Approximation of Solutions of DDEs Under Nonstandard Assumptions via Euler Scheme

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Abstract. We deal with approximation of solutions of delay differential equations (DDEs) via the classical Euler algorithm. We investigate the pointwise error of the Euler scheme under nonstandard assumptions imposed on the right-hand side function $f$. Namely, we assume that $f$ is globally of at most linear growth, satisfies globally one-side Lipschitz condition but it is only locally Hölder continuous. We provide a detailed error analysis of the Euler algorithm under such nonstandard regularity conditions. Moreover, we report results of numerical experiments.

Mathematics Subject Classification: 65L05, 65L70

1. Introduction

In this paper we deal with the problem of approximation of solutions $z : [0, +\infty) \to \mathbb{R}^d$ of delay differential equations (DDEs) of the following form

\[
\begin{cases}
  z'(t) = f(t, z(t), z(t-\tau)), & t \in [0, (n+1)\tau], \\
  z(t) = \eta, & t \in [-\tau, 0],
\end{cases}
\]

with a constant time lag $\tau \in (0, +\infty)$. Here $\eta \in \mathbb{R}^d$, $n \in \mathbb{Z}^+$ is a (finite and fixed) horizon parameter and a right-hand side function $f : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ that satisfies suitable regularity conditions. We investigate the error of the Euler scheme in the case when the right-hand side function is not necessarily globally Lipschitz continuous.

There are two main streams of common used methods for approximation of solutions of delay differential equations: connected with the theory of solving ordinary differential equations (ODE) and the theory of solving partial differential equations (PDE). The former group benefits e.g. from Runge-Kutta methods [1] Chapter 9 and [1] [7] [28] (also special cases as Euler, midpoint or Heun methods), continuous Runge-Kutta methods [3] [7], linear multistep methods [3] [18] (also Adams–Bashforth methods) or predictor–corrector methods, cf. [7]. The latter takes advantages from reformulating DDE into PDE and for instance uses the infinitesimal generator approach [8], an optimal asymptotic homotopy method [5] or method of lines [6] [7] [20]. All of the above and classical literature for solving DDEs [12] [13] [26] assumes mainly global Lipschitz regularity of the right-hand side function $f$ of the underlying DDE (1.1), alike to classical literature for ordinary differential equations [9] [17] [13].

Meanwhile, it turns out that real world applications need nonstandard assumptions. For example, a phase change of metallic materials can be described by a delay differential equation due to delay in the response to the change in processing conditions [11] [14] [23] [24] [25] [27]. A novel case with local Lipschitz condition and uniform boundedness of the right-hand side function was studied in [11]. In [11] [24] authors weakened this assumptions by considering a case when a right-hand side function is one-dimensional, locally Hölder continuous and monotone. In this paper we generalize the techniques used in [11] and study the error of the classical Euler scheme for a multidimensional case, when a right-hand side function is locally

Key words and phrases. delay differential equations, one-side Lipschitz condition, locally Hölder continuous right-hand side function, Euler scheme.
Hölder continuous and satisfies one-side Lipschitz conditions. According to our best knowledge till now there were no results in the literature on error analysis for the Euler scheme under such nonstandard assumptions. Our paper can be viewed as a first step into that direction. Moreover, we believe that our approach can be adopted for other numerical schemes, especially those of higher order.

The main contributions of the paper are as follows.

(i) We provide detailed and rigorous theoretical analysis of the error of the Euler scheme under nonstandard assumptions on the right-hand side function $f$. In particular, we show dependence of the error of the Euler scheme on Hölder exponents of the right-hand side function $f$ (see Theorem 1.2).

(ii) We perform many numerical experiments that confirm our theoretical conclusions.

The paper is organized as follows. The statement of the problem, basic notions and definitions are given in Section 2. In this section we also recall the definition of the Euler scheme. Auxiliary analytical properties of DDE (1.1) (such as existence, uniqueness of the solution) under nonstandard assumptions are settled in Section 3. Proof of the main result, that states the upper bound on the error of the Euler algorithm, is given in Section 4. Section 5 contains results of theoretical experiments performed for three exemplary DDEs. In Appendix (Section 6) we gathered and proved some auxiliary results concerning properties of solutions of ODEs in the case when the right-hand side function is of at most linear growth satisfies one-side Lipschitz condition, and is only locally Hölder continuous (Lemma 6.1). We also show the error estimate for the Euler scheme when applied to ODEs under such regularity conditions (Lemma 6.2). These results are used in the proof of Theorem 1.2.

2. Problem formulation

We denote $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $x_+ = \max\{0, x\}$ for all $x, y \in \mathbb{R}$.

For $x, y \in \mathbb{R}^d$ we take $\langle x, y \rangle = \sum_{k=1}^d x_k y_k$ and $\|x\| = \langle x, x \rangle^{1/2}$. For the right-hand side function $f : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ in the equation (1.1) we impose the following assumptions:

(F1) $f \in C([0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$.

(F2) There exists a constant $K \in (0, +\infty)$ such that for all $(t, y, z) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$\|f(t, y, z)\| \leq K(1 + \|y\|)(1 + \|z\|).$$

(F3) There exists a constant $H \in \mathbb{R}$ such that for all $(t, z) \in [0, +\infty) \times \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^d$

$$\langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z) \rangle \leq H \cdot (1 + \|z\|) \cdot \|y_1 - y_2\|^2.$$

(F4) There exist $L \in (0, +\infty)$, $\alpha, \beta_1, \beta_2, \gamma \in (0, 1]$ such that for all $t_1, t_2 \in [0, +\infty)$, $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$

$$\|f(t_1, y_1, z_1) - f(t_2, y_2, z_2)\| \leq L \left( (1 + \|y_1\| + \|y_2\|) \cdot (1 + \|z_1\| + \|z_2\|) \cdot |t_1 - t_2|^\alpha \\
+ (1 + \|z_1\| + \|z_2\|) \cdot \|y_1 - y_2\|^{\beta_1} \\
+ (1 + \|z_1\| + \|z_2\|) \cdot \|y_1 - y_2\|^{\beta_2} \\
+ (1 + \|y_1\| + \|y_2\|) \cdot \|z_1 - z_2\|^{\gamma} \right).$$

The assumptions above, especially (F3),(F4), are inspired by the real-life model describing evolution of dislocation density, see [11, 14, 23, 24, 25, 27] for detailed description and discussion of the DDE involved. In Lemma 3.1 below we prove that under the assumptions (F1)-(F3) the equation (1.1) has unique solution $z = z(t)$ on the whole interval $[-\tau, (n + 1)\tau]$. 
After \cite{7} we recall the definition of the Euler scheme for DDEs of the form (1.1). For the fixed horizon parameter $n \in \mathbb{Z}_+$ the Euler scheme that approximates a solution $z = z(t)$ of (1.1) for $t \in [0, (n + 1)\tau]$ is defined recursively for subsequent intervals in the following way \cite{7}. This iterative way of solving DDEs is often called a method of steps (see \cite{26}). We fix the discretization parameter $N \in \mathbb{Z}_+$ and set

$$ t^j_k = j\tau + kh, \quad k = 0, 1, \ldots, N, \quad j = 0, 1, \ldots, n, $$

where

$$ h = \frac{\tau}{N}. \quad \text{(2.1)} $$

Note that for each $j$ the sequence $\{t^j_k\}_{k=0}^N$ provides uniform discretization of the subinterval $[j\tau, (j+1)\tau]$. Discrete approximation of $z$ in $[0, \tau]$ is defined by

$$ y^0_0 = \eta, \quad y^0_{k+1} = y^0_k + h \cdot f(t^0_k, y^0_k, \eta), \quad k = 0, 1, \ldots, N - 1. \quad \text{(2.2)} $$

Then for $j = 2, 3, \ldots, n$ we take

$$ y^j_0 = y^{j-1}_N, \quad y^j_{k+1} = y^j_k + h \cdot f(t^j_k, y^j_k, y^{j-1}_k), \quad k = 0, 1, \ldots, N - 1. \quad \text{(2.3)} $$

So as the output we obtain the sequence $\{y^j_k\}_{k=0,1,\ldots,N, j = 0, 1, \ldots, n}$ that provides a discrete approximation of the values $\{z(t^j_k)\}_{k=0,1,\ldots,N, j = 0, 1, \ldots, n}$.

The aim is to investigate the error of Euler scheme under the (mild and nonstandard) assumptions (F1)-(F4), i.e.: upper bound on the following quantity

$$ \max_{0 \leq j \leq n} \max_{0 \leq k \leq N} \|z(t^j_k) - y^j_k\|. \quad \text{(2.6)} $$

Unless otherwise stated, all constants appearing in the estimates will only depend on $\tau$, $\eta$, $n$, $d$, $K, H, L, \alpha, \beta_1, \beta_2$, and $\gamma$, but not on the discretization parameter $N$. Moreover, we use the same symbol to denote different constants.

**Remark 2.1.** For a time-dependent delay, i.e. $\tau = \tau(t)$ or a state-dependent delay, i.e. $\tau = \tau(t, z(t))$ one can notice that it is not possible to use methods of steps. When we deal with a variable delay it is highly unlikely to hit the approximation mesh $\{t^j_k\}_{0 \leq k \leq N}$. Because of that, it is required to use an ODE solver with additional dense output (called continuous extension, cf. \cite{4} Chapter 9). However, the direct imitation of the method of steps with a vanishing lag case (a DDE is said to have a vanishing lag at the point $t_*$ if $\tau(t, z(t)) \to 0$ as $t \to t_*$) encounters a natural barrier beyond which the solution cannot proceed \cite{3}.

### 3. Analytical Properties of Solutions of DDEs

In the following lemma we show, by using results from the Appendix, that the delay differential equation (1.1) has a unique solution under the assumptions (F1)-(F3). Note that the assumptions are weaker than those known from the standard literature. Namely, we use only one-side Lipschitz assumption and local Hölder condition for the right-hand side function $f$ instead of a global Lipschitz continuity.

