On the degree sequences of uniform hypergraphs

A. Frosini * C. Picouleau † S. Rinaldi ‡

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Abstract

In hypergraph theory, determining a characterization of the degree sequence \(d = (d_1, d_2, \ldots, d_n)\) where \(d_1 \geq d_2 \geq \cdots \geq d_n\) are positive integers, of an \(h\)-uniform simple hypergraph \(H\), and deciding the complexity status of the reconstruction of \(H\) from \(d\), are two challenging open problems. They can be formulated in the context of discrete tomography: asks whether there is a matrix \(A\) with positive projection vectors \(H = (h, h, \ldots, h)\) and \(V = (d_1, d_2, \ldots, d_n)\) with distinct rows.

In this paper we consider the two subcases where the vector \(V\) is an homogeneous vector, and where \(V\) is almost homogeneous, i.e., \(d_1 - d_n = 1\). We give a simple characterization for these two subcases, and we show how to solve the related reconstruction problems in polynomial time. To reach our goal, we use the concepts of Lyndon words and necklaces of fixed density, and we apply some already known algorithms for their efficient generation.

keywords: Discrete Tomography, Reconstruction problem, Lyndon word, Necklace, Hypergraph degree sequence, Regular bipartite graph.

1 Introduction

The degree sequence, also called graphic sequence, of a simple graph (a graph without loop or parallel edges) is the list of vertex degrees, usually written in nonincreasing order, as \(d = (d_1, d_2, \ldots, d_n)\), \(d_1 \geq d_2 \geq \cdots \geq d_n\). The problem of characterizing the graphic sequences of graphs was solved by Erdős and Gallai (see [4]):

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*Università di Firenze, Dipartimento di Sistemi e Informatica, Viale Morgagni 65, 50134 Firenze, Italy
†CEDRIC, CNAM, 292 rue St-Martin 75141, Paris cedex 03, France
‡Università di Siena, Dipartimento di Ingegneria dell’informazione e scienze matematiche, Pian dei Mantellini 44, 53100 Siena, Italy
Theorem 1. (Erdős, Gallai) A sequence $d = (d_1, d_2, \ldots, d_n)$ where $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if $\Sigma_{i=1}^{n} d_i$ is even and
\[ \Sigma_{i=1}^{k} d_i \leq k(k-1) + \Sigma_{i=k+1}^{n} \min\{k, d_i\}, 1 \leq k \leq n. \]

An hypergraph $H = (\text{Vert}, \mathcal{E})$ is defined as follows (see [5]): $\text{Vert} = \{v_1, \ldots, v_n\}$ is a ground set of vertices and $\mathcal{E} \subseteq 2^{\text{Vert}} \setminus \emptyset$ is the set of hyperedges such that $e \not\subseteq e'$ for any pair $e, e'$ of $\mathcal{E}$. The degree of a vertex $v \in \text{Vert}$ is the number of hyperedges $e \in \mathcal{E}$ such that $v \in e$. An hypergraph $H = (\text{Vert}, \mathcal{E})$ is $h$-uniform if $|e| = h$ for all hyperedge $e \in \mathcal{E}$. Moreover $H = (\text{Vert}, \mathcal{E})$ has no parallel hyperedges, i.e., $e \neq e'$ for any pair $e, e'$ of hyperedges. Thus a simple graph (loopless and without parallel edges) is a 2-uniform hypergraph.

The problem of the characterization of the degree sequences of $h$-uniform hypergraphs is one of the most relevant among the unsolved problems in the theory of hypergraphs [5] even for the case of 3-uniform hypergraphs. For its last case Kocay and Li show that any two 3-uniform hypergraphs with the same degree sequence can be transformed into each other using a sequence of trades [9]. Furthermore the complexity status of the reconstruction problem is still open.

This problem has been related to a class of problems that are of great relevance in the field of discrete tomography. More precisely the aim of discrete tomography is the retrieval of geometrical information about a physical structure, regarded as a finite set of points in the integer lattice, from measurements, generically known as projections, of the number of atoms in the structure that lie on lines with fixed scopes. A common simplification is to represent a finite physical structure as a binary matrix, where an entry is 1 or 0 according to the presence or absence of an atom in the structure at the corresponding point of the lattice. One of the challenging problems in the field is then to reconstruct the structure, or, at least, to detect some of its geometrical properties from a small number of projections. One can refer to the books of G.T. Herman and A. Kuba [15, 16] for further information on the theory, algorithms and applications of this classical problem in discrete tomography.

