Gravitons, induced geometry and expectation value formalism at finite temperature

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After establishing the positivity constraint and spin content of the theory for gravitons interacting with a necessarily, and a priori, non-conserved external energy-momentum tensor, the expectation value formalism of the theory is developed at finite temperature in the functional differential treatment of quantum field theory. The necessity of having, a priori, a non-conserved external energy-momentum tensor is an obvious technical requirement so that its respective ten components may be varied independently in order to generate expectation values and non-linearities in the theory. The covariance of the induced Riemann curvature tensor, in the initial vacuum, is established even for the quantization in a gauge corresponding only to two physical states of the gravitons as established above. As an application, the induced correction to the metric and the underlying geometry is investigated due to a closed string arising from the Nambu action as a solution of a circularly oscillating string as, perhaps, the simplest generalization of a limiting point-like object. Finally it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized.

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1 Introduction

The graviton propagator [1–6] plays a central role in the quantum field theory treatment of gravitation. It mediates the gravitational interaction between all particles to the leading order in the gravitational coupling constant. It is well known that in the functional differential formalism of quantum field theory, pioneered by Schwinger [7], functional derivatives (e.g., [7–12]) are taken of the so-called vacuum-to-vacuum transition amplitude \( \langle 0_+ | 0_- \rangle \) with respect to external sources, via the application, in the process, of the quantum dynamical (action) principle (e.g., [8, 11, 12]) to generate non-linearities (interactions) in the theory and n-point functions leading finally to transition amplitudes for various physical processes. [For a recent modern and a detailed derivation of the quantum dynamical principle see [12].] For higher spin fields such as the electromagnetic vector potential \( A^\mu \), the gluon field \( A^{\mu a} \), and, of course, the gravitational field \( h^{\mu\nu} \), the respective external sources \( J_\mu \), \( J_a^\mu \), \( T^{\mu\nu} \), coupled to these fields, cannot a priori taken to be conserved so that their respective components may be varied independently in the functional differentiations process. A problem that may arise otherwise, may be readily seen from a simple example given in [1]: The functional derivative of an expression like \( a_{\mu\nu}(x) + b(x) \partial_\mu \partial_\nu T^{\mu\nu}(x) \) with respect to \( T^{\rho\lambda}(x') \) is \( (1/2)[a_{\mu\nu}(x) + b(x) \partial_\mu \partial_\nu] (\delta_\mu^\rho \delta_\nu^\lambda + \delta_\lambda^\mu \delta_\nu^\rho) \delta^4(x, x') \), where \( a_{\mu\nu}(x), b(x) \), for example, depend on \( x \), and not \( (1/2) a_{\mu\nu}(x)(\delta_\mu^\rho \delta_\nu^\lambda + \delta_\lambda^\rho \delta_\nu^\mu) \delta^4(x, x') \) as one may naively assume by, a priori,

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imposing a conservation law on $T_{\mu\nu}(x)$ prior to functional differentiation. The consequences of relaxing the conservation of the such external sources are highly non-trivial. For one thing the corresponding field propagators become modified. Also they have led to the rediscovery [8, 9] of Faddeev-Popov (FP) [13]-like factors in non-abelian gauge theories [8, 9] and the discovery of even further generalizations [9] of such factors, directly from the functional differential treatment, via the application of the quantum dynamical principle [12], in the presence of external sources, without making an appeal to path integrals, without using symmetry arguments which may be broken, and without even going into the well known complicated structures of the underlying Hamiltonians. An account of this procedure, which is also pedagogical, was given in the concluding section of [1] for the convenience of the reader and needs not to be repeated.

For higher spin fields, the propagator and time-ordered product of two fields do not, in general, coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification for the structures of the underlying Hamiltonians. An account of this procedure, which is also pedagogical, was given in the concluding section of [1] for the convenience of the reader and needs not to be repeated. For higher spin fields, the propagator and time-ordered product of two fields do not, in general, coincide as the former includes so-called Schwinger terms which, in general, lead to a simplification for the structures of the underlying Hamiltonians. An account of this procedure, which is also pedagogical, was given in the concluding section of [1] for the convenience of the reader and needs not to be repeated.

If we denote the vacuum-to-vacuum transition amplitude for the interaction of gravitons with the external source $T_{\mu\nu}$ by $\langle 0_+ | 0_- \rangle^T$, then the propagator of the gravitational field is defined by

$$\Delta_{\mu\nu}^{\sigma\lambda}(x, x') = i \left( -i \right) \frac{\delta}{\delta T_{\mu\nu}(x)} \left( -i \right) \frac{\delta}{\delta T_{\sigma\lambda}(x')} \frac{\langle 0_+ | 0_- \rangle^T}{\langle 0_+ | 0_- \rangle^T},$$

in the limit of the vanishing of the external source $T_{\mu\nu}$. In more detail we may rewrite (1.2) as

$$\Delta_{\mu\nu}^{\sigma\lambda}(x, x') = i \frac{\langle 0_+ | \left( h_{\mu\nu}(x) h_{\sigma\lambda}(x') \right) \rangle^T | 0_- \rangle}{\langle 0_+ | 0_- \rangle^T} + \frac{\langle 0_+ | \left[ \delta T_{\mu\nu}(x) h_{\sigma\lambda}(x') \right] \rangle^T | 0_- \rangle}{\langle 0_+ | 0_- \rangle^T}$$

in the limit of vanishing $T_{\mu\nu}$, where the first term on the right-hand side, up to the i factor, denotes the time-ordered product. In the second term, the functional derivative with respect to $T_{\mu\nu}(x)$ is taken by keeping the independent field components of $h_{\sigma\lambda}(x')$ fixed. The dependent field components depend on the external source and lead to extra terms on the right-hand side of (1.3) in addition to the time-ordered product and may be referred to as Schwinger terms. For a detailed derivation of the general identity in (1.3) is given in [12] (see also [11]). It is the propagator $\Delta_{\mu\nu}^{\sigma\lambda}$ that appears in this formalism and not the time-ordered product. The propagator $\Delta_{\mu\nu}^{\sigma\lambda}(x, x')$ has been derived in [1] and will be elaborated upon in Sect. 2. It includes 30 terms in contrast to the well known one involving only 3 terms when a conservation law of $T_{\mu\nu}$ is imposed. The positivity constraint of the vacuum persistence probability $| \langle 0_+ | 0_- \rangle |^2 \leq 1$, as well as the correct spin content of the theory is established in Sect.3 for, a priori, non-conserved external energy-momentum tensor.

