Four dimensional string solutions in Hořava-Lifshitz gravity

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Abstract

We investigate string-like solutions in four dimensions based on Hořava-Lifshitz gravity. For a restricted class of solutions where the Cotton tensor vanishes, we find that the string-like solutions in Einstein gravity including the BTZ black strings are solutions in Hořava-Lifshitz gravity as well. The geometry is warped in the same way as in Einstein gravity, but the “conformal” lapse function is not constrained in Hořava-Lifshitz gravity. It turns out that if $\lambda \neq 1$, there exist no other solutions. For the value of model parameter with which Einstein gravity recovers in IR limit (i.e., $\lambda = 1$), there exists an additional solution of which the conformal lapse function is determined. Interestingly, this solution admits a uniform BTZ black string along the string direction, which is distinguished from the warped BTZ black string in Einstein gravity. Therefore, it is a good candidate for the test of the theory.

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I. INTRODUCTION

Very recently Hořava proposed a new quantum gravity theory which is improved in renormalizability in UV [1, 2]. This theory treats space and time on an unequal footing, and the theory becomes nonrelativistic. This Hořava-Lifshitz theory attracted much attention in gravity theory [3, 4, 5, 6, 7, 8, 9] and particularly in cosmology [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and black-hole physics [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

In the Hořava-Lifshitz theory, the space and time are scaled differently,

\[ x \rightarrow b x, \quad t \rightarrow b^z t, \]

and according to this rescaling, the space and the time dimensions are

\[ [x] = -1, \quad [t] = -z \]

in mass dimension. The \( z = 3 \) case corresponds to the three spatial dimensions, and is power-counting UV renormalizable.

Using the ADM formalism, the four dimensional metric is written as

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \]

and the dimensions for each metric coefficients are

\[ [g_{ij}] = 0, \quad [N_i] = z - 1, \quad [N] = 0. \]

In this ADM formalism, the Einstein-Hilbert action is given by

\[ S_{EH} = \frac{1}{16 \pi G} \int d^4 x \sqrt{g} N (K_{ij} K^{ij} - K^2 + R - 2\Lambda), \]

where \( G \) is Newton’s constant, \( R \) is the 3D Ricciscalar, and \( K_{ij} \) is the extrinsic curvature defined by

\[ K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i). \]

In the Hořava-Lifshitz gravity for \( z = 3 \), the kinetic part for the action is

\[ S_K = \frac{2}{\kappa^2} \int dt d^3 x \sqrt{g} N (K_{ij} K^{ij} - \lambda K^2), \]
where $\kappa$ and $\lambda$ are dimensionless couplings, and for $\lambda = 1$ the kinetic part becomes that of the Einstein-Hilbert action. The remaining terms correspond to the nonrelativistic potential term which satisfy the so called “detailed balance” condition,

$$S_V = \frac{\kappa^2}{8} \int dt d^3 x \sqrt{g} N \mathcal{E}^{ij} \mathcal{G}_{ijkl} \mathcal{E}^{kl}, \quad \sqrt{g} \mathcal{E}^{ij} \equiv \frac{\delta W[g_{ij}]}{\delta g_{ij}}, \quad (8)$$

where $\mathcal{G}_{ijkl}$ is the inverse of the De Witt metric,

$$\mathcal{G}_{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl}, \quad (9)$$

and $W[g_{ij}]$ is a three dimensional Euclidean action,

$$W = \frac{1}{w^2} \int \omega_3(\Gamma) + \mu \int d^3 x \sqrt{g} \left( R - 2\Lambda_W \right), \quad (10)$$

where the first term is the gravitational Chern-Simons term with the dimensionless coupling $w$, and the second term is a three dimensional Einstein-Hilbert term with a coupling $\mu$ of dimension 1 and a three dimensional cosmological constant $\Lambda_W$ of dimension 2.

