Rolling tachyons for separated brane-antibrane systems

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Abstract: We consider tachyon condensation between a D-brane and an anti-D-brane in superstring theory, when they are separated in their common transverse directions. A simple rolling tachyon solution, that describes the time evolution of the process, is studied from the point of view of boundary conformal field theory. By computing the boundary beta-functions of the system, one finds that this theory is conformal, hence corresponds to an exact solution of the string theory equations of motion. By contrast, the time-reversal-symmetric rolling tachyon is not conformal. Using these results we study space-time effective actions that can describe the system in the vicinity of these exact solutions.

Keywords: Tachyon condensation, Conformal Field Theory

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1 Introduction

Annihilation of D-branes of opposite Ramond-Ramond charge is one of the fundamental processes of string theory. Tachyon condensation on brane-antibrane systems has also important cosmological applications, either as a tractable model of a time-dependent process in string theory, or concretely in D-brane inflation models [1]. It also appears in holographic models of QCD, to describe chiral symmetry breaking [2].

Whenever the distance between the branes is smaller than the critical value $r_c$, the ground state in the brane-antibrane open string sectors becomes tachyonic. It has been conjectured long ago that the condensation of this complex-valued tachyon leads to the closed string vacuum, corresponding to the minimum of the tachyon potential [3], and partially confirmed by string field theory computations [4].

In the case where the brane and the anti-brane are coincident in their common transverse directions, this system has been thoroughly studied using background-independent string field theory (BSFT) [5–7]. In this approach, one considers the two-dimensional worldsheet conformal field theory on the disk with marginal and relevant boundary perturbations. It allows to compute the exact off-shell tree-level tachyon potential [8, 9].

On-shell configurations corresponding to real-time tachyon condensation on unstable D-branes are also of interest, especially whenever the boundary conformal field theory (CFT) is known. For unstable D-branes, a first type of solution, known as the full S-brane was found by Sen and represents a time-reversal symmetric process [10]. The second type of solution, known as the half S-brane [11, 12], represents the more realistic case of a tachyon starting, from $t \to -\infty$, at the maximum of its potential. It is straightforward to extend these results to coincident brane-antibrane pairs.

Although the gradient of the tachyon field on the rolling tachyon solutions is very large, it should make sense to consider a spacetime effective action that describes slowly varying perturbations thereof. Remarkably, as was shown by Kutasov and Niarchos [13], it is possible to find unambiguously the effective action for the tachyon and its first derivative asking only that (i) the rolling tachyon discussed above is a solution to its equations of motion and that (ii) the on-shell Lagrangian on this solution is equal to the disk partition function with the time-like zero mode unintegrated.\(^1\) Upon a simple field redefinition, it coincides also with the "tachyon-DBI" action that was earlier proposed by Garousi [14], and is able to reproduce correctly N-point tachyon amplitudes [15].\(^2\)

Surprisingly, not much of this program has been carried out for the system of a D-brane and an anti-D-brane at finite distance – letting aside the even more interesting and challenging case of brane-antibrane scattering. The brane separation is a modulus at tree-level, even though a brane-antibrane potential is generated at one string loop [17]. Hence, we can ask whether tachyon condensation at fixed separation is possible. One may expect

\(^1\)This last statement is a conjecture that is mainly based on the boundary string field theory approach discussed above.

\(^2\)A different an interesting approach to tachyon effective actions on brane-antibrane pairs was given in [16].
different spacetime physics compared to the coincident case, especially in the limit where the absolute value of the tachyon mass is small in string units.

With cubic string field theory, an approximation of the tachyon potential as a function of the fixed brane-antibrane separation $r$ has been computed few years ago using level truncation at next-to-leading order in [18]. In BSFT, the framework for studying the T-dual configuration – a brane-antibrane pair compactified along a worldvolume direction, with a relative Wilson line – was set in the works [19] and [20]. There, the worldsheet action of the system, including the background spacetime gauge fields along with the complex tachyon, was set. Unfortunately, the Abelian gauge field T-dual to $r$ was set to zero in order to simplify the path integral computation.1

Finally, the 'half S-brane' rolling tachyon solution describing condensation at fixed, finite distance is not really understood, let alone the effective action of which it should be a solution. In [18] this problem was studied using conformal perturbation theory, which is expected to be valid, in spacetime terms, for very early times at the onset of tachyon condensation. Surprisingly, it was found that the boundary interaction corresponding to the rolling tachyon ceases to be marginal for a countable set of values of $|r|$ larger than $r_c/\sqrt{2}$.

In this note, we show that, taking in particular into account the effect of contact terms that are dictated by worldsheet supersymmetry, the rolling tachyon boundary interaction seems to be exactly marginal for all values of $|r|$ below the critical separation. Study of beta-functions for the system illuminates the crucial role of the contact term. The latter is able to cancel the power-like short distance singularity that arises at second order in perturbation theory for $|r| > 1/2$. At fourth order, it cancels all but one power-like singularity that is present for $|r| > \sqrt{7}/4$, for which an higher-order contact term (that can be viewed as a counter term in the RG analysis) is needed. Nevertheless, the potentially dangerous logarithmic singularities, that could occur for certain values of $|r|$, vanish by themselves without the help of the contact term.

We find that the beta-functions of the theory are are zero to all orders for $|r| < \sqrt{17}/6$, while for larger values of $|r|$ they vanish at least up to order five in perturbation theory. Thus, we expect that the perturbative expansion in the boundary tachyon perturbations does not break conformal invariance on the boundary, for any sub-critical separation.

Unexpectedly, we find that the 'full S-brane' rolling tachyon is not a boundary conformal field theory, for any non-zero separation between the branes. In that case the beta-function for the distance-changing boundary operator does not vanish. It implies that the corresponding space-time tachyon profile is not a solution of the equations of motion.

From these results we learn that there should exist a space-time effective action for the system, that is valid for any $0 \leq |r| < r_c$ (to be more precise, the effective action for the tachyon and distance field should admit a solution where the distance is a constant). Effective actions for this system were proposed in the past by Sen [22] and Garousi [23]. The latter

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1Using these results, the spacetime effective action with non-zero gauge field profiles was conjectured in [21] as a plausible covariantization, however it was not derived from first principles.
is obtained by a projection of the non-Abelian tachyon-DBI action for coincident non-BPS branes. However, as we will emphasize, the domain of validity of this action is not clear. Indeed, it does not allow as a solution a tachyon condensation at fixed distance, even in the regime of small brane separation in string units.

Imposing the existence of the 'half S-brane' solution at fixed distance gives strong constraints on the effective action. It is completely fixed up to second order in the tachyon field. In order to get the fully explicit effective action around this rolling tachyon at fixed distance without further hypothesis, we can proceed as in [13] and try to fix all the coefficients of a generic first-order Lagrangian expressed in power series. It fails to give a single answer for two reasons. First, as the 'full S-brane' solution seems not be allowed, the constraints from the tachyon equations of motion are weaker. Second, we would need to compute the disk partition function, to all orders in the tachyon coupling; for a generic distance analytical results for the perturbative integrals seem out of reach, from the fourth order. At a specific distance \( r = r_c/\sqrt{2} \) this computation is possible, but tedious. We carried the computation up to eighth order, hoping to recognize the Taylor expansion of a simple expression.

We finally give an example of an effective Lagrangian for the system, based on a sensible assumption, with the expected properties. However, comparison with the disk partition function seems to indicate that the exact effective Lagrangian is different.

This work is organized as follows. In section 2 we give some background on the brane-antibrane worldsheet action on the disk, emphasizing the role of the Fermi multiplets that realize the Chan-Paton degrees of freedom. In section 3 we discuss the role of contact terms in canceling the divergences that arises when tachyon perturbations collide. In section 4 we examine the system from the point of view of boundary renormalization group flow, and how the contact term arises in this context. Finally in section 5 we discuss space-time effective actions for the system, before the conclusions. Some lengthy computations are given in the appendices.

2 Brane-antibrane worldsheet action

In this section we discuss in detail the boundary worldsheet action of the brane/antibrane system, and set our conventions.

2.1 Superspace action on the disk

As a starting point, one considers the worldsheet action for coincident D1-brane and anti-D1-brane wrapped around a circle in a compactified direction \( Y \), T-dual to the system of interest. We set everywhere in the following \( \alpha' = 1 \).

The \( \mathcal{N} = (1, 1) \) superspace action on the disk was written in [19, 20, 24], including the coupling to background gauge and tachyon fields. In the present context one considers non-trivial Wilson Lines along the circle, T-dual to the brane positions \( x_1 \) and \( x_2 \) along \( X \), the T-dual of \( Y \). They naturally appear in the form \( x^{(\pm)} = x_1 \pm x_2 \).
Setting aside the ‘spectator’ dimensions, one considers a pair of $N = (1, 1)$ superfields on the disk, one time-like ($X_0$) and the other compactified on a circle ($Y$), with e.g. $X_0 = X_0 + \frac{1}{\sqrt{2}}(\theta \psi_0 + \bar{\theta} \bar{\psi}_0) + \theta \theta F_0$. The superspace coordinates are denoted as $\hat{z} = (z, \theta, \bar{\theta})$.

At the boundary of the disk, the Grassmann coordinates satisfy the boundary condition $\theta = \pm \bar{\theta}$. The algebra of the Chan-Paton factors for the brane-antibrane system is conveniently implemented by the canonical quantization of boundary fermions [25], see below. These boundary fermions are the bottom components of Fermi superfields of the boundary $N = 1$ superspace. With a Lorentzian signature target space, one needs a complex superfield

$$\Gamma^\pm = \eta^\pm + \theta F^\pm.$$  \hspace{1cm} (2.1)

with $\Gamma^- = (\Gamma^+)^*$. 

Then the worldsheet action on the disk, including the tachyon background as well as Wilson lines around the circle, reads:

$$S_{BCFT}(\lambda^+, \lambda^-) = \frac{1}{2\pi} \int_{D^2} d^2z \, d^2\theta \left( -DX_0 \overline{D}X_0 + D\bar{Y} \bar{D}Y \right) + i \oint_{S^1} du \, d\theta \frac{x^+(\gamma)}{4\pi} D_u \bar{Y}$$

$$- \oint_{S^1} du \, d\theta \left( \Gamma^+ \left( D_u + i \frac{x^-(\gamma)}{2\pi} D_u \bar{Y} \right) \Gamma^- - \Gamma^+ \gamma^+ - \Gamma^- \gamma^- \right) , \hspace{1cm} (2.2)$$

with the measure $d^2\theta = d\theta \, d\bar{\theta}$, the superspace holomorphic derivative $D = \partial \theta + \theta \partial$ and the superspace boundary derivative $D_u = \partial_u + \theta \partial_u$, with the boundary coordinate $u$ on $S^1$.\textsuperscript{1}

We consider simple rolling tachyon profiles of the form:

$$T^\pm = \frac{\lambda^\pm}{2\pi} e^{i\omega X_0}, \hspace{1cm} (2.3)$$

with $0 < \omega \leq 1/\sqrt{2}$. In order to get a real action, one chooses $(\lambda^+)^* = \lambda^-$. These are actually the tachyons that we are expecting to be solutions of the spacetime effective action. It is understood in this expression that the superfield $X$ is taken on the (super)boundary of the disk.

The space-time gauge field $A^{(-)} = -\frac{x^{(-)}(\gamma)}{4\pi} dy$ being locally pure gauge, its minimal coupling to the Fermi superfields can be absorbed by a ‘gauge’ transformation.\textsuperscript{2} One has to be careful with this transformation if $Y$-dependent insertions appear in the path-integral; a prescription must be chosen (see below).

$$\Gamma^\pm \rightarrow \Gamma^\pm e^{i \frac{x^{(-)}(\gamma)}{4\pi} Y}. \hspace{1cm} (2.4)$$

After this field redefinition, the boundary Fermi superfields are free, with the propagator on the real axis:

$$\langle \Gamma^+(\hat{z}) \Gamma^- (\hat{w}) \rangle = \hat{c}(\hat{z} - \hat{w}) = \epsilon (z - w) - 2 \theta \partial_u \delta (z - w) , \hspace{1cm} (2.5)$$

\textsuperscript{1} The boundary current superfield $D_u \bar{Y}$ is defined to be the boundary super-derivative of $Y$ first taken to the boundary (where $Y$ has Neumann boundary conditions).

\textsuperscript{2} This is a slight abuse of language, as this is not a gauge symmetry from the worldsheet perspective.
in terms of the sign function $\epsilon(z) = \Theta(z) - \Theta(-z)$. This implies that $\Delta(\Gamma^\pm) = 0$, i.e. vanishing conformal dimension.

In terms of these new variables the worldsheet action (2.2) reads:

$$S_{BCFT}(\lambda^+, \lambda^-) = \frac{1}{2\pi} \int_{D^2} d^2z \, d^2\theta \left( -D\bar{X}^0 D\bar{X}^0 + D\bar{Y} D\bar{Y} \right) + i \int_{S^1} du \, d\theta \frac{x^{(+)}}{4\pi} D_u \bar{Y}$$
$$- \oint_{S^1} du \, d\theta \left( \Gamma^+ D\Gamma^- - \Gamma^+ T^+ - \Gamma^- T^- \right)$$

(2.6)

where the tachyon fields have now the expression:

$$T^\pm = \lambda^\pm e^{\pm i \frac{(-)}{2\pi} X^0 + \omega X^0}.$$  

(2.7)

Conformal invariance of the action at leading order imposes then:

$$\omega^2 + \left( \frac{x^{(-)}}{2\pi} \right)^2 = \frac{1}{2}.$$  

(2.8)

This is the standard mass-shell condition of an open string tachyon with $U(1) \times U(1)$ Wilson lines turned on.

The world-sheet action that describes a system of separated brane and anti-branes is obtained from the previous one by a T-duality along $\eta$. In the bulk, the superfield $\bar{Y}$ is traded for the superfield $\bar{X}$ that has Dirichlet boundary conditions. Renaming $\bar{Y}$ as $\tilde{X}$, the tachyon interaction of interest reads

$$T^\pm = \lambda^\pm e^{\pm i \frac{(-)}{2\pi} \tilde{X} + \omega X^0}.$$  

(2.9)

Action (2.6) will be our starting point. In the free theory, one has two different boundary conditions on the disk boundary, related to the distinct positions of the branes: $X = x^{(1)}$ or $Y = x^{(2)}$. We introduce the notations

$$x^{(-)} = x^{(1)} - x^{(2)} = 2\pi r$$
$$x^{(+)} = x^{(1)} + x^{(2)} = 2x_{cm},$$  

(2.10)

where on the first line $r$ is such that $\omega^2 + r^2 = 1/2$. On the second line, $x_{cm}$ is simply the center of mass coordinate of the system.

**2.2 Action in components, quantization of the Fermi superfields**

Starting from the action (2.6), renaming $\bar{Y}$ as $\tilde{X}$, and integrating over the fermionic coordinates one gets the action:

$$S_{BCFT}(\lambda^+, \lambda^-) = \frac{1}{2\pi} \int_{D^2} d^2z \left( -\partial X^0 \bar{\partial} \bar{X}^0 + \partial X \bar{\partial} X \right) + i \int_{S^1} du \frac{x^{(+)}}{4\pi} \partial_u \tilde{X}$$
$$+ \oint_{S^1} du \left( \eta^+ \partial_u \eta^- - \frac{\lambda^+}{2\pi} \eta^+ \psi^+ T^+ - \frac{\lambda^-}{2\pi} \eta^- \psi^- T^- \right)$$
$$- \oint_{S^1} du \left( F^+ F^- - F^+ T^+ - F^- T^- \right),$$  

(2.11)
with:
\[ \psi^\pm = \pm ir\sqrt{2}\tilde{\psi}^2 + \omega\sqrt{2}\psi^0 \]
\[ T^\pm = e^{\pm ir\tilde{X} + \omega X^0} \]  
\[ (2.12) \]

Auxiliary fields \( F^{\pm} \) are then integrated to give:
\[ S_{BCFT}(\lambda^+, \lambda^-) = \frac{1}{2\pi} \int_D d^2 z \left( -\partial X^0 \bar{\partial} X^0 + \partial X \bar{\partial} X \right) + i \oint_{S^1} du \frac{x^{(+)}}{4\pi} \partial_u \tilde{X} \]
\[ + \oint_{S^1} du \left( \eta^+ \partial_u \eta^- - \frac{\lambda^+}{2\pi} \eta^+ \psi^+ T^+ - \frac{\lambda^-}{2\pi} \eta^- \psi^- T^- + e^{1-4r^2} \lambda^+ \lambda^- T^+ T^- \right) \]  
\[ (2.13) \]

A contact term at the end of the second line shows up, with a UV cutoff \( \varepsilon \). This term, that does not follow from the equations of motion contributes nevertheless to correlation functions when \( 1/2 < |r| < 1/\sqrt{2} \). Its role will be discussed in section 3.3.

Finally, as the center-of-mass perturbation completely factorizes and commutes with any operators in \( (2.13) \), one can set \( x^{(+)} = 0 \) without loss of generality.

