GROUND STATE SOLUTIONS FOR QUASILINEAR STATIONARY SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

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Abstract. We establish the existence of ground state solution for quasilinear Schrödinger equations involving critical growth. The method used here is minimizing the gradient integral norm in a manifold defined by integrals involving the primitive of the nonlinearity function.

1. Introduction. The study of the minimization problem

\[ \min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}, \text{ for } N \geq 3, \]

\[ \min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 0 \right\}, \text{ for } N = 2, \]

has a great importance for the existence of solutions for the equation

\[- \Delta u = g(u) \text{ in } \mathbb{R}^N, \quad u \neq 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \]

where \( g(u) = G'(u) \) and \( G \in C^1(\mathbb{R}, \mathbb{R}) \). In [4], for \( N \geq 3 \), and [3], for \( N = 2 \), the authors studied the problem (1) in order to establish the existence of ground state solution for (2) provided that \( g \in C(\mathbb{R}, \mathbb{R}) \) is odd and satisfies

\((g_1)\) For \( N \geq 3 \),

\[ -\infty < \liminf_{s \to 0^+} \frac{g(s)}{s} \leq \limsup_{s \to 0^+} \frac{g(s)}{s} \leq -m < 0. \]

For \( N = 2 \),

\[ \lim_{s \to 0} \frac{g(s)}{s} = -m < 0. \]

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(g2) For $N \geq 3$,
\[
\limsup_{s \to +\infty} \frac{|g(s)|}{s^{2^* - 1}} \leq 0.
\]

For $N = 2$, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that
\[
|g(s)| \leq C_\alpha e^{\alpha s^2} \text{ for all } s \geq 0.
\]

(g3) There exists $\xi > 0$ such that $G(\xi) > 0$.

In [2], the authors completed this study of (2) for a class of nonlinearities with critical growth. The aim of the present paper is to extend the method which has been used in [2] to obtain a ground state solution to the quasilinear Schrödinger equation
\[
- \Delta u + u - u \Delta (u^2) = h(u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),
\]
where $N \geq 2$ and we assume that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function which is odd and satisfies:

\((h_1)\) $\lim_{s \to 0^+} \frac{h(s)}{s} = 0$.

\((h_2)\) When $N \geq 3$, there exists $\rho \geq 0$ such that
\[
\limsup_{s \to \infty} \frac{h(s)}{s^{2(2^*) - 1}} = \rho, \text{ where } 2^* = \frac{2N}{N-2}.
\]

When $N = 2$, there exists $\alpha_o > 0$ such that
\[
\lim_{|s| \to \infty} \frac{h(s)}{e^{\alpha s^4}} = 0, \text{ if } \alpha > \alpha_o.
\]

\((h_3)\) $h(s)s - 2H(s) \geq 0$ for every $s \geq 0$.

\((h_4)\) There is $\lambda > 0$ and $q \in (4, 2(2^*))$ when $N \geq 3$, or $q > 4$ when $N = 2$, such that
\[
h(s) \geq \lambda s^{q-1}, \text{ for every } s \geq 0.
\]

Typical and relevant examples are given by $h(s) = |s|^{2(2^*)-2}s + \lambda |s|^{q-2}s$ with $q \in (4, 2(2^*))$, when $N \geq 3$, and $h(s) = e^{s^4} + \lambda |s|^{q-2}s$ with $q \in (4, \infty)$ when $N = 2$.

The condition $(h_2)$ says that $h$ has a critical growth at infinity. As observed in [14] (see also [19]), the number $2(2^*)$ behaves like a critical exponent for equation (3) when $N \geq 3$, while the exponential growth above is the critical growth for this kind of problem when $N = 2$. According to [10, 11], the critical growth when $N = 2$ is the following:

\((CG)\)
\[
\lim_{|s| \to \infty} \frac{h(s)}{e^{\alpha s^4}} = \begin{cases} 0, & \text{if } \alpha > \alpha_o, \\ \infty, & \text{if } \alpha < \alpha_o. \end{cases}
\]

Note that $(h_2)$ in the case $N = 2$ is a slightly weakened version of $(CG)$ and says that $h$ has a critical exponential growth at infinity. Consequently, the so-called Trudinger-Moser inequality [6, 17, 20] plays a crucial role in our arguments. We observe that the condition $(h_4)$ implies that $(g_3)$ holds. It is worth pointing out that $(g_3)$ is a necessary condition for the existence of a solution of (4), as we can see using the Pohozaev’s identity (see [4, Proposition 1]).

The quasilinear equation (3) has a great relevance because it appears in mathematical models associated with several physical phenomena (see [5, 9] or [18] for a complete bibliography about other contexts of physics where quasilinear Schrödinger equations arise).
Formally, the problem (3) is the Euler-Lagrange equation for the energy functional

\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} H(u) dx. \]

Because of the term \( u^2 |\nabla u|^2 \), the natural function space associated with the functional \( J \) is given by

\[ \mathcal{S} = \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx < \infty \}. \]

A function \( u \in \mathcal{S} \) is called a weak solution of (3) if

\[ \int_{\mathbb{R}^N} (1 + 2u^2) \nabla u \nabla \phi dx + 2 \int_{\mathbb{R}^N} |\nabla u|^2 \phi dx + \int_{\mathbb{R}^N} u \phi dx - \int_{\mathbb{R}^N} h(u) \phi dx = 0, \]

for all \( \phi \in C_0^\infty(\mathbb{R}^N) \). It is not difficult to verify that the derivative of \( J \) in the direction \( \phi \) at \( u \) is

\[ J'(u) \phi = \int_{\mathbb{R}^N} (1 + 2u^2) \nabla u \nabla \phi dx + 2 \int_{\mathbb{R}^N} |\nabla u|^2 \phi dx + \int_{\mathbb{R}^N} u \phi dx - \int_{\mathbb{R}^N} h(u) \phi dx. \]

Thus, \( u \in \mathcal{S} \) is a weak solution of (3) if, and only if, the derivative \( J'(u) \phi = 0 \) for every direction \( \phi \in C_0^\infty(\mathbb{R}^N) \). As defined in [8], we say that a weak solution \( u \) of (3) is a ground state if

\[ J(u) = \inf \{ J(w) : w \text{ is a nontrivial weak solution of (3)} \}. \]

In order to obtain a ground state solution of (3), we proceed as in [9, 14] by changing of variables \( v = f^{-1}(u) \), where \( f \) is defined by

\[ f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \text{ on } [0, +\infty) \quad \text{and} \quad f(-t) = -f(t) \text{ on } (-\infty, 0]. \]

The transformation \( f \) is the key in many arguments here, it was used first independently in [9] and [14]. Thus, we can write \( J(u) \) as

\[ I(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + f^2(v)) dx - \int_{\mathbb{R}^N} H(f(v)) dx. \]

In [9], it is proved that that \( I \) is well defined on the usual Sobolev space \( H^1(\mathbb{R}^N) \) and \( I \in C^1(\mathbb{H}^1(\mathbb{R}^N), \mathbb{R}) \). It is also proved in [9] that if \( v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N) \) is a solution to the Euler-Lagrange equation for \( I \)

\[ -\Delta v = h(f(v)) f'(v) - f(v) f''(v) \quad \text{in } \mathbb{R}^N \]

then \( u = f(v) \in \mathcal{S} \cap C^2(\mathbb{R}^N) \) is a solution of (3).

