BIPOLAR FUZZY SOFT SETS AND ITS APPLICATIONS IN DECISION MAKING PROBLEM

MUHAMMAD ASLAM, SALEEM ABDULLAH, AND KIFAYAT ULLAH

Abstract. In this article, we combine the concept of a bipolar fuzzy set and a soft set. We introduce the notion of bipolar fuzzy soft set and study fundamental properties. We study basic operations on bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solve decision making problems by using bipolar fuzzy soft set.

1. Introduction

Complicated problems in different field like engineering, economics, environmental science, medicine and social sciences, which arising due to classical mathematical modelling and manipulating of various type of uncertainty. While some of traditional mathematical tool fail to solve these complicated problems. We used some mathematical modelling like fuzzy set theory [1], rough set theory [2], interval mathematics [12] and probability theory are well-known and operative tools for handling with vagueness and uncertainty, each of them has its own inherent limitations; one major fault shared by these mathematical methodologies may be due to the inadequacy of parametrization tools [3].

Molodtsov, [4] adopted the notion of soft sets. Soft set is a new mathematical tool to describe the uncertainties. Soft set theory is powerful tool to describe uncertainties. Recently, researcher are engaged in soft set theory. Maji et al. [5] defined new notions on soft sets. Ali et al. [21] studied some new concepts of a soft set. Sezgin and Atagün [22] studied some new theoretical soft set operations. Majumdar and Samanta, worked on soft mappings [23]. Choudhure et al. defined the concept of soft relation and fuzzy soft relation and then applied them to solve a number of decision- making problems. In [7], Aktaş and Çağman applied the concept of soft set to groups theory and adopted soft group of a group. Feng et.al, studied and applied softness to semirings [8]. Recently, Acar studied soft rings [9]. Jun et. al, applied the concept of soft set to BCK/BCI-algebras [10,11]. Sezgin and Atagün initiated the concept of normalistic soft groups [13]. Zhan et.al, worked on soft ideal of BL-algebras [15]. In [10], Kazancı et. al, used the concept of soft set to BCH-algebras. Sezgin et. al, studied soft nearrings [17]. Çağman et al. considered two types of notions of a soft set with group, which is called group Soft intersection group softunion groups of a group [20]. see [14].

Fuzzy set originally proposed by Zadeh in [1] of 1965. After semblance of the concept of fuzzy set, researcher given much attention to developed fuzzy set theory. Maji et al. [36] introduced the concept of fuzzy soft sets. Afterwards, many

Key words and phrases. Soft set, bipolar fuzzy set, fuzzy soft set and bipolar fuzzy soft set.
researchers have worked on this concept. Roy and Maji [28] provided some results on an application of fuzzy soft sets in decision making problems. F. Feng et al. give application in decision making problem [31, 32].

Fuzzy set is a type of important mathematical structure to represent a collection of objects whose boundary is vague. There are several types of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. bi-polar-valued fuzzy set is another an extension of fuzzy set whose membership degree range is different from the above extensions. In 2000, Lee [35] initiated an extension of fuzzy set named bi-polar-valued fuzzy set. He gave two kinds of representations of the notion of bi-polar-valued fuzzy sets. In case of Bi-polar-valued fuzzy sets membership degree range is enlarged from the interval \([0,1]\) to \([-1,1]\). In a bi-polar-valued fuzzy set, the membership degree 0 indicate that elements are irrelevant to the corresponding property, the membership degrees on \((0,1]\) assigne that elements some what satisfy the property, and the membership degrees on \([-1,0)\) assigne that elements somewhat satisfy the implicit counter-property [35].

In this article, we combine the concept of a bipolar fuzzy set and a soft set. We introduce the notion of bipolar fuzzy soft set and study fundamental properties. We study basic operations on bipolar fuzzy soft set. We define extended union, intersection of two bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solve decision making problems by using bipolar fuzzy soft set.

2. Preliminaries

In this section we provide previous concept of bipolar fuzzy sets, soft sets and fuzzy soft sets.

**Definition 1.** [35] A bipolar fuzzy set \(A\) in a universe \(U\) is an object having the form, \(A = \{(x, \mu^+_A(x), \mu^-_A(x)) : x \in U\}\) where \(\mu^+_A : U \rightarrow [0,1]\), \(\mu^-_A : U \rightarrow [-1,0]\). So \(\mu^+_A\) denote for positive information and \(\mu^-_A\) denote for negative information.

**Definition 2.** [4] Let \(U\) be an initial universe, \(E\) be the set of parameters, \(A \subset E\) and \(P(U)\) is the power set of \(U\). Then \((F,A)\) is called a soft set, where \(F : A \rightarrow P(U)\).

**Definition 3.** [21] For two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\), we say that \((F,A)\) is a soft subset of \((G,B)\), denoted by \((F,A) \subseteq (G,B)\), if it satisfies.

\[
(1) \quad A \subset B \\
(2) \quad \forall a \in A, F(a) \text{ is a subset of } G(a).
\]

Similarly, \((F,A)\) is called a superset of \((G,B)\) if \((G,B)\) is a soft subset of \((F,A)\). This relation is denoted by \((F,A) \supseteq (G,B)\).

**Definition 4.** [6] If \((F,A)\) and \((G,B)\) are two soft sets over a common universe \(U\). The union of \((F,A)\) and \((G,B)\) is defined to be the soft set \((H,C)\) satisfying the following conditions: (i) \(C = A \cup B\): (ii) for all \(c \in C\),

\[
H(c) = \begin{cases} 
F(c) & \text{if } c \in A \setminus B \\
G(c) & \text{if } c \in B \setminus A \\
F(c) \cup G(c) & \text{if } c \in A \cap B 
\end{cases}
\]
This relation is denoted by \((H, C) = (F, A) \cup (G, B)\).

**Definition 5.** \[21\] Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \(U\) such that \(A \cap B \neq \emptyset\). The restricted intersection of \((F, A)\) and \((G, B)\) is defined to be the soft set \((H, C)\), \(C = A \cap B\) and \(\forall c \in C\), \(H(c) = F(c) \cap G(c)\). We write \((H, C) = (F, A) \cap (G, B)\).

**Definition 6.** \[36\] Let \(U\) be an initial universe, \(E\) be the set of all parameters, \(A \subset E\) and \(\bar{P}(U)\) is the collection of all fuzzy subsets of \(U\). Then \((F, A)\) is called fuzzy soft set, where \(F : A \rightarrow \bar{P}(U)\).

**Definition 7.** \[39\] If \((F, A)\) and \((G, B)\) are two fuzzy soft sets over a common universe \(U\), then the union of \((F, A)\) and \((G, B)\) is defined to be the fuzzy soft set \((H, C)\) satisfying the following conditions: \(i)\) \(C = A \cup B\); \(ii)\) \(c \in C\),

\[
H(c) = \begin{cases} F(c) & \text{if } c \in A \cap B \\ G(c) & \text{if } c \in B \setminus A \\ F(c) \cup G(c) & \text{if } c \in A \cap B \\
\end{cases}
\]

This relation is denoted by \((H, C) = (F, A) \cup (G, B)\).

**3. Bipolar Fuzzy Soft Sets.**

In this section we introduce the concept of bipolar fuzzy soft set, absolute bipolar fuzzy soft set, null bipolar fuzzy soft set and complement of bipolar fuzzy soft set.