Here we recall the seminal result in the field of the discrete tomography due to Ryser [20]. Let $H = (h_1, \ldots, h_m), h_1 \geq h_2 \geq \cdots \geq h_m$, and $V = (v_1, \ldots, v_n), v_1 \geq v_2 \geq \cdots \geq v_n$, be two nonnegative integral vectors, and $\mathcal{U}(H, V)$ be the class of binary matrices $A = (a_{ij})$ satisfying
\[ \Sigma_{j=1}^{n} a_{ij} = h_i \quad 1 \leq i \leq m \] (1)
\[ \Sigma_{i=1}^{m} a_{ij} = v_j \quad 1 \leq j \leq n \] (2)
In this context \( H \) and \( V \) are called the row, respectively column, projection of \( A \), as depicted in Fig. 1. Denoting by \( \bar{V} = (\bar{v}_1, \bar{v}_2, \ldots) \) the conjugate sequence, also called the Ferrer sequence, of \( V \) where \( \bar{v}_i = |\{ v_j : v_j \in V, v_j \geq i \}| \). Ryser gave the following [20]:

**Theorem 2. (Ryser)** \( U(H, V) \) is nonempty if and only if

\[
\begin{align*}
\sum_{i=1}^{m} h_i &= \sum_{i=1}^{n} v_i \\
\sum_{j=1}^{i} h_j &\geq \sum_{j=1}^{i} \bar{v}_j \quad \forall i \in \{1, \ldots, m\}
\end{align*}
\]

Moreover this characterization, and the reconstruction of \( A \) from its two projections \( H \) and \( V \), can be done in polynomial time (see [15]). Some applications in discrete tomography requiring additional constraints can be found in [1, 7, 2, 12, 17, 18, 19, 24].

As shown in [4] this problem is equivalent to the reconstruction of a bipartite graph \( G = (H, V, E) \) from its degree sequences \( H = (h_1, \ldots, h_m) \) and \( V = (v_1, \ldots, v_n) \). Numerous papers give some generalizations of this problem for the graphs with colored edges (see [3, 6, 10, 11, 14]).

So, in this context, the problem of the characterization of the degree sequence \( (d_1, d_2, \ldots, d_n) \) of an \( h \)-uniform hypergraph \( H \) (without parallel edges) asks whether there is a binary matrix \( A \in U(H, V) \) with nonnegative projection vectors \( H = (h, h, \ldots, h) \) and \( V = (d_1, d_2, \ldots, d_n) \) with distinct rows, i.e., \( A \) is the incidence matrix of \( H \) where rows and columns correspond to hyperedges and vertices, respectively. To our knowledge the problem of the reconstruction of a binary matrix with distinct rows has not been studied in discrete tomography.

In this paper, we carry on our analysis in the special case where the \( h \)-uniform hypergraph to reconstruct is also \( d \)-regular, i.e., each vertex \( v \) has the same degree \( d \), in other words the vector of the vertical projection is homogeneous, i.e., \( V = (d, \ldots, d) \). We also study the problem where the \( h \)-uniform hypergraph to reconstruct is almost \( d \)-regular, i.e., \( V = (d, \ldots, d, d-1, \ldots, d-1) \), in other words the hypergraph has span one.

We focus both on the decision problem, and on the related reconstruction problem, i.e., the problem of determining the existence of an element of \( U(H, V) \) consistent with \( H \) and \( V \), and in affirmative case, how to quickly reconstruct it. To accomplish these tasks, we will design an algorithm that runs in polynomial time with respect to the dimensions \( m \) and \( n \) of the matrix to reconstruct. The algorithm relies on the concepts of Lyndon words and necklaces of fixed density, and uses an already known algorithm for their efficient generation.
Figure 1: A binary matrix, used in discrete tomography to represent finite discrete sets, and its vectors $H$ and $V$ of horizontal and vertical projections, respectively.

2 Definitions and introduction of the problems

Let $A$ be a binary matrix having $m$ rows and $n$ columns, and let us consider the two integer vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ of its horizontal and vertical projections, respectively, as defined in Section 1 (see Fig. 1).

In this paper we will consider some specialized versions of the following general problems:

Consistency $(H, V, C)$

Input: two integer vectors $H$ and $V$, and a class of discrete sets $C$.

Question: does there exist an element of $C$ whose horizontal and vertical projections are $H$ and $V$, respectively?

Reconstruction $(H, V, C)$

Input: two integer vectors $H$ and $V$, and a class of discrete sets.

Task: reconstruct a matrix $A \in C$ whose horizontal and vertical projections are $H$ and $V$, respectively, if it exists, otherwise give failure.

