Bumping operators and insertion algorithms for queer supercrystals

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Abstract

Results of Morse and Schilling show that the set of increasing factorizations of reduced words for a permutation is naturally a crystal for the general linear Lie algebra. Hiroshima has recently constructed two superalgebra analogues of such crystals. Specifically, Hiroshima has shown that the sets of increasing factorizations of involution words and fpf-involution words for a self-inverse permutation are each crystals for the queer Lie superalgebra. In this paper, we prove that these crystals are normal and identify their connected components. To accomplish this, we study two insertion algorithms that may be viewed as shifted analogues of the Edelman-Greene correspondence. We prove that the connected components of Hiroshima’s crystals are the subsets of factorizations with the same insertion tableau for these algorithms, and that passing to the recording tableau defines a crystal morphism. This confirms a conjecture of Hiroshima. Our methods involve a detailed investigation of certain analogues of the Little map, through which we extend several results of Hamaker and Young.

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1 Introduction

This article is about combinatorial models for crystals for quantum queer Lie superalgebras. The crystals of primary interest will arise as sets of factorizations of reduced words for permutations. Our main results show how certain insertion algorithms that map words to pairs of shifted tableaux classify the connected components of these crystals and may be interpreted as crystal isomorphisms.

Crystal bases or (Kashiwara) crystals are combinatorial objects that arise in the representation theory of Lie algebras (or, more precisely, of quantum deformations of the corresponding universal enveloping algebras). The theory of crystals first appeared in independent work of Kashiwara [31, 32] and Lusztig [35, 36] in the 1990s; for a history of the relevant literature, see [7, § 1].

A theory of abstract crystals exists for any finite-dimensional Lie (super)algebra. We confine our attention in this paper to abstract crystals for the general linear Lie algebra $\mathfrak{gl}_n$ and the queer Lie superalgebra $\mathfrak{q}_n$. The latter object is the second super-analogue of $\mathfrak{gl}_n$, and has a number of interesting features. For example, all of its Cartan subalgebras are noncommutative, and this gives the highest weight space of any highest weight $\mathfrak{q}_n$-module the structure of a Clifford algebra. For background on $\mathfrak{q}_n$ and its somewhat complicated representation theory, see [12, 13, 14, 15].

The data of an abstract $\mathfrak{gl}_n$-crystal or $\mathfrak{q}_n$-crystal is equivalent to a certain directed weighted crystal graph. Under this identification, crystal isomorphisms correspond to graph isomorphisms. The weakly connected components of a crystal graph are called its full subcrystals. The categories of $\mathfrak{gl}_n$- and $\mathfrak{q}_n$-crystals are both equipped with a tensor product and a standard crystal $\mathbb{B}_n$, which is derived from the vector representation of the associated Lie superalgebra.

It is an interesting problem to determine whether or not an abstract crystal is normal in the sense of being isomorphic to a disjoint union of full subcrystals of tensor powers of $\mathbb{B}_n$, since such abstract crystals will correspond to actual representations of the associated Lie superalgebra. Stembridge [49] identified a set of local axioms that give a solution to this problem for $\mathfrak{gl}_n$-crystals. Gillespie, Hawkes, Poh, and Schilling [11], building on work of Assaf and Oguz [4], have recently extended Stembridge’s results to $\mathfrak{q}_n$-crystals.

The prototypical example of a normal $\mathfrak{gl}_n$-crystal is the set of semistandard Young tableaux with entries in $\{1, 2, \ldots, n\}$. The $\mathfrak{q}_n$-analagoues of these crystals have two combinatorial models, either as semistandard decomposition tableaux [13] or semistandard shifted tableaux [4] [24, 25]. We focus primarily on the second model in this article. For the definitions, see Section 2.4.
Another source of crystal constructions comes from reduced words. A reduced word for a permutation \( \pi \) is a minimal-length integer sequence \( i_1 i_2 \cdots i_l \) such that \( \pi = s_{i_1} s_{i_2} \cdots s_{i_l} \) where \( s_i := (i, i + 1) \). One can divide a word \( i_1 i_2 \cdots i_l \) into a sequence of \( n \) strictly increasing, possibly empty subwords (which we call an \( n \)-fold increasing factorization) if and only if there are fewer than \( n \) indices \( j \) with \( i_j \geq i_{j+1} \). Let \( \mathcal{R}_n(\pi) \) denote the set of all \( n \)-fold increasing factorizations of reduced words for a fixed permutation \( \pi \). Morse and Schilling [41] identified a natural \( \mathfrak{gl}_n \)-crystal structure on this set (see Section 3.1) and by checking Stembridge’s local axioms, proved the following:

**Theorem** (Morse and Schilling [41]; see Corollary 3.29). The \( \mathfrak{gl}_n \)-crystal \( \mathcal{R}_n(\pi) \) is normal.

Morse and Schilling also showed something more specific. The Edelman-Greene correspondence [10] is a well-known map that sends each \( w \in \mathcal{R}_n(\pi) \) to a pair of tableaux \((P_{\text{EG}}(w), Q_{\text{EG}}(w))\) with the same shape. This correspondence has many interesting and desirable properties.

**Theorem** (Morse and Schilling [41]; see Theorem 3.28). The full subcrystals of \( \mathcal{R}_n(\pi) \) are the fibers of \( w \mapsto P_{\text{EG}}(w) \). Moreover, the map \( w \mapsto Q_{\text{EG}}(w) \) defines an isomorphism from each full subcrystal of \( \mathcal{R}_n(\pi) \) to a (normal) \( \mathfrak{gl}_n \)-crystal of semistandard tableaux.

One application of this result is to give a crystal theoretic interpretation of the positive coefficients in the Schur expansion of the Stanley symmetric functions; see Corollary 3.30. The precise definitions of \( \mathcal{R}_n(\pi) \) and the Edelman-Greene correspondence appear in Section 3.

In the recent paper [26], Hiroshima has constructed two \( q_n \)-analogues of Morse and Schilling’s \( \mathfrak{gl}_n \)-crystals of factorized reduced words. The elements of Hiroshima’s \( q_n \)-crystals are \( n \)-fold increasing factorizations of the involution words and fpf-involution words of a self-inverse permutation \( \pi \). Such words have been studied under various names by several authors [9, 17, 23, 27, 45].

Whereas reduced words for permutations may be identified with maximal chains in the weak order on the symmetric group, involution words and fpf-involution words correspond to maximal chains in an analogous weak order on the finite set of orbits of the orthogonal and symplectic groups acting on the complete flag variety. For this reason, we denote Hiroshima’s \( q_n \)-crystals by \( \mathcal{R}_n^O(\pi) \) and \( \mathcal{R}_n^{Sp}(\pi) \). We review the definitions of these crystals in Sections 3.2 and 3.3. One of our main new results is the following theorem concerning these objects.

**Theorem** (Corollaries 3.33 and 3.37). For each \( K \in \{O, Sp\} \), the \( q_n \)-crystal \( \mathcal{R}_n^K(\pi) \) is normal.

This result is a corollary of a more effective theorem, which we sketch as follows. It turns out that one obtains natural “orthogonal” and “symplectic” analogues of the Edelman-Greene correspondence by restricting the shifted Hecke insertion and symplectic Hecke insertion algorithms introduced in [38, 42]. We denote these maps by \( w \mapsto (P_{\text{EG}}^O(w), Q_{\text{EG}}^O(w)) \) and \( w \mapsto (P_{\text{EG}}^{Sp}(w), Q_{\text{EG}}^{Sp}(w)) \). Both are generalizations of Sagan-Worley insertion [47, 50], and assign increasing factorizations to pairs of shifted tableaux with the same shape; the definitions are given in Section 3.4.

The following theorem provides more substance to the strong formal analogy between Morse and Schilling’s \( \mathfrak{gl}_n \)-crystals \( \mathcal{R}_n(\pi) \) and Hiroshima’s \( q_n \)-crystals \( \mathcal{R}_n^O(\pi) \) and \( \mathcal{R}_n^{Sp}(\pi) \).

**Theorem** (Theorems 3.32 and 3.36). Let \( K \in \{O, Sp\} \). The full subcrystals of \( \mathcal{R}_n^K(\pi) \) are the fibers of \( w \mapsto P_{\text{EG}}^K(w) \), and the map \( w \mapsto Q_{\text{EG}}^K(w) \) defines an isomorphism from each full subcrystal of \( \mathcal{R}_n^K(\pi) \) to a (normal) \( q_n \)-crystal of semistandard shifted tableaux.

An application of this result is to give a crystal theoretic interpretation of the positive coefficients in the Schur \( P \)-expansion of the involution Stanley symmetric functions studied in [18, 20]; see
Corollaries §3.34 and §3.38. As we work to show this result, we will end up proving a few conjectures from [18, 26, 38]. In particular, the claim that \( w \mapsto Q_{EG}^K(w) \) is a crystal morphism is equivalent to [26, Conjecture 5.1], as we explain in Section 5.4. Another way to frame the main results of this article is as an in-depth study of these operators. Ignoring all applications to crystals, our work serves to generalize several theorems of Hamaker and Young [21] and to clarify the relationship between involution Little bumps and shifted forms of the Edelman-Greene correspondence.

Our strategy to prove these results is to identify certain crystal isomorphisms between different instances of \( R_n^K(\pi) \), which commute with the map \( w \mapsto Q_{EG}^K(w) \). We use these isomorphisms to translate our crystals to a simpler form. The relevant maps will be composed of the involution Little bumping operators introduced in [19]. Another way to frame the main results of this article as an expository appendix provides some extra background on crystals of shifted tableaux.

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## 2 Preliminaries

In this section, we review some general background on crystals, words, and tableaux from [7, 13]. Let \( Z \) be the set of integers and write \( Z_{\geq 0} \) and \( Z_{> 0} \) for the sets of nonnegative and positive integers. For each \( m \in Z_{\geq 0} \), let \( [m] = \{i \in Z_{> 0} : i \leq m\} = \{1, 2, \ldots, m\} \), so that \([0] = \emptyset\).

### 2.1 Crystals

Fix a positive integer \( n \) and let \( e_1, e_2, \ldots, e_n \) be the standard unit basis vectors in \( \mathbb{Z}^n \).

**Definition 2.1** ([7, §2.2]). An abstract \( \mathfrak{g}_n \)-crystal is a set \( \mathcal{B} \) with maps \( wt : \mathcal{B} \to Z_{\geq 0}^n \) and \( e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \) for \( i \in [n-1] \), where \( 0 \notin \mathcal{B} \) is an auxiliary element, such that if \( i \in [n-1] \) then:

1. If \( b, c \in \mathcal{B} \) then \( e_i(b) = c \) if and only if \( f_i(c) = b \), in which case \( wt(c) = wt(b) + e_i - e_{i+1} \).

2. If \( b \in \mathcal{B} \) then \( \varepsilon_i(b) := \max \{k \geq 0 : e_i^k(b) \neq 0\} \) and \( \varphi_i(b) := \max \{k \geq 0 : f_i^k(b) \neq 0\} \) are both finite, and \( \varphi_i(b) - \varepsilon_i(b) = wt(b)_i - wt(b)_{i+1} \).

We refer to the function \( wt \) as the weight map, to \( e_i \) and \( f_i \) as the raising and lowering crystal operators, and to \( \varepsilon_i \) and \( \varphi_i \) as the string lengths of \( \mathcal{B} \).
This is slightly more specialized than the definition of a \( \mathfrak{gl}_n \)-crystal in [7]. In the terminology of [7], our definition describes the \( \mathfrak{gl}_n \)-crystals that are seminormal.

**Definition 2.2** ([13] §1.3). An abstract \( q_n \)-crystal (for \( n \geq 2 \)) is an abstract \( \mathfrak{gl}_n \)-crystal \( \mathcal{B} \) with queer raising and lowering operators \( e_T, f_T : \mathcal{B} \to \mathcal{B} \sqcup \{0\} \) satisfying the following conditions:

1. If \( b, c \in \mathcal{B} \) then \( e_T(b) = c \) if and only if \( f_T(c) = b \), in which case
   \[
   \wt(b) = \wt(c) + \epsilon_2 - \epsilon_1, \quad \varepsilon_i(b) = \varepsilon_i(c), \quad \text{and} \quad \varphi_i(b) = \varphi_i(c)
   \]
   for all \( 3 \leq i \leq n - 1 \), where \( \varepsilon_i, \varphi_i \) are the string lengths from Definition 2.1.

2. The operators \( e_T \) and \( f_T \) commute with \( e_i \) and \( f_i \) for each \( 3 \leq i \leq n - 1 \), under the convention that \( e_T(0) = f_T(0) = e_i(0) = f_i(0) = 0 \).

3. If \( b \in \mathcal{B} \) and we define \( \varepsilon_T(b) := \max \{ k \geq 0 : e^k_T(b) \neq 0 \} \) and \( \varphi_T(b) := \max \{ k \geq 0 : f^k_T(b) \neq 0 \} \), then we have \( \varepsilon_T(b) + \varphi_T(b) \leq 1 \), with equality if \( \wt(b)_1 \neq 0 \) or \( \wt(b)_2 \neq 0 \).

The original definition of an abstract \( q_n \)-crystal in [13] §1.3 omits condition (3). This condition holds in all examples of interest and will imply a desirable symmetry property. To simplify some later statements, we consider the empty set to be an abstract \( \mathfrak{gl}_n \)- and \( q_n \)-crystal and define an abstract \( q_1 \)-crystal to be any set with a weight map \( \wt \) taking values in \( \mathbb{Z}_{\geq 0} \).

The crystal graph of an abstract \( q_n \)-crystal \( \mathcal{B} \) is the weighted directed graph with vertex set \( \mathcal{B} \) that has an edge \( x \to y \) whenever \( y = f_i(x) \) for some \( i \in \{1, 2, \ldots, n - 1\} \). A weakly connected component of the crystal graph of an abstract \( \mathfrak{gl}_n \)- or \( q_n \)-crystal is called a full subcrystal.

**Example 2.3.** The standard \( q_n \)-crystal \( B_n \) has weight function \( \wt([i]) = \epsilon_i \) and crystal graph

\[
\begin{array}{cccccccc}
1 & \xrightarrow{T} & 2 & \xrightarrow{1} & 3 & \xrightarrow{2} & \cdots & \xrightarrow{n-1} & n
\end{array}
\]

The character of a finite \( \mathfrak{gl}_n \)-crystal \( \mathcal{B} \) is the polynomial \( \text{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x_1^{\wt(b)_1} x_2^{\wt(b)_2} \cdots x_n^{\wt(b)_n} \).

Let \( \Lambda_n \) be the ring of symmetric polynomials in \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \).

**Proposition 2.4** ([7] §2.6]). The character of a finite \( \mathfrak{gl}_n \)-crystal is in \( \Lambda_n \).

An element \( f \in \Lambda_n \) is supersymmetric if \( f(x_1, -x_1, x_3, \ldots, x_n) \in \mathbb{Z}[x_3, \ldots, x_n] \). Let \( \Gamma_n \) denote the subring of supersymmetric polynomials in \( \Lambda_n \) for \( n \geq 2 \), and set \( \Gamma_1 = \Lambda_1 \).

**Proposition 2.5.** The character of a finite \( q_n \)-crystal is in \( \Gamma_n \).

**Proof.** Fix \( n \geq 2 \). Let \( R \) be the set of \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) with \( f(x_1, -x_1, x_3, \ldots, x_n) \in \mathbb{Z}[x_3, \ldots, x_n] \). Suppose \( \mathcal{B} \) is a finite \( q_n \)-crystal and \( b \in \mathcal{B} \). If \( \wt(b)_1 = \wt(b)_2 = 0 \) then \( x^{\wt(b)} \in R_n \). Otherwise, Definition 2.2(3) implies that there exists a unique \( c \in \mathcal{B} \) with \( e_T(b) = c \) or \( f_T(c) = b \), and \( x^{\wt(b)} + x^{\wt(c)} \in (x_1 + x_2)\mathbb{Z}[x_1, x_2, \ldots, x_n] \subset R_n \). We conclude that \( \text{ch}(\mathcal{B}) \in R \cap \Lambda_n = \Gamma_n \). \( \square \)

A (strict) morphism \( \mathcal{B} \to \mathcal{C} \) of abstract \( \mathfrak{gl}_n \)- or \( q_n \)-crystals is a map \( \mathcal{B} \sqcup \{0\} \to \mathcal{C} \sqcup \{0\} \) with \( 0 \mapsto 0 \) that preserves weights and string lengths and commutes with all crystal operators. A morphism that is also a bijection is an isomorphism; such a map induces an isomorphism of crystal graphs.
2.2 Words

Given two abstract \( \mathfrak{g}_l \)- or \( \mathfrak{q}_n \)-crystals \( B \) and \( C \), one can form the tensor product crystal \( B \otimes C \); see [11, §3] and [13, §1.3] for the precise definitions. For our applications, it will suffice to describe the \( m \)-fold tensor product \( (\mathbb{B}_m)^{\otimes m} \) of the standard \( \mathfrak{q}_n \)-crystal from Example [2.3]. One can realize this object as the following crystal of words.

A word is a finite sequence \( w_1 w_2 \cdots w_n \) of integers. Fix \( m \in \mathbb{Z}_{>0} \) and let \( \mathcal{W}_n(m) \) be the set of \( m \)-letter words in the alphabet \( \{1, 2, \ldots, n\} \). Given \( w \in \mathcal{W}_n(m) \), define \( \text{wt}(w) \in (\mathbb{Z}_{\geq 0})^n \) to be the \( n \)-tuple whose \( i \)th entry is the number of occurrences of \( i \) in \( w \). For any \( i \in \mathbb{Z} \), there are operators \( f_i \) and \( e_i \) acting on \( \mathcal{W}_n(m) \) as follows. Consider the sequence formed by replacing each \( i \) in a word \( w \) by a right parenthesis and each \( i + 1 \) in \( w \) by a left parenthesis.

- If all right parentheses in this sequence belong to a balanced pair of left and right parentheses, then \( f_i(w) = 0 \). Otherwise, form \( f_i(w) \) from \( w \) by changing the letter \( i \) corresponding to the last unbalanced right parenthesis to \( i + 1 \).
- Similarly, if all left parentheses belong to a balanced pair, then \( e_i(w) = 0 \). Otherwise, form \( e_i(w) \) by changing the \( i + 1 \) in \( w \) corresponding to the first unbalanced left parenthesis to \( i \).

For example, if \( w = 1223313212 \) and \( i = 2 \) then the parenthesized word is \( 1)((1()1)(), so \( f_2(w) = 12\underline{2}3313212 \) and \( e_2(w) = 12223313212 \). We also define \( f_\lambda(w) \) and \( e_\lambda(w) \) for words \( w \in \mathcal{W}_n(m) \):

- If \( w \) has no 1’s or if its first 1 appears after its first 2, then \( f_\lambda(w) = 0 \). Otherwise, \( f_\lambda(w) \) is the word formed by changing the first 1 in \( w \) to 2.
- If \( w \) has no 2’s or if its first 2 appears after its first 1, then \( e_\lambda(w) = 0 \). Otherwise, \( e_\lambda(w) \) is the word formed by changing the first 2 in \( w \) to 1.

If \( w = 1223313212 \) then \( f_\lambda(w) = 2\underline{2}23313212 \) and \( e_\lambda(w) = 0 \).

Proposition 2.6 ([11, Remarks 2.3 and 2.4]). Relative to the maps \( \text{wt}, e_i, f_i \) just given, \( \mathcal{W}_n(m) \) is an abstract \( \mathfrak{q}_n \)-crystal and there is a \( \mathfrak{q}_n \)-crystal isomorphism \( \mathcal{W}_n(m) \cong (\mathbb{B}_m)^{\otimes m} \).

2.3 Tableaux

The Young diagram of an integer partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0) \) is the set of pairs \( D_\lambda = \{(i, j) \in [n] \times [\lambda_1] : 1 \leq j \leq \lambda_i\} \). We use the term tableau to mean a map \( D_\lambda \to \mathbb{Z} \) for some partition \( \lambda \); such a map is said to have shape \( \lambda \).

We draw tableaux in French notation, so that row indices increase going up. For example,

\[
\begin{array}{c c c}
3 & 4 \\
2 & 2 & 4
\end{array}
\quad \text{and} \quad
\begin{array}{c c c c}
3 & 4 & 4 & 4 \\
3 & 3 & 3 & 3
\end{array}
\quad \text{and} \quad
\begin{array}{c c c c}
3 & 5 & 5 & 5 \\
1 & 2 & 3 & 4
\end{array}
\]

all have shape \( \lambda = (3, 2) \). The pairs in the domain of a tableau are its boxes.

A tableau is semistandard if its rows are weakly increasing and its columns are strictly increasing. A tableau is increasing if its rows and columns are both strictly increasing. A tableau with \( m \) boxes is standard if it is increasing and it contains each of the numbers \( 1, 2, \ldots, m \) exactly once. The three tableaux drawn above are respectively semistandard, increasing, and standard. Let \( \text{Tab}_n(m) \) denote the set of semistandard tableaux with \( m \) boxes and entries in \([n]\). Let \( \text{Tab}_n(\lambda) \) denote the subset of \( T \in \text{Tab}_n(|\lambda|) \) of shape \( \lambda \).
The row reading word of a tableu \( T \) is the sequence \( \text{row}(T) \) formed by listing the entries of \( T \) row-by-row from left to right, starting with the top row. The row reading words of the tableaux in (2.1) are 34224, 34234, and 35124. The column reading word of \( T \) is the sequence \( \text{col}(T) \) formed by listing the entries of \( T \) down each column, starting with the first column. The column reading words of the tableaux in (2.1) are 32424, 32434, and 31524.

We introduce the term quasi-isomorphism to mean a morphism \( \psi : \mathcal{B} \rightarrow \mathcal{C} \) between abstract \( \mathfrak{gl}_n \)- or \( \mathfrak{q}_n \)-crystals with the property that for each full subcrystal \( \tilde{\mathcal{B}} \subset \mathcal{B} \), there is a full subcrystal \( \tilde{\mathcal{C}} \subset \mathcal{C} \) such that \( \psi \) restricts to an isomorphism \( \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{C}} \). Let \( m, n \in \mathbb{Z}_{>0} \).

**Theorem-Definition 2.7** ([7, §3.1]). There is a unique abstract \( \mathfrak{gl}_n \)-crystal structure on \( \text{Tab}_n(m) \) that makes the injective map \( \text{row} : \text{Tab}_n(m) \rightarrow \mathcal{W}_n(m) \) into a quasi-isomorphism. The full \( \mathfrak{gl}_n \)-subcrystals of \( \text{Tab}_n(m) \) are the sets \( \text{Tab}_n(\lambda) \) as \( \lambda \) ranges over all partitions of \( m \) with \( \leq n \) parts.

**Remark 2.8.** If \( s_\lambda \) denotes the Schur function of a partition \( \lambda \), then the character of the abstract \( \mathfrak{gl}_n \)-crystal \( \text{Tab}_n(\lambda) \) is the Schur polynomial \( s_\lambda(x_1, x_2, \ldots, x_n) \in \Lambda_n \) [7, Eq. (3.3)].

### 2.4 Shifted tableaux

If \( i \in \mathbb{Z} \) then we define \( i' := i - \frac{1}{2} \). Using this notation, we can write

\[
\frac{1}{2}\mathbb{Z} = \{ \ldots < 0' < 0 < 1' < 1 < 2' < 2 < \ldots \}.
\]

Since \( 1' = \frac{1}{2} \), we have \((i + 1)' = i' + 1 = i + 1' \) for all \( i \in \mathbb{Z} \). An element of \( \frac{1}{2}\mathbb{Z} \) is a primed number if it has the form \( i' \in \mathbb{Z} - \frac{1}{2} \) for some \( i \in \mathbb{Z} \). We sometimes refer to integers \( i \in \mathbb{Z} \) as unprimed numbers.

The shifted diagram of a strict partition \( \mu = (\mu_1 > \cdots > \mu_n > 0) \) is the set of pairs

\[
\text{SD}_\mu := \{(i, j) \in [n] \times [\mu_1] : i \leq j \leq \mu_i + i - 1\}.
\]

We use the term shifted tableau to mean a map from the shifted diagram \( \text{SD}_\mu \) of some strict to partition to \( \frac{1}{2}\mathbb{Z} \). A shifted tableau with domain \( \text{SD}_\mu \) has shape \( \mu \). The (main) diagonal of a shifted tableau consists of the boxes \((i, j)\) in its domain with \( i = j \).

A shifted tableau with positive entries is semistandard if its rows and columns are weakly increasing, no unprimed number appears more than once in a column, and no primed number appears in a diagonal position or more than once in a row. A shifted tableau is increasing if it contains no primed entries and its rows and columns are strictly increasing. A shifted tableau with \( m \) boxes is standard if it is semistandard and its boxes contain exactly one of \( i \) or \( i' \) for each \( i \in [m] \).

The following examples are respectively semistandard, increasing, and standard:

\[
\begin{array}{ccc}
3 & 4' & 2' \\
2 & 2 & 4
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
4 & 5 & 2 \quad 3' \\
2 & 3 & 4 \quad 1 \quad 2 \quad 1
\end{array}
\]

Fix \( m, n \in \mathbb{Z}_{>0} \) and let \( \text{ShTab}_n(m) \) denote the set of semistandard shifted tableaux with \( m \) boxes and entries in \( \{1' < 1 < 2' < 2 < \cdots < n' < n\} \). For each strict partition \( \mu \), let \( \text{ShTab}_n(\mu) \) denote the subset of shifted tableaux in \( \text{ShTab}_n(|\mu|) \) of shape \( \mu \). The row and column reading words of a shifted tableau are defined in the same way as for ordinary tableaux.

Results in [4, 24, 25] show that the set \( \text{ShTab}_n(m) \) carries a natural \( \mathfrak{q}_n \)-crystal structure, which we describe in Section A. Except for the proof of Theorem 5.11 we will not need to work with
the explicit formulas for the relevant crystal operators the appear in this appendix. Instead, it will suffice to use the following characterization of the $q_n$-crystal structure on $\text{ShTab}_n(m)$ in terms of Haiman’s notion of mixed (shifted) insertion \cite{16} Definition 6.7.

Mixed insertion is defined as an iterative algorithm, where at each stage a number $x$ is “inserted” into a row or column of a tableau. When this happens, $x$ replaces some other (usually larger) entry $y$, and we say that the number $y$ is “bumped.”

**Definition 2.9** \cite{16}. Given a word $w = w_1 w_2 \cdots w_m \in W_n(m)$, let \(0 = T_0, T_1, \ldots, T_m = P_{\text{HM}}(w)\) be the sequence of shifted tableaux in which $T_i$ for $i \in [m]$ is formed from $T_{i-1}$ by inserting $w_i$ according to the following procedure:

Start by inserting $w_1$ into the first row. At each stage, an entry $x$ is inserted into a row or column. Let $y$ be the first entry in the row going left to right (respectively, column going bottom to top) with $x < y$. If no such entry $y$ exists then $x$ is added to the end of the row or column. Otherwise, $x$ replaces $y$ and we continue by inserting $y$ into the next row if $y$ is unprimed or into the next column otherwise, with the exception that if $y$ is on the main diagonal (and therefore unprimed) then we insert the primed number $y'$ into the next column.

We call $P_{\text{HM}}(w)$ the mixed insertion tableau of $w$. The mixed recording tableau $Q_{\text{HM}}(w)$ is the shifted tableau with the same shape as $P_{\text{HM}}(w)$ which contains $i$ in the box added to $T_{i-1}$.

