HYPONORMAL TOEPLITZ OPERATORS WITH NON-HARMONIC SYMBOL
ACTING ON THE BERGMAN SPACE

MATTHEW FLEEMAN AND CONSTANZE LIAW

Abstract. The Toeplitz operator acting on the Bergman space $A^2(D)$, with symbol $\varphi$ is given by $T_\varphi f = P(\varphi f)$, where $P$ is the projection from $L^2(D)$ onto the Bergman space. We present some history on the study of hyponormal Toeplitz operators acting on $A^2(D)$, as well as give results for when $\varphi$ is a non-harmonic polynomial. We include a first investigation of Putnam’s inequality for hyponormal operators with non-analytic symbols. Particular attention is given to unusual hyponormality behavior that arises due to the extension of the class of allowed symbols.

1. Introduction

Let $H$ be a complex Hilbert space and $T$ be a bounded linear operator acting on $H$ with adjoint $T^*$. Operator $T$ is said to be hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$
\[ \langle [T^*, T]u, u \rangle \geq 0. \]

The study of hyponormal operators is strongly related to the spectral and perturbation theories of Hilbert space operators, singular integral equations, and scattering theory. The interested reader is referred to the monograph [10] by M. Martin and M. Putinar. One particularly interesting result for hyponormal operators, Putnam’s inequality, states that if $T$ is hyponormal, then
\[ \| [T^*, T] \| \leq \frac{\text{Area}(\sigma(T))}{\pi}, \]
where $\sigma(T)$ denotes the spectrum of $T$ (cf. [2]).

We study the hyponormality of certain operators acting on the Bergman space
\[ A^2(D) = \left\{ f \in \text{Hol}(D) : \int_D |f(z)|^2 \, dA(z) < \infty \right\}. \]
Let $\varphi \in L^\infty(D)$. The Toeplitz operator $T_\varphi$ is given by
\[ T_\varphi f = P(\varphi f) \quad f \in A^2(D), \]
where $P$ is the orthogonal projection from $L^2(D)$ onto $A^2(D)$.

2010 Mathematics Subject Classification. 47B35, 47B20.
Key words and phrases. Toeplitz operator, Bergman space, hyponormality, non-harmonic symbol.
The work of C. Liaw was supported by Simons Foundation Grant #426258.
In the Hardy space setting the question of when $T_\varphi$ is hyponormal for $\varphi \in L^\infty(\mathbb{T})$ was answered by C. Cowen in [4], who proved the following theorem:

**Theorem.** Let $\varphi \in L^\infty(\mathbb{T})$ be given by $\varphi = f + \bar{g}$, with $f, g \in H^2$. Then $T_\varphi$ is hyponormal if and only if

$$g = c + T_h f,$$

for some constant $c$ and some $h \in H^\infty(\mathbb{D})$, with $\|h\|_\infty \leq 1$.

This completely characterized hyponormal Toeplitz operators acting on the Hardy space. Cowen’s proof relies on a dilation theorem of D. Sarason [14, Theorem 1], and the fact that $(H^2)\perp$ is just the conjugates of $H^2$ functions which vanish at the origin.

In the Bergman space setting, where we lack an analog to Sarason’s dilation theorem, and where $(A^2)\perp$ is a much larger space, a similar characterization is lacking. One of the principle difficulties in exploring questions of hyponormality originates from the behavior of the self-commutator under operator addition. In particular, if we let $u$ be in a complex Hilbert space $H$, and $T$ and $S$ be operators on $H$, then we find

$$\langle ([T + S]^*, T + S] u, u \rangle = \langle Tu, Tu \rangle - \langle T^* u, T^* u \rangle + 2\operatorname{Re} \langle [Tu, Su] - \langle T^* u, S^* u \rangle + \langle Su, Su \rangle - \langle S^* u, S^* u \rangle. $$

(1.1)

As we shall see, the “cross-terms” $2\operatorname{Re} \langle [Tu, Su] - \langle T^* u, S^* u \rangle$ lead to many somewhat unexpected results which reveals a subtlety in the study of hyponormal operators. The explicit expressions in (1.1) lead to involved series computations. Our primary effort consists of extracting reasonable necessary and/or sufficient conditions from series corresponding to several different types of non-harmonic symbols. It is worth noting that if both $T$ and $S$ are Toeplitz operators with harmonic symbols, then these cross terms vanish, which leads to a smoother study of such operators, e.g. in [1], [8], and [13].

One of the central questions this paper explores is the following:

*Given a hyponormal Toeplitz operator $T_\varphi$ acting on $A^2(\mathbb{D})$ and a symbol $\psi \in L^\infty(\mathbb{D})$, when is $T_\varphi + \psi$ hyponormal?*

When $\psi$ is not harmonic, this question turns out to be particularly elusive. As we shall see in Section 3, even requiring that $T^*_\psi$ be self-adjoint is not enough to guarantee the hyponormality of $T_\varphi + \psi$.

We are also interested in some spectral properties of hyponormal $T_\varphi$, especially because the commutator has interesting interactions with the geometry of the image $\varphi(\mathbb{D})$. It is an immediate consequence of Putnam’s inequality and the spectral mapping theorem (cf. [12, p. 263]) that the norm of the commutator of $T^*_\varphi$ and $T_\varphi$ is bounded above by $\text{Area}(\varphi(\mathbb{D}))/\pi$ for analytic $\varphi$, and in [11] it was shown that this bound can be improved to $\text{Area}(\varphi(\mathbb{D}))/(2\pi)$ for analytic and univalent $\varphi$. In [7], it was conjectured that the hypothesis “univalent” is superfluous for this stronger bound. We extend this conjecture to non-analytic symbols.
The paper proceeds as follows: In Section 2, we give an overview of some known results for the hyponormality results of Toeplitz operators with harmonic symbols. This overview is by no means exhaustive, but gives a flavor for the types of results in this area to date. Of particular note is that questions of hyponormality even of operators with harmonic polynomials as symbols have still not been completely answered, as well as the elusiveness of both necessary and sufficient conditions for hyponormality. In Section 3, we focus on operators with symbols which are not harmonic. We give several sufficient conditions for the hyponormality of certain operators whose symbol is a non-harmonic polynomial, as well as several examples which indicate that the situation is rather subtle. In Section 4, we look at operators whose symbols satisfy

\[ \varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k}, \] where \( m_1 - n_1 = \ldots = m_k - n_k = \delta \geq 0 \). Finally, in Section 5, we show that the norm of the commutator of \( T^* \varphi \) and \( T \varphi \) is bounded by \( 1/2 \) for \( \varphi(z) = z^m \bar{z}^n \) with \( m > n \).

Acknowledgement. Many thanks to D. Khavinson for inspiring discussions, and to C. Cowen for his very helpful correspondence and encouragement.

2. Toeplitz operators with harmonic symbol

The study of hyponormal operators with harmonic symbols is greatly simplified by the lack of cross-terms. In particular, if \( \varphi = f + \bar{g} \) where \( f \) and \( g \) are holomorphic and bounded in \( \mathbb{D} \) then one may show that the cross-term \( 2\text{Re} \left[ \langle T_f u, T_g \rangle - \langle T_{\bar{f}} u, T_{\bar{g}} \rangle \right] \) vanishes. Thus, one can show the hyponormality of \( T_\varphi \) by showing that \( \| H_{\bar{f}} u \|^2 \geq \| H_{\bar{g}} u \|^2 \) for all \( u \) in the Bergman space, where \( H_{\bar{\varphi}} \) is the Hankel operator \( I - T_{\bar{\varphi}} \).

In [13], H. Sadraoui examined the hyponormality of Toeplitz operators \( T_\varphi \) acting on the Bergman space when \( \varphi \) is harmonic. One of his first results, [13, Prop. 1.4.3], gave a necessary boundary condition for \( f \) and \( g \) whenever \( f' \) is in the Hardy space. This result is particularly interesting because in the Bergman space, boundary value results are so rare.

**Theorem.** Let \( f \) and \( g \) be bounded analytic functions, such that \( f' \in H^2 \). If \( T_{f+\bar{g}} \) is hyponormal, then \( g' \in H^2 \) and \( |g'| \leq |f'| \) almost everywhere on \( \mathbb{T} \).

He also showed that this result is sharp, but not in general sufficient. In particular, he proved the following theorem [13, Prop. 1.4.4] for harmonic polynomials.

