Quiver matrix model of ADHM type and BPS state counting in diverse dimensions

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We review the problem of BPS state counting described by the generalized quiver matrix model of ADHM type. In four dimensions the generating function of the counting gives the Nekrasov partition function and we obtain generalization in higher dimensions. By the localization theorem, the partition function is given by the sum of contributions from the fixed points of the torus action, which are labeled by partitions, plane partitions and solid partitions. The measure or the Boltzmann weight of the path integral can take the form of the plethystic exponential. Remarkably after integration the partition function or the vacuum expectation value is again expressed in plethystic form. We regard it as a characteristic property of the BPS state counting problem, which is closely related to the integrability.

Subject Index
A10 Integrable systems and exact solutions, B27 Topological field theory

1. Introduction

It is well known the instantons (anti-self-dual connections) in four dimensional gauge theory allow ADHM construction [1],[2]. From the viewpoint of string theory the ADHM description can be obtained by considering $D4-D0$ system in type IIA string theory, where the matrices, which are basic dynamical variables in ADHM construction, come from the open strings connecting $D$-branes$^1$ [3],[4],[5],[6]. The low energy effective theory on the world volume of $D$-branes is the dimensional reduction of ten dimensional super Yang-Mills theory. The original anti-self-duality of the gauge field is translated to the BPS condition for the world volume theory on $D4$-branes, while the ADHM equations are obtained as the BPS condition on $D0$-branes. Since the world volume of $D0$-branes has no spacial direction, the theory is reduced to supersymmetric quantum mechanics (in fact matrix model), which we call ADHM matrix model.

The ADHM description of the four dimensional instantons (or the BPS solitons in five dimensional theory from the viewpoint of $M$-theory) also plays a significant role in the computation of the instanton partition function of Nekrasov [7],[8],[9], which provides a microscopic derivation of the Seiberg-Witten prepotential of four dimensional $\mathcal{N} = 2$ Yang-Mills theory. By introducing sufficiently large number of torus action on the ADHM moduli space, we can employ the Atiyah-Bott type localization formula to compute the path integral.

$^1$In this article we only consider $U(n)$ gauge theory.
The fixed points of the topic action are isolated and labeled by a tuple of partitions (or Young diagrams). Then the partition function is expressed as a summation over the contribution from each fixed point, which is in turn given by the equivariant character of the tangent space at the fixed point as a module of the torus action.

In the following we will argue some of intriguing aspects in generalizing this story to higher dimensions by replacing $D4$-branes with $Dp$ ($p = 2d = 6, 8$)-branes, where the fixed points are labeled by higher dimensional generalizations of the partition, called plane partition ($d = 3$) and solid partition ($d = 4$). The BPS condition on $D6$ and $D8$-branes can be identified with the higher dimensional instanton equation in six and eight dimensions, respectively [10], while the BPS condition on $D0$-branes gives what we call quiver matrix model of ADHM type. In the same manner as the four dimensional case, the moduli space $\mathcal{M}_{n,k}$ of the quiver matrix model is topologically labeled by the number $n$ of $Dp$-branes and the number $k$ of $D0$-branes. We call $k$ instanton number in analogy with the four dimensional case. To obtain the partition function of $U(n)$ gauge theory on $Dp$-brane, we will fix $n$ and take a summation of $k$ over non-negative integers.

1.1. Fixed points of the torus action and $(d-1)$-partitions

Let $t_i$ collectively denote equivariant parameters of the torus action on the moduli space $\mathcal{M}_{n,k}$ of matrix equations of ADHM type. In general, they consist of the equivariant parameters of the torus action on the (flat) space-time coordinate $(z_1, \cdots, z_d) \in \mathbb{C}^d$ (the $\Omega$ background parameters of Nekrasov), the Cartan subgroup of the gauge symmetry $G_C = U(n)$ (the Coulomb moduli parameters) and of the flavor symmetry $G_F$ (mass parameters). We can identify the equivariant $K$ group of a point with the ring of Laurent polynomials in the equivariant parameters $K_{T(\text{pt})} = \mathbb{C}[t_i^{\pm 1}]$. Hence the equivariant character at the fixed points takes the value in $K_{T(\text{pt})}$.

\begin{equation}
\pi = \begin{pmatrix}
5 & 3 & 2 & 2 & 1 \\
4 & 2 & 2 & 1 \\
2 & 1 \\
1
\end{pmatrix}, \quad |\pi| = 26.
\end{equation}

Recall that a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ is a non-increasing sequence of positive integers such that $\lambda_{\ell+1} = 0$ for finite $\ell$. It is useful to represent $\lambda$ in terms of the Young diagram. We denote $|\lambda| = \sum_{i=1}^\infty \lambda_i$, which is the total number of boxes (cells) in the corresponding Young diagram. We can consider higher dimensional generalization or a $(d-1)$-partition $\pi = \{\pi_{i_1, \cdots, i_{d-1}}\} (i_1, \cdots, i_{d-1}) \in \mathbb{Z}_{>0}^{d-1}$, which is an array of positive integers with (obvious) higher dimensional generalization of the non-increasing condition; for example $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$ when $d = 3$ and $\pi_{i_1, \cdots, i_{d-1}} \neq 0$ for only finite set of $(i_1, \cdots, i_{d-1})$.
(see Fig.1). When \( d = 3 \) and \( d = 4 \), it is usually called plane and solid partition, respectively. We define \( |\pi| = \sum_{(i_1, \ldots, i_{d-1})} \pi_{i_1, \ldots, i_{d-1}} \), which means the volume (the number of boxes, cubes \( \cdots \)) of the \((d-1)\)-partition \( \pi \). It turns out that the fixed points of the toric action on \( \mathcal{M}_{n,k} \) are isolated and in one to one correspondence with the set of \( n \)-tuples of \((d-1)\)-partitions \( \vec{\pi} = (\pi^a)_{a=1}^n \), and that \( |\vec{\pi}| := \sum_{a=1}^n |\pi^a| \) is identified with the instanton number \( k \). Thus in the sector of instanton number \( k \), we can reduce the quiver matrix model to a statistical model with the configuration space \( \Pi^k_n := \{ \vec{\pi} | |\vec{\pi}| = k \} \), where the equivariant character at each fixed point \( \vec{\pi} \) gives the Boltzmann weight of the model.

### 1.2. Partition function and plethystic exponential

It is interesting that the Boltzmann weight derived from the ADHM matrix model takes the form of the plethystic exponential (see section 2 for definition) \( \text{P.E.}[\chi_{\vec{\pi}}(t_i)] \). We define the topological partition function by a weighted sum over the total configuration space \( \cup_{k \geq 0} \Pi^k_n \), where introducing the box counting parameter \( q \), we multiply the volume (the number of boxes, cubes \( \cdots \)) of \((d-1)\)-partitions \(|\pi|\) as the additional Boltzman weight;

\[
Z_{\text{top}}(t_i; q) := \langle \text{P.E.}[\chi_{\vec{\pi}}(t_i)] \rangle = \sum_{\pi} q^{|\pi|} \text{P.E.}[\chi_{\vec{\pi}}(t_i)]. \tag{1.1}
\]

Namely if we identify the instanton number \( k \) as the particle number, the topological partition function corresponds to the grand canonical ensemble in statistical mechanics. The phenomena on which we will focus in this article is that in the computation of the topological partition function, the expectation value of the plethystic exponential is again expressed by the plethystic exponential;

\[
\langle \text{P.E.}[\chi_{\pi}(t_i)] \rangle = \text{P.E.}[F(t_i; q)]. \tag{1.2}
\]

Since the plethystic exponential can be regarded as the character of the symmetric algebra \( S^\bullet V \) of a \( G \)-module \( V \), this is an example of “super”-integrability that the expectation value of the character gives another character, which we encounter typically in matrix model and plays an important role for extending the realm of symmetric functions [11],[12].

When \( d = 3 \) with the computation of the topological partition function \( Z_{\text{top}}(t_i; q) \) we may associate equivariant (or \( K \)-theory) vertices [13], which are generalizations of the refined topological vertex [14],[15]. In fact in an appropriate limit of the \( \Omega \) background parameters \( q_i \), the equivariant vertex reduces to the refined topological vertex. Since the refined topological vertex is characterized as the intertwining operator of the quantum toroidal algebra of \( \mathfrak{gl}(1) \) [16], it is tempting to expect some quantum algebras behind the “super”-integrability (1.2). Furthermore, since physically the partition function (1.1) is nothing but the generation function of the numbers of BPS states, this seems to be along the same line of BPS/VOA correspondence, the correspondence of the algebra of BPS states with the chiral algebra of some 2 dimensional CFT. It may be interesting to look at cohomological Hall algebra associated with the quiver of ADHM type [17].