The expectation value formalism, pioneered by Schwinger [14], also known as the closed-time path formalism, in quantum field theory has been a useful tool in performing expectation values without first evaluating transition amplitudes. For a partial list of studies of the expectation value formalism, the reader may refer to [15, 16] in the functional differential formalism. See also related work in [17–20] emphasizing non-equilibrium phenomena and [21–23] emphasizing Feynman path integrals.

In order to study gravitational effects such as the induced geometry due to external sources and even due to fluctuating quantum fields, the expectation value formalism turns out to be of practical value. In Sect. 4, we develop the expectation value formalism for gravitons interacting with an external energy-momentum tensor $T_{\mu\nu}$ at finite temperature with a priori not conserved $T_{\mu\nu}$, so that variations with respect to its ten
components may be varied independently in order to generate expectation values. After all the relevant functional differentiations with respect to $T_{\mu\nu}$ are carried out, the conservation law on $T_{\mu\nu}$ may be then imposed. We establish the covariance of the induced Riemann curvature tensor, in the initial vacuum, due to the external source, in spite of the quantization carried out in a gauge which ensures only two polarization states for the graviton. As an application, we investigate the induced correction to the metric and the underlying geometry due a closed string arising from the Nambu action (e.g., [24–26]) as a solution of a circularly oscillating string [27–30] as, perhaps, the simplest generalization of a limiting point-like object. Finally, it is discussed on why the geometry of spacetime may, in general, depend on temperature due to radiative corrections and its physical significance is emphasized. The Minkowski metric is denoted by $[\eta_{\mu\nu}]=\text{diag}[-1,1,1,1]$, and we use units such that $\hbar = 1, c = 1$.

2 Graviton propagator and vacuum-to-vacuum transition amplitude

The action for the gravitational field $h^{\mu\nu}$ coupled to an external energy-momentum tensor source $T_{\mu\nu}$ is taken to be

$$A = \frac{1}{8\pi G} \int (dx) \mathcal{L}(x) + \int (dx) h^{\mu\nu}(x) T_{\mu\nu}(x), \quad (2.1)$$

with

$$\mathcal{L} = -\frac{1}{2} \partial^\alpha h^{\mu\nu} \partial_\alpha h_{\mu\nu} + \frac{1}{2} \partial^\alpha h^{\sigma\nu} \partial_\alpha h_{\nu\sigma} - \partial^\alpha h_{\alpha\mu} \partial_\mu h^{\sigma\sigma} - \frac{1}{2} \partial_\alpha h^{\alpha\mu} \partial_\beta h_{\mu\beta} + \frac{1}{2} \partial_\alpha h^{\alpha\nu} \partial_\mu h_{\mu\nu}, \quad (2.2)$$

and $G$ is Newton’s gravitational constant. The action part $\int (dx) \mathcal{L}$ is invariant under gauge transformations

$$h^{\mu\nu}(x) \rightarrow h^{\mu\nu}(x) + \partial^\mu \xi^\nu(x) + \partial^\nu \xi^\mu(x) + \partial^\rho \partial^\sigma \xi(x), \quad (2.3)$$

As mentioned above the external energy-momentum tensor $T_{\mu\nu}$ is, a priori, taken to be not conserved so that variations of its respective ten components may be varied independently - a necessary technical requirement. Details on dependent fields due to the gauge constraints are spelled out in [12] as well as in [1].

The vacuum-to-vacuum transition amplitude is then given by [1]

$$\langle 0_+ \mid 0_- \rangle^T = \exp \left[ 4\pi G i \int (dx)(dx') T_{\mu\nu}(x) \Delta_+^{\mu\nu,\sigma\lambda}(x, x') T_{\sigma\lambda}(x') \right], \quad (2.4)$$

$$(dx) = dx^0 dx^1 dx^2 dx^3. \quad (2.5)$$

Here we note that the exponent is scaled by the factor $8\pi G$ to satisfy the boundary condition that the gravitational attraction of two widely separated static sources is given by Newton’s law [2]. The graviton propagator $\Delta_+^{\mu\nu,\sigma\lambda}(x, x')$ contains 30 terms and not only just the first 3 terms as may be naively expected, and is given by

$$\Delta_+^{\mu\nu,\sigma\lambda}(x, x') = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} \left[ \Delta_1^{\mu\nu,\sigma\lambda}(k) \frac{k^2}{k^2 - i\epsilon} + \Delta_2^{\mu\nu,\sigma\lambda}(k) \right], \quad (2.6)$$
\[ \epsilon \rightarrow +0, \text{ where } (dk) = dk^0 dk^1 dk^2 dk^3, k^2 = k^0 k^0, \text{ and} \]
\[ \Delta_1^\mu\nu,\lambda\sigma (k) = \frac{(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\sigma})}{2} \]
\[ + \frac{1}{2k^2} \left[ \eta^{\mu\nu} k^\sigma k^\lambda + \eta^{\nu\lambda} k^\mu k^\nu - \eta^{\nu\sigma} k^\mu k^\lambda - \eta^{\nu\lambda} k^\mu k^\sigma \right] \]
\[ - \eta^{\mu\nu} k^\lambda k^\sigma + \eta^{\nu\lambda} k^\mu k^\sigma - \frac{k^\mu k^\nu k^\sigma k^\lambda}{k^2} \]
\[ - \frac{1}{2} \left( \eta^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right) \left( \frac{N^\sigma k^\lambda + N^\lambda k^\sigma}{k^2} \right) k^0 \]
\[ - \frac{1}{2} \left( \eta^{\nu\lambda} + \frac{k^\nu k^\lambda}{k^2} \right) \left( \frac{N^\mu k^\rho + N^\rho k^\mu}{k^2} \right) k^0 \]
\[ + \frac{1}{2} \left[ \eta^{\mu\nu} (N^\mu k^\lambda + N^\lambda k^\mu) + \eta^{\nu\lambda} (N^\mu k^\sigma + N^\sigma k^\mu) \right] \]
\[ + \eta^{\mu\nu} (N^\nu k^\lambda + N^\lambda k^\nu) + \eta^{\nu\lambda} (N^\nu k^\sigma + N^\sigma k^\nu) \]
\[ + \frac{k^\mu k^\nu}{k^2} N^\sigma N^\lambda \]
\[ + \frac{k^\sigma k^\lambda}{k^2} N^\mu N^\nu, \] (2.7)
\[ \Delta_2^\mu\nu,\lambda\sigma (k) = \frac{k^\mu k^\nu}{k^2} N^\sigma N^\lambda + \frac{k^\sigma k^\lambda}{k^2} N^\mu N^\nu. \] (2.8)

Here \((N^\nu) = (\eta^\nu_0) = (1, 0, 0, 0)\). The \(i\epsilon\) factor in (2.6) corresponds to the Schwinger-Feynman boundary condition.

