Winding number of a Brownian particle on a ring under stochastic resetting

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Abstract
We consider a random walker on a ring, subjected to resetting at Poisson-distributed times to the initial position (the walker takes the shortest path along the ring to the initial position at resetting times). In the case of a Brownian random walker the mean first-completion time of a turn is expressed in closed form as a function of the resetting rate. The value is shorter than in the ordinary process if the resetting rate is low enough. Moreover, the mean first-completion time of a turn can be minimised in the resetting rate. At large time the distribution of winding numbers does not reach a steady state, which is in contrast with the non-compact case of a Brownian particle under resetting on the real line. The mean total number of turns and the variance of the net number of turns grow linearly with time, with a proportionality constant equal to the inverse of the mean first-completion time of a turn.

Keywords: stochastic resetting, random walks, topological effects

1. Introduction
In equilibrium statistical mechanics, systems are assumed to have forgotten their initial state. One way to study the behaviour of a system out of equilibrium is to put it in contact with its initial configuration by resetting it. If the resetting occurs at Poisson-distributed times, the process exhibits a renewal structure. Technically, this renewal structure allows to work out the Laplace transform of the probability of the configurations in the system subjected to resetting, in terms of the probability of the configurations in the ordinary system (we will refer to processes without resetting as ordinary processes).
This approach, first taken in the case of a diffusive random walker on a line [1], has led to deep insights on the steady states of stochastic processes under resetting, and on mean first-passage times. In particular, the mean first-passage time of a diffusive random walker at a fixed target, which is known to be infinite in the ordinary case, becomes finite under resetting. Moreover, it can be expressed in closed form as a function of the resetting rate $r$, which exhibits a single minimum [2]. Resetting a Brownian random walker to its initial position cuts off long excursions in the wrong direction, which shortens the mean first-passage time on average. On the other hand, resetting may occur when the walker is reset when its position is very close to the target. Intuitively, stochastic resetting decreases the mean first-passage time because the amplitude of excursions in the wrong direction is unbounded. Stochastic resetting has found applications in a variety of fields, including statistical mechanics and active matter [3–6], population dynamics [7–10], reaction-diffusion systems [11, 12], search processes and stochastic processes [13–21]. For a review, see [22] and references therein.

In this work we revisit the case of a random walker in one dimension, and put it in a compact setting. The coordinate describing the position of a random walker on a circle is the polar angle, but it comes together with the winding number of the random walker (the number of turns it has completed since the beginning of the process). The distribution of the winding number for a Brownian walker has been studied in [23]. We first study the mean first-passage time until the completion of a turn (either clockwise or anti-clockwise). The motivation is the topological acceleration effect of resetting: indeed, with two absorbing boundaries at winding numbers $\pm 1$, the trajectory of the walker is bounded, and there is no obvious acceleration in the process due to cutting-off long excursions in a wrong direction. However, resetting the walker to its initial position (going along the shortest path on the circle), may help the random walker complete one turn: if it is within $\pi$ radians of one of the targets at resetting, it immediately moves to the target.

The paper is organised as follows. In section 2 we set up notations to describe the acceleration phenomenon. In section 3 we work out the corresponding renewal equations. The Laplace transform of the survival probability (with absorbing boundary conditions at winding number $\pm 1$) is obtained in closed form in terms of the ordinary process. In section 4 we work out the quantities needed for the application to a Brownian walker on a ring (some of which have already been worked out in [23]). In section 5 we use the explicit expression of the extinction rate of the process under resetting to characterise the long-time behaviour of the winding number of the diffusive random walker under resetting.

2. Notations and quantities of interest

Consider a random walker (or particle) on a circle with polar angle $\theta$, starting at $\theta = 0$. The process ends when the particle completes a turn (clockwise or anti-clockwise). The duration of the process is therefore the first-passage time of the coordinate $\theta$ at $\pm 2\pi$. The system is equivalent to a one-dimensional random walk on a segment with absorbing boundaries at $-2\pi$ and $2\pi$, with the two ends identified by periodic boundary conditions. For an investigation of the mean time to double absorption under resetting by two targets on either side of the origin, see [24].

When a resetting to the initial position occurs, the particle takes the shortest path to its initial position. Hence, if a resetting event occurs when the angle is in the domain

$$D := [-2\pi, -\pi] \cup [\pi, 2\pi],$$

(1)
Figure 1. Two possible configurations of the system. The process started at $\theta = 0$. The patch symbolises the absorbing boundary conditions at $\theta = \pm 2\pi$. In (a) the particle is in the safe zone, it goes to $\theta = 0$ if a resetting event occurs, and the process continues. In (b) the particle is in the dangerous zone: the shortest path to the initial configuration leads to $\theta = 2\pi$. If a resetting event occurs in this configuration, the process ends.

the process ends. On the other hand, if a resetting event occurs when the angle is in the domain

$$\mathcal{S} := [\pi, \pi],$$

the angle is reset to zero and the walker keeps going. We will refer to $\mathcal{D}$ and $\mathcal{S}$ as the dangerous zone and safe zone respectively. The situation is depicted on figure 1.

