LOCALLY COHEN-MACAULAY SPACE CURVES DEFINED BY CUBIC EQUATIONS AND GLOBALLY GENERATED VECTOR BUNDLES

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Abstract. We classify globally generated vector bundles with first Chern class $c_1$ at least 4 on the projective 3-space with the property that $E(-c_1 + 3)$ has a non-zero global section. This (seemingly) technical result allows one to reduce the classification of globally generated vector bundles with $c_1$ at most 7 on the projective 3-space to the classification of stable rank 2 reflexive sheaves with the same properties. The proof is based on a description of the monads of all locally Cohen-Macaulay space curves defined by cubic equations. We extend, then, this kind of classification to higher dimensional projective spaces. We use this extension to recuperate quickly the classification of globally generated vector bundles with $c_1 = 4$ on the projective $n$-space for $n$ at least 4, which is part of the main result of our previous paper [arXiv:1305.3464]. We provide, in the appendices to the paper, graded free resolutions for the homogeneous ideals and for the graded structural algebras of all nonreduced locally Cohen-Macaulay space curves of degree at most 4.

Introduction

Motivated, in part, by some geometric applications (cf., for example, the papers of Huh [19], Manivel and Mezzetti [21], Fania and Mezzetti [14]) the systematic study of globally generated vector bundles on projective spaces was initiated by Sierra and Ugaglia [32] who classified the bundles of this kind with the first Chern class $c_1 \leq 2$. Then Ellia [13] determined the Chern classes of globally generated vector bundles on $\mathbb{P}^2$, Chiodera and Ellia [9] determined the Chern classes of the globally generated vector bundles of rank 2 on $\mathbb{P}^3$ with $c_1 \leq 5$, while Anghel and Manolache [2] and, independently, Sierra and Ugaglia [33], classified the globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 3$.

The present paper is a sequel to our work [3], where we classified the globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 4$. We are here (mainly) concerned with the classification of globally generated vector bundles on $\mathbb{P}^3$. (From the point of view of the methods used in [3], this is a critical case: on one hand, the study of globally generated vector bundles on $\mathbb{P}^2$ presents some special features and may be treated separately. On the other hand, once one has a classification of globally generated vector bundles on $\mathbb{P}^3$ one can try to decide which of these bundles extend, as a globally generated vector bundle, to higher dimensional projective spaces. The latter idea has limited efficiency, but we are unaware of a direct method for studying globally generated bundles on $\mathbb{P}^n$, $n \geq 4$.) One natural
idea to approach the study this kind of bundles is to relate them to rank 2 reflexive sheaves, which were intensively studied. If $E$ is a globally generated vector bundle on $\mathbb{P}^3$, of rank $r$ and Chern classes $c_i = c_i(E)$, $i = 1, 2, 3$, then $r - 2$ general global sections of $E$ define an exact sequence:

$$0 \rightarrow (r - 2)\mathcal{O}_E \rightarrow E \rightarrow \mathcal{E}' \rightarrow 0$$

where $\mathcal{E}'$ is a globally generated rank 2 reflexive sheaf with $c_i(\mathcal{E}') = c_i$, $i = 1, 2, 3$. The classification problem splits, now, into two cases: (1) $\mathcal{E}'$ stable; (2) $\mathcal{E}'$ non-stable. We are concerned, in this paper, with the latter case. Let $\mathcal{E}'_{\text{norm}}$ denote the reflexive sheaf $\mathcal{E}' \left( - \left\lfloor \frac{c_1 + 1}{2} \right\rfloor \right)$, which has the first Chern class either 0 or $-1$. Then $\mathcal{E}'$ is non-stable if and only if $H^0(\mathcal{E}'_{\text{norm}}) \not= 0$ and this happens, of course, if and only if $H^0(E \left( - \left\lfloor \frac{c_1 + 1}{2} \right\rfloor \right)) \not= 0$.

Now, it is easy to show that if $H^0(E(-c_1)) \not= 0$ then $E \cong \mathcal{O}_E(c_1) \oplus (r - 1)\mathcal{O}_E$ (this is a particular case of a result of Sierra [31]). Sierra and Ugaglia [33, Prop. 2.2] (for $c_1 = 3$) and Anghel et al. [3 Prop. 2.4] (for $c_1 \geq 2$) described the globally generated vector bundles $E$ with $H^0(E(-c_1)) = 0$ and $H^0(E(-c_1 + 1)) \not= 0$. Moreover, we described in [3 Prop. 2.9] the globally generated vector bundles with $c_1 \geq 3$, $H^0(E(-c_1 + 1)) = 0$ and $H^0(E(-c_1 + 2)) \not= 0$. These results suffice, already, to classify globally generated vector bundles on $\mathbb{P}^3$ with $c_1 \leq 3$ (see [3 Remark 2.12]).

The main objective of the present paper is to classify globally generated vector bundles $E$ on $\mathbb{P}^3$ with $c_1 \geq 4$, $H^0(E(-c_1 + 2)) = 0$ and $H^0(E(-c_1 + 3)) \not= 0$ (the similar problem on $\mathbb{P}^2$ has been settled in [3 Prop. 3.2]). The associated rank 2 reflexive sheaf $\mathcal{E}'$ introduced above satisfies, also, the conditions $H^0(\mathcal{E}'(-c_1 + 2)) = 0$, $H^0(\mathcal{E}'(-c_1 + 3)) \not= 0$, hence any non-zero global section of $\mathcal{E}'(-c_1 + 3)$ defines an exact sequence:

$$0 \rightarrow \mathcal{O}_E(c_1 - 3) \rightarrow \mathcal{E}' \rightarrow \mathcal{I}_Z(3) \rightarrow 0$$

with $Z$ a locally Cohen-Macaulay (CM, for short) closed subscheme of $\mathbb{P}^3$, of pure codimension 2, locally complete intersection (l.c.i., for short) except at finitely many points, with $\mathcal{I}_Z(3)$ globally generated. One deduces an exact sequence:

$$(0.1) \quad 0 \rightarrow \mathcal{O}_E(c_1 - 3) \oplus (r - 2)\mathcal{O}_E \rightarrow E \rightarrow \mathcal{I}_Z(3) \rightarrow 0.$$  

Now, assume we know a Horrocks monad $B^\bullet$ of $\mathcal{I}_Z(3)$, i.e., a three terms complex of sheaves:

$$(0.2) \quad 0 \rightarrow B^{-1} \xrightarrow{d^{-1}} B^0 \xrightarrow{d^0} B^1 \rightarrow 0$$

with the $B^i$'s direct sums of invertible sheaves $\mathcal{O}_E(j)$, $j \in \mathbb{Z}$, such that $\mathcal{H}^0(B^\bullet) \cong \mathcal{I}_Z(3)$ and $\mathcal{H}^i(B^\bullet) = 0$ for $i \not= 0$. Then there exists a morphism $(\phi, \psi) : B^{-1} \rightarrow \mathcal{O}_E(c_1 - 3) \oplus (r - 2)\mathcal{O}_E$ such that:

$$(0.3) \quad 0 \rightarrow B^{-1} \xrightarrow{\begin{pmatrix} \phi \\ d^{-1} \\ \psi \end{pmatrix}} \mathcal{O}_E(c_1 - 3) \oplus B^0 \oplus (r - 2)\mathcal{O}_E \xrightarrow{(0, d^0, 0)} B^1 \rightarrow 0$$

is a monad for $E$. 

Thus, we are left with the problem of determining the monads of the ideal sheaves of locally CM space curves \( Z \) with \( \mathcal{I}_Z(3) \) globally generated. There are two ways to get such a monad. The first one is based on the observation that if:

\[
0 \longrightarrow L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \longrightarrow H^0_*(\mathcal{O}_Z) \longrightarrow 0
\]

is a free resolution of the graded \( S \)-module \( H^0_*(\mathcal{O}_Z) := \bigoplus_{i \in \mathbb{Z}} H^0(\mathcal{O}_Z(i)) \) (here \( S \) is the projective coordinate ring \( k[x_0, x_1, x_2, x_3] \) of \( \mathbb{P}^3 \)) then, removing the direct summand \( S \) of \( L_0 \) corresponding to the generator \( 1 \in H^0(\mathcal{O}_Z) \) of \( H^0_*(\mathcal{O}_Z) \) and sheafifying, one gets a monad for \( \mathcal{I}_Z \):

\[
0 \longrightarrow \widetilde{L}_2 \xrightarrow{\tilde{d}_2} \widetilde{L}_1 \xrightarrow{\tilde{d}_1} \widetilde{L}_0 \longrightarrow 0.
\]

The second way is based on a result, attributed by Peskine and Szpiro [28, Prop. 2.5] to D. Ferrand, asserting that if \( Z \) is linked to a curve \( Z' \) by a complete intersection of type \((a, b)\), if

\[
0 \longrightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \longrightarrow \mathcal{I}_{Z'} \longrightarrow 0
\]

is a resolution of \( \mathcal{I}_{Z'} \) with direct sums of invertible sheaves and if \( \psi : \mathcal{O}_P(-a) \oplus \mathcal{O}_P(-b) \to A_0 \) is a morphism lifting \( \mathcal{O}_Z(-a) \oplus \mathcal{O}_Z(-b) \to \mathcal{I}_{Z'} \); then:

\[
0 \longrightarrow A_0^\vee \xrightarrow{(\psi^\vee, d_1^\vee)} \mathcal{O}_P(a) \oplus \mathcal{O}_P(b) \xrightarrow{A_1^\vee, A_2^\vee} 0
\]

is a monad for \( \mathcal{I}_Z(a + b) \). One can also see that, conversely, if

\[
0 \longrightarrow B^{-1} \xrightarrow{d^{-1}} B^0 \xrightarrow{d^0} B^1 \longrightarrow 0
\]

is a monad for \( \mathcal{I}_Z \) and if \( \rho : \mathcal{O}_P(-a) \oplus \mathcal{O}_P(-b) \to B^0 \) is a morphism lifting \( \mathcal{O}_Z(-a) \oplus \mathcal{O}_Z(-b) \to \mathcal{I}_Z \) such that \( d^0 \circ \rho = 0 \) then there exists an exact sequence:

\[
0 \longrightarrow B^{1^\vee} \xrightarrow{d^{0^\vee}} B^{0^\vee} \xrightarrow{(\rho^{0^\vee}, d^{-1^\vee})} \mathcal{O}_P(a) \oplus \mathcal{O}_P(b) \oplus B^{-1^\vee} \longrightarrow \mathcal{I}_{Z'}(a + b) \longrightarrow 0.
\]

Notice that if one applies the functor \( H^0_*(\cdot) \) to this exact sequence (twisted by \(-a - b\)) one gets a graded free resolution of the homogeneous ideal \( I(Z') \subset S \) of \( Z' \).

Now, let \( d \) be the degree of such a curve \( Z \). Of course, \( d \leq 9 \). The description of the possible monads for \( \mathcal{I}_Z(3) \) when \( d \geq 5 \), which we accomplish in Section 4 using the concrete description of the stable rank 2 reflexive sheaves on \( \mathbb{P}^3 \) with \( c_1 = -1 \) and \( c_2 \leq 2 \) (due to Hartshorne [17], Hartshorne and Sols [18], Manolache [23], and Chang [8]), turns out to be easy. We determine, in Section 2, the monads of \( \mathcal{I}_Z(3) \) for \( d \leq 4 \), using a case by case analysis. We rely heavily, in this analysis, on the results from Appendix A where we describe, using the theory of Bănică and Forster [5], the generators of the homogeneous ideals and the monads of the ideal sheaves of the multiple lines in \( \mathbb{P}^3 \) of degree \( \leq 4 \), and on the results from Appendix B where we do the same thing for the reducible curves of degree \( \leq 4 \) having a multiple line as a component.
Finally, we use, in Section 3, the classification of the monads of the locally CM curves $Z$ with $\mathcal{I}_Z(3)$ globally generated to prove the main result of this paper which is the following:

**Theorem 0.1.** Let $E$ be a globally vector bundle on $\mathbb{P}^3$ with $c_1 \geq 4$ and such that $H^i(E^\vee) = 0$, $i = 0, 1$. If $H^0(E(-c_1 + 2)) = 0$ and $H^0(E(-c_1 + 3)) \neq 0$ then one of the following holds:

(i) $E \simeq \mathcal{O}_P(c_1 - 3) \oplus F$, where $F$ is a globally generated vector bundle with $c_1(F) = 3$;

(ii) $c_1 = 4$ and $E \simeq F(2)$, where $F$ is a nullcorrelation bundle or a 2-instanton;

(iii) $c_1 = 4$ and, up to a linear change of coordinates, $E$ is the kernel of the epimorphism

$$(x_0, x_1, x_2, x_3^2) : 3\mathcal{O}_P(2) \oplus \mathcal{O}_P(1) \to \mathcal{O}_P(3);$$

(iv) $c_1 = 4$ and, up to a linear change of coordinates, $E$ is the kernel of the epimorphism

$$(x_0, x_1, x_2^2, x_2x_3, x_3^2) : 2\mathcal{O}_P(2) \oplus 3\mathcal{O}_P(1) \to \mathcal{O}_P(3);$$

(v) $c_1 = 4$ and, up to a linear change of coordinates, $E$ is the cohomology of the monad

$$\mathcal{O}_P(-1) \to 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \oplus 4\mathcal{O}_P \xrightarrow{(p, 0)} \mathcal{O}_P(3)$$

where $\mathcal{O}_P(-1) \to 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \to \mathcal{O}_P(3)$ is a subcomplex of the Koszul complex defined by $x_0, x_1, x_2, x_3$ and $u : \mathcal{O}_P(-1) \to 4\mathcal{O}_P$ is defined by $x_0, x_1, x_2, x_3;

(vi) $c_1 = 5$ and $E \simeq F(3)$, where $F$ is a stable rank 2 vector bundle with $c_1(F) = -1$ and $c_2(F) = 2$.

(vii) $c_1 = 6$ and $E \simeq F(3)$, where $F$ is a properly semistable rank 2 vector bundle with $c_1(F) = 0$ and $c_2(F) = 3$.

Some comments are in order. As originally noticed by Sierra and Ugaglia, the condition $H^i(E^\vee) = 0$, $i = 0, 1$, is not too restrictive: any other globally generated vector bundle can be obtained from a bundle satisfying this additional condition by quotienting a trivial subbundle and, then, by adding a trivial direct summand. As we said at the beginning of the introduction, the bundles $F$ appearing in item (i) were classified in [2] and [33], independently.

Finally, we extend, in Prop. 3.5, the classification from Theorem 0.1 to higher dimensional projective spaces. In order to show that the bundle appearing in item (vii) of Thm. 0.1 cannot be extended to $\mathbb{P}^4$ we need a result of Barth and Elencwajg [7, Thm. 4.2] asserting that there is no stable rank 2 vector bundle on $\mathbb{P}^4$ with Chern classes $c_1 = 0$, $c_2 = 3$. We provide, in Thm. 3.3, a different proof of this fact, based on the results of Mohan Kumar, Peterson and Rao [20].

We use Prop. 3.5 to recover, in a different manner, in Thm. 3.8, the classification of globally generated vector bundles with $c_1 = 4$ on $\mathbb{P}^n$, $n \geq 4$, which is part of the main result of our previous paper [3]. It is very likely that Prop. 3.5 can be used to get a classification of globally generated vector bundles with $c_1 = 5$ on $\mathbb{P}^n$, $n \geq 5$. We intend to come back to this subject in another paper.

Finally, we use Prop. 4.1 to recover, in a different manner, in Thm. 4.8, the classification of globally generated vector bundles with $c_1 = 4$ on $\mathbb{P}^n$, $n \geq 4$, which is part of the main result of our previous paper [3]. It is very likely that Prop. 4.1 can be used to get a classification of globally generated vector bundles with $c_1 = 5$ on $\mathbb{P}^n$, $n \geq 5$. We intend to come back to this subject in another paper.
Note. We provide, in the appendices to the paper, graded free resolutions for the homogeneous ideals and for the graded structural algebras of all nonreduced locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree at most 4. We need only a small part of these results for the proof of Thm. 0.1 but we decided to give (at least in e-print form) the complete (and very long) list because it might be useful in some other contexts, too. Most of the results are at exercise level but our approach is almost algorithmic and, at least, some aspects of the description of quadruple structures on a line should be new. There are overlaps with the papers of Nollet [26] and Nollet and Schlessinger [27] but our emphasis is on the description of the individual curves and not on their deformations.

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Notation. (i) We denote by $S = k[x_0, \ldots, x_n]$ the projective coordinate ring of the projective $n$-space $\mathbb{P}^n$ over an algebraically closed field $k$ of characteristic 0. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, we denote by $H^i_*(\mathcal{F})$ the graded $S$-module $\bigoplus_{l \in \mathbb{Z}} H^i(l)$. (ii) If $X$ is a closed subscheme of $\mathbb{P}^n$, we denote by $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ its ideal sheaf. If $Y$ is a closed subscheme of $X$, we denote by $\mathcal{I}_{Y,X} \subset \mathcal{O}_X$ the ideal sheaf defining $Y$ as a closed subscheme of $X$. In other words, $\mathcal{I}_{Y,X} = \mathcal{I}_Y / \mathcal{I}_X$. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, we put $\mathcal{F}_X := \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_X$ and $\mathcal{F}|X := i^* \mathcal{F}$, where $i : X \to \mathbb{P}^n$ is the inclusion. (iii) By a point of a quasi-projective scheme $X$ we always mean a closed point. If $\mathcal{F}$ is a coherent sheaf on $X$ and $x \in X$, we denote by $\mathcal{F}(x)$ the reduced stalk $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$ of $\mathcal{F}$ at $x$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{x,x}$. (iv) We frequently write “CM” for “Cohen-Macaulay” and “l.c.i.” for “locally complete intersection”.

1. Curves of degree at least 5

Proposition 1.1. Let $Z$ be a locally CM space curve of degree $d \geq 4$. If $\mathcal{I}_Z(3)$ is globally generated and $H^0(\mathcal{I}_Z(2)) \neq 0$ then one of the following holds:

(i) $Z$ is a complete intersection of type $(2,3)$;
(ii) $d = 5$ and $\mathcal{I}_Z(3)$ admits a resolution of the form:
$$0 \to 2\mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}(1) \oplus 2\mathcal{O}_{\mathbb{P}} \to \mathcal{I}_Z(3) \to 0;$$
(iii) $d = 4$ and $\mathcal{I}_Z(3)$ admits a resolution of the form:
$$0 \to \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1) \to 2\mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}} \to \mathcal{I}_Z(3) \to 0;$$
(iv) $d = 4$ and $\mathcal{I}_Z(3)$ admits a resolution of the form:
$$0 \to 3\mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \oplus \Omega_{\mathbb{P}}(2) \to \mathcal{I}_Z(3) \to 0.$$

Proof. $Z$ is contained in a complete intersection of type $(2,3)$ hence $d \leq 6$.

Case 1. $d = 6$.

In this case $Z$ is a complete intersection of type $(2,3)$.
Case 2. \( d = 5 \).

In this case, \( Z \) is linked to a line by a complete intersection of type \((2, 3)\) hence, by Ferrand's result about liaison, \( J_Z(3) \) admits a resolution as in item (ii) of the statement.

Case 3. \( d = 4 \).

In this case, \( Z \) is linked to a curve \( X \) of degree 2. One has to divide this case into subcases.

Subcase 3.1. \( X \) is a complete intersection of type \((1, 2)\).

In this subcase, by Ferrand's result about liaison, \( I_Z(3) \) admits a resolution as in item (iii) of the statement.

Subcase 3.2. \( X \) is the disjoint union of two lines.

In this subcase, \( I_X \) admits a resolution of the form:

\[
0 \rightarrow 3\mathcal{O}_P \rightarrow 5\mathcal{O}_P(1) \rightarrow \mathcal{O}_P(2) \rightarrow 0
\]

Since the kernel of any epimorphism \( 5\mathcal{O}_P \rightarrow \mathcal{O}_P(1) \) is isomorphic to \( \mathcal{O}_P \oplus \mathcal{O}_P(1) \), it follows that \( I_Z(3) \) admits a resolution as in item (iv) of the statement.

Subcase 3.3 \( X \) is a double structure on a line \( L \).

\( X \) is defined by an epimorphism \( I_L/I_L^2 \rightarrow \mathcal{O}_L(l) \), for some \( l \geq -1 \). From the fundamental exact sequence of liaison:

\[
0 \rightarrow \mathcal{O}_P(-5) \rightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-3) \rightarrow \mathcal{I}_Z \rightarrow \omega_X(-1) \rightarrow 0
\]

it follows that \( \mathcal{I}_Z(3) \) is globally generated iff \( \omega_X(2) \) is globally generated. But, as we recalled in Subsection A.3 of Appendix A, \( \omega_X \simeq \mathcal{O}_X(-l - 2) \). One deduces that \( l \leq 0 \). If \( l = -1 \) then \( X \) is the divisor \( 2L \) on a plane \( H \supset L \) and this situation has been treated in Subcase 3.1, and if \( l = 0 \) then, as we recalled in Subsection A.3, \( \mathcal{I}_X \) admits a resolution having the (numerical) shape as that of the disjoint union of two lines hence, by the proof of Subcase 3.2, \( \mathcal{I}_Z(3) \) admits a resolution as in item (iv) of the statement. \( \square \)

We assume, from now and until the end of this section, that \( Z \) is a locally CM space curve of degree \( d \geq 5 \), such that \( \mathcal{I}_Z(3) \) is globally generated and \( H^0(\mathcal{I}_Z(2)) = 0 \). \( Z \) is linked by a complete intersection of type \((3, 3)\) to a locally CM curve \( Z' \) of degree \( d' = 9 - d \). The fundamental exact sequence of liaison:

\[
0 \rightarrow \mathcal{O}_P(-6) \rightarrow 2\mathcal{O}_P(-3) \rightarrow \mathcal{I}_Z \rightarrow \omega_{Z'}(-2) \rightarrow 0
\]

shows that the condition \( \mathcal{I}_Z(3) \) globally generated is equivalent to \( \omega_{Z'}(1) \) being globally generated, and the condition \( H^0(\mathcal{I}_Z(2)) = 0 \) is equivalent to \( H^0(\omega_{Z'}) = 0 \), which by Serre duality is equivalent to \( H^1(\mathcal{O}_{Z'}) = 0 \) hence to \( H^2(\mathcal{I}_{Z'}) = 0 \).

**Proposition 1.2.** Using the above hypotheses and notation, assume that \( Z' \) is l.c.i. except at finitely many points. Then one of the following holds:
(i) \( d = 7 \) and \( \mathcal{I}_Z(3) \) admits a resolution of the form:
\[
0 \to \mathcal{O}_F(-1) \oplus \mathcal{O}_P(-2) \to 3\mathcal{O}_P \to \mathcal{I}_Z(3) \to 0;
\]

(ii) \( d = 6 \) and \( \mathcal{I}_Z(3) \) admits a resolution of the form:
\[
0 \to 3\mathcal{O}_P(-1) \to 4\mathcal{O}_P \to \mathcal{I}_Z(3) \to 0;
\]

(iii) \( d = 5 \) and \( \mathcal{I}_Z(3) \) admits a resolution of the form:
\[
0 \to 2\mathcal{O}_P \oplus \mathcal{O}_P(-1) \to \Omega_F(2) \oplus \mathcal{O}_P \to \mathcal{I}_Z(3) \to 0;
\]

(iv) \( d = 5 \) and \( \mathcal{I}_Z(3) \) admits a resolution of the form:
\[
0 \to \mathcal{O}_P(1) \oplus \mathcal{O}_P \oplus \mathcal{O}_P(-1) \to K \oplus \mathcal{O}_P \to \mathcal{I}_Z(3) \to 0
\]
where \( K \) is the kernel of an epimorphism \( 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \to \mathcal{O}_P(3) \).

Proof. Assume, firstly, that \( H^0(\mathcal{I}_Z(1)) \neq 0 \). Then \( Z' \) is a complete intersection of type \((1, d')\). In this case \( \omega_{Z'} \cong \mathcal{O}_{Z'}(d' - 3) \) hence the two conditions \( \omega_{Z'}(1) \) globally generated and \( H^0(\omega_{Z'}) = 0 \) imply that \( d' = 2 \). In this case, by Ferrand’s result about liaison, \( \mathcal{I}_Z(3) \) admits a resolution as in item (i) of the statement.

Assume, from now on, that \( H^0(\mathcal{I}_Z(1)) = 0 \). Since \( Z' \) is l.c.i. except at finitely many points and since \( \omega_{Z'}(1) \) is globally generated, a general global section of this sheaf generates it except at finitely many points, hence it defines an extension:
\[
0 \to \mathcal{O}_P \to \mathcal{F}'(2) \to \mathcal{I}_{Z'}(3) \to 0
\]
where \( \mathcal{F}' \) is a rank 2 reflexive sheaf with Chern classes \( c_1(\mathcal{F}') = -1 \) and \( c_2(\mathcal{F}') = d' - 2 \). Our assumption \( H^0(\mathcal{I}_{Z'}(1)) = 0 \) implies that \( H^0(\mathcal{F}') = 0 \), hence \( \mathcal{F}' \) is stable. Now, Hartshorne’s results [17, Cor. 3.3] and [17, Thm. 8.2(d)] imply that \( c_2(\mathcal{F}') \geq 1 \) hence \( c_2(\mathcal{F}') \in \{1, 2\} \) because \( d' \leq 4 \) and that, moreover, if \( c_2(\mathcal{F}') = 1 \) then \( c_3(\mathcal{F}') = 1 \) and if \( c_2(\mathcal{F}') = 2 \) then \( c_3(\mathcal{F}') \in \{0, 2, 4\} \).

Case 1. \( c_2(\mathcal{F}') = 1 \).

In this case, by Hartshorne [17] Lemma 9.4], \( \mathcal{F}' \) can be realised as an extension:
\[
0 \to \mathcal{O}_P(-1) \to \mathcal{F}' \to \mathcal{I}_L \to 0
\]
where \( L \) is a line. One deduces that \( \mathcal{F}' \) and, then, \( \mathcal{I}_{Z'} \) have resolutions of the form:
\[
0 \to \mathcal{O}_P(-2) \to 3\mathcal{O}_P(-1) \to \mathcal{F}' \to 0
\]
\[
0 \to 2\mathcal{O}_P(-3) \to 3\mathcal{O}_P(-2) \to \mathcal{I}_{Z'} \to 0.
\]

Using Ferrand’s result about liaison, one gets that \( \mathcal{I}_Z(3) \) admits a resolution as in item (ii) of the statement.

Case 2. \( c_2(\mathcal{F}') = 2, c_3(\mathcal{F}') = 4 \).

In this case, by Hartshorne [17] Lemma 9.6], \( \mathcal{F}' \) can be realised as an extension:
\[
0 \to \mathcal{O}_P(-1) \to \mathcal{F}' \to \mathcal{I}_Y \to 0
\]
where $Y$ is a plane curve of degree 2. But in this case $H^2(I_{\mathcal{Y}}(-1)) \neq 0$, which implies $H^2(\mathcal{F}'(-1)) \neq 0$, hence $H^2(I_{\mathcal{Z}'}) \neq 0$ and this contradicts our hypothesis that $H^0(I_\mathcal{Z}(2)) = 0$ (see the discussion preceding the statement of the proposition).

**Case 3.** $c_2(\mathcal{F}') = 2$, $c_3(\mathcal{F}') = 2$.

In this case, a result of Chang [8, Lemma 2.4] implies that $\mathcal{F}'$ can be realized as an extension:

$$0 \rightarrow \mathcal{O}_P(-1) \rightarrow \mathcal{F}' \rightarrow \mathcal{I}_Y \rightarrow 0$$

where either $Y$ is the union of two disjoint lines or it is a double structure on a line $L$ defined by an epimorphism $I_L/I_L^2 \rightarrow \mathcal{O}_L$. In both cases, $Y$ admits a resolution of the form (see Subsection [A.5 and Lemma [B.1]):

$$0 \rightarrow \mathcal{O}_P(-4) \xrightarrow{d_2} 4\mathcal{O}_P(-3) \xrightarrow{(d_1)} \mathcal{O}_P(-1) \oplus 4\mathcal{O}_P(-2) \rightarrow \mathcal{F}' \rightarrow 0$$

One gets resolutions for $\mathcal{F}'$ and $\mathcal{I}_{\mathcal{Z}'}$ of the form:

$$0 \rightarrow \mathcal{O}_P(-5) \xrightarrow{(d_2)} \mathcal{O}_P(-3) \oplus 4\mathcal{O}_P(-4) \rightarrow \mathcal{O}_P(-2) \oplus 4\mathcal{O}_P(-3) \rightarrow \mathcal{I}_{\mathcal{Z}'} \rightarrow 0$$

Using Ferrand’s result about liaison, one deduces that $\mathcal{I}_{\mathcal{Z}}(3)$ has a monad of the form:

$$0 \rightarrow 4\mathcal{O}_P \oplus \mathcal{O}_P(-1) \rightarrow 4\mathcal{O}_P(1) \oplus 3\mathcal{O}_P \xrightarrow{(d_2'(-2), 0)} \mathcal{O}_P(2) \rightarrow 0$$

In this way, one gets an exact sequence:

$$0 \rightarrow 4\mathcal{O}_P \oplus \mathcal{O}_P(-1) \xrightarrow{\alpha} \Omega_P(2) \oplus 3\mathcal{O}_P \rightarrow \mathcal{I}_{\mathcal{Z}}(3) \rightarrow 0.$$

The component $4\mathcal{O}_P \rightarrow 3\mathcal{O}_P$ of $\alpha$ must have rank at least 2 because any monomorphism $3\mathcal{O}_P \rightarrow \Omega_P(2)$ degenerates along a surface in $\mathbb{P}^3$. It follows that at least 2 direct summands $\mathcal{O}_P$ from the above resolution of $\mathcal{I}_{\mathcal{Z}}(3)$ must cancel (see Remark [A.8 from Appendix [A]), hence $\mathcal{I}_{\mathcal{Z}}(3)$ has a resolution as in item (iii) of the statement.

**Case 4.** $c_2(\mathcal{F}') = 2$, $c_3(\mathcal{F}') = 0$.

In this case, according to the results of Hartshorne and Sols [18, Prop. 2.1] or of Manolache [23, Cor. 2], $\mathcal{F}'$ can be realized as an extension:

$$0 \rightarrow \mathcal{O}_P(-1) \rightarrow \mathcal{F}' \rightarrow \mathcal{I}_Y \rightarrow 0$$

where $Y$ is a double structure on a line $L$ defined by an epimorphism $I_L/I_L^2 \rightarrow \mathcal{O}_L(1)$. According to the results recalled in Subsection [A.5], $\mathcal{I}_Y$ has a resolution of the form:

$$0 \rightarrow \mathcal{O}_P(-5) \xrightarrow{d_2} 2\mathcal{O}_P(-3) \oplus 2\mathcal{O}_P(-4) \xrightarrow{d_1} 3\mathcal{O}_P(-2) \oplus \mathcal{O}_P(-3) \rightarrow \mathcal{I}_Y \rightarrow 0.$$
One deduces that $\mathcal{F}'$ and, then, $\mathcal{I}_{Z'}$ have resolutions of the form:

$$
0 \rightarrow \mathcal{O}_p(-5) \xrightarrow{d_2} 2 \mathcal{O}_p(-3) \oplus 2 \mathcal{O}_p(-4) \xrightarrow{(\ast)} \mathcal{O}_p(-1) \oplus 3 \mathcal{O}_p(-2) \oplus \mathcal{O}_p(-3) \rightarrow \mathcal{F}' \rightarrow 0
$$

$$
0 \rightarrow \mathcal{O}_p(-6) \xrightarrow{d_2(-1)} \mathcal{O}_p(-3) \oplus 2 \mathcal{O}_p(-4) \oplus 2 \mathcal{O}_p(-5) \rightarrow \mathcal{O}_p(-2) \oplus 3 \mathcal{O}_p(-3) \oplus \mathcal{O}_p(-4) \rightarrow \mathcal{I}_{Z'} \rightarrow 0.
$$

Now, the result of Ferrand about liaison implies that $\mathcal{I}_Z(3)$ has a monad of the form:

$$
0 \rightarrow \mathcal{O}_p(1) \oplus 3 \mathcal{O}_p \oplus \mathcal{O}_p(-1) \rightarrow 2 \mathcal{O}_p(2) \oplus 2 \mathcal{O}_p(1) \oplus 3 \mathcal{O}_p \xrightarrow{(d'_2(-2), 0)} \mathcal{O}_p(3) \rightarrow 0.
$$

Denoting by $K$ the kernel of the epimorphism $d'_2(-2) : 2 \mathcal{O}_p(2) \oplus 2 \mathcal{O}_p(1) \rightarrow \mathcal{O}_p(3)$, one gets an exact sequence:

$$
0 \rightarrow \mathcal{O}_p(1) \oplus 3 \mathcal{O}_p \oplus \mathcal{O}_p(-1) \xrightarrow{\alpha} K \oplus 3 \mathcal{O}_p \rightarrow \mathcal{I}_Z(3) \rightarrow 0.
$$

The component $3 \mathcal{O}_p \rightarrow 3 \mathcal{O}_p$ of $\alpha$ must have rank at least 2, because any monomorphism $\mathcal{O}_p(1) \oplus 2 \mathcal{O}_p \rightarrow K$ degenerates along a surface in $\mathbb{P}^3$. One deduces that at least two direct summands $\mathcal{O}_p$ from the above resolution of $\mathcal{I}_Z$ cancel, hence $\mathcal{I}_Z(3)$ admits a resolution as in item (iv) of the statement. \qed

**Proposition 1.3.** Using the hypotheses and notation established before Prop. 1.2, assume that $Z'$ is not l.c.i. except finitely many points. Then $d = 6$ and $\mathcal{I}_Z(3)$ admits a resolution of the form:

$$
0 \rightarrow 3 \mathcal{O}_p(-1) \rightarrow 4 \mathcal{O}_p \rightarrow \mathcal{I}_Z(3) \rightarrow 0.
$$

**Proof.** There are only three possibilities for $Z'$.

**Case 1.** $Z' = L^{(1)}$ = first infinitesimal neighbourhood of a line $L \subset \mathbb{P}^3$.

$Z'$ is the subscheme of $\mathbb{P}^3$ defined by $\mathcal{I}_L^2$, hence $\mathcal{I}_{Z'}$ admits a resolution of the form:

$$
0 \rightarrow 2 \mathcal{O}_p(-3) \rightarrow 3 \mathcal{O}_p(-2) \rightarrow \mathcal{I}_{Z'} \rightarrow 0
$$

which, by dualization, provides a resolution of $\omega_{Z'}$:

$$
0 \rightarrow \mathcal{O}_p(-4) \rightarrow 3 \mathcal{O}_p(-2) \rightarrow 2 \mathcal{O}_p(-1) \rightarrow \omega_{Z'} \rightarrow 0.
$$

One sees that, indeed, $\omega_{Z'}(1)$ is globally generated and that $H^0(\omega_{Z'}) = 0$. Applying Ferrand result about liaison to the above resolution of $\mathcal{I}_{Z'}$ one gets that $\mathcal{I}_Z(3)$ admits a resolution as in the statement.

**Case 2.** $Z' = L^{(1)} \cup L'$, with $L$ and $L'$ distinct lines.

If $L \cap L' = \emptyset$ then $\omega_{Z'}(1)$ is not globally generated because $\omega_{L'}(1) \simeq \mathcal{O}_{L'}(-1)$. If $L \cap L' \neq \emptyset$ then, choosing conveniently the homogeneous coordinates on $\mathbb{P}^3$, one may assume that $L$ has equations $x_2 = x_3 = 0$ and $L'$ has equations $x_1 = x_3 = 0$. Then $I(Z') = \ldots$
If \( L \cup L' \). The fundamental exact sequence of liaison:

\[
0 \rightarrow \mathcal{O}_P(-5) \rightarrow \mathcal{O}_P(-2) \oplus \mathcal{O}_P(-3) \rightarrow \mathcal{I}_{L \cup L'} \rightarrow \omega_Z(-1) \rightarrow 0
\]

and the fact that \( H^0(\mathcal{I}_{L \cup L'}(1)) \neq 0 \) implies that \( H^0(\omega_Z) \neq 0 \), which contradicts our hypothesis (see the discussion before Prop. 1.2).

