb{E}'[u_0(x)] + \int_0^t a_{ij}(s) \mathbb{E}' \left[ \frac{\partial^2 u_0}{\partial x^i \partial x^j} (x + X'_s) \right] ds$$

$$= u_0(x) + \int_0^t a_{ij}(s) u_{x^i x^j}(s,x) ds. \quad (3.13)$$

Thus $u(t,x)$ satisfies equation (3.1). Also note that

$$\int_0^t \|a_{ij}(s) u_{x^i x^j}\|_p ds \leq \int_0^t |a(s)| \|\mathbb{E}' \left[ \frac{\partial^2 u_0}{\partial x^i \partial x^j} (x - X'_s) \right]\|_p ds$$

$$\leq \|u_0\|_{H^2_p} \int_0^T |a(s)| ds < \infty.$$

Therefore, from (3.13) it easily follows that $u \in C([0,T];L_p)$.

(ii) Let $u_0 = 0$ and $f \in L_1((0,T);C^2 \cap H^2_p)$. Applying (a generalized) Itô’s formula (see e.g. Theorem 4.1.1 or Corollary 4.1.2 in [9]), we get for each $t > s$,

$$\mathbb{E}'[f(s,x + X'_t - X'_s)] = f(s,x) + \int_s^t a_{ij}(r) \mathbb{E}'[f_{x^i x^j}(s,x + X'_r - X'_s)] dr. \quad (3.14)$$
By integrating the above terms with respect to $s$ from 0 to $t$ and the Fubini theorem,

$$u(t, x) := \int_0^t E'[f(s, x + X'_s - X'_t)] ds$$

$$= \int_0^t f(s, x) ds + \int_0^t \int_s^t a^{ij}(r) E'[f_x^{xj}(s, x + X'_s - X'_t)] dr ds$$

$$= \int_0^t f(s, x) ds + \int_0^t a^{ij}(r) \int_0^s E'[f_x^{xj}(s, x + X'_r - X'_s)] ds dr$$

$$= \int_0^t f(s, x) ds + \int_0^t a^{ij}(r) u_{x^{xj}}(r, x) dr.$$  \hspace{1cm} (3.15)

Therefore $u(t, x)$ is a solution to equation (3.1). The inclusion $u \in C([0, T]; L_p)$ can be easily obtained from (3.15) as was shown in (i) if $f \in L_1((0, T); C^2 \cap H^2_p)$.

(iii) (General Case) Choose sequences

$$u^n_0 \in C_c^\infty(\mathbb{R}^d), \quad f^n \in C([0, T]; C^2 \cap H^2_p),$$

so that as $n \to \infty$,

$$u^n_0 \to u_0 \quad \text{in} \quad L_p \quad \text{and} \quad f^n \to f \quad \text{in} \quad L_p((0, T); L_p).$$

Then by (i) and (ii), for all $n \in \mathbb{N}$

$$u^n(t, x) := E'[u^n_0(x + X'_t)] + \int_0^t E'[f^n(s, x + X'_s - X'_t)] ds$$ \hspace{1cm} (3.16)

satisfies

$$u^n(t, x) = a^{ij}(t) u^n_{x^{xj}}(t, x) + f^n(t, x) \quad (t, x) \in (0, T] \times \mathbb{R}^d$$

$$u^n(0, x) = u^n_0(x).$$

Moreover, due to (3.7), for all $n, m \in \mathbb{N}$

$$\sup_{t \in [0, T]} \| (u^n - u^m)(t, \cdot) \|^p_{L_p} \leq N \left( \int_0^T \| (f^n - f^m)(t, \cdot) \|^p_{L_p} dt + \| u^n_0 - u^m_0 \|^p_{L_p} \right).$$

Thus $u^n$ becomes a Cauchy sequence in $C([0, T]; L_p)$ and thus there exists a $u \in C([0, T]; L_p)$ such that $u_n \to u$ in $C([0, T]; L_p)$ as $n \to \infty$. Also, using (3.7) corresponding to $(u_n, f_n, u^n_0)$, and then taking $n \to \infty$, we easily find that $u$ is a solution to equation (3.1). The theorem is proved.

\[ \square \]

**Remark 3.4.** (i) Due to the approximation used in the proof of Theorem 3.3 for general $u_0 \in L_p$ and $f \in L_p((0, T); L_p)$, the solution $u$ to (3.1) is given by

$$u(t, x) = E'[u_0(x + X'_t)] + \int_0^t E'[f(s, x + X'_s - X'_t)] ds.$$  \hspace{1cm} (3.17)

More generally, following the proof of the theorem, one can check that $u$ defined in (3.17) belongs to $C([0, T]; L_p)$ and becomes a solution to (3.1) under a weaker condition, that is, if $u_0 \in L_p$ and $f \in L_1((0, T); L_p)$.

Indeed, in the above approximation, we can take $u^n_0 \in C^2 \cap H^2_p$ and $f^n \in L_1((0, T); C^2 \cap H^2_p)$ so that, as $n \to \infty$,

$$u^n_0 \to u_0 \quad \text{in} \quad L_p, \quad f^n \to f \quad \text{in} \quad L_1((0, T); L_p).$$
Take $u^n$ from (3.18), then by Minkowski’s inequality and the translation invariant of the $L_p$-norm,
\[ \|u^n(t, \cdot) - u(t, \cdot)\|_{L_p} \]
\[ \leq \|E'[u^n_0 - u_0](\cdot + X^n_0)\|_{L_p} + \left\| \int_0^t E'[(f^n - f)(s, \cdot + X^n_s - X'_s)]ds \right\|_{L_p} \]
\[ \leq E' \left[ \|u^n_0 - u_0\|_{L_p} \right] + \int_0^t E' \left[ \|(f^n - f)(s, \cdot + X^n_s - X'_s)\|_{L_p} \right] ds \]
\[ = \|u^n_0 - u_0\|_{L_p} + \int_0^t (f^n - f)(s, \cdot) \|_{L_p} ds. \quad (3.18) \]

Also, by (3.18),
\[ \int_0^T |a(t)| \|u^n(t, \cdot) - u(t, \cdot)\|_{L_p} dt \]
\[ \leq \int_0^T |a(t)| dt \|u^n_0 - u_0\|_{L_p} + \int_0^T |a(t)| dt \int_0^T \|(f^n - f)(s, \cdot)\|_{L_p} ds. \quad (3.19) \]

Therefore for any $\phi \in C^\infty_{c}(\mathbb{R}^d)$ and $t \in (0, T)$, taking $n \to \infty$ to the equality
\[ (u^n(t, \cdot), \phi) = (u^n_0, \phi) + \int_0^t a^{ij}(s) (u^n(s, \cdot), \phi_{x^i \cdot x^j}) ds + \int_0^t (f^n(s, \cdot), \phi) ds, \]
we get
\[ (u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t a^{ij}(s) (u(s, \cdot), \phi_{x^i \cdot x^j}) ds + \int_0^t (f(s, \cdot), \phi) ds. \]

In other words, the function $u$ defined in (3.17) is a solution to (3.1) if $u_0 \in L_p$ and $f \in L_1([0, T]; L_p)$. Moreover, by (3.18), $\sup_{t \leq T} \|u^n(t) - u(t)\|_p \to 0$ as $n \to \infty$, and therefore
\[ u \in C([0, T]; L_p). \]

(ii) Let $h \in C^2_0(\mathbb{R}^d)$. Recall
\[ X^n_t - X^n_0 = \int_r^t \sigma'(s) dW_s'. \]

Note that, since $\sigma'$ is not random, both $X^n_t - X^n_0$ and $\int_0^{t-s} \sigma'(t-s) dW_s'$ have Gaussian distributions with mean zero and the same covariance, and therefore they have the same distribution. Thus by Itô’s formula and a change of variables,
\[ E[h(x + X^n_t - X^n_0)] = E[h \left( x + \int_r^t \sigma'(t-s) dW_s' \right) ] \]
\[ = h(x) + \int_r^t a^{ij}(t-s) E \left[ h_{x^i x^j} \left( x + \int_0^{s} \sigma'(t-t) dW_s' \right) \right] ds \]
\[ = h(x) + \int_r^t a^{ij}(s) E \left[ h_{x^i x^j} \left( x + \int_0^{t-s} \sigma'(t-t) dW_s' \right) \right] ds \]
\[ = h(x) + \int_r^t a^{ij}(s) E \left[ h_{x^i x^j} \left( x + X^n_t - X^n_0 \right) \right] ds. \]
This will be used later for the solution representation to SPDEs (see Remark 4.2(ii) below).

