Large cycles in generalized Johnson graphs

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Abstract
We count cycles of an unbounded length in generalized Johnson graphs. Asymptotics of the number of such cycles is obtained for certain growth rates of the cycle length.

KEYWORDS
Johnson graph, Kneser graph, large cycles, random walk

1 | INTRODUCTION AND NEW RESULTS

For integers \(i \leq j\), everywhere below we denote \([i, j] := \{i, i + 1, \ldots, j\}\) and \([i] := [1, i]\). For integers \(n, r, s\) such that \(0 \leq s < r < n\), a simple graph \(G(n, r, s)\) with the set of vertices

\[V := V(G(n, r, s)) = \{x \subseteq [n] : |x| = r\}\]

and the set of edges

\[E := E(G(n, r, s)) = \{\{x, y\} : |x \cap y| = s\}\]

is called a generalized Johnson graph.

Unfortunately, there is no established term for graphs \(G(n, r, s)\). In literature they appear as generalized Johnson graphs [1, 7, 33]; uniform subset graphs [9, 11, 41] and distance graphs [5, 6, 36, 45]. The family of \(G(n, r, s)\) graphs was initially (to the best of our knowledge) considered in [9], where they are called “uniform subset graphs.” However, this name did not become widespread. In our opinion, the term “generalized Johnson graph” is preferred as the most comprehensible, since, if we set \(s = r - 1\), then the definition of \(G(n, r, s)\) turns into the definition of the well-known Johnson graph. Note that the Kneser graph is also a special case of
$G(n, r, s)$ with $s = 0$. However, the term “generalized Kneser graph” is already used for another class of graphs [12, 16, 22, 25].

On the one hand, as we mentioned above, graphs $G(n, r, s)$ generalize Johnson graphs $G(n, r, r - 1)$ [2, 14, 17–19] and Kneser graphs $G(n, r, 0)$ [4, 10, 30, 31, 34, 35, 43], which are themselves of interest in the graph theory. On the other hand, they are a special case of distance graphs in $\mathbb{R}^n$ with the Euclidean metric, which are used to study problems of combinatorial geometry (Hadwiger–Nelson problem about the chromatic number $\chi(\mathbb{R}^n)$ [8, 13, 20, 21, 23, 28, 29], Borsuk problem about partitioning of a set in $\mathbb{R}^n$ into subsets of a smaller diameter [27], and various generalizations of these problems [37–39]).

Throughout the paper, we assume that $r$ and $s$ are constant and $n$ approaches infinity. The total number of vertices in this graph is denoted by $N$:

$N = |V| = \binom{n}{r} \sim \frac{n^r}{r!}.$

From the definition of $G(n, r, s)$ it is evident that this graph is vertex-transitive, that is, for any two vertices there exists an automorphism of the graph mapping the first vertex to the second one. In particular, $G(n, r, s)$ is regular. Let $N_1$ denote the degree of its vertex:

$N_1 = \binom{r}{s} \binom{n - r}{r - s} \sim \frac{n^r}{r!} \frac{n^r - s}{(r - s)!}.$

In [9] it is proved that the graph $G(n, r, s)$ is Hamiltonian for $s \in \{r - 1, r - 2, r - 3\}$, arbitrary $r$ and $n$ as well as for $s \in \{0, 1\}$, arbitrary $r$, and sufficiently large $n$. Hamiltonian cycles has been extensively studied in Kneser graphs $G(n, r, 0)$. It is known that they are hamiltonian for $n \geq 2.62r$ [10] and for all $r$ when $n \leq 27$ (except for the Petersen graph $G(5, 2, 0)$) [40]. Graphs $G(2r + 1, r, 0)$ are also known to be Hamiltonian for all $r \geq 3$ [34]. As for cycles of a constant length, the asymptotics of the number of their appearances in $G(n, r, s)$ is known for all constant $r$ and $s$ and given below in Theorem 1.

Let $H$ and $G$ be graphs. A map $\varphi : V(H) \to V(G)$ is called a homomorphism from $H$ to $G$ if, for any pair of vertices $x, y$ of $H$, $[x, y] \in E(H) \Rightarrow \{\varphi(x), \varphi(y)\} \in E(G)$. If a homomorphism is injective, then it is called a monomorphism. Let $\text{hom}(H, G)$ and $\text{mon}(H, G)$ denote, respectively, the number of homomorphisms and monomorphisms from $H$ to $G$. Throughout this paper we write simply $\text{hom}(H)$ and $\text{mon}(H)$ when $G = G(n, r, s)$.

Let $C_t$ be a cycle on $t$ vertices. The purpose of this paper is to find the asymptotic value of $\text{mon}(C_t)$ for different $t = t(n)$.

Burkin [5] found the asymptotics of $\text{mon}(C_t)$ for all $t = \text{const}$.

