Gauge Consistent Wilson Renormalization
Group I: Abelian Case

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Abstract

A version of the Exact Renormalization Group Equation consistent with gauge symmetry is presented. A discussion of its regularization and renormalization is given. The relation with the Callan-Symanzik equation is clarified.

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1 Introduction

The problem of the perturbative renormalization of a quantum field theory (possibly with gauge symmetry) is highly non-trivial. Therefore even after its complete rigorous solution in the BPHZL formalism in the seventies [1, 2, 3] there has been a big effort in the literature in order to find alternative and simpler approaches. In particular in recent years the Wilsonian point of view [4, 5, 6, 7] has gained more and more popularity and nowadays is commonly regarded by many theorists not only as the better alternative to the traditional formulation of quantum field theory, but maybe also as the correct way of thinking about quantum field theory [8, 9, 10]. The reasons of this success are clear: first, the approach is very physically appealing; second, it is well founded at the mathematical level. In particular, in recent years, our understanding of the technical aspects of the formalism is much improved; for instance the perturbative implementation of symmetries has been clarified [11] and the relation with the BPHZ approach has been understood [12]. Nevertheless, the practical implementation of the Wilsonian formalism suffers for a technical very annoying problem, i.e. the explicit breaking of gauge-invariance. This is due to an inconsistency between the Wilson’s Exact Renormalization Group Equation (ERGE) and the Ward-Takahashi identities. Therefore a lot of non-trivial work is needed to recover gauge-invariance on physical quantities. In particular the all perturbative machinery of Quantum Action Principle [13, 14] and fine-tuning conditions seems needed. This fact is very unpleasant, because it is unclear why in theories like QED or QCD, where there are traditional renormalization methods explicitly consistent with the gauge symmetry, the Wilsonian formalism should be so bad. In particular one could think that in a theory as simple as QED should be possible to implement a consistent Wilsonian formulation, at least at the perturbative level. However, to the best of our knowledge, no such a formulation appeared in the literature. Here we fill this gap.

Our basic idea is simply of introducing the Wilsonian infrared cutoff as a mass-like term for both photons and electrons: in this way the basic structure of Ward identities is preserved. This idea is quite natural, nevertheless its implementation is not straightforward. The point is that the mass cut-off does not sufficiently regularize the theory and to be properly managed requires an intermediate ultraviolet regularization of the evolution equation. In this paper we discuss in detail how to perform this task which is non-unusual in a Wilsonian context. Our result open the door to the application
of non-perturbative numerical approximation schemes consistent with Ward identities, thus giving a strong improvement with respect to the methods currently used in the Wilsonian literature [15, 16, 17, 18].

The plan of the paper is the following: in section 2 we give a short introduction about the Wilsonian point of view and the previous work on the gauge-invariance problem; in section 3 we fix the notation for the quantum electrodynamics and we write down the Ward identities; in section 4 we implement the Exact Renormalization Group Equation in a form suitable for the following analysis; in section 5 we briefly review how it can be perturbatively solved in the loop-wise expansion; in sections 6 and 7 some explicit one-loop computation are presented. A general analysis of Ward identities is given in section 8; in section 9 we explain how to extract the Callan-Symanzik equation directly from the ERGE. Section 10 contains our conclusions and the outlook: various possible extensions and physical application of the formalism are suggested. In particular we stress that the formulation is perfectly calculative: the framework should be considered not only useful to study formal questions, but also for practical purposes. Three appendices on technical questions close the paper.

2 The Wilsonian point of view and the problem of gauge invariance

We begin by summarizing the basics of the Wilsonian point of view, as needed for the applications to quantum field theory.

1. The fundamental object of the formalism is the effective action at the scale \( \Lambda \), obtained by integrating out the ultraviolet degrees of freedom.

2. The procedure of integrating degrees of freedom is converted into the problem of solving a differential equation in \( \Lambda \), the Wilson’s Exact Renormalization Group Equation. In this way, by the knowledge of the ultraviolet physics (i.e. of the effective action at some ultraviolet scale \( \Lambda = \Lambda_{UV} \)) one can deduce the infrared physics (i.e. the effective action at some infrared scale \( \Lambda = \Lambda_{IR} << \Lambda_{UV} \)) by solving the ERGE.

3. The infrared effective action is independent on the details of the ultraviolet physics, i.e. it depends only on a little number of relevant
parameters. This is the physical meaning of the renormalizability (universality in statistical language) property.

Points 1, 2, 3 are common to all the approaches based on the Wilson’s point of view; however the various technical implementation of the formalism are strongly author dependent and very different in practice. For instance the degrees of freedom integration can be done à la Wegner-Houghton, by integrating the momenta on a shell of thickness δΛ, or à la Polchinski, by introducing a smooth cutoff function which multiplies the free propagators of the theory. Moreover, one can take as fundamental effective action the Wilsonian action $S_{eff}(\Phi; \Lambda)$ or, alternatively, its Legendre transform $\Gamma(\Phi; \Lambda)$ (sometimes called effective average action [19] or simply effective cutoff action [20]). This latter formalism is better suited for a comparison with the traditional renormalization theory and will be adopted in this work.

As we said, all the usual formulation of the evolution equation are inconsistent with gauge-invariance, thus the flow does not preserve the symmetry: even if the ultraviolet action is gauge-invariant, the infrared is not. Conversely, in order to have a gauge-invariant infrared action, one is forced to start with a non-gauge-invariant ultraviolet action. There was a big effort in the literature to face this problem. Here we give a short review of various solutions proposed in the past, with no pretense of completeness. In particular we restrict ourself to the Polchinski’s formulation of the evolution equation or its Legendre transformed version, neglecting some work in other formalisms, as for instance [40].

1. Maybe the first attempt, following the original Polchinski formalism, was the work of Warr [36]. In this paper the idea is of using an explicit gauge-invariant Pauli-Villars regularization supplemented by higher derivative terms. This idea is quite simple in principle, but in practice the rigorous formulation is very technical and it needs as an intermediate step a pre-regulator, i.e. a momentum cutoff, which explicitly breaks gauge-invariance; moreover concrete computations are difficult to perform and, up to our knowledge, this approach never was pursued in the successive literature.

2. A second very important point of view was advocated by Becchi [11]. In this approach the attention is on rigorous proofs concerning the perturbative recovering of the symmetry for the physical objects. Put in
other way, the Wilson Renormalization Group in the Polchinski implementation is used to prove the Quantum Action Principle [13] of perturbative quantum field theory. In this way it is possible to show that the gauge symmetry can be recovered via a perturbative fine-tuning of a finite number of relevant couplings, provided that the theory is anomaly free. Unfortunately, the explicit solution of the fine-tuning conditions is extremely cumbersome beyond one-loop, even in simple models [30]. Moreover, even if this is the general situation in theories where there are no regularization methods consistent with the symmetries, one would expect to be possible to avoid this problem in QED and QCD.

3. A third approach was developed in a series of paper by Bonini, D’Attanasio and Marchesini [20, 34, 35]. Here the formalism of the Legendre transformed cutoff effective action was developed in order to give a proof of renormalizability simpler and closer to the usual one of quantum field theory. However the point of view about the symmetries is essentially that of Becchi (even if generalized to $\mu$—momentum prescriptions and directly extended to the $\Gamma(\Phi; A)$ functional in [38]).

4. A fourth approach was implemented by Reuter and Wetterich in the formalism of the effective average action [28]. Here the idea is of adding background gauge fields to the action in order to have explicit background gauge-invariance. However, this approach is quite cumbersome in concrete computations and, moreover, its perturbative implementation is not so efficient. In fact it is well known that a perturbative implementation (see for instance [24]) of the background field method requires fine tuning of both Slavnov-Taylor identities and background Ward identities.

5. A fifth approach was introduced by Ellwanger [37] (see also [38], where the relation between this approach and the Becchi’s point of view is clarified, and [39] for a careful analysis of the QED case). In this point of view, the attention is on the quantification of the gauge-breaking term, which can be estimated by using some modified Slavnov-Taylor identities. This approach is very appealing since the broken identities can be used to extract various non-trivial informations: for instance the form of the chiral anomaly both in non-supersymmetric and in supersymmetric chiral gauge theories [46, 47]. Nevertheless, an analytical
study of the breaking term is very difficult in general.

6. Finally, there is a recent proposal of Morris [50] based on a fully gauge-invariant formalism where the fundamental quantities are Wilson loops and Wilson lines. The idea is of combining the numerical methods available for the Exact Renormalization Group with the insights coming from the large $N_C$ expansion, where $N_C$ is the number of colors. However the analysis is not simple and the comparison with the perturbative results is difficult; moreover by construction this approach cannot say nothing about the Abelian case which is our concern in the present work.

In this paper we provide an explicitly Ward-identities-consistent formulation of the evolution equation for the case of Abelian QED-like gauge theories. This formulation, if extended to non-covariant gauges, is also suitable for the analysis of the non-Abelian case. This is left for a separate publication [53].

3 Tree Level Quantum Electrodynamics

As a typical example of Abelian gauge theory we will consider the quantum electrodynamics (QED) with fields $\Phi = (A, \psi, \bar{\psi})$. Our notations on metric and gamma matrices are as in [26] and the covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$. The electron mass is denoted by $m$ and the gauge fixing parameter by $\xi$; in explicit computations we will use the Feynman gauge $\xi = 1$. Some useful abbreviations on integrals are

$$\int_x = \int d^4x, \quad \int_p = \int \frac{d^4p}{(2\pi)^4}. \tag{1}$$

For the Euclidean momenta we use the notations

$$q_E = (iq_0, \vec{q}), \quad q_E^2 \equiv \delta_{\mu\nu} q_\mu^E q_\nu^E = -q^2. \tag{2}$$

If not otherwise specified, all the quantities should be intended in the Minkowski space.

The fundamental ingredients of our analysis are:

1. The functional operator

$$\mathcal{W}_f = -\int_x f(x) \left[ \partial_\mu \frac{\delta}{\delta A_\mu} + ie\bar{\psi} \frac{\delta}{\delta \psi} - ie\psi \frac{\delta}{\delta \bar{\psi}} \right] \tag{3}$$
which defines the gauge symmetry,

\[ W_f A_\mu = \partial_\mu f, \quad W_f \psi = i e f \psi, \quad W_f \bar{\psi} = -i e f \bar{\psi}. \]  

(4)

2. The classical gauge-invariant action

\[ S_{CL}(\Phi) = \int \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i (\not{\partial} - m) \psi \right), \quad W_f S_{CL} = 0 \]  

(5)

which specifies the theory.

3. The infrared cutoff functions \( K_{\Lambda \infty}(q) \) (for the photon propagator) and \( \tilde{K}_{\Lambda \infty}(q) \) (for the electron propagator) which specify the distinction between soft \( (q_E^2 << \Lambda^2) \) and hard \( (q_E^2 >> \Lambda^2) \) modes. In general they are smooth functions with the properties

\[ \lim_{\Lambda \to \infty} K_{\Lambda \infty}(q) = 0, \quad \lim_{\Lambda \to 0} K_{\Lambda \infty}(q) = 1, \]  

(6)

\[ \lim_{\Lambda \to \infty} \tilde{K}_{\Lambda \infty}(q) = 0, \quad \lim_{\Lambda \to 0} \tilde{K}_{\Lambda \infty}(q) = 1, \]  

(7)

i.e. soft momenta are dumped whereas hard momenta are unaffected.

