SPECTRAL GEOMETRY OF RIEMANNIAN LEGENDRE FOLIATIONS

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Abstract. We obtain geometric characterizations of isospectral minimal Riemannian Legendre foliations on compact Sasakian manifolds of constant \( \varphi \)-sectional curvature.

1. Preliminaries

Let \( \mathcal{F} \) be a Riemannian foliation on an \( m \)-dimensional compact Riemannian manifold \((M, g)\). We denote by \( L \) and \( Q = L^\perp \) the tangent and normal bundles of \( \mathcal{F} \), and that gives the decomposition of the tangent bundle \( TM = L \oplus L^\perp \). Let \( \Delta_g \) be the Laplace operator associated to \( g \) and let \( \nabla \) be the Bott connection of the normal bundle \( Q = TM / L \) (see [16, pp. 20-21]). The Jacobi operator \( J_\nabla \) of \( F \), defined by \( J_\nabla s = (d_\nabla^* \nabla d_\nabla - \rho_\nabla) s \) for any \( s \) section of the normal bundle, is a second order elliptic operator (see [15]). The compactness of \( M \) implies that the spectra of \( \Delta_g \) and \( J_\nabla \) are discrete. Using Gilkey’s theory ([8, 14]), one can write their associated asymptotic expansions:

\[
Tr e^{-t\Delta_g} = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sum_{s=0}^{\infty} t^s a_s(\Delta_g)
\]

\[
Tr e^{-tJ_\nabla} = \sum_{i=1}^{\infty} e^{-t\mu_i} \sum_{s=0}^{\infty} t^s b_s(J_\nabla)
\]

where

\[
a_s(\Delta_g) = \int_M a_s(x, \Delta_g) dv_g
\]

\[
b_s(J_\nabla) = \int_M b_s(X, J_\nabla) dv_g
\]

are invariants of \( \Delta_g \) and \( J_\nabla \) depending only on their corresponding discrete spectra

\[
Spec(M, g) = \{ 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots \uparrow \infty \}
\]

\[
Spec(\mathcal{F}, J_\nabla) = \{ \mu_1 \leq \mu_2 \leq \ldots \leq \mu_i \leq \ldots \uparrow \infty \}
\]

We restrict our attention to the first coefficients \( a_s \) and \( b_s \) for \( s \in \{0, 1, 2\} \) which encodes certain properties of the spectral geometry of \((M, \mathcal{F})\). We recall the following theorem from [8, 11].

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Theorem 1.1. Let $\mathcal{F}$ be a Riemannian foliation of codimension $q \geq 2$ on a compact Riemannian manifold $(M, g)$. Then
\begin{align*}
a_0(\Delta_g) &= a_0 = Vol_g(M) \\
a_1(\Delta_g) &= a_1 = \frac{1}{6} \int_M \tau dv_g \\
a_2(\Delta_g) &= a_2 = \frac{1}{360} \int_M (2\|R\|^2 - 2\|\rho\|^2 + 5\tau^2)dv_g \\
b_0(\mathcal{J}_\tau) &= b_0 = qVol_g(M) \\
b_1(\mathcal{J}_\tau) &= b_1 = qa_1 + \int_M \tau_\varsigma dv_g \\
b_2(\mathcal{J}_\tau) &= b_2 = qa_2 + \frac{1}{12} \int_M (2\tau_\varsigma + 6\|\rho_\varsigma\|^2 - \|R_\varsigma\|^2)dv_g,
\end{align*}
where $R, \rho$ are the Riemann and the Ricci tensor fields, $\tau$ is the scalar curvature of $g$, and $R_\varsigma, \rho_\varsigma, \tau_\varsigma$ are those associated to the Bott connection $\nabla$ of the transverse bundle $Q = TM/L$.

The following theorem, due to Nishikawa, Tondeur and Vanhecke \[11\], is fundamental for the spectral geometry of a Riemannian foliation.

Theorem 1.2. Let $(M, g)$ and $(M_0, g_0)$ be two compact Riemannian manifolds endowed with Riemannian foliations $\mathcal{F}$ and $\mathcal{F}_0$ of codimensions $q$ and $q_0$, respectively. If $\mathcal{F}$ and $\mathcal{F}_0$ are isospectral, that is

\[ Spec(M, g) = Spec(M_0, g_0), \quad Spec(\mathcal{F}, \mathcal{J}_\tau) = Spec(\mathcal{F}_0, \mathcal{J}_{\tau_0}), \]

then the following hold:
\begin{enumerate}
  i) $\dim M = \dim M_0$, $\text{Vol}(M) = \text{Vol}(M_0)$, $q = q_0$,
  ii) $\int_M \tau dv_g = \int_{M_0} \tau_\varsigma dv_{g_0}$, $\int_M \tau_\varsigma dv_g = \int_{M_0} \tau_\varsigma dv_{g_0}$,
  iii) $\int_M (2\|R\|^2 - 2\|\rho\|^2 + 5\tau^2)dv_g = \int_{M_0} (2\|R_\varsigma\|^2 - 2\|\rho_\varsigma\|^2 + 5\tau_\varsigma^2)dv_{g_0}$,
  iv) $\int_M (2\tau_\varsigma + 6\|\rho_\varsigma\|^2 - \|R_\varsigma\|^2)dv_g = \int_{M_0} (2\tau_\varsigma + 6\|\rho_\varsigma\|^2 - \|R_\varsigma\|^2)dv_{g_0}$.
\end{enumerate}

We recall that in the one-codimension case the isospectral Riemannian foliations are completely determined by the spectrum of $\Delta_g$ and for this reason we shall assume throughout the paper that the codimension $q \geq 2$.

