Interaction-induced interference for two independent Bose–Einstein condensates

Hongwei Xiong$^{1,2,3}$, Shujuan Liu$^{1,2,3}$ and Mingsheng Zhan$^{1,2}$

1 State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, People’s Republic of China
2 Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, People’s Republic of China
3 Graduate School of the Chinese Academy of Sciences, People’s Republic of China
E-mail: xionghongwei@wipm.ac.cn

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Abstract. After removal of the double-well potential trapping two initially independent Bose condensates, the density expectation value is calculated when both the exchange symmetry of identical bosons and interatomic interaction are considered. The density expectation value and evolution equations are obtained based on both the first-quantization and second-quantization methods. When the interatomic interaction is considered carefully, after the overlapping of two initially independent condensates, it is shown that there is a nonzero interference term in the density expectation value. It is found that the calculated density expectation value with this model agrees with the interference pattern observed in the experiment by Andrews et al (1997 Science 275 637). The nonzero interference term in the density expectation value physically arises from the exchange symmetry of identical bosons and interatomic interaction which make two initially independent condensates become coherent after the overlapping. For two initially independent condensates, our researches show that there is an interaction-induced coherence process.
1. Introduction

The coherence property plays an essential role in the wave nature of Bose–Einstein condensates (BECs), and it has been investigated intensively after the experimental realizations of BECs in dilute gases [2]–[4]. For a single BEC which is a perfect quantum fluid at zero temperature, the existence of a macroscopic wavefunction (or order parameter) means that spatial coherence is an intriguing property of the condensate. In the dilute Bose-condensed gases, the spatial coherence can be investigated directly by interfering two Bose condensates. For two coherently separated BECs, because the relative phase of two sub-condensates is locked, it is not surprising that there is a clear interference pattern when the two sub-condensates are allowed to overlap. In the celebrated experiment by Andrews et al [5], however, high-contrast fringes were observed even for two completely independent condensates at an initial time. In this experiment, to prepare two initially independent condensates, the dilute Bose gases were evaporatively cooled in a double-well potential created by splitting a magnetic trap in half with a far-off blue-detuned laser beam (see also [6]). In particular, the height of the external potential due to the laser beam is much larger than the chemical potential of two separated condensates, which means that the tunnelling current can be safely omitted. After switching off the double-well potential, the two initially independent condensates overlapped and high-contrast fringes were observed in [5]. This experimental result shows clearly that there is a spatial coherence property after the overlapping between two independent interacting condensates.

Although high-contrast fringes were observed for two initially independent condensates in the literature (see for example [7]–[9] and references therein), it is shown that there is no interference term in the density expectation value for two initially independent condensates.

4 See special issue Nature Insight [1].
To solve the contradiction between this result and the observed high-contrast fringes [5] for two initially independent condensates, the observed high-contrast fringes were interpreted with the aid of the high-order correlation function \( P(\mathbf{r}, \mathbf{r}', t) \) (which is an oscillation function of \( \mathbf{r} - \mathbf{r}' \)) and quantum measurement process in the present popular viewpoint. Several theories have been proposed to interpret the observed high-contrast fringes for two initially independent condensates such as the stochastic simulations of the photon detection for atoms [10], the expansion of the Fock state by the linear superposition of coherent states [11], and the continuous measurement theory [12].

In the present study, we calculate the density expectation value for two initially independent condensates by including carefully the interatomic interaction. Quite different from the simple derivation in other theories, it is found that for the case of two initially independent condensates, upon expansion, there is a nonzero interference term in the density expectation value when the interatomic interaction and exchange symmetry of identical bosons are both taken into account. To exclude any possible error in the cumbersome derivations, we calculate the density expectation value and evolution equations by both the first-quantization and second-quantization methods, and the same results are obtained. After removing the double-well potential trapping the two initially independent condensates, we give the theoretical result of the density expectation value which agrees with the interference pattern observed in [5].

To show the interaction-induced coherence process between two initially independent Bose condensates when both the interatomic interaction and exchange symmetry of identical bosons are considered carefully, the paper is organized as follows. In section 2, we give the general expression of the density expectation value for two initially independent condensates based on the many-body wavefunction of the whole system. It is shown that there is a nonzero interference term when the wavefunctions of two initially independent condensates are no more orthogonal after their overlapping. In section 3, we give the general expression of the overall energy of the whole system, and the evolution equations are given by the action principle. In section 4, we give a brief proof why the wavefunctions of two initially independent condensates become non-orthogonal after their overlapping when the interatomic interaction is considered. In this section, an effective order parameter is introduced to show further the interaction-induced coherence formation process for two initially independent condensates. In section 5, we give the numerical results for the evolution of the density expectation value according to the experimental parameters in [5]. It is shown clearly that our theoretical results of the density expectation value agree with the experimental results of the interference patterns. In section 6, the evolution of the density expectation value is given for different coupling constants. It is shown that increasing the coupling constant has the effect of enhancing the interference effect for two independent condensates. To show further the physical mechanism of the interaction-induced coherence process, in section 7, we prove that the theoretical results based on the second-quantization method is the same as the results derived from the many-body wavefunction in the previous sections. In section 8, the role of quantum fluctuations and the orthogonality of the many-body wavefunction are discussed. Finally, we give a brief summary and discussion in the last section.

### 2. The expression of the density expectation value for two initially independent BECs

First, we give a brief introduction to the scheme of observing the interference effect of two separated condensates. As shown in figure 1, the double-well potential can be created by...
superposing a far-off blue-detuned laser beam (which generating a repulsive optical dipole force for atoms) upon the magnetic trap. The combined double-well potential is

\[ V_{\text{ext}} = \frac{1}{2}m[\omega_x^2(x - x_0)^2 + \omega_y^2y^2 + \omega_z^2z^2] + U_0 e^{-(x-x_0)^2/w_x^2-(y^2+z^2)/w_y^2}, \]

where the first term is the trapping potential due to the magnetic trap, while the second term represents the potential due to the laser beam.

For the Bose gases confined in this double-well potential, there are two quite different cases. (i) If the height \( U_0 \) of the external potential due to the laser beam is smaller than the chemical potential of the system, the two sub-condensates can be regarded as coherently separated after the evaporative cooling. In this situation, the two coherently separated sub-condensates can be regarded as a single condensate. The relative phase of two sub-condensates is thus locked. (ii) If the height \( U_0 \) of the laser beam is much larger than the chemical potential of the system so that the tunnelling effect can be omitted safely, the two condensates are completely independent after the evaporative cooling.

For two coherently separated sub-condensates, every atom is described by the following normalization wavefunction

\[ \phi_c(\mathbf{r}, t) = \left[ \sqrt{N_1}\phi_{c1}(\mathbf{r}, t) + \sqrt{N_2}\phi_{c2}(\mathbf{r}, t) \right]/\sqrt{N}, \]

where \( \phi_{c1}(\mathbf{r}, t) \) and \( \phi_{c2}(\mathbf{r}, t) \) are the normalization wavefunctions accounting for the two sub-condensates. \( N = N_1 + N_2 \) is the total number of particles. Before removing the double-well
After removing the double-well potential, the average particle numbers in each condensate are \( N_1 \) and \( N_2 \), respectively. After removing the double-well potential, the evolution of the wavefunction \( \phi_c(\mathbf{r}, t) \) can be obtained based on the Gross–Pitaevskii (GP) equation [13]–[15]. The density expectation value is then
\[
n_c(\mathbf{r}, t) = N|^\phi_c(\mathbf{r}, t)|^2 = N_1|^\phi_{c1}(\mathbf{r}, t)|^2 + 2\sqrt{N_1N_2}\text{Re}[^\phi^*_{c1}(\mathbf{r}, t)\phi_{c2}(\mathbf{r}, t)] + N_2|^\phi_{c2}(\mathbf{r}, t)|^2. \tag{3}
\]
The second term in the above equation accounts for the interference effect when there is an overlapping between two sub-condensates upon expansion. For \( \phi_c(\mathbf{r}, t) \) satisfying the GP equation, the nonlinear effects in the interference pattern of two coherently separated sub-condensates were investigated in [16, 17].