A comment about the cutoff functions is in order here. Usually one does not distinguish between the cutoff function for photons and electrons, i.e. keeps \( K_{\Lambda \infty}(q) = \tilde{K}_{\Lambda \infty}(q) \). Nevertheless this distinction will be important in our analysis. In particular \( K_{\Lambda \infty}(q) \) is a scalar function whereas \( \tilde{K}_{\Lambda \infty}(q) \) is intended as a matrix in spinor space. One could also take \( K_{\Lambda \infty}(q) \) as a matrix in Lorentz indices, but this generalization is not needed here. From the cutoff functions one derives the following useful objects

\[ Q^{\mu\nu}_\Lambda (p) = -[K^{-1}_{\Lambda \infty}(p^2) - 1] p^2 g^{\mu\nu} - \frac{1}{\xi} p^\mu p^\nu \]  

(8)

and

\[ \tilde{Q}_\Lambda (p) = [\tilde{K}^{-1}_{\Lambda \infty}(p^2) - 1] (\not{p} - m). \]  

(9)

They enter in the gauge-fixed tree level cutoff action as follows:

\[ \Gamma^{(0)}(\Phi; \Lambda) = S_{CL}(\Phi) + \frac{1}{2} \bar{\Phi} Q_\Lambda \Phi, \]  

(10)
\[
\frac{1}{2} \Phi Q \Lambda \Phi = \frac{1}{2} \int_p A_\mu(-p)Q_A^{\mu\nu}(p)A_\nu(p) + \int_p \bar{\psi}(p)\tilde{Q}(p)\psi(p). \tag{11}
\]

In general there is an hard gauge-invariance problem because of the quadratic breaking term \(\frac{1}{2} \Phi Q \Lambda (p) \Phi\) (we remind that the \(Q_A(p)\)’s in \(x\)-space are in general complicate non-local differential operators):

\[
W_f \Gamma^{(0)} = \int_x A_\mu Q^{\mu\nu}_A(i\partial)\partial_\nu f + \partial_\mu fQ^{\mu\nu}_A(i\partial)A_\nu + \int_x i\bar{\psi}Q_A(i\partial)(f\psi) - i\bar{\psi}fQ_A(i\partial)\psi. \tag{12}
\]

The main idea of this paper is of solving this problem by using a particularly simple form for \(Q^{\mu\nu}_A(p)\) and \(\tilde{Q}_A(p)\). In particular we will take

\[
Q^{\mu\nu}_A(p) = \Lambda^2 g^{\mu\nu} - \frac{1}{\xi}p^\mu p^\nu, \tag{13a}
\]

\[
\tilde{Q}_A = -i\Lambda \gamma_5, \tag{13b}
\]

which corresponds to the following choice for the cutoff functions in Minkowski space,

\[
K_{\Lambda \infty}(p) = \frac{p^2}{p^2 - \Lambda^2} \tag{14a}
\]

\[
\tilde{K}_{\Lambda \infty}(p) = \frac{\not{p} - m}{\not{p} - m - i\Lambda \gamma_5}. \tag{14b}
\]

With these choices the explicit expressions for the tree level propagators of the theory are

\[
D_{\mu\nu}(k; \Lambda) = \frac{1}{\Gamma^{(0)}_{\mu\nu}(k; \Lambda)} = \left( (-k^2 + \Lambda^2)g_{\mu\nu} + (1 - \frac{1}{\xi})k_\mu k_\nu \right)^{-1}
\]

\[
= -\frac{g_{\mu\nu}}{k^2 - \Lambda^2 + i\varepsilon} + (1 - \xi)\frac{k_\mu k_\nu}{(k^2 - \Lambda^2 + i\varepsilon)(k^2 - \xi \Lambda^2 + i\varepsilon)} \tag{15}
\]

and

\[
S_{\alpha\beta}(p; \Lambda) = \frac{1}{\Gamma^{(0)}_{\alpha\beta}(p; \Lambda)} = \frac{1}{(\not{p} - m - i\Lambda \gamma_5)_{\alpha\beta}} = \frac{(\not{p} + m - i\Lambda \gamma_5)_{\alpha\beta}}{p^2 - m^2 - \Lambda^2 + i\varepsilon}, \tag{16}
\]

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where the causal $i\varepsilon$ prescription has been used. In section 7, we will discuss the technical advantages of the choice (13b) for the spinor “mass” term, involving the $\gamma_5$ matrix. Essentially, this choice will simplify the task of finding a suitable ultraviolet regularization of the evolution equation. However, notice that we cannot give a direct physical meaning to the effective action for $\Lambda \neq 0$ since parity symmetry is broken (the transformation law of the effective action under parity is $\Gamma(A', \psi', \bar{\psi}; A) = \Gamma(A, \psi, \bar{\psi}; -A)$).

The important point we want to stress here is the fact that with this choice of the infrared cutoff functions, the breaking of gauge invariance is innocuous, in the sense that the gauge breaking term is linear in the fields:

$$\mathcal{W}_f \Gamma^{(0)} = \int_x \left( \frac{\Box}{\xi} + \Lambda^2 \right) \partial_\mu A^\mu f(x)$$

As it is well known [8], this exceptional property guarantees Ward identities can be lifted to the quantum level to all orders in perturbation theory. More in detail, if we expand the effective action in powers or $\hbar$

$$\Gamma^{[\ell]} = \Gamma^{(0)} + \hbar \Gamma^{(1)} + \cdots + \hbar^\ell \Gamma^{(\ell)}$$

and we use a consistent renormalization procedure, we can have in general

$$\mathcal{W}_f \Gamma^{[\ell]} = \int_x \left( \frac{\Box}{\xi} + \Lambda^2 \right) \partial_\mu A^\mu f(x) \quad \forall \ell \geq 0.$$  \hspace{1cm} (19)

In other words, the breaking of gauge invariance is confined at the tree level and the perturbative corrections to $\Gamma^{(0)}(\Phi; \Lambda)$ are gauge invariant:

$$\mathcal{W}_f \Gamma^{(\ell)} = 0 \quad \forall \ell \geq 1;$$  \hspace{1cm} (20)

This fact can be proved with the standard techniques of perturbative quantum field theory [14]; however, here we will give a Wilsonian analysis based on the evolution equation.

### 4 The evolution equation

The Wilson’s evolution equation has a long history [4, 5, 6, 7] and there are many different formulations which describe the same physics. Here we are interested in the Legendre transformed version of the equation introduced in
Figure 1: Recursive expansion of the $\bar{\Gamma}_{AA_1...A_nB}$ vertices, denoted by the boxes. The black dots denote the full propagators and the ovals the full vertices.

$$\frac{2}{i} \Lambda \partial_\Lambda \begin{array}{c} A \\ A_1 \end{array} = \begin{array}{c} A \\ A_1 \end{array} - \begin{array}{c} CD \\ A_1 \end{array} - \begin{array}{c} A \\ A_1 \end{array}$$

Figure 2: Diagrammatic version of the exact evolution equation in Minkowski space. Here $X = \dot{Q}_\Lambda$.

A detailed explanation of the employed notations and some comments about the derivation are collected in appendix A. Here we report only the final form for the proper vertices, which reads

$$\dot{\Pi}_{A_1...A_n} = I_{A_1...A_n} = -\frac{i}{2} (-)^A (\Gamma_2^{-1} \dot{Q}_\Lambda \Gamma_2^{-1})^{BA} \Gamma_{AA_1...A_nB}$$

where the dot denotes the $\Lambda \partial_\Lambda$ derivative and

$$\Pi(\Phi; \Lambda) \equiv \Gamma(\Phi; \Lambda) - \frac{1}{2} (-)^A \Phi^A Q_{\Lambda,AB} \Phi^B.$$
ultraviolet regularizations consistent with gauge symmetry do exist (in the following we will introduce a Pauli-Villars-like regularization), this step gives no problems. The renormalizability property guarantees that the intermediate regularization can be removed at the end, once the correct subtractions are performed.

We notice also that in theories with better ultraviolet behavior, such as low dimensional or supersymmetric theories, this intermediate step can be skipped.

5 Boundary conditions and perturbative expansion

Now we review in brief how the evolution equation (21) can be solved iteratively, once having specified suitable boundary conditions. Since the fixing of the boundary condition is a non-trivial point, we report here some technical remarks (see also [20] and the original discussion of Polchinski [6]).

1. We split the effective action in a relevant and an irrelevant part

\[ \Gamma(\Phi; \Lambda) = \Gamma_{rel}(\Phi; \Lambda) + \Gamma_{irr}(\Phi; \Lambda) \]  

where by definition the relevant part contains only renormalizable interactions, i.e. terms with couplings of non-negative mass-dimension,

\[ \Gamma_{rel}(\Phi; \Lambda) = \sum_r c_r(\Lambda) \int \mathcal{O}_r[\Phi], \quad \text{dim} c_r(\Lambda) \geq 0, \]  

where \( \mathcal{O}_r[\Phi] \) denote the (finite) set of relevant local operators (dim \( \mathcal{O}_r[\Phi] \leq 4 \)) constructed with the fields and their derivatives which are consistent with the symmetries. In particular we extract the relevant part by using zero-momentum prescriptions, i.e. by a Taylor expansion in fields and momenta (see appendix B for details). In this way \( \Gamma_{irr}(\Phi; \Lambda) \) contains only couplings with negative mass dimension. However other prescriptions are possible, as for instance on-shell renormalization prescriptions [34] or prescriptions at momentum \( \mu [35] \).

2. Following Polchinski, we suppose of knowing the relevant part of the action, i.e. the relevant parameters, at some initial low-energy scale
\( \Lambda_R \), which can be thought as the typical energy scale accessible in everyday experiments\(^2\). On the other hand, the irrelevant parameters are fixed at some ultraviolet scale \( \Lambda_{UV} \gg \Lambda_R \), which is interpreted as the scale where new physics (unification, quantum gravity, etc) is expected to modify completely our field theory. By dimensional arguments one expects that the irrelevant couplings affect the low-energy Green functions only as inverse powers of \( \Lambda_{UV} \) and in fact this can be rigorously proved to all orders in perturbation theory, as done for the first time by Polchinski. Therefore we can safely take

\[
\Gamma_{irr}(\Phi; \Lambda_{UV}) = 0 \tag{25}
\]

for large \( \Lambda_{UV} \).

Having stated the boundary conditions we can write the exact evolution equation in its integral form \(^{[20]}\)

\[
\Gamma_{rel}(\Phi; \Lambda) = \left. \Gamma_{rel} \right|_{\Lambda=\Lambda_R} + \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda_1}{\Lambda_1} I_{rel}(\Phi; \Lambda_1) \tag{26a}
\]

\[
\Gamma_{irr}(\Phi; \Lambda) = \left. \Gamma_{irr} \right|_{\Lambda=\Lambda_{UV}} - \int_{\Lambda}^{\Lambda_{UV}} \frac{d\Lambda_1}{\Lambda_1} I_{irr}(\Phi; \Lambda_1). \tag{26b}
\]

Now we are in position to solve iteratively the ERGE by expanding the effective action in the loop-wise series \(^{[18]}\). In this way the integrated evolution equation (26), with the following boundary conditions on relevant and irrelevant couplings

\[
c_{r}^{(\ell)}|_{\Lambda=\Lambda_R} = \bar{c}_r \delta_{\ell 0}, \quad c_{i}^{(\ell)}|_{\Lambda=\Lambda_{UV}} = 0, \tag{27}
\]

can be solved iteratively:

\[
\Gamma^{(\ell+1)}(\Phi; \Lambda) = \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda_\ell}{\Lambda_\ell} I_{rel}^{(\ell)} - \int_{\Lambda}^{\Lambda_{UV}} \frac{d\Lambda_\ell}{\Lambda_\ell} I_{irr}^{(\ell)}, \quad \ell \geq 0. \tag{28}
\]

In particular, as we will prove in section 8, in the perturbative expansion one can safely replace \( \Lambda_{UV} = \infty \) and the dependence on the ultraviolet scale is completely lost.

