Developing solution algorithm for LR-type fully interval-valued intuitionistic fuzzy linear programming problems using lexicographic-ranking method

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Abstract
The current article is devoted to mathematically handling the inherent uncertainties of various practical problems by introducing the concept of LR-type interval-valued intuitionistic fuzzy numbers (LR-type IVIFN). The theoretic development of this concept has also been enhanced by providing various diagrammatic representations of LR-type IVIFNs and establishing the arithmetic operations among these fuzzy numbers. The notion of α, β-cuts and interval arithmetic have been employed to derive the expressions for the arithmetic operations on LR-type IVIFNs. In addition to this, the total order properties of lexicographic-ranking criteria along with considering seven distinct parameters have been used to define the ordering on LR-type IVIFNs. Further, a linear programming problem (LPP) with both equality and inequality type constraints, all parameters in the form of LR-type IVIFNs, and unrestricted decision variables has been modeled. In order to obtain a unique optimal solution for the proposed LPP, the lexicographic ranking-based solution methodology has been developed in which by introducing some binary variables, the original LPP is converted to an equivalent mixed 0-1 lexicographic non-linear programming problem having seven components in the objective function. Various theorems have also been proved to show the equivalence of the original problem and its different proposed constructions. The model formulation, algorithm, and discussed results have not only developed a new idea but also generalized various well-known related works existing in the literature. Moreover, a numerical illustration has been provided to clarify the step-wise procedure of the proposed solution approach. Additionally,
to establish the practical significance of the study, a real-world production planning problem is framed, solved, and analyzed under the \( LR \)-type interval-valued intuitionistic fuzzy scenario. In last, a comparative study has also been carried out to prove the relevancy of the developed algorithm.

**Keywords** Fuzzy mathematical programming · Interval-valued intuitionistic fuzzy number · \( LR \)-type fuzzy number · Lexicographic ranking criterion · Score function · Accuracy function

**Mathematics Subject Classification** 03F55 · 03E72 · 90C08

### 1 Introduction

Linear programming problems (LPPs) are the simplest kind of optimization problem that are widely used to solve many real-life problems. In conventional LPPs, all the parameters and decision variables are taken to be precise real numbers. However, in practical situations due to various uncontrollable factors such as measurement errors, inadequate statistical data estimation, market fluctuations, preference for human judgement, etc., the input data of a problem may not be available as crisp values. There may involve certain vagueness/ambiguity in all or some of the parameters and/or decision variables of the problem. Consequently, Zadeh (1965) developed the concept of fuzzy sets which incorporates imprecision in the data successfully. After that, the fuzzy sets are amalgamated greatly with optimization theory to assimilate the vague/ill-defined data in the mathematical formulation of an optimization problem. Motivated by Zadeh’s concept of fuzzy sets, Zimmermann (1975) initiated and developed the theory to solve fuzzy linear programming problems (FLPPs). Tanaka and Asai (1984) first introduced the FLPPs in which both the parameters and decision variables were represented by fuzzy numbers, which are described to be as fully fuzzy LPPs. In mathematical terms, a general fuzzy LPP can be modeled as follows:

\[
(P) \quad \text{max (or min)} \sum_{j=1}^{n} c_j x_j \\
\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \{\leq, =, \geq\} b_i, \quad i = 1, 2, \ldots, m, \\
x_j \geq 0, \quad j = 1, 2, \ldots, n
\]

where \( c_j, a_{ij}, b_i \) and \( x_j \) can be either crisp or fuzzy numbers depending on the available information for the problem under consideration. Here, it is to be emphasized that the inequalities “\( \leq \)” and “\( \geq \)” are fuzzy inequalities which essentially mean “approximately less than” and “approximately greater than”, respectively. Initially, the researchers had extended the classical methods which were used to solve crisp LPPs to deal with FLPPs such as simplex algorithm, two-phase approach, etc. But, later on, these were proven to be incompatible with fuzzy theory. After that, linear ranking functions have been widely employed to convert FLPPs into its equivalent crisp optimization problems. However, the ranking functions fail to order two such fuzzy numbers (FNs) which seem to be distinguished to a decision-maker. To overcome such limitations, the idea of lexicographic ranking criteria came which uses multiple parameters at a time associated with a FN and hence is a more effective and powerful ordering criterion.
The fuzzy theory was later extended to the intuitionistic fuzzy set theory, which is more
general and provide more flexibility to a decision-maker. The intuitionistic fuzzy sets assign
an additional rejection degree besides the acceptance degree to each element of the set. This
generalization has attracted the attention of researchers to explore the optimization theory
under the intuitionistic fuzzy environment to tackle various practical problems having hesitant
data. A detailed literature review on such problems in view of fuzzy theory and its extensions
can be referred to Sect. 2 of the article. It was observed that while handling the ambiguous
data, the intuitionistic fuzzy sets utilize exact or crisp real numbers to assign the membership
and non-membership degrees for each element of the set. However, in practical situations, a
decision-maker may fail to give these degrees with full confidence. Consequently, the notion
of intuitionistic fuzzy sets is extended to the interval-valued intuitionistic fuzzy (IVIF) sets,
which is the key motivation for our present study. The IVIF sets use an interval to define the
acceptance and rejection degrees for each element of the set.

On the other hand, the idea of LR-type fuzzy numbers came into existence to characterize
analytically any sort of behaviour (linear/non-linear) of the membership function. Moreover,
to represent a realistic situation, LR-type fuzzy numbers play a crucial role in optimization
theory since any variation in the input data can be reflected in the mathematical model
using an appropriate L and R shape functions. Therefore, to introduce a most generalized
class of fuzzy numbers, we first defined the concept of LR-type interval-valued intuitionistic
fuzzy (IVIF) numbers and explored their basic properties. After that, an LPP having all the
parameters and decision variables as LR-type IVIF numbers is taken into consideration and
further, an approach for solving such LPPs using a lexicographic ranking methodology has
been proposed.

To the best of our knowledge, there is no study in the literature, describing the arithmetic
operations on LR-type IVIF numbers (IVIFNs) and finding the unique optimal IVIF solution
for an interval-valued intuitionistic fuzzy linear programming problem (IVIFLPP) having
both linear equalities and inequalities with all the parameters represented by LR-type IVIFNs
and decision variables as unrestricted LR-type IVIFNs. Many practical problems occurring
in distinct real-world phenomena involve certain quantities like selling or purchasing of some
units, profit or loss, etc. often resulting in an optimization model having unrestricted decision
variables. Therefore, it is required to devise and solve a model of LPPs under uncertain
conditions which can handle the unrestricted behaviour of decision variables. To sum up, the
key features of the present work can be listed as follows:

1. Based on \((\alpha, \beta)\)-cut of LR-type IVIFNs, we define the score and accuracy indices of these
   numbers.
2. The basic arithmetic operations on unrestricted LR-type IVIFNs are developed by
   applying the \((\alpha, \beta)\)-cut.
3. Using the total order properties of the lexicographic criterion, a ranking of LR-type
   IVIFNs has been proposed.
4. Based on the introduced lexicographic ranking criterion, the LR-type IVIFLPP is con-
   verted to an equivalent mixed 0-1 lexicographic non-linear programming problem for
   finding a unique optimal IVIF solution of the fully LR-type IVIFLPP.
5. The underlying theorems are also proved at the relevant places to show the equivalence
   between the various problems obtained during the proposed algorithm.
6. A practical application in production planning is constructed, solved, and examined using
   the proposed technique.
7. A comparison study has also been undertaken to evaluate the efficacy of the developed
   methodology.
The rest of the paper is summarized as follows: In Sect. 2, a detailed literature review on fuzzy, intuitionistic fuzzy, and IVIF theory is given. Section 3 includes some basic definitions and arithmetic operations on $LR$-type IVIFNs. Further, a lexicographic ranking criterion is also proposed to rank two $LR$-type IVIFNs. The mathematical formulation of an IVIFLPP is described in Sect. 4 and a lexicographic method has been proposed to find the unique IVIF optimal solution of $LR$-type IVIFLPPs. In Sect. 5, the advantages of the proposed method are listed. Section 6 illustrates a numerical example to describe the steps of the proposed algorithm. A production planning problem along with its managerial insights and comparative results are discussed in Sect. 7. The last section sums up the conclusions and some interesting future directions.

2 Literature review

2.1 Fuzzy LPP

In the literature, several methods have been proposed to solve the model (P) (given in Sect. 1) depending on which parameters and/or decision variables are taken to be fuzzy. A comprehensive survey on fuzzy linear programming problems (FLPPs) can be found in Ebrahimnejad and Verdegay (2018). Hashemi et al. (2006) considered a fully FLPP with inequality constraints having all the parameters and decision variables to be given by symmetric $LR$-type FNs and proposed a two-phase solution approach by using the lexicographic comparison of the mean and standard deviation of FNs. Then, Allahviranloo et al. (2008) used a ranking function to develop a solution algorithm to deal with the fully FLPPs having inequality constraints. Later on, Kumar et al. (2011) proposed a methodology to solve the fully FLPPs with equality constraints where all the parameters and decision variables are given by non-negative triangular FNs. After that, Najafi and Edalatpanah (2013) proposed some corrections to the methodology of Kumar et al. (2011). Further, Khan et al. (2013) studied a fully FLPP where parameters and decision variables were taken to be triangular FNs and proposed a method by making use of some ranking function. Then, Ozkok et al. (2016) extended the method of Kumar and Kaur (2014) to solve fully FLPPs with all types of constraints having parameters as unrestricted and decision variables as non-negative triangular FNs. To enrich the fuzzy theory, Rahmani et al. (2016) gave a new defuzzification technique for fuzzy numbers employing the statistical Beta distribution and further proposed an approach to rank these fuzzy numbers.

Najafi et al. (2016) examined a fully FLPP having equality constraints with parameters as well as decision variables to be expressed by unrestricted triangular FNs and developed a solution technique by converting the original problem to a non-linear model. Later, Gong and Zhao (2018) considered a fully FLPP with equality constraints and proposed a methodology in which the problem is first transformed into a crisp multi-objective LPP and then solved by using different approaches. Arana-Jiménez (2018) presented a new method to find fuzzy optimal (nondominated) solutions of fully FLPPs having inequality constraints with triangular fuzzy numbers and not necessarily symmetric, via solving a multiobjective linear problem having crisp data. In this direction, Kaur and Kumar (2012) analyzed that by employing the existing approaches for solving fully FLPPs, the obtained optimal solution is not necessarily unique. To overcome this limitation, they have defined a lexicographic criterion for ranking trapezoidal FNs and introduced an approach to find the unique optimal solution of fully FLPPs having equality constraints with unrestricted parameters and non-negative decision.
variables. Along similar lines, Ezzati et al. (2015) introduced a lexicographic method to solve a fully FLPP with equality constraints having parameters and decision variables as triangular fuzzy numbers. Further, Mottaghi et al. (2015) solved a fully FLPP with inequality constraints by introducing non-negative fuzzy slack and surplus variables for converting the inequalities into fuzzy equality constraints.

Based on a lexicographic criterion for ranking of $LR$-type FNs, Hosseinzadeh and Edalatpanah (2016) devised a technique to handle fully FLPPs having only equality constraints where the parameters and decision variables were taken to be non-positive or non-negative $LR$-type FNs. After that, Kaur and Kumar (2016) introduced a lexicographic technique for obtaining the unique optimal solution of a fully FLPP with equality constraints having parameters and decision variables as unrestricted $LR$-type FNs. They pointed out that no method exists to obtain the unique optimal solution of a fully FLPP having inequalities in the set of constraints. In this context, various researchers (Gong and Zhao 2018; Mottaghi et al. 2015; Giri et al. 2015) solved fully FLPPs with inequality constraints by transforming them into equality constraints using fuzzy slack and surplus variables. However, in the case of FNs, such transformations are not correct mathematically and may lead to infeasible solutions for the considered FLPPs. On a different note, Das et al. (2017) proposed a lexicographic method to solve a fully FLPP with all types of constraints keeping decision variables as non-negative trapezoidal FNs. Later on, Ebrahimnejad and Verdegay (2018) demonstrated that this method is not suitable to deal with the fully FLPPs having inequality constraints as the authors utilized different order relation for inequality constraints than that was used for the objective function. Consequently, Ebrahimnejad and Verdegay (2018) suggested a correction by replacing the inequalities with a set of crisp linear inequalities.

Mahmoudi and Nasseri (2019) proposed a new approach to deal with the fully FLPPs. Thereafter, Pérez-Cañedo and Concepción-Morales (2019a) solved a fully FLPP having equality and inequality constraints with parameters and decision variables as unrestricted $LR$-type FNs, using the lexicographic ranking criterion for the objective function and the set of inequality constraints. Voskoglou (2020) first proposed a fuzzy theory-based group performance assessment technique and further introduced an algorithm to deal with the FLPPs in order to solve several related practical problems such as student and basketball players assessments, furniture manufacturing problems, and food mixing problems for feeding in a poultry farm. Das (2021) utilized the ranking function to optimize the class of fuzzy linear fractional programming problems. Next, Tadesse et al. (2021) described a geometrical approach to handle the fully FLLPs having non-negative decision variables. Kane et al. (2021) obtained the fuzzy optimal solution for semi-fully fuzzy LPPs by converting them into their equivalent semi-fully interval LPPs and applied it to solve various numerical cases. Further, some other significant applications of fuzzy theory can be found in the studies (Akram et al. 2021a; Hesamian and Akbari 2022; Enginoğlu and Arslan 2020; Aydın and Enginoğlu 2022). In another study, Akram et al. (2022a) defined the new arithmetic operations for trapezoidal fuzzy numbers by overcoming the limitations of the existing ones and thereby, proposed an improved version of the simplex algorithm to solve the fully fuzzy linear optimization models. Then, Akram et al. (2022b) introduced the concept of $LR$-flat picture fuzzy numbers and presented a generalized ranking function-based methodology to tackle the LPPs having parameters and decision variables in the form of $LR$-flat picture fuzzy numbers. Further, Ranjbar et al. (2022) devised an approach to solve the class of fully hesitant fuzzy LPPs by converting them into some equivalent interval LPP. Recently, Saghi et al. (2023) proposed the simplex algorithm to handle the hesitant fuzzy LPPs having cost coefficients given by trapezoidal hesitant fuzzy numbers.
2.2 Intuitionistic fuzzy LPP

Atanassov (1986) generalized Zadeh’s concept of fuzzy sets by introducing intuitionistic fuzzy sets (IFSs) in order to include hesitation in the involved parameters. Angelov (1997) was the first to apply the IFS theory to optimization problems. Mahapatra and Roy (2009) developed the arithmetic operations on triangular intuitionistic fuzzy numbers (IFNs) and did a reliability evaluation using these numbers. A linear programming problem (P) (Sect. 1) having equality and inequality constraints with all the parameters and decision variables expressed by IFNs is classified as a fully intuitionistic fuzzy linear programming problem (IFLPP). Nagoorgani and Ponnalagu (2012) introduced an algorithm for solving the IFLPPs having only inequality constraints. Furthermore, using a ranking function for IFNs, Suresh et al. (2014) introduced another methodology to solve the IFLPPs. Singh and Yadav (2015) suggested the modelling and optimization of multi-objective non-linear programming problems in an intuitionistic fuzzy environment.

Arefi and Taheri (2014) proposed the product of LR-type IFNs when both the numbers are either non-negative or non-positive or one is non-negative, and the other is non-positive. However, the remaining cases are not discussed. Then, Singh and Yadav (2017) introduced the product of unrestricted LR-type IFNs using (α, β)-cut and developed a solution approach for solving fully IFLPPs using score and accuracy indices of LR-type IFNs. An extensive review of IFS theory and its application to fully IFLPPs can be referred in the works (Atanassov 1999; Kumar and Hussain 2016; Niroomand 2018; Roy et al. 2018; Sidhu and Kumar 2019; Wan et al. 2015). Mahmoodirad et al. (2019) introduced a procedure to obtain the non-negative optimal solution of a fully intuitionistic fuzzy transportation problem. Thereafter, Pérez-Cañedo and Concepción-Morales (2019b) devised a method employing the total order properties of the lexicographic ranking criterion for finding the unique optimal intuitionistic fuzzy solution of a fully IFLPP having equality as well as inequality constraints with all the parameters and/or decision variables represented by unrestricted LR-type IFNs. Akram et al. (2021d) formulated a class of LR-type fully Pythagorean fuzzy LPPs with equality constraints and proposed a methodology to solve them. Next, Akram et al. (2021b) introduced two different techniques to deal with the LR-type fully Pythagorean fuzzy LPPs having mixed constraints and unrestricted decision variables. Then, Akram et al. (2021c) used the triangular Pythagorean fuzzy numbers for all the parameters and decision variables of a LPP and devised a linear ranking function based-solution technique for handling such problems along with demonstrating the applicability in solving the fertilizer mixing problem, food blending problem and farmer planning problem. Recently, Hosseinzadeh and Tayyebi (2023) considered a model of multi-objective LPPs under a neutrosophic environment and developed a ranking function algorithm to solve such problems. Furthermore, several other ways to describe the indeterminate or ambiguous information while solving different optimization problems under uncertainty can be determined by using plithogenic sets (Rezaei et al. 2022; Wang et al. 2023), neutrosophic hypersoft sets (Zhao et al. 2023; Jafar et al. 2022), neutrosophic sets (Tamilarasi and Paulraj 2022), and Pythagorean fuzzy hypersoft sets (Zulqarnain et al. 2023).

2.3 Interval-valued intuitionistic fuzzy LPP

In view of real-life situations, it is more flexible and viable to represent the membership and non-membership degrees of an element by intervals rather than crisp real numbers. Hence, Atanassov and Gargov (1989) proposed the concept of interval-valued intuitionistic
fuzzy (IVIF) sets. Optimizing a linear objective function over a set of linear constraints (P) where all the parameters and decision variables are expressed using the interval-valued intuitionistic fuzzy numbers (IVIFNs) is termed a fully interval-valued intuitionistic fuzzy linear programming problem (IVIFLPP). Several researchers had used the idea of IVIF theory for dealing with realistic decision-making problems. Ishibuchi and Tanaka (1990) were the first to solve a multi-objective programming problem in which the coefficients of the objective function are intervals instead of crisp numbers. Şahin (2016) suggested a ranking technique to define the ordering of IVIFNs. The basic theory and various rankings of interval-valued fuzzy numbers can be reviewed in the works (Chen and Lee 2010; Chen and Wang 2013; Chen et al. 2012; Hesamian 2016). Yang et al. (2009) studied the combination of interval-valued fuzzy assessments and unknown weights. Next, Garg et al. (2014) gave a technique for solving multi-criteria decision-making problems with LR fuzzy sets and soft sets.

An intuitionistic fuzzy optimization approach using an interval environment to solve the multi-criteria group-decision making problem. Bharati and Singh (2019) proposed a method to solve a multi-objective LPP in an IVIF environment. Then, Bharati and Singh (2020) developed an approach for solving the IVIFLPPs having unrestricted parameters while decision variables are taken to be non-negative quantities. Recently, Meekawy (2022) devised a close interval approximation based-methodology to solve the multi-objective linear fractional programming problems by taking all the parameters of the problem as piecewise quadratic fuzzy numbers.

A brief summarization of the various approaches to tackle the FLPPs, IFLPPs, and IVIFLPPs along with our proposed methodology is presented in Table 1.

### 3 Preliminaries

In this section, we have introduced the basic concepts related to LR-type IVIFNs followed by the arithmetic operations on them.

**Definition 3.1** (Atanassov and Gargov 1989) Let X be the universal set and Int[0,1] denote the set of all subintervals of the interval [0,1]. An interval-valued intuitionistic fuzzy set (IVIFS) is defined as a set \( \hat{A} = \{(x, \mu_{\hat{A}}(x), \nu_{\hat{A}}(x)) : x \in X\} \), where \( \mu_{\hat{A}} : X \rightarrow \text{Int[0,1]} \) and \( \nu_{\hat{A}} : X \rightarrow \text{Int[0,1]} \) represent the interval-valued membership and non-membership functions respectively, provided \( 0 \leq \text{Sup}(\mu_{\hat{A}}(x)) + \text{Sup}(\nu_{\hat{A}}(x)) \leq 1 \), \( \forall x \in X \).

**Definition 3.2** A set \( \hat{A} = \{(x, \mu_{\hat{A}}(x), \nu_{\hat{A}}(x)) : x \in X\} \) where \( \mu_{\hat{A}} = [\mu_{L_{\hat{A}}}, \mu_{U_{\hat{A}}} ] \) and \( \nu_{\hat{A}} = [\nu_{L_{\hat{A}}}, \nu_{U_{\hat{A}}} ] \) is called a convex IVIFS if \( \forall x_1, x_2 \in X, 0 \leq \lambda \leq 1 \), the following conditions are satisfied:

- \( \mu_{L_{\hat{A}}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{L_{\hat{A}}}(x_1), \mu_{L_{\hat{A}}}(x_2)\} \),
- \( \mu_{U_{\hat{A}}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{U_{\hat{A}}}(x_1), \mu_{U_{\hat{A}}}(x_2)\} \),
- \( \nu_{L_{\hat{A}}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\nu_{L_{\hat{A}}}(x_1), \nu_{L_{\hat{A}}}(x_2)\} \), and
- \( \nu_{U_{\hat{A}}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\nu_{U_{\hat{A}}}(x_1), \nu_{U_{\hat{A}}}(x_2)\} \).

**Definition 3.3** An IVIF set \( \hat{A} \) in \( X \) is called normal IVIFS if there exist \( x_1, x_2 \in X \) such that \( \mu_{\hat{A}}(x_1) = 1 \) and \( \nu_{\hat{A}}(x_2) = 1 \).
| References                        | Type of FN/IFN/IVIFN | Unrestricted variables | Ordering criterion | Type of constraints (equality and/or inequality) |
|----------------------------------|----------------------|------------------------|--------------------|-------------------------------------------------|
| Hashemi et al. (2006)            | LR-type FN           | ×                      | Lexicographic      | Inequality only                                 |
| Lotfi et al. (2009)              | Triangular FN        | ×                      | Lexicographic      | Equality only                                   |
| Kaur and Kumar (2012)            | Trapezoidal FN       | ×                      | Lexicographic      | Equality only                                   |
| Kaur and Kumar (2013)            | LR-type FN           | ✓                      | Ranking function   | Equality and inequality both                   |
| Hosseinzadeh and Edalatpanah (2016) | LR-type FN           | ×                      | Lexicographic      | Equality only                                   |
| Kaur and Kumar (2016)            | LR-type FN           | ✓                      | Lexicographic      | Equality only                                   |
| Pérez-Cañedo and Concepción-Morales (2019a) | LR-type FN           | ✓                      | Lexicographic      | Equality and inequality both                   |
| Tadesse et al. (2021)            | Triangular FN        | ×                      | Geometric approach | Inequality only                                 |
| Akram et al. (2021c)             | Pythagorean FN       | ✓                      | Ranking function   | Equality only                                   |
| Akram et al. (2021b)             | LR-type              | ✓                      | Ranking function   | Equality and inequality both                   |
| Akram et al. (2022a)             | Trapezoidal FN       | ✓                      | Ranking function   | Equality and inequality both                   |
| Nagoorgani and Ponnalagu (2012)  | Triangular IFN       | ×                      | Score function     | Inequality only                                 |
| Singh and Yadav (2017)           | LR-type IFN          | ✓                      | Weighted sum of score and accuracy indices | Equality and inequality both                   |
| Pérez-Cañedo and Concepción-Morales (2019b) | LR-type IFN          | ✓                      | Lexicographic      | Equality and inequality both                   |
| Bharati and Singh (2020)         | Triangular IVIFN     | ×                      | Expected value function | Equality and inequality both                   |
| Present study                    | LR-type IVIFN        | ✓                      | Lexicographic      | Equality and inequality both                   |
Definition 3.4 An IVIF set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)) : x \in \mathbb{R}\}$ is called an IVIFN if the following conditions hold:

- $\tilde{A}$ is a convex IVIFS in $\mathbb{R}$,
- $\tilde{A}$ is a normal IVIFS, and
- $\mu_{\tilde{A}}, \mu_{\tilde{A}}, v_{\tilde{A}}$ and $v_{\tilde{A}}$ are piecewise continuous functions from $\mathbb{R}$ to $[0,1]$.

