Ground states for planar axially Schrödinger–Newton system with an exponential critical growth

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Abstract
In this paper, we study the following planar Schrödinger–Newton system:

\[
\begin{align*}
-\Delta u + V(x)u + \lambda \phi u &= f(x, u) \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2,
\end{align*}
\]

where \( V, f \) are axially symmetric about \( x \), \( V \) is positive, and \( f \) is super-linear at zero and exponential critical at infinity. Using a weaker condition

\[
\frac{f(x, u)}{u} - \frac{f(x, tu)}{(tu)^3} \left\{ \text{sign}(1 - t) + \theta V(x) \right\} \frac{|1 - t|^2}{(tu)^2} \geq 0, \quad \forall x \in \mathbb{R}^2, t > 0, u \neq 0
\]

with \( \theta \in [0, 1) \) instead of the Nehari type monotonic condition on \( \frac{f(u)}{|u|^3} \), we obtain a ground state solution of the above problem via variational methods.

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Keywords: Schrödinger–Newton system; Axial symmetry; The exponential critical growth; Ground states

1 Introduction and main results
In the present paper, we are concerned with the wave solutions of the Schrödinger–Newton system

\[
\begin{align*}
-\Delta \psi + W(x)\psi + \lambda \phi \psi &= g(x, \psi) \quad \text{in } \mathbb{R}^d, \\
\Delta \phi &= |\psi|^2 \quad \text{in } \mathbb{R}^d,
\end{align*}
\]

where \( \psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C} \) is the wave function, \( W(x) \) is a real external potential, \( \lambda > 0 \) is a parameter. Problems of the type (1.1) arise in many problems from physics. We refer the readers to [15], therein (1.1) appears in a quantum mechanical context in the case \( d \leq 3 \).
A standing wave solution of (1.1) is a solution of the form $\psi(x,t) = e^{-iEt}u(x)$ and its existence reduces (1.1) to the system

$$
\begin{align*}
-\Delta u + V(x)u + \lambda \phi u &= f(x, u) \quad \text{in } \mathbb{R}^d, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^d,
\end{align*}
$$

(1.2)

where $V(x) = W(x) - E$, $g(x, e^{-iEt}u) = f(x, u)e^{-iEt}$. For the case $d = 3$, it is called the Schrödinger–Poisson system and it has been well studied. For the existence, multiplicity, and concentration, we refer the readers to [2, 3, 9, 10, 13, 20] and the references therein. For Kirchhoff type equations involving subcritical and critical growth in three dimensions, please see [19] and the references therein. We also quote the paper [12] for Hardy–Schrödinger–Kirchhoff systems.

However, much less is known about the case $d = 2$. For $\Delta \phi = u^2$, in $\mathbb{R}^2$, one has

$$
\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x - y|) |u(y)|^2 \, dy.
$$

(1.3)

Substituting it into (1.2), we obtain the integro-differential equation

$$
-\Delta u + V(x)u + \lambda \left(\ln(|\cdot|) * u^2\right)u = f(x, u) \quad \text{in } \mathbb{R}^2.
$$

(1.4)

For simplicity, throughout this paper, let $\lambda = 2\pi$. The approach for $d = 3$ cannot be easily adapted to $d = 2$ since

$$
\frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) |u(y)|^2 |u(x)|^2 \, dy \, dx,
$$

(1.5)

which is the functional associated with the third term in (1.4), is sign-changing, and is neither bounded from above nor from below on $H^1(\mathbb{R}^2)$. This difficulty has been overcome recently in [7] or [16]. For

$$
-\Delta u + \left(\ln(|\cdot|) * u^2\right)u = \mu u \quad \text{in } \mathbb{R}^2,
$$

(1.6)

by introducing the following subspace of $H^1(\mathbb{R}^2)$

$$
X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1 + |x|) u^2 \, dx < \infty \right\}
$$

endowed with the norm

$$
\|u\|^2 = \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2 + \ln(1 + |x|) u^2) \, dx,
$$

Stubbe considered the $L^2$-constraint minimization problem and proved that (1.6) admits a ground state.

