Three range measurements with multiplicative noises for single source localization problem

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Abstract

This purpose of this paper is to locate a single localized source from three range measurements with multiplicative noises. Although some minimization approaches for additive noise have been found, studies on the existence of solutions are rare. We analyzed a situation with one or two solutions for the same multiplicative noise at three measurement sensors. A strategy for finding the best localized source when there are no solutions for the same multilicative noise is suggested that involves adjusting the multiplicative noise ratio. The numerical simulation is conducted for three randomly generated measurement locations and their distances to the source.

1 Introduction

We consider the problem of locating a single radiating source from multiplicaive noisy range measurements collected using a network of passive sensors. This problem is relavant to Global Positioning Systems(GPS), wireless communications, surveillance etc. [3]. Most of the applications use many measurements, and only three measurements are used in the inverse scattering problem for phaseless far field patterns [4] and the references therein. Only three measurements cases will be examined in this paper.

Consider an array of \(m\) sensors, and let \(z_j \in \mathbb{R}^n\) denote the coordinate of the \(j\)-th sensor for \(j = 1, 2, 3\). Let \(x \in \mathbb{R}^n\) denote the unknown source’s coordinate vector, and let \(d_{0j} > 0\) be a noisy observation of the range between the source and the \(j\)-th sensor:

\[
d_{0j} = ||x - z_j||(1 + \epsilon_j), j = 1, 2, 3.
\]

This formulation differs from traditional additive noise formulation in [1, 2, 3]. If \(n \geq 3\) and the source is not located on the plane defined by \(z_1, z_2,\) and \(z_3,\) we can only approximate \(x\) as the orthogonal projection to this plane. Thus, let us assume in this paper that the source location is on the two-dimensional plane defined by \(z_1, z_2,\) and \(z_3.\) Without a loss of generality, let us assume \(d_{23}^0 \geq d_{1}^0 \geq d_{1}^0.\)

Let us consider the multiplicative noise \(\epsilon.\) If \(\epsilon\) is known exactly, \(x\) can be defined without difficulty. However, it is generally not exactly known due to the possible ill-posed nature of the measurements. Let us assume that the devices and surrounding environments for the three measurement sensors are similar; that is to say, that \(\epsilon\) is the solution to the following minimization problem:

\[
\epsilon = arg\min_{(\epsilon_1, \epsilon_2, \epsilon_3)} |\epsilon_3 - \epsilon_2| + |\epsilon_3 - \epsilon_1|.
\]

If there is a corresponding source \(x,\) the best candidate for \(\epsilon\) is \(\epsilon_1 = \epsilon_2 = \epsilon_3.\) In general, various minimization techniques can be used for these problems. However, in this paper, we will identify the exact source location

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x and related noise $\epsilon$ when the solution exists. In addition, we proposed an approximation strategy for finding the exact source location $x$ by controlling $\epsilon$ to minimize the objective function in (2).

In section 2, the case in which there exists a source location $x$ satisfying (1) when $\epsilon_1 = \epsilon_2 = \epsilon_3$ will be discussed. In section 3, an approximation strategy for when there does not exist $x$ satisfying (1) is proposed. Theorems supporting the strategies are also stated and proved, and numerical examples for all cases controlling $\epsilon$ are explained. The error for finding $x$ for randomly generated measurement data added by multiplicative noise is also investigated with increasing multiplicative noise.

2 The existence of source

Let us denote the followings for $i, j = 1, 2, 3$:

1. $d_j = \frac{d_i}{1+\epsilon_j}$
2. $z_{ij} = z_i - z_j$,
3. $d_{ij} = \|z_{ij}\|$
4. $b_{ij}^{in} = \frac{d_j}{d_j+d_i} z_i + \frac{d_i}{d_i+d_j} z_j$
5. $b_{ij}^{out} = \frac{d_j}{d_j-d_i} z_i + \frac{d_i}{d_i-d_j} z_j$ only when $d_i \neq d_j$
6. $b_{ij} = \frac{d_i^2}{d_i^2-d_j^2} z_i + \frac{d_j^2}{d_i^2-d_j^2} z_j$ only when $d_i \neq d_j$
7. $r_{ij} = \frac{d_i d_j}{d_i^2-d_j^2}$ only when $d_i \neq d_j$

