Abstract  Casimir W-algebras are shown to be exist in such a way that the conformal spins of primary (generating) fields coincide with the orders of independent Casimir operators. We show here that this coincidence can be extended further to the case that these generating fields have the same eigenvalues with the Casimir operators.
1. Introduction

Conformal invariance is of fundamental importance in two-dimensional field theories and hence it
finds remarkable applications in string theory [1] and in the study of critical phenomena in statistical
physics [2], as well as in mathematics [3]. Its underlying symmetry algebra is Virasoro algebra which
appears naturally in two-dimensional field theories. The idea to extend Virasoro algebra with the in-
truction of higher conformal spin generators is also seem to be relevant in these theories. A seminal
type of these extensions is the so-called $W_N$-algebras and Virasoro algebra is a $W_2$-algebra within this
framework. $W_2$ is, except superconformal algebras, one of the first non-trivial extensions of $W_2$ and it was
constructed first by Zamolodchikov [4,5]. There are several works dealing with the classification and also
the construction of this type of algebras [6-10]. To this end, a difficulty is the fact that, except $W_2$ , all
the extended ones are non-linear algebras. This poses great complications in explicit constructions of $W_N$
- algebras. One must emphasize here that very little is known beyond $W_N$ . There are, on the other hand,
some recent efforts to bring out a relation between these algebras and Casimir algebras [11,12]. The idea
is principally based on the Sugawara construction of $W_2$-algebra [13]. The purpose of our work is to es-
ablish a method in this direction. For this, we made use of construction known as Miura transformation
[14-19] with Feigin-Fuchs type of free massless scalar fields. It is seen that this gives us the possibility
to exploit a relation between $W_N$ - algebras and $A_{N-1}$-Lie algebras. This brings us to the fact that one
can define spin-2,3,4 primary fields of $W_4$-algebra [20] and also spin-5 primary field in such a way that
their eigenvalues are in one to one correspondence with 2,3,4 and 5 order Casimir operators of $A_{N-1}$ Lie
algebras [21].

A definition of Casimir W-Algebras is given definitively in [20] and also a relation is tried to be given
with the eigenvalues of Casimir operators in section 6 of the same paper, in examples only for second
and also third order Casimir operators. The studies for these orders are also given in [19]. We tried to
extend these studies beyond third orders by calculating explicitly the eigenvalues of the generating fields
and also the Casimir operators.

The paper is organized as follows. In section 2, we constructed $W_N$- algebra by utilizing a construction
known as Miura transformation with Feigin-Fuchs type of free massless scalar fields. In section 3, we show
our way of computing the eigenvalue spectrum of $W_N$-algebras on highest weight states of Fock space of
$A_{N-1}$ Lie algebras. In section 4, we obtain a relation between the eigenvalue spectrum of $W_4$-algebra by
adding pure spin-5 conformal field and related order Casimir eigenvalues for irreducible representations
of $A_{N-1}$-Lie algebras. In appendix-A we give an explicit form for our pure spin-5 conformal field. All these
have been possible with a dense application of Mathematica Package program [22].

2. The (quantum) Miura Transformation and $W_N$ -Algebras

The $W_N$-algebra is generated by a set of chiral currents $\{U_k(z)\}$ of conformal dimension $k$ ($k =$
$1, \cdots , N$). Let us define a differential operator $R_N(z)$ of degree $N$ [19]

$$ R_N(z) = - \sum_{k=0}^{N} U_k(z) (\alpha_0 \partial)^{N-k} = : \prod_{j=1}^{N} \nabla_j : , $$

(2.1)

where $\nabla_j = \alpha_0 \partial_z - h_j(z)$ and the symbol $: \cdot :$ shows the normal ordering of the fields $\varphi(z)$. Here $\varphi(z)$ is
an $N - 1$ component Feigin Fuchs-type of free massless scalar fields. This transformation is called the
(quantum) Miura transformation and it determines completely the fields $\{U_i(z)\}$ with

$$ h_j(z) = i \mu_j \partial \varphi(z) $$

(2.2)

