GENERIC REGULARITY OF MINIMAL HYPERSURFACES
IN DIMENSION 8

YANGYANG LI AND ZHIHAN WANG

Abstract. In this paper, we show that every 8-dimensional closed Riemannian manifold with $C^\infty$-generic metrics admits a smooth minimal hypersurface. This generalized previous results by N. Smale [Sma93] and Chodosh-Liokumovich-Spolaor [CLS20]. Different from their local perturbation techniques, our construction is based on a global perturbation argument in [Wan20] and a novel geometric invariant which counts singular points with suitable weights.

1. Introduction

The interior regularity theory of minimal hypersurfaces dates back to around 1970, when H. Federer [Fed70] applied his dimension reduction arguments to show that the singular set of any area-minimizing rectifiable hypercurrent has Hausdorff codimension at least 7 (away from its boundary). His proof relied on a previous work by J. Simons [Sim68] on the nonexistence of stable cones in $\mathbb{R}^{n+1}$ for $2 \leq n \leq 6$. In particular, in a Riemannian manifold $M^{n+1}$ with $2 \leq n \leq 6$, an area-minimizing integral $n$-cycle must be a smooth minimal hypersurface. However, even when $n = 7$, Bombieri-De Giorgi-Giusti [BDGG69] has proved that the famous Simons cone (introduced in [Sim68])

$$C = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| = |y|\},$$

is an area-minimizing hypercone in $\mathbb{R}^8$, which has a singular point at the origin.

The similar phenomenon occurs in the stable minimal hypersurface (varifold) setting with natural a priori assumption on the singular set, which was originally proved by Schoen-Simon [SS81] and later improved by N. Wickramasekara [Wie14]. This deep regularity theorem plays an important role in the Almgren-Pitts min-max theory [Alm65,Pit81] extending the original regularity result obtained from Schoen-Simon-Yau [SSY75] to higher dimensional closed manifolds. Quantitative versions of the partial regularity theorem and rectifiability of singular set can be found in [CN13,NV20].

In general, the existence of Simons cone has shown that even in dimension 8, a minimal hypersurface can have singular set. In fact, N. Smale [Sma89]...
has constructed explicit examples of minimal hypersurfaces with boundary containing arbitrarily many singular points virtually by applying bridge principle on stable cones. Later, in [Sma99], he also constructed a closed Riemannian manifold admitting a unique area-minimizer in a homology class, which contains 2 singular points. [Sim21] shown that any closed subset in \( \mathbb{R}^{n-7} \) can be realized as singular set of some stable minimal hypersurface in \( (\mathbb{R}^{n+1}, g) \) for some smooth metric \( g \) arbitrarily close to the Euclidean one, making the rectifiability result of singular sets by [NV20] sharp in some way.

This would naturally stimulate one to opine the opposite:

**Q 1:** Is it possible to resolve singularities of a minimal hypersurface by a small perturbation of the ambient metric?

The same question on area-minimizing hypersurfaces has been raised in [Yau82, Sma89]. An affirmative answer to this question together with the construction of minimal hypersurfaces from Almgren-Pitts min-max theory would directly lead to the generic existence of smooth minimal hypersurfaces.

**Q 2:** Does there exist a smooth minimal hypersurface in a closed Riemannian manifold with generic metrics?

In a closed manifold \( M \) of dimension 8, N. Smale [Sma93] confirmed the existence of such a small perturbation for an area minimizer in its homology classes and thus, the generic existence of smooth minimal hypersurfaces. Recently, Chodosh-Liokumovich-Spolaor [CLS20] justified the generic existence in the positive Ricci curvature case by constructing an optimal nested volume parametrized sweepout for 1-parameter min-max minimal hypersurfaces. Both of their arguments essentially employed the local analysis of isolated singular points by Hardt-Simon [HS85], which produces unique Hardt-Simon foliations of smooth minimal hypersurfaces in either side of a regular area-minimizing cone. Similar results were obtained for regular one-sided area-minimizing cone by F. Lin [Lin87] and Z. Liu [Liu19]. Nevertheless, by a calibration argument [BDGG69, Law91], the existence of such a smooth foliations implies that the regular minimal cone should be at least one-sided area-minimizing, and thus, it seems impossible to apply this local analysis directly on a regular stable minimal cone which is not area-minimizing one either side. To the best of the authors’ knowledge, the existence of such a cone is still widely open, and a negative answer to this will significantly simplify our arguments and other important analysis in the literature.

In this paper, we shall utilize a global perturbation argument from [Wan20] (See Lemma 2.24) and a novel geometric invariant, “singular capacity” (See Definition 4.1) to resolve singular points by small perturbations. As an application, we are able to settle the second question and prove the generic
existence of a smooth minimal hypersurface in a given closed Riemannian manifold.

**Theorem 1.1** (Main Theorem). Let $M$ be an $8$-dimensional closed Riemannian manifold with $H_7(M, \mathbb{Z}_2) = 0$. Then for any $C^k$-generic ($k \geq 4$ or $k = \infty$) Riemannian metric $g$, there exists a closed embedded smooth minimal hypersurface $\Sigma$ in $(M, g)$.

**Theorem 1.2.** Every $8$-dimensional closed Riemannian manifold $M$ with any $C^\infty$-generic metric admits a closed smooth minimal hypersurface $\Sigma$.

**Proof.** If $H_7(M, \mathbb{Z}_2) \neq 0$, then we can take a nontrivial homology class $\alpha \in H_7(M, \mathbb{Z}_2)$. The standard Federer-Fleming minimizing process [FF60] implies the existence of an area-minimizing multiplicity one minimal hypersurface $\Sigma \subset (M, g)$ with $[\Sigma] = \alpha$. In this case, by Lemma 2.22 below, one can show that there exists a metric $g'$ arbitrarily close to $g$ such that $(M, g')$ admits a smooth non-degenerate minimal hypersurface $\Sigma'$ close to $\Sigma$ and $[\Sigma'] = \alpha$ as well. The openness of the set of such $g'$'s follows immediately from the non-degenerateness and B. White’s structure theorem [Whi91, Whi17].

The $H_7(M, \mathbb{Z}_2) = 0$ case is exactly the main theorem. \qed

Our study only deals with minimal hypersurfaces in a close Riemannian manifold, but in a general setting, one may also be interested in generic regularity of minimal submanifolds with codimension greater than 1. In this direction, B. White [Whi85, Whi19] has proved generic smoothness and embeddedness of minimizing integral 2-cycles and J.D. Moore [Moo06, Moo07] has shown generic nonexistence of branch points for parametrized minimal surfaces.

Recently, N. Edelen [Ede21] proved that in a closed Riemannian manifold $(M^8, g)$, the number of diffeomorphism classes of minimal hypersurfaces of bounded area and Morse index is finite. More precisely, he proved the existence of model minimal hypersurfaces such that others are bi-Lipschitz to these models. This suggests the possibility to describe the structure of singular minimal hypersurfaces as in B. White’s structure theorem [Whi91], and inspires us to conjecture that the generic smoothness should hold for all minimal hypersurfaces with optimal regularity and finite Morse index. We expect to solve the 8-dimensional case in the near future.

### 1.1. Some conventions.

Throughout this paper, unless otherwise stated, a minimal hypersurface $\Sigma$ in an $(n + 1)$-dimensional closed manifold $M$ is referred to a smooth, locally stable $n$-dimensional submanifold with locally bounded area and optimal regularity, i.e. $\mathcal{H}^n(\Sigma \cap K) < +\infty$ for every compact subset $K \subset M$, and $\mathcal{H}^{n-2}(\Sigma \setminus \Sigma) = 0$. These are exactly the minimal hypersurfaces generated from area minimizing arguments or Almgren-Pitts min-max theory (See [Pit81, SS81, Li19]). We always identify $\Sigma$ with the regular part of $\Sigma$, and denote $\Sigma \setminus \Sigma$ by $\text{Sing}(\Sigma)$.
For a Caccioppoli set $A$, we shall identify $A$ with $\mathcal{H}^{n+1}$-density 1 part of $A$. We will use $\partial^\# A$ to denote its topological boundary to avoid confusion with the boundary map defined in Subsection 2.2.

1.2. Sketch of Proof. We define
\[ \mathcal{G} := \left\{ C^k \text{ Riemannian metrics on } M \right\} ; \]
\[ \mathcal{G}_F := \left\{ g \in \mathcal{G} \mid (M, g) \text{ has Frankel property} \right\} ; \]
\[ \mathcal{G}_{NF} := \mathcal{G} \setminus \mathcal{G}_F ; \]
\[ \mathcal{R} := \left\{ g \in \mathcal{G} \mid (M, g) \text{ admits a nondegenerate smooth minimal hypersurface} \right\} . \]

To prove our main theorem, it suffices to show that $\mathcal{R}$ is open and dense in $\mathcal{G}_F$. As one will see later, the most difficult part is to justify the denseness of $\mathcal{R}$ in the interior of $\mathcal{G}_F$, i.e., int($\mathcal{G}_F$).

To prove this density, we introduce in general a geometric invariant, singular capacity, denoted by $\text{SCap}$, on the space of 8-dimensional Riemannian manifolds paired with a locally stable minimal hypersurface. Roughly speaking, this invariant counts how many singularities “potentially” a minimal hypersurface contains. The key of this invariant is its upper semi-continuity:

\[ \text{SCap}(\Sigma; U, g) \geq \limsup_{j \to \infty} \text{SCap}(\Sigma_j; U, g_j) \]

where $g_j \to g$ in $C^k_{loc}$ and $\Sigma_j \to \Sigma$ in the varifold sense with some technical assumptions.

With the notion of singular capacity, the proof can be decomposed into three steps.

Step 1. For every $g \in \text{int}(\mathcal{G}_F)$, let $V = \kappa |\Sigma| \in \mathcal{R}$ be a 1-width min-max integral varifold generated from an ONVP sweepout. One can observe the following dichotomy:

- either $\mathcal{h}_{nm}(\Sigma) = \emptyset$;
- or $V = |\Sigma|$ and $\Sigma$ is connected, two-sided and separating with isolated singular points $\text{Sing}(\Sigma) = \{p_0, p_1, \ldots, p_k\}$.

Let’s consider the latter case. By constructing a metric perturbation by hand, we may further assume that $\Sigma$ is the unique and non-degenerate one realizing 1-width provided that $\mathcal{H}^0(\mathcal{h}_{nm}(\Sigma)) = 1$. Let $\nu$ denote a unit normal on $\Sigma$.

Step 2. Let $f \in \mathcal{F} \subset C^\infty_c(M - \text{Sing}(\Sigma))$ be a generically chosen function with $\nu(f)_{|\Sigma} \neq 0$. Let the metric perturbation be $g_t := g(1 + tf)$ and $\Sigma_t$ be a 1-width min-max minimal hypersurfaces in $(M, g_t)$.

It follows immediately from the uniqueness of $\Sigma$ that $\Sigma_t \to \Sigma$ in the varifold sense. By analyzing the associated Jacobi field $u \cdot \nu$ on $\Sigma$ generated by $\{\Sigma_t\}$ as in [Wan20, Section 4], we can show that at some $p \in \text{Sing}(\Sigma)$, the asymptotic rate of $u$ (See Subsection 2.4) is either $\gamma^+_1(C_p)$ or $\gamma^-_1(C_p)$.
This implies that $\Sigma_t$ is smooth in a small neighborhood $U_p$ of $p$ for $t$ close to 0.

A naive induction argument based on the $H^0(\text{Sing})$ would not work, because in a neighborhood $U_q$ of another singular point $q \in \Sigma$, $\Sigma_t$ might have more than 1 singular point inside. Fortunately, by upper semi-continuity of the singular capacity, we can conclude that

$$\text{SCap}(\Sigma_{t_j}; M, g_{t_j}) \leq \text{SCap}(\Sigma; M, g) - 1.$$ 

**Step 3.** Iterate Step 1 and Step 2, and then a backward induction argument would lead to the following dichotomy. For some $\tilde{g} \in \text{int}(\mathcal{G}_F)$ arbitrarily close to $g$, we have a minimal hypersurface $\Sigma \subset (M, \tilde{g})$ satisfying

- either $h_{nm}(\Sigma) = \emptyset$;
- or $\text{SCap}(\Sigma) = 0$.