In [21], Ryser gave a characterization of the instances of Consistency $(H, V, C)$, with $C$ being the class of the binary matrices, that admit a positive answer. He moved from the following trivial conditions that are necessary for the
existence of a matrix consistent with two generic vectors $H$ and $V$ of projections:

**Condition 1**: for each $1 \leq i \leq m$ and $1 \leq j \leq n$, it holds $h_i \leq n$ and $v_j \leq m$;

**Condition 2**: $\sum_{i=1}^{m} h_i = \sum_{j=1}^{n} v_j$,

and then he added a third one to obtain the characterization, as recalled in the Introduction.

The authors of [8], pointed out that these two conditions are also sufficient in case of homogeneous horizontal and vertical projections, by showing their maximality w.r.t. the cardinality of the related sets of solutions.

Ryser defined a well known greedy algorithm to solve $Reconstruction(H,V,C)$ that does not compare the obtained rows, and does not admit an easy generalization to perform this further task.

In the sequel, we are going to consider the class of binary matrices having no equal rows and homogeneous horizontal projections, due to its connections, as mentioned in the Introduction, with the characterization of the degree sequences of $h$-uniform hypergraphs. Among them, we restrict our analysis to those matrices that are also, first, $d$-regular, i.e., whose vertical projections are also homogeneous: $H = (h, \ldots, h)$ and $V = (v, \ldots, v)$; we denote this class by $\mathcal{E}$; and, second, almost $d$-regular, i.e., whose vertical projections are also almost homogeneous: $H = (h, \ldots, h)$ and $V = (v, \ldots, v, v - 1, \ldots, v - 1)$; we denote this class by $\mathcal{E}_1$.

Now, we state a third necessary condition for answering to $Consistency(H,V,\mathcal{E})$ (and also to $Consistency(H,V,\mathcal{E}_1)$ as we will see in Section 5):

**Condition 3**: If $Consistency(H,V,\mathcal{E})$ has positive answer, then

$$v \leq h/n \cdot \binom{n}{h}.$$

Condition 3 can be rephrased, in our setting, as follows: there does not exist a matrix having $H = (h, \ldots, h)$ and $V = (v, \ldots, v)$ as homogeneous projections, and more than $\binom{n}{h}$ different rows; otherwise at least two rows will be identical. We will prove that the three conditions 1, 2, and 3 are also sufficient to solve (in linear time) the problem $Consistency(H,V,\mathcal{E})$.

To this aim, we use an approach different from those standardly used in Discrete Tomography: we consider each row of a matrix in $\mathcal{E}$ as a binary word, and we group them into equivalence classes according to their cyclic shifts, as defined in the next section.
3 The problem $\text{Consistency}(H, V, \mathcal{E})$

Let us consider each row of a binary matrix as a binary finite word $u = u_1 u_2 \ldots u_n$, whose length $n$ is the number of columns of the matrix, and whose number $h$ of 1-elements is the value of the horizontal projection.

We note that applying a cyclic shift to the word $u$, denoted by $s(u)$, we obtain a different word $s(u) = u_2 u_3 \ldots u_n u_1$, unless the cases $u = (1)^n$ or $u = (0)^n$, of the same length, and having the same number of 1-elements inside. Iterating the shift of a word $u$, we obtain a sequence of different words that row wise arranged as a matrix, belong to $\mathcal{E}$. We indicate with $s^k(u)$, where $k \geq 0$, the application of $k$ times the shift operator to the word $u$.

Unfortunately the words repeat after at most $n$ shifts, and consequently the vertical projections of the obtained matrix are upper bounded by $n$, so, in general, only a submatrix of a solution of $\text{Reconstruction}(H, V, \mathcal{E})$ is achieved (see Fig. 2). The following trivial result holds:

**Proposition 1.** Let $u$ be a binary word of length $n$ having $h \leq n$ 1-elements inside. Let us consider the $n \times n$ matrix $A$ obtained by row wise arranging the $n$ cyclic shifts of $u$. Then, $A$ has the horizontal and vertical projections equal to $h$.

As already noticed, the rows of the matrix $A$ may not all be different. Throughout the paper we will denote by $M(u)$ the matrix obtained by row wise arranging all the different cyclic shifts of a word $u$. To establish how many different rows can be obtained by shifting a given binary word, we need to recall the definitions and main properties of necklaces and Lyndon words.