Mathematically, lower-upper membership and non-membership functions of an IVIFN $\tilde{A}$ can be represented as follows:

$$\mu_{\tilde{A}} = \begin{cases} 1, & \text{if } x = a, \\ g_1(x), & \text{if } a - l_{L}^\mu < x < a, \\ g_2(x), & \text{if } a < x < a + r_{L}^\mu, \\ 0, & \text{otherwise,} \end{cases} \quad \mu_{\tilde{A}} = \begin{cases} 1, & \text{if } x = a, \\ h_1(x), & \text{if } a - l_{U}^\mu < x < a, \\ h_2(x), & \text{if } a < x < a + r_{U}^\mu, \\ 0, & \text{otherwise,} \end{cases}$$

$$v_{\tilde{A}} = \begin{cases} 0, & \text{if } x = a, \\ l_1(x), & \text{if } a - l_{L}^v < x < a, \\ l_2(x), & \text{if } a < x < a + r_{L}^v, \\ 1, & \text{otherwise,} \end{cases} \quad v_{\tilde{A}} = \begin{cases} 0, & \text{if } x = a, \\ m_1(x), & \text{if } a - l_{U}^v < x < a, \\ m_2(x), & \text{if } a < x < a + r_{U}^v, \\ 1, & \text{otherwise} \end{cases}$$

where

- (i) $g_1, h_1, l_2$ and $m_2$ are piecewise continuous and strictly increasing functions,
- (ii) $g_2, h_2, l_1$ and $m_1$ are piecewise continuous and strictly decreasing functions,
- (iii) $g_1(x) \leq h_1(x), g_2(x) \leq h_2(x), l_1(x) \leq m_1(x), l_2(x) \leq m_2(x), \forall x \in \mathbb{R}$,
- (iv) $a$ is called the mean value of $\tilde{A}$,
- (v) $l_{L}^\mu, l_{U}^\mu, l_{L}^v$ and $l_{U}^v$ are respectively, the left spreads of $\mu_{\tilde{A}}, \mu_{\tilde{A}}, v_{\tilde{A}}$ and $v_{\tilde{A}}$, and
- (vi) $r_{L}^\mu, r_{U}^\mu, r_{L}^v$ and $r_{U}^v$ are respectively, the right spreads of $\mu_{\tilde{A}}, \mu_{\tilde{A}}, v_{\tilde{A}}$ and $v_{\tilde{A}}$.

This IVIFN can be represented as $\tilde{A} = (a; l_{L}^\mu, r_{L}^\mu, l_{U}^\mu, r_{U}^\mu; l_{L}^v, r_{L}^v, l_{U}^v, r_{U}^v)$. The graphical representation of an IVIFN $\tilde{A}$ is given in Fig. 1.

Definition 3.5 (Bharati and Singh 2018) A triangular IVIFN (TIVIFN) is denoted by $\tilde{A} = \{(a_1^L, a_1^U, a_2, a_2^L, a_2^U, b_1^L, b_1^U, a_2, b_2^L, b_2^U)\}$, and its membership and non-membership degrees are defined as follows:

- Lower and upper membership functions are respectively given by:

$$\mu_{\tilde{A}}^L = \begin{cases} 1, & \text{if } x = a_2, \\ \frac{x-a_1^L}{a_2-a_1^L}, & \text{if } a_1^L < x < a_2, \\ \frac{x-a_2}{a_2-a_2^L}, & \text{if } a_2 < x < a_2^L, \\ 0, & \text{otherwise} \end{cases} \quad \mu_{\tilde{A}}^U = \begin{cases} 1, & \text{if } x = a_2, \\ \frac{x-a_1^U}{a_2-a_1^U}, & \text{if } a_1^U < x < a_2, \\ \frac{x-a_2}{a_2-a_2^U}, & \text{if } a_2 < x < a_2^U, \\ 0, & \text{otherwise} \end{cases}$$
Lower and upper non-membership functions are respectively defined as:

\[
v_L \tilde{A}(x) = \begin{cases} 
0, & \text{if } x = a_2, \\
\frac{a_2 - x}{a_2 - b_1}, & \text{if } b_1 < x < a_2, \\
\frac{a_2 - x}{a_2 - b_3}, & \text{if } a_2 < x < b_3, \\
1, & \text{otherwise}
\end{cases}
\]

\[
v_U \tilde{A}(x) = \begin{cases} 
0, & \text{if } x = a_2, \\
\frac{x - a_2}{b_1 - a_2}, & \text{if } b_1 < x < a_2, \\
\frac{x - a_2}{b_3 - a_2}, & \text{if } a_2 < x < b_3, \\
1, & \text{otherwise}
\end{cases}
\]

where \( b_1 < b_3 < a_1^L \leq a_1^U \leq a_2^U \leq a_2^L \leq a_3^U \leq a_3^L \leq b_3^U \leq b_3^L \). The diagrammatic representation of a TIVIFN is shown in Fig. 2.

**Definition 3.6** (Singh and Yadav 2017) A function \( f : [0, \infty) \to [0, 1] \) is said to be shape function or reference function if it satisfies the following conditions:

(i) \( f(0) = 1 \),

(ii) \( f \) is invertible on \([0, \infty)\),

(iii) \( f \) is continuous function on \([0, \infty)\),

(iv) \( f \) is strictly decreasing on \([0, \infty)\), and

(v) \( \lim_{x \to \infty} f(x) = 0 \).

**Definition 3.7** An IVIFN \( \tilde{A} \) is said to be \( LR \)-type IVIFN if there exist shape functions \( L, R, L' \) and \( R' \), and positive real constants \( l_\mu^L, r_\mu^L, l_\mu^U, r_\mu^U, l_\nu^L, r_\nu^L, l_\nu^U, r_\nu^U \) and \( r_\nu^U \), such that its...
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Fig. 2 Triangular interval-valued intuitionistic fuzzy number

- Lower and upper membership functions, respectively are defined as

\[
\mu^L_A(x) = \begin{cases} 
L \left( \frac{a-x}{l^L_M} \right), & a - l^L_M \leq x \leq a, \\
R \left( \frac{x-a}{r^L_M} \right), & a \leq x \leq a + r^L_M, \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\mu^U_A(x) = \begin{cases} 
L' \left( \frac{a-x}{l^U_M} \right), & a - l^U_M \leq x \leq a, \\
R' \left( \frac{x-a}{r^U_M} \right), & a \leq x \leq a + r^U_M, \\
0, & \text{otherwise}
\end{cases}
\]

- Lower and upper non-membership functions, respectively are given by

\[
v^L_A(x) = \begin{cases} 
1 - L \left( \frac{a-x}{l^L_M} \right), & a - l^L_M \leq x \leq a, \\
1 - R \left( \frac{x-a}{r^L_M} \right), & a \leq x \leq a + r^L_M, \\
1, & \text{otherwise}
\end{cases}
\]

and

\[
v^U_A(x) = \begin{cases} 
1 - L' \left( \frac{a-x}{l^U_M} \right), & a - l^U_M \leq x \leq a, \\
1 - R' \left( \frac{x-a}{r^U_M} \right), & a \leq x \leq a + r^U_M, \\
1, & \text{otherwise}
\end{cases}
\]

where \( l^L_M \geq l^L_U \), \( r^L_M \geq r^L_U \), \( l^U_M \geq l^U_U \), \( r^U_M \geq r^U_U \), \( l^L_M \geq l^L_L \), \( r^L_M \geq r^L_L \), \( l^U_M \geq l^U_U \), \( r^U_M \geq r^U_U \) and \( 0 \leq \text{Sup}\{\mu_A(x)\} + \text{Sup}\{v_A(x)\} \leq 1 \), \( \forall x \in \mathbb{R} \). \( a \) is called the mean value of \( A \); \( l^L_M \), \( l^U_M \), \( l^L_U \) and \( l^U_U \) are respectively, the left spreads of \( \mu^L_A \), \( \mu^U_A \), \( v^L_A \) and \( v^U_A \); and
Definition 3.9

An LR-unrestricted $\alpha, \beta$-type IVIFN is denoted by $\tilde{\alpha}, \tilde{\beta}$-type IVIFN if $\tilde{\alpha} = (a; l^L_L, r^L_L, l^U_L, r^U_L; l^L_U, r^L_U, l^U_U, r^U_U)_{LR}$ and its possible general graphical representation is shown in Fig. 3. Let IV(\mathbb{R}) represents the set of all LR-type IVIFNs.

Remark 3.1

Taking $L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1 - x\}, \forall x \in \mathbb{R}$, the Definition 3.7 reduces to Definition 3.5.

Definition 3.8

An LR-type IVIFN $\tilde{\alpha} = (a; l^L_L, r^L_L, l^U_L, r^U_L; l^L_U, r^L_U, l^U_U, r^U_U)_{LR}$ is called an unrestricted LR-type IVIFN if $a$ is any real number.

Definition 3.9

An LR-type IVIFN $\tilde{\alpha} = (a; l^L_L, r^L_L, l^U_L, r^U_L; l^L_U, r^L_U, l^U_U, r^U_U)_{LR}$ is called non-negative (positive) if $a - l^U_U \geq (>) 0$ and non-positive (negative) if $a + r^L_U \leq (<) 0$.

Theorem 3.1

Let $\tilde{\alpha} = (a; l^L_L, r^L_L, l^U_L, r^U_L; l^L_U, r^L_U, l^U_U, r^U_U)_{LR}$ be an LR-type IVIFN. Then, \( \forall \alpha, \beta \in (0, 1) \) and $\alpha + \beta \leq 1$,

(i) its lower $\alpha$-cut for membership and lower $\beta$-cut for non-membership are, respectively, given by

$$A^L_\alpha = [a - l^L_L L^{-1}(\alpha), a + r^L_L R^{-1}(\alpha)] \quad \text{and} \quad A^L_\beta = [a - l^U_U L^{-1}(1 - \beta), a + r^L_U R^{-1}(1 - \beta)],$$

(ii) its upper $\alpha$-cut for membership and upper $\beta$-cut for non-membership, respectively, are

$$A^U_\alpha = [a - l^L_L (L')^{-1}(\alpha), a + r^L_L (R')^{-1}(\alpha)] \quad \text{and} \quad A^U_\beta = [a - l^U_U (L')^{-1}(1 - \beta), a + r^U_U (R')^{-1}(1 - \beta)],$$

(iii) its lower and upper $(\alpha, \beta)$-cut, respectively, are

$$A^L_{\alpha, \beta} = [a - l^L_L L^{-1}(\alpha), a + r^L_L R^{-1}(\alpha)] \cap [a - l^U_U L^{-1}(1 - \beta), a + r^U_U R^{-1}(1 - \beta)].$$
\[ a + r_L^v R^{-1}(1 - \beta) \]

and

\[ A^U_{\alpha, \beta} = [a - l^\mu_U (L')^{-1}(\alpha), a + r^\mu_U (R')^{-1}(\alpha)] \cap [a - l^v_U (L')^{-1}(1 - \beta), a + r^v_U (R')^{-1}(1 - \beta)]. \]

**Proof**  
(i) For \( \alpha \in (0, 1] \), \( \mu^L_A(x) \geq \alpha \) implies

\[
L\left(\frac{a - x}{l^\mu_L}ight) \geq \alpha \quad \text{and} \quad R\left(\frac{x - a}{r^\mu_L}ight) \geq \alpha.
\]

Since \( L \) and \( R \) are decreasing functions, therefore

\[
\frac{a - x}{l^\mu_L} \leq L^{-1}(\alpha), \quad \frac{x - a}{r^\mu_L} \leq R^{-1}(\alpha).
\]

It further yields

\[
a - l^\mu_L L^{-1}(\alpha) \leq x \leq a + r^\mu_L R^{-1}(\alpha).
\]

Hence,

\[ A^L_\alpha = [a - l^\mu_L L^{-1}(\alpha), a + r^\mu_L R^{-1}(\alpha)]. \]

Now, for \( \beta \in (0, 1] \) such that \( \alpha + \beta \leq 1 \), \( \nu^L_A(x) \leq \beta \) gives

\[
1 - L\left(\frac{a - x}{l^\nu_L}ight) \leq \beta \quad \text{and} \quad 1 - R\left(\frac{x - a}{r^\nu_L}ight) \leq \beta
\]

which implies

\[
\frac{a - x}{l^\nu_L} \leq L^{-1}(1 - \beta), \quad \frac{x - a}{r^\nu_L} \leq R^{-1}(1 - \beta).
\]

Thus,

\[ A^L_\beta = [a - l^\nu_L L^{-1}(1 - \beta), a + r^\nu_L R^{-1}(1 - \beta)]. \]

This proves (i).

(ii) Applying \( \alpha \)-cut on the upper membership function, that is, \( \mu^U_A(x) \geq \alpha, \alpha \in (0, 1] \), we get

\[
L'\left(\frac{a - x}{l'^\mu_U}\right) \geq \alpha \quad \text{and} \quad R'\left(\frac{x - a}{r'^\mu_U}\right) \geq \alpha.
\]

Using the fact that \( L' \) and \( R' \) are decreasing functions, it follows that

\[
\frac{a - x}{l'^\mu_U} \leq (L')^{-1}(\alpha) \quad \text{and} \quad \frac{x - a}{r'^\mu_U} \leq (R')^{-1}(\alpha).
\]

This, after simplification gives

\[
a - l'^\mu_U (L')^{-1}(\alpha) \leq x \leq a + r'^\mu_U (R')^{-1}(\alpha).
\]

Therefore,

\[ A^U_\alpha = [a - l'^\mu_U (L')^{-1}(\alpha), a + r'^\mu_U (R')^{-1}(\alpha)]. \]
Similarly, for $\beta \in (0, 1]$ such that $\alpha + \beta \leq 1$, the expression \( \nu^U(x) \leq \beta \) yields

\[
1 - L' \left( \frac{a - x}{l^U} \right) \leq \beta \quad \text{and} \quad 1 - R' \left( \frac{x - a}{r^U} \right) \leq \beta.
\]

This finally gives

\[
A^U_\beta = [a - l^U, (L')^{-1}(1 - \beta), a + r^U, (R')^{-1}(1 - \beta)].
\]

Hence proved part (ii).

(iii) From (i), the lower \( \alpha \)-cut for membership and the lower \( \beta \)-cut for non-membership of \( \tilde{A} \) are, respectively, given by

\[
A^L_\alpha = [a - l^L L^{-1}(\alpha), a + r^L R^{-1}(\alpha)] \quad \text{and} \quad A^L_\beta = [a - l^L L^{-1}(1 - \beta), a + r^L R^{-1}(1 - \beta)].
\]

It yields

\[
A^L_{\alpha, \beta} = A^L_\alpha \cap A^L_\beta = [a - l^L L^{-1}(\alpha), a + r^L R^{-1}(\alpha)] \cap [a - l^L L^{-1}(1 - \beta), a + r^L R^{-1}(1 - \beta)].
\]

On the same lines, the proof of \( A^U_{\alpha, \beta} \) can also be obtained. Hence, the result.

\[\square\]

**Definition 3.10** Let \( \tilde{A} = (a; l^L, r^L, l^U, r^U; l^\mu, r^\mu, l^\nu, r^\nu)_{LR} \) be an LR-type IVIFN. Then, the score and accuracy indices of \( \tilde{A} \) are denoted by \( S(\tilde{A}) \) and \( A(\tilde{A}) \), respectively and are defined as follows:

\[
S(\tilde{A}) := \frac{1}{4} \int_0^1 (a - l^L L^{-1}(\alpha) + a + r^L R^{-1}(\alpha) + a - l^U U^{-1}(\alpha) + a + r^U U^{-1}(\alpha)) \, d\alpha
\]

\[
+ a + r^L R^{-1}(\alpha))\, d\alpha - \frac{1}{4} \int_0^1 (a - l^L L^{-1}(1 - \beta) + a + r^L R^{-1}(1 - \beta) + a - l^U U^{-1}(1 - \beta) + a + r^U U^{-1}(1 - \beta))\, d\beta
\]

\[
A(\tilde{A}) := \frac{1}{4} \int_0^1 (a - l^L L^{-1}(\alpha) + a + r^L R^{-1}(\alpha) + a - l^U U^{-1}(\alpha) + a + r^U U^{-1}(\alpha)) \, d\alpha
\]

\[
+ a + r^L R^{-1}(\alpha))\, d\alpha + \frac{1}{4} \int_0^1 (a - l^L L^{-1}(1 - \beta) + a + r^L R^{-1}(1 - \beta) + a - l^U U^{-1}(1 - \beta) + a + r^U U^{-1}(1 - \beta))\, d\beta.
\]

**Remark 3.2** If \( l^L = l^U, r^L = r^U, l^\mu = l^\nu \) and \( r^\mu = r^\nu \), then the Definition 3.10 reduces to the corresponding definition for LR-type IFNs given in Singh and Yadav (2017).

**Theorem 3.2** Let \( \tilde{A} = (a; a - a^L, a^L - a, a - a^U, a^U - a, a - b^L, b^L - a, a - b^U, b^U - a)_{LR} \) be an LR-type TIVIFN. Then, the score and accuracy indices of LR-type TIVIFN \( \tilde{A} \) are, respectively, given by:

\[
S(\tilde{A}) = \frac{a^L + a^L + a^U + a^U + b^L - b^L - b^U - b^U}{8},
\]

\[
A(\tilde{A}) = \frac{a^L + a^L + a^U + a^U + 8a + b^L + b^L + b^U + b^U}{8}.
\]
This further implies $\alpha$ we have introduced the addition operator $LR$ in this subsection, the basic arithmetic operations on $\tilde{A}$.

From Definition 3.10, we have

$$S(\tilde{A}) = \frac{1}{4} \int_0^1 (a - l_1^\mu L^{-1}(\alpha) + a + r_2^\mu R^{-1}(\alpha) + a - l_1^\mu (L')^{-1}(\alpha) + a + r_2^\mu (R')^{-1}(\alpha)) \, d\alpha - \frac{1}{4} \int_0^1 (a - l_1^\mu L^{-1}(1 - \beta) + a + r_2^\mu R^{-1}(1 - \beta) + a - l_1^\mu (L')^{-1}(1 - \beta) + a + r_2^\mu (R')^{-1}(1 - \beta)) \, d\beta$$

$$A(\tilde{A}) = \frac{1}{4} \int_0^1 (a - l_1^\mu L^{-1}(\alpha) + a + r_2^\mu R^{-1}(\alpha) + a - l_1^\mu (L')^{-1}(\alpha) + a + r_2^\mu (R')^{-1}(\alpha)) \, d\alpha + \frac{1}{4} \int_0^1 (a - l_1^\mu L^{-1}(1 - \beta) + a + r_2^\mu R^{-1}(1 - \beta) + a - l_1^\mu (L')^{-1}(1 - \beta) + a + r_2^\mu (R')^{-1}(1 - \beta)) \, d\beta. \quad (1)$$

Now, since $\tilde{A}$ is a TIVIFN, therefore

$$L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1 - x\}, \quad \forall x \in \mathbb{R}.$$ Hence, for $\alpha \in (0, 1)$, we have

$$L(\alpha) = L'(\alpha) = R(\alpha) = R'(\alpha) = 1 - \alpha. \quad (3)$$

This further implies

$$L^{-1}(\alpha) = R^{-1}(\alpha) = (L')^{-1}(\alpha) = (R')^{-1}(\alpha) = 1 - \alpha. \quad (4)$$

Substituting the expressions from the Eqs. (3) and (4) in (1) and (2), we obtain

$$S(\tilde{A}) = \frac{a_1^L + a_2^L + a_1^U + a_2^U - b_1^L - b_2^L - b_1^U - b_2^U}{8} \quad \text{and}$$

$$A(\tilde{A}) = \frac{a_1^L + a_2^L + a_1^U + a_2^U + 8a + b_1^L + b_2^L + b_1^U + b_2^U}{8}.$$ Hence the result. \hfill \Box\hfill

### 3.1 Arithmetic operations on LR-type IVIFNs

In this subsection, the basic arithmetic operations on $LR$-type IVIFNs are discussed. Here, we have introduced the addition operator $(\oplus)$, subtraction operator $(\ominus)$ and product operator $(\odot)$ for $LR$-type IVIFNs. The following propositions discuss the detailed expressions for the addition, subtraction, scalar multiplication, and product operations on these numbers.

**Proposition 3.1** Let $\tilde{A}_1 = (a_1^L; l_{11L}^\mu, r_{11L}^\mu, l_{11U}^\mu, r_{11U}^\mu, l_{1U}^\mu, r_{1U}^\mu, l_{12U}^\mu, r_{12U}^\mu)_{LR}$ and $\tilde{A}_2 = (a_2^L; l_{22L}^\mu, r_{22L}^\mu, l_{21U}^\mu, r_{21U}^\mu, l_{2U}^\mu, r_{2U}^\mu, l_{12U}^\mu, r_{12U}^\mu)_{LR}$ be two $LR$-type IVIFNs. Then,

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2; l_{11L}^\mu + l_{22L}^\mu, r_{11L}^\mu + r_{22L}^\mu, l_{11U}^\mu + l_{21U}^\mu + l_{1U}^\mu + l_{2U}^\mu; r_{12U}^\mu, r_{12U}^\mu, r_{12U}^\mu, r_{12U}^\mu)_{LR},$

where the conditions for $LR$-type representation of $\tilde{A}_1 \oplus \tilde{A}_2$ are satisfied.

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - a_2; l_{11L}^\mu + r_{22L}^\mu, r_{11L}^\mu + l_{22L}^\mu, l_{11U}^\mu + r_{21U}^\mu + l_{1U}^\mu + r_{2U}^\mu; r_{12U}^\mu, r_{12U}^\mu, l_{12U}^\mu, r_{12U}^\mu)_{LR},$

where the conditions for $LR$-type representation of $\tilde{A}_1 \ominus \tilde{A}_2$ are fulfilled.
Proof In view of Theorem 3.1, the \( \alpha \) and \( \beta \)-cuts of \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are, respectively, given by

\[
A_{1\alpha}^L = \{a_1 - l_{1L}^\mu L^{-1}(\alpha), a_1 + r_{1L}^\mu R^{-1}(\alpha)\}, \\
A_{1\alpha}^U = \{a_1 - l_{1U}^\mu (L')^{-1}(\alpha), a_1 + r_{1U}^\mu (R')^{-1}(\alpha)\}, \\
A_{1\beta}^L = \{a_1 - l_{1L}^\mu (1 - \beta), a_1 + r_{1L}^\mu (1 - \beta)\}, \\
A_{1\beta}^U = \{a_1 - l_{1U}^\mu (L')^{-1}(1 - \beta), a_1 + r_{1U}^\mu (R')^{-1}(1 - \beta)\}.
\]

(5)

Also, choosing \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are \( LR \)-type IVIFNs, therefore

\[
l_{1U}^\mu \geq l_{1L}^\mu > 0, \quad r_{1U}^\mu \geq r_{1L}^\mu > 0, \quad l_{1L}^\mu \geq l_{1U}^\nu > 0, \quad r_{1L}^\mu \geq r_{1U}^\nu > 0,
\]

\[
l_{1L}^\nu \geq l_{1U}^\mu > 0, \quad r_{1L}^\nu \geq r_{1U}^\mu > 0, \quad l_{1U}^\nu \geq l_{1L}^\mu > 0, \quad r_{1U}^\nu \geq r_{1L}^\mu > 0,
\]

(6)

(i) From the Eqs. (5) and (6), we get

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_\alpha^L = A_{1\alpha}^L + A_{2\alpha}^L
= \{a_1 + a_2 - (l_{1L}^\mu + l_{2L}^\nu) L^{-1}(\alpha), a_1 + a_2 + (r_{1L}^\mu + r_{2L}^\nu) R^{-1}(\alpha)\}.
\]

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_\alpha^U = A_{1\alpha}^U + A_{2\alpha}^U
= \{a_1 + a_2 - (l_{1U}^\mu (L')^{-1}(\alpha), a_1 + a_2 + (r_{1U}^\mu + r_{2U}^\nu) (R')^{-1}(\alpha)\}.
\]

(7)

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_\beta^L = A_{1\beta}^L + A_{2\beta}^L
= \{a_1 + a_2 - (l_{1L}^\mu (1 - \beta), a_1 + a_2 + (r_{1L}^\mu + r_{2L}^\nu) R^{-1}(1 - \beta)\}.
\]

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_\beta^U = A_{1\beta}^U + A_{2\beta}^U
= \{a_1 + a_2 - (l_{1U}^\nu (L')^{-1}(1 - \beta), a_1 + a_2 + (r_{1U}^\nu + r_{2U}^\nu) (R')^{-1}(1 - \beta)\}.
\]

(8)

Since \( L, R, L' \) and \( R' \) are decreasing functions on \([0, \infty)\) with \( L(0) = R(0) = L'(0) = R'(0) = 1 \), there exists \( \alpha_0 \in (0, 1) \), such that \( L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = 1 \). Hence,

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_{\alpha_0}^L = A_{1\alpha_0}^L + A_{2\alpha_0}^L
= \{a_1 + a_2 - (l_{1L}^\mu + l_{2L}^\nu), a_1 + a_2 + (r_{1L}^\mu + r_{2L}^\nu)\}.
\]

(9)

Also, choosing \( \beta_0 = 1 - \alpha_0 \in (0, 1) \), we get

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_{\beta_0}^L = A_{1\beta_0}^L + A_{2\beta_0}^L
= \{a_1 + a_2 - (l_{1L}^\mu + l_{2L}^\nu), a_1 + a_2 + (r_{1L}^\mu + r_{2L}^\nu)\}.
\]

(10)

Further,

\[
(\tilde{A}_1 \oplus \tilde{A}_2)_{\alpha_1}^L = (\tilde{A}_1 \oplus \tilde{A}_2)_{\alpha_0}^L = (\tilde{A}_1 \oplus \tilde{A}_2)_{\beta_0}^L
= \{a_1 + a_2, a_1 + a_2\}.
\]

(11)
Using Eqs. (5) and (6), we can write
\[ l_{2L}^v \geq l_{2U}^\mu > 0, \quad r_{2L}^v \geq r_{2L}^\mu > 0, \quad l_{2U}^v \geq l_{2U}^\mu > 0, \quad r_{2U}^v \geq r_{2U}^\mu > 0. \]

Thus,
\[
\begin{align*}
&l_{1L}^\mu + l_{1L}^\mu \geq l_{1L}^\mu + l_{2L}^\mu > 0, \quad r_{1L}^\mu + r_{1L}^\mu \geq r_{1L}^\mu + r_{2L}^\mu > 0, \\
&l_{1L}^\mu + l_{2L}^\mu \geq l_{1L}^\mu + l_{2L}^\mu > 0, \quad r_{1L}^\mu + r_{2L}^\mu \geq r_{1L}^\mu + r_{2L}^\mu > 0, \\
&l_{1L}^\mu + l_{2L}^\mu \geq l_{1L}^\mu + l_{2L}^\mu > 0, \quad r_{1L}^\mu + r_{2L}^\mu \geq r_{1L}^\mu + r_{2L}^\mu > 0, \\
&l_{1L}^\mu + l_{2L}^\mu \geq l_{1L}^\mu + l_{2L}^\mu > 0, \quad r_{1L}^\mu + r_{2L}^\mu \geq r_{1L}^\mu + r_{2L}^\mu > 0.
\end{align*}
\]

Combining Eqs. (7)–(11), we have
\[ \tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2 = (a_1 + a_2) l_{1L}^\mu + l_{2L}^\mu, r_{1L}^\mu + r_{2L}^\mu, l_{1U}^\mu + l_{2U}^\mu, r_{1U}^\mu + r_{2U}^\mu, l_{1L}^\mu + l_{2L}^\mu, r_{1L}^\mu + r_{2L}^\mu, l_{1U}^\mu + l_{2U}^\mu, r_{1U}^\mu + r_{2U}^\mu)_{LR}, \]

where the conditions for LR-type form of \( \tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2 \) holds from (12).

