Quasi-cotilting modules and torsion-free classes

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Abstract

We prove that all quasi-cotilting modules are pure-injective and cofinendo. It follows that the class $\text{Cogen}_M$ is always a covering class whenever $M$ is a quasi-cotilting module. Some characterizations of quasi-cotilting modules are given. As a main result, we prove that there is a bijective correspondence between the equivalent classes of quasi-cotilting modules and torsion-free covering classes.

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1 Introduction and Preliminaries

Quasi-tilting modules were introduced by Colpi, Deste, and Tonolo in [9] in the study of contexts of $*$-modules and tilting modules, where it was shown that these modules are closely relative to torsion theory counter equivalences. Recently, quasi-tilting modules turn out to be new interesting after the study of support $\tau$-tilting modules by Adachi, Iyama and Reiten [1]. The second author proved that a finitely generated module over an artin algebra is support $\tau$-tilting if and only if it is quasi-tilting [19]. Moreover, Angeleri-Hügel, Marks and Vitória [2] proved that finendo quasi-tilting modules are also closely related to silting modules (which is a generalizations of support $\tau$-tilting modules in general rings).

It is natural to consider the dual of quasi-tilting modules, i.e., quasi-cotilting modules. As cotilting modules possess their own interesting properties not dual to tilting modules, we show in this paper that quasi-cotilting modules are not only to be the dual of quasi-tilting modules and they have their own interesting properties too. We prove that all quasi-cotilting modules are pure-injective and cofinendo. These are clearly new important properties of quasi-cotilting modules. Note that not all quasi-tilting modules are finendo. As a corollary, we obtain that the class $\text{Cogen}_M$ is a covering class whenever $M$ is quasi-cotilting. We give variant characterizations of quasi-cotilting modules and cotilting modules. As the main result, we prove that there is a bijection between the equivalent classes of quasi-cotilting modules and torsion-free covering classes.

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Throughout this paper, $R$ is always an associative ring with identity and subcategories are always full and closed under isomorphisms. We denote by $R$-Mod the category of all left $R$-modules. By Proj$R$ we denote the class of all projective $R$-module.

Let $M \in R$-Mod, we use following notations throughout this paper.

\[
\text{Adp}M := \{ N \in R \text{-Mod} \mid \text{there is a module } L \text{ such that } N \oplus L = M^X \text{ for some } X \};
\]
\[
\text{Cogen}M := \{ N \in R \text{-Mod} \mid \text{there is an exact sequence } 0 \to N \to M_0 \text{ with } M_0 \in \text{Adp}M \};
\]
\[
\text{Copres}M := \{ N \in R \text{-Mod} \mid \text{there is an exact sequence } 0 \to N \to M_0 \to M_1 \text{ with } M_0, M_1 \in \text{Adp}M \};
\]
\[
\overset{\perp 1}{M} := \{ N \in R \text{-Mod} \mid \text{Ext}^{1}_R(N, M) = 0 \};
\]
\[
\overset{\perp}{M} := \{ N \in R \text{-Mod} \mid \text{Ext}^{1}_R(M, N) = 0 \};
\]
\[
\overset{\circ}{M} := \{ N \in R \text{-Mod} \mid \text{Hom}_R(N, M) = 0 \};
\]
\[
M^{\circ} := \{ N \in R \text{-Mod} \mid \text{Hom}_R(M, N) = 0 \}.
\]

Note that $\text{Cogen}M$ is clearly closed under submodules, direct products and direct sums.

Let $\mathcal{T}$ be a class of $R$-modules, we denoted by $\text{Fac}(\mathcal{T})$ the classes forms by the factor modules of all modules in $\mathcal{T}$. An $R$-module $M \in \mathcal{T}$ is called $\text{Ext}$-injective in $\mathcal{T}$ if $\mathcal{T} \subseteq \overset{\perp 1}{M}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of $R$-Mod. The pair $(\mathcal{X}, \mathcal{Y})$ is said to be a torsion pair if it satisfies the following three condition (1) $\text{Hom}(\mathcal{X}, \mathcal{Y}) = 0$; (2) if $\text{Hom}(\mathcal{M}, \mathcal{Y}) = 0$, then $M \in \mathcal{X}$; (3) if $\text{Hom}(\mathcal{X}, N) = 0$, then $N \in \mathcal{Y}$. In the case, $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. Note that a subcategory $\mathcal{X}$ of $R$-Mod is torsion-free if and only if $\mathcal{X}$ is closed under direct products, submodules and extensions, see [11].

**Lemma 1.1** If an $R$-module $M$ is $\text{Ext}$-injective in $\text{Cogen}M$, then $(\overset{\circ}{M}, \text{Cogen}M)$ is a torsion pair.

**Proof.** It is easy to verify that $\overset{\circ}{M} = \overset{\circ}{(\text{Cogen}M)}$. We only need to prove that $(\overset{\circ}{M})^{\circ} = \text{Cogen}M$. If $T \in \text{Cogen}M$, then there is an injective homomorphism $i: T \to M^X$ for some set $X$. For any $N \in \overset{\circ}{M}$ and $f \in \text{Hom}_R(N, T)$, then $\text{Hom}_R(N, M^X) = (\text{Hom}_R(N, M))^X = 0$, hence $if = 0$. Then $f = 0$ since $i$ is injective. So $\text{Cogen}M \subseteq (\overset{\circ}{M})^{\circ}$.