Here, $b_{ij}^{in}$ and $b_{ij}^{out}$ are respectively the internally and externally dividing points with ratios $d_i$ and $d_j$ between two points $z_i$ and $z_j$. And $b_{ij}$ and $r_{ij}$ are the center and radius, respectively, of the corresponding Apollonius circle. Without a loss of generality, let us assume

$$d_3 \geq d_2 \geq d_1 > 0.$$

**Theorem 1.** Suppose that $\epsilon_1 = \epsilon_2 = \epsilon_3$. Then, there is a unique solution for (1) if and only if one of the following conditions hold:

$$d_3 = d_2 = d_1 : \quad z_1, z_2, z_3 \text{ are not colinear},$$
$$d_3 = d_2 > d_1 : \quad \frac{|z_{23} \cdot (b_{13} - b_{23}^{in})|}{d_{23}} = r_{13},$$
$$d_3 > d_2 \geq d_1 : \quad |b_{23} - b_{13}| = r_{23} + r_{13} \text{ or } |b_{23} - b_{13}| = |r_{23} - r_{13}|.$$

And there are exactly two solutions for (1) if and only if one of the following conditions hold

$$d_3 = d_2 > d_1 : \quad \frac{|z_{23} \cdot (b_{13} - b_{23}^{in})|}{d_{23}} < r_{13}$$
$$d_3 > d_2 \geq d_1 : \quad |r_{23} - r_{13}| < |b_{23} - b_{13}| < r_{23} + r_{13}$$

And there is no solution for (1) if and only if one of the following conditions hold

$$d_3 = d_2 > d_1 : \quad \frac{|z_{23} \cdot (b_{13} - b_{23}^{in})|}{d_{23}} > r_{13},$$
$$d_3 > d_2 \geq d_1 : \quad |b_{23} - b_{13}| < |r_{23} - r_{13}|,$$
$$d_3 > d_2 \geq d_1 : \quad |b_{23} - b_{13}| > r_{23} + r_{13}. $$
Proof. If \( z_1, z_2, \) and \( z_3 \) are not colinear, they make a triangle. Further, \( d_3 = d_2 = d_1 \) means that the solution is located at the intersection of three lines that are perpendicular to each side passing through the midpoint of the side. The existence of the solution is exactly the circumcenter. If \( z_1, z_2, \) and \( z_3 \) are colinear and \( d_3 = d_2 = d_1 \), it can be easily verified that there is no solution to \( (1) \). This proves that the noncollinearity is an equivalent condition of the unique solution of \( (1) \) when \( d_3 = d_2 = d_1 \).

If \( d_3 = d_2 > d_1 \), then the solution lies at the perpendicular line bisecting the side between \( z_2 \) and \( z_3 \) and the Apollonius circle with radius \( r \), centered at \( b_{13} \). The distance from point \( b_{13} \) to the perpendicular line bisecting the side \( z_2 z_3 \) is

\[
D = \frac{|z_{23} \cdot (b_{13} - b_{23}^n)|}{d_{23}}.
\]

Thus, the existence of the solution for \( (1) \) is equivalent to \( D \leq r_{13} \). In addition, there is a unique solution if and only if \( D = r_{13} \), while there exist exactly two solutions if and only if \( D < r_{13} \).

If \( d_3 > d_2 \geq d_1 \), the existence of the solution for \( (1) \) is equivalent to the existence of the meeting points of two Apollonius’s circles for the two sides \( z_2 z_3 \) and \( z_1 z_3 \). If \( |b_{23} - b_{13}| = r_{23} + r_{13} \), the two circles circumscribe and if \( |b_{23} - b_{13}| = |r_{23} - r_{13}| \), the two circles inscribe. If \( |r_{23} - r_{13}| < |b_{23} - b_{13}| < r_{23} + r_{13} \), the two circles meet at two points. These prove conditions \( (5), (7), (9), \) and \( (10) \).

Corollary 1. If \( \epsilon \) is exactly known, one of the following conditions hold:

\[
\begin{align*}
 d_3 = d_2 = d_1 & : z_1, z_2, z_3 \text{ are not colinear.} \quad (11) \\
 d_3 = d_2 > d_1 & : \frac{|z_{23} \cdot (b_{13} - b_{23}^n)|}{d_{23}} \leq r_{13} \quad (12) \\
 d_3 > d_2 \geq d_1 & : |r_{23} - r_{13}| \leq |b_{23} - b_{13}| \leq r_{23} + r_{13}. \quad (13)
\end{align*}
\]