Hence, (i) is proved.

(ii) Using Eqs. (5) and (6), we can write
\[
\begin{align*}
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)_\alpha^L &= A_{1\alpha}^L - A_{2\alpha}^L \\
&= [a_1 - a_2 - l_{1L}^\mu L^{-1}(\alpha) - r_{2L}^\mu R^{-1}(\alpha), \ a_1 - a_2 + r_{1L}^\mu R^{-1}(\alpha) + l_{2L}^\mu L^{-1}(\alpha)], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)_\alpha^U &= A_{1\alpha}^U - A_{2\alpha}^U \\
&= [a_1 - a_2 - r_{1L}^\mu (L')^{-1}(\alpha) - r_{2U}^\mu (R')^{-1}(\alpha), \ a_1 - a_2 + r_{1U}^\mu (R')^{-1}(\alpha) + l_{2U}^\mu (L')^{-1}(\alpha)], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)_\beta^L &= A_{1\beta}^L - A_{2\beta}^L \\
&= [a_1 - a_2 - l_{1L}^\mu L^{-1}(1 - \beta) - r_{2L}^\mu R^{-1}(1 - \beta), \ a_1 - a_2 + r_{1L}^\mu R^{-1}(1 - \beta) + l_{2L}^\mu L^{-1}(1 - \beta)], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)_\beta^U &= A_{1\beta}^U - A_{2\beta}^U \\
&= [a_1 - a_2 - l_{1U}^\mu (L')^{-1}(1 - \beta) - r_{2U}^\mu (R')^{-1}(1 - \beta), \ a_1 - a_2 + r_{1U}^\mu (R')^{-1}(1 - \beta) + l_{2U}^\mu (L')^{-1}(1 - \beta)].
\end{align*}
\]

Now, taking \( \alpha = 0 \) and \( \beta = 0 \in (0, 1], \) we get
\[
\begin{align*}
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^L_{\alpha=0} &= [a_1 - a_2 - l_{1L}^\mu - r_{2L}^\mu, a_1 - a_2 + r_{1L}^\mu + l_{2L}^\mu], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^U_{\alpha=0} &= [a_1 - a_2 - r_{1L}^\mu - r_{2L}^\mu, a_1 - a_2 + r_{1U}^\mu + r_{2L}^\mu], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^L_{\beta=0} &= [a_1 - a_2 - l_{1L}^\mu - r_{2L}^\mu, a_1 - a_2 + r_{1L}^\mu + l_{2L}^\mu], \\
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^U_{\beta=0} &= [a_1 - a_2 - l_{1U}^\mu - r_{2U}^\mu, a_1 - a_2 + r_{1U}^\mu + r_{2U}^\mu].
\end{align*}
\]

Further, on substituting \( \alpha = 1 \) and \( \beta = 0, \) we obtain
\[
(\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^L_{\alpha=1} = (\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^U_{\alpha=1} = (\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^L_{\beta=0} = (\tilde{\mathcal{A}}_1 \oplus \tilde{\mathcal{A}}_2)^U_{\beta=0} = [a_1 - a_2, a_1 - a_2].
\]

Also, using the fact \( \tilde{\mathcal{A}}_1 \) and \( \tilde{\mathcal{A}}_2 \) are LR-type IVIFNs, we have
\[ l_{1U}^\mu + r_{2L}^\mu \geq l_{1L}^\mu + l_{2L}^\mu > 0, \quad r_{1L}^\mu + r_{1U}^\mu \geq r_{1L}^\mu + l_{2L}^\mu > 0. \]
\[ l_L^v + r_L^v \geq l_U^v + r_U^v > 0, \quad r_L^v + r_U^v > 0, \quad r_U^v + r_L^v > 0, \quad l_U^v + r_L^v > 0, \quad r_U^v + r_L^v > 0, \quad l_L^v + r_U^v > 0, \quad r_L^v + r_U^v > 0. \]

Finally, from the Eqs. (13) and (14), we have
\[
\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - a_2; l_L^u + r_L^u, l_U^u + r_U^u, l_L^v + r_U^v, l_U^v + r_L^v; l_L^v + r_U^v, l_U^v + r_L^v)_{LR},
\]
along with \( \tilde{A}_1 \ominus \tilde{A}_2 \) retains the form of a \( LR \)-type IVFNF. This proves (ii).

\[\square\]

**Proposition 3.2** Let \( \tilde{A} = (a; l_L^u, l_L^v, l_U^u, l_U^v, r_L^u, r_L^v, r_U^u, r_U^v)_{LR} \) be an \( LR \)-type IVFNF and \( \lambda \) be any real number. Then,

\[
\lambda \tilde{A} = \begin{cases} 
(\lambda a; \lambda l_L^u, \lambda l_L^v, \lambda l_U^u, \lambda l_U^v, \lambda r_L^u, \lambda r_L^v, \lambda r_U^u, \lambda r_U^v)_{LR} & \text{if } \lambda \geq 0, \\
(\lambda a; -\lambda r_L^u, -\lambda l_L^u, -\lambda r_L^v, -\lambda l_L^v, -\lambda r_U^u, -\lambda l_U^v)_{LR} & \text{if } \lambda < 0.
\end{cases}
\]

**Proof** From Theorem 3.1, the \( \alpha \) and \( \beta \)-cuts of \( \tilde{A} \) are given by

\[
\begin{align*}
A_L^L &= [a - l_L^u L^{-1}(\alpha), a + r_L^u R^{-1}(\alpha)], \\
A_U^L &= [a - l_U^u (L')^{-1}(\alpha), a + r_U^u (R')^{-1}(\alpha)], \\
A_L^U &= [a - l_L^v L^{-1}(1 - \beta), a + r_L^v R^{-1}(1 - \beta)], \\
A_U^U &= [a - l_U^v (L')^{-1}(1 - \beta), a + r_U^v (R')^{-1}(1 - \beta)].
\end{align*}
\]

Using the expression (15), we have

\[
\begin{align*}
(\lambda \tilde{A})_L^L &= \lambda A_L^L = [\lambda a - l_L^u L^{-1}(\alpha), a + r_L^u R^{-1}(\alpha)], \\
(\lambda \tilde{A})_U^L &= \lambda A_U^L = [\lambda a - l_U^u (L')^{-1}(\alpha), a + r_U^u (R')^{-1}(\alpha)], \\
(\lambda \tilde{A})_L^U &= \lambda A_L^U = [\lambda a - l_L^v L^{-1}(1 - \beta), a + r_L^v R^{-1}(1 - \beta)], \\
(\lambda \tilde{A})_U^U &= \lambda A_U^U = [\lambda a - l_U^v (L')^{-1}(1 - \beta), a + r_U^v (R')^{-1}(1 - \beta)].
\end{align*}
\]

**Case 1.** \( \tilde{A} \) is a non-negative \( LR \)-type IVFNF.

**Sub-case 1.1.** \( \lambda \geq 0. \)

Since, \( \tilde{A} \) is a non-negative \( LR \)-type IVFNF, i.e., \( a - l_L^u \geq 0. \) Thus, \( a - l_L^u \geq 0, \) \( a - l_U^u \geq 0, \) \( a - l_L^v \geq 0, \) \( a - l_U^v \geq 0. \)

This further implies \( a - l_L^u L^{-1}(\alpha) \geq 0, \) \( a - l_U^u (L')^{-1}(\alpha) \geq 0, \) \( a - l_L^v (1 - \beta) \geq 0, \) \( a - l_U^v (L')^{-1}(1 - \beta) \geq 0. \) Hence, we get

\[
\begin{align*}
(\lambda \tilde{A})_L^L &= [\lambda a - l_L^u L^{-1}(\alpha), a + r_L^u R^{-1}(\alpha)], \\
&= [\lambda(a - l_L^u L^{-1}(\alpha)), \lambda(a + r_L^u R^{-1}(\alpha))], \\
(\lambda \tilde{A})_U^L &= [\lambda(a - l_U^u (L')^{-1}(\alpha)), \lambda(a + r_U^u (R')^{-1}(\alpha))], \\
(\lambda \tilde{A})_L^U &= [\lambda(a - l_L^v L^{-1}(1 - \beta)), \lambda(a + r_L^v R^{-1}(1 - \beta))], \\
(\lambda \tilde{A})_U^U &= [\lambda(a - l_U^v (L')^{-1}(1 - \beta)), \lambda(a + r_U^v (R')^{-1}(1 - \beta))].
\end{align*}
\]

Further, as \( L, R, L' \) and \( R' \) are decreasing functions on \([0, \infty)\) with \( L(0) = R(0) = L'(0) = R'(0) = 1, \) there exists \( \alpha_0 \in (0, 1], \) such that \( L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) =
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Hence, the expressions (19) and (20) finally yield
\[ \lambda \tilde{A} = (\lambda a; -\lambda r^\mu_L, -\lambda r^\mu_U, -\lambda r^\nu_L, -\lambda r^\nu_U, -\lambda I^\mu_L, -\lambda I^\nu_L)_{LR}. \]

Therefore, the result is proved for a non-negative \( LR \)-type IVIFN.

**Case 2.** \( \tilde{A} \) is an \( LR \)-type IVIFN such that \( a - l^\nu_L < 0 \) and \( a - l^\nu_U \geq 0 \).

Since, \( a - l^\mu_L \geq 0 \), \( a - l^\mu_U \geq 0 \), \( a - l^\nu_L \geq 0 \)
\[ \implies a - l^\mu_U L^{-1}(\alpha) \geq 0, \quad a - l^\mu_U (L')^{-1}(\alpha) \geq 0, \quad a - l^\nu_U (L')^{-1}(1 - \beta) \geq 0, \quad \forall \alpha, \beta \in [0, 1]. \]

So, expressions for \( (\lambda \tilde{A})^L_\beta \), \( (\lambda \tilde{A})^U_\beta \) and \( (\lambda \tilde{A})^L_\beta \) are same as derived in Case 1, and
\[ (\lambda \tilde{A})^L_\beta = \lambda A^L_\beta = [\lambda, \lambda][a - l^\nu_U L^{-1}(1 - \beta), a + r^\nu_L R^{-1}(1 - \beta)]. \]

Now, if \( a - l^\nu_L < 0 \), then either
\[ a - l^\nu_U L^{-1}(1 - \beta) < 0 \iff a - l^\nu_U \leq L^{-1}(1 - \beta) \iff \beta > 1 - L\left( \frac{a}{l^\nu_L} \right) \]
or
\[ a - l^\nu_L L^{-1}(1 - \beta) \geq 0 \iff a - l^\nu_L \geq L^{-1}(1 - \beta) \iff \beta \leq 1 - L\left( \frac{a}{l^\nu_L} \right). \]

Hence, \( a - l^\nu_L L^{-1}(1 - \beta) < 0 \) for \( \beta > 1 - L\left( \frac{a}{l^\nu_L} \right) \)
\[ \text{and} \quad a - l^\nu_L L^{-1}(1 - \beta) \geq 0 \text{ for } \beta \leq 1 - L\left( \frac{a}{l^\nu_L} \right). \]

(a) Let \( 1 - L\left( \frac{a}{l^\nu_L} \right) < \beta \leq 1 \) or \( a - l^\nu_L L^{-1}(1 - \beta) < 0 \).

Sub-case 2.1. \( \lambda \geq 0 \).

\[ (\lambda \tilde{A})^L_\beta = [\lambda, \lambda][a - l^\nu_L L^{-1}(1 - \beta), a + r^\nu_L R^{-1}(1 - \beta)] \]
\[ = [\lambda(a - l^\nu_L L^{-1}(1 - \beta)), \lambda(a + r^\nu_L R^{-1}(1 - \beta))]. \]

On the similar lines as in Case 1, we get
\[ (\lambda \tilde{A})^L_\beta = [\lambda(a - l^\nu_L), \lambda(a + r^\nu_L)] = [\lambda a - \lambda l^\nu_L, \lambda a + \lambda r^\nu_L]. \]

This yields
\[ \lambda \tilde{A} = (\lambda a; -\lambda l^\nu_L, -\lambda r^\mu_L, -\lambda r^\mu_U, -\lambda r^\nu_L, -\lambda r^\nu_U, -\lambda l^\nu_L, -\lambda l^\nu_L)_{LR}. \]

Sub-case 2.2. \( \lambda < 0 \).

\[ (\lambda \tilde{A})^L_\beta = [\lambda(a + r^\nu_L R^{-1}(1 - \beta)), \lambda(a - l^\nu_L L^{-1}(1 - \beta))]. \]

Further, we have
\[ (\lambda \tilde{A})^L_\beta = [\lambda(a + r^\nu_L), \lambda(a - l^\nu_L)] = [\lambda a + \lambda r^\nu_L, \lambda a - \lambda l^\nu_L] \]

which gives
\[ \lambda \tilde{A} = (\lambda a; -\lambda r^\mu_L, -\lambda r^\mu_U, -\lambda r^\nu_L, -\lambda r^\nu_U, -\lambda l^\nu_L, -\lambda l^\nu_L)_{LR}. \]

(b) Let \( 0 \leq \beta \leq 1 - L\left( \frac{a}{l^\nu_L} \right) \) or \( a - l^\nu_L L^{-1}(1 - \beta) \geq 0 \).
Sub-case 2.3. \( \lambda \geq 0 \).

\[
(\lambda \tilde{A})^L_{\beta} = [\lambda, \lambda](a - l^v_L L^{-1}(1 - \beta), \alpha + r^v_L R^{-1}(1 - \beta)]
\]

Following the steps of Case 1, we obtain

\[
(\lambda \tilde{A})^L_{\beta_0} = [\lambda(a - l^v_L), \lambda(a + r^v_L)] = [\lambda a - \lambda l^\mu_L, \lambda a + \lambda r^\nu_L]
\]

which yields

\[
\lambda \tilde{A} = (\lambda a; -\lambda r^\mu_L, -\lambda l^\mu_L, -\lambda r^\mu_L, -\lambda l^\mu_L; -\lambda r^\nu_L, -\lambda l^\nu_L) L_R.
\]

Sub-case 2.4. \( \lambda < 0 \).

\[
(\lambda \tilde{A})^L_{\beta} = [\lambda(a + r^v_L R^{-1}(1 - \beta)), \lambda(a - l^v_L L^{-1}(1 - \beta))].
\]

Proceeding on the lines of Case 1 and taking \( \beta = \beta_0 \), we get

\[
(\lambda \tilde{A})^L_{\beta_0} = [\lambda(a + r^v_L), \lambda(a - l^v_L)] = [\lambda a + \lambda r^\nu_L, \lambda a - \lambda l^\nu_L].
\]

Therefore, we have

\[
\lambda \tilde{A} = (\lambda a; -\lambda r^\mu_L, -\lambda l^\mu_L, -\lambda r^\mu_L, -\lambda l^\mu_L; -\lambda r^\nu_L, -\lambda l^\nu_L)(\beta_0).R.
\]

Thus, the result follows. Rest all the cases can also be proved on the similar lines. This completes the proof. \( \square \)

**Corollary 3.1** Proposition 3.2 can also be restated as:

If \( A \) be an LR-type IVIFN and \( \lambda \) be any arbitrary real number, then

\[
\lambda \tilde{A} = (\lambda, \lambda; \lambda r^\mu, \lambda l^\mu, \lambda r^\mu, \lambda l^\mu, \lambda r^\nu, \lambda l^\nu, \lambda r^\nu, \lambda l^\nu) L_R.
\]

**Proposition 3.3** Let \( \tilde{A}_1 = (a_1; l^\mu_{1L}, r^\mu_{1L}, l^\mu_{1U}, r^\mu_{1U}, l^\nu_{1L}, r^\nu_{1L}, l^\nu_{1U}, r^\nu_{1U}) L_R \) be an LR-type IVIFN such that \( a_1 - l^\nu_{1L} < 0 \), \( a_1 - l^\nu_{1L} \leq 0 \) and \( \tilde{A}_2 = (a_2; l^\mu_{2L}, r^\mu_{2L}, l^\mu_{2U}, r^\mu_{2U}, l^\nu_{2L}, r^\nu_{2L}, l^\nu_{2U}, r^\nu_{2U}) L_R \) be any LR-type IVIFN. Then,

\[
\tilde{A}_1 \odot \tilde{A}_2 = (a; l^\mu, r^\mu, l^\mu, r^\mu, l^\nu, r^\nu, l^\nu, r^\nu) L_R \]

where

\[
\begin{align*}
a & = a_1 a_2, \\
l^\mu & = a_1 a_2 - \min\{(a_1 - l^\mu_{1L})(a_2 - l^\mu_{2L}), (a_1 + r^\mu_{1L})(a_2 - l^\mu_{2L})\}, \\
r^\mu & = \max\{(a_1 - l^\mu_{1L})(a_2 + r^\mu_{2L}), (a_1 + r^\mu_{1L})(a_2 + r^\mu_{2L}) - a_1 a_2, \\
l^\mu & = a_1 a_2 - \min\{(a_1 - l^\mu_{1L})(a_2 - l^\mu_{2L}), (a_1 + r^\mu_{1L})(a_2 - l^\mu_{2L})\}, \\
r^\mu & = \max\{(a_1 - l^\mu_{1L})(a_2 + r^\mu_{2L}), (a_1 + r^\mu_{1L})(a_2 + r^\mu_{2L}) - a_1 a_2, \\
l^\nu & = a_1 a_2 - \min\{(a_1 - l^\nu_{1L})(a_2 - l^\nu_{2L}), (a_1 + r^\nu_{1L})(a_2 - l^\nu_{2L})\}, \\
r^\nu & = \max\{(a_1 + r^\nu_{1L})(a_2 + r^\nu_{2L}), (a_1 - l^\nu_{1L})(a_2 + r^\nu_{2L}) - a_1 a_2, \\
l^\nu & = a_1 a_2 - \min\{(a_1 - l^\nu_{1L})(a_2 - l^\nu_{2L}), (a_1 + r^\nu_{1L})(a_2 - l^\nu_{2L})\}, \\
r^\nu & = \max\{(a_1 + r^\nu_{1L})(a_2 + r^\nu_{2L}), (a_1 - l^\nu_{1L})(a_2 + r^\nu_{2L}) - a_1 a_2,
\end{align*}
\]

where the conditions for LR-type representation of \( \tilde{A}_1 \odot \tilde{A}_2 \) are satisfied.
Proof Let $\tilde{A}_1 = (a_1; l_{1L}^L, r_{1L}^L, l_{1U}^L, r_{1U}^L, l_{1L}^v, r_{1L}^v, l_{1U}^v, r_{1U}^v)_L$ and $\tilde{A}_2 = (a_2; l_{2L}^L, r_{2L}^L, l_{2U}^L, r_{2U}^L, l_{2L}^v, r_{2L}^v, l_{2U}^v, r_{2U}^v)_L$ be two LR-type IVFN with $a_1 - l_{1L}^v < 0$, $a_1 - l_{1U}^v \geq 0$ and $a_2 - l_{2L}^v, a_2 - l_{2U}^v, a_2 - l_{2L}^v, a_2 - l_{2U}^v, a_2 + r_{2U}^v, a_2 + r_{2U}^v, a_2 + r_{2L}^v$ be any real numbers. Then, in view of Theorem 3.1, we can write

\[
(\tilde{A}_1 \odot \tilde{A}_2)_L^L = A_{L \alpha}^L \times A_{L \beta}^L = \left[ a_1 - l_{1L}^L L^{-1}(\alpha), a_1 + r_{1L}^L R^{-1}(\alpha) \right] \times \left[ a_2 - l_{2L}^L L^{-1}(\alpha), a_2 + r_{2L}^L R^{-1}(\alpha) \right].
\]

\[
(\tilde{A}_1 \odot \tilde{A}_2)_L^U = A_{U \alpha}^L \times A_{U \beta}^L = \left[ a_1 - l_{1U}^L (L')^{-1}(\alpha), a_1 + r_{1U}^L (R')^{-1}(\alpha) \right] \times \left[ a_2 - l_{2U}^L (L')^{-1}(\alpha), a_2 + r_{2U}^L (R')^{-1}(\alpha) \right].
\]

Since $a_1 - l_{1L}^v < 0$ and $a_1 - l_{1U}^v \geq 0$, therefore

\[
a_1 - l_{1L}^L (L')^{-1}(1 - \beta) \geq 0 \quad \text{for } \beta \in [0, 1] \quad \text{and}
\]

\[
a_1 - l_{1L}^v L^{-1}(1 - \beta) \leq (\geq) 0 \quad \text{for } \beta \leq (\geq) 1 - L \left( \frac{a_1}{l_{1L}^v} \right).
\]

Now, to find the product $\tilde{A}_1 \odot \tilde{A}_2$, the following nine cases will arise and in each case, based on the sign of $a_1 - l_{1L}^v L^{-1}(1 - \beta)$, two sub-cases are there.

Case 1: $a_2 - l_{2L}^v \geq 0$.

Then, $a_2 - l_{2L}^v L^{-1}(1 - \beta) \geq 0$, $a_2 - l_{2U}^v (L')^{-1}(1 - \beta) \geq 0$, $a_2 - l_{2L}^v L^{-1}(\alpha) \geq 0$, $a_2 - l_{2U}^v (L')^{-1}(\alpha) \geq 0 \ \forall \ \alpha, \beta \in [0, 1]$.

Sub-case 1.1. $a_1 - l_{1L}^v L^{-1}(1 - \beta) \leq 0$.