**Theorem.** Consider the operator \( T_{z^n + \alpha z^m} \).

1. If \( m \leq n \), then \( T_{z^n + \alpha z^m} \) is hyponormal if and only if \( |\alpha| \leq \sqrt{m+1 \over n+1} \).
2. If \( m \geq n \), then \( T_{z^n + \alpha z^m} \) is hyponormal if and only if \( |\alpha| \leq {n \over m} \).

This leads to a host of examples where \( |g'| \leq |f'| \) on \( \mathbb{T} \), but \( T_{f+\bar{g}} \) is not hyponormal. In [11, Theorem 4], P. Ahern and Z. Čučković showed the following result giving another necessary, but not sufficient, condition for the hyponormality of \( T_\varphi \) when \( \varphi \) is harmonic.
Theorem. Suppose $f$ and $g$ are holomorphic in $D$ and $\varphi = f + \bar{g} \in L^\infty(D)$. If $T_\varphi$ is hyponormal then $Tu \geq u$ in $D$ where $u = |f|^2 - |g|^2$.

Using this, they were able to show, as a corollary, a more general version of Sadraoui’s result.

Corollary. Suppose $f$ and $g$ are holomorphic in $D$, that $\varphi = f + \bar{g}$ is bounded in $D$, and that $T_\varphi$ is hyponormal. Then $\lim_{z \to \zeta} \left( |f'(z)|^2 - |g'(z)|^2 \right) \geq 0$ for all $\zeta \in T$. In particular, if $f'$ and $g'$ are continuous at $\zeta \in T$, then $|f'(\zeta)| \geq |g'(\zeta)|$.

Finally, in [5], I.S. Hwang proved the following theorem as part of his study of hyponormal operators whose symbol is a harmonic polynomial. We note here that the condition deals only with the modulus of the coefficients of the given harmonic polynomial.

Theorem. Let $f(z) = a_m z^m + a_n z^n$ and $g(z) = a_{-m} z^m + a_{-n} z^n$, with $0 < m < n$. If $T_{f+\bar{g}}$ is hyponormal and $|a_n| \leq |a_{-n}|$, then we have

$$n^2 \left( |a_{-n}|^2 - |a_n|^2 \right) \leq m^2 \left( |a_m|^2 - |a_{-m}|^2 \right).$$

Work continues to this day on the study of hyponormal Toeplitz operators whose symbol is a harmonic polynomial. It is a testament to the subtlety of the topic that even in this case there is still much to be said about such symbols. Recently, in [5], Z. Ćučković and R. Curto proved the following result.

Theorem. Suppose $T_\varphi$ is hyponormal on $A^2(D)$ with $\varphi(z) = \alpha z^m + \beta z^n + \gamma \bar{z}^p + \delta \bar{z}^q$, where $m < n$ and $p < q$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Assume also that $n - m = q - p$. Then

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\alpha \beta mn - \gamma \delta p q|.$$

Note that in the above Theorems, only the moduli of the coefficients are taken into account. As we shall see in Section 4 this is not necessarily the case when $\varphi$ is not harmonic. We now turn our attention to such operators.

3. Toeplitz operators with non-harmonic symbol

So far, all of these results deal with Toeplitz operators whose symbol is harmonic. The study of operators whose symbol is not harmonic turns out to be more complicated because the cross-terms in equation (1.1) do not vanish.

3.1. Simple non-harmonic symbols. We begin our own investigations by looking at some simple examples. We did not have to look far for some results which we found surprising.

It seemed heuristically plausible that adding a symbol corresponding to a hyponormal Toeplitz operator to a symbol corresponding to a self-adjoint Toeplitz operator should generate a hyponormal Toeplitz operator. But this is not the case.
Example 1. Operator $T_{z+C|z|^2}$ is not hyponormal when $C < -2\sqrt{2}$.

Proof. We verify the statement in Example 1. Let $\varphi_n(z) = \sqrt{\frac{n+1}{n}} z^n$. The collection $\{\varphi_n\}_{n=0}^{\infty}$ is the standard orthonormal basis of $A^2(\mathbb{D})$. Given $u(z) = \sum_{n=0}^{\infty} u_n \varphi_n \in A^2(\mathbb{D})$, where $\{u_n\} \in l^2$ we have that

$$T_z u = \sum_{n=0}^{\infty} \frac{n+1}{n+2} u_n \varphi_{n+1}, \quad \text{and} \quad T_{|z|^2} u = \sum_{n=0}^{\infty} \frac{n+1}{n+2} u_n \varphi_n.$$

Thus, we have that the cross-terms are

$$\text{2Re} \left( \langle T_{|z|^2} T_z u, u \rangle - \langle T_z T_{|z|^2} u, u \rangle \right) = 2\text{Re} \left( \langle T_z u, T_{|z|^2} u \rangle - \langle T_{|z|^2} u, T_z u \rangle \right) = 2\text{Re} \left( \sum_{n=0}^{\infty} \frac{n+1}{n+2} \left( \frac{n+2}{n+3} - \frac{n+1}{n+2} \right) u_n \varphi_{n+1}, \sum_{n=0}^{\infty} u_n \varphi_n \right) \right) = 2\text{Re} \sum_{n=0}^{\infty} \frac{n+1}{n+2} \left( \frac{n+2}{n+3} - \frac{n+1}{n+2} \right) u_n \overline{u}_{n+1}.$$

Now, by [7] and [11] we have

$$\langle T_z u, T_z u \rangle - \langle T_z u, T_z u \rangle \leq \frac{1}{2} \|u\|^2,$$

and since $T_{|z|^2}$ is self adjoint we have

$$\langle T_{|z|^2} u, T_{|z|^2} u \rangle - \langle T_{|z|^2} u, T_{|z|^2} u \rangle = 0.$$

If we then replace $T_{|z|^2}$ with $T_{C|z|^2}$, with real $C$, we have the cross-terms

$$2\text{Re} \left( \langle T_z u, T_{C|z|^2} u \rangle - \langle T_{C|z|^2} u, T_z u \rangle \right) = 2C \text{Re} \sum_{n=0}^{\infty} \frac{n+1}{n+2} \left( \frac{n+2}{n+3} - \frac{n+1}{n+2} \right) u_n \overline{u}_{n+1}.$$

Thus we may choose $u \in A^2(\mathbb{D})$ and $C \in \mathbb{R}$, such that

$$\frac{1}{2} \|u\|^2 + 2C \text{Re} \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n+2}} \left( \frac{n+2}{n+3} - \frac{n+1}{n+2} \right) u_n \overline{u}_{n+1} < 0.$$

For such a choice of $C$ then, operator $T_{z+C|z|^2}$ would not be hyponormal. In particular if we choose $u(z) = \frac{1}{2} \varphi_0 + \frac{1}{2} \varphi_1$, then

$$\left( \left[T_{z+C|z|^2}^*, T_{z+C|z|^2} \right] u, u \right) = \frac{1}{6} + \frac{C}{12\sqrt{2}},$$

which will be negative whenever $C < -2\sqrt{2}$. Thus, whenever we have $C < -2\sqrt{2}$, we have that $T_{z+C|z|^2}$ is not hyponormal. \hfill \Box

At this point it is not known whether $-2\sqrt{2}$ is sharp. Because of the form of the cross-terms, a test function of the form $u(z) = u_0 \varphi_0(z) + u_1 \varphi_1(z)$ of a given norm will have the largest possible contribution to the final value of the self-commutator, however such a function function also will have a relatively large value for $\langle T_z u, T_z u \rangle - \langle T_z u, T_z u \rangle$, since $\langle T_z \varphi_n, T_z \varphi_n \rangle - \langle T_z \varphi_n, T_z \varphi_n \rangle \to 0$ as $n \to \infty$. 

In particular $\langle T_\varphi u, T_\varphi u \rangle - \langle T_\z u, T_\z u \rangle = \frac{1}{2} \|u\|^2$ only for $u = \varphi_0$. Yet the example came as a surprise to us. We had conjectured that the sum of a self-adjoint plus a hyponormal symbol would always correspond to a hyponormal operator, and the above simple counterexample was striking.

**Theorem 2.** Let $\varphi(z) = a_{m,n} z^m \bar{z}^n$, with $m \geq n$ and $a_{m,n} \in \mathbb{C}$. Then $T_\varphi$ is hyponormal.