Upon quantizing canonically the boundary fermions \( \eta^{\pm} \), one recovers the Chan-Patton algebra corresponding to the brane-antibrane system \[ 20 \]. It leads to the following identifications:
\[ \eta^+ \Leftrightarrow \sigma^+ = \frac{\sigma^1 + i\sigma^2}{2} \]
\[ \eta^- \Leftrightarrow \sigma^- = \frac{\sigma^1 - i\sigma^2}{2} \]
\[ \eta^+ \eta^- (z) \Leftrightarrow \frac{[\sigma^+, \sigma^-]}{2} = \frac{\sigma^3}{2} \]  
\[ (2.14) \]

where now the prescription for the path integral is \( Z = \text{Tr} \int D\psi^i D\psi^j P e^{-S[\tilde{X}^i, \psi^j]} \), which includes a path ordering for the operator insertions and a trace over the CP factors. In this context the tachyon becomes a boundary changing operator; when inserted on the boundary of the disk, it interpolates between the two distinct boundary conditions corresponding to the brane and to the anti-brane.

After quantizing canonically the boundary fermions \( \eta^{\pm} \), the worldsheet action on the disk takes finally the form
\[ S = S_{\text{bulk}} - \oint_{S^1} du \left( \frac{\lambda^+}{2\pi} \sigma^+ \otimes \psi^+ e^{ir\tilde{X} + \omega X^0} + \frac{\lambda^-}{2\pi} \sigma^- \otimes \psi^- e^{-ir\tilde{X} + \omega X^0} - \frac{\lambda^+ \lambda^-}{4\pi^2} e^{1-4r^2} e^{2\omega X^0} \right) \]  
\[ (2.15) \]

We saw in the previous sub-section that the action \( (2.6) \) is related by a simple redefinition of the boundary fermions to the action \( (2.2) \). The components expression of the latter reads,
after a T-duality:

$$S_{\text{BCFT}}(\lambda^+,\lambda^-) = \frac{1}{2\pi} \int_{D^2} d^2 z \left( -\partial X^0 \bar{\partial} X^0 + \partial X \bar{\partial} X \right) + i \oint_{S^1} du \frac{x_r^{(+)}}{4\pi} - \partial_u \bar{X}$$

$$+ \oint_{S^1} du \left( \frac{i\sigma^3}{4\pi} \partial_u \bar{X} - \frac{\lambda^+}{2\pi} \sigma^+ \psi^+ T - \frac{\lambda^-}{2\pi} \sigma^- \psi^- T + \frac{r^2}{\varepsilon} \right)$$

(2.16)

with now \( T = e^{\omega X^0} \) and \( \psi^\pm \) still given by (2.12). In this action a tachyon-tachyon contact term is present in principle, but its UV-regularized expression scales as \( \varepsilon^{2\omega^2} \). Hence, as \( \omega \) is real, it vanishes when one removes the UV cutoff. On the other hand, note the presence of the contact term \( r^2/\varepsilon \) that should regularize singular \( \partial X \) correlators, in accordance with [26].

It is remarkable that the actions (2.16) and (2.13) should give the same results. More remarkable is the absence of contact term in (2.16); its role is in fact ensured by divergences appearing from ordered products involving \( \sigma^\pm \psi^\pm T \) and \( \sigma^3 \partial X \).

In order to understand this relation, a remark is in order about the boundary fermions path-integration. The 'gauge' transformation from (2.2) to (2.6) is more subtle in the presence of insertions in the path-integral. As an example, consider two separated bosonic branes with tachyons switched off, and compute the following closed string tachyon one-point function on the disk:

$$\langle e^{ikX(0,0)} \rangle = \int \mathcal{D}X |_{X_{\text{bdy}}=0} \mathcal{D}\eta^+ \mathcal{D}\eta^- e^{-S_{\text{bulk}}-\oint \eta^+(\partial_u-i\frac{\varepsilon}{\pi^2} \partial_u \bar{X})\eta^-} e^{ikX(0,0)}$$

(2.17)

with \( X \) set to be 0 on the boundary in the path integral. The operator \( i\frac{\varepsilon}{\pi^2} \eta^+ \eta^- \partial_u \bar{X} \) then acts as a zero-mode 'translator' in each sector as \( X |_{\text{bdy}} = \pm \frac{r}{2} \)† so that (2.17) is:

$$\text{Tr} \int \mathcal{D}X |_{X_{\text{bdy}}=0} e^{-S_{\text{bulk}}+i\frac{\varepsilon}{\pi^2} \oint \eta^+ \partial_u \bar{X} + ikX(0,0) + ik \frac{\varepsilon}{\pi^2} \sigma^3} = \text{Tr} e^{ik \frac{\varepsilon}{\pi^2} \sigma^3} = e^{ik \frac{\varepsilon}{\pi^2}} + e^{-ik \frac{\varepsilon}{\pi^2}}$$

(2.18)

This is as expected: the zero mode is shifted either by \( r/2 \) or by \(-r/2 \). A problem appears when we transform the \( \eta \)-fields such that to absorb the \( r \)-dependent term. Indeed, by naive and straightforward transformation of the field, the distance information leaks out of the computation, thus leading to an unequal correction. The correct prescription is to perform the OPE's between all \( X \)-dependent operators before the field redefinition \( \eta^\pm \to e^{\mp ir \bar{X}} \eta^\pm \).

The following identity is obvious but illustrate this fact:

$$\int \mathcal{D}X |_{X_{\text{bdy}}=0} \mathcal{D}\eta^+ \mathcal{D}\eta^- e^{-S_{\text{bulk}}-\oint \eta^+(\partial_u-i\frac{\varepsilon}{\pi^2} \partial_u \bar{X})\eta^-} e^{ikX(0,0)}$$

$$= \int \mathcal{D}X |_{X_{\text{bdy}}=0} \mathcal{D}\eta^+ \mathcal{D}\eta^- e^{-S_{\text{bulk}}-\oint \eta^+(\partial_u-i\frac{\varepsilon}{\pi^2} \partial_u \bar{X})\eta^-+ikX(0,0)+k \frac{\varepsilon}{\pi^2} \oint \eta^+ \eta^-}$$

$$= \int \mathcal{D}X |_{X_{\text{bdy}}=0} \mathcal{D}\eta^+ \mathcal{D}\eta^- e^{-S_{\text{bulk}}-\oint \eta^+ \partial_u \eta^-+ikX(0,0)+k \frac{\varepsilon}{\pi^2} \oint \eta^+ \eta^-}$$

$$= \text{Tr} e^{ik \frac{\varepsilon}{\pi^2} \sigma^3}$$

(2.19)

†Indeed, we have the OPE: \( e^{iX(0,0)} e^{iX(0,0)} = e^{i \frac{\varepsilon}{\pi^2} \oint \eta^+ \eta^- \partial_u \bar{X} + ikX(0,0) + k \frac{\varepsilon}{\pi^2} \oint \eta^+ \eta^-} \).
Boundary normal ordering is implicit here; similar arguments apply in the supersymmetric case.¹

3 Perturbative integrals and contact terms

In this section we discuss in more detail the contact term, quadratic in the tachyon field, that appear in the action (2.6) after integrating out the auxiliary fields from the Fermi superfields Γ±, and quantizing their fermionic components. As was discussed long ago by Green and Seiberg [26] for closed string correlation functions, contact terms, dictated by worldsheet supersymmetry, can cancel unphysical divergences in correlation functions. We shall see below that it indeed cancels the short-distance singularity when two tachyons perturbations collide in the perturbative expansion.

3.1 Free field correlators

In order to fix the conventions, we use the following Green functions on the upper half-plane $H^+$ for a free-field $X$ with Dirichlet boundary conditions, and its T-dual field $\tilde{X}$:

\[
\langle X(z_1)X(z_2) \rangle = -\frac{\eta_{xx}}{2} \ln |z_{12}|^2 + \frac{\eta_{xx}}{2} \ln |\bar{z}_{12}|^2 \\
\langle \tilde{X}(z_1)\tilde{X}(z_2) \rangle = -\frac{\eta_{xx}}{2} \ln |\bar{z}_{12}|^2 - \frac{\eta_{xx}}{2} \ln |z_{12}|^2 \\
\langle X(z_1)\tilde{X}(z_2) \rangle = -\frac{\eta_{xx}}{2} \ln \frac{z_{12}}{z_{12}} - \frac{\eta_{xx}}{2} \ln \frac{\bar{z}_{12}}{\bar{z}_{12}}
\]

(3.1)

with e.g. $z_{12} = z_1 - z_2$ and $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$. Finally, the two-point function for fermions with Dirichlet b.c. read:

\[
\langle \psi^x(z_1)\psi^x(z_2) \rangle = \frac{\eta^{xx}}{z_1 - z_2} \\
\langle \bar{\psi}^x(\bar{z}_1)\bar{\psi}^x(\bar{z}_2) \rangle = \frac{\eta^{xx}}{\bar{z}_1 - \bar{z}_2} \\
\langle \psi^x(z_1)\bar{\psi}^x(\bar{z}_2) \rangle = -\zeta \frac{\eta^{xx}}{z_1 - \bar{z}_2}
\]

(3.2a, 3.2b, 3.2c)

where $\zeta = \pm 1$ corresponds to the spin structure. It corresponds to the boundary conditions for the supercurrent $G(z) - \zeta \bar{G}(\bar{z})|_{z=\bar{z}} = 0$. For the Virasoro superfield $G = G + \theta T$, this is naturally associated with the superspace boundary $(z, \theta) = (\bar{z}, \zeta \bar{\theta})$. With Neumann b.c., eq. (3.2c) gets a minus sign on the RHS.

Finally, the boundary Green function for a superfield $X$ with Neumann boundary conditions reads:

\[
\langle X(\hat{z}_1)X(\hat{z}_2) \rangle_{\hat{z}_{12} = 0, \hat{\theta}_{12} = \zeta \hat{\theta}} = -2\eta_{xx} \ln \hat{z}_{12} = -2\eta_{xx} \ln (z_{12} - \theta_1 \theta_2)
\]

(3.3)

while it vanishes with Dirichlet b.c..

¹Note that the redefinition of $\Gamma^\pm$ from (2.2) to (2.6) is not problematic, because in (2.2) the tachyon depends only in $X^0$, and $\oint \partial X$ commutes with $\oint \eta^+ \eta^- \partial X$. 

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3.2 Contact term in the worldsheet action

As has been explicited in section (2.2), upon integrating out the auxiliary fields $F^\pm$ that appear in the Fermi multiplets $\Gamma^\pm$, one obtains a contact term for the tachyon in the worldsheet action.

The auxiliary fields have the two-point function $\langle F^+(u)F^-(v) \rangle = 2\delta(u - v)$. It is regularized at short distances according to:

$$\langle F^+(t)F^-(s) \rangle = 2\delta(t - s) \rightarrow \delta(|t - s| - \varepsilon) \quad (3.4)$$

Then the contact term is given by the following non-local interaction on the disk (with $u = e^{it}$, $v = e^{is}$):

$$\frac{1}{2} \int_0^{2\pi} dt \int_0^{2\pi} ds \delta(|t - s| - \varepsilon) \hat{e}^{ir\hat{X} + \omega X^0(u)} \hat{e}^{-ir\hat{X} + \omega X^0(v)};$$

$$= \frac{1}{2} \int_0^{2\pi} dt \int_0^{2\pi} ds \delta(|t - s| - \varepsilon) |u - v|^{2(\omega^2 - r^2)} \hat{e}^{ir\hat{X} + \omega X^0(u)} \hat{e}^{-ir\hat{X} + \omega X^0(v)};$$

$$= \frac{1}{2} (2\sin \frac{\varepsilon}{2})^{1-4r^2} \int_0^{2\pi} ds \left( \hat{e}^{ir\hat{X} + \omega X^0(v + \varepsilon)} \hat{e}^{-ir\hat{X} + \omega X^0(v)}; + \hat{e}^{ir\hat{X} + \omega X^0(v)} \hat{e}^{-ir\hat{X} + \omega X^0(v + \varepsilon)}; \right) \varepsilon \rightarrow 0 \int_{S^1} du \hat{e}^{2\omega X^0(u)}; \quad (3.5)$$

By $;*$ we denote the boundary normal ordering (see e.g. [27]). We added a $\frac{1}{2}$ normalization such that to take account for the factor 2 coming from the trace over CP-factor, since the contact term is multiplied by the identity matrix.

We will use in a next section the contact term on the upper half plane. It is similarly written as:

$$\frac{e^{1-4r^2}}{2} \int_{-\infty}^{+\infty} dv \left( \hat{e}^{ir\hat{X} + \omega X^0(v + \varepsilon)} \hat{e}^{-ir\hat{X} + \omega X^0(v)}; + \hat{e}^{ir\hat{X} + \omega X^0(v)} \hat{e}^{-ir\hat{X} + \omega X^0(v + \varepsilon)}; \right) \varepsilon \rightarrow 0 \int_{\mathbb{R}} dv \hat{e}^{2\omega X^0(v)}; \quad (3.6)$$

In order to compute all the counterterms generated from this contact term one will need to work with its complete non-local expression, though the dominant term, here the only divergent one, in its Taylor expansion (in terms of local operators) is sufficient to compute most of them. Indeed, it is found that working directly with the dominant term, a local operator, seems to be equivalent to working with the complete non-local contact term. It may be explained by the fact that after Taylor expansion of $T^\pm(x + \varepsilon)$ and commutation of the sum and the integral, all other terms in the series of integrated local operators vanish as $\varepsilon$ goes to zero. One may object that we are forgetting sub-dominant terms, but, as the UV cut-off is an artifact signaling our lack of ability to manipulate infinite quantities; it is to be understood as being strictly equal to zero, from the very beginning. From this point
of view, we expect that only the divergent terms in (3.6) do contribute. Then it should be equivalent to use either the dominant (local) term or the complete (non-local) contact term. This statement seems to be confirmed numerically in the fourth order computations of section 4.4.

As one can see, in the limit \( \varepsilon \to 0 \) when one takes the UV cut-off to infinity, the contact term vanishes when \(|r| < 1/2\). Therefore, the results of the computations made in [22], where the contact term was not taken into account, remain unchanged.\(^1\) It can be seen also by working directly with the \( \mathcal{N} = 1 \) boundary superspace amplitudes; the contact terms contributions from the \( \Gamma^\pm \) correlators vanish for \(|r| < 1/2\).

However, the contact term diverges when \(|r| > 1/2\). This contact term may ensure that the amplitudes do not diverge for \(|r| > 1/2\). The divergence associated with the contact term, that arises from the fusion of two tachyon vertices, correspond to the unphysical integrated vertex operator

\[
\int du \int d\theta \theta \delta^2 x_0(u) \gamma = \int du \delta^2 x_0(u) \gamma, \tag{3.7}
\]

that is not supersymmetric. Hence, as in [26], one can understand the contact term as necessary to preserve worldsheet superconformal invariance on the boundary, when \(|r| > 1/2\).

### 3.3 Boundary one-point function

In order to illustrate more precisely the role of the contact term, we compute the one-point function on the disk for a tachyon boundary vertex operator. This one-point function does not have to vanish because of the rolling tachyon background. We will find that the contact term cancels the two-tachyon divergence for all values of \(r\) in the range \(1/2 < |r| \leq 1/\sqrt{2}\).

At first order in the couplings \(\lambda^\pm\), the one-point function for one of the boundary tachyon vertex operators is given by the integrated correlator

\[
\text{Tr} \left\langle \sigma^\pm \otimes \psi^\pm e^{+ir\hat{x} + \omega X^0(e^{i\theta})} \right\rangle \\
\sim \frac{\lambda^\mp}{2\pi} \text{Tr} \sigma^\pm \sigma^\mp \int_0^1 dt_2 \left\langle \psi^\pm e^{+ir\hat{x} + \omega X^0(e^{i\theta})} \psi^\mp e^{+ir\hat{x} + \omega X^0(e^{i\theta})} \right\rangle_0 \tag{3.8}
\]

The integration over \(t_2\) is not defined for \(|r| > 1/2\), nevertheless the result

\[
\text{Tr} \left\langle \sigma^\pm \otimes \psi^\pm e^{+ir\hat{x} + \omega X^0(e^{i\theta})} \right\rangle \sim \frac{\lambda^\mp}{2\pi} (1 - 4r^2) 2^{1-4r^2} \pi^{1/2} \Gamma(\frac{1}{2} - 2r^2) \int_{-\infty}^{+\infty} dx^0 e^{2\sqrt{1-r^2}x^0}. \tag{3.9}
\]

\(^1\)As a side remark, for the rolling tachyon on a non-BPS D-brane, it was already noticed in [28] that the contact terms, that were absent in the original computation of the partition function performed in [29], did not contribute to the final result.
is analytic for any $r \in [0, 1/\sqrt{2}]$.