It is far from obvious that with a non-conserved energy-momentum tensor, the vacuum-to-vacuum amplitude \(\langle 0_+ | 0_- \rangle\) in (2.4) satisfies the positivity constraint \(\langle 0_+ | 0_- \rangle^2 \leq 1\). This together with the correct spin content of the theory is established in the next section.

### 3 Positivity constraint and spin content

We rewrite the vacuum-to-vacuum transition amplitude \(\langle 0_+ | 0_- \rangle\) in (2.4) as
\[ \langle 0_+ | 0_- \rangle_T = \exp \left[ 4\pi G_3 \int (dx) T_{\mu\nu}(x) H^{\mu\nu}(x) \right], \] (3.1)
with
\[ T_{\mu\nu} H^{\mu\nu} = T_{00} H^{00} + 2 T_{0\nu} H^{0\nu} + T_{ij} H^{ij}, \] (3.2)
i, j = 1, 2, 3, and we may infer from Eq.(13) in [1] that
\[ H^{00} = -\frac{1}{\delta^2} \left[ T^{00} + \frac{T}{2} - \frac{1}{2\delta^2} \left( \partial^0 \partial^0 T_{00} + \partial^i \partial^j T_{ij} \right) \right], \] (3.3)
\[ T = T_{ii} - T_{00}, \text{ and } H^{00} \text{ is real. Also from Eq.(12) in [1], we may infer that} \]
\[ H^{0\nu} = -\frac{1}{\delta^2} \left[ \delta^{ij} - \frac{\partial^0 \partial^j}{\delta^2} \right] T_{0j}, \] (3.4)
which is again real. That is,

\[
\exp \left[ 4\pi Gi \int (dx) \left( T_{00}(x)H^{00}(x) + 2T_{0i}(x)H^{0i}(x) \right) \right]
\]

(3.5)
is a phase factor.

On the other hand, we may infer from Eq.(17) in [1] that

\[
H^{ij} = \frac{1}{(-\Box - i\epsilon)} A^{ij,lm} T_{lm} - \frac{1}{2} \frac{1}{\partial^2} \left( \delta^{ij} - \frac{\partial^i \partial^j}{\partial^2} \right) T_{00},
\]

(3.6)
and the second term above involving \(T_{00}\) is real, while \(A^{ij,lm}\) is given by

\[
A^{ij,lm} = \frac{(\delta^{ij} \delta^{lm} + \delta^{im} \delta^{jl} - \delta^{il} \delta^{jm})}{2} - \frac{1}{2\partial^2} \left[ \partial^i \partial^j \delta^{lm} + \partial^i \partial^l \delta^{jm} + \partial^i \partial^m \delta^{jl} 
\right.
\]

\[
\left. + \partial^j \partial^l \delta^{im} - \delta^{ij} \partial^l \partial^m - \frac{\partial^i \partial^j \partial^l \partial^m}{\partial^2} \right],
\]

(3.7)
where \(i, j, l, m = 1, 2, 3\).

Accordingly, from (3.1), (3.5)-(3.7), we may rewrite

\[
\langle 0_+ | 0_- \rangle^T = e^{iG[T]} \exp \left[ 4\pi Gi \int (dx) T_{ij}(x) \frac{1}{(-\Box - i\epsilon)} A^{ij,lm} T_{lm}(x) \right],
\]

(3.8)
where \(expG[T]\) is a phase factor.

By using the facts that the reality of \(T_{ij}(x)\) implies that \(T_{ij}(k)^* = T_{ij}(-k)\), where \((k^\mu) = (k^0, \mathbf{k})\), and the identity

\[
\frac{i}{2} \left( \frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 + i\epsilon} \right) = -\pi \delta(k^2) = -\pi \frac{1}{|k|} \left[ \delta(k^0 - |k|) + \delta(k^0 + |k|) \right]
\]

(3.9)
for \(\epsilon \to +0\), in the sense of distributions, we obtain that

\[
\left| \langle 0_+ | 0_- \rangle^T \right|^2 = \exp \left[ -8\pi G \int d\omega_k T_{ij}^*(k) B^{ij,lm}(k) T_{lm}(k) \right],
\]

(3.10)
where now \(k^0 = +|k|\), \(d\omega_k = k^3/(2\pi^2)|k|\), and

\[
B^{ij,lm}(k) = \frac{1}{2} \left[ (\delta^{ij} - \frac{k^j k^l}{k^2})(\delta^{lm} - \frac{k^i k^m}{k^2}) + (\delta^{im} - \frac{k^i k^m}{k^2})(\delta^{jl} - \frac{k^j k^l}{k^2}) 
\right]
\]

\[
\left. - (\delta^{ij} - \frac{k^i k^j}{k^2})(\delta^{lm} - \frac{k^l k^m}{k^2}) \right],
\]

(3.11)
with \(i, j, l, m = 1, 2, 3\) as before.

For a given 3-vector \(\mathbf{k}\), we introduce two orthonormal complex 3-vectors \(\mathbf{e}_+, \mathbf{e}_-\),

\[
\mathbf{e}_+ \cdot \mathbf{e}_+^* = 1 = \mathbf{e}_- \cdot \mathbf{e}_-^*, \quad \mathbf{e}_+ \cdot \mathbf{e}_-^* = 0
\]

(3.12)
such that \(\mathbf{k}/|\mathbf{k}|, \mathbf{e}_+, \mathbf{e}_-\) constitute three mutually orthonormal vectors. That is, in addition to the conditions in (3.12),

\[
\mathbf{k} \cdot \mathbf{e}_+ = 0, \quad \mathbf{k} \cdot \mathbf{e}_- = 0
\]

(3.13)
Upon writing
\[ \mathbf{k} = |\mathbf{k}|(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \] (3.14)
we may set
\[ \mathbf{e}_+ = \frac{1}{\sqrt{2}} (\cos \phi \cos \theta - i \sin \phi, \sin \phi \cos \theta + i \cos \phi, -\sin \theta), \] (3.15)
\[ \mathbf{e}_- = \frac{1}{\sqrt{2}} (\cos \phi \cos \theta + i \sin \phi, \sin \phi \cos \theta - i \cos \phi, -\sin \theta), \] (3.16)
and note that
\[ \mathbf{e}_- = \mathbf{e}_+^* \] (3.17)

