COHOMOGENEITY ONE KÄHLER-RICCI SOLITONS UNDER A HEISENBERG GROUP ACTION AND RELATED METRICS

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Abstract. We show that integrability of an almost complex structure in complex dimension $m$ is equivalent, in the presence of an almost hermitian metric, to $m(m-1)$ equations involving what we call shear operators. Inspired by this, we give an ansatz for Kähler metrics in dimension $m > 1$, for which only $m-1$ of these shear equations are non-trivial. The equations for gradient Kähler-Ricci solitons in this ansatz are frame dependent PDEs, which specialize to ODEs under extra assumptions. Metrics solving the latter system include a restricted class of cohomogeneity one metrics, and we find among them complete expanding gradient Kähler-Ricci solitons under the action of the $(2m-1)$-dimensional Heisenberg group, and some incomplete steady solitons. In another special case of the ansatz we present, for $m = 2$, a class of complete metrics of a more general type which we call gradient Kähler-Ricci skew-solitons, which are cohomogeneity one under the Euclidean plane group action. This paper continues research started in [MR, AM2].

1. Introduction

The main result of this paper is the construction of explicit complete expanding gradient Kähler-Ricci solitons of cohomogeneity one under the action of the Heisenberg group in any complex dimension $m > 1$. See Theorem 2 and formulas (56). We approach them via a framework in which we also show existence of incomplete steady Kähler-Ricci solitons, and in dimension four, complete cohomogeneity one metrics under the Euclidean plane group of a type we call gradient Kähler-Ricci skew-solitons. The definition of such a skew-soliton includes that of a gradient Kähler-Ricci soliton as a special case.

Ricci solitons have been studied extensively in the last few decades. They were introduced by Hamilton [H1] as solutions to the Ricci flow that vary only by rescaling of pull-backs of an initial metric under diffeomorphisms of the manifold. In the case of gradient solitons, that initial metric $g$ satisfies an equation of the form

$$\text{Ric} + \nabla df = \lambda g,$$

where $\text{Ric}$ is the Ricci curvature of $g$ and $\nabla df$ is the Hessian of a smooth function $f$ on the given manifold. The soliton constant $\lambda$ is negative in the expanding case, and the corresponding solution to the flow exists for finite negative time and all positive time. This constant is zero in the steady case.
Explicit examples of Kähler-Ricci solitons were given in [C2], and those are both expanding and $U(n)$-invariant. Other early and more recent examples appear in [K, C1, CV, G, PTV, FIK, Y, FW, CD]. Among metrics on homogeneous manifolds, there is a well-known correspondence between left-invariant Einstein metrics on solvable Lie groups and homogeneous Ricci solitons on their nilradicals [La1].

In the case of cohomogeneity one metrics, many compact and complete Ricci solitons, both Kähler and non-Kähler, where found relatively recently in [DW, BDW, BDGW, W1, W2], under compact group actions. Aside from being a non-compact group, the case of the Heisenberg group presented here features Kähler-Ricci solitons with a group action having no singular orbits.

We also examine some of the properties of these solitons of relevance to Ricci flow considerations. Their Ricci curvature is indefinite, the scalar curvature changes sign, but has a finite lower bound. The sectional curvatures and holomorphic bisectional curvatures are bounded and not all of them are nonnegative. We compute some of the invariants of the associated Ricci flow, and determine that it is of type III, in the sense of [H2].

We now describe the relation of this work to our earlier paper [MR], and to [AM2]. The former paper, which was greatly influenced by work of Dancer and Strachan [DS1], explored diagonal Kähler-Einstein metrics in dimension four under a cohomogeneity one action of unimodular Lie groups of dimension three, the so-called Bianchi type A class. A complete such metric was shown to exist for the Euclidean plane group, while [AM2] contained a similar result for the three dimensional Heisenberg group. Both of these results were obtained via ODE methods.

Now in [MR] we developed a more general framework within which these $m=2$ cohomogeneity one metrics are special cases, and another result there gives local existence of Kähler Ricci-flat metrics that are determined by PDEs (more precisely, generalized frame-dependent PDEs), and are not of cohomogeneity one. In the current paper, this framework is extended to all complex dimensions $m > 1$. However, while for $m=2$, diagonal cohomogeneity one metrics for all unimodular three dimensional Lie groups are special cases of our general ansatz, its application to cohomogeneity one metrics in dimension $m > 2$ is limited. In fact, only the Heisenberg group action fits as a special case when trying to construct expanding solitons of the type we consider. But just as in the $m=2$ case, for $m > 2$ one can ask whether more solutions will result from this ansatz with frame-dependent PDEs replacing the above mentioned ODEs. We leave the exploration of such questions to future work.

Here is one main ingredient of the said framework. As is well-known, integrability of an almost complex structure $J$ is equivalent to the vanishing of the Nijenhuis tensor. In the presence of an almost hermitian metric, we fix some orthogonal decomposition of the tangent bundle into rank 2 $J$-invariant subbundles. Extending the ideas of [AM2] and [MR], we show Nijenhuis vanishing can be given equivalently in the form of $m(m-1)$ so-called shear equations, relating the complex structure with shear operators, each of which acts on a pair of sections, one from each of two distinct such subbundles. The utility
of this relates to matters of integrability, as one can consider cases in which many of these shear operators are simply the trivial zero operator. The number of non-zero shear operators in the integrability equations of a given complex structure can be used as a measure of its complexity. In the scenario described in this paper, at most \( m - 1 \) of the shear equations are nontrivial. In [MR], in which \( m = 2 \), there was at most one nontrivial shear equation, and subsequent metrics constructed were simpler in case there were none. We expect this approach to integrability to have other applications.

As the shear operator in a local frame of each rank two subbundle can be given by expressions involving the metric and Lie brackets, we proceed to define our most general ansatz through Lie bracket relations of frame vector fields. Those are designed so that the above \( m - 1 \) shear equations hold, and also so that the metric making this frame orthonormal is Kähler. For the latter to hold \( m \) additional Lie bracket relations are assumed between frame vector fields forming sections of the same rank 2-subbundle. An analysis of this additional structure is given in the appendix.

Another notable point pertains to our method of proof of completeness of the metrics in the case of non-compact cohomogeneity one group actions. The degree of closeness of the group to being abelian can be measured by the number of vanishing Lie bracket relations in some basis for its Lie algebra. Representing this basis by a suitable left-invariant orthogonal frame on the manifold, this number corresponds to the number of closed, hence locally exact, 1-forms in the dual coframe. The primitives of these forms can be used as coordinate functions forming part of a coordinate system on an open and dense chart domain. We can call these the “trivial” coordinates as their values along a finite length curve are fairly easy to bound. This method was already used in [AM2, MR]. However, in the first of these references we were only able to prove completeness for a Kähler-Einstein metric under the action of a quotient of the dimension 3 Heisenberg group, with the quotient being employed in part to mod-out the remaining “nontrivial” coordinate function. In this paper, and for Kähler-Ricci solitons, we bound that coordinate function as well, in spite of the nontriviality of the coordinate expression of the associated coframe 1-form. One however can avoid the need to do so, see Remark 5.4.

With regard to the notion of gradient Kähler-Ricci skew-solitons, we define it for use on manifolds that do not admit gradient Kähler-Ricci solitons of the type we consider in this paper for prosaic reasons: the candidate for a soliton vector field we examine is not holomorphic. To our knowledge, this natural generalization of a Kähler-Ricci soliton does not seem to have been considered in the literature. See [La2] for related notions that are discussed in subsection 2.4.

In Section 2 we describe the framework (Theorem 1) and overall ansatz, and give the definition of skew-solitons. Section 3 gives the generalized PDEs for skew-solitons in this ansatz, their specialization to ODEs and further conditions required for a skew-soliton in this ansatz to be an actual Kähler-Ricci soliton (Prop. 3.1). In Section 4 we show how a certain class of cohomogeneity one metrics fits with our ansatz and derive the soliton and skew-soliton equations.
in this case (Prop. 4.1). In Section 5, which is our main section, we prove the existence of complete expanding gradient Kähler-Ricci solitons under a Heisenberg group action, show they are complete (Theorem 2), and describe some of their properties, including the singularity type for the associated Ricci flow. In Section 6 we describe our results on steady solitons for the specialization of the ansatz to ODEs. Section 7 is devoted to the demonstration of existence of complete Kähler-Ricci skew-solitons in complex dimension two under the action of the Euclidean plane group (Theorem 3), following a method employed in [MR]. The appendix contains an analysis of the closedness of the Kähler form from the point of view of the framework of Section 2.

2. The general framework

2.1. Shear and integrability. Let \((M, g, J)\) be an almost hermitian manifold of real dimension \(2m\). Assume a decomposition of the tangent bundle as an orthogonal sum of rank two \(J\)-invariant subbundles

\[ TM = \bigoplus_{i=1}^{m} \mathcal{H}_i. \]  

Let \(\pi_{\mathcal{H}_j} : TM \rightarrow \mathcal{H}_j\) denote the orthogonal projection. Consider the operator \(\pi_{\mathcal{H}_j} \circ \nabla : \Gamma(\mathcal{H}_j) \times \Gamma(\mathcal{H}_i) \rightarrow \Gamma(\mathcal{H}_j), \ j \neq i\), in which we restrict the Levi-Civita covariant derivative \(\nabla\) of \(g\) to sections of \(\mathcal{H}_j \times \mathcal{H}_i\). Define the shear operator of such a pair of subspaces by

\[ S_{ji} := \text{Sym}^0[\pi_{\mathcal{H}_j} \circ \nabla|_{\mathcal{H}_j \times \Gamma(\mathcal{H}_i)}], \]

where \(\text{Sym}^0\) denotes the trace-free symmetric component with respect to the linear action of \(\nabla X\) on \(\mathcal{H}_j\), for any fixed \(X \in \Gamma(\mathcal{H}_i)\).

See [AM1] for background on the relation to the shear operator in general relativity.

Our purpose here is to give a condition equivalent to the integrability of \(J\) in terms of shear operators.

**Theorem 1.** Given the above set-up, the almost complex structure \(J\) is integrable if and only if for all \(i = 1, \ldots m\) and any \(j \neq i\)

\[ JS_{ji}^v w = S_{ji}^v J w, \quad \text{for all } w \in \Gamma(\mathcal{H}_i) \text{ and } v \in \mathcal{H}_j. \]  

**Proof sketch.** The issue is local, so consider a frame domain for an orthonormal frame \(\{e_i\}\) such that locally \(\mathcal{H}_i = \text{span}(e_{2i-1}, e_{2i})\). It is enough to show the Nijenhuis tensor \(N\) vanishes on pairs of frame fields. Its defining formula clearly shows that it vanishes if both frame fields are in \(\mathcal{H}_i\). Because \(N(a, b) = JN(a, Jb)\) and \(N\) is antisymmetric, if \(j \neq i\) it is enough to check under what condition it vanishes on a pair \(e_{2i-1}, e_{2j-1}\). That calculation proceeds similarly to [AM1, Theorem 1], and will be omitted. \[\Box\]

We point out that the calculation just alluded to relies on the following expression of the matrix corresponding to the shear operator, evaluated on a
vector field $X \in \Gamma(H_i)$ in a local oriented orthonormal frame \{e_{2j-1}, e_{2j}\} on $\mathcal{H}_j$.

$$[S^i_j X]_{e_{2j-1}, e_{2j}} = \begin{bmatrix} -\sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 \end{bmatrix},$$

with shear coefficients:

$$2\sigma_1 := g([X, e_{2j-1}], e_{2j-1}) - g([X, e_{2j}], e_{2j}),$$
$$2\sigma_2 := -g([X, e_{2j-1}], e_{2j}) - g([X, e_{2j}], e_{2j-1}).$$

Note that to check integrability via Theorem 1, one has to verify $m(m-1)$ operator equations. Simpler special cases occur if some of the shear operators $S^i_j$ happen to be the zero operator, in which case the corresponding equation (2) holds automatically. In the next subsection we will define an ansatz for Kähler metrics for which $S^i_j = 0$ for all pairs $ji$ except for those of the form $ji_0$ for one particular index $i_0$. Thus there will only be $m-1$ non-trivial equations involving shear operators. All metrics discussed in this paper will be special cases of this ansatz.

2.2. Kähler frame system. Let $(M, g)$ be a Riemannian manifold of dimension $2m = 2n + 2$ admitting an orthonormal frame \{s_j \}_{j=1}^{2m} = \{k, t, \{x_i, y_i\}_{i=1}^{n}\}, defined over an open $U \subset M$, which satisfies the following Lie bracket relations:

$$[k, t] = L(k + t), \quad [x_i, y_i] = N_i(k + t), \quad (4)$$
$$[k, x_i] = A_i x_i + B_i y_i, \quad [k, y_i] = C_i x_i + D_i y_i, \quad (5)$$
$$[t, x_i] = E_i x_i + F_i y_i, \quad [t, y_i] = G_i x_i + H_i y_i, \quad (6)$$

for smooth functions $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, L, N_i$ on $U$ such that

$$A_i - D_i = F_i + G_i, \quad B_i + C_i = H_i - E_i, \quad (7)$$
$$N_i = A_i + D_i = -(E_i + H_i) \quad (8)$$

for $i = 1, \ldots, n$. Define an almost complex structure $J = J_{g, \{e_j\}}$ by linearly extending the relations $Jk = t, Jt = -k, Jx_i = y_i$ and $Jy_i = -x_i, i = 1, \ldots, n$.

Another proof of the following proposition, which employs the decomposition (1) in a different way, is outlined in the appendix.

Proposition 2.1. $(M, g, J)$ defined as above gives a Kähler structure on $U$.