In order to show how the divergence for $|r| > 1/2$ is canceled, we can compute directly this quantity in superspace, using the Fermi multiplets $\Gamma^\pm$. Letting aside for a moment the zero-mode integral over $x_0$, one considers the superspace integral

$$\int d\theta_1 \langle \Gamma^\pm e^{\pm i r \hat{x} + \omega x^0} (\hat{z}_1) \rangle$$

which becomes:

$$\sim -\frac{\lambda^\mp}{2\pi} \int d\theta_1 d\theta_2 \int d\tau (z_1 - \hat{z}_2) \langle e^{\pm i r \hat{x} + \omega x^0} (\hat{z}_1)e^{\mp i r \hat{x} + \omega x^0} (\hat{z}_2) \rangle_0$$

$$\sim \frac{\lambda^\mp}{2\pi} e^{2\omega x^0} \int d\theta_1 d\theta_2 \int d\tau [\epsilon(t_1 - t_2) - 2\theta_1 \theta_2 \delta(t_1 - t_2)] \times$$

$$\times \left( 2\sin \frac{t_1 - t_2}{2} \right)^{1-4r^2} - \theta_1 \theta_2 (1 - 4r^2) \epsilon(t_1 - t_2) \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2}$$

$$\sim -\frac{\lambda^\mp}{2\pi} e^{2\omega x^0} \int d\tau \left( 1 - 4r^2 \right) 2\sin \frac{t_1 - t_2}{2} \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2} + 2\delta(t_1 - t_2) \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2}. \quad (3.10)$$

Now we introduce a point splitting regularization, asking that $|t_1 - t_2| > \epsilon$. As we wish to keep the contact term in the computation, it is natural to include this point splitting in the $\Theta$ and $\delta$ distributions that appear in the above integral, as:

$$\Theta(|t_1 - t_2| - \epsilon) = \Theta(t_1 - t_2 - \epsilon) + \Theta(t_2 - t_1 - \epsilon)$$

$$\delta(|t_1 - t_2| - \epsilon) = \delta(t_1 - t_2 - \epsilon) + \delta(t_2 - t_1 - \epsilon). \quad (3.11)$$

In other words, we 'spread' the contact term at the boundary of the interval $|t_1 - t_2| < \epsilon$. Then the contribution to the one point-function becomes:

$$-\frac{\lambda^\mp}{2\pi} \int d\tau \left( 1 - 4r^2 \right) 2\sin \frac{t_1 - t_2}{2} \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2} + 2\delta(|t_1 - t_2| - \epsilon) \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2}$$

$$= -\frac{\lambda^\mp}{2\pi} (1 - 4r^2) \int_{t_1 - t_2 + \epsilon}^{t_1 - \epsilon} d\tau \left( 2\sin \frac{t_1 - t_2}{2} \right)^{-4r^2} - 2\frac{\lambda^\mp}{2\pi} \left( 2\sin \frac{\epsilon}{2} \right)^{-4r^2}$$

$$\sim -\frac{\lambda^\mp}{2\pi} (1 - 4r^2) 2^{-4r^2} \sqrt{\frac{\Gamma(\frac{1}{2} - 2r^2)}{\Gamma(1 - 2r^2)}} + 2\frac{\lambda^\mp}{2\pi} \varepsilon^{-4r^2} - 2\frac{\lambda^\mp}{2\pi} \left( 2\sin \frac{\epsilon}{2} \right)^{-4r^2}, \quad (3.12)$$

where two first terms in the last line come from the expansion of the following function:

$$(1 - 4r^2) 2^{-4r^2} \cos \frac{\varepsilon}{2} \binom{1 + 4r^2}{3} \binom{2}{2} \cos^2 \frac{\varepsilon}{2}. \quad (3.13)$$

The second term of eq. (3.12) is the only divergent one if $4r^2 > 1$. It simplifies to

$$-\frac{\lambda^\mp}{2\pi} (1 - 4r^2) 2^{-4r^2} \sqrt{\frac{\Gamma(\frac{1}{2} - 2r^2)}{\Gamma(1 - 2r^2)}} + 2\frac{\lambda^\mp}{2\pi} \varepsilon^{-4r^2} - 2\frac{\lambda^\mp}{2\pi} \varepsilon^{-4r^2}. \quad (3.14)$$
Divergences compensate correctly, so that we eventually have at first order:

\[
\text{Tr} \left( (\sigma^\pm \otimes \psi^\pm - F^\pm) e^{\pm i r \bar{X} + \omega X^0} (z_1) \right) \sim -\lambda^\pm \frac{\Gamma(2 - 4r^2)}{\Gamma^2(1 - 2r^2)} \int_{-\infty}^{+\infty} dx_0^\pm e^{2\sqrt{1 - r^2} x_0^\pm}.
\]

This quantity is UV-finite, but has an IR divergence due to the zero-mode integral. This divergence, that appears when \( x_0^\pm \to \infty \), simply signals the breakdown of perturbation theory in \( \lambda^\pm \). Note that for the homogeneous rolling tachyon on a non-BPS brane, for which the all orders computation is doable, summing up the whole perturbative expansion gives a finite zero-mode integral.\(^1\)

4 Computation of beta-functions

In this section, we argue that the theory defined in (2.6) is exactly conformal, with the rolling tachyon profile (2.9), for any value of \(|r|\) below \( r_c = 1/\sqrt{2} \). This will imply that for the spacetime effective action of the brane-antibrane system there exists a ‘half S-brane’ rolling tachyon solution at fixed separation of the equations of motion. This is an important point since the effective action proposed in [23] did not admit solution at fixed distance; in fact, in this action, for non-vanishing tachyon the distance field has an attractive potential towards the origin.

Our motivation for looking closely at this issue was in part due to the results of Bagchi and Sen [18]. They found that the boundary deformation corresponding to the tachyon (2.9) was only marginal in the range \( 0 \leq |r| < r_c/\sqrt{2} \). For \( r_c/\sqrt{2} \leq |r| < r_c \) it was found that for an infinite but countable set of distances the theory was not conformal. This is puzzling as we expect that everything goes smoothly up to the critical separation \( r_c \).

At the end of the day, the basic difference between those two approaches is the contact term, however the latter is not responsible for restoring marginality, since it cannot cancel the logarithmic divergences that could spoil conformal invariance as we shall see; rather, the actual computation of the possible conformal symmetry-violating terms in the path integral gives zero thanks to the different contributions that cancel among themselves at a given order. Nevertheless, the contact term is able, as expected, to cancel the power-like two-tachyon divergences in the perturbative integrals.

The cleanest way to show that the action (2.6), with the rolling tachyon perturbation (2.7) is a boundary CFT is to compute the boundary \( \beta \)-function for all the boundary couplings involved. On top of the coupling constants \( \lambda^\pm \) for the rolling tachyon perturbations, one needs to introduce in the computation a perturbation corresponding to the separation-changing boundary operator \( \sigma^3 \otimes i \partial_u X \).\(^2\) The brane-antibrane separation is classically fixed at some value \( r \), but still in the quantum theory one has to check that the corresponding beta-function vanishes for any \( r \), in other words that it is not ‘sourced’ by terms in \( \lambda^\pm \).

---

\(^1\)If we Wick-rotate the theory to an Euclidean target space, the integration on zero-mode gives \( \delta(2 \omega) \) which is zero for any value of \(|r| < 1/\sqrt{2}\).

\(^2\)To be exact we will have to add it in superspace as \( \Gamma^+ \Gamma^- DX \)
4.1 Generalities about boundary beta-functions

In order to compute the beta-functions for their boundary couplings, we follow mostly the clear presentation of \[30\].

One considers a conformal field theory on the upper half-plane \( H^+ = \{z, \Im z \geq 0\} \) perturbed by boundary operators that can be marginal or relevant. The action of the theory is defined to be
\[
S(\lambda^\mu) = S_{\text{bulk}} + \sum_{\mu} \ell^{-y_\mu} \lambda^\mu \int dx \phi_\mu(x) + S_{\text{ct}}, \tag{4.1}
\]
in terms of the renormalized dimensionless couplings \(\{\lambda^\mu\}\) and the anomalous dimensions \(y_\mu = 1 - h_\mu\). The renormalization scale is denoted by \(\ell\). The last term \(S_{\text{ct}}\) stands for boundary counterterms whenever they are necessary. The boundary fields \(\phi_\mu\) are normalized as
\[
(\phi^*_\mu(\infty) | \phi_\mu(0)) = 1 \tag{4.2}
\]
with \(\phi^*_\mu\) the conjugate field to \(\phi_\mu\).\(^2\)

At second order in perturbation theory, one encounters the integral (which lies inside a correlator with arbitrary other insertions):
\[
\frac{1}{2} \sum_{\mu, \nu} \ell^{h_\mu - h_\nu - \frac{1}{2}} \int dx_1 \int dx_2 \phi_\mu(x_1) \phi_\nu(x_2) \Theta(|x_1 - x_2| - \epsilon) \Theta(L - |x_1 - x_2|). \tag{4.3}
\]
This integral has been regularized by point-splitting with a UV cutoff \(\epsilon\), and with and an IR cutoff \(L\). In order to compute the integral one can use the boundary OPE
\[
\phi_\mu(x_1) \phi_\nu(x_2) = \sum_{\rho} \frac{D_{\mu\nu}^\rho}{(x_2 - x_1)^{h_\mu + h_\nu - h_\rho}} \phi_\rho(x_2) + \cdots \quad x_2 > x_1. \tag{4.4}
\]

Minimal substraction scheme

In this scheme, we aim to isolate the divergences that occur in the integral (4.3) when the two perturbations collide. One has to consider separately two cases. The subset of boundary fields \(\{\phi_\rho\}\) such that \(y_\mu + y_\nu - y_\rho < 0\) (which are all relevant), gives a divergent contribution to the action (4.1) of the form (after removing the IR cutoff)
\[
S_d = -\frac{1}{2} \sum_{\mu, \nu, \rho} \frac{D_{\mu\nu}^\rho}{y_\mu + y_\nu - y_\rho} \epsilon^{y_\mu + y_\nu - y_\rho} \ell^{y_\mu + y_\nu} \lambda^{\mu} \lambda^{\nu} \int dx \phi_\rho. \tag{4.5}
\]
In the minimal substraction scheme, this divergence is canceled by a similar counter term \(S_{\text{ct}} = -S_d\).

\(^1\)The Zamolodchikov correlators are defined as \((\phi_\alpha(\infty) | \phi_\beta(z_3)) = \lim_{z \to \infty} z^{2h_\alpha} \bar{z}^{2h_\beta} \langle \phi_\alpha(z) \phi_\beta(z_3) \rangle\).

\(^2\)In the case of theories with several boundary conditions, one has to trace over the Chan-Patton factors, which would be here included inside the fields, e.g. as \(\text{Tr} \ (\phi^*_\mu(\infty) | \phi_\mu(0)) = 1\). Considering deformations by boundary-changing operators, the CP factors induce selection rules.
The subset of boundary fields \( \{ \phi_\tau \} \) such that \( y_\mu + y_\nu - y_\tau = 0 \) gives logarithmic divergences, or resonances (cutting the integration at the renormalization scale \( \ell \)):

\[
S_d = -\frac{1}{2} \sum_{\mu, \nu, \tau} D^\tau_{\mu \nu} \ln(\varepsilon/\ell) \ell^{-y_\tau} \lambda^\mu \lambda^{\nu} \int dx \phi_\tau
\]  

This divergent piece is again canceled by an appropriate counterterm \( S_{ct} = -S_d \). Now equating the bare couplings to the two corresponding contributions from the renormalized action \( (4.1) \), one gets the beta-function at second order

\[
\beta^{\text{WIS}}_{\rho} := \frac{d}{d\ell} \ell \frac{d \mu_\rho}{d \ell} = y_\rho \lambda^\rho + \sum_{\mu, \nu} D^\rho_{\mu \nu} \lambda^\mu \lambda^{\nu}
\]  

So non-linear contributions at quadratic order occur only in the cases of resonances, if they exist.\(^1\) One can show that, in the minimal subtraction scheme, this property holds to all orders in perturbation theory.

**Wilsonian scheme**

In this scheme, we equate the renormalization scale \( \ell \) with the UV scale \( \varepsilon \), viewed as a fundamental high-energy scale. We demand that the renormalized theory does not depend on the UV cutoff scale, i.e. \( \varepsilon \partial_\varepsilon e^{S_{\text{bdy}}} = 0 \). Then the renormalized boundary couplings depend on the UV scale \( \varepsilon \) (as the regularized perturbative integrals do). At second order, the corresponding beta-functions read:

\[
\beta^{\text{WS}}_{\rho} := \varepsilon \frac{d}{d \varepsilon} \mu_\rho = y_\rho \lambda^\rho + \sum_{\mu, \nu} D^\rho_{\mu \nu} \lambda^\mu \lambda^{\nu}
\]  

In contrast with the minimal subtraction scheme, eq. \( (4.7) \), there is no restriction to ‘resonant’ boundary couplings in the sum giving the quadratic term of the beta-function \( (4.8) \).\(^2\)

We will see below that both schemes are useful in the study of the rolling tachyon perturbations, when it comes to understand the role of the contact terms.

### 4.2 Beta-functions for the brane-antibrane system

Coming back to the brane-antibrane system, we consider the following worldsheet action on the upper half-plane, as a function of the boundary couplings. So now we take the boundary variable to be \( u \in \mathbb{R} \). For convenience, we rescale the coupling according to \( \lambda^\pm \to 2\pi \lambda^\pm \).

\[
S = S_{\text{bulk}} - \int dx \left( \lambda^+ \sigma^+ \otimes \psi^+ e^{i\sigma X + \omega X^0} + \lambda^- \sigma^- \otimes \psi^- e^{-i\sigma X + \omega X^0} - i \frac{\delta r}{2} \sigma^3 \otimes \partial_u \tilde{X} \right)
\]  

We omitted for the moment the contact term, which will enter later on in the discussion.

\(^1\)Notice that, if the boundary perturbations in \( (4.3) \) are superficially marginal, the resonances correspond to the appearance of a marginal operator in the boundary OPE.

\(^2\)The linear term, as well as the resonant quadratic terms, that are common to both schemes, can be shown to be ‘universal’, i.e. independent of the scheme chosen for the computations.
Distance coupling

Let us start by discussing the beta-function for the distance perturbation. According to the general discussion above, one has

$$\beta_r = (1 - h_r) \frac{\delta r}{2} + (D^r_{r+} + D^r_{r-}) \lambda^- \lambda^+ + D^r_r \frac{\delta r}{2} \lambda^+ + D^r_r \frac{\delta r}{2} \lambda^- \cdots$$  \hspace{1cm} (4.10)

where the ellipsis here stands for higher order terms. The first term on the RHS vanishes because the conformal dimension of the distance perturbation is one. All the second order, all the structure constants for the three boundary operators under study appear, since, being all of conformal dimension one, they lead potentially to resonances.

Without much work, we have that $D^r_{\mp \mp} = 0$. The fusion of the tachyon vertex operators $T^\pm$ will never produce the current $\partial_u X$, as the $e^{\omega X_0}$ factors just add up. The structure constants $D^r_{r \mp}$ also have to vanish, since the fusion of $T^\pm$ with the boundary current $i\sigma^3 \partial_u X$ comes with the Chan-Paton factor $\sigma^\pm \sigma^3 = \mp \sigma^\pm$ hence the beta-function coefficient would be proportional to $\text{Tr}(\sigma^\pm \sigma^3) = 0$.

At higher orders in perturbation theory, we would find a similar behavior. Namely, the fusion of any number tachyon vertices cannot produce the distance-changing operator, hence the beta-function for $\delta r$ does not get tachyon 'source terms' (which would be proportional to $(\lambda^+ \lambda^-)^n$ at order $2n$). In other words, the distance coupling does not run in the rolling tachyon background (2.9).

If we consider instead of (2.9) the time-reversal-symmetric tachyon profile ('full S-brane'):

$$T^\pm = \frac{\lambda^\pm}{2\pi} e^{\pm irX} \cosh \omega X_0$$  \hspace{1cm} (4.11)

the conclusion can be different, as the structure constants $D^r_{\mp \mp}$ do not have to vanish by similar arguments. With an explicit computation, one gets indeed in this case

$$\beta_r = 2r \lambda^+ \lambda^- + O \left( (\lambda^+ \lambda^-)^2 \right)$$  \hspace{1cm} (4.12)

Hence the perturbation (4.11) implies a RG running of the distance coupling, unless $r = 0$. The beta-function (4.12) is scheme-independent, as the divergence is logarithmic.

This result has far-reaching consequences. Unlike the case of coincident brane-antibrane or of a non-BPS brane, the effective action of the brane-antibrane system at finite distance should be such that, while the 'half S-brane' rolling tachyon is allowed as a solution of its equations of motion, the 'full S-brane' should not.

4.3 Tachyon couplings at quadratic order

We now compute the beta-functions for the tachyon couplings $\lambda^\pm$ at order $\lambda^+ \lambda^-$, for the 'half S-brane' profile.1

---

1Let us remark in passing that, by shifting the zero-mode of the time-like field $X_0 \rightarrow X_0 + \alpha$, there is a common rescaling of the couplings $\lambda^\pm \rightarrow \lambda^\pm e^{-\alpha}$. This is a common feature of Liouville-like theories. For this reason, the perturbative expansion in $\lambda^\pm$ does strictly make sense only in the Euclidean theory obtained by $X_0 \rightarrow iX_0$. 

---
The boundary OPEs to consider at quadratic order are the distance-tachyon OPE
\[-i\sigma^3 \otimes \partial_\mu \tilde{X}(x_1) \sigma^\pm \otimes \psi^\pm e^{\omega X_0 \pm i r \tilde{X}}(x_2) \sim \frac{-2}{x_1 - x_2} (\pm \sigma^\pm \otimes (\pm r) \psi^\pm \bar{e}^{\omega X_0 \pm i r \tilde{X}} + \cdots) \] (4.13)
and the tachyon-tachyon OPE
\[\sigma^+ \otimes \psi^+ e^{\omega X_0 + i r \tilde{X}}(x_1) \sigma^- \otimes \psi^- e^{\omega X_0 - i r \tilde{X}}(x_2) \sim -\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right) (1 - 4r^2) \frac{1}{(x_1 - x_2)^2} \bar{e}^{2\omega X_0(x_1)} + \cdots \] (4.14)
both for \(x_1 > x_2\). The ellipsis stands for less singular terms.

**Beta-function for \(|r| < 1/2\).**
Whenever \(|r| < 1/2\) the OPE (4.14) does not lead to singularities when integrated. Hence, in the minimal substraction scheme, no corresponding counterterm is needed. This reflects the fact that the contact term is zero in this range.1 This extends to all orders in perturbation theory.