2. Spectral Invariants of a Riemannian Legendre foliation

In this section we compute the spectral invariants $a_s, b_s, s \in \{0, 1, 2\}$ of a Riemannian Legendre foliation with minimal leaves on a Sasakian manifold $M$ of constant $\varphi$-sectional curvature and then we obtain certain geometric properties of two such isospectral Riemannian foliations. First, we recall the notion of Riemannian Legendre foliation.

Definition 2.1. Let $M$ be a $(2n + 1)$-dimensional compact manifold endowed with a Sasakian structure $(\varphi, \xi, \eta, g)$ and let $\mathcal{D} = Ker\eta = Im\varphi$ be the $2n$-dimensional distribution on $M$ orthogonal to the 1-dimensional distribution generated by $\xi$. A Riemannian foliation $\mathcal{L}$ on $M$ is said to be a Riemannian Legendre foliation if the leaves are $n$-dimensional and $L_x \subset \mathcal{D}_x$, for each $x \in M$. Note that $\varphi(L) \subset L^\perp$. 

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold of constant $\varphi$-sectional curvature $c$ and of dimension $2n + 1 \geq 5$. We recall that $\varphi$ is an endomorphism of tangent bundle, $\xi$ is a vector field on $M$, $\eta$ is the 1-form dual to $\xi$ with respect to $g$, satisfying:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \varphi(\eta(X)) = 0,$$

$$g(X, Y) = g(\varphi(X), \varphi(Y)) + \eta(X)\eta(Y),$$

$$\nabla^M_X \xi = -\varphi(X), \quad (\nabla^M_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

for any vector fields $X, Y$. By [3], its curvature tensor is given by

$$R(X, Y)Z = \frac{c+3}{4} \{g(Y, Z)X - g(X, Z)Y\}
+ \frac{c-1}{4} \{g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y \}
+ 2g(X, \varphi Y)\varphi Z - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi
- \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y,$$

and its Ricci tensor and scalar curvature satisfy

$$\rho(X, Y) = \frac{n(c + 3) + c - 1}{2} g(X, Y) - \frac{(n + 1)(c - 1)}{2} \eta(X)\eta(Y),$$

$$\tau = \frac{n}{2} (2n + 1)(c + 3) + \frac{n}{2} (c - 1).$$

Let $L$ be a Riemannian Legendre foliation on $M$ and $(e_i, \varphi e_i, \xi), \ i \in \{1, \ldots, n\}$ be a local orthonormal basis of $TM$ adapted to the foliation $L$, which means that $(e_1, \ldots, e_n)$ is a local basis of $L$ and $(\varphi e_1, \ldots, \varphi e_n)$ is a basis of $\varphi(L)$. The curvature tensor writes as

$$R(e_i, \xi, e_k, \xi) = R(\varphi e_i, \xi, \varphi e_k, \xi) = \delta_{ik},$$

$$R(e_i, e_j, e_k, e_m) = R(\varphi e_i, \varphi e_j, \varphi e_k, \varphi e_m) = \frac{c+3}{4} (\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}),$$

$$R(e_i, e_j, \varphi e_k, e_m) = R(\varphi e_i, \varphi e_j, \varphi e_k, e_m) = \frac{c-1}{4} (\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}),$$

$$R(e_i, \varphi e_j, e_k, \varphi e_m) = \frac{c+3}{4} \delta_{ik}\delta_{jm} + \frac{c-1}{4} \delta_{im}\delta_{jk} + \frac{c-1}{2} \delta_{ij}\delta_{km},$$

the other expressions being equal to zero. The Ricci tensor is given by

$$\rho(e_i, \xi) = \rho(\varphi e_i, \xi) = \rho(e_i, \varphi e_j) = 0,$$

$$\rho(\xi, \xi) = 2n,$$

$$\rho(e_i, e_j) = \rho(\varphi e_i, \varphi e_j) = \frac{n(c + 3) + c - 1}{2} \delta_{ij}.$$
in any orthonormal basis, writes, in the fixed adapted basis, as
\[
\|R\|^2 = 2\left(\frac{c+3}{4} + \frac{c-1}{4}\right) \sum_{i,j,k,m}(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk})^2 + 8\sum_{ij}\delta_{ij}
+ 4\sum_{i,j,k,m}(\frac{c+3}{4}\delta_{ik}\delta_{jm} + \frac{c-1}{4}\delta_{im}\delta_{jk} + \frac{c-1}{2}\delta_{ij}\delta_{km})^2
= \left[\frac{(c+3)^2}{8} + \frac{(c-1)^2}{2}\right](2n^2 - 2n) + 8n + 4\left\{\frac{(c+3)^2}{16}n^2 + \frac{(c-1)^2}{16}n^2 \right\}
+ \frac{(c-1)^2}{4}n^2 + 2\frac{c+3}{4}(\frac{c-1}{4}n + \frac{c-1}{2}n) + \frac{(c-1)^2}{4}n\right\}
= \frac{(c-1)^2 + (c+3)^2}{4}n(n-1) + \frac{(c+3)^2}{4}n^2 + \frac{(c-1)^2}{4}(5n^2 + 4n) + \frac{3(c+3)(c-1)n}{2} + 8n
= \frac{(c+3)^2n(2n-1)}{4} + \frac{(c-1)^2n(6n+3)}{4} + \frac{3(c+3)(c-1)n}{2} + 8n.
\]

Furthermore, for the norm of the Ricci tensor, computing
\[
\|\rho\|^2 = \sum_{a,b}\rho(e_a, e_b)\rho(e_a, e_b),
\]
in the adapted basis, one has
\[
\|\rho\|^2 = 2\left(\frac{n(c + 3) + c - 1}{2}\right)^2 n + 4n^2.
\]

Summarizing the above computations, by Theorem 1.1 we get the following proposition.