Let us denote by $Q^{(i)}(\theta, t)$ the probability to find the particle at angle $\theta$ at time $t$ in the process with resetting rate $r$. The survival probability $q^{(i)}(t)$ of the particle at time $t$ satisfies

$$q^{(i)}(t) = \int_{-2\pi}^{2\pi} Q^{(i)}(\theta, t) d\theta. \quad (3)$$

Let us denote by $T^{(i)}$ the duration of the process. Its mean $\langle T^{(i)} \rangle$ is obtained by integrating the time variable against the absorption rate (or death rate) of the process:

$$\langle T^{(i)} \rangle = \int_0^\infty t \left(-\frac{dq^{(i)}}{dr}(t)\right) dt = -[tq^{(i)}(t)]_0^\infty + \int_0^\infty q^{(i)}(t) dt. \quad (4)$$

As the lifetime of the process is almost surely finite, it is enough to derive the Laplace transform $\tilde{q}^{(i)}$ of the survival probability to express the mean lifetime:

$$\langle T^{(i)} \rangle = \tilde{q}^{(i)}(0), \quad (5)$$

with the following notation for the Laplace transform of any function $f$ of time:

$$\tilde{f}(s) := \int_0^\infty f(t)e^{-st}dt. \quad (6)$$
3. Renewal equations

3.1. Extinction rate of the process

Let us take advantage of the renewal structure of the process by conditioning on the latest time of resetting in the interval \([0, t]\) (this approach has proven very efficient in a broad range of models under stochastic resetting, see section 2.2 of \([22]\) for a review).

If there is no resetting in \([0, t]\), which is the case with probability \(e^{-rt}\), the system evolves according to the ordinary process on \([0, t]\). Otherwise, consider the latest resetting event in this interval, which can be written as \(t - \tau\), for some \(\tau\) in \([0, t]\). Consider the time-derivative of the survival probability of the process (which is the extinction rate of the process). At fixed \(\tau\), there is a contribution to the extinction rate from histories in which the walker has survived in the process under resetting, undergone resetting at \(t - \tau\) while being in the safe zone \(S\), and evolved according to the ordinary process on \([t - \tau, t]\), to reach one of the targets at time \(t\).

Moreover, there is a contribution from resetting events at time \(t\) that happen when the walker is in the dangerous zone \(D\) (as in figure 1(b)). These three contributions add up to

\[
\frac{dq^0}{dt}(t) = e^{-rt} \frac{dq^0}{dt}(t) + r \int_0^t d\tau e^{-r\tau} \left( \int_S Q^0(\theta, t - \tau) d\theta \right) \frac{dq^0}{dt}(\tau) - r \int_D Q^0(\theta, t) d\theta. \tag{7}
\]

Each of the three above-described processes reduces the survival probability (note that the time derivative of \(q^0(0)\) present in the first two terms on the rhs of equation (7) is negative).

Let us denote by \(S^0(t)\) the probability of presence of the particle in the safe zone at time \(t\):

\[
S^0(t) := \int_{-\pi}^{\pi} Q^0(\theta, t) d\theta. \tag{8}
\]

With this notation,

\[
\int_D Q^0(\theta, t) d\theta = q^0(t) - S^0(t) \tag{9}
\]

and equation (7) reads

\[
\frac{dq^0}{dt}(t) = e^{-rt} \frac{dq^0}{dt}(t) + r \int_0^t d\tau e^{-r\tau} S^0(t - \tau) \frac{dq^0}{dt}(\tau) - rq^0(t) + rS^0(t). \tag{10}
\]

The Laplace transform of the renewal equation for the extinction rate therefore reads

\[
-1 + s\tilde{q}^0(s) = -1 + (r + s)\tilde{q}^0(r + s) + r\tilde{S}^0(s) \left[ -1 + (r + s)\tilde{q}^0(r + s) \right] - rq^0(s) + r\tilde{S}^0(s), \tag{11}
\]

\[
= -1 + (r + s)\tilde{q}^0(r + s) + r\tilde{S}^0(s). \hspace{1cm}
\]
where we have used the normalisation conditions \( q^{(r)}(0) = 1 \) and \( q^{(0)}(0) = 1 \). In particular, substituting 0 to \( s \) yields

\[
0 = r\tilde{q}^{(0)}(r) + r\tilde{S}^{(r)}(0) \left[ -1 + r\tilde{q}^{(0)}(r) \right] - r\tilde{q}^{(r)}(0) + r\tilde{S}^{(r)}(0),
\]
hence

\[
\tilde{q}^{(r)}(0) = \tilde{q}^{(0)}(r) \left( 1 + r\tilde{S}^{(r)}(0) \right).
\]

