Percolation for level-sets of Gaussian free fields on metric graphs

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Abstract

We study level-set percolation for Gaussian free fields on metric graphs. In two dimensions, we give an upper bound on the chemical distance between the two boundaries of a macroscopic annulus. Our bound holds with high probability conditioned on connectivity and the bound is sharp up to a poly-logarithmic factor with an exponent of one-quarter. This substantially improves a previous result by Li and the first author. In three dimensions and higher, we provide rather sharp estimates of percolation probabilities in different regimes which altogether describe a sharp phase transition.

1 Introduction

1.1 Gaussian free fields on metric graphs

In this paper, we study Gaussian free fields on metric graphs of integer lattices, which are closely related to (discrete) Gaussian free fields on integer lattices. Let \{S_t : t \geq 0\} be a continuous-time random walk on \(\mathbb{Z}^d\) with transition rates \(\frac{1}{2d}\). For \(d \geq 3\), the (discrete) Gaussian free field on \(\mathbb{Z}^d\), \(\{\phi_v : v \in \mathbb{Z}^d\}\), is defined as a mean-zero Gaussian process with covariance \(\mathbb{E}(\phi_u \phi_v)\) given by (denoting below by \(1_A\) the indicator function of the event \(A\))

\[
G(u, v) = \mathbb{E}_u \left[ \int_0^\infty 1_{S_t = v} dt \right] \quad u, v \in \mathbb{Z}^d.
\]  

(1)

It is clear that the preceding definition cannot extend to \(d = 2\) because simple random walk is recurrent in the two-dimensional lattice. For this reason, (as usual) for \(d = 2\) we define the Gaussian free field on a finite set \(V \subset \mathbb{Z}^2\) with Dirichlet boundary conditions,
denoted by \( \{ \phi_v : v \in V \} \), to be a mean zero Gaussian process with covariance \( \mathbb{E}(\phi_u \phi_v) \) given by
\[
G(u, v) = \mathbb{E}_u \left[ \int_0^\zeta 1_{S_t=v} dt \right] \quad u, v \in V,
\]
where \( \zeta = \inf \{ t \geq 0 : S_t \in \partial V \} \) is the hitting time of the internal boundary \( \partial V = \{ v \in V : \exists u \in V^c, |u - v| = 1 \} \).

Let \( \mathcal{G} = \mathcal{G}(V, E) \) be the subgraph of \( \mathbb{Z}^d \) on \( V \), where we usually let \( V \) be a finite box for \( d = 2 \) and we take \( V = \mathbb{Z}^d \) for \( d \geq 3 \). To each \( e \in E \) we associate a different compact interval \( I_e \) of length \( d \) and identify the endpoints of this interval with the two vertices adjacent to \( e \). The metric graph \( \tilde{\mathcal{G}} \) associated to \( V \) is defined to be \( \tilde{\mathcal{G}} = \bigcup_{e \in E} I_e \). With this definition, it was shown in [11] that the Gaussian free field on \( \tilde{\mathcal{G}} \), denoted by \( \{ \tilde{\phi}_v : v \in \tilde{\mathcal{G}} \} \), can be constructed in two equivalent ways. The first is by extending \( \phi \) to \( \tilde{\mathcal{G}} \) in the following manner: for adjacent vertices \( u, v \), the value of \( \tilde{\phi} \) on the edge \( e(u, v) \), conditioned on \( \phi_u \) and \( \phi_v \), is given by an independent bridge of length \( d \) of a Brownian motion with variance \( 2 \) at time \( 1 \). We note in passing that we have chosen the convention that each edge of \( \mathcal{G} \) has conductance \( \frac{1}{2d} \) in order to be consistent with [10].

Alternatively, one can construct \( \tilde{\phi} \) by first defining a Brownian motion \( \{ \tilde{B}_t : t \geq 0 \} \) on \( \tilde{\mathcal{G}} \) as in [11] Section 2]. \( \tilde{B} \) behaves like a standard Brownian motion in the interior of the edges, while on the vertices (i.e. lattice points) it chooses to do excursions on each incoming edge uniformly at random (see [11] for further details). By an abuse of notation, we let \( \zeta = \inf \{ t \geq 0 : \tilde{B}_t \in \partial V \} \) for \( d = 2 \), \( \zeta = \infty \) for \( d \geq 3 \), and \( \{ G(u, v) : u, v \in \tilde{\mathcal{G}} \} \) be the density of the 0-potential of \( \{ \tilde{B}_t : 0 \leq t < \zeta \} \) (with respect to the Lebesgue measure on \( \tilde{\mathcal{G}} \)), where \( u \) and \( v \) are now arbitrary points in \( \tilde{\mathcal{G}} \) (not necessarily vertices). It is shown in [11] that the trace of \( \tilde{B} \) on \( V \) (when parametrized by its local time at the vertices) is exactly the continuous-time simple random walk on \( V \) (killed at \( \partial V \) for \( d = 2 \)), and therefore the two definitions of \( \zeta \) coincide, and the two definitions of \( G \) coincide for \( u, v \in V \), justifying the abuse of notation. The Gaussian free field \( \{ \tilde{\phi}_v : v \in \tilde{\mathcal{G}} \} \) is then the continuous, mean-zero Gaussian field on \( \tilde{\mathcal{G}} \) with covariance given by \( \mathbb{E}[\tilde{\phi}_u \tilde{\phi}_v] = G(u, v) \). It was also shown in [11] that the value of \( G \) on the edges of \( \tilde{\mathcal{G}} \) can be obtained by interpolation from the value on the vertices. For two pairs of adjacent vertices \( (u_1, v_1) \) and \( (u_2, v_2) \) in \( V \), and two points \( w_1 \in e(u_1, v_1) \) and \( w_2 \in e(u_2, v_2) \) on the corresponding edges, taking the convention that either the edges are distinct or \( (u_1, v_1) = (u_2, v_2) \) and letting \( r_1 = |w_1 - u_1| \) and \( r_2 = |w_2 - u_2| \) (here we are measuring the standard Euclidean distance), we have (c.f. [11] Equation (2.1))
\[
G(w_1, w_2) = (1 - r_1)(1 - r_2)G(u_1, u_2) + r_1 r_2 G(v_1, v_2) + (1 - r_1)r_2 G(u_1, v_2) + r_1 (1 - r_2) G(v_1, u_2) + 2d(r_1 \land r_2 - r_1 r_2) \mathbb{1}_{(u_1, v_1) = (u_2, v_2)}.
\]
\[ (3) \]
1.2 Main results

The main goal of the present paper is to study level-set percolation for Gaussian free fields on metric graphs. For \( r \geq 1 \) we let \( V_r = [-r, r] \cap \mathbb{Z}^d \) be the points in the lattice contained in the box of side-length \( 2r \) centered at the origin (we choose this convention so that all boxes can be centered at the origin). For \( d = 2 \) we take a sequence \( \tilde{\phi}_N \) of fields defined on the metric graphs \( \tilde{G}_N \) associated to \( V_N \) (with Dirichlet boundary conditions). For \( \phi_v \geq h \) we let \( \tilde{E}_N^{\geq h} = \{ v \in \tilde{G}_N : \tilde{\phi}_N,v \geq h \} \) be the level set, or excursion set, of \( \tilde{\phi}_N \) above \( h \) — note that our choice of level set is different from that of [6] by a flipping symmetry, in order to be consistent with the majority of the literature. Further, for \( u,v \in \tilde{E}_N^{\geq h} \), we let the chemical distance \( D_{N,h}(u,v) \) be the graph distance between \( u \) and \( v \) in \( \tilde{E}_N^{\geq h} \), with \( D_{N,h}(u,v) = \infty \) if \( u \) and \( v \) are disconnected in \( \tilde{E}_N^{\geq h} \). For two subsets \( A,B \subset \tilde{G}_N \), we let \( D_{N,h}(A,B) = \inf \{ D_{N,h}(u,v) : u \in A, v \in B \} \). The following result is an upper bound on the chemical distance between two boundaries of a macroscopic annulus, conditioned on percolation.

**Theorem 1.** For any fixed \( h \in \mathbb{R} \), \( 0 < \alpha < \beta < \gamma < 1 \), and \( \epsilon > 0 \), there exists a constant \( C \) such that

\[
\limsup_{N \to \infty} \mathbb{P}(D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) > C N (\log N)^{1/4} | D_{N,h}(V_{\alpha N}, \partial V_{\gamma N}) < \infty) \leq \epsilon. \tag{4}
\]

**Remark 2.** Note that \( \mathbb{P}(D_{N,h}(V_{\alpha N}, \partial V_{\gamma N}) < \infty) \) stays above 0 uniformly in \( N \) (see Lemma 18). In a work in preparation by Aru–Lupu–Sepúlveda, it is expected that the following may be deduced as a consequence of their main results: \( \mathbb{P}(D_{N,h}(V_{\alpha N}, \partial V_{\gamma N}) < \infty) \) is continuous in \((\alpha, \gamma)\) uniformly for all \( N \). Provided with this continuity on percolation probability, one would then be able to derive from Theorem 1 that

\[
\limsup_{N \to \infty} \mathbb{P}(D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) > C N (\log N)^{1/4} | D_{N,h}(V_{\alpha N}, \partial V_{\gamma N}) < \infty) \leq \epsilon. \tag{5}
\]

For \( d \geq 3 \) we let \( V = \mathbb{Z}^d \), let 0 be the origin in \( \mathbb{Z}^d \), and let \( \tilde{G} \) be the metric graph associated to \( V \). In the present paper, we will focus on the behavior of \( P_{N,h} = \mathbb{P}(0 \leftrightarrow \partial V_N) \) as \( N \to \infty \), where \( \{0 \leftrightarrow \partial V_N\} \) denotes the event that 0 is connected to \( \partial V_N \) in \( \tilde{E}_N^{\geq h} = \{ v \in \tilde{G} : \tilde{\phi}_v \geq h \} \). We obtain the following results for supercritical, subcritical, and critical percolation, respectively.

**Theorem 3.** Let \( \sigma^2_d = \text{Var}[\tilde{\phi}_0] \). Then for any \( h > 0 \),

\[
\lim_{N \to \infty} P_{N,-h} = \mathbb{E}
\left[
\left(1 - e^{-2h(\tilde{\phi}_0 + h) / \sigma^2_d}\right)\mathbb{1}_{\tilde{\phi}_0 > -h}
\right]. \tag{6}
\]
The next result establishes the exponential decay of $p_{N,h}$ as $N \to \infty$ for $d > 3$, with an extra log factor for $d = 3$.

**Theorem 4.** For any $h > 0$ and $d \geq 3$, there exists a constant $c$ such that for $N \geq 1$,

\[
p_{N,h} \leq \exp(-ch^2N/\log N), \quad d = 3, \tag{7}
\]
\[
p_{N,h} \leq \exp(-ch^2N), \quad d > 3. \tag{8}
\]

The third result establishes the polynomial decay of $p_{0,N}$ as $N \to \infty$ and provides bounds on the exponent.

**Theorem 5.** For any $d \geq 3$, there exist constants $C, c > 0$ such that for $N \geq 1$,

\[
\frac{c}{\sqrt{N}} \leq p_{0,N} \leq C\sqrt{\frac{\log N}{N}}, \quad d = 3, \tag{9}
\]
\[
\frac{c}{N^{d/2-1}} \leq p_{0,N} \leq \frac{C}{\sqrt{N}}, \quad d > 3. \tag{10}
\]

Finally, for $d = 3$ we provide some bounds on the critical window for $h$. Below we take $h_N > 0$ to be a sequence of levels that converges to 0, and write $p_N^+$ for $p_{h_N,N}$ and $p_N^-$ for $p_{-h_N,N}$ to simplify notation.