**Proof.** It is a well known fact (cf. [6, Chapter 2, Lemma 6]) that

$$P(z^m \bar{z}^n) = \begin{cases} \sum_{k=0}^{m-n+1} \frac{m+1}{m+n+1} z^{m-n} & m \geq n \\ 0 & m < n. \end{cases}$$

Thus, if we let $u(z) = \sum_{k=0}^\infty u_k z^k \in A^2(\mathbb{D})$, then we have

$$P(z^m \bar{z}^n u) = \begin{cases} \sum_{k=0}^\infty \frac{m+k-n+1}{m+n+1} u_k z^{m-k} & m \geq n \\ \sum_{k=n-m}^\infty \frac{m+k-n+1}{m+n+1} u_k z^{m-n} & m < n. \end{cases}$$

Taking into account that $T_\varphi^* = T_\varphi$, we find that

$$\langle [T_\varphi^*, T_\varphi] u, u \rangle = \langle T_\varphi u, T_\varphi u \rangle - \langle T_\varphi^* u, T_\varphi^* u \rangle$$

$$= |a_{m,n}|^2 \left( \sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 + \sum_{k=m-n}^\infty \frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \right) |u_k|^2$$

(3.1) $$= |a_{m,n}|^2 \left( \sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 + \sum_{k=m-n}^\infty \left( \frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \right) |u_k|^2 \right)$$

Now,

$$\frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} = \frac{(n+k+1)^2 (m+k-n+1)-(m+k+1)^2 (n+k-m+1)}{(m+k+1)^2(n+k+1)^2}$$

(3.2) $$= \frac{(m^2-n^2)k + (m-n+1)(n+1)^2 + (m-n-1)(n+1)^2}{(m+k+1)^2(n+k+1)^2}.$$ 

This is clearly positive when $k = m-n \geq 1$.

Further, when we take the derivative of the numerator with respect to $k$, we find that it is positive whenever $m > n$, and so the numerator is increasing and thus always positive. Therefore we may conclude that

$$\sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} |u_k|^2 \geq 0$$

for all $u(z) = \sum_{k=0}^\infty u_k z^k \in A^2(\mathbb{D})$, and so $T_\varphi$ is hyponormal. 

$\square$
3.2. Non-harmonic polynomials. We now turn to an examination of two term non-harmonic polynomials.

**Theorem 3.** Suppose \( f = a_{m,n} z^m \bar{z}^n \) and \( g = a_{i,j} z^i \bar{z}^j \), with \( m > n \) and \( i > j \). Then \( T_{f+g} \) is hyponormal if for each \( k \geq 0 \) the term

\[
|a_{m,n}|^2 \frac{m-n+k+1}{(m+k+1)^2} + |a_{i,j}|^2 \frac{i-j+k+1}{(i+k+1)^2}
\]

is sufficiently large.

**Remark.** Here sufficiently large means that, under the assumption \( m-n > i-j \), we have the following four conditions:

\[
|a_{m,n}| \frac{m-k-n+1}{(m+k+1)^2} + |a_{i,j}| \frac{i-k-j+1}{(i+k+1)^2} \geq C_k
\]

for \( k \leq i-j-1 \), and

\[
|a_{m,n}| \left( \frac{m-k-n+1}{(m+k+1)^2} - \frac{n-k-m+1}{(n+k+1)^2} \right) + |a_{i,j}| \left( \frac{i-k-j+1}{(i+k+1)^2} - \frac{j-k-i+1}{(j+k+1)^2} \right) \geq C_k
\]

for \( i-j \leq k \leq m-n-1 \), and

\[
|a_{m,n}| \left( \frac{m-k-n+1}{(m+k+1)^2} - \frac{n-k-m+1}{(n+k+1)^2} \right) + |a_{i,j}| \left( \frac{i-k-j+1}{(i+k+1)^2} - \frac{j-k-i+1}{(j+k+1)^2} \right) \geq C_k + D_k
\]

for \( m-n \leq k \leq m-n+i-j-1 \), and

where

\[
C_k := \begin{cases} \frac{m-n+k+1}{(m+k+1)(m-n+j+k+1)}, & \text{for } 0 \leq k \leq i-j-1, \\ \frac{m-n+k+1}{(m+k+1)(m-n+j+k+1)} - \frac{j-i+k+1}{(j+k+1)(j-i+n+k+1)}, & \text{for } k \geq i-j. \end{cases}
\]

and

\[
D_k := \frac{j-i+k+1}{(j-i+n+k+1)(2j-i+k+1)} - \frac{2j-2i+n-m+k+1}{(2j-i+n-m+k+1)(2j-2i+n+k+1)}.
\]

**Proof.** Recall that for \( f, g \in L^\infty(\mathbb{D}) \), and \( u \in A^2 \), we have

\[
\langle [T_{f+g}, T_{f+g}]u, u \rangle = \|T_f u\|^2 - \|T_g u\|^2 - \|T_{f+g} u\|^2 + 2 \text{Re} \left[ \langle T_f u, T_g u \rangle - \langle T_f^* u, T_g^* u \rangle \right].
\]
We begin to calculate the cross-term \(2\text{Re} \left[ (T_f u, T_g u) - \langle T_f^* u, T_g^* u \rangle \right] \). Without loss of generality, we may assume that \(m - n > i - j \). Under this assumption, we find

\[
2\text{Re} \left[ (T_f u, T_g u) - \langle T_f^* u, T_g^* u \rangle \right] = 2\text{Re} \left( a_{m,n} a_{i,j} \right) \left[ \sum_{k=0}^{\infty} \frac{m + k - n + 1}{m + k + 1} u_k z^{m-k-n} \sum_{k=0}^{\infty} \frac{i + k - j + 1}{i + k + 1} u_{k-i-j} \right]
\]

\[
= 2 \sum_{k=0}^{\infty} C_k \text{Re} \left( a_{m,n} a_{i,j} u_k u_{k+m-n+i-j} \right),
\]

where, for the purposes of slightly less daunting expressions, we used \(C_k\) as defined by (3.3) in the above remark. We will also, for reasons that will soon be clear, use \(D_k\) as defined by (3.4).

Unfortunately, as we have seen, we cannot control the sign of these cross terms. Therefore, we will assume that we must always subtract them. Further, by the inequality \(2\text{Re} \left( \bar{a} \bar{b} \right) \leq |a|^2 + |b|^2\), we have

\[
2\text{Re} \left( a_{m,n} a_{i,j} u_k u_{k+m-n+i-j} \right) \leq |a_{m,n} a_{i,j}| \left( |u_k|^2 + |u_{k+m-n+i-j}|^2 \right).
\]

We combine equation (3.5) with the calculations performed in the proof of Theorem 2 to evaluate

\[
\|T_f u\|^2 - \|T_f^* u\|^2 + \|T_g u\|^2 - \|T_g^* u\|^2
\]

applied to our given \(f \) and \(g \). Thereby we may conclude that \(T_f \) will be hyponormal if

\[
|a_{m,n}|^2 \left( \sum_{k=0}^{m-n-1} \frac{m + k - n + 1}{(m + k + 1)^2} |u_k|^2 + \sum_{k=m-n}^{\infty} \frac{m + k - n + 1}{(m + k + 1)^2} - \frac{n + k - m + 1}{(n + k + 1)^2} |u_k|^2 \right)
\]

\[
+ |a_{i,j}|^2 \left( \sum_{k=0}^{i-j-1} \frac{i + k - j + 1}{(i + k + 1)^2} |u_k|^2 + \sum_{k=i-j}^{\infty} \frac{i + k - j + 1}{(i + k + 1)^2} - \frac{j + k - i + 1}{(j + k + 1)^2} |u_k|^2 \right)
\]

\[
\geq |a_{m,n} a_{i,j}| \sum_{k=0}^{\infty} C_k \left( |u_k|^2 + |u_{k+m-n+i-j}|^2 \right)
\]

\[
= \sum_{k=0}^{\infty} C_k |u_k|^2 + \sum_{k=m-n+i-j}^{\infty} D_k |u_k|^2.
\]

Thus, an appropriate term by term comparison of the coefficients of \(|u_k|^2\) will show that operator \(T_{f+g}\) is hyponormal, if the bounds given in the above remark hold.

In particular, we obtain the stronger estimate

\[
\langle \langle T_{f+g} f, T_{f+g} g \rangle, u \rangle \geq \sum_{k=0}^{\infty} A_k |u_k|^2,
\]
where $A_k$ is non-negative for all $k$.

The next theorem examines the case when one of the terms in our binomial is the symbol of a cohyponormal operator (i.e. an operator whose adjoint is hyponormal). This is in contrast to Theorem 3 where each term individually yielded a hyponormal operator.