**Proposition 2.2.** If \(\mathcal{L}\) is a Riemannian Legendre foliation on a compact \((2n + 1)\)-dimensional Sasakian manifold of constant \(\varphi\)-sectional curvature \(c\), then the spectral invariants satisfy:

\[
\begin{align*}
(2.9) \quad a_0(\Delta_g) & = a_0 = \text{Vol}_g(M) \\
(2.10) \quad a_1(\Delta_g) & = a_1 = \frac{n(2n + 1)(c + 3) + c - 1}{12} \text{Vol}_g(M) \\
(2.11) \quad a_2(\Delta_g) & = a_2 = \frac{n}{1440}(64 - 32n + (c + 3)^2(-2 + 9n + 16n^2 + 20n^3) \]
+ (c + 3)(c - 1)(12 + 2n + 20n^2) + (c - 1)^2(2 + 17n)\right) \text{Vol}_g(M) \\
(2.12) \quad b_0(\mathcal{J}_\tau) & = b_0 = (n + 1)\text{Vol}_g(M) \\
(2.13) \quad b_1(\mathcal{J}_\tau) & = b_1 = (n + 1)a_1 + \int_M \tau_{\mathcal{J}_\tau} dv_g \\
(2.14) \quad b_2(\mathcal{J}_\tau) & = b_2 = (n + 1)a_2 + \frac{1}{12} \int_M (2\tau_{\mathcal{J}_\tau} + \rho_{\mathcal{J}_\tau})^2 - \|R_{\mathcal{J}_\tau}\|^2) dv_g.
\]

Since the foliation \(\mathcal{L}\) is assumed to be Riemannian, there exists, locally, a Riemannian submersion whose vertical and horizontal distributions are \(L\) and \(L^\perp = \varphi(L) \oplus [\xi]\), respectively. Let \(A\) and \(T\) be the O'Neill tensors (see [7], [16, p. 49]).

**Proposition 2.3.** Let \(\mathcal{L}\) be a Riemannian Legendre minimal foliation on a Sasakian manifold with constant \(\varphi\)-sectional curvature \(c\). Then

\(\text{a) } \tau_{\mathcal{J}_\tau} = 3\|A\|^2 + \frac{p}{n}(c + 3)(n - 1) + 8)\);  
\[\text{b) } \|A\|^2 = \|T\|^2 + n(c + 1).\]
Proof. Let $X, Y \in \Gamma(\varphi(L) \oplus [\xi])$. From the theory of Riemannian submersions \[\ref{quadratic_eq1}\] we have
\begin{equation}
K(X, Y) = K_\nabla(X, Y) - 3\|A_X Y\|^2,
\end{equation}
where $\nabla$ is the connection associated to $TM/L \simeq \varphi(L) \oplus [\xi]$, which in the case of a Riemannian foliation coincides with the connection induced by the Levi-Civita connection of $M$ on the horizontal distribution.

Fixing an adapted local orthonormal basis of the foliation, by \[\ref{quadratic_eq2}\], we see the transverse scalar curvature can be written as
\begin{equation}
\tau_\nabla = \sum_{i \neq j=1}^n K(\varphi e_i, \varphi e_j) + 2 \sum_{i=1}^n K(\varphi e_i, \xi) + 3\|A\|^2
\end{equation}
which implies a).

Denoting by $\tau_{\text{mixed}}$ the mixed scalar curvature, defined by
\begin{equation}
\tau_{\text{mixed}} = \sum_{i=1}^n \sum_{j=n+1}^{2n} R(e_i, f_j, e_i, f_j) + \sum_{i=1}^n R(e_i, \xi, e_i, \xi),
\end{equation}
where $f_{n+i} = \varphi e_i$, we obtain:
\begin{equation}
\tau_{\text{mixed}} = \sum_{i,j} c\delta_{ij} + n = (c + 1)n.
\end{equation}

We denote by $H$ the mean curvature of the leaves. We recall Ranjan’s formula (\[\ref{quadratic_eq4}\])
\begin{equation}
\tau_{\text{mixed}} = \text{div}(H) + \|H\|^2 + \|A\|^2 - \|T\|^2.
\end{equation}
Specializing to the case $H = 0$, the relations (\[\ref{quadratic_eq5}\]) and (\[\ref{quadratic_eq6}\]) simply imply b). \hfill \Box

**Theorem 2.4.** Let $\mathcal{L}$ and $\mathcal{L}_0$ be two Riemannian Legendre minimal foliations on compact Sasakian manifolds $(M, \varphi, \xi, \eta, g)$ and $(M_0, \varphi_0, \xi_0, \eta_0, g_0)$. If $(M, g)$ and $(M, g_0)$ have constant $\varphi$, and $\varphi_0$-sectional curvature $c$ and $c_0$, and if $\mathcal{L}$ and $\mathcal{L}_0$ are isospectral, then
\begin{itemize}
  \item[a)] $\dim M = \dim M_0$, $\text{Vol}(M) = \text{Vol}(M_0)$, $c = c_0$,
  \item[b)] $\int_M \|A\|^2 dv_g = \int_{M_0} \|A_0\|^2 dv_{g_0}$, and
  \item[c)] $\int_M \|T\|^2 dv_g = \int_{M_0} \|T_0\|^2 dv_{g_0}$.
\end{itemize}