Soon afterwards, in [8], Cingolani and Weth processed successfully the two dimensional Schrödinger–Newton equations with nonlinear term $|u|^{p-2}u$, $p \geq 4$. Du and Weth [11] provided some results about $p > 2$ and $p \geq 3$. The key tool is Pohozaev type identity (see [11, Lemma 2.4]). Chen, Shi, and Tang [4] used the same idea to obtain a ground state.
but they could deal with the general nonlinearity \( f(u) \). Simultaneously, Chen and Tang [5] investigated the existence of an axially symmetric Nehari type ground state and nontrivial solution for

\[
-\Delta u + V(x)u + (\ln(\cdot) \ast u^2)u = f(x, u) \quad \text{in } \mathbb{R}^2,
\]

where \( V, f \) is axially symmetric about \( x \). Please see [6, 17] for further results about two dimensional Schrödinger–Newton equations with the axially symmetric assumptions. Recently, when \( V(x) = 1 \), Alves and Figueiredo [1] proved that (1.4) admits a positive ground state, where \( f \) is a continuous function with the exponential critical growth.

In this paper, motivated by the papers [1] and [5], we shall study the existence of ground state solutions of planar problem (1.1) with an exponential critical growth. In order to state our main result, we assume that

\( (V_1) \) \( V \in C(\mathbb{R}^2, \mathbb{R}) \), \( \inf_{x \in \mathbb{R}^2} V(x) > 0 \), \( V(x) := V(x_1, x_2) = V(|x_1|, |x_2|) \) for all \( x \in \mathbb{R}^2 \).

\( (V_2) \) There exists a sequence \( \{t_n\} \subset (0, \infty) \) such that \( t_n \to \infty \) and

\[
\sup_{x \in \mathbb{R}^2, n \in \mathbb{N}} \frac{V(t_n^{-1}x)}{V(x)} < \infty.
\]

\( (f_1) \) \( f \in C^1(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}) \), \( f(x, u) := f((x_1, x_2), u) = f(|x_1|, |x_2|), u) \).

\( (f_2) \) \( f(x, u) = o(|u|) \) as \( u \to 0 \), uniformly in \( x \in \mathbb{R}^2 \).

\( (f_3) \) There exists \( \alpha_0 > 0 \) such that

\[
\lim_{|u| \to \infty} \frac{f(x, u)}{\exp(\alpha u^2)} = 0 \quad \text{for } \alpha > \alpha_0, \quad \lim_{|u| \to \infty} \frac{f(x, u)}{\exp(\alpha u^2)} = +\infty \quad \text{for } \alpha < \alpha_0.
\]

\( (f_4) \) There exists \( \theta \in [0, 1) \) such that

\[
\left[ \frac{f(x, \tau)}{\tau^3} - \frac{f(x, \tau)}{(\tau t)^3} \right] \text{sign}(1 - t) + \theta V(x) \frac{|1 - t^2|}{(\tau t)^2} \geq 0, \quad \forall x \in \mathbb{R}^2, t > 0, \tau \neq 0;
\]

\( (f_5) \) \( \inf_{x \in \mathbb{R}^2, u \neq 0} \frac{F(x, u)}{u} > -\infty \), where \( F(u) = \int_0^u f(t) \, dt \).

Remark 1.1 A simple example of satisfying the hypotheses of \((V_1)-(V_2)\) is the function \( V(x) = 1 + |x_2|[1 + \sin(\pi |x_1|)] \) with \( t_n = n \). Here we also give an example which satisfies \((f_1)-(f_5)\):

\[
f(x, u) = (K(x)|u|^3u - V(x)|u|^3 u + V(x)|u|u) \exp\left(\frac{1 - \theta}{2 \pi u^2}\right),
\]

where \( K \in (\mathbb{R}^2, \mathbb{R}) \) is axially symmetric and \( \inf_{x \in \mathbb{R}^2} K(x) > 0 \), \( V \) satisfies \((V_1)\) and \((V_2)\). But it does not satisfy the Nehari type monotonic condition

\[
\frac{f(x, u)}{|u|^3} \text{ is a strictly increasing function of } u \in \mathbb{R} \setminus \{0\}.
\]

Now we state our main result as follows.
Theorem 1 For $d = 2$, suppose that $(V_1)$, $(V_2)$ and $(f_1) - (f_5)$ are satisfied. Then, for any $\alpha \in (0, \frac{\pi (1 - \theta)}{m})$, where $m$ is the least energy (it will be defined in (2.22)), $\theta$ is from $(f_5)$, (1.7) possesses a ground state solution.