**Theorem 4.** Suppose $f = a_{m,n}z^m\bar{z}^n$ and $g = a_{i,j}z^i\bar{z}^j$, with $m > n$ and $i > j$. Then $T_{f+g}$ is hyponormal if for each $k \geq 0$

$$|a_{m,n}|^2 \frac{m-n+k+1}{(m+k+1)^2} - |a_{i,j}|^2 \frac{i-j+k+1}{(i+k+1)^2}$$

is sufficiently large.

**Remark 5.** Here, as in Theorem 3, we can specify what sufficiently large means. To do so, we abbreviate

$$\tilde{A}_k := |a_{m,n}|^2 \left( \frac{m+k-n+1}{(m+k+1)^2} - \frac{n+k-m+1}{(n+k+1)^2} \right),$$

and

$$\tilde{B}_k := |a_{i,j}|^2 \left( \frac{i+k-j+1}{(i+k+1)^2} - \frac{j+k-i+1}{(j+k+1)^2} \right),$$

as well as

$$\tilde{C}_k := \frac{m-n+k+1}{(m+k+1)(m-n+i+k+1)} - \frac{i-j+k+1}{(i+k+1)(i-j+m+k+1)},$$

and

$$\tilde{D}_k := \frac{k+j-i+1}{(n-j+i+k+1)(j+k+1)} - \frac{k+n-m+1}{(m-n+j+k+1)(n+k+1)}.$$

Now sufficiently large means that the following four conditions are satisfied:

$$\left| \frac{a_{m,n}}{a_{i,j}} \right| \frac{m+k-n+1}{(m+k+1)^2} - \left| \frac{a_{i,j}}{a_{m,n}} \right| \frac{i+k-j+1}{(i+k+1)^2} \geq \tilde{C}_k$$

for $k \leq \min\{m-n,i-j\} - 1$, and

$$\tilde{C}_k \leq \begin{cases} \left| \frac{a_{m,n}}{a_{i,j}} \right| \frac{m+k-n+1}{(m+k+1)^2} - \frac{\tilde{B}_k}{|a_{m,n}a_{i,j}|} & \text{when } m-n > i-j \\ \frac{\tilde{A}_k}{|a_{m,n}a_{i,j}|} - \left| \frac{a_{i,j}}{a_{m,n}} \right| \frac{i+k-j+1}{(i+k+1)^2} & \text{when } m-n < i-j \end{cases}$$

for $\min\{m-n,i-j\} \leq k \leq \max\{m-n,i-j\} - 1$, and

$$\tilde{A}_k - \tilde{B}_k \geq \tilde{C}_k$$

for $\max\{m-n,i-j\} \leq k \leq m-n+i-j - 1$, and

$$\tilde{A}_k - \tilde{B}_k \geq \tilde{C}_k + \tilde{D}_k$$

for $k \geq m-n+i-j$. 
Proof. Recall that for $f,g \in L^\infty(D)$, and $u \in A^2(D)$, we have

$$\langle [T^*_f g, T_f u], u \rangle = \|T_f u\|^2 - \|T^*_f u\|^2 + \|T_g u\|^2 - \|T^*_g u\|^2 + 2\text{Re} \left[ \langle T_f u, T_g u \rangle - \langle T^*_f u, T^*_g u \rangle \right].$$

Again, the calculations performed in the proof of Theorem 2 applied to the current $f$ and $g$ show

$$\|T_f u\|^2 - \|T^*_f u\|^2 + \|T_g u\|^2 - \|T^*_g u\|^2 = \left| a_{m,n} \right|^2 m - n - 1 \sum_{k=0}^{m-n-1} \frac{m+k-n+1}{(m+k+1)^2} u_k^2 + \sum_{k=m-n}^\infty \tilde{A}_k |u_k|^2 \right.$$  

(3.10)

$$- \left| a_{i,j} \right|^2 \sum_{k=0}^{i-j-1} \frac{i+k-j+1}{(i+k+1)^2} u_k^2 - \sum_{k=i-j}^{m-n} \tilde{B}_k |u_k|^2,$$

where we used $\tilde{A}_k$ and $\tilde{B}_k$ as defined in (3.6) and (3.7).

This proof differs from that of Theorem 3 in the calculation for the cross-terms. Under the assumption $i > j$, we have

$$2\text{Re} \left[ \langle T_f u, T_g u \rangle - \langle T^*_f u, T^*_g u \rangle \right] = 2\text{Re} \left( a_{m,n} \overline{a_{i,j}} \right) \left[ \sum_{k=0}^{m-n-1} \frac{m+k-n+1}{m+k+1} u_k z^{m+k-n} \sum_{k=i-j}^{j+k-i} \frac{j+k-i+1}{j+k+1} u_k z^{j+k-i} \right]$$  

$$- \left[ \sum_{k=m-n}^{i-j-1} \frac{i+k-j+1}{i+k+1} u_k z^{i+k-j} \right].$$

(3.12)

via direct calculation and with $\tilde{C}_k$ from (3.8).

The argument now follows mutatis mutandis as in Theorem 3. In particular, with $\tilde{D}_k$ from (3.9) and once again taking advantage of the inequality $2\text{Re} (\overline{ab}) \leq |a|^2 + |b|^2$, we have that if the conditions given in Remark 5 hold, then operator $T_{f+g}$ will be hyponormal. □

Both of the above theorems are rather cumbersome to apply directly. Further, it is not immediately clear a priori that the relevant bounds are ever actually attainable. In the following example we look at a symbol which shows that the bounds in Theorem 4 can be attained. This shows that while a seemingly “nice” symbol like $T_{z-3|z|^2}$ might fail to be hyponormal even though it is the sum of a subnormal operator and a self-adjoint operator, the sum of a hyponormal and co-hyponormal operator might still produce an operator which is hyponormal.
Example 6. Consider $\varphi(z) = z^2 + \frac{1}{7}z^4z^3$. We can plug this into the relevant calculations from Theorem 4 to test for hyponormality. In particular, we find that

$$\tilde{A}_k = \frac{3k + 8}{(k + 3)^2(k + 2)^2},$$

and that

$$\tilde{B}_k + |a_{m,n}a_{i,j}| \left( \tilde{C}_k + \tilde{D}_k \right)$$

$$= \frac{1}{7} \left( \frac{7k + 32}{(k + 5)^2(k + 4)^2} + \frac{3k^3 + 21k^2 + 46k + 8}{(k + 6)(k + 5)(k + 4)(k + 3)(k + 2)(k + 1)} \right).$$

Thus, we find that $T_{\varphi}$ will be hyponormal if

$$\tilde{A}_k - \tilde{B}_k - |a_{m,n}a_{i,j}| \left( \tilde{C}_k + \tilde{D}_k \right)$$

$$= \frac{3k + 8}{(k + 3)^2(k + 2)^2} - \frac{1}{7} \left( \frac{7k + 32}{(k + 5)^2(k + 4)^2} + \frac{3k^3 + 21k^2 + 46k + 8}{(k + 6)(k + 5)(k + 4)(k + 3)(k + 2)(k + 1)} \right)$$

$$= \frac{119k^7 + 3475k^6 + 41785k^5 + 267977k^4 + 985764k^3 + 2061168k^2 + 2228760k + 927168}{49(k + 6)(k + 5)(k + 4)^2(k + 3)^2(k + 2)^2(k + 1)} > 0$$

for all $k \geq 2$, since the checks for $k = 0, 1$ show the desired inequalities hold.

However in fact, it is clear from observation that this rational function is positive for all $k > 0$, and in particular for $k \geq 2$. Thus $T_{\varphi}$ is hyponormal. This example will be explored more in depth in Theorem 7.

Note however that for our choice of $\varphi$, since we have that the expression in (3.13) is less than $\frac{3k + 8}{(k + 3)^2(k + 2)^2}$ for all $k > 0$, only one check was actually necessary to show that $T_{\varphi}$ is hyponormal. Indeed the construction of this example was based on ensuring a sufficiently quick decay of the expression in (3.13) while also ensuring that for small values of $k$ the required inequalities would still hold. In the following theorem, we generalize the idea of this construction to find a general construction for hyponormal operators whose symbol is of the form in the hypothesis of Theorem 4.

Theorem 7. Fix $\delta \in \mathbb{N}$. For every integer $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$, such that $T_{\varphi}$ with symbol $\varphi(z) = z^n + \delta z^n + \frac{1}{2j+\bar{3}}z^{j+\delta}z^j$ is hyponormal.

Proof. The idea of the proof lies in specifying the words “sufficiently large” in Theorem 4 in accordance with Remark 5.

