LARGE 2-COLOURED MATCHINGS IN 3-COLOURED COMPLETE HYPERGRAPHS

TAMÁS TERPAI

Abstract. We prove the generalized Ramsey-type result on large 2-coloured matchings in a 3-coloured complete 3-uniform hypergraph, supporting a conjecture by A. Gyárfás.

1. Introduction and statement of result

In [3], the authors consider generalisations of Ramsey-type problems where the goal is not to find a monochromatic subgraph, but a subgraph that uses “few” colours. In particular, the following theorem is proven:

Theorem 1 ([3, Theorem 13]). For \( k \geq 1 \), in every 3-colouring of a complete graph with \( f(k) = \left\lceil \frac{2k-1}{3} \right\rceil \) vertices there is a 2-coloured matching of size \( k \). This is sharp for every \( k \geq 2 \), i.e. there is a 3-colouring of \( K_{f(k)-1} \) that does not contain a 2-coloured matching of size \( k \).

The example that shows the sharpness of the estimate is close to the colouring obtained by first colouring the vertices with the available colours in proportion close to 1 : 2 : 4 and then colouring the edges by the lowest index colour among its endpoints. The analogous question and construction make sense in the case of complete hypergraphs instead of \( K_n \). At the 1. Emléktábla workshop held at Gyöngyöstatján in July 2010, the first nontrivial case of this question (with 3-uniform hypergraphs and 3 colours) was considered. The best known construction in this case is based on the proportion 1 : 3 : 9, and leads to the following conjecture by A. Gyárfás:

Conjecture 1.1. For any \( t \)-colouring of the complete \( r \)-uniform hypergraph on \( n \geq kr + \left\lceil \frac{(k-1)(t-s)}{1 + r + \cdots + rs-1} \right\rceil \) vertices, there exists a \( s \)-coloured matching of size \( k \).

While it is known that the conjecture fails for e.g. \( t = 6 \) and \( s = 2 \), several particular cases are open. We consider here only \( t = 3, r = 3 \) and \( s = 2 \), in which case the conjecture has the form

Theorem 2. For any 3-colouring of the complete 3-uniform hypergraph on \( n \geq 3k + \left\lceil \frac{k-1}{4} \right\rceil \) vertices, there exists a 2-coloured matching of size \( k \).

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The case $k = 4$ (the first case that is not a trivial consequence of the existing results for the monochromatic problem, see e.g. [1]) was confirmed at the workshop by a team consisting of N. Bushaw, A. Gyárfás, D. Gerbner, L. Merchant, D. Piguet, A. Riet, D. Vu and the author:

**Theorem 3 ([2])**. For any 3-colouring of the complete 3-uniform hypergraph on 12 vertices there exists a perfect matching that uses at most 2 colours.

In this paper, we prove Theorem 2 in the following equivalent form:

**Theorem 4.** For any 3-colouring of the complete 3-uniform hypergraph on $n$ vertices, there exists a 2-coloured matching of size

$$m(n) = \left\lfloor \frac{4(n+1)}{13} \right\rfloor .$$

It is easy to check that these indeed are formulations of the same result as

$$n = 3k + \left\lfloor \frac{k-1}{4} \right\rfloor = \left\lfloor \frac{13k-1}{4} \right\rfloor$$

is the smallest integer for which \( \left\lfloor \frac{4(n+1)}{13} \right\rfloor \geq k \) holds.

2. Proof

In the proof, the set of vertices of the hypergraph will be denoted by $V$, the colouring will be a function $c : \binom{\mathbb{V}}{3} \rightarrow \{1, 2, 3\}$, and $\alpha, \beta, \gamma$ will be an arbitrary permutation of the colours 1, 2, 3. The colours are shifted cyclically, e.g. if $\alpha = 3$, then $\alpha + 1$ denotes the colour 1 and if $\alpha = 1$, then $\alpha 1$ denotes the colour 3. A matching on $n$ vertices is near perfect, if it has size $\left\lfloor \frac{n}{3} \right\rfloor$.

