RECONSTRUCTION OF A VERTEX ALGEBRA IN HIGHER
DIMENSIONS FROM ITS ONE-DIMENSIONAL RESTRICTION

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Abstract. Vertex algebras in higher dimensions correspond to models of
quantum field theory with global conformal invariance. Any vertex algebra
in higher dimensions admits a restriction to a vertex algebra in any lower
dimension, and in particular, to dimension one. In this paper, we find
natural conditions under which the converse passage is possible. These
conditions include a unitary action of the conformal Lie algebra with a
positive energy, which is given by local endomorphisms and obeys certain
integrability properties.

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1. Introduction

In Whightman’s axiomatic approach to Quantum Field Theory (QFT), it is possible to restrict quantum fields to time-like submanifolds. This fact is a consequence of the most general and fundamental physical principles such as positivity of energy and locality. Thus, a natural question arises: which QFT models in low-dimensional space-time can be obtained in this way? In other words, which models in low dimensions can generate models in higher dimensions, and what additional structure is needed for this?

In this paper, we fully solve this problem within a special class of quantum field models, those with Global Conformal Invariance (GCI). These models were introduced in [NT01], where it was shown that they are characterized as conformally invariant models with rational correlation functions. This rationality allows the QFT models with GCI to be described in a purely algebraic way in terms of the vertex algebras in higher dimensions introduced in [N05a] and further developed in [BN06a]. One of the main results of [N05a] (see Theorems 9.2 and 9.3), is a precise 1-1 correspondence between conformal vertex algebras in higher dimensions with Hermitian structure and models of Whightman’s axioms that possess GCI.

The present work is done entirely within the formalism of vertex algebras in higher dimensions, which are briefly reviewed in Sect. 2.2 below. A vertex algebra in dimension $D$ admits a restriction to any lower dimension $D' < D$ (see Sect. 2.3). In particular, vertex algebras in dimension 1 are the same as the (chiral) vertex algebras introduced by Borcherds in [B86]; see Sect. 2.1. They are well studied in connection with representation theory [FLM88, K98, FB04, LL04] and 2-dimensional conformal field theory [BPZ84, G89, DMS97].

Our main result is the Reconstruction Theorem 4.1, which shows how one can construct vertex algebras in higher dimensions from chiral vertex algebras that possess additional symmetry structure. In a different form, this result was previously announced in [N05b, BN06b, BN08]. In this paper, we present a complete proof of the theorem.

Let us explain our construction in a heuristic way. Consider a vertex algebra $V$ in dimension $D$ with a state-field correspondence $Y$ (see Sect. 2.2). Suppose that $V$ is equipped with an action of the Lie algebra $\mathfrak{c}_D$ of infinitesimal conformal transformations of the complex Euclidean space $\mathbb{C}^D$, so that $Y$ is covariant under this action (see Sect. 2.4). Additionally, we assume that the action of $\mathfrak{c}_D$ on $V$ can be integrated to a group action $U(g)$ on $V$ for suitable elements $g$ of the conformal group (see Definition 4.1; for general elements $g$ one needs a Hilbert space completion of $V$). Intuitively, for $a, b \in V$, we can think of $Y(a, z)b$ as a conformally-covariant vector-valued function of $z \in \mathbb{C}^D$. If we consider the restriction $z \mapsto x e_1$ to a line, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^D$, we obtain that $V$ is a chiral vertex algebra with a state-field correspondence $Y_1(a, x) = Y'(a, xe_1)$.

Conversely, if we are given a chiral vertex algebra $(V, Y_1)$, which is a restriction of a vertex algebra $(V, Y)$ in dimension $D$, then we can reconstruct $Y$ from $Y_1$ and the action of $\mathfrak{c}_D$. Explicitly, $Y(a, z)b$ is obtained from the adjoint action $U(g)Y_1(U(g)^{-1}a, x)U(g)^{-1}b$ of a conformal transformation $g$ that maps the pair...
of points \((0, x_1)\) to the pair \((0, z)\); cf. Eq. (4.20) below. In order for this construction to be well defined, we need a certain invariance condition for the action of the stabilizer of a pair \((0, x_1)\) in the conformal group action. We have found two equivalent formulations of this invariance. The first consists of explicit commutation relations between the Lie algebra \(c_D\) and the vertex operators \(Y_1(a, x)\) of the initial chiral vertex algebra (cf. Proposition 3.4). The second condition is more abstract and is given in terms of the notion of a local endomorphism of a vertex algebra, introduced in Sect. 3. We further develop this notion, which may be of independent interest and is related to the pseudoderivations of Etingof and Kazhdan \([EK00]\) (see also \([L05]\)). In fact, local endomorphisms have already found applications in other works such as \([Ne15, Ne16a, Ne16b]\). We also remark that there is an analogy between our algebraic construction and the geometric dimensional reduction introduced in \([N87]\) (see also \([NP03a, NP03b]\)).

After we reconstruct the fields \(Y(a, z)\) from their restrictions \(Y(a, x_1)\), as outlined above, we have to show that the state-field correspondence \(Y\) endows \(V\) with the structure of a vertex algebra in dimension \(D\). One of the most important axioms to check is the locality of the fields. Let us explain heuristically the idea behind its proof (see Sect. 4.7). Locality is a statement about the product \(Y(a, z)Y(b, w)\) for three mutually non-isotropic points \(0, z, w\) in \(C^D\). It is known that the conformal group acts transitively on triples of mutually non-isotropic complex points. In particular, they can be mapped into a line, and in this way locality in dimension \(D\) can be derived from locality in dimension 1.

Another axiom of a vertex algebra requires that \(Y(a, z)b\) become a formal power series in \(z = (z_1, \ldots, z_D)\) after multiplication by some power of the Euclidean square-length \(z^2 = (z_1^2 + \cdots + z_D^2)\). However, our construction only defines \(Y(a, z)b\) as a doubly-infinite formal series in \(z\) and \(1/z^2\) (see \([N05a, BN06a]\) for more details about such series). Proving the bounds of the poles in \(z^2\) turned out to be challenging and solving this problem is one of the main contributions of this paper. This is where we essentially used unitarity (see Sect. 4.5), while the rest of our construction does not rely on any Hilbert space positivity.

It is remarkable that in low space-time dimensions (1 or 2) there are large classes of non-trivial QFT models, while in dimension 4 and higher only the free field models are mathematically fully developed within the established axiomatic frameworks of Whigtman and Haag–Kastler (see \([BLOT90, H96, A99]\)). With the present work, we suggest an intriguing bridge between lower- and higher-dimensional models. The long-term goal of constructing QFT models in higher dimensions is one of the main application prospects of our results.

**Notation.** All vector spaces in this paper are over the field \(C\) of complex numbers. We denote by \(Z\) and \(Z_{\geq 0}\) the sets of integers and non-negative integers, respectively. We will mainly follow the notation from our previous works \([N05a, BN06a]\) but for completeness we will list the most important notation here.

Throughout the paper, we fix a positive integer \(D\). Vectors in \(C^D\) are denoted by \(z = (z_1, \ldots, z_D)\), \(w = (w_1, \ldots, w_D)\), etc. We equip \(C^D\) with the complexified Euclidean scalar product \(z \cdot w = z_1w_1 + \cdots + z_Dw_D\) and set \(z^2 := z \cdot z\) to be the “light-cone” quadric. Let \(\partial_{z^\alpha} = \partial/\partial z^\alpha\) for \(\alpha = 1, \ldots, D\). For a vector
space $V$, we denote by $V[z]$ the space of formal power series in $z^1, \ldots, z^D$ with coefficients in $V$. The space $V[z]_{\leq N}$ is the space of series of the form $(z^2)^{-N} v(z)$ where $v(z) \in V[z]$ and $N \in \mathbb{Z}_{\geq 0}$. Similarly, we will use the spaces $V[z, w]_{\leq N}$.

A complete introduction to the various spaces of formal series that appear in the theory of vertex algebras in higher dimensions can be found in [N05a, Sect. 1] and [BN06a, Sect. 1]. In particular, we recall that $V[z]_{\leq N}$ is a module over $C[z]_{\leq N}$, as well as a differential module with derivations $\partial_{z^\alpha}$ ($\alpha = 1, \ldots, D$).

In dimension $D = 1$, we have $z \equiv z^1$ and we shall just denote it by $z$. In this case, the space $V[z]_{\leq N}$ reduces to the space of Laurent series $V((z)) := V[[z]][z^{-1}] \equiv V[z]_{\leq 1} \equiv V[z]_{\leq 1}$. 

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2. Vertex Algebras in Dimension 1 and $D$

In this section, we first briefly recall the definition and main properties of vertex algebras in dimension 1 (for more details, see [FLM88, K98, FB04, LL04]). Then we review the theory of vertex algebras in higher dimensions, which were introduced in [N05a] and developed further in [BN06a].

2.1. Vertex algebras in dimension 1. The notion of a vertex algebra introduced by Borcherds [B86] provides a rigorous algebraic description of two-dimensional chiral conformal field theory (see e.g. [BPZ84, G89, DMS97]). A vertex algebra is a vector superspace $V$ (space of states) with a distinguished even vector $1 \in V$ (vacuum vector), together with an even linear map (state-field correspondence)

$$Y(\cdot, z) : V \otimes V \rightarrow V((z)) = V[[z]][z^{-1}] .$$

For every state $a \in V$, we have the field $Y(a, z) : V \rightarrow V((z))$. This field can also be viewed as a formal power series from $(\text{End} V)[z, z^{-1}]$, which after applied to any vector involves only finitely many negative powers of $z$. The coefficients in front of powers of $z$ in this expansion are known as the modes of $a$:

$$Y(a, z)b = \sum_{n=-\infty}^{N_{a,b}-1} a_n b z^{-n-1}, \quad a, b \in V .$$

We can think of $a_n b$ as an infinite sequence of products that vanish for sufficiently large $n \geq N_{a,b}$ (depending on $a$ and $b$) and are subject to certain axioms.

First, the vacuum vector $1$ behaves as a partial identity:

$$a_{(m)} 1 = \delta_{m,-1} a , \quad 1_{(n)} a = \delta_{n,-1} a , \quad m, n \in \mathbb{Z} , \quad m \geq -1 .$$

Next, define the translation operator $T \in \text{End} V$ by $T a = a_{(-2)} 1$. Then all fields $Y(a, z)$ are required to be translation covariant:

$$[T, Y(a, z)] = \partial_z Y(a, z) .$$
Finally, all fields $Y(a, z)$ in a vertex algebra must be \textit{local} with each other, which means that for all $a, b \in V$ there is a non-negative integer $N_{a,b}$ such that
\[(z - w)^{N_{a,b}}[Y(a, z), Y(b, w)] = 0. \tag{2.5}\]
This completes the definition of a vertex algebra. We will denote a vertex algebra as $(V, Y, 1, T)$, or simply as $V$.

