Entropy of extremal black holes in asymptotically anti-de Sitter spacetime

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Abstract

Unlike the extremal Reissner - Nordström black hole in ordinary spacetime, the one in anti-de Sitter spacetime is a minimum of action and has zero entropy if quantization is carried out after extremalization. However, if extremalization is carried out after quantization, then the entropy is a quarter of the area as in the usual case.

While the entropy of ordinary (non-extremal) black holes has been known to be given by a quarter of the horizon area for a long time, there has been some uncertainty in the case of extremal black holes. The usual derivations do not go through in a straightforward manner, and because of the difference in topology of euclidean extremal black holes and euclidean non-extremal black holes, one cannot fall back on extrapolation. In fact, it has been suggested that extremal black holes should have zero entropy even though the horizon area is nonzero.

On the other hand, some microscopic models have indicated that extremal black holes could satisfy the area law just like non-extremal black

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holes. One way out of this mismatch would be to say that the microscopic model is wrong, but it is also possible to argue that the arguments of [1] are somewhat naïve. Usually, when one quantizes a classical theory, one tries to preserve the classical topology. In this spirit, [1] seeks to have a quantum theory of extremal black holes based exclusively on extremal topologies. As an alternative, one can try out a quantization where a sum over topologies is carried out. Then, in the consideration of the functional integral, classical configurations corresponding to both topologies must be included. The extremality condition can subsequently be imposed on the averages that result from the functional integration. It is convenient, following [2] and [3], to use a grand canonical ensemble. Here the temperature and the chemical potential for the charges are supposed to be specified as inputs, and the average mass $M$ and charges $Q$ of the black hole are outputs. So the actual definition of extremality that is involved here for a Reissner-Nordström black hole with one kind of charge is $Q = M$. This may be described as extremalization after quantization, as opposed to the usual approach of quantization after extremalization [4]. It was shown in [4] that extremalization after quantization does lead to an entropy equal to a quarter of the area.

Does the approach of quantization after extremalization in the case of the Reissner-Nordström black hole lead to zero entropy as suggested in [1]? Even that is not quite true [5]: the reason is that the semiclassical approximation fails because the action does not have a stable minimum there.

In view of recent interest in anti-de Sitter geometries, an investigation has been made to determine whether anything more interesting happens if an asymptotically anti-de Sitter version of the extremal Reissner-Nordström black hole is considered. It will be shown that a stable minimum does occur in this case. Consequently, there is a sensible semiclassical approximation, and as expected in [4], the entropy vanishes if quantization is carried out after extremalization. However, if quantization is carried out first, the entropy is once again given by a quarter of the area.

The Reissner-Nordström black hole solution of Einstein’s equations in free space with a negative cosmological constant $\Lambda = -\frac{3}{r^2}$ is given by (see e.g. [6])

$$ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\Omega^2, \quad A = \frac{Q}{r} dt,$$

(1)
with
\[ h = 1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \]  

(2)

The asymptotic form of this spacetime is anti-de Sitter. There is an outer horizon located at \( r = r_+ \). The mass of the black hole is given by
\[ M = \frac{1}{2} \left( r_+ + \frac{r_+^3}{l^2} + \frac{Q^2}{r_+} \right). \]  

(3)

It satisfies the laws of black hole thermodynamics with a temperature
\[ T_H = \frac{1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}}{4\pi r_+}. \]  

(4)

and a potential
\[ \phi = \frac{Q}{r_+}. \]  

(5)

In general \( r_+, Q \) are independent, but in the extremal case they get related:
\[ 1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2} = 0. \]  

(6)

The action for the euclidean version of the anti-de Sitter Reissner-Nordström black hole on a four dimensional manifold \( \mathcal{M} \) with a boundary is given by
\[ I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g}(R - 2\Lambda) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3x \sqrt{\gamma}(K - K_0) + \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu}. \]  

(7)

Here \( \gamma \) is the induced metric on the boundary \( \partial\mathcal{M} \) and \( K \) the extrinsic curvature of the boundary. \( K_0 \) is to be chosen to make the action finite.

