UNIQUENESS OF QUASI-EINSTEIN METRICS
ON 3-DIMENSIONAL HOMOGENEOUS MANIFOLDS

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Abstract. The study of 3-dimensional homogeneous Riemannian manifolds is done, in general, according to the dimension of its isometry group $\text{Iso}(M^3, g)$, which can be 3, 4 or 6. Following this trend we present here a complete description of $m$-quasi-Einstein metrics, when this manifold is compact or not compact provided $\text{dim} \, \text{Iso}(M^3, g) = 4$. In addition, we shall show the absence of such gradient structure on $\text{Sol}^3$, which corresponds to $\text{dim} \, \text{Iso}(M^3, g) = 3$. When $\text{dim} \, \text{Iso}(M^3, g) = 6$ it is well known that $M^3$ is a space form. In this case, its canonical structure gives a trivial example. Moreover, we prove that Berger’s spheres carry a non-trivial quasi-Einstein structure with non gradient associated vector field, this shows that Perelman’s Gradient Theorem cannot be extend to quasi-Einstein metrics. Finally, we prove that a 3-dimensional homogeneous manifold carrying a gradient quasi-Einstein structure is either Einstein or $H^2 \times \mathbb{R}$.

1. Introduction and statement of the results

One of the motivation to study $m$-quasi-Einstein metrics on a Riemannian manifold $(M^n, g)$ is its closed relation with warped product Einstein metrics, see e.g. [8], [7] and [12]. For instance, when $m$ is a positive integer, $m$-quasi-Einstein metrics correspond to exactly those $n$-dimensional manifolds which are the base of an $(n+m)$-dimensional Einstein warped product. One fundamental ingredient to understand the behavior of such a class of manifold is the $m$-Bakry-Emery Ricci tensor which is given by

\begin{equation}
Ric^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df,
\end{equation}

where $f$ is a smooth function on $M^n$ and $\nabla^2 f$ stands for the Hessian form.

This tensor was extended recently, independently, by Barros and Ribeiro Jr [3] and Limoncu [14] for an arbitrary vector field $X$ on $M^n$ as follows:

\begin{equation}
Ric^m_X = Ric + \frac{1}{2} \mathcal{L}_X g - \frac{1}{m} X^\flat \otimes X^\flat,
\end{equation}

where $\mathcal{L}_X g$ and $X^\flat$ denote, respectively, the Lie derivative on $M^n$ and the canonical 1-form associated to $X$.

With this setting we say that $(M^n, g)$ is a $m$-quasi-Einstein metric, if there exist a vector field $X \in \mathfrak{X}(M)$ and constants $0 < m \leq \infty$ and $\lambda$ such that

\begin{equation}
Ric^m_X = \lambda g.
\end{equation}

Classically the study of such metrics are considered when $X$ is a gradient of a smooth function $f$ on $M^n$. Noticing that the trace of $Ric^m_X$ is given by $R + \text{div} X - \frac{1}{m} |X|^2$, where $R$ denotes the scalar curvature, we deduce

\begin{equation}
R + \text{div} X - \frac{1}{m} |X|^2 = \lambda n.
\end{equation}

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On the other hand, when $m$ goes to infinity, equation (1.2) reduces to the one associated to a Ricci soliton, for more details in this subject we recommend the survey due to Cao [6] and the references therein. Whereas, when $m$ is a positive integer and $X$ is gradient, it corresponds to warped product Einstein metrics, for more details see [7]. Following the terminology of Ricci solitons, a $m$-quasi-Einstein metric $g$ on a manifold $M^n$ will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

**Definition 1.** A $m$-quasi-Einstein metric will be called trivial if $X \equiv 0$. Otherwise, it will be nontrivial.

We notice that the triviality implies that $M^n$ is an Einstein manifold. It is important to detach that gradient $1$-quasi-Einstein metrics satisfying $\Delta e^{-f} + \lambda e^{-f} = 0$ are more commonly called static metrics with cosmological constant $\lambda$. These static metrics have been studied extensively because their connection with scalar curvature, the positive mass theorem and general relativity, for more details see e.g. [1], [2] and [9]. On the other hand, it is well known that on a compact manifold $M^n$ a gradient $\infty$-quasi-Einstein metric with $\lambda \leq 0$ is trivial, see [10]. The same result was proved in [13] for gradient $m$-quasi-Einstein metric on compact manifold with $m$ finite. Besides, we known that compact shrinking Ricci solitons have positive scalar curvature, see e.g. [10]. An extension of this result for shrinking gradient $m$-quasi-Einstein metrics with $1 \leq m < \infty$ was obtained in [7]. Recently, in [5] Brozos-Vázquez et al. proved that locally conformally flat gradient $m$-quasi-Einstein metrics are globally conformally equivalent to a space form or locally isometric to a pp-wave or a warped product. In [12] it was given a classification for $m$-quasi-Einstein metrics where the base has non empty boundary. Moreover, they proved a characterization for $m$-quasi-Einstein metrics when the base is locally conformally flat. We point out that Case et al. in [7] proved that every compact gradient $m$-quasi-Einstein metric with constant scalar curvature is trivial, as well as we remember that in [15], Perelman proved that every compact Ricci soliton is gradient.

Here we shall show that Berger’s spheres carry naturally a non trivial structure of quasi-Einstein metrics. Since they have constant scalar curvature, their associated vector fields can not be gradient. In particular, we can not extend Perelman’s result to compact quasi-Einstein metrics. Moreover, these examples show that Theorem 4.6 of [12] can not be extended for a non gradient vector field.

