COMPUTING LIMITS OF QUOTIENTS OF MULTIVARIATE REAL ANALYTIC FUNCTIONS

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ABSTRACT. We present an algorithm for computing limits of quotients of real analytic functions. The algorithm is based on computation of a bound on the Łojasiewicz exponent and requires the denominator to have an isolated zero at the limit point.

Keywords: Multivariate function limit, symbolic limit computation.

1. INTRODUCTION

Computation of limits is one of the basic problems of computational calculus. In the univariate case computing limits of rational functions is easy, and the state of the art limit computation algorithms [3, 6] are applicable to large classes of functions. In the multivariate case computing limits of real rational functions is a nontrivial problem that has been a subject of recent research [1, 2, 8, 9, 10]. In [7] we compared five methods for computation of limits of real rational functions based on the Cylindrical Algebraic Decomposition (CAD) algorithm. Here we extend the methods to computation of limits of quotients of multivariate analytic functions.

The limit of a real function may not exist, however the lower limit and the upper limit always exist. A weak version of the limit computation problem consists of deciding whether the limit exists and, if it does, finding the value of the limit. A strong version consists of finding the values of the lower limit and the upper limit.

Let us state the problems precisely. Denote \( x = (x_1, \ldots, x_n), \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \). Let \( U \subseteq \mathbb{R}^n \) be an open set and let \( \mathcal{A}(U) \) denote the set of analytic functions on \( U \). Let \( g \in \mathcal{A}(U), h \in \mathcal{A}(U) \setminus \{0\}, D = \{ u \in U : h(u) \neq 0 \}, f : D \ni u \to \frac{g(u)}{h(u)} \in \mathbb{R}, \) and let \( c \in U \).

Problem 1. Find \( l \in \mathbb{R} \) such that \( l = \lim_{u \to c} f(u) \) or prove that such \( l \) does not exist.

Problem 2. Find \( l_1, l_2 \in \mathbb{R} \) such that \( l_1 = \liminf_{u \to c} f(u) \) and \( l_2 = \limsup_{u \to c} f(u) \).

Example 3. Let \( g = \sin(x^2 + y^2 + z^2) \) and \( h = 3 - \cos x - \cos y - \cos z \). Then

\[
\lim_{(x,y,z) \to (0,0,0)} \frac{g(x,y,z)}{h(x,y,z)} = 2
\]

Example 4. Let \( g = \sin(xy) \) and \( h = \cos x + \cos y - 2 \). Then

\[
\liminf_{(x,y) \to (0,0)} \frac{g(x,y)}{h(x,y)} = -1
\]

\[
\limsup_{(x,y) \to (0,0)} \frac{g(x,y)}{h(x,y)} = 1
\]

and \( \lim_{(x,y) \to (0,0)} \frac{g(x,y)}{h(x,y)} \) does not exist.
Recently two algorithms partially solving Problem 1 for rational functions have been proposed. The algorithm presented in [9] solves a modified version of the problem, namely it decides whether the limit exists and is finite. The negative answer includes both the case when the limit does not exist and the case when the limit exists and is infinite. The algorithm uses Wu’s elimination method, rational univariate representations, and requires adjoining two infinitesimal elements to the field. The algorithm presented in [1] (which generalizes algorithms of [2, 8]) solves Problem 1 under the additional assumption that $c$ is an isolated zero of $h$. The authors use the theory of Lagrange multipliers to reduce the problem to computing the limit along a real algebraic set, and solve the reduced problem using regular chains methods.

In [7] we presented five methods based on the CAD algorithm that solve both Problem 1 and Problem 2 for arbitrary rational functions. In this note we describe an algorithm that reduces computation of limits of quotients of multivariate analytic functions to computation of limits of rational functions. The algorithm can be combined with any of the algorithms described in [7] to solve both Problem 1 and Problem 2 for quotients of analytic functions.

2. THE ALGORITHM

Let $f = \frac{g}{h}$, where $g$ and $h$ are analytic functions in a neighbourhood of $c \in \mathbb{R}^n$. Without loss of generality we will assume that $c = 0$. If $h(0) \neq 0$ then $f$ is continuous at 0, and hence the limit can be obtained by evaluation. Therefore w.l.o.g. we will assume that $h(0) = 0$. The only computability assumption we make about functions $g$ and $h$ is that for any $d \in \mathbb{N}$ we can compute the Taylor polynomials $T_dg$ and $T_dh$ of total degree $d$. This is a typical case in a computer algebra system, where $g$ and $h$ are given as expressions obtained by composing analytic functions implemented in the system. Taylor polynomials of arbitrary degree can be readily computed, but for instance algorithms for testing whether an expression represents a function that is identically zero may not be available.