3 Approximation strategy

Given \( z_j, d_j, j = 1, 2, 3 \) and \( \epsilon_1 = \epsilon_2 = \epsilon_3 \), if one of conditions \( (3), (4), \) and \( (5) \) hold, then we can find the unique solution \( x \) for \( (1) \) using Theorem 1. If one of the two conditions \( (6) \) and \( (7) \) hold, there are two solutions for \( (1) \); let these two solutions be \( x_1 \) and \( x_2 \). Then we can choose

\[
x = \arg\min_{x, j=1,2} \sum_{j=1,2,3} ||z_j - x_i||(1 + \epsilon_j) - d_j^0 .
\]

If one of the three conditions \( (3), (9), \) and \( (10) \) hold, the assumption \( \epsilon_1 = \epsilon_2 = \epsilon_3 \) no longer holds, and we should change \( \epsilon \) to have a solution. We will find a solution \( x \) along with \( \epsilon \) of the minimization problem such as

\[
\min_{\epsilon = \epsilon(x)} ||\epsilon_3 - \epsilon_2| + |\epsilon_3 - \epsilon_1|| = \min_{x = x(\epsilon)} \left[ \frac{d_3}{|z_3 - x|} - \frac{d_2}{|z_2 - x|} \right] + \frac{d_3}{|z_3 - x|} - \frac{d_1}{|z_1 - x|} .
\]

Rather than attempting to directly solve this minimization problem using known minimization methods, we will try another strategy to obtain a solution by controlling \( \epsilon \), which results in \( d \) changing accordingly.

- The order \( d_3 \geq d_2 \geq d_1 \) should be fixed.
- The value \( d_3 \) is fixed and \( d_2 \) and/or \( d_1 \) increase(s).
- The values by which \( d_2 \) and \( d_1 \) increase should be as small as possible.

Bearing in mind that \( d_2 \) and \( d_1 \) increase into \( k_2 d_2 \) and \( k_1 d_1 \) for \( 1 \leq k_2 \leq \frac{d_2}{d_3} \) and \( 1 \leq k_1 \leq \frac{k_2 d_2}{d_3} \), let us define the followings:

1. \( b_{ij}^{kn}(k_i, k_j) = \frac{k_j d_j}{k_j d_j + k_i d_i} z_i + \frac{k_i d_i}{k_i d_i + k_j d_j} z_j \)
2. \( t_{ij}^\text{out}(k_i, k_j) = \frac{k_i d_i}{k_j d_j - k_i d_i} z_i + \frac{k_i d_i}{k_i d_i - k_i d_i} z_j \) only when \( k_i d_i \neq k_j d_j \)

3. \( b_{ij}(k_i, k_j) = \frac{k_j d_j^2}{k_j d_j - k_i d_i} z_i + \frac{k_i d_i^2}{k_i d_i - k_i d_i} z_j \) only when \( k_i d_i \neq k_j d_j \)

4. \( r_{ij}(k_i, k_j) = \frac{k_i k_j d_i d_j}{k_i d_i - k_i d_i} \) only when \( k_i d_i \neq k_j d_j \)

If condition (8) holds, there exists no solution if \( \epsilon_1 = \epsilon_2 = \epsilon_3 \). By the above strategy, we should increase \( d_1 \) up to \( d_2 \). Let

\[
D(k) := \|z_{23} \cdot (b_{13}(k, 1) - b_{13}^n(k, 1))\|/d_{23} - r_{13}(k, 1).
\]

Then, (8) implies that \( D(1) > 0 \). If there is a \( k \) such that \( 1 < k \leq \frac{d_2}{d_1} \) and \( D(k) \leq 0 \), then there is a solution \( x \) for (1) with \( d_3 = d_2 > k d_1 \). Let \( k_D = d_2/d_1 \).

**Theorem 2.** Suppose \( d_3 = d_2 > d_1 \) and \( D(1) > 0 \). Then, \( D(k) \) is a decreasing function for \( k \in [1, k_D] \) and there is a \( k \in [1, k_D] \) such that \( D(k) = 0 \) and \( d_3 = d_2 > k d_1 \).