In the former case, Hardt-Simon type metric perturbation would lead to a smooth minimal hypersurface as in $[\text{CLS20}]$; In the latter case, by definition, $\Sigma$ itself is smooth.

In summary, we conclude the main theorem in $\text{int}(\mathcal{G}_F)$.

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2. **Preliminaries**

We mainly focus on a closed Riemannian manifold $(M^{n+1}, g)$ with $H_n(M, \mathbb{Z}_2) = 0$. In this case, by Poincaré duality, its first cohomology group with $\mathbb{Z}_2$ coefficients $H^1(M, \mathbb{Z}_2) = 0$. Therefore, the Stiefel-Whitney class $\omega_1(M)$ vanishes and $M$ is orientable.

Let’s first list some notations in geometric measure theory and Almgren-Pitts’ min-max theory. Interested readers may refer to L. Simon’s lecture notes $[\text{Sim84}]$ and J. Pitts’ monograph $[\text{Pit81}]$.

- $\mathcal{H}^n$: $n$-dimensional Hausdorff measure;
- $I_k(M; \mathbb{Z}_2)$: the space of $k$-dimensional mod 2 flat chains in $M$;
- $Z_k(M; \mathbb{Z}_2)$: the space of $k$-dimensional mod 2 flat cycles in $M$;
- $M_U$: the mass norm in an open subset $U$ on the flat chain space $(M_M$ abbreviated to $M)$;
- $\mathcal{F}$: the flat metric on the flat chain space defined by

$$\mathcal{F}(S, T) = \inf \{M(R) + M(P) : S - T = \partial R + P\},$$
where \( S, T, P \in \mathcal{I}_k(M; \mathbb{Z}_2) \) and \( R \in \mathcal{I}_{k+1}(M; \mathbb{Z}_2) \). We always assume that \( \mathcal{I}_n(M; \mathbb{Z}_2) \) is endowed with \( \mathcal{F} \) metric, and so is \( \mathcal{Z}_n(M; \mathbb{Z}_2) \);

- \( \mathcal{C}(M) \): the space of Caccioppoli sets, i.e., subsets of \( M \) of finite perimeter in the Riemannian manifold setting.

notes \[
\text{More details in the Euclidean setting can be found in L. Simon’s lecture notes [Sim84] and multiple monographs written by H. Federer [Fed69, Chap. 4], F. Lin and X. Yang [LY02, Chap. 5], F. Maggi [Mag12], L.C. Evans and R.F. Gariepy [EG15], E. Giusti [Giu84], etc., whose statements can be easily extended to our setting.}
\]

Definition 2.1. A Lebesgue measurable set \( E \subset (M^{n+1}, g) \) is Caccioppoli (have finite perimeter) if its characteristic function \( \chi_E \in BV(M) \). In other words, \( \chi_E \in L^1(M) \) has bounded variation, i.e.,

\[
\| Dg\chi_E \|(M) := \sup \left\{ \int_E \text{div}_g X dx : X \in \Gamma^1(TM), |X|_g \leq 1 \right\} < \infty.
\]

Note here \( Dg\chi_E : \Gamma^1(TM) \mapsto \mathbb{R} \) is the associated functional with \( E \). Two sets will be identified if they differ by a measure zero set and the collection of all the Caccioppoli sets are denoted by \( \mathcal{C}(M) \).

Remark 2.2. It is easy to verify that \( \mathcal{C}(M) \) is independent of the choice of \( C^\infty \) metrics on \( M \).

Since \( M \) is assumed to be orientable, we can take a unique \((n+1)\)-vector \( \xi \) in \( \Lambda^{n+1}TM \) so that at each point \( p \in M \), \( \xi_p \) is the wedge product of an orthonormal basis of \( T_pM \). We will use \( \mathcal{L}^{n+1} \) to denote the Lebesgue measure on \( M \) and \( \mathbf{E}^{n+1} \) to denote the current \( \mathcal{L}^{n+1} \wedge \xi \). By definition, a set \( E \in \mathcal{C}(M) \) is equivalent to \( \mathbf{E}^{n+1} | E \in \mathcal{I}_{n+1}(M; \mathbb{Z}_2) \).
Definition 2.3. Given $E \in \mathcal{C}(M)$, by Riesz’s representation theorem, there is a unique $TM$-valued Radon measure $\mu_E$ so that

$$\int_E \text{div}_g X = \int_M X \cdot g \, d\mu_E, \quad \forall X \in \Gamma^1(TM).$$

Moreover, $|\mu_E|(M)$ is finite. $\mu_E$ is called the **Gauss-Green measure** of $E$. And we define the **relative perimeter** of $E$ in $F \subset M$, and the **perimeter** of $E$ to be

$$P(E; F) = |\mu_E|(F), \quad P(E) = |\mu_E|(M).$$

**Remark 2.4.** If $E \subset (M^{n+1}, g)$ is Caccioppoli, then so is $M \setminus E$. Furthermore,

$$\mu_{M \setminus E} = -\mu_E, \quad P(E) = P(M \setminus E).$$

Definition 2.5. Given $E \subset (M^{n+1}, g)$ and $x \in M$, if the limit

$$\theta_{n+1}(E)(x) = \lim_{r \to 0^+} \frac{|E \cap B_g(x, r)|}{|B_g(x, r)|}$$

exists, then $\theta_{n+1}(E)(x)$ is called the $(n + 1)$-dimensional density of $E$ at $x$. Given $t \in [0, 1]$, the **set of points of density $t$ of $E$** is defined to be

$$E(t) := \{ x \in M : \theta_{n+1}(E)(x) = t \}.$$

Definition 2.6. Given $E \in \mathcal{C}(M)$, the **reduced boundary** $\partial^* E$ is the set of those $x \in \text{supp } \mu_E$ such that

$$\nu_E = \lim_{r \to 0^+} \frac{\mu_E(B(x, r))}{|\mu_E|(B(x, r))} \quad \text{exists and belongs to } S^n.$$ 

Here $\nu_E$ is called the **measure-theoretic outer unit normal** to $E$.

**Remark 2.7.** By Lebesgue-Besicovitch differentiation theorem, $\mu_E = \nu_E|\mu_E| \circ \partial^* E$, where $\nu_E$ is a $|\mu_E|$ measurable vector field.

**Remark 2.8.** One can check that $\partial^* E \subset \overline{\partial E} = \text{supp } \mu_E \subset \partial^1 E$, where the last one is the topological boundary of $E$. Up to modification on sets of measure zero, we also have

$$\overline{\partial E} = \partial^1 E.$$ 

A Caccioppoli set is in fact an equivalence class of sets differed by a measure zero set, so for convenience, we will only consider all the points of density 1 which satisfies the inequality above for each Caccioppoli set.

Definition 2.9. For a set $E \subset (M^{n+1}, g)$, we define its **essential boundary** $\partial^e E$ to be

$$\partial^e E = M \setminus (E^{(0)} \cup E^{(1)})$$

In the monograph, H. Federer gave a criterion for Caccioppoli sets [Fed69, Theorem 4.5.11], and here we adapt it to the Riemannian manifold setting.
Theorem 2.10. For \( E \subset (M^{n+1}, g) \), the following two conditions are equivalent:

1. \( E \) is Lebesgue measurable and \( \partial (E^{n+1} \cup I) \) is representable by integration, i.e., \( E \) is a Caccioppoli set.
2. \( \mathcal{I}^n (\partial^* E) < \infty \).

Remark 2.11. By Nash embedding theorem, we can take a large integer \( N \) such that \( (M, g) \) can be isometrically embedded in \( \mathbb{R}^N \). Then \( \mathcal{I}^{n} \) denotes the \textit{n dimensional integral geometric measure} over \( \mathbb{R}^N \) (Interested readers may refer to [Fed69, 2.10.5, 2.10.15, 2.10.16] or [Mor16, 2.4] for more details).

Corollary 2.12. Suppose that a minimal hypersurface \( \Sigma \subset (M, g) \) separates \( M \) into two open connected components \( M_+ \) and \( M_- \), and then both \( M_+ \) and \( M_- \) are Caccioppoli sets.

Proof. It suffices to check that \( \mathcal{I}^n (\partial^* M_{\pm}) < \infty \) and we only focus on \( M_+ \). By definition, we have \( M_+ \subset M^{(1)}_+ \) and \( M_- \subset M^{(0)}_+ \). Since \( \Sigma \) is smooth almost everywhere, we can conclude that \( \partial^* M_\pm \approx \Sigma \), and \( \mathcal{I}^n (\partial^* E) = \mathcal{H}^n (\Sigma) < \infty \). Hence, \( M_+ \) is a Caccioppoli set. \( \square \)

Before concluding this subsection, we state an important theorem showing that \( C(M) \) is close under set operations and characterizing the Gauss-Green measure of set operations.

Theorem 2.13 ([Mag12, Theorem 16.3]). Given \( E, F \in C(M) \), let

\[
\{ \nu_E = \nu_F \} = \{ x \in \partial^* E \cap \partial^* F : \nu_E (x) = \nu_F (x) \},
\]

\[
\{ \nu_E = -\nu_F \} = \{ x \in \partial^* E \cap \partial^* F : \nu_E (x) = -\nu_F (x) \}.
\]

Then \( E \cap F, E \setminus F \) and \( E \cup F \) are all Caccioppoli, with

\[
\mu_{E \cap F} = \mu_{E \cap F}^{(1)} + \mu_{E \cap F}^{(0)} + \nu_E \mathcal{H}^n_\{ \nu_E = \nu_F \},
\]

\[
\mu_{E \setminus F} = \mu_{E \setminus F}^{(0)} - \mu_{E \cap F}^{(1)} + \nu_E \mathcal{H}^n_\{ \nu_E = -\nu_F \},
\]

\[
\mu_{E \cup F} = \mu_{E \cup F}^{(0)} + \mu_{E \cap F}^{(0)} + \nu_E \mathcal{H}^n_\{ \nu_E = \nu_F \},
\]

and in the measure-theoretic sense,

\[
\partial^* (E \cap F) \approx (F^{(1)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{ \nu_E = \nu_F \},
\]

\[
\partial^* (E \setminus F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{ \nu_E = -\nu_F \},
\]

\[
\partial^* (E \cup F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(0)} \cap \partial^* F) \cup \{ \nu_E = \nu_F \}.
\]

Moreover, for every borel set \( G \subset M \),

\[
P(E \cap F; G) = P(E; F^{(1)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^n (\{ \nu_E = \nu_F \} \cap G),
\]

\[
P(E \setminus F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(1)} \cap G) + \mathcal{H}^n (\{ \nu_E = -\nu_F \} \cap G),
\]

\[
P(E \cup F; G) = P(E; F^{(0)} \cap G) + P(F; E^{(0)} \cap G) + \mathcal{H}^n (\{ \nu_E = \nu_F \} \cap G).
\]
2.2. 1-parameter Min-max Theory. Let’s consider the boundary map \( \partial : \mathcal{C}(M) \to \mathbb{Z}_n(M; \mathbb{Z}_2) \): More precisely, for any \( A \in \mathcal{C}(M) \), we use \( \partial A \) to denote the mod 2 flat cycle \( \partial(E^{n+1} \lrcorner A) \), where \( E^{n+1} \) is defined in the previous subsection. It’s easy to see that

\[
P(A) = M(\partial A).
\]

Since we assume that \( H_n(M, \mathbb{Z}_2) = 0 \), the boundary map is surjective. By Constancy Theorem, this map is in fact a double cover. With this observation, we can utilise Caccioppoli sets instead of mod 2 flat cycles to define sweepouts on \( M \). This has been used in [Zho17, ZZ17, ZZ18, Zho19, CLS20].

