§0 Introduction.

Let $X$ be a scheme of finite type over Spec $\mathbb{Z}$, and Krull dimension $d$. Let $x$ denote a closed point of $X$, and let $N(x)$ be the order of the residue field $\kappa(x)$. Recall that $x$ is closed if and only if $\kappa(x)$ is finite. Let $G$ be any meromorphic function on $\mathbb{C}$, let $r$ be a rational integer, and let $a_r$ be the order of the zero of $G(s)$ at $s = r$. Let $G^*(r)$ be the limit as $s$ approaches $r$ of $G(s)(s - r)^{-a_r}$. If $G$ is a zeta-function, $G^*(r)$ is referred to as a special value of $G$.

Definition 0.1. [Se1]: The zeta function $\zeta(X, s)$ of $X$ is defined to be $\prod_x (1 - N(x)^{-s})^{-1}$, where $x$ runs over the closed points of $X$.

The product defining $\zeta(X, s)$ is well known to converge for $\text{Re}(s) > d$ and is conjectured to have a meromorphic continuation to the entire plane. We will tacitly assume this conjecture in what follows.

The goal of this paper is to give two different, but presumably equivalent, conjectural formulas for $\zeta^*(X, r)$ in terms of cohomology when $X$ is regular, and projective and flat over Spec $\mathbb{Z}$.

We will also show that modulo some very believable conjectures, our first formula is compatible with Serre’s conjectural functional equation ([Se2]) if $X$ is smooth over $\mathbb{Z}$. These conjectures are interesting in their own right, and we list them here:

Conjecture 1.1.3. The \'{e}tale motivic cohomology group $H^{2r+1}_{\text{et}}(X, \mathbb{Z}(r))$ is finite, and is dual to $H^{2(d-r)+1}_{\text{et}}(X, \mathbb{Z}(d-r))$, up to 2-torsion.

Let $X$ be projective and smooth over Spec $\mathbb{Z}$.
Conjecture 4.2.2a. Recall that the weight of the cohomology motive $H^j(X, \mathbb{Z}(r))$ is given by $j - 2r$. Let $M$ be a pure motive of weight $w(M)$. Let $B_M$ be the dimension of the Betti (or de Rham) realization of $M$. Then the determinant of the period map from $H_B(M)$ to $H_{DR}(M)$ is equal to $(2\pi i)^{-w(M)B_M/2}$.

Conjecture 4.3.1. The Euler characteristic $\chi(H^*_B(X, \mathbb{Z})_{tor})$ is equal to the Euler characteristic $\chi(H^*_{DR}(X/\mathbb{Z}))_{tor}$ if $X$ is projective and smooth over $\mathbb{Z}$.

Of course, we also need the conjectures that the relevant étale motivic cohomology groups are finitely generated, and that the zeta function extends to a meromorphic function satisfying the functional equation.

Both these formulas will involve the Weil-étale motivic cohomology groups of $X$. These will be defined directly in the cases that interest us, but they will not be defined as the cohomology groups of sheaves on a site. Our approach is related to that of Morin in [M], who mainly considers the case $r = 0$. We will relate these formulas to Soulé’s conjectured formula for $a_r$ ([So], Conj. 2.2), and to the work of Fontaine, Perrin-Riou, Bloch and Kato on similar questions.

These formulas are closely related to conjectured formulas which Fontaine ([Fo]) refers to as Deligne-Beilinson and Bloch-Kato, and in fact (at least for schemes smooth over $\mathbb{Z}$) implies Fontaine’s Deligne-Beilinson conjecture, which gives the special value up to a rational number. (As the reader will see, this should be taken with the usual grain of salt, since in the course of this paper, we will be assuming many things which everyone believes and no one can prove). (We note that Fontaine begins [Fo] by discussing Ext groups of mixed motives. Fontaine later conjectures that in addition these Ext-groups and the maps between them can be described in terms of algebraic K-theory and Beilinson regulators. When we refer to Fontaine’s conjectures, we mean these enhanced ones). What we refer to as “Fontaine’s conjectures” are taken from Fontaine’s Bourbaki talk [Fo], but are based on his work with Perrin-Riou [FP]. Whenever we refer to Fontaine the reader should keep in mind that we are really talking about the joint work of Fontaine and Perrin-Riou. The main differences are:

a) Fontaine works with the Hasse-Weil zeta-function, which only depends on the generic fiber of $X$, whereas the scheme zeta function depends also on the nature of the degenerate fibers.

b) Fontaine relies on a detailed local analysis, using $\ell$–adic and crystalline cohomology,
whereas our conjecture is entirely global.

c) Fontaine gives a conjecture for special values of the L-functions whose alternating product is the Hasse-Weil zeta-function, whereas we only consider the scheme zeta-function. In our situation, we do not believe it is possible to give formulas of this kind on the level on the level of L-functions, unless the motive of $X$ breaks up integrally as the sum of cohomology motives.

Our second formula gives the special values in terms of generalized Euler characteristics, (see Appendix A) which is exactly what happens in the geometric case, where one considers projective non-singular varieties over finite fields, as in [Ge]. In [Ge], the cohomology groups involved in the Euler characteristic are Weil-étale motivic cohomology groups, (cohomology groups on the Weil-étale site) which are conjectured to be finitely generated, and cohomology groups of sheaves of differentials, which are known to be finite. In our situation there is no satisfactory definition of the Weil-étale site, but we conjecture the existence of such a site, various sheaves on this site and maps between them. This formula involves many sheaves on the hypothetical Weil-étale site, and so leaves something to be desired. However it has the advantage of being natural and elegant. In any case, both formulas are entirely global, and avoid any detailed local constructions. Presumably it is not hard to show (under enough assumptions!) that the two formulas are equivalent, but we do not do that in this paper.

Now let $X$ again just be regular, and projective and flat over Spec $\mathbb{Z}$. The basic idea behind both our conjectured formulas is to start with Fontaine’s “Deligne-Beilinson” conjectures [Fo], which give the special values of to a rational number in terms of determinants of maps of complex vector spaces with given rational structures. These complex vector spaces come from Betti and de Rham cohomology, and from Weil-étale cohomology. We replace the rational structures by integral structures, and take determinants with respect to these. The Betti cohomology of course has a natural integral structure, and the Weil-étale groups conjecturally do also. We define an integral structure on the de Rham groups by using derived exterior powers.

We also introduce the orders of naturally occurring finite cohomology groups into the picture. Finally we replace the period maps in Fontaine’s picture by “modified” period maps, where we divide by a special value of the Gamma function.

Our first conjectured formula is:

**Conjecture 3.1.1.** $\zeta^*(X, r) = \chi(H^*_W(X, \mathbb{Z}(r))_{tor}) \chi(H^*_B(X, \mathbb{Z}(r))_{tor}) \chi(H^*(X, t(r))_{tor})$
multiplied by $\prod_{j=0}^{2d-2} (\det(C(M^{j,r}))^{(-1)^j} \prod_{j=0}^{2d-2} (\det(E(M^{j,r}))^{(-1)^j}$, up to sign and powers of 2.

(For explanation of the notation, see Section 3).

Now let $r$ be a rational integer. Our basic construction for the second formula involves three complexes of sheaves on the (hypothetical) Weil-étale site corresponding to $X (\mathbb{Z}(r), \mathbb{Z}^+_B(r), and t(r))$, whose cohomology groups are respectively. the Weil-étale motivic cohomology groups $H^j_W(X, \mathbb{Z}(r))$ defined in §1 below, the singular cohomology group $H^j_B(X_C, \mathbb{Z}(r))^+$, and the Zariski cohomology group $H^j(X, t(r)) = \bigoplus_{b<r} H^j_{Zar}(X, \lambda^b \Omega_{X/\mathbb{Z}})$, where $\lambda^b$ denotes derived exterior power (See Appendix B). These notations are explained more fully in §2 and §3 below.

We further conjecture that there exist maps of complexes on the Weil-étale site:

$g : \mathbb{Z}(r) \to \mathbb{Z}^+_B(r)$ and $h : t(r)[-3] \to \mathbb{Z}(r)$, whose composition is zero.

(These give respectively maps from higher Chow groups to singular cohomology and maps dual to maps from higher Chow groups to de Rham cohomology.)

We combine them to form the double complex $D(r)$ of Weil-étale sheaves on $X$:

$$
\begin{array}{ccc}
0 & \longrightarrow & t(r)[-3] \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}(r) \\
& & \longrightarrow \mathbb{Z}^+_B(r)
\end{array}
$$

We conjecture in addition that there are natural maps $\delta^j$ from $H^j_W(X, D(r)) \otimes \mathbb{C}$ to $H^{j+1}_W(X, D(r)) \otimes \mathbb{C}$ which combine to give an acyclic complex $H^j_W(X, D(r)) \otimes \mathbb{C}$, and that the cohomology groups $H^j_W(X, D(r))$ are finitely generated. This means that the Euler characteristic $\chi(X, D(r))$ is defined (See Appendix A).

Our second conjectured formula is:

**Conjecture 4.3.1b.** $\zeta^*(X, r) = \chi(X, D(r))$, up to sign and powers of 2.

The plan of this paper is as follows:

In Section 1, we define the various cohomology groups that we use. In Section 2, we discuss the maps between these groups, and get information about the total complex $D(r)$. In Section 3, we state the first conjectural formula, and give examples. In Section 4, we state the second conjectural formulas, and give examples. In Section 5, we give the proof of the compatibility with the functional equation. In Section 6, we explain how Soulé’s conjecture fits into our picture.
As we move along we will explain how our definitions of groups and maps relate to those of Fontaine and Perrin-Riou [FP].

The reader might wonder why we have chosen to put torsion subgroups of singular cohomology and de Rham cohomology groups into our formula, since as far as we know there is no numerical evidence for this, these groups are zero for \( d \leq 2 \), and if they occur in previous conjectures it is well hidden. We have done this because in every standard example, if a zeta-function formula involves a determinant of a map between two complex vector spaces which are the complexification of natural finitely generated groups, the torsion subgroups of these groups also enter into the formula. (See the conjecture of Birch and Swinnerton-Dyer for example). In our situation it is well understood that we have to consider period maps between singular and de Rham cohomology.

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§1. Fontaine’s DB (Deligne-Beilinson)-conjecture.

We begin by describing Fontaine’s DB-conjecture, with some minor modifications. This conjecture describes the values of certain L-functions up to rational numbers, in terms of generalized determinants of maps of real vector spaces with given \( \mathbb{Q} \)-structures. We will in fact tensor with \( \mathbb{C} \), which is harmless, but allows us to multiply by \( 2\pi i \) if necessary. If \( A \) is an abelian group, let \( A_{\mathbb{Q}} \) (resp. \( A_{\mathbb{C}} \)) be \( A \otimes_{\mathbb{Z}} \mathbb{Q} \) (resp. \( A \otimes_{\mathbb{Z}} \mathbb{C} \)).

Let \( X \) be a regular scheme of pure dimension \( d \), projective and flat over \( \text{Spec} \, \mathbb{Z} \), with generic fiber \( X_0 \). Fix an integer \( r \) and a non-negative integer \( j \), and let \( M = M^{j-r} \) be the motive \( H^j(X_0, \mathbb{Z}(r)) \). Recall that the weight \( w(M) \) of \( M \) is equal to \( j - 2r \). Let \( Z^r(X_0) \) be the group of codimension \( r \) cycles on \( X_0 \) modulo homological equivalence, and let \( C^r(X_0) \) be the group of codimension \( r \) cycles modulo rational equivalence which are homologically equivalent to zero. If \( Y \) is a scheme, let \( K_1(Y)^{(r)} \) be the weight \( r \) piece of \( K_1(Y)_{\mathbb{Q}} \). Let \( \tilde{K}_1(X)^{(r)}_{\mathbb{Q}} \) be the image of \( K_1(X)^{(r)}_{\mathbb{Q}} \) in \( K_1(X_0)^{(r)}_{\mathbb{Q}} \). In [Fo], Fontaine defines \( \mathbb{Q} \)-vector spaces as follows: \( H_f^j(M) = Z^r(X_0)_{\mathbb{Q}} \) if \( j = 2r \), and zero otherwise, \( H_f^j(M) = H_2r-j-1(X)^{(r)}_{\mathbb{Q}} \) if \( j \neq 2r - 1 \), and be \( C^r(X_0)_{\mathbb{Q}} \) if \( j = 2r - 1 \).

Let \( M^*(1) = H^{2d-2-j}(X_0, \mathbb{Z}(d-r)) \). (Remember that \( X_0 \) has dimension \( d - 1 \).) Fontaine defines \( H^i_c(M) = \text{Hom}(H_f^{2-i}(M^*(1)), \mathbb{Q}) \) for \( i = 1, 2 \).
Let $X_C = X \times_\mathbb{Z} \mathbb{C}$. Complex conjugation $c$ acts on $\mathbb{C}$, so it acts on $X_C$, and by transport of structure on the Betti cohomology group $H^j_B(X_C, \mathbb{Z})$. We define a subgroup $H^+_B(M) = (H^j_B(X_C, \mathbb{Z}(r)))^+$ to be the set of $x$ in $H^j_B(X_C, \mathbb{Z})$ such that $c(x) = x$ if $r$ is even, and the set of $x$ in $H^j_B(X_C, \mathbb{Z})$ such that $c(x) = -x$ if $r$ is odd.

Let $\Omega = \Omega_{X_0/\mathbb{Q}}$. Let $\tilde{H}_{DR}(M) = \bigoplus_{i<r} H^{j-i}(X_0, \Lambda^i \Omega)$, where as usual $\Lambda^i$ denotes the $i$-th exterior power of the locally free sheaf $\Omega$.

Let $\alpha_M$ be the period map from the singular cohomology $H^*_B(M)_\mathbb{C}$ to the de Rham cohomology $H^*_{DR}(M)_\mathbb{C}$, and let $\tilde{\alpha}_M$ be the map induced by $\alpha_M$ from $H^*_B(M)_\mathbb{C}^+$ to $\tilde{H}_{DR}(M)_\mathbb{C}$, viewed via the Hodge decomposition as a quotient of $H_{DR}(M)_\mathbb{C}$.

Fontaine considers the following two (conjecturally) exact sequences of complex vector spaces, where the maps are induced by Beilinson regulators and height pairings:

**A(M)**

\[ 0 \to H^0_B(M)_\mathbb{C} \to Ker(\tilde{\alpha}_M) \to H^1_c(M)_\mathbb{C} \to H^1_B(M)_\mathbb{C} \to Coker(\tilde{\alpha}_M) \to H^2_c(M)_\mathbb{C} \to 0 \]

**B(M)**

\[ 0 \to Ker(\tilde{\alpha}_M) \to (H^+_B(M))_\mathbb{C} \to (\tilde{H}_{DR}(M))_\mathbb{C} \to Coker(\tilde{\alpha}_M) \to 0 \]

In equation $B(M)$ the map from $(H^+_B(M))_\mathbb{C}$ to $(\tilde{H}_{DR}(M))_\mathbb{C}$ is given by $\tilde{\alpha}_M$.

In order to define the determinants of these sequences, we need to give $\mathbb{Q}$-bases of each term. Every term except the kernel and cokernel terms has a natural $\mathbb{Q}$-structure, and we choose arbitrary $\mathbb{Q}$-structures for the kernel and cokernel terms (but of course the same in both sequences).

If we then multiply the determinant of the first sequence by the determinant of the second sequence, the result is well defined in $\mathbb{C}^*/\mathbb{Q}^*$, independent of the choice of $\mathbb{Q}$-structures for the kernels and cokernels.

