Some inferences based on a mixture of power function and continuous logarithmic distribution

Ahmed M. T. Abd El-Bar a, Maria do Carmo S. Lima b and M. Ahsanullah c

aDepartment of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt; bDepartment of Statistics, Federal University of Pernambuco, Brazil; cDepartment of Management Sciences, Rider University, Lawrenceville, USA

ABSTRACT

In recent decades, many families of distributions and, consequently, new distributions are proposed in order to provide a good flexibility and fit to real data sets. However, several of these distributions have a complicated shape to express their probability density function (pdf) and cumulative distribution function (cdf). Examples of families that involve such functions are: beta-G (see Eugene et al. Beta normal distribution and its applications. Commun Stat - Theory Methods. 2006;31:497–512, for example) and gamma-G (for more details, see Nadarajah et al. The Zografos–Balakrishnan-G family of distributions: mathematical properties and application. Commun Stat - Theory Methods. 2015;44:186–215) families. In this sense, we introduce a new bouded distribution by using a mixture of power function and continuous logarithmic distribution, named as the power logarithmic (PL) distribution, that has a simple form in the expressions of its pdf and cdf. Various statistical and mathematical properties of the new model are obtained in closed form, which is a very positive aspect when we propose a new model. Based on the basic properties, two new characterizations of the new model will be given. Finally, the applicability of PL model to modelling real data is proved by two real data sets, showing the good fit of the new distribution, when compared with others already known in the literature.

1. Introduction

The power function is a simple distribution model. It is a special case of the beta model. It is one of the distributions to study the reliability of electric devices (see Meniconi [1]). This simple distribution is preferred by most of the engineers to find future rates and reliability info over other distributions. Many authors have studied various aspects of the power function distribution (see Meniconi [1], Rider [2], Lwin [3] and Arnold and Press [4]). Also, Kabir and Ahsanullah [5] obtained estimation location and scale parameter of power function model. Further, Ahsanullah [6] gave characterizations of the power function model. Tahmasbi and Rezaei [7] introduced a new two parameter decreasing failure rate distribution by mixing exponential and logarithmic distributions. Also, Athar and Abdel-Aty [8] studied characterization of general class of distributions by truncated moments. The statistical literature contain many extended forms of the distribution. For example, McDonald modified Burr-III distribution (Mukhtar et al. [9]) and weighted exponential Gompertz distribution (Abd El-Bar and Ragab [10]).

In recent years, many distributions have been proposed in order to adjusting several type of data. The odd log-logistic-Stacy distribution is proposed by Prataviera et al. [11] and deals with the regression model of that distribution with applications in survival analysis. Although the proposed distribution has bimodality with one of the forms of its probability density function (pdf), this distribution has a big expression for the pdf that involves complicated functions as gamma and incomplete gamma ratio functions.

Handique et al. [12] introduce the beta generated Kumaraswamy-G family of distributions that is a generalization of the Kumaraswamy-G family. Again, despite having some flexibility in its hazard rate function (hrf), the expression of pdf and cdf involve the beta and incomplete beta ratio functions.

The generalized Kumaraswamy-G family of distributions, proposed by Nofal et al. [13] and which extends the Kumaraswamy-G family, has simple expressions for the pdf and cdf and presented good flexibility in their pdf and hrf. However, the mathematical properties are not obtained in closed forms.

Bhatti et al. [14] introduce a new family called Burr III Marshall Olkin family and present some special submodels. Some of them have a bimodality shape in their pdf. However, like the other families mentioned above, the expressions for the mathematical properties of that family are not expressed in closed form.

In this sense, the goal of this paper is to propose a model, more flexible than the power function...
distribution, with support in (0,1) and having three parameters, that performs well in these situations.

Besides that we introduced a distribution that extends the power function model, but which does not have complicated functions in the forms of its pdf and cumulative distribution function (cdf), as can be seen in the following section.

The article is sketched into the following sections. We describe the new model and some of its important features in Section 2. Various statistical properties of the power logarithmic (PL) model are derived in Section 3, including moment generating function, moments, skewness, kurtosis and the distributions sum, products and ratio. Also, we discuss the residual life random variables for the PL model. In Section 4, we investigate Rényi and Shannon entropy. The maximum likelihood estimators of the PL parameters, asymptotic and expected information matrix are discussed in Section 5. The results for the new characterizations of the PL model by using truncated moments are obtained in Section 6. Section 7 is related to order statistics. We provide a simulation study in order to verify the asymptotic properties of the parameters vector, varying the true parameter vector and the sample size n in Section 8. Two real data applications for the PL model discuss in Section 9, while the concluding remarks are presented in Section 10.

2. The proposed model

A random variable $X$ is said to have a PL distribution if the cdf is

$$F(x) = \frac{x^{\alpha+1}[\beta + (\alpha + 1)\beta - \beta \ln(x)]}{\beta + \delta + \alpha \beta},$$

$$0 \leq x \leq 1, \quad \alpha, \beta, \delta > 0 \quad (1)$$

where $\alpha$ is a shape parameter and $\beta, \delta$ are scale parameters. The corresponding pdf takes the form

$$f(x) = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} x^\alpha \beta - \beta \ln(x)). \quad (2)$$

The pdf (2) can be defined as a two-component mixture

$$f(x) = p f_\beta(x) + (1 - p) f_\delta(x),$$

where $p = \frac{(\alpha + 1)\beta}{\delta + (\alpha + 1)\beta}$, $f_\beta(x) = (\alpha + 1)x^\alpha; 0 \leq x \leq 1$ is the pdf of power function distribution and $f_\delta(x) = (\alpha + 1)^2 x^\alpha \ln \left( \frac{1}{\delta} \right); 0 \leq x \leq 1$ is the pdf of logarithmic distribution.