**Definition 2.14.** A (1-parameter) sweepout on \( M \) is a continuous map \( \Phi : [0, 1] \to \mathcal{C}(M) \) with \( \Phi(0) = 0 \) and \( \Phi(1) = M \). The collection of all sweepouts is denoted by \( \mathcal{P} \).

**Definition 2.15.** The min-max width on \( M \) is defined to be

\[
\mathcal{W} = \inf_{\Phi \in \mathcal{P}} \sup_{t \in [0, 1]} M(\partial \Phi(t)) > 0.
\]

The Almgren-Pitts theory [Alm65, Pit81, SS81] has shown that the width \( \mathcal{W} \) could always be realized by a minimal hypersurface (possibly with multiplicities). Recently, O. Chodosh, Y. Liokumovich and L. Spolaor [CLS20] gave a refined description of such a minimal hypersurface in terms of its singularities and Morse index. Here, we adapt their results to our setting.

**Definition 2.16 (ONVP sweepouts).** A sweepout \( \Phi \) is called an optimal nested volume parametrized (ONVP) sweepout if it satisfies the following conditions.

1. **optimal:** \( \sup_{x \in [0, 1]} M(\partial \Phi(x)) = \mathcal{W} \);
2. **nested:** \( \Phi(x_1) \subset \Phi(x_2) \), for all \( 0 \leq x_1 \leq x_2 \leq 1 \);
3. **volume parametrized:** \( \text{Vol}(\Phi(x)) = x \cdot \text{Vol}(M, g) \), for every \( x \in [0, 1] \).

The critical domain of \( \Phi \) is the set

\[
m(\Phi) = \left\{ x \in [0, 1] : \limsup_{y \to x} M(\partial \Phi(y)) = \mathcal{W} \right\}.
\]

Similarly, the left (resp. right) critical domain of \( \Phi \) can be defined as the set \( m_L(\Phi) \) (resp. \( m_R(\Phi) \)) only involving \( y \nearrow x \) (resp. \( y \searrow x \)). Apparently, \( m(\Phi) = m_L(\Phi) \cup m_R(\Phi) \).

The critical set of \( \Phi \) is

\[
C(\Phi) := \{ V \in \mathcal{V}_n(M) : V = \lim_{j \to \infty} |\partial \Phi(x_j)|, \ x_j \in [0, 1] ; \ |V|\parallel(M) = \mathcal{W} \}.
\]

**Definition 2.17 (Excessive points).** A point \( x_0 \in [0, 1] \) is called left (resp. right) excessive for a sweepout \( \Phi \), if there exists a constant \( \varepsilon > 0 \) and
an interval \( I = [a, b], [a, b), (a, b) \) or \((a, b]\) with \((x_0 - \varepsilon, x_0] \subset I \) (resp. \([x_0, x_0 + \varepsilon) \subset I \)), satisfying the following replacement condition.

We can find a continuous map on \( I \), \( \{ \Phi^I(x) \}_{x \in I} \), such that \( \Phi^I(a) = \Phi(a) \) and \( \Phi^I(b) = \Phi(b) \) but for all \( x \in I \),

\[
\limsup_{y \to x} M(\partial \Phi^I(y)) < W.
\]

**Theorem 2.18** ([CLS20, Theorem 19]). For any closed Riemannian manifold \((M, g)\), there exists an (ONVP) sweepout \( \Psi \) such that every \( x \in m_L(\Psi) \) is not left-excessive and every \( x \in m_R(\Psi) \) is not right excessive. The Almgren-Pitts min-max theory implies that there exists a stationary integral varifold \( V \in C(\Psi) \) whose support is the closure of a minimal hypersurface \( \Sigma \), and \( \|V\|(M) = W \).

**Definition 2.19** (One-sided homotopy area-minimizing). Given a minimal hypersurface \( \Sigma \subset (M, g) \), \( p \in \Sigma \) and \( r > 0 \) small enough such that \( \Sigma \cap B_r(p) \) separates the open ball \( B_r(p) \) into two open connected components \( E_+ \) and \( E_- \). \( \Sigma \) is said to be one-sided homotopy area-minimizing (OSHAM) in \( B_r(p) \), if there does not exist a deformation \( \{\Omega(t) \subset C(M)\}_{t \in [0,1]} \) satisfying the following conditions.

1. \( \Omega(0) \cap B_r(p) = E_{\pm} \) and \( \Omega(t) \subset \Omega(s) \) for any \( t \geq s \);
2. \( \partial^* \Omega(t) \Delta \partial^* E_{\pm} \subset B_r(p) \);
3. \( M_{B_r(p)}(\partial \Omega(t)) \leq M_{B_r(p)}(\partial E_{\pm}) \) and \( M_{B_r(p)}(\partial \Omega(1)) < M_{B_r(p)}(\partial E_{\pm}) \).

We define

\[
h_{nm}(\Sigma) := \{ p \in \Sigma : \forall r > 0 \text{ small}, \Sigma \cap B_r(p) \text{ is not (OSHAM) in } B_r(p) \}. \tag{2.14}
\]

**Theorem 2.20** ([CLS20, Theorem 4, Proposition 29, Lemma 30]). Given a closed Riemannian manifold \((M^8, g)\) and a sweepout \( \Psi \) in Theorem 2.18, by possibly replacing \( \Phi(x) \) by \( M \setminus \Phi(1 - x) \), there exists a sequence \( x_i \nearrow x_0 \in m_L(\Phi) \) such that

\[
\lim_i |\partial \Phi(i)| \to V = \sum_i \kappa_i |\Sigma_i| \in C(\Phi),
\]

where \( \Sigma_i \)'s are pairwise disjoint minimal hypersurfaces with \( \kappa_i \in \{1, 2\} \) and

\[
\sum_i H^0(h_{nm}(\Sigma_i)) + \sum_i \text{Index}(\Sigma_i) \leq 1. \tag{2.15}
\]

Moreover, if there exists a \( \kappa_i = 2 \), then

\[
h_{nm}(\Sigma) = \emptyset. \tag{2.16}
\]

Otherwise, \( V \) is of multiplicity one and \( V = |\partial \Phi(x_0)| \).

One essential ingredient of the proof is the following interpolation lemma which follows from a result of K.J. Falconer [Fal80] (See also [Gut11, Appendix 6], [CL20, Lemma 5.3]).
Lemma 2.21 ([CLS20, Lemma 16]). On $(M^{n+1}, g)$, for every $L, \varepsilon > 0$, there exists a $\delta > 0$ satisfying the following property.

For any $\Omega_0, \Omega_1 \in \mathcal{C}(M)$ with $\Omega_0 \subset \Omega_1$, $P(\Omega_i) \leq L$ and $\text{Vol}(\Omega_1 \setminus \Omega_0) \leq \delta$, there exists a nested continuous family $\{\Omega_t\}$ with

$$P(\Omega_t) \leq \max\{P(\Omega_1), P(\Omega_0)\} + \varepsilon,$$

for all $t \in [0, 1]$.

2.3. Surgery Procedure à la Chodosh-Liokumovich-Spolaor. In this subsection, we will recall the surgery procedure in [CLS20] to perturb away singularities locally when $h_{nm}(\Sigma) = \emptyset$ for a minimal hypersurface $\Sigma^7 \subset (M^8, g)$. Since the surgery was done locally, from now on, we focus on a singular point $p \in \text{Sing}(\Sigma)$. WLOG, let’s assume that for $\varepsilon_0 > 0$, $\Sigma$ has only one singular point and is OSHAM in $B_{2\varepsilon_0}(p)$.

Lemma 2.22 (Surgery Procedure [CLS20, Proposition 31]). Let $M, g, \Sigma, p$ and $\varepsilon_0$ be as above. For every $k \geq 4$, $\delta > 0$, there exists a Riemannian metric $g'$ and a minimal hypersurface $\Sigma' \subset (M, g')$ satisfying the following conditions.

1. $\|g - g'\|_{C^k} < \delta$;
2. $g = g'$ and $\Sigma = \Sigma'$ outside $B_{2\varepsilon_0}(p)$;
3. $\text{Sing}(\Sigma') \cap B_{2\varepsilon_0} = \emptyset$.

2.4. Associated Jacobi Fields. We shall recall some notions and some results from [Wan20], which will be utilised later.

Let $\Sigma \subset (M^8, g)$ be a two-sided minimal hypersurface with a unit normal field $\nu$. It follows from [Fed69, Section 5.4] and [SS81, Sim83] that the singular set $\text{Sing}(\Sigma) := \Sigma \setminus \Sigma$ of $\Sigma$ consists of isolated points, at each of which $\Sigma$ has a unique and regular tangent cone.

On $\Sigma$, the space of functions that we are going to work on will be denoted by $\mathcal{B}(\Sigma)$ and defined as follows. By [Wan20, Lemma 3.1], one can observe that there exists $C_{\Sigma} > 0$ such that

$$\|\phi\|^2_{\mathcal{B}(\Sigma)} := Q_\Sigma(\phi, \phi) + C_{\Sigma}\|\phi\|^2_{L^2(\Sigma)} \geq \|\phi\|^2_{L^2(\Sigma)} \quad \forall \phi \in C^1_c(\Sigma),$$

where $Q_\Sigma(\phi, \phi) := \int_\Sigma |\nabla \phi|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu))\phi^2$ be the quadratic form associated to the second variation of area functional at $\Sigma$. Hence,

$$\mathcal{B}(\Sigma) := \overline{C^\infty_c(\Sigma)} \|\cdot\|_{\mathcal{B}(\Sigma)},$$

is a well defined Hilbert space and is naturally embedded in $L^2(\Sigma)$. Moreover, by [Wan20, Lemma 3.2], every $\phi \in \mathcal{B}(\Sigma)$ is locally $W^{1,2}$ on $\Sigma$, and by [Wan20, Proposition 3.5 & Lemma 3.9], $\mathcal{B}(\Sigma) \hookrightarrow L^2(\Sigma)$ is a compact embedding.

With $\mathcal{B}(\Sigma)$, we can define the Morse index via the Jacobi operator $L_\Sigma := \Delta_\Sigma + |A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu)$ associated to $Q_\Sigma$. Thanks to the compact embedding, we can define the $L^2$-eigenvalues and eigenfunctions for $-L_\Sigma$ and derive the spectral decomposition of $L^2(\Sigma)$ as well as $\mathcal{B}(\Sigma)$. Recall that the Morse index of $\Sigma$ has been defined in [MNS19, Dey19] as the...
maximal dimension of the linear subspace of smooth ambient vector fields decreasing its area functional at second order. With $\mathcal{B}(\Sigma)$, an equivalent definition [Wan20, Corollary 3.7] could be

$$\text{Index}(\Sigma) = \sum_{\lambda_j < 0} \dim E_j,$$

where $E_j \subset \mathcal{B}(\Sigma)$ is the $j$-th eigenspace of $-L_{\Sigma}$.

$\Sigma$ is called **non-degenerate** if 0 is not an eigenvalue of $-L_{\Sigma}$. When $\Sigma$ is non-degenerate, by [Wan20, Proposition 3.5] for every $f \in L^2(\Sigma)$, the equation $-L_{\Sigma}u = f$ has a unique solution $u \in \mathcal{B}(\Sigma)$, denoted by $L_{\Sigma}^{-1}(f)$.

Now, for each singular point $p \in \text{Sing}(\Sigma)$ with $B_r(p)$ small enough such that $-L_{\Sigma}$ is strictly positive on $B_0(B_r(p)) := C^\infty_c(B_r(p) \cap \Sigma)$, according to [Wan20, Subsection 3.3], we can define a unique (up to a normalization) Green’s function $G_p \in C^\infty_{\text{loc}}(B_r(p) \cap \Sigma)$ of $L_{\Sigma}$ which vanishes on $\partial B_r(p) \cap \Sigma$. We extend $G_p$ to $\Sigma$ by setting $G_p = 0$ outside $B_r(p)$.