**Theorem 1** (Burkin [5]). Let $r, s, t$ be fixed. Then

$$\text{mon}(C_t) \sim NN_1 \left( \frac{N_1}{r} \right)^{t-2}.$$  \hspace{1cm} (1)

We generalize this result to cycles of variable length, that is, $t = t(n)$. It turns out that for slow enough (sublogarithmic) growth of $t(n)$ the asymptotics of $\text{mon}(C_t)$ remains the same as in (1). In contrast, for superlogarithmic $t(n) = o(\min\{\sqrt{N}, N_1\})$ the asymptotics is different, namely, $\text{mon}(C_t) \sim N_1^t$. These results can be summarized in the following two theorems.
Theorem 2. For fixed $r, s$, as $n \to +\infty$, $\text{mon}(C_t) \sim \text{hom}(C_t)$ iff $t = o(\min\{\sqrt{N}, N_1\})$.

Theorem 2 is the trickiest result of our paper. Asymptotics of $\text{hom}(C_t)$ (stated below in Theorem 3) is a more or less direct corollary (modulo technical asymptotical computations) of the well-known representation of $\text{hom}(C_t)$ in terms of eigenvalues of $G(n, r, s)$. Let us fix an arbitrarily small $\varepsilon > 0$ and consider the partition of $\mathbb{N}$ obtained by excluding $\varepsilon$-neighborhoods of $\frac{\ln n}{\ln s-j}$, $j \in [0, s - 1]$, that is, the intervals $I_j = \left[\left(\frac{1 + \varepsilon}{\ln r - j} n\right) \left(\frac{1 - \varepsilon}{\ln r - j + 1} n\right), \frac{1}{s-j}\right]$, $j \in [s - 1]$,

$$I_s = \left[\frac{(1 - \varepsilon)\ln n}{\ln(r - s + 1)}, \infty\right], I_0 = \left[\frac{(1 + \varepsilon)\ln n}{\ln s}, \infty\right].$$

Theorem 3. For fixed $r, s$ and arbitrary $t = t(n) \in \mathbb{N}$,

$$\text{hom}(C_t) = N_1^{t} \left[1 + O\left(\frac{1}{n}\right) + \sum_{j=1}^{s} \frac{n^j}{j!} \left(\frac{r-j}{s-j}\right) + O\left(\frac{1}{n}\right)\right].$$

Moreover, for $j \in [0, s]$ and $t \in I_j$,

$$\text{hom}(C_t) \sim N_1^{t} \frac{n^j}{j!} \left(\frac{r-j}{s-j}\right)^t.$$

Note that for $t \in I_0$, $\text{hom}(C_t) \sim N_1^{t}$, while, for $t \in I_s$, $\text{hom}(C_t) \sim N_1^{t} \left(\frac{r}{s}\right)^t$, that is, (1) holds.

Theorems 2 and 3 immediately yield asymptotics of the number of copies of $C_t$ in $G(n, r, s)$ for all $t = o(\min\{\sqrt{N}, N_1\})$ since it equals $\frac{1}{2t}\text{mon}(C_t)$.

The rest of the paper is organized as follows. First, in Section 2, we discuss general properties of random walks on graphs (Section 2.1) and more specific properties of random walks on $G(n, r, s)$ (Sections 2.2 and 2.3). Second, in Section 3, we prove Theorem 3. Finally, in Sections 4 and 5 we prove that the condition in Theorem 2 is, respectively, sufficient and necessary.

The proof of the sufficiency provided in Section 4 uses exact expressions for the spectrum of $G(n, r, s)$. It should be noted that the proof in the case $r > 2s$ (in which $\sqrt{N} = o(N_1)$) as well as in the case $t = \omega(\ln N)$ can be considerably simplified by using a more general argument (which we omit in this paper) applicable to a wide subclass of spectral expanders (see Section 5). However, for an arbitrary $N_1$-regular graph $G$ on $N$ vertices, the property $\text{mon}(C_t, G) \sim \text{hom}(C_t, G)$ does not necessarily hold when $t = O(\ln N)$ and $N_1 = O(\sqrt{N})$, even if $G$ is a spectral expander. This fact can be demonstrated, for example, by considering the random regular graph $G(N, N_1)$ with $N_1 = [\ln^8 N]$, in which, for any $\varepsilon > 0$, the inequality $\text{mon}(C_t, G(N, N_1)) / \text{hom}(C_t, G(N, N_1)) < \varepsilon$ holds with probability approaching 1 as soon as $t = o(\ln N / \ln \ln N)$ and $t$ is even. This can be
shown by translating the same property from the binomial random graph $G(N, (1 + o(1))N_1/N)$ to $G(N, N_1)$ using the sandwich conjecture, which is true for $N_1 = \omega(\ln^2 N)$ [24]. It is easy to see why the property holds for the binomial random graph $G(N, (1 + o(1))N_1/N)$ by comparing the expectations of $\mu(C_t, G)$ and $\text{hom}(C_t, G)$. Obviously, $\mathbb{E}(\mu(C_t, G)) = \binom{N}{t} t! p^t \sim N_1^t$. Since a self-intersecting even-length cycle can be contained in a single edge, $\mathbb{E}(\text{hom}(C_t, G)) \geq \binom{N}{2} p - \frac{N_1}{2} > N$, which is asymptotically larger than $N_1^t$ if $t = o(\ln N / \ln \ln N)$.

