Conditional expectation and Bayes’ rule for quantum random variables and positive operator valued measures

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(F-K) *Conditional expectation and Bayes rule for quantum random variables and positive operator valued measures* by Douglas Farenick and MJK. *J. Math. Phys.*, 53:042201, 2012.
A measurement of a quantum system is represented mathematically by a positive operator valued measure \( \nu \) which is defined on a \( \sigma \)-algebra \( \mathcal{O}(X) \) of measurement events such that whenever a measurement is made with the system in state \( \rho \), the measurement event \( E \in \mathcal{O}(X) \) will occur with probability

\[
\text{Tr}(\rho \nu(E)).
\]

**Reference.** *The Quantum Theory of Measurement* by Busch, Lahti, Mittelstaedt, LNP, Springer, 1991.

In practice, quantum measurements of an actual physical system are made by way of some apparatus and so \( X \) is often assumed to be finite.

Mathematically, however, there is no need for such a restriction and so one of our goals is to approach the theory of quantum measurement under the assumption that \( X \) be arbitrary.
Some notation

$X$, a locally compact Hausdorff space

$\mathcal{O}(X)$, the Borel $\sigma$-algebra of subsets of $X$

$\mathcal{F}(X)$, a sub-$\sigma$-algebra of $\mathcal{O}(X)$

$\mathcal{H}$, a $d$-dimensional Hilbert space

$\mathcal{B}(\mathcal{H})$, the space of (bounded) linear operators on $\mathcal{H}$

$\text{Tr} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, the canonical trace functional

$\mathcal{B}(\mathcal{H})_+ = \{ a \in \mathcal{B}(\mathcal{H}) : \langle a\zeta, \zeta \rangle \geq 0 \ \forall \ \zeta \in \mathcal{H} \}$, the space of positive operators

$S(\mathcal{H})$, the state space of $\mathcal{H}$, namely the set of all density operators $\rho \in \mathcal{B}(\mathcal{H})_+$ with $\text{Tr}(\rho) = 1$

$\text{Eff}(\mathcal{H})$, the set of quantum effects, namely those positive operators $h \in \mathcal{B}(\mathcal{H})_+$ such that every eigenvalue of $h$ lies in $[0, 1]$. 

$S(\mathcal{H}) \subset \text{Eff}(\mathcal{H})$
A set function $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is called a positive operator valued measure on $(X, \mathcal{F}(X))$ if

1. $\nu(E) \in \text{Eff}(\mathcal{H})$ for every $E \in \mathcal{F}(X)$,

2. $\nu(X) \neq 0$, and

3. for every countable collection $\{E_k\}_{k \in \mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$ we have

$$\nu\left( \bigcup_{k \in \mathbb{N}} E_k \right) = \sum_{k \in \mathbb{N}} \nu(E_k)$$

where the convergence on the right side of the previous equality is with respect to the $\sigma$-weak topology of $\mathcal{B}(\mathcal{H})$.

If $\nu(X) = 1 \in \mathcal{B}(\mathcal{H})$, we call it a positive operator valued probability measure.

**Notation.** $\text{POVM}_{\mathcal{H}}(X)$ or $\text{POVM}^{\perp}_{\mathcal{H}}(X)$
Quantum random variables

A quantum random variable on $X$ is a function $\psi : X \to \mathcal{B} (\mathcal{H})$ such that

$$x \mapsto \text{Tr}(\rho \psi(x))$$

is a complex random variable on $X$ for every density operator $\rho \in S(\mathcal{H})$.

Recall. A random variable is a Borel-measurable function on a measure space $X$ with $\mu(X) = 1$. 
Quantum averaging (F-P-S, F-K)

**Theorem.** There is a definition of integral whereby a quantum random variable $\psi$ may be integrated against the positive operator valued probability measure $\nu$ to produce an operator

$$
\mathbb{E}_{\nu} [\psi] := \int_X \psi \, d\nu \in \mathcal{B}(\mathcal{H}).
$$

**Example.** If $X = \{x_1, x_2, \ldots, x_n\}$, then

$$
\mathbb{E}_{\nu} [\psi] = \sum_{j=1}^{n} h_j^{1/2} \psi(x_j) h_j^{1/2}
$$

where $h_j = \nu(x_j)$. 
The principal Radon-Nikodým derivative (F-P-S)

Let $\nu \in \text{POVM}_H(X, \mathcal{F}(X))$ so that $\mu(E) = \frac{\text{Tr}(\nu(E))}{d}$ is a Borel measure.

Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathcal{H}$.

Let $\nu_{ij} : \mathcal{F}(X) \to \mathbb{C}$ be defined by $\nu_{ij}(E) = \langle \nu(E)e_j, e_i \rangle$ so that $\nu_{ij} \ll_{ac} \mu$. By classical R-N Theorem, there exists a unique function

$$\frac{d\nu_{ij}}{d\mu} \in L^1(X, \mathcal{F}(X), \mu)$$

such that

$$\nu_{ij}(E) = \int_E \frac{d\nu_{ij}}{d\mu} d\mu.$$

The function

$$\frac{d\nu}{d\mu} = \sum_{i,j=1}^d \frac{d\nu_{ij}}{d\mu} \otimes e_{ij}$$

where $e_{ij} \in \mathcal{B}(\mathcal{H})$ sends $e_j$ to $e_i$ and $e_k$ to 0 is called the principal Radon-Nikodým derivative of $\nu$. 
The non-principal Radon-Nikodým derivative (F-P-S)

**Theorem.** If \( \nu_1, \nu_2 \in \text{POVM}_\mathcal{H}(X, \mathcal{F}(X)) \), then the following statements are equivalent.

1. \( \nu_2 \ll_{ac} \nu_1 \), i.e., \( \nu_2(E) = 0 \) whenever \( \nu_1(E) = 0 \).

2. There exists a bounded \( \nu_1 \)-integrable \( \mathcal{F}(X) \)-measurable function \( \varphi : (X, \mathcal{F}(X)) \to \mathcal{B}(\mathcal{H}) \), unique up to sets of \( \nu_1 \)-measure zero, such that
\[
\nu_2(E) = \int_E \varphi \, d\nu_1
\]
for every \( E \in \mathcal{F}(X) \).

Moreover, if the equivalent conditions above hold and if \( \mu_j = \mu_{\nu_j} \) is the finite Borel measure induced by \( \nu_j \), then \( \mu_2 \ll_{ac} \mu_1 \) and
\[
\varphi = \left( \frac{d\mu_2}{d\mu_1} \right) \left[ \left( \frac{d\nu_1}{d\mu_1} \right)^{-1/2} \left( \frac{d\nu_2}{d\mu_2} \right) \left( \frac{d\nu_1}{d\mu_1} \right)^{-1/2} \right]
\]
(2)
i.e., \( \varphi \) is the non-principal Radon-Nikodým derivative of \( \nu_2 \) wrt \( \nu_1 \) so we write
\[
\frac{d\nu_2}{d\nu_1} = \varphi.
\]
A non-commutative multiplication (F-K)

If \( a, b \in \mathcal{B}(\mathcal{H})_+ \) are both invertible, then the geometric mean of \( a \) and \( b \) is defined by setting

\[
a \# b = a^{1/2} (a^{-1/2}ba^{-1/2})^{1/2} b^{1/2}.
\]

**Definition.** Suppose that \( \nu_1, \nu_2 \in \text{POVM}_\mathcal{H}(X) \) with \( \nu_2 \ll_{ac} \nu_1 \), and let \( \mu_j = \mu_{\nu_j} \) be the induced Borel probability measures. If \( \psi : X \to \mathcal{B}(\mathcal{H}) \) is a quantum random variable, then

\[
\psi \boxtimes \frac{d\nu_2}{d\nu_1} = \left( \left( \frac{d\nu_1}{d\mu_1} \right)^{-1} \# \frac{d\nu_2}{d\nu_1} \right) \left( \frac{d\nu_1}{d\mu_1} \right)^{1/2} \psi \left( \frac{d\nu_1}{d\mu_1} \right)^{1/2} \left( \left( \frac{d\nu_1}{d\mu_1} \right)^{-1} \# \frac{d\nu_2}{d\nu_1} \right).
\]