**Proof.** By Theorem \[\ref{quadratic_eq3}\] and ii), we see that $n = n_0$ and $\text{Vol}(g) = \text{Vol}(g_0)$ and
\begin{align*}
\int_M \tau dv_g &= \int_{M_0} \tau_0 dv_{g_0}, \text{ and } \int_M \tau_\nabla dv_g = \int_{M_0} \tau_0 \nabla_0 dv_{g_0}.
\end{align*}
Therefore by \[\ref{quadratic_eq7}\], we get $c = c_0$, which by Proposition \[\ref{quadratic_eq8}\] simply implies a). Now, by Proposition \[\ref{quadratic_eq9}\]b) we get b). \hfill \Box

**Corollary 2.5.** Under the hypotheses of Theorem \[\ref{quadratic_eq10}\], the following statements hold:
\begin{itemize}
  \item[(a)] If $\varphi(L) \oplus [\xi]$ is integrable, then so is $\varphi(L_0) \oplus [\xi]$.
  \item[(b)] If $\mathcal{L}$ is totally geodesic, then so is $\mathcal{L}_0$.
\end{itemize}

**Proof.** It is sufficient to observe that $A = 0 \Rightarrow A_0 = 0$ and that $T = 0 \Rightarrow T_0 = 0$. \hfill \Box
3. The invariants $b_1$ and $b_2$

We shall explicitly compute $b_1, b_2$ of Proposition 2.2. By Theorem 1.1 and Proposition 2.3 we get

$$b_1(\mathcal{K}) = \frac{n}{12} [(c + 3)(2n^2 + 6n - 2) + 3(n + 1)(c - 1) + 2n] + 3 \int_M ||A||^2 dv_g.$$  \hfill (3.1)

Now, we proceed to the computation of each term involved in (2.14) of $b_2$. Let $(X_i, U_j)$ be an orthonormal basis adapted to the foliation with $X_i$ horizontal and $U_j$ vertical. We introduce the notations from [2]

$$(A_X, A_Y) = \sum_i g(A_X X_i, A_Y X_i) = \sum_j g(A_X U_j, A_Y U_j),$$  \hfill (3.2)

$$(TX, TY) = \sum_j g(T_U X, T_U Y).$$  \hfill (3.3)

Since $\mathcal{L}$ is assumed to be minimal, by Proposition 9.36 in [2], we have

$$\rho_\mathcal{K}(X, Y) = \rho(X, Y) + 2(A_X, A_Y) + (TX, TY),$$  \hfill (3.4)

Setting $f_i = \varphi e_i$, for $i \in \{1, \ldots, n\}$, and $f_{n+1} = \xi$, we have

$$\|\rho_\mathcal{K}\|^2 = \sum_{i,j=1}^n \rho_\mathcal{K}(f_i, f_j)^2 + \rho_\mathcal{K}(\xi, \xi)^2$$  \hfill (3.5)

and we obtain

$$\|\rho_\mathcal{K}\|^2 = \sum_{i,j=1}^{n+1} \rho(f_i, f_j)^2 + 2 \sum_{i,j=1}^{n+1} \rho(f_i, f_j)[2(A_{f_i}, A_{f_j}) + (T f_i, T f_j)]$$

$$+ \sum_{i,j=1}^{n+1} [2(A_{f_i}, A_{f_j}) + (T f_i, T f_j)]^2.$$  \hfill (3.6)

We easily see that

$$T_{e_i} \xi = 0, \quad T_{e_i} \varphi e_j = \varphi(T_{e_i} e_j), \quad A_{\varphi e_i} \varphi e_j = \varphi(A_{\varphi e_i} e_j), \quad A_\xi \varphi e_i = e_i,$$

$$2(A_\xi, A_\xi) + (T_\xi, T_\xi) = 2 \sum_{i=1}^n (A_\xi \varphi e_i, A_\xi \varphi e_i) = 2n.$$

We also know from (2.5) that

$$\rho(\xi, \xi) = 2n$$

$$\sum_{i,j=1}^n \rho(f_i, f_j)^2 = \left[ \frac{n(c + 3) + c - 1}{2} \right]^2 n.$$

Setting

$$l = \sum_{i,j=1}^{n+1} \rho(f_i, f_j)^2,$$
we obtain:
\[
\|\rho\|^2 = l + 2 \sum_{i,j} \frac{n(c+3)+c-1}{2} \delta_{ij} \left[ 2(A_{f_i}, A_{f_j}) + (T_{f_i}, T_{f_j}) \right] \\
+ 2(2n)^2 + \sum_{i,j=1}^{n+1} [2(A_{f_i}, A_{f_j}) + (T_{f_i}, T_{f_j})]^2
\]
\[(3.7)\]
\[= (l + 8n^2) + (n(c + 3) + c - 1)[2\|A\|^2 + \|T\|^2]
\[+ \sum_{i,j=1}^{n+1} [2(A_{f_i}, A_{f_j}) + (T_{f_i}, T_{f_j})]^2.
\]

**Proposition 3.1.** If \(\mathcal{L}\) is a Riemannian foliation on \((M, g)\), then
\[(3.8)\]
\[
\sum_{i=1}^{n} R(X, e_i, Y, e_i) = \frac{1}{2} \left[ g(\nabla^M_X H, X) + g(\nabla^M_Y H, Y) \right] + (A_X, A_Y) - (TX, TY),
\]
where \(\nabla^M\) is the Levi-Civita connection of \((M, g)\), \(H\) is the mean curvature of the leaves, \((e_1, \ldots, e_n)\) is a local basis of the vertical distribution \(L\), and \(X, Y\) are horizontal.