The on-shell action for the black hole with the boundary taken at \( r = r_B \) and euclidean time integrated over from 0 to \( \beta \) is
\[ \frac{\beta}{2l^2} (r_B^3 - r_+^3) - \frac{\beta}{2} \left( \sqrt{1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r_+ r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}} \right) \]  

3
\[
\times \frac{d}{dr} \left[ r^2 \sqrt{1 - \frac{r_+}{r} - \frac{r_+^3}{l^2 r} - \frac{Q^2}{r r^2 + \frac{r^2}{l^2}}} \right]_{r=r_B} + \\
\frac{\beta}{2} \frac{r_B^2}{r_B} \left[ 1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{Q^2}{r_B r_B} + \frac{Q^2}{r_B^2} + \frac{r_B^2}{l^2} (-K_0) \right] \\
- \frac{\beta}{2} Q^2 (r_+^{-1} - r_B^{-1}).
\]

To keep this finite in the limit \( r_B \to \infty \), it is necessary to take

\[
K_0 = -\frac{2}{l} - \frac{l}{r_B^2}.
\]}

With this choice, the \( r_B \to \infty \) limit of the action is

\[
\frac{\beta}{2} (M - Q\phi - \frac{r_+^3}{l^2}).
\]

The corresponding entropy is calculated by equating \( \beta \) times the free energy with the action in the leading semiclassical approximation:

\[
S = \beta (M - Q\phi) - I = \frac{\beta}{2} (M - Q\phi + \frac{r_+^3}{l^2}) = \frac{\beta}{4} (1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2})
\]

If \( \beta \) is taken to be the reciprocal of (4), this expression simplifies to a quarter of the area:

\[
S_{\text{non-ex}} = \pi r_+^2.
\]

In the extremal case, where there is no conical singularity in the euclidean metric, the (vanishing) expression for the temperature is not used for \( \beta \), which is allowed to be finite. Then one gets the entropy to be

\[
S_{\text{ex}} = \frac{\beta}{4} (1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2}) = 0
\]

because of (6). All this is very similar to what happens in the ordinary Reissner-Nordström case, to which everything reduces in the limit \( l \to \infty \).

Now we turn to a study of the action for off-shell configurations near the black hole solution. For simplicity, only a class of spherically symmetric metrics [3] is considered on \( \mathcal{M} \):

\[
ds^2 = b^2 d\tau^2 + \alpha^2 dr^2 + r^2 d\Omega^2,
\]

\( 3 \)
with the variable \( r \) ranging between \( r_+ \) (the horizon) and \( r_B \) (the boundary), and \( b, \alpha \) functions of \( r \) only. There are boundary conditions as usual [3, 4, 7]:

\[
b(r_+) = 0, \quad 2\pi b(r_B) = \beta.
\]

(15)

This corresponds to the convention of fixing the range of integration of the euclidean time \( \tau \) to be \( 2\pi \). \( \beta \) is the inverse temperature at the boundary of radius \( r_B \). There is another boundary condition involving \( b'(r_+) \): It reflects the extremal/non-extremal nature of the black hole and is therefore different for the two conditions:

\[
\frac{b'(r_+)}{\alpha(r_+)} = \begin{cases} 
1 & \text{in non-extremal case,} \\
0 & \text{in extremal case.} 
\end{cases}
\]

(16)

The vector potential is taken to be zero and the scalar potential satisfies the boundary conditions

\[
A_\tau(r_+) = 0, \quad A_\tau(r_B) = \frac{\beta \phi}{2\pi i}.
\]

(17)

The action (11) with this form of the metric depends on the functions \( b(r), \alpha(r) \) and \( A_\tau(r) \):

\[
I = \frac{1}{2} \int_0^{2\pi} d\tau \int_{r_+}^{r_B} dr \left( -\frac{2rb'}{\alpha} - \frac{b}{\alpha} - \alpha b + \Lambda \alpha br^2 \right) - \frac{1}{2} \int_0^{2\pi} d\tau \left[ \frac{(br^2)'}{\alpha} \right]_{r=r_+} + I_0 + \frac{1}{2} \int_0^{2\pi} d\tau \int_{r_+}^{r_B} dr \frac{r^2}{\alpha b} A_\tau^2.
\]

