SCREEN ALMOST SEMI-INVARIANT LIGHTLIKE SUBMANIFOLDS
OF INDEFINITE KAehler MANIFOLDS

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Abstract. In the present paper, we introduce screen almost semi-invariant (SASI) lightlike submanifolds of indefinite Kaehler manifolds. We obtain the necessary and sufficient condition for the induced connection to be a metric connection on SASI-lightlike submanifolds and construct an example for this manifold. Also we find some conditions for integrability of distributions and investigate certain characterizations.

1. INTRODUCTION

One of the most important issues of differential geometry is the Riemannian geometry of submanifolds [7]. Obviously, semi-Riemannian submanifolds have similar properties with Riemannian submanifolds, but the lightlike submanifolds [9] are different since (contrary to the nondegenerate cases) their normal vector bundle intersects with the tangent bundle. So, studying them becomes more difficult than studying non-degenerate submanifolds. When we think of hypersurfaces as submanifolds, we say that lightlike hypersurfaces of semi-Riemannian manifolds are important due to their physical applications in mathematical physics. Moreover, in physics, lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons. Lightlike hypersurfaces are also studied in the theory of electromagnetism.

The geometry of lightlike submanifolds of a semi-Riemannian manifold has been presented in [9] (see also [8]) by Duggal and Bejancu. In [14], Duggal and Sahin have introduced differential geometry of lightlike submanifolds and they have studied geometry of classes of lightlike submanifolds (see [10], [11], [12], [13]). Also, the geometry of lightlike submanifolds of indefinite Kaehler manifolds has been presented in a book by Duggal and Bejancu [12]. The notion of semi-invariant lightlike submanifolds has been studied some authors. For instance, authors have introduced screen semi-invariant lightlike submanifolds of semi-Riemannian product manifolds in [16] and they have given the following definition:

Let \((\overline{M}, \overline{g})\) be a semi-Riemannian product manifold and \(M\) be a lightlike submanifold of \(\overline{M}\). We say that \(M\) is screen semi-invariant (SSI)-lightlike submanifold of \(\overline{M}\) if the following statements are satisfied

1) There exists a non-null distribution \(D \subseteq S(TM)\) such that

\[ S(TM) = D \perp D^\perp, \quad FD = D, \quad FD^\perp \subset S(TM^\perp), \quad D \cap D^\perp = \{0\}, \]

where \(D^\perp\) is orthogonal complementary to \(D\) in \(S(TM)\).

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2) \( \text{Rad}(TM) \) is invariant with respect to \( F \), that is \( F \text{Rad}(TM) = \text{Rad}(TM) \).

Then we have

\[
F \ell tr(TM) = \ell tr(TM),
\]

\[
TM = D' \perp D', \quad D' = D \perp \text{Rad}(TM).
\]

Hence, it follows that \( D' \) is also invariant with respect to \( F \). We denote the orthogonal complement to \( FD' \) in \( S(TM) \) by \( D_0 \). Then, we have

\[
\ell tr(TM) = \ell tr(TM) \perp FD' \perp D_0,
\]

where \( F^2 = I \) and \( \overline{g}(FX, Y) = \overline{g}(X, FY) \), for any \( X, Y \in (TM) \).

Also, Bahadır has studied screen semi-invariant half-lightlike submanifolds of a semi-Riemannian product manifold \([2]\). In \([21]\), semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds have been introduced by Poyraz and Doğan. And, in \([1]\), authors have introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. Finally, Gupta and friends have given geometry of semi-invariant lightlike product manifolds in \([15]\).

On the other hand, in 1984, Bejancu and Papaghiuc \([5]\) have given the definition of almost semi-invariant submanifolds of a Sasakian manifold as follows:

Let \( \tilde{M} \) be a \((2n + l)\)-dimensional almost contact metric manifold with \((F, \xi, \eta, g)\) as the almost contact metric structure, where \( F \) is tensor field of type \((1,1)\), \( \xi \) is a vector field, \( \eta \) is a 1-from and \( g \) is a Riemannian metric on \( \tilde{M} \). These tensor fields satisfy

\[
F^2 = -I + \eta \otimes \xi, \quad F(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ F = 0
\]  

(1.1)

and

\[
g(FX, FY) = g(X, Y) - \eta(X)\eta(Y)
\]  

(1.2)

for all vector fields \( X, Y \) tangent to \( \tilde{M} \), where \( I \) designes the identity morphism on the tangent bundle \( T\tilde{M} \). It is well known that \( \tilde{M} \) is a Sasakian manifold if and only

\[
(\tilde{\nabla}_XF)Y = g(X, Y)\xi - \eta(Y)X
\]  

(1.3)

for all \( X, Y \) tangent to \( \tilde{M} \), where \( \tilde{\nabla} \) is the Riemannian connection with respect to \( g \). From \((1.3)\), we have

\[
\tilde{\nabla}_X\xi = -FX.
\]  

(1.4)

Now, let \( M \) be an \( m \)-dimensional manifold isometrically immersed in a Sasakian manifold \( \tilde{M} \). Denote by \( TM \) and \( TM^\perp \) the tangent bundle of \( M \) and the normal bundle to \( M \), respectively. Suppose the structure vector field \( \xi \) of \( \tilde{M} \) be tangent to the submanifold \( M \) and denote by \( \{\xi\} \) the 1-dimensional distribution spanned by \( \xi \) on \( M \) and by \( \{\xi\}^\perp \) the complementary orthogonal distribution to \( \{\xi\} \) in \( TM \).

