Polynomial SUSY in Quantum Mechanics and Second Derivative Darboux Transformation

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We give the classification of second-order polynomial SUSY Quantum Mechanics in one and two dimensions. The particular attention is paid to the irreducible supercharges which cannot be built by repetition of ordinary Darboux transformations. In two dimensions it is found that the binomial superalgebra leads to the dynamic symmetry generated by a central charge operator.

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1. Introduction

The Supersymmetrical Quantum Mechanics (SSQM) [1],[2] represents a concise algebraic form of spectral equivalence between different hamiltonian quantum systems realized by means of Darboux transformation [3],[4]. Certainly SSQM is used also for verification of some properties predicted by SUSY in the QFT [5].

On the other hand the variety of extensions of SUSY-algebra seems to be broader in QM than in the QFT. Recently the polynomial SUSY-algebra has been proposed and elaborated in the one-dimensional SSQM [6]. Its consequences for the scattering characteristics of spectrally-equivalent systems have been outlined in [7].

The aim of our paper is to give the complete classification of second-order polynomial SSQM in one-dimensional and to generalize the polynomial SUSY onto two-dimensional QM. After short description of related superalgebra we select out the subclass of reducible SUSY algebras which is built by repetition of two single Darboux transformations or equivalently by gluing of two ordinary SUSY systems [3], (see also [8]). We distinguish the irreducible subclass which cannot be factorized in this way. In the one-dimensional case both subclasses are rich of representatives as it is shown in Sect.2. The two-dimensional version of the polynomial SUSY is built in Sect.3 and it is found that the reducible subclass is restricted with potentials combining the harmonic oscillator and the centrifugal-type terms.

The irreducible class appears to be more significant and is generated by the second derivative supercharges. The set of nonlinear equations on potentials and coefficient functions of supercharges are derived. In Sect.4 the corresponding binomial superalgebra is analyzed and nontrivial operator of dynamic central charge is discovered. When the senior derivatives of supercharge form the Laplacian the separation of variables is provided in particular coordinate systems. The explicit form of potentials is described and the operators of supercharge and central charge are evaluated. In Sect.5 the second derivative supercharges with metric $g_{ik}(\vec{x})$ are introduced and their possible structure is analyzed. It is found that in the general case the polynomial superalgebra defines the dynamic symmetry operator of Hamiltonian system. Therefore the building of isospectral systems is tightly connected with another problem namely with a search for dynamic symmetry operator $R$.

2. Polynomial SUSY in one dimension

Let us remind the basic notations of the ordinary SUSY QM [4],[5] and describe its polynomial extension [6]. The intertwining relations between two Hamiltonians with equivalent spectra are realized by means of well-known Darboux transformation [3]:

$$
\begin{align*}
 h^{(1)} &= q^+ q^- = -\partial^2 + V^{(1)}(x); & h^{(1)} \Psi_n^{(1)}(x) &= E_n \Psi_n^{(1)}(x); \\
 h^{(2)} &= q^- q^+ = -\partial^2 + V^{(2)}(x); & h^{(2)} \Psi_n^{(2)}(x) &= E_n \Psi_n^{(2)}(x); \\
 h^{(1)} q^+ &= q^+ h^{(2)}; & q^- h^{(1)} &= h^{(2)} q^-; \\
 q^+ &= -\partial + \partial \chi(x); & q^- = (q^+)^\dagger &= \partial + \partial \chi(x); \\
 \Psi_n^{(2)}(x) &= q^- \Psi_n^{(1)}(x); & \Psi_n^{(1)}(x) &= q^+ \Psi_n^{(2)}(x),
\end{align*}
$$

(1)
where $\partial \equiv d/dx$.