### 3.2. Probability density of the position of the particle on the ring

Conditioning again on the latest time of resetting in \([0, t]\), denoted by \( t - \tau \), we must condition on the value \( \varphi \) of the angle at time \( t - \tau \), because only trajectories for which \( \varphi \) is in the safe zone survive the last resetting event:

\[
Q^{(r)}(\varphi, t) = e^{-r(t - \tau)} \int_0^{t - \tau} d\tau e^{-r\tau} \left( \int_S \! Q^{(r)}(\varphi', t - \tau) d\varphi' \right) Q^{(0)}(\varphi', t - \tau),
\]

\[
\theta \in (-\pi, \pi].
\]

Integrating equation (13) w.r.t. \( \theta \) over the safe zone yields the following renewal equation for the probability of presence in the safe zone:

\[
S^{(r)}(t) = e^{-rS^{(0)}(t)} + r \int_0^t d\tau e^{-r\tau} S^{(r)}(t - \tau) S^{(0)}(\tau).
\]

Taking the Laplace transform and substituting 0 to \( s \) yields the expression of \( \tilde{S}^{(r)}(0) \) in terms of quantities defined in the ordinary process:

\[
\tilde{S}^{(r)}(0) = \frac{\tilde{S}^{(0)}(r)}{1 - r\tilde{S}^{(0)}(r)}.
\]

Substituting into equation (12) yields the mean first-passage time at winding number \( \pm 1 \) in terms of quantities defined in the ordinary process:

\[
\tilde{q}^{(r)}(0) = \tilde{q}^{(0)}(r) \left( 1 + r\tilde{S}^{(r)}(0) \right).
\]

It is therefore enough for our purposes to calculate the probability density of the walker in the ordinary process \( Q^{(0)}(\theta, t) \) for all values of \( \theta \) in \([-\pi, \pi]\).

### 4. Probability density of a Brownian random walker without resetting

For a Brownian random walk (in units of length and time such that the radius of the ring and the diffusion constant are both equal to 1), the probability density \( Q^{(0)}(\theta, t) \) is expressed as a
\[ Q^{(0)}(\theta, t) = \int_{\Theta(0)=0}^{\Theta(t)=\theta} D[\Theta(\tau)] \exp \left[ -\frac{1}{2} \int_0^t \left( \frac{d\Theta}{d\tau}(\tau) \right)^2 d\tau \right] \times \prod_{\tau=0}^t 1 \left( \Theta(\tau) \in [\theta, 2\pi] \right), \tag{17} \]

where the last factor in the integrand constrains the angle \( \Theta(\tau) \) to stay in the interval \([\theta, 2\pi]\) at all time \( \tau \) in \([0, t]\). We can include it in the exponential integrand as the integral over \([0, t]\) of an infinite square potential well:

\[ Q^{(0)}(\theta, t) = \int_{\Theta(0)=0}^{\Theta(t)=\theta} D[\Theta(\tau)] \exp \left[ -\frac{1}{2} \int_0^t \left( \frac{d\Theta}{d\tau}(\tau) \right)^2 d\tau - \int_0^t V[\Theta(\tau)] d\tau \right], \tag{18} \]

where \( V(\varphi) = 0 \) if \( \varphi \in [\theta, 2\pi] \), \( V(\varphi) = +\infty \) otherwise.

Let us rewrite the probability \( Q^{(0)}(\theta, t) \) using bracket notations:

\[ Q^{(0)}(\theta, t) = \langle \theta | e^{-\hat{H}t} | 0 \rangle, \tag{19} \]

where \( \hat{H} = -\frac{1}{2} \frac{\partial^2}{\partial \theta^2} + V \).

A complete family of \( (\psi_n)_{n \geq 1} \) normalised eigenfunctions of this Hamiltonian is given by

\[ \langle \theta | n \rangle := \psi_n(\theta), \quad \hat{H}\psi_n = E_n\psi_n, \quad (n \in \mathbb{N}^*) \]

where

\[ \psi_n(\theta) = \frac{1}{\sqrt{2\pi}} \sin \left( \frac{n}{4}(\theta + 2\pi) \right), \tag{20} \]

so that \( E_n = \frac{n^2}{32} \) and \( \sum_{n \geq 1} |n\langle n| = 1. \)

