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ABSTRACT

The flow of an electrically conducting fluid through a curved channel with wavy boundaries is studied. The waviness of the curved boundaries is sinusoidal and periodic. The analytical results for the velocity field and the volumetric flow rate are obtained using the boundary perturbation method. The effects of the wavy boundaries, the channel radius of curvature, and the applied magnetic field on the flow field are analyzed. The study shows that the impact of the wavy boundaries on the flow decreases with the increase in the flow Hartmann number. However, the flow rate increases for any alignment of the wavy curved boundaries and for the wave numbers less than a threshold wavenumber (depending on the radius of curvature and the Hartmann number), and a further increase in the flow rate occurs with the increase in the phase difference between the wavy curved boundaries. On the other hand, the flow rate decreases with the increasing wavenumber, and for a sufficiently large wavenumber and Hartmann number, the phase difference between the wavy curved boundaries becomes irrelevant to the flow.

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I. INTRODUCTION

The experimental and theoretical research involving Magnetohydrodynamic (MHD) flow is continuously evolving with time because of its intense applications in applied sciences and engineering. The main importance lies in the development of non-mechanical pumps designed on the principle of MHD. The MHD pumps are found to be more reliable than classical mechanical pumps, which are more vulnerable to wearing off. Another pro is that the Lorentz force acting on electrically conducting fluid can be harnessed to control and optimize the fluid flow. The MHD pumping technology is used for fluid transport in industry concerning nuclear energy and metallurgy. In nuclear power plants, MHD pumps are used to transport coolants for the nuclear reactors, and in metallurgical processes, it is used to transport molten metals. In liquid chromatography, MHD is also used in purification and separation processes. Moreover, microscale MHD technologies including micro-pumps and micro-mixers have extensive applications in biomedical technology and space exploration.

A theoretical and experimental study of the MHD pump was presented by Jang and Lee, where the fluid was pumped by the Lorentz force. The effect of wall slippage in MHD pumps was analyzed by Rivero and Cuevas. Smolentsev studied MHD duct flows under hydrodynamic slip conditions. The microchannel flow control through a combined electromagneto-hydrodynamic transport was presented by Chakraborty and Paul. In addition, see the study by Ellahi et al. and Ahmed et al. for related works. In addition to the flows through smooth channels, several studies were made on fluid transport through rough channels, where the boundary roughness is reported to have a significant effect on the efficiency of the fluid transport. For example, the study of the effect of boundary corrugations on flow between two parallel plates was done by Wang. The main finding of this study was that an increase in the flow rate can be ensured through the phase difference between the corrugated channels. Another example is the viscous flow through a corrugated pipe investigated by Phan-Thien. Furthermore, Moradi and Floyan analyzed the flow in an annulus with groovy walls. Duan and Muzychka considered the effect of corrugations on developed...
laminar flow in microtubes. Chu\textsuperscript{15} studied the slip flow in an annulus with corrugated walls. Buren et al.\textsuperscript{16} studied electromagnetohydrodynamic flow through a microparallel channel with corrugated walls, when the flow is both driven by the pressure gradient and the Lorentz force due to an applied magnetic field. In this article, the authors analytically showed that an increase in the flow is only possible for out-of-phase corrugated walls. Okechi and Asghar\textsuperscript{17} considered the flow between the corrugated curved channels, studying the combined effects of curvature and the boundary corrugations. A significant result of this study among others is that the flow rate may be increased for any configuration of the corrugated curved boundaries; either with or without phase difference between the corrugated curved boundaries, depending on the channel radius of curvature and the corrugation wavelength.

The present study deals with the analysis of the flow of an electrically conducting fluid through a wavy curved channel, subjected to an applied magnetic field. This problem is yet to be considered in the literature. The existing MHD studies are mainly carried out for smooth curved channels. Therefore, the understanding of the flow situation in wavy curved channels is not as broad, compared to that of the smooth curved channels. Taking this into consideration, we initiate the investigation of the MHD flow through a wavy curved channel. The core objective of this study is to analyze the flow rate in wavy curved channels, studying the key parameters of the emerging parameters of the problem.

The fundamental mathematical model of the problem is formulated in Sec. II. The perturbation analysis for the velocity distribution and the volumetric flow rate of the problem are given in Sec. III. In Sec. IV, the discussion of the analytical results is provided, and the underlying conclusion in Sec. V ends the study.