For any vector bundle \( H \) on \( M \), we denote by \( \Gamma(H) \) the module of all differentiable sections of \( H \). For any \( X \in \Gamma(TM) \), we have \( g(FX, \xi) = 0 \). Then we put

\[
FX = bX + cX,
\]  

(1.5)

where \( bX \in \Gamma(\{\xi\}^\perp) \) and \( cX \in \Gamma(TM^\perp) \). Thus \( b \) is an endomorphism of the tangent bundle \( TM \) and \( c \) is a normal bundle valued 1-form on \( M \). Next, for each \( x \in M \) we define the following subspaces:

\[
D_x = \{X_x \in \{\xi\}_x^\perp : c(X_x) = 0\}
\]  

(1.6)
We note that $D_x$ and $D_x^\perp$ are two orthogonal subspaces of space $T_xM$. In fact, using (1.1), (1.2) and (1.5), for any $x \in D_x$ and $Y_x \in D_x^\perp$, we have

$$g(X_x, Y_x) = g(FX_x, FY_x) = g(bX_x, cY_x) = 0.$$  

Thus, the submanifold $M$ of the Sasakian manifold $\tilde{M}$ is said to be an almost semi-invariant submanifold if $\dim(D_x)$ and $\dim(D_x^\perp)$ are constant along $M$ and $D : x \to D_x \subset T_xM$, $D^\perp : x \to D_x^\perp \subset T_xM$ define differentiable distributions on $M$.

Now we denote by $\tilde{D}$ the complementary orthogonal to $D \oplus D^\perp \oplus \xi$ in $TM$. Thus, for the tangent bundle to the almost semi-invariant submanifold $M$, we have the orthogonal decomposition:

$$TM = D \oplus D^\perp \oplus \tilde{D} \oplus \xi,$$

where

i) $D$ is an invariant distribution on $M$, that is $FD = D$,
ii) $D^\perp$ is anti-invariant distribution on $M$, that is $FD^\perp \subset TM^\perp$,
iii) $\tilde{D}$ is neither an invariant nor an anti-invariant distribution on $M$, that is $bX_x \neq 0$ and $cX_x \neq 0$, for any $x \in M$ and $X_x \in \tilde{D}_x$.

In this paper, after giving some basic notions about lightlike submanifolds in the second Section, in Section 3 we give the definition of screen almost semi-invariant (SASI) lightlike submanifolds of indefinite Kähler manifolds and construct an example for this submanifold. After as a characteristic theorem, we obtain the necessary and sufficient condition for the induced connection to be a metric connection on SASI-lightlike submanifolds. Also, we give some certain characterizations for SASI-lightlike submanifolds. For instance, we investigate the totally geodesic of the SASI-lightlike submanifolds.

2. LIGHTLIKE SUBMANIFOLDS

Let $(\tilde{N}^{p+q}, \overline{\rho})$ be a semi-Riemannian manifold and let $\tilde{N}^p$ be an immersed submanifold in $\tilde{N}$. If $\tilde{N}$ is a lightlike submanifold with respect to the metric $\rho$ induced from $\overline{\rho}$ and the radical distribution $\text{Rad}(T\tilde{N})$ is of ranks, where $1 \leq s \leq p$, then $\tilde{N}$ is called a lightlike submanifold [9]. According to this definition, we recall basic notions about lightlike submanifolds as following. A semi-Riemannian complementary distribution of $\text{Rad}(T\tilde{N})$ in $T\tilde{N}$ is called screen distribution and it is denoted by $S(T\tilde{N})$, that is $T\tilde{N} = \text{Rad}(T\tilde{N}) \perp S(T\tilde{N})$. Now, we consider a screen transversal vector bundle $S(T\tilde{N}^\perp)$. This vector bundle is a semi-Riemannian complementary vector bundle of $\text{Rad}(T\tilde{N})$ in

$$T\tilde{N}^\perp = \bigcup_{x \in \tilde{N}} \{ u \in T_x\tilde{N} \mid \overline{\rho}(u, v) = 0, \forall v \in T_x\tilde{N} \}. \quad (2.1)$$

Let $tr(T\tilde{N})$ and $ltr(T\tilde{N})$ be complementary (but not orthogonal) vector bundles to $T\tilde{N}$ in $T\tilde{N}\vert_{\tilde{N}}$ and to $\text{Rad}(T\tilde{N})$ in $S(T\tilde{N}^\perp)$, respectively. Then we get

$$tr(T\tilde{N}) = ltr(T\tilde{N}) \perp S(T\tilde{N}^\perp) \quad (2.2)$$
and
\[ T\overline{N}|_{\overline{N}} = T\overline{N} \oplus \text{tr}(T\overline{N}) = (\text{Rad}(T\overline{N}) \oplus \text{ltr}(T\overline{N})) \perp S(T\overline{N}) \perp S(T\overline{N}^\perp). \tag{2.3} \]

Here, \( \text{ltr}(T\overline{N}) \) is called lightlike transversal vector bundle in \( T\overline{N} \) such that there exists a local null frame \( \{ N_i \} \) of sections with values in the orthogonal complement of \( S(T\overline{N}^\perp) \) in \( [S(T\overline{N})]^\perp \) and \( \text{ltr}(T\overline{N}) \) locally spanned by \( \{ N_i \} \). Then, take into account any local basis \( \{ \xi_i \} \) of \( \text{Rad}(T\overline{N}) \) and \( \{ N_i \} \) of \( \text{ltr}(T\overline{N}) \), we have \( \overline{p}(\xi_i, N_j) = \delta_{ij} \) and \( \overline{p}(N_i, N_j) = 0 \), \( i, j = 1, ..., s \).