The concise algebraic form of spectral equivalence is given by superalgebra for the partners $h^{(1)}$, $h^{(2)}$ and off-diagonal supercharges:

\[
H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix}; \quad Q^+ = (Q^-)^\dagger = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix}; \\
\{Q^+, Q^-\} = H; \quad (Q^+)^2 = (Q^-)^2 = 0; \quad [H, Q^\pm] = 0. \tag{2}
\]

The superpotential $\chi(x)$ is defined by an arbitrary solution $\Psi(x)$ of the Schrödinger equation $h^{(1)}\Psi(x) = E\Psi(x)$

\[
\chi(x) \equiv -\ln \Psi(x). \tag{3}
\]

If this solution is nodeless one has almost coinciding spectra of $h^{(1)}$ and $h^{(2)}$ or, equivalently, double degeneracy of the energy spectrum of $H$. The polynomial superalgebra is created by the intertwining operators of higher order in derivatives. Let us consider the second-derivative superalgebra generated by the following operators

\[
q^+ = \partial^2 - 2f(x)\partial + b(x); \quad q^- = (q^+)^\dagger. \tag{4}
\]

The conserving supercharges $[H, Q^\pm] = 0$ determine the Hamiltonian as follows:

\[
\{Q^+, Q^-\} = (H + a)^2 + d, \tag{5}
\]

where the potentials can be expressed in terms of a real coefficient function $f(x)$:

\[
V^{(1,2)} = \mp 2f'(x) + f(x)^2 + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{d}{4f(x)^2} - a. \tag{6}
\]

The intertwining relations for $H$ require that

\[
b(x) = -f'(x) + f(x)^2 - \frac{f''(x)}{2f(x)} + \left(\frac{f'(x)}{2f(x)}\right)^2 + \frac{d}{4f(x)^2}. \tag{7}
\]

Depending on the sign of $d$ one finds either the reducible algebra ($d < 0$) or the irreducible one ($d > 0$). In the reducible case there exists an intermediate Hamiltonian $h$ which is a superpartner of both $h^{(1)}$ and $h^{(2)}$ with the ordinary superalgebra, respectively the second-order Darboux transformation can be factorized into a product of two ordinary Darboux transformation (see details in [3], [4]):

\[
q^+ = q_1^+ q_2^+ = (-\partial + W_1)(-\partial + W_2); \\
W_1(x) \equiv \partial\chi_1(x); \quad W_2(x) \equiv \partial\chi_2(x); \\
h^{(1)} = q_1^+ q_1^- + c/2; \quad h = q_1^- q_1^+ + c/2 = q_2^+ q_2^- - c/2; \quad h^{(2)} = q_2^- q_2^+ - c/2; \\
W_{1,2} = \pm \frac{2f'(x) - c}{4f(x)} - f(x). \tag{8}
\]

which corresponds to $a = 0$, $c^2 = -4d$. In the irreducible case ($d > 0$) $c$ becomes imaginary and the hermitian intermediate Hamiltonian $h$ does not exist. In fact the analytic continuation from $c$ to $ic$ leads to a complex potential for $h$. Respectively $W_1$ and $W_2$ are complex
functions and \( q_i^\pm = \mp \partial + W_i \) \((i = 1, 2)\), but \( q_i^+ \neq (q_i^-)^\dagger \). Nevertheless the second-order supercharges are hermitian
\[
q_1^+ q_2^+ = (q_2^- q_1^-)^\dagger. \tag{9}
\]
From the hermiticity of \( h^{(1)}, h^{(2)} \) and Eqs.\( (8),(10) \) it follows that
\[
Re W_1 = -\frac{f'(x)}{2f(x)} + f(x); \quad Im W_1 = -\frac{c}{4f(x)},
\]
where \( f(x) \) is a real function as before and potentials are described by Eq.\( (8) \).

In general there is the large variety of pairs of potentials \( V^{(1)} \) and \( V^{(2)} \) which obey Eq.\( (8) \) for any sign of \( d \). The higher-order polynomial SUSY algebra can be created by composition of ordinary (first-order in derivatives) SUSY transformations and of irreducible ones, which are of second-order in derivatives:
\[
\{Q^+, Q^-\} = \prod_{i+2j=n} (H + c_j)((H + a_i)^2 + d_j); \quad d_j > 0. \tag{10}
\]