As shown in figure 1(b), if the intensity of the blue-detuned laser beam is sufficiently high so that the tunnelling effect can be omitted, the two condensates can be regarded to be completely independent. In this situation, the number of particles \( N \) is fixed. We assume that there are \( N_1 \) particles described by the wavefunction \( \phi_1 \) in the left well, while there are \( N_2 \) particles described by the wavefunction \( \phi_2 \) in the right well. To investigate clearly the role of the exchange symmetry of identical particles and interatomic interaction, we calculate in the following the density expectation value \( n_d(\mathbf{r}, t) \) directly from the many-body wavefunction. In section 7, we will also calculate the density expectation value based on the second-quantization method, and the same result is obtained.

We will prove in section 4 that in the presence of the interatomic interaction, \( \phi_1(\mathbf{r}, t) \) and \( \phi_2(\mathbf{r}, t) \) will become non-orthogonal after the overlapping between two initially independent condensates. Thus we consider here the following general case from the beginning
\[
\zeta(t) = \int \phi_1(\mathbf{r}, t)\phi^*_2(\mathbf{r}, t)\, dV = |\zeta(t)|e^{i\omega_N} \tag{4}
\]
After removing the double-well potential, the two initially independent condensates will overlap, and thus one should consider the indistinguishability of identical particles. When the exchange symmetry of identical bosons is considered, the many-body wavefunction is [18]
\[
\Psi_{N_1N_2}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = A_N\sqrt{\frac{N_1!N_2!}{N!}}\sum_P P[\phi_1(\mathbf{r}_1, t)\cdots\phi_1(\mathbf{r}_{N_1}, t)\times\phi_2(\mathbf{r}_{N_1+1}, t)\cdots\phi_2(\mathbf{r}_{N_1+N_2}, t)], \tag{5}
\]
where \( P \) denotes the \( N!/N_1!N_2! \) permutations for the particles in different single-particle state \( \phi_1 \) or \( \phi_2 \). \( A_N \) is a normalization factor to assure \( \int |\Psi_{N_1N_2}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)|^2\, d\mathbf{r}_1\, d\mathbf{r}_2\cdots\, d\mathbf{r}_N = 1 \). \( A_N \) is determined by the following equation:
\[
A_N\left[\sum_{i=0}^{\min(N_1, N_2)} \frac{N_1!N_2!|\zeta(t)|^2}{i!(N_1-i)!(N_2-i)!}\right]^{1/2} = 1. \tag{6}
\]
In this paper, to give a concise expression for various coefficients such as \( A_n \), we have introduced the rule \( 0^0 = 1 \). When the non-orthogonal property between \( \phi_1 \) and \( \phi_2 \) is considered, one should note that the normalization constant \( A_n \) is relevant to the parameter \( \zeta \). From the form of the many-body wavefunction (5), the quantum depletion originating from interparticle interaction is omitted. Thus, this form of quantum state is valid when \( a_n/\bar{l} \ll 1 \) with \( a_n \) and \( \bar{l} \) being respectively...
the scattering length and mean distance between particles. The role of quantum depletion will be discussed in section 8.

From the above many-body wavefunction, after straightforward derivations, the exact expression of the density expectation value takes the following form:

\[
\begin{align*}
n_d(r, t) &= N \int \Psi_{N_1 N_2}(r, r_2, \ldots, r_N, t) \Psi_{N_1 N_2}(r, r_2, \ldots, r_N, t) \, d^3r_2 \cdots d^3r_N \\
&= a_d |\phi_1(r, t)|^2 + 2b_d \times \text{Re}[e^{i\zeta} \phi_1^*(r, t) \phi_2(r, t)] + c_d |\phi_2(r, t)|^2,
\end{align*}
\]

where the coefficients are

\[
\begin{align*}
a_d &= \sum_{i=0}^{\min(N_1-1, N_2)} a_d(i), \\
b_d &= \sum_{i=0}^{\min(N_1-1, N_2-1)} b_d(i), \\
c_d &= \sum_{i=0}^{\min(N_1, N_2-1)} c_d(i).
\end{align*}
\]

In the above summations,

\[
\begin{align*}
a_d(i) &= \frac{A_n^2 N_1! N_2! |\zeta(t)|^{2i}}{i!i!(N_1 - i - 1)!(N_2 - i)!}, \\
b_d(i) &= \frac{A_n^2 N_1! N_2! |\zeta(t)|^{2i+1}}{i!(i + 1)!(N_1 - i - 1)!(N_2 - i - 1)!}, \\
c_d(i) &= \frac{A_n^2 N_1! N_2! |\zeta(t)|^{2i}}{i!(N_1 - i)!(N_2 - i - 1)!}.
\end{align*}
\]

For two independent ideal condensates, before the overlapping between the two condensates, we have \(\zeta(t) = 0\). Based on the Schrödinger equation, it is easy to verify that after the double-well potential separating the condensates is removed, we have \(\zeta(t) = 0\) at any further time. Thus \(b_d = 0\), and the density expectation value is given by

\[
n_d(r, t) = N_1 |\phi_1(r, t)|^2 + N_2 |\phi_2(r, t)|^2.
\]

In this situation, the interference term is zero in the density expectation value.

In the presence of the interatomic interaction, \(\zeta(t)\) can be a nonzero value after the two initially independent condensates begin to overlap. For \(\zeta(t)\) being nonzero, we see clearly from the coefficient \(b_d\) that the interference term in (7) gives a contribution to the density expectation value.
Based on the numerical calculations of equations (8), (9) and (10), (a) shows the relation between $b_d/c_d$ and $\zeta$, while (b) shows the relation between $b_d/c_d$ and $N_1 = N_2$ for $\zeta = 0.001$. It is shown clearly that the interference term can play an important role in the density expectation value for $|\zeta|$ being larger than $N_1^{-1}$.

It is easy to understand that when $N_1|\zeta(t)| > 1$ and $N_2|\zeta(t)| > 1$, $b_d$ cannot be omitted, and thus there would be clear interference patterns. This shows clearly that for large particle number, a small $|\zeta(t)|$ can play important role in the density expectation value when $|\zeta(t)| > N_1^{-1}$ and $|\zeta(t)| > N_2^{-1}$.

Generally speaking, even in the presence of the interatomic interaction, $|\zeta(t)|$ is much smaller than 1 because $\phi_1\phi_2^*$ is an oscillation function about the space coordinate. As shown above, however, a nonzero value of $\zeta(t)$ can give significant contribution to the density expectation value for large $N_1$ and $N_2$. In figure 2(a), we give the relation between $b_d/c_d$ and $\zeta$ for $N_1 = N_2 = 10^3$ based on equations (9) and (10). The relation between $b_d/c_d$ and $N_1 = N_2$ for $\zeta = 0.001$ is shown in figure 2(b). Generally speaking, for $N_1|\zeta| \gg 1$ and $N_2|\zeta| \gg 1$, one has $b_d/\sqrt{a_d c_d} \approx 1$. 

