Solutions of Quadratic First-Order ODEs applied to Computer Vision Problems
Plane Curve Reconstruction

UAH

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Abstract

The article proves the existence of a maximum of two possible solutions to the initial value problem composed by the planar-perspective equation and an initial condition. This initial value problem has a geometric interpretation. Solutions are curves that pass through the initial condition which is a point of the plane.

1 Description of the problem

Let \( C \) be a regular curve in the plane \( \mathbb{R}^2 \) parameterized by the perspective parameterization \( X(t) = \rho(t)(t,1) \), where \( t \in I \subset \mathbb{R} \) is an interval and \( \rho: I \mapsto \mathbb{R} \) is the objective function. A complete description of parametric curves is available in any generic differential geometric book as [Kreyszig (1991)]. Let \( U(t) \) be a function of class \( U(t) \in C^1(I,\mathbb{R}) \) defined in the interval \( I \) except for a finite number of points \( A \) where the function is of class \( U(t) \in C^\infty(A,\mathbb{R}) \) and where the first-order derivative of the \( \rho \) function is null. The function \( U(t) \) is positive in \( I \) and it has the geometric interpretation of being the square of the modulus of the velocity vector of the curve \( C \). It means that \( U(t) = ||\vec{v}(t)||^2 \). This function \( U(t) \) is known. Consider the next non-linear first-order ordinary differential equation (ODE) (strictly speaking, the equation is a differential algebraic equation DAE):

\[
\varepsilon \frac{d\rho^2}{dt} + \rho^2 = U, \tag{1}
\]

where \( \rho(t) \) is the function that we have to find which is positive or null \( \rho(t) \geq 0 \). The function \( \varepsilon = (1 + t^2)^2 \) is a polynomial of degree 4. The variable \( t \) is the independent variable. We consider the Initial Value Problem (IVP) (also called the Cauchy problem) composed by the equation (1) (also called planar-perspective reconstruction equation) with the initial condition \( \rho(t_0) = \rho_0 \) in a neighborhood \( t_0 \in J \subset I \). The formal description of an initial value problem appears in the book [Tenenbaum and Pollard (1985)].
2 Hypotheses

We want to prove that the IVP composed by the equation and any Initial Condition \((t_0, \rho_0)\), inside the domain of the equation, has a maximum of two possible solutions \(\rho_1(t)\) and \(\rho_2(t)\). In other words, given a point in the plane \(p = (t_0, \rho_0)\) compatible with the equation, there are only a mainum of two curves that fulfill the equation and pass through the point \(p = (t_0, \rho_0)\).

Figure shows empirical evidences that there are only two possible curves which passes trough different initial conditions. The red stright line represent the curve we want to reconstruct. We only know the square modulus of the velocity vector \(\vec{v}\). The magenta curve is called maximal curve. This curve is obtained vanishing the term \(\rho_t = 0\) in the original equation and solving \(\rho\). It has the property of contain all the points of solutions with derivative null. Any extrema of this curve is a point with null derivative. In the red stright line is tangent to the magenta maximal curve in a point \(p\) with null derivative (it will be critical points during the proof). All this kind of points have the normal of the curve parallel to the optical ray (the line that join \(p\) with the origin). The blue and green curves are the two solutions that we obtain with different non-null initial conditions. We can see that there exist solutions which pass trough the origin and go to the \(y < 1\) semi-plane.

Figure shows the reconstructed curves of the red straight line in an initial condition with null
derivative. We observe empirically that there exist only two possible analytic solutions or $4 \ C^1$ function if we mix all the branches.

![Reconstrucción Recta](image)

**Figure 2:** SfT of a straight line in a null initial condition.

**Remark 1** It is possible to have curves without any non-null initial condition. For instance, the red straight line in figure 2 with a positive grow has non-null initial conditions.

### 2.1 Case 1: The derivative in the initial condition is not null

Solving the equation taking $\rho(t_0) = \rho_0$, we obtain two different values of the derivatives in $t_0$. We call both values $\dot{\rho}(t_0) = \rho_{1}^{1}$ and $\dot{\rho}(t_0) = \rho_{2}^{1}$. By the hypothesis, the derivative in the initial condition is not null, therefore, both values have equal modulus and different sign.

