WEIGHTED ALEKSANDROV ESTIMATES: PDE AND STOCHASTIC VERSIONS

N.V. KRYLOV

Abstract. We prove several pointwise estimates for solutions of linear elliptic (parabolic) equations with measurable coefficients in smooth domains (cylinders) through the weighted $L_d (L_{d+1})$-norm of the free term. The weights allow the free term to blow up near the (lateral) boundary. We also present weighted estimates for occupation times of diffusion processes.

In the recent paper [4] the authors prove weighted and mixed-norm $L_p$ estimates for fully nonlinear elliptic and parabolic equations with relaxed convexity assumption and almost VMO assumption on the dependence on the space-time variables of the functions defining the equations. The full norm including the second order spatial derivatives is estimated, however, sometimes they are estimated through the weighted norm of the free term plus the weighted norm of the unknown function itself. Here we show how the unknown function can be estimated through the weighted norm of the free term by considering linear elliptic and parabolic equations with measurable coefficients. One knows that zeroth-order estimates for fully nonlinear elliptic and parabolic equations even not explicitly involving $x$ reduce to the estimates for linear elliptic and parabolic equations with measurable coefficients. Our estimates are given for $C^{1,1}$ domains and cylinders with weights that are powers of the distance to the boundary of the domain or to the lateral boundary of the cylinder. In [12] and the references therein one can find similar estimates in case the boundaries of domains have wedges with weights related to the wedges.

It is worth noting that for the case of linear and quasilinear elliptic equations with regular coefficients a rather detailed information about weighted estimates can be found in [3] and references therein.

It is also worth noting that for the case of linear parabolic equations with coefficients independent of $x$ a rather detailed information about weighted estimates can be found in [6] and the references therein. It is also worth noting [2], where an abstract treatment of weighted estimates is presented from the point of view of semigroups and special Riemannian manifolds. Then the coefficient of operators are necessarily smooth apart from some special singularities.

1991 Mathematics Subject Classification. 35J15, 35J60, 35K10, 35K55, 35K96, 60H05.
Our method in the elliptic case is an extension of the original Aleksandrov methods based on Monge-Ampère equations and is presented in Section 1. In Section 2 we apply the results of Section 1 to derive estimates for equations of main type in the unit ball. Section 3 contains our main analytic result about estimates of solutions of elliptic equations. In Section 4 we derive stochastic Aleksandrov estimates for functions which can blow up near the boundary. These provide better estimates than known before for the time spent by diffusion processes near the boundary of a domain before reaching it.

Our method in the parabolic case is an extension of the one introduced in [7], is based on considering the parabolic Monge-Ampère equations introduced in [7], and is presented in Section 5 where we also derive estimates for equations of main type in round cylinders. In Section 6 we apply the results of Section 5 to derive estimates for general parabolic equations in round cylinders. Section 7 contains our main analytic result about estimates of solutions of parabolic equations. Finally, in Section 8 we derive stochastic weighted Aleksandrov estimates for functions which can blow up near the boundary. These provide better estimates than known before for the time spent by diffusion processes near the lateral boundary of a cylinder before reaching its boundary.

In the elliptic part of the article we work in a $d$-dimensional Euclidean space $\mathbb{R}^d$ of points $x = (x^1, \ldots, x^d)$, $d \geq 2$. We use the notation

$$D_i = \frac{\partial}{\partial x^i}, \quad Du = (D_1 u, \ldots, D_d u), \quad D_{ij} = D_i D_j, \quad D^2 u = (D_{ij} u)^{d}_{i,j=1},$$

$$a_\pm = (1/2)(|a| \pm a), \quad a_\pm^p = (a_\pm)^p, \quad B_r(x) = \{ y \in \mathbb{R}^d : |y - x| < r \},$$

$$B_r = B_r(0), \quad B = B_1.$$

By $|\Gamma|$ we denote the volume of $\Gamma \subset \mathbb{R}^d$. If $\Omega$ is a domain in $\mathbb{R}^d$ with regular boundary by $W^2_d(\Omega)$ we mean the subset of $W^2_d(\Omega)$ consisting of continuous functions in $\Omega$ vanishing on $\partial \Omega$.

In the parabolic part of the article we fix $T \in (0, \infty)$ and use the notation

$$C = [0, T) \times B, \quad \partial' C = \partial C \setminus (\{0\} \times \bar{B}).$$

We call $\partial' C$ the parabolic boundary of $C$.

Everywhere below $\psi(x) = 1 - |x|^2$.

1. Elliptic equations of the main type in a ball

**Theorem 1.1.** Let $u \in W^2_{d, \text{loc}}(B) \cap C(\bar{B})$ be a convex function in $B$, and let $\alpha \in [0, (d + 1)/2)$. Then, for any $x_0 \in B$,

$$u(x_0) \geq \inf_{\partial B} u - N(d, \alpha) \psi^\beta(x_0) \left( \int_B \psi^\alpha \det D^2 u \, dx \right)^{1/d},$$

where

$$\beta = (d + 1 - 2\alpha)/(2d).$$
Corollary 1.2. Under the conditions of Theorem 1.1, if \( u = 0 \) on \( \partial B \), we have

\[
\sup_B |u| \leq N(\alpha, d) \left( \int_B \psi^\alpha \det D^2 u \, dx \right)^{1/d}.
\]

(1.2)

Remark 1.3. It might be that (1.1) also holds if \( \alpha = (d+1)/2 \). At least this is true indeed if \( d = 1 \). Generally, estimate (1.2) is close to be optimal in the following sense. Take \( \alpha > (d+1)/2 \) and a sequence \( x_n \in B \) such that \(|x_n| \to 1\) as \( n \to \infty \). Then one can construct a sequence of smooth in \( B \), convex functions \( u^n \), vanishing on the boundary, such that \( u^n(x_n) \to -\infty \) and the integral in the right-hand side of (1.2) stays bounded.

Our argument showing this is rather descriptive dropping some rigorous justifications. But the author is sure that the reader will be able to make it absolutely rigorous. To construct such a sequence of \( u^n \), define negative \( v^n(x_n) \) so that

\[
|v^n(x_n)|^d = \psi^{(d+1)/2 - \alpha}(x_n)
\]

and introduce a cone with vertex at \((x_n, v^n(x_n))\) and base \( \partial B \). Let this cone be the graph of a function which we call \( v^n(x) \). Then mollify \( v^n \) near \( x_n \), without changing it for \( x \) not close to \( x_n \) so that the new function, \( u^n \), will be smooth, convex, and close to \( v^n \), so that \( u^n(x_n) \to -\infty \) (observe that \( v^n(x_n) \to -\infty \)).

Note that, since \( u^n \) is smooth, by change of variables formula,

\[
\int_B \det D^2 u^n \, dx = |Du^n(B)|,
\]

where \( Du^n(B) = \{ Du^n(x) : x \in B \} \). It turns out that \( Du^n(B) \) and \(|Du^n(B)|\) are independent of what we did with \( v^n \) in the small neighborhood of \( x_n \). Indeed, if \( x \in B \) and \( p = Du^n(x) \), then the hyper-plane \( y = u^n(x) + p \cdot (z - x), z \in \mathbb{R}^d \), can be shifted down, if necessary, so that it will become a supporting plane for the graph of \( v^n \) at \((x_n, v^n(x_n))\). Obviously, all supporting planes for the graph of \( v^n \) at \((x_n, v^n(x_n))\) can be obtained in this way, so that \( Du^n(B) \) is just the collection of \( p \in \mathbb{R}^d \) such that \( b + p \cdot (z - x_n) \) is a supporting plane for the graph of \( v^n \) at \((x_n, v^n(x_n))\) for a \( b \in \mathbb{R} \).

In analytic terms this means that

\[ p \in Du^n(B) \iff (p, x_n) + b = v^n(x_n), \quad (p, x) + b \leq 0 \quad \forall x \in \partial B. \]

We can rewrite the latter conditions as

\[
\max_{|z| = 1} (x - x_n, p) \leq -v^n(x_n), \quad |p| - (x_n, p) \leq -v^n(x_n),
\]

\[
|p|^2 - (x_n, p)^2 + 2v^n(x_n)(x_n, p) \leq |v^n(x_n)|^2,
\]

and \( b = v^n(x_n) - (p, x_n) \). For \( x_n = |x_n| e_1 \) we have

\[
(1 - |x_n|^2) \left( p^1 + \frac{v^n(x_n)|x_n|}{1 - |x_n|^2} \right)^2 + \sum_{i \geq 2} |p_i|^2 \leq \frac{|v^n(x_n)|^2}{1 - |x_n|^2}.
\]
It follows that $Du^n(B)$ is an ellipsoid whose $d - 1$ principal semi-axes have length
\[ |v^n(x_n)|/\sqrt{1 - |x_n|^2} \]
and the remaining one is of length
\[ |v^n(x_n)|/(1 - |x_n|^2). \]
Hence,
\[ |Du^n(B)| = \omega_d |v^n(x_n)|^{d\psi-(d+1)/2}(x_n). \]
After that observe that $\det D^2u^n \neq 0$ only in the neighborhood of $x_n,$ where we changed $v^n.$ Then
\[ \int_B \psi^{n} \det D^2u^n \, dx \]
is close to
\[ J := \psi^n(x_n) \int_B \det D^2u^n \, dx = \psi^n(x_n)|Du^n(B)| = \omega_d \psi^{n-(d+1)/2}(x_n)|v^n(x_n)|^d = \omega_d, \]
and becomes as close to $J$ as we wish as we shrink the neighborhoods of $x_n,$ where we changed $v^n.$ This finishes the proof of the claim made at the beginning of this remark.

**Remark 1.4.** It suffices to prove (1.1) with $B_r$ in place of $B$ and $r^2 - |x|^2$ in place of $\psi$ for $r < 1$ close to 1. In that case we have $u \in W^2_d(B_r).$ This shows that in the proof of Theorem 1.1 without losing generality we may assume that $u \in W^2_d(B).

**Lemma 1.5.** Let $u \in W^2_d(B)$ be convex in $B,$ $0 \leq s < r \leq 1,$ and suppose that $\det D^2u \neq 0$ only on $B_r \setminus B_s$ and $u = 0$ on $\partial B.$ Then
(a) for $|x| \leq s$ we have
\[ |u(x)| \leq M \left( \frac{1 - s^2}{1 - r^2} \right)^{(d+1)/(2d)}, \tag{1.3} \]
where
\[ M = \omega^{-1/d}_d (1 - r^2)^{(d+1)/(2d)} \left( \int_{B_r \setminus B_s} \det D^2u \, dx \right)^{1/d}, \]
(b) for $|x| \geq r$ we have
\[ |u(x)| \leq M (1 - |x|)/(1 - r). \tag{1.4} \]

**Proof.** (a) Since in $B_s$ we have
\[ \inf_{a \in A} a^{ij} D_{ij} u = 0, \tag{1.5} \]
where $A$ is the set of $d \times d$ symmetric nonnegative matrices with unit trace, by the maximum principle $u$ in $B_s$ attains its minimum on $\partial B_s.$ Furthermore,
the classical Aleksandrov estimate (the derivation of which is just part of the arguments in Remark 1.3) says that, for any $x \in B$,

$$|u(x)| \leq \omega_d^{-1/d} (1 - |x|^2)^{(d+1)/(2d)} \left( \int_B \det D^2 u \, dx \right)^{1/d},$$  \hspace{1cm} (1.6)

where the integral can be restricted to $B_r \setminus B_s$. It follows that $|u(x)|$ on $\partial B_s$ is dominated by the right-hand side of (1.3) and proves (a).