**Proof.** Let \( 1 \leq k_1 < k_2 < k_D \) and let the blue and red circles in Fig. 1(a) respectively correspond to the Apollonius circles for \( k_1 \) and \( k_2 \). Let \( \theta \) be the angle \( \angle z_1 z_2 z_2 \) and assume that this angle is acute. Even though Fig. 1(a) is for acute angle \( \theta \), the following points are also true for an obtuse angle by using \( |\cos \theta| \). Under these assumptions, we have

\[
D(k_2) - D(k_1) = \|DG\| - |DA| - \|CF\| - |CB| = \|DG\| - |CF| - \|DA\| - |CB| = |CD| |\cos \theta| - ||AB| + |CD| = |CD| (|\cos \theta| - 1) - |AB| < 0. \tag{14}
\]

Therefore, \( D(k) \) is a decreasing function on \([1, k_D]\). Note that

\[
b_{13}(k, 1) = \frac{d_3^2}{d_3^2 - k^2 d_1^2} z_1 + \frac{k^2 d_1^2}{k^2 d_1^2 - d_3^2} z_3 = \frac{1}{1 - k^2 (d_1/d_3)^2} z_{13} \quad \text{and} \quad |b_{13}(k, 1) - \frac{1}{2}(z_1 + z_3)| = \frac{1 + k^2 (d_1/d_3)^2}{2(1 - k^2 (d_1/d_3)^2)} d_{13}
\]

and

\[
r_{13}(k, 1) = \frac{k (d_1/d_3) d_{13}}{1 - k^2 (d_1/d_3)^2} = |b_{13}(k, 1) - b_{13}^n(k, 1)|.
\]

Since \( b_{13}(k, 1), b_{13}^n(k, 1), \) and \( \frac{1}{2}(z_1 + z_3) \) lie on the same line, we have

\[
\left| \frac{z_1 + z_3}{2} - b_{13}^n(k, 1) \right| = |b_{13}(k, 1) - \frac{z_1 + z_3}{2}| = |b_{13}(k, 1) - b_{13}^n(k, 1)| = \frac{1 - k (d_1/d_3)}{1 + k (d_1/d_3)} d_{13}.
\]

Therefore \( k \) is less than and sufficiently close to \( k_D \) and \( b_{13}^n(k, 1) \) approaches \( \frac{z_1 + z_2}{2} \); that is to say, \( b_{13}^n(k_D - 0, 1) = \frac{z_1 + z_2}{2} \).

Thus, the Apollonius circle on the side \( z_{12} z_3 \) goes to the perpendicular line to \( z_{12} z_3 \) at \( \frac{z_1 + z_2}{2} \) as \( k \) goes to \( k_D - 0 \). Since the two perpendicular lines with respect to sides \( z_{21} z_3 \) and \( z_{12} z_3 \) meet at \( z_{12} z_3 \) meeting the perpendicular line with respect to \( z_{21} z_3 \) as shown in Fig. 1(b). \( \square \)

Next, we consider case (9).

**Lemma 3.** Let \( d_3 > d_2 \geq d_1 \) and \( |r_{23} - r_{13}| > |b_{23} - b_{13}| \). Then

\[
r_{23} > r_{13} \quad \text{and} \quad d_2 > d_1 \tag{15}
\]
Figure 1: Cases in which the Apollonius circle along $z_{13}$ does not meet the perpendicular line bisecting $z_{23}$: (a) Bigger and smaller Apollonius circles along $z_{13}$. (b) As $d_1$ increases to $d_2$, the Apollonius circle goes to the dotted perpendicular line bisecting $z_{12}$ and eventually meets the perpendicular line bisecting $z_{23}$.

**Proof.** Suppose that $r_{13} \geq r_{23}$. Then, by the assumption

$$r_{13} > r_{23} + |b_{23} - b_{13}|,$$

points $z_3, b_{23}, b_{13}$, and $b_{13}$ respectively correspond to $O, A, B, C, D$ in Fig. 2 (a). In addition, the above equation (16) becomes

$$|DC| > |BA| + |CA|,$$

$$|OD| = \frac{d_3}{d_3 + d_1} d_{13} = \left( \frac{d_3}{d_1} - 1 \right) r_{13} \geq \left( \frac{d_3}{d_2} - 1 \right) r_{13} > \left( \frac{d_3}{d_2} - 1 \right) r_{23} = \frac{d_3}{d_3 + d_2} d_{23} = |OB|.$$

(18)

Therefore, using (17) and (18), we have

$$|OC| = |OD| + |DC| > |OB| + |BA| + |CA| = |OA| + |CA|,$$

which contradicts the fact that $|OC|, |OA|,$ and $|CA|$ are three sides of a triangle. Therefore, we proved $r_{23} > r_{13}$.

