HOLOMORPHIC EXTENSION OF DECOMPOSABLE DISTRIBUTIONS FROM A CR SUBMANIFOLD OF $\mathbb{C}^L$

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Abstract

Given $N$ a non generic smooth CR submanifold of $\mathbb{C}^L$, $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$ where $\mathfrak{N}$ is generic in $\mathbb{C}^{L-n}$ and $h$ is a CR map from $\mathfrak{N}$ into $\mathbb{C}^n$. We prove, using only elementary tools, that if $h$ is decomposable at $p' \in \mathfrak{N}$ then any decomposable CR distribution on $N$ at $p = (p', h(p'))$ extends holomorphically to a complex transversal wedge. This gives an elementary proof of the well known equivalent for totally real non generic submanifolds, i.e. if $N$ is a smooth totally real submanifold of $\mathbb{C}^L$ any continuous function on $N$ admits a holomorphic extension to a complex transverse wedge.

1 Introduction

1.1 Statement of Results

Let $\mathfrak{N}$ be generic submanifold of $\mathbb{C}^{k+m}$ of CR dimension $k$ and $h$ a CR map from $\mathfrak{N}$ into some $\mathbb{C}^n$ verifying $dh(0) = 0$. Set $L = k + m + n$, we construct a CR submanifold $N$ of $\mathbb{C}^L$ near the origin as the graph of $h$ over $N$, that is $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$. It turns out that any non generic CR submanifold of $\mathbb{C}^L$ can be obtained in that fashion, see for example [2]. The main question we address in this paper is the possible holomorphic extension of a CR distribution of $N$ to some wedge $W$ in a complex transverse direction. The aim of this paper is to give a proof of an extension result using only elementary tools. The CR structure of $N$ is determined by $\mathfrak{N}$ hence any CR distribution on $N$ is a CR distribution on $\mathfrak{N}$.

Definition 1.1.1 (a) Let $\mathfrak{N}$ be a smooth generic submanifold of $\mathbb{C}^L$. A CR distribution $u$ on $\mathfrak{N}$ is decomposable at the point $p \in \mathfrak{N}$ if, near $p$, $u = \sum_{j=1}^{K} U_j$ where the $U_j$ are CR distributions extending holomorphically to wedges $W_j$ in $\mathbb{C}^{k+m}$ with edges $\mathfrak{N}$. We shall say a distribution $u$ on $N$ is decomposable at a point $p = (p', h(p'))$ if $u$ is decomposable at $p'$ on $\mathfrak{N}$.

(b) We say that $v$ is complex transversal to $N$ at $p$ if $v \notin \mathbb{C}\text{span}T_{p}N$.

Our main result is the following

Theorem 1.1.1 Let $N = \{(\mathfrak{N}, h(\mathfrak{N}))\}$ be a non generic smooth (C$^\infty$) CR submanifold of $\mathbb{C}^{k+m+n}$ such that the map $h$ is decomposable at some $p_0 \in \mathfrak{N}$. Let $v$ be a complex transversal vector to $N$ at $p_0 = (p'_0, h(p'_0))$. If $f$ is a decomposable CR distribution at $p_0 \in N$ then, near $p_0$, there exists a wedge $W$ of direction $v$ whose edge contains a neighborhood of $p_0$ in $N$ and $F \in \mathcal{O}(W)$ such that the boundary value of $F$ is $f$. Furthermore, there exist $\{F_\ell\}_{\ell=1}^{n}$, $F_\ell \in \mathcal{O}(W)$ such that $dF_1 \wedge ... \wedge dF_n \neq 0$ on $W$ and each $F_\ell$ vanishes to order one on $N$.
The boundary value of a holomorphic function $F$ is defined by $\lim_{\lambda \to 0^+} \int_\mathbb{R} F(x + \lambda v) \varphi(z) dx$, where $v \in \mathcal{W}$. It turns out that if $F$ has slow growth in a wedge $\mathcal{W}$ (there exists a constant $C > 0$ and a positive integer $\ell$ such that

$$|F(z)| \leq \frac{C}{\text{dist}(z, M)^\ell}$$

where $\text{dist}(z, M)$ denotes the distance from a point $z$ to $M$) then the boundary value of $F$ defines a CR distribution on $N$ (see, for example [1]). We call the integer $\ell$ above the growth degree of $F$.

