WELL-POSEDNESS OF THE TWO-PHASE FLOW PROBLEM IN INCOMPRESSIBLE MHD

CHANGYAN LI
School of Mathematical Sciences, Peking University
Beijing 100871, China

HUI LI*
Department of Mathematics, Zhejiang University
Hangzhou 310027, China

(Communicated by Chongchun Zeng)

Abstract. In this paper, we study the two phase flow problem in the ideal incompressible magnetohydrodynamics. We propose a Syrovatskij type stability condition, and prove the local well-posedness of the two phase flow problem with initial data satisfies such condition. This result shows that the magnetic field has a stabilizing effect on Kelvin-Helmholtz instability even the fluids on each side of the free interface have different densities.

1. Introduction.

1.1. Presentation of the problem. In this paper, we consider the free boundary problem in 3-D ideal incompressible MHD. The incompressible MHD system can be written as

\[
\begin{aligned}
\rho \partial_t u + \rho u \cdot \nabla u - h \cdot \nabla h + \nabla p &= 0, \\
\text{div} u &= 0, \\
\text{div} h &= 0, \\
\partial_t h + u \cdot \nabla h - h \cdot \nabla u &= 0,
\end{aligned}
\]

where \( u \) is the fluids velocity, \( h \) is the magnetic field, \( \rho \) is the density, and \( p \) denotes the pressure. For the two-phase flow, the fluids have different densities on each side of a smooth interface \( \Gamma(t) \) in a domain \( \Omega \). More precisely, we let

\[
\Omega = T^2 \times [-1,1] \subset \mathbb{R}^3, \quad \Gamma(t) = \{ x \in \Omega | x_3 = f(t,x'), x' = (x_1, x_2) \in T^2 \},
\]

\[
\Omega_{t}^{\pm} = \{ x \in \Omega | x_3 > (<) f(t,x'), x' \in T^2 \}, \quad Q_{t}^{\pm} = \bigcup_{t \in (0,T)} \{ t \} \times \Omega_{t}^{\pm}.
\]

For simplicity of notation we write \( \rho_{\Omega_{t}^{\pm}} = \rho^{\pm} \), where \( \rho^{\pm} \) are two constants, and

\[
u^{\pm} \overset{\text{def}}{=} u|_{\Omega_{t}^{\pm}}^{\pm}, \quad h^{\pm} \overset{\text{def}}{=} h|_{\Omega_{t}^{\pm}}^{\pm}, \quad p^{\pm} \overset{\text{def}}{=} p|_{\Omega_{t}^{\pm}}^{\pm},
\]
are smooth in $Q_T^\pm$ and satisfy
\[
\begin{cases}
\rho^\pm \partial_t u^\pm + \rho^\pm u^\pm \cdot \nabla u^\pm - h^\pm \cdot \nabla h^\pm + \nabla p^\pm = 0 & \text{in } Q_T^\pm, \\
\text{div} u^\pm = 0, \quad \text{div} h^\pm = 0 & \text{in } Q_T^\pm, \\
\partial_t h^\pm + u^\pm \cdot \nabla h^\pm - h^\pm \cdot \nabla u^\pm = 0 & \text{in } Q_T^\pm,
\end{cases}
\]
with the boundary conditions on the moving interface:
\[
[p]^\pm \overset{\text{def}}{=} p^+ - p^- = 0, \quad u^\pm \cdot N_f = \partial_t f, \quad h^\pm \cdot N_f = 0 \quad \text{on } \Gamma_t,
\]
where $N_f = (-\partial_1 f, -\partial_2 f, 1)$. We use $n = N_f/|N_f|$ to denote the outward unit normal to $\partial\Omega_f^\pm$. On the artificial boundary $\Gamma^\pm = T^2 \times \{\pm 1\}$, we impose the following boundary conditions on $(u^\pm, h^\pm)$:
\[
u^\pm_3 = 0, \quad h^\pm_3 = 0 \quad \text{on } \Gamma^\pm.
\]
The system (1) is supplement with the initial data:
\[
u^\pm(0, x) = u^\pm_0(x), \quad h^\pm(0, x) = h^\pm_0(x) \quad \text{in } \Omega^\pm_0,
\]
which satisfies
\[
\begin{cases}
\text{div} u^0_\pm = 0, \text{div} h^0_\pm = 0 & \text{in } \Omega^\pm_0, \\
u^0_\pm \cdot n_0 = u^0_\mp \cdot n_0, h^0_\pm \cdot n_0 = 0 & \text{on } \Gamma_0.
\end{cases}
\]
The system (1)-(3) is called the two-phase flow problem for incompressible MHD.

One of main goals in this paper is to study the local well-posedness of this system under some suitable stability conditions imposed on the initial data.

We remark that the divergence-free restriction on $h^\pm$ is a compatibility condition. Applying the divergence operator to the third equation of (1), we have
\[
\partial_t \text{div} h^\pm + u^\pm \cdot \nabla \text{div} h^\pm = 0.
\]
Therefore, if div$h^0_\pm = 0$, the solution of (1)-(3) will satisfies div$h^\pm = 0$ for $\forall t > 0$. A similar argument can be applied to yield that $h^\pm \cdot N_f = 0$ if $h^0_\pm \cdot N_{f_0} = 0$.

### 1.2. Background and related works

In inviscid flow, a surface across which there is a discontinuity in fluid velocity is called a vortex sheet. It has been known for over a century that vortex sheets may exhibit the Kelvin-Helmholtz instability \cite{12}. Researches have found that surface tension can stabilize the Kelvin-Helmholtz instability \cite{3, 4, 19}.

For electrically conducting fluids, e.g. plasmas and liquid metals, there may be shears in both the velocity field and magnetic field. Current-vortex sheet problem plays an important role in researches of tokamaks\cite{13}, magnetosphere\cite{10}, solar surges\cite{5}, and solar flares\cite{15}. In the mid-twentieth century, Syrovatskij\cite{16} and Axford\cite{1} found the necessary and sufficient condition for the stability of the current-vortex sheet when the fluids on each side of the free interface have the same density. The Syrovatskij stability condition can be expressed as:
\[
||u||^2 \leq 2(|h^+|^2 + |h^-|^2), \quad \text{on } \Gamma_t,
\]
\[
||u \times h^+||^2 + ||u \times h^-||^2 \leq 2|h^+ \times h^-|^2, \quad \text{on } \Gamma_t,
\]
where $[u] = (u^+ - u^-)$. In particular, if $h^+ \times h^- \neq 0$ and
\[
||u \times h^+||^2 + ||u \times h^-||^2 < 2|h^+ \times h^-|^2, \quad \text{on } \Gamma_t,
\]
then (4) is automatically satisfied.
During the past several decades, great progress has been made in studying the stabilizing effect of the Syrovatskij condition \((5)\). Morando, Trakhinin, and Trebeschi\([14]\) proved a priori estimates with a loss of derivatives for the linearized system. Furthermore, under a strong stability condition
\[
\max(|u| \times h^+, [u] \times h^-) < |h^+ \times h^-|, \quad \text{on } \Gamma_t, \tag{6}
\]
Trakhinin\([20]\) proved an a priori estimate for the linearized problem without loss of derivative. For the nonlinear current-vortex sheet problem, Coulombel, Morando, Secchi, and Trebeschi\([7]\) proved an a priori estimate under the strong stability condition \((6)\). Recently, Sun-Wang-Zhang\([17]\) gave the first rigorous confirmation of the stabilizing effect of the magnetic field on Kelvin-Helmholtz instability under the Syrovatskij stability condition \((5)\).

All the results above under the assumption that the fluids on each side of the free interface have the same density. However, it is inevitably to face the two-phase flow problems in many important physical systems. For example, in tokamak devices\([13]\), there are parallel flows of plasma and melt stream; during the observation of \([23]\), in solar prominence, the density of the plasma which is ejected from the stream is about 10 times that of the surrounding stream. The aim of this paper is to study the stability of the two-phase flow problem in the ideal incompressible magnetohydrodynamics. We find a natural condition under which the free boundary problem is local well-posed. The framework we used in this paper is developed in \([17]\). The basic idea is study the evolution equation of the free surface. This idea is very effective to study the free boundary problems of the incompressible Euler equations\([21, 22, 24, 18]\).

1.3. Main results. In this paper we find the following stability condition which is a Syrovatskij type stability condition of the two-phase flow system:
\[
\Lambda(h^\pm, [u]) \overset{\text{def}}{=} \inf_{x \in \Gamma^t} \inf_{\varphi_1 + \varphi_2 = 1} \frac{1}{\rho^+ + \rho^-} (h_1^+ \varphi_1 + h_2^+ \varphi_2)^2 + \frac{1}{\rho^+ + \rho^-} (h_1^- \varphi_1 + h_2^- \varphi_2)^2 - (v_1 \varphi_1 + v_2 \varphi_2)^2 \geq c_0,
\]
where \(v_i = \sqrt{\rho^+ + \rho^-}[u_i]\). This stability condition has similar formulation to the one to the two-phase flow problem in incompressible elastodynamics\([11]\).

Now, we state the main result.

**Theorem 1.1.** Assume \(s \geq 3\) is an integer and \(f_0 \in H^{s+\frac{1}{2}}(\mathbb{T}^2), u_0^\pm, h_0^\pm \in H^s(\Omega_0^\pm)\), moreover we assume that there exists \(c_0 \in (0, \frac{1}{2})\) so that
1. \(- (1 - 2c_0) \leq f_0 \leq (1 - 2c_0)\),
2. \(\Lambda(h_0^\pm, [u_0]) \geq 2c_0\).

Then there exists a time \(T\) such that system \((1)\) admits a unique solution \((f, u, h)\) in \([0, T]\) satisfying
1. \(f \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{T}^2))\),
2. \(u^\pm, h^\pm \in L^\infty(0, T; H^s(\Omega_t^\pm))\),
3. \(- (1 - c_0) \leq f \leq (1 - c_0)\),
4. \(\Lambda(h^\pm, [u]) \geq c_0\).

We organize this paper as follow: in Section 2, we introduce the reference domain, harmonic coordinate and Dirichlet-Neumann operator. In Section 3, derive an evolution equation for the free interface in which the effect of the stability condition can be reflected. In Section 4, we give the uniform estimates for the linearized
system. In Section 5, we prove the existence and uniqueness of the solution to the two-phase flow problem.

2. Reference domain, harmonic coordinate and Dirichlet-Neumann Operator. In this section, we recall some well-known results on the harmonic coordinate and Dirichlet-Neumann operators.

We first introduce some notations used throughout this paper. We use $x = (x_1, x_2, x_3)$ to denote the coordinates in the fluid region, and use $x' = (x_1, x_2)$ to denote the natural coordinates on the interface. For a function $g : \Omega_f^* \rightarrow \mathbb{R}$, we can define its trace $\overline{g}$ by $\overline{g}(x') = g(x', f(x'))$, so

$$\partial_1 g(x') = \partial_1 g(x', f(x')) + \partial_3 g(x', f(x')) \partial_1 f(x').$$

For simplicity of notation, we denote by $\| \cdot \|_{H^s}$ the Sobolev norm in $\mathbb{T}^2$, and $\| \cdot \|_{H^s(\Omega)}$ the Sobolev norm in $\Omega$.

In free boundary problems, the functions are defined in a domain changing with time. We always draw the moving domain back to a fixed domain which is called reference domain [18].