Actually, we prove in the Appendix the following proposition, which will be useful to study the equation (3).

**Proposition 1.** Let \( u \in \mathcal{S} \) and \( v = f^{-1}(u) \). Then, \( u \) is a weak solution for problem (3) if, and only if, \( v = f^{-1}(u) \in H^1(\mathbb{R}^N) \) is a critical point of the functional \( I \). Moreover, \( J(u) = I(v) \).

Our aim is to show the following results:

**Theorem 1.1.** Suppose \( N \geq 3 \) and \( h \) satisfies \((h_1)-(h_3)\) and \((h_4)\), with \( \lambda > \Lambda \) for some constant \( \Lambda > 0 \) sufficiently large. Then problem (1), for \( G(s) = H(f(s)) - f(s)^2/2 \), has a minimizing solution.
Theorem 1.1 provides a ground state solution of (4), that is, a nontrivial solution \( w \in H^1(\mathbb{R}^N) \) of (4) such that \( I(w) \leq I(v) \) for every nontrivial solution \( v \) of (4). As a consequence we have the following result:

**Corollary 1.** Under the hypotheses of Theorem 1.1, then problem (3) (respectively, the dual problem (4)) possesses a nontrivial ground state solution.

The results of Theorem 1.1 and Corollary 1 also hold for \( N = 2 \), more precisely:

**Theorem 1.2.** Suppose \( N = 2 \) and \( h \) satisfies \((h_1)-(h_3)\) and \((h_4)\), with \( \lambda > \Lambda \) for some constant \( \Lambda > 0 \) sufficiently large. Then problem (1), for \( G(s) = H(f(s)) - f(s)^2/2 \), has a minimizing solution.

**Corollary 2.** Under the hypotheses of Theorem 1.2, the problem (3) (respectively, the dual problem (4)) possesses a nontrivial ground state solution.

Existence of ground state solutions for (3) were established in the subcritical case by Colin, Jeanjean and Squassina in [8, Theorem 1.3]. Thus, Corollaries 1 and 2 extend this result to the critical case.

Positive and sign-changing ground state solutions were considered in [15, 16] for a class of quasilinear Schrödinger equations involving several types of potentials; however, the corresponding problems have been studied for the subcritical case with nonlinearities requiring the assumption \( h(s)/s \) is increasing in \((0, \infty)\) and the Ambrosetti-Rabinowitz condition.

The paper is organized as follows. The section 2 is dedicated to fix some notations. The case \( N \geq 3 \) is treated in Section 3, while the case \( N = 2 \) is considered in Section 4. In the Appendix, we prove Proposition 1.

### 2. Notation

We consider the functional \( I : H^1(\mathbb{R}^N) \to \mathbb{R} \) associated with (4) and defined by

\[
I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \int_{\mathbb{R}^N} G(v) dx
\]

or

\[
I(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + f(v)^2) dx - \int_{\mathbb{R}^N} H(f(v)) dx,
\]

where

\[
G(t) = \int_0^t g(\tau) d\tau = \int_0^t [h(f(\tau)))f'(\tau) - f(\tau)f'(\tau)] d\tau = H(f(t)) - \frac{f(t)^2}{2}
\]

and \( H(s) = \int_0^s h(\tau) d\tau \). Since \( f \) and \( h \) are odd functions, the function \( G \) is even.

Set

\[
m = \inf \{ I(v) : v \text{ is nontrivial solution of (4)} \}
\]

and

\[
A = \inf \{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx : \int_{\mathbb{R}^N} G(v) dx = 1 \}, \quad \text{if } N \geq 3,
\]

while

\[
A = \inf \{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx : \int_{\mathbb{R}^N} G(v) dx = 0 \}, \quad \text{if } N = 2.
\]

We also define the following minimax value

\[
b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0 \} \).
Observing that the problem (4) is autonomous, the Schwarz symmetrization can be used to reduce the minimizing problem (6) in the space $H^1_{rad}(\mathbb{R}^N)$ of the radial functions.

The main feature of the proof of our results is the verification that $A$ is attained and $m = A = b$, when $N = 2$, or

$$m = b = \frac{1}{N} \left( \frac{N - 2}{2N} \right)^{(N-2)/2} [2A]^{N/2}, \quad \text{if } N \geq 3.$$

In the remainder of this section we gather some properties satisfied by the function $f$. These were proved first independently in [9] and [14] (see also [1]).

1. $f$ is $C^\infty$, $f(0) = 0$, $f(-t) = -f(t)$, and $|f'(t)| \leq 1$, for all $t \in \mathbb{R}$.
2. $\lim_{t \to 0} \frac{f(t)}{t} = 1$.
3. $\lim_{t \to \infty} \frac{f(t)^4}{t^2} = 2$.
4. $\frac{1}{2} f(t) \leq t f'(t) \leq f(t)$, for all $t \geq 0$.
5. $f(t)^2 \leq t^2$, for all $t \in \mathbb{R}$.
6. There exists a constant $C > 0$ such that

$$|f(t)| \geq \begin{cases} C|t|, & \text{if } |t| \leq 1, \\ C|t|^{1/2}, & \text{if } |t| \geq 1. \end{cases}$$

Remark 1. From (f2) and (f3), we have

$$\lim_{t \to \infty} \frac{f(t)^{q-1} f'(t)t}{t^2} = 0, \quad \text{if } q > 4.$$

3. The case $N \geq 3$. Throughout this section, $S$ denotes the best constant of Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, that is

$$S \left[ \int_{\mathbb{R}^N} |v|^2 \right]^\frac{1}{2} dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 dx, \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N). \quad (9)$$

The proof of the following result is based on arguments found in [19].

Lemma 3.1. Suppose $h$ satisfies $(h_1)$ and $(h_2)$. Then, any minimizing sequence $(v_n)$ for (1) is bounded in $H^1(\mathbb{R}^N)$.

Proof. We have

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to A \quad \text{and} \quad \int_{\mathbb{R}^N} H(f(v_n)) dx = 1 + \int_{\mathbb{R}^N} \frac{f(v_n)^2}{2} dx.$$

Combining $(h_1) - (h_2)$ with $(f_1) - (f_2)$, there is $C > 0$ such that

$$H(f(t)) \leq \frac{1}{4} f(t)^2 + Ct^2^* \quad \text{for every } t \geq 0.$$

Hence

$$\int_{\mathbb{R}^N} |f(v_n)|^2 dx \leq C \int_{\mathbb{R}^N} |v_n|^2^* dx, \quad \text{for every } n. \quad (10)$$
Therefore, \( \int_{\mathbb{R}^N} |f(v_n)|^2 \, dx \) is bounded. From (f3), we have

\[
\int_{\{x \in \mathbb{R}^N : |v_n(x)| \leq 1\}} |v_n|^2 \, dx \leq C \int_{\mathbb{R}^N} |f(v_n)|^2 \, dx,
\]

\[
\int_{\{x \in \mathbb{R}^N : |v_n(x)| \geq 1\}} |v_n|^2 \, dx \leq C \int_{\mathbb{R}^N} |v_n|^2 \, dx \leq C \left( S^{-1} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{2/2}.
\]