**Definition 8.** Let \(U\) be a universe, \(E\) a set of parameters and \(A \subset E\). Define \(F : A \rightarrow BF^U\), where \(BF^U\) is the collection of all bipolar fuzzy subsets of \(U\). Then \((F, A)\) is said to be a bipolar fuzzy soft set over a universe \(U\). It is defined by

\[
(F, A) = \{(x, \mu^+(x), \mu^-(x)) : \text{for all } x \in U \text{ and } e \in A\}
\]

**Example 1.** Let \(U = \{c_1, c_2, c_3, c_4\}\) be the set of four cars under consideration and \(E = \{e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_3 = \text{Fuel Efficient}, e_4 = \text{Modern Technology}\}\) be the set of parameters and \(A = \{e_1, e_2, e_3\} \subseteq E\). Then,

\[
(F, A) = \begin{pmatrix}
F(e_1) = \{(c_1, 0.1, -0.5), (c_2, 0.3, -0.6)\}, \\
F(e_2) = \{(c_1, 0.3, -0.5), (c_2, 0.4, -0.2)\}, \\
F(e_3) = \{(c_1, 0.8, -0.11), (c_2, 0.3, -0.6)\}
\end{pmatrix}
\]

**Definition 9.** Let \(U\) be a universe and \(E\) a set of attributes. Then, \((U, E)\) is the collection of all bipolar fuzzy soft sets on \(U\) with attributes from \(E\) and is said to be bipolar fuzzy soft class.

**Definition 10.** A bipolar fuzzy soft set \((F, A)\) is said to be a null bipolar fuzzy soft set denoted by empty set \(\emptyset\), if for all \(e \in A\), \(F(e) = \emptyset\).

**Definition 11.** A bipolar fuzzy soft set \((F, A)\) is said to be an absolute bipolar fuzzy soft set. If for all \(e \in A\), \(F(e) = BF^U\).

**Definition 12.** The complement of a bipolar fuzzy soft set \((F, A)\) is denoted \((F, A)^c\) and is defined by \((F, A)^c = \{(x, 1 - \mu^+_A(x), -1 + \mu^-_A(x)) : x \in U\}\).
Example 2. Let $U = \{b_1, b_2, b_3, b_4\}$ be the set of four bikes under consideration and $E = \{e_1 = \text{Stylish}, e_2 = \text{Heavy duty}, e_3 = \text{Light}, e_4 = \text{Steel}\}$ be the set of parameters and $A = \{e_1, e_2\}$ be subset of $E$. Then,

$$(F, A) = \left\{ \begin{array}{l}
F(e_1) = \{(b_1, 0.1, -0.5), (b_2, 0.3, -0.6)\}, \\
F(e_2) = \{(b_3, 0.4, -0.2), (b_4, 0.7, -0.2)\}.
\end{array} \right.$$

The complement of the bipolar fuzzy soft set $(F, A)$ is

$$(F, A)^c = \left\{ \begin{array}{l}
F(e_1) = \{(b_1, 0.9, -0.5), (b_2, 0.7, -0.4)\}, \\
F(e_2) = \{(b_3, 0.6, -0.8), (b_4, 0.3, -0.8)\}.
\end{array} \right.$$

4. Bipolar Fuzzy Soft Subsets

Definition 13. Let $(F, A)$ and $(G, B)$ be two bipolar fuzzy soft sets over a common universe $U$. We say that $(F, A)$ is a bipolar fuzzy soft subset of $(G, B)$, if (1) $A \subseteq B$ and (2) $\forall e \in A, F(e)$ is a bipolar fuzzy subset of $G(e)$. We write $(F, A) \subseteq (G, B)$.

Remark 1. Every element of $(F, A)$ is presented in $(G, B)$ and do not depend on its membership or non-membership.

Example 3. Let $U = \{m_1, m_2, m_3, m_4\}$ be the set of four men under consideration and $E = \{e_1 = \text{Educated}, e_2 = \text{Government employee}, e_3 = \text{Businessman}, e_4 = \text{Smart}\}$ be the set of parameters and $A = \{e_1, e_2\}$, $B = \{e_1, e_2, e_3\}$ be subsets of $E$. Then,

$$(F, A) = \left\{ \begin{array}{l}
F(e_1) = \{(m_1, 0.1, -0.5), (m_2, 0.3, -0.6)\}, \\
F(e_2) = \{(m_3, 0.4, -0.2), (m_4, 0.7, -0.2)\}.
\end{array} \right.$$

$$(G, B) = \left\{ \begin{array}{l}
G(e_1) = \{(m_1, 0.2, -0.5), (m_2, 0.2, -0.6)\}, \\
G(e_2) = \{(m_3, 0.3, -0.3), (m_4, 0.7, -0.1)\}, \\
G(e_3) = \{(m_1, 0.8, -0.01), (m_2, 0.4, -0.6)\}.
\end{array} \right.$$

$A \subseteq B$ and for all $e \in A$, $F(e) \leq G(e)$. Then $(F, A) \subseteq (G, B)$.

Definition 14. Let $(F, A)$ and $(G, B)$ be two bipolar fuzzy soft sets over a common universe $U$. We say that $(F, A)$ and $(G, B)$ are bipolar fuzzy soft equal sets if $(F, A)$ is a bipolar fuzzy soft subset of $(G, B)$ and $(G, B)$ is a bipolar fuzzy soft subset of $(F, A)$.

5. Operations on Bipolar Fuzzy Soft Sets

Definition 15. An intersection of two bipolar fuzzy soft sets $(F, A)$ and $(G, B)$ is a bipolar fuzzy soft set $(H, C)$, where $C = A \cap B \neq \emptyset$ and $H : C \rightarrow BF^U$ is defined by $H(e) = F(e) \cap G(e) \forall e \in C$ and denoted by $(H, C) = (F, A) \cap (G, B)$. 
Example 4. Let \( U = \{b_1, b_2, b_3, b_4\} \) be the set of four bikes under consideration and \( E = \{e_1 = \text{Light}, e_2 = \text{Beautiful}, e_3 = \text{Good mileage}, e_4 = \text{Modern Technology}\} \) be the set of parameters and \( A = \{e_1, e_2\} \subseteq E \), \( B = \{e_1, e_3\} \subseteq E \). Then,

\[
(F, A) = \begin{cases}
F(e_1) = \{(b_1, 0.1, -0.5), (b_2, 0.3, -0.6), (b_3, 0.4, -0.2), (b_4, 0.7, -0.2)\},
F(e_2) = \{(b_1, 0.3, -0.5), (b_2, 0.4, -0.2), (b_3, 0.4, -0.4), (b_4, 0.4, -0.2)\},
\end{cases}
\]

\[
(G, B) = \begin{cases}
G(e_1) = \{(b_1, 0.2, -0.5), (b_2, 0.2, -0.6), (b_3, 0.2, -0.3), (b_4, 0.7, -0.1)\},
G(e_2) = \{(b_1, 0.3, -0.6), (b_2, 0.2, -0.5), (b_3, 0.5, -0.3), (b_4, 0.5, -0.2)\},
G(e_3) = \{(b_1, 0.8, -0.01), (b_2, 0.4, -0.6), (b_3, 0.2, -0.3), (b_4, 0.7, -0.2)\},
\end{cases}
\]

Then \((H, C) = (F, A) \cap (G, B)\), where \( C = A \cap B = \{e_1, e_2\}\).

\[
(H, C) = \begin{cases}
H(e_1) = \{(b_1, 0.1, -0.5), (b_2, 0.2, -0.6), (b_3, 0.2, -0.2), (b_4, 0.7, -0.1)\},
H(e_2) = \{(b_1, 0.3, -0.5), (b_2, 0.2, -0.2), (b_3, 0.4, -0.3), (b_4, 0.4, -0.2)\},
\end{cases}
\]

Definition 16. Union of two bipolar fuzzy soft sets over a common universe \( U \) is a bipolar fuzzy soft set \((H, C)\), where \( C = A \cup B \) and \( H : C \rightarrow BF^U \) is defined by

\[
H(e) = \begin{cases}
F(e) \text{ if } e \in A \setminus B, \\
G(e) \text{ if } e \in B \setminus A, \\
F(e) \cup G(e) \text{ if } e \in A \cap B,
\end{cases}
\]

and denoted by \((H, C) = (F, A) \cup (G, B)\).