With the above kinetic and potential terms, the 6th-order action for Hořava gravity becomes

$$S = \int dt d^3 x \left( \mathcal{L}_0 + \mathcal{L}_1 \right), \quad (11)$$

$$\mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3\Lambda_W^2)}{8(1 - 3\lambda)} \right\}, \quad (12)$$

$$\mathcal{L}_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) \right\}, \quad (13)$$

where $C^{ij}$ is the Cotton tensor defined by

$$C^{ij} = \epsilon^{ikl} \nabla_k \left( R^l_{\ell} - \frac{1}{4} R \delta^l_{\ell} \right) = \epsilon^{ikl} \nabla_k R_{ij} - \frac{1}{4} \epsilon^{ikl} \partial_k R. \quad (14)$$

Comparing $\mathcal{L}_0$ with that of general relativity in the ADM formalism, the speed of light, the Newton’s constant and the cosmological constant are related with the model parameters as

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1 - 3\lambda}}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \Lambda = \frac{3}{2} \Lambda_W. \quad (15)$$

In order for the speed of light to be real, $\Lambda_W < 0$ when $\lambda > 1/3$.Performing an analytic continuation, $\mu \rightarrow i\mu, w^2 \rightarrow -iw^2$, we can have $\Lambda_W > 0$ when $\lambda > 1/3$, which makes the action consistent. After the analytic continuation, the potential terms change their signature while the kinetic terms remain unchanged.
The field equations from the action (11) were derived in Refs. [11, 22]. We summarize the results in Appendix for further convenience.

In this work, we investigate an axially symmetric system in Hořava-Lifshitz gravity. We solve field equations and obtain static solutions with vanishing Cotton tensor. We discuss that a BTZ-type black-string solution is possible for a specific value of the model parameter in this theory.

II. BTZ BLACK STRINGS IN EINSTEIN GRAVITY WITH COSMOLOGICAL CONSTANT

In this section we briefly review the BTZ black-string solutions in four dimensional Einstein gravity with a cosmological constant $\Lambda_4$ [35]. The Einstein equation in four dimensional spacetime in the presence of the cosmological constant can be written as

$$R_{MN} = \Lambda_4 g_{MN}.$$ (16)

One can easily see that any metric satisfying the three dimensional Einstein equation in the presence of the 3D cosmological constant $\Lambda_3$,

$$\tilde{R}_{\mu\nu} = 2\Lambda_3 \gamma_{\mu\nu},$$ (17)

can be embedded into the 4D warped geometry given by

$$ds^2 = W^{-2}(z) \left[ \gamma_{\mu\nu}(x^\sigma) dx^\mu dx^\nu + dz^2 \right].$$ (18)

Here, the warp factor satisfies

$$\frac{\ddot{W}}{W} - \left( \frac{\dot{W}}{W} \right)^2 - \frac{\Lambda_4}{3W^2} = 0,$$ (19)

where $\dot{\equiv} \frac{d}{dz}$.

and $\Lambda_3$ is given by

$$\Lambda_3 = \frac{\dot{W}}{W}.$$ (20)

If the 4D bulk cosmological constant is negative, $\Lambda_4 < 0$, there are three types of solutions to Eq. (19),

$$W(z) = \begin{cases} 
\sqrt{-\frac{\Lambda_4}{3\Lambda_3}} \sinh \sqrt{\Lambda_3}(z - z_0) & \text{for } dS_3 (\Lambda_3 > 0), \\
\sqrt{-\frac{\Lambda_4}{3\Lambda_3}} (z - z_0) & \text{for } M_3 (\Lambda_3 = 0), \\
\sqrt{\frac{\Lambda_4}{3\Lambda_3}} \sin \sqrt{-\Lambda_3}(z - z_0) & \text{for } AdS_3 (\Lambda_3 < 0). 
\end{cases}$$ (21)
where $z_0$ is an integral constant, and $\Lambda_3$ is related with the other integration constant through Eq. (20). On the other hand, if the bulk cosmological constant is positive, $\Lambda_4 > 0$, the only possible solution is given by

$$W(z) = \sqrt{\frac{\Lambda_4}{3\Lambda_3}} \cosh \sqrt{\Lambda_3}(z - z_0),$$

(22)

where $\Lambda_3$ is positive only in this case.