**Theorem 6.** Suppose that there exists a constant $C > 0$ such that $h_N \leq CN^{-1/2}$ for $N \geq 1$. Then there exists a constant $c > 0$ such that for $N \geq 1$,

\[
p_N^+ \geq \frac{c}{\sqrt{N}}. \tag{11}
\]

Conversely, there exists a constant $C > 0$ such that when $\lim \inf_{N \to \infty} h_N \sqrt{\frac{N}{\log N \log \log N}} \geq C$, we have

\[
\lim_{N \to \infty} \frac{p_N^+}{p_{0,N}} = 0. \tag{12}
\]

Furthermore, if there exists a constant $C > 0$, $h_N \leq C(\log N/N)^{1/2}$ for $N \geq 1$. Then there exists a constant $c > 0$ such that for $N \geq 1$,

\[
p_N^- \leq c\sqrt{\frac{\log N}{N}}. \tag{13}
\]

Conversely, if $\lim_{N \to \infty} h_N \sqrt{\frac{N}{\log N}} = \infty$, we have

\[
\lim_{N \to \infty} \frac{p_N^-}{p_{0,N}} = \infty. \tag{14}
\]
1.3 Related work

The chemical distance on level sets for \( d = 2 \) has been previously studied in \([6]\) (we refer the reader to \([6]\) for another extensive discussion of related work), where it was proved that with positive probability the chemical distance between two boundaries of a macroscopic annulus is at most \( Ne^{(\log N)^{\alpha}} \) for any fixed \( \alpha > 1/2 \). Our Theorem 1 improves on \([6]\) in the following two ways:

- Instead of proving a positive probability bound as in \([6]\), Theorem 1 states that the upper bound on the chemical distance holds with high probability given connectivity. At the moment, we can only show the with high probability result as in \([4]\); as noted in Remark 2, it is possible that with some expected future input, one would be able to derive a stronger version as in \([5]\).

- The upper bound is sharpened from \( Ne^{(\log N)^{\alpha}} \) to \( N(\log N)^{1/4} \), which is somewhat surprising. In fact, the authors of the present article as well as a few people we talked to believed that the chemical distance should be at least \( N \log N \).

A major difference between our proof of Theorem 1 and the proof of the corresponding result in \([6]\) is that our proof does not rely on Makarov’s theorem (on the dimension for the support of planar harmonic measures) which was a fundamental ingredient in \([6]\). Instead of applying Makarov’s theorem, we study the intrinsic structure of the “exploration martingale” introduced in Section 2.

Additionally, we remark that the result proved in \([6]\) applies to level-set percolation for Gaussian free fields on the integer lattice (as well as on the metric graph). Since percolation on the metric graph is dominated by the percolation on the integer lattice, our Theorem 1 implies that with non-vanishing probability the chemical distance in the level-set cluster on the integer lattice between the boundary of the annulus is \( O(N(\log N)^{1/4}) \). We feel it is possible that the methods we employ in proving Theorem 1 together with some technical work might be sufficient to show that the chemical distance on the integer lattice is \( O(N(\log N)^{1/4}) \) with high probability given connectivity. However, we prefer not to consider this problem here to avoid further complications. Furthermore, we note that percolation clusters for level sets on the metric graph in two dimensions are of fundamental importance since their first passage sets converge to those of (continuous) Gaussian free fields \([1,2]\) (thus, we believe Theorem 1 is of substantial interest on its own). Finally, \([6]\) also established the chemical distance for critical random walk loop soup clusters. We chose not to consider proving an analogue of Theorem 1 for random walk clusters in the present paper.

For \( d \geq 3 \), level-set percolation for Gaussian free fields (on metric graphs) has been studied in \([11]\), which in particular computed the connectivity probability between any two points and showed that the critical threshold is at \( h = 0 \). Our methods allow us to improve on those results by deriving more quantitative information on the phase transition, especially when \( d = 3 \). In this case we can compute the connectivity exponent at criticality.
(we remark that the real contribution of the present paper is on its upper bound, since
the lower bound can be deduced easily from [11]), prove an almost exponential decay at
subcriticality (it is quite possible that by employing a renormalization technique we can
get rid of the log factor in the subcritical regime, but we chose not to consider that in
the present paper), and provide an explicit description of the percolation probability in the
supercritical regime (which can be rarely achieved in percolation models). Our results seem
to describe the phase transition of the percolation model for the metric graph Gaussian
free field in three dimensions in rather precise detail. This is somewhat interesting since
percolation models in three dimensions are in general rather difficult.

That being said, we would like to mention that level-set percolation for Gaussian free
fields on integer lattices for $d \geq 3$ has already been extensively studied (see [3, 5, 18, 16, 8,
14, 21, 22, 17, 7, 20]). Contrary to the case of two dimensions, percolation is substantially
different on metric graphs and on integer lattices for $d \geq 3$, (roughly speaking) for the
reason that there is a phase transition in higher dimensions but in two dimensions the
percolation has the same qualitative behavior for any fixed $h$ (see Theorem 1). We remark
that percolation on integer lattices is considerably more challenging than on metric graphs.
In fact, despite intensive research, it remains an open question what the exact critical
threshold is for $d \geq 3$ on integer lattices (but it was proved in [7] that the critical threshold
is strictly positive), as well as whether a sharp phase transition exists. It would be difficult
to apply methods in the present paper to prove something on the integer lattices for $d \geq 3$,
for the reason that we do not have a precise control over the “exploration martingale” (as
introduced in Section 2) in the case of integer lattices.

1.4 Discussions on future directions

Our work suggests a number of interesting directions for further research, which we list
below.

- The factor of $N(\log N)^{1/4}$ in Theorem 1 reinstates the (now even more intriguing)
  question of whether the chemical distance is linear or not. Our bound of $N(\log N)^{1/4}$
  strongly suggests that this is a highly delicate problem.

- Our method can give some non-trivial bounds on the exponent for chemical distances
  for $d \geq 3$ at criticality, but it seems challenging to compute the exact exponent.

- The difference of a factor of $\sqrt{\log N}$ between the upper and lower bounds of (9)
  hides important information about the geometry of critical clusters for $d = 3$. For example,
  whether the capacity of the critical cluster containing $0$ is of order $N$, conditioned on
  the cluster intersecting with $\partial V_N$. It would be interesting to prove an up-to-constants
  bound for the connectivity probability at criticality.

- It would be very interesting to construct an incipient infinite cluster measure for
critical percolation in three dimensions, as has been done for Bernoulli percolation
in two dimensions in [9] (see also [3] for a nicely streamlined presentation).

1.5 Notation conventions and organization

For a real vector \( \mathbf{x} \) (in any dimension), we denote by \( |\mathbf{x}| \) the Euclidean norm of \( \mathbf{x} \) and denote by \( |\mathbf{x}|_{\ell_1} \) the \( \ell_1 \)-norm of \( \mathbf{x} \). We will also use \( |A| \) to denote the cardinality of a finite set \( A \). The meaning will be clear from context. We will denote by \( A^c \) to denote the complement of the set (or event) \( A \).

Throughout, we will use \( \varphi \) and \( \Phi \) to denote the density function and distribution function of the standard normal distribution, and \( \bar{\Phi} \) to denote its survivor function. That is, \( \bar{\Phi}(x) = 1 - \Phi(x) \).

To simplify certain statements we use the following notation to describe the asymptotic behavior of functions of \( N \) as \( N \) tends to infinity. For two positive functions \( f \) and \( g \), we say \( f(N) = O(g(N)) \) as \( N \to \infty \) if there exists constants \( c > 0 \) and \( N_0 > 0 \) (possibly depending on \( d \), \( h \), or other parameters) such that for all \( N \geq N_0 \), \( f(N) \leq cg(N) \). We say \( f(N) = o(g(N)) \) if for every constant \( c > 0 \) there exists \( N_0 > 0 \) such that for all \( N \geq N_0 \), \( f(N) \leq cg(N) \). Similarly, we say \( f(N) = \Omega(g(N)) \) if \( g(N) = O(f(N)) \), and \( f(N) = \omega(g(N)) \) if \( g(N) = o(f(N)) \). Finally, we say \( f(N) = \Theta(g(N)) \) if \( f(N) = O(g(N)) \) and \( f(N) = \Omega(g(N)) \).

The rest of the paper is organized as follows. In Section 2 we introduce a family of martingales which is the key to proving all the results in the present paper. In Section 3 we prove the results concerning \( d \geq 3 \) (as the proof is substantially simpler than that for \( d = 2 \)), and in Section 4 we prove Theorem 1 concerning \( d = 2 \) (we remark that the proof of Theorem 1 encapsulates all the technical ideas of the present article).

2 Exploration Martingale

In this section, we introduce the “exploration martingale” and demonstrate some of its basic properties. We note that the approach of applying martingales in the study of percolation for Gaussian free fields has appeared before (c.f. [12, 1]).

The discussion in this section applies to all dimensions, and we will denote by \( \{\tilde{\phi}_v : v \in \tilde{G}\} \) the Gaussian free field under consideration, without further specifying \( \tilde{G} \). For a finite subset \( A \subset V \), we define the “observable” \( X_A \) to be the average of \( \tilde{\phi} \) on \( A \)

\[
X_A = \frac{1}{|A|} \sum_{v \in A} \tilde{\phi}_v.
\]

Let \( I_0 \) be a deterministic, closed, bounded, connected subset of \( \tilde{G} \) and let \( I_t = \{v : D_h(I_0, v) \leq t\} \) be the closed ball of radius \( t > 0 \) around \( I_0 \) with respect to \( D_h \), the graph
distance on $\tilde{E}^h$ (here we use the following convention: if $u$ and $v$ are distinct and $u \notin \tilde{E}^h$, we let $D_h(u, v) = D_h(v, u) = \infty$, but for any $u \in \tilde{G}$ we set $D_h(u, u) = 0$ even if $u \notin \tilde{E}^h$).

For $U \subset \tilde{G}$, let $\mathcal{F}_U$ be the $\sigma$-field generated by $\{\tilde{\phi}_v : v \in U\}$, and

$$
\mathcal{F}_{\mathcal{I}_t} = \{ \mathcal{E} \in \mathcal{F}_{\tilde{\mathcal{G}}} : \mathcal{E} \cap \{ \mathcal{I}_t \subset U \} \in \mathcal{F}_U \text{ for all open } U \supset \mathcal{I}_0 \}.
$$

We then define the continuous-time martingale $M_A$ by

$$
M_{A,t} = \mathbb{E}[X_A \mid \mathcal{F}_{\mathcal{I}_t}],
$$

(15)

We will call $M_A$ the exploration martingale with source $\mathcal{I}_0$ and target $A$. Before proceeding further, we show the following measurability property of $\mathcal{I}_t$.

**Proposition 7.** For any open subset $U$ of $\tilde{G}$ containing $\mathcal{I}_0$ and $t \geq 0$, we have $\{\mathcal{I}_t \subset U\} \in \mathcal{F}_U$.