Now,

\[
(\tilde{A}_1 \odot \tilde{A}_2)_L^L = [(a_1 - l_{1L}^L L^{-1}(\alpha))(a_2 - l_{2L}^L L^{-1}(\alpha)), (a_1 + r_{1L}^L R^{-1}(\alpha))]
\]

\[
(\tilde{A}_1 \odot \tilde{A}_2)_U^L = [(a_1 - l_{1U}^L (L')^{-1}(\alpha))(a_2 - l_{2L}^L (L')^{-1}(\alpha)), (a_1 + r_{1U}^L (R')^{-1}(\alpha))]
\]

\[
(\tilde{A}_1 \odot \tilde{A}_2)_L^U = [(a_1 - l_{1L}^v L^{-1}(1 - \beta))(a_2 + r_{2L}^v R^{-1}(1 - \beta)), (a_1 + r_{1L}^v R^{-1}(1 - \beta))]
\]

\[
(\tilde{A}_1 \odot \tilde{A}_2)_L^U = [(a_1 - l_{1U}^v (L')^{-1}(1 - \beta))(a_2 - l_{2L}^v (L')^{-1}(1 - \beta)), (a_1 + r_{1U}^v (R')^{-1}(1 - \beta))(a_2 + r_{2L}^v (R')^{-1}(1 - \beta))]
\]

Further, as $L$, $R$, $L'$ and $R'$ are decreasing functions on $[0, \infty)$ with $L(0) = R(0) = L'(0) = R'(0) = 1$, there exists $\alpha_0 \in (0, 1)$, such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = \alpha_0$.\]
\[(R')^{-1}(\alpha_0) = 1.\] Hence,

\[
\begin{align*}
\tilde{A}_1 \circ \tilde{A}_2 \biggr|_{\alpha_0}^L &= [(a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu), (a_1 + r_{l1}^\mu)(a_2 + r_{l2}^\mu)], \\
\tilde{A}_1 \circ \tilde{A}_2 \biggr|_{\alpha_0}^U &= [(a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu), (a_1 + r_{l1}^\mu)(a_2 + r_{l2}^\mu)].
\end{align*}
\] (21)

Also, choosing \(\beta_0 = 1 - \alpha_0 \in (0, 1]\), we have

\[
\begin{align*}
\tilde{A}_1 \circ \tilde{A}_2 \biggr|_{\beta_0}^L &= [(a_1 - l_{l1}^\nu)(a_2 + r_{l2}^\nu), (a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu)], \\
\tilde{A}_1 \circ \tilde{A}_2 \biggr|_{\beta_0}^U &= [(a_1 - l_{l1}^\nu)(a_2 - l_{l2}^\nu), (a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu)].
\end{align*}
\] (22)

Taking \(\alpha = 1\) and \(\beta = 0\), we obtain

\[
\tilde{A}_1 \circ \tilde{A}_2 \biggr|_{\alpha=1}^L = (\tilde{A}_1 \circ \tilde{A}_2) \biggr|_{\alpha=1}^L = (\tilde{A}_1 \circ \tilde{A}_2) \biggr|_{\beta=0}^U = a_1 a_2. \tag{23}
\]

Now, using the property that \(\tilde{A}_1\) and \(\tilde{A}_2\) are LR-type IVIFNs, we obtain

\[
\begin{align*}
a_1 a_2 - (a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu) &\geq a_1 a_2 - (a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu) \implies l_{l1}^\mu \geq l_{l2}^\mu. \\
(a_1 + r_{l1}^\mu)(a_2 + r_{l2}^\mu) - a_1 a_2 &\geq (a_1 + r_{l1}^\mu)(a_2 + r_{l2}^\mu) - a_1 a_2 \implies r_{l1}^\mu \geq r_{l2}^\mu. \\
(a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu) - a_1 a_2 &\geq (a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu) - a_1 a_2 \implies r_{l1}^\nu \geq r_{l2}^\nu. \\
(a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu) - a_1 a_2 &\geq (a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu) - a_1 a_2 \implies r_{l1}^\nu \geq r_{l2}^\mu.
\end{align*}
\]

Further, since \(a_2 - l_{l2}^\nu \leq a_2 + r_{l2}^\nu\), \(a_1 - l_{l1}^\nu \leq a_1 - l_{l1}^\nu\), and \(a_2 - l_{l2}^\nu \leq a_2 - l_{l2}^\nu\), therefore

\[
a_1 a_2 - (a_1 - l_{l1}^\nu)(a_2 - l_{l2}^\nu) \leq a_1 a_2 - (a_1 - l_{l1}^\nu)(a_2 + r_{l2}^\nu), \text{ that is, } l_{l1}^\nu \leq l_{l2}^\nu.
\]

and from \(a_1 - l_{l1}^\nu < 0, a_2 - l_{l2}^\nu \leq a_2 + r_{l2}^\nu, a_1 - l_{l1}^\nu \leq a_1 - l_{l1}^\nu, a_2 - l_{l2}^\nu \leq a_2 - l_{l2}^\nu\), we have

\[
a_1 a_2 - (a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu) \leq a_1 a_2 - (a_2 + r_{l2}^\nu)(a_1 - l_{l1}^\nu), \text{ that is, } l_{l1}^\mu \leq l_{l2}^\nu.
\]

Moreover, \(l_{l1}^\mu (a_1 - l_{l1}^\mu) + a_2 l_{l1}^\mu \geq 0\) and \(a_1 r_{l2}^\nu + a_2 r_{l2}^\nu + r_{l1}^\nu r_{l2}^\mu \geq 0\) yield \(l_{l1}^\mu \geq 0\) and \(r_{l2}^\mu \geq 0\), respectively.

Finally, in view of these inequalities and combining expressions (21), (22) and (23), we get

\[
\tilde{A}_1 \circ \tilde{A}_2 = (a_1 a_2; a_1 a_2 - (a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu), (a_1 + r_{l1}^\mu)(a_2 + r_{l2}^\mu) - a_1 a_2),
\]

\[
\begin{align*}
a_1 a_2 - (a_1 - l_{l1}^\mu)(a_2 - l_{l2}^\mu) &\leq a_1 a_2 - (a_2 + r_{l2}^\nu)(a_1 - l_{l1}^\nu), \text{ that is, } l_{l1}^\mu \leq l_{l2}^\nu. \\
a_1 a_2 &\geq (a_1 - l_{l1}^\nu)(a_2 + r_{l2}^\nu), \text{ that is, } l_{l1}^\nu \leq l_{l2}^\nu.
\end{align*}
\]

Sub-case 1.2. \(a_1 - l_{l1}^\nu L^{-1}(1 - \beta) \geq 0\).

Now,

\[
\begin{align*}
(\tilde{A}_1 \circ \tilde{A}_2) \biggr|_{\beta}^L &= [(a_1 - l_{l1}^\nu L^{-1}(1 - \beta))(a_2 - l_{l2}^\nu L^{-1}(1 - \beta)), \\
(a_1 + r_{l1}^\nu R^{-1}(1 - \beta))(a_2 + r_{l2}^\nu R^{-1}(1 - \beta))].
\end{align*}
\]

Choosing \(\beta = \beta_0\), we have

\[
(\tilde{A}_1 \circ \tilde{A}_2) \biggr|_{\beta_0}^L = [(a_1 - l_{l1}^\nu)(a_2 - l_{l2}^\nu), (a_1 + r_{l1}^\nu)(a_2 + r_{l2}^\nu)]. \tag{24}
\]
Hence, we have \( l_L^v = a_1 a_2 - (a_1 - l_{1L}^v) (a_2 - l_{2L}^v) \) and remaining spreads of \( \tilde{A}_1 \odot \tilde{A}_2 \) will be same as derived in Sub-case 1.1. Further,

\[
(a_1 - l_{1L}^v)(a_2 - l_{2L}^v) \leq (a_1 - l_{1U}^v)(a_2 - l_{2U}^v) \implies l_L^v \geq l_U^v \quad \text{and} \quad (a_1 - l_{1L}^v)(a_2 - l_{2L}^v) \leq (a_1 - l_{1L}^v)(a_2 - l_{2L}^v) \implies l_L^v \geq l_L^v .
\]

Finally, combining Sub-cases 1.1 and 1.2, we get

\[
\tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu; l_L^v, r_L^v, l_U^v, r_U^v)_{LR} \quad \text{where} \quad a = a_1 a_2 ,
\]

\[
l_L^v = a_1 a_2 - (a_1 - l_{1L}^v)(a_2 - l_{2L}^v) ,
\]

\[
r_L^\mu = (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1 a_2 ,
\]

\[
l_U^\mu = a_1 a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu) ,
\]

\[
r_U^\mu = a_1 a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu) .
\]

**Case 2.** \( a_2 - l_{2L}^v < 0 \) and \( a_2 - l_{2U}^v \geq 0 \).

Then, we have

\[
a_2 - l_{2L}^v L^{-1}(1 - \beta) \leq (\geq) 0 \quad \text{for} \quad \beta \leq (\geq) 1 - L \left( \frac{a_2}{l_{2L}^v} \right) .
\]

**Sub-case 2.1.** \( a_1 - l_{1L}^v L^{-1}(1 - \beta) \leq 0 \).

Then, \( (\tilde{A}_1 \odot \tilde{A}_2)^L_a, (\tilde{A}_1 \odot \tilde{A}_2)^U_a, (\tilde{A}_1 \odot \tilde{A}_2)^U_\beta \) remain same as in Sub-case 1.1, however, the expression for \( (\tilde{A}_1 \odot \tilde{A}_2)^L_\beta \) will be changed and is given by

\[
(\tilde{A}_1 \odot \tilde{A}_2)^L_\beta = A_{1\beta}^L \times A_{2\beta}^L
\]

\[
= \left[ \left( a_1 - l_{1L}^v L^{-1}(1 - \beta) \right), \left( a_1 + r_{1L}^v R^{-1}(1 - \beta) \right) \right] \times \left[ \left( a_2 - l_{2L}^v L^{-1}(1 - \beta) \right), \left( a_2 + r_{2L}^v R^{-1}(1 - \beta) \right) \right]
\]

\[
= \left[ \min \left\{ (a_1 - l_{1L}^v L^{-1}(1 - \beta)), (a_1 + r_{1L}^v L^{-1}(1 - \beta)) \right\}, (a_2 - l_{2L}^v L^{-1}(1 - \beta)), (a_2 + r_{2L}^v R^{-1}(1 - \beta)) \right] .
\]

Choosing \( \beta = \beta_0 \), we get

\[
(\tilde{A}_1 \odot \tilde{A}_2)^L_{\beta_0} = \min \left\{ (a_1 - l_{1L}^v)(a_2 + r_{2L}^v), (a_2 - l_{2L}^v)(a_1 + r_{1L}^v) \right\} .
\]

Hence,

\[
l_L^v = a_1 a_2 - \min \left\{ (a_1 - l_{1L}^v)(a_2 + r_{2L}^v), (a_2 - l_{2L}^v)(a_1 + r_{1L}^v) \right\} ,
\]

\[
r_L^v = \max \left\{ (a_1 - l_{1L}^v)(a_2 - l_{2L}^v), (a_1 + r_{1L}^v)(a_2 + r_{2L}^v) \right\} - a_1 a_2 .
\]
Further, since \( a_1 - l_U^v \leq a_1 - l_{1U}^v, \ a_2 - l_{2U}^v \leq a_2 + r_{2U}^v, \ a_2 + r_{2U}^v \leq a_2 + r_{2L}^v \) and \( a_1 - l_{1L}^v < 0 \); therefore,

\[
(1 - l_{1U}^v)(a_2 - l_{2U}^v) \geq (1 - l_{1L}^v)(a_2 + r_{2L}^v).
\]

(26)

Also, \( a_1 - l_{1U}^v \leq a_1 + r_{1U}^v \leq a_1 + r_{1L}^v \) and \( a_2 - l_{1L}^v < 0 \) implies

\[
(1 - l_{1U}^v)(a_2 - l_{2U}^v) \geq (1 + r_{1L}^v)(a_2 - l_{2L}^v).
\]

(27)

It follows from the inequalities (26) and (27) that

\[
(1 - l_{1U}^v)(a_2 - l_{2U}^v) \geq \min\{(a_1 - l_{1L}^v)(a_2 + r_{2L}^v), \ (a_1 + r_{1L}^v)(a_2 - l_{2L}^v)\},
\]

that is, \( l_{2U}^v \leq l_{1L}^v \).

Now, if \( (a_1 - l_{1U}^v)(a_2 - l_{2U}^v) \leq (a_1 + r_{1U}^v)(a_2 + r_{2U}^v) \), then

\[
(1 + r_{1L}^v)(a_2 - l_{2U}^v) \geq (1 + r_{1L}^v)(a_2 + r_{2L}^v)
\]

(28)

and if \( (a_1 - l_{1U}^v)(a_2 - l_{2U}^v) \geq (a_1 + r_{1U}^v)(a_2 + r_{2U}^v) \), then

\[
(1 + r_{1L}^v)(a_2 - l_{2U}^v) \leq (1 + r_{1L}^v)(a_2 + r_{2L}^v)
\]

(29)

Thus, the inequalities (28) and (29) yield

\[
(1 + r_{1L}^v)(a_2 + r_{2L}^v) \leq \max\{(a_1 + r_{1L}^v)(a_2 + r_{2L}^v), \ (a_1 - l_{1L}^v)(a_2 - l_{2L}^v)\},
\]

that is, \( r_{2L}^v \leq r_{1L}^v \).

Further, \( a_1 - l_{1L}^v \leq a_1 - l_{1L}^v, \ a_1 - l_{1L}^v < 0 \) give

\[
(a_1 - l_{1L}^v)(a_2 - l_{2U}^v) \geq (a_1 - l_{1L}^v)(a_2 + r_{2L}^v)
\]

(30)

and \( (a_1 - l_{1L}^v)(a_2 - l_{2U}^v) \geq (a_1 - l_{1L}^v)(a_2 - l_{2L}^v) \geq (a_1 + r_{1L}^v)(a_2 - l_{2L}^v) \).

(31)

It follows from the expressions (30) and (31) that

\[
(1 - l_{1U}^v)(a_2 - l_{2L}^v) \geq \min\{(a_1 - l_{1L}^v)(a_2 + r_{2L}^v), \ (a_1 + r_{1L}^v)(a_2 - l_{2L}^v)\},
\]

that is, \( l_{1L}^v \leq l_{1L}^v \).

Now, if \( (a_1 - l_{1U}^v)(a_2 - l_{2U}^v) \leq (a_1 + r_{1U}^v)(a_2 + r_{2U}^v) \), then

\[
(1 + r_{1L}^v)(a_2 - l_{2U}^v) \geq (1 + r_{1L}^v)(a_2 + r_{2L}^v)
\]

(32)

and if \( (a_1 - l_{1U}^v)(a_2 - l_{2U}^v) \geq (a_1 + r_{1U}^v)(a_2 + r_{2U}^v) \), then

\[
(1 + r_{1L}^v)(a_2 - l_{2U}^v) \leq (1 + r_{1L}^v)(a_2 + r_{2L}^v)
\]

(33)

Hence, the inequalities (32) and (33) give

\[
(1 + r_{1L}^v)(a_2 + r_{2L}^v) \leq \max\{(a_1 + r_{1L}^v)(a_2 + r_{2L}^v), \ (a_1 - l_{1L}^v)(a_2 - l_{2L}^v)\},
\]

that is, \( r_{2L}^v \leq r_{1L}^v \).

Sub-case 2.2. \( a_1 - l_{1L}^v L^{-1}(1 - \beta) \geq 0 \).

The proof of this part follows on the lines of Sub-case 1.1.

Hence, now combining Cases 1 and 2, we get

\[
\tilde{A}_1 \cap \tilde{A}_2 = (a; l_{1L}^v, r_{1L}^v, l_{1U}^v, r_{1U}^v, l_{2L}^v, r_{2L}^v, l_{2U}^v, r_{2U}^v)_{LR}
\]

where

\[
a = a_1 a_2,
\]

\[
l_{1L}^v = a_1 a_2 - (a_1 - l_{1L}^v)(a_2 - l_{2L}^v),
\]

\[
r_{1L}^v = (a_1 + r_{1L}^v)(a_2 + r_{2L}^v) - a_1 a_2.
\]
The proof of this part follows on the lines of Sub-cases 1.1 and 2.2.

Hence, it is concluded from Cases 1–3 that

\[
\tilde{A}_1 \circ \tilde{A}_2 = (a; \ l_{1L}^\mu, r_{1L}^\mu, l_{1U}^\mu, r_{1U}^\mu, l_{2L}^\nu, r_{2L}^\nu, l_{2U}^\nu, r_{2U}^\nu)_{LR}
\]

where
Hence, clubbing the Cases 1–4, we obtain

\[
\begin{align*}
a &= a_1a_2, \\
l_L^\mu &= a_1a_2 - (a_1 - l_1^\mu)(a_2 - l_2^\mu), \\
r_L^\mu &= (a_1 + r_1^\mu)(a_2 + r_2^\mu) - a_1a_2, \\
l_U^\mu &= a_1a_2 - (a_1 - l_U^\mu)(a_2 - l_U^\mu), \\
r_U^\mu &= (a_1 + r_U^\mu)(a_2 + r_U^\mu) - a_1a_2,
\end{align*}
\]

Then, \( a_2 - l_2^\mu(L')^{-1}(\alpha) \geq 0 \) for \( \alpha \geq (\leq) L' \left( \frac{a_2}{l_2^\mu} \right) \).

**Sub-case 4.1.** \( a_1 - l_L^\nu L^{-1}(1 - \beta) \leq 0. \)

Then, the expressions \((\tilde{A}_1 \circ \tilde{A}_2)^L_\alpha, (\tilde{A}_1 \circ \tilde{A}_2)^L_\beta, (\tilde{A}_1 \circ \tilde{A}_2)^U_\beta\) are same as in Cases 2 and 3 but \((\tilde{A}_1 \circ \tilde{A}_2)^U_\alpha\) is given by

\[
(\tilde{A}_1 \circ \tilde{A}_2)^U_\alpha = A^U_{1a} \times A^U_{2a} \\
= \left[ (a_1 + r_U^\mu(R')^{-1}(\alpha)) \left( a_2 - l_2^\mu(L')^{-1}(\alpha) \right), (a_1 + r_U^\mu(R')^{-1}(\alpha)) \right].
\]

Choosing \( \alpha = a_0 \), we get

\[
(\tilde{A}_1 \circ \tilde{A}_2)^U_{a_0} = \left[ (a_1 + r_U^\mu) \left( a_2 - l_2^\mu \right), (a_1 + r_U^\mu) \left( a_2 + r_2^\mu \right) \right].
\]

This further yields \( l_U^\mu = a_1a_2 - (a_1 + r_U^\mu)(a_2 - l_2^\mu) \).

Moreover, from the preceding Case 3, we have

\[
l_L^\mu = a_1a_2 - (a_1 - l_1^\mu)(a_2 - l_2^\mu), \quad l_U^\nu = a_1a_2 - (a_1 + r_1^\mu)(a_2 - l_2^\mu).
\]

Next, it remains to prove that \( l_L^\mu \geq l_L^\mu \) and \( l_U^\nu \geq l_U^\nu \).

Since \( a_2 - l_2^\mu < 0, a_2 - l_2^\mu \leq a_2 - l_2^\mu \) and \( a_1 - l_1^\mu \leq a_1 + r_1^\mu \); therefore, \( a_1a_2 - (a_2 - l_2^\mu)(a_1 - l_1^\mu) \leq a_1a_2 - (a_2 - l_2^\mu)(a_1 + r_1^\mu) \implies l_L^\mu \leq l_U^\nu \).

Further, \( a_2 - l_2^\mu \leq a_2 - l_2^\mu, a_2 - l_2^\mu \leq 0 \) yield

\[
a_1a_2 - (a_2 - l_2^\mu) \leq a_1a_2 - (a_2 - l_2^\mu)(a_1 + r_1^\mu) \implies l_U^\mu \leq l_U^\nu.
\]

**Sub-case 4.2.** \( a_1 - l_1^\nu L^{-1}(1 - \beta) \geq 0. \)

It follows on the lines of Sub-case 3.2.

Hence, clubbing the Cases 1–4, we obtain

\[
\tilde{A}_1 \circ \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu, l_U^\nu, r_U^\nu, l_U^\nu)_{LR} \quad \text{where} \quad a = a_1a_2,
\]

\[
l_L^\mu = a_1a_2 - (a_1 - l_1^\mu)(a_2 - l_2^\mu),
\]
\[ r_L^μ = (a_1 + r_{1L}^μ)(a_2 + r_{2L}^μ) - a_1a_2, \]
\[ l_L^μ = a_1a_2 - \min \left\{ (a_1 - l_{1L}^μ)(a_2 - l_{2L}^μ), (a_1 + r_{1L}^μ)(a_2 - l_{2L}^μ) \right\}, \]
\[ r_U^μ = (a_1 + r_{1U}^μ)(a_2 + r_{2U}^μ) - a_1a_2, \]
\[ l_U^μ = a_1a_2 - \min \left\{ (a_1 - l_{1U}^μ)(a_2 + r_{1L}^μ), (a_2 - l_{2U}^μ)(a_1 + r_{1L}^μ) \right\}, \]
\[ l_U^μ = \max \left\{ (a_1 - l_{1U}^μ)(a_2 - l_{2U}^μ), (a_1 + r_{1L}^μ)(a_2 + r_{2L}^μ) \right\} - a_1a_2, \]
\[ l_U^ν = a_1a_2 - \min \left\{ (a_1 - l_{1U}^ν)(a_2 - l_{2U}^ν), (a_1 + r_{1U}^ν)(a_2 - l_{2U}^ν) \right\}, \]
\[ l_U^ν = (a_1 + r_{1U}^ν)(a_2 + r_{2U}^ν) - a_1a_2. \]

Now, for the succeeding cases, we have only derived the expressions for newer (or changed) spreads of the product \( \tilde{A}_1 \odot \tilde{A}_2 \) because the proofs for the bounds on the other spreads can be carried out in the similar pattern as is discussed in the preceding four cases.

**Case 5.** \( a_2 - l_{2L}^μ < 0 \) and \( a_2 \in \mathbb{R} \).

Then, \( a_2 - l_{2L}^νL^{-1}(\alpha) \leq (\geq) 0 \) for \( \alpha \geq (\leq) L \left( \frac{a_2}{l_{2L}^μ} \right) \).

**Sub-case 5.1.** \( a_1 - l_{1L}^νL^{-1}(1 - \beta) \leq 0. \)

Then,

\[
(\tilde{A}_1 \odot \tilde{A}_2)_α^L = A_{1α}^L \times A_{2α}^L \\
= [(a_1 + r_{1L}^μR^{-1}(\alpha))(a_2 - l_{2L}^μL^{-1}(\alpha)), (a_1 + r_{1L}^μR^{-1}(\alpha))(a_2 + r_{2L}^μR^{-1}(\alpha))].
\]

Taking \( \alpha = a_0 \), we get

\[
(\tilde{A}_1 \odot \tilde{A}_2)_α^L = [(a_1 + r_{1L}^μ)(a_2 - l_{2L}^μ), (a_1 + r_{1L}^μ)(a_2 + r_{2L}^μ)].
\]

It further yields \( l_L^μ = a_1a_2 - (a_1 + r_{1L}^μ)(a_2 - l_{2L}^μ). \)

Moreover, from above cases, we have

\[
l_U^μ = a_1a_2 - \min \left\{ (a_1 - l_{1U}^μ)(a_2 - l_{2U}^μ), (a_1 + r_{1L}^μ)(a_2 - l_{2L}^μ) \right\}, \]
\[
l_U^ν = a_1a_2 - \min \left\{ (a_1 - l_{1U}^ν)(a_2 + r_{1L}^μ), (a_2 - l_{2U}^ν)(a_1 + r_{1L}^ν) \right\}. \]

Next, the claim that \( l_U^μ \geq l_L^μ \) and \( l_U^ν \geq l_L^ν \) can be proved following the lines of Sub-case 3.1.

**Sub-case 5.2.** \( a_1 - l_{1L}^νL^{-1}(1 - \beta) \geq 0. \)

The proof can be carried out following the lines of Sub-case 1.1.

**Case 6.** \( a_2 + r_{2L}^μ < 0 \) and \( a_2 + r_{2U}^ν \geq 0. \)

Then, \( a_2 + r_{2L}^μR^{-1}(\alpha) \leq (\geq) 0 \) for \( \alpha \leq (\geq) R \left( -\frac{a_2}{r_{2L}^ν} \right) \).

