Implication Zroupoids and Birkhoff Systems

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Abstract

An algebra $A = \langle A, \rightarrow, 0 \rangle$, where $\rightarrow$ is binary and 0 is a constant, is called an implication zroupoid (I-zroupoid, for short) if $A$ satisfies the identities: 
\[(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)]',\]
where $x' := x \rightarrow 0$, and $0'' \approx 0$. These algebras generalize De Morgan algebras and $\lor$-semilattices with zero. Let $I$ denote the variety of implication zroupoids. For details on the motivation leading to these algebras, we refer the reader to [San12] (or the relevant papers mentioned at the end of this paper). The investigations into the structure of the lattice of subvarieties of $I$, begun in [San12], have continued in [CS16a, CS16b, CS17a, CS17b, CS18a, CS18b, CS19] and [GSV19]. The present paper is a sequel to this series of papers and is devoted to making further contributions to the theory of implication zroupoids.

The identity (BR): $x \land (x \lor y) \approx x \lor (x \land y)$ is called the Birkhoff’s identity. The main purpose of this paper is to prove that if $A$ is an algebra in the variety $I$, then the derived algebra $A_{mj} := \langle A; \land, \lor \rangle$, where $a \land b := (a' \rightarrow b)'$ and $a \lor b := (a' \land b')'$, satisfies the Birkhoff’s identity. As a consequence, we characterize the implication zroupoids $A$ whose derived algebras $A_{mj}$ are Birkhoff systems. It also follows from the main result that there are bisemigroups that are not bisemilattices but satisfy the Birkhoff’s identity, which suggests a more general notion, than Birkhoff systems, of “Birkhoff bisemigroups” as bisemigroups satisfying the Birkhoff’s identity. The paper concludes with an open problem on Birkhoff bisemigroups.

Keywords: symmetric implication zroupoid, De Morgan algebra, semilattice, Birkhoff identity, Birkhoff system, lattice of subvarieties

2010 AMS subject class: Primary : 06D30, 06E75; Secondary : 08B15, 20N02, 03G10

1 Introduction

An algebra $A = \langle A, \rightarrow, 0 \rangle$, where $\rightarrow$ is binary and 0 is a constant, is called an implication zroupoid (I-zroupoid, for short) if $A$ satisfies the following identities: 

(1) \[(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)]',\]
where $x' := x \rightarrow 0$, and 

(2) $0'' \approx 0$. Let $I$ denote the variety of implication zroupoids.

These algebras generalize De Morgan algebras and $\lor$-semilattices with zero. For more details on the motivation leading to these algebras, we refer the reader to [San12] (or the relevant papers mentioned at the end of this paper).

The investigations into the structure of the lattice of subvarieties of $I$, begun in [San12], have continued in [CS16a, CS16b, CS17a, CS17b, CS18a, CS18b, CS19] and [GSV19]. (It should be

∗The authors wish to dedicate this work to children and their families who fight against cancer.
noted that in [CS17a] implication zroupoids were referred to as “implicator groupoids”.) The present paper is a sequel to this series of papers and is devoted to making further contributions to the theory of implication zroupoids.

Throughout this paper we use the following definitions:

\[(M) \quad x \land y := (x \to y)' \quad \text{and} \quad (J) \quad x \lor y := (x' \land y')'.\]

With each \(A \in I\), we associate the following algebras:

\[A^{mj} := \langle A, \land, \lor, 0 \rangle \quad \text{and} \quad A_{mj} := \langle A, \land, \lor \rangle.\]

**Theorem 1.1** [CS17b, Corollary 4.6] If \(A \in I\) then \(\langle A, \land \rangle\) and \(\langle A, \lor \rangle\) are semigroups. Hence, \(A_{mj}\) is a bisemigroup.

Two of the important subvarieties of \(I\) are: \(I_{2,0}\) and \(MC\) which are defined relative to \(I\), respectively, by the following identities:

\[(I_{2,0}) \quad x'' \approx x.\]

\[(MC) \quad x \land y \approx y \land x.\]

**Definition 1.2** Members of the variety \(I_{2,0}\) are called involutive, and members of \(MC\) are called meet-commutative. An algebra \(A \in I\) is symmetric if \(A\) is both involutive and meet-commutative.