For a lightlike submanifold \((\widetilde{N}, \rho, S(T\overline{N}), S(T\overline{N}^\perp))\), there are the following four cases:

(i) \( \widetilde{N} \) is called s-lightlike, if \( s < \min\{p, q\} \),

(ii) \( \widetilde{N} \) is called co-isotropic, if \( s = q < p \), that is, \( S(T\overline{N}^\perp) = \{0\} \),

(iii) \( \widetilde{N} \) is called isotropic, if \( s = p < q \), that is, \( S(T\overline{N}) = \{0\} \),

(iv) \( \widetilde{N} \) is called totally lightlike, if \( s = p = q \), that is, \( S(T\overline{N}) = \{0\} = S(T\overline{N}^\perp) \).

Now, let \( \nabla \) be the Levi-Civita connection on \( \overline{N} \). According to \( (2.3) \), for \( \forall X, Y \in \Gamma(T\overline{N}) \) and \( \forall V \in \Gamma(\text{tr}(T\overline{N})) \), the Gauss and Weingarten formula of \( \widetilde{N} \) are given by
\[
\nabla_X Y = \nabla_X Y + h(X, Y) \tag{2.4}
\]
and
\[
\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.5}
\]
where \( \{ \nabla_X Y, A_V X \} \) and \( \{ h(X, Y), \nabla^t_X V \} \) are belong to \( \Gamma(T\overline{N}) \) and \( \Gamma(\text{tr}(T\overline{N})) \), respectively. Also, \( \nabla \) and \( \overline{\nabla} \) are linear connections on \( \overline{N} \) and on the vector bundle \( \text{tr}(T\overline{N}) \), respectively. The second fundamental form \( h \) is a symmetric \( \mathcal{F}(\overline{N}) \)-bilinear form on \( \Gamma(T\overline{N}) \) with values in \( \Gamma(\text{tr}(T\overline{N})) \) and the shape operator \( A_V \) is a linear endomorphism of \( \Gamma(T\overline{N}) \). Then, for \( \forall X, Y \in \Gamma(T\overline{N}), N \in \Gamma(\text{tr}(T\overline{N})) \) and \( Z \in \Gamma(S(T\overline{N}^\perp)) \), we get
\[
\nabla_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.6}
\]
\[
\overline{\nabla}_X N = -A_N X + \nabla^t_X (N) + D^s(X, N), \tag{2.7}
\]
\[
\nabla_X Z = -A_Z X + D^l(X, Z) + \nabla^s_X (Z), \tag{2.8}
\]
where \( \{ \nabla^l_X (N), D^l(X, Z) \} \) and \( \{ D^s(X, N), \nabla^s_X (Z) \} \) are parts of \( \text{ltr}(T\overline{N}) \) and \( S(T\overline{N}^\perp) \), respectively. Also, \( h^l(X, Y) = L h(X, Y) \in \Gamma(\text{ltr}(T\overline{N})) \) and \( h^s(X, Y) = S h(X, Y) \in \Gamma(S(T\overline{N}^\perp)) \), where \( L \) and \( S \) are the projectors of transversal vector bundle \( \text{tr}(T\overline{N}) \) on \( \text{ltr}(T\overline{N}) \) and \( S(T\overline{N}^\perp) \). Denote the projection of \( T\overline{N} \) on \( S(T\overline{N}) \) by \( Q \). Thus, using \( (2.4), (2.6), (2.8) \) and considering the metric connection \( \overline{\nabla} \), we have
\[
\overline{p}(h^s(X, Y), Z) + \overline{p}(Y, D^l(X, Z)) = \rho(A_Z X, Y), \tag{2.9}
\]
\[
\overline{p}(D^s(X, N), Z) = \overline{p}(N, A_Z X) \tag{2.10}
\]
and
\[
\nabla_X QY = \nabla^s_X QY + h^s(X, QY), \tag{2.11}
\]
\[
\nabla_X \xi = -A^s_\xi X + \nabla^s_X \xi, \tag{2.12}
\]
where $\forall X, Y \in \Gamma(T\tilde{N})$, $\xi \in \Gamma(Rad(T\tilde{N}))$ and $\nabla^*$ is induced connection on $S(T\tilde{N})$ which is a metric connection and $\nabla^*$ is induced connection on $Rad(T\tilde{N})$. Using (2.9), (2.10), (2.11) and (2.12), we get

\[
\begin{align*}
&\mathcal{P}(h^l(X, QY), \xi) = \rho(A^*_l X, QY), \quad \mathcal{P}(h^r(X, QY), N) = \rho(A_N X, QY), \quad (2.13) \\
&\mathcal{P}(h^l(X, \xi), \xi) = 0 \text{ and } A^*_l \xi = 0. \quad (2.14)
\end{align*}
\]

Also, the induced connection on lightlike submanifold $\tilde{N}$ is torsion-free, but it is not metric connection and satisfies the following condition

\[
(\nabla_X \rho)(Y, Z) = \mathcal{P}(h^l(X, Y), Z) + \mathcal{P}(h^l(X, Z), Y). \quad (2.15)
\]

For more details and studies about lightlike submanifolds, one can see [9], [10], [17], [18], [19], [23], [24], [25], [26], [27].

On the other hand, an indefinite almost Hermitian manifold $(\overline{N}, \overline{\rho}, J)$ is a $2m$-dimensional semi-Riemannian manifold $\overline{N}$ with a semi-Riemannian metric $\rho$ of the constant index $q$, $0 < q < 2m$ and a $(1, 1)$ tensor field $\overline{J}$ on $\overline{N}$ such that for $\forall X, Y \in \Gamma(T\overline{N})$ the following conditions are satisfied

\[
\begin{align*}
&\overline{J}^2 X = -X, \quad (2.16) \\
&\mathcal{P}(\overline{J}X, \overline{J}Y) = \mathcal{P}(X, Y). \quad (2.17)
\end{align*}
\]

If $\overline{J}$ is parallel with respect to $\nabla$, that is,

\[
(\nabla_X \overline{J})Y = 0, \quad (2.18)
\]

then an indefinite almost Hermitian manifold $(\overline{N}, \overline{\rho}, \overline{J})$ is called an indefinite Kaehler manifold [3], where $\forall X, Y \in \Gamma(T\overline{N})$ and $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{\rho}$.