\(^2\)As a matter of fact, it is also possible to fix the couplings at the scale \( \Lambda = 0 \). But in this case and in presence of massless particles, one is forced to introduce a non-zero momentum scale \( \mu \) as a subtraction point. That scale plays the same role of \( \Lambda_R \).
6 The $\phi^4$ theory

In order to see how the previous general analysis works in a simple example, we consider here the paradigmatic case of the Euclidean massless $\phi^4$ theory, regularized in the infrared with a mass-like cutoff $\Lambda^2$ and in the ultraviolet through an higher derivative regularization. In addition, we introduce an external source $K(x)$ coupled to the composite operator $\phi^2(x)/2$.

We take as tree-level Euclidean action of the model

$$\Gamma^{(0)}(\phi, K; \Lambda, M_0) = \int_x \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \phi \frac{(\partial^2)^2}{M_0^2} \phi$$

$$+ \frac{1}{2} \Lambda^2 \phi^2 + \frac{1}{2} K(x) \phi^2.$$  \hfill (29)

The regularized free propagator reads

$$D_{\text{reg}}(q; \Lambda, M_0) = \frac{1}{q^2 + \Lambda^2 + q^4/M_0^2}$$  \hfill (30)

and satisfies the important property

$$\dot{D}_{\text{reg}}(q; \Lambda, M_0) \simeq -2 \Lambda^2 \frac{M_0^4}{q^8}, \quad q^2 >> M_0^2.$$ \hfill (31)

This fact is essential to verify that for any finite $M_0$ all the momentum integrals in the evolution equation are well defined in the ultraviolet.

In the following the $M_0$-dependence will be often understood. We will denote the proper vertices with $l$ insertions of the operator $\phi^2/2$ as

$$\Gamma_{2n,l}(x_i, y_j; \Lambda) \equiv \left. \frac{\delta \Gamma}{\delta \phi(x_1) \ldots \delta \phi(x_{2n}) \delta K(y_1) \ldots \delta K(y_l)} \right|_{\phi=K=0}$$ \hfill (32)

and their Fourier transforms as

$$\tilde{\Gamma}_{2n,l}(p_i, q_j; \Lambda) = (2\pi)^4 \delta \left( \sum_i p_i + \sum_j q_j \right) \Gamma_{2n,l}(p_i, q_j; \Lambda).$$ \hfill (33)

Notice that the derivative with respect to $\Lambda^2$ can be replaced with a derivative with respect to $K$ taken at zero momentum; for instance we have $\partial_{\Lambda^2} \Gamma^{(0)}_{2n}(p_i; \Lambda) = \Gamma^{(0)}_{2n,1}(p_i, 0; \Lambda)$. In general the relation

$$\partial_{\Lambda^2} \Gamma_{2n,l}(p_i, q_j; \Lambda) = \Gamma_{2n,l+1}(p_i, q_j, 0; \Lambda)$$ \hfill (34)
can be imposed to all orders in perturbation theory. We define the relevant coefficients

\begin{align}
c_m(\Lambda) &= \Gamma_2|_{p=0}, \quad \sigma_m(\Lambda) = \Gamma_{2,1}|_{p_i=0}, \quad (35a) \\
c_\phi(\Lambda) &= \partial_p \Gamma_2|_{p=0}, \quad c_\lambda(\Lambda) = \Gamma_4|_{p_i=0}, \quad (35b)
\end{align}

the relevant action

\[ \Gamma_{rel}(\phi, K; \Lambda, M_0) = \int \frac{1}{2} c_\phi \partial^\mu \phi \partial_\mu \phi + \frac{1}{4!} c_\lambda \phi^4 + \frac{1}{2} c_m \phi^2 + \frac{1}{2} \sigma_m K \phi^2 \]  
(36)

and the irrelevant vertices

\begin{align}
\Gamma_{2, irr}(p; \Lambda) &= \Gamma_2(p; \Lambda) - \Gamma_2|_{p=0} - p^2 \partial_\rho \Gamma_2|_{\rho=0}, \quad (37a) \\
\Gamma_{4, irr}(p_i; \Lambda) &= \Gamma_4(p_i; \Lambda) - \Gamma_4|_{p_i=0}, \quad (37b) \\
\Gamma_{2,1, irr}(p, q; \Lambda) &= \Gamma_{2,1}(p, q; \Lambda) - \Gamma_{2,1}|_{p=q=0}, \quad (37c) \\
\Gamma_{2n,l, irr}(p_i, q_j; \Lambda) &= \Gamma_{2n}(p_i, q_j; \Lambda), \quad n > 2, \ l > 1. \quad (37d)
\end{align}

The boundary conditions on the relevant couplings (renormalization prescriptions) at loop \( \ell \) are

\begin{align}
c_m^{(\ell)}(0) &= \Gamma_2^{(\ell)}|_{p=0} = 0, \quad \Gamma_{2,1}^{(\ell)}|_{p=0} = \delta_{00}, \quad (38a) \\
\sigma_m^{(\ell)}(\Lambda_R) &= \Gamma_{2,1}^{(\ell)}|_{p_{\Lambda=\Lambda}} = \delta_{00}, \quad (38b) \\
c_\phi^{(\ell)}(\Lambda_R) &= \partial_{p^2} \Gamma_2^{(\ell)}|_{p^2=0} = \delta_{00}, \quad (38c) \\
c_\lambda^{(\ell)}(\Lambda_R) &= \Gamma_4^{(\ell)}|_{p_i=0} = \lambda \delta_{00}, \quad (38d)
\end{align}

where the renormalization scale \( \Lambda_R \ll M_0 \) is non-zero in order to avoid infrared divergences. The irrelevant couplings are fixed at the ultraviolet scale \( \Lambda_{UV} \) which is of order \( M_0 \).

Now we make some specific one-loop computation. At leading order approximation the evolution equation for the vertices without insertions has the explicit form

\[ \tilde{\Pi}_{2n}^{(1)}(p_i; \Lambda) = -\int_q \frac{\Lambda^2}{(\Lambda^2 + q^2 + q^4/M_0^2)^2} \tilde{\Gamma}_{2n+2}^{(0)}(q, p_i, -q; \Lambda) \]  
(39)
where the recursive form of the $\Gamma^{(0)}_{2+2n}$ vertices is given by the condensed expression (see figure 1)

$$\Gamma^{(0)}_{2} = 0, \quad \Gamma^{(0)}_{4} = \lambda,$$

$$\Gamma^{(0)}_{2+2n} = \Gamma^{(0)}_{2+2n} - \sum_{k=1}^{n-1} \Gamma^{(0)}_{2+2k} (\Gamma^{(0)}_{2})^{-1} \Gamma^{(0)}_{2+2n-2k}, \quad n \geq 2.$$  \hfill (40)

In particular the one-loop two-point equation reads

$$\dot{\Pi}^{(1)}_{2}(p; \Lambda; M_0) = \int_{q} \frac{-\lambda \Lambda^2}{(q^2 + \Lambda^2 + q^4/M_0^2)^2},$$

and is logarithmically divergent,

$$\dot{\Pi}^{(1)}_{2}(p; \Lambda; M_0) = -\frac{\lambda \Lambda^2}{16\pi^2} \log \frac{M_0^2}{\Lambda^2} + O(1).$$ \hfill (42)

This divergence can be compensated if we introduce a mass renormalization coupling $Z_m^{(1)}(M_0/\Lambda_R)$ by defining

$$\Gamma^{(1)}_{2}(p; \Lambda) \equiv \Lambda^2 Z_m^{(1)}(M_0/\Lambda_R) + \Pi^{(1)}_{2}(p; \Lambda).$$ \hfill (43)

In this way the two-point equation for $\Gamma^{(1)}_{2}(p; \Lambda)$

$$\dot{\Gamma}^{(1)}_{2}(p; \Lambda) = 2Z_m^{(1)}(M_0/\Lambda_R) \Lambda^2 + 2\Lambda^2 \partial_{\Lambda} \Pi^{(1)}_{2}(p; \Lambda)$$ \hfill (44)

can be made finite by using (41) and imposing the normalization condition (38b),

$$\Gamma^{(1)}_{2}\big|_{p=0}^{\Lambda=\Lambda_R} = \partial_{\Lambda}\Gamma^{(1)}_{2}\big|_{p=0}^{\Lambda=\Lambda_R} = Z_m^{(1)}(M_0/\Lambda_R) + \partial_{\Lambda}\Pi^{(1)}_{2}\big|_{p=0}^{\Lambda=\Lambda_R} = 0.$$ \hfill (45)

In this way by using (41) we obtain

$$Z_m^{(1)}(M_0/\Lambda_R) = \frac{1}{2} \int_{q} \frac{\lambda}{(q^2 + \Lambda^2 + q^4/M_0^2)^2}$$ \hfill (46)

and the two-point evolution equation (44) has the explicit finite form

$$\lim_{M_0 \to \infty} \dot{\Gamma}^{(1)}_{2} = \lambda \lambda \int \left[ \frac{\Lambda^2}{(q^2 + \Lambda^2)^2} - \frac{\Lambda^2}{(q^2 + \Lambda_R^2)^2} \right] = \frac{\Lambda \Lambda^2}{16\pi^2} \ln \frac{\Lambda^2}{\Lambda_R^2}. $$ \hfill (47)
Notice that only after the imposition of the renormalization prescription (38b) the ultraviolet regularization can be removed: the situation here is more similar to the traditional approach to Quantum Field Theory than to the Wilsonian one. The difference is that in the usual Wilsonian formulation the $\Lambda \partial_\Lambda$ derivative of the cutoff function is strongly damped in the ultraviolet and (41) is automatically finite; at a consequence both $\Gamma(\phi; \Lambda)$ and $\Pi(\phi; \Lambda)$ are finite and the introduction of the renormalization constant $Z_m(M_0/\Lambda_R)$ is not needed. This simplifies for certain aspects the analysis, but the price to pay is the lost of gauge-invariance. For this reason the mass cutoff should be preferred in gauge-theories.

The higher points vertices are automatically finite at one-loop; for instance the four point vertex reads

$$\lim_{M_0 \to \infty} \Gamma_4^{(1)}(p_i; \Lambda, M_0) = -\frac{\lambda^2}{2} \sum_{i<j} \int_q \hat{D}(q; \Lambda) D(q + p_i + p_j; \Lambda)$$

$$= -\frac{\lambda^2}{32\pi^2} \sum_{i<j} \int_0^1 dx \frac{\Lambda^2}{(p_i + p_j)^2 x(1 - x) + \Lambda^2}.$$  \hspace{1cm} (48)

* A fortiori the finiteness property holds for the vertices with insertions.