We call a sextuple $A$ of points $\alpha$-dominated for a colour $\alpha = 1, 2, 3$, if for all splittings $A = B_1 \cup B_2$ into two disjoint triples at least one of $c(B_1) = \alpha$ or $c(B_2) = \alpha$ holds. If $A$ is not $\alpha$-dominated for any $\alpha$, we call it universal. Similarly, we call a set $X$ of 13 points universal if it admits a near perfect matching in any pair of colours.

The proof proceeds by taking a maximal set of disjoint universal sets $A_1, \ldots, A_i, X_1, \ldots, X_m$ with $|A_i| = 6$ and $|X_j| = 13$. If we can now construct a 2-colour matching on $W = V \setminus (A_1 \cup \cdots \cup X_m)$ of the size $m(|W|)$, then we can extend it by the appropriately coloured near perfect matchings in the universal sets $A_i$ and $X_j$ and keep the size of the matching at least $m(|V|)$. Indeed, in the case of an $A_i$ decreasing $n$ by 6 decreases $m(n)$ by at most $\left\lfloor 4 \cdot 6/13 \right\rfloor = 2$ and in the case of an $X_j$ decreasing $n$ by 13 decreases $m(n)$ by 4. Thus by switching to $W$ we may assume that there are no universal sets of size 6 or 13, and the resulting structural properties of the colouring will give us the necessary large 2-colour matching.

If a vertex sextuple is $\alpha$-dominated, and there are splittings of it into hyperedges of colours $\alpha$ and $\alpha + 1$ as well as into those of colours $\alpha$ and $\alpha + 2$, we call this sextuple a spread in colour $\alpha$, and the splittings are its demonstration splittings. Depending of whether the hyperedges of colour $\alpha$ in the demonstration splittings overlap in 1 or 2 vertices, we assign the spread (with a fixed demonstration splitting implied) a level of 1 or 2 respectively.

**Lemma 4.1.** Assume that there are two disjoint spreads $A$ in colour $\alpha$ and $B$ in colour $\beta$ such that $\alpha \neq \beta$, and let $v$ be an arbitrary vertex in the complement
Figure 1. If the substitution of a single vertex $v$ can change the colour of a dominant triple in two spreads of different colour, we have a universal set of size 13. Example: case when $c(E_1) = \beta$ and $c(E_2) = \alpha$.

$V \setminus (A \cup B)$ of their union. Then the following property holds for $X = A$ or for $X = B$ (or both): if we substitute $v$ for any vertex of the dominantly coloured triple in either demonstration splitting of the spread $X$, the colour of that triple stays the same, the dominating colour of $X$.

Proof. Indirectly assume that $A = M_1 \cup E_1$ and $B = M_2 \cup E_2$ are splittings which $v$ “spoils”. That is, $M_1$ has the dominant colour $\alpha$ of $A$ and $M_1 \cup \{v\}$ contains a triple $F_1$ of colour different from $c(M_1) = \alpha$, and analogously $c(M_2) = \beta$ and $M_2 \cup \{v\}$ contains a triple $F_2$ of colour different from $\beta$ (see example on Figure 1). Then $A \cup B \cup \{v\}$ is a universal set of size 13. Indeed, it has a matching of size 4 that contains only colours $\alpha$ and $\beta$ since both $A$ and $B$ admit perfect matchings in these colours. It also has a matching of size 4 that avoids the colour $\alpha$: the spread $B$ has such a matching of size 2, the triple $E_1$ has a colour different from $\alpha$, and the remainder $M_1 \cup \{v\}$ contains $F_1$, also a triple of a colour different from $\alpha$. The same argument with $A$ and $B$ reversed produces a near perfect matching that avoids the colour $\beta$, proving our claim and arriving at the contradiction that proves the lemma.

This coupling property implies a very rigid structure of the colouring:

**Proposition 4.1.** If there is a pair of disjoint spreads in two different colours, then there is a nearly perfect matching avoiding one colour.

Proof. Without loss of generality we may assume that the two colours are 1 and 2. Let the two spreads be $A^{(1)}$ (colour 1) and $A^{(2)}$ (colour 2), and out of all disjoint pairs of spreads of colours 1 and 2 this one contains the most level 2 spreads. Then each of them is either level 2 or it is level 1 and there are no level 2 spreads of their colour that would be disjoint from the other spread.

