CSS Equation, etc, Follow from Structure of TMD Factorization

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I show that the forms of the Collins-Soper-Sterman and renormalization-group equations for the evolution of transverse-momentum-dependent (TMD) parton densities in QCD follow from the structure of TMD factorization. A derivation does not need to directly use detailed properties of the operator definition of the TMD parton densities.

I. INTRODUCTION

An accurate description of the Drell-Yan process and other similar processes requires an understanding in QCD of the \( Q \) dependence of the transverse-momentum dependence when \( q_T \ll Q \). (Here, \( q_T \) and \( Q \) are the transverse momentum and invariant mass of the Drell-Yan pair.) An often-used way of treating this issue was found by Collins and Soper (CS) \([1, 2]\), and Collins, Soper and Sterman (CSS) \([3]\), who used factorization with transverse-momentum-dependent parton density functions (TMD pdfs). In contrast to ordinary “integrated pdfs”, the TMD pdfs of CS depend on an extra argument \( \zeta \). The phenomenological significance of \( \zeta \) can seen in the basic factorization formula, Eq. (12) below, where the two instances of \( \zeta \) obey \( \zeta A \zeta B = Q^4 \). CS obtained an equation for the \( \zeta \) dependence. This equation implies a strong \( Q \)-dependence of the shape of the measured \( q_T \) distribution, which is in agreement with data for the Drell-Yan and other processes — see, for example, Refs. \([4, 5]\).

However, there is a bewildering variety of treatments of TMD factorization in the literature, with what for the outsider appear to be contradictory claims about the need for and the use of the CS equation and other aspects of the CSS approach. This can be seen in work within an SCET framework by Becher and Neubert (BN) \([6]\) and by Echevarria, Idilbi, and Scimemi (EIS) \([7]\). Notably, the last authors state that their approach is “without the Collins-Soper like evolution equations”. In reality, however, the last authors do use an exact equivalent of the CS equation, but label it differently. To see this, one can merely compare Eq. (5.7) of Ref. \([6]\) with the results in Sec. 13.13.1 of \([7]\), which uses a modernized and improved version of the CSS method. One difference in the EIS treatment is the derivation of the results, which are shown to follow from more structural properties of TMD factorization, but within the particular approach of EIS. The strong \( Q \)-dependence of the transverse-momentum distribution is a property of QCD that cannot be avoided. The most that can be done is to change the method of derivation and the form of presentation of the results. (Another set of discrepancies between the treatments concerns the large-\( b_T \) region, where \( b_T \) is a variable Fourier-conjugate to transverse momentum. In this region, non-perturbative information is needed beyond that in the integrated pdfs. A discussion of the disagreements on this topic is not the concern of this paper.)

In this paper, motivated by a derivation by EIS, I show that the form of the CS equation and associated renormalization group (RG) properties follow from structural properties of TMD factorization (in its recent formulations). In contrast, the derivations by CS \([1, 2]\) and by Collins \([10]\) used specific properties of the operators defining the TMD pdfs to obtain their dependence on the \( \zeta \) parameter, and therefore appear to tie the CS equations to these properties of the operators. Thus these derivations are inapplicable, for example, to the rather different definition given by EIS \([6]\), even though the definitions give results numerically equivalent \([11]\) to those from the definitions in \([10]\). Of course, the validity of factorization strongly constrains which operator definitions of pdfs are suitable.

The results in the present paper are important step in unifying the various treatments of TMD factorization, as regards their phenomenological implications, following on from \([11]\). Note that the derivations by BN \([6]\) and by EIS \([7]\) are explicitly restricted to \( \Lambda_{QCD} \ll q_T \ll Q \), i.e., they exclude the region of the smallest \( q_T \) where the cross section is largest. In contrast the derivation of the form of the CS equation in this paper applies generally, and the TMD factorization property used is valid over the whole range of small transverse momentum \( q_T \ll Q \). Note also that BN \([6]\) did not give definitions of finite TMD pdfs within their approach: They obtained finite results only for the product of two TMD functions.

Among other results that are associated with the CS equation and its derivation are that the anomalous dimension of the TMD pdf has linear dependence on \( \ln(\zeta/\mu^2) \) (or equivalently \( \ln(Q^2/\mu^2) \)), where \( \mu \) is the RG scale. This logarithmic dependence is controlled by the \( \zeta \)-independent RG coefficient \( \gamma_K \) for the kernel of the CS equation. Another result is that the kernel of the CS equation does not depend on \( \zeta \) (or \( Q \)) or on the flavor of the partons and beam hadrons. Both of these results also follow from the derivation in this paper, which does not in any way invalidate the derivations in \([1, 2, 10]\).

\footnote{As explained, for example, in the CS and CSS work, TMD factorization can be matched with ordinary collinear factorization at large \( q_T \) to give results valid for all \( q_T \).}
II. WHAT THE CS AND COLLINS DERIVATIONS DID

CS [1, 2] defined TMD pdfs by matrix elements of appropriate non-gauge-invariant operators in a non-light-like axial gauge with gauge-fixing vector \( n \). This vector can be thought of as providing a standard for classifying momenta into left-movers and right-movers. They found it convenient to use the variable \( \zeta = 4(x P \cdot n)^2/|n|^2 \) for parameterizing the dependence on the vector \( n \), where \( x \) is the normal longitudinal momentum fraction in a pdf and \( P \) is the momentum of the target hadron.