Here, $\mu_j$’s, $(i = 1, \cdots , N)$ are the weights of the fundamental representation of $SU(N)$, satisfying
$\sum_{i=1}^{N} \mu_i = 0$ and $\mu_i, \mu_j = \delta_{ij} - \frac{1}{N}$. The simple roots of $SU(N)$ are given by $\alpha_i = \mu_i - \mu_{i+1}$,
$(i = 1, \cdots , N - 1)$. The Weyl vector of $SU(N)$ is denoted as $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha^+$ where $\alpha^+$ are the
positive roots of $SU(N)$. A free scalar field $\varphi(z)$ is a single-valued function on the complex plane and
its mode expansion [19] is given by

$$ i \partial \varphi(z) = \sum_{\alpha \in \mathbb{Z}} a_\alpha z^{-\alpha-1} . $$

(2.3)
Canonical quantization gives the commutator relations

\[ [a_m, a_n] = m\delta_{m+n,0}, \] (2.4)

and these commutator relations are equivalent to the contraction

\[ \partial \varphi(z) \partial \varphi(w) = -\frac{1}{(z-w)^2}. \] (2.5)

By using single contraction \( \partial \varphi(z) \partial \varphi(w) \), a contraction of \( h_i(z) \) with itself is given by [16]

\[ h_i(z) h_i(w) = \frac{\delta_{ij} - \frac{1}{N}}{(z-w)^2}. \] (2.6)

The fields \( \{ U_k(z) \} \) can be obtained by expanding \( R_N(z) \). We present a first few one as in the following

\[ U_0(z) = -1, \quad U_1(z) = \sum_i h_i(z) = 0, \quad U_2(z) = -\sum_{i<j} (h_i h_j)(z) + \alpha_0 \sum_i (i-1) \partial h_i(z) \] (2.7)

One can see that \( U_2(z) \equiv T(z) \) has spin-2, which is called the stress-energy tensor, \( U_k(z) \) has spin-k.

The standard OPE of \( T(z) \) with itself is

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \] (2.8)

where the central charge, for \( SU(N) \), is given by

\[ c = (N-1) \left(1 - N(N+1)\alpha_0^2\right). \] (2.9)

A primary field \( \phi_h(z) \) with conformal spin-\( h \) must provide the following OPE with \( T(z) \)

\[ T(z)\phi_h(w) = \frac{h \phi_h(w)}{(z-w)^2} + \frac{\partial \phi_h(w)}{(z-w)} + \cdots \] (2.10)

Therefore the fields \( \{ U_k(z) \} \) are not primary because

\[ T(z)U_k(w) = \frac{1}{2} \sum_{s=1}^{k} \frac{(N-k+s)!}{(N-k)!} a_0^{s-2} \left(((s-1)(N-1) + 2(k-1))a_0^2 - \frac{s-1}{N}\right) \frac{U_s(w)}{(z-w)^{s+2}} + \frac{kU_k(w)}{(z-w)^2} + \frac{\partial U_k(w)}{(z-w)} + (TU_k)(z) + \cdots \] (2.11)

Using above OPE, we want to construct \( W_4 \)-algebra. Therefore we must obtain spin-3 and spin-4 primary fields. Here we first write down the spin-3 primary field for \( SU(N) \) as

\[ \overline{U}_3(z) = U_3(z) - \frac{(N-2)}{2} \alpha_0 \partial T(z) \] (2.12)

and the spin-4 primary field as

\[ \overline{U}_4(z) = U_4(z) - \frac{(N-3)}{2} \alpha_0 \partial U_3(z) + \frac{(N-2)(N-3)}{4N(22+5c)} \left[-3 + N(13 + 3N + 2c) a_0^2\right] \partial^2 T(z) + \frac{(N-2)(N-3)}{2N(22+5c)} \left[5 - N(5N + 7) a_0^2\right] (TT)(z) \] (2.13)
To obtain OPE of the two primary fields \( \{ \overline{U}_3(z) \} \) and \( \{ \overline{U}_4(z) \} \) which gives the central term in the known form, we must take care of the normalized forms of all the primary fields \( \{ \overline{U}_3(z) \} \). Therefore the normalized form of the \( W_3 \)-algebra generators are given by the following expressions

\[
\overline{U}_3(z) = \sqrt{\theta_w} W(z) \quad , \quad \overline{U}_4(z) = \sqrt{\theta_L} L(z)
\]  

