Weyl-invariant gravity
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Abstract. A few implications of a classical Weyl-invariant scalar-tensor (WIST) version of general relativity (GR) are considered. This theory reduces to GR in a particular conformal frame in which the gravitational coupling and active gravitational masses are fixed. As an example, we recast the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) spacetime in the WIST form with static space and monotonically evolving Planck mass. Cosmological redshift is then explained by the time-dependent lapse function. The latter is proportional to the shrinking (in this particular WIST frame) Planck length squared. The conformal Hubble expansion rate is replaced in this description by the logarithmic (conformal) time derivative of the Planck mass.

Keywords. Gravitation, Weyl-invariance.

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1 Introduction

Our current best theory of the (classical) gravitational interaction is general relativity (GR). The latter has successfully passed numerous tests in our solar system. GR is one of the simplest of all metric theories; it obeys the metric condition, it is torsionless, it is described by a lagrangian linear in the curvature scalar, etc. It provides a diffeomorphism-invariant field theory that incorporates the principle of relativity applied to non-inertial observers.
and is realized on curved spacetimes. The curvature in the presence of a given energy-momentum distribution is governed by $G$, the universal coupling constant of gravitation. This specific coupling between matter and geometry was chosen so as to guarantee the consistency of GR with Newton law of gravitation in the weak field limit, assuming that the locally measured $G$ (within our solar system) is universal.

However, Newtonian gravity fails to adequately describe gravitational phenomena already on galactic scales, e.g. at the outskirts of galaxies and galaxy clusters. Consequently, GR suffers from this drawback as well when used in describing astrophysical systems where gravitational fields are small, e.g. in galaxies and clusters of galaxies). An straightforward remedy for this problem (and others in cosmology) involves introducing some form of non-relativistic (NR) invisible matter, i.e. ‘dark matter’ (DM). By far the leading candidate is some yet-to-be-determined beyond-the-standard-model (SM) particle. Another, less popular, possibility is that primordial black holes may account for the missing non-luminous matter.

As of yet, no DM particle candidate has been found in spite of decades of intensive search. In addition, the status of beyond-the-SM theories of particle physics – our best hope for new DM particles – is unclear with persistent null results coming from the large hadron collider (LHC), although certain anomalies have been reported very recently with mild indications for new physics. In any case, these are inconclusive as of yet, and even if corroborated considerable amount of effort is still required before the existence of DM particle that adequately accounts for DM phenomena on Hubble down to galactic scales is established.

Another tantalizing possibility is that particulate DM does not exist, and that on galactic scales or larger gravitation is not strictly described by GR, but rather by a modified version thereof. Indeed, the validity of GR has been only compellingly established within our own solar system. One possible extension of GR, invokes a higher symmetry – Weyl invariance (WI), where $G$ is determined by a scalar field. In GR this scalar field is replaced by a fixed $G$ throughout space and time. However, its local determination by Cavendish-type experiments does not necessarily imply its universality. In a WI version of GR, $G$ can vary in space and time, freely departing from its local value, and simultaneously inducing scalar metric perturbations. From this perspective, there is a priori no preferred conformal frame, and whereas the choice $G = \text{const}$ seems to work impressively well in our solar system it clearly fails on galactic scales and larger. It has been shown recently, that fractional variations of $G$ and active gravitational masses at the $O(10^{-3})$ or $O(10^{-4})$ level on galactic and galaxy cluster scales are sufficient for explain-
ing away DM phenomena without particulate DM at the reasonable ‘cost’ of endowing GR with WI [1]. Another possible application of WI version of GR is a possible resolution of the ‘Hubble tension’: the apparent statistically significant $\gtrsim 4\sigma$ tension between local and cosmological inference of the Hubble constant, $H_0$ [2].

The present work focuses on the foundations of the WI scalar-tensor (WIST) version of GR, hereafter referred to in abbreviation as ‘WIST’. Although WIST has been contemplated over the past fifty years, objections to this framework range between two extremes. On the one hand, it is often claimed that the WIST is ‘GR in a guise’, that extending the symmetry of GR to allow WI does not make it genuinely WI, in particular there is classically no preferred conformal frame, so WIST is equivalent to GR. On the other hand, it is sometimes implied that dressing GR with WI brings about a ghost scalar field that inflicts ‘disastrous instabilities’ on the theory, so much so to disqualify the theory (while the same is not similarly claimed about GR). It is also claimed that this ‘ghost problem’ is ‘typical’ to conformal theories of gravity (although, e.g. fourth order Weyl gravity and WIST are two completely distinct theories). We address these claims in the present work, and as a side we explicitly show that, not surprisingly, the action of the Friedmann-Robertson-Walker (FRW) spacetime, a legitimate solution of GR, can be presented in a WIST form. By doing so, we illustrate with a specific example, with well-known general-relativistic description of spacetime, that there are no ‘fatal instabilities’ associated with WIST that do not already exist in GR. Again, we stress that it is our assumption that WIST is entirely classical to the extent that GR is, and indeed the validity of GR has only been experimentally established on macroscopic down to $\mu$m, i.e. mesoscopic, scales, and not smaller.