(b) Observe that in $C_r := B \setminus \bar{B}$ the function $u$ satisfies (1.5). The function $v(x) := M(|x| - 1)/(1 - r)$ also satisfies this equation in $C_r$ and is less than $u$ on $\partial C_r$. By Theorem 4.1.18 of [11] we have $u \geq v$ in $C_r$ and this is (1.4). The lemma is proved.

Next, we need a special partition of unity in $B$. For $n = 0, 1, 2, \ldots$ introduce $r_n = 1 - e^{-n}$, $r_{-1} = 0$, and find nonnegative $C^\infty$-functions $\zeta_n$ on $[0, 1]$ such that $\zeta_n \leq 1$,

$$\zeta_n = 1 \text{ on } [r_n, r_{n+1}], \quad \zeta_n = 0 \text{ for } t < (r_{n-1} + r_n)/2 =: s_{n-1},$$

$$\zeta_n = 0 \text{ for } t \geq (r_{n+1} + r_{n+2})/2 =: s_{n+1}.$$

After that set

$$\eta = \sum_{n=0}^\infty \zeta_n, \quad \eta_n = \zeta_n/\eta.$$

Clearly, $\eta$ is infinitely differentiable on $[0, 1)$ and $0 \leq \eta \leq 3$.

**Lemma 1.6.** Let $u$ be four times continuously differentiable in $\bar{B}$ function, which is convex in $B$, is nonnegative on $\partial B$ and is such that $f := \det D^2 u > 0$ in $\bar{B}$. Introduce $u_n$ as $C^{1,1}(\bar{B})$-functions which are convex in $B$, vanish on $\partial B$ and satisfy

$$\det D^2 u_n = f \eta_n^d$$

in $B$ (a.e.). Then

$$u \geq \sum_{n=1}^\infty u_n.$$

Proof. First of all note that, since $g := f^{1/d}$ is at least in $C^{1,1}(\bar{D})$, the functions $u_n$ with the described properties exist and are unique according to §4 in [9].

Recall that $u$ is a smooth solution of

$$\inf_{a \in A} \left[ a^{ij} D_{ij} u - gd \sqrt[1/d]{\det a} \right] = 0$$

nonnegative on $\partial B$, and $u_n$ are unique $C^{1,1}$-solutions of

$$\inf_{a \in A} \left[ a^{ij} D_{ij} u_n - g_n d \sqrt[1/d]{\det a} \right] = 0$$

in $B$ (a.e.) vanishing on $\partial B$. Set $v_n = u_0 + \ldots + u_n$, $g_n = g \eta_0 + \ldots + g \eta_n$ and note that

$$\inf_{a \in A} \left[ a^{ij} D_{ij} u v_n - g_n d \sqrt[1/d]{\det a} \right] \geq 0$$
in $B$ (a.e.). It follows that
\[
\sup_{a \in A} \{a^{ij} D_{ij}(v_n - u) - (g_n - g)d^d \sqrt{\det a}\} \geq 0
\]
and there is an $A$-valued function $a = a(x)$ such that in $B$ (a.e.)
\[
a^{ij} D_{ij}(v_n - u) - (g_n - g)d^d \sqrt{\det a} \geq 0.
\]
By Theorem 3.3.4 of [11] (see also the end of the introduction into Chapter 3 in [11])
\[
v_n - u \leq N(d)\|g_n - g\|_{L_d(B)}.
\]
This proves the lemma since the norm on the right tends to zero as $n \to \infty$.

**Proof of Theorem 1.1.** According to Remark 1.4 we may assume that $u \in W^2_d(B)$. By having in mind approximations and adding to $u$ the function $-\varepsilon \psi$ and then letting $\varepsilon \downarrow 0$, we convince ourselves that we may also assume that $u$ is a strictly convex $C^4$-function. By replacing $u$ with $u - \inf_{\partial B} u$ we see that we may assume that $u \geq 0$ on $\partial B$ as well. Under these additional assumptions fix $x_0 \in B$ and define $n_0$ as the smallest $n \geq 1$ such that $|x_0| \leq s_n$, that is
\[
|x_0| \leq s_{n_0}, \quad |x_0| \geq s_{n_0-1}.
\]
Below we are going to use a few times that the ratio $(1 - s_{n+1})/(1 - s_{n+1})$ is bounded from above and away from zero by absolute constants independent of $n$ and that, for $|x| \in [s_{n-1}, s_{n+2}]$, the ratio $\psi^\alpha(x)/(1 - s_n)^\alpha$ is bounded from above and away from zero by constants independent of $n$ and depending only on $\alpha$.

Take $u_n$ from Lemma 1.6 and observe that, by Lemma 1.5 (and (1.6)), if $n \leq n_0$, then
\[
|u_n(x_0)| \leq N(d)(1 - |x_0|) (1 - s_n)^{-1} (1 - s_n^2)^{(d+1)/(2d)} \left( \int_B \eta_n^d \det D^2 u \, dx \right)^{1/d}
\]
\[
\leq N(d)(1 - |x_0|) (1 - s_n)^{(d+1)/(2d) - 1 - \alpha/(d-1)} \left( \int_B \eta_n^d \psi^\alpha \det D^2 u \, dx \right)^{1/d}
\]
\[
\leq N(d)(1 - |x_0|) e^{n(1-\beta)} \left( \int_B \eta_n^d \psi^\alpha \det D^2 u \, dx \right)^{1/d}.
\]
By Hölder’s inequality and in light of the fact that $1 - \beta > 0$ ($d \geq 2$),
\[
\sum_{n=0}^{n_0} |u_n(x_0)| \leq N(d)(1 - |x_0|) \left( \sum_{n=0}^{n_0} e^{n(1-\beta)d/(d-1)} \right)^{(d-1)/d}
\]
\[
\times \left( \int_B \psi^{\alpha} \det D^2 u \sum_{n=0}^{n_0} \eta_n^d \, dx \right)^{1/d}
\]
\[
\leq N(d, \alpha)(1 - |x_0|) e^{n_0(1-\beta)} \left( \int_B \psi^\alpha \det D^2 u \, dx \right)^{1/d}
\]
\[
\leq N(d, \alpha) \psi^\beta(x_0) \left( \int_B \psi^\alpha \det D^2 u \, dx \right)^{1/d}. \tag{1.7}
\]
If $n > n_0$, then by Lemma 1.5
\[
|u_n(x_0)| \leq N(d)(1 - s_n^2)^{(d+1)/(2d)} \left( \int_B \eta_n^d \det D^2u \, dx \right)^{1/d} \\
\leq N(d)e^{-n\beta} \left( \int_B \psi^\alpha \det D^2u \, dx \right)^{1/d}.
\]
Since $\beta > 0$,
\[
\sum_{n>n_0} |u_n(x_0)| \leq N(d) \left( \sum_{n>n_0} e^{-n\beta d/(d-1)} \right)^{(d-1)/d} \left( \int_B \psi^\alpha \det D^2u \, dx \right)^{1/d} \\
\leq N(d,\alpha)e^{-n_0\beta} \left( \int_B \psi^\alpha \det D^2u \, dx \right)^{1/d} \\
\leq N(d,\alpha)\psi^\beta(x_0) \left( \int_B \psi^\alpha \det D^2u \, dx \right)^{1/d}.
\]
Upon combining this with (1.7) and Lemma 1.6 we get (1.1). The theorem is proved.

**Theorem 1.7.** Let $a(x) = (a^i_j(x))$ be a $d \times d$-symmetric nonnegative definite matrix-valued measurable function on $B$ such that $\text{tr} \, a > 0$ in $B$. Let $\alpha \in [0, (d+1)/2)$ and $u \in W^{2,d}_{d,\text{loc}}(B) \cap C(\bar{B})$. Introduce
\[
L_0 u = a^{ij} D_{ij} u.
\]
Then, for any $x_0 \in B$, $(0/0 := 0)$
\[
u(x_0) \leq \sup_{\partial B} u + N(d,\alpha)\psi^\beta(x_0) \left( \int_B \psi^\alpha I_{L_0u<0}(\det a)^{-1/\alpha} |L_0u|^{d} \, dx \right)^{1/d}. \tag{1.8}
\]

Proof. As in Remark 1.4 we may assume that $u \in W^2_d(B)$. Since $\text{tr} \, a > 0$, by the homogeneity of estimate (1.8) we may assume that
\[
\text{tr} \, a \equiv 1. \tag{1.9}
\]

A particular case. Suppose that $\det a \geq \varepsilon$, where $\varepsilon > 0$ is a constant. In that case it is easy to pass to the limit in (1.8) from smooth $u$ to the ones in $W^2_d(B)$. Therefore, we assume that $u \in C^4(B)$. We may also assume that $a$ is infinitely differentiable so that
\[
f := -\frac{1}{d} (\det a)^{-1/\alpha} L_0 u
\]
is twice continuously differentiable in $\bar{B}$. In that case, as we know, there exists a unique $C^{1,1}(\bar{B})$-function $v$ which is concave in $B$, vanishes on $\partial B$, and satisfies
\[
\text{det}(-D^2v) = f^d_+ \tag{1.10}
\]
in $B$ (a.e.). By the way, it is proved in [5] that the same result holds if we replace $f^d_+$ with $g^{d-1}$, provided that $g \in C^{1,1}(B), g \geq 0$. Interestingly enough, since $f^d_+/(d-1)$ is not necessarily in $C^{1,1}$, the result in [5] is not applicable here.
Then we know that $v$ also satisfies
\[ \sup_{a \in A} [a^{ij} D_{ij} v + f d^{j} \sqrt{\det a}] = 0 \tag{1.11} \]
and for $\bar{u} = u - \sup_{\partial B} u$ we obviously have
\[ \sup_{a \in A} [a^{ij} D_{ij} \bar{u} + f d^{j} \sqrt{\det a}] \geq 0. \]
By Theorem 4.1.18 of [11] we have $\bar{u} \leq v$ and a reference to Theorem 1.1 completes considering this case.

**General case.** In order to drop the additional assumptions we take $\gamma, \delta > 0$, take $\psi$ from Lemma 3.1.8 of [11] introduce $L^\delta = L + \delta \Delta$, and apply the above result to $u^\gamma = u - \gamma (\psi + 1)$ and $L^\delta$. Then following almost word for word the appropriate parts of the proof of Lemma 3.2.4 of [11], we arrive at (1.8). The theorem is proved.

### 2. General elliptic equations in $B$

Here we generalize Theorem 1.7 for equation with lower order terms.

**Theorem 2.1.** Under the assumptions of Theorem 1.7 let $b = (b^i(x))$ be $\mathbb{R}^d$-valued measurable function on $B$ and $c = c(x)$ be a nonnegative bounded measurable function on $B$ and let $|b| \leq K \mu$ in $B$, where $K$ is a fixed constant and $\mu = \mu(x)$ is the smallest eigenvalue of $a = a(x)$. Then estimate (1.8) holds again with $N = N(d, \alpha, K)$ if we replace $L_0 u$ in (1.8) with
\[ Lu = a^{ij} D_{ij} u + b^i D_i u - cu \tag{2.1} \]
and $\sup_{\partial B} u$ with $\sup_{\partial B} u^+$. To prove the theorem we need a lemma.