Therefore, \( \int_{\mathbb{R}^N} |v_n|^2 \, dx \) is bounded and finally that \((v_n)\) is bounded in \( H^1(\mathbb{R}^N) \). \( \square \)

For simplicity of notation, we write

\[
\mathcal{M} = \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(v) \, dx = 1 \},
\]

\[
\mathcal{P} = \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : 2N \int_{\mathbb{R}^N} G(v) \, dx = (N - 2) \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \},
\]

\[
T(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \quad \text{and} \quad \Psi(v) = \int_{\mathbb{R}^N} G(v) \, dx.
\]

We have

\[
2A = \inf_{v \in \mathcal{M}} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx.
\]

The conditions (f3) and (h3) imply that the set \( \mathcal{M} \) is a \( C^1 \) manifold. Indeed, from (f3) and (h3),

\[
\Psi'(v)v = \int_{\mathbb{R}^N} [h(f(v))f'(v)v - f(v)f'(v)v] \, dx
\]

\[
= \int_{\mathbb{R}^N} [h(f(v))f(v) - f(v)^2] \frac{f'(v)v}{f(v)} \, dx
\]

\[
> \int_{\mathbb{R}^N} [2H(f(v)) - f(v)^2] \frac{1}{2} \, dx = \Psi(v) = 1.
\]

Thus, \( \Psi'(v) \neq 0 \) for every \( v \in \mathcal{M} \).

**Remark 2.** This last argument works when \( N = 2 \). In this case

\[
\mathcal{M} = \{ v \in H^1(\mathbb{R}^2) \setminus \{0\} : \Psi(v) = 0 \},
\]

and the last inequality is strict.

The first result involving \( A \) and \( b \) is given by the following:

**Lemma 3.2.** Suppose \( h \) satisfies (h1) and (h2). Then,

\[
\frac{1}{N} \left( \frac{N - 2}{2N} \right)^{(N-2)/2} (2A)^{N/2} \leq b.
\] (11)

**Proof.** For each \( \gamma \in \Gamma \) one has \( \gamma([0,1]) \cap \mathcal{P} \neq \emptyset \) (see [12] for more details). Hence, there exists \( t_0 \in [0,1] \) such that \( \gamma(t_0) \in \mathcal{P} \), and consequently

\[
\inf_{v \in \mathcal{P}} I(v) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \gamma(t_0)|^2 \, dx - \int_{\mathbb{R}^N} G(\gamma(t_0)) \, dx.
\]

Thus

\[
\inf_{v \in \mathcal{P}} I(v) \leq I(\gamma(t_0)) \leq \max_{t \in [0,1]} I(\gamma(t)) \leq b.
\]
As in the proof of Lemma 3.1 in [12], following Coleman-Glazer-Martin [7],

$$\inf_{v \in P} I(v) = \frac{1}{N} \left( \frac{N-2}{2N} \right)^{(N-2)/2} (2A)^{N/2},$$

and the lemma follows.

The proof of the identity

$$m = b = \frac{1}{N} \left( \frac{N-2}{2N} \right)^{(N-2)/2} (2A)^{N/2}$$

is based on arguments that will be used in the next section for the case $N = 2$, following Jeanjean and Tanaka [12].

**Lemma 3.3.** Suppose $h$ satisfies $(h_1)$ and $(h_2)$. Then, $A > 0$.

**Proof.** It is clear that $A \geq 0$. Assume by contradiction that $A = 0$ and let $(v_n)$ be a sequence in $H^1_{rad}(\mathbb{R}^N)$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to A \quad \text{with} \quad \int_{\mathbb{R}^N} G(v_n) dx = 1.$$

From (9),

$$\int_{\mathbb{R}^N} |v_n|^2 dx \to 0, \quad \text{as } n \to \infty.$$

As in the proof of Lemma 3.1,

$$\int_{\mathbb{R}^N} |f(v_n)|^2 dx \leq C \int_{\mathbb{R}^N} |v_n|^2 dx,$$

and so $\int_{\mathbb{R}^N} |f(v_n)|^2 dx \to 0$ as $n \to \infty$. On the other hand,

$$1 + \frac{1}{2} \int_{\mathbb{R}^N} |f(v_n)|^2 dx = \int_{\mathbb{R}^N} H(f(v_n)) dx \leq C \int_{\mathbb{R}^N} |f(v_n)|^2 dx + C \int_{\mathbb{R}^N} |v_n|^2 dx.$$

Taking $n \to \infty$ in the above expression, we obtain a contradiction. Thus $A > 0$. \qed

Since $\mathcal{M}$ is a $C^1$ manifold, from the Ekeland variational principle, there is $v_n \in \mathcal{M}$ and $\lambda_n \in \mathbb{R}$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to A \quad \text{and} \quad T'(v_n) - \lambda_n \Psi'(v_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

**Lemma 3.4.** The sequence $(\lambda_n)$ given by (13) is bounded and $\limsup_{n \to \infty} \lambda_n \leq 2A$.

**Proof.** Using (13) and $(f_3)$,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lambda_n \int_{\mathbb{R}^N} [h(f(v_n)) f(v_n) - f(v_n)^2] \frac{f'(v_n) v_n}{f(v_n)} dx + o_n(1)$$

$$\geq \frac{\lambda_n}{2} \int_{\mathbb{R}^N} \left( 2H(f(v_n)) - f(v_n)^2 \right) dx + o_n(1)$$

$$= \lambda_n \Psi(v_n) + o_n(1) = \lambda_n + o_n(1).$$

Since $\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \to 2A$, (14) shows that $\limsup_{n \to \infty} \lambda_n \leq 2A$. \qed
As \( A > 0 \) and \((v_n)\) is bounded in \(H^1(\mathbb{R}^N)\), the relation
\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lambda_n \int_{\mathbb{R}^N} (h(f(v_n))f'(v_n)v_n - f(v_n)f'(v_n)v_n)dx
\]
implies that \(\lambda_n\) does not converges to 0. The concentration-compactness principle [13] applied to the sequence \((v_n)\) guarantees the existence of positive finite measures \(\mu\) and \(\nu\) and nonnegative sequences \((\mu_i), (\nu_i)\) \(\in \mathbb{R}\) and \(x_i \in \mathbb{R}^N\) such that

(i) \(|\nabla v_n|^2 \to d\mu\geq |\nabla v|^2 + \sum_i \delta_{x_i}d\mu_i\),
(ii) \(|v_n|^{2^*} \to d\nu= |v|^{2^*} + \sum_i \delta_{x_i}d\nu_i\),
(iii) \(\mu_i \geq S v_i^{2/2^*}\), for every \(i\),
(iv) \(\sum_i \nu_i^{2/2^*} < \infty\).

where \(v\) is the weak limit of \(v_n\) in \(H^1(\mathbb{R}^N)\).

**Lemma 3.5.** Consider \(\rho \geq 0\) given by condition \((h_2)\). If \(\rho > 0\) and \(\nu_i > 0\), for some index \(i\), then \(\nu_i \geq \left(\frac{s}{2^{2^*/2^*}A}\right)^{N/2}\). If \(\rho = 0\) then \(\nu_i = 0\) for every index \(i\).