Let us introduce following notation. By $\phi_j = \phi_j(t)$ we denote the solution $z$ of (1.1) on the interval $t \in [j\tau, (j+1)\tau]$ for $j \in \mathbb{Z}_+ \cup \{0\}$. Let $\phi_{-1}(t) := \eta$ for all $t \in [-\tau, 0]$. 


Lemma 3.1. Let $\eta \in \mathbb{R}^d$ and let $f$ satisfy (F1)-(F3). Moreover, fix $\tau \in (0, +\infty)$ and $n \in \mathbb{Z}_+ \cup \{0\}$. Then the equation (1.1) has a unique solution

$$z \in C^1([0, (n+1)\tau]; \mathbb{R}^d).$$

Moreover, there exist $K_0, K_1, \ldots, K_n \geq 0$ such that for $j = 0, 1, \ldots, n$

$$\sup_{j\tau \leq t \leq (j+1)\tau} \|\phi_j(t)\| \leq K_j,$$

and, for all $t, s \in [j\tau, (j+1)\tau]$

$$\|\phi_j(t) - \phi_j(s)\| \leq \bar{K}_j|t - s|,$$

with $\bar{K}_j = K(1 + K_{j-1})(1 + K_j)$, where $K_{-1} := \|\eta\|$.

Proof. We proceed by induction with respect to $\bar{K}_j$. 

Let $\hat{\phi}_j(t) := f(t, z(t), \eta)$, $t \in [0, \tau]$, with the initial condition $z(0) = \eta$. Denoting by

$$g_0(t, y) := f(t, y, \eta), \quad t \in [0, \tau], y \in \mathbb{R}^d, \quad (3.4)$$

we get, by the properties of $f$ namely (F1) and (F2), that $g_0 \in C([0, \tau] \times \mathbb{R}^d)$,

$$\|g_0(t, y)\| \leq \bar{K}_0(1 + \|y\|), \quad (3.5)$$

with $\bar{K}_0 = K(1 + \|\eta\|)$, and by (F3) for all $y_1, y_2 \in \mathbb{R}^d$ and $t \in [0, \tau]$

$$\langle y_1 - y_2, g_0(t, y_1) - g_0(t, y_2) \rangle \leq H(1 + \|\eta\|) \cdot \|y_1 - y_2\|^2 \leq \hat{H}_0 \cdot \|y_1 - y_2\|^2, \quad (3.6)$$

where $\hat{H}_0 = H(1 + \|\eta\|)$. Therefore, by Lemma 6.1 we get that there exists a unique continuously differentiable solution $\phi_0 : [0, \tau] \to \mathbb{R}^d$ of the equation (3.4), such that

$$\sup_{t \in [0, \tau]} \|\phi_0(t)\| \leq K_0,$$

where

$$K_0 = (\|\eta\| + \bar{K}_0 \tau)e^{\bar{K}_0 \tau} = (\|\eta\| + K(1 + \|\eta\|)\tau)e^{K(1 + \|\eta\|)\tau} \geq 0,$$

and for all $t, s \in [0, \tau]$

$$\|\phi_0(t) - \phi_0(s)\| \leq \bar{K}_0|t - s|,$$

where

$$\bar{K}_0 = \bar{K}_0(1 + K_0) = K(1 + K_{-1})(1 + K_0)$$

and we set $K_{-1} := \|\eta\|$. In that way it depends only on values of $\|\eta\|, K, \tau$.

Let us now assume that there exists $0 \leq j \leq n - 1$ such that the statement of the lemma holds for the solution $\phi_j : [j\tau, (j+1)\tau] \to \mathbb{R}^d$. Consider the equation

$$z'(t) = f(t, z(t), \phi_j(t - \tau)), \quad t \in [(j + 1)\tau, (j + 2)\tau], \quad (3.8)$$

with the initial condition $z((j + 1)\tau) = \phi_j((j + 1)\tau)$. Let

$$g_{j+1}(t, y) := f(t, y, \phi_j(t - \tau)), \quad t \in [(j + 1)\tau, (j + 2)\tau], y \in \mathbb{R}^d. \quad (3.9)$$

We get by the induction assumption and from the properties of $f$ that $g_{j+1} \in C([(j + 1)\tau, (j + 2)\tau] \times \mathbb{R}^d, \mathbb{R}^d)$, for all $y \in \mathbb{R}^d$ we have

$$\|g_{j+1}(t, y)\| \leq K(1 + \sup_{j\tau \leq t \leq (j+1)\tau} \|\phi_j(t)\|) \leq \hat{K}_{j+1}(1 + \|y\|)$$

with $\hat{K}_{j+1} = K(1 + K_j)$, and for all $t \in [(j + 1)\tau, (j + 2)\tau], y_1, y_2 \in \mathbb{R}^d$

$$\langle y_1 - y_2, g_{j+1}(t, y_1) - g_{j+1}(t, y_2) \rangle \leq H(1 + \|\phi_j(t - \tau)\|) \cdot \|y_1 - y_2\|^2.$$
\[ \leq H_+ (1 + \| \phi_j (t - \tau) \|) \cdot \| y_1 - y_2 \|^2 \leq H_+ \left( 1 + \sup_{j \tau \leq t \leq (j+1) \tau} \| \phi_j (t) \| \right) \cdot \| y_1 - y_2 \|^2 \]

\[ \leq \hat{H}_{j+1} \cdot \| y_1 - y_2 \|^2, \quad (3.11) \]

where \( \hat{H}_{j+1} = H_+ (1+K_j) \). Hence, by Lemma 3.1 we get that there exists a unique continuously differentiable solution \( \phi_{j+1} : [(j + 1)\tau, (j + 2)\tau] \to \mathbb{R}^d \) of the equation (3.8), such that

\[ \sup_{t \in [(j+1)\tau, (j+2)\tau]} \| \phi_{j+1} (t) \| \leq K_{j+1}, \]

where

\[ K_{j+1} = (K_j + K_{j+1}) e^{K_{j+1} \tau} = (K_j + K (1 + K_j) \tau) e^{K (1 + K_j) \tau} \geq 0, \]

and for all \( t, s \in [(j + 1)\tau, (j + 2)\tau] \) we have

\[ \| \phi_{j+1} (t) - \phi_{j+1} (s) \| \leq K_{j+1} |t - s|, \]

where \( K_{j+1} = \hat{K}_{j+1} (1 + K_{j+1}) = K (1 + K_j)(1 + K_{j+1}) \).

From the above inductive construction we see that the solution of (1.1) is continuous. Moreover, due to the continuity of \( f, \phi_j \), and \( \phi_{j-1} \) we get for any \( 0 \leq j \leq n - 1 \) that

\[ \lim_{t \to (j + 1)\tau^-} z' (t) = \lim_{t \to (j + 1)\tau^-} \phi_j' (t) = \lim_{t \to (j + 1)\tau^-} f (t, \phi_j (t), \phi_{j-1} (t - \tau)) \]

\[ = f ((j + 1)\tau, \phi_j ((j + 1)\tau), \phi_{j-1} (j\tau)) = f ((j + 1)\tau, \phi_{j+1} ((j + 1)\tau), \phi_j (j\tau)) \]

\[ = \lim_{t \to (j + 1)\tau^+} f (t, \phi_{j+1} (t), \phi_j (t - \tau)) = \lim_{t \to (j + 1)\tau^+} \phi_{j+1}' (t) = \lim_{t \to (j + 1)\tau^+} z' (t). \]

Hence, the solution of (1.1) is continuously differentiable and the proof is completed. \( \square \)

**Lemma 3.2.** Let \( \eta \in \mathbb{R}^d \) and let \( f \) satisfy (F1)-(F4). Additionally, fix \( \tau \in (0, +\infty) \) and \( n \in \mathbb{Z}_+ \cup \{0\} \). For \( j = 0, 1, \ldots, n \) consider the functions \( g_j : [j\tau, (j + 1)\tau] \times \mathbb{R}^d \to \mathbb{R}^d \) defined by

\[ g_j(t, y) = f(t, y, \phi_{j-1} (t - \tau)), \quad (3.12) \]

where \( \phi_{-1} (t) := \eta \) for \( t \in [-\tau, 0] \). Then the following holds:

(i) \( g_j \in C([j\tau, (j + 1)\tau] \times \mathbb{R}^d; \mathbb{R}^d), \quad j = 0, 1, \ldots, n \).