**Example 2.10.** We compute $P_{\text{HM}}(w)$ and $Q_{\text{HM}}(w)$ for $w = 332332$:

$$
\begin{array}{cccc}
3 & 3 & 3 & \\
3 & 2' & 3 & \\
3 & 2' & 3 & 3 & \\
3 & 2' & 2 & 3 & 3 & = P_{\text{HM}}(w)
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
3 & 0 & & \\
1 & 2 & 4 & 5 & = Q_{\text{HM}}(w).
\end{array}
$$

The following theorem is due to Haiman \cite{16}, but the cited result in \cite{21} matches our notation.

**Theorem 2.11** \cite{16}; see \cite{21} Theorem 3.12]. The map $w \mapsto (P_{\text{HM}}(w), Q_{\text{HM}}(w))$ is a bijection from $W_n(m)$ to the set of pairs $(P, Q)$ of shifted tableaux of the same shape in which $P \in \text{ShTab}_n(m)$ and $Q$ is standard with no primed entries.

The $q_n$-crystal structure on $\text{ShTab}_n(m)$ is now determined by the following result.

**Theorem-Definition 2.12** \cite{11, 21, 25}. Fix integers $m, n \in \mathbb{Z}_{>0}$. Then:

(a) The full $q_n$-subcrystals of $W_n(m)$ are the sets on which $Q_{\text{HM}}$ is constant.

(b) There is a unique abstract $q_n$-crystal structure on the set $\text{ShTab}_n(m)$ that makes the surjective map $P_{\text{HM}} : W_n(m) \to \text{ShTab}_n(m)$ into a quasi-isomorphism.

(c) The full $q_n$-subcrystals of $\text{ShTab}_n(m)$ in this structure are the sets $\text{ShTab}_n(\mu)$ where $\mu$ ranges over all strict partitions of $m$ with at most $n$ parts.

**Proof sketch.** One can check directly that $Q_{\text{HM}}(w) = Q_{\text{HM}}(f_T(w))$ if $f_T(w) \neq 0$. The other properties follow from \cite{21} Theorems 3.13 and 4.3, \cite{25} Theorem 3.2, and \cite{21} Theorem 4.8. \hfill \square

It follows that the weight map for $\text{ShTab}_n(m)$ assigns to a shifted tableau $T$ the sequence $\text{wt}(T) \in \mathbb{Z}_{\geq 0}^n$ whose $ith$ entry is the number of times $i$ or $i'$ appears in $T$. For example,

$$
\text{wt}\left(\begin{array}{cc}
3 & 1 \\
2 & 2 & 4
\end{array}\right) = (0, 2, 1, 2, 0)
$$

when $n = 5$. For an example of the $q_n$-crystal $\text{ShTab}_n(m)$, see Figure 1. For more direct formulas for the operators $e_i$ and $f_i$ acting on this crystal, see Section A.

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Remark 2.13. If \( P_\mu \) denotes the Schur \( P \)-function of a strict partition \( \mu \) (see [37, III.8]), then the character of the abstract \( \mathfrak{q}_n \)-crystal \( \text{ShTab}_n(\mu) \) is the Schur \( P \)-polynomial \( P_\mu(x_1, x_2, \ldots, x_n) \in \Gamma_n \).

3 Crystals of factorizations

In this section, we first review three families of abstract crystals introduced in [26, 41]. The elements of each crystal are increasing factorizations of reduced words for certain permutations. We then present our main new results in Section 3.5. Throughout, \( n \in \mathbb{Z}_{>0} \) is a positive integer.

3.1 Reduced factorizations

Let \( S_Z \) denote the group of permutations of \( \mathbb{Z} \) fixing all but finitely many integers. The simple transpositions \( s_i := (i, i + 1) \in S_Z \) for \( i \in \mathbb{Z} \) generate \( S_Z \) and with respect to this generating set \( S_Z \) is a Coxeter group, whose length function \( \ell : S_Z \to \mathbb{Z}_{\geq 0} \) counts the inversions of a permutation.

Definition 3.1. A reduced word for a permutation \( \pi \in S_Z \) is a minimal-length sequence of integers \( i_1 i_2 \cdots i_l \) with \( \pi = s_{i_1} s_{i_2} \cdots s_{i_l} \). Let \( R(\pi) \) denote the set of reduced words for \( \pi \in S_Z \).

Let \( \pi \in S_Z \). It is well-known that \( \ell(\pi) \) is the length every word in \( R(\pi) \), which is a single equivalence class under the transitive closure of the Coxeter braid relations defined by \( i \cdots j \cdots \sim i j \cdots \) for \( |i - j| > 1 \) and \( \cdots i(i+1) \cdots \sim \cdots (i+1)i(i+1) \cdots \) for all \( i \in \mathbb{Z} \) [10, Lemma 6.18].

Definition 3.2. An increasing factorization of a word \( w \) is a finite sequence \( (w^1, w^2, \ldots, w^n) \) in which each \( w^i \) is a strictly increasing, possibly empty word and \( w = w^1 w^2 \cdots w^n \). A reduced factorization is an increasing factorization of a reduced word for some \( \pi \in S_Z \). Let \( R_n(\pi) \) denote the set of increasing factorizations with \( n \) factors of reduced words for \( \pi \in S_Z \).

Given increasing words \( a = a_1 a_2 \cdots a_p \) and \( b = b_1 b_2 \cdots b_q \), define a set \( \text{pair}(a, b) \) inductively as follows. If no \( (i, j) \in [p] \times [q] \) exists with \( a_i > b_j \), then \( \text{pair}(a, b) = \varnothing \). Otherwise, choose \( j \) to be maximal with \( \{i : a_i > b_j\} \neq \varnothing \), then choose \( i \) to be minimal with \( a_i > b_j \), and finally set

\[
\text{pair}(a, b) = \{(a_i, b_j)\} \cup \text{pair}(a_1 \cdots a_{i-1} a_{i+1} \cdots a_p, b_1 \cdots b_{j-1} b_{j+1} \cdots b_q).
\]

Equivalently, we form \( \text{pair}(a, b) \) by iterating over the letters in the second word from largest to smallest; at each iteration, the current letter \( b_j \) is paired with the smallest unpaired letter \( a_i \) in the first word with \( a_i > b_j \), if such a letter exists. For example, if \( u = 1, 3, 4, 5, 8, 10, 11 \) and \( v = 2, 6, 9, 12, 13 \) then we have \( \text{pair}(u, v) = \{(10, 9), (8, 6), (3, 2)\} \).

The set \( R_n(\pi) \) has an abstract \( \mathfrak{gl}_n \)-crystal structure [11], which we can describe as follows. Fix \( \pi \in S_Z \) and \( w = (w^1, w^2, \ldots, w^n) \in R_n(\pi) \). Let \( \text{wt}(w) = (\ell(w^1), \ell(w^2), \ldots, \ell(w^n)) \in \mathbb{Z}_{\geq 0}^n \). To define \( f_i(w) \) and \( e_i(w) \) for \( i \in [n - 1] \), we examine the set \( \text{pair}(w^i, w^{i+1}) \):

- If every letter in \( w^i \) belongs to \( \{a : (a, b) \in \text{pair}(w^i, w^{i+1})\} \), then \( f_i(w) = 0 \). Otherwise, form \( f_i(w) \) by removing from \( w^i \) its largest unpaired letter \( x \), and then adding to \( w^{i+1} \) the smallest integer \( y \geq x \) that is not already a letter (in the position yielding an increasing word).

- If every letter in \( w^{i+1} \) belongs to \( \{b : (a, b) \in \text{pair}(w^i, w^{i+1})\} \), then \( e_i(w) = 0 \). Otherwise, form \( e_i(w) \) by removing from \( w^{i+1} \) its smallest unpaired letter \( x \), and then adding to \( w^i \) the largest integer \( x \leq y \) that is not already a letter (in the position yielding an increasing word).
The following is equivalent to [41, Theorem 3.5]; see also [7, §10.2]. Both references work with factorizations into decreasing subwords that are indexed in reverse order as \((w^n, \ldots, w^2, w^1)\). Reading everything backwards translates the relevant statements to what is given here.

**Proposition 3.3 ([41]).** Relative to the maps \(w_t, e_i, f_i\) just given, the set of reduced factorizations \(\mathcal{R}_n(\pi)\) is an abstract \(\mathfrak{gl}_n\)-crystal for all \(\pi \in S_\mathbb{Z}\). (We call this a Morse-Schilling crystal.)

For examples of Morse-Schilling crystals, see Figures 2 and 3, ignoring any arrows \(x \rightarrow y\).

**Remark 3.4.** In [48], Stanley associates to each permutation \(\pi\) a certain homogeneous symmetric function \(F_\pi\). (In later references, the indexing convention for this power series is often inverted.) It is clear from the definition in [48] that the character of the crystal \(\mathcal{R}_n(\pi)\) is the polynomial \(F_\pi(x_1, x_2, \ldots, x_n) \in \Lambda_n\) obtained by specializing the Stanley symmetric function \(F_\pi\) to \(n\) variables.

Stanley derives a formula for the maximal term in the Schur expansion of \(F_\pi\) relative to dominance order on partitions [48, Theorem 4.1]. Because \(\mathfrak{s}_\lambda(x_1, x_2, \ldots, x_n)\) is nonzero if and only if the number of parts of \(\lambda\) satisfies \(\ell(\lambda) \leq n\), and because one has \(\mu \leq \lambda\) in dominance order only if \(\ell(\mu) \geq \ell(\lambda)\), Stanley’s formula implies that \(F_\pi(x_1, x_2, \ldots, x_n)\) is nonzero if and only if the numbers \(c_\lambda(\pi) := |\{i < j \in \mathbb{Z} : \pi(i) > \pi(j)\}|\) that make up the code of \(\pi\) are at most \(n\) for all \(i \in \mathbb{Z}\). Necessarily, the set \(\mathcal{R}_n(\pi)\) is nonempty if and only if the same condition holds.

### 3.2 Orthogonal factorizations

Let \(I_\mathbb{Z} = \{\pi \in S_\mathbb{Z} : \pi = \pi^{-1}\}\) be the set of involutions in \(S_\mathbb{Z}\). Such permutations are in bijection with the (incomplete) matchings on \(\mathbb{Z}\) with finitely many edges. Given \(\pi \in S_\mathbb{Z}\) and \(i \in \mathbb{Z}\), define

\[
\pi \circ s_i = \begin{cases} 
\pi s_i & \text{if } \pi(i) < \pi(i+1) \\
\pi & \text{if } \pi(i) > \pi(i+1)
\end{cases} \quad \text{and} \quad \pi \times s_i = \begin{cases} 
\pi s_i & \text{if } \pi s_i \neq \pi s_i \\
\pi & \text{if } \pi s_i = \pi s_i
\end{cases}
\]

The operation \(\circ\) extends to an associative product \(S_\mathbb{Z} \times S_\mathbb{Z} \to S_\mathbb{Z}\) [30, §7.1]. The operation \(\times\) is not associative, but one has \(\pi \times s_i \in I_\mathbb{Z}\) if and only if \(\pi \in I_\mathbb{Z}\).

**Lemma 3.5 ([17, §2]).** Let \(\pi \in S_\mathbb{Z}\) and \(i_1, i_2, \ldots, i_l \in \mathbb{Z}\). The following are equivalent:

(a) \(i_1 i_2 \cdots i_l\) is a minimal-length word with \(\pi = s_{i_l} \circ \cdots \circ s_{i_2} \circ s_{i_1} \circ 1 \circ s_{i_1} \circ s_{i_2} \cdots \circ s_{i_l}\).

(b) \(i_1 i_2 \cdots i_l\) is a minimal-length word with \(\pi = (\cdots ((1 \times s_{i_1}) \times s_{i_2}) \times \cdots) \times s_{i_l}\).

(c) \(i_j\) is not a descent of \((\cdots ((1 \times s_{i_1}) \times s_{i_2}) \times \cdots) \times s_{i_{j-1}}\) for each \(j \in [l]\).

Moreover, if these properties hold then we must have \(\pi \in I_\mathbb{Z}\).

**Definition 3.6.** An involution word for \(\pi \in I_\mathbb{Z}\) is a sequence of integers \(i_1 i_2 \cdots i_l\) with the equivalent properties in Lemma 3.5. Let \(\mathcal{R}_\mathbb{Z}(\pi)\) denote the set of involution words for \(\pi \in S_\mathbb{Z}\).

One can view \(\mathcal{R}_\mathbb{Z}(\pi)\) as an “orthogonal” analogue of the usual set of reduced words \(\mathcal{R}(\pi)\). Elements of the latter correspond to maximal chains in weak order on the set of Borel orbits in the type A flag variety \(\text{Fl}_n\). The orbits of the orthogonal group \(O_n(\mathbb{C})\) in \(\text{Fl}_n\) are in bijection with the set of permutations \(\pi \in I_\mathbb{Z}\) with support in \([n]\). Involution words may be identified with the set of maximal chains in an analogous weak order on these orbits; see [41, 45, 51].

Let \(=\) denote the transitive closure of the Coxeter braid relations plus the relation on words that has \(w_1 w_2 w_3 \cdots w_m = w_2 w_1 w_3 \cdots w_m\) for any choice of \(w_1, w_2, w_3, \ldots, w_m \in \mathbb{Z}\).
Theorem 3.7 ([27, Theorem 3.1]). If \( \pi \in I_\mathbb{Z} \) then \( \mathcal{R}^O(\pi) \) is a single equivalence class under \( =_O \).
Moreover, an equivalence class of words under \( =_O \) is equal to \( \mathcal{R}^O(\pi) \) for some \( \pi \in I_\mathbb{Z} \) if and only if no word in the class has equal adjacent letters.

The theorem implies that there is a finite set \( \mathcal{A}^0(\pi) \subset S_\mathbb{Z} \) with \( \mathcal{R}^O(\pi) = \bigcup_{\sigma \in \mathcal{A}^0(\pi)} \mathcal{R}(\sigma) \) for each \( \pi \in I_\mathbb{Z} \). In prior related work, the sets \( \mathcal{R}^O(\pi) \) and \( \mathcal{A}^0(\pi) \) have usually been denoted "\( \hat{\mathcal{R}}(\pi) \)" and "\( \mathcal{A}(\pi) \)" or sometimes "\( \mathcal{W}(\pi) \)". Our present convention follows [43].

Definition 3.8. An orthogonal factorization is an increasing factorization of an involution word for some element of \( I_\mathbb{Z} \). Let \( \mathcal{R}^O_n(\pi) \) denote the set of increasing factorizations with \( n \) factors of involution words for \( \pi \in I_\mathbb{Z} \).

Let \( \pi \in I_\mathbb{Z} \). Then \( \mathcal{R}^O_n(\pi) = \bigsqcup_{\pi \in \mathcal{A}^0(\pi)} \mathcal{R}_n(\pi) \) is a finite, disjoint union of Morse-Schilling crystals, so is an abstract \( gl_n \)-crystal. Hiroshima [26, Appendix B] has identified two new operators \( f^O \) and \( e^O \) acting on this set. Given \( w = (w^1, w^2, \ldots, w^n) \in \mathcal{R}^O_n(\pi) \), define \( f^O(w) \) and \( e^O(w) \) as follows:

- If \( w^1 \neq \emptyset \) and its first letter \( x \) is smaller than every letter in \( w^2 \), then \( f^O(w) \) is formed from \( w \) by moving \( x \) from \( w^1 \) to \( w^2 \). Otherwise, \( f^O(w) := 0 \).
- If \( w^2 \neq \emptyset \) and its first letter \( x \) is smaller than every letter in \( w^1 \), then \( e^O(w) \) is formed from \( w \) by moving \( x \) from \( w^2 \) to \( w^1 \). Otherwise, \( e^O(w) := 0 \).

Note that Theorem 3.7 implies that the smallest letters of \( w^1 \) and \( w^2 \) can never be equal.

Proposition 3.9 ([26, Theorem B.2]). Relative to the queer crystal operators \( e_\pi = e^O \) and \( f_\pi = f^O \) just given, the abstract \( gl_n \)-crystal \( \mathcal{R}^O_n(\pi) \) is an abstract \( q_n \)-crystal for all \( \pi \in I_\mathbb{Z} \).

For an example of the crystal \( \mathcal{R}^O_n(\pi) \), see Figure 2.

Remark 3.10. The character of the crystal \( \mathcal{R}^O_n(\pi) \) is the polynomial \( \hat{F}_\pi(x_1, x_2, \ldots, x_n) \in \Gamma_n \), where \( \hat{F}_\pi := \sum_{\sigma \in \mathcal{A}^0(\pi)} F_\sigma \) is the involution Stanley symmetric function studied in [18]. Because the definition of \( F_\sigma \) in [18] differs from the one in [38] by inverting indices, it is not obvious that the formula for \( \hat{F}_\pi \) just given matches what is in [18]. However, this follows from [38, Lemma 5.3 and Corollary 5.10], which show that \( \sum_{\sigma \in \mathcal{A}^0(\pi)} F_\sigma = \sum_{\sigma \in \mathcal{A}^0(\pi)} F_{\sigma^{-1}} \).

The maximal term in the Schur \( P \)-expansion of \( \hat{F}_\pi \) relative to dominance order is given in [18, Theorem 1.13]. Similar to Remark 3.1 since \( P_{\mu}(x_1, x_2, \ldots, x_n) \neq 0 \) if and only if \( \ell(\mu) \leq n \), this result characterizes when \( \hat{F}_\pi(x_1, x_2, \ldots, x_n) \neq 0 \) and hence also when \( \mathcal{R}^O_n(\pi) \neq \emptyset \). Specifically, [18, Theorem 1.13] implies that \( \mathcal{R}^O_n(\pi) \) is nonempty if and only if the numbers \( \hat{c}_i(\pi) := |\{i < j \in \mathbb{Z} : \min\{i, \pi(i)\} \geq \pi(j)\}| \) in the involution code of \( \pi \) [18, Definition 4.6] are at most \( n \) for all \( i \in \mathbb{Z} \).

3.3 Symplectic factorizations

Let \( I^\text{FPF}_\mathbb{Z} \) denote the \( S_\mathbb{Z} \)-conjugacy class of the permutation \( 1^\text{FPF} \) of \( \mathbb{Z} \) that maps \( i \mapsto i - (-1)^i \). The elements \( \pi \in I^\text{FPF}_\mathbb{Z} \) are the fixed-point-free involutions of \( \mathbb{Z} \) with \( \pi(i) = 1^\text{FPF}(i) \) whenever \( |i| \) is sufficiently large. If \( n \) is even and \( \pi \in I_\mathbb{Z} \) restricts to a fixed-point-free involution of \( [n] \), then there is a unique element \( \hat{\pi} \in I^\text{FPF}_\mathbb{Z} \) that restricts to \( \pi \) on \( [n] \) and to \( 1^\text{FPF} \) on \( \mathbb{Z} \setminus [n] \). In examples, for convenience, we usually make no distinction between the permutations \( \pi \) and \( \hat{\pi} \).
**Definition 3.11.** An fpf-involution word for an element \( \pi \in \mathcal{I}^\text{EFF}_Z \) is a minimal-length sequence of integers \( i_1i_2\cdots i_l \) such that \( \pi = s_{i_l} \cdots s_{i_2} s_{i_1} \cdot 1_{\text{EFF}} \cdot s_{i_1} s_{i_2} \cdots s_{i_l} \). Let \( \mathcal{R}^{\text{Sp}}(\pi) \) denote the set of fpf-involution words for \( \pi \in \mathcal{I}^\text{EFF}_Z \).

One can view \( \mathcal{R}^{\text{Sp}}(\pi) \) as a “symplectic” analogue of \( \mathcal{R}(\pi) \). When \( n \) is even, the orbits of the symplectic group \( \text{Sp}_n(\mathbb{C}) \) acting on the type A flag variety \( \text{Fl}_n \) are in bijection with the set of involutions \( \pi \in \mathcal{I}^\text{EFF}_Z \) with \( \pi(i) = 1_{\text{EFF}}(i) \) for all \( i \notin [n] \). Words in \( \mathcal{R}^{\text{Sp}}(\pi) \) may be identified with maximal chains in a weak order on these orbits \([9, 51]\). Fpf-involution words are also instances of involutions \( \pi \).

The following is helpful for checking whether a given word is an fpf-involution word:

**Lemma 3.12 ([20 §2.3]).** An integer sequence \( i_1i_2\cdots i_l \) is an fpf-involution word for some element of \( \mathcal{I}^\text{EFF}_Z \) if and only if \( i_j \) is not a descent of \( s_{i_{j-1}} \cdots s_{i_2} s_{i_1} \cdot 1_{\text{EFF}} \cdot s_{i_1} s_{i_2} \cdots s_{i_{j-1}} \) for all \( j \in [l] \).

Since every odd integer is a descent of \( 1_{\text{EFF}} \), an fpf-involution word must start with an even letter. Let \( =_{\text{Sp}} \) denote the transitive closure of the Coxeter braid relations plus the symmetric relation on words that has \( w_1w_1-1w_2\cdots w_m =_{\text{Sp}} w_1(w_1+1)w_2\cdots w_m \) for any choice of \( w_1, w_2, \ldots, w_m \).

**Theorem 3.13 ([33 Theorem 2.4]).** If \( \pi \in \mathcal{I}^\text{EFF}_Z \) then \( \mathcal{R}^{\text{Sp}}(\pi) \) is an equivalence class under the relation \( =_{\text{Sp}} \). Moreover, an equivalence class of words under \( =_{\text{Sp}} \) is equal to \( \mathcal{R}^{\text{Sp}}(\pi) \) for some \( \pi \in \mathcal{I}^\text{EFF}_Z \) if and only if no word in the class has equal adjacent letters or starts with an odd letter.

This theorem implies that there is a finite set \( \mathcal{A}^{\text{Sp}}(\pi) \subset S_Z \) with \( \mathcal{R}^{\text{Sp}}(\pi) = \bigcup_{\sigma \in \mathcal{A}^{\text{Sp}}(\pi)} \mathcal{R}(\sigma) \) for each \( \pi \in \mathcal{I}^\text{EFF}_Z \). Our notation again follows the convention of [43], which differs from prior related work where \( \mathcal{R}^{\text{Sp}}(\pi) \) and \( \mathcal{A}^{\text{Sp}}(\pi) \) have usually been denoted “\( \mathcal{R}^{\text{EFF}}(\pi) \)” and “\( \mathcal{A}^{\text{EFF}}(\pi) \).”

**Definition 3.14.** A symplectic factorization is a reduced factorization of an fpf-involution word for some element of \( \mathcal{I}^\text{EFF}_Z \). Let \( \mathcal{R}^{\text{Sp}}_n(\pi) \) denote the set of increasing factorizations with \( n \) factors of fpf-involution words for \( \pi \in \mathcal{I}^\text{EFF}_Z \).

Fix \( \pi \in \mathcal{I}^\text{EFF}_Z \). Then \( \mathcal{R}^{\text{Sp}}_n(\pi) \) is again a disjoint union of Morse-Schilling crystals. Hiroshima [20 §5] has defined two other operators \( f^{\text{Sp}} \) and \( e^{\text{Sp}} \) acting on this set, analogous to \( f^O \) and \( e^O \) in the previous section. Given \( w = (w_1, w_2, \ldots, w^n) \in \mathcal{R}^{\text{Sp}}_n(\pi) \), form \( f^{\text{Sp}}(w) \) and \( e^{\text{Sp}}(w) \) as follows:

- If \( w^1 = \emptyset \) or \( \min(w^2) \leq \min(w^1) \), then \( f^{\text{Sp}}(w) := 0 \). Otherwise, let \( x = \min(w^1) \). If \( x+1 \) is not in \( w^1 \) then form \( f^{\text{Sp}}(w) \) from \( w \) by moving \( x \) from \( w^1 \) to \( w^2 \). If \( x+1 \) is in \( w^1 \) then form \( f^{\text{Sp}}(w) \) from \( w \) by removing \( x+1 \) from \( w^1 \) and adding \( x-1 \) to the start of \( w^2 \).
- If \( w^2 = \emptyset \) or \( \min(w^1) \leq \min(w^2) \), then \( e^{\text{Sp}}(w) := 0 \). Otherwise, let \( x = \min(w^2) \). If \( x \) is even then form \( e^{\text{Sp}}(w) \) from \( w \) by moving \( x \) from \( w^2 \) to \( w^1 \). If \( x \) is odd then form \( e^{\text{Sp}}(w) \) from \( w \) by removing \( x \) from \( w^2 \) and adding \( x+2 \) to \( w^1 \) (in the position yielding an increasing word).

Some observations are warranted. As in the orthogonal case, if \( w^1 \) and \( w^2 \) are both nonempty then it follows from Theorem 3.13 that \( \min(w^1) \neq \min(w^2) \). Likewise, in the definition of \( e^{\text{Sp}}(w) \), if \( x \) is odd and \( e^{\text{Sp}}(w) \neq 0 \), then \( x+2 \) cannot be a letter in \( w^1 \).

**Proposition 3.15 ([26 Theorem 5.1]).** Relative to the queer crystal operators \( c_T = e^{\text{Sp}} \) and \( f_T = f^{\text{Sp}} \) just given, the abstract \( \mathfrak{gl}_n \)-crystal \( \mathcal{R}^{\text{Sp}}_n(\pi) \) is an abstract \( \mathfrak{q}_n \)-crystal for all \( \pi \in \mathcal{I}^\text{EFF}_Z \).

See Figure 3 for an example of the crystal \( \mathcal{R}^{\text{Sp}}_n(\pi) \).
Definition 3.20. Let $\pi \in I_Z$ and $w = (w^1, \ldots, w^n) \in \mathcal{R}_n^O(\pi)$. Suppose concatenating the factors in $w$ gives the $m$-letter word $w_1 w_2 \cdots w_m$. Let $\emptyset = T_0, T_1, \ldots, T_m = P_{\mathcal{E}G}(w)$ be the sequence of shifted tableaux in which $T_i$ for $i \in [m]$ is formed from $T_{i-1}$ by inserting $w_i$ as follows:

\begin{itemize}
  \item Start by inserting $w_1$ into the first row. At each stage, an entry $x$ is inserted into a row. Let $y$ be the smallest entry in the row with $x \leq y$. If no such entry exists then $x$ is added to the end of the row. If $x = y$ then the current row is unchanged and $y + 1$ is inserted into the next row. If $x < y$ then $y$ is replaced by $x$ and $y$ is inserted into the next row.
\end{itemize}

We call $P_{\mathcal{E}G}(w)$ the *EG-insertion tableau* of $w$. The *EG-recording tableau* $Q_{\mathcal{E}G}(w)$ is the tableau with the same shape as $P_{\mathcal{E}G}(w)$ that contains $j$ in each of the boxes added by inserting the letters in the factor $w^j$, for each $j \in [n]$.

Example 3.18. We compute $P_{\mathcal{E}G}(w)$ and $Q_{\mathcal{E}G}(w)$ for $w = (4, 23, 2)$:

\begin{align*}
  4 & \sim 4 \sim 4 \sim 3 = P_{\mathcal{E}G}(w) \quad \text{and} \quad 3 \quad 2 \quad 1 \quad 2 = Q_{\mathcal{E}G}(w). 
\end{align*}

When $w = w_1 w_2 \cdots w_m \in \mathcal{R}(\pi)$, we set $P_{\mathcal{E}G}(w) := P_{\mathcal{E}G}((w_1, w_2, \ldots, w_m))$ and $Q_{\mathcal{E}G}(w) := Q_{\mathcal{E}G}((w_1, w_2, \ldots, w_m))$. That is, we treat words as factorizations with all factors of length one.