II. PROBLEM FORMULATION

We examine the flow of an incompressible viscous electrically conducting fluid through a wavy curved channel. The channel is bounded by two curved periodic wavy boundaries, separated by a distance $2d$ with a radius of curvature $k$ ($>d$) (see Fig. 1). The channel is curved along the $x$-direction, with its center at $O$, and $(x, y, z)$ defines a mutually orthogonal curvilinear coordinate system for the curved channel. The outer wavy boundary is defined by

$$y_{O} = d + a \sin\left(\frac{2\pi}{\delta}x\right),$$  \hspace{1cm} (1)

and the inner wavy boundary is given by

$$y_{I} = -d + a \sin\left(\frac{2\pi}{\delta}z + \nu\right),$$  \hspace{1cm} (2)

where $a$ is the wavy boundary amplitude, $\delta > 0$ is the wavy boundary wavelength, $\nu \in [0, \pi]$ is the phase difference between the two wavy curved boundaries, and each $\nu$ corresponds to different configurations of the wavy curved channel. The wavy boundaries are in-phase when $\nu = 0$, while $\nu > 0$ denotes the out-of-phase wavy boundaries, and $\nu = \pi$ gives the completely out-of-phase wavy boundaries. The smooth boundaries are located at $y = d$ and $y = -d$. The dimensional continuity and momentum equations for the steady (Stokes) flow can be written as

$$\frac{k}{y + k} \frac{\partial u}{\partial x} + \frac{1}{y + k} \frac{\partial}{\partial y} ((y + k)v) + \frac{\partial w}{\partial z} = 0, \hspace{1cm} (3)$$

$$\mu \left(\frac{\partial}{\partial y} \left(\frac{1}{y + k} \frac{\partial}{\partial y} (y + k)u\right) + \frac{2k}{(y + k)^2} \frac{\partial v}{\partial x}\right) + \frac{k^2}{(y + k)^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + F_u = \frac{k}{y + k} \frac{\partial p}{\partial x}, \hspace{1cm} (4)$$

$$\mu \left(\frac{\partial}{\partial y} \left(\frac{1}{y + k} \frac{\partial}{\partial y} (y + k)v\right) - \frac{2k}{(y + k)^2} \frac{\partial u}{\partial x}\right) + \frac{k^2}{(y + k)^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} + F_v = \frac{\partial p}{\partial y}, \hspace{1cm} (5)$$

$$\mu \left(\frac{\partial}{\partial y} \left(\frac{1}{y + k} \frac{\partial}{\partial y} (y + k)w\right) + \frac{k^2}{(y + k)^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_w = \frac{\partial p}{\partial z}. \hspace{1cm} (6)$$

Equation (3) is the continuity equation, while Eqs. (4)–(6) are the components of the momentum in the $x$-, $y$-, and $z$-directions, respectively. $\mathbf{U} = (u, v, w)$ is the velocity in the $x$-, $y$-, and
$z$-directions, and $F = (F_x, F_y, F_z)$ represents the magnetic force. The constant dynamic viscosity of the fluid is denoted by $\mu$, and $p$ is the pressure. Suppose that the flow is generated by constant pressure gradient $-p_x = G$, in the $x$-direction, with velocity vector $U = (u(y, z), 0, 0)$, and the wavy curved channel is subjected to an applied magnetic field $B = (0, 0, B_0)$. The induced current density can be expressed as $J = \sigma(E + U \times B)$ by Ohm’s law, where $\sigma$ is the fluid of conductivity and $E$ is the electric field.

For a negligible electric field, it follows that $J \sim auB$, through Ohm’s law. Thus, Lorentz force defined by $F = J \times B = (-\sigma u B_0^2, 0, 0)$ is produced. The Lorentz force acts on the fluid, but in the opposite direction to the fluid motion. Now, scaling the lengths by $d$, the pressure gradient by $G$, and the velocity by $Gd^2/\mu$, we can now write the reduced dimensionless equation governing the MHD flow as

$$
\frac{1}{y} \frac{\partial}{\partial y} \left( y + k \right) \frac{\partial u}{\partial y} - \frac{u}{(y + k)^2} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{\sigma u} = -\frac{k}{(y + k)^2},
$$

(7)

with the boundary conditions

$$
u(y, z) = 0 \text{ at } y = y_0 = 1 + \varepsilon \sin(az)
$$

and

$$
y = y_1 = -1 + \varepsilon \sin(az + \phi),
$$

(8)

where $Ha = B_0 d / (\sigma u)$ is the Hartmann number, $\varepsilon$ is the dimensionless amplitude, and $\phi = 2\pi d / \delta$ is the wavenumber. The Hartmann number is the dimensionless number representing the ratio of the Lorentz force to the viscous force.

