A NEW ACTIVATION FOR NEURAL NETWORKS AND ITS APPROXIMATION

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ABSTRACT

Deep learning with deep neural networks (DNNs) has attracted tremendous attention from various fields of science and technology recently. Activation functions for a DNN define the output of a neuron given an input or set of inputs. They are essential and inevitable in learning non-linear transformations and performing diverse computations among successive neuron layers. Thus, the design of activation functions is still an important topic in deep learning research. Meanwhile, theoretical studies on the approximation ability of DNNs with activation functions have been investigated within the last few years. In this paper, we propose a new activation function, named as “DLU”, and investigate its approximation ability for functions with various smoothness and structures. Our theoretical results show that DLU networks can process competitive approximation performance with rational and ReLU networks, and have some advantages. Numerical experiments are conducted comparing DLU with the existing activations-ReLU, Leaky ReLU, and ELU, which illustrate the good practical performance of DLU.

keywords: Deep neural networks, Activation function, Approximation power, DLU activation

1 Introduction

Deep learning has attracted tremendous attention from various fields of science and technology recently. Wide applications of deep learning including those in image processing [17] [13] and speech recognition [26] [29] have received great successes. Theoretical explanations for the success of deep learning have been recently studied from some viewpoint of approximation.

A fully connected deep neural networks (DNNs) of input \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) with depth \( L - 1 \) is defined as

\[
\Phi = \mathcal{L}_L \circ \sigma \circ \mathcal{L}_{L-1} \circ \sigma \circ \cdots \circ \sigma \circ \mathcal{L}_2 \circ \sigma \circ \mathcal{L}_1,
\]

where \( \mathcal{L}_i(y) := W_i y + b_i \) are affine transforms with weight matrices \( W_i \in \mathbb{R}^{d_i \times d_{i-1}} \) and the bias vectors \( b_i \in \mathbb{R}^{d_i} \) with \( d_0 = d \), and \( \sigma : \mathbb{R} \to \mathbb{R} \) is an activation function acting componentwise on vectors. We call \( \sigma \circ \mathcal{L}_i \) in \( \Phi \) the \( i \)-th layer (hidden layer) with width \( d_i \) and \( \mathcal{L}_i \) the output layer. We say a neural network has width \( W \) if the maximum width \( \max_{1 \leq i \leq L} \{d_i\} \) of hidden layers is no more than \( W \). We also call the number of nonzero elements of all \( W_i \) and \( b_i \) in the neural network the number of weights of the neural network \( \Phi \).

The activation function \( \sigma \) is allowed to be general, but important and inevitable as they help in learning non-linear transformations and perform diverse computations among successive neuron layers. Latent patterns in data are learned by an appropriate activation function and presented in the output of neural networks.

Neuron networks in early days utilized sigmoid-like activations to train with backpropagation, which however suffers from gradient vanishing problems [2], especially as networks become deep. To overcome this phenomenon, a new activation was found and popularized [25] [10], called the rectified linear unit (ReLU), and from then on, artificial intelligence began to flourish with the development of deep learning techniques. Despite the remarkable success of

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ReLU, it has a significant limitation that ReLU squishes negative inputs to zeros, some dead neurons appear during training which may cause information loss at the feed-forward stage.

To resolve such issues, some modifications of ReLU were proposed and gaining more and more popularity, which include Leaky ReLU [20], Parametric Linear Unit(PReLU) [12], Softplus function [7], Exponential Linear Unit(ELU) [5] and Scaled Exponential Linear Unit(SELU) [14]. These modified activation functions show promising improvement on several tasks compared with ReLU [27].

Theoretical studies on the approximation ability of DNNs with various activation functions have never stopped and have been developed in a large literature. Contract to practical applications, it is still far lacking and mainly focused on sigmoid type and ReLU activations. When the activation function \( \sigma \) is a \( C^\infty \) sigmoid type function, which means \( \lim_{x \to -\infty} \sigma(x) = 1 \) and \( \lim_{x \to -\infty} \sigma(x) = 0 \), the approximation rates were given by Barron [11] for functions \( f \in L_2(\mathbb{R}^d) \) whose Fourier transforms \( \hat{f} \) satisfy a decay condition \( \int_{\mathbb{R}^d} |w||\hat{f}(w)|dw < \infty \). Another remarkable result (e.g. Mhaskar [23]) based on localized Taylor expansions asserts rates of approximation for functions from Sobolev spaces. These results were developed by the localized Taylor expansion approach and with \( \sigma \) satisfying \( \sigma^{(k)}(\mu) \neq 0 \) for some \( \mu \in \mathbb{R} \) and every \( k \in \mathbb{Z}_+ \). This condition is not satisfied by ReLU type activation. Therefore, the approximation theory developed previously does not apply to ReLU. Until recent years, for ReLU nets, approximation properties were established in [15, 21] for shallow nets and in [34, 3, 28] for deep nets and functions from Sobolev spaces and in [31] for general continuous functions. These results are obtained for fully connected nets.

Besides the above, related analysis from the perspective of approximation theory has been also investigated for some other kinds of activation functions. In [32], the authors proposed Elementary Universal Activation Function (EUAF) and proved that all functions represented by a EUAF fully connected neural network (FNN) with a fixed structure can be dense in the space of continuous functions, which was proved impossible with ReLU FNN [31]. Unfortunately, EUAF is partly a nonsmooth periodic function which makes it not applicable in practice. Another investigation about rational activation has been developed in [4]. It has been shown that the depth needed is exponentially smaller than ReLU neural networks [34] for approximating Sobolev functions while rational neural networks provide better approximation error. Meanwhile, some numerical experiments illustrated its potential power for solving PDEs and GAN.

Owing to the essential impact of activations on neural networks’ performance, the design of activation functions is still an important topic in deep learning research today. From numerous applications, it has been found that a nonlinear activation is usually expected to possess four characteristics [38]: differentiable almost everywhere, nearly linear mapping, few parameters, and not too complicated calculation.

In this paper, we construct a monotonic activation function with a stronger approximation ability which is potentially valuable for applications. Precisely, we define \( \rho : \mathbb{R} \to \mathbb{R} \) by

\[
\rho(x) = \begin{cases} 
  x, & \text{if } x \in [0, \infty), \\
  \frac{x}{1+e^{-x}}, & \text{if } x \in (-\infty, 0),
\end{cases}
\]

and refer to it as “DLU” activation. Actually, DLU can be categorized as a ReLU type having the above-mentioned four properties. Particularly, it enjoys a similar shape to ELU. The design on the negative part was motivated by EUAF.
which allows DLU to represent the division gate as well as product gate. These advantages lead to stronger approximation abilities and make it applicable for deep learning algorithms.

Given a target function, our main results in the following form quantify the structure (i.e., depth, layer sizes) of DLU neural networks that guarantee certain approximation accuracy.

**Form of approximation by DLU nets.** Let $\Omega \subset \mathbb{R}^d$ and $F$ be a class of functions on $\Omega$. For any $f \in F$ and $\epsilon > 0$, there exists a fully connected DLU neural network with $L$ layers and $N$ weights where $N = N(\epsilon; \Omega; F)$, producing a function $\Phi$ such that

$$|f(x) - \Phi(x)| \leq \epsilon, \quad \forall x \in \Omega.$$ 

Our results will show that the approximation ability of DLU nets can improve those of rational neural networks and ReLU neural networks. On the other hand, we prove that for functions with tensor form or from Korobov spaces, the approximation by DLU networks can alleviate the curse of dimensionality. Finally, some numerical experiments with DLU neural networks are conducted on classification and image denoising tasks, which illustrate competitive performance over the existing activations—ReLU, Leaky ReLU, and ELU. For the sake of convenience, we shall denote ReLU by $\sigma(x) = \max\{x, 0\}$ in the rest of this paper.

## 2 Theoretical results

In this section, we state our main theoretical results on DLU networks. All proofs are given in Appendix.

### 2.1 Basic properties of DLU neural networks

The subsection is devoted to some useful properties of DLU, which will be applied in our approximation theory.

The following lemma shows that DLU neural networks are able to produce product and division gates exactly.

**Lemma 1.** Let $0 < a < M$ be constants.

(i) There exists a DLU neural network $\Phi$ with depth 2, width 9 and number of weights 45 such that

$$\Phi(x, y) = xy, \quad \forall x, y \in [-M, M].$$

(ii) There exists a DLU neural network $\Phi$ with depth 3, width 9 and number of weights 51 such that

$$\Phi(x, y) = \frac{y}{x}, \quad \forall x \in [a, M], y \in [-M, M].$$

Next, we construct DLU networks to achieve polynomials and rational functions.

**Lemma 2.** Let $p, q$ be polynomials with degrees at most $n$ and $m$, respectively. If $q$ has no roots on $[-1, 1]$, there exists a DLU neural network $\Phi$ with depth $2 \max\{n, m\} + 3$, width 12 and number of weights $122 \max\{n, m\} + 51$ such that

$$\Phi(x) = \frac{p(x)}{q(x)}, \quad \forall x \in [-1, 1].$$

Lemma 2 shows the superiority of DLU for realizing polynomials, compared with ReLU, which has lower bounds in terms of depths and widths for approximating polynomials stated in Lemma 8 and Corollary 2 in Appendix.

The following lemma shows how ReLU can be approximated by DLU nets.

