Universal bifurcation property of two- or higher-dimensional dissipative systems in parameter space: Why does 1D symbolic dynamics work so well?

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Abstract

The universal bifurcation property of the Hénon map in parameter space is studied with symbolic dynamics. The universal-$L$ region is defined to characterize the bifurcation universality. It is found that the universal-$L$ region for relative small $L$ is not restricted to very small $b$ values. These results show that it is also a universal phenomenon that universal sequences with short period can be found in many nonlinear dissipative systems.

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I. INTRODUCTION

One of the standard ways of investigating the dynamics of physical systems is by exploiting the universal (system-independent) property\textsuperscript{1–8} of them. The best understood transition sequence is the period-doubling cascade, which has been observed in a variety of physical systems. Beyond the accumulation point for the period-doubling sequence there is chaos. Two decades ago Metropolis, Stein and Stein showed\textsuperscript{1} that there is an ordered sequence of distinct periodic windows, each of which occurs for some range of control parameter, within the chaotic region for unimodal maps, $x_{n+1} = f(\mu, x_n)$. They have called this sequence the U-sequence since the ordering of the windows is system independent. Remarkably, this universality is also observed in systems with many degrees of freedom both experimentally\textsuperscript{2,3,8} and theoretically\textsuperscript{4–7} although the phase portraits of these two- or high-dimensional system still exhibit very complex behaviour which is clearly not one-dimensional or close to one-dimensional. It has been found that the periodic windows interspersed in chaotic region for
these systems are ordered in a systematic way as those of one-dimensional (1D) maps. The most striking and detailed observation is obtained in the Lorenz equations

\[ \begin{align*}
\dot{x} &= 10(y - x), \\
\dot{y} &= rx - xz - y, \\
\dot{z} &= xy - 8z/3.
\end{align*} \tag{1} \]

On the parameter \( r \) axis with \( 45 < r < 400 \), all of the found 68 periodic windows of the Lorenz equations can fit into those of a 1D antisymmetrical map with only one exception\(^9\). Experimentally, even though the Belousov-Zhabotinskii reaction involves more than thirty chemical species, it exhibits rather complex bifurcation behaviour that is modeled well by 1D maps\(^3\).

Despite these numerical and experimental observations, the underlying mechanism for the universal property is not fully understood. The motivation of this paper is to present an approach towards interpreting all these experimental and numerical observations and exploring their limitations. We will take the Hénon map\(^10\)

\[ \begin{align*}
x_{n+1} &= 1 - ax_n^2 + y_n, \\
y_{n+1} &= bx_n,
\end{align*} \tag{2} \]

as an example. The bifurcation structure of the Hénon map in the two-dimensional parameter \((a, b)\) space has been extensively discussed\(^11,12\). In this paper, we will use symbolic dynamics\(^1,14-20\) of 1D mappings and 2D mappings to illustrate the universal topological property of the Hénon map at selected parameters by considering the unstable periodic orbits embedded in its chaotic attractor. Two topological quantities \( \delta \) and \( L \) are defined to characterize this universal topological property. Then we discuss the universal bifurcation property of the Hénon map in 2D parameter space \((a, b)\) by defining universal-\(L\) regions in which the Hénon map exhibits 1D bifurcation behavior to period \( L \). It is remarkable to find that the universal-\(L\) region for relative small \( L \) is not restricted to very small \( b \) values. We will also present two examples of ordinary differential equations (ODE’s), the Rössler equations\(^13\) and the forced Brusselator\(^4\), to demonstrate the validity and robustness of our approach. These results show that it is also a universal phenomenon that universal sequences with short period can be found in many experiments or numerical calculations on nonlinear dissipative systems.

The paper is organized as follows. In Section II, we review the basic property of 1D unimodal maps. The universal bifurcation property and its limitations of the Hénon map in the 2D parameter \((a, b)\) space is studied in Section III. To demonstrate the validity of the method presented in Section III, the universal bifurcation property of Rössler equations and the forced Brusselator in a definite parameter axis is investigated in Section IV. Finally, in Section V we give our conclusion.