Let us insert the identity operator into the propagator to express it in terms of the eigenfunctions:

\[ Q^{(0)}(\theta, t) = \sum_{n \geq 1} \langle \theta | e^{-iHt} | n \rangle \langle n | 0 \rangle = \sum_{n \geq 1} \langle \theta | e^{-E_nt} | n \rangle \langle n | 0 \rangle \]

\[ = \sum_{n \geq 1} e^{-E_nt} \psi_n(\theta)\psi_n(0). \tag{21} \]

We obtain the Laplace transform of the propagator in series form as

\[ \tilde{Q}^{(0)}(\theta, s) = \sum_{n \geq 1} \frac{1}{s - E_n} \psi_n(\theta)\psi_n(0). \tag{22} \]
The Laplace transform of the survival probability reads

$$\tilde{q}^{(0)}(s) = \int_{-2\pi}^{2\pi} \tilde{Q}^{(0)}(\theta, s) d\theta,$$

(23)

whose explicit value $\tilde{q}^{(0)}(s) = s^{-1} \left[ 1 - (\cosh (2\pi \sqrt{2} s))^{-1} \right]$ was reported in [23].

Moreover, the Laplace transform of the probability of presence in the safe zone reads

$$\tilde{S}^{(0)}(s) = \int_{-\pi}^{\pi} \tilde{Q}^{(0)}(\theta, s) d\theta.$$

(24)

The sums involved in the calculation of $\tilde{S}^{(0)}(s)$ can be expressed in terms of the Euler digamma function $\psi$ (the derivation is worked out in the appendix).

$$\tilde{S}^{(0)}(s) = \frac{\sqrt{2}}{8\pi^3} \Psi(s),$$

with $\Psi(s) := \Lambda(s) + \overline{\Lambda(s)}$,

and $\Lambda(s) := \psi \left( \frac{1 + i\sqrt{2}}{8} \right) - \psi \left( \frac{1}{8} \right) + \psi \left( \frac{3 + i\sqrt{2}}{8} \right) - \psi \left( \frac{3}{8} \right)$

$$- \psi \left( \frac{5 + i\sqrt{2}}{8} \right) + \psi \left( \frac{5}{8} \right) - \psi \left( \frac{7 + i\sqrt{2}}{8} \right) + \psi \left( \frac{7}{8} \right).$$

(25)

where the overline symbol denotes complex conjugation.

From equations (4) and (16), we obtain the expression of the mean first passage time at total winding number one:

$$\langle T^{(r)} \rangle = \frac{1}{r} \left( 1 - \frac{1}{\cosh(2\pi \sqrt{2} r)} \right) \left( 1 - \frac{1}{4\pi \sqrt{2}} \Psi(r) \right)^{-1}.$$

(26)

As $\Psi(0) = 0$, this mean first-passage time has a finite limit at zero resetting rate, which reads $\tilde{q}^{(0)}(0)$, the mean first-passage time in the ordinary process. At large resetting rate the mean first-passage time goes to infinity, as the walker becomes less and less likely to reach the dangerous zone before being reset. The mean first-passage time is plotted on figure 2, where the optimal resetting rate $r^*$ is apparent. The accelerating effect of the topology in our resetting prescription leads to a lower mean first-passage time at winding number $\pm 1$ than in the ordinary process if the resetting rate is low enough.

The process is slowed down by resetting events happening when the walker is in the safe zone. On the other hand it is accelerated by resetting events happening when the walker is in the dangerous zone. These two effects compensate each other exactly for one value $r_0$ of the resetting rate, for which $\langle T^{(r)} \rangle = \langle T^{(0)} \rangle$. Any value of the mean first passage time in $[T^{(r^*)}, T^{(0)}]$ is reached for two values of the resetting rate. These two values are in the intervals $[0, r^*]$ and $[r^*, r_0]$ respectively.
Figure 2. The mean-first passage time at winding number ±1, as a function of the resetting rate (log–log scale). Numerically the minimum is at \( r^* \approx 5.9956 \ldots \) (or log \( r^* \approx 1.7910 \ldots \)).

5. Distribution of winding numbers

5.1. Total winding number

The probability density of the first-passage time \( f^{(r)}(\theta) \) of the particle at \( \theta = \pm 2\pi \) (in the process with resetting rate \( r \)) is the opposite of the time derivative of the survival probability \( q^{(r)} \). Taking the Laplace transform of this relation yields

\[
\tilde{f}^{(r)}(s) = -s\tilde{q}^{(r)}(s) + 1.
\]  

(27)

The renewal equation (equation (11)) satisfied by the extinction rate of the process, which we have only used for \( s = 0 \) so far, can be rewritten, using equation (14), yielding

\[
\tilde{q}^{(r)}(s) = \tilde{q}^{(0)}(r + s) \left[ 1 + r \frac{S^{(0)}(r + s)}{1 - rS^{(0)}(r + s)} \right],
\]

\[
\tilde{f}^{(r)}(s) = 1 - \frac{s\tilde{q}^{(0)}(r + s)}{1 - rS^{(0)}(r + s)}.
\]  