**Case 3.** \( Z' \) is a thick structure of degree 4 on a line \( L \).

In this case, by the results recalled in Subsection A.3 of Appendix A one has an exact sequence:

\[
0 \rightarrow \mathcal{O}_L(l) \rightarrow \mathcal{O}_{Z'} \rightarrow \mathcal{O}_{L(1)} \rightarrow 0
\]

for some \( l \geq -2 \). Applying \( \mathcal{E}xt^2_{\mathcal{O}_P}(-, \omega_P) \), one gets an exact sequence:

\[
0 \rightarrow \omega_{L(1)} \rightarrow \omega_{Z'} \rightarrow \omega_L(-l) \rightarrow 0.
\]

But \( \omega_L(-l) \simeq \mathcal{O}_L(-l - 2) \) hence \( \omega_{Z'}(1) \) globally generated implies \( l \leq -1 \). Since we assumed that \( Z' \) is not l.c.i. except at finitely many points, Prop. A.14 implies that \( l \) must be even, hence \( l = -2 \). Since \( h^0(\mathcal{O}_{L(1)}) = 1 \) and \( H^1(\mathcal{O}_L(-2)) \neq 0 \), the above exact sequence involving \( \mathcal{O}_{Z'} \) shows that \( H^1(\mathcal{O}_{Z'}) \neq 0 \), which contradicts our hypothesis (see the discussion before Prop. 1.2).

\[\square\]

2. Curves of degree at most 4

We want to classify, in this section, the locally CM curves \( Z \) in \( \mathbb{P}^3 \), l.c.i. except at finitely many points, of degree \( d \leq 4 \), with \( \mathcal{I}_Z(3) \) globally generated. More precisely, we want to list the monads of the ideal sheaves of these curves. According to Prop. 1.1 we can assume, in the case \( d = 4 \), that \( H^0(\mathcal{I}_Z(2)) = 0 \).

**Lemma 2.1.** If \( d = 4 \) and \( \mathcal{I}_Z(3) \) is globally generated then \( h^0(\mathcal{I}_Z(3)) \geq 5 \).

**Proof.** The assertion is clear if \( H^0(\mathcal{I}_Z(2)) \neq 0 \). Assume, from now on, that \( H^0(\mathcal{I}_Z(2)) = 0 \).

Since \( Z \) is not a complete intersection of type (3,3) it follows that \( h^0(\mathcal{I}_Z(3)) \geq 3 \).

If \( h^0(\mathcal{I}_Z(3)) = 3 \) then one has an exact sequence:

\[
0 \rightarrow E \rightarrow 3\mathcal{O}_P \rightarrow \mathcal{I}_Z(3) \rightarrow 0
\]

with \( E \) a rank 2 vector bundle. Let us recall the following fact: if \( W \) is a closed subscheme of \( \mathbb{P}^3 \) of dimension \( \leq 1 \) and if \( \deg W \) is defined by \( \chi(\mathcal{O}_W(t)) = t\deg W + \chi(\mathcal{O}_W), \forall t \in \mathbb{Z} \), (such that \( \deg W = 0 \) if \( \dim W \leq 0 \) then:

\[
c_1(\mathcal{O}_W(t)) = t \quad \text{and} \quad c_2(\mathcal{O}_W(t)) = \deg W.
\]

(Indeed, it suffices to compute \( c_1(\mathcal{O}_W(t)) \) and \( c_2(\mathcal{O}_W(t)) \). If \( H \subset \mathbb{P}^3 \) is a general plane then the latter Chern classes are (numerically) equal to the corresponding Chern classes of \( \mathcal{O}_{H \cap W}(t) \) on \( H \simeq \mathbb{P}^2 \) and these can be computed as in [17, Lemma 2.7].)

One gets that \( c_1(E) = -3 \) and \( c_2(E) = 5 \) but this is not possible because the Chern classes of a rank 2 vector bundle on \( \mathbb{P}^3 \) must satisfy the relation \( c_1c_2 \equiv 0 \pmod{2} \) (by the Riemann-Roch formula). It thus remains that \( h^0(\mathcal{I}_Z(3)) \geq 4 \).
Assume, finally, that $h^0(I_Z(3)) = 4$. Eliminating this case turns out to be more difficult. One has an exact sequence:

$$0 \to F \to 4O_P \to I_Z(3) \to 0$$

where $F$ is a rank 3 vector bundle with Chern classes $c_1(F) = -3$ and $c_2(F) = 5$. By our assumption, $H^i(F) = 0$, $i = 0, 1$. The rank 3 vector bundle $G := F(1)$ has Chern classes $c_1(G) = 0$ and $c_2(G) = 2$ and $H^i(G(-1)) = 0$, $i = 0, 1$. Now, $Z$ is directly linked by a complete intersection of type $(3, 3)$ to a curve $Z'$ of degree 5. By Ferrand’s result about liaison, one gets an exact sequence:

$$0 \to 2O_P \to G'(1) \to I_{Z'}(3) \to 0. \quad (2.1)$$

One cannot have $H^0(I_{Z'}(2)) \neq 0$ because this would imply that $Z'$ is directly linked to a line by a complete intersection of type $(2, 3)$ which would imply in turn, by applying twice Ferrand’s result about liaison, that $H^0(I_{Z'}(2)) \neq 0$. It follows that $H^0(G') = 0$.

On the other hand, one cannot have $H^0(G) = 0$. Indeed, $G$ is semistable hence, by a result of Schneider [29] (see, also, Ein et al. [11, Thm. 1.6(a)]), its restriction to a general plane $H \subset \mathbb{P}^3$ is also semistable which means that $H^0(G_H(-1)) = 0$ and $H^0(G_H'(-1)) = 0$. By Serre duality, $H^2(G_H) \simeq H^0(G_H'(-3))^* = 0$ and by Riemann-Roch $\chi(G_H) = 3 - c_2(G_H) = 1$. One deduces that $H^1(G_H) \neq 0$. Using, now, the exact sequence:

$$0 \to G(-1) \to G \to G_H \to 0$$

one sees that the condition $H^0(G) = 0$ would contradict our assumption that $H^1(G(-1)) = 0$. It thus remains that $H^0(G) \neq 0$.

Since $H^0(G(-1)) = 0$, the scheme of zeroes $W$ of a non-zero global section of $G$ must have dimension $\leq 1$. One has an exact sequence:

$$0 \to F \to G' \to I_W \to 0 \quad (2.2)$$

where $F$ is a rank 2 reflexive sheaf with $c_1(F) = 0$. Since $H^0(F) = 0$, $F$ is stable. It follows that $c_2(F) \geq 1$ (for results about stable rank 2 reflexive sheaves one may consult Hartshorne [17], especially Sect. 7). On the other hand, one gets, using the exact sequence (2.2), that:

$$c_2(F) + \deg W = c_2(G') = 2. \quad (2.2)$$

If $c_2(F) = 1$ then $F$ is a nullcorrelation bundle. In particular, it is locally free. One deduces, from the exact sequence (2.2), that $W$ is locally CM of pure codimension 2 in $\mathbb{P}^3$. Since $\deg W = 1$, $W$ is a line. $F$ being a nullcorrelation bundle one has $h^2(F(-3)) = 1$. It follows that $H^2(G'(-3)) \neq 0$ hence, by Serre duality, $H^1(G(-1)) \neq 0$, a contradiction.

It remains that $c_2(F) = 2$. In this case $\deg W = 0$, i.e., $\dim W \leq 0$. Dualizing the exact sequence (2.2) and taking into account that $F' \simeq F$ (because $c_1(F) = 0$), one gets an exact sequence:

$$0 \to O_P \to G \to F \to 0. \quad (2.3)$$

The possible spectra of $F$ are $(0, 0)$, $(-1, 0)$ and $(-1, -1)$. For the first two spectra one has $H^1(F(-1)) \neq 0$ which is not possible because $H^1(G(-1)) = 0$. It remains that the spectrum of $F$ is $(-1, -1)$. In particular, $c_3(F) = 4$, hence, by [17, Prop. 2.6],
Proof. Assume that $H$ is a complete intersection of type $P_1$. Let $Y$ be a reduced, connected curve of degree $3$ in $\mathbb{P}^3$. Then either $Y$ is a complete intersection of type $(1,3)$ or $Y$ is directly linked to a line by a complete intersection of type $(2,2)$. \\
\textbf{Proof.} Assume that $H^0(\mathcal{I}_Y(1)) = 0$. Then one can choose three noncolinear points $P_1$, $P_2$, $P_3$ in $Y$ such that the plane $H$ determined by them intersects $Y$ properly. In this case the map $H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{O}_{P_1P_2P_3}(1))$ is surjective, hence the map $H^0(\mathcal{O}(1)) \to H^0(\mathcal{O}(H\cap Y)(1))$ is surjective, too. Using the exact sequences:

$$0 \to \mathcal{O}_H(i) \to \mathcal{O}_H(i + 1) \to \mathcal{O}_{H\cap Y}(i + 1) \to 0$$

one derives (when $i = 0$) that $h^0(\mathcal{O}_Y(1)) = 4$ and that $H^1(\mathcal{O}_Y(i)) \to H^1(\mathcal{O}_Y(i + 1))$ is injective for $i \geq 0$, hence $H^1(\mathcal{O}_Y) = 0$. Using the exact sequence:

$$0 \to \mathcal{I}_Y(1) \to \mathcal{O}_H(1) \to \mathcal{O}_Y(1) \to 0$$

one deduces that $H^1(\mathcal{I}_Y(1)) = 0$. Since $H^2(\mathcal{I}_Y) \simeq H^1(\mathcal{O}_Y) = 0$ it follows that $\mathcal{I}_Y$ is 2-regular, hence $\mathcal{I}_Y$ is globally generated. \\
\textbf{Lemma 2.3.} Let $Z$ be a reduced, connected curve of degree $4$ in $\mathbb{P}^3$. Then $H^0(\mathcal{I}_Z(2)) \neq 0$. \\
\textbf{Proof.} Let $H \subset \mathbb{P}^3$ be a plane intersecting $Z$ properly. Using the exact sequences:

$$0 \to \mathcal{O}_Z(i) \to \mathcal{O}_Z(i + 1) \to \mathcal{O}_{H\cap Z}(i + 1) \to 0,$$

for $i = 0$ and $1$, one deduces that:

$$h^0(\mathcal{O}_Z(2)) \leq h^0(\mathcal{O}_Z(1)) + h^0(\mathcal{O}_{H\cap Z}(2)) \leq h^0(\mathcal{O}_Z) + h^0(\mathcal{O}_{H\cap Z}(1)) + h^0(\mathcal{O}_{H\cap Z}(2)) = 9.$$

Using, now, the exact sequence:

$$0 \to \mathcal{I}_Z(2) \to \mathcal{O}_H(2) \to \mathcal{O}_Z(2) \to 0$$
and the fact that $h^0(\mathcal{O}_P(2)) = 10$ one derives that $h^0(\mathcal{I}_Z(2)) \geq 1$. \hfill \Box

**Lemma 2.4.** Let $X$ and $X'$ be two disjoint locally CM curves of degree 2 in $\mathbb{P}^3$ and let $Z = X \cup X'$. Then $Z$ admits a 4-secant hence $\mathcal{I}_Z(3)$ is not globally generated.

**Proof.** $X$ (and, similarly, $X'$) can be a complete intersection of type $(1,2)$, the union of two disjoint lines $L$ and $L'$, or a double structure on a line $L$. In the first case, $X$ is contained in a plane $H$ and every line in $H$ intersecting $X$ properly is a 2-secant to $X$. In the second case, every line joining a point of $L$ and a point of $L'$ is a 2-secant to $X$. In the third case, assume that $L$ is the line of equations $x_2 = x_3 = 0$. As it is well known, there exist an integer $l \geq -1$ and two coprime polynomials $a, b \in k[x_0, x_1]_{t+1}$ such that the holomogeneous ideal $I(X) \subset S$ of $X$ is generated by:

$$F_2 = -bx_2 + ax_3, x_2^2, x_2x_3, x_3^2$$

(this result is recalled in Subsection A.5 from Appendix A). The surface $\Sigma \subset \mathbb{P}^3$ of equation $F_2 = 0$ is nonsingular along $L$. If $L_1$ is another line meeting $L$ in a point $z$ then $L_1$ is a 2-secant to $X$, i.e., $\deg(L_1 \cap X) = 2$ if and only if $L_1$ is contained in the projective tangent plane $T_z\Sigma$ to $\Sigma$ at $z$. This tangent plane has equation:

$$-b(z)x_2 + a(z)x_3 = 0.$$

We will show that $X$ and $X'$ have a common 2-secant which is, consequently, a 4-secant to $Z$. We split the proof of this fact into several cases according to the nature of $X$ and $X'$.

**Case 1.** $X$ is a double structure on the line $L$ of equations $x_2 = x_3 = 0$ and $X'$ is a double structure on the line $L'$ of equations $x_0 = x_1 = 0$.

Consider the morphism $\phi : L \to L'$ defined by $\{\phi(z)\} = T_z\Sigma \cap L'$. One has:

$$\phi((z_0 : z_1 : 0 : 0)) = (0 : 0 : a(z_0, z_1) : b(z_0, z_1)).$$

For $z \in L$, the line joining $z$ to $\phi(z) \in L'$ is a 2-secant to $X$. Consider the similar morphism $\phi' : L' \to L$. The composite morphism $\phi' \circ \phi : L \to L$ has a fixed point $z$, hence, if $z' = \phi(z)$ then $\phi'(z') = z$. The line joining $z$ to $z'$ is a 2-secant to both $X$ and $X'$.

**Case 2.** $X$ is a double structure on the line $L$ of equations $x_2 = x_3 = 0$ and $X'$ is a complete intersection of type $(1,2)$.

Let $H \subset \mathbb{P}^3$ be the plane containing $X'$, let $\{z\} = H \cap L$ and let $L_1 = T_z\Sigma \cap H$. Then $L_1$ is a 2-secant to both $X$ and $X'$.

**Case 3.** $X$ is a double structure on the line $L$ of equations $x_2 = x_3 = 0$ and $X' = L' \cup L''$, where $L'$ and $L''$ are disjoint lines not intersecting $L$.

Consider the morphism $\phi : L \to L'$ defined in Case 1. Consider also the morphism $\psi : L \to L'$ defined by:

$$\{\psi(z)\} = \text{span}(\{z\} \cup L'') \cap L'.$$
ψ is, actually, an isomorphism. Then $\psi^{-1} \circ \phi : L \to L$ has a fixed point $z$. Let $w = \phi(z) = \psi(z)$. The line joining $z$ and $w$ is a 2-secant to $X$ and meets both $L'$ and $L''$, hence it is a 2-secant to $X'$, too.

**Case 4.** $X$ and $X'$ are both complete intersections of type $(1,2)$.

$X$ is contained in a plane $H$ and $X'$ is contained in a plane $H'$. The line $H \cap H'$ is a 2-secant to both $X$ and $X'$.

**Case 5.** $X$ is a complete intersection of type $(1,2)$ and $X'$ is the union of two disjoint lines $L'$ and $L''$ not intersecting $X$.

$X$ is contained in a plane $H$ and $L'$ (resp., $L''$) intersects $H$ in a point $z'$ (resp., $z''$). The line joining $z'$ and $z''$ is a 2-secant to both $X$ and $X'$.

**Case 6.** $Z$ is the union of four mutually disjoint lines.

This case is well-known: three of the lines are contained in a unique nonsingular quadric surface $Q \subset \mathbb{P}^3$. Let $z$ be a point from the intersection of $Q$ with the fourth line. Then the line contained in $Q$, passing through $z$, and not belonging to the ruling of $Q$ to which the first three lines belong intersects all the four lines. \square

**Lemma 2.5.** Let $Y$ be a locally CM curve of degree 3 in $\mathbb{P}^3$, $L' \subset \mathbb{P}^3$ a line not intersecting $Y$, and $Z = Y \cup L'$. If $\mathcal{I}_Z(3)$ is globally generated then $Y$ is directly linked to a line by a complete intersection of type $(2,2)$.

**Proof.** By Lemma 2.4, we can assume that $Y$ is connected. $Y$ cannot be a complete intersection of type $(1,3)$ because, in that case, any line contained in the plane $H \supset Y$ and passing through the intersection point of $H$ and $L'$ is a 4-secant to $Z$. If $Y$ is reduced then, by Lemma 2.2, $Y$ is directly linked to a line by a complete intersection of type $(2,2)$.

Consequently, we can assume that $Y$ is connected and nonreduced. There are two cases to be considered.

**Case 1.** $Y$ is a triple structure on a line.

In this case, since the first infinitesimal neighbourhood of a line in $\mathbb{P}^3$ is directly linked to a line by a complete intersection of type $(2,2)$, we can assume that $Y$ is a quasiprimitive triple structure on the line $L_1$ of equations $x_2 = x_3 = 0$ and that $L'$ is the line of equations $x_0 = x_1 = 0$. We use the notation and results from Subsection B.2 of Appendix B.

If $l = -1$ then we can assume, by Lemma B.14 that $m \geq 2$, i.e., that the hypothesis of Prop. B.16(a) is fulfilled. Let $\lambda \in k[x_0, x_1]$ be a divisor of $a'$. Then the line of equations $\lambda = x_2 = 0$ is a 4-secant to $Z$.

If $l = m = 0$ then, by Lemma B.15, $Y$ is the divisor $3L_1$ on a nonsingular quadric surface $Q \supset L_1$. If $P \in Q \cap L'$ then the line contained in $Q$, passing through $P$ and intersecting $L_1$ is a 4-secant to $Z$.

Finally, if $l = 0$ and $m \geq 1$ or if $l \geq 1$ then, by Prop. B.16(b) and by Lemma 2.1, $\mathcal{I}_Z(3)$ is not globally generated.
Lemma 2.6. Assume that $b$ and $l = -1$ then, by the proof of Prop. [B.31(a)], $Y$ is a complete intersection of type (1,3) and this case was excluded above.

Remark 2.7. If $Z$ is a primitive structure of degree 4 on the line $X \subset \mathbb{P}^3$ of equations $x_2 = x_3 = 0$, then $H^0(\mathcal{I}_Z(2)) = 0$ implies that $Z$ is not globally generated.

Proof. We use Prop. [A.6] and Prop. [A.12] from Appendix and the notation from that appendix. One has $l \geq 0$ because, in the case $l = -1$, $a$ and $b$ are constants, not simultaneously 0, $F_2 = -bx + ay$, and $G_1 = F_2^2$, hence $H^0(\mathcal{I}_Z(2)) \neq 0$. The conclusion follows, now, from Lemma 2.1 using Prop. [A.6] and Prop. [A.12]

Remark 2.7. If $Z$ is a primitive structure of degree 4 on the line $L \subset \mathbb{P}^3$ of equations $x_2 = x_3 = 0$, with $l = 0$ and such that $H^0(\mathcal{I}_Z(2)) = 0$, then $Z$ admits a 4-secant.

Indeed, the conditions $l = 0$ and $H^0(\mathcal{I}_Z(2)) = 0$ imply that $Z$ doesn’t satisfy the hypothesis of Prop. [A.6] but it satisfies the hypothesis of Prop. [A.12]. One may assume, now, that $a = x_0$ and $b = x_1$, hence:

$$F_2 = -x_1x_2 + x_0x_3, \quad F_3 = F_2 + v_0x_2^2 + v_1x_2x_3 + v_2x_3^2,$$

$$F_{40} = x_0F_3 + w_0x_2^3 + w_0x_2x_3 + w_2x_2x_3^2 + w_3x_3^3,$$

$$F_{41} = x_1F_3 + w_1x_2^3 + w_1x_2x_3 + w_2x_2x_3^2 + w_3x_3^3,$$

$$G_1 = x_2F_3, \quad G_2 = x_3F_3.$$

The surface $Q \subset \mathbb{P}^3$ of equation $F_3 = 0$ is a smooth quadric. Relation [A.22] becomes:

$$-x_1(w_0x_2^3 + w_0x_2x_3 + w_2x_2x_3^2 + w_3x_3^3) + x_0(w_1x_2^3 + w_1x_2x_3 + w_2x_2x_3^2 + w_3x_3^3)$$

$$= (\gamma_0x_2^2 + \gamma_1x_2x_3 + \gamma_2x_3^2)(-x_1x_2 + x_0x_3)$$

Since $w_{ij}$ and $\gamma_i$ are constants, one gets that:

$$F_{40} = x_0F_3 + x_3(\gamma_0x_2^2 + \gamma_1x_2x_3 + \gamma_2x_3^2), \quad F_{41} = x_1F_3 + x_2(\gamma_0x_2^2 + \gamma_1x_2x_3 + \gamma_2x_3^2).$$
The divisor \( \{ \gamma_0 x_2^2 + \gamma_1 x_2 x_3 + \gamma_2 x_3^2 = 0 \} \cap Q \) on \( Q \) contains \( 2L \) hence it is of the form \( 2L + L'_1 + L'_2 \), where \( L'_1, L'_2 \) are lines from the other ruling of \( Q \). Both of these lines are 4-secants to \( Z \).

**Proposition 2.8.** Let \( Z \) be a locally CM curve in \( \mathbb{P}^3 \) of degree 4. If \( H^0(\mathcal{I}_Z(2)) = 0 \) and \( \mathcal{I}_Z(3) \) is globally generated then \( \mathcal{I}_Z(3) \) admits a monad of the form:

\[
0 \rightarrow \mathcal{O}_\mathbb{P}(1) \oplus 2\mathcal{O}_\mathbb{P} \rightarrow 2\mathcal{O}_\mathbb{P}(2) \oplus 3\mathcal{O}_\mathbb{P}(1) \rightarrow \mathcal{O}_\mathbb{P}(3) \rightarrow 0.
\]

**Proof.** If \( Z \) is not connected then the conclusion of the proposition follows, even without assuming that \( H^0(\mathcal{I}_Z(2)) = 0 \), from Lemma 2.5 taking into account Lemma 2.4 (see the first method for getting monads stated in the Introduction).

Assume, from now on, that \( Z \) is, moreover, connected. By Lemma 2.6, \( Z \) cannot be a quasiprimitive structure of degree 4 on a line. The hypothesis \( H^0(\mathcal{I}_Z(2)) = 0 \) eliminates, also, many other cases as, for example, the case where \( Z \) is reduced (by Lemma 2.3) or the case where \( Z \) is the union of a double line and of two other (simple) lines, both of them intersecting the double line. It remains, actually, to analyse the following four cases:

**Case 1.** \( Z \) is a thick structure of degree 4 on a line \( L \).

We may assume that \( L \) is the line of equations \( x_2 = x_3 = 0 \). We use, in this case, the results and notation from Subsection A.3 of Appendix A (with \( W = Z \)). Since

\[
\mathcal{I}_Z/\mathcal{I}_Z^3 \simeq \mathcal{O}_L(-m-2) \oplus \mathcal{O}_L(-n-2)
\]

it follows that if \( \mathcal{I}_Z(3) \) is globally generated then \( m \leq 1 \) and \( n \leq 1 \). On the other hand, since the generators \( F \) and \( G \) of the homogeneous ideal \( I(Z) \subset S \) from Prop. A.13 have degrees \( m+2 \) and \( n+2 \), respectively, and since \( H^0(I(Z)(2)) = 0 \) it follows that, actually, \( m = n = 1 \). Since \( m + n = l + 2 \), one gets that \( l = 0 \). Prop. A.15 implies, now, that \( \mathcal{I}_Z(3) \) satisfies the conclusion of the proposition.

**Case 2.** \( Z \) is the union of a triple structure \( Y \) on a line and of another line intersecting it.

We may assume that \( Y \) is a triple structure on the line \( L_1 \) of equations \( x_2 = x_3 = 0 \) and that the second line is the line \( L_2 \) of equations \( x_1 = x_3 = 0 \). We use, in this case, the results and notation from Subsection B.2 of Appendix B. The hypothesis \( H^0(I(Z)(2)) = 0 \) implies that \( Y \) cannot be the first infinitesimal neighbourhood of \( L_1 \) in \( \mathbb{P}^3 \) hence it is a quasiprimitive triple structure on \( L_1 \).

Now, Lemma 2.4 and Prop. B.19 imply that \( l \leq 0 \). Since \( x_3 F_2 \in I(Z) \) and \( H^0(I(Z)(2)) = 0 \) it follows that \( l \geq 0 \). Consequently, \( l = 0 \).

In the cases (a) and (c) of Prop. B.19 one has, on one hand, that \( l + m + 2 = \deg F_3 \geq 3 \) (because \( H^0(I(Z)(2)) = 0 \)) and, on the other hand, since \( \mathcal{O}_{L_1}(-l-m-2) \) is a quotient of \( \mathcal{I}_Z \), that \( -l - m - 2 + 3 \geq 0 \). It follows that, in this cases, \( l = 0 \) and \( m = 1 \).

In the cases (b) and (d) of Prop. B.19 one has, on one hand, that \( l + m + 3 = \deg (x_1 F_3) \geq 3 \) (because \( H^0(I(Z)(2)) = 0 \)) and, on the other hand, since \( \mathcal{O}_{L_1}(-l-m-3) \) is a quotient of \( \mathcal{I}_Z \), that \( -l - m - 3 + 3 \geq 0 \). It follows that, in this cases, \( l = 0 \) and \( m = 0 \). We analyse, now, each one of the cases occuring in Prop. B.19.
In case (a), one must have \( b = x_1 \) hence \( b_1 = 1 \). It follows that:
\[
x_2F_2 = -x_1x_2^2 + ax_2x_3 \quad \text{and} \quad x_1x_2^3 = -x_2 \cdot x_2F_2 + a \cdot x_2^2x_3,
\]
hence \( I(Z) \) is generated by polynomials of degree 3. Using Prop. [B.20] one sees that \( \mathcal{I}_Z(3) \) satisfies the conclusion of the proposition.

In case (b), essentially the same argument shows that \( I(Z) \) is generated by polynomials of degree 3 and that \( \mathcal{I}_Z(3) \) satisfies the conclusion of the proposition.

In case (c), one must have \( p = x_1 \). By Lemma [B.18](c), one can assume that \( v_0 = c_0x_1 \), for some \( c_0 \in k \), hence:
\[
F_3 = x_1(-bx_2 + ax_3) + c_0x_1x_2^2 + v_1x_2x_3 + v_2x_3^2.
\]
If \( \mathcal{I}_Z(3) \) is globally generated then:
\[
x_1x_2^3 \in (F_3, x_3F_2, x_2x_3, x_3x_3^2, x_3^{3\text{sat}}).
\]
Let \( S' = S/Sx_3 = k[x_0, x_1, x_2] \). It follows that, working in \( S' \):
\[
x_1x_2^3 \in (S'x_1x_2(-b + c_0x_2))^{\text{sat}}.
\]
But \( S'x_1x_2(-b + c_0x_2) \) is already saturated in \( S' \) and \( x_1x_2^3 \notin S'x_1x_2(-b + c_0x_2) \). It follows that \( \mathcal{I}_Z(3) \) is not globally generated in case (c).

In case (d), \( p = 1 \), \( F_3 = F_2 + v_0x_2^2 + v_1x_2x_3 + v_2x_3^2 \), and the same kind of argument as that used in case (c) shows that \( \mathcal{I}_Z(3) \) cannot be globally generated (because \( x_1x_2^3 \notin S'x_1x_2(-b + v_0x_2) \)).

**Case 3.** \( Z \) is the union of a double structure \( X \) on a line, of another line intersecting it, and of a third line intersecting the second line but not the double line.

We may assume that \( X \) is a double structure on the line \( L_1 \) of equations \( x_2 = x_3 = 0 \), that the second line is the line \( L_2 \) of equations \( x_1 = x_3 = 0 \), and that the third line is the line \( L_1' \) of equations \( x_0 = x_1 = 0 \). We use, in this case, the results and notation from Subsection [B.3]. It follows, from Lemma [B.27] and Prop. [B.28] that, under the hypothesis of Prop. [B.28](b) one must have \( l = 1 \), and under the hypothesis of Prop. [B.28](c) one must have \( l = 0 \). Prop. [B.29] shows, now, that \( \mathcal{I}_Z(3) \) satisfies the conclusion of the proposition.

**Case 4.** \( Z \) is the union of a double structure \( X \) on a line and of a (nonsingular) conic \( C \) intersecting it.

Since \( \text{H}^0(\mathcal{I}_Z(2)) = 0 \), the plane containing the conic must not contain the support of the double line. One may assume, in this case, that \( X \) is a double structure on the line \( L_1 \) of equations \( x_2 = x_3 = 0 \) and that \( C \) is the conic of equations \( x_1 = x_0x_3 - x_2^2 = 0 \). One can use, now, an argument similar to that used in Case 3, based on Lemma [B.39], Prop. [B.40] and Prop. [B.41] from Subsection [B.4].

**Proposition 2.9.** Let \( Y \) be a locally CM curve in \( \mathbb{P}^3 \) of degree 3. If \( \mathcal{I}_Y(3) \) is globally generated then one of the following holds:
(i) $Y$ is a complete intersection of type $(1,3)$;
(ii) $Y$ is directly linked to a line by a complete intersection of type $(2,2)$;
(iii) $\mathcal{I}_Y(3)$ admits a monad of the form:
\[ 0 \to \mathcal{O}_P(1) \oplus \mathcal{O}_P \to 3\mathcal{O}_P(2) \oplus \mathcal{O}_P(1) \to \mathcal{O}_P(3) \to 0; \]
(iv) $\mathcal{I}_Y(3)$ admits a monad of the form:
\[ 0 \to \mathcal{O}_P(i + 1) \oplus 2\mathcal{O}_P(1) \to 2\mathcal{O}_P(i + 2) \oplus 4\mathcal{O}_P(2) \to \mathcal{O}_P(i + 3) \oplus \mathcal{O}_P(3) \to 0, \]
with $i = 0$ or 1;
(v) $\mathcal{I}_Y(3)$ admits a monad of the form:
\[ 0 \to \mathcal{O}_P(3) \oplus \mathcal{O}_P(2) \oplus \mathcal{O}_P(1) \to 2\mathcal{O}_P(4) \oplus 2\mathcal{O}_P(3) \oplus 2\mathcal{O}_P(2) \to \mathcal{O}_P(5) \oplus \mathcal{O}_P(4) \to 0. \]

Proof. We split the proof into several cases.

Case 1. $Y$ is not connected.

In this case $Y$ is the union of a curve $X$ of degree 2 and of a line $L'$ not intersecting $X$. If $X$ is a complete intersection of type $(1,2)$ then $\mathcal{I}_Y(3)$ satisfies the condition (iii) from the statement. If $X$ is the union of two disjoint lines then $\mathcal{I}_Y(3)$ satisfies condition (iv) from the statement with $i = 0$.

It remains to consider the situation where $X$ is a double structure on a line $L$ defined by an epimorphism $\mathcal{I}_L/\mathcal{I}_L^2 \simeq 2\mathcal{O}_L(-1) \to \mathcal{O}_L(l)$ with $l \geq 0$ (for $l = -1$, $X$ is a complete intersection of type $(1,2)$). We may assume that $L$ has equations $x_2 = x_3 = 0$ and that $L'$ has equations $x_0 = x_1 = 0$. In this case, using Prop. [B.9] one sees easily that one must have $l \leq 1$. It follows, from the results stated in Subsection A.5, that $\mathcal{I}_Y(3)$ satisfies condition (iv) from the statement with $i = l$.

Case 2. $Y$ is reduced and connected.

In this case, one applies Lemma [2.2].

Case 3. $Y$ is the union of a double structure $X$ on a line and of another line intersecting it.

We may assume that $X$ is a double structure on the line $L_1$ of equations $x_2 = x_3 = 0$ and that the other line is the line $L_2$ of equations $x_1 = x_3 = 0$. We use, in this case, the results and notation from Subsection B.1

Under the hypothesis of Prop. [B.11(a)] (with $c = 0$), $\mathcal{I}_Y(3)$ is globally generated if and only if $l \in \{-1, 0, 1\}$. If $l = -1$, then $\deg F_2 = 1$ hence $Y$ is a complete intersection of type $(1,3)$. If $l = 0$ then $b = x_1$ and $b_1 = 1$ hence, by Prop. [B.12(a)], $Y$ is directly linked to a line by a complete intersection of type $(2,2)$. Finally, if $l = 1$ then, by Prop. [B.12(a)], $\mathcal{I}_Y(3)$ satisfies condition (iii) from the statement.

On the other hand, under the hypothesis of Prop. [B.11(b)] (with $c = 0$), $\mathcal{I}_Y(3)$ is globally generated if and only if $l \in \{-1, 0\}$. If $l = -1$ then $b$ is a non-zero constant hence, by Prop. [B.12(b)], $Y$ is directly linked to a line by a complete intersection of type $(2,2)$. If $l = 0$ then, by Prop. [B.12(b)], $\mathcal{I}_Y(3)$ satisfies condition (iii) from the statement.

Case 4. $Y$ is a triple structure on a line $L$. 


We may assume that $L$ has equations $x_2 = x_3 = 0$. If $Y$ is the first infinitesimal neighbourhood of $L$ in $\mathbb{P}^3$ then $Y$ is directly linked to $L$ by a complete intersection of type $(2, 2)$. Consequently, we may assume that $Y$ is a \textit{quasiprimitive} triple structure on $L$. We use, in this case, the results and notation from Subsection A.4. Since $\mathcal{O}_L(-l - m - 2)$ is a quotient of $\mathcal{I}_Y$ (this fact is recalled at the beginning of Subsection B.2) it follows that if $\mathcal{I}_Y(3)$ is globally generated then $-l - m - 2 + 3 \geq 0$, i.e., $l + m \leq 1$. Conversely, if $l + m \leq 1$ then $I(Y)$ is generated by polynomials of degree 3 (for $l \leq 0$ this is clear, while for $l = 1$ and $m = 0$ $Y$ is a \textit{primitive} triple structure on $L$ hence $I(Y)$ is generated by $F_3, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3$).

If $l = 1$ and $m = 0$ (i.e., if $Y$ is the divisor $3L$ on a cubic surface $\Sigma \supset L$ nonsingular along $L$) then $\mathcal{I}_Y(3)$ satisfies condition (v) from the statement.

If $l = 0$ and $m = 1$ then $\mathcal{I}_Y(3)$ satisfies condition (iv) from the statement with $i = 1$.

If $l = 0$ and $m = 0$ then $\mathcal{I}_Y(3)$ satisfies condition (iv) from the statement with $i = 0$.

If $l = -1$ and $m = 2$ then one can assume that $a = 0$ and $b = -1$ hence $\mathcal{I}_Y(3)$ satisfies condition (iii) from the statement.

If $l = -1$ and $m = 1$ then, by Lemma B.14(b), $Y$ is directly linked to $L$ by a complete intersection of type $(2, 2)$.

Finally, if $l = -1$ and $m = 0$ then, by Lemma B.14(a), $Y$ is a complete intersection of type $(1, 3)$.

\begin{proposition}
Let $X$ be a locally CM curve in $\mathbb{P}^3$ of degree 2. If $\mathcal{I}_X(3)$ is globally generated then either $X$ is a complete intersection of type $(1, 2)$ or $\mathcal{I}_X(3)$ admits a monad of the form:

$$0 \to \mathcal{O}_\mathbb{P}(i + 1) \oplus \mathcal{O}_\mathbb{P}(1) \to 2\mathcal{O}_\mathbb{P}(i + 2) \oplus 2\mathcal{O}_\mathbb{P}(2) \to \mathcal{O}_\mathbb{P}(i + 3) \to 0$$

with $i = 0$ or 1.

\end{proposition}

\begin{proof}
If $X$ is not a complete intersection of type $(1, 2)$ then either $X$ is the union of two disjoint lines (in which case $\mathcal{I}_X(3)$ admits, by Lemma B.1 a monad as in the statement with $i = 0$) or it is a double structure on a line $L$. We may assume that $L$ has equations $x_2 = x_3 = 0$. We use, in this case, the results and notation from Subsection A.5. $\mathcal{I}_X(3)$ globally generated and $X$ not a complete intersection of type $(1, 2)$ turn out to be equivalent to $l \in \{0, 1\}$. It follows that $\mathcal{I}_X(3)$ admits a monad of the form from the statement with $i = l$.

\end{proof}

3. Globally generated vector bundles

In this section we prove Theorem 0.1 using the results from the Propositions 1.1, 1.2, 1.3, 2.8, 2.9 and 2.10. So, let $E$ be a globally generated vector bundle on $\mathbb{P}^3$ with $c_1 \geq 4$ such that $H^0(E^*) = 0$, $i = 0, 1$. Assume that $H^0(E(-c_1 + 2)) = 0$ and $H^0(E(-c_1 + 3)) \neq 0$. Recall, from the Introduction, the exact sequence (0.1) and the monad (0.3) of $E$ deduced from the monad (0.2) of $\mathcal{I}_Z(3)$. Let us assume that, moreover:

$$B^{-1} = B^+_{-1} \oplus m_{-1} \mathcal{O}_\mathbb{P} \oplus B^-_{-1} \quad \text{and} \quad B^0 = B^+_0 \oplus m_0 \mathcal{O}_\mathbb{P},$$

where $B^+$ and $B^-$ are locally free sheaves of rank $a$ and $b$, respectively.
with \( H^0(B_{1^V}^{-}) = 0, H^0(B_{-1}^+) = 0, \) and that \( H^0(B_{1}^V) = 0. \) Let \( d_{+}^{-1} : B_{+}^{-1} \to B_{+}^{0} \) be the restriction of \( d^{-1} : B^{-1} \to B^{0} \) and \( d_{+}^{0} : B_{+}^{0} \to B_{+}^{1} \) the restriction of \( d^{0}. \) The next lemma will allow us to handle all the cases occurring in the proof of Theorem 0.1.