**Lemma 2.2.** Take $\gamma \in [0, 1]$ and introduce
\[ w(x) = \int_{1}^{1/|x|} e^{Ks} \int_{0}^{s} \frac{e^{-Kt}}{(1-t^2)^{\gamma}} dt ds. \]
Then $w(x)$ has bounded first and second order derivatives in $B_r$ for any $r \in (0, 1)$, and satisfies there
\[ Lw + \frac{\mu}{(1 - |x|^2)^{\gamma}} \leq 0. \tag{2.2} \]
Proof. We have
\[
D_i w(x) = -\frac{x^i}{|x|} e^{K|x|} \int_{0}^{1} \frac{e^{-Kt}}{(1-t^2)^{\gamma}} dt,
\]
\[
D_{ij} w(x) = -\frac{x^i x^j}{|x|^2} \frac{1}{(1 - |x|^2)^{\gamma}} - \left( \delta^{ij} - \frac{x^i x^j}{|x|^2} + \frac{x^i x^j}{|x|^2} \right) e^{K|x|} \int_{0}^{1} \frac{e^{-Kt}}{(1-t^2)^{\gamma}} dt.
\]
Since $a^{ij}x^ix^j \geq \mu(x)|x|^2$ and $a^{ij}x^i \leq |x|^2a^{ij}\delta^{ij}$, it follows that
\[
Lw \leq -\frac{\mu}{(1 - |x|^2)^\gamma} - cw
\]
\[
-K_\mu e^{K|x|} \int_0^{|x|} \frac{e^{-Kt}}{(1 - t^2)^\gamma} dt - \frac{b^i x^i}{x^j} e^{K|x|} \int_0^{|x|} \frac{e^{-Kt}}{(1 - t^2)^\gamma} dt.
\]
This easily implies (2.2). The lemma is proved.

**Proof of Theorem 2.1.** As in the proof of Theorem 1.7 we assume (1.9), $u \in W^2_d(B)$, and first also assume that $\det a \geq \varepsilon$, where $\varepsilon > 0$ is a constant. This allows us to also assume that $u$, $a$, $b$, and $c$ are infinitely differentiable so that
\[
f := -\frac{1}{d}(\det a)^{-1/d}Lu
\]
is twice continuously differentiable in $\bar{B}$.

After that we define a function $v$ having the same properties as in the proof of Theorem 1.7 by solving (1.10). We also fix $x_0 \in B$, take the function $w$ from Lemma 2.2 with $\gamma = 1 - \beta$, take $N(d, \alpha)$ from (1.1) and set
\[
N(d, \alpha)\psi^\beta(x_0) \left(\int_B \psi^\alpha \det D^2 v \, dx\right)^{1/d} =: N_1\psi^\beta(x_0).
\]
Observe that, as for any concave function vanishing on $\partial B$, we have $|Dv| \leq v/(1 - |x|) \leq 2v/\psi$. Also note that for $\bar{u} = u - \sup_{\partial B} u_+$ we have $L\bar{u} \geq Lu$. Then for the function
\[
\phi = v + 2KN_1w
\]
owing to (1.1), (2.2), and (1.11), (also recall the definition of $f$) we get
\[
L\bar{u} \geq Lv - b^i D_i v + cv \geq Lv - 2|b|v/\psi \geq Lv - 2N_1|b|\psi^{-(1 - \beta)} \geq L\phi.
\]
By the maximum principle $\bar{u} \leq \phi$ and, to finish the proof in out particular case, it only remains to observe that by using l'Hospital’s rule it is easy to check that
\[
\lim_{|x| \uparrow 1} \frac{w(x)}{(1 - |x|^2)^\delta} = 2^{-\beta}e^{-K} \lim_{s \uparrow 1} (1 - s)^{1 - \beta} \int_0^s \frac{e^{-Kt}}{(1 - t^2)^{1 - \beta}} dt = 0. \tag{2.3}
\]
The general case is dealt with as at the end of the proof of Theorem 1.7. The theorem is proved.

### 3. The Case of General Smooth Domains

Let $\Omega$ be a $C^{1,1}$ bounded domain in $\mathbb{R}^d$, $a = (a^{ij}(x))$ be $d \times d$ symmetric matrix-valued function on $\mathbb{R}^d$, $b = (b^i(x))$ be $\mathbb{R}^d$-valued function on $\mathbb{R}^d$, and $c = c(x)$ be a nonnegative bounded function on $\mathbb{R}^d$. We assume that these functions are measurable, fix three numbers $\delta, K > 0$, and assume that for all values of arguments and $\lambda \in \mathbb{R}^d$
\[
|\lambda|^2 \geq a^{ij}\lambda^i \lambda^j \geq \delta|\lambda|^2, \quad |b| \leq K, \quad c \geq 0. \tag{3.1}
\]
Finally, take $\alpha \in [0, (d + 1)/2)$ and define $\rho_\Omega(x) = \text{dist}(x, \Omega^c)$. 

Theorem 3.1. Let \( u \in W^{2, \text{loc}}_{d}(\Omega) \cap C(\overline{\Omega}) \). Then, for any \( x \in \Omega \),
\[
u(x) \leq \sup_{\partial \Omega} u_{+} + N \rho_{\Omega}^{\theta}(x) \| \rho_{\Omega}^{\theta}(Lu) - \|_{L_{d}(\Omega)},
\] (3.2)
where \( L \) is taken from (2.1) and the constant \( N \) depends only on \( \Omega, \delta, \alpha, \) and \( K \).

To prove this theorem we need three lemmas.

Lemma 3.2. Set \( B_{r}^{+} = B_{r} \cap \{ x \in \mathbb{R}^{d} : x^{1} > 0 \} \) and suppose that \( u \in W^{2, \text{loc}}_{d}(B_{r}^{+}) \cap C(\overline{B}_{r}^{+}) \), \( u \leq 0 \) on \( \partial B_{r}^{+} \). Also assume that the coefficients of \( L \) are infinitely differentiable and \( f := Lu \in L_{d}(B_{r}^{+}) \). Then in \( B_{r}^{+} \) we have
\[
u(x) \leq N(x^{1})^{\beta} \| M^{\alpha} f - \|_{L_{d}(B_{r}^{+})},
\] (3.3)
where \( M^{\alpha}(x) = (x^{1})^{\alpha} \) and the constant \( N \) depends only on \( d, \delta, \alpha, \) and \( K \).

Proof. Find a \( C^{1,1} \) domain \( B' \) which contains \( B_{r}^{+} \), has \( B_{2} \cap \{ x \in \mathbb{R}^{d} : x^{1} = 0 \} \) as part of its boundary, and is \( C^{1,1} \)-diffeomorphic to \( B \).

Then, extend \( f \) as zero outside \( B_{r}^{+} \) and denote by \( v \) the function of class \( W^{2, \text{loc}}_{d}(B') \) satisfying
\[
u(x) = -(x^{1})\beta \| M^{\alpha} f - \|_{L_{d}(B_{r}^{+})},
\] (3.4)
in \( B' \) (a.e.). It is a classical fact that such a function exists and is unique. By the maximum principle \( v \geq 0 \), \( v \geq u \) in \( B_{r}^{+} \). Furthermore, is we apply the diffeomorphism mentioned above to (3.4) then we will see that the image \( v' \) of \( v \) will satisfy the equation
\[
u'(x') = -(x^{1})\beta \| M^{\alpha} f - \|_{L_{d}(B_{r}^{+})},
\] (3.5)
in \( B \) (a.e.), where \( f' \) is the image of \( f \) and \( L' \) is the image of \( L \).

By Theorem 2.1
\[
u'(x') \leq N(\psi(x'))^{\beta} \| \psi^{\alpha/d} f' - \|_{L_{d}(B)},
\] (3.6)
where \( \psi(x') \) is the distance from \( x' \) to the boundary of \( B \). If \( x' \) is the image of \( x \), \( \psi(x') \) is comparable to the distance of \( x \) to the boundary of \( B' \), since the diffeomorphism and its inverse are Lipschitz continuous.

It only remains to write down (3.5) in the original coordinates and use the fact that \( v \geq u \) and that for \( x \in B_{r}^{+} \) it distance to the boundary of \( B' \) equals \( x^{1} \). The lemma is proved.

Next we use a well-known fact (see, for instance Lemma 8.8 in [10]) that there exists a function \( \Psi \in C^{1,1}(\overline{\Omega}) \) such that, for a constant \( N \) depending only on \( \Omega, \delta, \) and \( K \), we have on \( \Omega \)
\[
N \rho_{\Omega} \geq \Psi, \quad \Psi \geq \rho_{\Omega},
\] (3.6)
\[
L\Psi + c\Psi \leq -1.
\] (3.7)
**Lemma 3.3.** Let the coefficients of $L$ be infinitely differentiable. Take $\rho_0 \in (0,1)$, $x_0 \in \Omega$ and suppose that $2\rho := \rho_\Omega(x_0) \geq \rho_0$. Let $\gamma \in (0,1]$ and let $\Phi$ be the classical solution of $L\Phi = 0$ in $B_\rho(x_0)$ with boundary condition $\Phi = \Psi^\gamma$ on $\partial B_\rho(x_0)$. Then

$$\Phi(x_0) \leq [1 - \varepsilon \rho_0]^{\gamma} \Psi^\gamma(x_0),$$

where $\varepsilon > 0$ depends only on $K$, $\delta$, and $\Omega$.

Proof. A simple argument based on the maximum principle shows that it suffices to concentrate on $\gamma = 1$. In that case first we note that by the maximum principle $\Phi \geq 0$. Therefore, $(L + c)\Phi \geq 0$ and $v := \Psi - \Phi$ satisfies $(L + c)v \leq -1$ and $v = 0$ on $\partial B_\rho(x_0)$. Then elementary barriers show that $v(x_0) \geq \varepsilon \rho^2$, where $\varepsilon > 0$ depends only on $K$, $\delta$, and the diameter of $\Omega$. Since $\rho_\Omega(x_0)$ and $\Psi(x_0)$ are comparable we have (we use $\varepsilon$ as a generic constant $> 0$ depending only on $K$, $\delta$, and $\Omega$)

$$\rho^2 = \rho(1/2)\rho_\Omega(x_0) \geq \varepsilon \rho_0 \Psi(x_0),$$

$$\Psi(x_0) - \Phi(x_0) \geq \varepsilon \rho_0 \Psi(x_0),$$

and the lemma is proved.

Below by $\rho_0$ we mean a number $> 0$ such that any point $\bar{x}_0 \in \partial \Omega$ is the only common point of $\partial \Omega$ and the closure of a ball, say $B_{\rho_0}(y_0)$, belonging to $\Omega$ with radius $\rho_0$. Since $\Omega \in C^{1,1}$, such $\rho_0 > 0$ exists. We further decrease $\rho_0$, if necessary, so that there is a $C^{1,1}$-diffeomorphism with its first- and second-order derivatives and the first- and second-order derivatives of its inverse bounded by a constant depending only on $\Omega$ and mapping $B_{2\rho_0}(\bar{x}_0) \cap \Omega$ onto $B_{\rho_0}^+ \cap \Omega$ and $B_{2\rho_0}(\bar{x}_0) \cap \partial \Omega$ onto $B_1^+ \cap \{x_1 = 0\}$.