Let \(S\) denote the variety of symmetric \(I\)-zroupoids. Thus, \(S = I_{2,0} \cap MC\). The identity

\[(BR) \quad x \land (x \lor y) \approx x \lor (x \land y).\]

is called the Birkhoff’s identity. This identity, a weakened form of the absorption identities, was introduced by Birkhoff in 1948. In fact, Birkhoff asked in [B48, Problem 7] for an investigation of algebras satisfying the lattice identities without absorption identities but with the identity (BR), which led to the following notion:

**Definition 1.3** A Birkhoff system is a bisemilattice satisfying the Birkhoff’s identity (BR).

Indeed, in response to Birkhoff’s problem, there have been a series of papers in the literature revealing the structure of the lattice of subvarieties of the variety of Birkhoff systems; for example, see [HWW12], [HR17a], [HR17b] and the references therein. More recently, it was proved in [CS17a, Theorem 7.3] that if \(A \in S\), then \(A^{mj}\) is a distributive Birkhoff system, from which it immediately follows that \(A_{mj}\) is a distributive Birkhoff system—a result which will be strengthened in this paper.

The main purpose of this paper is to prove the following result:

**Theorem 1.4** If \(A\) is an implication zroupoid, then the bisemigroup \(A_{mj}\) satisfies the Birkhoff identity.

As a consequence, we characterize the implication zroupoids \(A\) for which \(A_{mj}\) is a Birkhoff system.

It also follows from the main result, Theorem 1.4, that there are bisemigroups that are not bisemilattices but satisfy the Birkhoff’s identity, which naturally suggests a more general notion, than Birkhoff systems, of “Birkhoff bisemigroups” as bisemigroups satisfying the Birkhoff identity. This notion seems to be new. In this new terminology, we can now recast our main theorem as: If \(A \in I\), then \(A_{mj}\) is a Birkhoff bisemigroup.

The paper concludes with an open problem on Birkhoff bisemigroups.

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2 Preliminaries

In this section we present some preliminary results that will be useful later.

Lemma 2.1 [San12, Theorem 8.15] The following identities are equivalent in the variety $\mathcal{I}$:

(a) $0' \rightarrow x \approx x$,
(b) $x'' \approx x$,
(c) $(x \rightarrow x')' \approx x$,
(d) $x' \rightarrow x \approx x$.

The following theorem is proved in [CSI7a, Theorem 7.3].

Theorem 2.2 Let $A \in S$. Then $A_{mj}$ satisfies:

(a) $x \land x \approx x$,
(b) $x \lor x \approx x$,
(c) $x \lor y \approx y \lor x$,
(d) $x \land (x \lor y) \approx x \lor (x \land y)$.

Lemma 2.3 Let $A \in \mathcal{I}_{2,0}$. Then $A$ satisfies:

1. $x' \rightarrow 0' \approx 0 \rightarrow x$,
2. $0 \rightarrow x' \approx x \rightarrow 0'$,
3. $(x \rightarrow 0') \rightarrow (y \rightarrow z) \approx ((0 \rightarrow x) \rightarrow y) \rightarrow z$,
4. $(0 \rightarrow x) \rightarrow (0 \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$,
5. $0 \rightarrow (x \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$,
6. $0 \rightarrow (x' \rightarrow y)' \approx x \rightarrow (0 \rightarrow y')$,
7. $(x \rightarrow y) \rightarrow (y \rightarrow z) \approx (0 \rightarrow x') \rightarrow (y \rightarrow z)$,
8. $((x \rightarrow y) \rightarrow z) \rightarrow (z \rightarrow u) \approx (0 \rightarrow x) \rightarrow ((y \rightarrow z) \rightarrow (z \rightarrow u))$,
9. $x \rightarrow y \approx x \rightarrow (x \rightarrow y)$,
10. $(y \rightarrow x) \rightarrow y \approx (0 \rightarrow x) \rightarrow y$,
11. $(x \rightarrow y)' \rightarrow (0 \rightarrow x)' \approx y' \rightarrow x'$,
12. $(x \rightarrow y)' \rightarrow y \approx x \rightarrow y$,
13. $[x \rightarrow (y \rightarrow x)']' \approx (x \rightarrow y) \rightarrow x$,
14. $x \rightarrow ((0 \rightarrow x) \rightarrow y) \approx x \rightarrow y$,
15. $x \rightarrow (y \rightarrow x') \approx y \rightarrow x'$,
Let $A \in \mathcal{I}_{2,0}$. Then