(2.14)

where \( \theta_w \) and \( \theta_L \) are the normalization factors for \( SU(4) \) and their explicit form are given

\[
\theta_{w} = \frac{c + 7}{10} \quad , \quad \theta_{L} = \frac{(114 + 7c)(c + 7)(c + 2)}{300(22 + 5c)}
\]  

(2.15)

respectively. These results are being in line with those of ref.[9,20]. On the other hand, we must emphasize here that the stress-energy tensor \( T(z) \) is not given in the normalized form. A straightforward calculation gives us the first non-trivial OPE of \( W(z) \) with itself for \( SU(4) \)

\[
W(z) W(w) = \frac{c/3}{(z-w)^4} + \frac{2 T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \frac{2 \beta_\Lambda(w) + \frac{3}{10} \partial T(w) + \beta_L L(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \frac{\beta_\Lambda \partial \Lambda(w) + \frac{1}{15} \partial^3 T(w) + \frac{\beta_L}{2} \partial L(w) + (W W)(w) + \cdots}{(z-w)^2}
\]  

(2.16)

Previously, we have written the normalization factor \( \theta_w \) for \( SU(4) \) in equation (2.15). In addition to this, after some calculations it is also possible to write \( \theta_w \) for \( SU(N) \) in general form

\[
\theta_w = \frac{(N - 2)(-2 + 2c - N + cN + 3N^2)}{2(N - 1)N(N + 1)}
\]  

(2.18)

The second non-trivial OPE of \( W(z) \) with \( L(w) \) takes the form

\[
W(z) L(w) = \eta_w \left[ \frac{W(w)}{(z-w)^4} + \frac{1}{3} \frac{\partial W(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \frac{\eta_{T W} \left( (T W)(w) - \frac{3}{10} \partial^2 W(w) \right) + \eta_{W W} \partial^2 W(w)}{14} \right]
\]  

+ \frac{1}{(z-w)^2} \left[ \eta_{\partial T W} \partial \left( (T W)(w) - \frac{3}{10} \partial^2 W(w) \right) - \eta_{T W} \left( (T \partial W)(w) - \frac{c + 17}{84} \partial^3 W(w) \right) \right] + (W L)(w) + \cdots
\]  

(2.19)

where

\[
\eta_w^2 = \frac{3}{4} \beta_L \quad , \quad \eta_{T W} = \frac{39}{114 + 7c} \beta_L \quad , \quad \eta_{\partial T W} = \frac{15}{4} \frac{(22 + 5c)}{(c + 2)(114 + 7c)} \beta_L \quad , \quad \eta_{W W} = \frac{3}{4(c + 2)} \beta_L
\]  

(2.20)

Finally, the last non-trivial OPE of \( L(z) \) with itself takes the form

\[
L(z) L(w) = \frac{c/4}{(z-w)^8} + \frac{2 T(w)}{(z-w)^6} + \frac{\partial T(w)}{(z-w)^5} + \frac{1}{(z-w)^4} \frac{3 \partial^2 T(w) + 2 \rho_\Lambda \Lambda(w) - \rho_L L(w)}{10}
\]  

+ \frac{1}{(z-w)^3} \left[ \frac{1}{15} \partial^3 T(w) + \rho_\Lambda \partial \Lambda(w) - \frac{\rho_\Lambda}{2} \partial L(w) \right] + \frac{1}{(z-w)^2} \left[ \rho_{\partial L} \partial^2 L(w) + \rho_{\partial \Lambda} \rho_\partial L(T \partial^2 T)(w) + \rho_{\partial \partial \partial \partial T} (\partial T \partial T)(w) \right]
\]  

+ \rho_{\partial T W} \left( (T T)(w) + \rho_{W W} (W W)(w) + \rho_{L T} (L T)(w) + \rho_{\partial T \partial \partial T} (\partial T \partial T)(w) \right)
\]  

+ \frac{1}{(z-w)} \left[ \rho_{\partial L \partial L} \partial^3 L(w) + \rho_{\partial \Lambda \partial L} \partial (T \partial^2 T)(w) + \rho_{\partial \partial \partial \partial \partial T} \partial (\partial T \partial T)(w) + \frac{1}{2} \rho_{\partial T \partial T} \partial ((T T)(w) \right)
\]  