Remark 1.2 The condition $\alpha \in (0, \frac{\pi (1 - \theta)}{m})$ is used to prove the minimizing sequence of $m$ is bounded, and please see Lemma 3.3. Up to now, we have not been able to remove it.

The paper is organized as follows. Section 2 is to establish the variational setting and to give some preliminaries. Section 3 is to prove the existence of ground states. Throughout the paper, we always assume that $(V_1)$, $(V_2)$ and $(f_1) - (f_5)$ hold and make use of the following notations:

- $C, C_i$ ($i = 0, 1, 2, \ldots$) for positive constants (possibly different from line to line).
- $L^s(\mathbb{R}^2) := \{ u : \mathbb{R}^2 \to \mathbb{R} : \int_{\mathbb{R}^2} |u|^s \, dx < \infty \}$ and $\| \cdot \|_s$ denotes the usual $L^s$-norm in $L^s(\mathbb{R}^2)$.

2 Variational setting and preliminaries

In this section, we begin our study by establishing the variational setting for (1.7). Let $H^1(\mathbb{R}^2)$ be the usual fractional Sobolev space with the usual norm

$$\|u\|_{H^1} = \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx \right)^{\frac{1}{2}}$$

and

$$H^1_{ad}(\mathbb{R}^2) := \{ u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|), \forall x \in \mathbb{R}^2 \}.$$

By $(V_1)$ and $(f_1)$, similar to [5], let $E$ be defined as

$$E := \left\{ u \in H^1_{ad}(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_E = \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2 + \ln(1 + |x|)u^2) \, dx \right)^{\frac{1}{2}}.$$

Denote

$$\|u\| := \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \, dx \right)^{\frac{1}{2}}, \quad \|u\|_* := \left( \int_{\mathbb{R}^2} \ln(1 + |x|)u^2 \, dx \right)^{\frac{1}{2}}.$$

According to [1, Lemma 2.1], we have the following.

Proposition 2.1 $E \hookrightarrow L^t(\mathbb{R}^2)$ is compact for all $t \in [2, \infty)$.

We formally formulate problem (1.7) in a variational way as follows:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|) |u(y)|^2 |u(x)|^2 \, dy \, dx$$

$$- \int_{\mathbb{R}^2} F(x, u) \, dx, \quad u \in E. \quad (2.1)$$
For simplicity of notations, denote

\[ I_0(u) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x - y|)|u(y)|^2 |u(x)|^2 \, dy \, dx. \]

Similar to [8], using \( \ln(r) = \ln(1 + r) - \ln(1 + \frac{1}{r}) \), \( \forall r > 0 \), it holds that

\[
I_0(u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x - y|)|u(y)|^2 |u(x)|^2 \, dy \, dx \\
- \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{1}{|x - y|}\right)|u(y)|^2 |u(x)|^2 \, dy \, dx \\
:= I_1(u) - I_2(u).
\]

We give the following proposition which is used to estimate the nonlinearity.

**Proposition 2.2 ([1, Lemma 2.5])** For every \( \alpha > 0 \) and for all \( u \in H^1(\mathbb{R}^2) \), we have

\[
\exp(\alpha u^2) - 1 \in L^1(\mathbb{R}^2).
\] (2.2)

Moreover, if \( \|\nabla u\|_2 \leq 1, \|u\|_2 \leq M, \) and \( \alpha < 4\pi \), then there exists \( C > 0 \) independent of \( u \) such that

\[
\int_{\mathbb{R}^2} \left[ \exp(\alpha u^2) - 1 \right] \, dx \leq C.
\] (2.3)

**Lemma 2.3** \( I \in C^1(E, \mathbb{R}). \)

**Proof** Noting that \( \ln(1 + |x - y|) \leq \ln(1 + |x|) - \ln(1 + |y|), \forall x, y \in \mathbb{R}^2 \), we get

\[
|I_1(u)| \leq 2\|u\|_2^2 \|u\|^2_*.
\] (2.4)

In view of \( \ln(1 + r) \leq r, \forall r > 0 \), jointly with the Hardy–Littlewood–Sobolev inequality [14], we obtain

\[
|I_2(u)| \leq C\|u\|_\frac{4}{3}.
\] (2.5)

So \( I_0 \) is well defined in \( E \).