**II. UNIVERSAL SEQUENCES IN 1D UNIMODAL MAPS**

By using the symbolic dynamics of 1D mappings, Metropolis, Stein and Stein (MSS) had already shown that the dynamics of unimodal 1D maps of the interval \([-1, 1]\) is embodies
in the U-sequence of periodic windows\textsuperscript{1,14,15}. Fig. 1 shows a typical case. The extremum is denoted by a letter C. Each periodic window of the map can be labelled by a symbolic sequence of 0’s and 1’s that mark the location (to the left or right of C) of the successive iterates of the initial point C. For example, the only windows with period 5 are 101\textsuperscript{2}C, 10\textsuperscript{2}1C, and 10\textsuperscript{6}C (101\textsuperscript{2}C represents the periodic window (101\textsuperscript{2}C)\textsuperscript{∞} hereafter.) Indeed, we can define an ordering\textsuperscript{14,15} for these symbolic sequences referring to the natural order in the interval [-1, 1]. These ordering rules are consistent with the ordering of a real number \(\alpha\) defined for a sequence \(S(x)\) with an initial point \(x\) as following\textsuperscript{17}

\[
\alpha(S(x)) = \sum_{i=1}^{\infty} \mu_i 2^{-i},
\]

with \cite{16}

\[
\mu_i = \begin{cases} 
0 & \text{for } \sum_{j=1}^{i} s_j = 0 \\
1 & \text{for } \sum_{j=1}^{i} s_j = 1 \pmod{2}.
\end{cases}
\]

Since the symbolic sequence \(K=S(C)\), called also the kneading sequence, acquires a maximal \(\alpha\) in this metric representation, a symbolic sequence \(S(x)\) corresponds to a real trajectory if and only if it satisfies

\[
\alpha(\sigma^m(S(x))) \leq \alpha(K), \quad m = 0, 1, 2, \ldots,
\]

where \(\sigma\) denotes the shift operator. With this admissibility condition, we can generate all the admissible periodic orbits for a given kneading sequence \(K\). The kneading sequence changes as the controlling parameter alters. Since kneading sequences correspond to orbits coming from C, they should also satisfy the above condition. Thus we obtain the admissibility condition for \(K\) themselves: a symbolic sequence \(K\) can be a kneading sequence if and only if it satisfies

\[
\alpha(\sigma^m(K)) \leq \alpha(K), \quad m = 0, 1, 2, \ldots.
\]

When \(K\) is a periodic string, \(K\) corresponds to a periodic window. From Eq. 5, we can generate all the possible periodic windows. It can be checked that there are only three period 5 windows as those listed above.

With the ordering rules in equation (3), all periodic windows can be ordered to yield the U-sequence. In the logistic map, this U-sequence is consistent with the increasing \(\mu\) order which is listed in Table 1 up to period 7.

**III. UNIVERSAL SEQUENCES IN 2D HÉNON MAPS**

The Hénon map (2) has been extensively studied by using symbolic dynamics\textsuperscript{16–19}. The set of all “primary” tangencies between stable and unstable manifolds determines a binary generating partition which divides the attractor into two parts marked by letters 0 and 1. Any trajectory is encoded by a bi-infinite string \(S(x) = \ldots s_{-m} \ldots s_{-1} s_0 \bullet s_1 s_2 \ldots s_n \ldots\), where \(s_n\) denotes a letter for the \(n\)th image, \(s_{-m}\) a letter for the \(m\)th preimage, each is either 0 or 1,
Table 1: Symbolic sequences for periodic windows of the Hénon map along two different parameter axes and that for the forced Brusselator equations. The axis I (long dashed in Fig. 4) and axis II (dash-dotted) are two axes in and out of the universal-7 region of the Hénon map, on which a complete and incomplete U-sequence is found respectively.

| No. | Period | Word | $a$ range | Brusselator $\omega$ |
|-----|--------|------|-----------|----------------------|
|     |        |      | b=0       | Axis I               | Axis II              |
| 0   | 4      | C    | 0.749-1.164538 | 1.032565           | none                 | 0.45-0.544 |
| 2   | 1C     |      | 0.75-1.32034 | 1.164539           | none                 | 0.545-0.577 |
| 2×2 | 101C   |      | 1.25-1.367 | 1.332035           | none                 | 0.58249-0.58251 |
| 2   | 6      | 1011C| 1.4747-1.47973 | 1.499840           | none                 | 0.5845-0.5848 |
| 3   | 7      | 10111C| 1.57472-1.57541 | 1.585860           | 1.521515-1.522540 | none                 |
| 4   | 5      | 1011C| 1.6244-1.62843 | 1.635395           | 1.588650-1.594165  | none                 |
| 5   | 7      | 101101C| 1.67396-1.6744 | 1.679020           | 1.651665-1.652290  | none                 |
| 6   | 3      | 10C  | 1.75-1.76853 | 1.755474           | 1.776860-1.792900  | 0.5947-0.5975 |
| 7   | 2×3    | 10010C| 1.76854-1.77722 | 1.772255           | 1.792920-1.800525  | 0.6545-0.7025 |
| 8   | 5      | 1001C| 1.8059-1.86136 | 1.863083           | 1.843810-1.844705  | 0.7068-0.7115 |
| 9   | 7      | 10011C| 1.8848-1.88483 | 1.886869           | 1.878845-1.879760  | 0.718-0.7185 |
| 15  | 7      | 100001C| 1.95371-1.95371 | 1.952122           | 1.964545-1.964555  | none                 |
| 16  | 5      | 1000C | 1.98541-1.985468 | 1.984101           | 1.9897795-1.997891  | 0.8359-0.8675 |
| 17  | 7      | 100001C| 1.991814-1.991818 | 1.992925           | 1.982968-1.982969  | none                 |
| 18  | 6      | 10000C| 1.996375-1.996379 | 1.995150           | 1.995150-1.995153  | 0.9015-0.923 |
| 19  | 7      | 100000C| 1.9999096        | 1.997890           | 1.997891           | none                 |