Dependence on \( \zeta \) is equivalent to dependence on the gauge-fixing vector, and the CS equation was derived by using Ward identities associated with gauge-dependence to get a particularly simple form. The actual form of the CS equation was not known ahead of time, and the derivation amounted to a calculation of the \( \zeta \) dependence of a TMD pdf to leading power at large \( \zeta \). The form of the CS equation, and the various auxiliary properties, were outputs of the derivation.

Much more recently Collins [10] worked out an improved gauge-invariant definition of TMD pdfs, again with an auxiliary vector \( n \) which is now the direction of a certain Wilson line. With this definition, the CS equation arises from factorization properties of Wilson line operators when there are two Wilson lines of very different rapidities. An important quantity is the anomalous dimension \( \gamma_K \), for the kernel of the CS equation; it corresponds to a similar quantity in the work of Korchemsky and Radyushkin [12] for the dependence of the anomalous dimension of a Wilson line cusp on the hyperbolic angle of the cusp.

III. CORRESPONDING DERIVATION FOR RG EQUATION FOR OPERATORS

Before examining how TMD factorization implies the CS equation, I illustrate the principles of the method by constructing an analogous argument for a more well-known and simpler case, which is the RG equation for operators that appear in the short-distance operator-product expansion (OPE). A simple generalization of the argument allows one to obtain the DGLAP equation for integrated pdfs from an ordinary collinear factorization formula.

The material in this section is not original. I remember learning the method from someone else many years ago, but I don’t remember who, or if and where it has been published.

Let us examine the OPE for an object of the following form:

\[
F(Q, P, H, m(\mu), \mu, \alpha_s(\mu)) = \int d^4 x e^{i q \cdot x} \langle P, H | T j(x) j(0) | P, H \rangle,
\]

which is the Fourier transform of a current-current matrix element in a target state of momentum \( P \) and flavor \( H \). I restrict the situation to be simple enough that only a single operator appears in the OPE at leading power in \( Q = |q| \), and, as usual, choose the currents to be RG invariant. Also as usual, the renormalization scale is \( \mu \), the renormalized coupling of the theory is \( \alpha_s(\mu) \), and the renormalized mass parameter(s) of the theory are denoted by \( m(\mu) \).

To formulate the OPE, we take an essentially Euclidean limit where \( q \) is made large. Then, up to power-suppressed corrections:

\[
F = C(Q, \mu, \alpha_s) A(P, H, m, \mu, \alpha_s),
\]

where \( C \) is a short-distance coefficient function and \( A \) is the expectation value of some renormalized operator \( O \) in the state \( |P, H\rangle \): \( A = \langle P, H | O | P, H \rangle \). The operator is a product of fields (and possibly derivatives) at some space-time point, and of a renormalization factor.

Important properties of this statement of the OPE are that: (a) The coefficient function depends on \( Q \) and \( \mu \), but not on the masses or on \( P \) or on hadron flavor \( H \). (b) The operator matrix element is independent of \( Q \). Furthermore, if the currents defining \( F \) are changed such that the same operator \( O \) is needed in the OPE, then the value of \( C \) can depend on which currents are used, but the matrix element \( A \) is independent of the choice of currents. Thus the only arguments in common between the two factors are \( \mu \) and \( \alpha_s \). These properties are a consequence of deriving the OPE by a strict expansion in powers of \( Q \) and keeping only the leading power, and of using a mass-independent renormalization scheme like \( \overline{\text{MS}} \).

A. Standard derivation from renormalization properties

The standard derivation of the RG equation for the operator and its matrix element \( A \) uses the fact that the operator is multiplicatively renormalized, so that the bare operator and the renormalized operator are related by \( O_0 = Z_A O \). The renormalization factor \( Z_A \) is a function of \( \alpha_s \) and the dimensionless regulator parameter \( \epsilon \). Then the renormalization of the matrix element similarly obeys

\[
A_0 = Z_A A.
\]

Since bare operators are RG invariant, the RG equation for the renormalized matrix element is obtained by

\[
\frac{dA}{d \ln \mu} = d \ln \mu A = -\gamma_A(\alpha_s) A,
\]

where

\[
\gamma_A(\alpha_s) = \frac{d \ln Z_A}{d \ln \mu}
\]
Here a total derivative is used, as usual, to indicate that the differentiation with respect to \( \mu \) includes the running of the coupling and of the quark masses:

\[
\frac{d}{d \ln \mu} = \frac{\partial}{\partial \ln \mu} + \frac{d \alpha_s(\mu)}{d \ln \mu} \frac{\partial}{\partial \alpha_s} + \sum_i \frac{d m_i(\mu)}{d \ln \mu} \frac{\partial}{\partial m_i}. \quad (6)
\]

The anomalous dimension \( \gamma_A \) is a function of the renormalized coupling alone.