The paper is organized as follows. The WIST is derived in sect. 2. The FRW example is discussed in sect. 3 followed by a summary in sect. 4. Throughout, we adopt a mostly-positive signature for the spacetime metric $(-1,1,1,1)$, with the speed of light $c$ and reduced Planck constant $\hbar$ set to unity.

## 2 Weyl invariant scalar-tensor theory

In this section we formulate the idea that GR is fundamentally endowed with WI, within a WIST theory. The latter reduces to GR in a particular conformal frame. However, we emphasize that only gravitation is endowed with WI in the proposed framework. All three other fundamental interac-
tions are described by the SM of particle physics, with no modifications. Since any realistic physical systems are governed by both gravitational and non-gravitational interactions, and since all other fundamental interactions are minimally-coupled to gravitation, it then follows that this particular framework is clearly not merely a field re-definition of GR and the SM of particle physics.

The WIST theory readily follows from locally rescaling the spacetime metric \( g_{\mu\nu} \rightarrow \phi\phi^*g_{\mu\nu} \) everywhere in the Einstein-Hilbert (EH) action (in units where \( G \equiv \frac{3}{8\pi} \))

\[
I_{EH} = \int \left( \frac{1}{6} (R - 2\Lambda) + L_m \right) \sqrt{-g}d^4x, \tag{1}
\]

where \( \phi \) & \( \phi^* \) are a scalar field and its complex conjugate, \( R \) & \( \Lambda \) are the curvature scalar and cosmological constant, respectively, \( L_m \) is the lagrangian density of matter, and \( g \) is the metric determinant. The cosmological constant, \( \Lambda \), can be absorbed in \( L_m \) as a species with vacuum-like properties as will be described below. The resulting WIST theory is 

\[
I_{WIST} = \int \left( \frac{1}{6} |\phi|^2 R - \phi^*\Box\phi + L_m(|\phi|, \{\Psi\}) \right) \sqrt{-g}d^4x
= \int \left( \frac{1}{6} |\phi|^2 R + \phi_{\mu}\phi^{*\mu} + L_m(|\phi|, \{\Psi\}) \right) \sqrt{-g}d^4x, \tag{2}
\]

where \( \phi_{\mu} \equiv \frac{\partial\phi}{\partial x^{\mu}} \), \( L_m \) is now allowed to explicitly depend on \( |\phi| \) but not on its derivatives, and the second equality follows from integration by parts of the kinetic term associated with the scalar field. With \( L_m \) being explicitly \( \phi \)-dependent, (2) is now a scalar-tensor theory of the Bergmann-Wagoner type [3], [4]. It was first obtained by Deser [5] and later by [6] with \( L_M \) not necessarily depending on \( \phi \). In [5] \( \phi \) was assumed to be real. All other fields are collectively denoted by \( \{\Psi\} \). A similar procedure to the replacement \( g_{\mu\nu} \rightarrow \phi\phi^*g_{\mu\nu} \) was employed by Chamseddine & Mukhanov in their Mimetic Gravity [7] but in the latter case the scalar field was an irrotational velocity potential \( \varphi \), i.e. \( g_{\mu\nu} \rightarrow \varphi_{,\mu}\varphi^{,\mu}g_{\mu\nu} \).