**Sub-case 6.1.** \( a_1 - l_{1L}^νL^{-1}(1 - \beta) \leq 0. \)

Then,

\[
(\tilde{A}_1 \odot \tilde{A}_2)_α^L = A_{1α}^L \times A_{2α}^L \\
= [(a_1 + r_{1L}^μR^{-1}(\alpha))(a_2 - l_{2L}^μL^{-1}(\alpha)), (a_1 + r_{1L}^μR^{-1}(\alpha))(a_2 + r_{2L}^μR^{-1}(\alpha))].
\]

Choosing \( \alpha = a_0 \), we obtain

\[
(\tilde{A}_1 \odot \tilde{A}_2)_α^L = [(a_1 + r_{1L}^μ)(a_2 - l_{2L}^μ), (a_1 + r_{1L}^μ)(a_2 + r_{2L}^μ)]
\]

which gives \( r_L^μ = (a_1 - l_{1L}^μ)(a_2 + r_{2L}^μ) - a_1a_2. \)
Further, we have
\[
\begin{align*}
\rho_U^{i\mu} &= (a_1 + \rho_U^{i\mu})(a_2 + \rho_U^{i\mu}) - a_1 a_2 \quad \text{and} \\
\rho_L^{v} &= \max\{(a_1 - l_{1L}^{v})(a_2 - l_{2L}^{v}), (a_1 + r_{1L}^{v})(a_2 + r_{2L}^{v})\} - a_1 a_2.
\end{align*}
\]

The proof that \(\rho_U^{i\mu} \geq \rho_L^{i\mu}\) and \(\rho_L^{v} \geq \rho_U^{v}\) can be obtained on the lines of Sub-cases 2.1 and 3.1.

Sub-case 6.2. \(a_1 - l_{1L}^{v}L^{-1}(1 - \beta) \geq 0\).

This part can be proved on the lines of Sub-case 1.1.

Case 7. \(a_2 + \rho_U^{i\mu} < 0\) and \(a_2 + \rho_U^{v} \geq 0\).

Then, \(a_2 + \rho_U^{i\mu}(R')^{-1}(\alpha) \leq (\geq) 0\) for \(\alpha \leq (\geq) R'\left(-\frac{a_2}{\rho_U^{v}}\right)\).

Sub-case 7.1. \(a_1 - l_{1L}^{v}L^{-1}(1 - \beta) \leq 0\).

Then,
\[
\begin{align*}
(\tilde{A}_1 \circ \tilde{A}_2)^{U}_\alpha &= A^{U}_{1\alpha} \times A^{U}_{2\alpha} \\
&= [(a_1 + \rho_U^{i\mu})(a_2 - \rho_U^{v}), (a_1 - l_{1L}^{v})(a_2 + \rho_U^{v})].
\end{align*}
\]

It follows that \(\rho_U^{i\mu} = (a_1 - l_{1L}^{v})(a_2 + \rho_U^{v}) - a_1 a_2\).

Also, we have
\[
\begin{align*}
\rho_L^{v} &= \max\{(a_1 - l_{1L}^{v})(a_2 + \rho_U^{v}), (a_1 + r_{1L}^{v})(a_2 + \rho_U^{v})\} - a_1 a_2 \quad \text{and} \\
\rho_L^{v} &= (a_1 + \rho_U^{i\mu})(a_2 + \rho_U^{v}) - a_1 a_2.
\end{align*}
\]

Further, the inequalities \(\rho_U^{i\mu} \geq \rho_L^{i\mu}\) and \(\rho_L^{v} \geq \rho_U^{v}\) can be established on the lines of Sub-cases 2.1 and 3.1.

Sub-case 7.2. \(a_1 - l_{1L}^{v}L^{-1}(1 - \beta) \geq 0\).

This part can be proved on the similar lines as Sub-case 1.1.

Case 8. \(a_2 + \rho_U^{v} < 0\) and \(a_2 + \rho_U^{v} \geq 0\).

Then, \(a_2 + \rho_U^{v}(R')^{-1}(1 - \beta) \leq (\geq) 0\) for \(\beta \geq (\leq) 1 - R'\left(-\frac{a_2}{\rho_U^{v}}\right)\).

Sub-case 8.1. \(a_1 - l_{1L}^{v}L^{-1}(1 - \beta) \leq 0\).

Then,
\[
\begin{align*}
(\tilde{A}_1 \circ \tilde{A}_2)^{U}_\beta &= A^{U}_{1\beta} \times A^{U}_{2\beta} \\
&= [(a_1 + \rho_U^{v})(a_2 - \rho_U^{v}), (a_1 - l_{1L}^{v})(a_2 + \rho_U^{v})].
\end{align*}
\]

Choosing \(\beta = \beta_0\), we get
\[
(\tilde{A}_1 \circ \tilde{A}_2)^{U}_{\beta_0} = [(a_1 + \rho_U^{v})(a_2 - \rho_U^{v}), (a_1 - l_{1L}^{v})(a_2 + \rho_U^{v})].
\]

This further yields \(\rho_U^{v} = (a_1 - l_{1L}^{v})(a_2 + \rho_U^{v}) - a_1 a_2\).

Moreover, we have
\[
\rho_L^{v} = \max\{(a_1 - l_{1L}^{v})(a_2 - \rho_U^{v}), (a_1 + r_{1L}^{v})(a_2 + \rho_U^{v})\} - a_1 a_2 \quad \text{and}
\]

\[\text{The Springer logo.}\]
Next, \( r_U^v \geq r_U^v \) and \( r_L^v \geq r_L^v \) can be proved following the lines of Sub-cases 2.1 and 3.1. Sub-case 8.2, \( a_1 - l_{1L}^v R^{-1} (1 - \beta) \geq 0 \).

The proof of this part follows on the lines of Sub-case 1.1.

**Case 9.** \( a_2 + r_{2L}^v < 0 \).

Then, \( a_2 + r_{2L}^v R^{-1} (1 - \beta) \leq (\geq) 0 \) for \( \beta \geq (\leq) 1 - R \left( -\frac{a_2}{r_{2L}^v} \right) \).

**Sub-case 9.1.** \( a_1 - l_{1L}^v R^{-1} (1 - \beta) \leq 0 \).

Then,

\[
(\tilde{A}_1 \odot \tilde{A}_2)_{\beta_0}^L = A_{\beta_0}^L \times A_{2\beta}^L = [(a_1 + r_{1L}^v R^{-1} (1 - \beta))(a_2 - l_{2L}^v L^{-1} (1 - \beta)), (a_1 - l_{1L}^v L^{-1} (1 - \beta))]
\]

Taking \( \beta = \beta_0 \), we obtain

\[
(\tilde{A}_1 \odot \tilde{A}_2)_{\beta_0}^L = [(a_1 + r_{1L}^v)(a_2 - l_{2L}^v), (a_1 - l_{1L}^v)(a_2 - l_{2L}^v)].
\]

It follows that \( r_U^v = (a_1 - l_{1L}^v)(a_2 - l_{2L}^v) - a_1 a_2 \).

Further,

\[
r_U^v = \max \{(a_1 - l_{1L}^v)(a_2 + r_{2L}^v), (a_1 + r_{1L}^v)(a_2 + r_{2L}^v)\} - a_1 a_2 \quad \text{and}
\]

\[
r_L^v = \max \{(a_1 - l_{1L}^v)(a_2 + r_{2L}^v), (a_1 + r_{1L}^v)(a_2 + r_{2L}^v)\} - a_1 a_2.
\]

Finally, the proof of the inequalities \( r_U^v \geq r_U^v \) and \( r_L^v \geq r_L^v \) can be obtained on the lines of Sub-cases 2.1 and 3.1.

**Sub-case 9.2.** \( a_1 - l_{1L}^v L^{-1} (1 - \beta) \geq 0 \).

The proof of this part can be established on the lines of Sub-case 1.1.

Finally, combining all the Cases (1)--(9), we have

\[
\tilde{A}_1 \odot \tilde{A}_2 = (a; l_{1L}^u, l_{1U}^u, l_{1L}^v, l_{1U}^v, l_{2L}^u, l_{2U}^u, l_{2L}^v, l_{2U}^v)_{LR}
\]

where \( a = a_1 a_2 \),

\[
l_{1L}^u = a_1 a_2 - \min((a_1 - l_{1L}^u)(a_2 - l_{2L}^u), (a_1 + r_{1L}^u)(a_2 + r_{2L}^u)) \]

\[
r_{1L}^v = \max((a_1 - l_{1L}^v)(a_2 + r_{2L}^v), (a_1 + r_{1L}^v)(a_2 + r_{2L}^v)) - a_1 a_2,
\]

\[
l_{1U}^u = a_1 a_2 - \min((a_1 - l_{1U}^u)(a_2 - l_{2U}^u), (a_1 + r_{1U}^u)(a_2 - l_{2U}^u)) \]

\[
r_{1U}^v = \max((a_1 - l_{1U}^v)(a_2 - l_{2L}^v), (a_1 + r_{1U}^v)(a_2 + r_{2L}^v)) - a_1 a_2,
\]

\[
l_{2L}^u = a_1 a_2 - \min((a_1 + l_{1L}^v)(a_2 + r_{2L}^v), (a_1 + l_{1U}^v)(a_2 + r_{2L}^v)) \]

\[
r_{2L}^v = \max((a_1 + l_{1L}^v)(a_2 + r_{2L}^v), (a_1 + r_{1U}^v)(a_2 - l_{2U}^v)) - a_1 a_2,
\]

\[
l_{2U}^u = a_1 a_2 - \min((a_1 - l_{1U}^v)(a_2 - l_{2L}^v), (a_1 + r_{1U}^v)(a_2 - l_{2L}^v)) \]

\[
r_{2U}^v = \max((a_1 - l_{1U}^v)(a_2 + r_{2L}^v), (a_1 - l_{1U}^v)(a_2 + r_{2L}^v)) - a_1 a_2.
\]

where \( l_{1U}^u \geq l_{1L}^u > 0, \quad r_{1U}^v \geq r_{1L}^v > 0, \quad l_{1L}^u \geq l_{1U}^u > 0, \quad l_{1U}^v \geq l_{1L}^v > 0, \quad l_{2L}^u \geq l_{2U}^u > 0, \quad r_{1U}^v \geq r_{1L}^v > 0. \)

Hence, the result. \( \square \)
Proposition 3.4 Let $\tilde{A}_1 = (a_1; l^u_{IL}, r^u_{IL}, l^u_{IU}, r^u_{IU}, l^v_{I1}, r^v_{I1}, l^v_{IU}, r^v_{IU})_{LR}$ be an LR-type IVIFN such that $a_1 - l^u_{IU} < 0$, $a_1 - l^u_{IL} \geq 0$, $a_1 - l^u_{IU} \geq 0$ and $\tilde{A}_2 = (a_2; l^u_{2L}, r^u_{2L}, l^u_{2U}, r^u_{2U}, l^v_{2L}, r^v_{2L}, l^v_{2U}, r^v_{2U})_{LR}$ be an LR-type IVIFN. Then,

$$\tilde{A}_1 \circ \tilde{A}_2 = (a; l^u_{L}, r^u_{L}, l^u_{U}, r^u_{U}, l^v_{L}, r^v_{L}, l^v_{U}, r^v_{U})_{LR}$$

where $a = a_1a_2$,

$$l^u_{L} = a_1a_2 - \min\{(a_1 + r^u_{IL})(a_2 - l^u_{2L}), (a_1 - l^u_{1L})(a_2 - l^u_{2L})\},$$

$$r^u_{L} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^u_{U} = a_1a_2 - \min\{(a_1 + r^u_{IL})(a_2 - l^u_{2L}), (a_1 - l^u_{1L})(a_2 - l^u_{2L})\},$$

$$r^u_{U} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^v_{L} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{L} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$

$$l^v_{U} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{U} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$

where the conditions for LR-type representation of $\tilde{A}_1 \circ \tilde{A}_2$ are satisfied.

Proof Similar to the Proposition 3.3.

Proposition 3.5 Let $\tilde{A}_1 = (a_1; l^u_{IL}, r^u_{IL}, l^u_{IU}, r^u_{IU}, l^v_{I1}, r^v_{I1}, l^v_{IU}, r^v_{IU})_{LR}$ be an LR-type IVIFN such that $a_1 - l^u_{IU} < 0$, $a_1 - l^u_{IL} \geq 0$ and $\tilde{A}_2 = (a_2; l^u_{2L}, r^u_{2L}, l^u_{2U}, r^u_{2U}, l^v_{2L}, r^v_{2L}, l^v_{2U}, r^v_{2U})_{LR}$ be an LR-type IVIFN. Then,

$$\tilde{A}_1 \circ \tilde{A}_2 = (a; l^u_{L}, r^u_{L}, l^u_{U}, r^u_{U}, l^v_{L}, r^v_{L}, l^v_{U}, r^v_{U})_{LR}$$

where $a = a_1a_2$,

$$l^u_{L} = a_1a_2 - \min\{(a_1 - l^u_{IL})(a_2 + r^u_{2L}), (a_1 + r^u_{IL})(a_2 - l^u_{2L})\},$$

$$r^u_{L} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^u_{U} = a_1a_2 - \min\{(a_1 - l^u_{IL})(a_2 + r^u_{2L}), (a_1 + r^u_{IL})(a_2 - l^u_{2L})\},$$

$$r^u_{U} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^v_{L} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{L} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$

$$l^v_{U} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{U} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$

where the conditions for LR-type representation of $\tilde{A}_1 \circ \tilde{A}_2$ are satisfied.

Proof Similar to the Proposition 3.3.

Proposition 3.6 Let $\tilde{A}_1 = (a_1; l^u_{IL}, r^u_{IL}, l^u_{IU}, r^u_{IU}, l^v_{I1}, r^v_{I1}, l^v_{IU}, r^v_{IU})_{LR}$ be an LR-type IVIFN such that $a_1 - l^u_{IL} < 0$, $a_1 \geq 0$, $a_1 \geq 0$ and $A_2 = (a_2; l^u_{2L}, r^u_{2L}, l^u_{2U}, r^u_{2U}, l^v_{2L}, r^v_{2L}, l^v_{2U}, r^v_{2U})_{LR}$ be an LR-type IVIFN. Then,

$$\tilde{A}_1 \circ \tilde{A}_2 = (a; l^u_{L}, r^u_{L}, l^u_{U}, r^u_{U}, l^v_{L}, r^v_{L}, l^v_{U}, r^v_{U})_{LR}$$

where $a = a_1a_2$,

$$l^u_{L} = a_1a_2 - \min\{(a_1 - l^u_{IL})(a_2 + r^u_{2L}), (a_1 + r^u_{IL})(a_2 - l^u_{2L})\},$$

$$r^u_{L} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^u_{U} = a_1a_2 - \min\{(a_1 - l^u_{IL})(a_2 + r^u_{2L}), (a_1 + r^u_{IL})(a_2 - l^u_{2L})\},$$

$$r^u_{U} = \max\{(a_1 + r^u_{IL})(a_2 + r^u_{2L}), (a_1 - l^u_{1L})(a_2 + r^u_{2L})\} - a_1a_2,$$

$$l^v_{L} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{L} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$

$$l^v_{U} = a_1a_2 - \min\{(a_1 - l^v_{IL})(a_2 + r^v_{2L}), (a_1 + r^v_{IL})(a_2 + l^v_{2L})\},$$

$$r^v_{U} = \max\{(a_1 + r^v_{IL})(a_2 + r^v_{2L}), (a_1 - l^v_{1L})(a_2 + l^v_{2L})\} - a_1a_2,$$
\[ r_L^\mu = \max((a_1 + r_L^\mu)(a_2 + r_L^\mu), (a_1 - l_L^\mu)(a_2 - l_L^\mu)) - a_1a_2, \]
\[ l_L^\mu = a_1a_2 - \min((a_1 - l_L^\mu)(a_2 + r_L^\mu), (a_1 + r_L^\mu)(a_2 - l_L^\mu)), \]
\[ r_U^\mu = \max((a_1 + r_U^\mu)(a_2 + r_U^\mu), (a_1 - l_U^\mu)(a_2 - l_U^\mu)) - a_1a_2, \]
\[ l_U^\mu = a_1a_2 - \min((a_1 - l_U^\mu)(a_2 + r_U^\mu), (a_1 + r_U^\mu)(a_2 - l_U^\mu)), \]
\[ r_L^\nu = \max((a_1 + r_L^\nu)(a_2 + r_L^\nu), (a_1 - l_L^\nu)(a_2 - l_L^\nu)) - a_1a_2, \]
\[ l_L^\nu = a_1a_2 - \min((a_1 - l_L^\nu)(a_2 + r_L^\nu), (a_1 + r_L^\nu)(a_2 - l_L^\nu)), \]
\[ r_U^\nu = \max((a_1 + r_U^\nu)(a_2 + r_U^\nu), (a_1 - l_U^\nu)(a_2 - l_U^\nu)) - a_1a_2, \]
where the conditions for LR-type representation of \( \tilde{A}_1 \odot \tilde{A}_2 \) are satisfied.

**Proof** Similar to the Proposition 3.3.

**Proposition 3.7** Let \( \tilde{A}_1 = (a_1; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu; l_L^\nu, r_L^\nu, l_U^\nu, r_U^\nu)_{LR} \) be an LR-type IVIFN such that \( a_1 < 0 \) and \( \tilde{A}_2 = (a_2; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu; l_L^\nu, r_L^\nu, l_U^\nu, r_U^\nu)_{LR} \) be any LR-type IVIFN. Then,

\[ \tilde{A}_1 \odot \tilde{A}_2 = (a; l_L, r_L, l_U, r_U; l_L, r_L, l_U, r_U)_{LR} \]
where
\[ a = a_1a_2, \]
\[ l_L^\mu = a_1a_2 - \min((a_1 - l_L^\mu)(a_2 + r_L^\mu), (a_1 + r_L^\mu)(a_2 - l_L^\mu)), \]
\[ r_L^\mu = \max((a_1 - l_L^\mu)(a_2 + r_L^\mu), (a_1 + r_L^\mu)(a_2 - l_L^\mu)) - a_1a_2, \]
\[ l_U^\mu = a_1a_2 - \min((a_1 - l_U^\mu)(a_2 + r_U^\mu), (a_1 + r_U^\mu)(a_2 - l_U^\mu)), \]
\[ r_U^\mu = \max((a_1 - l_U^\mu)(a_2 + r_U^\mu), (a_1 + r_U^\mu)(a_2 - l_U^\mu)) - a_1a_2, \]
where the conditions for LR-type representation of \( \tilde{A}_1 \odot \tilde{A}_2 \) are satisfied.

**Proof** Similar to the Proposition 3.3.

**Proposition 3.8** Let \( \tilde{A}_1 = (a_1; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu; l_L^\nu, r_L^\nu, l_U^\nu, r_U^\nu)_{LR} \) be an LR-type IVIFN such that \( a_1 + r_L^\mu < 0 \) and \( a_1 \leq 0 \) and \( \tilde{A}_2 = (a_2; l_L^\mu, r_L^\mu, l_U^\mu, r_U^\mu; l_L^\nu, r_L^\nu, l_U^\nu, r_U^\nu)_{LR} \) be any LR-type IVIFN. Then,

\[ \tilde{A}_1 \odot \tilde{A}_2 = (a; l_L, r_L, l_U, r_U; l_L, r_L, l_U, r_U)_{LR} \]
where
\[ a = a_1a_2, \]
\[ l_L^\mu = a_1a_2 - \min((a_1 + r_L^\mu)(a_2 + r_L^\mu), (a_1 - l_L^\mu)(a_2 + r_L^\mu)), \]
\[ r_L^\mu = \max((a_1 - l_L^\mu)(a_2 + r_L^\mu), (a_1 + r_L^\mu)(a_2 - l_L^\mu)) - a_1a_2, \]
\[ l_U^\mu = a_1a_2 - \min((a_1 - l_U^\mu)(a_2 + r_U^\mu), (a_1 + r_U^\mu)(a_2 - l_U^\mu)), \]
\[ r_U^\mu = \max((a_1 - l_U^\mu)(a_2 + r_U^\mu), (a_1 + r_U^\mu)(a_2 - l_U^\mu)) - a_1a_2, \]
where the conditions for LR-type representation of \( \tilde{A}_1 \circ \tilde{A}_2 \) are satisfied.

**Proof** Similar to the Proposition 3.3. \( \Box \)

**Proposition 3.9** Let \( \tilde{A}_1 = (a_1; l_{1L}^\mu, l_{1U}^\mu, r_{1L}^\mu, r_{1U}^\mu, l_{1L}^\nu, l_{1U}^\nu, r_{1L}^\nu, r_{1U}^\nu)_LR \) be an LR-type IVIFN such that \( a_1 + r_{1U}^\nu < 0, a_1 + r_{1U}^\mu \geq 0 \) and \( \tilde{A}_2 = (a_2; l_{2L}^\mu, l_{2U}^\mu, r_{2L}^\mu, r_{2U}^\mu, l_{2L}^\nu, l_{2U}^\nu, r_{2L}^\nu, r_{2U}^\nu)_LR \) be any LR-type IVIFN. Then,

\[
\tilde{A}_1 \circ \tilde{A}_2 = (a; l_{L}^\mu, r_L^\mu, l_{U}^\mu, r_U^\mu, l_{L}^\nu, r_L^\nu, l_{U}^\nu, r_U^\nu)_LR
\]

where the conditions for LR-type representation of \( \tilde{A}_1 \circ \tilde{A}_2 \) are satisfied.

**Proof** Similar to the Proposition 3.3. \( \Box \)

**Proposition 3.10** Let \( \tilde{A}_1 = (a_1; l_{1L}^\mu, l_{1U}^\mu, r_{1L}^\mu, r_{1U}^\mu, l_{1L}^\nu, l_{1U}^\nu, r_{1L}^\nu, r_{1U}^\nu)_LR \) be an LR-type IVIFN such that \( a_1 + r_{1U}^\nu < 0, a_1 + r_{1U}^\mu \geq 0 \) and \( \tilde{A}_2 = (a_2; l_{2L}^\mu, l_{2U}^\mu, r_{2L}^\mu, r_{2U}^\mu, l_{2L}^\nu, l_{2U}^\nu, r_{2L}^\nu, r_{2U}^\nu)_LR \) be any LR-type IVIFN. Then,

\[
\tilde{A}_1 \circ \tilde{A}_2 = (a; l_{L}^\mu, r_L^\mu, l_{U}^\mu, r_U^\mu, l_{L}^\nu, r_L^\nu, l_{U}^\nu, r_U^\nu)_LR
\]

where the conditions for LR-type representation of \( \tilde{A}_1 \circ \tilde{A}_2 \) are satisfied.

**Proof** Similar to the Proposition 3.3. \( \Box \)
Proposition 3.11 Let $\tilde{A}_1 = (a_1; l_{1L}, r_{1L}, l_{1U}, r_{1U}, l_{11L}, r_{11L}, l_{11U}, r_{11U})_{LR}$ be an LR-type IVIFN such that $a_1 + r_{11U} < 0$ and $\tilde{A}_2 = (a_2; l_{2L}, r_{2L}, l_{2U}, r_{2U}, l_{21L}, r_{21L}, l_{21U}, r_{21U})_{LR}$ be any LR-type IVIFN. Then, 

$$\tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}, r_{L}, l_{U}, r_{U}, l_{L}, r_{L}, l_{U}, r_{U})_{LR}$$

where

$$a = a_1a_2,$$

$$l_{L} = a_1a_2 - \min((a_1 + l_{1L})(a_2 + l_{2L}), (a_1 - l_{11L})(a_2 + r_{2L})),$$

$$r_{L} = \max((a_1 - l_{1L})(a_2 - l_{2L}), (a_1 + r_{11L})(a_2 - l_{2L})),$$

$$l_{U} = a_1a_2 - \min((a_1 + r_{1U})(a_2 + r_{2U}), (a_1 - r_{11U})(a_2 + l_{2U})),$$

$$r_{U} = \max((a_1 - r_{1U})(a_2 - l_{2U}), (a_1 + l_{11U})(a_2 - r_{2U})),$$

where the conditions for LR-type representation of $\tilde{A}_1 \odot \tilde{A}_2$ are satisfied.

Proof Similar to the Proposition 3.3.