1. $((x \rightarrow (0 \rightarrow y)) \rightarrow z) \rightarrow u \approx (0 \rightarrow x) \rightarrow ((0 \rightarrow y') \rightarrow (z \rightarrow u))$,
2. $((0 \rightarrow x) \rightarrow y) \rightarrow z \approx (x \rightarrow y) \rightarrow (y \rightarrow z)$,
3. $((x \rightarrow y) \rightarrow z) \rightarrow (z \rightarrow u) \approx (0 \rightarrow x) \rightarrow ((0 \rightarrow y') \rightarrow (z \rightarrow u))$,
4. $((x \rightarrow y) \rightarrow z) \rightarrow x' \approx (y \rightarrow z) \rightarrow x'$,
5. $[x \rightarrow [(0 \rightarrow y') \rightarrow (z \rightarrow u)]] \rightarrow [(0 \rightarrow y) \rightarrow [(z \rightarrow u) \rightarrow (0 \rightarrow (x \rightarrow y))'] \approx (z \rightarrow u) \rightarrow (0 \rightarrow (x \rightarrow y))'$,
6. $(x \rightarrow (0 \rightarrow y)) \rightarrow z \approx (z \rightarrow (x \rightarrow y)) \rightarrow z$,
7. $[0 \rightarrow (x \rightarrow (y \rightarrow z))] \rightarrow u \approx (0 \rightarrow x') \rightarrow ((0 \rightarrow (y \rightarrow z)) \rightarrow u)$,
8. $(x \rightarrow y) \rightarrow ((0 \rightarrow y) \rightarrow z) \approx (x \rightarrow y) \rightarrow z$,
9. $x \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow (u \rightarrow z)) \approx (x \rightarrow y) \rightarrow (u \rightarrow z)$,
10. $[(0 \rightarrow y) \rightarrow z'] \rightarrow u \approx (y \rightarrow z') \rightarrow (z' \rightarrow u)$,
11. $(0 \rightarrow x) \rightarrow ((y \rightarrow x) \rightarrow z) \approx (y \rightarrow x) \rightarrow z$,
12. $x' \rightarrow (0 \rightarrow (y \rightarrow z))' \approx x' \rightarrow (x \rightarrow (y \rightarrow z))'$,
13. $0 \rightarrow (x \rightarrow (y \rightarrow z)) \approx 0 \rightarrow ((x' \rightarrow y) \rightarrow z)$,
14. $0 \rightarrow [x \rightarrow ((y \rightarrow z) \rightarrow u)] \approx 0 \rightarrow [((x \rightarrow y) \rightarrow z) \rightarrow u]$.
(15) \((x \to y') \to [y' \to (0 \to x')] \approx y' \to (0 \to x'))'\.

(16) \([x \to (x' \to y')]' \approx x' \to (0 \to y')'\.

Proof Let \(a, b, c, d \in A\).

1. \((0 \to a) \to ((0 \to b') \to (c \to d)) \overset{2.3 \text{b}}{=} (a' \to 0') \to ((0 \to b') \to (c \to d)) \overset{2.3 \text{b}}{=} [(0 \to a') \to (0 \to b')] \to (c \to d) \overset{4 \text{ and } 2.3 \text{a}}{=} [0 \to (a' \to b')] \to (c \to d) \overset{2.3 \text{b}}{=} [(a \to b) \to 0'] \to (c \to d) \overset{2.3 \text{b}}{=} [[0 \to (a \to b)] \to c] \to d \overset{2.3 \text{b}}{=} [[a \to (0 \to b)] \to c] \to d.