Proof. $J$ clearly makes $g$ into an almost hermitian metric. To see that $J$ is integrable, we verify the conditions of Theorem 1. Changing slightly the indexing of the previous subsection, let $\mathcal{H}_0 = \text{span}(k, t), \mathcal{H}_i = \text{span}(x_i, y_i), i = 1, \ldots, n$, with the frame indexing beginning with $k = e_{-1}, t = e_0$. In view of (3), the vanishing Lie bracket relations clearly show that the shear operators $S^i_j = 0$ if $i \neq 0, j \neq 0$ and $i \neq j$. Additionally $S^i_0 = 0$ for all $i = 1, \ldots, m - 1$ as well, since (3), with $X = x_i$ or $X = y_i$ and $e_{2j-1} = k, e_{2j} = t$ yields zero shear coefficients for $X$ in view of (5)-(6) and the orthonormality of our frame. Finally, to check (2) for $S^j_0$ we note that in terms of shear coefficients this equation takes the form $\sigma_1^t = \sigma_2^t, \sigma_3^t = -\sigma_4^t$, and these hold as they are equivalent, by (3), to the assumed relations (7).
To show that \( g \) is Kähler, define a connection on \( U \) by first setting
\[
\nabla_k k = -Lt, \quad \nabla_x x_i = A_i k + E_i t, \quad \nabla_x x_i = -A_i x_i + E_i y_i, \quad \nabla_x x_j = 0, i \neq j,
\]
and then having all other covariant derivative expressions on frame fields determined by the requirement that \( \nabla \) be torsion-free and make \( J \) parallel (here the definition of \( J \) and relations (4)-(8) are used repeatedly). It is easily checked that \( \nabla \), thus defined, is compatible with the metric \( g \), so that it is its Levi-Civita connection and hence \( J \) is \( g \)-parallel. This completes the proof. \( \Box \)

One additional object comes for free with the Kähler structure in Proposition 2.1. Namely, the Lie bracket relations (4)-(6) imply that the distribution spanned by \( k + t, x_i \) and \( y_i, i = 1, \ldots n \) is integrable. Since this distribution is orthogonal to \( k - t \), while the latter vector field has constant length and is easily seen to have geodesic flow, it follows that it is locally a gradient (cf. [ON, Cor. 12.33]). Thus, there exists a smooth function \( \tau \) defined in some open set \( V \subset U \), such that
\[
k - t = \nabla \tau.
\]

As in [MR], in searching for distinguished metrics we will assume that the functions \( A_1, \ldots H_k, N_k \), \( i = 1 \ldots n \) and \( L \) are each a composition with \( \tau \) of a smooth real-valued function defined on the image of \( \tau \), and will abuse notation by denoting the latter functions by the same respective letters as the former. In the case of non-Ricci-flat Kähler-Einstein metrics of dimension four that fit the ansatz of Prop. 2.1, we have shown in [MR] that this assumption always holds necessarily. We will not attempt to extend that result in this paper.

2.3. The Ricci form. The Ricci form of the Kähler metric \( g \) in Proposition 2.1 is computed as follows. Denote by \( w_0 = k - it, w_i = x_i - iy_i, i = 1, \ldots n \) the corresponding complex-valued frame, and compute the complex valued 1-forms \( \Gamma_i^j \) for which \( \nabla w_i = \Gamma_i^j \otimes w_j \), where here \( \nabla \) denotes the obvious complexification of the Levi-Civita connection of \( g \) and the summation convention was used. The formulas are deduced by computing the components \( \nabla_{\varepsilon_i} w_i \), where \( \varepsilon_i \) stands for one of the frame fields, using the covariant derivative frame formulas for the Levi-Civita connection \( \nabla \), given in the proof of Proposition 2.1. The 1-forms \( \Gamma_0^n, \Gamma_j^j, j = 1, \ldots n \) resulting from this calculation are given by
\[
\Gamma_0^n = -i L (k + \hat{t}), \quad \Gamma_j^j = -i (C_j - H_j) \hat{k} - i (A_j - F_j) \hat{t}, \quad j = 1 \ldots n,
\]
where the hatted quantities denote the non-metrically-dual coframe of \( \{ e_\ell \} \).

Citing, for example, Lemma 4.2 in [DM], the Ricci form of \( g \) is given by
\[
\rho = i (d\Gamma_0^n + \sum_{j=1}^n d\Gamma_j^j) = L (dk + dt) + \left( \sum_{j=1}^n (C_j - H_j) \right) dk
\]
\[
+ \left( \sum_{j=1}^n (A_j - F_j) \right) dt + dL \wedge (k + \hat{t}) + \left( \sum_{j=1}^n d(C_j - H_j) \right) \wedge \hat{k}
\]
\[
+ \left( \sum_{j=1}^n d(A_j - F_j) \right) \wedge \hat{t}. \tag{11}
\]

We now wish to write the Ricci components in our frame. Applying to our coframe the formula \( d\eta(a, b) = d_a(\eta(b)) - d_b(\eta(a)) - \eta([a, b]) \), valid for any
smooth 1-form \( \eta \), we have
\[
\begin{align*}
\hat{d}k(k, t) &= -L = \hat{d}(k, t), \\
\hat{d}k(x_i, y_i) &= -k([x_i, y_i]) = -\hat{k}(N_i(k + t)) = -N_i = \hat{d}(x_i, y_i), \\
\hat{d}k(k, x_i) &= d\hat{k}(k, y_i) = d\hat{k}(t, x_i) = d\hat{k}(t, y_i) = 0,
\end{align*}
\] (12)
whereas \( \hat{d}k, \hat{d}t \) vanish on pairs taken from \( x_i, y_i, x_j, y_j \) for \( i \neq j \).

Using this in (11) we have for \( i = 1, \ldots, n \)
\[
\begin{align*}
\rho(x_i, y_i) &= -N_i(2L + \sum_{j=1}^{n} (C_j - H_j + A_j - F_j)), \\
\rho(k, t) &= L(2L + \sum_{j=1}^{n} (C_j - H_j + A_j - F_j)) \\
&+ d_{k-t}L - \sum_{j=1}^{n} (d_{t}(C_j - H_j) + d_{k}(A_j - F_j)), \\
\rho(k, x_i) &= -d_{x_i}(L + \sum_{j=1}^{n} (C_j - H_j)), \\
\rho(k, y_i) &= -d_{y_i}(L + \sum_{j=1}^{n} (C_j - H_j)), \\
\rho(t, x_i) &= -d_{x_i}(L + \sum_{j=1}^{n} (A_j - F_j)), \\
\rho(t, y_i) &= -d_{y_i}(L + \sum_{j=1}^{n} (A_j - F_j)), \\
\rho(x_i, x_j) &= 0, \quad \rho(x_i, y_j) = 0, \quad \rho(y_i, y_j) = 0 \quad \text{for} \ i \neq j,
\end{align*}
\] (13)
where \( d_{e\ell} \) denotes the directional derivative with respect to \( e_{\ell} \).

2.4. **The Ricci soliton equation and its two generalizations.** We discuss here two soliton-type equations, one of which will be examined later for \((M, g)\) as in Proposition 2.1.

Consider on a Kähler manifold an equation of the form
\[
\rho + \frac{1}{2}\mathcal{L}_X \omega = \lambda \omega
\] (14)
where \( \rho \) is the Ricci form, \( \omega \) the Kähler form, \( \lambda \) is a constant and \( \mathcal{L}_X \) is the Lie derivative with respect to a smooth vector field \( X \). This is the Chern-Ricci soliton equation [La2], even on a Kähler manifold, as \( X \) is just smooth and possibly not holomorphic. Computing this Lie derivative term on any Kähler manifold, we have,
\[
\begin{align*}
\mathcal{L}_X \omega(a, b) &= (dt X \omega)(a, b) = (\nabla_a (tX \omega))(b) - (\nabla_b (tX \omega))(a) \\
&= \omega(\nabla_a X, b) - \omega(\nabla_b X, a) = g(\nabla_a (JX), b) - g(\nabla_b (JX), a),
\end{align*}
\] (15)
as \( \omega \) and \( J \) are parallel. Note that this expression is generally different from the skew-symmetric expression
\[
g(\nabla_{Ja} X, b) - g(\nabla_{Jb} X, a)
\] (16)
However, the two expressions are equal if \( X \) is holomorphic, since then
\[
0 = (\mathcal{L}_X J)(a) = \mathcal{L}_X (Ja) - J\mathcal{L}_X a = [X, Ja] - J[X, a] \\
\nabla X (Ja) - \nabla_{Ja} X - J\nabla X a + J\nabla a X = J\nabla a X - \nabla_{Ja} X,
\]
and this observation also shows that (15) and (16) are indeed generally not equal if \( X \) is smooth but not holomorphic, as \( \mathcal{L}_X J \) is skew-adjoint rather than self-adjoint for a hermitian metric.

The condition that \( X \) is holomorphic and not just smooth turns (14) into the standard Ricci soliton equation. Now if \( X = \nabla f \) is a gradient of a smooth function \( f \), then \( X \) is holomorphic exactly when \( JX \) is a Killing field (cf. [DM, Lemma 5.2]). In the following we will use the latter criteria to examine gradient Ricci solitons. However, we will also look at cases where \( X = \nabla f \) is only smooth, and between the two generalizations (15) and (16) of the gradient Ricci soliton condition, we will be examining the latter, i.e. the equation

\[
\rho(a, b) + \frac{1}{2}(\nabla df(Ja, b) - \nabla df(Jb, a)) = \lambda \omega(a, b),
\]

which corresponds to (16) since \( \nabla df(Ja, b) = g(\nabla_Ja \nabla f, b) \). We will call pairs \((g, f)\) satisfying (17) \textit{gradient Kähler-Ricci skew-solitons}.

3. The soliton equation in our frame

We now consider \((M, g, J)\) as in Proposition 2.1. Employing the formula

\[
\nabla df(e_i, e_j) = d_{e_i} d_{e_j} f - df(\nabla_{e_i} e_j)
\]

and noting that our Kähler form is just \( \omega = \hat{k} \wedge \hat{t} + \sum_{i=1}^n \hat{x}_i \wedge \hat{y}_i \), one calculates, using also (13) and the covariant derivative formulas stemming from (9), that the skew-soliton equation (17) is equivalent to the following system of frame-dependent PDEs, for each \( i = 1, \ldots, n \)

\[
\begin{align*}
- \mathcal{N}_i(2L + \sum_{j=1}^n (C_j - H_j + A_j - F_j)) + \frac{1}{2}(d_{x_i}^2 f + d_{y_i}^2 f - \mathcal{N}_i (d_k f - dt f)) &= \lambda, \\
- L(2L + \sum_{j=1}^n (C_j - H_j + A_j - F_j)) + d_{k-t} L - \sum_{j=1}^n (dt (C_j - H_j) + d_k (A_j - F_j)) + \frac{1}{2}(d_{k-t}^2 f + d_{t-t}^2 f - L (d_k f - dt f)) &= \lambda, \\
- d_{x_i} (L + \sum_{j=1}^n (C_j - H_j)) + \frac{1}{2}(d_{x_i} d_k f - d_{x_i} d_{y_i} f - B_i d_{x_i} f + A_i d_{y_i} f) &= 0, \\
- d_{y_i} (L + \sum_{j=1}^n (C_j - H_j)) + \frac{1}{2}(d_{y_i} d_k f + d_{k} d_{x_i} f - D_i d_{x_i} f + C_i d_{y_i} f) &= 0, \\
- d_{x_i} (L + \sum_{j=1}^n (A_j - F_j)) + \frac{1}{2}(-d_{x_i} d_t f - d_{t} d_{x_i} f - F_i d_{x_i} f + E_i d_{y_i} f) &= 0, \\
- d_{y_i} (L + \sum_{j=1}^n (A_j - F_j)) + \frac{1}{2}(-d_{y_i} d_k f + d_{k} d_{x_i} f - H_i d_{x_i} f + G_i d_{y_i} f) &= 0, \\
d_{y_i} d_{x_j} f - d_{y_j} d_{x_i} f &= 0, \\
d_{x_i} d_{y_j} f + d_{x_j} d_{x_i} f &= 0 \text{ for } i \neq j.
\end{align*}
\]

As for the additional conditions for this to be Kähler-Ricci soliton, we will compute them later only for special choices of \( f \).

3.1. The \( \tau \)-dependent case. We now make the assumption that the skew-soliton potential \( f \) is a composition of a function on the range of \( \tau \), with \( \tau \). By abuse of notation we write \( f = f(\tau) \).

Additionally, we assume the functions \( A_i, \ldots, H_i, N_i, L \) are also such composites. We call this setting the \( \tau \)-dependent case.
Now \( d_{\mathbf{x}, \tau} = g(\mathbf{x}, \mathbf{k} - \mathbf{t}) = 0 \) and similarly \( d_{\mathbf{y}, \tau} = 0 \), \( i = 1 \ldots n \), while \( d_{\mathbf{k}} \tau = 1 \) and \( d_{\mathbf{t}} \tau = -1 \). It follows that under these assumptions the equations represented in the last five lines of in (18) are satisfied trivially, while those in the first two lines become

\[
- N_i(2L + \sum_{j=1}^{n}(C_j - H_j + A_j - F_j)) - N_i f' = \lambda, \\
- L(2L + \sum_{j=1}^{n}(C_j - H_j + A_j - F_j)) + 2L' + \sum_{j=1}^{n}(C'_j - H'_j + A'_j - F'_j) + f'' - Lf' = \lambda,
\]

where the prime denotes differentiation with respect to \( \tau \).

Consider the case \( \lambda \neq 0 \). In that case, from (19), \( 2L + \sum_{j=1}^{n}(C_j - H_j + A_j - F_j) \) is nowhere zero, and isolating \( N_i \) from that equation, we see \( N_i \), which is also nonzero, is independent of \( i \). Hence in analyzing this case we denote

\[
N = N_i, \quad i = 1 \ldots n.
\]

Writing (19) in the form

\[
2L + \sum_{j=1}^{n}(C_j - H_j + A_j - F_j) = -\frac{\lambda}{N} - f'
\]

we see that (20) is just

\[
L\left(\frac{\lambda}{N} + f'\right) - \left(\frac{\lambda}{N} + f'\right)' + f'' - Lf' = \lambda
\]

which simplifies to

\[
\lambda\left(\frac{L}{N} + \frac{N'}{N^2} - 1\right) = f'' - Lf' - f'' + Lf' = 0
\]

However, we will now see that this equation is in fact an identity, so that we only need to consider (21).

