Continuity of plurisubharmonic envelopes in non-archimedean geometry and test ideals

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joint work with
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Boucksom, S.; Favre, C.; Jonsson, M.: **Solution to a non-Archimedean Monge-Ampère equation.** J. Amer. Math. Soc. 28 (2015), no. 3, 617–667.

Boucksom, S.; Favre, C.; Jonsson, M.: **Singular semipositive metrics in non-Archimedean geometry.** J. Algebraic Geom. 25 (2016), no. 1, 77–139.

Burgos Gil, J.; Gubler, W.; Jell, P.; Künemann, K.; Martin, F.: **Differentiability of non-archimedean volumes and non-archimedean Monge-Ampère equations** (with an appendix by R. Lazarsfeld). arXiv:1608.01919

Gubler, W.; Jell,P.; Künemann, K.; Martin, F: Continuity of **Plurisubharmonic Envelopes in Non-Archimedean Geometry and Test Ideals** (with an Appendix by J. Burgos Gil and M. Sombra). arXiv:1712.00980
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The archimedean Calabi-Yau problem

- $(M, \omega)$ Kähler manifold of dimension $n$,
- $\omega$ a real, closed, positive $(1, 1)$-form in $M$,
in holomorphic coordinates $(z_1, \ldots, z_n): U \to \mathbb{C}^n$ on $M$

\[ \omega|_U = \frac{-1}{2\pi i} \sum_{k,l=1}^{n} h_{kl} dz_k \wedge d\bar{z}_l \]

where the hermitian matrix $(h_{kl})_{kl}$ is positive definite.

**Theorem (Calabi (uniqueness 1957), Yau (existence 1978))**
Given a smooth positive volume form $\Omega$ with $\int_M \Omega = \int_M \omega^n$ there exists a unique real smooth closed $(1, 1)$-form $\alpha$ on $M$ with $[\alpha] = [\omega]$ in $H^2_{DR}(M)$ and

\[ \alpha^n = \Omega \quad \text{(Monge-Ampère equation)}. \]
Let $L$ holom. line bundle on $M$ with $c_1(L) = [\omega] \in H^2_{\text{DR}}(M)$.

Hodge theory yields a smooth hermitian metric $\| \|$ on $L$ (unique up to scaling) such that the curvature form satisfies $c_1(L, \| \|) = \alpha$.

Get equality of measures

$$c_1(L, \| \|)^n = \mu \quad \text{(Monge-Ampère equation)}$$

where Radon measure $\mu$ is defined by $\Omega$ and Monge-Ampère measure $c_1(L, \| \|)^n$ is given by

$$f \mapsto \int_M f \cdot c_1(L, \| \|)^n$$

with local density on holomorphic chart $(U, z)$ of $M$

$$\frac{1}{(2\pi i)^n} \det \left( \frac{\partial^2 \log \|s\|^2}{\partial z_k \partial \bar{z}_l} \right) dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.$$
**Bedford-Taylor Theory.** If metric $\| \|$ on $L$ is **semipositive** in the sense that it can be approximated uniformly by semipositive smooth metrics $(\| \|_i)_{i \in \mathbb{N}}$ on $L$ then measure $c_1(L, \| \|)^n$ can be defined as weak limit of the measures $c_1(L, \| \|_i)^n$.

**Theorem (Kołodziej 1998)** If measure $\mu$ on $M$ has (locally) $L^p$-density with respect to Lebesgue measure for some $p > 1$ then there exist a semipositive continuous metric $\| \|$ on $L$ such that

$$c_1(L, \| \|)^n = \mu.$$
The non-archimedean Calabi-Yau problem

Setup for the rest of this talk:

- \((K, | |)\) non-archimedean complete discretely valued field
- discrete valuation ring \(K^\circ = \{ x \in K | |x| \leq 1 \}\) (noetherian),
- maximal ideal \(K^{\circ\circ} = \{ x \in K | |x| < 1 \}\),
- residue class field \(\tilde{K} = K^\circ/K^{\circ\circ}\),
- examples \((\mathbb{C}((T)), \mathbb{C}[[T]], \mathbb{C})\) and \((\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z})\) for prime \(p\),
- \(X\) normal projective variety over \(K\) of dimension \(n\),
- \(X^{\text{an}} := \text{Berkovich analytification of } X\). Consists of pairs \((p, | |_p)\) where \(p \in X\) and \(| |_p\) is absolute value on \(\kappa(p) = \mathcal{O}_{X,p}/m_{X,p}\) which extends \(| |\) on \(K\). Equip \(X^{\text{an}}\) with coarsest topology such that \(\pi : X^{\text{an}} \to X, (p, | |_p) \mapsto p\)
- continuous and for all \(U\) open in \(X\) and \(f \in \mathcal{O}_X(U)\)

\[ |f| : U^{\text{an}} = \pi^{-1}(U) \to \mathbb{R}, (p, | |_p) \mapsto |f(p)| := |f + m_{X,p}|_p \]

is continuous as well.
**Metrics.** Let $L$ be a line bundle on $X$. A continuous metric $\|\| \, \|\|$ on $L$ associates with every $s \in \Gamma(U, L)$ for $U \subseteq X$ open a continuous function $\|s\| : U^{\text{an}} \to \mathbb{R}_{\geq 0}$ such that $\|f \cdot s\| = |f| \|s\|$ for all $f \in \mathcal{O}_X(U)$ and $\|s\| > 0$ if $s$ is a frame of $L$.

**Models.** A model $\mathcal{X}$ of $X$ over $K^\circ$ is a normal proper flat scheme over $S := \text{Spec} \, K^\circ$ with generic fibre $X$.

**Reduction.** Let $\mathcal{X}$ be a model of $X$ over $K^\circ$. There is a canonical reduction map

$$\text{red} : X^{\text{an}} \longrightarrow \mathcal{X}_s$$

where $\mathcal{X}_s := \mathcal{X} \otimes_{K^\circ} \tilde{K}$ denotes the special fibre of $\mathcal{X}$.

**Shilov Points.** A generic point $\eta_V$ of an irreducible component $V$ of $\mathcal{X}_s$ has a unique preimage $x_V$ in $X^{\text{an}}$ (Berkovich).
Model metrics. Let $X$ be a model of $X$ over $K^\circ$ and $L$ a line bundle on $X$ with $L|_X = L$. Pair $(X, L)$ determines a unique continuous metric $\| \|_L$ on $L$ such that a frame $s$ of $L$ over $U \subseteq X$ open satisfies $\| s \|_L = 1$ on $\text{red}^{-1}(U_s)$. Such a metric is called an algebraic metric. A continuous metric $\| \|_L$ on $L$ is a model metric if some power $(L^\otimes k, \| \|_L^\otimes k)$ is an algebraic metric.

Nef line bundles. A line bundle $L$ on model $X$ is called nef if

$$\text{deg}_{\bar{K}}(L|_C) \geq 0$$

for any proper curve $C$ in $X_s$.

Semipositive model metrics. A model metric $\| \|_L$ on $L$ is called semipositive if some power is induced by a nef line bundle.

Semipositive metrics. A uniform limit of semipositive model metrics on $L$ is called semipositive metric.
Chambert-Loir measures. There is unique way to associate with semipositive metrized line bundles $(L_1, \| \|_1), \ldots, (L_n, \| \|_n)$ a (positive) Radon measure $c_1(L_1, \| \|_1) \wedge \ldots \wedge c_1(L_n, \| \|_n)$ on $X^{\an}$ of total mass $\deg_X(c_1(L_1) \ldots c_1(L_n))$ which is multilinear and continuous in the $(L_i, \| \|_i)$, satisfies a projection formula, and is given as follows for model metrics.