For any $T \in (\overset{\circ}{M})^{\circ}$, consider the evaluation map $\alpha: T \to M^{\text{Hom}_R(T, M)}$ with $K = \ker \alpha$. Take canonical resolution of $\alpha$, i.e. $\alpha = i\pi$ with $\pi: T \to \text{Im} \alpha$ and $i: \text{Im} \alpha \to M^{\text{Hom}_R(T, M)}$. Clearly, $\text{Hom}_R(\pi, M)$ is surjective from the definition of $\alpha$. Applying the functor $\text{Hom}_R(-, M)$ to the exact sequence $0 \to K \to T \to \text{Im} \alpha \to 0$, we have an exact sequence

\[
0 \to \text{Hom}_R(\text{Im} \alpha, M) \to \text{Hom}_R(T, M) \to \text{Hom}_R(K, M) \to \text{Ext}^{1}_R(\text{Im} \alpha, M).
\]

Since $\text{Im} \alpha \in \text{Cogen}M \subseteq \overset{\perp 1}{M}$, $\text{Ext}^{1}_R(\text{Im} \alpha, M) = 0$. As $\text{Hom}_R(\pi, M)$ is surjective by above discussion, $\text{Hom}_R(K, M) = 0$, and hence, $K \in \overset{\circ}{M}$. Since $T \in (\overset{\circ}{M})^{\circ}$, we have that $\text{Hom}_R(K, T) = 0$ and $\alpha$ is injective. Thus $K = 0$, i.e. $T \in \text{Cogen}(M)$. \[\square\]

**Definition 1.2** [11] Definition 1.4 (1) Let $Q$ be an injective cogenerator of $R$-Mod. A module $M$ is called $Q$-cofinendo if there exist a cardinal $\gamma$ and a map $f: M^\gamma \to Q$ such that for any cardinal $\alpha$, all maps $M^\alpha \to Q$ factor through $f$.

(2) A module $M$ is cofinendo if there is an injective cogenerator $Q$ of $R$-Mod such that $M$ is $Q$-cofinendo.
Let $T$ be a class of $R$-modules and $M$ be an $R$-module. Then $f: X \to M$ with $X \in T$ is a $T$-precover of $M$ provided that $\text{Hom}(Y, f)$ is surjective for any $Y \in T$. A $T$-precover of $M$ is called $T$-cover of $M$ if any $g: X \to X$ such that $f = fg$ must be an isomorphism. A class $T$ of $R$-modules is said to be a precover class (cover class) provided that each module has a $T$-precover ($T$-cover).

Lemma 1.3 [Proof Proposition 1.6] The following are equivalent for a module $M$:

1. $M$ is cofinendo;
2. there is an $\text{Adp} M$-precover of an injective cogenerator $Q$ of $R$-$\text{Mod}$;
3. $\text{Cogen} M$ is a precovers class.

2 Quasi-cotilting modules

In this section, we introduce notion of quasi-cotilting modules and give some characterization of quasi-cotilting modules. In particular, we prove that all quasi-cotilting modules are pure injective and cofinendo.

Definition 2.1

1. An $R$-module $M$ is said to be a costar module if $\text{Cogen} M = \text{Copres} M$ and $\text{Hom}_R(-,M)$ preserves exactness of any short exact sequence in $\text{Cogen} M$.
2. An $R$-module $M$ is called a quasi-cotilting module, if it is a costar module and $M$ is $\text{Ext}$-injective in $\text{Cogen} M$.

Remark (1) Costar modules defined above had been studied in [16]. Moreover, their general version, i.e., $n$-costar modules, were studied by He [13] and Yao and Chen [22] respectively.

(2) Colby and Fuller [7] had defined another notion of costar modules which can be viewed as a special case of the above-defined costar modules.

For convenience, we say that a short exact sequence $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(-,M)$-exact ($(M \otimes_R -, \text{resp.})$-exact) if the functor $\text{Hom}_R(-,M)$ $(M \otimes_R -, \text{resp.})$ preserves exactness of this exact sequence.

The following result is well-known.

Lemma 2.2 Suppose that two short exact sequences $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C' \to 0$ are $\text{Hom}(-,M)$-exact with $B, B' \in \text{Adp} M$. Then $B \oplus C' \cong B' \oplus C$

Proof. It is dual to Lemma 2.2 in [20].

The following result presents a useful property of costar modules.

Lemma 2.3 Let $M$ be a co-star module. Suppose that the short exact sequence $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(-,M)$-exact and $A \in \text{Cogen} M$, then $B \in \text{Cogen} M$ if and only if $C \in \text{Cogen} M$.

Proof. $\Leftarrow$ Since $A$ and $C$ are in $\text{Cogen} M$, we have two monomorphisms $f: A \to M_A$ and $g: C \to M_C$ with $M_A$ and $M_C$ in $\text{Adp} M$. We consider the following diagram:

\[
\begin{array}{ccccccc}
0 & \to & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \to & 0 \\
\downarrow f & & \theta & & \downarrow (\theta,gb) & & \downarrow g \\
0 & \to & M_A & \xrightarrow{\alpha} & M_A \oplus M_C & \xrightarrow{\beta} & M_C & \to & 0
\end{array}
\]
where i and π is canonical injective and canonical projective respectively. Since the first row in above diagram is \( \text{Hom}_R(\cdot, M) \)-exact, there exists a morphism \( \theta \) such that \( f = \theta a \). It further induces a commutative diagram as above. It follows from Snake Lemma that \( (\theta, gb) \) is injective, i.e. \( B \in \text{Cogen}M \).