From now on, we shall consider $(M^3, g)$ a simply connected homogeneous Riemannian manifold. We recall that the classification of these manifolds is already well-known according to their isometry group $Iso(M^3, g)$, whose dimension can be 3, 4 or 6, detaching that 6-dimensional are space forms, which are Einstein. But Einstein structures are well known in dimension 3, see e.g. [4]. So, it remains to describe $m$-quasi-Einstein metrics on simply connected homogeneous spaces with isometry group of dimension 3 and 4. When this dimension is 3 they have the geometry of the Lie group $Sol^3$. Concerning to this manifold we have the next result.

**Theorem 1.** $Sol^3$ does not carry any gradient $m$-quasi-Einstein metric.

Proceeding, we consider $dim Iso(M^3, g) = 4$. In this case, such a manifold is a Riemannian fibration onto a 2-dimensional space form $N^2_\kappa$ with constant sectional curvature $\kappa$. In other words, denoting these manifolds by $E^3(\kappa, \tau)$ there is a Riemannian submersion $\pi: E^3(\kappa, \tau) \to N^2_\kappa$ with fibers diffeomorphic either to $S^3$ or to $\mathbb{R}$, depending whether $E^3(\kappa, \tau)$ is compact or not compact. One remarkable propriety of the vector field $E_3$ tangent to the fibers is that it is a Killing vector field for which $\nabla_X E_3 = \tau X \times E_3$ for all $X \in \mathfrak{x}(M)$, where $\tau$ is a constant, called curvature of the bundle, while $\times$ means cross product. Denoting Heisenberg’s space by $Nil_3(\kappa, \tau)$ and Berger’s sphere by $S^3_{\kappa, \tau}$ we have the following
alternatives for $E^3(\kappa, \tau)$:

$$E^3(\kappa, \tau) = \begin{cases} 
\mathbb{S}_\kappa^2 \times \mathbb{R}, & \kappa > 0, \tau = 0 \\
\mathbb{H}_\kappa^2 \times \mathbb{R}, & \kappa < 0, \tau = 0 \\
\text{Nil}_3(\kappa, \tau), & \kappa = 0, \tau \neq 0 \\
\text{PSL}_2(\kappa, \tau), & \kappa < 0, \tau \neq 0 \\
\mathbb{S}_\kappa^3, & \kappa > 0, \tau \neq 0.
\end{cases}$$

On the other hand, we shall show in Lemma 6 that for a 3-dimensional homogeneous Riemannian manifold $E^3(\kappa, \tau)$ whose group of isometries has dimension 4 its Ricci tensor satisfies

$$(1.5) \quad \text{Ric} - (4\tau^2 - \kappa)E^3_3 \otimes E^3_3 = (\kappa - 2\tau^2)g.$$ 

Now taking into account that $E_3$ is a Killing vector field, if the vector field $X = \sqrt{m(4\tau^2 - \kappa)}E_3$ is well defined, then we have $\frac{1}{2}\mathcal{L}_X g = 0$. Whence, letting $\lambda = \kappa - 2\tau^2$ we obtain

$$\text{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^3 \otimes X^3 = \lambda g,$$

which gives the next example:

**Example 1.** Let $E^3(\kappa, \tau)$ be a 3-dimensional homogeneous Riemannian manifolds with 4-dimensional isometry group such that $X = \sqrt{m(4\tau^2 - \kappa)}E_3$ is well defined. Letting $\lambda = \kappa - 2\tau^2$ we deduce that $(E^3(\kappa, \tau), g, X, \lambda)$ is a m-quasi-Einstein metric. We notice that in this case $X$ is not necessarily type gradient.

We point out that if $\lambda > 0$ we can prove by a similar argument used in [11] and [18] that $(E^3(\kappa, \tau), g)$ is compact. On the other hand, for $\mathbb{S}_\kappa^2 \times \mathbb{R}$ we have $4\tau^2 < \kappa$, therefore, in the previous example we must have $(E^3(\kappa, \tau), g) \neq \mathbb{S}_\kappa^2 \times \mathbb{R}$. Moreover, we shall show that $\mathbb{H}_\kappa^2 \times \mathbb{R}$ is the unique case for which the associated vector field is gradient, more precisely, $X$ is the gradient of $f$ given according to Example 3 and Example 4. In the others cases the associated vector fields are not gradient. Whence, we present the first examples of compact and not compact m-quasi-Einstein metrics with not gradient vector field making sense the general definition (1.2) of $E^3(\kappa, \tau)$.

Concerning to Berger’s sphere we detach that they admit shrinking, expanding and steady not gradient m-quasi-Einstein metrics, since $\lambda = \kappa - 2\tau^2$ can assume any sign.

Proceeding, it is important to detach that on $\mathbb{H}_\kappa^2 \times \mathbb{R}$ we have two examples of gradient quasi-Einstein structure. First, we have the following example for a Killing vector field.

**Example 2.** We consider $\mathbb{H}_\kappa^2 \times \mathbb{R}$ with its standard metric and the potential function $f(x, y, t) = \pm \sqrt{-\kappa t} + c$, where $c$ is a constant. It is easy to see that $\nabla f = \pm \sqrt{-\kappa} \partial_t$, hence $\text{Hess} f = 0$. Therefore $(\mathbb{H}_\kappa^2 \times \mathbb{R}, \nabla f, \kappa)$ is a quasi-Einstein metric.

Next we shall describe our second example on $\mathbb{H}_\kappa^2 \times \mathbb{R}$, where its associated vector field is not a Killing vector field.