Following the notation in [22], a *binary necklace* (briefly *necklace*) is an equivalence class of binary words under cyclic shift. We identify a necklace with the lexicographically least representative $u$ in its equivalence class, denoted by $[u]$. The set of all (the words representative of) the necklaces with length $n$ is denoted $N(n)$. For example,

$$N(4) = \{0000, 0001, 0011, 0101, 0111, 1111\}.$$

An important class of necklaces are those that are aperiodic. An aperiodic (i.e. period $\geq n$) necklace is called a *Lyndon word*. Let $L(n)$ denote the set of all Lyndon words with length $n$. For example, $L(4) = \{0001, 0011, 0111\}$.

We denote fixed-density necklaces, and Lyndon words in a similar manner by adding the parameter $d$ to represent the number of 1-elements in
the words. We refer to the number $d$ as the density of the word. Thus the set of necklaces with density $d$ is represented by $N(n,d)$, and the set of Lyndon words with density $d$ is represented by $L(n,d)$. For example, $N(4, 2) = \{0011, 0101\}$, and $L(4, 2) = \{0011\}$.

It is known from Gilbert and Riordan [13] that the number of fixed density necklaces and Lyndon words is

$$\begin{align*}
N(n,d) &= \frac{1}{n} \sum_{j \mid \gcd(n,d)} \phi\left(\frac{n}{j}\right), \\
L(n,d) &= \frac{1}{n} \sum_{j \mid \gcd(n,d)} \mu\left(\frac{n}{j}\right) \left(\frac{n}{j}\right)/d/j
\end{align*}$$

respectively, where the symbols $\phi$ and $\mu$ refer to the Euler and Möbius functions.

Now we enlighten the connection between these objects and our problem, refining Proposition 1:

**Proposition 2.** If $u$ is a word of length $n$ and density $h \leq n$, then the cardinality of $[u]$ (i.e. the number of rows of $M(u)$) is a divisor of $n$.

As a consequence, we have:

**Corollary 1.** If $u$ is a Lyndon word of length $n$ and density $h$, then the cardinality of $[u]$, i.e. the number of rows of $M(u)$, is equal to $n$, and the vertical projections of $M(u)$ are all equal to $h$.

The first 12 rows of the matrix in Fig. 2 are obtained by row wise arranging the 12 different cyclic shifts of the Lyndon word $u = (0)^6(1)^6$. Such a submatrix $M(u)$ has horizontal and vertical projections equal to 6, that is the density of $u$.

**Proposition 3.** If $u = v^k$ (i.e. $u = v\ldots v$, $k$ times), with $k = \gcd\{n, h\}$, is a necklace of length $n$ and density $h$, and $v$ a Lyndon word, then the cardinality of $[u]$ is equal to $n/k$, and the vertical projections of $M(u)$ are all equal to $h/k$.

Figure 2 shows the $12/3 = 4$ different cyclic shifts of the word $u = (0011)^3$ arranged from row 13 to row 16 of the matrix, and the $12/6 = 2$ different cyclic shifts of the word $v = (01)^6$ in rows 17 and 18. All the rows of $M(u)$ have horizontal projections equal to 6 and vertical projections equal to $6/3 = 2$, while the rows of $M(v)$ have horizontal projections equal to 6 and vertical projections equal to $6/6 = 1$.

In the following we will prove that a pair $H$ and $V$ of projections satisfy Conditions 1, 2, and 3 if and only if they are consistent with a matrix in $\mathcal{E}$, solving $\text{Consistency}(H, V, \mathcal{E})$. 

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Figure 2: A solution to Reconstruction $(H, V, \mathcal{E})$ when the horizontal projections have constant value 6, and the vertical projections 9. The submatrices $M(u)$ obtained by row wise arranging the elements of three necklaces are highlighted. Note that $M((0011)^3)$ and $M((01)^6)$ are the only two possible necklaces of length 12 and density 6 having 4 and 2 rows, respectively.
Let \( d_0 = 1, d_1, d_2, \ldots, d_t \) be the increasing sequence of the common divisors of \( n \) and \( h \). The following equation holds:

\[
\left( \frac{n}{h} \right) = \sum_{i=0}^{t} \frac{n}{d_i} L \left( \frac{n}{d_i}, \frac{h}{d_i} \right).
\]

This equation is an immediate consequence of the fact that each word of length \( n \) and density \( h \) belongs to exactly one necklace.

**Theorem 3.** Let \( H \) and \( V \) be two homogeneous vectors of projections of dimension \( m \) and \( n \), and elements \( h \) and \( v \), respectively, satisfying Conditions 1, 2, and 3, i.e., being a valid instance of \( \text{Consistency}(H, V, E) \). Then, there exists a Lyndon word \( L(n/d_i, h/d_i) \) such that \( n/d_i \leq m \).