The hypothesis ensures that both solutions are different and non-equal to 0 in $t_0$. Consequently, we have two different ODEs (one per sign):

$$\frac{d \rho}{dt} = \pm \sqrt{U - \frac{\rho^2}{\varepsilon}},$$

(2)

Calling the *Picard-Lindelöf* theorem for each ODE with the initial condition $\rho_0$, it is guaranteed the existence and the uniqueness of both two solutions $\rho_1(t)$ y $\rho_2(t)$. Of course, we have to fulfill
the conditions of Picard-Lindelöf theorem. Picard-Lindelöf theorem is formally described in the book [Tenenbaum and Pollard (1985)].

Now we wonder what happened when the initial condition is a critical point, that means, a point where the derivative vanishes. Given a critical point, there exist two and only two curves that pass through this point?

If the initial condition \( \rho(t_0) = \rho_0 \) is a critical point, the equation [3] becomes algebraic. To show that there exist only two solutions we cannot call directly Picard-Lindelöf theorem. To prove that there exist only two solutions we follow the next steps:

### 2.2 Case 2: The derivative in the initial condition is null

Firstly, we derivate both sides of the equation [1]. We will use the next notation: \( \rho^j_1 = "i"th derivative of \( \rho \) raised to the \( j \)th power".

\[
\varepsilon_1 \rho^2_{1j} + 2 \varepsilon_1 \rho_1 \rho_{2j} + 2 \rho_0 \rho_1 = U_{1j}
\]  

(3)

The obtained equation is second-order. Besides, observe how the highest-order of the equation is not quadratic. Therefore, any bifurcation appears. Since we supposed that the first-order derivative in the initial condition is null, the equation [3] is an identity and necessary it must fulfill \( U_{1j} = 0 \). Contrary, the equation would not have solution. We cannot solve the IVP with this equation and is null in the initial condition, the equation 3 is an identity and necessary it must fulfill not quadratic. Therefore, any bifurcation appears. Since we supposed that the first-order derivative in the initial condition is null, the equation 3 becomes a quadratic in \( t \):

\[
2 \varepsilon_2 \rho^2_{2j} + \rho_0 \rho_2 = U_{2j}
\]  

(5)

The previous equation [5] produce a bifurcation. Solving the equation, we obtain two different values for the second-order derivatives in the initial condition due to its quadratic character. In this point, we assume that the discriminant \( \rho^2_0 + 8U_{2j} \varepsilon > 0 \) in \( t = t_0 \) to get the two different values. We will refer to this pair of values of the second-order derivative as \( \theta_1 \) and \( \theta_2 \). Again, we repeat the process and derivate both sides of the previous equation again:

\[
\varepsilon_3 \rho^2_{3j} + 6 \varepsilon_2 \rho_1 \rho_{2j} + 6 \varepsilon_1 (\rho^2_{2j} + \rho_1 \rho_{3j}) + 2 \varepsilon (3 \rho_2 \rho_3 + \rho_1 \rho_{4j}) + 2 \varepsilon (3 \rho_1 \rho_{2j} + \rho_0 \rho_{3j}) = U_{3j}
\]  

(6)

The obtained equation is fourth-order. At this point, we prove the existence of only two possible solutions. We can calculate the value of the third-order derivative for each value of the initial condition (there are two because of the bifurcation) \((\rho_0, 0, \theta_1)\) and \((\rho_0, 0, \theta_2)\). Vanishing the first-order derivative, we obtain the next third-order equation:

\[
6 \varepsilon_1 \rho^2_{2j} + 6 \varepsilon_2 \rho_1 \rho_{3j} + 2 \rho_0 \rho_{3j} = U_{3j}
\]  

(7)

For each initial condition \((\rho_0, 0, \theta_1)\) y \((\rho_0, 0, \theta_2)\) we obtain an unique value of the third-order derivative (due to the fact that it is not going to appear a quadratic term in the last equation, at least in the highest-order term, in this case \( \rho_{3j} \)). We will prove at the end of the article. If we follow this procedure, we can build a Taylor serie for each initial condition calculates \( a = (\rho_0, 0, \theta_1, ...) \) y
b = (ρ₀, 0, θ₂, ...) in t₀. Assuming that the function \( U(t) \) is analytic in the critical points, that means, the Taylor serie of the function \( U(t) \) converge in a neighborhood (an interval) \( t₀ \in K \) around \( θ₀ \). As the operations that involves the equation (squares, sums, derivatives) are friendly with the analytic property of the functions, then, Taylor series converge. That means that, the solutions derived are analytic and particularly \( ρ₁(t) \) and \( ρ₂(t) \). Consequently, the solutions converge in a neighboothood of \( t₀ \).

\[
ρ₁(t) = \sum_{i=0}^{∞} \frac{a_{i}}{i!} (t - t₀)^i \quad t \in J
\]

\[
ρ₂(t) = \sum_{i=0}^{∞} \frac{b_{i}}{i!} (t - t₀)^i \quad t \in J
\]

Now, we prove the existence of only two solutions. As the obtained function analytic in \( t₀ \), the Taylor serie converges in a neighbourhood \( J \) arround \( t₀ \). For any point of the \( t \in J, t \neq t₀ \), the first-order derivative is not null and, consequently, que can apply the Picard-Lindellöf theorem to guarantee the existence of only 2 solutions that pass through the critical point.

