Invocations of a canonical curve.

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Abstract: We give a geometrical characterization of the ideal of quadrics containing a canonical curve with an involution. This implies to study involutions of rational normal scrolls and Veronese surfaces.

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Introduction: Let $C$ be a nonhyperelliptic smooth curve of genus $\pi$. An involution of $C$ is an automorphism $\varphi : C \rightarrow C$ such that $\varphi^2 = id$. It induces a double cover $\gamma : C \rightarrow C/\varphi = X$, where $X$ is a smooth curve of genus $g$. We say that $C$ has an involution of genus $g$. By Hurwitz formula, we know that $\pi \geq 2g - 1$. It is well known that a general smooth curve of genus $\pi \geq 3$ has no nontrivial automorphisms. In particular a smooth curve with an involution is not generic.

In this paper we give a geometric characterization of the ideal of quadrics containing the canonical model of a nonhyperelliptic curve $C$ with an involution. Let $C_K \subset P^{\pi-1}$ be the canonical model of $C$. We see that an involution of $C_K$ is a harmonic involution; that is, it can be extended to $P^{\pi-1}$.

An involution $\varphi$ of $P^n$ has two complementary spaces of base points $S_1$ and $S_2$. Moreover, $\varphi$ induces an involution $\varphi^*$ in the space of quadrics of $P^n$. This involution has two spaces of base points: the base quadrics, that is, quadrics containing the spaces $S_1$ and $S_2$ and the harmonic quadrics, that is, quadrics such that $S_1$ and $S_2$ are polar respect to them. A subspace $\Sigma \subset P(H^0(\mathcal{O}_{P^n}(2)))$ is called a base-harmonic system respect to $S_1, S_2$ when it is a fixed space of $\varphi^*$. In this case $\Sigma_b$ and $\Sigma_h$ will denote the base quadrics and the harmonic quadrics of $\Sigma$ respectively.

We prove the following Theorem:

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Theorem 2.4.5  Let $C_K \subset P^{\pi - 1}$ be the canonical curve of genus $\pi$, with $\pi > 4$. If $C_K$ has an involution of genus $g$ then $\pi \geq 2g - 1$ and the quadrics of $P(H^0(I_{C_K}(2)))$ are a base-harmonic system respect to the base spaces $P^{g-1}$, $P^{\pi - g - 1}$ that contains $(g - 1)(\pi - g - 2)$ independent base quadrics. Conversely, these conditions are sufficient to grant the existence of an involution, except when:

1. $\pi = 6, g = 2$ and $C_K$ has a $g_2$; or
2. $\pi = 2g, 2g + 1$ or $2g + 2$ and $C_K$ is trigonal.

First we prove that the conditions are sufficient when the curve $C_K$ is complete intersection of quadrics. The Enriqües-Babbage Theorem says that $C_K$ is the complete intersection of quadrics except when it is trigonal or $C$ is a quintic smooth curve. In these cases the quadrics containing the canonical curve intersect in a rational normal scroll and in the Veronese surface respectively.

In order to examine the special cases we make an study of the harmonic involutions of the rational normal scrolls and the Veronese surface. We compute the number of base quadrics on each case. From this calculus we obtain the Corollaries 2.4.2 and 2.4.3:

Corollary 2.4.2  The unique involutions on a trigonal canonical curve of genus $\pi$, $\pi > 4$ are of genus $\pi/2, \pi - 1/2$ or $\pi - 2/2$.

Corollary 2.4.3  The unique involutions on a smooth quintic plane curve are of genus 2.

Furthermore, we make a particular geometrical study of the canonical curves of genus 4 with an involution of genus 2 and genus 1.

Note that to compute the number of independent base quadrics containing the canonical curves with and involution we need the result about the projective normality of the canonical scrolls (see [2], §5). We will follow the notation of [1] and [3] to work with scrolls and ruled surfaces.

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1 Harmonic involutions.
Definition 1.1 Let $X \subset \mathbb{P}^n$ be a projective variety. An isomorphism $\varphi : X \to X$ is called an involution if $\varphi^2 = \text{Id}$. Moreover, if $\varphi$ is the restriction of an involution $\overline{\varphi} : \mathbb{P}^n \to \mathbb{P}^n$ then $\varphi$ is called a harmonic involution.

Proposition 1.2 Let $X \subset \mathbb{P}^n$ be a linearly normal projective variety. An involution $\varphi : X \to X$ is harmonic if and only if $\varphi^* (X \cap H) \sim X \cap H$ for all hyperplane $H$.

Proof: If $\varphi$ is harmonic we clearly have that $\varphi^* (X \cap H) \sim X \cap H$.

Conversely, if $\varphi^* (X \cap H) \sim X \cap H$ for all hyperplane, then we have an involution $\varphi^* : H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(1))$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathbb{P}(H^0(\mathcal{O}_X(1))) & \xrightarrow{\mathbb{P}(\varphi^*)} & \mathbb{P}(H^0(\mathcal{O}_X(1))) \\
\uparrow & & \uparrow \\
X & \xrightarrow{\varphi} & X
\end{array}
$$

so $\varphi$ extends to $\mathbb{P}^n$.

Examples:

1. Any involution of a rational curve $D_n \subset \mathbb{P}^n$ is harmonic.

   It is sufficient to note that $\varphi^* (D_n \cap H)$ has degree $n$; because $D_n$ is a rational curve, it follows that $\varphi^* (D_n \cap H) \sim D_n \cap H$.

2. Any involution of a canonical curve $C_\pi \subset \mathbb{P}^{\pi-1}$ of genus $\pi$ is harmonic.

   The linear system $|\varphi^* (C_\pi \cap H)|$ has degree $2\pi - 2$ and it has dimension $\pi$, so it is the canonical linear system and $\varphi^* (C_\pi \cap H) \sim C_\pi \cap H$.

3. Any involution of a normal rational scroll $R_{n-1} \subset \mathbb{P}^n$ with invariant $e > 0$ is harmonic.

   Let $S_e = \mathbb{P}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-e))$ be the ruled surface associated to $R_{n-1}$. We know that $H \cap R_{n-1} \sim X_0 + bf$. Since $\varphi$ is an isomorphism, $\varphi^* (X_0)^2 = X_0^2$ and $\varphi^* (f)^2 = f^2 = 0$. But, in $S_e$ with $e > 0$, $X_0$ is the unique curve with negative self-intersection and $f$ is the unique curve with self-intersection 0. From this $\varphi^* (X_0) \sim X_0$ and $\varphi^* (f) \sim f$. Then:

   $$
   \varphi^* (H \cap R_{n-1}) \sim \varphi^* (X_0 + bf) \sim \varphi^* (X_0) + bf \sim X_0 + bf \sim H \cap R_{n-1}
   $$

We recall some basic facts about involutions in a projective space.
Proposition 1.3 Any involution $\varphi$ of $\mathbb{P}^n$ has two complementary spaces $S_1, S_2 \subset \mathbb{P}^n$ of base points. In this way, the image of a point $P$ is the fourth harmonic of $P$, $l \cap S_1$ and $l \cap S_2$, where $l$ is the unique line passing through $P$ verifying $l \cap S_1 \neq \emptyset$ and $l \cap S_2 \neq \emptyset$.