### 3. Two-dimensional Darboux transformations of second-order in derivatives

The conventional two-dimensional SUSY algebra provides the equivalence of energy spectra between a pair of two scalar Hamiltonians \( h^{(1)}, \tilde{h}^{(1)} \) and \( 2 \times 2 \) matrix Hamiltonian \( h_{ik} \) \((i, k = 1, 2)\) [9]:
\[
\begin{align*}
 h^{(1)} &= -\partial_t^2 + V^{(1)}(x) = -\partial_t^2 + (\partial_\chi)^2 - \partial_\chi^2; \\
 \tilde{h}^{(1)} &= -\partial_t^2 + \tilde{V}^{(1)}(x) = -\partial_t^2 + (\partial_\chi)^2 + \partial_\chi^2; \\
 h_{ik} &= -\delta_{ik}\partial_t^2 + \delta_{ik}(\partial_\chi)^2 - \partial_t^2 + 2\partial_\chi \partial_\chi,
\end{align*}
\]
where the summation on index \( l \) is implied, \( \partial_\chi^2 \equiv \partial_t^2 + \partial_\chi^2 \).

The superalgebra is represented by intertwining relations:
\[
\begin{align*}
 h^{(1)} q_i^+ &= q_i^+ h_{ki}; \quad h_{ik} q_k^- = q_i^- h^{(1)}; \quad q_i^\pm \equiv \mp \partial_i + \partial_\chi(x); \\
 h_{ik} p_k^- &= p_i^- \tilde{h}^{(1)}; \quad p_k^+ h_{ki} = \tilde{h}^{(1)} p_i^+; \\
 q_i^\pm p_i^\mp &= 0; \quad p_i^\mp = (p_i^+)\dagger = \epsilon_{ik} q_k^+; \quad \epsilon_{ik} = -\epsilon_{ki}.
\end{align*}
\]

The energy spectra of \( h^{(1)} \) and \( \tilde{h}^{(1)} \) are different except for particular cases [11],[12]. However it is of interest to extend this superalgebra so that the spectral equivalence could be established between some scalar hamiltonians. One can attempt to build the polynomial superalgebra similarly to the one-dimensional reducible scheme:
\[
\begin{pmatrix} h^{(1)} & 0 \\ 0 & \tilde{h}^{(1)} \end{pmatrix} \longrightarrow h_{ik} = h_{ik}^{(*)} - 4\epsilon I \longrightarrow \begin{pmatrix} h^{(2)} & 0 \\ 0 & \tilde{h}^{(2)} \end{pmatrix}.
\]

In result we arrive to the following intertwining relations between upper components \( h^{(1)} \) and \( h^{(2)} \)
\[
 h^{(1)} q_+ = q_+ h^{(2)}; \quad h^{(1)} q_- = h^{(2)} q_-; \tag{11}
\]
where
\[ h^{(2)} = \tilde{q}_i q_i - 4cI; \quad q^+ = q^+_i q_i = (-\partial_i + \partial_i \chi)(\partial_i + \partial_i \tilde{\chi}); \quad q^- = \tilde{q}_i q_i. \] (12)

We have investigated the consistency conditions for superpotentials and have found the following solutions in this reducible case

\[ \chi = \frac{1}{2} a \rho^2 - \frac{c}{a} \ln \rho + \xi(\varphi); \]
\[ \tilde{\chi} = -\frac{1}{2} a \rho^2 - \frac{c}{a} \ln \rho + \xi(\varphi) + b; \]
\[ V^{(1)} = a^2 \rho^2 + \frac{1}{\rho^2} \left[ \xi'^2 - \xi'' + \frac{c^2}{a^2} \right] - 2(a + c); \]
\[ V^{(2)} = a^2 \rho^2 + \frac{1}{\rho^2} \left[ \xi'^2 - \xi'' + \frac{c^2}{a^2} \right] + 2(a - c), \]

where \( a, b \) are constants and \( \xi(\varphi) \) is arbitrary function of angular variable \( \varphi \).

In two dimensions the irreducible class of second-derivative superalgebra with intertwining components

\[ q^+ = -\triangle + C_i \partial_i + B; \quad q^- = (q^+)^\dagger; \quad \triangle \equiv \partial_i^2 \] (13)

can be constructed when imposing the conditions (11) irrespectively of the existence of superpartners \( \tilde{h}^{(1)} \) and \( \tilde{h}^{(2)} \). Thus we do not assume that \( q^+ = q^+_i q_i^- \) where \([q^+_i, q^-_j] = 0\) (the integrability condition). At this point the two-dimensional case is drastically different from the one-dimensional case.