**Lemma 3.** Let $m, n > 0$ be two integers.

(i) For any $\epsilon > 0$, there exists a neural network $\Phi$ activated by DLU with depth 1 and width 1 such that

$$|\sigma(x) - \Phi(x)| \leq \epsilon, \quad \forall x \in \mathbb{R}.$$ 

(ii) There exists a DLU neural network $\Phi$ with depth $m$ and width 1 with weights being chosen from $\{0, 1\}$ such that

$$|\sigma(x) - \Phi(x)| \leq m^{-1}, \quad \forall x \in \mathbb{R}.$$ 

(iii) There exists a DLU neural network $\Phi$ with depth $m$, width 1 with weights being chosen from $\left\{\frac{1}{n}, n\right\}$ such that

$$|\sigma(x) - \Phi(x)| \leq (mn)^{-1}, \quad \forall x \in \mathbb{R}.$$
Notice that from Lemma 1 of [4] if we use rational activation function \( \frac{p}{q} \) with degrees 3 and 2 respectively, there exists a neural network with \( \log(\log(\frac{1}{\epsilon})) \) nonzero free parameters and depth \( \log(\log(\frac{1}{\epsilon})) \) to approximate ReLU with accuracy \( \epsilon \). Although it is better in terms of the number of nonzero free parameters and depth with bounded parameters, our activation function can approximate ReLU uniformly on \( \mathbb{R} \) (Lemma 5) instead of \([-1, 1]\).

Although it is possible to approximate polynomials by DLU nets with fixed numbers of weights, we also want to get lower bounds for approximating with ReLU nets, which is important to understand the differences between ReLU and DLU. We postpone the relevant results to Appendix C. The results show that ReLU nets can never reproduce exactly polynomials and interestingly, the network depth is much more important than width for approximating polynomials.

2.2 Approximation of \( e^{-|x|} \)
In this subsection, by choosing the function \( y = e^{-|x|} \) on \( \mathbb{R} \) which is Lipschitz but not differential, we illustrate the strong approximation ability of DLU neural networks. It is shown that approximation rate can be exponential in terms of depth and number of weights of DLU neural networks.

**Theorem 1.** Let \( n \in \mathbb{N} \). There exists a DLU neural network \( \Phi \) with depth \( 2n + 4 \), width 12 and number of weights \( 122n + 55 \) such that

\[
|e^{-|x|} - \Phi(x)| \leq 3^{1-n}, \quad \forall x \in \mathbb{R}.
\]

Furthermore, there exists a DLU neural network \( \Phi_n \) with depth \( 2n + 4 \), width \( \max\{12, 2d\} \) and number of weights \( 122n + 4d + 51 \) such that

\[
|e^{-|x|} - \Phi_n(x)| \leq 3^{1-n}, \quad \forall x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]

where \( |x| = \sum_{j=1}^d |x_j| \).

**Remark 1.** Applying the same argument, for arbitrary real constants \( a, b \) and non-zero \( c \), a Gauss function \( f(x) = ae^{-(x-b)^2/c} \) can be approximated by DLU networks exponentially as well.

2.3 Approximation of weighted Sobolev smooth functions
For \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d \) and \( x = (x_1, \ldots, x_d) \in [-1, 1]^d \), let

\[
D^\mathbf{k}f = \frac{\partial^{||\mathbf{k}||}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}} \text{ and } w(x) = 2^d \prod_{j=1}^d (1 - x_j^2)^{-1/2}.
\]

We consider weighted Sobolev space \( W_p^r([-1, 1]^d, w) \) with \( r > 0 \) and \( 1 \leq p < \infty \) defined by locally integrable functions on \([-1, 1]^d\) with norm

\[
\|f\|_{W_p^r} := \sum_{||\mathbf{k}|| \leq r} \left[ \int_{[-1,1]^d} |D^\mathbf{k}f(x)|^p w(x) dx \right]^{1/p} < \infty,
\]

and \( W_p^r([-1, 1]^d) \) by functions \( f \in C[-1, 1]^d \) with norm

\[
\|f\|_{W_p^r} := \max_{\mathbf{k}:||\mathbf{k}|| \leq r} \esssup_{x \in [-1, 1]^d} |D^\mathbf{k}f(x)| < \infty,
\]

where \( \mathbf{k} = (k_1, \ldots, k_d) \in \{0, 1, \ldots, r\}^d, ||\mathbf{k}|| = k_1 + \ldots + k_d \).

In the following result, we shall show the approximation ability of DLU neural networks to functions in the weighted Sobolev spaces. It shows that with DLU activation an improved rate can be achieved.

**Theorem 2.** Let \( r \in \mathbb{N} \) and \( 1 \leq p \leq \infty \). Given \( f \) from the unit ball of \( W_p^r([0, 1]^d) \), for any \( \epsilon > 0 \), there exists a DLU neural network \( \Phi \) of depth \( O(e^{-\frac{1}{d^2}}) \), width \( O(e^{-\frac{1}{d}}) \) and number of weights \( C_d e^{-\frac{1}{d}} \), such that \( \|f - \Phi\|_p \leq \epsilon \).

Moreover, the constant factor \( C_d \) for the number of weights can be at most \( C_0 \ln d \ln(\ln d) \), where \( C_0 > 0 \) is an absolute constant.

**Remark 1.** The improvement on the constant factor \( C_d \) is significant. In fact, to achieve the same accuracy \( \epsilon \in (0, 1) \), Theorem 1 of [4] asserts that \( f \in W_p^r([0, 1]^d) \) can be approximated by a ReLU deep net with at most \( c(\log(1/\epsilon) + 1) \) layers and at most \( c e^{-c/d}(\log(1/\epsilon) + 1) \) computation units with a constant \( c = c(d, r) \). However, the constant \( c \) here increases much faster as \( d \) becomes large. To be more specific, as pointed out in [7], the main approach in [4] is to approximate \( f \) by a localized Taylor polynomial, which leads to the constant \( c \) at least \( (2^d + 1)^d \) when \( d \) is large.
2.4 To break the curse of dimensionality

In this subsection, we attempt to break the curse of dimensionality in approximating multivariate functions by employing a “tensor-friendly” structure. Here breaking the curse of dimensionality means the approximation rate or complexity rate of a model can be merely impacted by the dimension of the input domain. To achieve it, we will introduce a space of functions with finite mixed derivative norms, which is sometimes referred to as a Korobov space. Related investigations with ReLU activation have been achieved in [24][22].

We consider weighted Korobov space $K^w_r([-1,1]^d, w)$ with $r \in \mathbb{N}$ and $1 \leq p < \infty$ defined by locally integrable functions on $[-1,1]^d$ with norm

$$
\|f\|_{K^w_r} := \sum_{|k|_{\infty} \leq r} \left[ \int_{[-1,1]^d} |D^k f(x)|^p w(x) dx \right]^{1/p} < \infty,
$$

and $K^w([-1,1]^d)$ by functions $f \in C([-1,1]^d]$ with norm

$$
\|f\|_{K^w} := \max_{k:|k|_{\infty} \leq r} \text{ess sup}_{x \in [-1,1]^d} |D^k f(x)| < \infty,
$$

where $|k|_{\infty} = \max \{k_1, \ldots, k_d\}$.

The following theorem shows that the dominant term of complexity rate is free of the input dimension.

**Theorem 3.** Let $r \in \mathbb{N}_+$ and $1 \leq p \leq \infty$. Given $f$ from the unit ball of $K^w_r([-1,1]^d)$, for any $\epsilon > 0$, there exists a DLU neural network $\Phi$ of depth $O(\log \epsilon^{-1})$, width $O \left( \epsilon^{-\frac{1}{r}} (\log \epsilon^{-1})^{\beta(r,d)} \right)$ and number of weights

$$
O \left( \epsilon^{-\frac{1}{r}} (\log \epsilon^{-1})^{\beta(r,d)+1} \right)
$$

such that $\|f - \Phi\|_p \leq \epsilon$, where $\beta(r, d) = \frac{(d-1)(2+r-d)}{1+r-d}$.

Sparse grids are employed to break the curse of dimensionality of ReLU neural networks for approximating functions from Korobov space [24] for $r = 2$. Notice that our bound is better since for ReLU neural networks, the power of $\ln \epsilon^{-1}$ is $\frac{3}{2}(d-1) + 1$. In comparison, by DLU neural networks, it is improved to be less than $\frac{3}{2}(d-1) + 1$.

The above result can be further improved by restricting functions in the form of rank-one tensor. For $r = 1, 2, \ldots$, let $V^r[0,1]$ be the space of functions $g$ such that $g, g', \ldots, g^{(r-1)}$ are absolutely continuous on $[0,1]$ and $g^{(r)}$ is of bounded variation not exceeding 1. We denote

$$
\mathcal{V}^r_p := \left\{ f(x) = \prod_{i=1}^d f_i(x_i) : f_i \in V^r_p[0,1], \quad \|f_i\|_{\infty} \leq 1, r_i \leq r, x \in [0,1]^d \right\}.
$$

**Theorem 4.** Let $r \in \mathbb{N}_+$ and $d \in \mathbb{N}$. For any $\epsilon > 0$ and $f \in \mathcal{V}^r_p$, there exists a DLU neural network of depth $O(\epsilon^{-\frac{1}{r+1}})$, width $O(1)$ and number of weights $O(\epsilon^{-\frac{1}{r+1}})$ such that $\|f - \Phi\|_{\infty} < \epsilon$.

2.5 Approximation of general continuous functions

Next we consider the ability of DLU neural networks for approximating general continuous functions. The approximation error will be estimated in terms of the modulus of continuity. For $t > 0$ and $A \subset \mathbb{R}^d$,

$$
w^k_f(t) := \sup \{|f(x) - f(y)| : x_i = y_i, x, y \in A, \|x - y\|_2 \leq t\},
$$

which can be bounded by the classical modulus $w_f(t) := \sup \{|f(x) - f(y)| : x, y \in A, \|x - y\|_2 \leq t\}.