Chambert-Loir measures for model metrics. Let $\mathcal{X}$ be a model of $X$. For line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\mathcal{X}$ models of $L$ on $X$ the Chambert-Loir measure $c_1(L, \| \|_{\mathcal{L}_1}) \wedge \ldots \wedge c_1(L, \| \|_{\mathcal{L}_n})$ is the discrete signed measure

$$
\sum_{V \in \mathcal{X}_s^{(0)}} \ell_{\mathcal{O}_{\mathcal{X}_s}, V}(\mathcal{O}_{\mathcal{X}_s}, V) \deg_V(\mathcal{L}_1 \cdots \mathcal{L}_n) \delta_{x_V}
$$

on $X^{\an}$, where $x_V \in X^{\an}$ is Shilov point determined by $\text{red}(x_V) = \eta_V$ and $\delta_{x_V}$ is the Dirac measure in $x_V$. 
Assume in the following that the projective variety $X$ is smooth.

**SNC models.** An SNC model $\mathcal{X}$ of $X$ is a regular model such that $(\mathcal{X}_s)_{\text{red}}$ is simple normal crossing divisor in $\mathcal{X}$.

**Skeleta.** Each SNC model determines a skeleton $\Delta_{\mathcal{X}} \subset X^{\text{an}}$ (Berkovich, BFJ). The skeleton is geometric realization of a simplicial complex whose 0-dimensional vertices are given by the Shilov points associated with irreducible components of $\mathcal{X}_s$.

**Fact.** If $\text{char} \tilde{K} = 0$ then $K \cong \tilde{K}((T))$ and SNC models exist.

**Varieties over $K$ of geometric origin.** We say that $X$ is of geometric origin if there exists a normal curve $B$ over a field $k$ and a closed point $b$ on $B$ such that $K^o$ is the completion of the discrete valuation ring $\mathcal{O}_{B,p}$ and $X$ is defined over the function field $\kappa(B)$ of $B$. 
Theorem (Boucksom, Favre, Jonsson 2015). Fix \( L \) ample line bundle on \( X \) and \( \mu \) positive Radon measure on \( X^{\text{an}} \) of total mass \( \deg c_1(L)^n \). If

(i) \( \text{char}(\tilde{K}) = 0 \),

(ii) \( X \) is of geometric origin,

(iii) \( \mu \) is supported on skeleton of some SNC model of \( X \)

then there exists semipositive metric \( \| \| \) on \( L \) such that

\[
\left( c_1(L, \| \|) \right)^n = \mu.
\]

Remarks.

- Thuillier (2005) \( X \) curve without (i)-(ii)
- Y. Liu (2011) \( X \) totally degenerate abelian variety w.o. (i)-(ii)
- Yuan, Zhang (2016) uniqueness up to scaling without (i)-(iii)
- Thm. BFJ without (ii) (Burgos, Gubler, Jell, K., Martin 2016)
Strategy (Boucksom, Favre, Jonsson)

- Fix semipositive reference metric $\| \|_0$ on $L^\text{an}$,
- consider energy $E(\| \|, \| \|_0)$ defined as

$$\frac{-1}{n+1} \sum_{j=0}^{n} \int_{X^\text{an}} \log \frac{\| c_1(L, \| \|)^j \wedge c_1(L, \| \|_0)^{n-j}}{\| \|_0} \, d\mu,$$

and maximize

$$\| \| \longrightarrow E(\| \|, \| \|_0) - \int_{X^\text{an}} -\log \frac{\| c_1(L, \| \|)^j \wedge c_1(L, \| \|_0)^{n-j}}{\| \|_0} \, d\mu,$$

- use orthogonality principle + differentiability of $E \circ P$ where

$$P(\| \|) = \inf_{\text{pointwise}} \{ \| \|' \mid \| \|' \text{ semipos. model metric} \| \| \leq \| \|' \}$$

denotes the semipositive envelope of the metric,
Strategy (Boucksom, Favre, Jonsson)