Theorem 3.19 (10). If $\pi \in S_Z$ then $w \mapsto (P_{\mathcal{E}G}(w), Q_{\mathcal{E}G}(w))$ is a bijection

\[ \mathcal{R}_n(\pi) \xrightarrow{\sim} \left\{ (P, Q) \text{ of tableaux of the same shape with } \begin{array}{l}
\text{pairs } (P, Q) \text{ of tableaux of the same shape with } \\
\text{P increasing, row}(P) \in \mathcal{R}(\pi), \text{ and } Q \in \text{Tab}_n(\ell(\pi)) \end{array} \right\}. \]

Moreover, $Q_{\mathcal{E}G}(w)$ is a standard tableau if and only if all factors of $w$ have size one.

We turn to our first shifted analogue of Definition 3.17.

Definition 3.21 (12). Let $\pi \in S_Z$ and $w = (w^1, \ldots, w^n) \in \mathcal{R}_n^O(\pi)$. Suppose concatenating the factors in $w$ gives the $m$-letter word $w_1 w_2 \cdots w_m$. Let $\emptyset = T_0, T_1, \ldots, T_m = P_{\mathcal{E}G}(w)$ be the sequence of shifted tableaux in which $T_i$ for $i \in [m]$ is formed from $T_{i-1}$ by inserting $w_i$ as follows:
Start by inserting \( w_i \) into the first row. At each stage, an entry \( x \) is inserted into a row or column. Let \( y \) be the smallest entry in the row or column with \( x \leq y \). If no such entry \( y \) exists then add \( x \) to the end of the row or column. If \( x = y \) then the row (respectively, column) is unchanged and \( y + 1 \) is inserted into the next row (respectively, column), with one exception. If \( x < y \) then \( y \) is replaced by \( x \) and \( y + 1 \) is inserted into the next row (respectively, column), again with one exception. The exceptions are that if \( x \) is inserted into a row and \( y \) is on the main diagonal, then we insert \( y + 1 \) (if \( x = y \)) or \( y \) (if \( x < y \)) into the next column.

If the orientation of insertion changes from rows to columns during this process, then we say that \( w_i \) is column-inserted; otherwise, \( w_i \) is row-inserted. We call \( P_{\text{EG}}^O(w) \) the orthogonal-EG-insertion tableau of \( w \). The orthogonal-EG-recording tableau \( Q_{\text{EG}}^O(w) \) is the shifted tableau with the same shape as \( P_{\text{EG}}^O(w) \) that contains \( j \) (respectively, \( j' \)) in each of the boxes added by a row-inserted (respectively, column-inserted) letter from the factor \( w^j \), for each \( j \in [n] \).

This algorithm is called {	extit{involution Coxeter-Knuth insertion}} in [BS] §4.3.

**Example 3.21.** We compute \( P_{\text{EG}}^O(w) \) and \( Q_{\text{EG}}^O(w) \) for \( w = (4, 23, 2, 1) \):

\[
\begin{array}{c}
4 & \sim & 2 & \sim & 4 & \sim & 4 & \sim & 1 & 2 & 3 & 1 = P_{\text{EG}}^O(w) \\
\end{array}
\]

\[
\begin{array}{c}
2 & \sim & 2 & 3 & 4 & = Q_{\text{EG}}^O(w).
\end{array}
\]

If \( w = w_1 \cdots w_m \in \mathcal{R}_n^O(\pi) \) is an involution word then we let \( P_{\text{EG}}^O(w) := P_{\text{EG}}^O((w_1, \ldots, w_m)) \) and \( Q_{\text{EG}}^O(w) := Q_{\text{EG}}^O((w_1, \ldots, w_m)) \). Define \( \ell^O(\pi) \) for \( \pi \in I_{\mathbb{Z}} \) to be the length of any word in \( \mathcal{R}_n^O(\pi) \). One has \( \ell^O(\pi) = \frac{1}{2}(\ell(\pi) + \kappa(\pi)) \) where \( \kappa(\pi) \) is the number of 2-cycles of \( \pi \) [BS §3].

**Theorem 3.22 (§13 Theorem 5.19).** If \( \pi \in I_{\mathbb{Z}} \) then \( w \mapsto (P_{\text{EG}}^O(w), Q_{\text{EG}}^O(w)) \) is a bijection

\[
\mathcal{R}_n^O(\pi) \sim \left\{ \text{pairs } (P, Q) \text{ of shifted tableaux of the same shape with} \right. \\
\left. P \text{ increasing, } \text{row}(P) \in \mathcal{R}_n^O(\pi), \text{and } Q \in \text{ShTab}_n(\ell^O(\pi)) \right\}.
\]

Moreover, \( Q_{\text{EG}}^O(w) \) is a standard shifted tableau if and only if all factors of \( w \) have size one.

Our second shifted analogue of Definition 3.17 relates to fpf-involution words.

**Definition 3.23 (§13).** Let \( \pi \in I_{\mathbb{Z}}^\text{fpf} \) and \( w = (w^1, w^2, \ldots, w^n) \in \mathcal{R}_n^\text{Sp}(\pi) \). Suppose concatenating the factors in \( w \) gives the \( m \)-letter word \( w_1 w_2 \cdots w_m \). Let \( \emptyset = T_0, T_1, \ldots, T_m = P_{\text{EG}}^\text{Sp}(w) \) be the sequence of shifted tableaux in which \( T_i \) for \( i \in [m] \) is formed from \( T_{i-1} \) by inserting \( w_i \) as follows:

Start by inserting \( w_i \) into the first row. At each stage, an entry \( x \) is inserted into a row or column. Let \( y \) be the smallest entry in the row or column with \( x \leq y \). If no such entry \( y \) exists then \( x \) is added to the end of the row or column. If \( x = y \) then the current row (respectively, column) is unchanged, and \( y + 1 \) is inserted into the next row (respectively, column). If \( x < y \) then \( y \) is replaced by \( x \) and \( y + 1 \) is inserted into the next row (respectively, column), except when \( x \) is inserted into a row and \( y \) is on the main diagonal. In this case, if \( y > x + 1 \) then \( y \) is replaced by \( x \) and \( y \) is inserted into the next column, while if \( y = x + 1 \) then the current row is unchanged and \( y + 1 \) is inserted into the next column.

If the orientation of insertion changes from rows to columns during this process, then we again say that \( w_i \) is column-inserted; otherwise, \( w_i \) is row-inserted. We call \( P_{\text{EG}}^\text{Sp}(w) \) the symplectic-EG-insertion tableau of \( w \). The symplectic-EG-recording tableau \( Q_{\text{EG}}^\text{Sp}(w) \) is the shifted tableau with the same shape as \( P_{\text{EG}}^\text{Sp}(w) \) that contains \( j \) (respectively, \( j' \)) in each of the boxes added by a row-inserted (respectively, column-inserted) letter from the factor \( w^j \), for each \( j \in [n] \).

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This algorithm is called \textit{FPF-involution Coxeter-Knuth insertion} in [38 §4.3].

**Example 3.24.** We compute $P_{\text{SpEG}}^w(w)$ and $Q_{\text{SpEG}}^w(w)$ for $w = (4, 23, 12)$:

$$
\begin{array}{cc}
4 & \sim \\
\downarrow & \updownarrow \\
2 & 4 & \sim \\
\downarrow & \updownarrow & \updownarrow \\
1 & 2 & 3 & \sim \\
\downarrow & \updownarrow & \updownarrow & \updownarrow \\
1 & 2 & 3 & 4 & \sim \\
\downarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
= P_{\text{SpEG}}^w(w) \quad \text{and} \quad 2 \ 3 \ 1 \ 2 \ 3 \ 4 \ 5 \ = Q_{\text{SpEG}}^w(w).
$$

**Remark 3.25.** Consider the procedures adding $w_i$ to $T_{i-1}$ in Definitions 3.17, 3.20, and 3.23. Each iteration of these processes inserts an entry $x$ into a row or column of a tableau of the same shape as $T_{i-1}$. The analysis of these algorithms in [10, 22, 38, 42] implies some additional properties:

- Each intermediate (shifted) tableau is itself increasing.
- If the current row or column contains $x$ prior to insertion, then it must also contain $x + 1$.
- In both shifted algorithms, if $x$ is inserted into a row and the row’s smallest entry $y$ is on the diagonal, then $x = y$ only if the diagonal position in the next row is occupied by $y + 2$.
- In the symplectic algorithm, if $x$ is inserted into a row and the row’s smallest entry $y \geq x$ is on the main diagonal, then $x$ is even or $x = y - 1$. The case with $x = y - 1$ can only occur if the diagonal position in the next row is occupied by $y + 2$.

If $w = w_1 \cdots w_m$ is an fpf-involution word then we define $P_{\text{SpEG}}^w(w) := P_{\text{SpEG}}^w((w_1, \ldots, w_m))$ and $Q_{\text{SpEG}}^w(w) := Q_{\text{SpEG}}^w((w_1, \ldots, w_m))$. Given $\pi \in I_{\mathbb{Z}}^{\text{FPF}}$, write $\ell_{\text{Sp}}(\pi)$ for the length of any word in $R_{\text{Sp}}^\pi(\pi)$. To compute this number, choose $m \in \mathbb{Z}_{\geq 0}$ with $\pi(i) = 1_{\text{FPF}}(i)$ for all $i > m$ and $i \leq -m$. If $\sigma \in I_{\mathbb{Z}}$ has $\sigma(i) = \pi(i)$ for $-m < i \leq m$ and $\sigma(i) = i$ otherwise, then $\ell_{\text{Sp}}(\pi) = \frac{1}{2}(\ell(\sigma) - m)$ [20 §2.3].

**Theorem 3.26 ([38 Theorem 4.5]).** If $\pi \in I_{\mathbb{Z}}^{\text{FPF}}$ then $w \mapsto (P_{\text{SpEG}}^w(w), Q_{\text{SpEG}}^w(w))$ is a bijection

$$
\mathcal{R}_{\text{Sp}}^\pi(\pi) \simeq \left\{ \begin{array}{l}
\text{pairs } (P, Q) \text{ of shifted tableaux of the same shape with } \\
P \text{ increasing, row}(P) \in R_{\text{Sp}}^\pi(\pi), \text{ and } Q \in \text{ShTab}_{\mathbb{n}}(\ell_{\text{Sp}}(\pi))
\end{array} \right\}.
$$

Moreover, $Q_{\text{SpEG}}^w(w)$ is a standard shifted tableau if and only if all factors of $w$ have size one.

Ending this subsection, we include a few remarks about the relationship between our insertion algorithms and similar maps in the literature. To start, Edelman-Greene insertion is a special case of \textit{Hecke insertion} [6] and coincides with the \textit{RSK correspondence} [7 §7.1] when restricted to (factorizations of) \textit{partial permutations}, i.e., words without repeated letters.

Orthogonal Edelman-Greene insertion, in turn, is a special case of \textit{shifted Hecke insertion} [22, 42] and coincides with Sagan-Worley insertion [47, 50] when restricted to (factorizations of) partial permutations. Symplectic Edelman-Greene insertion is a special case of \textit{symplectic Hecke insertion} [38] and gives another generalization of Sagan-Worley insertion.

Haiman’s mixed insertion is also closely related to Sagan-Worley insertion (and therefore to our shifted forms of Edelman-Greene insertion). We will say more about this connection in Section 5.1.
3.5 Main results

This section summarizes our main results, which relate our two shifted forms of Edelman-Greene insertion to the abstract $q_n$-crystals $R_n^O(\pi)$ and $R_n^S(\pi)$.

As motivation and to provide some needed background, we start by reviewing two theorems about Edelman-Greene insertion. Let $ck$ be the involution on 3-letter words that acts by

$$acb \leftrightarrow cab, \quad bea \leftrightarrow bac, \quad \text{and} \quad a(a + 1)a \leftrightarrow (a + 1)a(a + 1), \quad \text{if} \ a < b < c,$$

while fixing all other words. Given a word $w$ and $i \in [\ell(w) - 2]$, define $ck_i(w)$ to be the word obtained from $w$ by replacing the subword $w_iw_{i+1}w_{i+2}$ by its image under $ck$. For integers $i \notin [\ell(w) - 2]$ set $ck_i(w) := w$. For example, $13541 = ck_1(13541) = ck_2(15341) = ck_4(13541)$. Coxeter-Knuth equivalence $\sim$ is the transitive closure of the reflexive relation with $w \sim w$ for all $w$ and $i$.

**Theorem 3.27** ([10] §6). If $v$ and $w$ are reduced words, then $v \sim w$ if and only if $P_{EG}(v) = P_{EG}(w)$.

Morse and Schilling [41] have shown that $w \mapsto Q_{EG}(w)$ is a crystal morphism:

**Theorem 3.28** ([41] Theorem 4.11; see [7] §10]). Consider a permutation $\pi \in S_Z$. Then:

(a) The full $gl_n$-subcrystals of $R_n(\pi)$ are the subsets on which $P_{EG}$ is constant.

(b) The map $Q_{EG}$ is a quasi-isomorphism $R_n(\pi) \to Tab_n(\ell(\pi))$.

This theorem has some noteworthy corollaries. An abstract $gl_n$-crystal is normal (or Stembridge [49]) if each of its full subcrystals is isomorphic to a full $gl_n$-subcrystal of $W_n(m)$ for some $m$.

**Corollary 3.29** ([41]). Each set of factorizations $R_n(\pi)$ for $\pi \in S_Z$ is a normal $gl_n$-crystal.

An element $b \in B$ in an abstract $gl_n$-crystal is a highest weight if $e_i(b) = 0$ for all $i \in [n - 1]$. A connected normal $gl_n$-crystal $B$ has a unique highest weight $b$, and the value of $\text{wt}(b)$ for this element (after discarding trailing zeros) is always an integer partition [7] §4.4.

Edelman and Greene [10] proved that each Stanley symmetric function $F_\pi$ is a nonnegative integer linear combination of Schur functions. One can interpret these coefficients as follows:

**Corollary 3.30** ([41]). Suppose $\pi \in S_Z$ and $n \geq \ell(\pi)$. Then $F_\pi = \sum a_{\pi\lambda} s_\lambda$ where $a_{\pi\lambda}$ is the number of highest weights $w$ in the abstract $gl_n$-crystal $R_n(\pi)$ with $\text{wt}(w) = \lambda$.

Our main new results are “orthogonal” and “symplectic” analogues of Theorems 3.27 and 3.28. We start with the orthogonal case. Let $ck^O_0$ denote the operator on words given by

$$ck^O_0(w_1w_2w_3w_4\cdots w_m) := w_2w_1w_3w_4\cdots w_m \quad (3.2)$$

for any letters $w_i \in Z$. If $\ell(w) \leq 1$ then set $ck^O_0(w) := w$. Define orthogonal Coxeter-Knuth equivalence $O\sim$ to be the transitive closure of $\sim^K$ and the relation with $w \sim^K ck^O_0(w)$ for all words $w$.

**Theorem 3.31.** If $v$ and $w$ are involution words, then $v \sim^K w$ if and only if $P_{EG}(v) = P_{EG}(w)$.

This affirms [18] Conjecture 5.24, which is also stated as [38] Conjecture 4.13].

**Theorem 3.32.** Consider an involution $\pi \in I_Z$. Then:
(a) The full $q_n$-subcrystals of $R_n^O(\pi)$ are the subsets on which $P_{EG}^{O}$ is constant.

(b) The map $Q_{EG}^O$ is a quasi-isomorphism $R_n^O(\pi) \to \text{ShTab}_n(\ell^O(\pi))$.

We give the proofs of both theorems in Section 5.2. These results lead to interesting analogues of Corollaries 3.29 and 3.30. Following [4, 11], we say that an abstract $q_n$-crystal is normal if each of its full subcrystals is isomorphic to a full $q_n$-subcrystal of $W_n(m) \cong (\mathbb{B}_n)^{\otimes m}$ for some $m$.

**Corollary 3.33.** Each set of factorizations $R_n^O(\pi)$ for $\pi \in I_Z$ is a normal $q_n$-crystal.

The notion of a highest weight of an abstract $q_n$-crystal is subtler than for $\mathfrak{g}l_n$-crystals; see [13, §1.3] for the precise definition. Each connected normal $q_n$-crystal $B$ has a unique highest weight $b$, and it holds that $\text{wt}(b)$ is a strict partition and $\text{ch}(B) = P_{\text{wt}(b)}(x_1, x_2, \ldots, x_n)$ [13 Theorem 1.14].

It is shown in [18] that the symmetric functions $\hat{F}_{\pi}$ from Remark 3.10 are $\mathbb{Z}_{\geq 0}$-linear combinations of Schur $P$-functions. We can give a new interpretation of these coefficients:

**Corollary 3.34.** Suppose $\pi \in I_Z$ and $n \geq \ell^O(\pi)$. Then $\hat{F}_{\pi} = \sum b_{\pi\lambda} P_{\lambda}$ where $b_{\pi\lambda}$ is the number of highest weights $\lambda$ in the abstract $q_n$-crystal $R_n^O(\pi)$ with $\text{wt}(w) = \lambda$.

Parallel results hold in the symplectic case. For a word $w = w_1 w_2 \cdots w_m$ with $m \geq 2$, define

$$c_k^{SP}(w) := \begin{cases} w_1(w_1 \mp 1)w_2w_4 \cdots w_m & \text{if } w_2 = w_1 \pm 1 \\ w_2w_1w_3w_4 \cdots w_m & \text{if } w_1 - w_2 \text{ is even} \\ w & \text{otherwise.} \end{cases} \quad (3.3)$$

For words with $\ell(w) \leq 1$ we set $c_k^{SP}(w) := w$. Define symplectic Coxeter-Knuth equivalence $^{SP}$ to be the transitive closure of $\sim$ and the relation with $w \sim c_k^{SP}(w)$ for all words $w$.

Hiroshima [26, Theorem 4.4] has recently proved the following analogue of Theorem 3.31. His methods consist of a self-contained but very complicated analysis of symplectic-EG insertion. We give a conceptually simpler (but less self-contained) alternate proof in Section 5.3.

**Theorem 3.35** ([26]). If $v$, $w$ are fpf-involution words, then $v \sim^{SP} w$ if and only if $P_{EG}^{SP}(v) = P_{EG}^{SP}(w)$.

Next, we have a symplectic version of Theorem 3.32. This result implies [26, Conjecture 5.1].

**Theorem 3.36.** Consider an involution $\pi \in I_Z^{FPF}$. Then:

(a) The full $q_n$-subcrystals of $R_n^{SP}(\pi)$ are the subsets on which $P_{EG}^{SP}$ is constant.

(b) The map $Q_{EG}^{SP}$ is a quasi-isomorphism $R_n^{SP}(\pi) \to \text{ShTab}_n(\ell^{SP}(\pi))$.

The proof is also in Section 5.3. There are again a few corollaries worth noting.

**Corollary 3.37.** Each set of factorizations $R_n^{SP}(\pi)$ for $\pi \in I_Z^{FPF}$ is a normal $q_n$-crystal.

It is shown in [20] that the symmetric functions $\hat{F}_{\pi}^{FPF}$ from Remark 3.16 are $\mathbb{Z}_{\geq 0}$-linear combinations of Schur $P$-functions. We can give a new interpretation of these coefficients:

**Corollary 3.38.** Suppose $\pi \in I_Z^{FPF}$ and $n \geq \ell^{SP}(\pi)$. Then $\hat{F}_{\pi}^{FPF} = \sum c_{\pi\lambda} P_{\lambda}$ where $c_{\pi\lambda}$ is the number of highest weights $w$ in the abstract $q_n$-crystal $R_n^{SP}(\pi)$ with $\text{wt}(w) = \lambda$.  

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4 Bumping operators

Rather than attempting to prove our main theorems directly, our strategy is to construct certain $q_m$-crystal isomorphisms that translate each result to an equivalent but more tractable setting. These isomorphisms will be formed as compositions of the \textit{involution Little bumping operators} introduced in \cite{L}. We study the properties of these operators in this section.

4.1 Little bumps

We begin by reviewing some “classical” results about Little bumping operators from \cite{HY,LY}. For related background, see also \cite{L}, whose notational conventions we follow.

Let $w$ be a word, choose an index $i \in [\ell(w)]$, and define $\del_i(w)$ to be the subword of $w$ formed by omitting the $i$th letter. The pair $(w, i)$ is a \textit{$\pi$-marked word} for a permutation $\pi \in S_Z$ if $\del_i(w) \in \mathcal{R}(\pi)$. A marked word $(w, i)$ is \textit{reduced} if $w$ is a reduced word.

\textbf{Lemma 4.1 (\cite{L} Lemma 21).} If $(w, i)$ is an un-reduced $\pi$-marked word, then there exists a unique index $i \neq j \in [\ell(w)]$ such that $(w, j)$ is a $\pi$-marked word.

\textbf{Definition 4.2.} Fix $\pi \in S_Z$ and suppose $(w, i)$ is a $\pi$-marked word of length $m$. If $w$ is reduced then let $j = i$, and otherwise let $j \in [m]$ be the unique index with $i \neq j$ such that $(w, j)$ is a $\pi$-marked word. Then define $\push(w, i) := (v, j)$ where $v := w_1 \cdots w_{j-1}(w_j + 1)w_{j+1} \cdots w_m$.

If $(w, i)$ is a $\pi$-marked word then $\push^N(w, i)$ is reduced for some sufficiently large $N > 0$ \cite{LY} Lemma 5]. The \textit{Strong Exchange Condition} \cite[Theorem 5.8]{S}, moreover, implies that if $\pi \in S_Z$ and $w$ is a fixed reduced word, then $(w, i)$ is a $\pi$-marked word for at most one choice of $i$.

\textbf{Definition 4.3 (\cite{LY} §5).} For each $\pi \in S_Z$, the \textit{Little bumping operator} $b_{\pi}$ acts on reduced words $w$ as follows. If there exists an index $i$ such that $(w, i)$ is a $\pi$-marked word and $N > 0$ is minimal such that $\push^N(w, i) =: (v, j)$ is reduced, then $b_{\pi}(w) := v$. Otherwise, $b_{\pi}(w) := w$.

Our definition of $b_{\pi}$ gives the inverse of operator described in \cite[Algorithm 2]{LY}. In \cite[Theorem 4.4]{LY}, bumping operators act by decrementing letters, but here it is convenient to adopt the opposite convention. The \textit{descent set} of a word $w$ is $\mathrm{Des}(w) := \{i \in [\ell(w) - 1] : w_i > w_{i+1}\}$. The following theorem gathers together several properties of $b_{\pi}$ from papers of Little \cite{LY} and of Hamaker and Young \cite{HY}.

\textbf{Theorem 4.4 (\cite{HY,LY}).} Let $\pi, \sigma \in S_Z$ and $w \in \mathcal{R}(\sigma)$.

(a) The operator $b_{\pi}$ is a bijection $\bigsqcup_{\tau \in S_Z} \mathcal{R}(\tau) \rightarrow \bigsqcup_{\tau \in S_Z} \mathcal{R}(\tau)$.

(b) It holds that $\Des(b_{\pi}(w)) = \Des(w)$.

(c) For all $i \in \mathbb{Z}_{>0}$ it holds that $\ck_i(b_{\pi}(w)) = b_{\pi}(\ck_i(w))$.

(d) It holds that $Q_{\text{EG}}(b_{\pi}(w)) = Q_{\text{EG}}(w)$.

\textbf{Proof.} Part (a) is a weaker form of \cite[Lemma 7]{LY}. Parts (b), (c), and (d) are equivalent to \cite[Corollary 1, Lemma 3, and Proposition 1]{HY}, respectively.

Since $b_{\pi}$ preserve descents, it extends to an operator on reduced factorizations as follows. Given $\pi, \sigma \in S_Z$ and $w = (w^1, w^2, \ldots, w^n) \in \mathcal{R}_m(\sigma)$, define $b_{\pi}(w) = (v^1, v^2, \ldots, v^n)$ where the words $v^i$ are such that $b_{\pi}(w^1w^2 \cdots w^n) = v^1v^2 \cdots v^n$ and $\ell(v^i) = \ell(w^i)$. Since $\Des(v^i) = \Des(w^i)$, each $v^i$ is again strictly increasing. The following theorem appears to be new.

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Theorem 4.5. Let $\pi \in S_\mathbb{Z}$. Then $b_\pi$ is an isomorphism of $\mathfrak{gl}_n$-crystals $\bigsqcup_{\sigma \in S_\mathbb{Z}} \mathcal{R}_n(\sigma) \rightarrow \bigsqcup_{\sigma \in S_\mathbb{Z}} \mathcal{R}_n(\sigma)$.

Proof. Suppose $B$ is a full $\mathfrak{gl}_n$-subcrystal of $X := \bigsqcup_{\sigma \in S_\mathbb{Z}} \mathcal{R}_n(\sigma)$. Theorems 3.27 and 3.28 imply that $B$ consists of all $n$-fold increasing factorizations of reduced words in a single Coxeter-Knuth equivalence class. Theorem 3.3(c) implies that $C := b_\pi(B)$ is another full subcrystal of $X$. Let $\Lambda$ be the partition that is the common shape of the EG-recording tableaux for all factorizations in $B$ and $C$. By Theorem 4.4(d), the map $B \xrightarrow{\pi} C \xrightarrow{\pi} \mathcal{T}_n(\lambda)$ coincides with $B \xrightarrow{\pi \circ \pi} \mathcal{T}_n(\lambda)$. Since both $B \xrightarrow{\pi \circ \pi} \mathcal{T}_n(\lambda)$ and $C \xrightarrow{\pi} \mathcal{T}_n(\lambda)$ are crystal isomorphisms by Theorem 3.28 we conclude that $b_\pi : B \rightarrow C$ is a crystal isomorphism. By Theorem 4.4(a), $b_\pi$ is an isomorphism $X \rightarrow X$. \qed 

4.2 Involution Little bumps

Our next goal is to prove analogues of Theorems 4.4 and 4.5 for involution words. Fix $\pi \in I_\mathbb{Z}$ and recall that $A^0(\pi) \subset S_\mathbb{Z}$ is the set with $R^0(\pi) = \bigsqcup_{\sigma \in A^0(\pi)} R(\sigma)$. A $\pi$-marked involution word is a pair $(w, i)$ in which $w$ is word and $i$ is an index such that $\text{del}_i(w) \in R^0(\pi)$. Equivalently, this is just an $\alpha$-marked word for some $\alpha \in A^0(\pi)$. A $\pi$-marked involution word $(w, i)$ is inv-reduced if $w$ is an involution word for some element of $I_\mathbb{Z}$.

Lemma 4.6 ([19] Lemma 3.34). If $(w, i)$ is a $\pi$-marked involution word that is not inv-reduced, then there is a unique index $i \neq j \in [\ell(w)]$ such that $(w, j)$ is also a $\pi$-marked involution word.

Remark 4.7. A $\pi$-marked involution word $(w, i)$ may fail to be inv-reduced in two ways: either $w$ is a reduced word that is not an involution word, or $w$ is not reduced. In the latter case, the index $j$ identified in Lemma 4.6 is necessarily the same as the one in Lemma 4.4.

Definition 4.8. Let $(w, i)$ be a $\pi$-marked involution word of length $m$. If $(w, i)$ is inv-reduced then let $j = i$, and otherwise let $i \neq j \in [m]$ be the unique index such that $(w, j)$ is a $\pi$-marked involution word. Then define $\text{ipush}(w, i) := (v, j)$ where $v := w_1 \cdots w_{j-1}(w_j + 1)w_{j+1} \cdots w_m$.