The kinetic term associated with the scalar field that appears in (2), \( L_\phi \equiv \phi_{,\mu}\phi^{*,\mu} \) or \( L_\phi \equiv -\phi^*\Box\phi \), can be considered as a new source of the gravitational interaction. This term is completely ignored in GR where \( \phi \) is set to a constant value, \( \phi_0 \equiv \sqrt{\frac{3}{8\pi G}} \). Invariance of (2) implies that \( L_m \rightarrow |\phi|^{-4}L_m \) under \( g_{\mu\nu} \rightarrow |\phi|^2g_{\mu\nu} \). This simple derivation also underscores the origin of the ‘ghost’ scalar field whose kinetic term, \( L_\phi \equiv \phi_{,\mu}\phi^{*,\mu} \),
appears with the ‘wrong’ sign in (2); it simply stems from locally stretching/squeezing the spacetime metric, or equivalently stretching/squeezing the yardstick with which distances are measured, in our case it is the Planck length $l_p \propto \sqrt{G}$ in vacuum, or $\propto (G\rho)^{-1/2}$ where $\rho$ is the energy density in non-vacuum configurations. This observation is crucial for evaluation of the significance of the ghost field. We see that $\phi$ is tightly related to $g_{\mu\nu}$ (merely a local stretch of the spacetime metric), and as $g_{\mu\nu}$ is treated classically so should be $\phi$; the ghostly nature of $\phi$ derives solely from the specific form in which $g_{\mu\nu}$ and its derivatives appear in the EH action. Below, we explicitly show that the classical field $\phi$ is not determined by any dynamical equation (which is not unexpected in a WI theory). In particular, it is not driven dynamically to unstable field configurations.

Since (2) was obtained from (1) by affecting the transformation $g_{\mu\nu} \to \phi\phi^*g_{\mu\nu}$, invariance of the former is guaranteed insofar $|\phi|^2g_{\mu\nu}$ is invariant, i.e.

$$
\phi \to \phi/\Omega \quad \text{g}_{\mu\nu} \to \Omega^2\text{g}_{\mu\nu},
$$

for an arbitrary $\Omega(x) > 0$. It is thanks to this freedom that $\phi = \phi_0/\Omega(x)$ is arbitrary, and so whereas it is spacetime-dependent it is not dynamical in the sense that it is not derived from a dynamical differential equation which has an attractor solution, that, e.g. drives the kinetic term, $L_\phi \equiv \phi_{,\mu}\phi^{*\mu}$, to arbitrary negative values, as is the case with generic ghost fields. Actually, we will see in sect. 3 that exactly the opposite takes place in our expanding universe when the action of the FRW spacetime is recast in the form of (2) instead of (1); the negative kinetic terms evolves from negative infinity at the big bang towards vanishingly negative values at the remote future assuming a monotonic expansion history, compatible with the concordance cosmological model.

The presence of the kinetic term in $L_\phi \equiv \phi_{,\mu}\phi^{*\mu}$ is expected once we promote $G$ to a field. More specifically, since the curvature scalar depends on derivatives of the metric field it transforms under (3) inhomogeneously

$$
R \to \Omega^{-2}\left(R - 6\Box\Omega/\Omega\right),
$$

where $\Box$ is the d’Alambertian. Invariance of (11) under (3) then requires the presence of the kinetic term $L_\phi \equiv \phi_{,\mu}\phi^{*\mu}$ that transform inhomogeneously and guarantees the mutual cancellation of derivatives of $\Omega$ (provided that the appropriate integration by parts has been carried out). The prefactor
in front of the curvature coupling term in (2) guarantees that the inhomogeneous term on the right hand side of (4) is compensated by a similar inhomogeneous term from the transformation of the kinetic term, $\phi^\mu \phi^\mu$. In other words, under the transformation (3) the combination of the first two terms in (2) transforms as, $\frac{1}{6}|\phi|^2 R - \phi \Box \phi^* \rightarrow \Omega^{-4} (\frac{1}{6}|\phi|^2 R - \phi \Box \phi^*)$, i.e. it has the appropriate well-defined conformal weight of $\text{length}^{-4}$, such that its product with the volume element in (2) is WI.

In the special case that $\phi$ is a real field and $L_M$ is independent of $\phi$ (2) can be cast in the form of a Brans-Dicke (BD) theory \[ I_{BD} = \int \left( \Phi R - \omega_{BD} \Phi_{\mu} \Phi_{\mu} + L_M \right) \sqrt{-g} d^4x \] with $\Phi$ & $\omega_{BD}$ being the BD scalar field and dimensionless BD parameter, respectively, where $\Phi \equiv \varphi^2/6$ & $\omega_{BD} \equiv -3/2$. Whereas tight lower limits, $\omega_{BD} > 40000$, have been reported in [9], it should be stressed that they were obtained from satellite observations directly exploring the impact of varying $G$ within our own solar system, and not beyond. In addition, in the (generally non-WI) BD framework the source term $L_M$ does not in general depend on $\phi$ while in WIST it does.