**Example 3.** We consider $\mathbb{H}_\kappa^2 \times \mathbb{R}$ with its standard metric and the potential function $f(x, y, t) = -m \ln \cosh \left[ \sqrt{-\frac{\kappa}{m}} (t + a) \right] + b$, where $a$ and $b$ are constants. Under these conditions $(\mathbb{H}_\kappa^2 \times \mathbb{R}, \nabla f, \kappa)$ is a quasi-Einstein metric.

Now, it is natural to ask what are the $m$-quasi-Einstein metrics on $E^3(\kappa, \tau)$? In fact, for gradient quasi-Einstein structure on not compact manifolds those presented in Example 3 and Example 4 are unique. Therefore, we deduce the following uniqueness theorem.

**Theorem 2.** Let $(E^3(\kappa, \tau), g, \nabla f, \lambda)$ be a 3-dimensional homogeneous gradient quasi-Einstein metric. Then this structure is either Einstein or is $\mathbb{H}_\kappa^2 \times \mathbb{R}$ such as in either Example 3 or Example 4. In particular, $g$ is a static metric provided $m = 1$. 


As a consequence of Theorem 2 we shall derive the following corollary.

**Corollary 1.** $S^2 \times \mathbb{R}, Nil_3(\kappa, \tau)$ and $\tilde{PSl}_2(\kappa, \tau)$ do not carry a gradient quasi-Einstein structure.

### 2. Preliminaries

In this section we shall develop a few tools concerning to 3-dimensional homogeneous Riemannian manifolds according to the dimension of their isometry group in order to prove our results. A good reference for this subject is the book of Thurston [17].

#### 2.1. 3-dimensional homogeneous manifold with isometry group of dimension 3

It is well-known that 3-dimensional homogeneous manifold with 3-dimensional isometry group has the geometry of the Lie group $Sol^3$. Moreover, we may consider $Sol^3$ as $\mathbb{R}^3$ endowed with the metric

$$g_{Sol^3} = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2.$$  \hfill (2.1)

Whence, we can check directly that the next set gives an orthonormal frame on $Sol^3$.

$$\{E_1 = e^{-t}\partial_x, E_2 = e^{t}\partial_y, E_3 = \partial_t\}. \hfill (2.2)$$

By using this frame we obtain the next lemma.

**Lemma 1.** Let us consider on $Sol^3$ the metric and the frame given, respectively, by (2.1) and (2.2). Then its Riemannian connection $\nabla$ obeys the rules:

$$\begin{align*}
\nabla_{E_1}E_1 &= -E_3 & \nabla_{E_1}E_2 &= 0 & \nabla_{E_1}E_3 &= E_1 \\
\nabla_{E_2}E_1 &= 0 & \nabla_{E_2}E_2 &= E_3 & \nabla_{E_2}E_3 &= -E_2 \\
\nabla_{E_3}E_1 &= 0 & \nabla_{E_3}E_2 &= 0 & \nabla_{E_3}E_3 &= 0.
\end{align*} \hfill (2.3)$$

Moreover, the Lie brackets satisfy:

$$[E_1, E_2] = 0, [E_1, E_3] = E_1 \text{ and } [E_2, E_3] = -E_2. \hfill (2.4)$$

Next we use this lemma in order to compute the Ricci tensor of $Sol^3$. More exactly, we have.

**Lemma 2.** The Ricci tensor of $Sol^3$ is given by $Ric = -2E_3 \otimes E_3$.

**Proof.** Computing $Ric(E_1, E_1)$ with the aid of (2.3) and (2.4) we obtain

$$Ric(E_1, E_1) = \langle \nabla_{E_2}\nabla_{E_1}E_1 - \nabla_{E_1}\nabla_{E_2}E_1 + \nabla_{[E_1, E_2]}E_1, E_2 \rangle$$

$$+ \langle \nabla_{E_3}\nabla_{E_1}E_1 - \nabla_{E_1}\nabla_{E_3}E_1 + \nabla_{[E_1, E_3]}E_1, E_3 \rangle$$

$$= \langle -\nabla_{E_2}E_3, E_2 \rangle + \langle \nabla_{E_1}E_1, E_3 \rangle = 0.$$

In a similar way we show that $Ric(E_i, E_j) = 0$, for $i \neq j$ as well as $i = j = 2$. Finally, we have

$$Ric(E_3, E_3) = \langle \nabla_{E_2}\nabla_{E_3}E_3 - \nabla_{E_3}\nabla_{E_2}E_3 + \nabla_{[E_3, E_2]}E_3, E_1 \rangle$$

$$+ \langle \nabla_{E_2}\nabla_{E_3}E_3 - \nabla_{E_3}\nabla_{E_2}E_3 + \nabla_{[E_3, E_2]}E_3, E_2 \rangle$$

$$= \langle \nabla_{[E_3, E_1]}E_1, E_1 \rangle + \langle \nabla_{[E_3, E_2]}E_3, E_2 \rangle = -2,$$

which completes the proof of the lemma.  \hfill □

Proceeding we have the following lemma for $Sol^3$.

**Lemma 3.** Suppose that $(Sol^3, g, X, \lambda)$ carries a m-quasi-Einstein metric. Then the following statements hold:
Lemma 3 to obtain
This proves the first part of the proof.

Now, we substitute the last three equations in (2.6) to arrive at
and
Proof. Firstly, computing $\text{Ric}^m_{\lambda}(E_1, E_1)$ in equation (1.2), we obtain

$$\text{Ric}(E_1, E_1) + \langle \nabla_{E_1} X, E_1 \rangle - \frac{1}{m}(X, E_1)^2 = \lambda.$$ 

We may use Lemma 2 to deduce $\text{Ric}(E_1, E_1) = 0$, thus, since $\nabla_{E_1} E_1 = -E_3$, we obtain the first assertion.