**Theorem 5.** For any continuous function $f$ on $[0,1]^d$ and $s \in \mathbb{N}$, there exists a DLU neural network $\Phi$ with depth $2 \log_2 (s + 2) + 4 \log_2 d + 1$, width $10d(s + 1)^d$ and number of weights $12d(s + 1)^d \log_2 (s + 2)$ such that $\|\Phi - f\|_{C([0,1]^d)} \leq \frac{1}{4} \sum_{i=1}^d w_f^i \left( \frac{1}{4} \right)$.
2.6 Approximation of piecewise smooth functions

Piecewise smooth functions are important in some practical applications, such as problems in image processing. In this subsection, we study the approximation of some special piecewise smooth functions. We consider the following piecewise functions, which depend on a function space $F$ on $[0,1]^d$ and a collection $A$ of subsets of $[0,1]^d$,

$$S(F, A) = \{ f_1 + f_2 \chi_{\Omega} : f_1, f_2 \in F, \Omega \subset A \}.$$ 

where $\chi_{\Omega}(x) = 1$ if $x \in \Omega$, and zero otherwise.

When functions in $F$ have some smoothness and $A$ is defined as level sets of $h$ for $h \in F$ by

$$A = \{ \Omega = \{ x \in [0,1]^d : h(x) \leq p(x) \} : h, p \in F \},$$

then we use the notation $S(F) := S(F, A)$ for short and call functions in $S(F)$ piecewise smooth functions induced by $F$. In the following, $B_1(F)$ denote the unit ball of $F$ centered at 0.

**Theorem 6.** Let $r \in \mathbb{N}$ and $1 \leq p \leq \infty$.

1. Let $f \in S \left( B_2(\mathbb{W}^p([0,1]^d)) \right)$. For any $\epsilon > 0$, there exists a DLU neural network $\Phi$ of depth $O(\epsilon^{-\frac{r}{p}})$, width $O(\epsilon^{-\frac{d}{p}})$ and number of weights $O(\epsilon^{-\frac{d}{p}})$ such that

$$\| f - \Phi \|_p \leq \epsilon.$$

2. Let $f \in S \left( B_2(K^p([0,1]^d)) \right)$. For any $\epsilon > 0$, there exists a DLU neural network $\Phi$ of depth $O(\log_2 \epsilon^{-1})$, width $O(\epsilon^{-\frac{d}{p}} \log_2 \epsilon^{-1})$ and number of weights $O(\epsilon^{-\frac{d}{p}} \log_2 \epsilon^{-1})$ such that $\| f - \Phi \|_\infty \leq \epsilon$, where

$$\beta(r, d) = \frac{d(2+r-d)-1}{1+r-d}.$$ 

Figure 2: Figure (a-c), (d-f) show MSEs and variances on the test sets of a three-layer fully connected neural network with different activation functions. X-axis represents the size of the training set and Y-axis gives mean MSE and variance over 10 independent trails.

3 Experiments

In this section, we conduct some numerical experiments to test the ability of DLU activation function for various learning tasks (regression, classification, image denoising). In each experiment, we utilize a neural network with the same architecture activated by different nonlinear activation functions, ReLU, LeakyReLU(negative slope = 0.01), ELU($\alpha = 1$), and DLU. The datasets we take include MNIST images with 60000 images for training and 10000 for testing, CIFAR10 images with 50000 for training and 10000 for testing, and Caltech101 images with 7677 for training and 1000 for testing.
3.1 Regression

We consider the following regression model on $[-10, 10]^d$

$$y = \frac{2 - 2z + 0.05z^3}{2 + z^2} + \xi,$$

where $z = \sum_{i=1}^{d} x_i$ and $\xi$ is the Gaussian noise with mean 0 and variance 0.25. See an illustration of the proposed regression model in Figure 3. We draw data with $x$ uniformly sampled with dimension $d$ varying in $\{50, 100, 1000\}$ and the size of the training set varying in $\{2000, 2500, 3000, 4000, \ldots, 10000\}$ for each $d$. We randomly generate 2000 samples for the test set in the same way as the training set without noise for all experiments in this subsection. For the network structure, we choose three hidden layers with widths all equal to 100.

Figure 2 depicts the results of 10 independent trials in terms of MSE. We observe that ELU and DLU outperform ReLU and LeakyReLU, and that ELU and DLU have similar performances for this regression model.

3.2 Classification

In the classification task \cite{9, 39}, we evaluate activation functions on MNIST and CIFAR10. For MNIST, we use a fully connected neural network which is the same as that used in the previous subsection on regression. For CIFAR10, we use a "small" Resnet18 with output channels for each convolutional layer to be 16, and max-pooling is applied after each residual block. Since the image is small, we also remove the first $7 \times 7$ kernel of Resnet18. The classification layer we employed is a fully connected layer with the input dimension 64. This neural network has no more than 40 thousand parameters. Test accuracy in Table 1 proves the superiority of the classification algorithm induced by DLU neural networks.

| Test acc(%) | ReLU | DLU | ELU | LeakyReLU |
|-------------|------|-----|-----|-----------|
| MNIST       | 97.82| 97.92| 97.12| 97.83     |
| CIFAR10     | 75.28| 76.57| 76.07| 74.62     |

3.3 Spherical image denosing

One more experiment we conduct follows that in \cite{16} for spherical image denoising with convolutional neural network activated by ReLU. Here we employ the neural network with the same architecture activated by various functions including Leaky ReLU, ELU, and DLU.

The generalization performance is evaluated by recovered PSNR over four typical images, while the neural network is trained on Caltech101, see Table 2 for reference. We find that DLU activation function can give the best denoised Hill and Man. On Barbara and Boat, DLU is as good as LeakyReLU. For all these four spherical images, DLU and LeakyReLU are better than ReLU and ELU. Finally, ELU and DLU have a similar shape but ELU is worst for rate = 0.2 trained on Caltech101, which shows the superiority of the proposed DLU activation function.

A Proofs of the results in Section 2.1

We first give the proof of Lemma 1, which follows some ideas of \cite{32}[Lemma 3.3 & Lemma 3.4].
Table 2: Average PSNRs on four images by trained NN over 5 independent trials.

| Image | Barbta | Boat | Hill | Man |
|-------|--------|------|------|-----|
| Noise Rate | 0.2 | 0.3 | 0.5 | 0.2 | 0.3 | 0.5 | 0.2 | 0.3 | 0.5 |
| ReLU | 23.748 | 22.552 | 21.374 | 26.031 | 24.560 | 22.839 | 26.008 | 24.735 | 23.230 | 26.267 | 24.881 | 23.215 |
| LeakyReLU | 23.675 | 22.568 | 21.392 | 25.837 | 24.537 | 22.852 | 25.933 | 24.730 | 23.269 | 26.128 | 24.871 | 23.269 |
| ELU | 21.550 | 22.568 | 21.315 | 23.540 | 24.537 | 22.762 | 23.373 | 24.758 | 23.204 | 23.614 | 24.901 | 23.167 |
| DLU | 23.786 | 22.570 | 21.387 | 25.970 | 24.612 | 22.838 | 26.019 | 24.751 | 23.266 | 26.270 | 24.911 | 23.271 |

Proof of Lemma [1] For any $x \in [-1, 1]$, we have $-x - 1 \leq 0, -x - 2 \leq 0$ and thereby

$$1 - 12\rho(-x - 1) + 12\rho(-x - 2) = 1 - 12\frac{-x - 1}{x + 2} + 12\frac{-x - 2}{x + 3} = 1 - \frac{12}{(x + 2)(x + 3)}.$$ 

But $(x + 2)(x + 3) \leq 12$ for $x \in [-1, 1]$. Hence,

$$12\rho(1 - 12\rho(-x - 1) + 12\rho(-x - 2)) = \left(1 - \frac{12}{(x + 2)(x + 3)}\right)(x + 2)(x + 3) = x^2 + 5x - 6.$$ 

As $\rho(x + 1) = x + 1$ for $x \in [-1, 1]$, we have

$$11\rho\left(\frac{11 - 5\rho(x + 1)}{11}\right) = 11\rho\left(\frac{6 - 5x}{11}\right) = 6 - 5x.$$

Hence,

$$12\rho\left(1 - 12\rho(-x - 1) + 12\rho(-x - 2)\right) + 11\rho\left(\frac{11 - 5\rho(x + 1)}{11}\right) = x^2 := \Phi_1(x).$$

The above formula corresponds to a realization of the quadratic polynomial $\Phi_1$ by a DLU neural network with depth 2, width 3, and number of weights 13. Then for $x, y \in [-M, M]$,

$$\Phi(x, y) := 2M^2\left(\Phi_1\left(\frac{x + y}{2M}\right) - \Phi_1\left(\frac{x}{2M}\right) - \Phi_1\left(\frac{y}{2M}\right)\right) = xy,$$

is a realization of the product function $x \cdot y$ by DLU neural network with depth 2, width 9 and number of weights 45. This proves the statement in (i).

To see (ii), noticing that $1 - x/a < 0$ for $x > a > 0$, we have that

$$\rho\left(1 - \frac{x}{a}\right) = \frac{1 - \frac{x}{a}}{1 - \rho\left(1 - \frac{x}{a}\right)} = -1 + \frac{a}{x}.$$

Thus

$$\frac{1}{a}\left(\rho\left(1 - \frac{a}{x}\right) + 1\right) = \frac{1}{x},$$

is realized by a neural network with depth 1, width 1 and number of weights 4. By Lemma [1], we have for any $x \in [a, M]$ and $y \in [-M, M]$, there exists a neural network $\Phi$ with depth 3, width 9 and number of weights 51 such that $\Phi(x, y) = \frac{y}{x}$. 