**Proof.** Set \(\varphi \in C_0^\infty(\mathbb{R}^N)\) and define \(\varphi_\varepsilon(x) = \varphi\left(\frac{x-x_i}{\varepsilon}\right)\) whatever the choice of \(i\). By (13), we obtain
\[
\int_{\mathbb{R}^N} \nabla v_n \nabla (\varphi_\varepsilon v_n) dx = \lambda_n \int_{\mathbb{R}^N} [h(f(v_n))f'(v_n)v_n - f(v_n)f'(v_n)v_n] \varphi_\varepsilon dx + o_n(1).
\]

From \((h_1) - (h_2), (f_1) - (f_3)\), for any \(\eta > \rho\), there is a constant \(C > 0\) such that
\[
h(f(t))f'(t)t \leq Ct^2 + \eta 2^{2^*/2^*}t^{2^*}\quad \text{for every } t \geq 0.
\]

Hence
\[
\int_{\mathbb{R}^N} |\nabla v_n|^{2^*} \varphi_\varepsilon dx + \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \varphi_\varepsilon dx \leq 2^{2^*/2} \eta \lambda_n \int_{\mathbb{R}^N} |v_n|^{2^*} \varphi dx + C \lambda_n \int_{\mathbb{R}^N} |v_n|^2 \varphi dx.
\]

By standard arguments, first letting \(n \to \infty\) and then letting \(\varepsilon \to 0\), and using Lemma 3.3, it follows that \(\mu_i \leq 2^{2^*/2} \eta A \nu_i\). Since \(\eta > \rho\) is arbitrary, \(\mu_i \leq 2^{2^*/2} \rho A \nu_i\).

From (iii), we get
\[
2^{2^*/2} \rho A \nu_i \geq \mu_i \geq S v_i^{2/2^*}.
\]

Hence, if \(\rho = 0\), then \(\nu_i = 0\) for every \(i\). If \(\rho > 0\) and \(\nu_i > 0\), for some index \(i\), we have from (15) \(\nu_i \geq \left(\frac{s}{2^{2^*/2^*}2\rho A}\right)^{N/2}\).

**Lemma 3.6.** If \(\rho > 0\) and \(\nu_i > 0\) for some index \(i\), then \(2A \geq \frac{s}{2^{2^*/2^*}}\).

**Proof.** From (9), we have
\[
\int_{\mathbb{R}^N} |v_n|^{2^*} dx \leq S^{2^*/2^*} \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)^{2^*/2}\quad \text{for } n \in \mathbb{N}.
\]

Combining \((iv)\) with Lemma 3.5, we deduce that there exists at most a finite number of points \(x_j\) on bounded subsets of \(\mathbb{R}^N\). Hence, given \(i\), there exists \(\delta_i > 0\) such that \(x_j\) does not belong to the ball \(B(x_i, \delta_i)\), except for \(j = i\). Let \(0 < \varepsilon < \delta_i\) and \(\varphi \in C_0^\infty(\mathbb{R}^N), \ 0 \leq \varphi(x) \leq 1, \ \varphi \equiv 1 \ on \ B(x_i, \varepsilon/2) \ and \ \varphi \equiv 0 \ on \ \mathbb{R}^N \setminus B(x_i, \varepsilon). \ By \ construction, \ we \ have \)
\[
\int_{\mathbb{R}^N} |v_n|^{2} \varphi dx \leq S^{2^*/2^*} \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 dx\right)^{2^*/2}\quad \text{for } n \in \mathbb{N}.
\]
Taking \( n \to \infty \) in the above inequality and taking \( \epsilon \to 0 \) in the resultant expression, we obtain

\[ \nu_i \leq S^{-2/2^*}(2A)^{2^*/2}. \]

By Lemma 3.5, \( \nu_i \geq \left( \frac{S}{2^{2^*/2^*}2^*A} \right)^{N/2} \). Thus,

\[ \left( \frac{S}{2^{2^*/2^*}2^*A} \right)^{N/2} \leq S^{-2/2^*}(2A)^{2^*/2}, \]

which implies \( 2A \geq \frac{S}{2^22^*A} \).

**Remark 3.** From Lemmas 3.2 and 3.5, if \( \rho > 0 \) and \( \nu_i > 0 \), for some index \( i \), then

\[ b \geq 1 - \frac{N(N-2)}{2^22^*A} \frac{S}{2^22^*A} \frac{N}{2}. \]

**Lemma 3.7.** Suppose \( h \) satisfies \((h_4)\) and \( \rho > 0 \). Then there exists \( \Lambda > 0 \), such that

\[ b < \frac{1}{N} \left( \frac{N-2}{2N} \right)^{(N-2)/2} \left( \frac{S}{2^22^*A} \right)^{N/2}, \]

for all \( \lambda > \Lambda \).

**Proof.** Let us fix \( \psi \in C_0^\infty (\mathbb{R}^N) \), such that \( 0 \leq \psi \leq 1 \). By \((f_6)\), \( f(t\psi(x)) \geq f(t)\psi(x) \) for all \( t \geq 0 \). Thus, using \((h_4)\) and \((f_4)\), we obtain

\[ b \leq \max_{t \geq 0} I(t\psi) \leq \max_{t \geq 0} \left\{ t^2 \left( \frac{||\psi||^2}{2} - \frac{\lambda f(t)^q}{q} \int_{\mathbb{R}^N} \psi^q dx \right) \right\}, \]

where

\[ ||\psi||^2 = \int_{\mathbb{R}^N} (|\nabla \psi|^2 + \psi^2) dx. \]

Let \( t_\lambda > 0 \) be the global maximum point of the function

\[ \sigma(t) = \frac{t^2}{2} - \lambda f(t)^q \int_{\mathbb{R}^N} \psi^q dx. \]

We can check that

\[ t_\lambda^2 ||\psi||^2 = \lambda f(t_\lambda)^{-1} f(t_\lambda) t_\lambda \int_{\mathbb{R}^N} \psi^q dx \] (16)

and from \((f_4)-(f_6)\), we have

\[ \frac{\lambda}{2} f(t_\lambda)^q \int_{\mathbb{R}^N} \psi^q dx \leq t_\lambda^2 ||\psi||^2 \leq \lambda f(t_\lambda)^q \int_{\mathbb{R}^N} \psi^q dx \] (17)

that implies

\[ \sigma(t_\lambda) \leq t_\lambda^2 (q-2) ||\psi||^2. \] (18)

On the other hand, since

\[ \frac{1}{\lambda} = \frac{f(t_\lambda)^{q-1} f'(t_\lambda) t_\lambda}{t_\lambda^2} ||\psi||^{-2} \int_{\mathbb{R}^N} \psi^q dx, \]

from Remark 1 and \((f_3)\), we can see that \( t_\lambda \to 0 \) as \( \lambda \to \infty \). To complete the proof, fix \( \Lambda > 0 \) such that

\[ t_\lambda^2 (q-2) ||\psi||^2 < \frac{1}{N} \left( \frac{N-2}{2N} \right)^{(N-2)/2} \left( \frac{S}{2^22^*A} \right)^{N/2}, \]

for all \( \lambda > \Lambda \).
We observe that the choice of Λ depends only on q, ψ and its norms in \( L^q(\mathbb{R}^N) \) and \( H^1(\mathbb{R}^N) \).

**Remark 4.** As a consequence of Remark 3 and Lemma 3.7, the cases \( \rho > 0 \) and \( \nu_i > 0 \), for some index \( i \), cannot coexist.