Figure 2. Based on the numerical calculations of equations (8), (9) and (10), (a) shows the relation between $b_d/c_d$ and $\zeta$, while (b) shows the relation between $b_d/c_d$ and $N_1 = N_2$ for $\zeta = 0.001$. It is shown clearly that the interference term can play an important role in the density expectation value for $|\zeta|$ being larger than $N_1^{-1}$. 

expectation value. Based on equations (11), (12) and (13), we have

$$\frac{b_d(i)}{c_d(i)} = \frac{|\zeta(t)|(N_1 - i)}{i + 1}, \quad (15)$$

and

$$\frac{b_d(i)}{a_d(i)} = \frac{|\zeta(t)|(N_2 - i)}{i + 1}. \quad (16)$$

It is easy to understand that when $N_1|\zeta(t)| > 1$ and $N_2|\zeta(t)| > 1$, $b_d$ cannot be omitted, and thus there would be clear interference patterns. This shows clearly that for large particle number, a small $|\zeta(t)|$ can play important role in the density expectation value when $|\zeta(t)| > N_1^{-1}$ and $|\zeta(t)| > N_2^{-1}$.
3. The evolution equations of the system

For the many-body wavefunction $\Psi_{N_1N_2}$, in the presence of the interatomic interaction, the evolution equation can be obtained based on the standard quantum mechanical principle. After the double-well potential is removed, the evolution equation is given by

$$i\hbar \frac{\partial \Psi_{N_1N_2}}{\partial t} = \hat{H}_f \Psi_{N_1N_2},$$

(17)

where the Hamiltonian of the whole system in the first-quantization method is

$$\hat{H}_f = \sum_{i=1}^{N} -\frac{\hbar^2}{2m} \nabla_i^2 + g \sum_{i<j} \delta(r_i - r_j).$$

(18)

Here the coupling constant $g = 4\pi \hbar^2 a_s/m$ with $a_s$ being the scattering length. In the above expression of the Hamiltonian, we have used the well-known two-body pseudopotentials.

It is well known that the action principle is quite useful to derive the GP equation for a single condensate (see for example [8]). Similarly, we consider here the evolution of $\Psi_{N_1N_2}$ based on the action principle. To get the evolution equation based on the action principle, we first give the general expression of the overall energy of the whole system. After removing the double-well potential, the overall energy of the whole system is

$$E = \int \Psi_{N_1N_2}^* \hat{H}_f \Psi_{N_1N_2} d^3r_1 \cdots d^3r_N.$$

(19)

After straightforward derivations, the exact expression of the overall energy is given by

$$E = E_{\text{kin}} + E_{\text{int}},$$

(20)

where the kinetic energy $E_{\text{kin}}$ is

$$E_{\text{kin}} = \int \Psi_{N_1N_2}^* \left( \sum_{i=1}^{N} -\frac{\hbar^2}{2m} \nabla_i^2 \right) \Psi_{N_1N_2} d^3r_1 \cdots d^3r_N = \int dV \left( \frac{a_d \hbar^2}{2m} \nabla \phi_1^* \cdot \nabla \phi_1 + \frac{b_d \hbar^2}{2m} e^{i\psi} \nabla \phi_1^* \cdot \nabla \phi_2 + \frac{b_d \hbar^2}{2m} e^{-i\psi} \nabla \phi_2^* \cdot \nabla \phi_1 + \frac{c_d \hbar^2}{2m} \nabla \phi_2^* \cdot \nabla \phi_2 \right).$$

(21)

In addition, the interaction energy $E_{\text{int}}$ of the whole system is given by

$$E_{\text{int}} = \int \Psi_{N_1N_2}^* \left( g \sum_{i<j} \delta(r_i - r_j) \right) \Psi_{N_1N_2} d^3r_1 \cdots d^3r_N = \frac{g}{2} \int dV |h_1|\phi_1|^4 + h_2|\phi_2|^4$$

$$+ h_3|\phi_1|^2|\phi_2|^2 + \Re(h_4|\phi_1|^2\phi_1^*\phi_2 e^{i\psi} + h_5(\phi_1^2\phi_2^2 e^{2i\psi} + h_6|\phi_2|^2\phi_1^2 e^{2i\psi})].$$

(22)
where the coefficients are given by

\[ h_1 = \sum_{i=0}^{\min(N_1-2,N_2)} \frac{A_n^2 N_1! N_2! |\zeta|^{2i}}{i!i!(N_1 - i - 2)!(N_2 - i - 1)!}, \]

\[ h_2 = \sum_{i=0}^{\min(N_1,N_2-2)} \frac{A_n^2 N_1! N_2! |\zeta|^{2i}}{i!i!(N_1 - i)!(N_2 - i - 2)!}, \]

\[ h_3 = \sum_{i=0}^{\min(N_1-1,N_2-1)} \frac{4A_n^2 N_1! N_2! |\zeta|^{2i}}{i!(N_1 - i - 1)!(N_2 - i - 1)!}, \]

\[ h_4 = \sum_{i=0}^{\min(N_1-2,N_2-1)} \frac{4A_n^2 N_1! N_2! |\zeta|^{2i+1}}{i!(i + 1)!(N_1 - i - 2)!(N_2 - i - 1)!}, \]

\[ h_5 = \sum_{i=0}^{\min(N_1-2,N_2-2)} \frac{2A_n^2 N_1! N_2! |\zeta|^{2i+2}}{i!(i + 2)!(N_1 - i - 2)!(N_2 - i - 2)!}, \]

\[ h_6 = \sum_{i=0}^{\min(N_1-1,N_2-2)} \frac{4A_n^2 N_1! N_2! |\zeta|^{2i+1}}{i!(i + 1)!(N_1 - i - 1)!(N_2 - i - 2)!}. \]  

By using the ordinary action principle and the energy of the whole system, one can get the following coupled evolution equations for \( \phi_1 \) and \( \phi_2 \):

\[ i\hbar \frac{\partial \phi_1}{\partial t} = \frac{1}{N_1} \frac{\delta E}{\delta \phi_1^*}, \]  

\[ i\hbar \frac{\partial \phi_2}{\partial t} = \frac{1}{N_2} \frac{\delta E}{\delta \phi_2^*}, \]

where \( \delta E/\delta \phi_1^* \) and \( \delta E/\delta \phi_2^* \) are functional derivatives.

4. The crossover from \(|\zeta| \ll N^{-1}\) to \(|\zeta| \gg N^{-1}\)

4.1. The case of \(|\zeta| \ll N^{-1}\)

In section 2, we have shown that the nonzero interference term in the density expectation value origins from the assumption that \( \zeta(t) \) can be a nonzero value after the overlapping between two initially independent condensates. In the last section, we have given the evolution equations about \( \phi_1 \) and \( \phi_2 \). Thus, an important question emerges naturally: is it physical for \( \zeta(t) \) being nonzero with the development of time?