The paper is organized as follows; In the next section we introduce the plethysite exponential. We can regard it as the character of the symmetric algebra and hence it plays a significant role in this article. After presenting ADHM type matrix model equations coming from the BPS condition of \( D \)-brane system in section 3, we discuss a matrix model formulation or the measure for eigenvalues of matrices in section 4. The measure is given in terms
of the plethystic exponential and hence naturally expressed by the power sum functions of eigenvalues. In section 5 we compute the equivariant character of the tangent space at the fixed points. Finally we present the plethysitic forms of the partition function inspection 6.

In each section after section 4, we first review the well-established case of $d = 2$ (the original ADHM equation) and then try to generalize it to higher dimensions. From the view point of mathematics, one of the crucial points is that though the fixed points of the torus action are still isolated and labeled by higher dimensional generalization of the partition, the tangent space at each fixed point is not smooth anymore and it is defined only virtually.

I would like to dedicate this article to the memory of Prof. Tohru Eguchi who passed away last year. My collaboration with him started when both of us participated the inaugural project at Newton Institute in summer of 1992. Our interest was an interplay of topological string as two dimensional TQFT and integrable systems such as $w_{1+\infty}$ algebra, Toda lattice hierarchy [18],[19],[20], which became one of main themes in my research afterwards. After almost a decade I had a second chance of collaboration on five dimensional lift of Seiberg-Witten theory, Nekrasov partition function and topological strings [21],[22],[23], which are closely related to the subject reviewed in the present paper. I am very grateful to Eguchi-san for these fruitful and inspiring collaborations. Though I was not his student, I learned how to enjoy the research through the collaboration with Eguchi-san.

2. Plethystic exponential

For a function $F(t_1, t_2, \cdots, t_\ell)$ we define the plethystic exponential by

$$\text{P.E.}[F(t_1, t_2, \cdots, t_\ell)] = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} F(t_1^k, t_2^k, \cdots, t_\ell^k) \right).$$

Let us assume that $F(t_1, t_2, \cdots, t_\ell)$ can be expanded as follows;

$$F(t_1, t_2, \cdots, t_\ell) = \sum_{n_1, \cdots, n_\ell \in \mathbb{Z}} a_{n_1, \cdots, n_\ell} t_1^{n_1} \cdots t_\ell^{n_\ell}$$

with $a_{0, \cdots, 0} = 0$. Then we see

$$\sum_{k=1}^{\infty} \frac{1}{k} F(t_1^k, t_2^k, \cdots, t_\ell^k) = \sum_{n_1, \cdots, n_\ell \in \mathbb{Z}} a_{n_1, \cdots, n_\ell} \sum_{k=1}^{\infty} \frac{1}{k} t_1^{kn_1} \cdots t_\ell^{kn_\ell}$$

$$= - \sum_{n_1, \cdots, n_\ell \in \mathbb{Z}} a_{n_1, \cdots, n_\ell} \log(1 - t_1^{n_1} \cdots t_\ell^{n_\ell}).$$

Thus the plethystic exponential factorizes as an infinite product;

$$\text{P.E.}[F(t_1, t_2, \cdots, t_\ell)] = \prod_{n_1, \cdots, n_\ell \in \mathbb{Z}} (1 - t_1^{n_1} \cdots t_\ell^{n_\ell})^{-a_{n_1, \cdots, n_\ell}}.$$ 

In fact when $F(t_1, t_2, \cdots, t_\ell)$ is a character of a $G$ module $V$, with $t_i$ parametrizing the Cartan subgroup of $G$;

$$F(t_1, t_2, \cdots, t_\ell) = \text{Tr}_V g,$$
the plethystic exponential computes the character of the symmetric algebra $S^kV$;

$$
\sum_{k=1}^{\infty} s^k \text{Tr}_{S^kV} g^k = \text{P.E.}[s \cdot F(t_1, t_2, \ldots, t_\ell)].
$$

The MacMahon function is a typical example of the plethystic exponential;

$$
M(t) := \prod_{n=1}^{\infty} (1 - t^n)^{-n} = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k |t|^k} \right),
$$

where we have introduced the notation

$$
[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}} = -[x^{-1}].
$$

Note that

$$
F(t) = \frac{1}{|t|^2} = \frac{\partial}{\partial t} \left( \frac{1}{1-t} \right) = \sum_{n=1}^{\infty} nt^n.
$$

Another example which is also ubiquitous in our computation is

$$
(x; q)_\infty = \prod_{n=0}^{\infty} (1 - x q^n) = \text{P.E.} \left[ -\frac{x}{1-q} \right] = \text{P.E.} \left[ \frac{x/\sqrt{q}}{|q|} \right].
$$

It is curious that the generating function of the counting of solid partitions does not seem to allow a plethystic expression. In fact the conjecture of MacMahon, which assumes a plethystic form, fails.

3. ADHM type equation as BPS condition

To write down the matrix equations of ADHM type, we introduce two vector spaces $N$ and $K$ with $\dim_{\mathbb{C}} N = n$ and $\dim_{\mathbb{C}} K = k$. They are associated with $Dp(p = 2d = 4, 6, 8)$ and $D0$-branes, respectively and the dimensions give the numbers of these branes. ADHM type equation is supposed to describe the BPS bound states of $D0$-branes (instantons) with the background $Dp$-branes. In all the cases the equation of motion is invariant under the gauge symmetry $U(k)$ acting on the vector space $K$. Note that since the matrix equations of ADHM type describes the theory on $D0$-branes the gauge symmetry is $U(k)$, while $U(n)$ symmetry on $Dp$-branes are regarded as the flavor symmetry. In the following we list the equations of quiver matrix model. There are two types of open string with boundary on $D0$ branes; one is $k \times k$ matrix in $\text{Hom}_{\mathbb{C}}(K, K)$, where both ends are attached to $D0$-branes and the other is $k \times n$ matrix in $\text{Hom}_{\mathbb{C}}(N, K)$ together with the conjugate which describes open strings stretching between $D0$ and $Dp$-branes.

1. $d = 2, X = \mathbb{C}^2$ ($D0$-$D4$ system, the original ADHM equation) [24];

\[
\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0, \\
\mu_{\mathbb{R}}(\zeta) = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J - \zeta \cdot E_{k \times k} = 0 \quad (\zeta > 0),
\]

where $B_{1,2} \in \text{Hom}_{\mathbb{C}}(K, K)$ and $I, J^\dagger \in \text{Hom}_{\mathbb{C}}(N, K)$. 


(2) $d = 3, X = \mathbb{C}^3$ (D0-D6 system) [25],[26],[27];

$$\mu_C = [B_i, B_j] + \frac{1}{2} \epsilon_{ijk} [B^\dagger_k, Y] = 0,$$

(3.3)

$$\mu_\mathbb{R}(\zeta) = \sum_{i=1}^{3} [B_i, B_i^\dagger] + [Y, Y^\dagger] + II^\dagger - \zeta \cdot E_{K \times K} = 0 \quad (\zeta > 0),$$

(3.4)

$$\mu_B = Y \cdot I = 0,$$

(3.5)

where $B_{1,2,3}, Y \in \text{Hom}_\mathbb{C}(K, K)$ and $I \in \text{Hom}_\mathbb{C}(N, K)$.

(3) $d = 4, X = \mathbb{C}^4$ (D0-D8 system) [28],[29];

$$\mu_C = [B_a, B_b] + \frac{1}{2} \Omega_{abcd} [B^\dagger_c, B^\dagger_d] = 0,$$

(3.6)

$$\mu_\mathbb{R}(\zeta) = \sum_{i=1}^{4} [B_i, B_i^\dagger] + II^\dagger - \zeta \cdot E_{K \times K} = 0 \quad (\zeta > 0),$$

(3.7)

where $B_{1,2,3,4} \in \text{Hom}_\mathbb{C}(K, K), I \in \text{Hom}_\mathbb{C}(N, K)$ and $\Omega_{abcd}$ is the component of the Calabi-Yau 4 form, with $\Omega \wedge \overline{\Omega} = \text{vol}_8$.

The origin of $Y$ in the case of $d = 3$ is rather subtle. But the equations can be obtained by a dimensional reduction of those for $d = 4$ by putting $Y = B_4$. Or we can regards it as a consequence of “tachyon condensation” [29]. The additional condition $\mu_B = 0$, which only appears for $d = 3$, means that D0 branes cannot escape along the normal direction to D6-branes.