While it is true that the kinetic term appears in (2) with the ‘wrong’ sign relative to $L_M(|\phi|)$ and $\phi$ is formally a ‘ghost’ field it is merely a gauge artifact that can be seen when set to a constant in the unitary gauge [10]. The ‘fatal instability’ usually ascribed to quantum ghost fields stems from the fact that in their presence there is either no stable vacuum state, or non-conservation of probability, e.g. [11]. However, GR is classical, and so is WIST. As was already mentioned above, in the special case of WIST, it is exactly WI that prevents even classical instability from taking place because there is essentially no equation of motion for $\phi$ to dynamically drive it towards instability. This is not surprising since at least classically, two conformally-related theories are equivalent, e.g. [12], [13]. Actually, it has been argued that WI is a sham/fake symmetry of GR [14], [15], i.e. that (2) is equivalent to GR. On face value, the conclusion that the scalar field is a ‘spurion’ [14] implies that its ghostly nature is of no practical significance. We re-iterate that in the framework proposed here gravitation is WI, whereas the SM of particle physics is not.

Ultimately, the appropriate conformal frame to be used in the description of a physical system should be determined by observations, much like the appropriate coordinate system employed in the description of, e.g., a black hole in GR is determined by observations, e.g. [13]. For example, a spherically symmetric static black hole as seen from the perspective of a static observer would be most naturally described by a Schwarzschild metric, while a freely falling observer towards the same black hole will likely employ
a different, time-dependent, metric in describing the same black hole. The
diffeomorphism symmetry of GR allows observers to describe the system in
the most symmetric fashion, which clearly depends on the \((a \text{ priori} \text{ arbitrary})\) observers states. In the same fashion, at least at the classical level,
theory cannot pre-select the ‘appropriate’ units to be used in the description
of a physical system. This freedom is clearly not a property of GR, but is
certainly a desirable tenet of its WI generalization described by (2).

In the quantum regime the situation generally changes as quantization
and Weyl transformations do not in general commute, e.g. \cite{12}, \cite{13}, and in
fact ghosts are known to violate unitarity, e.g. \cite{16}. However, quantization
of GR has never been carried out, nor are there available measurements indicating that quantum gravitational effects are in play so as to necessitate
its quantization. Actually, the standard theoretical arguments in favor of
its quantization might be strong, yet inconclusive, e.g. \cite{17} and references
within, and at the very best GR could be considered an effective low en-
ergy limit of a more fundamental quantum theory of gravity, if it should
indeed be ultimately quantized. Therefore, in the same fashion that the
non-renormalizability of GR does not disqualify it from being the backbone
of the standard cosmological model what might be a ghost ‘problem’ of (2)
at the quantum level is not an issue for physical systems that are adequately
described by GR. Therefore, the inequivalence between conformal frames at
the quantum level need not concern us in the present work. It has been
claimed in \cite{16} that fixing the scalar field (and thereby rendering (2) equiva-

tent to GR) in order to go around the ‘ghost problem’ in practice ‘robs’ (2)
from its claimed WI. Again, while this argument might be valid at the quan-
tum level it definitely does not apply classically. Moreover, and as discussed
below, the set of fields \(\phi \& g_{\mu\nu}\) that appears in (2) is under-determined and
\(\phi\) could be chosen to be any arbitrary function, not necessarily a fixed value,
thereby still ‘manifesting’ WI, while at the same time avoiding driving the
field configuration away from its equilibrium. This will be illustrated with
a concrete example in the next section. In summary, the ‘wrong’ sign of the
scalar field in a WI theory is not an issue at the classical level, which is the
only level at which (2), exactly as GR, can be trusted.

It is constructive at this point to explore a slightly more general case
of a dynamical scalar field which is non-minimally coupled to spacetime curvature

\[
I_{ST} = \int \left( \xi |\phi|^2 R + \phi^*_\mu \phi^\mu + \mathcal{L}_M(|\phi|) \right) \sqrt{-g} d^4 x,
\]

where \(\xi\) is an arbitrary dimensionless coupling parameter, thereby general-
The field equations derived from variation of (5) with respect to the metric and scalar field are, respectively

\[ 2\xi|\phi|^2 G_{\mu\nu} = T_{M,\mu\nu} + \Theta_{\mu\nu}, \]  
(6)

and

\[ \xi \phi R - \Box \phi + \frac{\partial L_M}{\partial \phi} = 0, \]  
(7)

where

\[ 3\Theta_{\mu\nu} \equiv 6\xi \phi^* \phi,_{\mu\nu} + (6\xi - 3)\phi^* \phi_{\mu\nu} - 6\xi g_{\mu\nu} \left[ \phi^* \Box \phi - \left( \frac{1}{4\xi} - 1 \right) \phi^*_\alpha \phi^\alpha \right] + c.c. \]  
(8)

