GAP THEOREMS FOR LOCALLY CONFORMALLY FLAT MANIFOLDS

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Abstract. In this paper, we prove a gap result for a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature and a scalar curvature average condition. We show that if it has positive Green function, then it is flat. This result is proved by setting up new global Yamabe flow. Other extensions related to bounded positive solutions to a Schrödinger equation are also discussed.

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1. Introduction

In the recent work of L. Ni [6], the author proves a very interesting gap theorem on a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature via the use heat equation technique. The remarkable feature in Ni’s result is the fast quadratic decay condition about the average of the scalar curvature in large balls. His result can be considered as some kind of positive mass theorem. In this paper, we consider other scalar curvature conditions and we prove the following new gap theorem on a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature.

Theorem 1. Assume that $(M^n, g_0)$ $(n \geq 3)$ is a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature and with the scalar curvature average condition for some $x_0 \in M^n,$

\[ \int_0^\infty \frac{r}{Vol_{x_0}(r)} \int_{B_{x_0}(r)} R_0(y) dv_{g_0} dr < \infty, \]

where $R_0(y)$ is the scalar curvature function of $(M^n, g_0)$ and $Vol_{x_0}(r)$ is the volume of the ball $B(x_0, r)$ in $(M^n, g_0).$ Assume further that it is non-parabolic, i.e., it has a positive Green function. Then it is flat.

We now give two remarks about the proof and the conditions of Theorem 1.

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Remark 1. According to heat kernel estimate of Li-Yau [5](1986) and the work of N. Varopoulos [13], we know that for the complete non-compact Riemannian manifold with non-negative Ricci curvature, the non-parabolic condition is equivalent to the ball volume assumption that

\[ \int_1^\infty \frac{r}{Vol_{x_0}(r)} dr < \infty, \]

where \( x_0 \in M := M^n \) is some point. We remark that on the non-parabolic manifold \((M, g_0)\), if the scalar curvature average condition at some point \( x_0 \), then it is also true at any point \( x \) of \( M \). Further extensions of our result above are stated in the last section.

Remark 2. The key part of our proof of Theorem 1 is our discovery that there is a remarkable relation between the Poisson equation (or Schrodinger equation) and the existence of the global Yamabe flow on complete non-compact Riemannian manifold with non-negative scalar curvature.

The idea of the proof of Theorem 1 above is below. Assume that \((M, g_0)\) is non-flat. Then the scalar curvature \( R_0(x) \) of it is non-trivial and non-negative. We use the non-parabolic condition and the average condition 1 to solve the Poisson equation

\[ -\Delta w(x) = R_0(x), \text{ in } M \]

and get a non-negative solution \( w \). Then we use the function \( w \) to get a lower barrier of the Yamabe flow to prevent it from locally collapsing and show that we have a global Yamabe flow. We then use the fundamental Harnack inequality of B. Chow [2] for locally conformally flat manifolds to show that the scalar curvature must be trivial. This then concludes the result of Theorem 1.

From our argument of Theorem 1, we can actually prove stronger results. See Theorems 7 and 8 in the last section.

The result above can be considered as a generalized positive mass theorem [11] in the sense that if \((M, g_0)\) is a locally conformally flat complete non-compact Riemannian manifold with bounded non-trivial non-negative Ricci curvature and the condition (1), then it is parabolic, i.e., the Green function must be negative somewhere. It is reasonable to believe that our result above should have more applications to locally conformally flat geometric structures. In our previous work [7], we can use the positive mass theorem to solve an open question posed in [3].

There are many gap theorems for complete non-compact Riemannian manifold with suitable assumption about curvatures stronger than non-negative Ricci curvature. For more references, one may see L.Ni’s paper [6]. See also [3] and [12]. Here we mention the following result of Ma-Cheng [8] as a comparison.

**Theorem 2.** Assume that \((M, g_0)\) is a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature. Assume further that it has a Ricci pinching condition in the sense that there
is a positive constant $\epsilon$ such that
\[ Rc(g_0)(x) \geq \epsilon R_0(x) g_0(x). \]
Then it is flat.