\[ \Rightarrow \] Since \( B \in \text{Cogen}M \) and \( M \) is a costar module, we have that \( 0 \to B \to M_0 \to L' \to 0 \) with \( M_0 \in \text{Adp}M \) and \( L' \in \text{Cogen}M \). There is a module \( M'_0 \) such that \( M_0 \oplus M'_0 = M^X \). So we obtain a new short exact sequence \( 0 \to B \to M^X \to L \to 0 \) with \( L = L' \oplus M'_0 \in \text{Cogen}M \). Consider the pushout of \( B \to C \) and \( B \to M^X \):

\[
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
0 & A & B & C & 0 & & & & \\
0 & A & M^X & N & 0 & & & & \\
& L & 1 & L & & & & & \\
& 0 & 0 & & & & & & \\
\end{array}
\]

Since the first row and second column are \( \text{Hom}_R(\cdot, M) \)-exact in above diagram, it is easy to see that the second row is also \( \text{Hom}_R(\cdot, M) \)-exact. Since \( A \in \text{Cogen}M \), similar to \( B \), there is a short exact sequence \( 0 \to A \to M^Y \to K \to 0 \) with \( K \in \text{Cogen}M \). By Lemma 2.2, we have that \( M^Y \oplus N \cong M^X \oplus K \), then it is easy to see that \( N \in \text{Cogen}M \). Consequently, \( C \in \text{Cogen}M \). □

Now we give some characterizations of quasi-cotilting modules.

**Proposition 2.4** Let \( M \) be an \( R \)-Module, the following statements are equivalent:

1. \( M \) is a quasi-cotilting module;
2. \( \text{Cogen}M = \text{Copres}M \) and \( M \) is Ext-injective in \( \text{Cogen}M \);
3. \( M \) is a costar module and \( \text{Cogen}M \) is a torsion-free class;
4. \( \text{Cogen}M = \text{Fac}(\text{Cogen}M) \cap \frac{1}{i}M \).

**Proof.** (1)⇒(2) By the involved definitions.

(2)⇒(1),(3) If \( M \) is Ext-injective in \( \text{Cogen}M \), then it is easy to see that the functor \( \text{Hom}_R(\cdot, M) \) preserves the exactness of any short exact sequences in \( \text{Cogen}M \). So \( M \) is a costar module by the assumption. By Lemma 1.1, we also have that \( \text{Cogen}M \) is a torsion-free class.

(3)⇒(4) Clearly, \( \text{Cogen}M \subseteq \text{Fac}(\text{Cogen}M) \). To see that \( \text{Cogen}M \subseteq \frac{1}{i}M \), we take any extension \( 0 \to M \to N \to L \to 0 \) with \( L \in \text{Cogen}M \). Since \( \text{Cogen}M \) is a torsion-free class, it is closed under extensions. It follows that \( N \in \text{Cogen}M \). But \( M \) is a costar module, the exact sequence is then \( \text{Hom}_R(\cdot, M) \)-exact. Thus, we can obtain that the exact sequence is actually split. Consequently we have that \( L \in \frac{1}{i}M \) for any \( L \in \text{Cogen}M \), i.e., \( \text{Cogen}M \subseteq \frac{1}{i}M \).

On the other hand, take any \( N \in \text{Fac}(\text{Cogen}M) \cap \frac{1}{i}M \). Then there is a module \( L \) in \( \text{Cogen}M \) and an epimorphism \( f: L \to N \). Set \( K = \ker f \), then \( K \in \text{Cogen}M \) since \( L \in \text{Cogen}M \). Then there is a short exact sequence \( 0 \to K \to M_0 \to A \to 0 \) with \( A \in \text{Cogen}M \) and \( M_0 \in \text{Adp}M \), as \( \text{Cogen}M = \text{Copres}M \). Since \( N \in \frac{1}{i}M \), we have the following communicative diagram:
0 \arrow{r} & K \arrow{r}{1} \arrow{d}{\alpha} & L \arrow{r}{f} \arrow{d}{\beta} & N \arrow{r} & 0 \\
0 \arrow{r} & K \arrow{r} & M_0 \arrow{r} & A \arrow{r} & 0

By Snake Lemma, we obtain that \( \ker \beta = \ker \alpha \in \text{Cogen}M = \text{Copres}M \) since \( L \in \text{Cogen}M \). From the third column in above diagram, we obtain a new short exact sequence \( 0 \to \ker \beta \to N \to \text{Im} \beta \to 0 \) with \( \text{Im} \beta \in \text{Cogen}M \) (since \( A \in \text{Cogen}M \)). By the assumption, \( \text{Cogen}M \) is a torsion-free class, so we have \( N \in \text{Cogen}M \). Hence, \( \text{Cogen}M = \text{Fac}(\text{Cogen}M) \cap \perp 1 M \).