The analysis of higher order correction is more involved. The general form of the evolution equation with $Q_\Lambda = Z_m(M_0/\Lambda_R)\Lambda^2$ is

$$\Pi_{2n}(p_i, K; \Lambda) = -\int_q \frac{Z_m \Lambda^2}{[Z_m \Lambda^2 + \Pi_2(q, K; \Lambda)]^2} \bar{\Gamma}_{2n+2}(q, p_i, -q, K; \Lambda)$$  \hspace{1cm} (49)

where the mass renormalization coupling

$$Z_m(M_0/\Lambda_R) = 1 + Z_m^{(1)}(M_0/\Lambda_R) + Z_m^{(2)}(M_0/\Lambda_R) + \ldots$$  \hspace{1cm} (50)

is fixed by the renormalization prescription (38b) i.e. by the self-consistent equation, to be solved in perturbation theory,

$$Z_m(M_0/\Lambda_R) = 1 + \frac{1}{2} \int_q \frac{Z_m \Gamma_4(q, 0, 0, -q; \Lambda_R)}{[Z_m \Lambda_R^2 + \Pi_2(q, \Lambda_R)]^2}.$$  \hspace{1cm} (51)

Apparently for $M_0 \to \infty$ the evolution equation (49) contains overlapping divergences at higher orders in perturbation theory. Actually, thanks to the renormalizability proof, they cancel. This can be shown for instance by using the Callan’s proof [32] presented in [8]. Actually our approach can be seen
as the bridge between the Wilsonian point of the view and the Field Theory methods based on the Callan-Symanzik equation. We refer to section 9 for more details on this point. We stress here that the relevance of the Wilsonian interpretation is the fact that there are numerical techniques, based on suitable truncations\(^3\), to solve the exact equation (49) non-perturbatively.

\section{The QED case}

The previous considerations generalize quite straightforwardly to the QED case, provided that we use a gauge-consistent ultraviolet regularization. Here we will use a kind of Pauli-Villars regularization that we shall call holomorphic Pauli-Villars regularization following \cite{52}.

In general, the Pauli-Villars approach consists in adding some very massive \((M_0 >> m)\) unphysical fields to the physical theory, in such a way of smoothing its ultraviolet behavior. In the case we are considering it is sufficient to take as tree level action

\begin{equation}
\Gamma^{(0)}(A, \psi, \bar{\psi}, A', \psi', \bar{\psi}'; \Lambda, M_0) = \int_x \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \frac{1}{2} \Lambda^2 A_\mu A^\mu + \int_x \bar{\psi} i \not{D} (A + A') \psi - \bar{\psi} (m + i\Lambda\gamma_5) \psi + \int_x \bar{\psi}' i \not{D} (A + A') \psi' - \bar{\psi}' (M_0 + i\Lambda\gamma_5) \psi' \right]
\end{equation}

where we have introduced an heavy photon field \(A'_\mu\) (commuting) and two heavy fermion fields \(\psi', \bar{\psi}'\) (commuting) interacting in a gauge-invariant way (the covariant derivative is \(D_\mu (A + A') = \partial_\mu - ieA_\mu - ieA'_\mu\)). The kinetic term of the heavy photon has a minus sign compared to the kinetic term of

\(^3\)There is a subtle point here. Even if the evolution equation (49) is well defined to all orders in perturbation theory as \(M_0 \rightarrow \infty\), this property relies on delicate cancellations of Feynman diagrams and could be lost by using a generic non-perturbative truncation. Nevertheless we checked explicitly that in the typical non-perturbative approximation used in the Wilsonian literature, i.e. the local potential approximation, there are no practical problems in renormalizing the potential, by using for instance the Coleman-Weinberg prescriptions.
the physical photon, thus the effect of the unphysical photon fields $A'_\mu$ is a
modification of the photon propagator at high energies

$$\left[ D_{\mu\nu}(k; \Lambda^2) \right]_{\text{reg}} = D_{\mu\nu}(k; \Lambda^2) - D_{\mu\nu}(k; \Lambda^2 + M_0^2) \quad (53)$$

which becomes more convergent ($[D_{\mu\nu}]_{\text{reg}} \sim M_0^2/k^4$). The unphysical fermions
are taken to be commuting therefore contributing with a plus sign to the
fermion loops. The net effect is of increasing the ultraviolet convergence
of the fermion bubble of a factor $M^2/q^2$. Notice that the infrared cutoff is
inserted by means of the $\gamma_5$ matrix in order to ensure this property. As an
intriguing additional bonus one has some analyticity properties on the depen-
dence on the complex masses $m + i\Lambda\gamma_5$ and $M_0 + i\Lambda\gamma_5$. This is very appealing
in view of a supersymmetric extension of this work. A similar observation
can be found in [52].

With the regularization we have introduced all the momentum integrals
in the evolution equation becomes well defined. Notice that in a more usual
context, in order to directly regularize the Feynman diagrams, a much more
complicate Pauli-Villars regularization with more unphysical fields is needed.
However, for our aims it is sufficient to regularize the ultraviolet behavior of
the evolution equation i.e. of the mass-derived Feynman diagrams, which are
more convergent of the standard ones. This simplifies our task.

At the end the intermediate regularization can be removed, provided that
we correctly subtract the divergences: this is done by imposing the zero-
momentum renormalization prescriptions.

It is very simple to write down the explicit form of the QED evolution

Figure 3: Graphic representation of the right hand side of equation (55). Each
diagram should be subtracted by an analogous diagram involving the Pauli-Villars
heavy fields.
equations (21) in the one-loop approximation. In fact by using the property
\[-\frac{1}{\Gamma_2^{(0)}} \dot{Q}_\Lambda \frac{1}{\Gamma_2^{(0)}} = \Lambda \partial_\Lambda \frac{1}{\Gamma_2^{(0)}} \] (54)
one easily realize that the right hand side of the one loop evolution equation
is given by the logarithmic derivative of the usual Feynman diagrams (minus
a zero-momentum subtraction for the two-point functions). Graphically the
situation is represented in figure 3. Since a detailed derivation can be found
in [34] here we just report the formulae for the simpler proper vertices:
\[ I^{(1)}_{\mu \nu} = i e^2 \int q \Lambda \partial_\Lambda \text{Tr} \left[ \gamma_\mu S(q; \Lambda) \gamma_\nu S(q + p; \Lambda) \right]_{PV} \]
\[ I^{(1)}_{\alpha \beta} = i e^2 \int q \Lambda \partial_\Lambda \left[ D^{\mu \nu}(q; \Lambda) \gamma_\nu S(q + p; \Lambda) \gamma_\mu e^2 \right]_{\alpha \beta, PV} \]
\[ I^{(1)}_{\mu} = i e^3 \int q \Lambda \partial_\Lambda \left[ \gamma_\nu D^{\nu \rho}(q; \Lambda) \gamma_\mu S(q + p; \Lambda) \gamma_\rho S(q + p'; \Lambda) \right]_{PV}. \] (55)
The foot \textit{PV} remind that we are using the Pauli-Villars regularization to
properly define the two-point vertices. As a concrete example, we can com-
pute in detail the evolution of the (inverse) photon propagator. That analy-
sis should be intended also as a practical introduction about checking Ward
identities and computing beta functions in this formalism.

The photon propagator can be decomposed in its transversal and longi-
tudinal components (for details on notation see appendix B)
\[ \Pi_T(p; \Lambda) = \frac{1}{3} \ell^{\mu \nu}(p) \Pi_{\mu \nu}(p; \Lambda), \quad \Pi_L(p; \Lambda) = \ell^{\mu \nu}(p) \Pi_{\mu \nu}(p; \Lambda). \] (56)
Doing the traces, using Feynman parameterization, and continuing to the
Euclidean space, we can explicitly compute \( \Pi_T(p_E; \Lambda) \) and \( \Pi_L(p_E; \Lambda) \). In
particular we obtain
\[ \Pi_T^{(1)}(p_E; \Lambda) = \int_0^1 dx \int_{q_E} \Lambda \partial_\Lambda \left[ f_T(q_E, p_E; \Lambda, m) - f_T(q_E, p_E; \Lambda, M_0) \right] \] (57)
and
\[ \Pi_L^{(1)}(p_E; \Lambda) = \int_0^1 dx \int_{q_E} \Lambda \partial_\Lambda \left[ f_L(q_E, p_E; \Lambda, m) - f_L(q_E, p_E; \Lambda, M_0) \right] \] (58)

\[ ^4\text{We remind that equation (54) is a condensed matrix relation and means that analogous relations holds both for the photon and the electron propagator.} \]
where
\[
f_T(q_E, p_E; \Lambda, M) = 4e^2 \frac{\frac{1}{2} q_E^2 - p_E^2 x(1 - x) + M^2 + \Lambda^2}{(q_E^2 + p_E^2 x(1 - x) + M^2 + \Lambda^2)^2}
\]
and
\[
f_L(q_E, p_E; \Lambda, M) = 4e^2 \frac{\frac{1}{2} q_E^2 + p_E^2 x(1 - x) + M^2 + \Lambda^2}{(q_E^2 + p_E^2 x(1 - x) + M^2 + \Lambda^2)^2}.
\]

Notice that, after the subtraction and the application of the \( \Lambda \partial_\Lambda \) operator, the integrals in \( q_E \) are perfectly convergent, and simple to compute. The final results are
\[
\dot{\Pi}^{(1)}_T(p_E; \Lambda) = -\frac{e^2}{\pi^2} \int_0^1 dx \frac{p_E^2 x(1 - x) \Lambda^2}{p_E^2 x(1 - x) + m^2 + \Lambda^2} + O(1/M_0^2)
\]
and
\[
\dot{\Pi}^{(1)}_L(p_E; \Lambda) \equiv 0, \quad \forall \, \Lambda, \, \forall \, p_E, \, \forall \, M_0 < \infty.
\]

The first result is remarkable because of the relation with the usual renormalization group. In fact, for large \( \Lambda \), the coefficient in front to \( I_T(p; \Lambda) \) is related to the one-loop QED beta function
\[
I_T(p^2; \Lambda^2) = -\frac{8}{3} \frac{e^2}{16\pi^2} p_E^2 + O \left( \frac{p_E^2}{\Lambda^2}, \frac{m^2}{\Lambda^2} \right).
\]

The second result is also remarkable, because it is a direct check of gauge-invariance, i.e. of the Ward identity
\[
p^\mu I^{(1)}_{\mu\nu}(p; \Lambda) \equiv 0.
\]

Technically equation (62) holds since the function \( f_L(q_E, p_E; \Lambda, m) - f_L(q_E, p_E; \Lambda, M_0) \) can be rewritten as a total derivative,
\[
\frac{\partial}{\partial q^\mu} \left[ \frac{q^\mu}{q^2 + p_E^2 x(1 - x) + m^2 + \Lambda^2} - \frac{q^\mu}{q^2 + p_E^2 x(1 - x) + M_0^2 + \Lambda^2} \right]
\]
and therefore its momentum integral is identically zero. Here one sees the importance of the intermediate ultraviolet regularization: had we not taken in account the Pauli-Villars fields, i.e. had we neglected the second piece in
(64), should we have obtained a finite but wrong (non-zero) result. Similar subtleties are well known in the literature. The same remark on the necessity of regularizing the evolution equation, even if in a very different formalism, can be found in [50].

Having explained how the machinery works on simple examples, now we can turn to the analysis of general questions, like gauge-invariance and renormalizability.