Proof of Lemma [2]. We start by proving that any polynomial on $[-1, 1]$ of degree at most $n$ can be achieved by a DLU neural network with depth $2n$, width 12, and number of weights $61n$. 


Recall that Legendre polynomials $p^n_i$ satisfy a three term recurrence relationship
\[ p^n_i(x) = (a_i x + b_i)p^n_{i-1}(x) - c_i p^n_{i-2}(x), \quad i = 2, 3, \ldots, \]  \hspace{1cm} (2)
where $p^n_0(x) = 1$, $p^n_1(x) = x$ and $a_n, b_n, c_n$ are real coefficients.

We need the $j$-th step realize four values: $x, p^j_n, p^j_{j-1}$ and any polynomial $P_{j-1}$ with degree $j$. Note that for the first step we only realize $p^n_1$ and $p^n_0$ and $x$. By \[ x, p^n_1 \text{ and } p^n_{j-1}, \] we can produce $p^n_j$ in the $j + 1$-th step. Furthermore, $x$ and $p^n_j$ in the $j$-th step are easy to be implemented in the $j + 1$-th step which only need an identity map.

By Lemma 1, we need a neural network with depth 2 and width 9 and number of weights 45 to output a multiplications in $\mathbb{R}$. Notice that orthogonal polynomials on $[-1, 1]$ is bounded, thus $\rho(x + 1) - 1$ is enough for identity map with width 1, depth 2 and number of weights 5. Notice that to output $P_j$ in the $j + 1$-th layer, we only need to add $c_j p^n_j$ to $P_{j-1}$ where $c_j$ is a constant. Thus we need a neural network $\rho(c_j P^n_j + P_{j-1} b)) + b$ where we choose $b = \min_{x \in [-1, 1]} c_j p^n_j + P_{j-1}$ This is actually a neural network with depth 2, width 1 and number of weights 6.

Recall that in each step we need to implement two identity maps, a multiplication operation, and a summation and we need to produce a polynomial with degree no more than $n$. Hence, there exists a neural network with depth $2n$, width 12, and number of weights $61n$ to produce the polynomial with degree no more than $n$.

By Lemma 1, it is easy to show that there exists a DLU neural network that produces a rational function with depth $2 \max\{n, m\} + 3$, width 12 and number of weights $122 \max\{n, m\} + 51$. This proves Lemma 2.

**Proof of Lemma 2** Define $\Phi(x) := \frac{\rho(x)}{n}$, it is easy to check that for any $x \in \mathbb{R}$, we have $|\sigma(x) - \Phi(x)| \leq \frac{1}{n}$. Then the first statement follows by taking $n \geq \frac{1}{\varepsilon}$.

To prove (ii), we define $\Phi_m(x) := \rho \circ \rho \circ \cdots \circ \rho(x)$ with $m$ compositions which is equal to $x$ when $x$ is nonnegative and $\frac{1}{1-x^2}$ otherwise. Then $\Phi_m(x) = \sigma(x)$ for $x \geq 0$ and $|\Phi_m(x) - \sigma(x)| = -e^{-x} \leq \frac{1}{m} = \frac{1}{1-mx} \leq \frac{1}{m}$ for $x < 0$. Hence the statement in (ii) follows.

For the last statement, we choose the network $\Phi(x) := \Phi_n \circ \Phi_n \circ \cdots \Phi_n(x) = \frac{1}{\frac{1}{m} - mx}$ with $m$ compositions. It is easy to see the conclusion.

**B Proof of Theorem 1**

**Proof.** Let $\psi(x) := \frac{1}{\lambda} (\rho(\lambda x) + \rho(-\lambda x))$ for $x \in \mathbb{R}$ and $\lambda > 1$. It is easy to see that
\[ \psi(x) = \begin{cases} x - \frac{x}{1+\lambda^2}, & \text{if } x \geq 0, \\ -x + \frac{x}{1-\lambda^2}, & \text{if } x < 0, \end{cases} \]
which implies that $0 \leq \psi(x) \leq |x|$ and $|\psi(x) - |x|| \leq \frac{1}{\lambda}$ for any $x \in \mathbb{R}$. Furthermore, since $1 - e^{-x} \leq x$ for all $x \in \mathbb{R}$ we can have
\[ |e^{-|x|} - e^{-\psi(x)}| \leq \left| 1 - e^{-|x|}\psi(x) \right| \]
\[ \leq 1 - e^{-1/\lambda} \leq \frac{1}{\lambda}, \quad \forall x \in \mathbb{R}. \]  \hspace{1cm} (3)
\[ \leq 1 - e^{-1/\lambda} \leq \frac{1}{\lambda}, \quad \forall x \in \mathbb{R}. \]  \hspace{1cm} (4)

On the other hand, by a classical result on rational approximation to $e^{-x}$, $x \geq 0$ (see, e.g. [18]), for any $n \in \mathbb{N}$, there exists a polynomial $q(x)$ of degree at most $n$ such that
\[ \left| \frac{1}{q(x)} - e^{-x} \right| \leq \sqrt{23}^{-n}, \quad \forall x \geq 0, \]
which combining with (3) yields
\[ \left| \frac{1}{q(\psi(x))} - e^{-|x|} \right| \leq \left| \frac{1}{q(\psi(x))} - e^{-\psi(x)} \right| + |e^{-\psi(x)} - e^{-|x|}| \]
\[ \leq \sqrt{23}^{-n} + \frac{1}{\lambda}, \quad \forall x \in \mathbb{R}. \]

Finally, taking $\lambda := 3^n$, by Lemma 2, we can construct a DLU neural network $\Phi(x)$ with depth $2n + 4$, width 12 and number of weights $122n + 55$ such that $\Phi(x) = \frac{1}{q(\psi(x))}$, which verifies the first statement.
For the second one, let $\psi_d(x) = \sum_{j=1}^d \psi(x_j)$ and $\Phi_d(x) = \frac{1}{q(\psi_d(x))}$. Applying the same arguments, we have that
\[
\left| e^{-\|x\|_1} - \frac{1}{\Phi_d(x)} \right| \leq \left| e^{-\|\psi_d(x)\|} - \frac{1}{q(\psi_d(x))} \right| + \left| e^{-\|x\|_1} - e^{-\psi_d(x)} \right| 
\leq \sqrt{23^{-n}} + \frac{d}{\lambda}.
\]
Then by taking $\lambda = 3^n d$, we can get the desired result in $d$-dimensional case.

### C Proof of Theorem 2 and Theorem 3

We first derive a lemma of orthogonal expansions, which will play a key role in sequential proofs.

**Lemma 4.** Let $r, d \in \mathbb{N}_+$ and $1 \leq p \leq \infty$. Denote $x^n = \prod_{j=1}^d x_j^{n_j}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$.

1. For any $f \in W_p([-1, 1]^d, w)$ and $N \in \mathbb{N}$, there exists $c_n \in \mathbb{R}$, $\|n\|_\infty \leq N$, such that
   \[
   \|f - \sum_{\|n\|_\infty \leq N} c_n x^n\|_{p,w} \leq C N^{-r} \|f\|_{W_p^r}.
   \]
   (5)

2. For any $f \in K_p^r([0, 1]^d)$ and $N \in \mathbb{N}$, there exists $c_n \in \mathbb{R}$, $\|n\|_\pi \leq N$, such that
   \[
   \|f - \sum_{\|n\|_\pi \leq N} c_n x^n\|_{p,w} \leq C N^{-r} (\log N)^{(d-1)(r+1)} \|f\|_{K_p^r},
   \]
   (6)
   where $\|n\|_\pi = \prod_{j=1}^d \max\{1, n_j\}$, $\|\cdot\|_{p,w}$ is the weighted $L_p$ norm with $w$.

**Proof.** For $f \in W_p([-1, 1]^d)$, we define a $2\pi$-periodic function $G_f$ by
\[
G_f(\theta) = f(\cos \theta_1, \ldots, \cos \theta_d), \quad \text{for } \theta_j \in [-\pi, \pi], \ j = 1, 2, \ldots, d.
\]
Noting that $2 \int_{[-1, 1]^d} f(x)(1 - x^2)^{-1/2} dx = \sum_{[-\pi, \pi]^d} G_f(\theta) d\theta$, we have
\[
\int_{[-1, 1]^d} f(x)w(x) dx = \int_{[-\pi, \pi]^d} G_f(\theta) d\theta,
\]
(7)
here recall that $w(x) = 2^d \prod_{j=1}^d (1 - x_j^2)^{-1/2}$. Since $G_f$ is even, we have that
\[
G_f(\theta) = \sum_{m=1}^{\infty} \sum_{|k|_\infty < m} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j)
\]
where $\hat{G}_f(k) = \int_{[-\pi, \pi]^d} G_f(\theta) \prod_{j=1}^d \cos(k_j \theta_j) d\theta$. For any $N \in \mathbb{N}$, setting $\Lambda_{\ell,N} = \{k \in \mathbb{N}^d : k_\ell \geq N\}$, $\ell = 1, \ldots, d$, then
\[
\int_{[-\pi, \pi]^d} \left| \sum_{|k|_\infty < N} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j) \right|^p d\theta 
\leq \sum_{\ell=1}^d \int_{[-\pi, \pi]^d} \left| \sum_{k \in \Lambda_{\ell,N}} k_\ell^{-r} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j) \right|^p d\theta 
\leq C \sum_{\ell=1}^d \left| \sum_{k \in \Lambda_{\ell,N}} k_\ell^{-r} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j) \right|^2 \left( \sum_{k \in \Lambda_{\ell,N}} k_\ell^{-r} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j) \right)^{1/2} d\theta 
\leq C d^{N-r} \|f\|_{W_p^r},
\]
where the second and first last step follows from the Littlewood-Paley inequalities with the function
\[
g_1(\theta) = \sum_{k \in \Lambda_{\ell,N}} k_\ell^{-r} \hat{G}_f(k) \prod_{j=1}^d \cos(k_j \theta_j)
\]
and \( g_2(\theta) = \sum_{k \in \Lambda_{d,N}} \hat{g}_f(k) \prod_{j=1}^d \cos(k_j \theta_j) \). Here recall the Littlewood-Paley inequalities