4. Stochastic linear equations with additive noises

In this section, we study the following SPDE with additive noises:

\[
du = (a^{ij}(t)u_{x^i x^j} + f)\,dt + g^k\,dw^k_t, \quad (t, x) \in (0, \tau] \times \mathbb{R}^d
\]

\[u(0, x) = u_0(x),\]

where \(\tau\) is a bounded stopping time. We assume that the coefficients \(a^{ij}\) are predictable functions of \((\omega, t)\) and satisfy

\[a^{ij}(t)\xi^i \xi^j \geq 0, \quad \forall (\omega, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^d. \tag{4.2}\]

We denote by \(H^\infty_c(\tau, l_2)\) the space of stochastic processes \(g = (g^1, g^2, \ldots)\) such that \(g^k = 0\) for all large \(k\) and each \(g^k\) is of the type

\[g^k(t, x) = \sum_{i=1}^{j(k)} 1_{(\tau_{i-1}, \tau_i]}(t)g^{ik}(x),\]

where \(j(k) \in \mathbb{N}\), \(g^{ik} \in C^\infty_c(\mathbb{R}^d)\), and \(\tau_i\) are stopping times with \(\tau_i \leq \tau\). Similarly, we denote by \(H^\infty_c(\tau)\) the space of stochastic processes \(g\) such that

\[g(t, x) = \sum_{i=1}^{j} 1_{(\tau_{i-1}, \tau_i]}(t)g^{i}(x),\]

where \(j \in \mathbb{N}\), \(g^{i} \in C^\infty_c(\mathbb{R}^d)\), and \(\tau_i\) are stopping times with \(\tau_i \leq \tau\). Also, we denote by \(H^\infty_c(\mathbb{R}^d)\) the space of random variables \(g_0\) of the type

\[g_0(\omega, x) = 1_A(\omega)g(x)\]

where \(g \in C^\infty_c(\mathbb{R}^d)\), and \(A \in \mathcal{F}_0\).

It is known that \(H^\infty_c(\tau, l_2)\) is dense in \(H^p_\gamma(\tau, l_2)\) for all \(p \in (1, \infty)\) and \(\gamma \in \mathbb{R}\) (for instance, see [11] Theorem 3.10]). In particular, \(H^\infty_c(\tau)\) is dense in \(H^p_\gamma(\tau)\) for all \(p \in (1, \infty)\) and \(\gamma \in \mathbb{R}\). Following the idea of [11] Theorem 3.10, one can also easily check that \(H^\infty_c(\mathbb{R}^d)\) is dense in \(H^p_\gamma(\mathbb{R}^d)\) for all \(p \in (1, \infty)\) and \(\gamma \in \mathbb{R}\).

**Theorem 4.1.** Let \(p \in [2, \infty), T \in [0, \infty), \gamma \in \mathbb{R},\) and \(\tau\) be a stopping time such that \(\tau \leq T\). Assume that the coefficients \(a^{ij}(t)\) are locally integrable in \(t\), that is,

\[\sum_{i,j} \int_0^t |a^{ij}(\omega, t)|\,dt < \infty, \quad \forall t \text{ (a.s.)}. \tag{4.3}\]

Then for all \(u_0 \in H^p_\gamma, f \in H^p_\gamma(\tau),\) and \(g \in H^p_\gamma(\tau, l_2)\), there exists a unique solution \(u \in L_p(\Omega, \mathcal{F}; C([0, \tau]; H^p_\gamma))\) to (4.1) such that

\[\mathbb{E} \sup_{t \in [0, \tau]} \|u(t, \cdot)\|_{H^p_\gamma}^p \leq N(p, T) \left(\|u_0\|_{H^p_\gamma}^p + \|f\|_{H^p_\gamma(\tau)}^p + \|g\|_{H^p_\gamma(\tau, l_2)}^p\right). \tag{4.4}\]

**Proof.** If the coefficients are bounded then the results were proved in [8].

Due to the isometry of the map \((1 - \Delta)^{\gamma/2}\) on \(H^p_\gamma\), we may assume \(\gamma = 0\).

**Part I.** (A priori estimate and the uniqueness).

First, we show that any solution \(u \in L_p(\Omega, C([0, \tau]; L_p))\) to (4.1) satisfies (4.4) following the proof of Lemma 3.1 but using Itô’s formula instead of the chain rule. Let \(u\) be a solution to (4.1) in \(L_p(\Omega, C([0, \tau]; L_p))\). Due to the argument of Sobolev’s
mollifier used in the proof of Lemma 3.1 we may assume that the given solution
u and the data f and g are sufficiently smooth with respect to x. Then by Itô’s
formula,
\[ d\langle |u|^p \rangle = p|u|^{p-2}u \left( a^{ij}(t)u_{x^ix^j} + f \right) dt + p|u|^{p-2}u_x g^k dw^k_t \]
\[ + \frac{1}{2} p(p-1)|u|^{p-2}|g^k_t|^2 dt. \]

Applying (stochastic) Fubini’s theorem, and the integration by parts, we have
\[
\int_{\mathbb{R}^d} |u(t, x)|^p dx
\]
\[ = \int_{\mathbb{R}^d} |u_0(x)|^p dx - \int_0^t \int_{\mathbb{R}^d} p(p-1)|u|^{p-2}(s, x)u_x(s, x)a^{ij}(t)u_{x^i}(s, x)ds dx \]
\[ + \int_0^t \int_{\mathbb{R}^d} p|u|^{p-2}(s, x)u(s, x)f(s, x)dx ds \]
\[ + \frac{1}{2} p(p-1) \int_0^t \int_{\mathbb{R}^d} |u|^{p-2}(s, x)|g^k_t|^2(s, x)dx ds \]
\[ + \int_0^t \int_{\mathbb{R}^d} p|u|^{p-2}(s, x)u(s, x)(g^k(s, x))dx dw^k_s \quad \forall t \in [0, T]. \]

By the BDG (Burkholder-Davis-Gundy) inequality, the Hölder inequality, and the
generalized Minkowski inequality,
\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \int_0^T \int_{\mathbb{R}^d} p|u|^{p-2}(s, x)u(s, x)(g^k(s, x))dx dw^k_s \right| \right]
\]
\[ \leq N(p) \mathbb{E} \left[ \left[ \int_0^T \int_{\mathbb{R}^d} |u|^{p-2}(s, x)u(s, x)(g^k(s, x))dx \right]^{1/2} ds \right] \]
\[ \leq N(p) \mathbb{E} \left[ \left[ \int_0^T \int_{\mathbb{R}^d} |u|^{p-1}(s, x)|g(s, x)| dx \right]^{1/2} ds \right] \]
\[ \leq N(p) \mathbb{E} \left[ \left[ \int_0^T \left( \|u^{p-1}(s, \cdot)\|_{L_q} \|g\|_{L_p(t_2)} \right)^2 ds \right]^{1/2} \right], \]
where \( q = \frac{p}{p-1} \). Due to \( \frac{q}{2} \),
\[
\int_0^t \int_{\mathbb{R}^d} p(p-1)|u|^{p-2}(s, x)u_x(s, x)a^{ij}(t)u_{x^i}(s, x)ds dx \geq 0.
\]
Thus
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |u(t, x)|^p dx \right]
\]
\[ \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} |u_0(x)|^p dx \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} p|u|^{p-2}(s, x)u(s, x)f(s, x)dx ds \right] \]
\[ + N \mathbb{E} \left[ \int_0^T \left( \|u^{p-1}(s, \cdot)\|_{L_q} \|g\|_{L_p(t_2)} \right)^2 ds \right]^{1/2}. \]
By Hölder’s inequality and Young’s inequality, for any constant $c > 0$