(ii) There exist \( K_0, K_1, \ldots, K_n \in [0, +\infty) \) such that for all \( j = 0, 1, \ldots, n \), \( (t, y) \in [j\tau, (j+1)\tau] \times \mathbb{R}^d \)

\[ \| g_j (t, y) \| \leq K_j (1 + \| y \|). \]

(iii) There exist \( \hat{H}_0, \hat{H}_1, \ldots, \hat{H}_n \in [0, +\infty) \) such that for all \( j = 0, 1, \ldots, n \), \( t \in [j\tau, (j + 1)\tau], y_1, y_2 \in \mathbb{R}^d \)

\[ \langle y_1 - y_2, g_j (t, y_1) - g_j (t, y_2) \rangle \leq \hat{H}_j \cdot \| y_1 - y_2 \|^2. \]

(iv) There exist \( \tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_n \in [0, +\infty) \) such that for all \( j = 0, 1, \ldots, n \), \( t_1, t_2 \in [j\tau, (j + 1)\tau], y_1, y_2 \in \mathbb{R}^d \)

\[ \| g_j (t_1, y_1) - g_j (t_2, y_2) \| \leq \tilde{L}_j \left( (1 + \| y_1 \| + \| y_2 \|) \cdot |t_1 - t_2|^{\alpha \wedge \gamma} + \| y_1 - y_2 \|^{\beta_1} + \| y_1 - y_2 \|^{\beta_2} \right). \]

**Proof.** Conditions (i), (ii), and (iii) follow from the proof of Lemma 3.1. By the assumption (F4) and Lemma 3.1 we get for all \( t_1, t_2 \in [j\tau, (j + 1)\tau], y_1, y_2 \in \mathbb{R}^d \) that

\[ \| g_j (t_1, y_1) - g_j (t_2, y_2) \| \]

\[ \leq \tilde{L} \left( (1 + 2K_{j-1}) (1 + \| y_1 \| + \| y_2 \|)|t_1 - t_2|^\alpha \right. \]

\[ + \left. (1 + 2K_{j-1}) (\| y_1 - y_2 \|^{\beta_1} + \| y_1 - y_2 \|^{\beta_2}) \right). \]
\[+\hat{K}_{j-1} \left(1 + \|y_1\| + \|y_2\|\right)|t_1 - t_2|^\gamma\]

\[\leq \hat{L}_j \left((1 + \|y_1\| + \|y_2\|) \cdot |t_1 - t_2|^{\alpha \wedge \gamma} + \|y_1 - y_2\|^{\beta_1} + \|y_1 - y_2\|^{\beta_2}\right),\]

where \(\hat{L}_j = L \max\{1 + 2K_{j-1}, \hat{K}_{j-1}\} \cdot 2(1 + \tau), \hat{L}_0 = L(1 + 2\|\eta\|) \cdot 2(1 + \tau),\) and \(\hat{K}_{-1} := 0, K_{-1} := \|\eta\|\).

4. Error of the Euler scheme for delay differential equations

In this section we provide a proof of the main result that consists of the upper bound on the error (2.6) for the Euler algorithm. In the proof we shall use the following lemma.

**Lemma 4.1.** Let \(\tau \in (0, +\infty), \eta \in \mathbb{R}^d\) and let \(f\) satisfy (F1)-(F4). For any \(n \in \mathbb{Z}_+ \cup \{0\}\) there exist \(\hat{K}_0, \ldots, \hat{K}_n \in (0, +\infty), \hat{K}_j = \hat{K}_j(K, \eta, \tau),\) such that for all \(N \in \mathbb{Z}_+\)

\[
\max_{0 \leq j \leq n} \max_{0 \leq k \leq N} \|y_k^0\| \leq \max_{0 \leq j \leq n} \hat{K}_j. \quad (4.1)
\]

**Proof.** We proceed by induction. Note that

\[
y_{k+1}^0 = y_k^0 + h \cdot g_0(t_k^0, y_k^0), \quad k = 0, 1, \ldots, N - 1, \quad (4.2)
\]

where \(y_0^0 = \eta\) and \(g_0(t, y) = f(t, y, \eta),\) so by Lemmas 3.2 [4.2] we get that

\[
\max_{0 \leq k \leq N} \|y_k^0\| \leq \hat{K}_0, \quad (4.3)
\]

where \(\hat{K}_0 = \hat{K}_0(K, \eta, \tau).\) Now, let us assume that there exist \(j = 0, 1, \ldots, n - 1\) and \(\hat{K}_j = \hat{K}_j(K, \eta, \tau) \in (0, +\infty)\) such that for all \(N \in \mathbb{Z}_+\)

\[
\max_{0 \leq k \leq N} \|y_k^j\| \leq \hat{K}_j, \quad (4.4)
\]

which is obviously satisfied for \(j = 0.\) By (2.5) and (F2) assumption we get for \(k = 0, 1, \ldots, N - 1\) that

\[
\|y_{k+1}^{j+1}\| \leq \|y_k^{j+1}\| + h\|f(t_k^{j+1}, y_{k+1}^{j+1}, y_k^j)\| \leq (1 + h\hat{C}_{j+1})\|y_k^{j+1}\| + h\tilde{C}_{j+1}, \quad (4.5)
\]

where \(\|y_{k+1}^{j+1}\| = \|y_{k}^{j}\| \leq \hat{K}_j\) and \(\hat{C}_{j+1} = \hat{K}(1 + \hat{K}_j).\) From the discrete version of Gronwall’s lemma we obtain

\[
\max_{0 \leq k \leq N} \|y_k^{j+1}\| \leq \hat{K}_{j+1}, \quad (4.6)
\]

with \(\hat{K}_{j+1} = e^{\tau\hat{C}_{j+1}}(\hat{K}_j + 1) - 1\) and \(\hat{K}_{j+1} = \hat{K}_{j+1}(K, \eta, \tau).\) This completes the proof. \(\square\)

Below we stated and prove the main result of the paper.

**Theorem 4.2.** Let \(\tau \in (0, +\infty), \eta \in \mathbb{R}^d\) and let \(f\) satisfy (F1)-(F4). For any \(n \in \mathbb{Z}_+ \cup \{0\}\) there exist \(C_0, C_1, \ldots, C_n \geq 0\) such that for \(N \geq 2\lfloor \tau \rfloor\) the following holds

\[
\max_{0 \leq k \leq N} \|\phi_0(t_k^0) - y_k^0\| \leq C_0(h^{\alpha \wedge \gamma} + h^{\beta_1} + h^{\beta_2}), \quad (4.7)
\]

and for \(j = 1, 2, \ldots, n\)

\[
\max_{0 \leq k \leq N} \|\phi_j(t_k^j) - y_k^j\| \leq C_j \sum_{l=1}^{j} \left(h^{l+1} + h^{l \wedge (\alpha \wedge \gamma)} + h^{\beta_1 \wedge l} + h^{\beta_2 \wedge l}\right), \quad (4.8)
\]

where \(\phi_j = \phi_j(t)\) is the solution of \(1.1\) on the interval \([j\tau, (j + 1)\tau].\) In particular, if \(\gamma = 1\) then for \(j = 1, 2, \ldots, n\)

\[
\max_{0 \leq k \leq N} \|\phi_j(t_k^j) - y_k^j\| \leq jC_j(h^{1/2} + h^\alpha + h^{\beta_1} + h^{\beta_2}). \quad (4.9)
\]
Proof. For \( t \in [0, \tau] \) we approximate the solution \( z \) of (1.1) by the Euler method

\[
\begin{align*}
y_0^0 &= \eta, \\
y_{k+1}^0 &= y_k^0 + h \cdot g_0(t_k^0, y_k^0), \quad k = 0, 1, \ldots, N - 1,
\end{align*}
\]

where \( g_0(t, y) = f(t, y, \eta) \). Applying Lemmas 3.2, 3.1 and 6.2 for the solution \( z \) of (1.1) by the auxiliary Euler scheme

\[
\begin{align*}
y_0^1 &= y_0^0 = y_N, \\
y_{k+1}^1 &= y_k^1 + h \cdot g_1(t_k^1, y_k^1), \quad k = 0, 1, \ldots, N - 1,
\end{align*}
\]

and from (4.12) we get

\[
\begin{align*}
\max_{0 \leq k \leq N} \| \phi_0(t_k^0) - y_k^0 \| &\leq \mathcal{C}_2(1 + \| \eta \|)(h^{\alpha + \gamma} + h^{\beta_1} + h^{\beta_2}).
\end{align*}
\]

Therefore \( (4.7) \) is proved.

Starting from the interval \([\tau, 2\tau]\) we proceed by induction. Namely, for \( j = 1 \) and \( t \in [\tau, 2\tau] \) the DDE (1.1) reduces to the following ODE

\[
z'(t) = g_1(t, z(t)), \quad t \in [\tau, 2\tau],
\]

with the initial value \( z(\tau) = \phi_0(\tau) = \phi_0(t_N^0) \) and \( g_1(t, y, \phi_0(t - \tau)) \). We approximate the solution \( z \) of (1.13) by the auxiliary Euler scheme

\[
\begin{align*}
y_0^1 &= y_0 = y_N, \\
y_{k+1}^1 &= y_k + h \cdot g_1(t_k^1, y_k), \quad k = 0, 1, \ldots, N - 1,
\end{align*}
\]

and from (4.12) we get

\[
\begin{align*}
\max_{0 \leq k \leq N} \| \phi_1(t_k^1) - y_k^1 \| &\leq C_0(h^{\alpha + \gamma} + h^{\beta_1} + h^{\beta_2}).
\end{align*}
\]

Therefore, we have for \( k = 0, 1, \ldots, N \)

\[
\begin{align*}
\| \phi_1(t_k^1) - y_k^1 \| &\leq \| \phi_1(t_k^1) - y_k^1 \| + \| y_k^1 - y_k^1 \| \leq C_0(h^{\alpha + \gamma} + h^{\beta_1} + h^{\beta_2}) + \| y_k^1 - y_k^1 \|
\end{align*}
\]

and we need to estimate \( \| y_k^1 - y_k^1 \| \). Let us denote by

\[
e_k^1 := y_k^1 - y_k^1, \quad k = 0, 1, \ldots, N,
\]

where, by (1.14), \( e_0^1 = y_0^1 - y_0^0 = 0 \). From (1.13) and (2.5) we have for \( k = 0, 1, \ldots, N - 1 \) that

\[
e_{k+1}^1 = e_k^1 + hR_k^1 + hL_k^1,
\]

where

\[
R_k^1 = f(t_k^1, y_k^1, \phi_0(t_k^1)) - f(t_k, y_k, \phi_0(t_k^1)),
\]

and

\[
L_k^1 = f(t_k, y_k, \phi_0(t_k^1)) - f(t_k, y_k, \phi_0(t_k^0)).
\]

From (4.21) we obtain that

\[
\| e_{k+1}^1 - hL_k^1 \|^2 = \| e_k^1 + hR_k^1 \|^2,
\]

where

\[
\begin{align*}
\| e_{k+1}^1 - hL_k^1 \|^2 &= \| e_{k+1}^1 \|^2 - 2h\langle e_{k+1}^1, L_k^1 \rangle + h^2\| L_k^1 \|^2 \\
\| e_k^1 + hR_k^1 \|^2 &= \| e_k^1 \|^2 + 2h\langle e_k^1, R_k^1 \rangle + h^2\| R_k^1 \|^2.
\end{align*}
\]