Let $M_\ell = H^j(X_0, \mathbb{Q}_\ell(r))$ be he $\ell$-adic realization of the motive $M$. If $p$ is a rational prime and $\ell$ is a prime different from $p$, let $P_p(T) = det(1 - Fr_p^{-1}T|M_\ell^{I_p}) \in \mathbb{Q}_\ell[T]$ where $I_p$ is the inertia group at $p$ and $Fr_p$ is a Frobenius element.

It is conjectured, (and known if $X$ has good reduction at $p$) that $P_p(T)$ lies in $\mathbb{Q}[T]$ and is independent of the choice of $\ell$.

We then define the Hasse-Weil $L$-function $L_{HW}(M, s)$ to be $\prod_p P_p(p^{-s})^{-1}$ which we assume (as usual) can be continued meromorphically to the entire plane.
Fontaine’s Deligne-Beilinson conjecture then asserts that if $r$ is an integer,

\begin{equation}
L^*_{HW}(M, r) = \text{det}(A(M))\text{det}(B(M))
\end{equation}

up to a non-zero rational number.

We are going to make a conjecture modifying Fontaine’s DB-conjecture which focuses on the model $X$ and should be valid up to sign and powers of 2. (Our methods give no information about the sign, but the power of 2 should be able to be described by looking at the behavior of cohomology at real primes. See for example [FM])

We will replace the cycle groups on $X_0$ by cycle groups on $X$, change all the groups $H_f$ and $H_c$ to Weil-étale motivic cohomology groups, which are conjecturally finitely generated, bring derived exterior powers into the de Rham cohomology, introduce a modified period map, and put in the orders of torsion subgroups of all our cohomology groups.

We also, as mentioned in the introduction, will replace Fontaine’s Hasse Weil-L-functions by the scheme zeta function, which will agree with the alternating product of the Hasse-Weil zeta-functions when $X$ is smooth.

§2. The groups involved in the conjectures.

Throughout Section 2 let $\pi$ be a projective, flat morphism from the connected regular scheme $X$ to $\text{Spec } \mathbb{Z}$. Let $d$ be the Krull dimension of $X$. Let $X_0$ be the generic fiber of $X$.

§2.1 Weil-étale motivic cohomology.

We will first define Weil-étale motivic cohomology groups and then discuss their relation to the groups defined by Fontaine and Perrin-Riou ([Fo] and [Fl]).

Let $r$ be an integer, and $j$ a non-negative integer. We would like to define a Weil-étale site and sheaves $\mathbb{Z}(r)$ on this site whose cohomology groups $H^j_W(X, \mathbb{Z}(r))$ would be Weil-étale motivic cohomology, but unfortunately we do not know how to do this. Instead, for $j \leq 2r$ we define $H^j_W(X, \mathbb{Z}(r))$ to be the hypercohomology groups $H^j_{\text{ét}}(X, \mathbb{Z}(r))$, where $\mathbb{Z}(r)$ denotes Bloch’s higher Chow group complex sheafified for the étale topology ([Bl1],[Le]). Sometimes these groups are referred to as étale motivic cohomology. For $j \geq 2r + 1$, we
define \( H^i_W(X, \mathbb{Z}(r)) \) to be \( h^i(R\text{Hom}(R\Gamma_{et}(X, \mathbb{Z}(d - r)), \mathbb{Z}[-2d - 1])) \), so we have the exact sequence

\[
(2.1.1) \quad 0 \to Ext^1(H^{2d+2-j}_{et}(X, \mathbb{Z}(d - r)), \mathbb{Z}) \to H^j_W(X, \mathbb{Z}(r)) \to Hom(H^{2d+1-j}_{et}(X, \mathbb{Z}(d - r)), \mathbb{Z}) \to 0
\]

If we had our hypothetical Weil-étale site, with section functor denoted by \( \Gamma_W \), this would follow, up to 2-torsion, from a duality theorem which asserted that \( R\Gamma_W(X, \mathbb{Z}(d - r)) \) was isomorphic to \( R\text{Hom}(R\Gamma_W(X, \mathbb{Z}(r)), \mathbb{Z}[-2d - 1]) \). The analogue of this theorem, assuming the usual conjectures, is true for Weil-étale cohomology in the geometric case, as shown in [Ge]). We note here that in [FM], Flach and Morin have constructed such a complex of abelian groups, which satisfies this duality theorem assuming that standard finiteness conjectures hold.

**Conjecture 2.1.1.** The groups \( H^j_{et}(X, \mathbb{Z}(r)) \) are finitely generated for \( j \leq 2r + 1 \), and finite for \( j = 2r + 1 \).

This implies

**Conjecture 2.1.2.** The cohomology groups \( H^j_W(X, \mathbb{Z}(r)) \) are finitely generated for all \( j \).

The two finite groups \( H^{2r+1}_{et}(X, \mathbb{Z}(r)) \) and \( H^{2d-2r+1}_{W}(X, \mathbb{Z}(d - r))_{tor} \) are dual to each other.

**Conjecture 2.1.3.** The finite group \( H^{2r+1}_{et}(X, \mathbb{Z}(r)) \) is dual to \( H^{2(d-r)+1}_{et}(X, \mathbb{Z}(d - r)) \), up to 2-torsion.

Conjecture 2.1.3 should be true in order for our main conjecture to be compatible with the functional equation (See §5. below).

Note that our groups \( H^{2r}_W(X, \mathbb{Z}(r)) \) are the étale codimension \( r \) cycle groups on \( X \), which agree with the usual cycle groups only after tensoring with \( \mathbb{Q} \) and agree (also after tensoring both with \( \mathbb{Q} \)) with the usual cycle groups on \( X_0 \) if \( X \) is smooth.

Our Weil-étale motivic cohomology groups roughly correspond (after tensoring with \( \mathbb{Q} \)) to Fontaine’s vector spaces \( H^j_f \) and \( H^i_c \). For simplicity, assume that \( X \) is smooth over \( \text{Spec } O_F \), where \( O_F \) is the ring of integers on the number field \( F \). Then the following relations should hold:
Fix an integer $r$. If $j \geq 0$ let $M = M^{j,r}$ be the motive $H^j(X_0, \mathbb{Z}(r))$. If $j < 0$, let $M^{j,r} = 0$. We need to consider these motives for $0 \leq j \leq 2(d - 1)$. If $X$ is a scheme, let $K^{(a)}_b(X)$ be the weight $a$ piece of $K_b(X)$. (This is only defined after tensoring with $\mathbb{Q}$).

To connect Fontaine’s picture with ours, we need two more plausible conjectures:

**Conjecture 2.1.4.** The motivic cohomology group $H^j_{\text{et}}(X, \mathbb{Z}(r))$ is isomorphic to $K^{(r)}_{2r-j}(X)$ after tensoring with $\mathbb{Q}$. (This was proved by Bloch [Bl1] in the case that $X$ is of finite type over a field, and it is also expected to be true for schemes of finite type over $\mathbb{Z}$.) Note that étale and Zariski motivic cohomology agree after tensoring with $\mathbb{Q}$.

**Conjecture 2.1.5.** The map $\tau$ from $K_n(X)$ to $K_n(X_0)$ has torsion kernel for $n > 0$ so $(K_n(X)) \otimes \mathbb{Q}$ is isomorphic to $\tau(K_n(X)) \otimes \mathbb{Q}$. (In the geometric case, Parshin’s conjecture says that $K_n(X)$ is torsion for $n > 0$, so this is a kind of generalization to the arithmetic situation).

Then the relation between Fontaine’s vector spaces $H^i_f$ and $H^i_c$ and our motivic cohomology groups should be:

- **a)** For $j \neq 2r$, $H^0_f(M^{j,r}) = 0$.
- **b)** For $j \geq 2r$, $H^1_f(M^{j,r}) = 0$. For $j \leq 2r - 2$, $H^1_f(M^{j,r}) = H^{j+1}_W(X, \mathbb{Z}(r))_{\mathbb{Q}}$.
- **c)** There is an exact sequence

$$0 \rightarrow H^1_f(M^{2r-1,r}) \rightarrow H^{2r}_W(X, \mathbb{Z}(r))_{\mathbb{Q}} \rightarrow H^0_f(M^{2r,r}) \rightarrow 0$$

- **d)** For $j = 2r - 2$, $H^2_f(M^{j,r}) = 0$.
- **e)** For $j \leq 2r - 2$, $H^1_c(M^{j,r}) = 0$. For $j \geq 2r$, $H^1_c(M^{j,r}) = H^{j+2}_W(X, \mathbb{Z}(r))_{\mathbb{Q}}$.
- **f)** There is an exact sequence

$$0 \rightarrow H^2_c(M^{2r-2,r}) \rightarrow H^{2r+1}_W(X, \mathbb{Z}(r))_{\mathbb{Q}} \rightarrow H^1_c(M^{2r-1,r}) \rightarrow 0$$

We now explain what our groups look like in the case when $d = 1$, i.e. when $X = \text{Spec} \ O_F$, with $O_F$ being the ring of integers in the number field $F$.

Assume that $F$ has $r_1$ real places and $r_2$ complex places. Let $d_F$ be the discriminant of $F$. The following computations are up to finite 2-torsion groups. The relationship between motivic cohomology and K-theory follows from [M].
Example 2.1.6. Let $X = \text{Spec } O_F$ and let $r = 0$. Recall that $\mathbb{Z}(0) = \mathbb{Z}$. Then:

$$H^0_W(X, \mathbb{Z}) = \mathbb{Z}$$

$$H^1_W(X, \mathbb{Z}) = 0$$

$H^2_W(X, \mathbb{Z})$ fits into an exact sequence

$$0 \rightarrow \text{Hom}(\text{Pic}(O_F), \mathbb{Q}/\mathbb{Z}) \rightarrow H^2_W(X, \mathbb{Z}) \rightarrow \text{Hom}(O_F^*, \mathbb{Z}) \rightarrow 0$$

$$H^3_W(X, \mathbb{Z}) = \text{Hom}(\mu_F, \mathbb{Q}/\mathbb{Z})$$

Here Pic($O_F$) is the class group of $F$, $O_F^*$ is the unit group of $O_F$, and $\mu_F$ is the group of roots of unity in $F$.

$$H^j_W(X, \mathbb{Z}) = 0 \quad \text{for} \quad j \neq 0, 1, 2, 3$$

Example 2.1.7. Let $X = \text{Spec } O_F$ and let $r = 1$. Recall that $\mathbb{Z}(1) = G_m[-1]$. Then

$$H^1_W(X, \mathbb{Z}(1)) = O_F^*$$

$$H^2_W(X, \mathbb{Z}(1)) = \text{Pic}(O_F)$$

$$H^3_W(X, \mathbb{Z}(1)) = \mathbb{Z}$$

$$H^j(X, \mathbb{Z}(1)) = 0 \quad \text{for} \quad j \neq 1, 2, 3$$

Example 2.1.8. Let $X = \text{Spec } O_F$ and let $r > 1$. Then

$$H^1_W(X, \mathbb{Z}(r)) = H^1_{et}(X, \mathbb{Z}(r)) = K_{2r-1}(O_F)$$

$$H^2_W(X, \mathbb{Z}(r)) = H^2_{et}(X, \mathbb{Z}(r)) = K_{2r-2}(O_F)$$

$$H^j_W(X, \mathbb{Z}(r)) = 0 \quad \text{for} \quad j \neq 1, 2$$
Example 2.1.9. Let \( X = \text{Spec} \, O_F \) and let \( r < 0 \). Then \( H^j_W(X, \mathbb{Z}(r)) \) fits into an exact sequence

\[
0 \to \text{Hom}(K_{-2r}(O_F), \mathbb{Q}/\mathbb{Z}) \to H^j_W(X, \mathbb{Z}(r)) \to \text{Hom}(K_{1-2r}(O_F), \mathbb{Z}) \to 0
\]

\[
H^j_W(X, \mathbb{Z}(r)) = \text{Hom}((K_{1-2r}(O_F))_{\text{tor}}, \mathbb{Q}/\mathbb{Z})
\]

\[
H^j_W(X, \mathbb{Z}(r)) = 0 \quad \text{for} \quad j \neq 2, 3
\]

The skeptical reader should remember that we are only working up to 2-torsion.

The computations of Examples 2.1.8 and 2.1.9 follow from the spectral sequence relating Zariski motivic cohomology to algebraic K-theory, the Beilinson-Lichtenbaum conjecture which implies that in the range considered here, étale motivic cohomology is the same as Zariski motivic cohomology, the Garland-Borel theorem ([Bo]) that \( K_{2m}(O_F) \) is finite for \( m > 0 \), and of course the definition of \( H^i_W \).

S 2.2. Betti, Hodge and de Rham cohomology

We now define the Betti cohomology groups \( H^j(X, \mathbb{Z}(r))^+ \). We let \( H^j_B(X, \mathbb{Z}(r)) \) be the usual singular cohomology groups \( H^j(X, \mathbb{Z}(r)) \). Denote by \( c \) complex conjugation acting on \( H^j_B(X, \mathbb{Z}) \), and let \( H^j_B(X, \mathbb{Z}(r))^+ \) be the set of \( x \) in \( H^j_B(X, \mathbb{Z}) \) such that \( c(x) = x \) if \( r \) is even, and the set of \( x \) such that \( c(x) = -x \) if \( r \) is odd. (Note that the usual factor of \((2\pi i)^r\) will appear when we consider period maps.)

Example 2.2.1. Let \( X = \text{Spec} \, O_F \). Let \( a_r = r_2 \) if \( r \) is even and \( a_r = r_1 + r_2 \) if \( r \) is odd. Let \( b_r = r_1 + r_2 \) if \( r \) is even and \( b_r = r_2 \) if \( r \) is odd, so \( a_r + b_r = r_1 + 2r_2 = [F : \mathbb{Q}] \). Then \( H^0_B(X_C, \mathbb{Z}(r))^+ \) is isomorphic to \( \mathbb{Z}^{b_r} \). \( H^1_B(X_C, \mathbb{Z}(r))^+ \) is zero if \( j \neq 0 \).

We next consider Hodge cohomology. If \( V \) is a smooth complex variety, let \( \Omega_V = \Omega_{V/\mathbb{C}} \) be the sheaf of Kahler differentials on \( V \). Let \( \Lambda^q \) denote \( q \)-th exterior power, and let \( \Omega^r \) denote \( r \)-th exterior power of \( \Omega \). We want to give a natural \( \mathbb{Z} \)-structure to the cohomology groups \( H^j(X_C, T(X_C, r)) \). Let \( \lambda^q \Omega_{X/\mathbb{Z}} \) denote the \( q \)-th derived exterior power of \( \Omega_{X/\mathbb{Z}} \). (See Appendix B). Let \( t(r) \) be the complex (in the derived category) of étale sheaves on \( X \) given by \( \prod_{q=0}^{r-1} \lambda^q \Omega_{X/\mathbb{Z}}[-q] \). It is immediate that the pullback of \( t(r) \) to \( X_C \) is \( T(X_C, r) \). If \( M \) denotes the motive \( H^j(X, \mathbb{Z}(r)) \), let \( \tilde{H}^j_{DR}(M) = \tilde{H}^j_{DR}(X, \mathbb{Z}(r)) = \prod_{i<r} H^{j-i}(X, \lambda^i \Omega_{X/\mathbb{Z}}) = H^j(X, t(r)) \).
Lemma 2.2.2. Let $X$ be as in Example 2.1.6. Then $\lambda^0\Omega_{X/Z} = O_F$. If $i > 0$, $(\lambda^i)\Omega_{X/Z} = \Omega_{X/Z}[i - 1]$.