Here and throughout \( T_{M,\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \delta(\sqrt{-g}L_M) \) is the energy-momentum tensor. Eq (6) is a generalization of Einstein equations with \( 2\xi|\phi|^2 \) replacing \( 1/(16\pi G) \), and \( \Theta_{\mu\nu} \) is an effective contribution to the energy-momentum tensor essentially due to gradients of \( G \) and active gravitational masses, the latter appear in the source term, \( L_M \). Multiplying (7) by \( \phi \), adding the result to its complex conjugate and to the trace of (6) results in

\[ \phi \frac{\partial L_M}{\partial \phi} + \phi^* \frac{\partial L_M}{\partial \phi^*} = T_M + (1 - 6\xi)\left( \phi^* \Box \phi + \phi^*_\alpha \phi^\alpha + c.c. \right). \]  
(9)

Together, (6) & (9) provide a system of dynamical equations for the ten metric components and the complex scalar field. In the BD form (\( \phi \) is real and \( L_M \) being independent of \( \phi \)) (9) reduces to the well-known form \((3 + 2\omega_{BD})\Box \Phi = T_M \). It follows from (9) that for any \( \xi \neq \frac{1}{6} \) (equivalently \( \omega_{BD} \neq -3/2 \)) the scalar field is dynamically determined by the matter distribution via a differential equation, thereby embodies the Machianity of BD theory of gravity.

In contrast, in the special case \( \xi = \frac{1}{6} \) (equivalently \( \omega_{BD} = -3/2 \) \([5],[18]–[22]\)) the theory becomes (2), and (9) reduces to a constraint that only determines the dependence of the matter lagrangian on the scalar field. In this special case, \( \phi \) is indeed non-dynamical; it can be set to any desired function, irrespective of the matter distribution. Although \( \phi \) still appears in (6) we now have only ten differential equations for ten metric components and a scalar field – this under-determined system simply reflects the freedom to locally rescale the fields in the special case \( \xi = 1/6 \), merely a manifestation
of WI, [4], [5], [18]–[25]. The latter implies invariance of (2) under local-rescaling of fields such that $L_m \rightarrow |\phi|^{-4}L_m$. This implies in particular that, e.g.,

$$\psi \rightarrow \Omega^{-\frac{3}{2}}\psi$$

$$A_\mu \rightarrow A_\mu,$$

where $\psi & A_\mu$ are the Dirac and vector fields, respectively, in addition to the transformation of the scalar and metric fields described in (3) – each field is rescaled by its mass/length dimension. The trivial transformation of the vector field $A_\mu \rightarrow A_\mu$, e.g. [26], is for example required by the transformation rule $D_\mu \rightarrow D_\mu$ of the gauge-invariant derivative $D_\mu = \partial_\mu - igA_\mu$, where $g$ is a dimensionless gauge-coupling constant. From (3) it then follows that $A \rightarrow \Omega^{-1}A$ as would be expected for a field of mass dimension 1, where $A \equiv \sqrt{A_\mu A^\mu}$.

Any mass terms that appear in $L_M$ are by definition active gravitational masses, which need not be equivalent to inertial or passive gravitational masses. While these three types of mass are not necessarily equivalent [27], the notion of a passive gravitational mass in any theory of gravity that satisfies the equivalence principle is vacuous. If the ratio of the later two mass types is a universal constant then the equivalence principle is satisfied – an assumption that we indeed make in the present work. The other three fundamental interactions are described by the lagrangian of the SM of particle physics, $L_{SM}$, where all masses are inertial, whether generated via the Higgs mechanism in the electroweak sector or via the explicitly broken chiral symmetry in QCD.