Since \( h^2\| L_k^1 \|^2 \geq 0 \), we get

\[
\| e_{k+1}^1 - hL_k^1 \|^2 \geq \| e_{k+1}^1 \|^2 - 2h\langle e_{k+1}^1, L_k^1 \rangle.
\]
while from the assumption (F3) and from Lemma 3.1 we get
\[
\langle e_k^1, R_k^1 \rangle = \langle \tilde{y}_k^1 - y_k^1, f(t_k^1, \tilde{y}_k^1, \phi_0(t_k^0)) - f(t_k^1, y_k^1, \phi_0(t_k^0)) \rangle
\]
\[
\leq H \left(1 + \|\phi_0(t_k^0)\| \right) \|e_k^1\| + L \left(1 + \|\phi_0(t_k^0)\| \right) \|e_k^1\| + L \left(1 + \|\phi_0(t_k^0)\| \right) \|e_k^1\|
\]
\[
\leq H + \left(1 + \sup_{0 \leq t \leq T} \|\phi_0(t)\| \right) \|e_k^1\| + L \left(1 + \|\phi_0(t_k^0)\| \right) \|e_k^1\| + L \left(1 + \|\phi_0(t_k^0)\| \right) \|e_k^1\|
\]
\[
\leq H + (1 + K_0) \|e_k^1\|^2.
\]

The fact above together with (4.25) implies
\[
\|e_k^1 + hR_k^1\|^2 \leq \|e_k^1\|^2 + 2hH + (1 + K_0) \|e_k^1\|^2 + h^2\|R_k^1\|^2.
\]

Hence, by using (4.24), (4.26), and (4.27) we obtain
\[
\|e_k^1 + hR_k^1\|^2 \leq (1 + 2hH + (1 + K_0)) \|e_k^1\|^2 + h^2\|R_k^1\|^2 + 2h \|e_{k+1}^1, \mathcal{L}_k^1\|.
\]

Moreover, by the Cauchy-Schwarz inequality
\[
\langle e_k^1, \mathcal{L}_k^1 \rangle \leq \langle e_k^1, e_k^1 \rangle \leq \frac{1}{2} \|e_k^1\|^2 + \|\mathcal{L}_k^1\|^2.
\]

Hence, combining (4.28) with (4.29) we have
\[
\|e_k^1\|^2 \leq \left(1 + \hat{C}_1 h\right) \|e_k^1\|^2 + h^2\|R_k^1\|^2 + h\|e_{k+1}^1\|^2 + h\|\mathcal{L}_k^1\|^2
\]
where \( \hat{C}_1 := 2H + (1 + K_0) \). For \( N \geq 2l\tau \) we have that \( h \in (0, 1/2) \), and by the Fact 6.6 we obtain
\[
\|e_{k+1}^1\|^2 \leq (1 + 2h)(1 + \hat{C}_1 h) \|e_k^1\|^2 + 2h^2\|R_k^1\|^2 + 2h\|\mathcal{L}_k^1\|^2
\]
for \( k = 0, 1, \ldots, N - 1 \). Recall the well-known facts that for all \( k \in (0, 1) \) and \( x, y \geq 0 \) it holds
\[
x^y \leq 1 + x.
\]
and
\[
(x + y)^y \leq x^y + y^y.
\]

Thereby, from the assumption (F4) and by Lemma 3.1 we have the following estimate
\[
\|R_k^1\|^2 = \|f(t_k^1, \tilde{y}_k^1, \phi_0(t_k^0)) - f(t_k^1, y_k^1, \phi_0(t_k^0))\|^2
\]
\[
\leq L \left[1 + 2\|\phi_0(t_k^0)\| \right] \|e_k^1\|^2 + 2h^2\|R_k^1\|^2 + 2h\|\mathcal{L}_k^1\|^2
\]
\[
\leq 2L \left(1 + 2\sup_{0 \leq t \leq T} \|\phi_0(t)\| \right) \|e_k^1\|^2 + 2L(1 + 2K_0) \|e_k^1\|. \tag{4.33}
\]

while by Lemma 4.1 and (4.12) we obtain
\[
\|\mathcal{L}_k^1\|^2 = \|f(t_k^1, \tilde{y}_k^1, \phi_0(t_k^0)) - f(t_k^1, y_k^1, y_k^0)\|^2
\]
\[
\leq L \|\phi_0(t_k^0)\| \langle e_k^1, e_k^1 \rangle \leq \hat{C}_1 ^2 \|\phi_0(t_k^0)\| \langle e_k^1, e_k^1 \rangle \leq \hat{C}_1 ^2 \|\phi_0(t_k^0)\| \|e_k^1\|^2 + h^2\|\mathcal{L}_k^1\|^2
\]
where \( \hat{C}_1 := 2H + (1 + K_0) \). Hence, by (4.31), (4.33) and (4.34)
\[
\|e_{k+1}^1\|^2 \leq (1 + \tilde{D}_1 h) \|e_k^1\|^2 + D_1 h^2 + M_1 h^2 (\|\phi_0(t_k^0)\| + h^2) \|e_{k+1}^1\|^2
\]
\[
\|e_k^1\|^2 \leq \left(\hat{C}_1 \right)^2 \left[h + h^2 (\|\phi_0(t_k^0)\| + h^2) \|e_{k+1}^1\|^2 + h^2\|\mathcal{L}_k^1\|^2
\]
for all \( h \in (0, \frac{1}{2}) \), \( k = 0, 1, \ldots, N - 1 \). We stress that \( \tilde{D}_1, D_1, M_1 \) do not depend on \( N \). Since \(\|e_0^1\|^2 = 0\), from the discrete Gronwall’s inequality we get for all \( k = 0, 1, \ldots, N \)
\[
\|e_k^1\|^2 \leq \left(\hat{C}_1 \right)^2 \left[h + h^2 (\|\phi_0(t_k^0)\| + h^2) \|e_{k+1}^1\|^2 + h^2\|\mathcal{L}_k^1\|^2
\]
where \( (\overline{C}_1)^2 = \frac{D_1 r - 1}{D_x} \cdot \max \{D_1, M_1 \} \). Hence

\[
\max_{0 \leq k \leq N} \|e_k^j\| \leq \overline{C}_1 \left( h^{\frac{j}{2}} + h^{\gamma(\alpha \wedge \gamma)} + h^{\beta_1 \gamma} + h^{\beta_2 \gamma} \right).
\]

From (4.19) and (4.36) we get

\[
\|\phi_1(t^j_k) - y^j_k\| \leq C_0(h^{\alpha \wedge \gamma} + h^{\beta_1} + h^{\beta_2}) + \|e_k^j\| \leq C_1 \left( h^{\frac{j}{2}} + h^{\gamma(\alpha \wedge \gamma)} + h^{\beta_1 \gamma} + h^{\beta_2 \gamma} \right),
\]

for \( k = 0, 1, \ldots, N \) where \( C_1 \) does not depend on \( N \).

Let us now assume that there exist \( j \in \{1, 2, \ldots, n-1\} \) and \( C_j \), which does not depend on \( N \), such that for all \( k = 0, 1, \ldots, N \) it holds

\[
\|\phi_j(t^j_k) - y^j_k\| \leq C_j \sum_{l=1}^{j} \left( h^{\frac{j}{2} l^{j-1} - 1} + h^{\gamma(l(\alpha \wedge \gamma))} + h^{\beta_1 l^{j-1} \gamma} + h^{\beta_2 \gamma} \right).
\]

(For \( j = 1 \) the statement has already been proven.) For \( t \in [(j + 1)\tau, (j + 2)\tau] \) we consider the following ODE

\[
z'(t) = g_{j+1}(t, z(t)), \quad t \in [(j + 1)\tau, (j + 2)\tau],
\]

with the initial value \( z((j + 1)\tau) = \phi_j((j + 1)\tau) = \phi_j(t^j_N) = \phi_j(t^j_{N}) \) and \( g_{j+1}(t, y) = f(t, y, \phi_j(t - \tau)) \). We approximate (4.39) by the following auxiliary Euler scheme

\[
y^j_{k+1} = y^j_k + h \cdot g_{j+1}(t^j_k, y^j_k), \quad k = 0, 1, \ldots, N - 1.
\]

Hence, from the induction assumption (4.38)

\[
\|z((j + 1)\tau) - y^j_{N+1}\| = \|\phi_j(t^j_N) - y^j_N\| \leq C_j \sum_{l=1}^{j} \left( h^{\frac{j}{2} l^{j-1} - 1} + h^{\gamma(l(\alpha \wedge \gamma))} + h^{\beta_1 l^{j-1} \gamma} + h^{\beta_2 \gamma} \right).
\]

Applying Lemmas 3.1, 3.2 and 6.2 for \( \xi := \phi_j(t^j_N) \), \( g := g_{j+1} \), \( [a, b] := [(j + 1)\tau, (j + 2)\tau] \),

\[
\Delta := C_j \sum_{l=1}^{j} \left( h^{\frac{j}{2} l^{j-1} - 1} + h^{\gamma(l(\alpha \wedge \gamma))} + h^{\beta_1 \gamma l^{j-1}} + h^{\beta_2 \gamma l^{j-1}} \right),
\]

we obtain

\[
\|\phi_j(t^j_N)\| \leq K_j,
\]

and

\[
\max_{0 \leq k \leq N} \|\phi_{j+1}(t^j_{k+1}) - y^j_k\| \leq \overline{C}_2 (1 + \|\phi_j(t^j_N)\|)/(\Delta + h^{\alpha \wedge \gamma} + h^{\beta_1} + h^{\beta_2})
\]

\[
\leq \overline{C}_2 \left( \sum_{l=1}^{j} h^{\frac{j}{2} l^{j-1} - 1} + h^{\gamma(l(\alpha \wedge \gamma))} + h^{\beta_1 \gamma l^{j-1}} + h^{\beta_2 \gamma l^{j-1}} \right).
\]