**Proof.** From the theory of Riemannian submersions, for any horizontal vectors \(X, Y\) and vertical vector \(U\), we have
\[
R(X, U, Y, U) = g((\nabla^M_X T)U, Y) - g(T_X T_Y + g((\nabla^M_A)X Y, U) + g(A_X U, A_Y U).
\]
Therefore
\[
\sum_{i=1}^{n} R(X, e_i, Y, e_i) = \sum_{i=1}^{n} (g((\nabla^M_X T)_{e_i} e_i, Y) - g(T_{e_i} X, T_{e_i} Y)
\[+ g((\nabla^M_{e_i} A)X Y, e_i) + g(A_X e_i, A_Y e_i)).
\]
The covariant derivative of \(T\) satisfies
\[
\sum_i g((\nabla^M_X T)_{e_i} e_i, Y) = g(\nabla^M_X H, Y) - \sum_i g(T_{v(\nabla^M_X e_i)} e_i, Y) - \sum_i g(T_{e_i} v(\nabla^M_X e_i), Y)
\[= g(\nabla^M_X H, Y) - 2 \sum_i g(T_{e_i} v(\nabla^M_X e_i), Y),
\]
since \(T_{v} W = T_{W} U\) and \(\sum_{i} T_{e_i} e_i = H\).
Setting
\[
v(\nabla^M_X e_i) = \sum_{j} h_{ij} e_j
\]
we note that
\[
h_{ij} = g(v(\nabla^M_X e_i), e_j) = X(g(e_i, e_j)) - g(e_i, \nabla^M_X e_j) = -g(e_i, v(\nabla^M_X e_j)) = -h_{ji};
\]
\[
\sum_{i,j} g(T_{e_i} h_{ij} e_j, Y) = \sum_{i,j} h_{ij} g(T_{e_i} e_j, Y) = 0;
\]
\[
\sum_i g((\nabla^M_{e_i} A)X Y, e_i) = \frac{1}{2} \left( g(\nabla^M_X H, X) - g(\nabla^M_X H, Y) \right).
\]
Therefore
\[
\sum_i R(X, e_i, Y, e_i) = g(\nabla^M_X H, Y) - (TX, TY) + (A_X, A_Y) \\
+ \frac{1}{2} (g(\nabla^M_X H, X) - g(\nabla^M_Y H, Y)),
\]
and (3.8) follows. \(\square\)

From (3.9), one can obtain the following proposition.

**Proposition 3.2.** If \(L\) is a Riemannian foliation with minimal leaves then
\[
\sum_{k=1}^{n} R(f_i, e_k, f_j, e_k) = (A_{f_i}, A_{f_j}) - (T f_i, T f_j).
\]

Now, we consider a Riemannian Legendre foliation with minimal leaves on a Sasakian manifold \(M\) of constant \(\varphi\)-sectional curvature \(c\) and we fix a local orthonormal basis \((e_i, f_i, f_{n+1})\) adapted to the foliation, that is \(f_{n+1} = \xi\) and \(f_i = \varphi e_i\) for any \(i \in \{1, \ldots, n\}\) and \(\{e_1, \ldots, e_n\}\) is a local basis of a leaf \(L\).

Setting
\[
S(f_i, f_j) = \sum_{k=1}^{n} R(f_i, e_k, f_j, e_k)
\]
and using relations (2.4), we obtain that
\[
S(\varphi e_i, \xi) = \sum_{k=1}^{n} R(\varphi e_i, e_k, \xi, e_k) = 0,
\]
\[
S(\xi, \xi) = n, \quad S(f_i, f_j) = S(\varphi e_i, \varphi e_j) = d\delta_{ij},
\]
where \(d = \frac{c+3}{4} n + \frac{3(c-1)}{4}\).

By Proposition 2.3b) and equations (3.7) and (3.10), it follows:
\[
\|\rho_{\nabla}\|^2 = (l + 8n^2) + [n(c + 3) + c - 1][3\|A\|^2 - c(n + 1)] \\
+ \sum_{i,j=1}^{n+1} [3(A_{f_i}, A_{f_j}) - S(f_i, f_j)]^2.
\]

Denoting by \(E\) the last sum of the previous relation, we have:
\[
E = \sum_{i=1}^{n} [3(A_{\varphi e_i}, A_{\varphi e_i}) - d]^2 + [2(A_{\xi}, A_{\xi}) + (T\xi, T\xi)]^2 + 2 \sum_{i<j}^{n+1} [3(A_{f_i}, A_{f_j})]^2 \\
= 9 \sum_{i=1}^{n} (A_{\varphi e_i}, A_{\varphi e_i})^2 + d^2 n - 6d \sum_{i=1}^{n} (A_{\varphi e_i}, A_{\varphi e_i}) + 4n^2 + 18 \sum_{i<j}^{n+1} (A_{f_i}, A_{f_j})^2.
\]

Summarizing, we conclude:
Proposition 3.3. Under the hypothesis of Proposition 2.2, the following holds

\[ \| \rho \|^2 = 9 \sum_{i=1}^{n} (A_{\varphi e_i}, A_{\varphi e_i})^2 + 18 \sum_{i<j} (A_{f_i}, A_{f_j})^2 + nd(d+6) - 6d\|A\|^2 + 16n^2 \]
\[ + (n(c+3)+c-1)(3\|A\|^2 - c(n+1)) + n \left( \frac{n(c+3)+c-1}{2} \right)^2, \]

where \( d = \frac{c+3}{4} n + \frac{3(c-1)}{4} \).