$$
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |u|^{p-2}(s, x) u(s, x) f(s, x) dx ds \right] \\
\leq \mathbb{E} \left[ \int_0^T \| cu^{p-1}(s, \cdot) \|_{L^p(\mathbb{R}^d)} \| c^{-1} f \|_{L^p(\mathbb{R}^d)} ds \right] \\
\leq \frac{p}{q} \mathbb{E} \left[ \int_0^T \| u(s, \cdot) \|_{L^p(\mathbb{R}^d)}^p ds \right] + c^{-p} \frac{1}{p} \mathbb{E} \left[ \int_0^T \| f \|_{L^p(\mathbb{R}^d)}^p ds \right] \\
\leq \frac{p}{q} \mathbb{E} \left[ \sup_{s \leq T} \| u(s, \cdot) \|_{L^p(\mathbb{R}^d)}^p \right] + c^{-p} \frac{1}{p} \mathbb{E} \left[ \int_0^T \| f \|_{L^p(\mathbb{R}^d)}^p ds \right].
$$

Similarly,

$$
\mathbb{E} \left[ \int_0^T \| u^{p-1}(s, \cdot) \|_{L^q(\mathbb{R}^d)}^q ds \right] \\
= \mathbb{E} \left[ \int_0^T \| cu^{p-1}(s, \cdot) \|_{L^q(\mathbb{R}^d)} \| c^{-1} g \|_{L^q(\mathbb{R}^d)}^q ds \right] \\
\leq \mathbb{E} \left[ \sup_{t \leq T} \| c^{-1} u(t, \cdot) \|_{L^p(\mathbb{R}^d)}^{p-1} \int_0^T \| c^{-1} g \|_{L^q(\mathbb{R}^d)}^q ds \right] \\
\leq \left( \mathbb{E} \left[ \sup_{t \leq T} \| c^{-1} u(t, \cdot) \|_{L^p(\mathbb{R}^d)} \right] \right)^{1/q} \left( \mathbb{E} \left[ \int_0^T \| c^{-1} g \|_{L^q(\mathbb{R}^d)}^q ds \right]^{p/2} \right)^{1/p} \\
\leq N \left( \mathbb{E} \left[ \sup_{t \leq T} \| c^{-1} u(t, \cdot) \|_{L^p(\mathbb{R}^d)} \right] \right)^{1/q} \left( \| c^{-1} g \|_{L^q(\mathbb{R}^d)}^p \right)^{1/p} \\
\leq N \left( \frac{c}{q} \mathbb{E} \left[ \sup_{t \leq T} \| u(t, \cdot) \|_{L_p} \right] + \frac{1}{c^p} \left( \| g \|_{L_q(\mathbb{R}^d)}^p \right)^{1/p} \right).
$$

Therefore taking $c > 0$ small enough, we obtain

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_{\mathbb{R}^d} |u(t, x)|^p dx \right] \leq N \left( \| u_0 \|_{L_p(\mathbb{R}^d)}^p + \| f \|_{L_p(\mathbb{R}^d)}^p + \| g \|_{L_q(\mathbb{R}^d)}^p \right),
$$

where $N$ depends only on $p$ and $T$. Obviously, this a priori estimate yields the uniqueness of the solution.

**Part II.** (Existence) We divide the proof of the existence into several steps.

(i) First, we assume that $u_0 = 0$, $f = 0$, and

$$
g \in H^\infty(\tau, t_2). \tag{4.5}
$$

For a while, we additionally assume that there exists a positive constant $M > 0$ such that

$$
\int_0^t |a(t)| dt \leq M, \quad \forall t \leq \tau \ (a.s.). \tag{4.6}
$$

Let $(\Omega', \mathcal{F}', P')$ be a probability space different from $(\Omega, \mathcal{F}, P)$ and $W'_t$ be a Wiener process on $(\Omega', \mathcal{F}', P')$. Take a symmetric matrix-valued process $\sigma'_t$
on $\Omega$ such that
\[(a^{ij}(\omega, t)) = \frac{1}{2} (\sigma')^2 (\omega, t).\]

For each fixed $\omega \in \Omega$, we define the stochastic process $X'_{t, \omega}$ on $\Omega' \times [0, \infty)$ by
\[X'_{t, \omega} = \int_0^t \sigma' (\omega, t) dW'_t,\]
where $W'_t$ is a Wiener process on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. Then by [21] IV, Theorem 63, the process $X'_{t, \omega}$ has a $\mathcal{F}' \otimes \mathcal{F} \otimes \mathcal{B}([0, \infty))$-measurable version and predictable for each fixed $\omega$. Set
\[v(t, x) := \int_0^t g^k (s, x) dW'_s,\]
and for each $\omega$, consider the deterministic PDE
\[z_t (t, x) = a^{ij}(\omega, t) z_{xx} (t, x) + y_\omega (t, x), \quad (t, x) \in (0, \tau(\omega)) \times \mathbb{R}^d,\]
\[z(0, x) = 0.\]

Note that by the Fubini Theorem, the BDG inequality, and the Hölder inequality,
\[
\mathbb{E} \left[ \sup_{t \leq \tau} \int_{\mathbb{R}^d} |v(x, x)|^p \, dx \right] \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} \left( \int_0^{\tau} |g_{xx}(x, x)|^2 \, dt \right)^{p/2} \, dx \right] \leq N \|g_{xx}\|_{L^p(\tau, \tau^2)}^p < \infty,
\]
and similarly,
\[
\mathbb{E} \left[ \sup_{t \leq \tau} \int_{\mathbb{R}^d} |v(t, x)|^p \, dx \right] \leq N \|g\|_{L^p(\tau, \tau^2)^p} < \infty.
\]

Note that
\[
\int_0^{\tau} \|a^{ij}(t) v_{xx}(t, \cdot)\|_{L^p} \, dt \leq \int_0^{\tau} |a(t)| \, dt \cdot \sup_{t \leq \tau} \|v_{xx}(t, \cdot)\|_{L^p} \leq M \sup_{t \leq \tau} \|v_{xx}(t, \cdot)\|_{L^p}.
\]

Applying [19] and Hölder’s inequality, we have
\[
\mathbb{E} \left[ \int_0^{\tau} \|a^{ij}(t) v_{xx}(t, \cdot)\|_{L^p} \, dt \right] \leq M \left( \mathbb{E} \left[ \sup_{t \leq \tau} \|v_{xx}(t, \cdot)\|_{L^p}^p \right] \right)^{1/p} < \infty.
\]

Thus, $y_\omega \in L^1((0, \tau(\omega)); L^p)$ (a.s.), and by Remark 3.4(i),
\[z_\omega = z_\omega (t, x) := \int_0^t \mathbb{E} [y_\omega (s, x + X'_{t, \omega} - X'_{s, \omega})] \, ds\]
is a solution to (4.8) such that $z_\omega \in C([0, \tau]; L^p)$, and
\[
\sup_{t \leq \tau} \|z_\omega\|_p \leq N \int_0^{\tau} \|y_\omega (t)\|_p \, dt.
\]

This, (4.10), (4.9), and (4.11) yield that $z$ is $\mathcal{F}_t$-adapted, $L^p$-valued predictable, and
\[z := z_\omega (t, x) \in L^p(\Omega, \mathcal{F}; C([0, \tau]; L^p)).\]
Define
\[ u(t, x) := z(t, x) + v(t, x) \]
\[ = \int_0^t E[y_\omega(s, x + X_{t,\omega}^s - X_{t,\omega}^s)]ds + \int_0^t g^k(s, x)dw^k_s, \]  
(4.12)
where
\[ y(t, x) := y_\omega(t, x) = a^{ij}(\omega, t)v_{x_{i,x}}(\omega, t, x). \]