**Lemma 3.1.** Under the above hypotheses and notation, \( r = 2 - m_{0} + m_{-1} + H^0(B_{-1}^V) \) and one has an exact sequence:

\[
0 \to B_{+}^{-1} \xrightarrow{(\phi_+, d_{+}^{-1})} \mathcal{O}_p(c_1 - 3) \oplus F \to E \to 0
\]

where \( F \) is the cohomology of a monad:

\[
0 \to B_{-1}^{-1} \xrightarrow{(0, u)} B_{+}^{0} \oplus m\mathcal{O}_p \xrightarrow{(d_{+}^{0}, \ast)} B_{+}^{1} \to 0
\]

with \( u : B_{-1}^{-1} \to m\mathcal{O}_p \) the dual of the evaluation \( H^0(B_{-1}^V) \otimes_k \mathcal{O}_p \to B_{-1}^V. \)

**Proof.** Put \( B_{<0}^{-1} = m_{-1}\mathcal{O}_p \oplus B_{-1}^{-1}. \) The monad \((0.3)\) of \( E \) from the Introduction can be written in the following form:

\[
0 \to B_{+}^{-1} \oplus B_{<0}^{-1} \xrightarrow{(\phi_+, d_{+}^{-1}, \beta)} \mathcal{O}_p(c_1 - 3) \oplus B_{+}^{0} \oplus m'\mathcal{O}_p \xrightarrow{(0, d_{+}^{0}, \rho)} B_{+}^{1} \to 0
\]

where \( m' = m_{0} + r - 2. \) The condition \( H^i(E^\vee) = 0, i = 0, 1, \) is equivalent to the fact that:

\[
H^0(\gamma^\vee) : H^0(m'\mathcal{O}_p) \to H^0(B_{<0}^{-1}^V)
\]

is an isomorphism. Up to an automorphism of \( m'\mathcal{O}_p, \) one can assume that \( \gamma \) is the dual of the evaluation morphism:

\[
H^0(B_{<0}^{-1}^V) \otimes_k \mathcal{O}_p \to B_{<0}^{-1}^V.
\]

It follows that \((\alpha, \beta)^t : B_{<0}^{-1} \to \mathcal{O}_p(c_1 - 3) \oplus B_{+}^{0} \) factorizes as:

\[
B_{<0}^{-1} \xrightarrow{\gamma} m'\mathcal{O}_p \xrightarrow{(\alpha', \beta')} \mathcal{O}_p(c_1 - 3) \oplus B_{+}^{0}.
\]

One deduces that the monad \((0.3)\) is isomorphic to:

\[
0 \to B_{+}^{-1} \oplus B_{<0}^{-1} \xrightarrow{(\phi_+, d_{+}^{-1}, \beta)} \mathcal{O}_p(c_1 - 3) \oplus B_{+}^{0} \oplus m'\mathcal{O}_p \xrightarrow{(0, d_{+}^{0}, \rho') B_{+}^{1} \to 0
\]

with \( \rho' = \rho + d_{+}^{0} \circ \beta'. \) It suffices, now, to cancel the direct summand \( m_{-1}\mathcal{O}_p \) of \( B_{<0}^{-1} \) and the corresponding direct summand of \( m'\mathcal{O}_p. \) \( \square \)
Lemma 3.2. Let $F$ be the kernel of an epimorphism $\varepsilon : \bigoplus_{i=0}^{3} \mathcal{O}_\mathbb{P}(-d_i) \to \mathcal{O}_\mathbb{P}$, with $1 \leq d_0 \leq \cdots \leq d_3$. Then $F(t)$ is globally generated if and only if $t \geq d_2 + d_3$.

Proof. Using the Koszul complex associated to $\varepsilon$ one sees easily that $F(d_2 + d_3)$ is globally generated.

On the other hand, let $f_i \in H^0(\mathcal{O}_\mathbb{P}(d_i))$, $i = 0, \ldots, 3$, be the polynomials defining $\varepsilon$. Restricting $F$ to the complete intersection $C$ defined by $f_0$ and $f_1$, one gets:

$$F|C \simeq \mathcal{O}_C(-d_0) \oplus \mathcal{O}_C(-d_1) \oplus \mathcal{O}_C(-d_2 - d_3)$$

from which one deduces that $F(d_2 + d_3 - 1)$ is not globally generated. \qed

Proof of Theorem 0.1. We split the proof into several cases according to the form of the monad of $\mathcal{I}_Z(3)$, where $Z$ is the curve occurring in the exact sequence (0.1) from the Introduction.

Case 0. $Z = \emptyset$.

In this case $E \simeq \mathcal{O}_\mathbb{P}(c_1 - 3) \oplus \mathcal{O}_\mathbb{P}(3)$.

Case 1. $Z$ is a line.

In this case, by Lemma 3.1, one has an exact sequence:

$$0 \to \mathcal{O}_\mathbb{P}(1) \to \mathcal{O}_\mathbb{P}(c_1 - 3) \oplus 2\mathcal{O}_\mathbb{P}(2) \to E \to 0.$$

Since we are on $\mathbb{P}^3$, this is possible only if $c_1 = 4$ and $E \simeq 2\mathcal{O}_\mathbb{P}(2)$. But, then, $H^0(E(-c_1 + 2)) \neq 0$, which contradicts our hypothesis, hence this case cannot occur.

Case 2. $Z$ is a complete intersection of type $(a, b)$, with $1 \leq a \leq b \leq 3$ and $a + b \geq 3$.

In this case, by Lemma 3.1, $\mathcal{O}_\mathbb{P}(c_1 - 3)$ is a direct summand of $E$.

Case 3. $Z$ is directly linked to a line by a complete intersection of type $(2, 2)$.

In this case, $\mathcal{I}_Z(3)$ admits a resolution of the form:

$$0 \to 2\mathcal{O}_\mathbb{P} \to 3\mathcal{O}_\mathbb{P}(1) \to \mathcal{I}_Z(3) \to 0,$$

hence, by Lemma 3.1, $\mathcal{O}_\mathbb{P}(c_1 - 3)$ is a direct summand of $E$.

Case 4. $Z$ is as in Prop. 1.1(ii)-(iv), or as in Prop. 1.2(i)-(iii), or as in Prop. 1.3.

In this case, by Lemma 3.1, $\mathcal{O}_\mathbb{P}(c_1 - 3)$ is a direct summand of $E$. Indeed, it suffices to notice that if $Z$ is as in Prop. 1.1(iv) then $\mathcal{I}_Z(3)$ admits a monad of the form:

$$0 \to 3\mathcal{O}_\mathbb{P} \to 5\mathcal{O}_\mathbb{P}(1) \to \mathcal{O}_\mathbb{P}(2) \to 0,$$

and if $Z$ is as in Prop. 1.2(iii) then $\mathcal{I}_Z(3)$ admits a monad of the form:

$$0 \to 2\mathcal{O}_\mathbb{P} \oplus \mathcal{O}_\mathbb{P}(-1) \to 4\mathcal{O}_\mathbb{P}(1) \oplus \mathcal{O}_\mathbb{P} \to \mathcal{O}_\mathbb{P}(2) \to 0.$$

Case 5. $Z$ is as (Y) in Prop. 2.9(iv)-(v) or as (X) in the second part of the conclusion of Prop. 2.10.
In all these cases it follows, from Lemma 3.1, that $E$ has rank 2 hence it can be realized as an extension:

$$0 \rightarrow \mathcal{O}_P(c_1 - 3) \rightarrow E \rightarrow \mathcal{I}_Z(3) \rightarrow 0.$$ 

One deduces that $Z$ is l.c.i. and that $\omega_Z \simeq \mathcal{O}_Z(2 - c_1)$. Since $\chi(\omega_Z) = -\chi(\mathcal{O}_Z)$ one deduces, from Riemann-Roch on $Z$, that

$$(c_1 - 2)\text{deg } Z = 2\chi(\mathcal{O}_Z).$$

Notice, also, that $c_2 = \text{deg } Z + 3(c_1 - 3)$ and that $\chi(\mathcal{O}_Z)$ depends only on the numerical shape of the monad of $\mathcal{I}_Z(3)$.

**Subcase 5.1.** $Z$ as $(Y)$ in Prop. 2.9(iv) with $i = 0$.

In this subcase, $\text{deg } Z = 3$ and, in order to compute $\chi(\mathcal{O}_Z)$, one may assume that $Z$ is a triple structure on a line $L$ with $l = 0$ and $m = 0$ (notation as in Subsection A.3), hence with $\mathcal{O}_Z \simeq 3\mathcal{O}_L$ as an $\mathcal{O}_L$-module. One gets $\chi(\mathcal{O}_Z) = 3$, hence $c_1 = 4$ and $c_2 = 6$. It follows that $c_1(E(-2)) = 0$, $c_2(E(-2)) = c_2 - 2c_1 + 4 = 2$ and, since $H^0(E(-2)) = 0$, $E$ is stable.

**Subcase 5.2.** $Z$ as $(Y)$ in Prop. 2.9(iv) with $i = 1$.

In this subcase, $\text{deg } Z = 3$ and, in order to compute $\chi(\mathcal{O}_Z)$, one may assume that $Z$ is a triple structure on a line $L$ with $l = 1$ and $m = 1$, hence with $\mathcal{O}_Z \simeq 2\mathcal{O}_L \oplus \mathcal{O}_L(1)$ as an $\mathcal{O}_L$-module. It follows that $\chi(\mathcal{O}_Z) = 4$ hence $3 \cdot (c_1 - 2) = 2 \cdot 4$, hence this subcase cannot occur.

**Subcase 5.3.** $Z$ as $(Y)$ in Prop. 2.9(v).

In this subcase, $\text{deg } Z = 3$ and, in order to compute $\chi(\mathcal{O}_Z)$, one may assume that $Z$ is a triple structure on a line $L$ with $l = 1$ and $m = 0$, hence with $\mathcal{O}_Z \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$ as an $\mathcal{O}_L$-module. It follows that $\chi(\mathcal{O}_Z) = 6$ hence $c_1 = 6$ and $c_2 = 12$ hence $c_1(E(-3)) = 0$ and $c_2(E(-3)) = 3$. The hypothesis $H^0(E(-4)) = 0$ and $H^0(E(-3)) \neq 0$ shows that $E(-3)$ is properly semistable.

We notice, at this point, that if $Z$ is a triple structure on a line $L$ with $l = 1$ and $m = 0$ then $\omega_Z \simeq \mathcal{O}_Z(-4)$ (by [5] Prop. 2.3 and par. 3.2) hence a global section of $\omega_Z(4)$ generating everywhere this sheaf defines an extension:

$$0 \rightarrow \mathcal{O}_P \rightarrow F \rightarrow \mathcal{I}_Z \rightarrow 0$$

with $F$ a properly semistable rank 2 vector bundle with $c_1(F) = 0$ and $c_2(F) = 3$. Notice also that, conversely, if $F$ is a properly semistable rank 2 vector bundle with these Chern classes then the zero locus $Z$ of the unique nonzero global section of $F$ is a triple structure on a line $L$ with $l = 1$ and $m = 0$. Indeed, $\text{deg } Z = 3$ and $\omega_Z \simeq \mathcal{O}_Z(-4)$. $Z$ cannot be reducible because, in that case, one would have $Z = X \cup L$, with $L$ a line and $X$ a curve of degree 2 such that the scheme $D = X \cap L$ is 0-dimensional (or empty) and in this case it is well-known that $\omega_Z \mid L \simeq \omega_L \otimes \mathcal{O}_L(D)$ and this would contradict the fact that $\omega_Z \simeq \mathcal{O}_Z(-4)$. It remains that $Z$ is irreducible. $Z$ cannot be reduced hence $Z$ is a triple
structure on a line $L$. Since $Z$ is l.c.i. it follows that $m = 0$ (by [5, Par. 3.3]) and the condition $\omega_Z \cong O_Z(-4)$ implies, now, that $l = 1$ (by [5, Prop. 2.3]).

**Subcase 5.4.** $Z$ as $(X)$ in Prop. 2.10 with $i = 0$.

In this subcase $\deg Z = 2$ and $Z$ is either the union of two disjoint lines or a double structure on a line $L$ with $l = 0$ (notation as in Subsection A.5). It follows that $\chi(O_Z) = 2$ hence $c_1 = 4$ and $c_2 = 5$ hence $c_1(E(-2)) = 0$ and $c_2(E(-2)) = 1$. Since, by hypothesis, $H^0(E(-2)) = 0$, $E(-2)$ is stable. It is, actually, a nullcorrelation bundle.

**Subcase 5.5.** $Z$ as $(X)$ in Prop. 2.10 with $i = 1$.

In this subcase, $\deg Z = 2$ and $Z$ is a double structure on a line $L$ with $l = 1$. It follows that $\chi(O_Z) = 3$ hence $c_1 = 5$ and $c_2 = 8$. One gets that $c_1(E(-3)) = -1$ and $c_2(E(-3)) = 2$. The hypothesis $H^0(E(-3)) = 0$ implies that $E(-3)$ is stable. These bundles were studied by Hartshorne and Sols [18] and, independently, by Manolache [23].

**Case 6.** $Z$ is as $(Y)$ in Prop. 2.9(iii).

In this case, by Lemma 8.1 one has an exact sequence:

$$0 \rightarrow O_\mathbb{P}(1) \xrightarrow{(\phi_+^2, d_+^1)} O_\mathbb{P}(c_1 - 3) \oplus F \rightarrow E \rightarrow 0$$

where $F$ is defined by an exact sequence:

$$0 \rightarrow F \rightarrow 3O_\mathbb{P}(2) \oplus O_\mathbb{P}(1) \xrightarrow{d_+^0} O_\mathbb{P}(3) \rightarrow 0.$$  

Since any global section of $F(-1)$ vanishes along a line in $\mathbb{P}^3$, it follows that $c_1 = 4$ and that $\phi_+$ is an isomorphism hence that $E \cong F$. Now, up to a linear change of coordinates, one can assume that the first three components of $d_+^0$ are $x_0$, $x_1$, $x_2$ and then, modulo an automorphism of $3O_\mathbb{P}(2) \oplus O_\mathbb{P}(1)$ invariating $3O_\mathbb{P}(2)$, one can assume that the fourth component of $d_+^0$ is $x_3^2$.

**Case 7.** $Z$ is as in Prop. 2.8.

In this case, an argument similar to that used in Case 6, shows that $c_1 = 4$ and that one has an exact sequence:

$$0 \rightarrow E \rightarrow 2O_\mathbb{P}(2) \oplus 3O_\mathbb{P}(1) \xrightarrow{d_+^0} O_\mathbb{P}(3) \rightarrow 0.$$  

Since, by Lemma 3.2 the kernel of an epimorphism $O_\mathbb{P}(2) \oplus 3O_\mathbb{P}(1) \rightarrow O_\mathbb{P}(3)$ is not globally generated, the first two components of $d_+^0$ must be linearly independent. Up to a linear change of coordinates, one can assume that they are $x_0$ and $x_1$. Let $L \subset \mathbb{P}^3$ be the line of equations $x_0 = x_1 = 0$. Since $E|L$ is globally generated, it follows that $H^0((d_+^0|L)(-1))$ induces an isomorphism $H^0(3O_L) \rightarrow H^0(O_L(2))$ (because, otherwise, $E|L \cong 2O_L(2) \oplus O_L(1) \oplus O_L(-1)$). Now, modulo an automorphism of $2O_\mathbb{P}(2) \oplus 3O_\mathbb{P}(1)$ invariating $2O_\mathbb{P}(2)$, one can assume that the last three components of $d_+^0$ are $x_2^2$, $x_2x_3$ and $x_3^2$. 


Case 8.  \(Z\) is as in Prop. \[1.2\](iv).

In this case, an argument similar to that used in Case 6, shows that \(c_1 = 4\) and that \(E\) is the cohomology of a monad:

\[
0 \rightarrow \mathcal{O}_P(-1) \xrightarrow{u} 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \oplus 4\mathcal{O}_P \xrightarrow{(d^0_+, v)} \mathcal{O}_P(3) \rightarrow 0
\]

where \(u : \mathcal{O}_P(-1) \rightarrow 2\mathcal{O}_P\) is defined by \(x_0, \ldots, x_3\) and \(d^0_+ : 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \rightarrow \mathcal{O}_P(3)\) is an epimorphism. Up to a linear change of coordinates and an automorphism of \(2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1)\), one can assume that \(d^0_+\) is defined by \(x_0, x_1, x_2^2, x_3^2\). Modulo an automorphism of \(4\mathcal{O}_P\), one can continue to assume that \(u\) is defined by \(x_0, \ldots, x_3\). Since \(H^0(d^0_+)\) is obviously surjective, \(v : 4\mathcal{O}_P \rightarrow \mathcal{O}_P(3)\) factorizes as:

\[4\mathcal{O}_P \rightarrow 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \xrightarrow{d^0_+} \mathcal{O}_P(3)\]

One deduces that, modulo an automorphism of \(2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \oplus 4\mathcal{O}_P\) invariating \(2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1)\), one can assume that \(v = 0\). \(E\) occurs, now, as the cohomology of a monad:

\[
0 \rightarrow \mathcal{O}_P(-1) \xrightarrow{u} 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \oplus 4\mathcal{O}_P \xrightarrow{(p, 0)} \mathcal{O}_P(3) \rightarrow 0
\]

where \(u\) is defined by \(x_0, \ldots, x_3\) and \(p = d^0_+\) by \(x_0, x_1, x_2^2, x_3^2\). Let \(K = \text{Ker } p\) and let \(s : \mathcal{O}_P(-1) \rightarrow K\) be the morphism induced by \(0, 0, -x_3^2, x_2^2 : \mathcal{O}_P(-1) \rightarrow 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1)\). Using the Koszul complex associated to \(x_0, x_1, x_2^2, x_3^2\), one sees that there exist a constant \(c \in k\) and a morphism \(\sigma : 4\mathcal{O}_P \rightarrow K\) such that \(s' = cs + \sigma \circ u\). It follows that the above monad is isomorphic to the monad:

\[
0 \rightarrow \mathcal{O}_P(-1) \xrightarrow{u} 2\mathcal{O}_P(2) \oplus 2\mathcal{O}_P(1) \oplus 4\mathcal{O}_P \xrightarrow{(p, 0)} \mathcal{O}_P(3) \rightarrow 0.
\]

If \(c = 0\) then one would get \(E \simeq K \oplus T_P(-1)\) which is not possible because, by Lemma 3.2, \(K\) is not globally generated. It remains that \(c \neq 0\) hence one can assume, actually, that \(c = 1\). \(\square\)

We extend, next, Theorem 0.1 to higher dimensional projective spaces. At a certain point of this extension we shall need the following result of Barth and Elencwajg [7 Thm. 4.2], for which we provide a different proof, based on the results of Mohan Kumar, Peterson and Rao [20].

**Theorem 3.3** (Barth, Elencwajg). There exists no stable rank 2 vector bundle \(E\) on \(\mathbb{P}^4\) with Chern classes \(c_1 = 0\) and \(c_2 = 3\).

**Proof.** Assume that such a vector bundle \(E\) exists. According to Barth [6], the restriction \(E_H\) of \(E\) to a general hyperplane \(H \subset \mathbb{P}^4\) is stable. If \(F\) is a stable rank 2 vector bundle on \(\mathbb{P}^3\) with \(c_1(F) = 0\) and \(c_2(F) = 3\) then the possible spectra (see [17 Sect. 7]) of \(F\) are \((0, 0, 0)\) and \((-1, 0, 1)\). In the former case \(H^1(F(l)) = 0\) for \(l \leq -2\) and \(h^1(F(-1)) = 3\),
while in the latter case $\text{H}^1(F(l)) = 0$ for $l \leq -3$, $\text{h}^1(F(-2)) = 1$ and $\text{h}^1(F(-1)) = 3$. Moreover, in the later case, $F$ is the cohomology of a monad of the form:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0
\]

(see Ein [10, Thm. 3.3(a)]) hence, in particular, the graded module $\text{H}^1(F)$ is generated by $\text{H}^1(F(-2))$. If $F$ is properly semistable then, as we saw in Subcase 5.3 of the proof of Thm. 0.1, $F$ can be realized as an extension:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow F \longrightarrow \mathcal{I}_Z \longrightarrow 0,
\]

where $Z$ is a triple structure on a line $L \subset \mathbb{P}^3$ such that $\mathcal{O}_Z \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$ as an $\mathcal{O}_L$-algebra. One deduces that in this case $\text{H}^1(F(l)) = 0$ for $l \leq -3$, $\text{h}^1(F(-2)) = 1$ and $\text{h}^1(F(-1)) = 3$.

**Claim 1.** $\text{H}^1(E(l)) = 0$ for $l \leq -2$.

Indeed, consider a hyperplane $H \subset \mathbb{P}^4$ such that $E_H$ is stable. By [3, Lemma 1.16(a)], $\text{H}^1(E(l)) = 0$ for $l \leq -3$ and $\text{H}^1(E(-2))$ injects into $\text{H}^1(E_H(-2))$. If $E_H$ has spectrum $(0,0,0)$ then $\text{H}^1(E_H(-2)) = 0$ hence $\text{H}^1(E(-2)) = 0$. If the spectrum of $E_H$ is $(-1,0,1)$ and $\text{H}^1(E(-2)) \neq 0$ then $\text{H}^1(E(-2)) \twoheadrightarrow \text{H}^1(E_H(-2))$. Since, as we recalled above, the module $\text{H}^1(E_H)$ is generated by $\text{H}^1(E_H(-2))$ it follows that the map $\text{H}^1(E(l)) \rightarrow \text{H}^1(E_H(l))$ is surjective for any $l \in \mathbb{Z}$ hence, by [3, Lemma 1.16(b)], $\text{H}^1_2(E) = 0$. But this contradicts Mohan Kumar et al. [20, Thm. 1] because $E$ is not decomposable.

**Claim 2.** $\text{H}^2(E(l)) = 0$ for $l \geq -1$.

Indeed, since $\text{H}^0(E(-1)) = 0$ and $\text{H}^1(E(-2)) = 0$, it follows that for every hyperplane $H \subset \mathbb{P}^4$ one has $\text{H}^0(E_H(-1)) = 0$, i.e., $E_H$ is semistable. One deduces, using Serre duality and the fact that $E \simeq E^\vee$, that $\text{H}^2(E_H(l)) = 0$ for $l \geq -1$, for any hyperplane $H \subset \mathbb{P}^4$. Moreover, $\text{h}^1(E_H(-1)) = 3$. It follows that for any $0 \neq h \in \text{H}^0(\mathcal{O}_{\mathbb{P}^4}(1))$, the multiplication by $h : \text{H}^2(E(-2)) \rightarrow \text{H}^2(E(-1))$ is surjective and that $\text{h}^2(E(-2)) \leq \text{h}^2(E(-1)) + 3$. Assume that $\text{H}^2(E(-1)) \neq 0$. Applying the Bilinear map lemma [17, Lemma 5.1] to $\text{H}^2(E(-1))^\vee \times \text{H}^0(\mathcal{O}_{\mathbb{P}^4}(1)) \rightarrow \text{H}^2(E(-2))^\vee$ one deduces that $\text{h}^2(E(-2)) \geq \text{h}^2(E(-1)) + 4$, which contradicts a previously established inequality. It thus remains that $\text{H}^2(E(-1)) = 0$. Since for any hyperplane $H \subset \mathbb{P}^4$ one has $\text{H}^2(E_H(l)) = 0$ for $l \geq -1$ one gets the claim.

**Claim 3.** $\text{H}^2_2(E) \simeq k(3) \oplus k(2)$.

Indeed, by Serre duality and the fact that $E \simeq E^\vee$, $\text{H}^2(E(l)) = 0$ for $l \leq -4$. Using Riemann-Roch (see, for example, [3, Thm. 7.3]), one gets $\text{h}^2(E(-2)) = \chi(E(-2)) = 1$. By Serre duality again, $\text{h}^2(E(-3)) = \text{h}^2(E(-2)) = 1$. It remains to show that, for every $h \in \text{H}^0(\mathcal{O}_{\mathbb{P}^4}(1))$, the multiplication by $h : \text{H}^2(E(-3)) \rightarrow \text{H}^2(E(-2))$ is the zero map. Recall that the cup product:

\[
- \cup - : \text{H}^2(E(-3)) \times \text{H}^2(E(-3)) \longrightarrow \text{H}^4((\Lambda^{2}E)(-6)) \simeq \text{H}^4(\mathcal{O}_{\mathbb{P}^4}(-6))
\]
is skew-symmetric (because \(- \wedge - : E \times E \to \wedge^2 E\) is). If \(0 \neq \xi \in H^2(E(-3))\) then:

\[ h\xi \cup \xi = h(\xi \cup \xi) = 0. \]

Since, on the other hand, the cup product \(H^2(E(-2)) \times H^2(E(-3)) \to H^4(\wedge^2 E(-5)) \simeq H^4(\mathcal{O}_{\mathbb{P}^4}(-5)) \simeq k\) is a perfect pairing (by Serre duality), it follows that \(h\xi = 0\). Claim 3 is proven.

At this point one can invoke Mohan Kumar et al. [20, Thm. 2] and deduce that the bundle \(E\) cannot exist. However, since our situation is very concrete, we shall also provide a slightly different argument. One has, by Riemann-Roch, \(h^1(E(-1)) = -\chi(E(-1)) = 2\) and \(h^1(E) = -\chi(E) = 6\). Since \(H^2(E(-1)) = 0\), \(H^3(E(-2)) \simeq H^1(E(-3))^\vee = 0\) and \(H^4(E(-3)) \simeq H^0(E(-2))^\vee = 0\) it follows from the Castelnuovo-Mumford lemma (see, for example, [21, Lemma 1.21]) that the graded module \(H^4_1(E)\) is generated in degrees \(\leq 0\). By Barth’s restriction theorem, there exists a 2-plane \(\Pi \subset \mathbb{P}^4\) such that \(E_\Pi\) is stable, i.e., such that \(H^0(E_\Pi) = 0\). It follows that, for every hyperplane \(H \supset \Pi\), \(H^0(E_H) = 0\). If \(h = 0\) is the equation of such a hyperplane, then the multiplication by \(h : H^1(E(-1)) \to H^1(E)\) is injective. Applying the Bilinear map lemma [17, Lemma 5.1] to \(\mu : H^1(E(-1)) \otimes H^0(\mathcal{O}_\Pi(1)) \to H^1(E)\) one gets that \(\dim \operatorname{Im} \mu \geq 3\). One deduces that the graded module \(H^4_1(E)\) has two minimal generators in degree \(-1\) and at most three minimal generators in degree 0, whence a surjection \(2S(1) \oplus 3S \to H^4_1(E)\). Since \(E^\vee \simeq E\), one deduces a surjection \(2S(1) \oplus 3S \to H^4_1(E^\vee)\). Using Horrocks’ method of “killing cohomology”, one gets that \(E\) is the cohomology of a monad of the form:

\[ 0 \to 3\mathcal{O}_{\mathbb{P}^4} \oplus 2\mathcal{O}_{\mathbb{P}^4}(-1) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} 2\mathcal{O}_{\mathbb{P}^4}(1) \oplus 3\mathcal{O}_{\mathbb{P}^4} \to 0, \]

where \(\mathcal{P}\) is a vector bundle of rank 12, with \(H^1_1(\mathcal{P}) = 0\), \(H^4_1(\mathcal{P}) = 0\) and \(H^2_1(\mathcal{P}) \simeq H^2_1(E)\). It follows that \(\mathcal{P} \simeq \Omega^2_{\mathbb{P}^4}(3) \oplus \Omega^2_{\mathbb{P}^4}(2)\), hence \(E\) is the cohomology of a monad of the form:

\[ 0 \to 3\mathcal{O}_{\mathbb{P}^4} \oplus 2\mathcal{O}_{\mathbb{P}^4}(-1) \xrightarrow{\alpha} \Omega^2_{\mathbb{P}^4}(3) \oplus \Omega^2_{\mathbb{P}^4}(2) \xrightarrow{\beta} 2\mathcal{O}_{\mathbb{P}^4}(1) \oplus 3\mathcal{O}_{\mathbb{P}^4} \to 0. \]

Claim 4. Such a monad cannot exist.

Indeed, \(\alpha\) maps \(3\mathcal{O}\) into \(\Omega^2(3)\) and \(\beta\) maps \(\Omega^2(3)\) into \(2\mathcal{O}(1)\). Consequently, the monad has a subcomplex of the form:

\[ 0 \to \mathcal{E} \xrightarrow{\alpha'} \Omega^2(3) \xrightarrow{\beta'} 2\mathcal{O}(1) \to 0. \]

Let \(F := \operatorname{Coker} \alpha'(-1)\). \(F\) is a Tango bundle on \(\mathbb{P}^4\). It is well-known that \(H^0(F^\vee) = 0\).

The usual argument is the following one: since \(\operatorname{rk} F = 3\) and \(c_1(F) = 0\) one has \(F^\vee \simeq \wedge^2 F\).

The second exterior power of the resolution:

\[ 0 \to \mathcal{E}(-3) \to 5\mathcal{E}(-2) \to 7\mathcal{E}(-1) \to F \to 0 \]
of $F$ is a resolution of $\bigwedge^2 F$ of the form:

\[
0 \to 5\mathcal{O}(-5) \to \bigoplus_{7\mathcal{O}(-4)} \to 35\mathcal{O}(-3) \to \bigwedge^2(7\mathcal{O}(-1)) \to \bigwedge^2 F \to 0
\]

from which one gets that $H^0(\bigwedge^2 F) = 0$. Since $H^0(F^\vee) = 0$ it follows that there is no non-zero morphism $\beta : \Omega^2(3) \to 2\mathcal{O}(1)$ such that $\beta' \circ \alpha' = 0$. Since there is no epimorphism $\Omega^2(2) \to 2\mathcal{O}(1) \oplus 3\mathcal{O}$, a monad of the above form cannot exist. \qed

Ballico and Chiantini [4, Prop. 3] showed, moreover, that there is no properly semistable rank 2 vector bundle on $\mathbb{P}^4$ with Chern classes $c_1 = 0$ and $c_2 = 3$. The key point of their proof is the following lemma, for which we provide a different argument.

**Lemma 3.4** (Ballico, Chiantini). Let $Y$ be a locally complete intersection subscheme of $\mathbb{P}^4$, of degree 3, supported on a plane $\Pi \subset \mathbb{P}^4$. Then $Y$ is a complete intersection of type (1,3).

**Proof.** Let $S = k[x_0, x_1, x_2, x, y]$ be the projective coordinate ring of $\mathbb{P}^4$ and assume that $\Pi$ is the plane of equations $x = y = 0$. Let $L \subset \mathbb{P}^4$ be the (complementary) line of equations $x_0 = x_1 = x_2 = 0$ and let $\pi : \mathbb{P}^4 \setminus L \to \Pi$ be the linear projection. If $\mathcal{F}$ is a locally CM coherent $\mathcal{O}_\mathbb{P}$-module supported on $\Pi$ then $F := \pi_*\mathcal{F}$ is a locally free $\mathcal{O}_\Pi$-module. The reduced stalk $F(z)$ of $F$ at a point $z \in \Pi$ can be described geometrically as follows: let $\Pi'$ be the plane spanned by $z$ and $L$. If $\Pi'$ has equations $\ell' = \ell'' = 0$ with $\ell', \ell'' \in k[x_0, x_1, x_2]$, then $\ell', \ell''$ generate the maximal ideal $m_{\Pi,z}$ of $\mathcal{O}_{\Pi,z}$ hence:

\[
F(z) := F_z/m_{\Pi,z}F_z \simeq \mathcal{F}_z/((\ell', \ell''))\mathcal{F}_z \simeq \mathcal{F} \otimes_{\mathcal{O}_\Pi} \mathcal{O}_{\Pi}.
\]

Notice also that if $\ell \in k[x_0, x_1, x_2]$ has the property that $\ell(z) \neq 0$ then $\ell, \ell', \ell''$ is a $k$-basis of $k[x_0, x_1, x_2]$, $\ell, x, y$ is a $k$-basis of $H^0(\mathcal{O}_\Pi(1))$ and $z$ is the point of $\Pi'$ of equations $x = y = 0$.

Let us, now, prove the lemma. Consider the rank-2 locally free $\mathcal{O}_\Pi$-module $E := \pi_*(\mathcal{I}_\Pi/\mathcal{I}_Y)$ and the two morphisms $\xi, \eta : E \to E(1)$ obtained by applying $\pi_*$ to the morphisms $x : - , y : - : \mathcal{I}_\Pi/\mathcal{I}_Y \to (\mathcal{I}_\Pi/\mathcal{I}_Y)(1)$, respectively. Let $z$ be a point of $\Pi$ and let $\Pi'$ be as above. Since any locally complete intersection subscheme of $\Pi'$ of degree 3, supported on $z$, is a complete intersection of type (1,3) in $\Pi'$ it follows that there exist linearly independent linear forms $x', y' \in k[x, y]_1$ such that $\Pi' \cap Y$ is the subscheme of $\Pi'$ whose homogeneous ideal is generated by $x'^d$ and $y'$. Working with ideals of the ring $\mathcal{O}_{\Pi,z}$, the observation from the beginning of this proof shows that:

\[
E(z) \simeq \frac{(x, y)}{(x^3, y')}, \quad \text{Im } \xi(z) + \text{Im } \eta(z) \simeq \frac{(x, y)^2 + (x^3, y')}{(x^3, y')} \simeq \frac{(x^2, y')}{(x^3, y')} \simeq k.
\]

It follows that $\text{Im } \xi + \text{Im } \eta \simeq ((\mathcal{I}_\Pi^2 + \mathcal{I}_Y)/\mathcal{I}_Y)(1)$ is a line subbundle of $E(1)$ hence:

\[
\frac{\mathcal{I}_\Pi^2 + \mathcal{I}_Y}{\mathcal{I}_Y} \simeq \mathcal{O}_\Pi(a), \quad \frac{\mathcal{I}_\Pi}{\mathcal{I}_\Pi^2 + \mathcal{I}_Y} \simeq \mathcal{O}_\Pi(b) \quad \text{and } E \simeq \mathcal{O}_\Pi(a) \oplus \mathcal{O}_\Pi(b)
\]
for some \(a, b \in \mathbb{Z}\). Since we have epimorphisms:

\[
2\mathcal{O}_\mathcal{H}(-1) \simeq \mathcal{F}_\mathcal{H} \rightarrow \mathcal{F}_\mathcal{H}^2 + \mathcal{J}_\mathcal{Y} \quad \text{and} \quad (x, y) : 2\mathcal{F}_\mathcal{H}^2 + \mathcal{J}_\mathcal{Y} \rightarrow \mathcal{J}_\mathcal{H} + \mathcal{J}_\mathcal{Y}(1)
\]

it follows that \(b = -1\) and \(a + 1 = b\), hence \(a = -2\) and \(E \simeq \mathcal{O}_\mathcal{H}(-2) \oplus \mathcal{O}_\mathcal{H}(-1)\). Using the exact sequence \(0 \rightarrow \mathcal{J}_\mathcal{Y} \rightarrow \mathcal{F}_\mathcal{H} \rightarrow \mathcal{F}_\mathcal{H}/\mathcal{J}_\mathcal{Y} \rightarrow 0\) one deduces, now, that \(H^0(\mathcal{J}_\mathcal{Y}(1)) \neq 0\), whence the conclusion of the lemma.

\[\square\]

**Proposition 3.5.** Let \(E\) be a globally generated vector bundle on \(\mathbb{P}^n\), \(n \geq 4\), with \(c_1 \geq 4\) and such that \(H^i(\mathcal{E}^\vee) = 0\), \(i = 0, 1\). Let \(\Pi \subset \mathbb{P}^n\) be a fixed 3-plane. Assume that \(H^0(E(-c_1 + 2)) = 0\) and that \(H^0(E(-c_1 + 3)) \neq 0\). Then one of the following holds:

(i) \(E \simeq \mathcal{O}_\mathcal{P}(c_1 - 3) \oplus E'\) where \(E'\) is a globally generated vector bundle with \(c_1(E') = 3\);

(ii) \(n = 4, c_1 = 4\) and \(E \simeq T_\mathcal{P}^4(-1) \oplus \Omega_{\mathcal{P}^4}(2)\);

(iii) \(n = 5, c_1 = 4\) and \(E \simeq \Omega_{\mathcal{P}^5}(2)\);

(iv) \(n = 4, c_1 = 4\) and, up to a linear change of coordinates, denoting by \((C_p, \delta_p)_{p \geq 0}\) the Koszul complex associated to the epimorphism \(\delta_1 : 4\mathcal{O}_\mathcal{P}^4(-1) \oplus \mathcal{O}_\mathcal{P}^4(-2) \rightarrow \mathcal{O}_\mathcal{P}^4\) defined by \(x_0, \ldots, x_3, x_4^2\), one has exact sequences:

\[
0 \longrightarrow \mathcal{O}_\mathcal{P}^4(-2) \xrightarrow{\delta_4} \mathcal{O}_\mathcal{P}^4(1) \oplus 6\mathcal{O}_\mathcal{P}^4 \longrightarrow E' \longrightarrow 0
\]

The bundle from item (iv) of Prop. 3.5 appeared for the first time, constructed differently, in the paper of Abo, Decker and Sasakura [1].