(28)

Following the reasoning of [23], let us consider the event where a Brownian particle complete exactly \( n \) turns (either clockwise or counterclockwise), under resetting at rate \( r \), in the time interval [0, \( t \)]. The respective durations of the turns are denoted by \( \tau_1, \tau_2, \ldots, \tau_n \), with \( \sum_{i=1}^{n} \tau_i < t \). The probability of this event coincides with \( n \) independent first passages at the absorbing boundaries (at times \( \tau_i \), for \( i \in \{1 \ldots n\} \)), followed by a survival of duration \( t - \sum_{i=1}^{n} \tau_i \). The joint probability density of the number of turns and their durations therefore reads

\[
P(n, \{\tau_1, \ldots, \tau_n\}; t) = q^{(r)} \left( t - \sum_{i=1}^{n} \tau_i \right) \prod_{i=1}^{n} f^{(r)}(\tau_i), \quad n \in \mathbb{N},
\]

(29)

where the case \( n = 0 \) is defined by substituting 1 to the product and by setting the empty sum of the durations \( (\tau_i)_{1 \leq i \leq n} \) to zero. The probability law of the total number of turns at time \( t \) is
obtained as a convolution product by integrating over the durations

$$\mathcal{P}^{(r)}(n, t) = \left( \prod_{i=1}^{n} \int_0^{t} d\tau_i \right) q^{(r)} \left( t - \sum_{i=1}^{n} \tau_i \right) \prod_{i=1}^{n} f^{(r)}(\tau_i). \quad (30)$$

Taking the Laplace transform of this probability law maps the \((n - 1)\) convolutions to ordinary products, yielding

$$\tilde{\mathcal{P}}^{(r)}(n, s) = \tilde{q}^{(r)}(s) \left( \tilde{f}^{(r)}(s) \right)^n. \quad (31)$$

We therefore obtain the probability of the total number of turns in Laplace space in terms of the Laplace transform close to \(s = 0\):

$$\tilde{\mathcal{P}}^{(r)}(n, s) = \frac{\tilde{q}^{(0)}(r + s)}{1 - r\tilde{S}^{(0)}(r + s)} \left[ 1 - \frac{s\tilde{q}^{(0)}(r + s)}{1 - r\tilde{S}^{(0)}(r + s)} \right]^n. \quad (32)$$

The mean total number of turns follows as

$$\langle n \rangle(s) = \sum_{n \geq 0} n \tilde{\mathcal{P}}^{(r)}(n, s)$$

\[
= \tilde{q}^{(r)}(s) \left( \frac{\tilde{f}^{(r)}(s)}{1 - \tilde{f}^{(r)}(s)} \right)^2 
= \frac{\tilde{q}^{(0)}(r + s)}{1 - r\tilde{S}^{(0)}(r + s)} \left[ 1 - \frac{s\tilde{q}^{(0)}(r + s)}{1 - r\tilde{S}^{(0)}(r + s)} \right]^2 
= \frac{1}{s^2\tilde{q}^{(0)}(r + s)} \left[ 1 - r\tilde{S}^{(0)}(r + s) - s\tilde{q}^{(0)}(r + s) \right]. \quad (33)
\]

The large-time behaviour of the mean number of turns can be extracted from the behaviour of the Laplace transform close to \(s = 0\):

$$\langle n \rangle(s) \approx \frac{1}{s^2\tilde{q}^{(0)}(r + s)} \left[ 1 - r\tilde{S}^{(0)}(r + s) - s\tilde{q}^{(0)}(r + s) \right] = \frac{1}{s^2\tilde{q}^{(0)}(r)}$$

\[
\times \left[ 1 - r\tilde{S}^{(0)}(r) - s \left( \tilde{q}^{(0)}(r) + r(\tilde{S}^{(0)}(r) + \frac{(\tilde{q}^{(0)}(r)'}{\tilde{q}^{(0)}(r)} + o(s) \right) \right], \quad (34)
\]

\[
\langle n \rangle(t) = Wt + w_r + o(1) \\
W = \frac{1}{\tilde{q}^{(0)}(r)} - \frac{r\tilde{S}^{(0)}(r)}{\tilde{q}^{(0)}(r)}, \quad w_r = -1 - r\left( \frac{(\tilde{S}^{(0)}(r)'}{\tilde{q}^{(0)}(r)} - \frac{(\tilde{q}^{(0)}(r)'}{\tilde{q}^{(0)}(r)} \right)^2. \quad (35)
\]