Hence, we have for \( k = 0, 1, \ldots, N \)

\[
\|\phi_{j+1}(t^j_{k+1}) - y^j_k\| \leq \|\phi_{j+1}(t^j_{k+1}) - y^j_k\| + \|e^j_k\|
\]

\[
\leq \overline{C}_2 \left( \sum_{l=1}^{j} h^{\frac{j}{2} l^{j-1} - 1} + h^{\gamma(l(\alpha \wedge \gamma))} + h^{\beta_1 \gamma l^{j-1}} + h^{\beta_2 \gamma l^{j-1}} \right) + \|e^j_k\|,
\]

where

\[
e^j_k := y^j_{k+1} - y^j_k, \quad k = 0, 1, \ldots, N,
\]

and \( e^j_0 = \tilde{y}^j_{N+1} - y^j_0 = 0 \) by (4.40). Using analogous arguments as in (4.21)-(4.34) we obtain for \( h \in (0, 1/2) \) and \( k = 0, 1, \ldots, N - 1 \) that

\[
\|e^j_k\|^2 \leq (1 + 2h)(1 + \overline{C}_{j+1}h)\|e^j_k\|^2 + 2h^2 \|R^j_{k+1}\|^2 + 2h\|L^j_k\|^2
\]

(4.44)
where $\hat{C}_{j+1}$ does not depend on $N$. Moreover
\[
\|R_{k}^{j+1}\| = \|f(t_{k}^{j+1}, y_{k}^{j+1}, \phi_{j}(t_{k}^{j})) - f(t_{k}^{j+1}, y_{k}^{j+1}, \phi_{j}(t_{k}^{j}))\| \leq 2L(1 + 2K_{j})(1 + \|e_{k}^{j+1}\|), \quad \text{(4.45)}
\]
and, in particular by using (4.38) and Lemma 4.1, we get
\[
\|L_{k}^{j+1}\| = \|f(t_{k}^{j+1}, y_{k}^{j+1}, \phi_{j}(t_{k}^{j})) - f(t_{k}^{j+1}, y_{k}^{j+1}, \phi_{k}(t_{k}^{j}))\| \leq L(1 + 2\|y_{k}^{j+1}\|)\|\phi_{j}(t_{k}^{j}) - \phi_{k}(t_{k}^{j})\| = L(1 + \|y_{k}^{j+1}\|)\|\phi_{j}(t_{k}^{j}) - \phi_{k}(t_{k}^{j})\| \leq \hat{C}_{j+1} \sum_{l=1}^{j} \left( h^{2j + 1} + h^{\gamma l + 1}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right), \quad \text{(4.46)}
\]
where $\hat{C}_{j+1} := L(1 + 2\hat{K}_{j+1})C_{j}^{\gamma}$. Hence, from (4.41), (4.45), and (4.46) we have for $h \in (0, \frac{1}{T})$ and $k = 0, 1, \ldots, N$
\[
\|e_{k}^{j+1}\|^{2} \leq (1 + \bar{D}_{j+1}h)\|e_{k}^{j+1}\|^{2} + D_{j+1}h^{2} + M_{j+1}h \left\{ \sum_{l=1}^{j} \left( h^{2j + 1} + h^{\gamma l + 1}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right) \right\}^{2}
\]
where $D_{j+1}$, $M_{j+1}$, $\bar{D}_{j+1}$ do not depend on $N$. Since $\|e_{0}^{j+1}\|^{2} = 0$, by the discrete Gronwall’s lemma we get that for all $k = 0, 1, \ldots, N$
\[
\|e_{k}^{j+1}\|^{2} \leq \left( \frac{\bar{C}_{j+1}}{C_{j+1}} \right)^{2} \left\{ h + \left[ \sum_{l=1}^{j} \left( h^{2j + 1} + h^{\gamma l + 1}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right) \right] \right\}^{2}
\]
where $\left( \frac{\bar{C}_{j+1}}{C_{j+1}} \right)^{2} = \frac{\bar{D}_{j+1}h - 1}{D_{j+1}} \cdot \max\{D_{j+1}, M_{j+1}\}$. Thus
\[
\max_{0 \leq k \leq N} \|e_{k}^{j+1}\| \leq \bar{C}_{j+1} \left[ h^{2j + 1} + \sum_{l=1}^{j} \left( h^{2j + 1} + h^{\gamma l + 1}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right) \right] \leq 2\bar{C}_{j+1} \sum_{l=1}^{j} \left( h^{2j + 1} + h^{\gamma l + 1}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right). \quad \text{(4.47)}
\]
Therefore, from (4.43) and (4.47) we get for $k = 0, 1, \ldots, N$
\[
\|\phi_{j+1}(t_{k}^{j+1}) - \phi_{j}(t_{k}^{j})\| \leq C_{j+1} \sum_{l=1}^{j+1} \left( h^{2l - 1} + h^{\gamma l}(\alpha + \gamma) + h^{b_{1}^{+}l + 1} + h^{b_{2}^{+}l + 1} \right),
\]
where $C_{j+1}$ does not depend on $N$. This ends the proof. \qed

**Remark 4.3.** Instead of (F4) we can consider the following general assumption:

(F4*) There exist $L \geq 0$, $p, q \in \mathbb{Z}_{+}$, $\alpha, \beta_{i}, \gamma_{j} \in (0, 1]$ for $i = 1, \ldots, p$, $j = 1, \ldots, q$ such that for all $t_{1}, t_{2} \in [0, +\infty)$, $y_{1}, y_{2}, z_{1}, z_{2} \in \mathbb{R}^{d}$
\[
\|f(t_{1}, y_{1}, z_{1}) - f(t_{2}, y_{2}, z_{2})\| \leq L \left( (1 + \|y_{1}\| + \|y_{2}\|) \cdot (1 + \|z_{1}\| + \|z_{2}\|) \cdot |t_{1} - t_{2}|^{\alpha} + (1 + \|z_{1}\| + \|z_{2}\|) \sum_{i=1}^{p} \|y_{1} - y_{2}\|^{\beta_{i}} + (1 + \|y_{1}\| + \|y_{2}\|) \sum_{j=1}^{q} \|z_{1} - z_{2}\|^{\gamma_{j}} \right).
\]
Because of the notational simplicity we considered only the case when (F4) is assumed. However, we stress that the general case, when (F4*) is assumed, can be treated by using the same proof technique as used in the proof of Theorem 4.2.
5. Numerical experiments

We have conducted the numerical experiments with the Python implementations of the method described in [24] and [25]. The numerical experiments were carried on using Intel® Xeon® CPU E5-2650 v4 @ 2.20GHz. The exact solutions for the analyzed problems are not known so we proceed as follows to estimate the empirical rate of convergence. Theorem 4.2 provides the theoretical convergence rate of the Euler method, hence, the approximation computed by the Euler method on the dense mesh will be used as the referential solution (provides the theoretical convergence rate of the Euler method, hence, the approximation known so we proceeded as follows to estimate the empirical rate of convergence. Theorem 4.2 = 1

\[ A(F1)-(F4), \text{ see Fact 6.4. For the numerical experiment we take following values of parameters } \]

\[ A, B, C, D, \bar{\rho}, \gamma, \tau, z \]

are presented in Figure 3. The test results regarding the convergence rate for all examples are always close to one. It can indicate that in the considered case lower bounds of the error are not sharp.

5.1. Metal phase change model. Firstly, we consider a modified model describing a phase change of metallic materials from [24] Chapter 3.3, see also [11 14 23 24 27]. It can be described by delay differential equation due to delay in the response to the change in processing conditions. We consider the following case

\[ f(t, y, z) = A - B \cdot \text{sgn}(y) \cdot |y| - C \cdot \text{sgn}(y) \cdot |y|^\rho \cdot |z|^\gamma + D \cdot y \cdot |z|^\gamma, \]  

(5.1)

where \( \rho, \gamma \in (0, 1], A, B, C > 0, \) and \( D \in \mathbb{R}. \) Note that the function (5.1) satisfies assumptions (F1)-(F4), see Fact 6.4. For the numerical experiment we take following values of parameters

\[ A = 1.7137, B = 0.7769, C = 0.5895, D = -0.82615, \rho = 0.973, \gamma = 0.714, \tau = 9.2603, \]

\( z_0 = 0.05854, t_0 = 0, \) and \( n = 5. \) The approximated solution of (5.1) is presented in Figure 1. Note that the solution graph is similar to the solutions from [11 24 27]. The test results for this case are given in Figure 1. The computation time is linear with respect to the number of computation points and for \( N \cdot n = 90 \) equals 0.0023s and for \( N \cdot n = 368640 \) equals 5.775s.

We also tested equation (5.1) with different values of the parameters, exemplary solutions are presented in Figure 3. The test results regarding the convergence rate for all examples are always close to one.

We also consider a similar case with subtle changes, i.e.

\[ f(t, y, z) = A - B \cdot \text{sgn}(y) \cdot |y| - C \cdot \text{sgn}(y) \cdot |y|^\rho \cdot |z| + D \cdot y \cdot z, \]  

(5.2)

where \( \rho, \gamma \in (0, 1], A, B, C > 0, \) and \( D \in \mathbb{R}. \) This function (5.2) also satisfies assumptions (F1)-(F4), see Fact 6.5. For the numerical experiment we take the same values of parameters \( A, B, C, D, \rho, \gamma, \tau, z_0, t_0, n \) like in the (5.1) case. The approximated solution of (5.2) is presented in Figure 1 and the test results for this case in Figure 2. As before the computation time behaves linear with respect to the number of computation points and for \( N \cdot n = 90 \) equals 0.002s and for \( N \cdot n = 368640 \) equals 5.3525s.