The fixed graph $\Gamma_*$ is given by

$$\Gamma_* = \{(y_1, y_2, y_3) : y_3 = f_*(y_1, y_2)\},$$

and the fixed reference domain is given by

$$\Omega_* = \mathbb{T}^2 \times (-1, 1), \quad \Omega_*^+ = \{y \in \Omega_* | y > f_*(y_1, y_2)\}, \quad \Omega_*^- = \{y \in \Omega_* | y < f_*(y_1, y_2)\}.$$

We only consider the functions lie in the neighborhood of $f_*$. We define

$$\Upsilon(\delta, k) \overset{\text{def}}{=} \{f \in H^k(\mathbb{T}^2) : \|f - f_*\|_{H^k(\mathbb{T}^2)} \leq \delta\}.$$

For $f \in \Upsilon(\delta, k)$, we define $\Gamma_f, \Omega_f^+, \Omega_f^-$ by

$$\Gamma_f \overset{\text{def}}{=} \{x \in \Omega_t | x_3 = f(t, x'), \int_{\mathbb{T}^2} f(t, x') dx' = 0\},$$

and

$$\Omega_f^+ = \{x \in \Omega_t | x_3 > f(t, x')\}, \quad \Omega_f^- = \{x \in \Omega_t | x_3 < f(t, x')\}.$$

We denote $N_f \overset{\text{def}}{=} (-\partial_1 f, -\partial_2 f, 1)$ the outward normal vector of $\partial \Omega_f^-$, and $n_f \overset{\text{def}}{=} N_f / \sqrt{1 + |\nabla f|^2}$ the unit normal.

Now we introduce the harmonic coordinate. For given $f \in \Upsilon(\delta, k)$, we define the harmonic extension $\Phi_f^\pm : \Omega_*^\pm \rightarrow \Omega_f^\pm$ as follows:

$$\begin{cases}
\Delta_y \Phi_f^\pm = 0 & y \in \Omega_*^\pm, \\
\Phi_f^+ (y', f_*(y')) = (y', f(y')) & y' \in \mathbb{T}^2, \\
\Phi_f^- (y', \pm 1) = (y', \pm 1) & y' \in \mathbb{T}^2.
\end{cases}$$

Given $\Gamma_*$, there exists $\delta_0 = \delta_0(\|f_*\|_{W^{1, \infty}}) > 0$ so that $\Phi_f^\pm$ is a bijection when $\delta \leq \delta_0$. Then one can define an inverse map $\Phi_f^{\pm -1} : \Omega_f^\pm \rightarrow \Omega_*^\pm$ such that

$$\Phi_f^{\pm -1} \circ \Phi_f^\pm = \Phi_f^\pm \circ \Phi_f^{\pm -1} = \text{Id}.$$

The following properties come from [17].
Lemma 2.1. Let \( f \in \mathcal{Y}(\delta_0, s - \frac{1}{2}) \) for \( s \geq 3 \). Then there exists a constant \( C \) depending only on \( \delta_0 \) and \( \| f \|_{H^{s-\frac{1}{2}}} \) so that
1. If \( u \in H^\sigma(\Omega_f^\pm) \) for \( \sigma \in [0, s] \), then
   \[
   \| u \circ \Phi_f^\pm \|_{H^\sigma(\Omega_f^\pm)} \leq C \| u \|_{H^\sigma(\Omega_f^\pm)}.
   \]
2. If \( u \in H^\sigma(\Omega_f^\pm) \) for \( \sigma \in [0, s] \), then
   \[
   \| u \circ \Phi_f^\pm^{-1} \|_{H^\sigma(\Omega_f^\pm)} \leq C \| u \|_{H^\sigma(\Omega_f^\pm)}.
   \]
3. If \( u, v \in H^\sigma(\Omega_f^\pm) \) for \( \sigma \in [2, s] \), then
   \[
   \| uv \|_{H^\sigma(\Omega_f^\pm)} \leq C \| u \|_{H^\sigma(\Omega_f^\pm)} \| v \|_{H^\sigma(\Omega_f^\pm)}.
   \]

In order to define the Dirichlet-Neumann operator, we need to introduce the harmonic extension \( H_f^\pm : H^k(\mathbb{T}^2) \rightarrow \Omega_f^\pm \):
\[
\begin{cases}
\Delta H_f^\pm g = 0 & x \in \Omega_f^\pm, \\
(H_f^\pm g)(x', f(x')) = g(x') & x' \in \mathbb{T}^2, \\
\partial_3 H_f^\pm g(x', \pm 1) = 0 & x' \in \mathbb{T}^2.
\end{cases}
\] (7)

Then we can define the Dirichlet-Neumann operator:
\[
N_f^\pm \overset{\text{def}}{=} +N_f \cdot (\nabla H_f^\pm g) |_{\Gamma_f},
\]
which maps the Dirichlet boundary value of a harmonic function to its Neumann boundary value. We will use the following properties of Dirichlet-Neumann operator from [2, 17].

Lemma 2.2. It holds that
1. \( N_f \) is a self-adjoint operator:
   \[
   (N_f \psi, \phi) = (\psi, N_f \phi), \quad \forall \phi, \psi \in H^{1/2}(\mathbb{T}^2).
   \]
2. \( N_f \) is a positive operator:
   \[
   (N_f \phi, \phi) = \| \nabla H_f \phi \|^2_{L^2(\Omega_f)} \geq 0, \quad \forall \phi \in H^{1/2}(\mathbb{T}^2).
   \]

Especially, if \( \int_{\mathbb{T}^2} \phi(x') dx' = 0 \), there exists \( c > 0 \) depending on \( c_0, \| f \|_{W^{1, \infty}} \) such that

3. \( N_f \) is a bijection from \( H^{k+1}_0(\mathbb{T}^2) \) to \( H^k_0(\mathbb{T}^2) \) for \( k \geq 0 \), where
   \[
   H^k_0(\mathbb{T}^2) \overset{\text{def}}{=} H^k(\mathbb{T}^2) \cap \{ \phi \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} \phi(x') dx' = 0 \}.
   \]

3. Reformulation of the problem. In this section, we derive a new system which is equivalent to the original system (1)-(3). The new system is the combination of evolution equations for the following quantities:
   
   - the height function of the interface: \( f \);
   - the scaled normal velocity on the interface: \( \theta = u^\pm \cdot N_f \);
   - the curl part of the velocity and magnetic field in the fluid region: \( \omega = \nabla \times u, \; \xi = \nabla \times h \);
   - the average of the tangential part of the velocity and the magnetic field on the top and bottom fixed boundary:
\[ \beta^\pm_i(t) = \int_{\mathbb{T}^2} u^\pm_i(t, x', \pm 1)dx', \quad \gamma^\pm_i(t) = \int_{\mathbb{T}^2} h^\pm_i(t, x', \pm 1)dx'(i = 1, 2). \]

3.1. Evolution of the scaled normal velocity. First we define

\[ \theta(t, x') \overset{\text{def}}{=} u^\pm(t, x', f(t, x')) \cdot N_f(t, x'), \]

thus we have

\[ \partial_t f(t, x') = \theta(t, x'). \tag{8} \]

In order to derive the evolution of \( \theta \), we need the following elementary lemma.

**Lemma 3.1.** \([17]\) For \( u = u^\pm, h^\pm \), we have

\[ (u \nabla u) \cdot N_f - \sum_{j=1,2,3} \partial_{x_j} u_j N_j (u \cdot N_f)|_{x_3 = f(t, x')} = \sum_{i,j=1,2} u_i \partial_i (u_j N_j) + \sum_{i,j=1,2} u_i u_j \partial_i \partial_j f. \]

It follows from the first equation of (1) and Lemma 3.1 that

\[ \partial_t \theta = (\partial_t u^\pm + \partial_t u^\pm \partial_t f) \cdot N_f + u^\pm \cdot \partial_t N_f|_{x_3 = f(t, x')} \]

\[ = (-u^\pm \cdot \nabla u^\pm + \frac{1}{\rho^+} h^\pm \cdot \nabla h^\pm - \frac{1}{\rho^+} \nabla p^+ + \partial_t u^\pm \partial_t f) \cdot N_f \]

\[ = (\partial_t \theta, \nabla \theta)|_{x_3 = f(t, x')} \]

\[ = -2(u^+_i \partial_i \theta + u^-_i \partial_i \theta) - \frac{1}{\rho^+} N_f \cdot \nabla p^+ - \sum_{i,j=1,2} u^+_i u^-_j \partial_i \partial_j f \]

\[ + \frac{1}{\rho^+} \sum_{i,j=1,2} h^+_i h^-_j \partial_i \partial_j f. \]

A similar derivation gives

\[ \partial_t \theta = -2(u^-_i \partial_i \theta + u^+_i \partial_i \theta) - \frac{1}{\rho^-} N_f \cdot \nabla p^- - \sum_{i,j=1,2} u^-_i u^-_j \partial_i \partial_j f + \frac{1}{\rho^-} \sum_{i,j=1,2} h^-_i h^-_j \partial_i \partial_j f. \]

Therefore, we have

\[ 2(u^+_i \partial_i \theta + u^-_i \partial_i \theta) + \frac{1}{\rho^+} N_f \cdot \nabla p^+ + \sum_{i,j=1,2} u^+_i u^+_j \partial_i \partial_j f - \frac{1}{\rho^+} \sum_{i,j=1,2} h^+_i h^+_j \partial_i \partial_j f \]

\[ = 2(u^-_i \partial_i \theta + u^-_i \partial_i \theta) + \frac{1}{\rho^-} N_f \cdot \nabla p^- + \sum_{i,j=1,2} u^-_i u^-_j \partial_i \partial_j f - \frac{1}{\rho^-} \sum_{i,j=1,2} h^-_i h^-_j \partial_i \partial_j f. \]

Taking the divergence to the first equation of (1), we get

\[ \Delta p^\pm = \text{tr}(\nabla h^\pm)^2 - \rho^\pm \text{tr}(\nabla u^\pm)^2. \]

Then we can rewrite \( p^\pm \) as

\[ p^\pm = H_f^\pm p^\pm + \rho^\pm p_{u^\pm} u^\pm - p_{h^\pm} h^\pm, \]
where \( \mathcal{H}_f^\pm \) denote the harmonic extension from \( \Gamma_f \) to \( \Omega_f^\pm \), and \( p_{u_1^\pm,u_2^\pm} \) is the solution of elliptic equation

\[
\begin{cases}
\Delta p_{u_1^\pm,u_2^\pm} = -\text{tr}(\nabla u_1^\pm \nabla u_2^\pm) & \text{in } \Omega_f^\pm, \\
p_{u_1^\pm,u_2^\pm} = 0 & \text{on } \Gamma_f, \\
e_3 \cdot \nabla p_{u_1^\pm,u_2^\pm} = 0 & \text{on } \Gamma^\pm.
\end{cases}
\]

As a result, one can derive the following equality from (10):

\[
\frac{1}{\rho^+} N_f \cdot \nabla \mathcal{H}_f^+ p^+ - \frac{1}{\rho^-} N_f \cdot \nabla \mathcal{H}_f^- p^-
= -2(u_1^\pm \partial_1 \theta + u_2^\pm \partial_2 \theta) + N_f \cdot \nabla (p_{u_1^+,u_2^+} - \frac{1}{\rho^+} \mathcal{H}_f^+ \partial_1 \theta) + \sum_{i,j=1,2} (u_1^+ u_1^j - \frac{1}{\rho^+} \mathcal{H}_f^+ \partial_i \partial_j f) + \sum_{i,j=1,2} (u_2^+ u_2^j - \frac{1}{\rho^+} \mathcal{H}_f^+ \partial_i \partial_j f)
\]

\[
def = -g^+ + g^-.
\]

Recalling the definition of the Dirichlet-Neumann operator, we have

\[-\frac{1}{\rho^+} N_f^+ p^+ - \frac{1}{\rho^-} N_f^- p^- = -g^+ + g^-.
\]

We define

\[\tilde{N}_f \defeq \frac{1}{\rho^+} N_f^+ + \frac{1}{\rho^-} N_f^-,
\]

it follows from the fact \([p] = 0\) that

\[p^\pm = \tilde{N}_f^{-1}(g^+ - g^-).
\]