8 The gauge-invariance proof

It is quite simple to prove that our formulation is consistent with the gauge symmetry, i.e. that the $\Gamma(\Phi; \Lambda)$ functional is gauge invariant

$$W_f \Gamma(\Phi; \Lambda) = 0 \quad \forall \Lambda$$

for any $\Lambda$. This fact is expected, since the perturbative expansion of the evolution equation gives the usual Feynman diagrams with massive propagators supplemented with the BPHZ zero-momentum subtractions and it is known that this approach is consistent with Ward-Takahashi identities [33]. However here we will give a more direct proof based on the evolution equation. The simpler way to proceed is from diagrammatic considerations, even if more formal non-diagrammatic proofs are possible [53].

At the level of proper vertices the functional Ward identity (65) corresponds to an infinite set of transversality constraints like

$$k^\mu \Pi_{\mu \nu}(k; \Lambda) = 0, \quad (k_1 + k_2 + k_3)^\mu \Gamma_{\mu \nu \rho \lambda}(k_1, k_2, k_3; \Lambda) = 0,$$

$$(p_2 - p_1)^\mu \Gamma_{\mu \alpha \beta}(p_1, -p_2; \Lambda) = e \Gamma_{\alpha \beta}(p_2; \Lambda) - e \Gamma_{\alpha \beta}(p_1; \Lambda),$$

$$(k + p_1 + p_2)^\mu \Gamma_{\mu \nu \alpha \beta}(k, p_1, p_2; \Lambda) = e \Gamma_{\nu \alpha \beta}(k, p_1; \Lambda) - e \Gamma_{\nu \alpha \beta}(k, p_2; \Lambda),$$

and infinite others. In general the transverse part of a vertex $\Gamma_{n_{A+1}, n_{\bar{\psi} \psi}}(k_1 \ldots k_{n_{A+1}}, p_1 \ldots p_{2n_{\bar{\psi} \psi}})$ is related to a difference of vertices $\Gamma_{n_{A}, n_{\bar{\psi} \psi}}(k_1 \ldots k_{n_A}, p_1 \ldots p_{2n_{\bar{\psi} \psi}})$ or, in absence of fermion legs, is zero.

It is clear that a proof of explicit gauge-invariance is doomed to fail for a generic choice of cutoff functions. In fact in the generic case the Ward identities are badly broken at tree level; for instance the vertex Ward identity
does not hold
\[
\Gamma_\alpha^\beta(p_1; \Lambda) - \Gamma_\alpha^\beta(p_2; \Lambda) = K_{\Lambda-1}^{-1}(p_1) (\not{p}_1 - m) - K_{\Lambda-1}^{-1}(p_2) (\not{p}_2 - m) \\
\neq (p_1^\mu - p_2^\mu) \Gamma_\mu^{(0)}(p_1, p_2; \Lambda)
\]

therefore there is no hope to recover gauge-invariance at any \(\Lambda\). On the other hand, with our choice of cutoff functions, the situation is much better and for instance the tree level vertex satisfies the correct transversality relation
\[
\Gamma_\alpha^\beta(p_1; \Lambda) - \Gamma_\alpha^\beta(p_2; \Lambda) = \left[ \not{p}_1 - m - i\Lambda\gamma_5 \right] - \left[ \not{p}_2 - m - i\Lambda\gamma_5 \right] \\
= (p_1^\mu - p_2^\mu) \Gamma_\mu^{(0)}(p_1, p_2; \Lambda)/e
\]

for any \(\Lambda\). This is obvious because a change of the fermion mass matrix from \(m \rightarrow m + i\Lambda\gamma_5\) does not break gauge-invariance. With our choice the only breaking of gauge-invariance is focused on the vertex \(\Gamma_\mu^{(0)}(k; \Lambda)\) which is not transverse,
\[
k^\mu \Gamma_\mu^{(0)}(k; \Lambda) = \left( -\frac{k^2}{\xi} + \Lambda^2 \right) k_\nu \neq 0.
\]

However the perturbative corrections \(\Gamma_\mu^{(l)}\) are transverse. Therefore we expect gauge-invariance be preserved for \(\Gamma_\mu^{(l)}(\Phi; \Lambda)\), \(l \geq 1\).

The logic of a formal proof is the following.

1. We suppose that the functional \(\Gamma(\Phi; \bar{\Lambda})\) is gauge-invariant (i.e. the proper vertices satisfy Ward identities) at some initial scale \(\bar{\Lambda}\).

2. We observe that in this hypothesis even the functional \(I(\Phi; \bar{\Lambda})\) is gauge invariant (i.e. the \(I_n(p_i; \bar{\Lambda})\) vertices satisfy Ward identities) at the scale \(\bar{\Lambda}\).

3. Therefore the evolution equation is gauge-invariant and, as a consequence, the Ward identities are satisfied to any \(\Lambda\).

One can convince himself of the transversality property of the \(I_n(p_i; \bar{\Lambda})\) vertices directly from their definition, by considering some specific case like \(I_{\mu\nu}, I_{\mu\alpha\beta}\), etc. and by using the Ward identities \([\Box]\) at the scale \(\bar{\Lambda}\). For instance one can prove the transversality relation \(k^\mu I_{\mu\nu}(k; \bar{\Lambda}) = 0\). To this
aim one must take in account all the pieces in $I_{\mu\nu}(k;\Lambda)$; moreover usual tricks, such as the use of the cyclic property of the trace and the possibility of doing translations in the momentum integrals must be applied. Notice that this latter translation can readily be done thanks to the intermediate regularization making convergent the integration.

The ultimate reason for the validity of all Ward identities is of geometric origin and it is completely elucidated in [33]. Notice that this proof does not require loop expansion and formally holds even non-perturbatively provided that all the momentum integrals implicit in the evolution equation are well defined. This is guaranteed by the ultraviolet regularization. The possibility of removing the regularization can be rigorously proved to all orders in perturbation theory, as shown in appendix C.

9 The Callan-Symanzik equation

In this section we explain the relation between our formulation of the Wilson renormalization group equation and the Callan-Symanzik equation [31]. Such a relation is expected because the two approaches are very similar: in both case we study the response of the field theory (i.e. of the functional $\Gamma(\Phi, K; \Lambda)$) under variations of a mass term.

In order to simplify the notation, initially we consider the Euclidean massless $\phi^4$ theory and then we extend to the QED case. The first step to convert the ERGE in a form suitable for comparison with the standard Callan-Symanzik equation consists in the introduction of the rescaling functions (at zero-momentum)

$$\hat{Z}_\phi(\Lambda) = \partial_{p^2}\Gamma_2|_{p^2=0}, \quad \hat{Z}_K(\Lambda) = \hat{Z}_\phi^{-1}(\Lambda)\Gamma_{2,1}|_{p_i=0},$$

and of the flowing coupling (at zero-momentum)

$$\hat{\lambda}(\Lambda) = \hat{Z}_\phi^{-2}(\Lambda)\Gamma_4|_{p^2=0}.$$  

(70)

(71)

Now we define the rescaled quantities

$$\hat{\phi} = \hat{Z}_\phi^{1/2}(\Lambda)\phi, \quad \hat{K} = Z_K(\Lambda)K, \quad \hat{Q}_\Lambda = \hat{Z}_\phi(\Lambda)^{-1}Q_\Lambda.$$  

(72)

With these redefinitions the relation $\hat{\Gamma}(\hat{\phi}, \hat{K}, \hat{\lambda}; \Lambda) = \Gamma(\phi, K; \Lambda)$, i.e.

$$\hat{\Pi}(\hat{\phi}, \hat{K}, \hat{\lambda}; \Lambda) + \frac{1}{2}\hat{\phi} \cdot \hat{Q}_\Lambda \hat{\phi} = \Pi(\phi, K; \Lambda) + \frac{1}{2}\phi \cdot Q_\Lambda \phi,$$

(73)
holds, therefore the proper vertices rescale as
\[ \hat{\Gamma}_{2n,l} = \hat{Z}_\phi^{-n} Z_k^{-l} \Gamma_{2n,l}, \quad \hat{\Gamma}_{2+2n,l} = \hat{Z}_\phi^{-n-1} \hat{Z}_K^{-l} \Gamma_{2+2n,l} . \] (74)

We point out that these redefinitions correspond to the imposition of the zero-momentum renormalization prescriptions for any \( \Lambda \),
\[ \frac{\partial}{\partial^2} \hat{\Gamma}_2 \bigg|_{p=0} \equiv 1, \quad \hat{\Gamma}_2 \bigg|_{p=0} \equiv \Lambda^2, \quad \hat{\Gamma}_4 \bigg|_{p_i=0} \equiv \hat{\lambda}(\Lambda), \quad \hat{\Gamma}_{2,1} \bigg|_{p_i=0} \equiv 1. \] (75)

The left hand side of the evolution equation for the rescaled functional \( \hat{\Gamma}(\hat{\phi}, \hat{K}, \hat{\lambda}; \Lambda) \) reads
\[ \Lambda \partial_\Lambda \Gamma = \frac{\partial}{\partial^2} \hat{\Gamma}_2, \quad \hat{\Gamma}_2 = \Lambda^2, \quad \hat{\Gamma}_4 = \hat{\lambda}(\Lambda), \quad \hat{\Gamma}_{2,1} = 1. \] (76)

where \( \Lambda \partial_\Lambda \) denotes the partial derivative with respect to the explicit \( \Lambda \) dependence of \( \hat{\Gamma}(\hat{\phi}, \hat{K}, \hat{\lambda}; \Lambda) \). It is also convenient to define
\[ \hat{\gamma}_\phi = -\frac{1}{2} \Lambda \partial_\Lambda \hat{Z}_\phi, \quad \hat{\gamma}_K = -\frac{\Lambda \partial_\Lambda \hat{Z}_K}{\hat{Z}_K}, \quad \hat{\beta} \equiv \Lambda \partial_\Lambda \hat{\lambda}. \] (77)

With these notations the left hand side of the evolution equation on proper vertices reads
\[ \Lambda \partial_\Lambda \hat{\Gamma}_{2n,l}(p_i, q_j; \Lambda) = \left( \Lambda \partial_\Lambda - 2n \hat{\gamma}_\phi - l \hat{\gamma}_K + \hat{\beta} \frac{\partial}{\partial \lambda} \right) \hat{\Gamma}_{2n,l}(p_i, q_j; \Lambda). \] (78)

In order to recover the Callan-Symanzik equation we observe that in the case of the mass cutoff \( (\hat{Q}_\Lambda = \Lambda^2) \) the relation \( \Lambda \partial_\Lambda \hat{Q}_\Lambda = 2(1 - \hat{\gamma}_\phi) \hat{Q}_\Lambda \) holds; therefore using \( \Lambda \partial_\Lambda \Gamma = -\Lambda \partial_\Lambda \ln Z \) and the path integral representation of the partition function \( Z(J, K; \Lambda) \) (suitably regularized in the ultraviolet) the right hand side of the evolution equation can be written as
\[ -\frac{\Lambda \partial_\Lambda \hat{Z}}{Z} = \int_x \frac{1}{2} \Lambda \partial_\Lambda \hat{Q}_\Lambda \langle \phi^2(x) \rangle_{J,K} = (1 - \hat{\gamma}_\phi) \Lambda^2 < \hat{\phi}^2(x) >_{J,K}. \] (79)

Replacing now the expectation value \( \Lambda^2 < \hat{\phi}^2(x) >_{J,K} \) with \( -2\Lambda^2 \int_x \delta \hat{V} \delta K(x) \) we immediately see that the Wilson evolution equation on proper vertices assumes the textbook Callan-Symanzik form
\[ \left( \Lambda \partial_\Lambda - 2n \hat{\gamma}_\phi - l \hat{\gamma}_K + \hat{\beta} \frac{\partial}{\partial \lambda} \right) \hat{\Gamma}_{2n,l} = 2\Lambda^2 (1 - \hat{\gamma}_\phi) \hat{\Gamma}_{2n,l+1}(p_i, q_j, 0; \Lambda). \] (80)
An alternative more explicit form for (80) is
\[
\left( \Lambda \delta_\Lambda - \hat{\gamma}_\phi \hat{\phi} \cdot \frac{\delta}{\delta \hat{\phi}} - \hat{\gamma}_K \hat{K} \cdot \frac{\delta}{\delta \hat{K}} + \hat{\beta} \frac{\partial}{\partial \hat{\lambda}} \right) \hat{\Gamma} = (1 - \gamma_\phi) \hat{I}(\hat{\phi}, \hat{K}; \hat{\lambda}; \Lambda)
\]
with
\[
\hat{I}(\hat{\phi}, \hat{K}; \hat{\lambda}; \Lambda) = \int_x \Lambda^2 \hat{\phi}^2(x) - \int_{xyz} \Lambda^2 \hat{\Gamma}^{-1}_2(x - y) \hat{\Gamma}_{\phi\phi}(y, z) \hat{\Gamma}^{-1}_2(z - x).
\]

We can consider separately the relevant and the irrelevant part of equation (81).