Example 5. Let \( U = \{c_1, c_2, c_3, c_4\} \) be the set of four cars under consideration and \( E = \{e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_3 = \text{Fuel Efficient}, e_4 = \text{Modern Technology}\} \) be the set of parameters and \( A = \{e_1, e_2, e_3\} \subseteq E \), \( B = \{e_1, e_2, e_3, e_4\} \subseteq E \). Then

\[
(F, A) = \begin{cases}
F(e_1) = \{(c_1, 0.1, -0.5), (c_2, 0.3, -0.6), (c_3, 0.4, -0.2), (c_4, 0.7, -0.2)\},
F(e_2) = \{(c_1, 0.3, -0.5), (c_2, 0.4, -0.2), (c_3, 0.5, -0.2), (c_4, 0.4, -0.2)\},
F(e_3) = \{(c_1, 0.8, -0.1), (c_2, 0.3, -0.6), (c_3, 0.4, -0.3), (c_4, 0.6, -0.2)\},
\end{cases}
\]

\[
(G, B) = \begin{cases}
G(e_1) = \{(c_1, 0.2, -0.5), (c_2, 0.2, -0.6), (c_3, 0.2, -0.3), (c_4, 0.7, -0.1)\},
G(e_2) = \{(c_1, 0.3, -0.6), (c_2, 0.2, -0.5), (c_3, 0.5, -0.3), (c_4, 0.5, -0.2)\},
G(e_3) = \{(c_1, 0.8, -0.01), (c_2, 0.4, -0.6), (c_3, 0.2, -0.3), (c_4, 0.7, -0.2)\},
G(e_4) = \{(c_1, 0.1, -0.6), (c_2, 0.3, -0.4), (c_3, 0.1, -0.6), (c_4, 0.0, -0.2)\},
\end{cases}
\]
Then $(H, C) = (F, A) \cup (G, B)$, where $C = A \cup B = \{e_1, e_2, e_3, e_4\}$

\[
(H, C) = \begin{cases} 
H(e_1) = \{(c_1, 0.2, -0.5), (c_2, 0.3, -0.6), (c_3, 0.4, -0.3), (c_4, 0.7, -0.2)\}, \\
H(e_2) = \{(c_1, 0.3, -0.6), (c_2, 0.4, -0.5), (c_3, 0.5, -0.3), (c_4, 0.5, -0.2)\}, \\
H(e_3) = \{(c_1, 0.8, -0.1), (c_2, 0.4, -0.6), (c_3, 0.4, -0.3), (c_4, 0.7, -0.2)\}, \\
H(e_4) = \{(c_1, 0.1, -0.6), (c_2, 0.3, -0.4), (c_3, 0.1, -0.6), (c_4, 0.0, -0.2)\}
\end{cases}
\]

**Definition 17.** Let $T = \{(F_i, A_i) : i \in I\}$ be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class $(U, E)$. Then the intersection of bipolar fuzzy soft sets in $T$ is a bipolar fuzzy soft set $(H, C)$, where $C = \cap A_i$ if for all $i \in I$, $H(e) = \cap F_i(e)$ for all $e \in C$.

**Definition 18.** Let $T = \{(F_i, A_i) : i \in I\}$ be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class $(U, E)$. Then the union of bipolar fuzzy soft sets in $T$ is a bipolar fuzzy soft set $(H, C)$, where $C = \cup A_i$ for all $i \in I$.

\[
H(e) = \begin{cases} 
F_i(e) & \text{if } e \in A_i \\
\emptyset & \text{if } e \notin A_i
\end{cases}
\]

**Definition 19.** Let $(F, A)$ and $(G, B)$ be two bipolar fuzzy soft sets over a common universe $U$. The extended intersection of $(F, A)$ and $(G, B)$ is defined to be the bipolar fuzzy soft set $(H, C)$, where $C = A \cap B$ and for all $e \in C$.

\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A \setminus B \\
G(e) & \text{if } e \in B \setminus A \\
F(e) \cap G(e) & \text{if } e \in A \cap B
\end{cases}
\]

This intersection is denoted by $(H, C) = (F, A) \cap_e (G, B)$.

**Definition 20.** Let $(F, A)$ and $(G, B)$ be two bipolar fuzzy soft sets over a common universe $U$. The restricted union of $(F, A)$ and $(G, B)$ is defined to be the bipolar fuzzy soft set $(H, C)$, where $C = A \cap B \neq \emptyset$ and for all $e \in C$.

\[
H(e) = F(e) \cup G(e)
\]

This union is denoted by $(H, C) = (F, A) \cup_R (G, B)$.

**Proposition 1.** Let $(F, A)$ be bipolar fuzzy soft set over a common universe $U$. Then,

1. $(F, A) \cup (F, A) = (F, A)$
2. $(F, A) \cap (F, A) = (F, A)$
3. $(F, A) \cup \emptyset = (F, A)$, where $\emptyset$ is a null bipolar fuzzy soft set.
4. $(F, A) \cap \emptyset = \emptyset$, where $\emptyset$ is a null bipolar fuzzy soft set.

**Proof.**

1. $(F, A) \cup (F, A) = (F, A)$.

A bipolar fuzzy soft set $(H, C)$ is union of two bipolar fuzzy soft sets $(F, A)$ and $(F, A)$ which is

(5.1) $(H, C) = (F, A) \cup (F, A)$ where $C = A \cup A$
Define by
\[ H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus \Delta A \\ F(e) & \text{if } e \in A \setminus A \\ F(e) \cup F(e) & \text{if } e \in A \cap A \end{cases} \]

L.H.S. There are three cases.

Case (1). If \( a \in A \setminus \Delta A \).
\[ H(a) = F(a) \text{ if } a \in A \setminus A = \emptyset \]

Case (2) If \( a \in A \setminus A \).
\[ H(a) = F(a) \text{ if } a \in A \setminus A = \emptyset \]

Case (3) If \( a \in A \cap A \).
\[ H(a) = F(a) \text{ if } a \in A \cap A = A \]
\[ H(a) = F(a) \text{ if } a \in A \]
\[ (H, C) = (F, A) \text{ from equation } 5.1 \]
\[ (F, A) \cup (F, A) = (F, A) \text{ from equation } 5.1 \]

It is satisfied in all three cases. Hence \( (F, A) \cup (F, A) = (F, A) \).