We let $m = n + \delta$ and $i = j + \delta$. Since $m - n = i - j = \delta$, the formulas from Remark 5 become somewhat simplified. Recall that we use $\tilde{A}_k = |a_{m,n}|^2 \left( \frac{m+n+i+1}{(m+n+i+1)^2} - \frac{n+k-m+1}{(n+k)^2} \right)$, as well as $\tilde{B}_k = |a_{i,j}|^2 \left( \frac{i+k+j+1}{(i+k+j+1)^2} - \frac{j+k-1+1}{(j+k+1)^2} \right)$. In particular, for $k \geq \delta$ and with $a_{m,n} = 1$, we can arrive at

$$\tilde{A}_k = \frac{(m + n)\delta k + (\delta + 1)(n + 1)^2 + (\delta - 1)(m + 1)^2}{(k + m + 1)^2(k + n + 1)^2}. $$


And, with $a_{i,j} = \frac{1}{i+j}$, we obtain
\[
\tilde{B}_k = \frac{(i+j) \delta + (\delta +1)(j+1)^2 + (\delta -1)(j+1)^2}{(i+j)^2(k+i+1)^2(k+j+1)^2}.
\]
Finally, we have
\[
\tilde{C}_k = \frac{\delta(i-m)(k+\delta+1)}{(k+m+1)(k+m+\delta+1)(k+i+1)(k+i+\delta+1)}, \quad \text{and}
\]
\[
\tilde{D}_k = \frac{\delta(i-m)(k-\delta+1)}{(k+m+1)(k+n-\delta+1)(k+j+1)(k+j-\delta+1)}.
\]

Recall that our aim is now to prove that for $k \geq 2\delta$ we have
\[
(i+j)\tilde{A}_k \geq (i+j)\tilde{B}_k + \tilde{C}_k + \tilde{D}_k,
\]
since $a_{i,j} = \frac{1}{i+j}$. This is a direct application of the bounds given in Theorem 4.

Our goal will be to prove that the numerator of $(i+j)\tilde{A}_k$ is larger than the sums of the numerators of $(i+j)\tilde{B}_k$, $\tilde{C}_k$, and $\tilde{D}_k$, while ensuring that the denominator of $\tilde{A}_k$ is smaller than each of the denominators of $(i+j)\tilde{B}_k$, $\tilde{C}_k$, and $\tilde{D}_k$. If we can show this we will have shown that (3.14) holds for all $k \geq 2\delta$, and in fact, the other required bounds of Theorem 4 will also necessarily follow immediately, guaranteeing the hyponormality of $T_\varphi$.

Looking first at the numerators then, we first wish to show
\[
(i+j)(m+n)\delta k \geq (2i-2m+1)k,
\]
for all $k \geq 2\delta$. Yet since clearly $(i+j)(m+n)\delta > (2i-2m+1)$, we have that (3.15) holds for all $k \geq 0$. Looking at the constant terms of the numerators, and multiplying through by $(i+j)$ to prevent a fraction in the constant term of the numerator $\tilde{B}_k$, it is equally clear that
\[
(i+j)^2 [(\delta +1)(n+1)^2 + (\delta -1)(m+1)^2] \geq (\delta +1)(j+1)^2 + (\delta -1)(j+1)^2 + 2\delta(i-m)(i+j),
\]
since the inequality
\[
(i+j)^2 [(\delta +1) + (\delta -1)] \geq (\delta +1)(j+1)^2 + (\delta -1)(j+1)^2,
\]
and the inequality
\[
(i+j)^2 ([(\delta +1)(n^2 + 2n) + (\delta -1)(m^2 + 2m)] \geq 2\delta(i-m)(i+j)
\]
both hold by inspection. So we have that the numerator of $\tilde{A}_k$ is larger than the sums of the numerators of $(i+j)\tilde{B}_k$, $\tilde{C}_k$, and $\tilde{D}_k$, as desired.

It remains to show our desired inequalities for the denominators. It is clear by inspection that if $j > m$, then we have that
\[
(k+m+1)^2(k+n+1)^2 \leq (k+i+1)^2(k+j+1)^2
\]
and 

\[(k + m + 1)^2 (k + n + 1)^2 \leq (k + m + 1)(k + m + \delta + 1)(k + i + 1)(k + i + \delta + 1).\]

We take a moment to show that it is possible to choose \(j\) large enough so that

\[
(k + m + 1)^2 (k + n + 1)^2 \leq (k + m + 1)(k + n - \delta + 1)(k + j + 1)(k + j - \delta + 1)
\]

for all \(k \geq 2\delta\). Since we have already assumed that \(j > m\), we have that \(j - \delta > n\), and thus (3.17) follows so long as

\[
(k + m + 1)(k + n + 1) < (k + n - \delta + 1)(k + j + 1).
\]

Or equivalently, since \(k \geq 2\delta\), inequality (3.17) follows so long as

\[
j > \frac{(k + m + 1)(k + n + 1)}{k + n - \delta + 1} - k - 1 = \frac{k(m + \delta) + mn + m + \delta}{k + n - \delta + 1} =: q(k).
\]

Since the rational function \(q(k)\) remains bounded for \(k \in [2\delta, \infty)\), it is possible to choose an appropriate \(j \in \mathbb{N}\). Thus (3.14) holds for all \(k \geq 2\delta\).

Thus, by Theorem 4, operator \(T_\varphi\) is hyponormal. \(\square\)

### 4. Polynomials of Fixed Relative Degree

We now turn to operators whose symbol is a polynomial of the form

\[
\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k}, \quad \text{with } m_1 - n_1 = \ldots = m_k - n_k = \delta \geq 0.
\]

We shall call these polynomials of fixed relative degree. Though working with non-harmonic symbols can be difficult, some results are known in these special cases. One which we will be interested in for this paper is due to Y. Liu and C. Lu in [9, Theorem 3.1]. There they make use of the Mellin transform of \(\varphi\).

**Definition.** Suppose \(\varphi \in L^1([0, 1], r\, dr)\). For \(\Re z \geq 2\), the *Mellin transform* of \(\varphi\), is given by

\[
\widehat{\varphi}(z) := \int_0^1 \varphi(x) x^{z-1} dx.
\]

For \(\varphi(re^{i\theta}) = e^{ik\theta} \varphi_0(r)\), with \(k \in \mathbb{Z}\) and \(\varphi_0\) radial, we can compute the action of \(T_\varphi\) on \(z^n\). Specifically,

\[
T_\varphi z^n = \begin{cases} 
2 (n + k + 1) \widehat{\varphi_0}(2n + k + 2) z^{n+k} & n + k \geq 0 \\
0 & n + k < 0,
\end{cases}
\]

and

\[
T_{\bar{\varphi}} z^n = \begin{cases} 
2 (n - k + 1) \widehat{\varphi_0}(2n - k + 2) z^{n-k} & n - k \geq 0 \\
0 & n - k < 0.
\end{cases}
\]
Using this, Y. Liu and C. Lu proved the following theorem in [9, Theorem 3.1].

**Theorem 8.** Let \( \varphi(re^{i\theta}) = e^{i\delta \theta} \varphi_0(r) \in L^\infty(D) \), where \( \delta \in \mathbb{Z} \) and \( \varphi_0 \) is radial. Then \( T_\varphi \) is hyponormal if and only if one of the following conditions holds:

1) \( \delta = 0 \) and \( \varphi_0 \equiv 0 \);
2) \( \delta = 0 \);
3) \( \delta > 0 \) and for each \( \alpha \geq \delta \),

\[
|\hat{\varphi}_0(2\alpha + \delta + 2)| \geq c_{\alpha,\delta} |\hat{\varphi}_0(2\alpha - \delta + 2)|,
\]

where we abbreviate
\[
c_{\alpha,\delta} := \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}}.
\]

The first situation immediately implies that if \( \varphi(z) \) is a polynomial in \( z \) and \( \bar{z} \) where the degree of \( \bar{z} \) is larger than the degree of \( z \) in each term, then \( T_\varphi \) cannot be hyponormal. The second situation is a consequence of the fact that whenever \( \varphi \) is real valued in \( D \), then \( T_\varphi \) is actually self-adjoint and thus trivially hyponormal. The final situation, when \( \delta > 0 \), will be of interest to us.

**Remark.** One can prove Theorem 2 by applying Theorem 8, however the proof is non-trivial.

The following is a corollary of Theorem 13. However, a direct proof is simple enough that we showcase it here for the convenience of the reader.