(18)

\( I_0 \) is the contribution of the \( K_0 \) term in the action. Variation of the functions \( b(r), \alpha(r) \) and \( A_\tau(r) \) with proper boundary conditions leads to reduced versions of the Einstein - Maxwell equations. The solution of a subset of these equations, namely the Gauss law and the Hamiltonian constraint, is given by [3, 4]

\[
\frac{1}{\alpha} = \left( 1 - \frac{r_+}{r} - \frac{r_+^3}{l^2r} - \frac{q^2}{r_+r} + \frac{q^2}{r^2} + \frac{r^2}{l^2} \right)^{1/2}, \quad A_\tau' = -\frac{iqb\alpha}{r^2}.
\]

(19)

with \( r_+ \) and \( q \) arbitrary at this stage. The reason why these parameters are not expressed as functions of \( \beta, \phi \) is that some of the equations of motion
and the corresponding boundary conditions have not yet been imposed on the solution. Instead of that, the action may be expressed in terms of \( r_+, q \) and then extremized with respect to \( r_+, q \) as in [3].

The value of the action corresponding to the solution depends on the boundary condition:

\[
I = -\beta \left( r_B \sqrt{1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{q^2}{r_+ r_B} + \frac{q^2}{r_B^2} + \frac{r_B^2}{l^2}} + q \phi \right) + I_0 - \pi r_+^2
\]

for non-extremal bc,

\[
I = -\beta \left( r_B \sqrt{1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{q^2}{r_+ r_B} + \frac{q^2}{r_B^2} + \frac{r_B^2}{l^2}} + q \phi \right) + I_0
\]

for extremal bc. (20)

The first line is analogous to [3, 7], where the non-extremal boundary condition was used in connection with a semiclassically quantized non-extremal black hole. The second line is similar to the consequence of the extremal boundary condition used in connection with a semiclassically quantized extremal black hole [5].

The above “reduced action” has to be extremized with respect to \( q, r_+ \) in order to impose the equations of motion ignored so far. The form of \( I_0 \) is not important for this as it does not involve \( q, r_+ \) when \( r_B \) is large. The extremization with respect to \( q \) yields the relation

\[
\frac{q}{r_+} - \frac{q}{r_B} = \phi,
\]

while extremization with respect to \( r_+ \) yields

\[
\beta (1 - \frac{r_+^2}{l^2} + \frac{3 r_+^2}{l^2})
\]

\[
\sqrt{1 - \frac{r_+}{r_B} - \frac{r_+^3}{l^2 r_B} - \frac{q^2}{r_+ r_B} + \frac{q^2}{r_B^2} + \frac{r_B^2}{l^2}} = 4\pi r_+ \text{ for non-extremal bc,}
\]

\[
\text{but } 0 \text{ for extremal bc. (22)}
\]

For non-extremal boundary conditions, these two relations can be used to fix \( q, r_+ \) in terms of the specified values of \( \beta, \phi \); they also show the expected forms of \( \beta, \phi \) as functions of \( q, r_+ \). The nature of the extremum has been discussed
in \[\pi r_+^2\]. The entropy can be calculated by standard thermodynamical methods and is found to be the expected \(\pi r_+^2\).

