Optimal Sensor Placement for Hybrid Source Localization Using Fused TOA–RSS–AOA Measurements

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Source localization techniques incorporating hybrid measurements improve the reliability and accuracy of the location estimate. Given a set of hybrid sensors that can collect combined time of arrival, received signal strength, and angle of arrival measurements, the localization accuracy can be enhanced further by optimally designing the placements of the hybrid sensors. In this article, we present an optimal sensor placement methodology, which is based on the principle of majorization–minimization (MM), for the hybrid localization technique. We first derive the Cramer–Rao lower bound of the hybrid measurement model, and formulate the design problem using the A-optimal criterion. Next, we introduce an auxiliary variable to reformulate the design problem into an equivalent saddle-point problem, and then, construct simple surrogate functions (having closed form solutions) over both primal and dual variables. The application of MM in this article is distinct from the conventional MM (that is usually developed only over the primal variable), and we believe that the MM framework developed in this article can be employed to solve many optimization problems. The main advantage of our method over most of the existing state-of-the-art algorithms (which are mostly analytical in nature) is its ability to work for both uncorrelated and correlated noise in the measurements. We also discuss the extension of the proposed algorithm for the optimal placement designs based on D and E optimal criteria. Finally, the performance of the proposed method is studied under different noise conditions and different design parameters.

1. INTRODUCTION AND RELEVANT LITERATURE

Accurate source localization has been one of the most well-studied techniques due to its wide range of applications in sonars, radars, and wireless sensor networks. A variety of source localization approaches are available in the literature, which can be classified under the following categories: time of arrival (TOA) [1], time difference of arrival (TDOA) [2], received signal strength (RSS) [3], frequency difference of arrival (FDOA) [4], and angle of arrival (AOA) [5]. The aforementioned techniques differ in the way they use sensor measurements to localize the source, and they also differ in their applicability depending on the range of the source to be localized. For instance, techniques like TOA and TDOA are generally preferred for long range source localization and RSS is usually preferred for short range localization. In the recent years, various hybrid source localization techniques, which are based on the combination of the aforementioned techniques, are proposed and are observed to be range independent. Some of the hybrid source localization techniques available in the literature are as follows: TOA–RSS [6], [7], [8], [9], [10], RSS–AOA [11], [12], [13], [14], [15], TOA–AOA [16], TDOA–AOA [17], [18], [19], and AOA–TDOA–FDOA [20], [21]. As the modern-day sensors collecting the measurements are generally capable to collect more than one type of measurements (they are capable of collecting both range and angle measurements), one can carefully design their placements, which will lead to improved localization accuracy [22], [23], [24].

The relation between the sensor locations and the Cramer–Rao lower bound (CRLB) matrix of the localization model has been well studied [25] and the optimal positions of sensors can indeed be designed by optimizing a related metric of the CRLB matrix. Based on the choice of the design metric, the optimal sensor placement techniques can be categorized as [26]: A-optimal design (minimizing the trace of CRLB matrix), D-optimal design (minimizing the determinant of CRLB matrix), and E-optimal design (minimizing the maximum eigenvalue of CRLB matrix). The most commonly used criterion is the A-optimal criterion since it denotes the average variance of the estimates, and therefore, minimizing the A-optimal criterion is equivalent to minimization of the mean square error (MSE).

A wide variety of literature on optimal sensor placement approaches for individual source localization techniques are available. For instance, in [27], the optimal sensor placement for target localization by bearing measurements was obtained via maximization of determinant of Fisher information matrix (FIM). The authors in [28] have derived the CRLB for sensor placement for TDOA measurements and found the optimal geometry to be centered platonic solids (cube, tetrahedron, etc.) with target at its center and sensors at each vertex. The D-optimal criterion was used for obtaining placement strategies for localization.
TABLE I

Comparison Between Related Previous Studies and Proposed Algorithm

| Paper | Hybrid measurements | Correlated noise | Non-uniform noise variances | Optimality criteria |
|-------|---------------------|------------------|-----------------------------|--------------------|
| [22], [24], [27], [29], [30] | No                  | No               | No                          | A                  |
| [31]–[33] | No                  | No               | Yes                         | D                  |
| [31]–[33] | No                  | No               | Yes                         | A                  |
| [40] | No                  | Yes              | No                          | D                  |
| [39], [41] | Yes                 | No               | No                          | A, D and E         |
| [42] | Yes                 | Yes              | No                          | A                  |
| [23], [43] | Yes                 | Yes              | Yes                         | A, D and E         |
| Proposed Algorithm | Yes                 | Yes              | Yes                         | A, D and E         |

using range measurements in [29] and [30], received signal strength difference (RSSD) measurements in [31], and TOA measurements in [32] and [33]. The estimation accuracy of elliptical TOA localization was improved by minimizing the trace of the CRLB in [34]. In [35] and [36], the optimal geometry is derived using a generalized inequality for the A-optimal criterion for AOA measurements. This approach was further extended to RSS in [37] and TOA in [38]. The authors in [39] derived the optimal sensing directions for target localization for different types of measurements while considering the uncertainty in the prior knowledge of the target position. A unified framework for designing optimal sensor orientations based on A, D, and E optimal criteria was developed for different types of measurement models in [40].

In the case of hybrid localization methods, very few methods are available to design optimal sensor placements; this may be due to complicated dependence of the hybrid CRLB matrix in terms of the sensor locations. In [41], optimal velocity configurations were derived for a sensor-target geometry to localize a stationary emitter using TDOA-FDOA measurements. Using D-optimal design, both the sensor placements and velocity configurations were derived in [42] for hybrid TDOA-FDOA sensors. The authors of [43] have designed optimal sensor placement strategies by considering the A-optimal design for the hybrid TOA–RSS–AOA model; however, their approach is restrictive as they assume the noise in the measurements to be uncorrelated with uniform noise variance, and moreover, the design metric is limited only to A-optimal criterion. A brief summary of the various works from the literature on optimal sensor placement techniques is given in Table I. As noted previously, most of the aforementioned methods for the hybrid model invariably assume the noise in the measurements to be uncorrelated and uniform, which is quite restrictive as there are many applications, like underwater surveillance [44], where the noise in the sensor measurements are correlated, and in almost all real-life scenarios, the noise variance in the measurements will not be equal. So, there is a need to devise a method that can effectively design optimal sensor geometries in the case of hybrid localization models for both correlated and uncorrelated noise (uniform and nonuniform variances) and include all the optimal designs (A, D, and E).

In this article, for the hybrid TOA–RSS–AOA measurement model, an optimal sensor placement algorithm that can optimize all the three design criteria and handle correlated noise measurements has been proposed. The associated nonconvex design problem is first reformulated using either the Fenchel or the Lagrangian formulation into a saddle-point problem. The equivalent saddle-point problem is then solved using the principle of majorization–minimization (MM), which is applied over both primal and dual variables. The proposed algorithm can work efficiently for all types of noise models.

We would like to note that the ADMM-based approach proposed in [40] is not extendable for hybrid source localization model, as extending the framework in [40] to handle the hybrid CRLB is not straightforward due to the presence of additional rotation matrix in the FIM of the AOA model. On the other hand, the proposed algorithm can easily handle the design objective based on CRLB for the hybrid measurement model.

The novelty and the key contributions of our work are listed as follows:

1) We derive the CRLB of the hybrid TOA–RSS–AOA model. The CRLB is derived for general nondiagonal noise covariance matrices (assuming the noise in the measurements to be correlated). Although the derivation of CRLB for the hybrid model is standard and can be found in the literature, unlike the commonly used parameterization of expressing the elements of CRLB in terms of highly nonlinear trigonometric functions, we parameterize the elements of the CRLB matrix in terms of some unit norm vectors, which will ease the complexity of the design criterion and help us to devise a numerical algorithm. The aforementioned reparameterization can swiftly handle the variations in the expressions of the CRLB matrices associated with different measurement models.

2) We then formulate the design objective by considering A, D, and E optimal criteria based on the hybrid CRLB. For each of the design objective, we derive a monotonic algorithm based on the MM principle. The MM technique developed in this article is novel and unlike the conventional MM methods, which work only on the primal optimization variable, it exploits both primal and dual variable updates in its steps. More precisely, we reformulate each of the optimal design problem (A, D, and E) into a
saddle-point problem and design novel upperbounds over primal and dual variables. The general framework of our primal-dual MM algorithm can inspire researchers to solve many different optimization problems where devising the conventional MM is not easy.