### 5.2. Net winding number

If the walker has completed \(n_+\) counterclockwise turns and \(n_-\) clockwise turns (in any order), the total winding number is \(n := n_+ + n_-\), and the net winding number is \(k := n_+ - n_-\). The probability of having net winding number \(k\) and total winding number \(n\) at time \(t\) (denoted by \(\mathcal{P}^{(r)}(n, k, t)\)) is the sum of the probabilities of any sequence of turns of independent signs,
exactly \( n_+ = (n + k)/2 \) of which are counterclockwise. Each of the signs in the sequence has probability 1/2 to occur, hence

\[
\mathcal{P}(n, k, t) = \left( \frac{n}{n+k} \right) \left( \frac{1}{2} \right)^n \mathcal{P}(n, t), \quad (n \in \mathbb{N}, k \in [-n..n]).
\] (36)

The marginal probability \( P(k, t) \) of the net winding number \( k \) at time \( t \) is even in \( k \), hence:

\[
\mathcal{P}(k, t) = \sum_{m \geq 0} \mathcal{P}(2m + |k|, k, t) = \sum_{m \geq 0} \left( \frac{2m + |k|}{|k|} \right) \left( \frac{1}{2} \right)^{2m+|k|} \times \mathcal{P}(2m + |k|, t), \quad (k \in \mathbb{Z}).
\] (37)

In Laplace space, equation (32) yields

\[
\tilde{\mathcal{P}}(k, s) = \sum_{m \geq 0} \left( \frac{2m + |k|}{|k|} \right) \left( \frac{1}{2} \right)^{2m+|k|} \frac{\tilde{q}^{(0)}(r + s)}{1 - r \tilde{S}^{(0)}(r + s)} \times \left[ 1 - \frac{s \tilde{q}^{(0)}(r + s)}{1 - r \tilde{S}^{(0)}(r + s)} \right]^{2m+|k|} \times \sum_{m \geq 0} \left( \frac{2m + |k|}{|k|} \right) \left[ \frac{1}{4} \left( 1 - \frac{s \tilde{q}^{(0)}(r + s)}{1 - r \tilde{S}^{(0)}(r + s)} \right) \right]^{2m} \times \frac{\tilde{q}^{(0)}(r + s)}{1 - r \tilde{S}^{(0)}(r + s)} \zeta(s)^{|k|} \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^{|k|} (k \in \mathbb{Z}),
\] (38)

with the notation

\[
z(s) = \frac{1}{4} \left( 1 - \frac{s \tilde{q}^{(0)}(r + s)}{1 - r \tilde{S}^{(0)}(r + s)} \right)^2.
\] (39)

We used the identity borrowed from [23], where it was used in the same context (for \(|z| < 1/4\):

\[
\sum_{m \geq 0} \left( \frac{2m + |k|}{|k|} \right) z^m = \frac{1}{\sqrt{1-4z}} \left( \frac{1 - \sqrt{1-4z}}{2z} \right)^{|k|}.
\] (40)
By symmetry $\langle k \rangle$ is identically zero. Hence the variance of the net winding number is obtained as

$$\langle k^2 \rangle(s) = \frac{2\tilde{q}(r+s)}{1-rS(r+s)} \frac{1}{\sqrt{1-4z(s)}} \sum_{k \geq 0} k^2 \left( \frac{1 - \sqrt{1 - 4z(s)}}{2\sqrt{z(s)}} \right)^k$$

$$= \frac{2\tilde{q}(r+s)}{1-rS(r+s)} \frac{1}{\sqrt{1-4z(s)}} Z(s)(1+Z(s))^{3/2}, \quad \text{with}$$

$$Z := \frac{1 - \sqrt{1 - 4z}}{2\sqrt{z}} = \frac{1 - \sqrt{S(s)(2 - sK(s))}}{1 - sK(s)}, \quad K(s) := \frac{\tilde{q}(r+s)}{1-rS(r+s)}.$$  

Substituting and factorising the leading contribution at small $s$ yields

$$\langle k^2 \rangle(s) \sim \frac{1}{s^2K(s)} \left[ 1 - sK(s)/2 - \sqrt{\frac{K(s)}{2}} \left( \frac{1 - \sqrt{K(s)}}{2} \right) \right] \left[ 1 - \sqrt{S(s)(2 - sK(s))} \right]$$

$$\sim \frac{1}{s^2} \frac{1 - rS(0)(r)}{\tilde{q}(0)(r)}.$$

Inverting the leading term of the Laplace transform close to $s = 0$ yields the large-time equivalent

$$\langle k^2 \rangle(t) \sim W_r t.$$  

For any underlying random walk in the ordinary process (not just the diffusive random walk), the large-time rate of increase of the variance of the net winding number (or the mean total winding number), is given by the inverse of the mean first-completion time of a turn:

$$W_r = \frac{1}{\langle T^0 \rangle}.$$  

In the case of a diffusive random walk, the coefficient $W_r$ has a finite limit when the resetting rate $r$ goes to zero:

$$\lim_{r \to 0^+} W_r = \frac{1}{4\pi^2},$$

which corresponds to the large-time equivalents of $\langle n \rangle(t)$ and $\langle k^2 \rangle(t)$ reported in [23].