Suppose that $d_2 = d_1$. In this case, $A = E$ and $C = F$ in Fig. 2 (a). Since

$$r_{23} - r_{13} = \frac{d_2 d_3}{d_3^2 - d_2^2} (d_{23} - d_{13}) < \frac{d_2^2 d_{13}}{d_3^2 - d_2^2} (d_{23} - d_{13})$$

and

$$|b_{23} - b_{13}| = |AC| = \frac{d_2^2 d_{12}}{d_3^2 - d_2^2}$$

we have from the assumption $r_{23} - r_{13} > |b_{23} - b_{13}|$

$$d_{23} - d_{13} > d_{12},$$

which contradicts the fact that $d_{12}, d_{13},$ and $, d_{23}$ are the lengths of the sides of a triangle. Thus, we proved $d_2 \neq d_1$ and $d_2 > d_1$. 

\[ \square \]
Thus, \( \triangle ABC \) has sides of \( k \) such that \( 1 < k < k_2 < k_E \). Let \( b_{23}, b_{13}(k_1, 1), b_{13}(k_2, 1), b_{13}^0(k_1, 1), \) and \( b_{13}^0(k_2, 1) \) be \( A, B, C, D, \) and \( E \) in Fig. (2) (c), respectively. Then, using \( |AB|, |BC|, \) and \( |CA|'s \) which are the lengths of the three sides of a triangle \( \Delta ABC \), we have

\[
E(k_2) - E(k_1) = -r_{13}(k_2, 1) + r_{13}(k_1, 1) - |b_{23} - b_{13}(k_2, 1)| + |b_{23} - b_{13}(k_1, 1)|
\]

\[
= -|CE| + |BD| - |AC| + |AB| = -|ED| - |BC| - |AC| + |AB|
\]

\[
= -|ED| - (|BC| + |CA| - |AB|) < 0.
\]

Thus, \( E \) is a decreasing function for \( k \in [1, k_E) \).

The Apollonius circle along \( \frac{z_2 - z_1}{z_2 - z_3} \) contains \( z_2 \) and \( z_1 \), and therefore also contains the line joining \( z_2 \) and \( z_1 \). If \( k \) goes to \( k_E \) from below, the Apollonius circle along \( \frac{z_1 - z_2}{z_1 - z_3} \) converges to the perpendicular line bisecting \( \frac{z_1 - z_2}{z_1 - z_3} \). Therefore, two Apollonius circles along \( \frac{z_1 - z_2}{z_1 - z_3} \) and \( \frac{z_2 - z_3}{z_2 - z_3} \) meet for some \( k \in [1, k_E) \), meaning that two Apollonius circles along \( \frac{z_1 - z_2}{z_1 - z_3} \) and \( \frac{z_2 - z_3}{z_2 - z_3} \) also meet at the same value \( k \). Therefore, there is some \( k \in [1, k_E) \) such that \( E(k) = 0 \).

Finally, let us consider case (10). In this case, \( d_1 \) and \( d_2 \) increase at the same rate upto \( k_F = \frac{d_2}{d_2} \) times.

Let

\[
F(k_1, k_2) := |b_{23}(k_1, 1) - b_{13}(k_2, 1)| - (r_{23}(k_1, 1) + r_{13}(k_2, 1)).
\]

Theorem 5. Suppose \( d_3 > d_2 \geq d_1 \) and \( F(1) > 0 \). \( F(k) \) is a decreasing function with respect to \( k \in [1, k_F) \).

If \( d_2 = d_1 \), there is a \( k \in [1, k_F) \) such that \( F(k) = 0 \) and \( d_3 > kd_2 = kd_1 \).

Proof. Let \( 1 < k_1 < k_2 < k_F \). Note that

\[
F(k_2) - F(k_1) = [F(k_2) - F(k_1, k_2)] + [F(k_1, k_2) - F(k_1)].
\]
Figure 3: Flow chart of the proposed algorithm

We can prove that
\[ F(k2) - F(k1, k2) < 0 \text{ and } F(k1, k2) - F(k1) < 0 \]

in a similar manner to the proofs of Theorem 2 and Theorem 4.