**Lemma 3.8.** The weak limit \( v \) of the minimizing sequence \( (v_n) \) is nontrivial.

**Proof.** Suppose, by contradiction, that \( v = 0 \). Consider \( \rho \geq 0 \) given by \( (h_2) \). If \( \rho = 0 \), by Lemma 3.5,

\[
v_n \to 0 \text{ in } L^{2^*}(\mathbb{R}^N).
\]

We now assume \( \rho > 0 \). Since \( v_n \in H^1_{rad}(\mathbb{R}^N) \), \( v_n(x) \) are uniformly bounded in \( |x| \geq \delta \), for every \( \delta > 0 \). As a consequence, every number \( \nu_i \) is null, except eventually the corresponding number associated with the atom at the origin. Let \( \nu_0 \) denote this unique number which can be assumed nonnegative. Invoking Remark 4, we get \( \nu_0 = 0 \), and we conclude that (19) holds. Arguing as in the proof of Lemma 3.3, we obtain the same contradiction. Therefore, \( v \neq 0 \).

**Lemma 3.9.** The constant \( A \) is attained by the weak limit \( v \) of \( v_n \).

**Proof.** Since \( v_n \to v \) in \( H^1(\mathbb{R}^N) \), we obtain

\[
T(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \leq \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = A.
\]

In case \( \rho = 0 \), Lemma 3.5 implies that

\[
v_n \to v \text{ in } L^{2^*}(\mathbb{R}^N).
\]

On the other hand, if \( \rho > 0 \) then Lemmas 3.2, 3.6 and 3.7 imply that \( \nu_i = 0 \) for every \( i \). Thus, we obtain that (20) holds. Recall that

\[
\int_{\mathbb{R}^N} H(f(v_n)) \, dx = \int_{\mathbb{R}^N} \frac{f(v_n)^2}{2} \, dx + 1.
\]

We claim that

\[
\int_{\mathbb{R}^N} H(f(v)) \, dx \geq \int_{\mathbb{R}^N} \frac{f(v)^2}{2} \, dx + 1,
\]

which gives

\[
\int_{\mathbb{R}^N} G(v) \, dx \geq 1.
\]

In effect, set \( R(t) = [H(f(t)) - t^{2^*/2^*}]^+ = \max\{0, H(f(t)) - t^{2^*/2^*}\} \). It is easy to check that \( (h_1) \) and \( (h_2) \) imply that

\[
\lim_{s \to 0^+} \frac{R(t)}{s^{2^*} + s^{2^*}} = 0 = \lim_{s \to +\infty} \frac{R(t)}{s^{2^*} + s^{2^*}},
\]

and by using the compactness lemma of Strauss [4, Theorem A.I, p.338]

\[
\int_{\mathbb{R}^N} R(v_n) \, dx \to \int_{\mathbb{R}^N} R(v) \, dx.
\]

Now, taking \( Q(t) = R(t) - [H(f(t)) - t^2/2 - t^{2^*/2^*}] \), we have \( Q(t) \geq 0 \) and \( H(f(t)) - t^2/2^* = R(t) - Q(t) \), for all \( t > 0 \). Observe that

\[
1 + \int_{\mathbb{R}^N} [Q(v_n) + \frac{f(v_n)^2}{2}] \, dx = \frac{1}{2} \int_{\mathbb{R}^N} |v_n|^2 \, dx + \int_{\mathbb{R}^N} R(v_n) \, dx.
\]
By Fatou lemma, after passing to the limit in above equality, we find
\[ 1 + \int_{\mathbb{R}^N} (Q(v) + \frac{f(v)^2}{2}) dx \leq \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx + \int_{\mathbb{R}^N} R(v) dx \]
and the claim (21) holds. We claim now that \( v \in \mathcal{M} \). Indeed, if \( v \notin \mathcal{M} \), one would have
\[ \int_{\mathbb{R}^N} G(v) dx > 1. \]
Define \( \phi : [0,1] \to \mathbb{R} \) by \( \phi(t) = \int_{\mathbb{R}^N} G(tv) dx \). From \((h_1)-(h_2)\) and \((f_1)-(f_2),\)
\( \phi(0) = 0 \) and \( \phi(1) = \int_{\mathbb{R}^N} G(v) dx > 1. \) Hence, there is \( t_0 \in (0,1) \) such that
\( \phi(t_0) = 1 \). Observing that \( \phi(t_0) = 1 \) if, and only if, \( t_0 v \in \mathcal{M} \), we have \( T(t_0 v) \geq A \).
On the other hand,
\[ 0 < T(t_0 v) = t_0^2 T(v) \leq t_0^2 A < A, \]
which is a contradiction. Therefore, \( v \in \mathcal{M} \) and \( T(v) = A \). Theorem 1.1 is proved. \( \Box \)

We are now ready to prove Corollary 1. In fact Corollary 1 may be proved in much the same way as [4, Theorem 3, p.331]. By Theorem 1.1, there is \( w \in H^1_{rad}(\mathbb{R}^N)\backslash \{0\} \) such that
\[ \int_{\mathbb{R}^N} |\nabla w|^2 dx = A \quad \text{and} \quad \int_{\mathbb{R}^N} G(w) dx = 1. \]
Since \( \mathcal{M} \) a \( C^1 \) manifold (see Remark 2), by Lagrange multipliers, there is \( \theta \in \mathbb{R} \) such that
\[ \int_{\mathbb{R}^N} \nabla w \nabla v dx = \theta \int_{\mathbb{R}^N} g(w) v dx, \quad \text{for every} \quad v \in H^1(\mathbb{R}^N), \]
Since
\[ \int_{\mathbb{R}^N} |\nabla w|^2 dx = \theta \int_{\mathbb{R}^N} g(w) w dx, \]
we have \( \theta > 0 \). Setting \( v(x) = w\left(\frac{x}{\sigma}\right) \), we see that \( v \) is also a nontrivial solution of (4).

In the following, we prove that \( v \) is a ground state solution for (4). From Pohožaev’s identity (see [4, Proposition 1]), for any solution \( \phi \in H^1(\mathbb{R}^N) \) of (4), we have:
\[ I(\phi) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx. \] (23)
In particular,
\[ I(v) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 dx = \frac{\theta N - 2}{N} \int_{\mathbb{R}^N} |\nabla w|^2 dx. \] (24)
Choose \( \sigma > 0 \) such that \( \sigma^N \int_{\mathbb{R}^N} G(\phi) dx = 1 \) and observe that \( \phi_\sigma(x) = \phi(x/\sigma) \) belongs to \( \mathcal{M} \) and
\[ \int_{\mathbb{R}^N} |\nabla \phi_\sigma|^2 dx = \sigma^{N-2} \int_{\mathbb{R}^N} |\nabla \phi|^2 dx. \] (25)
Since \( w \) solution of \( -\Delta w = \theta g(w) \) in \( \mathbb{R}^N \), from Pohožaev’s identity again we have
\[ \int_{\mathbb{R}^N} |\nabla w|^2 dx = \frac{2N \theta}{N - 2} \int_{\mathbb{R}^N} G(w) dx = \frac{2N \theta}{N - 2} \] (26)
and (from \( -\Delta \phi = g(\phi) \))
\[ \int_{\mathbb{R}^N} |\nabla \phi_\sigma|^2 dx = \frac{2N \sigma^{-2}}{N - 2}. \] (27)
From (26) and (27)

$$\int_{\mathbb{R}^N} |\nabla w|^2 dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 dx$$

implies that $\sigma^2 \leq \theta^{-1}$. \hfill (28)

On the other hand, combining (23), (24), (25), (26) and (27), we have

$$\frac{I(\phi)}{I(v)} = \frac{\sigma^{-\frac{N}{p}}}{\theta} \geq 1,$$

thanks to (28). This implies that $v$ is a ground state solution for problem (4).