+ \rho_{W W} \partial (W W)(w) + \frac{1}{2} \rho_{L T} \partial (L T)(w) + \rho_{\partial \Lambda \partial L} \partial^3 L(w)] + (LL)(w) + \cdots
\]  

(2.21)
where

\[ \rho_\Delta = \frac{21}{22 + 5c}, \quad \rho_L = \sqrt{\frac{27(c^2 + c + 218)^2}{(7c + 114)(5c + 22)(c + 7)(c + 2)}} \]

\[ \rho_{\partial_\Delta} = \frac{4194 + 137c - 5c^2}{48(7c + 114)(5c + 22)(c + 7)(c + 2)}, \quad \rho_{\partial_\Delta T} = \frac{-1596 - 2068c + 25c^2}{70(c + 2)(114 + 7c)} \]

\[ \rho_{\partial T \partial T} = \frac{1806 - 907c + 10c^2}{28(7c + 114)(c + 2)}, \quad \rho_{(\partial T \partial T)} = \frac{96(9c - 2)}{7c + 114)(5c + 22)(c + 2)}, \quad \rho_{\partial W} = \frac{45(22 + 5c)}{2(7c + 114)(c + 2)} \]

\[ \rho_{\partial_\Delta} = \frac{48(114 + 7c)}{(5c + 22)(c + 7)(c + 2)}, \quad \rho_{\partial_\Delta T} = \frac{-23016 + 12948c - 3190c^2 + 25c^3}{280(7c + 114)(c + 2)} \]

\[ \rho_{\partial T \partial T} = \frac{3(144 - 1592c + 5c^2)}{280(7c + 114)(c + 2)} \]

\[ \rho_{\partial W} = \frac{3(-28568 + 15676c - 1934c^2 + 5c^3)}{112(7c + 114)(5c + 22)(c + 2)} \] (2.22)

These are also the same as in ref.[9].

3. The Eigenvalue Spectrum of \( W_N \)-Algebras

We denote the Fock space of a free massless scalar field \( h_i(z) = i \mu \partial \varphi(z) \) in (2.2) by \( F_\Lambda \), where \( \Lambda \) is the dominant weight of the Lie algebra \( A_{N - 1} \), which can be expressed as \( \Lambda = \sum_{i=1}^{N-1} r_i \lambda_i \) where \( \lambda_i \)'s are fundamental dominant weights which are defined by \( \lambda_i \). \( \alpha_j \)'s are dual vectors to simple roots \( \alpha_j \). Cartan matrices \( C_{ij} \) are then defined by \( C_{ij} = \alpha_i \cdot \alpha_j \) being in accordance with the Dynkin diagrams

\[ o-o-o-o-o\cdots o-o-o \]

\[ 1 \quad 2 \quad 3 \quad \cdots \quad N-1 \quad N \] (3.1)

of \( A_{N - 1} \) chain where \( N = 1, 2, 3, \ldots \). The parameters \( r_i \)'s are taken to be positive integers including zero, and also \( \Lambda \) labels the eigenvalue of the scalar zero modes \( a_0^\Lambda = p^\Lambda \) on the Fock space vacuum \( |\Lambda > \) [19]. This can be written as

\[ a_0^\Lambda |\Lambda >= \Lambda |\Lambda > . \] (3.2)

The eigenvalues \{\( U_k(\Lambda) \)\} of the zero mode of \{\( U_k(z) \)\} on the highest weight states of Fock space \( F_\Lambda \); are given by [19]

\[ U_k(\Lambda) = (-1)^{k-1} \sum_{i_1 < i_2 < \ldots < i_k} \prod_{j=1}^{k} ((\Lambda, \mu_{i_j}) + (k - j) \alpha_o) \] (3.3)