Since the current-current matrix element \( F \) is RG invariant, so is the leading term \( CA \) in its expansion in powers of \( Q \). Hence the RG equation of the coefficient \( C \) has the opposite anomalous dimension to \( A \):

\[
\frac{d C}{d \ln \mu} = \gamma_A(\alpha_s) C. \quad (7)
\]

For applications, the important result is that the anomalous dimension depends only on \( \alpha_s \). Thus the condition that a perturbative calculation of \( \gamma_A \) is accurate is simply that \( \alpha_s \) is small enough; there are no large logarithms of kinematic variables to worsen the perturbation expansion, unlike the coefficient function \( C \) with its logarithms of \( Q/\mu \). We apply the RG equation to express the product \( CA \) in terms of \( C \), with \( \mu \) set equal to \( Q \), and of \( A \) with a fixed reference value \( \mu_0 \) of the renormalization scale:

\[
CA = C(Q, Q, \alpha_s(Q)) A(P, H, m, \mu_0, \alpha_s) \times
\times \exp \left\{ - \int_{\mu_0}^Q \frac{d \mu}{\mu} \gamma_A(\alpha_s(\mu)) \right\}. \quad (8)
\]

On the right-hand-side, \( C \) has no large logarithms, so that it may be calculated perturbatively if \( \alpha_s(Q) \) is small, i.e., if \( Q \) is large enough, because of the asymptotic freedom of QCD. Since \( A \) is at a fixed renormalization scale, the formula illustrates the universality properties of non-perturbative part embodied in the operator matrix element.

All the above is (or should be) well-known and can be found in textbooks.

**B. Derivation from OPE**

Instead of the above derivation, let us simply define the anomalous dimensions of \( C \) and \( A \) by derivatives of the logarithms of \( C \) and \( A \) with respect to \( \ln \mu \):

\[
\gamma_C(Q, \mu, \alpha_s) = - \frac{d \ln C}{d \ln \mu}, \quad (9)
\]

\[
\gamma_A(P, H, m, \mu, \alpha_s) = - \frac{d \ln A}{d \ln \mu}. \quad (10)
\]

Since we do not use the connection to the renormalization factor \( Z_A \), we are no longer guaranteed that the anomalous dimensions are independent of \( Q, P, H, m, \) and \( \mu \) : Whatever arguments are needed for \( C \) and \( A \) are also needed for \( \gamma_C \) and \( \gamma_A \) respectively. If this dependence actually existed, then the solution \( \gamma_C \) of the RG equation would not be very useful, since large logarithms could appear in the calculation of \( \gamma_A \) in the exponent in the solution. Moreover, the RG evolution could depend on hadron flavor.

But the product \( CA \) is RG invariant, from which it follows that the sum of the anomalous dimensions is zero:

\[
\gamma_C(Q, \mu, \alpha_s) + \gamma_A(P, H, m, \mu, \alpha_s) = 0. \quad (11)
\]

Since \( \gamma_A \) is independent of \( Q \), differentiation with respect to \( Q \) shows that \( \partial \gamma_C/\partial Q = 0 \), i.e., that \( \gamma_C \) is independent of \( Q \). Similarly \( \gamma_A \) is independent of \( P, H, \) and \( m \), because \( \gamma_C \) is. Dimensional analysis then shows that neither anomalous dimension depends on the explicit argument \( \mu \).

In this derivation, the statement of the RG equation for \( A \) is a triviality: it is merely a definition. The non-trivial result is that \( \gamma_A \) is a function only of \( \alpha_s \). From the point of view of applications, it is this last property that is the critical one, since it allows a perturbative calculation of \( \gamma_A(\alpha_s(\mu)) \) in Eq. (8) provided only that \( \alpha_S(\mu) \) is large enough, i.e., that \( Q \) and \( \mu_0 \) are in what is often called the perturbative region of mass scales for QCD.

In the first derivation, this same property followed from the multiplicative renormalizability of the operator \( \mathcal{O} \) and from properties of \( \overline{\text{MS}} \) renormalization. But in the second derivation, renormalization properties are not used. The contrast between these two facts becomes less strange when one observes that (a) the derivation of the OPE more-or-less determines which operator is used, and (b) the requirement that the coefficient function correspond to behavior that is purely leading-power in \( Q \) requires the use of a mass-independent renormalization scheme. Furthermore the proof of multiplicative renormalizability of the operator has a lot in common with the proof of the OPE.

**IV. STATEMENT OF TMD FACTORIZATION, ETC**

In this section, I will give all the statements of TMD factorization and of the associated evolution equations together with the other results that are needed for full phenomenological application. They will be presented in the generalized CSS form that was derived (to all orders) in [10]. The forms given by BN [8] and by EIS [9] are similar (except that BN do not define individually finite TMD pdfs). The justifications given by BN and EIS are, of course, different, but that does not affect the structural properties stated here.