The other ones are obtained by the same way. Computing $\text{Ric}^m_{\lambda}(E_2, E_2)$, $\text{Ric}^m_{\lambda}(E_3, E_3)$, $\text{Ric}^m_{\lambda}(E_2, E_1)$, $\text{Ric}^m_{\lambda}(E_3, E_1)$ and $\text{Ric}^m_{\lambda}(E_3, E_2)$ we arrive at (2), (3), (4), (5) and (6). We left its check in for the reader. So, we finish the proof of the lemma. $\square$

2.1.1. Proof of Theorem 7

Proof. Let us suppose the existence of a gradient quasi-Einstein structure on $S^3$. From item (3) of Lemma 3 we have

$$(2.5) \quad \partial_t \langle \nabla f, \partial_t \rangle = \frac{1}{m}(\nabla f, \partial_t)^2 + \lambda + 2,$$

where $f$ is the potential function. Under this condition (2.5) is a separable ODE and we may use the differentiability of $\langle \nabla f, \partial_t \rangle$ to obtain the solutions $\langle \nabla f, \partial_t \rangle = \pm \sqrt{-m(\lambda + 2)}$ and

$$\langle \nabla f, \partial_t \rangle = -\sqrt{-m(\lambda + 2)} \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right],$$

where $\psi$ is a function that does not depend on $t$. Therefore, we may say that the potential function is either $f = \varphi \pm \sqrt{-m(\lambda + 2)}t$ or $f = \varphi - m \ln \cosh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right]$, where $\varphi$ does not depend on $t$. Now, we divide our proof in two cases.

Our first case is when $f = \varphi \pm \sqrt{-m(\lambda + 2)}t$. In this case, we can admit without loss generality, that $f = \varphi + \sqrt{-m(\lambda + 2)}t$. Thus, item (5) of Lemma 3 gives $E_1(f) = 0$, which implies from the first item of Lemma 3 that $0 \leq \sqrt{-\frac{\lambda + 2}{m}} \lambda \leq -2$, giving a contradiction. This proves the first part of the proof.

Otherwise, if $f = \varphi - m \ln \cosh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right]$, we may substitute $f$ in equation (5) of Lemma 3 to obtain

$$(2.6) \quad \partial^2_{xx} f = \left( \frac{1}{m} \partial_t f + 1 \right) \partial_x f.$$ 

On the other hand, we have

$$(2.7) \quad \partial^2_{xx} f = (\lambda + 2) \partial_x \varphi \sech^2 \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right],$$

$$(2.8) \quad \partial_x f = \partial_x \varphi - \sqrt{-m(\lambda + 2)} \partial_x \psi \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right]$$

and

$$(2.9) \quad \partial_t f = -\sqrt{-m(\lambda + 2)} \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right].$$

Now, we substitute the last three equations in (2.6) to arrive at
(2.10) \[ \sqrt{m}(\partial_x \varphi - (\lambda + 2)\partial_x \psi) = \sqrt{-m}(\partial_x \varphi + m\partial_x \psi) \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right]. \]

From what it follows that
(2.11) \[ \partial_x \varphi - (\lambda + 2)\partial_x \psi = 0 \]

and
(2.12) \[ \partial_x \varphi + m\partial_x \psi = 0. \]

In a similar way, we may substitute \( f \) in item (6) of Lemma 3 to obtain
(2.13) \[ \partial_{yt}^2 f = \left( \frac{1}{m} \partial_t f - 1 \right) \partial_y f. \]

On the other hand, it easy see that
(2.14) \[ \partial_{yt}^2 f = (\lambda + 2)\partial_y \psi \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right], \]
(2.15) \[ \partial_y f = \partial_y \varphi - \sqrt{-m(\lambda + 2)} \partial_y \psi \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right] \]
and
(2.16) \[ \partial_t f = -\sqrt{-m(\lambda + 2)} \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right]. \]

Therefore, substituting (2.14), (2.15) and (2.16) in item (4) of Lemma 3 we have
\[ \sqrt{m}[\partial_y \varphi + (\lambda + 2)\partial_y \psi] = \sqrt{-m}[\partial_y \varphi - m\partial_y \psi] \tanh \left[ \sqrt{-\frac{\lambda + 2}{m}}(\psi + t) \right], \]
which implies
(2.17) \[ \partial_y \varphi + (\lambda + 2)\partial_y \psi = 0 \]
and
(2.18) \[ \partial_y \varphi - m\partial_y \psi = 0. \]

Now, we consider \( \lambda \neq -m - 2 \), therefore, from (2.11), (2.12), (2.17) and (2.18) we have
\[ \partial_x \varphi = \partial_y \varphi = \partial_x \psi = \partial_y \psi = 0, \]
which implies that \( f \) do not depend of \( x \) and \( y \). Moreover, by using items (1) and (2) of Lemma 3 we conclude that \( f \) does not depend on \( t \) and then \( f \) is constant, which is a contradiction.