**Remarks on the smoothness of $N$**

Note that one does not need $N$ to be smooth to be able to define a decomposable CR distribution on $N$. Indeed, suppose $F$ is a holomorphic function of slow growth of growth degree $\ell$, then one can prove, following theorem 7.2.6 in [1] the next result.

**Proposition 1.1.2** Let $F$ be as above and suppose that the edge of the wedge $N$ is of regularity $\ell + 1$, then the boundary value of $F$ defines a CR distribution of order $\ell + 1$ on $N$.

We thus define for $u$ a decomposable distribution, $u = \sum bvF_j$, the growth degree of $u$ to be the maximum of $\ell$ growth degrees of the $F_j$. We see that if the growth degree of a decomposable distribution $u$ is $\ell$ it makes sense to speak of a decomposable distribution on a manifold of smoothness $\ell + 1$. Hence the hypothesis of smoothness on $N$ in theorem[1,1,1] can be replaced by $\ell + 1$.

If instead of CR distribution we wish to consider functions, we can reduce the smoothness hypothesis in theorem[1,1,1]. A $C^0$ function $u$ that is decomposable near a point $p$ is NOT the sum of $C^0$ functions $U_j$ extending holomorphically. On the other hand if $u \in C^\alpha$, $\alpha \not\in \mathbb{N}$, if $u$ extends holomorphically at some $p$ into a wedge $\mathcal{W}_j$ of direction $w_j$ then the wedge $\mathcal{W}_j$ can be written as $(\mathcal{M} \cap V_p) + i\Gamma_j$ where $V_p$ is a neighborhood of $p$ and $\Gamma_j$ is a conical neighborhood of $w_j$ in the normal space to $N$ at $p$. The wave front set of $u$ (with respect to any micro-local class) at $p$ is contained in the dual cone of $\Gamma_j$ denoted by $\Gamma_j^\circ$. Hence if $u \in C^\alpha$ is decomposable at some $p$, since the $\Gamma_j^\circ$ are pairwise disjoint the regularity of the sum of the $U_j$ cannot be any better than the regularity of each $U_j$. Therefore if $u$ is a $C^\alpha$ decomposable function, $u = \sum U_j$, then each $U_j$ has at least the same regularity as $U$. Hence if we wish to study the problem of holomorphic extension from the point of view of functions rather than distributions we can replace the smoothness of the manifold $N$ by $C^{\ell + \alpha}$ and study the extension of $C^{\ell + \alpha}$ decomposable CR functions. We get the following result.

**Theorem 1.1.3** Let $N = \{ (\mathcal{M}, h(\mathcal{M})) \}$ be a non generic $C^{\ell + \alpha}$ CR submanifold of $C^{k_m + m + n}$ such that the map $h$ is decomposable at some $p_0 \in \mathcal{M}$. Let $v$ be a complex transversal vector to $N$ at $p_0 = (p_0', h(p_0'))$. If $f$ is a $C^{\ell + \alpha}$ decomposable CR function at $p_0$, then, near $p_0$, there exists a wedge $\mathcal{W}$ of direction $v$ whose edge contains a neighborhood of $p_0$ in $N$ and $F \in \mathcal{O}(W)$ such that $F|_N = f$. Moreover, there exist $\{ F_i \}_{i=1}^n$, $F_i \in \mathcal{O}(W)$ such that $dF_1 \wedge \ldots \wedge dF_n \neq 0$ on $W$ and each $F_i$ vanishes to order one on $N$.

We also obtain, using theorem[1,1,1] the following corollary.

**Corollary 1.1.4** Let $M$ be a $C^\infty$ generic submanifold of $C^L$ containing through some $p_0 \in M$ a proper $C^\infty$ CR submanifold $N = \{ (\mathcal{M}, h(\mathcal{M})) \}$ of same CR dimension, $p_0 = (p_0', h(p_0'))$ with $p_0' \in \mathcal{M}$. Assume that the function $h$ decomposable at $p_0$. Let $v \in T_{p_0}M \setminus \text{span}_\mathbb{C} T_{p_0}N$. If $f$ is a CR distribution on $N$ that is decomposable at $p_0$, then there exists a wedge $\mathcal{W}$ in $M$ of direction $v$ whose edge contains a neighborhood of $p_0$ in $N$ and $F$ a $C^\infty$ CR function on $W$ such that $F|_N = f$. Furthermore, there exists a collection of $C^\infty$ CR functions $\{ g_i \}_{i=1}^n$ vanishing to order one on $N$ and such that $dg_1 \wedge \ldots \wedge dg_n \neq 0$ on $W$. 