Proposition 3.12 Let $\tilde{A}_1 = (a_1; l_{1L}, r_{1L}, l_{1U}, r_{1U}, l_{11L}, r_{11L}, l_{11U}, r_{11U})_{LR}$ be an LR-type IVIFN such that $a_1 - l_{11L} \geq 0$ and $\tilde{A}_2 = (a_2; l_{2L}, r_{2L}, l_{2U}, r_{2U}, l_{21L}, r_{21L}, l_{21U}, r_{21U})_{LR}$ be any LR-type IVIFN. Then, 

$$\tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}, r_{L}, l_{U}, r_{U}, l_{L}, r_{L}, l_{U}, r_{U})_{LR}$$

where

$$a = a_1a_2,$$

$$l_{L} = a_1a_2 - \min((a_1 - l_{1L})(a_2 + l_{2L}), (a_1 + r_{11L})(a_2 - l_{2L})),$$

$$r_{L} = \max((a_1 - l_{1L})(a_2 - l_{2L}), (a_1 + r_{11L})(a_2 + r_{2L})),$$

$$l_{U} = a_1a_2 - \min((a_1 + r_{1U})(a_2 + r_{2U}), (a_1 - r_{11U})(a_2 + l_{2U})),$$

$$r_{U} = \max((a_1 - r_{1U})(a_2 - l_{2U}), (a_1 + l_{11U})(a_2 - r_{2U})),$$

where the conditions for LR-type representation of $\tilde{A}_1 \odot \tilde{A}_2$ are satisfied.

Proof Similar to the Proposition 3.3.

3.2 Proposed lexicographic criteria for the ranking of LR-type IVIFNs

In the literature, there exist several ranking criteria to define the ordering of FNs, IFNs, and IVIFNs. Most of the researchers have used the linear ranking functions which map the set of these numbers to real-line and then a comparison is carried out on the basis of the usual ordering of real numbers. However, as pointed out in Pérez-Cañedo and Conception-Morales (2019b), it may be possible that two of these numbers may look different for the
In this section, we discuss the lexicographic criteria for the ordering of IVIFNs. However, one may consider a different permutation ranking of the functions in deciding the ordering of the IVIFNs. To overcome this limitation, we may consider a different permutation ranking of the decision-maker but can be mapped to the same real number. To overcome this limitation, lexicographic ranking criteria with total order properties seem to be more logical and relevant (Kaur and Kumar 2016; Pérez-Cañedo and Concepción-Morales 2019a; Farhadinia 2009). In this section, we discuss the lexicographic criteria for the ordering of LR-type IVIFNs.

Definition 3.11 (Pérez-Cañedo and Concepción-Morales 2019a) For \( x, y \in \mathbb{R}^n \), the strict lexicographic inequality \( x \prec_{\text{lex}} y \) holds if there is \( 1 \leq i \leq n \) so that \( x_j = y_j \) holds for \( j < i \) and \( x_i < y_i \). The weak lexicographic inequality \( x \preceq_{\text{lex}} y \) holds if \( x \prec_{\text{lex}} y \) or \( x = y \).

Definition 3.12 Let \( \tilde{A}_1 = (a_1; l_{1L}^\mu, r_{1L}^\mu, l_{1U}^\mu, r_{1U}^\mu, l_{1L}^\nu, r_{1U}^\nu) \) and \( \tilde{A}_2 = (a_2; l_{2L}^\mu, r_{2L}^\mu, l_{2U}^\mu, r_{2U}^\mu, l_{2L}^\nu, r_{2U}^\nu) \) be two LR-type IVIFNs, then \( \tilde{A}_1 \prec \tilde{A}_2 \) if
\[
\begin{align*}
& a_1 = a_2, l_{1L}^\mu = l_{2L}^\mu, r_{1L}^\mu = r_{2L}^\mu, l_{1U}^\mu = l_{2U}^\mu, r_{1U}^\mu = r_{2U}^\mu, \\
& l_{1L}^\nu = l_{2L}^\nu, r_{1L}^\nu = r_{2L}^\nu, l_{1U}^\nu = l_{2U}^\nu, r_{1U}^\nu = r_{2U}^\nu.
\end{align*}
\]

Theorem 3.3 The order relation on \( \text{IV}(\mathbb{R}) \) given in Definition 3.13, has the total order properties and yields a complete ranking on the set of all LR-type IVIFNs of the same type.

Proof The order relation given in Definition 3.13 is a total order due to its following properties:
1. (reflexivity) \( \tilde{A}_1 \leq \tilde{A}_1 \) \( \forall \tilde{A}_1 \in \text{IV}(\mathbb{R}) \).
2. (anti-symmetry) \( \tilde{A}_1 \leq \tilde{A}_2 \) and \( \tilde{A}_2 \leq \tilde{A}_1 \) \( \Rightarrow \tilde{A}_1 = \tilde{A}_2 \) \( \forall \tilde{A}_1, \tilde{A}_2 \in \text{IV}(\mathbb{R}) \).
3. (transitivity) \( \tilde{A}_1 \leq \tilde{A}_2 \) and \( \tilde{A}_2 \leq \tilde{A}_3 \)
\[ \implies \tilde{A}_1 \leq \tilde{A}_3 \quad \forall \ \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in \text{IV}(\mathbb{R}). \]

4. (comparability) \( \tilde{A}_1 \leq \tilde{A}_2 \) or \( \tilde{A}_2 \leq \tilde{A}_1 \) \( \forall \ \tilde{A}_1, \tilde{A}_2 \in \text{IV}(\mathbb{R}). \)

Further, it is to be noted that the system of linear equations \( \mathbf{S}(\tilde{A}_1) = \mathbf{S}(\tilde{A}_2), \mathbf{A}(\tilde{A}_1) = \mathbf{A}(\tilde{A}_2), \mathbf{M}(\tilde{A}_1) = \mathbf{M}(\tilde{A}_2), \mathbf{C}(\tilde{A}_1) = \mathbf{C}(\tilde{A}_2), \mathbf{D}(\tilde{A}_1) = \mathbf{D}(\tilde{A}_2), \mathbf{G}(\tilde{A}_1) = \mathbf{G}(\tilde{A}_2) \) and \( \mathbf{H}(\tilde{A}_1) = \mathbf{H}(\tilde{A}_2) \) has a unique solution, that is, \( \tilde{A}_1 = \tilde{A}_2 \) as the absolute value of the determinant of this linear system is \( 2bd \) where \( b = \int_0^1 R^{-1}(x) \, dx > 0 \) and \( d = \int_0^1 R^{-1}(1-x) \, dx > 0. \) Hence, the result.

\[ \square \]

4 Model formulation and solution algorithm

Consider a linear programming problem in which the parameters \( \tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i \) and decision variables \( \tilde{x}_j \) are taken to be in the form of \( L.R \)-type IVIFNs. The model of such a fully \( L.R \)-type interval-valued intuitionistic fuzzy linear programming problem along with unrestricted decision variables can be mathematically formulated as follows:

\[(P1) \ \text{max} \quad \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j \]
\[\text{s.t.} \quad \sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j = \tilde{b}_i, \quad \text{for} \ i \in I_1 := \{1, 2, \ldots, m_1\}, \]
\[\sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j \leq \tilde{b}_i, \quad \text{for} \ i \in I_2 := \{m_1 + 1, m_1 + 2, \ldots, m_2\}, \]
\[\sum_{j=1}^{n} \tilde{a}_{ij} \odot \tilde{x}_j \geq \tilde{b}_i, \quad \text{for} \ i \in I_3 := \{m_2 + 1, m_2 + 2, \ldots, m\}, \]
\[\tilde{x}_j \ \text{are unrestricted in sign}, \quad \text{for} \ j \in J := \{1, 2, \ldots, n\} \]

where the inequalities “\( \leq \)” and “\( \geq \)” in the problem (P1) is in accordance with the lexicographic ranking given in Definition 3.13.

**Definition 4.1** Let \( X \) denote the set of all feasible solutions of (P1). A vector \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in X \) is said to be an optimal solution of (P1) if

\[ \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j \leq \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j, \quad \text{for all} \ \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in X. \]

Next, we propose a method to find the unique optimal solution of (P1) by employing the proposed lexicographic ranking criteria for ordering the \( L.R \)-type IVIFNs. The steps of the proposed approach are as follows:

**Step 1.** Let \( \tilde{a}_{ij} = (a_{ij}^L, a_{ij}^U, \alpha_{ij}^L, \alpha_{ij}^U, \beta_{ij}^L, \beta_{ij}^U, \gamma_{ij}^L, \gamma_{ij}^U, \delta_{ij}^L, \delta_{ij}^U)_{L.R}, \)

\[ \tilde{c}_j = (c_j^L, c_j^U, \sigma_j^L, \sigma_j^U, \rho_j^L, \rho_j^U, \gamma_j^L, \gamma_j^U, \delta_j^L, \delta_j^U)_{L.R}, \]

\[ \tilde{b}_i = (b_i^L, b_i^U, \gamma_i^L, \gamma_i^U, \delta_i^L, \delta_i^U)_{L.R}, \quad \text{and} \]

\[ \tilde{x}_j = (x_j^L, x_j^U, \eta_j^L, \eta_j^U, \xi_j^L, \xi_j^U)_{L.R}. \]
Then, (P1) can be recast as follows:

\[
\begin{align*}
\max & \quad \sum_{j=1}^{n} (c_j; \sigma_j^{\mu}; \rho_j^{\mu}; \sigma_j^{\nu}; \rho_j^{\nu}; \sigma_j^{v}; \rho_j^{v}, \phi_{ij}^{L}, \phi_{ij}^{U})LR \circ \\
\text{s.t.} & \quad \sum_{j=1}^{n} (a_{ij}; \alpha_{ij}^{\mu}; \beta_{ij}^{\mu}; \beta_{ij}^{v}; \alpha_{ij}^{v}; \beta_{ij}^{v}, \phi_{ij}^{L}, \phi_{ij}^{U})LR \circ \\
& = (b_i; \gamma_i^{\mu}; \delta_i^{\mu}; \gamma_i^{v}; \delta_i^{v}, \gamma_i^{v}; \delta_i^{v})LR, \quad \text{for } i \in I_1,
\end{align*}
\]

Step 2. After applying the multiplication operation (Sect. 3.1), let

\[
\tilde{c}_j \circ \tilde{x}_j = (p_j; \tau_j^{\mu}; \omega_j^{\mu}; \tau_j^{\nu}; \omega_j^{\nu}; \tau_j^{v}, \omega_j^{v})LR \quad \text{and}
\tilde{a}_{ij} \circ \tilde{x}_j = (m_{ij}; \iota_{ij}^{\mu}; \lambda_{ij}^{\mu}; \lambda_{ij}^{v}; \lambda_{ij}^{v}; \delta_{ij}^{v}, \lambda_{ij}^{v}; \delta_{ij}^{v})LR.
\]

Then, the problem (P1) in Step 1 becomes:

\[
\text{(P2)} \quad \max \sum_{j=1}^{n} (p_j; \tau_j^{\mu}; \omega_j^{\mu}; \tau_j^{\nu}; \omega_j^{\nu}; \tau_j^{v}, \omega_j^{v})LR
\]
\[
\text{s.t.} \quad \sum_{j=1}^{n} (m_{ij}; \iota_{ij}^{\mu}; \lambda_{ij}^{\mu}; \lambda_{ij}^{v}; \lambda_{ij}^{v}; \delta_{ij}^{v}, \lambda_{ij}^{v}; \delta_{ij}^{v})LR
\]
\[
\leq (b_i; \gamma_i^{\mu}; \delta_i^{\mu}; \gamma_i^{v}; \delta_i^{v}, \gamma_i^{v}; \delta_i^{v})LR, \quad \text{for } i \in I_1,
\]
\[
\sum_{j=1}^{n} (m_{ij}; \iota_{ij}^{\mu}; \lambda_{ij}^{\mu}; \lambda_{ij}^{v}; \lambda_{ij}^{v}; \delta_{ij}^{v}, \lambda_{ij}^{v}; \delta_{ij}^{v})LR
\]
\[
\leq (b_i; \gamma_i^{\mu}; \delta_i^{\mu}; \gamma_i^{v}; \delta_i^{v}, \gamma_i^{v}; \delta_i^{v})LR, \quad \text{for } i \in I_2,
\]
\[
\sum_{j=1}^{n} (m_{ij}; \iota_{ij}^{\mu}; \lambda_{ij}^{\mu}; \lambda_{ij}^{v}; \lambda_{ij}^{v}; \delta_{ij}^{v}, \lambda_{ij}^{v}; \delta_{ij}^{v})LR
\]
\[
\leq (b_i; \gamma_i^{\mu}; \delta_i^{\mu}; \gamma_i^{v}; \delta_i^{v}, \gamma_i^{v}; \delta_i^{v})LR, \quad \text{for } i \in I_3.
\]
Define Step 3. Define \( \tilde{t}_i := \sum_{j=1}^{\eta_i} (m_{ij} + s_{ijL} + \lambda_{ijL} + s_{ijU} + \lambda_{ijU}) \) and \( \tilde{r}_i := (b_i + \gamma_{iL} - \delta_{iL}) \). Further, by taking the order relation \( \leq_{\text{lex}} \) as in Definition 3.13, according to the addition operation from Sect. 3.1 and the equality between LR-type IVFNs given in Definition 3.12, we can write the constraint set of (P2) as:

\[
\sum_{j=1}^{n} m_{ij} = b_i, \quad \sum_{j=1}^{n} s_{ijL} = \gamma_{iL}, \quad \sum_{j=1}^{n} \lambda_{ijL} = \delta_{iL}, \quad \sum_{j=1}^{n} s_{ijU} = \gamma_{iU},
\]

\[
\sum_{j=1}^{n} \lambda_{ijU} = \delta_{iU}, \quad \sum_{j=1}^{n} s_{ijL} = \gamma_{iL}, \quad \sum_{j=1}^{n} \lambda_{ijL} = \delta_{iL}, \quad \sum_{j=1}^{n} s_{ijU} = \gamma_{iU},
\]

\[
\sum_{j=1}^{n} \lambda_{ijU} = \delta_{iU}, \quad \text{for } i \in I_1, \tag{3a}
\]

\[
(S(\tilde{t}_i), A(\tilde{t}_i), M(\tilde{t}_i), C(\tilde{t}_i), D(\tilde{t}_i), G(\tilde{t}_i), H(\tilde{t}_i)) \leq_{\text{lex}} (S(r_i), A(r_i), M(r_i), C(r_i), D(r_i), G(r_i), H(r_i)), \quad \text{for } i \in I_2, \tag{3b}
\]

\[
(S(\tilde{t}_i), A(\tilde{t}_i), M(\tilde{t}_i), C(\tilde{t}_i), D(\tilde{t}_i), G(\tilde{t}_i), H(\tilde{t}_i)) \geq_{\text{lex}} (S(r_i), A(r_i), M(r_i), C(r_i), D(r_i), G(r_i), H(r_i)), \quad \text{for } i \in I_3, \tag{3c}
\]

\[
\xi^{\mu}_{jU} \geq \xi^{\mu}_{jL}, \quad \eta^{\mu}_{jU} \geq \eta^{\mu}_{jL}, \quad \xi^{\nu}_{jU} \geq \xi^{\nu}_{jL}, \quad \eta^{\nu}_{jU} \geq \eta^{\nu}_{jL}, \tag{3d}
\]

The objective function of the problem (P2) can be recast as:

\[
\text{lex max } (S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(\tilde{Z}), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z}))
\]

where \( \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j = \sum_{j=1}^{n} (p_j; \epsilon^\mu_{jL}, \omega^\mu_{jL}, \epsilon^\mu_{jU}, \omega^\mu_{jU}, \epsilon^\nu_{jL}, \omega^\nu_{jL}, \epsilon^\nu_{jU}, \omega^\nu_{jU}) \).

Hence, the problem (P1) is finally transformed into the following optimization problem (P3):

\[
(P3) \quad \text{lex max } (S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(\tilde{Z}), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z}))
\]

subject to constraints (3a)–(3d).

Step 4. By introducing new binary variables \( z_i^S, z_i^A, z_i^M, z_i^C, z_i^P, z_i^D, z_i^G, z_i^H \) for \( i \in I_2 \cup I_3 \) following Pérez-Cañedo and Concepción-Morales (2019b) along with a new constraint (4p), for positive real values of \( k \) and \( K \) sufficiently small and large, respectively, the lexicographic constraints (3b) and (3c) can be converted into the following set of constraints (4a)–(4p):

\[
kz^S \leq S(\tilde{r}_i) - S(\tilde{t}_i) \leq Kz^S, \quad \text{for } i \in I_2, \tag{4a}
\]

\[
-Kz^S + kz^A \leq A(\tilde{r}_i) - A(\tilde{t}_i) \leq Kz^A, \quad \text{for } i \in I_2, \tag{4b}
\]

\[
-K(z^S + z^D) + kz^M \leq M(\tilde{r}_i) - M(\tilde{t}_i) \leq Kz^M, \quad \text{for } i \in I_2, \tag{4c}
\]
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\[ K(z_i^C + z_i^M + z_i^G) + k_i^H \leq H(\tilde{r}_i) - H(\bar{r}_i) \leq Kz_i^H, \quad \text{for } i \in I_2, \]  
\[ k_i^N \leq S(\tilde{r}_i) - S(\bar{r}_i) \leq Kz_i^N, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A \leq A(\tilde{r}_i) - A(\bar{r}_i) \leq Kz_i^A, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A + k_i^M \leq M(\tilde{r}_i) - M(\bar{r}_i) \leq Kz_i^M, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A + k_i^M + k_i^C \leq C(\tilde{r}_i) - C(\bar{r}_i) \leq Kz_i^C, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A + k_i^M + k_i^C + k_i^D \leq D(\tilde{r}_i) - D(\bar{r}_i) \leq Kz_i^D, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A + k_i^M + k_i^C + k_i^D + k_i^G \leq G(\tilde{r}_i) - G(\bar{r}_i) \leq Kz_i^G, \quad \text{for } i \in I_3, \]  
\[ k_i^S + k_i^A + k_i^M + k_i^C + k_i^D + k_i^G \leq H(\tilde{r}_i) - H(\bar{r}_i) \leq Kz_i^H, \quad \text{for } i \in I_3, \]  
\[ z_i^S, z_i^A, z_i^M, z_i^C, z_i^D, z_i^G, z_i^H \in \{0, 1\}, \quad \text{for } i \in I_2 \cup I_3, \]  
\[ z_i^S \leq z_i^A \leq z_i^M \leq z_i^C \leq z_i^D \leq z_i^G \leq z_i^H, \quad \text{for } i \in I_2 \cup I_3. \]  

**Step 5.** Convert problem (P3) into the following mixed 0-1 lexicographic non-linear programming problem (P4):

\[ \text{(P4)} \quad \text{lex max } (S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(Z), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z})) \]

subject to constraints (3a), (3d), (4a)–(4p).

**Step 6.** Solve the problem (P4) by optimizing orderly one objective at a time subject to constraints (3a), (3d), (4a)–(4p); including all previously optimized objectives in the constraint set. Thus, by using the lexicographic method of multi-objective optimization (Greco et al. 2016) and a suitable optimization solver, we can find the optimal solution \( x_j, \xi_1^H, \eta_1^H, \xi_2^H, \eta_2^H, \xi_3^H, \eta_3^H, \xi_4^H, \eta_4^H \) for \( j \in J \). Hence, \( \tilde{x}_j = (x_j; \xi_1^H, \eta_1^H, \xi_2^H, \eta_2^H, \xi_3^H, \eta_3^H, \xi_4^H, \eta_4^H)_{LR} \) for \( j \in J \) can be obtained.

**Step 7.** Finally, by substituting the values of \( \tilde{x}_j \)'s into \( \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j \), we obtain the optimal interval-valued intuitionistic fuzzy value of the problem (P1).

Now, we present Theorems 4.1 and 4.2 to prove the equivalence of the models (P1), (P3) and (P3), (P4), respectively. Similar results for fuzzy LPP were discussed by Pérez-Cañedo and Concepción-Morales (2019a). However, we have shown the results for LR-type IVIFLPPs.

**Theorem 4.1** If \( \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_n) \) is an optimal solution of problem (P3), then it is also an optimal solution of (P1).

**Proof** We will prove the result by the method of contradiction. Let \( \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_n) \) be an optimal solution of problem (P3) but not an optimal solution of (P1). Then, there exists a feasible solution \( \tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \ldots, \tilde{x}_n^*) \) of the problem (P1), such that

\[ \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j < \sum_{j=1}^{n} \tilde{c}_j \odot \tilde{x}_j^*. \]
In view of Definition 3.13, $\hat{x}$ is a feasible solution of the problem (P3) for which

$$
\begin{align*}
&\left( S\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right), A\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right), M\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right), \right. \\
&D\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right), G\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right), H\left( \sum_{j=1}^{n} \tilde{c}_j \odot \hat{x}_j \right) \left. \right)
\end{align*}
$$

that is, there exists a feasible solution of (P3) with the higher objective function value. This contradicts the fact that $\hat{x}$ is an optimal solution of (P3). Hence, the result.

\begin{theorem}
The optimal solution of the problem (P4) is also optimal for (P3). The converse of the statement is also true.
\end{theorem}

\begin{proof}
Let $\tilde{I}_i := \sum_{j=1}^{n} (m_{ij}; s_{ijL}, \lambda_{ijL}; s_{ijU}, \lambda_{ijU}; s_{ijL}^v, \lambda_{ijL}^v; s_{ijU}^v, \lambda_{ijU}^v)_{LR}$,

$\tilde{r}_i := (b_i; \gamma_i^L, \delta_{ij}^L, \gamma_i^U, \delta_{ij}^U, \gamma_i^L^{v}, \delta_{ij}^{vL}, \gamma_i^U^{v}, \delta_{ij}^{vU})_{LR}$ and $z_i := (A_i^{k}, M_i^{k}, D_i^{k}, G_i^{k}, H_i^{k}).$

In order to prove this theorem, we need to establish the equivalence between constraint sets (3)(3b), (3c)) and (4)(4a)–(4p) for positive real values of $k$ and $K$ sufficiently small and large, respectively. Here, we will prove the result for the $I_2$ set of constraints only. For the remaining set of constraints, the result can be proved on the same lines.

To show that any solution satisfying constraint set (4) also satisfies set (3), we consider the following cases:

1. If $z_i = (1, \star, \star, \star, \star, \star, \star)$, where $\star = 0$ or 1, then by substituting into constraint set (4), we get $k \leq S(\tilde{r}_i) - S(\tilde{I}_i) \leq K$, which implies $S(\tilde{I}_i) < S(\tilde{r}_i)$; hence, according to Definition 3.13, set of constraints (3) is satisfied.

2. If $z_i = (0, 1, \star, \star, \star, \star, \star)$, where $\star = 0$ or 1, then by the constraint set (4) we get $0 \leq S(\tilde{r}_i) - S(\tilde{I}_i) \leq 0$, and $k \leq A(\tilde{r}_i) - A(\tilde{I}_i) \leq K$, which implies $S(\tilde{I}_i) = S(\tilde{r}_i)$ and $A(\tilde{I}_i) < A(\tilde{r}_i)$; hence, from Definition 3.13, the constraint set (3) is satisfied.

The remaining cases $z_i = (0, 0, 1, \star, \star, \star, \star)$, $z_i = (0, 0, 0, 0, 1, \star, \star)$, $z_i = (0, 0, 0, 0, 0, 1, \star)$, $z_i = (0, 0, 0, 0, 0, 0, 0)$ and $z_i = (0, 0, 0, 0, 0, 0, 0, 0)$ can be proved similarly.

Now, it is to be shown that any solution satisfying constraint set (3) also satisfies constraints (4). For this, let us consider the following cases:

1. If $S(\tilde{r}_i) = S(\tilde{I}_i)$, $A(\tilde{r}_i) = A(\tilde{I}_i)$, $M(\tilde{r}_i) = M(\tilde{I}_i)$, $C(\tilde{r}_i) = C(\tilde{I}_i)$, $D(\tilde{r}_i) = D(\tilde{I}_i)$, $G(\tilde{r}_i) = G(\tilde{I}_i)$ and $H(\tilde{r}_i) = H(\tilde{I}_i)$, then by constraint set (4), we get $z_i = (0, 0, 0, 0, 0, 0, 0)$.

2. If $S(\tilde{r}_i) > S(\tilde{I}_i)$, i.e., $S(\tilde{r}_i) - S(\tilde{I}_i) = s_i > 0$, then by constraint (4a), we get $kz_i^S \leq s_i \leq Kz_i^S$, this inequality is satisfied for $z_i^S = 1$ and for positive real values of $k$ and $K$, sufficiently small and large. Further, (4o) and (4p) give $(z_i^A, z_i^M, z_i^C, z_i^D, z_i^G, z_i^H) = (1, 1, 1, 1, 1, 1)$, due to which rest of the constraints (4b)–(4g) are automatically satisfied.