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Remark 5. This specification is not sufficient to always solve Problem 1 or 2. For instance, let \( \varphi_0 = x^2 + y^{2^k} \), let \( \varphi_m = x^2 \), and suppose that \( g \) and \( h \) are expressions that as functions (but not as expressions) satisfy equalities \( g = \varphi_k \) and \( h = \varphi_m \) for some (unknown) \( k, m \in \mathbb{N} \cup \{\infty\} \). If \( k = m \) then \( \lim_{(x,y) \to 0} \frac{g(x,y)}{h(x,y)} = 1 \) otherwise the limit does not exist. Suppose that no algorithms are available that would simplify \( g \) and \( h \) to explicit polynomials. Computation of the Taylor polynomials \( T_d g \) and \( T_d h \) for any finite \( d \) does not allow to decide whether \( k \) and \( m \) are finite, but greater than \( d/2 \), or are infinite. Hence Problem 1 or 2 cannot always be solved. Similarly, we cannot always decide whether the zero of \( h \) at \( (0,0) \) is isolated.

The algorithm we present here solves Problem 1 and 2 if \( h \) has an isolated zero at \( 0 \). Otherwise the algorithm does not terminate. Considering Remark 5 this is the best we can hope for with the given input specification.

Let \( U \subseteq \mathbb{R}^n \) be an open neighbourhood of \( 0 \), and let \( \|u\| \) denote the Euclidean norm of \( u \in \mathbb{R}^n \). The algorithm is based on the following theorem, which is a special case of the Łojasiewicz inequality [5].

**Theorem 6.** Let \( h \in \mathcal{A}(U) \) and suppose that \( \{u : \|u\| < \rho \land h(u) = 0\} = \{0\} \) for some \( \rho > 0 \). Then there exist positive constants \( r, c, \) and \( \alpha \) such that if \( \|u\| < r \) then \( |h(u)| \geq c\|u\|^\alpha \).

**Algorithm 7. (MLIM)**

**Input:** \( g \in \mathcal{A}(U), h \in \mathcal{A}(U) \setminus \{0\}, \) such that \( h(0) = 0 \).

**Output:** \( \liminf_{u \to 0} \frac{g(u)}{h(u)} \).

1. Set \( d = 2 \).
2. Compute \( \lim_{u \to 0} \frac{u^2 + \cdots + u^2}{T_{2d-1} h(u)} \).
3. If the limit does not exist or is not zero, set \( d = d + 1 \) and go to step 2.
4. Return \( \liminf_{u \to 0} \frac{T_{2d-1} g(u)}{T_{2d-1} h(u)} \).

**Theorem 8.** If \( h \) has an isolated zero at \( 0 \) then Algorithm 7 terminates and returns

\[
\liminf_{u \to 0} \frac{g(u)}{h(u)}
\]

Otherwise the algorithm does not terminate.

**Proof.** Suppose that \( h \) has an isolated zero at \( 0 \). By Theorem 6 there exist positive constants \( r, c, \) and \( \alpha \) such that if \( \|u\| < r \) then \( |h(u)| \geq c\|u\|^\alpha \). To prove that Algorithm 7 terminates it suffices to show that if \( 2d > \alpha \) then

\[
\lim_{u \to 0} \frac{u^2 + \cdots + u^2}{T_{2d-1} h(u)} = 0
\]

Let \( R_{2d-1} h(u) = h(u) - T_{2d-1} h(u) \). We have

\[
\frac{u^2 + \cdots + u^2}{|T_{2d-1} h(u)|} \leq \frac{\|u\|^{2d}}{|h(u) - R_{2d-1} h(u)|} \leq \frac{1}{\frac{|h(u)|}{\|u\|^{\alpha}} - \frac{|R_{2d-1} h(u)|}{\|u\|^{2d}}}
\]

Since \( R_{2d-1} h(u) \) is an analytic function whose Taylor series does not contain terms of degree lower than \( 2d \), \( \frac{|R_{2d-1} h(u)|}{\|u\|^{2d}} \) is bounded in a neighbourhood of \( 0 \). Moreover, if \( \|u\| < r \),

\[
\frac{|h(u)|}{\|u\|^{\alpha}} \geq \frac{c\|u\|^{\alpha}}{\|u\|^{2d}} = c\|u\|^{-2d} \to \infty
\]
hence
\[ \lim_{u \to 0} \frac{1}{\|u\|^{2d}} \frac{|R_{2d-1}h(u)|}{\|u\|^{2d}} = 0 \]

which proves that
\[ \lim_{u \to 0} \frac{u_1^{2d} + \cdots + u_n^{2d}}{T_{2d-1}h(u)} = 0 \]