- fix continuous metric $\| \| \|$ on $L^\text{an}$,
- use Assumption (i) and the theory of multiplier ideals to show continuity of the semipositive envelope

$$P(\| \|) = \inf_{\text{pointwise}} \{ \| \|' | \| \|' \text{ semipos. model metric} \| \| \leq \| \|' \} ,$$

- use Assumption (ii) that $X$ is of geometric origin to show:

**Orthogonality Principle.** If the line bundle $L$ is ample then

$$\int_{X^\text{an}} \log \frac{P(\| \|)}{\| \|} c_1(L, P(\| \|))^{\wedge n} = 0,$$

i.e. the support of the Chambert-Loir measure is orthogonal to the locus where the metric differs from its envelope.
Strategy (Burgos, Gubler, Jell, K., Martin 2016).

**Definition.** For \( \| \|_1, \| \|_2 \) on \( L \) define non-archimedean volume

\[
\text{vol} (L, \| \|_1, \| \|_2) = \lim_{m \to \infty} \sup_{n} \frac{n!}{m^{n+1}} \text{length}_{K^0} \left( \frac{\hat{H}^0(X, L^\otimes m, \| \|_1^m)}{\hat{H}^0(X, L^\otimes m, \| \|_2^m)} \right).
\]

**Remark.** Relate \( \text{vol} \) to energy \( E \) + show \( \text{vol}(L, P(\| \|), \| \|) = 0 \).

**Theorem (Burgos, Gubler, Jell, K., Martin)** If \( \| \| \) continuous semipositive metric on \( L \) and \( f : X^{an} \to \mathbb{R} \) continuous then \( t \in \mathbb{R} \mapsto \text{vol} (L, \| \| e^{-tf}, \| \|) \) is differentiable at \( t = 0 \) and

\[
\left. \frac{d}{dt} \right|_{t=0} \text{vol} (L, \| \| e^{-tf}, \| \|) = \int_{X^{an}} f \ c_1 (L, \| \|)^n.
\]

**Remark.** Theorem gives orthogonality principle without assumption \( X \) of geometric origin if \( P(\| \|) \) is continuous.
Results in positive characteristic

Assume for the rest of this talk that

- char $K = p > 0$,
- $X$ a smooth projective surface over $K$,
- $L$ an ample line bundle on $X$.

**Theorem (Gubler, Jell, K. , Martin).** Let $X$ be of geometric origin from perfect ground field $k$. If $||$ is continuous metric on $L^{an}$ then $P(||)$ is continuous semipositive metric on $L^{an}$.

**Corollary (Gubler, Jell, K. , Martin).** In situation of Theorem let $\mu$ be a positive Radon measure on $X^{an}$ with $\mu(X^{an}) = \text{deg } c_1(L)^2$ supported on skeleton of projective SNC-model of $X$. Then the non-archimedean Monge-Ampère equation $c_1(L, ||)^2 = \mu$ has continuous semipositive solution $||$.
Remarks.

- Continuity of envelopes holds for curves in any characteristic (Gubler, Jell, K., Martin).
- Proof of our Theorem uses results about asymptotic test ideals (Mustață) and resolution of singularities in positive characteristic (Cossart–Piltant).
- Under the assumption of resolution of singularities in positive characteristic our results generalize from surfaces $X$ to varieties $X$ of arbitrary dimension.
- Our results generalize furthermore to varieties coming by base change in codimension one from families over higher dimensional varieties.
Asymptotic test ideals