\[
\left\| \sum_{i \in D} c_i \psi_i \right\|_p \sim \left( \left\| \sum_{i \in D} [c_i \psi_i]_i^2 \right\|_p \right)^{1/2},
\]

where \( D \) is an index set and \( \{\psi_i\}_{i \in D} \) is an orthogonal system, \( A \sim B \) means \( c_1 B \leq A \leq c_2 B \) for some positive constants \( c_1, c_2 \). By substitution \( \theta_j = \arccos(x_j) \) and (7), we have

\[
\left\| f - \sum_{|n|x < N} c_n x^n \right\|_{p,w} \leq C_d N^{-r} \|f\|_{W^r},
\]

by making

\[
\sum_{|n|x < N} c_n x^n = \sum_{|n|x < N} \hat{g}_f(n) \prod_{j=1}^d \cos(n_j \arccos(x_j)).
\]

For \( f \in K^r_p \) and \( G_f \) as defined above, by [8, Thm4.4.1, 4.4.2] and the fact that \( G_f \) is even in each variable, we have that

\[
\|G_f - T_N(G_f)\|_p \leq N^{-r} (\log N)^{(d-1)(r+1)} \|f\|_{K^r_p},
\]

where \( T_N(G_f)(\theta) = \sum_{|n|_N \leq N} \hat{G}_f(n) \prod_{j=1}^d \cos(n_j \theta_j) \). Using the same argument above, we can have (6).

To prove Theorem 2 and 3, we also need the following lemma.

**Lemma 5.** Let \( n, d \in \mathbb{N} \) and \( \{\beta_i, i = 1, \ldots, n\} \) be a set of nonnegative integers with \( \sum_i \beta_i \leq n \). Then

(i) there exists a DLU neural network with depth \( 2[\log_2 d] \), width \( 5d \) and the number of weights \( 90 \log_2 d \log_2(\log_2 d) \) to output \( \prod_{i=1}^d x_i \) for \( x \in [-1,1]^d \);

(ii) there exists a neural network \( \Phi \) with depth \( 2 \log_2(n + d) + 2[\log_2 d] \), width \( 5d \) and the number of weights \( 13d \log_2(\frac{n + d}{d}) + 90 \log_2 d \log_2(\log_2 d) \) such that \( \Phi(x) = x_1^{\beta_1} x_2^{\beta_2} \cdots x_d^{\beta_d}, \) for \( x \in [0,1]^d \).

**Proof.** We need first show the result for \( x^n \) when \( x \in [-1,1] \). \( x^n \) is \( x^2 \) composed with itself or \( x \) at most \( \log_2 n \) times. Let \( \Phi(x) = x^2 \) in Lemma 1. Then we have \( \log_2 n \) times composition of \( \Phi(x) \) with reassigned parameters of last few layers so that those layers output identity mapping to output \( x^n \). This neural network has depth at most \( 2[\log_2 n] \), width 3 and the number of weights \( 13[\log_2 n] \).

Given any input \( x \in [0,1]^d \), we can construct a hierarchical partition of \( x_1, \ldots, x_d \) with each level \( \ell \) a clustering of lower level partition where each cluster has no more than 2 variables. Without loss of generality, we assume that there is a constant \( k \) such that \( 2^k = d \). Then we need \( 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \) multiplications to realize \( \prod_{i=1}^d x_i \). By Lemma 1, we can output multiplication within each group, and for those groups with only one elements we can use an identity mapping. It means that in the first step, we construct a neural network with depth 2 and width \( 9(\lceil \frac{d}{2} \rceil + 1) \) to achieve this goal. Iteratively, we can construct a neural network with depth \( 2 \log_2 d \) and width \( 9(\lceil \frac{d}{2} \rceil + 1), \ldots, 1 \) to exactly output multiplication \( \prod_{i=1}^d x_i \). Hence we need at most \( \sum_{j=1}^k \log_2 d \) binary multiplications, which means that there is the number of weights \( 90 \log_2 d \log_2(\log_2 d) \).

For the last statement, firstly let’s consider \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \), \( \alpha_i > 0 \). Notice that we can construct a neural network to output \( x_1^{\alpha_1} \cdots x_m^{\alpha_m} \) first and then compute their multiplication. By using the above results, there exists a neural network with depth \( 2 \max\{\log_2 \alpha_i : i = 1, \ldots, m\} + 2[\log_2 m] \), width \( 5m \) and number of weights \( 13 \sum_{i=1}^m \log_2 \alpha_i + 90 \log_2 d \log_2(\log_2 d) \).

Now we can prove our result. If we output \( x_1^{\beta_1} x_2^{\beta_2} \cdots x_d^{\beta_d} \) by a neural network, then it is obvious that it is contained in a neural network that outputs \( x_1^{\beta_1 + 1} x_2^{\beta_2 + 1} \cdots x_d^{\beta_d + 1} \). This means that degree is not larger than \( n + d \). Thus to output this monomial we need neural network with depth \( 2 \log_2(n + d) + 2[\log_2 d] \), width \( 5d \) and the number of weights \( 13d \log_2(\frac{n + d}{d}) + 90 \log_2 d \log_2(\log_2 d) \).
Proof of Theorem 2 and Theorem 2. The key idea is to employ Lemma 2 to construct a network that outputs a polynomial with each item to be a tensor product of some univariate polynomials. To approximate the Sobolev function \( f \), each univariate polynomial needed has degree no greater than \( N \). Lemma 2 shows that we can have \( d \) "columns" in the neural network which output univariate polynomials for each dimension. These columns is of depth \( 2N \), width \( 12d \) and the number of weights \( 61N^d \). To construct the tensor product, we need to keep all these polynomials to the degree \( N \) layer. As discussed in Lemma 2 such an identity mapping is a neural network with depth 2, width 1 and the number of univariate polynomials. 

Proof of Theorem 6. Proof of the result in Section 2.6. The key idea is to employ Theorem 3.1 [30]. To realize the Bernstein mapping, by Lemma 5 we have a neural network \( \Phi \) with depth \( 2\log_2 d \), width \( 5dN^d \) and number of weights \( 90N^d \log_2 d \log_2(\log_2 d) \). Thus we construct a neural network \( \Phi \) with depth \( 2N + 2\log_2 d \), width \( \max\{12d + N^d, 5dN^d\} \leq 12dN^d \) and number of weights \( 61N^d + 6N(N+1)d \) + \( 90N^d \log_2 d \log_2(\log_2 d) \). If we choose \( N = \epsilon^{-\frac{1}{4}} \), then we have \( \Phi \) of depth \( O(\epsilon^{-\frac{1}{4}}) \), width \( O(\epsilon^{-\frac{1}{4}}) \) and the number of weights \( O(\epsilon^{-\frac{1}{4}}) \) such that \( \| f - \Phi \| \leq \epsilon \).

D Proof of Theorem 4

Proof of Theorem 4. Set \( n \leq 1 \). Notice that Theorem 7.2 [18] shows that given \( f \in \mathcal{V}^r([0, 1]) \), there exists a rational function \( R \) of type \( (n, n) \), such that \( \| f - R \|_{\infty} \leq \frac{C(r)}{\sqrt{n^r}} \). Thus if \( f \in \mathcal{V}^r_d \) with \( f(x) = \prod_{i=1}^d f_i(x_i) \) and \( f_i \in \mathcal{V}^r([0, 1]) \), we can find \( R_1, \ldots, R_d \) of type \( (n, n) \) such that

\[
|f(x) - \prod_{i=1}^d R_i(x_i)| = \left| \sum_{i=1}^d \prod_{j=1}^{i-1} f_j(x_j) (f_i(x_i) - R_i(x_i)) \prod_{k=i+1}^d R_k(x_k) \right| \leq \sum_{i=1}^d \prod_{j=1}^{i-1} \| f_j \|_{\infty} \| f_i(x) - R_i(x) \|_{\infty} \prod_{k=i+1}^d (\| f_k \|_{\infty} + \| f_k - R_k \|_{\infty}) \leq \frac{C(r, d)}{n^r + 1},
\]

where \( f_0 \equiv 1 \).

Lemma 5 implies that we need a DLU neural network with 2[log2 d], width 5d and number of weights 90log2 d log2(2 log2 d) to output x1x2⋯xd. Together with Lemma 2 there exists a DLU neural network \( \Phi \) with depth \( 2[\log_2 d] + 2n + 3 \), width \( 12d \) and number of weights \( 122nd + 90\log_2 d \log_2(\log_2 d) + 51d \) to output \( \prod_{i=1}^d R_i(x_i) \). This implies \( \Phi \) is of depth \( O(\epsilon^{-\frac{1}{4r}}) \), width \( O(1) \) and the number of weights \( O(\epsilon^{-\frac{1}{4r}}) \). 