**Proof.** It follows from [3, Prop. 2.11] that \(H^0(E_\mathcal{H}(-c_1 + 2)) = 0\). One has \(E_\mathcal{H} \simeq \mathcal{G} \oplus t\mathcal{O}_\mathcal{H}\), with \(\mathcal{G}\) defined by an exact sequence \(0 \rightarrow s\mathcal{O}_\mathcal{H} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0\) where \(\mathcal{F}\) is a globally generated vector bundle on \(\Pi\) such that \(H^i(F^\vee) = 0\), \(i = 0, 1\) (see, for example, [3, Sect. 1]). It follows that \(\mathcal{F}\) is one of the bundles described in the statement of Theorem 0.1.

**Case 1.** \(F \simeq \mathcal{O}_\mathcal{H}(c_1 - 3) \oplus F'\) with \(c_1(F') = 3\).

As we said in the Introduction, the bundles \(F'\) were classified by Anghel and Manolache [2] and, independently, by Sierra and Ugaglia [33]. A concise description of this classification can be found in [3, Thm. 0.1]. Except when \(F'\) contains \(\Omega_{\mathcal{H}}(2)\) as a direct summand, [3, Lemma 1.18] implies that \(\mathcal{E}\) is as in item (i) of the statement of the proposition.

If \(F' \simeq \mathcal{O}_\mathcal{H}(1) \oplus \Omega_{\mathcal{H}}(2)\) then [3, Lemma 1.19] (and [3, Remark 1.20(c)]) implies that \(n = 4\) and \(E \simeq \mathcal{O}_{\mathcal{P}^4}(c_1 - 3) \oplus \Omega_{\mathcal{P}^4}(2)\) or \(c_1 = 4, n = 5\) and \(E \simeq \Omega_{\mathcal{P}^5}(2)\).

If \(F' \simeq T_\mathcal{H}(-1) \oplus \Omega_{\mathcal{H}}(2)\) then [3, Lemma 1.19] (and [3, Remark 1.20(c)]) implies that \(n = 4\) and \(E \simeq \mathcal{O}_{\mathcal{P}^4}(c_1 - 3) \oplus \Omega_{\mathcal{P}^4}(3)\) or \(c_1 = 4, n = 4\) and \(E \simeq T_\mathcal{P}^4(-1) \oplus \Omega_{\mathcal{P}^4}(2)\).

**Case 2.** \(F\) as \(E\) in Theorem 0.1(ii).
This case cannot occur. Indeed, assume the contrary. Then [3, Cor. 1.5] would imply that there exists a rank 2 vector bundle on \( \mathbb{P}^4 \) with Chern classes \( c_1 = 0 \) and \( c_2 = 1 \) or with Chern classes \( c_1 = 0 \) and \( c_2 = 2 \). But this would contradict Schwarzenberger’s congruence asserting that the Chern classes of a coherent sheaf on \( \mathbb{P}^4 \) satisfy the relation:

\[
(2c_1 + 3)(c_3 - c_1c_2) + c_2^2 + c_2 \equiv 2c_4 \quad (\text{mod } 12)
\]

(see, for example, [3, Cor. 7.4]).

Case 3. \( F \) as \( E \) in Theorem 0.1(iii).

This case cannot occur. Indeed, in order to show this, one can assume that \( n = 4 \). Since \( H^1(F^\vee) = 0 \), [3, Lemma 1.17(a)] implies that \( t = 0 \). Moreover, \( s = 0 \) because \( c_3(F) = 2 \neq 0 \). Now, applying [3, Lemma 1.16(a)] and [3, Lemma 1.14(b)] to \( F^\vee \) one derives the existence of an exact sequence

\[
0 \rightarrow E \rightarrow 3O_{\mathbb{P}^4}(2) \oplus O_{\mathbb{P}^4}(1) \rightarrow O_{\mathbb{P}^4}(3) \rightarrow 0.
\]

But this is not possible because there is no epimorphism \( 3O_{\mathbb{P}^4}(2) \oplus O_{\mathbb{P}^4}(1) \rightarrow O_{\mathbb{P}^4}(3) \).

Case 4. \( F \) as \( E \) in Theorem 0.1(iv).

This case cannot occur. Indeed, in order to prove this, one can assume that \( n = 4 \). It follows, as in Case 3, that \( t = 0 \). Moreover, \( F \) has rank 4 and \( c_3(F) = 4 \neq 0 \) hence \( s \leq 1 \).

If \( s = 0 \) then, as in Case 3, there exists an exact sequence:

\[
0 \rightarrow E \rightarrow 2O_{\mathbb{P}^4}(2) \oplus 3O_{\mathbb{P}^4}(1) \xrightarrow{\varepsilon} O_{\mathbb{P}^4}(3) \rightarrow 0.
\]

Let \( C \subset \mathbb{P}^4 \) be the complete intersection 1-dimensional subscheme of \( \mathbb{P}^4 \) with the property that \( \mathcal{I}_C(3) \) is the image of the restriction \( 2O_{\mathbb{P}^4}(2) \oplus O_{\mathbb{P}^4}(1) \rightarrow O_{\mathbb{P}^4}(3) \) of \( \varepsilon \). It follows that \( E|C \simeq 2O_C(2) \oplus O_C(1) \oplus O_C(-1) \) hence \( E|C \) is not globally generated, a contradiction.

If \( s = 1 \) then \( E \) is a rank 3 vector bundle on \( \mathbb{P}^4 \) with Chern classes \( c_1 = 4 \), \( c_2 = 7 \), \( c_3 = 4 \). But this would contradict Schwarzenberger’s congruence.

Case 5. \( F \) as \( E \) in Theorem 0.1(v).

In this case, the proof of [3, Prop. 7.11] (more precisely, the Cases 7 and 8 of that proof) shows that \( n = 4 \) and that \( E \) is as in item (iv) of the statement of our proposition.

Case 6. \( F \) as \( E \) in Theorem 0.1(vi).

This case cannot occur: one uses the same kind of argument as in Case 2.

Case 7. \( F \) as \( E \) in Theorem 0.1(vii).

This case cannot occur. Indeed, assume the contrary. Choose a 4-plane \( \mathbb{P}^4 \subset \mathbb{P}^n \) containing \( \Pi \). The last part of [3, Cor. 1.5] would imply that there exists a rank 2 vector bundle \( E' \) on \( \mathbb{P}^4 \) with Chern classes \( c_1(E') = 0 \) and \( c_2(E') = 3 \) such that \( E'|\Pi \) is properly semistable. The result of Barth and Elencwajg recalled above (Theorem 3.3) implies that \( E' \) must be properly semistable. But, according to Ballico and Chiantini [4, Prop. 3], this is not
possible, either. Their argument runs as follows: one must have an exact sequence:

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^4} \rightarrow E' \rightarrow \mathcal{I}_Y \rightarrow 0 \]

where \( Y \) is a locally complete intersection closed subscheme of \( \mathbb{P}^4 \), of codimension 2 and degree 3. As we noticed in Subcase 5.3 of the above proof of Theorem 0.1, if \( H \subset \mathbb{P}^4 \) is a hyperplane cutting \( Y \) properly then the subscheme \( H \cap Y \) of \( H \simeq \mathbb{P}^3 \) must be supported on a line. One deduces that \( Y \) is supported on a plane in \( \mathbb{P}^4 \). Lemma 3.4 implies, now, that \( Y \) is a complete intersection of type (1, 3), hence \( E' \) is a direct sum of two line bundles. But this is not possible because the system of equations \( a + b = 0 \) and \( ab = 3 \) has no integer solutions. □

Remark 3.6. As we recalled in Case 7 of the proof of Prop. 3.5, Ballico and Chiantini used their Lemma 3.4 to show, in connection with Thm. 3.3 of Barth and Ellencwajg, that there exists no rank 2 vector bundle \( E \) on \( \mathbb{P}^4 \), with Chern classes \( c_1(E) = 0, c_2(E) = 3 \) such that \( H^0(E(-1)) = 0 \). One can push this kind of results one step further by showing that there exists no rank 2 vector bundle \( E \) on \( \mathbb{P}^4 \) with those Chern classes such that \( H^0(E(-2)) = 0 \).

Indeed, assume that such a bundle exists. Then it can be realized as an extension:

\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(1) \rightarrow E \rightarrow \mathcal{I}_Y(-1) \rightarrow 0 \]

where \( Y \) is a l.c.i. closed subscheme of \( \mathbb{P}^4 \), of pure codimension 2, of degree 4, with \( \omega_Y \simeq \mathcal{O}_Y(-7) \). Let \( H \subset \mathbb{P}^4 \) be a hyperplane cutting properly \( Y_{\text{red}} \) and \( X := H \cap Y \). One has an exact sequence:

\[ 0 \rightarrow \mathcal{O}_H(1) \rightarrow E_H \rightarrow \mathcal{I}_{X,H}(-1) \rightarrow 0. \]

\( X \) is a l.c.i. curve in \( H \simeq \mathbb{P}^3 \), of degree 4, with \( \omega_X \simeq \mathcal{O}_X(-6) \). The last condition implies that \( X \) cannot be reduced and irreducible. Moreover, \( X \) cannot have a reduced component. Indeed, if this would be the case then \( X \) would have as a component a line or a (nonsingular) conic and the argument used in Subcase 5.3 of the proof of Thm. 0.1 would show that this contradicts the condition \( \omega_X \simeq \mathcal{O}_X(-6) \). It thus remains that \( X \) is either the union of two double lines or a double conic or a quadruple line. It follows that one of the following holds:

(i) \( Y_{\text{red}} \) is the union of two planes;
(ii) \( Y_{\text{red}} \) is a double structure on a nonsingular quadric surface;
(iii) \( Y_{\text{red}} \) is a double structure on a quadratic cone;
(iv) \( Y_{\text{red}} \) is a plane.

In case (i) the two planes should intersect along a line (see, for example, [12, Thm. 18.12]). Consequently, in the first three cases, \( Y_{\text{red}} \) is a reduced complete intersection of type (1, 2) in \( \mathbb{P}^4 \) and \( Y \) is a double structure on it. Let \( Y' \) be the residual of \( Y_{\text{red}} \) with respect to the l.c.i. scheme \( Y \). One has, by definition:

\[ \mathcal{I}_{Y'}/\mathcal{I}_Y \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{Y_{\text{red}}}, \mathcal{O}_Y) \simeq \omega_{Y_{\text{red}}} \otimes \omega_Y^{-1} \simeq \mathcal{O}_{Y_{\text{red}}}(5). \]
Using the exact sequence $0 \to \mathcal{O}_{Y_{\text{red}}}(-5) \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0$ and Hilbert polynomials one deduces that $\deg Y' = \deg Y - \deg Y_{\text{red}} = 2$. Since $Y_{\text{red}} \subseteq Y'$ and since $Y'$ is locally CM it follows that $Y' = Y_{\text{red}}$. Since there is no epimorphism:

$$\mathcal{J}_{Y_{\text{red}}} / \mathcal{J}_{Y_{\text{red}}}^2 \simeq \mathcal{O}_{Y_{\text{red}}}(-1) \oplus \mathcal{O}_{Y_{\text{red}}}(-2) \longrightarrow \mathcal{O}_{Y_{\text{red}}}(5)$$

none of the first three cases can occur.

In case (iv) the results of Manolache [25] show that $Y$ is a complete intersection which implies that $E$ is a direct sum of line bundles. But this is not possible because $E$ has Chern classes $c_1 = 0$ and $c_2 = 3$.

**Lemma 3.7.** Let $E$ be a globally generated vector bundle on $\mathbb{P}^4$ with Chern classes $c_1 = 4$ and $c_2 \leq 8$. Then there exists a hyperplane $H \subset \mathbb{P}^4$ such that $H^0(E_H(-1)) \neq 0$.

**Proof.** Assume the contrary, namely that $H^0(E_H(-1)) = 0$ for any hyperplane $H \subset \mathbb{P}^4$. For each hyperplane $H \subset \mathbb{P}^4$ one has $E_H \simeq G \oplus t\mathcal{O}_H$ with $G$ defined by an exact sequence $0 \to s\mathcal{O}_H \to F \to G \to 0$, where $F$ is a globally generated vector bundle on $H \simeq \mathbb{P}^3$ such that $H^0(F^\vee) = 0, i = 0, 1$. Moreover, one has an exact sequence:

$$0 \longrightarrow (r - 2)\mathcal{O}_H \longrightarrow F \longrightarrow \mathcal{F}(2) \longrightarrow 0$$

where $r = \text{rk} F$ and with $\mathcal{F}$ a rank 2 reflexive sheaf on $H$ with $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = c_2 - 4 \leq 4$ and $c_2(\mathcal{F}) = c_3(\mathcal{F}) = c_3(E) = c_3$. The condition $H^0(E_H(-1)) = 0$ is equivalent to $H^0(\mathcal{F}(1)) = 0$. In particular, $\mathcal{F}$ is stable (that is, $H^0(\mathcal{F}) = 0$).

By [17, Thm. 8.2(b)], $H^2(\mathcal{F}(l)) = 0$ for $l \geq c_2(\mathcal{F}) - 3$. In particular, $H^2(\mathcal{F}(1)) = 0$. Since, by Serre duality and the fact that $\mathcal{F}^\vee \simeq \mathcal{F}$, $H^3(\mathcal{F}(1)) = 0$, the Riemann-Roch theorem implies that:

$$2\chi(\mathcal{O}_H(1)) - 3c_2(\mathcal{F}) + \frac{1}{2}c_3(\mathcal{F}) = \chi(\mathcal{F}(1)) = -h^1(\mathcal{F}(1)) \leq 0, \text{ hence } \frac{1}{2}c_3 \leq 3c_2 - 20.$$

Recall that $c_3 \geq 0$ and that $c_3 \equiv 0 \pmod{2}$. It follows that either $c_2 = 7$ and $c_3 \leq 2$ or $c_2 = 8$ and $c_3 \leq 8$. Looking at the proof of [3, Prop. 5.1] and of [3, Prop. 6.3] one sees that one of the following holds:

(i) $c_2 = 7, c_3 = 0$ and $\mathcal{F}$ has spectrum $(0, 0, 0)$;
(ii) $c_2 = 7, c_3 = 2$ and $\mathcal{F}$ has spectrum $(0, 0, -1)$;
(iii) $c_2 = 8, c_3 = 0$ and $\mathcal{F}$ has spectrum $(0, 0, 0, 0)$;
(iv) $c_2 = 8, c_3 = 0$ and $\mathcal{F}$ has spectrum $(1, 0, 0, -1)$;
(v) $c_2 = 8, c_3 = 2$ and $\mathcal{F}$ has spectrum $(0, 0, 0, -1)$;
(vi) $c_2 = 8, c_3 = 4$ and $\mathcal{F}$ has spectrum $(0, 0, -1, -1)$;
(vii) $c_2 = 8, c_3 = 6$ and $\mathcal{F}$ has spectrum $(0, -1, -1, -1)$;
(viii) $c_2 = 8, c_3 = 8$ and $\mathcal{F}$ has spectrum $(-1, -1, -1, -1)$.

(We saw, at the beginning of Case 6 in the proof of [3, Prop. 6.3], that if $c_2 = 8, c_3 = 8$ and $\mathcal{F}$ has spectrum $(0, -1, -1, -2)$ then $h^0(F(-1)) = 1$). The cases (iii) and (iv) can be, however, eliminated using [5, Cor. 1.5] and Schwarzenberger’s congruence (see, for example, [3, Cor. 7.4]). Let us assume, from now on, that one of the remaining cases holds.
Claim 1. \( H^1(E(l)) = 0 \) for \( l \leq -2 \).

Indeed, if \( h \) is a non-zero linear form on \( \mathbb{P}^4 \) and \( H \subset \mathbb{P}^4 \) is the hyperplane of equation \( h = 0 \) then one has an exact sequence:

\[
0 = H^0(E_H(-1)) \rightarrow H^1(E(-2)) \xrightarrow{h} H^1(E(-1)) \rightarrow H^1(E_H(-1)).
\]

The Bilinear map lemma [17, Lemma 5.1] implies that if \( H^1(E(-2)) \neq 0 \) then \( h^1(E(-1)) \geq h^1(E(-2)) + 4. \) On the other hand:

\[
h^1(E(-1)) - h^1(E(-2)) \leq h^1(E_H(-1)) = h^1(\mathcal{F}(1)) = 3c_2 - 20 - \frac{1}{2}c_3 \leq 3
\]

and this is a contradiction. It thus remains that \( H^1(E(-2)) = 0 \).

Using the exact sequence \( H^0(E_H(l + 1)) \rightarrow H^1(E(l)) \rightarrow H^1(E(l + 1)) \) one shows now, by descending induction, that \( H^1(E(l)) = 0 \) for \( l \leq -2 \).

Claim 2. \( H^2(E(-3)) = 0 \).

Indeed, using the notation from the proof of Claim 1, one has an exact sequence:

\[
0 \rightarrow H^1(E_H(-3)) \rightarrow H^2(E(-4)) \xrightarrow{h} H^2(E(-3)) \rightarrow H^2(E_H(-3)).
\]

But \( H^2(E_H(-3)) \cong H^2(\mathcal{F}(-1)) = 0 \) (by the definition of the spectrum). The Bilinear map lemma [17, Lemma 5.1] implies, now, that if \( H^2(E(-3)) \neq 0 \) then \( h^2(E(-4)) \geq h^2(E(-3)) + 4 \). On the other hand:

\[
h^2(E(-4)) - h^2(E(-3)) = h^1(E_H(-3)) = h^1(\mathcal{F}(-1)) \leq 3
\]

and this is a contradiction. It remains that \( H^2(E(-3)) = 0 \).

Claim 3. \( H^3(E(l)) = 0 \) for \( l \geq -4 \).

Indeed, \( H^2(E_H(l)) \cong H^2(\mathcal{F}(l + 2)) = 0 \) for \( l \geq -3 \) hence, by [3] Lemma 1.16(b)], \( H^3(E(l)) = 0 \) for \( l \geq -4 \).

Finally, \( H^4(E(-5)) \cong H^0(E^\vee)^\vee = 0 \). Taking into account the above claims it follows that \( E \) is \((-1)\)-regular. But this contradicts the fact that \( H^0(E(-1)) = 0 \).

\[\square\]

Theorem 3.8. Let \( E \) be an indecomposable globally generated vector bundle on \( \mathbb{P}^n \), \( n \geq 4 \), with \( c_1 = 4 \) and such that \( H^i(E^\vee) = 0 \), \( i = 0, 1 \). Then one of the following holds:

(i) \( E \cong \mathcal{O}_p(4) \);
(ii) \( E \cong P(\mathcal{O}_p(4)) \);
(iii) \( n = 4 \) and \( E \) as in item (iv) of Prop. 3.7;
(iv) \( n = 5 \) and \( E \cong \Omega_{p^5}(2) \);
(v) \( n = 5 \) and \( E \cong \Omega_{p^5}^3(4) \).

Proof. Taking into account Lemma 3.7, this follows from Prop. 3.5 above and from [3] Prop. 2.4 and Cor. 2.5] and [3] Prop. 2.11. Notice that \( c_2(\Omega_{p^5}(2)) = 7 < 8 \), that \( \Omega_{p^5}^3(4) \cong P(\Omega_{p^5}(2)) \) and that \( c_2(\Omega_{p^5}^3(4)) = 9 > 8 \).

\[\square\]
APPENDIX A. Multiple structures on a line

We denote, in this appendix, the homogeneous coordinates on \( \mathbb{P}^3 \) by \( x_0, x_1, x, y \) hence the projective coordinate ring of \( \mathbb{P}^3 \) is \( S = k[x_0, x_1, x, y] \). Let \( L \subset \mathbb{P}^3 \) be the line of equations \( x = y = 0 \) and let \( L' \) be the (complementary) line of equations \( x_0 = x_1 = 0 \). Our aim is to describe, using the results of Bănică and Forster [5], the locally CM space curves \( Z \), of degree at most 4, supported on \( L \). More precisely, we describe a set of generators of the homogeneous ideal \( I(Z) \subset S \) and a free resolution of the graded \( S \)-module \( H^0_*(\mathcal{O}_Z) \).

We begin by recalling, in this particular context, the results of Bănică and Forster on multiple structures on smooth curves embedded in threefolds. Let \( \pi : \mathbb{P}^3 \setminus L' \to L \) be the linear projection. The functor \( \pi \) induces an equivalence of categories between the category of coherent \( \mathcal{O}_L \)-modules supported on \( L \) and the category of coherent \( \mathcal{O}_L \)-modules \( \mathcal{F} \) endowed with two commuting twisted endomorphisms \( \xi, \eta : \mathcal{F} \to \mathcal{F}(1) \) (corresponding to the \( \mathcal{O}_L \)-module multiplication by \( x \) and \( y \)). Under this equivalence, locally CM \( \mathcal{O}_L \)-modules correspond to locally free \( \mathcal{O}_L \)-modules. Recall that if \( \mathcal{F} \) is a locally free \( \mathcal{O}_L \)-module and \( \mathcal{F}' \) is an \( \mathcal{O}_L \)-submodule then the saturation \( \mathcal{F}'^{\text{sat}} \) of \( \mathcal{F}' \) is defined by \( \mathcal{F}'^{\text{sat}}/\mathcal{F}' = (\mathcal{F}/\mathcal{F}')_{\text{tors}} \). If \( \mathcal{F} \) has, moreover, an \( \mathcal{O}_F \)-module structure as above and if \( \mathcal{F}' \) is an \( \mathcal{O}_F \)-submodule then \( \mathcal{F}'^{\text{sat}} \) is an \( \mathcal{O}_F \)-submodule of \( \mathcal{F} \).

According to [5], a locally CM multiple structure \( Z \) on \( L \) is quasi-primitive if the morphism \( \mathcal{I}_Z \to \mathcal{I}_L/\mathcal{I}_L^2 \) is non-zero, and it is thick if \( \mathcal{I}_Z \subseteq \mathcal{I}_L^2 \). We consider, firstly, the quasi-primitive case. In this case, there exists a nonempty open subset \( U \) of \( L \) such that, \( \forall z \in U \), there exists a system of parameters \( (t, u, v) \) of \( \mathcal{O}_{\mathbb{P}, z} \) such that \( \mathcal{I}_{L,z} = (u, v) \) and \( \mathcal{I}_{Z,z} = (u, t v^d) \), where \( d \) is the degree of \( Z \). Bănică and Forster define the Cohen-Macaulay filtration \( \mathcal{O}_Z \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots \supset \mathcal{I}_d = (0) \) of \( \mathcal{O}_Z \) by \( \mathcal{I}_i := (\mathcal{I}_L^i)_Z^{\text{sat}} \). This is a multiplicative filtration in the sense that \( \mathcal{I}_i \) are ideals of \( \mathcal{O}_Z \) and \( \mathcal{I}_i \mathcal{I}_j \subseteq \mathcal{I}_{i+j} \). \( \mathcal{I}_i \mathcal{I}_{i+1} \) is annihilated by \( \mathcal{I}_L \) hence it is already an \( \mathcal{O}_L \)-module and, in the quasi-primitive case under consideration, \( \mathcal{I}_i/\mathcal{I}_{i+1} \) is an invertible \( \mathcal{O}_L \)-module for \( i = 0, \ldots, d - 1 \).

A.1. Quasiprimitive structures of degree 4. We describe, firstly, following [5], the \( \mathcal{O}_L \)-module structure of \( \mathcal{O}_Z \). The CM filtration has, in this case, four steps:

\[
\mathcal{O}_Z \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \mathcal{I}_3 \supset (0) .
\]

Moreover, there exists an epimorphism of filtered \( \mathcal{O}_L \)-algebras \( \varepsilon : \mathcal{O}_F/\mathcal{I}_L^4 \to \mathcal{O}_Z \). Recall that, as an \( \mathcal{O}_L \)-module,

\[
\mathcal{O}_F/\mathcal{I}_L^4 \cong \mathcal{O}_L \oplus 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \oplus 4\mathcal{O}_L(-3) .
\]

Now, \( \mathcal{I}_1/\mathcal{I}_2 \cong \mathcal{O}_L(l) \), for some \( l \in \mathbb{Z} \). Since there exists an epimorphism \( 2\mathcal{O}_L(-1) \cong \mathcal{I}_L/\mathcal{I}_L^2 \to \mathcal{I}_1/\mathcal{I}_2 \) it follows that \( l \geq -1 \). Since the multiplication morphisms:

\[
\mathcal{I}_1/\mathcal{I}_2 \otimes \mathcal{O}_L \mathcal{I}_1/\mathcal{I}_2 \to \mathcal{I}_2/\mathcal{I}_3 , \quad \mathcal{I}_1/\mathcal{I}_2 \otimes \mathcal{O}_L \mathcal{I}_2/\mathcal{I}_3 \to \mathcal{I}_3
\]

are nonzero (due to the local structure of the algebra \( \mathcal{O}_Z \)) it follows that \( \mathcal{I}_2/\mathcal{I}_3 \cong \mathcal{O}_L(2l + m) \) with \( m \geq 0 \) and \( \mathcal{I}_3 \cong \mathcal{O}_L(3l + m + n) \) with \( n \geq 0 \). The exact sequence:

\[
0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_2/\mathcal{I}_3 \to 0
\]
must split in the category of $\mathcal{O}_L$-modules, hence $\mathcal{I}_2 \simeq \mathcal{O}_L(2l + m) \oplus \mathcal{O}_L(3l + m + n)$ as $\mathcal{O}_L$-modules. The exact sequence:

$$0 \longrightarrow \mathcal{I}_2 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_1/\mathcal{I}_2 \longrightarrow 0$$

splits in the category of $\mathcal{O}_L$-modules if $l \geq 0$. It splits, also, for $l = -1$ because, in this case, the epimorphism $\varepsilon$ considered above induces a composite epimorphism:

$$2\mathcal{O}_L(-1) \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_1/\mathcal{I}_2 \simeq \mathcal{O}_L(-1).$$

The same kind of argument shows that $\mathcal{O}_Z/\mathcal{I}_1 = \mathcal{O}_L$ is a direct summand of the $\mathcal{O}_L$-module $\mathcal{O}_Z$ hence:

$$\mathcal{O}_Z \simeq \mathcal{O}_L \oplus \mathcal{O}_L(l) \oplus \mathcal{O}_L(2l + m) \oplus \mathcal{O}_L(3l + m + n).$$

The multiplicative structure of $\mathcal{O}_Z$ is defined by two morphisms of $\mathcal{O}_L$-modules:

$$\mu_{11} : \mathcal{O}_L(l) \otimes \mathcal{O}_L(l) \xrightarrow{(p,p')} \mathcal{O}_L(2l + m) \oplus \mathcal{O}_L(3l + m + n),$$

$$\mu_{12} : \mathcal{O}_L(l) \otimes \mathcal{O}_L(2l + m) \xrightarrow{q} \mathcal{O}_L(3l + m + n),$$

with $0 \neq p \in H^0(\mathcal{O}_L(m))$, $p' \in H^0(\mathcal{O}_L(l + m + n))$ and $0 \neq q \in H^0(\mathcal{O}_L(n))$.

As a graded $H^0(\mathcal{O}_L) = k[x_0, x_1]$-module, $H^0(\mathcal{O}_Z)$ has a minimal set of generators $1 \in H^0(\mathcal{O}_Z)$, $e_1 \in H^0(\mathcal{O}_Z(-l))$, $e_2 \in H^0(\mathcal{O}_Z(-2l - m))$, $e_3 \in H^0(\mathcal{O}_Z(-3l - m - n))$. The multiplicative structure of $H^0(\mathcal{O}_Z)$ is defined by the relations:

\begin{equation}
(A.1)
\begin{align*}
e_1^2 &= p'e_2 + p'e_3, \quad e_1e_2 = qe_3.
\end{align*}
\end{equation}

In order to get the complete $S$-module structure of $H^0(\mathcal{O}_Z)$ it remains to consider the multiplications by $x$ and by $y$. One must have:

\begin{equation}
(A.2)
\begin{align*}
x \cdot 1 &= ae_1 + a'e_2 + a''e_3, \quad y \cdot 1 = be_1 + b'e_2 + b''e_3,
\end{align*}
\end{equation}

with $a, b \in H^0(\mathcal{O}_L(l+1))$, $a', b' \in H^0(\mathcal{O}_L(2l+m+1))$ and $a'', b'' \in H^0(\mathcal{O}_L(3l+m+n+1))$. The rest of the multiplications by $x$ and $y$ can be recovered from the multiplicative structure of $H^0(\mathcal{O}_Z)$:

\begin{align*}
x \cdot e_1 &= x \cdot 1 \cdot e_1 = pae_2 + (p'a + qa')e_3, \quad y \cdot e_1 = pbe_2 + (p'b + qb')e_3
\end{align*}

\begin{align*}
x \cdot e_2 &= qae_3, \quad y \cdot e_2 = qbe_3.
\end{align*}

One deduces that the epimorphism $\varepsilon_1 : \mathcal{I}_L/\mathcal{I}_L^2 \rightarrow \mathcal{I}_1$ induced by the epimorphism $\varepsilon : \mathcal{O}_Z/\mathcal{I}_L^2 \rightarrow \mathcal{O}_Z$ is defined by the matrix:

\[
\begin{pmatrix}
a & b & 0 & 0 & 0 & 0 & 0 & 0 \\
a' & b' & pa^2 & pab & pb^2 & 0 & 0 & 0 \\
a'' & b'' & p'a^2 + 2qaa' & p'ab + q(ab' + a'b) & p'b^2 + 2qbb' & pqa^2b & pqab^2 & pqb^3
\end{pmatrix}
\]

Consider the following two determinants:

\begin{equation}
(A.3)
\begin{align*}
\Delta_1 := \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}, \quad \Delta_2 := \begin{vmatrix} a & b & 0 \\ a' & b' & p \\ a'' & b'' & p' \end{vmatrix}
\end{align*}
\end{equation}
One has $\Delta_1 \in H^0(\mathcal{O}_L(3l + m + 2))$ and $\Delta_2 \in H^0(\mathcal{O}_L(4l + 2m + n + 2))$. One can easily prove the following:

**Lemma A.1.** The above $3 \times 9$ matrix defines an epimorphism:

$$\varepsilon_4 : 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \oplus 4\mathcal{O}_L(-3) \rightarrow \mathcal{O}_L(l) \oplus \mathcal{O}_L(2l + m) \oplus \mathcal{O}_L(3l + m + n)$$

if and only if the following three conditions are satisfied:

(i) $a$ and $b$ have no common zero on $L$;
(ii) $\Delta_1$ and $p$ have no common zero on $L$;
(iii) $\Delta_2$ and $q$ have no common zero on $L$.

**Proof.** It is helpful to notice the following relations:

$$-b(p'a'^2 + 2qaa') + a(p'ab + q(ab' + a'b)) = aq\Delta_1,$$
$$-b(p'ab + q(ab' + a'b)) + a(p'b^2 + 2qbb') = bq\Delta_1. \quad \square$$

Our method of finding a system of generators (and, actually, even a free resolution) for the homogeneous ideal $I(Z) \subset S$ is based on the following observation: one has an exact sequence $0 \rightarrow \mathcal{I}_L^4 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_Z/\mathcal{I}_L^4 \rightarrow 0$, $\mathcal{I}_Z/\mathcal{I}_L^4$ is the kernel of the epimorphism $\varepsilon_4 : \mathcal{I}_L/\mathcal{I}_L^4 \rightarrow \mathcal{I}_1$ and $H^4(\mathcal{I}_L^4) = 0$, whence an exact sequence of graded $S$-modules:

$$0 \rightarrow I(L)^4 \rightarrow I(Z) \rightarrow H^0_*(\text{Ker } \varepsilon_4) \rightarrow 0. \quad (A.4)$$

It follows that if one knows the structure of $\text{Ker } \varepsilon_4$ as an $\mathcal{O}_L$-module (i.e., its Grothendieck decomposition as a direct sum of invertible sheaves $\mathcal{O}_L(i)$) and one can lift the generators of the graded $H^0_*(\mathcal{O}_L) = k[x_0, x_1]$-module $H^0_*(\text{Ker } \varepsilon_4)$ to elements of $I(Z)$ then one can complete the system of generators of $I(L)^4$ to a system of generators of $I(Z)$.

As steps towards the description of $\text{Ker } \varepsilon_4$, we describe, firstly, the kernels of the epimorphisms $\varepsilon_2 : \mathcal{I}_L/\mathcal{I}_L^2 \rightarrow \mathcal{I}_1/\mathcal{I}_2$ and $\varepsilon_3 : \mathcal{I}_L/\mathcal{I}_L^3 \rightarrow \mathcal{I}_1/\mathcal{I}_3$ induced by $\varepsilon_4$.

**Description of $\text{Ker } \varepsilon_2$.** The morphism $\varepsilon_2 : 2\mathcal{O}_L(-1) \rightarrow \mathcal{O}_L(l)$ is defined by the matrix $(a, b)$ and one has an exact sequence:

$$0 \rightarrow \mathcal{O}_L(-l - 2) \xrightarrow{(b, a)} 2\mathcal{O}_L(-1) \xrightarrow{(a, b)} \mathcal{O}_L(l) \rightarrow 0 \quad (A.5)$$

hence $\text{Ker } \varepsilon_2 \simeq \mathcal{O}_L(-l - 2)$. Let us denote by $\nu_2$ the morphism $(-b, a)^t : \mathcal{O}_L(-l - 2) \rightarrow 2\mathcal{O}_L(-1)$.

**Description of $\text{Ker } \varepsilon_3$.** The morphism $\varepsilon_3 : 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \rightarrow \mathcal{O}_L(l) \oplus \mathcal{O}_L(2l + m)$ is defined by the matrix:

$$\begin{pmatrix}
a & b & 0 & 0 & 0
a' & b' & pa^2 & pab & pb^2
\end{pmatrix}.$$ 

Consider the (composite) morphism $\varepsilon'_3 : \mathcal{O}_L(-l - 2) \oplus 3\mathcal{O}_L(-2) \xrightarrow{\sim} \text{Ker } \varepsilon_2 \oplus 3\mathcal{O}_L(-2) \rightarrow \mathcal{O}_L(2l + m)$ induced by $\varepsilon_3$. Since $\text{Ker } \varepsilon_3 \subseteq \text{Ker } \varepsilon_2 \oplus 3\mathcal{O}_L(-2)$ it follows that $\nu_2 \oplus \text{id}_{3\mathcal{O}_L(-2)} : \mathcal{O}_L(-l - 2) \oplus 3\mathcal{O}_L(-2) \rightarrow 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2)$ induces an isomorphism $\text{Ker } \varepsilon'_3 \xrightarrow{\sim} \text{Ker } \varepsilon_3$. 


Now, $\varepsilon_3'$ is defined by the matrix:

$$(\Delta_1, pa^2, pab, pb^2).$$

Using the exact sequence:

$$0 \to 2\mathcal{O}_L(-l - 3) \xrightarrow{(a, b)} 3\mathcal{O}_L(-2) \xrightarrow{(a^2, ab, b^2)} \mathcal{O}_L(2 S) \to 0$$

one deduces that $\varepsilon_3'$ factorizes as:

$$\mathcal{O}_L(-l - 2) \oplus 3\mathcal{O}_L(-2) \to \mathcal{O}_L(-l - 2) \oplus \mathcal{O}_L(2 S) \xrightarrow{(\Delta_1, p)} \mathcal{O}_L(2 S + m).$$

One has an exact sequence:

$$0 \to \mathcal{O}_L(-l - m - 2) \xrightarrow{(-\Delta_1)} \mathcal{O}_L(-l - 2) \oplus \mathcal{O}_L(2 S) \xrightarrow{(\Delta_1, p)} \mathcal{O}_L(2 S + m) \to 0.$$ Since $\Delta_1 \in H^0(\mathcal{O}_L(3l + m + 2))$ and since $-l - 3 + (l + m + 2) = m - 1 \geq -1$ one deduces, using the exact sequence (A.6), that there exist polynomials $v_0, v_1, v_2 \in H^0(\mathcal{O}_L(l + m))$ such that:

$$(A.7) \quad -\Delta_1 = v_0a^2 + v_1ab + v_2b^2.$$ One derives an exact sequence:

$$0 \to \mathcal{O}_L(-l - m - 2) \oplus 2\mathcal{O}_L(-l - 3) \xrightarrow{\nu_3'} \mathcal{O}_L(-l - 2) \oplus 3\mathcal{O}_L(-2) \xrightarrow{\varepsilon_3'} \mathcal{O}_L(2 S + m) \to 0$$

with $\nu_3'$ defined by the matrix:

$$\begin{pmatrix}
  p & 0 & 0 \\
  v_0 & -b & 0 \\
  v_1 & a & -b \\
  v_2 & 0 & a
\end{pmatrix},$$

from which one gets an exact sequence:

$$0 \to \mathcal{O}_L(-l - m - 2) \oplus 2\mathcal{O}_L(-l - 3) \xrightarrow{\nu_3} 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \xrightarrow{\varepsilon_3} \mathcal{O}_L(l + \mathcal{O}_L(2 S + m) \to 0$$

with $\nu_3 := (\nu_2 \oplus \text{id}_{3\mathcal{O}_L(-2)}) \circ \nu_3'$ defined by the matrix:

$$\begin{pmatrix}
  -pb & 0 & 0 \\
  pa & 0 & 0 \\
  v_0 & -b & 0 \\
  v_1 & a & -b \\
  v_2 & 0 & a
\end{pmatrix}.$$ Description of $\ker \varepsilon_4$. Consider the (composite) morphism: $\varepsilon_4' : \mathcal{O}_L(-l - m - 2) \oplus 2\mathcal{O}_L(-l - 3) \oplus 4\mathcal{O}_L(-3) \xrightarrow{\varepsilon_4} \ker \varepsilon_3 \oplus 4\mathcal{O}_L(-3) \to \mathcal{O}_L(3l + m + n)$.
induced by $\varepsilon_4$. Since $\text{Ker} \varepsilon_4 \subseteq \text{Ker} \varepsilon_3 \oplus 4\mathcal{O}_L(-3)$ it follows that $\nu_3 \oplus \text{id}_{4\mathcal{O}_L(-3)} : \mathcal{O}_L(-1-m-2) \oplus 2\mathcal{O}_L(-l-3) \oplus 4\mathcal{O}_L(-3) \to 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \oplus 4\mathcal{O}_L(-3)$ induces an isomorphism $\text{Ker} \varepsilon_4' \sim \to \text{Ker} \varepsilon_4$. 