2. \((0 \to a) \to b \overset{2.3 \text{b}}{=} (a \to 0') \to (b \to c) \overset{2.3 \text{b}}{=} (0 \to a') \to (b \to c) \overset{2.3 \text{b}}{=} (a \to b) \to (b \to c).

3. \((a \to b) \to c \overset{2.3 \text{b}}{=} (c \to d) \overset{2.3 \text{b}}{=} (0 \to a) \to ((b \to c) \to (c \to d)) \overset{2.3 \text{b}}{=} (0 \to a) \to ((0 \to b') \to (c \to d)).

4. \((a \to b) \to c \overset{2.3 \text{b}}{=} a' \overset{\{(a' \to b) \to (c \to d)\}}{=} ((a' \to (a \to b)) \to (c \to d)) \overset{2.3 \text{b}}{=} (a \to a') \to (b \to c) \overset{2.3 \text{b}}{=} (a \to b) \to (b \to c) \overset{2.3 \text{b}}{=} (a \to b) \to (0 \to (a \to b))'.

5. \((c \to d) \to (0 \to (a \to b))' \overset{\{(\text{with } x = 0 \to (a \to b)\), y = c, z = d\}}{=} [(0 \to (a \to b)) \to c] \to d \to (0 \to (a \to b))' \overset{\{(\text{with } x = 0 \to (a \to b)\), y = c, z = d\}}{=} [[a \to (0 \to b') \to (c \to d)] \to (0 \to a)] \to ((0 \to b') \to (c \to d)) \overset{2.3 \text{b}}{=} (0 \to a) \to ((0 \to b') \to (c \to d)).

6. \((a \to (0 \to b)) \to [((0 \to b') \to (c \to d)) \to (0 \to (a \to b))]' \overset{\{(\text{with } x = b, y = 0, z = c \to d, u = 0 \to (a \to b))'\}}{=} [a \to ((0 \to b') \to (c \to d)) \to (0 \to (a \to b))]' \overset{\{(\text{with } x = b, y = 0, z = c \to d, u = 0 \to (a \to b))'\}}{=} [a \to ((0 \to b') \to (c \to d)) \to (0 \to b) \to (0 \to (a \to b))'] \overset{\{(\text{with } x = b, y = 0, z = c \to d, u = 0 \to (a \to b))'\}}{=} [a \to ((0 \to b') \to (c \to d)) \to (0 \to b) \to (0 \to (a \to b))'] \overset{2.3 \text{b}}{=} (a \to (0 \to b)) \to [((0 \to b') \to (c \to d)) \to (0 \to (a \to b))]'.