As with the earlier push operator, one can show that if $(w, i)$ is a $\pi$-marked involution word then $\text{ipush}(w, i)$ is inv-reduced for some sufficiently large $N > 0$ [19] Lemma 3.37. By [19] Theorem 3.4 (see also [29] Theorem 2.8), it holds that if $\pi \in I_\mathbb{Z}$ and $w$ is a fixed involution word, then there exists at most one index $i$ such that $(w, i)$ is a $\pi$-marked involution word.

Definition 4.9 ([19] §3.3). The involution Little bumping operator $ib_\pi$ of $\pi \in I_\mathbb{Z}$ acts on involution words $w$ as follows. If there exists $i$ such that $(w, i)$ is a $\pi$-marked involution word and $N > 0$ is minimal such that $\text{ipush}^N(w, i) =: (v, j)$ is inv-reduced, then $ib_\pi(w) := v$. Otherwise, $ib_\pi(w) := w$.

The map $ib_\pi$ is the inverse of the operator $B_\pi$ in [19] Theorem 3.40.

Example 4.10. Let $\pi = (2, 5) \in I_\mathbb{Z}$, $\sigma = (1, 5) \in I_\mathbb{Z}$, and $w = 2134 \in R^0(\sigma)$. Then $234 \in R^0(\pi)$, so to compute $ib_\pi(w)$ we must find the minimal $N > 0$ such that $\text{ipush}^N(2134, 2)$ is inv-reduced. The following pictures show that $N = 4$:

```
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot \\
\end{array} \xrightarrow{\text{ipush}} \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot \\
\end{array} \xrightarrow{\text{ipush}} \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot \\
\end{array} \xrightarrow{\text{ipush}} \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot \\
\end{array} \xrightarrow{\text{ipush}} \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot \\
\end{array}
```

The third marked word is reduced but not inv-reduced, as $3234 \not\equiv 2334$. Thus $ib_\pi(w) = 3245$. 

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Lemma 4.11. Let \( \pi \in I_z \). For any involution word \( w \), there is a finite sequence of elements \( \alpha_1, \alpha_2, \ldots, \alpha_l \in \mathcal{A}^O(\pi) \) such that \( \mathbf{i}_\pi(w) = b_{\alpha_1} \cdots b_{\alpha_l}b_{\alpha_1}(w) \). Moreover, this sequence is the same for all involution words in a single Coxeter-Knuth equivalence class.

In Example 4.10 where \( \pi = (2, 5) \) and \( w = 2134 \) we have \( \mathbf{i}_\pi(w) = b_{\alpha_2}b_{\alpha_1}(w) \) for \( \alpha_1 = s_2s_3s_4 = (2, 3, 4, 5) \) and \( \alpha_2 = s_3s_2s_4 = (2, 4, 5, 3) \).

Proof. The first assertion holds by the remark after Lemma 4.6. Write \( \mathbf{\prec} \) for the covering relation in the Bruhat order on \( S_z \). Fix an involution word \( w \) and let \( \alpha_1, \alpha_2, \ldots, \alpha_l \in \mathcal{A}^O(\pi) \) be the sequence with \( \mathbf{i}_\pi(w) = b_{\alpha_l} \cdots b_{\alpha_2}b_{\alpha_1}(w) \). Define \( \sigma_i \in S_z \) such that \( w \in \mathcal{R}(\sigma_1) \) and \( b_{\alpha_{l-1}} \cdots b_{\alpha_2}b_{\alpha_1}(w) \in \mathcal{R}(\sigma_i) \) for \( 1 < i \leq l \). Lemma 4.6 implies that \( \alpha_1 \) is the unique element of \( \mathcal{A}^O(\pi) \) with \( \alpha_1 \prec \sigma_1 \). Write \( \mathbf{\prec} \) for the covering relation in the Bruhat order on \( A \). Lemma 4.12. Let \( \mu \) be a strict partition.

(1) Suppose we know that \( w \) is the row (respectively, column) reading word of an increasing shifted tableau of shape \( \mu \). Then we can recover \( \mu \) from \( \ell(w) \) and \( \text{Des}(w) \).

(2) For each strict partition \( \mu \), there are indices \( i_1, i_2, \ldots, i_p \in \mathbb{Z}_{>0} \) (depending only on \( \mu \)) such that \( c_{ik_1}c_{ik_2} \cdots c_{ik_p}(\text{row}(T)) = \text{col}(T) \) and \( c_{ik_p} \cdots c_{ik_2}c_{ik_1}(\text{col}(T)) = \text{row}(T) \) for all increasing shifted tableaux \( T \) of shape \( \mu \). We define

\[
\tau_{\mu}^{\text{col}} := c_{ik_1}c_{ik_2} \cdots c_{ik_p} \quad \text{and} \quad \tau_{\mu}^{\text{row}} := c_{ik_p} \cdots c_{ik_2}c_{ik_1}. \tag{4.1}
\]

Now suppose \( u = u_1u_2 \cdots u_n \) is a strictly increasing word with \( n > 0 \) and \( x \in \mathbb{Z} \).

(3) Assume \( ux \) is a reduced word and \( u_0 > x \). If \( i \) is minimal such that \( y := u_i \geq x \) then

\[
c_{ik_1}c_{ik_2} \cdots c_{kn-1}(ux) = \begin{cases} 
(y+1) \cdot u_1u_2 \cdots u_n & \text{if } x = y \\
u_1 \cdots u_{i-1} \cdot x \cdot u_{i+1} \cdots u_n & \text{if } x < y.
\end{cases}
\]

In particular, if \( x < u_1 \) then \( c_{ik_1}c_{ik_2} \cdots c_{kn-1}(ux) = u_1 \cdot x \cdot u_2 \cdots u_n \).

(4) If \( ux \) is an involution word (respectively, fpf-involution word) then \( x < u_1 \) if and only if a descent in position \( n \) occurs in \( c_{kn-2} \cdots c_{ik_1}c_{k_0}(ux) \) (respectively, \( c_{kn-2} \cdots c_{ik_1}c_{k_0}(ux) \)).

Finally, let \( v = v_1v_2 \cdots v_n \) be a strictly decreasing word with \( n > 0 \) and \( x \in \mathbb{Z} \).

(5) Assume \( xv \) is a reduced word and \( x < v_1 \). If \( i \) is maximal such that \( y := v_i \geq x \) then

\[
c_{i_k}c_{i_{k-1}} \cdots c_{i_1}(vx) = \begin{cases} 
v_1v_2 \cdots v_n \cdot (y+1) & \text{if } x = y \\
v_1 \cdots v_{i-1} \cdot x \cdot v_{i+1} \cdots v_n \cdot y & \text{if } x < y.
\end{cases}
\]

(6) If \( w := v_1xv_2 \cdots v_n \) is reduced and \( v_1 < x \), then \( c_{kn-1} \cdots c_{ik_1}(w) = vx \).
Proof. The derivation of each of these assertions is a straightforward exercise. The claims in part (2) follow from the discussion in [38, §2.2]. Part (4) is a consequence of Theorems 3.7 and 3.13. □

Theorem 4.13. Let \( \pi, \sigma \in I_Z \) and \( w \in R^O(\sigma) \). Then:

(a) The operator \( \mathfrak{i} \beta_\pi \) is a bijection \( \bigsqcup_{z \in I_Z} R^O(z) \to \bigsqcup_{z \in I_Z} R^O(z) \).

(b) It holds that \( \text{Des}(\mathfrak{i} \beta_\pi(w)) = \text{Des}(w) \).

(c) It holds that \( O^\pi_\mathfrak{i} (\mathfrak{i} \beta_\pi(c_k^O(w))) = \mathfrak{i} \beta_\pi(c_k^O(c_k^O(w))) \) and \( O^\pi_\mathfrak{i} (\mathfrak{i} \beta_\pi(c_k^O(w))) = \mathfrak{i} \beta_\pi(c_k^O(c_k^O(w))) \) for all \( i > 0 \).

(d) It holds that \( Q^O_{\mathfrak{Eg}}(\mathfrak{i} \beta_\pi(w)) = Q^O_{\mathfrak{Eg}}(w) \).

Proof. Part (a) is equivalent to [19, Theorem 3.40]. Part (b) is immediate from Theorem 4.13(b) and Lemma 4.11.

When \( i \) is a positive integer, the identity in part (c) follows from Theorem 4.13(c) and Lemma 4.11. It remains to show that \( \mathfrak{i} \beta_\pi \) commutes with \( c_k^O \) and \( c_k \) for \( i > 0 \). Hence, to prove part (d), it suffices to give an algorithm to compute \( Q^O_{\mathfrak{Eg}}(w) \) that only relies on the descent sets of words in the \( \sim \)-equivalence class of \( w \) as inputs. Such an algorithm will produce the same output for \( \mathfrak{i} \beta_\pi(w) \) as for \( w \) by parts (b) and (c), showing that \( Q^O_{\mathfrak{Eg}}(\mathfrak{i} \beta_\pi(w)) = Q^O_{\mathfrak{Eg}}(w) \).

Suppose \( w \) has the form \( w = w_1 w_2 \cdots w_{n+1} \) where \( n \in Z \geq 0 \). Specifically, we will describe an inductive procedure to produce the tableau \( Q^O_{\mathfrak{Eg}}(w) \) along with an operator \( \mathfrak{p} \) such that \( \mathfrak{p}(w) = \text{row}(P^O_{\mathfrak{Eg}}(w)) \). Each step in this algorithm will depend only on \( n \) and the descent sets of words obtained by applying certain fixed sequences orthogonal Coxeter-Knuth operators to \( w \). To follow our discussion it will be helpful to consult Examples 4.14, 4.16 and 4.15, which illustrate our notation in some particular cases.

If \( n = 0 \) then we always have \( Q^O_{\mathfrak{Eg}}(w) = 1 \) and \( \mathfrak{p} \) is given by the identity operator. Assume \( n > 0 \) and let \( P = P^O_{\mathfrak{Eg}}(w_1 w_2 \cdots w_n) \) and \( Q = Q^O_{\mathfrak{Eg}}(w_1 w_2 \cdots w_n) \). By induction, we may assume that \( Q \) is given and that we have an operator \( \sigma \) such that \( \sigma(w) = \text{row}(P)w_{n+1} \).

Let \( \mu = (\mu_1 > \mu_2 \cdots > \mu_r > 0) \) be the strict partition of \( n \) that is the shape of \( Q \). By Lemma 4.12(1), this partition can be computed from \( \text{Des}(\sigma(w)) \). For each \( i \in [r] \), let

\[
d_i := n - \mu_1 - \mu_2 - \cdots - \mu_{i-1} \quad \text{and} \quad \rho_i := c_k^O(c_k^O(\cdots c_k^O(1) \cdots c_k^O(0))) = c_k^O(c_k^O(\cdots c_k^O(0))) \quad \text{for } i > 1.
\]

Also set

\[
\rho_{r+1} := \begin{cases} 
\rho_r & \text{if } \mu_r = 1 \\
c_k^O(c_k^O(\cdots c_k^O(1) \cdots c_k^O(0))) \circ \rho_r & \text{if } \mu_r \geq 2.
\end{cases}
\]

It follows from Lemma 4.12(3) that if \( d_i \notin \text{Des}(\rho_i \circ \sigma(w)) \) for some \( i \in [r] \), and \( i \) is the minimal index with this property, then we have \( \text{row}(P^O_{\mathfrak{Eg}}(w)) = \mathfrak{p}(w) \) for the operator

\[
\mathfrak{p} := \rho_i \circ \sigma
\]
and $Q_{EG}(w)$ is formed from $Q$ by adding $n + 1$ to box $(i, i + \mu_i)$.

Assume $d_i \in \text{Des}(\rho_i \circ \sigma(w))$ for all $i \in [r]$. By Lemma 4.12(3)-(4), adding $w_{n+1}$ to $P$ ends in row-insertion if and only if $d_r \notin \text{Des}(\rho_{r+1} \circ \sigma(w))$, in which case $\text{row}(P_{EG}(w)) = \rho(w)$ for

$$p := \rho_{r+1} \circ \sigma$$

and $Q_{EG}(w)$ is formed from $Q$ by adding $n + 1$ to the diagonal box $(r + 1, r + 1)$. Otherwise, adding $w_{n+1}$ to $P$ must end in column-insertion. Assume we are in this case. For each $i \in [r - 1]$, let

$$\Delta(i) := 1 + 2 + \cdots + i$$

and set $\text{row-insertion if and only if }\Delta(i) \leq n + 1 \text{ inserts } y_{i+1}$ and define

$$x_i := \frac{\Delta(i) + 1}{2} := 1 + 2 + \cdots + (i + 1)$$

and $\phi_i := (\Delta(i) + 1) \cdots \Delta(i) + \Delta(i + 1)$. We compose these operators to define

$$\text{row-insert} := \psi_i \circ \cdots \circ \psi_1,$$

$$\text{reverse} := (ck_{r-1} \cdots ck_2ck_1)\sigma_{r}^{r},$$

$$\text{column-insert} := \phi_{r+1} \circ \cdots \circ \phi_1.$$ 

Finally, let $\nu$ be the strict partition formed by subtracting one from each part of $\mu$, suppose $i_1, i_2, \ldots, i_p \in \mathbb{Z}_{\geq 0}$ are such that $\tau_{\nu}^{\col} = ck_{i_1}ck_{i_2} \cdots ck_{i_p}$, and define

$$\text{reorient} := ck_{i_1+1}ck_{i_2+1} \cdots ck_{i_p+1}.$$ 

Here is what these operators do. Consider the insertion process outlined in Definition 3.20 that adds $w_{n+1}$ to $P$. At exactly one iteration in this process, a number $x$ is inserted into a row (of an increasing shifted tableau $T$ with shape $\mu$) whose smallest entry $y$ has $x \leq y$, and the next iteration proceeds by inserting either $y$ or $y + 1$ into the next column. Suppose this diagonal bump happens in row $j \in [r]$. Let $t_1t_2 \cdots t_r$ be the main diagonal of $T$ read bottom-to-top, so $x < y = t_j$. Write $U$ for the tableau of shape $\nu$ formed by removing the main diagonal of $T$ and let

$$\tilde{x} = \tilde{x} = \begin{cases} x & \text{if } x < y \\ y & \text{if } x = y \end{cases} \quad \text{and} \quad \tilde{y} = \tilde{y} = \begin{cases} y & \text{if } x < y \\ y + 1 & \text{if } x = y \end{cases} \quad \text{(4.2)}$$ 

(We introduce the symbols $\tilde{x}$ and $\tilde{y}$ in order to reuse the following identities in the symplectic case, where these variables will have a different meaning.) Using Lemma 4.12(3), we compute that

$$\sigma(w) = \text{row}(P)w_{n+1},$$

$$\text{row-insert} \circ \sigma(w) = t_r t_{r-1} \cdots t_{j+1} \cdot \tilde{y} \cdot t_j t_{j+1} \cdot \cdots \cdot \cdot \cdot t_1 \cdot \text{row}(U),$$

$$\text{reverse} \circ \text{row-insert} \circ \sigma(w) = t_1 t_2 \cdots t_{j-1} \cdot \tilde{x} \cdot \tilde{y} \cdot t_j t_{j+1} \cdot \cdots \cdot \cdot \cdot t_r \cdot \text{row}(U),$$

$$\text{reorient} \circ \text{reverse} \circ \text{row-insert} \circ \sigma(w) = t_1 t_2 \cdots t_{j-1} \cdot \tilde{x} \cdot \tilde{y} \cdot t_j t_{j+1} \cdot \cdots \cdot \cdot \cdot t_r \cdot \text{col}(U).$$

The insertion process adding $w_{n+1}$ to $P$ lasts for at least $r + 1$ iterations, and every iteration after the $j$th proceeds by column insertion. Suppose $z$ is the number and $V$ is the shifted tableau of shape $\mu$ such that iteration $r + 1$ inserts $z$ into column $r + 1$ of $V$. If we write $v_1v_2 \cdots v_n = \text{col}(V)$ and set $\Delta := 1 + 2 + \cdots + r$ then it follows from Lemma 4.12(5)-(6) that

$$\text{column-insert} \circ \text{reorient} \circ \text{reverse} \circ \text{row-insert} \circ \sigma(w) = v_1 v_2 \cdots v_{\Delta} \cdot z \cdot v_{\Delta + 1} v_{\Delta + 2} \cdots v_n. \quad \text{(4.3)}$$
Let $q = \mu_1$ and write $h_i$ for the number of boxes in column $i$ of $\text{SD}_\mu$, so that $h_i = i$ for $i \in [r]$ and $h_i = 0$ for $i > q$. Continue to let $\Delta := 1 + 2 + \cdots + r$. For $r + 1 \leq i \leq q + 1$ define
\[
\text{insert up to column}(i) := ck h_1 + h_2 + \cdots + h_{i-1} + 1 \cdot c k_{\Delta + 2} c k_{\Delta + 1}.
\]
Note that this gives the identity operator if $i = r + 1$ or if $i = r + 2$ and $h_{r+1} = 1$. Applying $\text{insert up to column}(i)$ to (4.3) has the effect of continuing the column insertion process described in Definition 3.20 past columns $r + 1, r + 2, \ldots , i - 1$. Then taking the column reading word; compare again with Example 4.15.

Define
\[
\text{total insertion}(i) := \text{insert up to column}(i) \circ \text{column insert} \circ \text{reorient} \circ \text{reverse} \circ \text{row insert}.
\]

Let $m \in \{r + 1, r + 2, \ldots , q\}$ be minimal with
\[
h_1 + h_2 + \cdots + h_{m-1} + 1 \in \text{Des(\text{total insertion}(m) \circ \sigma(w))},
\]
or if no such $m$ exists then set $m := q + 1$. It follows from Lemma 4.12(5) and (4.3) that $Q_{\text{EG}}^\mu(w)$ is formed from $Q$ by adding $n + 1'$ to the box $(h_m + 1, m)$, and if $\lambda$ is the strict partition shape of $Q_{\text{EG}}^\mu(w)$ then $\text{row}(P_{\text{EG}}^\mu) = \pi(w)$ for the operator
\[
\sigma := \tau_{\lambda}^{\text{ROW}} \circ \text{total insertion}(m) \circ \sigma.
\]

The shifted tableau $Q_{\text{EG}}^\mu(w)$ and the operator $\sigma$ have now been determined for all cases. As the outputs of the preceding algorithm are exchanged if the starting word $w$ is replaced by $ib_{\pi}(w)$, we conclude that $Q_{\text{EG}}^\mu(w) = Q_{\text{EG}}^\mu(ib_{\pi}(w))$ as desired. \hfill \Box

The following worked examples demonstrate the algorithm just given to compute $Q_{\text{EG}}^\mu(w)$. Our notation maintains the conventions in the proof above. The first example shows the relatively simple case when adding the final letter $w_{n+1}$ to $P_{\text{EG}}^\mu(w_1 w_2 \cdots w_n)$ ends in row insertion.

**Example 4.14.** Suppose $w = 354251234$ so that $n = \ell(w) - 1 = 8$. We have
\[
P := P_{\text{EG}}^\mu(w_1 w_2 \cdots w_8) = \begin{array}{ccccccc}
& & & & & 5 & \\
& & & 3 & 4 & & \\
& 1 & 2 & 3 & 4 & 5 & \\
\end{array}
\quad \text{and} \quad Q := Q_{\text{EG}}^\mu(w_1 w_2 \cdots w_8) = \begin{array}{ccccccc}
& & & & & 8 & \\
& & & 3 & 7 & & \\
& 1 & 2 & 3 & 4 & 5 & \\
\end{array}
\]
so $\mu = (5, 2, 1), r = 3, q = 5,$ and $d_1 = 8 > d_2 = 3 > d_3 = 1$. We assume that the operator $\sigma$ with
\[
\sigma(w) = \text{row}(P)w_{n+1} = \text{row} \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \ \\
\end{array} \quad 7 \quad \begin{array}{cccccc}
3 & 4 & 5 & 6 \ \\
\end{array}
\]
is given. Then
\[
\rho_1 \circ \sigma(w) = \text{row} \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \ \\
\end{array} \quad 4 \quad \begin{array}{cccccc}
5 & \ \\
3 & 4 \ \\
\end{array} = 534123454 \quad \text{has a descent at } d_1 = 8, \text{ but}
\]
\[
\rho_2 \circ \sigma(w) = \text{row} \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \ \\
\end{array} \quad 5 \quad \begin{array}{cccccc}
5 & \ \\
3 & 4 \ \\
\end{array} = 534512345 \quad \text{has no descent at } d_2 = 3.
\]
Therefore \( \text{row}(P_{\text{EG}}^O(w)) = p(w) \) for \( p := \rho_2 \circ o \) and \( Q_{\text{EG}}^O(w) \) is formed from \( Q \) by adding \( 9 = n + 1 \) to box \((2, 4) = (2, 2 + \mu_2) \). This is consistent with Definition 3.20 which gives

\[
P_{\text{EG}}^O(w) = \begin{pmatrix} 5 \\ 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \text{and} \quad Q_{\text{EG}}^O(w) = \begin{pmatrix} 8 \\ 3 & 7 & 9 \\ 1 & 2 & 4 & 5 & 6 \end{pmatrix}.
\]

The next example shows the case where inserting \( w_{n+1} \) into \( P_{\text{EG}}^O(w_1 w_2 \cdots w_n) \) adds a new row.

**Example 4.15.** Suppose \( w = 35425123 \) so that \( n = \ell(w) - 1 = 7 \). We have

\[
P := P_{\text{EG}}^O(w_1 w_2 \cdots w_7) = \begin{pmatrix} 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \text{and} \quad Q := Q_{\text{EG}}^O(w_1 w_2 \cdots w_7) = \begin{pmatrix} 3 & 7' \\ 1 & 2 & 4 & 5 & 6' \end{pmatrix}
\]

so \( \mu = (5, 2), r = 2, q = 5, \) and \( d_1 = 7 > d_2 = 2 \). We assume that the operator \( o \) with

\[
o(w) = \text{row}(P)w_{n+1} = \text{row} \begin{pmatrix} 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 3 \end{pmatrix} = 3512345
\]

is given. Then

\[
\rho_1 \circ o(w) = \text{row} \begin{pmatrix} 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 3 \end{pmatrix} = 35123453 \text{ has a descent at } d_1 = 7,
\]

\[
\rho_2 \circ o(w) = \text{row} \begin{pmatrix} 3 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 4 \end{pmatrix} = 35412345 \text{ has a descent at } d_2 = 2, \text{ but}
\]

\[
\rho_3 \circ o(w) = \text{row} \begin{pmatrix} 5 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 4 \end{pmatrix} = 53412345 \text{ has no descent at } d_2 = 2.
\]

Therefore \( \text{row}(P_{\text{EG}}^O(w)) = p(w) \) for \( p := \rho_3 \circ o \) and \( Q_{\text{EG}}^O(w) \) is formed from \( Q \) by adding \( 8 = n + 1 \) to the diagonal box \((3, 3) = (r+1, r+1) \). This is consistent with Definition 3.20 which gives

\[
P_{\text{EG}}^O(w) = \begin{pmatrix} 5 \\ 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \text{and} \quad Q_{\text{EG}}^O(w) = \begin{pmatrix} 8 \\ 3 & 7 \\ 1 & 2 & 4 & 5 & 6 \end{pmatrix}.
\]

We now consider a case where adding \( w_{n+1} \) to \( P_{\text{EG}}^O(w_1 w_2 \cdots w_n) \) ends in column insertion.

**Example 4.16.** Suppose \( w = 1528639742 \) so that \( n = \ell(w) - 1 = 9 \). We have

\[
P := P_{\text{EG}}^O(w_1 w_2 \cdots w_9) = \begin{pmatrix} 8 & 9 \\ 5 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad Q := Q_{\text{EG}}^O(w_1 w_2 \cdots w_9) = \begin{pmatrix} 6 & 9 \\ 3 & 5 & 8 \\ 1 & 2 & 4 & 7 \end{pmatrix},
\]

so \( \mu = (4, 3, 2), r = 3, q = 4, d_1 = 9 > d_2 = 5 > d_3 = 2 \). We assume that the operator \( o \) with

\[
o(w) = \text{row}(P)w_{n+1} = \text{row} \begin{pmatrix} 8 & 9 \\ 5 & 6 & 7 \\ 1 & 2 & 3 & 4 \\ 2 \end{pmatrix} = 8956712342
\]
is given. Then

\[ \rho_1 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & 9 & & \\ 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 2 \end{array} \right) = 8956712342 \text{ has a descent at } d_1 = 9, \]

\[ \rho_2 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & 9 & & \\ 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 3 \end{array} \right) = 8956731234 \text{ has a descent at } d_2 = 5, \]

\[ \rho_3 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & 9 & & \\ 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 5 \end{array} \right) = 8953671234 \text{ has a descent at } d_3 = 2, \text{ and} \]

\[ \rho_4 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & 9 & & \\ 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 \end{array} \right) = 9853671234 \text{ has a descent at } d_3 = 2. \]

This means that the process inserting \( w_{n+1} \) into \( P \) will end in column-insertion rather than row-insertion. Successively applying the operators \( \psi_i \) to \( \sigma(w) \) gives

\[ \psi_1 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & & & \\ 5 & 9 & & \\ 3 & 6 & 7 & \\ 1 & 2 & 3 & 4 \end{array} \right) = 8593671234, \]

\[ \psi_2 \psi_1 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & & & \\ 5 & 3 & 9 & \\ 1 & 6 & 7 & \\ 2 & 3 & 4 \end{array} \right) = 8539167234, \]

\[ \psi_3 \psi_2 \psi_1 \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & & & \\ 5 & 3 & 1 & 9 \\ 6 & 7 & \\ 2 & 3 & 4 \end{array} \right) = 8531967234, \]

so we have

\[ \text{row}_\text{insert} \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 8 & 5 & 3 & 1 \\ 6 & 7 & & \\ 2 & 3 & 4 & 9 \end{array} \right) = 8531967234, \]

\[ \text{reverse} \circ \text{row}_\text{insert} \circ \sigma(w) = \text{row} \left( \begin{array}{cccc} 1 & 3 & 5 & 8 \\ 6 & 7 & & \\ 2 & 3 & 4 & 9 \end{array} \right) = 1358967234, \]

\[ \text{reorient} \circ \text{reverse} \circ \text{row}_\text{insert} \circ \sigma(w) = \text{col} \left( \begin{array}{cccc} 1 & & & \\ 3 & 9 & 6 & 7 \\ 5 & & & \\ 8 & 2 & 3 & 4 \end{array} \right) = 1358263974. \]

Successively applying the operators \( \phi_i \) to the last word gives

\[ \text{reorient} \circ \text{reverse} \circ \text{row}_\text{insert} \circ \sigma(w) = \text{col} \left( \begin{array}{cccc} 3 & & & \\ 5 & 9 & 6 & 7 \\ 8 & 6 & 7 & \\ 1 & 2 & 3 & 4 \end{array} \right) = 1358263974, \]

\[ \phi_1 \circ \text{reorient} \circ \text{reverse} \circ \text{row}_\text{insert} \circ \sigma(w) = \text{col} \left( \begin{array}{cccc} 5 & 8 & 9 & \\ 3 & 6 & 7 & \\ 1 & 2 & 3 & 4 \end{array} \right) = 1325863974, \]

\[ \phi_2 \phi_1 \circ \text{reorient} \circ \text{reverse} \circ \text{row}_\text{insert} \circ \sigma(w) = \text{col} \left( \begin{array}{cccc} 8 & 9 & 6 & \\ 3 & 5 & 7 & \\ 1 & 2 & 3 & 4 \end{array} \right) = 1328536974, \]
so we have

\[
\text{column_insert} \circ \text{reorient} \circ \text{reverse} \circ \text{row_insert} \circ \circ(w) = \text{col} \left( \begin{array}{c}
6 \\
8 \\
9 \\
3 \\
5 \\
7 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \right) = 1328536974.
\]

Since \( r + 1 = q = 4 \) in this example, the operator \( \text{total_insertion}(i) \) is only defined for \( i \in \{4, 5\} \), and this gives

\[
\text{total_insertion}(4) \circ \circ(w) = \text{col} \left( \begin{array}{c}
6 \\
8 \\
9 \\
3 \\
5 \\
7 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \right) = 1328536974,
\]

\[
\text{total_insertion}(5) \circ \circ(w) = \text{col} \left( \begin{array}{c}
6 \\
8 \\
9 \\
3 \\
5 \\
6 \\
1 \\
2 \\
3 \\
4 \\
\end{array} \right) = 1328539647.
\]

As \( h_1 + h_2 + \cdots + h_r + 1 = 1 + 2 + 3 + 1 = 7 \) is not a descent of \( \text{total_insertion}(4) \circ \circ(w) \), we have \( m = q+1 = 5 \). Thus \( Q^O_{\text{EG}}(w) \) is formed from \( Q \) by adding \( 10' = n+1' \) to the box \( (1, 5) = (h_m+1, m) \), meaning that

\[
Q^O_{\text{EG}}(w) = \begin{array}{ccccccc}
6 & 9 \\
3 & 5 & 8 \\
1 & 2 & 4 & 7 & 10'
\end{array}.
\]

Additionally, we have \( \text{row}(P^O_{\text{EG}}(w)) = \text{p}(w) \) for \( \text{p} := \tau^\text{row} \circ \text{total_insertion}(5) \circ \circ \) where \( \lambda = (5, 3, 2) \) is the shape of \( Q^O_{\text{EG}}(w) \). This agrees with Definition 3.20 which gives \( P^O_{\text{EG}}(w) = \begin{array}{ccccccc}
8 & 9 \\
3 & 5 & 6 \\
1 & 2 & 3 & 4 & 7
\end{array} \).