Now, $\varepsilon_4'$ is defined by the matrix:

$$(-\Delta_2 + qv, aq\Delta_1, bq\Delta_1, pqa^3, pqa^2b, pqab^2, pqb^3)$$

where:

$$v := 2v_0aa' + v_1(ab' + a'b) + 2v_2bb'.$$

Using the exact sequence:

$$(A.9) \quad 0 \to 3\mathcal{O}_L(-l-4) \xrightarrow{(a^3, a^2b, ab^2, b^3)} 4\mathcal{O}_L(-3) \xrightarrow{\varepsilon_4''} \mathcal{O}_L(3l) \to 0$$

one deduces that $\varepsilon_4'$ factorizes as:

$$\mathcal{O}_L(-l-m-2) \oplus 2\mathcal{O}_L(-l-3) \oplus 4\mathcal{O}_L(-3) \xrightarrow{\varepsilon_4''} \mathcal{O}_L(3l + m + n)$$

with $\varepsilon_4''$ is defined by $(-\Delta_2 + qv, aq\Delta_1, bq\Delta_1, pq)$. Consider the morphism:

$$\eta := (a\Delta_1, b\Delta_1, p) : 2\mathcal{O}_L(-l-3) \oplus \mathcal{O}_L(3l) \to \mathcal{O}_L(3l + m).$$

One has a commutative diagram:

$$\begin{array}{ccc}
2\mathcal{O}_L(-l-3) \oplus \mathcal{O}_L(-m-2) & \xrightarrow{(a,b,p)} & \mathcal{O}_L(-2) \\
\text{id} \oplus \text{id} \oplus \Delta_1 & & \text{id} \oplus \text{id} \oplus \Delta_1 \\
2\mathcal{O}_L(-l-3) \oplus \mathcal{O}_L(3l) & \xrightarrow{\eta} & \mathcal{O}_L(3l + m)
\end{array}$$

Since $p$ and $\Delta_1$ are coprime, it follows that $\text{id} \oplus \text{id} \oplus \Delta_1$ maps isomorphically the kernel of $(a,b,p) : 2\mathcal{O}_L(-l-3) \oplus \mathcal{O}_L(-m-2) \to \mathcal{O}_L(-2)$ onto $\text{Ker} \eta$. One needs, now, the following easy:

**Lemma A.2.** Let $K$ be the kernel of an epimorphism:

$$(a_1, a_2, a_3) : \mathcal{O}_L(-i_1) \oplus \mathcal{O}_L(-i_2) \oplus \mathcal{O}_L(-i_3) \to \mathcal{O}_L.$$  

Assume that $i_1 \leq i_2$ and that $a_3 \neq 0$. Then $K \simeq \mathcal{O}_L(-j_1) \oplus \mathcal{O}_L(-j_2)$ with $i_1 \leq j_1 \leq j_2 \leq i_2 + i_3$. In particular, $H^1(K(t)) = 0$ for $t \geq i_2 + i_3 - 1$.

**Proof.** $K$ is locally free of rank 2 hence one can write $K \simeq \mathcal{O}_L(-j_1) \oplus \mathcal{O}_L(-j_2)$ with $j_1 \leq j_2$. It follows from our hypothesis that $H^0(K(-i_1 + 1)) = 0$, hence $j_1 \geq i_1$. On the other hand, $c_1(K) = -i_1 - i_2 - i_3$ hence $j_2 = i_1 + i_2 + i_3 - j_1 \leq i_2 + i_3$. \hfill \square
According to this lemma, one has an exact sequence:

$$0 \to \mathcal{O}_L(-l_1) \oplus \mathcal{O}_L(-l_2) \xrightarrow{(f_1 \quad f_2) \quad (g_1 \quad g_2) \quad (u_1 \quad u_2)} 2\mathcal{O}_L(-l - 1) \oplus \mathcal{O}_L(-m) \xrightarrow{(a, b, p)} \mathcal{O}_L \to 0,$$

with $l + 1 \leq l_1 \leq l_2 \leq l + m + 1$ and $l_1 + l_2 = 2l + m + 2$. One derives an exact sequence:

$$0 \to \mathcal{O}_L(-l_1 - 2) \oplus \mathcal{O}_L(-l_2 - 2) \xrightarrow{(f_1 \quad f_2) \quad (g_1 \quad g_2) \quad (u_1 \Delta_1 \quad u_2 \Delta_1)} 2\mathcal{O}_L(-l - 3) \oplus \mathcal{O}_L(3l) \xrightarrow{\eta} \mathcal{O}_L(3l + m) \to 0,$$

Now, $\varepsilon''_n$ factorizes as:

$$\mathcal{O}_L(-l - m - 2) \xrightarrow{\oplus \mathcal{O}_L(-l - 3) \oplus \mathcal{id \oplus \eta}} \mathcal{O}_L(-l - m - 2) \oplus \mathcal{O}_L(3l + m) \xrightarrow{(\Delta_2 + qv, q)} \mathcal{O}_L(3l + m + n).$$

One has an exact sequence:

$$0 \to \mathcal{O}_L(-l - m - n - 2) \xrightarrow{(\Delta_2 - qv)} \mathcal{O}_L(-l - m - 2) \oplus \mathcal{O}_L(3l + m) \xrightarrow{(\Delta_2 + qv, q)} \mathcal{O}_L(3l + m + n) \to 0.$$

Since $\Delta_2 - qv \in H^0(\mathcal{O}_L(4l + 2m + n + 2))$ and since 

$$-l_i - 2 + (l + m + n + 2) \geq -(l + m + 1) - 2 + (l + m + n + 2) = n - 1 \geq -1, \ i = 1, 2,$$

the exact sequence (A.11) shows that there exist polynomials $f, g \in H^0(\mathcal{O}_L(m + n - 1))$ and $w \in H^0(\mathcal{O}_L(4l + m + n + 2))$ such that:

$$\Delta_2 - qv = fa\Delta_1 + gb\Delta_1 + wp$$

($v$ has been defined in (A.8)). One deduces the existence of an exact sequence:

$$0 \to \mathcal{O}_L(-l - m - n - 2) \oplus \mathcal{O}_L(-l - m - 2) \xrightarrow{\oplus \mathcal{O}_L(-l_1 - 2) \oplus \mathcal{O}_L(-l_2 - 2)} 2\mathcal{O}_L(-l - 3) \oplus \mathcal{O}_L(3l + m + n) \to 0$$

with $\nu_4''$ defined by the matrix:

$$\begin{pmatrix}
q & 0 & 0 \\
f & f_1 & f_2 \\
g & g_1 & g_2 \\
w & u_1 \Delta_1 & u_2 \Delta_1
\end{pmatrix}.$$
Since $u_i \in H^0(O_L(l_i - m))$, $\Delta_1 \in H^0(O_L(3l + m + 2))$ and $-l - 4 + (l_i - m) + m + 2 \geq -l - 4 + (l + 1 - m) + l + 2 = -1$, $i = 1, 2$, one derives, using the exact sequence (A.9), the existence of polynomials $v_{ij} \in H^0(O_L(l_i - 1))$, $j = 0, \ldots, 3$, such that:

\begin{equation}
\tag{A.13}
u_i \Delta_1 = v_{i0}a^3 + v_{i1}a^2b + v_{i2}ab^2 + v_{i3}b^3, \quad i = 1, 2.
\end{equation}

One gets immediately:

**Lemma A.3.** If the polynomial $w \in H^0(O_L(4l + m + n + 2))$ (defined in (A.12)) can be written as a combination:

\[w = w_0a^3 + w_1a^2b + w_2ab^2 + w_3b^3\]

with $w_0, \ldots, w_3 \in H^0(O_L(l + m + n - 1))$ then one has an exact sequence:

\[
0 \longrightarrow O_L(-l - m - n - 2) \oplus O_L(-l_1 - 2) \oplus O_L(-l_2 - 2) \oplus 3O_L(-l - 4) \xrightarrow{\nu'_4} 2O_L(-l - 3) \oplus 4O_L(-3) \longrightarrow O_L(3l + m + n) \longrightarrow 0
\]

with $\nu'_4$ defined by the matrix:

\[
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 \\
0 & f & f_1 & f_2 & 0 & 0 \\
g & g_1 & g_2 & 0 & 0 & 0 \\
0 & w_0 & v_{10} & v_{20} & -b & 0 \\
w_1 & v_{11} & v_{21} & a & -b & 0 \\
w_2 & v_{12} & v_{22} & 0 & a & -b \\
w_3 & v_{13} & v_{23} & 0 & 0 & a
\end{pmatrix}
\]

**Remark A.4.** Since $w \in H^0(O_L(4l + m + n + 2))$ and $-l - 4 + (l + m + n + 2) = m + n - 2$, the exact sequence (A.9) shows that if $m \neq 0$ or if $n \neq 0$ then $w$ can be always written as a combination of $a^3, a^2b, ab^2, b^3$. This is also true when $m = n = 0$ and $l = -1$ because, in this case, $w \in H^0(O_L(-2)) = 0$.

**Remark A.5.** We want to emphasize, for later use, a relation between the polynomials $v_0, v_1, v_2 \in H^0(O_L(l + m))$ defined by relation (A.7) and the polynomials $v_{ij} \in H^0(O_L(l_i - 1))$, $i = 1, 2$, $0 \leq j \leq 3$ defined by relation (A.13). Firstly, we notice that, by considering the Eagon-Northcott complex associated to the matrix appearing on the left side of the exact sequence (A.10) as in the proof of the Hilbert-Burch theorem, one may assume that:

\begin{equation}
\tag{A.14}
a = \begin{vmatrix}
g_1 \\
u_1 & u_2
\end{vmatrix}, \quad b = -\begin{vmatrix}
f_1 \\
u_1 & u_2
\end{vmatrix}, \quad p = \begin{vmatrix}
f_1 & f_2 \\
g_1 & g_2
\end{vmatrix}.
\end{equation}

Now, since $a = -g_2u_1 + g_1u_2$ then, multiplying relation (A.7) by $a$ and the two relations in (A.13) by $-g_2$ and $g_1$, respectively, one gets:

\[a(v_0a^2 + v_1ab + v_2b^2) - g_2(v_{10}a^3 + \ldots + v_{13}b^3) + g_1(v_{20}a^3 + \ldots + v_{23}b^3) = 0.\]
Using, now, the exact sequence \( \text{(A.9)} \), one derives the existence of polynomials \( \alpha_0, \alpha_1, \alpha_2 \in \text{H}^0(\mathcal{O}_L(m-1)) \) such that:

\[
\begin{pmatrix}
 v_0 \\
 v_1 \\
 v_2 \\
 0
\end{pmatrix} - g_2 \begin{pmatrix}
 v_{10} \\
 v_{11} \\
 v_{12} \\
 v_{13}
\end{pmatrix} + g_1 \begin{pmatrix}
 v_{20} \\
 v_{21} \\
 v_{22} \\
 v_{23}
\end{pmatrix} = \begin{pmatrix}
 -b \\
 a \\
 0 \\
 0
\end{pmatrix} \begin{pmatrix}
 \alpha_0 \\
 \alpha_1 \\
 \alpha_2
\end{pmatrix} .
\]

Similarly, using the relation \( b = f_2u_1 - f_1u_2 \), one derives the existence of polynomials \( \beta_0, \beta_1, \beta_2 \in \text{H}^0(\mathcal{O}_L(m-1)) \) such that:

\[
\begin{pmatrix}
 0 \\
 v_0 \\
 v_1 \\
 v_2
\end{pmatrix} + f_2 \begin{pmatrix}
 v_{10} \\
 v_{11} \\
 v_{12} \\
 v_{13}
\end{pmatrix} - f_1 \begin{pmatrix}
 v_{20} \\
 v_{21} \\
 v_{22} \\
 v_{23}
\end{pmatrix} = \begin{pmatrix}
 -b \\
 a \\
 0 \\
 0
\end{pmatrix} \begin{pmatrix}
 \beta_0 \\
 \beta_1 \\
 \beta_2
\end{pmatrix} .
\]

**Proposition A.6.** Assume that the polynomial \( w \) (defined in \( \text{(A.12)} \)) can be written as a combination of \( a^3, \ldots, b^3 \) as in Lemma \( \text{(A.3)} \) (which happens automatically if \( m \neq 0 \) or if \( n \neq 0 \) or if \( m = n = 0 \) and \( l = -1 \) according to Remark \( \text{(A.4)} \)). Consider the polynomials (in \( S \)):

\[
F_2 = \begin{bmatrix} a \\ b \\ x \\ y \end{bmatrix} , \quad F_3 = pF_2 + v_0x^2 + v_1xy + v_2y^2 .
\]

with \( v_0, v_1, v_2 \) defined in \( \text{(A.7)} \). Then the homogeneous ideal \( I(Z) \subset S \) of \( Z \) is generated by the following polynomials:

\[
F_4 = qF_3 + (f + gy)F_2 + w_0x^3 + w_1x^2y + w_2xy^2 + w_3y^3 ,
\]

\[
G_1 = (f_1x + g_1y)F_2 + v_{10}x^3 + v_{11}x^2y + v_{12}xy^2 + v_{13}y^3 ,
\]

\[
G_2 = (f_2x + g_2y)F_2 + v_{20}x^3 + v_{21}x^2y + v_{22}xy^2 + v_{23}y^3 ,
\]

\[
x^2F_2 , \quad xyF_2 , \quad y^2F_2 , \quad x^4 , \quad x^3y , \quad x^2y^2 , \quad xy^3 , \quad y^4.
\]

with \( f, g \) defined in \( \text{(A.12)} \), the \( f_i \)'s and the \( g_i \)'s defined in \( \text{(A.10)} \) and the \( v_{ij} \)'s defined in \( \text{(A.13)} \).

Notice that \( \deg F_2 = l + 2, \quad \deg F_3 = l + m + 2, \quad \deg F_4 = l + m + n + 2, \quad \deg G_1 = l_1 + 2 \) and \( \deg G_2 = l_2 + 2, \) with \( l + 1 \leq l_1 \leq l_2 \leq l + m + 1 \) and \( l_1 + l_2 = 2l + m + 2 \).

The case where \( w \) cannot be written as a combination of \( a^3, \ldots, b^3 \) will be analyzed in Prop. \( \text{(A.12)} \) below.

**Proof.** It follows, from Lemma \( \text{(A.3)} \), that one has an exact sequence:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{O}_L(l - m - n - 2) & \oplus & 2\mathcal{O}_L(-1) & \oplus & \mathcal{O}_L(l) \\
& & \mathcal{O}_L(l - 1 - 2) & \oplus & & & \\
& \rightarrow & \mathcal{O}_L(l - 2 - 2) & \oplus & 3\mathcal{O}_L(-2) & \oplus & \mathcal{O}_L(2l + m) & \rightarrow & 0 \\
& & \oplus & & 4\mathcal{O}_L(-3) & & \mathcal{O}_L(3l + m)
\end{array}
\]
where \( \nu_4 := (\nu_3 \oplus \text{id}_{\Theta(-3)}) \circ \nu'_4 \). Since \( \nu_3 \) and \( \nu'_4 \) are defined by explicit matrices, one deduces that \( \nu_4 \) is defined by the matrix:

\[
\begin{pmatrix}
-pbq & 0 & 0 & 0 & 0 \\
paq & 0 & 0 & 0 & 0 \\
v_0q - bf & -bf_1 & -bf_2 & 0 & 0 \\
v_1q + af - bg & af_1 - bg_1 & af_2 - bg_2 & 0 & 0 \\
v_2q + ag & ag_1 & ag_2 & 0 & 0 \\
w_0 & v_{10} & v_{20} & -b & 0 \\
w_1 & v_{11} & v_{21} & a & -b \\
w_2 & v_{12} & v_{22} & 0 & a \\
w_3 & v_{13} & v_{23} & 0 & a
\end{pmatrix}.
\]

One uses, now, the exact sequence (A.4) and the fact that:

\[\mathcal{I}_L/\mathcal{I}_L^4 = \Theta_L(-1)x \oplus \Theta_L(-1)y \oplus \Theta_L(-2)x^2 \oplus \cdots \oplus \Theta_L(-3)y^3.\]

\[
\mathcal{I}_L/\mathcal{I}_L^4 = \Theta_L(-1)x \oplus \Theta_L(-1)y \oplus \Theta_L(-2)x^2 \oplus \cdots \oplus \Theta_L(-3)y^3. \quad \square
\]

A graded free resolution of \( I(Z) \) under the hypothesis of Prop. A.6. One has a filtration by homogeneous ideals:

\[I(Z) \supset J_2 \supset J_3 \supset I(L)^4 \supset (0)\]

where \( J_2 \) is the ideal generated by \( G_1, G_2, x^2F_2, xyF_2, y^2F_2, x^4, \ldots, y^4 \) and \( J_3 \) is the ideal generated by \( x^2F_2, xyF_2, y^2F_2, x^4, \ldots, y^4 \). Notice that:

\[(x, y)I(Z) \subseteq J_2, (x, y)J_2 \subseteq J_3, (x, y)J_3 \subseteq I(L)^4.\]

If \( \nu_4 \) is the morphism from the proof of Prop. A.6 then \( H^6_x(\nu_4) \) induces isomorphisms:

\[
H^6_x(\Theta_L(-l_1 - 2) \oplus \Theta_L(-l_2 - 2) \oplus 3\Theta_L(-l - 4)) \xrightarrow{\sim} J_2/I(L)^4, \\
H^6_x(3\Theta_L(-l - 4)) \xrightarrow{\sim} J_3/I(L)^4.
\]

One deduces isomorphisms of \( S \)-modules:

\[I(Z)/J_2 \simeq S(L)(-l - m - n - 2), \quad J_2/J_3 \simeq S(L)(-l_1 - 2) \oplus S(L)(-l_2 - 2), \]

\[J_3/I(L)^4 \simeq 3S(L)(-l - 4), \]

where \( S(L) := S/I(L) \). Now, using the well-known resolutions:

\[
0 \rightarrow S(-2) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} 2S(-1) \xrightarrow{\begin{pmatrix} -y & 0 & 0 & 0 \\ x & -y & 0 & 0 \\ 0 & x & -y & 0 \\ 0 & 0 & x & -y \end{pmatrix}} S \rightarrow S(L) \rightarrow 0
\]

\[
0 \rightarrow 4S(-5) \xrightarrow{\begin{pmatrix} x^4, x^3y, x^2y^2, xy^3, y^4 \end{pmatrix}} 5S(-4) \rightarrow I(L)^4 \rightarrow 0
\]
one deduces that $I(Z)$ has a (not necessarily minimal) graded free resolution of the form:

$$
\begin{array}{cccccc}
S(-l-m-n-4) & 2S(-l-m-n-3) & S(-l-m-n-2) \\
\oplus & \oplus & \oplus \\
S(-l_1-4) & 2S(-l_1-3) & S(-l_1-2) \\
\oplus & \oplus & \oplus \\
S(-l_2-4) & 6S(-l-5) & 3S(-l-4) \\
\oplus & \oplus & \oplus \\
3S(-l-6) & & & & & \\
\oplus & & & & & \\
0 & d_2 & d_1 & d_0 & I(Z) & 0
\end{array}
$$

with $d_0$ defined by the generators of $I(Z)$ enumerated in the statement of Prop. [4.6] and with the linear parts of $d_1$ and $d_2$ deduced from the above resolutions of $S/I(L)$ and $I(L)^3$. The rest of the matrices defining $d_1$ and $d_2$ can be easily guessed. For example, $d_1$ is defined by the matrix of relations between the generators of $I(Z)$. The module of these relations is generated by the relations of the following form: one multiplies each generator of $I(Z)$ by $x$ and by $y$ and one expresses the results as combinations of the next generators. One gets the following matrix for $d_1$:

$$
\begin{pmatrix}
x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-qg_2 & qf_2 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-qg_1 & -qf_1 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-f-q\alpha_0 & -q\beta_0 & -f_1 & 0 & -f_2 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g-q\alpha_1 & f-q\beta_1 & -g_1 & -f_1 & -g_2 & -f_2 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g-q\alpha_2 & g-q\beta_2 & 0 & -g_1 & 0 & -g_2 & 0 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w_0 & 0 & -v_{10} & 0 & v_{20} & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w_1 & -w_0 & -v_{11} & -v_{10} & -v_{21} & -v_{20} & -a & b & b & 0 & 0 & 0 & x & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w_2 & -w_1 & -v_{12} & -v_{11} & -v_{22} & -v_{21} & 0 & -a & -a & b & b & 0 & 0 & x & -y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-w_3 & -w_2 & -v_{13} & -v_{12} & -v_{23} & -v_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -w_3 & 0 & -v_{13} & 0 & -v_{23} & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Only the first two columns of this matrix need an explanation. For that, we use Remark [4.5]. Using the determinantal expression of $p$ from (4.4), one gets the relations:

$$
\begin{align*}
xp - g_2(f_1x + g_1y) + g_1(f_2x + g_2y) &= 0 \\
yp + f_2(f_1x + g_1y) - f_1(f_2x + g_2y) &= 0.
\end{align*}
$$

On the other hand, multiplying to the left the matrix relation (4.5) by $(x^3, x^2y, xy^2, y^3)$ one obtains:

$$
\begin{align*}
x(v_0x^2 + v_1xy + v_2y^2) - g_2(v_{10}x^3 + \cdots + v_{13}y^3) + g_1(v_{20}x^3 + \cdots + v_{23}y^3) = \\
(\alpha_0 x^2 + \alpha_1 xy + \alpha_2 y^2)(-bx + ay).
\end{align*}
$$
One deduces, similarly, from the matrix relation (A.16), the following polynomial relation:
\[y(v_0x^2 + v_1xy + v_2y^2) + f_2(v_{10}x^3 + \cdots + v_{13}y^3) - f_1(v_{20}x^3 + \cdots + v_{23}y^3) = (\beta_0x^2 + \beta_1xy + \beta_2y^2)(-bx + ay)\].

One thus obtains the following relations:
\[xF_3 - g_2G_1 + g_1G_2 - \alpha_0x^2F_2 - \alpha_1xyF_2 - \alpha_2y^2F_2 = 0\]
\[yF_3 + f_2G_1 - f_1G_2 - \beta_0x^2F_2 - \beta_1xyF_2 - \beta_2y^2F_2 = 0\]

which explain the first two columns of the matrix of \(d_1\).

The matrix of \(d_2\) is the matrix of relations between the columns of the matrix of \(d_1\). It looks like this:

\[
\begin{pmatrix}
-y & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 & 0 \\
-qf_2 & -y & 0 & 0 & 0 & 0 \\
-qg_2 & x & 0 & 0 & 0 & 0 \\
qf_1 & 0 & -y & 0 & 0 & 0 \\
qg_1 & 0 & x & 0 & 0 & 0 \\
q\beta_0 & 0 & 0 & -y & 0 & 0 \\
-f - q\alpha_0 & -f_1 & -f_2 & x & 0 & 0 \\
f + q\beta_1 & f_1 & f_2 & 0 & -y & 0 \\
-g - q\alpha_1 & -g_1 & -g_2 & 0 & x & 0 \\
g + q\beta_2 & g_1 & g_2 & 0 & 0 & -y \\
-q\alpha_2 & 0 & 0 & 0 & x \\
w_0 & v_{10} & v_{20} & -b & 0 & 0 \\
w_1 & v_{11} & v_{21} & a & -b & 0 \\
w_2 & v_{12} & v_{22} & 0 & a & -b \\
w_3 & v_{13} & v_{23} & 0 & 0 & a
\end{pmatrix}
\]

Again, one needs to motivate only the fact that the first column of this matrix is a relation between the columns of the matrix of \(d_1\). For example, if one multiplies the 9th row of the matrix of \(d_1\) against the first column of the matrix of \(d_2\) one gets:

\[(*) \quad qf_2v_{12} + qg_2v_{11} - qf_1v_{22} - qg_1v_{21} + qa\alpha_0 - qa\beta_1 - qb\alpha_1 + qb\beta_2.\]

But identifying the coefficient of \(x^2y\) in the polynomial relation deduced above from the matrix relation (A.15) and the coefficient of \(xy^2\) in the polynomial relation deduced from (A.16) one obtains:

\[v_1 - g_2v_{11} + g_1v_{21} = -b\alpha_1 + a\alpha_0\]
\[v_1 + f_2v_{12} - f_1v_{22} = a\beta_0 - b\beta_2\]

from which one deduces that \((*) = 0\).

**Proposition A.7.** No matter whether the polynomial \(w\) (defined in (A.12)) is a combination of \(a^3, \ldots, b^3\) as in Lemma A.3 or not, the graded \(S\)-module \(H^0_S(O_Z)\) has a graded
free resolution of the form:

\[
\begin{array}{ccc}
S(-2) & 2S(-1) & S \\
\oplus & \oplus & \oplus \\
S(l-2) & 2S(l-1) & S(l) \\
\oplus & \oplus & \oplus \\
S(2l+m-2) & 2S(2l+m-1) & S(2l+m) \\
\oplus & \oplus & \oplus \\
S(3l+m+n-2) & 2S(3l+m+n-1) & S(3l+m+n) \\
0 \rightarrow & \delta_2 \rightarrow & \delta_1 \rightarrow & \delta_0 \rightarrow H^0_s(\mathcal{O}_Z) \rightarrow 0
\end{array}
\]

with \( \delta_0 \) defined by the generators 1, \( e_1, e_2, e_3 \) of \( H^0_s(\mathcal{O}_Z) \) considered at the beginning of this subsection and with \( \delta_1, \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
x & y & 0 & 0 & 0 & 0 & 0 \\
-b & -b & x & y & 0 & 0 & 0 \\
-a & -b' & -pa & -pb & x & y & 0 \\
-a'' & -b'' & -pa-a'q' & -pb' & qa & qb & x & y
\end{pmatrix},
\begin{pmatrix}
-y & 0 & 0 & 0 \\
x & 0 & 0 & 0 \\
b & -y & 0 & 0 \\
-a & x & 0 & 0 \\
b' & pb & -y & 0 \\
-a' & -pa & x & 0 \\
b'' & p'b+qb' & qb & -y \\
-a'' & -pa-a'q' & qa & x
\end{pmatrix}
\]

**Proof.** \( H^0_s(\mathcal{O}_Z) \) admits a filtration by graded \( S \)-submodules with the successive quotients isomorphic to \( S(L) := S/I(L), S(L)(l), S(L)(2l+m) \) and \( S(L)(3l+m+n) \), respectively. Using the well-known graded free resolution of \( S(L) \) over \( S \) one deduces that \( H^0_s(\mathcal{O}_Z) \) has a graded free resolution of the numerical shape from the statement. Moreover, the linear parts of the differentials \( \delta_1 \) and \( \delta_2 \) can be deduced from the resolution of \( S(L) \). The rest of the matrices defining \( \delta_1 \) and \( \delta_2 \) can be easily guessed. For example, \( \delta_1 \) is defined by the matrix of relations between the generators 1, \( e_1, e_2, e_3 \) of \( H^0_s(\mathcal{O}_Z) \). The module of these relations is generated by the relations of the following form: one multiplies each generator of \( H^0_s(\mathcal{O}_Z) \) by \( x \) and by \( y \) and one expresses the results as combinations of the next generators. \( \square \)

**Remark A.8.** We recall, for completeness, an algorithmic procedure for getting a minimal free resolution of a graded \( S \)-module \( M \) from a non-minimal one. Consider a complex:

\[
F_* : \cdots F_{s+1} \xrightarrow{d_{s+1}} F_s \xrightarrow{d_s} F_{s-1} \xrightarrow{d_{s-1}} F_{s-2} \cdots
\]

of graded free \( S \)-modules of finite rank. Assume that the \( (i, j) \) entry of the matrix of \( d_s \) is a non-zero constant \( c \in k \). Performing elementary operations on the columns of the matrix of \( d_s \), one can turn into 0 all the other entries of its \( i \)th row. One gets, in this way, a new morphism \( d'_s : F_s \rightarrow F_{s-1} \). Then:

(i) deleting the \( j \)th rank 1 direct summand of \( F_s \) and the \( i \)th rank 1 direct summand of \( F_{s-1} \),
(ii) deleting the \( i \)th row and the \( j \)th column of the matrix of \( d'_s \),
(iii) deleting the \( i \)th column of the matrix of \( d_{s-1} \),
(iv) deleting the \( j \)th row of the matrix of \( d_{s+1} \).
one obtains a complex $F'_\ast$ such that $F'_\ast$ is isomorphic to the direct sum of $F'_\ast$ and of a complex of the form: \[ \cdots \rightarrow \cdots \rightarrow 0 \rightarrow S(-a) \rightarrow S(-a) \rightarrow 0 \rightarrow \cdots \]

A.2. **Primitive structures of degree 4.** We assume, in this subsection, using the notation introduced in the previous one, that $m = n = 0$ and that $l \geq 0$ (if $m = n = 0$ and $l = -1$ then $Z$ is the divisor $4L$ on the plane $H \supset L$ of equation $-bx + ay = 0$ as it follows from Prop. A.6). One has, in this case, $p = q = 1$ and the exact sequence \((A.10)\) becomes:

\[
0 \rightarrow 2\mathcal{O}_L(-l - 1) \xrightarrow{\begin{pmatrix} 1 & 0 \\ -a & -b \end{pmatrix}} 2\mathcal{O}_L(-l - 1) \oplus \mathcal{O}_L(a, b, 1) \rightarrow \mathcal{O}_L \rightarrow 0.
\]

In particular, $l_1 = l_2 = l + 1$. The elements $v_0, v_1, v_2 \in H^0(\mathcal{O}_L(l))$ from relation \((A.7)\) are uniquely determined and the relations \((A.13)\) become:

\[
-a\Delta_1 = v_0a^3 + v_1a^2b + v_2ab^2,
\]

\[
b\Delta_1 = v_0a^2b + v_1ab^2 + v_2b^3.
\]

In relation \((A.12)\) one must have $f = g = 0$, and this relation becomes:

\[
(A.17) \quad \Delta_2 - 2v_0aa' - v_1(ab' + a'b) - 2v_2bb' = w,
\]

with $w \in H^0(\mathcal{O}_L(4l + 2))$. Prop. A.6 becomes in this special case:

**Corollary A.9.** Assume that $m = n = 0$, $l \geq 0$ and that $w$ (defined in \((A.17)\)) can be written as a combination $w = w_0a^3 + w_1a^2b + w_2ab^2 + w_3b^3$ with $w_0, \ldots, w_3 \in H^0(\mathcal{O}_L(l - 1))$ (uniquely determined, taking into account \((A.9)\)). Recalling the notation:

\[
F_2 = \begin{vmatrix} a \\ x \\ y \end{vmatrix}, \quad F_3 = F_2 + v_0x^2 + v_1xy + v_2y^2,
\]

the homogeneous ideal $I(Z) \subset S$ is generated by:

\[
F_4 = F_3 + w_0x^3 + w_1a^2y + w_2xy^2 + w_3y^3, \quad x^4, \quad x^3y, \quad x^2y^2, \quad xy^3, \quad y^4.
\]

**Proof.** It suffices to notice that the polynomials $G_1$ and $G_2$ occurring in the statement of Prop. A.6 are equal, in the particular case under consideration, to $xF_3$ and $yF_3$, respectively, and that these polynomials and the polynomials $x^2F_2, xyF_2, y^2F_2$ belong to the ideal generated by the polynomials from the statement of the corollary. \(\square\)

**A graded free resolution of $I(Z)$ under the hypothesis of Cor. A.9.** Using Remark A.8 to cancel redundant direct summands of the terms of the resolution of $I(Z)$ described after Prop. A.6 and taking into account that the polynomials $\alpha_i, \beta_i, i = 0, 1, 2$, introduced in Remark A.5 are all zero, one gets that the ideal $I(Z)$ has, under the hypothesis of Cor. A.9, a free resolution of the form:

\[
0 \rightarrow 3S(-l - 6) \xrightarrow{d'_2} 4S(-l - 5) \oplus 4S(-l - 2) \oplus 5S(-4) \rightarrow I(Z) \rightarrow 0.
\]
with $d'_1$ and $d'_2$ defined by the matrices:

\[
\begin{pmatrix}
 x^3 & x^2 y & xy^2 & y^3 & 0 & 0 & 0 & 0 \\
b - v_0 x - w_0 x^2 & 0 & 0 & 0 & - y & 0 & 0 & 0 \\
a - v_1 x - w_1 x^2 & b - v_0 x - w_0 x^2 & - w_0 y & - w_0 y^2 & x - y & 0 & 0 & 0 \\
v_2 x - w_2 x^2 & - a - v_1 x - w_1 x^2 & b - v_0 x - w_1 x y & - v_0 y - w_1 y^2 & 0 & x - y & 0 & 0 \\
v_3 x^2 & - v_2 x - w_3 x^2 & - a - v_1 x - w_2 x y & b - v_1 y - w_2 y^2 & 0 & 0 & x - y & 0 \\
0 & - w_3 x^2 & - v_2 x - w_3 x y & - a - v_2 y - w_3 y^2 & 0 & 0 & 0 & x \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
 - y & 0 & 0 \\
x & - y & 0 \\
0 & x & - y \\
0 & 0 & x \\
-b + v_0 x + w_0 x^2 & 0 & 0 \\
 a + v_1 x + w_1 x^2 & - b + v_0 x & 0 \\
v_2 x + w_2 x^2 & a + v_1 x & - b \\
v_3 x^2 & v_2 x & a \\
\end{pmatrix}
\]

**Lemma A.10.** Assume that $m = n = 0$, $l \geq 0$ and that the polynomial $w$ (defined in (A.17)) cannot be written as a combination of $a^3, \ldots, b^3$ as in Cor. A.9. Since $x_0 w, x_1 w \in H^0(L(4l + 3))$, one deduces, using the exact sequence (A.9), that there exist uniquely determined polynomials $w_{ij} \in H^0(L(l))$, $i = 0, 1, j = 0, \ldots, 3$, such that:

\[
x_i w = w_0 a^3 + w_1 a^2 b + w_2 a b^2 + w_3 b^3, \quad i = 0, 1.
\]

Then the graded $H^0_*(L) = k[x_0, x_1]$-module $H^0_*(\text{Ker} \varepsilon'_4)$ is generated by the columns of the matrix:

\[
\begin{pmatrix}
x_0 & x_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
w_0 & w_{10} & v_0 & 0 & - b & 0 & 0 \\
w_{01} & w_{11} & v_1 & v_0 & a & - b & 0 \\
w_{02} & w_{12} & v_2 & v_1 & 0 & a & - b \\
w_{03} & w_{13} & 0 & v_2 & 0 & 0 & a \\
\end{pmatrix}
\]

**Proof.** Recall, from the previous subsection A.1, that, under the hypothesis of the lemma:

\[
\text{Ker} \varepsilon'_4 \subset L(-l - 2) \oplus 2L(-l - 3) \oplus 4L(3l)
\]

and that, recalling the exact sequence (A.9), one has an exact sequence:

(A.18) \quad 0 \rightarrow 3L(-l - 4) \rightarrow \text{Ker} \varepsilon'_4 \rightarrow \text{Ker} \varepsilon''_4 \rightarrow 0.

where $\text{Ker} \varepsilon''_4$ is the (isomorphic) image of the morphism:

\[
\nu''_4 : L(-l - 2) \oplus 2L(-l - 3) \rightarrow L(-l - 2) \oplus 2L(-l - 3) \oplus L(3l)
\]
Remark A.11. We want to emphasize, for later use, a relation between the polynomials \( w_{ij} \in H^0(\mathcal{O}_L(l)) \) defined in Lemma A.10 (under the assumption that \( w \) cannot be written as a combination of \( a^3, \ldots, b^3 \)). One deduces, from the defining relations of the \( w_{ij} \)'s, that

\[
-w_1(w_{00}a^3 + w_{01}a^2b + w_{02}ab^2 + w_{03}b^3) + x_0(w_{10}a^3 + w_{11}a^2b + w_{12}ab^2 + w_{13}b^3) = 0,
\]

hence, using the exact sequence (A.9), one gets a matrix relation:

\[
(A.19) \quad -x_1 \begin{pmatrix} w_{00} \\ w_{01} \\ w_{02} \\ w_{03} \end{pmatrix} + x_0 \begin{pmatrix} w_{10} \\ w_{11} \\ w_{12} \\ w_{13} \end{pmatrix} = \begin{pmatrix} -b & 0 & 0 \\ a & -b & 0 \\ 0 & a & -b \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}
\]

with \( \gamma_0, \gamma_1, \gamma_2 \in k \). At least one of these three constants must be non-zero because, otherwise, \( x_0 \) would divide each of the polynomials \( w_{00}, \ldots, w_{03} \) and this would contradict our assumption that \( w \) cannot be written as a combination of \( a^3, \ldots, b^3 \).