To compute \( \| R \rho \|^2 \), we notice that

\[ R \rho (f_i, f_j, f_k, f_l) = R(f_i, f_j, f_k, f_l) + 2g(A_{f_i}, A_{f_j}, A_{f_k}, f_l) \]
\[ - g(A_{f_j}, f_k, A_{f_i}, f_l) - g(A_{f_k}, f_i, A_{f_j}, f_l), \]

we consider the following tensor of type (0,4) associated to the horizontal distribution

\[ V(X, Y, Z, Z') = 2g(A_X Y, A_Z Z') - g(A_Y Z, A_X Z') - g(A_Z X, A_Y Z') \]

and we set

\[ \| V \|^2 = \sum_{i,j,k,l=1}^{n+1} V(f_i, f_j, f_k, f_l)^2. \]

We can write:

\[ \| R \rho \|^2 = \sum_{i,j,k,l=1}^{n+1} (R \rho (f_i, f_j, f_k, f_l))^2 \]
\[ = \sum_{i,j,k,l=1}^{n+1} (R(f_i, f_j, f_k, f_l))^2 + 2 \sum_{i,j,k,l=1}^{n+1} R(f_i, f_j, f_k, f_l) \{ 2g(A_{f_i}, A_{f_j}, A_{f_k}, f_l) \}
\[ - g(A_{f_j}, f_k, A_{f_i}, f_l) - g(A_{f_k}, f_i, A_{f_j}, f_l) \} + \| V \|^2. \]

Then

\[ l' = \sum_{i,j,k,l=1}^{n+1} R(f_i, f_j, f_k, f_l)^2 \]
\[ = \sum_{i,j,k,l=1}^{n} R(\varphi e_i, \varphi e_j, \varphi e_k, \varphi e_l)^2 + 4 \sum_{i,k=1}^{n} R(\varphi e_i, \xi, \varphi e_k, \xi)^2 \]
\[ = \left( \frac{c+3}{4} \right)^2 \sum_{i,j,k,l=1}^{n} (\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})^2 + 4 \sum_{i,k}^{n} \delta_{ik}^2 \]
\[ = \frac{(c+3)^2}{16} [2n^2 - 2 \sum_{i,j} \delta_{ij}] + 4n = \frac{(c+3)^2(n-1)n}{8} + 4n, \]
and thus,

\[
\| R_{\nabla} \|_2 = l' + 2 \sum_{i,j,k,l=1}^{n} \frac{c + 3}{4} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \left[ 2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_k}\varphi e_l) - g(A_{\varphi e_i}\varphi e_k, A_{\varphi e_j}\varphi e_l) \right] \\
- g(A_{\varphi e_j}\varphi e_k, A_{\varphi e_i}\varphi e_l) - g(A_{\varphi e_k}\varphi e_i, A_{\varphi e_j}\varphi e_l) ] \\
+ 8 \sum_{i,k} R(\varphi e_i, \xi, \varphi e_k) \left[ 2g(A_{\varphi e_i}\xi, A_{\varphi e_k}\xi) - g(A_{\xi}\varphi e_k, A_{\xi}\varphi e_i) \right] \\
= l' + \| V \|_2 + 2 \left[ \frac{c + 3}{4} \sum_{i,j} \left[ 2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_i}\varphi e_j) \\
- g(A_{\varphi e_j}\varphi e_i, A_{\varphi e_i}\varphi e_j) - g(A_{\varphi e_i}\varphi e_i, A_{\varphi e_j}\varphi e_i) \right] \\
- 2 \left[ \frac{c + 3}{4} \sum_{i,j} \left[ 2g(A_{\varphi e_i}\varphi e_j, A_{\varphi e_j}\varphi e_i) \\
- g(A_{\varphi e_j}\varphi e_j, A_{\varphi e_i}\varphi e_i) - g(A_{\varphi e_j}\varphi e_i, A_{\varphi e_j}\varphi e_i) \right] + 24n \\
= l' + 24n - 3n(c + 3) + 3(c + 3)\| A \|^2 + \| V \|^2 .
\]

Therefore

\[
(3.15) \quad \| R_{\nabla} \|^2 = \frac{(c + 3)^2 (n - 1)n}{8} - 3n(c + 3) + 28n + 3(c + 3)\| A \|^2 + \| V \|^2 .
\]