From relations (12), we have, returning momentarily to the notation \( N_i \),

\[
d\hat{\mathbf{k}} = -\sum_{i=1}^{n} N_i \hat{\mathbf{x}}_i \wedge \hat{\mathbf{y}}_i, d\hat{\mathbf{k}} = -L\hat{\mathbf{k}} \wedge \hat{\mathbf{t}}.
\]

Expanding \( d^2 \mathbf{k} = 0 \) and substituting

\[
dN_i = d_k N_i \hat{\mathbf{k}} + d_t N_i \hat{\mathbf{t}} + \sum_{i=1}^{n}(d_{\mathbf{x}_i} N_i \hat{\mathbf{x}}_i + d_{\mathbf{y}_i} N_i \hat{\mathbf{y}}_i),
\]

we get a complicated expression, where the coefficients of \( \hat{\mathbf{k}} \wedge \mathbf{x}_i \wedge \mathbf{y}_i \) and \( \hat{\mathbf{t}} \wedge \mathbf{x}_i \wedge \mathbf{y}_i \) are, respectively

\[
-d_k N_i + N_i A_i + N_i D_i - L N_i = 0, \\
-d_t N_i + N_i E_i + N_i H_i + L N_i = 0,
\]

Subtracting the second of these from the first, then using \( d_{\mathbf{k} - \mathbf{t}} = 2\partial_\tau \) along with the relations (8) and dividing by \(-2N_i^2\), gives (22).

We now check the extra conditions needed for the skew-soliton to be a Kähler-Ricci soliton, by examining when \( X = J\nabla f \) is a Killing vector field. As \( f = f(\tau) \), we have \( \nabla f = f'(\tau)(\mathbf{k} - \mathbf{t}) \) so \( J\nabla f = f'(\tau)(\mathbf{k} + \mathbf{t}) \). Computing

\[
(\mathcal{L}_X g)(a, b) = g([a, X], b) + g([b, X], a)
\]
on our frame, we find
\[ \frac{1}{2} \mathcal{L}_X g = (f''(\tau) + Lf'(\tau))(k^2 - \dot{t}^2) \]
\[ - f'(\tau) \left( \sum_{i=1}^n \left[ (A_i + E_i)\hat{x}_i^2 \right. \right. \]
\[ + \left. \left. (B_i + C_i + F_i + G_i)\hat{x}_i \circ \hat{y}_i + (D_i + H_i)\hat{y}_i^2 \right] \right). \]

Therefore, at points where \( f'(\tau) \neq 0 \), \( J\nabla f \) is Killing when
\[ f''(\tau) + Lf'(\tau) = 0, \quad A_i + E_i = 0, \quad D_i + H_i = 0, \]
\[ B_i + C_i + F_i + G_i = 0, \]
for \( i = 1, \ldots, n \).

We summarize the results of this subsection.

**Proposition 3.1.** Let \((M, g, J)\) be a Kähler manifold as in Proposition 2.1, with \( A_i, \ldots, H_i, N_i, i = 1, \ldots, n \) and \( L \) functions of \( \tau \). Then \( g \) is a gradient Ricci skew-soliton with soliton potential \( f = f(\tau) \) and nonzero soliton constant \( \lambda \) if and only if all functions \( N_i \) are equal to a single function \( N \) and equation (21) holds. Furthermore, \((g, f)\) is a gradient Ricci soliton if and only if additionally equations (23)-(26) hold for all \( i = 1 \ldots n \) away from critical points of \( f(\tau) \) and every such critical point is degenerate.

### 4. The Cohomogeneity One Subclass

In this section we begin a study of a fairly explicit class of examples. We first discuss cohomogeneity one metrics admitting a frame satisfying conditions (4)-(8). Our discussion follows [MR] closely.

Let \((M, g)\) is a Riemannian manifold of dimension \( 2m = 2n + 2 \) admitting a proper isometric action by a Lie group \( \mathcal{G} \) with cohomogeneity one. Then there is a subgroup \( \mathcal{H} < \mathcal{G} \) so that \( \mathcal{G}/\mathcal{H} \) is the \( 2n + 1 \)-dimensional principal orbit type. Let \( p \in M \) be a point with isotropy group \( \mathcal{K} \) satisfying \( \mathcal{H} \leq \mathcal{K} < \mathcal{G} \). Then the orbit \( \mathcal{G} \cdot p \) through \( p \) is isomorphic to \( \mathcal{G}/\mathcal{K} \). Consider the following complementary facts. For the principal \( \mathcal{K} \)-bundle \( \mathcal{G} \rightarrow \mathcal{G}/\mathcal{K} \), one has the an associated bundle \( \mathcal{G} \times_\mathcal{K} \nu_p \), where \( \nu_p \) is the normal space to the orbit at \( p \). The differential of the action mapping identifies this bundle with the full normal bundle \( \nu \) to the orbit. On the other hand the normal exponential map \( \exp_p^\perp \) at \( p \) sends an \( \varepsilon \)-disk in \( \nu_p \) to a slice for the action of \( \mathcal{G} \)
\[ S' = \{ \exp_p(rX) \mid 0 \leq r < \varepsilon, |X| = 1, X \perp \mathcal{G} \cdot p \}. \]

and induces, by the tubular neighborhood theorem, a map from a neighborhood of the zero section in \( \nu \) to a neighborhood of the orbit. Putting these facts together we obtain an equivariant diffeomorphism,
\[ \mathcal{G} \times_\mathcal{K} D^{\ell+1} \cong \mathcal{G} \cdot S', \]
where \( \ell = \dim \mathcal{K} - \dim \mathcal{H} \) (cf. [Pt, Section 5.6]).
The isotropy action of $K$ in fact preserves length, so on $S'$, we see that the spheres

$$S_r = \{ \exp_p(rX) \mid |X| = 1, X \perp G \cdot p \}$$

are preserved by the induced action of $K$. Since points on one of these spheres have isotropy type $H$, we must have

$$K/H \cong S^\ell.$$

Regarding the metric as residing on $G \times K D^{\ell+1}$, it can be written in the form

$$dr^2 + g_r. \quad (27)$$

In the case of a unimodular group, we will now consider the special case of a diagonal metric, in the form inspired by the four-dimensional case appearing, for example, in [DS1]. Namely, we write

$$g = \left( \prod_{i=1}^n (a_ib_i)^2 \right) c^2 dt^2 + c^2 \zeta^2 + \sum_{i=1}^n (a_i^2 \sigma_i^2 + b_i^2 \rho_i^2), \quad (28)$$

for functions $a_i, b_i$ and $c$ of $t$ and left invariant 1-forms $\sigma_i, \rho_i$ and $\zeta$.

In terms of the frame $\partial_t, Z, \{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n$ non-metrically dual, respectively, to $dt, \zeta, \{\sigma_i\}_{i=1}^n, \{\rho_i\}_{i=1}^n$ (which is assumed to be a coframe), we restrict the possible groups $G$ by requiring repeated Bianchi type A structure constants:

$$[\partial_t, X_i] = [\partial_t, Y_i] = [\partial_t, Z] = 0, \quad i = 1, \ldots n,$$

$$[X_i, Y_i] = -\Gamma_i^k Z, \quad i = 1, \ldots n,$$

$$[Y_i, Z] = -\Gamma_i^k X_i, \quad i = 1, \ldots n,$$

$$[Z, X_i] = -\Gamma_i^k Y_i, \quad i = 1, \ldots n,$$

$$[X_i, X_j] = [X_i, Y_j] = [Y_i, X_j] = 0, \quad [X_i, Y_j] = 0 \text{ for } i \neq j, \quad i, j = 1, \ldots n,$$

with all other structure constants vanishing or determined by the Lie algebra requirements. Note that the summation convention is not used here. For $n > 1$ these are very stringent conditions: the Jacobi identity for $X_i, Y_i, X_j$ and for $X_i, Y_i, Y_j$ with $j \neq i$ implies that either $\Gamma_i^z = 0$ or both $\Gamma_j^z$ and $\Gamma_j^z$ vanish for all $j \neq i$.

The almost complex structure will be determined in this frame by

$$J\partial_t = \left( \prod_{i=1}^n a_i b_i \right) Z \quad \text{and} \quad JX_i = \frac{a_i}{b_i} Y_i, \quad i = 1 \ldots n. \quad (30)$$

In the next subsection we show how this model fits within the framework of Section 2.

4.1. A frame as in Section 2.2. In this subsection we show how the metric $g$ of the previous subsection gives rise to data satisfying (4)-(8) and (18).
Set $\alpha = (\prod_{i=1}^{n} (a_i b_i)) c$, and consider the orthonormal frame and dual coframe

$$
k = (Z/c + \partial_t/\alpha) \sqrt{2}, \quad \hat{k} = (c \zeta + \alpha dt) \sqrt{2},$$
$$
t = (Z/c - \partial_t/\alpha) \sqrt{2}, \quad \hat{t} = (c \zeta - \alpha dt) \sqrt{2},$$
$$
x_i = X_i/a_i, \quad \hat{x}_i = a_i \sigma_i, \quad i = 1 \ldots n$$
$$
y_i = Y_i/b_i, \quad \hat{y}_i = b_i \rho_i, \quad i = 1 \ldots n.$$

It can be checked that this frame satisfies (4)-(6) for the functions

$$
A_i = -E_i = -\frac{a_i'}{\sqrt{2} \alpha a_i} = -\frac{1}{\alpha} \frac{da_i}{a_i d\tau}, \quad B_i = F_i = -\frac{1}{\sqrt{2}} \Gamma^i_{zi} c \frac{b_i}{a_i},
$$
$$
D_i = -H_i = -\frac{b_i'}{\sqrt{2} \alpha b_i} = -\frac{1}{\alpha} \frac{db_i}{b_i d\tau}, \quad C_i = G_i = \frac{1}{\sqrt{2}} \Gamma^i_{zi} c \frac{a_i}{a_i},
$$
$$
L = -\frac{c'}{\sqrt{2} \alpha c} = -\frac{1}{\alpha} \frac{dc}{c d\tau}, \quad N_i = -\frac{1}{\sqrt{2}} \Gamma^i_{zi} c \frac{a_i}{a_i b_i}.
$$

Here the prime denotes differentiation with respect to $t$, and

$$
\hat{k} - \hat{t} = dt = \sqrt{2} \alpha dt,
$$

so that

$$
\frac{d}{dt} = \frac{1}{\sqrt{2} \alpha} \frac{d}{d\tau}.
$$

We note that for each $i = 1 \ldots n$, requiring the functions $A_i$, $B_i$, $D_i$, $C_i$, and $L$, to fulfill the four relations in (7)-(8) implies that in this model the Kähler condition imposes only two additional relations here, say $A_i + D_i = N_i$ and $B_i + C_i = H_i - E_i$, giving for each $i = 1 \ldots n$

$$
\frac{a_i'}{a_i} + \frac{b_i'}{b_i} = \Gamma^z_{zi} \frac{ac}{a_i b_i},
$$
$$
\frac{b_i'}{b_i} - \frac{a_i'}{a_i} = \left( \Gamma^i_{zi} \frac{a_i}{b_i} - \Gamma^i_{zi} \frac{b_i}{a_i} \right) \frac{c}{\alpha}.
$$

We now impose the skew-soliton and soliton conditions of Proposition 3.1. We assume the soliton constant is nonzero and the soliton potential depends on $\tau$. This implies all functions $N_i$ equal the same function $N$, i.e.

$$
\Gamma^z_{zi}(a_i b_i)^{-1} \text{ is independent of } i.
$$

Then equation (21) can be written with the help of (32) in the form

$$
\frac{c'}{c} = \alpha \left[ \sum_{j=1}^{n} \frac{1}{2a_j b_j} \left( \frac{1}{c} \left( \Gamma^j_{jz} a_j^2 + \Gamma^j_{jj} b_j^2 \right) - \Gamma^z_{zj} c \right) \right] + \frac{\lambda}{2} \frac{a_i b_i}{c}.
$$

We now consider the additional conditions for a Ricci soliton, that must hold away from critical points of $f(\tau)$. Two of them, (24) and (25) always hold in our model. Condition (26) simplifies to

$$
\Gamma^i_{zi} a_i^2 - \Gamma^i_{zj} b_j^2 = 0, \quad i = 1, \ldots n,
$$

and
so only one of the terms in this difference is retained in (35), but we also see from (33) that

\[ b_i = \ell_i a_i \text{ for a constant } \ell_i > 0 \text{ satisfying } \Gamma^i_{ix} - \ell^2_i \Gamma^i_{zi} = 0, \ i = 1, \ldots, n. \]  

(37)

Finally, condition (23) simplifies to \( f'' - \frac{\alpha c'}{\alpha c} f' = 0 \), which is equivalent to

\[ f' = k \alpha c \text{ for a constant } k. \]  

(38)

Substituting the consequences of these two conditions in (35), we arrive at the following equation for a gradient Kähler-Ricci soliton in this model:

\[ \frac{c'}{c} = \alpha \left[ \sum_{j=1}^{n} \left( \frac{1}{2\ell_j a_j^2} \left( 2\Gamma^j_{xz} a_j^2 - \Gamma^z_{jz} c \right) \right) - \frac{\lambda \ell_i a_i^2}{\Gamma^z_{ii}} c + \frac{1}{2} kc \right]. \]  

(39)

To this we can add the modification of equation (32):

\[ \frac{a_i'}{a_i} = \frac{\Gamma^z_{ii}}{2 \ell_i a_i^2} \alpha c, \quad i = 1, \ldots, n. \]  

(40)

We summarize

**Proposition 4.1.** Let \((M^{2n+2}, g)\) be a cohomogeneity one manifold under a proper action of a Lie group \(G\) whose Lie algebra relations in a given frame are as in the last four lines of (29), while \(g\) has the form (28). If \(\tau = \tau(t)\) is a solution of \(\tau'(t) = \sqrt{2} (\prod_{i=1}^{n} a_i b_i)c\), then \((g, f(\tau))\) is a Kähler-Ricci skew-soliton with nonzero soliton constant if (32)-(35) hold. If additionally (36) holds, and (38) also holds for a constant \(k\) and \(f = f(\tau(t))\), then (37), (39) and (40) also hold, and if, furthermore, every critical point of \(f(\tau)\) is degenerate, then \(g\) is a Kähler-Ricci soliton metric with soliton potential \(f\).