Proof. This follows by induction on $j$ from Lemma B.4 and the fact that there exists an exact sequence $0 \to D_F \to O_F \to \Omega_{X/Z} \to 0$, where $D_F$ is the different of $F$ over $\mathbb{Q}$.

Example 2.2.3. Let $X$ be as in Example 2.1.6. Then $H^0_W(X, t(r)) = 0$ for all $n$ if $r \leq 0$. If $r = 1$, $H^0_W(X, t(1)) = O_F$, and if $r > 1$, the order of $H^1_W(X, t(r))$ is $|d_F|^r - 1$.

Proof. This follows from Lemma 2.2.2. Recall that $|d_F|$ is the norm of $D_F$ and equal to $|\Omega_{X/Z}|$.

§3.1 The first version of our conjecture.

The formulas in [Fo] and [Fl] involve the four families of conjecturally finite-dimensional $\mathbb{Q}$-vector spaces we discussed in §1 called $H_f$, $H_c$, $H^+_B$ and $\tilde{H}_{DR}$. These formulas, up to a rational number, are given by considering generalized determinants of maps between the complexifications of these vector spaces. In [Fo], this type of result is referred to as the ’’Deligne-Beilinson conjecture’’. In order to get formulas valid up to signs and powers of 2, referred to as the ’’Bloch-Kato conjecture” , a detailed local analysis is necessary, using Fontaine’s B-rings.

Our point of view is to eliminate the detailed local analysis by replacing rational vector spaces with finitely generated abelian groups. Roughly speaking this is a question of describing canonical integral structures on Fontaine’s vector spaces, and then modifying the resulting formula by multiplying and dividing by the orders of finite abelian groups. We will explain the relation in the next few paragraphs. As in §1, let $X$ be a regular scheme of Krull dimension $d$, projective and flat over Spec $\mathbb{Z}$. Then $d - 1$ is the relative dimension of $X$ over Spec $O_F$, or over Spec $\mathbb{Z}$. Let $M$ be the “motive” $H^3(X_0, \mathbb{Z}(r))$.

Recall from §1. that Fontaine considers the following two (conjecturally) exact sequences of complex vector spaces:

$$A(M)$$

$$0 \to H^0_f(M)_\mathbb{C} \to Ker(\tilde{\alpha}_M) \to H^1_c(M)_\mathbb{C} \to H^1_f(M)_\mathbb{C} \to Coker(\tilde{\alpha}_M) \to H^2_c(M)_\mathbb{C} \to 0$$

$$B(M)$$

$$0 \to Ker(\tilde{\alpha}_M) \to (H^+_B(M))_\mathbb{C} \to (\tilde{H}_{DR}(M))_\mathbb{C} \to Coker(\tilde{\alpha}_M) \to 0$$
In equation $B(M)$ the map from $(H^+_B(M))_C$ to $(\hat{H}_{DR}M)_C$ is given by $\tilde{\alpha}_M$.

In order to get a formula that makes sense integrally we need to do several things. 1) We will replace the period map $\alpha_M$ by our modified version $\gamma_M$, 2) We choose integral structures for Fontaine’s $\mathbb{Q}$-rational vector spaces. 3) We include the orders of some finite groups. 4) We modify some of the spaces slightly to move from varieties over $\mathbb{Q}$ to schemes over $\mathbb{Z}$.

We begin by constructing a “modified period map”. Let $V$ be a projective and smooth variety over $\mathbb{C}$. Let $H^j_B(V,\mathbb{Z}(r)) = H^j_B(V,\mathbb{Z})$. Recall that $H^j_B(V,\mathbb{Z}(r))$ is the set of $x$ in $H^j_B(V,\mathbb{Z})$ such that $c(x) = x$ if $r$ is even, and the set of $x$ in $H^j_B(V,\mathbb{Z})$ such that $c(x) = -x$ if $r$ is odd. Let $H^j_{DR}(V,\mathbb{C}(r)) = H^j_{DR}(V,\mathbb{C})$. Recall that the Hodge filtration $F^q$ on $H^j_{DR}(V,\mathbb{C}(r))$ is related to the Hodge filtration $F_q$ on $H^j_{DR}(V,\mathbb{C})$ by $F^q = F_q + r$. Let $\alpha_M$ be the usual period map from $H^j_B(V,\mathbb{Z})_C$ to $H^j_{DR}(V,\mathbb{C})$.

Let $M = M'^r = H^j(V,\mathbb{Z}(r))$. We define a modified period map $\gamma_M$ from $H^j_B(M)_C$ to $H^j_{DR}(M)_C$ as follows: $H^j_{DR}(M)_C$ has a decreasing Hodge filtration $F_q(M)$. Let $H^q = F_q/F_{q+1}$. Then $H^j_{DR}(M)_C$ has the direct sum Hodge decomposition $H^j_{DR}(M)_C = \amalg H^q$. We can decompose $\alpha_M$ into the direct sum of the maps $\alpha^q(M)$ where $\alpha^q$ is the map $\alpha_M$ followed by the projection onto $H^q$. Let $\Gamma$ denote the usual gamma-function. Recall that $\Gamma^*$ is defined in the first paragraph of this paper, and $w(M) = j - 2r$. Now let $\gamma^q(M) = ((2\pi i)^r \Gamma^*(-w(M) - q))^{-1} \alpha^q(M)$, and let $\gamma_M = \amalg_q \gamma^q(M)$. Replace $B(M)$ by a new sequence $E(M)$ whose groups are identical to those in $B(M)$ but where the map from $H^+_B(M)_C$ to $t(M)_C$ is given by $\gamma_M$ instead of $\alpha_M$. Note that this does not change the kernel or cokernel, so the sequence remains exact.

We now replace Fontaine’s sequence $A(M)$ by a modified sequence $C(M)$. We first describe the changes in the groups, and then give the requisite integral structures. We replace the cycle groups on the generic fiber by the cycle groups on $X$, so that $\hat{H}^0_f(M^{2r,.r})$ is now defined to be $H^{2r}_\text{et}(X,\mathbb{Z}(r))$ (the codimension $r$ étale cycles on $X$), modulo homological equivalence. Similarly, $\hat{H}^j_f(M^{j,.r})$ is defined to be $H^{j+1}_\text{et}(X,\mathbb{Z}(r))$ for $j < 2r - 1$ (by Conjectures 2.1.5 and 2.1.6, our definition here should be the same as Fontaine’s if we tensor with $\mathbb{Q}$) and to be the subgroup of $H^{2r}_\text{et}(X,\mathbb{Z}(r))$ consisting of étale cycles homologically equivalent to zero, if $j = 2r - 1$. The integral structure on $\hat{H}^i_c$ is the dual of the integral structure on the appropriate $\hat{H}^{2-i}_c$.

So we have:
\[ C(M) \to \tilde{H}^0_f(M) \to \ker(\tilde{\gamma}_M) \to \tilde{H}^1_c(M) \to \tilde{H}^1_f(M) \to \ker(\tilde{\gamma}_M) \to \tilde{H}^2_c(M) \to 0 \]

\[ E(M) \to \ker(\tilde{\gamma}_M) \to (H^+_B(M))_C \to (\tilde{H}_{DR}(M))_C \to \ker(\tilde{\gamma}_M) \to 0 \]

In equation \( E(M) \) the map from \((H^+_B(M))_C\) to \((\tilde{H}_{DR}(M))_C\) is given by \( \tilde{\gamma}_M \).

We take the obvious integral structure on \( H^j(X_C, T(X_C, r)) \) coming from usual singular cohomology.

We take the integral structure on \( H^j(X, T(X_C, r)) \) coming from \( H^j(X, t(r)) \).

If \( H^i \) is a sequence of finite abelian groups which are zero for \( i < 0 \) and for \( i \) sufficiently large, let the Euler characteristic \( \chi(H^*) \) be \( \prod |H^i|^{(-1)^i} \).

The first version of our conjecture now becomes:

**Conjecture 3.1.1.** :

\[ \zeta^*(X, r) = \chi(H^*_W(X, \mathbb{Z}(r))_{tor}) \chi(H^*_B(X, \mathbb{Z}(r))^{+}_{tor}) \chi(H^*_DR(X, \mathbb{Z}(r))_{tor}) \]

multiplied by

\[ \prod_{j=0}^{2d-2} (\det(C(M^{j,r}))^{(-1)^j} \prod_{j=0}^{2d-2} (\det(E(M^{j,r}))^{(-1)^j} ) \]

up to sign and powers of 2.

Remember that we are considering the scheme zeta-function, which in general is not the alternating product of Fontaine’s L-functions, because of problems at the bad primes.

For later computations, let \( \chi_W(r) = \chi(H^*_W(X, \mathbb{Z}(r))_{tor}) \), \( \chi_B(r) = \chi((H^*_B(X, \mathbb{Z}(r))^{+}_{tor}) \), and \( \chi_{DR}(r) = \chi((\tilde{H}^*_S(X, t(r))_{tor}) \).

Fontaine also has a more precise conjecture, which he calls the Bloch-Kato conjecture. The actual Bloch-Kato conjecture stated in [BK] requires that the weight be less than -2. But even then, although it seems highly likely that Fontaine’s Bloch-Kato conjecture is equivalent in that case to the actual Bloch-Kato conjecture, it is not clear that the proof is complete. Both versions of Bloch-Kato involve making extensive p-adic corrections by looking at Fontaine’s B-rings \( B_{DR}, B_{cris}, \text{ etc.} \) It is a very interesting and important problem to discuss whether this conjecture is equivalent to our conjecture, even for \( X \) smooth over \( \mathbb{Z} \).
Example 3.1.2. Let $X$ be as in Example 2.1.6, and $r = 0$. We have $\chi_{W}(0) = h/w$, $\text{det}(C(M_{0,0})) = R$ and $\text{det}(E(M_{0,0})) = 1$.

So our conjectured formula gives $\zeta^{*}(X,0) = hR/w$, which is compatible with the known answer $-hR/w$.

Example 3.1.3. Let $X$ be as in Example 2.1.6 and $r = 1$. $\chi_{W}(1) = h/w$. $\chi_{B} = \chi_{DR} = 1$

Let $M = M_{0,1}$.

We have the two exact sequences:

\[
C(M) \quad 0 \to O_{F}^* \otimes \mathbb{C} \to \text{Coker}(\gamma_{M}) \to \mathbb{C} \to 0.
\]

\[
E(M) \quad 0 \to H_{B}^{+}(X,\mathbb{Z}(1))_{\mathbb{C}} \to (O_{F})_{\mathbb{C}} \to \text{Coker}(\gamma_{M}) \to 0
\]

We regard Coker $(\gamma_{M}) = \text{Coker}(\alpha_{M})$ as having the integral structure which is defined by $H_{B}^{-}(\mathbb{Z}(1)) = \mathbb{Z}^{r_{1}+r_{2}}$.

The determinant of $C(M)$ is equal to $R$ (the classical regulator). The determinant of $E(M)$ is given by $(2\pi i)^{r_{2}}(\sqrt{d_{F}})^{-1}$, where the $(2\pi i)^{r_{2}}$ comes from changing integral structures from $H_{B}^{+}(1)$ to $H_{B}^{-}$ and the $\sqrt{d_{F}}$ comes from changing integral structures from $O_{F}$ to $H_{B}$. This gives us the formula $\zeta^{*}(X,1) = (hR/w)(2\pi i)^{r_{2}}/\sqrt{d_{F}}$, which agrees up to sign and powers of 2 with the classical formula $\zeta^{*}(X,1) = (hR/w)2^{r_{1}}(2\pi)^{r_{2}}/\sqrt{|d_{F}|}$ in view of the well-known relation $i^{r_{2}} = \sqrt{d_{F}/|d_{F}|}$ up to sign.

§4. The second version of the conjecture.

This version of the conjecture is more conceptual since it gives the zeta-value as a generalized Euler characteristic. The difficulty here is that the generalized Euler characteristic involves cohomology in a hypothetical Weil-étale site, whose existence is not known. However, the hypothetical properties possessed by this site are so natural, and the formulas so simple, that this point of view should not be neglected.

S 4.1 Complexes of Weil-étale sheaves $(A(r), C(r)$ and $D(r))$.

Let $X_{\mathbb{C}} = X \times \text{Spec} \mathbb{C}$. Let a rational integer $r$ be given. We assume the existence of a Grothendieck topology associated with $X$ (the Weil-étale site), of various complexes of sheaves on this site, and of maps between these complexes having certain properties.
We first assume the existence of the Weil-étale motivic complex \( \mathbb{Z}(r) \). This should have the property that the Weil-étale cohomology groups \( H^i_W(X, \mathbb{Z}(r)) \) are the ones defined in an ad hoc fashion in §2.1. We next assume the duality conjecture mentioned in §2.1, which asserts that we may identify \( R\Gamma_W(X, \mathbb{Z}(d - r)) \) with \( R\text{Hom}(R\Gamma_W(X, \mathbb{Z}(r)), \mathbb{Z}[-2d - 1]) \).

We also assume the existence of a Weil-étale complex of sheaves \( t(r) \) such that the cohomology groups \( H^i_W(X, \mathbb{Z}(r)) \) are naturally the groups \( H^i_B(X, \mathbb{Z}(r)) \) defined in §1, and a Weil-étale complex of sheaves \( t(r) \) such that the cohomology groups \( H^i_W(X, t(r)) \) are the groups \( \tilde{H}_{DR}(X, \mathbb{Z}(r)) = H^j(X, t(r)) \) defined in §2.2.

There should be a map \( g_{et} \) of complexes of Weil-étale sheaves on \( X \) from \( t_{\leq 2r} \mathbb{Z}(r) \) to \( t_{\leq 2r} \mathbb{Z}_B(r)^+ \), and a map \( g \) of complexes of Weil-étale sheaves from \( \mathbb{Z}(r) \) to \( \mathbb{Z}_B(r)^+ \) inducing \( g_{et} \) and inducing the zero map on \( t_{\geq 2r+1} \mathbb{Z}(r) \).

We let \( A(r) \) be the single complex associated with the map \( g \) of complexes.

There should exist a map \( \beta_{et} \) of complexes of étale sheaves on \( X \) mapping \( \mathbb{Z}(r) \) to a complex representing the derived exterior power \( \lambda^r\Omega_X/\mathbb{Z}[-r] \) such that

1) If \( r = 0 \), \( \beta_{et} \) is the natural map from \( \mathbb{Z} \) to \( \Omega_X \)

2) If \( r = 1 \), \( \beta_{et} \) is the map induced by the map \( \Omega_X^* \) to \( \Omega_X/\mathbb{Z} \) given by \( x \mapsto dx/x \), which obviously maps \( \mathbb{Z}(1) = \mathbb{G}_m[-1] \) to \( \Omega_X/\mathbb{Z}[-1] \).

3) For general \( r \), \( \beta_{et} \) induces the usual cycle map from \( H^d_{\mathrm{Zar}}(X, \mathbb{Z}(r)) \) to \( H^r(X_C, \Lambda^r\Omega_{X_C}) \), extended to the étale cycle group \( H^d_{et}(X, \mathbb{Z}(r)) \).