To show that Algorithm\textsuperscript{[7]} returns \( \lim_{u \to 0} \frac{g(u)}{h(u)} \) note that
\[ g(u) = \frac{T_{2d-1}g(u) + R_{2d-1}g(u)}{T_{2d-1}h(u)} = \frac{T_{2d-1}g(u)}{T_{2d-1}h(u)} \frac{T_{2d-1}g(u) + R_{2d-1}g(u)}{T_{2d-1}h(u)} \]

We have
\[ \frac{R_{2d-1}g(u)}{T_{2d-1}h(u)} = \frac{R_{2d-1}g(u)}{\|u\|^{2d}} \frac{\|u\|^{2d}}{T_{2d-1}h(u)} \]

Since \( R_{2d-1}g(u) \) is an analytic function whose Taylor series does not contain terms of degree lower than 2d, \( \frac{R_{2d-1}g(u)}{\|u\|^{2d}} \) is bounded in a neighbourhood of 0. Moreover,
\[ \frac{\|u\|^{2d}}{|T_{2d-1}h(u)|} \leq n^d \frac{u_1^{2d} + \cdots + u_n^{2d}}{|T_{2d-1}h(u)|} \xrightarrow{u \to 0} 0 \]

hence
\[ \lim_{u \to 0} \frac{R_{2d-1}g(u)}{T_{2d-1}h(u)} = 0 \]

Similarly
\[ \lim_{u \to 0} \frac{R_{2d-1}h(u)}{T_{2d-1}h(u)} = 0 \]

and therefore
\[ \liminf_{u \to 0} \frac{g(u)}{h(u)} = \liminf_{u \to 0} \frac{T_{2d-1}g(u)}{T_{2d-1}h(u)} \]

Suppose now that the zero of \( h \) at 0 is not isolated. Let \( d \geq 2 \).

If the zero of \( T_{2d-1}h \) at 0 is not isolated, then
\[ \frac{u_1^{2d} + \cdots + u_n^{2d}}{T_{2d-1}h(u)} \]

attains arbitrarily large values in any neighbourhood of 0, and hence
\[ \lim_{u \to 0} \frac{u_1^{2d} + \cdots + u_n^{2d}}{T_{2d-1}h(u)} \]

does not exist or is not zero.

If the zero of \( T_{2d-1}h \) at 0 is isolated, then \( \{u : \|u\| \leq \rho \land T_{2d-1}h(u) = 0\} = \{0\} \) for some \( \rho > 0 \). Since \( R_{2d-1}h(u) \) is an analytic function whose Taylor series does not contain terms of degree lower than 2d, there exists \( M > 0 \) such that \( \frac{|R_{2d-1}h(u)|}{\|u\|^{2d}} \leq M \) for all \( \|u\| \leq \rho \).

Let \( Z = \{u : \|u\| \leq \rho \land h(u) = 0\} \). For \( u \in Z \setminus \{0\} \) we have
\[ T_{2d-1}h(u) = -R_{2d-1}h(u) \]

and hence
\[ \frac{u_1^{2d} + \cdots + u_n^{2d}}{|T_{2d-1}h(u)|} \geq n^{-d} \frac{\|u\|^{2d}}{|R_{2d-1}h(u)|} \geq n^{-d}M^{-1} > 0 \]
Since 0 is a limit point of $Z \setminus \{0\}$,

$$\lim_{u \to 0} \frac{u_1^{2d} + \cdots + u_n^{2d}}{T_{2d-1}h(u)}$$

does not exist or is not zero. Therefore Algorithm 7 does not terminate. \hfill \square

Remark 9. Algorithm 7 can be adapted to compute the upper limit or the limit of $f(u)$ by returning the upper limit or the limit of $\frac{T_{2d-1}h(u)}{h(a)}$ in step (4).

Example 10. Algorithm 7 may not stop for the first $d$ such that $T_{2d-1}h$ has an isolated zero at 0, even if $T_{2d-1}h = h$. Let $h(x,y) = (x^n)^2 + (x-y)^2$ (cf. Example 1). Then $T_{2d-1}h = h$ for $d \geq n + 1$. However, since $h(t^n, t) = t^{2n^2}$

$$\lim_{u \to 0} \frac{x^{2d} + y^{2d}}{T_{2d-1}h(x,y)}$$

will not be zero for any $d \leq n^2$.