It’s easy to see that

\[N_f^+ = \left(\frac{1}{\rho^+} + \frac{1}{\rho^-}\right)^{-1}(\tilde{N}_f + \frac{1}{\rho^-}(N_f^+ - \tilde{N}_f)),
\]

\[N_f^- = \frac{1}{\rho^+} + \frac{1}{\rho^-}^{-1}(\tilde{N}_f - \frac{1}{\rho^+}(N_f^+ - \tilde{N}_f)),
\]

which implies

\[
\frac{1}{\rho^+} N_f^+ \tilde{N}_f^{-1} g^- + \frac{1}{\rho^-} N_f^- \tilde{N}_f^{-1} g^+
= \rho^+ g^+ + \rho^- g^- - \frac{1}{\rho^+ + \rho^-} (N_f^+ - \tilde{N}_f) \tilde{N}_f^{-1} (g^+ - g^-).
\]
Accordingly, thus we rewrite (9) to
\[
\partial_t \theta = \frac{1}{\rho^+} N^+_\tau p^+ - g^+ \\
= \frac{1}{\rho^+} N^+_{\tau f} \tilde{N}_f^{-1}(g^+ - g^-) - g^+ \\
= -\frac{1}{\rho^+} N^+_{\tau f} \tilde{N}_f^{-1} g^- - \frac{1}{\rho^-} N^-_{\tau f} \tilde{N}_f^{-1} g^+ \\
= -\rho^+ g^+ + \rho^- g^- + \frac{1}{\rho^+ + \rho^-} (N^+_{\tau f} - N^-_{\tau f}) \tilde{N}_f^{-1} (g^+ - g^-) \\
= -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u_1^+ + \rho^- u_1^-) \partial_t \theta + (\rho^+ u_2^+ + \rho^- u_2^-) \partial_2 \theta \right) \\
\left( -\frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} (\rho^+ u_i^+ u_j^+ + \rho^- u_i^- u_j^- - \theta^+ \delta_{ij} - \theta^- \delta_{ij}) \partial_i \partial_j f \right) \\
+ \frac{1}{\rho^+ + \rho^-} (N^+_{\tau f} - N^-_{\tau f}) \tilde{N}_f^{-1} P \left( \sum_{i,j=1,2} (u_i^+ u_j^+ + u_i^- u_j^- - \frac{1}{\rho^+} \theta^+ \delta_{ij} + \frac{1}{\rho^-} \theta^- \delta_{ij}) \partial_i \partial_j f \right) \\
+ \frac{2}{\rho^+ + \rho^-} (N^+_{\tau f} - N^-_{\tau f}) \tilde{N}_f^{-1} P \left( (u_i^+ - u_i^-) \partial_i \theta + (u_i^+ - u_i^-) \partial_2 \theta \right) \\
- \frac{1}{\rho^+ + \rho^-} N_{\tau f} \cdot \nabla (\rho^+ p_{a+} + \rho^- p_{a-} - \rho_{p_{a+}} + \rho_{p_{a-}}) \\
+ \frac{1}{\rho^+ + \rho^-} (N^+_{\tau f} - N^-_{\tau f}) \tilde{N}_f^{-1} P \left( \nabla (p_{a+} - \frac{1}{\rho^+} \rho_{p_{a+}} - \frac{1}{\rho^-} \rho_{p_{a-}}) \right) \\
\right)
\] 
where \( P : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2) \) denote the projection operator which is defined by \( \mathcal{P}g = g - \langle g \rangle \).

and \( \langle g \rangle \) \(df \) \( gdx'\).

Here we introduce \( \mathcal{P} \) for the reason that \( \tilde{N}_f^{-1} \) can only apply to functions with a mean value of 0. However, as \( \mathcal{P}g = g^\pm \), it does not change the formulation of this system.

3.2. Equations for \( (\omega^\pm, \xi^\pm, \beta_i^\pm, \gamma_i^\pm) \). Now we derive the evolution equations for \( \omega^\pm = \nabla \times u^\pm, \quad \xi^\pm = \nabla \times h^\pm \),

\[
\beta_i^\pm(t) = \int_{\mathbb{T}^2} u_i^\pm(t, x', \pm 1) dx', \quad \gamma_i^\pm(t) = \int_{\mathbb{T}^2} h_i^\pm(t, x', \pm 1) dx'.
\]

Applying the curl operator to (1), we have

\[
\begin{cases}
\partial_t \omega^\pm + \nabla u^\pm \cdot \nabla \omega^\pm + \frac{1}{\rho^\pm} \nabla h^\pm \cdot \nabla \omega^\pm = \omega^\pm \cdot \nabla u^\pm - \frac{1}{\rho^\pm} \xi^\pm \cdot \nabla h^\pm, \\
\partial_t \xi^\pm + \nabla u^\pm \cdot \nabla \xi^\pm - \nabla \omega^\pm = \xi^\pm \cdot \nabla u^\pm - \frac{1}{\rho^\pm} \xi^\pm \cdot \nabla h^\pm - 2 \sum_{i=1,2,3} \nabla u_i^\pm \times \nabla h_i^\pm, \\
\end{cases}
\]

As \( u_i^\pm(t, x', \pm 1) \equiv 0, \) for \( i = 1, 2, \) it holds that

\[
\rho^\pm \partial_t u_i^\pm + \rho^\pm \sum_{j=1,2,3} u_j^\pm \partial_j u_i^\pm - \sum_{j=1,2,3} h_j^\pm \partial_j h_i^\pm - \partial_i p^\pm = 0 \quad \text{on} \quad \Gamma^\pm.
\]

After integrating the above equation on \( \Gamma^\pm \), we deduce that

\[
\partial_t \beta_i^\pm + \int_{\Gamma^\pm} \left( \sum_{j=1,2,3} u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm} h_j^\pm \partial_j h_i^\pm \right) dx',
\]

which gives

\[
\beta_i^\pm(t) = \beta_i^\pm(0) - \int_0^t \int_{\Gamma^\pm} \left( \sum_{j=1,2,3} u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm} h_j^\pm \partial_j h_i^\pm \right) dx' \, dt.
\]
For the same reason, we have
\[ \gamma_i^\pm(t) = \gamma_i^+(0) - \int_0^t \int_{\Gamma^\pm} \left( \sum_{j=1,2,3} u_j^\pm \partial_j h_i^\pm - \sum_{j=1,2,3} h_j^\pm \partial_j u_i^\pm \right) dx'dt. \quad (15) \]

3.3. Solvability conditions for the div-curl system. In the above 2 subsections, we deduce a new evolution system \((8),(12)-(15)\) from the original system. Now we will show that one can recover the velocity field and the magnetic field from the quantities in the new system. We need to solve the following div-curl system:

\[
\begin{align*}
\begin{cases}
\text{curl} \ u^\pm = \omega^\pm, & \quad \text{in } \Omega_f^\pm, \\
\text{div} \ u^\pm = 0 & \quad \text{in } \Omega_f^\pm, \\
u^\pm \cdot N_f = \partial_t f & \quad \text{on } \Gamma_f, \\
u^\pm \cdot e_3 = 0, & \quad \int_{\Gamma^\pm} u_idx' = \beta_i^\pm, \ (i = 1, 2) \quad \text{on } \Gamma^\pm.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\text{curl} \ h^\pm = \xi^\pm, & \quad \text{in } \Omega_f^\pm, \\
\text{div} \ h^\pm = 0 & \quad \text{in } \Omega_f^\pm, \\
h^\pm \cdot N_f = 0 & \quad \text{on } \Gamma_f, \\
h^\pm \cdot e_3 = 0, & \quad \int_{\Gamma^\pm} h_idx' = \gamma_i^\pm, \ (i = 1, 2) \quad \text{on } \Gamma^\pm.
\end{cases}
\end{align*}
\]

The above Div-Curl systems are solvable under the following compatibility condition:

\[ C1. \quad \text{div} \omega^\pm = 0, \quad \text{div} \xi^\pm = 0, \quad \text{in } \Omega_f^\pm, \]

\[ C2. \quad \int_{\Gamma_f^+} \omega_3^\pm dx' = 0, \quad \int_{\Gamma_f^-} \xi_3^\pm dx' = 0, \]

\[ C3. \quad \int_{\partial T^2} \partial_t f dx' = 0. \]

**Proposition 1.** [17] Let \( \sigma \in [2, s] \) be an integer, \( c_0 \in (0, 1) \) be a constant. Given \( f \in H^{s+\frac{1}{2}}(T^2), \omega, g \in H^{s-1}(\Omega_f^+), \vartheta \in H^{s-1}(\Gamma_f) \) satisfying

\[
\int_{\Omega_f^+} gdx = \int_{\Gamma_f} \partial ds, \quad \int_{\Gamma_f^+} \omega_3 dx' = 0, \quad \int_{\Gamma_f^-} \omega_3 dx' = 0, \]

\[ \text{div} \omega = 0 \text{ in } \Omega_f^+, \quad -(1 - c_0) \leq f \leq (1 - c_0) \text{ on } \Gamma_f. \]

The div-curl system

\[
\begin{align*}
\begin{cases}
\text{curl} \ u = \omega, & \quad \text{in } \Omega_f^+, \\
\text{div} \ u = g & \quad \text{in } \Omega_f^+, \\
\mathbf{u} \cdot N_f = \vartheta & \quad \text{on } \Gamma_f, \\
\mathbf{u} \cdot e_3 = 0, & \quad \int_{\partial T^2} u_idx' = \alpha_i(i = 1, 2) \quad \text{on } \Gamma^+, \quad \text{(16)}
\end{cases}
\end{align*}
\]

admits a unique solution \( \mathbf{u} \in H^s(\Omega_f^+) \) so that

\[
\| \mathbf{u} \|_{H^s(\Omega_f^+)} \leq C(c_0, \| f \|_{H^{s+\frac{1}{2}}}) \left( \| \omega \|_{H^{s-1}(\Omega_f^+)} + \| g \|_{H^{s-1}(\Omega_f^+)} + \| \vartheta \|_{H^{s-\frac{1}{2}}(\Gamma_f)} + |\alpha_1| + |\alpha_2| \right). \]

4. Uniform estimates for the linearized system. In this section, we linearize the equivalent system derived in Section 3 around given functions \((f, u^\pm, h^\pm)\), and establish the uniform energy estimates for the linearized system. We assume that
Recalling the definition of \( \Lambda(\, t, u^\pm, h^\pm) \) and the stability condition (22), the stability condition (22) ensures that \( \tilde{f} \) satisfies a strictly hyperbolic equation. Based on this observation, we introduce the linearized system:

\[
\begin{align*}
&\text{div}u^\pm = \text{div}h^\pm = 0 \quad \text{in} \quad \Omega^\pm, \\
&h^\pm \cdot N_f = 0, \quad u^\pm \cdot N_f = \partial_t f \quad \text{on} \quad \Gamma_f, \\
&u_3^\pm = h_3^\pm = 0 \quad \text{on} \quad \Gamma^\pm.
\end{align*}
\]

Here \( T, L_0, L_1, L_2, c_0, \delta_0 \) are positive constants.