The solid dot indicates the “present” position. In order to extend the grammar for unimodal maps to this map, a “backward” variable is defined as:

$$\beta(S(x)) = \sum_{i=1}^{\infty} \nu_i 2^{-i+1},$$

with

$$\nu_i = \begin{cases} 
1 & \text{for } \sum_{j=0}^{i} (1 - s_j) = 0 \pmod{2} \text{ for } b > 0, \\
0 & \text{for } \sum_{j=0}^{i} (1 - s_j) = 1 \pmod{2} \text{ for } b < 0,
\end{cases}$$

For this 2D map, each primary tangency $C$ is associated with a bi-infinite kneading sequence $K$ (with the first backward letter $s_0$ undetermined which may be 0 or 1) and two symmetrical points $(\alpha(K), \beta_-(K))$ and $(\alpha(K), \beta_+(K) = 1 - \beta_-(K))$ in the symbolic plane corresponding to $s_0 = 0$ and 1 respectively. Analogously to those in unimodal maps, for all admissible points $(\alpha, \beta)$ with $\beta \in [\beta_-(K), \beta_+(K)]$, $\alpha$ should be less than $\alpha(K)$ and thus the
pruning front\textsuperscript{17} is obtained by cutting out rectangles $\{\alpha, \beta | \alpha > \alpha(K), \beta \in [\beta_-(K), \beta_+(K)]\}$ for all points on the partition. The union of these rectangles gives fundamentally forbidden zone. Consequently, the grammar for a word admissible or forbidden in this map can be expressed as: A bi-infinite word is admissible if and only if all its shifts never fall into the fundamentally forbidden zone\textsuperscript{17,18}. It is clear that there are infinitely many kneading sequences (corresponding to infinitely many primary tangencies) in a 2D map to determine the admissibility condition for a word, while there is only one kneading sequence in a 1D map.

**Universality in the Hénon map.** Fig. 3 shows a typical symbolic plane, $(a, b) = (1.4, 0.16)$. The corresponding attractor is shown in Fig. 2 which has a rather complicated structure. Its fractal dimension is 1.16±0.03. Numerically 203 kneading sequences are found as shown in Fig. 2. It is found that the minimal and maximal of all the forward parts of these kneading sequences start with $K_{\min} = 101111010101$ and $K_{\max} = 101111011111$ respectively, corresponding to a minimal and maximal $\alpha$-values $\alpha_{\min} = 0.837560$ and $\alpha_{\max} =0.838466$ of all these kneading sequences. We define two quantities $\delta$ and $L$ as

$$\delta = \alpha_{\max} - \alpha_{\min} = 0.000906,$$

$$L = -[\log_2 \delta] = 10,$$

where $[\log_2 \delta]$ denotes the integer part of $\log_2 \delta$. It is clear that $\delta = 0$ and $L \to +\infty$ in the 1D limit ($b = 0$). For $(a, b) = (1.4, 0.16)$, an unstable periodic orbit with length $n \leq L = 10$ can not tell the difference between these kneading sequences. Indeed, no symbolic string with length $n \leq L$ lies in the interval between $K_{\min}$ and $K_{\max}$. Thus for the unstable periodic orbits with length $n \leq L$, the grammar is completely determined by a symbolic string $K_f$, which is 10111101101 or 101111011011, the first 10 letters of $K_{\min}$ or $K_{\max}$, that is, a word $S(x)$ corresponds to an unstable periodic orbit of the Hénon map for $(a, b) = (1.4, 0.16)$ if and only if it satisfies

$$\alpha(\sigma^m(S(x))) \leq \alpha(K_f), \quad m = 0, 1, 2, \cdots.$$  

This is just the grammar for unimodal maps with a kneading sequence $K_f$. Consequently the unstable periodic orbits of the Hénon map for $(a, b) = (1.4, 0.16)$ can be generated as that of unimodal maps with a kneading sequence $K_f$ (see Eq. 4). The only exception is the unstable periodic orbit $K_f^\infty$ which can not be determined by Eq. 9. We here noted that the Hénon map is divergent for $a = 1.4$ and $b > 0.315$.