3) We discuss the computational complexities of the proposed algorithm for the three design problems, and discuss the proof of convergence.

4) We present extensive numerical simulations under different noise models (correlated, uncorrelated, uniform, and nonuniform) to prove the effectiveness of proposed method in designing optimal sensor locations for the hybrid model. We also present studies based on MSE for the optimal orientations obtained via our approach.

The rest of this article is organized as follows: Section II discusses the data model for the hybrid measurements and formulates the design problem for optimal sensor placement. Section III gives a brief overview of the MM principle, introduces the proposed MM-based algorithm, presents its extensions to deal with D and E optimality criteria, and discusses its computation complexity and convergence. Section IV gives the details of numerical simulations and the results. Finally, Section V concludes this article.

Notations: Vectors, matrices, and scalars are represented by bold lowercase letters (e.g., \( \mathbf{a} \)), bold uppercase letters (e.g., \( \mathbf{A} \)), and italic letters (e.g., \( a \), \( A \)), respectively. The \( i \)th element of vector \( \mathbf{a} \) is denoted as \( a_i \). The \( i \)th column and the \((i, j)\)th element of the matrix \( \mathbf{A} \) are denoted as \( \mathbf{a}_i \) and \( A_{i,j} \), respectively. For a vector \( \mathbf{a} \), its Euclidean norm is denoted as \( \| \mathbf{a} \| \). The transpose, inverse, square, and square root of a matrix are defined as \( \mathbf{A}^T \), \( \mathbf{A}^{-1} \), \( \mathbf{A}^2 \), and \( \sqrt{\mathbf{A}} \), respectively. The notations \( \text{Tr}(\mathbf{A}) \), \( \lambda_{\text{max}}(\mathbf{A}) \), \( \lambda_{\text{min}}(\mathbf{A}) \), and \( \| \mathbf{A} \|_F \) represent trace, maximum eigenvalue, minimum eigenvalue, and Frobenius norm of a matrix, respectively. In \( \cdot \) represents natural logarithm and \( \mathbf{I}_n \) represents the identity matrix of size \( m \times n \). The subscript \( \mathbf{a}_k \), \( \mathbf{a}_k \) denotes the estimate of the vector \( \mathbf{a} \) at the \( k \)th iteration. A diagonal matrix is represented as \( \text{diag}(\mathbf{a}) \), where \( \mathbf{a} \) is the vector containing the diagonal elements, and a block diagonal matrix is represented as \( \text{blkdiag}(\mathbf{A}, \mathbf{B}) \), where \( \mathbf{A} \) and \( \mathbf{B} \) are the smaller submatrices. The notation \( \langle \cdot, \cdot \rangle \) denotes inner product, \( S_n \) denotes the set of symmetric positive semidefinite matrices of size \( n \times n \), and \( \mathbb{N} \) denotes the set of natural numbers. The notation \( \mathcal{N}(\mu, \sigma^2) \) denotes Gaussian distribution with mean \( \mu \) and variance \( \sigma^2 \), \( \mathcal{U}(a, b) \) denotes uniform distribution taking values between \( a \) and \( b \), and \( p(x) \) denotes the probability density function of the random variable \( x \).

II. PROBLEM FORMULATION

In this section, we formulate the optimization problem for optimal sensor placement to accurately localize a static target using hybrid measurements from \( m \) stationary sensors in an \( n \)-dimensional space. For convenience, we present the derivation for the 2-D case and mention the necessary changes to be adopted for the 3-D case as a separate remark later. From hereon, we will use the terms source and target interchangeably.

Fig. 1 shows the hybrid localization geometry in which four sensors collect TOA, RSS, and AOA measurements of a source. Let the position coordinates of the source/target and the \( i \)th sensor be denoted by \( \mathbf{r} = [r_x, r_y]^T \) and \( \mathbf{s}_i = [s_{xi}, s_{yi}]^T \), respectively. Assuming each sensor obtain its corresponding TOA, RSS, and AOA measurements, the TOA measurement at the \( i \)th sensor is given as follows:

\[
t_i = \frac{\| \mathbf{r} - \mathbf{s}_i \|}{c} + \mu_i, \; \forall i
\]  

where \( c \) is the speed of the signal (we have considered only line of sight propagation) and \( d_i \triangleq \| \mathbf{r} - \mathbf{s}_i \| \) is the distance between the \( i \)th sensor and the target. The measurements are assumed to be corrupted by zero-mean Gaussian noise \( (\mu_i) \). Computing the corresponding range measurements from (1), we have

\[
z_i = d_i + c\mu_i \; \forall i. \tag{2}
\]

Integrating the TOA measurements from \( m \) sensors, we get the following measurement model:

\[
\mathbf{z} = \mathbf{g}_{\text{TOA}}(\mathbf{r}) + \boldsymbol{\xi}_{\text{TOA}} \tag{3}
\]

where \( \mathbf{z} = [z_1, \ldots, z_m]^T \) are the measurements, \( \mathbf{g}_{\text{TOA}}(\mathbf{r}) = [d_1, \ldots, d_m]^T \) and the error vector is \( \boldsymbol{\xi}_{\text{TOA}} = [\epsilon_{\mu_1}, \ldots, c\epsilon_{\mu_m}]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\text{TOA}}) \).

The RSS measurement (average power \( p_i \) [in decibel]) received at the \( i \)th sensor is given as follows [45]:

\[
p_i = p_0 - 10\alpha \log_{10}(d_i) + v_i, \; \forall i \tag{4}
\]

where \( p_0 \) is the signal power at the target and \( \alpha \) is the path loss constant, which depends on the medium. It is assumed that the parameters \( p_0 \) and \( \alpha \) are known (determined using calibration [45]). The noise \( (v_i) \) in the measurements is
assumed to be Gaussian noise. Integrating the RSS measurements from \( m \) sensors, we get

\[
p = \eta \mathbf{g}_{\text{RSS}}(\mathbf{r}) + \xi_{\text{RSS}}
\]

where \( p = [p_1 - p_0, \ldots, p_m - p_0]^T \), \( \eta = \frac{-10\alpha}{\ln(10)} \), \( \mathbf{g}_{\text{RSS}}(\mathbf{r}) = [\ln(d_1), \ldots, \ln(d_m)]^T \), and \( \xi_{\text{RSS}} = [v_1, \ldots, v_m]^T \sim \mathcal{N}(\mathbf{0}, \Sigma_{\text{RSS}}) \).

The AOA measurement at the \( n \)th sensor is given by

\[
\theta_n = \tan^{-1}\left( \frac{r_y - s_{ni}}{r_x - s_{xi}} \right) + \omega_n \quad \forall i
\]

where \( \tan^{-1}(\cdot) \) is the 4-quadrant arctangent. If we assume the measurements are corrupted by Gaussian noise \( (\omega_n) \), then the AOA measurement model for \( m \) sensors is given as

\[
\theta = \mathbf{g}_{\text{AOA}}(\mathbf{r}) + \xi_{\text{AOA}}
\]

where \( \theta = [\theta_1, \ldots, \theta_m]^T \), \( \mathbf{g}_{\text{AOA}}(\mathbf{r}) = \left[ \tan^{-1}\left( \frac{r_y - s_{n1}}{r_x - s_{x1}} \right), \ldots, \tan^{-1}\left( \frac{r_y - s_{nm}}{r_x - s_{xm}} \right) \right]^T \), and \( \xi_{\text{AOA}} = [\omega_1, \ldots, \omega_m]^T \sim \mathcal{N}(\mathbf{0}, \Sigma_{\text{AOA}}) \).