- The relevant part is described by the functions \( \hat{\gamma}_\phi, \hat{\gamma}_K \) and \( \hat{\beta} \). In fact, since the rescaling factors were chosen in such a way to canonically normalize the kinetic term and the \( \hat{\phi}^2 / 2 \) insertion we can compute \( \hat{\gamma}_\phi \) and \( \hat{\gamma}_K \) from (81) and (75), by obtaining
  \[
  \hat{\gamma}_\phi = \frac{-\frac{1}{2} \partial_{\hat{p}^2} \hat{I}_2|_0}{1 - \frac{1}{2} \partial_{\hat{p}^2} \hat{I}_2|_0},
  \]
  and
  \[
  \hat{\gamma}_K = -(1 - \gamma_\phi) \left. \hat{I}_{2,1} \right|_{p_i=0} - 2 \gamma_\phi.
  \]
  Moreover \( \hat{\beta} \) can be computed from the four point vertex,
  \[
  \hat{\beta} = 4 \hat{\lambda} \lambda + (1 - \hat{\delta}) \hat{I}_4|_{p_i=0}.
  \]

It is a simple exercise to compute \( \hat{\gamma}_\phi, \hat{\gamma}_K \) and \( \hat{\beta} \) at the lowest order in perturbation theory, obtaining the usual results
\[
\hat{\gamma}^{(1)}_\phi = 0, \quad \hat{\gamma}^{(1)}_K = -\frac{1}{16\pi^2} \hat{\lambda}, \quad \hat{\beta}^{(1)} = \frac{3}{16\pi^2} \hat{\lambda}^2.
\]

- The irrelevant part of the evolution equation (81) in the critical regime \( \Lambda^2 \ll p_i^2 \) can be simply neglected. In fact, by dimensional analysis, we see that the irrelevant (i.e. zero-momentum subtracted) vertices are suppressed as inverse powers of the momenta. Therefore in this regime the usual asymptotic Callan-Symanzik equation [31] holds
  \[
  \left( \Lambda \delta_\Lambda - 2n \hat{\gamma}_\phi - l \hat{\gamma}_K + \hat{\beta} \frac{\partial}{\partial \hat{\lambda}} \right) \hat{\Gamma}_{2n,\text{irr}}(p_i, \hat{\lambda}; \Lambda) \simeq 0, \quad \frac{\Lambda^2}{p_i^2} \to 0.
  \]
The deduction of the scaling equation (87) from the Exact Renormalization Equation, even if particularly transparent in our approach, is actually quite general and independent of the technical implementation of the evolution equation. In fact, in the asymptotic regime the right hand side of equation (87) vanishes by power counting independently of the specific form of the evolution equation. Moreover the $\hat{\gamma}_\phi$, $\hat{\gamma}_K$ and $\hat{\beta}$ functions are universal at first order in perturbation theory (actually the beta function is universal up to the second order in perturbation theory [8]).

We point out that an extension to gauge theories is straightforward only within our gauge-consistent formalism. Otherwise we are forced to follow the running of spurious couplings, related to the non-gauge-invariant operators, in terms of the physical one’s with the cumbersome mechanism of broken Ward identities or fine-tuning conditions. As a matter of fact, at one-loop the spurious couplings are finite in the large $-\Lambda$ limit (except the mass coupling which is quadratically divergent) therefore in practice in this approximation a similar approach is suitable also with generic cutoffs [11, 12]. Nevertheless, at higher loops, also the spurious couplings develop logarithmic divergences, even if they are sub-leading as a consequence of the (broken) Ward identities, and should be considered. All these complications are avoided in our gauge-invariant formulation and one obtains the expected asymptotic Callan-Symanzik equation for QED in the critical regime

$$m^2 << \Lambda^2 << p_i^2 << \Lambda^2_{UV}$$

i.e. when $\Lambda$ is large with respect to $m$ but small with respect to the momenta and the ultraviolet cutoff. The explicit formula is the usual one [31]

$$\left(\Lambda \delta_\Lambda - \hat{\gamma}_A N_A - \hat{\gamma}_\psi (N_\psi + N_\bar{\psi}) + \hat{\beta}_e \partial_\hat{\psi} + \hat{\beta}_\xi \partial_\hat{\xi}\right) \hat{\Gamma}_{irr}(p_i; \Lambda) \xrightarrow{\Lambda \to 0} 0$$

(89)

where

$$N_A = \int_x A_\mu \frac{\delta}{\delta A_\mu}, \quad N_\psi = \int_x \psi \frac{\delta}{\delta \psi}, \quad N_{\bar{\psi}} = \int_x \bar{\psi} \frac{\delta}{\delta \bar{\psi}}$$

(90)

and, as a consequence of Ward identities,

$$\hat{e} = e / \hat{Z}_A^{1/2}, \quad \hat{\xi} = \xi \hat{Z}_A, \quad \hat{\beta}_e = \hat{\beta}_A, \quad \hat{\beta}_\xi = -2 \hat{\xi} \hat{\gamma}_A.$$ (91)

We stress that in our approach all the usual consequence of gauge-invariance apply and in particular the unitarity property holds. Moreover one has the
usual control on the gauge-parameter dependence \[8\] and can prove that the flowing beta function is gauge-independent to all orders. On the contrary all these important properties do not have any simple analytic control with generic Wilsonian procedures. Equation \((89)\) can be also seen as a starting point for an improved perturbation theory in the sense of \([41, 42]\).

Finally, we remind the reader that there is a one-to-one correspondence between the Wilson or Callan-Symanzik renormalization group functions \(\beta\) and \(\gamma\) and the corresponding \(\beta\) and \(\gamma\) of the Gell-Mann and Low renormalization group, obtained by imposing the independence of the bare (ultraviolet) objects from the renormalization point \(\mu\). The interested reader is referred to \([20]\) for the general formulae and some explicit computation in YM theories (at one-loop) and in \(\phi^4\) (at two-loops).

\section{Conclusions and outlook}

In this paper we succeeded to give an explicitly gauge consistent Wilson Renormalization Group formulation of Quantum Electrodynamics. The solution of the problem is based on a specific choice of the infrared cutoff, corresponding to a mass term for both the photon and the electron, supplemented with a gauge-invariant ultraviolet regularization. In this context the Callan-Symanzik equation is equivalent to the Wilson’s equation and can be used in the study both of perturbative and non-perturbative applications.

On the perturbative side, a possible application of the scheme we propose is in the problem of the renormalization of composite operators in gauge theories. Here the mass cutoff is very convenient since it avoids the problem of the usual Wilsonian approaches where gauge-invariance is broken and gauge-invariant operators unavoidably mix with non-gauge invariant operators. Actually, in the issue of the perturbative computation of anomalous dimensions the formalism we present has the same level of efficiency of dimensional regularization.

The most important application of the Wilsonian approach is in the study of non-perturbative aspects. There are various well studied numerical methods of solution of the ERGE in the literature, based on some truncations of the effective action \([19]\) or the derivative expansion \([19]\). The important point is that since our version of the ERGE is gauge-consistent, no gauge-variant terms are generated by the evolution, as instead happens in generic Wilsonian approaches \([18]\). In general, non-perturbative methods can be applied
to our formulation, provided that we renormalize correctly the theory in the ultraviolet. We also mention that there is a paper in the literature in which the mass cutoff is introduced and used in order to study non-perturbative aspects of the $\phi^4$ theory at next to leading order in the gradient expansion [51].

There are a number of possible extensions of this work to other theories.

• A trivial extension is the application to scalar QED. The procedure works exactly as in the spinor case, provided that we use the following cutoff function for the scalars:

$$\tilde{K}_{\Lambda\infty}(q) = \frac{q^2 - m^2}{q^2 - m^2 - \Lambda^2}.$$  

In this way the quadratic term $\phi^* \tilde{Q}_\Lambda \phi = \Lambda^2 \phi^* \phi$ is explicitly gauge invariant and the Ward identities breaking is exactly as in (19). The three-dimensional case has been studied non-perturbatively in [15] but without control of gauge-invariance and it would be a very practical model to test the method since no ultraviolet regularization is needed.

• Another straightforward extension is the application to supersymmetric Abelian gauge theories, because the Wilsonian formulation is consistent with supersymmetry [47, 48]. In this case it is sufficient to add to the usual classical gauge-invariant action of super QED $S_{CL}(V, \phi_{\pm}, \phi_{\pm}^\dagger)$ the quadratic term

$$\int_{x,\theta,\bar{\theta}} \left[ -\frac{1}{32}\Lambda^2 V^2 - \frac{1}{128\xi} D^2 V \bar{D}^2 V \right] + \frac{1}{32} \int_{x,\theta} i\Lambda \phi_+ \phi_- + \text{h.c.}$$

In this way $\Gamma^{(0)}(V, \phi_{\pm}, \phi_{\pm}^\dagger, \Lambda)$ is well behaved under the infinitesimal gauge transformation

$$\mathcal{W}_f V = f + f^*, \quad \mathcal{W}_f \phi_{\pm} = \pm ie f \phi_{\pm}, \quad \mathcal{W}_f \phi_{\pm}^\dagger = \mp ie f \phi_{\pm}^\dagger$$

in the sense that the breaking term is linear in the field $V$, i.e. we can maintain to all orders the Ward identity

$$\mathcal{W}_f \Gamma = \int_{x,\theta,\bar{\theta}} (f + f^*) \left( -\frac{1}{16}\Lambda^2 + \frac{\bar{D}^2 D^2 + D^2 \bar{D}^2}{128\xi} \right)$$
which is the supersymmetric generalization of (19). We remark that even the analysis of supersymmetric theories is particularly simple since in this case mass divergences automatically cancel and therefore the step of an intermediate regularization can be avoided. This is ultimately related to the supersymmetric solution of the naturalness problem [4] i.e. the absence of quadratic divergences.

- From these examples it is clear that our procedure formally works for any theory characterized by linearly broken Ward identities. In particular the formalism applies to non-Abelian gauge theories in algebraic non-covariant gauges. However, in this case, one expects some difficulty related to the presence of a gluon “mass” $\Lambda^2 \neq 0$. A detailed study of the question is given in [53].