In the following lemma we consider the case that $\rho_\Omega(x_0) \leq \rho_0$ and denote by $\bar{x}_0$ a point on $\partial \Omega$ such that $\rho_\Omega(x_0) = |\bar{x}_0 - x_0|$. By assumption there exists $y_0 \in \Omega$ such that

$$B_{\rho_0}(y_0) \cap \partial \Omega = \{\bar{x}_0\},$$

and since both balls $B_{\rho_0}(y_0)$ and $B_{\rho_\Omega(x_0)}(x_0)$ touch $\partial \Omega$ at $\bar{x}_0$ and the former ball has a smaller radius, the points $\bar{x}_0, x_0$, and $y_0$ lie on the same line and

$$\text{dist}(x_0, B_{\rho_0}^c(y_0)) = \rho_\Omega(x_0).$$

(3.8)

**Lemma 3.4.** Let the coefficients of $L$ be infinitely differentiable. Take $\rho_0 \in (0,1)$, $x_0 \in \Omega$ and suppose that $\rho := \rho_\Omega(x_0) \leq \rho_0$. Take $y_0$ introduced before the lemma. Let $\gamma \in (0,1]$ and let $\Phi$ be the classical solution of $L\Phi = 0$ in $B_{\rho_0}(y_0)$ with boundary condition $\Phi = \Psi^\gamma$ on $\partial B_{\rho_0}(y_0)$. Then

$$\Phi(x_0) \leq [1 - \varepsilon \rho_0]^{\gamma} \Psi^\gamma(x_0),$$

where $\varepsilon > 0$ depends only on $K$, $\delta$, and $\Omega$.

Proof. As in the proof of Lemma 3.3, we concentrate on the case that $\gamma = 1$ and we have $(L + c)v \leq -1$. Simple barriers show that (recall that $\rho_0$ is fixed) $v(x) \geq \varepsilon(\rho_0 - |x - y_0|)$ in $B_{\rho_0}(y_0)$, where $\varepsilon > 0$ depends only on $d$,
\[ v(x_0) \geq \varepsilon \rho_\Omega(x_0) \]
and we are done because \( \rho_\Omega(x_0) \) and \( \Psi(x_0) \) are comparable. The lemma is proved.

**Proof of Theorem 3.1.** An argument similar to the one in the beginning of the proof of Theorem 1.7 shows that without losing generality we may assume that \( u \in W^{2,2}_d(\Omega) \). After that we can certainly concentrate on the case that \( u \) and the coefficients of \( L \) are infinitely differentiable in \( \Omega \). Then define \( v \) as a unique classical solution of \( Lv = Lu \) in \( \Omega \) with zero boundary condition. By the maximum principle \( u - v \leq \sup_{\partial \Omega} u_+ \). It follows that we only need to estimate \( v \) and consequently we may assume that \( u = 0 \) on \( \partial \Omega \).

In that case the function
\[
v = \frac{u}{\Psi^\beta}
\]
is continuous in \( \bar{\Omega} \), equals zero on \( \partial \Omega \) and, hence, attains its maximum value at a point \( x_0 \in \Omega \):
\[
M := \frac{u(x_0)}{\Psi^\beta(x_0)} \geq \frac{u(x)}{\Psi^\beta(x)} \quad \forall x \in \Omega.
\]
If \( M \) is less than zero, we have nothing to prove. Therefore, we assume that
\[
u(x_0) > 0
\]
and consider two cases:
(a) \( \rho_\Omega(x_0) \geq \rho_0 \),
(b) \( \rho_\Omega(x_0) < \rho_0 \).

In case (a), in \( B' := B_{\rho_\Omega(x_0)/2}(x_0) \) we have \( u = v + h \), where \( v \) is the classical solution of \( Lv = Lu \) in \( B' \) with zero boundary value and \( h \) is the classical solution of \( Lh = 0 \) in \( B' \) with boundary condition \( h = u \) on \( \partial B' \).

We apply the Aleksandrov estimate to \( v \) and take into account that on \( B' \), \( \rho_\Omega \) and \( \Psi \) are comparable to a constant one. Then we see that
\[
v(x_0) \leq N \| (Lu)_- \|_{L_d(B')} \leq N \Psi^\beta(x_0) \| \rho^\beta_\Omega(Lu)_- \|_{L_d(\Omega)}.
\]

In what concerns \( h \), observe that owing to (3.9), by the maximum principle, it is less than the solution \( w \) of the equation \( Lw = 0 \) in \( B' \) with boundary condition \( M \Psi^\beta \). By Lemma 3.3, \( h(x_0) \leq w(x_0) \leq M \varepsilon \Psi^\beta(x_0) \), where \( \varepsilon \in (0,1) \) depends only on \( K, \delta, \alpha, \) and \( \Omega \). It follows that
\[
M \leq N \| \rho^\beta_\Omega(Lu)_- \|_{L_d(\Omega)} + M \varepsilon, \quad M \leq N \| \rho^\beta_\Omega(Lu)_- \|_{L_d(\Omega)}
\]
which yields (3.2).

In case (b), in the ball \( B_{\rho_\Omega}(y_0) \) introduced before Lemma 3.4 we have \( u = v + h \), where \( v \) is the solution of \( Lv = Lu \) in \( B_{\rho_\Omega}(y_0) \) with zero boundary condition and \( h \) satisfies \( Lh = 0 \) and equals \( u \) on \( \partial B_{\rho_\Omega}(y_0) \). As in case (a) by the maximum principle and Lemma 3.4 we have \( h(x_0) \leq M \varepsilon \Psi^\beta(x_0) \).

In what concerns \( v \) observe that in \( B_{\rho_\Omega}(y_0) \) by the maximum principle it is less than \( w \) defined as \( W^{2,2}_{d,loc}(B') \cap C(\bar{B}') \)-solution of \( Lw = -f_- \) in
\[
B' := B_{2\rho_\Omega}(\bar{x}_0) \cap \Omega
\]
vanishing on its boundary, where

\[ f := I_{B_{\rho_0}(y_0)} Lu. \]

By an argument similar to the one used in the proof of Lemma 3.2 we obtain that

\[ w(x_0) \leq N\rho_0^\beta(x_0)\|\rho_0^\alpha f - \|_{L_d(B')} \leq N\rho_0^\beta(x_0)\|\rho_0^\alpha f - \|_{L_d(\Omega)}. \]

Since \( \rho_0 \) and \( \Psi \) are comparable we conclude

\[ M \leq M\varepsilon + N\|\rho_0^\alpha f - \|_{L_d(\Omega)} \]

and this proves the theorem.

4. Estimates for stochastic integrals

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space, \(\{\mathcal{F}_t, t \geq 0\}\) be an increasing filtration of \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\) each of which is complete relative to \(\mathcal{F}, P\). Suppose that on \((\Omega, \mathcal{F}, P)\) we are given a \(d_1\)-dimensional Wiener process which is \(\mathcal{F}_t\)-adapted and such that \(w_t - w_s\) are independent of \(\mathcal{F}_s\) as long as \(0 \leq s \leq t < \infty\).

Fix some constants \(\delta \in (0, 1]\) and \(K \geq 0\). Let \(\sigma_t = \sigma_t(\omega)\) be a progressively measurable with respect to \(\{\mathcal{F}_t, t \geq 0\}\), \(d \times d_1\)-matrix valued process such that

\[ \delta^{-1}|\lambda|^2 \geq a_{ij}^t \lambda^i \lambda^j \geq \delta |\lambda|^2 \]

for all \(\lambda \in \mathbb{R}^d\), \(t \geq 0\), and \(\omega \in \Omega\), where \(a = (1/2)\sigma\sigma^*\).

Let \(b_t\) be a progressively measurable with respect to \(\{\mathcal{F}_t, t \geq 0\}\), \(\mathbb{R}^d\)-valued process such that \(|b_t| \leq K\) for all \(t \geq 0\) and \(\omega \in \Omega\).

Introduce

\[ x_t = \int_0^t \sigma_s dw_s + \int_0^s b_s ds. \]

Let \(G\) be a bounded \(C^{1,1}\) domain in \(\mathbb{R}^d\) containing the origin, set

\[ \rho_G(x) = \text{dist}(x, G^c), \]

take a nonnegative Borel function \(f\) on \(G\), a number

\[ \alpha \in [0, (d + 1)/2), \]

and set

\[ \beta = (d + 1 - 2\alpha)/(2d). \]

Here is the result of this section.

**Theorem 4.1.** There exists a constant \(N\), depending only on \(G, \delta, \alpha,\) and \(K\), such that

\[ E \int_0^\tau f(x_t) dt \leq N\rho_G^\beta(0) \left( \int_G \rho_G^\alpha f^d(x) dx \right)^{1/d}, \quad (4.1) \]

where \(\tau\) is the first exit time of \(x_t\) from \(G\).
Proof. A usual measure-theoretic argument shows that it suffices to prove (4.1) for bounded \( f \). Let \( A \) be the collection of couples \((a,b)\), where \( a \) is a symmetric \( d \times d \) matrix and \( b \in \mathbb{R}^d \) such that
\[
|b| \leq K, \quad \delta^{-1} |\lambda|^2 \geq a^{ij} \lambda^i \lambda^j \geq \delta |\lambda|^2
\]
for all \( \lambda \in \mathbb{R}^d \).

As it follows from [13] or [11] there exists a unique \( u \in W^2_d(G) \) vanishing on \( \partial G \) and satisfying
\[
\sup_{(a,b) \in A} [a^{ij} D_{ij} u + b^i D_i u + f] = 0
\]
in \( G \) (a.e.). By Itô’s formula (see, for instance, [8])
\[
u(0) = E \int_0^\tau [-a^{ij}_t D_{ij} u(x_t) - b^i_t D_i u(x_t)] dt \geq E \int_0^\tau f(x_t) dt.
\]
After that it only remains to apply Theorem 3.1 first observing that there exist a measurable \( A \)-valued function \((a(x), b(x))\) such that
\[
a^{ij}(x) D_{ij} u(x) + b^i(x) D_i u(x) + f(x) = 0
\]
in \( G \) (a.e.). The theorem is proved.

If Borel \( \Gamma \subset G \), then
\[
G(\Gamma) = E \int_0^\tau I_{\Gamma}(x_t) dt
\]
(the so-called Green’s measure) is the mean time that the process \( x_t, t \in [0, \tau], \) spends in \( \Gamma \), the mean time it occupies \( \Gamma \) before exiting from \( G \). By taking \( f = I_{\Gamma} \) in (4.1) we come to the following.

**Corollary 4.2.** We have
\[
G(\Gamma) \leq N \rho_0^\beta \left( \int_{\Gamma} \rho_0^\alpha(x) dx \right)^{1/d}.
\]

**Remark 4.3.** By analyzing the arguments in Remark 1.3 and the proof of Theorem 1.7, it is not hard to show that in the whole class of processes satisfying the conditions of Theorem 4.1 with arbitrary \( \delta \) and \( K = 0 \) estimate (4.1) fails to hold if one replaces \( d \) with \( p < d \) even if \( \alpha = 0 \) and also fails to hold for \( \alpha > (d+1)/2 \).

5. **Parabolic equations of the main type in a round cylinder**

If \( u \) is a \( d \times d \) symmetric matrix, by \( A_{ij}[u] \) we denote the co-factor of \( u_{ij} \) in \( \det u \).