**Lemma 2.23** ([Wan20, Theorem 4.2]). Suppose that an 8-dimensional closed Riemannian manifold $(M, g)$ admits a minimal hypersurface $\Sigma$ with a unit normal $\nu$. Let $f$ be a smooth function defined on $M$ such that $\nu(f)|_\Sigma \not\equiv 0$, $\{c_j\}$ be a sequence of positive real numbers with $c_j \to 0$, and $\{f_j\}$ be a sequence of smooth functions with $f_j \to f$ in $C^4$. Let’s further assume that for each metric $g_j := (1+c_j f_j) g$, there exists a minimal surface $\Sigma_j \subset (M, g_j)$ with $\text{Index}(\Sigma_j) = \text{Index}(\Sigma)$ and $\Sigma_j \to \Sigma$ in the varifold sense with multiplicity 1.

Then after passing to a subsequence, there exists a generalized Jacobi field $0 \neq u \in C^2(\Sigma)$ associated to the subsequence which will still be denoted by $\{\Sigma_j\}_{j \geq 1}$. More precisely, there exist functions $u_j \in C^2(\Sigma)$ and positive real numbers $t_j \to 0_+$ such that

- for every open subset $W \subset\subset M \setminus \text{Sing}(\Sigma)$ and $j$ sufficiently large, $\text{graph}_\Sigma(u_j) \cap W = \Sigma_j \cap W$;
- $u_j/t_j \to u$ in $C^0_{\text{loc}}(\Sigma)$;
- $L_{\Sigma}u = c\nu(f)$ for some real number $c \geq 0$;
- $u \in \mathcal{B}(\Sigma) \oplus \mathbb{R}_{L_{\Sigma}}(\text{Sing}(\Sigma)) := \mathcal{B}(\Sigma) \oplus \bigoplus_{p \in \text{Sing}(\Sigma)} \mathbb{R} G_p$.

The Jacobi field generated above could help us understand the behavior of $\Sigma_j$ near $\Sigma$. In particular, it depicts a picture where generically, one of the singular points of $\Sigma$ can be perturbed away as $\Sigma_j$.

**Lemma 2.24.** In Lemma 2.23, if we further assume that $\Sigma$ is nondegenerate, then there exists an open dense subset $\mathcal{F} \subset C^\infty_c(M \setminus \text{Sing}(\Sigma))$ depending only on $M, g$ and $\Sigma$ with the following property.
For every \( f \in \mathcal{F} \) and every sequence \( c_j \to 0_+ \), if \((M,g_j)\) admits \( \Sigma_j \) as described above, then there exists a small neighborhood \( U_p \subset M \) of some \( p \in \text{Sing}(\Sigma) \) such that \( \text{Sing}(\Sigma_j) \cap U_p = \emptyset \) for infinitely many \( j \).

**Proof.** Let \( u \) be an associated generalized Jacobi fields generated in the previous lemma. It follows from [Wan20, Corollary 3.15 & 3.17] that the asymptotic rate of \( u \) at \( p \in \text{Sing}(\Sigma) \) satisfies

\[
\mathcal{A}\mathcal{R}_p(u) := \sup \left\{ \sigma : \lim_{t \to 0^+} \int_{A_t, x_t(p) \cap \Sigma} u^2(x) \text{dist}(x, p)^{-n-2\sigma} = 0 \right\} \geq \gamma_1^-(C_p),
\]

where \( C_p \) is the tangent cone of \( \Sigma \) at \( p \). \( \gamma_1^-(C_p) \) is a growth rate spectrum for Jacobi field on \( C_p \) (See also [Sim82, CHS84, HS85]).

If \( \mathcal{A}\mathcal{R}_p(u) > \gamma_1^+(C_p) \) for every \( p \in \text{Sing}(\Sigma) \), then again by [Wan20, Corollary 3.15 & 3.17], we have \( u \in \mathcal{B}(\Sigma) \). Since \( L_\Sigma u = cv(f) \) in \( \mathcal{B}(\Sigma) \) with \( c \neq 0 \). However, by [Wan20, Lemma 3.21], the set \( \mathcal{E} \) of \( f \in C^\infty_c(M \setminus \text{Sing}(\Sigma)) \) such that \( \mathcal{A}\mathcal{R}_p(L_\Sigma^{-1}(\nu(f))) > \gamma_1^+(C_p) \) for some \( p \in \text{Sing}(\Sigma) \) is nowhere dense in \( C^\infty_c(M \setminus \text{Sing}(\Sigma)) \). Hence, as long as we choose \( \mathcal{F} := C^\infty_c(M \setminus \text{Sing}(\Sigma)) \setminus \mathcal{E} \), this case could not happen.

Therefore, for \( f \in \mathcal{F} \), there exists a singular point \( p \) at which \( \mathcal{A}\mathcal{R}_p(u) \leq \gamma_1^+(C_p) \), i.e., either \( \mathcal{A}\mathcal{R}_p(u) = \gamma_1^-(C_p) \) or \( \mathcal{A}\mathcal{R}_p(u) = \gamma_1^+(C_p) \) ( [Wan20, Lemma 3.14]). Then it follows from [Wan20, Corollary 4.12] that for some neighborhood \( U_p \supset p \), \( \text{Sing}(\Sigma_j) \cap U_p = \emptyset \) for infinitely many \( j \). \( \square \)

3. Generation of Candidate Minimal Hypersurfaces

In this section, we shall discuss how to generate a candidate minimal hypersurface in a given Riemannian manifold \((M^{n+1}, g)\) with \( H_n(M, \mathbb{Z}_2) = 0 \). For simplicity, whenever it is clear, we shall abuse the use of set relations \( = \) and \( \subset \) for Caccioppoli sets in the measure-theoretic sense, i.e., up to a measure zero set.

3.1. Manifolds with Frankel Property.

**Definition 3.1.** A closed Riemannian manifold \((M^{n+1}, g)\) is said to have **Frankel property**, if any pair of minimal hypersurfaces has nonempty intersections.

Theorem 2.18 and Theorem 2.20 together imply the existence of an (ONVP) sweepout \( \Phi \) and a sequence \( x_i \not\rightarrow x_0 \in \text{nil}(\Phi) \) such that \( |\partial \Phi(x_i)| \rightarrow V \in \mathcal{R} \), where \( V = \sum_i \kappa_i |\Sigma_i| \) for some pairwise disjoint minimal hypersurfaces \( \Sigma_i \), where \( \kappa_i \in \{1, 2\} \).

If the ambient manifold \((M^8, g)\) has Frankel property, then \( V = \kappa |\Sigma| \) for \( \kappa \in \{1, 2\} \). Moreover, one of the following conditions holds:

1. either \( h_{nm}(\Sigma) = \emptyset \);
2. or \( h_{nm}(\Sigma) \neq \emptyset \), \( \kappa = 1 \), and \( \Sigma = \partial^* \Phi(x_0) \) is stable.
In the second case, we will modify the sweepout such that near \( \Sigma \) the mass of each slice is strictly smaller than \( \text{Area}(\Sigma) \), which provides a room for us to perturb the ambient metric without breaking the optimality of the sweepout.

**Lemma 3.2.** For a closed ambient manifold \((M^8, g)\) with Frankel property, let \( \Sigma \) be a minimal hypersurface generated from an (ONVP) sweepout \( \Phi \) via \( x_i \nearrow x_0 \in \mathbf{m}_L(\Phi) \). If \( \mathfrak{h}_{nm}(\Sigma) = \{p\} \), we can construct a new (ONVP) sweepout \( \Psi \) satisfying the following property.

For any \( r > 0 \) small enough, there exists \( \varepsilon_0 > 0 \), an open set \( U \supset \Sigma \), and a compact set \( K \subset U \cap B_r(p) \) containing \( p \) such that for any \( x \in [0, 1] \), we have

\[
\mathbf{M}(\partial \Psi(x)) < \text{Area}(\Sigma) - \varepsilon_0,
\]

provided that \( \partial^* \Psi(x) \cap U \setminus (K \cup \Sigma) \neq \emptyset \).

**Proof.** By the definition of \( \mathfrak{h}_{nm}(\Sigma) \), we can take a geodesic ball \( B_r(p) \) with \( r \) small enough such that \( \partial B_r(p) \) is strictly convex, \( \partial B_r(p) \cap \Sigma \) is a smooth codimension 2 submanifold, and \( \Sigma \) is not OSHAM on either side in \( B_r(p) \).

Since \( H_7(M, \mathbb{Z}_2) = 0 \), \( \Sigma \) is two-sided and separates \( M \) into two connected open component \( M_+ \) and \( M_- \). By Corollary 2.12, they are both Caccioppoli sets. Because \( \partial^* \Phi(x_0) = \Sigma \), by constancy theorem, w.l.o.g., we may assume that \( \Phi(x_0) = M_+ \) and thus, \( \Phi(x) \subset M_+ \) for \( x \in [0, x_0] \). We will only focus on \( M_+ \), since the same process can be performed on \( M_- \) as well.

By the existence of homotopic minimizers [CLS20, Lemma 13] and the mean convexity of \( \partial B_r(p) \), there exists a nested map \( E : [0, 1] \to C(M) \) with \( E(0) = \Omega_1, E(1) = M_+ \) and \( \Omega_1 \Delta M_+ \subset M_+ \cap B_r(p) \) satisfying

- \( \partial \Omega_1 \cap B_r(p) \) is minimal and strictly one-sided area minimizing in \( M_+ \setminus \Omega_1 \) ([CLS20, Lemma 15]);
- \( \mathbf{M}(\partial E(x)) \leq \text{Area}(\Sigma) \);
- \( \mathbf{M}(\partial \Omega_1) < \text{Area}(\Sigma) - 2\varepsilon_0 \), where \( \varepsilon = \varepsilon_0(r) > 0 \).

We can construct an intermediate nested sweepout \( \{\Phi'(x)\}_{x \in [0, x_0]} \) by concatenating \( \{\Phi(x) \cap \Omega_1\}_{x \in [0, x_0]} \) and \( E(x) \), up to reparametrization. By the first bullet above, we have for \( x \in [0, x_0] \),

\[
\mathbf{M}(\partial (\Phi(x) \cap \Omega_1)) \leq \mathbf{M}(\partial \Phi(x)).
\]

So together with the second bullet and a similar construction for \( \{\Phi'(x)\}_{x \in [x_0, 1]} \) on \( M_- \), \( \{\Phi'(x)\}_{x \in [0, 1]} \) is still an optimal nested sweepout.

Then, let \( \Sigma_1 = \partial^d \Omega_1 \). Let \( \tau > 0 \) small enough depending on \( \varepsilon_0 \) and \( U_0 = B_{\tau}(\Sigma_1) \cap \Omega_1 \). By compactness of Caccioppoli sets, we can find a perimeter minimizer \( \Omega_2 \) with the constraint that \( \Omega_1 - U_0 \subset \Omega_2 \subset \Omega_1 \).

**Claim 1.** \( \partial^d \Omega_2 \cap \partial^d \Omega_1 = \emptyset \).

**Proof of Claim 1.** As mentioned in Remark 2.8, we always assume that \( \partial^d \Omega_1 = \overline{\partial^* \Omega_1} \) and \( \partial^d \Omega_2 = \overline{\partial^* \Omega_2} \).
We first note that \( \Omega_2 \) is also a perimeter minimizer with the constraint that \( \Omega_1 - U_0 \subset \Omega_2 \subset M_+ \). Indeed, if this is not true, we can find a perimeter minimizer \( \Omega'_2 \) with \( \Omega_1 - U_0 \subset \Omega'_2 \subset M_+ \) such that
\[
\mathcal{H}^8(\Omega'_2 \cap (M_+ \setminus \Omega_1)) > 0.
\]
In other words, \( \Omega'_2 \neq (\Omega_2 \cap \Omega_1) \). However, by the first bullet above that \( \partial \Omega_1 \) is strictly one-sided area-minimizing in \( M_+ \setminus \Omega_1 \), one can conclude
\[
P(\Omega'_2 \cap \Omega_1) < P(\Omega'_2),
\]
giving a contradiction.

As \( \Omega_2 \) lies on one side of \( M_+ \), T. Ilmanen’s strong maximum principle [Ilm96] and Solomon-White strong maximum principle [SW89] together imply that \( \partial^2 \Omega_2 \cap \Sigma = \emptyset \).