2: \( (F, A) \cap (F, A) = (F, A) \)

A bipolar fuzzy soft set \( (H, C) \) is intersection of two bipolar fuzzy soft sets \( (F, A) \) and \( (F, A) \) which is
\[ (H, C) = (F, A) \cap (F, A) \text{ where } C = A \cap A \]

Define by
\[ H(e) = F(e) \cap G(e) \text{ if } e \in C = A \cap A \]

L.H.S. Let \( a \in C = A \cap A \).
\[ H(e) = F(e) \cap F(e) \text{ if } e \in C = A \cap A = A \]
\[ H(e) = F(e) \text{ if } e \in C = A \]
\[ H(e) = F(e) \text{ if } e \in A \]
\[ (H, C) = (F, A) \text{ from equation } 5.2 \]
\[ (F, A) \cap (F, A) = (F, A) \text{ from equation } 5.2 \]

Hence \( (F, A) \cap (F, A) = (F, A) \).

\[ \square \]

**Lemma 1.** Absorption property of bipolar fuzzy soft sets \((F, A)\) and \((G, B)\).

1. \((F, A) \cup ((F, A) \cap (G, B)) = (F, A)\)
2. \((F, A) \cap ((F, A) \cup (G, B)) = (F, A)\)

**Proof.** (1). \((F, A) \cup ((F, A) \cap (G, B)) = (F, A)\)

Let bipolar fuzzy soft set \((H, C)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((G, B)\), where \( C = A \cap B \)
\[ (H, C) = (F, A) \cap (G, B) \text{ where } C = A \cap B \]

Define by
\[ H(e) = F(e) \cap G(e) \text{ if } e \in C = A \cap B \]
Let bipolar fuzzy soft set \((K, M)\) is union of two bipolar fuzzy soft sets \((F, A)\) and \((H, C)\) which is

\[(5.4) \quad (K, M) = (F, A) \cup (H, C) \quad \text{where} \quad M = A \cup C\]

Define by

\[
K(e) = \begin{cases} 
F(e) & \text{if } e \in A \setminus C \\
H(e) & \text{if } e \in C \setminus A \\
F(e) \cup H(e) & \text{if } e \in A \cap C
\end{cases}
\]

\[
K(e) = \begin{cases} 
F(e) & \text{if } e \in A \setminus C \\
H(e) & \text{if } e \in C \setminus A \\
F(e) \cup H(e) & \text{if } e \in A \cap C
\end{cases}
\]

\[
(K, M) = (F, A) \quad \text{from equation } 5.4
\]

L.H.S. There are three cases.

Cases (1). If \(e \in A \setminus C\).

\[
K(e) = \begin{cases} 
F(e) & \text{if } e \in A \setminus C \\
H(e) & \text{if } e \in C \setminus A \\
F(e) \cup H(e) & \text{if } e \in A \cap C
\end{cases}
\]

Case (2). If \(e \in C \setminus A = A \cap B - A = 0\).

\[
K(e) = \begin{cases} 
H(e) & \text{if } e \in C \setminus A = 0 \\
\emptyset & \text{if } e \in \emptyset
\end{cases}
\]

\[
K(e) = \emptyset \quad \text{from equation } 5.4
\]

Case (3). If \(e \in A \cap C\).

\[
K(e) = \begin{cases} 
F(e) \cup H(e) & \text{if } e \in A \cap C \quad \text{and} \quad C = A \cap B \\
F(e) & \text{if } e \in (F(e) \cap G(e)) \quad \text{from equation } 5.3
\end{cases}
\]

\[
K(e) = \begin{cases} 
F(e) \cup H(e) & \text{if } e \in A \cap C \quad \text{and} \quad C = A \cap B \\
F(e) & \text{if } e \in (F(e) \cap G(e)) \quad \text{from equation } 5.3
\end{cases}
\]

\[
K(e) = F(e) \quad \text{from equation } 5.4
\]

\[
(K, M) = (F, A) \quad \text{from equation } 5.4
\]

It is satisfied in three cases. Hence \((F, A) \cup ((F, A) \cap (G, B)) = (F, A)\).

\[\tag{2}\] same as above.

Theorem 1. Commutative property of bipolar fuzzy soft sets \((F, A)\) and \((G, B)\).

\[
(1) \quad (F, A) \cap (G, B) = (G, B) \cap (F, A)
\]

\[
(2) \quad (F, A) \cup (G, B) = (G, B) \cup (F, A)
\]

Proof. (1). To show that \((F, A) \cap (G, B) = (G, B) \cap (F, A)\).

A bipolar fuzzy soft set \((H, C)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((G, B)\), where \(C = A \cap B\)

\[
(5.5) \quad H(e) = F(e) \cap G(e) \quad \text{if } e \in C = A \cap B
\]

A bipolar fuzzy soft set \((K, D)\) is an intersection of two bipolar fuzzy soft sets \((G, B)\) and \((F, A)\), where \(D = B \cap A\)

\[
(5.6) \quad K(e) = G(e) \cap F(e) \quad \text{if } e \in D = B \cap A
\]

To show that \((H, C) = (K, D)\)
L.H.S

\[ H(e) = \begin{cases} F(e) \cap G(e) & \text{for all } e \in C = A \cap B \\ G(e) \cap F(e) & \text{since } F(e) \cap G(e) = G(e) \cap F(e) \\ G(e) \cap F(e) & \text{for all } e \in C = A \cap B = B \cap A = D \\ K(e) & \text{for all } e \in B \cap A = D \end{cases} \]

\[ H(e) = K(e) \]

\((H, C)\) = \((K, D)\) from equation 5.5, equation 5.6

\((F, A) \cap (G, B)\) = \((G, B) \cap (F, A)\) from equation 5.5, equation 5.6

Hence \((F, A) \cap (G, B)\) = \((G, B) \cap (F, A)\).

(2). To show that \((F, A) \cup (G, B)\) = \((G, B) \cup (F, A)\).

L.H.S.

A bipolar fuzzy soft set \((H, C)\) is union of two bipolar fuzzy soft sets \((F, A),\) \((G, B)\) over a common universe \(U\)

\[(5.7) \quad (H, C) = (F, A) \cup (G, B) \quad \text{where } C = A \cup B\]

Define by

\[(5.8) \quad H(e) = F(e) \quad \text{if } e \in A \setminus B\]
\[(5.9) \quad H(e) = G(e) \quad \text{if } e \in B \setminus A\]
\[(5.10) \quad H(e) = F(e) \cup G(e) \quad \text{if } e \in A \cap B\]

There are three cases.

Case (1). If \(e \in A \setminus B\)

\[(5.11) \quad H(e) = F(e) \quad \text{if } e \in A \setminus B\] from equation 5.8

Case (2). If \(e \in B \setminus A\)

\[(5.12) \quad H(e) = G(e) \quad \text{if } e \in B \setminus A\] from equation 5.9

Case (3). If \(e \in A \cap B\)

\[(5.13) \quad H(e) = F(e) \cup G(e) \quad \text{if } e \in B \cap A\] from equation 5.10

Combine equation 5.11, equation 5.12 and equation 5.13. We get

\[ H(e) = \begin{cases} G(e) & \text{if } e \in B \setminus A \\ F(e) & \text{if } e \in A \setminus B \\ G(e) \cup F(e) & \text{if } e \in B \cap A \end{cases} \]

\((H, C)\) becomes

\[ (H, C) = (G, B) \cup (F, A) \quad \text{where } C = B \cup A \]

= \(R.H.S\)

Hence \((F, A) \cup (G, B)\) = \((G, B) \cup (F, A)\). \(\square\)

**Theorem 2.** Associative law of bipolar fuzzy soft sets \((F, A), (G, B)\) and \((H, C)\).

(i) \((F, A) \cap ((G, B) \cap (H, C))\) = \(((F, A) \cap (G, B)) \cap (H, C)\)

(ii) \((F, A) \cup ((G, B) \cup (H, C))\) = \(((F, A) \cup (G, B)) \cup (H, C)\)
Proof. (i) \( (F, A) \cap ((G, B) \cap (H, C)) = ((F, A) \cap (G, B)) \cap (H, C) \).