Conversely, any pair of complementary spaces $S_1, S_2$ of $\mathbb{P}^n$ defines an involution of $\mathbb{P}^n$.

Remark 1.4 If we take a base of $\mathbb{P}^n$, $W = \{P_1, \ldots, P_{k+1}, P'_1, \ldots, P'_{k'+1}\}$ where $\langle P_1, \ldots, P_{k+1} \rangle = S_1$ and $\langle P'_1, \ldots, P'_{k'+1} \rangle = S_2$ then the involution $\varphi$ is given by the matrix:

$$M_\varphi = \begin{pmatrix} 1d & 0 \\ 0 & -1d \end{pmatrix}$$

Definition 1.5 Under the above assumptions, we say that $\varphi$ is harmonic respect to $S_1$ and $S_2$. Moreover, $S_1$ and $S_2$ are called the base spaces of $\varphi$.

Remark 1.6 The linear isomorphism $\varphi$ induces an isomorphism:

$$\varphi^* : \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(2))) \rightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(2)))$$

Because $\varphi$ is an involution, $\varphi^*$ is an involution too. Therefore, it has two complementary spaces of base points. Taking coordinates respect to the base $W$, a quadric $Q \subset \mathbb{P}^n$ has a matrix:

$$M_Q = \begin{pmatrix} A & C \\ C^t & B \end{pmatrix}$$

We see that $\varphi^*(Q) = Q$ if and only if $M_{\varphi}(Q) = M_\varphi M_Q M_\varphi = \lambda M_Q$ for some $\lambda \neq 0$, if and only if $A = B = 0$ or $C = 0$. In the first case $Q$ is called a harmonic quadric and in the second case $Q$ is called a base quadric. The set of harmonic quadrics will be denoted by $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(2)))_h$ and the set of base quadrics by $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(2)))_b$. We have that:

1. $Q$ is a harmonic quadric if and only if $P^n M_Q P = 0$ for each $P \in S_1, P' \in S_2$; that is, if $S_1$ and $S_2$ are polar respect to $Q$.
2. $Q$ is a base quadric if and only if $S_1, S_2 \subset Q$.
Definition 1.7 Let $\Sigma \subset P(H^0(\mathcal{O}_{P^n}(2)))$ be a projective subspace $\Sigma = P(V)$ and let $\varphi : P^n \rightarrow P^n$ a harmonic involution respect to two spaces $S_1, S_2$. $\Sigma$ is called a base-harmonic system respect to $S_1, S_2$ when it is a fixed space of $\varphi^\ast$.

Remark 1.8 If we denote $\Sigma_h = \Sigma \cap P(H^0(\mathcal{O}_{P^n}(2))_h)$ and $\Sigma_b = \Sigma \cap P(H^0(\mathcal{O}_{P^n}(2))_b)$, we see that $\Sigma$ is a base-harmonic system if and only if $\Sigma = \Sigma_h + \Sigma_b$; that is, $V$ has a base composed by harmonic and base quadrics. ■

Proposition 1.9 Let $X \subset P^n$ be a projective variety.

1. If $X$ has a harmonic involution $\varphi$ respect to two spaces $S_1, S_2$ then the system $P(H^0(I_X(2)))$ is base-harmonic system respect to $S_1, S_2$.

2. Suppose that $X$ is the complete intersection of quadrics. If $P(H^0(I_X(2)))$ is a base-harmonic system respect to $S_1, S_2$, then $X$ has a harmonic involution respect to two spaces $S_1, S_2$.

Proof:

1. Let $Q \in P(H^0(I_X(2)))$, that is, $X$ is contained on $Q$. It is sufficient to show that $\varphi^\ast(Q)$ contains $X$. Since $\varphi(X) = X$ and $X \subset Q$, $X = \varphi(X) \subset \varphi(Q)$ and the conclusion follows.

2. Let $\varphi : P^n \rightarrow P^n$ the harmonic involution defined by the spaces $S_1, S_2$. If $X$ is the complete intersection of quadrics, then $X = Q_1 \cap \ldots \cap Q_k$ where $\{Q_1, \ldots, Q_k\}$ is a base of $P(H^0(I_X(2)))$. If $P(H^0(I_X(2)))$ is a base-harmonic system respect to $S_1, S_2$ we can choose a base of fixed quadrics, so

$$\varphi(X) = \varphi(Q_1 \cap \ldots \cap Q_k) = \varphi(Q_1) \cap \ldots \cap \varphi(Q_k) = Q_1 \cap \ldots Q_k = X$$

and we can restrict $\varphi$ to $X$. ■

Proposition 1.10 Let $X \subset P^n$ be a projective variety and let $\varphi : X \rightarrow X$ a harmonic involution of $X$ respect to spaces $S_1, S_2$.

Let $F = \{P \in P^n / P \in \{x, \varphi(x)\}, x \in X\}$ be the variety of lines joining points of $X$ related by the involution. Then:

$$h^0(I_{X,P^n}(2))_b = h^0(I_{F,P^n}(2))_b = h^0(I_{F,P^n}(2)) - h^0(I_{F \cap S_1,S_1}(2)) - h^0(I_{F \cap S_2,S_2}(2))$$

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**Proof:** Let us first prove that $H^0(I_{X,P^n(2)})_b = H^0(I_{F,P^n(2)})_b$.

Since $X \subset F$, a quadric containing $F$ contains $X$ too, so $H^0(I_{F,P^n(2)})_b \subset H^0(I_{X,P^n(2)})_b$.

Conversely, if $Q \in H^0(I_{X,P^n(2)})_b$, $X, S_1, S_2 \subset Q$. Therefore, each line $l$ of $F$ meets $Q$ in four points: $(x, \varphi(x), l \cap S_1, l \cap S_2)$ and then it is contained on $Q$. Thus, $F \subset Q$ and $H^0(I_{X,P^n(2)})_b \subset H^0(I_{F,P^n(2)})_b$.

Now, let us consider the following exact sequence:

$$0 \rightarrow H^0(I_{F \cup S_1, P^n(2)}) \rightarrow H^0(I_{F, P^n(2)}) \rightarrow \alpha \rightarrow H^0(I_{F \cap S_1, S_2}(2))$$

Let us see that $\alpha$ is a surjective map. Let $Q_1 \subset S_1$ be a quadric which contains $F \cap S_1$. Taking the cone of $Q_1$ over $S_2$, we obtain a quadric of $P^n$ that contains the lines joining $Q_1$ and $S_2$ so it contains $F$. From this, we deduce that:

$$h^0(I_{F \cup S_1, P^n(2)}) = h^0(I_{F, P^n(2)}) - h^0(I_{F \cap S_1, S_2}(2))$$

Similarly, we have

$$0 \rightarrow H^0(I_{F \cup S_1 \cup S_2, P^n(2)}) \rightarrow H^0(I_{F \cup S_1, P^n(2)}) \rightarrow \beta \rightarrow H^0(I_{F \cap S_2, S_2}(2))$$

where $H^0(I_{F \cup S_1 \cup S_2, P^n(2)}) = H^0(I_{F, P^n(2)})_b$ and $\beta$ is a surjective map. Therefore:

$$h^0(I_{F, P^n(2)})_b = h^0(I_{F \cup S_1, P^n(2)}) - h^0(I_{F \cap S_2, S_2}(2)) = h^0(I_{F, P^n(2)}) - h^0(I_{F \cap S_1, S_2}(2)) - h^0(I_{F \cap S_2, S_2}(2))$$

---

We finish this section by computing the dimension of the spaces of harmonic and base quadrics in an involution over a normal rational curve.