Using \((f_1)-(f_3)\), for each \( \varepsilon > 0 \), we have

\[
|\mathcal{F}(x,u)| \leq \varepsilon |u|^2 + C(\varepsilon)|u|^p\left[ \exp(\alpha|u|^2) - 1 \right],
\] (2.6)

where \( p > 2 \). Thus, using Hölder’s inequality with \( s > 1, \frac{1}{s} + \frac{1}{s'} = 1 \), we get

\[
\int_{\mathbb{R}^2} \mathcal{F}(x,u) \, dx \leq \varepsilon \int_{\mathbb{R}^2} |u|^2 \, dx + C(\varepsilon) \int_{\mathbb{R}^2} |u|^p\left[ \exp(\alpha|u|^2) - 1 \right] \, dx \\
\leq \varepsilon \int_{\mathbb{R}^2} |u|^2 \, dx + C(\varepsilon) \left( \int_{\mathbb{R}^2} |u|^{ps} \, dx \right)^{\frac{1}{s'}} \left( \int_{\mathbb{R}^2} \left[ \exp(s'\alpha|u|^2) - 1 \right] \, dx \right)^{\frac{1}{s'}}.
\]
By Propositions 2.1 and 2.2, \( I \) is well defined in \( E \). By [8, Lemma 2.2], \( I_0 \in C^1(E, \mathbb{R}) \). It is easy to check that \( \int_{\mathbb{R}^2} F(x, u) \, dx \) belongs to \( C^1(E, \mathbb{R}) \). Thus, \( I \in C^1(E, \mathbb{R}) \). \( \square \)

Based on Lemma 2.3, we have

\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^2} \nabla u \nabla v + V(x)uv \, dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) |u(y)|^2 u(x)v(x) \, dy \, dx
- \int_{\mathbb{R}^2} f(x, u)v \, dx. \tag{2.7}
\]

**Lemma 2.4** For every \( u \in E \), we have

\[
I(u) \geq I(tu) + \frac{1-t^4}{4} \langle I'(u), u \rangle + \frac{(1-\theta)(1-t^2)^2}{4} \|u\|^2, \quad \forall t \geq 0. \tag{2.8}
\]

**Proof** Since the proof is similar to [5, Lemma 2.3], we omit it here. \( \square \)

Now, we define the Nehari manifold

\[
\mathcal{N} := \{ u \in E \setminus \{0\} : \langle I'(u), u \rangle = 0 \}.
\]

Since the Nehari type monotonic condition on \( \frac{f(x, u)}{|u|^3} \) and super-cubic condition are not satisfied, we need to prove that \( \mathcal{N} \neq \emptyset \). To the end, we introduce the following new set:

\[
\mathcal{E} := \left\{ u \in E \setminus \{0\} : \int_{\mathbb{R}^2} V(x)u^2 \, dx + I_0(u) < \int_{\mathbb{R}^2} f(x, u)u \, dx \right\}.
\]

**Lemma 2.5** \( \mathcal{E} \neq \emptyset \).

**Proof** Let \( u \in E \) with \( u \neq 0 \). \( u_t(x) := u(tx) \). By \((V_2)\), there exists \( C_1 > 0 \) such that

\[
V(t_n^{-1}x) \leq C_1 V(x), \quad \forall x \in \mathbb{R}^2, n \in \mathbb{N}. \tag{2.9}
\]

It follows that

\[
\int_{\mathbb{R}^2} V(x)(t_n u_n)^2 \, dx + I_0(t_n u_n) - \int_{\mathbb{R}^2} f(x, t_n u_n)t_n u_n \, dx
\leq C_1 \|u\|^2 + I_0(u) - \ln t_n \|u\|^2 - \int_{\mathbb{R}^2} \frac{f(t_n^{-1}x, t_n u)n}{t_n} \, dx.
\]