3. SCREEN ALMOST SEMI-INVARIANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAELHLER MANIFOLDS

In this section, we introduce screen almost semi-invariant (SASI) lightlike submanifolds, give an example and obtain some characterizations. We give the necessary and sufficient condition for the induced connection which is not metric connection in general. Finally, we investigate the notion of mixed geodesic for (SASI) lightlike submanifolds.

**Definition 1.** Let $(\tilde{N}, \rho, S(T\tilde{N}))$ be a lightlike submanifold of an indefinite Kaehler manifold $(\overline{N}, \overline{\rho}, \overline{J})$. Then we say that $\tilde{N}$ is a SASI-lightlike submanifold of $\overline{N}$, if the following conditions are satisfied:

i) $D_0, D_1, D$ and $D^*$ are orthogonal distributions on $S(T\tilde{N})$ such that

\[
S(T\tilde{N}) = D \oplus_D D^* \oplus_D D^{'},
\]

where $\overline{J}D = D$, $D^*$ anti-invariant and $D^{''} = D_0 \oplus_D D_1$. Also $\overline{J}D^{''} \nsubseteq S(T\tilde{N})$ and $\overline{J}D^{''} \nsubseteq S(T\overline{N})$ such that the distribution $D^{''}$ is neither invariant nor anti-invariant.

ii) $\overline{J}(\text{Rad}(T\tilde{N})) = \text{Rad}(T\tilde{N})$, that is $\text{Rad}(T\tilde{N})$ is invariant respect to $\overline{J}$.

iii) $S(T\tilde{N}^{''}) = \overline{J}D^* \oplus_D \mu$, $\mu = D_2 \oplus_D D_3$, $\overline{J}\mu \neq \mu$.

Also, we have $\overline{J}ltr(T\tilde{N}) = ltr(T\tilde{N})$ and the following decomposition

\[
T\tilde{N} = D' \oplus_D D^* \oplus_D D^{'},
\]
where $D' = D \oplus_\perp \text{Rad}(T\tilde{N})$ and $D'$ is invariant with respect to $\tilde{\mathcal{J}}$.

We denote the orthogonal complement to $\tilde{\mathcal{J}}D^*$ in $S(T\tilde{N}^\perp)$ by $\mu$. Then, we have

$$\text{tr}(T\tilde{N}) = \text{ltr}(T\tilde{N}) \oplus_\perp \tilde{\mathcal{J}}D^* \oplus_\perp \mu.$$ 

**Proposition 1.** A SASI-lightlike submanifold $\tilde{N}$ of an indefinite Kaehler manifold $\overline{N}$ is a SSI-lightlike submanifold if and only if $D_0$ and $D_1$ (similarly $D_2$ and $D_3$) are invariant distributions.

**Proof.** Let $\tilde{N}$ be a SSI-lightlike submanifold of $\overline{N}$. Then form [10], we have $S(T\tilde{N}) = D_0 \perp D_1 \subset S(T\tilde{N}^\perp)$ and $D \cap D^* = \{0\}$. In this case, either $D_0$ and $D_1$ are zero or $D_0$ and $D_1$ are invariant. If $D_0$ and $D_1$ are zero, then $\mu$ is zero. But, since $\tilde{N}$ is the SSI-lightlike submanifold, $\mu$ cannot be zero. Therefore, $D_0$ and $D_1$ must be invariant. This implies that $\mu = D_2 \oplus_\perp D_3$ is invariant. Conversely, assume that $D_0$ and $D_1$ (similarly $D_2$ and $D_3$) are invariant distributions. Then, we get $D' = D \oplus_\perp \text{Rad}(T\tilde{N}) \oplus_\perp D_0 \oplus_\perp D_1$. Thus, we obtain

$$T\tilde{N} = D' \oplus_\perp D^*.$$ 

Since we have $\mu = D_2 \oplus_\perp D_3$, $\mu \subset S(T\tilde{N}^\perp)$. Here, we get $S(T\tilde{N}) = \tilde{D} \perp D^*$, where $\tilde{D} = D \oplus_\perp D_0 \oplus_\perp D_1$. Then, $\tilde{N}$ is a SSI-lightlike submanifold of $\overline{N}$. Thus, the proof completes. \(\square\)

Now, we will construct an example of SASI-lightlike submanifold in $\mathbb{R}^{12}_2$.

**Example 1.** Let $(\mathbb{R}^{12}_2, \tilde{p}, \tilde{\mathcal{J}})$ be an indefinite Kaehler manifold with signature $(-, -, +, +, +, +, +, +, +, +, +, +)$ and let $\tilde{N}$ be a submanifold of $\mathbb{R}^{12}_2$ given by

$$F(u, v, t, m, n, l, w) = (u, v, -\cos t, \sin t, -t \cos m, -t \sin m, n \cos \beta - w \sin \beta, n \sin \beta + w \cos \beta, \cos l, \sin l, u \cos \alpha - v \sin \alpha, u \sin \alpha + v \cos \alpha).$$