### III. Perturbation Analysis

#### A. Velocity

For small amplitudes of the waves, in other words, $\varepsilon \ll 1$, the fluid velocity $u$ can be expanded as

$$u(y, z) = u_0(y) + \varepsilon u_1(y, z) + \varepsilon^2 u_2(y, z) + \cdots.
$$

(9)

Using the regular perturbation expansion Eq. (9), the Taylor expansion of the boundary conditions at $y = y_0$ and $y = y_1$ about the points $y = 1$ and $y = -1$, respectively, gives

$$
u_{y_1 = 1 + \varepsilon \sin(az)} = \left( u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \right)_{y_1 = 1} + \varepsilon \sin(az) \left( \frac{d u_0}{d y} + \varepsilon u_1 \right)_{y_1 = 1} + \frac{1}{2} \varepsilon^2 \sin^2(az) \left( \frac{d^2 u_0}{d y^2} \right)_{y_1 = 1} + \cdots = 0,
$$

(10)

$$
u_{y_1 = 1 + \varepsilon \sin(az + \phi)} = \left( u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \right)_{y_1 = 1} + \varepsilon \sin(az + \phi) \left( \frac{d u_0}{d y} + \varepsilon u_1 \right)_{y_1 = 1} + \frac{1}{2} \varepsilon^2 \sin^2(az + \phi) \left( \frac{d^2 u_0}{d y^2} \right)_{y_1 = 1} + \cdots = 0.
$$

(11)

To obtain the analytical solution for the problem, Eqs. (9)–(11) are substituted in Eqs. (7) and (8), and grouped to obtain the following perturbation problems up to the second-order in $\varepsilon$.

#### 1. $\mathcal{O}(1)$ Problem

The zeroth-order equation with the boundary conditions is

$$
\frac{1}{y} \frac{\partial}{\partial y} \left( y + k \right) \frac{\partial u_0}{\partial y} - \frac{u_0}{(y + k)^2} - \frac{u_0}{\sigma u} = -\frac{k}{y + k},
$$

(12)

$$
u_{y_1 = 0} = 0 \text{ and } \nu_{y_1 = 1} = 0.
$$

The solution of Eq. (12) is obtained as

$$u_0(y) = c_0 I_1(\text{Ha}(y + k)) + c_1 K_1(\text{Ha}(y + k)) + k[\text{Ha}^2(y + k)]^{-1}.
$$

(13)

Equation (13) gives the zeroth-order velocity of a conducting fluid flow through a curved channel with smooth walls, where $I_1$ and $K_1$ are modified Bessel function of order one and of the first and the second kind, respectively.

#### 2. $\mathcal{O}(\varepsilon)$ Problem

The first-order problem and with the wavy boundary conditions can be written as

$$
\frac{1}{y} \frac{\partial}{\partial y} \left( y + k \right) \frac{\partial u_1}{\partial y} + \frac{\partial^2 u_1}{\partial z^2} - \frac{u_1}{(y + k)^2} - \frac{u_1}{\sigma u} = 0,
$$

$$
u_{y_1 = 1} = -\frac{du_1}{dy} \bigg|_{y_1 = 1} \sin(az + \phi),
$$

(14)

$$
u_{y_1 = -1} = -\frac{du_1}{dy} \bigg|_{y_1 = -1} \sin(az).
$$

The form of the differential equation with the boundary conditions suggests the solution

$$u_1(y, z) = U_1(y) \sin(az) + U_2(y) \cos(az).
$$

(15)