Based on the evolution equations (24) and (25) given in the last section, one can understand easily that \( \zeta(t) \) becomes nonzero after the overlapping between two initially independent condensates in the presence of the interatomic interaction. Before removing the double-well potential, \( N_1|\zeta| = 0 \) and \( N_2|\zeta| = 0 \) because there is no overlapping between \( \phi_1 \) and \( \phi_2 \). After removing the double-well potential and at the beginning of the overlapping between two condensates, \( N_1|\zeta| \ll 1 \) and \( N_2|\zeta| \ll 1 \). For \( N_1|\zeta| \ll 1 \) and \( N_2|\zeta| \ll 1 \), the overall energy can
be approximated as
\[
E \approx \frac{\hbar^2}{2m} \int dV (N_1 \nabla \phi_1^* \cdot \nabla \phi_1 + N_2 \nabla \phi_2^* \cdot \nabla \phi_2) + \frac{g}{2} \int dV (N_1 (N_1 - 1) |\phi_1|^4 \\
+ N_2 (N_2 - 1) |\phi_2|^4 + 4N_1 N_2 |\phi_1|^2 |\phi_2|^2).
\]  
(26)

The above overall energy is obtained by setting \( \zeta = 0 \) in equation (20).

Based on equations (24) and (25), the approximate evolution equations for \( \phi_1(r, t) \) and \( \phi_2(r, t) \) are then
\[
i\hbar \frac{\partial \phi_1}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi_1 + V_1 \phi_1,
\]
and
\[
i\hbar \frac{\partial \phi_2}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi_2 + V_2 \phi_2,
\]
where
\[
V_1 = (N_1 - 1)g|\phi_1|^2 + 2N_2g|\phi_2|^2,
\]
and
\[
V_2 = (N_2 - 1)g|\phi_2|^2 + 2N_1g|\phi_1|^2.
\]

We see that \( V_1 \) is not equal to \( V_2 \) for \( g \neq 0 \). This leads to the important result that \( \zeta(t) \) can be nonzero when there is an overlapping between two initially independent condensates. One can understand this result further through the following equation which determines the evolution of \( \zeta(t) \):
\[
i\hbar \frac{d\zeta(t)}{dt} = \int f(r, t) \phi_1 \phi_2^* dV,
\]
(31)
where \( f(r, t) \) is a nonzero function given by
\[
f(r, t) = (N_2 + 1)g|\phi_2|^2 - (N_1 + 1)g|\phi_1|^2.
\]

The above analyses show clearly that why \( \zeta(t) \) becomes nonzero after the overlapping between two initially independent condensates for \( g \neq 0 \). Obviously, for \( g \) being zero, \( \zeta(t) = 0 \) at any further time because \( d\zeta(t)/dt = 0 \). Our numerical calculations in the following section also show that \( \zeta(t) \) can be a nonzero value in the presence of the interatomic interaction.

In the presence of the interatomic interaction, it is inconsistent to assume that \( \zeta(t) \) is always zero with the development of time. If \( \zeta(t) = 0 \), the evolution equations (24) and (25) are exact. However, based on these two evolution equations, \(|\zeta|\) will increase from zero after the overlapping.
4.2. The case of $|\xi| \gg N^{-1}$

As shown above, the evolution of $\phi_1$ and $\phi_2$ is determined by the expression of the overall energy and the evolution equations (24) and (25). It seems that there is no exact solution for these evolution equations considering the fact that even there is no exact solution for three-dimensional (3D) nonlinear Schrödinger equation. However, for $|\xi| \gg N_1^{-1}$ and $|\xi| \gg N_2^{-1}$, we find that there is a quite simple evolution equation by introducing an effective parameter order. This would also contribute to our understanding of the interaction-induced coherence process for two initially independent condensates.

For the cases of $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$ and $N_1 \sim N_2$, first we introduce the following effective order parameter $\Phi_e(r, t)$ which is given by

$$\Phi_e(r, t) = \sqrt{N_1} \phi_1(r, t) + \sqrt{N_2} e^{i\xi} \phi_2(r, t).$$

(33)

Based on this effective order parameter, the density expectation value can be approximated well as

$$n_d(r, t) \simeq \Phi_e^*(r, t) \Phi_e(r, t) = a_d^* |\phi_1(r, t)|^2 + 2b_d^* \times \text{Re}[e^{i\xi} \phi_1^*(r, t) \phi_2(r, t)] + c_d^* |\phi_2(r, t)|^2,$$

(34)

where

$$a_d^* = N_1, \quad b_d^* = \sqrt{N_1 N_2}, \quad c_d^* = N_2.$$

(35)

To compare with the exact expression of the density expectation value given by equation (7), figure 3 shows the ratio $\lambda_1 = a_d/a_d^*$, $\lambda_2 = b_d^*/b_d$ and $\lambda_3 = c_d/c_d^*$. We see that for $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$, the approximate density expectation value $\Phi_e^*(r, t) \Phi_e(r, t)$ agrees very well with the exact expression of the density expectation value.

Based on the effective order parameter, for $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$ and $N_1 \sim N_2$, we also find that the overall energy of the whole system can be approximated very well as

$$E_{\text{app}} = E_{\text{kin}} + E_{\text{int}}^\prime,$$

(36)

where

$$E_{\text{kin}}^\prime = \frac{\hbar^2}{2m} \int \nabla \Phi_e^* \cdot \nabla \Phi_e \text{d}V,$$

(37)

and

$$E_{\text{int}}^\prime = \frac{g}{2} \int \text{d}V |\Phi_e|^4.$$

(38)

For the cases of $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$ and $N_1 \sim N_2$, it is easy to verify that $E_{\text{kin}}^\prime \approx E_{\text{kin}}$ based on the analogous analyses about the density expectation value. For $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$ and $N_1 \sim N_2$, one can also prove the result of $E_{\text{int}}^\prime \approx E_{\text{int}}$. Based on equation (38), $E_{\text{int}}^\prime$ can be expanded as

$$E_{\text{int}}^\prime = \frac{g}{2} \int \text{d}V \left[ \beta_1 |\phi_1|^4 + \beta_2 |\phi_2|^4 + \beta_3 |\phi_1|^2 |\phi_2|^2 + \text{Re}(\beta_4 |\phi_1|^2 \phi_1^* \phi_2 e^{i\xi}) \right.$$

$$+ \beta_5 (\phi_1^*)^2 |\phi_2|^2 e^{2i\xi} + \beta_6 |\phi_2|^2 \phi_1^* \phi_2 e^{i\xi}) \right].$$

(39)
The values of $\lambda_1$, $\lambda_2$ and $\lambda_3$. We see that for $N_1|\xi| \gg 1$, $N_2|\xi| \gg 1$ and $N_1 \sim N_2$, the density expectation value given by the effective order parameter agrees well with the exact expression of the density expectation value given by equation (7).

where

$$
\beta_1 = N_1^2, \quad \beta_2 = N_2^2, \quad \beta_3 = 4N_1N_2, \quad \beta_4 = 4N_1\sqrt{N_1N_2},
\beta_5 = 2N_1N_2, \quad \beta_6 = 4N_2\sqrt{N_1N_2}.
$$

(40)

To compare with the exact expression of the interaction energy given by equation (22), figure 4 shows the relation between $h_i/\beta_i$ ($i = 1, \ldots, 6$) and $\xi$ for $N_1 = N_2 = 10^3$. It is shown clearly that for $N_1|\xi| \gg 1$ and $N_2|\xi| \gg 1$, $h_i/\beta_i \approx 1$, and thus $E'_\text{int} \approx E_{\text{int}}$.