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The above corollary does not hold in the abstract CR structure case, we finish the paper by constructing an abstract CR structure on which there is no CR extension. More precisely, set \( \mathcal{L} = \partial + \partial \overline{\partial} + C(z, s, t) \partial_s + tD(z, s, t) \partial_t \) and define \( \mathcal{L}^0 = \mathcal{L}|_{t=0} \). The question we now address is the following: if \( f = f(z, s) \) is such that \( \mathcal{L}^0(f) = 0 \), does there exist \( g \) such that \( \mathcal{L}(f + tg) = 0 \)? The answer in general is negative.

**Proposition 1.1.5** There exist \( \mathcal{L} \) as above and \( h \) a real analytic function, with \( \mathcal{L}^0(h) = 0 \) and such that the equation \( \mathcal{L}(h + tg) = 0 \) has no solution for \( g \in \mathbb{C}^1 \).

### 1.2 Remarks

As pointed out in [3], this type of extension result was well known in the totally real case and is essentially due to Nagel [7]. It can be stated in the following way:

**Theorem 1.2.1** Let \( N \) be a non-generic totally real smooth submanifold of \( \mathbb{C}^L \). Let \( v \in \mathbb{C}^L \) be complex transversal to \( N \) at \( p \). Then for any continuous function \( f \) on a neighborhood of \( p \) in \( N \) there exists \( W_v \), a wedge of direction \( v \) whose edge contains \( N \) such that \( f \) has a holomorphic extension to \( W_v \).

This paper provides the easiest and simplest proof of this result since any continuous function on a totally real submanifold is decomposable.

We wish to point out the main differences between this paper and [3]. In [3] we obtain similar results, but the techniques we use are not at all the same and they yield extension results for non-decomposable CR distributions, also, the size of the wedges obtained are much larger than the one obtained here, roughly speaking, in [3] we obtain wedges that contain \( (\mathbb{R} \times \mathbb{R}^n) \cap \{ t_1 > 0 \} \).

As it is noted in [11] on most CR submanifolds of \( \mathbb{C}^L \) all CR functions are decomposable, hence the hypotheses of theorem 1.1.1 hold in a general sense for CR distributions. However, they are examples of CR submanifolds of \( \mathbb{C}^L \) on which indecomposable CR functions exist.

Theorem 1.1.1 implies that the extension obtained is not unique, which differs greatly with the holomorphic extension results obtained for generic submanifolds where the extension if it exists is always unique.

One should note that the question of CR extension can be viewed as a Cauchy problem with Cauchy data on a characteristic set \( N \).

### 1.3 Background

For a general background on CR geometry, we refer the reader to the books of Baouendi, Ebenfelt and Rothschild [1], Boggess [2] and Jacobowitz [6].

Most of the results on holomorphic extension deal with generic submanifolds of \( \mathbb{C}^n \). In a general way, these results imply a forced unique extension of CR functions under certain hypothesis on the manifold \( M \) such as Lewy non degenerateness or more generally, minimality. Under these hypothesis one does show that it is possible to fill a wedge with edge \( M \) with analytic discs attached to \( M \). Using the fact that continuous CR functions are uniform limits of polynomials and the maximum principle one obtains a unique extension for continuous CR functions, see for example the survey paper by Trépreau [10].

The subject of decomposable CR functions has been studied by many authors and it was believed that all CR were decomposable, however Trépreau produced examples
of non decomposable CR functions (an elementary explanation of this can be found in a paper by Rosay [8]). However one should note that on most CR submanifolds of \(\mathbb{C}^n\) any CR function is decomposable.

The subject of CR extension has not been studied in as much depth as the holomorphic extension has. When studying CR extension from a submanifold of lower CR dimension, the tools involving analytic discs still work, see for example [11] and [12], however this is not the case when the CR dimensions are equal see [13].