Since \( \lambda = -m - 2 \), we may derive item (1) of Lemma 3 with respect to \( t \), item (5) with respect to \( x \) and we compare the results to arrive at
(2.19) \[ (\partial_t f + m)\partial_{xx}^2 f - \partial_x f\partial_{xt}^2 f = -m[2(\partial_t f - \lambda) + \partial_{tt}^2 f]e^{2t}. \]

Next, we substitute item (5) of Lemma 3 in (2.19) to obtain
\[ (\partial_t f + m) \left( \partial_{xx}^2 f - \frac{1}{m}(\partial_x f)^2 \right) = -m[2(\partial_t f - \lambda) + \partial_{tt}^2 f]e^{2t}, \]
thus we may use again item (1) of Lemma 3 to conclude
\[ (\partial_t f - m)(\partial_t f - \lambda) = m\partial_{tt}^2 f. \]
Finally, it suffices to use item (3) of Lemma 3 to obtain $\partial_t f = m$, which is a contradiction. So, we finish the proof of the theorem.

2.2. Non compact 3-dimensional homogeneous manifold with isometry group of dimension 4. We recall that the projection $\pi : E^3(\kappa, \tau) \to N^2_\kappa$, given by $\pi(x, y, t) = (x, y)$ is a submersion, where $N^2_\kappa$ is endowed with its canonical metric $ds^2 = \rho^2(dx^2 + dy^2)$, where $\rho = 1$ or $\rho = \frac{2}{1+\kappa(x^2+y^2)}$ according to $\kappa = 0$ or $\kappa \neq 0$, respectively. The natural orthonormal frame on $N^2_\kappa$ is given by $\{e_1 = \rho^{-1}\partial_x, e_2 = \rho^{-1}\partial_y\}$. Moreover, translations along the fibers are isometries, therefore $E_3$ is a Killing vector field. Thus, considering horizontal lifting of $\{e_1, e_2\}$ we obtain $\{E_1, E_2\}$, which jointly with $E_3$ gives an orthonormal frame $\{E_1, E_2, E_3\}$ on $E^3(\kappa, \tau)$. In addition, since $\{\partial_t, \partial_y\}$ is a natural frame for $N^2_\kappa$, then a natural frame for $E^3(\kappa, \tau)$ is $\{\partial_x, \partial_y, \partial_t\}$, where $\partial_t$ is tangent to the fibers. Using this frame we have the following lemma for a non compact 3-dimensional homogeneous manifold which can be found in [17].

**Lemma 4.** Rewriting the referential $\{E_1, E_2, E_3\}$ in terms of $\{\partial_x, \partial_y, \partial_t\}$, we have:

1. If $\kappa \neq 0$, then $E_1 = \frac{1}{\rho}\partial_x + 2\kappa\tau y \partial_t$, $E_2 = \frac{1}{\rho}\partial_y - 2\kappa\tau x \partial_t$ and $E_3 = \partial_t$.

2. If $\kappa = 0$, then $E_1 = \partial_x - \tau y \partial_t$, $E_2 = \partial_y + \tau x \partial_t$ and $E_3 = \partial_t$.

Moreover, endowing $E^3(\kappa, \tau)$ with the metric

$$g = \begin{cases} dx^2 + dy^2 + [\tau(xdy - ydx) + dt]^2, & \kappa = 0 \\ \rho^2(dx^2 + dy^2) + [2\kappa\tau\rho(ydx - xdy) + dt]^2, & \kappa \neq 0. \end{cases}$$

we have the following identities for its Riemannian connection $\nabla$:

\begin{align}
\nabla_{E_1} E_1 &= \kappa y E_2 \\
\nabla_{E_1} E_2 &= -\kappa y E_1 + \tau E_3 \\
\nabla_{E_1} E_3 &= -\tau E_2 \\
\nabla_{E_2} E_1 &= -\kappa x E_2 - \tau E_3 \\
\nabla_{E_2} E_2 &= \kappa x E_1 \\
\nabla_{E_2} E_3 &= \tau E_1 \\
\nabla_{E_3} E_1 &= -\tau E_2 \\
\nabla_{E_3} E_2 &= \tau E_1 \\
\nabla_{E_3} E_3 &= 0.
\end{align}

In particular, we obtain from the above identities the following relations for the Lie brackets:

\begin{align}
\{E_1, E_2\} &= -\kappa y E_1 + \kappa x E_2 + 2\tau E_3 \\
\{E_1, E_3\} &= [E_2, E_3] = 0.
\end{align}

Moreover, up to isometries, we may assume that $\kappa = -1, 0$ or 1.

2.3. Compact 3-dimensional homogeneous manifold with isometry group of dimension 4. Firstly, we recall that compact homogeneous Riemannian manifolds $E^3(\kappa, \tau)$ are Berger’s spheres. For sake of completeness and to keep the same notation we shall choose the next construction for Berger’s sphere, for more details see [4]. In what follow, Berger’s sphere is a standard 3-dimensional sphere

$$S^3 = \{(z, w) \in \mathbb{C}^2; |z|^2 + |w|^2 = 1\}$$

endowed with the family of metrics

$$g_{\kappa, \tau}(X, Y) = \frac{4}{\kappa} \left[ \langle X, Y \rangle + \left( \frac{4\tau^2}{\kappa} - 1 \right) \langle X, V \rangle \langle Y, V \rangle \right],$$

where $\langle \cdot, \cdot \rangle$ stands for the standard metric on $S^3$, $V_{(z, w)} = (iz, iw)$ for each $(z, w) \in S^3$ and $\kappa, \tau$ are real numbers with $\kappa > 0$ and $\tau \neq 0$. In particular, $g_{1, 1}$ is the round metric. In addition, Berger’s sphere $(S^3, g_{\kappa, \tau})$ will be denoted by $S^3_{\kappa, \tau}$, which is a model for a homogeneous space $E^3(\kappa, \tau)$ when $\kappa > 0$ and $\tau \neq 0$. In this case the vertical Killing vector
field is given by $E_3 = \frac{1}{\sqrt{2}} V$. In order to obtain an orthonormal frame we choose $E_1(z, w) = \frac{1}{\sqrt{2}} (-z, w)$ and $E_2(z, w) = \frac{1}{\sqrt{2}} (-w, z)$.