### 4.1. Linearized system of \((f, \theta)\)

From (8) and (12), we introduce the following linearized system:

\[
\begin{cases}
\partial_t \tilde{f} &= \tilde{\theta}, \\
\partial_t \tilde{\theta} = -\frac{2}{\rho^+ + \rho^-} ((\rho^+ u_i^+ + \rho^- u_i^-) \partial_i \tilde{\theta} + (\rho^+ u_j^+ + \rho^- u_j^-) \partial_j \tilde{\theta}) \\
&\quad - \frac{1}{\rho^+ + \rho^-} \sum_{i,j=1,2} (\rho^+ u_i^+ u_j^+ - h_i^+ h_j^+ + \rho^- u_i^- u_j^- - h_i^- h_j^-) \partial_i \partial_j \tilde{f} + g,
\end{cases}
\]

where

\[
g = \frac{1}{\rho^+ + \rho^-}(N_f^+ - N_f^-)N_f^{-1}P\left( \sum_{i,j=1,2} (u_i^+ u_j^+ - 1/\rho^+ h_i^+ h_j^+ - u_i^- u_j^- + 1/\rho^- h_i^- h_j^-) \partial_i \partial_j f \right)
+ \frac{2}{\rho^+ + \rho^-}(N_f^+ - N_f^-)N_f^{-1}P((u_i^+ - u_i^-) \partial_i \theta + (u_j^+ - u_j^-) \partial_j \theta)
- \frac{1}{\rho^+ + \rho^-}N_f \cdot \nabla(\rho^+ p_{u^+ - u^-} + \rho^+ h^+ - \rho^- p_{-h} - \rho^- h^- - \rho^+ p_{-h} + \rho^- h^-)
+ \frac{1}{\rho^+ + \rho^-}(N_f^+ - N_f^-)N_f^{-1}P\frac{1}{\rho^+ + \rho^-}N_f \cdot \nabla(p_{u^+ - u^-} + \rho^+ h^+ - \rho^- p_{-h} - \rho^- h^- - \rho^+ p_{-h} + \rho^- h^-)
= g_1 + g_2 + g_3 + g_4.
\]

One need be careful that \( \int_{T_2} \tilde{\theta} dx' \) may not equal to 0.

We can rewrite the evolution equation of \( \tilde{f} \) to

\[
(\partial_t + w_1 \partial_t + w_2 \partial_x)^2 \tilde{f} = \sum_{i,j=1}^2 \left( -v_i v_j + \frac{1}{\rho^+ + \rho^-}(h_i^+ h_j^+ + h_i^- h_j^-) \right) \partial_i \partial_j f + \text{low order terms},
\]

where

\[
w_i = \frac{\rho^+}{\rho^+ + \rho^-}(\rho^+ u_i^+ + \rho^- u_i^-), \quad v_i = \frac{\rho^+ \rho^-}{\rho^+ + \rho^-}(u_i^+ - u_i^-).
\]

Recalling the definition of \( \Lambda(\, t, u^\pm, [u]) \), the stability condition (22) ensures that \( \tilde{f} \) satisfies a strictly hyperbolic equation. Based on this observation, we introduce the
following energy functional:

\[
E_s(\partial_t \tilde{f}, \tilde{f}) \overset{\text{def}}{=} \| (\partial_t + 2 \sum_{i=1}^{2} w_i \partial_i) (\nabla)^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2 - \| \sum_{i=1}^{2} v_i \partial_i (\nabla)^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2 \\
+ \frac{1}{\rho^+ + \rho^-} \| \sum_{i=1}^{2} h_i^+ \partial_i (\nabla)^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2 + \frac{1}{\rho^+ + \rho^-} \| \sum_{i=1}^{2} h_i^- \partial_i (\nabla)^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2,
\]

where

\[
(\nabla)^{s} f = \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \hat{f}).
\]

It’s clear that there exists \(C(L_0) > 0\) such that

\[
E_s(\partial_t \tilde{f}, \tilde{f}) \leq C(L_0)(\| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2).
\]

On the other hand, it follows from the stability condition (22) that there exists \(C(c_0, L_0) > 0\) such that

\[
\| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2 \leq C(c_0, L_0)\{E_s(\partial_t \tilde{f}, \tilde{f}) + \| \partial_t \tilde{f} \|_{L^2}^2 + \| \tilde{f} \|_{L^2}^2 \}. \tag{25}
\]

**Proposition 2.** Assume that \( g \in L^\infty(0, T; H^{s-\frac{1}{2}}(\mathbb{T}^2)) \). Given the initial data \((\tilde{\theta}_0, \tilde{f}_0) \in H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}(\mathbb{T}^2)\), there exists a unique solution \((\bar{f}, \bar{\theta}) \in C([0, T]; H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}(\mathbb{T}^2))\) to the system (23) so that

\[
\sup_{t \in [0,T]} (\| \partial_t \bar{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \bar{f} \|_{H^{s+\frac{1}{2}}}^2) \leq C(c_0, L_0)(\| \tilde{\theta}_0 \|_{H^{s-\frac{1}{2}}}^2 + \| \tilde{f}_0 \|_{H^{s+\frac{1}{2}}}^2 \\
+ \int_0^T \| g \|_{H^{s-\frac{1}{2}}} \, dt) e^{C(c_0, L_1, L_2)T}.
\]

**Proof.** As \( \bar{f} \) satisfies a strictly hyperbolic equation, it suffices to prove the uniform estimates.
From (23), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \| (\partial_t + \sum_{i=1}^{2} w_i \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} f \|^2_{L^2} = \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \partial_t^2 \langle \nabla \rangle^{s-\frac{1}{2}} f \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} f \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} f \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f} \rangle
\]

\[\frac{1}{\rho^* + \rho} \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \sum_{i,j=1,2} (h_i^+ h_j^- + h_i^- h_j^+) \langle \nabla \rangle^{s-\frac{1}{2}} \partial_i \partial_j \bar{f} \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \langle \nabla \rangle^{s-\frac{1}{2}} \bar{g} \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, 2 \sum_{i=1}^{2} \langle \nabla \rangle^{s-\frac{1}{2}} \partial_i \bar{f} \rangle + \langle \partial_t + \sum_{i=1}^{2} w_i \partial_i \rangle \langle \nabla \rangle^{s-\frac{1}{2}} f, \left[ \sum_{i,j=1,2} (w_i w_j + v_i v_j) - \frac{1}{\rho^* + \rho} h_i^- h_j^- - \frac{1}{\rho^* + \rho} h_i^+ h_j^+ \right] \langle \nabla \rangle^{s-\frac{1}{2}} \partial_i \partial_j \bar{f} \rangle \]

\[= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\]

Here we use \([a, \langle \nabla \rangle^s]\) to denote commutator

\[[a, \langle \nabla \rangle^s] f = a \langle \nabla \rangle^s f - \langle \nabla \rangle^s (af).\]

By using Lemma A.1, we have

\[I_5 \leq CE_s(\partial_t \bar{f}, \bar{f}) \frac{1}{2} \| \nabla \|^2_{H^{s-\frac{1}{2}}} \| \partial_t \bar{f} \|_{H^{s-\frac{1}{2}}},\]

\[I_6 \leq CE_s(\partial_t \bar{f}, \bar{f}) \left( \| \nabla \|^2_{H^{s-\frac{1}{2}}} + \| \bar{g} \|^2_{H^{s-\frac{1}{2}}} \right) \| \bar{f} \|_{H^{s+\frac{1}{2}}}.\]

It is clear that

\[I_4 \leq CE_s(\partial_t \bar{f}, \bar{f}) \frac{1}{2} \| \bar{g} \|_{H^{s-\frac{1}{2}}},\]

\[I_7 \leq CE_s(\partial_t \bar{f}, \bar{f}) \frac{1}{2} \| \partial_t \bar{w} \|_{L^\infty} \| \bar{f} \|_{H^{s+\frac{1}{2}}}.\]
Using integration by parts we obtain
\[
\langle \partial_t \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \partial_i \tilde{f} \rangle = -\frac{1}{2} \langle \sum_{i=1}^{2} \partial_i w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \partial_t \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \rangle
\]
\[
\leq C \| \sum_{i=1}^{2} \partial_i w_i \|_{L^\infty} \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}},
\]
and
\[
\langle \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \partial_i \tilde{f} \rangle
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 - \langle \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i=1}^{2} \partial_i w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \rangle.
\]
Therefore, it holds that
\[
I_1 \leq -\frac{d}{dt} \| \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 + (1 + \| \partial_t w \|_{L^\infty} + \| w \|_{W^{1,\infty}}) \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}},
\]
In much the same way, we have
\[
- \langle \partial_t \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i,j=1,2} w_i w_j \langle \nabla \rangle^{s - \frac{1}{2}} \partial_i \partial_j \tilde{f} \rangle
\]
\[
= \sum_{i,j=1,2} \langle \partial_t (w_i w_j \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \langle \nabla \rangle^{s - \frac{1}{2}} \partial_j \tilde{f} \rangle + \sum_{i,j=1,2} \langle w_i \partial_i \partial_t \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, w_j \partial_j \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \rangle
\]
\[
\leq \frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 + \| w \|_{L^\infty} (\| \partial_t w \|_{L^\infty} + \| w \|_{W^{1,\infty}}) \| \tilde{f} \|_{H^{s + \frac{1}{2}}} + \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}},
\]
and
\[
- \langle \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i,j=1,2} v_i v_j \langle \nabla \rangle^{s - \frac{1}{2}} \partial_i \partial_j \tilde{f} \rangle
\]
\[
\leq C \| w \|_{L^2}^2 \| w \|_{W^{1,\infty}} \| \tilde{f} \|_{H^{s + \frac{1}{2}}},
\]
\[
- \langle \partial_t \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f}, \sum_{i,j=1,2} v_i v_j \langle \nabla \rangle^{s - \frac{1}{2}} \partial_i \partial_j \tilde{f} \rangle
\]
\[
\leq -\frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} v_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 + \| v \|_{L^\infty} (\| \partial_t v \|_{L^\infty} + \| v \|_{W^{1,\infty}}) \| \tilde{f} \|_{H^{s + \frac{1}{2}}} + \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}}).
\]
From this, we conclude that
\[
I_2 \leq \frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} w_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} v_i \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[
+ C(1 + \| (u^\pm, h^\pm) \|_{W^{1,\infty}} + \| \partial_t (u^\pm, h^\pm) \|_{L^\infty}) \| \tilde{f} \|_{H^{s + \frac{1}{2}}} + \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}}).
\]
Similarly, it holds that
\[
I_3 \leq -\frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} h_i^\pm \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2 - \frac{1}{2} \frac{d}{dt} \| \sum_{i=1}^{2} h_i^\pm \partial_i \langle \nabla \rangle^{s - \frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[
+ C(1 + \| (u^\pm, h^\pm) \|_{W^{1,\infty}} + \| \partial_t (u^\pm, h^\pm) \|_{L^\infty}) \| \tilde{f} \|_{H^{s + \frac{1}{2}}} + \| \partial_t \tilde{f} \|_{H^{s - \frac{1}{2}}}).
\]
Putting the estimates of $I_1, I_2, ..., I_7$ together, we conclude that
\[
\frac{d}{dt} E_s(\partial_t \bar{f}, \bar{f}) \leq \|g\|_{H^{s-\frac{1}{2}}}^2 + C(L_0)(1 + \|u^\pm, h^\pm\|_{H^{s-\frac{1}{2}}}
+ \|\partial_t (u^\pm, h^\pm)\|_{L^\infty})^3(\|\bar{f}\|_{H^{s+\frac{1}{2}}}^2 + \|\partial_t \bar{f}\|_{H^{s-\frac{1}{2}}}^2).
\]

Besides, an easy computation shows that
\[
\frac{d}{dt}(\|\partial_t \bar{f}\|_{L^2}^2 + \|\bar{f}\|_{L^2}^2) \leq C(L_0)(\|\|\|_{H^{s+\frac{1}{2}}}^2 + \|\partial_t \bar{f}\|_{H^{s-\frac{1}{2}}}^2) + \|g\|_{L^2}^2.
\]

Let $\mathcal{E}(t) \equiv \|\partial_t \bar{f}\|_{H^{s-\frac{1}{2}}}^2 + \|\bar{f}\|_{H^{s-\frac{1}{2}}}^2$, we get by (25) that
\[
\mathcal{E}(t) \leq C(c_0, L_0)(\|\bar{g}_0\|_{H^{s-\frac{1}{2}}}^2 + \|\bar{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \|g\|_{H^{s-\frac{1}{2}}}^2 d\tau
+ \int_0^t (1 + \|u^\pm, h^\pm\|_{H^{s-\frac{1}{2}}} + \|\partial_t (u^\pm, h^\pm)\|_{L^\infty})^3 \mathcal{E}(\tau) d\tau)
\leq C(c_0, L_0)(\|\bar{g}_0\|_{H^{s-\frac{1}{2}}}^2 + \|\bar{f}_0\|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \|g\|_{H^{s-\frac{1}{2}}}^2 d\tau + C(L_1, L_2) \int_0^t \mathcal{E}(\tau) d\tau).
\]

By using Gronwall’s inequality, we arrive at the conclusion of this proposition.