**A. TMD factorization in CS notation**

The Drell-Yan process is production of a high-mass lepton pair via a virtual electroweak boson of momentum
The hard scattering depends only on $Q$, $\mu$ and the strong coupling $\alpha_s(\mu)$, and on the flavors of the annihilating partons. If $\mu$ is chosen to be of order $Q$, then perturbative calculations are appropriate for it, with the lowest order just being the parton model calculation for whatever version of the Drell-Yan process is being used. In contrast, the TMD pdfs contain all the contributions to the cross section (at leading power in $Q$) from the non-perturbative domain of QCD. They depend, therefore, on quark masses and the flavors of the partons and hadrons.

B. Evolution equations, and small $b_T$ expansion

The TMD pdfs in Eq. (12) depend on two auxiliary parameters $\zeta$ and $\mu$. To obtain useful predictions for experiments at different energies, equations are needed to relate the TMD functions at different values of the auxiliary parameters. In addition there is an equation that is a kind of generalized operator product expansion (OPE) that expresses the TMD pdfs in terms of the ordinary integrated pdfs when $b_T$ is small enough to be considered to be in the region where perturbative calculations are appropriate in QCD.

The CS equation is for the evolution with respect to $\zeta$ of a TMD pdf for a parton of flavor $i$ in a hadron $A$. In the form appropriate to the definitions in Ref. [10] it is

$$\frac{\partial \ln \tilde{f}_{i/A}(x_A, b_T, \zeta; \mu)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu),$$

with an exactly similar equation for the TMD in hadron $B$, of course. The quantity $\tilde{K}$ is a new entity, which I call the kernel of the CS equation. In Ref. [10], it was constructed in terms of a vacuum matrix element of a certain Wilson-loop operator. It is independent of $x$ and $\zeta$, and of the flavor of the parton and the hadron in the pdf. But it does depend on the color representation for the parton, i.e., there is one $\tilde{K}$ for quarks and another for the gluon.

(It is worth observing that the TMD pdfs and $\tilde{K}$ depend also on the coupling $\alpha_s(\mu)$ and the quark masses, as well as on the parameters explicitly indicated.)

In the CS style of derivation, the CS equation is obtained by differentiating with respect to the auxiliary rapidity parameter $y_n$, because this can be exploited to make a relatively simple derivation, using factorization properties and Ward identities. This appears to tie the CS equation to the use of a particular axial gauge, in Ref. [1, 2], or to the use of certain non-light-like Wilson lines, in Ref. [10]. But a physical interpretation is more easily made if one considers holding $y_n$ fixed, e.g., at the rapidity of the Drell-Yan pair. Then the derivative with respect to $\zeta$ becomes a derivative with respect to the physical momentum of the hadron beam, actually with

$q$ in a high-energy collision of two hadrons $A$ and $B$. The factorization formula for the cross section has the form

$$\frac{d\sigma}{dq^2} = \sum_{ij} H_{ij}(Q^2/\mu^2, \alpha_s(\mu)) \int d^2b_T e^{iq_T \cdot b_T} \times$$

$$\times \hat{f}_{i/A}(x_A, b_T; \zeta, \mu) \hat{f}_{j/B}(x_B, b_T; \zeta, \mu) + Y,$$

and is valid up to corrections suppressed by a power of $Q$. Here, $H_{ij}$ is the hard-scattering associated with the process at the partonic level, which is just a quark-antiquark annihilation to the lepton pair (or a gluon-gluon hard-scattering for Higgs production). The sum is over all relevant parton flavors. The quantities $\hat{f}$ are the TMD pdfs Fourier transformed into transverse coordinate space, while $x_A$ and $x_B$ are longitudinal momentum fractions, $x_A = Qe^{-y_n}/\sqrt{2}P_A^+$, $x_B = Qe^{-y_n}/\sqrt{2}P_B^+$, where standard light-front coordinates are used$^2$ $y_n = \frac{1}{2} \ln (q^+/q^-)$ is the rapidity of the lepton pair, and $P_A$ and $P_B$ are the 4-momenta of the beam hadrons. The quantity $\mu$ is the renormalization scale, which we will assume to be in the $\overline{\text{MS}}$ scheme. The cross section is RG invariant, i.e., it is independent of the choice of $\mu$.

The parameters $\zeta_A$ and $\zeta_B$ are defined by

$$\zeta_A = 2x_A^2(P_A^+)^2 e^{-2y_n} = Q^2 e^{2(y_n-y_n)},$$

$$\zeta_B = 2x_B^2(P_B^-)^2 e^{-2y_n} = Q^2 e^{2(y_n-y_n)},$$

where $y_n$ is an arbitrarily chosen quantity, which in the CS and Collins derivations of factorization can be though of as a rapidity defining a separation between left-moving quanta from right-moving quanta. Technically, $y_n$ is the rapidity of a vector $n^\mu$ that implements the separation, either by a choice of gauge-fixing or as the direction of certain Wilson lines.