The intertwining between \( h^{(1)} \) and \( h^{(2)} \) leads to six equations on five real functions \((C_{1,2}, B, V^{(1,2)})\) :

\[ -\partial_\mu C_\nu - \partial_\nu C_\mu + (V^{(2)} - V^{(1)}) \delta_{\mu\nu} = 0; \]
\[ \triangle C_\nu - 2\partial_\nu (V^{(2)} - B) - (V^{(1)} - V^{(2)}) C_\nu = 0; \]
\[ -\triangle (V^{(2)} - B) - (V^{(1)} - V^{(2)}) B + C_\nu \partial_\nu V^{(2)} = 0. \] (14)

Certainly they cannot be satisfied for arbitrary potentials \( V^{(1)} \) or \( V^{(2)} \). Indeed the solutions can be found explicitly and read :

\[ C^2 \equiv (C_1 + iC_2)^2 = \alpha z^2 + 8\beta z + \gamma; \quad z \equiv x_1 + ix_2; \]
\[ V^{(2)} - B = \frac{1}{4} \alpha \left| z \right|^2 + z \bar{\beta} + \bar{z} \beta + \frac{1}{4} \left| C \right|^2 - \eta; \]
\[ V^{(2)} - V^{(1)} = \partial_\zeta C + \partial_{\bar{\zeta}} \bar{C}, \] (15)

where \( \alpha, \eta \) are real constants and \( \beta, \gamma \) are complex constants. The further evaluation of solutions for \( V^{(1,2)} \) is based on the equation :

\[ (C \partial_\zeta + \bar{C} \partial_{\bar{\zeta}})(B \mid C \mid^2) = G \mid C \mid^2, \] (16)

where

\[ G = \alpha + (\partial_\zeta C)(\partial_{\bar{\zeta}} \bar{C}) - \frac{\alpha}{2} (\bar{z}C + z \bar{C}) - 2(\bar{\beta}C + \beta \bar{C}). \] (17)
In terms of variables
\[ \tau_1 = \int \frac{dz}{C} + \int \frac{d\bar{z}}{\bar{C}}; \quad i\tau_2 = \int \frac{dz}{C} - \int \frac{d\bar{z}}{\bar{C}}, \]
one obtains
\[ B = \frac{1}{2} \left| \frac{C}{C} \right|^2 \int G \left| C \right|^2 d\tau_1 + \frac{F(\tau_2)}{\left| C \right|^2}, \] (18)
where \( F(\tau_2) \) is an arbitrary function. Thus we find that the irreducible second-derivative SUSY algebra is realized in the class of potentials which is much broader comparing to the reducible case.

4. The second-derivative superalgebra

Let us describe superalgebra relations for the irreducible Darboux transformations (11). Since supercharges \( Q^\pm \) (with components (13)) commute with the Hamiltonian \( H \) one expects that the closing of superalgebra leads to the symmetry operator \( R \) (central charge):
\[ \{ Q^+, Q^- \} = F(H, R); \quad [H, R] = 0. \]

For the supercharges (13) one finds that
\[ F(H, R) = H^2 + R + 2\eta H, \] (19)
where \( R \) is a diagonal operator:
\[ R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}. \]
Its components
\[ R_1 = 2(\alpha \left| z \right|^2 + 4(\beta z + \beta \bar{z}))\partial_z \partial_{\bar{z}} - C^2 \partial_z^2 - \bar{C}^2 \partial_{\bar{z}}^2 - C(\partial_z C)\partial_z - \bar{C}(\partial_{\bar{z}} \bar{C})\partial_{\bar{z}} + (C\partial_z + \bar{C}\partial_{\bar{z}})(B + V^{(1)}) + B^2 - V^{(1)^2} - 2\eta V^{(1)}; \] (20)
\[ R_2 = 2(\alpha \left| z \right|^2 + 4(\beta z + \beta \bar{z}))\partial_z \partial_{\bar{z}} - C^2 \partial_z^2 - \bar{C}^2 \partial_{\bar{z}}^2 - C(\partial_z C)\partial_z - \bar{C}(\partial_{\bar{z}} \bar{C})\partial_{\bar{z}} - (C\partial_z + \bar{C}\partial_{\bar{z}})(B - V^{(2)}) + B^2 - V^{(2)^2} - 2\eta V^{(2)} \] (21)
are the symmetry operators of Hamiltonians \( h^{(1)}, h^{(2)} \) respectively.