In this situation, the overall energy can be approximated well as

$$
E'_\text{app} = \frac{\hbar^2}{2m} \int \nabla \Phi_e^* \cdot \nabla \Phi_e \, dV + \frac{g}{2} \int dV |\Phi_e|^4.
$$

(41)

The evolution of the effective order parameter can be then obtained by using the action principle. Based on this approximate energy, after removing the double-well potential, it is quite interesting to note that the evolution of the effective order parameter can be described very well by the ordinary GP equation

$$
i\hbar \frac{\partial \Phi_e}{\partial t} \simeq -\frac{\hbar^2}{2m} \nabla^2 \Phi_e + g|\Phi_e|^2 \Phi_e.
$$

(42)
Based on equations (34) and (42), we see that the emergence of the effective order parameter $\Phi_1$ gives us strong evidence that the coherence is formed in the interaction process between two initially independent condensates, and thus results in the emergence of high-contrast interference fringes. The effective order parameter and the approximate density expectation value suggest strongly that a full coherence is formed between two initially independent condensates for the cases of $N_1|\varsigma| \gg 1$ and $N_2|\varsigma| \gg 1$.

5. The evolution of the density expectation value according to the experimental parameters

Now we turn to give the theoretical results of the density expectation value according to the experimental parameters in [5] where clear interference patterns were observed for two initially independent condensates. In the experiment of [5], $N = 5 \times 10^6$ condensed sodium atoms were confined in a magnetic trap with $\omega_x = 2\pi \times 18$ Hz, $\omega_y = \omega_z = 2\pi \times 320$ Hz. A blue-detuned laser beam of wavelength 514 nm was focused into a light sheet with a cross-section of $12 \mu m \times 67 \mu m$. The long axis of the laser beam was perpendicular to the long $x$-axis of the condensate. For a laser power of 14 mW, the barrier height is about $1.4 \mu K$, which is much larger than the chemical potential $\mu = 0.03 \mu K$. For this laser beam, the two condensates can be regarded to be independent because they are well separated and the tunnelling effect can be omitted. With these experimental parameters and the $s$-wave scattering length $a_s = 2.75 \text{nm}$, the initial profile of the two condensates is shown in figure 5. At the initial time, the overlapping between two condensates can be omitted safely, and thus $\varsigma(t = 0) = 0$. After the double-well potential is removed, the evolution of the density expectation value $n_{d-1}(x, t) = \int n_d(\mathbf{r}, t) \, dy \, dz$ (in unit of $N/2$) is given in figure 5 through the numerical calculations of equations (24), (25)
Figure 5. The evolution of the density expectation value calculated using the experimental parameters in [5]. It is shown that there is a clear interference pattern after the overlapping between two independent interacting condensates. The inset shows the evolution of $|\zeta(t)|$ for the experimental parameters in [5].

and (7). We see that there is a clear interference pattern in the density expectation value which agrees with the experimental result. The inset shows the evolution of $|\zeta|$ for these parameters. We see that $|\zeta|$ increases from zero with the development of time. For the expansion time of 40 ms, the numerical result of $|\zeta|$ shows that $b_0/c_0 \approx 1$. To check further that the nonzero value of $|\zeta|$ does not originate from numerical error, with the same initial conditions, we have verified in the numerical calculations that $|\zeta|$ is always zero (smaller than $10^{-10}$) if the scattering length is assumed as zero.

Based on the evolution of the density expectation value shown in figure 5, we see that the overall width (about 300 µm) of the ultracold gases in the $x$-direction does not increase obviously. This is due to the fact that the initial density distribution is cigar-shaped, and thus the expansion in the $x$-direction is very slow, while the expansion in the $y$- and $z$-directions is much quick. In the experiment of [19], one can see clearly that there is no obvious expansion in the long $x$-axis for a cigar-shaped condensate. For two initially independent condensates, when the double-well potential is switched off, one should note that in the regime close to $x = 250 \mu m$ shown in figure 5, the ultracold gases expand rapidly in the $x$-direction because in this regime the ultracold gases have higher kinetic energy. Thus, although the total width of the system in the $x$-direction does not increase obviously, the rapid expansion in the central regime leads to the overlapping between two initially independent condensates, and results in the interference
effect. The overall width of the system shown in figure 5 is smaller than the experimental result of about 500 μm [5]. This difference may come from the expansion of thermal cloud in this experiment [5]. After 40 ms expansion, the numerical result in [16] for two coherently separated condensates also showed that the overall width of the system is about 300 μm.

6. The evolution of the density expectation value for different coupling constants

We see that the interatomic interaction plays an essential role in the emergence of the interference effect for two initially independent condensates. Generally speaking, increasing the particle number will enhance the effect of the interference term in the density expectation value. Based on equations (24) and (25), increasing the coupling constant $g$ has the effect of increasing $\zeta(t)$. Together with the relation between $b_d/c_d$ and $\zeta$ illustrated in figure 2(a), this shows that increasing the interatomic interaction will enhance the effect of the interference term. To show more clearly the interaction-induced coherence process between two initially independent condensates, in this section we consider the density expectation value for different coupling constants.

We consider here the evolution of the density expectation value for 1D case. At $t = 0$, to give a general comparison, the initial wavefunctions for two independent condensates are assumed to be identical for different coupling constants. The initial wavefunctions are respectively given by

$$\phi_1(x_1, t = 0) = \frac{1}{\pi^{1/4}\sqrt{\Delta_1}} \exp \left[-\frac{(x_1 - x_{1l})^2}{2\Delta_1^2}\right],$$

$$\phi_2(x_1, t = 0) = \frac{1}{\pi^{1/4}\sqrt{\Delta_2}} \exp \left[-\frac{(x_1 - x_{12})^2}{2\Delta_2^2}\right].$$

(43)

(44)

In the above wavefunctions, we have introduced a dimensionless variable $x_1 = x/l$ with $l$ being a length. In the present study, we assume that $\Delta_1 = \Delta_2 = 0.5$ and $x_{12} - x_{1l} = 4.5$. For these parameters, at $t = 0$, the two condensates are well separated. In the numerical calculations of the coupled equations given by equations (24) and (25), it is useful to introduce the dimensionless variable $\tau = E_1 t/\hbar$ with $E_1 = \hbar^2/2ml^2$, and dimensionless coupling constant $g_1 = N_1 g/E_1 l$. In addition, the particle number is assumed as $N_1 = N_2 = 1.0 \times 10^5$. In real experiments, interatomic interaction plays very important role in the initial ground-state wavefunction of the condensates. However, in principle, one can prepare the state given by equations (43) and (44) by adjusting the trapping potential for different coupling constants. In this section, the identical initial wavefunctions for different coupling constants would be helpful in the comparison of the density expectation value for different coupling constants.

For $t > 0$, we consider the evolution of the density expectation value in free space. The evolution of $\phi_1$ and $\phi_2$ is obtained based on the numerical calculations of equations (24) and (25). From $\phi_1$ and $\phi_2$, we can get $\zeta$, and thus the density expectation value based on equation (7). Shown in figure 6 is the evolution of the density expectation value for different coupling constants. It is shown clearly that increasing the coupling constant has the effect of enhancing the coherence effect, and results in higher contrast in the interference patterns.
Figure 6. After two initially independent condensates are allowed to expand freely, shown is the evolution of the density expectation value \( n_d(x_1, \tau) \) (in unit of \( N_1 + N_2 \)) for different coupling constants \( g_1 \). The inset of each figure shows the relation between \( b_d/c_d \) and dimensionless time \( \tau \). For two ideal condensates shown in (a), we see that there is no interference pattern even there is an overlapping between two condensates. For the case of \( g_1 = 1 \) shown in (b), we see that low-contrast interference patterns begin to emerge due to the interaction-induced coherence process. In (c) and (d), we see that there are high-contrast interference patterns. In particular, in (d) for \( g_1 = 20 \), two initially independent condensates can be regarded to be fully coherent near \( \tau = 1 \).