Proposition A.12. Assume that \( m = n = 0, l \geq 0 \) and that the polynomial \( w \) (defined in (A.17)) cannot be written as a combination of \( a^3, \ldots, b^3 \) as in Cor. A.9. Then the homogeneous ideal \( I(Z) \subset S \) is generated by the polynomials:

\[
F_{40} = x_0F_3 + w_{00}x^3 + w_{01}x^2y + w_{02}xy^2 + w_{03}y^3,
\]

\[
F_{41} = x_1F_3 + w_{10}x^3 + w_{11}x^2y + w_{12}xy^2 + w_{13}y^3,
\]

\[
G_1 = xF_3, \quad G_2 = yF_3, \quad x^4, \quad x^3y, \quad x^2y^2, \quad xy^3, \quad y^4
\]

with the \( w_{ij} \)'s as in Lemma A.10.

Proof. Recall, from Subsection A.1 that (under the hypothesis of the proposition) if

\[
\nu_3 : \mathcal{O}_L(-l - 2) \oplus 2\mathcal{O}_L(-l - 3) \rightarrow 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2)
\]
is the morphism defined by the matrix:

\[
\begin{pmatrix}
-b & 0 & 0 \\
a & 0 & 0 \\
v_0 & -b & 0 \\
v_1 & a & -b \\
v_2 & 0 & a
\end{pmatrix}
\]

then \(\nu_3 \oplus \text{id}_{4\mathcal{O}(-3)}\) induces an isomorphism \(\text{Ker} \varepsilon'_4 \xrightarrow{\sim} \text{Ker} \varepsilon_4\).

Now, multiplying to the left the matrix from the statement of Lemma \(\text{A.10}\) by the matrix of \(\nu_3 \oplus \text{id}_{4\mathcal{O}(-3)}\) one gets the matrix:

\[
\begin{pmatrix}
-bx_0 & -bx_1 & 0 & 0 & 0 & 0 \\
ax_0 & ax_1 & 0 & 0 & 0 & 0 \\
v_0x_0 & v_0x_1 & -b & 0 & 0 & 0 \\
v_1x_0 & v_1x_1 & a & -b & 0 & 0 \\
v_2x_0 & v_2x_1 & 0 & a & 0 & 0 \\
w_00 & w_10 & v_0 & 0 & -b & 0 \\
w_01 & w_11 & v_1 & v_0 & a & -b \\
w_02 & w_12 & v_2 & v_1 & 0 & a & -b \\
w_03 & w_{13} & 0 & v_2 & 0 & 0 & a
\end{pmatrix}.
\]

It follows that the columns of this matrix generate the graded \(H^0_*(\mathcal{O}_L)\)-module \(H^0_*(\text{Ker} \varepsilon_4)\). Using the exact sequence \((\text{A.4})\) and the fact that:

\[
\mathcal{J}_L/\mathcal{J}_L^4 = \mathcal{O}_L(-1)x \oplus \mathcal{O}_L(-1)y \oplus \mathcal{O}_L(-2)x^2 \oplus \cdots \oplus \mathcal{O}_L(-3)y^3
\]

one deduces that \(I(Z)\) is generated by the polynomials from the statement and by \(x^2F_2, xyF_2, y^2F_2\). But the latter polynomials belong to the ideal generated by \(G_1, G_2, x^4, \ldots, y^4\).

\[\square\]

**A graded free resolution of \(I(Z)\) under the hypothesis of Prop. \(\text{A.12}\)**

One deduces, from the exact sequence \((\text{A.18})\) in the proof of Lemma \(\text{A.10}\) an exact sequence:

\[
0 \longrightarrow 2\mathcal{O}_L(-l - 3) \oplus 3\mathcal{O}_L(-l - 4) \xrightarrow{\rho} \text{Ker} \varepsilon'_4 \xrightarrow{\pi} \mathcal{O}_L(-l - 2) \longrightarrow 0
\]

where \(\rho\) is the corestriction of the morphism:

\[
2\mathcal{O}_L(-l - 3) \oplus 3\mathcal{O}_L(-l - 4) \longrightarrow \mathcal{O}_L(-l - 2) \oplus 2\mathcal{O}_L(-l - 3) \oplus 4\mathcal{O}_L(-3)
\]

defined by the last five columns of the matrix from Lemma \(\text{A.10}\) and \(\pi\) is the restriction of the projection:

\[
\text{pr}_1 : \mathcal{O}_L(-l - 2) \oplus 2\mathcal{O}_L(-l - 3) \oplus 4\mathcal{O}_L(-3) \longrightarrow \mathcal{O}_L(-l - 2).
\]

Since, as we recalled in the proof of Prop. \(\text{A.12}\), \(\nu_3 \oplus \text{id}_{4\mathcal{O}(-3)}\) induces an isomorphism \(\text{Ker} \varepsilon'_4 \xrightarrow{\sim} \text{Ker} \varepsilon_4\), one derives an exact sequence:

\[
\begin{equation}
(\text{A.20}) \quad 0 \longrightarrow 2\mathcal{O}_L(-l - 3) \oplus 3\mathcal{O}_L(-l - 4) \xrightarrow{\tilde{\rho}} \text{Ker} \varepsilon_4 \xrightarrow{\tilde{\pi}} \mathcal{O}_L(-l - 2) \longrightarrow 0
\end{equation}
\]
where \( \tilde{\rho} \) is the corestriction of the morphism:
\[
R : 2\mathcal{O}_L(-l - 3) \oplus 3\mathcal{O}_L(-l - 4) \longrightarrow 2\mathcal{O}_L(-1) \oplus 3\mathcal{O}_L(-2) \oplus 4\mathcal{O}_L(-3)
\]
defined by the matrix:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-b & 0 & 0 & 0 & 0 \\
a & -b & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 \\
v_0 & 0 & -b & 0 & 0 \\
v_1 & v_0 & a & -b & 0 \\
v_2 & v_1 & 0 & a & -b \\
0 & v_2 & 0 & 0 & a
\end{pmatrix}
\]
and where \( \tilde{\pi} \) maps the elements:
\[
(-bx_i, ax_i, v_0x_i, v_1x_i, v_2x_i, w_{i0}, w_{i1}, w_{i2}, w_{i3})^t, \ i = 0, 1,
\]
of \( H^0((\text{Ker} \varepsilon_4)(l + 3)) \) into the elements \( x_0, x_1 \) of \( H^0(\mathcal{O}_L(1)) \).

It is easy to check that the morphism \( R \) considered above can be identified with the morphism:
\[
F_3 : - : (\mathcal{I}_L/\mathcal{I}_L^3)(-l - 2) \longrightarrow \mathcal{I}_L^2/\mathcal{I}_L^4.
\]
Applying, now, \( H^0(\cdot) \) to the exact sequence (A.20) one gets an exact sequence of graded \( S \)-modules:
\[
(A.21) \quad 0 \longrightarrow (I(L)/I(L)^3)(-l - 2) \xrightarrow{F_3 \cdot -} I(Z)/I(L)^4 \longrightarrow S(L_+(-l - 2) \longrightarrow 0
\]
where \( S(L) = S/I(L) \) and \( S(L_+) = x_0S(L) + x_1S(L) = \bigoplus_{i \geq 1} S(L)_i \).

We provide, now, explicit minimal free resolutions of the graded \( S \)-modules \( S(L)_+ \) and \( I(L)/I(L)^3 \). Using the exact sequence:
\[
0 \longrightarrow S(L)(-2) \xrightarrow{(-x_1, x_0)} 2S(L)(-1) \xrightarrow{(x_0, x_1)} S(L)_+ \longrightarrow 0
\]
on one sees that the tensor product of the complexes:
\[
S(-2) \xrightarrow{(-x_1, x_0)} 2S(-1), \ S(-2) \xrightarrow{(y, x)} 2S(-2) \xrightarrow{(x, y)} S
\]
is the following minimal free resolution of \( S(L)_+ \):
\[
0 \longrightarrow S(-4) \xrightarrow{(-y, x)} 4S(-3) \xrightarrow{(-x_1, x_0, 0, 0)} 5S(-2) \xrightarrow{(-x_1, x, y, 0, 0)} 2S(-1)
\]
As for $I(L)/I(L)^3$, one uses the standard free resolutions of $I(L)/I(L)^2 \cong 2S(L)(-1)$ and of $I(L)^2/I(L)^3 \cong 3S(L)(-2)$ to get a non-minimal free resolution of $I(L)/I(L)^3$ over $S$ and then, cancelling redundant terms (see Remark A.8), one gets the following minimal free resolution:

$$
\begin{array}{c}
0 \to 3S(-4) \xrightarrow{\begin{pmatrix} 0 & 0 & -y^2 \\
-\gamma_0 x - \gamma_1 y & -x_0 & x \end{pmatrix}} S(-2) \oplus \begin{pmatrix} -y & x^2 & xy & y^2 \\
x & 0 & 0 & y^2 \end{pmatrix} \to 2S(-1) \xrightarrow{(x, y)} I(L) \to I(L)^3
\end{array}
$$

One deduces, now, from the exact sequences (A.4) and (A.21), that $I(Z)$ admits a non-minimal free resolution of the form:

$$
0 \longrightarrow S(-l - 6) \xrightarrow{d_3} 4S(-l - 5) \oplus \begin{pmatrix} 0 \\
3S(-l - 6) \end{pmatrix} \xrightarrow{d_2} 4S(-l - 5) \oplus \begin{pmatrix} 0 \\
4S(-5) \end{pmatrix} \xrightarrow{d_1} 4S(-l - 3) \oplus \begin{pmatrix} 0 \\
5S(-4) \end{pmatrix} \longrightarrow I(Z) \longrightarrow 0.
$$

This resolution has a filtration with successive quotients the above minimal free resolutions of $S(L)_+(-l - 2)$, $(I(L)/I(L)^3)(-l - 2)$ and $I(L)^4$, respectively. Consequently, large parts of the matrices of $d_1$, $d_2$ and $d_3$ are known. In order to get the entire matrix of $d_1$ one has to extend the known parts of its columns to relations between the generators of $I(Z)$ from Prop. [A.12]. This presents some difficulty only for the first column. Multiplying to the left the matrix relation (A.19) by $(x^3, x^2y, xy^2, y^3)$ one gets the following relation:

$$(A.22) \quad -x_1(w_{00}x^3 + w_{01}x^2y + w_{02}xy^2 + w_{03}y^3) + x_0(w_{10}x^3 + w_{11}x^2y + w_{12}xy^2 + w_{13}y^3) = (\gamma_0 x^2 + \gamma_1 xy + \gamma_2 y^2)(-\delta x + a y),$$

from which one deduces that:

$$-x_1F_{40} + x_0F_{41} - (\gamma_0 x^2 + \gamma_1 xy + \gamma_2 y^2)F_3 + (\gamma_0 x^2 + \gamma_2 y^2)(v_0 x^2 + v_1 xy + v_2 y^2) = 0.$$

Now, the matrices of $d_1$, $d_2$ and $d_3$ are the following ones:

$$
\begin{pmatrix}
-x_1 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_0 x - \gamma_1 y & -x_0 & 0 & -x_1 & 0 & -y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_2 y & 0 & -x_0 & 0 & -x_1 & x & 0 & 0 & 0 & y^2 & 0 & 0 & 0 & 0 & 0 \\
\theta_0 & -w_{00} & 0 & -w_{10} & 0 & 0 & b \cdot v_0 x & 0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 0 \\
\theta_1 & -w_{01} & -w_{00} & -w_{11} & -w_{10} & 0 & -a \cdot v_1 x & b \cdot v_0 x & 0 & 0 & x & -y & 0 & 0 & 0 & 0 & 0 \\
\theta_2 & -w_{02} & -w_{01} & -w_{12} & -w_{11} & 0 & -v_2 x & -a \cdot v_1 x & b \cdot v_0 x & -v_0 y & 0 & x & -y & 0 & 0 & 0 & 0 \\
\theta_3 & -w_{03} & -w_{02} & -w_{13} & -w_{12} & 0 & 0 & -v_2 x & -a \cdot v_1 x & b \cdot v_1 y & 0 & 0 & x & -y & 0 & 0 & 0 & 0 \\
\theta_4 & 0 & -w_{03} & 0 & -w_{13} & 0 & 0 & -v_2 x & -a \cdot v_2 y & 0 & 0 & 0 & x & -y & 0 & 0 & 0 & 0
\end{pmatrix}
$$
is defined by three polynomials follows that, as an \( O_l \) for some integer \( (A.23) \)

\[
0 \to \gamma_{2} y \to x_{0} \to x_{1} \to x \to 0 \to 0.
\]

One deduces an exact sequence:

\[
\gamma(L) \text{ the following shape:}
\]

In order to check that \( d_{1}d_{2} = 0 \) and \( d_{2}d_{3} = 0 \) one has to use the above polynomial relation \((A.22)\). Using Remark \( A.8 \), one can cancel the redundant terms of the above free resolution of \( I(Z) \). One has to consider three cases: (I) \( \gamma_{0} = 0 \), (II) \( \gamma_{0} = 0 \) and \( \gamma_{1} \neq 0 \), (III) \( \gamma_{0} = \gamma_{1} = 0 \) and \( \gamma_{2} \neq 0 \). In all the cases, one gets a free resolution of \( I(Z) \) having the following shape:

\[
0 \to 2S(-l-5) \oplus 2S(-l-6) \oplus 2S(-l-6) \oplus 2S(-l-5) \oplus 4S(-l-3) \oplus 4S(-5) \to I(Z) \to 0.
\]

**A.3. Thick structures of degree 4.** According to Bănică and Forster \([5, \S \text{4}]\), if \( W \) is such a structure supported on the line \( L \) then \( \mathcal{O}_{W}^{3} \subset \mathcal{I}_{W} \subset \mathcal{I}_{L}^{2} \) and one has an exact sequence of the form:

\[
0 \to \mathcal{I}_{W} / \mathcal{I}_{L}^{3} \to \mathcal{I}_{L}^{2} / \mathcal{I}_{L}^{3} \xrightarrow{\varepsilon} \mathcal{O}_{L}(l) \to 0
\]

for some integer \( l \). Since \( \mathcal{I}_{L}^{2} / \mathcal{I}_{L}^{3} \simeq 3\mathcal{O}_{L}(-2) \), one must have \( l \geq -2 \). The epimorphism \( \varepsilon \) is defined by three polynomials \( p_{0}, p_{1}, p_{2} \in H^{0}(\mathcal{O}_{L}(l+2)) \) having no common zero on \( L \). One deduces an exact sequence:

\[
(A.23) \quad 0 \to \mathcal{O}_{L}(l) \to \mathcal{O}_{W} \to \mathcal{O}_{L(l)} \to 0
\]

where \( L^{(1)} \) is the first infinitesimal neighbourhood of \( L \) in \( \mathbb{P}^{3} \), defined by the ideal \( \mathcal{I}_{L}^{2} \). It follows that, as an \( \mathcal{O}_{L} \)-module:

\[
\mathcal{O}_{W} \simeq \mathcal{O}_{L} \oplus 2\mathcal{O}_{L}(-1) \oplus \mathcal{O}_{L}(l).
\]
The graded $H^0(\mathcal{O}_L)$-module $H^0(\mathcal{O}_W)$ has a system of generators consisting of $1 \in H^0(\mathcal{O}_W)$, $e_0, e_1 \in H^0(\mathcal{O}_W(1))$ and $e_2 \in H^0(\mathcal{O}_W(-l))$. The multiplicative structure of $H^0(\mathcal{O}_W)$ is defined by:

$$e_0^2 = p_0 e_2, \ e_0 e_1 = p_1 e_2, \ e_1^2 = p_2 e_2,$$

and its graded $S$-module structure by:

$$x \cdot 1 = e_0, \ y \cdot 1 = e_1, \ x \cdot e_0 = x \cdot 1 \cdot e_0 = e_0^2 = p_0 e_2, \ y \cdot e_0 = x \cdot e_1 = p_1 e_2, \ y \cdot e_1 = p_2 e_2.$$

One has an exact sequence:

$$0 \longrightarrow \mathcal{O}_L(-m) \oplus \mathcal{O}_L(-n) \xrightarrow{(f_0 \ g_0, f_1 \ g_1, f_2 \ g_2)} 3\mathcal{O}_L \xrightarrow{(p_0, p_1, p_2)} \mathcal{O}_L(l+2) \longrightarrow 0 \tag{A.24}$$

with $0 \leq m \leq n$ and $m+n = l+2$. Since $\mathcal{I}_W / \mathcal{I}_L^3$ is annihilated by $\mathcal{I}_L$ it is already an $\mathcal{O}_L$-module. Using the above exact sequence, it follows that, actually:

$$\mathcal{I}_W / \mathcal{I}_L^3 \simeq \mathcal{O}_L(-m-2) \oplus \mathcal{O}_L(-n-2).$$

Since $H^1(\mathcal{I}_L^3) = 0$, one deduces an exact sequence of graded $S$-modules:

$$0 \longrightarrow I(L)^3 \longrightarrow I(W) \longrightarrow S(L)(-m-2) \oplus S(L)(-n-2) \longrightarrow 0$$

with $S(L) = S/I(L)$, from which one gets immediately:

**Proposition A.13.** The homogeneous ideal $I(W) \subset S$ of $W$ is generated by the following polynomials:

$$F = f_0 x^2 + f_1 xy + f_2 y^2, \ G = g_0 x^2 + g_1 xy + g_2 y^2, \ x^3, \ x^2 y, \ xy^2, \ y^3.$$

Moreover, one also gets that $I(W)$ has a graded free resolution of the form:

$$0 \longrightarrow S(-m-4) \oplus 2S(-m-3) \oplus S(-m-2) \xrightarrow{d_2} S(-n-4) \oplus 2S(-n-3) \oplus S(-n-2) \xrightarrow{d_1} 3S(-4) \oplus 4S(-3) \longrightarrow I(W) \longrightarrow 0$$

with $d_1$ and $d_2$ defined by the matrices:

$$\begin{pmatrix}
    x & y & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x & y & 0 & 0 & 0 \\
    -f_0 & 0 & -g_0 & 0 & -y & 0 & 0 \\
    -f_1 & -f_0 & -g_1 & -g_0 & x & -y & 0 \\
    -f_2 & -f_1 & -g_2 & -g_1 & 0 & x & -y \\
    0 & -f_2 & 0 & -g_2 & 0 & 0 & x
\end{pmatrix}, \begin{pmatrix}
    -y & 0 & \\
    0 & -y & \\
    0 & x & \\
    f_0 & g_0 & \\
    f_1 & g_1 & \\
    f_2 & g_2
\end{pmatrix}.$$
Proposition A.14. Assume that \( W \) is properly locally CM everywhere, i.e., it is not l.c.i. except at finitely many points. Then \( l \) is even, \( l = 2l' \) with \( l' \geq -1 \), and there exist two polynomials \( q_0, q_1 \in H^0(\mathcal{O}_L(l' + 1)) \) having no common zero on \( L \) such that \( I(W) \) is generated by:

\[
\begin{vmatrix}
  x & q_0 \\
  y & q_1 
\end{vmatrix}_{2 \times 2}, \quad \begin{vmatrix}
  x & x^2 \\
  y & xy 
\end{vmatrix}_{2 \times 2}, \quad \begin{vmatrix}
  x & xy \end{vmatrix}_{2 \times 1}, \quad \begin{vmatrix}
  y & y^2 
\end{vmatrix}_{2 \times 1}.
\]

Notice that if \( X \) is the double structure on \( L \) defined by the epimorphism \((q_0, q_1): \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_L(l') (\text{see Subsection A.3 below})\) then \( I(W) = I(L)I(X) \).

Proof. By [5, Prop. 4.3], the hypothesis of the proposition implies that \( p_1^2 = p_0p_2 \). Since \( p_0, p_1, p_2 \) have no common zero on \( L \), it follows that \( p_0 \) and \( p_2 \) have no common zero on \( L \). One deduces, now, that \( p_0 \) and \( p_2 \) must be “perfect squares”, i.e., that there exist polynomials \( q_0 \) and \( q_1 \) such that \( p_0 = q_0^2 \) and \( p_2 = q_1^2 \). Then \( p_1 = q_0q_1 \) and the exact sequence \((A.24)\) becomes:

\[
0 \rightarrow 2\mathcal{O}_L(-l' - 1) \rightarrow 3\mathcal{O}_L(q_0^2, q_0q_1; q_1^2) \rightarrow \mathcal{O}_L(l + 2) \rightarrow 0.
\]

One can apply, now, Prop. A.13.

Applying \( H^0_s(\mathcal{O}_W) \) to the exact sequence \((A.24)\) one gets an exact sequence:

\[
0 \rightarrow S(L)(l) \rightarrow H^0_s(\mathcal{O}_W) \rightarrow S/I(L)^2 \rightarrow 0
\]

from which one gets readily the following:

Proposition A.15. The graded \( S \)-module \( H^0_s(\mathcal{O}_W) \) admits the following free resolution:

\[
0 \rightarrow \bigoplus_{S(-l - 2)} S(-l - 1) \rightarrow \bigoplus_{2S(-l - 1)} S(-l) \rightarrow \bigoplus_{S(-l)} S \rightarrow H^0_s(\mathcal{O}_W).
\]

A.4. Quasiprimitive structures of degree 3. We record, quickly, for reference, the similar but easier results concerning quasiprimitive multiple structures \( Y \) of degree 3 on the line \( L \). We use the notation introduced in Subsection A.1 restricted in an obvious way to the degree 3 case. As an \( \mathcal{O}_L \)-module:

\[
\mathcal{O}_Y \simeq \mathcal{O}_L \oplus \mathcal{O}_L(l) \oplus \mathcal{O}_L(2l + m).
\]

Recalling that \( F_2 = -bx + ay \) and the definition \((A.7)\) of the polynomials \( v_0, v_1, v_2 \), one has that the homogeneous ideal \( I(Y) \subset S \) of \( Y \) is generated by:

\[
F_3 = pF_2 + v_0x^2 + v_1xy + v_2y^2, \quad xF_2, \quad yF_2, \quad x^3, \quad x^2y, \quad xy^2, \quad y^3.
\]
I(Y) admits a graded free resolution over $S$ of the form:

$$
0 \longrightarrow S(-l-m-4) \oplus 2S(-l-m-3) \oplus S(-l-m-2) \oplus 2S(-l-5) \oplus 4S(-l-4) \longrightarrow 4S(-l-3) \longrightarrow I(Y) \longrightarrow 0
$$

with $d_1$ and $d_2$ defined by the matrices:

$$
\begin{pmatrix}
  x & y & 0 & 0 & 0 & 0 & 0 & 0 \\
  -p & 0 & x & y & 0 & 0 & 0 & 0 \\
  0 & -p & 0 & 0 & x & y & 0 & 0 \\
  -v_0 & 0 & b & 0 & 0 & 0 & -y & 0 \\
  -v_1 & -v_0 & -a & b & b & 0 & x & -y \\
  -v_2 & -v_1 & 0 & -a & -a & b & 0 & x & -y \\
  0 & -v_2 & 0 & 0 & -a & 0 & 0 & x \\
\end{pmatrix},
\begin{pmatrix}
  -y & 0 & 0 \\
  x & 0 & 0 \\
  0 & -y & 0 \\
  -p & x & 0 \\
  p & 0 & -y \\
  v_0 & -b & 0 \\
  v_1 & a & -b \\
  v_2 & 0 & a \\
\end{pmatrix}.
$$

Notice that in the primitive case, i.e., when $m = 0$ hence $p = 1$, $I(Y)$ is generated only by $F_3$, $x^3$, $x^2y$, $xy^2$, $y^3$ and one can cancel a number of direct summands of the terms of the above free resolution of $I(Y)$.

The graded $S$-module $H^0_*(\mathcal{O}_Y)$ admits the following free resolution:

$$
0 \longrightarrow S(-2) \oplus 2S(-1) \oplus S \oplus S(l-2) \oplus 2S(l-1) \oplus S(l) \longrightarrow H^0_*(\mathcal{O}_Y) \longrightarrow 0,
$$

with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x & y & 0 & 0 & 0 & 0 \\
  -a & -b & x & y & 0 & 0 \\
  -a' & -b' & -pa & -pb & x & y \\
\end{pmatrix},
\begin{pmatrix}
  -y & 0 & 0 \\
  x & 0 & 0 \\
  b & -y & 0 \\
  -a & x & 0 \\
  b' & pb & -y \\
  -a' & -pa & x \\
\end{pmatrix}.
$$

A.5. **Primitive structures of degree 2.** Let $X$ be a locally CM curve of degree 2 supported by the line $L$. Then $X$ is automatically a primitive structure, $\mathcal{O}_X \simeq \mathcal{O}_L \oplus \mathcal{O}_L(l)$ as an $\mathcal{O}_L$-module, $I(X) \subset S$ is generated by $F_2 = -bx+ay$, $x^2$, $xy$, $y^2$, and one has graded
free resolutions:

\[
\begin{array}{cccccccc}
0 & \rightarrow & S(-l-4) & \rightarrow & 2S(-l-3) & \oplus & 2S(-3) & \oplus & S(-l-2) & \rightarrow & I(X) & \rightarrow & 0 \\
\end{array}
\]

Moreover, by a general result of Ferrand [15], one has \( \omega_X \cong \mathcal{O}_X(-l-2) \).

**APPENDIX B. CURVES WITH A MULTIPLE LINE AS A COMPONENT**

We shall denote, in this appendix, the projective coordinates on \( \mathbb{P}^3 \) by \( x_0, x_1, x_2, x_3 \), hence the projective coordinate ring of \( \mathbb{P}^3 \) is \( S = k[x_0, x_1, x_2, x_3] \). We consider the following points an lines in \( \mathbb{P}^3 \):

\[
P_0 = (1 : 0 : 0 : 0), \ P_1 = (0 : 1 : 0 : 0), \ P_2 = (0 : 0 : 1 : 0), \ P_3 = (0 : 0 : 0 : 1)
\]

\[
L_1 = P_0 \cdot P_1 = \{x_2 = x_3 = 0\}, \ L_1' = P_2 \cdot P_3 = \{x_0 = x_1 = 0\}
\]

\[
L_2 = P_0 \cdot P_2 = \{x_1 = x_3 = 0\}, \ L_2' = P_1 \cdot P_3 = \{x_0 = x_2 = 0\}
\]

\[
L_3 = P_0 \cdot P_3 = \{x_1 = x_2 = 0\}, \ L_3' = P_1 \cdot P_2 = \{x_0 = x_3 = 0\}
\]

We shall need the following easy results:

**Lemma B.1.** Let \( Z \) and \( W \) be closed subschemes of \( \mathbb{P}^3 \) such that \( Z \cap W = \emptyset \). Assume that \( Z \) is arithmetically CM of pure codimension 2. Consider minimal graded free resolutions:

\[
0 \rightarrow A_1 \rightarrow A_0 \rightarrow I(Z) \rightarrow 0
\]

\[
0 \rightarrow B_2 \rightarrow B_1 \rightarrow B_0 \rightarrow I(W) \rightarrow 0.
\]

Then:

(a) \( A_1 \otimes_S B_1 \) is a minimal graded free resolution of the ideal \( I(Z)I(W) \).

(b) \( I(Z \cup W) = I(Z)I(W) \) if, moreover, \( W \) is arithmetically CM of pure codimension 2.

**Proof.** (a) Tensorizing by \( \mathcal{I}_W \) the exact sequence:

\[
0 \rightarrow \tilde{A}_1 \rightarrow \tilde{A}_0 \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_Z \rightarrow 0
\]

one gets an exact sequence:

\[
0 \rightarrow \tilde{A}_1 \otimes \mathcal{I}_W \rightarrow \tilde{A}_0 \otimes \mathcal{I}_W \rightarrow \mathcal{I}_W \rightarrow \mathcal{O}_Z \rightarrow 0.
\]
But $\ker (\mathcal{I}_W \to \mathcal{O}_Z) = \mathcal{I}_{Z \cup W}$, whence an exact sequence:

$$0 \to A_1 \otimes \mathcal{I}_W \to A_0 \otimes \mathcal{I}_W \to \mathcal{I}_{Z \cup W} \to 0.$$ 

Applying $H^0(\mathcal{I}_W)$ to this exact sequence one gets an exact sequence:

$$0 \to A_1 \otimes_S I(W) \to A_0 \otimes_S I(W) \to I(Z \cup W).$$

But the image of $A_0 \otimes_S I(W) \to I(Z \cup W)$ is $I(Z)I(W)$.

(b) Since $H^1(\mathcal{I}_W) = 0$ it follows that $A_0 \otimes_S I(W) \to I(Z \cup W)$ is surjective. Notice that, under the hypothesis of (b), one has $B_2 = 0$.

**Remark B.2.** Let $R = k[x_0, x_1]$ and let $J \subset R$ be a homogeneous, $R_+$-primary ideal. Then $JS$ is the homogeneous ideal of an arithmetically CM subscheme of $\mathbb{P}^3$ supported on the line of equations $x_0 = x_1 = 0$.

Indeed, $x_2, x_3$ is an $S/JS$-regular sequence.

**Remark B.3.** (a) If $I$ (resp., $J$) is the ideal of $S$ generated by the monomials $m_1, \ldots, m_s$ (resp., $n_1, \ldots, n_t$) then $I \cap J$ is generated by the monomials $[m_i, n_j], 1 \leq i \leq s, 1 \leq j \leq t$, where $[f, g]$ denotes the least common multiple of $f$ and $g$.

(b) If $I$ be the ideal of $S$ generated by the monomials $m_1, \ldots, m_s$ then:

$$(I : (m_1, m_2)) = \bigcap_{i=3}^s \left( S\frac{[m_i, m_1]}{m_i} + S\frac{[m_i, m_2]}{m_i} \right).$$

**Remark B.4.** We recall, from Lazarsfeld and Rao [22], the notion of **basic double linkage** (which is a particular case of the notion of **liaison addition** introduced by Schwartau [30]).

Let $Y$ be a locally CM subscheme of $\mathbb{P}^3$, of pure codimension 2, and let $f, h \in S_+$ be coprime homogeneous polynomials, of degree $a$ and $c$, respectively, such that $f \in I(Y)$. Let $Z$ be the closed subscheme of $\mathbb{P}^3$ defined by the homogeneous ideal $J := Sf + I(Y)h$. Set theoretically, one has $Z = Y \cup \{f = h = 0\}$. Consider a graded free resolution of the homogeneous ideal $I(Y) \subset S$:

$$0 \to K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \xrightarrow{d_0} I(Y) \to 0.$$ 

One has an exact sequence:

$$0 \to S(-a - c) \xrightarrow{(f, h)} S(-a) \oplus I(Y)(-c) \xrightarrow{(f, h)} J \to 0.$$ 

The morphism $S(-a) \to I(Y)$ defined by $f$ lifts to a morphism $\phi : S(-a) \to K_0$. One deduces that the sequence:

$$0 \to K_2(-c) \xrightarrow{d_2(-c)} S(-a - c) \oplus S(-a) \xrightarrow{(f, h)} J \to 0$$ 

is exact hence $Z$ is locally CM and $J = I(Z)$. 


Lemma B.5. Let \( X \subseteq Y \subseteq Z \) be closed subschemes of a scheme \( P \) and let \( T \) be another closed subscheme of \( P \). Let us denote by \( \phi \) the composite morphism \( \mathcal{I}_{Y\cup T} := \mathcal{I}_Y \cap \mathcal{I}_T \hookrightarrow \mathcal{I}_Y \twoheadrightarrow \mathcal{I}_Y / \mathcal{I}_Z \). Then:

(a) One has an exact sequence and a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{I}_{Z\cup T} & \rightarrow & \mathcal{I}_{Y\cup T} & \rightarrow & \mathcal{I}_Y / \mathcal{I}_Z \\
0 & \rightarrow & \text{Im} \phi & \rightarrow & \mathcal{O}_{Z\cup T} & \rightarrow & \mathcal{O}_{Y\cup T} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_Y / \mathcal{I}_Z & \rightarrow & \mathcal{O}_Z & \rightarrow & \mathcal{O}_Y & \rightarrow & 0
\end{array}
\]

(b) If \( Z \cap T = \emptyset \) then \( \phi \) is an epimorphism.
(c) If \( \mathcal{I}_X \mathcal{I}_Y \subseteq \mathcal{I}_Z \) then \( \text{Coker} \phi \) is an \( \mathcal{O}_{X\cap T} \)-module.

Proof. (a) \( \text{Ker} \phi = \mathcal{I}_Z \cap (\mathcal{I}_Y \cap \mathcal{I}_T) = \mathcal{I}_Z \cap \mathcal{I}_T =: \mathcal{I}_{Z\cup T} \).
(b) \( \text{Supp}(\mathcal{I}_Y / \mathcal{I}_Z) \subseteq Z \) and \( \mathcal{I}_T \) coincides with \( \mathcal{O}_P \) on the open neighbourhood \( P \setminus T \) of \( Z \).
(c) \( \text{Coker} \phi \simeq \mathcal{I}_Y / (\mathcal{I}_Z + (\mathcal{I}_Y \cap \mathcal{I}_T)) \) is annihilated by \( \mathcal{I}_X + \mathcal{I}_T =: \mathcal{I}_{X\cap T} \). \( \square \)

Remark B.6. Keeping the notation from Lemma B.5, \( \mathcal{O}_{Z\cup T} \) embeds into \( \mathcal{O}_Z \oplus \mathcal{O}_T \). Using the commutative diagram:

\[
\begin{array}{cccccc}
\mathcal{O}_{Z\cup T} & \rightarrow & \mathcal{O}_{Y\cup T} \\
& \downarrow & & \downarrow \\
\mathcal{O}_T & \rightarrow & \mathcal{O}_T
\end{array}
\]

one deduces that the image of \( \text{Ker} (\mathcal{O}_{Z\cup T} \rightarrow \mathcal{O}_{Y\cup T}) \simeq \text{Im} \phi \) in \( \mathcal{O}_T \) is 0.