Let $D_n \subset P^n$ be a rational normal curve of degree $n$ and let $\varphi : D_n \rightarrow D_n$ an involution of degree 2. The lines joining points related by the involution generate a rational normal ruled surface $R_{n-1} \subset P^n$. This ruled surface has two directrix curves $C_1, C_2$ on the base spaces $S_1, S_2$ of the involution.

We now ([3], IV, 2.17 and 2.19) that $R_{n-1} \cong P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-e))$ for some $e \geq 0$. In this way the divisor of hyperplane sections of $R_{n-1}$ is $H \sim X_0 + (n-1 + e)/2f$. Moreover the curve $D_n$ corresponds to a 2-secant curve on the surface, so $D_n \sim 2X_0 + kf$. Since $\deg(D_n) = D_n.H = n$, we obtain $k = e + 1$. Because $D_n$ is irreducible, $D_n.X_0 \geq 0$ and we obtain $e \leq 1$. We deduce that $e = 0$ if $n$ is odd and $e = 1$ if $n$ is even.

In this way we see that if $n$ is odd, the directrix curves $C_1, C_2$ have degree $(n-1)/2$ and if $n$ is even, the directrix curves have degree $(n-2)/2$ and $n/2$.

By applying Proposition 6.10 we have that:

$$h^0(I_{D_n}(2))_b = h^0(I_{R_{n-1}}(2)) - h^0(I_{C_1}(2)) - h^0(I_{C_2}(2))$$
The dimension of these spaces are well known. Thus, we obtain:

\[ h^0(I_{D_n}(2))_b = \begin{cases} 
\frac{(n-1)^2}{2} & \text{if } n \text{ is odd} \\
\frac{n(n-2)}{4} & \text{if } n \text{ is even}
\end{cases} \]

Note that when \( n \) is odd, \( D_n \sim 2X_0 + f \), so \( D_n.X_0 = 1 \) with \( C_1, C_2 \sim X_0 \). Thus, the base points of the involution of \( D_n \) are \( D_n \cap C_1 \) and \( D_n \cap C_2 \). When \( n \) is even, \( D_n \sim 2X_0 + 2f \), so \( D_n.X_0 = 0 \) and \( D_n.X_0 + f = 2 \) where \( C_1 \sim X_0 \) and \( C_2 \sim X_0 + f \). In this case the base points of the involutions of \( D_n \) are the two points of \( D_n \cap C_2 \).

2 Involutions of the canonical curve.

Let \( C_K \) be the canonical curve of genus \( \pi \) and let \( \varphi : C_K \to C_K \) be an involution. We saw that it is a harmonic involution. We will use the results obtained in [2]. The scroll generated by the involution is a canonical scroll \( R_b \). We call genus of the involution \( \varphi \) to the genus of the ruled surface \( R_b \). We have a 2 : 1-morphism \( \gamma : C_K \to X \). \( R_b \) has a canonical directrix curve \( X_0 \) of genus \( g \), and a nonspecial curve \( X_1 \) with degree \( \pi - 2(g - 1) \) base points that are the ramifications of \( \gamma \). We denote them by \( B \) and we know that \( B \sim C_K \cap X_1 \).

Let us compute the number of base quadrics containing \( C_K \), that is, the dimension of \( H^0(I_{C_K}(2))_b \). By Proposition [1.10] we know that:

\[ h^0(I_{C_K}(2))_b = h^0(I_{R_b}(2)) - h^0(I_{X_0,P_g-1}(2)) - h^0(I_{X_1,P_{\pi-g-1}}(2)) \]

We can compute this dimension because we know that (see [2]):

\[
\begin{align*}
    h^0(I_{R_b}(2)) &= h^0(\mathcal{O}_{P^{\pi-1}}(2)) - h^0(\mathcal{O}_{S_b}(2H)) + \text{dim}(s(H, H)) \\
    h^0(I_{X_0}(2)) &= h^0(\mathcal{O}_{P^{\pi-1}}(2)) - h^0(\mathcal{O}_X(2\mathcal{K})) + \text{dim}(s(\mathcal{K}, \mathcal{K})) \\
    h^0(I_{X_1}(2)) &= h^0(\mathcal{O}_{P^{\pi-g-1}}(2)) - h^0(\mathcal{O}_X(2b)) + \text{dim}(s(b, b)) \\
\end{align*}
\]

\[ \text{dim}(s(H, H)) = \text{dim}(s(\mathcal{K}, \mathcal{K})) + \text{dim}(s(b, b)) \]

From this, we obtain:

\[ h^0(I_{C_K}(2))_b = h^0(\mathcal{O}_{P^{\pi-1}}(2))_b - h^0(\mathcal{O}_X(b + \mathcal{K})) \]
Thus the number of base quadrics containing $C_K$ is:

$$h^0(C_K(2))_b = (g - 1)(\pi - g - 2)$$

**Theorem 2.1** Let $C_K \subset P^{\pi-1}$ be a nonhyperelliptic canonical curve of genus $\pi$ that it is complete intersection of quadrics. $C_K$ has an involution of degree 2 if and only if $P(H^0(I_{C_K}(2)))$ is harmonic respect to two disjoint complementary subspaces of dimensions $k$ and $\pi - k - 2$, with $k \leq \pi - k - 1$. Moreover the involution has genus $g = k + 1$ if and only if $h^0(I_{C_K}(2))_b = (g - 1)(\pi - g - 2)$.

**Proof:** The first assertion follows from Proposition 1.9.

By the above discussion we know the number of base quadrics $h^0(I_{C_K}(2))_b = (g' - 1)(\pi - g' - 2)$ where $g'$ is the genus of the involution. This genus is $k + 1$ or $\pi - k - 1$. Suppose that $(g' - 1)(\pi - g' - 2) = (g - 1)(\pi - g - 2)$ with $g = k + 1$. If $g' = \pi - k - 1 = \pi - g$ then we obtain $\pi = 2g$ so the genus $g'$ of the involution is $k + 1$.

**Remark 2.2** It is well known that a nonhyperelliptic canonical curve of genus $\pi \geq 5$ is the complete intersection of quadrics, except when it is trigonal or when $\pi = 6$ and it has a $g^2_5$. In the first case the intersection of the quadrics of $P(H^0(I_{C_K}(2)))$ is a rational normal scroll; in the second case it is the Veronese surface of $P^5$.

2.1 **Involutions of the canonical curve of genus $\pi = 4$.**

Let $C_K \subset P^3$ be a canonical curve of genus 4. It is well known that this curve is the complete intersection of a quadric and a cubic surface (see [3],IV,Example 5.2.2). Suppose that $C_K$ has an involution $C_K \rightarrow X$ where $X$ is a smooth curve of genus $g$. We have that $\pi - 1 \geq 2g - 2$, so then genus of $X$ can be 1 or 2 (if $g = 0$ the curve $C_K$ is hyperelliptic).