The survival and hazard rate (hr) functions of $X$ are given, respectively, by

$$S(x) = \frac{x^{\alpha+1}}{\beta + \delta + \alpha \beta} \times \left[ (\alpha + 1)\delta \ln(x) - (\beta(\alpha + 1) + \delta)(1 - x^{-\alpha-1}) \right], \quad (3)$$

$$h(x) = \frac{(\alpha + 1)^2 (\beta - \delta \ln(x))}{x[(\alpha + 1)\delta \ln(x) - (\beta(\alpha + 1) + \delta)(1 - x^{-\alpha-1})]}. \quad (4)$$

If a random variable has cdf given by Equation (1) with parameters $\alpha, \beta$ and $\delta$, we denoted it by $\text{PL} (\alpha, \beta, \delta)$. Besides that Equations (1) and (2) do not involve any complicated function. It is a good point of this model.

Following the Qian idea (see Qian [15]), from Equation (4), we have

$$\ln[h(x)] = 2\ln(\alpha + 1) + \ln(\beta - \delta \ln(x))$$

$$- \ln(x) - \ln(\alpha + 1)\delta \ln(x)$$

$$- (\beta(\alpha + 1) + \delta)(1 - x^{-\alpha-1})].$$

Taking derivative, we obtain

$$\frac{h'(x)}{h(x)} = -\frac{\delta}{x[\beta - \delta \ln(x)]} - \frac{1}{x}$$

$$= \frac{((\alpha + 1)\delta)/(x) - (\beta(\alpha + 1) + \delta)(\alpha + 1)x^{-\alpha-2}}{(\alpha + 1)\delta \ln(x) - (\beta(\alpha + 1) + \delta)(1 - x^{-\alpha-1})}.$$ 

So, we can rewrite the last equations as $h'(x) = h(x) s(x)$, where

$$s(x) = \frac{-\delta}{x[\beta - \delta \ln(x)]} - \frac{1}{x}$$

$$= \frac{(((\alpha + 1)\delta)/(x) - (\beta(\alpha + 1) + \delta)(\alpha + 1)x^{-\alpha-2}}{(\alpha + 1)\delta \ln(x) - (\beta(\alpha + 1) + \delta)(1 - x^{-\alpha-1})}.$$ 

The sign of $h'(x)$ is the same as the sign of $s(x)$ since $h(x) > 0$. Thus, we have two important shapes to the hrf if we analyze the plot of the function $s(.)$ as presented in Figure 1.

Since our proposed model has only one shape parameter ($\alpha$), the shapes of the hazard depend on this parameter. It is easy to verify that if $\alpha > 0.2$, the hazard is increasing (solid line in Figure 1) and otherwise, the hazard has the bathtub shape (dotted line in Figure 1).

The pdf and hrf for different parameter values are shown in Figure 2. From Figure 2, we note that the plot (Figure 2(a)) indicates how the three parameter $\alpha$, $\beta$ and $\delta$ affect on the PL density and show that the density can take various forms, which are increasing, decreasing, constant, right-skewed, left-skewed and upside-down bathtub stapled. While the plot (Figure 2(b)) represents

**Figure 1.** The graph of $S(x)$ for two defined regions: (solid line) Region I and (dotted line) Region II.
Lemma 2.1: Let

\[\psi(z; r, \alpha, \beta, \delta)\]

\[= \int_0^z x^{r} f(x) \, dx = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} \]
\[\times \int_0^z x^{r+1} (\beta - \delta \ln(x)) \, dx, \quad r = 1, 2, \ldots,\]

hence, we have

\[\psi(z; r, \alpha, \beta, \delta)\]

\[= \frac{(\alpha + 1)^2}{(\beta + \delta + \alpha \beta) (\alpha + r + 1)^2} \]
\[\times [\delta + (\alpha + r + 1)(\beta - \delta \ln(z))] z^{r+1}.\]

And the used integrations are

\[\int_0^1 x^\gamma \ln(x) (\beta - \delta \ln(x)) \, dx\]

\[= \frac{\beta}{\beta + \delta + \alpha \beta} - \frac{2}{\alpha + 1}.\]

and

\[\int_0^1 (\alpha + 1)^2 x^\alpha (\beta - \delta \ln(x)) (\beta - \delta \ln(x)) \, dx\]

\[\delta - \delta e^\gamma \left[ \frac{(\alpha + 1)^2}{(\beta + \delta + \alpha \beta) (\beta + \delta + \alpha \beta)} \right] \]

Finally, the used special functions are

(i) The exponential integral function defined by

\[Ei(z) = -\int_z^\infty \frac{e^{-u} \, du}{u}\]

Table 1. Special cases of the PL distribution.

| Submodel | Parametric values in Equation (2) |
|----------|----------------------------------|
| Power function distribution \((\alpha)\) | \(\delta = 0\) |
| Logarithmic distribution \((\beta, \delta)\) | \(\alpha + 1 = \sigma\) and \(\delta = 1\) |
| Transformed gamma distribution \((\sigma)\) | \(\alpha + 1 = \sigma, \beta = 0\) and \(\delta = 1\) |
| Weighted log-Lindley distribution \((\alpha, \beta, c = 1)\) [New] | \(\delta = 1\) |