**Proof.** Since $\mathcal{I}_0$ is deterministic, we assume without loss of generality that $t > 0$. Write $U_h = \tilde{U} \cap \tilde{E}^h$, let $D_{U_h}$ be the graph distance on $U_h$ and $\mathcal{I}_{U,t} = \{ u \in \tilde{U} : D_{U_h}(\mathcal{I}_0, u) \leq t \}$ be the closed ball of radius $t$ around $\mathcal{I}_0$ with respect to $D_{U_h}$. We will show that $\{\mathcal{I}_t \subset U\} = \{ \mathcal{I}_{U,t} \subset U \} \in \mathcal{F}_U$. First, it is clear that for any $u, v \in \tilde{G}$, $D_h(u, v) \leq D_{U_h}(u, v)$ so we have $\mathcal{I}_{U,t} \subset \mathcal{I}_t$ and $\{\mathcal{I}_t \subset U\} \subseteq \{ \mathcal{I}_{U,t} \subset U \}$. Now, assume $\mathcal{I}_{U,t} \subset U$ and that there exists $v \in \mathcal{I}_t$ with $v \notin U$. Then there exists $0 < s \leq t$ and a path $\Gamma : [0, s] \to \tilde{G}$ parametrized by $D_h$ with $\Gamma(0) = u_0 \in \partial \mathcal{I}_0$, $\Gamma(s) = v$, and $\Gamma([0, s]) \subseteq \tilde{E}^h$. Setting $s' = \sup\{ x \in [0, s] : \Gamma(x) \in \mathcal{I}_{U,t}\}$ we have $\Gamma(s') \in \mathcal{I}_{U,t} \subset U$ and thus $s' < s$. Hence, there exists an open neighborhood $(a, b)$ of $s'$ such that $\Gamma((a, b)) \subset U$. By assumption, $\Gamma((a, b)) \subset \tilde{E}^h$, so we get $\Gamma((a, b)) \subset U_h$ and thus $\Gamma((a, b)) \subset \mathcal{I}_{U,t}$, which contradicts the maximality of $s'$. This concludes the proof that $\{\mathcal{I}_t \subset U\} = \{ \mathcal{I}_{U,t} \subset U \}$ and thus the proposition follows. \qed

Proposition 7 together with the following strong Markov property of the Gaussian free field (see [19] Theorem 4 in Chapter 2, Section 2.4 for a proof) will provide useful formulas for $M_A$ and its quadratic variation $\langle M_A \rangle$.

**Theorem 8.** Let $\mathcal{K}$ be a random compact connected subset of $\tilde{G}$ such that for every deterministic open subset $U$ of $\tilde{G}$, the event $\{ \mathcal{K} \subset U \} \in \mathcal{F}_U$. Then conditioned on $\mathcal{K}$ and $\mathcal{F}_K$, $\{ \tilde{\phi}_v : v \in \tilde{G} \setminus K \}$ is equal in distribution to $\{ \mathbb{E}[\tilde{\phi}_v \mid \mathcal{F}_K] + \tilde{\psi}_v : v \in \tilde{G} \setminus K \}$ where $\tilde{\psi}$ is a Gaussian free field (with Dirichlet boundary condition) on $\tilde{G} \setminus K$. Additionally, for $v \in \tilde{G} \setminus K$ and $T = \inf\{ t \geq 0 : \tilde{B}_t \in K \}$

$$
\mathbb{E}[\tilde{\phi}_v \mid \mathcal{F}_K] = \mathbb{E}_v[\tilde{\phi}_{\tilde{B}_T} \mathbb{1}_{T < \zeta} \mid \mathcal{F}_K] = \sum_{u \in \partial K} \mathbb{H}_m(v, u; K) \tilde{\phi}_u.
$$

(16)

Here the harmonic measure $\mathbb{H}_m$ is given by $\mathbb{H}_m(v, u; K) = \mathbb{P}_v(T < \zeta, \tilde{B}_T = u)$. 

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Note that if \( v \in \mathcal{K} \), the harmonic measure is a point mass of mass 1 at \( v \). To avoid having to account for this case separately, we will sometimes let the sum in (16) range over all \( u \in \mathcal{K} \). The summation notation is justified since in either case the harmonic measure is supported on a finite number of points. Writing \( H_m(t, v, u) \) for \( H_m(v, u; \mathcal{I}_t) \) and

\[
H_m(A, u) = \frac{1}{|A|} \sum_{v \in A} H_m(v, u),
\]

we get from (16)

\[
M_{A,t} = \sum_{u \in \mathcal{I}_t} H_m(A, u) \bar{\phi}_u.
\]

Writing \( G_t \) for the Green’s function on \( \mathcal{G} \setminus \mathcal{I}_t \) we get from Theorem 8 that

\[
\text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] = \frac{1}{|A|^2} \sum_{v, v' \in A} G_t(v, v').
\]

The following lemma establishes the continuity of \( M_A \) and will allow us to compute its quadratic variation. We defer the proof to the end of this section.

**Lemma 9.** \( M_{A,t} \) and \( \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \) are almost-surely continuous as functions of \( t \).

Recall that for a continuous martingale \( M \) its quadratic variation \( \langle M \rangle \) is the unique increasing continuous process vanishing at zero such that \( M^2 - \langle M \rangle \) is a martingale (see [15] Theorem 1.3 in Chapter IV). From this we deduce the following.

**Corollary 10.** The quadratic variation of \( M_A \) is given by

\[
\langle M_A \rangle_t = \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_0}] - \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}].
\]

**Proof.** By definition the process \( \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_0}] - \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \) is adapted to \( \mathcal{F}_{\mathcal{I}_t} \) and vanishes at zero. It is increasing because \( \mathcal{I}_t \) is increasing (so \( \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \) is decreasing), and it is continuous by Lemma 9. Therefore by the characterization of \( \langle M_A \rangle \) we only need to show that the process \( \{Y_t : t \geq 0\} \) is a martingale, where

\[
Y_t = M_{A,t}^2 - \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_0}] + \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}].
\]

To this end, note that for any times \( 0 \leq s < t \),

\[
\mathbb{E}[M_{A,t}^2 \mid \mathcal{F}_{\mathcal{I}_s}] = M_{A,s}^2 + \mathbb{E}[(M_{A,t} - M_{A,s})^2 \mid \mathcal{F}_{\mathcal{I}_s}],
\]

\[
\mathbb{E}[\text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \mid \mathcal{F}_{\mathcal{I}_s}] = \mathbb{E}[(X_A - M_{A,t})^2 \mid \mathcal{F}_{\mathcal{I}_s}]
\]

\[
= \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_s}] - \mathbb{E}[(M_{A,t} - M_{A,s})^2 \mid \mathcal{F}_{\mathcal{I}_s}].
\]
We can then calculate \( \mathbb{E}[Y_t \mid \mathcal{F}_s] \) as follows

\[
\mathbb{E}[Y_t \mid \mathcal{F}_s] = \mathbb{E}[M^2_{A,t} \mid \mathcal{F}_s] + \mathbb{E}[\text{Var}[X_A \mid \mathcal{F}_s] \mid \mathcal{F}_s] - \text{Var}[X_A \mid \mathcal{F}_0]
\]

\[
= M^2_{A,s} + \text{Var}[X_A \mid \mathcal{F}_s] - \text{Var}[X_A \mid \mathcal{F}_0]
\]

\[
= Y_s,
\]

completing the verification that \( \{Y_t : t \geq 0\} \) is a martingale.

Next, we recall that the quadratic variation relates \( M \) to Brownian motion. In particular [15, Theorem 1.7 in Chapter V], stated below, gives the appropriate extension of the Dubins-Schwarz theorem for martingales of bounded quadratic variation.

**Theorem 11.** Let be \( M \) a continuous martingale, \( T_t = \inf\{s : \langle M \rangle_s > t\} \), and \( W \) be the following process

\[
W_t = \begin{cases} 
M_{T_t} - M_0 & t < \langle M \rangle_{\infty}, \\
M_{\infty} - M_0 & t \geq \langle M \rangle_{\infty}.
\end{cases}
\]

Then \( W \) is a Brownian motion stopped at \( \langle M \rangle_{\infty} \).

When applying this theorem, we will generally denote by \( B \) a Brownian motion which satisfies \( B_t = M_{T_t} - M_0 \) for \( t < \langle M \rangle_{\infty} \) but is not stopped at \( \langle M \rangle_{\infty} \), so that \( W_t = B_{t \wedge \langle M \rangle_{\infty}} \).

Next, we state a result which will be used in the later sections to relate the quadratic variation \( \langle M_A \rangle_t \) to the harmonic measure of \( I_t \) as functions of \( t \). Before proceeding we introduce some more notation. We let \( \pi(v, I_t) = \sum_{u \in I_t} H_m(t, u) \) be the probability that a Brownian motion started at \( v \) hits \( I_t \) before \( \partial V \), and extend this to \( A \) by

\[
\pi(A, I_t) = \frac{1}{|A|} \sum_{v \in A} \pi(v, I_t).
\]

**Proposition 12.** For \( q \geq 1 \), write \( g^{-}_{A,q} = \inf \left\{ \frac{1}{|A|} \sum_{u \in A} G(u, v) : u \in V_q \right\} \) and \( g^{+}_{A,q} = \sup \left\{ \frac{1}{|A|} \sum_{v \in A} G(u, v) : u \in V_q \right\} \). Then for \( 0 \leq s \leq t \leq D_h(I_0, \partial V_q) \), we have

\[
g^{-}_{A,q}[\pi(A, I_s) - \pi(A, I_t)] \leq \langle M_A \rangle_t - \langle M_A \rangle_s \leq g^{+}_{A,q}[\pi(A, I_t) - \pi(A, I_s)].
\]

**Proof.** Corollary [10] and [18] imply

\[
\langle M_A \rangle_t - \langle M_A \rangle_s = \frac{1}{|A|^2} \sum_{v, v' \in A} \left[ G_s(v, v') - G_t(v, v') \right].
\]
For any \( v, v' \in A \) we have
\[
G_t(v, v') = G(v, v') - \sum_{u \in \mathcal{I}_t} H_m_t(v, u) G(u, v'),
\]
(20)
and similarly for \( G_s(v, v') \). Letting \( \overline{H}_m_t(u', u) = H_m(u', u; \tilde{G} \setminus \mathcal{I}_t) \) we get from the strong Markov property that for any \( u' \in \mathcal{I}_s \subseteq \mathcal{I}_t \)
\[
G(u', v') = \sum_{u \in \mathcal{I}_t} \overline{H}_m_t(u', u) G(u, v').
\]
Applying (20) to both \( G_t \) and \( G_s \), and combining with the preceding equality, we get that
\[
G_s(v, v') - G_t(v, v') = \sum_{u \in \mathcal{I}_t} H_m_t(v, u) \left[ H_m_t(v, u) - \sum_{u' \in \mathcal{I}_s} H_m_s(v, u') \overline{H}_m_t(u', u) \right].
\]
Plugging this into (19) and using the definition of \( g_{\mathcal{A}, q}^{+} \) together with (3) to handle non-lattice points, we conclude
\[
\langle M_r \rangle_t - \langle M_r \rangle_s \leq g_{\mathcal{A}, q}^{+} \frac{1}{|\mathcal{A}|} \sum_{v \in \mathcal{A}} \sum_{u \in \partial \mathcal{I}_t} H_m_t(v, u) \left[ H_m_t(v, u) - \sum_{u' \in \mathcal{I}_s} H_m_s(v, u') \overline{H}_m_t(u', u) \right],
\]
where we have used the fact that \( \sum_{u \in \partial \mathcal{I}_t} \overline{H}_m_t(u', u) = 1 \). We can deduce the lower bound similarly, thereby completing the proof of the proposition.

Finally, we prove that \( M_{A,t} \) and \( \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \) are indeed continuous.