5.2. Model of releasing mature cells into the blood stream. Another example is modeling the release of mature cells into the blood stream, so called Mackey–Glass equation (see Example 1.1.7, page 7 in [7] and originally introduced in [21]),

\[ z'(t) = \frac{bz(t - \tau)}{1 + [z(t - \tau)]^m} - az(t) \]  

(5.3)

with parameters \( a = 0.1, b = 0.2, m = 10, \tau = 20, \) and \( n = 500. \) One can see that the equation (5.3) does not fulfill the assumptions (F1)-(F4), still we present the example as a
5.3. Epidemiology model. Widely known usage of delay differential equations is a Susceptible-Infectious-Recovered (SIR) model and its modifications. This model family describes spread classical test problem where the method behaves properly. The solution of (5.3) is presented in Figure 4 and the test results for this case in Figure 5. As before the computation time is linear with respect to the number of computation points and for $N \cdot n = 900$ equals 0.01793s and for $N \cdot n = 3686400$ equals 25.6793s.
(i) An approximated solution of (5.1) for 
\[ A = 3.27, \quad B = 5.62, \quad C = 9.89, \quad D = -7.31, \quad \varrho = 0.88, \quad \gamma = 0.89, \quad \tau = 1.03, \quad z_0 = 1, \quad t_0 = 0, \quad n = 5. \]

(ii) An approximated solution of (5.1) for 
\[ A = 5, \quad B = 6.62, \quad C = 0.52, \quad D = 4, \quad \varrho = 0.32, \quad \gamma = 0.33, \quad \tau = 8.55, \quad z_0 = 0.22, \quad t_0 = 0, \quad n = 5. \]

(iii) An approximated solution of (5.1) for 
\[ A = 5.16, \quad B = 0.42, \quad C = 3.61, \quad D = -6.74, \quad \varrho = 0.99, \quad \gamma = 0.11, \quad \tau = 5.69, \quad z_0 = 0.17, \quad t_0 = 0, \quad n = 5. \]

(iv) An approximated solution of (5.1) for 
\[ A = 6.75, \quad B = 2.79, \quad C = 4.7, \quad D = -0.01, \quad \varrho = 0.86, \quad \gamma = 0.02, \quad \tau = 1.58, \quad z_0 = 0.31, \quad t_0 = 0, \quad n = 5. \]

**Figure 3.** Other exemplary reference solutions of the metal phase change model (5.1)

**Figure 4.** An approximated solution computed on the dense mesh, treated as the reference solution of the model of releasing mature cells into the blood stream (5.3).
of the disease. By [22], we consider the following multidimensional delay differential equation

\[
\begin{aligned}
S'(t) &= -\beta(1-u)\frac{S(t)I_a(t)}{N_{\text{pop}}} \\
I'_a(t) &= \beta \epsilon(1-u)a(t)S(t-\tau_1)I_a(t-\tau_1) - \alpha I_a(t) - (1-\alpha)(\mu_s + \eta_s)I_a(t) \\
I'_b(t) &= \beta(1-\epsilon)(1-u)\frac{S(t-\tau_1)I_a(t-\tau_1)}{N_{\text{pop}}} - \eta_a I_b(t) \\
F'_b(t) &= \alpha \gamma_b I_a(t - \tau_2) - (\mu_b + \tau_5)F_b(t) \\
F'_g(t) &= \alpha \gamma_g I_a(t - \tau_2) - (\mu_g + \tau_4)F_g(t) \\
F'_c(t) &= \alpha \gamma_c I_a(t - \tau_2) - (\mu_c + \tau_c)F_c(t) \\
R'(t) &= \eta_a(1-\alpha)I_a(t - \tau_3) + \eta_b I_b(t - \tau_3) + \eta_c F_c(t - \tau_4) + r_F g(t - \tau_4) + r_c F_c(t - \tau_4) \\
M'(t) &= \mu_a(1-\alpha)I_a(t - \tau_3) + \mu_b F_b(t - \tau_4) + \mu_g F_g(t - \tau_4) + \mu_c F_c(t - \tau_4)
\end{aligned}
\]

(5.4)

with values of parameters \( \beta = 0.4517, \epsilon = 0.794, \gamma_b = 0.8, \gamma_g = 0.15, \gamma_c = 0.05, \alpha = 0.06, \eta_a = \frac{1}{2}, \eta_b = \frac{0.8}{27}, \mu_s = \frac{0.01}{27}, \mu_b = 0, \mu_g = 0, \mu_c = \frac{0.4}{13.5}, r_b = \frac{1}{13.5}, r_g = \frac{1}{13.5}, r_c = \frac{0.6}{13.5}, \tau_1 = 5.5, \tau_2 = 7.5, \tau_3 = 21, \tau_4 = 13.5, N_{\text{pop}} = 35280000, \) and

\[
u = \begin{cases} 
0.2, & t \in (0, 8], \\
0.3, & t \in (8, 18], \\
0.4, & t \in (18, 35], \\
0.8, & t \in (35, 240]. 
\end{cases}
\]

As initial value vector we take \( \eta = (35280000, 20, 0, 0, 0, 0, 0, 0, 0). \)

It can be noticed that in (5.4) we have to deal with 4 different values of delay. In order to compute the approximation we take one common delay \( \tau = 0.5 \) and we compute a solution with that value of time lag for \( n = 480 \) (which is equivalent to 240 days). Simultaneously we had to slightly change an algorithm to take proper delay values as arguments in the right-hand side function. A solution of SIR model can be analyzed in Figure 6 (without presenting values of \( S \) because of the big difference in order of magnitude) and the convergence rate is presented in Figure 7. The computation time is linear with respect to the number of computation points and for \( N \cdot n = 8640 \) equals 0.586s and for \( N \cdot n = 4423680 \) equals 85.902s.

Figure 5. \( -\log_{10}(\text{err}) \) vs. \( \log_{10}N \) for the model of releasing mature cells into the blood stream [5,3].
Figure 6. An approximated solution computed on the dense mesh treated as a reference solution of the SIR model \((5.4)\), values of \(S\) can be computed as a difference between \(N_{pop}\) and values presented on the figure.

Figure 7. \(-\log_{10}(\text{err})\) vs. \(\log_{10}N\) for the SIR model \((5.4)\).

6. Appendix - analytical properties of solutions of ODEs and its Euler approximation

This section consist of some auxiliary results for solutions of ordinary differential equations and its Euler approximation that are used in the paper, especially for proving the main Theorem 4.2.
Lemma 6.1. Let us consider the following ODE

\[ z'(t) = g(t, z(t)), \quad t \in [a, b], \quad z(a) = \xi, \]  

(6.1)

where \(-\infty < a < b < +\infty, \; \xi \in \mathbb{R}^d\) and \(g : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d\) satisfies the following conditions

(G1) \(g \in C([a, b] \times \mathbb{R}^d, \mathbb{R}^d)\).

(G2) There exists \(K \in (0, +\infty)\) such that for all \((t, y) \in [a, b] \times \mathbb{R}^d\)

\[ \|g(t, y)\| \leq K(1 + \|y\|). \]

(G3) There exists \(H \in \mathbb{R}\) such that for all \(t \in [a, b], \; y_1, y_2 \in \mathbb{R}^d\)

\[ \langle y_1 - y_2, g(t, y_1) - g(t, y_2) \rangle \leq H\|y_1 - y_2\|^2. \]

Then we have what follows.

(i) The equation (6.1) has a unique solution \(z \in C^1([a, b]; \mathbb{R}^d)\) such that

\[ \sup_{t \in [a, b]} \|z(t)\| \leq (\|\xi\| + K(b - a))e^{K(b - a)}, \]

(6.2)

and for all \(t, s \in [a, b]\)

\[ \|z(t) - z(s)\| \leq \bar{K}|t - s|, \]

(6.3)

where \(\bar{K} = K\left(1 + (\|\xi\| + K(b - a))e^{K(b - a)}\right)\).

(ii) Let us consider \(u : [a, b] \to \mathbb{R}^d\) the solution of

\[ u'(t) = g(t, z(t)), \quad t \in [a, b], \quad u(a) = \zeta, \]

(6.4)

with \(\zeta \in \mathbb{R}^d\). Then for all \(t \in [a, b]\) we have

\[ \|u(t) - z(t)\| \leq e^{H(t - a)}\|\xi - \zeta\|. \]

(6.5)

Proof. Since the right-hand side function \(g\) is continuous and it is of at most linear growth, Peano’s theorem guarantees existence of the \(C^1\)-solution (e.g. see Theorem 70.4, page 292 in [10]). Now, we show that the uniqueness follows from the one-sided Lipschitz assumption (G3). Namely, let us assume that (6.1) has two solutions \(z = z(t)\) and \(x = x(t)\) with the same initial-value \(z(a) = x(a) = \xi\). By (G3) we have for all \(t \in [a, b]\) that

\[
\frac{d}{dt}\|z(t) - x(t)\|^2 = 2 \langle z(t) - x(t), g(t, z(t)) - g(t, x(t)) \rangle
\]

\[
\leq 2H\|z(t) - x(t)\|^2 \leq 2H_+\|z(t) - x(t)\|^2.
\]

(6.6)

Let us consider the \(C^1\)-function \([a, b] \ni t \to \varphi(t) = \|z(t) - x(t)\|^2 \in [0, +\infty)\), where \(\varphi(a) = 0\). Integrating two sides of the preceding inequality we get

\[ \varphi(t) \leq 2H_+ \int_a^t \varphi(s)ds. \]

Hence, from the Gronwall’s lemma we obtain that \(\varphi(t) = 0\) for all \(t \in [a, b]\), which in turns implies that \(z(t) = x(t)\) for all \(t \in [a, b]\).