## 2 Proof of the Extension Theorem

**Proposition 2.0.1** Let \(N = \{(\mathfrak{N}, 0)\}\) be a smooth non generic CR submanifold of \(\mathbb{C}^{k+m+n}\). Let \(\tilde{W}\) be a wedge in \(\mathbb{C}^{k+m}\) with edge \(\mathfrak{N}\) near \(p_0 \in \mathfrak{N}\). Suppose \(F\) is a holomorphic function in \(\tilde{W}\) which is of slow growth, denote by \(f\) its boundary value on \(\mathfrak{N}\). For any \(\nu\) complex transversal to \(N\) at \(p_0\), there exists a wedge \(\tilde{W}_\nu\) of direction \(\nu\) whose edge contains \(\mathfrak{N}\) such that \(f\) extends holomorphically to \(\tilde{W}_\nu\).

**Proof of Proposition 2.0.1** We begin with a choice of local coordinates on \(\mathfrak{N}\). \(\mathfrak{N}\) is a generic manifold in \(\mathbb{C}^{k+m}\). We introduce local coordinates near \(p_0\). We may choose a local embedding so that \(p_0 = 0\) and \(\mathfrak{N}\) is parameterized in \(\mathbb{C}^{k+m} = \mathbb{C}_x \times \mathbb{C}^m\) by

\[
\mathfrak{N} = \{(z, w') \in \mathbb{C}^k \times \mathbb{C}^m : I\nu(w') = a(z, Re(w')), \ a(0) = da(0) = 0\}. \tag{2.0.1}
\]

We will denote by \(s = Re(w') \in \mathbb{R}^m\), we thus have

\[
\mathfrak{N} = \{(z, s + ia(z, s)) \in \mathbb{C}^k \times \mathbb{C}^m, \ T_0 \mathfrak{N} = \mathbb{C}^k \times \mathbb{R}^m\}. \tag{2.0.2}
\]

Define \(CT_p \mathfrak{N} = T_p \mathfrak{N} \otimes \mathbb{C}\) and \(T^0 \mathfrak{N} = T^{0,1} \mathfrak{N} \cap CT_p \mathfrak{N}\). We say that \(\mathfrak{N}\) is a CR manifold if \(\text{dim}_\mathbb{C} T^{0,1} \mathfrak{N}\) does not depend on \(p\). The CR vector fields of \(\mathfrak{N}\) are vector fields \(L\) on \(\mathfrak{N}\) such that for any \(p \in \mathfrak{N}\) we have \(L_p \in T_p^{0,1} \mathfrak{N}\). One can choose a basis \(L\) of \(T^{0,1} \mathfrak{N}\) near the origin consisting of vector fields \(L_j\) of the form

\[
L_j = \frac{\partial}{\partial z_j} + \sum_{\ell=1}^n F_j \frac{\partial}{\partial s_\ell}, \tag{2.0.3}
\]

The wedge \(\tilde{W}\) in \(\mathbb{C}^{k+m}\) with edge \(\mathfrak{N}\) on which \(F\) is defined is given, in a neighborhood of the origin by

\[
\tilde{W} = (\mathfrak{U} + i\mathfrak{F}),
\]

where \(\mathfrak{U}\) is a neighborhood of the origin in \(\mathfrak{N}\) and \(\mathfrak{F}\) is a conic neighborhood of some vector \(\mu\) in \(\mathbb{R}^m \setminus \{0\}\). Note then that \(F\) admits (trivially) a holomorphic extension to the region \(W \times \mathbb{C}^n \subset C^{k+m+n}\). This region is much more than a wedge. But it clearly contains a wedge in \(C^{k+m+n}\) with direction \(u\) whenever \(u\) is a vector of the \(u = (u', u'') \in C^{k+m} \times C^n\) with \(u' \in \tilde{W}\). By 2.0.1 complex transversility of \(v = (v', v'')\) means that \(v'' \neq 0\). Fix a vector \(u \in \tilde{W}\). Consider a \(\mathbb{C}\) linear change of variables \(T\) that is the identity on \(C^{k+m} \times \{0\}\) and such that \(T(v) = (u, v'')\). The desired extension of \(f\) to a wedge of direction \(v\) is then given by \(F(T(z, w))\). We now need to show that the boundary value of \(F(T(z, w))\) on \(\mathfrak{N}\) is \(f\). The boundary value of \(F\) on the wedge \(\tilde{W}\) is defined to be

\[
<f, \varphi> = \lim_{\lambda \to 0^+} \int_{\mathfrak{N}} F(x + \lambda \gamma) \varphi(x) dx. \tag{2.0.4}
\]