In the special case of interest to the present work, $\xi = 1/6$, (9) only implies that for a given equation of state (EOS), $w$, that the energy density scales $\rho_M \propto |\phi|^{1-3w}$, as will be discussed below. For example, the lagrangian densities that describe NR matter and cosmological constant (CC) are $L_{NR} \propto |\phi|$ and $L_{CC} \propto |\phi|^4$, respectively. The lagrangian density of radiation is independent of $\phi$. Since in this case $\phi(x) = \Omega^{-1}(x)\phi_0$, where $\phi_0$ is the fixed GR value, then for any given such a choice $\Theta_{\mu\nu}$ can be calculated according to (8) and used in (6) to solve for the corresponding $g_{\mu\nu}$. In practice, however, it is easier to start from any known solution of the Einstein equations, $\phi_0 & g_{\mu\nu}$, and transform it to $\Omega^{-1}(x)\phi_0 & \Omega^2(x)g_{\mu\nu}$ with any arbitrary $\Omega(x)$ by virtue of WI.

In the theory described by (2), the scalar field $\phi$ determines not only $G$ but also active gravitational masses. Only pure radiation $T_{rad} = 0$ is
consistent with the theory described by (5) with $\xi = \frac{1}{6}$ unless $L_M$ explicitly depends on $|\phi|$. In the case of perfect fluid $T_M = L_M$ on shell [28], and with EOS $w_M$ the trace is $T_M = -\rho_M(1 - 3w_M)$. It then follows from (5) that $L_M \propto |\phi|^{1-3w_M}$ in case that $\xi = 1/6$, i.e. it is linear and quartic in $|\phi|$ in cases of NR and vacuum-like matter respectively, and is independent of $|\phi|$ in case of pure radiation. Linearity of $L_M$ in $|\phi|$ in the case vanishing EOS, $w_M = 0$, suggests that active gravitational masses are regulated by $|\phi|$. Not only that the same $\phi$ determining both Planck mass and active gravitational masses is necessary for the consistency of non-radiation sources with this WI theory, it is also a conceptually natural “conclusion” as the concept of active gravitational mass is meaningless unless it couples to $G$, and in this sense it seems natural that both quantities are determined by the same scalar field. This clearly does not have necessarily to be the case in general (as in e.g. BD theory), but it is a nice merit of the WI model described by (2).

We emphasize that in the original BD proposal [8] the matter lagrangian, $L_M$, does not explicitly depend on the scalar field. Notably, Brans & Dicke required $\omega_{BD} > 0$ to guarantee the positivity of the Hamiltonian in their original proposal [8], a fact that was emphasized and reinterpreted in [29]. The instability of BD theories with $\omega_{BD} < 0$ was further emphasized in [30], but as we argued above, the non-positive kinetic term of the scalar field is not an issue in the special case $\omega_{BD} = -3/2$ and $L_M = L_M(|\phi|)$ due to WI. The observational lower limit $\omega_{BD} \geq 40000$ reported in [9] implies that the BD field $\Phi$ is very nearly a constant, essentially reducing the theory to GR. But, again, this conclusion applies to the BD theory, where $L_M$ is independent of the scalar field, and (perhaps not less important and relevant to astrophysical and cosmology) this tight limit has been obtained in our solar system and clearly does not apply on, e.g., galactic scales.

3 Example: The case of FRW spacetime

In the present section we cast the FRW action in the form (2) in a particular conformal frame where spacetime is static. The conventional initial curvature singularity at the big bang is replaced in this alternative description by the vanishing of $\phi$, i.e. of Planck and active gravitational masses. The redshifting universe is then a reflection not of space expansion, but rather of temporal evolution of $G$, i.e. the redshifting universe is a manifestation of shrinking Planck length. The FRW action

$$I_{FRW} = \frac{(3/\kappa)}{\int} \left[-a'^2 + Ka^2 - \Lambda a^4/3 + a^4\kappa L_M(a)/3\right] \sqrt{\gamma} d^4x,\ (11)$$
is obtained from (1) assuming the metric \( g_{\mu\nu} = a^2 \gamma_{\mu\nu} \), where \( \gamma_{\mu\nu} \equiv \text{diag}(-1, \frac{1}{-K r^2}, r^2, r^2 \sin^2 \theta) \), \( \gamma \equiv \text{det}(\gamma_{\mu\nu}) \), and after the term proportional to the corresponding curvature scalar \( a^4 R = 6a^4(\frac{a''}{a} + K)/a^2 \) is integrated by parts. Here, \( K \) is the spatial curvature, \( a(\eta) \) is the purely time-dependent scale factor, a prime denotes derivatives with respect to conformal time \( \eta \), and \( d^4 x = d\eta dr d\theta d\phi \).