Observe that \( \beta_{et} \) induces a map from \( t_{\leq 2(d-r)}\mathbb{Z}(d-r) \) to \( \lambda^{d-r}\Omega_X/\mathbb{Z}[r-d] \), and so a map:

\[
R\text{Hom}(R\Gamma(X, \lambda^{d-r}\Omega_X/\mathbb{Z})), \mathbb{Z}[1-d][d-r][-d-2]) \rightarrow R\text{Hom}(R\Gamma_{et}(X, \mathbb{Z}(d-r)), \mathbb{Z}[-2d-1])
\]

Using on the one hand the duality which comes from our definition of Weil-étale cohomology, and the other hand Grothendieck-Serre duality for coherent sheaves of differentials (and assuming that derived exterior power behaves well with regard to duality), we get a map from \( R\Gamma_{et}(X, (\lambda^{r-1}\Omega_X/\mathbb{Z})[-r-2]) \) to \( R\Gamma_W(X, \mathbb{Z}(r)) \), and so a map from \( R\Gamma_{et}(X, \lambda^{r-1}\Omega_X/\mathbb{Z}[1-r][-3]) \) to \( R\Gamma_W(X, \mathbb{Z}(r)) \). We conjecture that this map is in fact induced by a map \( h \) of complexes of Weil-étale sheaves \( h : t(r)[-3] \rightarrow \mathbb{Z}(r) \) recalling that \( t(r) = \prod_{q=0}^{r-1} \lambda^q\Omega_X/\mathbb{Z}[-q] \).

We conjecture further that \( g \circ h = 0 \). Let \( C(r) \) be the single complex associated with the map \( h \) of complexes.
As usual we have the long hypercohomology exact sequences:

\[(4.1.1) \quad \cdots \rightarrow H^j_W(X, \mathbb{Z}(r)) \rightarrow H^{j+1}_W(X, C(r)) \rightarrow H^{j-2}_W(X, t(r)) \rightarrow H^{j+1}_W(X, \mathbb{Z}(r)) \rightarrow \cdots \]

\[(4.1.2) \quad \cdots \rightarrow H^j_W(X, \mathbb{Z}_B^+(r)) \rightarrow H^{j+1}_W(X, A(r)) \rightarrow H^{j+1}_W(X, \mathbb{Z}(r)) \rightarrow H^{j+1}_W(X_C, \mathbb{Z}_B^+(r)) \rightarrow \cdots \]

The map from \(H^{j-2}_W(X, t(r))\) to \(H^{j+1}_W(X, \mathbb{Z}(r))\) should be induced from the map described above if \(j \geq 2r\) and should be the zero map otherwise.

If \(j > 2r\), the sequence (4.1.1) splits (after tensoring with \(\mathbb{Q}\)) into short exact sequences

\[(4.1.3) \quad 0 \rightarrow H^j_W(X, \mathbb{Z}(r))_{\mathbb{Q}} \rightarrow H^{j+1}_W(X, C(r))_{\mathbb{Q}} \rightarrow H^{j-2}(X, t(r))_{\mathbb{Q}} \rightarrow 0 \]

If \(j < 2r\), the sequence (4.1.2) splits (after tensoring with \(\mathbb{Q}\)) into short exact sequences

\[(4.1.4) \quad 0 \rightarrow H^j_B(X, \mathbb{Z}(r))_{\mathbb{Q}} \rightarrow H^{j+1}_B(X, A(r))_{\mathbb{Q}} \rightarrow H^{j+1}_B(X, \mathbb{Z}(r))_{\mathbb{Q}} \rightarrow 0 \]

since the map from \(H^j_W(X, \mathbb{Z}(r)) \rightarrow H^j_B(X, \mathbb{Z}(r))^+\) is zero (up to torsion) by weight considerations.

Let \(D'(r)\) be the single complex obtained from \(A(r)\) and \(C(r)\) as follows:

\[
\begin{array}{cccccc}
0 & \rightarrow & t(r)[−3] & \xrightarrow{h} & \mathbb{Z}(r) & \xrightarrow{g} \mathbb{Z}_B^+(r) & \rightarrow & 0
\end{array}
\]

Let \(D(r) = D'(r)[1]\).

Then we also have the following two hypercohomology exact sequences:

\[(4.1.5) \quad \cdots \rightarrow H^j_W(X, A(r)) \rightarrow H^j_W(X, D(r)) \rightarrow H^{j-2}(X, t(r)) \rightarrow H^{j+1}_W(X, A(r)) \rightarrow \cdots \]

\[(4.1.6) \quad \cdots \rightarrow H^{j-1}_B(X, \mathbb{Z}(r))^+ \rightarrow H^j_W(X, D(r)) \rightarrow H^{j+1}_W(X, C(r)) \rightarrow H^j_B(X, \mathbb{Z}(r))^+ \rightarrow \cdots \]
Conjecture 4.1.1.

Let \( r \) be a rational integer. Then the complex of Weil-étale sheaves \( D(r) \) has the following properties:

a) The hypercohomology groups \( H^j_W(X, D(r)) \) satisfy the hypotheses of Definition A.2 needed to define an Euler characteristic. Specifically, \( H^j_W(X, D(r)) \) is finitely generated, zero for \( j < 0 \) and for all but finitely many \( j \), and there exist maps \( \delta_j \) from \( H^j_W(X, D(r))_\mathbb{C} \) to \( H^{j+1}_W(X, D(r))_\mathbb{C} \) such that \( (H^*_*W(X, D(r))_\mathbb{C}, \delta^*_C) \) is an acyclic complex.

We will give evidence and partial definitions for the maps \( \delta_M \) in the next section.

§4.2. Beilinson Chern class maps, \( H^1_\zeta \), and \( H^2_\zeta \)

We begin with some generalities about regulators. In the literature, the word "regulator" is used for both a map from an algebraic K-group to a Deligne cohomology group and for the determinant of such a map with respect to integral bases for the domain and range. We will try to avoid confusion by calling the maps "logarithm maps" and reserving the term "regulator" for determinants.

First, Beilinson defines logarithm maps from the weight \( r \) piece of algebraic K-theory to integral Deligne cohomology (the hypercohomology of the integral Deligne complex). Various people (Bloch [Bl2], Fan, [Fa], Goncharov, [Go] Kerr-Lewis-Müller-Stach [KLM]) have defined maps of complexes from motivic cohomology complexes (also called higher Chow group complexes) to Deligne complexes. Using the conjectured natural isomorphism between algebraic K-groups and motivic cohomology, we believe that these maps all induce the same maps from the motivic cohomology groups to Deligne cohomology, and we call these maps \( B_M \), so if \( M = M^{j,r} \) with \( j \leq 2r - 2 \), \( B_M \) maps \( H^{j+1}_\text{et}(X, \mathbb{Z}(r)) \) to Coker \( (\tilde{\gamma}_M) \).

In addition, in the number field case, Zagier defines polylogarithm maps from certain "higher Bloch groups" defined by generators and relations to finite-dimensional complex vector spaces. These vector spaces may be identified with the appropriate Deligne cohomology, and it is assumed that the "higher Bloch groups" give algebraic K-theory when both are tensored with \( \mathbb{Q} \). (Here we need to make use of the identification of \( O_F \otimes \mathbb{C} \) with \( \bigsqcup \sigma \mathbb{C} \) where the sum is taken over the distinct embeddings \( \sigma \) of the number field \( F \) into \( \mathbb{C} \). This introduces a factor of the square root of the discriminant.)

We also believe that, again making the indicated identifications, Zagier’s polylogarithm maps agree with the logarithm maps.
In order to define the regulators (determinants) we would like to have integral structures for the two complex vector spaces which are the domain and range of the logarithm maps. Since the domain is obtained by tensoring the conjecturally finitely-generated K-group with $\mathbb{C}$ there is a natural integral structure. However, since the map from Betti cohomology to de Rham cohomology is transcendental, there is no natural integral structure on Deligne cohomology. Beilinson gets around this by using highest exterior powers, where there is an integral structure induced from the Betti and de Rham integral structures. If $M$ is a pure motive, we call the "regulator" ($R(M)$) the determinant obtained by using this integral structure. We would like to call this the "Beilinson regulator" but that term has already been used for a different way of introducing integral structures. The basic problem in defining regulators is that the regulator which has the best relation to values of Dedekind zeta-functions at positive values $r$ is not the same as the one which has the best relation to values of zeta-functions at negative integers $1 - r$. (It is the same for $r = 0$ and $r = 1$, which leads to confusion). Traditionally, the Beilinson and (modified) Borel regulators ($R_B$) are defined so as to be best for negative integers. However the logarithm regulators, which are best for positive integers, are more natural.

More precisely, let again $M = M^{j-r}$ be the motive $H^j(X, \mathbb{Z}(r))$. Let $\tilde{H}_{DR}(M) = H^j_{DR}(X, \mathbb{Z}(r))/F^0$. Let $H_W(M) = H^{j+1}_W(X, \mathbb{Z}(r))$. We have the modified period map $\gamma_M : H^+_B(M) \to \tilde{H}_{DR}(M)$. If the weight $w(M) = j - 2r$ of $M$ is $\leq -3$ (resp. $\geq 1$) we define the Deligne cohomology (resp. homology) associated with $M$ to be the cokernel (resp. kernel) of $\gamma_M$.

If $w(M) \leq -3$, we define the regulator $R(M)$ as follows: Let $\log_M$ be the logarithm map from motivic cohomology $H_W(M)$ to Deligne cohomology. Then define $R(M)$ to be the determinant of $\log_M$ with respect to the integral bases coming from $H_W(M), H^+_B(M)$, and the derived exterior powers of $\tilde{H}_{DR}(M)$.

If $w(N) \geq 0$, define the regulator $R(N)$ as follows. Consider the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^+_B(N)_\mathbb{C} & \longrightarrow^i & H_B(N)_\mathbb{C} & \longrightarrow^j H_B(N) & \longrightarrow 0 \\
& & \downarrow^{\gamma_N} & & & \\
0 & \longrightarrow & F_0(N)_\mathbb{C} & \longrightarrow^k & H_{DR}(N)_\mathbb{C} & \longrightarrow t(N)_\mathbb{C} & \longrightarrow 0
\end{array}
$$

Let $M = N^*(1)$. Let $\beta_N : F_0(N)_\mathbb{C} \to H_B(N) = j \circ \gamma_N^{-1} \circ k$. We may identify $H_B(N)$ with the dual of $H_B(M)$ and $H_{DR}(N)$ with the dual of $H_{DR}(M)$ Under this identification,
we have \( \gamma_N = (\gamma_M^*)^{-1} \). Then we see that \( \text{Ker}(\beta_N) \) is the dual of \( \text{coker}(|\alpha_M^*|) \) and also \( \gamma_N \) induces an isomorphism from \( \text{Ker}(\alpha_N) \) to \( \text{Ker}(\beta_N) \). By definition \( H_W(N)_C \) is the dual of \( H_W(M)_C \). Let \( \log^{-1}(N) : \text{Ker}(\alpha_N) \to H_W(N)_C \) be the transpose of \( \log_M \). then define \( R(N) \) to be the determinant of \( \log^{-1}(N) \) with respect to the integral bases coming from \( H_W(N) \), \( H^+_B(N) \) and the derived highest exterior power of \( t(N) \).

We now examine the particular case of number rings. Let as above \( F \) be a number field, \( O_F \) the ring of integers in \( F \), and \( X = \text{Spec} \, O_F \). Let \( r > 1 \) be an integer, and let \( M \) be the motive \( H^0(X, \mathbb{Z}(r)) \). Recall that the logarithm map \( \log_M \) maps \( H^1_W(X, \mathbb{Z}(r))_C \) to \( t(M)_C = (O_F)_C/\gamma_M(H_B^j(M))_C \). We now identify \( (O_F)_C \) as usual with \( H^0_B(X_C) = \prod_\sigma \mathbb{C} \) and consequently \( t(M)_C \) with \( H_B^j(M) = \mathbb{C}^{a_r} \). Define \( R_r \) to be the determinant of the map from \( H^1_W(X, \mathbb{Z}(r))_C \) to \( \mathbb{C}^{a_r} \) with respect to the integral bases coming from \( H^1_W(X, \mathbb{Z}(r)) \) and \( \mathbb{Z}^{a_r} \).

Now look at \( N = H^0(X, \mathbb{Z}(1-r)) \). Our definition of \( R_{1-r} \) is as follows: We define \( \log^{-1}_N \) to be the transpose map \( \log^*_M : \mathbb{C}^{a_r} \to H^1(X, \mathbb{Z}(r))_C \). We have the map \( \gamma_N \), which maps \( H_B^j(N) \) to the dual space of \( \mathbb{C}^{a_r} \). Recall that if we identify \( \mathbb{C}^{a_r} \) with de Rham cohomology the map \( \gamma_N \) is given by multiplication by \((2\pi i)^{1-r}/\Gamma^*(1-r)\). We take a basis for \( \mathbb{C}^{a_r} \) given by \( \gamma_N(H_B^j(N)) \), and a basis for \( H^1_W(X, \mathbb{Z}(r))^* \) coming from the dual group \( H^2_W(X, \mathbb{Z}(1-r)) \). Then we let \( R_{1-r} \) be the determinant of \( \log^{-1}_N \) with respect to these bases, which is \(((2\pi i)^{1-r}(r-1)!)^{a_r} R_r \).

Note that, up to a rational number, \( R_{1-r} \) is what is usually called the Beilinson (or Borel) regulator. On the other hand, \( R_r \) is (presumably) the regulator defined by Zagier [Z] using polylogarithms. In [Z] Zagier says that this ought to be the same as the Borel regulator, but this is not accurate.

In this section we will explain how our conjectured maps \( \delta_i \) from \( H^i_W(X, D(r))_C \) to \( H^{i+1}_W(X, D(r))_C \), relate to logarithm maps and to Fontaine’s maps.

Let \( V = X_C \). First let \( j \leq 2r-2 \). We have the logarithm map \( \log_M \) mapping \( H_{et}^j(X, \mathbb{Z}(r)) \) to the Deligne cohomology group \( \text{Coker}(\gamma_M) \).

\[
\begin{array}{cccccc}
0 & \longrightarrow & (H^+_B(M)) & \overset{\tilde{\gamma}_M}{\longrightarrow} & H_{DR}(M)_C & \longrightarrow & \text{Coker}(\gamma_M) & \longrightarrow & 0 \\
& & & & & & & \uparrow_{C_M} & & \\
& & & & & & & H^{j+1}(X, \mathbb{Z}(r)) & & \\
\end{array}
\]

where \( \tilde{\gamma}_M \) is induced by the modified period map from Betti to de Rham cohomology defined in §3.1.
Now take the pullback of the top sequence, to obtain a short exact sequence

\[(4.2.1) \quad 0 \to H_B^1(M) \to H_\zeta^1(M) \to H^{j+1}(X, \mathbb{Z}(r)) \to 0\]

We conjecture that \(H_\zeta^1(M)\mathbb{Q}\) may be identified with \(H^{j+1}(X, A(r))\mathbb{Q}\), and that after tensoring with \(\mathbb{Q}\) (4.2.1) may be identified with (4.1.4).

We now obviously get a map \(\phi_1\) from \(H_\zeta^1(M)\mathbb{C}\) to \(\tilde{H}_{DR}(M)\mathbb{C}\). If \(j \leq 2r - 3\), this map is an isomorphism if we believe, as Beilinson conjectures, that for \(j \leq 2r - 3\), the logarithm map \(\log_M\) induces an isomorphism from \(H_{et}^{j+1}(X, \mathbb{Z}(r))\mathbb{C}\) to \(\text{Coker}(\tilde{\alpha}_M) = \text{Coker}(\tilde{\gamma}_M)\). For \(j \leq 2r - 3\), define \(H_\zeta^2(M)\) to be \(\tilde{H}_{DR}(M)\), so we have a conjectured isomorphism from \(H_\zeta^1(M)\mathbb{C}\) to \(H_\zeta^2(M)\mathbb{C}\).