7. \((a \to (0 \to b)) \to (c \to d) \overset{2.3 \text{b}}{=} (0 \to (a \to b)) \to (c \to d) \overset{2.3 \text{b}}{=} (c \to d) \overset{2.3 \text{b}}{=} (c \to d) \overset{2.3 \text{b}}{=} (0 \to (a \to b)) \to (0 \to (b \to c)) \to d \overset{2.3 \text{b}}{=} (a \to 0') \to ((0 \to (b \to c)) \to d) \overset{2.3 \text{b}}{=} [(0 \to a) \to (0 \to (b \to c)) \to d] \overset{\{(\text{with } x = 0 \to (a \to b)\), y = c, z = d\}}{=} [a \to (0 \to (b \to c)) \to d].
\[(a \rightarrow b) \rightarrow ((0 \rightarrow b) \rightarrow (d \rightarrow c)) \overset{(I)}{=} (a \rightarrow b) \rightarrow [[(d \rightarrow c) \rightarrow c] \rightarrow [b \rightarrow (d \rightarrow c)]]' \overset{2.3}{=} (a \rightarrow b) \rightarrow [[(d \rightarrow c) \rightarrow c] \rightarrow [b \rightarrow (d \rightarrow c)]]'
\]
\[(a \rightarrow b) \rightarrow [(d \rightarrow c) \rightarrow [b \rightarrow (d \rightarrow c)]]' \overset{2.3}{=} (a \rightarrow b) \rightarrow [[(d \rightarrow c) \rightarrow b] \rightarrow (d \rightarrow c)]' \overset{2.3}{=} (a \rightarrow b) \rightarrow [(d \rightarrow c) \rightarrow [b \rightarrow (d \rightarrow c)]]'
\]
\[(0 \rightarrow b) \rightarrow (c \rightarrow 0) \rightarrow d \overset{2.3}{=} (b \rightarrow 0') \rightarrow ((c \rightarrow 0) \rightarrow d) \overset{2.3}{=} (0 \rightarrow b') \rightarrow ((c \rightarrow 0) \rightarrow d) \overset{2.3}{=} (b \rightarrow c') \rightarrow (c' \rightarrow d).
\]
\[(0 \rightarrow a) \rightarrow ((b \rightarrow a) \rightarrow c) \overset{2.3}{=} (a' \rightarrow 0') \rightarrow ((b \rightarrow a) \rightarrow c) \overset{2.3}{=} (0 \rightarrow a') \rightarrow (b \rightarrow a)] \rightarrow c \overset{2.3}{=} [(a \rightarrow a') \rightarrow (b \rightarrow a)] \rightarrow c \overset{2.3}{=} [a' \rightarrow (b \rightarrow a)] \rightarrow c \overset{2.3}{=} (b \rightarrow a) \rightarrow c.
\]
\[(a' \to (a \to (b \to c)))' \]
\[
= [(a \to (b \to c)) \to a]' \to [0 \to (a \to (b \to c))]' \\
\text{by Lemma 2.3(11) with } x := a \to (b \to c), y := a \\
= [(0 \to b) \to ((0 \to c') \to a')] \to [0 \to (a \to (b \to c))]' \\
\text{by (4)} \text{ with } x := b, y := c, z := a, u := 0 \\
= [(0 \to b) \to ((0 \to c') \to a')] \to [(0 \to a') \to (0 \to (b \to c))]' \\
\text{by (4)} \text{ with } x := b, y := (0 \to c'), z := (0 \to a') \to (0 \to (b \to c))' \\
= (b \to ((0 \to c') \to a')) \to [(0 \to a') \to (0 \to (b \to c))'] \\
\text{by (2)} \text{ with } x := b, y := (0 \to c'), z := a \to (0 \to a') \to (0 \to (b \to c))' \\
= (b \to ((0 \to c') \to a')) \to [(0 \to c) \to [(0 \to a') \to (0 \to (b \to c))'] \\
\text{by (4)} \text{ with } x := c, y := a', z := 0, u := (0 \to (b \to c))' \\
= (b \to ((0 \to c') \to a')) \to [(0 \to c) \to [(0 \to c) \to [(0 \to a') \to (0 \to (b \to c))']] \\
\text{by Lemma 2.1(4)} \\
= (b \to ((0 \to c') \to a')) \to [(0 \to c) \to (a' \to (0 \to (b \to c))'] \\
\text{by Lemma 2.1(4)} \\
= (a \to 0) \to (0 \to (b \to c))' \\
\text{by (4)} \text{ with } x := b, y := c, z := a, u := 0 \\
= a' \to (0 \to (b \to c)) '.
\]
\[ \begin{align*}
\text{(16)} & \quad [a \rightarrow (a' \rightarrow b)']' = \left[ [(a' \rightarrow 0) \rightarrow (a' \rightarrow b)']' \right]' \\
& = \left[ [((a' \rightarrow b))' \rightarrow a'] \rightarrow (0 \rightarrow (a' \rightarrow b))']'' \right]' \\
& \quad \text{by (1)} \\
& = \left[ (0 \rightarrow b) \rightarrow a'] \rightarrow (0 \rightarrow (a' \rightarrow b))']' \\
& \quad \text{by Lemma 2.3 (10)} \\
& = \left[ (a' \rightarrow 0) \rightarrow (a \rightarrow 0 \rightarrow (0 \rightarrow (a' \rightarrow b))') \right]' \\
& \quad \text{by Lemma 2.3 (5)} \\
& = \left[ a' \rightarrow (0 \rightarrow (0 \rightarrow (a' \rightarrow b))') \right]' \\
& \quad \text{by (9) with } x := 0 \rightarrow b, y := a', z := 0, \ u := 0 \\
& = (b \rightarrow a') \rightarrow \left[ a' \rightarrow (0 \rightarrow (0 \rightarrow (a' \rightarrow b))') \right]' \\
& \quad \text{by (10) with } y := b, z := a, u := 0 \rightarrow (0 \rightarrow b)' \\
& = a' \rightarrow (0 \rightarrow b)' \\
& \quad \text{by (15) with } x := b, y := a.
\end{align*} \]