Note that we can take the same $N_{a,b}$ in Eqs. (2.2) and (2.5). It follows from the vacuum and translation axioms (2.3), (2.4) that
\[Y(a, z)1 = e^{zT}a, \quad Y(1, z)a = a. \tag{2.6}\]

The locality condition (2.5) implies the Borcherds \textbf{commutator formula} $(a, b \in V, m, n \in \mathbb{Z})$
\[ [a^{(m)}, b^{(n)}] = \sum_{j=0}^{N_{a,b}} \binom{m}{j} (a^{(j)}b)(m+n-j), \tag{2.7}\]
which can be rewritten equivalently as
\[ [a^{(m)}, Y(b, w)] = \sum_{j=0}^{N_{a,b}} \binom{m}{j} w^{m-j} Y(a^{(j)}b, w). \tag{2.8}\]

Among other important results of the theory of vertex algebras, let us mention Dong’s Lemma ([K98, Lemma 3.2]), the Goddard Uniqueness Theorem ([K98, Theorem 4.4]), and the Kac Existence Theorem ([K98, Theorem 4.5]). The last theorem allows one to generate a vertex algebra from a collection of translation covariant local fields.

### 2.2. The definition of vertex algebras in dimension $D$

Let us recall from [N05a, Definition 2.1], [BN06a, Definition 4.1] the notion of a \textit{vertex algebra} over $\mathbb{C}^D$, also called a vertex algebra in dimension $D$. This is a vector superspace $V$, endowed with an even linear map $V \otimes V \rightarrow V[z,w]$, $a \otimes b \mapsto Y(a, z)b$, called a \textbf{state-field correspondence}, a system of commuting even endomorphisms $T_1, \ldots, T_D$ of $V$ called \textbf{translation operators}, and an even vector $1 \in V$ called the \textbf{vacuum}, subject to the following axioms.

\begin{itemize}
\item[(a)] \textbf{Locality}: The formal series $f_{a,b,c}(z, w) := (z - w)^{N_{a,b}}Y(a, z)Y(b, w)c$ takes values in $V[z,w]_{z,w}$ for some non-negative integer $N_{a,b} = N_{b,a}$, and $f_{a,b,c}(z, w) = (-1)^{p_a p_b} f_{b,a,c}(w, z)$ for all $a, b, c \in V$, where $p_a$ and $p_b$ are the parities of $a$ and $b$, respectively.
\item[(b)] \textbf{Translation covariance}: $[T_\alpha, Y(a, z)] = \partial_z^{\alpha} Y(a, z)$ for all $a \in V, \alpha = 1, \ldots, D$.
\item[(c)] \textbf{Vacuum}: $T_\alpha 1 = 0$, $Y(1, z)a = a$ and $Y(a, z)1 \in V[z]$ with the property that $Y(a, z)1|_{z=0} = a$, for all $a \in V$ and $\alpha = 1, \ldots, D$.
\end{itemize}

We will denote a vertex algebra as $(V, Y, 1, T)$ or simply as $V$, where $T$ is the collection $(T_1, \ldots, T_D)$. In the case $D = 1$, the above definition reduces to that
of the well-known chiral vertex algebras; see [B86, FLM88, K98, FB04, LL04] and Sect. 2.1.

It is convenient to expand the series \( Y(a, z)b \) in a basis similarly to Eq. (2.2). To this end, we use bases of harmonic polynomials \( \{ h_{m,\sigma} \} \subset \mathbb{C}_{\text{har}}[z] \) for each degree \( m \) of homogeneity (cf. [N05a, Sect. 1] or [BN06a, Sect. II.B]):

\[
(\partial_{z_1}^2 + \cdots + \partial_{z_D}^2) h_{m,\sigma}(z) = 0, \quad (z^1 \partial_{z_1} + \cdots + z^D \partial_{z_D}) h_{m,\sigma}(z) = m h_{m,\sigma}(z).
\]

Then we have

\[
Y(a, z)b = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_{mD}} a_{\{n,m,\sigma\}} b (z^2)^n h_{m,\sigma}(z) \quad (2.9)
\]

for some elements \( a_{\{n,m,\sigma\}} b \in V \) (cf. [N05a, Eq. (2.1)], [BN06a, Eq. (2.9)]).

Remark 2.1. The expansion (2.9) is an extension of the well-known expansion of an arbitrary homogeneous polynomial \( f(z) \in \mathbb{C}[z] \) of degree \( m \) in terms of harmonic polynomials:

\[
f(z) = \sum_{0 \leq n \leq m} \sum_{\sigma=1}^{b_{m-2n}} f_{n,m-2n,\sigma} (z^2)^n h_{m-2n,\sigma}(z),
\]

for unique coefficients \( f_{n,m-2n,\sigma} \in \mathbb{C} \) (see e.g. [N05a, Lemma 1.1]).

Many important results from the theory of 1-dimensional vertex algebras have generalizations to higher dimensions. In particular, we have

\[
Y(a, z)1 = e^z T a, \quad Y(1, z)a = a, \quad (2.11)
\]

where \( z \cdot T \) stands for \( \sum_{\alpha=1}^{D} z^\alpha T_\alpha \) (cf. [N05a, Proposition 3.2]). The analog of Goddard’s Uniqueness Theorem is Theorem 3.1 of [N05a], and it is in fact the vertex algebra counterpart of the Reeh–Schlieder Theorem in the Wightman approach to quantum field theory. There are also analogs of Dong’s Lemma and the Kac Existence Theorem, which are Proposition 3.4 and Theorem 4.1 of [N05a], respectively.

2.3. Restriction to lower dimensions. Now let us assume that \( D \geq 2 \) and fix a positive integer \( D' < D \). We will denote vectors in \( \mathbb{C}^{D'} \) by \( z', w' \), etc. Notice that

\[
(z')^2 = z^2, \quad (z' - w')^2 = (z - w)^2 \quad \text{for} \quad z = (z',0,\ldots,0), \quad w = (w',0,\ldots,0).
\]

This induces a restriction morphism \( V[z]_{z,z} \rightarrow V[z']_{z',z'} \), which agrees with the module actions of \( \mathbb{C}[z]_{z,z} \) and \( \mathbb{C}[z']_{z',z'} \) under the algebra homomorphism \( \mathbb{C}[z]_{z,z} \rightarrow \mathbb{C}[z']_{z',z'} \) induced also by the restriction. Moreover, the operations of taking a partial derivative \( \partial_{z_\alpha} \) (\( \alpha = 1, \ldots, D' \)) and evaluation \( z = (z',0,\ldots,0) \) commute. Hence, the restriction of the state-field correspondence

\[
Y_{D'}(a, z')b := Y(a, z)b|_{z=(z',0,\ldots,0)} \quad (2.12)
\]
makes sense, and \( V \) endowed with the restricted state-field correspondence (2.12), the translation endomorphisms \( T_1, \ldots, T_D \) and the same vacuum \( 1 \in V \) is a vertex algebra over \( \mathbb{C}^D' \). It is called a \( D' \)-dimensional \textbf{restriction} of \( V \).

2.4. \textbf{Conformal vertex algebras.} The Lie algebra \( \mathfrak{c}_D \) of \textbf{infinitesimal conformal transformations} of \( \mathbb{C}^D \) is spanned by the infinitesimal translations \( T_\alpha \), dilation \( H \), rotations \( \Omega_{\alpha \beta} = -\Omega_{\beta \alpha} \), and special conformal transformations \( C_\alpha \). These generators satisfy the following relations:

\[
[H, \Omega_{\alpha \beta}] = [T_\alpha, T_\beta] = [C_\alpha, C_\beta] = 0,
\]

\[
[\Omega_{\alpha \beta}, T_\gamma] = \delta_{\alpha \gamma} T_\beta - \delta_{\beta \gamma} T_\alpha,
\]

\[
[H, T_\alpha] = T_\alpha, \quad [H, C_\alpha] = -C_\alpha,
\]

\[
[T_\alpha, C_\beta] = 2\delta_{\alpha \beta}H - 2\Omega_{\alpha \beta},
\]

\[
[\Omega_{\alpha_1 \beta_1}, \Omega_{\alpha_2 \beta_2}] = \delta_{\alpha_1 \alpha_2}\Omega_{\beta_1 \beta_2} + \delta_{\beta_1 \beta_2}\Omega_{\alpha_1 \alpha_2} - \delta_{\beta_1 \alpha_2}\Omega_{\alpha_1 \beta_2} - \delta_{\beta_1 \alpha_2}\Omega_{\alpha_1 \beta_2}.
\]

Note that we have an isomorphism of Lie algebras \( \mathfrak{c}_D \cong \mathfrak{o}(D + 2, \mathbb{C}) \), and the infinitesimal rotations \( \Omega_{\alpha \beta} \) span a subalgebra isomorphic to \( \mathfrak{o}(D, \mathbb{C}) \).

\textbf{A conformal vertex algebra} \cite[Definition 7.1]{N05a} is a vertex algebra whose underlying vector superspace is a \( \mathfrak{c}_D \)-module, which extends the action of the infinitesimal translations \( T_\alpha \) and satisfies the following properties.

(a) \textit{Integrability}: Every vector is contained in a finite-dimensional subspace invariant under all \( \Omega_{\alpha \beta} \).

(b) \textit{Energy positivity}: \( H \) is diagonalizable with non-negative eigenvalues.

(c) \textit{Conformal equivariance}:

\[
[H, Y(a, z)] = Y(Ha, z) + z \cdot \partial_z Y(a, z),
\]

\[
[\Omega_{\alpha \beta}, Y(a, z)] = Y(\Omega_{\alpha \beta}a, z) + (z^\alpha \partial_z - z^\beta \partial_{z^\alpha})Y(a, z),
\]

\[
[C_\alpha, Y(a, z)] = Y(C_\alpha a, z) - 2z^\alpha Y(Ha, z) - 2 \sum_{\beta = 1}^D z^\beta Y(\Omega_{\alpha \beta}a, z)
\]

\[
+ (z^2 \partial_{z^\alpha} - 2z^\alpha z \cdot \partial_z)Y(a, z).
\]

(d) \textit{Vacuum}: \( \mathfrak{c}_D 1 = 0 \).

When we want to specify all the structure of a conformal vertex algebra in dimension \( D \), we will denote it as \( (V, Y, 1, \mathfrak{c}_D) \). If we restrict the above definition to the case \( D = 1 \), then \( \mathfrak{c}_1 \cong \mathfrak{o}(3, \mathbb{C}) \cong \mathfrak{so}(2, \mathbb{C}) \) and our notion of a conformal vertex algebra coincides with that of a Möbius vertex algebra from \cite{K98}.

\textbf{Remark 2.2.} It follows from Theorem 3.1 below that the above condition (d) is a consequence of condition (c).

We introduce a real form of \( \mathfrak{c}_D \) by the following anti-linear anti-involution:

\[
H^* = H, \quad \Omega_{\alpha \beta}^* = -\Omega_{\beta \alpha}, \quad T_\alpha^* = C_\alpha, \quad C_\alpha^* = T_\alpha,
\]

so that \( (\lambda A)^* = \lambda A^* \), \( A^* = A \), and \([A, B]^* = [B^*, A^*]\) for all \( \lambda \in \mathbb{C} \) and \( A, B \in \mathfrak{c}_D \). A representation of \( \mathfrak{c}_D \) on a complex vector space \( V \) is called \textbf{unitary} if \( V \) is equipped with a positive-definite Hermitian product such that the above anti-involution coincides with the Hermitian conjugation.
3. Local Endomorphisms of Vertex Algebras

In this section, we introduce the notion of a local endomorphism of a vertex algebra (in any dimension) and present a formula for its commutator with the fields of the vertex algebra. In the case of dimension 1, local endomorphisms are related to the pseudoderivations of [EK00]. We consider the example of a local action of $c_D$, which will be important for the rest of the paper.