**Corollary 9.** Let \( \varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k} \), with \( m_1 - n_1 = \ldots = m_k - n_k = \delta \geq 0 \), and \( a_i \) all lying along the same ray for \( 1 \leq i \leq k \) (i.e. \( \arg(a_1) = \ldots = \arg(a_k) \)), then \( T_\varphi \) is hyponormal.

**Proof.** Write \( \varphi = \varphi_1 + \ldots + \varphi_k \), where \( \varphi_i = a_i e^{i\delta \theta} \hat{\varphi}_{0,i} \). Recall that \( c_{\alpha,\delta} = \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}} \). By Theorem 8 and Theorem 2 we have that for each \( \alpha \geq \delta \)

\[
|a_i \hat{\varphi}_{0,i}(2\alpha + \delta + 2)| \geq c_{\alpha,\delta} |a_i \hat{\varphi}_{0,i}(2\alpha - \delta + 2)|.
\]

Since the \( a_i \)'s all lie along the same ray, we have that for each \( n \geq \delta \)

\[
\left| \sum_{i=1}^k a_i \hat{\varphi}_{0,i}(2\alpha + \delta + 2) \right| = \sum_{i=1}^k |a_i| |\hat{\varphi}_{0,i}(2\alpha + \delta + 2)| \geq \sum_{i=1}^k c_{\alpha,\delta} |a_i| |\hat{\varphi}_{0,i}(2\alpha - \delta + 2)|.
\]

The claim now follows by Theorem 8. \( \square \)

One is tempted to conjecture that the argument of these coefficients should not matter. However the following example shows that this is not the case.

**Example 10.** Let \( \varphi(z) = z^2 \bar{z} - z^3 \bar{z}^2 \). Then \( \hat{\varphi}_0(k) = \frac{1}{k+3} - \frac{1}{k+5} \), and we find that

\[
\frac{1}{2\alpha + 6} - \frac{1}{2\alpha + 8} < \sqrt{\frac{\alpha}{\alpha + 2} \left( \frac{1}{2\alpha + 4} - \frac{1}{2\alpha + 6} \right)},
\]

whenever \( \alpha \geq 2 \). This violates the conditions of Theorem 8 and so \( T_\varphi \) cannot be hyponormal.
So, can we find sufficient conditions, beyond all coefficients lying along the same ray, to guarantee that such functions yield hyponormal operators? The answer is yes, and depends somewhat on the number of terms, as well as the relative position of the coefficients, as the following two theorems demonstrate.

**Theorem 11.** Let \( \varphi(z) = a_1 z^m \bar{z}^n + a_2 z^i \bar{z}^j \), with \( m - n = i - j = \delta \geq 0 \). Then \( T_\varphi \) is hyponormal if \( a_1 \) and \( a_2 \) lie in the same quarter-plane (i.e. \( |\arg(a_1) - \arg(a_2)| \leq \frac{\pi}{2} \)). Further, under the additional condition that

\[
0 \leq \frac{|a_1|}{\alpha + m + 1} - \frac{|a_2|}{\alpha + i + 1} < c^2_{\alpha, \delta} \left( \frac{|a_1|}{\alpha + n + 1} - \frac{|a_2|}{\alpha + j + 1} \right) \quad \text{for all } \alpha,
\]

the requirement that \( |\arg(a_1) - \arg(a_2)| \leq \frac{\pi}{2} \) is also necessary for the hyponormality of \( T_\varphi \).

**Proof.** We begin with some general observations. Without loss of generality, we may assume that \( a_1 \) is a positive real number and that \( a_2 = r_2 e^{\theta i} \) with \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). We have \( \hat{\varphi}_0(k) = a_1^m + a_2^i + k \).

Recall that \( c_{\alpha, \delta} = \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}} \). By Theorem 8, \( T_\varphi \) will be hyponormal if and only if

\[
|\hat{\varphi}_0(2\alpha + \delta + 2)|^2 \geq c^2_{\alpha, \delta} |\hat{\varphi}_0(2\alpha - \delta + 2)|^2,
\]

which is equivalent to

\[
\left| \frac{a_1}{\alpha + m + 1} + \frac{a_2}{\alpha + i + 1} \right|^2 \geq c^2_{\alpha, \delta} \left| \frac{a_1}{\alpha + n + 1} + \frac{a_2}{\alpha + j + 1} \right|^2,
\]

as well as to

\[
\left( \frac{a_1}{\alpha + m + 1} + \frac{r_2 \cos \theta}{\alpha + i + 1} \right)^2 + \frac{r_2^2 \sin^2 \theta}{\alpha + i + 1} \geq c^2_{\alpha, \delta} \left[ \left( \frac{a_1}{\alpha + n + 1} + \frac{r_2 \cos \theta}{\alpha + j + 1} \right)^2 + \frac{r_2^2 \sin^2 \theta}{(\alpha + j + 1)^2} \right],
\]

for all \( \alpha \geq \delta \).

Let us focus on proving the first statement. By the hypothesis that \( i = \delta + j \), we can verify

\[
\frac{r_2^2 \sin^2 \theta}{(\alpha + i + 1)^2} \geq c^2_{\alpha, \delta} \frac{r_2^2 \sin^2 \theta}{(\alpha + j + 1)^2}
\]

for all \( \alpha \geq \delta \). Similarly,

\[
\left( \frac{a_1}{\alpha + m + 1} + \frac{r_2 \cos \theta}{\alpha + i + 1} \right)^2 \geq c^2_{\alpha, \delta} \left( \frac{a_1}{\alpha + n + 1} + \frac{r_2 \cos \theta}{\alpha + j + 1} \right)^2
\]

so long as \( \cos \theta \geq 0 \). That is, when \( a_2 \) is in the closed right half-plane. Thus, it follows that when \( |\arg(a_1) - \arg(a_2)| \leq \frac{\pi}{2} \), then the estimate in equation (4.2) holds for all \( \alpha \geq \delta \). And so \( T_\varphi \) is hyponormal by Theorem 8.
To show the converse, we assume the extra condition (4.1). We will show that if \( \frac{\pi}{2} < \theta \leq \pi \), then there exists an \( \alpha \) for which (4.2) fails, and consequently \( T_{\varphi} \) must fail to be hyponormal by Theorem 8.

First, fix \( \alpha \geq \delta \). We construct two circles.

\[
C_1 := \left\{ z : \left| z - \frac{a_1}{\alpha + m + 1} \right| = \frac{r_2}{\alpha + i + 1} \right\},
\]

centered at \( \frac{a_1}{\alpha + m + 1} \) with radius \( \frac{r_2}{\alpha + i + 1} \), and

\[
C_2 := \left\{ z : \left| z - c_{\alpha, \delta}^2 \frac{a_1}{\alpha + n + 1} \right| = c_{\alpha, \delta}^2 \frac{r_2}{\alpha + j + 1} \right\},
\]

centered at \( c_{\alpha, \delta}^2 \frac{a_1}{\alpha + n + 1} \) with radius \( c_{\alpha, \delta}^2 \frac{r_2}{\alpha + j + 1} \). Without loss of generality, we may always assume that both of these circles lie in the right half-plane.

So long as the difference of their centers is bounded by the difference of their radii, i.e.

\[
\frac{a_1}{\alpha + m + 1} - c_{\alpha, \delta}^2 \frac{a_1}{\alpha + n + 1} < \frac{r_2}{\alpha + i + 1} - c_{\alpha, \delta}^2 \frac{r_2}{\alpha + j + 1},
\]

we have that \( C_2 \) lies completely in the region bounded by \( C_2 \). Such a scenario is illustrated in Figure 4.1 for one value of \( \alpha = 6 \).

![Figure 4.1](image_url)

**Figure 4.1.** The situation when \( \alpha = 6, m = 5, i = 9, \) and \( \delta = 4 \).

In this case, it is clear that there exists a \( \frac{\pi}{2} < \theta < \pi \) such that

\[
(4.3) \quad \left( \frac{a_1}{\alpha + m + 1} + \frac{r_2 \cos \theta}{\alpha + i + 1} \right)^2 - c_{\alpha, \delta}^2 \left( \frac{a_1}{\alpha + n + 1} + \frac{r_2 \cos \theta}{\alpha + j + 1} \right)^2 = 0.
\]
Then if \( \theta \to \pi \), the left hand side of (4.3) will converge to a negative real number by condition (4.1).