With the individual data models in (3), (5), and (7), the hybrid TOA–RSS–AOA measurement model can be obtained by stacking the individual models as follows:

\[
\mathbf{q} = \begin{bmatrix} \mathbf{z} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{\text{TOA}}(\mathbf{r}) \\ \eta \mathbf{g}_{\text{RSS}}(\mathbf{r}) + \xi_{\text{RSS}} \\ \mathbf{g}_{\text{AOA}}(\mathbf{r}) + \xi_{\text{AOA}} \end{bmatrix}.
\]

The FIM matrix for the hybrid model is defined as

\[
\mathbf{F}_{\text{Hybrid}} = \mathbb{E} \left[ \left( \frac{\partial \ln p(\mathbf{q}; \mathbf{r})}{\partial \mathbf{r}} \right) \left( \frac{\partial \ln p(\mathbf{q}; \mathbf{r})}{\partial \mathbf{r}} \right)^T \right] \in \mathbb{R}^{n \times n}
\]

where \( p(\mathbf{q}; \mathbf{r}) \) is the joint probability density function of the hybrid model. If we assume the noise in the three localization models \( (\xi_{\text{TOA}}, \xi_{\text{RSS}}, \text{and } \xi_{\text{AOA}}) \) are independent\(^1\) of each other, the joint probability density function will be proportional to the product of their individual probability density functions

\[
p(\mathbf{q}; \mathbf{r}) \propto p(\mathbf{z}; \mathbf{r}) \times p(\mathbf{p}; \mathbf{r}) \times p(\theta; \mathbf{r}).
\]

The FIM for the hybrid measurements can therefore be written as [36]

\[
\mathbf{F}_{\text{Hybrid}}(\mathbf{r}) = \mathbf{F}_{\text{TOA}}(\mathbf{r}) + \mathbf{F}_{\text{RSS}}(\mathbf{r}) + \mathbf{F}_{\text{AOA}}(\mathbf{r})
\]

where \( \mathbf{F}_{\text{TOA}}, \mathbf{F}_{\text{RSS}}, \) and \( \mathbf{F}_{\text{AOA}} \) are the individual FIM matrices and are given by (9) with the joint probability density function replaced by their individual probability density functions.

\(^1\)Every hybrid sensor comprises of different mechanisms to collect the TOA, RSS, and AOA measurements. Therefore, it is natural to assume the noise in the different measurements to be independent of each other. However, the noise in individual measurements may be correlated, which we have already modelled by taking the individual noise covariance matrices to be non-diagonal.

We first derive the CRLB matrix for TOA measurement. The probability density function of the observation vector \( z \) for the TOA model is given as

\[
p(z; \mathbf{r}) = \exp \left\{ -\frac{1}{2} \left[ \mathbf{z} - \mathbf{g}_{\text{TOA}}(\mathbf{r}) \right]^T \Sigma_{\text{TOA}}^{-1} \left[ \mathbf{z} - \mathbf{g}_{\text{TOA}}(\mathbf{r}) \right] \right\}
\]

(12)

Let \( \hat{\mathbf{r}} \) be an unbiased estimator of the true target position, then the covariance matrix satisfies the following well-known inequality [46]

\[
\mathbb{E} [ (\hat{\mathbf{r}} - \mathbf{r})(\hat{\mathbf{r}} - \mathbf{r})^T ] \geq \mathbf{C}_{\text{TOA}}(\mathbf{r}) = \mathbf{F}_{\text{TOA}}^{-1}(\mathbf{r})
\]

(13)

where \( \mathbf{C}_{\text{TOA}}(\mathbf{r}) \) is the CRLB matrix and is given as

\[
\mathbf{C}_{\text{TOA}}(\mathbf{r}) = \mathbf{F}_{\text{TOA}}^{-1}(\mathbf{r}) = (J^T \Sigma_{\text{TOA}}^{-1} J)^{-1}
\]

(14)

where

\[
J = \begin{bmatrix} \cos \theta_1 \sin \theta_1 \\ \vdots \\ \cos \theta_m \sin \theta_m \end{bmatrix} = \begin{bmatrix} (r_{x-y})/|r-x| \\ (r_{y-z})/|r-z| \end{bmatrix}.
\]

(15)

The matrix \( J \) is referred to as the orientation matrix and each row of this matrix denotes the unit vector pointing in the direction of the line connecting the \( i \)th sensor and the target.

From hereon, we will express the CRLB (or the FIM) matrix only in terms of \( J \) (as \( \mathbf{C}_{\text{TOA}}(J) \)) as it can be seen from (15) that the rows of \( J \) are only dependent on the orientation and not the actual position of the sensors or the target.

**Remark 1** The parameterization of orientation matrix \( J \) in (15) is stated in both the cartesian and polar forms. Most of the literature utilizes the polar form, which is specified in terms of “azimuth” angle, thereby, resulting in complex trigonometric expressions for the elements of the CRLB matrix. In order to simplify the design problem associated with the CRLB matrix, we will prefer the elements of the \( J \) to be in the cartesian coordinates. We can easily derive the respective azimuth angles from the cartesian representation, if required.

Similar to the derivation of \( \mathbf{C}_{\text{TOA}} \), the CRLB for RSS and AOA can be derived, and their FIM are given by [40]

\[
\mathbf{F}_{\text{RSS}}(\mathbf{J}) = \eta^2 J^T D \Sigma_{\text{RSS}}^{-1} DJ
\]

(16)

\[
\mathbf{F}_{\text{AOA}}(\mathbf{J}) = U^T J^T D \Sigma_{\text{AOA}}^{-1} DJ U
\]

(17)

where \( D \) is the distance matrix and \( U \) is the unitary matrix, which are given by

\[
D = \text{diag} (1/d_1, \ldots, 1/d_m); \quad U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

(18)

**Remark 2** The CRLB for RSS and AOA depends on the distance matrix \( D \) and the CRLB of the AOA model also involves an extra unitary matrix \( U \) (however, its elements are known). Before venturing to derive the CRLB for the hybrid measurement model, it is important to note that any design objective that is based on the CRLB (for RSS and...
AOA) will depend on the target-sensor distances \((d_i)\) (which is not known). So, a usual approach is to perform the optimal placement design based on an initial coarse target estimate. This initial target position (which maybe very rough) is assumed to be known by some alternate means and using which the orientation of the sensors are obtained. Although, this design of optimal placements by using a rough target location looks heuristic, it works very well in practice even if there is a slight mismatch between assumed target location in design and the true underlying unknown target location. In the numerical section, we will present MSE analysis for this mismatch issue.

With the individual FIMs mentioned in (14), (16), and (17), the FIM of the hybrid measurements \((F_{\text{Hybrid}})\) is given by

\[
F_{\text{Hybrid}}(J) = J^T \Sigma_{\text{TOA}}^{-1} J + \eta^2 J^T D \Sigma_{\text{RSS}}^{-1} DJ
+ U^T J^T \Sigma_{\text{AOA}}^{-1} DJU
\]

and the corresponding hybrid CRLB matrix is given as

\[
C_{\text{Hybrid}}(J) = (J^T \Sigma_{\text{TOA}}^{-1} J + \eta^2 J^T D \Sigma_{\text{RSS}}^{-1} DJ
+ U^T J^T \Sigma_{\text{AOA}}^{-1} DJU)^{-1}.
\]

We would like to note that the hybrid CRLB matrix is the function of the orientation matrix, and we can optimize a scalar metric of CRLB to come up with optimal orientation function of the orientation matrix, and we can optimize a

\[
\text{arg min}_J \text{Tr}\left[(J^T \Sigma_{\text{TOA}}^{-1} J + \eta^2 J^T D \Sigma_{\text{RSS}}^{-1} DJ\right.
+ U^T J^T \Sigma_{\text{AOA}}^{-1} DJU)^{-1}\]
\]

s.t. \(J^T J_i = 1, i = 1, \ldots, m\)

\[(21)\]

where \(J_i\) is the \(i\)th column of \(J^T\). The problem (21) can be rewritten as

\[
\min_{J \in \mathcal{D}} \text{Tr}\left[(J^T (\Sigma_{\text{TOA}}^{-1} + \eta^2 D \Sigma_{\text{RSS}}^{-1} D) J
+ U^T J^T \Sigma_{\text{AOA}}^{-1} DJU)^{-1}\right]\]

\[(22)\]

where the set \(\mathcal{D}\) denotes the constraint \(J^T J_i = 1 \forall i\). Using \(H = [J^T, U^T J^T]^T\) and \(R = \text{blkdiag}(R_1, R_2)\), where \(R_1 = \Sigma_{\text{TOA}}^{-1} + \eta^2 D \Sigma_{\text{RSS}}^{-1} D\) and \(R_2 = \Sigma_{\text{AOA}}^{-1} D\), (22) can be compactly written as the following minimization problem:

\[
\min_{H, J \in \mathcal{D}} \text{Tr}\left[(H^T RH)^{-1}\right] \quad \text{s.t.} \quad H = \begin{bmatrix} J & JU \end{bmatrix}
\]

\[(23)\]

The optimization problem (23) has a nonconvex objective function and a nonconvex constraint set. Moreover, the presence of inverse of the quadratic term inside \(\text{Tr}()\) makes (23) a challenging optimization problem to solve. In the next section, we will propose an MM-based algorithm to solve this nonlinear optimization problem.