On the other hand, we get

\[
\sum_j V(f_i, f_j, f_k, f_j) = \sum_j \left[ 2g(A_{f_i} f_j, A_{f_k} f_j) - g(A_{f_j} f_k, A_{f_i} f_j) - g(A_{f_k} f_i, A_{f_j} f_j) \right] \\
= 3 \sum_j g(A_{f_i} f_j, A_{f_k} f_j) = 3(A_{f_i}, A_{f_k}) ,
\]

and thus

\[
(3.16) \quad (C_{24} V)(f_i, f_k) = 3(A_{f_i}, A_{f_k}) ,
\]

where \( C_{24} \) denotes the contraction of the tensor with respect to the indices 2 and 4. The Hilbert-Schmidt norm of the (0, 2) tensor \( C_{24} V \) along the horizontal distribution satisfies

\[
(3.17) \quad \| C_{24} V \|^2 = 9 \sum_{i=1}^{n} (A_{\varphi e_i}, A_{\varphi e_i})^2 + 18 \sum_{i<j}^{n+1} (A_{f_i}, A_{f_j})^2 + 9n^2 .
\]
Summarizing, by Proposition 2.2 we obtain that

\[(3.18) \quad b_2 = (n + 1) a_2 + \frac{1}{12} \int_M 2 \tau \nu + 6 \| \nu \|^2 - \| R \nu \|^2 dv_g \]

\[= (n + 1) a_2 + \frac{1}{12} \int_M 6 \| (C_{24} V) \|^2 - \| V \|^2 dv_g \]

\[+ \frac{\text{Vol}_g(M)}{12} \left( 42 n^2 - 28 n + (c + 3)n(3 + 11n + 4n^2) \right) \]

\[+ \frac{1}{8} (c + 3)^2 n(-11 - 15n + 13n^2 + 4n^3) \]

\[+ (c - 1)n(27 + 2n) + \frac{1}{8} (c - 1)^2 (-36 + 3n) \]

\[+ \frac{1}{4} (c + 3)(c - 1)(-6 - 24n + 2n^2 + n^3) \).

By (3.18) and Theorems 1.1, 1.2, 2.4 we now get our main result.

**Theorem 3.4.** Let \((M^{2n+1}, \varphi, \xi, \eta, g)\) and \((M_0^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)\) be compact isospectral Sasakian manifolds with constant \(\varphi\)-sectional curvature \(c\) and constant \(\varphi_0\)-sectional curvature \(c_0\) respectively. If \(\mathcal{L}\) and \(\mathcal{L}_0\) are Riemannian minimal Legendre foliations on \(M\) and \(M_0\) such that \(\text{Spec}(\mathcal{L}, \mathcal{J}_\varphi) = \text{Spec}(\mathcal{L}_0, \mathcal{J}_{\varphi_0})\), then

1) \(\dim M = \dim M_0\), \(\text{Vol}(M) = \text{Vol}(M_0)\), \(c = c_0\),

2) \(\int_M \| A \|^2 dv_g = \int_{M_0} \| A_0 \|^2 dv_{g_0}\), \(\int_M \| T \|^2 dv_g = \int_{M_0} \| T_0 \|^2 dv_{g_0}\),

3) \(\int_M [6 \| C_{24} V \|^2 - \| V \|^2] dv_g = \int_{M_0} [6 \| C_{24} V_0 \|^2 - \| V_0 \|^2] dv_{g_0}\).

4. **Concluding remarks**

Let \(\mathcal{L}\) be a Riemannian Legendre foliation with totally geodesic leaves on \((M, g)\) and assume that \(M\) has the constant curvature \(c = 1\) and that \(2n + 1 = \dim M\). In this particular case, we would like to point that the condition 3) of Theorem 3.4 is implied by 1). Indeed, for any Riemannian totally geodesic foliation \(\mathcal{L}\) on a constant curvature space \(M\) with \(\dim Q = \dim L + 1\), one can see that

\[g(A_Y W, A_Y W) = g(Y, Y)g(W, W)\]

for any \(Y \in L\) and for any \(Y \in Q\),

\[A_X : L \to Q^\perp = \{ Y \in Q \mid g(Y, X) = 0 \}\] is a bijection for any unit vector \(X\), and \(R \nu\) has the constant curvature 4 (see the argument of [H Prop. 4.6]). Thus, we simply have

\[g(\varphi e_i, \varphi e_i) = \sum_{k=1}^n g(\varphi e_i, \varphi e_i) g(e_k, e_k) = n,\]

\[g(f_i, f_j) g(e_k, e_k) = 0, \text{ for any } i < j,\]
and therefore, by (3.17), $\|C_{24}V\|^2 = 18n^2$. We easily see that

$$\|V\|^2 = \sum_{i,j,k,l=1}^{n+1} V(f_i, f_j, f_k, f_l)^2 = \sum_{i,j,k,l=1}^{n+1} (R_N(f_i, f_j, f_k, f_l) - R(f_i, f_j, f_k, f_l))^2$$

$$= \sum_{i,j,k,l=1}^{n+1} [(4 - 1)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})]^2 = 18n^2 + 18n.$$

This concludes that

$$(4.1) \int_M [6\|C_{24}V\|^2 - \|V\|^2]dv_g = (90n^2 - 18n)Vol(M).$$

**Example 1.** Let $(S^3, \varphi, \xi, \eta, g)$ be the standard contact metric structure on $S^3$, which we now recall. Let $\mathbb{H} = \{x_1 + ix_2 + jx_3 + kx_4 \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the algebra of quaternion numbers, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. The set of unit quaternions is identified with $S^3$. Let $(I, J, K)$ be the quaternionic structure on $\mathbb{H}$ given for $I, J, K : \mathbb{H} \to \mathbb{H}$, by $I(h) = ih, J(h) = jh, K(h) = kh$ for any $h \in \mathbb{H}$. Let $N$ be the unit outer normal vector field on $S^3$ and let $g$ be the Riemann metric with constant curvature $c = 1$. We set $\xi = -IN, \eta$ the dual form of $\xi, \varphi(Z)$ the projection of $I(Z)$ onto tangent space of $S^3$, for any vector field $Z$ of $S^3$. Note that $(\varphi, \xi, \eta, g)$ is the standard contact metric on $S^3$ and the its $\varphi$-sectional curvature is $c = 1$.