The first term on the right-hand side of Eq. (12) is the TMD factorization result. It gives a good approximation to cross-section when $q_T \ll Q$. The second term, $Y$, was proposed by CS to provide a matching of TMD factorization with ordinary collinear factorization for large $q_T$. That is, it allows one to combine TMD factorization with collinear factorization, so that Eq. (12) is valid up to corrections suppressed by a power of $Q$ for any $q_T$. For the rest of this paper, the $Y$ term (or some other matching method) will be ignored. The angle of the lepton pair has been integrated over, and the hadrons are assumed to be unpolarized. Simple generalizations of TMD factorization obeying the same principles can be readily worked out to deal with angular dependence and polarization (e.g., 13).

$\zeta_A^2$ I use the convention that $V^\pm = (V^0 \pm V^3)/\sqrt{2}$. Note that up to corrections suppressed by a power of $q_T/Q$, $x_A = q^+ / P_A^+$, $x_B = q^- / P_B^+$. Using this last two equations to define $x_A$ and $x_B$ would keep the structure of TMD factorization, but would require a change in the calculation used in the $Y$ term.
VI. PROPERTIES TO BE PROVED

Suppose we ignored the particular derivations of the CS and RG equations, \([13], [15],\) and \([17]\). Then we could just define the CS kernel \(K\) by Eq. \((14)\) and the anomalous dimensions by Eqs. \((10)\) and \((17)\). But that would allow the CS kernel \(K\) to depend on \(x, \zeta\) and the flavors of the parton and hadron. It would allow the RG coefficients \(\gamma_f\) and \(\gamma_K\) to depend on \(x, b_T,\) masses, and the hadron flavor. It would also not restrict the anomalous dimension \(\gamma_f\) to be linear in \(\ln \zeta\) and \(\gamma_K\) to be independent of \(\zeta\). The extra dependencies, particularly of \(K\) would remove much of the predictive power of TMD factorization. General properties of renormalization do require the anomalous dimensions to be independent of masses in the MS scheme.

We therefore need to derive the follow properties:

1. \(\hat{K}\) is independent of \(x\) and \(\zeta\), and of parton and hadron flavor.

2. \(\gamma_K\) is independent of these same variables.

3. \(\gamma_f\) is independent of \(x\), of masses, and of hadron flavor.

4. Its dependence on \(\zeta\) is linear in \(\ln \zeta\), as in Eq. \((18)\) with the coefficient being \(-\frac{1}{2}\gamma_K\). Note that since \(\gamma_f\) is independent of quark masses, the dependence on \(\zeta\) and \(\mu\) is only via the ratio \(\zeta/\mu^2\), by dimensional analysis, and also \(\gamma_K\) is independent of masses.

The second property is a trivial consequence of the first, by the definition \((15)\) of \(\gamma_K\).

VII. DERIVATION OF FORM OF CS EQUATION AND RG KERNEL

The TMD pdfs depend not only on parameters determined by the kinematic variables of the process, but also on the auxiliary parameter \(y_n\), via the values of the \(\zeta\) parameters in Eqs. \((13)\). But the cross section does not depend on \(y_n\). Moreover, if one uses the form of TMD factorization derived in \([10]\) or the similar factorization used in \([8]\) and \([6]\), the hard factor is also independent of \(y_n\); it depends only on \(Q\), on the renormalization scale \(\mu\), and on the QCD coupling. There is also no separate soft factor in the factorization formula, unlike many of the earlier formalisms, notably that of CSS \([13]\). I now show how to use these results to demonstrate the properties listed in Sec. \(\text{V}\).

The starting point of the derivation is not an absolutely necessary property of a TMD factorization formalism. For example, there can be a soft factor. Thus in some formalisms, e.g., that of Korchemsky \([14]\) for the Sudakov form factor, it is the soft factor that carries the \(Q\)-dependence that in Eq. \((12)\) is given by the \(\zeta\) dependence of the TMD pdfs. Moreover, in the formalism of
Ji, Ma, and Yuan \[13\] there appear several kinds of non-light-like Wilson lines, and not only does that formalism have a soft factor but the hard scattering has dependence on an extra auxiliary parameter \( \rho \).

The new formalisms with a structure like that listed in Sec. \[14\] are in some sense minimal (as in “minimal subtraction”). The structure applies equally to the EIS formalism \[8\] and (modulo their lack of finite individually defined TMD pdfs) to the BN formalism \[8\]. In both these last two cases, the TMD factorization given by the authors is obtained by setting \( \xi_A = \xi_B = Q^2 \) in Eq. \[12\]. The existence of an equivalent of a \( \gamma_n \) parameter is not obvious, but, at least in the EIS case, can be considered a consequence of an implicit choice of coordinate frame \[11\].

Common themes of the new formalisms are that the Wilson lines (or their equivalent) in the definitions of the various factors are made light-like as far as possible (but with regulation at intermediate stages), and that any soft factor is absorbed into the collinear factors (i.e., the TMD pdfs).