**Proof of Lemma 2.** It suffices to show that for any \( v, v' \in V \), \( \mathbb{E}[\tilde{\phi}_v \mid \mathcal{F}_{\mathcal{I}_t}] \) and \( \text{Cov}[\tilde{\phi}_v, \tilde{\phi}_{v'} \mid \mathcal{F}_{\mathcal{I}_t}] \) are continuous (then it is clear that \( M_{A,t} \) and \( \text{Var}[X_A \mid \mathcal{F}_{\mathcal{I}_t}] \) are averages of a finite number of continuous functions and are thus continuous). Since both functions are constant for \( t \geq D_h(\mathcal{I}_0, v) \), we let \( 0 \leq t \leq D_h(\mathcal{I}_0, v) \). By (10) and (20) it suffices to show that for any continuous function \( f \) on \( \tilde{G} \), the following function is continuous
\[
F(t) = \sum_{u \in \partial \mathcal{I}_t} H_m_t(v, u) f(u).
\]
Let \( D_{\ell_1} \) denote the graph distance on \( \tilde{G} \) (i.e., \( \ell_1 \) distance on \( \mathbb{R}^d \)). Since \( K = |\partial \mathcal{I}_t| < \infty \),
\[
\delta_1 = \min\{D_{\ell_1}(u, (\partial \mathcal{I}_t \cup V) \setminus \{u\}) : u \in \partial \mathcal{I}_t \} > 0.
\]
For $s$ such that $|t - s| < \delta_1/2$ and $u \in \partial I_t$, let $\psi_s(u) = \partial I_s \cap B_{\ell_1}(u, \delta_1/2)$ (here $B_{\ell_1}(u, \delta_1/2)$ is the open ball of radius $\delta_1/2$ around $u$ with respect to $D_{\ell_1}$). The sets $\psi_s(u)$ are non-empty and disjoint. If $s > t$, $\partial I_s = \cup_{u \in \partial I_t} \psi_s(u)$; if $s < t$, we let $R_s = \partial I_s \setminus (\cup_{u \in \partial I_t} \psi_s(u))$. We have

$$|F(t) - F(s)| \leq \sum_{u \in \partial I_t} |\text{Hm}_t(v, u)f(u) - \sum_{u' \in \psi_s(u)} \text{Hm}_s(v, u')f(u')| + 1_{s < t} \sum_{u' \in R_s} \text{Hm}_s(v, u')f(u').$$

Since $I_t$ is compact, $M = \max\{f(u) : u \in I_t\} < \infty$. Since $\partial I_t$ is finite, for any $\epsilon > 0$ there exists $0 < \delta_2 \leq \delta_1/2$ such that $|f(u) - f(u')| < \epsilon/2$ for $u \in \partial I_t$ and $u' \in B_{\ell_1}(u, \delta_2)$. Thus, for $|t - s| < \delta_2$

$$|F(t) - F(s)| \leq M \sum_{u \in \partial I_t} |\text{Hm}_t(v, u) - \text{Hm}_s(v, \psi_s(u))| + 1_{s < t} M \text{Hm}_s(v, R_s) + \frac{\epsilon}{2}.$$

Finally, it follows from the construction of $\tilde{B}_t$ (by considering the excursions of a standard Brownian motion) that for any $u \in \tilde{G}$, $b \leq D_{\ell_1}(u, V \setminus \{u\})$, and $u' \in \tilde{G}$ such that $a = |u - u'|_{\ell_1} \leq b$,

$$\text{Hm}(u, u'; \{u'\} \cup \partial B_{\ell_1}(u, b)) \geq \frac{b}{b + (2d - 1)a}.$$

Combining this with the previous bound it follows from a straightforward calculation that there exists $\delta_3 \leq \delta_2$ such that if $|t - s| < \delta_3$,

$$|F(t) - F(s)| \leq \epsilon.$$

\[3\] **Percolation in three and higher dimensions**

In the case $d \geq 3$, we let the vertex set $V = \mathbb{Z}^d$ be the whole lattice and study the behavior of $p_{N,h} = \mathbb{P}(\bf{0} \xrightarrow{\geq h} \partial V_N^d)$ as $N \to \infty$. The main idea of the proof, encapsulated in the following proposition, is to apply Theorem 14 to the exploration martingale with source set $I_0 = \{\bf{0}\}$ and target set $A = \{kN\bf{e}_1\}$, where $\bf{e}_1 = \bf{1}_{j=1}$ (i.e. $\bf{e}_1$ is the standard basis vector on the first coordinate axis) and $k > 1$ is a constant. This will allow us to relate the harmonic measure of the explored set at the time the exploration stops (or hits $\partial V_N$) to the time it takes a Brownian motion with drift to fall below a certain level. For ease of notation we will write $M_{kN}$ for the exploration martingale instead of $M_{kN\bf{e}_1}$, and similarly for $\sigma_{kN}^2$ (as introduced in Proposition 14). Recall $\sigma_d^2 = G(\bf{0}, \bf{0})$ and note that by translation invariance $\sigma_d^2 = G(u, u)$ for all $u \in \mathbb{Z}^d$. 

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Proposition 13. Let $g_{kN,N}^{\pm}$ be as in Proposition 12 and $s = \text{sign}(h)$. Define the stopping times $\tau_{N,h}^{\pm}$ by

$$
\tau_{N,h}^{\pm} = \inf \left\{ t : B_t \leq \frac{h}{g_{kN,N}^{\pm}} t - (\tilde{\phi}_0 - h) \frac{g_{kN,N}^{\pm}}{\sigma_d^2} \right\},
$$

where $B$ is a Brownian motion such that $B_t = M_{kN,T_t} - M_{kN,0}$ for $t < \langle M_{kN} \rangle_\infty$ (Recall $T_t = \inf\{ T : \langle M_{kN} \rangle_T \geq t \}$). Additionally, let

$$
\pi_{k,N} = \inf_{\mathcal{T}} \{ \pi(\text{ker} \{kN\mathbf{e}_1, \mathcal{T}\}) - \pi(kN\mathbf{e}_1, 0), \quad \pi_{k,N}^+ = \pi(kN\mathbf{e}_1, V_N) - \pi(kN\mathbf{e}_1, 0),
$$

where the infimum is taken over all closed, connected sets $\mathcal{T}$ containing $0$ and intersecting $\partial V_N$. Then the following inclusions hold

$$
\{ \tau_{N,h}^- > g_{kN,N}^{\pm} \pi_{k,N}^+ \} \subseteq \{ 0 \leftrightarrow_{\partial V_N} \} \subseteq \{ \tau_{N,h}^+ \geq g_{kN,N} \pi_{k,N}^- \}.
$$

Proof. Trivially, if $\tilde{\phi}_0 \leq h$ then almost surely $\tau_{N,h}^- = \tau_{N,h}^+ = 0$ and $0 \leftrightarrow_{\partial V_N} \partial V_N$. Therefore, we assume $\tilde{\phi}_0 > h$. Recalling (17), we let $\eta_h = \inf\{ t : M_{kN,t} = h\pi(kN\mathbf{e}_1, \mathcal{T}_t) \}$ and note that for any $t \geq 0$

$$
M_{kN,t} - M_{kN,0} \geq h[\pi(kN\mathbf{e}_1, \mathcal{T}_t) - \pi(kN\mathbf{e}_1, 0)] - (\tilde{\phi}_0 - h)\pi(kN\mathbf{e}_1, 0),
$$

with equality if and only if $t \geq \eta_h$, in which case $M_{kN,t} = M_{kN,\eta_h}$. Noting $G(v,u) = \pi(v,u)G(u,u)$, we have

$$
\frac{g_{kN,N}^-}{\sigma_d^2} \leq \pi(kN\mathbf{e}_1, 0) \leq \frac{g_{kN,N}^+}{\sigma_d^2}.
$$

Applying Proposition 12 we obtain

$$
M_{kN,t} - M_{kN,0} > \frac{h}{g_{kN,N}^+} \langle M_{kN} \rangle_t - (\tilde{\phi}_0 - h) \frac{g_{kN,N}^+}{\sigma_d^2}, \quad 0 \leq t \leq \eta_h \land D_h(0, \partial V_N).
$$

Or equivalently, $\tau_{N,h}^+ \geq \langle M_{kN} \rangle_{D_h(0, \partial V_N)}$. By Proposition 12 we also have $\{ 0 \leftrightarrow_{\partial V_N} \} \subseteq \{ \langle M_{kN} \rangle_{D_h(0, \partial V_N)} \geq g_{kN,N} \pi_{k,N}^- \}$ so it follows that $\{ 0 \leftrightarrow_{\partial V_N} \} \subseteq \{ \tau_{N,h}^+ \geq g_{kN,N} \pi_{k,N}^- \}$. For the other inclusion, Proposition 12 and (21) imply that on $\{ 0 \leftrightarrow_{\partial V_N} \}$,

$$
M_{kN,\eta_h} - M_{kN,0} \leq \frac{h}{g_{kN,N}^-} \langle M_{kN} \rangle_{\eta_h} - (\tilde{\phi}_0 - h) \frac{g_{kN,N}^-}{\sigma_d^2},
$$

and $\langle M_{kN} \rangle_{\eta_h} \leq g_{kN,N} \pi_{k,N}^+$. This implies $\{ 0 \leftrightarrow_{\partial V_N} \} \subseteq \{ \tau_{N,h}^- \leq \langle M_{kN} \rangle_{\eta_h} \leq g_{kN,N} \pi_{k,N}^+ \}$, completing the proof of the proposition. \qed
In order to use Proposition 13 we need bounds on $g_{k,N,N} \pm\$, $\pi_{k,N} \pm\$, and to understand the distribution of $\tau_{N,h} \pm\$. The following proposition (c.f. [4, Equation 2.0.2 in Part II]) gives the distribution of $\tau_{N,h} \pm\$.

**Proposition 14.** For $m \in \mathbb{R}$ and $b > 0$, let $\tau = \inf\{t > 0 : B_t \leq mt - b\}$. Then
\[
\mathbb{P}(\tau \leq T) = \Phi\left(\frac{b}{\sqrt{T}} - m\sqrt{T}\right) + e^{2bm}\Phi\left(\frac{b}{\sqrt{T}} + m\sqrt{T}\right), \text{ for any } T > 0.
\]

Since it is clear that $\mathbb{P}(\tau \leq T)$ is continuous in $T$, in what follows we will not distinguish between $\mathbb{P}(\tau \leq T)$ and $\mathbb{P}(\tau < T)$. The following proposition is an immediate consequence of [10, Theorem 4.3.1].

**Proposition 15.** For $d \geq 3$ and $k \geq 2$, there exist $c_{d,k}^-, c_{d,k}^+$ where $c_{d,k}^- \leq c_{d,k}^+$ that is increasing in $k$ and $c_{d,k}^+$ is decreasing, such that
\[
\lim_{k \to \infty} c_{d,k}^- = \lim_{k \to \infty} c_{d,k}^+ := c_d \text{ and } g_{k,N,N} = \frac{c_{d,k}^+}{(kN)^{d-2}} \left(1 + O\left(\frac{1}{N}\right)\right),
\]
where the suppressed constant depends only on $d$.

Note that one can obtain explicit formulas for $c_d$ and $c_{d,k}^+$ from [10, Theorem 4.3.1], but we will not need these. The proof of the next lemma is postponed until Section 3.3.

**Lemma 16.** For $d \geq 3$ and $k \geq 2$, $\pi_{k,N}$ satisfies the following
\[
\pi_{k,N}^- = \begin{cases} 
\Omega\left(\frac{Ng_{k,N,N}}{\log N}\right), & d = 3, \\
\Omega\left(\frac{Ng_{k,N,N}}{\log N}\right), & d > 3,
\end{cases}
\]
where the suppressed constant depends only on $d$.

For $\pi_{k,N}^+$ we note the trivial bound $\pi_{k,N}^+ \leq 1$.

### 3.1 Proof of main theorems

In this subsection we prove Theorems 3, 4, and 5. In each case, we apply Proposition 14 to $\tau_{N,h} \pm\$ with appropriate values for $T$, $m$, and $b$.