**Lemma 5.1.** Let \( u, v \in W^{1,2}_{d+1}(C) \), \( u = v = 0 \) on \( \partial' C \), \( u \) and \( v \) be convex with respect to \( x \) and satisfying \( u \geq v \) in \( C \). Then
\[
\int_C \partial_t u \det D^2 u \, dx dt \leq \int_C \partial_t v \det D^2 v \, dx dt.
\]
Proof. By having in mind approximations, we may assume that \( u \) and \( v \) are infinitely differentiable in \( \bar{C} \). Then define \( v_\tau = (1 - \tau)v + \tau u \) and observe that, to prove (5.1), it suffices to prove that

\[
\frac{d}{d\tau} \int_C \partial_\tau v_\tau \det D^2 v_\tau \, dx dt \leq 0.
\]  

(5.2)

By denoting \( \dot{v}_\tau = dv_\tau/d\tau \) \((u - v \geq 0)\), we see that the left-hand side of (5.2) equals

\[
\int_C \partial_\tau \dot{v}_\tau \det D^2 v_\tau \, dx dt + \int_C \partial_\tau v_\tau D_{ij} \dot{v}_\tau A_{ij} [D^2 v_\tau] \, dx dt =: I_1 + I_2,
\]

By taking into account that

\[
\sum_i \frac{\partial}{\partial x^i} A_{ij} [D^2 u] = \sum_i \frac{\partial}{\partial x^i} A_{ji} [D^2 u] = 0
\]

for any \( j \) and smooth \( u \) and that \( \partial_\tau v_\tau(t, x) = 0 \) for \(|x| = 1\), we conclude that

\[
I_2 = \int_C \partial_\tau D_{ij} v_\tau \dot{v}_\tau A_{ij} [D^2 u] \, dx dt = \int_C \dot{v}_\tau \partial_\tau \det D^2 v_\tau \, dx dt.
\]

It follows that the left-hand side of (5.2) equals

\[
\int_C \partial_\tau [\dot{v}_\tau \det D^2 v_\tau] \, dx dt = - \int_B \dot{v}_\tau \det D^2 v_\tau(0, x) \, dx.
\]

Since \( \dot{v}_\tau \geq 0 \) and \( \det D^2 v_\tau \geq 0 \) we come to (5.2) thus proving the lemma.

**Theorem 5.2.** Let \( v \in W^{1,2}_{d+1}(C) \), \( u = 0 \) on \( \partial C \), and \( v \) be convex with respect to \( x \). Then, for any \( x_0 \in B \),

\[
|v(0, x_0)|^{d+1} \leq (d + 1)\omega_d^{-1}(1 - |x_0|^2)^{(d+1)/2} \int_C \partial_\tau v \det D^2 v \, dx dt,
\]

(5.3)

where \( \omega_d \) is the volume of \( B \).

Proof. We may assume that \( v \) is smooth. Define \( u(x_0) = -1 \) and introduce a cone with vertex at \((x_0, u(x_0))\) and base \( \partial B \). Let this cone be the graph of a function which we call \( u(x) \). Then mollify \( u \) in \( x \) near \( x_0 \) so that it remains convex, becomes smooth, larger than \( u \) and coincides with \( u \) in \( B \) apart from a small neighborhood of \( x_0 \). Call the resulting function \( w \) and observe that \( w(x_0) \) is as close to \(-1\) as we wish.

Then set

\[
w(t, x) = |v(t, x_0)|w(x).
\]

By convexity of \( v \) we have \( w(t, x) \geq |v(t, x_0)|u(x) \geq v(t, x) \). Also

\[
\partial_\tau w \det D^2 w(t, x) = -|v(t, x_0)|^d \partial_t v(t, x_0)|w(x)| \det D^2 w(x).
\]

It follows that

\[
\int_C \partial_\tau w \det D^2 w \, dx dt = \frac{1}{d+1} |v(0, x_0)|^{d+1} \int_B |w(x)| \det D^2 w(x) \, dx,
\]
where the integral on the right can be restricted to the small neighborhood of $x_0$ where we modified $u$ and hence this integral is as close as we wish to
\[ \int_B \det D^2 w(x) \, dx. \]
The value of the last integral is well known to be $\omega_d (1 - |x_0|^2)^{-(d+1)/2}$ (see, for instance, Remark 1.3).

After that it only remains to remember that by construction and Lemma 5.1
\[ \int_C \partial_t w \, dt \, \det D^2 w \, dx \leq \int_C \partial_t v \, dt \, \det D^2 v \, dx. \]
The theorem is proved.

Recall that $\psi(x) = 1 - |x|^2$.

**Theorem 5.3.** Let $u \in W^{1,2}_{d+1,\text{loc}}(C) \cap C(\overline{C})$, $u$ be convex with respect to $x$ and increasing in $t$. Let $\alpha \in [0, (d + 1)/2)$. Then, for any $x_0 \in B$,
\[ u(0, x_0) \geq \inf_{\partial C} u - N(d, \alpha) \psi^{\beta}(x_0) \left( \int_C \psi^\alpha \partial_t u \, dt \, \det D^2 u \, dx \right)^{1/(d+1)}, \tag{5.4} \]
where
\[ \beta = \frac{1}{2} - \frac{\alpha}{d+1}. \]

**Corollary 5.4.** Under the conditions of Theorem 5.3, if $u = 0$ on $\partial' C$, we have
\[ \sup_C |u| \leq N(d, \alpha) \left( \int_C \psi^\alpha \partial_t u \, dt \, \det D^2 u \, dx \right)^{1/(d+1)}. \tag{5.5} \]

We prove Theorem 5.3 after some preparations. For $1 \geq r > 0$ denote
\[ B_r = \{ x : |x| < r \}, \quad C_r = [0, T) \times B_r. \]

**Lemma 5.5.** Let $v \in W^{1,2}_{d+1}(C)$, $v = 0$ on $\partial' C$, $v$ be convex with respect to $x$ and increasing in $t$. Let $s \in [0, r)$ and suppose that $\partial_t v \, \det D^2 v \neq 0$ only on $C_r \setminus C_s$. Then
\( (a) \) for $t \geq 0, |x| \leq s$ we have
\[ |v(t, x)| \leq M \left( \frac{1 - s^2}{1 - r^2} \right)^{1/2}, \tag{5.6} \]
where
\[ M = \omega_d^{-1/(d+1)} (1 - r^2)^{1/2} \left( \int_{C_r \setminus C_s} \partial_t v \, dt \, \det D^2 v \, dx \right)^{1/(d+1)}, \]
\( (b) \) for $|x| \geq r$ we have
\[ |v(t, x)| \leq M (1 - |x|)/(1 - r). \tag{5.7} \]

**Proof.** (a) Observe that in $B_s$ we have
\[ \inf_{(r, a) \in A} \left[ r \partial_t v + a^{ij} D_{ij} v \right] = 0, \tag{5.8} \]
where $A$ is the set of couples $(r, a)$, where $a$ are $d \times d$ symmetric nonnegative matrices, $r \geq 0$, and $r + \text{tr} a = 1$. By the maximum principle $u$ in $C_s$ attains its minimum on $[0, \infty) \times \partial B_s$. Furthermore, by Theorem 5.2 for any, $t_0 \geq 0, x_0 \in B$, the quantity $|v(t_0, x_0)|^{d+1}$ is less than the right-hand side of (5.3) where the integral can be restricted to $C_r \setminus C_s$. It follows that $|v(t_0, x_0)|$ on $[0, \infty) \times \partial B_s$ is dominated by the right-hand side of (5.6) and proves (a).

(b) Observe that in $G_r := C \setminus \bar{C}_r$ the function $v$ satisfies (5.8). The function $w(t, x) := M(|x| - 1)/(1 - r)$ also satisfies this equation in $C_r$ and is less than $v(t, x)$ if $|x| = r$ or $|x| = 1$. Also $v(T, x) = 0 \geq w(x)$. By Theorem 4.1.11 of [11] we have $u \geq w$ in $G_r$ and this is (5.7). The lemma is proved.

Next, we use the special partition of unity in $B$ introduced before Lemma 1.6.

**Lemma 5.6.** Let $u \in W^{1,2}_{d+1}(C)$, $u$ be convex with respect to $x$ and increasing in $t$. Assume that

\[ \partial_t u \det D^2u = f_+^{d+1}, \]

in $C$ (a.e.), where $f \in L^{d+1}_{d+1}(C) \cap W^{1,2}_{\infty}(C_r)$ for any $r \in (0,1)$, and $f(T, x) = 0$ in $B$. Introduce $u_n$ as $W^{1,2}_{\infty}(C)$-functions which are convex in $x$, increase in $t$, vanish on $\partial' C$, and satisfy

\[ \partial_t u_n \det D^2u_n = f_+^{d+1}\eta_n^{d+1} \]

in $C$ (a.e.). Then

\[ u \geq \inf_{\partial' C} u + \sum_{n=1}^{\infty} u_n. \]

Proof. First of all note that, since $\psi^{-1/(d+1)} f \eta_n$ is in $W^{1,2}_{\infty}(C)$ and vanish for $t = T$, the functions $u_n$ with the described properties exist and are unique according to §5 in [9].

Recall that $\bar{u} := u - \inf_{\partial' C} u$ satisfies

\[ \inf_{(r, a) \in A} \left[ r \partial_t \bar{u} + a^{ij} D_{ij} \bar{u} - f(d + 1)^{d+1}\sqrt{r \det a} \right] = 0 \]

in $C$, and $u_n$ are unique $W^{1,2}_{\infty}(C)$-solutions of

\[ \inf_{(r, a) \in A} \left[ r \partial_t u_n + a^{ij} D_{ij} u_n - f \eta_n(d + 1)^{d+1}\sqrt{r \det a} \right] = 0 \]

in $C$ vanishing on $\partial' C$. Set $v_n = u_0 + \ldots + u_n$, $f_n = f \eta_0 + \ldots + f \eta_n$ and note that

\[ \inf_{(r, a) \in A} \left[ r \partial_t v_n + a^{ij} D_{ij} v_n - f_n(d + 1)^{d+1}\sqrt{r \det a} \right] \geq 0 \]

in $C$. It follows that

\[ \sup_{(r, a) \in A} \left[ r \partial_t (v_n - \bar{u}) + a^{ij} D_{ij} (v_n - \bar{u}) - (f_n - f)(d + 1)^{d+1}\sqrt{r \det a} \right] \geq 0 \]
and there is an $A$-valued function $(r, a) = (r(x), a(x))$ such that in $C$ (a.e.)

$$r\partial_t(v_n - \bar{u}) + a^{ij}D_{ij}(v_n - \bar{u}) - (f_n - f)(d + 1)^{d + 1/2} \partial \det a \geq 0.$$  

By Theorem 3.2.3 of [11] (see also the end of the introduction into Chapter 3 in [11])

$$v_n - \bar{u} \leq N(d)\|f_n - f\|_{L_{d+1}(C)}.$$  

This proves the lemma since the norm on the right tends to zero as $n \to \infty$.

**Lemma 5.7.** Let $u \in W_{d+1}^{1,2}(C)$, $u$ be convex with respect to $x$ and increasing in $t$. Assume that

$$\partial_t u \det D^2 u = f_d^{d+1}$$

in $C$ (a.e.), where $f \in L_{d+1}(C) \cap W^{1,2}_\infty(C)$ for any $r \in (0, 1)$, and $f(T, x) = 0$ in $B$. Let $\alpha \in [0, (d + 1)/2)$. Then, for any $x_0 \in B$, estimate (5.4) holds.

Proof. Fix $x_0 \in B$ and define $n_0$ as the smallest $n \geq 1$ such that $|x_0| \leq s_n$, that is

$$|x_0| \leq s_{n_0}, \quad |x_0| \geq s_{n_0-1}. \quad (5.9)$$

Below we are going to use a few times that the ratio $(1 - s_n)/(1 - s_{n+1})$ is bounded from above and away from zero by absolute constants independent of $n$ and that, for $x \in [s_{n-1}, s_{n+1}]$, the ratio $\psi^\alpha(x)/(1 - s_n)^\alpha$ is bounded from above and away from zero by constants independent of $n$ and depending only on $d$.