In view of \((f_3)\), \( t \geq 0, \tau \neq 0 \), it holds that

\[
\frac{1-t^4}{4} tf(x, \tau) + F(x, \tau) - F(x, \tau) + \frac{\theta V(x)}{4} (1-t^2)^2 \tau^2
= \int_1^\tau \left[ \frac{f(x, s\tau)}{s^3} - \frac{f(x, \tau)}{(s\tau)^3} \right] \sign(1-s) + \theta V(x) \frac{|1-t^2|}{(s\tau)^2} \tau^3 \tau^4 \, ds \geq 0. \tag{2.10}
\]

Taking \( t = 0 \), we obtain

\[
\frac{1}{4} tf(x, \tau) + \frac{\theta V(x)}{4} \tau^2 \geq 0, \quad \forall x \in \mathbb{R}^2, \tau \in \mathbb{R}. \tag{2.11}
\]
By (f₂), one has

\[ F(x, \tau) \geq -C_2 \tau^2, \quad \forall x \in \mathbb{R}^2, \tau \in \mathbb{R}. \]  

(2.12)

Thus, we get

\[
\int_{\mathbb{R}^2} \frac{f(t_n^{-1} x, t_n u) t_n u}{t_n^2} \, dx \geq \int_{\mathbb{R}^2} \left[ \frac{4F(t_n^{-1} x, t_n u)}{t_n^2} - \theta V(t_n^{-1} x) u^2 \right] \, dx \\
\geq -4C_2 \int_{\mathbb{R}^2} u^2 \, dx - \theta C_1 \int_{\mathbb{R}^2} V(x) u^2 \, dx.
\]  

(2.13)

So

\[
\int_{\mathbb{R}^2} V(x)(t_n u t_n) u^2 \, dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(|x-y|) |t_n u t_n(x)|^2 |t_n u t_n(y)|^2 \, dx \, dy \\
- \int_{\mathbb{R}^2} f(x, t_n u t_n) t_n u t_n \, dx \rightarrow -\infty,
\]

which implies that \( \mathcal{E} \neq \emptyset \).  

The following lemma shows that \( \mathcal{N} \neq \emptyset \).

**Lemma 2.6** For any \( u \in \mathcal{E} \), there exists unique \( t > 0 \) such that \( tu \in \mathcal{N} \).

**Proof** Given \( u \in \mathcal{E} \), let \( \gamma_u(t) := \langle I'(tu), tu \rangle \) for \( t > 0 \). Then \( tu \in \mathcal{N} \) if and only if \( \gamma_u(t) = 0 \).

Taking \( \varepsilon > 0 \) sufficiently small, jointly with Sobolev embedding, we obtain

\[
\gamma_u(t) \geq t^2 \|u\|^2 - t^4 I_2(u) - \int_{\mathbb{R}^2} f(x, tu) tu \, dx \\
\geq t^2 \|u\|^2 - t^4 C_1 \|u\|^2 - t^2 \varepsilon C_2 \|u\|^2 - t^6 C(\varepsilon) \int_{\mathbb{R}^2} |u|^p \left[ \exp\left( \alpha |tu|^2 \right) - 1 \right] \, dx \\
\geq t^2(1 - \varepsilon C_2) \|u\|^2 - t^4 C_1 \|u\|^3 \\
- t^6 C(\varepsilon) \left( \int_{\mathbb{R}^2} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2} \left[ \exp\left( \alpha \|tu\|^2 \left( \frac{u}{\|u\|} \right)^2 \right) - 1 \right) \, dx \right)^{\frac{1}{p}}.
\]

Choosing \( t > 0 \) small such that \( \alpha \|tu\|^2 < 4\pi \), it follows from Proposition 2.2 that there exists \( \bar{t} > 0 \) small enough such that

\[ \gamma_u(t) > 0 \quad \text{for all } 0 < t < \bar{t}. \]  

(2.14)

Now, by (f₄), one has

\[ f(x, t\tau) t\tau \geq f(x, \tau) t^4 - \theta V(x)(t^2 - 1)(t\tau)^2, \quad \forall x \in \mathbb{R}^2, t \geq 1, \tau \in \mathbb{R}, \]  