There exist a degenerate case where all the coefficients of the Taylor series are null except \( ρ₀ \). In this case, the function \( ρ(t) = ρ₀ \) is a constant. It is the only case where there is only one solution that matches with the circumference.

Next points are important to generalize the proof for all order of the derivative. In the previous proof, we assumed that the bifurcation appeared in the second-order derivative, but, in general, the bifurcation may appear in the \( n \)th derivative.

**a.** Whatever the number of times we derivate the equation, the coefficient of the highest-order of the derivative is always \( 2ερ₁ \). We can prove it easily using the Leibniz rule:

\[
(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \quad (9)
\]

Substituting \( f = g = ρ^{(1)} \), we find that:

\[
x^{(n)} = (ρ^{(1)} \cdot ρ^{(1)})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} ρ^{(k+1)} ρ^{(n-k+1)} \quad (10)
\]

We do the same for the non-derivative term:

\[
y^{(n)} = (ρ \cdot ρ)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} ρ^{(k)} ρ^{(n-k)} \quad (11)
\]

Now, the original equation \( \varepsilon x + y = U \) can be rewritten as follow:

\[
\varepsilon x + y = U \quad (12)
\]

The successive derivatives that it will be appear can be expressed as follow:

\[
\sum_{k=0}^{n} \binom{n}{k} \varepsilon^{(k)} x^{(n-k)} + y^{(n)} = U^{(n)} \quad n \in \mathbb{N} \quad (13)
\]
Fixing the iteration $n$, we always find that the highest-order derivative is obtained from the term $k = 0$ of the summatory of the equation $x^{(n)}$ in (13) and the terms $k = 0$ and $k = n$ of the summatory of the equation $x^{(n)}$ in (10) yielding to:

$$
\varepsilon \left( \binom{n}{0} \rho^{(1)} \rho^{(n+1)} + \binom{n}{1} \rho^{(n)} \rho^{(1)} \right) = 2 \varepsilon \rho^{(1)} \rho^{(n+1)}
$$

(14)

Consequently, this term vanishes in critical points reducing the order of the differential equation in one.

b. Before continuing the proof, we show some first iterations of the expressions $x^{(n)}$, $y^{(n)}$ and the equation 13.

c. We show $x^{(n)}$:

\[
\begin{array}{l|l}
 n = 0 & \rho^{(1)} \rho^{(1)} \\
 n = 1 & 2(0)\rho^{(1)} \rho^{(2)} \\
 n = 2 & 2(2)\rho^{(1)} \rho^{(3)} + (2)\rho^{(2)} \rho^{(2)} \\
 n = 3 & 2(3)\rho^{(1)} \rho^{(4)} + 2(3)\rho^{(2)} \rho^{(3)} \\
 n = 4 & 2(4)\rho^{(1)} \rho^{(5)} + 2(4)\rho^{(2)} \rho^{(4)} + (4)\rho^{(3)} \rho^{(3)} \\
 n = 5 & 2(5)\rho^{(1)} \rho^{(6)} + 2(5)\rho^{(2)} \rho^{(5)} + 2(5)\rho^{(3)} \rho^{(4)} \\
 n = 6 & 2(6)\rho^{(1)} \rho^{(7)} + 2(5)\rho^{(2)} \rho^{(6)} + 2(6)\rho^{(3)} \rho^{(5)} + (6)\rho^{(4)} \rho^{(4)} \\
 \vdots & \vdots \\
 n = i - 1 & 2(i-1)\rho^{(1)} \rho^{(i)} + 2(i-1)\rho^{(2)} \rho^{(i-1)} + \cdots + 2(i-1)\rho^{(i-1)} \rho^{(i)} + 2(i-1)\rho^{(1)} \rho^{(i-1)} \\
 n = i & 2(i)\rho^{(1)} \rho^{(i+1)} + 2(i)\rho^{(2)} \rho^{(i)} + \cdots + 2(i)\rho^{(i)} \rho^{(i)} + (i)\rho^{(i+1)} \rho^{(i)} \\
 n = i + 1 & 2(i+1)\rho^{(1)} \rho^{(i+2)} + 2(i+1)\rho^{(2)} \rho^{(i+1)} + \cdots + 2(i+1)\rho^{(i+1)} \rho^{(i+2)} + 2(i+1)\rho^{(1)} \rho^{(i+1)} \\
 \vdots & \vdots \\
\end{array}
\]