Take $u_n$ from Lemma 5.6 and observe that, by Lemma 5.5 (and Theorem 5.2), if $n \leq n_0$, then

$$|u_n(0, x_0)| \leq N(d)(1 - |x_0|)(1 - s_n)^{1 - s_n^2}^{1/2}\left(\int_C \eta_n^{d+1} f_d^{d+1} dxdt\right)^{1/(d+1)}$$

$$\leq N(d)(1 - |x_0|)(1 - s_n)^{-1/2 - \alpha/(d+1)}\left(\int_C \psi^\alpha \eta_n^{d+1} f_d^{d+1} dxdt\right)^{1/(d+1)}$$

$$\leq N(d)(1 - |x_0|)e_n^{(1-\beta)}\left(\int_C \psi^\alpha \eta_n^{d+1} f_d^{d+1} dxdt\right)^{1/(d+1)}.$$  

By Hölder’s inequality and in light of the fact that $1 - \beta > 0$

$$\sum_{n=0}^{n_0} |u_n(0, x_0)| \leq N(d)(1 - |x_0|)\left(\sum_{n=0}^{n_0} e_n^{(1-\beta)(d+1)/d}\right)^{d/(d+1)}$$

$$\times \left(\int_C \psi^\alpha f_d^{d+1} \sum_{n=0}^{n_0} \eta_n^{d+1} dxdt\right)^{1/(d+1)}$$

$$\leq N(d, \alpha)(1 - |x_0|)e_{n_0}^{(1-\beta)}\left(\int_C \psi^\alpha f_d^{d+1} dxdt\right)^{1/(d+1)}$$

$$\leq N(d, \alpha)\psi^\beta(x_0)\left(\int_C \psi^\alpha f_d^{d+1} dxdt\right)^{1/(d+1)}.$$  

(5.10)

If $n > n_0$, then by Lemma 5.5

$$|u_n(0, x_0)| \leq N(d)(1 - s_n^2)^{1/2}\left(\int_C \eta_n^{d+1} f_d^{d+1} dxdt\right)^{1/(d+1)}$$
\[ \leq N(d)e^{-n\beta} \left( \int_C \psi^\alpha \eta_n^d f^d_+ \, dx \right)^{1/(d+1)} \].

Since \( \beta > 0 \),
\[
\sum_{n > n_0} |u_n(0, x_0)| \leq N(d) \left( \sum_{n > n_0} e^{-n\beta(d+1)/d} \right)^{d/(d+1)}
\times \left( \int_C \psi^\alpha f^d_+ \, dx \right)^{1/(d+1)} \leq N(d, \alpha) e^{-n_0\beta} \left( \int_C \psi^\alpha \det D^2 u \, dx \right)^{1/(d+1)}.
\]

Upon combining this with (5.10) and Lemma 5.6 we get (5.4). The lemma is proved.

**Proof of Theorem 5.3.** For small \( \varepsilon > 0 \) set \( C^\varepsilon = [\varepsilon, T - \varepsilon) \times B_{1-\varepsilon} \) and observe that \( u \in W^{1,2}_{d+1}(C^\varepsilon) \). If the obvious version of (5.4) is true with the objects with \( \varepsilon \), then setting \( \varepsilon \downarrow 0 \) we obtain (5.4) as is. Hence, we may assume that \( u \in W^{1,2}_{d+1}(C) \). After that, as usual, we may assume that \( u \) is a smooth function. Then take \( \varepsilon > 0 \) and set \( u_\varepsilon = u - \varepsilon(\psi + T - t) \), so that \( v_\varepsilon \) is strictly convex and strictly increasing. Observe that on \( \bar{C} \)
\[
f_\varepsilon := (\partial_t u_\varepsilon \det D^2 u_\varepsilon)^{1/(d+1)} \geq \varepsilon^{2d/(d+1)},
\]
and \( f_\varepsilon \) is smooth. We extend it for \( t \in (T, T + 1], x \in \bar{B} \), so that it remains nonnegative, smooth, becomes zero for \( t = T + 1 \), and
\[
\int_{(T, T+1) \times B} f^d_+ \, dx \leq \varepsilon.
\]
Then define \( C' = [0, T + 1] \times B \) and introduce \( v_\varepsilon \) as a unique \( W^{1,2}_{\infty}(C') \)-function which is convex in \( x \), increases in \( t \), vanishes on \( \partial' C' \) and satisfies
\[
\partial_t v_\varepsilon \det D^2 v_\varepsilon = f^d_+ \in C' \text{ (a.e.)}. \]
In light of Lemma 5.7, to prove the theorem, it suffices to prove that in \( C \)
\[ \bar{u}_\varepsilon := u_\varepsilon - \inf_{\partial' C} \geq v_\varepsilon \quad (5.11) \]
Since \( v_\varepsilon \leq 0 \), (5.11) holds on \( \partial' C \). Furthermore, both \( u_\varepsilon \) and \( v_\varepsilon \) satisfy the same equation
\[
\inf_{(r,a) \in A} \left\{ r\partial_t w + a^{ij} D_{ij} w - (d+1) f_\varepsilon \sqrt{r \det a} \right\} = 0
\]
in \( C \). Hence, (5.11) follows by the maximum principle and the theorem is proved.

We now turn to estimates for parabolic operators of main type.

**Theorem 5.8.** Let \( a(t, x) = (a^{ij}(t, x)) \) and \( r(t, x) \) be a \( d \times d \)-symmetric nonnegative definite matrix-valued measurable function and a nonnegative
measurable function on $C$, respectively, such that $r + \text{tr}a > 0$ in $C$. Let $\alpha \in [0, (d + 1)/2)$ $u \in W^{1,2}_{d+1,\text{loc}}(C) \cap C(\bar{C})$. Introduce
\[ L_0u = r\partial_t u + a^{ij}D_{ij}u. \]
Then, for any $x_0 \in B$, $(0/0 := 0)$
\[ u(0, x_0) \leq \sup_{\partial C} u + N(d, \alpha)\psi^\beta(x_0)\left( \int_C \psi^\alpha I_{u < 0}(r \det a)^{-1/2}|L_0u|^{d+1} d\mu dt \right)^{1/(d+1)}. \]  
(5.12)

Proof. As in the proof of Theorem 5.3, we may assume that $u \in W^{1,2}_{d+1}(C)$. Since $r + \text{tr}a > 0$, by the homogeneity of estimate (5.12) we may also assume that
\[ r + \text{tr}a \equiv 1. \]  
(5.13)

A particular case. Suppose that $r \det a \geq \varepsilon$, where $\varepsilon > 0$ is a constant. Set
\[ f := -\frac{1}{(d + 1)}(r \det a)^{-1/2}L_0u \]
and find a sequence of $f_n \in L_{d+1}(C)$ such that $f_n \rightarrow f$ in $L_{d+1}(C)$, $f_n$ are smooth, and vanish near $\partial C$. Then $(f_n)^{d+1}_+ = \psi(f_n)^{d+1}_+$, where $f_n = f_n\psi^{-1/(d+1)}$ is a smooth function vanishing for $t = T$.

In that case, as we know, for each $n$, there exists a unique $W^{1,2}_{\infty}(C)$-function $v_n$ which is concave in $x$, decreases in $t$, vanishes on $\partial C$, and satisfies
\[ -\partial_t v_n \det(-D^2v) = (f_n)^{d+1}_+ \]  
(5.14)
in $C$ (a.e.). We also know that $v_n$ also satisfies
\[ \sup_{(r,a) \in A} \left[ \partial_t v_n + a^{ij}D_{ij}v_n + f_n(d + 1)^{d+1/2}\sqrt{r \det a} \right] = 0 \]  
(5.15)
and for $u$ we obviously have
\[ \sup_{(r,a) \in A} \left[ \partial_t u + a^{ij}D_{ij}u + f(d + 1)^{d+1/2}\sqrt{r \det a} \right] \geq 0. \]
It follows that for $w_n = u - \sup_{\partial C} u - v_n$ we have
\[ \sup_{(r,a) \in A} \left[ \partial_t w_n + a^{ij}D_{ij}w_n + (f - f_n)(d + 1)^{d+1/2}\sqrt{r \det a} \right] \geq 0 \]
and there exists an $A$-valued function $(r(t, x), a(t, x))$ such that in $C$ (a.e.)
\[ \partial_t w_n + a^{ij}D_{ij}w_n + (f - f_n)(d + 1)^{d+1/2}\sqrt{r \det a} \geq 0. \]

By Theorem 3.2.2 of [11]
\[ w_n \leq N\|f_n - f\|_{L_{d+1}(C)}, \]
where $N$ is independent of $n$. This and Theorem 5.3 show that
\[ u(0, x_0) - \sup_{\partial C} u \leq \lim_{n \to \infty} v_n(0, x_0) \leq N(d, \alpha)\psi^\beta(x_0)\|\psi^\alpha f_+\|_{L_{d+1}(C)}, \]
which completes considering this case.

General case. In order to drop the additional assumptions we take \( \gamma, \delta > 0 \), take \( \psi \) from Lemma 3.1.8 of [11] introduce \( L^\delta = L + \delta(\partial_t + \Delta) \), and apply the above result to \( u^\gamma = u - \gamma(\psi + 1) \) and \( L^\delta \). Then following almost word for word the appropriate parts of the proof of Lemma 3.2.4 of [11], we arrive at \( 5.12 \). The theorem is proved.

6. General parabolic equations in \( C \)

Here we generalize Theorem 5.8 for equation with lower order terms.

**Theorem 6.1.** Under the assumptions of Theorem 5.8 let \( b = (b^i(t,x)) \) be \( \mathbb{R}^d \)-valued measurable function on \( C \) and \( c = c(t,x) \) be a nonnegative measurable bounded function on \( C \) and let \( |b| \leq K \mu \) in \( B \), where \( K \) is a fixed constant and \( \mu = \mu(t,x) \) is the smallest eigenvalue of \( a = a(t,x) \). Then estimate \( 5.12 \) holds again with \( N = N(d, \alpha, K) \) if we replace \( L_0 u \) in \( 5.12 \) with

\[
Lu = r\partial_t u + a^{ij}D_{ij}u + b^i D_i u - cu,
\]

and \( \sup_{\partial^C u} u \) with \( \sup_{\partial^C u} u^+ \).