7. The calculations based on the second-quantization method

The essential reason for the emergence of the interference term of two initially independent condensates lies in that because of the exchange symmetry of identical bosons and interatomic interaction, the two initially independent condensates become coherent after the overlapping between the two condensates. The physical mechanism of this interaction-induced coherence can be understood further based on the second quantization method. Thus, in this section, we give the density expectation value and evolution equations based on the second-quantization method. The merit of the second-quantization method lies in that the indistinguishability of identical bosons is satisfied when the correct commutation relation of the field operator is used.
7.1. The density expectation value

For two initially independent condensates comprising particle number \( N_1 \) and \( N_2 \), the corresponding quantum state is (see for example [7, 8]):

\[
|N_1, N_2\rangle = \frac{\Xi_n}{\sqrt{N_1!N_2!}}(\hat{a}_1^\dagger)^{N_1}(\hat{a}_2^\dagger)^{N_2}|0\rangle,
\]

where \( \Xi_n \) is a normalization constant to assure \( \langle N_1, N_2|N_1, N_2\rangle = 1 \). \( \hat{a}_1^\dagger \) (\( \hat{a}_2^\dagger \)) is a creation operator which creates a particle described by the single-particle state \( \phi_1 \) (\( \phi_2 \)) in the left (right) condensate. Similarly to equation (5), the quantum depletion is omitted in this sort of operator which creates a particle described by the single-particle state \( \phi \).

In the above equation, the field operator \( \hat{\Psi} \) is expanded as \( \hat{\Psi} = \hat{a}_1\phi_1 + \hat{a}_2\phi_2 + \cdots \) with \( \hat{a}_1 \) and \( \hat{a}_2 \) being the annihilation operators. One should note that, to get (47) from (46), there is an implicit assumption that \( \hat{a}_1 \) and \( \hat{a}_2^\dagger \) are commutative. This holds when \( \int \phi_1(\mathbf{r}, t)\phi_2^\ast(\mathbf{r}, t) \, dV = 0 \). When \( [\hat{a}_1, \hat{a}_2^\dagger] = 0 \), it is easy to understand that the interference term (the last term in equation (46)) is zero in \( n_d(\mathbf{r}, t) \).

As shown in the previous sections, \( \phi_1 \) and \( \phi_2 \) will become non-orthogonal in the presence of the interatomic interaction. The operators \( \hat{a}_1 \) and \( \hat{a}_2 \) can be written as

\[
\hat{a}_1 = \int \hat{\Psi}\phi_1^\ast \, dV,
\]

and

\[
\hat{a}_2 = \int \hat{\Psi}\phi_2^\ast \, dV.
\]

Here \( \hat{\Psi} \) is the field operator. By using the commutation relations of the field operators \( [\hat{\Psi}(\mathbf{r}_1, t), \hat{\Psi}(\mathbf{r}_2, t)] = 0 \) and \( [\hat{\Psi}(\mathbf{r}_1, t), \hat{\Psi}^\dagger(\mathbf{r}_2, t)] = \delta(\mathbf{r}_1 - \mathbf{r}_2) \), it is easy to get the commutation relation

\[
[\hat{a}_1, \hat{a}_2^\dagger] = \zeta^\ast.
\]

We see that \( \hat{a}_1 \) and \( \hat{a}_2^\dagger \) are not commutative any more for \( \int \phi_1\phi_2^\ast \, dV \) being a nonzero value. In this situation, it is obvious that one cannot get the result (47) from (46) any more. This means that one should be very careful to get the correct density expectation value for two initially independent condensates.

It is well-known that the field operator should be expanded in terms of a complete and orthogonal basis set. Generally speaking, the field operator \( \hat{\Psi} \) can be expanded as

\[
\hat{\Psi} = \hat{a}_1\phi_1 + \hat{a}_2\phi_2 + \cdots,
\]
Although it seems that the expressions of the coefficients given by equations (56)–(59) are quite different from the results calculated from the many-body wavefunction, we have proven based on the conditions \( \int \phi_1^* \phi_2^* \, dV = 0 \) and \( \int |\phi_2|^2 \, dV = 1 \), we have \( |\beta| = (1 - |\zeta|^2)^{-1/2} \) and \( \alpha = - \zeta^* \). Based on

\[
\hat{k} = \int \hat{\Psi}(\phi_2^*) \, dV, \tag{52}
\]

we have

\[
\hat{a}_2 = \frac{\hat{k}}{\beta^*} + \zeta \hat{a}_1. \tag{53}
\]

It is easy to get the following commutation relations:

\[
[\hat{k}, \hat{k}] = [\hat{k}^\dagger, \hat{k}^\dagger] = 0, \quad [\hat{k}, \hat{a}_1] = [\hat{a}_1^\dagger, \hat{a}_1^\dagger] = 0, \quad
[\hat{a}_1, \hat{a}_1^\dagger] = 1, \quad [\hat{k}, \hat{a}_1] = [\hat{k}, \hat{a}_1^\dagger] = 0. \tag{54}
\]

Because \( \hat{k} \) and \( \hat{a}_1^\dagger \) are commutative, it is convenient to calculate the density expectation value \( n_d(r, t) \) by using the operators \( \hat{k} \) and \( \hat{a}_1^\dagger \). After straightforward derivations, the exact expression of the density expectation value is

\[
n_d(r, t) = \frac{\Xi_n^2}{N_1! N_2!} \left( \frac{\hat{k}}{\beta} + \zeta \hat{a}_1^\dagger \right)_{N_2} \hat{a}_1^\dagger [\hat{a}_1^\dagger \psi_1^* + \hat{k}^\dagger (\phi_2^*)^*] [\hat{a}_1 \phi_1 + \hat{k} \phi_2^*] [\hat{a}_1^\dagger]_{N_1} \left[ \frac{\hat{k}^\dagger}{\beta} + \zeta^* \hat{a}_1^\dagger \right]_{N_2} \langle 0 \rangle
= \alpha_d |\phi_1(r, t)|^2 + 2 \beta_d \times \text{Re}(e^{i\psi}\phi_1^*(r, t)\phi_2(r, t)) + \gamma_d |\phi_2(r, t)|^2, \tag{55}
\]

where the coefficients are

\[
\alpha_d = \sum_{i=0}^{N_2} \Xi_n^2 N_2! (N_1 + i - 1)! N_1 (1 - |\zeta|^2)^{N_2-i} |\zeta|^{2i} / i! (N_1 - 1)! (N_2 - i)!, \tag{56}
\]

\[
\beta_d = \sum_{i=0}^{N_2-1} \Xi_n^2 N_2! (N_1 + i)! (1 - |\zeta|^2)^{N_2-i-1} |\zeta|^{2i+1} / i! (i+1)! (N_1 - 1)! (N_2 - i - 1)!, \tag{57}
\]

\[
\gamma_d = \sum_{i=0}^{N_2-1} \Xi_n^2 N_2! (N_1 + i)! (1 - |\zeta|^2)^{N_2-i-1} |\zeta|^{2i} / i! i! N_1! (N_2 - i - 1)!. \tag{58}
\]