Then
\[ u(t, x) = \int_0^t (a^{ij}(s)z_{x_{i,x}}(s, x) + y(s, x))\,ds + v(t, x) \]
\[ = \int_0^t a^{ij}(s)u_{x_{i,x}}(s, x)ds + \int_0^t g^k(s, x)dw^k_s. \]

Therefore \( u \) becomes a solution to (4.1) in \( L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p)) \) if (4.6) holds. To remove the bounded condition (4.6), consider stopping times
\[ \tau_n := \inf \{ t \leq \tau : \sum_{i,j} \int_0^t |a^{ij}(t)|dt > n \}. \]

Then, since (4.10) holds with \( \tau_n \) and \( n \), by the above result there exists a solution \( u_n \) to (4.1) with \( \tau_n \) in \( L_p(\Omega; C([0, \tau_n]; L_p)) \). By the uniqueness of a solution and a priori estimate (4.4) obtained in Part I, \( u_n = u_m \) a.e. on \( \{ (\omega, t) : t \in [0, \tau_n(\omega)) \} \) for all \( m \geq n \), and
\[ E \sup_{t \in [0, \tau_n]} \| u_n(t, \cdot) \|_{L_p}^p \leq N(p, T)E\| g \|_{L_p(\tau, l_2)}^p. \]
(4.13)

Define
\[ \tilde{u}(t) := \lim_{n \to \infty} u_n(t), \]
where the limit is the point-wise limit on a subset of \( \{ (\omega, t) : t \in [0, \tau(\omega)) \} \). Since \( \tau_n \to \tau \) (a.s.) as \( n \to \infty \), we have \( \tilde{u} \in C([0, \tau]; L_p) \) (a.s.), and \( \tilde{u} \) becomes a (distribution-valued) solution to (4.1) for \( t < \tau \). Also, since \( \tilde{u} = u_n \) for \( t \leq \tau_n \), we have
\[ \sup_{t \leq \tau_n} \| u_n(t) \|_{L_p}^p = \sup_{t \leq \tau_n} \| \tilde{u}(t) \|_{L_p}^p, \]
and therefore, applying Fatou’s lemma to (4.13), we conclude
\[ E \sup_{t \in [0, \tau]} \| \tilde{u}(t, \cdot) \|_{L_p}^p \leq N(p, T)E\| g \|_{L_p(\tau, l_2)}^p. \]

Note also that, since \( u_n \) is defined as in (4.12) for \( t \leq \tau_n \), it follows that if \( t < \tau \) then \( \tilde{u} \) is equal to the right hand side of (4.12), which is adapted and continuous \( L_p \)-valued process on \([0, \tau]\). Therefore, we conclude that there exists a continuous extension \( u \) which is a version of \( \tilde{u} \) and a solution to (4.1) in the class \( L_p(\Omega; C([0, \tau]; L_p)) \).

(ii) Second, we assume
\[ u_0 \in \mathbb{H}_c^\infty (\mathbb{R}^d), \quad f \in \mathbb{H}_c^\infty (\tau), \quad g \in \mathbb{H}_c^\infty (\tau, l_2). \]
For each $\omega$, consider the equation
\[ z(t, x) = a^{ij}(t)z_{x^ix^j}(t, x) + f(t, x) \quad (t, x) \in (0, \tau(\omega)) \times \mathbb{R}^d \]
\[ z(0, x) = u_0(x). \quad (4.14) \]

Take $X'_{t,\omega}$ from (117), and define
\[ z_{\omega}(t, x) = \mathbb{E}[u_0(0, x + X'_{t,\omega})] + \int_0^t \mathbb{E}[f(s, x + X'_{t,\omega} - X'_{s,\omega})]ds. \quad (4.15) \]

Then by Remark 3.4(i), $z_{\omega}$ is a solution to (4.14) and $z_{\omega} \in C([0, \tau(\omega)]; L_p)$ for each $\omega$. Moreover, by the generalized Minkowski inequality,
\[ z := z_{\omega}(t) \in L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p)). \]

Moreover, due to (i), formula (4.12) gives a unique solution $\bar{v}$ to the equation
\[ d\bar{v} = a^{ij}(t)\bar{v}_{x^i}dt + g^k dw^k_t \quad (t, x) \in (0, \tau) \times \mathbb{R}^d \]
\[ \bar{v}(0, x) = 0 \in \mathbb{R}_p, \quad f^n \to f \quad \text{in} \quad L_p(\tau), \quad g^n \to g \quad \text{in} \quad L_p(\tau, t_2) \]
as $n \to \infty$. Then for each $n$, by (ii) there exists a solution
\[ u^n \in L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p)) \]
to the equation
\[ du^n(t, x) = (a^{ij}(t)u^n_{x^i}x^j(t, x) + f^n(t, x))dt + (g^n)^k(t, x)dw^k_t \]
\[ u^n(0, x) = u^n_0(x) \quad (\omega, t, x) \in \Omega \times (0, \tau) \times \mathbb{R}^d. \]

and thus for all $n, m$
\[ d(u^n - u^m)(t, x) = (a^{ij}(t)(u^n - u^m)_{x^i}x^j(t, x) + (f^n - f^m)(t, x))dt \]
\[ + (g^n - g^m)^k(t, x)dw^k_t \quad (\omega, t, x) \in \Omega \times (0, \tau) \times \mathbb{R}^d \]
\[ (u^n - u^m)(0, x) = (u^n_0 - u^m_0)(x). \]

Due to a priori estimate (14), we have
\[ \mathbb{E} \sup_{t \in [0, \tau]} \|u^n - u^m\|_{L_p}^p \]
\[ \leq N(p, T) \left( \|f^n - f^m\|_{L_p(\tau)}^p + \|g^n - g^m\|_{L_p(\tau, t_2)}^p + \|u^n_0 - u^m_0\|_{L_p}^p \right) \]

Thus $u^n$ becomes a Cauchy sequence in $L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p))$ and by taking the limit, we have a solution $u \in L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p))$ to equation (4.14). The theorem is proved. \(\square\)

The results of the following remark will not be used anywhere in this article.
Remark 4.2. (i) Based on the approximation used in the above proof, one can easily check that if \( u_0 \in L_p, f \in L_p(\tau) \), and \( g \in H^2(\tau, l_2) \) then the solution to the equation

\[
du = (a^{ij}(t)u_{x',x} + f) dt + g^k dw^k_t, \quad t > 0; \quad u(0, x) = u_0
\]

is given by

\[
u(t, x) = E[u_0(\omega, x + X^l_{t, \omega})] + \int_0^t E[f(\omega, s, x + X^l_{t, \omega} - X^l_{s, \omega})] ds
\]

\[
+ \int_0^t E[y_\omega(s, x + X^l_{t, \omega} - X^l_{s, \omega})] ds + \int_0^t g^k(s, x) dw^k_s. \tag{4.16}
\]

The additional assumption \( g \in H^2(\tau, l_2) \) is needed to make sense of \( y_\omega \) which is defined by

\[
y_\omega(t, x) := a^{ij}(t) \int_0^t g^k_{x',x}(s, x) dw^k_s.
\]

(ii) If coefficients \( a^{ij}(t) \) are not random, then for any \( u_0 \in L_p, f \in L_p(\tau) \), and \( g \in L_p(\tau, l_2) \) then solution to the equation

\[
du = (a^{ij}(t)u_{x',x} + f) dt + g^k dw^k_t, \quad t > 0; \quad u(0, \cdot) = u_0
\]

is given by

\[
u(t, x) = E[u_0(x + X^l_t)] + \int_0^t E[f(s, x + X^l_t - X^l_s)] ds
\]

\[
+ \int_0^t E[g^k(s, x + X^l_t - X^l_s)] dw^k_s. \tag{4.17}
\]

Actually this is a well known result if the coefficients are bounded and have uniform ellipticity condition. The general case can be proved based on Ito’s formula. For simplicity, we only consider the case \( u_0 = 0 \) and \( f = 0 \). Considering an approximation argument we may assume \( g \in H^2(\tau, l_2) \). This is possible because there are no derivatives of \( g \) in formula (4.17).