In (1+2) dimensions, the static circularly-symmetric solution to the 3D Einstein equation (17) is given by

$$ds_3^2 = - \left( -M - \Lambda_3 r^2 \right) dt^2 + \frac{dr^2}{-M - \Lambda_3 r^2} + r^2 d\theta^2 + dz^2.$$

(23)

For $\Lambda_3 < 0$ with $M > 0$, this is the so called BTZ black hole. The corresponding 4D solution does not have a translation symmetry along the extra $z$-direction. Due to the warp factors in Eqs. (21)-(22), for the case of $\Lambda_3 < 0$, the warped geometry is the BTZ black string in 4D Einstein gravity,

$$ds^2 = \frac{3\Lambda_3 / \Lambda_4}{\sin^2 \sqrt{-\Lambda_3}(z - z_0)} \left[ - \left( -M - \Lambda_3 r^2 \right) dt^2 + \frac{dr^2}{-M - \Lambda_3 r^2} + r^2 d\theta^2 + dz^2 \right].$$

(24)

### III. GRAVITY OF STRING MODEL

In this section, we consider a string-like object in Hořava-Lifshitz gravity. We consider a static system with axial symmetry in four dimensions. The metric ansatz is then

$$ds^2 = W^{-2}(z) \left[ - \tilde{N}(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + dz^2 \right],$$

(25)

where $W(z)$ is the warp factor. In the four dimensional Einstein gravity with vanishing cosmological constant, it is known that there is no stationary black-string solution. The topology theorem for stationary black holes in 4D simply contradicts with the existence of such a black-string configuration. Presumably, this is also related with the fact that there is no black-hole solution in the 3D Einstein gravity so that it is impossible to obtain a black-string configuration in 4D by arranging 3D black holes along one spatial direction. In the presence of a cosmological constant, however, black-hole solutions exist in 3D, for example, BTZ black holes. By foliating these 3D BTZ black holes along one direction, one can obtain black-string solutions in four dimensions as shown above. In this section, we search for such BTZ-type black-string solutions in the 4D Hořava-Lifshitz gravity.
Since we are dealing with a static system without a momentum flow \( N^i = 0 \), the extrinsic curvature vanishes, \( K_{ij} = 0 \). Therefore, the field equations in Appendix are simplified, and the equation from \( \delta N^i \) is absent. Since the kinetic terms involved with the extrinsic curvature are missing, the resulting field equations are the same regardless of the ranges of \( \Lambda_W \) and \( \lambda \) (i.e., after the analytic continuation, the equations remain unchanged).

Now, we solve field equations (A.1) and (A.4). Unfortunately, however, we are unable to solve those equations fully analytically. Instead, we consider the situation in which the Cotton tensor vanishes. Note that the Cotton tensor identically vanishes for the static system with spherical symmetry studied in Ref. [22]. For the system with axial symmetry with the metric ansatz (25), however, the Cotton tensor does not vanish identically, and the nonvanishing components are

\[
C_{\theta z} = C_{z\theta} = -\frac{1}{4} W \sqrt{f} \left(f'' - \frac{f'}{r}\right),
\]

where the prime denotes the derivative with respect to \( r \). The solution to \( C_{ij} = 0 \) is

\[
f(r) = -M - \alpha r^2,
\]

where \( M \) and \( \alpha \) are integration constants. Later, we shall see that \( M \) is arbitrary playing the role of the mass, and that \( \alpha \) plays like an effective cosmological constant of the \((1 + 2)\) dimensional spacetime transverse to the \( z \)-direction.

By substituting the metric function \( f(r) \) in Eqs. (A.1) and (A.4) with Eq. (27), we obtain a simpler set of equations shown in Eqs. (A.11)-(A.17). Although Eq. (A.11) provides a decoupled equation involving the warp factor \( W(z) \) only, it is unlikely to be solved easily. Other equations consist of both \( \tilde{N}(r) \) and \( W(z) \) functions and their derivatives, but all of them are separable. One can easily check that all solutions for the warp factor in Einstein gravity given in Eqs. (21)-(22) with \( \tilde{N}^2 = -M - \Lambda_3 r^2 \) and \( \alpha = \Lambda_3 \), become the solutions in Hořava-Lifshitz gravity as well. We shall show below that indeed there exists only this class of solutions in Hořava-Lifshitz gravity, but the conformal lapse function \( \tilde{N} \) is not necessarily constrained. In addition, for the special case of \( \lambda = 1 \), there exists a BTZ-type black-string solution, but interestingly a constant warp factor is also allowed; the space without being warped along the string direction also exists.