The plan of this paper is below. In section 2, we discuss the existence of a non-negative weak solution to the Poisson equation in $(M, g_0)$ under the condition 2. In section 3, we introduce the global Yamabe flow with the initial data $g_0$. In the last section, we prove the main result and discuss other extensions. We may use $C$ to denote various uniform constants.

2. POISSON EQUATION

Let $(M, g_0)$ be a complete non-compact Riemannian manifold of dimension $n \geq 3$ and with non-negative Ricci curvature and the scalar curvature average condition (1).

We assume (2). Then according to Li-Yau [5] (1986) we know that $(M, g_0)$ is non-parabolic, i.e., there is a positive Green function $G(x, y)$ such that
\[ C^{-1} \int_{d(x, y)}^{\infty} \frac{r}{Vol_{x_0}(r)} dr \leq G(x, y) \leq C \int_{d(x, y)}^{\infty} \frac{r}{Vol_{x_0}(r)} dr \]
for some uniform dimension constant $C > 0$, where $d(x, y)$ is the distance function between the points $x$ and $y$ in $(M, g_0)$.

Given any non-negative continuous function $f : M \to \mathbb{R}$ such that for any $x \in M$,
\[ \int_{0}^{\infty} \frac{r}{Vol_{x}(r)} \int_{B_{r}(x)} f(y) dv_{g_0} dr < \infty. \]
Then the non-negative function
\[ w(x) = \int_{M} G(x, y) f(y) dv_{g_0} \]
is well-defined and is the non-negative solution in the class $C^{1, \alpha}_{loc}(M) \cap W^{2, p}_{loc}(M)$ ($\alpha \in (0, 1)$ and $1 < p < \infty$) to the Poisson equation
\[ -\Delta_{g_0} w = f, \text{ in } M. \]

In fact, using the Fubini theorem, we know that
\[ \int_{M} G(x, y) f(y) dv_{g_0} = \int_{0}^{\infty} dr \int_{\partial B(x, r)} G(x, y) f(y) d\sigma_y, \]
which is no bigger than
\[ C \int_{0}^{\infty} dr \int_{r}^{\infty} \frac{s}{Vol_{x}(s)} ds \int_{\partial B(x, r)} f(y) d\sigma_y, \]
where $C$ is some uniform constant. Exchanging the order of $s$ and $r$, the latter term can be written as
\[ C \int_{0}^{\infty} ds \int_{0}^{s} \frac{s}{Vol_{x}(s)} \int_{\partial B(x, r)} f(y) d\sigma_y dr, \]
which is
\[ C \int_0^\infty \frac{s}{\text{Vol}_x(s)} \int_{B_x(s)} f(y) dv_{g_0} ds < \infty. \]

Since for any \( \phi \in C^2_0(M) \), we have
\[ \int_M \Delta \phi(x) \int_M G(x,y) f(y) dv_{g_0} = \int_M f(y) \int_M G(x,y) \Delta \phi(x) dv_{g_0}, \]
and the latter is
\[ \int_M f(y) \int_M \Delta G(x,y) \phi(x) dv_{g_0} = -\int_M f(y) \phi(y) \]
after a use of integration by part. Hence we have
\[ -\Delta w = f \]
holds weakly. Using the Schauder theory we know that \( u \in C^{1,\alpha}_{loc}(M) \) for any \( \alpha \in (0,1) \). Using the Calderon-Zugmund \( L^p \) theory we know that \( u \in W^{2,p}_{loc}(M) \). Higher regularity of \( w \) can be obtained by assumption of higher regularity of the function \( f \).

In summary we have proved the following result, which is well known to experts in geometric analysis.