Our next result is a variant of Theorem 4.5.

**Theorem 4.17.** Let \( \pi \in I_Z \). Then \( \text{ib}_\pi \) is an isomorphism of \( q_n \)-crystals \( \bigsqcup_{\sigma \in I_Z} R^O_n(\sigma) \rightarrow \bigsqcup_{\sigma \in I_Z} R^O_n(\sigma) \).

**Proof.** Theorems 3.17, 3.27 and 3.28 show that each full \( q_n \)-subcrystal of \( \mathcal{Y} := \bigsqcup_{\sigma \in I_Z} R^O_n(\sigma) \) is the set of all \( n \)-fold increasing factorizations of reduced words in a single Coxeter-Knuth equivalence class. Thus, it follows by combining Theorem 4.5, Lemma 4.11 and Theorem 4.12(a) that \( \text{ib}_\pi : \mathcal{Y} \rightarrow \mathcal{Y} \) is at least an isomorphism of abstract \( q_n \)-crystals.

Let \( w = (w^1, w^2, \ldots, w^n) \in \mathcal{Y} \). To show that \( \text{ib}_\pi \) is a \( q_n \)-crystal morphism, it is enough to check that \( f^O(w) \neq 0 \) if and only if \( f^O(\text{ib}_\pi(w)) \neq 0 \), and that in this case \( \text{ib}_\pi(f^O(w)) = f^O(\text{ib}_\pi(w)) \). Let \( p = \ell(w^1) \) and \( q = \ell(w^2) \). From the definitions in Section 3.2, it is easy to work out that \( f^O(w) \neq 0 \) if and only if \( p \) is neither zero nor a descent of \( v := \text{ck}_{p-2} \cdots \text{ck}_1 \text{ck}_1^O(w^1 w^2) \). In this case, if \( v^1 \) and \( v^2 \) are the words of length \( p-1 \) and \( q+1 \) such that \( v = v^1 v^2 \), then \( f^O(w) = (v^1, v^2, w^3, \ldots, w^n) \). As \( \text{ib}_\pi \) preserves descents and commutes with every \( \text{ck}_0^O \) and \( \text{ck}_i \) by Theorem 4.13 the claim follows.

### 4.3 Fixed-point-free Little bumps

The results in the previous section have a parallel story for ffp-involutory words, which we present here. Fix \( \pi \in I_Z^{\text{FPF}} \) and recall that \( A^{5p}(\pi) \subset S_Z \) is such that \( R^{5p}(\pi) = \bigsqcup_{\sigma \in A^{5p}(\pi)} R(\sigma) \). A \( \pi \)-marked
A **fpf-involution word** is a pair \((w, i)\) in which \(w\) is a word and \(i\) is an index such that \(\text{del}_i(w) \in \mathcal{R}^{S_p}(\pi)\). Equivalently, this is just an \(\alpha\)-marked word for some \(\alpha \in \mathcal{A}^{S_p}(\pi)\).

A \(\pi\)-marked fpf-involution word is **fpf-reduced** if \(w\) is an fpf-involution word. If \((w, i)\) is not fpf-reduced but \(w \in \mathcal{R}(\sigma)\) for some \(\sigma \in S_\mathbb{Z}\) with \(\sigma^{-1} \cdot \mathcal{I}_{FPF} \cdot \sigma = \pi\), then \((w, i)\) is semi-reduced.

**Lemma 4.18 ([19 Lemma 4.21]).** If \((w, i)\) is a \(\pi\)-marked fpf-involution word of length \(m\) that is neither fpf-reduced nor semi-reduced, then there is a unique index \(i \neq j \in [l(w)]\) such that \((w, j)\) is also a \(\pi\)-marked fpf-involution word; moreover, in this event \((w, j)\) is also not semi-reduced.

**Definition 4.19.** Let \((w, i)\) be a \(\pi\)-marked fpf-involution word of length \(m\). If \((w, i)\) is semi- or fpf-reduced, then let \(j = i\), and otherwise let \(i \neq j \in [m]\) be such that \((w, j)\) is a \(\pi\)-marked fpf-involution word. Then define \(\text{fpf}(w, i) := (v, j)\) where \(v := w_1 \ldots w_{j-1}(w_j + 1)w_{j+1} \ldots w_m\).

As with \(\text{push}\) and \(\text{ipush}\), if \((w, i)\) is a \(\pi\)-marked fpf-involution word then \(\text{fpf}(w, i)\) is fpf-reduced for some sufficiently large \(N > 0\) [19 Lemma 4.26]. Also, for any fixed \(\pi \in I_Z^{FPF}\) and fpf-involution word \(w\), at most one index \(i\) exists such that \((w, i)\) is a \(\pi\)-marked fpf-involution word.

**Definition 4.20 ([19 §4.3]).** The **fpf-involution Little bumping operator** \(\text{fb}_\pi\) of \(\pi \in I_Z^{FPF}\) acts on fpf-involution words \(w\) as follows. If \((w, i)\) is a marked fpf-involution word for some \(i\), and \(N > 0\) is minimal such that \(\text{fpf}^N (w, i) =: (v, j)\) is fpf-reduced, then \(\text{fb}_\pi(w) := v\). Otherwise, \(\text{fb}_\pi(w) := w\).

The map \(\text{fb}_\pi\) is the inverse of the operator \(\hat{\mathcal{I}}_{FPF}\) in [19 Theorem 4.29].

**Example 4.21.** Let \(\pi = (1, 2)(3, 6)(4, 5) \in I_Z^{FPF}, \sigma = (1, 4)(2, 5)(3, 6) \in I_Z^{FPF}\), and \(w = 243 \in \mathcal{R}^{S_p}(\sigma)\), so that \(43 \in \mathcal{R}^{S_p}(\pi)\). The values of \(\text{fpf}^N (243, 1)\) for \(0 \leq N \leq 6\) are as follows:

\[
\begin{array}{cccccccc}
2 & 4 & 3 & \text{fpf} & 3 & 4 & 3 & \text{fpf} & 4 & 5 & 3 & \text{fpf} & 4 & 5 & 4 & \text{fpf} & 4 & 5 & 5 & \text{fpf} & 4 & 6 & 5 \\
\end{array}
\]

The last marked word in this sequence is fpf-reduced, the second and fifth are semi-reduced, and the fourth is reduced but not fpf-reduced. We conclude that \(\text{fb}_\pi(w) = 465\).

We have an analogue of **Lemma 4.2** with almost the same proof.

**Lemma 4.2**. Let \(\pi \in I_Z^{FPF}\). For any fpf-involution word \(w\), there is a finite sequence of elements \(\alpha_1, \alpha_2, \ldots, \alpha_l \in \mathcal{A}^{S_p}(\pi)\) such that \(\text{fb}_\pi(w) = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot (w)\). Moreover, this sequence is the same for all fpf-involution words in a single Coxeter-Knuth equivalence class.

In Example 4.21 where \(\pi = (1, 2)(3, 6)(4, 5)\) and \(w = 243\) we have \(\text{fb}_\pi(w) = \alpha_4 \cdot \alpha_3 \cdot \alpha_2 \cdot \alpha_1 \cdot (w)\) for \(\alpha_1 = \alpha_2 = \alpha_3 = s_4 s_3 = (3, 5, 4)\) and \(\alpha_4 = s_4 s_5 = (4, 5, 6)\).

**Proof.** Fix an fpf-involution word \(w\) and let \(\alpha_1, \ldots, \alpha_l \in \mathcal{A}^{S_p}(\pi)\) be such that \(\text{fb}_\pi(w) = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot (w)\). Define \(\sigma_i \in S_\mathbb{Z}\) for \(i = 1, 2, \ldots, l\) as in the proof of **Lemma 4.2**. Then \(\alpha_1\) is the unique element of \(\mathcal{A}^{S_p}(\pi)\) with \(\alpha_1 \preceq \sigma_1\), and \(\alpha_i\) for \(i > 1\) is either \(\alpha_{i-1}\) when \(\sigma_{i-1}^{-1} \cdot \mathcal{I}_{FPF} \cdot \sigma_i = \pi\) or else the unique element of \(\mathcal{A}^{S_p}(\pi) \setminus \{\alpha_{i-1}\}\) with \(\alpha_i \preceq \sigma_i\). It follows that any word \(v \in \mathcal{R}(\sigma_1)\) with \(\beta_{\sigma_{i-1}} \cdot \alpha_{i-1} \cdot \beta_{\sigma_i} (v) = \mathcal{I}_{FPF}(v)\) for all \(1 \leq i \leq l\) also has \(\text{fb}_\pi(v) = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot (v)\). This holds if \(v \sim w\) by Theorem 4.4.

Recall the definition of \(\mathcal{C}^{S_p}_0\) from (3.3). Theorem 4.4 has another analogue for the maps \(\text{fb}_\pi\).
Theorem 4.23. Let $\pi, \sigma \in I^\text{EPP}_Z$ and $w \in R^\text{Sp}(\sigma)$. Then:

(a) The operator $fb_\pi$ is a bijection $\bigcup_{z \in I^\text{EPP}_Z} R^\text{Sp}(z) \to \bigcup_{z \in I^\text{EPP}_Z} R^\text{Sp}(z)$.

(b) It holds that $\text{Des}(fb_\pi(w)) = \text{Des}(w)$.

(c) For all $i \in \mathbb{Z}_{>0}$ it holds that $ck^\text{Sp}_0(fb_\pi(w)) = fb_\pi(ck^\text{Sp}_0(w))$ and $ck_i(fb_\pi(w)) = fb_\pi(ck_i(w))$.

(d) It holds that $Q^\text{Sp}_{\text{EG}}(fb_\pi(w)) = Q^\text{Sp}_{\text{EG}}(w)$.

Proof. Part (a) is equivalent to [19] Theorem 4.29, while part (b) is immediate from Theorem 4.4(b) and Lemma 4.22. Our proof of part (c) is similar to argument given for Theorem 4.13(c), but the details to check are more complicated. It is clear from Theorem 4.4(c) and Lemma 4.22 that if $i > 0$ then $fb_\pi(ck_i(w)) = fb_\pi(ck_i(w))$ for all fpf-involution words $w$.

It remains to consider the operator $ck^\text{Sp}_0$. Fix a word $w = w_1 w_2 \cdots w_m$ with $m \geq 2$, and suppose $i \in [m]$ is such that $(w, i)$ is a $\pi$-marked fpf-involution word. If $i = 1$ then $w_2$ must be even; if $i > 1$ then $w_1$ must be even; and if $i > 2$ then $w_2$ must either be even or equal to $w_1 \pm 1$. Let $w^\bullet := ck^\text{Sp}_0(w)$ and define $i^\bullet$ to be $3 - i$ if $w_2$ is even and $i \in \{1, 2\}$, and otherwise set $i^\bullet := i$. If $w$ is an fpf-involution word, so that $(w, i)$ is fpf-reduced, then it follows from Theorem 3.13 that $(w^\bullet, i^\bullet)$ is another $\pi$-marked fpf-involution word that is fpf-reduced.

Suppose the subword $w_1 w_2$ is an fpf-involution word. It suffices to show that there exist $N, N^\bullet > 0$ with the following properties: neither $\text{fpush}^M(w, i)$ nor $\text{fpush}^{M^*}(w^\bullet, i^\bullet)$ is fpf-reduced for any $0 < M < N$ or $0 < M^* < N^\bullet$, and if $\text{fpush}^N(w, i) = (v, j)$ then $v_1 v_2$ is an fpf-involution word and $\text{fpush}^{N^*}(w^\bullet, i^\bullet) = (v^\bullet, j^\bullet)$ where $v^\bullet := ck^\text{Sp}_0(v)$ and $j^\bullet$ is defined to be either $3 - j$ when $j \in \{1, 2\}$ and $v_2$ is even or else $j^\bullet := j$. If this holds then by repeating the same claim for $(v, j)$ and $(v^\bullet, j^\bullet)$, we deduce by induction that $fb_\pi(ck^\text{Sp}_0(w)) = ck^\text{Sp}_0(fb_\pi(w))$.

It is a straightforward exercise to check this claim. There are four possibilities for $N$ and $N^\bullet$:

- $N = N^\bullet = 1$ if $i = i^\bullet \notin \{1, 2\}$.
- $N = N^\bullet = 2$ if $i = 3 - i^\bullet \in \{1, 2\}$ and either $w_2 > w_1 + 1$ or $w_1 < w_2 - 1$.
- $N = 3$ and $N^\bullet = 1$ if $i = 1$ and $w_1 = w_2 - 2$, or $i = 2$ and $w_2 = w_1 + 1$.
- $N = 1$ and $N^\bullet = 3$ if $i = 2$ and $w_2 \in \{w_2 - 2, w_2 - 1\}$.

If $i = 1$ then $w_2 \notin \{w_2 \pm 1\}$ so these cases are exhaustive. The last two cases are more interesting. For example, $\text{fpush}^3(24, 1) = \text{fpush}^2(34, 1) = \text{fpush}(44, 1) = (45, 2)$ and $\text{fpush}(42, 2) = (43, 2)$ as

and the situation is analogous if we replace 24 with any word $w_1 w_2$ that has $w_2 = w_1 + 2$. Likewise, we have $\text{fpush}(23, 2) = (24, 2)$ and $\text{fpush}^3(21, 2) = \text{fpush}^2(22, 2) = \text{fpush}(32, 1) = (42, 1)$ as
and something similar happens if we replace 23 with any word \( w_1w_2 \) that has \( w_2 \in \{ w_1 - 2, w_1 + 1 \} \). We leave the details of the other cases to the reader.

Parts (a), (b), and (c) show that \( \mathbf{f}_\pi \) preserves descents and commutes with \( \text{ck}_0^\text{Sp} \) and \( \text{ck}_i \) for all \( i > 0 \). Choose an fpf-involution word \( w \). To prove part (d), it suffices to give an algorithm that computes \( Q_{\text{EG}}^\text{Sp}(w) \) using only the descent sets of words in the \( \text{Sp} \)-equivalence class of \( w \) (plus the sequences of symplectic Coxeter-Knuth moves transforming \( w \) to each word) as inputs.

An algorithm of this form exists with exactly the same description as the one given in the proof of Theorem 4.13(d); one just needs to replace all instances of the symbols \( \pi, \text{ck}_0, P_{\text{EG}}^\text{Sp}, Q_{\text{EG}}^\text{Sp} \) with \( \mathbf{f}_\pi, \text{ck}_0^\text{Sp}, P_{\text{EG}}^\text{Sp}, Q_{\text{EG}}^\text{Sp} \), respectively, and redirect any references to Definition 3.20 to Definition 3.23. After making these substitutions, every step in the proof of Theorem 4.13(d) holds verbatim, with the exception that one should also change (1.2) to define

\[
\tilde{x} = \begin{cases} x & \text{if } x < y \\ y & \text{if } x = y - 1 \end{cases}
\quad \text{and} \quad
\tilde{y} = \begin{cases} y & \text{if } x < y \\ y + 1 & \text{if } x = y - 1 \\ y + 3 & \text{if } x = y \end{cases}
\]

while keeping the same values of \( \tilde{x} \) and \( \tilde{y} \). The task of walking through the steps in the proof of Theorem 4.13(d) a second time, with these minor modifications, is straightforward and left to reader. The moral is that we can compute the tableau \( Q_{\text{EG}}^\text{Sp}(w) \) with an algorithm that executes in exactly the same way when \( w \) is replaced by \( \mathbf{f}_\pi(w) \), so we must have \( Q_{\text{EG}}^\text{Sp}(w) = Q_{\text{EG}}^\text{Sp}(\mathbf{f}_\pi(w)) \).

Finally, we have a symplectic version of Theorem 4.17.

**Theorem 4.24.** Let \( \pi \in I_n^\text{FFP} \). Then \( \mathbf{f}_\pi \) is an isomorphism of \( q_n \)-crystals

\[
\bigsqcup_{\sigma \in I_n^\text{FFP}} \mathcal{R}_n^\text{Sp}(\sigma) \rightarrow \bigsqcup_{\sigma \in I_n^\text{FFP}} \mathcal{R}_n^\text{Sp}(\sigma).
\]

**Proof.** By Theorems 3.13, 3.27, and 3.28 each full \( \mathfrak{gl}_n \)-subcrystal of \( \mathcal{Z} := \bigsqcup_{\sigma \in I_n^\text{FFP}} \mathcal{R}_n^\text{Sp}(\sigma) \) is the set of all \( n \)-fold increasing factorizations of reduced words in a single Coxeter-Knuth equivalence class. Theorem 4.17, Lemma 4.22, and Theorem 4.23(a) therefore imply that \( \mathbf{f}_\pi : \mathcal{Z} \rightarrow \mathcal{Z} \) is an isomorphism of abstract \( \mathfrak{gl}_n \)-crystals.

Let \( w = (w_1, w_2, \ldots, w_n) \in \mathcal{Z} \). To show that \( \mathbf{f}_\pi \) is a \( q_n \)-crystal morphism, it is enough to check that \( f^\text{Sp}(w) \neq 0 \) if and only if \( f^\text{Sp}(\mathbf{f}_\pi(w)) \neq 0 \), which case \( \mathbf{f}_\pi(f^\text{Sp}(w)) = f^\text{Sp}(\mathbf{f}_\pi(w)) \). Let \( p = \ell(w_1) \) and \( q = \ell(w_2) \). In the orthogonal case, we have \( f^\text{Sp}(w) \neq 0 \) if and only if \( p > 0 \) and \( p \) is not a descent of \( v := \text{ck}_{p-2} \cdots \text{ck}_1 \text{ck}_0^\text{Sp}(w_1w_2) \), which we interpret as \( v := w_1w_2 \) when \( p = 1 \). If this happens, then it follows by definition that \( f^\text{Sp}(w) = (v_1, v_2, w_3, \ldots, w_n) \) where \( v_1 = v_2 = \) the words of length \( p - 1 \) and \( q + 1 \) such that \( v = v_1v_2 \). Since \( \mathbf{f}_\pi \) preserves descents and commutes with \( \text{ck}_0^\text{Sp} \) and \( \text{ck}_i \), we have \( f^\text{Sp}(w) = f^\text{Sp}(\mathbf{f}_\pi(w)) = 0 \) or \( \mathbf{f}_\pi(f^\text{Sp}(w)) = f^\text{Sp}(\mathbf{f}_\pi(w)) \neq 0 \), as needed.

## 5 Proofs of the main results

Here, we leverage our results in the previous section to give complete proofs of Theorems 3.31, 3.32, 3.35, and 3.36. We then describe some other applications and open problems.
5.1 Reduction to permutations

In this section, a permutation of a list of distinct numbers $a_1, a_2, \ldots, a_m$ is a word of length $m$ containing each $a_i$ as a letter exactly once.

Fix $m,n \in \mathbb{Z}_{\geq 0}$. Let $\text{Perm}(m) \subset \mathcal{W}_m(m)$ be the set of permutations of $1, 2, \ldots, m$ and define $\text{Perm}_n(m)$ to be the set of all $n$-fold increasing factorizations of words in $\text{Perm}(m)$. Let $\text{Even}(m)$ be the set of permutations of $2, 4, 6, \ldots, 2m$ and write $\text{Even}_n(m)$ for the set of $n$-fold increasing factorizations of words in $\text{Even}(m)$. Finally, let $\Sigma(m)$ be the set of involutions $\sigma \in I_Z$ of the form

$$\sigma = (1, m) \quad \text{or} \quad \sigma = (1, i_1)(i_1 - 1, i_2)(i_2 - 1, i_3) \cdots (i_k - 1, m)$$

for some $1 < i_1 - 1 < i_1 < i_2 - 1 < i_2 < \cdots < i_k - 1 < i_k < m$. Both sets are $q_n$-crystals:

**Proposition 5.1.** One has $\text{Perm}_n(m) = \bigsqcup_{\sigma \in \Sigma(m)} \mathcal{R}_n^O(\sigma)$ and $\text{Even}_n(m) = \mathcal{R}_n^O(\tau) = \mathcal{R}_n^{sp}(\pi)$ where $\tau = s_2 s_4 s_6 \cdots s_{2n} \in I_Z$ and $\pi \in I_Z^{FPF}$ is the involution with $\pi(i) = 1_{FPF}(i)$ for all $i \notin [2m]$ that maps

$$i \mapsto \begin{cases} i + 2 & \text{if } i \in \{1, 2m - 2\} \\
-1 & \text{if } i \in \{3, 2m\} \
\end{cases} \quad \text{and} \quad i \mapsto \begin{cases} i + 3 & \text{if } i \text{ is even and } 1 < i < 2m - 2 \\
-3 & \text{if } i \text{ is odd and } 3 < i < 2m. \
\end{cases}$$

Moreover, the abstract $q_n$-crystal structures on $\mathcal{R}_n^O(\tau)$ and $\mathcal{R}_n^{sp}(\pi)$ coincide.

**Proof.** Note that $\pi = s_2 s_4 s_6 \cdots s_{2n} \cdot 1_{FPF} \cdot s_2 s_4 s_6 \cdots s_{2n} \in I_Z^{FPF}$. Clearly $\text{Perm}(m)$ is a union of equivalence classes under the relation $=_{O}$, while $\text{Even}(m)$ is a single equivalence class under $=_{O}$ and $=_{sp}$. Theorems 3.7 and 3.13 imply that $\text{Perm}_n(m) = \bigsqcup_{\sigma \in \Sigma(m)} \mathcal{R}_n^O(\sigma)$ for a finite set $\Sigma(m) \subset I_Z$ while $\text{Even}_n(m) = \mathcal{R}_n^O(\tau) = \mathcal{R}_n^{sp}(\pi)$. Checking that $\Sigma(m)$ is the given set is straightforward. \qed

Fix $w = (w^1, w^2, \ldots, w^n) \in \text{Perm}_n(m)$. Define $w^{-1} \in \mathcal{W}_n(m)$ to be the word of length $m$ whose $i$th letter is the index $j \in [n]$ of the factor $w^j$ that contains $i$ as a letter. Then form $2[w] \in \text{Even}_n(m)$ by doubling the letters in each component of $w$. For example, $(245, 0, 1, 3)^{-1} = 31411$. We write $\text{inv}$ and $\text{dbl}$ for the corresponding maps $\text{Perm}_n(m) \to \mathcal{W}_n(m)$ and $\text{Perm}_n(m) \to \text{Even}_n(m)$.

The following lemma shows that the $q_n$-crystal structures on $\text{Perm}_n(m)$ and $\text{Even}_n(m)$ afforded by Proposition 5.1 are isomorphic to the crystal of words $\mathcal{W}_n(m) \cong (\mathcal{B}_n)\bigotimes^m$.

**Lemma 5.2.** Fix $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$, and $w \in \text{Perm}_n(m)$. The following properties hold:

(a) The map $\text{inv} : \text{Perm}_n(m) \to \mathcal{W}_n(m)$ is a $q_n$-crystal isomorphism.

(b) The map $\text{dbl} : \text{Perm}_n(m) \to \text{Even}_n(m)$ is a $q_n$-crystal isomorphism.

(c) We have $P_{\text{HEG}}(w) = Q_{\text{HM}}(w^{-1})$ and $Q_{\text{HEG}}(w) = Q_{\text{HM}}(2[w]) = Q_{\text{HEG}}^{sp}(2[w]) = P_{\text{HM}}(w^{-1})$.

**Proof.** The inverse of $\text{inv}$ is the map that sends an $m$-letter word $v = v_1 v_2 \cdots v_m \in \mathcal{W}_n(m)$ to the tuple $(w^1, w^2, \ldots, w^n)$ in which $w^j$ is the increasing word whose letters are the positions of $j$ in $v$. To prove part (a), it suffices to check, for each $i \in \{1, 2, \ldots, n - 1\}$ and $w = (w^1, w^2, \ldots, w^n) \in \text{Perm}_n(m)$, that $f_i(w) \neq 0$ if and only if $f_i(w^{-1}) \neq 0$, in which case $f_i(w)^{-1} = f_i(w^{-1})$.

If $i = 1$ so that $f_1$ acts on $\text{Perm}_n(m)$ as the operator $f^O$, then this is clear by definition. Assume $i \in [n - 1]$. If $f_i(w)$ is nonzero then it is formed from $w$ by removing some letter from $w^i$ and adding the same letter to $w^{i+1}$. On the other hand, we have $(a, b) \in \text{pair}(w^i, w^{i+1})$ if and only if $a$ and $b$ are the positions of matching left and right parentheses in the word formed from $w^1 w^2 \cdots w^n$
by replacing each $i$ by “$j$” and each $i+1$ by “($i$). After comparing these observations with the definitions of $f_i$ for each crystal, the desired claim is evident. This completes the proof of part (a).

Part (b) and the identity $Q_{\text{EG}}(w) = Q_{\text{EG}}(2[w]) = Q_{\text{Sp}}(2[w])$ are obvious from the definitions. Let $w = (w^1, w^2, \ldots, w^n) \in \text{Perm}_n(m)$ and define $\underline{w} \in \text{Perm}_n(m)$ to be the factorization obtained by dividing $w^1 w^2 \cdots w^n$ into subwords of length one. Let $\phi : \{1' < 1 < \cdots < m' < m\} \rightarrow \{1' < 1 < \cdots < n' < n\}$ be the map that assigns $i \mapsto j$ and $i' \mapsto j'$ if the $i$th letter of $w^1 w^2 \cdots w^n$ is part of $w^j$. Then $P_{\text{EG}}(w) = P_{\text{EG}}(\underline{w})$ and $Q_{\text{EG}}(w) = \phi \circ Q_{\text{EG}}(\underline{w})$ by definition, and it is easy to see that $Q_{\text{HM}}(w^{-1}) = Q_{\text{HM}}(\underline{w}^{-1})$ and $P_{\text{HM}}(w^{-1}) = \phi \circ P_{\text{HM}}(\underline{w}^{-1})$. Thus, it suffices to show that $P_{\text{EG}}(w) = Q_{\text{HM}}(w^{-1})$ and $Q_{\text{EG}}(w) = P_{\text{HM}}(\underline{w}^{-1})$, but this is [16, Theorem 6.10] as orthogonal-EG insertion applied to $\underline{w}$ coincides with Sagan-Worley insertion [16, Definition 6.1].