Similar condition appears in the BPS condition for the spiked instanton [30],[31],[32],[33]. Thus it might be more natural to consider D8-D6-D0 system as a generalization of the spiked instanton. It has been argued that a constant $B$ field (a background flux) is required for the existence of bound states of D0-D6 and D0-D8 systems [34],[35],(see also [33] for a related discussion). We implicitly assume that such a flux is turned on, if necessary.

In each case we can discard the $D$ term condition $\mu_{\mathbb{R}}(\zeta) = 0$ (or the real component of the hyper-Kähler moment map ) with $\zeta > 0$ in favor of the following stability condition;

$$\text{If a subspace } K' \subset K \text{ satisfies } I(N) \subset K' \text{ and } B_a(K') \subset K', \text{ then } K' = K,$$

with the gauge symmetry being complexified to $GL(k, \mathbb{C})$. We can show that the $F$-term condition $\mu_C = 0$ implies that $B_a$ are commuting $[B_a, B_b] = 0$. In the case of $d = 4$ it follows from $\text{Tr} (\mu_C)^2 = 0$. In other cases we use the stability condition to show the vanishing of $J$ or $Y$. Then the stability condition implies that the vector space $K$ is spanned by action of $B_a$ on the subspace (‘vacuum’) $I(N)$;

$$K = \mathbb{C}[B_a] \cdot I(N).$$

(3.8)

We will use this property, when we compute the equivariant character of the tangent space in section 5.

The formal complex dimensions of the moduli space are computed by subtracting the gauge degrees of freedom and constraints from the total number of components of matrices;

(1) $d = 2$

$$2k^2 + 2nk - k^2 - k^2 = 2nk,$$

(2) $d = 3$

$$3k^2 + k^2 + nk - k^2 - 3k^3 - nk = 0,$$
(3) $d = 4$

$4k^2 + nk - k^2 - 3k^3 = nk.$

Note that if we did not introduce $Y$ in $d = 3$, the computation was

$3k^2 + nk - k^2 - 3k^3 = (n - k)k$

and we cannot have a good moduli space. Since the dimensions are not necessarily even for $d = 4$, the moduli space cannot be hyperKähler. In fact, for $d > 2$ the moduli space is not smooth and the tangent space only has a virtual meaning.

When $n = 1$ which corresponds to abelian gauge theory on $Dp$-branes, we expect the moduli space is mathematically equivalent to the Hilbert scheme of $k$ points on $\mathbb{C}^d$. It is known when $d > 2$ it is qualitatively different from the case of $\mathbb{C}^2$ [36]. It is desirable to clarify the meaning of the generalized ADHM conditions from the viewpoint of the Hilbert scheme of $k$ points on $\mathbb{C}^d$.

4. Matrix model description

One can construct a cohomological matrix model by imposing ADHM type BPS conditions as gauge fixing condition of cohomological matrix model, which is achieved in BRST manner. In the case of $d = 2$ ADHM constraints are obtained as hyperKähler moment maps and this leads to integration over the Higgs branch of supersymmetric quantum mechanics [37], [38]. Equivariant localization of topological (BRST) symmetry allows us to compute the partition function as a residue integral over the eigenvalues (diagonal elements) of the matrix. It turns out that the poles of the residue integral are labeled by partitions and after the residue integral, we obtain a summation over the partitions.

4.1. $d = 2$ (From localization to Macdonald polynomials)

Let $\{x_i\}_{i=1}^k$ be the Cartan variables of $GL(k)$ or the eigenvalues of $k$ by $k$ matrices. The equivariant integration over the instanton moduli space $\mathcal{M}_{n,k}$ is reduced to a contour integral

$$Z_k = \frac{1}{k!} \int \prod_{i=1}^k \frac{dx_i}{2\pi \sqrt{-1}x_i} z_k(x_i, u_\alpha, q_1, q_2),$$

(4.1)

where we have divided the integral by the order of the Weyl group (we will order the eigenvalues) and $\{u_\alpha = e^{a_\alpha}\}_{\alpha=1}^n$ is the Cartan variables for $GL(n)$ symmetry coming from $n$ D4 branes. The parameters $q_i = e^{t_i}$ are $\Omega$-background parameters or the equivariant parameters of the torus action $(z_1, z_2) \rightarrow (q_1 z_1, q_2 z_2)$ on $\mathbb{C}^2$. The full partition function is

$$Z^{4D}(u_\alpha, q; q) = 1 + \sum_{k=1}^{\infty} q^k Z_k$$

(4.2)

and we will see by introducing the power sum function $p_n(x)$ of the eigenvalues, the integrand $z_k(x_i, u_\alpha, q_1, q_2)$ allows a plethystic expression. Note that we may identify $\log z_k(x_i, u_\alpha, q_1, q_2)$ as an effective action of the matrix model. The contributions to $z_k(x_i, u_\alpha, t_1, t_2)$ are evaluated as follows:2

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2 These contributions are in one to one correspondence with the terms in the equivariant character to be given in the next section.
we can write the integrand as follows;
\[ \prod_{i \neq j} \left( 1 - \frac{x_i}{x_j} \right). \] (4.3)

This factor is also regarded as the contribution of \( GL(k) \) gauge symmetry of ADHM constraints.

- Contribution of ADHM constraints;
\[ \prod_{i,j} \left( 1 - q_1 q_2 \frac{x_i}{x_j} \right) = (1 - q_1 q_2)^k \prod_{i \neq j} \left( 1 - q_1 q_2 \frac{x_i}{x_j} \right). \] (4.4)

- Contribution of matrix variables \( B_{1,2}, I, J; \)
\[ \prod_{i,j} \left( 1 - q_\alpha \frac{x_i}{x_j} \right)^{-1} = (1 - q_\alpha)^{-k} \prod_{i \neq j} \left( 1 - q_\alpha \frac{x_i}{x_j} \right), \] (4.5)

from \( B_\alpha \) and
\[ \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - \frac{x_i}{u_\alpha} \right)^{-1}, \quad \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - q_1 q_2 \frac{u_\alpha}{x_i} \right)^{-1}. \] (4.6)

Let us rescale the variable \( u_\alpha \to \sqrt{q_1 q_2} u_\alpha \) to make the last two contributions symmetric;
\[ \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - \sqrt{q_1 q_2} \frac{x_i}{u_\alpha} \right)^{-1}, \quad \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - \sqrt{q_1 q_2} \frac{u_\alpha}{x_i} \right)^{-1}. \] (4.7)

In terms of the function
\[ S(z) := \frac{(1-z)(1-q_1 q_2 z)}{(1-q_1 z)(1-q_2 z)} \] (4.8)

we can write the integrand as follows;
\[ z_k(x_i, a_\alpha, q_1, q_2) = \left( \frac{1 - q_1 q_2}{(1-q_1)(1-q_2)} \right)^k \prod_{i \neq j} \frac{S \left( \frac{x_i}{x_j} \right)}{\prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - \sqrt{q_1 q_2} \frac{x_i}{u_\alpha} \right) \left( 1 - \sqrt{q_1 q_2} \frac{u_\alpha}{x_i} \right)}. \] (4.9)

Now in terms of the power sum variables \( p_m = \sum_{i=1}^{k} x_i^m \), we can rewrite the measure \( z_k(x_i, u_\alpha, q_1, q_2) \) for the contour integral in a plethystic form;
\[ \log(z_k(x_i, u_\alpha, q_1, q_2)) = \sum_{m=1}^{\infty} \frac{1}{m} (q_1^m + q_2^m - q_1^m q_2^m) \sum_{i,j=1}^{k} \left( \frac{x_i}{x_j} \right)^m - \sum_{m=1}^{\infty} \frac{1}{m} \sum_{i \neq j} \left( \frac{x_i}{x_j} \right)^m \]
\[ + \sum_{m=1}^{\infty} \frac{(\sqrt{q_1 q_2})^m}{m} \sum_{i=1}^{k} \sum_{\alpha=1}^{n} \left[ \left( \frac{x_i}{u_\alpha} \right)^m + \left( \frac{u_\alpha}{x_i} \right)^m \right] \]
\[ = \sum_{m=1}^{\infty} \frac{1}{m} - \sum_{m=1}^{\infty} \frac{1}{m} (1-q_1^m)(1-q_2^m)p_{mP-m} \]
\[ + \sum_{m=1}^{\infty} \frac{(\sqrt{q_1 q_2})^m}{m} \sum_{\alpha=1}^{n} (p_m u_\alpha^m + p_{-m} u_\alpha^m). \] (4.10)
Using the holonomy variables

\[ U_m := \sum_{\alpha = 1}^{n} u^m_{\alpha} \]  