In both $A^{(1)}$ and $A^{(2)}$, fix two demonstration splittings

$$A^{(i)} = M_+^{(i)} \cup P^{(i)} = M_-^{(i)} \cup N^{(i)}$$

such that $c(M_+^{(i)}) = c(M_-^{(i)}) = i$, $c(P^{(i)}) = i + 1$ and $c(N^{(i)}) = i - 1$. Depending on the level of $A^{(i)}$, we can label the vertices of $A^{(i)} = \{a_1^{(i)}, \ldots, a_6^{(i)}\}$ to satisfy the following equalities:
That is, $\hat{A}^i$ is bounded by $D$ sets exist (for example, following properties:

- The vertex set $A$ there is a vertex $\hat{A}$ for the spreads $\hat{A}$ for $\hat{A}$ we have by definition of $\hat{A}$ the property that $z$ forms triples of colour $i$ with all the critical vertex pairs of $C \cup \{x, y\}$, in particular, with $\{x, y\}$, and we are done.

Proof. We suppress for brevity the indices $i$. If $A$ is level 2, then for any $\{x, y, z\} \subseteq A$ we have by definition of $\hat{A}$ the property that $z$ forms triples of colour $i$ with all the critical vertex pairs of $C \cup \{x, y\}$, in particular, with $\{x, y\}$, and we are done.

We will call the vertex $v_1^i$ the dominating vertex and the rest of the vertices the core vertices.

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In both cases, $D^{(i)}$ will denote the set of the dominating vertices and $C^{(i)}$ will denote the set of the core vertices. A pair of vertices will be called critical, if they are contained in either $M_+^{(i)}$ or $M_-^{(i)}$.

We choose sets $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ to be a maximal disjoint pair of sets satisfying the following properties:

- $D^{(i)} \subseteq \hat{A}^{(i)} \subseteq V \setminus (C^{(i)} \cup A^{(3-i)})$ for $i = 1, 2$.
- For any subset $D$ of $\hat{A}^{(i)}$ of size $|D| = |D^{(i)}|$, the triples of the sextuple $C^{(i)} \cup D = (A^{(i)} \setminus D^{(i)}) \cup D$ complementary to $P^{(i)}$ and $N^{(i)}$ have colour $i$.
- For any subset $D$ of $\hat{A}^{(i)}$ of size $|D| = |D^{(i)}|$, any pair of vertices $(u, v) \in V$ that is covered by the complement of either $P^{(i)}$ or $N^{(i)}$ in the sextuple $C^{(i)} \cup D = (A^{(i)} \setminus D^{(i)}) \cup D$, and any vertex $w \in \hat{A}^{(i)} \setminus D$, the triple $\{u, v, w\}$ has colour $i$.

That is, $\hat{A}^{(i)}$ is a maximal set of vertices (outside of $A^{(3-i)}$) extending the set of dominant vertices of $A^{(i)}$ such that we can switch the dominant vertices of $A^{(i)}$ with any two vertices of $\hat{A}^{(i)}$ and still be unable to change the colour of the dominant triples of the modified sextuple by a single vertex change within the set $\hat{A}^{(i)}$. Such sets exist (for example, $D^{(i)}$ satisfies the requirements for $A^{(i)}$), and their total size is bounded by $|V|$, so we can choose a maximal pair.

We claim that the sets $\hat{A}^{(1)} \cup \hat{A}^{(2)}$ cover $V \setminus (C^{(1)} \cup C^{(2)})$. Indeed, assume that there is a vertex $w \in V \setminus \left( C^{(1)} \cup C^{(2)} \cup \hat{A}^{(1)} \cup \hat{A}^{(2)} \right)$ such that it cannot be added to either $\hat{A}^{(1)}$ or $\hat{A}^{(2)}$ without violating their defining properties. This means that for $i = 1, 2$ we can switch the vertices in $D^{(i)}$ to some other vertices in $\hat{A}^{(i)}$ in such a way that for the resulting spread $\hat{A}^{(i)}$ there is a pair of vertices $(u^{(i)}, v^{(i)})$ of a dominating triple such that $c(u^{(i)}, v^{(i)}, w) \neq i$.