A bipolar fuzzy soft set \((L, D)\) is an intersection of two bipolar fuzzy soft sets \((G, B)\) and \((H, C)\) which is

\[
(G, B) \cap (H, C) = (L, D) \quad \text{where} \quad D = B \cap C
\]

Define by

\[
L(e) = G(e) \cap H(e) \quad \text{if} \quad e \in D = B \cap C
\]

A bipolar fuzzy soft set \((M, X)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((L, D)\) which is

\[
(F, A) \cap (L, D) = (M, X) \quad \text{where} \quad X = A \cap D
\]

Define by

\[
M(e) = F(e) \cap L(e) \quad \text{if} \quad e \in X = A \cap D
\]

L.H.S:

\[
M(e) = F(e) \cap L(e) \quad \text{for all} \quad e \in X = A \cap D \quad \text{from equation} \, 5.17
\]

\[
= F(e) \cap (G(e) \cap H(e)) \quad \text{from equation} \, 5.15
\]

\[
= (F(e) \cap G(e)) \cap H(e) \quad \text{for all} \quad e \in X = A \cap D = A \cap (B \cap C)
\]

\[
M(e) = (F(e) \cap G(e)) \cap H(e) \quad \text{for all} \quad e \in A \cap (B \cap C)
\]

\[
(M, X) = ((F, A) \cap (G, B)) \cap (H, C) \quad \text{from equation} \, 5.16
\]

\[
(F, A) \cap ((G, B) \cap (H, C)) = ((F, A) \cap (G, B)) \cap (H, C) \quad \text{from equation} \, 5.14
\]

Hence

\[
(F, A) \cap ((G, B) \cap (H, C)) = ((F, A) \cap (G, B)) \cap (H, C).
\]

(ii). same as above.

Hence \((F, A) \cup ((G, B) \cup (H, C)) = ((F, A) \cup (G, B)) \cup (H, C).

\[\square\]

Theorem 3. Distributive law of bipolar fuzzy soft sets \((F, A)\), \((G, B)\) and \((H, C)\).

1. \((F, A) \cap ((G, B) \cup (H, C)) = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C))\)

2. \((F, A) \cup ((G, B) \cap (H, C)) = ((F, A) \cup (G, B)) \cap ((F, A) \cup (H, C))\)

Proof. (1) \((F, A) \cap ((G, B) \cup (H, C)) = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C))\)

A bipolar fuzzy soft set \((L, D)\) is union of two bipolar fuzzy soft sets \((G, B)\) and \((H, C)\) over a common universe \(U\).

\[
(G, B) \cup (H, C) = (L, D) \quad \text{where} \quad D = B \cup C
\]

Define by

\[
L(e) = G(e) \quad \text{if} \quad e \in B \setminus C
\]

\[
= H(e) \quad \text{if} \quad e \in C \setminus B
\]

\[
= G(e) \cup H(e) \quad \text{if} \quad e \in B \cap C
\]

A bipolar fuzzy soft set \((M, V)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((L, D)\).

\[
(F, A) \cap (L, D) = (M, V) \quad \text{where} \quad V = A \cap D
\]

Define by

\[
M(e) = F(e) \cap L(e) \quad \text{if} \quad e \in V = A \cap D
\]
L.H.S

\[ M(e) = F(e) \cap L(e) \text{ for all } e \in V = A \cap D \text{ from equation 5.20} \]

(5.21) \[ M(e) = F(e) \cap L(e) \text{ for all } e \in A \cap D \text{ so } e \in A, e \in D \]

If \( e \in D = B \cup C \) from equation 5.18 Then there are three cases.

Case (1) If \( e \in B \backslash C \)

(5.22) \[ L(e) = G(e) \text{ if } e \in B \backslash C \text{ from equation } 5.18 \]

Case (2) If \( e \in C \backslash B \)

(5.23) \[ L(e) = H(e) \text{ if } e \in C \backslash B \text{ from equation } 5.18 \]

Case (3) If \( e \in C \cap B \)

(5.24) \[ L(e) = G(e) \cup H(e) \text{ if } e \in B \cap C \text{ from equation } 5.18 \]

Put equation 5.22, equation 5.23 and equation 5.24 in equation 5.21

\[ M(e) = F(e) \cap G(e) \text{ for all } e \in A, e \in B \backslash C \]

\[ = F(e) \cap H(e) \text{ if } e \in A, e \in C \backslash B \]

\[ = F(e) \cap (G(e) \cup H(e)) \text{ if } e \in A, e \in B \cap C \]

\[ = (F(e) \cap G(e)) \cup (F(e) \cap H(e)) \]

(5.21) \[ M(e) = (F(e) \cap G(e)) \cup (F(e) \cap H(e)) \]

( \( M, V \) ) \[ = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C)) \text{ from equation } 5.19 \]

( \( F, A \) ) \[ \cap (L, D) = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C)) \text{ from equation } 5.19 \]

Hence \( (F, A) \cap ((G, B) \cup (H, C)) = ((F, A) \cap (G, B)) \cup ((F, A) \cap (H, C)). \)

(2). same as above. \( \square \)

**Lemma 2.** : \((F, A) \text{ and } (G, B) \text{ are two bipolar fuzzy soft sets.}\)

(1) \((F, A) \subset (G, B) \Rightarrow (F, A) \cap (G, B) = (F, A)\)
(2) \((F, A) \subset (G, B) \Rightarrow (F, A) \cup (G, B) = (G, B)\).

6. De Morgan’s Law of Bipolar Fuzzy Soft Sets

**Theorem 4.** : De Morgan’s law of bipolar fuzzy soft sets \((F, A) \text{ and } (G, B).\)

(1) \(( (F, A) \cup (G, B))^c = (F, A)^c \cap (G, B)^c \).
(2) \(( (F, A) \cap (G, B))^c = (F, A)^c \cup (G, B)^c \).

**Proof.** (1).

Let \((F, A) \text{ and } (G, B)\) be a two bipolar fuzzy soft sets over a common universe \(U\). Then the union of two bipolar fuzzy soft sets \((F, A) \text{ and } (G, B)\) is a bipolar fuzzy soft set \((H, C)\) where \(C = A \cup B\) and \((H, C) = (F, A) \cup (G, B)\) is define by

(6.1) \[ H(e) = F(e) \text{ if } e \in A \backslash B \]

\[ = G(e) \text{ if } e \in B \backslash A \]

\[ = F(e) \cup G(e) \text{ if } e \in A \cap B \]
The extended intersection of two bipolar fuzzy soft sets $(F, A)$ and $(G, B)$ is bipolar fuzzy soft set $(K, D)$ where $D = A \cap B$ and $(K, D) = (F, A) \cap_r (G, B)$. is define by

$$K (e) = \begin{cases} F (e) & \text{if } e \in A \backslash B \\ G (e) & \text{if } e \in B \backslash A \\ F (e) \cap G (e) & \text{if } e \in A \cap B \end{cases}$$

(6.2)

To show that $((F, A) \cup (G, B))^c = (F, A)^c \cap_r (G, B)^c$.

L.H.S: There are three cases.