Note that, by the assumption (G2), for all \(t \in [a, b]\) it holds

\[ \|z(t)\| \leq \|\xi\| + \int_a^t \|g(s, z(s))\|ds \leq \|\xi\| + K(b - a) + K \int_a^t \|z(s)\|ds. \]

(6.7)

Again by the Gronwall’s lemma we obtain for all \(t \in [a, b]\) that

\[ \|z(t)\| \leq (\|\xi\| + K(b - a))e^{K(t - a)}. \]

(6.8)

This implies (6.2).
By (G2) and (6.2) we obtain for all $t, s \in [a, b]$

$$\|z(t) - z(s)\| = \|z(t \lor s) - z(t \land s)\| \leq \int_{t \lor s}^{t \land s} \|g(u, z(u))\| du$$

$$\leq K(1 + \sup_{t \in [a, b]} \|z(t)\|)(t \land s - t \lor s) \leq \tilde{K}|t - s|,$$  

(6.9)

where $\tilde{K} = K\left(1 + (\|\xi\| + K(b - a))e^{K(b-a)}\right)$. Hence, the proof of (6.3) is completed.

As in (6.6) we get that for all $t \in [a, b]$

$$\frac{d}{dt}\|z(t) - u(t)\|^2 \leq 2H_+\|z(t) - u(t)\|^2$$  

(6.10)

which implies that

$$\|z(t) - u(t)\|^2 \leq \|\xi - \zeta\|^2 + 2H_+ \int_a^t \|z(s) - u(s)\|^2 ds.$$  

(6.11)

By using Gronwall’s lemma we get (6.5). □

**Lemma 6.2.** Let us consider the following ordinary differential equation

$$z'(t) = g(t, z(t)), \quad t \in [a, b], \quad z(a) = \xi,$$  

(6.12)

where $-\infty < a < b < +\infty$, $\eta \in \mathbb{R}^d$ and $g : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following conditions:

(G1) $g \in C([a, b] \times \mathbb{R}^d, \mathbb{R}^d)$.

(G2) There exists $K \in (0, +\infty)$ such that for all $(t, y) \in [a, b] \times \mathbb{R}^d$

$$\|g(t, y)\| \leq K(1 + \|y\|).$$

(G3) There exists $H \in \mathbb{R}$ such that for all $t \in [a, b], y_1, y_2 \in \mathbb{R}^d$

$$\langle y_1 - y_2, g(t, y_1) - g(t, y_2) \rangle \leq H\|y_1 - y_2\|^2.$$  

(G4) There exist $L \geq 0$, $p \in \mathbb{N}$, $\alpha, \beta_1, \beta_2, \ldots, \beta_p \in (0, 1]$, such that for all $t_1, t_2 \in [a, b], y_1, y_2 \in \mathbb{R}^d$

$$\|g(t_1, y_1) - g(t_2, y_2)\| \leq L\left((1 + \|y_1\| + \|y_2\|) \cdot |t_1 - t_2|^\alpha + \sum_{i=1}^p \|y_1 - y_2\|^{\beta_i}\right).$$

Let us consider the Euler method based on equidistant discretization. Namely, for $n \in \mathbb{Z}_+, \Delta \in [0, +\infty)$ we set $h = (b - a)/n$, $t_k = a + kh, k = 0, 1, \ldots, n$, and let $y_0 \in \mathbb{R}^d$ be any vector from the ball $B(\xi, \Delta) = \{y \in \mathbb{R}^d : \|\xi - y\| \leq \Delta\}$. We take

$$y_{k+1} = y_k + h \cdot g(t_k, y_k), \quad k = 0, 1, \ldots, n - 1.$$  

(6.13)

Then the following holds

(i) There exists $\tilde{C}_1 = \tilde{C}_1(b - a, K) \in (0, +\infty)$ such that for all $n \in \mathbb{Z}_+, \Delta \in [0, +\infty)$, $\xi \in \mathbb{R}^d, y_0 \in B(\xi, \Delta)$ we have

$$\max_{0 \leq k \leq n} \|y_k\| \leq \tilde{C}_1(1 + \|\xi\|)(1 + \Delta).$$  

(6.14)

(ii) There exists $\tilde{C}_2 = \tilde{C}_2(b - a, \Delta, K, H, p, \alpha, \beta_1, \ldots, \beta_p) \in (0, +\infty)$ such that for all $n \in \mathbb{Z}_+, \Delta \in [0, +\infty)$, $\xi \in \mathbb{R}^d, y_0 \in B(\xi, \Delta)$ we have

$$\max_{0 \leq k \leq n} \|z(t_k) - y_k\| \leq \tilde{C}_2(1 + \|\xi\|)(\Delta + h^\alpha + \sum_{i=1}^p h^{\beta_i}).$$  

(6.15)
**Proof.** Fix \( n \in \mathbb{Z}_+, \Delta \in [0, +\infty), \xi \in \mathbb{R}^d \) and \( y_0 \in B(\xi, \Delta) \). By the assumption (G2) we have that for all \( k = 0, 1, \ldots, n - 1 \)

\[
\|y_{k+1}\| \leq \|y_k\| + h\|g(t_k, y_k)\| \leq (1 + hK)\|y_k\| + hK
\]

and, since \( y_0 \in B(\xi, \Delta), \|y_0\| \leq \Delta + \|\xi\| \). Hence, by the discrete version of Gronwall’s lemma we get that for all \( k = 0, 1, \ldots, n \) that

\[
\|y_k\| \leq e^{K(b-a)}(\Delta + \|\xi\|) + e^{K(b-a)} - 1 \leq e^{K(b-a)}(1 + \|\xi\|)(1 + \Delta).
\]

This proves (6.14) with \( \tilde{C}_1 = e^{K(b-a)}. \)

We now prove (6.15). For \( k = 0, 1, \ldots, n - 1 \) we consider the following local ODE

\[
z_k'(t) = g(t, z_k(t)), \quad t \in [t_k, t_{k+1}], \quad z_k(t_k) = y_k.
\] (6.16)

By Lemma 6.1 there exists unique solution \( z_k : [t_k, t_{k+1}] \to \mathbb{R}^d \) of (6.16). From the assumption (G2) and by (6.14) we get for all \( t \in [t_k, t_{k+1}], \) \( k = 0, 1, \ldots, n - 1 \) that

\[
\|z_k(t)\| \leq \tilde{C}_1(1 + \|\xi\|)(1 + \Delta) + K(b - a) + K \int_{t_k}^t \|z_k(s)\| ds.
\]

The use of Gronwall’s lemma yields

\[
\max_{0 \leq k \leq n-1} \sup_{t \in [t_k, t_{k+1}]} \|z_k(t)\| \leq C_2(1 + \|\xi\|)(1 + \Delta),
\] (6.17)

where \( C_2 = e^{K(b-a)}(\tilde{C}_1 + K(b - a)) \). By (G2) and (6.17) we get for all \( t \in [t_k, t_{k+1}], \) \( k = 0, 1, \ldots, n - 1 \)

\[
\|z_k(t) - y_k\| \leq \int_{t_k}^t \|g(s, z_k(s))\| ds \leq hK \left( 1 + \sup_{t \in [t_k, t_{k+1}]} \|z_k(t)\| \right)
\]

\[
\leq hK \left( 1 + C_2(1 + \|\xi\|)(1 + \Delta) \right) \leq hC_3(1 + \|\xi\|)(1 + \Delta),
\] (6.18)

with \( C_3 = K(1 + C_2) \). From Lemma 6.1 (ii) we arrive at

\[
\|z(t_{k+1}) - y_{k+1}\| \leq \|z(t_{k+1}) - z_k(t_{k+1})\| + \|z_k(t_{k+1}) - y_{k+1}\|
\]

\[
\leq e^{hH+}\|z(t)-y_k\| + \|z_k(t_{k+1}) - y_{k+1}\|
\] (6.19)

for \( k = 0, 1, \ldots, n - 1 \). Using (G4), (6.14), (6.17) and (6.18) we get

\[
\|z_k(t_{k+1}) - y_{k+1}\| \leq \int_{t_k}^{t_{k+1}} \|g(s, z_k(s)) - g(t_k, y_k)\| ds
\]

\[
\leq L \int_{t_k}^{t_{k+1}} \left( 1 + \|z_k(s)\| + \|y_k\| \right) \cdot (s - t_k)^\alpha + \sum_{i=1}^{p} \|z_k(s) - y_k\|^{\beta_i} \right) ds
\]

\[
\leq Lh \left( 1 + (\tilde{C}_1 + C_2)(1 + \|\xi\|)(1 + \Delta) \right)^{\frac{1}{\alpha+1}} \cdot (1 + \sum_{i=1}^{p} h^{\beta_i} C_3^{\beta_i} (1 + \|\xi\|)^{\beta_i}(1 + \Delta)^{\beta_i})
\]

It is easy to see that for all \( x \in \mathbb{R}_+ \cup \{0\}, \varrho \in (0, 1], \)

\[
(1 + x)^\varrho \leq 2(1 + x).
\] (6.20)

Hence

\[
\|z_k(t_{k+1}) - y_{k+1}\| \leq C_4(1 + \|\xi\|)(1 + \Delta)h \left( h^\alpha + \sum_{i=1}^{p} h^{\beta_i} \right),
\] (6.21)
where $C_4 = L \max \left\{ \frac{1 + \tilde{C}_1 + C_2}{\alpha + 1}, 4 \max_{i \leq j \leq p} C_3^{\beta_j} \right\}$. From the above considerations we see that $C_4$ depends only on $\alpha, \beta_1, \ldots, \beta_p, p, L, K, b - a$. By (6.19) and (6.21) we get
\[
\|z(t_{k+1}) - y_{k+1}\| \leq e^{h\|H\|} \cdot \|z(t_k) - y_k\| + C_4 (1 + \|\xi\|)(1 + \Delta) h \left( h^\alpha + \sum_{i=1}^p h^{\beta_i} \right). \tag{6.22}
\]