Substituting Eq. (13) in Eq. (14), we have the undermentioned ordinary differential equations (ODEs) with the boundary conditions at $O(\varepsilon)$, namely,

$$
(y + k)^2 \frac{d^2 U_1}{dy^2} + (y + k) \frac{d U_1}{dy} - [n^2(y + k)^2 + 1] U_1 = 0,
$$

$$
U_1 \bigg|_{y_1 = 1} = -\cos(\phi) \frac{du_0}{dy} \bigg|_{y_1 = 1},
$$

(16)

$$
U_1 \bigg|_{y_1 = -1} = -\frac{du_0}{dy} \bigg|_{y_1 = -1},
$$

and

$$
(y + k)^2 \frac{d^2 U_2}{dy^2} + (y + k) \frac{d U_2}{dy} - [n^2(y + k)^2 + 1] U_2 = 0,
$$

$$
U_2 \bigg|_{y_1 = 1} = -\sin(\phi) \frac{du_0}{dy} \bigg|_{y_1 = 1},
$$

(17)

$$
U_2 \bigg|_{y_1 = -1} = 0.
$$

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On solving Eqs. (16) and (17), we obtain the following exact solutions:

\[ U_1(y) = c_1I_1(n_1(y + k)) + c_3K_1(n_1(y + k)), \]
\[ U_2(y) = c_4I_1(n_1(y + k)) + c_5K_1(n_1(y + k)), \]

where \( n_1^2 = \alpha^2 + 4 \), \( I_1 \) and \( K_1 \) are modified Bessel function of order one and of the first and second kind, respectively. The solution Eq. (15) with Eqs. (18) and (19) constitutes the first-order solution.

### 3. \( \ell^2 \) problem

The second-order problem is obtained as

\[
\frac{1}{y + k} \frac{\partial}{\partial y} \left( (y + k) \frac{\partial U_2}{\partial y} \right) + \frac{\partial^2 U_2}{\partial z^2} - \frac{u_2}{(y + k)^2} - Ha^2 u_2 = 0,
\]

\[
u_2|_{y=-1} = -\frac{1}{2} \frac{d^2 u_0}{dy^2}\bigg|_{y=-1} \sin^2(\alpha z + v) - \frac{\partial u_1}{\partial y}\bigg|_{y=-1} \sin(\alpha z + v),
\]

\[
u_2|_{y=1} = -\frac{1}{2} \frac{d^2 u_0}{dy^2}\bigg|_{y=1} \sin^2(\alpha z) - \frac{\partial u_1}{\partial y}\bigg|_{y=1} \sin(\alpha z).
\]

Similarly, the form of the differential equation together with the boundary conditions admits the solution

\[ u_2(y, z) = U_5(y) + U_4(y)\sin(2\alpha z) + U_5(y)\cos(2\alpha z). \]

The expression in Eq. (21) is a superposition of a periodic and a non-periodic solution.

Substituting Eq. (21) in Eq. (20), we also have the ODEs with the boundary conditions at \( O(\ell^2) \) as

\[
(y + k)^2 \frac{d^2 U_5}{dy^2} + (y + k) \frac{d U_5}{dy} - \left[ Ha^2(y + k)^2 + 1 \right] U_5 = 0,
\]

\[
u_5|_{y=-1} = -\frac{1}{2} \frac{d^2 u_0}{dy^2}\bigg|_{y=-1} \frac{1}{2} \cos(v) \frac{d U_5}{dy}\bigg|_{y=-1} - \frac{1}{2} \sin(v) \frac{d U_5}{dy}\bigg|_{y=-1},
\]

\[
u_5|_{y=1} = -\frac{1}{2} \frac{d^2 u_0}{dy^2}\bigg|_{y=1} \frac{1}{2} \frac{d U_5}{dy}\bigg|_{y=1}, \quad (22)
\]

Thus, the exact solutions for Eqs. (22)–(24) after some work are

\[ U_5(y) = c_6 I_1(Ha(y + k)) + c_7 K_1(Ha(y + k)), \]

\[ U_4(y) = c_8 I_1(n_2(y + k)) + c_9 K_1(n_2(y + k)), \]

\[ U_5(y) = c_{10} I_1(n_2(y + k)) + c_{11} K_1(n_2(y + k)), \]

where \( n_2^2 = \alpha^2 + 4 \), and as before \( I_1 \) and \( K_1 \) are modified Bessel functions of order one and of the first and second kind, respectively. The solution Eq. (21) with Eqs. (25)–(27) constitutes the second-order solution. The expressions for coefficients \( c_{0-11} \) are given in Appendix.