The case of the OPE (4.13) is different, as it leads to a logarithmic divergence for any \(r \neq 0\). From (4.7) the relevant \(\beta\)-functions are of the form:
\[\beta_\pm = (1 - h_\pm)\lambda^\pm + (D^\pm_\pm + D^\pm_\mp) \frac{\delta r}{2} \lambda^\pm + \cdots \] (4.15)
We get at second order that
\[\beta_\pm = \left(\frac{1}{2} - r^2 - \omega^2 - 2r \delta r\right) \lambda^\pm \] (4.16)
this is valid in any scheme, as only universal quantities appear. If one keeps the distance perturbation to zero (\(\delta r = 0\)) then the rolling tachyon background is marginal at second order provided that the on-shell condition \(\omega^2 + r^2 = 1/2\), as expected.

Otherwise, the marginality of the perturbation is restored, at this order, if we use instead the on-shell condition
\[\omega^2 + (r + \delta r)^2 = 1/2 \] (4.17)
This is compatible with the interpretation of the boundary perturbation \(\sigma^3 \otimes i \partial_\mu \tilde{X}\), that changes the relative position of the D-brane and the anti D-brane. It is T-dual to the relative Wilson line that appears in the action (2.2).2 One checks that the normalization of this coupling in (2.2) is compatible, through T-duality, with relation (4.17). This analysis shows that, at least at this order, the rolling tachyon perturbations \(T^\pm\) 'adjust themselves' to a change ofbrane-antibrane separation in order to stay marginal.

---
1 In the Wilsonian scheme the contact term is an *irrelevant* operator in this range.
2 To be more correct, as auxiliary fields from the Fermi superfield couples to this perturbation, some \(\pm i \delta r \lambda^\pm \psi^\pm e^{\pm i r \tilde{X} + \omega X_0}\) correction should be included. We verify that it doesn’t modify the \(\beta\)-function at quadratic order. Moreover, this term shows up naturally if we work directly with the superspace distance perturbation \(i \delta r \Gamma^+ \Gamma^- D_\mu \tilde{X}\).
**Beta function for $1/2 < |r| < r_c$.**

When $1/2 < |r| < r_c$ the situation is different. The operator $\exp 2\omega X_0$ (that appears also in the contact term) becomes relevant, hence should be considered in the discussion. As stated earlier, this operator in unphysical from the superstring theory point of view (at zero superghost number).

The corresponding boundary coupling is denoted by $\mu_c$. The tachyon-tachyon OPE (4.14) gives a singular perturbative integral at second order:

$$
\int dx_1 \int_{x_1-L}^{x_1-\varepsilon} dx_2 \psi^+ e^{\omega X_0 + ir\tilde{X}}(x_1) \psi^- e^{\omega X_0 - ir\tilde{X}}(x_2) \neq \frac{1}{2} \nabla^2 \int dx_1 \psi^2 e^{2\omega X_0}(x),
$$

after removing the IR cutoff ($L \to \infty$ limit).

In the minimal subtraction scheme, the following local counterterm is needed at this order to cancel the divergence:

$$S_{ct} = -\lambda^+ \lambda^- \varepsilon^{1-4r^2} \int dx \psi^2 e^{2\omega X_0}(x).$$

Naturally, it agrees precisely with the expression of the contact term in the action (2.15). Since this divergence is power-like, it does not add any non-linear term in the minimal scheme beta-function $\beta_{c}^{\text{BS}}$ for the coupling $\mu_c$. Hence, the latter can be consistently set to zero in the renormalized theory at this order.

For the distance $|r| = 1/2$, amplitudes are finite without the counterterm, so it is not strictly needed\(^1\), but it contributes nevertheless finitely to the amplitudes.

In the Wilsonian scheme, the beta-function reads, at second order:

$$\beta_{c}^{\text{WS}} = (1 - 4r^2)\mu_c + (1 - 4r^2) \lambda^+ \lambda^-$$

One sees here an interesting phenomenon. The operator $\exp 2\omega X_0$ is relevant at linear order, but the RG flow gives an IR fixed point for this coupling at quadratic order, for $\mu_c = -\lambda^+ \lambda^-$. Comparing the outcomes of both schemes, one gets the same results but the interpretation is different. In the minimal subtraction scheme the contact term appears as a counterterm, but the corresponding renormalized coupling is consistently set to zero. On the contrary, in the Wilsonian scheme, the RG flow has a fixed point with non-zero renormalized coupling $\mu_c$. Both points of view are 'non-supersymmetric', as in the superspace formulation this term is present from the beginning and removes the divergence under discussion.

### 4.4 Marginality beyond quadratic order ($1/2 \leq |r| < r_c$)

Part of the quadratic order results generalizes immediately to higher orders. Indeed only the fusion of distance perturbations with, say, $T^+$ can produce $T^+$ itself (since the fusion of $n$ tachyons goes as $e^{n\omega X_0}$, as far as the $X_0$ dependence is concerned). Hence, if we set

\(^1\)But partition function appears to be discontinuous at $|r| = 1/2$ without its contribution.
\( \delta r = 0 \) from the very beginning, we expect that the beta-functions \( \beta_{\pm} \) vanish to all orders in perturbation theory. With the same reasoning, the operator \( \exp(2\omega X_0) \) that we had to consider for \( |r| > 1/2 \) cannot receive higher-order contributions to its beta-function.

However, study of the marginality at higher orders is quite messy when \( |r| \) is getting closer to the critical distance, as the fusion of tachyon vertex operators produces more and more relevant boundary operators. For a given value of \( r \), these operators, of the form \( e^{2n\omega X_0} \) with \( n \in \mathbb{Z}_+ \), become (superficially) relevant if \( n < (2 - 4r^2)^{-1/2} \), and are of dimension one when they saturate this bound. These resonances occur all for \( 1/2 \leq |r| < 1/\sqrt{2} \); this range was excluded by Bagchi and Sen in their analysis \([18]\) for this precise reason.

A given operator \( e^{2n\omega X_0} \) appears first at order \( 2n \) in the perturbative expansion in the tachyon perturbations, hence the beta-function \( \beta_n \) for its coupling \( \lambda_n \) is of the form:

\[
\beta_n = (1 - 4n^2\omega^2)\lambda_n + \mathcal{O}((\lambda^+\lambda^-)^n) \tag{4.21}
\]

It is easier then to work in the minimal subtraction scheme, where one just has to worry about logarithmic divergences, i.e. resonances. As we emphasized above, if the fusion of (superficially) marginal operators produces a (superficially) marginal operator, it generates a source term in the corresponding minimal scheme beta-function. It is nevertheless interesting to consider whether power-like divergences are also present.

At second order the potentially marginal operator is nothing but the contact term itself, \( e^{2\omega X_0} \), for the distance \( |r| = 1/2 \). Fortunately, thanks to its fermionic part the OPE \((4.14)\) vanishes, hence there is no logarithmic divergence to cancel. The contact term, that is not requested for this purpose, is nonetheless non-zero and contribute finitely to the amplitudes.

**Marginality for \( \sqrt{7}/4 < |r| < \sqrt{17}/6 \)**

The next possible resonance occurs when the operator \( e^{4\omega X_0} \) becomes of dimension one, i.e. for \( \omega = 1/4 \) (equivalently, \( |r| = \sqrt{7}/4 \). The potential logarithmic divergence would occur at fourth order in perturbation theory. In order to investigate this issue we compute below all the possible divergent terms that occur at order \((\lambda^+\lambda^-)^2\) from the perturbative integrals, that involve both the tachyon and contact term vertex operators. In the computations of this subsection, we use the full non-local contact term \((3.6)\), as even the sub-leading terms contribute *a priori* to the divergences.

The first contribution comes from two contact term insertions (symbolically CC). Using the notations \( a = 4\omega^2 \) and \( T^\pm = e^{\pm ir\tilde{X}+\omega X_0} \), it reads

\[
CC = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \frac{\varepsilon^{2a-2}}{4} \int \frac{dx_1}{x_1-L+\varepsilon} \int \frac{dx_2}{x_2-L+\varepsilon} \left( ;T^+(x_1+\varepsilon)T^-(x_1) ; + ;T^-(x_1+\varepsilon)T^+(x_1) ; \right) \times \left( ;T^+(x_2+\varepsilon)T^-(x_2) ; + ;T^-(x_2+\varepsilon)T^+(x_2) ; \right) \tag{4.22}
\]

The contact term being multiplied by the Chan-Patton identity matrix. The short-distance regularization chosen here prevents any operator to approach another one at less that \( \varepsilon \), before
integration of the auxiliary fields. The most natural IR cutoff prescription is to constraint two ordered operators not to move away from each other by more that $L$, also before integration of auxiliary fields. One gets then

$$
CC \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left( \frac{1}{2a+1} \left( \frac{L}{\varepsilon} \right)^{2-2a} - \left( \frac{L}{\varepsilon} \right)^{1-2a} \right)
- \frac{5 - 6a - (2a - 1)2^{2a+2} \text{F}_1(1-a,-a - \frac{1}{2}; -a + \frac{1}{2}, \frac{3}{4})}{4(2a + 1)(2a - 1)} \left( \frac{L}{\varepsilon} \right)^{1-4a} 
\times L^{4a-1} \int dx_1 \varepsilon^{4\omega} X_0(x_1). \tag{4.23} \right)
$$

The second contribution, from two tachyons and a contact term, is more involved as one has to integrate over two operator positions, leading to various type of singularities. One has to be careful with path ordering of the contact term with the tachyon; we have to distinguish three contributions, symbolically noted CTT, TCT and TTC. One finds that the contributions of CTT and TTC are equal, but TCT is different. We have to sum these three contributions together. Using the notation $C(x) = \varepsilon^T(x+\varepsilon)T^{-}(x) + \varepsilon^T(x+\varepsilon)T^{+}(x)$, one has:

$$
CTT + TCT + TTC = 
- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon^{a-1} \left( \int dx_1 \int_{x_1-L+\varepsilon}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \varepsilon^T(x_1) \varepsilon^T(x_2) \varepsilon^T(x_3) 
+ \int dx_1 \int_{x_1-L}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \varepsilon^T(x_1) \varepsilon^T(x_2) \varepsilon^T(x_3) 
+ \int dx_1 \int_{x_1-L}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \varepsilon^T(x_1) \varepsilon^T(x_2) \varepsilon^T(x_3) \right) \tag{4.24} \right)
$$

Here, the whole computation is multiplied by the upper part of the identity matrix, since $T^+$ and $T^-$ are themselves multiplied by $\sigma^+$ and $\sigma^-$ respectively. One should also take into account the permuted version of (4.24) which has ordering $T^-T^+$ instead of $T^+T^-$. From symmetry of the OPE’s under this permutation, it contributes the same result but multiplied by the lower part of the identity matrix. Thus, the computation of the divergent terms gives the result, see appendix B:

$$
CTT + TCT + TTC \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[ - \frac{2}{1+2a} \left( \frac{L}{\varepsilon} \right)^{2-2a} + \frac{1}{a} \left( \frac{L}{\varepsilon} \right)^{1-2a} 
+ \frac{2(a - 1)}{3a} \left( \frac{L}{\varepsilon} \right)^{1-a} \left( \frac{2}{a + 1} \text{F}_1(-a,a+1,a+2,-1) \right) \right. 
\left. + \frac{2}{a + 1} \text{F}_1(2-a,a+1,a+2,-1) \right]
\times L^{4a-1} \int dx_1 \varepsilon^{4\omega} X_0(x_1). \tag{4.25} \right)
The coefficient $V(a)$ is given by (we did not find a closed form for it):

$$
V(a) = (a-1) \sum_{n=0}^{\infty} \sum_{s=0}^{1} \frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)(3a-s-n)} \left( 2F_1(n-a,1+n-2a;2+n-2a;-1) \int \frac{\Gamma(a)}{1+n-2a} \right.
$$

$$
+ \frac{2F_1(s-a,1+s-2a;2+s-2a;-1)}{1+s-2a} \int + \frac{2F_1(n-a,s+n-1-2a;s+n-2a;-1)}{s+n-1-2a} \int
$$

$$
+ (a-1) \sum_{n,p=0}^{\infty} \frac{\Gamma(a)\Gamma(a-1)}{\Gamma(a-n)\Gamma(1+n)\Gamma(a-1-p)\Gamma(1+p)} \left( 2F_1(1-a,n+p-3a,n+p+1-3a,-1) \int \right.
$$

$$
\times \frac{2F_1(2+p-a,n+p+1-2a,n+p+2-2a,-1)}{n+p+1-2a} \int
$$

$$
+ (a-1) \sum_{p=0}^{\infty} \sum_{s,t=0}^{1} \frac{\Gamma(a-1)}{\Gamma(a-1-p)\Gamma(1+p)(3a-s-t-p)} \left( 2F_1(2+p-a,s+p+1-2a,s+p+2-2a,-1) \int \right.
$$

Finally, one has to consider the contribution from four tachyon insertions in the path integral (TTTT). The method of computation of the multiple integral is explained in appendix C. After a lengthy computation one gets\(^1\)

$$
TTTT = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \int dx_1 \int_{x_1}^{x_1-\epsilon} dx_2 \int_{x_2}^{x_2-\epsilon} dx_3 \int_{x_3}^{x_3-\epsilon} dx_4 \psi^+T^+(x_1)\psi^-T^-(x_2)\psi^+T^+(x_3)\psi^-T^-(x_4):
$$

$$
\sim \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \left\{ \frac{1}{2a+1} \left( \frac{L}{\epsilon} \right)^{2-2a} + \frac{a-1}{a} \left( \frac{L}{\epsilon} \right)^{1-2a} \right\}
$$

$$
- \frac{2(a-1)}{3a} \left( \frac{L}{\epsilon} \right)^{1-a} \left( 2F_1(-a,a+1,a+2,-1) \int \frac{2F_1(-a,a-1,a,-1)}{a+1} + \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \right)
$$

\(^1\)The term with ordering $T^-T^+T^-T^+$ contributes the same result thus the total computation is directly multiplied by the identity matrix as in (4.25).
\[ + \left( \frac{4}{a} \right)^{1-4a} \frac{1}{1-4a} \left[ (a-1)^2 \left( \frac{\text{2F1} (1 - 2a, a - 1, 1, -1)}{a - 1} + \frac{\text{2F1} (1 - 2a, 2 - 3a, 3 - 3a, -1)}{2 - 3a} \right) \times \left( \frac{\text{2F1} (-a, a - 1, -1)}{a - 1} + \frac{\text{2F1} (-a, 1 - 2a, 2 - 2a, -1)}{1 - 2a} \right) + \frac{\text{2F1} (2 - a, a + 1, a + 2, -1)}{a + 1} + \frac{\text{2F1} (2 - a, 1 - 2a, 2 - 2a, -1)}{1 - 2a} \right) + (2(a - 1)^2 - 1) \left( \frac{\text{2F1} (1 - a, a + 1, -1)}{a} + \frac{\text{2F1} (1 - a, 1 - 2a, 2 - 2a, -1)}{1 - 2a} \right) \times \left( \frac{\text{2F1} (1 - 2a, a + 1, -1)}{a} + \frac{\text{2F1} (1 - 2a, 1 - 3a, 2 - 3a, -1)}{1 - 3a} \right) \left\{ U(a) \left( \frac{L}{\varepsilon} \right)^{1-4a} \right\} L^{4a-1} \int dx_1 e^{4\omega X_0^*(x_1)} \right) (4.27) \]

with \( U(a) \) a numerical coefficient which is not singular at \( a = 1/4 \).

As in the previous computation, the coefficient \( U(a) \) is known only as a series expansion

\[ U(a) = \frac{(a-1)^2}{4a-1} \left( \frac{\text{2F1} (1 - 2a, a - 1; a; -1)}{a - 1} + \frac{\text{2F1} (1 - 2a, -3a; 1 - 3a; -1)}{3a} \right) \times \left( \frac{\text{2F1} (-a, 1 - 2a; 2 - 2a; -1)}{1 - 2a} + \frac{\text{2F1} (-a, a - 1; a; -1)}{a} \right) + \frac{\text{2F1} (2 - a, 1 - 2a; 2 - 2a; -1)}{1 - 2a} + \frac{\text{2F1} (2 - a, a + 1; a + 2; -1)}{a + 1} \right) + \frac{(2(a - 1)^2 - 1)}{4a-1} \left( \frac{\text{2F1} (1 - 2a, 1 - 3a; 2 - 3a; -1)}{3a - 1} + \frac{\text{2F1} (1 - 2a, a + 1; a + 1; -1)}{a} \right) \times \left( \frac{\text{2F1} (1 - a, 1 - 2a; 2 - 2a; -1)}{1 - 2a} + \frac{\text{2F1} (1 - a, a + 1; a + 1; -1)}{a} \right) \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(a + 1)}{\Gamma(n + 1)\Gamma(a - n + 1)(a + n + 1)3a - n} \right\} \left( \frac{\text{2F1} (n - a, -2a + n + 1; -2a + n + 2; -1)}{-2a + n + 1} + \frac{\text{2F1} (n - a, -2a + n - 1; n - 2a; -1)}{-2a + n - 1} \right) + \frac{(a - 1)^2}{4a-1} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(a - 1)}{\Gamma(n + 1)\Gamma(a - n - 1)(a + n + 1)3a - n} \right\} \left( \frac{\text{2F1} (-a + n + 2, -2a + n + 1; -2a + n + 2; -1)}{-2a + n + 1} + \frac{\text{2F1} (-a + n + 2, -2a + n + 3; -2a + n + 4; -1)}{-2a + n + 3} \right) \]
\[ + 2 \left(2(a - 1)^2 - 1\right) \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(n+1)\Gamma(a-n)(a+n)(3a-n-1)} \frac{2F_1(-a+n+1, -2a+n+1; -2a+n+2; -1)}{-2a+n+1} \]

(4.28)

The last three sums are rapidly converging, thus \(U(a)\) is known with good accuracy, for any value of \(a \leq 1/4\) (or \(\omega \leq 1/4\)).