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6. Discussion

We have obtained a system of renewal equations for the probability density and survival rate of a random walker on a ring with absorbing boundary conditions at winding number $\pm 1$, subjected to Poisson-distributed resetting to its initial position. The process is accelerated if the walker is reset when its polar angle is within $\pi$ radians of the completion of a turn. We have expressed the mean completion time of a turn in terms of the probability density of the random walker in the ordinary process (without resetting). This relation holds for any process underlying the random walk. In the Brownian case we have obtained the mean completion time of a turn, and put in a closed form involving the digamma function. The inverse of this time is the rate of increase of the mean total winding number (and of the variance of the net winding number) at long time.

We have chosen a system of units in which both the radius $R$ of the ring and the diffusion constant $D$ are set to 1. For dimensional reasons, we can restore these parameters and find the optimal value of the resetting rate close to $5.9956R^{-2}D$. In the limit of a large radius, this optimal value of the resetting rate becomes small, because the topological benefits of acceleration are felt only when at least one half of a turn has been completed during a renewal period (which becomes less probable at fixed resetting rate if the radius becomes larger).

There are two major differences with the case of a diffusive random walker on a non-compact one-dimensional space: the mean first-passage time is finite in the ordinary case on a ring, and the system under resetting does not reach a steady state. The acceleration of the process occurs for topological reasons: in our resetting prescription, the walker takes the shortest possible path to the physical starting point of the process (whose polar coordinate can take any value among the integer multiples of $2\pi$). Possible generalisations of the present work include absorbing boundary conditions at fixed winding numbers $-w_-$ and $w_+$, with $w_\pm > 1$ (the dangerous zone still being within $\pi$ radians of each boundary). Moreover, it would be interesting to estimate the behaviour of the winding number of random walks in the plane with a singularity at the origin, subjecting to resetting the models of polymers studied in [25–28].

The resetting prescription we have defined, taking the shortest path to the resetting position, induces a correlation between runs (defining a run as a realisation of the process between resetting times). Indeed, the resetting time may or may not be a first-passage time depending on the position of the walker. The search processes considered in [29] (to provide a unified approach to stochastic processes under resetting) are renewal processes, as they assume independence between consecutive runs. They do not contain the present prescription as a particular case. It would be interesting to look for corrections that would account for the lack of renewal structure in terms of probability of presence in a dangerous zone in the ordinary process.

Developments on the experimental realisation of stochastic resetting (using optical techniques) have been reported in [30–32], with extension to two dimensions and investigation of periodic resetting protocols. An experimental realisation of our resetting prescription for a random walker on a ring could be provided by a one-armed mechanical clock, suspended vertically. The random walker would sit at the tip of the arm, the process would start in a configuration where the arm indicates 6 o’clock. The arm would turn by independent random amounts generated by the random walk and transmitted through a rod at the centre of the clock. At resetting times the connection between the arm and the rod would become loose, and the
arm would go back to its initial position under the influence of gravity (this move of the arm is considered to be instantaneous in our model). It would be natural to compare this mechanism to the realisation of resetting by confining potentials, as introduced in [33]. It would be interesting to generalise the model to random walks on higher-dimensional non-simply-connected spaces.

Data availability statement

No new data were created or analysed in this study.

Appendix

Let us work out the expression of the probability density of the position of the diffusive random walker without resetting, starting from the Laplace transform of the equation (22) and the definitions of equation (20):

\[
\hat{Q}^{(0)}(\theta, s) = \sum_{n>1} \frac{1}{s + E_n} \psi_n(\theta) \psi_n(0)
\]

\[
= \frac{1}{2\pi} \sum_{n \neq 1} \frac{1}{s + \frac{n\pi}{32}} \sin \left( \frac{n\theta}{4} + \frac{n\pi}{2} \right) \sin \left( \frac{n\pi}{2} \right)
\]

\[
= \frac{1}{4\pi} \sum_{n \neq 1} \frac{1}{s + \frac{n\pi}{32}} \left[ \cos \left( \frac{n\theta}{4} \right) - \cos \left( \frac{n\theta}{4} + \frac{n\pi}{2} \right) \right]
\]

\[
= \frac{16}{\pi} \sum_{m \geq 0} \frac{1}{32s + (2m + 1)^2} \cos \left( \frac{(2m + 1)\theta}{4} \right).
\]