Note that $G(N, N_1)$ is a spectral expander [44]. For the definition and properties of binomial random graphs and regular random graphs see [26].

The proof of the necessity in Theorem 2 provided in Section 5 does not rely upon the whole spectrum of $G(n, r, s)$ but rather uses its spectral expansion property. The necessity of the condition $t = o(N_1)$ follows from the fact that a random walk starts backtracking with positive probability if $t > cN_1$ for a constant $c$, which is proved in Section 5.2 using almost solely the regularity of $G(n, r, s)$. The necessity of $t = o(\sqrt{N})$ is proved in Section 5.1 using a high convergence rate of a random walk on an expander, which is discussed in Section 2.1. Therefore, in Section 5 we formulate a generalization of Theorem 2 to a class of spectral expanders.

2 | RANDOM WALKS

Counting cycles in $G(n, r, s)$ can be reduced to analyzing the distribution of a random walk on $G(n, r, s)$.

2.1 | Distribution and adjacency matrix

Let $G$ be an arbitrary regular connected graph on the vertex set $[N]$ with every vertex having degree $N_1$. Let $A = (A_{ij}, i, j \in [N])$ be its adjacency matrix ($A_{ij} = 1$ if and only if $i$ and $j$ are adjacent in $G$). Moreover, let $\lambda_j, j \in [0, r], be all distinct eigenvalues of $A$, and let $m_j$ be the multiplicity of $\lambda_j$.

Let $Z_+$ denote the set of nonnegative integers. Recall that a random walk on $G$ is a discrete-time random process $X_n \in [N], n \in Z_+$, where $X_0$ is a vertex chosen uniformly at random from $[N], and, for every $n \in Z_+, X_{n+1}$ is chosen uniformly at random from the neighbors of $X_n$ in $G$. For $x, y \in [N], let$

$$P^t(x, y) := \mathbb{P}(X_t = y | X_0 = x)$$

and $P^t = (P^t(x, y), x, y \in [N])$ be the $k$-step transition probabilities matrix.

For a positive integer $t$, $\text{hom}(C_t, G)$ is exactly the trace of $A^t$. Since the trace of $A^t$ equals the sum of its eigenvalues and the eigenvalues of $A^t$ can be computed as the $t$th power of the eigenvalues of $A$ (see, e.g., [32]), we get

$$\text{hom}(C_t, G) = \sum_{j=0}^{r} m_j \lambda_j^t. \quad (3)$$

If $G$ is vertex-transitive, then, clearly, all $P^t(z, z), z \in [N], are equal to each other. Then, for every $x \in [N],$

$$NP^t(x, x) = \sum_{z \in [N]} P^t(z, z) = \frac{\text{hom}(C_t, G)}{N_1^t}. \quad (4)$$
Therefore, (3) implies that

\[
P^t(x, x) = \frac{1}{N} \sum_{j=0}^{r} m_j \left( \frac{\lambda_j}{N_1} \right)^t. \tag{5}
\]

Notice that (due to regularity of \(G\)) the distribution \(\pi = (1/N, \ldots, 1/N) \in \mathbb{R}^N\) is stationary meaning that \(\pi P^t = \pi\). Let us here assume that \(\lambda_0\) is the largest eigenvalue and \(\lambda_1\) is the largest in absolute value eigenvalue distinct from \(\lambda_0\). From the regularity of \(G\) it follows that \(\lambda_0 = N_1\) and from its connectedness, that \(m_0 = 1\) [3]. Let us also assume that \(|\lambda_1| < \lambda_0\) (which is equivalent, for a connected graph, to the graph being nonbipartite [3]). Fix \(v \in [N]\) and \(\varepsilon > 0\). Let us recall that the variation distance at time \(t \in \mathbb{Z}_+\) with initial state \(v\) is

\[
\Delta_v(t) = \frac{1}{2} \sum_{u \in [N]} \left| P^t(v, u) - \frac{1}{N} \right|.
\]

It is very well known [42] that the mixing time \(\tau_v(\varepsilon) := \min\{t : \Delta_v(t') \leq \varepsilon\text{ for all }t' \geq t\}\) satisfies

\[
\tau_v(\varepsilon) \leq \left(1 - \frac{|\lambda_1|}{N_1}\right)^{-1} \ln \frac{N}{\varepsilon}. \tag{6}
\]

### 2.2 Eigenvalues of \(G(n, r, s)\)

Let \(A\) be the adjacency matrix of \(G(n, r, s)\). The eigenvalues of this matrix are known [15] (although the cited paper deals with Johnson association schemes, the adjacency matrix of \(G(n, r, s)\) is exactly the \((r-s)\)th relation in the Johnson association scheme with \(r\) classes). They are equal to

\[
\lambda_j = \sum_{\ell = \max\{0, j-s\}}^{\min\{j, r-s\}} (-1)^{\ell} \binom{j}{\ell} \binom{r-j}{r-s-\ell} \binom{n-r-j}{r-s-\ell} j \in [0, r], \tag{7}
\]

and the multiplicity of the eigenvalue \(\lambda_j\) equals (we let \(\binom{n}{-1} = 0\))

\[
m_j = \binom{n}{j} - \binom{n}{j-1}.
\]

To prove Theorem 3, we need to analyze asymptotical behavior of the expression to the right in (5).