\( (4) \Rightarrow (2) \) We need only to prove that \( \text{Cogen}M \subseteq \text{Copres}M \). Take any \( N \in \text{Cogen}M \) and consider the evaluation map \( u : N \to M^X \) with \( X = \text{Hom}_R(N, M) \). It is easy to verify that \( u \) is injective. Thus we have a short exact sequence \( 0 \to N \to M^X \to C \to 0 \), so it is enough to prove that \( C \in \text{Cogen}M \). Applying the functor \( \text{Hom}_R(-, M) \) to the sequence, we have an exact sequence \( 0 \to \text{Hom}_R(C, M) \to \text{Hom}_R(M^X, M) \to \text{Hom}_R(N, M) \to \text{Ext}^1_R(C, M) \to 0 \). It follows from \( u \) is the evaluation map that \( \gamma \) is surjective. So that \( \text{Ext}^1_R(C, M) = 0 \), i.e. \( C \in \perp 1 M \). It follows that \( C \in \text{Fac}(\text{Cogen}M) \cap \perp 1 M = \text{Cogen}M \) by \( (4) \).

The above result suggests the following definition.

**Definition 2.5** The torsion-free class \( \mathcal{T} \) is called a quasi-cotilting class if \( \mathcal{T} = \text{Cogen}M \) for some quasi-cotilting module.

Let \( M \) be an \( R \)-module. We denoted by \( \text{Ann}M \) the ideal of \( R \) consisting of all elements \( r \in R \) such that \( rM = 0 \). If \( \text{Ann}M = 0 \), then \( M \) is called faithful.

**Lemma 2.6** The following statements are equivalent for an \( R \)-module \( M \).

1. \( M \) is faithful;
2. \( R \in \text{Cogen}M \);
3. \( \text{Proj}R \subseteq \text{Cogen}M \);
4. \( \text{Fac}(\text{Cogen}M) = R-\text{Mod} \);
5. \( Q \in \text{Fac}(\text{Adp}M) \);
6. \( Q \in \text{Fac}(\text{Cogen}M) \).

**Proof.**

(1)\( \Leftrightarrow \) (2) Note that \( \text{Ann}M \) is just the kernel of the evaluation map \( R \to M^{\text{Hom}_R(R, M)} \), so the result follows from the universal property of the evaluation map.

(2)\( \Leftrightarrow \) (3) This is followed from the fact that \( \text{Cogen}M \) is closed under direct sums and direct summands.

(3)\( \Rightarrow \) (4) Using the fact that every module is a quotient of a projective module.

(4)\( \Rightarrow \) (3) If \( P \) is any projective module and \( \text{Fac}(\text{Cogen}M) = R-\text{Mod} \), then there is an exact sequence \( 0 \to P \to C \to P \to 0 \) with \( C \in \text{Cogen}M \). But \( P \) is projective implies that the exact sequence is split, it follows that \( P \) is a direct summand of \( C \) and consequently, \( P \in \text{Cogen}M \). Thus, (3) follows.

(4)\( \Rightarrow \) (5)\( \Rightarrow \) (6) Obviously.

(6)\( \Rightarrow \) (4) Clearly, \( \text{Fac}(\text{Cogen}M) \subseteq R-\text{Mod} \). On the other hand, since \( Q \in \text{Fac}(\text{Cogen}M) \), there exists an \( H \in \text{Cogen}M \) such that \( f : H \to Q \) is surjective. For any \( L \in R-\text{Mod} \), we have a monomorphism \( L \to Q^X \) for some \( X \), since \( Q \) an injective cogenerator. Then \( f^X \) is surjective and \( Q^X \cong H^X/K \) with \( K = \ker f^X \). Consequently, there exists a submodule \( H_1 \) of \( H^X \) such that \( L \cong H_1/K \). It is easy to see that \( H_1 \in \text{Cogen}M \), so \( L \in \text{Fac}(\text{Cogen}M) \) and (4) holds. \( \square \)
Recall that an $R$-module $M$ is called (1-)cotilting if it satisfies the following three conditions: (1) the injective dimension of $M$ is not more than 1, i.e., $\text{id}M \leq 1$; (2) $\text{Ext}^1_R(M^\lambda, M) = 0$ for any set $\lambda$; (3) There is an exact sequence $0 \to M_1 \to M_0 \to Q \to 0$ where $M_0, M_1 \in \text{Adp}M$ and $Q$ is an injective cogenerator. Note that $M$ is cotilting is equivalent to that $\text{Cogen}M = \perp^1 M$, see for instance [3]. We will freely use these two equivalent definition of cotilting modules.

A torsion-free class $\mathcal{T}$ is called a cotilting class if $\mathcal{T} = \text{Cogen}M$ for some cotilting module $M$.

We have the following characterizations of cotilting modules. Some of them were obtained in [16] (in Chinese). For reader’s convenience, we include here a complete proof.

**Proposition 2.7** Let $M$ be an $R$-module and $Q$ be an injective cogenerator of $R\text{-Mod}$, then the following statements are equivalent:

1. $M$ is a cotilting module;
2. $M$ is a quasi-cotilting module and $Q \in \text{Fac}(\text{Adp}M)$;
3. $M$ is a quasi-cotilting module and $\text{Proj}R \subseteq \text{Cogen}M$;
4. $M$ is a quasi-cotilting module and $\text{Fac}(\text{Cogen}M) = R\text{-Mod}$;
5. $M$ is a faithful quasi-cotilting module.
6. $M$ is a quasi-cotilting module and $\text{Cogen}M$ is a cotilting torsion-free class.
7. $M$ is a costar module and $Q \in \text{Fac}(\text{Adp}M)$;
8. $M$ is a costar module and $\text{Proj}R \subseteq \text{Cogen}M$;
9. $M$ is a costar module and $\text{Fac}(\text{Cogen}M) = R\text{-Mod}$;
10. $M$ is a faithful costar module.
11. $M$ is a costar module and $\text{Cogen}M$ is a cotilting torsion-free class.