5. **Cohomogeneity one Ricci solitons under the Heisenberg group**

5.1. **The equations.** We now consider the special case of gradient Kähler-Ricci solitons for the action of the Heisenberg group. For this group, one can take

\[ \Gamma^i_{iz} = \Gamma^i_{zi} = 0, \quad \Gamma^z_{ii} = 1, \quad i = 1, \ldots, n, \]

so that (36) holds automatically. The constants \(\ell_i\) of (37) can now be chosen freely, and we take them all to equal 1. Also, from (34) we see that for all \(i = 1, \ldots, n\), \(a_i^2 = a_1^2\). Thus \(\alpha = a_1^{2n+2}c\) and equations (39)-(40) take the form

\[ 2 \frac{a_i'}{a_i} = a_1^{2n-2} c^2, \]  

(41)

\[ 2 \frac{c'}{c} = -n a_1^{2n-2} c^2 - 2\lambda a_1^{2n+2} + k a_1^{2n} c^2. \]  

(42)

The metric has the form

\[ g = a_1^2 \sum_{i=1}^{n} (\sigma_i^2 + \rho_i^2) + c^2 \xi^2 + a_1^{4n} c^2 dt^2. \]
As in [AM2], we now make the change of variables $a_2^2 dt = dq$. Then, setting $\phi(q) := a_1^2$, we see from (41) that, with the prime denoting from now on the derivative with respect to $q$,

$$\phi'(q) = 2a_1 \frac{da_1}{dq} = 2a_1 \frac{da_1}{dt} \frac{dt}{dq} = 2a_1 \frac{1}{a_1^2} = a_1^{2n-2} c^2,$$

and hence

$$\phi'(q) dq^2 = a_1^{2n+2} c^2 dt^2.$$

It follows that the metric takes the form

$$g = \sum_{i=1}^{n} \phi(q) (\sigma_i^2 + \rho_i^2) + \frac{\phi'(q)}{\phi(q)^{n-1}} \zeta^2 + \phi(q)^{n-1} \phi'(q) dq^2. \tag{43}$$

with Kähler form

$$\omega = d(\phi(q) \zeta).$$

Note that equation (41) becomes an identity with these choices, while equation (42) yields

$$\left( \frac{\phi'(q)}{\phi(q)^{n-1}} \right)' \left/ \frac{\phi'(q)}{\phi(q)^{n-1}} \right. = 2 \frac{dc}{dt} \frac{dt}{dq} \frac{1}{c^2} = \frac{2}{\phi(q)} \frac{\phi'(q)}{\phi(q)^{n-1}} \frac{1}{c^2}$$

$$\quad = -n \frac{\phi'(q)}{\phi(q)} - 2\lambda \phi(q)^n + k\phi'(q),$$

or equivalently,

$$\frac{(\phi^2)''}{(\phi^2)'} = -2\lambda \phi^n + k\phi'.$$ \tag{44}

Also, from (38)

$$\frac{df}{dt} = f' \frac{dq}{dt} = f' \phi = k \phi'.$$ \tag{45}

In summary

**Proposition 5.1.** Let $(M^{2n+2}, g)$ be a cohomogeneity one manifold under a proper action of the Heisenberg group $\mathcal{H}_{2n+1}$ with $g$ of the form (43). If $\phi$ satisfies (44) then $g$ is a gradient Kähler-Ricci soliton metric with soliton potential $f$ given as an affine function of $\phi$, so long as $f$ is either constant, or $\phi$, when considered as a function of $\tau$ such that $q'(\tau) = (2\phi(q)^{n-1} \phi'(q))^{-1/2}$, has only degenerate critical points.

The last clause simply involves computing the change of variable from $\tau$ to $q$.

5.2. A completeness theorem. We now look for complete expanding gradient Kähler-Ricci soliton metrics in the case where $G = \mathcal{H}_{2n+1}$ is the $2n + 1$-dimensional Heisenberg group. In this simple case of the construction given in the beginning of Section 4, the cohomogeneity one manifold $M$ will only have regular orbits, and at any point $p$ the isotropy is trivial, giving $M = \mathcal{H}_{2n+1} \times I$ for some open interval $I$, with the action given by left multiplication on the first factor.
To set the stage, consider the metric $g$ given by (43) for $\phi$ satisfying (44) with $\lambda = -1$ and $k = -1$, i.e.

$$\frac{(\phi^2)''}{(\phi^2)'} = 2\phi^n - \phi'.$$

This has a first integral

$$\phi' = 2(-1)^{n+1}(n + 1)! \frac{1}{\phi} \left[ \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} \phi^k - e^{-\phi} \right] =: F_n(\phi), \quad (46)$$

where the coefficient of the exponential is a particular choice of an integration constant. The right hand side of this equation will be denoted $F_n(\phi)$. Now, positive definiteness of $g$ holds if both $\phi > 0$ and $\phi' > 0$. But $H_n(\phi) = \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} \phi^k - e^{-\phi}$ is zero at $\phi = 0$, and has a positive/negative derivative for $\phi > 0$ if $n$ is odd/even, as can be shown by induction starting at $n = -1$. Thus $\phi' > 0$ if $\phi > 0$ and therefore $g$ is positive definite whenever $\phi > 0$.

Fix $\phi_0 > 0$. Since $F_n(\phi) = 2(-1)^{n+1}(n + 1)!H_n(\phi)/\phi$ is positive and continuous on $(0, \infty)$, there exists a unique solution $\phi(q)$ to (46) with initial condition $\phi(q_0) = \phi_0 \in (0, \infty)$ defined on the interval

$$(q_a, q_b) = \left( \int_{\phi_0}^{q_a} 1/F_n(\phi) d\phi, \int_{\phi_0}^{q_b} 1/F_n(\phi) d\phi \right). \quad (47)$$

Note that on $(q_a, q_b)$, both $\phi$ and $\phi'$ are positive, so the metric $g$ is well-defined for $q$ in this interval. Note also that employing L’Hôpital’s rule to $F_n(\phi)$ shows that $\phi'(q)$, like $\phi(q)$, approaches zero as $q \to q_a$. Additionally, we have

**Lemma 5.2.** $\phi'(q)$, like $\phi(q)$, is increasing in $(q_a, q_b)$.

**Proof.** We consider the easily verifiable formula

$$F_n'(\phi) = (n + 1)F_{n-1}(\phi) - F_n(\phi)/\phi \quad (48)$$

(where the prime is the $\phi$-derivative). As $\phi''(q) = F_n'(\phi(q))F_n(\phi(q))$, to prove the latter claim it is enough to show $F_n'(\phi) > 0$ for $\phi > 0$. Unpacking this using (48) and the definition of $F_n$ reveals that in case $n$ is odd, it suffices to show that

$$P_{n+1}(\phi) + \phi P_n(\phi) < (\phi + 1)e^{-\phi}, \quad (49)$$

where $P_n$ is the $n$-th Taylor polynomial of $e^{-\phi}$. But the left hand side is just the $(n + 1)$-th Taylor polynomial of the right hand side. The two sides have $n + 1$ equal derivatives at $\phi = 0$, whereas for $\phi > 0$ we have

$$[P_{n+1}(\phi) + \phi P_n(\phi)]^{(n+1)} = (-1)^{n+1} + (-1)^n(n + 1) = -n \quad < (-1)^{n+1} (\phi - n)e^{-\phi} = [(\phi + 1)e^{-\phi}]^{(n+1)}.$$

Working backwards inductively, each $k$-th derivative, $k = n, n - 1, \ldots, 1, 0$ of the left hand side of (49) is smaller than the $k$-th derivative of the right hand side for $\phi > 0$. Thus (49) holds as required. The proof for $n$ even is similar. □

In the proof of the next theorem we will make use of the fact that, as $\phi$ is monotone on $(q_a, q_b)$, its inverse $q(\phi)$ is well defined on $(0, \infty)$. 
Theorem 2. Let \( \phi(q) \) be a solution to \((46)\) with initial condition \( \phi(q_0) = \phi_0 \), with \( \phi_0 > 0 \), defined on the interval \((47)\). Then there exists a complete expanding gradient Kähler-Ricci soliton \((g, f)\) on \( M = \mathcal{H}_{2n+1} \times (q_a, q_b) \), such that \( g \) is given by \((43)\), while up to an additive constant, \( f = -\phi \).

The proof will be given in the next subsection. A fully explicit form for the metric appears later, see \((56)\).

5.3. Length of an escaping curve. Aside from the properties of \( f \) and its gradient, all that remains is to prove the completeness claim for \( g \). For this we have

Proposition 5.3. A finite length curve on \((M, g)\) of Theorem 2 cannot leave every compact set.

Proof. It follows from \((47)\) that \( \lim_{q \to q_b} \phi(q) = \infty \). Moreover, for the inverse \( q(\phi) \) of \( \phi \) on \((0, \infty)\), we have

\[
\int_{q_0}^{q_b} \sqrt{\phi^{n-1}(q)\phi'(q)} \, dq = \int_{q_0}^{\infty} \sqrt{\phi^{n-1}(\phi)} \, d\phi = \int_{q_0}^{\infty} \sqrt{\phi^{n-1}/F_n(\phi)} \, d\phi = \infty
\]

as the Taylor remainder estimate for \( H_n(\phi) \) easily shows \( \sqrt{\phi^{n-1}/F(\phi)} \) is larger than a constant multiple of \( 1/\phi \). For the same reason

\[
\int_{q_a}^{q_0} \sqrt{\phi^{n-1}(q)\phi'(q)} \, dq = \int_{q_0}^{\phi_0} \sqrt{\phi^{n-1}(\phi)} \, d\phi = \int_{q_0}^{\phi_0} \sqrt{\phi^{n-1}/F_n(\phi)} \, d\phi = \infty.
\]

We now choose a left-invariant frame for \( \mathcal{H}_{2n+1} \) which is given in coordinates \( x_i, y_i, z \) by \( X_i = \partial x_i, Y_i = \partial y_i + x_i \partial z, Z = -\partial z \), to which we will add on \( M \) the vector field \( \partial_q \). The domain of these coordinate functions is open and dense in \( M \). The corresponding left invariant coframe is \( \sigma_i = dx_i, \rho_i = dy_i, \zeta = \sum_i x_idy_i - dz, dq \). Given a smooth curve \( \gamma(s) : I \to M \), with \( I = [p, r] \) a closed interval, having coordinate presentation \((x(s), y(s), z(s), q(s))\), we have

\[
\gamma' = \sum_{i=1}^{n} (x'_i \partial_{x_i} + y'_i \partial_{y_i}) + z' \partial_z + q' \partial_q
\]

so that for each \( s \in I \) and each \( i = 1, \ldots, n \)

\[
\begin{align*}
g\left(\gamma'(s), \frac{X_i}{|X_i|}_{\gamma(s)}\right) &= x'_i(s) \sqrt{\phi(q(s))}, \\
g\left(\gamma'(s), \frac{Y_i}{|Y_i|}_{\gamma(s)}\right) &= y'_i(s) \sqrt{\phi(q(s))}, \\
g\left(\gamma'(s), \frac{\partial_q}{|\partial_q|}_{\gamma(s)}\right) &= q'(s) \sqrt{\phi(q(s))^{n-1}\phi'(q(s))}.
\end{align*}
\]
It follows via the Cauchy-Schwartz inequality that for the length $L(\gamma)$ of $\gamma$ one has lower bounds

\[
L(\gamma) = \int_p^r |\gamma'(s)| \, ds \geq \inf_I \left( \sqrt{\phi(q(s))} \right) \left| \int_p^r x_i'(s) \, ds \right|,
\]

\[
L(\gamma) \geq \inf_I \left( \sqrt{\phi(q(s))} \right) \left| \int_p^r y_i'(s) \, ds \right|,
\]

\[
L(\gamma) \geq \left| \int_p^r \sqrt{\phi(q(s))^{n-1} \phi'(q(s))q'(s)} \, ds \right|,
\]

for $i = 1, \ldots, n$. Now if $\gamma$ has finite length, (54) via the change of variable $q = q(s)$, along with (50) and (51), imply that $q(s)$ is bounded away from $q_a$ and $q_b$. Thus $\phi(q)$ is bounded away from zero along the curve. Therefore, (52)-(53) imply that $x_i(s)$ and $y_i(s)$ are bounded, for any $i = 1, \ldots, n$.

For the coordinate $z$, we have

\[
g \left( \gamma'(s), \frac{Z}{|Z|} \right) = \sqrt{F_n(\phi(q(s))) / \phi^{n-1}(q(s))} \zeta(\gamma'(s))
\]

\[
= \sqrt{F_n(\phi(q(s))) / \phi(q(s))^{n-1}} \left( \sum_{i=1}^n x_i(s)y_i'(s) - z'(s) \right).
\]

So

\[
L(\gamma) \geq \inf_I \sqrt{F_n(\phi(q(s))) / \phi(q(s))^{n-1}} \int_p^r \left| \sum_{i=1}^n (x_i(s)y_i'(s)) - z'(s) \right| \, ds.
\]

First, the infimum of the square root term in this expression is bounded away from zero as before: we already know that $q(s)$ is bounded away from $q_a$ and hence $\phi(q(s))$ is bounded away from zero, whereas $\sqrt{F_n(\phi)} / \phi^{n-1}$ is positive for $\phi > 0$, and is increasing as a function of $\phi$ due to an argument similar to the proof of Lemma 5.2.