**Proposition 2.1.1** A canonical curve of genus 4 has an elliptic involution if and only if it is the complete intersection of an elliptic cubic cone $S$ and a quadric that doesn’t pass through the vertex of $S$.

**Proof:** Suppose that $C_K$ has an elliptic involution. We know that the involution generates a scroll $R$. In this case the directrix curve $X_0$ has degree $g - 1 = 0$, so it is a point. Then $R$ is an elliptic cone. If $Q$ is the unique quadric that contains
$C_K$, then necessary $C_K = Q \cap R$. Moreover, we know that $C_K \cap X_0 = \emptyset$, so $Q$ does not pass through the vertex of $R$. Conversely, if $C_K = Q \cap R$ and $Q$ does not pass through the vertex of $R$, the generators of $S$ provide an elliptic involution of $C_K$.

**Theorem 2.1.2** A canonical curve of genus 4 has an involution of genus 2 if and only if it is the complete intersection of a quadric and a cubic surface which has a harmonic involution respect to polar lines respect to the quadric.

**Proof:**

1. Suppose that $C_K = Q_2 \cap Q_3$ where $Q_2$ is a quadric and $Q_3$ is a cubic surface with a harmonic involution $\varphi : Q_3 \rightarrow Q_3$. Let $l$ and $l'$ the base spaces. Suppose that they are polar respect to $Q_2$. From this, $\varphi(Q_2) = Q_2$ so $C_K$ has a harmonic involution. Because the base spaces have dimension 1, the involution has genus 2.

2. Suppose that $C_K$ has a harmonic involution $\varphi$ of genus 2. From this, the base spaces are two lines $l$ and $l'$. Let $R \subset \mathbb{P}^3$ the ruled surface generated by the involution. We know that $\varphi(Q_2) = Q_2$ for any cubic surface $Q_2 \in \mathbb{P}(H^0(I_{C_K}(3)))$. Consider the involution $\overline{\varphi} : \mathbb{P}(H^0(I_{C_K}(3))) \rightarrow \mathbb{P}(H^0(I_{C_K}(3)))$ induced by $\varphi$. Let us see that there is a fixed irreducible element. Let $V$ be the set of reducible elements of $\mathbb{P}(H^0(I_{C_K}(3)))$. Because $Q_2$ is the unique quadric containing $C_K$ then $V = \{Q_2 + H; H \subset \mathbb{P}^3\}$ and $\dim(V) = 3$. Since the base points of $\overline{\varphi}$ generate $\mathbb{P}(H^0(I_{C_K}(3)))$, then there exist at least a fixed irreducible element.

2.2 Involutions of the Veronese surface.

Let $v_{2,n}$ be the Veronese map of $\mathbb{P}^n$:

$$v_{2,n} : \mathbb{P}^n^* \rightarrow V_{2,n} \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(2)))$$

$$[x_0 : \ldots : x_n] \rightarrow [x_0^2 : x_0x_1 : \ldots : x_n^2]$$

We will denote the image of this map by $V_{2,n}$. If $n = 2$, it is the Veronese surface and we will denote it by $V_2$.

**Proposition 2.2.1** The involutions of the Veronese variety $V_{2,n}$ are harmonic respect to two base spaces which are the harmonic and base quadrics of two subspaces $S_1, S_2$ of $\mathbb{P}^n$. 

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Proof: A harmonic involution $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ respect to spaces $S_1, S_2$ induces a harmonic involution $\overline{\varphi}$ in $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))$ respect to $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))_h$ and $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))_b$.

Because $\overline{\varphi} \circ v_{2, n} = v_{2, n} \circ \overline{\varphi}$, we see that $\overline{\varphi}$ restricts to $V_{2, n}$. In this way we have a harmonic involution $\overline{\varphi} : V_{2, n} \rightarrow V_{2, n}$ respect to the spaces $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))_h$ and $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))_b$.

Conversely, given an involution $\eta$ of $V_2$. Applying the isomorphism $v_{2, n}$, we obtain an involution $\varphi^* = \varphi^{**}$ provides an involution $\overline{\varphi}$ of $\mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))$ such that $\overline{\varphi} | V_{2, n} = \eta$.

We will denote the Veronese surface by $V_2$.

Corollary 2.2.2 Any nontrivial involution of the Veronese surface $V_2$ is harmonic respect to a line which corresponds to the conics of $\mathbb{P}^2$ passing through a point $P$ and a line $r$, and a 3-dimensional space $V$ corresponding to the polar conics respect to $P$ and $r$.

Proof: It is sufficient to note that an involution $\eta$ of $V_2$ is induced by and involution $\varphi$ of $\mathbb{P}^2$. If $\varphi$ is nontrivial, then its base spaces are a point $P$ and a line $r$.

Proposition 2.2.3 Let $\eta$ be a nontrivial harmonic involution of the Veronese surface. Then $h^0(O_{V_2}(2)) = 2$.

Proof: Consider the Veronese map of $\mathbb{P}^2$:

$\nu_2 : \mathbb{P}^{n*} \longrightarrow V_2 \subset \mathbb{P}(H^0(O_{\mathbb{P}^n}(2)))$

$[x_0 : x_1 : x_2] \longrightarrow [x_0^2 : x_0 x_1 : \ldots : x_2^2] = [y_0 : y_1 : \ldots : y_5]$  

Let $Y$ be the matrix:

$$
\begin{pmatrix}
y_0 & y_1 & y_2 \\
y_1 & y_3 & y_4 \\
y_2 & y_4 & y_5
\end{pmatrix}
$$

We know that $H^0(O_{V_2}(2))$ is generated by the quadrics of $\mathbb{P}^5$ whose equations are defined by the minors of order 2 of the matrix $Y$.

Moreover the base spaces of $\varphi$ are a line $l$ corresponding to the conics of $\mathbb{P}^2$ passing through a line $r$ and a point $P$ and a space $V$ corresponding to the polar conics respect to $P$ and $r$.

Taking an adequate system of coordinates we can consider $P$ generated by the equations $\{x_1 = x_2 = 0\}$ and $r$ generated by the equation $x_0 = 0$.  

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A conic containing $P$ and $r$ has an equation $ax_0x_1 + bx_0x_2 = 0$. Then the equations of $l$ are $\{y_0 = y_3 = y_4 = y_5 = 0\}$.

A polar conic respect to $P$ and $r$ has an equation $ax_0^2 + bx_1^2 + cx_1x_2 + dx_2^2 = 0$. Then the equations of $V$ are $\{y_1 = y_2 = 0\}$.

Applying the conditions to contain $V$ and $l$ to the equations of $H^0(\mathcal{O}_{V_2}(2))$, we obtain that the quadrics containing $V$ and $l$ are generated by $\{y_1y_4 - y_2y_3 = y_1y_5 - y_2y_4 = 0\}$. From this, $h^0(\mathcal{O}_{V_2}(2))_0 = 2$.

### 2.3 Harmonic involutions of the rational ruled surfaces.

Let $R_{n-1} \subset \mathbb{P}^n$ a rational normal ruled surface of degree $n-1$, with $n > 3$. Let $\varphi : R_{n-1} \rightarrow R_{n-1}$ a harmonic involution of the surface. Then $\varphi$ conserves the degree of the curves. From this, it applies generators into generators if $n > 3$.