Figure 2. Plots of the PL density and hazard functions. (a) \(\alpha = 0.05, \beta = 0.1, \delta = 0.1\) (black), \(\alpha = 0.5, \beta = 0.3, \delta = 0.4\) (red), \(\alpha = 0.6, \beta = 0.1, \delta = 20\) (dashes-yellow), \(\alpha = 1, \beta = 1, \delta = 0.01\) (yellow), \(\alpha = 2, \beta = 0.9, \delta = 0.5\) (blue), \(\alpha = 3, \beta = 0.1, \delta = 5\) (pink), \(\alpha = 5, \beta = 0.001, \delta = 20\) (dashes-green). (b) \(\alpha = 0.01, \beta = 0.01, \delta = 5\) (black), \(\alpha = 0.003, \beta = 0.02, \delta = 0.1\) (red), \(\alpha = 0.01, \beta = 5, \delta = 8\) (yellow), \(\alpha = 2, \beta = 0.9, \delta = 0.5\) (blue), \(\alpha = 0.5, \beta = 0.3, \delta = 0.4\) (dashes-green).
(ii) The hyperbolic sine integral function, often called “Shi function” is defined by 
\[ \text{Shi}(z) = \int_0^z \sinh t \, dt, \]
where \( \gamma \) is the Euler constant. It has the series expansion
\[ \text{Shi}(z) = \gamma + \ln(z) + \frac{z^2}{4} + \frac{z^4}{96} + \frac{z^6}{4320} + \frac{z^8}{322560} + \frac{z^{10}}{36288000} + O(z^{12}). \]

3. Mathematical properties

In this section, we introduce some important mathematical properties for the PL model, including moment generating function, moments, skewness, kurtosis, the distributions of sums, products and ratios and residual lifetime moments.

3.1. Moments

The moment-generating function (mgf) and the rth moment for the PL distribution can be defined as
\[ M_X(t) = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} \sum_{k=0}^{\infty} \frac{k!}{\beta^k} (1 + k + \alpha)\beta + \delta, \]
and
\[ \mu_r = E(X^r) = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} \frac{(1 + r + \alpha)\beta + \delta}{(1 + k + \alpha)^2}, \]
respectively. In particular
\[ \mu_1 = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} (\alpha + 2) = \mu_2, \]
\[ \mu_3 = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} (\alpha + 4) = \mu_4, \]
\[ \mu_5 = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} (\alpha + 5) = \mu_6, \]
and
\[ V(X) = \frac{(\alpha + 1)^2}{\beta + \delta + \alpha \beta} \left[ \frac{(\alpha + 3)\beta + \delta}{(\alpha + 3)^2} - \frac{(\alpha + 1)^2((\alpha + 2)\beta + \delta)}{(\alpha + 2)^4} \right]. \]

Now, the skewness \( (\gamma_1) \) and kurtosis \( (\gamma_2) \) can be obtained using the following relations
\[ \gamma_1 = \frac{\mu_3^2}{\sqrt{\mu_2(\mu_4)}}, \]
and
\[ \gamma_2 = \frac{\mu_4^2}{\mu_2} - 3. \]

where
\[ \mu_3 = \mu_2^2 - 3, \quad \mu_4 = \mu_2^3 + 6(\mu_2^2) \mu_2 - 3(\mu_2^3). \]

In Table 2, we introduce the mean, variance, \( \gamma_1 \) and \( \gamma_2 \) for different values of the parameters \( \alpha, \beta, \) and \( \delta. \) It is observed that the mean of PL increases as \( \alpha \) increases, while the variance decreases as \( \alpha \) increases for fixed \( \beta \) and \( \delta. \) Also, for fixed \( \alpha \) and \( \delta, \) the mean and the variance increase as \( \beta \) increases. Further, the mean and the variance decrease by increasing \( \delta \) for fixed \( \alpha \) and \( \beta. \)

Additionally, Table 2 clearly shows that for fixed \( \beta \) and \( \delta, \) the skewness decreases as \( \alpha \) increases, while the kurtosis increases as \( \beta \) increases for fixed \( \alpha \) and \( \delta. \) Also, Table 2 reveals that for fixed \( \alpha \) and \( \beta, \) the skewness increases as \( \delta \) increases while the kurtosis increases as \( \delta \) decreases. Hence, Table 2 indicates that \( \alpha, \beta \) and \( \delta \) affect the shape of the PL model.

3.2. Sums, products and ratios

In this subsection, we derive the exact distributions of sums, products and ratios of PL variables.

Let \( X_i, i = 1, 2 \) be independent random variables having the PL model with parameters \( (\alpha_i, \beta_i, \delta_i), i = 1, 2. \) Then
\[ f_5(s) = \left( \frac{(\alpha_1 + 1)^2(\alpha_2 + 1)^2}{(\beta_1 + \delta + \alpha_1 \beta_1)(\beta_2 + \delta + \alpha_2 \beta_2)} \times \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{\mu_2^k} s^{\alpha_1 + \alpha_2 + k} \right) \right) \times \left[ (-\mu \delta_1 \delta_2 \psi(1, 2 + k + \alpha_2) + \delta_1 \delta_2 \beta_1 \psi(0, \nu) + \frac{1}{\nu} (\psi(1, 1 + \gamma \delta_1 - \nu \delta_1 \log s)(\nu \beta_2 + \delta_2 - \nu \delta_2 \log s)). \right] \]