This contradicts Lemma 4.1 for the spreads $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ and the vertex $w$.

Additionally, these sets are already easy to colour with 2 colours:

**Lemma 4.2.** The vertex set $\hat{A}^{(i)}$ is a clique of colour $i$.

Proof. We suppress for brevity the indices $i$. If $A$ is level 2, then for any $\{x, y, z\} \subseteq A$ we have by definition of $\hat{A}$ the property that $z$ forms triples of colour $i$ with all the critical vertex pairs of $C \cup \{x, y\}$, in particular, with $\{x, y\}$, and we are done.
If $A$ is level 1, recall first that we also assume that there are no spreads of colour $i$ and level 2 in $A \cup C$. Indirectly assume furthermore that there is a triple $X = \{x_1, x_2, x_3\} \subseteq \hat{A}$ such that its colour is not $i$. For symmetry reasons it is enough to check the case when $c(X) = i + 1$. Then $P \cup X$ is covered by two disjoint triples of colour $i + 1$ and must be therefore $i + 1$-dominated - otherwise it would form a universal sextuple contrary to our assumptions. But $P \setminus N = \{v_4, v_5\}$ is a critical pair of vertices and hence $\{x_1, v_4, v_5\}$ has colour $i$; therefore its complement $Y = \{x_2, x_3, v_6\}$ has colour $i + 1$. This implies that the sextuple $X \cup N = Y \cup \{x_1, v_2, v_3\}$ can be split into colours $c(Y) = i + 1$ and $c(N) = i - 1$ as well as into colours $c(Y) = i + 1$ and $c(\{x_1, v_2, v_3\}) = i$ (the set $\{v_2, v_3, x_1\}$ is the complement of $P$ in $C \cup \{x_1\}$ with $x_1 \in \hat{A}$), so this sextuple is $i + 1$-dominated.

Now use the fact that $\{x_2, v_3\}$ is a critical pair of vertices in $C \cup \{x_2\}$ (it lies in the complement of $P$). This implies that $c(\{x_2, x_3, v_3\}) = i$, and consequently its complement in $X \cup N$ has colour $i + 1$:

$$c(\{x_1, v_2, v_6\}) = i + 1.$$ 

By definition of $\hat{A}$, the sextuple $\{x_1\} \cup C$ is $i$-dominant as it cannot be dominant in any other colour. Hence the complement of $\{x_1, v_2, v_6\}$ in it has to have colour $i$:

$$c(\{v_3, v_4, v_5\}) = i.$$ 

Also, $\{v_4, v_5\}$ is a critical pair of vertices, so we have

$$c(\{x_1, v_4, v_5\}) = i$$

as well. But this means that $C \cup \{x_1\}$ is a level 2 spread of colour $i$ as evidenced by splitting into $\{x_1, v_4, v_5\} \cup N$ (colours $i$ and $i - 1$ respectively) and into $\{v_3, v_4, v_5\} \cup \{x_1, v_2, v_6\}$ (colours $i$ and $i + 1$ respectively) - a contradiction with our initial assumption, hence $\hat{A}$ is indeed a clique of colour $i$ as claimed.

This also implies that $\hat{A}^{(1)} \cup M^{(1)}_1$ is a clique of colour 1 and $\hat{A}^{(2)} \cup M^{(2)}_1$ is a clique of colour 2 (we are adding a vertex or a critical pair of vertices to the appropriate $\hat{A}^{(i)}$). Notice that their complement is the union of the 2-coloured hyperedge $P^{(1)}$ and the 1-coloured hyperedge $N^{(2)}$.

Lemma 4.3. If $U$ and $W$ are disjoint cliques of colours 1 and 2 respectively such that $|U| \geq 3$ and $|W| \geq 3$, then there exists an almost perfect matching in $U \cup W$ in colours 1 and 2.