Then we have $T\tilde{N} = \text{Sp}\{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$ such that

$$F_1 = \partial x_1 + \cos \alpha \partial x_{11} + \sin \alpha \partial x_{12},$$
$$F_2 = \partial x_2 - \sin \alpha \partial x_{11} + \cos \alpha \partial x_{12},$$
$$F_3 = \sin t \partial x_3 + \cos t \partial x_4 - \cos m \partial x_5 - \sin m \partial x_6,$$
$$F_4 = t \sin m \partial x_5 - t \cos m \partial x_6,$$
$$F_5 = \cos \beta \partial x_7 + \sin \beta \partial x_8,$$
$$F_6 = -\sin \beta \partial x_7 + \cos \beta \partial x_8,$$
$$F_7 = -\sin l \partial x_9 + \cos l \partial x_{10}.$$

Since $\text{Rad}(T\tilde{N}) = \text{Sp}\{F_1, F_2\}$ and $\tilde{\mathcal{J}}F_1 = F_2$, one can easily see that $\tilde{N}$ is a 2-lightlike submanifold. Now, we have $D = \text{Sp}\{F_5, F_6\}$ such that $\tilde{\mathcal{J}}F_5 = F_6$ which implies that $D$ is invariant with respect to $\tilde{\mathcal{J}}$. Also, we get $D^* = \text{Sp}\{F_7\}$ and $D'' = \text{Sp}\{F_3, F_4\}$. On the other hand, we obtain the lightlike transversal bundle $\text{ltr}(T\tilde{N}) = \text{Sp}\{N_1, N_2\}$, where

$$N_1 = \frac{1}{2} \{-\partial x_1 + \cos \alpha \partial x_{11} + \sin \alpha \partial x_{12}\},$$
$$N_2 = \frac{1}{2} \{-\partial x_2 - \sin \alpha \partial x_{11} + \cos \alpha \partial x_{12}\}.$$
and the screen transversal bundle $S(T\tilde{N}^\perp) = Sp\{W_1, W_2, W_3\}$, where
\[ W_1 = -\cos t\partial x_3 + \sin t\partial x_4, \quad W_2 = -\cos l\partial x_9 + \sin l\partial x_{10}, \]
\[ W_3 = \sin t\partial x_3 + \cos t\partial x_4 + \cos m\partial x_5 + \sin m\partial x_6. \]

Here we get $\mathcal{J}F_7 = W_2$, $\mathcal{J}N_1 = N_2$ and $\mu = Sp\{W_1, W_3\}$. Then, since
\[ \mathcal{J}F_3 = W_1 + \frac{1}{t}F_1, \quad \mathcal{J}F_4 = \frac{t}{2}W_3 - \frac{t}{2}F_3 \]
and
\[ \mathcal{J}W_1 = -\frac{1}{2}F_3 - \frac{1}{2}W_3, \quad \mathcal{J}W_3 = W_1 - \frac{1}{t}F_4 \]
we obtain that $\tilde{N}$ is a SASI-lightlike submanifold of $\mathbb{R}^2_{12}$.

Now, let $P_0, P_1, P_2, P_3$ and $S_1$ be the projections on $D, D^s, D_0, D_1$ and $Rad(T\tilde{N})$ in $T\tilde{N}$, respectively. Similarly, let $R_1, R_2$ and $R_3$ be the the projection on $\mathcal{J}D^s, D_2$ and $D_3$ in $tr(T\tilde{N})$, respectively. Then for any $X \in \Gamma(T\tilde{N})$, we get
\[ X = PX + RX + QX = P_0X + P_1X + P_2X + P_3X + S_1X, \]
where $PX \in \Gamma(D'), RX \in \Gamma(D^s)$ and $QX \in \Gamma(D^\perp)$.

On the other hand, one can write that
\[ \mathcal{J}X = TX + wX, \]
where $TX$ and $wX$ are the tangential and transversal components of $\mathcal{J}X$, respectively. Then applying $\mathcal{J}$ to equation (3.1) and substituting $T_0, w_1, T_2 + w_2, T_3 + w_3, T_1$ for $\mathcal{J}P_0, \mathcal{J}P_1, \mathcal{J}P_2, \mathcal{J}P_3, \mathcal{J}S_1$, respectively, we have
\[ \mathcal{J}X = T_0X + w_1X + T_2X + w_2X + T_3X + w_3X + T_1X, \]
where it is clear that $T_0X \in \Gamma(\mathcal{J}D), w_1X \in \Gamma(\mathcal{J}D^s), (T_2X + w_2X) \in \Gamma(\mathcal{J}D_0 = D_1 \oplus D_2), (T_3X + w_3X) \in \Gamma(\mathcal{J}D_1 = D_0 \oplus D_3), T_1X \in \Gamma(\mathcal{J}(Rad(T\tilde{N}))).$

Similarly, for any $\Omega \in \Gamma(tr(T\tilde{N}))$, it is known that
\[ \mathcal{J}\Omega = B\Omega + C\Omega, \]
where $B\Omega$ and $C\Omega$ are the sections of $T\tilde{N}$ and $tr(T\tilde{N})$, respectively. Thus, differentiating (3.3) and using the equations Gauss-Weingarten in (2.6, 2.8) and (3.4), we obtain the following equations
\[ (\nabla_X T)Y = A_{w_1Y}X + A_{w_2Y}X + A_{w_3Y}X + B\h(X, Y), \]
\[ D^l(X, w_1Y) + D^l(X, w_2Y) + D^l(X, w_3Y) = Ch^l(X, Y) - h^l(X, TY) \]
and
\[ h^s(X, TY) + \nabla^s_X w_1Y + \nabla^s_X w_2Y + \nabla^s_X w_3Y - w(\nabla_X Y) - Ch^s(X, Y) = 0. \]

**Theorem 1.** Let $\tilde{N}$ be a SASI-lightlike submanifold (not proper) of an indefinite Kaehler manifold $N$. Then, we have the following statements

**i)** If $\tilde{N}$ is isotropic, coisotropic or totally lightlike submanifold, then $\tilde{N}$ is invariant,

**ii)** If $\tilde{N}$ is an isotropic lightlike submanifold, then $\tilde{N}$ is also a totally lightlike submanifold.
Theorem 2. Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. The induced connection $\nabla$ is a metric connection if and only if
\[
(A_{\tilde{N}}^*X - \nabla_X^*JY) \in \Gamma(\text{Rad}(T\tilde{N})) \quad \text{and} \quad Bh(X, JY) = 0,
\]
where $X \in \Gamma(T\tilde{N})$ and $Y \in \Gamma(\text{Rad}(T\tilde{N}))$.