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1 This case corresponds to the Hořava-Lifshitz gravity having no contribution from the gravitational Chern-Simons term.
A. Solution with $\tilde{N}(r)$ unconstrained

Assuming that $\lambda \neq 1$, one can use Eq. (A.11) to replace $\ddot{W}$ and its higher derivatives in other equations. Then the subtraction between Eqs. (A.12) and (A.13) presents our master equation

$$\left[ 1 + \lambda \left( -3 \pm \sqrt{2(3\lambda - 1)} \right) \right] \left[ \dot{W}^2 - \alpha W^2 + \Lambda_W \right] \left[ r \left( -M - \alpha r^2 \right) \tilde{N}'' + M \tilde{N}' \right] = 0 \quad (28)$$

The first term in the above equation vanishes when $\lambda = 1/3, 1/2, 1$. Thus, if $\lambda \neq 1/3, 1/2, 1$, we have either

$$\dot{W}^2 - \alpha W^2 + \Lambda_W = 0, \quad (29)$$

or

$$r \left( -M - \alpha r^2 \right) \tilde{N}'' + M \tilde{N}' = 0. \quad (30)$$

Both equations can be solved analytically.

The solution to Eq. (29) is given by

$$W(z) = C_1 e^{\sqrt{\alpha}z} + C_2 e^{-\sqrt{\alpha}z} \quad \text{with} \quad C_1 C_2 = \frac{\Lambda_W}{4\alpha}. \quad (31)$$

Plugging this into the rest of equations, we find that all of them are satisfied. Thus, the warping function given in Eq. (31) is indeed a solution. Interestingly, this is true for any function of $\tilde{N}(r)$. Therefore, the conformal lapse function $\tilde{N}(r)$ is unconstrained. (A similar situation arises also in the spherically symmetric system investigated in Ref. [22]. This particular feature arises due to the specific choice of coefficients to satisfy the detailed-balance condition.)

For $\Lambda_W < 0$, depending on the signature of $\alpha$, the warp factor in Eq. (31) can be rewritten as follows;

$$W(z) = \begin{cases} \sqrt{-\Lambda_W/\alpha} \sinh \sqrt{\alpha}(z - z_0) & \text{for } \alpha > 0, \\ -\sqrt{-\Lambda_W}(z - z_0) & \text{for } \alpha = 0, \\ \sqrt{\Lambda_W/\alpha} \sin \sqrt{-\alpha}(z - z_0) & \text{for } \alpha < 0. \end{cases} \quad (32)$$

The second solution in Eq. (32) can also be seen by taking the limit of the third one. For $\Lambda_W > 0$, we have the solution only when $\alpha > 0$

$$W(z) = \sqrt{\Lambda_W/\alpha} \cosh \sqrt{\alpha}(z - z_0). \quad (33)$$
By comparing these results with the cases in Einstein gravity in Eqs. (21)-(22), one may identify parameters as
\[ \alpha \to \Lambda_3, \quad \text{and} \quad \Lambda_W \to \Lambda_4/3. \] (34)

Therefore, the warp factors seem to be same both in the Einstein and the Hořava gravity theories. As it was mentioned above, however, the conformal lapse function is unconstrained in Hořava gravity.

Note that Eq. (29) has another type of solutions, namely,
\[ W(z) = \sqrt{\Lambda_W/\alpha}. \] (35)

However, it turns out that there is no \( \tilde{N}(r) \) for which all the rest of equations are satisfied. \(^3\)

Note also that, in the case of \( \Lambda_W = 0 \), \(^4\) the solution \( W(z) \) can be reexpressed as
\[ W(z) = \frac{1}{\sqrt{\alpha}} e^{\pm\sqrt{\alpha}(z-z_0)}. \] (36)

Now let us consider the case that Eq. (30) is satisfied. The solution for this equation is in general given by
\[ \tilde{N}(r) = n_1 \sqrt{-M - \alpha r^2} + n_2, \] (37)

where \( n_1 \) and \( n_2 \) are integration constants. With this conformal lapse function, however, it turns out that there exists no \( W(z) \) for which all remaining equations are satisfied, other than the solutions given in Eq. (30). \(^5\) Therefore, we conclude that the warp factor \( W(z) \) given by Eq. (30) with the conformal lapse function \( \tilde{N}(r) \) unconstrained is the solution to the field equations.

**B. BTZ black-string solution**

In this subsection, we consider the special cases of \( \lambda = 1/3, 1/2, \) and \( 1, \) which were excluded in the previous subsection. For \( \lambda = 1/3, \) the theory itself is not defined well. Thus

\(^2\) The second relation has a factor 2 difference from that in Eq. (15), which is similar to the spherical case in Ref. \( 22. \)

\(^3\) For the value of \( \lambda = 1, \) however, we have a constant warp-factor solution as shall be shown below.