As $b$ decreases, $L$ increases. It had already shown\textsuperscript{19} that $L = 32$ for $(a, b) = (1.4, 0.05)$. In the 2D phase space, even for $(a, b) = (1.4, 0.05)$ the attractor has a clear hook indicating that the map is two-dimensional. We emphasize that though the attractors reveal very complicated structure in 2D phase space, the topologies for these attractors may be very close to those in 1D maps that the unstable periodic orbits can be generated with only one kneading sequence to some degree.

Now we consider the universal bifurcation property of the Hénon map in parameter space. Fig. 4(a,b) show the isoperiodic lines\textsuperscript{11,12} for all the nine period 7 windows. Numerically we find that $L \leq 7$ for all the parameters $a$ and $b$ in the region between the two heavy solid lines shown in the figure\textsuperscript{21}. We call this region the universal $- 7$ region hereafter. Thus all the periodic orbits with length $\leq 7$ of the Hénon map in this Universal-7 region can be
determined with only one kneading sequences as those of 1D unimodal maps. Consequently, in this region there is a perfect MSS-sequence up to period ≤ 7 along any axis provided that the axis is never tangent with any isoperiodic lines. These axes are in a sense the same as the axis of $b = 0$ (corresponding to the Logistic map). We present an example of these axes in Fig. 4 (line I, long dashed). The periodic windows on this axis are listed in Table 1. It is clear that they share the universal feature as that of 1D unimodal maps up to period 7. We can also obtain universal – M region numerically for $M = 5, 6, 8, 9, \cdots$ in which there are MSS-sequences for period ≤ M along any axes provided that they are never tangent with any isoperiodic lines for period ≤ M.

In Fig. 4 we also show the borders for the Hénon map exhibit an attracting set with initial points $(x_0, y_0)$ very close to original point (0,0). Comparing to this borders we can say that the universal-7 region is not restricted to very small $b$ values. Thus it is rather likely to get a MSS-sequence up to a relative short period (say, period 7) in the full 2D parameter plane of the Hénon map.

Incomplete U – sequence in the Hénon map. In fact, even on a axis out of the universal region, the Hénon map can exhibit approximately 1D behaviour if the axis is never tangent with any isoperiodic lines. In table 1 we also show the periodic windows on the axis represented by dash-dotted line (II) in Fig. 4. It is clear that all of these words increase monotonically as $a$ increases except the word 10001C and the period windows 10111C, 1000C, 10000C and 100000C are missing.

IV. APPLICATIONS TO ODE’s

The above idea can be extended to many other two- or higher-dimensional systems. Here we only take the Rössler’s equations $^{13}$

$$\begin{align*}
\dot{x} &= -y - x, \\
\dot{y} &= x + ay, \\
\dot{z} &= b - z(x - c),
\end{align*}$$

and the forced Brusselator $^{4}$

$$\begin{align*}
\dot{x} &= A - (B + 1)x - x^2y + \alpha \cos(\omega t), \\
\dot{y} &= Bx - x^2y,
\end{align*}$$

as examples.

The 2D attractor of the Rössler’s equations is usually taken from a section of the 3D flow on the half-plane $y = 0, x < 0$ $^{13}$. It has already shown $^{19}$ that the unstable periodic orbits of the attractor can be generated with only one kneading sequence up to period 12 for parameters $c = 2, d = 4$ and $a = 0.408$ (corresponding to $L \geq 12$). We find similar results ($L \geq 9$) for $c = 2, d = 4$ and $0.125 < a < 0.415$. Table 2 shows the periodic windows up to period 9 in descending $a$ order along with their periods, words and locations on the parameter axis. They are exactly consistent with part of the U-sequence from word C up to 1001011C.
Table 2: The symbolic sequences for the periodic windows of the Rössler’s equations