| \( \alpha \) | \( \beta = 0.5 \) | \( \delta = 2 \) | \( \gamma_1 \) | \( \gamma_2 \) |
|---|---|---|---|---|
| 0.2 | 0.35473 | 0.068874 | 0.597127 | 2.33322 |
| 0.5 | 0.42545 | 0.068873 | 0.331014 | 2.0667 |
| 0.8 | 0.48451 | 0.066990 | 0.129401 | 1.98737 |
| 1.2 | 0.54889 | 0.061003 | 0.078966 | 2.0194 |
| 1.9 | 0.63306 | 0.051033 | 0.365258 | 2.11428 |
| 0.6 | 0.38181 | 0.060015 | 0.457445 | 2.27627 |
| 0.9 | 0.45138 | 0.067644 | 0.365258 | 2.11428 |
| 1.3 | 0.54889 | 0.070091 | 0.29881 | 2.02704 |
| 1.9 | 0.73536 | 0.073536 | 0.223721 | 1.95198 |
| 1.9 | 0.73536 | 0.073536 | 0.223721 | 1.95198 |
| 2.7 | 0.43849 | 0.070091 | 0.301119 | 2.02973 |
| 3.5 | 0.42127 | 0.069203 | 0.345646 | 2.08607 |
where 0 < s < 1, \( \nu = \alpha_2 + k + 1 \), \( \gamma \) is Euler’s constant with numerical value = 0.577216 and \( \psi(n, z) \) is the \( n \)th derivative of the digamma function \( \psi^{(n)}(z) \).

\[ f_\nu(p) = \frac{(\alpha_1 + 1)^2(\alpha_2 + 1)^2p^{\alpha_1}}{\rho^3\beta_1 + \delta_1 + \alpha_1\beta_1(\beta_2 + \delta_2 + \alpha_2\beta_2)} \times [(\alpha_1\beta_1 - \alpha_2\beta_1 + 2\delta_1)\delta_2 + \rho\delta_2(\alpha_1\beta_2 - \alpha_2\beta_2 - \delta_2)\log p - \rho\beta_2(\alpha_1\beta_2 - \alpha_2\beta_1 + \delta_1)], \]

where 0 < \( \rho < 1 \), \( \rho = (\alpha_1 - \alpha_2) \), and

\[ f_\nu(r) = \frac{(\alpha_1 + 1)^2(\beta_2 + 1)^2r^{\beta_2}}{\xi^3(\beta_1 + \alpha_1\beta_1)(\beta_2 + \delta_2 + \alpha_2\beta_2)}, \]

where 0 < \( r < 1 \), \( \xi = (\alpha_1 + \alpha_2 + 2) \).

**Remark 1:** The results above come directly using the known definitions of

\[ f_5(s) = \int_0^s f(s - x) f(x) dx, \]

\[ f_\nu(p) = \int_0^1 \frac{1}{x} f\left(\frac{p}{x}\right) f(x) dx \quad \text{and} \]

\[ f_\nu(r) = \int_0^1 xf(x) f(x) dx. \]

### 3.3. Residual life function

Some measures of residual lifetime of the PL distribution are obtained in this section, including survival, density, hrf, mean, and variance.

The survival function of the residual lifetime \( \xi_1 \) for the PL model is

\[ S_{\xi_1}(x) = \frac{S(x + t)}{S(t)} = \frac{[(\alpha_1 + 1)\delta \ln(x + t) - (\delta + \beta(\alpha_1 + 1))]}{[(\alpha_1 + 1)\delta \ln(t) - (\delta + \beta(\alpha_1 + 1))(1 - t^{-\alpha - 1})]} \]

\[ \times \frac{(x + t)^{\alpha + 1}}{t^{\alpha + 1}}, \quad 0 \leq x \leq 1. \]

The corresponding pdf of \( \xi_1 \) takes the form

\[ f_{\xi_1}(x) = \frac{(\alpha_1 + 1)^2(\beta - \delta \ln(x + t))}{[(\alpha_1 + 1)\delta \ln(t) - (\delta + \beta(\alpha_1 + 1))(1 - t^{-\alpha - 1})]} \times \frac{(x + t)^{\alpha + 1}}{t^{\alpha + 1}}. \]

Based on Equations (9) and (10), the hrf of \( \xi_1 \) is

\[ h_{\xi_1}(x) = \frac{(\alpha_1 + 1)^2(\beta - \delta \ln(x + t))}{[(\alpha_1 + 1)\delta \ln(x + t) - (\delta + \beta(\alpha_1 + 1))]} \times \frac{(x + t)^{\alpha + 1}}{t^{\alpha + 1}}. \]

The mean residual lifetime (MRL) \( \xi_1 \) for the PL model is

\[ \Omega(t) = E(\xi_1) = \frac{1}{S(t)} \int_0^t x f(x) dx - t \]

\[ = \frac{1}{S(t)} [\mu_1 - \psi(t; 1, \alpha, \beta, \delta)] - t, \quad t \geq 0, \]

where \( \mu_1 \) can be obtained using (8), and \( \psi(t; 1, \alpha, \beta, \delta) \) is obtained by Lemma 2.1 for \( r = 1 \).

