Solutions of Einstein’s field equations related to
Jacobi’s inversion problem

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Abstract

A new class of exact solutions to the axisymmetric and stationary vacuum Einstein equations containing \( n \) arbitrary complex parameters and one arbitrary real solution of the axisymmetric three–dimensional Laplace equation is presented. The solutions are related to Jacobi’s inversion problem for hyperelliptic Abelian integrals.
1 Introduction

The inversion problem for hyperelliptic integrals (integrals over a rational function \( R[x, W(x)] \) with \( W(x) \) being the square root of a polynomial of order \( 2n + 1 \) or \( 2n + 2, \ n = 2, 3, 4, \ldots \) was formulated by Jacobi in 1832 [1]. Its solution in the ultraelliptic case (\( n = 2 \)) was presented by Göpel [2] and Rosenhain [3]. They generalized Jacobi’s theta functions to theta functions depending on two variables. Riemann and Weierstrass solved the general problem in terms of theta functions of \( n \) variables, cf. [4], [5], [6].

In the course of solving a physical problem within Einstein’s theory of general relativity [7], [8] — the problem of a rigidly rotating disk of dust — we were lead exactly to the ultraelliptic case of Jacobi’s inversion problem [9]. In this letter we will show that a whole class of solutions to the axisymmetric and stationary vacuum Einstein equations may be associated with Jacobi’s inversion problem in the general case.

2 The class of solutions

As is well known the axisymmetric and stationary vacuum Einstein equations may be reduced to the Ernst equation

\[
(\Re f) \triangle f = (\nabla f)^2
\]

for the complex function \( f(\rho, \zeta) \) called the Ernst potential [10], [11]. The operators \( \triangle \) and \( \nabla \) have their usual three–dimensional meaning applied to an axisymmetric function depending on the cylindrical coordinates \( \rho \) and \( \zeta \):

\[
\triangle = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2}, \quad \nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \zeta}).
\]
Any solution of our class is given by

\[ f = \exp \left\{ \sum_{m=1}^{n} \frac{K^{(m)}_{m} dK}{W} - u_{n} \right\} \quad (3) \]

with

\[ W = \sqrt{(K + iz)(K - iz) \prod_{j=1}^{n}(K - K_{j})(K - \bar{K}_{j})}. \quad (4) \]

The \( K_{m} (m = 1, 2, \ldots, n) \) are arbitrary complex constants; a bar denotes complex conjugation. The coordinates \( \rho \) and \( \zeta \) are combined to the complex coordinate

\[ z = \rho + i\zeta. \quad (5) \]

The upper integration limits \( K^{(m)} (m = 1, 2, \ldots, n) \) are functions of \( \rho \) and \( \zeta \) and have to be determined as the solution of the following inversion problem:

\[ \sum_{m=1}^{n} \int_{K_{m}}^{K^{(m)}} \frac{K^{j} dK}{W} = u_{j}, \quad j = 0, 1, 2, \ldots, n - 1. \quad (6) \]

The quantities \( u_{j} (j = 0, 1, 2, \ldots, n) \) in (6) and (3) are defined as follows: Let \( u_{0} \) be any real solution to the (axisymmetric) Laplace equation

\[ \Delta u_{0} = 0. \quad (7) \]

Then the \( u_{j} (j = 1, 2, \ldots, n) \) are real solutions to

\[ iu_{j,z} = \frac{u_{j-1,z}}{2} + z u_{j-1,z}, \quad j = 1, 2, \ldots, n \quad (8) \]

where an index \( \cdot, z \) denotes partial derivation with respect to \( z \). As a consequence, all \( u_{j} \) are solutions to the Laplace equation. The calculation of the \( u_{j} \), starting from \( u_{0} \), is simply a (path–independent) line integration. If one starts from a solution \( u_{0} \) of the Laplace equation that vanishes at infinity one may choose the integration constants such that all \( u_{j} \) vanish at infinity as well. In
this case one obtains $f \to 1$ at infinity, i.e. an asymptotically flat solution to the Einstein equations. It should be noted that the integrations in (3) and (6) have to be performed along the same curves in the two-sheeted Riemann surface (of genus $n$) associated with $W(K)$ defined by (4). Using a result of Riemann [cf. [4], p. 311, Eq. (5)] it is possible to express $f$ in terms of theta functions of $n$ variables. On the symmetry axis, $\rho = 0$, the two zeros $-iz$ and $iz$ of the polynomial $W^2$ coincide. This leads, for $n \geq 2$, to expressions for the axis values of the Ernst potential in terms of theta functions of $n - 1$ variables. Hence, the class of solutions considered cannot be obtained by applying usual Bäcklund transformations repeatedly to an arbitrary Weyl metric.