**Example 5.3.** If $w = (\emptyset, 36, 1245) \in \text{Perm}_3(6)$ then $w^{-1} = 332332$ and we compute

\[
\begin{align*}
\begin{array}{ccccccccccc}
3 & \sim & 3 & 6 & \sim & 1 & 3 & 6 & \sim & 1 & 2 & 4 & 6
\end{array}
\end{align*}
\]

\[
\begin{align*}
&= P_{\text{EG}}(w) \quad \text{and} \quad
\begin{array}{ccccccccccc}
3 & 3 & 6 & \sim & 1 & 2 & 4 & 6 & \sim & 1 & 2 & 4 & 6
\end{array}
\end{align*}
\]

Comparing with Example 2.10 shows that $P_{\text{EG}}(w) = Q_{\text{HM}}(w^{-1})$ and $Q_{\text{EG}}(w) = P_{\text{HM}}(\underline{w}^{-1})$.

### 5.2 Proofs in the orthogonal case

This section contains our proofs of Theorems 3.31 and 3.32.

**Definition 5.4.** An involution $\pi \in I_Z \setminus \{1\}$ is inv-Grassmannian if

$$
\pi = (m + 1, m + r + \mu_r)(m + 2, m + r + \phi_{r-1}) \cdots (m + r, m + r + \mu_1)
$$

for some $m \in \mathbb{Z}$ and some strict partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_r > 0)$. In this case, the strict partition $\mu$ is the shape of $\pi$. We also consider $\pi = 1$ to be inv-Grassmannian with shape $\mu = \emptyset$.

The map $\pi \mapsto (\mu, m)$ is a bijection from nontrivial inv-Grassmannian involutions to nonempty strict partitions paired with nonnegative integers.

Given a word $w = w_1 w_2 \cdots w_m$ and a map $\pi : Z \rightarrow Z$, let

$$
w^* = (-w_1)(-w_2) \cdots (-w_m) \quad \text{and} \quad \pi^* : i \mapsto 1 - \pi(1 - i).
$$

Then $\pi \mapsto \pi^*$ is the unique automorphism of $S_Z$ sending $s_i \mapsto s_{-i}$, while $w \mapsto w^*$ is a bijection $\mathcal{R}(\pi) \rightarrow \mathcal{R}(\pi^*)$. Also, if $\pi \in I_Z$ then $\pi^* \in I_Z$ and $\mathcal{R}^O(\pi^*) = \{w^* : w \in \mathcal{R}^O(\pi)\}$.

An element $\pi \in I_Z$ is inv-Grassmannian if and only if $\pi^*$ is I-Grassmannian in the sense of [18, §4.1]. It follows from [18, §4.1] that if $\pi \in I_Z$ is inv-Grassmannian of shape $\mu$ then $\ell^O(\pi) = |\mu|$. On the other hand, the operator $i_b \pi$ for $\pi \in I_Z$ is the inverse of the involution Little map in [19, §3.3]. After adjusting for these symmetries, the results in [18, 19, 38] imply the following:

**Lemma 5.5** ([18, 19, 38]). Suppose $w$ is an involution word for some permutation in $I_Z$ and $\pi \in I_Z$ is an inv-Grassmannian involution of shape $\mu$.

(a) There is a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_l \in I_Z$ such that $i_b \sigma_1 i_b \sigma_2 \cdots i_b \sigma_l(w)$ is an involution word for an inv-Grassmannian element of $I_Z$.

(b) The set $\mathcal{R}^O(\pi)$ is a single equivalence class under $\sim$, and $Q_{\text{EG}}$ is a bijection from $\mathcal{R}^O(\pi)$ to the set of standard shifted tableaux of shape $\mu$.  

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Proof. The inverse of \( \text{ib}_\sigma \) for \( \sigma \in I_Z \) is the operator given by \( \text{ib}_\sigma^{-1} : v \mapsto (\text{ib}_\sigma^{-1}(v^*))^* \). Combining [19] Theorem 3.40] (where \( \text{ib}_\sigma^{-1} \) is denoted \( \check{B}_\sigma \)) and [18] Theorem 4.36] shows that for any involution word \( v \), there exists a finite sequence \( y_1, y_2, \ldots, y_l \in I_Z \) and an inv-Grassmannian involution \( \gamma \in I_Z \) with \( (\text{ib}_{y_1} \text{ib}_{y_2} \cdots \text{ib}_{y_l}(v^*))^* = \text{ib}_{y_l}^{-1} \text{ib}_{y_{l-1}}^{-1} \cdots \text{ib}_{y_1}^{-1}(v) \in \mathcal{R}_O(\gamma^*). \) Taking \( v = w^* \) and \( \sigma_i = y_i^* \) for \( i \in [l] \) proves part (a). For part (b), we observe that [38] Corollaries 5.9 and 5.10 and [18] Theorem 4.20] imply that exactly one increasing shifted tableau \( P \) exists with \( \text{row}(P) \in \mathcal{R}_O(\pi) \), and this tableau has shape \( \mu \). The desired claims hold by Theorem [38], Corollary 4.12], which asserts that if involution words \( v \) and \( w \) have \( P_{\text{E}_G}^O(v) = P_{\text{E}_G}^O(w) \) then \( v \sim w \).

This lemma lets us prove Theorem 3.31 from Section 3.5.

Proof of Theorem 3.31] Let \( v \) and \( w \) be involution words. If \( P_{\text{E}_G}^O(v) = P_{\text{E}_G}^O(w) \), then \( v \sim w \) holds by [38] Corollary 4.12]. We must prove the converse. Assume \( v \sim w \). By Lemma 5.5(a), a finite sequence of operators \( \text{ib}_{\sigma_1}, \text{ib}_{\sigma_2}, \ldots, \text{ib}_{\sigma_l} \) transforms \( v \) to \( \tilde{v} \in \mathcal{R}_O(\pi) \) for some inv-Grassmannian permutation \( \pi \in I_Z \). Let \( \tilde{w} \) be the word obtained by applying the same operators to \( w \). Theorem 4.13 implies that \( Q_{\text{E}_G}^O(v) = Q_{\text{E}_G}^O(\tilde{v}) \), \( Q_{\text{E}_G}^O(w) = Q_{\text{E}_G}^O(\tilde{w}) \), and \( \tilde{v} \sim \tilde{w} \). Therefore \( \tilde{w} \in \mathcal{R}_O(\pi) \), so by Lemma 5.5(b) the shifted tableaux \( Q_{\text{E}_G}^O(v) \) and \( Q_{\text{E}_G}^O(w) \) must have the same shape. Hence, by Theorem 3.22] there is a unique involution word \( u \) such that \( P_{\text{E}_G}^O(u) = P_{\text{E}_G}^O(v) \) and \( Q_{\text{E}_G}^O(u) = Q_{\text{E}_G}^O(w) \). It suffices to show that \( u = w \). Let \( \tilde{u} = \text{ib}_{\sigma_1} \text{ib}_{\sigma_2} \cdots \text{ib}_{\sigma_l}(u) \). Since \( u \sim v \sim w \), Theorem 4.13] implies that \( \tilde{u} \in \mathcal{R}_O(\pi) \) and \( Q_{\text{E}_G}^O(\tilde{u}) = Q_{\text{E}_G}^O(u) = Q_{\text{E}_G}^O(w) = Q_{\text{E}_G}^O(\tilde{w}) \). It therefore follows from Lemma 5.5(b) that \( \tilde{u} = \tilde{w} \), so \( u = w \) since each \( \text{ib}_\sigma \) is injective.

Let \( t_m \) for \( m \in \mathbb{Z} \) be the operator acting on words \( w = w_1 w_2 \cdots w_n \) and maps \( \pi : \mathbb{Z} \to \mathbb{Z} \) by

\[
t_m(w) = (w_1 + m)(w_2 + m) \cdots (w_n + m) \quad \text{and} \quad t_m(\pi) : i \mapsto \pi(i - m) + m.
\]

(5.1)

If \( w = (w^1, w^2, \ldots, w^n) \) is an \( n \)-tuple of words then set \( t_m(w) = (t_m(w^1), t_m(w^2), \ldots, t_m(w^n)) \). As an operator on words, \( t_m \) preserves descents and commutes with \( c_{k_0} \) and \( c_k \) for all \( i > 0 \). If \( \pi \in I_Z \) is any involution, then \( t_m(\pi) \in I_Z \) and the map \( w \mapsto t_m(w) \) is obviously an isomorphism of abstract \( q_n \)-crystals \( \mathcal{R}_n^O(\pi) \to \mathcal{R}_n^O(t_m(\pi)) \). Moreover, we clearly have \( Q_{\text{E}_G}(w) = Q_{\text{E}_G}(t_m(w)) \) for all \( w \in \mathcal{R}_n^O(\pi) \) and \( t_m(\text{ib}_\pi) = \text{ib}_m(\pi) t_m \) for all \( \pi \in I_Z \).

We can now upgrade Lemma 5.5 to the following statement.

Lemma 5.6. If \( w \) is an involution word then there is a finite sequence \( \sigma_1, \sigma_2, \ldots, \sigma_l \in I_Z \) and an integer \( m \in \mathbb{Z} \) such that the word \( \text{ib}_{\sigma_1} \text{ib}_{\sigma_2} \cdots \text{ib}_{\sigma_l} t_m(w) \) is a permutation of \( 2, 4, 6, \ldots, 2l(w) \).

Proof. Let \( \mu = (\mu_1 > \mu_2 > \cdots > \mu_r > 0) \) be a strict partition of \( n \) and consider the inv-Grassmannian permutation \( \pi_\mu : (1, r + \mu_r)(2, r + \mu_{r-1}) \cdots (r, r + \mu_1) \in I_Z \). In view of Theorem 4.13] and Lemma 5.5] it suffices to produce a finite sequence \( \sigma_{\mu,1}, \sigma_{\mu,2}, \ldots, \sigma_{\mu,l} \in I_Z \) such that \( \text{ib}_{\sigma_{\mu,1}} \text{ib}_{\sigma_{\mu,2}} \cdots \text{ib}_{\sigma_{\mu,l}}(v) \in \text{Even}(n) \) for some (and therefore every) word \( v \in \mathcal{R}_O(\pi_\mu) \).

If \( r = 0 \) then we set \( l = 0 \). Suppose \( r > 0 \), let \( q = \mu_1 \geq r \), and write \( h_i \) for the number of boxes in the \( i \)th column of \( SD_\mu \). Let \( \nu \) be the strict partition with \( SD_\nu = SD_\mu \setminus \{(h_q,q)\} \). Assume by induction that \( \sigma_{\nu,1}, \sigma_{\nu,2}, \ldots, \sigma_{\nu,k} \in I_Z \) are given such that \( \text{ib}_{\sigma_{\nu,1}} \text{ib}_{\sigma_{\nu,2}} \cdots \text{ib}_{\sigma_{\nu,k}}(v) \in \text{Even}(n - 1) \) for all involution words \( v \in \mathcal{R}_O(\pi_\nu) \). We claim that the desired sequence is

\[
\begin{align*}
(\sigma_{\mu,1}, \sigma_{\mu,2}, \ldots, \sigma_{\mu,l}) &= (s_{2n}\sigma_{\nu,1}, s_{2n}\sigma_{\nu,2}, \ldots, s_{2n}\sigma_{\nu,k}, \pi_{\mu_1}, \pi_{\mu_2}, \ldots, \pi_{\mu_r}) \\
&\quad\text{for } 2n - q - r + 1 \text{ times}
\end{align*}
\]

(5.2)
To prove this, let
\[
w^i := \begin{cases} (2i - 1)(2i - 2) \cdots i & \text{for } 1 \leq i \leq r \\ (2i + 1)(2i + 2) \cdots (2i - h_i) & \text{for } r < i \leq q \end{cases}
\]
and define \( w := w^1w^2 \cdots w^q \). Using Lemma 3.35 one can check that \( w \in \mathcal{R}^O(\pi_\mu) \). We claim more specifically that if \( \sigma_{\mu,i} \) is defined as in (5.2) then
\[
i^{\sigma_{\mu,i}}: \mathbb{Z}^{\mu,i} \to \mathbb{Z}^{\mu,i}(w) = a_1^1a_2^2 \cdots a_q^q \in \text{Even}(n)
\]
where \( a_i^j \) is the word formed by adding \( 2h_{i,j} \) to \( w \). It is a straightforward exercise to show that \( w(0) = \text{ib}_{\pi_\nu}(w) \) and \( w(i) = \text{ib}_{\pi_\nu}(w(i-1)) \) for each \( i > 0 \). Thus, the word \( w' := (\text{ib}_{\pi_\nu})^{2n - q - r + 1}(w) \) is obtained from \( w \) by replacing either its last letter or its largest letter by \( 2n \), and then moving \( 2n \) to be in position \( n - h_q + 1 \). Moreover, removing \( 2n \) from \( v \) yields a word \( v \in \mathcal{R}^O(\pi_\nu) \).

Let \( \sigma'_{\pi,i} := s_{2n} \sigma_{\pi,i} \). Our key observation is now that the words \( \text{ib}_{\pi_{\nu,i}} \text{ib}_{\pi_{\nu,i+1}} \cdots \text{ib}_{\pi_{\nu,k}}(v) \) and \( \text{ib}_{\pi_{\nu,i}} \text{ib}_{\pi_{\nu,i+1}} \cdots \text{ib}_{\pi_{\nu,k}}(v') \) have exactly the same relationship as \( v \) and \( v' \) for all \( i \in [k] \); the second word is the same as the first but with \( 2n \) inserted in position \( n - h_q + 1 \). Given this fact, the desired identity (5.3) follows by induction, which completes our proof of the lemma.

Given \( n \)-tuples of words \( u = (u^1, u^2, \ldots, u^n) \) and \( v = (v^1, v^2, \ldots, v^n) \), write \( u \sim v \) if it holds that \( u^1u^2 \cdots u^n \sim v^1v^2 \cdots v^n \). Using all of our results so far, we can now prove Theorem 3.32.

**Proof of Theorem 3.32.** Recall that \( \pi \in \mathcal{I}_\sigma \). Choose a factorization \( w \in \mathcal{R}^O(\pi) \) and let \( \mathcal{C} \) be the full \( q_n \)-subcrystal of \( \mathcal{R}^O(\pi) \) containing \( w \). From Theorem 5.27, Theorem 3.28(a), and the definitions of \( f^O \) and \( e^O \), it is clear that a factorization \( v \in \mathcal{R}^O(\pi) \) belongs to \( \mathcal{C} \) only if \( v \sim w \). Part (a) of Theorem 3.32 is equivalent to the converse statement, which is not yet evident.

By Lemma 5.6, we have \( \text{ib}_{\pi_\sigma_1} \text{ib}_{\pi_\sigma_2} \cdots \text{ib}_{\pi_\sigma_l} t_m(w) \in \text{Even}_n(f^O(\pi)) \) for some \( \sigma_1, \sigma_2, \ldots, \sigma_l \in \mathcal{I}_\sigma \) and \( m \in \mathbb{Z} \). Write \( i := \text{ib}_{\pi_\sigma_1} \text{ib}_{\pi_\sigma_2} \cdots \text{ib}_{\pi_\sigma_l} t_m \). Since each \( \text{ib}_{\pi_\sigma} \) and \( t_m \) preserves \( \mathcal{Q} \), it follows that \( i(v) \sim i(w) \) and \( i(v) \in \text{Even}_n(f^O(\pi)) \) for all \( v \in \mathcal{C} \). Thus, by Theorem 4.13 and Lemma 5.2, the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & \text{Even}_n(f^O(\pi)) \\
\downarrow Q_{\mathcal{E}C} & & \downarrow Q_{\mathcal{E}C} \\
\text{ShTab}_n(f^O(\pi)) & \xrightarrow{\text{dbl}^{-1}} & \text{Perm}_n(f^O(\pi)) \\
\downarrow & & \downarrow \\
\text{Perm}_n(f^O(\pi)) & \xrightarrow{\text{inv}} & \mathcal{W}_n(f^O(\pi)) \\
\end{array}
\]

commutes. The map \( i : \mathcal{C} \to \text{Perm}_n(f^O(\pi)) \) is a quasi-isomorphism of abstract \( q_n \)-crystals by Theorem 4.17 both \( \text{dbl}^{-1} \) and \( \text{inv} \) are isomorphisms by Lemma 5.2 and the map \( P_{\mathcal{HM}} : \mathcal{W}_n(m) \to \)
ShTab_{n}(m) is a quasi-isomorphism by construction. Therefore $Q_{\text{EG}}^O : R_{\text{O}}^O(\pi) \to \text{ShTab}_{n}(\ell^O(\pi))$ is a quasi-isomorphism of abstract $q_n$-crystals. This proves (b).

To prove (a), suppose $v \in R_{\text{O}}(\pi)$ has $O \sim w$. Let $\tilde{v} := \text{dbl}^{-1} \circ i(v)$ and $\tilde{w} := \text{dbl}^{-1} \circ i(w)$. Then $\tilde{v} \sim \tilde{w}$, so $Q_{\text{HM}}(\tilde{v}^{-1}) = P_{\text{EG}}^O(\tilde{v}) = P_{\text{EG}}^O(\tilde{w}) = Q_{\text{HM}}(\tilde{w}^{-1})$. By Theorem-Definition 2.12, we deduce that $\tilde{v}^{-1}$ and $\tilde{w}^{-1}$ belong to the same full $q_n$-subcrystal of $W_n(\ell^O(\pi))$. Since the maps $i$, $\text{dbl}^{-1}$, and $\text{inv}$ send full subcrystals to full subcrystals, we must have $v \in C$. \hfill \square

5.3 Proofs in the symplectic case

This section contains our proofs of Theorems 3.35 and 3.36.

Definition 5.7. Given $\pi \in I_{Z}^\text{FFP}$, define $\hat{\pi} \in I_{Z}$ to be the involution with

$$\hat{\pi}(i) = \begin{cases} i & \text{if no } j \in Z \text{ with } \min\{i, \pi(i)\} < j < \max\{i, \pi(i)\} \text{ has } j < \pi(j) \\ \pi(i) & \text{otherwise} \end{cases}$$

for each $i \in Z$. An element $\pi \in I_{Z}^\text{FFP} \setminus \{1_{\text{FFP}}\}$ is fpf-Grassmannian if $\hat{\pi} \in I_{Z}$ is inv-Grassmannian. In this case, if $\hat{\pi}$ has shape $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, then the shape of $\pi$ is $\nu = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_r - 1)$. We also consider $1_{\text{FFP}}$ to be fpf-Grassmannian with shape $\nu = \emptyset$.

To recover $\pi \in I_{Z}^\text{FFP}$ from $\hat{\pi} \in I_{Z}$, let $i \mapsto f_i$ be an order-preserving map from $Z$ to the fixed-points of $\hat{\pi}$ such that $i \equiv f_i \pmod{2}$ for all sufficiently large $i$. Then $\pi(f_i) = f_{i+1}$ and $\pi(f_{i+1}) = f_i$ for all odd $i \in Z$, while $\pi(i) = \hat{\pi}(i)$ for all $\hat{\pi}(i) \neq i \in Z$.

If $\pi \in I_{Z}^\text{FFP}$ then $\pi^* : i \mapsto 1 - \pi(1 - i)$ is an element of $I_{Z}^\text{FFP}$ and $w \mapsto w^*$ is again a bijection $R_{\text{Sp}}(\pi) \to R_{\text{Sp}}(\pi^*)$. An involution $\pi \in I_{Z}^\text{FFP}$ is fpf-Grassmannian if and only if $\pi^*$ is fpf-Grassmannian in the sense of [20] §4.

Lemma 5.8 ([20] [19] [38]). Suppose $w$ is an fpf-involution word for some permutation in $I_{Z}^\text{FFP}$ and $\pi \in I_{Z}^\text{FFP}$ is an fpf-Grassmannian involution of shape $\nu$.

(a) There is a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_l \in I_{Z}^\text{FFP}$ such that $f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_l}(w)$ is an fpf-involution word for an fpf-Grassmannian element of $I_{Z}^\text{FFP}$.

(b) The set $R_{\text{Sp}}(\pi)$ is a single equivalence class under $\sim_{\text{Sp}}$, and $Q_{\text{EG}}^\text{Sp}$ defines a bijection from $R_{\text{Sp}}(\pi)$ to the set of standard shifted tableaux of shape $\nu$.

Proof. The structure of the proof is the same as for Lemma 5.5. The inverse of $f_{\sigma}$ for $\sigma \in I_{Z}^\text{FFP}$ is the operator $f_{\sigma}^{-1} : v \mapsto (f_{\sigma^*}(v^*))^*$. Combining [19] Theorem 4.29] (where $f_{\sigma^{-1}}$ is denoted $B_{\sigma}$) and [20] Theorem 5.21] shows that for any fpf-involution word $v$, there exists a finite sequence $y_1, y_2, \ldots, y_l \in I_{Z}^\text{FFP}$ and an fpf-Grassmannian involution $\gamma \in I_{Z}^\text{FFP}$ such that $(f_{y_1} f_{y_2} \cdots f_{y_l}(v^*))^* = f_{y_l^{-1}} f_{y_{l-1}}^{-1} \cdots f_{y_1}^{-1}(v) \in R_{\text{Sp}}(\gamma^*)$. Taking $v = w^*$ and $\sigma_i = y_i^*$ for $i \in [l]$ proves part (a). Next, [38] Corollaries 5.9 and 5.10] and [20] Lemma 4.16 and Theorem 4.19] imply that exactly one increasing shifted tableau $P$ exists with $\text{row}(P) \in R_{\text{Sp}}(\pi)$, and this tableau has shape $\nu$. Part (b) therefore follows from Theorem 3.26 and [38] Corollary 3.22]. \hfill \square

We can give a second proof of Theorem 3.35 using the preceding lemma.
Proof of Theorem 3.35. If $v$ and $w$ are fpf-involution words with $P_{\mathcal{E}_G}^{\mathcal{S}_P}(v) = P_{\mathcal{E}_G}^{\mathcal{S}_P}(w)$, then Corollary 3.22 implies that $v \sim w$. The converse follows by same argument as in the proof of Theorem 3.31, using Theorem 4.23 and Lemma 5.8 in place of Theorem 1.13 and Lemma 5.5, and substituting the respective symbols $I_{\mathcal{E}_G}^{\mathcal{S}_P}, P_{\mathcal{E}_G}^{\mathcal{S}_P}, Q_{\mathcal{E}_G}^{\mathcal{S}_P}$, $\mathcal{R}_n^{\mathcal{S}_P}, \mathcal{O}_n^{\mathcal{S}_P}, \mathcal{R}_n^{\mathcal{S}_P}$.

Recall the operator $t_m$ from (5.1). We have $t_m(I_{\mathcal{E}_G}^{\mathcal{S}_P}) = I_{\mathcal{E}_G}^{\mathcal{S}_P}$ if and only if $m \in \mathbb{Z}$ is even. Assume this is the case; then it is easy to see that $t_m$ commutes with $c_{\mathcal{S}_P}$ and that $w \mapsto t_m(w)$ is an isomorphism of abstract $q_n$-crystals $\mathcal{R}_n^{\mathcal{S}_P}(\pi) \to \mathcal{R}_n^{\mathcal{S}_P}(t_m(\pi))$ for all $\pi \in I_{\mathcal{E}_G}^{\mathcal{S}_P}$.

Moreover, we clearly have $Q_{\mathcal{E}_G}^{\mathcal{S}_P}(w) = Q_{\mathcal{E}_G}^{\mathcal{S}_P}(t_m(w))$ for all $w \in \mathcal{R}_n^{\mathcal{S}_P}$ and $t_m f_{\pi} = f_{t_m(\pi)} t_m$.

There is a symplectic analogue of Lemma 5.6.

Lemma 5.9. Suppose $w$ is an fpf-involution word. There is a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_l \in I_{\mathcal{E}_G}^{\mathcal{S}_P}$ and an even integer $m \in 2\mathbb{Z}$ such that $f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_l} t_m(w)$ is a permutation of $2, 4, 6, \ldots, 2\ell(w)$.

Proof. Let $\mu = (\mu_1 > \mu_2 > \cdots > \mu_r > 0)$ be a strict partition of $n$ and define $\theta_\mu$ to be the element of $I_{\mathcal{E}_G}^{\mathcal{S}_P}$ with $\theta_\mu = (1, r + 1 + \mu_r) / 2$ so that $\theta_\mu$ is fpf-Grassmannian with shape $\mu$. It suffices by Theorem 4.23(c) and Lemma 5.5 to construct a finite sequence $\sigma_{\mu,1}, \sigma_{\mu,2}, \ldots, \sigma_{\mu,l} \in I_{\mathcal{E}_G}^{\mathcal{S}_P}$ such that $f_{\sigma_{\mu,1}} f_{\sigma_{\mu,2}} \cdots f_{\sigma_{\mu,l}} (v) \in \text{Even}(n)$ for some (and therefore every) word $v \in \mathcal{R}_n^{\mathcal{S}_P}(\theta_\mu)$.

Our argument is similar to one in the proof of Lemma 5.6. If $r = 0$ then we again set $l = 0$. Suppose $r > 0$, let $q = \mu_1$, write $h_i$ for the number of boxes in the $i$th column of $\text{SD}_\mu$, and define $\nu$ to be the strict partition with $\text{SD}_\nu = \text{SD}_\mu \setminus \{(h_q, q)\}$. Let $\epsilon \in \{0, 1\}$ be such that $q + r + \epsilon$ is odd, and assume that $\sigma_{\nu,1}, \sigma_{\nu,2}, \ldots, \sigma_{\nu,k} \in I_{\mathcal{E}_G}^{\mathcal{S}_P}$ are given such that $f_{\sigma_{\nu,1}} f_{\sigma_{\nu,2}} \cdots f_{\sigma_{\nu,k}} (v) \in \text{Even}(n - 1)$ for all fpf-involution words $v \in \mathcal{R}_n^{\mathcal{S}_P}(\theta_\mu)$. We claim that the desired sequence is

$$ (\sigma_{\mu,1}, \sigma_{\mu,2}, \ldots, \sigma_{\mu,l}) = (\sigma_{\nu,1}^\prime, \sigma_{\nu,2}^\prime, \ldots, \sigma_{\nu,k}^\prime, \pi_{\nu,1}, \pi_{\nu,2}, \ldots, \pi_{\nu,l}) \quad (5.4) $$

where $\sigma_{\nu,i}^\prime := s_{2n} \cdot \sigma_{\nu,u} \cdot s_{2n}$. To prove this, let

$$ w^i := \begin{cases} (2i)(2i - 1) \cdots (i + 1) & \text{for } 1 \leq i \leq r \\ (r + i)(r + i - 1) \cdots (r + i - h_i + 1) & \text{for } r < i \leq q \end{cases} $$

and set $w := w^1 w^2 \cdots w^q$. This is the same as the analogous word in the proof of Lemma 5.6, but with all letters incremented by one. Using Lemma 3.12 one can check that $w \in \mathcal{R}_n^{\mathcal{S}_P}(\theta_\mu)$. Then, specifically, we claim that if $\sigma_{\mu,i}$ is defined as in (5.4) then

$$ f_{\sigma_{\nu,1}} f_{\sigma_{\nu,2}} \cdots f_{\sigma_{\nu,l}} (w) = a^i a^2 \cdots a^q \in \text{Even}(n) \quad (5.5) $$

where $a^i$ is again the word formed by adding $2h_1 + 2h_2 + \cdots + 2h_{i-1}$ to $(2h_i) \cdots 642$.