Proof. If $|U| + |W| \mod 3 = |U| \mod 3 + |W| \mod 3$, that is, $|U| \mod 3 + |W| \mod 3 \leq 2$, then taking maximal disjoint sets of hyperedges in $U$ and $W$ separately gives an almost perfect matching in colours 1 and 2.

If this is not the case, then both $|U| \mod 3$ and $|W| \mod 3$ are at least 1 and at least one of them is equal to 2; without loss of generality, we may assume that $|U| \equiv 2 \mod 3$. We claim that there is a hyperedge $E \subseteq U \cup W$ of colour 1 or 2 with the property that $|U \cap E| = 2$. Indeed, assume indirectly that all triples intersecting $U$ in 2 vertices and $W$ in 1 vertex have colour 3. Since $|U| \geq 3$ and $|U| \mod 3 = 2$, we have $|U| \geq 5$. Consider any four distinct vertices $u_1, u_2, u_3, u_4 \in U$ and any two distinct vertices $w_1, w_2 \in W$. Then the set $X = \{u_1, u_2, u_3, u_4, w_1, w_2\}$ is covered by the triples $\{u_1, u_2, w_1\}$ and $\{u_3, u_4, w_3\}$, both of which have to have colour 3. Hence $X$ can only be 3-dominated, consequently at least one of the members of the matching $\{u_1, w_1, w_2\} \cup \{u_2, u_3, u_4\}$ has colour 3. But the triple $\{u_2, u_3, u_4\}$ lies in
Figure 2. If there are no spreads, then only two colours may be used.

the clique \( U \) and therefore has colour 1, so \( c(\{u_1, w_1, w_2\}) = 3 \). This implies that for any choice of a vertex \( w_3 \in W \setminus \{w_1, w_2\} \) we have on one hand
\[
c(\{u_1, w_1, w_2\}) = 3 \quad \text{and} \quad c(\{u_2, u_3, w_3\}) = 3
\]
due to the latter triple intersecting \( U \) in 2 vertices, and on the other hand
\[
c(\{u_1, u_2, u_3\}) = 1 \quad \text{and} \quad c(\{w_1, w_2, w_3\}) = 2
\]
due to \( U \) and \( V \) being cliques. Hence \( \{u_1, u_2, u_3, w_1, w_2, w_3\} \) would be a universal sextuple, a contradiction that proves our claim.

Given a hyperedge \( E \subset U \cup W \) of colour 1 or 2 with the property that \( |U \cap E| = 2 \), we can just add it to the union of any maximal matching of \( U \setminus E \) and any maximal matching of \( W \setminus E \) to get a nearly perfect matching of \( U \cup W \) in colours 1 and 2.

\[\square\]

Applying Lemma 4.3 to the cliques \( \tilde{A}^{(1)} \cup M^{(1)}_+ \) and \( \tilde{A}^{(2)} \cup M^{(2)}_- \) and adding the triples \( P^{(1)} \) and \( N^{(2)} \) yields a near perfect matching in colours 1 and 2 on \( V \). This finishes the proof of Proposition 4.1.

\[\square\]

Once we can exclude two disjoint spreads of different colours, we have two possibilities: either there are no spreads at all, or there is a spread of, say, colour 1, and any spread in its complement is also of colour 1. We will also assume that \( |V| \geq 9 \) as otherwise the 2-colour condition is trivially fulfilled by any near perfect matching.

**Case 1: there are no spreads.** If there are no spreads, then no sextuple can contain triples of all three colours: one of them would be dominating, and any two instances of the other two colours could be chosen to be \( P \) and \( N \) of a spread. We will first look for a pair of triples of different colours that share two vertices, \( c(A) \neq c(B) \), \( |A \cap B| = 2 \). If there are no such pairs, then all triples have the same colour and any nearly perfect matching is monochromatic, we are done. If, on the other hand, such triples \( A \) and \( B \) exist, we may assume without loss of generality that \( c(A) = 1 \) and \( c(B) = 2 \). Could there be triples of colour 3 (see Figure 2)? Any such triple \( C \) would have to be disjoint from \( A \cup B \), because otherwise their union \( A \cup B \cup C \) (together with any other vertex if it has only 5 elements) would form a sextuple of vertices that contains all the three colours. Then for any vertex \( v \in C \) the triple \( T = (A \cap B) \cup \{v\} \) is covered by both \( A \cup C \) (covering only triples of colour 1 and 3) and \( B \cup C \) (covering only triples of colour 2 and 3) and therefore
can only be of colour 3. But then $A \cup B \cup \{v\}$ together with any other vertex form a sextuple that contains triples of all three colours, $A$, $B$ and $T$ - a contradiction.