Proof. For almost complex structure $\overline{J}$, one can write from (2.18)
\[
\nabla_X Y = -\overline{J}(\nabla_X JY),
\]
where $X, Y \in \Gamma(T\overline{N})$. Then, using (2.6) and (2.12), for $X \in \Gamma(T\tilde{N})$ and $Y \in \Gamma(\text{Rad}(T\tilde{N}))$, we have
\[
\nabla_X Y = -\overline{J}(A_{\tilde{N}}^*X - \nabla_X^*JY) - Bh(X, JY) - Ch(X, JY)
\]
\[
= TA_{\tilde{N}}^*X + wA_{\tilde{N}}^*X - T \nabla_X^*JY - w \nabla_X^*JY - Bh(X, JY) - Ch(X, JY). \quad (3.8)
\]

Taking the tangential parts of (3.8), we get
\[
\nabla_X Y = TA_{\tilde{N}}^*X - T \nabla_X^*JY - Bh(X, JY). \quad (3.9)
\]

According to (3.9), if $T(A_{\tilde{N}}^*X - \nabla_X^*JY) \in \Gamma(\text{Rad}(T\tilde{N}))$ and $Bh(X, JY) = 0$, then $\nabla_X Y \in \Gamma(\text{Rad}(T\tilde{N}))$. Consequently, the proof is completed. $\square$

Theorem 3. Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. Then distribution $D'$ is integrable if and only if
\[
h(X, JY) = h(JX, Y)
\]
holds for $\forall X, Y \in \Gamma(D')$.

Proof. $D'$ is integrable if and only if for $\forall X, Y \in \Gamma(D')$, $[X, Y] \in \Gamma(D')$. Here for $\forall X, Y \in \Gamma(D')$, obviously $wX = wY = 0$. From the equations (3.6) and (3.7), we obtain
\[
-Ch^l(X, Y) + h^l(X, TY) + h^s(X, TY) - w(\nabla_X Y) - Ch^s(X, Y) = 0.
\]

Thus, we get
\[
w(\nabla_X Y) = h(X, TY) - Ch(X, Y). \quad (3.10)
\]

Now, using (3.10) and symmetry property of the second fundamental form $h$, we have $w([X, Y]) = h(X, TY) - h(TX, Y)$. Then the proof completes. $\square$

Theorem 4. Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. Then distribution $D^*$ is integrable if and only if for $\forall V, W \in \Gamma(D^*)$
\[
A_{JW}V = A_{JW}V.
\]
Proof. Since $TV = TW = 0$ and from (3.5), we obtain $-T\nabla_V W = A_{wV} V + Bh(V, W)$ for $\forall V, W \in \Gamma(D')$. Then, we have $T([V, W]) = A_{wV} V - A_{wV} W$. Thus, the proof completes. □

**Theorem 5.** Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. Then, the distribution $D'$ is integrable if and only if the following properties hold for $\forall Z, U \in \Gamma(D')$

(i) $(h(Z, TU) - h(U, TZ) + \nabla^s_U wZ - \nabla^s_Z wU) \in \Gamma(\mu)$ and $D'(U, wZ) = D'(Z, wU)$,

(ii) $(\nabla Z TU - \nabla_U TZ - A_{wZ} Z + A_{wU} Z) \in \Gamma(D'')$.

Proof. Since $D''$ is neither an invariant nor an anti-invariant distribution on $\tilde{N}$, we have $T Z \neq 0 \neq T U$ and $wZ \neq 0 \neq wU$. Using (3.5)-(3.6)-(3.7), we obtain

$$\nabla ZTU + h(Z, TU) + \{-A_{wZ} Z + \nabla^s_U wZ + D'(Z, wU)\} = T\nabla ZU + w(\nabla ZU) + Bh(Z, U) + Ch(Z, U).$$

Then, using symmetry property of the second fundamental form $h$, we complete the proof. □

Now, we can examine the parallelism of distributions $D'$, $D^*$ and $D''$ on $\tilde{N}$.

**Theorem 6.** Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. Then, the distribution $D'$ is parallel if and only if the following properties hold for $\forall X, Y \in \Gamma(D')$, $\forall V \in \Gamma(D^*)$ and $\forall Z \in \Gamma(D'')$

(i) For $Y \in \Gamma(D)$, $A_{\nabla X} X$ and $\nabla X TZ - A_{wZ} X$ have no component on $\Gamma(D)$.

(ii) For $Y \in \Gamma(\text{Rad}(T\tilde{N}))$, $h'(X, TZ) = -D'(X, wZ)$ and $\overline{\nabla}(D'(X, wV), \overline{Y}) = 0$.