The two metrics \( g_{\mu\nu} \& \gamma_{\mu\nu} \) are conformally related. Since null geodesics are blind to conformal metric transformations we expect light to still be redshifted in the WIST description with the spatially-static metric \( \gamma_{\mu\nu} \) now replacing the standard expanding space metric \( g_{\mu\nu} \). This is explained by the fact that \( -d\eta^2 = \frac{dt^2}{a^2} \), and so the the lapse function \( \gamma_{00} \) is \( a^{-2} \) if the metric \( \gamma_{\mu\nu} \) is presented in cosmic time, \( t \), coordinates, rather than conformal \( \eta \). Incoming photon wavelengths are not stretched by space expansion in this description (space is static) but rather by the temporarily-evolving gravitational potential, \( \gamma_{00}(t) \), which is \( \propto \phi^{-2}(\eta) \). In other words, cosmological redshift is due to the growing (decreasing) planck mass (length).

It can be readily verified that the Euler-Lagrange equation for the scale factor \( a(\eta) \) that extremizes the action \( I_{\text{FRW}} \) is indeed the Friedmann equation \( H^2 + K = \frac{8\pi G}{3} a^2 \rho_M + \frac{\Delta a^2}{a^2} + \text{const.} / a^2 \), where \( H \equiv a' / a \) is the conformal Hubble function, and it is assumed that \( \mathcal{L}_M(a) \) is a power-law in \( a \), as \( \rho_M \) is a power-law in standard cosmology. Here, \( \mathcal{L}_M(a) \) accounts for all sources (e.g. dust, radiation, etc.) other than spatial curvature and cosmological constant (which are characterized by effective equations of state \(-1/3 \& -1\), respectively). Completing the derivation requires that the integration constant is related to the energy density of radiation \( \rho_r = \text{const.} / a^4 \). Since \( \rho_M > 0 \) it is clear that the kinetic and potential terms in (11) have the ‘wrong’ relative sign (unless \( K > 0 \) or \( \Lambda < 0 \)), yet the GR-based FRW model is not regarded as ‘plagued with ghosts’, or ‘disastrous instabilities’ at the classical level. On the contrary, FRW is the backbone of the remarkably successful standard cosmological model.

Re-defining the scale factor \( \phi^2 \equiv \frac{3a^2}{8\pi G} \), the FRW action is reformulated as a WIST theory with a non-positive kinetic term, defined on a static background

\[
I_{\text{FRW}} = \int \left( \frac{1}{6} R \phi^2 - \phi'^2 - \lambda \phi^4 + \tilde{\mathcal{L}}_M(\phi) \right) \sqrt{-\gamma} d^4 x.
\]  

(12)

Here, \( \tilde{\mathcal{L}}_M = a^4 \mathcal{L}_M, R = 6K \) and \( \lambda \equiv \kappa \Lambda / 9 \). The single degree of freedom of standard FRW spacetime, \( a(\eta) \), is here replaced by the scalar field \( \phi(\eta) \). Clearly, renaming the scale factor, \( a^2 \to \frac{8\pi G \phi^2}{3} \), in going from the gen-
eral relativistic (11) to the WIST form, (12), does not introduce any new dynamics, in particular no unacceptable new instabilities that are not already present in the FRW spacetime. The relative ‘wrong’ sign between the kinetic and potential term is simply manifested in that the integration of the Friedmann equation results in $a(\eta)$ (or equivalently $\phi(\eta)$) that evolves monotonically with $\eta$ rather than having oscillatory behavior.

As (11) & (12) are equivalent any perturbation is either $a(\eta)$ of (11) or $\phi(\eta)$ of (12) will have exactly the same dynamics. Exactly as in the standard cosmological model $\delta a$ is never considered as a dynamical degree of freedom, so should be the case with $\delta \phi$. This is so since $g_{\mu\nu} = a^2 \gamma_{\mu\nu}$, and therefore any perturbation in $a$ can be absorbed in $\delta \gamma_{\mu\nu}$, a perturbation of $\gamma_{\mu\nu}$. Considering scalar metric perturbations only – these are the Newtonian potentials. Therefore, invoking WI implies that any perturbation in $G$ induces scalar metric perturbations, and vice versa any excessive gravitational potential could be explained by corresponding perturbations in $G$.