**Proof.** (2)$\iff$(3)$\iff$(4)$\iff$(5)$\iff$(10)$\iff$(9)$\iff$(8)$\iff$(7) By Lemma 2.6 and the definitions.

(1)$\Rightarrow$(2) Since $M$ is cotilting, we have that and $\text{Cogen}M = \perp^1 M$ and $\text{Fac}(\text{Cogen}M) = R\text{-Mod}$ by the above arguments. It follows $\text{Cogen}M = \text{Fac}(\text{Cogen}M) \cap \perp^1 M$. Hence, $M$ is quasi-cotilting by Proposition 2.4.

(8)$\Rightarrow$(1) Since $\text{Proj}R \subseteq \text{Cogen}M$, we have that $\text{Fac}(\text{Cogen}M) = R\text{-Mod}$. For any $T \in \text{Cogen}M$, there is a short exact sequence $0 \to K \to P_0 \to T \to 0$ with $P_0 \in \text{Proj}R$. Note that the sequence is indeed in $\text{Cogen}M$, so $\text{Hom}_R(-, M)$ preserves exactness of this short exact sequence, as $M$ is a costar module. It follows that $\text{Ext}^1_R(T, M) = 0$ since $\text{Ext}^1_R(P_0, M) = 0$. Thus $\text{Cogen}M \subseteq \perp^1 M$ and $\text{Cogen}M$ is a torsion-free class by Lemma 1.1. Furthermore, we have that $M$ is a quasi-cotilting module by Proposition 2.4. Consequently, $\text{Cogen}M = \perp^1 M$ by Proposition 2.4 again, since $\text{Fac}(\text{Cogen}M) = R\text{-Mod}$. So $M$ is a cotilting module.

(1)$\Rightarrow$(6)$\Rightarrow$(11) It is obvious now.

(11)$\Rightarrow$(8). If $\text{Cogen}M$ is a cotilting torsion-free class, i.e., $\text{Cogen}M = \text{Cogen}T$ is for some cotilting module $T$, Then $\text{Proj}R \subseteq \text{Cogen}T$ by the above argument. Thus (8) follows. 


The above result yields the following characterization of costar modules.

**Proposition 2.8** The following statements are equivalent for an $R$-module $M$.

1. $M$ is a costar $R$-module;
2. $M$ is a costar $\bar{R}$-module, where $\bar{R} = R/\text{Ann}M$;
3. $M$ is a costar $R/I$-module for any ideal $I$ of $R$ such that $IM = 0$;
4. $M$ is a cotilting $\bar{R}$-module.
Note that the category of $R/I$-modules can be identified to the full subcategory $\{M \in R\text{-Mod} \mid IM = 0\}$. Under this identification, it is not difficult to verify that $\text{Cogen}_{R/I}M = \text{Cogen}_R M = \text{Cogen}_{\bar{R}}M$. Therefore, (1) $\iff$ (2) $\iff$ (3) is obvious from the definitions.

(2) $\Rightarrow$ (4) Note that $M$ is always faithful as an $\bar{R}$-module, so the conclusion follows from Proposition 2.7.

(4) $\Rightarrow$ (2) By Proposition 2.7.

A short exact sequence $0 \to A \to B \to C \to 0$ is called pure exact if it is $(M \otimes_R -)$-exact for any right $R$-module $M$. In this case, $C$ is called a pure quotient module of $B$. A module $M$ is called pure injective if any pure exact sequence is $\text{Hom}_R(-, M)$-exact.

Lemma 2.9 The following statements are equivalent:

1. $M$ is a pure injective $R$-module;
2. For any $X$, the short exact sequence $0 \to M^{(X)} \to M^X \to C \to 0$ is $\text{Hom}_R(-, M)$ exact;
3. $M$ is a pure injective $\bar{R}$-module, where $\bar{R} = R/\text{Ann}M$.

Proof. (1) $\iff$ (2) by [5, Lemma 2.1]. Similar to (1) $\iff$ (2), it is easy to see that (2) $\iff$ (3) since $M^{(X)}$, $M^X$ and $C$ are in $\bar{R}\text{-Mod}$.

From the above discussion, we can get the following important properties of quasi-cotilting modules.

Proposition 2.10 All costar modules are pure injective and cofinendo. Specially, all quasi-cotilting modules are pure injective and cofinendo.

Proof. Let $M$ be a costar $R$-module. We obtain that $M$ is a cotilting $\bar{R}$-module by Proposition 2.8. It follows that $M$ is a pure injective $\bar{R}$-module since all cotilting modules are pure injective [5]. Thus $M$ is a pure injective $R$-module by Lemma 2.9.

To prove that $M$ is cofinendo, we only need to prove that $\text{Cogen}M$ is closed under direct sums and pure quotient modules by Lemma 2.11 and Lemma 1.3. Obviously, $\text{Cogen}M$ is closed under direct sums. Suppose the short exact sequence $0 \to A \to B \to C \to 0$ is pure exact and $B \in \text{Cogen}M$. It follows that $A \in \text{Cogen}M$ and that $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(-, M)$ exact from $M$ is pure injective. So $C \in \text{Cogen}M$ by Proposition 2.8, i.e. $\text{Cogen}M$ is closed under pure quotient modules.