Example 11. The second part of the proof does need two cases, that is the zero of $T_{2d-1}h$ at 0 may be isolated even if the zero of $h$ at 0 is not isolated. Let $h(x,y) = y^4 + (y - x^2)^2 - x^6 - y^6$. Then $T_4h = y^4 + (y - x^2)^2$ has an isolated zero at 0. However, the zero of $h$ at 0 is not isolated. In a neighbourhood of zero there are two analytic solutions of $h(x,y) = 0$ with initial series terms given by

$$y_1(x) = x^2 - x^3 + \frac{x^5}{2} - 2x^6 + \frac{25x^7}{8} + \cdots$$

$$y_2(x) = x^2 + x^3 - \frac{x^5}{2} - 2x^6 - \frac{25x^7}{8} + \cdots$$

Remark 12. In some cases it is possible to detect that the zero of $h$ at 0 is not isolated and terminate the algorithm. For instance if $h_m$ is the lowest degree nonzero form of the Taylor series of $h$ and $h_m(a) > 0$ and $h_m(b) < 0$ for some $a, b \in \mathbb{R}^n$, then $h(ta) > 0$ and $h(tb) < 0$ for $t \in (0, \varepsilon)$ with some $\varepsilon > 0$, and hence the zero of $h$ at 0 is not isolated.

The number of limit computations in step (2) can be reduced by using fast negative criteria to decide that the zero of $T_{2d-1}h(u)$ at 0 is not isolated.

3. Example

Let us compute the lower limit and the upper limit of $g(x,y,z) = h(x,y,z)$ at 0, where

$$g = \exp(\sin(x^2 + y^4 + z^6)) - 1$$

$$h = \sqrt{\cos(x) - \sin(y^2) - z^4 - 1}$$

For $d = 2$ in step (2) we have $T_{2d-1}h = -\frac{1}{2}x^2$. Since $T_{2d-1}h(0,0,z) = 0$, the limit

$$\lim_{(x,y,z) \to 0} \frac{x^4 + y^4 + z^4}{T_{2d-1}h}$$

does not exist, and so in step (3) we set $d = 3$ and go back to step (2). Now

$$T_{2d-1}h = -\frac{1}{96}x^4 - 12x^2y^2 - 24x^2 - 12y^4 - 48y^2 - 48z^4$$

and

$$\lim_{(x,y,z) \to 0} \frac{x^6 + y^6 + z^6}{T_{2d-1}h} = 0$$
hence we move on to step (4). We have $T_{2d-1}g = \frac{x^4 + 2z^2 + 2y^4}{2}$ and the returned value is

$$\liminf_{(x,y,z)\to0} g(x,y,z) = \liminf_{(x,y,z)\to0} \frac{T_{2d-1}g(x,y,z)}{T_{2d-1}h(x,y,z)} = -4$$

To find the upper limit we just need to compute the upper limit in step (4)

$$\limsup_{(x,y,z)\to0} g(x,y,z) = \limsup_{(x,y,z)\to0} \frac{T_{2d-1}g(x,y,z)}{T_{2d-1}h(x,y,z)} = 0$$

To compute limits of rational functions Algorithm [7] can use any of the algorithms described in [7]. We used the implementation of these algorithms in Mathematica. In this example methods based on topological properties performed better. Algorithm 15 (TLIM) took 77 seconds when using Algorithm 14 (ZCQ2) and 323 seconds when using Algorithm 13 (ZCQ1). Methods based on optimization were not able to complete the computation in 12 hours. The most time-consuming part of the computation is the limit in step (2) with $d = 3$.  

REFERENCES

[1] P. Alvandi, M. Kazemi, and M. Moreno Maza. Computing limits of real multivariate rational functions. In Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC 2016, pages 39–46. ACM, 2016.
[2] C. Cadavid, S. Molina, and J. D. Velez. Limits of quotients of bivariate real analytic functions. J. Symbolic Comp., 50:197–207, 2013.
[3] D. Gruntz. On computing limits in a symbolic manipulation system. PhD thesis, ETH, 1996.
[4] K. Kurdyka and S. Spodzieja. Separation of real algebraic sets and the Łojasiewicz exponent. Proceedings of the AMS, 142(9):3089–3102, 2014.
[5] S. Łojasiewicz. Ensembles semi-analytiques. I.H.E.S., 1964.
[6] B. Salvy and J. Shackell. Symbolic asymptotics: Multiseries of inverse functions. J. Symb. Comput., 27:543–563, 1999.
[7] A. Strzeboński. Comparison of cad-based methods for computation of rational function limits. In Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC 2018, pages 375–382. ACM, 2018.
[8] J. D. Velez, J. P. Hernandez, and C. A. Cadavid. Limits of quotients of real polynomial functions of three variables, 2015. arXiv 1505.04121.
[9] S. J. Xiao and G. X. Zeng. Determination of the limits for multivariate rational functions. Science China Mathematics, 57(2):397–416, 2014.
[10] S. J. Xiao, X. N. Zeng, and G. X. Zeng. Real valuations and the limits of multivariate rational functions. Journal of Algebra and Its Applications, 14(5):1550–1567, 2015.