At the same time, since

\[
\lim_{\theta \to \pi} \left( \frac{r_2^2 \sin^2 \theta}{(\alpha + i + 1)^2} - \frac{r_2^2 \sin^2 \theta}{(\alpha + j + 1)^2} \right) = 0,
\]

there exists some \( \theta \) for which (4.2) fails. Define

\[
\theta_\alpha := \inf \{ \theta : \text{equation (4.2) fails} \}.
\]

We will now show that \( \theta_\alpha \to \pi/2 \) as \( \alpha \to \infty \). Define

\[
F_\alpha(\theta) := \left( \frac{a_1}{\alpha + m + 1} + \frac{r_2 \cos \theta}{\alpha + i + 1} \right)^2 + \frac{r_2^2 \sin^2 \theta}{(\alpha + i + 1)^2} \left[ \frac{a_1}{\alpha + n + 1} + \frac{r_2 \cos \theta}{\alpha + j + 1} \right]^2 + \frac{r_2^2 \sin^2 \theta}{(\alpha + j + 1)^2}.
\]

As shown above, there exists a \( \theta \) such that \( F_\alpha(\theta) = 0 \). It must be the case that \( \theta = \theta_\alpha \) is a root, since \( F_\alpha(\theta) > 0 \) for \( \theta < \theta_\alpha \), and since \( F_\alpha(\theta) < 0 \) for \( \theta > \theta_\alpha \). Solving for this \( \theta_\alpha \), we find that

\[
\theta_\alpha = \arccos \left( \frac{\mathcal{O}(\alpha^5)}{2r_2(1 + r_2^2)\alpha^7} \right).
\]

Since

\[
\frac{\mathcal{O}(\alpha^5)}{2r_2(1 + r_2^2)\alpha^7} \to 0 \quad \text{as} \quad \alpha \to \infty,
\]

this means that \( \theta_\alpha \to \pi/2 \) when \( \alpha \to \infty \), as claimed. In particular, this shows that for all \( \pi/2 < \theta \leq \pi \), there exists an \( \alpha \) for which \( F_\alpha(\theta) < 0 \). For such \( \theta \) then, the Toeplitz operator with the symbol \( a_1 z^m \bar{z}^n + r_2 e^{i\theta} z^i \bar{z}^j \) is not hyponormal.

The next example will demonstrate that the extra conditions we used for necessity in Theorem 11 cannot be completely dropped.

Example 12. Let \( \varphi_\theta(z) = z^2 \bar{z} + \frac{1}{10} e^{i\theta} z^3 \bar{z}^2 \). Here again, thinking in terms of two circles as in the proof of Theorem 11, we see that in this case the interiors of the two circles are disjoint for small \( \alpha \) as shown in Figure 4.2.
And indeed, as $\alpha \to \infty$, and these two circles come together, we find that $$\left( \frac{1}{\alpha + 3} + \frac{\cos \theta}{10(\alpha + 4)} \right)^2 + \frac{\sin^2 \theta}{100(\alpha + 4)^2} - c_{\alpha,\delta}^2 \left[ \left( \frac{1}{\alpha + 2} + \frac{\cos \theta}{10(\alpha + 3)} \right)^2 + \frac{\sin^2 \theta}{100(\alpha + 3)^2} \right] > 0$$ for all $\theta \in [0, \pi]$ and all $\alpha \geq 1$. Thus, the Toeplitz operator with symbol $\varphi_\theta$ is hyponormal for all choices of $\theta$.

The next theorem improves slightly on the conditions of Corollary 9.

**Theorem 13.** Let $\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \cdots + a_k z^{m_k} \bar{z}^{n_k}$, with $m_1 - n_1 = \cdots = m_k - n_k = \delta \geq 0$, and $a_i$ all lying in the same quarter-plane $1 \leq i \leq k$ (i.e. $\max_{1 \leq i,j \leq k} |\arg(a_i) - \arg(a_j)| \leq \frac{\pi}{2}$), then $T_\varphi$ is hyponormal.

**Proof.** The proof follows *mutatis mutandis* the proof of Theorem 11. Recall again that $c_{\alpha,\delta} = \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}}$.

We assume without loss of generality that $a_1$ is a positive real number with largest modulus among the $a_i$'s, and with $a_j = r_j e^{i\theta_j}$ for $2 \leq j \leq k$. The only other change is that instead of condition (4.2), we have hyponormality if and only if

\[
(4.4) \quad \left( \frac{a_1}{\alpha + m_1 + 1} + \sum_{i=2}^{k} \frac{r_i \cos \theta_i}{\alpha + m_i + 1} \right)^2 + \left( \sum_{i=2}^{k} \frac{r_i \sin \theta_i}{\alpha + n_i + 1} \right)^2 \geq c_{\alpha,\delta}^2 \left( \frac{a_1}{\alpha + n_1 + 1} + \sum_{i=2}^{k} \frac{r_i \cos \theta_i}{\alpha + n_i + 1} \right)^2,
\]

for all $\alpha \geq \delta$.

Thus, in addition to needing all $a_i$ in the right-half plane to guarantee

\[
\left( \frac{a_1}{\alpha + m_1 + 1} + \sum_{i=2}^{k} \frac{r_i \cos \theta_i}{\alpha + m_i + 1} \right)^2 \geq c_{\alpha,\delta}^2 \left( \frac{a_1}{\alpha + n_1 + 1} + \sum_{i=2}^{k} \frac{r_i \cos \theta_i}{\alpha + n_i + 1} \right)^2,
\]

for all $\alpha \geq \delta$, we also need all $a_i$ in the upper half-plane to guarantee

\[
\left( \sum_{i=2}^{k} \frac{r_i \sin \theta_i}{\alpha + m_i + 1} \right)^2 \geq c_{\alpha,\delta}^2 \left( \sum_{i=2}^{k} \frac{r_i \sin \theta_i}{\alpha + n_i + 1} \right)^2.
\]

Thus, as long as $0 \leq \theta_i \leq \frac{\pi}{2}$, for all $i$, $T_\varphi$ is hyponormal.

By rotation, it is sufficient to have $\max_{1 \leq i,j \leq k} |\arg(a_i) - \arg(a_j)| \leq \frac{\pi}{2}$. \qed

It is not known whether or not this condition is necessary. It may be possible *a priori* to construct $\varphi$ in such a way that condition (4.4) holds, while allowing one of the $a_i$ to be outside the given quarter-plane. We expect that the techniques of Example 12 can be modified to yield the desired outcome.
5. Some spectral estimates

We discuss some question of the spectral properties of hyponormal $T_\varphi$. By Putnam’s inequality, and the spectral mapping theorem, we know that when $\varphi$ is analytic we have that

$$\|[T^*_\varphi, T_\varphi]\| \leq \frac{\text{Area}(\varphi(\mathbb{D}))}{\pi}.$$ 

However, in [11], it was shown that if $\varphi$ is also univalent in $\mathbb{D}$, that

$$\|[T^*_\varphi, T_\varphi]\| \leq \frac{\text{Area}(\varphi(\mathbb{D}))}{2\pi},$$

and it is a standing conjecture that the univalent condition can be dropped. Evidence for this conjecture was given in [7] where it was showed that

$$\|[T^*_z, T_z]\| = \frac{1}{2}.$$ 

An examination of Theorem 2 lets us show the following.

**Theorem 14.** Let $\varphi(z) = z^m \bar{z}^n$ with $m > n$. Then

$$\|[T^*_{z^n \bar{z}^m}, T_{z^n \bar{z}^m}]\| \leq \frac{1}{2}.$$

**Proof.** Recall that by Theorem 2, we have that $T_{z^n \bar{z}^m}$ is hyponormal. We will write here $u(z) = \sum_{k=0}^{\infty} u_k \varphi_k(z)$, where $\varphi_k(z) = \sqrt{\frac{n+1}{\pi} z^k}$, so that $\{\varphi_k\}_{k=0}^{\infty}$ is the standard orthonormal basis of $A^2(\mathbb{D})$, and $\|u\|^2_{A^2(\mathbb{D})} = \sum |u_k|^2$. We obtain

$$\|[T^*_{z^n \bar{z}^m}, T_{z^n \bar{z}^m}]\| = \sup_{u \in A^2(\mathbb{D}), \|u\| = 1} \langle [T^*_{z^n \bar{z}^m}, T_{z^n \bar{z}^m}]u, u \rangle$$

$$= \sum_{k=0}^{m-n-1} \frac{(k + m - n + 1)(k + 1)}{(k + m + 1)^2} |u_k|^2$$

$$+ \sum_{k=m-n}^{\infty} \left( \frac{k + m - n + 1}{(k + m + 1)^2} - \frac{k + n - m + 1}{(k + n + 1)^2} \right)(k + 1) |u_k|^2.$$

The goal will be to show

(5.1) $$\frac{(k + m - n + 1)(k + 1)}{(k + m + 1)^2} \leq \frac{1}{2}$$

for all $0 \leq k \leq m - n - 1$, and that

(5.2) $$\left( \frac{k + m - n + 1}{(k + m + 1)^2} - \frac{k + n - m + 1}{(k + n + 1)^2} \right)(k + 1) \leq \frac{1}{2}.$$
for all \( k \geq m - n \). If we accomplish these two estimates \[(5.1)\] and \[(5.2)\], then we will have

\[
\sup_{u \in A^2(D)} \langle [T^*_{z^m\bar{z}^n}, T_{z^m\bar{z}^n}]u, u \rangle \leq \frac{1}{2} \sum_{k=0}^{\infty} |u_k|^2,
\]

and the result will be proved.