(2.15)

which implies that

\[
\int_{\mathbb{R}^2} \left[ \theta V(x)(tu)^2 - f(x, tu)tu \right] \, dx \leq t^4 \int_{\mathbb{R}^2} \left[ \theta V(x)u^2 - f(x, u)u \right] \, dx, \quad \forall t \geq 1.
\]  

(2.16)
Therefore,
\[
\gamma_u(t) = t^2 \|u\|^2 + t^4 I_0(u) - \int_{\mathbb{R}^2} f(x, tu) tu \, dx
\]
\[
\leq t^2 \|u\|^2 + t^4 \left[ \int_{\mathbb{R}^2} [V(x)u^2 - f(x, u)u] \, dx + I_0(u) \right]
\]
\[
- \theta t^2 \int_{\mathbb{R}^2} V(x)u^2 \, dx, \quad \forall t \geq 1. \quad (2.17)
\]
Thus, we have \(\gamma_u(t) \to -\infty\), as \(t \to \infty\). So there exists \(t_0 > 0\) such that \(\gamma_u(t_0) = 0\). Next, we shall prove that \(t_0\) is unique. Suppose to the contrary that there are \(t_1, t_2 > 0\) with \(t_1 \neq t_2\) such that \(\gamma_u(t_1) = \gamma_u(t_2) = 0\). For \(t_1 u \in E\), using Lemma 2.4, for all \(t > 0\), we have
\[
I(t_1 u) \geq I(tt_1 u) + \frac{(1 - \theta)(1 - t^2)^2 t_1^2 \|u\|^2}{4}. \quad (2.18)
\]
Taking \(t = \frac{t_2}{t_1}\), it yields that
\[
I(t_1 u) \geq I(t_2 u) + \frac{(1 - \theta)(1 - \left(\frac{t_2}{t_1}\right)^2)^2 t_1^2 \|u\|^2}{4}. \quad (2.19)
\]
Similarly, one has
\[
I(t_2 u) \geq I(t_1 u) + \frac{(1 - \theta)(1 - \left(\frac{t_1}{t_2}\right)^2)^2 t_2^2 \|u\|^2}{4}. \quad (2.20)
\]
We obtain \(t_1 = t_2\), so it is absurd. \qed

Since \(u \in \mathcal{N}\), by Lemma 2.4, one has
\[
I(u) = I(u) - \frac{1}{4} \langle I'(u), u \rangle \geq \frac{1 - \theta}{4} \|u\|^2. \quad (2.21)
\]
So we can define
\[
m := \inf_{u \in \mathcal{N}} I(u). \quad (2.22)
\]
Up to this stage, preparations have been made. We point out that we can define \(m\) without using the condition \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\). In the next section, taking full advantage of the condition \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\), we shall prove the existence of ground state solutions of (1.7).

3 Existence of ground states

In this section, with the additional condition \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\), we are devoted to showing that \(m\) is achieved and the minimizer is a ground state solution of equation (1.7).

Lemma 3.1 There exists \(C > 0\) such that \(\|u\| \geq C\) for all \(u \in \mathcal{N}\); furthermore, \(m > 0\).

Proof Assume by contradiction that there is \(\{u_n\} \subset \mathcal{N}\) such that \(\|u_n\| \to 0\). Obviously,
\[
\|u_n\|^2 + 4\langle I'_1(u_n), u_n \rangle = 4\langle I'_2(u_n), u_n \rangle + \int_{\mathbb{R}^2} f(x, u_n)u_n \, dx.
\]


In view of \((f_1) - (f_3)\), combining Hölder’s inequality, it follows that
\[
\left| \int_{\mathbb{R}^2} f(x, u_n) u_n \, dx \right| \\
\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx + C(\varepsilon) \int_{\mathbb{R}^2} |u_n|^p \left[ \exp(\alpha |u_n|^2) - 1 \right] \, dx \\
\leq \varepsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx \\
+ C(\varepsilon) \left( \int_{\mathbb{R}^2} |u_n|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2} \left[ \exp\left( \alpha \|u_n\|^2 \left( \frac{u_n}{\|u_n\|} \right) \right) - 1 \right] \, dx \right)^{\frac{1}{p}}.
\]