3.1. Local endomorphisms.

Definition 3.1. Let $(V, Y, 1, T)$ be a vertex algebra over $\mathbb{C}$ (including the case $D = 1$). We say that a linear operator $X \in \text{End} V$ is local if $[X, Y(a, z)]$ is local with respect to $Y(b, z)$ for all $a, b \in V$.

Note that $[X, Y(a, z)]$ is again a field for any $X \in \text{End} V$, i.e., $[X, Y(a, z)]v \in V[z]_2$ for every $v \in V$. Explicitly, the condition that $X$ is local means that, for all $a, b \in V$, there is a non-negative integer $N_{a,b}^X$ such that

$$((z-w)^2)^{N_{a,b}^X} [[X, Y(a, z)], Y(b, w)] = 0. \quad (3.1)$$

Remark 3.1. (a) In general, the space of all local endomorphism of $V$ is not a Lie superalgebra under the commutator. However, under some additional assumptions like the conditions of Theorem 3.1 below, one obtains a Lie superalgebra.

(b) In any vertex algebra the translation operators $T_\alpha$ are local. More generally, any derivation of $V$ is a local endomorphism.

(c) Other examples of local endomorphisms in dimension $D = 1$ are provided by all modes of fields: the commutator formula (2.8) implies that $a_{(m)} \in \text{End} V$ are local for $a \in V$, $m \in \mathbb{Z}$.

(d) Suppose that the vertex algebra $V$ is generated by a collection of fields $\{\varphi_a(z)\}$. It follows from Dong’s Lemma that a linear operator $X \in \text{End} V$ is local if and only if $[X, \varphi_a(z)]$ is local with respect to $\varphi_b(z)$ for all $a, b$.

Theorem 3.1. For a vertex algebra $V$ and a linear operator $X$ of $V$, the following two conditions are equivalent:

(i) $X$ is a local endomorphism of $V$ such that $X1 = 0$ and $(e^{-\text{ad}(zT)}X)a \in V[z]$ for every $a \in V$.

(ii) There is a linear map $X(z) : V \to V[z]$ such that $X(0) = X$ and $[X, Y(a, z)] = Y(X(z)a, z)$ for all $a \in V$.

In any of these two cases, one has $X(z) = e^{-\text{ad}(zT)}X$ and

$$[X, Y(a, z)] = Y((e^{-\text{ad}(zT)}X)a, z). \quad (3.2)$$

Proof. If (i) holds, then $X(z) := e^{-\text{ad}(zT)}X$ obviously satisfies $X(0) = X$ and $X(z)a \in V[z]$ for all $a \in V$ by assumption. Hence, (ii) will follow from formula (3.2). To prove Eq. (3.2), we observe that $[X, Y(a, z)]$ is a field on $V$, which is
local with respect to all fields \( Y(b, z) \) for \( b \in V \). Furthermore, by Eq. (2.11), we have 
\[
[X, Y(a, z)]1 = XY(a, z)1 = Xe^{zT}a = e^{zT}(e^{-\text{ad}(zT)}X)a.
\]
Hence, Eq. (3.2) follows from the higher-dimensional analog of Goddard’s Uniqueness Theorem [N05a, Theorem 3.1].

Conversely, suppose that \((ii)\) holds. Then \([X, Y(a, z)] = Y(X(z)a, z)\) implies that \(X\) is a local endomorphism of \(V\). Setting \(a = 1\) in this formula, we get
\[
0 = [X, id] = [X, Y(1, z)] = Y(X(z)1, z).
\]
Hence, applying the right-hand side to \(1\) and setting \(z = 0\), we obtain \(X1 = X(0)1 = 0\). To finish the proof, it remains to show that \((e^{-\text{ad}(zT)}X)a \in V[z]\) for all \(a \in V\).

Fix an index \(\alpha\), apply the operator \(\text{ad} T_\alpha\) to both sides of formula \([X, Y(a, z)] = Y(X(z)a, z)\) and use the translation covariance \([T_\alpha, Y(a, z)] = \partial z_\alpha Y(a, z)\). We obtain:
\[
\left[[T_\alpha, X], Y(a, z)\right] + [X, \partial z_\alpha Y(a, z)] = \partial w_\alpha Y(X(z)a, w)|_{w=z}.
\]
The second summand above is equal to
\[
[X, \partial z_\alpha Y(a, z)] = \partial z_\alpha[X, Y(a, z)] = \partial z_\alpha(Y(X(z)a, z)).
\]
Therefore,
\[
\left[[T_\alpha, X], Y(a, z)\right] = -Y((\partial z_\alpha X(z)a), z).
\]
Repeating this process, we see that
\[
[(\text{ad} T_{\alpha_1}) \cdots (\text{ad} T_{\alpha_k})X, Y(a, z)] = (-1)^k Y((\partial z_{\alpha_1} \cdots \partial z_{\alpha_k} X(z)a), z) \quad (3.3)
\]
for any \(\alpha_1, \ldots, \alpha_k \in \{1, \ldots, D\}\). Since \(X(z)a\) is a polynomial in \(z\), there exists a positive integer \(N_a\) (depending on \(a\)) such that the right-hand side above is zero for all \(k \geq N_a\) and any choice of \(\alpha_1, \ldots, \alpha_k\). This implies that
\[
[(\text{ad} T_{\alpha_1}) \cdots (\text{ad} T_{\alpha_k})X, Y(a, z)] = 0, \quad k \geq N_a.
\]
Applying this identity to the vacuum vector \(1\) and setting \(z = 0\), we obtain
\[
((\text{ad} T_{\alpha_1}) \cdots (\text{ad} T_{\alpha_k})X)a = 0, \quad k \geq N_a,
\]
where we used that \(X1 = T_\alpha1 = 0\) and \(Y(a, z)1|_{z=0} = a\). Hence, \((e^{-\text{ad}(zT)}X)a \in V[z]\).

As customary in the theory of vertex algebras, let us denote by \(\epsilon_{z,u}\) the Taylor expansion
\[
\epsilon_{z,u}f(z + u) = e^{u \partial_z}f(z), \quad (3.4)
\]
for polynomials \(f(z)\) or more generally Laurent series in \(z\). The next result is a consequence of Eq. (3.3) from the proof of Theorem 3.1.

**Corollary 3.2.** With the notation of Theorem 3.1, we have:
\[
[T_\alpha, X(u)] = -\partial u_\alpha X(u), \quad \alpha = 1, \ldots, D, \quad (3.5)
\]
\[
[X(u), Y(a, z)] = \epsilon_{z,u}Y(X(z + u)a, z). \quad (3.6)
\]

In dimension \(D = 1\), this corollary means that the map \(X(u)\) is a pseudo-derivation of \(V\) in the sense of Etingof and Kazhdan [EK00] (see also [L05]).
3. Local action of $\mathfrak{c}_D$. In any conformal vertex algebra $V$ over $\mathbb{C}^D$, the conformal equivariance (2.14)−(2.16) implies that the action of $\mathfrak{c}_D$ on $V$ is a local action, i.e., any $X \in \mathfrak{c}_D$ acts as a local endomorphism of $V$. In fact, we have:

**Proposition 3.3.** Let $(V, Y, 1, T)$ be a vertex algebra over $\mathbb{C}^D$ equipped with an action of $\mathfrak{c}_D$ such that each $T_\alpha \in \mathfrak{c}_D$ acts as the translation operator $T_\alpha$ of $V$ ($\alpha = 1, \ldots, D$). Then the action of $\mathfrak{c}_D$ is local and annihilates the vacuum ($\mathfrak{c}_D \cdot 1 = 0$) if and only if the conformal equivariance relations (2.14)−(2.16) hold.

**Proof.** This follows from Theorem 3.1 and the commutation relations (2.13). Indeed, from (2.13), we find:

$$e^{-\text{ad}(x \cdot T)} H = H + x \cdot T,$$
$$e^{-\text{ad}(x \cdot T)} \Omega_{\alpha \beta} = \Omega_{\alpha \beta} + z^\alpha T_\beta - z^\beta T_\alpha,$$
$$e^{-\text{ad}(x \cdot T)} C_\alpha = C_\alpha - 2z^\alpha H - 2 \sum_{\beta=1}^D z^\beta \Omega_{\alpha \beta} + x^2 T_\alpha - 2z^\alpha (x \cdot T),$$

which together with $Y(T_\alpha a, x) = \partial_x Y(a, x)$ completes the proof.\[\square\]

We will also investigate the case of a local action of $\mathfrak{c}_D$ on a vertex algebra in dimension 1.

**Proposition 3.4.** Let $(V, Y_1, 1, T_1)$ be a vertex algebra equipped with a local action of $\mathfrak{c}_D$ such that it annihilates the vacuum ($\mathfrak{c}_D \cdot 1 = 0$) and $T_1 \in \mathfrak{c}_D$ is the translation operator of $V$. Then we have ($2 \leq \alpha, \beta \leq D$):

$$[T_\alpha, Y_1(a, x)] = Y_1(T_\alpha a, x), \quad [H, Y_1(a, x)] = Y_1(Ha, x) + x\partial_x Y_1(a, x),$$
$$[\Omega_{\alpha \beta}, Y_1(a, x)] = Y_1(\Omega_{\alpha \beta} a, x), \quad [\Omega_{1 \alpha}, Y_1(a, x)] = Y_1(\Omega_{1 \alpha} a, x) + xY_1(T_1 a, x),$$
$$[C_\alpha, Y_1(a, x)] = Y_1(C_\alpha a, x) + 2xY_1(\Omega_{1 \alpha} a, x) + x^2 Y_1(T_1 a, x),$$
$$[C_1, Y_1(a, x)] = Y_1(C_1 a, x) - 2xY_1(Ha, x) - x^2 \partial_x Y_1(a, x).$$

Conversely, the above commutation relations imply that the action of $\mathfrak{c}_D$ is local and annihilates the vacuum.

**Proof.** We can apply Theorem 3.1 with $z = x$, $T = T_1$ and any $X \in \mathfrak{c}_D$, as the action of $\text{ad} T$ on $\mathfrak{c}_D$ is nilpotent. Using the commutation relations (2.13), we find:

$$e^{-x \text{ad} T_\alpha} T_\alpha = T_\alpha,$$
$$e^{-x \text{ad} T} \Omega_{\alpha \beta} = \Omega_{\alpha \beta},$$
$$e^{-x \text{ad} T_1} C_\alpha = C_\alpha + 2x\Omega_{1 \alpha} + x^2 T_\alpha,$$
$$e^{-x \text{ad} T_1} C_1 = C_1 - 2xH - x^2 T.$$

Now the proof follows immediately from Theorem 3.1 and the property $Y_1(T_\alpha a, x) = \partial_x Y_1(a, x)$.

\[\square\]

4. **The Reconstruction Theorem**

In this section, we state and prove the main result of the paper, the Reconstruction Theorem 4.1.
4.1. **Formulation of the theorem.** We start by listing the necessary conditions satisfied by the action of $\mathfrak{c}_D$.

**Definition 4.1.** Let $V$ be a vector space equipped with an action of the Lie algebra $\mathfrak{c}_D$ of infinitesimal conformal transformations.