The proof that $f$ given by (3) – (8) satisfies indeed the Ernst equation (1) consists of two steps:

(i) Combining Eqs. (8) and their complex conjugates with (6) and (3) one obtains

\[ \sum_{m=1}^{n} \frac{K^{(m)} + iz}{W^{(m)}} (K^{(m)})^j K^{(m)}_{,z} = 0, \quad j = 0, 1, 2, \ldots, n - 2, \quad (9) \]

\[ \sum_{m=1}^{n} \frac{K^{(m)} - iz}{W^{(m)}} (K^{(m)})^j K^{(m)}_{,\bar{z}} = 0, \quad j = 0, 1, 2, \ldots, n - 2, \quad (10) \]

and

\[ (\ln f)_{,z} = \sum_{m=1}^{n} \frac{K^{(m)} + iz}{W^{(m)}} (K^{(m)})^{n-1} K^{(m)}_{,z}, \quad (11) \]

\[ (\ln f)_{,\bar{z}} = \sum_{m=1}^{n} \frac{K^{(m)} - iz}{W^{(m)}} (K^{(m)})^{n-1} K^{(m)}_{,\bar{z}} \quad (12) \]

where

\[ W^{(m)} \equiv W(K^{(m)}). \quad (13) \]

Purely algebraic manipulations of Eqs. (9)–(12) lead to

\[ K^{(j)}_{,z} = (\ln f)_{,z} \frac{W^{(j)}}{K^{(j)} + iz} \prod_{m=1}^{n} \frac{1}{(K^{(j)} - K^{(m)})}. \quad (14) \]
$$K^{(j)}_{z \bar{z}} = (\ln f)_z \frac{W^{(j)}}{K^{(j)} - i\bar{z}} \prod_{m=1, m \neq j}^{n} \frac{1}{K^{(j)} - K^{(m)}}. \quad (15)$$

The integrability conditions

$$K^{(j)}_{z \bar{z}} = K^{(j)}_{\bar{z} z} \quad (16)$$

yield, for all $j$,

$$\Delta (\ln f) = \alpha (\nabla \ln f)^2 \quad (17)$$

with

$$\alpha = \sum_{m=1}^{n} \left\{ \frac{W^{(m)}}{(K^{(m)} + i\bar{z})(K^{(m)} - i\bar{z})} \prod_{j=1, j \neq m}^{n} \frac{1}{K^{(m)} - K^{(j)}} \right\}. \quad (18)$$

(ii) Due to the reality of the $u_j$, it can easily be verified from Eqs. (6) that the $K^{(m)}$ and $\bar{K}^{(m)}$ ($m = 1, 2, \ldots, n$) are related such that they form the $2n$ zeros of a characteristic polynomial

$$\prod_{j=1}^{n} (K - K_j)(K - \bar{K}_j) - (K + i\bar{z})(K - i\bar{z}) \left( \sum_{j=0}^{n-1} \alpha_j K^j \right)^2 = (1 - \alpha^2_{n-1}) \prod_{j=1}^{n} (K - K^{(j)})(K - \bar{K}^{(j)}). \quad (19)$$

The (purely imaginary) quantities $\alpha_j$ may be determined from the system of linear algebraic equations

$$\sum_{j=0}^{n-1} \alpha_j (K^{(m)})^j = \frac{W^{(m)}}{(K^{(m)} + i\bar{z})(K^{(m)} - i\bar{z})}, \quad m = 1, 2, \ldots, n \quad (20)$$

following from (19) for $K = K^{(m)}$ ($m = 1, 2, \ldots, n$) and (4). In particular, it turns out that

$$\alpha_{n-1} = \alpha \quad (21)$$

with $\alpha$ defined according to (18). Using (19) together with (3) one finds

$$\frac{f - \bar{f}}{f + \bar{f}} = \alpha_{n-1} \quad (22)$$
and consequently, because of (17),

$$\triangle (\ln f) = \frac{\tilde{f} - \bar{f}}{f + \bar{f}} (\nabla \ln f)^2 \tag{23}$$

which is equivalent to the Ernst equation (1).

### 3 Discussion

For $n = 0$ the class of solutions (3) reduces to Weyl’s class with $f = \exp(-u_0)$. The case $n = 1$ leads to solutions in terms of elliptic functions. A nontrivial and physically meaningful example for $n = 2$ is the solution of the problem of a rigidly rotating disk of dust [9]. This example provides some justification for the hope that the class of solutions presented will lead to further physical applications in the context of the rotating body problem.

An interesting mathematical question concerns the relation to Korotkin’s finite–gap solutions [12]. It seems to us that those solutions can be obtained for a particular choice of the potential function $u_0$ and form therefore a subclass of the solutions presented here.
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