III. PROPOSED MM-BASED ALGORITHM

In this section, we first briefly discuss the MM technique, then give a detailed explanation of the proposed algorithm for the A-optimal design, and later discuss the computational complexity and the convergence of the proposed algorithm. Finally, we present the extension of the proposed algorithm for D and E optimal design.

A. MM Technique

For the hybrid model, solving the optimal sensor placement problem via analytical approach is not possible (for the 3-D case and as well as for the case of correlated and nonuniform noise variances in 2-D) because of its nonconvex nature, and moreover, in the case of E-optimal design, the objective is also not differentiable. This motivates us to use the MM framework to develop an iterative algorithm for optimal sensor placement that is independent of the type of the noise and applicable to all design criteria. The MM technique develops problem-driven algorithms by exploiting the problem structure, and it works in two steps: a majorization step and a minimization step. In the majorization step, a surrogate function \(g(x|x_t)\) that globally upperbounds the objective function \(f(x)\) (that is to be minimized) at a point \(x_t\) and satisfies the following properties is constructed:

\[
g(x|x_t) \geq f(x) \quad \text{and} \quad g(x_t|x_t) = f(x_t) \quad \text{for} \quad x_t \in \mathcal{D}.
\]

In the minimization step, the surrogate function is minimized to give the next update:

\[
x_{t+1} = \arg \min \limits_x g(x|x_t).
\]

This leads to an iterative algorithm, which is repeated until convergence to generate a nonincreasing sequence \([f(x_t)]_{t \in \mathbb{N}}\), since

\[
f(x_{t+1}) \leq g(x_{t+1}|x_t) \leq g(x_t|x_t) = f(x_t).
\]

The idea of MM can also be extended to maximization problems by replacing the upperbound minimization step to lowerbound maximization step. For more details on MM, its applications, and the construction of surrogate functions, refer to [47] and the references therein. We would like to point out here that the general description on the MM technique included previously is for problems having only primal variable. However, when we develop our algorithm later, we will show how the idea of MM can be extended to handle saddle point problems, which requires constructing MM upperbounds on both primal and dual variables.

B. Proposed Algorithm for the A-Optimal Design Problem

Let us restate the optimization problem for hybrid TOA–RSS–AOA

\[
\min_{H, J \in \mathcal{D}} \text{Tr}\left[(H^T RH)^{-1}\right] \quad \text{s.t.} \quad H = \begin{bmatrix} J & JU \end{bmatrix}
\]

\[(27)\]

To deal with the complicated objective function in (27), we introduce an auxiliary variable \(\Phi\) and reformulate the
original problem (27) using the Fenchel conjugate representation [48] of the objective function into the following equivalent saddle-point problem:

$$\max_{H,J \in \mathcal{D}} \min_{\Phi \geq 0} \text{Tr}(\Phi H^T RH) - \text{Tr}(\sqrt{\Phi}) \quad \text{s.t.} \quad H = \begin{bmatrix} J \\ JU \end{bmatrix}. \quad (28)$$

Here, the variables $H$ and $J$ can be interpreted as the primal variables, and the variable $\Phi$ as the dual variable of the optimization problem. The equivalence of (27) and (28) is stated as Lemma 9 in the Appendix.

We now proceed to derive an algorithm to solve the maximin problem in (28). Let

$$f(H) = \min_{\Phi \geq 0} \text{Tr}(\Phi H^T RH) - \text{Tr}(\sqrt{\Phi}) \quad (29)$$

be a function only in the primal variable $H$ and the optimization problem in (28) be compactly written as

$$\max_{H,J \in \mathcal{D}} f(H) \quad \text{s.t.} \quad H = \begin{bmatrix} J \\ JU \end{bmatrix}. \quad (30)$$

For a fixed $\Phi$ and $R \in S^m$, the term $\text{Tr}(\Phi H^T RH)$ in (29) is a convex function in $H$. Since (30) is a maximization problem over $H$, the MM technique can be utilized to lowerbound $f(H)$ (at a given $H$, which of course can be obtained using a given $J$), and iteratively maximize the lowerbound. We obtain the lowerbound via the following Lemma.

**Lemma 3**: Given $H$, the convex function $\text{Tr}(\Phi H^T RH)$ can be lowerbounded as

$$\text{Tr}(\Phi H^T RH) \geq 2\text{Tr}(\Phi H_i^T RH_i) - \text{Tr}(\Phi H_i^T RH_i) \quad (31)$$

with equality achieved at $H = H_i^2$.

Using (31), $f(H)$ can be lowerbounded as

$$f(H) \geq \min_{\Phi \geq 0} 2\text{Tr}(\Phi H_i^T RH_i) - \text{Tr}(\Phi H_i^T RH_i) - \text{Tr}(\sqrt{\Phi}) \quad \triangleq \ g_f(H|H_i) \quad (32)$$

giving rise to the following surrogate maximization problem:

$$\max_{H,J \in \mathcal{D}} \min_{\Phi \geq 0} 2\text{Tr}(\Phi H_i^T RH_i) - \text{Tr}(\Phi H_i^T RH_i) - \text{Tr}(\sqrt{\Phi}) \quad \text{s.t.} \quad H = \begin{bmatrix} J \\ JU \end{bmatrix}. \quad (33)$$

On substituting for $R$ and $H$ into (33), we get the following optimization problem:

$$\max_{J \in \mathcal{D}} \min_{\Phi \geq 0} 2\text{Tr}(\Phi J_i^T R_iJ_i) + 2\text{Tr}(\Phi U^T J_i^T R_iJ_i) - \text{Tr}(\sqrt{\Phi}) \quad (36)$$

$$- \text{Tr}(\Phi J_i^T R_iJ_i) - \text{Tr}(\Phi U^T J_i^T R_iJ_i) \quad (34)$$

The constraint set $\mathcal{D}$ for $J$ can be relaxed to $\mathcal{D}_r$ (which is defined as $(j_i)^*$ satisfying $\sum_{i} j_i^* = 1$) as the objective in (34) is linear in $J$, and the optimum over $J$ will occur only on the boundary, and hence, the relaxation will always be tight. With this relaxation, we can now reformulate the maximin problem in (34) into a minimax problem using the following minimax theorem.