(a) The action of $\mathfrak{c}_D$ is called **integrable** if every vector is contained in a finite-dimensional subspace invariant under all infinitesimal rotations $\Omega_{\alpha\beta} \in \mathfrak{so}(D, \mathbb{C})$ (as in the definition of a conformal vertex algebra in Sect. 2.4).

(b) The action of $\mathfrak{c}_D$ is called **strongly integrable** if, in addition, the infinitesimal dilation operator $H$ is diagonalizable and we have that the spectrum of

$$H + \imath(\Omega_{12} + \Omega_{34} + \cdots + \Omega_{k,k+1})$$

consists of even integers, where $k$ is the integer part of $\frac{D}{2}$ (cf. Remark 4.1 below). In other words,

$$(-1)^{H+\imath(\Omega_{12}+\Omega_{34}+\cdots+\Omega_{k,k+1})} = 1 \text{ on } V.$$  \hspace{1cm} (4.2)

(Note that in the $\mathfrak{so}(D+2, \mathbb{C})$ realization of $\mathfrak{c}_D$, the generator $H$ is $\imath \Omega_{-1,0}$ and then in (4.1) we have the sum over all of the Cartan basis.)

(c) We say that $V$ has a **positive energy** if the infinitesimal dilation operator $H$ is diagonalizable with non-negative eigenvalues (again as in the definition of a conformal vertex algebra in Sect. 2.4).

**Remark 4.1.** The meaning of the above strong integrability condition is that in the even (bosonic) case, we will have a unitary representation of the geometric group of rotations and dilations, which is $\text{SO}(D, \mathbb{R}) \times_{\mathbb{Z}/2\mathbb{Z}} \text{SO}(2, \mathbb{R})$, where the $\mathbb{Z}/2\mathbb{Z}$-quotient identifies $-1$ from $\text{SO}(D, \mathbb{R})$ and from the group $\text{SO}(2, \mathbb{R})$ of dilations. The latter identification will be crucial for Corollary 4.4 below, where we will establish the poles’ integrality. In the general case (including fermions), the $\text{SO}(D, \mathbb{R})$ group is replaced by the Euclidean spinor group $\text{Spin}(D, \mathbb{R})$. This is natural to expect, since unitary vertex algebras are in one-to-one correspondence with Wightman quantum field theories with global conformal invariance [N05a], and in globally conformal invariant quantum field theories the spatial unitary symmetry is provided by the (connected) geometric conformal group $\text{SO}_0(D, 2)$ (again in the bosonic case, while in the general case $\text{Spin}_0(D, 2)$).

The following is the main result of the paper.

**Theorem 4.1.** Let $D$ be an even positive integer. Let $(V, Y_1, 1, T_1)$ be a vertex algebra equipped with a local, strongly integrable, positive-energy unitary action of $\mathfrak{c}_D$ such that $\mathfrak{c}_D 1 = 0$ and $T_1 \in \mathfrak{c}_D$ is the translation operator of $V$. Then $V$ can be endowed with a unique structure $(V, Y_D, 1, \mathfrak{c}_D)$ of a conformal vertex algebra in dimension $D$, so that $Y_1$ is the restriction of $Y$.

The proof of this theorem is contained in the rest of this section. Here is a brief outline. We start in Sect. 4.2 with the reconstruction problem for $Y_D$. Initially, we construct $Y_D(a, z) b$ as a generalized series, possibly containing not
only infinitely may negative powers of \( z^2 \) but also half-integer powers. Then in Sects. 4.3 and 4.5, we show that in \( Y_D(a, z) b \) there are only integer powers of \( z^2 \) and the powers are bounded from below. Between these subsections, in Sect. 4.4, we consider the special case \( D = 2 \), which is not only an illustration of the reconstruction scheme of Sect. 4.2 but also serves as an important technical preparation for the pole bounds in Sect. 4.5. Then, in Sects. 4.6 and 4.7, we establish all the remaining axioms for the vertex algebra \((V, Y_D, \mathbf{1}, \mathfrak{c}_D)\). Our construction is more geometric, while in Appendix A, we give an alternative representation-theoretic interpretation of the construction of \( Y_D \).

### 4.2. Reconstructing the state-field correspondence.

Suppose that \( V \) is a vertex algebra endowed with a strongly integrable local action of the conformal Lie algebra \( \mathfrak{c}_D \). Let us write

\[
Y^1(a, x) b = \sum_{n = -N^1_{a,b}}^{\infty} \mu^1_n(a \otimes b) x^n ,
\]

so that \( \mu^1_n(a \otimes b) = a_{(-n-1)} b \) in the usual notation of Eq. (2.2). If we introduce the \( \frac{1}{2} \mathbb{Z}_{\geq 0} \)-grading of \( V \) provided by \( H, V = \bigoplus_{\Delta \in \frac{1}{2} \mathbb{Z}_{\geq 0}} V_\Delta, \)

it follows that

\[
\mu^1_n : V_{\Delta'} \otimes V_{\Delta''} \rightarrow V_{\Delta'+\Delta''+n},
\]

i.e., \( \mu^1_n \) is a degree \( n \) map \( (n \in \mathbb{Z}) \). (Note that, in general, the map \( \mu^1_n \) is a \( \mathbb{Z} \)-graded map \( V \otimes V \rightarrow V \) but by a slight abuse of notation we will use the same notation for its restrictions (4.5), which implicitly depend on \( \Delta' \) and \( \Delta'' \).)

Similarly, if we had a vertex algebra structure \( Y_D \) on \( V \), then we could write

\[
Y_D(a, x) b = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{\infty} \mu^D_{n,m,\sigma}(a \otimes b) (x^2)^{-m} h_{m,\sigma}(x) ,
\]

where the relation to the expansion (2.9) is via

\[
\mu^D_{2n+m,\sigma}(a \otimes b) = a_{(n,m,\sigma)} b .
\]

Furthermore, the commutation relations (2.14) with \( H \) require that

\[
\mu^D_{n,m,\sigma} : V_{\Delta'} \otimes V_{\Delta''} \rightarrow V_{\Delta'+\Delta''+n} .
\]

On the other hand, relations (2.15) are equivalent to the condition that all linear maps

\[
\mathfrak{C}_{\text{char}}^m[z] \rightarrow \text{Hom}(V_{\Delta'}, V_{\Delta''}, V_{\Delta'+\Delta''+n}), \quad h_{m,\sigma}(z) \mapsto \mu^D_{n,m,\sigma},
\]

are homomorphisms of \( \mathfrak{so}(D, \mathbb{C}) \)-modules, where \( \Omega_{\alpha\beta} \) acts on \( \mathbb{C}[x] \) (and by restriction on \( \mathbb{C}_{\text{char}}^m[z] \)) as \( \partial_z \alpha - z^\beta \partial_z \).

**Remark 4.2.** There is a natural representation of the above maps \( \mu^1_n : V \otimes V \rightarrow V \) and \( \mu^D_{n,m,\sigma} : V \otimes V \rightarrow V \) (Eqs. (4.5), (4.8)) by the residue map \( \text{res}_x : V[x]_x \rightarrow V \).
and its higher-dimensional analog $\text{Res}_z: V[z]_{xz} \to V$ introduced in [BN06a, Sect. III]:

$$
\mu_1^0(a \otimes b) = a_{(-n-1)}b = \text{res}_x x^{-n-1}Y_1(a, x)b,
$$

$$
\mu_{2n+m,m,\sigma}^D(a \otimes b) = a_{(n,m,\sigma)}b = \text{Res}_z (z^2)^{-(D/2)-m-n} h_{m,\sigma}(z) Y_D(a, z)b
$$

(cf. [BN06a, Eq. (3.8)]). This brings further clarity to the $\mathfrak{so}(D, \mathbb{C})$–equivariance of the map (4.9) in a conformal vertex algebra over $\mathbb{C}^D$.

In this subsection, we give an explicit construction of the maps $\mu_{n,m,\sigma}^D$ from the 1-dimensional vertex algebra $V_1$ and the action of $\epsilon_D$, satisfying the conditions of Theorem 4.1. In the subsequent subsections, we will derive the properties of $Y_D$. In particular, in Sect. 4.3 we prove that the powers of $z^2$ that appear in Eq. (4.6) are integral; in Sect. 4.5 we prove that these powers are bounded from below; in Sect. 4.6 we prove that the so-obtained series (4.6) is the only $\mathfrak{so}(D, \mathbb{C})$–equivariant series whose restriction to $z = (x, 0, \ldots, 0)$ gives (4.3).

For the sake of simplicity, from now on we shall make the additional assumption that the eigenspaces $V_\Delta$ of $H$ are finite dimensional. In this way, the linear maps $\mu_1^0$ and $\mu_{n,m,\sigma}^D$ will become maps between finite-dimensional spaces (see (4.5), (4.8)). However, this assumption is not crucial, since due to the integrability condition (a) from Definition 4.1, for any two elements $a \in V_{\Delta'}$ and $b \in V_{\Delta''}$, one can find finite-dimensional subspaces $W' \subseteq V_{\Delta'}$ and $W'' \subseteq V_{\Delta''}$ with $a \in W'$, $b \in W''$, which are invariant under the $\mathfrak{so}(D, \mathbb{C})$-subalgebra. Then the image $W := \mu_1^0(W' \otimes W'')$ will be a finite-dimensional subspace of $V_{\Delta'+\Delta''}$, which is also invariant under the $\mathfrak{so}(D, \mathbb{C})$-subalgebra. The maps $\mu_{n,m,\sigma}^D$ will be constructed again as linear maps $W' \otimes W'' \to W$. Furthermore, the resulting $\mu_{n,m,\sigma}^D(a \otimes b)$ will not depend on the choice of $W'$ and $W''$. Indeed, if we choose other subspaces $U'$ and $U''$ with $a \in U'$, $b \in U''$, then without loss of generality we can assume that $U' \subseteq W'$ and $U'' \subseteq W''$; hence setting $U := \mu_1^0(U' \otimes U'') \subseteq W$ we can restrict the construction from $W' \otimes W'' \to W$ to $U' \otimes U'' \to U$ due to the $\mathfrak{so}(D, \mathbb{C})$–equivariance of $\mu_1^0$. However, the proofs bellow will become more transparent if we assume instead that all $V_\Delta$ are finite dimensional, which in fact is the most relevant case in physics.

We pass now to the construction of the maps $\mu_{n,m,\sigma}^D: V_{\Delta'} \otimes V_{\Delta''} \to V_{\Delta'+\Delta''}$.

Let us consider $u \in \mathbb{C}^D$ with $u^2 = u \cdot u = 1$ as a complex vector parameter (not as a formal variable). We can also write it as

$$
u = (\cos \vartheta, \sin \vartheta, u'_\perp)
$$

with $\vartheta \in \mathbb{C}$ and $u'_\perp \in \mathbb{C}^{D-1}$ where again $(u'_\perp)^2 = 1$. Define a group element $g_u \in \text{SO}(D, \mathbb{C})$ by

$$
\Omega_{1, u'_\perp} := \sum_{\alpha=2}^D \Omega_{\alpha, u'_\perp}. 
$$

Then $g_u$ has the following properties:

1. $g_u(e_1) = u$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^D$. 