It suffices to verify that \( \partial^2 \Omega_2 \cap (\partial^2 \Omega_1 \cap B_r(p)) = \emptyset \). Indeed, if this is not true, these maximum principles again imply
\[
\partial^2 \Omega_1 \cap B_r(p) \subset \partial^2 \Omega_2,
\]
so \( \emptyset \neq \partial B_r(p) \cap \Sigma \subset \partial^2 \Omega_2 \cap \Sigma \) contradicting to \( \partial^2 \Omega_2 \cap \Sigma = \emptyset \).

By taking \( \tau > 0 \) small enough, the interpolation lemma (Lemma 2.21) induces a nested map \( E' : [0, 1] \to \mathcal{C}(M) \) with \( E'(0) = \Omega_2, E'(1) = \Omega_1 \) and \( M(\partial E'(x)) \leq \text{Area}(\Sigma) - \varepsilon_0 \), for any \( x \in [0, 1] \).

The desired sweepout \( \{\Psi(x)\}_{x \in [0,x_0]} \) on \( M_+ \) is the reparameterized concatenation of \( \{\Psi'(x) \cap \Omega_2\}_{x \in [0,x_0]}, \{E'(x)\}_{x \in [0,1]} \) and \( \{E(x)\}_{x \in [0,1]} \). The conclusion holds on \( M_+ \), if \( U \cap M_+ := M_+ \setminus \Omega_2 \) and \( K \cap M_+ = M_+ \setminus \text{int}(\Omega_1) \) with \( r \) and \( \varepsilon_0 \) chosen above.

The similar process can be done on \( M_- \), and the new (ONVP) sweepout by \( \{\Psi(x)\}_{x \in [0,1]} \), which satisfies the property. \( \square \)

**Remark 3.3.** Due to Frankel property and monotonicity formula for minimal hypersurfaces, the open subset \( U \setminus K \) can be chosen such that every minimal hypersurface \( \Sigma' \) other than \( \Sigma \) intersects \( U \setminus K \).

With the modification above, we can perturb the Riemannian metric to obtain the unique realization property of min-max width.

**Lemma 3.4** (Unique realization of min-max width). *Given a closed Riemannian manifold \( M^8 \) with \( H_7(M, \mathbb{Z}_2) = 0 \), let \( g \) be a metric in
\[
\text{int}(\mathcal{G}_F) = \text{int} \{ g \in \mathcal{G}(M, g) \text{ has Frankel property} \}.
\]
Let \( \Sigma \) be a minimal hypersurface realizing the min-max width \( \mathcal{W} \) generated from an (ONVP) sweepout as in Theorem 2.20. If \( h_{nm}(\Sigma) = \{p\} \), then \( \forall \varepsilon > 0 \), there exists a metric \( g' \in \text{int}(\mathcal{G}_F) \) with \( \|g - g'\|_{C^0} < \varepsilon \) satisfying that \( \mathcal{W}(M, g') \) is uniquely realized by \( \Sigma \) if generated from an (ONVP) sweepout. Furthermore, \( \Sigma \) can be taken nondegenerate in \( (M, g') \).
Proof. By Lemma 3.2, we obtain a new sweepout $\Psi$ for $\Sigma$, a positive constant $\varepsilon_0 > 0$, and an open subset $\tilde{U} = U \setminus (K \cup \Sigma)$ therein.

Firstly, let’s choose a smooth function $f_1 \in C^\infty(M)$ such that $f_1$ is positive in $\tilde{U}$, vanishing outside $\tilde{U}$ and

$$
(3.4) \quad (1 + f_1)^7 \text{Area}_g(\Sigma) \leq \text{Area}_g(\Sigma) + \varepsilon_0/2.
$$

Note that if $\varepsilon_1 \in (0, 1)$ small enough, $g_1 = (1 + \varepsilon_1 f_1)^2 g$ is still inside $\text{int}(\mathcal{G}_F)$.

Moreover, $\Psi$ is also an (ONVP) sweepout on $(M, g')$. Indeed, the metric only changes in $\tilde{U}$, so any slice intersecting $\tilde{U}$ will now has mass no greater than

$$
(3.5) \quad (1 + \varepsilon_1 f_1)^7 (\text{Area}_g(\Sigma) - \varepsilon_0) \leq \text{Area}_g(\Sigma) - \varepsilon_0/2.
$$

Hence, $W(M, g_1) \leq W(M, g)$. Because $g_1 \geq g$, by definition, $W(M, g_1) \geq W(M, g)$ and we can conclude that $W(M, g_1) = W(M, g)$.

Then, Let $q \in \Sigma$ and $r > 0$ small enough such that $B_r(q) \cap \Sigma$ is regular and $B_r(q) \subset \tilde{U}$. We can choose another nonnegative smooth function $f_2 \in C^\infty(B_r(q))$ such that

$$
f_2(x) = \text{dist}(x, \Sigma)^2 \eta(\text{dist}(x, q)),
$$

where $\eta$ is a standard cut-off function. It is not hard to check that for $\varepsilon_2 > 0$ small enough, $g_2 = \exp(\varepsilon_2 f_2) g_1 \in \text{int}(\mathcal{G}_F)$ and similarly, $W(M, g_2) = W(M, g)$. Furthermore, the same proof of [MNS19, Lemma 4] together with the choice of $f_2$ implies that $\Sigma$ is nondegenerate in $(M, g_2)$.

Finally, let $\Psi'$ be another (ONVP) sweepout in $(M, g_2)$ with $y_i \nearrow y_0 \in \text{m}_L(\Psi')$, such that

$$
(3.6) \quad |\partial \Psi'(y_i)| \to \kappa'|\Sigma'| \in \mathcal{R}, \quad \kappa' \in \{1, 2\}.
$$

It suffices to show that $\Sigma' = \Sigma$, and thus $\kappa' = 1$.

Suppose by contradiction that $\Sigma' \neq \Sigma$; Also note that $\Sigma' \cap \Sigma \neq \emptyset$ since $g' \in \text{int}(\mathcal{G}_F)$ and thus, $\Sigma$ does not lie on either side of $\Sigma'$ by strong maximum principles [Ilm96, SW89]. Let’s take $\Psi'$ back to the original metric $g$ and obviously, $\Psi'$ is still an optimal nested sweepout, albeit $\kappa'|\Sigma'|$ may not be inside the critical set. Theorem 2.20 implies that there exists a sequence $y'_i \to y'_0$ with $\kappa''|\Sigma''| = \lim_i |\partial \Psi(y'_i)| \in \mathcal{R}$ realizing the min-max width with $\kappa'' \in \{1, 2\}$. By the nested property, we know that $\Sigma''$ should lie on one side of $\Sigma'$ or coincide with $\Sigma'$. Since $\Sigma' \neq \Sigma$ and $\Sigma$ does not lie on either side of $\Sigma'$, we see that $\Sigma'' \neq \Sigma$. By Frankel property of $(M, g)$, $\Sigma'' \cap \Sigma \neq \emptyset$. As mentioned in Remark 3.3, $\Sigma'' \cap \tilde{U} \neq \emptyset$ and thus,

$$
(3.7) \quad \|\Sigma''\|(M, g_2) > \|\Sigma''\|(M, g) = W(M, g) = W(M, g_2),
$$

contradicting the optimality of $\Psi'$.

In summary, $g' = g_2$ is the desired metric. \hfill \Box
3.2. Manifolds without Frankel Property. If $M$ doesn’t have Frankel property, then there exist two minimal hypersurfaces $\Sigma_1$ and $\Sigma_2$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$. The topological assumption $H_7(M^8, \mathbb{Z}_2) = 0$ implies that $\Sigma_1$ and $\Sigma_2$ are two-sided and each of them separates the ambient manifold.

According to Ilmanen’s strong maximum principle [Ilm96], we can further obtain that

$$(3.8) \quad \overline{\Sigma_1 \cap \Sigma_2} = \emptyset.$$  

The goal of this subsection is to prove the following result.

**Proposition 3.5.** Given a closed Riemannian manifold $(M^{n+1}, g)$ without Frankel property but with $H_n(M, \mathbb{Z}_2) = 0$, there exists a locally one-sided area-minimizing hypersurface $\Sigma$ in $M$.

**Proof.** Take $\Sigma_1$ and $\Sigma_2$ as above. If either of them is locally one-sided area-minimizing, then we are done.

Suppose neither of them is locally one-sided area-minimizing, and due to the separateness, there exists a connected component $N$ in $M \setminus (\Sigma_1 \cup \Sigma_2)$, such that $\partial N = \overline{\Sigma_1 \cup \Sigma_2}$. Let $\Sigma$ be an area minimizer in $[\Sigma_1] \in H_7(N, \mathbb{Z}_2)$. We claim that $\Sigma \subset \text{int}(N)$.

On the one hand, $\Sigma$ can not be either $\Sigma_1$ or $\Sigma_2$ since they are not one-sided homologically area-minimizing.

On the other hand, if $\Sigma$ touches $\overline{\Sigma_1}$ or $\overline{\Sigma_2}$, then the intersection set should not be entirely inside $\text{Sing}(\Sigma_1) \cup \text{Sing}(\Sigma_2)$ due to Ilmanen’s strong maximum principle again. However, if $\Sigma \cap \Sigma_i \neq \emptyset$, Solomon-White’s strong maximum principle [SW89] implies that $\Sigma_i \subset \Sigma$, contradicting the area-minimizing property of $\Sigma$.

In summary, $\Sigma \subset \text{int}(N)$ and is homologically area-minimizing in $N$. Therefore, $\Sigma$ is also locally area-minimizing in $M$. \hfill $\Box$

4. Singular Capacity of Minimal Hypersurfaces

Let $\mathcal{C}$ be the space of stable minimal hypercones in $\mathbb{R}^8$. By the standard dimension reduction argument, every cone in $\mathcal{C}$ has a smooth cross section with $S^7(1)$.

Let $\mathcal{M}$ be the space of triples $(\Sigma; M, g)$, where $(M, g)$ is an open subset of a Riemannian manifold and $\Sigma$ is a minimal hypersurface in $(M, g)$ with finitely many singular points. The topology on $\mathcal{M}$ is induced by $C^{4,\text{loc}}$ convergence in $g$ with fixed $M$ and multiplicity one varifold convergence in $\Sigma$.

**Definition 4.1.** A map $\text{SCap} : \mathcal{M} \to \mathbb{N}$ is called a singular capacity, if

(i) For every nontrivial $C \in \mathcal{C}$ and every open subset $U \subset \mathbb{R}^8$ containing the origin, we have

$1 \leq \text{SCap}(C; \mathbb{R}^8, g_{\text{Euc}}) = \text{SCap}(C; U, g_{\text{Euc}}) < +\infty$;

where $g_{\text{Euc}}$ is the Euclidean metric. We abbreviate for simplicity $\text{SCap}(C; \mathbb{R}^8, g_{\text{Euc}})$ to $\text{SCap}(C)$;
For every \((\Sigma; M, g) \in \mathcal{M}\),

\[
\text{SCap}(\Sigma; M, g) := \sum_{p \in \text{Sing}(\Sigma)} \text{SCap}(C_p),
\]

where \(C_p\) is the unique tangent cone of \(\Sigma\) at \(p\) (conventionally, \(\text{SCap}(\Sigma; M, g) := 0\) if \(\Sigma\) is smooth);

(iii) If \((\Sigma_j; M, g_j) \to (\Sigma; M, g)\) in \(\mathcal{M}\) and \(\Sigma_j\) is stable in \((M, g_j)\), then for every open subset \(U \subset M\) with \(\partial U \cap \text{Sing}(\Sigma) = \emptyset\),

\[
\text{SCap}(\Sigma; U, g) \geq \limsup_{j \to \infty} \text{SCap}(\Sigma_j; U, g_j).
\]

The main goal of this section is to prove the existence of singular capacity on \(\mathcal{M}\).