**Proof.** Let us proceed by contradiction assuming that there does not exist a Lyndon word whose length is \( n/d < m \), for each \( d \in d_1, \ldots, d_t \). Since \( H \) and \( V \) are homogeneous, and satisfy Conditions 1 and 2, then there exists a matrix \( A \) having \( H \) and \( V \) as projections (a consequence of Ryser’s characterization of solvable instances, as stated in [8], Theorem 3).

Let us assume that \( d = d_t = \gcd\{n, h\} \), \( h' = h/d \), and \( n' = n/d \); from Condition 1, it holds

\[
h'm = vn'
\]

with \( n' \) and \( h' \) coprime, so \( v = h'(m/n') \), and \( n' \) divides \( m \). The hypothesis \( n' > m \) leads to a contradiction.

Theorem 3 can be rephrased saying that if \( H \) and \( V \) are homogeneous consistent vectors of projections, then there exists a solution that contains all the elements of a necklace \([u]\). The solution in linear time of \( \text{Consistency}(H, V, E) \) is a neat consequence:

**Corollary 2.** Let \( H \) and \( V \) be two homogeneous vectors satisfying Conditions 1, 2, and 3. There always exists a matrix having different rows, and \( H \) and \( V \) as projections.

The result of Theorem 3 together with the following proposition that point out a property of the necklace whose representant is \( u = (0)^{n-h}(1)^h \), will be used in the next section to solve \( \text{Reconstruction}(H, V, E) \).

**Proposition 4.** Let \( u' \) be an element of the class \([u]\), with \( u = (0)^{n-h}(1)^h \). The elements \( u', s^h(u'), s^{2h}(u'), \ldots, s^{(k-1)h}(u') \), with \( k = n/\gcd\{n, h\} \), forms a subclass of \([u]\), and they can be arranged in a matrix \( A' \) such that

1. the vertical projections of \( A' \) are homogeneous and equal to \( h/\gcd\{n, h\} \);
2. $A'$ is minimal with respect to the number of rows among the matrices having $H$ as horizontal projections, and homogeneous vertical projections.

The proof directly follows from the properties of the greatest common divisor. Let us denote with $M_0(u)$ the matrix $A'$ defined in Proposition 4 and with $M_t(u)$ the matrix defined in the same way starting from the word $u = (1)^i(0)^{n-h}(1)^{h-i}$, with $0 \leq i < \gcd\{n, h\} (= n/k)$.

4 An algorithm to solve Reconstruction($H, V, \mathcal{E}$)

We start recalling that in [23] a constant amortized time (CAT) algorithm FastFixedContent for the exhaustive generation of necklaces $N(n, h)$ of fixed length and density is presented. The author then shows that a slight modification of his algorithm can also be applied for the CAT generation of the Lyndon words $L(n, h)$. In particular, his algorithm –here denoted GenLyndon$(n, h)$– constructs a generating tree of the words, and since the tree has height $h$, the computational cost of generating $k$ words of $L(n, h)$ is $O(k \cdot h \cdot n)$.

Let us consider the following algorithm that reconstructs an element of $\mathcal{E}$ from a couple of homogeneous horizontal and vertical projections $H$ and $V$:

Rec($H, V, \mathcal{E}$)

Input : Two homogeneous vectors: $H = (h, \ldots, h)$ of length $m$, and $V = (v, \ldots, v)$ of length $n$, satisfying Conditions 1, 2, and 3.

Output : An element of the class $\mathcal{E}$ having $H$ and $V$ as horizontal and vertical projections, respectively.

Step 1: Let compute the sequence $d_0 = 1 < d_1 < d_2 < \cdots < d_t$ of the common divisors of $n$ and $h$, and initialize the matrix $A_{-1} = \emptyset$.

Step 2: For $i = 0$ to $t$ do:

Step 2.1: By applying GenLyndon$(n, h)$, generate the sequence of $q = \min \{[v/h], L(n, h)\}$ Lyndon words, denoted $u_1, \ldots, u_q$. If $q \neq L(n, h)$, then do not include in the sequence the Lyndon word $(0)^{n-h}(1)^h$. 
Step 2.2: Create the matrix $A_i$, obtained by row wise arranging the matrices $A_{i-1}$ and $M((u_j)^{d_i})$, for $j = 1, \ldots, q$.

Update $v = v - q \cdot h$.

If $v = 0$ then output $A_i$,

else if $q \neq L(n, h)$, create the matrix $A$ obtained by row wise arranging the matrix $A_i$ with the column wise arranging of $d_i$ times the matrices $M(u_j)$, with $u = (0)^{n-h}(1)^h$, $j = 0, \ldots, q'-1$, and $q' = v \cdot \gcd\{n, h\}/h$,

else update $n = n/d_{i+1}$, and $h = h/d_{i+1}$.