\textbf{Lemma 4.1.} It holds that
\[
\|g\|_{H^{s-\frac{1}{2}}} \leq C(L_1).
\]

\textbf{Proof.} By using Proposition 6 and Proposition 7, we get
\[
\|g_1\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \| \sum_{i,j=1}^2 (w_i \cdot u_j^+ - w_i \cdot u_j^- - \frac{1}{\rho^-} h_i^+ h_j^- + \frac{1}{\rho^-} h_i^- h_j^+) \partial_t \partial_j f\|_{H^{s-\frac{1}{2}}}
\leq C(L_1) \| (u^\pm, h^\pm) \|_{H^{s-\frac{1}{2}}} \| f\|_{H^{s+\frac{1}{2}}} \leq C(L_1),
\]
\[
\|g_2\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \| (u^\pm, h^\pm) \|_{H^{s-\frac{1}{2}}} \| \theta\|_{H^{s-\frac{1}{2}}} \leq C(L_1),
\]
\[
\|g_3, g_4\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \| (\nabla (\rho^+ p a^u a^+ - \rho^- p h^-) \|_{H^{s-\frac{1}{2}}} + \| (\nabla (\rho^- p a^u a^- - \rho^- p h^-) \|_{H^{s-\frac{1}{2}}} + \| (\nabla (\rho^+ p a^u a^+ - \rho^- p h^-) \|_{H^{s+\frac{1}{2}}} + \| (\nabla (\rho^- p a^u a^- - \rho^- p h^-) \|_{H^{s+\frac{1}{2}}} + \| (\nabla (\rho^+ p a^u a^+ - \rho^- p h^-) \|_{H^{s+\frac{1}{2}}} + \| (\nabla (\rho^- p a^u a^- - \rho^- p h^-) \|_{H^{s+\frac{1}{2}}}) \leq C(L_1) \| (u^\pm, h^\pm) \|_{H^{s+\frac{1}{2}}} \leq C(L_1).
\]

This ends the proof.

\textbf{4.2. The Linearized System of } $(\omega, \xi)$. From (13), we introduce the following linearized vorticity system:
\[
\begin{align*}
threealign{\partial_t \bar{\omega} + u^\pm \cdot \nabla \bar{\omega} - \frac{1}{\rho^\pm} h^\pm \cdot \nabla \xi^\pm = \omega^\pm \cdot \nabla u^\pm - \frac{1}{\rho^\pm} \bar{\xi}^\pm \cdot \nabla h^\pm,}{\partial_t \bar{\xi} + u^\pm \cdot \nabla \bar{\xi} - h^\pm \cdot \nabla \bar{\omega} = \xi^\pm \cdot \nabla u^\pm - \omega^\pm \cdot \nabla h^\pm - 2 \sum_{i=1}^3 \nabla u_i^\pm \times \nabla h_i^\pm,}{= (\sqrt{\rho^\pm} \omega^\pm + \bar{\xi}^\pm) \cdot \nabla u^\pm - \frac{1}{\sqrt{\rho^\pm}} (\sqrt{\rho^\pm} \omega^\pm + \bar{\xi}^\pm) \cdot \nabla h^\pm - 2 \sum_{i=1}^3 \nabla u_i^\pm \times \nabla h_i^\pm.}
\end{align*}
\]
Therefore, we introduce \( \omega^\pm = \sqrt{\rho^\pm} \omega^\pm + \xi^\pm \) which satisfies

\[
\partial_t \omega^\pm + (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla \omega^\pm = \omega^\pm \cdot \nabla (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) - 2 \nabla u^\pm \times \nabla h^\pm. \tag{27}
\]

We define

\[
\begin{cases}
\frac{dX^\pm(t,x)}{dt} = (u - \frac{1}{\sqrt{\rho^\pm}} h)^\pm (t, X^\pm(t,x)), & x \in \Omega^\pm_f, \\
X^\pm(0, x) = \text{Id}, & x \in \Omega^\pm_f,
\end{cases}
\tag{28}
\]

where \( \text{Id} \) is the identity map. Recalling that \( h^\pm \cdot N_f = 0 \), one can see that \( X^\pm(t, \cdot) \) is a flow map from \( \Omega^\pm_f \) to \( \Omega^\pm_f(t) \). Then we have

\[
\frac{d\omega^\pm(t, X^\pm(t,x))}{dt} = (\omega^\pm \cdot \nabla (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) - 2 \nabla u^\pm \times \nabla h^\pm)(t, X^\pm(t,x)), \quad x \in \Omega^\pm_f.
\]

This is a linear ODE system, and the existence of \( \sqrt{\rho^\pm} \omega^\pm - \xi^\pm \) follows immediately. So do \( \sqrt{\rho^\pm} \omega^\pm - \xi^\pm \). Next, we give the energy estimates for \( (\omega^\pm, \xi^\pm) \).

**Proposition 3.** It holds that

\[
\sup_{t \in [0, T]} (\|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2) \leq (1 + \|\omega^\pm\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm\|_{H^{s-1}(\Omega_f^\pm)}^2) e^{C(L)(T)}. \]

**Proof.** Using the fact that \( u^\pm \cdot N_f = \partial_f f \) and \( h^\pm \cdot N_f = 0 \), we deduce from (27) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_f^\pm} |\nabla^{s-1} \omega^\pm(t, x)|^2 \, dx
\]

\[
= \int_{\Omega_f^\pm} \nabla^{s-1} \omega^\pm \cdot \nabla^{s-1} \partial_t \omega^\pm \, dx + \frac{1}{2} \int_{\Gamma_f^\pm} |\nabla^{s-1} \omega^\pm|^2 (u^\pm \cdot n) \, d\sigma
\]

\[
\leq \int_{\Omega_f^\pm} \nabla^{s-1} \omega^\pm \cdot \nabla^{s-1} (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla \omega^\pm \, dx + \frac{1}{2} \int_{\Gamma_f^\pm} |\nabla^{s-1} \omega^\pm|^2 (u^\pm \cdot n) \, d\sigma
\]

\[
+ C(L_1)(1 + \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2)
\]

\[
\leq \frac{1}{2} \int_{\Omega_f^\pm} (u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm) \cdot \nabla ((|\nabla^{s-1} \omega^\pm|^2) \, dx + \frac{1}{2} \int_{\Gamma_f^\pm} |\nabla^{s-1} \omega^\pm|^2 (u^\pm \cdot n) \, d\sigma
\]

\[
+ C(L_1)(1 + \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2)
\]

\[
= \frac{1}{2} \int_{\Omega_f^\pm} \text{div}(u^\pm - \frac{1}{\sqrt{\rho^\pm}} h^\pm)|\nabla^{s-1} \omega^\pm|^2 \, dx + C(L_1)(1 + \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2)
\]

\[
+ \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2
\]

\[
\leq C(L_1)(1 + \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2).
\]

Similarly, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Gamma_f^\pm} |\nabla^{s-1} (\sqrt{\rho^\pm} \omega^\pm - \xi^\pm)|^2 \, dx \leq C(L_1)(1 + \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2 + \|\xi^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)}^2).
\]

The desired estimate follows from the Gronwall’s inequality. \( \Box \)

To solve velocity and magnetic field from the vorticity and current \( (\omega^\pm, \xi^\pm) \), we need to certify the following compatibility conditions.

**Lemma 4.2.** It holds that

\[
\frac{d}{dt} \int_{\Gamma^\pm} \omega^\pm \, dx = 0, \quad \frac{d}{dt} \int_{\Gamma^\pm} \xi^\pm \, dx = 0.
\]
Definition 5.1. Given two constant $c_0 > 0$, we introduce the projection operator
\[
\Pi_{c_0} f = \begin{cases} f & \text{if } \|f\|_{H^s(T,M)} \leq c_0, \\ 0 & \text{otherwise}, \end{cases}
\]
for some constant $c_0 > 0$.

Let $f_s = f_0$, and $\Omega_s^± = \Omega_{f_0}^±$ be the reference region. The initial data
\[
(f(t), (\partial_t f)_t, \omega_t^±, \xi_t^±, \beta_t^±, \gamma_t^±) \text{ for the equivalent system is defined as follows:}
\]
\[
f(t) = f_0, \quad (\partial_t f)_t = u_t^0(x', f_0(x')) \cdot (-\partial_1 f_0, -\partial_2 f_0, 1);
\]
\[
\omega_t^± = \text{curl} u_t^0, \quad \xi_t^± = \text{curl} h_t^0;
\]
\[
\beta_t^± = \int_{T^2} u_t^0(x', \pm 1) dx', \quad \gamma_t^± = \int_{T^2} h_t^0(x', \pm 1) dx',
\]
which satisfy
\[
\|f(t)\|_{H^{s+\frac{1}{2}}} + \|(\omega_t^±, \xi_t^±)\|_{H^{s-\frac{1}{2}}(\Omega_s^±)} + \|\Phi_t f\|_{H^{s-\frac{1}{2}}} + |\beta_t^±| + |\gamma_t^±| \leq M_0,
\]
for some constant $M_0 > 0$. Then we define the following functional space.

Definition 5.1. Given two constant $M_1, M_2 > 0$, we define the space $X = X(T, M_1, M_2)$ be the collection of ($f, \omega_t^±, \xi_t^±, \beta_t^±, \gamma_t^±$) that satisfies
\[
(f(0), (\partial_t f)_t, \omega_t^±(0), \xi_t^±(0), \beta_t^±(0), \gamma_t^±(0)) = (f_1, (\partial_t f)_t, \omega_1^±, \xi_1^±, \beta_1^±, \gamma_1^±),
\]
\[
\|f(t) - f_0\|_{H^{s+\frac{1}{2}}} \leq \delta_0,
\]
\[
\sup_{t \in [0, T]} \|f(t)\|_{H^{s+\frac{1}{2}}} + \|(\omega_t^±, \xi_t^±)\|_{H^{s-\frac{1}{2}}(\Omega_s^±)} + \|\Phi_t f\|_{H^{s-\frac{1}{2}}} + |\beta_t^±(t)| + |\gamma_t^±(t)| \leq M_1,
\]
\[
\sup_{t \in [0, T]} \|\beta_t^±(t)\|_{H^{s-2}(\Omega_s^±)} + \|\gamma_t^±(t)\|_{H^{s-\frac{1}{2}}} + |\partial_t \beta_t^±(t)| + |\partial_t \gamma_t^±(t)| \leq M_2,
\]
\[
\int_{T^2} \partial_t f(t, x') dx' = 0.
\]

Next, we will construct an iteration map
\[
\mathcal{F} : \mathcal{X}(T, M_1, M_2) \to \mathcal{X}(T, M_1, M_2),
\]
\[
\mathcal{F}(f, \omega_t^±, \xi_t^±, \beta_t^±, \gamma_t^±) \text{ def } (\tilde{f}, \tilde{\omega}_t^±, \tilde{\xi}_t^±, \tilde{\beta}_t^±, \tilde{\gamma}_t^±),
\]
with suitable constants $T, M_1, M_2$.