We find the Laplace transform of the survival probability as

\[
\hat{q}^{(0)}(s) = \int_{-2\pi}^{2\pi} \hat{Q}^{(0)}(\theta, s) d\theta
\]

\[
= \frac{1}{2\pi} \sum_{m \geq 0} \frac{1}{s + \frac{12m+11}{4}} \frac{8}{2m+1} \sin \left( \frac{(2m + 1)\pi}{2} \right)
\]

\[
= \frac{4}{\pi} \sum_{m \geq 0} \frac{32}{32s + (2m + 1)^2} \frac{(-1)^m}{2m+1},
\]

which coincides with the value \( \frac{1}{s} \left[ 1 - \frac{1}{\cosh(2\pi \sqrt{s})} \right] \) obtained in [23].
The Laplace transform of the probability of presence in the safe zone reads

\[
\tilde{S}^{(0)}(s) = \int_{-\pi}^{\pi} \tilde{Q}^{(0)}(\theta, s) d\theta
\]

\[
= \frac{1}{2\pi} \sum_{m \neq 0} \frac{1}{s + \frac{2m + 1}{2m + 1}} \cdot \frac{8}{2m + 1} \sin \left( \frac{(2m + 1)\pi}{4} \right)
\]

\[
= \frac{4}{\pi \sqrt{2}} \sum_{m \in \{4k, 4k+1\}, k \geq 0} \frac{32}{32s + (2m + 1)^2} \cdot \frac{1}{2m + 1}
\]

\[
- \frac{4}{\pi \sqrt{2}} \sum_{m \in \{4k+2, 4k+3\}, k \geq 0} \frac{32}{32s + (2m + 1)^2} \cdot \frac{1}{2m + 1}
\]

\[
= \frac{64\sqrt{2}}{\pi} \sum_{k \geq 0} \left[ \frac{1}{32s + (8k + 1)^2} \frac{1}{(8k + 1)} + \frac{1}{32s + (8k + 3)^2} \frac{1}{(8k + 3)} - \frac{1}{32s + (8k + 5)^2} \frac{1}{(8k + 5)} \frac{1}{32s + (8k + 7)^2} \frac{1}{(8k + 7)} \right]
\]

\[(48)\]

To work out the sums involved in equation (46), let us start from the identity (labelled 6.3.16 in [34])

\[
\psi(z) = -\gamma + \sum_{n \geq 0} \left( \frac{1}{n + 1} - \frac{1}{n + z} \right),
\]

\[(49)\]

where \(\psi = \Gamma'/\Gamma\) denotes the Euler digamma function. Consider the following combination:

\[
C_k(u):= \psi \left( \frac{k + i\sqrt{u}}{8} \right) + \psi \left( \frac{k - i\sqrt{u}}{8} \right) - 2\psi \left( \frac{k}{8} \right),
\]

\[(50)\]

where for our purposes \(u\) will be a positive number and \(k\) will be an integer.

\[
C_k(u) = \sum_{n \geq 0} \left[ \frac{8}{8n + k + i\sqrt{u}} + \frac{8}{8n + k - i\sqrt{u}} - \frac{16}{8n + k} \right]
\]

\[
= \sum_{n \geq 0} \left( \frac{16u}{u + (8n + k)^2} \right) \frac{1}{8n + k}
\]

\[(51)\]
Equation (48) therefore reads

\[
\tilde{S}(0)(s) = \frac{\sqrt{2}}{8\pi s} [C_1(32s) + C_3(32s) - C_5(32s) - C_7(32s)] \\
= \frac{\sqrt{2}}{8\pi s} \left[ \psi \left( \frac{1 + i\sqrt{u}}{8} \right) + \psi \left( \frac{1 - i\sqrt{u}}{8} \right) - 2\psi \left( \frac{1}{8} \right) \\
+ \psi \left( \frac{3 + i\sqrt{u}}{8} \right) + \psi \left( \frac{3 - i\sqrt{u}}{8} \right) - 2\psi \left( \frac{3}{8} \right) \\
- \psi \left( \frac{5 + i\sqrt{u}}{8} \right) - \psi \left( \frac{5 - i\sqrt{u}}{8} \right) + 2\psi \left( \frac{5}{8} \right) \\
- \psi \left( \frac{7 + i\sqrt{u}}{8} \right) - \psi \left( \frac{7 - i\sqrt{u}}{8} \right) + 2\psi \left( \frac{7}{8} \right) \right].
\]

Using the identity \(\psi(z) = \overline{\psi(-\bar{z})}\) (which is manifest from equation (49)) yields for the expression reported in equation (25).

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