### B. Volumetric flow rate

The dimensionless volumetric flow rate per unit cross-sectional area is defined as

\[ Q = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} \int_{y_1}^{y_0} u(y, z) dy dz. \]

The analytical expression of the integral Eq. (28) up to second-order in \( \epsilon \) using Taylor expansion about the unperturbed walls, at \( y = 1 \) and \( y = -1 \) is obtained as

\[
Q = \int_0^{2\pi/\alpha} \int_{y_1}^{y_0} u(y, z) dy dz = \int_0^{y_0} u_0(y) dy + \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} \int_1^{y_0} u_1(y, z) dy dz + \frac{\alpha}{2\pi} \left\{ \int_0^{2\pi/\alpha} \int_1^{y_0} u_2(y, z) dy dz + \int_0^{2\pi/\alpha} \left[ \sin(\alpha z) u_1(y, z) \right]_{y=1} - \sin(\alpha z + v) u_1(y, z) \right\} dy dz
\]

\[
+ \frac{1}{2} \int_0^{2\pi/\alpha} \left[ \sin^2(\alpha z) \frac{d u_0(y)}{dy} \right]_{y=1} - \sin^2(\alpha z + v) \frac{d u_0(y)}{dy} \right]_{y=1} \right\} dz + \cdots. \]

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After some calculations, we present the analytical expression for the volumetric flow rate up to the order of $O(\epsilon^2)$ as

$$Q(k, \alpha, \nu, Ha; \epsilon) = Q_0(k, Ha)(1 + \epsilon^2 B(k, \alpha, \nu, Ha) + O(\epsilon^4)), \tag{30}$$

where

$$Q_0(k, Ha) = \frac{1}{Ha^2} \left( c_2 Ha(l_0(\text{Ha}(k + 1)) - l_0(\text{Ha}(k - 1))) - c_1 Ha(K_0(\text{Ha}(k + 1)) - K_0(\text{Ha}(k - 1))) + k \ln \left( \frac{k + 1}{k - 1} \right) \right), \tag{31}$$

and

$$B(k, \alpha, \nu, Ha) = \frac{1}{4} \left( c_2 Ha \left( l_1(\text{Ha}(k + 1)) - l_1(\text{Ha}(k - 1)) \right) - c_1 Ha \left( K_0(\text{Ha}(k + 1)) + K_1(\text{Ha}(k + 1)) \right) + \frac{k}{Ha^2(k + 1)^2} \right)$$

$$+ \frac{1}{2} \left[ c_2 l_1(n_1(k - 1)) + c_2 K_1(n_1(k - 1)) \right] - \left[ c_2 l_1(n_1(k - 1)) + c_3 K_1(n_1(k - 1)) \right] \cos(\nu)$$

$$- \frac{1}{2} \left[ c_2 l_1(n_1(k - 1)) + c_3 K_1(n_1(k - 1)) \right] \sin(\nu) + \frac{1}{Ha} \left[ c_2 Ha(l_0(\text{Ha}(k + 1)) - l_0(\text{Ha}(k - 1))) \right] - c_1 Ha(K_0(\text{Ha}(k + 1)) - K_0(\text{Ha}(k - 1))) \right). \tag{32}$$

The expression $Q_0$ is the volumetric flow rate of a smooth curved channel with no waviness, and $B$ is the waviness function, which determines the effect of the waviness on the total flow $Q$ rate up to $O(\epsilon^2)$.

IV. RESULTS AND DISCUSSION

This section deals with the physical interpretation of the analytical results obtained in Sec. III. The velocity and the volumetric flow rate are analyzed. The influence of the Lorentz force on the electrically conducting viscous fluid flow in the corrugated curved channel is particularly of interest. This analysis provides the physical flow behavior of the MHD flow and quantifies the effect of the magnetic field on the flow. The emerging non-dimensional parameters are now $k, \alpha, \nu$, and $Ha$. Therefore, to investigate the effects of the wavy boundaries, and the channel curvature on the flow subjected to an applied magnetic field, the subsequent discussions will be concerned with the effects of $k, \alpha, \nu$, and $Ha$ on the flow.

A. The velocity distribution

The axial velocity variation with the radius of curvature $k$ in Fig. 2 indicates that the axial velocity increases with $k$ for a given $Ha$; this observation is similar to that in the absence of an applied magnetic field. To understand the influence of the magnetic field; we recall that $Ha$ being the ratio of the magnetic field to the viscous force determines the relative strength of the two forces. Increasing $Ha$ will increase the Lorentz force, which will eventually dominate the viscous force. Figure 3 shows that the axial velocity decreases with an increase of $Ha$. The physical explanation lies in: the flow tends to drag the magnetic field and the magnetic field exerts the force in the opposite direction of the fluid flow, reducing the axial velocity in turn.