**Theorem 4.2.** There exists a singular capacity \(\text{SCap}\) on \(\mathcal{M}\) satisfying the following condition.

For every \(\Lambda \geq 1\), there exists \(N(\Lambda) \in \mathbb{N}\) such that

\[
\text{SCap}(C) \leq N(\Lambda)
\]

for every \(C \in \mathcal{C}\) with density at 0 less than or equal to \(\Lambda\).

We start with a quantitative cone rigidity lemma, inspired by Cheeger-Naber [CN13, Theorem 7.3]. Let

\[
\theta(x, r; \mu) := \frac{\mu(B_r(x))}{r^7},
\]

\[
\theta(x; \mu) := \lim_{r \to 0^+} \theta(x, r; \mu),
\]

\[
B_r := B^8_r(0) \subset \mathbb{R}^8,
\]

\[
\mathcal{C}_\Lambda := \{C \in \mathcal{C} : \theta(0; \|C\|) \leq \Lambda\}.
\]

For simplicity, given two varifold \(V_1\) on \((M, g)\) and \(V_2\) on \((M, g_{\text{Euc}})\) with uniform volume bound, as long as \(g\) and \(g_{\text{Euc}}\) are close enough, we will view \(V_2\) as a varifold on \((M, g_{\text{Euc}})\) and define \(F(\{V_1, V_2\})\) in \((M, g_{\text{Euc}})\).

**Lemma 4.3.** For any \(\Lambda, \varepsilon > 0\), there exists \(\delta_1 = \delta_1(\Lambda, \varepsilon) > 0\) such that if \(\Sigma\) is a stable minimal hypersurface in \((\mathbb{B}_5, g)\) with \(\|\Sigma\|((\mathbb{B}_5) \leq \Lambda\) and \(0 \in \Sigma\) satisfying

(i) \(\theta(0, 4; \|\Sigma\|) - \theta(0, 1; \|\Sigma\|) \leq \delta_1;\)

(ii) \(\|g - g_{\text{Euc}}\|_{C^4} \leq \delta_1;\)

Then there exists \(C \in \mathcal{C}\), \(m \geq 1\) such that \(\Sigma\) is \(C^2\) \(\varepsilon\)-close to \(m\) pieces of \(C\) in \(A_{2,3}\) and \(|\Sigma|\) is \(F_{\mathbb{B}_4}\) \(\varepsilon\)-close to \(m|C|\).

**Proof.** This is essentially a corollary of [SS81]. Indeed, if this is false, we can find a sequence \(\{\Sigma_i\}\) with \(\delta_1(\Sigma_i) \leq \frac{1}{i}\) but \(\varepsilon\) far away from any multiple pieces of any cone \(C \in \mathcal{C}\). By Schoen-Simon’s compactness theorem [SS81, Theorem 2], we know that \(|\Sigma_i| \to V \in \mathcal{R}\) in the varifold sense in \(\mathbb{B}_9^2\), and
By the monotonicity formula for minimal hypersurfaces, we have
\begin{equation}
\theta(0, 4; \|V\|) - \theta(0, 1; \|V\|) = 0,
\end{equation}
which implies that \( C = \text{supp}(V) \) is a stable minimal hypercone in \( \mathbb{R}^8 \). Thus, 
\( C \) is smooth outside the origin.

It follows immediately from [SS81, Theorem 1] that for \( i \) large enough, \( \Sigma_i \) is \( C^2 \) \( \varepsilon \)-close to \( m \) pieces of \( C \) in \( A_{2,3} \) and \( |\Sigma| \) is \( F_{B_4} \) \( \varepsilon \)-close to \( m|C| \), contradicting to our assumption at the beginning of the proof. \( \square \)

**Lemma 4.4.** For every \( \Lambda > 1 \), there exists \( \varepsilon(\Lambda) > 0 \) such that for any pair \( C, C' \in \mathcal{C}_\Lambda \) and every \( m \geq 2 \),
\[ F_{B_4}(|C'|, m|C|) \geq \varepsilon(\Lambda). \]

**Proof.** Otherwise, for some \( \Lambda > 1 \), there are \( C_j, C'_j \in \mathcal{C}_\Lambda \) and \( m_j \geq 2 \) such that
\begin{equation}
F_{B_4}(|C'_j|, m_j|C_j|) \to 0.
\end{equation}
By the monotonicity formula for minimal hypersurfaces, we have
\[ 2 \leq m := \limsup_j m_j < \infty. \]

By Schoen-Simon’s compactness [SS81] again, up to a subsequence, \( |C'_j| \to m'|C_\infty| \) for some \( C_\infty \in \mathcal{C}_\Lambda \) and \( m' \geq 1 \). Hence \( |C'_j| \) subconverges to \( mm'|C_\infty| \) multi-graphically near the cross section \( C_\infty \cap \mathbb{S}^7 \). By Sharp’s compactness [Sha17], \( \{C_j\} \) induces a positive Jacobi field over \( S_\infty := C_\infty \cap \mathbb{S}^7 \subset \mathbb{S}^7 \). This implies \( S_\infty \subset \mathbb{S}^7 \) is stable, which is impossible since \( \mathbb{S}^7 \) has positive Ricci curvature. \( \square \)

**Corollary 4.5.** For every \( \varepsilon \in (0, 1) \), there exists \( \delta_2 = \delta_2(\Lambda, \varepsilon) \in (0, 1) \) such that if \( \Sigma \subset (B_5, g) \) is a stable minimal hypersurface with \( \|g - g_{Euc}\|_{C^4} \leq \delta_2 \) and \( F_{B_5}(|\Sigma|, |C|) \leq \delta_2 \) for some \( C \in \mathcal{C}_\Lambda \), then \( \text{Sing}(\Sigma) \cap B_4 \subset B_{\varepsilon} \). Moreover, for any \( x \in B_1 \setminus \Sigma \), we have
- either \( \theta(x; |\Sigma||) \leq \theta(0; |C||) - 2\delta_2; \)
- or \( \text{Sing}(\Sigma) \cap B_4 \subset \{x\} \).

**Proof.** For the first claim, suppose otherwise, there exist \( \Lambda \geq 1, \varepsilon \in (0, 1) \), a family of stable minimal hypersurfaces \( \{\Sigma_j \subset (B_5, g_j)\} \), a family of stable minimal hypercones \( \{C_j \in \mathcal{C}_\Lambda \} \) such that \( \|g_j - g_{Euc}\|_{C^4} \to 0 \), \( F_{B_4}(|\Sigma_j|, |C_j|) \to 0 \) but \( \text{Sing}(\Sigma_j) \cap B_4 \setminus B_{\varepsilon} \neq \emptyset \).

By Lemma 4.4, \( |C_j| \to |C_\infty| \in \mathcal{C}_\Lambda \). Hence,
\[ F_{B_5}(|\Sigma_j|, |C_\infty|) \leq F_{B_5}(|\Sigma_j|, |C_j|) + F_{B_5}(|C_j|, |C_\infty|) \to 0. \]
[SS81, Theorem 1] implies that for \( j \) sufficiently large, \( \text{Sing}(\Sigma_j) \subset B_{\varepsilon} \) contradicting to our assumption.

For the second claim, we also argue by contradiction that there exists a family of stable minimal hypersurfaces \( \{\Sigma_j \subset (B_5, g_j)\} \) as above with
\{ x_j \in \Sigma_j \cap B_1 \} \text{ such that } \limsup_{j \to \infty} \theta(x_j; \|\Sigma_j\|) - \theta(0; \|C_j\|) \geq 0 \text{ and } x_j' \in \text{Sing}(\Sigma_j) \setminus \{x_j\} \neq \emptyset. \text{ Let } C_\infty \text{ be the same limit cone as above.}

If \( C_\infty \) is a hyperplane, then by Allard regularity theorem [All72], we have \( \text{Sing}(\Sigma_j) \cap B_4 = \emptyset \) for \( j \gg 1 \), which violates our assumption.

If \( C_\infty \) is a nontrivial minimal cone, then by upper semi-continuity of density, Allard regularity and the fact that

\[
\limsup_j \theta(x_j; \|\Sigma_j\|) \geq \theta(0; \|C_\infty\|) > 1,
\]
we have \( x_j \to 0 \) and \( x_j' \to 0 \). Hence, by monotonicity formula, Lemma 4.3 can be applied to \( (\eta_{x_j,r_j})_\sharp(\Sigma_j) \), where \( r_j = 2\text{dist}(x_j, x_j')/5 \) to deduce that for sufficiently large \( j \), \( x_j' \notin \text{Sing}(\Sigma_j) \), which also violates our assumption. \( \square \)

**Lemma 4.6.** For every \( \Lambda > 1 \), there exists \( N(\Lambda) \geq 1 \) such that, for every \( C \in \mathcal{C}_\Lambda \) and any sequence of stable minimal hypersurfaces \( \Sigma_j \subset (B_6, g_j) \) satisfying \( (\Sigma_j; B_6, g_j) \to (C; B_6, g_{\text{Euc}}) \) in \( \mathcal{M} \), we have

\[
\limsup_{j \to \infty} \sharp(\text{Sing}(\Sigma_j) \cap B_4) \leq N(\Lambda).
\]

**Proof.** The lemma essentially follows from [NV20] in dimension 8. For completeness, here we give a simpler and more self-contained proof.

Let \( \varepsilon_1 = \varepsilon(2\Lambda) \) given in Lemma 4.4 and \( \delta_3 := \min\{\delta_1(\varepsilon_1/10, 2\Lambda), \delta_2(\varepsilon_1/10, 2\Lambda)\}/10 \), where \( \delta_1 \) is given by Lemma 4.3 and \( \delta_2 \) is given by Corollary 4.5.

Clearly, it suffices to prove inductively that for each integer \( 0 \leq k \leq 1 + (\Lambda - 1)/\delta_3 \),

\[
\sup\{\limsup_{j \to \infty} \sharp(\text{Sing}(\Sigma_j) \cap B_4) \} < +\infty,
\]
where the supremum is taken among all the sequences of stable minimal hypersurfaces \( \{\Sigma_j \subset (B_6, g_j)\}_{j \geq 1} \) such that \( (\Sigma_j; B_6, g_j) \to (C; B_6, g_{\text{Euc}}) \) in \( \mathcal{M} \) for some \( C \in \mathcal{C}_{1+k\delta_3} \).

For \( k = 0 \), by volume monotonicity formula and Allard regularity, (4.3) holds and the upper bound could be taken to be 1.

Suppose (4.3) holds for \( k - 1 \) but fails for \( k \), and then there exists a family of stable minimal hypersurfaces \( \Sigma_j \subset (B_6, g_j) \), with \( (\Sigma_j; B_6, g_j) \to (C; B_6, g_{\text{Euc}}) \) in \( \mathcal{M} \) for some \( C \in \mathcal{C}_{1+k\delta_3} \) but \( \sharp(\text{Sing}(\Sigma_j) \cap B_4) \to \infty \). It follows from Corollary 4.5 that the first bullet holds in \( B_1 \cap \Sigma_j \) for \( j \gg 1 \).