The above allows us to introduce the completeness relation
\[ \delta_{ij} = \sum_{\lambda = \pm} \mathbf{e}_\lambda^{ij} \mathbf{e}^{* \lambda j} + \sum_{\lambda = \pm} \mathbf{e}_\lambda^{* ij} \mathbf{e}^{\lambda j} = \frac{|\mathbf{k}|^2}{2} \] (3.18)

In turn, we may define polarization 3x3 tensors by
\[ e^{ij}_{\lambda \sigma} = \frac{1}{2} \left[ e^{i \lambda}_{\alpha} e^{* j \sigma} + e^{* i \lambda}_{\alpha} e^{j \sigma} - \delta_{\lambda \sigma} e^{i \lambda}_{\alpha} e^{* j \sigma} \right] \] (3.19)
with \( \lambda, \sigma, \alpha = \pm \), and a summation over the repeated index \( \alpha \) is assumed, and note that after some algebra,
\( B^{ij, lm} \) in (3.11) may be rewritten as
\[ B^{ij, lm} = \sum_{\lambda, \sigma = \pm} e^{ij}_{\lambda \sigma} e^{lm \sigma} \] (3.20)

Using, in the process, (3.19), we note that
\[ e^{++}_{ij} = 0, \quad e^{++}_{-} = 0, \] (3.21)
and
\[ e^{++}_{ij} = e^{+}_{ij}, \quad e^{--}_{ij} = e^{-}_{ij} \] (3.22)
\[ e^{++}_{ij} = e^{+}_{ij}, \quad e^{--}_{ij} = e^{-}_{ij} \] (3.23)
thus defining the two 3x3 tensors \( e^{ij}_{++}, e^{ij}_{--} \), and rewrite (3.20) as
\[ B^{ij, lm} = \sum_{\lambda = \pm} e^{ij}_{\lambda} e^{lm \lambda} \] (3.24)

From (3.10), (3.11), (3.24), we conclude that
\[ \left| \langle 0_+ | 0_- \rangle^T \right|^2 = \exp \left[ -8\pi G \int \omega_k \sum_{\lambda = \pm} \left( T^{* ij}_{\lambda} \right) \left( T^{ij \lambda}_{lm} \right) \right] \leq 1 \] (3.25)
with equality holding in the limit of vanishing \( T_{\mu\nu} \), thus establishing the underlying positivity constraint, as well as the correct spin content of the theory with the graviton having only two polarization states described by \( \epsilon_1^\mu, \epsilon_2^\mu \) for a theory with, in general, a not necessarily conserved external energy-momentum tensor.

The scalar product in (3.25) may be rewritten from (3.24) as follows:

\[
\int d\omega_k \sum_{\lambda=\pm} (T_{ij}^{\lambda} \epsilon_\lambda^{ij}) (\epsilon_\lambda^{lm} T_{lm}) = \int d\omega_k T_{ij}^{\lambda} B_{ij}^{\lambda} T_{lm}
\]

\[
= \int (dx)(dx')T_{\mu\nu}(x)C^{\mu\nu,\sigma\rho}(x,x')T_\sigma(x'),
\]

(3.26)

where

\[
C^{\mu\nu,\sigma\rho}(x,x') = \int d\omega_k e^{ik(x-x')} \pi^{\mu\nu,\sigma\rho}(k),
\]

(3.27)

\[
\pi^{\mu\nu,\sigma\rho}(k) = \frac{1}{2} (\beta^{\mu\nu} \beta^{\sigma\rho} + \beta^{\rho\sigma} \beta^{\mu\nu} - \beta^{\nu\sigma} \beta^{\mu\rho}),
\]

(3.28)

\[
\beta^{\mu\nu}(k) = \left[ 3 T_{\mu\nu} - \frac{k_\mu k_\nu}{(Nk)^2} - \frac{N^{\mu\nu}}{(Nk)} \right],
\]

(3.29)

\[
Nk = N_\alpha k^\alpha = -k^0 = -|k|.
\]

(3.30)

4 Gravitons and expectation value formalism at finite temperature

For book-keeping purposes, we use the notation

\[
\sqrt{8\pi G} T_{ij}^{lm} T_{lm} (k) \equiv S(k, \lambda),
\]

(4.1)

and conveniently introduce a discrete notation \([2, 31]\) for the momentum variable \( k \) by writing, in the process, \((k, \lambda) \equiv r\) for these pairs of variables and in turn use the notation \( S_r \) for \( S(k, \lambda) \). A scalar product as in (3.25) then becomes simply replaced as follows:

\[
8\pi G \int d\omega_k \sum_{\lambda=\pm} (T_{ij}^{\lambda} \epsilon_\lambda^{ij}) \left( \epsilon_\lambda^{lm} T_{lm} \right) \rightarrow \sum_r S_r^* S_r.
\]

(4.2)

With the above notation, and for any two, a priori, independent, not necessarily conserved, sources \( T^1_{\mu\nu}, T^2_{\mu\nu} \), we introduce the functional

\[
\mathcal{F}[T^1, T^2] = \sum_N \sum_{N_1 + N_2 + \ldots = N} \langle 0_\ldots | N_1, N_2, \ldots | 0_\ldots \rangle T^2 \langle N_1, N_2, \ldots | 0_\ldots \rangle T^1,
\]

(4.3)

where \( N \) denotes number of gravitons, \( N_1 \) of which have momentum-polarization index \( r_1 \), and so on, with \( \langle N_1, N_2, \ldots | 0_\ldots \rangle T^1 \) denoting the amplitude that these \( N \) gravitons are emitted by the source \( T^1 \), and is given by

\[
\langle N_1, N_2, \ldots | 0_\ldots \rangle T^1 = \langle 0_\ldots | 0_\ldots \rangle T^1 \frac{(iS^1)}{\sqrt{N_1!}} \frac{(iS^1)}{\sqrt{N_2!}} \ldots
\]

(4.4)

The expression for the functional \( \mathcal{F}[T^1, T^2] \) may be summed exactly by using, in the precess, (4.4), to give

\[
\mathcal{F}[T^1, T^2] = \left( \langle 0_\ldots | T^2 \rangle \right)^* \left( \langle 0_\ldots | T^1 \rangle \right) \exp \left[ 8\pi G \int d\omega_k \sum_{\lambda=\pm} \left( T^{\lambda}_{ij} \epsilon_\lambda^{ij} \right) \times \left( \epsilon_\lambda^{lm} T_{lm} \right) \right],
\]

(4.5)
where we have restored the integration signs. From (4.3), we realize that for the special case that $T_{\mu\nu}^1$ and $T_{\mu\nu}^2$ are equal, we have by unitarity

$$\mathcal{F}[T, T] = \langle 0_- | 0_- \rangle^T = 1,$$  \hspace{1cm} (4.6)

which also follows readily from (4.5) and the left-hand side equality in (3.25).