(ii) Let \( U(g_u) \) be the representation of \( g_u \) on \( V \) after the integration of the \( \mathfrak{so}(D, \mathbb{C}) \)-action (according to the strong integrability condition of Definition 4.1 (b)). Then each \( V_\Delta \) is invariant under \( U(g_u) \) and in a basis these are represented by matrices whose elements are polynomials in \( u \).

We then set
\[
\sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_m^D} \mu_{n,m,\sigma}^D(a \otimes b) h_{m,\sigma}(u) := U(g_u) \mu_n^1(U(g_u)^{-1}a \otimes U(g_u)^{-1}b), \tag{4.10}
\]
the left-hand side being a finite sum as the right-hand side is a polynomial in \( u \) whose degree depends on \( a, b \) and \( n \). Note that in the right-hand side of (4.10), we have the natural finite-dimensional action of \( g_u \in \mathfrak{so}(D, \mathbb{C}) \) on \( \mu_n^1 \in \text{Hom}(V_{\Delta} \otimes V_{\Delta'}, V_{\Delta'+\Delta'+n}) \). Eq. (4.10) determines unique coefficients \( \mu_{n,m,\sigma}^D(a \otimes b) \), because for fixed \( m \) the harmonic polynomials \( h_{m,\sigma}(u) \) form a basis of the space of homogenous polynomials of degree \( m \) on the sphere \( \{u \in \mathbb{C}^D | u^2 = 1\} \) (see Remark 2.1).

4.3. Parity property.

**Lemma 4.2.** Let \( f_u \in \text{SO}(D, \mathbb{C}) \) be a function of \( u \in \mathbb{C}^D \), where \( u^2 = 1 \), which satisfies the above two conditions (i) and (ii). Then
\[
U(g_u) \mu_n^1(U(g_u)^{-1}a \otimes U(g_u)^{-1}b) = U(f_u) \mu_n^1(U(f_u)^{-1}a \otimes U(f_u)^{-1}b). \tag{4.11}
\]

**Proof.** Since \( g_u^{-1}f_u(e_1) = e_1 \), we have that \( g_u^{-1}f_u \in \text{SO}(D-1, \mathbb{C}) \), which is the stabilizer group of \( e_1 \). The Lie algebra of this subgroup is the \( \mathfrak{so}(D-1, \mathbb{C}) \)-subalgebra of \( \mathfrak{so}(D, \mathbb{C}) \) spanned by \( \Omega_{\alpha\beta} \) for \( 2 \leq \alpha, \beta \leq D \). Recall from Proposition 3.4 that
\[
[\Omega_{\alpha\beta}, Y_1(a,x)] = Y_1(\Omega_{\alpha\beta}a, x), \quad 2 \leq \alpha, \beta \leq D.
\]
This implies that the map \( \mu_n^1 \) is \( \mathfrak{so}(D-1, \mathbb{C}) \)-equivariant, i.e., \( \mathfrak{so}(D-1, \mathbb{C}) \) acts trivially on \( \mu_n^1 \in \text{Hom}(V_{\Delta} \otimes V_{\Delta'}, V_{\Delta'+\Delta'+n}) \). Hence, \( \mu_n^1 \) is \( \text{SO}(D-1, \mathbb{C}) \)-invariant, and this implies (4.11). \( \square \)

**Lemma 4.3.** For \( g \in \text{SO}(D, \mathbb{C}) \), we have
\[
U(g) \sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_m^D} \mu_{n,m,\sigma}^D(U(g)^{-1}a \otimes U(g)^{-1}b) h_{m,\sigma}(u) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_m^D} \mu_{n,m,\sigma}^D(a \otimes b) h_{m,\sigma}(g(u)). \tag{4.12}
\]

**Proof.** We apply Lemma 4.2 to \( f_u := g^{-1}g_{\beta(u)} \) together with Eq. (4.10). \( \square \)

**Corollary 4.4.** We have \( \mu_{n,m,\sigma}^D = 0 \) if \( n - m \) is odd.

**Proof.** We need to prove that the left-hand side of (4.10) as a function of \( u \) has parity \((-1)^n\). This follows if we apply the previous lemma for
\[
g = (-1)^{k(\Omega_{12}+\Omega_{34}+\cdots+\Omega_{k-1,k})}, \quad k := \frac{D}{2}
\]
(recall that $D$ is even), and use that according to Eq. (4.2) we have $U(g) = (-1)^H$.

\[ \square \]

4.4. **The case $D = 2$**. Before continuing with the general case of Theorem 4.1, it will be useful to consider the special case $D = 2$. In this case, we have a chiral decomposition of the conformal Lie algebra $\mathfrak{e}_2 \cong \mathfrak{so}(4, \mathbb{C})$, which becomes isomorphic to a direct sum of two copies of the Möbius Lie algebra $\mathfrak{e}_1 \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. To write this explicitly, let us introduce:

\[
L^\pm_{-1} := \frac{1}{2}(T_1 \mp iT_2), \quad L^\pm := \frac{1}{2}(C_1 \pm iC_2) = (L^\pm_{-1})^*, \quad L^\pm_0 := \frac{1}{2}(H \mp i\Omega_{12}) = (L^\pm_0)^*.
\]

Then commutation relations (2.13) become:

\[ [L^\pm_m, L^\pm_n] = (m - n)L^\pm_{m+n}, \quad [L^+_m, L^-_n] = 0, \quad (4.14) \]

for $m, n \in \{-1, 0, 1\}$.

Introduce the chiral coordinates

\[ z^\pm := z^1 \pm iz^2, \quad \text{so that} \quad z^2 = z^+ z^- . \]

Then in the representation of $\mathfrak{e}_2$ on $\mathbb{C}[z] = \mathbb{C}[z^1, z^2] = \mathbb{C}[z^\pm]$ by differential operators, where $T_\alpha$ acts as $\partial_{z^\alpha}$ for $\alpha = 1, 2$, we have that $L^\pm_{-1}$ are represented by $\partial_{z^\pm}$. In this representation, $H$ acts as $z^1 \partial_{z^1} + z^1 \partial_{z^2} = z^+ \partial_{z^+} + z^- \partial_{z^-}$, while $\Omega_{12}$ acts as $z^1 \partial_{z^1} - z^2 \partial_{z^2} = i(z^+ \partial_{z^+} - z^- \partial_{z^-})$; hence, $L^\pm_0$ are represented by $z^\pm \partial_{z^\pm}$.

Next, the integrability condition (a) of Definition 4.1 just means that $i\Omega_{12}$ is diagonalizable, while the strong integrability condition (b) further implies that $V$ decomposes into a direct sum of eigenspaces:

\[ V = \bigoplus_{(\Delta^+, \Delta^-) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} V_{\Delta^+, \Delta^-}, \quad L^\pm_0 \big|_{V_{\Delta^+, \Delta^-}} = \Delta^\pm id_{V_{\Delta^+, \Delta^-}}, \quad (4.15) \]

where we use notation similar to Eq. (4.4). In particular, $L^\pm_0$ are simultaneously diagonalizable on $V$ and have integral eigenvalues.

Now, let us see how Eq. (4.10) works in this case. Note that in the chiral coordinates $z^\pm$, there is a natural basis of homogeneous harmonic polynomials of degree $m$:

\[ h_{m,\pm}(z^+, z^-) := (z^\pm)^m \quad \text{for} \quad m > 0, \quad h_0(z^+, z^-) := 1. \quad (4.16) \]

It is convenient to set

\[ h_m := h_{m,+}, \quad h_{-m} := h_{m,-}, \quad m > 0. \]

Then, for $u = z/(z^2)^{1/2}$, we have that $g_u = e^{i\Omega_{12}}$ for $\cos \vartheta = z^1/(z^2)^{1/2}$, i.e.,

\[ e^{2i\vartheta} = z^+ / z^- = (u^\pm)^{1/2} . \]

Hence,

\[ h_m(u) = \left( \frac{z^+}{z^-} \right)^{m/2}, \quad m \in \mathbb{Z}. \]
and Eq. (4.10) reads
\[
\sum_{m \in \mathbb{Z}} \mu_{n,m}^2(a \otimes b) \left( \frac{z^+}{z^-} \right)^\frac{1}{2} = \mu_n^1(a \otimes b)
\]
for \( a \in V_{\Delta^+,\Delta^-} \) and \( b \in V_{\Delta^+,\Delta^-} \). In other words, if we denote by \( \text{Pr}_{\Delta^+,\Delta^-} \) the projection onto \( V_{\Delta^+,\Delta^-} \) in the direct sum (4.15), then
\[
\mu_{n,m}^2(a \otimes b) = \text{Pr}_{\Delta^+,\Delta^-} \left( \Delta^+ + \Delta^- + n \right) \left( \mu_n^1(a \otimes b) \right),
\]
where \( n^\pm := (n \pm m)/2 \), i.e., \( m = n^+ - n^- \) and \( n = n^+ + n^- \).

4.5. Pole bounds. In this subsection, we shall specifically single out if\( \Omega_{12} \) as in the previous subsection and diagonalize it simultaneously with \( H \).

**Lemma 4.5.** Let \( V \) be a unitary representation of the conformal Lie algebra \( \mathfrak{e}_D \), which is strongly integrable and has positive energy (in the sense of Definition 4.1 (b) and (d)). If \( (\Delta, j) \) is a pair of common eigenvalues for \( H \) and \( i\Omega_{12} \), respectively, then it obeys the inequality \( \Delta \geq |j| \).

**Proof.** Since \( V \) is a unitary positive energy representation of \( \mathfrak{e}_D \), it is generated as a \( \mathbb{C}[T_1, \ldots, T_D] \)-module by the subspace of quasi-primary elements, i.e., those \( a \in V \) for which \( C_\alpha a = 0 \) for all \( \alpha = 1, \ldots, D \). We shall prove the lemma first for these quasi-primary elements.

Let us consider the subrepresentation of the algebra \( \mathfrak{e}_2 \) and introduce the Möbius generators of Eqs. (4.13), (4.14). If \( a \) is a nonzero quasi-primary element that has eigenvalues \( \Delta^\pm \) for \( L_0^\pm \), respectively, then we have
\[
0 \leq \|L^\pm_1 a\|^2 = \langle L^\pm_1 a|L^\pm_1 a \rangle = \langle a|L_1^+ L_1^- a \rangle = 2\Delta^\pm \|a\|^2,
\]
since \( L_1^\pm a = 0 \). Hence, \( \Delta^\pm = (\Delta \mp j)/2 \) are both non-negative.

A general eigenvector \( a \in V \) with eigenvalues \( (\Delta, j) \) can be obtained as a linear combination of products \( X_1 \cdots X_T \) acting on quasi-primary vectors, where \( X_1, \ldots, X_T \in \{L_1^+, L_1^-, T_1, \ldots, T_D\} \). Then the bound \( \Delta \geq |j| \) is established by a straightforward induction on \( T \). \( \square \)

**Lemma 4.6.** For every \( a, b \in V \), there exists \( N_{a,b}^{D^0} \in \mathbb{Z}_{\geq 0} \) such that \( \mu_n^{D^0}(a \otimes b) = 0 \) for all possible \( m \) and \( \sigma \) if \( \frac{m-n}{2} > N_{a,b}^{D^0} \) (cf. Eqs. (4.6), (4.7)).