Let \( x_j \in \text{Sing}(\Sigma_j) \subset B_1 \), and we can define

\[
r_j := \inf\{r > 0 : \theta(x_j, 4; \|\Sigma_j\|) - \theta(x_j, r; \|\Sigma_j\|) \leq 2\delta_3 \} > 0.
\]

By Schoen-Simon compactness, \( x_j \to 0 \in \mathbb{R}^8 \) and \( r_j \to 0_+ \). By the choice of \( \delta_3 \) and \( r_j \) as well as the volume monotonicity formula, for \( j \gg 1 \),

\[
\theta(x_j, 4s; \|\Sigma_j\|) - \theta(x_j, s; \|\Sigma_j\|) \leq \delta_1(\varepsilon_1/10, 2\Lambda)/2, \quad \forall r_j < s \leq 1.
\]

Hence by Lemma 4.3, for \( j \gg 1 \) and each \( s \in (r_j, 1] \),

\[
\mathbf{F}_{B_4}(\eta_{x_j,s})_\sharp|\Sigma_j|, m_j^s|C_j^s| \leq \varepsilon_1/10,
\]
for some \( C_j^s \in \mathcal{C}_{2\Lambda} \) and \( m_j^s \in \mathbb{N} \). Moreover, \( \text{Sing}(\Sigma_j) \subset B_{2r_j} \).
On the one hand, for $s \in [1/2, 1]$, since $|\Sigma_j| \to |C|$, we have $m^s_j = 1$ and $C^2_j$ can be chosen to be $C$ for $j$ even larger. On the other hand, since for every pair of varifolds $V_1, V_2$ and every $r \in (0, 1)$ we have
\[ F_{B_4}((\eta_{0,r})_2 V_1, (\eta_{0,r})_2 V_2) \leq F_{B_4}(V_1, V_2)/r, \]
thus we have
\[ F_{B_4}(m^s_j |C^s_j|, m^{2s}_j |C^{2s}_j|) \leq F_{B_4}(m^s_j |C^s_j|, (\eta_{x_j,s})_2 |\Sigma_j|) + F_{B_4}(m^{2s}_j |C^{2s}_j|, (\eta_{x_j,s})_2 |\Sigma_j|) \]
\[ \leq \varepsilon_1/5 + 2F_{B_4}(m^{2s}_j |C^{2s}_j|, (\eta_{x_j,s})_2 |\Sigma_j|) < \varepsilon_1, \]
where we utilise the dilation invariance of the cone $C^{2s}_j$. By the choice of $\varepsilon_1$ and lemma 4.4, we can conclude that $m^s_j = m^{2s}_j \equiv 1$ for $j >> 1$ and $s \in (r_j, 1]$.

By Lemma 4.3, \( \tilde{\Sigma}_j := (\eta_{x_j,r_j})_2 \Sigma_j \) subconverges to some stable minimal hypersurface $\Sigma_\infty \subset (\mathbb{R}^8, g_{Euc})$. By Lemma 4.4, the limit varifold should have multiplicity one due to the fact that $m^{K_{r_j}} = 1$ for all $K > 1$ and $j >> 1$. Moreover, $\text{Sing}(\Sigma_\infty) \subset B_3$ is a finite set containing 0, and by volume monotonicity formula,
\[ \theta(0, \infty; \|\Sigma_\infty\|) \leq \theta(0; \|C\|), \]
\[ \theta(0, 1; \|\Sigma_\infty\|) \leq \theta(0; \|C\|) - 2\delta_3. \]
Hence, by Corollary 4.5, only the case in the first bullet occurs, i.e., for every $x' \in \text{Sing}(\Sigma_\infty)$, \( \theta(x'; \|\Sigma_\infty\|) \leq \theta(0; \|C\|) - 2\delta_3. \)

Since $\sharp(\text{Sing}(\Sigma_j) \cap B_4) \to \infty$ but $\Sigma_\infty$ only has finitely many singular points, there exists $\hat{x} \in \text{Sing}(\Sigma_\infty) \cap B_4$ and $r_j \to 0_+$ such that $\sharp(\mathbb{B}_{r_j}(\hat{x}) \cap \text{Sing}(\tilde{\Sigma}_j)) \to \infty$. As the tangent cone of $\Sigma_\infty$ at $\hat{x}$ has density bounded above by $\theta(0; \|C\|) - 2\delta_3 \leq 1 + (k - 1)\delta_3$, the blow-up picture contradicts to the inductive assumption.

By induction, we conclude the existence of such finite $N(\Lambda)$. \hfill \square

**Proof of theorem 4.2**. By (i) and (ii) of definition 4.1, it suffices to define $\text{SCap}$ restricted to $\mathcal{C}$ and then verify (iii). Let $\{\Lambda_k\}_{k \geq 0}$ be an increasing family of real numbers given by
- $\Lambda_0 := 1$;
- $\Lambda_k := \Lambda_{k-1} + \delta_2(2 + \Lambda_{k-1}, 1)$, where $\delta_2$ is given by Corollary 4.5 and can be assumed WLOG to be monotonically decreasing in $\Lambda$.

For a trivial hyperplane $P \subset \mathbb{R}^8$, we have no choice but define $\text{SCap}(P) := 0$.

For a non-trivial $C \in \mathcal{C}$ with $\theta(0; \|C\|) \in [\Lambda_{k-1}, \Lambda_k)$, let’s define $\text{SCap}(C) := \prod_{j=0}^k (1 + N(\Lambda_j))$, where $N$ is given by lemma 4.6.

To verify Definition 4.1 (iii), by [SS81, Theorem 1] again, it suffices to show that if $\Sigma_j \subset (\mathbb{B}_6, g_j)$ are stable minimal hypersurfaces, $C \in \mathcal{C}$ and
\((\Sigma_j; B_6, g_j) \to (C; B_6, g_{Euc})\) in \(\mathcal{M}\), then
\[
\limsup_j \text{SCap}(\Sigma_j; B_4, g_j) \leq \text{SCap}(C).
\]

Suppose \(\theta(0; ||C||) \in [\Lambda_{k-1}, \Lambda_k]\), and we have the following three cases (up to subsequences).

**Case 1.** If \(\#(\text{Sing}(\Sigma_j) \cap B_4) > 1\) for \(j >> 1\), then by Corollary 4.5, each singularity \(p\) of \(\Sigma_j\) has density bounded above by
\[
\theta(0; ||C||) - 2\delta_2(2 + \Lambda_{k-1}, 1) < \Lambda_{k-1}
\]
and thus,
\[
\text{SCap}(C_p) \leq \prod_{j=0}^{k-1} (1 + N(\Lambda_j)).
\]

Therefore,
\[
\limsup_{j \to \infty} \text{SCap}(\Sigma_j; B_4, g_j) \leq \prod_{j=0}^{k-1} (1 + N(\Lambda_j)) \limsup_{j \to \infty} \#(\text{Sing}(\Sigma_j) \cap B_4) \leq \text{SCap}(C).
\]

**Case 2.** If \(\text{Sing}(\Sigma_j) \cap B_4\) is a single point \(x_j\) for \(j >> 1\), then by volume monotonicity formula, \(\limsup_j \theta(x_j; ||\Sigma_j||) \leq \theta(0; ||C||) < \Lambda_k\). Hence by definition, for \(j >> 1\), \(\theta(x_j; ||\Sigma_j||) < \Lambda_k\) and
\[
\limsup_{j} \text{SCap}(\Sigma_j; B_4, g_j) = \limsup_{j} \text{SCap}(C_{x_j}) \leq \text{SCap}(C).
\]

**Case 3.** If \(\text{Sing}(\Sigma_j) \cap B_4 = \emptyset\) for \(j >> 1\), then apparently,
\[
\limsup_{j} \text{SCap}(\Sigma_j; B_4, g_j) = 0 \leq \text{SCap}(C).
\]

\(\square\)

We will end this section with the following application of Singular Capacity.

**Lemma 4.7.** Let \(\mathcal{G}\) be the set of all \(C^k\) (\(k \geq 4\) or \(k = \infty\)) metrics on a closed Riemannian manifold \(M^8\) and \(\tilde{\mathcal{M}}\) be a subspace of \(\mathcal{M}\), consisting of triples \((\Sigma; M, g)\) satisfying the following

1. \(\forall (\Sigma; M, g) \in \tilde{\mathcal{M}}, \Sigma\) is nondegenerate.
2. For any sequence \(\left\{ (\Sigma_j; M, g_j) \in \tilde{\mathcal{M}} \right\}_{j=1}^{\infty}\) and \((\Sigma_{\infty}; M, g_{\infty}) \in \tilde{\mathcal{M}}\), with \(g_j \to g_{\infty}\) in \(C^k(M)\), we have
   \[
   \liminf_j \text{index}(\Sigma_j) = \text{index}(\Sigma_{\infty}),
   \]
   and
   \[
   |\Sigma_j| \to |\Sigma_{\infty}|,
   \]
in the varifold sense.

(3) The projection map \( \Pi : \mathcal{M} \rightarrow \mathcal{I} \) onto the third variable is injective.

Then for every metric \( g \) in the interior of \( \overline{\Pi(\mathcal{M})} \subset \mathcal{I} \), there is a family of triples \((\Sigma_i; M, g_i) \in \mathcal{M} \) such that \( g_i \rightarrow g \) in \( C^k \) and \( \text{Sing}(\Sigma_i) = \emptyset \).

Proof. Let \( g \) be a metric in the interior of \( \overline{\Pi(\mathcal{M})} \) and \( \mathcal{I} \) is an arbitrary \( C^k \) neighborhood of \( g \) which is also contained in \( \overline{\Pi(\mathcal{M})} \). Let \((\Sigma_0; M, g_0) \in \mathcal{M} \) such that \( g_0 \in \mathcal{I} \). Since each singular point of \( \Sigma_0 \) is isolated, we have \( \text{SCap}(\Sigma_0; M, g_0) < +\infty \).

We shall prove inductively that there exists a sequence \( \{(\Sigma_i; M, g_i)\}_{i \in \mathbb{N}} \subset \mathcal{M} \cap \Pi^{-1}(\mathcal{I}) \) such that for each \( l \),

- either \( \text{Sing}(\Sigma_i) = \emptyset \) (and take \( (\Sigma_{i+1}; g_{i+1}) := (\Sigma_i; g_i) \));
- or \( \text{SCap}(\Sigma_{i+1}; M, g_{i+1}) \leq \text{SCap}(\Sigma_i; M, g_i) - 1 \).

Note that by definition 4.1, \( \text{SCap}(\Sigma; M, g) = 0 \iff \text{Sing}(\Sigma) = \emptyset \). Hence for sufficiently large \( N \geq \text{SCap}(\Sigma; M, g) \), \( \Sigma_N \) is a closed smooth minimal hypersurface in \( g_N \) and \((\Sigma_N; M, g_N)\) can be one item in the desired sequence.

Suppose \((\Sigma_l, M, g_l)\) for some \( l \in \mathbb{N} \) has been constructed, and WLOG \( \text{Sing}(\Sigma_l) \neq \emptyset \). Let \( \mathcal{F} \subset C^\infty_c(M \setminus \text{Sing}(\Sigma_l)) \) depending on \( \Sigma_l, M \) and \( g_l \) be specified as in Lemma 2.24 and fix an \( f \in \mathcal{F} \). Since \( \Pi(\mathcal{M}) \) is \( C^k \)-dense in \( \mathcal{I} \), there exists a sequence of \( (\Sigma^{(j)}; M, g^{(j)}) \in \mathcal{M} \) so that \( g^{(j)} = g_l(1 + f^{(j)}/j) \) for some smooth functions \( f^{(j)} \rightarrow f \) in \( C^k \).