Lemma 2.11 [15, Theorem 2.5] If a class $\mathcal{A}$ is closed under pure quotient modules, then the following statements are equivalent:

1. $\mathcal{A}$ is closed under arbitrary direct sums;
2. $\mathcal{A}$ is a precover class;
3. $\mathcal{A}$ is a cover class.

Corollary 2.12 If $M$ is a costar module, then $\text{Cogen}M$ is a cover class. In particular, $\text{Cogen}M$ is a cover class for any quasi-cotilting module $M$.

Proof. By the proof of Proposition 2.10, we obtain that $\text{Cogen}M$ is closed under pure quotient and direct sums. Thus $\text{Cogen}M$ is a cover class by Lemma 2.11.

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7
3 Quasi-cotilting torsion-free class

Lemma 3.1 If $M$ is a quasi-cotilting module, then $\text{Adp}M = \ker \text{Ext}^1_R(\text{Cogen}M, -) \cap \text{Cogen}M$.

Proof. Suppose that $N \in \text{Adp}M$. Clearly we get $N \in \ker \text{Ext}^1_R(\text{Cogen}M, -) \cap \text{Cogen}M$, from (3) in Proposition 2.4. For the inverse inclusion, take any $L \in \ker \text{Ext}^1_R(\text{Cogen}M, -) \cap \text{Cogen}M$. Then there is a short exact sequence $0 \to L \to M_0 \to C \to 0$ with $M_0 \in \text{Adp}M$ and $C \in \text{Cogen}M$ since $L \in \text{Cogen}M = \text{Copres}M$. Since $\text{Ext}^1(C, L) = 0$, we can obtain that this exact sequence is split and hence $L \in \text{Adp}M$. \hfill \Box

Theorem 3.2 Let $Q$ be an injective cogenerator of $R$-Mod. The following statements are equivalent:

1. $M$ is a quasi-cotilting module;
2. $M$ is Ext-injective in $\text{Cogen}M$ and there is an exact sequence
   $$0 \to M_1 \to M_0 \to \alpha Q$$

with $M_0$ and $M_1$ in $\text{Adp}M$ and $\alpha$ a $\text{Cogen}M$-precover.

Proof. (1)$\Rightarrow$(2) By Proposition 2.10, $M$ is a cofinendo module. Then there is a morphism $\alpha: M_0 \to Q$ with $\alpha$ an $\text{Adp}M$-precover by Lemma 1.3. It can be shown that $\alpha$ is also a $\text{Cogen}M$-precover. Indeed, suppose that $M' \in \text{Cogen}M$, we proof that $f$ can factor through $\alpha$ for any $f: M' \to Q$. There exists a monomorphism $i: M' \to M^X$ since $M' \in \text{Cogen}M$. We have a morphism $g: M^X \to Q$ such that $f = gi$ since $Q$ is injective. It follows from $\alpha$ is $\text{Adp}M$-precover that there is a morphism $h: M^X \to M_0$ such that $g = ah$. Thus $f = gi = ah$. So $\alpha$ is a $\text{Cogen}M$-precover. Now set $M_1 = \ker \alpha$, we only need to prove that $M_1 \in \text{Adp}M$. Consider the exact sequence $0 \to M_1 \to M_0 \to \pi \text{Im} \alpha \to 0$, it is easy to prove that $\pi$ is also $\text{Cogen}M$-precover by the definition. For any $N \in \text{Cogen}M$, applying the functor $\text{Hom}_R(N, -)$ to this exact sequence, we have that

$$0 \to \text{Hom}_R(N, M_1) \to \text{Hom}_R(N, M_0) \to \text{Hom}_R(N, \text{Im} \alpha) \to \text{Ext}^1_R(N, M_1) \to \text{Ext}^1_R(N, M_0) = 0.$$ 

It follows from $\pi$ is $\text{Cogen}M$-precover that $\text{Ext}^1_R(N, M_1) = 0$. Obviously, $M_1 \in \text{Cogen}M$. Thus $M_1 \in \text{Adp}M$ by Lemma 3.1.

(2)$\Rightarrow$(1) We only need to prove that $\text{Cogen}M = \text{Copres}M$ by Proposition 2.4. Suppose that $N \in \text{Cogen}M$. Consider the evaluation map $a: N \to M^X$ with $X = \text{Hom}_R(N, M)$, which is injective since $N \in \text{Cogen}M$. Set $C = \text{coker} \ a$, next we prove that $C \in \text{Cogen}M$. There is a monomorphism $f: C \to Q^Y$ for some $Y$ since $Q$ is an injective cogenerator. Now consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \to & N & \xrightarrow{a} & M^X & \xrightarrow{b} & C & \xrightarrow{f} & 0 \\
& & \downarrow{h} & & \downarrow{s_1} & & \downarrow{g} & & \downarrow{s_0} \\
0 & \to & M_1^Y & \xrightarrow{\beta} & M_0^Y & \xrightarrow{\alpha} & Q^Y & & \\
\end{array}
$$

Since $\alpha$ is $\text{Cogen}M$-precover, there is a morphism $g$ such that $\alpha g = fb$, and then we have that $\beta h = ga$. Since $a$ is evaluation map and $M_1^Y \in \text{Adp}M$, we have $s_1$ such that $h = s_1 a$. It is easy to see that $(g - \beta s_1)a = 0$, and then we have $s_0$ such that $g - \beta s_1 = s_0 b$. So $\alpha s_0 b = \alpha(g - \beta s_1) = \alpha g = fb$, thus $f = \alpha s_0$ since $b$ is surjective. Since $f$ is injective, $s_0$ is injective. Consequently, $C \in \text{Cogen}M$ and $\text{Cogen}M = \text{Copres}M$. Then the proof is completed.
Let $\mathcal{D}$ be a class of $R$-modules. A module $M \in \mathcal{D}$ is called an injective object of $\mathcal{D}$ if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{D}$ is $\text{Hom}_R(-, M)$-exact. A module $M \in \mathcal{D}$ is called a cogenerator of $\mathcal{D}$ if for any $N$ in $\mathcal{D}$, there is a monomorphism $i: N \to M^X$ for some set $X$.