In this way, we have the following induced harmonic involutions:

$$
\varphi_0 : \mathbb{P}^1 \rightarrow \mathbb{P}^1,
\begin{align*}
P & \mapsto Q, \\
/\varphi(Pf) & = Qf
\end{align*}
$$

where $\mathbb{P}^1$ parameterizes the generators.

$$
\varphi_l : |D_l| \rightarrow |D_l|,
\begin{align*}
D & \mapsto \varphi(D)
\end{align*}
$$

where $|D_l|$ is the linear system of curves of degree $l$.

Let $k$ be the degree of the curve of minimum degree of $R_{n-1}$:

1. If $k = \frac{n-1}{2}$, then there is a 1-dimensional family of irreducible curves of degree $k$. The involution $\varphi_k$ has at least 2 base points, so there are two disjoint curves $D_k$ that are invariant by $\varphi$.

2. If $k < \frac{n-1}{2}$, then there is a unique curve of minimum degree, so it is invariant by $\varphi$. Moreover, if $l = n - 1 - k$ the linear system $|D_l|$ has dimension $l - k$. Its generic curve is an irreducible curve disjoint from $D_k$. In particular, the set of reducible curves of $|D_l|$ are an hyperplane composed by curves of the form $D_k + \sum f_i$. Thus, we have a harmonic involution:

$$
\varphi_l : \mathbb{P}^{l-k} \cong |D_l| \rightarrow |D_l| \cong \mathbb{P}^{l-k}
$$

We know that $\varphi_l$ has two disjoint spaces of base points. Both of them can not be contained on the hyperplane (because they generate $\mathbb{P}^{n-k}$), so necessary there exists an irreducible curve in $|D_l|$ that is fixed by the involution $\varphi_l$; that is, it is invariant by $\varphi$. 

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We conclude the following proposition:

**Proposition 2.3.1** Given a harmonic involution on a rational normal ruled surface of degree \( n - 1 \), there exist two disjoint rational normal curves \( D_k, D_l \) with degrees \( k \) and \( l \), \( k + l = n - 1 \), that are invariant by the involution.

Let \( D_k \subset P^k \) and \( D_l \subset P^l \) be the two invariant curves. The involution \( \varphi \) restricts to these spaces. Thus, we have a harmonic involution \( \varphi_k : P^k \longrightarrow P^k \). It has two invariant spaces \( P^{k_1}, P^{k_2} \) with \( k_1 + k_2 + 1 = k \). Similarly, the harmonic involution \( \varphi_l : P^l \longrightarrow P^l \) has two invariant spaces \( P^{l_1}, P^{l_2} \) with \( l_1 + l_2 + 1 = l \). From this, we have two possibilities for the base spaces \( S_1, S_2 \) of the involution \( \varphi \):

\[
S_1 = \langle P^{k_1}, P^{l_1} \rangle = P^{k_1+l_1+1} \quad S_1 = \langle P^{k_1}, P^{l_2} \rangle = P^{k_1+l_2+1} \\
S_2 = \langle P^{k_2}, P^{l_1} \rangle = P^{k_2+l_1+1} \quad S_2 = \langle P^{k_2}, P^{l_2} \rangle = P^{k_2+l_2+1}
\]

Conversely if we have two harmonic involutions in \( P^k \) and \( P^l \) we can recuperate an involution in \( P^n \). Note that this involution is not unique, because we have two ways to define it. Moreover, in order to restrict the involution to the ruled surface \( q : R_{n-1} \longrightarrow P^1 \) we need that the involutions in \( P^k \) and \( P^l \) restrict to \( D_k \) and \( D_l \) and that they are compatible, that is, the images of the points on the same generator lay on the same generator: \( q(\varphi_k(D_k \cap Pf)) = q(\varphi_l(D_l \cap Pf)), \forall P \in P^1 \).

Thus, if \( \varphi_k \) and \( \varphi_l \) verify these conditions we have a harmonic involution \( \varphi \) that restricts to \( R_{n-1} \).

**Proposition 2.3.2** A harmonic involution on a normal rational ruled surface \( R_{n-1} \) defines two harmonic involutions \( \varphi_k, \varphi_l \) on two disjoint rational curves \( D_k, D_l \) that generate the surface. Moreover, they make commutative the diagram (1).

Conversely, if two harmonic involutions \( \varphi_k, \varphi_l \) on two rational curves generating a rational ruled surface \( R_{n-1} \) verifying \( q(\varphi_k(D_k \cap Pf)) = q(\varphi_l(D_l \cap Pf)), \forall P \in P^1 \). then they define two possible harmonic involutions on \( R_{n-1} \), taking the space bases generate by the space bases of \( \varphi_k \) and \( \varphi_l \).

**Remark 2.3** In order to obtain compatible involutions \( \varphi_k, \varphi_l \) it is sufficient to define a involution \( \eta \) on \( P^1 \) and to translate it to \( D_k \) and \( D_l \):

\[
\varphi_k(D_k \cap Pf) := D_k \cap \eta(P)f \\
\varphi_l(D_l \cap Pf) := D_l \cap \eta(P)f
\]

for all \( P \in P^1 \).

Moreover, if \( \varphi_k \) and \( \varphi_l \) are compatible involutions and one of them is the identity, then the other one is the identity too.
We saw how are the (nontrivial) harmonic involutions on a normal rational curve $D_m \subset \mathbb{P}^m$:

1. If $m = 2\mu$ the involution is defined by two base spaces $\mathbb{P}^\mu, \mathbb{P}^{\mu-1}$ such that $\mathbb{P}^\mu \cap D_m = P \cup Q$ and $\mathbb{P}^{\mu-1} \cap D_m = \emptyset$ ($P, Q$ base points).

2. If $m = 2\mu + 1$ then involution is defined by two base spaces $\mathbb{P}_1^\mu, \mathbb{P}_2^\mu$ such that $\mathbb{P}_1^\mu \cap D_m = P$ and $\mathbb{P}_2^\mu \cap D_m = Q$ ($P, Q$ base points).

In both cases the involution generates a normal rational ruled surface $R_{m-1}$ of degree $m - 1$, whose directrix curves lay on the space bases. We call them base curves.

Thus, let $\varphi$ be a harmonic involution on $R_{n-1}$. Let $\varphi_k, \varphi_l$ be the harmonic involutions induced on the directrix curves $D_k, D_l$. Let $\mathbb{P}^{k_1}, \mathbb{P}^{k_2}, \mathbb{P}^{l_1}, \mathbb{P}^{l_2}$ be the base spaces of $\varphi_k$ and $\varphi_l$. We know that the base spaces of $\varphi$ are $S_1 = \langle \mathbb{P}^{k_1}, \mathbb{P}^{l_1} \rangle$, $S_2 = \langle \mathbb{P}^{k_2}, \mathbb{P}^{l_2} \rangle$. Let $C_{k_1}, C_{k_2}, C_{l_1}, C_{l_2}$ the corresponding base curves. Let $F$ be the variety of lines that join the points of the involution: $F = \{ P \in \mathbb{P}^n/ P \in \langle x, f(x) \rangle, x \in R_{n-1} \}$. Let us identificate $F \cap S_1$ and $F \cap S_2$.