**Remark.** In the commutative setting— and, in particular, in the classical case of \( \mathcal{H} = \mathbb{C} \)— the multiplication defined by \( \boxtimes \) reduces to ordinary multiplication. That is, if \( a, b \in \mathcal{B}(\mathcal{H})_+ \) commute, then \( a \# b = a^{1/2} b^{1/2} = b^{1/2} a^{1/2} = b \# a \). Thus, if \( \psi, \frac{d\nu_1}{d\mu_1}, \) and \( \frac{d\nu_2}{d\nu_1} \) are pairwise commuting, then

\[
\psi \boxtimes \frac{d\nu_2}{d\nu_1} = \psi \frac{d\nu_2}{d\nu_1} = \frac{d\nu_2}{d\nu_1} \psi.
\]
Theorem. Suppose that $\nu_1, \nu_2 \in \text{POVM}_H^1(X)$ with $\nu_2 \ll_{ac} \nu_1$, and let $\mu_j = \mu_{\nu_j}$ be the induced Borel probability measures. If $\psi : X \to B(H)$ is a $\nu_2$-integrable quantum random variable, then

$$\psi \boxtimes \frac{d\nu_2}{d\nu_1}$$

is a $\nu_1$-integrable quantum random variable and

$$\mathbb{E}_{\nu_2} [\psi] = \mathbb{E}_{\nu_1} \left[ \psi \boxtimes \frac{d\nu_2}{d\nu_1} \right]$$

or, equivalently,

$$\int_X \psi \, d\nu_2 = \int_X \psi \boxtimes \frac{d\nu_2}{d\nu_1} \, d\nu_1.$$
Theorems for the Radon-Nikodým derivatives (F-K)

Theorem (Chain Rule). If $\nu_1, \nu_2, \nu_3 \in \text{POVM}_H^1(X)$ with $\nu_1 \ll_{ac} \nu_2 \ll_{ac} \nu_3$, then
\[
\frac{d\nu_1}{d\nu_2} \otimes \frac{d\nu_2}{d\nu_3} = \frac{d\nu_1}{d\nu_3}.
\]

Corollary. If $\nu_1, \nu_2 \in \text{POVM}_H^1(X)$ with $\nu_2 \ll_{ac} \nu_1$ and $\nu_1 \ll_{ac} \nu_2$, then
\[
\frac{d\nu_1}{d\nu_2} \otimes \frac{d\nu_2}{d\nu_1} = \frac{d\nu_2}{d\nu_1} \otimes \frac{d\nu_1}{d\nu_2} = 1.
\]
Quantum conditional expectation (F-K)

**Theorem.** Suppose that \( \nu \in \text{POVM}_1^{(H)}(X) \) and that \( \psi : X \to \mathcal{B}(\mathcal{H})_+ \) is a \( \nu \)-integrable quantum random variable with \( \mathbb{E}_\nu [\psi] \neq 0 \). If \( \mathcal{F}(X) \) is a sub-\( \sigma \)-algebra of \( \mathcal{O}(X) \), then there exists a function \( \varphi : X \to \mathcal{B}(\mathcal{H}) \) such that

1. \( \varphi \) is \( \mathcal{F}(X) \)-measurable,
2. \( \varphi \) is \( \nu \)-integrable, and
3. \( \mathbb{E}_\nu [\psi \chi_E] = \mathbb{E}_\nu [\varphi \chi_E] \) for every \( E \in \mathcal{F}(X) \).

Moreover, if \( \tilde{\varphi} \) is any other \( \nu \)-integrable \( \mathcal{F}(X) \)-measurable function satisfying \( \mathbb{E}_\nu [\psi \chi_E] = \mathbb{E}_\nu [\tilde{\varphi} \chi_E] \) for every \( E \in \mathcal{F}(X) \), then \( \nu(\{x \in X : \varphi(x) \neq \tilde{\varphi}(x)\}) = 0 \).

We write

\[ \varphi = \mathbb{E}_\nu [\psi | \mathcal{F}(X)] . \]
Quantum Bayes’ rule (F-K)

**Theorem.** Let $\nu_1, \nu_2 \in \text{POVM}_\mathcal{H}^1(X)$ with $\nu_2 \ll_{ac} \nu_1$ and $\nu_1 \ll_{ac} \nu_2$.

If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a quantum random variable with $\mathbb{E}_{\nu_2}[\psi] \neq 0$ and $\mathcal{F}(X)$ is a sub-$\sigma$-algebra of $\mathcal{O}(X)$, then

$$
\mathbb{E}_{\nu_2}[\psi|\mathcal{F}(X)] \otimes \mathbb{E}_{\nu_1}\left[\frac{d\nu_2}{d\nu_1}|\mathcal{F}(X)\right] = \mathbb{E}_{\nu_1}\left[\psi \otimes \frac{d\nu_2}{d\nu_1}|\mathcal{F}(X)\right].
$$