**Theorem 4** [49] Let $\mathcal{X}$ and $\mathcal{Y}$ be compact convex sets. If $f: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ is a continuous function that is concave–convex, i.e., $f(\cdot, y) : \mathcal{X} \rightarrow \mathbb{R}$ is concave for fixed $y$, and $f(x, \cdot) : \mathcal{Y} \rightarrow \mathbb{R}$ is convex for fixed $x$, then we have

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} f(x, y) = \min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} f(x, y). \quad (35)$$

Since the constraint sets (after relaxation) are compact convex sets, and the objective function in (34) is continuous, and convex in $\Phi$ for a fixed $J$ (the term $-\text{Tr}(\sqrt{\Phi})$ is a convex function in $\Phi$ via the Lieb’s concavity theorem [50]) and linear in $J$ for a fixed $\Phi$, we can switch the max and min term using Theorem 4 to arrive at the following minimax problem:

$$\min_{\Phi \geq 0} \sum_{J \in \mathcal{D}_r} 2\text{Tr}(\Phi J_i^T R_iJ_i) + 2\text{Tr}(\Phi U^T J_i^T R_iJ_i)$$

$$- \text{Tr}(\sqrt{\Phi}) - \text{Tr}(\Phi J_i^T R_iJ_i) - \text{Tr}(\Phi U^T J_i^T R_iJ_i) \quad (36)$$

The minimax problem (36) has an inner maximization problem in $J$ and an outer minimization problem in $\Phi$. We will first solve the inner maximization problem in (36). Using $Q \triangleq U^T U^T$, the inner maximization problem be written as

$$\max_{J \in \mathcal{D}_r} 2\text{Tr}(J_i^T R_iJ_i) + 2\text{Tr}(J_i^T R_iJ_i) \quad (37)$$

Let

$$Z = \Phi \tilde{A}_1 + Q \tilde{A}_2 \quad (38)$$

with $\tilde{A}_1 \triangleq J_i^T R_i$ and $\tilde{A}_2 \triangleq J_i^T R_2$, then the optimization problem in (37) can be rewritten as

$$\max_{J \in \mathcal{D}_r} 2 \sum_{i=1}^m J_i z_i \quad (39)$$

where $z_i$ denotes the $i^{th}$ column $Z$. Problem (39) has the following closed-form solution:

$$J_i^* = \frac{z_i}{\|z_i\|} \quad (40)$$

Please note that the optimum occurs on the boundary of the constraint set $\mathcal{D}_r$ as noted before.

Eliminating the variable $J$ from (36) by substituting for the optimal minimizer over $J$ (i.e., $J_i^* = \frac{z_i}{\|z_i\|}$) gives the following minimization problem in the dual variable $\Phi$:

$$\min_{\Phi \geq 0} \sum_{i=1}^m \|z_i\| - \text{Tr}(\Phi J_i^T R_iJ_i)$$

$$- \text{Tr}(\Phi J_i^T R_iJ_i) - \text{Tr}(\sqrt{\Phi}) \quad (41)$$

Substituting for $z_i$ from (38) into (41), we get

$$\min_{\Phi \geq 0} \sum_{i=1}^m \|\Phi \tilde{a}_i^+ + Q \tilde{a}_i^+\| - \text{Tr}(\sqrt{\Phi})$$

$$- \text{Tr}(\Phi J_i^T R_iJ_i) - \text{Tr}(\sqrt{\Phi}) \quad (42)$$

The proof is based on the details in [47], Sec. III-A.
where $\tilde{a}_i^1$ and $\tilde{a}_i^2$ denote the $i^{th}$ column of $\tilde{A}_1$ and $\tilde{A}_2$, respectively. Problem (42) can be rewritten as
\[
\min_{\Phi \succeq 0} 2 \sum_{i=1}^{m} \left\| \Phi a_i \right\| - \text{Tr} (H_i^T R H_i + \Phi) - \text{Tr}(\sqrt{\Phi}).
\] (43)

Using $\Phi = [\Phi, Q]$ and $a_i = [a_i^1, a_i^2]$, the optimization problem (43) can be compactly written as
\[
\min_{\Phi \succeq 0} 2 \sum_{i=1}^{m} \left\| \Phi a_i \right\| - \text{Tr} (H_i^T R H_i + \Phi) - \text{Tr}(\sqrt{\Phi}) \triangleq h(\Phi).
\] (44)

The minimization problem (44) is a convex problem [semidefinite programming problem (SDP)] in $\Phi$ and can be solved using some off-the-shelf interior point solvers. However, solving such SDP via the standard solvers will be computationally inefficient and will limit the application of our approach to only low-dimensional problems. Therefore, we once again use “MM” to solve (44), which results in a double loop MM algorithm (by double loop we mean we loop inside a loop). To this end, we construct a surrogate for the objective in (44) at some $\Phi = \Phi_k$. Since, (44) is a minimization problem, we first upperbound the term $\left\| \Phi a_i \right\|$ in $h(\Phi)$ as per the following Lemma:

**Lemma 5** Given $\Phi_k$, the $\ell_2$ norm function $\left\| \Phi a_i \right\|$ can be upperbounded as
\[
\left\| \Phi a_i \right\| \leq \frac{\left\| \Phi a_i \right\|^2}{2\|\Phi_k a_i\|} + \text{const.}
\] (45)
given that $\left\| \Phi_k a_i \right\| \neq 0$. Equality is achieved at $\Phi = \Phi_k$.

Using Lemma 5, we can upperbound $h(\Phi)$ as
\[
h(\Phi) \leq \sum_{i=1}^{m} \frac{\left\| \Phi a_i \right\|^2}{\|\Phi_k a_i\|} - \text{Tr} (H_i^T R H_i + \Phi) - \text{Tr}(\sqrt{\Phi}).
\] (46)

which can also be written as
\[
h(\Phi) \leq \sum_{i=1}^{m} \frac{a_i^T \Phi \Phi_k a_i}{\|\Phi_k a_i\|} - \text{Tr} (H_i^T R H_i + \Phi) - \text{Tr}(\sqrt{\Phi}).
\] (47)

Using $A = \sum_{i=1}^{m} \frac{a_i a_i^T}{\|\Phi_k a_i\|^2}$, (47) can be rewritten as
\[
h(\Phi) \leq \text{Tr}(\Phi A \Phi^T) - \text{Tr} (H_i^T R H_i) - \text{Tr}(\sqrt{\Phi}) \triangleq g_{\Phi}(\Phi | \Phi_k).
\] (48)

and the associated surrogate minimization problem can be written as
\[
\min_{\Phi \succeq 0} g_{\Phi}(\Phi | \Phi_k).
\] (49)

Problem (49) does not have a closed-form solution, and therefore, we once again upperbound $g_{\Phi}(\Phi | \Phi_k)$ at $\Phi = \Phi_k$.

Let $\lambda_1 = \lambda_{\max}(A)$ and $\hat{A} = A - \lambda_1 I$, then (49) can be rewritten as
\[
\min_{\Phi \succeq 0} \text{Tr}((\hat{A} \Phi^T) + \lambda_1 \text{Tr} (\hat{A} \Phi^T) - \text{Tr} (H_i^T R H_i) - \text{Tr}(\sqrt{\Phi}).
\] (50)

The first term $\text{Tr}(\hat{A} \Phi^T)$ is a concave function in $\Phi$ and can be linearized using the first-order Taylor series expansion as
\[
\text{Tr}(\hat{A} \Phi^T) = -\text{Tr}(\hat{A} \Phi_k^T) + 2\text{Tr}(\hat{A} \Phi_k^T)
\] (51)

Using (51), $g_{\Phi}(\Phi | \Phi_k)$ can be upperbounded as
\[
g_{\Phi}(\Phi | \Phi_k) \leq -\text{Tr}(\hat{A} \Phi_k^T) + 2\text{Tr}(\hat{A} \Phi_k^T)
\] (52)

It is to be noted that $g_{\Phi}(\Phi | \Phi_k)$ will also tightly upperbound $h(\Phi)$ at $\Phi = \Phi_k$. Thus, we arrive at the following surrogate minimization problem:
\[
\min_{\Phi \succeq 0} 2\text{Tr} \left( [B_1 B_2] \begin{bmatrix} \Phi \\ Q \end{bmatrix} \right) - \text{Tr} (H_i^T R H_i) + 2\lambda_1 \text{Tr}(\Phi^T) - \text{Tr}(\sqrt{\Phi}).
\] (53)

Substituting for $\Phi$ in (53) and using $B = [B_1, B_2] = \Phi_k \hat{A}$, we get the following optimization problem:
\[
\min_{\Phi \succeq 0} 2\text{Tr} \left( [B_1 B_2] \begin{bmatrix} \Phi \\ Q \end{bmatrix} \right) - \text{Tr} (H_i^T R H_i) + 2\lambda_1 \text{Tr}(\Phi^T) - \text{Tr}(\sqrt{\Phi}).
\] (54)

With $\lambda = 2\lambda_1$ and substituting $Q$ into (54), we get
\[
\min_{\Phi \succeq 0} \text{Tr} \left( [B_1 + B_1^T] \Phi \right) + \text{Tr} \left( [U^T (B_2 + B_2^T)] U \Phi \right)
\] (55)