The 3D-graphical illustration of the modification of the axial velocity in the channel is provided in Fig. 4, taking one wavelength.

In Figs. 4(a)–4(d), it is observed that the phase difference $\nu$ (between the wavy boundaries) determines the pattern of the velocity distribution in the channel. Geometrically, the channel height remains constant in the x-direction, while varying periodically in the z-direction. The axial velocity increases and decreases when the periodically varying height of the channel in the z-direction increases and decreases, respectively.

This trend is further depicted by the axial velocity contours in Fig. 5. Note that we have also given the velocity contours of a smooth curved channel with no waviness ($\epsilon = 0$ in Fig. 5(a) to show the distinguishing effects of the waviness on the velocity distribution. In

![FIG. 2](image-url)
FIG. 3. The axial velocity profiles in the wavy curved channel for different Hartmann numbers \( H_a \); for \( H_a = 0 \) (dotted line), \( H_a = 1.5 \) (dashed-dotted line), \( H_a = 2 \) (dashed line); and \( H_a = 5 \) (solid line), when \( z = \pi/2 \), \( \alpha = 1 \), \( \nu = \pi \), \( \epsilon = 0.1 \), and \( k = 1.5 \).

Fig. 5(b), the channel height is periodically constant when \( \nu = 0 \), but skews toward the inner wall of the channel, this is due to the effects of the channel curvature. Similarly, the skewness of the velocity is seen in Figs. 5(c)–5(e), when \( \nu > 0 \). However, Fig. 6 clearly indicates that the skewness of the velocity toward the inner wall of the channel diminishes as \( k \) is increased; i.e., the curvature effects diminish as \( k \) increases. The decreasing effects of the magnetic field are illustrated in Fig. 7, where it is evident that the peak of the velocity decreases with the increase in the Hartman number, and the peak remains approximately the same irrespective of the phase difference between the corrugated walls.

B. The volumetric flow rate analysis

The total volumetric flow rate \( Q \) (up to the second-order) given by Eq. (30) can be discussed through the function \( \theta(\alpha, \nu, k, H_a) \): The flow rate \( Q \) will be increased (decreased) by a positive (negative) waviness function \( \theta \). Therefore, in what follows, we concentrate on the variations of \( \theta \) [see Eq. (32)] with the parameters \( \alpha \), \( \nu \), and \( k \) as well as the Hartmann number \( H_a \).

In Fig. 8, we have shown the variation of \( \theta \) with \( \alpha \), for different values of \( \nu \) and \( H_a \), and a fixed \( k \). It can be seen that \( \theta \) increases with the phase difference \( \nu \). Thus, the completely out-of-phase wavy boundaries will give the maximum flow rate. This is because the flow resistance is least when the wavy curved walls are completely out of phase. However, the flow rate decreases with increasing \( \alpha \) (i.e., decreasing wavelength) as the function \( \theta \) becomes negative. For sufficiently large values of \( \alpha \), the \( \theta \) curves reach the same asymptote for all values of \( \nu \), hence the phase difference becomes immaterial. The frequency of waviness (or roughness) of the channel is
determined by the wavenumber $\alpha$, such that when $\alpha$ increases, the wall roughness of the channel increases, which physically leads to an increasing flow resistance. The implications of a growing flow resistance are that it results in a decreasing flow rate. Furthermore, increasing $Ha$ decreases $\theta$, and at a large value of $Ha$, the influence of $\nu$ on $\theta$ also becomes negligible even for small $\alpha$. This phenomenon is well understood from Fig. 8(c), where $\theta$ appears to have a single curve irrespective of $\nu$. Although the flow rate of the problem can be enhanced by the roughness for some sufficiently small $\alpha$, the flow enhancement diminishes with the increasing magnitude of $Ha$, i.e., from Figs. 8(a)–8(c).