- Introduced Hochster, Huneke (1990) theory of tight closure,
- \( Y \) smooth variety over a perfect field \( k \) with \( p := \text{char} \ k > 0 \),
- \( F : Y \to Y \) Frobenius, i.e. \( F|_X = \text{id}|_X \) and \( F^*(s) = s^p \),
- \( \omega_{Y/k} = \det \Omega^1_{Y/k} = \mathcal{O}_Y(K_{Y/k}) \) for canonical divisor \( K_{Y/k} \),
- fix \( e \in \mathbb{Z} \) and ideal (sheaf) \( \mathfrak{a} \) of \( \mathcal{O}_Y \),
- there exists unique ideal \( \mathfrak{a}^{[p^e]} \) of \( \mathcal{O}_Y \) with
  \[
  \mathfrak{a}^{[p^e]}(U) = \langle u^{p^e} | u \in \mathfrak{a}(U) \rangle_{\text{ideal}} \quad (U \text{ open affine in } Y),
  \]
- Cartier isomorphism yields trace map
  \( \text{Tr} : F_* (\omega_{Y/k}) \to \omega_{Y/k} \),
- (Def. Mustață) there exists unique ideal \( \mathfrak{a}^{[1/p^e]} \) of \( \mathcal{O}_Y \) with
  \[
  \text{Tr}^e (F_*^e (\mathfrak{a} \cdot \omega_{Y/k})) = \mathfrak{a}^{[1/p^e]} \cdot \omega_{Y/k}.
  \]
Definition. (i) The test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ of $a$ is

$$\tau(a^\lambda) = \bigcup_{e \in \mathbb{N}_{>0}} (a^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq \mathcal{O}_Y.$$ 

(ii) A graded sequence of ideals $a_\bullet$ in $\mathcal{O}_X$ is a family $(a_m)_{m \in \mathbb{Z}_{>0}}$ of ideals in $\mathcal{O}_X$ such that $a_m \cdot a_n \subseteq a_{m+n}$ for all $m, n \in \mathbb{Z}_{>0}$ and $a_m \neq (0)$ for some $m > 0$.

(iii) The asymptotic test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ of a graded sequence of ideals $a_\bullet$ is

$$\tau(a_\bullet^\lambda) := \bigcup_{m \in \mathbb{Z}} \tau(a_{m}^\lambda/m) \subseteq \mathcal{O}_Y.$$ 

Properties. (i) Have $a \subseteq \tau(a)$ and $\tau(a_m) \subseteq \tau(a_\bullet^m)$ for all $m \in \mathbb{N}$.

(ii) Subadditivity Property: For all $m \in \mathbb{N}$ have

$$\tau(a_\bullet^{m\lambda}) \subseteq \tau(a_\bullet^\lambda)^m.$$ 

(iii) Uniform generation property.
Definition Let $D$ be a divisor on the smooth variety $Y$ with $H^0(Y, \mathcal{O}_Y(mD)) \neq 0$ for some $m > 0$. Define the asymptotic test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ associated with $Y$ and $D$ as

$$\tau(\lambda \cdot \|D\|) := \tau(a_\lambda)$$

where $a_\bullet$ denotes the graded sequence of base ideals for $D$, i.e. $a_m$ is the image of the natural map

$$H^0(Y, \mathcal{O}(mD)) \otimes_k \mathcal{O}_Y(-mD) \to \mathcal{O}_Y.$$
Theorem (Mustață’s uniform generation property). Let $R$ be a $k$-algebra of finite type, $Y$ an integral scheme of dimension $n$ projective over the spectrum of $R$ and smooth over $k$. Let $D$, $E$, and $H$ be divisors on $Y$ and $\lambda \in \mathbb{Q}_{\geq 0}$ such that

(i) $\mathcal{O}_Y(H)$ is an ample, globally generated line bundle,

(ii) $H^0(Y, \mathcal{O}_Y(mD)) \neq 0$ for some $m > 0$, and

(iii) the $\mathbb{Q}$-divisor $E - \lambda D$ is nef.

Then the sheaf $\mathcal{O}_Y(K_{Y/k} + E + dH) \otimes_{\mathcal{O}_Y} \tau(\lambda \cdot \|D\|)$ is globally generated for all $d \geq n + 1$.