Therefore in this case only two colours may be used at all, so any near perfect matching automatically satisfies our desired condition.

**Case 2:** there exists a spread (of colour 1, say). We first investigate what happens if there are no spreads of other colour at all. This results in a highly ordered structure:

**Proposition 4.2.** If a colouring is such that all spreads are of colour 1, then either
- there exists a near perfect matching avoiding colour 2 or colour 3, or
- there are no triples of colour 1 at all.

**Proof.** First note that the condition on the spreads means that whenever a sextuple contains triples of all three colours, it is 1-dominated. In particular, if a triple is covered by a disjoint union of a 2-coloured and a 3-coloured triple, it cannot have colour 1 - the union in question can only be 2- or 3-dominated. We show that this statement can also be used for non-disjoint pairs of triples of colours 2 and 3.

**Lemma 4.4.** Assume that all spreads in the colouring are of colour 1. Then either
- there are no triples of colour 1 covered by the union of a triple of colour 2 and a triple of colour 3, or
- there exists a near perfect matching avoiding colour 2 as well as one avoiding colour 3.

**Proof.** Assume $A = \{v_1, v_2, v_3\}$ is a colour 1 triple that is covered by triples $B$ and $C$ of colours 2 and 3 respectively. At least one of these has to cover 2 vertices of $A$, so after a renumbering of colours, triples and vertices we may assume that $B = \{v_2, v_3, v_4\}$ and $v_1 \in C$. We now have three cases for the situation of $C$ with respect to $A$ and $B$:

1. $C \cap B = \emptyset$. Then $B$ and $C$ are disjoint triples of colour 2 and 3 respectively which cover $A$, a triple of colour 1 - a contradiction.
2. $C \cap B = \{v_2, v_4\}$ (or analogously $\{v_3, v_4\}$): that is, $C$ is covered by $A \cup B$ (see Figure 3). The union $A \cup B = A \cup B \cup C$ has 4 elements and contains all three colours, so adding any pair of vertices $x$, $y$ makes it a 1-dominated sextuple. In this sextuple, the triples $\{x, y, v_1\}$ and $\{x, y, v_3\}$ have non-1-coloured complements, so they have to have colour 1 themselves. Now assume there is a triple $D = \{w_1, w_2, w_3\}$ of colour 2 disjoint from $A \cup B$ (the case of $c(D) = 3$ is similar). Then $D \cup C$ is a disjoint union of a
2-coloured triple and a 3-coloured one, and it covers the 1-coloured triple \( \{ w_1, w_2, v_1 \} \) - a contradiction. Hence all triples disjoint from \( A \cup B \) have colour 1. Consequently we can choose a near perfect matching either in colour 1 only, or at will in colours 1 and 2, or in colours 1 and 3 - if the total number of vertices is congruent to 1 or 2 modulo 3, we take a near perfect matching in the complement of \( A \cup B \) and add \( A \), otherwise we take a near perfect matching in the complement of \( A \cup B \), add the triple \( B \) or \( C \) depending on which colour out of 2 and 3 is wanted and match up the remaining two vertices with either \( v_1 \) or \( v_3 \) (whichever is left out).