Using this frame we have the following lemma for $\mathbb{S}^3_{\kappa, \tau}$ which can be found in [5].

**Lemma 5.** The Riemannian connection $\nabla$ on $\mathbb{S}^3_{\kappa, \tau}$ is determined by

\[
\begin{align*}
\nabla_{E_1} E_1 &= 0 \\
\nabla_{E_2} E_1 &= \tau E_3 \\
\nabla_{E_3} E_1 &= \frac{\kappa - 2\tau^2}{2\tau} E_2 - \frac{\kappa - 2\tau^2}{2\tau} E_3 \\
\nabla_{E_2} E_2 &= 0 \\
\nabla_{E_2} E_3 &= -\tau E_1 \\
\nabla_{E_3} E_2 &= \frac{\kappa - 2\tau^2}{2\tau} E_1 \\
\nabla_{E_3} E_3 &= 0.
\end{align*}
\]

It is immediate to verify that the Lie brackets satisfy:

\[
[E_1, E_2] = -2\tau E_3, \quad [E_2, E_3] = -\frac{\kappa}{2\tau} E_1, \quad [E_1, E_3] = \frac{\kappa}{2\tau} E_2.
\]

### 3. Key Results

As a consequence of Lemmas 4 and 5 we can explicit the Ricci tensor of a 3-dimensional homogeneous Riemannian manifold with 4-dimensional isometry group according to next lemma.

**Lemma 6.** Let $E^3(\kappa, \tau)$ be a 3-dimensional homogeneous Riemannian manifold with 4-dimensional isometry group. Then, each frame $\{E_1, E_2, E_3\}$ constructed before on $E^3(\kappa, \tau)$ diagonalizes the Ricci tensor. More precisely, we have

\[
\text{Ric} = (\kappa - 2\tau^2)g - (\kappa - 4\tau^2)E_3^\flat \otimes E_3^\flat.
\]

**Proof.** Firstly, we consider $E^3(\kappa, \tau)$ a non compact 3-dimensional homogeneous Riemannian manifold with 4-dimensional isometry group. Since we can write the Ricci tensor as follows

\[
\text{Ric}(X, Y) = \sum_{j, k=1}^{3} \langle X, E_j \rangle \langle Y, E_k \rangle \text{Ric}(E_j, E_k),
\]

in order to find $\text{Ric}(E_j, E_k)$ we shall show that $\text{Ric}(E_j, E_k) = \lambda_j \delta_{jk}$. Indeed, using Lemma 5 we have

\[
\begin{align*}
\text{Ric}(E_1, E_1) &= \langle \nabla_{E_2} \nabla_{E_1} E_1 - \nabla_{E_1} \nabla_{E_2} E_1 + \nabla_{[E_1, E_2]} E_1, E_2 \rangle \\
&+ \langle \nabla_{E_3} \nabla_{E_1} E_1 - \nabla_{E_1} \nabla_{E_3} E_1 + \nabla_{[E_1, E_3]} E_1, E_3 \rangle \\
&= \langle \nabla_{E_2} (\kappa y E_2) + \kappa x \nabla_{E_1} E_1 + \kappa x \nabla_{E_2} E_1 + 2\tau \nabla_{E_3} E_1, E_2 \rangle \\
&+ \langle \nabla_{E_3} (\kappa y E_2) + \nabla_{E_1} (\tau E_2), E_3 \rangle \\
&= \frac{2\kappa}{\rho} - \kappa^2 (x^2 + y^2) - 2\tau^2 = \kappa - 2\tau^2.
\end{align*}
\]

In a similar way, we have $\text{Ric}(E_2, E_2) = \kappa - 2\tau^2$ and $\text{Ric}(E_3, E_3) = 2\tau^2$.

Now we claim that $\text{Ric}(E_j, E_k) = 0$ for $j \neq k$. In fact, let us compute only $\text{Ric}(E_1, E_2)$, since the others term follow mutatis mutandis.

\[
\begin{align*}
\text{Ric}(E_1, E_2) &= \langle \nabla_{E_3} \nabla_{E_1} E_2 - \nabla_{E_1} \nabla_{E_3} E_2 + \nabla_{[E_1, E_3]} E_2, E_2 \rangle \\
&= \langle \nabla_{E_1} (\kappa y E_1 + \tau E_3) - \nabla_{E_1} (\tau E_3), E_3 \rangle \\
&= \kappa y \langle \nabla_{E_1} E_1, E_3 \rangle - \tau \langle \nabla_{E_1} E_1, E_3 \rangle = 0,
\end{align*}
\]

which finishes our claim. Therefore, using (3.2), we deduce

\[
\text{Ric} = (\kappa - 2\tau^2)g - (\kappa - 4\tau^2)E_3^\flat \otimes E_3^\flat.
\]
which completes the proof in this case. We now point out that using Lemma 4 straightforward computations as above give the same result for Berger’s spheres. Then we complete the proof of the lemma.