(4.11)
of \( U(n) \) gauge fields, we have

\[
\log(z_k(x_i, u, q_1, q_2)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} \left[ - (1 - q_1^m)(1 - q_2^m)p_m p_{-m} + (\sqrt{q_1 q_2})^m(p_m U_m + p_{-m} U_{-m}) \right],
\]

(4.12)

where we have introduced \( \log \Lambda := \sum_{m=1}^{\infty} \frac{1}{m} \). By the change of variables

\[ \alpha_m := \frac{\sqrt{q_1 q_2})^m}{1 - q_1^m} U_m - (1 - q_2^m) p_m, \]

(4.13)

we can eliminate linear terms in \( p_m \) to obtain

\[
\log(z_k(x_i, u, q_1, q_2)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} \left[ - (1 - q_1^m)(1 - q_2^m) \alpha_m \alpha_{-m} + \frac{1}{(1 - q_1^m)(1 - q_2^m)} \right].
\]

(4.14)

Thus we have

\[
Z_k = \frac{1}{k!} \left( \prod_{i,j=1}^{\infty} (1 - q_1^i q_2^j) \right) \int \prod_{i=1}^{k} \frac{dx_i}{2\pi \sqrt{-1} x_i} \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{1 - q_1^m}{1 - q_2^m} \right) \alpha_m \alpha_{-m} \right) \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{(1 - q_1^m)(1 - q_2^m)} \right).
\]

(4.15)

We may eliminate \( \Lambda \) by the renormalization of the instanton expansion parameter \( q \). The universal factor \( \prod_{i,j=1}^{\infty} (1 - q_1^i q_2^j) \) should be identified with the perturbative factor.

In the abelian case the contour integral (4.15) is related to the inner product for Macdonald polynomials [39]. To see it, we should note that the poles of the contour integral are labeled by partitions \( \lambda \) with \( |\lambda| = k \) and the position of poles are given by

\[ x_i = u \cdot q_1^{-\frac{1}{2}} q_2^{-\frac{1}{2}}, \quad (a, b) \in \lambda, \]

(4.16)

where \( u = U_1 \) and we have \( U_m = u^m \) for the abelian case. Hence the power sum takes the following values at the poles;

\[ p_1^{(\lambda)} = u \sum_{a=1}^{\ell^{(\lambda)}} \sum_{b=1}^{\lambda_a} q_1^{-\frac{1}{2}} q_2^{-\frac{1}{2}}, \]

(4.17)

and

\[ \alpha_1^{(\lambda)} = u \sqrt{q_1 q_2} \sum_{i=1}^{\infty} q_1^{i-1} q_2^{\lambda_i}. \]

(4.18)

Thus we recover the topological locus;

\[ \xi_i = u q_1^{\frac{i}{2}} q_2^{\lambda_i + \frac{1}{2}}. \]

(4.19)

This also explains an implication of the change of variables (4.13). In summary after the contour integration we have

\[
Z_k = \frac{1}{k!} \left( \prod_{i,j=1}^{\infty} (1 - q_1^i q_2^j) \right) \sum_{|\lambda| = k} \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{(1 - q_1^m)(1 - q_2^m)} \alpha_m^{(\lambda)} \alpha_{-m}^{(\lambda)} \right).
\]

(4.20)
Note that the measure factor coincides with the \((q, t)\)-deformed Vandermonde determinant
\[
\Delta_{q, t}(\xi)^2 := \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} \frac{1 - q^k}{1 - t^k} \right) \left( N - \alpha_k \alpha_{-k} \right)
\]
\[
= \prod_{n=1}^{\infty} \prod_{1 \leq a < b \leq N} \frac{1 - t^n \xi_a / \xi_b}{1 - q^n \xi_a / \xi_b}
\]
(4.21)
with \((q, t) = (q_1, q_2^{-1})\). This is employed to define the inner product on the space of symmetric polynomials that leads to Macdonald polynomials [40].

The integrand of the residue integral can be expressed in term of the plethystic exponential and taking the logarithm we may recognize “effective” action for eigenvalues, which is in turn expressed by the power sum. Then the integral can be related to the inner product for the Macdonald polynomials. This also means the effective action is bilinear in the power sums (the free boson operators). To construct refined topological vertex we have to insert a vertex operator. It is curious that the insertion induces the interaction term in the effective action.

4.2. \(d = 3\)

From the ADHM type conditions, we can similarly obtain a contour integral representation of the partition function with instanton number \(k\);
\[
Z_k = \frac{1}{k!} \oint \prod_{i=1}^{k} \frac{dx_i}{2\pi i} z_k(x_i, u_\alpha, q_a),
\]
(4.22)

where
\[
z_k(x_i, u_\alpha, q_a) = \prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left( 1 - q_1 q_2 q_3 u_\alpha x_i \right) \prod_{i\neq j} \left( 1 - \frac{x_i}{x_j} \right) \prod_{1 \leq a < b \leq 3} \prod_{i, j} \left( 1 - q_a q_b \frac{x_i}{x_j} \right)
\]
(4.23)

It is curious to see the role of the pole at \(x_j = q_1 q_2 q_3 x_i\) in the contour integral. An analogous computation to the case of \(d = 2\) leads the following plethystic form of the measure;
\[
\log(z_k(x_i, u, q_a)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} \left[ -(1 - q_1^m)(1 - q_2^m)(1 - q_3^m) p_m p_{-m} + (\sqrt{q_1 q_2 q_3})^m (p_m U_{-m} - p_{-m} U_m) \right].
\]
(4.24)

The crucial change here is the relative sign in the linear terms, which prevents us to make a complete square by the change of variable like (4.13). The flip of the relative sign causes an asymmetry in exchanging the positive modes and the negative modes. As will be discussed in the next section this seems to be related to the fact in contract to the case of \(d = 2\), we do not have hyperKähler (holomorphic symplectic) structure any more when \(d = 3\).
4.3. $d = 4$

We obtain

$$z_k(x_i, u_\alpha, q_a) = \frac{\prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \prod_{1 \leq a < b \leq 3} \prod_{i,j} \left(1 - q_a q_b \frac{x_i}{x_j}\right)}{\prod_{i=1}^{k} \prod_{\alpha=1}^{n} \left(1 - \frac{x_i}{u_\alpha}\right) \prod_{a=1}^{3} \prod_{i,j} \left(1 - q_a^{-1} \frac{x_i}{x_j}\right)}. \quad (4.25)$$

As we will argue in the next section, we have to choose a “chiral-half” of the full Euler character, which depends on the ordering of the set $\{(ab)| 1 \leq a \neq b \leq 4\}$. Here we chose $\{(12), (13), (23); (14), (24), (34)\}$ by taking $z_4$ as a “preferred” direction. As argued in [33], due to the choice of the ordering, we should be careful with the order of the contour integral.

Using the Calabi-Yau condition $q_1q_2q_3q_4 = 1$, we can obtain a plethystic form of $z_k(x_i, u_\alpha, q_a)$ as follows;

$$\log(z_k(x_i, u_\alpha, q_a)) = \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} [q_1^m + q_2^m + q_3^m + q_4^{-m}] p_m p_{-m}$$

$$- \sum_{m=1}^{\infty} \frac{1}{m} [1 + q_1^m q_2^m + q_1^m q_3^m + q_2^m q_3^m] p_m p_{-m} + \sum_{m=1}^{\infty} \frac{1}{m} p_m U_{-m}$$

$$= \log \Lambda^k + \sum_{m=1}^{\infty} \frac{1}{m} p_m U_{-m} - \sum_{m=1}^{\infty} \frac{1}{m} (1 - q_1^m)(1 - q_2^m)(1 - q_3^m)p_m p_{-m}. \quad (4.26)$$

Remarkably this is quite close to (4.24). Since this is a “chiral-half” of the full Euler character, only the negative modes $U_{-m}$ appear.

5. Equivariant character of (virtual) tangent space at fixed points

The fixed points of the toric action of $T^d$ on $\mathbb{C}^d$ and the Cartan subalgebra of the gauge symmetry $G_C$ are labelled by $n$-tuple of $(d-1)$ partitions. In terms of the equivariant parameters $u_\alpha := e^{a_\alpha}$ of the Cartan subgroup of $U(n)_C$, the character of the vector space $N$ (the Chan-Paton bundle for the background $Dp$ branes) is\(^3\)

$$N = \sum_{\alpha=1}^{n} u_\alpha \quad (5.1)$$

Then from the structure of the vector space $K$ (3.8), its equivariant character at the fixed point $\{\pi_\alpha\}_{\alpha=1}^{n}$ is

$$K_\pi = \sum_{\alpha=1}^{n} u_\alpha \left( \sum_{(i,j,k) \in \pi_\alpha} q_1^{1-i} q_2^{1-j} q_3^{1-k} \right), \quad (5.2)$$

where for illustration we write the formula for $d = 3$, but generalization to other cases, where $\pi$ stands for partition $(d = 2)$ and solid partition $(d = 4)$, should be clear. With these basic ingredients we can compute the (Euler) characters of the deformation complex for the ADHM type equation in each dimension.