E Proofs of the result in Section 2.5

Proof of Theorem 5. The key idea is to employ Theorem 3.1 [30]. To realize the Bernstein mapping, by Lemma 5 we actually need a neural network \( \phi \) with depth \( 2[\log_2 d] + 4[\log_2 d] + 1 \), width \( 10d(s + 1)^d \) and number of weights \( (s + 1)^d \) \( (26d \log_2 (s + 2) + 100 \log_2 d \log_2(\log_2 d)) \). Then we have \( \| f - \Phi \|_{L_\infty([0, 1]^d)} \leq \frac{3}{4} \sum_{i=1}^d w_i(\frac{1}{2}) \).

F Proof of the result in Section 2.6

Proof of Theorem 6. Notice that for each function \( f = f_1 + f_2 \chi_{\Omega} \in S(B_1(W_2^r([0, 1]^d))) \), we have the approximation results for Sobolev smooth functions. Now we only need to study \( \chi_{\Omega} \). Since there exist some \( h, p \in W_s^p([0, 1]^d) \) such that \( \Omega = \{ x \in [0, 1]^d : h(x) \leq p(x) \} = \{ x \in [0, 1]^d : \max\{h(x) - p(x), 0\} = 0 \} \), we have \( \chi_{\Omega} = \chi_{\Omega^c} \circ (p(x) - h(x)) \).
when \( x \in [0,1]^d \). By Lemma 3, we can replace ReLU by DLU to approximate piecewise linear functions. That is, we have an approximator \( \chi_{\{0,\infty\}}(x) := \min\{\rho(nx) - \rho(nx - 1) \} \) which equals one when \( x \geq \frac{1}{n} \) and \( \lim_{n \to \infty} \chi_{\{0,\infty\}}(x) = \chi_{\{0,\infty\}}(x) \). In the following, we use notations \( \chi_\Omega := \chi_{\{0,\infty\}} \circ \sigma(p(x) - h(x)) \) and \( \tilde{\chi}_\Omega := \tilde{\chi}_{\{0,\infty\}} \circ \tilde{\rho}(\tilde{p}(x) - \tilde{h}(x)) \) where \( \tilde{\rho}(x) \) and \( \tilde{h}(x) \) are two network approximators and \( \tilde{\rho}(x) := \frac{1}{n} \rho(nx) \).

Since we have \( \hat{f}_1, \hat{f}_2 \) and \( \hat{\chi}_{\{0,\infty\}} \) which are realized by neural networks to approximate \( f \), then we have the following error to use a neural network \( \Phi = \hat{f}_1 + \hat{f}_2 \tilde{\chi}_\Omega \) for approximation (product is realized by Lemma 1).

\[
\|\Phi - f\| \leq \|f_1 - \hat{f}_1\| + \|f_2 \tilde{\chi}_\Omega - f_2 \chi_\Omega\| \\
\leq \|f_1 - \hat{f}_1\| + \|f_2 \tilde{\chi}_\Omega - f_2 \chi_\Omega\| \\
+ \|f_2 \chi_\Omega - f_2 \tilde{\chi}_\Omega\| \\
\leq 2\epsilon + \|f_2 \chi_\Omega - f_2 \tilde{\chi}_\Omega\| 
\]

Let us first consider \( \|f_2 \tilde{\chi}_\Omega - f_2 \chi_\Omega\|^p \).

\[
\|f_2 \tilde{\chi}_\Omega - f_2 \chi_\Omega\|^p \\
= \int_{\{x\in[0,1]^d: 0 \leq p(x) - h(x) \leq \frac{1}{n}\}} |f_2(\tilde{\chi}_\Omega - 1)|^p dx \\
+ \int_{\{x\in[0,1]^d: p(x) - h(x) < 0\}} |f_2 \chi_\Omega|^p dx \\
\leq \int_{\{x\in[0,1]^d: 0 \leq p(x) - h(x) \leq \frac{1}{n}\}} \frac{3}{2} |f_2|^p dx + \int_{[0,1]^d} |f_2 \tilde{\chi}_\Omega|^p dx \\
\to 0, \quad n \to \infty. 
\]

The same is true for the \( L_{\infty} \) norm, where the tendency only depend on the value of the coefficients. By choosing a proper integer \( n \), we have \( \|f_2 \tilde{\chi}_\Omega - f_2 \chi_\Omega\| \leq \epsilon \).

Then for the term \( \|\hat{f}_2 \hat{\chi}_\Omega - f_2 \hat{\chi}_\Omega\| \), we can use the following bound

\[
|\hat{\chi}_\Omega - \chi_\Omega| \\
= |\chi_{\{0,\infty\}} \circ \tilde{\rho}(\tilde{p}(x) - \tilde{h}(x)) - \chi_{\{0,\infty\}} \circ \sigma(p(x) - h(x))| \\
\leq |\chi_{\{0,\infty\}} \circ \tilde{\rho}(\tilde{p}(x) - \tilde{h}(x)) - \chi_{\{0,\infty\}} \circ \tilde{\rho}(\tilde{p}(x) - \tilde{h}(x))| \\
+ |\chi_{\{0,\infty\}} \circ \tilde{\rho}(\tilde{p}(x) - \tilde{h}(x)) - \chi_{\{0,\infty\}} \circ \sigma(p(x) - h(x))| \\
\leq \sup_{x \in \mathbb{R}} |\tilde{\rho}(x) - \text{ReLU}(x)| + |\tilde{\rho}(x) - p(x)| + |\tilde{h}(x) - h(x)| \\
\leq 3\epsilon, 
\]

where we employ the fact that \( \chi_{\{0,\infty\}}, \sigma \) is a Lipschitz function and the results in Lemma 3.

This implies \( \|\hat{f}_2 \hat{\chi}_\Omega - f_2 \hat{\chi}_\Omega\| \leq 3\epsilon^2 \leq \epsilon \). Thus we prove the error bound \( \|\Phi - f\|_p \leq \epsilon \).

Now we can verify the complexity of the neural networks. We assume that to approximate smooth functions(\( f_1, f_2, p \) and \( h \)), neural networks are of at most depth \( D \), width \( w \) and number of weights \( N \). At the first step, we output \( f_1, f_2 \) and \( p - h \). Next, we keep \( f_1 \) and \( f_2 \) and use \( p - h \) to output \( \hat{\chi}_\Omega \). Finally, we use DLU to generate \( \hat{f}_2 \times \hat{\chi}_\Omega \) by Lemma 1 and add \( \hat{f}_1 \), we get our \( \Phi \). Notice that all these steps only some constants introduced to the network \( \Phi \) regardless of the \( \hat{f}_1, \hat{f}_2, \tilde{\rho} \) and \( \tilde{h} \). We conclude that \( \Phi \) is of depth \( O(D) \), width \( O(W) \), number of weights \( O(N) \). We finish the proof.

### G. Upper bounds and lower bounds of ReLU networks

**Lemma 6** ([34]). There exists a ReLU neural network \( \Phi_\sigma \) with depth \( m \), width \( 4 \) and number of weights \( 10m \) such that \( \|\Phi_\sigma - x^2\|_{L_{\infty}([0,1])} \leq 2^{-2(m+2)} \). Furthermore, given \( M \geq 0 \), there exists a ReLU neural network \( \Phi_\sigma(x,y) \) with depth \( m + 1 \), width \( 12 \) and number of weights \( 30m + 17 \) such that it approximates \( xy \) with accuracy \( \frac{M^2}{2m+1} \) on \([-M,M]\) and \( |\Phi_\sigma(x,y)| \leq M^2 \).
Proof. Denote $g(x) = 2\sigma(x) - 4\sigma(x-\frac{1}{2}) + 2\sigma(x-1)$ and $g_m = g \circ g \circ \cdots \circ g$. Then $f_m(x) := x - \sum_{k=1}^{m} \frac{g_k(x)}{2^k}$ is a piecewise linear interpolation of $f(x) = x^2$ with $2^m + 1$ uniformly distributed breakpoints on $[0,1]$. Thus $\|f - f_m\|_{L_\infty([0,1])} \leq 2^{-2m+2}$ and $f_m$ can be realized by a ReLU neural network with depth $m$, width 4 and number of weights $10m$.

Let $M > 0$ and $g_m(x,y) = 2M^2(f_m \left( \frac{x+y}{2M} \right) - f_m \left( \frac{x}{2M} \right) - f_m \left( \frac{y}{2M} \right))$ where $x, y \in [-M, M]$. Since $|x| = \sigma(x)+\sigma(-x)$ and $\|g_m(x,y) - xy\|_{L_\infty([-M,M]^2)} \leq \frac{3M^2}{2^{2m+2}}$, we can find a ReLU neural network $g_m(x,y)$ with depth $m + 1$, width 12 and number of weights $30m + 17$ such that it approximates $xy$ with accuracy $\frac{3M^2}{2^{2m+2}}$. \hfill $\Box$

Lemma 7. Let $P_n(x)$ be a polynomial with degree no greater than $n$. Then there exists a ReLU neural network $\tilde{P}_n$ with depth $n(m+1)$, width 18 and number of weights $n(36m + 29)$ such that $\|P_n(x) - \tilde{P}_n(x)\|_{L_\infty([0,1])} \leq \frac{M}{2^{2m+2}}$ where $M$ is the upper bound of coefficients of $P_n(x)$.