If $q > r$ then the subword of $w$ with the last letter omitted belongs to $\mathcal{R}_n^{\mathcal{S}_P}(\theta_\nu)$, while if $q = r$ then the subword of $w$ with the largest letter $2r$ omitted is in $\mathcal{R}_n^{\mathcal{S}_P}(\theta_\nu)$. For each $i \in \mathbb{Z}_{\geq 0}$, define

$$ w^i := \begin{cases} (q + r + 2i + \epsilon + 1)(q + r)(q + r - 1) \cdots (q + r - h_q + 2) & \text{if } q > r \\ (2r + 2i + 2)(2r - 1)(2r - 2) \cdots (r + 1) & \text{if } q = r \end{cases} $$

$$ \mu = (\mu_1, \mu_2, \ldots, \mu_r, 0) $$

$$ \theta_\mu = (1, r + 1 + \mu_r) / 2 $$

$$ \text{SD}_\nu = \text{SD}_\mu \setminus \{(h_q, q)\} $$

$$ \epsilon \in \{0, 1\} $$

$$ \sigma_{\nu,1}, \sigma_{\nu,2}, \ldots, \sigma_{\nu,k} \in I_{\mathcal{E}_G}^{\mathcal{S}_P} $$

$$ f_{\sigma_{\nu,1}} f_{\sigma_{\nu,2}} \cdots f_{\sigma_{\nu,k}} (v) \in \text{Even}(n - 1) $$

$$ \mathcal{R}_n^{\mathcal{S}_P}(\theta_\mu) $$

$$ a^i $$

$$ (q + r + 2i + \epsilon + 1)(q + r)(q + r - 1) \cdots (q + r - h_q + 2) $$

$$ (2r + 2i + 2)(2r - 1)(2r - 2) \cdots (r + 1) $$

$$ \text{Even}(n) $$
and let \( w_{(i)} := w^1 w^2 \cdots w^{i-1} u^i \). Then \( w_{(0)} = \text{fb}_{\theta_r}(w) \) and \( w_{(i)} = \text{fb}_{\theta_r}(w_{(i-1)}) \) for each \( i > 0 \), so we deduce that the word \( v' := (\text{fb}_{\theta_r})^{-\frac{r+1}{2}}(w) \) is obtained from \( v \) by replacing either its last letter or its largest letter by \( 2n \), and then moving \( 2n \) to be in position \( n - h_q + 1 \). Moreover, removing \( 2n \) from \( v' \) yields a word \( v \in \mathcal{R}^{\text{Sp}}(\theta_r) \).

As in the proof of Lemma 5.9, we now observe that the words \( \text{fb}_{\sigma_{v,1}} \cdot \text{fb}_{\sigma_{v,2}} \cdot \cdots \cdot \text{fb}_{\sigma_{v,k}} (v') \) and \( \text{fb}_{\sigma_{v,1}} \cdot \text{fb}_{\sigma_{v,2}} \cdot \cdots \cdot \text{fb}_{\sigma_{v,k}} (\nu, k) \) have exactly the same relationship as \( v \) and \( v' \) for all \( i \in [k] \): the second word is the same as the first but with the letter \( 2n \) inserted in position \( n - h_q + 1 \). From this, the desired identity (5.5) follows immediately by induction, which completes the proof.

We can now prove our last main result from Section 3.5.

**Proof of Theorem 3.36.** Our argument is similar to the proof of Theorem 3.32. We extend the definition of \( \text{Sp} \) from words to \( n \)-tuples of words exactly as we did with \( \sim \). Recall that \( \pi \in I_{\mathbb{Z}}^{\text{FPF}} \).

Choose a factorization \( w \in \mathcal{R}^{\text{Sp}}(\pi) \) and let \( C \) be the full \( q_n \)-subcrystal of \( \mathcal{R}^{\text{Sp}}(\pi) \) containing \( w \). It is again straightforward to check that every \( v \in C \) has \( v \sim \).

By Lemma 5.9, we have \( \text{fb}_{\sigma_{v,1}} \cdot \text{fb}_{\sigma_{v,2}} \cdot \cdots \cdot \text{fb}_{\sigma_{v,m}} (w) \in \text{Even}_n(\ell^{\text{Sp}}(\pi)) \) for some \( \sigma_1, \sigma_2, \ldots, \sigma_l \in I_{\mathbb{Z}}^{\text{FPF}} \) and \( m \in 2\mathbb{Z} \). Write \( f := \text{fb}_{\sigma_{v,1}} \cdot \text{fb}_{\sigma_{v,2}} \cdot \cdots \cdot \text{fb}_{\sigma_{v,m}} \). Since each \( \text{fb}_{\sigma_{v}} \) preserves \( \text{Sp} \), it follows that \( f(v) \in \text{Even}_n(\ell^{\text{Sp}}(\pi)) \) for all \( v \in C \). Therefore, by Theorem 4.23 and Lemma 5.2, the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & \text{Even}_n(\ell^{\text{Sp}}(\pi)) & \xrightarrow{\text{dbl}^{-1}} & \text{Perm}_n(\ell^{\text{Sp}}(\pi)) & \xrightarrow{\text{inv}} & \mathcal{W}_n(\ell^{\text{Sp}}(\pi)) \\
\text{ShTab}_n(\ell^{\text{Sp}}(\pi)) & \xleftarrow{\text{inv}} & \mathcal{W}_n(\ell^{\text{Sp}}(\pi)) & \xrightarrow{\text{dbl}^{-1}} & \text{Perm}_n(\ell^{\text{Sp}}(\pi)) & \xrightarrow{\text{inv}} & \text{Even}_n(\ell^{\text{Sp}}(\pi))
\end{array}
\]

commutes. The map \( f : C \rightarrow \text{Even}_n(\ell^{\text{Sp}}(\pi)) \) is a quasi-isomorphism of abstract \( q_n \)-crystals by Theorem 4.23 both \( \text{dbl}^{-1} \) and \( \text{inv} \) are isomorphisms by Lemma 5.2, and the map \( P_{\text{HM}} : \mathcal{W}_n(m) \rightarrow \text{ShTab}_n(m) \) is a quasi-isomorphism for all \( m \) by construction. We conclude that \( Q_{\text{EG}}^{\text{Sp}} : \mathcal{R}^{\text{Sp}}(\pi) \rightarrow \text{ShTab}_n(\ell^{\text{Sp}}(\pi)) \) is a quasi-isomorphism of abstract \( q_n \)-crystals. This proves (b).

To prove (a), suppose \( v \in \mathcal{R}^{\text{Sp}}(\pi) \) has \( v \sim w \). Then \( f(v) \sim \).

\[
f(w), \quad \text{so } f(v) \in \text{Even}_n(\ell^{\text{Sp}}(w)) \text{ and } \text{dbl}^{-1} \circ f(v) \sim \text{dbl}^{-1} \circ f(w) \text{ since } f(w) \text{ has only even letters. As in the proof of Theorem 3.36 we deduce that } \text{inv} \circ \text{dbl}^{-1} \circ f(v) \text{ and } \text{inv} \circ \text{dbl}^{-1} \circ f(w) \text{ are in the same full } q_n \text{-subcrystal of } \mathcal{W}_n(\ell^{\text{Sp}}(\pi)), \text{ so } v \in C \text{ as the maps } f, \text{ dbl}^{-1}, \text{ and inv send full subcrystals to full subcrystals.} \]

\[\square\]

**5.4 Dual equivalence operators**

As an application of Theorems 3.32 and 3.36, we can describe precisely how the Coxeter-Knuth operators interact with the orthogonal- and symplectic-EG-recording tableaux.

Consider a standard shifted tableau \( T \) with \( n \) boxes. Given \( i \in [n] \), let \( \square_i \) be the unique box of \( T \) containing \( i \) or \( i' \). For each index \( i \in [n - 1] \), let \( s_i \ast T \) be formed from \( T \) as follows:

- If \( \square_i \) and \( \square_{i+1} \) are in the same row or same column then do both of the following:
  - Interchange \( i \) and \( i' \) if the box \( \square_i \) is not on the main diagonal.
  - Interchange \( i + 1 \) and \( i + 1' \) if the box \( \square_{i+1} \) is not on the main diagonal.

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• Otherwise, interchange $i$ and $i + 1$ and then interchange $i'$ and $i' + 1$.

Although $s_1 \ast (s_1 \ast T) = T$, this operation does not extend to an action of the symmetric group. We always have $\square_1 = (1, 1)$ and $\square_2 = (1, 2)$, so $s_1 \ast T$ is obtained from $T$ by interchanging 2 and 2'.

Choose an integer $q > 0$ such that the domain of $T$ is a subset of $[q] \times [q]$. For $i \in [q]$ let $C_i$ be the sequence of primed entries in column $i$ of $T$, read in order, and let $R_i$ be the sequence of unprimed entries in row $i$ of $T$, read in order. The shifted reading word of $T$ is the sequence $\text{shword}(T)$ formed by removing all primes from $C_q R_q \cdots C_3 R_3 C_2 R_2 C_1 R_1$. For example, if

$$T = \begin{array}{c} 7 \\ 8 \ 0 \ 6 \\ 1 \ 2 \ 4 \ 9 \end{array}$$

then $\text{shword}(T) = 467238159$ (5.6)

since the nonempty sequences $C_i R_i$ are $C_1 R_1 = 159$, $C_2 R_2 = 2'38$, and $C_3 R_3 = 4'6'7$.

**Definition 5.10.** Let $\varnothing_i$ for $0 \leq i \leq n - 2$ be the operator on standard shifted tableaux with

$$\varnothing_i(T) := \begin{cases} s_i \ast T & \text{if } i + 2 \text{ is between } i \text{ and } i + 1 \text{ in } \text{shword}(T), \\ s_{i+1} \ast T & \text{if } i = 0 \text{ or if } i \text{ is between } i + 1 \text{ and } i + 2 \text{ in } \text{shword}(T), \\ T & \text{if } i + 1 \text{ is between } i \text{ and } i + 2 \text{ in } \text{shword}(T). \end{cases}$$

For convenience set $\varnothing_i(T) := T$ for all integers $i$ with $i + 1 \notin [n - 1]$.

When we say “$b$ is between $a$ and $c$” in some word, we mean that $abc$ or $cba$ occurs as a subword. It is easy to show that if $T$ is standard then $\varnothing_i(T)$ is also standard and that $\varnothing_i(\varnothing_i(T)) = T$. The letter $\varnothing$ in this notation stands for dual equivalence operator.

Define the descent set of a standard shifted tableau $T$ to be $\text{Des}(T) := \text{Des}(\text{shword}(T))$, so that the tableau in (5.6) has $\text{Des}(T) = \{1, 3, 5\}$. It is a standard exercise to check that an integer $i$ belongs to $\text{Des}(T)$ if and only if either (a) $i$ and $i + 1$ both appear in $T$ with $i + 1$ in a row strictly after $i$, (b) $i'$ and $i + 1'$ both appear in $T$ with $i + 1'$ in a column strictly after $i'$, or (c) $i$ and $i + 1'$ both appear in $T$. If $w$ is an involution word then $\text{Des}(w) = \text{Des}(Q^O_{\text{EG}}(w))$ [22, Proposition 2.24] and if $w$ is an fpf-involution word then $\text{Des}(w) = \text{Des}(Q^\text{Sp}_{\text{EG}}(w))$ [38, Theorem 4.4].

**Theorem 5.11.** Let $i$ be a positive integer.

(a) If $w$ is an involution word for some element of $I_Z$ then

$$Q^O_{\text{EG}}(c_{k_0}^{Q}(w)) = \varnothing_0(Q^O_{\text{EG}}(w)) \quad \text{and} \quad Q^O_{\text{EG}}(c_{k_0}^{Q}(w)) = \varnothing_i(Q^O_{\text{EG}}(w)).$$

(b) If $w$ is an fpf-involution word for some element of $I_Z^{\text{SPF}}$ then

$$Q^\text{Sp}_{\text{EG}}(c_{k_0}^{\text{Sp}}(w)) = \varnothing_0(Q^\text{Sp}_{\text{EG}}(w)) \quad \text{and} \quad Q^\text{Sp}_{\text{EG}}(c_{k_0}^{\text{Sp}}(w)) = \varnothing_i(Q^\text{Sp}_{\text{EG}}(w)).$$

Proof. The identities $Q^O_{\text{EG}}(c_{k_0}^{Q}(w)) = \varnothing_0(Q^O_{\text{EG}}(w))$ and $Q^\text{Sp}_{\text{EG}}(c_{k_0}^{\text{Sp}}(w)) = \varnothing_0(Q^\text{Sp}_{\text{EG}}(w))$ are easy to observe directly from the definitions of $c_{k_0}^Q$, $Q^O_{\text{EG}}$, $c_{k_0}^{\text{Sp}}$, and $Q^\text{Sp}_{\text{EG}}$.

The remaining identities in part (a) are trivial unless $0 < i \leq \ell(w) - 2$ and exactly one of $i$ or $i + 1$ is descent of $w$ (equivalently, $Q^O_{\text{EG}}(w)$), since otherwise we have $c_{k_0}(w) = w$ and $\varnothing_i(Q^O_{\text{EG}}(w)) = Q^O_{\text{EG}}(w)$. Assume this is the case and view $w$ as an orthogonal factorization by placing each
letter in its own factor. Then, as explained in the proof of [7 Proposition 10.14], exactly one of \( f_i f_{i+1} e_i e_{i+1}(w) \) or \( f_{i+1} f_i e_i e_{i+1}(w) \) is nonzero and the nonzero factorization may be identified with \( ck_i(w) \).

Similarly, the remaining identities in part (b) are trivial unless \( 0 < i \leq \ell(w) - 2 \) and exactly one of \( i \) or \( i + 1 \) is descent of \( w \) (equivalently, \( Q_{Eg}^{Sp}(w) \)). When this is the case and we view \( w \) as an symplectic factorization by placing each letter in its own factor, it follows by the same argument from [7] that exactly one of \( f_i f_{i+1} e_i e_{i+1}(w) \) or \( f_{i+1} f_i e_i e_{i+1}(w) \) is nonzero and the nonzero factorization may be identified with \( ck_i(w) \).

Since \( Q_{Eg}^{O} \) and \( Q_{Eg}^{Sp} \) are crystal quasi-isomorphisms by Theorems \( 3.32 \) and \( 3.36 \) it suffices to check that if \( T \) is standard shifted tableau with \( n \) boxes and \( |\text{Des}(T) \cap \{i, i + 1\}| = 1 \), then exactly one of \( f_i f_{i+1} e_i e_{i+1}(T) \) or \( f_{i+1} f_i e_i e_{i+1}(T) \) is nonzero and the nonzero shifted tableau is \( \delta_i(T) \). These assertions are immediate from Lemmas \( A.11 \) and \( A.12 \) in the appendix.

These identities are shifted analogues of a similar formula for \( Q_{Eg}(ck_i(w)) \) in terms of \( Q_{Eg}(w) \) when \( w \) is any reduced word; see [1] Definition 5.1.3 and Theorem 5.1.4, for example.

**Example 5.12.** The word \( w = 2343 \) is both an involution word and an fpf-involution word (for different permutations). We have

\[
P_{Eg}^{O}(w) = P_{Eg}^{Sp}(w) = \begin{array}{ccc} 4 & 2 & 3 \\ 4 & 2 & 3 \end{array} \quad \text{and} \quad Q_{Eg}^{O}(w) = Q_{Eg}^{Sp}(w) = \begin{array}{ccc} 4 & 1 & 2 \\ 3 & 1 & 2 \\ 4 \end{array}
\]

As predicted by the theorem, it holds that

\[
Q_{Eg}^{O}(ck_0^O(w)) = Q_{Eg}^{O}(3243) = Q_{Eg}^{Sp}(ck_0^Sp(w)) = Q_{Eg}^{Sp}(2143) = \begin{array}{ccc} 4 & 2 & 3 \\ 4 & 2 & 3 \end{array} = \delta_0(T),
\]

\[
Q_{Eg}^{O}(ck_2^O(w)) = Q_{Eg}^{O}(2434) = Q_{Eg}^{Sp}(ck_2^Sp(w)) = Q_{Eg}^{Sp}(2434) = \begin{array}{ccc} 3 & 1 & 2 \\ 1 & 2 & 4 \end{array} = \delta_2(T).
\]

Suppose \( \mathcal{A} \) is a finite set and \( \text{Des} \) is a map from \( \mathcal{A} \) to the set of subsets of \( [n - 1] \). A dual equivalence for \( \mathcal{A} \) with respect to \( \text{Des} \) is a family of involutions \( \{ \psi_i : \mathcal{A} \rightarrow \mathcal{A} \}_{1 \leq i < n} \), called dual equivalence operators, satisfying two technical conditions; see [2] Definition 4.1. As explained in [2] §4.1, if one is given a dual equivalence on \( \mathcal{A} \), then there is a natural way to turn \( \mathcal{A} \) into a dual equivalence graph as axiomatized in [2] 46.

Hamaker and Young have shown that if \( \pi \in S_n \) has \( \ell(\pi) = n \), then taking \( \varphi_i = ck_{i-1} \) for \( 1 < i < n \) gives a dual equivalence for \( \mathcal{A} = \mathcal{R}(\pi) \) with the usual descent set [2] Theorem 3. It follows that the same maps give a dual equivalence for \( \mathcal{R}^O(\pi) \) when \( \pi \in I_Z \) has \( \ell^O(\pi) = n \) and for \( \mathcal{R}^{Sp}(\pi) \) when \( \pi \in I_Z^{FPF} \) has \( \ell^{O}(\pi) = n \). On the other hand, one can check that the operators \( \delta_{i-1} \) for \( 1 < i < n \) are the same as the involutions on standard shifted tableaux that Assaf denotes by \( \psi_i \) in [3] Definition 6.1. The following theorem from [3] is therefore a corollary of our results:

**Corollary 5.13** ([3 Theorem 6.3]). Let \( \mu \) be a strict partition of \( n \). The maps \( \psi_i = \delta_{i-1} \) for \( 1 < i < n \) give a dual equivalence for the set \( \mathcal{A} \) of standard shifted tableaux of shape \( \mu \).

**Proof.** Let \( \pi \in I_Z \) be inv-Grassmannian of shape \( \mu \). It follows from Lemma \( 5.5 \) and Theorem \( 5.11 \) that \( \{ \psi_i \}_{1 < i < n} \) is the dual equivalence on \( \mathcal{A} \) induced by the bijection \( Q_{Eg}^{O} : \mathcal{R}^O(\pi) \rightarrow \mathcal{A} \). \( \square \)
5.5 Open problems

We mention some related questions and conjectures. Little proved the following in [34]:

**Proposition 5.14** ([34, Lemma 5]). Let \( \pi \in S_n \), let \( w = w_1 w_2 \cdots w_n \) be a reduced word, and suppose \( b_\pi(w) = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n \). Then \( \bar{w}_i - w_i \in \{0, 1\} \) for all \( i \in [n] \).

In turn, this general fact holds for Edelman-Greene insertion:

**Proposition 5.15.** Suppose \( w = w_1 w_2 \cdots w_n \) and \( \bar{w} = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n \) are reduced words with \( \bar{w}_i - w_i \in \{0, 1\} \) for all \( i \in [n] \). Then \( Q_{\text{EG}}(w) = Q_{\text{EG}}(\bar{w}) \).

**Proof.** It suffices to show that for each \( i \), the tableau \( P_{\text{EG}}(\bar{w}_i \bar{w}_{i+1} \cdots \bar{w}_n) \) is formed by adding one to a subset of entries in \( P_{\text{EG}}(w_1 w_2 \cdots w_i) \). This follows by induction using the observations in Remark 3.25. The details are left to the reader. \( \square \)

Combining these propositions gives an immediate proof of Theorem 4.4(d). Using the same sort of arguments, one can derive a similar property of orthogonal-EG insertion:

**Proposition 5.16.** Suppose \( w = w_1 w_2 \cdots w_n \) and \( \bar{w} = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n \) are involution words with \( \bar{w}_i - w_i \in \{0, 1\} \) for all \( i \in [n] \). Then \( Q^O_{\text{EG}}(w) = Q^O_{\text{EG}}(\bar{w}) \).

**Proof.** Using Remark 3.25 it is a straightforward exercise to show by induction that \( P^O_{\text{EG}}(\bar{w}_i \bar{w}_{i+1} \cdots \bar{w}_n) \) is formed by adding one to a subset of entries in \( P^O_{\text{EG}}(w_1 w_2 \cdots w_i) \) for all \( i \), and that a letter \( \bar{w}_i \) is row-inserted according to Definition 3.20 if and only if \( w_i \) is row-inserted. We omit the details. \( \square \)

Computations support the following analogue of Proposition 5.15. If this conjecture were true, then we would get an immediate proof of the most difficult part of Theorem 4.13.

**Conjecture 5.17.** Let \( \pi \in I_n \), let \( w = w_1 w_2 \cdots w_n \) be an involution word, and suppose \( b_\pi(w) = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n \). Then \( \bar{w}_i - w_i \in \{0, 1\} \) for all \( i \in [n] \).

A weaker version of this conjecture seems to hold for the \( \mathfrak{b}_\pi \) operators. However, this does not lead to a simple proof of Theorem 4.13 since letters may be incremented twice.

**Conjecture 5.18.** Let \( \pi \in I_n^{\text{FF}} \), let \( w = w_1 w_2 \cdots w_n \) be an ffp-involution word, and suppose \( \mathfrak{b}_\pi(w) = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n \). Then \( \bar{w}_i - w_i \in \{0, 1, 2\} \) for all \( i \in [n] \).

Little bumping operators are naturally described in terms of wiring diagrams for permutations; see [21, 34]. Moreover, if \( \pi \in S_n \) has a single descent, then there is simple way of reading off the tableau \( Q_{\text{EG}}(w) \) for any \( w \in R(\pi) \) from the associated wiring diagram [21, Lemma 5].

**Problem 5.19.** Identify good definitions of wiring diagrams for (ffp-)involution words, and describe the operators \( \mathfrak{b}_\pi \) and \( \mathfrak{b}_\pi^\ast \) in terms of these diagrams. Is there is an efficient way to compute \( Q^O_{\text{EG}}(w) \) (respectively, \( Q^{\mathfrak{s}_\pi}_{\text{EG}}(w) \)) from the wiring diagram associated to an involution word \( w \) for an inv-Grassmannian (respectively, ffp-Grassmannian) permutation?

Involution words arise naturally when studying the combinatorics of the \( O_n \)- and \( \mathfrak{S}_n \)-actions on the type A flag variety \( \mathfrak{F}_n \). These actions correspond to two of the three families of type A symmetric varieties. The third family comes from the action of \( GL_p \times GL_{n-p} \) on \( \mathfrak{F}_n \). There is a natural weak order on the corresponding set of orbits [8]. The maximal chains in this order give another variant of reduced words, which are studied under the name clan words in [8].

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Problem 5.20. Are there interesting crystal structures on factorizations of clan words? Is there an analogue of Edelman-Greene insertion for clan words that can be interpreted as a crystal morphism?

Morse and Schilling’s original aim in [41] was to identify crystal structures on cyclically decreasing factorizations of reduced words for affine permutations. They describe an abstract $\mathfrak{gl}_2$-crystal on the subset of such factorizations with exactly two factors [41, Theorem 3.14]. It remains an open problem to further extend the constructions in Section 3.1 to the affine case.

There are versions of involution words for affine permutations, with many of the same combinatorial properties as involution words for elements of $I_\mathbb{Z}$ [39, 40, 52]. This suggests the following:

Problem 5.21. Are there interesting crystal structures on factorizations of reduced words or involution words for affine permutations? Is there an affine analogue of Edelman-Greene insertion that can be interpreted as a crystal morphism?

The papers [3, 5] develop a notion of shifted dual equivalence graphs based on standard shifted tableaux with no primed entries. The set of reduced words for a signed permutation, connected by type B Coxeter-Knuth moves, is an example of such a graph [5, Theorem 1.3]. The results in Section 5.4 suggest the existence of another interesting kind of shifted dual equivalence.

Problem 5.22. Is there a version of shifted dual equivalence graphs based on standard shifted tableaux with primed entries that includes $R_{O}(\pi)$ and $R_{Sp}(\pi)$ as examples?

A Crystal operators on shifted tableaux

Fix a positive integer $n$ and a strict partition $\lambda$ with at most $n$ parts. By Theorem-Definition 2.12, the set $\text{ShTab}_n(\lambda)$ of semistandard shifted tableaux of shape $\lambda$ with all entries at most $n$ has a $q_n$-crystal structure. In this appendix we review the explicit formulas from [4, 24, 25] for the raising and lowering operators in this crystal. Recall that the weight map for $\text{ShTab}_n(\lambda)$ is given by (2.2). We include this material both for completeness and to assist the proof of Theorem 5.11.

The formulas below have already appeared in [4, 24, 25]. The $\mathfrak{gl}_n$-crystal operators $e_i$ and $f_i$ for $i \in [n-1]$ acting on $\text{ShTab}_n(\lambda)$ were described first, in [24]. About a year later [4] and [25] independently supplied the queer operators $f^T$ and $e^T$. Our exposition mostly follows the conventions of [4], which will let us correct some minor errors in the published version of that paper. We continue to draw all tableaux in French notation.

A.1 Shifted tableau pairing

If $\mu$ and $\nu$ are strict partitions, then we write $\mu \subset \nu$ to indicate that $\text{SD}_\mu \subset \text{SD}_\nu$. In this case we set $\text{SD}_{\nu/\mu} := \text{SD}_\nu \setminus \text{SD}_\mu$ and define a skew shifted tableau of shape $\nu/\mu$ to be a map

$$\text{SD}_{\nu/\mu} \to \frac{1}{2}\mathbb{Z} = \{\cdots < 1' < 1 < 2' < 2 < \cdots\}.$$

If $T$ is a semistandard shifted tableau and $i \leq j$ are positive integers, then

$$T^{-1}(\{i' < i < \cdots < j' < j\})$$

is equal to $\text{SD}_{\nu/\mu}$ for some strict partitions $\mu \subset \nu$, and we write $T|_{[i,j]}$ for the skew shifted tableau obtained by restricting $T$ to this subdomain.
We say that a skew shifted tableau is a rim if its domain has no positions \((i_1, j_1), (i_2, j_2)\) with \(i_1 < i_2\) and \(j_1 < j_2\). If \(T\) is a semistandard shifted tableau then \(T|_{[i,j]}\) is always a rim. A rim whose domain is connected is a ribbon. In French notation, the domain of a ribbon must appear as

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

or some analogous sequence of contiguous boxes.

Suppose \(T\) is a semistandard shifted tableau. Then \(T|_{[i,i]}\) is always a rim. A rim whose domain is connected is a ribbon. In French notation, the domain of a ribbon must appear as

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

or some analogous sequence of contiguous boxes.