Also, $S(\tilde{r}_i) > S(\tilde{I}_i)$ implies that the constraint (3b) is satisfied.
3. If \( S(\tilde{r}_i) = S(\tilde{l}_i) \) and \( A(\tilde{r}_i) > A(\tilde{l}_i) \) i.e., \( A(\tilde{r}_i) - A(\tilde{l}_i) = a_i > 0 \), then by constraints \((4a)\) and \((4b)\), we get \( s_i = 0 \) and \( k z_i^{A} \leq a_i \leq K z_i^{A} \), this inequality is satisfied for \( z_i^{s} = 0 \), \( z_i^{A} = 1 \) and for positive real values of \( k \) and \( K \), sufficiently small and large. Further, \((4o)\) and \((4p)\) yield \((z_{i}^{M}, z_{i}^{C}, z_{i}^{D}, z_{i}^{G}, z_{i}^{H}) = (1, 1, 1, 1, 1)\), due to which rest of the constraints \((4c) - (4g)\) are obviously satisfied. Further, \( S(\tilde{r}_i) = S(\tilde{l}_i) \) and \( A(\tilde{r}_i) > A(\tilde{l}_i) \) implies \((3b)\) holds.

The remaining cases can be similarly obtained. This establishes that the problems \((P3)\) and \((P4)\) are equivalent.

Hence proved. \(\Box\)

### 5 Advantages of the proposed method

Some of the main advantages of the proposed algorithm over the existing approaches are listed below as follows:

1. The existing method (Bharati and Singh 2020) can only be used to solve IVIFLPPs in which all the decision variables are taken to be non-negative crisp parameters. However, our proposed method can be employed successfully to handle IVIFLPPs having all the decision variables represented by unrestricted IVIFNs.

2. The existing study (Singh and Yadav 2017) defined a new product operator and the basic arithmetic operations on unrestricted LR-type IFNs. But, there is no study on LR-type IVIFNs. Consequently, we have introduced the definition of LR-type IVIFNs and developed the arithmetic operations on unrestricted LR-type IVIFNs.

3. The existing models (Kaur and Kumar 2016; Pérez-Cañedo and Concepción-Morales 2019a; Singh and Yadav 2017; Pérez-Cañedo and Concepción-Morales 2019b) can be used only to deal with LR-type FLPPs and IFLPPs. But, in this article, we have solved the IVIFLPP in which all the parameters and decision variables are represented by LR-type IVIFNs, which is more general. Hence, the proposed method can be successfully reduced to solve IFLPPs as well as FLPPs.

4. In the present study, we have considered the model parameters and decision variables to be LR-type IVIFNs, as a result, the proposed method can also be utilized to solve LPPs where parameters and variables are represented by triangular/trapezoidal IVIFNs.

5. The proposed algorithm can be used to solve the models having some/all decision variables as unrestricted LR-type IVIFNs or non-negative LR-type IVIFNs.

### 6 Numerical illustration

In this section, we present a numerical example to demonstrate the steps involved in the proposed algorithm.

Consider the following LPP having all the parameters as LR-type IVIFN and unrestricted crisp variables:

\[
\begin{align*}
\text{(S1) } \max & \quad 5 \odot x_1 + 8 \odot x_2 \\
\text{s.t.} & \quad 12 \odot x_1 + 4 \odot x_2 = 100, \\
& \quad 6 \odot x_1 + 10 \odot x_2 \leq 150, \\
& \quad x_1 \text{ and } x_2 \text{ are unrestricted in sign}
\end{align*}
\]
where
\[
\tilde{5} = (5; 2, 2, 3, 3; 5, 5, 5, 4)_{LR}, \quad \tilde{8} = (8; 1, 1, 2, 2; 4, 4, 2, 3)_{LR},
\]
\[
\tilde{l}_2 = (12; 2, 3, 4, 4; 6, 8, 4, 4)_{LR}, \quad \tilde{4} = (4; 1, 1, 2, 2; 4, 4, 2, 2)_{LR},
\]
\[
\tilde{6} = (6; 3, 4, 4, 4; 6, 6, 4, 4)_{LR}, \quad \tilde{10} = (10; 3, 4, 4, 5; 6, 8, 5, 5)_{LR},
\]
\[
\tilde{100} = (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR},
\]
\[
\tilde{150} = (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR}.
\]
\[
L(x) = R(x) = L'(x) = R'(x) = \max \{0, 1 - x\} \quad \forall \ x \in \mathbb{R}.
\]

**Solution:**

**Step 1.** Substituting the expressions of various parameters, the problem (S1) can be re-written as follows:

\[
\begin{align*}
\max & \quad \tilde{Z} = (5; 2, 2, 3, 3; 5, 5, 5, 4)_{LR} \odot x_1 \oplus (8; 1, 1, 2, 2; 4, 4, 2, 3)_{LR} \odot x_2 \\
\text{s.t.} & \quad (12; 2, 3, 4, 4; 6, 8, 4, 4)_{LR} \odot x_1 \oplus (4; 1, 1, 2, 2; 4, 4, 2, 2)_{LR} \odot x_2 \\
& \quad = (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR}, \\
& \quad (6; 3, 4, 4, 4; 6, 6, 4, 4)_{LR} \odot x_1 \oplus (10; 3, 4, 4, 5; 6, 8, 5, 5)_{LR} \odot x_2 \\
& \quad \leq (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR}, \\
& \quad x_1 \text{ and } x_2 \text{ are unrestricted in sign.}
\end{align*}
\]

**Step 2.** Using the multiplication operation (Corollary 3.1) on LR-type IVIFNs along with the fact that

\[
\max \{a, b\} = \frac{1}{2} (a + b + |a - b|),
\]

the problem (S1) is converted to the following equivalent problem:

\[
\text{(S2) max } \tilde{Z} = (5x_1; 2|x_1|, 2|x_1|, 3|x_1|, 3|x_1|; 5|x_1|, 5|x_1|, \frac{1}{2}(x_1 + 9|x_1|), \\
\frac{1}{2}(-x_1 + 9|x_1|))_{LR} \odot (8x_2; |x_2|, |x_2|, 2|x_2|, 2|x_2|; 4|x_2|, 4|x_2|, \\
\frac{1}{2}(-x_2 + 5|x_2|), \frac{1}{2}(x_2 + 5|x_2|))_{LR} \\
\text{s.t. } \left( 12x_1; \frac{1}{2}(-x_1 + 5|x_1|), \frac{1}{2}(x_1 + 5|x_1|), 4|x_1|, 4|x_1|; -x_1 + 7|x_1|, \\
x_1 + 7|x_1|, 4|x_1|, 4|x_1| \right)_{LR} \odot (4x_2; |x_2|, |x_2|, 2|x_2|, 2|x_2|; 4|x_2|, 4|x_2|, \\
2|x_2|, 2|x_2|)_{LR} = (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR}, \\
(6x_1; \frac{1}{2}(-x_1 + 7|x_1|), \frac{1}{2}(x_1 + 7|x_1|), 4|x_1|, 4|x_1|, 6|x_1|, 6|x_1|, 4|x_1|, \\
4|x_1|)_{LR} \odot (10x_2; \frac{1}{2}(-x_2 + 7|x_2|), \frac{1}{2}(x_2 + 7|x_2|), \frac{1}{2}(-x_2 + 9|x_2|), \\
\frac{1}{2}(x_2 + 9|x_2|); -x_2 + 7|x_2|, x_2 + 7|x_2|, 5|x_2|, 5|x_2|)_{LR} \\
\leq (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR}, \\
x_1 \text{ and } x_2 \text{ are unrestricted in sign.}
\]
Step 3. By employing the lexicographic ordering as in Definition 3.13, using the addition operation, equality between $LR$-type IVIFNs and the corresponding definitions of $S, A, M, C, D, G$ and $H$, we get the following non-linear programming problem:

$$\textbf{(S3) lex max } \left( \frac{1}{8} (x_1 - x_2), \frac{1}{8} (79x_1 + 129x_2), 5x_1 + 8x_2, 5x_1 + 8x_2 - 2|x_1| - |x_2|, \right.$$  
$$5x_1 + 8x_2 - 3|x_1| - 2|x_2|, \frac{9}{2} x_1 + \frac{17}{2} x_2 - \frac{9}{2} |x_1| - \frac{5}{2} |x_2|, \right.$$  
$$5x_1 + 8x_2 - 5|x_1| - 4|x_2| \right)$$  
$s.t.$  
$$3x_1 + x_2 = 25, \quad -x_1 + 5|x_1| + 2|x_2| = 50, \quad x_1 + 5|x_1| + 2|x_2| = 70, \quad 2|x_1| + |x_2| = 25, \quad -x_1 + 7|x_1| + 4|x_2| = 80, \quad x_1 + 7|x_1| + 4|x_2| = 100,$  
$$\left( \frac{x_1}{8}, \frac{1}{8} (97x_1 + 164x_2), 6x_1 + 10x_2, \frac{13}{2} x_1 + \frac{21}{2} x_2 - \frac{7}{2} |x_1| - \frac{7}{2} |x_2|, \right.$$  
$$6x_1 + \frac{21}{2} x_2 - 4|x_1| - \frac{9}{2} |x_2|, 6x_1 + 10x_2 - 4|x_1| - 5|x_2|, \right.$$  
$$6x_1 + 11x_2 - 6|x_1| - 7|x_2| \right) \leq \text{lex } (7.5, 300, 150, 100, 70, 30),$$  
$x_1$ and $x_2$ are unrestricted in sign.

Step 4. To convert the lexicographic inequality “$\leq \text{lex}$” into its equivalent crisp form, introduce the binary variables $z^S_1, z^A_1, z^M_1, z^C_1, z^D_1, z^G_1$ and $z^H_1$ and for $k = 10^{-4}$ and $K = 1000$, we have the following set of constraints:

$$kz^S_1 \leq 7.5 - \frac{1}{8} (x_1) \leq Kz^S_1,$$  
$$-Kz^S_1 + kz^A_1 \leq 300 - \frac{1}{8} (97x_1 + 164x_2) \leq Kz^A_1,$$  
$$-K(z^S_1 + z^A_1) + kz^M_1 \leq 150 - 6x_1 - 10x_2 \leq Kz^M_1,$$  
$$-K(z^S_1 + z^A_1 + z^M_1) + kz^C_1 \leq 100 - \frac{13}{2} x_1 - \frac{21}{2} x_2 + \frac{7}{2} |x_1| + \frac{7}{2} |x_2| \leq Kz^C_1,$$  
$$-K(z^S_1 + z^A_1 + z^M_1 + z^C_1) + kz^D_1 \leq 100 - 6x_1 - \frac{21}{2} x_2 + 4|x_1| + \frac{9}{2} |x_2| \leq Kz^D_1,$$  
$$-K(z^S_1 + z^A_1 + z^M_1 + z^C_1 + z^D_1) + kz^G_1 \leq 70 - 6x_1 - 10x_2 + 4|x_1| + 5|x_2| \leq Kz^G_1,$$  
$$-K(z^S_1 + z^A_1 + z^M_1 + z^C_1 + z^D_1 + z^G_1) + kz^H_1 \leq 30 - 6x_1 - 11x_2 + 6|x_1| + 7|x_2| \leq Kz^H_1,$$  
$$z^S_1, z^A_1, z^M_1, z^C_1, z^D_1, z^G_1, z^H_1 \in \{0, 1\},$$  
$$z^S_1 \leq z^A_1 \leq z^M_1 \leq z^C_1 \leq z^D_1 \leq z^G_1 \leq z^H_1.$$  

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Step 5. Finally, the equivalent mixed 0-1 lexicographic non-linear optimization problem is presented as follows:

\[(S4)\]  
\[
\text{lex max } \left( \frac{1}{8}(x_1 - x_2), \frac{1}{8}(79x_1 + 129x_2), 5x_1 + 8x_2, 5x_1 + 8x_2 - 2|x_1| - |x_2|, \\
5x_1 + 8x_2 - 3|x_1| - 2|x_2|, \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2|, \\
5x_1 + 8x_2 - 5|x_1| - 4|x_2\right)
\]

s.t.  
\[
3x_1 + x_2 = 25, -x_1 + 5|x_1| + 2|x_2| = 50, x_1 + 5|x_1| + 2|x_2| = 70, \\
2|x_1| + |x_2| = 25, -x_1 + 7|x_1| + 4|x_2| = 80, x_1 + 7|x_1| + 4|x_2| = 100, \\
\]
constraints (5a)–(5i),  
\[x_1 \text{ and } x_2 \text{ are unrestricted in sign.}\]

Step 6. To solve the problem (S4), we start by optimizing the first component of the objective function along with imposing all the constraints of problem (S4). As a result, we solve the single-objective non-linear programming problem (S4-1) given as follows:

\[(S4-1)\]  
\[
\text{max } S = \frac{1}{8}(x_1 - x_2) \\
\text{s.t. all the constraints of (S4).}
\]

Here, we have solved all the crisp optimization models by using a software “LINGO-17.0” on a MacBook Air system with 1.8 GHz Dual-Core Intel Core i5 processor and 8 GB RAM. The optimal solution of (S4-1) gives the optimal objective value as \(S = 1.875\). Next, for optimizing the second component of the objective function of (S4), we solve the following problem (S4-2):

\[(S4-2)\]  
\[
\text{max } A = \frac{1}{8}(79x_1 + 129x_2) \\
\text{s.t. all the constraints of (S4),} \\
\frac{1}{8}(x_1 - x_2) \geq 1.875.
\]

The optimal objective value of (S4-2) is \(A = 18.125\) and for optimizing the third objective, the problem (S4-3) is solved.

\[(S4-3)\]  
\[
\text{max } M = 5x_1 + 8x_2 \\
\text{s.t. all the constraints of (S4),} \\
\frac{1}{8}(x_1 - x_2) \geq 1.875, \\
\frac{1}{8}(79x_1 + 129x_2) \geq 18.125.
\]

The optimal solution of (S4-3) gives \(M = 10\). Further, the fourth objective is optimized by solving the problem (S4-4).

\[(S4-4)\]  
\[
\text{max } C = 5x_1 + 8x_2 - 2|x_1| - |x_2| \\
\text{s.t. all the constraints of (S4),} \\
\frac{1}{8}(x_1 - x_2) \geq 1.875,
\]
\[
\frac{1}{8}(79x_1 + 129x_2) \geq 18.125,
\]
\[
x_1 + 8x_2 \geq 10.
\]

The optimal objective value of (S4-4) is \( C = -15 \) and for optimizing the fifth objective, we solve the problem (S4-5).

**(S4-5)** \[
\begin{align*}
\text{max} & \quad D = 5x_1 + 8x_2 - 3|x_1| - 2|x_2| \\
\text{s.t.} & \quad \text{all the constraints of (S4)}, \\
& \quad \frac{1}{8}(x_1 - x_2) \geq 1.875, \\
& \quad \frac{1}{8}(79x_1 + 129x_2) \geq 18.125, \\
& \quad 5x_1 + 8x_2 \geq 10, \\
& \quad 5x_1 + 8x_2 - 2|x_1| - |x_2| \geq -15.
\end{align*}
\]

The optimal solution of (S4-5) gives the optimal objective value as \( D = -30 \) and to optimize the sixth objective, the problem (S4-6) is solved.

**(S4-6)** \[
\begin{align*}
\text{max} & \quad G = \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2| \\
\text{s.t.} & \quad \text{all the constraints of (S4)}, \\
& \quad \frac{1}{8}(x_1 - x_2) \geq 1.875, \\
& \quad \frac{1}{8}(79x_1 + 129x_2) \geq 18.125, \\
& \quad 5x_1 + 8x_2 \geq 10, \\
& \quad 5x_1 + 8x_2 - 2|x_1| - |x_2| \geq -15, \\
& \quad 5x_1 + 8x_2 - 3|x_1| - 2|x_2| \geq -30.
\end{align*}
\]

The optimal solution of (S4-6) gives \( G = -55 \). Finally, the seventh objective is optimized by solving the problem (S4-7).

**(S4-7)** \[
\begin{align*}
\text{max} & \quad H = 5x_1 + 8x_2 - 5|x_1| - 4|x_2| \\
\text{s.t.} & \quad \text{all the constraints of (S4)}, \\
& \quad \frac{1}{8}(x_1 - x_2) \geq 1.875, \\
& \quad \frac{1}{8}(79x_1 + 129x_2) \geq 18.125, \\
& \quad 5x_1 + 8x_2 \geq 10, \\
& \quad 5x_1 + 8x_2 - 2|x_1| - |x_2| \geq -15, \\
& \quad 5x_1 + 8x_2 - 3|x_1| - 2|x_2| \geq -30, \\
& \quad \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2| \geq -55.
\end{align*}
\]

The optimal objective value and optimal solution of problem (S4-7) are, respectively, equal to \( H = -60 \), and \( x_1 = 10, \ x_2 = -5 \).

**Step 7.** By putting the optimal solution values as obtained in Step 6 (\( x_1 = 10 \) and \( x_2 = -5 \)), we get the unique optimal IVIF objective function value as:

\[
\tilde{Z} = (10; 25, 25, 40, 40; 70, 70, 65, 50)_{LR}.
\]
Table 2  Data related to the requirement of skilled labour hours per bicycle

| Type of bicycle | Time in which no unit prepared completely | Estimated time range to prepare each unit | Peak production time | Time unacceptable by the company for each unit a |
|-----------------|------------------------------------------|------------------------------------------|---------------------|-----------------------------------------------|
| Road bike       | 1.5                                      | 1.75–2.25                                | 2                   | 2.5                                           |
| Mountain bike   | 3                                        | 3.75–4.5                                 | 4                   | 4.7                                           |

aThis represents the labour hours which, if employed for a unit of the bicycle then it will reduce the efficiency of the production run.

Table 3  Raw material requirement (in kilograms) per unit of bicycle

| Type of bicycle | Type of material | Amount of material that can’t produce one complete frame of bicycle | Estimated range of material required per unit of bike | Amount of material giving maximum production rate | Units of material can’t be utilized per bicycle a |
|-----------------|------------------|---------------------------------------------------------------------|------------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| Road bike       | Steel alloy      | 5                                                                   | 6–8                                                  | 7                                                | 8.5                                              |
|                 | Rubber           | 1.5                                                                 | 1.75–2.3                                             | 2                                                | 2.7                                              |
| Mountain bike   | Steel alloy      | 6.5                                                                 | 7.2–8.9                                              | 8                                                | 10                                               |
|                 | Rubber           | 3                                                                   | 3.5–4.7                                              | 4                                                | 5                                                |

aIt describes the units of the raw material which if used per frame of the bicycle then it will reduce the profit of the company.

7 An application in production planning

A bicycle manufacturing company produces two models of bicycles, namely road bikes and mountain bikes. Steel alloy and rubber constitute the primary raw materials required in the production process of the main body of the bicycles. It is observed from past experiences that no complete unit of a road bicycle can be manufactured if 1.5 h of skilled labourers are employed per unit. On the other hand, the company can’t bear to spend more than 2.5 h of labour time per unit of the road bike as it will reduce the efficiency of the process. From the past data, it is estimated that the complete manufacturing of each unit of a road bike requires about 1.75–2.25 h of skilled labourers. It was also judged that the production curve of road bikes peaks near 2 h of labour time per unit. Further, it is known from the judgement of the decision-maker that values of all the parameters follow linear variations and involve uncertainty as well as some inherent hesitation. In this context, all the information related to the skilled labour hours is summarized in Table 2 while the data referring to the requirement of raw material is tabulated in Table 3.

The manager of the company has approximated that about 100 h of labour time is available for the production process. Further, due to various uncontrollable factors, there occurs a fluctuation in the supply and consumption of Steel alloy and rubber in the manufacturing firm. Thus, the manager has estimated that around 300 units of Steel alloy and nearly 120 units of rubber will be available for this production run. However, the manufacturing firm can purchase the additional required units of Steel alloy at a price of about $10 or can sell leftover units at the same price. The company estimated the selling price per unit of road bikes at a price nearly $80 and that of mountain bikes at about $120. Finally, the firm manager wants...
to find the number of optimal units of Steel alloy that need to be purchased or sold and units of road bikes and mountain bikes to be produced so as to maximize the total profit.

**Problem formulation:** Since all the parameters of the problem are known to be in the estimated/uncertain form, it is more relevant to represent these estimated numbers by $LR$-type IVIFNs. Following the data given in Tables 2 and 3, the given estimated parameters can be presented as follows:

1. Labour time (h):
   (a) For road bike: $\bar{2} = (2; 0.1, 0.1, 0.25, 0.25; 0.5, 0.5, 0.3, 0.3)_{LR}$.
   (b) For mountain bike: $\bar{4} = (4; 0.2, 0.2, 0.25, 0.5; 1, 0.7, 0.5, 0.6)_{LR}$.

2. Raw material (kg):
   (a) For road bike:
      (i) Steel alloy: $\bar{7} = (7; 0.5, 0.5, 1, 1; 2, 1.5, 1.5, 1)_{LR}$.
      (ii) Rubber: $\bar{2} = (2; 0.1, 0.2, 0.25, 0.3; 0.5, 0.7, 0.3, 0.5)_{LR}$.
   (b) For mountain bike:
      (i) Steel alloy: $\bar{8} = (8; 0.5, 0.5, 0.8, 0.9; 1.5, 2, 1.2, 1.5)_{LR}$.
      (ii) Rubber: $\bar{4} = (4; 0.3, 0.3, 0.5, 0.7; 1, 1, 0.7, 0.8)_{LR}$.

3. Availability:
   (a) Labour time (h): $\bar{100} = (100; 8, 8, 10, 10; 20, 22, 15, 15)_{LR}$.
   (b) Steel alloy (kg): $\bar{300} = (300; 10, 12, 15, 15; 30, 30, 20, 25)_{LR}$.
   (c) Rubber (kg): $\bar{120} = (120; 10, 8, 15, 15; 30, 30, 18, 20)_{LR}$.

4. Estimated cost ($):
   (a) For steel alloy: $\bar{10} = (10; 1, 1.5, 2, 2; 4, 5, 3, 3.5)_{LR}$.
   (b) For road bike: $\bar{80} = (80; 5, 5, 7, 7; 10, 10, 8, 9)_{LR}$.
   (c) For mountain bike: $\bar{120} = (120; 8, 7, 10, 10; 15, 15, 12, 11)_{LR}$.

Based on these data and as per the statement of the problem, it can be formulated as follows:

**(M1)** $\max \quad \bar{Z} = \bar{80} \odot \bar{x}_1 \oplus \bar{120} \odot \bar{x}_2 \oplus \bar{10} \odot \bar{y}_1$

$s.t. \quad \bar{7} \odot \bar{x}_1 \oplus \bar{8} \odot \bar{x}_2 \oplus \bar{y}_1 = \bar{300}$,
\[ \bar{2} \odot \bar{x}_1 \oplus \bar{4} \odot \bar{x}_2 \leq \bar{120}, \]
\[ \bar{2} \odot \bar{x}_1 \oplus \bar{4} \odot \bar{x}_2 \leq \bar{100}, \]
\[ \bar{x}_1, \bar{x}_2 \geq 0, \quad \bar{y}_1 \text{ unrestricted} \]

where

\[ \bar{x}_1 = (x_1; \xi_{1L}^{\mu}, \eta_{1L}^{\mu}, \xi_{1U}^{\mu}, \eta_{1U}^{\mu}; \xi_{1L}^{\nu}, \eta_{1L}^{\nu}, \xi_{1U}^{\nu}, \eta_{1U}^{\nu})_{LR} \]

= Number of units of road bike to be produced,

\[ \bar{x}_2 = (x_2; \xi_{2L}^{\mu}, \eta_{2L}^{\mu}, \xi_{2U}^{\mu}, \eta_{2U}^{\mu}; \xi_{2L}^{\nu}, \eta_{2L}^{\nu}, \xi_{2U}^{\nu}, \eta_{2U}^{\nu})_{LR} \]

= Number of units of mountain bike to be produced,

\[ \bar{y}_1 = (y_1; \pi_{1L}^{\mu}, \theta_{1L}^{\mu}, \pi_{1U}^{\mu}, \theta_{1U}^{\mu}; \pi_{1L}^{\nu}, \theta_{1L}^{\nu}, \pi_{1U}^{\nu}, \theta_{1U}^{\nu})_{LR} \]

= Number of additional units of Steel alloy to be purchased or sold.
Solution: Substituting the values of the parameters, the problem (M1) can be further expressed as:

(M2) \[ \max \tilde{Z} = (80; 5, 5, 7, 7; 10, 10, 8, 9)_{LR} \odot \tilde{x}_1 \oplus \\
(120; 8, 7, 10, 10; 15, 15, 12, 11)_{LR} \odot \tilde{x}_2 \oplus \\
(10; 1, 1.5, 2, 2; 4, 5, 3, 3.5)_{LR} \odot \tilde{y}_1 \]
\[ \text{s.t.} \quad (7; 0.5, 0.5, 1, 1; 2, 1.5, 1.5, 1)_{LR} \odot \tilde{x}_1 \oplus \\
(8; 0.5, 0.5, 0.8, 0.9; 1.5, 2, 1.2, 1.5)_{LR} \odot \tilde{x}_2 \oplus \tilde{y}_1 \\
= (300; 10, 12, 15, 15; 30, 30, 20, 25)_{LR}, \\
(2; 0.1, 0.2, 0.25, 0.3; 0.5, 0.7, 0.3, 0.5)_{LR} \odot \tilde{x}_1 \oplus \\
(4; 0.3, 0.3, 0.5, 0.7; 1, 1, 0.7, 0.8)_{LR} \odot \tilde{x}_2 \\
\leq (120; 10, 8, 15, 15; 30, 30, 18, 20)_{LR}, \\
(2; 0.1, 0.1, 0.25, 0.25; 0.5, 0.5, 0.3, 0.3)_{LR} \odot \tilde{x}_1 \oplus \\
(4; 0.2, 0.2, 0.25, 0.5; 1, 0.7, 0.5, 0.6)_{LR} \odot \tilde{x}_2 \\
\leq (100; 8, 8, 10, 10; 20, 22, 15, 15)_{LR}, \\
\tilde{x}_1, \tilde{x}_2 \geq 0, \quad \tilde{y}_1 \text{ unrestricted.} \]

Now, since the input data follow the linear trend, \( L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1 - x\} \forall x \in \mathbb{R} \).