Let us now investigate the possible logarithmic divergences, that can only occur from the \(TTTT\) integral. Since we have that

\[\frac{(L)^{1-4a}-1}{1-4a} \xrightarrow{a \to 1/4} \log \frac{L}{\varepsilon},\]

(4.29)

only the last but one term in (4.27) could lead to a logarithmic divergence at \(\omega = 1/4\). It turns out that, in this limit the coefficient of this term vanishes exactly. Looking more closely at this computation, one sees that each multiple integral that one gets from the three different fermionic contractions – see eq. (5.34) – has a logarithmic term as expected, however the sum of them precisely cancels. Hence, the same occurs as at order two; the coefficient in front of the potentially resonant term in the beta-function vanishes.\(^1\)

In order to check whether power-like divergences remain at fourth order, one has to resum the three contributions obtained above. The full contribution at order \((\lambda^+\lambda^-)^2\) is given by \(CC + CTT + TCT + TTC + TTTT\). Comparing (4.22), (4.24) and (4.27), one sees that the coefficients in front of all divergent terms vanish exactly for any value of \(\omega \geq 1/4\). Hence, in this range, if one includes the two-tachyon contact term dictated by worldsheet supersymmetry, perturbative expansion is finite.\(^2\)

As said before we were not able to compute the coefficient associated to the term of order \(\varepsilon^{1-4a}\), which becomes divergent for \(\omega < 1/4\) in a closed form. Using a numerical evaluation, we find that the sum of the contributions gives a non-zero coefficient for any \(\omega < 1/4\). Hence, a power-like divergence remains in this range. By dimensional counting, this uncanceled divergence corresponds to four tachyon operators coming close together at the same point. It is not unexpected that this divergence is not canceled by the contact term, as the latter

\(^1\)This is confirmed by a direct evaluation of the TTTT integral at \(\omega = 1/4\) (with Mathematica) which gives

\[TTTT = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) 4 \int dx_1 \int^{x_1-\varepsilon} dx_2 \int^{x_2-\varepsilon} dx_3 \int^{x_3-\varepsilon} dx_4 \psi^+ e^{X_0/4 + i\epsilon X}(x_1) \psi^- e^{X_0/4 - i\epsilon X}(x_2) \times \]

\[\times \psi^+ e^{X_0/4 + i\epsilon X}(x_3) \psi^- e^{X_0/4 - i\epsilon X}(x_4); \]

\[\sim \left[2 \frac{L}{\varepsilon} \right]^{3/2} + \left[\frac{7\sqrt{\pi} \Gamma\left(\frac{3}{2}\right)}{3\Gamma\left(\frac{1}{2}\right)} - \alpha \right] \left[\frac{L}{\varepsilon}\right]^{3/4} - \left[3 \left(\frac{L}{\varepsilon}\right)^{1/2}\right] \int dx_1 \psi^0(x_1); \]

with \(\alpha \simeq 1.24\ldots\). Logarithmic divergences are again found to vanish.

\(^2\)Not considering into account possible operator renormalization if there are operator insertions in the path integral.
corresponds to a two-tachyon collision. Since this remaining divergence is non-logarithmic, it does not mean that the boundary theory is not conformal, but rather that it should be renormalized at quartic order. It should be possible to cancel this divergence with higher-order contact-term. They may correspond to additional non-linear terms in the superspace action (2.6) (a four-aiduxiliary field vertex is needed then).

As mentioned in section 3.2, we also obtained an unexpected result. If we assume that the computations of CTT and CC type terms could be equivalently done with the use of the simple dominant term $\varepsilon^{a-1}e^{2\omega X_0}$ in (3.6), then we get the following contribution

$$CC + CTT + TCT + TTC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left\{ -\frac{1}{1+2a} \left( \frac{L}{\varepsilon} \right)^{2-2a} + \frac{1}{a} \left( \frac{L}{\varepsilon} \right)^{1-2a} \right\} + \frac{2(a-1)}{3a} \left( \frac{L}{\varepsilon} \right)^{1-a} \left( \frac{2F_1(-a,a+1,a+2,-1)}{a+1} + \frac{2F_1(-a-1,a-1,-1)}{a-1} \right)$$

$$+ \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \left( \frac{L}{\varepsilon} \right)^{1-4a} + \left( \frac{L}{\varepsilon} \right)^{1-a} \left[ \frac{a-1}{3a} \left( \frac{2F_1(-a+1,1-2a;1-2a;-1)}{1-2a} + \frac{2F_1(-a+1,2-2a;2-2a;-1)}{1-2a} \right) \right. \right.$$ 

$$\left. - \frac{2F_1(-a,-1-2a;2-2a;-1)}{1+2a} \right] + \frac{1}{2a+1} \right\} \int d^4x_1 \; e^{4\omega X_0 X_0^*} (x_1)$$

One recognizes the coefficients of the three first divergences; these are precisely the ones appearing in the sum of (4.22) and (4.24). Moreover, numerical comparison of the coefficients gives almost identical results; this tiny difference could reasonably originates from the approximated evaluation of the infinite sums. This seems to show the equivalence of the two computations – (4.30) being of course significantly easier to perform – and then of the two (local and non-local) expressions of the contact term.

**Marginality to all orders**

Computations become intractable for the next resonance, which occurs for $\omega = 1/6$ (or equivalently $|r| = \sqrt{17}/6$), as we have to consider sixth order perturbation theory, with contributions from both counterterms found so far. However we assume that the same occurs; the coefficient in front of the logarithmic six-tachyon divergence should vanish as well.

To summarize, we have found that, to all orders in perturbation theory, the theory defined by the boundary action (2.15) is a boundary conformal field theory when $|r| < \sqrt{17}/6$. In the range $\sqrt{17}/6 < |r| < 1/\sqrt{2}$, the theory is conformal at least up to order five. We naturally expect that the theory is conformal to all orders in this range as well.

As a side remark, the theory defined by the limit $r \to r_c^-$ seems not well-defined. In this case, all the operators $e^{2\omega X_0}$ are relevant, and by doing the perturbative expansion in the tachyon couplings we would need an infinite number of counter-terms. By contrast, the theory
defined directly at \( r = r_c = 1/\sqrt{2} \) seems fine. The boundary interaction (with \( T^\pm \sim e^{\pm i\lambda/\sqrt{2}} \)) is similar to a boundary sine-Gordon theory, with additional CP factors. Other puzzling features of the \( r \to r_c^- \) limit will be discussed in the next section.

5 Towards an effective action

We argued in the last section that, for all values of the brane-anti-brane distance below the critical value \( r_c \), the homogeneous rolling tachyon solution with a fixed separation is an exact boundary conformal field theory. Thus, a spacetime effective action that is valid around this particular solution (for all the range \( 0 \leq |r| < r_c \)) should have such tachyon profile as a solution of its equations of motion.

An effective action for this same was proposed by Garousi [23]. In a different parameterization of the tachyon field,\(^1\), it reads:

\[
L_g(T, \dot{T}, r, \dot{r}) = -\frac{2}{\cosh \frac{\pi |T|}{\sqrt{1 + 4\pi^2 r^2}} \sqrt{1 + 4\pi^2 r^2 |T|^2 - |\dot{T}|^2 - \pi^2 \dot{r}^2}}
\]  

One checks readily that, with \( \dot{r} = 0 \), \( \delta_r L_g \neq 0 \) for any non-zero separation. Hence, this Lagrangian cannot admit solutions with constant brane-antibrane separation. This is not to be unexpected, since it was obtained by a fermion number orbifold of the non-Abelian tachyon-DBI action for a pair of coincident non-BPS D-branes. Therefore it could only be valid for an infinitesimal brane separation. Since \( \delta_r L_g \) is linear in \( r \), it seems not even to be valid in this limit.

For \( r = 0 \) one should recover the effective action for coincident brane and anti-brane, which is the same as the effective action for a non-BPS brane, up to a projection that makes the tachyon real. As our approach is similar, we expect to find an effective Lagrangian found by Kutasov and Niarchos [13]. In terms of the complex tachyon field\(^2\) \( \tau(t) \), the Lagrangian for coincident D0/anti-D0-brane is expected to be (setting the brane tension to one for convenience)

\[
L = -\frac{2}{1 + \frac{|\tau|^2}{2}} \sqrt{1 + \frac{|\tau|^2}{2} - |\dot{\tau}|^2}
\]  

5.1 Spacetime approach

One considers the spacetime Lagrangian of the D0-\( \bar{D}0 \) system, depending on the complex tachyon field \( \tau \), its first derivative, the distance field \( r \) and its first derivative. Without loss of generality, as the phase of the tachyon in the solutions under study is constant, we take \( \tau(t) \) real. Since, as we argued before, rolling tachyon solutions at constant separation exist, the effective Lagrangian describing nearby field configurations should satisfy the condition

\[
\frac{\delta L(\tau, \dot{\tau}, r, \dot{r})}{\delta r} \bigg|_{\dot{r}=0, \dot{\tau}=\omega \tau} = 0
\]  

\(^1\)This field redefinition was discussed in [13] for the \( r = 0 \) case.

\(^2\)By reality arguments we can always choose \( \lambda^+ = \lambda = (\lambda^-)^* \).
where $\omega = \sqrt{\frac{1}{2} - r^2}$, as well as the equation of motion for the tachyon with a profile of the form $\tau = \mu e^{\omega t}$.

This constraint is actually quite strong. Let us expand the Lagrangian in powers of the tachyon, for any distance; as in the worldsheet analysis, this expansion breaks down for $t \to \infty$. As the action should be invariant under the reflexion $r \to -r$, one gets at second order in the tachyon (but to all orders in the separation $0 \leq r < r_c$)

$$L = A(r^2, \dot{r}^2) + B(r^2, \dot{r}^2)\tau^2 + C(r^2, \dot{r}^2)\dot{\tau}^2 + O(\tau^4) \quad (5.4)$$

Using the equation of motion (5.3), as well as the e.o.m. for the tachyon field $\tau(t)$, one gets the following on-shell relations:

$$\begin{cases}
\frac{\partial}{\partial r} (A(r^2, 0) + B(r^2, 0)) + (\frac{1}{2} - r^2) \frac{\partial C(r^2, 0)}{\partial r^2} = 0 \\
B(r^2, 0) - (\frac{1}{2} - r^2)C(r^2, 0) = 0
\end{cases} \quad (5.5)$$

Then one can solve this simple equations, and set the integration constants in order to match (5.2) for $r = 0$. One obtains the following result:

$$L = -2 + \sqrt{1 - 2r^2} \left( \frac{\tau^2}{2} + \frac{\dot{\tau}^2}{1 - 2r^2} \right) + \ldots \quad (5.6)$$

Hence, the action is completely determined up to quadratic order.

### 5.2 Recurrence relations

In order to move further, we would like to adapt the computation of Kutasov and Niarchos [13] to the more general case of non-zero D0-D0 distance.

More precisely, one is looking for an effective Lagrangian for the complex tachyon $\tau(t)$, as well as for the distance field $r(t)$. We assume that, by the symmetries of the problem, only even powers of the fields and their derivative appear, i.e one has $L(|\tau|^2, |\dot{\tau}|^2, r^2, \dot{r}^2)$. In order to simplify the problem, we would like to consider only solutions at fixed distance as we have found in the BCFT analysis, i.e. the quantity $L(|\tau|^2, |\dot{\tau}|^2, r^2, \dot{r}^2 = 0)$.

We start by assuming that the effective Lagrangian is a generic (analytic) function of the tachyon and its first time derivative:

$$L = \sum_{n=0}^{\infty} L_{2n}$$

$$L_{2n} = \sum_{m=0}^{n} a_m^{(n)} |\dot{\tau}|^{2m} |\tau|^{2(n-m)} \quad (5.7)$$

Furthermore, we impose that the equations of motion coming from this Lagrangian should have a solution of the form

$$\tau = \lambda e^{\omega t}, \quad (5.8)$$
at fixed distance \( r \). Naturally the values of \( a_m^{(n)} \) must be completely independent from \( \lambda \) and \( \lambda^* \), but they are \textit{a priori} functions of \( r^2 \), or equivalently of \( \omega \), as well as of \( \dot{r}^2 \).

This requirement partially fixes field redefinitions ambiguities in the effective action, as it prevents non-linear redefinitions of the tachyon field, however there is still a residual freedom in rescaling the tachyon with a functional of the distance, as \( \tau(t) \to \mu(r)\tau(t) \).

Considering the tachyon equations of motion, requiring an 'half S-brane' solution is far less restrictive as one can do in the coincident case. Indeed, for \( r = 0 \), the most general rolling tachyon solution that corresponds to a boundary CFT reads:

\[
\tau(t) = \lambda e^{\sqrt{2}t} + \zeta e^{-\sqrt{2}t}
\]

with \( \lambda \) and \( \zeta \) arbitrary. As said earlier, this more general solution is not valid for \( r \neq 0 \) since the corresponding worldsheet theory is not marginal.

The equations of motion coming from (5.7) are:

\[
\sum_{m=0}^{n} m a_m^{(n)} \frac{d}{dt} \left[ |\dot{\tau}|^{2(m-1)} (\dot{\tau})^* |\tau|^{2(n-m)} \right] - \sum_{m=0}^{n} (n - m) a_m^{(n)} |\tau|^{2(n-m)} \tau^* |\dot{\tau}|^{2m} = 0
\]

(5.10)
and its complex conjugate.

Using the on-shell relation \( \dot{\tau} = \omega \tau \) one gets, for each level \( n \neq 0 \), a linear relation between the coefficients \( a_m^{(n)} \), \( m = 0, \ldots n \):

\[
\sum_{m=0}^{n} (-\omega^2)^m (2m - 1) a_m^{(n)} = 0.
\]

(5.11)

Since the original Lagrangian contains only even powers of \( \dot{r} \) by assumption, the equation of motion for the distance field is equivalent to

\[
\frac{\delta L}{\delta \omega} = 0,
\]

(5.12)
when evaluated on a solution at fixed distance (i.e. with \( \dot{r} = 0 \)). This equation leads to the following condition on the coefficients \( a_m^{(n)} (\omega) \) (becoming functions of \( \omega \) only for the solutions of interest):

\[
\sum_{m=0}^{n} \frac{d a_m^{(n)}(\omega)}{d \omega} \omega^{2m} = 0 \quad \forall n
\]

(5.13)

Because of the restrictive rolling tachyon ansatz (5.8), the constraints from (5.11) are quite weak. One possible solution (among many others) is the same as found in [13] for an unstable brane:\footnote{There, this recurrence relation seem to be the only possible solution, given the more general tachyon profile (5.9) that should be solution of the equations of motion.}

\[
a_{m+1}^{(n)} = a_m^{(n)} \frac{(n - m)(2m - 1)}{\omega^2(2m + 1)(m + 1)} \quad \forall n \geq 1, m \geq 0
\]

(5.14)

\[
a_0^{(0)} \neq 0
\]
giving

\[ d_{m}^{(n)} = - \frac{a_0^{(n)}}{(2m-1)\omega^{2m}} \binom{n}{m} \]  

(5.15)

Then, using the distance e.o.m. (5.12), as well as continuity with (5.2) in the limit \( r \to 0 \), one gets a unique effective Lagrangian for the D0-\overline{D}0 system at fixed distance:

\[ \mathcal{L}(|\tau|^2, |\dot{\tau}|^2, r^2, \dot{r}^2) = -2 \sqrt{1 - 2r^2} \sqrt{1 + \frac{\tau^2}{2} + \frac{|\dot{\tau}|^2}{1 - 2r^2} + 2\sqrt{1 - 2r^2} - 2} \]  

(5.16)

However, one should not rejoice too soon. By construction, this Lagrangian admits a 'full S-brane' solution of the form \( \tau \sim \cosh \omega t \), while the corresponding worldsheet boundary perturbation is not conformal. Another test that this Lagrangian fails to pass will be given below.

Therefore, it seems that the coefficients \( a_m^{(n)} \) are not given for \( r \neq 0 \) by the same recurrence (5.14) as in the case of coincident branes. Nevertheless, the resulting effective Lagrangian is expected to be continuous in the limit \( r \to 0 \).