**Proof.** We prove the statement in the converse direction: assuming that some \( \mu_{n,m}^{D^0}(a \otimes b) \neq 0 \) we shall find an upper bound \( N_{a,b}^{D^0} \) for
\[
N := \frac{m-n}{2},
\]
which depends only on \( a \) and \( b \). We can find irreducible \( \mathfrak{g}(D, \mathbb{C}) \)-subrepresentations \( W' \subseteq V_{\Delta'} \) and \( W'' \subseteq V_{\Delta''} \) such that \( a \in W' \) and \( b \in W'' \). We set
\[
W := \mu_n^1(W' \otimes W'') \subseteq V_{\Delta' + \Delta'' + n},
\]
which is again an \( \mathfrak{g}(D, \mathbb{C}) \)-subrepresentation.

Recall that the element \( \Omega_{0,3} \in \mathfrak{g}(D, \mathbb{C}) \) acts on \( \mathbb{C}[z] \) as \( z^a \partial_z^a - z^a \partial_z^a \). As \( i\Omega_{12} \) is Hermitian (see (2.17)), we can choose the basis \( \{h_{m,\sigma}(z)\}_{\sigma=1}^{2^D} \) of \( \mathfrak{c}_m^{\text{char}}[z] \)
so that it consists of eigenvectors for $i\Omega_{12}$. One of the eigenvalues is $m$, and we let it correspond to the index $\sigma_m$. In fact, explicitly we can set (cf. (4.16)): 

$$h_{m,\sigma_m}(z) := (z^1 - iz^2)^m,$$

and then we have 

$$i\Omega_{12} h_{m,\sigma_m}(z) = m h_{m,\sigma_m}(z).$$

As a consequence of Lemma 4.3, the linear map 

$$C_{m}^{\text{hat}}[z] \rightarrow \text{Hom}(W' \otimes W'', W), \quad h_{m,\sigma}(z) \mapsto \mu_{m,\sigma}^{D}, \quad (4.17)$$

is a homomorphism of $\mathfrak{so}(D, \mathbb{C})$--modules (cf. (4.9)). This map is nonzero, because by assumption $\mu_{m,\sigma}^{D} \neq 0$. Since $C_{m}^{\text{hat}}[z]$ is an irreducible $\mathfrak{so}(D, \mathbb{C})$--module (cf. Appendix A), it follows from Schur’s Lemma that the map (4.17) is injective.

In particular, there exist $c \in W'$ and $d \in W''$ such that 

$$\mu_{m,\sigma_m}(c \otimes d) \neq 0.$$ 

Without loss of generality, we can assume that $c$ and $d$ are eigenvectors for $i\Omega_{12}$ with eigenvalues $k'$ and $k''$, respectively. Hence, $\mu_{m,\sigma_m}(c \otimes d)$ has eigenvalues $(\Delta' + \Delta'' + n, k' + k'' + m)$ for $H$ and $i\Omega_{12}$, respectively. By Lemma 4.6, we have that 

$$\Delta' + \Delta'' + n \geq k' + k'' + m$$

and so, $m - n \leq \Delta' + \Delta'' - k' - k''$, which finally gives us 

$$N \leq \frac{1}{2}(\Delta' + J' + \Delta'' + J''), \quad (4.18)$$

where $-J'$ and $-J''$ are the minimal eigenvalues of $i\Omega_{12}$ on $W'$ and $W''$, respectively. This proves the lemma. 

**Remark 4.3.** In the physically most interesting case of $D = 4$, there is a local isomorphism between $\text{SO}(4, \mathbb{C})$ and $\text{SO}(3, \mathbb{C}) \times \text{SO}(3, \mathbb{C})$. Suppose that in the above proof the weights of $W'$, $W''$ are 

$$(j_1', j_2'), \quad (j_1'', j_2''),$$

respectively, as irreducible $\mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$--representations. Then $J' = j_1' + j_2'$ and $J'' = j_1'' + j_2''$, and estimate (4.18) becomes: 

$$N \leq \frac{1}{2}(\Delta' + j_1' + j_2' + \Delta'' + j_1'' + j_2''),$$

which corresponds to an earlier result of [NT01, Proposition 4.3] (there the estimate is a bit stronger but is derived only for $D = 4$).

Now, we define $Y_D(a, z)b$ via Eqs. (4.6), (4.10), and as a direct corollary of the above lemma, $Y_D(a, z)b$ becomes a series from the space $V[z]_{\infty}$. Since $y_u = 1$ for $u = e_1$, it follows that 

$$\sum_{m=0}^{\infty} \sum_{\sigma=1}^{b_D} \mu_{m,\sigma}^{D}(a \otimes b) h_{m,\sigma}(u) \bigg|_{u=e_1} = \mu_{1}^{1}(a \otimes b),$$

and hence 

$$Y_D(a, z)b \big|_{z=x_{e_1}} = Y_1(a, x)b. \quad (4.19)$$
Note also that
\[ Y_D(a, z)b = \sum_{n \geq -N^D_{a,b}} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{h^D_m} \mu^D_{n,m,\sigma}(a \otimes b) (z^2)^{\frac{\sigma}{2}} h_{n,\sigma}(u) \]
\[ = \sum_{n \geq -N^D_{a,b}} U(g_u(z)) \mu^1_n \left( U(g_u(z))^{-1} a \otimes U(g_u(z))^{-1} b \right) (z^2)^{\frac{\sigma}{2}} \]
\[ = U(g_u(z)) Y_1 \left( U(g_u(z))^{-1} a, (z^2)^{\frac{\sigma}{2}} \right) U(g_u(z))^{-1} b \] (4.20)

for \( u(z) = z/(z^2)^{\frac{1}{2}} \), where the second and the third lines can be viewed as generating series for the definition of \( \mu^D_{n,m,\sigma} \) in the first line.

In fact, note that if we restrict \( Y_D(a, z)b \) for \( a \in V_{\Delta'} \) and \( b \in V_{\Delta''} \) and then project the result onto \( V_{\Delta} \) in the direct sum \( V = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V_{\Delta} \), then it will give rise to a map between finite-dimensional spaces, whose matrix has polynomial coefficients in \( z \) and \( (z^2)^{-\frac{1}{2}} \), but due to the above results all half-integer powers of \( z^2 \) will be cancelled.

### 4.6. Covariance properties.

**Lemma 4.7.** The series \( Y_D(a, z)b \), constructed by Eqs. (4.6), (4.10), is \( SO(D, \mathbb{C}) \)-equivariant, and is the unique \( SO(D, \mathbb{C}) \)-equivariant series whose restriction to \( z = xe_1 \) is equal to \( Y_1(a, x)b \).

**Proof.** The \( SO(D, \mathbb{C}) \)-equivariance of \( Y_D(a, z)b \) is equivalent to the \( SO(D, \mathbb{C}) \)-equivariance of the left-hand side of Eq. (4.10). The latter is established in Lemma 4.3. Then the \( SO(D, \mathbb{C}) \)-equivariance of the left-hand side of Eq. (4.10) and the fact that its restriction to \( u = e_1 \) is \( Y_1(a, x)b \) implies Eq. (4.10). \( \Box \)

**Lemma 4.8.** The translation operators \( T_\alpha \) are derivations of all maps \( \mu^D_{n,m,\sigma} \), i.e., we have
\[ T_\alpha \mu^D_{n,m,\sigma}(a \otimes b) = \mu^D_{n,m,\sigma}(T_\alpha a \otimes b) + \mu^D_{n,m,\sigma}(a \otimes T_\alpha b), \]
for all \( a, b \in V \) and all possible values of \( \alpha, n, m \) and \( \sigma \). Equivalently, we have
\[ [T_\alpha, Y_D(a, z)] = Y_D(T_\alpha a, z). \] (4.21)

**Proof.** Consider the linear combination \( T(v) := \sum_{\beta=1}^D v^\beta T_\beta \) for arbitrary \( v = (v^\beta) \in \mathbb{C}^D \). We need to prove that all \( T(v) \) are derivations of \( \mu^D_{n,m,\sigma} \), and we already know that \( T(v) \) are derivations of \( \mu^1_n \) for every \( v \) and \( n \).
Note that $T(v') U(g_a) = U(g_a) T(v')$ where $v' := g_a^{-1}(v)$. Then we apply Eq. (4.10) and compute:

\[
\sum_{m=0}^{\infty} \sum_{s=1}^{b_m} \left( T(v) \mu_{m,s}(a \otimes b) \right) h_{m,s}(n)
\]

\[
= T(v) U(g_a) \mu_1^1(U(g_a))^{-1} a \otimes U(g_a) b
\]

\[
= U(g_a) T(v') \mu_1^1(U(g_a))^{-1} a \otimes U(g_a) b
\]

\[
= U(g_a) \mu_1^1(U(g_a))^{-1} T(v) a \otimes U(g_a) b
\]

\[
+ U(g_a) \mu_1^1(U(g_a))^{-1} a \otimes U(g_a) T(v) b
\]

\[
= \sum_{m=0}^{\infty} \sum_{s=1}^{b_m} \left( \mu_{m,s}(T(v) a \otimes b) + \mu_{m,s}(a \otimes T(v) b) \right) h_{m,s}(n),
\]

which completes the proof of the lemma. \qed

**Lemma 4.9.** We have

\[
Y_D(T_a, a, z) b = \partial_{z^a} Y_D(a, z) b
\]

for all $a, b \in V$ and $\alpha = 1, \ldots, D$.

**Proof.** Let us differentiate with respect to $z^a$ the third line of Eq. (4.20) of the generating series of $Y_D(a, z) b$:

\[
\partial_{z^a} Y_D(a, z) b = \partial_{z^a} \left( U(g_a) Y_1(U(g_a))^{-1} a, (x^2)^{\frac{1}{n}} \right) U(g_a) b
\]

\[
= \partial_{z^a} \left( U(g_a) Y_1(U(g_a))^{-1} a, (x^2)^{\frac{1}{n}} \right) U(g_a) b
\]

\[
+ U(g_a) Y_1(U(g_a))^{-1} a, (x^2)^{\frac{1}{n}} \right) \partial_{z^a} U(g_a) b
\]

\[
+ U(g_a) \left( z^{-a} x^{-1} \partial_{x^a} Y_1(U(g_a))^{-1} a, x \right) \right)_{x := (x^2)^{\frac{1}{n}} U(g_a) b}
\]