Notice that \(\lambda_0 = N_1\) and \(m_0 = 1\). Also, for \(j \in [1, s]\),

\[
\frac{\lambda_j}{N_1} = \binom{r-j}{s-j} \binom{r}{s} + O\left(\frac{1}{n}\right),
\]

\[
m_j = \frac{n^j}{j!} + O\left(\frac{1}{n}\right) \tag{8}
\]

and, for \(j \in [s + 1, r]\).
\[
\left| \frac{\lambda_j}{N_i} \right| \sim \frac{(j)(r-s)!}{(s)(r-j)!} n^{-(j-s)}, \quad m_j = \frac{n^j}{j!} + O\left(\frac{1}{n}\right).
\]

Therefore, for \( t \geq 2 \) and \( j \in [s+1, r], \)
\[
m_j \frac{\lambda_j^t}{m_s \lambda_s^t} = (O(1))^t \cdot n^{-(j-s)(t-1)} = (O(n^{-(j-s)}))^t = O\left(\frac{1}{n}\right)
\]
implicating that
\[
P^t(x, x) = \frac{1 + O\left(\frac{1}{n}\right)}{N} \sum_{j=0}^{s} m_j \left(\frac{\lambda_j}{N_t}\right)^t = \frac{1}{N} \left[ 1 + O\left(\frac{1}{n}\right) + \sum_{j=1}^{s} m_j \left(\frac{(r-s)}{s-j}\right) + O\left(\frac{1}{n}\right) \right]^t.
\]

### 2.3 Random walk on \( G(n, r, s) \)

Here, we consider the random walk \( (X_n, n \in \mathbb{Z}_+) \) on \( G(n, r, s) \). Since \( G(n, r, s) \) is vertex-transitive, for any \( x \in V \),
\[
\text{mon}(C_t) = NN'P(X_t = x, X_0 \neq X_t \neq ... \neq X_{t-1} | X_0 = x).
\]

To prove Theorem 2, we bound the deviation of \( \frac{\text{mon}(C_t)}{\text{hom}(C_t)} \) from 1. For convenience, in this section, we express the bound in terms of the diagonal elements of \( P^t \).

Due to (4), we get
\[
0 \leq \frac{\text{hom}(C_t) - \text{mon}(C_t)}{\text{hom}(C_t)} = \frac{P^t(x, x) - P(X_t = x, X_0 \neq X_t \neq ... \neq X_{t-1} | X_0 = x)}{P^t(x, x)} = \frac{P(X_t = x, \exists i, j \in [0, t-1]: i \neq j, X_i = X_j | X_0 = x)}{P(X_t = x | X_0 = x)}.
\]

Note that the expression to the right is exactly the probability that the random walk meets itself somewhere on \([0, t-1]\) subject to \( X_0 = x \) and \( X_t = x \).

By the union bound,
\[
\frac{\text{hom}(C_t) - \text{mon}(C_t)}{\text{hom}(C_t)} \leq \sum_{0 \leq i < j < t} \frac{P(X_j = X_t, X_t = x | X_0 = x)}{P(X_t = x | X_0 = x)} = \sum_{0 \leq i < j < t} \sum_{z \in V} \frac{P(X_i = x, X_j = z)P(X_j = z | X_i = z)P(X_i = z | X_0 = x)}{P(X_t = x | X_0 = x)}.
\]
Due to vertex-transitivity of $G(n, r, s)$ the probabilities $P(X_j = z | X_i = z)$ are equal for all $z$. Therefore,

\[
\frac{\text{hom}(C_i) - \text{mon}(C_i)}{\text{hom}(C_i)} \leq \sum_{0 \leq i < j < t} \frac{\sum_{z \in V} P(X_{i-j+i} = x | X_i = z) P_{j-i}^{j-i}(x, x) P(X_i = z | X_0 = x)}{P(X_i = x | X_0 = x)}
\]

\[
= \sum_{0 \leq i < j < t} \frac{P^{l-j+i}(x, x) P^{l-i}(x, x)}{P^l(x, x)} = \sum_{k=1}^{l-1} (t-k) \frac{P^k(x, x) P^{l-k}(x, x)}{P^l(x, x)}
\]

\[
\leq \sum_{k=1}^{l-1} \frac{p^k(x, x) P^{l-k}(x, x)}{P^l(x, x)} = \sum_{k=2}^{l-2} \frac{p^k(x, x) P^{l-k}(x, x)}{P^l(x, x)}.
\tag{12}
\]

3 | PROOF OF THEOREM 3

From (4) and (11), we get

\[
\text{hom}(C_i) = N^i \sum_{j=0}^{s} T_j,
\]

\[
T_0 = 1 + O\left(\frac{1}{n}\right),
\]

\[
T_j = n^j \left(\frac{r-j}{s-j} \left(\frac{r}{s}\right)^j + O\left(\frac{1}{n}\right)\right), j \in [s].
\tag{13}
\]