A. Derivation of CS equation

To derive the form of the CS equation, I will follow the pattern used in Sec. \[11\] but now applied to \( \gamma_n \) dependence instead of \( \mu \) dependence. A complication is that there was one term in the OPE used in Sec. \[11\] but there are generally multiple terms in TMD factorization \[12\], with its sum over parton flavors. The complication can be avoided by using the following trick inspired by how flavor-separated pdfs can be obtained from the structure functions of deep-inelastic scattering with charged currents.

The derivation of TMD factorization applies equally when we use a general current correlation function

\[
W^{\mu \nu} = s \int d^4 y e^{-i q \cdot z} \langle AB; \ln|j^{\mu}(z)j^{\nu}(0)|AB; \ln \rangle.
\]  

(20)

The initial state is the same high-energy state of two incoming hadrons as before, but the current can be any that we choose to use, not just one that carries the coupling to an electroweak boson. In the following, let us use the following chiral current giving a transition between two particular quark flavors:

\[
j^{\mu} = \bar{\psi}_j \gamma^{\mu}(1 - \gamma_5) \psi_i,
\]

(21)

with the quark flavor indices being unequal: \( i \neq j \). In this case, TMD factorization involves only two terms in \( \xi \), which use the following combinations of TMD pdfs:

\[
\tilde{f}_{ij/A}(x_A, b_T; \zeta_A, \mu) \tilde{f}_{ij/B}(x_B, b_T; \zeta_B, \mu),
\]

\[
\tilde{f}_{ij/A}(x_A, b_T; \zeta_A, \mu) \tilde{f}_{ij/B}(x_B, b_T; \zeta_B, \mu).
\]

(22a)

(22b)

That is, the \( i \) quark may come from the \( A \) hadron and the anti-\( j \) quark from the \( B \) hadron, or vice versa.

If the current were invariant (up to a sign change) under charge conjugation invariance, then the two combinations of TMD pdfs in \( \xi \) would simply be added, with equal coefficients. But this is not the case here. Let us decompose the hadronic tensor \( W^{\mu \nu} \) into a sum of scalar structure functions multiplying kinematic basis tensors. With a standard choice of basis tensors, we can find among the structure functions one for which factorization uses the sum of the terms in \( \xi \), and one for which factorization uses the difference. So we can construct a combination of structure functions that uses just one term in its TMD factorization formula:

\[
F = H(Q^2/\mu^2, \alpha_s(\mu)) \int d^2 b_T e^{i a_T \cdot b_T} \tilde{f}_{ij/A}(x_A, b_T; \zeta_A, \mu) \tilde{f}_{ij/B}(x_B, b_T; \zeta_B, \mu).
\]

(23)

(Alternatively, we could adjust the set of basis tensors so that this is the coefficient of one of the new basis tensors.) We work at low transverse momentum, so we ignore the \( Y \) term that would otherwise be needed. If the pair of flavors \( (i,j) \) is changed, the graphs for the hard scattering change only by a permutation of quark flavor labels, and have the same values, since masses are neglected in the hard scattering. Thus the value of \( H \) is the same for all cases.

Let us define

\[
L_{ij/A}(x_A, \zeta_A, b_T; \mu) = \frac{\partial \ln \tilde{f}_{ij/A}(x_A, b_T; \zeta_A, \mu)}{\partial \ln \sqrt{\zeta_A}} = - \frac{d \ln \tilde{f}_{ij/A}(x_A, b_T; \zeta_A, \mu)}{dy_n},
\]

(24a)

\[
L_{ij/B}(x_B, \zeta_B, b_T; \mu) = \frac{\partial \ln \tilde{f}_{ij/B}(x_B, b_T; \zeta_B, \mu)}{\partial \ln \sqrt{\zeta_B}} = \frac{d \ln \tilde{f}_{ij/B}(x_B, b_T; \zeta_B, \mu)}{dy_n},
\]

(24b)

where I have used no assumptions or knowledge about the dependence of the left-hand sides on the \( x \) and \( \zeta \) parameters, and on the parton and hadron flavors. (There is also dependence the QCD coupling and possibly on masses that is not indicated explicitly.)

Now neither the structure function \( F \) nor the hard
scattering factor \( H \) in Eq. (23) depends on \( y_n \). So differentiating Eq. (23) with respect to \( y_n \) gives
\[
0 = -L_{i/A}(x_A, \zeta_A, \beta_T^i; \mu) + L_{j/B}(x_B, \zeta_B, \beta_T^j; \mu),
\]
i.e., equality of the two kernels.

Next we observe that one \( L \) depends on \( x_A \) and \( \zeta_A \), but not on \( x_B \) and \( \zeta_B \), while the reverse is true for other \( L \). Since these quantities can be varied independently, it follows each \( L \) is independent of its \( x \) and \( \zeta \) arguments. Similarly, the hadron flavors can be varied independently, so that the \( L \) functions are independent of hadron flavor. Finally, by taking all possible values of \( i \) and \( j \), we find that the \( L \) functions for different quark flavors are also equal, i.e., \( L \) does not depend on quark flavor. So we can simply rename \( L \) to \( \tilde{K} \) to give the kernel in Eq. (14).