**Lemma 2.3.3** The variety $F \cap S_1$ ($F \cap S_2$) is a normal rational ruled surface of degree $k_1 + l_1 = n_1$ ($k_2 + l_2 = n_2$) generated by the directrix curves $C_{k_1}$ and $C_{l_1}$ ($C_{k_2}$ and $C_{l_2}$). We call it base ruled surface $R_{n_{1-1}}$ ($R_{n_{2-1}}$).

**Proof:** Given a point $P \in R_{n-1}$, consider the line $r = \langle P, f(P) \rangle$ of $F$. $r$ meets $S_1$ in a point $P_1$ that corresponds to project $P$ from $S_2$ onto $S_1$. Thus, given a generator $f \in R_{n-1}$, the lines of $F$ defined by the points of $f$ meet $S_1$ in a line $f_1$; this line is the projection of $f$ from $S_2$. Moreover, since $f$ meets $D_k$ and $D_l$, its projection on $S_1$ meets $C_{k_1}$ and $C_{l_1}$. In this way we see that the generator of $R_{n-1}$ project into lines joining $C_{k_1}$ and $C_{l_2}$, so $F \cap S_1$ is the rational ruled surface defined by these directrix curves.

We saw that a harmonic involution on a normal rational ruled surface is defined by the involutions of the directrix curves $D_k$ and $D_l$. From this, we distinguish several types of involutions:

1. $\varphi_k$ and $\varphi_l$ are the identity.

   Then the base spaces of $\varphi$ are the spaces $\mathbb{P}^k$ and $\mathbb{P}^l$ that contain the directrix curves. All the generators are invariants by $\varphi$ and the variety $F$ is the ruled surface $R_{n_{-1}}$.

2. $\varphi_k$ and $\varphi_l$ are not trivial.

   2.1. $n - 1 = 2\lambda$ (even).
2.1.1. \( k = 2\mu, l = 2(\lambda - \mu) \).

Then the involutions on \( D_k \) and \( D_l \) have the following base spaces and base curves:

\[ C_{\mu} \in \mathbb{P}^{\mu}, \ C_{\mu-1} \in \mathbb{P}^{\mu-1}, \text{ with } P_k, Q_k \in C_{\mu} \text{ base points of } D_k. \]

\[ C_{\lambda-\mu} \in \mathbb{P}^{\lambda-\mu}, \ C_{\lambda-\mu-1} \in \mathbb{P}^{\lambda-\mu-1}, \text{ with } P_l, Q_l \in C_{\lambda-\mu} \text{ base points of } D_l. \]

Then, the base spaces of \( \varphi \) are:

2.1.1.1 Case A:

\[ S_1 = (\mathbb{P}^{\mu}, \mathbb{P}^{\lambda-\mu}) = \mathbb{P}^{\lambda+1} \ni P_k, Q_k, P_l, Q_l. \]

\[ S_2 = (\mathbb{P}^{\mu-1}, \mathbb{P}^{\lambda-\mu-1}) = \mathbb{P}^{\lambda-1}. \]

Where the generators \( f_P, f_Q \in \mathbb{P}^{\lambda+1} \) are fixed.

2.1.1.2 Case B:

\[ S_1 = (\mathbb{P}^{\mu}, \mathbb{P}^{\lambda-\mu-1}) = \mathbb{P}^{\lambda} \ni P_k, Q_k. \]

\[ S_2 = (\mathbb{P}^{\mu-1}, \mathbb{P}^{\lambda-\mu}) = \mathbb{P}^{\lambda} \ni P_l, Q_l. \]

Where the generators \( f_P, f_Q \) are invariant (not fixed).

2.1.2. \( k = 2\mu + 1, l = 2(\lambda - \mu) - 1 \).

Then the involutions on \( D_k \) and \( D_l \) have the following base spaces and base curves:

\[ P_k \in C_{\mu} \in \mathbb{P}^{\mu}, \ Q_k \in C_{\mu} \in \mathbb{P}^{\mu}, \text{ with } P_k, Q_k \text{ base points of } D_k. \]

\[ P_l \in C_{\lambda-\mu} \in \mathbb{P}^{\lambda-\mu}, \ Q_l \in C_{\lambda-\mu} \in \mathbb{P}^{\lambda-\mu-1}, \text{ with } P_l, Q_l \text{ base points of } D_l. \]

Then, the base spaces of \( \varphi \) are:

2.1.2.1 Case C:

\[ S_1 = (\mathbb{P}^{\mu}, \mathbb{P}^{\lambda-\mu-1}) = \mathbb{P}^{\lambda} \ni P_k, P_l. \]

\[ S_2 = (\mathbb{P}^{\mu-1}, \mathbb{P}^{\lambda-\mu}) = \mathbb{P}^{\lambda} \ni Q_k, Q_l. \]

Where the generators \( f_P \in \mathbb{P}^{\mu}, f_Q \in S_2 \) are fixed.

2.1.2.2 Case B: (similar to case 2.1.1.2):

\[ S_1 = (\mathbb{P}^{\mu}, \mathbb{P}^{\lambda-\mu-1}) = \mathbb{P}^{\lambda} \ni P_k, Q_l. \]

\[ S_2 = (\mathbb{P}^{\mu-1}, \mathbb{P}^{\lambda-\mu}) = \mathbb{P}^{\lambda} \ni P_l, Q_k. \]

Where the generators \( f_P, f_Q \) are invariant (not fixed).

2.2. \( n - 1 = 2\lambda - 1 \) even.

Then the curves \( D_k \) and \( D_l \) have degrees \( k = 2\mu \) and \( l = 2(\lambda - \mu) - 1 \). The base spaces and base curves are:

\[ C_{\mu} \in \mathbb{P}^{\mu}, \ C_{\mu-1} \in \mathbb{P}^{\mu-1}, \text{ with } P_k, Q_k \in C_{\mu} \text{ base points of } D_k. \]

\[ P_l \in C_{\lambda-\mu-1} \in \mathbb{P}^{\lambda-\mu-1}, \ Q_l \in C_{\lambda-\mu-1} \in \mathbb{P}^{\lambda-\mu-1}, \text{ with } P_l, Q_l \text{ base points of } D_l. \]

In any case, the base spaces of \( \varphi \) are:

2.2.1. Case D:

\[ S_1 = (\mathbb{P}^{\mu}, \mathbb{P}^{\lambda-\mu-1}) = \mathbb{P}^{\lambda} \ni P_k, Q_k, P_l. \]

\[ S_2 = (\mathbb{P}^{\mu-1}, \mathbb{P}^{\lambda-\mu}) = \mathbb{P}^{\lambda} \ni Q_l. \]

Where \( f_P \) is a fixed generator and \( f_Q \) are an invariant generator.
Let \( \varphi : R_{n-1} \rightarrow R_{n-1} \) a harmonic involution over the rational normal ruled surface \( R_{n-1} \subset P^n \). By the proposition \ref{prop1.10} we know that \( H^0(I_{R_{n-1}}(2)) \) is a base-harmonic system; that is, \( H^0(I_{R_{n-1}}(2)) = H^0(I_{R_{n-1}}(2))_h \oplus H^0(I_{R_{n-1}}(2))_b \).