Using (4.10) and applying the integration by parts, the Fubini Theorem, and the stochastic Fubini theorem, we have

\[
\int_{\mathbb{R}^d} \int_0^t E[y(s, x + X^l_t - X^l_s)] ds \phi(x) dx
\]

\[
= \int_0^t \int_{\mathbb{R}^d} a^{ij}(s)E^r \left[ \int_0^s g^k(r, x + X^l_r - X^l_s) dw^k_r \right] \phi_{x'(x)}(x) dx ds
\]

\[
= \int_0^t \int_0^t \int_{\mathbb{R}^d} a^{ij}(s)E^r \left[ g^k(r, x + X^l_r - X^l_s) \right] \phi_{x'(x)}(x) dx ds dw^k_r
\]

\[
= \int_0^t \int_{\mathbb{R}^d} a^{ij}(s)E^r \left[ g^k_{x'(x)}(r, x + X^l_r - X^l_s) \right] ds \phi(x) dx dw^k_r \tag{4.18}
\]

for all \( t \in [0, \tau] \) (a.s.). By Itô’s formula (cf. Remark 4.1(ii)), for all \( t \geq r \) and \( \omega \), we have

\[
E[g^k(r, x + X^l_t - X^l_s)] = g^k(r, x) + \int_r^t a^{ij}(s)E^r \left[ g^k_{x'(x)}(r, x + X^l_r - X^l_s) \right] ds \tag{4.19}
\]
Thus from (4.18), (4.19), and the stochastic Fubini theorem, we have
\[
\int_{\mathbb{R}^d} u(t, x) \phi(x) dx \\
\int_{\mathbb{R}^d} \left( \int_0^t \mathbb{E}'[y(s, x + X^i_s - X^j_s)] ds + \int_0^t g^k(s, x) dw^k_s \right) \phi(x) dx \\
= \int_{\mathbb{R}^d} \int_0^t \mathbb{E}'[g^k(r, x + X^i_r - X^j_r)] dw^k_r \phi(x) dx
\]
for all \( t \leq \tau \) (a.s.). Hence, the claim is proved.

From now on, we focus on higher regularity of solution to equation (4.1).

**Lemma 4.3.** Suppose there are constants \( \kappa, M > 0 \) such that
\[
|a^{ij}(t)| \leq M, \quad \forall \omega \in \Omega, t > 0
\]
and
\[
a^{ij}(t) \xi^i \xi^j \geq \kappa |\xi|^2, \quad \forall \omega \in \Omega, t > 0, \xi \in \mathbb{R}^d.
\]
Let \( p \geq 2 \), \( \tau \) be a stopping time, \( u_0 \in \mathbb{B}^{2-2/p}_p \), \( f \in \mathbb{L}^p_\tau(\Omega) \) and \( g \in \mathbb{H}^1_2(\tau, l_2) \). Then equation (4.1) has a unique solution \( u \in \cap_{T > 0} L^p_\Omega(\Omega, \mathcal{F}; C([0, \tau \wedge T]; L^p_\mathbb{R})) \), and for this solution we have
\[
\|u_{xx}\|_{L^p_\mathbb{R}(\tau \wedge T)} \leq N \left( \|u_0\|_{\mathbb{B}^{2-2/p}_p} + \|f\|_{L^p_\mathbb{R}(\tau \wedge T)} + \|g_x\|_{L^p_\mathbb{R}(\tau \wedge T, l_2)} \right),
\]
where \( N = N(d,p,\kappa,M) \) is independent of \( \tau \).

**Proof.** The existence and uniqueness are consequence of Theorem 4.1. Estimate (4.22) was proved by Krylov (11, 12), however we give some details below because Krylov used \( \mathbb{B}^{2-2/p}_p \) for the space of initial data in place of \( \mathbb{B}^{2-2/p}_p \).

Step 1. Let \( u_0 = 0 \). Then, by (12, Theorem 2.1), for any \( T > 0 \),
\[
\|u_{xx}\|_{L^p_\mathbb{R}(\tau \wedge T)} \leq N(d,p,\kappa,M) \left( \|f\|_{L^p_\mathbb{R}(\tau \wedge T)} + \|g_x\|_{L^p_\mathbb{R}(\tau \wedge T, l_2)} \right),
\]
and thus one gets (4.22) by taking \( T \to \infty \).

Step 2. In general, take a solution \( v \) (cf. 11, Theorem 5.1) to equation
\[
dv = \Delta v \, dt, \quad t > 0; \quad v(0, x) = u_0
\]
such that \( v \in \mathbb{H}^2_\mathbb{R}(\tau \wedge T) \cap L^p_\Omega(\Omega, \mathcal{F}; C([0, \tau \wedge T]; L^p_\mathbb{R})) \) for any \( T > 0 \). Then using a classical result in PDE (see e.g. 15) for each \( \omega \),
\[
\|v_{xx}\|_{L^p_\mathbb{R}(\tau \wedge T)}^p \leq N(d,p) \|u_0\|_{\mathbb{B}^{2-2/p}_p}^p.
\]
Thus, taking the expectation and letting \( T \to \infty \), we get
\[
\|v_{xx}\|_{L^p_\mathbb{R}(\tau)}^p \leq N(d,p) \|u_0\|_{\mathbb{B}^{2-2/p}_p}^p.
\]
Finally, note that \( \tilde{u} := u - v \in \cap_{T > 0} L^p_\Omega(\Omega, \mathcal{F}; C([0, \tau \wedge T]; L^p_\mathbb{R})) \) and satisfies
\[
d\tilde{u} = (a^{ij} \tilde{u}_{x^i x^j} + \tilde{f}) dt + g^k dw^k, \quad t > 0; \quad \tilde{u}(0, x) = 0,
\]
where \( \tilde{f} := a^{ij} v_{x^i x^j} - \Delta v + f \). By the result of Step 1 and (4.24),
\[
\|u_{xx}\|_{L^p_\mathbb{R}(\tau)} \leq N(d,p,\kappa,M) \left( \|\tilde{f}\|_{L^p_\mathbb{R}(\tau)} + \|g_x\|_{L^p_\mathbb{R}(\tau, l_2)} \right) + \|v_{xx}\|_{L^p_\mathbb{R}(\tau)}
\leq N(d,p,\kappa,M) \left( \|f\|_{L^p_\mathbb{R}(\tau)} + \|g_x\|_{L^p_\mathbb{R}(\tau, l_2)} + \|u_0\|_{\mathbb{B}^{2-2/p}_p} \right).
\]
The lemma is proved. \qed

In the following lemma we show that the boundedness of coefficients is not needed for estimate (4.22).

**Lemma 4.4.** Let $p \in [2, \infty)$, $\tau$ be a stopping time, $u_0 \in \mathbb{H}^{2(1-1/p)}_p$, $f \in L_p(\tau)$, $g \in \mathbb{H}^{2}_p(\tau, l_2)$, and $u \in \bigcap_{T>0} L_p(\Omega, \mathcal{F};C([0, \tau \wedge T];L_p))$ be a solution to (4.11). Assume that (4.20) and (4.21) hold and coefficients $a^{ij}$ are predictable for all $i, j$. Then there exists a positive constant $N = N(d, p)$ such that

$$
\|u_{xx}\|_{L^p(\tau)} \leq N \left( \kappa^{-1/p} \|u_0\|_{\mathbb{H}^{2(1-1/p)}_p} + \kappa^{-1} \|f\|_{L^p(\tau)}^p + \kappa^{-1/2} \|g\|_{L^p(\tau, l_2)} \right). \quad (4.25)
$$

**Proof.** Due to the approximation used in Theorem 4.1, we may assume that $u_0 \in \mathbb{H}^{\infty}_c(\mathbb{R}^d)$, $f \in \mathbb{H}^{\infty}_c(\tau)$, and $g \in \mathbb{H}^{\infty}_c(\tau, l_2)$. We use the idea in the proof of [11] Theorem 4.10.