**Lemma 3.3** Let $\mathcal{T}$ be a class of $R$-modules and $Q$ be an injective cogenerator of $R$-$\text{Mod}$. Suppose that an exact sequence $0 \to A \to B \to^\alpha Q \to^\pi K \to 0$ satisfies that $\alpha$ is a $\mathcal{T}$-precover and that $A$ is Ext-injective in $\mathcal{T}$. Denote $\mathcal{D}=\{N \in R$-$\text{Mod} | \text{Hom}_R(N, \pi) = 0\}$. Then $\mathcal{T} \subseteq \mathcal{D}$ and $M := \text{Im} \alpha$ is an injective cogenerator in $\mathcal{D}$.

**Proof.** For any $C \in \mathcal{T}$ and any morphism $f: C \to Q$, there is a morphism $g: C \to B$ such that $f = \alpha g$ since $\alpha$ is a $\mathcal{T}$-precover. Thus $\text{Hom}_R(C, \pi) = 0$ and $\mathcal{T} \subseteq \mathcal{D}$.

Take any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{D}$ and any morphism $h: X \to M$. Consider the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & X & \overset{a}{\longrightarrow} & Y & \overset{b}{\longrightarrow} \ Z & \longrightarrow & 0 \\
& & \downarrow{h} & & \downarrow{\beta} & & \downarrow{c} & & \\
0 & \longrightarrow & M & \overset{b}{\longrightarrow} & Q & \overset{\pi}{\longrightarrow} & K & \longrightarrow & 0
\end{array}
$$

It follows that from $Q$ is injective that there is a morphisms $c$ such that $bh = ca$. Since $Y \in \mathcal{D}$, we have a morphism $\beta$ such that $c = b\beta$. Thus $bh = b\beta a$ and $h = \beta a$ since $b$ is injective. So the exact sequence is $\text{Hom}(-, M)$-exact, i.e. $M$ is injective in $\mathcal{D}$. For any $N \in \mathcal{D}$, we have a monomorphism $i: N \to Q^I$ since $Q$ is an injective cogenerator. Consider the exact sequence $0 \to M^I \to Q^I \to K^I \to 0$. Since $N \in \mathcal{D}$, we have that $\text{Hom}_R(N, \pi^I) \cong (\text{Hom}_R(N, \pi))^I = 0$, i.e., $\pi^I i = 0$. So there is a morphism $\gamma: N \to M^I$ such that $i = b^I \gamma$ and $\gamma$ is injective since $i$ is injective. Consequently, $M$ is a cogenerator in $\mathcal{D}$. \hfill $\square$

The following result is usually called Wakamatsu’s lemma, see for instance [12].

**Lemma 3.4** (Wakamatsu’s lemma) If $\mathcal{T}$ is a class of modules closed under extensions and if $\varphi: T \to M$ is a $\mathcal{T}$-cover, then $\ker \varphi \in T^{\perp \perp}$.

**Theorem 3.5** Let $\mathcal{T}$ be a torsion-free classes in $R$-$\text{Mod}$. The following statements are equivalent:

(1) $\mathcal{T}$ is quasi-cotilting torsion-free. i.e. there exists a quasi-cotilting module $M$ such that $\mathcal{T} = \text{Cogen} M$;

(2) $\mathcal{T}$ is a cover class;

(3) For any $R$-module $N$, there is an exact sequence

$$0 \to A \to B \to^\alpha N$$

with $\alpha$ a $\mathcal{T}$-precover and $A$ Ext-injective in $\mathcal{T}$.

**Proof.** (1$\Rightarrow$ 2) By Corollary 2.12.

(2$\Rightarrow$ 3) For any $R$-module $N$, there is an exact sequence

$$0 \to A \to B \to^\alpha N$$
with $\alpha$ $\mathcal{T}$-cover by (2). By Wakamatsu’s lemma, we have that $A \in \mathcal{T}^{-1}$. Since $B \in \mathcal{T}$ and $\mathcal{T}$ is a torsion-free class, $A$ is in $\mathcal{T}$. Thus $A$ is Ext-injective in $\mathcal{T}$.