In Fig. 9, $\theta$ is plotted against $Ha$, for several values of $\alpha$ and $\nu$, fixing $k$. The function $\theta$ is positive (increasing the flow rate) for all values of the phase difference $\nu$ in Fig. 9(a), but $\theta$ becomes negative (decreasing the flow rate) for all values of $\nu$ in Fig. 9(c), when $\alpha$ is large. Figure 9 further shows that for $Ha > 3$, the phase difference $\nu$ becomes immaterial. Again for a large Hartmann number, the effect of the applied magnetic field on the total flow rate $Q$ is seen to be the same, regardless of the phase difference between the wavy curved boundaries.

To discuss the effects of $Ha$ on $\theta$ against $k$, Fig. 10 is provided. Figure 10(a) indicates that the function $\theta$ increases with...
increasing $k$, for $\nu > 0$, i.e., when the wavy boundaries are out of phase. In Fig. 10(b), the function $\vartheta$ increases with $k$, for the completely out of phase wavy boundaries only. The effect of a larger value of the Hartmann number is also shown in Fig. 10(c); the curves of $\vartheta$ for all values of $\nu \geq 0$ become indistinguishable as they merge into a single curve, which decreases with $k$. In general, for a large radius of curvature ($k \to \infty$), Fig. 10 shows that the function $\vartheta$ tends to exhibit a constant behavior for any $\nu$ and $Ha$, and the flow rate is always decreased when the wavy boundaries are in phase. This phenomenon agrees with Buren et al.\textsuperscript{16} for a wavy straight channel flow ($k \to \infty$) in a magnetic field. It is important to emphasize that for the flow in a wavy curved channel in an applied magnetic field, the in-phase wavy boundaries may not decrease the flow rate as we have observed in this study, unlike the case of the flow through a wavy straight channel discussed by Buren et al.\textsuperscript{16}

From our previous observations, we understand that the flow rate can be increased or decreased depending on the contribution of the waviness function $\vartheta$ through the expression in Eq. (30). Therefore, to ensure an increment in the flow rate, $\vartheta$ must be positive. Now, there exists a wavenumber $\alpha_T$ below which $\vartheta$ is positive and beyond which $\vartheta$ is negative. This wavenumber is considered as
FIG. 7. Axial velocity contours in the wavy curved channel, when \( k = 1.5, Ha = 3, \alpha = 1, \) and \( \epsilon = 0.1. \) (a) \( \nu = 0; \) (b) \( \nu = \pi/3; \) (c) \( \nu = \pi/2; \) (d) \( \nu = \pi. \)

FIG. 8. The variation of \( \vartheta \) with wavenumber \( \alpha, \) for \( k = 1.5. \) When \( \nu = 0 \) (solid lines); \( \nu = \pi/3 \) (dashed lines); \( \nu = \pi/2 \) (dashed-dotted lines); \( \nu = \pi \) (dotted lines). (a) \( Ha = 0; \) (b) \( Ha = 1; \) (c) \( Ha = 3. \)
FIG. 9. The variation of $\vartheta$ with the Hartman number $Ha$, for $k = 1.5$. When $\nu = 0$ (solid lines); $\nu = \pi/3$ (dashed lines); $\nu = \pi/2$ (dashed-dotted lines); $\nu = \pi$ (dotted lines). (a) $\alpha = 0.3$; (b) $\alpha = 1$; (c) $\alpha = 3$.

FIG. 10. The variation of $\vartheta$ with radius of curvature $k$, for $\alpha = 0.5$. When $\nu = 0$ (solid lines); $\nu = \pi/3$ (dashed lines); $\nu = \pi/2$ (dashed-dotted lines); $\nu = \pi$ (dotted lines). (a) $Ha = 0$; (b) $Ha = 0.5$; (c) $Ha = 3$. 
for sufficiently large $k$.\(^{16}\) For a larger value of $Ha$, i.e., $Ha = 4$, $\alpha_T$ decreases rapidly with increasing $k$ for each $\nu$.

V. CONCLUSION

In the study of magnetohydrodynamic flow through a wavy curved channel, we elicit the following conclusions from the analytical results. The magnetic field is found to retard the fluid motion. For magnetohydrodynamic flow through wavy straight channels, the flow rate can only be enhanced when the channel wavy boundaries are out of phase. Conversely, the present study shows that the flow rate in the wavy curved channel may be enhanced, for both in-phase and out-of-phase wavy boundaries for an appropriate wavenumber $\alpha$, Hartmann number $Ha$, and radius of curvature $k$. However, for sufficiently large $k$, the results of this study tend to that of a wavy straight channel; in this case, the flow rate can only be increased by the out-of-phase wavy boundaries.