**Step 1.** Assume $A(t) = (a^{ij}(t)) = \frac{1}{2} I_{d \times d}$, where $I_{d \times d}$ is the $d \times d$ identity matrix. Thus, $u$ is a solution to

$$
du(t, x) = \left( \frac{\kappa}{2} \Delta u + f \right) dt + g^k dw^k_t, \quad 0 < t \leq \tau; \quad u(0, x) = u_0.
$$

Define $\bar{v}(t, x) = u(t, \sqrt{\kappa} x)$.

Then $\bar{v}$ satisfies

$$
d\bar{v} = \left( \frac{1}{2} \Delta \bar{v}(t, x) + f(t, \sqrt{\kappa} x) \right) dt + g^k(t, \sqrt{\kappa} x) dw^k_t, \quad 0 < t \leq \tau
$$

with initial data $\bar{v}(0, x) = u_0(\sqrt{\kappa} x)$. Since $\frac{1}{2} I_{d \times d}$ satisfies both (4.20) and (4.21) with $\kappa = M = \frac{1}{2}$, by (4.22) applied to $\bar{v}$, we get

$$
\kappa \|u_{xx}\|_{L^p(\tau)} \leq (4.22),
$$

which certainly leads to (4.25).

**Step 2.** (General case).

Let $W'_t$ be a $d$-dimensional Wiener process on a probability space $(\Omega', \mathcal{F}', P')$ different from $(\Omega, \mathcal{F}, P)$. Consider the product probability space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P \times P')$. Denote

$$
\hat{\mathcal{F}}_t := \{ A \times \Omega : A \in \mathcal{F}_t \}, \quad \hat{\sigma}(W'_s : s \leq t) := \{ \Omega \times \sigma(W'_s : s \leq t) \},
$$

and by $\hat{\mathcal{F}}_t$ we denote the smallest $\sigma$-field on $\Omega \times \Omega'$ containing above two $\sigma$-fields, that is

$$
\hat{\mathcal{F}}_t := \mathcal{F}_t \vee \hat{\sigma}(W'_s : s \leq t).
$$

Considering $\bigcap_{s \geq t} \hat{\mathcal{F}}_s$ in place of $\mathcal{F}_s$, we may assume that $\hat{\mathcal{F}}$ satisfies the usual condition. For a stopping time $\tau$ relative to $\hat{\mathcal{F}}_t$, we define the corresponding Banach spaces $\mathbb{H}^{\gamma}_p(\tau), \hat{\mathbb{L}}_p(\hat{\tau}), \hat{\mathbb{L}}^{\gamma}_p(\hat{\tau}, l_2), \hat{\mathbb{L}}_p(\hat{\tau}, l_2), \hat{\mathbb{B}}^{\gamma}_p$, and $\hat{\mathbb{B}}^{\gamma}_p$. Then, since any stochastic process defined on $\Omega$ can be considered as a stochastic processes on $\Omega \times \Omega'$, for any stopping $\tau$ relative to $\mathcal{F}_t$ we have

$$
\mathbb{H}^{\gamma}_p(\tau) \subset \hat{\mathbb{H}}^{\gamma}_p(\hat{\tau}), \quad \mathbb{H}^{\gamma}_p(l_2, \tau) \subset \hat{\mathbb{H}}^{\gamma}_p(l_2, \hat{\tau}), \quad \mathbb{B}^{\gamma}_p \subset \hat{\mathbb{B}}^{\gamma}_p, \quad \hat{\mathbb{B}}^{\gamma}_p \subset \mathbb{B}^{\gamma}_p.
$$
Write
\[ A(t) = (a^{ij}(t)), \quad \bar{A}(t) := (\bar{a}^{ij}(t)) := \left( a^{ij}(t) - \frac{\kappa}{2} \delta^{ij} \right), \]
where \( \delta^{ij} \) denotes the Kronecker delta. Then
\[ A(t) = \left( \bar{A}(t) - \frac{\kappa}{2} I_d \times d \right) + \frac{\kappa}{2} I_d \times d = \bar{A}(t) + \frac{\kappa}{2} I_d \times d. \]

Recall that \( W'_t(\omega') \) and \( w^k_t(\omega) \) can be considered as Wiener processes relative to \( \tilde{\mathcal{F}}_t \). Take a \( d \times d \) symmetric matrix \( \bar{\sigma}(\omega, t) \) such that \( 2\bar{A} = (\bar{\sigma})^2 \). Then, since \( \bar{\sigma} \) can be considered as a predictable process defined on \( \Omega \times \Omega' \times [0, \infty) \), we can define the stochastic integral
\[ X_t := \int_0^t \bar{\sigma}(s) dW'_s. \]

Since \( L_p \)-norms are translation invariant, we have
\[ f(t, x + X_t) \in \mathbb{L}_p(\tau), \quad g(t, x + X_t) \in \mathbb{L}_p^1(\tau, l_2). \]

Therefore, by Lemma 4.3, the equation
\[ dv(t, x) = \left( \frac{\kappa}{2} \Delta v(t, x) + f(t, x + X_t) \right) dt + g^k(t, x + X_t) dw^k_t, \quad 0 \leq t \leq \tau \]
with initial data \( v(0, x) = u_0 \) has a unique solution
\[ v(t, x) = v(t, x - X_t), \quad t \geq 0. \]

Then, using (4.30) and \( L_p \)-continuity (i.e. \( \lim_{|y| \to 0} \| h(x - y) - h(x) \|_p = 0 \)), we conclude
\[ z(t, x) := v(t, x - X_t). \]

By the Itô-Wentzell formula (cf. [13 Theorem 1.1]), \( z \) satisfies the equation
\[ dz(t, x) = (a^{ij}(t)z_{x',x'}(t, x) + f(t, x)) dt + g^k(t, x) dw^k_t - z_{x'}(t, x) \bar{\sigma}^{ij} dW'^j_t, \quad t < \tau; \quad z(0, x) = u_0. \]

Let \( \hat{E} [\cdot | \tilde{\mathcal{F}}_t] \) denote the conditional expectation with respect to \( \tilde{\mathcal{F}}_t \). Note that
\[ \hat{E} \left[ \int_0^t z_{x'}(s, x) \bar{\sigma}^{ij} dW'^j_s | \tilde{\mathcal{F}}_t \right] = 0 \]
because the process \( W'_t \) is independent of \( (\mathcal{F}_r)_{r>0} \). Denote
\[ \bar{u}(t) := \hat{E} [z(t) | \tilde{\mathcal{F}}_t] \in \bigcap_{T>0} L_p(\Omega \times \Omega', \mathcal{F} \times \tilde{\mathcal{F}}^i; C([0, \tau \wedge T]; L_p)). \]
The inclusion above is due to conditional Jensen’s inequality. Then, by [22 Theorem 1.4.7], for each $t$,
\[
\mathbb{E} \left[ \int_0^t a^{ij}(s) z_{x^j}(s) \, ds \Big| \mathcal{F}_t \right] = \int_0^t \mathbb{E} \left[ a^{ij}(s) z_{x^j}(s) \right] \, ds \quad (a.s.)
\]
\[
= \int_0^t a^{ij}(s) \bar{u}_{x^j} \, ds \quad (a.s.).
\] (4.31)
Thus, taking the conditional expectation to equation (4.28) with respect to $\bar{F}$, using (4.29), (4.31), and (4.30), we conclude that $\bar{u}$ satisfies
\[
d\bar{u}(t, x) = (a^{ij}(t) \bar{u}_{x^j}(t, x) + f(t, x)) \, dt + g^k(t, x) \, dw^k, \quad 0 < t \leq \tau
\]
\[
\bar{u}(0, x) = u_0(x).
\]
In other words, both $u$ and $\bar{u}$ are solutions to (4.11) in the class $\cap_{t>0} L_p(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}'; C([0, \tau \wedge T]; L_p))$. By the uniqueness result of Theorem 4.1 we get $u = \bar{u}$. Therefore,
\[
\|u_{xx}\|_{L_p} = \|u_{xx}\|_{L_p(\tau)} = \|\bar{u}_{xx}\|_{L_p(\tau)} = \|z_{xx}\|_{L_p(\tau)} = \|v_{xx}\|_{L_p(\tau)}.
\]
This and (4.27) finish the proof of the lemma.