Fix $j \in [s], i \in [0, j - 1], \varepsilon > 0$. If $t < \frac{(1-\varepsilon) \ln n}{\ln (r-j+1)/r-j+1}$, then

\[
\frac{T_i}{T_j} = O\left(\frac{1}{n^{l-i}}\left(\frac{r-i}{s-i} \frac{r-j}{s-j}\right)^{l-j} \left(\frac{r-i}{s-i} \frac{r-i-1}{s-i-1} \cdots \frac{r-j+1}{s-j+1}\right)^l\right)
\]

\[
= O\left(\frac{1}{n^{l-i}}\left(\frac{r-j+1}{s-j+1}\right)^{l-j} \left(\frac{r-j+1}{s-j+1}\right)^{(j-1)l}\right) = O(n^{-(j-1)\varepsilon}) = o(1).
\]

Similarly, fix $j \in [0, s - 1], i \in [j + 1, s], \varepsilon > 0$. If $t > \frac{(1+\varepsilon) \ln n}{\ln (r-j+1)/r-j+1}$, then it is easy to show that, again, $T_i/T_j = O(n^{-(j-1)\varepsilon}) = o(1)$. Thus, when $t \in I_j$, the term $T_j$ is asymptotically dominant. More precisely,
which proves Theorem 3.

□

4 | PROOF OF SUFFICIENCY IN THEOREM 2

Here we prove that if \( t = o(\min\{\sqrt{N}, N_i\}) \), then

\[
\frac{\text{hom}(C_t) - \text{mon}(C_t)}{\text{hom}(C_t)} \to 0, \; n \to \infty.
\]

Since this fraction is nonnegative, it is sufficient to prove that the upper bound from (12) approaches 0. Clearly, we may assume that \( t \geq 4 \).

By (5), for every \( x \in V \),

\[
\sum_{k=2}^{t-2} p^k(x, x) P^{t-k}(x, x) = \frac{1}{N^2} \sum_{k=2}^{t-2} \sum_{l,j=0}^r m_i m_j \left( \frac{\lambda_j}{N_1} \right)^k \left( \frac{\lambda_i}{N_1} \right)^{t-k} \leq \frac{t}{N^2} \sum_{i=0}^r m_i^2 \left| \frac{\lambda_i}{N_1} \right|^f + \frac{1}{N^2} \sum_{i \neq j} m_i m_j \sum_{k=2}^{t-2} \left| \frac{\lambda_j}{\lambda_i} \right|^k. \tag{14}
\]

Note that \( |\lambda_j|^i \sum_{k=2}^{t-2} \left| \frac{\lambda_j}{\lambda_i} \right|^k = |\lambda_j|^i \sum_{k=2}^{t-2} \left| \frac{\lambda_i}{\lambda_j} \right|^k \). Therefore, for \( n \) large enough, due to (8) and (9), we get

\[
\sum_{i \neq j} m_i m_j \left| \frac{\lambda_i}{N_1} \right|^f \sum_{k=2}^{t-2} \left| \frac{\lambda_j}{\lambda_i} \right|^k \leq 2 \sum_{0 \leq i < j \leq r} m_i m_j \sum_{k=2}^{t-2} \left| \frac{\lambda_j}{\lambda_i} \right|^k \leq 2 \sum_{0 \leq i < j \leq r} m_i m_j \left| \frac{\lambda_j}{\lambda_i} \right|^f \left( 1 - \left| \frac{\lambda_i}{\lambda_j} \right|^f \right).
\]

Moreover, from (8) and (9) we get that, for \( j > i \) and \( j > s \), \( \frac{\lambda_j}{\lambda_i} = O\left( \frac{1}{n} \right) \), while, for \( i < j \leq s \),

\[
\frac{\lambda_j}{\lambda_i} = \frac{(s-i) \ldots (s-j+1)}{(r-i) \ldots (r-j+1)} \left( 1 + O\left( \frac{1}{n} \right) \right).
\]
The latter expression is less than \( \frac{s}{r} \) if \( j \neq 1 \) and \( n \) is large enough. If \( i = 0, j = 1 \), then, from (7), we get

\[
\frac{\lambda_i}{\lambda_0} = \frac{(r-1)(n-r-1)(r-s-1)}{r(n-r)} \leq \frac{(r-1)(n-r-1)}{r(n-r-s)} = \frac{s(n-2r)}{r(n-r)} < \frac{s}{r}.
\]

Then, for \( n \) large enough,

\[
\sum_{i \neq j} m_i m_j \left| \frac{\lambda_i}{N_1} \right|^t \leq \frac{2}{1-s/r} \sum_{0 \leq i < j \leq r} m_i m_j \left| \frac{\lambda_i}{N_1} \right|^{t-2} \left| \frac{\lambda_j}{N_1} \right|^2
\]

\[
\leq \frac{2}{1-s/r} \sum_{i=0}^r m_i m_j \left| \frac{\lambda_i}{N_1} \right|^{t-2} \left| \frac{\lambda_j}{N_1} \right|^2
\]

\[
= \frac{2}{1-s/r} \left( \sum_{i=0}^r m_i \left| \frac{\lambda_i}{N_1} \right|^{t-2} \right) \left( \sum_{j=0}^r m_j \left| \frac{\lambda_j}{N_1} \right|^2 \right).
\]