Let us derive the explicit solution of Eqs. (14) and the related operators \( H, Q^\pm, R \). They have different form depending on values of \( \alpha \) and \( \beta \) in Eqs. (15).

i) If \( \alpha = 0, \beta \neq 0 \), then \( \gamma \) can be eliminated by translation \( z \to z - (\gamma/8\beta) \). In this case the variables
\[ \tau_1 = \sqrt{\frac{z}{2\beta}} + \sqrt{\frac{\bar{z}}{2\beta}}; \quad \tau_2 = i\left(\sqrt{\frac{\bar{z}}{2\beta}} - \sqrt{\frac{z}{2\beta}}\right) \] (22)
can be referred to the conventional parabolic coordinates \([12]\). From the Eq.(18) one obtains that
\[
B = \frac{2\tau_1 - |\beta|^2 \tau_1^4 - 2|\beta|^2 \tau_1^2 \tau_2^2 + F(\tau_2)}{\tau_1^2 + \tau_2^2}
\]
and from Eqs.(15) one finds the following expressions for potentials
\[
V^{(1)} = \frac{-2\tau_1 + |\beta|^2 \tau_1^4 + F(\tau_2)}{\tau_1^2 + \tau_2^2} - \eta; \quad (24)
\]
\[
V^{(2)} = \frac{2\tau_1 + |\beta|^2 \tau_1^4 + F(\tau_2)}{\tau_1^2 + \tau_2^2} - \eta. \quad (25)
\]

In terms of variables \(\tau_1, \tau_2\) the Laplacian is separable
\[
\Delta = \frac{1}{|\beta|^2 (\tau_1^2 + \tau_2^2)} (\partial_{\tau_1}^2 + \partial_{\tau_2}^2)
\]
with the same factor as in Eqs.(24),(25). Hence the spectral problem for both Hamiltonians \(h\Psi = E\Psi\) can be solved by \(R\)-separation \([12]\) of variables (further on we omit the indexes of \(h, R, \Psi\) for brevity). Namely one can decompose the corresponding eigenfunctions for a given \(E\) into a sum
\[
\Psi = \sum_n \nu_n \phi_{1n}(\tau_1) \phi_{2n}(\tau_2), \quad (26)
\]
where \(\nu_n\) are constants and \(\phi_{1n}(\tau_1), \phi_{2n}(\tau_2)\) are solutions of one-dimensional equations :
\[
-\phi''_{1n}(\tau_1) + |\beta|^2 [-(E + \eta)\tau_1^2 \mp 2\tau_1 + |\beta|^2 \tau_1^4] \phi_{1n}(\tau_1) = \frac{\lambda_n}{4} \phi_{1n}(\tau_1); \quad (27)
\]
\[
-\phi''_{2n}(\tau_2) + |\beta|^2 [-(E + \eta)\tau_2^2 + F(\tau_2)] \phi_{2n}(\tau_2) = -\frac{\lambda_n}{4} \phi_{2n}(\tau_2). \quad (28)
\]

where the upper (lower) sign in Eq.(27) corresponds to \(h^{(1)}(h^{(2)})\) and \(\lambda_n\) is a separation constant which serves as a spectral parameter for the symmetry operator \(R : R\phi_{1n}(\tau_1) \phi_{2n}(\tau_2) = (\lambda_n - \eta^2) \phi_{1n}(\tau_1) \phi_{2n}(\tau_2)\). Depending on properties of the function \(F(\tau_2)\) one can obtain either the finite-dimensional subspace \(\{\lambda_1\}\) of degeneracy for the energy spectrum or even the infinite-dimensional one. However the spectral values of \(h\) and \(R\) form the complete set of quantum numbers (the integral of motion) which characterize uniquely the quantum state.

ii) Let us describe the second case \(\beta = 0, \alpha > 0\) when the appropriate coordinates are elliptic ones:
\[
\tau_1 = \frac{1}{\sqrt{\alpha}} ln\left(z + \sqrt{z^2 + \frac{\gamma}{\alpha}}\right) \left(\bar{z} + \sqrt{\bar{z}^2 + \frac{\gamma}{\alpha}}\right); \quad (29)
\]
\[
\tau_2 = -\frac{i}{\sqrt{\alpha}} ln\frac{z + \sqrt{z^2 + \frac{\gamma}{\alpha}}}{\bar{z} + \sqrt{\bar{z}^2 + \frac{\gamma}{\alpha}}}. \quad (30)
\]