We claim that $u = f(v)$ is a ground state solution for problem (3). Effectively, let $\phi \in \mathcal{S}$ be any nontrivial solution of (3) and set $w = f^{-1}(\phi)$. According to [8] (see also [1]),

$$J(\phi) = J(f(w)) = I(w).$$

From the properties of $f$ and Proposition 1, $w$ is a nontrivial critical point of $I$ and then $I(w) \geq I(v)$. Consequently,

$$J(\phi) = J(f(w)) = I(w) \geq I(v) = I(f^{-1}(u)) = J(u).$$

Hence, $u$ is a ground state solution in $\mathcal{S}$, which completes the proof of Corollary 1.

4. The case $N = 2$. We begin by recalling that Pohožaev’s identity (see [4, Proposition 1]) shows that $\int_{\mathbb{R}^2} G(v) dx = 0$ is a necessary condition for the existence of a solution of (4). Throughout the section, we continue to write $M$ for the set \{ $v \in H^1(\mathbb{R}^2) \setminus \{0\}$ : $\int_{\mathbb{R}^2} G(v) dx = 0$ \}. Hence, in this case, $P = M$. Set

$$A = \min \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx : \int_{\mathbb{R}^2} G(v) dx = 0, \quad v \neq 0 \right\} = \inf_{v \in P} \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx. \hfill (29)$$

Define the following minimax value:

$$c = \inf_{0 \neq v \in H^1(\mathbb{R}^2)} \max_{t \geq 0} I(tv). \hfill (30)$$

The first result in this section shows a sufficient condition on a sequence $\{v_n\}$ to get a convergence like $H(f(v_n)) \to H(f(v))$ in $L^1(\mathbb{R}^2)$.

Lemma 4.1. Suppose $h$ satisfies (h1) and (h2). Let $(v_n)$ be a sequence in $H^1_{rad}(\mathbb{R}^2)$ such that

$$\|\nabla v_n\|^2_{L^2} \leq m < \frac{2\pi}{\alpha_\sigma} \quad \text{and} \quad \|v_n\|^2_{L^2} \leq M,$$

for some $M, m > 0$. Then

$$\int_{\mathbb{R}^2} H(f(v_n)) dx \to \int_{\mathbb{R}^2} H(f(v)) dx,$$

provided $(v_n)$ is weakly convergent to $v$ in $H^1(\mathbb{R}^2)$.

Proof. Since $(v_n)$ is bounded in $H^1(\mathbb{R}^2)$ and is weakly convergent to $v$, we may assume that $v_n(x) \to v(x)$ almost everywhere in $\mathbb{R}^2$. We can also assume that $\lim_{|x| \to +\infty} v_n(x) = 0$, uniformly in $n$, because $(v_n)$ is a bounded sequence in $H^1_{rad}(\mathbb{R}^2)$. By a version of Trudinger-Moser inequality [6], given positive numbers $\tilde{m} < 1$ and $\tilde{M}$, there exists a constant $C = C(\tilde{m}, \tilde{M}) > 0$ such that

$$\int_{\mathbb{R}^2} (e^{4\pi w^2} - 1) dx < C(\tilde{m}, \tilde{M}), \hfill (31)$$

for every $w \in H^1(\mathbb{R}^2)$ such that $\|\nabla w\|^2_{L^2} \leq \tilde{m} < 1$ and $\|w\|_{L^2} \leq \tilde{M}$. 


Fix $\alpha > \alpha_o$ and $\varepsilon > 0$ such that $(1 + \varepsilon)^2 m < 2\pi / \alpha_o$ and $\alpha \leq (1 + \varepsilon)^2 \alpha_o$, for $\alpha_o$ given by $(h_2)$. Observe that $(h_1)$-$(h_2)$ and $(f_1)$-$(f_2)$ imply that

$$\lim_{s \to 0^+} \frac{H(f(s))}{(e^{2s\alpha^2} - 1)} = 0 = \lim_{s \to +\infty} \frac{H(f(s))}{(e^{2s\alpha^2} - 1)}. \tag{32}$$

Write

$$z_n = \frac{\sqrt{\alpha_o} (1 + \varepsilon) v_n}{\sqrt{2\pi}}.$$

Thus,

$$\|\nabla z_n\|^2_{L^2} \leq \frac{(1 + \varepsilon)^2 \alpha_o m}{2\pi} = \tilde{m} < 1.$$

From (31), we find

$$\int_{\mathbb{R}^2} (e^{2\varepsilon z_n^2} - 1) dx \leq \int_{\mathbb{R}^2} (e^{4\varepsilon z_n^2} - 1) dx \leq C(\tilde{m}, \tilde{M}) \tag{33}$$

Combining (32) with (33), we can use the compactness lemma of Strauss [4, Theorem A.I, p.338] to conclude the proof of Lemma 4.1. \hfill \Box

As in the preceding section, we derive some results involving the levels $A$ and $c$.

**Lemma 4.2.** Suppose $h$ satisfies $(h_1)$, $(h_2)$ and $(h_4)$. Then, $A \leq c$.

*Proof.* For each $v \in H^1(\mathbb{R}^2) - \{0\}$, define the function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \int_{\mathbb{R}^2} G(tv) dx = \int_{\mathbb{R}^2} [H(f(tv)) - \frac{f(tv)^2}{2}] dx.$$

From $(h_1)$ and $(h_4)$, we conclude that $\phi(t) < 0$ for $t$ sufficiently small and $\phi(t) > 0$ for $t$ sufficiently large. Thus, exists $t_0 > 0$ such that $\phi(t_0) = 0$, hence that $t_0 v \in \mathcal{P}$ and finally

$$A \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla (t_0 v)|^2 = I(t_0 v) \leq \max_{t \geq 0} I(tv).$$

Therefore, $A \leq c$ and the lemma follows. \hfill \Box

**Lemma 4.3.** Suppose $h$ satisfies $(h_1)$, $(h_2)$ and $(h_4)$. Then $A > 0$.

*Proof.* It is clear that $A \geq 0$. Assume by contradiction that $A = 0$ and let $(w_n)$ be a sequence in $H^1(\mathbb{R}^2)$ satisfying

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla w_n|^2 dx \to A = 0 \quad \text{with} \quad \int_{\mathbb{R}^2} G(w_n) dx = 0.$$

For each $\lambda_n > 0$, the function $v_n(x) = w_n(\frac{x}{\lambda_n})$ satisfies

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_n|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla w_n|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^2} G(v_n) dx = 0.$$