Using $C = [B_1 + B_1^T + U^T (B_2 + B_2^T)] U - H_i^T R H_i$, (55) can be compactly written as
\[
\min_{\Phi \succeq 0} \text{Tr}(C \Phi) + \lambda \text{Tr}(\Phi^2) - \text{Tr}(\sqrt{\Phi}).
\] (56)

The optimal solution of the aforementioned problem can be computed easily based on the eigenvalue decomposition of $C$ and solving scalar cubic equations. Suppose $V$ and $E$ denote the matrices containing the eigenvectors and eigenvalues of $C$, then the minimizer for the problem will be $V X V^T$, where any diagonal element of the matrix $X$ will be computed by solving the following scalar problems:
\[
\min_{X_{ii} > 0} E_{ii} X_{ii} + \lambda X_{ii}^3 - \sqrt{X_{ii}} \forall i.
\] (57)

The Karush–Kuhn–Tucker (KKT) condition for the above problem will be
\[
E_{ii} + 2\lambda X_{ii}^2 - \frac{1}{2X_{ii}} = 0 \forall i.
\] (58)

Using $\hat{X}_{ii} = \sqrt{X_{ii}}$, and then, multiplying both sides of (58) with $2X_{ii}$ gives
\[
4\lambda (\hat{X}_{ii})^3 + 2E_{ii} \hat{X}_{ii} - 1 = 0 \forall i
\] (59)
which is nothing but a cubic equation whose positive real root will give the optimum $X^*$.

C. Solving the D and E Optimal Design Problems

The algorithm proposed in the previous subsection can be extended to D and E optimal criteria as follows.

In D-optimal design, the problem is to solve the following maximization problem:

$$
\max_{\mathbf{H}, J \in \mathcal{D}} \log |\mathbf{H}^T \mathbf{R}^H|, \quad \text{s.t. } \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (60)
$$

The presence of quadratic term inside $\log |\cdot|$ makes (60) a challenging problem to solve. Similar to the A-optimal case, we use the Fenchel conjugate representation of the log determinant function, which is given as

$$
\log |\mathbf{A}| = \min_{\mathbf{B} \succeq 0} \text{Tr} (\mathbf{A} \mathbf{B}) - \log |\mathbf{B}| \quad (61)
$$

to reformulate the problem (60) into the following equivalent saddle-point problem:

$$
\max_{\mathbf{H}, J \in \mathcal{D}} \min_{\Phi \succeq 0} \text{Tr} (\Phi \mathbf{H}^T \mathbf{R}^H) - \log |\Phi| \\
\text{s.t. } \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (62)
$$

Similar to the proof of Lemma 9 given in Appendix, it can be shown that the maximin problem (62) is equivalent to the problem (60).

In the case of E-optimal design, the objective is to solve the following maximization problem:

$$
\max_{\mathbf{H}, J \in \mathcal{E}} \lambda_{\min}(\mathbf{H}^T \mathbf{R}^H) \quad \text{s.t. } \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (63)
$$

(Typically, in the E optimal design, the objective is to minimize the maximum eigen value of the matrix $(\mathbf{H}^T \mathbf{R}^H)^{-1}$, which is equivalent to maximizing the minimum eigen value of the matrix $(\mathbf{H}^T \mathbf{R}^H)$). Unlike A and D optimal case, there is no such Fenchel representation readily available for the $\lambda_{\min}(\cdot)$ function. So, we resort to the Lagrangian dual formulation to arrive at the saddle-point formulation. We first consider the epigraph reformulation of the problem in (63)

$$
\max_{\mathbf{H}, J \in \mathcal{D}, \alpha} \alpha \\
\text{s.t. } \mathbf{H}^T \mathbf{R}^H \succeq \alpha \mathbf{I}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (64)
$$

The Lagrangian dual of the problem in (64) is given by

$$
\max_{\mathbf{H}, J \in \mathcal{D}, \alpha} \min_{\Phi \succeq 0} \alpha + \text{Tr} (\Phi \mathbf{H}^T \mathbf{R}^H) - \alpha \text{Tr}(\Phi) \\
\text{s.t. } \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (65)
$$

where $\Phi$ is the Lagrangian dual variable. On rearranging the terms in (65), we get

$$
\max_{\mathbf{H}, J \in \mathcal{D}, \alpha} \min_{\Phi \succeq 0} \text{Tr} (\Phi \mathbf{H}^T \mathbf{R}^H) + \alpha [1 - \text{Tr}(\Phi)] \\
\text{s.t. } \mathbf{H} = \begin{bmatrix} \mathbf{J} \\ \mathbf{JU} \end{bmatrix}. \quad (66)
$$

Here, we would like to note that in the saddle-point formulations in (28), (62), and (66), the term containing the primal variable “$\mathbf{H}$” remains the same, and the difference is in the second term (which is a function only in the dual variable “$\Phi$”). Thus, the difference between the algorithms for the different optimal criteria will only be in the update for “$\Phi$.” Therefore, in the following, we will discuss only necessary changes in the update of $\Phi$ for D and E optimal design problems.

In the D-optimal design, to update $\Phi$, we construct surrogate using MM (at $\Phi = \Phi_k$) in a manner similar to the A-optimal design and get the following minimization problems:

$$
\min_{\mathbf{E}, \lambda > 0} \mathbf{E}_{ii} X_{ii} + \lambda X_{ii}^2 - \log(X_{ii}) \quad \forall i. \quad (67)
$$

The KKT condition for the aforementioned problem is

$$
\mathbf{E}_{ii} + 2 \lambda X_{ii}^* - \frac{1}{\lambda^*} = 0 \quad \forall i \quad (68)
$$

which is a quadratic equation and the optimal solution will be its positive root

$$
X_{ii}^* = -\mathbf{E}_{ii} + \sqrt{\mathbf{E}_{ii}^2 + 8 \lambda} \quad \forall i. \quad (69)
$$

In the E-optimal design case, following the same steps as in the A-optimal design, we arrive at the following surrogate (minimax) problem:

$$
\min_{\mathbf{E}, \lambda > 0} \max_{\Phi \succeq 0} 2 \text{Tr}(\mathbf{J}_i^T \mathbf{R}_i \mathbf{J}_i \Phi) - \text{Tr}(\mathbf{J}_i^T \mathbf{R}_i \mathbf{J}_i) \\
+ 2 \text{Tr}(\mathbf{J}_i^T \mathbf{R}_i \mathbf{J}_i \mathbf{Q}) \quad \text{s.t. } \mathbf{Tr}(\Phi) \geq 1. \quad (71)
$$

The constraint in (71) will be tight as the objective in (71) is linear in $\Phi$, and therefore, the optimum will occur at the boundary of the constraint set. Following the steps similar to the A-optimal design for eliminating $\mathbf{J}$, we arrive at the following optimization problem:

$$
\min_{\Phi \succeq 0} 2 \sum_{i=1}^m \|\mathbf{a}_i\|_2 - \text{Tr}(\Phi \mathbf{H}_i^T \mathbf{R}_i) \\
\text{s.t. } \mathbf{Tr}(\Phi) = 1. \quad (72)
$$

The problem in (72) is a convex optimization problem (an SDP). Unlike the A and D optimal case, solving the problem in (72) via MM is very challenging, so we prefer to solve
From (26), we know that the sequence \( n, H, \phi \) (which has a total complexity \( \Phi_1 + \phi_n \) (which has a complexity of \( \Phi_1 \)) depends on the convergence of the MM update over dual \( \tilde{C} \) and the dual variable \( \phi \).

The iterative steps of the MM update over dual \( \phi \) in each iteration requires \( \text{O}(n^3) \) flops in each inner loop iteration. As \( m > n \), the overall complexity of the proposed algorithm for A and D optimal design is, therefore, of the order \( \text{O}(m^2n) \). For the E-optimal design, the dual problem is an SDP and the dual variable \( \phi \) is updated using interior point methods. According to [52], the average worst case computation complexity of semidefinite programming problem is of the order \( \text{O}(m^3) \). Table II summarizes the overall computation complexity of the proposed algorithm using the three different optimal designs.