By (11) and the definition of \( P^2 \),

\[
\sum_{j=0}^r m_j \left| \frac{\lambda_j}{N_1} \right|^2 = NP^2(x, x) = \frac{N}{N_1}.
\]

Moreover, by (8), (10), and (11),

\[
\sum_{i=0}^r m_i \left| \frac{\lambda_i}{N_1} \right|^{t-2} \sim \sum_{i=0}^s m_i \left| \frac{\lambda_i}{N_1} \right|^{t-2} \sim \sum_{i=0}^s m_i \left( \frac{\lambda_i}{N_1} \right)^{t-2} = O \left( \sum_{i=0}^s m_i \left( \frac{\lambda_i}{N_1} \right)^t \right)
\]

\[
= O \left( \sum_{i=0}^r m_i \left( \frac{\lambda_i}{N_1} \right)^t \right) = O(NP^t(x, x)).
\]

It remains to estimate the first summand in the rightmost expression in (14). From (8) and (9), we get that, for \( j \in [s+1, r] \),

\[
\frac{m_j^2 \lambda_j^t}{m_s^2 \lambda_s^t} = (O(1))^t \cdot n^{-(j-s)(t-2)} = (O(n^{-(j-s)}))^t = o(1)
\]

implying that

\[
\sum_{i=0}^r m_i^2 \left| \frac{\lambda_i}{N_1} \right|^t \sim \sum_{i=0}^s m_i^2 \left( \frac{\lambda_i}{N_1} \right)^t.
\]

Putting everything together and applying (11), we conclude that, for every \( x \in V \),
\[ t \sum_{k=2}^{t-2} \frac{p^k(x, x)p^{t-k}(x, x)}{p^t(x, x)} \leq \frac{t^2}{N} \sum_{i=0}^{S} m_i^2 \left( \frac{\lambda_i}{N_i} \right)^t (1 + o(1)) + O \left( \frac{t}{N_1} \right). \]

We finish with proving that the condition \( t = o(\sqrt{N}) \) implies
\[ \frac{t^2}{N} \sum_{i=0}^{S} m_i \left( \frac{\lambda_i}{N_i} \right)^t = o(1). \]

If \( t > C \ln n \), where \( C \) is sufficiently large constant, then, by (8), for every \( i \in [s] \),
\[ m_i \left( \frac{\lambda_i}{N_i} \right)^t = o(1) \text{ and } m_i^2 \left( \frac{\lambda_i}{N_i} \right)^t = o(1). \]
Therefore,
\[ \frac{t^2}{N} \sum_{i=0}^{S} m_i \left( \frac{\lambda_i}{N_i} \right)^t \sim \frac{t^2}{N} = o(1). \]

If \( t \leq C \ln n \), then, by (8),
\[ \frac{t^2}{N} \sum_{i=0}^{S} m_i \left( \frac{\lambda_i}{N_i} \right)^t \leq \frac{t^2 m_i}{N} (1 + o(1)) = O \left( \frac{n^t \ln^2 n}{N} \right) = O \left( \frac{\ln^2 n}{n^{t-2}} \right) = o(1). \]

The sufficiency in Theorem 2 is proved.

5 | PROOF OF NECESSITY IN THEOREM 2

As was already noted, the necessity of the condition in Theorem 2 follows from a more general fact about spectral expanders. Consider a sequence of graphs \( \{G_N, N \in \mathbb{N} \} \) such that \([N]\) is the set of vertices of \( G_N \), and \( G_N \) is nonbipartite, connected, and \( N_1 \)-regular \( (N_1 \text{ depends on } N) \). Let \( \lambda_1 = \lambda_1(N) \) be the second largest in absolute value eigenvalue of the adjacency matrix of \( G_N \). We call the sequence \( \{G_N, N \in \mathbb{N} \} \) a spectral expander, if there exists \( \delta > 0 \) such that, for all large enough \( N \), \( \frac{\ln \lambda_i}{N_1} < 1 - \delta \).

**Theorem 4.** Let \( \{G_N, N \in \mathbb{N} \} \) be a spectral expander such that \( N_1 = \omega(\ln N) \). Then for any \( c > 0 \) there exists \( \varepsilon > 0 \) such that, for sufficiently large \( N \), if \( t > c \min \{\sqrt{N}, N_1\} \), then \( \text{mon}(C_t, G_N) < (1 - \varepsilon) \text{hom}(C_t, G_N) \).

Obviously, Theorem 4 implies the necessity of the condition \( t = o(\min \{\sqrt{N}, N_1\}) \) for \( \text{mon}(C_t, G_N) \sim \text{hom}(C_t) \) stated in Theorem 2.

To prove Theorem 4, we introduce a random walk on \( G_N \) (see Section 2.1).