Case (i): If $e \in A \backslash B$. Then $e \in C$. Such that

$H (e) = F (e)$ if $e \in A \backslash B$ from equation 6.1

Taking complement of above. So

(6.3)

$\begin{cases} (H (e))^c = (F (e))^c & \text{if } e \in A \backslash B \\ (H (e))^c = (G (e))^c & \text{if } e \in B \backslash A \end{cases}$

Case (ii) If $e \in B \backslash A$. Then $e \in C$. Such that

$H (e) = G (e)$ if $e \in B \backslash A$ from 6.1

Taking complement of above. So

(6.4)

$\begin{cases} (H (e))^c = (F (e))^c \cap G (e)^c & \text{if } e \in B \backslash A \\ (H (e))^c = (G (e))^c \cap F (e)^c & \text{if } e \in A \cap B \end{cases}$

Case (iii) If $e \in A \cap B$. Then $e \in C$. Such that

$H (e) = F (e) \cup G (e)$ if $e \in A \cap B$ from 6.1

Taking complement of above. So

(6.5)

$\begin{cases} (H (e))^c = (F (e))^c \cup (G (e))^c & \text{if } e \in A \cap B \end{cases}$

We define $F (e)$ and $G (e)$ as

(6.6)

$F (e) = \{(u, \mu_A^+ (u), \mu_B^- (u)) : u \in U\}$

and

(6.7)

$G (e) = \{(u, \mu_A^- (u), \mu_B^+ (u)) : u \in U\}$

Putting equation 6.6, equation 6.7 in equation 6.5 We get

(6.8)

$\begin{cases} (H (e))^c = (u, \max (\mu_A^+ (u), \mu_B^- (u))) \cup (u, \mu_A^- (u), \mu_B^+ (u))) & \text{if } e \in A \cap B \\ (u, 1 - \min (\mu_A^- (u), \mu_B^- (u))) & \text{if } e \in A \cap B \\ (u, 1 - \min (\mu_A^- (u), \mu_B^- (u))) & \text{if } e \in A \cap B \\ (u, 1 - \min (\mu_A^- (u), \mu_B^- (u))) & \text{if } e \in A \cap B \end{cases}$

From equation 6.3, equation 6.4 and equation 6.8 We get

(6.9)

$\begin{cases} (H (e))^c = (F (e))^c \cap (G (e))^c & \text{if } e \in A \cap B \end{cases}$

Then

$\begin{cases} (H (e))^c = (F (e))^c \cap_r (G (e))^c & \text{if } e \in A \cap B \end{cases}$
Thus
\[(F, A) \cup (G, B) = (F, A)^c \cap (G, B)^c\]

Hence it is proved.
(2). same as above. \(\square\)

7. OR AND AND Operations on Bipolar Fuzzy Soft Sets

**Definition 21.** Let \((F, A)\) and \((G, B)\) be two bipolar fuzzy soft sets over a common universe \(U\). Then,

1. \((F, A) \wedge (G, B)\) is a bipolar fuzzy soft set defined by \((F, A) \wedge (G, B) = (H, A \times B)\) where \(H(a, b) = F(a) \cap G(b)\) for all \((a, b) \in C = A \times B\), where \(\cap\) is the intersection operation of sets.
2. \((F, A) \vee (G, B)\) is a bipolar fuzzy soft set defined by \((F, A) \vee (G, B) = (H, A \times B)\) where \(H(a, b) = F(a) \cup G(b)\) for all \((a, b) \in C = A \times B\), where \(\cup\) is the intersection operation of sets.

**Definition 22.** Let \(T = \{(F_i, A_i) : i \in I\}\) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the intersection of bipolar fuzzy soft sets in \(T\) is bipolar fuzzy soft set \((H, C)\), where \(C = \times A_i\) for all \(i \in I\), \(H(e) = \wedge F_i(e)\) for all \(e \in C\), \((H, C) = \wedge (F_i, A_i)\) for all \(i \in I\).

**Definition 23.** Let \(T = \{(F_i, A_i) : i \in I\}\) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the union of bipolar fuzzy soft sets in \(T\) is bipolar fuzzy soft set \((H, C)\), where \(C = \times A_i\) for all \(i \in I\), \(H(e) = \vee F_i(e)\) for all \(e \in C\), \((H, C) = \vee (F_i, A_i)\) for all \(i \in I\).

**Example 6.** Let \(U = \{m_1, m_2, m_3, m_4\}\) be the set of four man under consideration and \(E = \{e_1 = \text{Educated}, e_2 = \text{Government employee}, e_3 = \text{Businessman}, e_4 = \text{Smart}, e_5 = \text{Weak}\}\) be the set of parameters and \(A = \{e_1, e_2\} \subseteq E\), \(B = \{e_4, e_5\} \subseteq E\). Then

\[
(F, A) = \begin{cases} 
F(e_1) = \{(m_1, 0.1, -0.5), (m_2, 0.3, -0.6) \} \\
F(e_2) = \{(m_3, 0.4, -0.2), (m_4, 0.7, -0.2) \} 
\end{cases}
\]

\[
(G, B) = \begin{cases} 
G(e_4) = \{(m_1, 0.1, -0.6), (m_2, 0.3, -0.4) \} \\
G(e_5) = \{(m_3, 0.4, -0.1), (m_4, 0.7, -0.2) \}
\end{cases}
\]

Then \((F, A) \wedge (G, B) = (H, A \times B)\) and \(C = A \times B = \{e_1, e_2\} \times \{e_4, e_5\} = \{(e_1, e_4), (e_1, e_5), (e_2, e_4), (e_2, e_5)\}\) define by \(H(a, b) = F(a) \cap G(b)\) for all \((a, b) \in C\)

\[
(H, C) = (F, A) \wedge (G, B) = \begin{cases} 
H(e_1, e_4) = \{(m_1, 0.1, -0.5), (m_2, 0.3, -0.4) \} \\
H(e_1, e_5) = \{(m_3, 0.1, -0.1), (m_4, 0.0, -0.2) \} \\
H(e_2, e_4) = \{(m_1, 0.1, -0.5), (m_2, 0.3, -0.2) \} \\
H(e_2, e_5) = \{(m_3, 0.3, -0.1), (m_4, 0.2, -0.2) \}
\end{cases}
\]
Example 7. Let \( U = \{h_1, h_2, h_3, h_4\} \) be the set of four houses under consideration and \( E = \{e_1 = \text{Expensive}, e_2 = \text{Beautiful}, e_3 = \text{Wooden}, e_4 = \text{In the green surrounding}, e_5 = \text{Convenient traffic}\} \) be the set of parameters and \( A = \{e_1, e_2\} \subseteq E, B = \{e_4, e_5\} \subseteq E \). Then,

\[
(F, A) = \begin{cases}
F(e_1) = \{(h_1, 0.1, -0.5), (h_2, 0.3, -0.6), (h_3, 0.4, -0.2), (h_4, 0.7, -0.2)\}, \\
F(e_2) = \{(h_1, 0.3, -0.5), (h_2, 0.4, -0.2), (h_3, 0.5, -0.2), (h_4, 0.4, -0.2)\}.
\end{cases}
\]

\[
(G, B) = \begin{cases}
G(e_4) = \{(h_1, 0.1, -0.6), (h_2, 0.3, -0.4), (h_3, 0.1, -0.6), (h_4, 0.0, -0.2)\}, \\
G(e_5) = \{(h_1, 0.4, -0.1), (h_2, 0.2, -0.4), (h_3, 0.6, -0.4), (h_4, 0.7, -0.0)\}.
\end{cases}
\]