5. Construction and Contraction of the Iteration Map. Assume that
\[
f_0 \in H^{s+\frac{1}{2}}(T^2), \quad u_0^±, h_0^± \in H^s(\Omega_0^±),
\]
which satisfy
\[
1. -(1 - 2c_0) \leq f(x') \leq (1 - 2c_0);
\]
\[
2. \Lambda(h_0^±, |u_0^±|) \geq 2c_0,
\]
for some constant $c_0 > 0$.

Proof. The proof is straightforward, we refer the readers to [17].

5.1. Recover the bulk region, velocity and magnetic field. We define
\[
\tilde{\omega}_t^±, \tilde{\xi}_t^± \text{ def } P_f^{\text{div}}(\omega_t^± \circ \Phi_f^{-1}), \quad \tilde{\xi}_t^± \text{ def } P_f^{\text{div}}(\xi_t^± \circ \Phi_f^{-1}),
\]
where $\Phi_f^± : \Omega_0^± \to \Omega_f^±$ is the harmonic coordinate map, and $P_f^{\text{div}} \omega_0^± = \omega_0^± - \nabla \phi_0^±$ with
\[
\begin{cases}
\Delta \phi_0^± = \text{div} \omega_0^± \text{ in } \Omega_f^±; \\
\partial_\nu \phi_0^± = 0 \text{ on } \Gamma_f^±; \quad \phi_0^± = 0 \text{ on } \Gamma_f.
\end{cases}
\]
We introduce the projection operator $P_f^{\text{div}}$ to ensure that $(\tilde{\omega}_t^±, \tilde{\xi}_t^±)$ satisfy conditions (C1) and (C2) defined in Section 3.3. It is obvious that
\[
\|((\tilde{\omega}_t^±, \tilde{\xi}_t^±))(t)\|_{H^{s-1}(\Omega_f^±)} \leq C(M_1),
\]
Then, we define $u^\pm$ and $h^\pm$ as the solution of

\[
\begin{cases}
\text{curl } u^\pm = \tilde{\omega}^\pm, & \text{div } u^\pm = 0 \quad \text{in } \Omega_f^+, \\
u^\pm \cdot N_f = \partial_t f & \text{on } \Gamma_f, \\
u^\pm \cdot e_3 = 0, & \int_{\Gamma^\pm} u_i dx' = \beta^\pm_i (i = 1, 2) \quad \text{on } \Gamma^\pm, \\
\text{curl } h^\pm = \xi^\pm, & \text{div } h^\pm = 0 \quad \text{in } \Omega_f^+, \\
h^\pm \cdot N_f = 0 & \text{on } \Gamma_f, \\
h^\pm \cdot e_3 = 0, & \int_{\Gamma^\pm} h_i dx' = \gamma^\pm_i (i = 1, 2) \quad \text{on } \Gamma^\pm,
\end{cases}
\]

with initial data

$$u^\pm(0) = u_0^\pm, \quad h^\pm(0) = h_0^\pm.$$ 

It follows from Proposition 1 that

$$\|u^\pm\|_{H^s(\Omega_f^+)} \leq C(M_1)(\|\tilde{\omega}^\pm\|_{H^{s-1}(\Omega_f^+)} + \|\partial_t f\|_{H^{s-\frac{1}{2}}} + |\beta_1^\pm(t)| + |\beta_2^\pm(t)|) \leq C(M_1),$$

$$\|h^\pm\|_{H^s(\Omega_f^+)} \leq C(M_1)(\|\xi^\pm\|_{H^{s-1}(\Omega_f^+)} + |\gamma_1^\pm(t)| + |\gamma_2^\pm(t)|) \leq C(M_1).$$

Using the same argument to treat $\partial_t u^\pm$ and $\partial_t h^\pm$, we deduce that

$$\|\partial_t u^\pm\|_{H^{s-1}(\Omega_f^+)} \leq C(M_1, M_2), \quad \|\partial_t h^\pm\|_{H^{s-1}(\Omega_f^+)} \leq C(M_1, M_2),$$

which implies

$$\|u^\pm(t)\|_{L^\infty(\Gamma_f^+)} \leq \|u_0^\pm\|_{L^\infty(\Gamma_f^+)} + \int_0^t \|\partial_t u^\pm(t)\|_{L^\infty(\Gamma_f^+)} dt \leq \frac{M_0}{2} + TC(M_1, M_2),$$

$$\|h^\pm(t)\|_{L^\infty(\Gamma_f^+)} \leq \frac{M_0}{2} + TC(M_1, M_2).$$

Besides, it holds that

$$\|f(t) - f_0\|_{L^\infty} \leq \|f(t) - f_0\|_{H^{s-\frac{1}{2}}} \leq T\|\partial_t f\|_{H^{s-\frac{1}{2}}} \leq T M_1,$$

$$|\Lambda(h^\pm, u^\pm) - \Lambda(h_0^\pm, u_0^\pm)| \leq TC(\|\partial_t u^\pm(t)\|_{L^\infty(\Gamma_f^+)} , \|\partial_t h^\pm(t)\|_{L^\infty(\Gamma_f^+)}) \leq TC(M_1, M_2).$$

We can ensure

$$TM_1 \leq \min\{\delta_0, c_0\}, \quad TC(M_1) + TC(M_1, M_2) \leq \frac{M_0}{2}, \quad TC(M_1, M_2) \leq c_0,$$

by taking $T$ small enough. Taking $L_0 = M_0, L_1 = C(M_1), L_2 = C(M_1, M_2)$, we conclude that for $\forall t \in [0, T]$:

1. $- (1 - c_0) \leq f(t, x') \leq (1 - c_0),$
2. $\Lambda(h^\pm, u^\pm) \geq c_0,$
3. $\|(u^\pm, h^\pm)(t)\|_{L^\infty(\Gamma_f^+)} \leq L_0,$
4. $\|f(t) - f_0\|_{H^{s-\frac{1}{2}}} \leq \delta_0,$
5. $\|f(t)\|_{H^{s-\frac{1}{2}(\Omega_f^+)}} + \|\partial_t f(t)\|_{H^{s-\frac{1}{2}(\Omega_f^+)}} + \|u^\pm(t)\|_{H^s(\Omega_f^+)} + \|h^\pm(t)\|_{H^s(\Omega_f^+)} \leq L_1,$
6. $\|\partial_t u^\pm, \partial_t h^\pm)(t)\|_{L^\infty(\Gamma_f^+)} \leq L_2.$
5.2. Defining the iteration map. Given \((f, u^\pm, h^\pm)\) which is constructed from \((f, \omega^\pm, \xi^\pm, \beta^\pm, \gamma^\pm)\). Let \(f_1 \) and \((\omega^\pm, \xi^\pm)\) be the solutions of the linearized systems (23) and (26) with initial data
\[
(f_1(0), \bar{\theta}(0), \omega^\pm(0), \xi^\pm(0)) = (f_0, (\partial_t f)_I, \omega^\pm_{i_I}, \xi^\pm_{i_I})
\]
We define
\[
\omega^\pm_t = \omega^\pm \circ \Phi^\pm_t, \quad \xi^\pm_t = \xi^\pm \circ \Phi^\pm_t,
\]
\[
\beta^\pm_i(t) = \beta^\pm_i(0) - \int_0^t \int_{\Gamma^\pm} \sum_{j=1}^3 (u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm_j} \partial_j h_i^\pm) dx' d\tau,
\]
\[
\gamma^\pm_i(t) = \gamma^\pm_i(0) - \int_0^t \int_{\Gamma^\pm} \sum_{j=1}^3 (u_j^\pm \partial_j h_i^\pm - h_i^\pm \partial_j u_i^\pm) dx' d\tau.
\]
Then we have the iteration map \(F\) as follows
\[
F(f, \omega^\pm, \xi^\pm, \beta^\pm, \gamma^\pm) \overset{\text{def}}{=} (\bar{f}, \bar{\omega}^\pm, \bar{\xi}^\pm, \bar{\beta}^\pm, \bar{\gamma}^\pm),
\]
where \(\bar{f}(t, x') = \mathcal{P} f_1(t, x') + \langle f_0 \rangle\). Hence, \(\langle \bar{f} \rangle = \langle f_0 \rangle\) and \(\int_{\mathbb{T}^d} \partial_t \bar{f}(t, x') dx' = 0\) for \(t \in [0, T]\).

**Proposition 4.** There exists \(M_1, M_2, T > 0\) depending on \(c_0, \delta_0, M_0\) so that \(F\) is a map from \(\mathcal{X}(T, M_1, M_2)\) to itself.

**Proof.** According to Definition 5.1, the initial conditions are automatically satisfied. It follows from Proposition 2 and Proposition 3 that
\[
\sup_{t \in [0, T]} \left( \left\| \bar{f}(t) \right\|_{H^{s+\frac{1}{2}}} + \left\| \bar{\omega}^\pm \right\|_{H^{s-1}(\Omega^\pm)} + \left\| \bar{\xi}^\pm \right\|_{H^{s-1}(\Omega^\pm)} + \left\| (\partial_t \bar{f})(t) \right\|_{H^{s-\frac{1}{2}}} \right) \leq C(c_0, M_0) e^{C(M_1, M_2)T}.
\]
After taking time derivative of (23), (26), one can get the evolution equations of \(\partial_t \bar{f}\) and \(\partial_t \bar{\omega}^\pm, \partial_t \bar{\xi}^\pm\). One can deduce in a similar way to Proposition 2 and Proposition 3 that
\[
\sup_{t \in [0, T]} \left( \left\| (\partial_t \bar{f})(t) \right\|_{H^{s-\frac{1}{2}}} + \left\| \partial_t \bar{\omega}^\pm \right\|_{H^{s-2}(\Omega^\pm)} + \left\| \partial_t \bar{\xi}^\pm \right\|_{H^{s-2}(\Omega^\pm)} \right) \leq C(M_1).
\]
It is clear that
\[
|\bar{\beta}^\pm_i(t)| + |\bar{\gamma}^\pm_i(t)| \leq M_0 + TC(M_1),
\]
\[
|\partial_t \bar{\beta}^\pm_i(t)| + |\partial_t \bar{\gamma}^\pm_i(t)| \leq C(M_1),
\]
\[
\left\| \bar{f}(t) - f_* \right\|_{H^{s-\frac{1}{2}}} \leq \int_0^t \left\| (\partial_t \bar{f})(t) \right\|_{H^{s-\frac{1}{2}}} d\tau.
\]
We first take \(M_2 = C(M_1)\) and then take \(M_1\) large enough such that \(C(c_0, M_0) < \frac{M_2}{M_1}\). Then, we can take \(T\) small enough such that all the conditions in Definition 5.1 are satisfied. This complete the proof. \(\square\)

5.3. Contraction of the iteration map. Now, we will show that \(F\) is contract in \(\mathcal{X}(T, M_1, M_2)\). Suppose \((f^A, \omega^A, \xi^A, \beta^A, \gamma^A)\) and \((f^B, \omega^B, \xi^B, \beta^B, \gamma^B)\) are two elements in \(\mathcal{X}(T, M_1, M_2)\) and
\[
(f^C, \omega^C, \xi^C, \beta^C, \gamma^C) = F(f^C, \omega^C, \xi^C, \beta^C, \gamma^C)
\]
for \(C = A, B\). We denote by \(g^D\) the difference \(g^A - g^B\).
Proof. By elliptic estimates, we have

\[ E^D \overset{def}{=} \sup_{t \in [0,T]} \left( \| f^D(t) \|_{H^{s-\frac{1}{2}}} + \| \bar{\omega}_s^D(t) \|_{H^{s-1}(\Omega_s^Z)} + \| \bar{\xi}_s^D(t) \|_{H^{s-2}(\Omega_s^Z)} \right) \]