With Proposition 2.2 in hand, using the Sobolev embedding, it leads to
\[
\int_{\mathbb{R}^2} f(x, u_n) u_n \, dx = o_\varepsilon(1).
\]
By direct calculation, it holds that
\[
4[I_2'(u_n), u_n] \leq C\|u_n\|^4 = o_\varepsilon(1).
\]
Thus, one has
\[
[I_1'(u_n), u_n] = o_\varepsilon(1).
\]
Therefore, we obtain
\[
\|u_n\|^2 \leq 4[I_1'(u_n), u_n] + \int_{\mathbb{R}^2} f(x, u_n) u_n \, dx \\
\leq o_\varepsilon(1) + \varepsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx + C(\varepsilon) \int_{\mathbb{R}^2} |u_n|^p \left[ \exp(\alpha |u_n|^2) - 1 \right] \, dx. \tag{3.1}
\]
That is,
\[
(1 - \varepsilon C)\|u_n\|^2 \\
\leq o_\varepsilon(1) + C(\varepsilon) \left( \int_{\mathbb{R}^2} |u_n|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2} \left[ \exp\left( \alpha \|u_n\|^2 \left( \frac{u_n}{\|u_n\|} \right) \right) - 1 \right] \, dx \right)^{\frac{1}{p}}. \tag{3.2}
\]
Noting that \(\|u_n\| \to 0\), using Proposition 2.2 again, we get
\[
(1 - \varepsilon C)\|u_n\|^2 \leq C(\varepsilon)\|u_n\|^p, \tag{3.3}
\]
which is ridiculous. Combining with (2.21), we have \(m > 0\). \qed

Next, we give the following lemma which shall be used later.

\textbf{Lemma 3.2} \textit{For every} \(u \in E\), \textit{it holds that} \(I_1(u) \geq \frac{1}{16}\|u\|^2\|u\|_0^2\).
Proof The proof is similar to [5, Lemma 2.2]. Let

\[ A_1 := \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0, x_2 \geq 0\}, \quad A_3 := \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0, x_2 \leq 0\}. \]

For any \((x, y) \in A_1 \times A_3\), it holds that

\[ |x - y| = \sqrt{|x|^2 + |y|^2} - 2x \cdot y \geq \sqrt{|x|^2 + |y|^2} \geq |x|. \]

Thus,

\[ I_1(u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1 + |x - y|) |u(y)|^2 |u(x)|^2 \, dy \, dx \]
\[ \geq \int_{A_3} \int_{A_1} \ln(1 + |x - y|) |u(y)|^2 |u(x)|^2 \, dy \, dx \]
\[ \geq \int_{A_3} |u(y)|^2 \, dy \int_{A_1} \ln(1 + |x|) |u(x)|^2 \, dx \]
\[ = \frac{1}{16} \|u\|^2_2 \|u\|^2_\ast. \]

Let \(\{u_n\} \subset \mathcal{N}\) be a minimizing sequence of \(m\). On the additional condition \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\), we want to prove that \(\{u_n\}\) is bounded in \(E\).

Lemma 3.3 If \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\), we have \(\{u_n\}\) is bounded in \(E\).

Proof Similar to (2.21), \(\|u_n\|\) is bounded. Similar to (2.5), \(\{I_2(u_n)\}\) is bounded. Next, we want to estimate the \(\{I_1(u_n)\}\). Note that

\[ \left| \int_{\mathbb{R}^2} f(x, u_n) u_n \, dx \right| \leq \epsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx + C(\epsilon) \int_{\mathbb{R}^2} |u_n|^p \left[ \exp(\alpha |u_n|^2) - 1 \right] \, dx. \quad (3.4) \]

For the second term on the right, using Hölder’s inequality with \(s' > 1\) and \(s' \approx 1\), it holds that

\[ \int_{\mathbb{R}^2} |u_n|^p \left[ \exp(\alpha |u_n|^2) - 1 \right] \, dx \]
\[ \leq \left( \int_{\mathbb{R}^2} |u_n|^p \, dx \right)^{\frac{1}{s'}} \left( \int_{\mathbb{R}^2} \left[ \exp \left( \alpha s' \|u_n\|^2 \left( \frac{u_n}{\|u_n\|} \right)^2 \right) - 1 \right] \, dx \right)^{\frac{1}{s'}}. \]