Suppose \(T\) is a semistandard shifted tableau. Then \(T|_{[i,i]}\) is a disjoint union of ribbons, which we call the \(i\)-ribbons of \(T\). Each entry in an \(i\)-ribbon is \(i\) or \(i'\), and all of these are uniquely determined except for the top left entry, for which there are two possibilities as in these examples:

\[
\begin{array}{cccc}
4 & 5 & 5 & \\
4 & 4 & 5 & 5 \\
4 & 4 & 4 & 5 \\
\end{array}
\]

\[
\begin{array}{cccc}
2 & 3 & 2 & \\
2 & 2 & 2 & \\
2 & 2 & 2 & \\
\end{array}
\]

Recall the definition of the shifted reading word \(\text{shword}(T)\) of \(T\) from (5.6). This definition extends to skew shifted tableaux with no changes. Fix a positive integer \(i\) and assume the domain of \(T|_{[i,i+1]}\) has size \(N\). Let \(\alpha_1, \alpha_2, \ldots, \alpha_N\) be the positions in this domain, ordered such that \(\alpha_j\) is the position contributing the \(j\)th letter of \(\text{shword}(T|_{[i,i+1]}))\).

**Definition A.1.** Consider the word formed by replacing each \(i\) in \(\text{shword}(T|_{[i,i+1]}))\) by a right parenthesis “)" and each \(i + 1\) in \(\text{shword}(T|_{[i,i+1]}))\) by a left parenthesis “(". If \(j\) and \(k\) are the indices of a matching set of parentheses in this word then we say that \(\alpha_j\) and \(\alpha_k\) are paired. Remove all paired positions from \((\alpha_1, \alpha_2, \ldots, \alpha_N)\) and let \(\text{unpaired}_i(T)\) denote the resulting subsequence.

**Example A.2.** Suppose \(i = 4\) and \(T|_{[4,5]}\) is the skew shifted tableau

\[
\begin{array}{cccc}
4 & 5 & 5 & \\
4 & 4 & 5 & 5 \\
4 & 4 & 4 & 5 \\
\end{array}
\]

Then \(\text{shword}(T|_{[4,5]})) = 55445545445\) and the corresponding ordering of the boxes in \(T|_{[4,5]}\) is

\[
\begin{array}{cccc}
4 & 6 & 7 & \\
4 & 8 & 2 & 9 \\
3 & 10 & 11 & 12 \\
\end{array}
\]

The paired positions are \((\alpha_2, \alpha_3), (\alpha_1, \alpha_4), (\alpha_6, \alpha_11), (\alpha_7, \alpha_8), (\alpha_9, \alpha_{10})\), so

\[
\text{unpaired}_4(T) = (\alpha_5, \alpha_{12}) = ((3, 3), (1, 9)).
\]

**A.2 Lowering operators**

The queer lowering operator \(f_1\) for \(\text{ShTab}_n(\lambda)\) from Theorem-Definition 2.11 has the following description. This appears as both [4, Definition 4.4] and [25, Lemma 3.2].

**Proposition A.3 ([4, 25]).** Let \(T \in \text{ShTab}_n(\lambda)\). If no box of \(T\) contains 1 or some box of \(T\) contains 2', then \(f_T(T) = 0\). Otherwise \(f_T(T)\) is formed from \(T\) by changing the rightmost 1 in the first row of \(T\) to be 2 if the rightmost 1 is on the diagonal, or else 2'.

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Example A.4. Thus \( f_1 \left( \begin{array}{ccc} 3 & 1 & 2 \\ 1 & 2 & 1 \end{array} \right) = f_1 \left( \begin{array}{ccc} 3 & 1 & 2 \\ 2 & 2 & 2 \end{array} \right) = 0 \) while

\[
\begin{array}{ccc}
2 & 1 & 2 \\
1 & 2 & 1
\end{array}
\] and

\[
\begin{array}{ccc}
3 & 1 & 2 \\
1 & 2 & 1
\end{array}
\]}

The lowering operators \( f_i \) on \( \text{ShTab}_n(\lambda) \) for \( i \in [n-1] \) are more complicated, and were first described in [24, §4]. The theorem below reproduces [4] Definition 3.5, which is equivalent to the formulation in [24] by [4] Proposition 3.19.

Checking that the formula for \( f_i \) in the next theorem gives a map \( \text{ShTab}_n(\lambda) \rightarrow \text{ShTab}_n(\lambda) \sqcup \{0\} \) is already nontrivial (see [4] Theorem 3.8). Showing that \( f_i \) commutes with \( P_{\text{HM}} \) in the sense required for part (b) of Theorem-Definition 2.11 is even harder (see [24] Theorem 4.3).

Theorem A.5 ([4] [24]). Let \( i \in [n-1] \) and \( T \in \text{ShTab}_n(\lambda) \). Consider the positions \((x,y)\) in \( \text{unpaired}_i(T) \) with \( T_{x,y} \in \{i',i\} \). If there are no such positions then \( f_i(T) = 0 \). Otherwise, let \((x,y)\) be the last such position. Then \( f_i(T) \neq 0 \) is formed from \( T \) by the following procedure.

(L1) First assume \( T_{xy} = i \).

a. If \( T_{x,y+1} = i + 1 \) then \( f_i(T) \) is formed by changing \( T_{xy} \) to \( i + 1 \) and \( T_{x,y+1} \) to \( i + 1 \):

\[
\begin{array}{ccc}
T_{x+1,y} & = & ? \\
T_{xy} & +1 \\
T_{x,y+1} & i & i+1' \\
\rightarrow & i+1' & i+1
\end{array}
\]

b. If \( T_{x,y+1} \neq i + 1 \) and \( T_{x+1,y} \notin \{i + 1', i + 1\} \) then \( f_i(T) \) is formed by changing \( T_{xy} \) to \( i + 1 \):

\[
\begin{array}{ccc}
T_{x+1,y} & = & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array} \\
T_{xy} & \neq \neq \\
T_{x,y+1} & i & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array} \\
\rightarrow & i+1 & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array}
\]

(c. Suppose \( T_{x,y+1} \neq i + 1 \) and \( T_{x+1,y} \in \{i + 1', i + 1\} \). Let \((\tilde{x},\tilde{y})\) be the position farthest northwest in the \((i + 1)\)-ribbon containing \((x + 1, y)\). If \( \tilde{x} \neq \tilde{y} \), then it holds that \( T_{\tilde{x}y} = i + 1 \) and \( f_i(T) \) is formed by changing \( T_{xy} \) to \( i + 1 \) and \( T_{\tilde{x}y} \) to \( i + 1 \):

\[
\begin{array}{ccc}
T_{\tilde{x}y} & = & i+1' \\
\cdot \cdot \cdot \cdot \cdot \\
T_{x+1,y} & = & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array} \\
T_{xy} & \neq \neq \\
T_{x,y+1} & i & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array} \\
\rightarrow & i+1 & \begin{array}{c}
\text{not} \\
i+1 \\
or \\
i+1'
\end{array}
\]

If \( \tilde{x} = \tilde{y} \), then \( f_i(T) \) is formed by just changing \( T_{xy} \) to \( i + 1 \).

(L2) Next assume \( T_{xy} = i' \).
a. If $T_{x+1,y} = i$ then $f_i(T)$ is formed by changing $T_{xy}$ to $i$ and $T_{x+1,y}$ to $i + 1'$:

\[
\begin{array}{c|cc|c}
T_{x+1,y} & i & \text{?} & i + 1' \\
T_{xy} & i' & \text{?} & i + 1' \\
T_{x,y+1} & \text{?} & \text{?} & \text{?} \\
\end{array}
\]

b. If $T_{x+1,y} \neq i$ and $T_{x,y+1} \notin \{i,i+1\}$ then $f_i(T)$ is formed by changing $T_{xy}$ to $i + 1'$:

\[
\begin{array}{c|cc|c}
T_{x+1,y} & \text{not } i & \text{not } i & \text{not } i + 1' \\
T_{xy} & i' & \text{not } i & \text{not } i \text{ nor } i + 1' \\
T_{x,y+1} & \text{?} & \text{?} & \text{?} \\
\end{array}
\]

c. Suppose $T_{x+1,y} \neq i$ and $T_{x,y+1} \in \{i,i+1\}$. Let $(\tilde{x}, \tilde{y})$ be the first position in the $i$-ribbon containing $(x,y)$ that is southeast of $(x,y)$ and has $T_{\tilde{x}\tilde{y}} = i$ and $T_{\tilde{x},\tilde{y}+1} \notin \{i,i+1\}$. Such a positions always exists, and $f_i(T)$ is formed by changing $T_{xy}$ to $i$ and $T_{\tilde{x}\tilde{y}}$ to $i + 1'$:

\[
\begin{array}{c|cc|c}
T_{x+1,y} & \text{not } i & \text{not } i \text{ or } i + 1' & \text{not } i \text{ nor } i + 1' \\
T_{xy} & i' & \text{not } i \text{ or } i + 1' & \text{not } i \text{ nor } i + 1' \\
T_{x,y+1} & \text{?} & i + 1' & \text{?} \\
\end{array}
\]

Remark A.6. The adjectives “northeastern” and “southwestern” in cases L1(c) and L2(c) of [4, Definition 3.5] should be “northwestern” and “southeastern,” respectively.

### A.3 Raising operators

The queer raising operator $e_\bar{1}$ for $\text{ShTab}_n(\lambda)$ from Theorem-Definition 2.11 also has a relatively simple description. This appears as both [4, Definition 4.5] and [25, Lemma 3.1].

**Proposition A.7** ([4, 25]). Let $T \in \text{ShTab}_n(\lambda)$. If the first entry in the first row of $T$ is equal to 2 then $e_{\bar{1}}(T)$ is formed by changing this entry to 1. If the first row of $T$ has a (necessarily unique) entry equal to 2', then $e_{\bar{1}}(T)$ is formed by changing this entry to 1. Otherwise $e_{\bar{1}}(T) = 0$.

**Example A.8.** Thus $e_\bar{1} \left( \begin{array}{c|ccc} \text{3} \hline 1 & 2' & 2 \\
\end{array} \right) = e_\bar{1} \left( \begin{array}{c|ccc} \text{4} \hline 1 & 4 & 4 \\
\end{array} \right) = 0$ while $e_\bar{1} \left( \begin{array}{c|ccc} \text{3} \hline 2 & 2 & 2 \\
\end{array} \right) = \begin{array}{c|ccc} \text{3} \hline 1 & 2 & 2 \\
\end{array}$ and $e_\bar{1} \left( \begin{array}{c|ccc} \text{3} \hline 1 & 2' & 2 \\
\end{array} \right) = \begin{array}{c|ccc} \text{3} \hline 1 & 1 & 2 \\
\end{array}$.

Remarks similar to above apply to the remaining raising operators for $\text{ShTab}_n(\lambda)$. These were first described in [24, §4]. The theorem below reproduces [4, Definition 3.9], which is equivalent to the formulas in [24] by [4, Proposition 3.19].
Theorem A.9 \(4, 24\). Let \(i \in [n - 1]\) and \(T \in \text{ShTab}_n(\lambda)\). Consider the positions \((x, y)\) in unpaired\(_i\)(\(T\)) with \(T_{xy} \in \{i + 1', i + 1\}\). If there are no such positions then \(e_i(T) = 0\). Otherwise, let \((x, y)\) be the first such position. Then \(e_i(T) \neq 0\) is formed from \(T\) by the following procedure.

(R1) First assume \(T_{xy} = i + 1\).
   a. If \(T_{x-1,y} = i + 1'\) then \(e_i(T)\) is formed by changing \(T_{xy}\) to \(i + 1'\) and \(T_{x,y-1}\) to \(i\):

   \[
   \begin{array}{c|c|c|c|c}
   T_{x,y-1} & T_{xy} & \text{\(i + 1'\)} & \text{\(i + 1\)} & \text{\(i\)} \\
   \hline
   T_{x-1,y} & \text{\(?\)} & \text{\(?\)} & \text{\(?\)} & \text{\(?\)} \\
   \end{array}
   \]

   b. If \(T_{x,y-1} \neq i + 1'\) and \(T_{x-1,y} \notin \{i, i + 1\}'\) then \(e_i(T)\) is formed by changing \(T_{xy}\) to \(i\):

   \[
   \begin{array}{c|c|c|c|c}
   T_{x,y-1} & T_{xy} & \text{\(\text{not } i + 1\)'} & \text{\(i + 1\)} & \text{\(i\)}' \\
   \hline
   T_{x-1,y} & \text{\(\text{not } i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} \\
   \end{array}
   \]

   c. Suppose \(T_{x,y-1} \neq i + 1'\) and \(T_{x-1,y} \notin \{i, i + 1\}'\). Let \((\bar{x}, \bar{y})\) be the first position in the \((i + 1)'\)-ribbon containing \((x, y)\) that is southeast of \((x, y)\) with \(T_{\bar{x} \bar{y}} = i + 1'\) and \(T_{\bar{x}-1,\bar{y}} \notin \{i, i + 1\}'\). Such a position exists, and \(e_i(T)\) is formed from \(T\) by changing \(T_{xy}\) to \(i + 1'\) and \(T_{\bar{x} \bar{y}}\) to \(i\):

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   T_{x,y-1} & T_{xy} & \text{\(\text{not } i + 1\)'} & \text{\(i + 1\)} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} \\
   \hline
   T_{x-1,y} & \text{\(\text{not } i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} \\
   \end{array}
   \]

(R2) Next assume \(T_{xy} = i + 1'\).
   a. If \(T_{x-1,y} = i\) then \(e_i(T)\) is formed by changing \(T_{xy}\) to \(i\) and \(T_{x,y-1}\) to \(i'\):

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   T_{x,y-1} & T_{xy} & \text{\(?\)} & \text{\(i + 1'\)} & \text{\(i\)} & \text{\(i'\)} & \text{\(i'\)} & \text{\(i'\)} & \text{\(i'\)} \\
   \hline
   T_{x-1,y} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} & \text{\(i\)} \\
   \end{array}
   \]

   b. If \(T_{x-1,y} \neq i\) and \(T_{x,y-1} \notin \{i', i\}\) then \(e_i(T)\) is formed by changing \(T_{xy}\) to \(i + 1'\):

   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   T_{x,y-1} & T_{xy} & \text{\(\text{not } \text{nor } i\) or \(i + 1\)'} & \text{\(i + 1\)} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} \\
   \hline
   T_{x-1,y} & \text{\(\text{not } i\) nor \(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} & \text{\(i\) or \(i + 1\)'} \\
   \end{array}
   \]
c. Suppose $T_{x-1,y} \neq i$ and $T_{x,y-1} \in \{i', i\}$. Let $(\tilde{x}, \tilde{y})$ be the position that is farthest northwest in the $i$-ribbon containing $(x, y - 1)$. If $\tilde{x} \neq \tilde{y}$, then it holds that $T_{\tilde{x}\tilde{y}} = i$ and $e_i(T)$ is formed by changing $T_{xy}$ to $i$ and $T_{\tilde{x}\tilde{y}}$ to $i'$:

If $\tilde{x} = \tilde{y}$, then $e_i(T)$ is formed by just changing $T_{xy}$ to $i$.

**Remark A.10.** As in Remark A.6 in cases R1(c) and R2(c) of [4, Definition 3.9] the directions “southwest” and “northeast” should be “southeast” and “northwest,” respectively. In addition, the phrase “changes $x$ to $\bar{i}$” in case R2(c) of [4, Definition 3.9] should be “changes $x$ to $i$.” Finally, the picture illustrating R2(c) in [4, Figure 18] has an extra box on the left; this picture should be

The marked numbers $\bar{T}$ and $\bar{y}$ in these tableaux are what we would write as $1'$ and $2'$.

**A.4 Finishing the proof of Theorem 5.11**

As an application of the preceding discussion, we can now give a self-contained derivation of the two lemmas cited at the end of the proof of Theorem 5.11. Recall that if $i \in \mathbb{Z}$ then $i' + 1 = (i+1)' = i+1'$ and $i' - 1 = (i-1)' \neq i - 1'$ since $i' := i - \frac{1}{2}$ and $1' := \frac{1}{2}$.

**Lemma A.11.** Let $T$ be a standard shifted tableau with $n$ boxes. For each $j \in [n]$, let $\square_j$ be the unique box of $T$ containing $j$ or $j'$. Suppose $i \in [n - 1]$.

(a) If $i \in \text{Des}(T)$ then $e_i(T) = f_i(T) = 0$.

(b) If $i \notin \text{Des}(T)$ then $f_i(T)$ is formed from $T$ by adding 1 to the entry in box $\square_i$.

(c) If $i \notin \text{Des}(T)$ then $e_i(T)$ is formed from $T$ by subtracting 1 from the entry in box $\square_{i+1}$.

**Proof.** If $i \in \text{Des}(T)$ then $\text{unpaired}_i(T)$ is empty, which implies part (a). If $i \notin \text{Des}(T)$ then $\text{unpaired}_i(T) = (\square_i, \square_{i+1})$ and parts (b) and (c) follow from Theorems A.5 and A.9. 

**Lemma A.12.** Let $T$ be a standard shifted tableau with $n$ boxes. Suppose $i \in [n - 2]$.

(a) If $\text{Des}(T) \cap \{i, i + 1\} = \{i\}$ then $f_i f_{i+1} e_i e_{i+1}(T) = \delta_i(T)$.

(b) If $\text{Des}(T) \cap \{i, i + 1\} = \{i + 1\}$ then $f_{i+1} f_i e_{i+1} e_i(T) = \delta_i(T)$.
Proof. For each \( j \in [n] \), let \( \square_j \) be the unique box of \( T \) containing \( j \) or \( j' \). If \( f_if_{i+1}e_ie_{i+1}(T) \) is nonzero then \( f_if_{i+1}e_ie_{i+1}e_{i+1}(T) = T \). Thus parts (a) and (b) are equivalent since \( \varnothing_i \) acts as an involution and one can check that \( \text{Des}(T) \cap \{i, i+1\} = \{i\} \) if and only if \( \text{Des}(\varnothing_i(T)) \cap \{i, i+1\} = \{i+1\} \). We can therefore assume \( \text{Des}(T) \cap \{i, i+1\} = \{i\} \) and just prove part (a). Then \( \text{shword}(T) \) contains either \( i + 1, i, i + 2 \) or \( i + 1, i + 2, i \) as a subword. We consider these cases in turn:

(1) Suppose \( i + 1, i, i + 2 \) is a subword of \( \text{shword}(T) \). Then \( \varnothing_i(T) = s_{i+1} \ast T \) and \( \square_{i+2} \) must come after both \( \square_{i+1} \) and \( \square_i \) in the order corresponding to the shifted reading word of \( T \). By the previous lemma \( e_{i+1}(T) \) is formed from \( T \) by subtracting one from the entry in position \( \square_{i+2} \), which is subsequently the only position in \( \text{unpaired}_i(e_{i+1}(T)) \).

If \( \square_{i+1} \) and \( \square_{i+2} \) are in different rows and different columns, then it is easy to see from Theorems [A.5] and [A.9] that applying \( e_i \) to \( e_{i+1}(T) \) subtracts one from the entry in position \( \square_{i+2} \), while applying \( f_{i+1} \) to \( e_ie_{i+1}(T) \) adds one to \( \square_{i+1} \), while applying \( f_i \) to \( f_{i+1}e_ie_{i+1}(T) \) adds one to \( \square_{i+2} \), as illustrated by the following picture where \( i = 3: \)

\[
\begin{array}{c|c|c}
3 & 4 & 5 \\
\hline
4 & 3 \\
\end{array} \quad \xrightarrow{\varnothing_3} \quad \begin{array}{c|c|c}
4 & 3 & 5 \\
\hline
4 & 3 \\
\end{array} \quad \xrightarrow{\text{shword}} \quad \begin{array}{c|c|c|c}
4 & 3 & 5 & 6 \\
\hline
4 & 3 & 5 & 6 \\
\end{array} \quad \xrightarrow{\varnothing_3} \quad \begin{array}{c|c|c|c}
4 & 3 & 6 & 5 \\
\hline
4 & 3 & 6 & 5 \\
\end{array} \quad \xrightarrow{\varnothing_3} \quad \begin{array}{c|c|c|c}
4 & 3 & 5 & 6 \\
\hline
4 & 3 & 5 & 6 \\
\end{array} \quad \xrightarrow{\varnothing_3} \quad \begin{array}{c|c|c|c}
4 & 3 & 6 & 5 \\
\hline
4 & 3 & 6 & 5 \\
\end{array} \\
\end{array}
\]

In this case the aggregate effect of applying \( f_if_{i+1}e_ie_{i+1} \) to \( T \) is to subtract one from \( \square_{i+2} \) and add one to \( \square_{i+1} \), which gives \( s_{i+1} \ast T = \varnothing_i(T) \) as desired.

Assume instead that \( \square_{i+1} \) and \( \square_{i+2} \) are in the same row. Then it is only possible for \( i + 1, i, i + 2 \) to be a subword of \( \text{shword}(T) \) if the entry in \( \square_{i+1} \) is primed, the entry in \( \square_{i+2} \) is unprimed, and \( \square_i \) belongs to the region strictly left of and weakly above \( \square_{i+1} \). It follows from Theorems [A.5] and [A.9] that the effect of applying \( e_{i+1}, e_i, f_{i+1}, \) and \( f_i \) successively to \( T \) is represented by the following picture, and ultimately gives \( s_{i+1} \ast T = \varnothing_i(T) \) as needed:

\[
T|_{[i,i+2]} = \begin{array}{c|c|c|c}
i & i+1 & i+2 & e_{i+1} \\
\hline
i & i+1 & i & e_i \\
\end{array} \quad \xrightarrow{f_{i+1}} \quad \begin{array}{c|c|c|c}
i & i+1 & i & e_i \\
\hline
i & i+1 & i & e_i \\
\end{array} \quad \xrightarrow{f_i} \quad \begin{array}{c|c|c|c}
i & i+1 & i & e_i \\
\hline
i & i+1 & i & e_i \\
\end{array} \quad = \varnothing_i(T)|_{[i,i+2]}.
\]

Here, box \( \square_i \) need not be in the same row as \( \square_{i+1} \) and \( \square_{i+2} \); this position may be anywhere weakly northwest of the one shown. Also, the application of \( e_i \) invokes case R1(a) of Theorem [A.9] which interchanges which of \( \square_{i+1} \) or \( \square_{i+2} \) contains a primed entry.

Finally assume that \( \square_{i+1} \) and \( \square_{i+2} \) are in the same column. Then it is only possible for \( i + 1, i, i + 2 \) to be a subword of \( \text{shword}(T) \) if the entry in \( \square_{i+1} \) is primed, the entry in \( \square_{i+2} \) is unprimed, and \( \square_i \) belongs to the region strictly left of and strictly above \( \square_{i+1} \). It follows from Theorems [A.5] and [A.9] that the effect of applying \( e_{i+1}, e_i, f_{i+1}, \) and \( f_i \) successively to \( T \) is represented by the following picture, and again gives \( s_{i+1} \ast T = \varnothing_i(T) \):

\[
T|_{[i,i+2]} = \begin{array}{c|c|c|c}
i+2 & i+1 & e_{i+1} \\
\hline
i+1 & i+1 & e_i \\
\end{array} \quad \xrightarrow{f_{i+1}} \quad \begin{array}{c|c|c|c}
i+2 & i+1 & e_i \\
\hline
i+1 & i+1 & e_i \\
\end{array} \quad \xrightarrow{f_i} \quad \begin{array}{c|c|c|c}
i+2 & i+1 & e_i \\
\hline
i+1 & i+1 & e_i \\
\end{array} \quad = \varnothing_i(T)|_{[i,i+2]}.
\]
The box $\square_i$ again might occur anywhere weakly northwest of the position shown. The application of $e_i$ now invokes case R1(c) of Theorem \[A.9\].

(2) Suppose $i + 1, i + 2, i$ is a subword of $\text{shword}(T)$. Our arguments are similar. Now we have $\varrho_i(T) = s_i \ast T$ and $\square_i$ must come after both $\square_{i+1}$ and $\square_{i+2}$ in the order corresponding to the shifted reading word. By the previous lemma $e_{i+1}(T)$ is formed from $T$ by subtracting one from the entry in position $\square_{i+2}$, which leaves $\square_{i+1}$ as the only position in $\text{unpaired}_i(e_{i+1}(T))$.

Similar to the previous case, if $\square_i$ and $\square_{i+1}$ are in different rows and different columns then it follows from Theorems \[A.5\] and \[A.9\] that applying $e_i$ to $e_{i+1}(T)$ subtracts one from the entry in position $\square_{i+1}$, while applying $f_{i+1}$ to $e_i e_{i+1}(T)$ adds one to $\square_{i+2}$, while applying $f_i$ to $f_{i+1} e_i e_{i+1}(T)$ adds one to $\square_i$, as illustrated by the following picture where $i = 3$:

Thus we have $f_i f_{i+1} e_i e_{i+1}(T) = s_i \ast T = \varrho_i(T)$ as desired.

Assume instead that $\square_i$ and $\square_{i+1}$ are in the same row. Then it possible for $\square_i$ and $\square_{i+2}$ to be in consecutive diagonal positions if the entry in $\square_{i+1}$ is primed, in which case the effect of applying $e_{i+1}$, $e_i$, $f_{i+1}$, and $f_i$ successively to $T$ is represented by the following picture, and gives $s_i \ast T = \varrho_i(T)$ as needed:

Here the application of $e_i$ invokes case R2(c) of Theorem \[A.9\]. If $\square_i$ and $\square_{i+1}$ are in the row but the preceding situation does not occur, then it is only possible for $i + 1, i + 2, i$ to be a subword of $\text{shword}(T)$ if the entry in $\square_i$ is unprimed, the entry in $\square_{i+1}$ is primed, and $\square_{i+2}$ belongs to the region strictly above and weakly left of $\square_i$. In this event Theorems \[A.5\] and \[A.9\] imply that the effect of applying $e_{i+1}$, $e_i$, $f_{i+1}$, and $f_i$ successively to $T$ is represented by the following picture, and gives $s_i \ast T = \varrho_i(T)$:

The application of $e_i$ here again invokes case R2(c) of Theorem \[A.9\] and the actual location of $\square_{i+2}$ may be anywhere weakly northwest of the box shown.
Finally assume that □i and □i+1 are in the same column. Then it is only possible for i + 1, i + 2, i to be a subword of shword(T) if the entry in □i is unprimed, the entry in □i+1 is primed, and □i+2 belongs to the region strictly above and weakly left of □i+1. It follows from Theorems A.5 and A.9 that the effect of applying ei+1, ei, fi+1, and fi successively to T is represented by the following picture, and again gives s₁ ∗ T = d₁(T):

\[
T|_{[i,i+2]} = \begin{array}{c|c|c|c}
\text{□i} & \text{□i+1} & \text{□i+2} \\
\hline
i & i' & \text{□i+1}' \\
\text{□i+1}' & i & i' \\
\text{□i+2}' & i' & i' \\
\text{□i+2} & \text{□i+1}' & i \\
\text{□i+1} & i & i' \\
\hline
\end{array}
\]

Here, the application of ei invokes case R2(a) of Theorem A.9 and once again the actual location of □i+2 may be anywhere weakly northwest of the box shown.

This completes the proof of part (a), which suffices to prove the lemma.

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Figure 1: The $q_3$-crystal graph of $\text{ShTab}_3(\lambda)$ for $\lambda = (3, 1)$. 

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Figure 2: The $q_3$-crystal graph of $\mathcal{R}_3^O(\pi)$ for $\pi = (1, 3)(2, 5) \in I_2$. 
Figure 3: The $q_3$-crystal graph of $\mathcal{R}_{3}^{Sp}(\pi)$ for $\pi = (1, 4)(2, 6)(3, 5) \in I_{Z}^{FPF}$. 