The proof of our main result (Theorem 4.2), given in the next section, depends on the following theorem.

**Theorem 3.2** Let \( A \in \mathcal{I}_{2,0} \). Then \( A_{m3} \) satisfies the Birkhoff identity.

**Proof** Let \( a, b \in A \). Then

\[ a \land (a \lor b) = (a \rightarrow (a' \rightarrow b))' \]
\[ = a' \rightarrow (0 \rightarrow (0 \rightarrow (a' \rightarrow b))')' \]
\[ = a' \rightarrow (a \rightarrow b')' \]
\[ = \left[ (a' \land (a' \land b))' \right]' \]
\[ = a \lor (a \land b) \]

by definition of \( \lor \) and \( \land \)

by Lemma 3.1 (16)

by Lemma 3.1 (12) with \( z := 0 \)

by definition of \( \land \)

by definition of \( \land \).

\( \square \)

4 Main theorem

In this section, the main theorem of this paper is proved. For this we need one more crucial result proved in [CS17b].

**Theorem 4.1 (Transfer Theorem)** [CS17b]

Let \( t_i(\overline{x}), i = 1, \ldots, 6 \), be terms, where \( \overline{x} \) denotes the sequence \( \langle x_1, \cdots, x_n \rangle \), \( x_i \) being variables. Let \( \mathcal{V} \) be a subvariety of \( \mathcal{I} \).
Let

\[ \mathcal{V} \cap \mathcal{I}_{2,0} = (t_1(\overline{x}) \rightarrow t_2(\overline{x})) \rightarrow t_3(\overline{x}) \approx (t_4(\overline{x}) \rightarrow t_5(\overline{x})) \rightarrow t_6(\overline{x}), \]

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then
\[ V \vdash (t_1(x) \rightarrow t_2(x)) \rightarrow t_3(x) \approx (t_4(x) \rightarrow t_5(x)) \rightarrow t_6(x). \]

We are now ready to present the main result of this paper (i.e., Theorem 1.4 of Introduction).

**Theorem 4.2** Let \( A \in \mathcal{I} \) then \( A_{mj} \) satisfies the Birkhoff’s identity:

\[ (BR) \quad x \land (x \lor y) \approx x \lor (x \land y). \]

**Proof** Apply Theorem 3.2 and Theorem 4.1. \( \square \)

Recall that \( \mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{MC} \). The following corollary, which characterizes the implication zroupoids \( A \) for which \( A_{mj} \) is a Birkhoff system, is an improvement on [CS17a, Theorem 7.3].

**Corollary 4.3** Let \( A \in \mathcal{I} \). Then the algebra \( A_{mj} \) is a Birkhoff system if and only if \( A \in \mathcal{S} \).

**Proof** Let \( A_{mj} \) be a Birkhoff system, then \( A \) satisfies (MC) and the identity: \( x \land x \approx x \), which, in view of Lemma 2.1, implies that \( A \models x \approx x'' \). Hence \( A \in \mathcal{S} \). For the converse, assume that \( A \in \mathcal{S} \). Then \( A \) satisfies the identities \( x \land y \approx y \land x \) and \( x \approx x'' \). We know also by Theorem 1.4 that the operations \( \land \) and \( \lor \) are associative. Hence, from Theorem 4.2 (or Theorem 3.2) we conclude that \( A_{mj} \) is a Birkhoff system. \( \square \)

Recall that implication zroupoids that satisfy the associative identity:

\[ (A) \quad x \rightarrow (y \rightarrow z) \approx (x \rightarrow y) \rightarrow z \]

were called *implication semigroups* in [GSV19]. Let \( IS \) denote the variety of implication semigroups. The following special case of our main result may be of interest to the researchers in semigroup theory.