Then \((F, A) \triangledown (G, B) = (H, A \times B)\) and \(C = A \times B = \{e_1, e_2\} \times \{e_4, e_5\} = \{(e_1, e_4), (e_1, e_5), (e_2, e_4), (e_2, e_5)\}\) define by \(H(a) = F(a) \cup G(a)\), for all \(a \in C = A \times B\)

\[
(H, C) = (F, A) \triangledown (G, B) = \begin{cases}
H(e_1, e_4) = \{(h_1, 0.1, -0.6), (h_2, 0.3, -0.6), (h_3, 0.4, -0.6), (h_4, 0.7, -0.2)\}, \\
H(e_1, e_5) = \{(h_1, 0.4, -0.5), (h_2, 0.3, -0.6), (h_3, 0.6, -0.4), (h_4, 0.7, -0.2)\}, \\
H(e_2, e_4) = \{(h_1, 0.3, -0.6), (h_2, 0.4, -0.4), (h_3, 0.5, -0.6), (h_4, 0.4, -0.2)\}, \\
H(e_2, e_5) = \{(h_1, 0.4, -0.5), (h_2, 0.4, -0.4), (h_3, 0.6, -0.4), (h_4, 0.7, -0.2)\}.
\end{cases}
\]

Proposition 2. Idempotent Property. If \((F, A), (G, B)\) are two bipolar fuzzy soft sets over \(U\). Then,

1. \((F, A) \hat{\wedge} (F, A) = (F, A)\)
2. \((F, A) \hat{\triangledown} (F, A) = (F, A)\)

Proof. (1). \((F, A) \hat{\wedge} (F, A) = (F, A)\)

Suppose that \((F, A) \hat{\wedge} (F, A) = (H, C)\), where \(C = A \times A\). Let \(a \in A\)

\[
H(a, a) = F(a) \cap F(a) \quad \text{since} \quad F(a) \cap F(a) = F(a)
\]

\[
H(a, a) = F(a) \\
H(a, a) = F(a) \\
(H, C) = (F, A)
\]

\[
(F, A) \hat{\wedge} (F, A) = (F, A)
\]

Hence \((F, A) \hat{\wedge} (F, A) = (F, A)\)

(2). \((F, A) \hat{\triangledown} (F, A) = (F, A)\)

Suppose that \((F, A) \hat{\triangledown} (F, A) = (H, C)\), where \(C = A \times A\). Let \(a \in A\)

\[
H(a, a) = F(a) \cap F(a) \quad \text{since} \quad F(a) \cup F(a) = F(a)
\]

\[
H(a, a) = F(a) \\
H(a, a) = F(a) \\
(H, C) = (F, A)
\]

\[
(F, A) \hat{\triangledown} (F, A) = (F, A)
\]

Hence \((F, A) \hat{\triangledown} (F, A) = (F, A)\). \(\Box\)
Example 8. Let \( U = \{c_1, c_2, c_3, c_4\} \) be the set of four cars under consideration, \( E = \{e_1=\text{Costly}, e_2=\text{Beautiful}, e_3=\text{Fuel efficient}, e_4=\text{Luxurious}\} \) be the set of parameters and \( A = \{e_1, e_2, e_3\} \subseteq E \) then \( F : A \rightarrow BFU \) define by

\[
F(e_1) = \{(c_1, 0.2, -0.5), (c_2, 0.3, -0.6),
\quad (c_3, 0.4, -0.3), (c_4, 0.7, -0.2)\}
\]

\[
F(e_2) = \{(c_1, 0.1, -0.6), (c_2, 0.3, -0.5),
\quad (c_3, 0.6, -0.1), (c_4, 0.4, -0.4)\}
\]

\[
F(e_3) = \{(c_1, 0.2, -0.8), (c_2, 0.4, -0.3),
\quad (c_3, 0.5, -0.3), (c_4, 0.7, -0.3)\}
\]

Hence \( F(e_1), F(e_2) \) and \( F(e_3) \) are the elements of bipolar fuzzy soft set over a universe \( U \).

8. An Application of Bipolar Fuzzy Soft Sets in Decision Making

Bipolar fuzzy soft set has several applications to deal with uncertainties from our different kinds of daily life problems. Here, we discuss such an application for solving a socialistic decision making problem. We apply the concept of bipolar fuzzy soft set for modelling of a socialistic decision making problem and then we give an algorithm for the choice of optimal object based upon the available sets of information.

Suppose that \( U = \{c_1, c_2, c_3, c_4\} \) be the set of four cars under consideration say \( U \) is an initial universe and \( E = \{e_1=\text{Costly}, e_2=\text{Beautiful}, e_3=\text{Fuel efficient}, e_4=\text{Modern technology}, e_5=\text{Luxurious}\} \) be a set of parameters. Suppose a man Mr. \( X \) is going to buy a car on the basis of his wishing parameter among the listed above. Our aim is to find out the attractive car for Mr. \( X \).

Suppose the wishing parameters of Mr. \( X \) be \( A \subset E \) where \( A = \{e_1, e_2, e_3\} \). Consider the bipolar fuzzy soft set as below.

\[
F(e_1) = \{(c_1, 0.4, -0.5), (c_2, 0.6, -0.3),
\quad (c_3, 0.8, -0.2), (c_4, 0.5, -0.2)\}
\]

\[
F(e_2) = \{(c_1, 0.5, -0.5), (c_2, 0.3, -0.1),
\quad (c_3, 0.4, -0.4), (c_4, 0.7, -0.3)\}
\]

\[
F(e_3) = \{(c_1, 0.7, 0), (c_2, 0.5, -0.3),
\quad (c_3, 0.6, -0.3), (c_4, 0.4, -0.4)\}
\]

Definition 24. (Comparison table). It is a square table in which number of rows and number of columns are equal and both are labeled by the object name of the universe such as \( c_1, c_2, c_3, \ldots, c_n \) and the entries \( d_{ij} \) where \( d_{ij} \) is the number of parameters for which the value of \( d_i \) exceeds or equal to the value of \( d_j \).

Algorithm.

1. Input the set \( A \subset E \) of choice of parameters of Mr. \( X \).
2. Consider the bipolar fuzzy soft set in tabular form.
3. Compute the comparison table of positive information function and negative information function.
4. Compute the positive information score and negative information score.
5. Compute the final score by subtracting positive information score from negative information score.
Find the maximum score, if it occurs in i-th row, then Mr. X will buy to $d_i$, $1 \leq i \leq 4$.

|   | $e_1$ | $e_2$ | $e_3$ | $e_5$ |
|---|-------|-------|-------|-------|
| $c_1$ | 0.4   | 0.5   | 0.7   |       |
| $c_2$ | 0.6   | 0.3   | 0.5   |       |
| $c_3$ | 0.8   | 0.4   | 0.6   |       |
| $c_4$ | 0.5   | 0.7   | 0.4   |       |

Table 1. Tabular representation of positive information function.

|   | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|-------|-------|-------|-------|
| $c_1$ | 3     | 2     | 2     | 1     |
| $c_2$ | 1     | 3     | 0     | 2     |
| $c_3$ | 1     | 3     | 3     | 2     |
| $c_4$ | 2     | 1     | 1     | 3     |

Table 2. Comparison table of the above table.

|   | Row sum(a) | Column sum(b) | Membership score(a-b) |
|---|------------|---------------|-----------------------|
| $c_1$ | 8          | 7             | 1                     |
| $c_2$ | 6          | 9             | −3                    |
| $c_3$ | 9          | 6             | 3                     |
| $c_4$ | 7          | 8             | −1                    |

Table 3. Membership score table.

|   | $e_1$ | $e_2$ | $e_3$ |
|---|-------|-------|-------|
| $c_1$ | −0.5 | −0.5 | 0     |
| $c_2$ | −0.3 | −0.1 | −0.3  |
| $c_3$ | −0.2 | −0.4 | −0.3  |
| $c_4$ | −0.2 | −0.3 | −0.4  |