The definition of \( \mathcal{M} \) implies that \( |\Sigma^{(j)}| \rightarrow |\Sigma_l| \). Let \( \{U_p \ni p\}_{p \in \text{Sing}(\Sigma_l)} \) be a finite pairwise disjoint family of open subsets of \( M \) such that

\[
\text{index}(\Sigma_l \setminus \bigcup U_p) = \text{index}(\Sigma_l).
\]

By Lemma 2.24, after passing to a subsequence, there exists some \( p^* \in \text{Sing}(\Sigma_l) \) such that \( \text{Sing}(\Sigma^{(j)}) \cap U_{p^*} = \emptyset \) for \( j >> 1 \). In addition, Condition (2) and the choice of \( U_p \) implies that for sufficiently large \( j \), in each \( U_p, \Sigma^{(j)} \) is stable. Hence by Definition 4.1,

\[
\limsup_{j \rightarrow \infty} \text{SCap}(\Sigma^{(j)}; M, g^{(j)}) = \limsup_{j \rightarrow \infty} \sum_{p \in \text{Sing}(\Sigma_l)} \text{SCap}(\Sigma^{(j)}; U_p, g^{(j)})
\]

\[
\leq \sum_{p \neq p^* \in \text{Sing}(\Sigma_l)} \text{SCap}(\Sigma_l; U_p, g_l)
\]

\[
\leq \text{SCap}(\Sigma_l; M, g_l) - \text{SCap}(\Sigma_l; U_{p^*}, g_l)
\]

\[
\leq \text{SCap}(\Sigma_l; M, g_l) - 1.
\]

We can then choose \( (\Sigma_{l+1}; g_{l+1}) := (\Sigma^{(j)}, g^{(j)}) \) for a sufficiently large \( j \) such that \( g^{(j)} \in \mathcal{I} \) and \( \text{SCap}(\Sigma^{(j)}; M, g^{(j)}) \leq \text{SCap}(\Sigma_l; M, g_l) - 1. \) \(\Box\)
5. Proof of Theorem 1.1

Given a closed Riemannian manifold \( M^8 \) with \( H_7(M, \mathbb{Z}_2) = 0 \) and \( k \geq 4 \) \((k \text{ could be } \infty)\), we define
\[
\mathcal{G} := \left\{ C^k \text{ Riemannian metrics on } M \right\}; \\
\mathcal{G}_F := \left\{ g \in \mathcal{G} | (M, g) \text{ has Frankel property} \right\}; \\
\mathcal{G}_{NF} := \mathcal{G} \setminus \mathcal{G}_F; \\
\mathcal{R} := \left\{ g \in \mathcal{G} | (M, g) \text{ admits a nondegenerate smooth minimal hypersurface} \right\}.
\]

By B. White’s structure theorem of smooth minimal hypersurfaces [Whi91, Whi17], \( \mathcal{R} \) is an open subset in \( \mathcal{G} \). It suffices to show that \( \mathcal{R} \) is dense in \( \mathcal{G} \).

Observe that for any given \( g \in \mathcal{G}_{NF} \), the candidate minimal hypersurface \( \Sigma \) generated in Section 3 is at least locally one-sided area-minimizing. Therefore, by Lemma 2.22, \( \mathcal{R} \) is dense in \( \mathcal{G}_{NF} \).

Now, let’s focus on its complement \( \text{int}(\mathcal{G}_F) \) and we shall show that \( \mathcal{R} \) is dense in \( \text{int}(\mathcal{G}_F) \).

Proof of theorem 1.1. As mentioned above, it suffices to show that \( \mathcal{R} \) is dense in \( \text{int}(\mathcal{G}_F) \). Let
\[
\mathcal{G}_F^s := \left\{ g \in \text{int}(\mathcal{G}_F) : \exists! \text{ minimal hypersurface } \Sigma \text{ generated from an (ONVP) sweepout and realizing } W(M, g), \text{ and } \Sigma \text{ is non-degenerate stable} \right\}
\]

By Lemma 3.4, \( \forall g \in \text{int}(\mathcal{G}_F) \setminus \overline{\mathcal{G}_F^s} \), there exists a minimal hypersurface \( \Sigma \) in \( (M, g) \) with \( h_{nm}(\Sigma) = \emptyset \). Therefore, by Lemma 2.22,
\[
\text{int}(\mathcal{G}_F) \setminus \text{int}(\overline{\mathcal{G}_F^s}) \subset \text{int}(\mathcal{G}_F) \setminus \overline{\mathcal{G}_F^s} \subset \mathcal{R}. \tag{5.1}
\]

On the other hand, it’s easy to verify that the set
\[
\mathcal{M} := \left\{ (\Sigma; M, g) : g \in \overline{\mathcal{G}_F^s}, \Sigma \text{ is the nondegenerate stable minimal hypersurface generated from an (ONVP) sweepout and realizing } W(M, g) \right\}
\]
satisfies the assumptions in Lemma 4.7. Hence,
\[
\text{int}(\overline{\mathcal{G}_F^s}) \subset \mathcal{R}. \tag{5.2}
\]

Combining (5.1) and (5.2), we have \( \text{int}(\mathcal{G}_F) \subset \mathcal{R} \). This completes the proof of Theorem 1.1. \( \square \)

References

[All72] W. K. Allard. On the First Variation of a Varifold. Annals of Mathematics, 95(3):417–491, 1972.

[Alm65] F. J. Almgren. The Theory of Varifolds: A Variational Calculus in the Large for the k-Dimensional Area Integrand. Princeton; Institute for Advanced Study, 1965.

[BDGG69] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. Inventiones mathematicae, 7(3):243–268, September 1969.

[CHS84] L. Caffarelli, R. Hardt, and L. Simon. Minimal surfaces with isolated singularities. Manuscripta mathematica, 48(1):1–18, 1984.
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[CL20] G. R. Chambers and Y. Liokumovich. Existence of minimal hypersurfaces in complete manifolds of finite volume. Inventiones mathematicae, 219(1):179–217, January 2020.

[CLS20] O. Chodosh, Y. Liokumovich, and L. Spolaor. Singular behavior and generic regularity of min-max minimal hypersurfaces. arXiv:2007.11560 [math], August 2020.

[CN13] J. Cheeger and A. Naber. Quantitative Stratification and the Regularity of Harmonic Maps and Minimal Currents. Communications on Pure and Applied Mathematics, 66(6):965–990, June 2013.

[Dey19] A. Dey. Compactness of certain class of singular minimal hypersurfaces. arXiv:1901.05840 [math], January 2019.

[Ede21] N. Edelen. Degeneration of 7-dimensional minimal hypersurfaces which are stable or have bounded index. arXiv:2103.13563 [math], March 2021.

[EG15] L. C. Evans and R. F. Gariepy. Measure Theory and Fine Properties of Functions. Textbooks in Mathematics. CRC Press, Boca Raton, rev. ed edition, 2015.

[Fal80] K. J. Falconer. Continuity properties of k-plane integrals and Besicovitch sets. Mathematical Proceedings of the Cambridge Philosophical Society, 87(2):221–226, March 1980.

[Fed69] H. Federer. Geometric Measure Theory. Springer, Berlin; Heidelberg; New York, 1969.

[Fed70] H. Federer. The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. Bulletin of the American Mathematical Society, 76(4):767–771, July 1970.

[FF60] H. Federer and W. H. Fleming. Normal and Integral Currents. Annals of Mathematics, 72(3):458–520, 1960.

[Giu84] E. Giusti. Minimal Surfaces and Functions of Bounded Variation. Birkhäuser Boston, Boston, MA, 1984.

[Gut11] L. Guth. Volumes of balls in large Riemannian manifolds. Annals of Mathematics, 173(1):51–76, January 2011.

[HS85] R. Hardt and L. Simon. Area minimizing hypersurfaces with isolated singularities. Journal für die reine und angewandte Mathematik, 362:102–129, 1985.

[IIm96] T. Ilmanen. A strong maximum principle for singular minimal hypersurfaces. Calculus of Variations and Partial Differential Equations, 4(5):443–467, August 1996.

[Law91] G. R. Lawlor. A sufficient criterion for a cone to be area-minimizing, 1991.

[LY02] F. Lin and X. Yang. Geometric Measure Theory: An Introduction. Number 1 in Advanced Mathematics. Science Press [u.a.], Beijing, 2002.

[Mag12] F. Maggi. Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory. Number 135 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York, 2012.

[MNS19] F. C. Marques, A. Neves, and A. Song. Equidistribution of minimal hypersurfaces for generic metrics. Inventiones mathematicae, 216(2):421–443, May 2019.
[Moo06] J. D. Moore. Bumpy Metrics and Closed Parametrized Minimal Surfaces in Riemannian Manifolds. *Transactions of the American Mathematical Society*, 358(12):5193–5256, 2006.

[Moo07] J. D. Moore. Correction for: "Bumpy Metrics and Closed Parametrized Minimal Surfaces in Riemannian Manifolds". *Transactions of the American Mathematical Society*, 359(10):5117–5123, 2007.

[Mor16] F. Morgan. *Geometric Measure Theory : A Beginner’s Guide*. Tokyo: Academic Press, fifth edition. edition, 2016.

[NV20] A. Naber and D. Valtorta. The singular structure and regularity of stationary varifolds. *Journal of the European Mathematical Society*, 22(10):3305–3382, July 2020.

[Pit81] J. T. Pitts. *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*. Princeton University Press, 1981.

[Sha17] B. Sharp. Compactness of minimal hypersurfaces with bounded index. *Journal of Differential Geometry*, 106(2):317–339, June 2017.

[Sim68] J. Simons. Minimal Varieties in Riemannian Manifolds. *Annals of Mathematics*, 88(1):62–105, 1968.

[Sim82] L. Simon. On isolated singularities of minimal surfaces. *Miniconference on Partial Differential Equations*, pages 70–100, January 1982.

[Sim83] L. Simon. Asymptotics for a Class of Non-Linear Evolution Equations, with Applications to Geometric Problems. *Annals of Mathematics*, 118(3):525–571, 1983.

[Sim84] L. Simon. *Lectures on Geometric Measure Theory*. Number 3 in Proceedings of the Centre for Mathematical Analysis / Australian National University. Centre for Mathematical Analysis, Australian National University, Canberra, 1984.

[Sim21] L. Simon. Stable minimal hypersurfaces in $\mathbb{R}^{N+1+k}$ with singular set an arbitrary closed $k$ in $\{0\} \times \mathbb{R}^t$. *arXiv preprint arXiv:2101.06401*, 2021.

[Sma89] N. Smale. Minimal Hypersurfaces with Many Isolated Singularities. *Annals of Mathematics*, 130(3):603–642, 1989.

[Sma93] N. Smale. Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds. *Communications in Analysis and Geometry*, 1(2):217–228, 1993.

[Sma99] N. Smale. Singular homologically area minimizing surfaces of codimension one in Riemannian manifolds. *Inventiones mathematicae*, 135(1):145–183, January 1999.

[SS81] R. Schoen and L. Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, November 1981.

[SSY75] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134:275–288, 1975.

[SW89] B. Solomon and B. White. A Strong Maximum Principle for Varifolds that are Stationary with Respect to Even Parametric Elliptic Functionals. *Indiana University Mathematics Journal*, 38(3):683–691, 1989.

[Wan20] Z. Wang. Deformations of Singular Minimal Hypersurfaces I, Isolated Singularities. *arXiv:2011.00548 [math]*, November 2020.

[Whi85] B. White. Generic Regularity of Unoriented Two-Dimensional Area Minimizing Surfaces. *The Annals of Mathematics*, 121(3):595, May 1985.

[Whi91] B. White. The Space of Minimal Submanifolds for Varying Riemannian Metrics. *Indiana University Mathematics Journal*, 40(1):161–200, 1991.

[Whi17] B. White. On the bumpy metrics theorem for minimal submanifolds. *American Journal of Mathematics*, 139(4):1149–1155, 2017.
[Whi19] B. White. Generic Transversality of Minimal Submanifolds and Generic Regularity of Two-Dimensional Area-Minimizing Integral Currents. \textit{arXiv:1901.05148 [math]}, December 2019.

[Wic14] N. Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. \textit{Annals of Mathematics}, 179(3):843–1007, May 2014.

[Yau82] S.-T. Yau. Problem section. In \textit{Seminar on Differential Geometry. (AM-102)}, pages 669–706. Princeton University Press, 1982.

[Zho17] X. Zhou. Min–max hypersurface in manifold of positive Ricci curvature. \textit{Journal of Differential Geometry}, 105(2):291–343, February 2017.

[Zho19] X. Zhou. On the Multiplicity One Conjecture in Min-max theory. \textit{arXiv:1901.01173 [math]}, January 2019.

[ZZ17] X. Zhou and J. J. Zhu. Min-max theory for constant mean curvature hypersurfaces. \textit{arXiv:1707.08012 [math]}, July 2017.

[ZZ18] X. Zhou and J. J. Zhu. Existence of hypersurfaces with prescribed mean curvature I - Generic min-max. \textit{arXiv:1808.03527 [math]}, August 2018.

\textsc{Department of Mathematics, Princeton University, Fine Hall, 304 Washington Road, Princeton, NJ 08540, USA}
\textit{Email address: yl15@math.princeton.edu}

\textsc{Department of Mathematics, Princeton University, Fine Hall, 304 Washington Road, Princeton, NJ 08540, USA}
\textit{Email address: zhihanw@math.princeton.edu}