Additionally, the variance residual lifetime \( \xi_1 \) for the PL model is defined by

\[ \Xi(t) = \text{Var}(\xi_1) = \frac{2}{S(t)} \int_0^t x S(x) dx - 2 \cdot t \cdot (\Omega(t))^2 \]

\[ = \frac{1}{S(t)} \left[ \mu_2 - \psi(t; 2, \alpha, \beta, \delta) \right] - t^2 - 2 \cdot t \cdot (\Omega(t))^2, \]

where \( \mu_2 \) is given by (8), and \( \psi(t; 2, \alpha, \beta, \delta) \) is obtained by Lemma 2.1 for \( r = 2 \). The pdf of \( \xi_1 \) and MRL for the PL model are shown in Figure 3. These plots present the possible shapes of the pdf of \( \xi_1 \). Also, we note that the MRL decreases with increasing the time \( t \).

### 4. Entropy

In this section, we introduce the Rényi and Shannon entropy for the PL distribution.

**Theorem 4.1:** Let \( X \) have the pdf given by Equation (2).

Then, the Rényi and Shannon entropy of \( X \) are

\[ \sigma(\gamma) = \frac{1}{1 - \gamma} \left[ 2\gamma \log(\alpha + 1) + (\gamma + 1) \log(\beta) \right. \]

\[ - \log(\delta) - \gamma \log(\beta + \delta + \alpha \beta) + \frac{\beta + \alpha \beta \gamma}{\delta} + \log \left( \frac{E\left[ -\gamma, \frac{\beta + \alpha \beta \gamma}{\delta} \right] \right) \]

and

\[ \eta_x = E[-\log f(x)] = \log \left( \frac{\beta + \delta + \alpha \beta}{(\alpha + 1)^2} \right) \]

\[ + \frac{2\alpha}{\alpha + 1} - \log(\beta) - \frac{\alpha \beta}{\beta + \delta + \alpha \beta} \]

\[ + \delta \frac{e^{\frac{(\alpha + 1)\beta}{\delta}} - 1}{\beta + \delta + \alpha \beta} - \delta \left[ E\left[ -\frac{(\alpha + 1)\beta}{\delta} \right] \right], \]

respectively.

**Proof:** The Rényi entropy and Shannon entropy are defined as \( \sigma(\gamma) = \frac{1}{1 - \gamma} \log \left( \int f^\gamma(x) dx \right) \) and \( \eta_x = E[-\log f(x)] \), respectively, where \( \gamma > 0 \) and \( \gamma \neq 1 \). Then, the proof comes directly using Equations (5)–(7). \( \blacksquare \)
5. Statistical inferences

Here, we discuss the maximum likelihood estimates (MLEs) of the parameters of PL model and construct the expected Fisher’s information matrix.

5.1. Maximum likelihood estimates

The log-likelihood function for our distribution parameters from a size $n$ is

$$
\ell = 2n \log(\alpha + 1) - n \log(\beta + \delta + \alpha \beta) + \alpha \sum_{i=1}^{n} \log(x_i) + \frac{n}{\beta - \delta \ln(x_i)}. (12)
$$

Differentiating Equation (12) with respect to $\alpha$, $\beta$, and $\delta$, we have the following equations:

$$
\frac{\partial \ell}{\partial \alpha} = 2n \left( \frac{1}{\alpha + 1} - \frac{n \beta}{\beta + \delta + \alpha \beta} \right) + \frac{\sum_{i=1}^{n} \log(x_i)}{\beta - \delta \ln(x_i)} = 0, \quad (13)
$$

$$
\frac{\partial \ell}{\partial \beta} = -\frac{n(\alpha + 1)}{\beta + \delta + \alpha \beta} + \frac{1}{\beta - \delta \ln(x_i)} = 0, \quad (14)
$$

$$
\frac{\partial \ell}{\partial \delta} = -\frac{n}{\beta + \delta + \alpha \beta} - \frac{\sum_{i=1}^{n} \log(x_i)}{\beta - \delta \ln(x_i)} = 0. \quad (15)
$$

The MLEs $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\delta}$ of $\alpha$, $\beta$, and $\delta$, respectively, can be obtained by solving the above nonlinear equations numerically using the statistical package Mathematic package.

However, note that from Equation (13), we can obtain a semi closed form to the MLE of $\alpha$. Let $T = (\alpha + 1)(\beta + \delta + \alpha \beta) \sum_{i=1}^{n} \log(x_i)$. Thus, we can write:

$$
\hat{\alpha}(\beta, \delta) = \frac{T - 3n\beta - 2n\delta}{3n\beta}. (16)
$$

5.2. Fisher’s information matrix

Here, we construct the expected Fisher’s information matrix. First, we introduce the following lemma to help us to provide Fisher’s information matrix.

**Lemma 5.1**: Let $X$ have the pdf (2). Then, the expectations of $\frac{1}{\beta - \delta \ln(x)}$, $\frac{\log(x)}{\beta - \delta \ln(x)}$, and $\frac{\log^2(x)}{\beta - \delta \ln(x)}$ are, respectively, given by

$$
(i) \quad E \left[ \frac{1}{\beta - \delta \ln(x)} \right] = \frac{(\alpha + 1)^2}{\delta^{(\beta + \delta + \alpha \beta)}} - \frac{\alpha + 1}{\delta}, \quad (17)
$$

$$
(ii) \quad E \left[ \frac{\log(x)}{\beta - \delta \ln(x)} \right] = -\frac{\alpha + 1}{\delta} - \left( \frac{(\alpha + 1)^2}{\delta} \right) \left( \frac{\delta}{\delta - (\alpha + 1)\delta} \right), \quad (18)
$$

$$
(iii) \quad E \left[ \frac{\log^2(x)}{\beta - \delta \ln(x)} \right] = -\frac{(\alpha + 1)^2}{\delta} + \frac{\delta^2}{\delta - (\alpha + 1)\delta} \left( \frac{(\alpha + 1)^2}{\delta} \right) - \frac{\delta(\alpha + 1)^2}{\delta - (\alpha + 1)\delta}, \quad (19)
$$

The expected Fisher’s information matrix for sample size $n$ is given by

$$
J = -E \left( \begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \delta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \delta} \\ \frac{\partial^2 \ell}{\partial \delta \partial \alpha} & \frac{\partial^2 \ell}{\partial \delta \partial \beta} & \frac{\partial^2 \ell}{\partial \delta^2} \end{bmatrix} \right) = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},
$$

where each element can be found in Appendix 1.