In Section 2.3, we note that the proportion of self-intersecting cycles in \( G(n, r, s) \) is exactly the probability that the random walk meets itself somewhere on \([0, t-1]\) subject to \( X_0 = x \) and...
$X_t = x$. It is easy to see that (almost) the same is true for $G_N$ since $X_0$ is chosen uniformly at random from $[N]$. Indeed,

$$\text{hom}(C_t, G_N) = \sum_{x\in[N]} P^t(x, x)N_1' = NN_1' \sum_{x\in[N]} P^t(x, x)\mathbb{P}(X_0 = x) = NN_1' \mathbb{P}(X_t = X_0).$$

In the same way, $\text{mon}(C_t, G) = NN_1' \mathbb{P}(X_t = X_0, X_0 \neq X_1 \neq \ldots \neq X_{t-1})$. Therefore,

$$\frac{\text{hom}(C_t, G) - \text{mon}(C_t, G)}{\text{hom}(C_t, G)} = \mathbb{P}(\exists i, j \in [0, t-1] : i \neq j, X_i = X_j | X_t = X_0).$$

(15)

Further, we separately consider cases $\sqrt{N} = o(N_1)$ and $N_1 = O(\sqrt{N})$. Although other cases are possible for an arbitrary spectral expander $G_N$, it is enough to consider only these two cases. Indeed, assume that the theorem holds for spectral expanders falling into these two cases, but does not hold for some spectral expander $G_N$. Then it is possible to choose a subsequence $G_{N_k}$, for which one of the two cases holds, which leads to a contradiction.

5.1 $\sqrt{N} = o(N_1)$

Let us first bound from above $P^t(x, y)$ for arbitrary $x, y$ from $[N]$ and an integer $t \geq 1$. Since $P^t(x, y) = \sum_{v\in[N]} P^{t-1}(x, v)P^1(v, y)$, we get that

$$P^t(x, y) \leq \max_{v\in[N]} P^1(v, y) \leq \frac{1}{N_1}.$$   (16)

Let us also notice that from (6) it immediately follows that, for every $c > 0$, there exists $\kappa$ such that, for all $t \geq \kappa \ln N$ and all $x, y \in [N]$, we have

$$\left| P^t(x, y) - \frac{1}{N} \right| < \frac{1}{N^2}.$$   (17)

Let us fix a positive $\bar{c} < \min\{c, 1\}$ and prove that the random walk subject to $X_t = X_0$ intersects itself during the first $\bar{c} \sqrt{N}$ steps with probability bounded away from zero. Let

$$\mathcal{J} := \{(i, j) \in [t]^2 : \kappa \ln N < i < j - 3\kappa \ln N < j < \bar{c} \sqrt{N}\}.$$

Note that $|\mathcal{J}| = \frac{\bar{c}^2 + o(1)}{2} N$ and that $j < t - \kappa \ln N$ for every $(i, j) \in \mathcal{J}$.

For all $(i, j) \in \mathcal{J}$, we have that

$$\frac{\mathbb{P}(X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} = \sum_{x, u\in[N]} P^t(x, u)P^{j-l}(u, u)P^{l-j}(u, x) \mathbb{P}(y, y) \sum_{y\in[N]} P^t(y, y) = \frac{1 + o(1)}{N}.$$
due to (17). Note that \( o(1) \) in the expression above converges to 0 uniformly over all \((i, j) \in \mathcal{J}\). Therefore,

\[
\sum_{(i,j) \in \mathcal{J}} \frac{\mathbb{P}(X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} = \frac{\tilde{c}^2}{2} + o(1).
\]

Let

\[
\mathcal{J}_0 := \left\{((i_1, j_1), (i_2, j_2)) \in \binom{\mathcal{J}}{2} : (\forall \{i, j\} \subset \{i_1, i_2, j_1, j_2\} : |i - j| \geq \kappa \ln N)\right\},
\]

\[
\mathcal{J}_1 := \left\{((i_1, j_1), (i_2, j_2)) \in \binom{\mathcal{J}}{2} : (\exists! \{i, j\} \subset \{i_1, i_2, j_1, j_2\} : |i - j| < \kappa \ln N)\}, 1 - |j_2| < \kappa \ln N.
\]

\[
\mathcal{J}_2 := \left\{((i_1, j_1), (i_2, j_2)) \in \binom{\mathcal{J}}{2} : \max|i_i - i_2|, |j_j\right\}
\]

It is clear from the definition of \(\mathcal{J}\) that \(\mathcal{J}_0 \cap \mathcal{J}_1 \cap \mathcal{J}_2 = \binom{\mathcal{J}}{2}\) (recall that \(j - i > 3\kappa \ln N\) for every \((i, j) \in \mathcal{J}\)). Moreover,

\[
|\mathcal{J}_0| = \frac{1}{2} \binom{|\mathcal{J}|}{2}(1 + o(1)) = \frac{\tilde{c}^4 + o(1)}{8} N^2, |\mathcal{J}_1| = O(\kappa \ln N (\tilde{c} \sqrt{N})^3), |\mathcal{J}_2| = O((\kappa \ln N \tilde{c} \sqrt{N})^2).
\]