By Proposition 5. There exists \( T = T(c_0, \delta_0, M_0) > 0 \) so that

\[ \frac{1}{2} \sup_{t \in [0,T]} \left( \| f^D(t) \|_{H^{s-\frac{1}{2}}} + \| \bar{\omega}_s^D(t) \|_{H^{s-1}(\Omega_s^Z)} + \| \bar{\xi}_s^D(t) \|_{H^{s-2}(\Omega_s^Z)} \right) \overset{def}{=} E^D. \]

Proof. By elliptic estimates, we have

\[ \| \Phi_{f_c}^+ - \Phi_{f_c}^- \|_{H^{s-1}(\Omega_s^Z)} \leq C(M_1) \| f^A - f^B \|_{H^{s-\frac{1}{2}}} \leq CE^D. \]

For \( C = A, B \) we define

\[ u_s^C = u^C \circ f_c, \quad h_s^C = h^C \circ f_c, \]

and claim that

\[ \| u_s^C \|_{H^{s-1}(\Omega_s^Z)} + \| h_s^C \|_{H^{s-1}(\Omega_s^Z)} \leq CE^D. \]

Indeed, for a vector field \( v_s^\Phi \) defined on \( \Omega_s^Z \), we define

\[ \text{curl}_C v_s^\Phi = \left( \text{curl}(v_s^\Phi \circ (\Phi_{f_c})^{-1}) \right) \circ \Phi_{f_c}, \]

\[ \text{div}_C v_s^\Phi = \left( \text{div}(v_s^\Phi \circ (\Phi_{f_c})^{-1}) \right) \circ \Phi_{f_c}. \]

Thus, for \( C = A, B \), it holds that

\[ \begin{cases} \text{curl}_C u_s^C = \bar{\omega}_s^C, & \text{in } \Omega_s^Z, \\ \text{div}_C u_s^C = 0, & \text{in } \Omega_s^Z, \\ u_s^C \cdot N_{f_c} = \partial_t f_c & \text{on } G_s, \\ u_s^C \cdot e_3 = 0, & \int_{\Omega_s^\pm} u_s^C dx' = \beta_s^C(i = 1, 2) & \text{on } \Gamma_s^\pm. \end{cases} \]

Accordingly, we deduce that

\[ \begin{cases} \text{curl}_A u_s^A = \bar{\omega}_s^A + (\text{curl}_B - \text{curl}_A) u_s^B & \text{in } \Omega_s^A, \\ \text{div}_A u_s^A = (\text{div}_B - \text{div}_A) u_s^B & \text{in } \Omega_s^A, \\ u_s^A \cdot N_{f_A} = \partial_t f^D + u_s^B \cdot (N_{f_B} - N_{f_A}) & \text{on } G_s, \\ u_s^A \cdot e_3 = 0, & \int_{\Omega_s^\pm} u_s^A dx' = \beta_s^D(i = 1, 2) & \text{on } \Gamma_s^\pm. \end{cases} \]

Direct calculation shows that

\[ \| (\text{curl}_B - \text{curl}_A) u_s^B \|_{H^{s-2}(\Omega_s^Z)} \leq C \| \Phi_{f_A}^+ - \Phi_{f_B}^- \|_{H^{s-1}(\Omega_s^Z)} \leq C(M_1) \| f^D \|_{H^{s-\frac{1}{2}}} \leq CE^D; \]

\[ \| (\text{curl}_B - \text{curl}_A) u_s^B \|_{H^{s-2}(\Omega_s^Z)} \leq CE^D; \]

\[ \| u_s^B \cdot (N_{f_B} - N_{f_A}) \|_{H^{s-\frac{1}{2}}} \leq CE^D. \]

Then, we get by Proposition 1 that

\[ \| u_s^D \|_{H^{s-1}(\Omega_s^Z)} \leq C(\| \bar{\omega}_s^D \|_{H^{s-2}(\Omega_s^Z)} + \| \partial_t f^D \|_{H^{s-1}} + E^D) \leq CE^D. \]

Similarly, we have

\[ \| h_s^D \|_{H^{s-1}(\Omega_s^Z)} \leq CE^D. \]
Recalling (12), we deduce that
\[
\begin{aligned}
\partial_t \bar{f}_1^D &= \bar{g}^D, \\
\partial_t \bar{g}^D &= -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ \bar{u}_1^+ + \rho^- \bar{u}_1^-) \partial_1 \bar{g}^D + (\rho^+ \bar{u}_2^+ + \rho^- \bar{u}_2^-) \partial_2 \bar{g}^D \right) \\
&\quad - \frac{1}{\rho^+ + \rho^-} \sum_{i,j,1,2} \left( (\rho^+ \bar{u}_1^+ \bar{u}_j^+ - h_i^+ A_i h_j^+ + \rho^- \bar{u}_2^- \bar{u}_j^- - h_i^- A_i h_j^-) \right) \partial_1 \partial_j \bar{f}_1^D \\
&\quad + \mathcal{R},
\end{aligned}
\]
where
\[
\mathcal{R} = -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ \bar{u}_1^+ + \rho^- \bar{u}_1^-) \partial_1 \bar{g}^B + (\rho^+ \bar{u}_2^+ + \rho^- \bar{u}_2^-) \partial_2 \bar{g}^B \right) \\
- \frac{1}{\rho^+ + \rho^-} \sum_{i,j,1,2} \left( (\rho^+ \bar{u}_1^+ \bar{u}_j^+ - h_i^+ A_i h_j^+ + \rho^- \bar{u}_2^- \bar{u}_j^- - h_i^- A_i h_j^-) \right) \\
- \left( (\rho^+ \bar{u}_1^+ + \bar{u}_2^-) - h_i^+ h_j^+ + \rho^- \bar{u}_2^- h_i^- - h_i^- h_j^- \right) \partial_1 \partial_j \bar{f}_1^B \\
+ \bar{g}^A - \bar{g}^B,
\]
and for \( C = A, B \),
\[
\bar{g}^C = + \frac{1}{\rho^+ + \rho^-} (N^C_f - N^C_j) N^C_f \]
\[
\begin{aligned}
&\partial_t \left( \sum_{i,j,1,2} (\bar{u}_i^+ \bar{u}_j^C - \frac{1}{\rho^+} h_i^+ h_j^C - \bar{u}_i^- \bar{u}_j^C - \frac{1}{\rho^-} h_i^- h_j^C) \partial_1 \partial_j f^C \right) \\
&\quad + \frac{2}{\rho^+ + \rho^-} (N^C_f - N^C_j) N^C_j \]
\[
\begin{aligned}
&\partial_t \left( \sum_{i,j,1,2} (\bar{u}_i^+ \bar{u}_j^C - \frac{1}{\rho^+} h_i^+ h_j^C - \bar{u}_i^- \bar{u}_j^C - \frac{1}{\rho^-} h_i^- h_j^C) \partial_1 \partial_j f^C \right) \\
&\quad - \frac{1}{\rho^+ + \rho^-} (N^C_f \cdot \nabla (\rho^+ \bar{u}_1^+ \bar{u}_j^C - \rho^- \bar{u}_2^- \bar{u}_j^C + N^C_j \cdot \nabla (\rho^- \bar{u}_1^+ \bar{u}_j^C - \rho^- \bar{u}_2^- \bar{u}_j^C) \]
\[
\begin{aligned}
&\quad + \frac{1}{\rho^+ + \rho^-} (N^C_f \cdot \nabla (\rho^+ \bar{u}_1^+ \bar{u}_j^C - \rho^- \bar{u}_2^- \bar{u}_j^C + N^C_j \cdot \nabla (\rho^- \bar{u}_1^+ \bar{u}_j^C - \rho^- \bar{u}_2^- \bar{u}_j^C) \]
\end{aligned}
\]
In a similar way to Lemma 4.1, one can easily seen that
\[
\| \mathcal{R} \|_{H^{s-\frac{1}{2}}} \leq CE^D.
\]
Now, we define
\[
\tilde{F}_s(\partial_t \bar{f}_1^D, \bar{f}_1^D) = \| (\partial_1 + \sum_{i=1,2} \bar{w}_i^A \partial_i) \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}_1^D \|_{L^2}^2 - \| \sum_{i=1,2} \bar{v}_i^A \partial_1 \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}_1^D \|_{L^2}^2 \\
+ \frac{1}{\rho^+ + \rho^-} \| \sum_{i=1,2} h_i^+ A_i \partial_1 \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}_1^D \|_{L^2}^2 \\
+ \frac{1}{\rho^+ + \rho^-} \| \sum_{i=1,2} h_i^- A_i \partial_1 \langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}_1^D \|_{L^2}^2.
\]
Then, by following the proof of Proposition (2), we have
\[
\frac{d}{dt} \left( \tilde{F}_s(\partial_t \bar{f}_1^D, \bar{f}_1^D) + \| \bar{f}_1^D \|_{L^2}^2 + \| \partial_t \bar{f}_1^D \|_{L^2}^2 \right) \leq C(E^D + \bar{E}_1^D),
\]
where
\[
\bar{E}_1^D = \sup_{t \in [0,T]} \left( \| \bar{f}_1^D(t) \|_{H^{s-\frac{1}{2}}} + \| \partial_t \bar{f}_1^D(t) \|_{H^{s-\frac{1}{2}}} \right).
\]
As \((f^A, \omega^\pm_A, \xi^\pm_A, \beta^\pm_A, \gamma^\pm_A) \in X(T, M_1, M_2)\) which satisfies \(\Lambda(h^{A\pm}, [u^A]) \geq c_0\), it holds that
\[
\|\bar{f}_1^D(t)\|_{H^{s-\frac{3}{2}}} + \|\partial_t \bar{f}_1^D(t)\|_{H^{s-\frac{3}{2}}} \leq C \left( \|\bar{F}_s(\partial_t \bar{f}_1^D, \bar{f}_1^D)\|_{L^2} + \|\partial_t \bar{f}_1^D\|_{L^2}^2 \right).
\]
Applying Gronwall’s inequality, we get
\[
\sup_{t \in [0, T]} \left( \|\bar{f}_1^D(t)\|_{H^{s-\frac{3}{2}}} + \|\partial_t \bar{f}_1^D(t)\|_{H^{s-\frac{3}{2}}} \right) \leq C(e^{CT} - 1)E^D,
\]
which imply
\[
\sup_{t \in [0, T]} \left( \|\bar{f}_p^D(t)\|_{H^{s-\frac{3}{2}}} + \|\partial_t \bar{f}_p^D(t)\|_{H^{s-\frac{3}{2}}} \right) \leq C(e^{CT} - 1)E^D.
\]
Similarly,
\[
\sup_{t \in [0, T]} \left( \|\bar{\omega}_F^D\|_{H^{s-2}(\Omega^\pm)} + \|\bar{\xi}_F^D\|_{H^{s-2}(\Omega^\pm)} \right) \leq C(e^{CT} - 1)E^D.
\]
From the equation
\[
\tilde{\beta}_{i}^{\pm C}(t) = \tilde{\beta}_{i}^{\pm C}(0) - \int_{0}^{t} \int_{\tau}^{t} \sum_{j=1}^{3} (u_{j}^{\pm C} \partial_{i} u_{j}^{\pm C} - \frac{1}{\rho_{\pm}^{C}} \partial_{i} h_{j}^{\pm C} \partial_{j} h_{i}^{\pm C}) dx' d\tau,
\]
we have
\[
|\tilde{\beta}_{i}^{\pm D}(t)| \leq |\beta_{ii}^{\pm D}| + TCE^D.
\]
Similarly,
\[
|\tilde{\gamma}_{i}^{\pm D}(t)| \leq |\gamma_{ii}^{\pm D}| + TCE^D.
\]
As a conclusion, we arrive at
\[
E^D \leq C(e^{CT} - 1 + T)E^D.
\]
We get the desired estimate by taking \(T\) small enough.