In the expression for $\mathcal{F}[T^1, T^2]$, we write $T^1 = T_1 + T_1'$, $T^2 = T_2 + T_2'$, where $T_1'$ is switched on after $T_1$ is switched off, and $T_2'$ is switched on after $T_2$ is switched off, to obtain from (4.3) and (4.5), respectively,

$$\mathcal{F}[T_1 + T_1', T_2 + T_2'] = \sum_{(N)} \langle 0_- | N; N_1, N_2, \ldots \rangle T_2 + T_1' \langle N; N_1, N_2, \ldots | 0_- \rangle T_1 + T_1'$$

$$= \sum_{(N),(M)} \langle 0_- | N; N_1, N_2, \ldots \rangle T_2 \langle N; N_1, N_2, \ldots | M; M_1, M_2, \ldots \rangle T_2' T_1'$$

$$\times \langle M; M_1, M_2, \ldots | 0_- \rangle T_1 ,$$  \hspace{1cm} (4.7)

where

$$\langle N; N_1, N_2, \ldots | M; M_1, M_2, \ldots \rangle T_2 T_1 = \sum_{(L)} \langle N; N_1, N_2, \ldots | L; L_1, L_2, \ldots \rangle T_2$$

$$\times \langle L; L_1, L_2, \ldots | M; M_1, M_2, \ldots \rangle T_1 ,$$  \hspace{1cm} (4.8)

with $\sum_{(N)}$ denoting a sum over non-negative integers $N, N_1, N_2, \ldots$ such that $N_1 + N_2 + \ldots = N$, and similarly for $\sum_{(M)}, \sum_{(L)}$, and

$$\mathcal{F}[T_1 + T_1', T_2 + T_2'] = \mathcal{F}[T_1', T_2'] \exp[S_2^* S_1] \left( \langle 0_+ | 0_- \rangle^T \right) \times \exp[-(S_1^* - S_2^*) S_1],$$  \hspace{1cm} (4.9)

where the scalar product $S_2^* S_1$, for example, is defined as on the right-hand side of (4.2) with a sum over $r$. Upon comparison of the two equivalent expressions for $\mathcal{F}[T_1 + T_1', T_2 + T_2']$ in (4.7) and (4.9), we obtain, in particular, for the diagonal term $\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle T_2^2 T_1^2$, valid for any two, \textit{a priori}, independent and not necessarily conserved sources $T_{\mu\nu}^1, T_{\mu\nu}^2$, the expression:

$$\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle T_2^2 T_1^2 = \langle N_1! N_2! \ldots \rangle \mathcal{F}[T^1, T^2]$$

$$\times \sum_{m_i}^* \prod_i \frac{[-(S_1^{1*} - S_2^{1*})(S_2^{1} - S_2^{2})]^{N_i - m_i}}{m_i! (N_i - m_i)!^2}. \hspace{1cm} (4.10)$$

where $\sum_{m_i}^*$ stands for a summation over all non-negative integers $m_1, m_2, \ldots$ such that $0 \leq m_i \leq N_i$, $i = 1, 2, \ldots$.

We now perform a thermal average [16] of $\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle T_2^2 T_1^2$ by multiplying, in the process, the latter by the Boltzmann factor $\prod_i (\exp - \beta |k_i|)$ and summing over $(N)$, where $\beta = 1/K_T$, and we have used the notation $K$ for the Boltzmann constant and $T$ for temperature in order not to confuse it with the trace $T$ of an energy-momentum tensor. This gives the statistical thermal average:

$$\mathcal{F}[T^1, T^2; \tau] = \mathcal{F}[T^1, T^2; 0] \exp \left[-8\pi G \int d\omega_k \sum_{\lambda = \pm} \frac{(T_{1ij}^{1*} - T_{2ij}^{1*}) \epsilon^{1*}_{\lambda} \epsilon^{1*}_{\lambda} (T_{1im}^{1} - T_{2im}^{1})}{(\beta |k|)} \right]. \hspace{1cm} (4.11)$$
In particular, we note from (4.5), (4.6), (4.11) that for the special case that \( T^1_{\mu\nu}, T^2_{\mu\nu} \) are identical, we have the consistent normalization condition

\[
\mathcal{F}[T, T; \tau] \equiv 1. \tag{4.12}
\]

We also verify directly from (4.11) that

\[
\mathcal{F}[T^1, T^2; 0] = \mathcal{F}[T^1, T^2], \tag{4.13}
\]

as expected.

As we have not imposed conservation laws on \( T^1_{\mu\nu}, T^2_{\mu\nu} \), we may vary each of their respective ten components independently to obtain from the quantum dynamical principle [7, 12, 14] as applied, respectively, and in the process to \( \langle L; L_1, \ldots | M; M_1, \ldots \rangle T^1 \) and \( \langle N; N_1, \ldots | L; L_1, \ldots \rangle T^2 \) in (4.8) with \( T^1, T^2 \) in it replaced by \( T^1, T^2 \), the thermal average \( \langle h^{\mu\nu}(x) \rangle_T \) of the gravitational field

\[
\langle h^{\mu\nu}(x) \rangle_T = (-i) \frac{\delta}{\delta T^1_{\mu\nu}(x)} \mathcal{F}[T^1, T^2; \tau] \bigg|_{T^1 = T^2 = T} = (i) \frac{\delta}{\delta T^2_{\mu\nu}(x)} \mathcal{F}[T^1, T^2; \tau] \bigg|_{T^1 = T^2 = T}, \tag{4.14}
\]

generalizing the expression for \( \langle 0_+ | h^{\mu\nu}(x) | 0_- \rangle^T \) given by

\[
\langle 0_+ | h^{\mu\nu}(x) | 0_- \rangle^T = (-i) \frac{\delta}{\delta T^1_{\mu\nu}(x)} \mathcal{F}[T^1, T^2] \bigg|_{T^1 = T^2 = T} = (i) \frac{\delta}{\delta T^2_{\mu\nu}(x)} \mathcal{F}[T^1, T^2] \bigg|_{T^1 = T^2 = T}, \tag{4.15}
\]

from zero to finite temperature.