In addition, the normalization constant \( \Xi_n \) is determined by

\[
\Xi_n^2 \left( \sum_{i=0}^{N_2} N_2! (N_1 + i)! (1 - |\zeta|^2)^{N_2-i} |\zeta|^{2i} / i! i! N_1! (N_2 - i)! \right) = 1. \tag{59}
\]

The above density expectation value is obtained based on the second quantization method. Although it seems that the expressions of the coefficients given by equations (56)–(59) are quite different from the results calculated from the many-body wavefunction, we have proven that \( \alpha_d = a_d, \beta_d = b_d, \gamma_d = c_d \) and \( \Xi_n = A_n \). Thus, the density expectation value given by equation (55) is the same as the result calculated from the many-body wavefunction \( \psi_{N_1 N_2} \) which satisfies the exchange symmetry of identical bosons.
After straightforward calculations, the overall energy of the whole system is

\[
H = \int dV \left( \frac{\hbar^2}{2m} \nabla \hat{\psi}^* \cdot \nabla \hat{\psi} + \frac{g}{2} \hat{\psi}^* \hat{\psi} \hat{\psi}^* \hat{\psi} \right).
\]  

(60)

After straightforward calculations, the overall energy of the whole system is

\[
E = E_{\text{kin}} + E_{\text{int}},
\]

(61)

where the kinetic energy \(E_{\text{kin}}\) is given by

\[
E_{\text{kin}} = \int \langle N_1, N_2, t | \frac{\hbar^2}{2m} \nabla \hat{\psi}^* \cdot \nabla \hat{\psi} | N_1, N_2, t \rangle dV = \int dV \left( \frac{\alpha_1 \hbar^2}{2m} \nabla \phi_1^* \cdot \nabla \phi_1 + \frac{\beta_1 \hbar^2}{2m} e^{i\omega_c} \nabla \phi_2^* \cdot \nabla \phi_2 + \frac{\beta_2 \hbar^2}{2m} e^{i\omega_c} \nabla \phi_2^* \cdot \nabla \phi_1 + \frac{\gamma_1 \hbar^2}{2m} \nabla \phi_2^* \cdot \nabla \phi_2 \right).
\]

(62)

In addition, the interaction energy \(E_{\text{int}}\) of the whole system is given by

\[
E_{\text{int}} = \int \langle N_1, N_2, t | \frac{g}{2} \hat{\psi}^* \hat{\psi} \hat{\psi}^* \hat{\psi} | N_1, N_2, t \rangle dV = \frac{g}{2} \int dV [\alpha_1 |\phi_1|^4 + \alpha_2 |\phi_2|^4 + \alpha_3 |\phi_1|^2 |\phi_2|^2 + \text{Re}(\alpha_4 |\phi_1|^2 \phi_2^* e^{i\omega_c} + \alpha_5 (\phi_1^*)^2 \phi_2^* e^{2i\omega_c} + \alpha_6 |\phi_2|^2 \phi_2^* e^{i\omega_c})],
\]

(63)

where the coefficients are given by

\[
\alpha_1 = \sum_{i=0}^{N_2} \frac{\Xi^2 N_2!(N_1 + i - 2)!N_1(N_1 - 1)}{i!(N_1 - 2)!(N_2 - i)!} (1 - |\xi|^2)^{N_2 - i} |\xi|^2i,
\]

\[
\alpha_2 = \frac{\sum_{i=0}^{N_2} \Xi^2 N_2!(N_1 + i)!}{i!(N_1 - 2)!} (1 - |\xi|^2)^{N_2 - i} |\xi|^2i,
\]

\[
\alpha_3 = \sum_{i=0}^{N_2 - 1} \frac{4 \Xi^2 N_2!(N_1 + i - 1)!N_1}{i!(N_1 - 1)!(N_2 - i - 1)!} (1 - |\xi|^2)^{N_2 - i - 1} |\xi|^2i,
\]

\[
\alpha_4 = \sum_{i=0}^{N_2 - 1} \frac{4 \Xi^2 N_2!(N_1 + i - 1)!N_1}{i!(i + 1)!(N_1 - 2)!(N_2 - i - 1)!} (1 - |\xi|^2)^{N_2 - i - 1} |\xi|^{2i+1},
\]

\[
\alpha_5 = \frac{\sum_{i=0}^{N_2 - 2} \Xi^2 N_2!(N_1 + i)!}{i!(i + 2)!(N_1 - 2)!(N_2 - i - 2)!} (1 - |\xi|^2)^{N_2 - i - 2} |\xi|^{2i+2},
\]

\[
\alpha_6 = \frac{\sum_{i=0}^{N_2 - 2} \Xi^2 N_2!(N_1 + i)!}{i!(i + 1)!(N_1 - 1)!(N_2 - i - 2)!} (1 - |\xi|^2)^{N_2 - i - 2} |\xi|^{2i+1}.
\]

(64)
We have proven that the above overall energy is equal to the results based on the many-body wavefunction, by checking that $h_i = \alpha_i$ for $i = 1, 2, \ldots, 6$.

When the non-orthogonal property between $\phi_1$ and $\phi_2$ are considered, because the derivations of the density expectation value and overall energy are quite cumbersome, the same results based on the first-quantization method and second-quantization method give us strong evidence that our derivations are correct.

8. Quantum fluctuations and orthogonality of the whole quantum states

In both the first-quantization and second-quantization methods of the previous calculations, the quantum depletion originating from the elementary excitations at zero temperature is omitted in the quantum state of the whole system. Based on the Bogoliubov theory of elementary excitations, for the initial quantum state, the number of particles due to the quantum depletion is of the order of $(\alpha_i/\beta)^{3/2}$ and thus the quantum depletion is negligible for Bose condensate in dilute gases considered in the present study. With the development of time, the role of quantum depletion can also be omitted. Because of the factor $e^{\pm i k \cdot r}$ ($|k|$ is the wavenumber of the elementary excitations) in the wavefunction $\phi_k$ of the elementary excitations, a simple analysis shows qualitatively that $\langle \phi_k | \phi_1 \rangle$ and $\langle \phi_k | \phi_2 \rangle$ are of the order of $|\zeta| e^{-|k|L^2}$ with $L$ being the spatial size of the system. This exponential decay of $\langle \phi_k | \phi_1 \rangle$ and $\langle \phi_k | \phi_2 \rangle$ originates from the integral where there is spatially oscillating phase factor in the wavefunctions of the elementary excitations and condensates. Thus, the contribution to the effective order parameter and density expectation value due to elementary excitations can be omitted safely.

In the preceding paragraph, it is shown that in calculating the evolution of the density expectation value, the role of quantum depletion can be omitted. However, the quantum depletion plays important role in the consistency of our theory. Here, we discuss mainly the orthogonality for the quantum state of the whole system, especially about the subtle problem that whether the nonorthogonality between $\phi_1$ and $\phi_2$ violates the orthogonality of the whole system which must be satisfied.