### 5.3 A proposal for the effective Lagrangian

In order to move further, we have to make some assumptions. Without loss of generality, we express the full Lagrangian as

\[ \mathcal{L} = -K(\tau^2, r^2) \sqrt{f(\tau^2, \dot{\tau}^2, \omega^2, \dot{\omega}^2)} \]  

(5.17)

Where we have used for convenience \( \omega \) instead of \( r \). This Lagrangian should reproduce (5.2) in the \( r, \dot{r} \to 0 \) limit:

\[ K(\tau^2, 0) = \frac{2}{1 + \frac{\tau^2}{2}} \quad , \quad f(\tau^2, \dot{\tau}^2, 0, 0) = 1 + \frac{\tau^2}{2} - \dot{\tau}^2 \]  

(5.18)

and satisfy \( K(0, \omega) = 2 \), \( f(0, 0, \omega, 0) = 1 \). From the equations of motion one gets:

\[ \frac{\partial}{\partial \omega} K + \frac{K}{2} \frac{\partial}{\partial \omega} \ln f \bigg|_{\tau^2, \omega^2, \dot{\omega}^2, \dot{\omega}^2, 0} = 0 \]  

(5.19a)

\[ \frac{\partial}{\partial \tau} K + \frac{K}{2} \frac{\partial}{\partial \tau} \ln f \bigg|_{\tau^2, \omega^2, \dot{\tau}^2, \dot{\omega}^2, 0} = \frac{1}{\sqrt{f}} \frac{\partial}{\partial \tau} \left[ \left( \frac{K}{2\sqrt{f}} \right) \frac{\partial f}{\partial \tau} \right] \bigg|_{\tau^2, \omega^2, \dot{\tau}^2, \dot{\omega}^2, 0} \]  

(5.19b)

We now make an important assumption. We assume that, as at \( r = 0 \), the 'kinetic' part of the action is constant when evaluated on the rolling tachyon solution. Explicitly we demand that

\[ f(\tau^2, \omega^2, \dot{\tau}^2, \dot{\omega}^2, 0) = 1 \]  

(5.20)

This hypothesis, which is quite natural (if we view this Lagrangian as a deformation of the Lagrangian for coincident branes) was also made \cite{31} in order to get the tachyon effective
action in a linear dilaton background. This constraint gives relations between the derivatives of $f$ evaluated 'on-shell':

$$\left(\frac{\partial f}{\partial \tau} + \omega \frac{\partial f}{\partial \dot{\tau}}\right)\big|_{\dot{\tau} = \omega\tau, \dot{\omega} = 0} = 0, \quad \left(\frac{\partial f}{\partial \omega} + \tau \frac{\partial f}{\partial \dot{\tau}}\right)\big|_{\dot{\tau} = \omega\tau, \dot{\omega} = 0} = 0$$  \quad (5.21)

We define $g(\tau, \omega) = \frac{\partial f}{\partial \tau}\big|_{\tau, \omega\tau, \omega\dot{\tau}}$. Then one gets from the e.o.m. (5.19a)

$$\frac{\partial K}{\partial \omega} + \frac{\tau}{2\omega} K g = 0$$  \quad (5.22)

For the tachyon one gets from (5.19b)

$$\frac{\partial K}{\partial \tau} + \frac{K}{2} g = \frac{d}{dt} \left(\frac{K}{2\omega} g\right) = -\frac{\tau}{2} \frac{\partial (K g)}{\partial \tau}$$  \quad (5.23)

This is solved with

$$K = \omega G(\tau) + H(\omega)$$  \quad (5.24)

Using the boundary conditions (5.18) one has

$$\frac{G}{\sqrt{2}} + H(1/\sqrt{2}) = \frac{2}{1 + \frac{\tau^2}{2}} \quad \text{and} \quad \omega G(0) + H(\omega) = 2$$  \quad (5.25)

It leads to the unique solution, in terms of $\tau$ and $r$

$$K(\tau, r) = 2 \frac{2 + (1 - \sqrt{1 - 2r^2})\tau^2}{2 + \tau^2}$$  \quad (5.26)

One notices that, at the critical distance, $K$ is constant as expected since the tachyon becomes massless.

Let us consider now the kinetic part of the Lagrangian. One finds, using e.g. eq. (5.22) that

$$g(\tau, \omega) = \frac{2\omega\tau}{1 + (1 - \sqrt{2\omega})\frac{\tau^2}{2}}$$  \quad (5.27)

where $g$ is $\partial f / \partial \tau$, evaluated on shell ($\dot{\tau} = \omega\tau$). If we assume that $f$ is the sum of a function of $\tau$ and a function of $\dot{\tau}$, we find a unique Lagrangian for the system (for constant brane separation)

$$\mathcal{L}(\tau, \dot{\tau}, r, 0) = -2 \frac{2 + (1 - \sqrt{1 - 2r^2})\tau^2}{2 + \tau^2} \sqrt{1 + \frac{\sqrt{1 - 2r^2}}{1 - \sqrt{1 - 2r^2}} \log \left(\frac{1 + (1 - \sqrt{1 - 2r^2})\frac{\tau^2}{2}}{1 + (1 - \sqrt{1 - 2r^2})\frac{\tau^2}{2r}}\right)}$$  \quad (5.28)

As a first check, the limit $r \to 0$ naturally reproduces (5.2). We remark also that the second order expansion in terms of the tachyon reproduces (5.6), as expected. The limit $r \to r_c$ is interesting. Not only the potential terms for the tachyon vanishes, as expected, but the kinetic term also goes to zero; one gets $\mathcal{L} \to 1$, independently of $\tau$. 

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The two-dimensional static potential for the tachyon field $\tau$ and the distance $r$ that follows from this action (which is of limited meaning since it is very far from the on-shell configuration in field space) is given by $V(r, \tau) = -L(\tau, 0, r, 0)$. We notice that for $r > 0$ this potential has a global minimum for some finite value of the tachyon $\tau$ (whereas it is at infinity in the $r = 0$ case).

We do not know at this stage how to include the kinetic term for the distance $r$ in this action; in any case this issue may be meaningless, since we aim to reproduce the physics nearby exact open string backgrounds corresponding to constant separation. Would exact solutions with varying distance exist, the corresponding effective action would be different anyway. We can nevertheless study slowly varying configurations with this action, in this case the non-linearities of the distance kinetic term are not significant.

### 5.4 Worldsheet approach

The worldsheet theory contains more information about the tachyon effective action, besides imposing that the rolling tachyon background of interest should be a solution of its equations of motion. Following [13], we expect to get the effective Lagrangian evaluated on-shell to be given by the disk partition function, with the time-like zero modes kept unintegrated:

$$L|_{\tau, \omega, r, 0}(x_0) = -Z(r \mid x_0)_{\text{disk}} \quad (5.29)$$

This relation does not of course determine completely the off-shell effective Lagrangian, however, when one gets recurrence relations of the form (5.15) for its series expansion, it allows to determine the coefficients $a^{(n)}_0$, at each order. One can also test whether any proposal for the effective Lagrangian of the system is compatible with the worldsheet description.

The disk partition function of interest can be expressed as a series:

$$Z(r \mid x_0) = \sum_{n=0}^{\infty} \left( -\lambda^+ \lambda^- e^{2\omega x_0} \right)^n I_n \quad (5.30)$$

with $I_n$ a coefficient that is equal to the sum time-ordered integrals that appear at order $n$ in the perturbative expansion. We can express it in a condensed form as:

$$I_n = \int [dt]^{2n} \left| \frac{1}{2} \frac{3}{4} \ldots \frac{2n-1}{2n} \right| \left( -4r^2 \right)^{2n-1} \sum_{\text{perm} P} \left( -1 \right)^P |a_1 a_2 a_3 a_4 \ldots a_{2n-1} a_{2n}| \left( 1 - 4r^2 \right)^{\frac{2}{3} - \frac{2}{3} \sum_{i=1}^{n-1} a_{2i-1} - a_{2i}}$$

$$\sum_{i=1}^{n-1} \sum_{i=1}^{n} \Theta(t_i - t_{i+1}) \quad (5.31)$$

with the time-ordered measure

$$[dt]^{2n} = \prod_{i=1}^{2n} \frac{dt_i}{2\pi} \prod_{i=1}^{2n-1} \Theta(t_i - t_{i+1}) \quad (5.32)$$

$^1$ More checks can also be made from the worldsheet, using S-matrix computations in the tachyon background [15].
We have also introduced convenient notations for the integrand, defined as:

\[
\begin{vmatrix}
  a_1 & a_2 \\
  a_3 & a_4 \\
  \vdots \\
  a_{2n-1} & a_{2n}
\end{vmatrix} = \prod_{i=1}^{n} \prod_{j=2i+1}^{2n} S(a_{2i-1}, a_j) S(a_{2i}, a_j),
\]

\[
\begin{vmatrix}
  i_1 & i_2 \ldots i_p \\
  j_1 & j_2 \ldots j_n
\end{vmatrix} = \prod_{a=1}^{p} \prod_{a=1}^{n} S(i_a, j_a)
\]

(5.33)

where \( S(i, j) = \left| 2 \sin \frac{\theta_i - \theta_j}{2} \right| \). The sum in (5.31) is done over all permutations within the set \( \{1, 2, 3 \ldots 2n\} \).

Up to \( n = 2 \), the partition function, for given \( |r| < 1/\sqrt{2} \), reads:

\[
Z(r|x_0) = 2 - 2 \lambda^+ \lambda^- e^{2\omega x_0} \int [dt]_2 \left| \frac{1}{2} \right|^{-4r^2} (1 - 4r^2)
\]

\[
+ 2 \left( \lambda^+ \lambda^- e^{2\omega x_0} \right)^2 \int [dt]_4 \left| \frac{1}{2} \right|^{-4r^2} \left( 1 - 4r^2 \right)^2 \left| \frac{1}{2} \left( \frac{3}{4} \right) \right| - \left| \frac{1}{2} \left( \frac{3}{4} \right) \right| (1 - 4r^2)^2 \left| \frac{1}{2} \left( \frac{3}{4} \right) \right|
\]

\[
+ \ldots
\]

(5.34)

The computation at second order in \( T \), for \( r < 1/2 \), gives the result

\[
Z(r|x_0) = 2 - \frac{\Gamma(2 - 4r^2)}{\Gamma^2(1 - 2r^2)} \lambda^+ \lambda^- e^{2\omega x_0} + \mathcal{O}\left( (\lambda^+ \lambda^-)^2 \right)
\]

(5.35)

where we used the Dyson integral [32]:

\[
\int_0^{2\pi} \prod_{i=1}^{n} \frac{dt_i}{2\pi} \prod_{i<j}^{n} |e^{it_i} - e^{it_j}|^{2a} = \frac{\Gamma(1 + n\alpha)}{\Gamma^n(1 + \alpha)}
\]

(5.36)

We notice that the result (5.35) is analytic in \( r \) for all values below the critical distance \( r_c = 1/\sqrt{2} \). The reason for this property should now be familiar to the reader. For \( |r| < 1/2 \), the contact term vanishes, hence give no contribution to (5.35). The value \( r = 1/2 \) is particular. We see that the prefactor of the second order integral in (5.34) vanishes; at the same time, the contact term gives a finite contribution, ensuring the continuity of the result in (5.35). For any \( 1/2 < |r| < r_c \), the second-order integral in (5.35) is divergent. As we explained in subsec. 3.3, the divergence is canceled by the contribution from the contact term, that appear in the worldsheet action (2.15), where \( \varepsilon \) is chosen to be the same as the short distance cutoff in (5.34). The finite part that remains agrees precisely with (5.35). Hence, the presence of the contact term gives (at least at this order) a continuous result all the way to the critical distance.

Finding the complete expression of the disk partition function at any \( |r| < 1/\sqrt{2} \) seems to be out of reach, since integrals involve complicated highly coupled multidimensional integrals with path-ordering (see appendix A).

\[\text{To be precise, we have } P(\{1, 2, 3 \ldots 2n\}) = \{a_1, a_2, a_3 \ldots a_{2n}\}.\]
Spacetime vs. Worldsheet

It is interesting to compare the outcome of the spacetime and worldsheet approaches, up to second order where the worldsheet results are available. The second order spacetime Lagrangian, given by eq. (5.6), was obtained assuming that there exists a solution of the form \( \tau \sim \mu e^{\sqrt{\frac{1}{2} - r^2}} \), where the parameter \( \mu \) could, but did not have to, depend on the brane separation \( r \). Both this computation and the worldsheet one, eq. (5.35), are normalized such as reproducing the expected results in the limit \( r \to 0 \). According to eq. (5.29) they should be equal, when (5.6) is taken on-shell.

Unexpectedly, these two easy computations seem to give two different results. One may advocate that field redefinitions for the tachyon may account for this fact; as argued before, one is left with the possibility of having a rescaling by a function of \( r \) in the map between the worldsheet and spacetime tachyon profiles. Explicitly, in order to match the second-order results, one should have

\[
\tau(t) = \sqrt{\frac{1}{\sqrt{1 - 2r^2}} \frac{\Gamma(2 - 4r^2)}{\Gamma^2(1 - 2r^2)}} \lambda^+ e^{\omega t} \tag{5.37}
\]

As one can see, with this definition, the spacetime tachyon vanishes at the critical distance \( r = 1/\sqrt{2} \), for any finite value of the worldsheet coupling \( \lambda^+ \). This sounds strange, because at the critical distance the tachyon should be massless at tree level in the string perturbative expansion, hence a classical modulus. This strange feature may be related to the observation made above, that the non-linear spacetime action (5.28) becomes field-independent in the limit \( r \to r_c \).

It could be the way string theory deals with a strange situation. When \( r \to r_c \) the tachyon becomes a light field, and we could wonder how a local action along the brane worldvolume dimensions – that is a priori well-defined as the tachyon is lighter than all string modes – would make sense, since the separation between the brane and the antibrane is significant in this regime.

The validity of the field redefinition (5.37) should be tested beyond quadratic order. Unfortunately on both sides our knowledge of the system is not complete. On the spacetime side, the first effective Lagrangian that we considered, given by eq. (5.16) admits 'full S-brane' solutions which should not be allowed. Our other proposal (5.28) for the higher order completion of the Lagrangian relies on an extra (but sensible) assumption that may or may not be true.

On the worldsheet side, we do not know in general how to compute analytically the perturbative 'screening integrals' at higher order.\(^1\)

For the special value of the distance \( r = 1/2 \) the worldsheet computations are tractable hence can be compared to the effective Lagrangian candidates.\(^2\) We present the details of the

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\(^1\)For \( r > 1/2 \), it would be a doubtful task to use numerical calculations, as the integrals are divergent before we include contact terms contributions.

\(^2\)We remind that this is the special value for which the contact term gives finite, non-zero contribution.
computations, up to order eight in the tachyon amplitude in appendix A. Let us quote here the result:

\[
Z(1/2, x_0) = 2 \left( 1 - \frac{\lambda^+ \lambda^- e^{\omega x_0}}{2\pi} + \left( 1 - \frac{\pi^2}{6} \right) \left( \frac{\lambda^+ \lambda^- e^{\omega x_0}}{2\pi} \right)^2 - \left( 1 - \frac{128}{3\pi^2} \right) \left( \frac{\lambda^+ \lambda^- e^{\omega x_0}}{2\pi} \right)^3 \right.
\]

\[+ \left( 1 + \frac{205}{108} + \frac{3575}{162\pi^2} + \frac{\pi^2}{2} + \frac{\pi^4}{70} \right) \left( \frac{\lambda^+ \lambda^- e^{\omega x_0}}{2\pi} \right)^4 \]

\[+ O \left( \left( \frac{\lambda^+ \lambda^- e^{\omega x_0}}{2\pi} \right)^5 \right) \right)
\]

One can partially resum terms coming from the contributions to the perturbative integrals \(I_n\) with the maximal insertions of contact terms, giving an overall factor \((1 + \lambda^+ \lambda^- e^{\omega x_0} / 2\pi)^{-1}\), to all orders. However the terms that remain unfactorized are still difficult to trace back to the Taylor expansion of any known function.

Both Lagrangian candidates, eqs. (5.16) and (5.28), when evaluated on-shell at \(r = 1/2\), give Taylor expansions different from (5.38), beyond second order. Hence, the effective action for the system is still unknown.

6 Conclusions

In this work we have studied on-shell tachyon condensation in the system of a D-brane and an anti D-brane at finite but constant separation, below the critical value \(r_c = 1/\sqrt{2}\) for which the tachyon instability appears.

From the point of view of the worldsheet theory, we have argued that the simple ‘half S-brane’ tachyon time-dependent profile gives a boundary conformal field theory for all sub-critical values of the brane separation \(r\). In the range \(0 \leq |r| < r_c/\sqrt{2}\) this result was already established; no complications occur because the perturbative integrals on the disk all converge. At larger values of \(r\) the analysis becomes more complicated. On the one hand, perturbative integrals diverge; on the other hand, the contact term that arises in the worldsheet action by worldsheet supersymmetry is also divergent. At quadratic order, as expected these divergences cancel each other, ensuring continuity of the partition function at this order for all \(0 \leq |r| < r_c\). We have found a similar behavior for the boundary one-point function.

In order to investigate the question of marginality in more detail, we have analyzed the system from the point of view of the renormalization group equations. At second order we gave a different interpretation of the contact term, that arises by demanding that the beta-function of the system vanishes. We have also shown how the tachyon perturbation ‘reacts’ to a distance perturbation in order to restore marginality.

We have observed that the ‘full S-brane’ rolling tachyon is not marginal anymore whenever the distance is non-zero. We don’t have a physical explanation for this fact: why the time-reversal-symmetric rolling tachyon is a solution for coincident brane and anti-brane, and not when they are separated from each other?
Marginality of the 'half S-brane' rolling tachyon at higher orders in perturbation theory, in the range $r_c/\sqrt{2} < |r| < r_c$, is more difficult to study since, on top of the contact term, there are further relevant operators, that can be produced by the fusion of tachyon perturbations and appear above fourth order. The potentially dangerous divergences are logarithmic ones, since they add non linear 'source-terms' to the beta-functions in any scheme.

The pattern we have found at second and fourth order – and that we can safely assume to be similar at higher orders – is the following. The various contributions to the perturbative expansion at this order, involving both the tachyon vertices and the contact term, give all power-like divergences, that cancel mostly among themselves. The logarithmic divergence could come only from the term involving only tachyons and no contact terms, occurring when all the operators collide at the same point. We have seen that the various pieces of this OPE give indeed logarithmic divergences when a marginal operator can be produced, however the sum of these terms precisely cancel. Hence marginality is ensured, without the help of the contact term. Some power-like divergences, corresponding to the collision of more than two tachyons at the same point, still remain, but can be canceled by higher-order contact terms.