Lemma 7. Let \((E^3(\kappa, \tau), g, X, \lambda)\) be a non compact 3-dimensional homogeneous m-quasi-Einstein metric with 4-dimensional isometry group. If \(E_1, E_2\) and \(E_3\) are given by Lemma 4 then hold:

\[
\begin{align*}
(3.4) \quad & E_1(X, E_1) - \kappa y(X, E_2) = \frac{1}{m} (X, E_1)^2 + \lambda - (\kappa - 2\tau^2). \\
(3.5) \quad & E_2(X, E_2) - \kappa x(X, E_1) = \frac{1}{m} (X, E_2)^2 + \lambda - (\kappa - 2\tau^2). \\
(3.6) \quad & E_3(X, E_3) = \frac{1}{m} (X, E_3)^2 + \lambda - 2\tau^2. \\
(3.7) \quad & E_2(X, E_1) + E_1(X, E_2) + \kappa (g(X, E_1) + x(X, E_2)) = \frac{2}{m} (X, E_1)(X, E_2). \\
(3.8) \quad & E_3(X, E_1) + E_1(X, E_3) + 2\tau(X, E_2) = \frac{2}{m} (X, E_1)(X, E_3). \\
(3.9) \quad & E_3(X, E_2) + E_2(X, E_3) - 2\tau(X, E_1) = \frac{2}{m} (X, E_2)(X, E_3).
\end{align*}
\]

Proof. We notice that by using equation (1.2) we can write

\[
(3.10) \quad \mathcal{L}_X g(E_i, E_j) = 2 \left( \lambda \delta_{ij} - \text{Ric}(E_i, E_j) + \frac{1}{m} (X, E_i)(X, E_j) \right).
\]

Taking into account that \(\mathcal{L}_X g(E_i, E_j) = \langle \nabla E_i, X, E_j \rangle + \langle \nabla E_j, X, E_i \rangle\) we use the compatibility of the metric \(g\) to infer

\[
(3.11) \quad E_i(X, E_j) + E_j(X, E_i) - \langle X, \nabla E_i, E_j + \nabla E_j, E_i \rangle = 2 \left( \lambda \delta_{ij} - R_{ij} + \frac{1}{m} X_i X_j \right).
\]

Therefore, using Lemma 4 and (3.11) straightforward computations give the desired statements.

4. Proof of Theorem 2

Proof. Since \(M^3\) is a 3-dimensional homogeneous manifold, its isometry group has dimension 3, 4 or 6. Making use of Theorem 1 we can discard \(\text{Sol}^3\). Moreover, when \(\dim \text{Iso}(E^3(\kappa, \tau), g) = 6\) we have space forms, which give Einstein metrics. Therefore, it remains to describe gradient quasi-Einstein structures of homogeneous spaces with isometry group of dimension 4, which were denoted by \(E^3(\kappa, \tau)\).

We start solving the ODE of (3.6) to conclude that either

\[
\langle \nabla f, E_3 \rangle = \pm \sqrt{-m(\lambda - 2\tau^2)}
\]

or

\[
\langle \nabla f, E_3 \rangle = -\sqrt{-m(\lambda - 2\tau^2)} \tanh \left[ \sqrt{-\frac{\lambda - 2\tau^2}{m}} (\psi + t) \right],
\]

where \(\psi \in C^\infty(E^3(\kappa, \tau))\) does not depend of \(t\). Taking into account this two possibilities to \(\langle \nabla f, E_3 \rangle\) we conclude that \(f\) is given by either

\[
f = \varphi \pm \sqrt{-m(\lambda - 2\tau^2)} t
\]
or

\[ f = \varphi - m \log \cosh \left( \sqrt{\frac{\lambda - 2\tau^2}{m}} (\psi + t) \right), \]

where \( \varphi \in C^\infty(E^3(\kappa, \tau)) \) does not depend of \( t \).

Now, we shall divide this part of the proof in two cases.

First, if \( f = \varphi \pm \sqrt{-m(\lambda - 2\tau^2)} t \), then we substitute \( f \) in equations (3.8) and (3.9), respectively, to arrive at

\[
\tau E_2(f) = \pm \sqrt{\frac{\lambda - 2\tau^2}{m}} E_1(f)
\]

and

\[
\tau E_1(f) = \mp \sqrt{\frac{\lambda - 2\tau^2}{m}} E_2(f).
\]

On the other hand, by using equation (3.12) of Lemma 3.2 and item (b) of Proposition 3.6, both in [7], we conclude that \( \lambda \) and \( \tau \) can not be zero simultaneously.

Therefore, our two possibilities (4.1) gives

\[
E_1(f) = E_2(f) = 0,
\]

which substituted in (3.21) implies \( \lambda = \kappa - 2\tau^2 \). We notice that \( \mathbb{S}_k^2 \times \mathbb{R} \) has \( \tau = 0 \), therefore, if \( (E^3(\kappa, \tau), g) = \mathbb{S}_k^2 \times \mathbb{R} \) we use Qian’s Theorem [16] to conclude that \( \mathbb{S}_k^2 \times \mathbb{R} \) is compact, which is a contradiction, see also [13] and [11]. On the other hand, since \( [E_1, E_2] = -\kappa y E_1 + \kappa x E_2 + 2\tau E_3 \), we can use (4.3) to obtain \( 2\tau E_3(f) = 0 \). From what it follows that \( (E^3(\kappa, \tau), g) \) can not be \( Nil_3(\kappa, \tau) \) and \( \mathbb{PSL}_2(\kappa, \tau) \). Therefore, \( E^3(\kappa, \tau) = \mathbb{H}_k^2 \times \mathbb{R} \) and \( \lambda = \kappa \), which finishes the first case.