This sort of problem about orthogonality exists also for a single condensate. For a single condensate, if the quantum depletion is omitted, the many-body wavefunction is

$$\Psi(r_1, \ldots, r_N, t) = \phi(r_1, t) \cdots \phi(r_N, t).$$

(65)

Based on the many-body Schrödinger equation and the action principle, it is easy to get the following GP equation

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi + V_{\text{ext}}(r, t)\phi + g(N - 1)|\phi|^2\phi.$$  

(66)

Assume that there are two orthogonally initial quantum states $\Psi_A(t = 0) = \phi_A(r_1, t = 0) \cdots \phi_A(r_N, t = 0)$ and $\Psi_B(t = 0) = \phi_B(r_1, t = 0) \cdots \phi_B(r_N, t = 0)$. With the development of time, because of the nonlinearity of the GP equation, $\phi_A$ and $\phi_B$ may become nonorthogonal in the presence of interatomic interaction. It is obvious that $\int \Psi_A^\dagger \Psi_B \, dr_1 \cdots dr_N = [\int \phi_A^\dagger(r, t)\phi_B(r, t) \, dr]^N$. Although for large $N$, this integral can be approximated as zero, based on the consideration of consistancy, the physical mechanism of the exact orthogonality between $\Psi_A$ and $\Psi_B$ is an interesting problem.
We think that the effect of quantum depletion omitted in equation (65) is the essential reason for this sort of unphysical nonorthogonality. For the Hamiltonian (18) of the whole system, the term $g \sum_{i<j}^N \delta(r_i - r_j)$ is non-factorable about the coordinate, and thus the exact solution of the many-body wavefunction is non-factorable too. For a single condensate, the exact many-body wavefunction can be assumed as

$$\Psi_{\text{exact}}(r_1, \ldots, r_N, t) = c\phi(r_1, t) \cdots \phi(r_N, t) F_N(r_1, \ldots, r_N, t).$$  \hspace{1cm} (67)

Here $F_N$ accounts for the non-factorable component, and $c$ is a normalization constant. In the hard-sphere approximation, $F_N = 0$ if $|r_i - r_j| \leq 2r$ for any $i \neq j$ ($r$ is the hard-sphere radius), and $F_N = 1$ for otherwise situation. $F_N$ represents the quantum depletion, and was successfully used to calculate the quantum depletion of superfluid liquid $^4$He in [20]. The omission of the quantum depletion means that we approximate $\Psi_{\text{exact}}$ should be used. The non-factorable factor $F_N$ assures the orthogonality of the many-body quantum state, because $\Psi_{\text{exact}}$ is the exact solution of the many-body Schrödinger equation. For dilute Bose condensed gases, these analyses lead to two results: when the dynamic evolution of dilute Bose condensed gases is considered, omitting the quantum depletion can give us quite good description; when the consistency especially the orthogonality is considered, one should consider the role of quantum depletion.

It is natural to generalize the above analyses to two initially independent condensates. The nonorthogonality between $\phi_1$ and $\phi_2$ does not mean in any sense the violation of the orthogonality of the quantum state of the whole system. We stress here again that when orthogonality is considered, we should check whether the quantum state of the whole system satisfies the orthogonality. When the quantum depletion is considered, the many-body wavefunction takes the following form

$$\Psi_{N_1N_2}(r_1, r_2, \ldots, r_N, t) = c\Psi_{N_1N_2}(r_1, r_2, \ldots, r_N, t) F_N(r_1, \ldots, r_N, t).$$  \hspace{1cm} (68)

Here $\Psi_{N_1N_2}$ is given by equation (5). $\Psi_{N_1N_2}$ satisfies exactly the many-body Schrödinger equation, and thus this sort of wavefunction satisfies the orthogonality condition. Assume that $\Psi_{N_1N_2}$ is constructed by $\phi_1^A$ and $\phi_2^A$, while $\Psi_{N_1N_2}$ is constructed by $\phi_1^B$ and $\phi_2^B$. Assume further that $\phi_1^A$, $\phi_2^A$, $\phi_1^B$ and $\phi_2^B$ are orthogonal to each other at the initial time, so that $\Psi_{N_1N_2}$ and $\Psi_{N_1N_2}$ are orthogonal at the initial time. Omitting the quantum depletion and roughly speaking, the integral between $\Psi_{N_1N_2}$ and $\Psi_{N_1N_2}$ is of the order of $\int (\phi_1^A)^* \phi_1^B \, dr \int (\phi_2^A)^* \phi_2^B \, dr$ (here $\alpha + \beta + \gamma + \delta = N$). We see that similarly to a single condensate, the orthogonality between $\Psi_{N_1N_2}$ and $\Psi_{N_1N_2}$ is quite good for large $N$ when the quantum depletion is omitted. But to assure the exact orthogonality of the whole quantum state, we must consider the role of quantum depletion. Although we do not provide here a fully rigorous proof, these analyses lead to the result that: although for dilute Bose condensed gases, the omission of the quantum depletion will not play important contribution to the evolution of $\phi_1$ and $\phi_2$, it’s the quantum depletion that assure the orthogonality of the whole quantum state.

To date, we have mainly used the bases $\phi_1$ and $\phi_2$ to study the dynamic evolution and density expectation value. Considering the nonorthogonality between $\phi_1$ and $\phi_2$ in the presence of interparticle interaction, it is attractive to discuss the physical picture using the orthogonal bases $\phi_1$ and $\phi_2$ introduced in section 7. If we use the orthogonal bases $\phi_1$ and $\phi_2$, the quantum state of the...
whole system is \( |N_1, N_2 \rangle \sim (\hat{a}_1^\dagger)^{N_1}(\hat{k}\dagger/\beta + \xi^* \hat{a}_1^\dagger)^{N_2}|0\rangle \sim \sum_{m=0}^{N_2} C_m (\hat{a}_1^\dagger)^{N_1+m}(\hat{k}\dagger)^{N_2-m}|0\rangle \). We see that the number of particles in the orthogonal modes \( \phi_1 \) and \( \phi_2' \) are no longer definite. This quantum state becomes a superposition of different number of particles in the orthogonal modes \( \phi_1 \) and \( \phi_2' \). This result is very natural because the coupling (interaction) between two initially independent condensates leads to the coherent transfer of particles between two condensates. In [21], it was shown that for a single condensate, a coherent state description of the Bose condensed system is a robust state in the presence of the interactions between the condensate and its environment. In the present study, our research shows that the interaction between two condensates makes each condensate become a coherent superposition of different particle number. The results in [21] about the robustness of the coherent state imply that \( \sum_{m=0}^{N_2} C_m (\hat{a}_1^\dagger)^{N_1+m}(\hat{k}\dagger)^{N_2-m}|0\rangle \) is a robust quantum state even when the coupling with the environment exists.

9. Summary and discussion

In summary, upon expansion, we calculate the density expectation value of two initially independent condensates. It is found that there is a nonzero interference term when the inter-atomic interaction and the exchange symmetry of identical bosons are both considered carefully. In fact, it is well-known that the interaction plays an essential role in the formation of the order parameter of Bose-condensed gases, i.e. the formation of a stable coherent property. Here, we provide an example in which the interaction induces coherent evolution between two initially independent condensates.

In the present popular viewpoint, the high-order correlation function and quantum measurement process are used to interpret the interference patterns observed for two initially independent condensates. Although we find that there is already high-contrast interference patterns in the density expectation value for the experimental parameters in [5], it is still possible that the high-order correlation function and thus the mechanism of the interference due to the measurement process play an important role in the emergence of the interference patterns. Thus, to investigate more clearly the interference pattern due to the measurement process alone, we believe an experimental investigation of two independent ideal condensates would be very interesting, because there is no interference term for ideal condensates in the density expectation value. In the last few years, the rapid experimental advances of Feshbach resonance where the scattering length can be tuned from positive to negative make this sort of experiment feasible.

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Note added in proof. This paper is an expansion of our two previously unpublished works [22, 23]. Also, after submission of this paper, we noticed a paper by Cederbaum et al [24], where the role of interaction in the emergence of interference patterns for two initially independent Bose condensates is stressed.
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