**Corollary 4.4** Let \( A \in IS \). Then the algebra \( A_{mj} \) satisfies the Birkhoff identity.

In the next section we will improve the above corollary.

## 5 Concluding Remarks

As mentioned in the introduction, it follows from the main result, Theorem 4.2 that the algebras \( A \in \mathcal{I} \setminus \mathcal{S} \) such that \( A_{mj} \) are bisemigroups that are not bisemilattices and satisfy the Birkhoff’s identity, which suggests naturally a generalization of Birkhoff systems, which we will call “Birkhoff bisemigroups”. To the best of our knowledge, the algebras defined in the following definition seem to be new.

**Definition 5.1** A bisemigroup \( A \) is a Birkhoff bisemigroup if \( A \models (BR) \).

In this new terminology, we can now recast our main theorem as: If \( A \in \mathcal{I} \), then \( A_{mj} \) is a Birkhoff bisemigroup.

Thus, the class of algebras \( A_{mj} \), where \( A \in \mathcal{I} \) provide a large class of examples of Birkhoff bisemigroups.
Another class of examples of Birkhoff bisemigroups arise from semigroups themselves as follows: Let \( A = \langle A, \wedge \rangle \) be a semigroup. Then the algebra \( \langle A, \wedge, \wedge \rangle \) is clearly a bisemigroup and satisfies the Birkhoff identity trivially as the two binary operations are the same. Let us call such a bisemigroup arising from a semigroup “essentially a semigroup”. We shall now improve and clarify Corollary 4.4. For this we need the following lemma:

**Lemma 5.2** Let \( A \in IS \). Then \( A \) holds:

1. \( 0 \to 0' \approx 0 \),
2. \( 0 \to x' \approx x' \),
3. \( 0' \approx 0 \),
4. \( x \lor y \approx x \land y \).

**Proof** Let \( a, b \in A \).

1. \( 0 = 0'' = (0 \to 0) \to 0 \overset{(A)}{=} 0 \to (0 \to 0) = 0 \to 0' \).
2. \( a' = a \to 0 \overset{(A)}{=} a \to (0 \to 0') \overset{(A)}{=} a \to (0 \to 0') = (0' \to a) \to (0 \to 0') = (0 \to a) \to 0'' = (0 \to a) \to 0 \overset{(A)}{=} 0 \to (a \to 0) = 0 \to a' \).
3. \( 0' \overset{(A)}{=} 0 \to 0' \approx 0 \).
4. \( a \lor b \overset{\text{def of } \lor}{=} a \to (b \to (0 \to (0 \to 0))) \overset{(A)}{=} a \to (0 \to (b \to 0)) \).
5. \( a \to (b \to 0) \overset{(A)}{=} a \to (b \to (0 \to 0)) \overset{(A)}{=} a \to [(b \to 0) \to 0] \overset{(A)}{=} [a \to (b \to 0)] \to 0 \overset{\text{def of } \land}{=} a \land b \).

In view of the preceding lemma, Corollary 4.4 can be improved to the following.

**Corollary 5.3** Let \( A \in IS \). Then the algebra \( A_{mj} \) is essentially a semigroup.

We conclude this paper with the following open problem.

**PROBLEM:** Investigate Birkhoff bisemigroups; in particular, describe the structure of the lattice of subvarieties of the variety of Birkhoff bisemigroups.

**Acknowledgements**

The first author wants to thank the institutional support of CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas) and Universidad Nacional del Sur.. The authors also wish to acknowledge that [Mc] was a useful tool during the research phase of this paper.

**Compliance with Ethical Standards:**

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Conflict of Interest: The first author declares that he has no conflict of interest. The second author declares that he has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Funding: The work of Juan M. Cornejo was supported by CONICET (Consejo Nacional de Investigaciones Cientificas y Tecnicas) and Universidad Nacional del Sur.

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