Now let \(j = 2r - 2\). In this case the Beilinson conjecture says that the logarithm map induces an isomorphism between \(H_{et}^{j+1}(X, \mathbb{Z}(r))\mathbb{C}\) and the kernel of the map from \(\text{Coker}(\tilde{\alpha}_M)\) to \(H_\zeta^2(M)\mathbb{C}\) where \(H_\zeta^2(M)\) is defined to be the dual of \(H_{et}^{2d-2r}(X, \mathbb{Z}(d-r))\). (This is a variant of Beilinson’s conjecture which is implicit in Fontaine’s six-term exact sequence). As before, Beilinson’s conjecture implies that there is an induced isomorphism from \(H_\zeta^1(M)\mathbb{C}\) to the kernel of the induced map from \(\tilde{H}_{DR}(M)\mathbb{C}\) to \(H_\zeta^2(M)\mathbb{C}\). In fact, this map comes from a map from \(\tilde{H}_{DR}(M)\mathbb{C}\) to \(H_\zeta^2(M)\mathbb{C}\), and in this case we call this kernel of this map \(H_\zeta^2(M)\).

So again we have the conjectured isomorphism from \(H_\zeta^1(M)\mathbb{C}\) to \(H_\zeta^2(M)\mathbb{C}\).

We now consider the dual situation. Let \(j \geq 2r\). Let \(N = M^*(1) = H^{2d-2-j}(X, \mathbb{Z}(d-r))\). Then the dual \(C_N^*\) of the logarithm map for \(N\) maps the Deligne homology \(\text{Ker}(\tilde{\gamma}_M)\mathbb{C}\) to \(H_\tilde{W}^j(X, \mathbb{Z}(r))\mathbb{C}\). We look at the exact sequence

\[0 \to \text{Ker}(\tilde{\gamma}_M) \to H_B^+(M) \to \tilde{H}_{DR}(M)\mathbb{C} \to 0\]

and take the pushout from the map \(C_N^*\) to obtain

\[(4.2.2) \quad 0 \to H_{\tilde{W}}^{j+2}(X, \mathbb{Z}(r))\mathbb{C} \to H_\zeta^2(M) \to \tilde{H}_{DR}(M)\mathbb{C} \to 0\]

We conjecture that \(H_\zeta^2(M)\) may be identified with \(H^{j+3}(X, C(r))\mathbb{C}\) and that after tensoring with \(\mathbb{C}\), (4.2.2) may be identified with (4.1.3).
We now obviously get a map $\phi_2$ from $H^+_B(M)_C$ to $H^2_\zeta(M)$ If $j \geq 2r + 1$, this map is an isomorphism modulo Beilinson’s conjectures. For $j \geq 2r + 1$, define $H^1_\zeta(M)$ to be $H^+_B(M)$, so we have a conjectured isomorphism again from $H^1_\zeta(M)_C$ to $H^2_\zeta(M)_C$.

Now let $j = 2r$. Analogously to what we did before, we define $H^1_\zeta(M)$ to be the cokernel of the map from $H^{2r}_{et}(X,\mathbb{Z}(r))$ to $H^{2r}_{et}(X,\mathbb{Z}(r))^+$, and again get a conjectured isomorphism from $H^1_\zeta(M)_C$ to $H^2_\zeta(M)$.

Finally, if $j = 2r - 1$, let $H^1_\zeta(M) = H^{2r}(X,A(r))$ and let $H^2_\zeta(M) = H^{2r+2}(X,C(r))$.

Then there should be exact sequences:

\begin{align}
(4.2.3) & \quad 0 \to H^{2r}_{et}(X,\mathbb{Z}(r))_\mathbb{Q}^+ \to H^1_\zeta(M)_\mathbb{Q} \to H^{2r}_{et}(X,\mathbb{Z}(r))_\mathbb{Q} \to 0 \\
(4.2.4) & \quad 0 \to H^{2r+1}_{et}(X,\mathbb{Z}(r))_\mathbb{Q} \to H^2_\zeta(M)_\mathbb{Q} \to \tilde{H}_{DR}(M)_\mathbb{Q} \to 0
\end{align}

and isomorphisms $\theta_M$ mapping $H^1_\zeta(M)_C$ to $H^2_\zeta(M)_C$.

These isomorphisms should induce period isomorphisms from $H^+_B(M)_C$ to $\tilde{H}_{DR}(M)_C$ and height-pairing isomorphisms from $H^{2r+1}_{et}(X,\mathbb{Z}(r))_\mathbb{C}$ to $H^{2r}_{et}(X,\mathbb{Z}(r))_\mathbb{C}$.

(See [Li2] for a more detailed description of this using 1-motives if $r = 1$).

Summing up, let $M = M^{j,r}$. We believe that the following picture holds:

The sequence 4.1.5 breaks up after tensoring with $\mathbb{C}$ into short exact sequences:

\begin{align}
(4.2.5) & \quad 0 \to H^1_\zeta(M)_C \to H^{j+1}(X,D(r))_C \to H^2_\zeta(M^{j-1,r})_C \to 0 \\
& \quad \text{for } j \leq 2r.
\end{align}

the sequence 4.1.6 breaks up after tensoring with $\mathbb{C}$ into short exact sequences:

\begin{align}
(4.2.6) & \quad 0 \to H^1_\zeta(M)_C \to H^{j+1}(X,D(r))_C \to H^2_\zeta(M^{j-1,r})_C \to \\
& \quad \text{for } j \geq 2r + 1.
\end{align}

We can choose splittings of these sequences such that

\begin{align}
(4.2.7) & \quad H^{j+1}_{et}(X,D(r))_\mathbb{C} \simeq H^1_\zeta(M)_C \oplus H^2_\zeta(M^{j-1,r})_C
\end{align}
for all $j$.

We hope that these splittings can in fact be made canonical.

Also, we have isomorphisms $\theta_M: H^1_\zeta(M)_{\mathbb{C}} \simeq H^2_\zeta(M)_{\mathbb{C}}$ for all $j$. We immediately see that we can define maps $\delta^j$ from $H^j(X, D(r))_{\mathbb{C}}$ to $H^{j+1}(X, D(r))_{\mathbb{C}}$ such that the complex $H^*_W(X, D(r))_{\mathbb{C}}, \delta^*$ is acyclic.

We observe that it follows from the definitions of the groups $H^1_\zeta(M)$ and $H^2_\zeta(M)$ that we have the exact sequences:

\begin{equation}
0 \to H^0_f(M)_{\mathbb{Q}} \to H^+_B(M)_{\mathbb{Q}} \to H^1_\zeta(M)_{\mathbb{Q}} \to H^1_f(M)_{\mathbb{Q}} \to 0
\end{equation}

\begin{equation}
0 \to H^1_c(M)_{\mathbb{C}} \to H^2_\zeta(M)_{\mathbb{C}} \to \tilde{H}_{DR}(M)_{\mathbb{C}} \to H^2_c(M)_{\mathbb{C}} \to 0
\end{equation}

of (conjecturally) finitely-dimensional vector spaces. Taking into account the isomorphism $\theta_M$ from $H^1_\zeta(M)_{\mathbb{C}}$ to $H^2_\zeta(M)_{\mathbb{C}}$, we obtain by diagram-chasing the existence of the modified Fontaine sequence $C(M)$ defined in §3. Note that all the maps in (4.2.8) and (4.2.9) are defined integrally, and only $\theta_M$ is transcendental. The reader can easily verify that the induced map from $H^+_B(M)_{\mathbb{C}}$ to $\tilde{H}_{DR}(M)_{\mathbb{C}}$ is our modified period map $\tilde{\gamma}_M$.

All computations below are valid only up to finite 2-groups.

**Example 4.2.1.** Let $X$ be as in Example 2.1.6. Let $r = 0$. Then

\[ H^0_W(X, D(0)) = 0 \]

\[ H^1_W(X, D(0)) = (H^0_B(X, \mathbb{C}, \mathbb{Z}))^+ / H^0_W(X, \mathbb{Z}) \simeq \mathbb{Z}^{r_1+r_2}/\mathbb{Z} \]

\[ H^j_W(X, D(0)) = H^j_W(X, \mathbb{Z}), \text{ for } j \geq 2 \text{ so we have:} \]

\[ 0 \to \text{Hom}(\text{Pic}(O_F), \mathbb{Q}/\mathbb{Z}) \to H^2_W(X, D(0)) \to \text{Hom}(O_F^*, \mathbb{Z}) \to 0 \]

\[ H^3_W(X, D(0)) = \text{Hom}(\mu_F, \mathbb{Q}/\mathbb{Z}) \]

\[ H^j(X, D(0)) = 0 \text{ for } j \neq 0, 1, 2, 3 \]


Example 4.2.2. Let $X$ be as in Example 2.1.6. Let $r < 0$. Then

$$H^1_W(X, D(r)) = H^0_B(X_C, \mathbb{Z}(r))^+ \simeq \mathbb{Z}^{br}$$

$H^0_W(X, D(r))$ fits into an exact sequence

$$0 \to \text{Hom}(K_{-2r}(O_F), \mathbb{Q}/\mathbb{Z}) \to H^2_W(X, D(r)) \to \text{Hom}(K_{1-2r}(O_F), \mathbb{Z}) \to 0$$

$$H^3_W(X, D(r)) = \text{Hom}((K_{1-2r}(O_F))_{tor}, \mathbb{Q}/\mathbb{Z})$$

$$H^j_W(X, D(r)) = 0 \quad \text{for} \quad j \neq 1, 2, 3$$

Example 4.2.3. Let $X$ be as in Example 2.1.6, and let $r = 1$. $H^1_W(X, A(1))$ fits into an exact sequence:

$$0 \to H^0_B(X_C, \mathbb{Z}(1))^+ \to H^1_W(X, A(1)) \to O_F^* \to 0$$

$$(H^0_B(X_C, \mathbb{Z}(1))^+ \simeq \mathbb{Z}^{r_2})$$

$$H^2_W(X, A(1)) = \text{Pic}(O_F)$$

$$H^3_W(X, A(1)) = \mathbb{Z}$$

$$H^j_W(X, A(1)) = 0 \quad \text{for} \quad j \neq 1, 2, 3$$

$$H^1_W(X, D(1)) = H^1_W(X, A(1))$$

$H^2_W(X, D(1))$ and $H^3_W(X, D(1))$ fit into an exact sequence

$$0 \to \text{Pic}(O_F) \to H^2_W(X, D(1)) \to O_F \to \mathbb{Z} \to H^3_W(X, D(1)) \to 0$$

where the map from $O_F$ to $\mathbb{Z}$ is the trace.

$$H^j_W(X, D(1)) = 0 \quad \text{for} \quad j \neq 1, 2, 3$$

Example 4.2.4. Let $X$ be as in Example 2.1.6, and let $r > 1$. $H^1_W(X, A(r))$ fits into an exact sequence

$$0 \to H^0_B(X_C, \mathbb{Z}(r))^+ \to H^1_W(X, A(r)) \to K_{2r-1}(O_F) \to 0$$

$$H^3_W(X, A(r)) = K_{2r-2}(O_F)$$
\[ H^j_W(X, A(r)) = 0 \quad \text{for} \quad j \neq 1, 2 \]

\[ H^1_W(X, D(r)) = H^1_W(X, A(r)) \]

\[ H^2_W(X, D(r)) \] fits into an exact sequence

\[ 0 \to K_{2r-2}(O_F) \to H^2_W(X, D(r)) \to H^0(X, t(r)) \to 0 \]

where \( H^0(X, t(r)) \) is isomorphic to \( O_F \).

\[ H^3_W(X, D(r)) = H^1(X, t(r)) \cong \Omega^{(r-1)}_{X/\mathbb{Z}} \]

(See Example 2.2.2).

\[ H^j_W(X, D(r)) = 0 \quad \text{for} \quad j \neq 1, 2, 3 \]

§ 4.3. The second conjectural formula for special values of the zeta-function.

Assume that \( X \) is regular and projective, flat, (but not necessarily smooth or geometrically connected) over the ring of integers \( \mathbb{Z} \). Let \( d \) be the dimension of \( X \). Let \( X_C = X \times_{\mathbb{Z}} \mathbb{C} \). Let \( H^m_B(X_C, \mathbb{C}) \) have the Hodge decomposition \( \bigoplus_{p+q=m} H^{p,q} \) where \( H^{p,q} = H^p(X_C, \Lambda^q \Omega_{X_C/\mathbb{C}}) \), and let \( h^{p,q} \) be the dimension of \( H^{p,q} \).

**Conjecture 4.3.1.**

Let \( r \) be a rational integer. Then the complex of Weil-étale sheaves \( D(r) \) whose existence we postulated in Section 3 has the following properties:

a) The hypercohomology groups \( H^j_W(X, D(r)) \) satisfy the hypotheses of Definition A.2 needed to define an Euler characteristic. Specifically, \( H^j_W(X, D(r)) \) is finitely generated, zero for \( j < 0 \) and for all but finitely many \( j \), and the maps \( \theta_j \) mentioned in §4.3 induce \( \mathbb{C} \)-linear maps \( \theta^j_C \) from \( H^j_W(X, D(r))_C \) to \( H^{j+1}_W(X, D(r))_C \) such that \( (H^*_W(X, D(r))_C, \theta^*_C) \) is an acyclic complex.

b) \( \zeta^*(X, r) = \chi_W(X, D(r)) \), up to sign and powers of 2.

**Example 4.3.2.**

Let \( X \) be as in Example 2.1.6, and let \( r = 0 \). Let \( v \) denote an infinite place of \( F \).

The map \( \theta_1 \) from \( (H^1_W(X, D(0)))_C \cong (\bigoplus_v \mathbb{Z}/\mathbb{Z})_C \) to \( (H^2_W(X, D(0)))_C = Hom(O_F^*, \mathbb{C}) \) is given by \( 1_v \to (u \to \log |u|_v) \). This vanishes on \( \mathbb{Z} \) by the product formula. Its determinant is given by the classical regulator \( R \). Since the order of \( (H^2_W(X, D(0)))_{tor} = \mathbb{Z}/\mathbb{Z} \) is 1, \( \theta_1 \) is an isomorphism. Hence \( \chi_W(X, D(0)) \) is equivalent to \( \log |\cdot|_v \).
\( \text{Hom}(\text{Pic}(O_F), \mathbb{Q}/\mathbb{Z}) \) is the class number \( h \), and the order of \( H^3_W(X, D(0)) \) is the number \( w \) of roots of unity in \( F \), we obtain the classical formula \( \zeta^*(F, 0) = -hR/w \), up to sign and powers of 2.

**Example 4.3.3.** Let \( X \) be as in example 2.1.6, and let \( r = 1 \).

We recall from Example 4.2.3 that we have the two exact sequences

(4.3.1) \[ 0 \to H^0_B(X, \mathbb{Z}(1)) \to H^1_W(X, D(1)) \to O_F^* \to 0 \]

(4.3.2) \[ 0 \to \text{Pic}(O_F) \to H^2_W(D(1)) \to O_F \to \mathbb{Z} \to H^3_W(X, D(1)) \to 0 \]

and \( H^j_W(X, D(r)) = 0 \) for \( j \neq 1, 2, 3 \).