Let us see the dimension of these spaces. We know that \( h^0(I_{R_{n-1}}(2)) = \binom{n-1}{2} \).

We will treat each case separated:

1. All generators are invariant by the involution.

   We use the proposition \ref{prop1.10}. In this case \( F = R_{n-1}, F \cap P^a = D_k \) and \( F \cap P^b = D_l \). Thus,
   
   \[
   h^0(I_{R_{n-1}}(2))_b = h^0(I_{R_{n-1}}(2)) - h^0(I_{D_k}(2)) - h^0(I_{D_l}(2)) = kl
   \]
   
   and
   
   \[
   h^0(I_{R_{n-1}}(2))_h = h^0(I_{R_{n-1}}(2)) - h^0(I_{R_{n-1}}(2))_b = \binom{n-1}{2} - kl
   \]

2. The generic generator is not invariant by the involution.

   We use the proposition \ref{prop1.10}. But in this case, \( F \cap S_1 = R_{n-1} \) and \( F \cap S_2 = R_{n_2} \). Then we have:
   
   \[
   h^0(I_{R_{n-1}}(2))_b = h^0(I_{F}(2)) - h^0(I_{R_{n-1}}(2)) - h^0(I_{R_{n_2}}(2))
   \]  

A quadric containing \( R_{n-1} \cup R_{n_1} \) meets each line of \( F \) in three points. Then such quadric contains \( F \), so \( H^0(I_{F}(2)) = H^0(I_{R_{n-1}}(2)) \). Consider the exact sequence:

\[
0 \rightarrow H^0(I_{R_{n-1}}(2)) \rightarrow H^0(I_{R_{n-1}}(2)) \rightarrow H^0(O_{R_{n-1}}(2 - Y))
\]

where \( Y = R_{n-1} \cap R_{n_1} \). Then:

\[
h^0(I_{F}(2)) \geq h^0(I_{R_{n-1}}(2)) - h^0(O_{R_{n-1}}(2 - R_{n-1} \cap R_{n_1})))
\]

and applying (2) we obtain in each case:

A. \( (n - 1 = 2\lambda, S_1 = P^{\lambda-1}, S_2 = P^{\lambda+1}, f_Q, f_P \) fixed generators.)

\[
h^0(I_{R_{n-1}}(2))_b \geq \lambda(\lambda - 1)
\]

B. \( (n - 1 = 2\lambda, S_1 = P^{\lambda}, S_2 = P^{\lambda}, f_Q, f_P \) invariant (not fixed) generators, with \( f_P \cap P^{\lambda} = P^i, f_Q \cap P^{\lambda} = Q^i \).)

\[
h^0(I_{R_{n-1}}(2))_b \geq \lambda(\lambda - 1)
\]

C. \( (n - 1 = 2\lambda, S_1 = P^{\lambda}, S_2 = P^{\lambda}, f_Q, f_P \) fixed generators, with \( f_P \in P^{\lambda}, f_Q \in P^{\lambda} \).)

\[
h^0(I_{R_{n-1}}(2))_b \geq \lambda(\lambda - 1) + 1
\]
D. \((n - 1 = 2\lambda - 1, S_1 = P^{\lambda - 1}, S_2 = P^\lambda, f_Q \text{ fixed generator in } P^\lambda, \text{ and } f_q \text{ invariant generator, with } f_Q \cap P^{\lambda - 1} = Q_1,)\)

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq (\lambda - 1)^2\]

Now, let us compute the harmonic quadrics.

Let \(E_k\) the set of \(k + 1\) generic points in \(D_k\) and \(E_l\) the set of \(l + 1\) generic points on \(D_l\). Note that a harmonic quadric that passes through a point of \(R_{n-1}\) passes through the image point too. From this, a quadric passing through \(E_k(D_l)\) meets \(D_k(D_l)\) in \(2k + 2(2l + 2)\) points because \(D_k(D_l)\) is invariant by the involution. Moreover, a harmonic quadric that contains \(D_k\) and \(D_l\), contains the invariant (not fixed) generators too, because they meet each space base in a point.

Finally, a harmonic quadric containing \(D_k\) and \(D_l\) and passing through \(m\) generic points of \(R_{n-1}\) contains their images (\(2m\) points) and the corresponding \(2m\) generators. Let \(E_m\) be \(m\) generic points of \(R_{n-1}\) and let \(E\) be \(E_k \cup E_l \cup E_m\).

If \(Q\) is a harmonic quadric passing through the points of \(E\), then \(D_k \cup D_l \cup \{\text{invariant generators}\} \cup 2mf \subset Q \cap R_{n-1}\). If \(2m > 2(n - 1) - (k + l)\)–number of invariant generators, then \(R_{n-1} \subset Q\) and we have the exact sequence:

\[0 \rightarrow H^0(\mathcal{I}_{R_{n-1}}(2))_b \rightarrow H^0(\mathcal{I}_{P^{n-1}}(2))_b \rightarrow H^0(\mathcal{O}_E(2))\]

From this:

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq h^0(\mathcal{O}_{P^{n-1}}(2))_b - (n + 1 + m)\]

In each case we obtain:

A. There are not invariant generators. Taking \(m = \lambda + 1\) we have:

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq (\lambda + 1 + \frac{3}{2}) - 3\lambda - 3\]

B. There are two invariant generators. Taking \(m = \lambda\) we have:

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq (\lambda + 2 + \frac{3}{2}) - 3\lambda - 2\]

C. There are not invariant generators. Taking \(m = \lambda + 1\) we have:

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq (\lambda + 2 + \frac{3}{2}) - 3\lambda + 3\]

D. There is an invariant generator. Taking \(m = \lambda\) we have:

\[h^0(\mathcal{I}_{R_{n-1}}(2))_b \geq (\lambda + 1 + \frac{3}{2}) - 3\lambda + 1\]
We see that the sum of the bounds computed for the harmonic and base quadrics is the quadrics of $H^0(I_{R_{n-1}}(2))$ in all cases, so these bounds are reached in all cases, and we have the number of base quadrics:

**Proposition 2.3.4** Let $R_{n-1} \subset \mathbb{P}^n$ be a rational normal scroll of degree $n-1$. Let $\varphi : R_{n-1} \rightarrow R_{n-1}$ be a harmonic involution. Then we have the following cases:

1. All the generators are invariant. There are two directrix curves of base point $D_k$, $D_l$ with $k + l = n - 1$. They lay on the base spaces $\mathbb{P}^k$, $\mathbb{P}^l$.

\[ h^0(I_{R_{n-1}}(2))_b = kl \]

2. There are two invariant (fixed or not):
   (1) $n - 1 = 2\lambda$ (n even).
   A. The base spaces are $\mathbb{P}^{\lambda - 1}, \mathbb{P}^{\lambda + 1}$. There are two fixed generators in $\mathbb{P}^{\lambda + 1}$.

\[ h^0(I_{R_{n-1}}(2))_b = \lambda(\lambda - 1) \]

B. The base spaces are $\mathbb{P}^{\lambda}, \mathbb{P}^{\lambda}$. There is a fixed generator in each of them.

\[ h^0(I_{R_{n-1}}(2))_b = \lambda(\lambda - 1) + 1 \]

C. The base spaces are $\mathbb{P}^{\lambda}, \mathbb{P}^{\lambda}$. There are not fixed generators.

\[ h^0(I_{R_{n-1}}(2))_b = \lambda(\lambda - 1) \]

(2) $n - 1 = 2\lambda - 1$ (n odd).