We now briefly discuss the proof of convergence of the proposed algorithm. As the proposed algorithm is a double loop MM algorithm, we will state the convergence of the MM updates over the primal variable \( H \) and the dual variable \( \phi \) separately. Furthermore, since the convergence of the MM update over \( H \) depends on the convergence of the MM update over \( \phi \), we will first state the following Lemma:

**Lemma 7** The iterative steps of the MM update over dual variable \( \phi \) in the proposed algorithm converge to the global minimum of the dual problem (44).

**Proof** From (26), we know that the sequence \( \{\Phi_k\} \) generated by the MM update of dual variable \( \phi \) will monotonically decrease the dual objective function \( h(\phi) \). The

| Optimal Design Criterion | A-optimal | D-optimal | E-optimal |
|--------------------------|-----------|-----------|-----------|
| Computation Complexity   | \( \mathcal{O}(m^2n) \) | \( \mathcal{O}(m^2n) \) | \( \mathcal{O}(m^3) \) |

(72) via standard interior point methods like CVX [51]. The pseudocode of the proposed algorithm for TOA–RSS–AOA measurements is given in Algorithm 1, with Algorithms 2 and 3 describing the \( \Phi \) update for the three optimal designs.

**Remark 6** The proposed method can be extended for the 3-D case, however, the AOA measurement model for the 3-D case cannot be handled in the current framework as the CRLB expressions in the case of the AOA model for the 3-D will have sum of two terms (corresponding to the azimuth and elevation angles) and reformulating the hybrid CRLB matrix into the form similar to the problem in (23) is not possible. Nonetheless, we can extend the proposed
convergence of the sequence \( \{ h(\Phi_k) \} \) to a finite value at a stationary point of the dual problem can be proved using the details in Section II-C of [47] and the references therein. Moreover, since the dual problem is a convex problem, the stationary point will also be the global minimizer of \( h(\Phi) \).

Now that the convergence of the MM update over dual variable is established in Lemma 7, we move on to state the convergence of the MM update over the primal variable.

**Lemma 8** The iterative steps of the MM update over primal variable \( H \) in the proposed algorithm converge to a KKT point of the primal problem (27).

**Proof** Since, the primal problem is a minimization problem, the sequences \( \{ H_t \} \) generated by the MM update of primal variable \( H \) will monotonically decrease the objective function \( f(H) \). Similar to the details mentioned in Lemma 7, the sequence \( \{ f(H_t) \} \) generated can also be proved to converge to a finite value, and that will be a KKT point of the primal problem.\(^4\)

IV. NUMERICAL SIMULATIONS AND RESULTS

In this section, we discuss various numerical simulations under different noise settings and design parameters to evaluate the performance of the proposed method.

A. Convergence of the Proposed Algorithm

First, we check the convergence of the proposed algorithm and compare its performance with the naive way of placing sensors uniformly on a circle of chosen radius encompassing the target. Let us consider that there are four sensors, which are to be optimally placed on the circle of radius \( r \) with the target at its center. The sensors are initialized to be uniformly placed on the circumference of this circle. The noise covariance matrices corresponding to TOA, RSS, and AOA are taken to be non diagonal with its elements randomly sampled from uniform distribution \( U(0, 1) \).

In Fig. 2, we show the objective (A, D, and E design criteria, for the D-optimal, we plot \(-\log|\langle H^T R H \rangle|\), and for E-optimal, we plot \( \lambda_{\max}((H^T R H)^{-1}) \) versus iterations.\(^5\) The initial value of the primal variable \( J \) in the proposed algorithm (for all three designs) is initialized with the uniform configuration and the variable \( \Phi \) is initialized as \( \Phi_0 = \frac{1}{2} (H_0^T R H_0)^{-2} \), which is the solution of the Fenchel dual problem over the variable \( \Phi \) (as given in proof of Lemma 9 in the Appendix), where \( H_0 = [J_0^T, U^T J_0^T]^T \) and \( J_0 \) is the initial value of the primal variable. It can be seen from the plots in Fig. 2 that the proposed algorithm monotonically decreases the design objective and eventually converges to an optimum. The optimal configurations obtained are also shown in Fig. 2.

The initialization using the uniform placement also gives a comparison (in terms of the design objective) between this uniform geometry and the optimal sensor geometry obtained using the proposed algorithm. The optimal sensor placement obtained via proposed algorithm using A, D, and E optimal designs show an improvement of 40% – 70% with respect to the uniform placement. This shows that improvement in localization accuracy can be achieved using sensor-target geometry obtained by the proposed algorithm.

B. Proposed Algorithm for the 3-D Case

For the 3-D case, we have undertaken the design of placing six sensors on the surface of sphere of radius \( r \) meters with target at its center. The sensors are initialized to be uniformly placed on the surface of this sphere. As stated in Remark 6, the proposed method here is only for

\(^4\)The detailed proof of Lemmas 7 and 8 can be found in the supplementary material.

\(^5\)By iterations, we mean the iterations over the primal variable \( J \), or in other words, the outer loop of our primal dual MM algorithm.
hybrid TOA–RSS model. The measurements are corrupted with correlated noise of zero-mean and positive definite covariance matrices, which are generated similar to the 2-D case.

The proposed method is initialized using the uniform placement and the optimal sensor placements are obtained. The variation of objective with iterations are shown in Fig. 3 for all three designs. These plots show monotonic decrease in the objectives, which validates the convergence of the proposed algorithm for the 3-D case.

C. Performance in the Presence of Uncorrelated Noise Measurements

Next, we will study the performance of the proposed algorithm for the uncorrelated noise case. To this end, we have assumed that the sensors are to be placed such that the distance from the target (at origin) is always one meter and the measurements are affected by diagonal noise covariance matrices given by

$$\Sigma_{TOA} = \text{diag}[^2, \ldots, ^2],$$
$$\Sigma_{RSS} = \text{diag}[^2, \ldots, ^2],$$
$$\Sigma_{AOA} = \text{diag}[^2, \ldots, ^2].$$

(75)

For the uniform noise case, the optimal orientation matrix satisfies the following relation [40]:

$$J^T J = \frac{m}{2} I_4.$$  

(76)

The corresponding theoretical value of CRLB can also be easily calculated for this special case using a generalized linear inequality as shown in [43]. This theoretical value is given by

$$\text{Tr}(C_{\text{theo}}) = \frac{4m}{\eta^2 \sigma^2 + 1/\tau^2 + 1/\gamma}.$$ 

(77)

In the following, we present the value of the trace of the CRLB matrix obtained at convergence of our algorithm for different values of “m” with the choice of uniform noise variances ($^2 = 1$, $^2 = 1$, $^2 = 1$). All the sensor-target ranges are taken to be equal ($d_i = 1$ $\forall i$). The trace of the CRLB matrix obtained by the proposed algorithm (Tr($C_{\text{algo}}$)) and the trace of the theoretical CRLB matrix (Tr($C_{\text{theo}}$)) using (77) for different number of sensors are listed in Table III. From the table, it can be seen that the CRLB value obtained using the proposed algorithm is equivalent to the theoretical CRLB value proving the optimality of the proposed algorithm. The optimal orientation matrix obtained via our algorithm also satisfied the relation in (76).

1) Optimal Design for Uniform Sensor-Target Ranges:

In this subsection, we assume the distances between the sensors and the target are equal ($d_i = d$) and generate the optimal orientations for different values of $d$. We have considered two different configuration examples. In example 1, we assume $m = 2$, $d = 1000$ m, $^1 = ^1 = 1$ m, $^1 = ^1 = 1$ W, $\tau = \tau = 1$ rad, $\eta = -4.343$, and the initial positions (used to initialize our algorithm) for the two sensors are taken as $(0, 0.7)$ km and $(-1, 0)$ km. In example 2, we have assumed three sensors with initial...
Fig. 4. Optimal sensor trajectories of (a) example 1 and (b) example 2. ○: sensor initial position, ∗: sensor final position, □: target position.

Fig. 5. Optimal sensor trajectories and the final geometries with different sensor initial positions for \( m = 2 \) and uniform \( d_i \) considered in Section IV-C1.

Fig. 6. Optimal sensor trajectories and the final geometries with different sensor start positions for \( m = 3 \) and uniform \( d_i \) considered in Section IV-C1.