\(^4\) In this limiting case, Hořava gravity does not have the Einstein-Hilbert piece becoming a pure higher-order gravity theory.

\(^5\) If \( \lambda = 1 \) is allowed, there exists a solution of constant warping factor with \( n_2 = 0 \) though.
we do not consider this case. For $\lambda = 1/2$, it turns out that the solution is again the warp factor exactly given in Eq. (31) with the unconstrained conformal lapse function.

For the case of $\lambda = 1$, there exist two classes of solutions. The first class is the same as that in the previous section; the warp factor is given by Eq. (31) with the unconstrained conformal lapse function. Therefore, this class of solutions exists for the Hořava-Lifshitz gravity theory with any value of $\lambda$. In fact, the theory parameter $\lambda$ does not appear in the metric functions at all.

The second class of solutions is given by

$$W(z) = \sqrt{\frac{\Lambda_W}{\alpha}}, \quad \text{and} \quad \tilde{N}^2(r) = -M - \alpha r^2. \quad (38)$$

Here, an integration constant was absorbed by rescaling the $t$-coordinate. The conformal lapse function $\tilde{N}(r)$ is not unconstrained, but is determinative in this case. Note that $\alpha$ must be negative (positive) if $\Lambda_W$ is negative (positive). In other words, when the four dimensional cosmological constant $\Lambda_W$ is negative, an effective three dimensional cosmological constant of positive value $\alpha$ is not allowed. This property differs from that of the previous solutions given in Eq. (31) in which both vanishing and positive effective three dimensional cosmological constants were allowed.

In order to see how the value of $\lambda = 1$ is picked out, one may assume that the warping factor is constant, i.e., $W(z) = W_c$. The solution to Eq. (A.11) is given by

$$W_c^2 = \frac{-1 \pm \sqrt{2(3\lambda - 1)\Lambda_W}}{2\lambda - 1} \alpha. \quad (39)$$

With this value of $W_c$ the equation $E_{rr} = 0$ for the conformal lapse function can easily be solved, giving

$$\tilde{N}^2(r) = -M - \alpha r^2. \quad (40)$$

The metric functions $f(r)$, $\tilde{N}(r)$, and, $W_c$ obtained above are the solutions to the remaining components of the field equation $E_{ij} = 0$, while the left-hand side of $E_{zz} = 0$ equation does not vanish in general, but it is proportional to $(\lambda - 1)$ for the case of plus sign in Eq. (39). Therefore, the solution exists only for $\lambda = 1$, and it becomes,

$$ds^2 = \frac{\alpha}{\Lambda_W} \left[ - (M - \alpha r^2) \, dt^2 + \frac{dr^2}{M - \alpha r^2} + r^2 d\theta^2 + dz^2 \right]. \quad (41)$$

After rescaling the coordinates, $(T, R, Z) \equiv \sqrt{\alpha/\Lambda_W}(t, r, z)$, the solution becomes

$$ds^2 = - (M - \Lambda_W R^2) \, dT^2 + \frac{dR^2}{M - \Lambda_W R^2} + R^2 d\theta^2 + dZ^2. \quad (42)$$
Note that the cosmological constant is not the four dimensional one rather than the three dimensional one. As it was mentioned below Eq. (15), since $\lambda = 1$ now, $\Lambda_W < 0$. (For the model with the analytic continuation, $\Lambda_W > 0$.) If $M > 0$, the above solution represents a BTZ black string which possesses a translational symmetry along the string direction. If $M < 0$, the geometry is static everywhere.