**Proof of Theorem 3** Since we are considering $p_{-h,N}$ with $h > 0$, we have $s = -1$. For the lower bound, we wish to apply Proposition 14 with $T = g_{k,N,N}^+, b = (\tilde{\phi}_0 + h)g_{k,N,N}^-/\sigma_d^2$, and $m = -h/g_{k,N,N}^+$. By Proposition 15 we have for fixed $k$, as $N \to \infty$
\[
\frac{b}{\sqrt{T}} \to 0, \quad m\sqrt{T} \to -\infty, \quad bm \to -\frac{h(\tilde{\phi}_0 + h)c_{d,k}^-}{\sigma_d^2 c_{d,k}^+}, \quad \text{a.s.}
\]
Therefore, Proposition 14 gives that
\[
\lim_{N \to \infty} P(\tau_{-h,N} \geq g_{k,N,N}^+ | F_0) = \left[ 1 - \exp \left( -2 \frac{h(\tilde{\phi}_0 + h)c_{d,k}^-}{\sigma_d^2 c_{d,k}^+} \right) \right] \mathbb{1}_{\tilde{\phi}_0 > -h} \quad \text{a.s.}
\]
By dominated convergence,
\[
\lim_{N \to \infty} P(\tau_{-h,N} \geq g_{k,N,N}^+) = \mathbb{E} \left[ \left( 1 - \exp \left( -2 \frac{h(\tilde{\phi}_0 + h)c_{d,k}^-}{\sigma_d^2 c_{d,k}^+} \right) \right) \mathbb{1}_{\tilde{\phi}_0 > -h} \right].
\]
By Proposition 13, \(c_{d,k}^-/c_{d,k}^+ \to 1\) as \(k \to \infty\), and by Proposition 13, \(P(\tau_{-h,N} \geq g_{k,N,N}^+)\) for any \(k > 1\) so we conclude
\[
\lim_{N \to \infty} p_{-h,N} \geq \mathbb{E} \left[ \left( 1 - \exp \left( -2 \frac{h(\tilde{\phi}_0 + h)}{\sigma_d^2} \right) \right) \mathbb{1}_{\tilde{\phi}_0 > -h} \right].
\]
The upper bound follows from a similar argument by applying Proposition 14 with \(T = g_{k,N,N}^+ \pi_{k,N}, b = (\tilde{\phi}_0 + h)g_{k,N,N}^+ / \sigma_d^2\), and \(m = -h/g_{k,N,N}^+\). Note that Proposition 15 and Lemma 16 imply that \(\pi_{k,N} \to \infty\) as \(N \to \infty\), so we have as before
\[
\frac{b}{\sqrt{T}} \to 0, \quad m\sqrt{T} \to -\infty, \quad bm \to -\frac{h(\tilde{\phi}_0 + h)c_{d,k}^+}{\sigma_d^2 c_{d,k}^-}, \quad \text{a.s.}
\]
From here the argument is exactly the same as for the lower bound. \qed

Proof of Theorem 4. The idea, as in the supercritical case, is to apply Proposition 14. For the rest of the section, we fix \(k = 2\) and write \(g_N^\pm\) for \(g_{2,N}^\pm\) and \(\pi_N^-\) for \(\pi_{2,N}^-\). Adopting the notation of Proposition 14, we have
\[
P(\tau \geq T) \leq \Phi \left( m\sqrt{T} - \frac{b}{\sqrt{T}} \right) \leq \exp \left( -\frac{m^2 T}{2} + bm \right).
\]
Letting \(\tau = \tau_{N,h}^+, T = g_{N}^- \pi_{N}^-, b = (\tilde{\phi}_0 - h)g_{N}^+ / \sigma_d^2\), and \(m = h/g_{N}^+\) (the flipped signs in comparison with the supercritical case are due to the fact that \(s = 1\) in the subcritical regime) we get from Proposition 15 and Lemma 16
\[
m^2 T = \frac{h^2 g_{N}^- \pi_{N}^-}{(g_{N}^+)^2} = \Omega \left( \frac{h^2 N}{(\log N)^{2d=3}} \right).
\]
Noting that \(\exp(bm) = \exp(h(\tilde{\phi}_0 - h)/\sigma_d^2)\) has finite first moment and recalling \(p_{N,h} \leq P(\tau_{N,h}^+ \geq g_{N}^- \pi_{N}^-)\) we conclude
\[
p_{N,h} = \exp \left[ -\Omega \left( \frac{h^2 N}{(\log N)^{2d=3}} \right) \right]. \quad \square
Proof of Theorem 5. For the lower bound, we apply Proposition 14 with $T = g_N^+, b = \tilde{\phi}_0 g_N/\sigma_d^2$, and $m = 0$ to obtain

$$p_{0,N} \geq \mathbb{P}(\tau_{0,N} \geq g_N) = \mathbb{E} \left[ \left( \Phi \left( \frac{b}{\sqrt{T}} \right) - \Phi \left( -\frac{b}{\sqrt{T}} \right) \right) I_{\tilde{\phi}_0 > 0} \right] = \Omega \left( \frac{1}{N^{d/2 - 1}} \right),$$

where we have used the fact that $b/\sqrt{T} = \Theta(\tilde{\phi}_0 N^{-d/2+1})$. For the upper bound we let $T = g_N \pi N^{-d/2}$, $b = \tilde{\phi}_0 g_N/\sigma_d^2$, and again $m = 0$. This gives

$$p_{0,N} \leq \mathbb{P}(\tau_{0,N} \geq g_N \pi N) = \mathbb{E} \left[ \left( \Phi \left( \frac{b}{\sqrt{T}} \right) - \Phi \left( -\frac{b}{\sqrt{T}} \right) \right) I_{\tilde{\phi}_0 > 0} \right] = O \left( \sqrt{\log N} / N^{d/3} \right),$$

where we have used the fact that $b/\sqrt{T} = \Theta(\tilde{\phi}_0 \sqrt{(\log N)^{d/3}}/N)$ (by Proposition 15 and Lemma 16).

3.2 Critical window in three dimensions

In this section we prove Theorem 6. That is, we give rates of decay for $h$ (now considered as a function of $N$) such that $p_{\pm h,N}$ is of the same order as $p_{0,N}$. We will only consider the case $d = 3$ in this subsection. Throughout, we let $h_N > 0$ be a sequence such that $h_N \to 0$. To simplify notation, we will write $\sigma^2$ for $\sigma_3^2$, $p_{\pm N,h}$ for $p_{\pm h_N,N}$, and similarly with other quantities. Additionally, we write $(a)_+$ for $\max\{a,0\}$ and will use $(\tilde{\phi}_0 \mp h_N)_+$ instead of $(\tilde{\phi}_0 \mp h_N)$ when applying Proposition 13 to avoid writing $\Phi(1_{\tilde{\phi}_0 > \pm h_N})$ when taking expectations.

We first prove (11). Letting $T = g_N^+, b = (\tilde{\phi}_0 - h_N)_+ g_N/\sigma^2$, and $m = h_N/g_N^-$ we have from Propositions 13 and 14 that

$$p_{\pm N}^+ \geq \mathbb{P}(\tau_N^- \geq g_N^+) = \mathbb{E} \left[ \tilde{\Phi} \left( m \sqrt{T} - \frac{b}{\sqrt{T}} \right) - e^{2bm} \tilde{\Phi} \left( m \sqrt{T} + \frac{b}{\sqrt{T}} \right) \right]. \quad (23)$$

To bound the right hand side of the preceding inequality we use the following lemma.

Lemma 17. Let $x \in \mathbb{R}$ and $y \geq 0$. Define $f$ and $h$ by

$$f(x,y) = \tilde{\Phi}(x - y) - e^{2xy} \tilde{\Phi}(x + y) \quad \text{and} \quad h(x,y) = 1 - \frac{x \tilde{\Phi}(x + y)}{\varphi(x + y)}.$$
We have \( h(x, y) \geq 0 \) and
\[
2h(x, 0) [\Phi(x) - \Phi(x-y)] \leq f(x, y) \leq 2h(x, y) [\Phi(x) - \Phi(x-y)] \quad x \geq 0,
\]
\[
2h(x, y) [\Phi(x) - \Phi(x-y)] \leq f(x, y) \leq 2h(x, 0) [\Phi(x) - \Phi(x-y)] \quad x \leq 0.
\]

Proof. It is clear that \( h(x, y) > 0 \) for \( x \leq 0 \). For \( x > 0 \), the fact that \( h(x, 0) \geq 0 \) is equivalent to the well-known (and straightforward to check) bound \( \bar{\Phi}(x) \leq \varphi(x)/x \) for all \( x > 0 \) and it directly implies \( h(x, y) \geq h(x + y, 0) \geq 0 \).

To prove (24), note that
\[
\frac{\partial f}{\partial y}(x, y) = 2\varphi(x-y) - 2xe^{2xy}\bar{\Phi}(x+y) = 2h(x, y)\varphi(x-y).
\]
Using the fact that \( \bar{\Phi}(x)/\varphi(x) \) is decreasing in \( x \) (for all values of \( x \)) we conclude that \( h(x, y) \) is decreasing in \( y \) for \( x < 0 \) and increasing in \( y \) for \( x > 0 \). The desired bounds follow by integrating \( \partial f/\partial y \). For instance, for \( x > 0 \) we have
\[
f(x, y) = \int_0^y \frac{\partial f}{\partial y}(x, s)ds \leq 2h(x, y) \int_0^y \varphi(x-s)ds = 2h(x, y) [\Phi(x) - \Phi(x-y)].
\]
The other three bounds follow by similar arguments.

Note now that \( h_N = O(N^{-1/2}) \) implies \( m\sqrt{T} = O(1) \). Since \( b/\sqrt{T} = \Theta((\tilde{\varphi}_0 - h_N)_{+}N^{-1/2}) \), recalling (23) and applying Lemma 17 with \( x = m\sqrt{T} \) and \( y = b/\sqrt{T} \) gives
\[
p_N^+ \geq 2h(m\sqrt{T}, 0) \left( \Phi \left( m\sqrt{T} \right) - \mathbb{E} \left[ \Phi \left( m\sqrt{T} - \frac{b}{\sqrt{T}} \right) \right] \right) = \Omega \left( \frac{1}{\sqrt{N}} \right),
\]
which proves (11).

We next prove (12). Let \( T = g_N \pi_N \), \( b = (\tilde{\varphi}_0 - h_N)_{+}g_N^+/\sigma^2 \), and \( m = h_N/g_N^+ \). We have as before
\[
p_N^+ \leq \mathbb{P}(\tau_N^+ \geq g_N \pi_N) \leq 2 \mathbb{E} \left[ h \left( m\sqrt{T}, \frac{b}{\sqrt{T}} \right) \left[ \Phi \left( m\sqrt{T} \right) - \Phi \left( m\sqrt{T} - \frac{b}{\sqrt{T}} \right) \right] \right]
\leq \mathbb{E} \left[ \frac{2b}{\sqrt{T}} \varphi \left( \left( m\sqrt{T} - \frac{b}{\sqrt{T}} \right)_{+} \right) \right]
\leq \mathbb{E} \left[ \frac{b}{\sqrt{T}} e^{mb} \right] e^{-m^2T/2},
\]
Under the assumption \( \liminf_{N \to \infty} \frac{h_N\sqrt{N}}{\sqrt{\log N \log \log N}} \geq C \) for a large enough constant \( C \), we have \( e^{-m^2T/2} = e^{-\Omega(h_N^2N/\log N)} = o(\sqrt{\log N}) \). Since \( b/\sqrt{T} = O((\tilde{\varphi}_0 - h_N)_{+}\sqrt{\log N/N}) \) and \( mb = O(h_N(\tilde{\varphi}_0 - h_N)_{+}) \) we conclude \( p_N^+ = o \left( \frac{1}{\sqrt{N}} \right) \) as required for (12).
We next bound \( p_N^- \) by a similar argument. For the upper bound, let \( T = g_N^- \pi_N^- \), \( b = (\tilde{\phi}_0 + h_N) + g_N^+ / \sigma^2 \), and \( m = -h_N / g_N^- \). We have as above

\[
p_N^- \leq \mathbb{P}(\tau_N^+ \geq g_N^+ \pi_N^-) \leq 2h(m\sqrt{T}, 0)\mathbb{E}\left[\left(\Phi\left(m\sqrt{T}\right) - \Phi\left(m\sqrt{T} - \frac{b}{\sqrt{T}}\right)\right)\right]
\]

\[
\leq 2[\varphi(m\sqrt{T}) - m\sqrt{T}]\mathbb{E}\left[\frac{b}{\sqrt{T}}\right].
\]

If \( h_N = O(\sqrt{\log N/N}) \), then \(-m\sqrt{T} = O(h_N \sqrt{N/\log N}) = O(1)\). Since \( b/\sqrt{T} = O((\tilde{\phi}_0 + h_N) + \sqrt{\log N/N}) \) we conclude \( p_N^- = O\left(\frac{\log N}{N}\right) \), as required for (13).