Proof. The idea is similar to Lemma 2. For any $x \in [0,1]$, Legendre polynomials satisfy a three term recurrence relationship:

$$ (n+1)p_{n+1}^*(x) = (2n+1)xp_n^*(x) - np_{n-1}^*(x), \quad n = 1, 2, 3, \ldots, $$

with $p_0^*(x) = 1$ and $p_1^*(x) = x$. By Lemma 6 we denote $\Phi(x,y)$ as the approximating network of $xy$ with depth $m + 1$ and the approximation of $p_n^*(x)$ by ReLU neural network as $\tilde{p}_n(x)$ which has the same structure in Lemma 2 and has approximation error $|p_n^* - \tilde{p}_n|_{L_\infty([0,1])} \leq \epsilon_n$. This implies $|\Phi(x, \tilde{p}_n(x)) - \Phi(x, \tilde{p}_n(x))| \leq \frac{3(1+\epsilon_n)^2}{2^{2m+2}}$ since $|\tilde{p}_n(x)| \leq \epsilon_n + |p_n^*(x)| \leq 1 + \epsilon_n$. Then we have for any $x \in [0,1]$ and $n \geq 1$,

$$ |p_{n+1}^*(x) - \tilde{p}_{n+1}(x)| = \frac{2n+1}{n+1}xp_n^*(x) - \frac{n}{n+1}p_n^*(x) - \left( \frac{2n+1}{n+1}x\tilde{p}_n(x) - \frac{n}{n+1}\tilde{p}_n(x) \right) + \left( \frac{2n+1}{n+1}\Phi(x, \tilde{p}_n(x)) - \frac{n}{n+1}\Phi(x, \tilde{p}_n(x)) \right) \leq \epsilon_n + 2\epsilon_n + \frac{3(1+\epsilon_n)^2}{2^{2m+2}} \leq \epsilon_{n+1}. $$

One choice for $\epsilon_n$ is $\epsilon_n = 2^{2n-2m-2}$. Notice that $\epsilon_0 = \epsilon_1 = 0$ and $\epsilon_2 = 2^{-2m-2}$ are all satisfied under this choice. Let $\tilde{P}_n(x) = a_n\tilde{p}_n(x) + a_{n-1}\tilde{p}_{n-1}(x) + \cdots + a_1\tilde{p}_1(x) + a_0\tilde{p}_0(x)$ and $|a_i| \leq M$. Denote ReLU neural network by $\tilde{P}_n(x) = a_n\tilde{p}_n(x) + a_{n-1}\tilde{p}_{n-1}(x) + \cdots + a_1\tilde{p}_1(x) + a_0\tilde{p}_0(x)$. Then $|P_n(x) - \tilde{P}_n(x)| \leq M \sum_{i=2}^{n} 2^{2i-2m-2} \leq \frac{M}{2^{2m+2}}$.

Now we calculate the layers and width needed for $\tilde{P}_n(x)$. In the $j$-th step, we need ReLU neural network to approximate $\tilde{P}_{j-1}, \tilde{p}_j, \tilde{p}_{j-1}$ and $x$. That is we need two identity map, one summation and one multiplication. The network approximating multiplication, by Lemma 6 is of depth $m + 1$, width 12 and number of weights $30m + 17$. For identity map, we only need a neural network with depth $m + 1$, width 2 and number of weights $2(m + 1)$ and for summation operator, we need a neural network with depth $m + 1$, width 2 and number of weights $2m + 8$. Note that We need $n$ such steps, which means that $\tilde{P}_n$ is a neural network with depth $n(m+1)$, width 18 and number of weights $n(36m + 29)$.

Let $C_{m}^{PL} (0,1)$ be a set of continuous piecewise linear functions with breakpoints $0 < x_1 < \cdots < x_m < 1$. Then the set $\{x_i : i = 1, \ldots, m\}$ defines a partition $\Delta := \{\Delta_i = [x_i, x_{i+1}) : i = 0, \ldots, m\}$ of $[0,1]$ with $m+1$ intervals, where $x_0 = 0$ and $x_{m+1} = 1$. Notice that there exists a point $x_0' \in \Delta_i$ such that $f'(x_0') = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ for any differentiable function $f$. In the following, we denote $y_i = f'(x'_0)(x_i - x'_0) + f(x'_0)$.

Proposition 1. Any ReLU neural network $\Phi$ of depth $L - 1$ and width $d = [d_1, \ldots, d_{L-1}, 1]$ belongs to $C_m^{PL} (0,1)$ where $m = 3^{L-1} \prod_{i=1}^{L-1} d_i$. 

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We denote the inputs of activation function of the neurons in the second layer by $\Phi$. We have the following lower bound:

$$\Delta \sup_{i=0,\ldots,m} \frac{|y_i - f(x_i)|}{2}, \text{ where } m = 3^L W^L.$$

Let us consider how many break points $g_2$ can express when $g_2$ is a two-layer neural network with width $[d_1, d_2, 1]$. We denote the inputs of activation function of the neurons in the second layer by $g_{2i}$, $i = 1, \ldots, d_2$, then we can rewrite $g_2$ as $\sum_{i=1}^{d_2} w_i \sigma(g_{2i}) + b$. Notice that $g_{2i}$ is a one layer ReLU neural network. It means that $g_{2i}$ has at most $d_1$ break points. Hence $\sigma(g_{2i})$ has at most $d_1 + d_1 + 1 = 2d_1 + 1$ break points. Thus $g_2$ has at most $d_2 (2d_1 + 1) \leq 3d_1 d_2$ break points. It means that if we have a ReLU neural network with depth $L - 1$ and width $d = [d_1, \ldots, d_{L-1}, 1]$, then it can express continuous piecewise linear functions with at most $3^{L-1} \prod_{i=1}^{L-1} d_i$ break points.

**Theorem 7.** For any differentiable convex function $f$ on $[0, 1]$, and any ReLU neural network $\Phi$ with depth $L$, width $W$, we have $\|f - \Phi\|_{\infty, \{0, 1\}} \geq \inf_{h} \sup_{x \in [0, 1]} |y_1 - f(x)|/2$, where $m = 3^L W^L$.

**Proof.** We denote $\mathcal{N}^m_{L, W}$ to be all ReLU neural networks with depth $L$ and width $W$, and $P_{L, m}(0, 1)$ be piecewise linear functions with $m$ break points on $(0, 1)$. By Proposition 1 when we choose $m = 3^L W^L$, the following inequalities hold:

$$\inf_{f \in \mathcal{N}^m_{L, W}} \|f - \Phi\|_{\infty, \{0, 1\}} \geq \inf_{g \in \mathcal{C}^m_{L, \{0, 1\}}} \|f - g\|_{\infty, \{0, 1\}} \geq \inf_{h \in P_{L, m}(0, 1)} \|f - h\|_{\infty, \{0, 1\}}.$$

Thus we first consider a linear function $y = ax + b$ to approximate $f$ on $[x_1, x_1 + h] \subset [0, 1]$ for any $x_1 \in [0, 1)$ and $h > 0$.

Note that there exists a point $x_0 \in [x_1, x_1 + h]$ such that $f'(x_0) = \frac{f(x_1 + h) - f(x_1)}{h}$. Denote $y_1 = f'(x_0)(x_1 - x_0) + f(x_0)$. Then it is easy to see

$$\inf_{a, b} \|f - \Phi\|_{\infty, \{0, 1\}} \geq \frac{|y_1 - f(x_1)|}{2}.$$

Applying the linear approximaton of $f$ on each interval determined by $\Delta$, the following equality holds

$$\Phi \in \mathcal{N}^m_{L, W} \sup_{a, b} \inf_{\Delta \in [0, 1]} \|f - (ax + b)\|_{\infty, \{0, 1\}}.$$

Combining (8) and (9) and Proposition 1 we have

$$\inf_{f \in \mathcal{N}^m_{L, W}} \|f - \Phi\|_{\infty, \{0, 1\}} \geq \inf_{\Delta \in [0, 1]} \sup_{a, b} \inf_{\Delta \in [0, 1]} \|f - (ax + b)\|_{\infty, \{0, 1\}}.$$

**Corollary 1.** Let $f_k(x) = x^k$, $k \geq 2$. For any ReLU neural network $\Phi$ with depth $L$, width $W$, we have $\|f_k - \Phi\|_{\infty, \{0, 1\}} \geq (\frac{1}{2} - \frac{1}{2k}) (m + 1)^{-k}$, where $m = 3^L W^L$.

**Proof.** Let the interval $[x_1, x_1 + h] \subset [0, 1]$ and $x_0 \in (x_1, x_1 + h)$ which satisfies $f'(x_0) = \frac{f(x_1 + h) - f(x_1)}{h}$. Denote $y_1 = f'(x_0)(x_1 - x_0) + f(x_0)$ and $\delta := \frac{f(x_1) - y_1}{2}$. Notice that $f(x_1) - y_1$ is the vertical distance between $y = f'(x_0)(x - x_0) + f(x_0)$ and $y = \frac{f(x_1 + h) - f(x_1)}{h}(x - x_1) + f(x_1)$. Thus we have $\delta \geq \frac{1}{2} (f(x_1) + f(x_1 + h)) - f(x_1 + h/2)$. Then we have the following lower bound:

$$\begin{align*}
\delta &\geq \frac{1}{2} (f(x_1) + f(x_1 + h)) - f(x_1 + h/2) \\
&= \frac{1}{2} x^k + \frac{1}{2} (x_1 + h)^k - (x + h/2)^k \\
&= \sum_{i=1}^{k-1} \binom{k}{i} x_1^i \left( \frac{1}{2} h^{k-i} - \left( \frac{1}{2} h \right)^{k-i} \right) + \left( \frac{1}{2} - \frac{1}{2k} \right) h^k \\
&\geq \left( \frac{1}{2} - \frac{1}{2k} \right) h^k.
\end{align*}$$

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Combining Lemma 7 and 10, we have

\[
\inf_{\Phi \in \mathcal{NN}_{L,W}} \| f - \Phi \|_{L^\infty([0,1])} \geq \inf_{\Delta} \sup_{i=0,\ldots,m} \frac{|y_i - f(x_i)|}{2} \geq \left( \frac{1}{2} - \frac{1}{2^k} \right) \sup_{i=0,\ldots,m} h^k_i. \tag{11}
\]

Since larger interval means larger error, we have that \( \inf_{\Delta} \) in the last inequality is achieved when \( \{x_i : i = 0, \ldots, m+1\} \) is uniformly distributed on interval \([0, 1]\). It means that

\[
\inf_{\Phi \in \mathcal{NN}_{L,W}} \| f - \Phi \|_{L^\infty([0,1])} \geq \left( \frac{1}{2} - \frac{1}{2^k} \right) (m + 1)^{-k}.
\]

\[\blacksquare\]

**Remark 2.** Based on the above analysis, it is easy to follow the same idea to give lower bounds for any polynomial \( p(x) = \sum_{i=0}^k a_i x^i \) with \( a_i \geq 1 \). Similar result holds for differentiable concave functions.