Proof. Assume that $D'$ is parallel distribution. Then for $\forall X, Y \in \Gamma(D')$, $\forall V \in \Gamma(D^*)$ and $\forall Z \in \Gamma(D'')$, we get

$$\rho(\nabla_X Y, V) = 0 = \rho(\nabla_X Y, Z).$$

Using $\nabla$ is a metric connection, from (2.6), (2.8) and (3.2), we have

$$\rho(\nabla_X Y, V) = -\overline{\rho}(-A_{wV} X + D'(X, wV) + \nabla^s_X wV, \overline{Y}).$$

On the other hand, as a result of the necessary calculations, we get

$$\rho(\nabla_X Y, Z) = -\overline{\rho}(\nabla_X TZ + h'(X, TZ) + h^s(X, TZ) - A_{wZ} X + D'(X, wZ) + \nabla^s_X wZ, \overline{Y}).$$

Then, form (3.11) and (3.12), we prove that for $Y \in \Gamma(D)$ and $Y \in \Gamma(\text{Rad}(T\tilde{N}))$ the conditions (i) and (ii) are true. The converse of the proof of is obvious. □

**Theorem 7.** Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\overline{N}$. Then, the distribution $D^*$ is parallel if and only if

(i) For $\forall V, W \in \Gamma(D^*)$, $A_{wV} W$ has no components on $\Gamma(D)$, $\Gamma(D'')$ and $\Gamma(\text{Rad}(T\tilde{N}))$,

(ii) For $Z \in \Gamma(D'')$, $\nabla^s_X wV$ has no components on $\Gamma(\mu)$.

Proof. Assume that $D^*$ is parallel distribution. Then for $\forall X \in \Gamma(D)$, $\forall V, W \in \Gamma(D^*)$ and $N \in \Gamma(\text{ltr}(T\tilde{N}))$, we get

$$\rho(\nabla_W V, X) = \overline{\rho}(\nabla_W V, N) = \rho(\nabla_W V, Z) = 0.$$
Then using (2.17), (2.6) and (2.8), we obtain
\[ g(\nabla_W V, X) = \overline{g}(-A_{wV} W + D^i(W, wV) + \nabla^s_W wV, JX). \] 
(3.13)

Also, it is easy to see that
\[ \rho(\nabla_W V, N) = \overline{\rho}(-A_{wV} W + D^i(W, wV) + \nabla^s_W wV, JN) \] 
(3.14)

and
\[ \rho(\nabla_W V, Z) = \overline{\rho}(-A_{wV} W + \nabla^s_W wV, JZ). \] 
(3.15)

Thus, taking into account the equations (3.13) and (3.14), the condition (i) of the proof is complete. From (3.15), (ii) is obvious. \( \square \)

**Theorem 8.** Let \( \tilde{N} \) be a SASI-lightlike submanifold of an indefinite Kaehler manifold \( \overline{N} \). Then, the distribution \( D'' \) is parallel if and only if the following conditions are satisfied

i) \( \nabla_Z T U - A_{wU} Z \) has no components on both \( \Gamma(D) \) and \( \Gamma(\text{Rad}(T\tilde{N})) \),

ii) \( h^s(Z, T U) = -\nabla_Z^s wU, \) where \( U, Z \in \Gamma(D'') \).

**Proof.** Assume that \( D'' \) is parallel distribution. Then, for \( \forall X \in \Gamma(D), \forall Z, U \in \Gamma(D'') \) and \( N \in \Gamma(ltr(T\tilde{N})) \), we get
\[ \rho(\nabla_Z U, X) = \rho(\nabla_Z U, V) = \overline{\rho}(\nabla_Z U, N) = 0. \]

Then using (2.17), (2.6) and (2.8), we obtain
\[ \rho(\nabla_Z U, X) = \overline{\rho}(\nabla_Z U, V) = \overline{\rho}(\nabla_Z U, N). \] 
(3.16)

Now, with easy calculations, we have
\[ \rho(\nabla_Z U, N) = \overline{\rho}(\nabla_Z U, V) = \overline{\rho}(\nabla_Z UT - A_{wU} Z, JX). \] 
(3.17)

Hence, the equations (3.16) and (3.17) prove (i).

On the other hand, for \( \forall V \in \Gamma(D'') \), we get
\[ \rho(\nabla_Z U, V) = \overline{\rho}(h^s(Z, T U) + \nabla_Z^s wU, JV) \] 
(3.18)

and this proves (ii). \( \square \)

**Definition 2.** Let \( \tilde{N} \) be a SASI-lightlike submanifold of an indefinite Kaehler manifold \( \overline{N} \). If its second fundamental form \( h \) satisfies
\[ h(X, Y) = 0, \quad \forall X, Y \in \Gamma(D), \] 
(3.19)

then \( \tilde{N} \) is called a \( D \)-geodesic. Obviously, \( \tilde{N} \) is a \( D \)-geodesic SASI-lightlike submanifold if
\[ h^i(X, Y) = h^s(X, Y) = 0, \quad \forall X, Y \in \Gamma(D). \]
(3.20)

Also, if \( h \) satisfies
\[ h(X, Y) = 0, \quad \forall X \in \Gamma(D), \forall Y \in \Gamma(D') \text{ (or } \Gamma(D''), \) 
(3.21)

then \( \tilde{N} \) is called a mixed geodesic SASI-lightlike submanifold.