A brief explanation of Step 2.2 is needed: for each common divisor $d_i$ of $n$ and $h$, the algorithm considers all the Lyndon words of $L(n/d_i, h/d_i)$; if the matrices obtained from them can be stuffed inside the solution matrix, then the algorithm performs this action, and starts again the step with $i = i + 1$, otherwise the algorithm sets aside the word $u = (0)^{n-h}(1)^h$, and stuffs the matrices obtained from the other Lyndon words in the solution matrix. Since the remaining vertical projections $v'$ are less than $h/d_i$, then the matrices $M(u_j)$, with $1 \leq j \leq q'$ as defined in Proposition 4 can be used to fill the gap, without going on generating the elements of $L(n/d_{i+1}, h/d_{i+1})$.

To better understand the reconstruction algorithm, we first propose a simple example with the instance $H = (2, \ldots, 2)$ of length $m = 15$, and $V = (5, \ldots, 5)$ of length $n = 6$. In Step 1 the values $d_0 = 1$, and $d_1 = 2$ are set.

In Step 2, GenLyndon$(6, 2)$ generates $q = 2$ Lyndon words, i.e. the words $000011$, and $000101$; since $L(6, 2) = 2$, then the word $000011$ is included in the sequence. Now the matrix $A_0$, depicted in Fig. 3 on the left, is created. Finally, the values $v = 5 - 2 \cdot 2 = 1$, $n = 6/2 = 3$, and $h = 2/2 = 1$ are updated.

The second run of Step 2 starts, and GenLyndon$(3, 1)$ generates the Lyndon word $001$. The final matrix $A_1$ is created by row wise arranging $A_0$ with the matrix $M((001)^2)$ as shown in Fig. 3 on the right.

A second example concerns the use of the word $(0)^{n-h}(1)^h$ that in certain cases is set aside from the sequence of Lyndon words generated in Step 2: the instance we consider is $H = (3, \ldots, 3)$ of length $m = 15$, and $V = (5, \ldots, 5)$ of length $n = 9$. In Step 1 the values $d_0 = 1$, and $d_1 = 3$ are set.

In Step 2, GenLyndon$(9, 3)$ generates $q = \min\{\lfloor 5/3 \rfloor, L(9, 3)\} = 1$ Lyndon words, i.e. the word $000010111$; since $q \neq L(9, 3)$, then the word $000001111$ is not included in the sequence. Now the matrix $A_0$, depicted in Fig. 4 on the left, is created. The value $v = 5 - 3 \cdot 1 = 2$ is set.
Now, since \( q \neq L(9, 3), q' = 2(= 2 \cdot \gcd\{9, 3\}/3) \) submatrices of \( M(000000111) \) are computed, as defined in Proposition \( 4 \) and row wise arranged with \( A_0 \), obtaining the matrix in Fig. 4 on the right.

Note that without the use of the Lyndon word 000000111, the procedure is not able to reach the solution since in the second run of Step 2, \( \text{GenLyndon}(3, 1) \) generates only one Lyndon word, i.e. 001, whose matrix \( M((001)^3) \) has homogeneous vertical projections equal to 1, not enough to reach the desired value 2.

The validity of \( \text{Rec}(H, V, E) \) is a simple consequence of Theorem \( 3 \). Clearly, the obtained matrix has homogeneous horizontal and vertical projections, equal to \( h \) and \( v \), respectively, and, by construction, all the rows are distinct. Moreover, the algorithm always terminates since at each iteration, we add as many rows as possible to the final solution. Concerning the complexity analysis, we need to generate \( O(m) \) different Lyndon words and shift each of them \( O(n) \) times. So, since the algorithm \( \text{GenLyndon}(n, h) \) requires \( O(k \cdot h \cdot n) \) steps to generate \( k \) words of \( L(n, h) \), the whole process takes polynomial time.

**Remark:** Let us consider the special case where \( H = V = (h, \ldots, h) \) with \( 0 < h < n \). The step 2.1 of Rec gives \( q = 1 \) and \( \text{Rec}(H, H, E) \) returns the matrix \( A_0 \) with the first row \((0)^{n-h}(1)^h\). Hence we remark that \( ^tA_0 = A_0 \) and so any two columns are different.