From these results we have found that the boundary action corresponding to the 'half S-brane' rolling tachyon is conformal to all orders in the range $0 < |r| < \sqrt{\frac{17}{6}}$, and at least up to fifth order in the range $\sqrt{\frac{17}{6}} < |r| < 1/\sqrt{2}$. We assume that, would we were able to compute the higher-order corrections to the beta-functions there, we would find that the theory is conformal on the boundary for all distances below the critical value $r_c$. This is a sensible assumption, as we don’t expect from the space-time perspective any qualitative difference in the physics of the system when $|r|$ crosses the value $\sqrt{\frac{17}{6}} r_c$.

Finally, we have considered a space-time effective action of the tachyon and distance fields, that should have the 'half S-brane' rolling tachyon at constant separation as a solution of its equations of motion, and should reproduce quadratic fluctuations around it. The available effective action for the system does not allow solutions at constant separations, hence we had to look for an alternative one.

The constraint of having the requested form of solution completely fixes the action up to quadratic order in the tachyon (at all orders in the separation). The higher order terms are not completely fixed since, unlike for the non-BPS brane tachyon decay, only the 'half S-brane' solution should be allowed. We made a sensible extra assumption in order to be able to solve the system. We obtain from this constraint a unique effective action, that reproduces naturally the known action in the zero distance limit. Interestingly, the minimum of the tachyon potential is at finite distance for $r > 0$. In order to test the validity of this result, we compare with the worldsheet partition function at the special value $r = 1/2$, where the computations are tractable. Unfortunately, we found a quite complicated expansion (up to eighth order) that do not match with the space-time Lagrangian, but could not be obtained either from the expansion of a simple functional of the tachyon.

From this analysis we would expect that tachyon condensation at fixed sub-critical separation is possible, all the way to the tachyon vacuum (at infinity in field space). However a heuristic argument seems to contradict this conclusion. Following [33, 34], one could
describe the result of this condensation by studying the closed string emission from the time-dependent boundary state. It was found in [33] that, knowing the one-point function on the disk $B(E) = \langle e^{iEX_0} \rangle$, one can compute the density of closed string states emitted by the decay of a non-BPS brane which goes as

$$\rho_c \sim \sum_N \frac{1}{E_N} D(N) |B(E_N)|^2$$

(6.1)

where the asymptotic Hagedorn density of closed string states at level $N$ has the form $D(N) \sim N^{-\alpha} \exp(4\pi \sqrt{N})$, with $\alpha > 0$, and $E_N \sim 2\sqrt{N}$. The one-point function for an unstable non-BPS D-brane goes as $|B(E)|^2 \sim \exp(-2\pi E)$. Therefore, in this case, the sum is governed by the sub-leading power-like corrections to the Hagedorn density and typically diverge, giving the so-called 'tachyon dust' of massive closed strings. In the case of non-zero separation, by dimensional analysis we may expect that $|B(E)| \sim \exp(-\sqrt{2\pi E} |m_{tach}(r)|)$. This would lead to a convergent closed string production when $|m_{tach}| < 1/\sqrt{2}$ (i.e. for $r \neq 0$), signaling that the tachyon does not condense completely at finite distance.

This motivates to study the boundary string field theory associated to this system. Even though it does not contain information about the dynamics of the system, it allows to find the exact tachyon potential (as well as the appearance of lower-dimensional branes), hence can illuminate the fate of the tachyon. These computations seem not to be out of reach. We plan to come back to these issues in the near future.

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A Partition function to eighth order

We want to compute the partition function in the special case where $\omega = 1/2$ with action given by (2.6):

$$Z(r, \lambda^+, \lambda^-) = \langle e^{-\delta S} \rangle$$

$$= \langle e^{-\frac{\lambda^+}{4\pi} \int d\hat{t} \Gamma^+(\hat{t}) + \frac{\lambda^-}{4\pi} \int d\hat{t} \Gamma^-(\hat{t})} \rangle$$

$$= \sum_{n,p=0}^{\infty} (-1)^{n+p} \frac{(\lambda^+)^n (\lambda^-)^p}{n! p!} \sum_{\text{perm}} \int [dT]_{n+p} \langle \Gamma^+(\hat{t}_1) \ldots \Gamma^+(\hat{t}_{n+p}) \rangle \times$$

$$\times \langle T^+(\hat{t}_1) \ldots T^+(\hat{t}_{n+p}) \rangle$$

(A.1)

with $n$ and $p$ of the same parity and $T^\pm = e^{\pm i\hat{X} + \frac{\omega X_0}{2}}$. 

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Due to the Fermi multiplets correlators, the only non-vanishing terms are the ones which have as much $+$ as $−$. The correlator of the Fermi multiplets are easy to compute with Wick theorem and the Green function (2.5). It leads to one product of supersymmetric sign functions $\prod \hat{\epsilon}(2i, 2i + 1)$, which decomposes into a sum of $2(n!)^2$ supersymmetric path orderings. One find that these path orderings are all equivalent under permutations of $T^+$'s ($T^−$'s) with $T^+$'s ($T^−$'s) and permutations of integration variable. So one choose one path ordering, symbolically $\hat{t}_1 > \hat{t}_2 > \ldots > \hat{t}_{2n}$ multiplied by a factor $2(n!)^2$. We should then compute:

$$Z(r, \lambda^+, \lambda^-) = 2 \sum_{n=0}^{\infty} (\lambda^+ \lambda^-)^n \int [dt]_{2n} \langle T^+(\hat{t}_1) T^−(\hat{t}_2) \ldots T^−(\hat{t}_{2n}) \rangle$$

$$= 2 \sum_{n=0}^{\infty} (\lambda^+ \lambda^-)^n e^{inx} \int [d\hat{t}]_{2n} \prod_{i<j} \hat{S}(2i, 2j) \hat{S}(2i - 1, 2j - 1)$$

$$= 2 \sum_{n=0}^{\infty} (\lambda^+ \lambda^-)^n e^{inx} I_n \quad (A.2)$$

with $\hat{S}(i, j) = [2 \sin \frac{\hat{t}_i - \hat{t}_j}{2} - \epsilon(i, j) \theta_i \theta_j]$

The computation of the integrals $I_n$ is as follows, using the notation of eq. (5.33). We have first

$$I_1 = -\frac{1}{2\pi} \int [dt]_1 = -\frac{1}{2\pi} \quad (A.3)$$

Then $I_2$ which is still easy

$$I_2 = \frac{1}{(2\pi)^2} \int \frac{[dt]_2}{2!} \left| \begin{array}{c} 1 \\ 2 \\ 1 \\
\end{array} \right| = \frac{1}{(2\pi)^2} \int \frac{[dt]_4}{4!} = \frac{1}{(2\pi)^2} - \frac{1}{4!} \quad (A.4)$$

and

$$I_3 = \frac{1}{2\pi} \int \frac{[dt]_5}{5!} C_1^{5!} \left| \begin{array}{c} 1 \\ 2345 \end{array} \right| = \frac{1}{(2\pi)^2} \int \frac{[dt]_3}{3!} \left| \begin{array}{c} 1 \\ 2 \\ 3 \\
\end{array} \right| = \frac{2}{4!(2\pi)^5} - \frac{1}{(2\pi)^3} = \frac{16}{3\pi^5} - \frac{1}{8\pi^3} \quad (A.5)$$

$I_4$ is a bit more complicated to compute, but in terms of integrals, we find:

$$I_4 = \int [dt]_8 \left[ \begin{array}{c} 13 \\ 57 \\
\end{array} \right] \left[ \begin{array}{c} 24 \\ 68 \\
\end{array} \right] - \frac{1}{(2\pi)^2} \int \frac{[dt]_6}{6!} C_2^{6!} \left| \begin{array}{c} 1 \\ 2 \\ 3 \\
\end{array} \right| = \frac{1}{1120} + \frac{143}{2592\pi^6} - \frac{55}{192\pi^4} + \frac{13}{480\pi^2} - \left( \frac{1001}{2592\pi^6} - \frac{175}{432\pi^4} - \frac{1}{240\pi^2} \right) + \frac{1}{16\pi^4}$$

$$= \frac{1}{1120} + \frac{3575}{2592\pi^6} + \frac{205}{1728\pi^4} + \frac{1}{32\pi^2} + \frac{1}{16\pi^4} \quad (A.6)$$
where we introduced the totally antisymmetric form:

\[
\begin{bmatrix}
  ab\ldots \\
  cd\ldots
\end{bmatrix} = \sum_P (-1)^P \begin{pmatrix}
  p(a)p(b)\ldots \\
  p(c)p(d)\ldots
\end{pmatrix} = (ab\ldots) - (ac\ldots) + (ad\ldots) + \ldots
\]  

(A.7)

with the partially anti-symmetric form:

\[
\begin{vmatrix}
  abc\ldots \\
  def\ldots
\end{vmatrix} = \epsilon(a,b)\epsilon(a,c)\epsilon(b,c) \times \ldots \times \epsilon(d,e)\epsilon(d,f)\epsilon(e,f) \times \ldots \times \begin{vmatrix}
  abc\ldots \\
  def\ldots
\end{vmatrix}
\]  

(A.8)

The bigger \( n \) is, the more complicated is the corresponding term in the partition function. This is because more and more contribution of the contact term appear and that the path ordering can’t be always removed. For the special value \( r = 1/2 \) the contact term has indeed a non-zero, but finite contribution to the final result.

We end up with the following expansion. The terms coming from pure 'non-contact' contributions are underlined:

\[
\begin{align*}
\frac{Z(x)}{2} &= 1 - \lambda^+\lambda^- \frac{e^{ix}}{2\pi} + \frac{(\lambda^+\lambda^-)^2}{4\pi^2} e^{2ix} \left( 1 - \frac{\pi^2}{6} \right) - \frac{(\lambda^+\lambda^-)^3}{8\pi^3} e^{3ix} \left( 1 - \frac{128}{3\pi^2} \right) \\
&\quad + \frac{(\lambda^+\lambda^-)^4}{16\pi^4} e^{4ix} \left( 1 + \frac{175}{27} + \frac{1001}{162\pi^2} + \frac{\pi^2}{15} \frac{55}{12} + \frac{143}{9\pi^2} + \frac{13\pi^2}{30} + \frac{\pi^4}{70} \right) \ldots
\end{align*}
\]  

(A.9)

where we recognize the trivial expansion:

\[
1 - \lambda^+\lambda^- \frac{e^{ix}}{2\pi} + \frac{(\lambda^+\lambda^-)^2}{4\pi^2} e^{2ix} - \frac{(\lambda^+\lambda^-)^3}{8\pi^3} e^{3ix} + \ldots = \frac{1}{1 + \frac{\lambda^+\lambda^-}{2\pi} e^{ix}}
\]  

(A.10)

In fact, this factorization is exact to all orders; by looking at the integrals \( I_n \), one can see that the maximal contact term is always present and has a standard form, which we recognize as a Vandermonde determinant.

The remaining terms should come from a non-trivial function that multiplies (A.10):

\[
\begin{align*}
Z(x) &= \frac{2}{1 + \frac{\lambda^+\lambda^-}{2\pi} e^{ix}} \left( 1 - \frac{(\lambda^+\lambda^-)^2}{2\pi} \frac{\pi^2}{6} e^{2ix} + \frac{128}{3\pi^2} \frac{\pi^2}{6} \frac{(\lambda^+\lambda^-)^3}{2\pi} e^{3ix} \right) \\
&\quad + \left( \frac{205}{108} + \frac{10487}{162\pi^2} + \frac{\pi^2}{2} + \frac{\pi^4}{70} \right) \frac{(\lambda^+\lambda^-)^4}{2\pi} e^{4ix} + \ldots
\end{align*}
\]  

(A.11)

This doesn’t seem to come from the Taylor expansion of a simple expression.
B Computation of the divergences in CTT-type terms

We give below one example of computation of the divergence occuring in an integral involving one contact operator insertion. We study here the CTT term. With a bit of care, one can compute them exactly. With the expression of \( C \) given in (3.6), the CTT term is

\[
- \frac{\varepsilon^{a-1}}{2} \int dx_1 \int_{x_1-L}^{x_1-\varepsilon} \int_{x_2-L}^{x_2-\varepsilon} dx_2 \int_{x_3}^{x_3-\varepsilon} dx_3 \cdot C(x_1) \cdot T^+(x_2) \cdot T^-(x_3) = (a-1) \frac{\varepsilon^{a-1}}{2} \int dx_1 \int_{x_1-L}^{x_1-\varepsilon} \int_{x_2-L}^{x_2-\varepsilon} dx_2 \int_{x_3}^{x_3-\varepsilon} dx_3 \left( (x_1-x_2+\varepsilon)(x_1-x_3+\varepsilon)^{a-1}(x_1-x_2)^{a-1}(x_1-x_3)(x_2-x_3)^{a-2} \right.

\]

\[+ (x_1-x_2+\varepsilon)^{a-1}(x_1-x_3+\varepsilon)(x_1-x_2)(x_1-x_3)^{a-1}(x_2-x_3)^{a-2} \right) (B.1) \]

with \( a = 4\omega^2 \). Note that the IR cut-off is chosen such that two ordered operator do not move away from each other more that \( L \). Then, since \( C(x) \sim T^\pm(x+\varepsilon)T^\mp(x) \) the cut-off for \( x_2 \) in relation to \( x_1 \) is \( L - \varepsilon \). One can get read of the path ordering with the following change of variable:

\[
x_2 = -L\delta_1 + x_1 \\
x_3 = -L\delta_2 + x_2 \tag{B.2}
\]

such that it gives, introducing \( \eta = \varepsilon/L \):

\[
(a-1)L^{a-1} \frac{\varepsilon^{a-1}}{2} \int_{\eta}^{1-\eta} d\delta_1 \int_{\eta}^{1} d\delta_2 \left( (\delta_1+\eta)(\delta_1+\delta_2+\eta)^{a-1}\delta_1^{a-1}(\delta_1+\delta_2)\delta_2^{a-2} \right.

\[+ (\delta_1+\eta)^{a-1}(\delta_1+\delta_2+\eta)\delta_1(\delta_1+\delta_2)^{a-1}\delta_2^{a-2} \right) \int dx_1 e^{4\omega X^0}(x_1) \tag{B.3}
\]

The integral over \( \delta_1 \)'s can be done with the use of the series representation of \( (1 + \frac{\eta}{\delta_1+\delta_2})^\alpha \) since \( \delta_1 + \delta_2 > \eta \), and \( (1 + \frac{\eta}{\delta_1})^\beta \) since \( \delta_1 > \eta \). These are given by:

\[
(1 + x)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)} \frac{x^n}{\Gamma(1+n)} \quad \text{with } |x| < 1 \tag{B.4}
\]

Convergence of the series all along the domain of integration allows us to commute integral
and sum sign\(^1\), such that one has:

\[
(a - 1) \sum_{s=0}^{1} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a - n) \Gamma(1 + n)} \eta^{a-1+s+n} \times \int_{\eta}^{1-\eta} d\delta_1 \int_{\eta}^{1-\eta} d\delta_2 \left( \delta_1^{a-s} \delta_2^{a-2} (\delta_1 + \delta_2)^{a-n} + \delta_1^{a-n} \delta_2^{a-2} (\delta_1 + \delta_2)^{a-s} \right) \tag{B.5}
\]

As one can see, the two integral to compute are symmetric by permutation of \(s\) and \(n\). We then only focus on the first one. There are two ways to proceed now. Integrate directly and exactly since it is possible, or use an indirect method that reintroduce some path ordering. We use the second and apparently more complicated method, because it is needed to compute TTTT integrals. Indeed, one will see that hypergeometric functions will receive argument \(z\) which has absolute value less than 1, much more easier to handle for approximations, since the series representation is known exactly. We separate the first integral of (B.5) into:

\[
\int_{\eta}^{1-\eta} d\delta_1 \int_{\eta}^{1-\eta} d\delta_2 \frac{\delta_1^{a-s}}{\delta_2^{a-2}} (1 + \frac{\delta_2^{a-s}}{\delta_1^{a-2}})^{a-n} + \int_{\eta}^{1-\eta} d\delta_1 \int_{\eta}^{1-\eta} d\delta_2 \frac{\delta_1^{a-s}}{\delta_2^{a-2}} (1 + \frac{\delta_1^{a-s}}{\delta_2^{a-2}})^{a-n}
\]

\[
= \int_{\eta}^{1-\eta} d\delta_1 \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{\delta_1^{a-1}}{\delta_2^{a-2}} \right] \left[ \frac{2F_1 \left( n-a, a-1, a, -1 \right)}{1+n-2a} + \frac{2F_1 \left( n-a, 1+n-2a, 2 + n - 2a, -1 \right)}{1+n-2a} \right] \right] \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{\delta_1^{a-1}}{\delta_2^{a-2}} \right]
\]

\[
= \int_{\eta}^{1-\eta} d\delta_1 \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{2F_1 \left( n-a, a-1, a, -1 \right)}{a-1} \right] \left[ \frac{2F_1 \left( n-a, 1+n-2a, 2 + n - 2a, -1 \right)}{1+n-2a} \right] \right] \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{\delta_1^{a-1}}{\delta_2^{a-2}} \right]
\]

\[
\int_{\eta}^{1-\eta} d\delta_1 \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{2F_1 \left( n-a, a-1, a, -1 \right)}{a-1} \right] \left[ \frac{2F_1 \left( n-a, 1+n-2a, 2 + n - 2a, -1 \right)}{1+n-2a} \right] \right] \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{\delta_1^{a-1}}{\delta_2^{a-2}} \right]
\]

\[
\text{(B.6)}
\]