From (4.11), (4.5), (3.26), the generating functional \( \mathcal{F}[T^1, T^2; \tau] \) may be rewritten as

\[
\mathcal{F}[T^1, T^2; \tau] = \langle 0_+ | 0_- \rangle^T (\mathcal{F}[T^1, T^2])^* \times \exp \left[ 8\pi G \int (dx)(dx') T^1_{\mu\nu}(x) C^{\mu\nu,\sigma\rho}(x, x') T^1_{\sigma\rho}(x') \right] \times \exp \left[ -8\pi G \int (dx)(dx') (T^1_{\mu\nu}(x) - T^2_{\mu\nu}(x)) D^{\mu\nu,\sigma\rho}(x, x'; \tau) (T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x')) \right], \tag{4.16}
\]

where \( C^{\mu\nu,\sigma\rho}(x, x') \) is defined in (3.27), and

\[
D^{\mu\nu,\sigma\rho}(x, x'; \tau) = \int d\omega_k e^{ik(x-x')} \frac{\pi^{\mu\nu,\sigma\rho}(k)}{(e^{-\beta(Nk)} - 1)}, \tag{4.17}
\]

\( Nk = N^\alpha k_\alpha = -k^0 = -|k| \), where \( \pi^{\mu\nu,\sigma\rho}(k) \) is given in (3.28).

We note that the temperature dependence occurs only in the last exponential in (4.16) through \( D^{\mu\nu,\sigma\rho}(x, x'; \tau) \). We eventually set \( T^1_{\mu\nu} = T^2_{\mu\nu} \) after the relevant functional differentiations with respect to these sources are taken. For \( \tau \to 0 \), the last exponential in (4.16) is equal to one, giving the relation in (4.13).
5 Covariance of the induced Riemann curvature tensor

The thermal average \( \langle h_{\mu\nu}(x) \rangle^T \) may be obtained from (4.14), (4.16) to give

\[
\langle h_{\mu\nu}(x) \rangle^T = 8\pi G i \int (dx') T^{\sigma\rho}(x') \int d\omega_k \pi_{\mu\nu,\sigma\rho}(k)e^{ik(x-x')}
- 8\pi G i \int (dx') T^{\sigma\rho}(x') \int d\omega_k \pi_{\sigma\rho,\mu\nu}(k)e^{ik(x'-x)}
- 16\pi G \int (dx') T^{\sigma\rho}(x') \int d\omega_k \sin k(x-x')\pi_{\mu\nu,\sigma\rho}(k)
\equiv \langle 0_\omega | h_{\mu\nu}(x) | 0_- \rangle^T
\]

(5.1)

for \( x^0 > x^0 \), where after the functional differentiation was carried out with respect to, say, \( T^{1\mu\nu} \), we have set \( T^{2\mu\nu} = T^{1\mu\nu} = T^{\mu\nu} \). We learn that the above expectation value is independent of temperature in the leading linearized theory as a consequence of the fact that the exponent in the last exponential in (4.16) does not contribute if a single functional differentiation w.r.t. \( T^{1\mu\nu} \) is carried out and then by finally setting \( T^{\mu\nu}_0 - T^{1\mu\nu}_0 = 0 \). Radiative corrections and explicit temperature dependence will be discussed in Sect. 7.

In more detail, we may rewrite (5.1) as:

\[
\langle 0_\omega | h_{\mu\nu}(x) | 0_- \rangle^T = \left\{ 8\pi G i \int d\omega_k e^{i k z} \left[ T_{\mu\nu}(k) - \frac{\eta_{\mu\nu}}{2} T(k) \right] + \text{c.c.} \right\}
+ \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) + \partial_\nu \partial_\mu \xi(x)
\]

(5.2)

\[
\xi_\mu(x) = \left\{ 4\pi G \int d\omega_k e^{i k z} N_\mu T - 2T^{\rho\sigma}_\mu N_\sigma \frac{(N k)^2}{(N k)^2} + \text{c.c.} \right\}
\]

(5.3)

\[
\xi(x) = \left\{ \frac{4\pi G}{1} \int d\omega_k e^{i k z} \frac{T + 2T^{\rho\sigma}_\mu N_\nu N_\sigma}{(N k)^2} + \text{c.c.} \right\}
\]

(5.4)

and \( \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \partial_\nu \partial_\mu \xi \) are the so-called gauge terms (see (2.3)) and are non-covariant depending on the vector \( N^\mu \). Also in this section, since we are not carrying out further functional differentiations with respect to the sources we have finally imposed the conservation law \( \partial_\mu T^{\mu\nu} = 0 \) in (5.2).

The induced Riemann curvature tensor in the leading theory is given by

\[
\langle 0_\omega | R_{\mu\nu\sigma\lambda}(x) | 0_- \rangle^T = \langle 0_\omega | \partial_\mu \partial_\sigma h_{\nu\lambda} + \partial_\nu \partial_\lambda h_{\mu\sigma} - \partial_\mu \partial_\lambda h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\lambda} | 0_- \rangle^T
\]

(5.5)

By substituting the expression (5.2) in (5.5), we see that all the terms depending on \( \xi^{\mu} \), \( \xi \) cancel in the induced Riemann curvature tensor \( \langle 0_\omega | R_{\mu\nu\sigma\lambda}(x) | 0_- \rangle^T \) thus establishing its covariance. This means that one may restrict \( \langle 0_\omega | h_{\mu\nu}(x) | 0_- \rangle^T \) to its covariant gauge-independent part

\[
\langle 0_\omega | h_{\mu\nu}(x) | 0_- \rangle^T = \left\{ 8\pi G i \int d\omega_k e^{i k z} [T_{\mu\nu}(k) - \frac{\eta_{\mu\nu}}{2} T(k)] + \text{c.c.} \right\} \equiv h^{\omega}_{\mu\nu}(x)
\]

(5.6)

in applications. The expression for the latter may be further simplified to

\[
h^{\omega}_{\mu\nu}(x) = \left\{ 8\pi G i \int (dx') \int d\omega_k e^{i k (x-x')} [T_{\mu\nu}(x') - \frac{\eta_{\mu\nu}}{2} T(x')] + \text{c.c.} \right\}
\]

(5.7)

The \( k \)-integration as well as the \( x' \)-one may be explicitly carried out leading to

\[
h^{\omega}_{\mu\nu}(x) = 2G \int \frac{d^3x'}{|x - x'|} \left[ T_{\mu\nu}(x^0 - |x - x'|, x') - \frac{\eta_{\mu\nu}}{2} T(x^0 - |x - x'|, x') \right]
\]

(5.8)
6 The induced correction to the metric: Application to a Nambu string

The metric of spacetime to the leading contribution in our notation here is defined [2] by

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} + 2h_{\mu\nu}^o(x), \quad (6.1) \]

with the 2 factor, where \( h_{\mu\nu}^o(x) \) is given in (5.8). The leading contribution to the inverse \( g^{\mu\nu} \) is then given by \( g^{\mu\nu} = \eta^{\mu\nu} - 2h_{\mu\nu}^o \).