As above, uniformly over all \(((i_1, j_1), (i_2, j_2)) \in \mathcal{J}_0\), (17) implies

\[
\frac{\mathbb{P}(X_i = X_j, X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} = \frac{1 + o(1)}{N^2}.
\]

Uniformly over all \(((i_1, j_1), (i_2, j_2)) \in \mathcal{J}_1\), the relations (16) and (17) imply

\[
\frac{\mathbb{P}(X_i = X_j, X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} \leq \frac{1 + o(1)}{NN_1}.
\]

Uniformly over all \(((i_1, j_1), (i_2, j_2)) \in \mathcal{J}_2\), (16) implies

\[
\frac{\mathbb{P}(X_i = X_j, X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} \leq \frac{1 + o(1)}{N_i^2}.
\]

Summing up and recalling that, in the current case, \(\sqrt{N} = o(N_i)\),

\[
\sum_{((i_1,j_1),(i_2,j_2)) \in \binom{\mathcal{J}}{2}} \frac{\mathbb{P}(X_i = X_j, X_i = X_j, X_t = X_0)}{\mathbb{P}(X_t = X_0)} \leq \frac{\tilde{c}^4 + o(1)}{8} + O\left(\frac{\sqrt{N} \ln N}{N_i}\right) + O\left(\frac{N \ln N}{N_i^2}\right) = \frac{\tilde{c}^4 + o(1)}{8}.
\]
From (15), (18), and (19) we get

\[
\frac{\text{hom}(C_t, G) - \text{mon}(C_t, G)}{\text{hom}(C_t, G)} \geq \sum_{(i,j) \in \mathcal{J}} \frac{p(X_i = X_j, X_t = X_0)}{p(X_t = X_0)} - \sum_{((i,0),(i,0)) \in \binom{\mathcal{J}}{2}} \frac{p(X_i = X_j, X_i = X_j, X_t = X_0)}{p(X_t = X_0)} = \frac{\tilde{\sigma}^2}{2} - \frac{\tilde{\sigma}^4}{8} + o(1).
\]

Since \(\frac{\tilde{\sigma}^2}{2} - \frac{\tilde{\sigma}^4}{8} > 0\), we conclude that \(\frac{\text{mon}(C_t, G)}{\text{hom}(C_t, G)}\) is bounded away from 1 as needed.

### 5.2 \(N_1 = O(\sqrt{N})\)

W.l.o.g. we may assume that \(c < 1\) and prove that the random walk subject to \(X_t = X_0\) intersects itself during the first \(cN_t\) steps with nonzero probability. In the same way, as in Section 5.1, we use (15). However, here we consider all \((i,j)\) such that \(i\) is even and \(j = i + 2 \leq cN_t\). Let \(\mathcal{J} := \{2i : i \in \left[0, \left\lfloor \frac{cN_t - 2}{2} \right\rfloor \right]\}\). For every \(i \in \mathcal{J}\), we have

\[
\frac{p(X_i = X_{i+2}, X_t = X_0)}{p(X_t = X_0)} = \frac{1}{N} \sum_{x,u \in [N]} p^i(x, u)p^2(u, u)p^{t-i-2}(u, x)
\]

since \(p^2(u, u) = \frac{1}{N}\) for all \(u \in [N]\) and \(p(X_{i-2} = X_0) \sim p(X_t = X_0) \sim 1/N\) due to (17). Thus,

\[
\sum_{i \in \mathcal{J}} \frac{p(X_i = X_{i+2}, X_t = X_0)}{p(X_t = X_0)} = \frac{c}{2} + o(1).
\]

Now, let \(i_1, i_2 \in \mathcal{J}, i_1 < i_2\). Then, similarly,

\[
\frac{p(X_{i_1} = X_{i_1+2}, X_{i_2} = X_{i_2+2}, X_t = X_0)}{p(X_t = X_0)} = \frac{1}{N} \sum_{x,u,v \in [N]} p^{i_1}(x, u)p^2(u, u)p^{t-i_1-2}(u, v)p^2(v, v)p^{t-i_2-2}(v, x)
\]

\[
= \frac{1}{N} \sum_{x,u,v \in [N]} p^{i_1}(x, u)p^{i_2-i_1-2}(u, v)p^{t-i_2-2}(v, x)
\]

\[
= \frac{p(X_{i-4} = X_0)}{N_1^2 p(X_t = X_0)} \sim \frac{1}{N_1^2}.
\]
Therefore,

\[
\frac{\text{hom}(C_t, G) - \text{mon}(C_t, G)}{\text{hom}(C_t, G)} \geq \sum_{i \in J} \frac{\mathbb{P}(X_i = X_{i+2}, X_i = X_0)}{\mathbb{P}(X_i = X_0)} - \sum_{i, i' \in J, i' < i} \frac{\mathbb{P}(X_{i'} = X_{i'+2}, X_i = X_{i+2}, X_i = X_0)}{\mathbb{P}(X_i = X_0)} = \frac{c}{2} - \frac{c^2}{8} + o(1). 
\]

Since \(c/2 - c^2/8 > 0\), this finishes the proof of Theorem 4 and therefore of Theorem 2. □

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