We have the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0_B(X, \mathbb{Z}(1)) & \longrightarrow & H^1_W(X, D(1))_C & \longrightarrow & O_F^* \otimes \mathbb{C} & \longrightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\psi} & & \downarrow{\rho} & & \\
0 & \longrightarrow & H^0_B(X, \mathbb{Z}) & \longrightarrow & (H^0_B(X, \mathbb{Z})_C)^0 & \longrightarrow & (H^0_B(X, \mathbb{Z})_C^+)^0 & \longrightarrow & 0
\end{array}
\]

where \( ((H^0_B(X, \mathbb{Z})_C)^0) = \text{Ker}(H^0_B(X, \mathbb{Z})_C) \to \mathbb{Z}) \),

and \( (H^0_B(X, \mathbb{Z})_C^0) = \text{Ker}(H^0_B(X, \mathbb{Z})_C) \to \mathbb{Z}) \).

The determinant of \( \rho \) is the classical regulator \( R \), and the determinant of \( \beta \) is \((2\pi i)^r \), so the determinant of \( \psi \) is equal to \( R \) multiplied by \((2\pi i)^r \).

In order to get the determinant of the map \( \theta_1 \) from \((H^1_W(X, D(1)))_C \) to \((H^2_W(X, D(1)))_C \) which appears in our zeta-function formula, we have to compose \( \psi \) with the map \( \phi \) from \((H^0_B(X, \mathbb{Z})_C^0 \) to the kernel of the map from \((O_F)_C \to \mathbb{Z}_C \) induced by the trace. The determinant of \( \phi \) is easily seen to be \( \sqrt{d_F^{-1}} |\text{Coker}(Tr)| \), so the determinant of \( \theta_1 \) is given by \( R((2\pi i)^r)^2 \sqrt{d_F^{-1}} \).

Putting everything together we get that in the case \( d = 1, r = 1 \) the second zeta-function conjecture says

(4.3.3) \[ \zeta^*(\text{Spec} O_F, 1) = (hR/w)(2\pi i)^2 \sqrt{d_F^{-1}} \]
up to sign and powers of 2, which agrees with the classical formula, taking into account the well-known fact that $i^{r/2} = \pm \sqrt{d_F}/\sqrt{|d_F|}$. 

follows from Example 4.2.2 that $H^2_W(X, D(r))_{\text{tor}} \simeq \text{Hom}(K_{-2r}(O_F), \mathbb{Q}/\mathbb{Z})$. We also find that $H^3_W(X, D(r))_{\text{tor}} = H^3_W(X, D(r)) \simeq \text{Hom}(K_{1-2r}(O_F))_{\text{tor}}, \mathbb{Q}/\mathbb{Z})$. We then look at the map $(\rho_r)^t$ from $\mathbb{C}^{b_r}$ to $H^2_W(X, D(r))_C$. This latter is the dual space to $K_{1-2r}(O_F) \otimes \mathbb{C}$, so we take as its basis the dual basis to a $\mathbb{Z}$-basis for $K_{1-2r}(O_F)/\text{torsion}$, and the standard basis for $\mathbb{C}^{b_r}$. But we want our map $\theta_r$ to map $(H^1_W(X, D(r)))_C$ to $(H^2_W(X, D(r))_C = (H^0_B(X, \mathbb{Z}(r)^+)\mathbb{C})$. The dual of the logarithm map for $1-r$ becomes our map $\theta_r$, but we have to change the basis by $(2\pi i)^r(r-1)!$. Our general conjecture then says that

\begin{equation}
\zeta^*(\text{Spec } O_F, r) = |K_{-2r}(O_F)|/(K_{1-2r}(O_F))_{\text{tor}}|R_r = 
\end{equation}

$$= |K_{-2r}(O_F)|/(K_{1-2r}(O_F))_{\text{tor}}|R_{1-r}((2\pi i)^r(r-1)!)^{b_r}$$

up to sign and powers of 2 (since $a_r = b_{1-r}$, so $R_{1-r} = ((2\pi i)^r(r-1)!)^{-b_r}R_r$. This is in agreement with our long-ago conjecture ([Li3]) about $\zeta^*(F, r)$, remembering that we allowed some appropriate "noraiiization" of the regulator.

**Example 4.3.5.**

Let $X$ be as in Example 2.1.6, and let $r > 1$. Recall from §4.3 that we have the two exact sequences

\begin{equation}
0 \rightarrow H^0_B(X, \mathbb{Z}(r))^+ \rightarrow H^1_W(X, D(r)) \rightarrow K_{2r-1}(O_F) \rightarrow 0
\end{equation}

\begin{equation}
0 \rightarrow K_{2r-2}(O_F) \rightarrow H^2_W(X, D(r)) \rightarrow H^0_B(X, t(X, r)) \rightarrow 0
\end{equation}

$H^3_W(X, D(r)) \simeq H^1_{\text{et}}(X, t(X, r)) \simeq (\Omega_{X/\mathbb{Z}})^{r-1}$ and $H^i_W(X, D(r)) = 0$ for $i \neq 1, 2, 3$.

We have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0_B(X, \mathbb{Z}(r))^+_C & \rightarrow & H^1_W(X, D(r))_C & \rightarrow & H^1_W(X, \mathbb{Z}(r))_C & \rightarrow & 0 \\
& | \beta_r | & & | \psi | & & | B_r | & & & \\
0 & \rightarrow & H^0_B(X, \mathbb{Z})^+_C & \rightarrow & H^1_B(X, \mathbb{Z})_C & \rightarrow & H^0_B(X, \mathbb{Z})^-_C & \rightarrow & 0
\end{array}
\]
Let \( v = |(H^1_W(X,\mathbb{Z}(r))_{tor}|. \)

Our conjecture asserts that \( \zeta(X,r) = \chi(X,D(r)) = |K_{2r-2}(O_F)|\det(\psi)/vd_F^{r-1}. \) Lemma A4 tells us that \( v/|K_{2r-1}(O_F)_{tor}| = \det(\psi)/R_r(2\pi i^r)/(r - 1)!)^{br}. \)

In order to get the determinant of the map \( \theta_r \) from \( (H^1_W(X,D(r)))_C \) to \( (H^2_W(X,D(r)))_C \) which appears in our zeta-function formula, we have to compose \( \psi \) with the map \( \phi \) from \( H^0_B(X,\mathbb{Z})_C \) to \( H^0_{et}(X,O_X)_C. \) The determinant of \( \phi \) is well-known to be \( \sqrt{d_F}^{-1}. \)

Putting everything together we get that in the case \( d = 1, r > 1 \) the zeta-function conjecture says

\[
\zeta^*(\text{Spec}O_F, r) = \chi(D(r)) = |K_{2r-2}(O_F)|/|K_{2r-1}(O_F)_{tor}|.R_r((2\pi i^r)/(r - 1)!)^{br}.\sqrt{d_F^{-1-2r}}
\]

up to sign and powers of 2.

**Proposition 4.3.6.** Let \( r \geq 2 \) be an integer. Formulas (4.3.4) and (4.3.7) are compatible with the functional equation for the Dedekind zeta-function.

**Proof:** We first recall the functional equation. Define \( \phi(s) \) by

\[
\phi(s) = \Gamma(s/2)^{r_1}\Gamma(s)^{r_2}(2^{-r}2\sqrt{|d_F|\pi^{-n/2}})^s\zeta(\text{Spec}O_F, s)
\]

where \( n = r_1 + 2r_2 = [F : \mathbb{Q}]. \) Then \( \phi(s) = \phi(1 - s) \)

The compatibility of (4.3.4) and (4.3.7) immediately reduces to proving that

\[
((2\pi i^r)/(r - 1)!)^{br}.\sqrt{d_F^{-1-2r}}\Gamma(r/2)^{r_1}\Gamma(r)^{r_2} = 
\]

\[
= \Gamma^*((1 - r)/2)^{r_1}\Gamma^*(1 - r)^{r_2}(2^{-r}\sqrt{|d_F|\pi^{-n/2}})^{1-2r}((2\pi i)^{1-r}(r - 1)!)^{a_r}
\]

up to sign and powers of 2. \( \Gamma^*(r) \) is the leading term of the Laurent series expansion of \( \Gamma(z) \) at \( z = r). \)

**Lemma 4.3.7.** \( \Gamma(r)\Gamma(r/2)^{-1}\Gamma^*((1 - r)/2) = \sqrt{\pi} \) if \( r \) is even, and \( = \sqrt{\pi}^{-1} \) if \( r \) is odd, up to sign and powers of 2.

**Proof:** This is an immediate consequence of the functional equation for the \( \Gamma \)-function.
\[ (4.3.9) \quad \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z \]

and the duplication formula:

\[ (4.3.10) \quad \Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\Gamma(2z)\sqrt{\pi} \]

We also recall the classical formula from algebraic number theory

\[ (4.3.11) \quad \sqrt{d_F}/\sqrt{|d_F|} = \pm i^{r^2} \]

Then, considering the cases when \( r \) is even and \( r \) is odd separately, and using the functional equation one more time, Lemma 4.3.7 and equation \( (4.3.11) \) yield the proposition.

§5. Compatibility of the first conjecture with the functional equation for \( X \) smooth over \( \mathbb{Z} \).

In this section we discuss the compatibility of the first conjecture with the well-known conjectured functional equation due to Serre [Se2]. Let \( X \) as usual be a regular scheme projective and flat over \( \text{Spec} \mathbb{Z} \) and of Krull dimension \( d \), so that the relative dimension over \( \text{Spec} \mathbb{Z} \) is \( d - 1 \).

We first remind the reader that our conjecture is for special values of the scheme zeta function \( \zeta(X, s) \), which is defined to be the product of the scheme zeta-functions \( \zeta(X_p) \) of the fibers \( X_p \). We have \( \zeta(X_p, s) = \prod_{j=0}^{2d-2} L_j^j(X_p, s)(-1)^j \) and if \( p \) is a good prime \( L_j^j(X_p, s) \) is equal to the factor at \( p \) of the Hasse-Weil L-function \( L_{HW}^j(X, s) \). So if \( X \) is smooth over a number ring \( O_F \) we have \( \zeta(X, s) = \prod_{j=0}^{2d-2} L_{HW}^j(X, s)(-1)^j \).

Serre’s functional equation relates \( L_{HW}^j(X, s) \) to \( L_{HW}^j(X, j+1-s) \). In order to use this to get a conjectured functional equation for \( \zeta(X, s) \) we must first assume that \( X \) is smooth over \( O_F \), and then use the fact that in this case Poincaré duality and the Riemann hypothesis for varieties over finite fields imply that \( L_{HW}^j(X, s) = L_{HW}^{2d-2-j}(X, s+d-1-j) \) so the functional equation relates \( L_{HW}^j(s) \) to \( L_{HW}^{2d-2-j}(X, d-s) \), and we can multiply them together to get a functional equation relating \( \zeta(X, s) \) to \( \zeta(X, d-s) \).

From now on in this section we will assume that \( X \) is smooth over \( \text{Spec} \mathbb{Z} \), and we will write \( L_j^j(X, s) \) instead of \( L_{HW}^j(X, s) \). Everything should work similarly in the case where
is a certain positive integer, and \( \Gamma \)

\[ (s) \]

\[ \]
We observe that it is an easy computation that $\Gamma_v^j(s) = \Gamma_v^{2d-2-j}(s+d-j-1)$, so that with our earlier observation that at least in the smooth case $L^j(s) = L^{2d-2-j}(s+d-j-1)$ we obtain that Serre’s functional equation is equivalent to the functional equation $\phi^j(s) = \pm \phi^{2d-2-j}(d-s)$.

**Theorem 5.1.1.** Let $v$ be a real place of $F$.

If $j$ is even, $(\Gamma_v^j)^*(r)/(\Gamma_v^{2d-j})^*(d-r)$ is equal up to sign and powers of 2 to $\prod_p (\Gamma^*(r-p))^{h_v(p,q)\pi - B_v^j(r-j/2)}$. (The products run over $0 \leq p \leq B_v^j$).

If $j$ is odd, $(\Gamma_v^j)^*(r)/(\Gamma_v^{2d-j})^*(d-r)$ is equal up to sign and powers of 2 to $\prod_p (\Gamma^*(r-p))^{h_v(p,q)\pi - B_v^j(r-(j+1)/2)}$.

Proof. We consider the case when $j$ is even. (The case when $j$ is odd is similar, but simpler). Let $j = 2n$. We compute the terms not involving $n$ or $d-1-n$. Fix $v$ and let $q = j - p$. Let $p' = d - 1 - p$ and $q' = d - 1 - q$, so $p' + q' = 2d - 2 - j$. We have

$$\prod_{p<q} \Gamma_C^*(r-p)^{h_v(p,q)} / \prod_{p'<q'} \Gamma_C^*(d-r-p')^{h_v(p',q')} =$$

$$\prod_{p<q} \Gamma_C^*(r-p)^{h_v(p,q)} / \prod_{p>q} \Gamma_C^*(1-r+p)^{h_v(p,q)}$$

since $h_v(p,q) = h_v(p',q')$ by Serre duality.

By definition of $\Gamma_C$, this is equal to

$$\prod_{p<q} \Gamma^*(r-p)^{h_v(p,q)} / \prod_{p>q} \Gamma^*(1-r+p)^{h_v(p,q)}$$

multiplied by

$$(2\pi)^{-\left(\sum_{p<q}(h_v(p,q)(r-p) - \sum_{p>q}(h_v(p,q)(1-r+p))\right)}$$

This product is then equal to

$$\pm \prod_{p \neq n} \Gamma^*(r-p)^{h_v(p,q)}(2\pi)^{-\left(\sum_{p,q}(h_v(p,q)((r-p)-(1-r+j-p))\right)}$$

because of the relation $\Gamma^*(r) = \pm \Gamma^*(1-r)^{-1}$ for integral $r$ which follows immediately from the functional equation for the Gamma function. We then obtain:
\[(5.1.1) \quad \pm \prod_{p \neq n} \Gamma^*(r - p)^{h_v(p,q)}(2\pi)^{-\left(B_j^v - h_v(n,n)\right)(r-(j+1)/2)} \]

We now look at the terms involving \(n\) with \(v\) still fixed.

We first observe that the functional equation for the Gamma function implies that 
\[\Gamma^*\left(\frac{a}{2}\right)\Gamma^*\left(\frac{2-a}{2}\right) = \pm \pi\] if \(a\) is an odd integer and equals \(\pm 1\) if \(a\) is an even integer.