\(^{3}\)By the abuse of notation we use the same notation for the character.
5.1. $d = 2$

The fixed points are labelled by $n$-tuple of partitions (colored Young diagrams) $\lambda_\alpha$ and

$$\chi_{4D}(u_\alpha, q_i) = N^* K + q_1 q_2 K^* N - (1 - q_1)(1 - q_2) K^* K,$$

where the positive contributions $N^* K$ and $t_1 t_2 K^* N$ come from $I$ and $J$, $(q_1 + q_2) K^* K$ from $B_{1,2}$, while the negative ones $-q_1 q_2 K^* K$ from the $F$-term constraint and $-K^* K$ from the gauge symmetry. The difference of the numbers of positive coefficients $(+1)$ and negative coefficients $(-1)$ is $2nk$, which is exactly the (complex) dimensions of the tangent space.

After cancellations only positive term survive and when $n = 1$ it has a nice combinatorial formula [24]:

$$\chi_{4D}(q_i) = \sum_{s \in \lambda} \left( q_1^{-\ell(s)} q_2^{a(s)+1} + q_1^{\ell(s)+1} q_2^{-a(s)} \right),$$

(5.4)

where $a(s)$ and $\ell(s)$ are the arm and the leg length of the box $s$ in the Young diagram $\lambda$.

Note that in abelian case the dependence on $u_\alpha$ disappears. In the non-abelian ($n > 1$) case the fixed points are labeled by $n$-tuple of Young diagrams $\vec{\lambda} = (\lambda^\alpha)$ and we need the arm and the leg length of the box $s = (i, j) \in \lambda$ with respect to a second Young diagram $\mu$:

$$a_\mu(i, j) := \nu_i - j, \quad \ell_\mu(i, j) := \nu_j^\vee - i.$$  

(5.5)

Then an explicit formula for the equivariant character is

$$\chi_{4D}(u_\alpha, q_i) = \sum_{\alpha, \beta = 1}^n N_{\alpha \beta},$$

$$N_{\alpha \beta}(u_\alpha, q_i) = \frac{u_\beta}{u_\alpha} \left( \sum_{s \in \lambda^\alpha} q_1^{\ell_\mu(s)} q_2^{a_\mu(s)+1} + \sum_{t \in \lambda^\beta} q_1^{\ell_\mu(s)+1} q_2^{-a_\mu(s)} \right).$$  

(5.6)

---

**Fig. 2** ADHM quiver [right] as the double of the Jordan (framed $\hat{A}_0$) quiver [left]

Introducing the following polarization

$$P_2(u_\alpha, q_i) = N^* K + (q_1 - 1) K^* K,$$

(5.7)

---

$^4$By definition a polarization of a symplectic manifold $X$ is an equivariant $K$ theory class $P = T^{1/2} X \in K_T(X)$, such that the tangent space is represented as $TX = P + hP^*$.
and the notation $h := q_1 q_2$ we can express the character as follows:

$$P_2 + h P_2^* = N^* K - (1 - q_1) K^* K + q_1 q_2 (K^* N - (1 - q_1^{-1}) K^* K) = \chi_{4D}.$$  (5.8)

Note that $h$ is the scaling factor of the symplectic form $\omega = dz_1 \wedge dz_2$. This decomposition of the equivariant character $\chi_{4D}$ reflects the fact that the moduli space of ADHM matrix model is an example of Nakajima quiver varieties, which is defined as a hyperKähler quotient. The relevant quiver is called Jordan quiver which consists of a single vertex with a single loop (Fig.2). More precisely it is the framed Jordan quiver with a framing of $\mathbb{C}^n$. When $n = 1$ or $U(1)$ gauge theory, the associated quiver variety is nothing but the Hilbert scheme $\text{Hilb}_k \mathbb{C}^2$ of $k$-points on $\mathbb{C}^2$, where $k$ is physically the number of $D0$-branes or the instanton number of anti-self-dual connection. From this viewpoint the moduli space has the structure of a cotangent bundle and the polarization $P_2$ represents the contribution of the base space described by the Jordan quiver, where we subtract $K^* K$ coming from the gauge symmetry. Then the second term corresponds to the fiber of the cotangent bundle and the multiplication of the weight $h$ is necessary.

The equivariant character (5.3) is also derived from the equivariant Chern character of the universal bundle $E$ [41] [42]. The virtue of this derivation is that it is applicable for more general gauge groups of type $SO$ and $Sp$ [43]. To construct the universal bundle $E$, let $m^I$ be local coordinates on the moduli space of instantons. The tangent space of the moduli space is spanned by solutions to the linearized equations with a gauge fixing condition. Let $\{ \psi^I_\mu(x, m) \}$ denote a basis of the tangent space at $m \in \mathcal{M}_{\text{inst}}$. For a family of instantons $A_\mu(x, m)$ parametrized by $m$, we have

$$\frac{\partial A_\mu(x, m)}{\partial m_I} = h_{IJ} \psi^J_\mu + D_\mu \alpha_I.$$  (5.9)

Since the derivative of $A_\mu(x, m)$ does not necessarily satisfy the gauge fixing condition we need a compensating gauge transformation $D_\mu \alpha_I$. With an appropriate choice of the gauge fixing condition, for example $(D^*)^\mu \psi^I_\mu(x, m) = 0$, we can find a unique $\alpha_I$. Combining $A_\mu$ with the parameter of the compensating gauge transformation $\alpha_I$, we can define a one form $A(x, m) = A_\mu dx^\mu + \alpha_I dm^I$ which can be regarded as a connection of the universal bundle $E$ on $\mathbb{R}^4 \times \mathcal{M}_{\text{inst}}$ whose fiber is the fundamental representation $\mathbb{C}^n$ of $U(n)$. In the following we fix a complex structure of the space-time $\mathbb{R}^4$ and identify $\mathbb{R}^4 \simeq \mathbb{C}^2$. Then the spinor bundle $S^+ \oplus S^-$ on $\mathbb{R}^4$ is naturally identified with the space of $(0, k)$ forms $\Lambda^{(0,0)} \oplus \Lambda^{(0,1)} \oplus \Lambda^{(0,2)}$ on $\mathbb{C}^2$ [5]. With this identification the Dirac operator is translated to $\bar{\partial}$ operator.

The equivariant Chern character of the universal bundle $E$ is computed as the Euler character of the complex

$$0 \longrightarrow K \otimes \Lambda^{(0,0)} \xrightarrow{\tau_z} K \otimes \Lambda^{(0,1)} \oplus N \otimes \Lambda^{(0,2)} \xrightarrow{\sigma_z} K \otimes \Lambda^{(0,2)} \longrightarrow 0,$$  (5.10)

where

$$\tau_z = \begin{pmatrix} B_1 - z_1 \\ B_2 - z_2 \\ J \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} - (B_2 - z_2) & B_1 - z_1 & I \end{pmatrix},$$  (5.11)

In general the spinors on a Calabi-Yau manifold are equivalent to $(0, k)$ forms, where the chirality of the spinor corresponds to the parity of $k$. 