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A The exact evolution equation

For sake of completeness, and in order to fix the notation, here we briefly review the deduction of the ERGE in the formalism of the cutoff effective action $\Gamma(\Lambda)$. The standard deduction can be easily generalized in order to manage the problem of the renormalization of composite operators, simply by introducing sources $K$ associated with the operators of interest. The simplest case, as seen in section 6, is the operator $\frac{1}{2} \phi^2(x)$ in the $\phi^4$ theory. In the following, for sake of convenience, we directly work in the Minkowski space even if an analytic continuation in the Euclidean space should be understood in order to give a rigorous meaning to the momentum integrals, and also to have a more clear Wilsonian interpretation.

The starting point is the cutoff generating functional

$$Z(J, K; \Lambda) = \int [d\Phi] e^{iS_B(\Phi, K) + \frac{i}{2} \Phi Q\Phi + i J \cdot \Phi}$$

(93)
in which, since the quadratic term cuts the infrared modes, in practice we are integrating only the degrees of freedom over $\Lambda$, i.e. the ultraviolet modes.

In equation (93), $S_B(\Phi, K)$ denotes the bare action and some kind of ultraviolet regularization is understood even if it is not explicitly displayed. The evolution equation is derived simply by studying the behavior of the generating functional under variation of the infrared cutoff $\Lambda$. One readily obtains

$$\Lambda \partial_{\Lambda} Z(J, K; \Lambda) = \frac{i}{2} (-)^A \dot{Q}_{\Lambda,AB} \frac{\delta}{\delta i J_B} \frac{\delta}{\delta i J_A} Z(J, K; \Lambda),$$

(94)

where the de Witt notation is used, i.e. the indices $A, B$ represents both continuous and discrete indices and sums and integrals are understood. The symbol $(-)^A$ gives a plus sign for bosonic fields and a minus sign for fermionic fields. The functional derivative operator $\frac{\delta}{\delta i J_B}$ acts from the left whereas $\frac{\delta}{\delta i J_A}$ acts from the right. We also use the abbreviations

$$\tilde{\Phi} Q_A \Phi = (-)^A \Phi^A Q_{\Lambda,AB} \Phi^B, \quad \tilde{\Phi}^A = (-)^A \Phi^A.$$

(95)

From equation (94) one can obtain the evolution of the generating functional of cutoff connected Green functions $W(J; \Lambda) = -i \log Z(J; \Lambda),

$$\Lambda \partial_{\Lambda} W = -\frac{i}{2} (-)^A \dot{Q}_{\Lambda,AB} \left( \frac{\delta}{\delta J_B} \frac{\delta}{\delta J_A} W + i \frac{\delta W}{\delta J_B} \frac{\delta W}{\delta J_A} \right).$$

(96)

We recall that $W(J, 0; \Lambda)$ is directly related to the Wilsonian effective action \cite{20, 21}. Here we are interested in the evolution equation for the Legendre transformed effective action (simply called cutoff effective action or $\Lambda$-RG action)

$$\Gamma(\Phi, K; \Lambda) = -J_A \Phi^A + W(J, K; \Lambda), \quad \Phi^A = \frac{\delta W}{\delta J_A}.$$

(97)

With some simple manipulation one obtains

$$\Lambda \partial_{\Lambda} \left( \Gamma - \frac{1}{2} \tilde{\Phi} Q_A \tilde{\Phi} \right) = I(\Phi, K; \Lambda) = \frac{i}{2} \text{Str} \dot{Q}_A \Gamma^{-1}_{\tilde{\Phi} \tilde{\Phi}}$$

(98)

where the supertrace notation is used and we have defined

$$\left( \Gamma^{-1}_{\tilde{\Phi} \tilde{\Phi}} \right)^{BA} = -\left. \frac{\delta}{\delta J_B} \frac{\delta}{\delta J_A} W \right|_{J=J(\Phi)}.$$

(99)
Although this form of the evolution equation is well suited for non-perturbative studies, in particular to perform truncation of the evolution equation such as for instance the Local Potential Approximation [19, 49], nevertheless in order to extract the perturbative expansion and to give renormalizability proofs another form of (98) is more convenient. To this aim we introduce an auxiliary functional \( \bar{\Gamma}_{\Phi} \), implicitly defined by the relation

\[
\bar{\Gamma}_{\Phi} = \bar{\Gamma}_{\Phi} - \frac{1}{2} \bar{\Gamma}_{\Phi} - \frac{1}{2} \bar{\Gamma}_{\Phi} \bar{\Gamma}_{\Phi}^{-1} \bar{\Gamma}_{\Phi}^{-1}, \quad \text{(100)}
\]

where \( \bar{\Gamma}_{\Phi} \equiv \bar{\Gamma}_{\Phi}|_{\Phi=0,K=0} \) is the two-point function. Now the right hand side of equation (98) can be rewritten in the condensed form

\[
I(\Phi, K; \Lambda) \equiv -\frac{i}{2} S\text{Tr} \dot{Q}\Lambda^{-1}(\bar{\Gamma}_{\Phi} \bar{\Gamma}_{\Phi}^{-1} - 1). \quad \text{(101)}
\]

It is convenient to introduce the following condensed notation for the proper vertices without insertions of operators (\( K = 0 \))

\[
\Gamma(\Phi; \Lambda) = \frac{1}{n!} \sum_{n=2}^{\infty} \Phi A_n \cdots \Phi A_1 \Gamma A_1 \cdots A_n
\]

\[
I(\Phi; \Lambda) = \frac{1}{n!} \sum_{n=2}^{\infty} \Phi A_n \cdots \Phi A_1 I A_1 \cdots A_n \quad \text{(102)}
\]

\[
\bar{\Gamma}_{\Phi,AB}(\Phi; \Lambda) = \frac{1}{n!} \sum_{n=1}^{\infty} \Phi A_n \cdots \Phi A_1 \bar{\Gamma} A_{A_1} \cdots A_n B
\]

(for instance \( \Gamma_{A_1 A_2} \) in a more explicit notation corresponds both to \( \Gamma_{\mu\nu}(p; \Lambda) \cdot (2\pi)^4 \delta^4(p + q) \) and to \( \Gamma_{\alpha\beta}(p; \Lambda)(2\pi)^4 \delta^4(p + q) \)). In this way the auxiliary functional \( \bar{\Gamma}_{\Phi} \) introduced in (100) can be explicited by using the recursive formula

\[
\bar{\Gamma}_{A_{A_1} \cdots A_n B} = \Gamma_{A_{A_1} \cdots A_n B} - \sum_{k=1}^{n-1} \Gamma_{A_{A_1} \cdots A_k C} (\Gamma_2^{-1} C D) \bar{\Gamma}_{D A_{k+1} \cdots A_n B}. \quad \text{(103)}
\]

The graphical representation of equation (103) is reported in figure 1. See also [20, 34] for explicit examples. The vertices with insertions of operators are obtained straightforwardly by deriving successively the evolution equation with respect to the sources \( K \) and by taking \( K = 0 \).
B  Explicit form of relevant and irrelevant functionals for QED

In the QED case the relevant effective action has the following Ward-identities-consistent form,

\[ \Gamma_{rel}(\Phi; \Lambda) = \int_x \frac{1}{2} \Lambda^2 A \cdot A - \frac{1}{2\xi} (\partial \cdot A)^2 + \Gamma'_{rel}(\Phi; \Lambda), \]  

(104)

where

\[ \Gamma'_{rel}(\Phi; \Lambda) = \int_k -\frac{1}{2} c_A(\Lambda) A_\mu(-k) k^\mu k^\nu A_\nu(k) + \int_x e c_\psi(\Lambda) \bar{\psi} A \psi \]
\[ \int_p \bar{\psi}(p) \left[p c_\psi(\Lambda) - c_1(\Lambda) - ic_2(\Lambda) \gamma_5 \right] \psi(p), \]

(105)

For sake of brevity, we omitted the analogous contributions for Pauli-Villars fields and we defined

\[ t_{\mu\nu}(k) = g_{\mu\nu} - \ell_{\mu\nu}(k), \quad \ell_{\mu\nu}(k) = k_\mu k_\nu / k^2. \]

(106)

The relevant couplings are

\[ c_A(\Lambda) = - \partial k^2 \frac{1}{3} t^{\mu\nu} \Gamma_{\mu\nu} \bigg|_{k=0}, \]

(107a)

\[ c_\psi(\Lambda) = \partial_{\mu\nu} \frac{1}{16} \gamma^{\mu,\beta\alpha} \Gamma_{\alpha\beta} \bigg|_{p=0}, \]

(107b)

\[ c_1(\Lambda) = - \frac{1}{4} \delta^{\beta\alpha} \Gamma_{\alpha\beta} \bigg|_{p=0}, \]

(107c)

\[ c_2(\Lambda) = \frac{i}{4} \gamma^{\beta\alpha} \Gamma_{\alpha\beta} \bigg|_{p=0}, \]

(107d)

where \( \Gamma_{\mu\nu} \) and \( \Gamma_{\alpha\beta} \) are the photon and electron two-point functions. The renormalization prescriptions are

\[ c_A(\Lambda_R) = 1, \quad c_\psi(\Lambda_R) = 1, \quad c_1(0) = m, \quad c_2(0) = 0. \]

(108)

The irrelevant part of the photon two-point function is given by the formula

\[ \Gamma_{\mu\nu, irr}(k; \Lambda) = \Gamma_{T, irr}(k^2; \Lambda) t_{\mu\nu}(k) + \Gamma_{L, irr}(k^2; \Lambda) \ell_{\mu\nu}(k), \]

(109)
The irrelevant part of the electron two-point function is
\[ \Gamma_{\alpha\beta, \text{irr}}(p) = \Gamma_{\alpha\beta}(p) - \Gamma_{\alpha\beta}(\bar{p})\bigg|_{\bar{p}=0} - p_{\mu} \frac{\partial}{\partial \bar{p}_{\mu}} \Gamma_{\alpha\beta}(\bar{p})\bigg|_{\bar{p}=0}. \]

The irrelevant part of the photon-electron-positron vertex is
\[ \Gamma_{\mu\alpha\beta, \text{irr}}(p, p') = \Gamma_{\mu\alpha\beta}(p, p') - \Gamma_{\mu\alpha\beta}(0, 0). \]

The same decomposition into relevant and irrelevant parts holds for the vertices of the functional \( I(\Phi; \Lambda) \) and also in the rescaled case, i.e. for the functionals \( \hat{\Gamma}(\hat{\Phi}; \hat{\lambda}; \Lambda) \) and \( \hat{I}(\hat{\Phi}; \hat{\lambda}; \Lambda) \).

As we remarked in section 6, to properly renormalize the theory one should also consider the renormalization of the composite gauge-invariant operators \( O_1(x) = \bar{\psi}\psi \) and \( O_2(x) = i\bar{\psi}\gamma_5\psi \), which can be performed straightforwardly by adding the corresponding external sources \( K_1(x) \) and \( K_2(x) \). This gives two new dimensionless relevant couplings \( \sigma_1(\Lambda) \) and \( \sigma_2(\Lambda) \) and two new renormalization prescriptions and subtractions. We still stress that differently from other Wilsonian procedures spurious couplings corresponding to non-gauge-invariant operators are never generated.