6. Characterization of PL model

Here, we derive two characterization of the PL model by truncated moment. For this, we will provide the following assumption and lemmas to prove our characterization theorems.

**Assumption 6.1**: Suppose the random variable $X$ is absolutely continuous with cdf $F(x)$ and pdf $f(x)$. Also, we assume $E(X)$ and $f(x)$ is differentiable. We further assume that $\alpha = \text{Sup}\{x : F(x) > 0\}$ and $\beta = \text{inf}\{x : F(x) < 1\}$.
Lemma 6.1: Under Assumption 6.1, If \( E(X|X \leq x) = g(x) \frac{f(x)}{F(x)} \), where \( g(x) \) is a continuous differentiable function in \((\alpha, \beta)\), then \( f(x) = c e^{\frac{g(x) - g(x)}{\beta - \delta \ln(x)}} \), where \( c \) is a constant.

Proof: Proof follows from Ahsanullah et al. [16]. ■

Lemma 6.2: Under Assumption 6.1, If \( E(X|X \geq x) = h(x) \frac{f(x)}{1 - F(x)} \), where \( h(x) \) is a continuous differentiable function in \((\alpha, \beta)\), then \( f(x) = c e^{-\frac{g(x) - g(x)}{\beta - \delta \ln(x)}} \), where \( c \) is determined by the condition \( \int_a^b f(x) \, dx = 1 \).

Proof: It follows from Ahsanullah et al. [16]. ■

Theorem 6.1: If the random variable \( X \) satisfies Assumption 6.1 with \( \alpha = 0 \) and \( \beta = 1 \). Then \( E(X|X \leq x) = g(x) \tau(x) \), where \( \tau(x) = f(x) \frac{p(x)}{r(x)} \) and

\[
g(x) = \frac{p(x)}{(\alpha + 1)^2 (\beta - \delta \ln(x))}
\]

and

\[
p(x) = x^{\alpha+2} [(\delta + (\alpha + 1)\beta - \delta \ln(x)) - ((\delta + (\alpha + 1)\beta + (\delta(\alpha + 1)) \ln(x)) + \frac{\delta(\alpha + 1)}{(\alpha + 1)^2}]
\]

if and only if \( X \) has the PL model with pdf defined by Equation (2).

Proof: See Appendix 2. ■

Theorem 6.2: Suppose that \( X \) satisfies the conditions of Assumption 6.1 with \( \alpha = 0 \) and \( \beta = 1 \). Then \( E(X|X \geq x) = h(x)\tau(x) \), where \( h(x) = \frac{f(x) - p(x)}{1 - F(x)} \) and \( r(x) = \frac{f(x)}{p(x)} \), if and only if \( X \) has the PL model with pdf defined by Equation (2).

Proof: See Appendix 3. ■

7. Order statistics

If \( X_1, \ldots, X_n \) is a random sample from the PL distribution and let \( X_{1\cdots n} < \ldots < X_{n\cdots n} \) be the order statistic. Then the pdf \( f_{2n}(x) \) of the ith order statistic \( X_{i\cdots n} \) is given by

\[
f_{2n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \frac{(-1)^j}{(i+j)} (n-i+j) f(x) F(x)^{i+j-1}
\]

and \( h_{i+j}(x) = (i+j)f(x)F(x)^{i+j-1} \) denote the pdf of the exp-PL \((i+j)\) distribution.

8. Simulation study

In this section, we provide a simulation study in order to verify the asymptotic properties of the parameters. In this sense, we did a Monte Carlo simulation with 1000 replications and considering \( n = \{50, 150, 250\} \) and \((\alpha, \beta, \delta) = (1, 2, 3)\). The random numbers are generated based on the quantile function of the PL distribution, say \( Q(u) \), obtained taking the inverse function of the cdf. Using software as Mathematica, we obtain

\[
Q(u) = \text{Exp} \left[ \left( \frac{\delta}{\alpha \beta + \beta + \delta} \right) W \left( \frac{-u(\alpha \beta + \beta + \delta) e^{-\frac{u(\alpha \beta + \beta + \delta)}{\delta}}}{\delta} \right) + \frac{\alpha \beta + \beta + \delta}{(\alpha + 1)\delta} \right]
\]

where \( W(.) \) indicates the Lambert function.

Table 3 shows that was expected: when the sample size increases, the average estimates (AEs) tend to the true parameters and the mean square error (MSE) decrease. Thus, the asymptotic properties of the parameters are satisfied.

9. Applications

Here, we provide two applications to show the performance of the new model. To illustrate the good performance of our proposed distribution, we use some packages from R software: Adequacy Model, Gen SA and Mass. We use the goodness fit function (from the first one package) in order to provide the \( W^* \) (Crámer-von Mises) and \( A^* \) (Anderson Darling) statistics. Further, we obtain the MLEs and the respective standard errors.