Table 4. Tabular representation of negative information function.

|   | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|-------|-------|-------|-------|
| $c_1$ | 3     | 2     | 2     | 2     |
| $c_2$ | 1     | 3     | 2     | 1     |
| $c_3$ | 1     | 2     | 3     | 2     |
| $c_4$ | 1     | 2     | 2     | 3     |

Table 5. Comparison table of the above table.

|   | Row sum(c) | Column sum(d) | Non-membership score(c-d) |
|---|------------|---------------|---------------------------|
| $c_1$ | 9          | 6             | 3                         |
| $c_2$ | 7          | 9             | −2                        |
| $c_3$ | 8          | 9             | −1                        |
| $c_4$ | 8          | 8             | 0                         |

Table 6. Final Score table.

|   | Positive information score(m) | Negative information score(n) | Final score(m-n) |
|---|-------------------------------|-------------------------------|-----------------|
| $c_1$ | 1                             | 3                            | −2              |
| $c_2$ | −3                            | −2                            | −1              |
| $c_3$ | 3                             | −1                            | 4               |
| $c_4$ | −1                            | 0                             | −1              |
Table 7. Final score table.

Clearly the maximum score is 4 scored by the car $c_3$.

**Decision:** Mr. X will buy $c_3$. If he does not want to buy $c_3$ due to certain reason, his second choice will be $c_2$ or $c_4$, so the score of $c_2$ or $c_4$ are same.

9. Conclusion

We combine the concept of bipolar fuzzy set and soft set to introduced the concept of bipolar fuzzy soft sets. We examine some operations on bipolar fuzzy softs. We study basic operations on bipolar fuzzy soft set. We define extended union, intersection of two bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solved decision making problems by bipolar fuzzy soft set. Therefore, this paper gives an idea for the beginning of a new study for approximations of data with uncertainties. We will focus on the following problems: bipolar fuzzy soft relations, bipolar fuzzy soft matrix, bipolar fuzzy soft function and bipolar fuzzy soft graphs, and applications in artificial intelligence and general systems.

References

[1] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[2] Z. Pawlak, Rough sets, International Journal of Computing and Information Sciences 11 (1982) 341–356.
[3] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, 1991.
[4] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19–31.
[5] D. Molodtsov, The theory of soft sets, URSS Publishers, Moscow, 2004 (in Russian).
[6] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Computers and Mathematics with Applications 45, 2003, 555-562.
[7] H. Aktas, N. Cagman, Soft sets and soft groups. Inform Sci 177, 2007, 2726-2735.
[8] F. Feng, Y.B. Jun, X. Zhao, Soft semirings. Comput Math Appl 56, 2008, 2621-2628.
[9] U. Acar, F. Koyuncu and B. Tanay Soft sets and soft rings. Comput Math Appl, (2010), 59: 3458-3463.
[10] Y.B. Jun, Soft BCK/BCI-algebras. Comput Math Appl, 56, (2008), 1408-1413.
[11] Y.B. Jun, C.H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras. Inform Sci, 178, 2466-2475.
[12] Y.B. Jun , K. J. Lee and J. Zhan, Soft p-ideals of soft BCI-algebras. Comput Math Appl, 58, 2009, 2060-2068.
[13] A. Sezgin and A.O. Atagün, Soft groups and normalistic soft groups. Comput Math Appl 62, 2, 2011, 685-698.
[14] A. Sezgin, A.O. Atagün and N. Cagman, Soft intersection nearing with its applications, Neural Computing and Applications, 2011.
[15] J. Zhan and Y.B. Jun Soft BL-algebras based on fuzzy sets. Comput. Math. Appl. 59, 6, 2010, 2037-2046.
[16] Ş. O. Kazancı, Yılmaz and S. Yamak, Soft sets and soft BCH-algebras. Hacet J Math Stat 39 (2), (2010), 205-217.
[17] A. Sezgin and A.O. Atagün, E. Aygün, A note on soft nearrings and idealistic soft nearrings. Filomat 25(1), (2011), 53-68.
[18] A.O. Atagün and A. Sezgin, Soft substructures of rings, fields and modules. Comput Math Appl, 61(3), (2011), 592-601.
[19] A. Sezgin and A.O. Atagün, N. Çağman, Union soft substructures of nearrings and N-groups, Neural Comput. Appl, 2011, DOI: 10.1007/s00521-011-0732-1.
[20] N. Çağman, F. Çıtak, H. Aktas, Soft intersection-groups and its applications to group theory; Neural Comput. Appl., DOI: 10.1007/s00521-011-0752-x.
[21] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput Math Appl 57, 2009, 1547-1553.
[22] A. Sezgin, A. O. Atagün, On operations of soft sets, Comput. Math. Appl. 61, 5, 2011, 1457-1467.
[23] K. V. Babitha and J. J. Sunil, Soft set relations and functions. Comput Math Appl 60, 7, 2010, 1840-1849.
[24] P. Majumdar and S.K. Samanta, On soft mappings, Comput Math Appl 60(9), (2010), 2666-2672.
[25] N. Çağman and S. Enginoğlu, Soft set theory and its decision making. Comput Math Appl 59, (2010), 3308-3314.
[26] N. Çağman and S. Enginoğlu, Soft set theory and uni-intersection decision making, Eur J Oper Res 207, (2010), 848-855.
[27] P. K. Maji, A. R. Roy, B. Biswas, An application of soft sets in a decision making problem. Comput Math Appl 44, (2002), 1077-1083.
[28] A.R.Roy,P.K.Maji, A fuzzy soft set theoretical approach to decision making problems, Journal of Computational and Applied Mathematics, 203, (2007), 412-418.
[29] F. Feng, X.Y. Liu, V. Leoreanu-Fotea, Y. B. Jun, Soft sets and soft rough sets. Inform Sci 181, 6, 2011, 1125-1137.
[30] F. Feng, C. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Comput 14, 6, (2010), 899-911.
[31] F. Feng, Y. M. Li, V. Leoreanu-Fotea, Application of level soft sets in decision making based on interval-valued fuzzy soft sets, Comput Math Appl 60, (2010), 1756-1767.
[32] F. Feng , Y B. Jun , X.Y. Liu , L F. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234, (2010), 10-20.
[33] B. Ahmad and A. Kharal, On fuzzy soft sets, Advances in Fuzzy Systems, (2009), Article ID 586507, 6 pages.
[34] B. Ahmad and A. Kharal, Mappings on fuzzy soft classes, Advances in Fuzzy Systems, (2009), Article ID 407890, 6 pages.
[35] K.M. Lee, Bipolar-valued fuzzy sets and their basic operations, Proceeding International Conference, Bangkok, Thailand, (2000), 307-317.
[36] P.K. Maji, R. Biswas and A.R. Roy, Fuzzy soft sets, Journal of Fuzzy Mathematics, 9(2001), 589-602.
[37] P. Majumdar and S.K. Samanta, Generalised fuzzy soft sets, Computers and Mathematics with Applications, 59(2010), 1425-1432.
[38] T.J. Neog and D.K. Sut, On union and intersection of fuzzy soft set, International Journal of Computer and Technology with Applications, 2(2005), 1160-1176.
[39] W.R. Zhang, bipolar fuzzy sets and bipolar fuzzy relation, Industrial Fuzzy Control and Intelligent Systems Conference, 1(1994), 305-309. DOI: 10.1109/IJCF.1994.375115.