Conversely, letting \( T = g_N^+ \), \( b = (\tilde{\phi}_0 + h_N) + g_N^+ / \sigma^2 \), and \( m = -h_N / g_N^+ \) we have

\[
p_N^- \geq \mathbb{P}(\tau_N^+ \geq g_N^+) \geq \mathbb{E}\left[2h\left(m\sqrt{T}, \frac{b}{\sqrt{T}}\right)\left(\Phi\left(m\sqrt{T}\right) - \Phi\left(m\sqrt{T} - \frac{b}{\sqrt{T}}\right)\right)\right]
\]

\[
\geq \mathbb{E}\left[-2mb e^{2mb} \Phi\left(m\sqrt{T} + \frac{b}{\sqrt{T}}\right)\right].
\]

If \( h_N = \omega(\sqrt{\log N/N}) \) it follows \(-mb = o((\tilde{\phi}_0 + h_N) + \sqrt{\log N/N}) \) and \(-m\sqrt{T} = \omega(\sqrt{\log N}) \). Since \( b/\sqrt{T} = O((\tilde{\phi}_0 + h_N) + N^{-1/2}) \) and \(-mb = o((\tilde{\phi}_0 + h_N) +) \), taking expectation with respect to \( \tilde{\phi}_0 \) gives \( p_N^- = \Omega(h_N) = \omega\left(\frac{\log N}{N}\right) \), as required for (14).

### 3.3 Proof of Lemma 16

By Lemma 15 and 22 we have \( g_{k,N}^- = \Theta(g_{k,N}^+) \) and \( \pi(kNe_1, 0) = \Theta(g_{k,N}^+) \), so it suffices to show

\[
\inf_{\mathcal{I}} \pi(kNe_1, \mathcal{I}) = \Omega\left(\frac{Ng_{k,N}^-}{(\log N)^{d-3}}\right),
\]

where the infimum is taken over all closed, connected sets \( \mathcal{I} \) containing the origin \( 0 \) and intersecting \( \partial V_N \). Any such set \( \mathcal{I} \) contains a sequence of lattice points \( U_N = \{u_j\}_{j=0}^N \) such that \( |u_j|_\infty = j \). We will bound \( \pi(kNe_1, \mathcal{U}_N) \) by the second moment method. Let \( \tau_j = \inf\{t > 0 : B_t = u_j\} \) be the hitting time of \( u_j \) by a Brownian motion on \( \bar{G} \), and let \( Y_j = 1_{\tau_j < \infty} \) be the indicator that \( u_j \) is hit in finite time. Let \( S_N = \sum_{j=0}^N Y_j \) be the number of points in \( U_N \) that are hit, where the distribution of \( S_N \) is with respect to a Brownian motion \( B_t \) on \( \bar{G} \) with \( B_0 = kNe_1 \). For the first moment we have (using a bound similar to (22), where we replace \( 0 \) with \( u_j \))

\[
\mathbb{E}[S_N] = \sum_{j=0}^N \pi(kNe_1, u_j) = \Omega(Ng_{k,N}^-).
\]

(25)
For the second moment we have by the same reasoning
\[ E[S_N^2] = \sum_{i,j=0}^{N} \pi(kNe_1, \{u_i, u_j\}) \pi(u_i, u_j) = O \left( g_{kN,N}^+ \sum_{i,j=0}^{N} \pi(u_i, u_j) \right). \]

Let \( \pi^*_d,j = \sup \{ \pi(0, u) : |u| \geq j \} \). Using [10, Theorem 4.3.1] again we obtain
\[ \sum_{j=0}^{N} \pi^*_d,j \leq 2(N + 1) \sum_{j=0}^{N} \pi^*_d,j = O \left( N (\log N)^{1/d-3} \right). \]

Therefore, \( E[S_N^2] = O \left( g_{kN,N}^+ N (\log N)^{3/d-3} \right) \). Combined with (25) and Proposition 15 we complete the proof by a straightforward application of second moment method (note that \( \pi(kNe_1, U_N) = P(S_N > 0) \geq E[S_N^2] \)).

4 Chemical distance in two dimensions

This section is devoted to the proof of Theorem 1. Recall that \( \tilde{\phi}_N \) is the Gaussian free field on the metric graph of \( V_N \) with Dirichlet boundary conditions, and that \( G_N \) is Green’s function on \( V_N \) as in [2]. The proof employs the same type of exploration martingale as in the case of \( d \geq 3 \). Below we prove Theorem 1 while postponing proofs of a few lemmas to later subsections.

**Proof of Theorem 1.** For \( h \in \mathbb{R} \) and \( 0 < \alpha < \gamma < 1 \), define \( E_{N,1} = D_{N,h}(V_\alpha N, \partial V_\gamma N) < \infty \). That is, \( E_{N,1} \) is the event that \( V_\alpha N \) is connected to \( \partial V_\gamma N \) in \( \tilde{E}_N \geq h \).

**Lemma 18.** We have
\[ c_1 = \inf \{ P(E_{N,1}) : N \geq 1 \} > 0, \]
where \( c_1 \) depends on \( h, \alpha, \) and \( \gamma. \)

**Remark 19.** Despite the fact that the statement of Lemma 18 is formally slightly stronger than [6, Proposition 4] (since percolation on metric graph is a sub-event of percolation on discrete lattice), the proof of [6, Proposition 4] adapts with essentially no change. Thus, we omit further details of the proof.

Now, let \( \mu = (1 + \gamma)/2 \) and \( M_{\mu N} \) be the exploration martingale with target set \( \partial V_\mu N \) and source set \( I_0 = V_\alpha N, \) as defined in [15]. That is to say,
\[ X_{\mu N} = \frac{1}{|\partial V_\mu N|} \sum_{v \in \partial V_\mu N} \tilde{\phi}_N,v \] and \( M_{\mu N,t} = E[X_{\mu N} | \mathcal{F}_t], \)

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where $\mathcal{I}_t = \{ v \in V_N : D_{N,h}(V_{\alpha N}, v) \leq t \}$. From now on, we take $N$ large enough that the boxes $V_{\alpha N}, V_{\beta N}, V_{\gamma N}, V_{\mu N}$ are distinct. For $t \geq 0$ we let $\partial \mathcal{I}_t^+ = \partial \mathcal{I}_t \cap \tilde{E}_N^{>h}$ be the points on $\partial \mathcal{I}_t$ where $\tilde{\phi}$ is strictly above $h$, which we will refer to as the active points at time $t$ (by active here we mean that these are the points from which the metric ball exploration can proceed further), and let $\partial \mathcal{I}_t^- = \partial \mathcal{I}_t \setminus \tilde{E}_N^{\geq h}$ be the points on $\partial \mathcal{I}_t$ where $\tilde{\phi}$ is less than $h$ (note that $\partial \mathcal{I}_t^- = \partial \mathcal{I}_0^-$ for all $t$). We then define the “positive” and “negative” parts of $M_{\mu N}$ (which we denote by $M_{\mu N}^\pm$) as

$$M_{\mu N,t}^\pm = \sum_{u \in \partial \mathcal{I}_t^\pm} H_{\mu,N,t}(\partial V_{\mu N}, u) (\tilde{\phi}_u - h),$$

where $H_{\mu,N,t}(u, v) = H_m(u, v; I_t \cup \partial V_N)$. For $c \in \mathbb{R}$, define

$$\mathcal{E}_{N,2}(c) = \{ M_{\mu N,t}^+ \geq c \ \text{for all } 0 \leq t \leq D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) \}.$$

**Lemma 20.** There exists a constant $c_2 = c_2(h, \beta, \gamma) > 0$ such that

$$\mathbb{P}(\mathcal{E}_{N,2}(c_2 \epsilon) \mid \mathcal{E}_{N,1}) \geq 1 - \epsilon$$

for all $\epsilon > 0$.

For the rest of the section we let $\mathcal{E}_{N,2} = \mathcal{E}_{N,2}(c_2 \epsilon/2)$ for convenience. The core idea in proving Theorem 1 is to bound from below the rate at which the quadratic variation increases as a function of $M_{\mu N,t}^+$. Combined with an upper bound on $\langle M_{\mu N} \rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})}$, this then yields an upper bound on $D_{N,h}(V_{\alpha N}, \partial V_{\beta N})$. In order to carry out the proof, we first give the upper bound on the quadratic variation of $M_{\mu N}$ (which is easier than the lower bound).

**Lemma 21.** [6, Lemma 2] For $0 < \mu < 1$, there exist constants $c, c' > 0$ such that

$$\sum_{v \in \partial V_{\mu N}} G_N(u, v) \leq cN, \ \forall u \in \partial V_{\mu N};$$

$$G_N(u, v) \geq c', \ \forall u, v \in V_{\mu N}. \tag{26}$$

By (19) and (27), we get that for some constant $c_3 > 0$ which depends on $\mu$

$$\langle M_{\mu N} \rangle_{\infty} \leq \frac{1}{|\partial V_{\mu N}|^2} \sum_{v,v' \in \partial V_{\mu N}} G(v, v') \leq c_3. \tag{28}$$

The remaining main task for proving Theorem 1 is to show that on some event $\mathcal{E}_{N}^*$ with $\mathbb{P}(\mathcal{E}_{N}^* \mid \mathcal{E}_{N,1}) \geq 1 - \epsilon$, we have

$$\langle M_{\mu N} \rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})} \geq \kappa \frac{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})^2}{N^2 \sqrt{\log N}}, \tag{29}$$

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for some constant $\kappa = \kappa(\epsilon, \alpha, \beta, \gamma, h) > 0$. Indeed, assuming (29), we can then combine it with (28), and conclude that on $E_N^*$

$$D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) \leq c_3 \kappa^{-1/2} N (\log N)^{1/4},$$

completing the proof of Theorem 1.

It remains to show (29). To this end, we first bound the quadratic variation from below in terms of the $\ell_2$-norm of the harmonic measure on the active points, as in the next lemma.