Suppose \( d_3 > d_2 = d_1 \). Since \( kd_2 = kd_1 \), the solution should lie on the perpendicular line bisecting \( \frac{z_1 + z_2}{2} \). If \( k \) goes to \( k_F \), the Apollonius circle along \( \frac{z_2 z_3}{2} \) goes to the perpendicular line bisecting \( \frac{z_2 z_3}{2} \). Thus, there must exist a \( k \in [1, k_F] \) such that a perpendicular line bisects \( \frac{z_1 z_2}{2} \) and the Apollonius circle along \( \frac{z_2 z_3}{2} \).

If \( d_2 \neq d_1 \), there might or might not be a \( k \in [1, k_F] \) such that \( F(k) = 0 \). If there is such a \( k \), we have \( d_3 > kd_2 > kd_1 \). Otherwise, we should try the case \( d_3 = d'_2 = k_F d_2 \) and increase \( d'_1 \) upto \( d_1 \), which is the case in Theorem 2.

To summing up, our strategy for finding \( x \) is shown in Fig. 3. We excluded the case in which \( d_3 = d_2 = d_1 \) and \( z_1, z_2, z_3 \) are colinear, as there is no solution to (1) in such a case.

4 Numerical test

In this section, we numerically tested the proposed algorithm in two ways: numerical illustration of all cases is shown in Fig. 3 and approximation error analysis is used for 50 random samples with respect to the increasing multiplication error.

First, we randomly generated \( z_i \) on \((0, 1) \times (0, 1)\) and \( r_i \) on \((0, 1)\) for \( i = 1, 2, 3 \) with uniform probability density, then reordered them to satisfy \( d_3 \geq d_2 \geq d_1 \). We classified the cases as follows

- Case 003 : \( d_3 = d_2 = d_1 \) : There is a solution if \( z_1, z_2, \) and \( z_3 \) are not colinear by (3).
\[ RRE = \frac{|\epsilon_3 - \epsilon_2| + |\epsilon_3 - \epsilon_1|}{1 + \epsilon_3}, \]

which measures how close the three values \( \epsilon_1, \epsilon_2, \) and \( \epsilon_3 \) are to each other.

To find a solution of (1) where \( \epsilon \) is Gaussian noise, we have computed the mean and standard deviation for 50 samples with multiplicative Gaussian noise from 0% to 100%. The simulated data are obtained through the following procedures: First, let us assume that the source location \( x^0 = (0.5, 0.5) \) is fixed. Second, make uniform random measurement points \( z_j \in [0, 1] \times [0, 1], j = 1, 2, 3. \) Third, add the multiplicative Gaussian noises with 100 \( \times \) \% \( \eta \), \( \eta \in [0, 1] \) to the distances from \( x^0 \) to \( z_j \), resulting in \( d_j = \max(0.01, ||z_j - x^0|| (1 + \eta \ast G(0, 1))) \), where \( G(0, 1) \) is a standard Gaussian noise. Fourth, approximate \( x^0 \) using the proposed algorithm. Fifth, calculate the mean and standard deviation of 50 approximation errors \( ||x - x^0|| \), where \( x \) is the approximation of \( x^0 \), for given \( \eta \). The step size for \( \eta \) is chosen to be 2%. The computed errors with respect to \( \eta \) are shown in Fig. 5(a). Removing the four highest standard deviation points, we can see the approximately increasing behavior shown in Fig. 5(b).
Figure 4: Process of finding the source location $x$ for call cases (a)003, (b)013, (c)012013, (d)113, (e)112+113, (f)112+013, and (g)112-113 from Fig. 3. Magenta lines are the lines from $x$ to the three measurement points $z$. 
5 Conclusions

In this paper, the theoretical background on locating a singular source from three range measurements with multiplicative noise was exploited. When the multiplicative noise was the same for the points of three measurement data, the equivalent condition for the existence of the singular source was presented and proved using the idea of Apollonius circles. When there existed solutions, there were one or two. When two solutions existed, we chose the closest point whose distance to $z_3$ was more similar to the longest distance $d_3$ as a possible approximation of the source. When no solution existed for the same $\epsilon$, we proposed an algorithm with which to find the best approximation with respect to RRE by controlling $\epsilon$. The algorithm preserves the distance length order and minimizes RRE; that is to say, it minimizes the ratio difference among $d_j / |z_j - x|$, $j = 1, 2, 3$. Numerical examples for all cases in the algorithm are shown including the measurement triangle, Apollonius circle, and perpendicular line bisecting one of the sides, as well as the approximated solution. Finally, we showed that, as the multiplicative noise ratio increased from 0% to 100%, the mean and standard deviations’ for the 50 samples increased asymptotically.

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