In graph \( G \) a **twin** is a pair of vertices \( \{u, v\} \) such that \( u \) and \( v \) have
the same neighborhood \( N_G(u) = N_G(v) \). Moreover a graph \( G \) is regular if each vertex has same number of neighbors. Using a straightforward counting argument we have that if \( G = (X,Y,E) \) is a bipartite regular graph then \(|X| = |Y|\). Hence if \( G = (X,Y,E) \) is a bipartite regular graph without twins its incidence matrix \( A_G \) satisfies: (1) the horizontal and vertical projections satisfy \( H = V \), (2) both the horizontal and vertical rows are pairwise distinct.

The next result follows directly from the algorithm \( \text{Rec}(H,V,E) \) we designed above.

**Corollary 3.** Given \( n \) and \( k \) two positive integers, the construction of a \( k \)-regular bipartite graph \( G = (X,Y,E), |X| = |Y| = n, \) without twins, if any, can be done in polynomial time. Moreover the following condition characterizes the degree sequence of a \( k \)-regular bipartite graphs without twins: \( d_i = k, 0 < k < n, \) for each vertex \( v_i \in X \cup Y \).

**Proof.** It directly follows from the remark just above.

5 Reconstruction of an \( h \)-uniform hypergraph with span one

Let us consider the case where the \( h \)-uniform hypergraph to reconstruct is almost \( d \)-regular, in other words its degree sequence has a span one, i.e., its vertical projections are \( V = (v, \ldots, v, v - 1, \ldots, v - 1) \). So, let us indicate

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]
with $\mathcal{E}_1$, the set of matrices having different rows, homogeneous horizontal projections, and vertical span one projections. In order to solve this problem we will use the algorithm $\text{Rec}(H, V, \mathcal{E})$ we designed in the previous section.

Again in [8], it has been proved that also for span one projections vectors, Conditions 1 and 2 are sufficient to ensure the existence of a compatible matrix; again Condition 3, formulated with $\mathcal{E}_1$ instead of $\mathcal{E}$, succeeding in forcing that matrix to belong to the set $\mathcal{E}_1$. So, let us consider the following algorithm that relies on $\text{Rec}(H, V, \mathcal{E})$:

$\text{RecSpan1}(H, V, \mathcal{E}_1)$

**Input**: Two vectors: $H = (h, \ldots, h)$ of length $m$, and $V = (v, \ldots, v, v - 1, \ldots, v - 1)$ of length $n$, satisfying Conditions 1, 2, and 3.

**Output**: An element $A_1$ of the class $\mathcal{E}_1$ having $H$ and $V$ as horizontal and vertical projections, respectively.

**Step 1**: let $n_0$ and $n_1$ be the number of elements $v$ and $v - 1$ of $V$, respectively, and set $k$ to be the least integer such that it is both multiple of $h$ and $n$, and greater than $h \cdot m$. Create the homogeneous vectors of projections $H'$ and $V'$ such that $H' = (h, \ldots, h)$ has length $m' = k/h > m$, and $V' = (v', \ldots, v')$, of length $n$ and $v' = k/n$.

**Step 2**: run $\text{Rec}(H', V', \mathcal{E})$, and let $A$ be its output matrix.

**Step 3**: act on the submatrix $M_0(u)$ of $A$, as defined in Proposition 4, by deleting the rows $s^{i\cdot h}(u)$, with $0 \leq i < t$, and $t = (n(v' - v) + n_1)/h$. Give the obtained matrix $A_1$ as output, after rearranging the columns in order to obtain the desired sequence of vertical projections.

Again a simple example will clarify the algorithm: we consider the instance $H = (3, \ldots, 3)$ of length 13, and $V = (5, 5, 5, 4, 4, 4, 4, 4, 4)$. Step 1 sets $n_0 = 3$, $n_1 = 6$, and $k = 45$ is the least integer multiple of $h = 3$, $n = 9$, and greater than $3 \cdot 13 = 39$.

Step 2 runs $\text{Rec}(H', V', \mathcal{E})$, on the homogeneous instance $H' = (3, \ldots, 3)$ of length 15, and $V' = (5, \ldots, 5)$ of length 9. The output matrix $A$ is given by Fig. 4 on the right.

Then, Step 3 deletes the rows $s^0(u) = u = (0)^6(1)^3$, and $s^3(u) = (1)^3(0)^6$ of the submatrix $M_0(u)$, being $t = (9 \cdot (5 - 5) + 6)/3 = 2$, as in Fig. 5 on the left.

Finally, a rearrangement of the columns is needed in order to make the matrix compatible with the starting vector $V$, as in Fig. 5 on the right.
Figure 5: On the left, two rows of $M_0(u)$ are deleted from the matrix given on the right side of Fig. 4, which is the output of $Rec(H', V', \mathcal{E})$. On the right, a rearrangement of its columns makes it compatible with the initial sequence $V$.