Proof. As in the proof of Theorem 5.8 we assume \( 5.13 \), \( u \in W^{1,2}_d(C) \), and first also assume that \( r \det a \geq \varepsilon \), where \( \varepsilon > 0 \) is a constant. Set

\[
f := -\frac{1}{d+1} (r \det a)^{-1/(d+1)} L u
\]

and define \( f_n \) and \( v_n \) in the same way as in the proof Theorem 5.8. We also take the function \( w \) from Lemma 2.2 with \( \gamma = 1 - \beta \), take \( N(d, \alpha) \) from \( 5.4 \) and set

\[
N(d, \alpha)\psi^\beta(x_0)\left( \int_C \psi^\alpha(f_n)^{d+1} \, dx dt \right)^{1/(d+1)} =: N_n\psi^\beta(x_0).
\]

Observe that, as for any concave function vanishing on \( \partial B \), we have \( |Dv_n| \leq v_n/(1 - |x|) \leq 2v_n/\psi \). Also note that for \( \bar{u} = u - \sup_{\partial B} u^+ \) we have \( L\bar{u} \geq Lu \). Then for the function

\[
\phi_n = v_n + 2KN_n w
\]

owing to \( 5.4, 2.2, \) and \( 5.15 \), we get

\[
L\bar{u} \geq -(d + 1)(r \det a)^{1/(d+1)}(f - f_n) + r\partial_t v_n + a^{ij}D_{ij}v_n
\]

\[
= g_n + Lv_n - b^i D_i v_n + cv_n \geq g_n + Lv_n - 2|b| v_n/\psi
\]

\[
\geq g_n + Lv_n - 2N_n|b| \psi^{-(1-\beta)} \geq g_n + L\phi_n.
\]

By Theorem 3.2.2 of [11]

\[
\bar{u}(0,x_0) \leq \phi_n(0,x_0) + N\|f - f_n\|_{L_d(C)},
\]

where \( N \) is independent of \( n \). After that it only remains to use Theorem 5.2, let \( n \to \infty \), and recall that, as follows from \( 2.3 \), \( w \leq N\psi^\beta \), where \( N \) depends only on \( d, \alpha, K \). This proves the theorem in our particular case.
The general case is dealt with as at the end of the proof of Theorem 5.8. The theorem is proved.

7. The case of general smooth cylinders

Let \( \Omega \) be a \( C^{1,1} \) bounded domain in \( \mathbb{R}^d \), \( a = (a^{ij}(t,x)) \) be \( d \times d \) symmetric matrix-valued function on \( \mathbb{R}^{d+1} \), \( b = (b^i(t,x)) \) be \( \mathbb{R}^d \)-valued function on \( \mathbb{R}^{d+1} \), and \( c = c(t,x) \) be a nonnegative bounded function on \( \mathbb{R}^{d+1} \). Suppose that these functions are measurable. We fix two numbers \( \delta, K > 0 \) and assume that, for all values of arguments and \( \lambda \in \mathbb{R}^d \),

\[
\delta^{-1}|\lambda|^2 \geq a^{ij}\lambda^i\lambda^j \geq \delta|\lambda|^2, \quad |b| \leq K, \quad c \geq 0. \tag{7.1}
\]

Introduce \( \Pi = [0,T) \times \Omega, \partial'\Pi = \partial\Pi \setminus \{(0) \times \bar{\Omega}\} \),

\[
Lu = \partial_t u + a^{ij} D_{ij} u + b^i D_i u - cu.
\]

Finally, take \( \alpha \in [0,(d + 1)/2] \) and define \( \rho_\Omega(x) = \text{dist}(x,\Omega^c) \).

**Theorem 7.1.** Let \( u \in W^{1,2}_{d+1, \text{loc}}(\Pi) \cap C(\bar{\Pi}) \). Then, for any \( (t,x) \in \Pi \),

\[
u(t,x) \leq \sup_{\partial'\Pi} u + N \rho_\Omega^\beta(x) \|\rho_\Omega^\alpha(Lu)\|_{L_{d+1}(\Pi)}, \tag{7.2}
\]

where the constant \( N \) depends only on \( \Omega, \delta, \) and \( K \).

To prove this theorem we need three lemmas.

**Lemma 7.2.** Set \( B^+_r = B_r \cap \{x \in \mathbb{R}^d : x^1 > 0\} \), \( B^+ = B^+_1 \), \( C^+ = [0,T) \times B^+, \partial' C^+ = \partial C^+ \setminus \{(0) \times B^+\} \) and suppose that \( u \in W^{1,2}_{d+1, \text{loc}}(C^+) \cap C(C^+) \), \( u \leq 0 \) on \( \partial' C^+ \). Also assume that the coefficients of \( L \) are infinitely differentiable and \( f := Lu \in L_{d+1}(C^+) \). Then in \( C^+ \) we have

\[
u(t,x) \leq N(x^1)^\beta \|M^\alpha f_-\|_{L_{d+1}(C^+)}, \tag{7.3}
\]

where \( M^\alpha(x) = (x^1)^\alpha \) and the constant \( N \) depends only on \( d, \delta, \) and \( K \).

Proof. Find a \( C^{1,1} \) domain \( B' \) which contains \( B^+_2 \), has \( B_2 \cap \{x \in \mathbb{R}^d : x^1 = 0\} \) as part of its boundary, and is \( C^{1,1} \)-diffeomorphic to \( B \). Set \( C' = (0,T) \times B' \).

Then, extend \( f \) as zero outside \( C^+ \) and denote by \( v \) the function of class \( W^{1,2}_{d+1}(C') \) satisfying

\[
Lv = -f_- \tag{7.4}
\]

in \( C' \) (a.e.) and vanishing on \( \partial' C' \). It is a classical fact that such a function exists and is unique. By the maximum principle \( v \geq 0, v \geq u \) in \( C^+ \). Furthermore, is we apply the diffeomorphism mentioned above to (7.4) then we will see that the image \( v' \) of \( v \) will satisfy the equation

\[
L'v' = -f'_- \tag{7.4'}
\]

in \( C \) (a.e.), where \( f'_- \) is the image of \( f_- \) and \( L' \) is the image of \( L \).

By Theorem 6.1

\[
v'(x') \leq N\psi^\beta(x') \|\psi^\alpha/f_-\|_{L_{d+1}(C)}, \tag{7.5}
\]
where $\psi(x')$ is the distance from $x'$ to the boundary of $B$. If $x'$ is the image of $x$, $\psi(x')$ is comparable to the distance of $x$ to the boundary of $B'$, since the diffeomorphism and its inverse are Lipschitz continuous.

It only remains to write down (7.5) in the original coordinates and use the fact that $v \geq u$ and that for $x \in B^+$ its distance to the boundary of $B'$ equals $x^1$. The lemma is proved.

Next we use a well-known fact (see, for instance, Lemma 8.8 in [10]) that there exists a function $\Psi \in C^{1,1}(\Omega)$ such that, for a constant $N$ depending only on $\Omega$, $\delta$, and $K$, we have in $\Omega$ (for any $t$)

$$N\rho_\Omega \geq \Psi, \quad \Psi \geq \rho_\Omega,$$

$$L\Psi + c\Psi \leq -1.$$  \hfill (7.6) \hfill (7.7)

**Lemma 7.3.** Let the coefficients of $L$ be infinitely differentiable. Take $\rho_0 \in (0, 1)$, $x_0 \in \Omega$ and suppose that $2\rho := \rho_\Omega(x_0) \geq \rho_0$. Let $\gamma \in (0, 1)$ and let $\Phi$ be the classical bounded solution of $L\Phi = 0$ in $[0, \infty) \times B_\rho(x_0)$ with boundary condition $\Phi = \Psi^t$ for $|x - x_0| = \rho$. Then

$$\Phi(0, x_0) \leq [1 - \varepsilon \rho_0]^{\gamma}\Psi^\gamma(x_0),$$

where $\varepsilon > 0$ depends only on $K$, $\delta$, and $\Omega$.

Proof. A simple argument based on the maximum principle shows that it suffices to concentrate on $\gamma = 1$. In that case first we note that by the maximum principle $\Phi \geq 0$. Therefore, $(L + c)\Phi \geq 0$ and $v := \Psi - \Phi$ satisfies $(L + c)v \leq -1$ and $v = 0$ for $|x - x_0| = \rho$. Then elementary barriers show that $v(t, x_0) \geq \varepsilon \rho^2$, where $\varepsilon > 0$ depends only on $K$, $\delta$, and the diameter of $\Omega$. Since $\rho_\Omega(x_0)$ and $\Psi(x_0)$ are comparable we have (we use $\varepsilon$ as a generic constant $> 0$ depending only on $K$, $\delta$, and $\Omega$)

$$\rho^2 = \rho(1/2)\rho_\Omega(x_0) \geq \varepsilon \rho_0\Psi(x_0),$$

$$v(0, x_0) = \Psi(x_0) - \Phi(0, x_0) \geq \varepsilon \rho_0\Psi(x_0),$$

and the lemma is proved.

Below by $\rho_0$ we mean a number $> 0$ such that any point $\bar{x}_0 \in \partial\Omega$ is the only common point of $\partial\Omega$ and the closure of a ball, say $B_{\rho_0}(y_0)$, belonging to $\Omega$ with radius $\rho_0$. Since $\Omega \in C^{1,1}$ such $\rho_0 > 0$ exists. We further decrease $\rho_0$, if necessary, so that there is a $C^{1,1}$-diffeomorphism with its first- and second-order derivatives and the first- and second-order derivatives of its inverse bounded by a constant depending only on $\Omega$ and mapping $B_{2\rho_0}(\bar{x}_0) \cap \Omega$ onto $B^1_1$ and $\bar{B}_{2\rho_0}(\bar{x}_0) \cap \partial\Omega$ onto $B^1_1 \cap \{x^1 = 0\}$.

In the following lemma we consider the case that $\rho_\Omega(x_0) \leq \rho_0$ and denote by $\bar{x}_0$ a point on $\partial\Omega$ such that $\rho_\Omega(x_0) = |\bar{x}_0 - x_0|$. By assumption there exists $y_0 \in \Omega$ such that

$$\bar{B}_{\rho_0}(y_0) \cap \partial\Omega = \{\bar{x}_0\},$$

and since both balls $B_{\rho_0}(y_0)$ and $B_{\rho_\Omega(x_0)}(x_0)$ touch $\partial\Omega$ at $\bar{x}_0$ and the former ball has smaller radius, the points $\bar{x}_0$, $x_0$, and $y_0$ lie on the same line and

$$\text{dist}(x_0, B^c_{\rho_0}(y_0)) = \rho_\Omega(x_0).$$ \hfill (7.8)
Lemma 7.4. Let the coefficients of $L$ be infinitely differentiable. Take $\rho_0 \in (0, 1)$, $x_0 \in \Omega$ and suppose that $\rho := \rho_0(x_0) \leq \rho_0$. Take $y_0$ introduced before the lemma. Let $\gamma \in (0, 1)$ and let $\Phi$ be the classical bounded solution of $L\Phi = 0$ in $(0, \infty) \times B_{\rho_0}(y_0)$ with boundary condition $\Phi = \Psi^\gamma$ on $(0, \infty) \times \partial B_{\rho_0}(y_0)$. Then
\[ \Phi(0, x_0) \leq [1 - \varepsilon \rho_0]^\gamma \Psi^\gamma(x_0), \]
in which yields $\Phi(0, x_0) \leq (7.2)$.

Proof. As in the proof of Lemma 7.3, we concentrate on the case that $\gamma = 1$ and we have $(L + c)v \leq -1$. Simple barriers show that (recall that $\rho_0$ is fixed) $v(t, x) \geq \varepsilon(\rho_0 - |x - y_0|)$ in $B_{\rho_0}(y_0)$, where $\varepsilon > 0$ depends only on $d$, $\delta$, $K$, and $\rho_0$. Owing to (7.8) this shows that $v(0, x_0) \geq \varepsilon \rho_0(x_0)$ and we are done because $\rho_0(x_0)$ and $\Psi(x_0)$ are comparable. The lemma is proved.