We give a general definition to the \( \Theta_{(n_1, n_2, \ldots, n_k)} \),

\[ \Theta_{(n_1, n_2, \ldots, n_k)} = \sum_{i_1 < i_2 < \ldots < i_k} \prod_{j=1}^{k} \theta_{i_j} \] (3.4)

where \( \theta_i = (\Lambda + \alpha_o \rho , \mu_i) \) and \( \Theta_{(1)} = \sum_{i=1}^{N} \theta_i = 0 \) since \( \sum_{i=1}^{N} \mu_i = 0 \). Therefore, we give the results \( k = 2, 3, 4 \) and 5 respectively

\[ U_1(\Lambda) = \Theta_{(1)} = 0 \quad , \quad U_2(\Lambda) = -\Theta_{(1,1)} - \frac{1}{4} \left( \begin{array}{c} N + 1 \\ 3 \end{array} \right) \alpha_o^2 \] (3.5)

\[ U_3(\Lambda) = \Theta_{(1,1,1)} + (N - 2) \alpha_o \Theta_{(1,1)} + \left( \begin{array}{c} N + 1 \\ 4 \end{array} \right) \alpha_o^3 \] (3.6)
where the polynomial $W$ of give explicit expression of the eigenvalues for the Casimir operators in the following form:

$$U_4(\Lambda) = -\Theta_{(1,1,1,1)} - \frac{3}{2} (N-3) \alpha_o \Theta_{(1,1,1,1)}$$

$$-\frac{1}{24} (N-2)(N-3)(N+23) \alpha_o^3 \Theta_{(1,1,1,1)} - \frac{(223 + 5N)}{48} \left( \frac{N+1}{5} \right) \alpha_o^4$$

(3.7)

and

$$U_5(\Lambda) = \Theta_{(1,1,1,1,1)} + 2(N-4) \alpha_o \Theta_{(1,1,1,1,1)} + \frac{(N-3)(N-4)(N+43)}{24} \alpha_o^2 \Theta_{(1,1,1,1,1)}$$

$$-\frac{1}{12} (N-2)(N-3)(N-4)(N+11) \alpha_o^3 \Theta_{(1,1,1,1)} + \frac{(103 + 5N)}{4} \left( \frac{N+1}{6} \right) \alpha_o^5$$

(3.8)

where

$$\Theta_{(1)} = 0 , \quad \Theta_{(1,1)} = -\frac{1}{2} \Theta_{(2)} , \quad \Theta_{(1,1,1,1)} = \frac{1}{3} \Theta_{(3)}$$

$$\Theta_{(1,1,1,1,1)} = -\frac{1}{4} \Theta_{(4)} + \frac{1}{8} \Theta_{(2)}^2 , \quad \Theta_{(1,1,1,1,1,1)} = \frac{1}{5} \Theta_{(5)} - \frac{1}{6} \Theta_{(2)} \Theta_{(3)}$$

(3.9)

After some calculations it can be shown that the eigenvalues of the primary fields are $U_4(z) \equiv T(z) \equiv \mathcal{U}_4(z)$, spin-3 $\mathcal{U}_3(z)$, spin-4 $\mathcal{U}_4(z)$ and spin-5 $\mathcal{U}_5(z)$. As an example, the eigenvalue of primary field spin-5 $\mathcal{U}_5(z)$ is calculated in appendix-A. Finally, we now write down all the eigenvalues of the primary fields in the following form:

$$\mathcal{U}_2(\Lambda) = \frac{1}{2} \Theta_{(2)} - \frac{1}{4} \left( \frac{N+1}{3} \right) \alpha_o^2 , \quad \mathcal{U}_3(\Lambda) = \frac{1}{3} \Theta_{(3)}$$

(3.10)

$$\mathcal{U}_4(\Lambda) = \frac{1}{4} \Theta_{(4)} + \frac{3}{4N(22 + 5c)} \left[ 5 - 7N + N(3N^2 - 7) \alpha_o^2 \Theta_{(2)}^2 \right]$$

$$-\frac{(N-2)(N-3)}{4N(22 + 5c)} \left[ 9 - N(11N+15) \alpha_o^2 \Theta_{(2)} + \frac{(N-1)(N-2)(N-3)(N+1)}{240(22 + 5c)} \right] \Theta_{(3)}$$