**Theorem 9.** Let \( \tilde{N} \) be a SASI-lightlike submanifold of an indefinite Kaehler manifold \( \overline{N} \). Then, the distribution \( D' \) of \( \tilde{N} \) is a totally geodesic foliation in \( \overline{N} \) if and only if the following conditions are satisfied

i) \( \tilde{N} \) is \( D' \)-geodesic,

ii) \( D' \) is parallel respect to \( \nabla \) on \( \tilde{N} \).
Proof. If $D'$ is a totally geodesic foliation in $\tilde{N}$, then for $\forall X, Y \in \Gamma(D')$, we get $\nabla_X Y \in \Gamma(D')$. For $\forall \xi \in \Gamma(Rad(T\tilde{N}))$, $V \in \Gamma(D^*)$, $Z \in \Gamma(D^\perp)$ and $E \in \Gamma(S(T\tilde{N}^\perp))$, we have

$$\overline{\rho}(\nabla_X Y, \xi) = \overline{\rho}(h^l(X, Y), \xi),$$

$$\overline{\rho}(\nabla_X Y, E) = \overline{\rho}(h^s(X, Y), E),$$

$$\overline{\rho}(\nabla_X Y, V) = 0,$$

$$\overline{\rho}(\nabla_X Y, Z) = 0.$$

Then we obtain the following results:

$$\overline{\rho}(\nabla_X Y, \xi) = \overline{\rho}(h^l(X, Y), \xi), \quad (3.22)$$

$$\overline{\rho}(\nabla_X Y, E) = \overline{\rho}(h^s(X, Y), E), \quad (3.23)$$

$$\overline{\rho}(\nabla_X Y, V) = 0, \quad (3.24)$$

$$\overline{\rho}(\nabla_X Y, Z) = 0. \quad (3.25)$$

From (3.22) and (3.23), we can write that

$$h^l(X, Y) = 0 \quad \text{and} \quad h^s(X, Y) = 0, \quad \text{for} \forall X, Y \in \Gamma(D').$$

Thus, it is clear that $\tilde{N}$ is $D'$-geodesic, which implies that $D'$ is parallel respect to $\nabla$ on $\tilde{N}$. Then, the (i) and (ii) conditions are proved.

The converse of proof is clear. \hfill \Box

Theorem 10. Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\tilde{N}$. Then $\tilde{N}$ is mixed geodesic with respect to $D^*$ if and only if the following assertions hold:

(i) $\overline{\rho}(D^l(X, wV), \xi) = 0$,

(ii) For $E \in \Gamma(JD^*)$, $\overline{\rho}(\nabla^*_X wV, wE) = 0$ and for $E \in \Gamma(wD^\perp)$, $\rho(A_{wV} X, TE) = \overline{\rho}(\nabla^*_X wV, wE)$,

where $\forall X \in \Gamma(D')$, $V \in \Gamma(D^*)$, $\xi \in \Gamma(Rad(T\tilde{N}))$, $E \in \Gamma(S(T\tilde{N}^\perp))$.

Proof. Assume that $\tilde{N}$ is mixed geodesic. From (3.20) for $\forall X \in \Gamma(D')$, $V \in \Gamma(D^*)$, $\xi \in \Gamma(Rad(T\tilde{N}))$, we get $\overline{\rho}(h^l(X, V), \xi) = 0$. Then, we have $\overline{\rho}(\nabla_X V, \xi) = 0$. Here, taking $J_\xi = \xi$, we get $\overline{\rho}(\nabla_X V, J_\xi) = 0$. Thus, since $J\nabla_X J = -\nabla_X J$ and (3.2), we obtain $\overline{\rho}(D^l(X, wV), \xi) = 0$.

On the other hand, since $\tilde{N}$ is mixed geodesic, we get $\overline{\rho}(h^s(X, V), E) = 0$, where $\forall X \in \Gamma(D')$, $V \in \Gamma(D^*)$, $E \in \Gamma(S(T\tilde{N}^\perp))$. Then, we have $\overline{\rho}(\nabla_X V, E) = 0$. Similarly, since $J\nabla_X J = -\nabla_X J$ and (3.2), we obtain

$$\rho(A_{wV} X - \nabla_X TV, TE) = \overline{\rho}(\nabla^*_X wV, wE). \quad (3.26)$$

So, for $E \in \Gamma(JD^*)$ and $E \in \Gamma(wD^\perp)$, we have our assertion. \hfill \Box

Theorem 11. Let $\tilde{N}$ be a SASI-lightlike submanifold of an indefinite Kaehler manifold $\tilde{N}$. Then $\tilde{N}$ is mixed geodesic with respect to $D^\perp$ if and only if the following assertions hold:

(i) $\overline{\rho}(D^l(X, wZ), \xi) = 0$,

(ii) For $E \in \Gamma(JD^*)$, $\overline{\rho}(\nabla_X TZ - A_{wZ} X, TE) = 0$ and for $E \in \Gamma(wD^\perp)$, $\overline{\rho}(A_{wZ} X - \nabla_X TZ, TE) = \overline{\rho}(\nabla^*_X wZ, wE)$,

where $\forall X \in \Gamma(D')$, $Z \in \Gamma(D^\perp)$, $\xi \in \Gamma(Rad(T\tilde{N}))$, $E \in \Gamma(S(T\tilde{N}^\perp))$.

Proof. Getting $Z$ instead of $V$ in the previous theorem and its proof, one can prove the theorem. \hfill \Box
CONCLUDING REMARK. One can recall that a lightlike hypersurface \( M \) of a semi-Riemannian manifold is totally geodesic if only if \( \text{Rad}(T M) \) is a Killing distribution on \( M \). In our study, Theorem 9, Theorem 10 and Theorem 11 of Section 3 are important in this sense. Also, Theorem 2, Theorem 6, Theorem 7 and Theorem 8 are about the screen shape operator or the second fundamental form which are related to the notion of Killing horizon of general relativity. Indeed, “Solutions of the highly non-linear Einstein’s field equations require the assumption that they admit Killing or homothetic vector fields. This is due to the fact that the Killing symmetries leave invariant the metric connection, all the curvature quantities and the matter tensor of the Einstein field equations of a spacetime. Most explicit solutions (see pp. 267) have been found by using one or more Killing vector fields. Related to the theme of this section, here we show that the Killing symmetry has an important role in the most active area of research on Killing horizons in general relativity.” And it is known that the theory of lightlike geometry has an important relation with general relativity.

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