Notice that the argument would apply in hypothesized extensions of QCD (e.g., supersymmetric extensions) to include for example color-triplet scalars, and color-octet quarks and scalars. In each of these cases, we can construct gauge-invariant operators that couple one of the new fields to one of the old fields, and it then follows that the \( L \) function for a new field is the same as \( \tilde{K} \) for an old field with the same color. Of course, fields with new color representations, like color-sextet quarks, are not covered by this argument; each color representation needs a new \( \tilde{K} \) function.

The result of this section is that all the TMD pdfs obey
\[
\frac{\partial \ln \tilde{f}_{i/A}(x, b_T; \zeta, \mu)}{\partial \ln \sqrt{\mu}} = \tilde{K}(b_T; \mu),
\]
where \( \tilde{K} \) depends only on \( b_T \), on the parameters of QCD, including masses, and on the color representation for parton \( i \).

The RG equation for \( \tilde{K} \) will be derived in Sec. V. We observe that the RG equation for the TMD pdf in hadron \( A \) can depend on the flavor of its beam hadron and its quark, on the value of \( x \), and on masses: e.g., \( \gamma_{i/A}(x_A, \alpha_s(\mu); \zeta_A, \mu^2, m(\mu)) \). We will generalize the argument used for the elementary OPE in Sec. III. It will show that \( \gamma_{i/A} \) is independent of the quark and hadron flavor, and that it is independent of \( x \) and of quark masses. Mass independence of an anomalous dimension is also a consequence of the infra-red safety of renormalization in a mass-independent scheme, and infra-red safety is a prerequisite for reliable perturbative calculations. However, unlike the previous case, dependence on \( \zeta \) and \( Q \) remains and needs to be analyzed.

As with our derivation of the CS equation, we use a structure which has a single term in its TMD factorization, (23). Then, from the RG invariance of the cross section, it follows that the sum of the anomalous dimensions of the two TMD pdfs and of the hard scattering is zero:
\[
\gamma_{i/A}(x_A, \zeta_A) + \gamma_{j/B}(x_B, \zeta_B) + \gamma_H(Q^2) = 0,
\]
where \( \gamma_H = dH/d\ln \mu \), and the arguments have been omitted for all but the dependence on kinematic- and flavor-related variables. We can change \( x_A \) and \( x_B \) independently of the other variables, so differentiating with respect to \( x_A \) and \( x_B \) shows that each anomalous dimension of the TMD pdfs is independent of its \( x \) variable. Similarly, we can change the hadron flavors independently, so that \( \gamma_{i/A} \) is independent of \( A \).

We have already seen that \( H \) is independent of which values of \( i \) and \( j \) are used (provided that \( i \neq j \)). So considering all the possible values of \( i \) and \( j \), shows that the anomalous dimensions are equal for all flavors of quark and for all flavors of antiquark. The charge-conjugation invariance of QCD shows equality between the anomalous dimensions for quarks and antiquarks. Hence all the anomalous dimensions of the quark and antiquark TMD pdfs are equal: \( \gamma_{i/A} = \gamma_f \), with our previous notation.

---

3 The obvious derivation of this equality assumes that the TMD pdfs are always non-zero. However, it is conceivable that they have nodes, i.e., zeros on a submanifold in the space of their arguments. In that case \( L_{i/A} = L_{j/B} \) is established by analytic continuation from where the TMD pdfs are nonzero.

4 The four quantities \( x_A, \zeta_A, x_B, \) and \( \zeta_B \) can be changed independently by varying the following four quantities \( s = (P_A + P_B)^2 \), \( Q \), the center-of-mass rapidity of the lepton pair, and \( y_n \).
To show independence of $\gamma_{i/A}$ of masses, we set $\zeta_A = H = Q^2$, and use flavor independence in Eq. (29). This gives

$$\gamma_f(\alpha_s(\mu); Q^2, \mu^2, m(\mu)) = -\frac{1}{2} \gamma_H(\alpha_s(\mu), Q^2/\mu^2).$$

Since $H$ is independent of masses, it follows that $\gamma_f$ is. Thus in all cases, the functional dependence is $\gamma_{i/A} = \gamma_f(\alpha_s(\mu), \zeta_A/\mu^2)$, as already asserted in Sec. IV.

Unlike the case of the kernel of the CS equation, the proof of flavor-independence only works within the set of color-triplet spin-half quarks, since the hard scattering would be different in the case of hypothesized scalar quarks, for example. Then the anomalous dimension of the hard scattering could be different, which would require a changed value of $\gamma_{i/A}$ for the scalar quark.

To analyze the $\zeta$ dependence of $\gamma_f$ and the $Q$ dependence of $\gamma_H$, we rewrite Eq. (29) with all the allowed functional dependence exhibited:

$$\gamma_f(\alpha_s(\mu); Q^2 e^{(y_n-y_n)/\mu^2}) = \gamma_f(\alpha_s(\mu); Q^2 e^{2(y_n-y_n)/\mu^2})$$

$$= -\gamma_H(\alpha_s(\mu), Q^2/\mu^2).$$

The right-hand-side does not depend on $y_n$, because $H$ does not. Therefore differentiating the twice with respect to $y_n$ and setting $y_n = y_q$ gives

$$\frac{\partial^2 \gamma_f(\alpha_s(\mu); \zeta/\mu^2)}{\partial (\ln \zeta)^2} = 0$$

Hence the dependence of $\gamma_f$ on $\ln \zeta$ is linear.