Let us denote
\[
e_k = z(t_k) - y_k, \quad k = 0, 1, \ldots, n.
\]
Of course $\|e_0\| = \|\xi - y_0\| \leq \Delta$. Hence, we arrive at the following recursive inequality
\[
\|e_{k+1}\| \leq e^{h\|H\|} \|e_k\| + C_4 (1 + \|\xi\|)(1 + \Delta) h \left( h^\alpha + \sum_{i=1}^p h^{\beta_i} \right),
\]
for $k = 0, 1, \ldots, n - 1$. Applying the Gronwall’s lemma we get when $H > 0$
\[
\|e_k\| \leq e^{H \|H\| \Delta + e^{H \|H\| - 1} C_4 (1 + \|\xi\|)(1 + \Delta) h \left( h^\alpha + \sum_{i=1}^p h^{\beta_i} \right) \]
\[
\leq e^{H (b-a) \Delta + \frac{e^{H (b-a) - 1}}{H} C_4 (1 + \|\xi\|)(1 + \Delta) \left( h^\alpha + \sum_{i=1}^p h^{\beta_i} \right)} \tag{6.25}
\]
for $k = 0, 1, \ldots, n$, and when $H = 0$
\[
\|e_k\| \leq \Delta + C_4 (1 + \|\xi\|)(1 + \Delta)(b - a) \left( h^\alpha + \sum_{i=1}^p h^{\beta_i} \right) \tag{6.26}
\]
for $k = 0, 1, \ldots, n$. By elementary calculations we arrive at (6.15).

The following fact is well-known, see, for example, pages 3-4 in [15].

**Lemma 6.3.** For all $\varrho \in (0, 1]$ and $x, y \in \mathbb{R}$ it holds
\[
|x|^{\varrho} \leq 1 + |x|, \tag{6.27}
\]
and
\[
\left| |x|^{\varrho} - |y|^{\varrho} \right| \leq |x - y|^{\varrho}. \tag{6.28}
\]

Below we establish main properties of the functions (5.1) and (5.2).

**Fact 6.4.** Let $A, B, C \geq 0, D \in \mathbb{R}$, $\varrho, \gamma \in (0, 1]$ and consider $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follow
\[
f(t, y, z) = A - B \cdot \text{sgn}(y) \cdot |y| - C \cdot \text{sgn}(y) \cdot |y|^{\varrho} \cdot |z|^\gamma + D \cdot y \cdot |z|^\gamma.
\]
Then the function $f$ satisfies the assumptions (F1)-(F4).

**Proof.** Let us define $h_1(y) = - \text{sgn}(y) \cdot |y| = -y$, $h_2(y) = - \text{sgn}(y) \cdot |y|^{\varrho}$ for all $y \in \mathbb{R}$. Then we can write that
\[
f(t, y, z) = A + B \cdot h_1(y) + C \cdot h_2(y) \cdot |z|^\gamma + D \cdot y \cdot |z|^\gamma. \tag{6.29}
\]
Of course $h_1 \in C(\mathbb{R})$. Moreover,
\[
\lim_{y \to 0^-} h_2(y) = \lim_{y \to 0^-} (-y)^{\varrho} = 0 = h_2(0) = \lim_{y \to 0^+} h_2(y) = \lim_{y \to 0^+} (-1) \cdot y^{\varrho},
\]
therefore $h_2 \in C(\mathbb{R})$. In particular, this implies that $f \in C([0, \infty) \times \mathbb{R} \times \mathbb{R})$, so $f$ satisfies (F1).

\[\text{sgn}(x) = 1 \text{ if } x \geq 0 \text{ and } \text{sgn}(x) = -1 \text{ if } x < 0\]
By Lemma 6.3 we have \(|h_1(y)| = |y| \leq 1 + |y|, |h_2(y)| = |y|^\gamma \leq 1 + |y|\) for all \(y \in \mathbb{R}\). Hence, for \((t, y, z) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R}\)
\[
|f(t, y, z)| = A + B \cdot |h_1(y)| + C \cdot |h_2(y)| \cdot |z|^{\gamma} + D |y| |z|^{\gamma}
\leq A + B(1 + |y|) + C(1 + |y|)(1 + |z|) + D(1 + |y|)(1 + |z|)
\leq (A + B + C + D)(1 + |y|)(1 + |z|),
\]
and therefore \(f\) satisfies (F2).

For all \(t \geq 0, z, y_1, y_2 \in \mathbb{R}\)
\[
(y_1 - y_2) \left( \frac{f(t, y_1, z) - f(t, y_2, z)}{y_1 - y_2} \right) = B(y_1 - y_2) \left( h_1(y_1) - h_1(y_2) \right) + C \cdot |z|^{\gamma} (y_1 - y_2) \left( h_2(y_1) - h_2(y_2) \right) + D \cdot |z|^{\gamma} (y_1 - y_2)^2
\]
Since \(h_1, h_2\) are decreasing, it holds for all \(y_1, y_2 \in \mathbb{R}, i = 1, 2\) that \((y_1 - y_2) (h_1(y_1) - h_1(y_2)) \leq 0\). Moreover \(B, C, |z|^{\gamma} \geq 0\), hence
\[
(y_1 - y_2) \left( \frac{f(t, y_1, z) - f(t, y_2, z)}{y_1 - y_2} \right) \leq D \cdot |z|^{\gamma} (y_1 - y_2)^2 \leq |D| \cdot (1 + |z|) (y_1 - y_2)^2,
\]
and \(f\) satisfies (F3).

For all \(y_1, y_2 \in \mathbb{R}\) we have that \(|h_1(y_1) - h_1(y_2)| = |y_1 - y_2|\). We now justify that \(h_2\) satisfies for all \(y_1, y_2 \in \mathbb{R}\)
\[
|h_2(y_1) - h_2(y_2)| \leq 2 |y_1 - y_2|^\rho.
\]
When \(y_1, y_2 < 0\) or \(y_1, y_2 \geq 0\), by Lemma 6.3 we have
\[
|h_2(y_1) - h_2(y_2)| = ||y_1|^\rho - |y_2|^\rho| \leq |y_1 - y_2|^\rho \leq 2 |y_1 - y_2|^\rho.
\]
For the case when \(y_1 < 0, y_2 \geq 0\) (the case \(y_1 \geq 0, y_2 < 0\) is analogous) we have
\[
|h_2(y_1) - h_2(y_2)| = ||y_1| + y_2^\rho| = |y_1|^\rho + y_2^\rho \leq 2 |y_1 - y_2|^\rho,
\]
since \(-y_1 > 0, |y_1 - y_2| = y_2 + (-y_1) \geq y_2 \geq 0, |y_1 - y_2| = y_2 + (-y_1) \geq -y_1 = |y_1| \geq 0,\) and \([0, +\infty) \ni x \rightarrow x^\rho\) is increasing. Combining the facts above we obtain for all \(t_1, t_2 \geq 0, y_1, y_2, z_1, z_2 \in \mathbb{R}\)
\[
|f(t_1, y_1, z_1) - f(t_2, y_2, z_2)|
\leq B |h_1(y_1) - h_1(y_2)| + C \cdot |z_1|^\gamma \cdot |h_2(y_1) - h_2(y_2)| + D \cdot |y_1| |z_1|^\gamma - |y_2| |z_2|^\gamma
\leq B |y_1 - y_2| + C \cdot |z_1|^\gamma \cdot |h_2(y_1) - h_2(y_2)| + C \cdot |h_2(y_2)| \cdot |z_1|^\gamma - |z_2|^\gamma
+ D \cdot |y_2| \cdot |z_1|^\gamma - |z_2|^\gamma + D \cdot |z_1|^\gamma \cdot |y_1 - y_2|
\leq B |y_1 - y_2| + 2C (1 + |z_1|) |y_1 - y_2|^\rho + C (1 + |y_2|) \cdot |z_1 - z_2|^\gamma
+ |D| \cdot |y_2| \cdot |z_1 - z_2|^\gamma + |D| \cdot (1 + |z_1|) \cdot |y_1 - y_2|
\leq \max\{B + |D|, 2C, C + |D|\} \left[ (1 + |z_1| + |z_2|)(1 + |y_1| + |y_2|)|t_1 - t_2|
+ (1 + |z_1| + |z_2|) \cdot |y_1 - y_2| + (1 + |z_1| + |z_2|) \cdot |y_1 - y_2|^\rho + (1 + |y_1| + |y_2|) \cdot |z_1 - z_2|^\gamma \right].
\]
That ends the proof. \(\square\)

The proof of the fact below is analogous to the proof of Fact 6.4 and is omitted.

**Fact 6.5.** Let \(A, B, C \geq 0, D \in \mathbb{R}, \rho, \gamma \in (0, 1)\) and define a function \(f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) as follows
\[
f(t, y, z) = A - B \cdot \text{sgn}(y) \cdot |y| - C \cdot \text{sgn}(y) \cdot |y|^\rho \cdot |z| + D \cdot y \cdot z.
\]
Then the function \(f\) satisfies the assumptions (F1)-(F4).
The proof of the following fact is straightforward.

**Fact 6.6.** For all \( h \in (0, \frac{1}{2}) \) it holds

\[
0 < \frac{1}{1 - h} \leq 1 + 2h \leq 2.
\]

**Statements and Declarations**

The authors declare that they have no conflict of interest

The datasets generated during and analysed during the current study are available from the corresponding author on reasonable request.

**Acknowledgments**

This research was partly supported by the National Science Centre, Poland, under project 2017/25/B/ST1/00945.

We would like to thank two anonymous referees for their valuable comments and suggestions that helped to improve the presentation of the results and quality of this paper.

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