In a full analogy to case i) one can find
\[
B = \frac{1}{2(f_1 + f_2)} (2\partial_{\tau_1} f_1 - \frac{1}{2} f_1^2 - f_1 f_2 + F(\tau_2)); \quad (31)
\]
\[ V^{(1)} = \frac{1}{2(f_1 + f_2)}(-2\partial_{\tau_1}f_1 + \frac{1}{2}f_1^2 + F(\tau_2)) - \eta; \quad (32) \]
\[ V^{(2)} = \frac{1}{2(f_1 + f_2)}(2\partial_{\tau_1}f_1 + \frac{1}{2}f_1^2 + F(\tau_2)) - \eta, \quad (33) \]

where

\[ f_1 = \frac{1}{4} \left( \alpha \cdot \exp(\sqrt{\alpha}\tau_1) + \frac{|\gamma|^2}{\alpha} \exp(-\sqrt{\alpha}\tau_1) \right); \quad (34) \]
\[ f_2 = \frac{1}{4} \left( \bar{\gamma} \cdot \exp(i\sqrt{\alpha}\tau_2) + \gamma \cdot \exp(-i\sqrt{\alpha}\tau_2) \right). \quad (35) \]

This coordinates allow to separate variables as well that leads to the solution for the wave function in the form (26) where now

\[ -\phi''_{1n}(\tau_1) + \frac{1}{4} \left( \mp\partial_{\tau_1}f_1 + \frac{1}{2}f_1^2 - (E + \eta)f_1 \right) \phi_{1n}(\tau_1) = \frac{\lambda_n}{4} \phi_{1n}(\tau_1); \quad (36) \]
\[ -\phi''_{2n}(\tau_2) + \frac{1}{4} \left( \frac{F(\tau_2)}{2} - (E + \eta)f_2 \right) \phi_{2n}(\tau_2) = -\frac{\lambda_n}{4} \phi_{2n}(\tau_2) \quad (37) \]

and \( \lambda_n \) is again eigenvalue of the symmetry operator Eqs.(20),(21).

The analysis of the case \( \beta = 0, \alpha < 0 \) is similar and its result can be formulated as follows. One has to replace real \( (\tau_1, \tau_2) \) by imaginary \( (-i\tau_2, -i\tau_1) \) in Eqs.(29), (30) and respectively \( (f_1(\tau_1), f_2(\tau_2)) \) by \(-f_2(\tau_2), -f_1(\tau_1)\) in Eqs.(34), (35). Then the relations for potentials \( B, V_1, V_2 \) are given again by Eqs.(31) - (33).

We remark that the supercharge is factorizable into a composition of two ordinary SS-operators \( (q_i + i\tilde{q}_i) \) for the special choice \( \alpha > 0, \beta = \gamma = 0 \) that corresponds to the separation of variables in the polar coordinates (see Sect.3). We conclude that from the supersymmetry or equivalently from the intertwining of two Hamiltonians one inevitably recovers the hidden dynamical symmetry realized by \( R \) and furthermore the \( R \)-separation of variables.

5. General case and discussion

Let us describe the natural extension of the second derivative Darboux transformation which is generated by the operator with metric

\[ q^+ = g_{ik}(\vec{x})\partial_i\partial_k + \tilde{C}_i\partial_i + \tilde{B}. \quad (38) \]

The intertwining relations (11) determined completely the form of metric \( g_{ik}(\vec{x}) \) which satisfies the following equation:

\[ \partial_i g_{ik}(\vec{x}) + \partial_i g_{ik}(\vec{x}) + \partial_k g_{id}(\vec{x}) = 0. \]