Let us remark at this stage that \(z\) argument in \(2F_1(a, b, c, z)\) verifies \(|z| < 1\) in the above integrals. The first one is trivial and gives:

\[
I_1 = \frac{(1-\eta)^{3a-s-n} - \eta^{3a-s-n}}{3a-s-n} \left( \frac{2F_1 \left( n-a, a-1, a, -1 \right)}{a-1} + \frac{2F_1 \left( n-a, 1+n-2a, 2 + n - 2a, -1 \right)}{1+n-2a} \right) \right] \left[ \frac{\delta_1^{a-s}}{\delta_2^{a-2}} \right] \left[ \frac{\delta_1^{a-1}}{\delta_2^{a-2}} \right]
\]

\[
= 1 - \eta^{3a-s-n} \left( \frac{2F_1 \left( n-a, a-1, a, -1 \right)}{a-1} + \frac{2F_1 \left( n-a, 1+n-2a, 2 + n - 2a, -1 \right)}{1+n-2a} \right) + o(\eta)
\]  

\[
\text{(B.7)}
\]

\(^1\)It is true at least \textit{a fortiori} from the convergence of the integrals and the series of the integrals. Note besides that we do not integrate over any pole.
On the last line we used the series representation of $2F_1$:

$$2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$$

for $|z| < 1$. In particular, for $c = b + 1$ we have:

$$2F_1(-a, b, b + 1, -z) = \sum_{k=0}^{\infty} \frac{\Gamma(1+a) b}{\Gamma(1+a-k) \Gamma(1+k)(b+k)} z^k$$

Finally, the third one is:

$$I_3 = \frac{1}{2a - 1 - n} \left[ - \frac{\delta_1^{a+1-s} (1 + n - 2a)}{s + n - 3a} \left( \frac{2F_1(n-a, 1 + n - 2a, 2 + n - 2a, -\delta_1)}{1 + n - 2a} - \frac{2F_1(n-a, 1 + a - s, 2 + a - s, -\delta_1)}{1 + a - s} \right) \right]^{1-\eta}$$

$$= (1 - \delta_1)^{a+1-s} \left( \frac{2F_1(n-a, 1 + n - 2a, 2 + n - 2a, -1 + \eta)}{1 + n - 2a} - \frac{2F_1(n-a, 1 + a - s, 2 + a - s, -1 + \eta)}{1 + a - s} \right) - o(\eta^{a+1-s})$$

$$= -\frac{1}{3a - s - n} \left( \frac{2F_1(n-a, 1 + n - 2a, 2 + n - 2a, -1)}{1 + n - 2a} \right) \left( \frac{2F_1(n-a, 1 + a - s, 2 + a - s, -1)}{1 + a - s} \right)$$

$$+ o(\eta^{a+1-s}) + o(\eta)$$  (B.11)
Collecting these results one finally get the sum:

\[
\frac{a - 1}{2} \sum_{s=0}^{1} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)} \eta^{a-1+s+n} \int_{\eta}^{1-\eta} d\delta_1 \int_{\eta}^{1} d\delta_2 \left( \delta_1^{a-s} \delta_2^{a-2}(\delta_1+\delta_2)^{a-n} + \delta_1^{a-n} \delta_2^{a-2}(\delta_1+\delta_2)^{-s} \right)
\]

\[
= a - 1 \sum_{s=0}^{1} \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)} \eta^{a-1+s+n} (I_1 + I_2 + I_3 + (s \leftrightarrow n))
\]

\[
\sim \frac{\eta^{2a-2}}{2a+1} + \frac{\eta^{2a-1}}{2a}
\]

\[
- \frac{\eta^{a-1} a - 1}{3a} \left( 2F_1(-a, a + 1, a + 2, -1) + \frac{2F_1(-a - 1, a, -1)}{a - 1} \right)
\]

\[
\sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)(3a - s - n)} \left( \frac{2F_1(n - a, 1 + n - 2a, 2 + n - 2a, -1)}{1 + n - 2a} \right)
\]

\[
+ \frac{2F_1(s - a, 1 + s - 2a, 2 + s - 2a, -1)}{1 + s - 2a} + \frac{2F_1(n - a, s + n - 1 - 2a, s + n - 2a, -1)}{s + n - 1 - 2a}
\]

\[
\left( \frac{2F_1(s - a, s + n - 1 - 2a, s + n - 2a, -1)}{s + n - 1 - 2a} \right)
\]

(B.12)

A similar computation was done for the $TCT$ and $TTC$ terms, with the correct cut-off prescriptions. Note however that $CTT = TTC$.

**C Computation of the divergences in the TTTT term**

The computation of an amplitude with four tachyon insertions is clearly a lot more involved than the above one, since three integrations have to be done. The straightforward OPE of the four tachyons is doable and gives, from (5.34):

\[
\int dx_1 \int_{x_1-L}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \int_{x_3-L}^{x_3-\varepsilon} dx_4 \psi^+ T^+ (x_1) \cdots \psi^- T^- (x_4):
\]

\[
= \int dx_1 e^{4\omega X^a} \int_{x_1-L}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \int_{x_3-L}^{x_3-\varepsilon} dx_4
\]

\[
\left( (a - 1)^2 (x_1 - x_2)^{a-2} (x_1 - x_3) (x_1 - x_4)^{a-1} (x_2 - x_3)^{a-1} (x_3 - x_4)^{a-2} 
\]

\[
- (x_1 - x_2)^{a-1} (x_1 - x_4)^{a-1} (x_2 - x_3)^{a-1} (x_3 - x_4)^{a-1} 
\]

\[
+ (a - 1)^2 (x_1 - x_2)^{a-1} (x_1 - x_3) (x_1 - x_4)^{a-2} (x_2 - x_3)^{a-2} (x_2 - x_4) (x_3 - x_4)^{a-1} \right) \quad (C.1)
\]

This integrand is too much coupled in its variables and not analytically computable in this form. But one can show using the identity

\[
(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) + (x_1 - x_4)(x_2 - x_3) = 0 \quad (C.2)
\]
that the integrand can be reexpressed as

\[ \int d\varepsilon_1 e^{i\omega x_0} \int_{x_1-L}^{x_1-\varepsilon} dx_2 \int_{x_2-L}^{x_2-\varepsilon} dx_3 \int_{x_3-L}^{x_3-\varepsilon} dx_4 \]

\[ (a-1)^2(x_1-x_2)^{a-2}(x_2-x_3)^a(x_3-x_4)^{a-2}(x_1-x_4)^a \]

\[ + (2(a-1)^2 - 1) (x_1-x_2)^{a-1}(x_1-x_4)^{a-1}(x_2-x_3)^{a-1}(x_3-x_4)^{a-1} \]

\[ + (a-1)^2(x_1-x_2)^a(x_2-x_3)^{a-2}(x_3-x_4)^a(x_1-x_4)^{a-2} \] \hfill (C.3)

If we use the change of variable

\[ x_2 = -L\delta_1 + x_1 \]
\[ x_3 = -L\delta_2 + x_2 \]
\[ x_4 = -L\delta_3 + x_3 \] \hfill (C.4)

the integral becomes:

\[ \int dx_1 e^{i\omega x_0} \int_\eta^1 d\delta_1 \int_\eta^1 d\delta_2 \int_\eta^1 d\delta_3 \]

\[ (a-1)^2\delta_1^{a-2}\delta_2^{a-2}(\delta_1 + \delta_2 + \delta_3)^a + (a-1)^2\delta_1\delta_2\delta_3^{a-2}(\delta_1 + \delta_2 + \delta_3)^{a-2} \]

\[ + (2(a-1)^2 - 1) \delta_1^{a-1}\delta_2^{a-1}\delta_3^{a-1}(\delta_1 + \delta_2 + \delta_3)^{a-1} \] \hfill (C.5)

It is possible to extract the divergences by analytic integration but we need to be careful since we will need at some point to commute the integrals and sums. For this reason, the \( z \)-argument in the \( _2F_1(a, b, c, z) \) should satisfy \(|z| < 1 \).

We will not develop the whole computation, but give as an example one of the three integrals. Let us study the following one:

\[ \int_\eta^1 d\delta_1 \int_\eta^1 d\delta_2 \int_\eta^1 d\delta_3 \delta_1^{a-2}\delta_2^{a-2}\delta_3^{a-2}(\delta_1 + \delta_2 + \delta_3)^{a-2} \] \hfill (C.6)

Integration of \( \delta_3 \) imposes to separate the domain of integration in three parts:

\[ \delta_1 + \delta_2 > 1 \text{ and } \delta_3 \in [\eta; 1] < \delta_1 + \delta_2 \]
\[ \delta_1 + \delta_2 < 1 \text{ and } \delta_3 \in [\eta; \delta_1 + \delta_2] < \delta_1 + \delta_2 \]
\[ \delta_1 + \delta_2 < 1 \text{ and } \delta_3 \in [\delta_1 + \delta_2; 1] > \delta_1 + \delta_2 \] \hfill (C.7)
This makes three integrals:

\[ I_1 = \int_{\eta}^{1} d\delta_1 \int_{1-\delta_1}^{1} d\delta_2 \int_{\eta}^{1} d\delta_3 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2}(1 + \frac{\delta_3}{\delta_1 + \delta_2})^{a-2} \]

\[ I_2 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \int_{\eta}^{1} d\delta_3 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2}(1 + \frac{\delta_3}{\delta_1 + \delta_2})^{a-2} \]

\[ I_3 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \int_{\eta}^{1} d\delta_3 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2}(1 + \frac{\delta_3}{\delta_1 + \delta_2})^{a-2} \]  

which integrate to:

\[ I_1 = \int_{\eta}^{1} d\delta_1 \int_{1-\delta_1}^{1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{\delta_3^{a+1}}{a+1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

\[ I_2 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{\delta_3^{a+1}}{a+1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

\[ I_3 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{\delta_3^{a+1}}{a+1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

We will not develop the computations for all the three integrals. Let us focus on the third, which is easier to present. The method is similar for the two other ones.

\[ I_3 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{1}{2a-1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

These are two different integrations to do. We have:

\[ I_3^1 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{1}{2a-1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

\[ I_3^2 = \int_{\eta}^{1} d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \ 3 \delta_3^{a-2}(\delta_1 + \delta_2)^{a-2} \left[ \frac{1}{2a-1} \binom{2}{a} \binom{2}{a+1} \frac{\delta_3}{\delta_1 + \delta_2} \right] \]

Each of these separates again in three parts:

\[ \delta_1 \in [\eta; \frac{1}{2}] \text{ and } \delta_2 \in [\eta; \delta_1] \]

\[ \delta_1 \in [\eta; \frac{1}{2}] \text{ and } \delta_2 \in [\delta_1; 1 - \delta_1] \]

\[ \delta_1 \in [\frac{1}{2}; 1] \text{ and } \delta_2 \in [\eta; 1 - \delta_1] \]

There are no known expression for the integration of \( I_3^1 \), but it is not much of a problem since we only want to extract divergences. Because \(|\delta_1 + \delta_2| < 1\), one can express \( \binom{2}{a} \binom{2}{a+1} \) as its
series expansion given in (B.10). Since the series is convergent everywhere in the integration domain, we can commute the sum and the integral, such that

\[
I_3^1 = -\sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(a-1-n)\Gamma(1+n)(1-2a+n)}
\times \left( \int_{\eta}^{1/2} d\delta_1 \int_{\eta}^{\delta_1} d\delta_2 \delta_1^{a-1} \delta_2^{a-2}(\delta_1 + \delta_2)^n + \int_{\eta}^{1/2} d\delta_1 \int_{\delta_1}^{1-\delta_1} d\delta_2 \delta_1^{a} \delta_2^{a-2}(\delta_1 + \delta_2)^n \right. \\
\left. + \int_{1/2}^1 d\delta_1 \int_{\eta}^{1-\delta_1} d\delta_2 \delta_1^{a} \delta_2^{a-2}(\delta_1 + \delta_2)^n \right) \quad (C.13)
\]

These integrals are very similar to the ones studied in appendix B. Following the method presented there, and with a careful power analysis in \(\eta\), we can obtain:

\[
I_3^1 = -\sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(a-1-n)\Gamma(1+n)(1-2a+n)}
\times \left( -\frac{2^{-a-1-n}\eta^{a-1}}{(a+1+n)(a-1)} + o(1) + \frac{(2^{-a-1-n}-1)\eta^{a-1}}{(a+1+n)(a-1)} \right) \\
= -\frac{\eta^{a-1}}{3a(a-1)} \left( \frac{2F_1(2-a, 1-2a, 2-2a, -1)}{2a-1} + \frac{2F_1(2-a, a+1, a+2, -1)}{a+1} \right) + o(1) \quad (C.14)
\]

The computation of \(I_3^2\) is less difficult. With method of appendix B and (C.12), it gives:

\[
I_3^2 = \int_{\eta}^{1/2} d\delta_1 \delta_1^{4a-2} \left( \frac{2F_1(1-2a, a-1, a-1)}{a-1} + \frac{2F_1(1-2a, 2-3a, 3-3a, -1)}{2-3a} \right) \\
\times \frac{2F_1(2-a, 1-2a, 2-2a, -1)}{1-2a} + \frac{\eta^{a-1}}{3a(a-1)(4a-1)} \left( 3a_2 F_1(1-2a, a-1, a-1) + (a-1) F_1(1-2a, -3a, 1-3a, -1) \right) \\
\times \frac{2F_1(2-a, 1-2a, 2-2a, -1)}{1-2a} + \frac{\eta^{a-1} 2F_1(2-a, 1-2a, 2-2a, -1)}{a-1} + o(1) \quad (C.15)
\]

We do not integrate explicitely the first term so that the logarithm appears unambiguously at \(a = 1/4\). This has to be compared to the second term which does not become a logarithm, since it is finite at \(a = 1/4\). Indeed, for this precise value \(a - 1 = -3a\) and one gets \(\eta^{\delta}\).
Finally, summing up $I_3^1$ with $I_3^2$, one obtains:

\[
I_3 = \int_{\eta}^{1/2} d\delta_1 \delta_1^{4a-2} \left( \frac{2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{2F_1(1-2a,2-3a,3-3a,-1)}{2-3a} \right) \\
\times \frac{2F_1(2-a,1-2a,2-2a,-1)}{1-2a} \\
+ \frac{\eta^{4a-1}}{3a(a-1)(4a-1)} \left( 3a \cdot 2F_1(1-2a,a-1,a,-1) + (a-1) \cdot 2F_1(1-2a,-3a,1-3a,-1) \right) \\
\times \frac{2F_1(2-a,1-2a,2-2a,-1)}{1-2a} \\
- \frac{\eta^{4a-1}}{3a} \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} + o(1) \tag{C.16}
\]

Similarly one computes $I_1$ and $I_2$, for which we obtain:

\[
I_1 = o(1) \tag{C.17}
\]

and

\[
I_2 = \int_{\eta}^{1/2} d\delta_1 \delta_1^{4a-2} \left( \frac{2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{2F_1(1-2a,2-3a,3-3a,-1)}{2-3a} \right) \\
\times \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \\
+ \frac{\eta^{4a-1}}{3a(a-1)(4a-1)} \left( 3a \cdot 2F_1(1-2a,a-1,a,-1) + (a-1) \cdot 2F_1(1-2a,-3a,1-3a,-1) \right) \\
\times \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \\
+ \eta^{4a-1} \sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(n+1)\Gamma(a-n-1)(a+n+1)(3a-n-2)} \\
\left( \frac{2F_1(-2a+n+1,-a+n+2;2a+n+2,-2a+n+2;1)}{-2a+n+1} + \frac{2F_1(-2a+n+3,-a+n+2;2a+n+4;1)}{-2a+n+3} \right) \\
- \frac{\eta^{4a-1}}{3a} \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} + o(1) \tag{C.18}
\]
One expresses then the whole integral (C.6) as

\[
\int_1^{1} d\delta_1 \int_1^{1} d\delta_2 \int_1^{1} d\delta_3 \ \delta_1^a \delta_2^a \delta_3^a \ (\delta_1 + \delta_2 + \delta_3)^{a-2} \\
\sim \int_1^{1/2} d\delta_1 \ \delta_1^{4a-2} \left( \frac{2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{2F_1(1-2a,2-3a,3-3a,-1)}{2-3a} \right) \\
\times \left( \frac{2F_1(2-a,1-2a,2-2a,-1)}{1-2a} + \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \right) \\
+ \eta^{4a-1} \left[ \frac{1}{4(a-1)} \left( \frac{2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{2F_1(1-2a,-3a,1-3a,-1)}{3a} \right) \right. \\
\times \left( \frac{2F_1(2-a,1-2a,2-2a,-1)}{1-2a} + \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} \right) \\
+ \sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(n+1)\Gamma(a-n-1)(a+n+1)(3a-n-2)} \left( \frac{2F_1(-2a+n+1,-a+n+2;-2a+n+2;-1)}{-2a+n+1} \right. \\
\left. + \frac{2F_1(-2a+n+3,-a+n+2;-2a+n+4;-1)}{-2a+n+3} \right) \right] \\
- \frac{2}{3a} \eta^{4a-1} \frac{2F_1(2-a,a+1,a+2,-1)}{a+1} + o(1) \quad (C.19)
\]

Similar techniques apply to the two other kind of integrals.

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