Fig. 7. Optimal sensor trajectories and the final geometries for nonuniform \( d_i \) in examples 3 and 4 considered in Section IV-C2.

positions given by \((-0.75, -0.6) \text{ km}, (-1, -0.2) \text{ km}, \) and \((0.2, -1) \text{ km}. \) The initial and final sensors positions (obtained via our proposed algorithm), for example, 1 and 2 are shown in Fig. 4. The optimal value of the A-optimal objective obtained is 2 and 1.33 m\(^2\), which is also the theoretical minimum CRLB value calculated using (77).

Next, we have designed the optimal geometries for different initial positions of the sensors for the parameter settings mentioned in examples 1 and 2. The optimal orientations obtained via our proposed approach are shown in Figs. 5 and 6. Some of the optimal configurations obtained via our proposed approach (in Figs. 5 and 6) are same as the configurations obtained via the analytical approach of [43]. The angles corresponding to the optimal sensor-target geometries shown in Figs. 4–6 are given in Table IV.

2) Optimal Design for Nonuniform Sensor-Target Range: For the analysis of the optimal geometry in case of nonuniform \( d_i \), we have considered design problems of placing 2 and 3 sensors. In the first case, the initial sensor positions and noise variances are taken to be same as in example 1 but the sensor-target ranges are taken to be \( d_1 = 2 \text{ km} \) and \( d_2 = 1 \text{ km} \). Similarly, in the second case, the initial sensor positions and noise variances are same as in example 2 but the the sensor-target ranges are taken to be \( d_1 = 2 \text{ km}, d_2 = 1 \text{ km}, \) and \( d_3 = 1.5 \text{ km} \). The optimal geometries obtained using the proposed algorithm for these examples are given in Fig. 7. The minimum of the A-optimal objective obtained are 1.99 and 1.33 m\(^2\), respectively, which is equal to the minimum theoretical values in (77). The angles corresponding to the optimal sensor-target geometries shown in Fig. 7 are given in Table IV.

D. Mean Square Error (MSE) Analysis

Here, we study the improvement in the target localization accuracy using the sensor positions obtained by the proposed approach. We will compare the MSE of the target location obtained using the optimal sensor placement given by our approach with the MSE of estimates obtained using the uniform and random sensor placements, and the sensor placements obtained using the method described in [40], which obtains optimal configurations considering only individual measurements. The target location estimate is obtained by the maximum likelihood estimation (MLE) approach, which is implemented through a 2-D grid search of the hybrid likelihood function to arrive at the most probable target location, and then, performing one step of the Gauss–Newton update on the MLE estimate. The configuration used in the simulation is given as follows: The target is considered to be present at \([0, 0]^T\) and the

| Figure | No of sensors | Distance | \( \text{Angle 1 (degrees)} \) | \( \text{Angle 2 (degrees)} \) | \( \text{Angle 3 (degrees)} \) |
|--------|---------------|----------|-----------------------------|-----------------------------|-----------------------------|
| Fig. 4(a) | 2            | Uniform | 104.95                     | 14.95                       | -                           |
| Fig. 4(b) | 3            | Uniform | 50.19                      | -9.54                       | 110.46                     |
| Fig. 5(a) | 2            | Uniform | -135.14                   | -45.14                      | -                           |
| Fig. 5(b) | 2            | Uniform | -134.14                   | 135.62                      | 33.83                      |
| Fig. 6(a) | 3            | Uniform | 93.77                      | 153.76                      | -                           |
| Fig. 6(b) | 2            | Non-uniform | 3.66                   | 114.04                      | -                           |
| Fig. 7(a) | 2            | Non-uniform | 104.25                  | 14.04                       | -                           |
| Fig. 7(b) | 3            | Non-uniform | 53.27                   | -6.84                       | 113.23                     |

TABLE IV

Angles (In Degrees) of the Optimal Sensor-Target Geometries Obtained
TABLE V
Comparison of the MLE Performance for Different Placement

| No of sensors | Placement | MSE (in sq. meters) |
|---------------|-----------|---------------------|
| m = 3         | Random    | 0.4371              |
|               | Uniform   | 0.3931              |
| (uniform noise) | TOA       | 0.2910              |
|               | AOA       | 0.1971              |
|               | RSS       | 0.1774              |
|               | Optimal   | 0.1384              |
| m = 5         | Random    | 0.2763              |
|               | Uniform   | 0.1417              |
| (uniform noise) | TOA       | 0.1387              |
|               | AOA       | 0.1252              |
|               | RSS       | 0.1143              |
|               | Optimal   | 0.1082              |
| m = 7         | Random    | 0.5830              |
|               | Uniform   | 0.0962              |
| (uniform noise) | TOA       | 0.0944              |
|               | AOA       | 0.0928              |
|               | RSS       | 0.0892              |
|               | Optimal   | 0.0867              |

TABLE VI
Comparison of the MLE Performance of Sensor Placement for Target Location Mismatch

| No of sensors | Target Position | MSE (in sq. meters) |
|---------------|-----------------|---------------------|
| m = 3         | True position   | 0.1792              |
| (Uniform noise) | Shifted position 1 | 0.1855          |
|               | Shifted Position 2 | 0.1894          |
| m = 4         | True position   | 0.0304              |
| (Non-Uniform noise) | Shifted position 1 | 0.0334          |
|               | Shifted Position 2 | 0.0362          |
| m = 5         | True position   | 0.0202              |
| (Non-Uniform noise) | Shifted position 1 | 0.0303          |
|               | Shifted Position 2 | 0.0342          |

of target location mismatch. For this, we consider sensors placed on the circumference of a unit circle with target at its center. The measurements are assumed to be corrupted by Gaussian noise. The covariance matrix of the noise is taken to be $I_m$ for the uniform noise case, and in the case of nonuniform noise, the diagonal elements of it are randomly drawn from uniform distribution of $U(0, 1)$. The target positions used in the design are taken to be $[0.1, -0.2]^T$ and $[-0.3, -0.2]^T$ for the two mismatched cases. The MSE results for different number of sensors are given in Table VI. The MSE for the target mismatch cases are only slightly higher than that for the case that assumes the true target position in the design, this proves the robustness of the design under any mismatch in the design parameters.

V. CONCLUSION

In this article, a novel MM-based algorithm for optimal sensor placement for a hybrid TOA–RSS–AOA measurement model is proposed. We first derive the design objectives based on A, D, and E-optimal criteria, and then, reformulate the design problems into saddle-point problems, which are then solved via MM by constructing simple surrogate functions (with closed-form optimizers) over both primal and dual variables. The convergence, estimation efficiency, and robustness of the algorithm in designing optimal sensor locations for the hybrid model are verified using various numerical simulations. Unlike the state-of-the-art methods, our proposed algorithm works efficiently for both uncorrelated and correlated noise in the measurements, and as well as for different sensor-target ranges. Furthermore, the MSE for the optimal orientations obtained via our proposed algorithm are found to be smaller than that of the uniform and random placements, as well as the sensor placements based on individual measurements. As a part of the future work, we would like to model the noise in the measurements to be distance dependent and try and develop a numerical procedure to obtain optimal sensor locations.

APPENDIX

LEMMA 9 The saddle-point problem (28) is equivalent to the original optimization problem (27).
PROOF Let us consider the inner minimization problem of (28) over the dual variable $\Phi$

$$\min_{\Phi \succeq 0} \text{Tr} (\Phi H^T R H) - \text{tr} (\sqrt{\Phi}).$$

(78)

The KKT condition for (78) is given by

$$H^T R H - \frac{1}{2} \left( \Phi^* \right)^{-1/2} = 0.$$  

(79)

Solving (79), we get the following optimal value:

$$\Phi^* = \frac{1}{4} (H^T R H)^{-2}.$$  

(80)

Substituting back $\Phi^*$ in (28), we get

$$\max_{H \in \mathbb{R}} -\text{Tr} \left( \frac{1}{4} H^T R H \right)^{-1}, \quad \text{s.t.} \quad H = \begin{bmatrix} J \\ JU \end{bmatrix}.$$  

(81)

Leaving out the constant $\frac{1}{4}$ and converting (81) into a minimization problem gives the original problem (27). Hence, the equivalence of (27) and (28) is proved.

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