To do this, we use the CG method. About the initial kicks, we use the Generalized Simulated Annealing function (from the second and third packages, mentioned above). In order to illustrate the performance of the new model, we compare it with another well-known distributions, listed below. This study is important since the main point of this section is proof that our proposed...
model performs better than others well known in the literature in terms of fitting the data sets in question. To do this, we choose some competitive families: beta-G, gamma-G, extended generalized-G and Kumaraswamy-G (for some specific baselines in each application). These generators are very used in many applications and have good adjustments in several cases. But, the most of them presents its density functions involving some complicated functions, which implies, for example, difficulties in finding some properties in closed form. On the other hand, note that the pdf of the proposed model does not involve any complicated function and presents some mathematical properties in closed form.

The competitive models are:

1. Power function distribution (PF) – explored in Meniconi and Barry [17], for example;
2. Extended generalized exponential distribution (EG-exp) – see the generator of Cordeiro et al. [18];
3. Gamma Weibull distribution (GW) – Provost et al. [20];
4. Beta Lindley distribution (β-L) – see MirMostafae et al. [21];
5. Kumaraswamy Nadarajah-Haghighi (Kw-NH) – submitted in Chilean Journal of Statistics by MirMostafaee and Kumaraswamy-Haghighi [22];
6. Extended generalized Nadarajah-Haghighi (EG-NH) – submitted in Chilean Journal of Statistics by VedoVatto et al. ; see the generator presented in Nadarajah et al. [23];
7. Gamma–Gamma distribution (GG) – see the generator in Eugene

9.1. First data set

For the first data set, we have a “data on proportion of income spent on food for a random sample of 38 households in a large US city” (from betareg package). We choose the variable “food”, that indicates the households in a large US city” (from betareg package). We choose the variable “food”, that indicates the number of observations of the data. Table 4 gives some descriptive statistics.

Table 4. Descriptive statistics – first data.

| Min.  | Max.   | Mean   | Median | Skewness | Kurtosis | Variance |
|-------|--------|--------|--------|----------|----------|----------|
| 0.1956 | 0.7626 | 0.4198 | 0.3903 | 0.5464   | 2.6960   | 0.0219   |

9.2. Second data set

Now, we fit the PL model to the data about the total milk production in the first birth of 107 cows from SINDI race. The original data are not in the interval (0, 1), and we must make the transformation 

\[ x_i = \frac{y_i - \min(y_i)}{\max(y_i) - \min(y_i)}, \]

for \( i = 1, \ldots, 107 \).

These data can be found in [26], for example. Table 7 gives some descriptive statistics for the second data.

Table 5. Statistics for the first application.

| Models | W*      | A*      |
|--------|---------|---------|
| PL     | 0.0431  | 0.3343  |
| PF     | 0.0434  | 0.3359  |
| EG-exp | 0.0711  | 0.4205  |
| β – L  | 1.1061  | 0.5031  |
| GW     | 0.0450  | 0.3423  |
| Kw-NH  | 0.0566  | 0.4252  |
| EG-NH  | 0.0549  | 0.3356  |
| GG     | 0.0568  | 0.4259  |
| β – β  | 0.0592  | 0.4558  |
| β – W  | 0.1600  | 1.1277  |

Table 6. MLE’s and the respective standards errors in parenthesis – first application.

| Models  | Parameters | α        | β        | δ        |
|---------|------------|----------|----------|----------|
| PL      |            | 1.1522   | 0.9902   | 91535.8500 |
|         |            | (0.2468) | (0.2759) | (38131.8500) |
| PF      |            | 0.0761   | 7394.9400| –         |
|         |            | (0.1745) | (0.0024) | (–)       |
| EG-exp  |            | 4.7822   | 15.9492  | 1.6780   |
|         |            | (0.7585) | (6.1778) | (0.1467) |
| GW      |            | 0.0262   | 0.8665   | 10.9660  |
|         |            | (0.0016) | (0.0515) | (1.4158) |
| β – L   |            | 73.4858  | 0.3535   | 18.0168  |
|         |            | (< 0.0001) | (0.0999) | (1.8205) |
| Kw-NH   |            | 4.7346   | 1.2480   | 1.2480   |
|         |            | (0.0084) | (0.0102) | (0.0100) |
| EG-NH   |            | 9.6334   | 0.1015   | 5.6996   |
|         |            | (0.0009) | (< 0.0001) | (0.3889) |
| GG      |            | 49.7131  | 19.8741  | 0.7045   |
|         |            | (0.1743) | (0.0002) | (0.2783) |
| β – β   |            | 495.742  | 118.2907 | 0.0043   |
|         |            | (122.443) | (33.4076) | (0.0012) |
| β – W   |            | 0.1428   | 2.1450   | 7.5816   |
|         |            | (< 0.0001) | (< 0.0001) | (0.1011) |
|         |            |          |          | (0.0172) |
Table 7. Descriptive statistics – second data.

|        |       |
|--------|-------|
| Min.   | 0.0168|
| Max.   | 0.8781|
| Mean   | 0.4689|
| Median | 0.4741|
| Skewness | -0.3306|
| Kurtosis | -0.3638|

Table 8. Statistics for the second application.

| Models  | $W^*$ | $A^*$ |
|---------|-------|-------|
| PL      | 0.2310| 1.4605|
| EG-exp  | 0.8190| 4.8927|
| GW      | 0.2462| 1.6127|
| $\beta - L$ | 2.6467| 15.3731|
| Kw-NH   | 0.3157| 2.0353|
| EG-NH   | 0.2528| 1.6648|
| $\beta - G$ | 0.2600| 1.6258|
| $\beta - W$ | 0.4641| 2.8836|
|         | 0.8634| 5.1169|

We note that these data have negative skewness and kurtosis.