To this end, we look first at the case where \( 0 \leq k \leq m - n - 1 \). Differentiating \( \frac{(k+m-n+1)(k+1)}{(k+m+1)^2} \) with respect to \( k \), we find this derivative is

\[
\frac{(k + m + 1)^2 (2k + 2 + m - n) - 2 (k + m + 1) (k + m - n + 1) (k + 1)}{(k + m + 1)^4}.
\]

Since \( (k + m + 1)^2 \geq (k + m + 1) (k + m - n + 1) \), and \( (2k + 2 + m - n) \geq 2 (k + 1) \), we have that this derivative is positive for all \( k \geq 0 \). Thus it suffices to show that \[(5.1)\] holds when \( k = m - n - 1 \).

In this case we have

\[
\frac{2 (m - n)^2}{(2m - n)^2}.
\]

But then since \( (2m - n)^2 = (m + m - n)^2 = m^2 + 2m (m - n) + (m - n)^2 \) is clearly greater than \( 4 (m - n)^2 \), we have that \[(5.1)\] is bounded above by

\[
\frac{2 (m - n)^2}{4 (m - n)^2} = \frac{1}{2}
\]

as desired. It remains to show that \[(5.2)\] holds for \( k \geq m - n \). To show this we note that

\[
\left( \frac{k + m - n + 1}{(k + m + 1)^2} \right) \left( \frac{k + m - n + 1}{(k + m + 1)^2} \right) \left( k + 1 \right) \leq \frac{2 (m - n) (k + 1)}{(k + m + 1)^2},
\]

since \( 
\frac{k + m - n + 1}{(k + m + 1)^2} \geq \frac{k + m - n + 1}{(k + m + 1)^2} \). Now, \( (k + m + 1)^2 = (k + 1)^2 + 2m (k + 1) + m^2 \). It is clear that \( 2m (k + 1) \geq 2 (m - n) (k + 1) \). If we can then show that

\[
(k + 1)^2 + m^2 \geq 2 (m - n) (k + 1),
\]

the claim will follow. But indeed, we obtain

\[
(k + 1 - m)^2 = (k + 1)^2 - 2m (k + 1) + m^2 \geq 0
\]

for all \( k \), so we see

\[
(k + 1)^2 + m^2 \geq 2m (k + 1) \geq 2 (m - n) (k + 1).
\]

And it follows then that

\[
\frac{2 (m - n) (k + 1)}{(k + m + 1)^2} \leq \frac{2 (m - n) (k + 1)}{4 (m - n) (k + 1)} = \frac{1}{2},
\]

and so, as claimed, the theorem is proved. \( \Box \)

The result is, perhaps, not so surprising, given the results of \[7\] and \[11\], but it leads us to conjecture:
If \( \varphi \in L^\infty(D) \), and if \( T_\varphi \) is hyponormal, then \( \| [T_\varphi^*, T_\varphi] \| \leq \frac{\text{Area}(\varphi(D))}{2\pi} \).

### 6. Final remarks

Our studies have focused on finding sufficient conditions for the hyponormality of Toeplitz operators having certain non-harmonic polynomials as symbols, with our methods invariably focusing on what can only be described as “hard” analysis. We would be interested in finding more function theoretic results akin to P. Ahern and Z. Čučković in [1], which would generate softer proofs and more qualitative results. For example, something along the lines of the following conjecture:

If \( T_f \) is hyponormal and \( T_g \) is co-hyponormal, then \( T_{f+g} \) is hyponormal implies that \( |f_z| \geq |g_{\bar{z}}| \) in \( D \).

So far, all the examples we have conform to this prediction, but given the subtlety of hyponormality, this evidence is certainly not overwhelming.

We would also be interested in looking at necessary conditions, along the lines of much of the work that has been done by others studying operators with harmonic symbols such as Z. Čučković and R. Curto’s recent work in [3].

Finally, in [3], Ch. Chu and D. Khavinson proved the following theorem for hyponormal Toeplitz operators acting on the Hardy space.

**Theorem.** If \( \varphi = f + \overline{T_h f} \) for \( f, h \in H^\infty \), with \( \| h \|_\infty \leq 1 \) and \( h(0) = 0 \), that is, if \( T_\varphi \) is a hyponormal Toeplitz operator on the Hardy space \( H^2 \), then we have that

\[
\| [T_\varphi^*, T_\varphi] \| \geq \| P_+ (\varphi) - \varphi (0) \|_2^2 .
\]

where \( P_+ \) is the orthogonal projection from \( L^2(T) \) onto the Hardy space.

Combining this result with Putnam’s inequality they arrive immediately at the following corollary:

**Corollary.** If \( T_\varphi \) is a hyponormal Toeplitz operator acting on the Hardy space \( H^2 \), then

\[
\text{Area} (\sigma (T_\varphi)) \geq \pi \| P_+ (\varphi) - \varphi (0) \|_2^2 .
\]

Although a classification of hyponormal Toeplitz operators remains elusive for the Bergman space, it would be interesting to see under what conditions a similar lower bound could be obtained in the Bergman space setting. A cursory examination of the proof of Theorem [4] combined with Putnam’s inequality shows that

\[
\| P(z^m \bar{z}^n) \|_2^2 = \frac{(m - n + 1) \pi}{(m + 1)^2} \leq \text{Area} (\sigma (z^m \bar{z}^n)) .
\]

### References

[1] P. Ahern, Z. Čučković, *A mean value inequality with applications to Bergman space operators*, Pacific J. Math. 173 (1996), no. 2, 295–305.

[2] S. Axler, J.H. Shapiro, *Putnam’s theorem, Alexander’s spectral area estimate, and VMO*, Math. Ann. 271, 1985, 161–183.
[3] Ch. Chu, D. Khavinson, *A note on the spectral area of Toeplitz operators*, Proc. Amer. Math. Soc., Vol. 144, No. 6 (2016), 2533–2537.

[4] C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Surveys of some recent results in operator theory, Vol. I, 155–167, Pitman Res. Notes Math. Ser., 171, Longman Sci. Tech., Harlow, 1988.

[5] Z. Čučković, R. Curto, *A New Necessary Condition for the Hyponormality of Toeplitz Operators on the Bergman Space*, arXiv:1610.09596, 2016.

[6] P. Duren, A. Schuster, *Bergman Spaces*, Mathematical Surveys and Monographs v. 100, 2004.

[7] M. Fleeman, D. Khavinson, *Extremal domains for self-commutators in the Bergman space*, Complex Anal. Oper. Theory 9 (2015), no. 1, 99–111.

[8] I.S. Hwang, *Hyponormal Toeplitz operators on the Bergman space*, J. Korean Math. Soc. 42 (2005), no. 2, 387–403.

[9] Y. Lu, C. Liu, *Commutativity and Hyponormality of Toeplitz operators on the weighted Bergman space*, J. Korean Math. Soc. 46 (2009), no. 3, 621–642.

[10] M. Martin, M. Putinar, *Lectures on Hyponormal Operators*, Operator Theory: Advances and Applications, 39, 1989.

[11] J.-F. Olsen, M. Reguera, *On a sharp estimate for Hankel operators and Putnam’s inequality*, Rev. Mat. Iberoam. 32 (2016), no. 2, 495–510.

[12] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.

[13] H. Sadraoui, *Hyponormality of Toeplitz operators and Composition operators*, Thesis, Purdue University, 1992.

[14] D. Sarason, *Generalized Interpolation in $H^\infty$*, Trans. Amer. Math. Soc. 127 (1967) 179–203.

M. Fleeman: Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798, USA

E-mail address: Matthew_Fleeman@baylor.edu

C. Liaw: Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, DE 19716, USA. And CASPER, Baylor University, One Bear Place #97328, Waco, TX 76798, USA

E-mail address: Liaw@udel.edu