Much the same thing can be done for the extremal boundary condition, where, however, the second equation is homogeneous and \(\beta\) disappears from the relations. This is not surprising: \(q, r_+\) are not independent in this case, but are related to each other by \(\beta\), and the temperature is undetermined as there is no conical singularity. The first relation can be written as

\[
\frac{1 + \frac{3r_+^2}{l^2}}{1 + \frac{3r_+^2 + 2r_+r_B + r_B^2}{l^2}} = \phi^2.
\]

(23)

This is reminiscent of the fact that \(|\phi|\) has to be unity in the usual extremal case. In the anti-de Sitter situation the restriction on \(|\phi|\) is only that it has to be less than unity: \(r_+\) can then be sought to be determined in terms of \(\phi\) by solving the quadratic:

\[
r_+ = \frac{\phi^2 r_B \pm \sqrt{\phi^4 r_B^2 - 3(1 - \phi^2)(l^2 - \phi^2 l^2 - \phi^2 r_B^2)}}{3(1 - \phi^2)}.
\]

(24)

There are values of \(\phi\) for which this equation has only complex solutions, and even when there are real solutions, one solution may be negative. A positive solution \(r_+\) does not necessarily mean that the extremum of the action is a minimum.

The matrix of second derivatives of the action with respect to \(q, r_+\) is real, symmetric and equal to

\[
\begin{pmatrix}
\frac{\beta \phi}{q} + \frac{\beta \phi^2 \xi}{(1 - \xi)} & -\frac{\beta \phi}{r_+ \xi} \\
-\frac{\beta \phi}{r_+ \xi} & \frac{\beta \phi (\frac{2}{r_+} + \frac{3r_+^2}{l^2})}{q(1 - \xi)}
\end{pmatrix},
\]

where \(\xi \equiv \frac{r_+}{r_B}\). For the action to be a minimum at the extremum, this matrix has to be positive definite, i.e., both of its eigenvalues have to be positive. In view of the reality of the eigenvalues, this is equivalent to the requirement that both the trace and the determinant have to be positive. The trace is seen to be positive if \(r_+ < r_B\). We shall consider this condition to be imposed. The determinant is, up to a positive factor,

\[
\frac{3r_+^2}{l^2} - (1 - \phi^2)\xi(1 + \frac{6r_+^2}{l^2}),
\]

(25)

which again can be made positive by making \(r_B\) large enough, i.e., \(\xi\) small enough. \(r_+\) is to be held fixed if an adjustment of \(r_B\) has to be made, which
means that the value of $\phi$ is such that $r_+$ turns out to be appropriately small in comparison with $r_B$. Large solutions for $r_+$ also exist for appropriate $\phi$: they do not correspond to minima of the action. Note that the above expression vanishes in the limit $l \to \infty$ if the corresponding condition $|\phi| = 1$ is imposed, thus confirming that the extremal Reissner - Nordström black hole in asymptotically flat spacetime is not a minimum of the action.

The entropy corresponding to the saturation of the action by this minimum is zero. This follows from the fact that [1] the action continues to be proportional to $\beta$ after the extremizing values of $q, r_+$ are plugged in. Hence,

$$S = \beta^2 \frac{\partial (I/\beta)}{\partial \beta} = 0. \quad (26)$$

The above statements refer to the quantized extremal black hole. As indicated above, there is a possibility of quantizing the black hole before extremizing it, i.e., the two topologies may be summed over in the functional integral [4] and the extremality condition imposed afterwards on the averaged quantities. The partition function is of the form

$$\sum_{\text{topologies}} \int d\mu(r_+) \int d\mu(q) e^{-I(q, r_+)}, \quad (27)$$

with $I$ given by (20) as appropriate for non-extremal/extremal topology.

The semiclassical approximation involves replacing the double integral by the maximum value of the integrand, i.e., by the exponential of the negative of the minimum $I$. We consider the variation of $I$ as $q, r_+$ vary in both topologies. It is clear from (20) that the non-extremal action can be made lower than the extremal one because of the extra term $-\pi r_+^2$. Consequently, the partition function is to be approximated by $e^{-I_{\text{min}}}$, where $I_{\text{min}}$ is the classical action for the non-extremal case, minimized with respect to $q, r_+$. As in the non-extremal case, this leads to an entropy equal to a quarter of the horizon area. Extremality is imposed eventually through the condition (6) on $q, r_+$.

Thus the entropy depends very significantly on whether quantization is carried out first or extremalization [4]: in the former case, the answer is a quarter of the area, and in the latter, zero.

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