(3) \(| C \cap B | = 1 \); let \( w \) denote the single vertex in \( C \setminus (A \cup B) \) (see Figure 4). By the same argument as before, for any vertex \( x \) not in \( A \cup B \cup C \) we have that \( A \cup B \cup C \cup \{ x \} \) is 1-dominated, so the complements of the non-1-coloured triples \( B \) and \( C \) must have colour 1:

\[
\begin{align*}
c(\{ x \} \cup ((A \cup B \cup C) \setminus B)) &= 1, \\
c(\{ x \} \cup ((A \cup B \cup C) \setminus C)) &= 1.
\end{align*}
\]

This makes it impossible to have triples of colour other than 1 disjoint from \( A \cup B \cup C \), as they would cover a 1-coloured triple together with either \( B \) or \( C \) (whichever has the colour other from that of the selected triple). Now taking a maximal matching outside \( A \cup B \cup C \), we can extend it to a near perfect matching avoiding the colour 2 or the colour 3 as follows. If there are no vertices left outside the matching, add \( A \) to get a 1-coloured matching. If there is 1 vertex left, join it to \( (A \cup B \cup C) \setminus B \) and add \( B \) to avoid the colour 3; do the same with \( B \) and \( C \) switched to avoid the colour 2. Finally, if there are 2 vertices left, join them respectively to the disjoint vertex pairs \( (A \cup B \cup C) \setminus B \) and \( (A \cup B \cup C) \setminus C \) to obtain a matching of colour 1.

\[\square\]

Thus we may restrict our attention to the case when the union of a triple of colour 2 and a triple of colour 3 cannot cover a triple of colour 1, even if they are not disjoint.

We now try to find a vertex such that all triples containing it are of colour 1; we will call such a vertex 1-\textit{forcing}. If there are no triples of colours 1 and 2 or 1 and 3 such that they intersect in two vertices, then either there are no triples of colour 1 - in which case there is a near perfect matching in colours 2 and 3 - or there are no triples of colour different from 1 - in which case there is a near perfect matching in colour 1. Hence we might assume that there is a pair of triples of the form \( A = \{ v_1, v_2, v_3 \} \), \( B = \{ v_2, v_3, v_4 \} \) with \( c(A) = 1 \) and \( c(B) = 2 \), say. By our

![Figure 4. Case \(| C \cap B | = 1 \).](image-url)
previous lemma there are no triples of colour 3 that contain \( v_1 \). If there are no such triples disjoint from \( A \cup B \) either, then any near perfect matching containing \( A \) or \( B \) will be 1 and 2-coloured. Assume therefore that there is a triple \( C \) of colour \( c(C) = 3 \) in the complement of \( A \cup B \). No triple covered by \( B \cup C \) can have colour 1, in particular the triple \( D = \{v_2, v_3, x\} \) with some \( x \in C \) has to have colour 2 or 3. If \( c(D) = 3 \), then we can repeat the argument with \( A \) and \( D \) instead of \( A \) and \( B \) to get that no colour 2 triples contain \( v_1 \) either, so \( v_1 \) is 1-forcing. If \( c(D) = 2 \), then \( A \cup C \) contains triples of all three colours and thus is 1-dominated, in particular the triple \( E = \{v_1\} \cup (C \setminus \{x\}) = (A \cup C) \setminus D \) has to have colour 1 due to its complement having colour 2. In this case, the application of the same argument to \( E \) and \( C \) gives us the same result of no triples of colour 2 containing \( v_1 \) and \( v_1 \) is 1-forcing again.

Putting such a 1-forcing vertex aside and repeating the procedure, we end up with a set of 1-forcing vertices and a remainder set where either there are no triples of colour 1 or there is a near perfect matching in colours 1 and 2 or 3. In the latter case, we can just complete the matching with 1-forcing vertices at will, so we assume now that the remainder, denoted henceforth by \( R \), only has triples of colours 2 and 3.

If \( R = V \), then we get the second conclusion of our proposition, so we may assume that there is at least one 1-forcing vertex. If, moreover, \( R \) had three disjoint pairs of triples of colours 2 and 3 that intersect in 2 vertices, we could add a 1-forcing vertex and get a universal 13-vertex set (see Figure 5) - a contradiction. If there are no three disjoint pairs like that, then after picking at most two of them the rest (denoted by \( R' \)) has to be a clique, of colour 2, say. We can then take a 1-forcing vertex and add to it those vertices of the triples of colour 3 among the chosen pairs of colour 2 and 3 that are not covered by the corresponding triples of colour 2, and add another vertex from \( R' \) if we still don’t have three vertices. Choose a near perfect matching from the rest of \( R \) containing the selected triples of colour 2 and then cover the rest with 1-forcing vertices if there are any left. This yields a near perfect matching in colours 1 and 2, and finishes the proof of the proposition.

