ON THE FINITENESS PROBLEM FOR CLASSES OF MODULAR LATTICES

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Dedicated to the memory of Rudolf Wille

Abstract. The Finiteness Problem is shown to be unsolvable for any sufficiently large class of modular lattices.

Given a class $A$ of algebraic structures, the Finiteness Problem is to decide for any given finite presentation, that is a list of generator symbols and relations, whether or not there is a finite bound on the size of members of the class which 'admit the presentation', that is a system of generators satisfying the given relations; if $A$ is a quasi-variety, this means finiteness of the free $A$-algebra given by the presentation. Due to Slavik [6], the finiteness problem is algorithmically solvable for the class of all lattices, due to Wille [7] for any class of modular lattices, containing the subspace lattice of an infinite projective plane, if one allows only order relations between the generators. The present note relies on the unsolvability of the Triviality Problem for modular lattices [4] which in turn relies on the result of Adyan [1, 2] and Rabin [5] for groups. For a vector space $V$ let $L(V)$ denote the lattice of subspaces.

Theorem 1. Let $A$ a class of modular lattices such that $L(V) \in A$ for some $V$ of infinite dimension. Then the Finiteness Problem for $A$ is algorithmically unsolvable.

The following restates the relevant part of Lemma 10 in [4].

Lemma 2. There is a recursive set $\Sigma$ of conjunctions $\varphi(\bar{x}, x_\perp, x_\top)$ of lattice equations such that $\forall \bar{x}\forall x_\perp \forall x_\top. \varphi(\bar{x}, x_\perp, x_\top) \Rightarrow \bigwedge_i x_\perp \leq x_i \leq x_\top$ is valid in all modular lattices and such that the following hold where $\varphi^3$ denotes the sentence $\exists \bar{x} \exists x_\perp \exists x_\top. \varphi(\bar{x}, x_\perp, x_\top) \land x_\perp \neq x_\top$.

(i) If, for $\varphi \in \Sigma$, $\varphi^3$ is valid in some modular lattice, then it is so within $L(V)$ for any $V$ of infinite dimension. Moreover, one can choose $x_\perp = 0$ and $x_\top = V$.

(ii) The set of all $\varphi \in \Sigma$ with $\varphi^3$ valid in some modular lattice is not recursive.

1991 Mathematics Subject Classification. 06C05, 03D35.
Key words and phrases. Finiteness problem, modular lattice.
Consider the conjunction $\pi(\bar{y}, y_\perp, y_\top)$ of the following lattice equations

$$y_i \cdot y_j = y_\perp \quad (1 \leq i < j \leq 4), \quad y_i + y_j = y_\top \quad (1 \leq i < j \leq 4, j \neq 2)$$

We use $x, y, \ldots$ both as variables and generator symbols and also to denote their values under a particular assignment. In [3], $\text{FM}(J_4^1)$ was defined as the modular lattice freely generated under the presentation $\pi(\bar{y}, y_\perp, y_\top)$ (equivalently, by the partial lattice $J_4^1$ arising from the 6-element height 2 lattice $M_4$ with atoms $y_1, y_2, y_3, y_4$ keeping all joins and meets except the join of $\{y_1, y_2\}$). The following was shown (to prove (i) consider $V$ the direct sum of infinitely many subspaces of dimension $\aleph_0$).

**Lemma 3.** Up to isomorphism, $M_4$ and singleton are the only proper homomorphic images of $\text{FM}(J_4^1)$. Moreover, $\text{FM}(J_4^1)$ has the following properties:

1. $\text{FM}(J_4^1)$ embeds into $\text{L}(V)$ for any $V$ of infinite dimension. Moreover, the embedding can be chosen such that any prime quotient has infinite index.
2. $\text{FM}(J_4^1)$ has infinite height.
3. $\text{FM}(J_4^1)$ has prime quotient $y_\top/(y_1 + y_2)$, generating the unique proper congruence relation $\theta$.
4. $\text{FM}(J_4^1)/\theta$ is isomorphic to $M_4$.

**Proof.** of Theorem 1. Given $\varphi \in \Sigma$ from Lemma 2 consider the presentation $\varphi^\#$ with generators $\bar{x}, x_\perp, x_\top, \bar{y}, y_\perp, y_\top$ and the relations from $\varphi$, $\pi$, and in addition $x_\top = y_\top$ and $x_\perp = y_1 + y_2$. Considering a modular lattice $L$ with generators and relations according to $\varphi^\#$, the following are equivalent in view of Lemma 3:

1. $x_\perp = x_\top$.
2. $L$ is singleton or $M_4$.
3. $L$ is finite.
4. $L$ is of finite height.

Clearly, if $x_\perp = x_\top$ in every modular lattice admitting presentation $\varphi$ then the same applies to the presentation $\varphi^\#$. On the other hand, assume that $\varphi^3$ is valid in some modular lattice. Given any vector space $V$, embed $\text{FM}(J_4^1)$ into $\text{L}(V)$ as in (i) of Lemma 3 and denote $U = y_1 + y_2$. By (i) of Lemma 2 one can evaluate $\bar{x}$ in $\text{L}(V/U)$ such that $\varphi(\bar{x}, x_\perp, x_\top)$ holds where $x_\perp = U$ and $x_\top = V$. This results into generators of a sublattice $L$ of $\text{L}(V)$ satisfying the relations of $\varphi^\#$ and such that $x_\perp \neq x_\top$. Thus, to decide whether $x_\perp = x_\top$ for all modular lattices admitting presentation $\varphi$ reduces to deciding whether (i)–(iv)
apply to all $L \in \mathcal{A}$ admitting presentation $\varphi^\#$. Undecidability of the latter problems follows now from (ii) of Lemma 2.

**Corollary 4.** For no quasi-variety $\mathcal{A}$ as in Theorem 1 there is an algorithm to decide, given a finite presentation, whether or not the lattice freely generated in $\mathcal{A}$ under that presentation is of finite height.

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