Proof of Theorem 7.1. We can certainly concentrate on the case that $u$ and the coefficients of $L$ are infinitely differentiable in $\Pi$. In that case the function
\[ v = \frac{u}{\Psi^\beta} \]
is continuous in $\Pi$, equals zero on $\partial \Pi$ and, hence, attains its maximum value in $\Pi$ at a point $(t_0, x_0) \in \Pi$:
\[ M := \frac{u(t_0, x_0)}{\Psi^\beta(x_0)} \geq \frac{u(t, x)}{\Psi^\beta(x)} \quad \forall (t, x) \in \Pi. \tag{7.9} \]
If $M$ is less than zero, we have nothing to prove. Therefore, we assume that $u(t_0, x_0) > 0$ and consider two cases:
(a) $\rho_0(x_0) \geq \rho_0$,
(b) $\rho_0(x_0) < \rho_0$.
In case (a), in $C' := [t_0, T) \times B_{\rho_0'(x_0)/2}(x_0)$ we have $u = v + h$, where $v$ is the classical solution of $Lv = Lu$ in $C'$ with zero boundary value and $h$ is the classical solution of $Lh = 0$ in $C'$ with boundary condition $h = u$ on $\partial C'$. We apply the Aleksandrov estimate to $v$ and take into account that on $C'$, $\rho_0$ and $\Psi$ are comparable to constant one. Then we see that
\[ v(t_0, x_0) \leq N\|(Lu)_{-}\|_{L^{1}(C')} \leq N\Psi^\beta(x_0)\|\rho_0^\alpha(Lu)_{-}\|_{L^{1}(\Pi)}. \tag{7.10} \]
In what concerns $h$, observe that owing to (7.9) and the fact that $u(T, x) = 0$ on $\Omega$, by the maximum principle, $h$ is less than the bounded classical solution $\Phi$ of the equation $L\Phi = 0$ in $[t_0, \infty) \times B_{\rho_0'(x_0)/2}(x_0)$ with boundary condition $M\Psi^\beta$. By Lemma 7.3, $h(t_0, x_0) \leq \Phi(t_0, x_0) \leq M\varepsilon \Psi^\beta(x_0)$, where $\varepsilon \in (0, 1)$ depends only on $K$, $\delta$, $\alpha$, and $\Omega$. It follows that
\[ M \leq N\|\rho_0^\alpha(Lu)_{-}\|_{L^{1}(\Pi)} + M\varepsilon, \quad M \leq N\|\rho_0^\alpha(Lu)_{-}\|_{L^{1}(\Pi)}, \]
which yields (7.2).
In case (b), take the ball $B_{\rho_0}(y_0)$ introduced before Lemma 7.4 and set $C' = [t_0, T) \times B_{\rho_0}(y_0)$. Then in $C'$ we have $u = v + h$, where $v$ is the solution
of $Lv = Lu$ in $C'$ with zero boundary condition and $h$ satisfies $Lh = 0$ and equals $u$ on $\partial' C'$. As in case (a) by the maximum principle and Lemma 7.4 we have $h(t_0, x_0) \leq M\varepsilon \Psi^\beta(x_0)$.

In what concerns $v$ observe that in $C'$ by the maximum principle it is less than $w$ defined as $W_{d+1, \text{loc}}^2(C'') \cap C(\tilde{C}'')$-solution of $Lw = -f_-$ in

$$C'' = [t_0, T] \times (B_{2\rho_0}(\tilde{x}_0) \cap \Omega)$$

vanishing on its parabolic boundary, where

$$f := I_{C'} Lu.$$

By an argument similar to the one used in the proof of Lemma 7.2 we obtain that

$$w(t_0, x_0) \leq N\rho_{\Omega}(x_0)\|\rho_{\Omega} f_-\|_{L^{d+1}(C'')} \leq N\rho_{\Omega}(x_0)\|\rho_{\Omega} f_-\|_{L^{d+1}(\Pi)}.$$

Since $\rho_{\Omega}$ and $\Psi$ are comparable, we conclude

$$M \leq M\varepsilon + N\|\rho_{\Omega} f_-\|_{L^{d+1}(\Pi)}$$

and this proves the theorem.

8. Estimates for stochastic integrals

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$ each of which is complete relative to $\mathcal{F}, P$. Suppose that on $(\Omega, \mathcal{F}, P)$ we are given a $d_1$-dimensional Wiener process which is $\mathcal{F}_t$-adapted and such that $w_t - w_s$ are independent of $\mathcal{F}_s$ as long as $0 \leq s \leq t < \infty$.

Fix some constants $\delta \in (0, 1]$ and $K \geq 0$. Let $\sigma_t = \sigma_t(\omega)$ be a progressively measurable with respect to $\{\mathcal{F}_t, t \geq 0\}$, $d \times d_1$-matrix valued process such that

$$\delta^{-1} |\lambda|^2 \geq a_{ij} \lambda^i \lambda^j \geq \delta |\lambda|^2$$

for all $\lambda \in \mathbb{R}^d$, $t \geq 0$, and $\omega \in \Omega$, where $a = (1/2)\sigma \sigma^*$.

Let $b_t$ be a progressively measurable with respect to $\{\mathcal{F}_t, t \geq 0\}$, $\mathbb{R}^d$-valued process such that $|b_t| \leq K$ for all $t \geq 0$ and $\omega \in \Omega$.

Introduce

$$x_t = \int_0^t \sigma_s dw_s + \int_0^s b_s ds.$$

Let $G$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ containing the origin, set

$$\rho_G(x) = \text{dist}(x, G^c),$$

take a nonnegative Borel function $f$ on $(0, T) \times G$, a number

$$\alpha \in [0, (d+1)/2),$$

and set

$$\beta = \frac{1}{2} - \frac{\alpha}{d+1}.$$

Here is the result of this section.
Theorem 8.1. There exists a constant $N$, depending only on $G$, $\delta$, $\alpha$, and $K$, such that

$$E \int_0^\tau f(t,x_t) \, dt \leq N \rho_G^\beta(0) \left( \int_{(0,\infty) \times G} \rho_G^\alpha(x) f^{d+1}(t,x) \, dx \, dt \right)^{1/(d+1)},$$

(8.1)

where $\tau$ is the first exit time of $x_t$ from $G$.

Proof. A usual measure-theoretic argument shows that it suffices to prove (8.1) for bounded $f$ vanishing for $t \geq T$ with arbitrary $T \in (0, \infty)$. Let $A$ be the collection of couples $(a,b)$, where $a$ is a symmetric $d \times d$ matrix and $b \in \mathbb{R}^d$ such that

$$|b| \leq K, \quad \delta^{-1} |\lambda|^2 \geq a^{ij} \lambda^i \lambda^j \geq \delta |\lambda|^2$$

for all $\lambda \in \mathbb{R}^d$.

As it follows from [11] there exists a unique $u \in W^{1,2}_{d+1}((0,T) \times G)$ vanishing on the parabolic boundary of $(0,T) \times G$ and satisfying

$$\sup_{(a,b) \in A} [\partial_t u + a^{ij} D_{ij} u + b^i D_i u + f] = 0$$

in $(0,T) \times G$ (a.e.). By Itô’s formula (see, for instance, [8])

$$u(0) = -E \int_0^{\tau \wedge T} \left[ \partial_t u(t,x_t) + a^{ij} D_{ij} u(t,x_t) + b^i D_i u(t,x_t) \right] dt$$

$$\geq E \int_0^{\tau \wedge T} f(t,x_t) \, dt = E \int_0^\tau f(t,x_t) \, dt.$$

After that it only remains to apply Theorem 7.1 first observing that there exist a measurable $A$-valued function $(a(x),b(x))$ such that

$$\partial_t u(t,x) + a^{ij}(t,x) D_{ij} u(t,x) + b^i(t,x) D_i u(t,x) + f(t,x) = 0$$

in $(0,T) \times G$ (a.e.). The theorem is proved.

If Borel $\Gamma \subset (0,\infty) \times G$, then

$$G(\Gamma) = E \int_0^\tau I_T(t,x_t) \, dt$$

(the so-called Green’s measure) is the mean time that the trajectory $(t,x_t), t \in [0,\tau]$, spends in $\Gamma$, the mean time it occupies $\Gamma$ before time $\tau$. By taking $f = I_T$ in (8.1) we come to the following.

Corollary 8.2. We have

$$G(\Gamma) \leq N \rho_G^\beta(0) \left( \int_\Gamma \rho_G^\alpha(x) \, dx \, dt \right)^{1/d}.$$

Acknowledgement. The article was finished during the author’s visit to the University of Bielefeld on the invitation of Michael Roeckner. This is greatly appreciated. The author is also sincerely grateful to A.I. Nasarov for providing important information on the subject of the paper.
REFERENCES

[1] A. D. Aleksandrov, Certain estimates of solutions of the Dirichlet problem, Vestnik Leningrad. Univ., Vol. 22 (1967), No. 7, 19–29 in Russian.

[2] H. Amann, Parabolic equations on uniformly regular Riemannian manifolds and degenerate initial boundary value problems, Recent developments of mathematical fluid mechanics, 43–77, Adv. Math. Fluid Mech., Birkhäuser/Springer, Basel, 2016.

[3] D.E. Apushkinskaya and A.I. Nazarov, The elliptic Dirichlet problem in weighted spaces, Zapiski Nauchn. Semin. POMI, Vol. 288 (2002), 14–33 (Russian); English transl.: J. Math. Sci., Vol. 123 (2004), No. 6, 4527–4538.

[4] Hongjie Dong and N.V. Krylov, Fully nonlinear elliptic and parabolic equations in weighted and mixed-norm Sobolev spaces, submitted, arxiv.org/abs/1806.00077

[5] Pengfei Guan, N.S. Trudinger, and Xu-Jia Wang, On the Dirichlet problem for degenerate Monge-Ampère equations, Acta Math., 182 (1999), No. 1, 87–104.

[6] V. Kozlov and A. Nazarov, The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients in a wedge, Math. Nachr., Vol. 287 (2014), No. 10, 1142–1165.

[7] N.V. Krylov, Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation, Sibirski Matematicheski Jurnal, Vol. 17 (1976), No. 2, 290–303 in Russian; English transl.: Siberian Math. J., Vol. 17 (1976), No. 2, 226–236.

[8] N.V. Krylov, “Controlled diffusion processes”, Nauka, Moscow, 1977 in Russian; English translation by Springer, 1980.

[9] N.V. Krylov, On control of a diffusion process up to the time of first exit from a region, Izvestiya Akademii Nauk SSSR, seriya matematicheskaya, Vol. 45 (1981), No. 5, 1029–1048 in Russian; English translation in Math. USSR Izvestija, Vol. 19 (1982), No. 2, 297–313.

[10] N.V. Krylov, On a representation of fully nonlinear elliptic operators in terms of pure second order derivatives and its applications, Problemy Matemat. Analiza, Vol. 59, July 2011, p. 3–24 in Russian; English translation: Journal of Mathematical Sciences, New York, Vol. 177 (2011), No. 1, 1-26.

[11] N.V. Krylov, “Sobolev and viscosity solutions for fully nonlinear elliptic and parabolic equations”, Amer. Math. Soc., Providence, RI, 2018.

[12] A.I. Nazarov, Estimates for the maximum of solutions of elliptic and parabolic equations in terms of weighted norms of the right-hand side, Algebra & Analysis, Vol. 13 (2001), No. 2, 151–164 (Russian); English transl.: St.Petersburg Math. J., Vol. 13 (2002), No. 2, 269–279.

[13] N. Winter, W^{2,p} and W^{1,p}-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations, Z. Anal. Anwend., Vol. 28 (2009), No. 2, 129–164.

(N. V. Krylov) 127 VINCENT HALL, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455

E-mail address: nkrylov@umn.edu