(3.11)

and

$$\mathcal{U}_5(\Lambda) = \frac{1}{5} \Theta_{(5)} + \frac{1}{720N(114 + 7c)} \left[ 10080 + N(7N^4 - 71N^3 + 173N^2 + 191N - 19500) \right]$$

$$-N^2(N-1)(N+1)(7N^3 - 36N^2 - 7N - 684) \alpha_o^2 \Theta_{(2)} \Theta_{(3)}$$

$$+ \frac{(N-3)(N-4)}{1036800N(114 + 7c)} \left[ (N-5)(N-1)^2 N^3 (7N + 13)(N^2 + 5N + 24) \alpha_o^4 \right]$$

$$-2N^2(N-1)(N+1)(7N^5 - 22N^4 - 72N^3 - 398N^2 - 31735N - 56580) \alpha_o^2$$

$$+ 7N^7 - 57N^6 + 38N^5 + 382N^4 - 66285N^3 + 209915N^2 + 604800N - 6220800 \Theta_{(3)}$$

(3.12)

4. A Relation Between Casimir Eigenvalues of $A_{N-1}$-Lie Algebras and Eigenvalue Spectrum of $W_N$-Algebras

In this section, we will try to establish a relation between Casimir eigenvalues of $A_{N-1}$ Lie algebras and the eigenvalues of zero modes of generating fields. Let $\Lambda$ be a dominant weight for $A_{N-1}$ Lie algebra. For an irreducible representation $Rep[\Lambda]$, the eigenvalues of a Casimir operator of order $N$ are given in ref. [21] as in the following form

$$C_N[Rep[\Lambda]] = \dim Rep[\Lambda] P_N[Rep[\Lambda]]$$

(4.1)

where the polinomial $P_N[Rep[\Lambda]]$ is a $N$ order polinomial of $\theta_i$ ($i = 1, 2, \ldots, N$). In the following we will give explicit expression of the eigenvalues for the Casimir operators
\[ C_2[\text{Rep}[\Lambda]] = -\dim \text{Rep}[\Lambda] \left( \frac{12}{(N - 1)N(N + 1)} \Theta(2) - 1 \right) \]  
\[ C_3[\text{Rep}[\Lambda]] = \dim \text{Rep}[\Lambda] \Theta(3) \]  
\[ C_4[\text{Rep}[\Lambda]] = \dim \text{Rep}[\Lambda] (\alpha_4 P_4[\text{Rep}[\Lambda]] + \alpha_3 P_{2,2}[\text{Rep}[\Lambda]]) \]  
\[ C_5[\text{Rep}[\Lambda]] = \dim \text{Rep}[\Lambda] (\beta_5 P_5[\text{Rep}[\Lambda]] + \beta_3 P_{3,2}[\text{Rep}[\Lambda]]) \]  
and \[ P_1[\text{Rep}[\Lambda]] = \frac{720N[(N^2 + 1)\Theta(4) + (3 - 2N^2)\Theta(2)]}{(N - 3)(N - 2)(N - 1)N^2(N + 1)(N + 2)(N + 3)} + 1 \]  
\[ P_{2,2}[\text{Rep}[\Lambda]] = \frac{720[2N(2N^2 - 3)\Theta(4) - (N^4 - 6N^2 + 18)\Theta(2)]}{(N - 3)(N - 2)(N - 1)N^2(N + 1)(N + 2)(N + 3)(6 + 5N^2)} + \frac{120N}{(N - 1)(N + 1)(6 + 5N^2)} \Theta(2) - 1 \]  
\[ P_5[\text{Rep}[\Lambda]] = \frac{N(5 + N^2)}{5(2 - N^2)} \Theta(5) + \Theta(3) \Theta(2) \]  
and \[ P_{3,2}[\text{Rep}[\Lambda]] = \frac{72N(N^2 - 2)\Theta(6) - 12(N^4 + 24)\Theta(3)\Theta(2)}{(N - 4)(N - 3)N^3(N + 3)(N + 4)} + \Theta(3) \]  
with some arbitrary constants \( \alpha_i \) and \( \beta_i \).