Since

$$\int_{\mathbb{R}^2} f(v_n)^2 dx = \lambda_n^2 \int_{\mathbb{R}^2} f(w_n)^2 dx,$$

we choose $\lambda_n^2 = 1/\int_{\mathbb{R}^2} f(w_n)^2 dx$. In this way,

$$\int_{\mathbb{R}^2} f(v_n)^2 dx = 1.$$

Consequently,

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_n|^2 dx \to A = 0, \quad \int_{\mathbb{R}^2} f(v_n)^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} G(v_n) dx = 0.$$
We stress that we may consider the sequence \((v_n)\) in the space \(H^1_{\text{rad}}(\mathbb{R}^2)\). From Lemma 4.1,
\[
\int_{\mathbb{R}^2} H(f(v_n))dx \to \int_{\mathbb{R}^2} H(f(v))dx. \tag{34}
\]
This gives,
\[
\int_{\mathbb{R}^2} H(f(v))dx = \frac{1}{2}, \tag{35}
\]
because \(G(s) = H(f(s)) - f(s)^2/2\). Hence, \(v \neq 0\). On the other hand, from \((h_4)\), there exists \(q > 4\) and \(\lambda > 0\) such that
\[
\frac{\lambda}{q} \int_{\mathbb{R}^2} |f(v_n)|^q dx \leq \int_{\mathbb{R}^2} H(f(v_n))dx = \frac{1}{2}.
\]
Using \((f_5)\), there exists a positive constant \(C\) such that
\[
\int_{\{x \in \mathbb{R}^2 : |v_n(x)| \leq 1\}} |v_n|^2 dx \leq \frac{1}{C} \int_{\mathbb{R}^2} |f(v_n)|^q dx \leq \frac{1}{C},
\]
and
\[
\int_{\{x \in \mathbb{R}^2 : |v_n(x)| \geq 1\}} |v_n|^2 dx \leq \int_{\{x \in \mathbb{R}^2 : |v_n(x)| \geq 1\}} (|v_n|^2)^{q/4} dx
\leq C \int_{\mathbb{R}^2} |f(v_n)|^q dx
\leq \frac{q}{2\lambda C},
\]
where we have used that \(q > 4\). Therefore, \(\int_{\mathbb{R}^2} |v_n|^2 dx\) is bounded and finally that \((v_n)\) is bounded in \(H^1(\mathbb{R}^2)\). Hence there exist \(v \in H^1_{\text{rad}}(\mathbb{R}^2)\) and a subsequence, still denoted by \((v_n)\), such that \((v_n)\) is weakly convergent to \(v\) in \(H^1_{\text{rad}}(\mathbb{R}^2)\). By the weak convergence,
\[
0 = A = \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_n|^2 dx \geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx.
\]
Thus,
\[
\int_{\mathbb{R}^2} |\nabla v|^2 dx = 0,
\]
and hence that \(v\) is almost everywhere a constant function. Using that \(v(x) = 0\) as \(|x| \to +\infty\), we conclude that \(v = 0\), which is a contradiction with \((35)\). Therefore, \(A > 0\) and the lemma follows. \(\square\)

**Lemma 4.4.** Suppose \((h_2)\) and \((h_4)\). Then there exists \(\Lambda > 0\), such that, \(c < \frac{\pi}{\alpha}\), for all \(\lambda > \Lambda\).

**Proof.** It is the same proof of Lemma 3.7 \(\square\)

**Lemma 4.5.** Suppose \(h\) satisfies \((h_1)\), \((h_2)\) and \((h_4)\). Then, the infimum \(A\) is attained.

**Proof.** Let \((v_n)\) be a minimizing sequence in \(H^1_{\text{rad}}(\mathbb{R}^2)\) for \(A\), that is,
\[
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_n|^2 dx \to A \quad \text{and} \quad \int_{\mathbb{R}^2} G(v_n)dx = 0. \tag{36}
\]
As in the proof of Lemma 4.3, we can assume that
\[ \int_{\mathbb{R}^2} f(v_n)^2 \, dx = 1. \]

From (36) and \((h_4)\), it follows that
\[ \lambda \int_{\mathbb{R}^2} f(v_n)^q \, dx \leq \int_{\mathbb{R}^2} H(f(v_n)) \, dx = \frac{1}{2} \int_{\mathbb{R}^2} f(v_n)^2 \, dx = \frac{1}{2}. \]

Since \(q > 4\), by a similar argument used in the proof of Lemma 4.3, the sequence \((v_n)\) is bounded in \(H^1(\mathbb{R}^2)\). Without loss of generality we can assume that \((v_n)\) is weakly convergent in \(H^1(\mathbb{R}^2)\) to some function \(v\). On the other hand, from (36) and Lemmas 4.2 and 4.4, it follows that
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^2} |\nabla v_n|^2 \, dx \leq 2A \leq 2c < \frac{2\pi}{\alpha_0}. \]

From Lemma 4.1, we conclude that
\[ \int_{\mathbb{R}^2} H(f(v_n)) \, dx \to \int_{\mathbb{R}^2} H(f(v)) \, dx. \quad (37) \]

By (36) and (37), we get
\[ \int_{\mathbb{R}^2} H(f(v)) \, dx = \frac{1}{2}, \]

hence that \( v \neq 0 \) and \( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx \leq A \). In order to show that \( A \) is attained, it remains to prove that \( \int_{\mathbb{R}^2} G(v) \, dx = 0 \). Indeed, from Fatou Lemma, we begin by observing that
\[ \int_{\mathbb{R}^2} G(v) \, dx = \int_{\mathbb{R}^2} H(f(v)) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} f(v)^2 \, dx = \frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^2} f(v)^2 \, dx \geq 0. \]

Hence, if \( \int_{\mathbb{R}^2} G(v) \, dx \neq 0 \), we would have \( \int_{\mathbb{R}^2} G(v) \, dx > 0 \). With the notation \( \phi(t) = \int_{\mathbb{R}^2} G(tv) \, dx, \ t \in \mathbb{R} \), from \((h_1)\), \((h_2)\) and \((f_1) - (f_2)\), we have \( \phi(t) < 0 \) for \( t \) sufficiently small and \( \phi(1) > 0 \). Therefore, there exists \( t_0 \in (0,1) \) such that \( \phi(t_0) = 0 \). As a consequence,
\[ A \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla (t_0v)|^2 \, dx. \quad (38) \]

Since \( v \) can be assumed in \( H^1_{rad}(\mathbb{R}^2) \) and \( v \neq 0 \), we obtain \( \int_{\mathbb{R}^2} |\nabla v|^2 \, dx > 0 \). Thus,
\[ 0 < \frac{1}{2} \int_{\mathbb{R}^2} |\nabla (t_0v)|^2 \, dx = \frac{t_0^2}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx \leq \frac{t_0^2}{2} A < A, \]

contrary to (38). Hence, \( \int_{\mathbb{R}^2} G(v) \, dx = 0 \), which completes the proof. \( \square \)

**Proof of Theorem 1.2 and Corollary 2:** We begin by proving that \( m = b = A \).

The proof of this fact is based on an argument used in [12] to prove a similar result in dimensions \( N \geq 3 \). By Lemma 4.5, there is \( v \in H^1_{rad}(\mathbb{R}^2) - \{0\} \) such that
\[ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx = A \quad \text{and} \quad \int_{\mathbb{R}^2} G(v) \, dx = 0. \]