Its solution can be easily found

\[ g_{11} = \bar{\alpha}y^2 + \bar{a}_1y + \bar{b}_1; \]
\[ g_{22} = \bar{\alpha}x^2 + \bar{a}_2x + \bar{b}_2; \]
\[ g_{22} = -\frac{1}{2}(2\bar{\alpha}xy + \bar{a}_1x + \bar{a}_2y) + \bar{b}_3. \]
Thus one can see that in senior derivatives the possible supercharges belong to the $E(2)$-universal enveloping algebra $U\mathfrak{e}(2)$ in which, at the level of second order, we distinguish three different possibilities:

\begin{align}
q^{(1)+} &= \alpha P_1^2 + \gamma \Delta + \tilde{C}_i \partial_i + \tilde{B}; \\
q^{(2)+} &= \alpha \{J, P_1\} + \gamma \Delta + \tilde{C}_i \partial_i + \tilde{B}; \\
q^{(3)+} &= \alpha J^2 + \beta P_1^2 + \gamma \Delta + \tilde{C}_i \partial_i + \tilde{B},
\end{align}

where $J$ and $\vec{P}$ are rotation and translation generators, respectively, and $\alpha \neq 0$.

The coefficients of supercharge Eq.(38) and the potentials $V^{(1),(2)}$ obey the modified equations:

\begin{align}
\partial_i \tilde{C}_k + \partial_k \tilde{C}_i + \Delta g_{ik} - (V^{(1)} - V^{(2)}) g_{ik} &= 0; \\
\Delta \tilde{C}_i + 2 \partial_i \tilde{B} + 2 g_{ik} \partial_k V^{(2)} - (V^{(1)} - V^{(2)}) \tilde{C}_i &= 0; \\
\Delta \tilde{B} + g_{ik} \partial_k \partial_i V^{(2)} + \tilde{C}_i \partial_i V^{(2)} - (V^{(1)} - V^{(2)}) \tilde{B} &= 0.
\end{align}

The generalized superalgebra evidently yields to the symmetry operator for the Hamiltonian $H$:

$$\{Q^+, Q^-\} = \tilde{R}; \quad [\tilde{R}, H] = 0.$$  

However this operator creates the dynamical symmetry of higher order which cannot be in general represented by a polynomial of the Hamiltonian and of second order symmetry operator (similar to Eq.(19)). But in any case the closing of SUSY algebra leads to the $R$-separation of variables in the spectral problem for the Hamiltonian $H$ (to the integrability of the corresponding dynamical system).

This work was supported partially by the Russian Foundation for Fundamental Research. One of us (M.V.I.) is indebted to the International Science Foundation (G.Soros Foundation) and to the American Physical Society for the financial support. One of us (D.N.N.) is grateful to Prof. A.Kostava for encouragement and support.

**References**

[1] L.E. Gendenshtein and I.V. Krive, Sov. Phys. Usp. 28(1985)645.

[2] A. Lahiri, P.K.Roy and B.Bagghi, Int. J. Mod. Phys., A5(1990) 1383.

[3] Th. F. Moutard, C. R. Acad. Sci. Paris, 80 (1875) 729; J. de L’Ecole Politech. Chier, 45 (1879) 1;

G. Darboux, C. R. Acad. Sci. Paris, 94(1882) 1456.

[4] L.Infeld and T.E. Hull, Rev. Mod. Phys., 23(1951)21.

[5] E. Witten, Nucl. Phys., B188(1981)513; *ibid*. B202 (1982)253.

[6] A.A. Andrianov, M.V. Ioffe and V.P. Spiridonov, Phys. Lett. A174 (1993) 273.
[7] A.A. Andrianov, F. Cannata, J.-P. Dedonder and M.V. Ioffe, "Second Order Derivative SUSY and Scattering Problem", preprint SPbU-IP-94-03 (1994).

[8] L.U. Ancarani and D. Baye, Phys. Rev. A46(1992)206.

[9] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, Phys. Lett. A105 (1984)19.
A.A. Andrianov, N.V. Borisov, M.I. Eides and M.V. Ioffe, Phys. Lett. A109 (1985)143.

[10] A.A. Andrianov and M.V. Ioffe, Phys. Lett. B205 (1988)507.

[11] A.A. Andrianov, M.V. Ioffe and Tsu Zhun-Pin, Vestnik Leningradskogo Universiteta 4(1988)3 (in Russian).

[12] W. Miller, Jr., "Symmetry and Separation of Variables", Addison-Wesley Publishing Company, London (1977), 342 p.