(15)
d. We show \( y^{(n)} \):

\[
\begin{array}{|c|c|}
\hline
n = 0 & \rho^{(0)} \rho^{(0)} \\
\hline
n = 1 & 2(\frac{\rho}{1}) \rho^{(0)} \rho^{(1)} \\
\hline
n = 2 & 2(\frac{\rho}{2}) \rho^{(0)} \rho^{(2)} + (\frac{\rho}{1}) \rho^{(1)} \rho^{(1)} \\
\hline
n = 3 & 2(\frac{\rho}{3}) \rho^{(0)} \rho^{(3)} + 2(\frac{\rho}{2}) \rho^{(1)} \rho^{(2)} \\
\hline
n = 4 & 2(\frac{\rho}{4}) \rho^{(0)} \rho^{(4)} + 2(\frac{\rho}{3}) \rho^{(1)} \rho^{(3)} + (\frac{\rho}{2}) \rho^{(2)} \rho^{(2)} \\
\hline
n = 5 & 2(\frac{\rho}{5}) \rho^{(0)} \rho^{(5)} + 2(\frac{\rho}{4}) \rho^{(1)} \rho^{(4)} + 2(\frac{\rho}{3}) \rho^{(2)} \rho^{(3)} \\
\hline
n = 6 & 2(\frac{\rho}{6}) \rho^{(0)} \rho^{(6)} + 2(\frac{\rho}{5}) \rho^{(1)} \rho^{(5)} + 2(\frac{\rho}{4}) \rho^{(2)} \rho^{(4)} + (\frac{\rho}{3}) \rho^{(3)} \rho^{(3)} \\
\hline
\vdots & \vdots \\
\hline
n = i - 1 & 2(\frac{i-1}{0}) \rho^{(0)} \rho^{(i-1)} + 2(\frac{i-1}{1}) \rho^{(1)} \rho^{(i-2)} + \ldots + 2(\frac{i-1}{i-2}) \rho^{(i-1)} \rho^{(i-2)} + 2(\frac{i-1}{i-3}) \rho^{(i-2)} \rho^{(i-3)} \\
\hline
n = i & 2(\frac{i}{0}) \rho^{(0)} \rho^{(i)} + 2(\frac{i}{1}) \rho^{(1)} \rho^{(i-1)} + \ldots + 2(\frac{i}{i-2}) \rho^{(i-2)} \rho^{(i-3)} + (\frac{i}{i-1}) \rho^{(i-1)} \rho^{(i-2)} \\
\hline
n = i + 1 & 2(\frac{i+1}{0}) \rho^{(0)} \rho^{(i+1)} + 2(\frac{i+1}{1}) \rho^{(1)} \rho^{(i)} + \ldots + 2(\frac{i+1}{i+1}) \rho^{(i)} \rho^{(i+1)} + 2(\frac{i+1}{i+2}) \rho^{(i+2)} \\
\hline
\vdots & \vdots \\
\hline
\end{array}
\]

(16)

e. Now, we show the next iteration:

\[
\begin{array}{|c|c|}
\hline
n = 1 & \varepsilon^{(1)} x^{(0)} + \varepsilon^{(0)} x^{(1)} + y^{(1)} = U^{(1)} \\
\hline
n = 2 & \varepsilon^{(2)} x^{(0)} + 2 \varepsilon^{(1)} x^{(1)} + \varepsilon^{(0)} x^{(2)} + y^{(2)} = U^{(2)} \\
\hline
n = 3 & \varepsilon^{(3)} x^{(0)} + 3 \varepsilon^{(2)} x^{(1)} + 3 \varepsilon^{(1)} x^{(2)} + \varepsilon^{(0)} x^{(3)} + y^{(3)} = U^{(3)} \\
\hline
n = 4 & \varepsilon^{(4)} x^{(0)} + 4 \varepsilon^{(3)} x^{(1)} + 6 \varepsilon^{(2)} x^{(2)} + 4 \varepsilon^{(1)} x^{(3)} + \varepsilon^{(0)} x^{(4)} + y^{(4)} = U^{(4)} \\
\hline
n = 5 & \varepsilon^{(5)} x^{(0)} + 5 \varepsilon^{(4)} x^{(1)} + 10 \varepsilon^{(3)} x^{(2)} + 10 \varepsilon^{(2)} x^{(3)} + 5 \varepsilon^{(1)} x^{(4)} + \varepsilon^{(0)} x^{(5)} + y^{(5)} = U^{(5)} \\
\hline
n = 6 & \varepsilon^{(6)} x^{(0)} + 6 \varepsilon^{(5)} x^{(1)} + 15 \varepsilon^{(4)} x^{(2)} + 20 \varepsilon^{(3)} x^{(3)} + 15 \varepsilon^{(2)} x^{(4)} + 6 \varepsilon^{(1)} x^{(5)} + \varepsilon^{(0)} x^{(6)} + y^{(6)} = U^{(6)} \\
\hline
\vdots & \vdots \\
\hline
n = i - 1 & (\frac{i-1}{0}) \varepsilon^{(i-1)} x^{(0)} + (\frac{i-2}{1}) \varepsilon^{(i-2)} x^{(1)} + \ldots + (\frac{i-2}{i-2}) \varepsilon^{(1)} x^{(i-2)} + (\frac{i-1}{i-1}) \varepsilon^{(0)} x^{(i-1)} + y^{(i-1)} = U^{(i-1)} \\
\hline
n = i & (\frac{i}{0}) \varepsilon^{(i)} x^{(0)} + (\frac{i-1}{1}) \varepsilon^{(i-1)} x^{(1)} + \ldots + (\frac{i-1}{i-1}) \varepsilon^{(1)} x^{(i-1)} + (\frac{i}{i}) \varepsilon^{(0)} x^{(i)} + y^{(i)} = U^{(i)} \\
\hline
n = i + 1 & (\frac{i+1}{0}) \varepsilon^{(i+1)} x^{(0)} + (\frac{i+1}{1}) \varepsilon^{(i+1)} x^{(1)} + \ldots + (\frac{i+1}{i+1}) \varepsilon^{(1)} x^{(i+1)} + (\frac{i+1}{i+2}) \varepsilon^{(0)} x^{(i+1)} + y^{(i+1)} = U^{(i+1)} \\
\hline
\vdots & \vdots \\
\hline
\end{array}
\]

(17)
f. The first time that we derivate the equation (1) and evaluate the initial condition \((\rho_0, 0)\), we obtain an identity and we cannot solve the second-order derivative. If we derivate again, we obtain a second-order equation (5) in \(\rho^{(2)}\) variable. We can obtain two different roots or one double root. Derivating again and substituting the value of \(\rho(1) = 0\) yields to the equation (7). It is possible to solve the the value of \(\rho^{(3)}\) if we know all the previous derivatives. The only constrain is that the coefficient of this derivative has to be non-null \(2(\rho^{(0)} + 3\varepsilon\rho^{(2)}) \neq 0\) in order to solve it. But, there is a value of \(\rho^{(2)}\) that vanishes the coefficient of this derivative and obeys \(2(\rho^{(0)} + 3\varepsilon\rho^{(2)}) = 0\). We can prove that after the second order derivative, for each iteration of the equation, the highest-order of the equation has the next coefficient:

\[
2(\rho^{(i)} + (i + 1)\rho^{(2)})
\]

If this coefficient does not vanish for any iteration, then, we will achieve all the coefficients of Taylor Series. We can prove it also that if for an iteration, this coefficient vanishes, then, in the rest of iteration this coefficient is not vanished and we obtain a system of more variables than equations. But they are singular cases. For instance, if \(\rho^{(2)} \geq 0\) then this coefficient will never vanishes for any iteration.

The relation that has to obey \(\rho^{(0)}\) and \(\rho^{(2)}\) is for becoming in a special case is not common. It is related with a countable infinite set.

g. Assuming that we can obtain all the coefficients, we can build a Taylor serie and imposing analyticity in singular points. Taylor series of functions in analytical points converges in a neighbourhood of the point to the function (see the book [Can (2008)]). Calling Picard-Lindelöf to any point of the neighbourhood of the point proves the existence and the uniqueness as we show in case of derivative non-null.

*Quod erat demonstrandum*

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