(3)$\Rightarrow$ (1) Take $N = Q$ with $Q$ an injective cogenerator of $R$-$\text{Mod}$, then we have an exact sequence $0 \to A \to B \to ^\alpha Q$ with $\alpha$ a $\mathcal{T}$-precover and $A$ Ext-injective in $\mathcal{T}$ by assumption. Set $M = A \oplus B$. Then $\text{Cogen}M \subseteq \mathcal{T}$ since $\mathcal{T}$ is a torsion-free class. On the other hand, for any $L \in \mathcal{T}$, we have a monomorphism $f: L \to Q^X$. There is a morphism $g: L \to B^X$ such that $f = \alpha^X g$ since $\alpha^X$ is clearly a $\mathcal{T}$-precover. It follows from $f$ is injective that $g$ is injective. Thus $L \in \text{Cogen}M$ and $\text{Cogen}M = \mathcal{T}$. It remains to show that $M$ is Ext-injective in $\text{Cogen}M$ by Theorem 3.2. In fact, by assumption, we have to verify this only to $B$. Take any exact sequence $0 \to B \to N \to T \to 0$ with $T \in \text{Cogen}M$. Then $N$ is in $\text{Cogen}M$ since $\text{Cogen}M = \mathcal{T}$ is a torsion-free class. Set $C = \text{Im} \alpha$, consider the pushout of $B \to N$ and $B \to C$:

![Diagram]

By Lemma 3.3, the exact sequence $0 \to B \to N \to T \to 0$ is $\text{Hom}(-, C)$-exact. So there is a morphism $s_1$ such that $\pi = s_1 \alpha$. It is easy to see that $(d - cs_1)a = 0$. So we have $d = cs_1 + s_0 b$ in the above diagram for some $s_0$. Since $A$ is Ext-injective in $\text{Cogen}M$, there is a morphism $\delta: T \to N$ such that $s_0 = d \delta$. So $b = ed = e(s_0 b + cs_1) = es_0 b$ and $es_0 = 1$ since $b$ is surjective. But $b \delta = ed \delta = es_0 = 1$, thus, $b$ is split. Consequently, $B$ is Ext-injective in $\text{Cogen}M$. \qed

We say that two quasi-cotilting modules $M_1$ and $M_2$ are equivalent if $\text{Adp}M_1 = \text{Adp}M_2$.

**Corollary 3.6** There are bijections between

1. equivalence classes of quasi-cotilting modules;
2. torsion-free cover classes;
3. torsion-free classes $\mathcal{T}$ in $R$-$\text{Mod}$ such that every module has a $\mathcal{T}$-precover with Ext-injective kernel.

**Proof.** Let $M_1$ and $M_2$ be two quasi-cotilting modules, it is easy to prove that $\text{Adp}M_1 = \text{Adp}M_2$ if and only if $\text{Cogen}M_1 = \text{Cogen}M_2$ by Lemma 3.1. Now this correspondences can be defined as follows:

1. $\Rightarrow$ (2): $M \mapsto \text{Cogen}M$
2. (2)$\Rightarrow$ (3): $\mathcal{T} \mapsto \mathcal{T}$
3. (3)$\Rightarrow$ (1): $\mathcal{T} \mapsto M$ with $\text{Cogen}M = \mathcal{T}$. \qed

**References**

[1] T. Adachi, O. Iyama, I. Reiten, $\tau$-tilting theory, Compos. Math. 2014, 150 (3): 415-452.

[2] L. Angeleri-Hügel, F. Marks and J. Vitória, Silting module, arxiv: 1405.2531v1.
[3] L. Angeleri Hügel, A. Tonolo and J. Trlifaj, Tilting Preenvelopes and Cotilting Precovers, Algebras and Representation Theory, 2001, 4(2): 155-170.

[4] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, Adv. Math., 1991, 86: 111-152.

[5] S. Bazzoni, Cotilting modules are pure injective, Proceedings of the American Mathematical Society, 2003, 131(12): 3665-3672.

[6] S. Brenner, M. Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors, in: Proc. ICRA III, in: Lecture Notes in Math., vol. 832, Springer-Verlag, 1980, pp.: 103-169.

[7] R.R. Colby and K.R. Fuller, Costar modules, Journal of Algebra, 2001, 242: 146-159.

[8] R. Colpi, Some remarks on equivalences between categories of modules, Comm. Algebra, 1990, 18: 1935-1951.

[9] R. Colpi, G. Deste, and A. Tonolo, Quasi-Tilting Modules and Counter Equivalences, J. Algebra, 1997, 191: 461-494.

[10] R. Colpi, J. Trlifaj, Tilting modules and tilting torsion theories, J. Algebra 1995, 178: 614-634.

[11] S. E. Dickson, A torsion theory for Abelian categories, Trans. Amer. Math. Soc. 1966: 223-235.

[12] Edgar E. Enochs and Overtoun M. G. Jenda, Relative homological algebra,

[13] D. Happel, C. Ringel, Tilted algebras, Trans. Amer. Math. Soc., 1976, 215: 81-98.

[14] D. He, n-Costar modules and n-cotilting modules, Master Dissertion, 2009.

[15] H. Holm and P. Jørgensen, Covers, precovers and purity, Illinois Journal of Mathematics, 2008, 52(2): 691-703.

[16] H. Liu and S. Zhang, costar modules and 1-cotilting modules (in Chinese), Chinese Annals of Mathematics, 2009, 30A(1): 63-72.

[17] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z., 1986, 193: 113-146.

[18] C. Menini, A. Orsatti, Representable equivalences between categories of modules and applications, Rend. Sem. Mat. Univ. Padova 1989, 82: 203-231.

[19] J. Wei, τ-theory and ∗-modules, J. Algebra 2014, 414, 1-5.

[20] J. Wei, n-Star modules and n-tilting modules, Journal of Algebra, 2005, 283: 711-722.

[21] J. Wei, Z. Huang, W. Tong and J. Huang, Tilting modules of finite projective dimension and a generalization of ∗-modules, Journal of Algebra, 2003, 268: 404-418.

[22] L. Yao and J. Chen, Co-∗-modules, Algebra Colloq, 2010, 17(3): 447-456.