### 5.4. The limit system.

Proposition 5 shows that there exists a unique fixed point \((f^\pm, \omega^\pm, \xi^\pm, \beta^\pm, \gamma^\pm)\) of the map \(\mathcal{F}\) in \(X(T, M_1, M_2)\). Now, we will finish the proof of Theorem 1.1, and show that one can recover \((u^\pm, h^\pm, p^\pm)\) from \((f^\pm, \omega^\pm, \xi^\pm, \beta^\pm, \gamma^\pm)\) which is the unique solution to the original system (1)-(3).

From the construction of \(\mathcal{F}\), the fixed point \((f^\pm, \omega^\pm, \xi^\pm, \beta^\pm, \gamma^\pm)\) satisfies
\[
\partial_t f = \mathcal{F}_f,
\]
\[
\partial_t \theta = - \frac{2}{\rho^s + \rho^r} \left( (\rho^s u^s + \rho^r u^r) \partial_1 \theta + (\rho^s u^s + \rho^r u^r) \partial_2 \theta \right)
\]
\[
- \frac{1}{\rho^s + \rho^r} \sum_{i,j=1}^{2} \left( (\rho^s u^s_j - \rho^r u^r_j) h^s_i + (\rho^s u^s_i - \rho^r u^r_i) h^r_j \right) \partial_1 f
\]
\[
+ \frac{1}{\rho^s + \rho^r} (N^f_j - N^f_i) \bar{N}_j^{-1} \mathcal{P} \left( \sum_{i,j=1}^{2} (u^s_i u^s_j - u^r_i u^r_j) \partial_1 \theta + (u^s_i - u^r_i) \partial_2 \theta \right)
\]
\[
- \frac{1}{\rho^s + \rho^r} N_{j} \cdot \nabla (\rho^s p^s_u + \rho^r p^r_u - p_{h^+ + h^+ -} - p_{h^+ + h^- -})
\]
\[
+ \frac{1}{\rho^s + \rho^r} (N^f_j - N^f_i) \bar{N}_j^{-1} \mathcal{P} \mathcal{N}_j \cdot \nabla (p_{h^+ + h^+ -} - \frac{1}{\rho^s} p_{h^+ + h^+ -} - p_{h^+ - h^- -} + \frac{1}{\rho^s} p_{h^+ - h^- -}).
\]
Here $p_{u^+_1,u^+_2}$ is defined in (11), $(u^\pm,h^\pm)$ is the solution to
\[
\begin{cases}
\text{curl} u^\pm = P_f^\text{div} \omega^\pm, \quad \text{div} u^\pm = 0 & \text{in } \Omega_f^\pm; \\
u^\pm \cdot N_f = \partial_t f & \text{on } \Gamma_f; \\
u^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} u_i dx' = \beta^\pm_i (i = 1, 2) & \text{on } \Gamma^\pm; \\
\partial_t \beta^\pm_i = - \int_{\Gamma^\pm} \sum_{j=1}^3 (u_j^\pm \partial_j u_i^\pm - \frac{1}{\rho^\pm} h_j^\pm \partial_j h_i^\pm) dx',
\end{cases}
\]
and
\[
\begin{cases}
\text{curl} h^\pm = P_f^\text{div} \xi^\pm, \quad \text{div} h^\pm = 0 & \text{in } \Omega_f^\pm; \\
h^\pm \cdot N_f = 0 & \text{on } \Gamma_f; \\
h^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} h_i dx' = \gamma^\pm_i (i = 1, 2) & \text{on } \Gamma^\pm; \\
\partial_t \gamma^\pm_i = - \int_{\Gamma^\pm} \sum_{j=1}^3 (u_j^\pm \partial_j h_i^\pm - h_j^\pm \partial_j u_i^\pm) dx',
\end{cases}
\]
and $(\omega^\pm, \xi^\pm)$ satisfies
\[
\partial_t \omega^\pm + u^\pm \cdot \nabla \omega^\pm - \frac{1}{\rho^\pm} h^\pm \cdot \nabla \xi^\pm = \omega^\pm \cdot \nabla u^\pm - \frac{1}{\rho^\pm} \xi^\pm \cdot \nabla h^\pm,
\]
\[
\partial_t \xi^\pm + u^\pm \cdot \nabla \xi^\pm - h^\pm \cdot \nabla \omega^\pm = \xi^\pm \cdot \nabla u^\pm - \omega^\pm \cdot \nabla h^\pm - 2 \sum_{i=1}^3 \nabla u_i^\pm \times \nabla h_i^\pm.
\]
Next, we will show that the above system is equivalent to the origin system (1)-(3).
We introduce the pressure $p^\pm$ by
\[
p^\pm = \mathcal{H}^\pm p^\pm + \rho^\pm p_{u^\pm,u^\pm} - p_{h^\pm,h^\pm},
\]
where
\[
p^\pm = \mathcal{N}_f^{-1} P(g^+ - g^-),
\]
with
\[
g^\pm = 2(u_1^\pm \partial_1 \theta + u_2^\pm \partial_2 \theta) + N_f \cdot \nabla (pu^\pm,u^\pm - 1 + \rho u_{h^\pm,h^\pm}) + \sum_{i,j=1,2} (u_i^\pm u_j^\pm - \frac{1}{\rho^\pm} u_i^\pm u_j^\pm) \partial_i \partial_j f.
\]
Then, following the procedures in Section 9 of [17], one can check that
\[
\begin{cases}
\text{curl} k^\pm = 0, \quad \text{div} k^\pm = 0 & \text{in } \Omega_f^\pm; \\
k^\pm \cdot N_f = 0 & \text{on } \Gamma_f; \\
k^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} w_i dx' = 0 (i = 1, 2) & \text{on } \Gamma^\pm,
\end{cases}
\]
for
\[
k^\pm \overset{\text{def}}{=} \rho^\pm \partial_t u^\pm + \rho^\pm u^\pm \cdot \nabla u^\pm - h \cdot \nabla h^\pm + \nabla p^\pm = 0,
\]
or
\[
k^\pm \overset{\text{def}}{=} \partial_t h^\pm + u^\pm \cdot \nabla h^\pm - h^\pm \cdot \nabla u^\pm.
\]
This complete the proof.
Appendix A. Commutator estimate.

Lemma A.1. If \( s > 1 + \frac{d}{2} \), then we have
\[
\left\| [a, \langle \nabla \rangle^s] u \right\|_{L^2} \leq C \|a\|_{H^s} \|u\|_{H^{s-1}}.
\]

A.2. Sobolev estimates of DN operator.

Proposition 6. If \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then it holds that for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| \mathcal{N}_f^+ \psi \|_{H^\sigma} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^{s+1}}.
\]
Moreover, it holds that for any \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| (\mathcal{N}_f^+ - \mathcal{N}_f^-) \psi \|_{H^\sigma} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^s},
\]
where \( K_{s+\frac{1}{2},f} \) is a constant depending on \( c_0 \) and \( \|f\|_{H^{s+1/2}} \).

Proposition 7. If \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then it holds that for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\| \mathcal{G}_f^+ \psi \|_{H^{\sigma+1}} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^\sigma},
\]
where \( \mathcal{G}_f^+ \overset{\text{def}}{=} (\mathcal{N}_f^+)^{-1} \).

Acknowledgments. The authors wish to express their thanks to Prof. Zhifei Zhang for suggesting the problem and for many helpful discussions.

REFERENCES

[1] W. I. Axford, Note on a problem of magnetohydrodynamic stability, Canad. J. Phys., 40 (1962), 654–655.
[2] T. Alazard, N. Burq and C. Zuily, On the Cauchy problem for gravity water waves, Invent. Math., 198 (2014), 71–163.
[3] D. M. Ambrose and N. Masmoudi, Well-posedness of 3D vortex sheets with surface tension, Comm. Math. Sci., 5 (2007), 391–430.
[4] C. Cheng, D. Coutand and S. Shkoller, Navier-Stokes equations interacting with a nonlinear elastic biofluid shell, SIAM J. Math. Anal., 39 (2007), 742–800.
[5] V. Carbone, G. Einaudi and P. Veltri, Effects of turbulence development in solar surges, Solar. Phys., 111 (1987), 31–44.
[6] D. Christodoulou and H. Lindblad, On the motion of the free surface of a liquid, Comm. Pure Appl. Math., 53 (2000), 1536–1602.
[7] J.-F. Coulombel, A. Morando, P. Secchi and P. Trebeschi, A priori estimates for 3D incompressible current-vortex sheets, Comm. Math. Phys., 311 (2012), 247–275.
[8] S. Friedlander and D. Serre, Handbook of Mathematical Fluid Dynamics, North Holland/Elsevier, 2004.
[9] F. Jiang, S. Jiang and Y. Wang, On the Rayleigh-Taylor instability for the incompressible viscous magnetohydrodynamic equations, Comm. Partial Differential Equations, 39 (2014), 399–438.
[10] A. L. La Belle-Hamer, Z. F. Fu and L. C. Lee, A mechanism for patchy reconnection at the dayside magnetopause, Geophysical Research Letters, 15 (1988), 152–155.
[11] H. Li, W. Wang and Z. Zhang, Well-posedness of the free boundary problem in incompressible elastodynamics, J. Differential Equations, 267 (2019), 6604–6643.
[12] A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge Texts in Applied Mathematics, 27, Cambridge University Press, Cambridge, 2002.
[13] G. V. Miloshevsky and A. Hassanein, Modelling of Kelvin-Helmholtz instability and splashing of melt layers from plasma-facing components in tokamaks under plasma impact, Nuclear Fusion, 50 (2010), 115005.
[14] A. Morando, Y. Trakhinin and P. Trebeschi, Stability of incompressible current-vortex sheets, *J. Math. Anal. Appl.*, 347 (2008), 502–520.

[15] L. Ofman, X. L. Chen, P. J. Morrison and et al., Resistive tearing mode instability with shear flow and viscosity, *Physics of Fluids B: Plasma Physics*, 3 (1991), 1364.

[16] S. I. Syrovatskij, The stability of tangential discontinuities in a magnetohydrodynamic medium, *Z. Eksperim. Teoret. Fiz.*, 24 (1953), 622–629.

[17] Y. Sun, W. Wang and Z. Zhang, Nonlinear stability of the current-vortex sheet to the incompressible MHD equations, *Comm. Pure Appl. Math.*, 71 (2018), 356–403.

[18] J. Shatah and C. Zeng, Geometry and a priori estimates for free boundary problems of the Euler’s equation, *Comm. Pure Appl. Math.*, 61 (2008), 698–744.

[19] J. Shatah and C. Zeng, A priori estimates for fluid interface problems, *Comm. Pure Appl. Math.*, 61 (2008), 848–876.

[20] Y. Trakhinin, On the existence of incompressible current-vortex sheets: study of a linearized free boundary value problem, *Math. Methods Appl. Sci.*, 28 (2005), 917–945.

[21] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, *Invent. Math.*, 130 (1997), 39–72.

[22] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, *J. Amer. Math. Soc.*, 12 (1999), 445–495.

[23] H. Yang, Z. Xu, E. K. Lim, et al. Observation of the Kelvin-Helmholtz Instability in a Solar Prominence, *The Astrophysical Journal*, 857 (2018), 115.

[24] P. Zhang and Z. Zhang, On the free boundary problem of three-dimensional incompressible Euler equations, *Comm. Pure Appl. Math.*, 61 (2008), 877–940.

Received September 2020; revised April 2021.

E-mail address: lcy941024@pku.edu.cn
E-mail address: lihui92@zju.edu.cn