We compute:
\[\Gamma^*_\mathbb{R}\left((r - n)\right)^{h_v(n,+)}\Gamma^*_\mathbb{R}\left((r - n + 1)\right)^{h_v(n,-)} \]
multiplied by
\[\Gamma^*_\mathbb{R}\left((d - r - (d - 1 - n))\right)^{h_v(n,+)}\Gamma^*_\mathbb{R}\left(d + 1 - r - (d - 1 - n)\right)^{h_v(n,-)} \]
which we rewrite as

\[(5.1.2) \quad (\Gamma^*\left(\frac{r - n}{2}\right)\Gamma^*\left(\frac{1 - r + n}{2}\right)^{-1})^{h_v(n,+)} \]
multiplied by

\[(5.1.3) \quad \Gamma^*\left(\frac{r - n + 1}{2}\right)\left(\Gamma^*\left(\frac{n + 2 - r}{2}\right)^{-1}\right)^{h_v(n,-)} \]
multiplied by

\[(5.1.4) \quad \pi^{-(h_v(n,+)(2r-2n-1)/2 + h_v(n,-)(2r-2n-1)/2)} \]

First assume that \(r - n\) is odd. By the functional equation,

\[(5.1.5) \quad \Gamma^*\left(\frac{1 - r + n}{2}\right)^{-1} = \pm \Gamma^*\left(\frac{r - n + 1}{2}\right) \]

\[(5.1.6) \quad \Gamma^*\left(n + 2 - r\right)^{-1} = \pm \pi^{-1} \Gamma^*\left(r - n\right)/2 \]
Now recall the duplication formula for the Gamma function;

\[(5.1.7) \quad \Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)/\sqrt{\pi}\]

Then (5.1.2) becomes (using (5.1.5) and (5.1.7))

\[
(\pm 2^{n-r+1}\Gamma^*(r - n)\sqrt{\pi})^{h_v(n, +)}
\]

and (5.1.3) becomes

\[
(\pm 2^{n-r}\Gamma^*(r - n)(\sqrt{\pi})^{-1})^{h_v(n, -)}
\]

while (5.1.4) is

\[(5.1.8) \quad \pi^{-h_v(n,n)(r-(j+1)/2)}\]

So, up to sign and powers of 2, our product has become

\[(5.1/9) \quad \Gamma^*(r - n)^{h_v(n,n)}\pi^{(h_v(n,n)(r-(j+1)/2)+h_v(n,n)/2-h_v(n,-)/2)\}

Multiplying (5.1.1) by (5.1.9) we get

\[(5.1.10) \quad \prod_p \Gamma^*(r - p)^{h_v(p,q)}\pi^{-(B_v^j(r-(j+1)/2))\pm h_v(n,\pm)+h_v(n,-)/2}\]

which equals

\[(5.1.11) \quad \prod_p \Gamma^*(r - p)^{h_v(p,q)}\pi^{-(B_v^j(r-j/2)\pm (B_v^j)^\pm)}\]

if \(n\) is even (so \(r\) odd) and equals

\[(5.1.12) \quad \prod_p \Gamma^*(r - p)^{h_v(p,q)}\pi^{-(B_v^j(r-j/2)+ (B_v^j)^\pm)}\]

if \(n\) is odd (so \(r\) even). since \((B_j - h_v(n,n)/2) + h_v(n,n)^+\) is equal to \((B_v^j)^+\) if \(n\) is even and \((B_v^j)^-\) if \(n\) is odd.

The proof for \(r - n\) even is identical, except for switching \(h_v(n, +)\) and \(h_v(n, -)\).
Theorem 5.1.2. Let \( v \) be a complex place of \( F \). Then \((\Gamma_v^j)^*(r) / (\Gamma_v^{2d-2-j})^*(d-r)\) is equal up to sign and powers of 2 to \((\prod_p \Gamma^*(r-p))^{2h(p,q)}\pi^{-B^j_2(2r-(j+1))}\)

Proof. Let \( q = j - p, p' = d - 1 - p \) and \( q' = d - 1 - q \).

We write \((\Gamma_v^j)^*(r) / (\Gamma_v^{2d-2-j})^*(d-r) = \)

\[
\prod_p \Gamma^*(r - \text{Inf}(p,q))^{h_v(p,q)} / \prod_{p'} \Gamma^*(d - r - \text{Inf}(p',q'))^{h_v(p',q')}
\]

which is equal to

\[
(5.1.13) \quad \prod_p \Gamma^*(r - \text{Inf}(p,q))^{h_v(p,q)} / \prod_{p'} \Gamma^*(d - r - \text{Inf}(p',q'))^{h_v(p',q')}
\]

multiplied by

\[
(5.1.14) \quad (2\pi)^{-(\Sigma_p h_v(p,q)(r-\text{Inf}(p,q)) - \Sigma_{p'} (h_v(p',q')(d-r-\text{Inf}(p',q')))}
\]

Since \((h_v(p', q') = h_v(p, q))\), the functional equation transforms (5.1.13) into (up to sign)

\[
(5.1.15) \quad \prod_p \Gamma^*(r - \text{Inf}(p,q))^{h_v(p,q)} \prod_p \Gamma^*(r - \text{Sup}(p,q))^{h_v(p,q)}
\]

which in turn equals

\[
(\prod_p \Gamma^*(r - p)^{h_v(p,q)})^2
\]

Now (5.1.14) easily transforms into \((\pi^{2r-j+1})^{\Sigma_p h_v(p,q)}\), which becomes \(\pi^{B^j_2(2r-(j+1))}\).

§ 5.2 The ratio of the conjectural formulas.

Let \( r \) be an integer. Our first zeta-conjecture computes the values of \( \zeta^*(X, r) \) in terms of the orders of finite groups and the determinants \( E_j(X, r) \) of isomorphisms of finitely-dimensional complex vector spaces, with respect to bases given in terms of finitely generated abelian groups.

We will assume in this section that the scheme \( X \) is projective and smooth over \( \text{Spec} \ Z \), and of Krull dimension \( d \).
What we will do is show that the determinant part of the conjecture for $\zeta(X, r)$ divided by the determinant part of the conjecture for $\zeta(X, d - r)$ gives the product of the terms predicted by Serre’s functional equation. In fact we will do this at the level of L-functions. We will be ignoring the finite groups involved. (Their contributions should cancel by various duality theorems, as we will explain in the next section. However, the cancellation may only exist completely at the level of zeta-functions, and we are not convinced that there is a precise conjecture of the type we consider here which only looks at one L-function at a time).

It is easiest to understand what is going on if we shift back to the Fontaine-Perrin-Riou picture. Remember that where they compute determinants of maps of complex vector spaces with respect to $\mathbb{Q}$-rational structures, and so get a formula for the special value of the $L$-function up to a rational number, we will try to compute these same determinants with respect to integral structures, and so get a conjecture valid up to sign. (In fact, we also have to ignore powers of 2, which can be dealt with but would introduce a whole other level of complexity.)

Let $M$ be the motive $H^j(X, \mathbb{Z}(r))$. Then we recall the two sequences of Fontaine, as modified by us in §3.:

$$
C(M) \quad 0 \to H^0_f(M)_C \to \text{Ker}(\tilde{\gamma}_M) \to H^1_c(M)_C \to H^1_f(M)_C \to \text{Coker}(\tilde{\gamma}_M) \to H^2_c(M)_C \to 0
$$

$$
E(M) \quad 0 \to \text{Ker}(\tilde{\gamma}_M) \to (H^+_B(M))_C \to (t_M)_C \to \text{Coker}(\tilde{\gamma}_M) \to 0
$$

Let $N$ be the dual motive $M^*(1) = H^{2d-2-j}(X, \mathbb{Z}(d - r))$.

Then as we have remarked earlier, Fontaine’s DB-conjecture says that up to a rational number,

$$(L^j)^*(X, r) = \text{det}(C(M))\text{det}(E(M))$$

We need to give integral structures to the various terms in $C(M)$ and $E(M)$. The sequence $C(M)$ has four Weil-étale motivic cohomology terms which all have a natural integral structure, since they are the complexifications of finitely generated abelian groups. We choose arbitrary bases for the terms $\text{Ker}(\tilde{\gamma}_M)$ and $\text{Coker}(\tilde{\gamma}_M)$ in $C(M)$, being careful
to choose the same bases for the corresponding terms in $E(M)$. The other two terms in $E(M)$ have natural integral structures coming from Betti cohomology and the de Rham cohomology of the scheme over $\mathbb{Z}$. We also take $\tilde{\gamma}_M$ to be the map induced by our modified period map from Section 1.3.

The basic point here is that the sequence $C(N)$ is the transpose of the sequence $C(M)$, so they have the same determinant.

In order to examine the relation between the determinants of $E(M)$ and $E(N)$ we need the following linear algebra lemma, whose proof we leave to the reader.

**Lemma 5.2.1.** Let $0 \to A_1 \to A_2 \to A_3 \to 0$ and $0 \to A'_1 \to A'_2 \to A'_3 \to 0$ be exact sequences of finitely generated abelian groups. Let $\rho$ be an isomorphism from $(A_2)_C$ to $(A'_2)_C$. Consider the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (A_1)_C & \overset{i_1}{\longrightarrow} & (A_2)_C & \overset{i_2}{\longrightarrow} & (A_3)_C & \longrightarrow & 0 \\
\rho & \downarrow & & & & & \\
0 & \longrightarrow & (A'_1)_C & \overset{j_1}{\longrightarrow} & (A'_2)_C & \overset{j_2}{\longrightarrow} & (A'_3)_C & \longrightarrow & 0
\end{array}
$$

Let $\theta = j_2 \circ \rho \circ i_1$ and let $\psi = i_2 \circ \rho^{-1} \circ j_1$, so $\theta : (A_1)_C \to (A'_3)_C$ and $\psi : (A'_1)_C \to (A_3)_C$.

Then $\rho$ induces isomorphisms of $\text{Ker}(\theta)$ with $\text{Ker}(\psi)$ and $\text{Coker}(\theta)$ with $\text{Coker}(\psi)$. If we take compatible bases of the kernels and cokernels and bases of the other terms coming from the indicated integral structures, we have $\tilde{\det}(\theta) = \det(\rho) \tilde{\det}(\psi)$, where

$$
\tilde{\det}(\theta) = \det(0 \to \text{Ker}\theta \to A_1 \to A'_3 \to \text{Coker}\theta \to 0)
$$

$$
\tilde{\det}(\psi) = \det(0 \to \text{Ker}\psi \to A'_1 \to A_3 \to \text{Coker}\psi \to 0)
$$

To apply our lemma, let $A_1 = H^+_B(M), A_2 = H_B(M), A_3 = H^-_B(M)$. Let $B_M$ (resp. $B^+_M$) (resp. $B^-_M$) be the dimension of $H_B(M)$ (resp. $H^+_B(M)$) (resp. $H^-_B(M)$).

Let $A'_1 = F_0(M), A'_2 = H_{DR}(M), A'_3 = t(M)$, and let $\rho = \gamma_M$ be the modified period isomorphism from $(H_B)_C$ to $(H_{DR})_C$. Then $\det(\theta) = \det(E(M))$. On the other hand, the sequence

$$
0 \to \text{Ker}(\psi) \to F_0(M)_C \to H^-_B(M)_C \to \text{Coker}(\psi) \to 0
$$
is almost but not quite the dual of the sequence \( E(N) \), the only difference being that the dual of \( H_B^+(N) \) is \( (H_B^j(X,\mathbb{Z}(r-1)))^+ \) instead of \( (H_B^j(X,\mathbb{Z}(r)))^- \). (Here we use Poincaré duality on singular cohomology and Serre duality on de Rham cohomology). We compute that \( \gamma_M = \pm (2\pi i)(\gamma_N)^{-1} \), recalling that \( \Gamma^*(r)\Gamma^*(1-r) = \pm 1 \). This changes the determinant by a factor of \( (2\pi i)^{B_N} = (2\pi i)^{B_M} \), so using Lemma 5.2.1, we find that

\[
5.2.1 \quad \det(E(M))/\det(E(N)) = \det(\gamma_M)(2\pi i)^{-B_M}
\]

To compute \( \gamma_M \) we need (still assuming \( X \) is smooth over \( \mathbb{Z} \), and \( M = H^j(X,\mathbb{Z}(r)) \))

**Conjecture 5.2.2.** a) Recall that the weight of the cohomology motive \( H^j(X,\mathbb{Z}(r)) \) is given by \( j - 2r \). Let \( M \) be a pure motive of weight \( w \). Let \( B_M \) be the dimension of the Betti (or de Rham) realization of \( M \). Then the determinant of the period map \( \alpha_M \) from \( H_B(M) \) to \( H_{DR}(M) \) is equal to \( (2\pi i)^{-w(M)B_M/2} \). This obviously implies

\[
b) \text{ Let } F_r(M) \text{ be the (decreasing) Hodge filtration on } H_{DR}(M). \text{ Let } h_q = h_q(M) \text{ be the dimension of } F_q(M)/F_{q+1}(M), \text{ so that } B_M = \sum h_q(M).
\]

The determinant of the modified period map \( \gamma_M \) from \( H_B(M) \) to \( H_{DR}(M) \) is given by \( (2\pi i)^{-(w(M)/2)B_M} \prod q \Gamma^*(-w+q)^{h_q} \), where we take the integral structure on \( H_B(M,\mathbb{C}) \) coming from \( H_B(M,\mathbb{Z}) \) and the integral structure on \( H_{DR}(M,\mathbb{C}) \) given by \( H_{DR}(X) = \oplus H^q(X,\Lambda^p(\Omega_{X/\mathbb{Z}})) \) (Recall that if \( w \) is odd, \( B_M \) is even, so \( (w(M)/2)B_M \) is always an integer).

We have shown that

\[
5.2.2 \quad \det E(M)/\det E(N) = (2\pi i)^{-w(M)(B_M)/2} - B_M/\prod q'(\Gamma^*(-w+q')^{b(q')})
\]

The conjectures in the next section imply that the torsion terms in the formulas for \( \zeta(X,r) \) and \( \zeta(X,d-r) \) cancel, and we have also seen that \( \det(C(M)) = \det(C(N)) \) so our conjecture becomes \( \zeta(X,r)/\zeta(X,d-r) = \prod_{j=0}^{2d-2}(\det(E(H^j(X,\mathbb{Z}(r))))/\det(E(H^{2d-2-j}(X,\mathbb{Z}(d-r)))) \).

On the other hand, according to Serre’s conjectured functional equation and since in our case \( X \) is smooth over \( \text{Spec } \mathbb{Z} \), the constant \( A \) in the functional equation is equal to 1, and we have
(5.2.3) \[ L^j X, r / \mathbb{L}^{2d-2-j}(X, d-r) = \Gamma^{2d-2-j}(X, d-r) / \Gamma^j(X, r) \].

Since \( X \) is over \( \mathbb{Z} \) there is only one real place \( v \) to consider, and we will omit it in the notation. Let \( j \) be even (the case when \( j \) is odd is similar). From Theorem 5.1.1 we see that the right-hand side of (5.2.3) is equal to

\[ \left( \prod_p \Gamma^*(r - p)^{-h(p, q)} \right) \pi^{B^j(r-j/2) + (B^j, r)} \]

Since \( M = H^j(X, \mathbb{Z}(r)) \), \( B_M = B^j \), \( (B^j, r) = B_M^- \) and \( w(M) = j - 2r \).

Also, the numbers \( q' \) are the numbers where the Hodge filtration jumps. Since the Hodge filtration on \( H^j(X, \mathbb{Z}(r)) \) shifts by \( r \) from the Hodge filtration on \( H^j(X, \mathbb{Z}) \), we have \( q' = q - r \), where \( q \) runs through the numbers with \( h(p, q) \neq 0 \), and \( h(p, q) = h(q') \). The product in the denominator of (5.2.2) gets converted to \( \prod \Gamma^*(r - j + q)^{h(p, q)} \), but \( p = j - q \).

Therefore we have proven that (modulo the indicated conjectures) that if \( X \) is smooth over \( \mathbb{Z} \), Conjecture 4.1b is compatible with Serre’s functional equation, up to sign and powers of 2.