For \(\varphi \in C^\infty_0(\mathfrak{N})\) and \(x + \lambda \gamma \in \tilde{W}\). Write \(T = (T', T'') \in C^{k+m} \times C^n\). Let \(\tau = \tau(x, \lambda, \eta)\) be defined by \(T'(x + \lambda \eta) = (x + \tau(x, \lambda, \eta))\). Since \(T\) is the identity on \(C^{k+m} \times \{0\}\), we have \(\lim_{\lambda \to 0^+} \lambda \tau(x, \lambda, \eta) = 0\). The boundary value of \(F(T(z, w))\) on \(\mathfrak{N}\) on the wedge \(\tilde{W}\) is then given by

\[
\lim_{\lambda \to 0^+} \int_{\mathfrak{N}} F(T(x + \lambda \eta)) \varphi(x) dx, \tag{2.0.5}
\]
where \((x + \lambda \eta) \in W_v\). We then define
\[
G_r(\lambda) = \int_{\eta} F(x + \lambda \tau(x, \lambda, \eta))\varphi(x)dx,
\]
\[
F_\gamma(\lambda) = \int_{\eta} F(x + \lambda \gamma)\varphi(x)dx.
\]
Then by proposition 7.2.22 p189 of [1] we have
\[
G_r(\lambda) - F_\gamma(\lambda) = O(\lambda), \quad \lambda \to 0^+.
\]
Hence the boundary value defined by 2.0.3 is equal to the boundary value defined by 2.0.4. \(\blacksquare\)

We immediately note that if the boundary value is at least continuous then the proof of the proposition yields the following

**Proposition 2.0.2** Let \(N = \{(\mathfrak{N}, 0)\}\) be a \(C^1\) non generic CR submanifold of \(C^{k+m+n}\). Let \(W\) be a wedge in \(C^{k+m}\) with edge \(\mathfrak{N}\) near \(p_0 \in \mathfrak{N}\). Suppose \(F\) is a holomorphic function in \(W\) which has a continuous boundary value on \(N\), denote by \(f\) its boundary value on \(\mathfrak{N}\). For any \(v\) complex transversal to \(N\) at \(p_0\), there exists a wedge \(W_v\) of direction \(v\) whose edge contains \(\mathfrak{N}\) such that \(f\) extends holomorphically to \(W_v\).

Using proposition 2.0.1 we obtain a special case of theorem 1.1.1 from which we will deduce the later, namely,

**Proposition 2.0.3** Let \(N = \{(\mathfrak{N}, 0)\}\) be a \(C^\infty\) (resp \(C^{1+\alpha}\)) CR submanifold of \(C^{k+m+n}\). Let \(v\) be a complex transversal vector to \(N\) at \(p_0\). If \(f\) is a decomposable distribution at \(p_0\) (resp a \(C^{0+\alpha}\) decomposable function), then, near \((p_0, 0)\) in \(N\), there exists a wedge \(W\) of direction \(v\) whose edge contains a neighborhood of \((p_0, 0)\) in \(N\) and \(F \in \mathcal{O}(W)\) such that \(bf = f\).

**Proof of Proposition 2.0.3** Let \(v\) be a complex transversal vector and let \(u\) be a CR distribution on \(\mathfrak{N}\). By hypothesis, \(u = \sum_{j=1}^{K} U_j\) where each \(U_j\) is a boundary value of \(F_j, F_j \in O(\hat{W}_j), F_j\) of slow growth (or \(F_j \in C^{0+\alpha}(N)\)), where \(\hat{W}_j\) are wedges with edge \(\mathfrak{N}\) in \(C^{k+m}\). We thus apply proposition 2.0.1 (or proposition 2.0.2) to each \(U_j\) to obtain a holomorphic extension to wedges \(W_j\) all in the direction \(v\). Let \(W = \cap_{j=1}^{K} W_j\) we conclude that the function \(\sum_{j=1}^{K} U_j\) extends holomorphically to \(W\) and \(\sum_{j=1}^{K} U_j = u\) on \(\mathfrak{N}\). This concludes the proof of proposition 2.0.3. \(\blacksquare\)