Next, if $|x_i(s)| \leq M_i$ then

\[
\left| \sum_{i=1}^n (x_i(s)y_i'(s)) - z'(s) \right| \geq \left| \sum_{i=1}^n x_i(s)y_i'(s) \right| \geq \left| z'(s) \right| - \sum_{i=1}^n M_i |y_i'(s)|
\]

Now noting that in fact $\int_p^r |y_i'(s)| \, ds \leq L(\gamma) / \inf_I \left( \sqrt{\phi(q(s))} \right)$ for all $i = 1, \ldots, n$, we see that

\[
\left| \int_p^r z'(s) \, ds \right| \leq \int_p^r |z'(s)| \, ds
\]

\[
\leq L(\gamma) / \inf_I \sqrt{F_n(\phi(q(s))) / \phi(q(s))^{n-1}} + \left( L(\gamma) / \inf_I \sqrt{\phi(q(s))} \right) \sum_{i=1}^n M_i.
\]

Thus $z(s)$ is bounded as well.

It follows that any curve with domain a closed interval of length $\leq L$ starting at a given point has domain contained in a (compact) four dimensional coordinate cube. Hence the same holds if the curve is instead defined on a semi-closed bounded or unbounded interval.
By Proposition 5.3, a curve that leaves every compact set has length bounded below by any finite number, i.e. its length is infinite. Hence the metric $g$ in Theorem 2 is complete.

As for the remaining characterization of $f$, we know from (45) that $f'(q) = -\phi'(q)$, so that up to an additive constant $f$ coincides with $\phi$. Therefore $f'(\tau) = -\phi'(\tau) = -\phi'(q)dq = -\sqrt{\phi'(q)/(2\phi^{n-1}(q))} = -\sqrt{F_n(\phi)/(2\phi^{n-1})} < 0$, so $f(\tau)$ has no critical points.

**Remark 5.4.** Note that a simpler version of the proof of Theorem 2 will also apply to the case in which the group acting is the quotient group considered in \[\text{[AM2]}, \text{namely } H_{2n+1}/\mathbb{Z}, \text{where } \mathbb{Z} \text{ is the discrete subgroup of the center of } H_{2n+1}, \text{given by}
\]

$$
\mathbb{Z} := \left\{ \begin{bmatrix} 1 & 0 & 2\pi k \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid k \in \mathbb{Z} \right\}.
$$

For this quotient group the center $K$ is isomorphic to $SO(2)$, whose transitive action on the unit circle $S^1$ extends to a linear action on $V \setminus \{0\} := \mathbb{R}^2 \setminus \{0\}$. Employing the notation of Section 4, the manifold in that case has the form $(H_{2n+1}/\mathbb{Z}) \times_K (D^2 \setminus \{0\})$, where $D^2 \subset V$ is the 2-disk. However, $H_{2n+1} \times I$, $I$ an open interval, is the universal covering for this manifold, so its completeness would then be implied as well from such a proof. This alternative route to a proof would save the need to explicitly bound the $z$ coordinate as we have done.

It is known that for a complete gradient soliton metric, the soliton vector field is also complete [Zh]. We give an independent proof of this for completeness, and because we will later allude to one of the ingredients of the proof.

**Proposition 5.5.** For $f$ as in Theorem 2, $\nabla f$ is complete.

**Proof.** We have

$$
\nabla f = f'(\tau)(k - t) = \frac{1}{\sqrt{2\alpha}}f'(t)(k - t) = -\frac{1}{\sqrt{2\alpha}}c \alpha \frac{\sqrt{2}}{\alpha} \frac{d}{dt} \phi q = - \frac{c}{\alpha} \frac{dq}{dt} \phi q = - \frac{1}{a_{2n-2}^2} \phi q = \frac{1}{\phi(q)n^{-1}} \phi q.
$$

For a curve $\gamma$ as before, the only limitation on the range of the parameter $s$ may come from the equation $q'(s) = 1/\phi(q(s))^{n-1}$. But

$$
\tilde{s} - s_0 = \int_{s_0}^{\tilde{s}} \phi(q(s))^{n-1} q'(s) ds = \int_{q_0}^{\tilde{q}} \phi(q)^{n-1} dq = \int_{\phi_0}^{\tilde{\phi}} \phi_{n-1}(\phi) d\phi,
$$

with $\tilde{\phi} = \phi(\tilde{q}) = \phi(q(\tilde{s}))$. For $\tilde{\phi} = \infty$ or $\tilde{\phi} = 0$, the integral on the right diverges, respectively, to $\pm \infty$, as can be seen by showing that

$$
\phi^{n-1}/F_n(\phi) > 1/(2\phi).
$$

This follows due to the formula

$$
F_n(\phi) = -(n + 1)F_{n-1}(\phi) + 2\phi^n
$$
which is immediately verifiable from the definition of $F_n(\phi)$. From this we see that
\[
\frac{\phi^{n-1}}{F_n(\phi)} = \frac{\phi^{n-1}}{-(n+1)F_{n-1}(\phi) + 2\phi^n} > \frac{\phi^{n-1}}{2\phi^n} = \frac{1}{2\phi},
\]
because both $F_{n-1}$ and $F_n$ are positive.

This completes the proof of Theorem 2.

5.4. Some properties of the $H_{2n+1}$ Kähler-Ricci solitons. Note that $\phi(q)$ is obviously smooth, and as it is increasing in $(q_a, q_b)$, one can use $\phi$ as a variable, giving an explicit form for these solitons:

\[
g = \sum_{i=1}^{n} \phi(\sigma_i^2 + \rho_i^2) + \frac{F_n(\phi)}{\phi^n-1} \zeta^2 + \frac{\phi^{n-1}}{F_n(\phi)} \, d\phi^2,
\]
\[
F_n(\phi) = 2(-1)^n+1(n + 1)\frac{1}{\phi} \left[ \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} \phi^k - e^{-\phi} \right],
\]
\[
f = -\phi + \text{constant}.
\]

Combining (13), (19), (20) and (23), we see that the only nonzero components of the Ricci form satisfy
\[
\rho(x_i, y_i) = \lambda + NF'(\tau), \quad \rho(k, t) = \lambda - 2f''(\tau).
\]

One computes
\[
\rho = \frac{1}{2} \left( \frac{F_n(\phi)}{\phi^n} - 2 \right) \sum_{i=1}^{n} \hat{x}_i \wedge \hat{y}_i + \frac{1}{2} \left( \frac{F_n'(\phi)}{\phi^n-1} - (n - 1) \frac{F_n(\phi)}{\phi^n} - 2 \right) \hat{k} \wedge \hat{t}
\]
\[
= \frac{1}{2} \left( \frac{F_n(\phi)}{\phi^n} - 2 \right) \phi \sum_{i=1}^{n} \sigma_i \wedge \rho_i + \frac{1}{2} \left( \frac{F_n'(\phi)}{\phi^n-1} - (n - 1) \frac{F_n(\phi)}{\phi^n} - 2 \right) d\phi \wedge \zeta,
\]
where the prime is the $\phi$-derivative, so that the scalar curvature is
\[
\text{Scal} = \frac{F_n(\phi)}{\phi^n} + \frac{F_n'(\phi)}{\phi^n-1} - 2n - 2
\]
\[
= \frac{F_n(\phi)}{\phi^n-1} + 2(\phi - n - 1),
\]
where the last expression follows from the easily verifiable formula $F_n'(\phi) = -\frac{2n+1}{\phi} F_n(\phi) + 2\phi^n$. Thus the Ricci curvature is not definite and the scalar curvature changes sign as well. However the Hessian $\nabla df$ is negative definite, so that $\text{Ric} > -g$ and thus $\text{Scal} > -2m$, with this lower bound also being the $\phi \to 0^+$ limit of $\text{Scal}$.

Our bound (54) holds for any curve, hence in particular for a minimizing geodesic, so that it gives a lower bound on the distance function:
\[
d_g(\alpha, p) \geq A_n \log \phi|_{\phi_0}^{\phi_p},
\]
where $A_n = \sqrt{\frac{n+2}{2}}$. In the region $\phi > n + 2$ this can be improved via (55). Thus $f$ is not asymptotic to a multiple of the square of the distance function,
so that the Ricci curvature does not decay quadratically, as one can also deduce from the classification results in [CDS].

Similarly, Scal is increasing as a function of \( \phi \), as we have mentioned that \( F_n(\phi)/\phi^{n-1} \) is increasing. It then follows immediately that the asymptotic scalar curvature ratio \( \limsup_{d_Y(a,p) \to \infty} \text{Scal}(p)d_Y(a,p)^2 \) is infinite.

Writing the metric in the form \( dr^2 + g_r \), with \( r \) a primitive of \( \sqrt{\phi^{n-1}/F_n(\phi)} \), we see that \( f'(r) = -\frac{1}{\phi'/\phi} = -\sqrt{F_n(\phi)/\phi^{n-1}} < 0 \), and \( f''(r) < 0 \) also holds, due to an argument mentioned earlier analogous to the proof of Lemma 5.2. This is similar to soliton potential estimates obtained in [BDW, BDGW, W2] under the assumption that the manifold has a singular orbit.

5.5. Singularity type. As the solitons given by (56) are expanding, the associated solution to the Ricci flow, evolving by rescalings of pull-backs of the metric under diffeomorphisms, is either of type IIb or of type III, in the sense of [H2]. Which one of these it is, depends on the behavior of the curvature norm along the flow solution. We outline a proof that it is of the latter type.

The sectional curvatures of \( g \) on the frame fields are given by

\[
\begin{align*}
\text{Sec}(x_i, y_i) &= -2N^2 = -F_n(\phi)/\phi^{n+1}, \\
\text{Sec}(k, x_i) &= (N^2 - 2LN)/2 = ((n + 1 + \phi)F_n(\phi)/\phi^{n+1} - 2)/4, \\
\text{Sec}(k, t) &= 2(L' - L^2) = -[n(n + 1)/2 + n\phi + \phi^2/2]F_n(\phi)/\phi^{n+1} + n - 1 + \phi,
\end{align*}
\]

where the prime denotes \( \partial_r \), and the second of these expressions also equals \( \text{Sec}(k, y_i) = \text{Sec}(t, x_i) = \text{Sec}(t, y_i) \), and additionally is a constant multiple of the curvature tensor when evaluated on the quadruple \( k, t, x_i, y_i \). All other curvature values, unless determined by (57) and the curvature symmetries, in fact vanish.

These formulas are obtained as follows. The first expressions in (57), in terms of \( N \) and \( L \) are obtained as follows. Applying (4)-(8), we see that the connection form is

\[
\theta = \begin{pmatrix}
\theta_{00} & \theta_{01} & \theta_{02} & \cdots & \theta_{0n} \\
\theta_{10} & \theta_{11} & 0 & \cdots & 0 \\
\theta_{20} & 0 & \theta_{22} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
\theta_{n0} & 0 & \cdots & 0 & \theta_{nn}
\end{pmatrix}
\]

where the \( \theta_{ij} \) are \( 2 \times 2 \) blocks satisfying \( \theta_{ij} = -\theta_{ji}^T \). Since \( B_i = C_i = F_i = G_i = 0 \) in the case of the Heisenberg group, and \( a_i = b_i \) together with (8) imply \( A_i = D_i = -E_i = -H_i = N/2 \), we get

\[
\begin{align*}
\theta_{00} &= \begin{pmatrix} 0 & L(\hat{k} + \hat{t}) \\ -L(\hat{k} + \hat{t}) & 0 \end{pmatrix}, \\
\theta_{ii} &= \begin{pmatrix} 0 & \frac{N}{2}(\hat{k} + \hat{t}) \\ -\frac{N}{2}(\hat{k} + \hat{t}) & 0 \end{pmatrix},
\end{align*}
\]
\[ \theta_{0i} = \left( \frac{N}{2} (\hat{x}_i - \hat{y}_i) \quad \frac{N}{2} (\hat{x}_i + \hat{y}_i) \qquad -\frac{N}{2} (\hat{x}_i + \hat{y}_i) \quad \frac{N}{2} (\hat{x}_i - \hat{y}_i) \right). \]

The curvature matrix is then given by

\[ \Omega = \theta \wedge \theta + d\theta. \]

Using (22) for the \( \tau \) derivative of \( N \) this gives

\[
\Omega = \begin{pmatrix}
\Omega_{00} & \Omega_{01} & \Omega_{02} & \cdots & \Omega_{0n} \\
\Omega_{10} & \Omega_{11} & \theta_{10} \wedge \theta_{02} & \cdots & \theta_{10} \wedge \theta_{0n} \\
\Omega_{20} & \theta_{20} \wedge \theta_{01} & \Omega_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\Omega_{n0} & \theta_{n0} \wedge \theta_{01} & \cdots & \theta_{n0} \wedge \theta_{0(n-1)} & \Omega_{nn}
\end{pmatrix}
\]

where

\[
\Omega_{00} = \sum_{i=0}^{n} \theta_{0i} \wedge \theta_{i0} + d\theta_{00},
\]

\[
= (N^2 - 2LN) \begin{pmatrix} 0 & 1 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \end{pmatrix} \sum_{i=1}^{n} (\hat{x}_i \wedge \hat{y}_i + (2L' - 2L^2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{k} \wedge \hat{t},
\]

\[
\Omega_{ii} = \theta_{i0} \wedge \theta_{0i} + \delta_{ij} \wedge \theta_{ii} + \cdots + d\theta_{ii},
\]

\[
= -N^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sum_{j=1}^{n} (\delta_{ij} \hat{x}_j \wedge \hat{y}_j + (N^2 - 2LN) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{k} \wedge \hat{t},
\]

\[
\Omega_{0i} = \theta_{0i} \wedge \theta_{0i} + \theta_{0i} \wedge \theta_{ii} + \cdots + d\theta_{0i},
\]

\[
= \frac{N^2 - 2LN}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\hat{k} \wedge \hat{x}_i + \hat{t} \wedge \hat{y}_i) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\hat{k} \wedge \hat{y}_i - \hat{t} \wedge \hat{x}_i) \right).
\]

Since the terms \( \theta_{0i} \wedge \theta_{0j} \) for \( i \neq j \) are proportional to \( N^2 \), this shows that the only terms appearing in the full curvature form are multiples of the sectional curvatures given in (57), and from this the first equalities of (57) follows easily.