Channel roughness or waviness are either practically induced to enhance fluid transport or incurred in the fabrication process. Surface roughness also occurs overtime as result of the adsorption of molecules. Nevertheless, we have shown that the completely out-of-phase wavy curved boundaries will give the maximal flow, when the pressure driven flow is along the $x$-direction, as shown in Fig. 1.

For sufficiently large $\alpha$, the flow rate is decreased by the waviness. Hence, a smooth curved channel (without roughness or waviness) would give a higher flow rate and maybe the best in this situation as compared to a wavy curved channel for the same $Ha$ and $k$. The influence of the wavy boundaries on the flow decreases with the $Ha$, for any $k$. Moreover, by increasing $Ha$ and $\alpha$, the effect of the phase difference on the flow rate becomes irrelevant. The results of this study demonstrate the flow characteristics in a rough curved channel and may have potential applications in optimizing fluid pumping and mixing.

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APPENDIX: THE COEFFICIENTS

The expressions of the coefficients $c_0$ to $c_{11}$ in Sec. III are given as follows:

\[
c_0 = \frac{k((k+1)K_i(Ha(k+1)) - (k-1)K_i(Ha(k-1)))}{Ha^2(k^2-1)[I_1(Ha(k+1))K_i(Ha(k+1)) - K_i(Ha(k+1))I_1(Ha(k+1))]},
\]

\[
c_1 = \frac{k((k-1)I_1(Ha(k+1)) - (k+1)I_1(Ha(k-1)))}{Ha^2(k^2-1)[I_1(Ha(k+1))K_i(Ha(k+1)) - K_i(Ha(k+1))I_1(Ha(k+1))]},
\]

\[
c_2 = \frac{U_{1|j=0}K_i(n_1(k+1)) - U_{1|j=j}K_i(n_1(k+1))}{K_i(n_1(k+1))I_1(n_1(k+1)) - I_1(n_1(k+1))K_i(n_1(k+1))},
\]
\[c_3 = \frac{U_{1|y=n-1}I_1(n_1(k+1)) - U_{1|y=n}I_1(n_1(k-1))}{K_1(n_1(k-1))I_1(n_1(k+1)) - I_1(n_1(k-1))K_1(n_1(k+1))},\]  
(A4)

\[c_4 = \frac{-U_{2|y=n-1}K_1(n_1(k-1))}{K_1(n_1(k-1))I_1(n_1(k+1)) - I_1(n_1(k-1))K_1(n_1(k+1))},\]  
(A5)

\[c_5 = \frac{U_{2|y=n-1}I_1(n_1(k+1))}{K_1(n_1(k-1))I_1(n_1(k+1)) - I_1(n_1(k-1))K_1(n_1(k+1))},\]  
(A6)

\[c_6 = \frac{U_{1|y=n}K_1(Ha(k+1)) - U_{1|y=n}K_1(Ha(k-1))}{K_1(Ha(k+1))I_1(Ha(k-1)) - I_1(Ha(k+1))K_1(Ha(k-1))},\]  
(A7)

\[c_7 = \frac{U_{1|y=n}I_1(Ha(k-1))}{U_{1|y=n}K_1(Ha(k+1)) - U_{1|y=n}K_1(Ha(k-1))},\]  
(A8)

\[c_8 = \frac{U_{1|y=n}K_1(n_2(k+1)) - U_{1|y=n}K_1(n_2(k-1))}{K_1(n_2(k+1))I_1(n_2(k+1)) - I_1(n_2(k+1))K_1(n_2(k-1))},\]  
(A9)

\[c_9 = \frac{-U_{1|y=n}K_1(n_2(k+1))}{K_1(n_2(k+1))I_1(n_2(k-1)) - I_1(n_2(k+1))K_1(n_2(k-1))},\]  
(A10)

\[c_{10} = \frac{U_{1|y=n}I_1(n_2(k-1))}{U_{1|y=n}K_1(n_2(k+1)) - U_{1|y=n}K_1(n_2(k-1))},\]  
(A11)

\[c_{11} = \frac{-U_{1|y=n}I_1(n_2(k+1))}{U_{1|y=n}K_1(n_2(k+1)) - U_{1|y=n}K_1(n_2(k+1))},\]  
(A12)

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