### C. RG for kernel of CS equation

In this section, I relate the $Q$ and $\zeta$ dependence of the anomalous dimensions $\gamma_f$ and $\gamma_H$ to the RG properties of $K$, and derive the RG equation for $K$. This involves an energy-independent anomalous dimension $\gamma_K$.

Now derivatives with respect to different variables commute:

$$\frac{d}{d \ln \mu} \frac{\partial}{\partial \ln \sqrt{\zeta}} \ln f_{i/A}(x, b_T; \zeta, \mu)$$

$$= \frac{\partial}{\partial \ln \sqrt{\zeta}} \frac{d}{d \ln \mu} \ln f_{i/A}(x, b_T; \zeta, \mu).$$

From the RG and CS equations for the TMD pdfs we then get

$$\frac{dK(b_T; \mu)}{d \ln \mu} = \frac{\partial \gamma_f(\alpha_s(\mu); \zeta/\mu^2)}{\partial \ln \sqrt{\zeta}}.$$

The left-hand-side is independent of $\zeta$. The right-hand side is independent of masses, and therefore the left-hand-side is a function just of $\alpha_s(\mu)$. We therefore have the form of the RG equation for $K$, Eq. (13). The $\zeta$ dependence of the anomalous dimension of the TMD pdf immediately follows, as given in Eq. (18). Then for the anomalous dimension of the hard scattering we get

$$\gamma_H(g(\mu), Q^2/\mu^2) = -2\gamma_f(g(\mu); 1) + \gamma_K(g(\mu)) \ln \frac{Q^2}{\mu^2}. $$

### VII. CONCLUSIONS

In this paper I have shown that the CS and RG equations for TMD pdfs follow from structural properties of TMD factorization. The primary assumptions are that the formulation of TMD factorization uses only the pure leading power of $Q$ and that no soft factor appears in the factorization formula (unlike the case for the formulations in [14, 13]). In the new derivation, no direct use is made of properties of the operators whose matrix elements define the TMD pdfs. Of course, which operators are used is strongly constrained by the requirement that the resulting TMD pdfs can be used in a correct and useful TMD factorization property obeying the listed assumptions. The critical property underlying the proof is that the product of TMD pdfs in Eq. (12) is boost invariant even though the individual TMD pdfs are not: See Eqs. (27) and (28) and the associated comments.

As compared with the analogous and simpler case of the operators that are used in the OPE, the TMD case is notable for anomalous dimensions that depend on $Q$ (either directly or through the $\zeta_A$ and $\zeta_B$ parameters). This property is strange given the normal situation that anomalous dimensions are independent of kinematic variables.

The $Q$-dependence of the anomalous dimensions is intimately tied to the fact that perturbative calculations of processes for which TMD factorization applies have two “Sudakov” logarithms of $Q$ per loop instead of the one logarithm that occurs in situations to which the standard OPE applies. This can be seen by expanding the hard scattering to first order in $\alpha_s$. When the anomalous dimension is $Q$-independent, consistency with the RG equation (7) implies that there is at most one logarithm of $Q$ in the one-loop calculation.

Since one-loop vertex graphs in gauge theories have a double logarithm of $Q$, something more general is needed with TMD factorization. This can be provided if the anomalous dimension of the TMD pdf depends on $Q$ (or $\zeta$). This then entails that the TMD pdfs themselves depend on $Q$ (or $\zeta$) unlike the case for the operator matrix element $A$ in the OPE (and its generalizations to collinear factorization). If a different definition of TMD pdfs were to be constructed such that they have no $Q$ dependence, then the formulation of TMD factorization must be generalized from Eq. (12), e.g., to put the relevant $Q$ dependence in a soft factor, as in [14].

Once one finds that the anomalous dimension is $Q$-dependent, there is a danger that the exponent in the solution of the RG is no longer usefully calculable. This
is because of the presence of logarithms of $Q/\mu$ in the perturbative expansion of the anomalous dimension, and the typical phenomenon that the number of logarithms increases with the number of loops. This problem does not in fact occur because the anomalous dimension of the TMD pdfs has at most one logarithm independently of the order of perturbation theory. This property, Eq. (15), was known in the earliest work by CS [1, 2], and in this paper it is also derived from TMD factorization.

In all of the proofs in this paper, it was assumed that all quark masses are light, so that they can be legitimately neglected (to leading power in $m/Q$) in the hard scattering. The independence on masses of the hard scattering was important to the proof. But in reality there are heavy quarks in QCD, whose masses may be comparable to or larger than $Q$. Now the standard derivations of TMD factorization assume that all quark masses are small compared with $Q$. Evidently, a useful project that needs to be tackled is to generalize the results to the case that there are heavy quarks.

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