**Lemma 22.** There exists a constant $c_4 > 0$ such that for any integer $K \geq 1$,

$$\langle M_{\mu N} \rangle_K \geq c_4 \sum_{k=1}^K \sum_{u \in \partial I^+_k} H_{N,k}(\partial V_{\mu N}, u)^2.$$

Let $I^+ = \bigcup_{k=1}^{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})} \partial I^+_k$. Since the sets $\{\partial I^+_k : k \geq 1\}$ are disjoint, for $u \in I^+$ we can define

$$W(u) = H_{N,k}(\partial V_{\mu N}, u),$$

where $k$ is the unique positive integer such that $u \in \partial I^+_k$. We rewrite the conclusion of Lemma 22 as

$$\langle M_{\mu N} \rangle_{D_{N}(V_{\alpha N}, \partial V_{\beta N})} \geq c_4 \sum_{u \in I^+} W(u)^2. \quad (30)$$

In order to bound the right-hand side of (30) from below by $M_{\mu N}^+$, we need some control on the empirical profile of $\{\hat{\phi}_{N,u} : u \in I^+\}$. To this end, we define

$$B_0 = \{u \in I^+ : \hat{\phi}_{N,u} - h \leq \sqrt{\log N}\},$$

$$B_j = \{u \in I^+ : 2^{j-1} \sqrt{\log N} < \hat{\phi}_{N,u} - h \leq 2^j \sqrt{\log N}\}, \quad j \geq 1.$$

Here the scale $\sqrt{\log N}$ is chosen to match the order of $\sqrt{E[\hat{\phi}_u^2]}$ for $u \in V_{\beta N}$. Letting $W_j = \sum_{u \in B_j} W(u)$, we get from the Cauchy-Schwartz inequality that

$$\sum_{u \in I^+} W(u)^2 \geq \sum_{j=0}^{\infty} \frac{W_j^2}{|B_j|}, \quad (31)$$

where we use the convention $0/0 = 0$. The appearance of $|B_j|$ in the denominator in the preceding inequality calls for an upper bound on $|B_j|$, as incorporated in the next lemma (the reason for the specific form of the bound will be made clear below).
**Lemma 23.** Let $f$ be a function on the positive integers such that $f(j) = O(c^j)$ for some positive constant $c$. Then there exists $c_5 = c_5(\alpha, \beta, h, c) > 0$ such that

$$\sup\{f(j)\mathbb{E}[|B_j|] : j \geq 0\} \leq c_5 \frac{N^2}{\sqrt{\log N}}.$$  

We are now ready to give a lower bound on $\langle M_{\mu N} \rangle$. By definition of $W_j$, we have

$$W_j \geq 2^{-j}(\log N)^{-\frac{1}{2}} \sum_{u \in B_j} W(u)(\hat{\phi}_{N,u} - h).$$

In addition, on the event $\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2}$ we have

$$\sum_{j=0}^{\infty} \sum_{u \in B_j} W(u)(\hat{\phi}_{N,u} - h) = \sum_{u \in \mathcal{I}^{+}} W(u)(\hat{\phi}_{N,u} - h) \geq c_2 c_5 D_{N,h}(V_{\alpha N}, \partial V_{\beta N}).$$

Letting $c_6 = 6/\pi^2$ so that $c_6 \sum_{j=0}^{\infty} (j + 1)^{-2} = 1$, we see that

$$\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2} \subseteq \bigcup_{j=0}^{\infty} \left\{ W_j \geq c_6 c_2 c_5 D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) \right\}. \quad (32)$$

Letting $c_7 = 20/(c_1 \epsilon)$, we define $\mathcal{E}_{N,3} = \bigcap_{j=0}^{\infty} \mathcal{E}_{N,3,j}$, where

$$\mathcal{E}_{N,3,j} = \{|B_j| \leq c_7 \mathbb{E}[|B_j|](1.1)^{j+1}\}, \quad j \geq 0.$$  

By Markov’s inequality, we get that

$$\mathbb{P}(\mathcal{E}_{N,3}^c \mid \mathcal{E}_{N,1}) \leq \frac{\mathbb{P}(\mathcal{E}_{N,3}^c)}{c_1} \leq \frac{\epsilon}{2}. \quad (33)$$

Let $\mathcal{E}_N^* = \mathcal{E}_{N,1} \cap \mathcal{E}_{N,2} \cap \mathcal{E}_{N,3}$. By Lemma 20 and (33), we get that

$$\mathbb{P}(\mathcal{E}_N^* \mid \mathcal{E}_{N,1}) \geq 1 - \epsilon. \quad (34)$$

We deduce from (30), (31) and (32) that on $\mathcal{E}_N^*$

$$\langle M_{\mu N} \rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})} \geq \frac{c_6^2 c_4 c_5^2 c_1 \epsilon^3}{20} \inf_{j \geq 0} (4.4)^{j+1}(j+1)^4 \mathbb{E}[|B_j|] \log N.$$  

Combined with Lemma 23 this gives that on $\mathcal{E}_N^*$

$$\langle M_{\mu N} \rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})} \geq \frac{c_6^2 c_4 c_5^2 c_1 \epsilon^3}{20} \frac{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})^2}{N^2 \sqrt{\log N}}.$$  

Combining with (34), we have completed the verification of (29) as promised. \hfill \square
4.1 Proof of Lemma \([20]\)

We first give the main intuition behind the proof of Lemma \([20]\) in the case when \(h = 0\). On the event \(\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2}(\epsilon)^c\), we have \(M_{\mu N,s} \leq \epsilon + M_{\mu N,s}^-\) for some \(s \leq D_{N,h}(V_{\alpha N}, \partial V_{\beta N})\). However, we also have \(M_{\mu N,t} \geq M_{\mu N,s}^-\) for all \(t \geq s\). Since \(D_{N,h}(V_{\alpha N}, \partial V_{\beta N}) < \infty\) on \(\mathcal{E}_{N,1}\), the martingale must stay above \(M_{\mu N,s}^-\) after time \(s\) and yet accumulate an order 1 amount of quadratic variation — this happens with small probability. The case for general \(h\) is similar but a bit more complicated. We carry out a detailed proof below.

In this subsection and the ones that follow, we let \(c > 0\) be an arbitrary constant whose value may change each time it appears, and which may depend on \(h, \alpha, \beta, \gamma\) but not on \(N\). Since \(M_{\mu N,t}^-\) is increasing in \(t\) (recalling \(\partial I_t^- = \partial I_0^-\) for all \(t\)), we have that for \(0 \leq s < t < D_{N,h}(V_{\alpha N}, \partial V_{\mu N})\)

\[
M_{\mu N,t}^- - M_{\mu N,s}^- \geq h[\pi_N(\partial V_{\mu N}, \mathcal{I}_t) - \pi_N(\partial V_{\mu N}, \mathcal{I}_s)] - M_{\mu N,s}^+.
\]

with equality if and only if \(M_{\mu N,t}^+ = M_{\mu N,s}^+ = 0\). Additionally, we obtain from Proposition \([12]\) and Lemma \([21]\) that for some constants \(c, c'\) and any \(0 \leq s < t < D_{N,h}(V_{\alpha N}, \partial V_{\mu N})\),

\[
c[\pi_N(\partial V_{\mu N}, \mathcal{I}_t) - \pi_N(\partial V_{\mu N}, \mathcal{I}_s)] \leq \langle M_{\mu N}\rangle_t - \langle M_{\mu N}\rangle_s \leq c[\pi_N(\partial V_{\mu N}, \mathcal{I}_t) - \pi_N(\partial V_{\mu N}, \mathcal{I}_s)].
\]

Altogether, this implies

\[
M_{\mu N,t}^- - M_{\mu N,s}^- \geq c\langle (M_{\mu N})_t - \langle M_{\mu N}\rangle_s \rangle - M_{\mu N,s}^+.
\]

(35)

For any \(x > 0\), let \(\eta_x = \inf\{t : M_{\mu N,t}^+ \leq x\}\) and define the martingale \(\tilde{M}_{\mu N}^x\) (with respect to \(\mathcal{G}_t = \mathcal{F}_{\tau_{\eta_x}+t}\)) by

\[
\tilde{M}_{\mu N}^x_{\eta_x,t} = \begin{cases} M_{\mu N,\eta_x+t} - M_{\mu N,\eta_x} & \eta_x < \infty, \\ 0 & \eta_x = \infty. \end{cases}
\]

Let \(\Delta = D_{N,h}(V_{\alpha N}, \partial V_{\gamma N}) - \eta_x\) and note that on \(\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2}(x)^c\) we get from (35) that

\[
\tilde{M}_{\mu N,\eta_x,t} > c\langle \tilde{M}_{\mu N}^x\rangle_t - x, \quad 0 \leq t < \Delta.
\]

Recalling Proposition \([12]\) we see that we can adapt the proof of [6] Proposition 4] to show that for some constant \(c''\) the following bound holds almost surely on \(\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2}(x)^c\),

\[
\langle \tilde{M}_{\mu N}^x\rangle_\Delta \geq \langle M_{\mu N}\rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\gamma N})} - \langle M_{\mu N}\rangle_{D_{N,h}(V_{\alpha N}, \partial V_{\beta N})} \geq c'.
\]

(36)

Write \(T_t = \inf\{s : \langle \tilde{M}_{\mu N}^x\rangle_s > t\}\) and let \(B\) be a standard Brownian motion that satisfies \(B_t = \tilde{M}_{\mu N,T_t}^x - \tilde{M}_{\mu N,0}^x\) for \(t < \langle \tilde{M}_{\mu N}^x\rangle_\infty\). Letting \(\tau_{h,x} = \inf\{t : B_t \leq cht - x\}\), it follows from Propositions \([13, 14]\) and Lemma \([17]\) that for some \(c''\)

\[
\mathbb{P}(\mathcal{E}_{N,1} \cap \mathcal{E}_{N,2}(x)^c) \leq \mathbb{P}(\tau_{h,x} \geq c') \leq c''x.
\]

Since \(\mathbb{P}(\mathcal{E}_{N,1})\) is bounded away from 0 by Lemma \([18]\) the conclusion follows.
4.2 Proof of Lemma 22

Define $d_k = \langle M_{\mu N} \rangle_k - \langle M_{\mu N} \rangle_{k-1}$ (throughout this section $k$ and $K$ are positive integers), and $A_k = \partial I_k \setminus I_{k-1}$. Let $I^+ = \bigcup_{k=1}^K \partial I^+_k$, $A = \bigcup_{k=1}^K A_k$, and note that for all $k \geq 0$, $\partial I^+_k \subset A_k \cap V$. By Corollary 10 we have

$$d_k = \frac{1}{|\partial V_{\mu N}|} \sum_{v \in \partial V_{\mu N}} \sum_{u \in A_k} H_{N,k}(\partial V_{\mu N},u) G_{N,k-1}(u,v)$$

$$\geq \sum_{u \in A_k} \left( H_{N,k}(\partial V_{\mu N},u) \right)^2 G_{N,k-1}(u,u),$$

where the inequality follows from $G_{N,k-1}(u,v) \geq H_{N,k}(v,u) G_{N,k-1}(u,u)$. Consequently,

$$\langle M_{\mu N} \rangle_K \geq \sum_{k=1}^K \sum_{u \in A_k} \left( H_{N,k}(\partial V_{\mu N},u) \right)^2 G_{N,k-1}(u,u). (37)$$

Comparing (37) to the desired inequality in Lemma 22 we see two differences: (1) the summation in (37) is over $A_k$ as opposed to $\partial I^+_k$; (2) there is a term $G_{N,k-1}(u,u)$ in (37) which we need to bound from below. To address this, we will define a function $\psi : I^+ \rightarrow A$ which, roughly speaking, allows us to bound $(H_{N,k}(\partial V_{\mu N},u))^2 G_{N,k-1}(u,u)$ from below by $(H_{\mu N}(\partial V_{\mu N},\psi(u)))^2$ (where $\psi(u) \in A_\tau$). We next carry out the details.