**Proposition 3.** Let \((M, g_0)\) be a complete non-compact Riemannian manifold of dimension \( n \geq 3 \) and with non-negative Ricci curvature and the assumption (2). Then for any non-negative continuous function \( f : M \rightarrow \mathbb{R} \) such that for any \( x \in M \),
\[ \int_0^\infty \frac{\int_{B_x(r)} f(y) dv_{g_0}}{\text{Vol}_x(r)} dr < \infty, \]
there is the non-negative solution \( w \) in the class \( C^{1,\alpha}_{loc}(M) \cap W^{2,p}_{loc}(M) \) (\( \alpha \in (0,1) \) and \( 1 < p < \infty \)) to the Poisson equation \( -\Delta_{g_0} w = f \) in \( M \). Furthermore, if \( f \in C^\infty \), then \( w \in C^\infty \).

### 3. Global Yamabe Flow with Positive Scalar Curvature

In this section, we assume that \((M, g_0)\) is a complete non-compact Riemannian manifold with non-negative scalar curvature \( R_0 \). Assume that the Poisson equation \( -\Delta w = \frac{n^2}{4(n-1)} R_0 \) on \( M \) has a non-negative solution \( w \). We emphasize that in this section, we don’t need the conditions that the scalar curvature is bounded and the manifold is locally conformally flat. See also [1].

We may assume that \((M, g_0)\) is given with \( R(g_0) \geq 0 \) being non-trivial.

Recall that the Yamabe flow is a family of pointwise conformally equivalent metrics \( g(t) \) \((t \in [0,T), T > 0)\) evolved by the evolution equation
\[ \partial_t g(t) = -R(t)g(t), \text{ on } M \times (0,T), \]
with the initial metric \( g(0) = g_0 \). Here \( R(t) = R(g(t)) \) is the scalar curvature of the metric \( g(t) \). We shall write \( R_0(x) \) by the scalar curvature of \( g_0 \). Under the Yamabe flow, we know that

\[
\partial_t R(t) = (n-1)\Delta g(t)R + R^2, \text{ in } M \times (0, T).
\]

Let \( g(t) = u(x, t)^{4/(n-2)} g_0 \) for positive smooth functions \( u(x, t) \) and let \( p = \frac{n+2}{n-2} \). Then we have that

\[
R(t) = u^{-p}[-\frac{4(n-1)}{n-2} \Delta u + R_0 u].
\]

Then the evolution equation (5) becomes the parabolic equation for the positive functions \( u(x, t) \)

\[
\partial_t u^p = (n-1)p[\Delta u - \frac{n-2}{4(n-1)} R_0(x) u], \text{ on } M \times (0, T),
\]

with the initial value \( u(x, 0) = 1 \). Here \( \Delta \) is the Laplace operator of the metric \( g_0 \). We shall write by \( \nabla u \) the gradient of the function \( u \) in the metric \( g_0 \).

We know from the standard parabolic theory that for any bounded smooth domain \( \Omega \subset M \), there exists some small \( T > 0 \) such that the equation (5) on \( \Omega \times (0, T) \) always has a positive solution \( \Omega \times (0, T) \) with the initial and boundary conditions \( u(x, t) = 1 \) either for \( t > 0, x \in \partial\Omega \) or \( t = 0, x \in \Omega \). Since \( R_0 \geq 0 \), we have

\[
\partial_t u^p \leq (n-1)p\Delta u, \text{ in } \Omega \times (0, T)
\]

which implies that \( u(x, t) \leq 1 \) in \( \Omega \times [0, T) \).

With the abuse notation, we still use \( g(t) = u(x, t)^{4/(n-2)} g_0 \) on \( \Omega \times (0, T) \) for the locally defined solution \( u \).