Since \( \mathcal{M} \) a \( C^1 \) manifold (see Remark 2), by Lagrange multipliers, there is \( \theta > 0 \) (for the positivity of \( \theta \) we refer the reader to [4]) such that
\[ \int_{\mathbb{R}^2} \nabla v \nabla w \, dx = \theta \int_{\mathbb{R}^2} g(v)w \, dx, \quad \text{for every} \ w \in H^1(\mathbb{R}^2). \]
Setting \( v_\sqrt{n}(x) = v\left(\frac{x}{\sqrt{n}}\right) \), we see that \( v_\sqrt{n} \) is also a solution of (4) and satisfies
\[
\int_{\mathbb{R}^2} |\nabla v_\sqrt{n}|^2 dx = \int_{\mathbb{R}^2} |\nabla v|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^2} G(v_\sqrt{n}) dx = 0.
\]
Thus,
\[
m \leq I(v_\sqrt{n}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_\sqrt{n}|^2 dx - \int_{\mathbb{R}^2} G(v_\sqrt{n}) dx = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 dx = A.
\] (39)

For each \( \gamma \in \Gamma \) one has \( \gamma([0,1]) \cap \mathcal{P} \neq \emptyset \) (see [12]). Hence, there exists \( t_0 \in (0,1] \) such that \( \gamma(t_0) \in \mathcal{P} \), and consequently
\[
A \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \gamma(t_0)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \gamma(t_0)|^2 dx - \int_{\mathbb{R}^2} G(\gamma(t_0)) dx.
\]
Thus,
\[
A \leq I(\gamma(t_0)) \leq \max_{t \in [0,1]} I(\gamma(t)) \quad (40)
\]
which implies \( A \leq b \). From (39) and (40), we find
\[
m \leq A \leq b. \quad (41)
\]

On the other hand, let \( w \in H^1(\mathbb{R}^2) \) be a nontrivial solution of (4). Using the construction made in [12], there exists a path \( \gamma_w \in \Gamma \) such that \( w \in \gamma_w([0,1]) \) and \( \max_{t \in [0,1]} I(\gamma(t)) = I(w) \). Consequently, \( b \leq I(w) \). Therefore,
\[
b \leq m. \quad (42)
\]

Combining (41) with (42), we obtain \( m = A = b \). Thus \( I(v_\sqrt{n}) = m \) and it completes the proof of Theorem 1.2. The same reasoning applied to the case \( N \geq 3 \) shows that \( u = f(v) \) is a ground state solution of (3), which proves Corollary 2. \( \square \)

5. **Appendix.** This section is devoted to prove Proposition 1. We first show the case \( N \geq 3 \). Let \( u \) be a weak solution for the problem (3). We begin observing that the test functions can be any function \( \phi \) in \( \mathcal{S} \) such that
\[
\int_{\mathbb{R}^N} u^2 |\nabla \phi|^2 dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} \phi^2 |\nabla u|^2 dx.
\] (43)
In fact, we should verify that \( J'(u) \phi = 0 \) for every functions satisfying (43). It is enough to show that there exists \( \varphi_n \in C_0^\infty(\mathbb{R}^N) \) such that
\[
\varphi_n \to \phi \quad \text{in} \quad H^1(\mathbb{R}^N),
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} u^2 \nabla u \nabla \varphi_n dx = \int_{\mathbb{R}^N} u^2 \nabla u \nabla \phi dx, \quad (44)
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u|^2 u \varphi_n dx = \int_{\mathbb{R}^N} |\nabla u|^2 u \phi dx.
\]
In the case where \( \phi \in L^\infty(\mathbb{R}^N) \) has compact support, we can obtain (44) by \( \varphi_n = \rho_n \ast \phi \), where \( \rho_n \) is a mollifiers sequence. In the general case, we can verify (44) by two different approximations. Firstly, we approximate \( \phi \) (in the (44) sense) by the \( L^\infty(\mathbb{R}^N) \) functions: \( \phi_n = \phi_n \), in \( |\phi| < n \); \( \phi_n = n \phi |\phi| \), in \( |\phi| \geq n \). After this, we can approximate each \( \phi_n \) by the \( L^\infty(\mathbb{R}^N) \) functions, with compact support: \( \psi_m = \sigma_m \phi_n \), where \( \sigma_m \in C_0^\infty(\mathbb{R}^N) \), satisfying \( \sigma_m \equiv 1 \) in \( |x| \leq m \) and \( \sigma_m = 0 \) in
Moreover, \( \exists C > 0 \leq \frac{1}{2} \) for all \( 2 \leq p < 2(2^*) \). In fact, let \( u \in S \) and define \( z = u^2 \). Since \(|\nabla z| = 2|u|\nabla u| \in L^2(\mathbb{R}^N)\), \( z \in D^{1,2}(\mathbb{R}^N) \), and so
\[
\left( \int_{\mathbb{R}^N} |z|^{2^*/2} \, dx \right)^2 \leq S \int_{\mathbb{R}^N} |\nabla z|^2 \, dx.
\]
Then, \( u \in L^{2(2^*)}(\mathbb{R}^N) \). Using that \( u \in L^2(\mathbb{R}^N) \cap L^{2(2^*)}(\mathbb{R}^N) \), we have \( u \in L^p(\mathbb{R}^N) \) for all \( 2 \leq p \leq 2(2^*) \). In particular, \( u \in L^4(\mathbb{R}^N) \). Now, from (\( f_1 \)) and (\( f_2 \)), there exists \( C > 0 \) such that
\[
t^2 \leq C(f(t)^2 + f(t)^4),
\]
for every \( t \in \mathbb{R} \). Thus,
\[
\int_{\mathbb{R}^N} |v|^2 \leq C \int_{\mathbb{R}^N} (|u|^2 + |u|^4) \, dx. \tag{45}
\]
Hence, \( v \in L^2(\mathbb{R}^N) \). Moreover, \( v \) can be approximated by the \( H^1(\mathbb{R}^N) \) functions: \( v_n = v, \) in \( |u| < n; \) \( v_n = nn/v |v|, \) in \( |v| \geq n \). This implies that \( v \in H^1(\mathbb{R}^N) \). Fix \( w \in C^\infty_0(\mathbb{R}^N) \). We can verify that
\[
\phi = \frac{w}{\sqrt{1 + 2u^2}} \in H^1(\mathbb{R}^N) \quad \text{and} \quad \nabla \phi = \frac{\nabla w}{\sqrt{1 + 2u^2}} - \frac{2uw}{(1 + 2u^2)^{3/2}} \nabla u.
\]
Moreover, \( \phi \in S \) satisfies (43). It is readily seen that \( I'(v)w = J'(u)\phi \). Since \( C^\infty_0(\mathbb{R}^N) \) is dense in \( H^1(\mathbb{R}^N) \), we have proved that \( v \) is a critical point for the functional \( I \). We now consider the case \( N = 2 \). Let \( u \in S \) be a weak solution of (3). As \( H^1(\mathbb{R}^2) \subset L^r(\mathbb{R}^2) \) for every \( r \in [2, +\infty) \), we have \( u \in L^4(\mathbb{R}^2) \) and we can now proceed analogously to the proof of case \( N \geq 3 \) from (45) to conclude that \( v = f^{-1}(u) \) is a critical point of \( I \). To prove the sufficient part of the proposition and the identity \( J(u) = I(v) \), we refer the reader to [9].

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