5.3 Duality for torsion in motivic cohomology, Betti cohomology and de Rham cohomology.

The (conjecturally) finite group \( H^j_W(X, \mathbb{Z}(r))_{\text{tor}} \) is dual to \( H^{2d+2-j}_W(X, \mathbb{Z}(d-r))_{\text{tor}} \) (up to \( 2 \)-torsion). (This is true by definition unless \( i = 2r + 1 \), when we need Conjecture 1.1.3).

a) The finite group \( H^j_B(X_C, \mathbb{Z}(r))_{\text{tor}} \) is dual to \( H^{2d-1-j}_B(X_C, \mathbb{Z}(d-1-r))_{\text{tor}} \). This follows from Poincaré duality. But Poincaré duality respects complex conjugation, so \( H^j_B(X_C, \mathbb{Z}(r))_{\text{tor}}^+ \) is dual to \( H^{2d-1-j}_B(X_C, \mathbb{Z}(d-1-r))_{\text{tor}}^+ \), which is isomorphic to \( H^{2d-1-j}_B(X_C, \mathbb{Z}(d-r))_{\text{tor}}^+ \). Therefore the ratio \( \chi(H^*_B(X_C, \mathbb{Z}(r))_{\text{tor}}^+)/\chi(H^*_B(X_C, \mathbb{Z}(d-r))_{\text{tor}}^+) \) is equal to \( \chi(H^*_B(X_C, \mathbb{Z})_{\text{tor}}) \).

b) Recall that \( t(r) = \prod_{q < r} \lambda^q \Omega_{X/\mathbb{Z}}[-q] \). Let \( u(r) = \prod_{q \geq r} \lambda^q \Omega_{X/\mathbb{Z}}[-q] \) So we have \( (H^j(X, t(r))_C \oplus H^j(X, u(r)))_C = H^j_{DR}(X_C/C) \). If \( X \) is smooth over \( \mathbb{Z} \), the finite group \( H^j(X, t(r))_{\text{tor}} \) is dual to the group \( H^{2d-1-j}(X, u(d-r))_{\text{tor}} \), by Serre duality, so

\[ \chi(H^*(X, t(r))_{\text{tor}})/\chi(H^*(X, t(d-r))_{\text{tor}}) = \chi(H^*_{DR}(X, Z)_{\text{tor}}) \]
This leads us to:

**Conjecture 5.3.1.** The Euler characteristic \( \chi(H^*_B(X_\mathbb{C}, \mathbb{Z})_{tor}) \) is equal to the Euler characteristic \( \chi(H^*_{DR}(X/\mathbb{Z})_{tor}) \) if \( X \) is projective and smooth over \( \mathbb{Z} \).

Perhaps even we should expect:

**Conjecture 5.3.2.** Let \( X \) be projective and smooth over the ring of integers \( \mathcal{O}_F \) of the number field \( F \). Then \( \chi(H^*_B(X_\mathbb{C}, \mathbb{Z}))_{tor} = \chi(H^*_{DR}(X/\mathcal{O}_F)_{tor}) \).

Is it possible that in fact the Betti torsion and the de Rham torsion are actually equal for each \( H^j \)?

§6. Soulé’s Conjecture.

Soulé conjectures [So] that if \( X \) is any irreducible scheme of finite type over \( \text{Spec} \, \mathbb{Z} \) and dimension \( d \), and \( r \) is any integer, then the order \( a_r \) of the zero of the zeta-function \( \zeta(X, s) \) at \( s = r \) is given by \( \prod_{j=0}^{\infty} (-1)^{j+1}rk(K'_j(X))^{(d-r)} \). If \( X \) is regular and quasi-projective over \( \text{Spec} \, \mathbb{Z} \), \( K'(X) = K(X) \), and we assume, up to torsion, that \( K_i(X)^{(r)} = H^2_{et}(X, \mathbb{Z}(r)) \).

So in this case, Soulé’s conjecture can be restated as \( a_r = \prod_{j=0}^{\infty} (-1)^{j+1}rkH^j_{et}(X, \mathbb{Z}(d-r)) \).

We will explain how this fits in with the general Euler characteristic picture.

As is well-known, formulas giving the order of the zero of a zeta-function \( \zeta(s) \) at \( s = r \) in terms of the ranks of some finitely-generated groups ought to be accompanied by formulas given the leading term of \( \zeta(s) \) at \( s = r \) in terms of the orders of torsion subgroups of these groups and determinants of pairings involving these groups. We will give a picture of how this should work for Soulé’s conjecture.

We have conjectured that the cohomology groups of our complex \( H^*_W(X, D(r)) \) satisfy the conditions of Appendix A for defining a rank and Euler characteristic.

We first examine the rank. It is a straightforward calculation that the rank of \( H^*_W(X, D(r)) \) is given by \( \prod_{j=0}^{\infty} (-1)^{j}rkH^j_{et}(X, \mathbb{Z}(d-r)) + \prod_{j=0}^{\infty} (-1)^{j}rkH^j(X, t(X, r)) \).

In the geometric case \( t(x, r) \) does not exist, and the rank becomes the negative of Soulé’s rank. But why do we have an extra Euler characteristic appearing in the arithmetic case? In the geometric case, even though the order of the zero may be (conjecturally) computed using only motivic cohomology, in order to compute the leading term we must combine the motivic cohomology Euler characteristic with the Milne factor, which is a combination of Euler characteristics of coherent cohomology of sheaves of exterior powers of differentials. In this case, these cohomology groups are finite, and so do not contribute to the rank.
In the arithmetic case, the answer seems to be that the ranks of the cohomology groups of \( t(X, r) \) are not zero in general, and we have two different natural integral structures on these cohomology groups. We first assume for simplicity that \( X \) is projective over Spec \( O_F \), with \( O_F \) a principal ideal domain. Then the coherent cohomology groups \( H^j(X, t(X, r)) \) are finitely generated \( O_F \)-modules so free modulo torsion. One natural basis is obtained by just taking a basis of \( (H^j(X, t(X, r))/\text{torsion}) \) as a \( \mathbb{Z} \)-module. The other is obtained by observing that \( H^j(X, t(X, r)) \otimes \mathbb{C} \) is isomorphic to \( \coprod_\sigma H^j(X, t(X, r)) \otimes_{O_F} \mathbb{C} \), where the sum is over all embeddings \( \sigma \) of \( F \) into \( \mathbb{C} \), and the tensor product component corresponding to \( \sigma \) uses \( \sigma \) to map \( O_F \) to \( \mathbb{C} \). We then have to fix an \( O_F \)-basis of \( (H^j(X, t(X, r))/\text{torsion}) \) and take its image in \( H^j(X, t(X, r)) \otimes \mathbb{C} \) using in turn all the different \( \sigma \)'s. Changing to a different basis over \( O_F \) changes the determinant of the transformation by the norm of the determinant of the \( O_F \)-matrix, which must be \( \pm 1 \).

If \( O_F \) is not a principal ideal domain, we can define the comparison determinant by localizing and then patching together.

So we have to view the complex \( t(r) \to t(r) \) (via the identity map, but with the two different choices of bases for \( t(r) \)) also as an Euler characteristic complex. Its rank is the usual alternating sum of the ranks of \( H^j(X, t(r)) \), which is just the difference between the rank of \( D(r) \) and Soulé’s formula for \( a_r \).

Now let’s look at the leading term. To get our original conjecture we need to take \( H^j(X, t_X) \) with its natural \( \mathbb{Z} \)-basis. This corresponds to what is done in [Fo]. But to make everything analogous to the geometric case, we should take instead the amalgamation of the \( O_F \)-bases, and then correct by the determinant of our \( t(r) \)-complex in the preceding paragraph. This gives a formula (agreeing with Soulé) for the order of the zero in terms of the ranks of Euler-characteristic complexes, and, as should be the case, a formula for the leading term on terms of the Euler characteristics of these same complexes, analogous to the correction by the Milne factor in the geometric case.

Appendix A. Euler characteristics

We begin with a description of generalized Euler characteristics.

**Definition A.1.** Let \( F \) be a field, and let \( n \) be a positive integer. Let

\[
0 \to V_0 \to V_1 \to \ldots V_n \to 0
\]

be an exact sequence of finite-dimensional vector spaces over \( F \) with bases \( B_0, B_1, \ldots B_n \).

We give an inductive definition of the determinant of (\(*\)):: If \( n = 1 \) we just take the standard
linear algebra definition of the determinant. If \( n = 2 \), let \( T \) be the map from \( V_0 \) to \( V_1 \) and let \( U \) be the map from \( V_1 \) to \( V_2 \). Let \( B'_2 \) be any set of elements in \( V_1 \) mapping bijectively to \( B_2 \) by \( U \). Let \( B'_1 \) be the basis \((T(B_0), B'_2)\) of \( V_1 \). Define the determinant of \((*)\) to be the determinant of the identity map on \( V_1 \) relative to the bases \( B'_1 \) and \( B_1 \).

Having defined the determinant for exact sequences of length \( n \), we define it for sequences of length \( n + 1 \) as follows. Let \( I \) be the image of \( V_{n-2} \) in \( V_{n-1} \). Then \( \det(0 \to V_0 \to \cdots \to V_n \to 0) = \det(0 \to V_0 \to \cdots \to V_{n-2} \to I \to 0) \) \((\det(0 \to I \to V_{n-1} \to V_n \to 0))^{(-1)^n}\). (We may choose any basis \( B_I \) for \( I \), as long as it is the same basis in both exact sequences, since the product will be independent of \( B_I \).)

Note for later reference that if \( n = 2 \) and \( V_0 = 0 \) then the determinant of \((0 \to V_1 \to V_2 \to 0)\) is the inverse of the classical determinant.

Now let \( H^i \) be finitely generated abelian groups such that \( H^i = 0 \) for all but finitely many \( i \) and for \( i < 0 \). Assume that we are given \( \mathbb{C} \)-linear maps \( \theta_i \) from \( V_i = H^i \otimes_{\mathbb{Z}} \mathbb{C} \) to \( V_{i+1} = H^{i+1} \otimes_{\mathbb{Z}} \mathbb{C} \) such that \((V_*, \theta_*)\) form an acyclic complex.

**Definition A.2.** The Euler characteristic \( \chi(H^*, \theta_*) \) in \( \mathbb{C}^*/\pm 1 \) is given by the formula

\[
\prod_i ((H^i)_{tor})^{(-1)^i} / \det(V_*, \theta_*)
\]

Note that we take bases for \( H^i \otimes \mathbb{C} \) coming from the free finitely-generated groups \( H^i / H^i_{tor} \), and changing the basis of a free abelian group can only change the determinant by a sign.

**Definition A.3.** The rank \( r(H^*, \theta_*) \) is given by the "derived Euler characteristic" (Note that the usual alternating sum of the ranks is zero).

\[
\prod (-1)^j(rkH^j).
\]

**Lemma A.4.** Let \( 0 \to A_1 \to A_2 \to A_3 \to 0 \) and \( 0 \to B_1 \to B_2 \to B_3 \to 0 \) be exact sequences of finitely generated abelian groups. Assume that the groups \( B_i \) are torsion-free. Let \( \phi_i \) be isomorphisms from \((A_i)_{\mathbb{C}}\) to \((B_i)_{\mathbb{C}}\) such that the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & (A_1)_{\mathbb{C}} & \overset{i_1}{\longrightarrow} & (A_2)_{\mathbb{C}} & \overset{i_2}{\longrightarrow} & (A_3)_{\mathbb{C}} & \longrightarrow & 0 \\
\phi_1 & \downarrow & \phi_2 & \downarrow & \phi_3 & & & & \\
0 & \longrightarrow & (B_1)_{\mathbb{C}} & \overset{j_1}{\longrightarrow} & (B_2)_{\mathbb{C}} & \overset{j_2}{\longrightarrow} & (B_3)_{\mathbb{C}} & \longrightarrow & 0
\end{array}
\]
commutes. Let $w_i$ be the order of $(A_i)_{tor}$. Let $z_i$ be the determinant of $\phi_i$ calculated with respect to bases coming from $A_i$ and $B_i$. Then $w_2/w_1w_3 = z_2/z_1z_3$, up to sign.

Appendix B. Derived Exterior Powers

Let $A$ be an abelian category. Let $SA$ denote the category of simplicial objects of $A$ and $CA$ denote the category of homological chain complexes of objects of $A$ ending in degree zero. There are well-known functors $N: SA \to CA$ and $K: CA \to SA$ such that $NK$ is the identity and $KN$ is naturally equivalent to the identity functor. $N$ and $K$ also preserve homotopies. Let $\Lambda^k$ denote $k$-th exterior power. Let $X$ be a scheme and $A$ be the category of coherent locally free sheaves on $X$. If $Q_n$ is in $SA$ with $Q_n$ a locally free sheaf on $X$ for all $n$, we define $\Lambda^kQ$ to be $\Lambda^k(Q_n)$ in $SA$.

**Proposition B.1.**

Let $X$ be a regular scheme projective over $\text{Spec } \mathbb{Z}$. Write $X$ as a closed subscheme of a projective space $P = (P^n)_{\mathbb{Z}}$ such that $I$ is the sheaf of ideals defining $X$. Then the complex of locally free sheaves $C_{X,P} = I/I^2 \to \Omega_{P/\mathbb{Z}}$ defines an element in the derived category of locally free sheaves on $X$ which is independent of the choice of embedding of $X$ into $P$.

Proof. If we have two different embeddings of $X$ in $P_1$ and $P_2$, take the Segre embedding of $P_1 \times P_2$ in $P_3$, and compare successively the complexes defined by the embeddings into $P_1$ and $P_2$ with the product embedding into $P_3$. (For details, see [LS]).

**Definition B.2.** $\lambda^k\Omega_{X/\mathbb{Z}} = N\Lambda^kKC_{X,P}$.

We see easily that this is independent of the choice of embedding.

**Definition B.3.** $t(X,r) = \prod_{k=0}^{r-1} \lambda^k\Omega_{X/\mathbb{Z}}[-k]$.

We begin by recalling the following fact ([H], Exercise 5.16 (d)):

**Lemma B.4.** Let $(X,O_X)$ be a ringed space, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of locally free sheaves of $O_X$-modules. Then there exists a finite filtration of $\Lambda^rF$:

$$\Lambda^rF = G^0 \supset G^1 \supset \cdots \supset G^r \supset G^{r+1} = 0$$

with quotients $G^n/G^{n+1} = \Lambda^pF'/\Lambda^{p-r}F''$.

**Theorem B.5.** Let $A$ be a ring and $M$ an $A$-module. a) $\lambda^0M$ is canonically homotopic to the complex consisting only of $A$ in degree 0.
b) \( \lambda^1 M \) is canonically homotopic to the complex consisting only of \( M \) in degree 0.

c) If \( M \) has projective dimension \( r \) then \( \lambda^k M \) is represented by a complex of length \( kr \).

Proof: a) and b) are obvious and c) follows immediately from a theorem of Dold and Puppe ([DP])

**Definition B.6.** Let \( X \) be a regular noetherian scheme and let \( F \) be a coherent sheaf of \( O_X \)-modules. Since \( X \) is regular \( H^i(\lambda^k(F)) \) is zero for all but finitely many values of \( i \), so we may define the Euler characteristic \( \chi^k(F) \) to be \( \sum (-1)^i [H^i(\lambda^k(F))] \) in the Grothendieck group \( G_0(X) \) of coherent sheaves of \( O_X \)-modules.

**Theorem B.7.** Assume that \( X \) is quasi-projective and regular, so that the Grothendieck group \( K_0(X) \) of coherent locally free sheaves on \( X \) is naturally isomorphic to \( G_0(X) \). If \( 0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \) is an exact sequence of coherent sheaves on \( X \), we have the equality in \( K_0(X) : \chi^n(F) = \sum_{p+q=n} \chi^p(F')\chi^q(F'') \).

Proof, This follows easily from Lemma B.4.

**Corollary B.8.** If \( 0 \rightarrow I \rightarrow F \rightarrow G \rightarrow 0 \) is an exact sequence of coherent sheaves with \( I \) invertible, then \( \chi(\lambda^k+1 F) = \chi(\lambda^k+1 G)\chi(I \otimes \lambda^k G) \).

**Corollary B.9.** The class of \( \lambda^k(F) \) in the Grothendieck group agrees with the usual definition of \( \lambda^k([F]) \), using the \( \lambda \)-operations in \( K \)-theory.

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