Next, the final formulas in (57) in terms of \( F_n \) and \( \phi \) are obtained in a long calculation starting from the expressions for \( L \) and \( N \) in (31) using \( N_i = N \), \( a_1 = a_i = b_i \), \( a_i^2 = \phi \), \( c^2 = F_n(\phi)/\phi^{n-1} \), \( \partial_t = 2^{-1/2} a_1^{2n-1} c^{-1} \partial_t \) and \( \theta_{t} = \phi F_n(\phi) \partial_{\phi} \).

We now claim that all these expressions are bounded for \( \phi \in (0, \infty) \). First, \( F_n(\phi)/\phi^{n+1} \) is positive, and bounded above by \( 2/(n+2) \) via the Taylor remainder estimate for \( e^{-\phi} \). Next, \( F_n(\phi)/\phi^n \) is again positive, and is bounded above by \( 2, \) since we have seen that \( \phi^{n-1}/F_n(\phi) > 1/(2\phi) \). This shows that the first two sectional curvatures above are bounded. For the third, it remains to show that the expression \( -(\phi^2/2)F_n(\phi)/\phi^{n+1} + \phi = \phi - F_n(\phi)/(2\phi^{n-1}) \) is bounded. It is bounded below by \( \phi - \frac{1}{2}(2\phi) = 0 \). Then the Taylor remainder estimate shows its absolute value is no larger than the polynomial \( \phi + \frac{\phi^2}{2(n+2)} \), making it bounded on any finite interval of the form \((0, s)\). Finally its exact expression shows it equals \( \phi + (n + 1) \phi = n + 1 \) plus terms that are bounded above on any interval of the form \((r, \infty)\), \( r > 0 \). This concludes the proof of this claim.
It follows that all sectional curvatures of \( g \) are bounded. Hence the corresponding Ricci flow solution has sectional curvatures which are \( O(t^{-1}) \), where \( t \) is the flow parameter, and thus it is of type III (cf. [Lo]).

Note that one can improve these bounds to show that the frame sectional curvatures are negative. The holomorphic bisectional curvature is similarly bounded, and negative for each pair of the form \( k + it, x_i + iy_i, i = 1, \ldots n \).

6. Steady solitons

We now develop the case of steady solitons (\( \lambda = 0 \)). Our setting is that of \((M, g, J)\) of Proposition 2.1, with \( A_i, \ldots H_i, N_i, L \) and \( f \) all functions of \( \tau \). For skew-solitons, equations (19)-(20) hold, in which we set \( \lambda = 0 \). Setting

\[
Q := 2L + \sum_{j=1}^n (C_j - H_j + A_j - F_j),
\]
equation (19) yields two main cases to consider, which we will denote from now on (I) and (II):

(I) \( Q = f'(\tau) \),
(II) \( N_i = 0, \quad i = 1, \ldots n \). \hspace{1cm} (58)

These cases need not be mutually exclusive. Also, we will not consider hypothetical mixed cases where (I) and all or parts of (II) hold on different subsets.

For these two cases, equation (20) becomes

(I) \( f''(\tau) - Lf'(\tau) = 0 \),
(II) \( -L(Q + f'(\tau)) + (Q + f'(\tau))' = 0 \). \hspace{1cm} (59)

To these we add the conditions for existence of a Kähler-Ricci soliton, which we reproduce here:

\[
f''(\tau) + Lf'(\tau) = 0, \hspace{1cm} (60)
A_i + E_i = 0, \hspace{1cm} (61)
D_i + H_i = 0, \hspace{1cm} (62)
B_i + C_i + F_i + G_i = 0 \hspace{1cm} (63)
f(\tau) \text{ has only degenerate critical points}, \hspace{1cm} (64)
\]

where the first four of these conditions only need to hold away from the critical points of \( f(\tau) \).

Note that case (I) in (59), together with (60) can be given alternatively as

\[
f''(\tau) = 0, \hspace{1cm} Lf'(\tau) = 0 \hspace{1cm} (65)
\]
and the first of these implies that away from the critical points of \( f(\tau) \), and hence everywhere, \( f \) is affine in \( \tau \). Thus if we are not in the trivial soliton case where \( f \) is constant, \( L \) must vanish.

Finally, one should remember that to this list of conditions and equations one must add the Kähler conditions (4)-(8).
6.1. **Cohomogeneity one steady solitons.** For cohomogeneity one solitons, consider first the Kähler conditions (32)-(33). Away from the zeros of $f'(\tau)$, (33) together with (31) and (63) implies as before (37), i.e.
\[ b_i = \ell_i a_i \] for a constant $\ell_i > 0$ satisfying $\Gamma_{i\bar{z}}^i - \ell_i^2 \Gamma_{z\bar{z}}^i = 0$, $i = 1, \ldots n$. (66)

Next we have in our two cases

(I) $c = c_0$ is constant,

(II) $a_i$ are constant $i = 1, \ldots n$,

the first since $L = 0$, while the second since $N_i = 0$ implies $\Gamma_{zi}^z = 0$ in (32).

Also, for case (II) (38) remains in effect:
\[ (\text{II}) \quad f' = k\alpha c \] for a constant $k$, (68)

where the prime in this subsection denotes differentiation with respect to $t$.

Now the formula for $Q$ reads, due to (31), (66) and (32),
\[ Q = -\sqrt{2} \frac{1}{\alpha} \frac{c'}{c} + \frac{1}{\sqrt{2}c} \sum_{i=1}^{n} 2\ell_j^{-1} \Gamma_j^j - \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \Gamma_{jj}^z \frac{c}{\ell_j a_j^2}, \]

where the first term on the right hand side is zero in case (I), while the last sum is zero in case (II). Therefore (58)-(I) and (59)-(II) become, in view of (58) and (67), and together with (32) in case (I) and taking into account (68) in case (II), after minor simplification

(I) $\frac{1}{c_0} \sum_{i=1}^{n} 2\ell_j^{-1} \Gamma_j^j - c_0 \sum_{j=1}^{n} \Gamma^z_{ji} \frac{1}{\ell_j a_j^2} = K, \quad (a_i^2)'' = c_0^2 \Gamma_{ii}^z \prod_{k=1}^{n} a_k^2,$

(II) $\frac{c'}{c} \left( k\frac{c}{c} + \frac{1}{c} \sum_{i=1}^{n} 2\ell_j^{-1} \Gamma_j^j - \frac{2}{\alpha} c' \right) + \left( k\frac{c}{c} + \frac{1}{c} \sum_{i=1}^{n} 2\ell_j^{-1} \Gamma_j^j - \frac{2}{\alpha} c' \right)' = 0,$

where $K$ is a constant and $i = 1, \ldots n$.

In the first equation of (69) in case (I), the first sum is just a constant. Moreover, recall that $\Gamma_j^j = 0$ for all $j \neq i$ if $\Gamma_{ii}^z \neq 0$. Thus that first sum contains at most one term.

In case (II), the equation in (69) simplifies to
\[ \text{(II)} \quad k\beta c' + \frac{(c')^2}{c^3} - \frac{c''}{c^2} = 0, \]

where $\beta = \prod_{i=1}^{n} a_i^2$ is constant by (67)-(II).

6.2. **No solutions in case (I).** Assume without loss of generality that $\Gamma_{ii}^z$ are nonzero for $i = 1, \ldots k$, where $1 \leq k \leq n$, and any others are zero. Differentiating the first equation in case (I) of (69) while substituting the last $n$ gives
\[ c_0^3 \prod_{i=1}^{n} a_i^2 \sum_{j=1}^{k} (\Gamma_{ji}^z)^2 \ell_j^{-2} a_j^{-4} = 0 \]

which obviously cannot hold as $c_0$ and $a_i$, $i = 1, \ldots n$ are nonzero for a metric. Thus case (I) yields no solutions, unless it is a special case of case (II).
6.3. Solutions in case (II). For given $k, \beta$, the general nonconstant positive solution to (70) is
\[ c = \left( k_2 e^{k_1 t} + k \beta / k_1 \right)^{-1/2} \]
where $k_1 \neq 0$, $k_2$ are integration constants. The steady soliton metric has the form
\[ g = \frac{\beta^2}{k_2 e^{k_1 t} + k \beta / k_1} dt^2 + \sum_{i=1}^{n} \left( 1 + \ell_i^2 \right) a_i^2 \left( \sigma_i^2 + \rho_i^2 \right), \]
where we recall that $a_i, i = 1, \ldots, n$ and $\beta$ are constants. The length of a geodesic normal to the orbits is given by
\[ 2 \sqrt{\frac{\beta}{kk_1}} \tanh^{-1} \left( \sqrt{\frac{k_1}{k \beta}} \sqrt{k_2 e^{k_1 t} + k \beta / k_1} \right) \bigg|_{t_0}^{t} \]
As the argument of the inverse hyperbolic tangent is nonnegative and monotone, this function will approach a finite value at its infimum. Hence such a metric, when defined on a manifold with no singular orbits, will be incomplete. By adopting the methods of [VZ] one can show the same holds even with a singular orbit present.

7. Complete cohomogeneity one Kähler-Ricci skew-solitons under $E(2)$

In this section we show the existence of Kähler-Ricci skew-solitons in dimension four, on a space that does not admit gradient Kähler-Ricci solitons of the kind we are considering in this paper, namely the $\tau$-dependent case defined in subsection 3.1.

For our space we take $(M, g)$ to be a cohomogeneity one 4-manifold under the action of the Euclidean group $E(2)$. For $g$ of the form (28) we have $n = 1$, and denote $a := a_1$ and $b := b_1$. We represent the Lie algebra of $E(2)$ via a frame with $\Gamma_i^{11} = \Gamma_i^{1+} = 1$, and $\Gamma_i^{1-} = 0$, so the condition $\Gamma_i^{1+} - \ell_i^2 \Gamma_i^{1-} = 0$ from (37) is violated, showing that indeed a $\tau$-dependent gradient Kähler-Ricci soliton does not exist in this case. Finally, as we will be considering expanding skew-solitons, we take $\lambda = -1$.

To state our theorem, we make the following definition.
Let $\epsilon = \epsilon(b) : [0, \infty) \to \mathbb{R}$ be a $C^\infty$ function satisfying
\begin{align*}
 &i) \quad \epsilon(0) = 0, \quad 0 \leq \epsilon(b), \\
 &ii) \quad \epsilon'(0) = 0, \quad \epsilon'(b) > -2b, \\
 &iii) \quad \epsilon(b) \text{ has a smooth extension to an even function on } \mathbb{R}.
\end{align*}

\textbf{Theorem 3.} Let $M$ be a 4-manifold admitting a cohomogeneity one action of $E(2)$ with a single singular orbit and an $E(2)$-invariant metric of the form (28) with $n = 1$. Then for any function $\epsilon(b)$ as in (71), there exists on $M$ a complete expanding gradient Kähler-Ricci skew-soliton which is cohomogeneity
one under this action, of the form \((g, f)\), where
\[
g = (abc)^2 dt^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2,
\]
(72)
and \(a, b, c\) are functions of \(t\).

The proof follows very closely the proof of existence of Kähler-Einstein metrics on such manifolds in [MR], but with more difficult estimates. The latter proof, in turn, took its inspiration from [DS1]. It will be given over a number of separate steps.

7.1. The system. We consider the system
\[
2a'/a = -a^2 + c^2, \quad (73)
\]
\[
2b'/b = a^2 + c^2, \quad (74)
\]
\[
2c'/c = a^2 - c^2 + 2(ab)^2 + 2\epsilon(b)a^2, \quad (75)
\]
\[
f' = 2\epsilon(b)a^2, \quad (76)
\]
for \(\epsilon\) as in (71). The first two equations are easily derived from the Kähler conditions (32)-(33), while the third is just the specialization of (35) to this case, once we make the choice of \(f'\) as in (76). Thus existence of a solution to this system for positive \(a, b, c\) will give the skew-solitons, and the main task is to find complete solution metrics.

The system (73)-(75) decouples from the final equation (76). Using (71)i), ii), one of its equilibrium points is \((q, 0, q), q > 0\), and its linearization is the same as when \(\epsilon \equiv 0\), with one negative, one zero and one positive eigenvalue. By the Center Manifold Theorem, its unstable curve is smooth and has domain \((−∞, η)\). We will give a condition on \((a, b, c)\), all positive, which yields such a curve. But first we eliminate other candidate conditions. To this end, we first record a few useful relations.

**Lemma 7.1.** For the system (73)-(75),
\[
(ab)' = abc^2,
\]
\[
(bc)' = bca^2 (1 + b^2 + \epsilon(b)),
\]
\[
(ac)' = a^3c(b^2 + \epsilon(b)),
\]
\[
\left(\frac{a}{b}\right)' = -\frac{a^3}{b},
\]
\[
(c^2 - a^2)' = -(c^2 - a^2)(a^2 + c^2) + 2a^2c^2(b^2 + \epsilon(b))
\]

7.2. Solutions.

**Proposition 7.2.** There are no complete metrics corresponding to solutions of (73)-(75) with maximal interval \((\xi, η)\), when \(\xi\) is finite. Furthermore, the only solutions with maximal interval \((−∞, η)\) are the unstable curves of the
equilibrium points \((q, 0, q), q > 0\), and these solutions satisfy \(0 \leq c^2 - a^2 \leq 2a^2(b^2 + \epsilon(b))\).

Proof. For an initial time \(t_0\), we let \((\xi, \eta)\) be a maximal solution interval for the initial value problem for \((73)-(75)\) with \(a(t_0) = a_0, b(t_0) = b_0,\) and \(c(t_0) = c_0\).