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and the ADHM condition guarantees (5.10) is a complex; $\sigma_z \circ \tau_z = 0$. One can also check \( \text{Ker} \, \sigma_z = \text{Coker} \, \tau_z = 0 \) [24]. Taking the alternating sum, we obtain

$$\text{Ch}_q(\mathcal{E})(u; q_i) = N(u) - (1 - q_1)(1 - q_2)K(u; q_i). \quad (5.12)$$

Now the equivariant version of the index theorem tells the equivariant index of the Dirac operator coupled with the adjoint bundle $\mathcal{E} \otimes \mathcal{E}^*$ is $^6$

$$\text{Ind}_q \, \bar{\partial}_{\mathcal{E} \otimes \mathcal{E}^*} = \int_{\mathbb{C}^2} \text{Ch}_q(\mathcal{E} \otimes \mathcal{E}^*) \text{Td}_q(\mathbb{C}^2), \quad (5.13)$$

where the equivariant version of the Todd class is

$$\text{Td}_q(\mathbb{C}^2) = \frac{x_1 x_2}{(1 - e^{x_1})(1 - e^{x_2})}, \quad (5.14)$$

where

$$x_i = \epsilon_i + \delta(z_i) \frac{dz_i \wedge d\bar{z}_i}{2\pi \sqrt{-1}} \quad (5.15)$$

are the equivariant Chern roots of the tangent bundle to $\mathbb{R}^4 \simeq \mathbb{C}^2$, given by equivariantly closed two forms. We should use the Chern class of $\mathcal{E} \otimes \mathcal{E}^*$, because we consider the adjoint bundle whose fibre is the adjoint representation of $U(n)$. It should be easy to generalize the computation to the bi-fundamental representation. The integration over the space-time $\mathbb{C}^2 \simeq \mathbb{R}^4$ corresponds to the push-forward for the projection $\pi: \mathbb{R}^4 \times M_{\text{inst}} \to M_{\text{inst}}$ and can be evaluated by the localization by the torus action $(z_1, z_2) \to (q_1 z_1, q_2 z_2)$, whose unique fixed point is the origin $z_1 = z_2 = 0$. The Hamiltonian of the torus action is $\epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$ and the localization theorem for the equivariant closed forms gives

$$\int_{\mathbb{C}^2} \text{Ch}_q(\mathcal{E} \otimes \mathcal{E}^*) \text{Td}_q(\mathbb{C}^2) = \frac{\text{Ch}_q(\mathcal{E} \otimes \mathcal{E}^*) \text{Td}_q(\mathbb{C}^2)|_{(0,0)}}{\epsilon_1 \epsilon_2} = - \frac{N^* N}{(1 - q_1)(1 - q_2)} + \chi_{4D}, \quad (5.16)$$

where the first term, which survives even $k = 0$, is regarded as a perturbative part.

5.2. \( d = 3 \)

The fixed points are labelled by $n$-tuple of plane partitions and

$$\chi_{6D}(u; q_i) = N^* K - q_1 q_2 q_3 K^* N - (1 - q_1)(1 - q_2)(1 - q_3)K^*K, \quad (5.17)$$

where $N^* K, (q_1 + q_2 + q_3)K^* K$ and $q_1 q_2 q_3 K^* K$ come from dynamical matrix variables $I, B_{1,2,3}$ and $Y$, while $-(q_1 q_2 + q_2 q_3 + q_3 q_1)K^* K$ from the $F$-term constraints and $-K^* K$ from the gauge symmetry. Finally $-q_1 q_2 q_3 K^* N$ comes from the constraint $\mu_B = 0$. When we impose the Calabi-Yau condition $q_1 q_2 q_3 = 1$, the character is anti-self dual $\chi_{6D} + \chi_{6D}^* = 0$, which is a consequence of the Serre duality. By the anti-self duality, the measure on the space of plane partitions becomes uniform (up to sign), in fact it is $(-1)^{nk}$. Hence the partition function reduces to the MacMahon function.

$^6$ The Dirac operator on complex manifold is related to the the $\bar{\partial}$ operator by the twist of the square root of the determinant of the tangent bundle, which is trivial for Calabi-Yau case.
Now the analogue of the polarization (5.7) in \( d = 3 \) is
\[
P_3(u_\alpha, q_i) := N^*K + (q_1 + q_2 + q_3 - 1)K^*K, \tag{5.18}
\]
and we set \( h = q_1q_2q_3 \), then we have
\[
\chi_{6d} = P_3 - hP_3^*. \tag{5.19}
\]
Note that the relative sign between \( P \) and \( P^* \) should be negative for odd \( d \). Consequently the interpretation of the decomposition (5.19) is rather different from the case \( d = 2 \). Namely (5.19) reflects what is called symmetric obstruction theory in mathematics, where the first term corresponds to the deformation space of matrix variables coming from the framed quiver with a single vertex and three loops with the subtraction of gauge symmetry, while the second term is the contributions from the obstruction space, which are represented by anti-ghosts for constraints and the secondary ghost for the gauge symmetry. The symmetric obstruction theory gives a moduli space of virtual dimension zero.

5.3. \( d = 4 \)
The fixed points are labelled by \( n \)-tuple of solid partitions and
\[
\chi_{8D}(u_\alpha, q_i) = N^*K - (1 - q_1)(1 - q_2)(1 - q_3)K^*K = P_4, \tag{5.20}
\]
where \( N^*K \) and \( (q_1 + q_2 + q_3 + q_1q_2q_3)K^*K \) come from \( I \) and \( B_{1,2,3,4} \), while \( -(q_1q_2 + q_2q_3 + q_3q_1)K^*K \) from the \( F \)-term constraints and \( -K^*K \) from the gauge symmetry. Note that what we called the polarization in lower dimensional cases is obtained as the character of the deformation complex of ADHM type condition. Namely (5.20) is a “chiral-half” of the full Euler character \( \chi_E(\mathcal{E}) = \sum_{i=0}^{d} (-1)^i \text{Ext}^i(\mathcal{E}, \mathcal{E}) \); \( \chi_E(u_\alpha, q_i) = N^*K + K^*N - (1 - q_1)(1 - q_2)(1 - q_3)(1 - q_4)K^*K = \chi_{8D} + \chi_{8D}^* \), where we have used \( h = q_1q_2q_3q_4 = 1 \). Contrary to the odd dimensional case, the full character is self dual \( \chi_E^* = \chi_E \). To define a “chiral-half” of the full Euler character, we use the real structure of \( \text{Ext}^2(\mathcal{E}, \mathcal{F}) \), which is allowed by the Serre duality of \( \text{Ext}^i(\mathcal{E}, \mathcal{E}) \). This seems consistent with the idea that \( d = 4 \) theory is a holomorphic version of the Donaldson theory [10]. By taking a “chiral-half” of the full Euler character we consider a square root of the tangent space and hence there is an ambiguity of the choice of sign. We can fix it locally, but the global consistency is a non-trivial issue. Mathematically this is the problem of the orientability of the moduli space.

6. Topological partition function
As we have emphasized in introduction, all the partition function in the following can be expressed as plethystic exponentials.

6.1. \( d = 2 \) (Abelian \( \mathcal{N} = 2^* \) theory)
Using the formula (5.4) the partition function of \( U(1) \) theory with adjoint matter is
\[
Z_{U(1), \text{adj}}^{4D}(q_\alpha, \mu; q) = \sum_{\lambda} q^{\ell(\lambda)} \prod_{s \in \lambda} \frac{1 - \mu q_1^{-\ell(s)} q_2^{a(s) + 1}}{1 - q_1^{-\ell(s)} q_2^{a(s) + 1}} \frac{1 - \mu q_2^{\ell(s) + 1} q_1^{-a(s)}}{1 - q_1^{\ell(s) + 1} q_2^{-a(s)}}, \tag{6.1}
\]
where the parameter \( \mu := e^{-m} \) is the equivariant (mass) parameter for the \( U(1) \) flavor symmetry of the adjoint matter. Thus, physically \( Z_{U(1), \text{adj}}^{4D} \) is the Nekrasov partition function of
\( \mathcal{N} = 2^* \) theory. We can show that it has the following plethystic form \([44],[45],[46],[39]\):

\[
Z_{U(1), \text{adj}}^{AD}(q_a, \mu; q) = \text{P. E.} \left[ F(q_a, \mu; q) \right],
\]

\[
F(q_a, \mu; q) := -\sqrt{\mu q_a}[\mu q_1][\mu q_2] = \frac{q}{1 - \mu q} \frac{(1 - \mu q_1)(1 - \mu q_2)}{(1 - q_1)(1 - q_2)}.
\]

There are two natural limits for \( \mu \): the decoupling limit \( \mu \to 0 \) and the massless \( (\mathcal{N} = 4) \) limit \( \mu \to 1 \). In the latter case the measure on the space of partitions is uniform and we obtain the generating function of the counting of partitions:

\[
Z_{U(1), \mathcal{N}=4}^{AD}(q) = \text{P. E.} \left[ \frac{q}{1 - q} \right] = \frac{1}{(q; q)_\infty}.
\]

On the other hand, in the former limit the measure becomes the (refined) Plancherel measure and corresponds to the pure \( U(1) \) theory, which is geometrically engineered by the conifold geometry:

\[
Z_{U(1), \text{adj}}^{AD}(q_a; q) = \text{P. E.} \left[ \frac{q^{\sqrt{\mu q_a q_2}}}{[q_1][q_2]} \right].
\]

When \( q_1 = q_2^{-1} = t \) it gives a generalized McMahon function

\[
Z_{U(1), \text{adj}}^{AD}(t; q) = \text{P. E.} \left[ \frac{q}{t^2} \right] = \prod_{n=1}^\infty \left( 1 + qt^n \right)^{-n},
\]

where \( q \) plays the role of the Kähler parameter of the conifold. Thus this example gives a kind of interpolation between the counting of partitions and plane partitions.