**Lemma 4.5.** Let $p \in [2, \infty)$, $\tau$ be a stopping time, $u_0 \in \mathbb{H}^{2(1-1/p)}_p$, $f \in \mathbb{L}^{\delta_1-p}_p$, $g \in \mathbb{H}^{\delta_1-p/2}_p$, and $u \in \bigcap_{t>0} L_p(\Omega, \mathcal{F}; C([0, \tau \wedge T]; L_p))$ be a solution to equation (4.1). Assume that coefficients $a^{ij}(t)$ are predictable,
\[
\int_0^\tau |a^{ij}(t)| \, dt < \infty \quad (a.s.)
\] (4.32)
for all $i, j \in \{1, \ldots, d\}$, and
\[
a^{ij}(t) \xi_i \xi_j \geq 0, \quad \forall (\omega, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^d.
\]
Then
\[
\|u_{xx}\|_{L_p(\tau, \delta)} \leq N(d, p) \left( \|u_0\|_{\mathbb{H}^{2(1-1/p)}_p} + \|f\|_{L_p(\tau, \delta_1-p)} + \|g_x\|_{L_p(\tau, \delta_1-p/2)} \right),
\] (4.33)
where $\delta(t)$ is the smallest eigenvalue of the matrix $(a^{ij}(t))$.

**Proof.** **Step 1.** First we assume
\[
u_{xx} \in \mathbb{L}_p(\tau, \delta) \cap \mathbb{L}_p(\tau),
\]
and there exists a positive constant $\varepsilon \in (0, 1]$ such that
\[
\delta(t) \geq \varepsilon > 0 \quad \forall t, \omega.
\] (4.34)
For $t > 0$, denote
\[
\beta(t) = \int_0^t \delta(s) \, ds,
\]
and let $\psi(t)$ be the inverse of $\beta(t)$. Then
\[
\psi(\beta(t)) = t
\]
and thus
\[
\psi'(\beta(t))\beta'(t) = \psi'(\beta(t))\delta(t) = 1, \quad \forall (\omega, t).
\] (4.35)
Since for each fixed \( \omega \in \Omega \), \( \beta(t) \) is a strictly increasing continuous function with respect to \( t \), we have
\[
\psi(t) = \inf \{ s \in [0, \infty) : \beta(s) > t \}.
\]

Thus for each \( \omega \), \( \psi(t) \) is a strictly increasing continuous function with respect to \( t \)
and
\[
\beta(t) = \inf \{ s \in [0, \infty) : \psi(s) > t \}.
\]

In particular, both \( \psi(t) \) and \( \beta(t) \) are stopping times. Define
\[
\tilde{\mathcal{F}}_t := \mathcal{F}_{\psi(t)}, \quad m_{\psi(t)}^k = w_{\psi(t)}^k.
\]

Then \( m_{\psi(t)}^k \) is a square integrable continuous martingale relative to \( \tilde{\mathcal{F}}_t \) such that
\[
[m_{\psi(t)}^k]_t = \psi(t), \quad d[m_{\psi(t)}^k]_t = \psi'(t)dt = \frac{1}{\delta(\psi(t))} dt.
\]

Thus there exist \( \tilde{\mathcal{F}}_t \)-adapted independent Wiener processes \( \tilde{\mathcal{W}}_t^k \) such that
\[
m_{\psi(t)}^k := w_{\psi(t)}^k = \int_0^t 1/\sqrt{\delta(\psi(s))}d\tilde{\mathcal{W}}_s^k.
\]

Recall that \( u \) is a solution to (4.1) and consider the function \( v(t, x) := u(\psi(t), x) \). Then \( v \) satisfies
\[
dv(t, x) = \left( a^{ij}(\psi(t))u_{x_i x_j}(\psi(t), x)v'(t) + f(\psi(t), x)v'(t) \right) dt + g^k(\psi(t), x)dw^k_{\psi(t)}
\]
with initial condition \( v(0, x) = u_0 \), where
\[
\tilde{a}^{ij}(t) = a^{ij}(\psi(t)) / \delta(\psi(t)),
\]
\[
\tilde{f}(t, x) = f(\psi(t), x) / \delta(\psi(t)),
\]
and
\[
\tilde{g}(t, x) = g(\psi(t), x) / \sqrt{\psi'(t)} = g(\psi(t), x) / \sqrt{\delta(\psi(t))}.
\]

Since \( \delta(\psi(t)) \) is the smallest eigenvalue of \( a^{ij}(\psi(t)) \),
\[
\tilde{a}^{ij}(t)\xi^i \xi^j = a^{ij}(\psi(t)) \frac{1}{\delta(\psi(t))} \xi^i \xi^j \geq |\xi|^2 \quad \forall \xi \in \mathbb{R}^d,
\]
i.e. the ellipticity constant of the coefficients \( \tilde{a}^{ij}(t) \) is 1. Thus by Lemma 4.3 a change of variables, and (4.35),
\[
\|u_{xx}\|_{L_p(\tau, \delta)} = \|v_{xx}\|_{L_p(\beta(\tau))}
\]
\[
\leq N(d, p) \left( \|u_0\|_{L_p^{2(1-1/p)}} + \|\tilde{f}\|_{L_p(\tilde{\beta}(\tau))} + \|\tilde{g}\|_{L_p(\tilde{\delta}(\tau), l_2)} \right)
\]
\[
= N(d, p) \left( \|u_0\|_{L_p^{2(1-1/p)}} + \|f\|_{L_p(\beta(\tau), l_2)} + \|g\|_{L_p(\delta(\tau), l_2)} \right).
\]

**Step 2.** Second, we only assume that
\[
u_{xx} \in L_p(\tau, \delta) \cap L_p(\tau).
\]

In other words, we remove condition (4.34) in this step. For \( \varepsilon > 0 \), denote
\[
a^{ij}_\varepsilon(t) = a^{ij}(t) + \varepsilon I, \quad \delta_\varepsilon(t) := \delta(t) + \varepsilon.
\]
Then \( u \) satisfies
\[
du = (a^{ij}(t)u_{x^ix^j} + f - \varepsilon \Delta u) \, dt + g^k \, dw^k_t, \quad 0 < t \leq \tau, \\
u(0, x) = u_0(x).
\]

By Step 1 and the inequalities that \( \delta_p^1 \leq \delta_p^1 - \varepsilon \) and \( \delta_p^1 - \varepsilon/2 \leq \delta_p^1 - \varepsilon/2 \),
\[
\mathbb{E} \int_0^\tau \| u_{xx}(t, \cdot) \|^p \delta(t) \, dt \\
\leq N \| u_0 \|_{L_p}^p + N \mathbb{E} \int_0^\tau \left( \| \frac{1}{\delta(t)}(f - \varepsilon \Delta u)(t, \cdot) \|_{L_p}^p + \left\| \frac{1}{\sqrt{\delta(t)}} g_x(t, \cdot) \right\|_{L_p}^p \right) \delta(t) \, dt \\
\leq N \| u_0 \|_{L_p}^p + N \left( \| f \|_{L_p(\tau, 1)}^p + \| g \|_{L_p(\tau, 1)}^p \right) \mathbb{E} \int_0^\tau \| \Delta u(t, \cdot) \|_{L_p}^p \mathbb{E} \varepsilon \| \delta(t) \|_{L_p}^p \, dt.
\]

Observe that
\[
\varepsilon \delta_p^1 - \varepsilon \leq 1
\]
and recall \( u_{xx} \in L_p(\tau) \). Thus as \( \varepsilon \downarrow 0 \), we have
\[
\mathbb{E} \int_0^\tau \| \Delta u(t, \cdot) \|_{L_p}^p \delta_p^1 - \varepsilon \, dt \to 0.
\]

Therefore by Fatou’s lemma and (4.37), we finally obtain (4.33).