Uniqueness of solutions to \((73)-(75)\) implies that if any of \(a, b,\) or \(c\) are zero anywhere in \((\xi, \eta)\) then it is zero everywhere. Accordingly we thus assume that \(a, b,\) and \(c\) are all positive on \((\xi, \eta)\). Then we see from Lemma 7.1 and \((74)\) that \(ab, bc, ac,\) and \(b\) are all increasing on \((\xi, \eta)\).

We consider the following cases:

Case 1: \(c_0^2 - a_0^2 < 0\). We first claim the following.
Claim: In this case \(a \to \infty\) as \(t \to \xi^+\).

Proof of claim: Since
\[
(c^2 - a^2)' = -(c^2 - a^2)(c^2 + a^2) + 2a^2c^2(b^2 + \epsilon(b)),
\]
if \(c^2 - a^2 < 0\) then \((c^2 - a^2)' > 0\), thus \(c^2 - a^2 < 0\) for all \(\xi < t < t_0\). Therefore,
\[
\begin{align*}
a' &= \frac{a}{2}(c^2 - a^2), \\
a'' &= \frac{a}{4}[(c^2 - a^2)^2 - 2(c^2 - a^2)(c^2 + a^2) + 4a^2c^2(b^2 + \epsilon(b))],
\end{align*}
\]
showing that \(a\) is decreasing and concave up on \((\xi, t_0)\). Next, we always have \(b' > 0\), while
\[
c' = \frac{c}{2}(a^2 - c^2 + 2a^2(b^2 + \epsilon(b))) > 0,
\]
shows that \(c\) is increasing on \((\xi, t_0)\). Therefore \(b\) and \(c\) are bounded on \((\xi, t_0)\). Thus, as \((\xi, \eta)\) is the maximal solution interval, \(a\) could be bounded as \(t \to \xi^+\) only if \(\xi = -\infty\). But since \(a\) is concave up, \(a \to \infty\) as \(t \to \xi^+\) even when \(\xi = -\infty\). \(\square\)

Since \(ab\) and \(ac\) are increasing, they are bounded as \(t \to \xi^+\) and \(a \to \infty\), so \(b \to 0, c \to 0,\) and \(ab \to k\) for some constant \(k\). Near \(b = 0\) we can expand \(\epsilon(b)\) so that for some constants \(l\) and \(m_0,\)
\[
|\epsilon(b) - lb^2| \leq m_0b^4.
\]
Multiplying by \(a^2\) this becomes
\[
|a^2\epsilon(b) - la^2b^2| \leq m_0a^2b^4.
\]
This show that as \(t \to \xi^+, a^2\epsilon(b) \to lk^2\). Then as \(t \to \xi^+\) the equations will take the asymptotic form
\[
\begin{align*}
a' &= -\frac{1}{2}a^3 \\
b' &= \frac{1}{2}ba^2 \\
c' &= \frac{1}{2}c(a^2 + 2k^2(l + 1))
\end{align*}
\]
the solution of which has asymptotic form

\[
\begin{align*}
a &\simeq (t - \xi)^{-\frac{1}{2}}, \\
b &\simeq b_0(t - \xi)^{\frac{1}{2}}, \\
c &\simeq c_0(t - \xi)^{\frac{1}{2}},
\end{align*}
\]

for some constants \(b_0\) and \(c_0\). This shows that \(\xi\) is finite in this case and

\[
\int_{\xi}^{t_0} abc \, dt < \infty,
\]

hence the metric is not complete.

**Case 2:** \(c_0^2 - a_0^2 > 2a_0^2(b_0^2 + \epsilon(b_0))\). We have, again, a similar claim.

**Claim:** In this case \(c \to \infty\) as \(t \to \xi^+\).

**Proof of claim:** Since

\[
(c^2 - a^2 - 2a^2(b^2 + \epsilon(b)))' = -(c^2 - a^2)(c^2 + a^2) - 2a^2b^2c^2
\]

\[
+ 2a^4\epsilon(b) - b\epsilon'(b)/2 - a^2c^2b\epsilon'(b),
\]

\[
= -(c^2 - a^2 - 2a^2(b^2 + \epsilon(b)))(c^2 + a^2) - 4a^2b^2c^2
\]

\[
- 2a^4b^2 - 2a^2c^2\epsilon(b) - a^4b\epsilon'(b) - a^2c^2b\epsilon'(b),
\]

\[
= -(c^2 - a^2 - 2a^2(b^2 + \epsilon(b)))(c^2 + a^2) - 2a^2b^2c^2
\]

\[
- 2a^2c^2\epsilon(b) - a^2b\epsilon'(b) + 2b(a^2 + c^2),
\]

if \(c^2 - a^2 - 2a^2(b^2 + \epsilon(b)) > 0\) then \(c^2 - a^2 - 2a^2b^2 > 0\) so that via (71)ii) \((c^2 - a^2 - 2a^2(b^2 + \epsilon(b)))' < 0\), thus in this case, \(c^2 - a^2 > 2a^2(b^2 + \epsilon(b))\) for all \(\xi < t \leq t_0\). Therefore,

\[
c' = \frac{c}{2}(a^2 - c^2 + 2a^2(b^2 + \epsilon(b)),
\]

\[
c'' = \frac{c}{4}((a^2 - c^2 + 2a^2(b^2 + \epsilon(b)))^2 + 2(c^2 - a^2 - 2a^2(b^2 + \epsilon(b)))(c^2 + a^2) + 4a^2b^2c^2 + 4a^2c^2\epsilon(b) + 2a^2b\epsilon'(b) + 2b)(a^2 + c^2)],
\]

showing via (71)ii) that \(c\) is decreasing and concave up on \((\xi, t_0)\). Next, we always have \(b' > 0\), while \(a\) is increasing on \((\xi, t_0)\). Therefore \(a\) and \(b\) are bounded on \((\xi, t_0)\). Thus, as \((\xi, \eta)\) is the maximal solution interval, this time \(c\) could be bounded as \(t \to \xi^+\) only if \(\xi = -\infty\). But since \(c\) is concave up, \(c \to \infty\) as \(t \to \xi^+\) even when \(\xi = -\infty\). \(\square\)
Since $ac$ and $bc$ are increasing, they are bounded as $t \to \xi^+$ and $c \to \infty$, so $a \to 0$, $b \to 0$. Then as $t \to \xi^+$ the equations will take the asymptotic form

\[
\begin{align*}
    a' &= \frac{1}{2} ac^2 \\
    b' &= \frac{1}{2} bc^2 \\
    c' &= -\frac{1}{2} c^3
\end{align*}
\]

which have the solution

\[
\begin{align*}
    a &= a_0 (t - \xi)^{\frac{1}{2}} \\
    b &= b_0 (t - \xi)^{\frac{1}{2}} \\
    c &= (t - \xi)^{-\frac{1}{2}}
\end{align*}
\]

for some constants $a_0$ and $b_0$. This shows that $\xi$ is finite in this case and

\[
\int_{\xi}^{t_0} abc \, dt < \infty,
\]

so the metric is not complete.

If $c^2 - a^2 < 0$ or $c^2 - a^2 > 2a^2b^2$ at any time, then a constant shift in $t$ will give one of the previous cases. In both previous cases, $\xi$ is finite, but we know that the unstable curve of the equilibrium points $(q,0,q)$ must have $\xi = -\infty$. The existence of such curves is guaranteed by the Center Manifold theorem. Therefore we consider the final case:

**Case 3:** $0 \leq c^2 - a^2 \leq 2a^2(b^2 + \epsilon(b))$ for all $t \in (\xi, \eta)$. Here $(\xi, \eta)$ is a maximal domain for the solution.

**Claim:** In this case $\xi = -\infty$.

**Proof of claim:** If $t_0$ is in the maximum interval, as $a, b, c$ are all increasing, they are therefore all bounded on $(\xi, t_0)$. Since $(\xi, \eta)$ is the maximal solution interval, we see that $\xi = -\infty$. \[\square\]

As $a, b, c$ are all increasing, it must be that they all approach finite non-negative limits as $t \to -\infty$. Thus $(a, b, c)$ must approach an equilibrium point. If $(a, b, c) \to (0, q, 0)$ with $q > 0$, then $a/b \to 0$ as $t \to -\infty$, but $a/b$ is decreasing and positive, hence this cannot happen.

Therefore, when $t \to -\infty$ we see that $(a, b, c) \to (q, 0, q)$ in this case, but we still need to rule out the possibility that $q = 0$. For this we compute the variation of $ac$ with respect to $b$:

\[
\frac{d(ac)}{db} = \frac{2(\frac{a}{c})^2(ac)(b + \frac{\epsilon(b)}{b})}{(\frac{a}{c})^2 + 1}.
\]
Our assumption of an equilibrium point \((q,0,q)\) implies that \(a/c \to 1\) when \(b \to 0\). Since in our case \(c^2 - a^2 \geq 0\), we have
\[
\frac{a}{c} \leq 1.
\]
Employing this inequality in equation (77) yields
\[
\frac{d(ac)}{db} \leq (ac)\left(b + \frac{\epsilon(b)}{b}\right).
\]
By Grönwall’s inequality, if \(ac \to 0\) when \(b \to 0\) then \(ac = 0\) identically. As the latter is not possible, neither is \(q = 0\).

\[\square\]

### 7.3. Distance computations.

In the following proposition in particular, the estimates diverge significantly from those in [MR].

**Proposition 7.3.** Let \(g\) be a Riemannian metric of the form (72) on a manifold \(M\), with \(a, b, c\) a solution to (73)-(75) along an unstable curve of an equilibrium point \((q,0,q)\), \(q > 0\), having maximal domain \(I = (-\infty, \eta)\). Assume that the latter interval is also the range of the coordinate function \(t\) on \(M\). Then for any point \(p_0 \in M\) with orbit through \(p_0\) of principal type and \(M_t\) a level set of \(t\),
\[
\lim_{t \to -\infty} d_g(p_0, M_t) < \infty, \quad \lim_{t \to \eta} d_g(p_0, M_t) = \infty,
\]
where \(d_g\) is the distance function induced by \(g\).

**Proof.** The union of the principal orbits forms an open dense set, \(\tilde{M}\), so that \(\tilde{M}/\mathcal{G}\) is a smooth manifold of dimension 1. The function \(t\) is a smooth submersion from \(\tilde{M}\) to \(\tilde{M}/\mathcal{G}\). The metric
\[
(abc)^2 dt^2
\]
makes this into a Riemannian submersion. The level sets of \(t\) are orbits of \(\mathcal{G}\) and for \(t_0 = t(p_0)\)
\[
d_g(p_0, M_{t_0}) = d_g(M_{t_0}, M_{t_1}),
\]
is the distance in the quotient manifold, where
\[
d_g(M_{t_0}, M_{t_1}) = \left| \int_{t_0}^{t_1} abc \, dt \right|.
\]

Asymptotically as \(t \to -\infty\),
\[
a \simeq q, \quad b \simeq k e^{q^2 t}, \quad c \simeq q
\]
This gives the asymptotic metric
\[
g \simeq k^2 q^4 e^{2q^2 t} dt^2 + q^2 \sigma_1^2 + k^2 e^{2q^2 t} \sigma_2^2 + q^2 \sigma_3^2,
\]
and for \(v = ke^{q^2 t}\) this is just
\[
g \simeq (dv^2 + v^2 \sigma_2^2) + q^2 (\sigma_1^2 + \sigma_3^2).\]
In this coordinate, the endpoint \( \xi = -\infty \) is at \( v = 0 \), and we see that
\[
\lim_{t \to -\infty} d_g(p_0, M^t) = \int_{-\infty}^{v_0} abcdt = \int_0^{v_0} dv < \infty,
\]
as claimed.

Now to understand the behavior at the \( \eta \) side of the solution interval, we examine the derivative of \( \frac{a}{c} \) with respect to \( b \)
\[
\frac{d\left(\frac{a}{c}\right)}{db} = 2\left(\frac{a}{c}\right) \left(1 + \left(\frac{a}{c}\right)^2(1 + b^2 + \epsilon(b))\right),
\]
ignoring the case \( a/c = 0 \), this equation has nullcline
\[
\frac{a}{c} = \sqrt{\frac{1}{1 + b^2 + \epsilon(b)}},
\]
\[\text{Since (71)ii) easily implies that the nullcline is always decreasing, and our solution starts at } \frac{a}{c} = 1 \text{ when } b = 0, \text{ we have, using also (71)i) }
\]
\[
\frac{a}{c} \leq \sqrt{\frac{1}{1 + b^2 + \epsilon(b)}},
\]
\[\text{To find a better upper bound than 1 for } \frac{a}{c}, \text{ we consider the curve } \frac{a}{c} = \sqrt{\frac{k^2}{k^2 + b^2}} \text{ and plug its expression into the slope field. This gives slope}
\]
\[
\frac{2k^2 + 2b^2}{2k^2 + b^2(k^2 - 1)} \frac{d}{db}\sqrt{\frac{k^2}{k^2 + b^2}} - 2\left(\frac{k^2}{k^2 + b^2}\right)^{3/2} \frac{b^2}{b^2 + \frac{k^2}{k^2 + b^2} + 1} \frac{\epsilon(b)}{k^2 + b^2}.
\]
So for \( k^2 > 2 \), the slopes of the solutions along this curve are less, i.e. more negative, than the slope of the curve. Since for our (non-equilibrium) solution, there is some \( b_p \) in the range of \( b \) for which one has \( \left(\frac{a}{c}\right)(b_p) < 1 \), the graph of \( a/c \) is below the graph of \( \sqrt{\frac{k^2}{k^2 + b^2}} \) for some \( k = k_p > 2 \) at \( b_p \), and hence for all \( b \geq b_p \). Therefore for \( b \geq b_p \)
\[
\sqrt{\frac{1}{1 + b^2 + \epsilon(b)}} \leq \frac{a}{c} \leq \sqrt{\frac{k^2}{k^2 + b^2}}.
\]
Next we deduce an estimate for \( ac \) in terms of \( b \), for \( b > b_p \). Using (77) and the last inequalities, we have
\[
\frac{2(b + \frac{\epsilon(b)}{b})}{2 + b^2 + \epsilon(b)} \leq \frac{d(ln(ac))}{db} \leq \frac{2(b + \frac{\epsilon(b)}{b})}{2k^2 + b^2},
\]
and integrating this from \( b = 0 \) to \( b \), exponentiating and multiplying by \( q^2 \) gives
\[
q^2 \exp\left(\int_0^b \frac{2(b + \frac{\epsilon(b)}{b})}{2 + b^2 + \epsilon(b)} db\right) \leq ac \leq q^2 \exp\left(\int_0^b 2(b + \frac{\epsilon(b)}{b}) \frac{k^2}{2k^2 + b^2} db\right).
\]
Note here that our assumption (71) (ii) does imply the integrands are integrable. Now since the integrand on the left-hand side is larger than $b/(2 + b^2)$, we in fact have

$$ac \geq q^2(2 + b^2)^{1/2}$$

for $b > b_p$. We thus finally arrive at an estimate for $b'$ as

$$b' = b \frac{a^2 + c^2}{2} = \frac{b}{2}(ac) \left( \frac{a}{c} + \frac{c}{a} \right).$$