The formula (6.2) can be deduced as follows\(^7\): First we note the “removable” boxes of a non-empty partition have vanishing leg and arm length \( \ell(s) = a(s) = 0 \), because if they have non-empty leg or arm, we cannot remove them from the diagram. From the measure factor in (6.1), we see the measure on non-empty partition has zeros at \( \mu q_1 = 1 \) and \( \mu q_2 = 1 \). Note that these zeros are preserved under \( (q_1, q_2, \mu) \to (q_1^2, q_2^2, \mu^2) \). Thus we conclude that \( F \) has the factor \([\mu q_1] \cdot [\mu q_2]\). Moreover, when \( \mu = 1 \) the measure is independent of \( q_1 \) and \( q_2 \). Hence we arrive at

\[
F(q_a, \mu; q) \sim \frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]}.
\]

Now let us specialize \( q_2 = q_1^{-1} \) and take the limit \( q_1 \to 0 \). Then

\[
Z_{U(1), \text{adj}}^{AD}(q_a, \mu; q) = \sum_\lambda q^{\lambda} \prod_{s \in \lambda} \frac{q^{h(s)} - 1 - \mu q_1^{h(s)}}{1 - q_1^{h(s)}} \to \sum_\lambda (\mu q)^{\lambda},
\]

where \( h(s) = \ell(s) + a(s) + 1 \) is the hook length. On the other hand

\[
\frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]} = \frac{[\mu q_1] \cdot [\mu^{-1} q_1]}{[q_1]^2} \to 1
\]

in this limit. Hence we find

\[
F(q_a, \mu; q) \sim \frac{\mu q}{1 - \mu q} \frac{[\mu q_1] \cdot [\mu q_2]}{[q_1] \cdot [q_2]} = -\frac{\sqrt{\mu q}[\mu q_1][\mu q_2]}{[\mu q][q_1][q_2]}.
\]

We can also prove (6.2) by assuming the invariance of the topological string amplitudes under the change of the preferred direction of the refined topological vertex [46]. When we

\(^7\) Strictly speaking, we assume that the partition function has a plethystic form.
deduce the refined topological vertex from the equivariant vertex the preferred direction is related to the ways of the limit \(|q_i| \to \infty\) keeping the Calabi-Yau combination of \(q_i\) constant (see the next subsection). This reminds us of the fact the perturbative string theory can be obtained by taking appropriate limits of \(M\) theory.

The \(\mathcal{N} = 2^*\) theory can be regarded as \(\hat{\mathcal{A}}_0\) quiver gauge theory. As we have seen in the last section the quiver for the ADHM equation is the double of the framed \(\hat{\mathcal{A}}_0\) quiver, and when the framing is \(U(1)\) the Nakajima variety is nothing but the Hilbert scheme of point on \(\mathbb{C}^2\). This coincidence seems to be the origin of symmetric property of the topological partition function derived above.

6.2. \(d = 3\)

For abelian case \(n = 1\) by the localization theorem the partition function is given by the summation over the plane partitions\(^8\);

\[
Z_{U(1)}^{6D}(q_a; q) = \sum_{\pi} (-q)^{|\pi|} \hat{\mathcal{a}}(\chi_\pi) = \left\langle \mathcal{P.E.}[\chi_{6D}] \right\rangle
\]

(6.10)

where \(\hat{\mathcal{a}}\) is defined by

\[
\hat{\mathcal{a}}(\sum_i m_i w_i) = \prod_i [w_i]^{m_i}, \quad m_i \in \mathbb{Z}, w_i \in T^\vee.
\]

(6.11)

\(m_i\) is the multiplicity of the character (weight) \(w_i\) of the torus \(T\) that acts on the moduli space. Note that \(\hat{\mathcal{a}}\) gives the character of symmetrized symmetric products;

\[
\hat{\mathcal{a}}(-\text{character of } V) = \text{character of } \hat{S}^\ast V,
\]

(6.12)

where \(\hat{S}^\ast V = (\det V)^{\frac{1}{2}} \cdot S^\ast V\).

It turns out that the partition function allows a plethystic expression \([25],[36]\);

\[
Z_{U(1)}^{6D}(q_a; q) = \mathcal{P.E.}[F_1(q_a, q)],
\]

(6.13)

where

\[
F_1(q_a, q) = \frac{[q_1 q_2][q_2 q_3][q_3 q_1]}{[q_1][q_2][q_3][\sqrt{q}][\sqrt{q}]}.
\]

(6.14)

with \(h := q_1q_2q_3\). It is tempting to identify the parameter \(h\) with the mass parameter \(\mu\) in the previous example. In fact both are related to the weight of the line bundle over \(X = \mathbb{C}^2\) and \(X = \mathbb{C}^3\), where the total space is six and ten dimensions, respectively. However, an important difference here is the decoupling limit is not well defined, while the Calabi-Yau limit \(h \to 1\) is still well-defined. It seems this is related to the fact that the tangent space is smooth in \(d = 2\) case, but it is singular (the tangent space only has a meaning as virtual bundle) for \(d > 2\). In the Calabi-Yau limit the partition function reduces to the MacMahon function;

\[
F(q_a; q) = \frac{[q_1^{-1}][q_2^{-1}][q_3^{-1}]}{[q_1][q_2][q_3][q][q^{-1}]} = \frac{1}{|q|^2}.
\]

(6.15)

Another interesting limit is the “refined topological vertex limit”, where we take \(q_1, q_3 \to 0\) with \(|q_1| << |q_3|\) and \(q_2 \to \infty\) keeping \(h\) constant. In such a limit \(q_3\) corresponds to a preferred

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\(^8\)In six dimensional case the natural counting parameter is \((-1)^n q\), because with this choice the partition function of \(U(n)\) theory reduces to the \(n\)-th power of the MacMahon function \([27],[47]\) in the Calabi-Yau limit \(h \to 1\).
direction of the refined topological vertex. From the relation

$$
\frac{[ht]}{[t]} \to \begin{cases} h^{\frac{1}{2}} & t \to \infty \\ h^{-\frac{1}{2}} & t \to 0 \end{cases}
$$

we find

$$
F(q_α; q) = \frac{-[q_1 h][q_2 h][q_3 h]}{[q_1][q_2][q_3]|\sqrt{h}|[\sqrt{h}|q]} \to -h^{-\frac{1}{2}} = -1/\sqrt{|q_4 q_5|},
$$

which can be identified with the refined conifold amplitude with the Kähler parameter $-1$.

When $n > 1$ (non-abelian case) the fixed points are labeled by $n$-tuple of plane partitions (colored partitions) $\vec{π} = (π^n)_{α=1}^n$ and the topological partition function is

$$
Z^{6D}_{U(n)}(u_α, q_α; q) = \sum_{π}((-1)^n q)^{|π|} \prod_{α,β=1}^n \hat{a}(V_{αβ}),
$$

where

$$
V_{αβ}(u_α, q_α) = \frac{u_α}{u_β} \left( \sum_{(i,j,k)\in π^α} q_1^{1-i} q_2^{1-j} q_3^{1-k} - \sum_{(r,s,t)\in π^α} q_1^r q_2^s q_3^t \right)
$$

$$
- (1-q_1)(1-q_2)(1-q_3) \sum_{(r,s,t)\in π^α} q_1^{r-i} q_2^{s-j} q_3^{t-k}
$$

One of the significant properties of $Z^{6D}_{U(n)}(u_α, q_α; q)$ so defined is that it is completely independent of the equivariant parameters $u^α$ for the framing torus, or the Coulomb branch moduli which physically means the distance of D6-branes. For lower instanton numbers this crucial property in proved in [47] by checking the vanishing of residues at the possible poles of $Z^{6D}_{U(n)}(q_α; q)$ as a rational function in the equivariant parameters $u^α$. Quite recently it is proved for arbitrary $k$ by examining the contour integral representation of the partition function discussed in the last section [49]. Once we know $Z^{6D}_{U(n)}(q_α; q)$ is independent of $u^α$, we can evaluate the partition function in a judicious scaling of $u^α$, for example $u^α = L^α$ with $L \to \infty$ [47]. Then we can see for $α < β$ [49];

$$
\lim_{L \to \infty} \hat{a}(V_{αβ})\hat{a}(V_{βα})|_{u_α = L^α} = (-h^{\frac{1}{2}})^{|π^β|−|π^α|},
$$

which implies

$$
\lim_{L \to \infty} Z^{6D}_{U(n)}(L^α, q_α; q) = \sum_{π}((-1)^n q)^{|π|} \prod_{α=1}^n \hat{a}(V_{αα}) \prod_{1≤α<β≤n} (-h^{\frac{1}{2}})^{|π^β|−|π^α|}
$$