For the last term in the right-hand side, we apply the equation $\partial_{x^a} Y_1(a', x) b' = Y_1(T_1 a', x) b'$, which hold for arbitrary $a', b' \in V$. For the previous three terms, we first calculate:

\[
\partial_{z^a} U(g_a) = U(g_a) \partial_{z^a} \left( U(g_a)^{-1} g_a(x) \right) \big|_{x := z} = \left. U(g_a) \tilde{\Omega}, \right.
\]

\[
\partial_{z^a} \left( U(g_a)^{-1} \right) = -\tilde{\Omega} U(g_a)^{-1},
\]

where $\tilde{\Omega}$ is a linear combination of $\Omega_{\alpha, \beta}$ with polynomial coefficients in $z$ and $(x^2)^{-\frac{1}{2}}$. Then we apply the commutation relations with $\Omega_{\alpha, \beta}$ from Proposition 3.4 to obtain:

\[
\tilde{\Omega} Y_1(a', x) b' - Y_1(\tilde{\Omega} a', x) b' - Y_1(a', x) \tilde{\Omega} b' = Y_1(T a', x)
\]
Lemma 4.11. Let \( \alpha \) be homogeneous of degree 0 and 1, respectively, for any \( a, b \in V \), where \( \tilde{T} \) is a linear combination of \( T_{\alpha} \) again with polynomial coefficients in \( z \) and \((z^2)^{\frac{1}{2}}\). Thus, the right-hand side of (4.23) becomes:

\[
\partial_z Y_D(a, z) b = U(g_{u(z)}) Y_1 (\tilde{T} U(g_{u(z)})^{-1} a, (z^2)^{\frac{1}{2}}) U(g_{u(z)})^{-1} b \\
+ U(g_{u(z)}) z^\alpha (z^2)^{\frac{1}{2}} Y_1 (T_1 U(g_{u(z)})^{-1} a, (z^2)^{\frac{1}{2}}) U(g_{u(z)})^{-1} b \\
= U(g_{u(z)}) Y_1 (U(g_{u(z)})^{-1} T_{\alpha} a, (z^2)^{\frac{1}{2}}) U(g_{u(z)})^{-1} b.
\]

The second equality above is a consequence of the relation

\[
U(g_{u(z)}) \left( \tilde{T} + z^\alpha (z^2)^{\frac{1}{2}} T_1 \right) U(g_{u(z)})^{-1} = T_{\alpha},
\]

which can be derived from an explicit computation of \( \tilde{\Omega} \) and \( \tilde{T} \). This computation is a bit cumbersome, but fortunately there is a simplifying argument. Since both sides of Eq. (4.22) are \( \text{SO}(D, \mathbb{C}) \)-equivariant, it is enough to compare them for \( z = \alpha e_1 \). The latter is equivalent to establishing Eq. (4.24) for \( z = \alpha e_1 \). To this end, we first find \( \tilde{\Omega}|_{z = \alpha e_1} = x^{-1} \Omega_{1\alpha} \), and then it follows that

\[
\tilde{T}|_{z = \alpha e_1} = (1 - \delta_{\alpha, 1}) T_{\alpha}.
\]

Therefore, \( \tilde{T} + z^\alpha (z^2)^{\frac{1}{2}} T_1 \) restricts to \( T_{\alpha} \) for \( z = \alpha e_1 \), as claimed. \( \Box \)

Lemma 4.10. The commutation relation (2.14) for \( H \) is valid, i.e.,

\[
[ H, Y_D(a, z) ] = Y(Ha, z) + z \cdot \partial_z Y_D(a, z).
\]

Proof. As in the proof of the previous lemma, we use (4.20) and apply the Euler differential operator to the expression in the third line of (4.20):

\[
z \cdot \partial_z Y_D(a, z) b = z \cdot \partial_z \left( U(g_{u(z)}) Y_1 (U(g_{u(z)})^{-1} a, (z^2)^{\frac{1}{2}}) U(g_{u(z)})^{-1} b \right) \\
= U(g_{u(z)}) \left( z \partial_z Y_1 (U(g_{u(z)})^{-1} a, x) \right)|_{x := (z^2)^{\frac{1}{2}}} U(g_{u(z)})^{-1} b \\
= U(g_{u(z)}) \left( [ H, Y_1 (U(g_{u(z)})^{-1} a, (z^2)^{\frac{1}{2}}) ] \\
- Y_1 (H U(g_{u(z)})^{-1} a, (z^2)^{\frac{1}{2}}) \right) U(g_{u(z)})^{-1} b \\
= \left( [ H, Y_D(a, z) ] - Y_D(Ha, z) \right) b,
\]

where we use that \( u(z) \) and \((z^2)^{\frac{1}{2}}\) are homogeneous of degree 0 and 1, respectively, as well as that \( H \) commutes with \( U(g_{u(z)}) \). \( \Box \)

Lemma 4.11. The commutation relations (2.16) for \( C_{\alpha} \) hold.

Proof. We have already established (2.14) and (2.15). Under them, Eq. (2.16) is equivalent to the following:

\[
[C(v), Y_D(a, z)] - Y_D(C(v) a, z) \\
+ [H(z \cdot w), Y_D(a, z)] + Y_D(H(v \cdot z) a, z) \\
+ [\Omega(z, v), Y_D(a, z)] + Y_D(\Omega(v, z) a, z) = 0,
\]

(4.25)
where we have set
\[ C(v) := \sum_{\alpha = 1}^{D} v^{\alpha} C_{\alpha}, \quad H(v \cdot z) = H \sum_{\alpha = 1}^{D} v^{\alpha} z^{\alpha}, \quad \Omega(v, z) := \sum_{\alpha, \beta = 1}^{D} v^{\alpha} z^{\beta} \Omega_{\alpha \beta}. \]

Now, Eq. (4.25) is valid for \( Y_{1} \) in the restriction \( z = xe_{1} \) (as a similar consequence of the relations in Proposition 3.4). Hence, it also holds for \( Y_{D} \) and general \( z \) with a “rotation” by similar methods as those used above in Lemma 4.8. \( \Box \)

As a consequence of Lemma 4.9 (and Eq. (4.22)), we have:
\[ \iota_{w,z} Y_{D}(a, z+w)b = Y_{D}(e^{zT}a, w)b \]
where the expansion \( \iota_{w,z} \) is defined by Eq. (3.4). Then setting \( w = xe_{1} \), we obtain the identity
\[ \iota_{xe_{1},z} Y_{D}(a, z+xe_{1})b = Y_{1}(e^{zT}a, x)b, \quad (4.26) \]
in which the left-hand side is defined as
\[ \iota_{xe_{1},z} Y_{D}(a, z+xe_{1})b := e^{z \cdot \partial a} Y_{D}(a, w)b|_{w=xe_{1}}. \]
We remark that formula (4.26) could be used for an alternative construction of \( Y_{D} \) if we could prove that the right-hand side is an \( \iota_{xe_{1},z} \)-expansion of a Laurent polynomial in \( z + xe_{1} \).

4.7. Locality. Throughout this subsection, \( u, v, w, z \) will denote \( D \)-dimensional formal variables, while \( x, y \) will be 1-dimensional formal variables. For each \( a, b \in V, \) fix a non-negative integer \( N_{a,b} (:= N_{a,b}^{D}) \), cf. Lemma 4.6) such that
\[ (z^{2})^{N_{a,b}} Y_{D}(a, z)b \in V[z]. \quad (4.27) \]

Applying \( e^{u \partial_{k}} \) to the left side of (4.27) and using the translation covariance (4.22), we obtain
\[ ((u+z)^{2})^{N_{a,b}} Y_{D}(e^{uT}a, z)b \in V[u+z] \subset V[u, z]. \]
Setting \( z = xe_{1} \), we get that
\[ ((u+xe_{1})^{2})^{N_{a,b}} Y_{1}(e^{uT}a, x)b \in V[u, x]. \quad (4.28) \]

Lemma 4.12. For every \( a, b, c \in V, \) we have
\[ ((z + xe_{1})^{2})^{N_{a,c}} \left( (w + ye_{1})^{2} \right)^{N_{b,c}} \left( (z - w + xe_{1} - ye_{1})^{2} \right)^{N_{a,b}} \times Y_{1}(e^{zT}a, x)Y_{1}(e^{wT}b, y)c \]
\[ = ((z + xe_{1})^{2})^{N_{a,c}} \left( (w + ye_{1})^{2} \right)^{N_{b,c}} \left( (z - w + xe_{1} - ye_{1})^{2} \right)^{N_{a,b}} \times Y_{1}(e^{zT}b, y)Y_{1}(e^{wT}a, x)c \in V[z, w, x, y]. \quad (4.29) \]

Proof. We start from the commutator formula (2.7) or (2.8) for the vertex algebra \( (V, Y_{1}, 1, T_{1}) \), written in the form
\[ [Y_{1}(a, x), Y_{1}(b, y)] = (\iota_{x,y} - \iota_{y,x})Y_{1}(Y_{1}(a, x-y)b, y) \]
(see [FLM88] and (3.4)). First, we replace \( a \) with \( e^{uT}a \) and get
\[ [Y_{1}(e^{uT}a, x), Y_{1}(b, y)] = (\iota_{x,y} - \iota_{y,x})Y_{1}(Y_{1}(e^{uT}a, x-y)b, y). \]
Second, we multiply both sides by \((u + x e_1 - ye_1)^2\)^{N_{a,b}}. Notice that the \(\nu\)-
expansions commute with the multiplication by a polynomial. But the term

\[(u + x e_1 - ye_1)^2\]^{N_{a,b}} Y_1(e^{\mu T}a, x - y) \in V[u, x - y] \subseteq V[u, x, y]\n
by (4.28). The expansions \(\ell_{x,y}\) and \(\ell_{y,x}\) of a formal power series in \(x, y\) are equal; hence their difference is 0. Therefore,

\[(u + x e_1 - ye_1)^2 \cdot \[Y_1(e^{\mu T}a, x), Y_1(b, y)\] = 0.

Recall that, by Proposition 3.4, we have \([T_\alpha, Y_1(a, x)] = Y_1(T_\alpha a, x)\), which
implies that all \(T_\alpha\) are derivations of \(Y_1\). If we apply \(e^x\) to the above equation,
we can rewrite Eq. (4.29) as follows:

\[(u + x e_1 - ye_1)^2 \cdot \[Y_1(e^{\mu x}a, x), Y_1(e^{\mu y}b, y)\] = 0.

Now let us apply the left side to a vector \(c \in V\) and multiply it by \((u + x +
xe_1)^2\)^{N_{a,c}}((v + ye_1)^2)^{N_{b,c}}. We obtain

\[(u + v + x e_1)^2\]^{N_{a,c}}((v + ye_1)^2)^{N_{b,c}}((u + x e_1 - ye_1)^2)^{N_{a,b}}
\times Y_1(e^{(u+v)x}a, x)Y_1(e^{(v+y)e}b, y)c
\]

Notice that, by (4.28), the left-hand side has only non-negative powers of \(y\) while
the right-hand side has only non-negative powers of \(x\). Therefore, both sides lie
in the space \(V[u, v, x, y]\). Making the substitution \(u = z - w, v = w\) completes
the proof of (4.29).

Lemma 4.13. For every \(a, b \in V\), we have the locality condition

\[(z - w)^2 \cdot \[Y_D(a, z), Y_D(b, w)\] = 0. \hspace{1cm} (4.30)

Proof. Using (4.26) and the fact that \(\nu\)-
expansions commute with the multiplication by a polynomial, we can rewrite Eq. (4.29) as follows:

\[\ell_{x e_1, x} \ell_{y e_1, w} ((z + x e_1)^2)^{N_{a,c}}((v + ye_1)^2)^{N_{b,c}}((z - w + xe_1 - ye_1)^2)^{N_{a,b}}
\times Y_D(a, z + x e_1)Y_D(b, w + ye_1)c
\]

But as both sides lie in \(V[z, w, x, y]\), the expansions are redundant, so we can drop \(\ell_{x e_1, x} \ell_{y e_1, w}\) from the above formula. Moreover, since both sides are in \(V[z, w, x, y]\), it makes sense to set \(x = y = 0\) in them, which gives

\[(z^2)^{N_{a,c}}((w^2)^{N_{b,c}}((z - w)^2)^{N_{a,b}}Y_D(a, z)Y_D(b, w)c
\]

Now we can divide both sides by \((z^2)^{N_{a,c}}((w^2)^{N_{b,c}}\) to obtain

\[(z - w)^2 \cdot \[Y_D(a, z), Y_D(b, w)\] = ((z - w)^2)^{N_{a,b}}Y_D(b, w)Y_D(a, z)c.
\]

This proves (4.30). □
Combining the results of Sects. 4.2–4.7 completes the proof of Theorem 4.1.