Namely, for some positive constant $K_1$ and $b = b(t) > b_p$,

$$b' \geq K_1 b^3,$$

showing that $\eta$ is finite, and so $b \to \infty$ as $t \to \eta^-$. Additionally, we see that for some constant $K_2 > 0$, denoting $b_0 := \max\{b(t_0), b_p\} > 0$, we have

$$\lim_{t \to \eta} d_g(p_0, M^t) = \int_{t_0}^\eta abc \, dt \geq \int_{b_0}^\infty \frac{2}{\frac{a}{c} + \frac{c}{a}} \, db \geq \int_{b_0}^\infty K_2 \frac{b}{b} \, db = \infty. \quad (79)$$

\[\square\]

7.4. The Bolt. Recall, for example from [MR], that “attaching a bolt” refers to replacing a 4-manifold with a cohomogeneity one action with only regular orbits over an open interval with one admitting a similar action for the same group over a semi-closed interval with a 2-dimensional singular orbit (the bolt) over the endpoint of the interval. For the case at hand, the latter 4-manifold can then be described as

$${E}(2) \times {SO}(2) \mathbb{R}^2 = (0, \infty) \times {E}(2) \amalg \{0\} \times \mathbb{R}^2,$$

where the right $SO(2)$-action is $(g, (T, x)) \to (Tg, g^{-1}x)$. So this is an instance of the set-up described in the beginning of Section 4.

**Proposition 7.4.** The metric and Kähler form corresponding to solutions of (73)-(75) along the unstable curves of the equilibrium points $(q, 0, q)$, $q > 0$, defined on $(-\infty, \eta)$, can be smoothly extended to $E(2) \times {SO}(2) \mathbb{R}^2$, where the bolt fibers over $\xi = -\infty$.

**Proof.** Consider $M = E(2) \times {SO}(2) \mathbb{R}^2$. This has a left action by $E(2)$ with regular orbit $E(2)$ and a singular orbit $E(2)/{SO}(2)$. For any $E(2)$-invariant metric $g$ on $M$, with $r$ the distance along a geodesic perpendicular to the singular orbit,

$$g = dr^2 + g_r.$$

For a metric $g$ of the form (72), let $r = \int_{-\infty}^t a(s)b(s)c(s) \, ds$, then

$$g = dr^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2.$$
The ODE’s (73)-(75) in this coordinate become
\[
\begin{align*}
\frac{da}{dr} &= \frac{a}{2} \left( -\frac{a}{bc} + \frac{c}{ab} \right), \\
\frac{db}{dr} &= \frac{1}{2} \left( \frac{a}{c} + \frac{c}{a} \right), \\
\frac{dc}{dr} &= \frac{c}{2} \left( \frac{a}{bc} - \frac{c}{ab} + \frac{ab}{c} + \frac{2a\epsilon(b)}{bc} \right).
\end{align*}
\]
(80)
(81)
(82)

From these it is seen that if \(\epsilon(b)\) is extended as an even function of \(b\), then \(a, b, \) and \(c\) can be extended at \(r = 0\) so that \(a\) and \(c\) are even and \(b\) is odd. Following the notations of Verdiani and Ziller [VZ], the tangent space for \(r \neq 0\) splits as
\[
T_pM = \mathbb{R}\partial_r \oplus \mathfrak{k} \oplus \mathfrak{m},
\]
where
\[
\mathfrak{k} = \text{span}\{X_2\},
\]
\[
\mathfrak{m} = \text{span}\{X_1, X_3\} =: \ell_1,
\]
and we set
\[
V = \text{span}\{\partial_r, X_2\} =: \ell'_{-1}.
\]

Since \(\exp(\theta X_2)\) acts on both \(V\) and \(\mathfrak{m}\) as a rotation by \(\theta\), we have weights \(a_1 = d_1 = 1\) (with the former not to be confused with its use in previous sections). The smoothness conditions for \(V\) is that \(b\) can be extended to an odd function and \(b'(0) = 1\). Since we know that \(b\) can be extended to be odd, we just check from (81) that
\[
\left. \frac{db}{dr} \right|_{r=0} = \frac{1}{2} \left( q + \frac{q}{q} \right) = 1.
\]

As \(\ell'_{-1}\) and \(\ell_1\) are orthogonal, the smoothness conditions in table C of [VZ] are automatically satisfied, while those in table B there, are
\[
\begin{align*}
a^2 + c^2 &= \phi_1(r^2), \\
a^2 - c^2 &= r^2\phi_2(r^2),
\end{align*}
\]
(83)
(84)
for some smooth functions \(\phi_1\) and \(\phi_2\). Now to see that (83) is satisfied, note that
\[
a^2 + c^2 = 2ac \frac{db}{dr}.
\]

Since \(a, c,\) and \(\frac{db}{dr}\) are even, it just remains to check (84). As \(a, c\) are even while \(b\) is odd, \(a/c\) is even as a function of \(b\). We have to second order
\[
a^2 - c^2 = c^2 \left( \frac{a^2}{c^2} - 1 \right) \propto b^2 c^2,
\]
and since \(b(0) = 0\) and \(\left. \frac{db}{dr} \right|_{r=0} = 1\), we get that \(g\) extends to a smooth metric on \(M\).

Finally we check that the Kähler form also extends smoothly to the singular orbit at \(r = 0\). Following Verdiani and Ziller we analyze the eigenspaces for the action of \(SO(2)\) on \(T_pM\). We find that \(\partial_r + \frac{\epsilon}{r} X_2\) is an eigenvector with eigenvalue \(e^{ia_1\theta}\), and \(X_1 + iX_3\) is an eigenvector with eigenvalue \(e^{id_1\theta}\), and
likewise for their complex conjugates. Dualizing gives eigenspaces of $T^*_pM$: $dr - ir\sigma_2$ has eigenvalue $e^{ia_1\theta}$, and $\sigma_1 - i\sigma_3$ has eigenvalue $e^{id_1\theta}$. Thus the eigenspaces of $\Lambda^2 T^*_pM$ are given by

$$E_1 = \text{span}\{ rdr \wedge \sigma_2, \sigma_1 \wedge \sigma_3 \}$$

$$E_{e^{i(a_1-d_1)\theta}} = \text{span}\{ dr \wedge \sigma_1 + r\sigma_2 \wedge \sigma_3 + i(dr \wedge \sigma_3 + r\sigma_1 \wedge \sigma_2) \}$$

$$E_{e^{i(a_1+d_1)\theta}} = \text{span}\{ dr \wedge \sigma_1 - r\sigma_2 \wedge \sigma_3 + i(dr \wedge \sigma_3 - r\sigma_1 \wedge \sigma_2) \}$$

According to [VZ], the smoothness condition is just the equivariance condition $\omega(e^{a_1\theta}p) = \exp(\theta X_2)^*\omega$. This requires that the coefficient of $E_1$ is

$$\phi_1(r^2), \quad (85)$$

$$E_{e^{\pm i(a_1-d_1)\theta}} \text{ is } r \frac{|a_1-d_1|}{a_1} \phi_2(r^2), \quad (86)$$

$$E_{e^{\pm i(a_1+d_1)\theta}} \text{ is } r \frac{|a_1\pm d_1|}{a_1} \phi_3(r^2). \quad (87)$$

But we have

$$\omega = cdr \wedge \sigma_3 + ab\sigma_1 \wedge \sigma_3$$

$$= \frac{c}{2}[(dr \wedge \sigma_3 + r\sigma_1 \wedge \sigma_2) + (dr \wedge \sigma_3 - r\sigma_1 \wedge \sigma_2)] + \frac{ab}{2r}[(dr \wedge \sigma_3 + r\sigma_1 \wedge \sigma_2) - (dr \wedge \sigma_3 - r\sigma_1 \wedge \sigma_2)]$$

$$= \frac{cr + ab}{2r}(dr \wedge \sigma_3 + r\sigma_1 \wedge \sigma_2) + \frac{cr - ab}{2r}(dr \wedge \sigma_3 - r\sigma_1 \wedge \sigma_2),$$

and therefore the smoothness conditions can now be written as

$$cr + ab = r\phi_2(r^2),$$

$$cr - ab = r^3\phi_3(r^2).$$

The first of these is clear; for the second, expand to get $a/c = 1 + \alpha b^2 + O(b^4)$ for some constant $\alpha$ and $b = r + O(r^3)$, so

$$cr \left(1 - \frac{ab}{cr}\right) = cr \left(1 - \frac{b + \alpha b^3 + O(b^5)}{r}\right) = r^3\phi_3(r^2).$$

Therefore $\omega$ extends as a smooth form on all of $M$. \qed

### 7.5. Completeness

**Proposition 7.5.** For the metrics of Proposition 7.4, all finite length curves remain inside some compact set.

The proof is exactly the same as in [MR]. This completes the proof of Theorem 3.
APPENDIX A. THE KÅHLER CONDITION

We review here the Kåhler condition in terms of the decomposition (1). For a Kåhler form $\omega$ on a manifold $M^m, m > 1$, the vanishing of $d\omega$ on orthonormal frame vector fields $\{e_{2j-1}, e_{2j}\} \subset \mathcal{H}_j, j = 1, \ldots m$ amounts to

$$\binom{2m}{3} = 2m(m-1) + 2^3 \binom{m}{3}$$

independent equations, with $2m(m-1)$ involving frame fields only from two $J$-invariant subbundles $\mathcal{H}_i, \mathcal{H}_j, i \neq j$ while the remaining $8 \binom{m}{3}$ equations involve vector fields belonging to three subbundles $\mathcal{H}_i, \mathcal{H}_j, \mathcal{H}_k, i \neq j \neq k \neq i$ (and $\binom{m}{3}$ is taken to be zero if $m = 2$). For each fixed ordered pair $i, j$ there are two equations of the first of these two types, the first having the form

$$\hat{\epsilon}_{2j}([e_{2i-1}, e_{2i}]) + \hat{\epsilon}_{2i-1}([e_{2i-1}, e_{2j-1}]) + \hat{\epsilon}_{2i}([e_{2i}, e_{2j-1}]) = 0 \quad (88)$$

while the second obtained by exchanging $e_{2j-1}$ with $e_{2j}$ (and their hatted counterparts) in the first one. A typical example of the second type has the form

$$\hat{\epsilon}_{2k}([e_{2i-1}, e_{2j-1}]) + \hat{\epsilon}_{2i}([e_{2j-1}, e_{2k-1}]) + \hat{\epsilon}_{2j}([e_{2k-1}, e_{2i-1}]) = 0 \quad (89)$$

All these facts are easily obtained from the coboundary formula for the exterior derivative of a 2-form. Note that the sum $\binom{2m}{3} + 2m(m-1)$, where the second term now corresponds to the number of vector equations needed for integrability of the complex structure, yields a quantity still smaller than the $2m \binom{2m}{2}$ unknowns needed to specify all Lie brackets of pairs of frame vector fields.

For the frame with Lie brackets given by the first paragraph of subsection 2.2, it is clear that equations of type $(89)$ will hold. The same is true for equations of type $(88)$, which require knowledge of $[\Gamma(\mathcal{H}_i), \Gamma(\mathcal{H}_j)]$ for $i = j$, as opposed to integrability of the complex structure, which only required such bracket relations with $i \neq j$. For example, we have

$$\hat{y}_i([k, t]) + \hat{k}([k, x_i]) + \hat{t}([t, x_i])$$

$$= \hat{y}_i(L(k + t)) + \hat{k}(A_i x_i + B_i y_i) + \hat{t}(E_i x_i + F_i y_i)$$

$$= 0,$$

$$\hat{t}([x_i, y_i]) + \hat{x}_i([x_i, k]) + \hat{y}_i([y_i, k])$$

$$= \hat{t}(N_i (k + t)) + \hat{x}_i(-A_i x_i - B_i y_i) + \hat{y}_i(-C_i x_i - D_i y_i)$$

$$= N_i - A_i - D_i = 0,$$

etc. This provides another proof for Prop. 2.1, and explains the need to require relations $(8)$. One can also draw some limited conclusions regarding possible Lie bracket values for Kåhler structures. For example, if $m = 2$, we see that with an indexing $\mathcal{H}_0 = \text{span}(k, t), \mathcal{H}_1 = \text{span}(x_1, y_1)$ and shear relations as in $(5)-(6)$ and $(7)$, $[\Gamma(\mathcal{H}_0), \Gamma(\mathcal{H}_0)]$ cannot have a component in $\mathcal{H}_1$, whereas $[\Gamma(\mathcal{H}_1), \Gamma(\mathcal{H}_1)]$ can have a component in $\mathcal{H}_1$, without preventing the metric from being Kåhler.
Acknowledgements

We thank Ramiro Lafuente, whose message alerted us to an incorrect description in an earlier version of the manifold in Theorem 2.

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