The BTZ black string solution with a translation symmetry is very unique in Hořava gravity. This type of solution is not possible in Einstein gravity. The reason why this type of solution is possible can be explained by considering the effective-matter stress of the higher-order terms in the action. When there are only Einstein-Hilbert terms, the spacetime must be isotropic in the presence of a cosmological constant. The components of the effective-stress tensor $T_{ij}$ evaluated from the higher-order terms $E^{(4)}_{ij}$ to $E^{(6)}_{ij}$ in Eqs. (A.8)-(A.10) have a relation,

$$T_r^r = T_\theta^\theta = -\frac{1}{3}T_z^z = -\frac{1}{3}\hat{T}_i^i,$$

where $\hat{T}_i^i$ is the effective-stress tensor given solely by the cosmological constant. The $T_r^r$ and $T_\theta^\theta$ components reduce the effect of the cosmological constant in the transverse directions, while $T_z^z$ component adds the effect on the longitudinal direction. As a result, the components of the total effective-stress tensor, $T_j^i = T_j^i + \hat{T}_j^i$, arrange in such a way that the translational symmetry is possibly restored,

$$T_r^r = T_\theta^\theta = \frac{1}{3}T_z^z.$$

We can make coordinate transformations further by rescaling $(\tau, \rho) \equiv (\sqrt{|M|}|T, R/\sqrt{|M|})$, then the metric becomes

$$ds^2 = -(\pm 1 - \Lambda_W \rho^2) d\tau^2 + \frac{d\rho^2}{\pm 1 - \Lambda_W \rho^2} + |M|\rho^2 d\theta^2 + dZ^2,$$

where the upper sign stands for the $M < 0$ case. This metric exhibits the role of the 3D mass-density parameter $M$ (dimensionless); the transverse geometry is conical. There exists a deficit angle $\Delta = 2\pi(1 - \sqrt{|M|})$ when $|M| < 1$. Note that $M$ is an integration constant of which scale is not limited by the theory. Therefore, when $|M| > 1$, the angle $\Delta$ becomes negative, which implies a “surplus angle”.  

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6 This transformation may be regarded as setting the integration constant $\alpha$ to be $\alpha = \Lambda_W$.
7 For the case of analytic continuation ($\Lambda_W > 0$), the solution with $M > 0$ represents a nonstatic geometry everywhere, and the one with $M < 0$ represents a symmetrically translated $dS_3$ along the $z$-direction.
8 A similar situation arises for the monopole in Hořava gravity investigated in Ref. [17].
IV. CONCLUSIONS

We searched for string-like solutions in four dimensions based on Hořava-Lifshitz gravity. For a restricted class of solutions where the Cotton-tensor vanishes, we found that there exist two types of solutions. The first type of the solutions is warped along the string string direction, and the conformal lapse function is not constrained. The well-known warped string-like solutions in Einstein gravity including the warped BTZ black strings, are the solutions of this type in Hořava–Lifshitz gravity. In other words, the solutions in Einstein gravity become the solutions in Hořava–Lifshitz gravity, but the reverse is not true in general since the lapse function is not specified. The parameter $\lambda$ introduced in Hořava–Lifshitz gravity does not appear in the solution functions at all. For $\lambda \neq 1$, there exists no other type of solutions than this one.

The second type of solutions exists additionally only for $\lambda = 1$. In this case, the conformal lapse function is determined. This solution is uniform along the string direction. Interestingly, this type of solutions allows a uniform BTZ black string which is absent in Einstein gravity. The higher-derivative terms specifically chosen by the detailed-balance condition in the Hořava–Lifshitz theory makes this type of solutions possible. Unlike the spherical case studied in Ref. [22], the class of $\lambda$-dependent solutions does not exist in the axial case.

It is interesting to see if similar properties hold in the Hořava-Lifshitz gravity theory in spacetime dimensions higher than four. The existence of a uniform black string even in the presence of a bulk cosmological constant is particularly interesting. This uniformity is highly nontrivial. In the four dimensional point of view, this is a good candidate to test the theory of Hořava-Lifshitz gravity. In the context of brane world model it would be very interesting because the warped geometry is essential to have the Newtonian gravity on a brane.

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APPENDIX

The equations of motion for the action (11) are derived by variation. The equation from the variation of the lapse function, $\delta N$, is given by

$$
\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1 - 3\lambda)} + \frac{\kappa^2\mu^2(1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2w^4} Z_{ij}Z^{ij} = 0, \quad (A.1)
$$

where

$$
Z_{ij} \equiv C_{ij} - \frac{\mu w^2}{2} R_{ij}. \quad (A.2)
$$

The equation from the variation of the shift function, $\delta N^i$, is given by

$$
\nabla_k(K^{k\ell} - \lambda Kg^{k\ell}) = 0. \quad (A.3)
$$

The equations of motion from the variation $\delta g^{ij}$ are given by

$$
E_{ij} = \frac{2}{\kappa^2} E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2} E_{ij}^{(2)} + \frac{\kappa^2\mu^2\Lambda_W}{8(1 - 3\lambda)} E_{ij}^{(3)} + \frac{\kappa^2\mu^2(1 - 4\lambda)}{32(1 - 3\lambda)} E_{ij}^{(4)} - \frac{\mu^2}{4w^2} E_{ij}^{(5)} - \frac{\kappa^2}{2w^4} E_{ij}^{(6)}, \quad (A.4)
$$