Let $(x_1, x_2, x_3, x_4)$ be the Cartesian coordinate system on $\mathbb{R}^4 = \mathbb{H}$. It is easy to see that

$$\xi = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}.$$ 

Setting $W = -IN$ and $Y = -KN$, we have

$$W = x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4};$$

$$Y = x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}.$$ 

One can easily compute the Lie brackets between these vectors: $[W, \xi] = -2Y$, $[Y, \xi] = 2W$, $[W, Y] = 2\xi$ (see [10]). The distributions $L = \text{span}\{W\}$ and $L' = \text{span}\{Y\}$ define two non-degenerate Legendre foliations (see [10] Example 7.1)). Since $[W, Y] = 2\xi$ and $\varphi(W) = Y$, by [10] Lemma 6.6], $L$ and $L'$ are Riemannian Legendre foliations on the Sasakian space form $S^3$ and both of them are totally geodesic. By theorem 3.4, any Riemannian minimal Legendre foliations on a compact Sasakian space form $M_0$ with $\varphi_0$-sectional curvature $c_0$, isospectral to the foliation $\mathcal{L}$ (defined above) on the standard Sasakian space form $(S^3, \varphi, \xi, \eta)$ is totally geodesic, $c_0 = 1$, $\dim M_0 = 3$, and

$$\int_M [6\|C_{24}V\|^2 - \|V\|^2]dv_g = 72\text{Vol}(M) = 72\text{Vol}(M),$$

$$\int_M \|A\|^2dv_g = 2\text{Vol}(M).$$

It is well known that a typical example of a Sasakian space form is a $D$-homothetic deformation of the standard contact metric structure of an odd-dimensional sphere $S^{2n+1}$.
which we now recall (see [4, Example 7.4.1]). For a contact metric structure \((\varphi, \xi, \eta, g)\), one defines the \(\mathcal{D}\)-homothetic deformation \((\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})\) by

\[
\bar{\varphi} = \varphi, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\eta} = a\eta, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,
\]

where \(a\) is a positive constant (see [4, p. 114]). By [4, Theorem 7.15], a compact simply connected Sasakian space form with \(\varphi\)-sectional curvature \(c > -3\) is a \(\mathcal{D}\)-homothetic deformation of the standard contact metric structure on \(S^{2n+1}\) and \(c = \frac{4}{a} - 3\) (for some \(a > 0\)). Since \(\text{Ker} \bar{\eta} = \text{Ker} \eta\), the problem of finding Riemannian Legendre foliations on such a compact Sasakian space form (with \(c > -3\)) reduces to the one on the standard sphere \(S^{2n+1}\) (i.e. \(c=1\)).

Case \(n = 1\). One can apply a \(\mathcal{D}\)-homothetic deformation to Example 1 to obtain an example for any \(c > -3\).

Case \(n = 2\). From [9], there are no Riemannian foliations with two-dimensional leaves on a standard sphere, which in particular means that there are no Riemannian Legendre foliations on \(S^5\).

Case \(n = 3\). By [9, Theorem 5.3], we get, in particular, that any Riemannian foliation with 3-dimensional leaves on \(S^7\) is given uniquely (up to equivalence) by a direct sum of irreducible unitary representations of \(SU(2)\), namely \(\rho_1 \oplus \rho_1\), or by \(\rho_3\), where \(\rho_k\) is the action of \(SU(2)\) on the set of complex homogeneous polynomials in two variables of degree \(k\).

Note that \(\rho_1 \oplus \rho_1\) corresponds to the Hopf fibration \(S^7 \to S^4\) (see [9]), which is not a Legendre foliation. In fact, no leaf of \(\rho_1 \oplus \rho_1\) is Legendrian, simply because \(L\), the tangent distribution of the leaves, is generated by \(-\xi = IN, JN, KN\), where \((I, J, K)\) is the standard quaternionic structure on \(\mathbb{H}^2 = \mathbb{R}^8\) and \(N\) is the unit outer vector field to \(S^7\) (see [6, p. 265]).

In [12, p. 365], Ohnita constructed a unique minimal Legendrian orbit \(S^7\) under the action of \(\rho_3\), which means that only one leaf of the Riemannian foliation given by \(\rho_3\) on \(S^7\) is both minimal and Legendrian. This concludes that \(\rho_3\) does not provide a Riemannian Legendre foliation with minimal leaves.

Finally, another typical example of a Riemannian Legendre foliation with totally geodesic leaves is given by the tangent sphere bundle \(\pi: T_1P \to P\) of a Riemannian manifold \((P, h)\). Assume that \(T_1P\) is endowed with the standard contact metric structure.

If \(\dim P > 2\), then \(T_1P\) is never a Sasakian space form (see [5]). Note that if \(P\) has constant curvature, then \(T_1P\) admits a non-Sasakian contact metric structure of constant \(\varphi\)-sectional curvature \(c^2\) if and only if \(c = 2 \pm \sqrt{5}\) (see [4, Theorem 9.9]).

If \(\dim P = 2\) and if \(T_1P\) is a Sasakian space form, then \(P\) has constant curvature \(c = 1\) (see [4, Theorem 9.3]). Note that \(T_1S^2 \simeq \mathbb{R}P^3\) (see [4, p. 142]) and the Riemannian Legendre foliation on the universal cover of \(T_1S^2\) is equivalent to Example 1.

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