To specify $\psi$, let $D_{\ell_1}$ be the graph distance on $\hat{G}_N$ (i.e., $\ell_1$-distance on $\mathbb{R}^2$) and $u \in \partial I^+_k \subseteq A_k$ be an active point at time $k$. If $D_{\ell_1}(u, I_{k-1}) \geq 1/2$, then $G_{N,k-1}(u,u) \geq 1/2$ (see 11) and we let $\psi(u) = u$. If, on the other hand, $D_{\ell_1}(u, I_{k-1}) < 1/2$, there exist at most four points on $\partial I_{k-1} \cap B_{\ell_1}(u,1/2)$ (here $B_{\ell_1}(u,r)$ denotes the open ball of radius $r$ centered at $u$ with respect to $D_{\ell_1}$). For every $w \in \partial I_{k-1} \cap B_{\ell_1}(u,1/2)$ there is a unique (random) integer $\tau_w \leq k-1$ such that $w \in A_{\tau_w}$. We let $\psi(u) = w$ be the point in $\partial I_{k-1} \cap B_{\ell_1}(u,1/2)$ that minimizes $\tau_w$ (that is, the “oldest” $w$), breaking ties by distance to $u$ (choosing the $w$ closest to $u$) and if need be by lexicographical ordering. With this choice, $\partial I_{\tau_w} \cap B_{\ell_1}(u,|u-w|_{\ell_1}) = \emptyset$ so $H_{N,\tau_w}(u,w) \geq 1/4$ and hence

$$H_{N,\tau_w}(\partial V_{\mu N},w) \geq \frac{1}{4} H_{N,k}(\partial V_{\mu N},u). (38)$$

Also, $D_{\ell_1}(u, I_{\tau_w-1}) \geq 1/2$ so by (3) we have for $\delta = D_{\ell_1}(w,u) < 1/2$

$$G_{N,\tau_w-1}(w,w) = 4\delta(1-\delta) + (1-\delta)^2 G_{N,\tau_w-1}(u,u) \geq \frac{1}{2}. (39)$$

Finally, for distinct $u, u' \in I^+, B_{\ell_1}(u,1/2) \cap B_{\ell_1}(u',1/2) = \emptyset$ so $\psi$ is injective. Recalling
we get that
\[
\langle M_{\mu N} \rangle_k \geq \sum_{k=1}^{K} \sum_{u \in \partial I_k^+} (H_{\mu N, \tau \psi(u)}(\partial V_{\mu N}, \psi(u)))^2 G_{N, \tau \psi(u)-1}(\psi(u), \psi(u))
\]
\[
\geq \frac{1}{32} \sum_{k=1}^{K} \sum_{u \in \partial I_k^+} H_{N, k}(\partial V_{\mu N}, u)^2 ,
\]
where the factor of \(\frac{1}{32}\) comes from \((\frac{1}{4})^2\) (which accounts for the ratio on the square of harmonic measures; see \((38)\)) and \(\frac{1}{2}\) (which accounts for the Green function term; see \((39)\)).

4.3 Proof of Lemma \[23\]
Let \(A_N = V_{\beta N} \setminus V_{\alpha N}\). We will bound \(E[|B_j|]\) by bounding the probability that each vertex \(v \in A_N\) belongs to \(B_j\). Note that
\[
I^+ \subseteq \{v \in A_N \cap \tilde{E}_{N, h}^{\geq h}, D_{N, h}(V_{\alpha N}, v) \leq D_{N, h}(V_{\alpha N}, \partial V_{\beta N}), D_{N, h}(V_{\alpha N}, v) < \infty\}.
\]
Let \(\tilde{E}_{N, h}^{\geq h} = \tilde{E}_{N}^{\geq h} \cap \tilde{\phi}_{\beta N}\) be the subgraph of \(\tilde{E}_{N}^{\geq h}\) induced by the metric graph on \(V_{\beta N}\), and \(D_{N, h, \beta}\) be the graph distance on \(\tilde{E}_{N, \beta}^{\geq h}\). With these definitions we have
\[
I^+ \subseteq \{v \in A_N \cap \tilde{E}_{N, \beta}^{\geq h}, D_{N, h, \beta}(V_{\alpha N}, v) < \infty\}.
\]
By Lemma \[5\] \(\Pr(D_{N, h, \beta}(V_{\alpha N}, v) < \infty | \tilde{\phi}_{N, v})\) is increasing in \(\tilde{\phi}_{N, v}\). Therefore letting \(a_j = h + 2j^{-1} \sqrt{\log N} 1_{j > 0}\) and \(b_j = h + 2j \sqrt{\log N}\) we conclude that for \(j \geq 0\) and \(v \in A_N\)
\[
\Pr(v \in B_j) \leq \Pr(\tilde{\phi}_{N, v} > a_j) \Pr(D_{N, h, \beta}(V_{\alpha N}, v) < \infty | \tilde{\phi}_{N, v} = b_j).
\]
Since (see, e.g., \[10\] Theorem 4.4.4, Proposition 4.6.2])
\[
\text{Var}[\tilde{\phi}_{N, v}] = G_N(v, v) = \left(1 + O\left(\frac{1}{\log N}\right)\right) \frac{2}{\pi} \log N ,
\]
we see that there exists a constant \(c = c(h, \beta) > 0\) such that for all \(j \geq 0\) and \(v \in A_N\)
\[
\Pr(\tilde{\phi}_{N, v} > a_j) \leq e^{-c4j} .
\]
We will bound the second term of \((40)\) in terms of \(k\) for \(v \in \partial V_{\alpha N+k}\). We state the result here and defer the proof to the end of this section.
Lemma 24. Let \( k^* = (\beta - \alpha)N/\sqrt{\log N} \). There exists a positive constant \( c = c(h, \beta, \alpha) \) such that for all \( j \geq 0, k \geq k^* \) and \( v \in \partial V_{\alpha N + k} \),

\[
\mathbb{P}(D_{N,h,\beta}(V_{\alpha N}, v) < \infty \mid \tilde{\phi}_{N,v} = b_j) \leq \frac{c^2 j}{\sqrt{\log N}} \sqrt{\log N - \log k}.
\]

Using Lemma 24 and (42), we have

\[
E[|B_j|] = \sum_{v \in A_N} \mathbb{P}(v \in B_j) \leq c^2 j e^{-c' j} \left[ \frac{N^2}{\sqrt{\log N}} + \frac{N}{\sqrt{\log N}} \sum_{k=k^*}^{(\beta-\alpha)N} \sqrt{\log N - \log k} \right],
\]

where we used the fact that there are \( O(N^2/\sqrt{\log N}) \) lattice points in \( V_{\alpha N + k^*} \backslash V_{\alpha N} \). Finally, we note

\[
\sum_{k=k^*}^{(\beta-\alpha)N} \sqrt{\log N - \log k} \leq O(1) \int_{k^*}^{(\beta-\alpha)N} \sqrt{\log x} dx
\]

\[
= O(N) \int_{(\beta-\alpha)^{-1}}^{(\beta-\alpha)^{-1}} \sqrt{\log u} du = O(N).
\]

This completes the proof of Lemma 23.

Proof of Lemma 24. Since the event is decreasing in \( h \) we assume from now on that \( h > 0 \) and work with \( \tilde{E}_{N,\beta}^{\geq h} \). Let

\[
\tilde{M}_{\alpha N} = \mathbb{E}[X_{\alpha N} \mid I_t],
\]

be the exploration martingale with source \( I_0 = \{v\} \) and target \( \partial V_{\alpha N} \) on \( \tilde{E}_{N,\beta}^{\geq h} \). It follows from Lemma 21 and (41) that there exist positive constants \( c_- = c_-(\alpha, \beta) \) and \( c_+ = c_+(\alpha, \beta) \) such that for any \( w \in A_N \),

\[
c_- \leq \frac{1}{|\partial V_{\alpha N}|} \sum_{u \in \partial V_{\alpha N}} G(u, w) \leq c_+, \quad \text{and} \quad \pi_N(\partial V_{\alpha N}, w) \leq \frac{c_+}{\log N}.
\]

Let \( m = \frac{1}{c_-} \), and \( b = c_+(b_j + h)/\log N = O(2^j/\sqrt{\log N}) \). Write \( T_t = \inf\{s : \langle \tilde{M}_{\alpha N} \rangle_s > t\} \) and let \( B \) be a standard Brownian motion such that \( B_t = \tilde{M}_{\alpha N, T_t} - \tilde{M}_{\alpha N, 0} \) for \( t < \langle \tilde{M}_{\alpha N} \rangle_{\infty} \).

We define

\[
\tau_{-h,N} = \inf\{t : B_t \leq -mt - b\},
\]

\[
\pi_{N,k}^- = \inf_{I} \{\pi_N(\partial V_{\alpha N}, I) - \pi_N(\partial V_{\alpha N}, v)\},
\]

26
where the infimum is taken over all closed, connected subsets of $\tilde{\Gamma}_{\beta N}$ containing $v$ and intersecting $V_{\alpha N}$. For notational convenience, write $T = c_{-} \pi_{N,k}^{-}$. Applying Proposition 13 and Lemma 17, we get that

$$P(D_{N,-h,\beta}(v, \partial V_{\alpha N}) < \infty \mid \tilde{\phi}_{N,v} = b_j) \leq P(\tau_{-h,N} \geq T)$$

$$\leq 2h(-m\sqrt{T}, 0) \left[ \Phi \left( m\sqrt{T} + \frac{b}{\sqrt{T}} \right) - \Phi \left( m\sqrt{T} \right) \right]$$

$$\leq 2 \left[ \varphi(m\sqrt{T}) + m\sqrt{T} \right] \frac{b}{\sqrt{T}}.$$

To conclude the proof we use the following bound on $\pi_{N,k}^{-}$

$$\pi_{N,k}^{-} \geq c(\log N - \log k)^{-1}. \quad (44)$$

Provided with (44), we get that

$$m\sqrt{T} = O(1), \quad \frac{b}{\sqrt{T}} = O \left( \frac{2^j \sqrt{\log N - \log k}}{\sqrt{\log N}} \right).$$

Therefore,

$$P(D_{N,-h,\beta}(v, \partial V_{\alpha N}) < \infty \mid \tilde{\phi}_{N,v} = b_j) = O \left( \frac{2^j \sqrt{\log N - \log k}}{\sqrt{\log N}} \right).$$

It remains to prove (44). To this end, let $u$ be a point on $I \cap V_{\alpha N + k/2}$, and let $B_u, B'_u, B''_u$ be boxes centered at $u$ of side length $k/4, k/8, k/16$ respectively. By [10 Proposition 6.4.1], we get that there exists $c' = c'(\alpha) > 0$ such that for any $x \in V_{\alpha N},$

$$\pi_{N}(x, B'_u) \geq c'(\log N - \log k)^{-1}.$$

It is also obvious that once the random walk arrives at $\partial B'_u$, there is a probability bounded uniformly from below that the random walk range before exiting $B_u$ will contain a contour in $B_u \setminus B''_u$. In this case, the random walk will hit at least one point in $I \cap B_u$. Therefore, we get that

$$\pi_{N}(x, I \cap B_u) \geq c''(\log N - \log k)^{-1},$$

where $c'' > 0$ depends on $c'$. In addition, for any $w \in I \cap B_u$, we have

$$H_{\beta}(w, \partial V_{\alpha N} ; \partial V_{\alpha N} \cup \{v\}) \geq 1/2.$$
(see e.g., [10, Theorem 4.4.4., Proposition 4.6.2]). Altogether, this means that for any \( x \in \partial V_{\alpha N} \), we have

\[
\Delta \pi(x, I) := \mathbb{P}_x( \text{ random walk hits } I \text{ but not } v \text{ before it hits } \partial V_N ) \geq c(\log N - \log k)^{-1}
\]

for a constant \( c > 0 \). Since \( \pi_N(\partial V_{\alpha N}, I) - \pi_N(\partial V_{\alpha N}, v) = \frac{1}{|\partial V_{\alpha N}|} \sum_{x \in \partial V} \Delta \pi(x, I) \), this completes the verification of (44).

\[ \square \]

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