Further, using “LINGO-17.0” software for solving the equivalent crisp model obtained after applying Steps 1–7 of the proposed algorithm, we get the following optimal solution of problem (M1):

\( \tilde{x}_1 = (1.71; 0, 0, 0, 0; 1.7, 0, 0.64, 0)_{LR}, \)
\( \tilde{x}_2 = (3.05; 1.02, 1.13, 1.5, 1.18; 2.08, 2.13, 1.5, 1.97)_{LR} \)
and
\( \tilde{y}_1 = (263.69; 0, 0, 0, 0; 0, 0, 0, 0)_{LR} \)

with the optimal IVIF value of the objective function equals

\( \tilde{Z} = (3138.94; 410.44, 569.18, 735.66, 723.92; 1454.84, 1669.39, 1050.39, 1229.97)_{LR}. \)

7.1 Results and discussion

The optimal solution to the problem (M1) calls for selling nearly 263 kg of leftover Steel alloy, manufacturing about 2 units of the road bikes and 3 units of mountain bikes to gain the maximum profit in the given scenario. The graphical representation of the objective function value \( \tilde{Z} \) as an LR-type IVIFN is shown in Fig. 4. The interpretation of the profit function can be viewed as follows:

The company’s acceptance increases if the profit value increases from nearly $2403.28 to $3138.94 while the degree of attainability of profit decreases if profit further increases from $3138.94 to $3862.86. The manager is fully satisfied at a profit of $3138.94. However, when the profit increases from nearly $1684.1 to $3138.94, the degree of non-attainability (or rejection) decreases continuously while the non-attainability degree increases if the profit grows from $3138.94 to $4808.33. Further, the company’s profit cannot go below $1684.1 and a profit above $4808.33 is also not achievable.
7.2 Managerial insights

Our modelling of linear optimization problems in LR-type IVIF environment integrates two significant variations in the data to solve the realistic problems. First, this modelling allows different types of variation in the input data using the suitable choice of $L$ and $R$ functions. Second, the model parameters are taken as IVIFNs with interval degrees which handle the uncertain data in the most appropriate manner. Hence, it is crucial for a policy-maker to understand and judge how the optimal strategy varies using different $L$, $R$ functions and to evaluate the optimal solution which suits best the concerned organization.

The formulation, solution and analysis of the production planning problem (M1) successfully incorporate the uncertain and vague data in the model and further provides a flexible and optimal production strategy to the company manager.

7.3 Comparison with other cases

The problem (M1) is also solved by transforming the inequality constraints of the model into equality by introducing the non-negative LR-type IVIF slack variables. Additionally, the model is also solved using the usual order relation “$\leq$” in place of “$\leq_{lex}$”. The values obtained are mentioned in Table 4.

Further, by comparing the optimal values of $\tilde{Z}$, it is observed that the solution obtained by the proposed methodology yields better results. However, it may also be noticed that the optimal solutions are close to each other. Moreover, using the Definition 3.13 for comparing the objective function values, we get

$$S(\tilde{Z}_{slack}) = -75.34 < S(\tilde{Z}_{order \ relation \ \leq}) = -58.96 < S(\tilde{Z}_{proposed}) = -30.89$$

implying that $\tilde{Z}_{slack} \prec \tilde{Z}_{order \ relation \ \leq} \prec \tilde{Z}_{proposed}$. Therefore, it can be inferred that the proposed algorithm yields better results than the two alternative approaches.
Table 4  Optimal solution obtained using other cases and proposed algorithm

| Solution | Proposed method | Adding non-negative $L_R$-type IVIF slack variables | Using order relation $\preceq$ instead of $\preceq_{lex}$ |
|----------|-----------------|----------------------------------------------------|--------------------------------------------------|
| $\hat{x}_1$ | (1.71; 0, 0, 0, 0; \ 1.7, 0, 0.64, 0)$_{L_R}$ | (0; 0, 0, 0, 0; \ 0, 0, 3, 0.875)$_{L_R}$ | (0; 0, 0, 0, 0.1; \ 0, 0.794, 0, 0.687)$_{L_R}$ |
| $\hat{x}_2$ | (3.05; 1.02, 1.13, 1.5, 1.18; \ 2.08, 2.13, 1.5, 1.97)$_{L_R}$ | (3.682; 0, 0, 0, 0; \ 0, 0, 0, 0)$_{L_R}$ | (5.75; 0, 0, 0, 0; \ 1.5, 0, 0.294, 0)$_{L_R}$ |
| $\tilde{y}_1$ | (263.69; 0, 0, 0, 0; \ 0, 0, 0, 0)$_{L_R}$ | (246.67; 7, 8.67, 9.67, 9; \ 16.67, 20, 12, 15)$_{L_R}$ | (240; 6.25, 8.25, 9, 8.25; \ 9, 18.25, 9, 15.25)$_{L_R}$ |
| $\hat{Z}$ | (3138.94; 410.44, 569.18, 735.66, 723.92; \ 1454.84, 1669.39, 1050.39, 1229.97)$_{L_R}$ | (2908.54; 339.13, 495.48, 607.52, 638.16; \ 1141.93, 1615.58, 868.19, 1184.22)$_{L_R}$ | (3090; 342.25, 495.12, 609.5, 645.2; \ 1257.75, 1631.46, 883.75, 1170.27)$_{L_R}$ |
8 Conclusions and future research scope

In this study, the concept of LR-type interval-valued intuitionistic fuzzy numbers are introduced and further various fundamental properties are explored. First, the expressions of $\alpha$-cut and $\beta$-cut for these fuzzy numbers are obtained in terms of crisp intervals and then, two significant defuzzification functions, namely score and accuracy indices are defined. In succession to this, the fundamental arithmetic operations, viz., addition, subtraction, and multiplication operations have been developed for the LR-type IVIFNs by employing the technique of $\alpha$-cut and $\beta$-cut along with interval arithmetic. After that, a lexicographic ranking criteria-based ordering technique is proposed to rank the LR-type IVIFNs. The presented lexicographic ranking of the LR-type IVIFNs uses an ordered tuple of seven parameters $(S, A, M, C, D, G, H)$. However, the choice of this ranking criterion is not fixed. There are a total of $7!$ permutations possible. It totally depends on the decision-maker’s attitude towards the preferences of various parameters involved in the lexicographic ranking. Furthermore, the mathematical formulation of a linear programming problem having unrestricted decision variables is put forward under a fully LR-type IVIF environment. To find a unique optimal solution of the proposed optimization problem, a lexicographic solution methodology is introduced which converts the fuzzy constraints into their equivalent set of crisp constraints and involves the optimization of a deterministic objective function having seven components that are solved in a lexicographic manner. The key significance of the current article is that it generalizes the results and theory of fuzzy, intuitionistic fuzzy, and LR-type fuzzy/intuitionistic fuzzy numbers. Additionally, it is to be pointed out that all the existing models of LPPs under uncertainty (Kaur and Kumar 2016; Pérez-Cañedo and Concepción-Morales 2019a; Nagoorgani and Ponnalagu 2012; Singh and Yadav 2017; Pérez-Cañedo and Concepción-Morales 2019b) can also be handled using the proposed algorithm. However, many real-world problems which fail to have crisp parameters can be solved efficiently by representing their uncertain data in terms of the LR-type IVIFNs and further, the proposed technique can be successfully applied to optimize them. Moreover, the practical applicability of the proposed model and solution algorithm is demonstrated by solving the production planning problem of a manufacturing company. Subsequently, the optimal solution values given by the developed approach are also compared with the two alternative approaches that can be used to solve the concerned problem. The comparative analysis concludes that the proposed technique gives better results than both the alternate methods.

Although the proposed solution methodology successfully reaches a unique optimal solution for the linear optimization problems under uncertain conditions, this modelling and development of the optimization algorithm still have certain limitations and complexities involved in it. Some of the drawbacks of the developed algorithm and consequently the related future directions of the present work are mentioned as follows:

1. To convert the fuzzified constraints into their equivalent deterministic/crisp counterparts, a conversion is applied using lexicographic ranking followed by the introduction of various binary variables. This transformation results in the involvement of a large number of constraints and variables in the final model. In the future, one may try to find an alternative technique/modification to the proposed conversion approach so as to significantly reduce the associated computation work.
2. To obtain the final optimal value, the methodology requires solving seven mixed 0-1 integer non-linear programming problems which leads to a large number of computation steps and hence, increases the calculation complexity (number of solver iterations used,
CPU runtime, etc.) of the problem. Therefore, it will be interesting to devise a more computationally efficient approach to handle such problems.

3. In the future, different optimization problems such as supply chain problems, portfolio optimization problems, transportation problems, etc. can be solved under the \(LR\)-type IVIF environment.

4. The current article focuses on solving the single-objective LPPs, however, many realistic problems involve multiple conflicting objectives. As a result, an important futuristic research aspect would be to formulate and devise a solution algorithm for multi-objective LPPs under the IVIF scenario.

5. To handle the realistic problems of prospective strategy and decision-making, logistics, groundwater management, and many more, one needs to revoke the notion of quadratic and fractional optimization models. Thus, in the future, the methodology can be extended to solve quadratic programming problems and non-linear/fractional programming problems having parameters/variables as \(LR\)-type IVIFNs.

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References

Akbari MG, Hesamian G (2018) Signed-distance measures oriented to rank interval-valued fuzzy numbers. IEEE Trans Fuzzy Syst 26:3506–3513

Akram M, Allahviranloo T, Pedrycz W, Ali M (2021a) Methods for solving \(LR\)-bipolar fuzzy linear systems. Soft Comput 25:85–108

Akram M, Ullah I, Alharbi MG (2021b) Methods for solving \(LR\)-type Pythagorean fuzzy linear programming problems with mixed constraints. Math Probl Eng 2021:1–29

Akram M, Ullah I, Allahviranloo T, Edalatpanah S (2021c) Fully Pythagorean fuzzy linear programming problems with equality constraints. Comput Appl Math 40:1–30

Akram M, Ullah I, Allahviranloo T, Edalatpanah SA (2021d) \(LR\)-type fully Pythagorean fuzzy linear programming problems with equality constraints. J Intell Fuzzy Syst 41:1975–1992

Akram M, Ullah I, Allahviranloo T (2022a) A new method for the solution of fully fuzzy linear programming models. Comput Appl Math 41(1):55

Akram M, Ullah I, Allahviranloo T (2022b) A new method to solve linear programming problems in the environment of picture fuzzy sets. Iran J Fuzzy Syst 19:29–49

Allahviranloo T, Lotti FH, Kiasary MK, Kiani N, Alizadeh L (2008) Solving fully fuzzy linear programming problem by the ranking function. Appl Math Sci 2:19–32

Angelov PP (1997) Optimization in an intuitionistic fuzzy environment. Fuzzy Sets Syst 86:299–306

Arana-Jiménez M (2018) Nondominated solutions in a fully fuzzy linear programming problem. Math Methods Appl Sci 41:7421–7430

Arefi M, Taheri SM (2014) Least-squares regression based on Atanassov’s intuitionistic fuzzy inputs-outputs and Atanassov’s intuitionistic fuzzy parameters. IEEE Trans Fuzzy Syst 23:1142–1154

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96
Atanassov KT (1999) Intuitionistic fuzzy sets: theory and applications. Studies in fuzziness and soft computing. Physica-Verlag, Heidelberg
Atanassov K, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31:343–349
Aydn T, Enginoğlu S (2022) Interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft matrices and their application to performance-based value assignment to noise-removal filters. Comput Appl Math 41:192
Bharati SK, Singh S (2018) Transportation problem under interval-valued intuitionistic fuzzy environment. Int J Fuzzy Syst 20:1511–1522
Bharati SK, Singh S (2019) Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment. Soft Comput 23:77–84
Bharati SK, Singh S (2020) Interval-valued intuitionistic fuzzy linear programming problem. New Math Nat Comput 16:53–71
Chen SM, Lee LW (2010) Fuzzy multiple attributes group decision-making based on the ranking values and the arithmetic operations of interval type-2 fuzzy sets. Expert Syst Appl 37:824–833
Chen SM, Wang CY (2013) Fuzzy decision making systems based on interval type-2 fuzzy sets. Inf Sci 242:1–21
Chen SM, Yang MW, Lee LW, Yang SW (2012) Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets. Expert Syst Appl 39:5295–5308
Das S (2021) Optimization of fuzzy linear fractional programming problem with fuzzy numbers. Big Data Comput Vis 1(1):30–35
Das SK, Mandal T, Edalatpanah S (2017) A mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers. Appl Intell 46:509–519
Ebrahimnejad A, Verdegay JL (2018) Fuzzy sets-based methods and techniques for modern analytics. Studies in fuzziness and soft computing, vol 364, 1st edn. Springer, Cham. https://doi.org/10.1007/978-3-319-73903-8
Enginoğlu S, Arslan B (2020) Intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices and their application in decision-making. Comput Appl Math 39:325
Ezzati R, Khorram E, Enayati R (2015) A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem. Appl Math Model 39:3183–3193
Farhadinia B (2009) Ranking fuzzy numbers based on lexicographical ordering. Int J Appl Math Comput Sci 5:248–251
Garg H, Rani M, Sharma S, Vishwakarma Y (2014) Intuitionistic fuzzy optimization technique for solving multi-objective reliability optimization problems in interval environment. Expert Syst Appl 41:3157–3167
Giri PK, Maiti MK, Maiti M (2015) Fully fuzzy fixed charge multi-item solid transportation problem. Appl Soft Comput 27:77–91
Gong Z, Zhao W (2018) A novel approach for solving fully fuzzy linear programming problem with LR flat fuzzy numbers. J Comput Anal Appl 24:11–22
Greco S, Figueira J, Ehrgott M (2016) Multiple criteria decision analysis. International series in operations research and management science, vol 233. Springer, New York. https://doi.org/10.1007/978-1-4939-3094-4
Hashemi SM, Modarres M, Nasrabadi E, Nasrabadi MM (2006) Fully fuzzified linear programming, solution and duality. J Intell Fuzzy Syst 17:253–261
Hesamian G (2016) Measuring similarity and ordering based on interval type-2 fuzzy numbers. IEEE Trans Fuzzy Syst 25:788–798
Hesamian G, Akbari MG (2022) A fuzzy empirical quantile-based regression model based on triangular fuzzy numbers. Comput Appl Math 41:267
Hosseinzadeh A, Edalatpanah S (2016) A new approach for solving fully fuzzy linear programming by using the lexicography method. Adv Fuzzy Syst
Hosseinzadeh A, Tayyebi J (2023) A compromise solution for the neutrosophic multi-objective programming problem and its application in transportation problem. J Appl Res Ind Eng 10(1):1–10
Ishibuchi H, Tanaka H (1990) Multiobjective programming in optimization of the interval objective function. Eur J Oper Res 48:219–225
Jafar MN, Saeed M, Khan KM, Alamri FS, Khalifa HAEW (2022) Distance and similarity measures using maximum operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. IEEE Access 10:11220–11235
Kane L, Bado H, Diakite M, Konate M, Kane S, Traore K (2021) Solving semi-fuzzy linear programming problems. Int J Res Ind Eng 10(3):251–275
Kaur J, Kumar A (2012) Unique fuzzy optimal value of fully fuzzy linear programming problems. Control Cybern 41:497–508

Springer
Kaur, J., & Kumar, A. (2013). Mehar’s method for solving fully fuzzy linear programming problems with LR fuzzy parameters. *Appl Math Model*, 37:7142–7153.

Kaur, J., & Kumar, A. (2016). Unique fuzzy optimal value of fully fuzzy linear programming problems with equality constraints having *LR* flat fuzzy numbers. In: An introduction to fuzzy linear programming problems: theory, methods and applications, vol 340. Springer, Cham, pp 109–118.

Khan, I. U., Ahmad, T., & Maan, N. (2013). A simplified novel technique for solving fully fuzzy linear programming problems. *J Optim Theory Appl*, 159:536–546.

Kumar, A., & Kaur, J. (2014). Fuzzy optimal solution of fully fuzzy linear programming problems using ranking function. *J Intell Fuzzy Syst*, 26:337–344.

Kumar, A., & Hussain, R. J. (2016). Computationally simple approach for solving fully intuitionistic fuzzy real life transportation problems. *Int J Syst Assur Eng Manag*, 7:90–101.

Kumar, A., & Singh, P. (2011). A new method for solving fully fuzzy linear programming problems. *Appl Math Model*, 35:817–823.

Lotfi, F. H., Allahviranloo, T., Jondabeh, M. A., & Alizadeh, L. (2009). Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. *Appl Math Model*, 33:3151–3156.

Mahapatra, G., & Roy, T. (2009). Reliability evaluation using triangular intuitionistic fuzzy numbers arithmetic operations. *World Acad Sci Eng Technol*, 50:574–581.

Mahmodiirad, A., Allahviranloo, T., & Niroomand, S. (2019). A new effective solution method for fully intuitionistic fuzzy transportation problem. *Soft Comput*, 23:4521–4530.

Mahmoudi, F., & Nasseri, S. H. (2019). A new approach to solve fully fuzzy linear programming problem. *J Appl Ind Res Eng*, 6(2):139–149.

Mekawy, I. (2022). A novel method for solving multi-objective linear fractional programming problem under uncertainty. *J Fuzzy Ext Appl*, 3(2):169–176.

Mottaghi, A., Ezzati, R., & Khorram, E. (2015). A new method for solving fuzzy linear programming problems based on the fuzzy linear complementary problem (FLCP). *Int J Fuzzy Syst*, 17:236–245.

Nagoorgani, A., & Ponnalagu, K. (2012). A new approach on solving intuitionistic fuzzy linear programming problems. *Appl Math Sci*, 6:3467–3474.

Najafi, H. S., & Edalatpanah, S. (2013). A note on “a new method for solving fully fuzzy linear programming problems”. *Appl Math Model*, 37:7865–7867.

Najafi, H. S., Edalatpanah, S., & Dutta, H. (2016). A nonlinear model for fully fuzzy linear programming with fully unrestricted variables and parameters. *Alex Eng J*, 55:2589–2595.

Niroomand, S. (2018). A multi-objective based direct solution approach for linear programming with intuitionistic fuzzy parameters. *Int J Fuzzy Syst*, 35:1923–1934.

Ozkok, B. A., Albayrak, I., Kocken, H. G., & Ahlatcioglu, M. (2016). An approach for finding fuzzy optimal and approximate fuzzy optimal solution of fully fuzzy linear programming problems with mixed constraints. *J Intell Fuzzy Syst*, 31:623–632.

Pérez-Cañedo, B., & Concepción-Morales, E. R. (2019a). A method to find the unique optimal fuzzy value of fully fuzzy linear programming problems with inequality constraints having unrestricted LR fuzzy parameters and decision variables. *Expert Syst Appl*, 123:256–269.

Pérez-Cañedo, B., & Concepción-Morales, E. R. (2019b). On LR-type fully intuitionistic fuzzy linear programming with inequality constraints: solutions with unique optimal values. *Expert Syst Appl*, 128:246–255.

Rahmani, A., Hosseinzadeh Lotfi, F., & Stoczynski, M. A. (2016). A new method for defuzzification and ranking of fuzzy numbers based on the statistical beta distribution. *Adv Fuzzy Syst*.

Ranjarb, M., & Effati, S. (2022). Fully hesitant fuzzy linear programming with hesitant fuzzy numbers. *Eng Appl Intell Sci*, 114:105047.

Rezaei, A., Oner, T., & Katican, T. (2022). A short history of fuzzy, intuitionistic fuzzy, neutrosophic and phlogistic sets. *Int J Neutrosophic Sci*, 18(1):99–116.

Roy, S. K., & Ebrahimi Nejad, A. (2018). New approach for solving intuitionistic fuzzy multi-objective transportation problem. *Sadhanã*, 43:1–12.

Saghi, S., Nazemi, A., & Effati, S. (2023). Simplex algorithm for hesitant fuzzy linear programming problem with hesitant cost coefficient. *Iran J Fuzzy Syst*, 20:137–152.

Saheb, R. (2016). Fuzzy multicriteria decision making method based on the improved accuracy function for interval-valued intuitionistic fuzzy sets. *Soft Comput*, 20:2557–2563.

Sidhu, S. K., & Kumar, A. (2019). Mehar methods to solve intuitionistic fuzzy linear programming problems with trapezoidal intuitionistic fuzzy numbers. In: Deep K., Jain M., Salhi S. (eds) Performance prediction and analytics of fuzzy, reliability and queuing models. *Asset Analytics*. Springer Berlin, pp 265–282.

Singh, S. K., & Yadav, S. P. (2015). Modeling and optimization of multi objective non-linear programming problem in intuitionistic fuzzy environment. *Appl Math Model*, 39:4617–4629.

Singh, V., & Yadav, S. P. (2017). Development and optimization of unrestricted LR-type intuitionistic fuzzy mathematical programming problems. *Expert Syst Appl*, 80:147–161.
Suresh M, Vengataasalam S, Prakash KA (2014) Solving intuitionistic fuzzy linear programming problems by ranking function. J Intell Fuzzy Syst 27:3081–3087
Tadesse A, Acharya M, Sahoo M, Acharya S (2021) Fuzzy linear programming problem with fuzzy decision variables: a geometrical approach. J Stat Manag Syst 24:853–863
Tamilarasi G, Paulraj S (2022) An improved solution for the neutrosophic linear programming problems based on Mellon’s transform. Soft Comput 26(17):8497–8507
Tanaka H, Asai K (1984) Fuzzy solution in fuzzy linear programming problems. IEEE Trans Syst Man Cybern 2:325–328
Voskoglou M (2020) Assessment and linear programming under fuzzy conditions. https://doi.org/10.22105/jfea.2020.253436.1024
Wan SP, Wang F, Lin LL, Dong JY (2015) An intuitionistic fuzzy linear programming method for logistics outsourcing provider selection. Knowl Based Syst 82:80–94
Wang P, Lin Y, Fu M, Wang Z (2023) VIKOR method for plithogenic probabilistic linguistic MAGDM and application to sustainable supply chain financial risk evaluation. Int J Fuzzy Syst 25:780–793
Yang X, Lin TY, Yang J, Li Y, Yu D (2009) Combination of interval-valued fuzzy set and soft set. Comput Math Appl 58:521–527
Zadeh LA (1965) Information and control. Fuzzy Sets 8:338–353
Zhang SF, Liu SY, Zhai RH (2011) An extended GRA method for MCDM with interval-valued triangular fuzzy assessments and unknown weights. Comput Ind Eng 61:1336–1341
Zhao J, Li B, Rahman AU, Saeed M (2023) An intelligent multiple-criteria decision-making approach based on sv-neutrosophic hypersoft set with possibility degree setting for investment selection. Manag Decis 61:472–485
Zimmermann HJ (1975) Description and optimization of fuzzy systems. Int J Gen Syst 2:209–215
Zulqarnain RM, Siddique I, Ali R, Jarad F, Iampan A (2023) Aggregation operators for interval-valued Pythagorean fuzzy hypersoft set with their application to solve MCDM problem. CMES Comput Model Eng Sci 135(1):619–651

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