4 Summary

Symmetry plays a central role in modern understanding of the fundamental interactions. The SM of particle physics is based on a local $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ gauge symmetry, and GR is diffeomorphism-invariant. In the present work, the idea that GR accommodates a broader symmetry, WI, is explored. The underlying idea is that not only the gravitational interaction (and by induction also all the other interactions which take place in the spacetime arena) does not depend of the observer reference frame (as manifested by diffeomorphism invariance of, e.g., the Einstein and geodesic equations) but it also does not depend on (a priori arbitrary) choices of our yardsticks, e.g. the Planck length scale, $l_P$. The latter is the natural yardstick in vacuum. In the non-vacuum case the natural meter stick would be the dynamical gravitational time scale, $(G \rho_M)^{-1/2} \propto l_P^{-1} \rho_M^{-1/2}$ where $G$ is Newton constant and $\rho_M$ is the energy density. The standard units choice adopted in GR is that both $G$ and particle masses (active gravitational masses included) are fixed constants. Newton constant, $G$, has been measured in our solar system at a reasonable precision, and is purportedly also deduced outside of our solar system from big bang nucleosynthesis (BBN), etc. However, all these inferences of $G$ are based on the premise that all other universal constants are truly constants, as is clearly manifested in the Brans-Dicke program. In addition, Newtonian gravity fails in accounting for a few observed phenomena on galactic scales and beyond in spite of the
fact that the gravitational interaction is weak in these systems and Newtonian gravity should naively provide a reasonably accurate description. The standard solution to this problem has involved positing the existence of a non-luminous DM substance, presumably made of exotic electrically neutral particles. Another way, albeit less popular, to tackle the problem has involved modifying the Newtonian dynamics on sufficiently large scales. Both remedies assume that $G$ is a fixed universal constant across space and time; in the absence of compelling direct evidence for the constancy of $G$ this has to be assumed. However, positing WI as an additional symmetry of (what is usually considered GR) opens up new possibilities. Promoting $G$ to a scalar field necessitates that active gravitational masses are regulated by the same field as well. Clearly, it is assumed here that this symmetry applies only within the domain of validity of GR. This is of course not the case on microscopic scales. While GR is assumed here to be WI, the SM of particle physics is assumed to be as is, with no modifications. The effective energy and pressure contained in spatio-temporal variations of the scalar field, i.e. $G$ and active gravitational masses, then provide additional (non-particulate) source for curving spacetime, thereby accounting for ‘DM phenomena’ without actually recusing to DM. In this paper we explored a few key aspects of the underlying Weyl-invariant scalar-tensor generalization of GR.

Whereas GR is obtained from WIST is a given conformal frame (in which $G$ is a constant), the additional functional degree of freedom (obtained by promoting $m_p$ to a scalar field $\phi$) provides an ample freedom that allows doing away with clustered DM as mentioned above. A notable property of $\phi$ is that it is formally a ‘ghost’ field, i.e. its kinetic and potential terms have the ‘wrong’ relative sign. However, this property by itself does not disqualify the theory because the problem with ghost-afflicted theory is that the field configuration dynamically runaway from its ground state, or equilibrium, by ever lowering the kinetic energy while increasing the potential term. However, as was explicitly shown in sect. 2, and as is equally clear from the underlying WI, the spacetime-dependent $\phi$ is non-dynamical; it is simply not required to obey any specific Euler-Lagrange equation. Consequently, physical systems described by WIST are not doomed to runaway from stable field configurations. We also illustrated this conclusion with the redshifting universe. The FRW spacetime can be recast in the canonical WIST form on a static background. Cosmological redshift is then explained not by space expansion but rather by contraction of our fundamental yardsticks which are regulated by $\phi$. Whereas $\phi$ can be locally rescaled along with a corresponding rescaling of $g_{\mu\nu}$, adopting a specific conformal frame in which $\gamma_{\mu\nu}$ is static implies that $\phi$ satisfies exactly the same Friedmann
equation which is normally satisfied by $a$. The conformal Hubble function $\mathcal{H} = a'/a$ is now replaced by $\phi'/\phi$. In the WIST version of the FRW action the negative kinetic term has been actually increasing ($\phi'/\phi$ monotonically decreases) asymptotically towards zero over the entire cosmic history between the ‘big bang’ and the present time. This is a counter-example to often-raised concerns about instabilities associated with WIST. It is true that on the face of it we could have found ourselves in a contracting universe with an ever decreasing negative kinetic term of either $a$ or $\phi$ (in the GR and WIST formulations respectively) culminating in a ‘big crunch’, but even in this case WIST dynamics is not worse than GR.

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