**Proof of Theorem 1.1.1** Denote the coordinates \((z, w', w'')\) in \(C^k_x \times C^m_w \times C^n_w\). Recall that \(N = \{(\mathfrak{N}, h(\mathfrak{N}))\}\). Consider the CR map \(h : \mathfrak{N} \to \mathbb{C}^n\). By proposition 2.0.3 each \(h_j\) extends holomorphically to some wedge \(W_j\) in any complex transversal direction \(v\). Set \(W = \cap_{j=1}^{N} W_j, W \neq \emptyset\) since \(v \in W\). Define \(F : (\mathfrak{N}, 0) \to (\mathfrak{N}, sh(\mathfrak{N}))\) where \(\kappa \in \mathbb{R}^*\) by
\[
F(z, w', w'') = (z, w', w'' + sh(z, w')).
\]
Clearly, there exists \(\kappa \neq 0\) so that on \(\mathfrak{N}\), the Jacobian of \(F\) is non zero. Hence \(F\) is a biholomorphism from \(W\) to \(F(W)\) extending to a \(C^{1+\alpha}\) diffeomorphism from \(W \cup (\mathfrak{N} \times \{0\})\) to \(F(W) \cup (\mathfrak{N} \times \{0\})\). Note that since \(dh(0) = 0\), \(F\) is tangent to the identity at the origin, hence, there exists \(W'\) a wedge in \(C^{k+m+n}\) of direction \(v\) such that \(W' \subset F(W)\). Thus any decomposable distribution (resp \(C^{0+\alpha}\) function) on \(N\) extends holomorphically to the complex transversal wedge \(W'\). Note then that the functions \(f_j = w_j'' - h_j\) are holomorphic on a wedge \(W_v\) and null on \(N\) and they clearly verify the desired conclusions.
Proof of Corollary 1.1.4. Let $M$ and $N$ be as in the hypothesis of the corollary. After a linear of variables, we may assume that $p_0 = 0$ and that near the origin, $M$ is parametrized by

$$M = \{z, u + iv(z, u) : (z, u) \in \mathbb{C}^k \times \mathbb{R}^{p-k}\}.$$ 

By the implicit function theorem, we may assume that $N$ is given as a subset of $M$ by

$$\begin{cases}
u_{p-k-n} = \mu_1(z, u_1, ..., u_{p-k-n-1}), ..., \nu_{p-k} = \mu_n(z, u_1, ..., u_{p-k-n-1}), \\
\mu(0) = d\mu(0) = 0.
\end{cases}$$

Denote by $s = (u_1, ..., u_{p-k-n-1}) \in \mathbb{R}^m$ and $t = (u_{p-k-n}, ..., u_{p-k}) \in \mathbb{R}^n$. Setting $t' = t - \mu$, in the $(z, s, t')$ coordinates, we have $N$ given as a subset of $M$ by $t' = 0$ and

$$N = \{(z, w'(z, s), h(z, w') : (z, s) \in \mathbb{C}^k \times \mathbb{R}^m\},$$

where $h$ is a CR map from $\mathfrak{M} := \{(z, w'(z, s)\}$. We can now apply theorem 1.1.5 to obtain the CR extension as the restriction of the holomorphic extension of $f$ to $W \cap M$. The second part of the corollary follows in the same manner. ■

3 Non Extension Example

We will now construct an example of an abstract CR structure $(M, V)$ in which there are no local CR extension property.

Set $L = \frac{\partial}{\partial x} + C(z, s, t)\frac{\partial}{\partial t} + tD(z, s, t)\frac{\partial}{\partial m}$ and define $L^0 = L|_{t=0}$. Proposition 1.1.5 states that there exist $L$ as above and $h$ a real analytic function, with $L^0(h) = 0$ and such that the equation $L(h + tg) = 0$ has no solution for $g \in \mathcal{C}^1$.

Proof of Proposition 1.1.5. We first construct $L^0$. Let $f : \mathbb{C} \rightarrow \mathbb{R}$ be a real analytic function such that there exists $g \in \mathcal{C}^\omega(B_r(0))$ (the neighborhood is taken in $\mathbb{R}^3$) where the equation

$$L^0(u) = [\frac{\partial}{\partial x} - ifz\frac{\partial}{\partial s}](u) = g$$

is not solvable in any neighborhood of the origin in $\mathbb{R}^3$. (Hörmander’s Theorem)\[\text{p157}].