We investigate the contribution to the metric, the induced geometry and corresponding spacetime measurements due to a string. The dynamics of the string is described as follows. The trajectory of the string is described by a vector function \( R(\sigma, t) \), where \( \sigma \) parametrizes the string. The equation of motion of the closed string considered is taken to be

\[ \frac{\partial^2}{\partial t^2} R(\sigma, t) - \frac{\partial^2}{\partial \sigma^2} R(\sigma, t) = 0, \quad (6.2) \]

with constraints

\[ \partial_t R \cdot \partial_\sigma R = 0, \quad (\partial_t R)^2 + (\partial_\sigma R)^2 = 1, \quad R(\sigma + \frac{2\pi}{\omega}, t) = R(\sigma, t), \quad (6.3) \]

for a constant \( \omega \). The general solution to (6.2), (6.3) is given by

\[ R(\sigma, t) = \frac{1}{2}[\Phi(\sigma - t) + \Psi(\sigma + t)], \quad (6.4) \]

where \( \Phi, \Psi \), in particular, satisfy the normalization conditions \( (\partial_\sigma \Phi)^2 = (\partial_\sigma \Psi)^2 = 1 \). For the system (6.2)-(6.4), we consider a solution of the form [27–30]

\[ R(\sigma, t) = (\cos \omega \sigma, \sin \omega \sigma, 0) \frac{\sin \omega t}{\omega}, \quad (6.5) \]

describing a radially oscillating circular string in a plane. The general expression for the energy-momentum tensor of the string is given by

\[ T^{\mu\nu}(x) = \frac{M \omega^2}{2\pi} \int_0^{2\pi/\omega} d\sigma (\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu) \delta^3(r - R(\sigma, t)), \quad (6.6) \]

where \( R^0 = t, \quad r = r(\cos \phi, \sin \phi, 0) \), and \( M \) provides a mass scale. The various components of the energy-momentum tensor are worked out to be [27–30]

\[ T^{00} = \frac{M}{2\pi r} \delta\left(r - \frac{|\sin \omega t|}{\omega}\right) \delta(z), \quad (6.7) \]

\[ T^{0i} = \frac{M}{2\pi r} (\cos \phi, \sin \phi, 0) \delta\left(r - \frac{|\sin \omega t|}{\omega}\right) \delta(z) \cos \omega t \text{ sgn}(\sin \omega t), \quad (6.8) \]

\[ T^{11} = \frac{M}{2\pi r} \delta\left(r - \frac{|\sin \omega t|}{\omega}\right) \delta(z) [\cos^2 \omega t - \sin^2 \phi], \quad (6.9) \]

\[ T^{12} = \frac{M}{2\pi r} \delta\left(r - \frac{|\sin \omega t|}{\omega}\right) \delta(z) \frac{\sin 2\phi}{2}, \quad (6.10) \]

\[ T^{22} = \frac{M}{2\pi r} \delta\left(r - \frac{|\sin \omega t|}{\omega}\right) \delta(z) [\cos^2 \omega t - \cos^2 \phi], \quad (6.11) \]

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\( T^{\mu 3} = 0 \), \( (6.12) \)

where \( \text{sgn}(\alpha) = \pm 1 \) for \( \alpha \geq 0 \) is the sign function, \( i = 1, 2, 3 \).

We note the normalization condition

\[
\int d^3x T^{00}(x) = M.
\]

(6.13)

Also for the trace \( T^{\mu \mu}(x) \) of the energy-momentum tensor we have

\[
T = -\frac{M}{\pi r} \delta \left( r - \frac{\sin \omega t}{\omega} \right) \delta(z) \sin^2 \omega t.
\]

(6.14)

It is most interesting to consider spacetime measurements along the most symmetrical direction in the problem, that is, along the \( z - (x^3 -) \) axis perpendicular to the plane of oscillations. Before doing so, we note that in the plane of oscillations of the string, \( g_{\phi \phi} \) cannot be a function of \( \phi \) by symmetry. Also no cross term \( g_{r \phi} \) can occur in this plane, i.e., \( g_{r \phi} = 0 \). The metric contributions \( h_{rr}, h_{00} \), in the plane of oscillations, are readily obtained. To this end \((5.8), (6.7)-(6.12), (6.14) \) lead for \( r \gg 1/\omega \)

\[
2h_{11}(x) \simeq \frac{4G}{r} \int d^3x' \left[ T_{11}(x^0 - r, x') - \frac{T(x^0 - r, x')}{2} \right] = \frac{2GM}{r}
\]

\[
\simeq 2h_{22}(x), \quad h_{12} \simeq 0,
\]

(6.15)

where \( 1/\omega \) is the maximum radial extension of the string. Using the identity \( h_{rr} = \cos^2 \phi h_{11} + \sin^2 \phi h_{22} + \sin 2\phi h_{12} \), it leads to

\[
g_{rr} \simeq \left( 1 + \frac{2GM}{r} \right).
\]

(6.16)

On the other hand,

\[
2h_{00}(x) \simeq \frac{4GM}{r} \int d^3x' \left[ T_{00}(x^0 - r, x') + \frac{T(x^0 - r, x')}{2} \right]
\]

\[
= \frac{4GM}{r} \cos^2 \omega(t - r)
\]

(6.17)

or

\[
g_{00}(x) \simeq -\left( 1 - \frac{4GM}{r} \cos^2 \omega(t - r) \right),
\]

(6.18)

where we recall that the Minkowski metric is taken to be \([\eta_{\mu \nu}] = \text{diag}[-1,1,1,1] \).