Proceeding we consider \( f = \varphi - m \log \cosh \left[ \sqrt{\frac{\lambda - 2\tau^2}{m}} (\psi + t) \right] \). In this case, we start supposing that \( E^3(\kappa, \tau) = Nil_3(\kappa, \tau) \). Therefore, from (3.8) we obtain

\[
E_1 E_3(f) + \tau E_2(f) = \frac{1}{m} E_1(f) E_3(f).
\]

From what it follows that

\[
\partial^2_{tt} f - \tau y E_3 E_3(f) + \tau E_2(f) = \frac{1}{m} [\partial_x f - \tau y E_3(f)] E_3(f),
\]

which compared with (3.6) gives

\[
\partial^2_{tt} f - \tau(\lambda - 2\tau^2)y + \tau E_2(f) = \frac{1}{m} \partial_x f \partial_t f.
\]

Substituting the value of \( f \) in (4.3) we obtain

\[
\sqrt{-(\lambda - 2\tau^2)} [m \tau (\partial_y \psi + \tau x) - \partial_x \varphi] \tanh \left( \sqrt{\frac{\lambda - 2\tau^2}{m}} (\psi + t) \right)
\]

\[
= \sqrt{m} \left[ (\lambda - 2\tau^2)(\partial_x \psi - \tau y) + \tau \partial_y \varphi \right]
\]

Now, we notice that the right hand side of the previous expression does not depend on \( t \), thus, since \( \lambda - 2\tau^2 \neq 0 \), we have \( \tanh \left[ \sqrt{\frac{-(\lambda - 2\tau^2)}{m}} (\psi + t) \right] \neq 0 \), which implies that

\[
\partial_x \varphi - m \tau \partial_y \psi = m \tau^2 x
\]

and

\[
\tau \partial_y \varphi + (\lambda - 2\tau^2) \partial_x \psi = \tau(\lambda - 2\tau^2)y.
\]
In a similar way we use equations (3.6) and (3.9) to obtain

\[ \partial_y \varphi + m \tau \partial_x \psi = m \tau^2 y \]

and

\[ \tau \partial_x \varphi - (\lambda - 2 \tau^2) \partial_y \psi = \tau (\lambda - 2 \tau^2) x. \]

Now, we may combine (4.6) with (4.9) and (4.7) with (4.8) to obtain, respectively, \[ \partial_y \psi = -\tau x \] and \[ \partial_x \psi = \tau y, \] which gives \( \tau = 0 \). So, we obtain a contradiction.

Therefore, since \( E^3(\kappa, \tau) \neq Nil_3(\kappa, \tau) \), we can use (3.8) and (3.6) to arrive at

\[ \frac{1}{\rho} \frac{\partial^2_{xy} f}{m} + 2 \kappa \tau (\lambda - 2 \tau^2) x + \tau E_2(f) = \frac{1}{m \rho} \partial_x f \partial_t f. \]

Now, we substitute the value of \( f \) in (4.10) to obtain

\[ \sqrt{-\left(\lambda - 2 \tau^2\right)} \left\{ \frac{1}{\rho} \left( m \tau \partial_y \psi - \partial_x \varphi \right) - 2 m \kappa \tau^2 x \right\} \tanh \left\{ \sqrt{-\frac{(\lambda - 2 \tau^2)}{m}} (\psi + t) \right\} \]

by using a similar argument used previously we conclude

\[ \partial_x \varphi - m \tau \partial_y \psi = -2 m \kappa \tau^2 x \rho \]

and

\[ \tau \partial_y \varphi + (\lambda - 2 \tau^2) \partial_x \psi = -2 \kappa \tau (\lambda - 2 \tau^2) y \rho. \]

Analogously, from (3.6) and (3.9) we have

\[ \frac{1}{\rho} \frac{\partial^2_{yx} f}{m} - 2 \kappa \tau (\lambda - 2 \tau^2) x - \tau E_1(f) = \frac{1}{m} \partial_y f \partial_t f. \]

Substituting the value of \( f \) we obtain

\[ \sqrt{-\left(\lambda - 2 \tau^2\right)} \left\{ \frac{1}{\rho} \left( m \tau \partial_x \psi + \partial_y \varphi \right) + 2 m \kappa \tau^2 y \right\} \tanh \left\{ \sqrt{-\frac{(\lambda - 2 \tau^2)}{m}} (\psi + t) \right\} \]

\[ = -\sqrt{m} \left\{ \frac{1}{\rho} \left[ (\lambda - 2 \tau^2) \partial_y \psi - \tau \partial_x \varphi \right] - 2 \kappa \tau (\lambda - 2 \tau^2) \right\}, \]

which gives

\[ \partial_y \varphi + m \tau \partial_x \psi = -2 m \kappa \tau^2 y \rho \]

and

\[ \tau \partial_x \varphi - (\lambda - 2 \tau^2) \partial_y \psi = -2 \kappa \tau (\lambda - 2 \tau^2) x \rho. \]

Finally, we may combine (4.11) with (4.15) and (4.12) with (4.14), respectively, to obtain \( \partial_y \psi = 2 \kappa \tau x \rho \) and \( \partial_x \psi = -2 \kappa \tau y \rho \), hence \( \tau = 0 \). Since \( \tau = 0 \) we can use (4.11), (4.12), (4.14) and (4.15) to conclude that \( \varphi \) and \( \psi \) are constants. Thus, \( f \) depend only on \( t \) and then, from (3.9) we have \( \lambda = \kappa < 0 \) and \( E^3(\kappa, \tau) = E^2_\kappa \times \mathbb{R} \), which is in accordance to Example 3.

So, we finish the proof of the theorem.

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