$$
= \sum_{π} \prod_{α=1}^n ((-1)^n q)^{|π^α|} \hat{a}(V_{αα})(-h^{\frac{1}{2}})^{−n+1+2α}|π^α|
$$

$$
= \prod_{α=1}^n Z^{6D}_{U(1)}(q_α; q h^{α-\frac{n+1}{2}}).
$$

Hence we have a factorization of $U(n)$ partition function (6.18) into a product of $n U(1)$ partition functions with shifted instanton number counting parameters [48],[29]. This factorization is surely relies on the independence of $Z^{6D}_{U(n)}(q_α; q)$ of the Coulomb moduli and is related to the orbifold action of $Z_n$ on the transverse direction to D6-branes.
In fact we can use the following identity for generic variables $z_1, z_2$ to derive the $U(n)$ partition function (6.13) from (6.21);

$$
\sum_{\ell=1}^{n} \frac{1}{[z_1^{n+1-\ell}z_2^{-\ell}][z_1^{-\ell}z_2^{\ell}]} = \frac{1}{n} \sum_{\ell=0}^{n-1} \frac{1}{[\omega^\ell z_1][\omega^{-\ell} z_2]} = \frac{[(z_1 z_2)^n]}{[z_1 z_2][\omega^n][z_2^n]},
$$

(6.22)

where $\omega$ is an $n$-th root of unity $\omega^n = 1$. The second equality of (6.22) is just a simple consequence of

$$
\frac{1}{n} \sum_{\ell=0}^{n-1} \omega^{k\ell} = \delta_{k,0} + \delta_{k,n} (\text{mod } n).
$$

(6.23)

But, as is pointed out in [29], it is amusing to note that the first equality of (6.22) allows a geometric interpretation, though it can be also checked by induction on $n$. To see the geometric meaning let us consider the ALE resolution $\tilde{S}_n \rightarrow \mathbb{C}^2/Z_n$ of the $\mathbb{Z}_n$-orbifold of $\mathbb{C}^2$ where $Z_n$ acts by $(z_1, z_2) \rightarrow (\omega z_1, \omega^{-1} z_2)$. We can compare the equivariant index of the Dirac operator before and after the resolution. Recall that the equivariant index of the Dirac operator on $\mathbb{C}^2$ is simply

$$
\text{Ind}_q D = \frac{1}{[z_1][z_2]}.
$$

(6.24)

Then we can recognize the middle of (6.22) as the orbifold version of the equivariant index. On the other hand on the ALE space $\tilde{S}_n$ there appear $n$ fixed points over the origin which is the original fixed point of the torus action. For example, when $n = 2$, the ALE space $\tilde{S}_2$ is nothing but the Eguchi-Hanson space [50], which is isomorphic to the cotangent bundle of $\mathbb{CP}^1$. Thus we find two fixed points; the north and the south poles of $\mathbb{CP}^1$. The weights at these fixed points are exactly those appear on the left hand side of (6.22). Hence it is the blow up version of the index. Mathematically the equality of two versions follows from the fact that the fibre of the resolution is compact [29].

Since we already know $U(1)$ partition function has a plethystic form (6.13), we can compute a plethystic form of $U(n)$ partition function (6.13) as follows

$$
Z_{U(n)}^D(q_0; q) = \text{P. E. } [F_n(q_0, q)],
$$

(6.25)

where

$$
F_n(q_0; q) := \sum_{\alpha=1}^{n} F_1(q_0; q^{h^{\alpha - \frac{n+1}{2}}})
$$

(6.26)

$$
= \frac{[q_1 q_2][q_2 q_3][q_3 q_1]}{[q_1][q_2][q_3]} \sum_{\alpha=1}^{n} \frac{1}{[q^{h^{\alpha - \frac{n}{2}}}] [q^{-1} h^{1 - \alpha + \frac{n}{2}}]}.
$$

By applying the formula (6.22) with $z_1 = h^{\frac{1}{2}} q^{-\frac{1}{2}}$, $z_2 = h^{\frac{1}{2}} q^{\frac{1}{2}}$, we finally obtain

$$
F_n(q_0; q) = \frac{[q_1 q_2][q_2 q_3][q_3 q_1]}{[q_1][q_2][q_3]} \frac{[h^n]}{[h][h^{\frac{1}{2}} q][h^{\frac{1}{2}} q^{-1}]}.
$$

(6.27)

In [13] the equivariant ($K$ theory or $M$ theory) vertex is defined by

$$
V(\lambda, \mu, \nu) = \sum_{\pi \rightarrow (\lambda, \mu, \nu)} (-q)^{|\pi|} \hat{a}(|\pi|),
$$

(6.28)

where the summation is taken for the plane partitions with a fixed asymptotic condition $(\lambda, \mu, \nu)$. Note that $|\pi|$ and $\chi_\pi$ have to be regularized by taking the edge contributions into
account. Recently it has been show that if one of three partitions \((\lambda, \mu, \nu)\) is empty, \(V(\lambda, \mu, \nu)\) has a plethystic expression [51]. It is very interesting to see if this property holds for the full vertex.

6.3. \(d = 4\)

The following plethysitic form of the partition function is conjectured in [28], [29];

\[
Z^{8D}_{U(n)}(q_\alpha, \nu_\alpha, \mu_\alpha; q) = \text{P.E.} \left[ F(q_\alpha, \prod \frac{\nu_\alpha}{\mu_\alpha}, -q) \right],
\]

where

\[
F(q_\alpha, s; q) := \frac{[q_1 q_2] [q_2 q_3] [q_3 q_1] [s]}{[q_1] [q_2] [q_3] [q_4] [\sqrt{s q}] [q/\sqrt{s}]}. \tag{6.29}
\]

Note that \([q_4] = -[q_1 q_2 q_3]\) due to the Calabi-Yau condition \(q_1 q_2 q_3 q_4 = 1\). \(\nu_\alpha = e^{i a_\alpha}\) is the Coulomb branch parameters (associated with the position of \(D8\)-branes) for the gauge symmetry \(U(n)_C\) and \(\mu_\alpha = e^{-m_\alpha}\) is the mass parameter (associated with the position of \(\overline{D8}\)-branes) for the flavor symmetry \(U(n)_F\). It is remarkable that the final result only depends on the ratio \(\prod \nu_\alpha/\prod \mu_\alpha\), which is comparable to the fact that the partition function is independent of the Coulomb moduli \(u_\alpha\) in six dimensions.

For \(U(1)\) theory we can take \(\nu = 1\) by choosing the position of a single brane as the origin. With \(\mu = e^{-m}\) we find

\[
F(q_\alpha, \mu; q) = \frac{[q_1 q_2] [q_2 q_3] [q_3 q_1] [\mu]}{[q_1] [q_2] [q_3] [q_4] [\sqrt{s q}] [q/\sqrt{s}]}. \tag{6.29}
\]

It seems that we cannot produce the uniform measure on the space of solid partitions by tuning parameters. This is consistent with the fact that there is no known plethystic formula for the generating function of the counting of solid partitions. For example the massless limit \(\mu \to 1\) gives a trivial result \(F = 0\).

Let us put \(q_\alpha = e^{-R \alpha a}\) and \(\mu = e^{-R m}\) and take the limit \(R \to 0\), then

\[
F(t_\alpha, \mu; q) \to \exp \left( \frac{m (e_1 + e_2)(e_2 + e_3)(e_3 + e_4)}{e_1 e_2 e_3 e_4} \sum_{n=1}^{\infty} \frac{1}{n (1 - q^n)} \right)
\]

\[
= M(q)^{\frac{m (e_1 + e_2)(e_2 + e_3)(e_3 + e_4)}{e_1 e_2 e_3 e_4}}. \tag{6.29}
\]

It is the MacMahon function that appears in this limit. The generating function of the counting of the solid partition never appears.

It is interesting that a reduction to six dimensions is achieved by tuning the mass parameters (the positions of \(\overline{D8}\)-branes) which triggers a tachyon condensation of \(D8\)-\(\overline{D8}\) system to \(D6\)-branes [29]. The condition is \(\nu_\alpha = q_4 \mu_\alpha\), which gives \(s = q_4^n\) and we obtain;

\[
F(q_\alpha; q) := \frac{[q_1 q_2] [q_2 q_3] [q_3 q_1] [h^n]}{[q_1] [q_2] [q_3] [h] [h^{-\frac{1}{2}} q] [q h^{-\frac{1}{2}}]}, \tag{6.29}
\]

where \(h = q_1 q_2 q_3 = q_4^{-1}\). Up to sign this agrees with (6.27).
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