Lemma 3.0.1. There exists $\eta = \eta(z,s) \in \mathcal{C}^\omega$ such that $L^0(\eta) \neq 0$ and the equation $L^0(u) = e^\eta g$ is solvable nowhere.

Proof of the Lemma 3.0.1. Let $\eta \in \mathcal{C}^\omega$ such that $L^0(\eta) \neq 0$. We note that if $h \in \mathcal{C}^\omega$ is such that $L^0(h) = 0$ and $h$ does not vanish in some neighborhood of the origin in $\mathbb{R}^3$, then one of the two equations is not locally solvable

$$\begin{cases}L^0(u) = e^\eta g, \\
L^0(u) = (e^\eta + h)g.
\end{cases}$$

Indeed, if both of these equations were solvable, with solutions $u_1$ and $u_2$, on some neighborhoods $U_1$ and $U_2$ of the origin, then on $U_1 \cap U_2$ we would have, by setting $u = u_2 - u_1$

$$L^0(u) = L^0(h\frac{h}{h}) = hL^0(\frac{h}{h}) = hg.$$ 

So we conclude that $L^0(\frac{h}{h}) = g$, contradicting our choice of $L^0$.

Without loss of generality, assume $L^0(u) = (e^\eta + h)g$ is not locally solvable. To finish the proof of the lemma, we wish to find $h$ such that $\not= 0$, $L^0(h) = 0$ and $e^\eta + h \not= 0$, then we will set $e^\eta + h = e^\eta$. By Cauchy-Kovalevsky we solve the equation

$$L^0(u) = e^\eta g.$$
\[
\begin{cases}
L^0(v) = 0, \\
v \big|_{\text{Re}[z]=0} = e^\eta \big|_{\text{Re}[z]=0}.
\end{cases}
\]

We thus have \( v = e^{\eta(0, \text{Im}[z], s)} + \text{Re}[z]\xi \) and consequently \((e^\eta + v)(0, 0, 0) \neq 0 \). Therefore by eventually shrinking our neighborhoods we get the desired function. This completes the proof of Lemma 3.0.1.

We are now ready to define \( L \) from \( L^0 \), set

\[ L = L^0 + t\frac{\partial}{\partial s} + L^0(\eta)\frac{\partial}{\partial t} \]

Claim: The function \( h = s + if(z) \) admits no CR extension to \((M, L)\).

Indeed, \( L^0(h) = 0 \). Suppose there exists \( v \in C^1 \) such that \( L(h + tv) = 0 \) has a local solution, then, we note that \( L(h) = -tg \), thus we have \( L(h + tv) = -tg + tL(v) + tvL^0(\eta) = 0 \). So \(-g + L(v) + vL^0(\eta) = 0 \). Set \( v = v_0(x, s) + tv_1(x, s, t) \), then \( L(v) = L^0(v_0) + tG \), thus equating terms with no \( t \) and multiplying by \( e^\eta \), we get

\[
e^\eta(L^0(v_0) + v_0L^0(\eta)) = L^0(v_0e^\eta) = e^\eta g
\]

Contradicting the Lemma 3.0.1.

Remarks: (a) There are plenty of non zero CR functions on \((M, V)\), any holomorphic function of \( z \) is CR and we can also find functions of \( z \) and \( t \) that are CR, indeed, if \( f = f(z, t) \) then by Cauchy-Kovalevsky one can solve \( L(f) = 0 \) with non zero Cauchy data.

(b) The CR structure \((M, V)\) defined above is not realizable in \( \mathbb{C}^3 \).

By realizable, we mean that there does not exist \( \Phi_1, \Phi_2, \Phi_3 \), complex valued functions, such that \( L(\Phi_i) = 0 \) and \( d\Phi_1 \wedge d\Phi_2 \wedge d\Phi_3 \neq 0 \) in a neighborhood of the origin.

If \((M, V)\) was realizable, then any real analytic CR function on \((N, V_0)\) would admit a CR extension to \( M \), since \( L^0 \) is real analytic.

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