| No. | Period | Word       | a range       |
|-----|--------|------------|---------------|
| 0   | 0      | C          | 0.125-0.335   |
| 2   | 2×2    | 101C       | 0.336-0.3834  |
| 2²×2| 101101C| 0.3836-0.3852|
| 2   | 6      | 10111C     | 0.390668-0.39097|
| 3   | 8      | 101111C    | 0.393624-0.393638|
| 4   | 9      | 1011111C   | 0.395446-0.395449|
| 5   | 7      | 10111111C  | 0.396638-0.396676|
| 6   | 9      | 10111101C  | 0.398034-0.398041|
| 7   | 5      | 1011C      | 0.399948-0.399966|
| 8   | 9      | 10110101C  | 0.402190-0.402194|
| 9   | 7      | 101101C    | 0.403530-0.403564|
| 01  | 9      | 10110111C  | 0.404690-0.404691|
| 00  | 8      | 1011011C   | 0.406054      |
| 12  | 3      | 10C        | 0.40912-0.41091|
| 2×3 |        | 10010C     | 0.41092-0.41175|
| 13  | 8      | 1001011C   | 0.414432      |

The forced Brusselator had been extensively studied with symbolic dynamics of 1D maps. An incomplete U-sequence up to period six along the axis $A = 0.46 - 0.2\omega$ had already been found by Hao et. al which is also listed in Table 1. Only the periodic window 10001C was missing. Our investigation on the Poincaré map with symbolic dynamics shows that $L = 2$ for the parameter range $0.8056 < \omega < 0.8194$ so that the U-sequence up to period 6 might be incomplete. Recently, J.X. Liu has confirmed that the missing period 10001C is pruned.

V. CONCLUSION AND DISCUSSION

In this paper the universal bifurcation property and its limitations of the Hénon map in 2D parameter space $(a, b)$ is discussed with symbolic dynamics. Two topological quantities $\delta$ and $L$ are defined to characterize this topological universality. In the universal-$L$ region, as that of 1D unimodal map, there is a perfect MSS-sequence up to period $\leq L$ along any axis provided that the axis is never tangent with any isoperiodic lines, though the phase portraits of the Hénon map exhibit very complicated 2D behaviour. Extending this idea to many other two- or higher-dimensional systems ensures that the symbolic dynamics of 1D mappings is an effective technique to investigate the universality in these two- or higher-dimensional systems and then the parameter for definite periodic motion may be predicted. We have presented two examples of ordinary differential equations (ODE’s), the Rössler equations and the forced Brusselator, to demonstrate the validity and robustness of our approach.
It should be noted that only the short period is considered although the theory presented in this paper is also valid for higher period. In fact, in real experiment (or numerical study on ODE’s or PDE’s), only short periodic orbit can be obtained. Our investigation shows that it is not a surprising result that universal sequences with short period are found in many experiments. Moreover, our result shows that it is also a universal phenomenon that universal sequences with short period can be found in many nonlinear dissipative systems. This observation ensures that the parameter of many periodic motion for many dynamical systems (such as some fluid system, e.g. ref. 23) can be well predicted.

In this paper we also show that even on a axis out of the universal region, the Hénon map can exhibit approximately 1D behaviour. This observation interprete the numerical results that in some nonlinear dynamical systems only incomplete U-sequences had been found. Anyway, our defined universal-M region gives the background to interprete the experimental and numerical observations that complete or incomplete U-sequences with short period can be found in many dissipative systems, and understand the limitations that 1D symbolic dynamics can be used to study two- or high-dimensional dissipative systems.

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**Figure Captions**

Fig. 1 The Logistic map $x_{n+1} = 1 - \mu x_n^2$ with $\mu = 1.75488$ exhibits a 3-cycle of the type 10C.

Fig. 2 The strange attractor of the Hénon map for $(a, b) = (1.4, 0.16)$. The heavy lines outline the strange attractor. The diamonds are the “primary” tangencies. The dotted line connecting them divides the full attractor into two subsets marked by 0 and 1.

Fig. 3 The symbolic plane of the Hénon map for $(a, b) = (1.4, 0.16)$. (b) an enlarged part of the symbolic plane.
Fig. 4 The isoperiodic lines together with the universal-7 region (the region between two heavy lines) and the borders for the Hénon map exhibit an attracting set with initial points \((x_0, y_0)\) very close to original point \((0, 0)\) (the two heavy short dashed lines). The long dashed (I) and dash-dotted (II) lines represent two axes in and out of the universal-7 region, on which a complete and incomplete U-sequence is found respectively. (b) an enlarged part.