Taking into account \(\alpha \in (0, \frac{\pi(1-\theta)}{m})\), jointly with

\[ \frac{1 - \theta}{4} \|u_n\|^2 \leq I(u_n) \to m, \]

for \(n\) large enough, we obtain \(\alpha s' \|u_n\|^2 < 4\pi\). So, by Proposition 2.2, we get

\[ \left| \int_{\mathbb{R}^2} f(x, u_n) u_n \, dx \right| \leq \epsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx + C(\epsilon) \left( \int_{\mathbb{R}^2} |u_n|^p \, dx \right)^{\frac{1}{s'}}. \quad (3.6) \]
Since
\[ \|u_n\|^2 + I_1(u_n) = I_2(u_n) + \int_{\mathbb{R}^2} f(x, u_n)u_n\,dx, \] (3.7)
which yields that \{I_1(u_n)\} is bounded. And it follows from Lemma 3.2 that \{u_n\} is bounded in \(E\).

Next, we claim that there are \(R, \eta > 0\) such that
\[ \liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2\,dx \geq \eta. \] (3.8)
If it is false, using Lions’ lemma (see [18, Lemma 1.21]), we get \(u_n \rightharpoonup 0\) in \(L^t(\mathbb{R}^2)\) for all \(t \in [2, \infty)\). Noting that
\[ |I_1(u_n)| \leq 2\|u_n\|_2^2\|u_n\|_2^2 = o_n(1), \quad |I_2(u_n)| \leq C\|u_n\|_4^4 = o_n(1), \] (3.9)
similar to (3.5), it holds that
\[
\|u_n\|^2 = o_n(1) + \int_{\mathbb{R}^2} f(x, u_n)u_n\,dx
\leq o_n(1) + \varepsilon \int_{\mathbb{R}^2} |u_n|^2\,dx + C(\varepsilon)C\left(\int_{\mathbb{R}^2} |u_n|^p\,dx\right)^{\frac{1}{p}}
= o_n(1), \] (3.10)
which contradicts Lemma 3.1.

**Lemma 3.4** \(m\) is achieved and the minimizer is a weak solution of (1.7).

**Proof** Now, we can assume that \(u_n \to u_0 \neq 0\) in \(E\), \(u_n \to u_0\) in \(L^t(\mathbb{R}^2)\) for all \(t \in [2, \infty)\) and \(u_n(x) \to u_0(x)\) a.e. in \(\mathbb{R}^2\). By a standard argument, one can deduce that \(I'(u_0) = 0\). Obviously, we have
\[
\int_{\mathbb{R}^2} F(x, u_n)\,dx = \int_{\mathbb{R}^2} F(x, u_0)\,dx + o_n(1), \] (3.11)
\[
\int_{\mathbb{R}^2} f(x, u_n)u_n\,dx = \int_{\mathbb{R}^2} f(x, u_0)u_0\,dx + o_n(1). \] (3.12)

Here, we only check (3.12) since (3.11) is similar. We have already known that
\[
|f(x, u_n)u_n| \leq \varepsilon |u_n|^2 + C(\varepsilon)|u_n|^p \left[ \exp\left(\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right) - 1 \right]. \] (3.13)
Noting that \(\alpha \in (0, \frac{\pi^{(1-\theta)}}{m})\) and (3.5), we obtain that \(\alpha\|u_n\|^2 < 4\pi\) for \(n\) large enough. By Proposition 2.2, there exists \(C > 0\) independent of \(n\) such that
\[
\int_{\mathbb{R}^2} \left[ \exp\left(\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right) - 1 \right]\,dx \leq C.
\]
It follows from [18, Lemma A.1] and Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} f(x, u_n)u_n \, dx = \int_{\mathbb{R}^2} f(x, u_0)u_0 \, dx.$$  \hspace{1cm} (3.14)

Thus, we have

$$m = \lim_{n \to \infty} \left[ I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \right] \geq \frac{1}{4} \|u_0\|^2 + \int_{\mathbb{R}^2} \left[ \frac{1}{4} f(x, u_0) - F(x, u_0) \right] \, dx$$

$$= I(u_0) - \frac{1}{4} \langle I'(u_0), u_0 \rangle$$

$$\geq m. \quad \Box$$

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