For an observer at a fixed \( r \gg 1/\omega \) in the plane of oscillations of the string, then time slows down by a factor

\[
\frac{1}{(T_2 - T_1)} \int_{T_1}^{T_2} \frac{dt}{\sqrt{g_{00} dt}} = 1 - \frac{GM}{r} \left( 1 + \cos \omega(T_1 + T_2 - 2r) \frac{\sin \omega(T_2 - T_1)}{\omega(T_2 - T_1)} \right)
\]

(6.19)

relative to a time lapsed of length \((T_2 - T_1) \) in empty space.

For spacetime measurements along the \( z \)-axis, we have explicitly

\[
2h_{33}(x) = 4GM \int_0^{\infty} \frac{dr'}{\sqrt{r'^2 + z^2}} \delta \left( r' - \frac{\sin \omega(t - \sqrt{r'^2 + z^2})}{\omega} \right) r'^2 \omega^2.
\]

(6.20)
Again, since \( v' \) does not exceed \( 1/\omega \), we have for an observer at \( |z| \gg 1/\omega \)
\[
g_{33}(x) \simeq 1 + \frac{4GM}{|z|} \sin^2 \omega (t - |z|),
\]
showing an interesting oscillatory behaviour in the space metric with a relative expansion of length.

Similarly, we obtain
\[
g_{00}(x) \simeq - \left( 1 - \frac{4GM}{|z|} \cos^2 \omega (t - |z|) \right).
\]

7 Conclusion

The positivity constraint as well as the spin content of the theory of gravitons interacting with \textit{a priori} non-conserved external energy-momentum tensor was established. As emphasized throughout, relaxing this conservation law is necessary so that variations of the ten components of the energy-momentum tensor may be varied independently which goes to the heart of the functional differential formalism of quantum field theory. The expectation value formalism of the theory within the above context was derived at finite temperature for gravitons. Thermal averages of the generated gravitational field and their correlations may be then obtained by functional differentiations of the resulting generating functional at finite temperature which coincide with the corresponding expectation values \( \langle 0_- | \cdot | 0_- \rangle \) at zero temperature. The covariance of the induced Riemann curvature tensor was established in spite of the gauge constraint which ensures only two polarization states of the graviton. An application was carried out to determine the induced correction to the Minkowski metric resulting from a closed string arising from the Nambu action as a solution of a circularly oscillating string. Radiative corrections play an important role as the induced geometry may, in general, depend on temperature. Technically, this may be seen as follows. The multiplicative factor in the generating functional \( \mathcal{F}[T^1, T^2; \tau] \) in (4.16) depending on temperature is given by
\[
\exp \left[ -8\pi G \int (dx)(dx')(T^1_{\mu
u}(x) - T^2_{\mu
u}(x))D^{\mu\nu,\sigma\rho}(x, x'; \tau)(T^1_{\sigma\rho}(x') - T^2_{\sigma\rho}(x')) \right],
\]
where \( D^{\mu\nu,\sigma\rho}(x, x'; \tau) \) is defined in (4.17), (3.28)-(3.30). Consider a familiar correction to the leading order in the Lagrangian density given by \( h^{\mu\nu}(x) \left( \tau_{\mu\nu} + T^{(m)}_{\mu\nu} \right) \), where \( \tau_{\mu\nu}, T^{(m)}_{\mu\nu} \) are energy-momentum tensors of the gravitational field and matter, respectively. For example, if \( T^{(m)}_{\mu\nu} \) corresponds to a real scalar field coupled in turn to an external source \( K(x) \), then the multiplicative factor in the corresponding generating functional of the scalar field depending on temperature is clearly given by
\[
\exp \left[ - \int (dx)(dx')(K^1(x) - K^2(x)) \Delta^+(x, x'; \tau)(K^1(x') - K^2(x')) \right],
\]
where
\[
\Delta^+(x, x'; \tau) = \int \frac{d^3k e^{i\mathbf{k} \cdot (x - x')}}{(2\pi)^{3/2} \sqrt{k^2 + m^2}} (\exp \frac{k^0}{\sqrt{k^2 + m^2}} - 1)^{-1},
\]
\( k^0 = +\sqrt{k^2 + m^2} \), and \( m \) is the mass of the scalar field. Now both \( \tau_{\mu\nu} \) and \( T^{(m)}_{\mu\nu} \) are \textit{quadratic} in their respective fields. To generate the term \( h^{\mu\nu} \tau_{\mu\nu} \), we then need to functionally differentiate (7.1), say, with the external source \( T^{(m)}_{\mu\nu} \) \textit{three} times, also additively w.r.t. \( T^2_{\mu\nu} \) according to the quantum dynamical principle [14–16]. On the other hand, to generate \( T^{(m)}_{\mu\nu} \), we have to functionally differentiate (7.2) twice with respect to the external sources \( K^{1,2} \) of the scalar field. Finally to generate the thermal average of
\( h_{\mu\nu} \), we have to functionally differentiate once more w.r.t. \( T_{\mu\nu}^1 \) and then set \( T_{\mu\nu}^1 = T_{\mu\nu}^2 \equiv T_{\mu\nu} \), and \( K^1 = K^2 \equiv K \). That is, all in all, we have an even number of functional differentiations w.r.t. the corresponding external sources to generate the thermal average \( \langle h_{\mu\nu} \rangle^T_\tau \) before setting the equality of the sources just mentioned and thus generate a temperature dependence in \( \langle h_{\mu\nu} \rangle^T_\tau \). This is unlike the situation in the leading order in which we have to differentiate only once w.r.t. \( T_{\mu\nu}^1 \) to generate \( \langle h_{\mu\nu} \rangle^T_\tau \) before setting \( T_{\mu\nu}^1 - T_{\mu\nu}^2 = 0 \), resulting no temperature dependence in the former expression as seen in (5.1).

The study of higher orders, however, requires a detailed analysis of Faddeev-Popov-like factors of the type discovered in [1, 9], as generated in the functional differential treatment (see Sect.3 in [1, 8, 9], [12]) which would in turn lead to extra vertices coming from the second term on the right-hand side of (1.3) and its generalizations and complicates matter quite a bit in gravitation. This formidable problem as well as convergence aspects [32] will be investigated in a future report. Physically, temperature dependence of the underlying induced geometry is also clear. When we perform a thermal average, we introduce in the process, a background of gravitons, and in general other particles depending on the matter fields considered. These particles in turn would then act as additional sources of gravitation contributing to the net induced gravitational field and this happens only when non-linearities as field interactions are considered, and corresponding radiative corrections are taken into account.

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