- Mustață’s original theorem requires $Y$ projective over ground field $k$,
- we follow Mustață’s proof with two modifications,
- Castelnuovo-Mumford regularity holds for the projective scheme $X$ over $R$ (Brodmann, Sharp),
- replace use of Fujita’s vanishing theorem by application of Keeler’s vanishing theorem.
Resolution of singularities holds over field $k$ in dimension $m$ if for every quasi-projective variety $Y$ over $k$ of dimension $m$ there exists regular variety $\tilde{Y}$ over $k$ and projective morphism $\tilde{Y} \to Y$ which is isomorphism over the regular locus of $Y$.

Embedded resolution of singularities over a field $k$ in dimension $m$ holds if for every quasi-projective regular variety $Y$ over $k$ of dimension $m$ and every proper closed subset $Z$ of $Y$, there is a projective morphism $\pi : Y' \to Y$ of quasi-projective regular varieties over $k$ such that $\pi^{-1}(Z)$ is the support of a normal crossing divisor and $\pi$ is isomorphism over $Y \setminus Z$.

**Theorem (Cossart-Piltant).** Resolution of singularities and embedded resolution of singularities hold in dimension three over any perfect field.
Continuity of the envelope (ideas of the proof)

- \{ f : X^{\text{an}} \to \mathbb{R} \mid f = - \log \|1\| \text{ for some model metric on } \mathcal{O}_X \} \text{ is the } \mathbb{Q}\text{-vector space } \mathcal{D}(X) \text{ of model functions},
- \mathcal{N}^1(\mathcal{X}/S) \text{ denotes space of } \mathbb{Q}\text{-line bundles on } \mathcal{X} \text{ modulo numerical equivalence in special fiber: }
  [\mathcal{L}] = 0 \in \mathcal{N}^1(\mathcal{X}/S) \iff \deg(\mathcal{L}|_C) = 0 \text{ for all curves } C \text{ in } \mathcal{X}_S.
- If \(\|\|\) is model metric on \(L\) then
  \[ c_1(L, \|\|) \in Z^{1,1}(X) := \lim_{\mathcal{X} \to \text{model}} \mathcal{N}^1(\mathcal{X}/S) \otimes_{\mathbb{Q}} \mathbb{R}. \]

- Call \(\theta \in Z^{1,1}(X)\) semipositive if \(\theta\) is induced by a nef element in \(\mathcal{N}^1(\mathcal{X}/S)\) for some model \(\mathcal{X}\).
- Have \(dd^c : \mathcal{D}(X) \to Z^{1,1}(X), \ f \mapsto c_1(\mathcal{O}_X, \|\|_{\text{triv}} \cdot e^{-f})\).
The space of $\theta$-psh model functions is defined as
\[
\text{PSH}(X, \theta) = \{ f \in \mathcal{D}(X) \mid \theta + dd^c f \text{ is semipositive} \}.
\]

Say that $\theta \in Z^{1,1}(X)$ has ample de Rham class if restriction to $X$ of representative of $\theta$ is $\mathbb{R}_{>0}$-linear combination of classes induced by ample line bundles on $X$.

Fix $\theta \in Z^{1,1}(X)$ with ample de Rham class. Given $u : \mathbb{A}^\text{an} \to \mathbb{R}$ continuous define $\theta$-psh envelope $P_\theta(u) : X^\text{an} \to \mathbb{R}$ by
\[
P_\theta(u) = \sup_{\text{pointwise}} \{ \varphi \mid \varphi \in \text{PSH}(X, \theta) \land \varphi \leq u \}.
\]

Continuity of $\theta$-psh envelope is equivalent continuity of semipositive envelope.

We have $P_\theta(u) - v = P_{\theta + dd^c v}(u - v)$ for each $v \in \mathcal{D}(X)$.