Since \( R_0 \geq 0 \) on \( M \), by the equation (3) on \( \Omega \times (0, T) \) and the maximum principle [10] we know that \( R(t) > 0 \) on \( \Omega \times (0, T) \). Fix any \( t \in (0, T) \). Using the formula (4) we have

\[
-n \frac{4(n-1)}{n-2} \Delta u + R_0 u > 0, \text{ in } \Omega.
\]

We may rewrite this as

\[
\frac{n-2}{4(n-1)} R_0 > \frac{\Delta u}{u}.
\]

Set \( v = \log u \). Then we have

\[
\Delta v = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \leq \frac{\Delta u}{u}.
\]

Hence, we have

\[
\Delta v < \frac{n-2}{4(n-1)} R_0, \text{ in } \Omega.
\]
We now let \( f(x) = \frac{n-2}{4(n-1)}R_0 \) solve the Poisson equation \( \Delta(-w) = f \) for \( w \) in the whole manifold \( M \). Then we have
\[
\Delta(v + w) < 0, \text{ in } \Omega
\]
with the boundary condition \( v + w = w > 0 \) on \( \partial \Omega \). By the local maximum principle \([10]\) in \( \Omega \) we know that \( v + w > 0 \) in \( \Omega \). This implies that \( u > \exp(-w) \) in \( \Omega \). Hence, we have the uniform bound
\[
\exp(-w) < u \leq 1, \text{ in } \Omega \times (0, T).
\]
Then we can extend the solution \( u \) beyond any finite time \( T > 0 \). That is to say, \( u \) is a global solution to the equation \([5]\). We denote this solution by \( u = u_\Omega \).

We now write \( M = \bigcup_{j=1}^\infty \Omega_j, \Omega_j \subset \subset \Omega_{j+1}, \) the exhaustion of bounded smooth domains of the manifold \( M \). Let \( u_j = u_{\Omega_j} \). With the help of the uniform bound \([7]\), we can extract a convergent subsequence in \( C_\infty^0(M \times (0, \infty)) \) with the limit \( u \), which is the solution to \([5]\) on the whole manifold \( M \) with the initial value \( u(x, 0) = 1 \).

Again we denote by \( g(t) = u(x, t)^{4/(n-2)}g_0 \) for the limit solution \( u \). Then by the maximum principle, we know that \( R(t) > 0 \) in \( M \times (0, \infty) \).

In conclusion we have proved the following result.

**Theorem 4.** Assume that \((M, g_0)\) is a complete non-compact Riemannian manifold with non-negative scalar curvature \( R_0 \). Assume that the Poisson equation \(-\Delta w = \frac{n-2}{4(n-1)}R_0 \) on \( M \) has a non-negative solution \( w \). Then the Yamabe flow \([5]\) has a global positive solution.

From the argument above we can also prove the following result.

**Theorem 5.** Assume that \((M, g_0)\) is a complete non-compact Riemannian manifold with non-negative scalar curvature \( R_0 \). Assume that the Schrodinger equation \(-\Delta v + \frac{n-2}{4(n-1)}R_0 v = 0 \) on \( M \) has a bounded positive solution \( v \). Then the Yamabe flow \([7]\) has a global positive solution.

In fact, we may assume that \( v(x) \in (0, 1) \). Then, using the equation \([6]\) to compare \( u \) and \( v \), we conclude that \( u(x, t) > v(x) \). Then as above, we can get a global Yamabe flow \( g(t) = u(x, t)^{4/(n-2)}g_0 \) such that \( v(x) < u(x, t) \leq 1 \).

4. PROOF OF MAIN RESULT

In the first part of this section, we assume that the manifold \( M \) satisfies the assumptions in Theorem \([1]\). Since the initial metric has bounded curvature, the global Yamabe flow obtained in previous section has bounded curvature at any finite time interval (see Theorem 1 in \([9]\)). This property makes sure that one can use Hamilton’s tensor maximum principle \([4]\).

To prove our main result Theorem \([1]\) we need to state the fundamental Harnack inequality of B.Chow \([2]\) for Yamabe flow \( g = g(t) \) defined on locally conformally flat manifold \((M, g_0)\). As we noticed in \([8]\), B.Chow proved this result for the compact case. However, since one can use Hamilton’s tensor
maximum principle, B.Chow’s argument can be carried out into complete non-compact case. Recall that the Harnack quantity defined by B.Chow is

\[ Z(g, X) = (n - 1)\Delta g R + g(\nabla_g R, X) + \frac{1}{2(n - 1)}Rc(X, X) + R^2 + \frac{R}{t}, \]

for any metric \(g\) and 1–form \(X\). Then the Harnack inequality of Yamabe flow due to B.Chow is

**Theorem 6.** Let \((M, g_0)\) be a locally conformally flat manifold with positive Ricci curvature, then under the Yamabe flow,

\[ Z(g, X) \geq 0, \quad t > 0, \]

for any 1–form \(X\).