B.1. A double line union a line. Let \( X \) be a (primitive) double structure on the line \( L_1 \subset \mathbb{P}^3 \). Recall, from Appendix A especially from Subsection A.3, that, as an \( \mathcal{O}_{L_1} \)-module, \( \mathcal{O}_X \simeq \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_1}(l) \), for some \( l \geq -1 \), and that, considering the corresponding generators \( 1 \in \mathbb{H}^0(\mathcal{O}_X) \) and \( e_1 \in \mathbb{H}^0(\mathcal{O}_X(1)) \) of the graded \( \mathbb{H}^0(\mathcal{O}_{L_1}) \)-module \( \mathbb{H}^0(\mathcal{O}_X) \), the \( \mathcal{O}_X \)-module structure of \( \mathcal{O}_X \) is defined by:

\[
x_2 \cdot 1 = ae_1 \, , \, x_3 \cdot 1 = be_1 \, , \, x_2 \cdot e_1 = 0 \, , \, x_3 \cdot e_1 = 0
\]

with \( a, b \in \mathbb{H}^0(\mathcal{O}_{L_1}(l+1)) \) coprime. The homogeneous ideal \( I(X) \) of \( X \) is generated by \( F_2 := -bx_2 + ax_3 \, , \, x_2^2, x_2x_3 \) and \( x_3^2 \). One has exact sequences:

\[
0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_{L_1} \overset{\varepsilon}{\rightarrow} \mathcal{O}_{L_1}(l) \rightarrow 0
\]

\[
0 \rightarrow \mathcal{I}_{L_1(1)} \rightarrow \mathcal{I}_X \overset{\eta}{\rightarrow} \mathcal{O}_{L_1}(-l-2) \rightarrow 0
\]

where \( \varepsilon \) is the composite morphism \( \mathcal{I}_{L_1} \rightarrow \mathcal{I}_{L_1} / \mathcal{I}_{L_1}^2 \simeq 2\mathcal{O}_{L_1}(-1) \overset{(a,b)}{\rightarrow} \mathcal{O}_{L_1}(l) \) and where \( \eta \) maps \( F_2 \in \mathbb{H}^0(\mathcal{I}_X(l+2)) \) to \( 1 \in \mathbb{H}^0(\mathcal{O}_{L_1}) \) and \( x_2^2, x_2x_3, x_3^2 \), which belong to \( \mathbb{H}^0(\mathcal{I}_{L_1(1)}(2)) \), to 0.
Consider another line \( L \subset \mathbb{P}^3 \) and let \( \phi \) (resp., \( \psi \)) denote the composite morphism:

\[
\mathcal{I}_{L_1 \cup L} \rightarrow \mathcal{I}_{L_1} \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(l) \quad \text{(resp., \( \mathcal{I}_{X \cup L} \rightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2) \)).
\]

It follows, from Lemma B.5, that one has exact sequences:

\[
0 \longrightarrow \mathcal{I}_{L_1 \cup L}^{(1)} \longrightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2)
\]

\[
0 \longrightarrow \text{Im } \phi \longrightarrow \mathcal{O}_{X \cup L} \longrightarrow \mathcal{O}_{L_1 \cup L} \longrightarrow 0.
\]

Moreover, \( \text{Coker } \psi \) is an \( \mathcal{O}_{L_1 \cap L} \)-module. Lemma B.7 implies that:

**Lemma B.7.** The homogeneous ideal \( I(L_1^{(1)} \cup L_1') \) of \( L_1^{(1)} \cup L_1' \) admits the following graded free resolution:

\[
0 \longrightarrow 2S(-5) \xrightarrow{d_2} 7S(-4) \xrightarrow{d_3} 6S(-3) \xrightarrow{d_0} I(L_1^{(1)} \cup L_1') \longrightarrow 0
\]

with the differentials \( d_0, d_1, d_2 \) defined by the matrices:

\[
\left(\begin{array}{cccccc}
-x_3 & 0 & 0 & 0 & -x_1 & 0 \\
-x_3 & 0 & 0 & 0 & -x_1 & 0 \\
0 & x_2 & 0 & 0 & 0 & -x_1 \\
0 & 0 & -x_3 & 0 & x_0 & 0 \\
0 & 0 & x_2 & -x_3 & 0 & x_0 \\
0 & 0 & 0 & x_2 & 0 & 0 \\
\end{array}\right),
\]

\[
\left(\begin{array}{c}
-x_1 \\
x_0 \\
x_0 \\
x_3 \\
x_2 \\
0 \\
\end{array}\right).
\]

**Lemma B.8.** If \( l = -1 \) then \( X \) is the divisor \( 2L_1 \) on some plane \( H \supset L_1 \).

**Proof.** One can assume that \( a = 0 \) and \( b = -1 \), hence that \( F_2 = x_2 \). It follows that \( I(X) = (x_2, x_3^2) \).

\( \square \)

**Proposition B.9.** If \( l \geq 0 \) then \( I(X \cup L_1') = SF_2 + (x_0, x_1)(x_2, x_3)^2 \).

**Proof.** By what has been said at the beginning of the subsection, one has an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(1)} \cup L_1'} \longrightarrow \mathcal{I}_{X \cup L_1'} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0.
\]

Our hypothesis implies that \( F_2 \in I(X \cup L_1') \) from which one deduces that the sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L_1') \longrightarrow I(X \cup L_1') \xrightarrow{H^0(\psi)} S(L_1)(-l - 2) \longrightarrow 0
\]

is exact. Lemma B.7 implies that \( I(L_1^{(1)} \cup L_1') = (x_0, x_1)(x_2, x_3)^2 \). Notice that the last exact sequence allows one to get a graded free resolution of \( I(X \cup L_1') \).

\( \square \)

**Lemma B.10.** Let \( L \) be a line contained in the plane \( x_3 = 0 \) and different from \( L_1 \). It is given by equations of the form \( \ell = x_3 = 0 \), where \( \ell = c_0 x_0 + c_1 x_1 + c_2 x_2 \) and at least one of the coefficients \( c_0, c_1 \) is non-zero. Using a linear change of coordinates invariorating \( x_2 \),
x_3 and the vector space kx_0 + kx_1, one can assume that ℓ = x_1 + cx_2, c ∈ k. Then the homogeneous ideal I(L^{(1)}_1 ⊔ L) of L^{(1)}_1 ⊔ L admits the following graded free resolution:

\[ 0 \rightarrow S(-3) \oplus \begin{pmatrix} -x_3 - \ell x_2 \\ x_2 \\ 0 \\ x_3 \end{pmatrix} \rightarrow 2S(-2) \oplus \begin{pmatrix} x_2 x_3, x_3^2, \ell x_2^2 \end{pmatrix} \rightarrow I(L^{(1)}_1 ⊔ L) \rightarrow 0. \]

**Proof.** One has I(L^{(1)}_1 ⊔ L) = \((x_2^2, x_2 x_3, x_3^2) \cap (ℓ, x_3) = (x_2 x_3, x_3^2, ℓ x_2^2)\). If Z' is the curve directly linked to L^{(1)}_1 ⊔ L by the complete intersection defined by x_3^2 and ℓ x_2^2 then I(Z') = (x_3, ℓ x_2) hence Z' = L_1 ⊔ L. One can apply, now, Ferrand’s result about liaison.

**Proposition B.11.** Let X be the double structure on the line L_1 considered at the beginning of this subsection and let L be the line of equations ℓ = x_3 = 0, ℓ := x_1 + cx_2, considered in Lemma [B.10]

(a) If \(x_1 \mid b\), i.e., if \(b = x_1 b_1\) then \(I(X ⊔ L) = (F_2 - cb_1 x_2^2, x_2 x_3, x_3^2, ℓ x_2^2)\).

(b) If \(x_1 ∤ b\) then \(I(X ⊔ L) = (ℓ F_2, x_2 x_3, x_3^2, ℓ x_2^2)\).

**Proof.** By what has been said at the beginning of the subsection, one has an exact sequence:

\[ 0 \rightarrow \mathcal{J}_{L^{(1)}_1 ⊔ L} \rightarrow \mathcal{J}_{X ⊔ L} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2). \]

Moreover, \(\text{Coker } \psi\) is an \(\mathcal{O}_{L_1 \cap L} = \mathcal{O}_{\{P_b\}}\)-module. Actually, since \(ℓ F_2 ∈ I(X ⊔ L)\) and \(ψ(ℓ F_2) = η(ℓ F_2) = x_1 ∈ H^0(\mathcal{O}_{L_1}^{(1)})\), it follows that \(x_1 \mathcal{O}_{L_1}(-l - 3) ⊆ \text{Im } \psi ⊆ \mathcal{O}_{L_1}(-l - 2)\).

(a) Since \(b = x_1 b_1, F_2 - cb_1 x_2^2\) vanishes on L hence belongs to \(I(X ⊔ L)\). Since \(ψ(F_2 - cb_1 x_2^2) = η(F_2 - cb_1 x_2^2) = 1 ∈ H^0(\mathcal{O}_{L_1})\), it follows that \(ψ\) is an epimorphism and that one has an exact sequence:

\[ (B.1) \quad 0 \rightarrow I(L^{(1)}_1 ⊔ L) \rightarrow I(X ⊔ L) \xrightarrow{\text{H}^0(ψ)} S(L_1)(-l - 2) \rightarrow 0. \]

(b) We prove, firstly, the following:

**Claim.** \(\text{Im } ψ = x_1 \mathcal{O}_{L_1}(-l - 3)\).

Indeed, assume that \(\text{Im } ψ = \mathcal{O}_{L_1}(-l - 2)\). Since \(H^1_*(\mathcal{J}_{L^{(1)}_1 ⊔ L}) = 0\) by Lemma [B.10] there exists an element \(f ∈ H^0(\mathcal{J}_{X ⊔ L}^{(1)}(l + 2))\) such that \(ψ(f) = 1 ∈ H^0(\mathcal{O}_{L_1})\). One must have \(f = F_2 + f_0 x_2^2 + f_1 x_2 x_3 + f_2 x_3^2\), with \(f_0, f_1, f_2 ∈ S_1\). Since \(f \mid L = 0\) it follows that:

\[ (\ast) \quad -(b \mid L) x_2 + (f_0 \mid L) x_2^2 = 0. \]

The composite map \(k[0, x_2] \hookrightarrow S \rightarrow S(L)\) is bijective and, with respect to this identification, \(b \mid L = b(x_0, -cx_2)\). Since \(x_1 ∤ b\) it follows that \(x_2 ∤ b(x_0, -cx_2)\) which contradicts relation (\(\ast\)).
According to the Claim, $\psi$ can be written as a composite morphism $\mathcal{I}_{X \cup L} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2)$ with $\psi'(\ell F_2) = 1 \in H^0(\mathcal{O}_{L_1})$. One deduces the existence of an exact sequence:

\[(B.2) \quad 0 \to I(L_1^{(1)} \cup L) \to I(X \cup L) \xrightarrow{H^0_0(\psi')} S(L_1)(-l - 3) \to 0.\]

Notice that the exact sequences \[(B.1)\] and \[(B.2)\] can be used not only to describe a system of generators of $I(X \cup L)$ but also to get a graded free resolution of this ideal. \qed

**Proposition B.12.** Under the hypothesis of Prop. \[(B.11)\]:

(a) If $x_1 \mid b$ then $H^0_*(\mathcal{O}_{X \cup L})$ admits the following graded free resolution:

\[
\begin{align*}
0 & \to S(-3) \oplus S(-2) \oplus S(-1) \oplus S(l-1) \oplus 2S(l-2) \\
& \xrightarrow{\begin{pmatrix}
-x_2 & 0 \\
x_3 & 0 \\
-b & -x_3 \\
ax & x_2
\end{pmatrix}} \xrightarrow{\begin{pmatrix}
x_3 & \ell x_2 & 0 & 0 \\
-b & -ax & x_2 & x_3
\end{pmatrix}} S \oplus (1, x_1 e_1) \to H^0_*(\mathcal{O}_{X \cup L}) \to 0.
\end{align*}
\]

(b) If $x_1 \nmid b$ then $H^0_*(\mathcal{O}_{X \cup L})$ admits the following graded free resolution:

\[
\begin{align*}
0 & \to S(-3) \oplus S(-2) \oplus S(l-1) \oplus 2S(l-2) \\
& \xrightarrow{\begin{pmatrix}
-x_2 & 0 \\
x_3 & 0 \\
-b & -x_3 \\
ax & x_2
\end{pmatrix}} \xrightarrow{\begin{pmatrix}
x_3 & \ell x_2 & 0 & 0 \\
-b & -ax & x_2 & x_3
\end{pmatrix}} S \oplus (1, e_1) \to H^0_*(\mathcal{O}_{X \cup L}) \to 0.
\end{align*}
\]

**Proof.** By what has been said at the beginning of this subsection, one has an exact sequence:

\[0 \to \text{Im} \phi \to \mathcal{O}_{X \cup L} \to \mathcal{O}_{L_1 \cup L} \to 0.\]

Since $I(L_1 \cup L) = (x_3, \ell x_2)$, the image of $\phi$ coincides with the image of the composite morphism:

\[\mathcal{O}_p(-(1)) \oplus \mathcal{O}_p(-(2)) \xrightarrow{(x_3, \ell x_2)} \mathcal{I}_{L_1} \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(l)\]

which, in turn, coincides with the image of the composite morphism:

\[\mathcal{O}_{L_1}(-(1)) \oplus \mathcal{O}_{L_1}(-(2)) \xrightarrow{0 x_1 \atop 1 0} 2\mathcal{O}_{L_1}(-1) \xrightarrow{(a, b)} \mathcal{O}_{L_1}(l)\]

i.e., with the image of the morphism $(b, x_1 a): \mathcal{O}_{L_1}(-(1)) \oplus \mathcal{O}_{L_1}(-(2)) \to \mathcal{O}_{L_1}(l)$.

(a) In this case $\text{Im} \phi = x_1 \mathcal{O}_{L_1}(l - 1)$. Since the graded $S$-module $H^0_*(\mathcal{O}_{L_1 \cup L})$ is generated by $1 \in H^0(\mathcal{O}_{L_1 \cup L})$, the morphism $H^0_*(\mathcal{O}_{X \cup L}) \to H^0_*(\mathcal{O}_{L_1 \cup L})$ is surjective. One deduces the existence of an exact sequence:

\[0 \to S(L_1)(l - 1) \to H^0_*(\mathcal{O}_{X \cup L}) \to H^0_*(\mathcal{O}_{L_1 \cup L}) \to 0\]
where the left morphism maps $1 \in S(L_1)$ to the element of $H^0(\mathcal{O}_{X\cup L}(-l+1))$ whose image into $H^0(\mathcal{O}_X(-l+1)) \oplus H^0(\mathcal{O}_L(-l+1))$ is $(x_1 e_1, 0)$.

(b) In this case $\text{Im } \phi = \mathcal{O}_{L_1}(l)$ and one deduces, as in (a), the existence of an exact sequence:

$$0 \rightarrow S(L_1)(l) \rightarrow H^0_*(\mathcal{O}_{X\cup L}) \rightarrow H^0_*(\mathcal{O}_{L_1\cup L}) \rightarrow 0$$

where the left morphism maps $1 \in S(L_1)$ to the element of $H^0(\mathcal{O}_{X\cup L}(-l))$ whose image into $H^0(\mathcal{O}_X(-l)) \oplus H^0(\mathcal{O}_L(-l))$ is $(e_1, 0)$. $\square$

B.2. A triple line union a line. Let $Y$ be a quasiprimitive triple structure on the line $L_1 \subset \mathbb{P}^3$. Recall, from Appendix A especially from Subsection A.4, that, as an $\mathcal{O}_{L_1}$-module,

$$\mathcal{O}_Y \simeq \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_1}(l) \oplus \mathcal{O}_{L_1}(2l + m)$$

for some integers $l \geq -1$ and $m \geq 0$, and that if $1 \in H^0(\mathcal{O}_Y)$, $e_1 \in H^0(\mathcal{O}_Y(-l))$, $e_2 \in H^0(\mathcal{O}_Y(-2l - m))$ are the corresponding generators of the $H^0_*(\mathcal{O}_{L_1})$-module $H^0_*(\mathcal{O}_Y)$ then the $\mathcal{O}_Y$-module structure of $\mathcal{O}_Y$ is defined, modulo the isomorphism $\varepsilon$.

Let $X$ be the double structure on $L_1$ defined by the epimorphism $(a, b) : \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^2 \simeq 2\mathcal{O}_{L_1}(-1) \rightarrow \mathcal{O}_{L_1}(l)$, and let $W$ be the thick quadruple structure on $L_1$ defined by the ideal sheaf $\mathcal{I}_W = \mathcal{I}_{L_1}, \mathcal{I}_X$ (see the last part of Prop. A.14). One has $I(W) = I(L_1)I(X)$.

Applying the Snake Lemma to the diagram:

$$
\begin{array}{cccc}
0 & \rightarrow & \mathcal{I}_Y/\mathcal{I}_{L_1}^2 & \rightarrow & \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^2 & \rightarrow & \mathcal{O}_{L_1}(l) \oplus \mathcal{O}_{L_1}(2l + m) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \text{pr}_1 & & \\
0 & \rightarrow & \mathcal{I}_X/\mathcal{I}_{L_1}^3 & \rightarrow & \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^3 & \rightarrow & \mathcal{O}_{L_1}(l) & \rightarrow & 0
\end{array}
$$

with $\alpha$ and $\beta$ defined, modulo the isomorphism $\mathcal{I}_{L_1}/\mathcal{I}_{L_1}^3 \simeq 2\mathcal{O}_{L_1}(-1) \oplus 3\mathcal{O}_{L_1}(-2)$, by the matrices:

$$
(a, b, 0, 0, 0) \text{ and } \begin{pmatrix}
a & b & 0 & 0 & 0 \\
a' & b' & pa^2 & pab & pb^2
\end{pmatrix}
$$

one derives an exact sequence:

(B.3)

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_X \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(2l + m) \rightarrow 0$$

where $\varepsilon$ maps $F_2 \in H^0(\mathcal{I}_X(l + 2))$ to $\Delta_1 \in H^0(\mathcal{O}_{L_1}(3l + m + 2))$ and $x_2^2, x_2 x_3, x_3^2 \in H^0(\mathcal{I}_X(2))$ to $pa^2, pab, pb^2 \in H^0(\mathcal{O}_{L_1}(2l + m + 2))$, respectively.
On the other hand, \( \mathcal{I}_W/\mathcal{I}_{L_1}^3 = x_2(\mathcal{I}_X/\mathcal{I}_{L_1}^3) + x_3(\mathcal{I}_X/\mathcal{I}_{L_1}^3) \subseteq \mathcal{I}_{L_1}/\mathcal{I}_{L_1}^3 \) is the image of the morphism:

\[
\begin{pmatrix}
-a & 0 \\
0 & a
\end{pmatrix}
\]

whence an exact sequence:

(B.4) \[ 0 \to \mathcal{I}_{L_1}^{(2)} \to \mathcal{I}_W \xrightarrow{\eta_1} 2\mathcal{O}_{L_1}(-l - 3) \to 0 \]

where \( \eta_1 \) maps \( x_2F_2 \in H^0(\mathcal{I}_W(l + 3)) \) to \((1, 0) \in H^0(2\mathcal{O}_{L_1}) \) and \( x_3F_2 \in H^0(\mathcal{I}_W(l + 3)) \) to \((0, 1) \in H^0(2\mathcal{O}_{L_1}) \). One deduces also an exact sequence:

\[ 0 \to \mathcal{I}_W \xrightarrow{\rho} \mathcal{I}_{L_1}^2 \to \mathcal{O}_{L_1}(2l) \to 0 \]

with \( \rho \) the composite morphism \( \mathcal{I}_{L_1}^2 \to \mathcal{I}_{L_1}^3 \to 3\mathcal{O}_{L_1}(-2) \to \mathcal{O}_{L_1}(2l) \).

Finally, applying the Snake Lemma to the diagram:

\[
\begin{array}{ccc}
0 & \to & \mathcal{I}_W/\mathcal{I}_{L_1}^3 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{I}_Y/\mathcal{I}_{L_1}^3
\end{array}
\]

\[
\begin{array}{ccc}
\quad & \quad & \quad \\
\gamma & \quad & \theta \\
\quad & \quad & \quad \\
\mathcal{O}_{L_1}(-1) \oplus \mathcal{O}_{L_1}(2l) & \to & \mathcal{O}_{L_1}(l) \oplus \mathcal{O}_{L_1}(2l + m)
\end{array}
\]

with \( \gamma \) and \( \theta \) defined by the matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & a^2 & ab & b^2
\end{pmatrix}
\]

and taking into account the exact sequence:

\[
0 \to \mathcal{O}_{L_1}(-l - m - 2) \xrightarrow{\phi} 2\mathcal{O}_{L_1}(-1) \oplus \mathcal{O}_{L_1}(2l) \xrightarrow{\theta} \mathcal{O}_{L_1}(l) \oplus \mathcal{O}_{L_1}(2l + m) \to 0
\]

one deduces an exact sequence:

(B.5) \[ 0 \to \mathcal{I}_W \to \mathcal{I}_Y \xrightarrow{\eta_2} \mathcal{O}_{L_1}(-l - m - 2) \to 0 \]

where \( \eta_2 \) maps \( F_3 \in H^0(\mathcal{I}_Y(l + m + 2)) \) to \((1 \in H^0(\mathcal{O}_{L_1}) \) and the other generators of \( H(Y) \), which actually belong to \( H(W) \), to 0.

Consider another line \( L \subset \mathbb{P}^3 \). Let us denote by \( \psi_1, \psi_2 \) and \( \phi \) the following composite morphisms:

\[
\begin{array}{cc}
\mathcal{I}_{W \cup L} & \to \mathcal{I}_W \xrightarrow{\eta_1} 2\mathcal{O}_{L_1}(-l - 3), \\
\mathcal{I}_{Y \cup L} & \to \mathcal{I}_Y \xrightarrow{\eta_2} \mathcal{O}_{L_1}(-l - m - 2)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{I}_{X \cup L} \to \mathcal{I}_X \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(2l + m)
\end{array}
\]
It follows from Lemma [B.3] that one has exact sequences:

\[ 0 \rightarrow \mathcal{I}_{L_1(2) \cup L} \rightarrow \mathcal{I}_{W \cup L} \xrightarrow{\psi_1} 2\mathcal{O}_{L_1}(-l - 3), \]

\[ 0 \rightarrow \mathcal{I}_{W \cup L} \rightarrow \mathcal{I}_{Y \cup L} \xrightarrow{\psi_2} \mathcal{O}_{L_1}(-l - m - 2), \]

\[ 0 \rightarrow \text{Im } \phi \rightarrow \mathcal{O}_{Y \cup L} \rightarrow \mathcal{O}_{X \cup L} \rightarrow 0. \]

Moreover, \( \text{Coker } \psi_1, \text{Coker } \psi_2 \) and \( \text{Coker } \phi \) are \( \mathcal{O}_{L_1 \cap L} \)-modules.

The following lemma follows easily from Lemma [B.1].

**Lemma B.13.** Let \( L_1^{(2)} \) denote the second infinitesimal neighbourhood of \( L_1 \) in \( \mathbb{P}^3 \) defined by the ideal sheaf \( \mathcal{I}_{L_1}^3 \). Then the homogeneous ideal \( I(L_1^{(2)} \cup L_1') \) of \( L_1^{(2)} \cup L_1' \) admits the following graded free resolution:

\[
0 \rightarrow 3S(-6) \xrightarrow{d_2} 10S(-5) \xrightarrow{d_1} 8S(-4) \xrightarrow{d_0} I(L_1^{(2)} \cup L_1') \rightarrow 0
\]

with \( d_0, d_1, d_2 \) defined by the matrices:

\[
\begin{pmatrix}
-x_3 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 \\
x_2 & -x_3 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 \\
0 & x_2 & -x_3 & 0 & 0 & 0 & 0 & -x_1 & 0 \\
0 & 0 & x_2 & 0 & 0 & 0 & 0 & -x_1 & 0 \\
0 & 0 & 0 & -x_3 & 0 & 0 & x_0 & 0 & 0 \\
0 & 0 & 0 & x_2 & -x_3 & 0 & 0 & x_0 & 0 \\
0 & 0 & 0 & 0 & x_2 & -x_3 & 0 & 0 & x_0 \\
0 & 0 & 0 & 0 & 0 & x_2 & -x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
-x_1 & 0 & 0 \\
0 & -x_1 & 0 \\
x_0 & 0 & 0 \\
0 & x_0 & 0 \\
0 & 0 & x_0 \\
x_3 & 0 & 0 \\
-x_2 & x_3 & 0 \\
0 & -x_2 & x_3 \\
0 & 0 & -x_2
\end{pmatrix}
\]

**Lemma B.14.** (a) If \( l = -1 \) and \( m = 0 \) then \( Y \) is the divisor \( 3L_1 \) on some plane \( H \supset L_1 \).

(b) If \( l = -1 \) and \( m = 1 \) then \( Y \) is directly linked to \( L_1 \) by a complete intersection of type \((2, 2)\) hence it is, in particular, arithmetically CM.

**Proof.** We can assume that \( a = 0, b = -1 \), hence that \( F_2 = x_2 \). Moreover, \( \Delta_1 = \alpha' \), hence one can take, in relation \((\text{A.7})\), \( v_0 = v_1 = 0 \) and \( v_2 = -\alpha' \). In this case, \( F_3 = px_2 - \alpha'x_3^2 \), with \( p \in H^0(\mathcal{O}_{L_1}(m)) \) and \( \alpha' \in H^0(\mathcal{O}_{L_1}(m - 1)) \) coprime.

It follows that \( I(Y) = (px_2 - \alpha'x_3^2, x_2^2, x_2x_3, x_3^3) \). Notice that \( I(X) = (x_2, x_3^2) \) and \( I(W) = I(L_1)I(\mathcal{O}) = (x_2^2, x_2x_3, x_3^3) \). Moreover, \( W \) is directly linked to \( X \) by the complete intersection defined by \( x_2^2 \) and \( x_3^3 \), hence \( I(W) \) admits the following graded free resolution:

\[
0 \rightarrow \bigoplus S(-3) \xrightarrow{\begin{pmatrix}
-x_3 & 0 \\
x_2 & -x_3^2 \\
0 & x_2
\end{pmatrix}} \bigoplus 2S(-2) \xrightarrow{\begin{pmatrix}
x_2^2 & x_2x_3 & x_3^3
\end{pmatrix}} I(W) \rightarrow 0.
\]
(a) In this case \( p = 1 \) and \( a' = 0 \), hence \( I(Y) = (x_2, x_3^3) \).

(b) In this case \( p \) is a linear form on \( L_1 \) and \( a' \) is a constant. Since \( p \) and \( a' \) are coprime it follows that \( a' \neq 0 \), hence one can assume that \( a' = 1 \). It follows that \( I(Y) = (px_2 - x_3^2, x_2^2, x_3x_2) \). \( Y \) is directly linked to \( L_1 \) by the complete intersection defined by \( px_2 - x_3^2 \) and \( x_2^2 \). One gets, consequently, the following graded free resolution:

\[
\begin{pmatrix}
  x_2 & 0 \\
 -p & -x_3 \\
 x_3 & x_2
\end{pmatrix}
\]

\[0 \longrightarrow 2S(-3) \longrightarrow 3S(-2) \longrightarrow I(Y) \longrightarrow 0.\]

Lemma B.15. If \( l = m = 0 \) then \( Y \) is the divisor \( 3L_1 \) on some nonsingular quadric surface \( Q \supset L_1 \).

Proof. One can assume that \( a = x_0 \) and \( b = x_1 \) hence \( F_2 = -x_1x_2 + x_0x_3 \) and \( F_3 = F_2 + v_0x_2^2 + v_1x_2x_3 + v_2x_3^2 \), with \( v_0, v_1, v_2 \in k \). It follows that:

\[ I(Y) = (F_3, x_2^3, x_2x_3, x_2^2x_3, x_3^3) \]

hence \( Y \) is the divisor \( 3L_1 \) on the nonsingular quadric \( Q \) of equation \( F_3 = 0 \). □

Proposition B.16. (a) If \( l = -1 \) and \( m \geq 2 \) then:

\[ I(Y \cup L'_1) = S(px_2 - a'x_3^2) + (x_0, x_1)(x_2^2, x_2x_3, x_3^3). \]

(b) If \( l \geq 0 \) then:

\[ I(Y \cup L'_1) = SF_3 + Sx_2F_2 + Sx_3F_2 + (x_0, x_1)(x_2, x_3)^3. \]

except when \( l = m = 0 \) and \( L'_1 \) is not contained in the quadric surface \( Q \) of equation \( F_3 = 0 \) (which means that at least one of the constants \( v_0, v_1, v_2 \) is non-zero) in which case:

\[ I(Y \cup L'_1) = Sx_0F_3 + Sx_1F_3 + Sx_2F_2 + Sx_3F_2 + (x_0, x_1)(x_2, x_3)^3. \]

Proof. One uses the exact sequence:

\[ 0 \longrightarrow \mathcal{I}_{W \cup L'_1} \longrightarrow \mathcal{I}_{Y \cup L'_1} \xrightarrow{\psi_2} \mathcal{O}_{L_1}(-m - 1) \longrightarrow 0 \]

(see at the beginning of this subsection). One has \( I(W) = (x_2^2, x_2x_3, x_3^3) \) (see the proof of Lemma [B.14]) and \( I(W \cup L'_1) = I(W)I(L'_1) \) (by Lemma [B.1]). Since \( m \geq 2 \) it follows that \( F_3 \) vanishes on \( L'_1 \) hence \( F_3 \in I(Y \cup L'_1) \) hence \( \psi_2(F_3) = \eta_2(F_3) = 1 \in \mathbb{H}^0(\mathcal{O}_{L_1}) \). One deduces the exactness of the sequence:

\[ 0 \longrightarrow I(W \cup L'_1) \longrightarrow I(Y \cup L'_1) \xrightarrow{\mathbb{H}^0(\psi_2)} S(L_1)(-m - 1) \longrightarrow 0. \]

(b) As we saw at the begining of this subsection, one has exact sequences:

\[ 0 \longrightarrow \mathcal{I}_{L'_1(2) \cup L'_1} \longrightarrow \mathcal{I}_{W \cup L'_1} \xrightarrow{\psi_1} 2\mathcal{O}_{L_1}(-l - 3) \longrightarrow 0 \]

\[ 0 \longrightarrow \mathcal{I}_{W \cup L'_1} \longrightarrow \mathcal{I}_{Y \cup L'_1} \xrightarrow{\psi_2} \mathcal{O}_{L_1}(-l - m - 2) \longrightarrow 0. \]
Since \( l \geq 0 \), \( F_2 \) vanishes on \( L_1' \) hence \( x_2F_2, x_3F_2 \in I(W \cup L_1') \) hence the sequence:

\[
0 \to I(L_1^{(2)} \cup L_1') \to I(W \cup L_1') \xrightarrow{H_0^0(\psi_1)} 2S(L_1)(-l - 3) \to 0
\]

is exact.

On the other hand, if \( l \geq 1 \), or if \( l = 0 \) and \( m \geq 1 \) or if \( l = m = 0 \) and \( v_0 = v_1 = v_2 = 0 \) then \( F_3 \) vanishes on \( L_1' \) hence \( F_3 \in I(Y \cup L_1') \) hence the sequence:

\[
0 \to I(W \cup L_1') \to I(Y \cup L_1') \xrightarrow{H_0^0(\psi_2)} S(L_1)(-l - m - 2) \to 0
\]

is exact.

Finally, if \( l = m = 0 \) and at least one of the constants \( v_0, v_1, v_2 \) is non-zero then \( F_3 \) doesn’t vanish on \( L_1' \) hence \( H^0(I_{Y \cup L_1'}(2)) = 0 \). On the other hand, \( x_0 F_3 \) and \( x_1 F_3 \) vanish on \( L_1' \) hence they belong to \( H^0(I_{Y \cup L_1'}(3)) \) and \( \psi_2(x_i F_3) = \eta_2(x_i F_3) = x_i \in H^0(\mathcal{O}_{L_1}(1)) \), \( i = 0, 1 \). One deduces the exactness of the sequence:

\[
0 \to I(W \cup L_1') \to I(Y \cup L_1') \xrightarrow{H_0^0(\psi_2)} S(L_1)_+(2) \to 0.
\]

Notice that the exact sequences appearing in this proof can be used to get (easily) a concrete graded free resolution of \( I(Y \cup L_1') \). (A minimal graded free resolution of \( S(L_1)_+ \) can be found in the discussion following the proof of Prop. \[A.12\]) \( \square \)

**Lemma B.17.** The homogeneous ideal \( I(L_1^{(2)} \cup L_2) \) of \( L_1^{(2)} \cup L_2 \) admits the following graded free resolution:

\[
0 \to 2S(-4) \oplus \begin{pmatrix} -x_3 & 0 & -x_1 x_2 \\ x_2 & -x_3 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \to 3S(-3) \oplus \begin{pmatrix} x_2^3 x_3, x_2 x_3^2, x_3^3, x_1 x_2^3 \end{pmatrix} \to I(L_1^{(2)} \cup L_2) \to 0.
\]

**Proof.** Using Remark [B.3], one gets:

\[
I(L_1^{(2)} \cup L_2) = (x_2^3, x_2 x_3, x_3^3, x_1 x_2^3) \cap (x_1, x_3) = (x_2^3 x_3, x_2 x_3^2, x_3^3, x_1 x_2^3).
\]

If \( Z' \) is the curve directly linked to \( L_1^{(2)} \cup L_2 \) by the complete intersection defined by \( x_3^3 \) and \( x_1 x_2^3 \) then, using Remark [B.3]:

\[
I(Z') = (x_2^3, x_1 x_2) \cap (x_3, x_1 x_2^2) = (x_2^2 x_3, x_2 x_3^2, x_3^3, x_1 x_2^3).
\]

If \( Z'' \) is the curve directly linked to \( Z' \) by the complete intersection defined by \( x_3^3 \) and \( x_1 x_2^2 \) then \( I(Z'') = (x_2, x_3) \), i.e., \( Z'' = L_1 \). Using Ferrand’s result about linkage one deduces that the free resolution of \( I(L_1^{(2)} \cup L_2) \) has the numerical shape from the statement and its differentials are easy to guess. \( \square \)
Lemma B.18. Using the notation recalled at the beginning of this subsection:
(a) If $x_1 \mid b$ then $x_1 \mid b'$ if and only if $x_1 \mid v_0$.
(b) If $x_1 \nmid b$ then one can assume that $x_1 \mid b'$.
(c) If $x_1 \nmid b$ and $m \geq 1$ then one can assume that $x_1 \mid v_0$.

Proof. (a) This follows from relation $[A.7]: ab' - a'b + v_0 a^2 + v_1 ab + v_2 b^2$.
(b) Let $\alpha \in H^0(\mathcal{O}_{L_1}(l + m))$. Replacing the generators $1, e_1, e_2$ of the $H^0_s(\mathcal{O}_{L_1})$-module $H^s(\mathcal{O}_f)$ by the generators $1, e'_1 = e_1 + ae_2, e'_2 = e_2$ one has:
\[
x_2 \cdot 1 = ae'_1 + (a' - \alpha a)e'_2, \quad x_3 \cdot 1 = be'_1 + (b' - \alpha b)e'_2
\]
\[
x_2 \cdot e'_1 = pae'_2, \quad x_3 \cdot e'_1 = pbe'_2, \quad x_2 \cdot e'_2 = 0, \quad x_3 \cdot e'_2 = 0.
\]
Consequently, $(a, a' - \alpha a, b, b' - \alpha b, p)$ and $(a, a', b, b', p)$ define the same triple structure on $L_1$.

If $l = -1$ and $m = 0$ then $b' = 0$ because it belongs to $H^0(\mathcal{O}_{L_1}(-1)) = 0$. Assume, now, that $l = -1$ and $m \geq 1$ or that $l \geq 0$. If $x_1 \nmid b$ then, since $b'$ has degree $2l + m + 1 \geq l + 1$, there exist $\alpha \in H^0(\mathcal{O}_{L_1}(l + m))$ and $b'_1 \in H^0(\mathcal{O}_{L_1}(2l + m))$ such that $b' = ab + b'_1 x_1$.

(c) $v_0, v_1, v_2$ are (any) elements of $H^0(\mathcal{O}_{L_1}(l + m))$ satisfying relation $[A.7]$ recalled in the proof of (a). If $x_1 \nmid b$ then $x_1 a$ and $b$ are coprime. Since $b'$ has degree $2l + m + 1 \geq 2l + 2$ (we used, here, the hypothesis $m \geq 1$), one can find elements $v'_0 \in H^0(\mathcal{O}_{L_1}(l + m - 1))$ and $v'_1 \in H^0(\mathcal{O}_{L_1}(l + m))$ such that:
\[
b' = v'_0 x_1 a + v'_1 b.
\]
One gets, similarly, elements $v''_0, v_2 \in H^0(\mathcal{O}_{L_1}(l + m))$ such that $a' = v''_0 a + v_2 b$. One may take $v_0 = -x_1 v'_0$ and $v_1 = -v'_1 + v''_0$. \hfill $\square$

Proposition B.19. Using the notation recalled at the beginning of this subsection and assuming that the conditions from the conclusion of Lemma B.18 are fulfilled, one has:
(a) If $x_1 \mid b$ and $x_1 \mid b'$ then $I(Y \cup L_2) = (F_3, x_2 F_2, x_3 F_2, x_2^2 x_3, x_2 x_3^2, x_3^2, x_1 x_3^2)$.
(b) If $x_1 \mid b$ and $x_1 \nmid b'$ then $I(Y \cup L_2) = (x_1 F_3, x_2 F_3, x_3 F_2, x_2^2 x_3, x_2 x_3^2, x_3^2, x_1 x_3^2)$.
(c) If $x_1 \nmid b$ and $x_1 \mid p$ then $I(Y \cup L_2) = (F_3, x_3 F_2, x_1 x_2 F_2, x_2^2 x_3, x_2 x_3^2, x_3^2, x_1 x_3^2)$.
(d) If $x_1 \nmid b$ and $x_1 \nmid p$ then $I(Y \cup L_2) = (x_1 F_3, x_3 F_2, x_1 x_2 F_2, x_2^2 x_3, x_2 x_3^2, x_3^2, x_1 x_3^2)$.