**Lemma 8.** Let \( p_k(x) = \sum_{i=0}^k a_i x^i \) with \( a_i \geq 1, k \geq 2 \). For any \( L, W \in \mathbb{N} \), there exists a ReLU neural network \( \Phi \) with depth \( L \), width \( W \) that satisfies

\[
\| p_k - \Phi \|_{L^\infty([0,1])} \geq \frac{1}{4(m + 1)^k},
\]

where \( m = 3^L W^L \).

**Corollary 2.** We have the following upper bound and lower bound for approximating \( x^2 \) by ReLU neural networks

\[
12^{-2L-1} \leq \inf_{\Phi \in \mathcal{NN}_{L,4}} \| p_k - \Phi \|_{L^\infty([0,1])} \leq 4^{-L-1}.
\]

This result shows that the construction of Yarotsky [34] is nearly optimal.

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**References**

[1] A. R. Barron, “Universal approximation bounds for superpositions of a sigmoidal function,” IEEE Trans. Inform. Theory 39 (1993), 930–945.

[2] Y. Bengio, P. Simard, and P. Frasconi, “Learning long-term dependencies with gradient descent is difficult,” IEEE transactions on neural networks, vol. 5, no. 2, pp. 157–166, 1994.

[3] H. Bölcskei, P. Grohs, G. Kutyniok, and P. Petersen, “Optimal approximation with sparsely connected deep neural networks,” SIAM Journal on Mathematics of Data Science 1 (2019), 8–45.

[4] N. Boullé, Y. Nakatsukasa, and A. Townsend, “Rational neural networks,” arXiv preprint arXiv:2004.01902 [math] 2020.

[5] D.-A. Clevert, T. Unterthiner, and S. Hochreiter, “Fast and accurate deep network learning by exponential linear units (elus),” arXiv preprint arXiv:1511.07289 [cs.LG] 2015.

[6] P. J. Davis, Interpolation and approximation. Courier Corporation, 1975.

[7] C. Dugas, Y. Bengio, F. Bérisle, C. Nadeau, and R. García, “Incorporating second-order functional knowledge for better option pricing,” Advances in neural information processing systems, vol. 13, 2000.
[8] D. D. Őng, V. Temlyakov, and T. Ullrich, *Hyperbolic cross approximation*. Springer, 2018.

[9] H. Feng, S. Z. Hou, L. Y. Wei, and D. X. Zhou, “CNN models for readability of Chinese texts,” Math. Found. Comp., online first.

[10] X. Glorot, A. Bordes, and Y. Bengio, “Deep sparse rectifier neural networks,” in *Proceedings of the fourteenth international conference on artificial intelligence and statistics*. JMLR Workshop and Conference Proceedings, 2011, pp. 315–323.

[11] I. Goodfellow, Y. Bengio, and A. Courville, *Deep learning*. MIT press, 2016.

[12] K. He, X. Zhang, S. Ren, and J. Sun, “Delving deep into rectifiers: Surpassing human-level performance on imagenet classification,” in *Proceedings of the IEEE international conference on computer vision*, 2015, pp. 1026–1034.

[13] L. Jiao and J. Zhao, “A survey on the new generation of deep learning in image processing,” *IEEE Access*, vol. 7, pp. 172231–172263, 2019.

[14] G. Klambauer, T. Unterthiner, A. Mayr, and S. Hochreiter, “Self-normalizing neural networks,” *Advances in neural information processing systems*, vol. 30, 2017.

[15] J. Klusowski and A. Barron, “Approximation by combinations of ReLU and squared ReLU ridge functions with $\ell^1$ and $\ell^0$ controls,” *IEEE Transactions on Information Theory* 64 (2018), 7649–7656.

[16] J. Li, H. Feng, and X. Zhuang, “Convolutional neural networks for spherical signal processing via area-regular spherical haar tight framelets,” *IEEE Transactions on Neural Networks and Learning Systems*, 2022.

[17] X. Liu, L. Song, S. Liu, and Y. Zhang, “A review of deep-learning-based medical image segmentation methods,” *Sustainability*, vol. 13, no. 3, p. 1224, 2021.

[18] G. G. Lorentz, M. v. Golitschek, and Y. Makovoz, *Constructive approximation: advanced problems*. Springer, 1996, vol. 304.

[19] J. Lu, Z. Shen, H. Yang, and S. Zhang, “Deep network approximation for smooth functions,” *SIAM Journal on Mathematical Analysis*, vol. 53, no. 5, pp. 5465–5506, 2021.

[20] A. L. Maas, A. Y. Hannun, A. Y. Ng et al., “Rectifier nonlinearities improve neural network acoustic models,” in *Proc. icml*, vol. 30, no. 1. Citeseer, 2013, p. 3.

[21] T. Mao and D. X. Zhou, “Rates of approximation by ReLU shallow neural networks,” preprint, 2022.

[22] T. Mao, and D. X. Zhou, “Approximation of functions from Korobov spaces by deep convolutional neural networks,” Adv. Comput. Math., to appear.

[23] H. N. Mhaskar, “Approximation properties of a multilayered feedforward artificial neural network,” *Advances in Computational Mathematics*, vol. 1, no. 1, pp. 61–80, 1993.

[24] H. Montanelli and Q. Du, “New error bounds for deep relu networks using sparse grids,” *SIAM Journal on Mathematics of Data Science*, vol. 1, no. 1, pp. 78–92, 2019.

[25] V. Nair and G. E. Hinton, “Rectified linear units improve restricted boltzmann machines,” in *Icml*, 2010.

[26] A. B. Nassif, I. Shahin, I. Attiti, M. Azzeh, and K. Shaalan, “Speech recognition using deep neural networks: A systematic review,” *IEEE access*, vol. 7, pp. 19143–19165, 2019.

[27] C. Nwankpa, W. Ijomah, A. Gachagan, and S. Marshall, “Activation functions: Comparison of trends in practice and research for deep learning,” *arXiv preprint arXiv:1811.03378*, 2018.

[28] P. Petersen and F. Voigtlaender, “Optimal approximation of piecewise smooth functions using deep relu neural networks,” *Neural Networks*, vol. 108, pp. 296–330, 2018.

[29] A. Santhanavijayan, D. Naresh Kumar, and G. Deepak, “A semantic-aware strategy for automatic speech recognition incorporating deep learning models,” in *Intelligent system design*. Springer, 2021, pp. 247–254.

[30] M. H. Schultz, “$L^p$-multivariate approximation theory,” *SIAM Journal on Numerical Analysis*, vol. 6, no. 2, pp. 161–183, 1969.

[31] Z. Shen, H. Yang, and S. Zhang, “Deep network approximation characterized by number of neurons,” *arXiv preprint arXiv:1906.05497*, 2019.
[32] Z. Shen, H. Yang, and S. Zhang, “Deep network approximation with accuracy independent of number of neurons,” arXiv e-prints, pp. arXiv–2107, 2021.

[33] B. Xu, N. Wang, T. Chen, and M. Li, “Empirical evaluation of rectified activations in convolutional network,” arXiv preprint arXiv:1505.00853, 2015.

[34] D. Yarotsky, “Error bounds for approximations with deep relu networks,” Neural Networks, vol. 94, pp. 103–114, 2017.

[35] M. D. Zeiler, M. Ranzato, R. Monga, M. Mao, K. Yang, Q. V. Le, P. Nguyen, A. Senior, V. Vanhoucke, J. Dean et al., “On rectified linear units for speech processing,” in 2013 IEEE International Conference on Acoustics, Speech and Signal Processing. IEEE, 2013, pp. 3517–3521.

[36] H. Zheng, Z. Yang, W. Liu, J. Liang, and Y. Li, “Improving deep neural networks using softplus units,” in 2015 International Joint Conference on Neural Networks (IJCNN). IEEE, 2015, pp. 1–4.

[37] D. X. Zhou, “Universality of deep convolutional neural networks,” Appl. Comput. Harmonic Anal. 48 (2020), 787-794.

[38] H. Zhu, H. Zeng, J. Liu, and X. Zhang, “Logish: A new nonlinear nonmonotonic activation function for convolutional neural network,” Neurocomputing, vol. 458, pp. 490–499, 2021.

[39] X. N. Zhu, Z. Y. Li, and J. Sun, “Expression recognition method combining convolutional features and Transformer,” Math. Found. Comp., online first.