## C The renormalizability proof

In this section we give a simple renormalizability proof which generalizes the analysis of [20] to a quite large class of cutoff functions. For notational commodity we first present the method for the massless euclidean \( \phi^4 \) theory. We denote by \( M_0 \) the mass scale where the ultraviolet regularization becomes effective and we give the proof for cutoff functions such as the integral
\[ \lim_{M_0 \to \infty} \int_q |\hat{\Delta}_{\Lambda^\infty}(q; M_0)\Delta_{\Lambda^\infty}^{-1}(q; M_0)| = c\Lambda^4 \]  

is finite. For instance the exponential cutoff \( K_{\Lambda^\infty}(q) = 1 - \exp(-q^2/\Lambda^2) \) satisfies [111] with coefficient \( c = \zeta(3)/(4\pi^2) \). Notice that the mass cutoff does
not belong to this class and the renormalizability proof requires a different analysis \[32\]. For this reason we prefer to give the proof in the additional hypothesis (111), which is not needed from a rigorous point of view, but it is technically very convenient.

The renormalizability proof is essentially based on the perturbative evolution equation (28) which gives the proper vertices at loop \( \ell + 1 \) in terms of integrals containing the proper vertices at lower loops. We simply prove that these integrals are well defined when the ultraviolet regularization is removed, i.e. \( M_0 \to \infty \). Notice that the infrared behavior is safe by construction, because the infrared cutoff \( \Lambda \) at this level is assumed non-zero. It is convenient to define the norms at loops \( \ell' = 0, 1, 2, \ldots \),

\[
|||\Gamma^{(\ell')}_{2n}|||_\Lambda \equiv \lim_{M_0 \to \infty} \max_{p_i^2 < \Lambda^2} |\Gamma^{(\ell')}_{2n}(p_1 \ldots p_{2n}; \Lambda, M_0)|. \tag{112}
\]

The tree level vertices have finite norm since \( |||\Gamma^{(0)}_{2n}|||_\Lambda \sim \Lambda^{4-2n} \) is finite for \( \Lambda \neq 0 \). It is also convenient to introduce the functions

\[
X^{(\ell')}_{2n+2} = \frac{1}{2} [\Gamma^{-1}_2(q; \Lambda, M_0) \tilde{\Gamma}_{2n+2}(q, p_i, -q; \Lambda, M_0) \Gamma^{-1}_2(q; \Lambda, M_0) \Delta^{-1}_\Lambda(q; M_0)]^{(\ell')}
\]

and their norms

\[
|||X^{(\ell')}_{2n+2}|||_\Lambda \equiv \lim_{M_0 \to \infty} \max_{q, p_i^2 < \Lambda^2} |X^{(\ell')}_{2n+2}(q, p_i, -q; \Lambda, M_0)|. \tag{113}
\]

At tree level \( |||X^{(0)}_{2n+2}|||_\Lambda \sim \Lambda^{-2n} \) is finite for \( \Lambda \neq 0 \). In general, if the norms \( |||\Gamma^{(\ell')}_{2n}|||_\Lambda \) are finite for all loops \( \ell' \leq \ell \), then the norms \( |||X^{(\ell')}_{2n+2}|||_\Lambda \) are finite, since they are obtained from functions \( \Gamma^{(\ell')}_{2m} \) with \( 2m \leq 2n + 2 \) and \( \ell' \leq \ell \), by using the recursive relation (10) between functions \( \tilde{\Gamma}_{2m} \) and vertices \( \Gamma_{2m} \).

With these notations the evolution equation reads

\[
\dot{\Gamma}^{(\ell+1)}_{2n} = \int_q \Delta_\Lambda(q; M_0) \Delta^{-1}_\Lambda(q; M_0) X^{(\ell)}_{2n+2}(q, p_i, -q; \Lambda, M_0). \tag{114}
\]

We split the renormalizability proof in four steps.

1. Inductive hypothesis at loop \( \ell \). Due to Lorentz-invariance the proper vertices \( \Gamma_{2n}(p_i; \Lambda) \) only depend of the invariant combinations \( s_k = \)
(p_i^2, p_i \cdot p_j) \text{ (there are } n(2n - 1) \text{ independent invariants)}. \ We take as inductive hypothesis}

\[ ||\partial_{s_{k_1}} \ldots \partial_{s_{k_m}} \Gamma^{(\ell')}_{2n}||_\Lambda \sim \Lambda^{4-2n-2m} < \infty, \quad \ell' = 0, \ldots, \ell. \]  \tag{115}

From (115) we have

\[ ||\partial_{s_{k_1}} \ldots \partial_{s_{k_m}} X^{(\ell')}_{2n+2}||_\Lambda \sim \Lambda^{-2n-2m} < \infty \quad \ell' = 0, \ldots, \ell. \]  \tag{116}

2. Lemma 1. As a direct consequence of property (111) and definitions (112), (113), the inequality

\[ ||\int \hat{\Delta}_{\Lambda}^{-1}(q) \Delta_{\Lambda}^{-1}(q) X^{(\ell')}_{2n+2}(q, p_1 \ldots p_{2n}, -q)||_\Lambda \leq c \Lambda^4 ||X^{(\ell')}_{2n+2}||_\Lambda \]  \tag{117}

holds.

3. Lemma 2. There are important bounds for the irrelevant vertices \( \Gamma^{(\ell')}_{2, \text{irr}} \) and \( \Gamma^{(\ell')}_{4, \text{irr}} \). As a consequence of the identities

\[ f(z) - f(0) - zf'(0) = z^2 \int_0^1 dx(1 - x)f''(zx) \]  \tag{118}

and

\[ f(z) - f(0) = z \int_0^1 dx f'(zx), \]  \tag{119}

which hold for any analytic function, the inequalities

\[ ||\Gamma^{(\ell')}_{2, \text{irr}}||_\Lambda = ||\Gamma^{(\ell')}_{2}(s) - \Gamma^{(\ell')}_{2}(0) - s\partial_s \Gamma^{(\ell')}_{2}(0)||_\Lambda \leq \frac{\Lambda^4}{2} ||\partial_s^2 \Gamma^{(\ell')}_{2}||_\Lambda \]  \tag{120}

and

\[ ||\Gamma^{(\ell')}_{4, \text{irr}}||_\Lambda = ||\Gamma^{(\ell')}_{4}(p_i) - \Gamma^{(\ell')}_{4}(0)||_\Lambda \leq \Lambda^2 ||\partial_{s_n} \Gamma^{(\ell')}_{4}||_\Lambda \]  \tag{121}

hold.

\footnote{Actually we expect some logarithmic behavior, and a better \textit{Ansätze} should be}

\[ ||\Gamma^{(\ell')}_{2n}||_\Lambda = \Lambda^{4-2n} P^{(\ell')}_{2n} \left( \log \frac{\Lambda^2}{\Lambda_{R}^2} \right), \]

where \( P^{(\ell')}_{2n} \) is a polynomial of degree increasing with the loop number \( \ell' \). However this does not change our conclusions about the convergence of integrals. One can easily prove that this \textit{Ansätze} is consistent with the evolution equation, i.e. assuming the \textit{Ansätze} at loop \( \ell \), it holds at loop \( \ell + 1 \).
4. Inductive hypothesis at loop $\ell + 1$. We have to prove
\[
||\Gamma^{(\ell+1)}_{2n}||_\Lambda \leq ||\Gamma^{(\ell+1)}_{2n,rel}||_\Lambda + ||\Gamma^{(\ell+1)}_{2n,irr}||_\Lambda < \infty. \tag{122}
\]
The finiteness of $||\Gamma^{(\ell+1)}_{2n,rel}||_\Lambda$, i.e. of relevant coefficients, comes from the inductive hypothesis and lemma 1:
\[
|c^{(\ell+1)}_{\text{rel}}(\Lambda)| \leq \int_0^\Lambda \frac{d\Lambda_\ell}{\Lambda_\ell} \left| \int_q \hat{\Delta}_{\Lambda_\ell,\infty} \Delta_{\Lambda_\ell,\infty}^{-1} X^{(\ell)}_4 \right|_0 \sim \Lambda^2 \tag{123a}
\]
\[
|c^{(\ell+1)}_{\text{irr}}(\Lambda)| \leq \int_\Lambda^\infty \frac{d\Lambda_\ell}{\Lambda_\ell} \left| \int_q \hat{\Delta}_{\Lambda_\ell,\infty} \Delta_{\Lambda_\ell,\infty}^{-1} \partial_s X^{(\ell)}_4 \right| \sim \Lambda^0 \tag{123b}
\]
\[
|c^{(\ell+1)}_{\text{irr}}(\Lambda)| \leq \int_\Lambda^\infty \frac{d\Lambda_\ell}{\Lambda_\ell} \left| \int_q \hat{\Delta}_{\Lambda_\ell,\infty} \Delta_{\Lambda_\ell,\infty}^{-1} X^{(\ell)}_6 \right|_0 \sim \Lambda^0. \tag{123c}
\]
For the irrelevant vertices by using lemma 1 and lemma 2 we obtain
\[
||\Gamma^{(\ell+1)}_{2n,irr}||_\Lambda \leq \frac{\Lambda^2}{2} \int_0^\Lambda \frac{d\Lambda_\ell}{\Lambda_\ell} \left| \int_q \hat{\Delta}_{\Lambda_\ell,\infty} \Delta_{\Lambda_\ell,\infty}^{-1} \partial_s^2 X_4 \right|_0 \sim \Lambda^2 \tag{124}
\]
and analogously
\[
||\Gamma^{(\ell+1)}_{4,irr}||_\Lambda \leq \frac{\Lambda^2}{2} \int_\Lambda^\infty \frac{d\Lambda_\ell}{\Lambda_\ell} \Lambda_\ell^4 \Lambda_\ell^{-6} \sim \Lambda^2 \tag{125}
\]
\[
||\Gamma^{(\ell+1)}_{2n}||_\Lambda \leq \frac{\Lambda^2}{2} \int_\Lambda^\infty \frac{d\Lambda_\ell}{\Lambda_\ell} \left| \int_q \hat{\Delta}_{\Lambda_\ell,\infty} \Delta_{\Lambda_\ell,\infty}^{-1} \partial_s X_6 \right|_0 \Lambda_\ell \sim \Lambda^0 \tag{126}
\]
The convergence of $\Lambda_\ell$-integrals is guaranteed for the power counting and the subtractions (120), (121); therefore the proper vertex at loop $\ell + 1$ are well defined. A fortiori that holds for the derivatives $||\partial_{s_{k_1}} \ldots \partial_{s_{k_m}} \Gamma^{(\ell+1)}_{2n,irr}||_\Lambda < \infty$. Therefore the inductive hypothesis (115) holds at loop $\ell + 1$ also for irrelevant vertices. By induction, the renormalizability proof holds to any finite order $\ell$.

The same approach can be applied to the QED case: one easily prove that all the $\Lambda_\ell$-integrals are well defined by using the subtractions and the behavior expected by dimensional analysis,
\[
\lim_{M_0 \to \infty} \max_{p^2_i \leq \Lambda^2} ||\Gamma^{(\ell)}_{n_\bar{\psi} \psi, n_A}(p_i; \Lambda)| \sim \Lambda^{4-3n_\bar{\psi}-n_A}, \tag{127}
\]
where \( n_A \) is the number of external photon lines and \( n_{\bar{\psi}\psi} \) the number of external fermion-antifermion lines.

Actually, one can easily convince himself that this kind of proof holds for any theory respecting the power counting criterium.

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