**Step 3.** (General case). Finally we remove condition (4.36) in this step. Consider the mollification \( u^\varepsilon(t, x) \) used in the proof of Theorem 3.3. Then \( u^\varepsilon \) satisfies
\[
du^\varepsilon = (a^{ij}(t)u^\varepsilon_{x^ix^j} + f^\varepsilon(t, x)) \, dt + (g^k)^\varepsilon(t, x) \, dw^k_t, \quad 0 < t \leq \tau, \\
u^\varepsilon(0, x) = u_0^\varepsilon(x).
\]

Since \( \delta(t) \) is the smallest eigenvalue of \( a^{ij}(t) \), by applying Young’s convolution inequality and (4.32),
\[
\int_0^\tau \| u^\varepsilon_{x^ix^j}(s, \cdot) \|_{L_p}^p \delta(s) \, ds \\
\leq \sup_{s \leq \tau} \| u(s, \cdot) \|_{L_p}^p \| \phi^\varepsilon_{x^ix^j} \|_{L_1}^p \| \delta \|_{L_1} \int_0^\tau a^{11}(s) \, ds < \infty \quad (a.s.)
\]
for all \( i, j \). Similarly,
\[
\| u^\varepsilon_{xx} \|_{L_p(\tau \wedge T)} < \infty
\]
for all \( T > 0 \). Then denoting
\[
\tau_n = \inf \left\{ s < \tau \wedge T : \sum_{i,j} \int_0^s \| u^\varepsilon_{x^ix^j}(s, \cdot) \|_{L_p}^p \delta(s) \, ds > n \right\},
\]
we have
\[
u^\varepsilon_{xx} \in L_p(\tau_n, \delta) \cap L_p(\tau_n).
\]
and \( \lim_{n \to \infty} \tau_n = \tau \) (a.s.). Thus by Step 2, for all \( n \in \mathbb{N}, \varepsilon_1, \varepsilon_2 > 0, \)
\[
\|u_{\varepsilon_1}^x - u_{\varepsilon_2}^x\|_{L_p(\tau_n, \delta)} \\
\leq N(d, p) \left( \|u_{\varepsilon_1}^x - u_0^x\|_{L_p(\tau - \varepsilon_1, \delta)} + \|f_{\varepsilon_1} - f_{\varepsilon_2}\|_{L_p(\tau, \delta^1 - \varepsilon)} + \|g_{\varepsilon_1}^x - g_{\varepsilon_2}^x\|_{L_p(\tau, \delta^1 - \varepsilon, l_2)} \right),
\]
and
\[
\|u_{\varepsilon_1}^x\|_{L_p(\tau_n, \delta)} \\ \leq N(d, p) \left( \|u_0\|_{L_p(\tau - \varepsilon_1, \delta)} + \|f\|_{L_p(\tau, \delta^1 - \varepsilon)} + \|g_x\|_{L_p(\tau, \delta^1 - \varepsilon, l_2)} \right).
\]
Finally by Fatou’s lemma and the approximation argument, we obtain (4.33). The lemma is proved.

\[\square\]

5. PROOF OF THEOREM 2.7

Since \((1 - \Delta)^{\gamma/2}\) is an isometry both on Sobolev spaces and Besov spaces, we may assume that \(\gamma = 0\).

First observe that for any \(\phi \in C_c^\infty(\mathbb{R}^d)\) and \(u \in L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p))\),
\[
\int_0^\tau \sum_{k=1}^\infty |\sigma^{ik}(t)|^2 \int_{\mathbb{R}^d} u(t, x) \phi_{x^k}(x) dx|^2 dt \\
\leq N(d) \int_0^\tau |\sigma^2(t)||u(t, \cdot)|^2_{L_p} \|\phi_x\|_{L_q}^2 dt \\
\leq N(d) \|\phi_x\|_{L_q}^2 \int_0^\tau |\sigma(t)|^2 dt \left( \sup_{t \in [0, \tau]} \|u(t, \cdot)\|_{L_p(l_2)} \right)^2 < \infty \quad \text{(a.s.),}
\]
where \(q = \frac{p}{p-1}\). Thus
\[
\int_0^t \sigma^{ik}(u_{x^k}, \phi) dw_t^k := - \int_0^t \sigma^{ik}(u, \phi_x) dw_t^k
\]
is well-defined. Denote
\[
x_t^k = \int_0^t \sigma^{ik}(s) dw_s^k, \quad x_t = (x_1^1, \ldots, x_d^d).
\]
By the Itô-Wentzell formula (cf. [13 Theorem 1.1]), \(u(t, x)\) is a solution to (2.1) if and only if \(v(t, x) = u(t, x - x_t)\) is a solution to the equation
\[
dv = (\alpha^{ij}(t)u_{x^j} + f(t, x - x_t) - g_{x^k}^\ell(t, x - x_t) \sigma^{ik}(t)) dt \\
+ g^k(t, x - x_t) dw_t^k, \quad 0 < t \leq \tau; \quad v(0, x) = u_0. \tag{5.1}
\]
By the assumption that \(g_x \in L_p(\tau, |\sigma|^p, l_2) \cap L_p(\tau, |\sigma|^{p, \delta^1 - p}, l_2)\),
\[
\mathbb{E} \int_0^\tau \int_{\mathbb{R}^d} |g_{x^k}^\ell(t, x - x_t) \sigma^{ik}(t)|^p dx dt + \mathbb{E} \int_0^\tau \int_{\mathbb{R}^d} |g_x^k(t, x - x_t) \sigma^{ik}(t)|^p \delta^{1-p} dx dt
\]
\[
\leq \mathbb{E} \int_0^\tau \int_{\mathbb{R}^d} |g_{x^k}^\ell(t, x)|^p dx \sigma(t)|^p dt + \mathbb{E} \int_0^\tau \int_{\mathbb{R}^d} |g_x|_l^p(t, x) dx \sigma(t)|^p \delta^{1-p} dt < \infty.
\]
Thus
\[
g_{x^k}^\ell(t, x - x_t) \sigma^{ik}(t) \in L_p(\tau) \cap L_p(\tau, \delta^{1-p}),
\]
and by Theorem 4.4 and Lemma 4.3, there exists a unique solution \(v\) to (5.1) such that
\[
v \in L_p(\Omega, \mathcal{F}; C([0, \tau]; L_p)),
\]
\[ E \sup_{t \in [0,T]} \| v(t, \cdot) \|_{L^p} \leq N(p, T) \left( \| f \|_{L^p(\tau)} + \| g \|_{L^p(\tau, \gamma)} + \| g_x \|_{L^p(\tau, |\gamma|^p, \gamma)} + \mathbb{E} \| u_0 \|_{L^p}^p \right), \]

and
\[ \| v_{xx} \|_{L^p(\tau, \gamma)} \leq N(d, p) \left( \| u_0 \|_{L^2(1-1/n)} + \| f \|_{L^p(\tau, \gamma^1 - p)} + \| g_x \|_{L^p(\tau, |\gamma|^1 - p, \gamma)} + \| g_x \|_{L^p(\tau, \gamma^1 - p/2, \gamma)} \right). \]

Therefore, \( u(t, \cdot) := v(t, x + \sigma_t) \in L^p(\Omega, \mathcal{F}; C ([0, \tau]; L^p)) \) becomes a unique solution to equation (2.1) and satisfies (2.11) and (2.12). The theorem is proved. \( \square \)

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