where

$$
E_{ij}^{(1)} = N_i \nabla_k K^{kj} + N_j \nabla_k K^{ki} - K^k_i \nabla_j N_k - K^k_j \nabla_i N_k - N^k \nabla_k K_{ij}, \quad (A.5)
$$

$$
E_{ij}^{(2)} = \frac{1}{2} N K^k g_{ij} + N_i \partial_j K + N_j \partial_i K - N^k(\partial_k K)g_{ij} + \dot{K}g_{ij}, \quad (A.6)
$$

$$
E_{ij}^{(3)} = N(R_{ij} - \frac{1}{2} R g_{ij} + \frac{3}{2} \Lambda_W g_{ij}) - (\nabla_i \nabla_j - g_{ij} \nabla_k \nabla^k)N, \quad (A.7)
$$

$$
E_{ij}^{(4)} = NR(2R_{ij} - \frac{1}{2} R g_{ij}) - 2(\nabla_i \nabla_j - g_{ij} \nabla_k \nabla^k)(NR), \quad (A.8)
$$

$$
E_{ij}^{(5)} = \nabla_k [\nabla_j(NZ^k) + \nabla_i(NZ^k)] - \nabla_k \nabla^k(NZ_{ij}) - \nabla_k \nabla^\ell(NZ^{k\ell})g_{ij}, \quad (A.9)
$$

$$
E_{ij}^{(6)} = -\frac{1}{2} N Z_{k\ell}Z^{k\ell}g_{ij} + 2NZ_{ik}Z^k - N(Z_{ik}C^k_j + Z_{jk}C^k_i) + N Z_{k\ell}C^{k\ell} g_{ij} \quad (A.10)
$$

$$
-\frac{1}{2} \nabla_k [N\epsilon^{m\ell}(Z_{mi}R_{j\ell} + Z_{mj}R_{i\ell})] + \frac{1}{2} R_{n\ell} \nabla_n [N\epsilon^{m\ell}(Z_{mi}g_{kj} + Z_{mj}g_{ki})]
$$

$$
-\frac{1}{2} \nabla_n [N Z_m \epsilon^{m\ell}(g_{kj}R_{i\ell} + g_{kj}R_{i\ell})] - \frac{1}{2} \nabla_n \nabla_k [N\epsilon^{m\ell}(Z_{mi}g_{j\ell} + Z_{mj}g_{i\ell})]
$$

$$
+ \frac{1}{2} \nabla_n [\nabla_i \nabla_k(N Z_m \epsilon^{m\ell})g_{j\ell} + \nabla_j \nabla_k(N Z_m \epsilon^{m\ell})g_{i\ell}]
$$

$$
+ \frac{1}{2} \nabla_i \nabla_j(N Z_m \epsilon^{m\ell}) - \nabla_n \nabla_i \nabla_k(N Z_m \epsilon^{m\ell})g_{ij}. \quad (A.10)
$$
By plugging the function $f(r) = -M - \alpha r^2$ into the above equations with $N(r, z) = \tilde{N}(r)/W(z)$, the constraint equation (A.11) associated with lapse function becomes

$$\frac{\kappa^2 \mu^2}{8(3\lambda - 1)} \left[ \alpha^2(2\lambda - 1)W^4 - 3(W^2 + \Lambda W)^2 - 4\alpha\lambda W^3\dot{W} + 4W(\dot{W}^2 + \Lambda W)\ddot{W} 
+ 2W^2 \left[ \alpha(\dot{W}^2 + \Lambda W) + (\lambda - 1)\dot{W}^2 \right] \right] = 0.$$  (A.11)