**Proof of Theorem 1:** Assume that the scalar curvature \(R_0\) is non-trivial. Using Proposition 3 we can solve the Poisson equation

\[-\Delta w = \frac{n - 2}{4(n - 1)}R_0 \]

on \((M, g_0)\) to get the non-negative solution \(w\). We then run the Yamabe flow with the initial metric \(g_0\). By Theorem 4 we have a global Yamabe flow \(g(t)\). As noted in [2] (see also [3]), the non-negativity property of the Ricci curvature along the Yamabe flow is preserved. Note that

\[(n - 1)\Delta g R + R^2 = \partial_t R\]

along the Yamabe flow (see [2]. The relation (8) can be written as

\[\partial_t R + g(\nabla_g R, X) + \frac{1}{2(n - 1)}Rc(X, X) + \frac{R}{t} \geq 0.\]

Choose \(X = -d\log R\). We then get

\[\partial_t R + \frac{R}{t} \geq \frac{|\nabla_g R|^2}{2R}.\]

Hence, \(\partial_t(tR) \geq 0\). Then, for \(t \geq 1\) and \(\tau \in [\sqrt{t}, t]\),

\[\tau R(x, \tau) \geq \sqrt{t}R(x, \sqrt{t}).\]

Recall that the Yamabe flow can be written as \(\frac{4}{n-2} \partial_t \log u = -R\). Then using the bound (7) we have for some uniform constant \(C(n) > 0\),

\[\int_0^t R d\tau = -C(n) \log u(x, t) \leq C(n)w(x)\].
Then using (9) and (10), we have
\[
\sqrt{t} R(x, \sqrt{t}) \log t = \int_{\sqrt{t}}^{t} \frac{\sqrt{t} R(x, \sqrt{\tau})}{\sqrt{\tau}} d\tau \leq \int_{\sqrt{t}}^{t} \frac{\tau R(x, \tau)}{\tau} d\tau \leq \int_{0}^{t} R(x, \tau) d\tau \leq C(n) w(x).
\]
Hence,
\[
0 \leq \sqrt{t} R(x, \sqrt{t}) \leq \frac{w(x)}{\log t}.
\]
Then as \( t \to \infty \), we have for any \( x \in M \), the monotone non-decreasing quantity has its limit 0, i.e.,
\[
\sqrt{t} R(x, \sqrt{t}) \to 0.
\]
Then we have \( R(x, t) = 0 \) on \( M \) at \( t = 0 \). This implies that \( (M, g_0) \) is flat. This completes the proof of Theorem 1.

We now discuss some extensions of Theorem 1. From the argument above, we observe that we have actually proved the following stronger result.

**Theorem 7.** Assume that \((M^n, g_0) \ (n \geq 3)\) is a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature. Assume further that the Poisson equation \(-\Delta w = \frac{n-2}{4(n-1)} R_0\) has a non-negative smooth solution, where \( R_0 \) is the scalar curvature of the metric \( g_0 \). Then \((M, g_0)\) is flat.

By using Theorem 5 and the argument above, we can also prove the following result.

**Theorem 8.** Assume that \((M^n, g_0) \ (n \geq 3)\) is a locally conformally flat complete non-compact Riemannian manifold with bounded non-negative Ricci curvature. Assume further that the Schrodinger equation
\[
-\Delta v + \frac{n-2}{4(n-1)} R_0 v = 0
\]
on \( M \) has a positive bounded solution, where \( R_0 \) is the scalar curvature of the metric \( g_0 \). Then \((M, g_0)\) is flat.

The detail of proofs of these results are omitted.

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