By last equality it suffices to show continuity of $P_\theta(0)$.
Proposition (Boucksom, Favre, Jonsson) Let $L$ be ample line bundle on $X$, $\mathcal{L}$ extension to a model $\mathcal{X}$ of $X$ and $\theta = c_1(\mathcal{L}, ||L||) \in Z^{1,1}(X)$. For $m > 0$ let

$$a_m := \text{Im} \left( H^0(\mathcal{X}, \mathcal{L}^\otimes m) \otimes_{\mathcal{O}_X} \mathcal{L}^\otimes -m \longrightarrow \mathcal{O}_{\mathcal{X}} \right)$$

be the $m$-th base ideal of $\mathcal{L}$. Let $\varphi_m := m^{-1} \log |a_m|$ be $1/m$-times model function determined by $\mathcal{O}(E)$ for exceptional divisor $E$ of blowup of $\mathcal{X}$ along vertical ideal $a_m$. Then $\varphi_m \in \text{PSH}_D(X, \theta)$ and pointwise on $X^{an}$

$$\lim_m \varphi_m = \sup_m \varphi_m = P_\theta(0).$$

Embedded resolution of singularities shows that projective models are dominated by SNC-models.

Assumption that $X$ is of geometric origin construction says that $X$ comes by base change from variety over perfect field $k$ with a fibration to curve.
**Proposition (GJKM).** There exist normal curve $B$ over a perfect field $k$, closed point $b \in B^{(1)}$, a projective regular integral scheme $X_B$ over $B$, and line bundles $\mathcal{L}_B$ and $\mathcal{A}_B$ over $X_B$ such that there exist

- a flat morphism $h: \text{Spec } K^\circ \rightarrow \text{Spec } \mathcal{O}_{B,b} \rightarrow B$,
- an isomorphism $X_B \otimes_B \text{Spec } K^\circ \overset{\sim}{\rightarrow} X$,
- an isomorphism $h^* \mathcal{L}_B \overset{\sim}{\rightarrow} \mathcal{L}$ over the isomorphism above,
- and an isomorphism $\mathcal{A}_B|_{X_B,\eta} \overset{\sim}{\rightarrow} \mathcal{L}_B|_{X_B,\eta}$ where $\eta$ is the generic point of $B$ and the line bundle $\mathcal{A}_B$ on $X_B$ is ample.

Read all isomorphisms above as identifications. Have cartesian diagram

$$
\begin{tikzcd}
X \arrow{r}{g} \arrow{d} & X_B \arrow{d} \\
\text{Spec } K^\circ \arrow{r}{h} & B \arrow{r} & \text{Spec } k.
\end{tikzcd}
$$
Observe $\mathcal{X}_B$ is smooth variety over $k$. Write

$$a_{B,m} = \text{Im}(H^0(\mathcal{X}_B, \mathcal{L}_B^m) \otimes_k \mathcal{L}_B^{-m} \to \mathcal{O}_{\mathcal{X}_B})$$

for $m$-th base ideal of $\mathcal{L}_B$ with $a_m = g^*a_{B,m}$ for all $m \in \mathbb{Z}_{>0}$.

$a_{B,\bullet} = (a_{B,m})_{m>0}$ defines graded sequence of ideals. Let $b_{B,m} := \tau(a_{B,m}^m)$ be asymptotic test ideal of exponent $m$ and define $b_m := g^*b_{B,m}$.

These ideals have the following properties:

(i) We have $a_m \subset b_m$ for all $m \in \mathbb{Z}_{>0}$.

(ii) We have $b_{ml} \subset b_m^l$ for all $l, m \in \mathbb{Z}_{>0}$.

(iii) There is $m_0 \geq 0$ such that $\mathcal{A}^\otimes m_0 \otimes \mathcal{L}^\otimes m \otimes b_m$ is globally generated for all $m > 0$.

Following BFJ, it is now a formal consequence of these properties that $\lim_m \varphi_m = P_\theta(0)$ converges uniformly.
Thank you for your attention!