Table 8 gives the $W^*$ and $A^*$ statistics for the competitive distributions. We can notice again that our proposed model has the better fit. Besides that, we obtain the MLEs and the respective standards errors (in parenthesis) – Table 9.

Figure 5 shows the TTT plot and the empirical and fitted density functions for the second data set. In this case, again, the TTT plot indicates an increasing hrf. Besides that, we plot only the PL density and the GW density since the last one is competitive with our proposed model.

10. Concluding and discussion

The power logarithmic distribution is introduced by using a mixture power function and continuous logarithmic distribution. The main motivation for introducing this distribution is to propose a model with simple expressions, but that brings with it the flexibility that other more robust models also have. In addition, since
such a distribution has many expressions in closed form, this is a positive and important point when comparing it with other distributions that are part of families already known. We also know that proposing new distributions has been a challenge in the area, since much has already been developed. But we understand that it is necessary to discuss distributions that are as simple as possible and that fit as well as other distributions with real data sets. And our proposal has these positive aspects and is, therefore, a great option for the good adjustment to these data. We hope, therefore, that this proposal can motivate the introduction of other distributions like it. In this sense, statistical properties, estimation, information matrix and characterizations of this new model are obtained. The new model has various shapes of density and failure rate functions. Additionally, the new model includes submodels as special cases. A simulation study is performed and we prove that the asymptotic properties of the parameters are satisfied. Finally, two applications are developed in order to illustrate the performance of the new model. As the previous discussion, the PL model had the best performance.

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No potential conflict of interest was reported by the author(s).

ORCID
Maria do Carmo S. Lima http://orcid.org/0000-0002-5480-3103

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Appendices
Appendix 1: The elements of Fisher’s information matrix
Using Lemma 5.1, we have

\[ J_{11} = \frac{2n}{(\alpha + 1)^2} - \frac{n \beta^2}{(\beta + \alpha + \delta)^2} \]

\[ J_{12} = \frac{n \beta}{(\beta + \alpha + \delta)} - \frac{n (\alpha + 1)\beta}{(\beta + \alpha + \delta)^2} \]

\[ J_{13} = -\frac{n \beta}{(\beta + \alpha + \delta)^2} \]

\[ J_{22} = -\frac{n (\alpha + 1)^2}{(\beta + \alpha + \delta)^2} + \frac{n(\alpha + 1)^2}{\delta(\beta + \delta + \alpha \beta)} \]
By using Lemma 6.1, we have \( \frac{f(x)}{X(x)} = \frac{\alpha}{x} - \frac{\delta}{x(\beta - \delta \ln x)} \).

Integrating both sides of the above equation with respect to \( x \), we obtain
\[
f(x) = cx^\alpha (\beta - \delta \ln x),
\]
where \( c \) is a constant to be determined. Using the condition \( \int_0^\infty f(x) \, dx = 1 \), we obtain \( c = \left(\frac{(\alpha + 1)^2}{(\beta + \delta + \alpha \beta)}\right) \) and then \( f(x) = \left((\alpha + 1)^2x^\alpha (\beta - \delta \ln x))/((\beta + \delta + \alpha \beta)) \) which is the pdf of PL distribution, the proof is completed.

### Appendix 3: Proof of Theorem 6.2
**Proof:** Firstly, if the random variable \( X \) satisfies Assumption 6.1 and has the PL density function (2), then we have
\[
f(x) h(x) = \int_x^\infty \frac{(\alpha + 1)^2u^{\alpha+1}(\beta - \delta \ln u)}{\beta + \delta + \alpha \beta} \, du
= E(X) - \int_0^x \frac{(\alpha + 1)^2u^{\alpha+1}(\beta - \delta \ln u)}{\beta + \delta + \alpha \beta} \, du
= E(X) - p(x).
\]

Thus
\[
h(x) = \frac{(\beta + \delta + \alpha \beta) E(X) - p(x)}{(\alpha + 1)^2x^\alpha (\beta - \delta \ln x)}.
\]

Conversely, we assume that \( h(x) = \frac{(\beta + \delta + \alpha \beta) E(X) - p(x)}{(\alpha + 1)^2x^\alpha (\beta - \delta \ln x)} \). Then, after differentiation, we obtain
\[
h'(x) = \frac{-x - \frac{(\beta + \delta + \alpha \beta) E(X) - p(x)}{(\alpha + 1)^2x^\alpha (\beta - \delta \ln x)}}{\frac{x}{\beta - \delta \ln x}} \times \frac{\delta}{x(\beta - \delta \ln x)}
= x - h'(x)
= \frac{x}{\beta - \delta \ln x} \times \frac{\delta}{x(\beta - \delta \ln x)}.
\]

Thus
\[
\frac{x - h'(x)}{h(x)} = \frac{\delta}{x(\beta - \delta \ln x)}.
\]

By Lemma 6.2, we obtain
\[
\frac{f'(x)}{f(x)} = \frac{\alpha}{x} - \frac{\delta}{x(\beta - \delta \ln x)}.
\]

Integrating both sides of the above equation with respect to \( x \), we obtain \( f(x) = cx^\alpha(\beta - \delta \ln x) \), where \( c \) is a constant. Using the condition \( \int_0^\infty f(x) \, dx = 1 \), we obtain the pdf of PL model, the proof is achieved.