Since in both cases of Proposition 4.2 we get a near perfect matching in 2 colours, we only need to consider the case where there exist spreads of a different colour. By symmetry, assume that \( U \) is a spread of colour 1 and \( W \) is a spread of colour 2. By Proposition 4.1, we can apply Proposition 4.2 to \( V \setminus U \), so we either get a near perfect matching avoiding colour 2 or 3 or no triples of colour 1 at all. In the first case, we can add one of the demonstration splittings of \( U \) to get a near perfect matching of \( V \) avoiding either colour 2 or colour 3. The same argument of applying Proposition 4.2 to \( V \setminus W \) yields either a near perfect matching on \( V \) in 2 colours or
no triples of colour 2 at all. We may hence assume that we got the second result in both attempts, and the colouring is such that all triples of colour 1 intersect $U$ while all triples of colour 2 intersect $W$.

We see that in such a setup, the vertex set $V \setminus (U \cup W)$ is a clique in colour 3. Additionally, there is at least one triple of colour 3 in $U$ (and in $W$, but it may not be disjoint from those in $U$), so if we take a maximal matching in colour 3 extending an exhausting of $V \setminus (U \cup W)$, we end up with at most $2 + |U| + |W| - |U \cap W| - 3 \leq 10$ vertices not covered by this matching and consequently only containing triples of colours 1 and 2. We distinguish between three possibilities for the number $m$ of vertices left out:

- $m \leq 8$ and $m \neq 6$. By the theorem of Alon, Frankl and Lovász ([1]) there is an almost perfect monochromatic matching in this 2-coloured subgraph: 3 vertices needed for 1 triple, 7 for two triples. Adding it to the initial colour 3 matching, we obtain a near perfect matching of $V$ in 2 colours.
- $m = 6$ or $m = 9$. In this case $|V|$ is a multiple of 3, so either it is at most 12, in which case we apply Theorem 3 or $|V|$ is at least 15, hence the prediction ([1]) gives a size at least one less than that of a perfect matching. In this latter case, the result of ([1]) is sufficient (a size 1 matching for $m = 6$ and a size 2 matching for $m = 9$).
- $m = 10$. Here all of our estimates have to be sharp, that is, $|U \cap W| = 1$ and we must have 2 vertices from $V \setminus (U \cup W)$ and 8 vertices from $U \cup W$ not covered by the matching in colour 3. If choosing a different maximal matching in colour 3 leads to a different case, we are done, so we may assume that no matter which 2 vertices $a$ and $b$ of $V \setminus (U \cup W)$ are left out from the initial matching, there do not exist 2 disjoint triples of colour 3 in $U \cup W \cup \{a, b\}$. But any vertex in $U \cup W \setminus (U \cap W)$ lies in the complement of a triple of colour 3 - the elements of $U \setminus W$ miss the colour 3 triple in $W$ and vice versa. Therefore any vertex in $U \cup W \setminus (U \cap W)$ together with any two vertices in $V \setminus (U \cup W)$, and any two vertices in $U \setminus W$ (or $W \setminus U$) together with any vertex in $V \setminus (U \cup W)$, give a hyperedge of colour 1 or 2.

If now $|V| \geq 31$, we can cover all of $V \setminus (U \cap W)$ (at most 30 vertices) by at most 10 such hyperedges (adding a suitable splitting of $W$ or applying Theorem 3 if $|V| = 13$). If, on the other hand, $|V| \geq 32$, then the formula ([1]) predicts a matching at least 1 less than a near perfect one. Such a matching can be found with direct application of ([1]) to the 10-vertex remainder as before.

In all three cases we arrive at a matching in 2 colours of size at least that predicted by ([1]), finishing the proof of Theorem 4.

References

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Université de Genève, rue du Lièvre 2-4, Case postale 64, 1211 Genève 4, Switzerland
E-mail address: terpai@math.elte.hu