APPENDIX A. REPRESENTATION THEORY POINT OF VIEW

In this appendix, we present an alternative construction of \( Y_D(a, z)b \). Fix an integer \( n \) and two elements \( a \in V_{\Delta'} \) and \( b \in V_{\Delta''} \). As in Sect. 4.2, due to the integrability assumption from Definition 4.1 (a), without loss of generality we can assume that all eigenspaces \( V_{\Delta} \) of \( H \) are finite dimensional; since we can choose finite-dimensional \( \mathfrak{so}(D, \mathbb{C}) \)--representations \( W' \subseteq V_{\Delta'} \) and \( W'' \subseteq V_{\Delta''} \) such that \( a \in W' \) and \( b \in W'' \).

Then the map \( \mu_n^1 \), defined by Eq. (4.3), restricts to an element \( \mu_n^1 \in \text{Hom}(V_{\Delta'}, V_{\Delta'+\Delta''+n}) \) as in Eq. (4.5). Recall from the proof of Lemma 4.2 that \( \mu_n^1 \) is \( \mathfrak{so}(D-1, \mathbb{C}) \)--equivariant, where the \( \mathfrak{so}(D-1, \mathbb{C}) \)--subalgebra of \( \mathfrak{so}(D, \mathbb{C}) \) is spanned by \( \Omega_{\alpha\beta} \) for \( 2 \leq \alpha, \beta \leq D \). Thus, \( \mu_n^1 \) is an invariant element of \( \text{Hom}(V_{\Delta'}, V_{\Delta'+\Delta''+n}) \) under the subalgebra \( \mathfrak{so}(D-1, \mathbb{C}) \), i.e., the action of \( \mathfrak{so}(D-1, \mathbb{C}) \) on it is trivial. Hence, we can apply the following lemma.

**Lemma A.1.** Let \( F \) be a representation of \( \mathfrak{so}(D, \mathbb{C}) \), which is decomposable into a direct sum of finite-dimensional irreducible \( \mathfrak{so}(D, \mathbb{C}) \)--representations. Assume that \( v \in F \) is invariant under the action of the subalgebra \( \mathfrak{so}(D-1, \mathbb{C}) \). Then \( v \) is contained in a subrepresentation of the type \( \bigoplus_{m=0}^{\infty} Q_m \otimes C_m^\text{har}[\alpha] \), where \( C_m^\text{har}[\alpha] \) are \( \mathfrak{so}(D, \mathbb{C}) \)--representations of degree \( m \) homogeneous harmonic polynomials, and \( Q_m \) are multiplicity spaces with only a finite number of them nonzero.

**Proof.** Recall that the Lie algebra \( \mathfrak{so}(D, \mathbb{C}) \) is of type \( D_l \) for \( D = 2l \) and of type \( B_l \) for \( D = 2l + 1 \). In either case it has rank \( l \). Denote by \( R(\Lambda) \) the irreducible highest weight \( \mathfrak{so}(D, \mathbb{C}) \)--module with highest weight \( \Lambda \), and denote by \( \pi_1, \ldots, \pi_l \) the fundamental weights. Then \( R(\pi_1) \cong \mathbb{C}^D \) is the vector representation. Its symmetric powers are (see e.g. [OV90]):

\[
S^m \mathbb{C}^D \cong \bigoplus_{0 \leq k \leq \frac{m}{2}} R((m-2k)\pi_1).
\]  

(A.1)

Therefore, an irreducible \( \mathfrak{so}(D, \mathbb{C}) \)--module \( R(\Lambda) \) is contained in \( S^* \mathbb{C}^D \cong \mathbb{C}[\alpha] \) if and only if \( \Lambda = m\pi_1 \) for some \( m \in \mathbb{Z}_{\geq 0} \). It is well known that the space \( C_m^\text{har}[\alpha] \) is an irreducible \( \mathfrak{so}(D, \mathbb{C}) \)--representation with highest weight \( m\pi_1 \), i.e.,

\[
C_m^\text{har}[\alpha] \cong R(m\pi_1).
\]  

(A.2)

In fact, the decomposition (A.1) corresponds to the expression (2.10) of a degree \( m \) homogeneous polynomial in terms of \( \alpha^2 \) and harmonic polynomials. The proof now follows immediately from the next lemma.

**Lemma A.2.** Let \( R \) be an irreducible \( \mathfrak{so}(D, \mathbb{C}) \)--module. Assume that there exists a nonzero vector \( v \in R \) annihilated by the subalgebra \( \mathfrak{so}(D-1, \mathbb{C}) \). Then \( R \cong C_m^\text{har}[\alpha] \) for some \( m \in \mathbb{Z}_{\geq 0} \). Moreover, the vector \( v \) is unique up to a scalar multiple.
Proof. For $D = 2l$, in accordance with the branching rule, the restriction of $R(\lambda)$ to the subalgebra $\mathfrak{so}(2l-1, \mathbb{C})$ is given by
\[
R(\lambda)|_{\mathfrak{so}(2l-1, \mathbb{C})} \cong \bigoplus_{\nu} R'(\nu) .
\] (A.3)
Here $R'(\nu)$ is the irreducible finite-dimensional representation of $\mathfrak{so}(2l-1, \mathbb{C})$ with highest weight $\nu$, and the sum is taken over all weights $\nu$ satisfying the inequalities
\[
\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_{l-1} \geq \nu_{l-1} \geq |\lambda_l| ,
\] (A.4)
with all the $\nu_i$ being simultaneously integers or half-integers together with the $\lambda_i$.

For completeness, let us also consider the case $D = 2l+1$. Then, in accordance with the branching rule, the restriction of $R(\lambda)$ to the subalgebra $\mathfrak{so}(2l, \mathbb{C})$ is given by
\[
R(\lambda)|_{\mathfrak{so}(2l, \mathbb{C})} \cong \bigoplus_{\nu} R'(\nu) ,
\] (A.5)
where $R'(\nu)$ is the irreducible finite-dimensional representation of $\mathfrak{so}(2l, \mathbb{C})$ with highest weight $\nu$, and the sum is taken over all weights $\nu$ satisfying the inequalities
\[
\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_{l-1} \geq \nu_{l-1} \geq \lambda_l \geq |\nu_l| ,
\] (A.6)
with all the $\nu_i$ being simultaneously integers or half-integers together with the $\lambda_i$.

Now let us assume that the restriction $R(\lambda)|_{\mathfrak{so}(D-1, \mathbb{C})}$ contains the trivial representation $R'(0)$. Then inequalities (A.6) and (A.4) imply that for $\nu = (0, \ldots, 0)$, one has $\lambda = (\lambda_1, 0, \ldots, 0)$, where $\lambda_1$ is a non-negative integer. Therefore $\lambda = m\pi_1$ for some $m \in \mathbb{Z}_{\geq 0}$. This means that $R \cong R(m\pi_1) \cong \mathbb{C}^{\text{har}}[z]$.

Finally, the uniqueness of $\nu$ follows from the fact that the decompositions (A.3) and (A.5) are multiplicity free; in particular, the trivial $\mathfrak{so}(D-1, \mathbb{C})$–module $R'(0)$ appears only once in them.

Note that the unique (up to a scalar multiple) $\mathfrak{so}(D-1, \mathbb{C})$–invariant element $h_m(z) \in \mathbb{C}^{\text{har}}[z]$ can be obtained as the image of the projection of $(z^1)^m \in S^m \mathbb{C}^D$ onto the summand $R(m\pi_1) \cong \mathbb{C}^{\text{har}}[z]$ in the decomposition (A.1). This implies that $h_m(z)$ can be normalized so that $h_m(e_1) = z^m$.

Remark A.1. A more explicit expression for $h_m(z)$ can be derived from [BN06a, Eqs. (3.25), (3.29)]. In more details, a generating series for $h_m(z)$ is the expansion
\[
\ell_{e_1, x} \left( (e_1 + z)^2 \right)^{-(D-2)/2} := e^{x_0^2} (w^2)^{-(D-2)/2} \bigg|_{w = e_1} = \sum_{m=0}^{\infty} \binom{-D + 2}{m} h_m(z) .
\]

We can further write
\[
h_m(z) = \binom{z^2}{-D+2} \binom{z^2}{-D+2} C_m((D-2)/2) \left( -z^2 \right)^{-1/2} .
\]
in terms of the Gegenbauer polynomials $C_m^{(\alpha)}(x)$, which are defined by the expansion

$$(1 - 2xt + t^2)^{-\alpha} = \sum_{m=0}^{\infty} C_m^{(\alpha)}(x) t^m, \quad 0 \leq |x| < 1, \quad |t| \leq 1, \quad \alpha > 0.$$  

Let us fix a basis $\{h_{m,\sigma}(z)\}_{\sigma=1}^{b_m^D}$ for $\mathbb{C}^{\text{char}}_m[z]$ such that $h_{m,1}(z) = h_m(z)$ and $h_{m,\sigma}(xe_1) = \delta_{\sigma,1}x^m$; (A.7)

As a consequence of Lemmas A.1 and A.2, we obtain that in the space $\text{Hom}(V_{\Delta'} \otimes V_{\Delta''}, V_{\Delta'+\Delta''}+n)$ there is a system of linearly independent elements $\{f^n_{k,m,\sigma}\}_{k,m,\sigma}$ such that:

(a) For every fixed $n$, $k$ and $m$, the subsystem $\{f^n_{k,m,\sigma}\}_{\sigma=1}^{b_m^D}$ is a basis of an irreducible $\mathfrak{so}(D,\mathbb{C})$-subrepresentation isomorphic to $\mathbb{C}^{\text{char}}_m[z]$, corresponding to the fixed basis $\{h_{m,\sigma}(z)\}_{\sigma=1}^{b_m^D}$.

(b) There is a decomposition $\mu^n_1\mid_{V_{\Delta'} \otimes V_{\Delta''}} = \sum_{k,m} \gamma^n_{k,m} f^n_{k,m,1}$ where the $\gamma$'s are complex numbers.

Then we can define

$$Y_D(a, z)b := \sum_{n,k,m,\sigma} \gamma^n_{k,m} f^n_{k,m,\sigma}(a \otimes b) \left(x^2\right)^{n-m} h_{m,\sigma}(z).$$  

(A.8)

By construction, this expression is $\text{SO}(D,\mathbb{C})$-equivariant; hence, it does not depend on the choice of basis $\{h_{m,\sigma}(z)\}_{\sigma=1}^{b_m^D}$. Moreover, $Y_D(a, xe_1) = Y_1(a, x)$ by the assumption (A.7). Thus, (A.8) agrees with our previous definition, due to the uniqueness from Lemma 4.7.

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