Finally, we will show that there is a relation between the eigenvalues of the primary fields and \( P_N[\text{Rep}[\Lambda]] \) polynomials. In other words, it is possible that the eigenvalues of the primary fields can be written with respect to \( P_N[\text{Rep}[\Lambda]] \), and the resulting expressions will be as in following:

\[ \mathcal{U}_2(\Lambda) = \frac{(N - 1)N(N + 1)}{24} (P_1[\text{Rep}[\Lambda]] - 1) \]  
\[ \mathcal{U}_3(\Lambda) = \frac{1}{3} P_1[\text{Rep}[\Lambda]] \]  
\[ \mathcal{U}_4(\Lambda) = \gamma_1 P_4[\text{Rep}[\Lambda]] + \gamma_2 P_{2,2}[\text{Rep}[\Lambda]] + \gamma_3 P_2[\text{Rep}[\Lambda]] + \gamma_4 \]  
and \[ \mathcal{U}_5(\Lambda) = \sigma_1 P_5[\text{Rep}[\Lambda]] + \sigma_2 P_{3,2}[\text{Rep}[\Lambda]] + \sigma_3 P_3[\text{Rep}[\Lambda]] \]

where \[ \gamma_1 = \frac{(N - 1)(N - 2)(N - 3)(N + 1)}{2880(22 + 5c)N} \left[ 36 + N(5N^2 + 42N + 66) + N(N + 2)(N + 3)(6 - 5N^2) \alpha_2^2 \right] \]  
\[ \gamma_2 = \frac{(N - 1)(N - 2)(N - 3)(N + 1)(6 + 5N^2)}{2880(22 + 5c)N} [6 + 11N + N(N + 2)(N + 3) \alpha_2^2] \]  
\[ \gamma_3 = \frac{(N - 1)(N - 2)(N - 3)(N + 1)}{288(22 + 5c)N} [54 + 6N + 11N^2 + N(N^3 + 5N^2 - 60N - 90) \alpha_2^2] \]  
\[ \gamma_4 = \frac{N(N - 1)(N - 2)(N - 3)(N + 1)(6 + 5N)(\alpha_2^2 - 1)}{240(22 + 5c)} \]  
\[ \sigma_1 = \frac{(N^2 - 2)}{24(114 + 7c)N^3(N + 3)(N + 4)} [-3456 - 5602N - 3788N^2 - 89N^3 \]  
\[ + 22N^4 - 7N^5 + (N^3 - N)(336 + 310N + 154N^2 + 13N^3 + 7N^4) \alpha_6^2] \]  
\[ \sigma_2 = \frac{(N - 3)(N - 4)}{8640(114 + 7c)} [8640 + 14005N + 110N^2 + 30N^3 + 22N^4 - 7N^5] \]  
\[ + (N^3 - N)(-840 - 775N + 35N^2 + 13N^3 + 7N^4) \alpha_6^2] \]  
\[ \sigma_3 = \frac{(N - 3)(N - 4)}{1036800(114 + 7c)} [-6220800 - 432000N - 1470685N^2 - 79485N^3 \]  
\[ - 3218N^4 - 2602N^5 + 2N(N^3 - N)(106980 + 78235N - 1702N^2 - 708N^3 \]  
\[ - 398N^4 - 7N^5) \alpha_6^2 + (N - 5)(N - 1)^2 N^3 (13 + 7N)(24 + 5N + N^2) \alpha_6^2] \]
ACKNOWLEDGEMENT  I would like to thank H. R. Karadayı for his valuable discussions and excellent guidance throughout this research.

APPENDIX-A

The primary field $U(z)$, which was used in (3.12) with eigenvalue $U(Λ)$, is in the form of

$$U(z) = U(z) - \frac{(N - 4)}{2} a_0 \partial U(z) +$$

$$+ \frac{3}{4} \frac{(N - 3)(N - 4)}{N(114 + 7c)} [\partial^2 U(z)] +$$

$$+ \frac{(N - 2)(N - 3)(N - 4)}{12N(114 + 7c)} [9 - N(31 + 9N) a_0^3] \partial^3 U(z) +$$

$$+ \frac{(N - 2)(N - 3)(N - 4)}{2N(114 + 7c)} [-7 + N(137 + 7N) a_0^3] \partial^2 U(z) +$$

$$+ \frac{(N - 2)(N - 3)(N - 4)}{2N(114 + 7c)} [-7 + N(137 + 7N) a_0^3] \partial U(z) +$$

$$+ \frac{(N - 2)(N - 3)(N - 4)}{2N(114 + 7c)} [-7 + N(137 + 7N) a_0^3] \partial U(z) +$$

$$+ \frac{(N - 2)(N - 3)(N - 4)}{2N(114 + 7c)} [-7 + N(137 + 7N) a_0^3] \partial U(z) +$$

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