More precisely columns 4, 5, and 6 are shifted in the first three positions, preserving their order.

Remark: a rearrangement of the columns of a matrix causes a related rearrangement of the elements of the vector of the vertical projections, without modifying the values of its elements. Furthermore, it is straightforward that such a rearrangement also preserves the inequality relation between the rows.

The correctness of $RecSpan_1(H, V, \mathcal{E}_1)$ follows after observing that:

i) by definition of $k$, it holds:
\[ \frac{k}{n} - v < \frac{h}{gcd\{n, h\}}. \]

In words, this means that the reconstructed matrix $A$ compatible with the homogeneous vectors $H'$ and $V'$ is a minimal one, w.r.t. the dimensions, including $A_1$. Furthermore, the difference between the number of rows of $A_1$ and $A$ is less than the rows of $M_0(u)$;

ii) in Step 2, the vectors $H'$ and $V'$ satisfy Condition 3 by definition of $k$, and since $H$ and $V$ do. As a consequence the call of $Rec(H', V', \mathcal{E})$ always reconstructs a matrix $A$;

iii) it is straightforward that the algorithm $Rec(H', V', \mathcal{E})$ always inserts the submatrix $M_0(u)$, with $u = (0)^{n-h}(1)^h$, in $A$. So, the deletion

| 0 0 0 0 0 1 0 1 | | 0 0 1 0 0 0 1 1 |
| 1 0 0 0 0 1 0 1 | | 0 0 1 0 0 1 0 1 |
| 1 1 0 0 0 0 0 10 | | 0 0 0 1 1 0 0 10 |
| 0 1 1 0 0 0 0 1 | | 0 0 0 0 1 1 0 0 1 |
| 1 1 0 1 0 0 0 0 0 | | 1 1 0 1 0 1 0 0 0 |
| 0 1 0 1 1 0 0 0 0 | | 0 1 0 1 0 0 0 1 0 |
| 0 0 0 1 0 1 1 0 0 | | 0 1 0 0 0 0 1 1 0 |
| 0 0 0 0 1 0 1 1 0 | | 1 1 1 0 0 0 0 0 |
| 0 0 0 0 0 0 0 1 1 | | 0 0 1 0 0 1 0 0 1 |
| 1 1 0 0 0 0 0 1 | | 0 1 0 0 0 1 0 0 |
| 0 0 1 1 1 0 0 0 0 |


of the rows $s(u)^i_h$, according to the consecutive values of $i$ ranging from 1 to $t$, forces the vertical projections to maintain two different consecutive values at each time, and reaching the desired values.

It is straightforward that the complexity of $\text{RecSpan1}(H, V, E_1)$ is the same as $\text{Rec}(H, V, E)$.

6 Conclusion

The question of necessary and sufficient conditions for the existence of a simple hypergraph $H = (\text{Vert}, E), |\text{Vert}| = n, |E| = m$, with a given degree sequence is a long outstanding open question even in the case of a 3-uniform hypergraph ($|e| = 3$ for each $e \in E$). In this paper, we answered to this question in the special case where $H$ is $h$-uniform and $d$-regular or $H$ is $h$-uniform and almost $d$-regular, i.e. the degree sequence of $\text{Vert}$ is $(d_1 = v, d_2 = v, \ldots, d_n = v), (d_1 = v, \ldots, d_{n_0} = v, d_{n_0+1} = v - 1, \ldots, d_{n_0+n_1} = v - 1)$, respectively. Merging the results of the three previous sections we can state the following:

**Theorem 4.** $H = (\text{Vert}, E), |\text{Vert}| = n, |E| = m$, is a $h$-uniform $d$-regular, respectively $h$-uniform almost $d$-regular, hypergraph if and only if

1. $mh = nv$, resp. $mh = nv - n_1$;
2. $h \leq n, v \leq m$;
3. $v \leq h/n \cdot \left(\begin{array}{c} n \\ h \end{array}\right)$.

Moreover, given a degree sequence satisfying the conditions of this theorem, we give two linear time (in the size of the incidence matrix) algorithms that construct a $h$-uniform $d$-regular hypergraph or a $h$-uniform almost $d$-regular hypergraph.

A next step to the characterization of the degree sequence of a simple hypergraph would be its study for the subclass of uniform hypergraphs (in particular three uniform hypergraphs) with span $k$, i.e. the degree of any vertex ranges from \{v - k, v - k + 1, \ldots, v\} a set of $k$ successive values, where $k \geq 2$ is a fixed integer.

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