D. The base spaces are $\mathbb{P}^{\lambda - 1}, \mathbb{P}^{\lambda}$. There is a fixed generator in $\mathbb{P}^{\lambda}$.

\[ h^0(I_{R_{n-1}}(2))_b = (\lambda - 1)^2 \]

2.4 **Involutions of the canonical curve of genus $\pi > 4$**.

We have investigated all the possible cases where the quadrics that contain a canonical curve are a base-harmonic system:
Theorem 2.4.1 The unique cases where the system of quadrics containing a canonical curve $C_K$ of genus $\pi$, $\pi > 4$ are a base-harmonic system respect to base spaces $P^{g-1}$, $P^{\pi-g-1}$ ($\pi \geq 2g-1 > 0$) with $b$ independent base quadrics are:

1. $b = (g-1)(\pi - g - 2)$
   
   (a) If $\pi = 2g, 2g+1, 2g+2$ and the curve has a $g_3^1$ or an involution of genus $g$, or both of them; except if $\pi = 6$ and $g = 2$ when $C_K$ can have a $g_2^2$.
   
   (b) If $\pi \neq 2g, 2g+1, 2g+2$ and the curve $C_K$ has a $\gamma_1^2$ of genus $g$.

2. $b = (g-1)(\pi - g - 2) + 1$, $\pi = 2g-1$ or $\pi = 2g$ and $C_K$ has a $g_3^1$ (not a $\gamma_1^2$).

3. $b = (g-1)(\pi - g - 1)$ and $C_K$ has a $g_3^1$ (not a $\gamma_1^2$).

Corollary 2.4.2 The unique involutions on a trigonal canonical curve of genus $\pi$, $\pi > 4$ are of genus $\frac{\pi}{2}, \frac{\pi-1}{2}$ or $\frac{\pi-2}{2}$.

Proof: Let $C_K \subset P^n$ be a trigonal canonical curve and let $R_{\pi-2}$ be the ruled surface of trisecants. Suppose that $C_K$ has an involution of genus $g$. Then the system of quadrics $P(H^0(I_{C_K}(2)))$ is a base-harmonic system with $(g-1)(\pi - g - 2)$ independent base quadrics. Since $R_{\pi-2} = \cap_{Q \subset C_K} Q$, we have a harmonic involution over $R_{\pi-2}$. By Proposition 2.3.4 we know that $\pi = 2g$, $\pi = 2g+1$ or $\pi = 2g+2$ and the conclusion follows.

Corollary 2.4.3 The unique involutions on a smooth quintic plane curve are of genus 2.

Theorem 2.4.4 Let $C_K \subset P^{\pi-1}$ a canonical curve of genus $\pi$, $\pi > 4$. Then $C_K$ has an involution of genus 1 if and only if the quadrics of $P(H^0(I_{C_K}(2)))$ are a base-harmonic system respect to a point and a space $P^{\pi-2}$ without base quadrics.

Proof: If $C_K$ has an involution of genus 1, we know that the involution generates an elliptic cone, the system of quadrics $P(H^0(I_{C_K}(2)))$ is harmonic and it hasn’t base quadrics respect to $P^0$ and $P^{\pi-2}$.

Conversely, if the system of quadrics $P(H^0(I_{C_K}(2)))$ is harmonic respect to $P^0$ and $P^{\pi-2}$, necessary it hasn’t base quadrics, because the quadric containing $C_K$ are reducible. If $C_K$ is not trigonal we have an involution of genus 1 in $C_K$. If $C_K$ is trigonal, by Corollary 2.4.2, $\pi = 2, 3, 4$. But we have supposed that $\pi > 4$.\[\Box\]
Theorem 2.4.5 Let \( C_K \subseteq \mathbb{P}^{\pi-1} \) be the canonical curve of genus \( \pi \), with \( \pi > 4 \). If \( C_K \) has an involution of genus \( g \) then \( \pi \geq 2g - 1 \) and the quadrics of \( \mathbb{P}(H^0(I_{C_K}(2))) \) are a base-harmonic system respect to the base spaces \( \mathbb{P}^{g-1} \). Conversely, these conditions are sufficient to grant the existence of an involution, except when:

1. \( \pi = 6, g = 2 \) and \( C_K \) has a \( g_5^2 \); or
2. \( \pi = 2g, 2g+1 \) or \( 2g+2 \) and \( C_K \) is trigonal.

Remark 2.4 Let us study what happens at the two exceptions:

1. Suppose that \( C_K \) is a canonical curve of genus 6 with a \( g_5^2 \), that is, it is isomorphic to a smooth plane curve of degree 5. Suppose that the quadrics of \( \mathbb{P}(H^0(I_{C_K}(2))) \) are a base-harmonic system respect to the base spaces \( \mathbb{P}^1 \) and \( \mathbb{P}^3 \). It induces a harmonic involution on the Veronese surface and then, an involution on the plane. Obviously, the generic plane curve of degree 5 of the plane is not invariant by this involution. So in this case the hypothesis of the above theorem are not sufficient.

However, there are smooth quintic plane curves invariant by an involution. For example, we can take the quintic curve \( f(x_0, x_1) - x_2^2x_0 = 0 \) on the plane, where \( f(x_0, x_1) \) is a generic homogeneous polynomial of degree 5. This curve is smooth an it’s invariant by the involution

\[
\begin{align*}
x_0 &\mapsto x_0; \\
x_1 &\mapsto x_1; \\
x_2 &\mapsto -x_2
\end{align*}
\]

2. Now, suppose that \( C_K \) is trigonal. Then it lies on a rational ruled surface \( S_e = \mathbb{P} (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \) in the linear systems \( |3X_0 + af| \). The canonical embedding is obtained by the linear system \( X_0 + (a - e - 2)f \) on the ruled surface.

If \( \mathbb{P}(H^0(I_{C_K}(2))) \) is a base-harmonic system then it defines a harmonic involution on the ruled surface \( S_e \). Moreover, we have an induced involution in the linear system \( |3X_0 + af| \). The generic curve of this linear system is not invariant by the involution. We see that the hypothesis of the theorem are not sufficient.

On the other hand, there are smooth curves on these linear systems invariant by the involution. Let us see an example. Consider the rational ruled surface \( S_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) with coordinates \([ (x_0, x_1), (y_0, y_1) ] \). We can take the curve on \( S_0 \) with equation:

\[
x_0^3y_0^3 - x_0^ny_0y_1^2 + x_1^n y_1^3 + x_1^ny_0y_1 = 0
\]

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with \( n \geq 5 \) even. This is a smooth curve of type \((3, n)\) on the linear system \(3X_0 + nf\). Moreover it is invariant by the involution

\[
\begin{align*}
  x_0 &\rightarrow x_0; & x_1 &\rightarrow -x_1; \\
y_0 &\rightarrow y_0; & y_1 &\rightarrow -y_1
\end{align*}
\]

References

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