The nonvanishing components in Eq. (A.4) become

$$E_{rr} \times \left( \frac{\kappa^2 \mu^2 \tilde{N}}{16(3\lambda - 1)(-M - \alpha r^2)W^3} \right)^{-1}$$

$$= (-M - \alpha r^2)\tilde{N}' \frac{2W^2}{r\tilde{N}} \left[ -\Lambda W - \alpha(2\lambda - 1)W^2 - \dot{W}^2 + 2\lambda W\ddot{W} \right]$$

$$- \alpha^2(2\lambda - 1)W^4 - 3 \left( \Lambda W + \dot{W}^2 \right)^2 + 4W \left( \Lambda W - (2\lambda - 3)\dot{W}^2 \right) \ddot{W}$$

$$+ 2(\lambda - 1)W^2 \left[ \alpha\dot{W}^2 + 3\dot{W}^2 + 3\dot{W}\ddot{W} \right] - 2(\lambda - 1)W^3 \left( \alpha\dddot{W} + \dddot{W} \right) = 0,$$  (A.12)

$$E_{\theta\theta} \times \left( \frac{\kappa^2 \mu^2 r^2 \tilde{N}}{16(3\lambda - 1)W^3} \right)^{-1}$$

$$= (-M - \alpha r^2)\tilde{N}'' - \alpha r\tilde{N}' \frac{2W^2}{\tilde{N}} \left[ -\Lambda W - \alpha(2\lambda - 1)W^2 - \dot{W}^2 + 2\lambda W\ddot{W} \right]$$

$$- \alpha^2(2\lambda - 1)W^4 - 3 \left( \Lambda W + \dot{W}^2 \right)^2 + 4W \left( \Lambda W - (2\lambda - 3)\dot{W}^2 \right) \ddot{W}$$

$$+ 2(\lambda - 1)W^2 \left[ \alpha\dot{W}^2 + 3\dot{W}^2 + 3\dot{W}\ddot{W} \right] - 2(\lambda - 1)W^3 \left( \alpha\dddot{W} + \dddot{W} \right) = 0,$$  (A.13)

$$E_{zz} \times \left( \frac{\kappa^2 \mu^2 \tilde{N}}{16(3\lambda - 1)W^3} \right)^{-1}$$

$$= - \frac{r(-M - \alpha r^2)\tilde{N}'' - (M - 2\alpha r^2)\tilde{N}'}{r\tilde{N}} \frac{2W^2}{\tilde{N}} \left[ \Lambda W - \alpha\lambda W^2 + \dot{W}^2 + (\lambda - 1)W\ddot{W} \right]$$

$$+ \alpha^2(2\lambda - 1)W^4 - 3 \left( \Lambda W + \dot{W}^2 \right)^2 - 8(\lambda - 1)W\dot{W}^2\ddot{W}$$

$$+ 2W^2 \left[ \alpha\Lambda W + \alpha(2\lambda - 1)\dot{W}^2 - (\lambda - 1)\dddot{W}^2 + 2(\lambda - 1)\dot{W}\ddot{W} \right] = 0,$$  (A.14)

$$E_{r\theta} \times \left( \frac{\kappa^2 \mu W}{8w^2 \sqrt{-M - \alpha r^2}} \right)^{-1} = \left[ r(-M - \alpha r^2)\tilde{N}'' + MW \right] \left[ \ddot{W} - \alpha\dot{W} \right] = 0,$$  (A.15)

$$E_{rz} \times \left( \frac{\kappa^2 \mu^2}{8(3\lambda - 1)W} \right)^{-1} = (\lambda - 1)\tilde{N}' \left[ -2\dot{W}\ddot{W} + W \left( \alpha\dot{W} + \dddot{W} \right) \right] = 0,$$  (A.16)
\[ E_{\theta z} \times \left( -\kappa^2 \mu W \sqrt{M - \alpha r^2} \right)^{-1} \]
\[ = \left[ r^2 (-M - \alpha r^2) \tilde{N}'' + r (-M - 4\alpha r^2) \tilde{N}' + M \tilde{N}' \right] \left( \tilde{W} - \alpha \tilde{W} \right) = 0, \quad (A.17) \]

By subtracting Eq. (A.12) from Eq. (A.13), one obtains
\[ \left[ r (-M - \alpha r^2) \tilde{N}'' + M \tilde{N}' \right] \left[ -\Lambda_W - \alpha (2\lambda - 1) W^2 - \tilde{W}^2 + 2\lambda W \tilde{W} \right] = 0. \quad (A.18) \]

Therefore, one may solve this equation instead of solving Eq. (A.13) equivalently.

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