Proof. We use the exact sequences:
\[
0 \longrightarrow \mathcal{I}_{L_1(2l)} \longrightarrow \mathcal{I}_{W \cup L_2} \stackrel{\psi_1}{\longrightarrow} 2\mathcal{O}_L(-l - 3)
\]
\[
0 \longrightarrow \mathcal{I}_{W \cup L_2} \longrightarrow \mathcal{I}_{Y \cup L_2} \stackrel{\psi_2}{\longrightarrow} \mathcal{O}_L(-l - m - 2)
\]
introduced at the beginning of this subsection. Since $x_1 x_2 F_2, x_3 F_2$ and $x_1 F_3$ vanish on $L_2$ they belong to $I(Y \cup L_2)$ hence:
\[
x_1 \mathcal{O}_L(-l - 4) \oplus \mathcal{O}_L(-l - 3) \subseteq \text{Im } \psi_1 \subseteq 2\mathcal{O}_L(-l - 3)
\]
\[
\text{and } x_1 \mathcal{O}_L(-l - m - 3) \subseteq \text{Im } \psi_2 \subseteq \mathcal{O}_L(-l - m - 2).
\]

Claim 1. \quad $\text{Im } \psi_2 = 2\mathcal{O}_L(-l - 3)$ if and only if $x_1 \mid b$. 


Indeed, the “if” part is clear (because, in this case, \(x_2F_2\) vanishes on \(L_2\)). For the “only if” part assume that \(\text{Im } \psi_1 = 2\mathcal{O}_{L_1}(-l - 3)\). Since, by Lemma 17 \(\text{H}^1(\mathcal{I}_{L_1(2)} \cup L_2) = 0\) it follows that there exists \(f \in \text{H}^0(\mathcal{I}_{W \cup L_2}(l + 3))\) such that \(\psi_1(f) = (1, 0) \in \text{H}^0(2\mathcal{O}_{L_1})\). Since \(\psi_1(f) = \eta_1(f)\) it follows that \(f\) must be of the form:

\[
f = x_2F_2 + f_0x_2^2 + f_1x_2^2x_3 + f_2x_2x_3^2 + f_3x_3^3
\]

with \(f_0, \ldots, f_3 \in S_1\). One has:

\[
0 = f \mid L_2 = -b(x_0, 0)x_2^2 + f_0(x_0, 0, x_2, 0)x_2^3
\]

hence \(b(x_0, 0) = 0\) hence \(x_1 \mid b\).

One deduces, from Claim 1, that if \(x_1 \mid b\) then one has an exact sequence:

\[
0 \longrightarrow I(L_1^{(2)} \cup L_2) \longrightarrow I(W \cup L_2) \xrightarrow{\text{H}^0(\psi_1)} 2S(L_1)(-l - 3) \longrightarrow 0,
\]

and if \(x_1 \nmid b\) then \(\psi_1\) factorizes as:

\[
\begin{align*}
\mathcal{I}_{W \cup L_2} & \xrightarrow{\psi_1'} \mathcal{O}_{L_1}(-l - 4) \oplus \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1 \oplus \text{id}} 2\mathcal{O}_{L_1}(-l - 3)
\end{align*}
\]

with \(\psi'(x_1x_2F_2) = (1, 0) \in \text{H}^0(\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_1}(1))\), \(\psi'(x_3F_2) = (0, 1) \in \text{H}^0(\mathcal{O}_{L_1}(-1) \oplus \mathcal{O}_{L_1})\) and one has an exact sequence:

\[
0 \longrightarrow I(L_1^{(2)} \cup L_2) \longrightarrow I(W \cup L_2) \xrightarrow{\text{H}^0(\psi_1')} S(L_1)(-l - 4) \oplus S(L_1)(-l - 3) \longrightarrow 0.
\]

**Claim 2.** If \(x_1 \mid b\) then \(\text{Im } \psi_2 = \mathcal{O}_{L_1}(-l - m - 2)\) if and only if \(x_1 \mid b'\).

Indeed, the “if” part is clear (using Lemma 18(a)). For the “only if” part, assume that \(\text{Im } \psi_2 = \mathcal{O}_{L_1}(-l - m - 2)\). Using the exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(2)} \cup L_2} \longrightarrow \mathcal{I}_{W \cup L_2} \xrightarrow{\psi_1} 2\mathcal{O}_{L_1}(-l - 3) \longrightarrow 0
\]

and Lemma 17 one gets that \(\text{H}^1(\mathcal{I}_{W \cup L_2}(l + m + 2)) = 0\). One deduces that there exists an element \(f \in \text{H}^0(\mathcal{I}_{Y \cup L_2}(l + m + 2))\) such that \(\psi_2(f) = 1 \in \text{H}^0(\mathcal{O}_{L_1})\). Since \(\psi_2(f) = \eta_2(f)\) it follows that \(f\) must have the form:

\[
f = F_3 + f_0x_2F_2 + f_1x_3F_2 + g_0x_2^3 + g_1x_2^2x_3 + g_2x_2x_3^2 + g_3x_3^3
\]

with \(f_0, f_1 \in S_{m-1}\) and \(g_0, \ldots, g_3 \in S_{l+m-1}\). Taking into account that \(F_2 \mid L_2 = 0\) (because \(x_1 \mid b\)), one deduces that:

\[
0 = f \mid L_2 = v_0(x_0, 0)x_2^2 + g_0(x_0, 0, x_2, 0)x_2^2
\]

It follows that \(v_0(x_0, 0) = 0\) hence \(x_1 \mid v_0\) hence, by Lemma 18(a), \(x_1 \mid b'\).

(a) It follows from Claim 2 that one has an exact sequence:

\[
0 \longrightarrow I(W \cup L_2) \longrightarrow I(Y \cup L_2) \xrightarrow{\text{H}^0(\psi_2)} S(L_1)(-l - m - 2) \longrightarrow 0.
\]
(b) It follows from Claim 2 that $\psi_2$ factorizes as:

$$\mathcal{I}_{Y \cup L_2} \xrightarrow{\psi'_2} \mathcal{O}_{L_1}(-l - m - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - m - 2)$$

with $\psi'_2(x_1 F_3) = 1 \in H^0(\mathcal{O}_{L_1})$ and that one has an exact sequence:

$$0 \to I(W \cup L_2) \to I(Y \cup L_2) \xrightarrow{H^0(\psi'_2)} S(L_1)(-l - m - 3) \to 0.$$  

(c) Since $x_1 \mid p$ one has $m \geq 1$. Since $x_1 \nmid b$, Lemma [3.18] implies that one can assume that $x_1 \mid v_0$. It follows that $F_3 | L_2 = 0$ hence $F_3 \in I(Y \cup L_2)$. One deduces that $\text{Im} \, \psi_2 = \mathcal{O}_{L_1}(-l - m - 2)$ and that one has an exact sequence:

$$0 \to I(W \cup L_2) \to I(Y \cup L_2) \xrightarrow{H^0(\psi_2)} S(L_1)(-l - m - 2) \to 0.$$  

(d) Any element $f \in I(Y)$ can be written as:

$$f = f_0F_3 + f_1x_2F_2 + f_2x_3F_2 + f_3x_2^3 + f_4x_2^2x_3 + f_5x_2x_3^2 + f_6x_3^3$$

with $f_0, f_1, f_2 \in k[x_0, x_1], f_3 \in k[x_0, x_1, x_2]$ and $f_4, f_5, f_6 \in S$. One has:

$$f \mid L_2 = -f_0(x_0, 0)p(x_0, 0)b(x_0, 0)x_2 + f_0(x_0, 0)v_0(x_0, 0)x_2^2 -$$

$$-f_1(x_0, 0)b(x_0, 0)x_2^2 + f_3(x_0, 0, x_2)x_2^3.$$  

Since $x_1 \nmid b$ and $x_1 \nmid p$ it follows that $b(x_0, 0) \neq 0$ and $p(x_0, 0) \neq 0$. One deduces that if $f \in I(Y \cup L_2)$ (i.e., if $f \mid L_2 = 0$) then, firstly, $f_0(x_0, 0) = 0$ then $f_1(x_0, 0) = 0$ and, finally, $f_3(x_0, 0, x_2) = 0$. This means that $x_1 \mid f_0, x_1 \mid f_1$ and $x_1 \mid f_3$. One derives that $I(Y \cup L_2)$ is generated by the polynomials indicated in the statement. Moreover, it follows that $\psi_2$ factorizes as:

$$\mathcal{I}_{Y \cup L_2} \xrightarrow{\psi'_2} \mathcal{O}_{L_1}(-l - m - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - m - 2)$$

with $\psi'_2(x_1 F_3) = 1 \in H^0(\mathcal{O}_{L_1})$ and that one has an exact sequence:

$$0 \to I(W \cup L_2) \to I(Y \cup L_2) \xrightarrow{H^0(\psi'_2)} S(L_1)(-l - m - 3) \to 0.$$  

Notice that the exact sequences appearing in the above proof can be used to get (easily) concrete graded free resolutions of $I(W \cup L_2)$ and of $I(Y \cup L_2)$.

**Proposition B.20.** With the notation recalled at the beginning of this subsection:

(a) If $x_1 \mid b$ and $x_1 \mid b'$, i.e., if $b = x_1b_1$ and $b' = x_1b'_1$, then $H^0(\mathcal{O}_{Y \cup L_2})$ admits the following graded free resolution:

$$
\begin{array}{cccccccccc}
S(-3) & \oplus & S(-1) & \oplus & S & \oplus \\
0 \to & S(l - 3) & \xrightarrow{\delta_2} & S(-2) & \xrightarrow{\delta_1} & S(l - 1) & \xrightarrow{\delta_0} & H^0(\mathcal{O}_{Y \cup L_2}) & \to 0 \\
\oplus & 2S(l - 2) & \oplus & \oplus \\
S(2l + m - 3) & \oplus & S(2l + m - 1) \\
2S(2l + m - 2)
\end{array}
$$
with $\delta_0 = (1, x_1e_1, x_1e_2)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
  -b_1 & -a & x_2 & x_3 & 0 & 0 \\
  -b_1' & -a' & -pa & -pb & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_1x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
  -b & -x_3 & 0 \\
  a & x_2 & 0 \\
  -b' & pb & -x_3 \\
  a' & -pa & x_2
\end{pmatrix}.
$$

(b) If $x_1 \mid b$ and $x_1 \nmid b'$ then $H^0_*(\mathcal{O}_{Y \cup L_2})$ admits the following graded free resolution:

$$
0 \to S(-3) \oplus S(-2) \oplus S(-1) \oplus \cdots \oplus S(l-1) \oplus S(l) \oplus \cdots \oplus S(2l+m-1) \to 0
$$

with $\delta_0 = (1, x_1e_1, e_2)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
  -b_1 & -a & x_2 & x_3 & 0 & 0 \\
  -b_1' & -x_1a' & -x_1pa & -x_1pb & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_1x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
  -b & -x_3 & 0 \\
  a & x_2 & 0 \\
  -x_1b' & x_1pb & -x_3 \\
  x_1a' & -x_1pa & x_2
\end{pmatrix}.
$$

(c) If $x_1 \nmid b$ and $x_1 \mid p$ then $H^0_*(\mathcal{O}_{Y \cup L_2})$ admits the following graded free resolution:

$$
0 \to S(-3) \oplus S(-2) \oplus S(-1) \oplus \cdots \oplus S(l-1) \oplus S(l) \oplus \cdots \oplus S(2l+m-3) \oplus S(2l+m-2) \to 0
$$

with $\delta_0 = (1, e_1, x_1e_2)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
  -b & -x_1a & x_2 & x_3 & 0 & 0 \\
  -b_1' & -a' & -p_1a & -p_1b & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_1x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
  -x_1b & -x_3 & 0 \\
  x_1a & x_2 & 0 \\
  -b' & p_1b & -x_3 \\
  a' & -p_1a & x_2
\end{pmatrix}.
(d) If \(x_1 \nmid b\) and \(x_1 \nmid p\) then \(H^0_*(\mathcal{O}_{Y \cup L_2})\) admits the following graded free resolution:

\[
\begin{array}{cccccc}
& & S(-3) & \oplus & S(-2) & \oplus & S \\
0 \rightarrow & S(l-2) & \xrightarrow{\delta_2} & 2S(l-1) & \xrightarrow{\delta_1} & S(l) & \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{Y \cup L_2}) \rightarrow 0 \\
& & \oplus & \oplus & \oplus & \oplus & \\
& & S(2l+m-2) & \oplus & S(2l+m) & \oplus & S(2l+m-1)
\end{array}
\]

with \(\delta_0 = (1, e_1, e_2)\) and with \(\delta_1\) and \(\delta_2\) defined by the matrices:

\[
\begin{pmatrix}
x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
-\frac{b}{1} & -\frac{x_1a}{2} & x_2 & x_3 & 0 & 0 \\
-\frac{b'}{1} & -\frac{x_1a'}{2} & -pa & -pb & x_2 & x_3
\end{pmatrix}, \quad \begin{pmatrix}
-x_1x_2 & 0 & 0 \\
x_3 & 0 & 0 \\
-x_1b & -x_3 & 0 \\
x_1a & x_2 & 0 \\
-x_1b' & pb & -x_3 \\
x_1a' & -pa & x_2
\end{pmatrix}.
\]

**Proof.** Recall, from the beginning of this subsection, that one has an exact sequence:

\[
0 \rightarrow \text{Im} \phi \rightarrow \mathcal{O}_{Y \cup L_2} \rightarrow \mathcal{O}_{X \cup L_2} \rightarrow 0
\]

with \(\phi\) the composite morphism \(\mathcal{I}_{X \cup L_2} \rightarrow \mathcal{I}_X \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(2l+m)\) where the epimorphism \(\varepsilon : \mathcal{I}_X \rightarrow \mathcal{O}_{L_1}(2l+m)\) is characterized by:

\[
\varepsilon(F_2) = \Delta_1, \quad \varepsilon(x_2^2) = pa^2, \quad \varepsilon(x_2x_3) = pab, \quad \varepsilon(x_3^2) = pb^2.
\]

Moreover, Coker \(\phi\) is an \(\mathcal{O}_{L_1 \cap L_2} = \mathcal{O}_{\{P_0\}}\)-module which implies that \(x_1\mathcal{O}_{L_1}(2l+m-1) \subseteq \text{Im} \phi \subseteq \mathcal{O}_{L_1}(2l+m)\).

It follows, from Prop. **B.11** that if \(x_1 \nmid b\) then \(I(X \cup L_2) = (F_2, x_2x_3, x_3^2, x_1x_2^2)\) hence \(\text{Im} \phi\) coincides with the image of the morphism:

\[
(\Delta_1, pab, pb^2, px_1a^2) : \mathcal{O}_{L_1}(-l-2) \oplus 2\mathcal{O}_{L_1}(-2) \oplus \mathcal{O}_{L_1}(-3) \rightarrow \mathcal{O}_{L_1}(2l+m)
\]

and if \(x_1 \nmid b\) then \(I(X \cup L_2) = (x_1F_2, x_2x_3, x_3^2, x_1x_2^2)\) hence \(\text{Im} \phi\) coincides with the image of the morphism:

\[
(x_1\Delta_1, pab, pb^2, px_1a^2) : \mathcal{O}_{L_1}(-l-3) \oplus 2\mathcal{O}_{L_1}(-2) \oplus \mathcal{O}_{L_1}(-3) \rightarrow \mathcal{O}_{L_1}(2l+m).
\]

Recall, also, from Prop. **B.12** the graded free resolution of \(H^0_*(\mathcal{O}_{X \cup L_2})\).

(a) In this case \(x_1 \nmid \Delta_1\) and \(x_1 \nmid b\) hence \(\text{Im} \phi = x_1\mathcal{O}_{L_1}(2l+m-1)\). According to Prop. **B.12(a)**, the graded \(S\)-module \(H^0_*(\mathcal{O}_{Y \cup L_2})\) is generated by \(1 \in H^0_*(\mathcal{O}_{Y \cup L_2})\) and by \(x_1e_1 \in H^0_*(\mathcal{O}_{Y \cup L_2}(-1+1))\). Since \(H^1((\text{Im} \phi)(-l+1)) = 0\) it follows that the morphism \(H^0_*(\mathcal{O}_{Y \cup L_2}) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2})\) is surjective. One deduces an exact sequence:

\[
0 \rightarrow S(L_1)(2l+m-1) \rightarrow H^0_*(\mathcal{O}_{Y \cup L_2}) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2}) \rightarrow 0
\]

where the left morphism maps \(1 \in S(L_1)\) to the element of \(H^0_*(\mathcal{O}_{Y \cup L_2}(-2l-m+1))\) whose image into \(H^0_*(\mathcal{O}_{Y}(-2l-m+1)) \oplus H^0_*(\mathcal{O}_{L_2}(-2l-m+1)) = (x_1e_2, 0)\).
(b) In this case, since \( x_1 \nmid \Delta_1 \) it follows that \( \text{Im} \phi = \mathcal{O}_{L_1}(2l + m) \). One derives, as in case (a), an exact sequence:

\[
0 \longrightarrow S(L_1)(2l + m) \longrightarrow H^0_X(\mathcal{O}_{Y \cup L_2}) \longrightarrow H^0_X(\mathcal{O}_{X \cup L_2}) \longrightarrow 0
\]

where the left morphism maps 1 \( \in S(L_1) \) to the element of \( H^0(\mathcal{O}_{Y \cup L_2}(-2l - m)) \) whose image into \( H^0(\mathcal{O}_Y(-2l - m)) \oplus H^0(\mathcal{O}_{L_2}(-2l - m)) \) is \( (e_2, 0) \).

(c) In this case \( x_1 \nmid p \) hence \( \text{Im} \phi = x_1 \mathcal{O}_{L_1}(2l + m - 1) \). By Prop. [B.12](a), the graded \( S \)-module \( H^0_X(\mathcal{O}_{X \cup L_2}) \) is generated by \( 1 \in H^0(\mathcal{O}_{X \cup L_2}) \) and \( e_1 \in H^0(\mathcal{O}_{Y \cup L_2}(-l)) \). Since \( x_1 \nmid p \) it follows that \( m \geq 1 \) which implies that \( H^1((\text{Im} \phi)(-l)) = 0 \). It follows that the morphism \( H^0_X(\mathcal{O}_{Y \cup L_2}) \rightarrow H^0_X(\mathcal{O}_{X \cup L_2}) \) is surjective. One deduces an exact sequence:

\[
0 \longrightarrow S(L_1)(2l + m - 1) \longrightarrow H^0_X(\mathcal{O}_{Y \cup L_2}) \longrightarrow H^0_X(\mathcal{O}_{X \cup L_2}) \longrightarrow 0
\]

where the left morphism maps 1 \( \in S(L_1) \) to the element of \( H^0(\mathcal{O}_{Y \cup L_2}(-2l - m + 1)) \) whose image into \( H^0(\mathcal{O}_Y(-2l - m + 1)) \oplus H^0(\mathcal{O}_{L_2}(-2l - m + 1)) \) is \( (x_1e_2, 0) \).

(d) In this case, since \( x_1 \nmid pb^2 \) it follows that \( \text{Im} \phi = \mathcal{O}_{L_1}(2l + m) \). One derives an exact sequence:

\[
0 \longrightarrow S(L_1)(2l + m) \longrightarrow H^0_X(\mathcal{O}_{Y \cup L_2}) \longrightarrow H^0_X(\mathcal{O}_{X \cup L_2}) \longrightarrow 0
\]

where the left morphism maps 1 \( \in S(L_1) \) to the element of \( H^0(\mathcal{O}_{Y \cup L_2}(-2l - m)) \) whose image into \( H^0(\mathcal{O}_Y(-2l - m)) \oplus H^0(\mathcal{O}_{L_2}(-2l - m)) \) is \( (e_2, 0) \). \( \Box \)

B.3. A double line union two lines. We use the notation from the beginning of Subsection 3.1. In particular, \( X \) denotes a double structure on the line \( L_1 \). We want to add two other line to \( X \) and to describe the homogeneous ideal and the graded structural algebra of the resulting configuration. Up to a linear change of coordinates in \( \mathbb{P}^3 \) it suffices to consider only the following 5 configurations:

(i) \( X \cup L_2 \cup L'_2 \);

(ii) \( X \cup L_2 \cup L_3 \);

(iii) \( X \cup L_2 \cup L'_1 \);

(iv) \( X \cup L \cup L'_1 \) where \( L \) is the line of equations \( x_1 + cx_2 = x_3 = 0, c \in k \setminus \{0\} \);

(v) \( X \cup L'_1 \cup L''_1 \) where \( L''_1 \) is the line of equations \( x_0 - x_2 = x_1 - x_3 = 0; L_1, L'_1, L''_1 \)

are mutually disjoint lines contained in the quadric surface \( Q \subset \mathbb{P}^3 \) of equation \( x_0x_3 - x_1x_2 = 0 \).

The first four configurations can be treated in an unitary manner as follows: notice, fistly, that if one takes \( c = 0 \) in the first configuration of \( L \) one gets \( L_2 \). Let us denote \( x_1 + cx_2, c \in k, \) by \( \ell \). One has \( I(L_1 \cup L) = (x_3, \ell x_2) \). Let us denote by \( L' \) any of the lines \( L'_1, L'_2 \) and \( L_3 \). We have to consider two subcases.

(I) If \( x_1 \nmid b \), i.e., if \( b = x_1b_1 \) then, according to the proof of Prop. [B.11](a) and of Prop. [B.12](a), \( I(X \cup L) = S(F_2 - cb_1x_2^2) + I(L'_1 \cup L) \) and one has exact sequences:

\[
\begin{align*}
0 & \longrightarrow \mathcal{I}_{L'_1 \cup L} \longrightarrow \mathcal{I}_{X \cup L} \psi \longrightarrow \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0 \\
0 & \longrightarrow \mathcal{I}_{X \cup L} \longrightarrow \mathcal{I}_{L_1 \cup L} \phi' \longrightarrow \mathcal{O}_{L_1}(l - 1) \longrightarrow 0
\end{align*}
\]
with $\psi(F_2 - cb_1x_2^3) = 1 \in H^0(\mathcal{O}_{L_1})$, and with $\phi'(x_3) = b_1$, $\phi'(lx_2) = a$. Let us denote by $\psi_1$ and $\phi'_1$ the composite morphisms:

$$\mathcal{I}_{X \cup L \cup L'} \rightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2) \text{ and } \mathcal{I}_{L_1 \cup L \cup L'} \rightarrow \mathcal{I}_{L_1 \cup L} \xrightarrow{\phi'_1} \mathcal{O}_{L_1}(l - 1).$$

Then, by Lemma [B.35] one has exact sequences:

$$0 \rightarrow \mathcal{I}_{L_1(-l - 2)} \rightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi_1} \mathcal{O}_{L_1}(-l - 2),$$
$$0 \rightarrow \text{Im } \phi'_1 \rightarrow \mathcal{O}_{X \cup L \cup L'} \rightarrow \mathcal{O}_{L_1 \cup L \cup L'} \rightarrow 0.$$

Moreover, $\text{Coker } \psi_1$ and $\text{Coker } \phi'_1$ are $\mathcal{O}_{L_1 \cap L'}$-modules.

(II) If $x_1 \nmid b$ then, according to the proof of Prop. [B.11(b)] and of Prop. [B.12(b)], $I(X \cup L) = \alpha F_2 + I(L_1^{(1)} \cup L)$ and one has exact sequences:

$$0 \rightarrow \mathcal{I}_{L_1(-l - 3)} \rightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3) \rightarrow 0,$$
$$0 \rightarrow \mathcal{I}_{X \cup L} \rightarrow \mathcal{I}_{L_1 \cup L} \xrightarrow{\phi} \mathcal{O}_{L_1}(l) \rightarrow 0$$

with $\psi'(\ell F_2) = 1 \in H^0(\mathcal{O}_{L_1})$, and with $\phi(x_3) = b$, $\phi(\ell x_2) = x_1a$. Let us denote by $\psi'_1$ and $\phi_1$ the composite morphisms:

$$\mathcal{I}_{X \cup L \cup L'} \rightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi'_1} \mathcal{O}_{L_1}(-l - 3) \text{ and } \mathcal{I}_{L_1 \cup L \cup L'} \rightarrow \mathcal{I}_{L_1 \cup L} \xrightarrow{\phi} \mathcal{O}_{L_1}(l).$$

Then, by Lemma [B.35] one has exact sequences:

$$0 \rightarrow \mathcal{I}_{L_1(-l - 3)} \rightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi'_1} \mathcal{O}_{L_1}(-l - 3),$$
$$0 \rightarrow \text{Im } \phi_1 \rightarrow \mathcal{O}_{X \cup L \cup L'} \rightarrow \mathcal{O}_{L_1 \cup L \cup L'} \rightarrow 0.$$

Moreover, $\text{Coker } \psi'_1$ and $\text{Coker } \phi_1$ are $\mathcal{O}_{L_1 \cap L'}$-modules.

**Lemma B.21.** The homogeneous ideal $I(W) \subset S$ of $W = L_1^{(1)} \cup L_2 \cup L_2'$ is generated by $x_2x_3$, $x_0x_2^3$, $x_1x_2^3$ and admits the following graded free resolution:

$$0 \rightarrow 2S(-4) \xrightarrow{\left(\begin{array}{c} x_0x_3 & -x_1x_2 \\ x_2 & 0 \\ 0 & x_3 \end{array}\right)} S(-2) \oplus 2S(-3) \rightarrow I(W) \rightarrow 0.$$

**Proof.** $I(W)$ is, by definition:

$$(x_2^2, x_2x_3, x_3^2) \cap (x_1, x_3) \cap (x_0, x_2) = (x_2x_3, x_0x_3^2, x_1x_2^3) \cap (x_0, x_2) = (x_2x_3, x_0x_3^2, x_1x_2^3).$$

If $W'$ is the curve directly linked to $W$ by the complete intersection defined by $x_0x_3^2$ and $x_1x_2^3$ then $I(W') = (x_0x_3, x_1x_2)$. One can apply, now, Ferrand’s result about liaison. □
Proposition B.22. Let $X$ be the double structure on the line $L_1$ considered at the beginning of Subsection B.1.

(a) If $x_1 \mid b$ and $x_0 \mid a$ then $I(X \cup L_2 \cup L'_2) = SF_2 + I(L^{(1)}_1 \cup L_2 \cup L'_2)$.
(b) If $x_1 \mid b$ and $x_0 \not\mid a$ then $I(X \cup L_2 \cup L'_2) = Sx_0F_2 + I(L^{(1)}_1 \cup L_2 \cup L'_2)$.
(c) If $x_1 \not\mid b$ and $x_0 \mid a$ then $I(X \cup L_2 \cup L'_2) = Sx_1F_2 + I(L^{(1)}_1 \cup L_2 \cup L'_2)$.
(d) If $x_1 \not\mid b$ and $x_0 \not\mid a$ then $I(X \cup L_2 \cup L'_2) = Sx_0x_1F_2 + I(L^{(1)}_1 \cup L_2 \cup L'_2)$.

Proof. We use the exact sequences defined at the beginning of this subsection. Since $L_1 \cap L'_2 = \{P_1\}$ one deduces that:

$$x_0\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im } \psi_1 \subseteq \mathcal{O}_{L_1}(-l - 2)$$

$$x_0\mathcal{O}_{L_1}(-l - 4) \subseteq \text{Im } \psi_1' \subseteq \mathcal{O}_{L_1}(-l - 3).$$

Claim 1. If $x_1 \mid b$ then $\text{Im } \psi_1 = \mathcal{O}_{L_1}(-l - 2)$ if and only if $x_0 \mid a$.

Indeed, the “if” part is clear because if $x_0 \mid a$ then $F_2 \mid L'_2 = 0$ hence $F_2 \in I(X \cup L_2 \cup L'_2)$.

For the “only if” part, assume that $\text{Im } \psi_1 = \mathcal{O}_{L_1}(-l - 2)$. Since, by Lemma B.21, $H^1(\mathcal{I}_{L_1 \cup L_2 \cup L'_2}) = 0$, it follows that there exists $f \in H^0(\mathcal{I}_{X \cup L_2 \cup L'_2}(l + 2))$ such that $\psi_1(f) = 1 \in H^0(\mathcal{O}_{L_1})$. Since $\psi_1(f) = \psi(f)$ one deduces that $f$ must have the form:

$$f = F_2 + f_0x_2x_3 + f_1x_3^2 + f_2x_1x_2^2.$$ 

But $0 = f \mid L'_2 = a(0, x_1)x_3 + f_1(0, x_1, 0, x_3)x_3^2$ implies that $a(0, x_1) = 0$ hence $x_0 \mid a$.

Claim 2. If $x_1 \not\mid b$ then $\text{Im } \psi_1' = \mathcal{O}_{L_1}(-l - 3)$ if and only if $x_0 \mid a$.

Indeed, the “if” part is clear because if $x_0 \mid a$ then $F_2 \mid L'_2 = 0$ hence $x_1F_2 \in I(X \cup L_2 \cup L'_2)$.

For the “only if” part, assume that $\text{Im } \psi_1' = \mathcal{O}_{L_1}(-l - 3)$. Since, by Lemma B.21, $H^1(\mathcal{I}_{L_1 \cup L_2 \cup L'_2}) = 0$, it follows that there exists $f \in H^0(\mathcal{I}_{X \cup L_2 \cup L'_2}(l + 3))$ such that $\psi_1'(f) = 1 \in H^0(\mathcal{O}_{L_1})$. Since $\psi_1'(f) = \psi(f)$ one deduces that $f$ must have the form:

$$f = x_1F_2 + f_0x_2x_3 + f_1x_3^2 + f_2x_1x_2^2.$$ 

But $0 = f \mid L'_2 = a(0, x_1)x_1x_3 + f_1(0, x_1, 0, x_3)x_3^2$ implies that $a(0, x_1) = 0$ hence $x_0 \mid a$.

(a) One deduces, from Claim 1, the existence of an exact sequence:

$$0 \rightarrow I(L^{(1)}_1 \cup L_2 \cup L'_2) \rightarrow I(X \cup L_2 \cup L'_2) \xrightarrow{\text{H}^0(\psi_1)} S(L_1)(-l - 2) \rightarrow 0.$$ 

(b) Claim 1 implies that $\psi_1$ factorizes as:

$$\mathcal{I}_{X \cup L_2 \cup L'_2} \xrightarrow{\psi_2} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_0} \mathcal{O}_{L_1}(-l - 2)$$

with $\mathcal{I}_{x_0F_2} = 1 \in H^0(\mathcal{O}_{L_1})$. One deduces the existence of an exact sequence:

$$0 \rightarrow I(L^{(1)}_1 \cup L_2 \cup L'_2) \rightarrow I(X \cup L_2 \cup L'_2) \xrightarrow{\text{H}^0(\psi_2)} S(L_1)(-l - 3) \rightarrow 0.$$
(c) One deduces, from Claim 2, the existence of an exact sequence:

\[ 0 \to I(L_1^{(1)} \cup L_2 \cup L_2') \to I(X \cup L_2 \cup L_2') \xrightarrow{H^0_*(\psi'_1)} S(L_1)(-l - 3) \to 0. \]

(d) Claim 2 implies that \( \psi'_1 \) factorizes as:

\[ \mathcal{I}_{X \cup L_2 \cup L_2'} \xrightarrow{\psi'_2} \mathcal{O}_{L_1}(-l - 4) \xrightarrow{x_0} \mathcal{O}_{L_1}(-l - 3) \]

with \( \psi'_2(x_0x_2F_2) = 1 \in H^0(\mathcal{O}_{L_1}) \). One deduces the existence of an exact sequence:

\[ 0 \to I(L_1^{(1)} \cup L_2 \cup L_2') \to I(X \cup L_2 \cup L_2') \xrightarrow{H^0_*(\psi'_2)} S(L_1)(-l - 4) \to 0. \]

Notice that the exact sequences appearing in the above proof can be used to get a concrete graded free resolution of \( I(X \cup L_2 \cup L_2') \).

\[ \square \]

**Proposition B.23.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection [B.7]

(a) If \( x_1 \mid b \) and \( x_0 \mid a \), i.e., if \( b = x_1b_1 \) and \( a = x_0a_0 \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \) admits the following graded free resolution:

\[
0 \to 2S(-3) \xrightarrow{\delta_2} 3S(-2) \xrightarrow{\delta_1} S \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \to 0
\]

with \( \delta_0 = (1, x_0x_1e_1) \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
x_0x_2 & x_1x_2 & x_2x_3 & 0 & 0 \\
-b_1 & -a_0 & 0 & x_2 & x_3
\end{pmatrix}, \quad \begin{pmatrix}
x_0 & x_2 & 0 & 0 & 0 \\
0 & -x_3 & 0 & x_0 & x_1 \\
-x_1 & 0 & -x_3 & -b_1 & 0 \\
0 & -a_0 & x_2
\end{pmatrix}.
\]

(b) If \( x_1 \mid b \) and \( x_0 \nmid a \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \) admits the following graded free resolution:

\[
0 \to 2S(-3) \xrightarrow{\delta_2} 3S(-2) \xrightarrow{\delta_1} S \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \to 0
\]

with \( \delta_0 = (1, x_1e_1) \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
x_0x_3 & x_1x_2 & x_2x_3 & 0 & 0 \\
x_0b_1 & -a & 0 & x_2 & x_3
\end{pmatrix}, \quad \begin{pmatrix}
x_0 & x_2 & 0 & 0 & 0 \\
0 & -x_3 & 0 & x_0 & x_1 \\
x_0 & x_2 & 0 & -x_3 & -b_1 \\
0 & -a & x_2
\end{pmatrix}.
\]
(c) If $x_1 \not| b$ and $x_0 \mid a$ then the graded $S$-module $H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'})$ admits the following graded free resolution:

$$
0 \rightarrow 2S(-3) \oplus S(l-3) \xrightarrow{\delta_2} 3S(-2) \oplus 2S(l-2) \xrightarrow{\delta_1} S \oplus S(l) \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \rightarrow 0
$$

with $\delta_0 = (1, x_0e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_0x_3 & x_1x_2 & x_2x_3 & 0 & 0 \\
  -b & -x_1a & 0 & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_2 & 0 & 0 \\
  0 & x_3 & 0 \\
  x_0 & x_1 & 0 \\
  -b & 0 & -x_3 \\
  0 & -x_1a & x_2
\end{pmatrix}.
$$

(d) If $x_1 \not| b$ and $x_0 \not| a$ then the graded $S$-module $H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'})$ admits the following graded free resolution:

$$
0 \rightarrow 2S(-3) \oplus S(l-2) \xrightarrow{\delta_2} 3S(-2) \oplus 2S(l-1) \xrightarrow{\delta_1} S \oplus S(l) \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup L_2 \cup L_2'}) \rightarrow 0
$$

with $\delta_0 = (1, e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_0x_3 & x_1x_2 & x_2x_3 & 0 & 0 \\
  -x_0b & -x_1a & 0 & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_2 & 0 & 0 \\
  0 & x_3 & 0 \\
  x_0 & x_1 & 0 \\
  -x_0b & 0 & -x_3 \\
  0 & -x_1a & x_2
\end{pmatrix}.
$$

Proof. $I(L_1 \cup L_2 \cup L_2') = (x_2, x_3) \cap (x_1, x_3) \cap (x_0, x_2) = (x_0x_3, x_1x_2, x_2x_3)$. $L_1 \cup L_2 \cup L_2'$ is directly linked by the complete intersection defined by $x_0x_3$ and $x_1x_2$ to the curve whose homogeneous ideal is $(x_0, x_1, \text{i.e.},$ to the line $L_1'$. Using Ferrand’s result about liaison one gets a minimal graded free resolution:

$$
0 \rightarrow 2S(-3) \xrightarrow{\delta_2} 3S(-2) \xrightarrow{\delta_1} I(L_1 \cup L_2 \cup L_2') \rightarrow 0.
$$

If $x_1 \mid b$ we use the exact sequence (defined at the beginning of this subsection):

$$
0 \rightarrow \text{Im} \phi'_1 \rightarrow \mathcal{O}_{X \cup L_2 \cup L_2'} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_2'} \rightarrow 0
$$

with $\phi'_1$ the composite morphism $\mathcal{J}_{L_1 \cup L_2 \cup L_1'} \rightarrow \mathcal{J}_{L_1 \cup L_2} \xrightarrow{\phi} \mathcal{O}_{L_1}(l-1)$. Recall, from the proof of Prop. [B.12] that $\phi'(x_3) = b_1$ and $\phi'(x_1x_2) = a$ hence :

$$
\phi'_1(x_0x_3) = x_0b_1, \phi'_1(x_1x_2) = a, \phi'_1(x_2x_3) = 0.
$$

If $x_1 \not| b$ we use the exact sequence :

$$
0 \rightarrow \text{Im} \phi_1 \rightarrow \mathcal{O}_{X \cup L_2 \cup L_2'} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_2'} \rightarrow 0
$$
with $\phi_1$ the composite morphism $\mathcal{I}_{L_1 \cup L_2 \cup L'_2} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\phi} \mathcal{O}_{L_1}(l)$. Recall, from the proof of Prop. \textbf{B.12} that $\phi(x_3) = b$ and $\phi(x_1x_2) = x_1a$ hence:

$$\phi_1(x_0x_3) = x_0b, \quad \phi'_1(x_1x_2) = x_1a, \quad \phi'_1(x_2x_3) = 0.$$ (a) In this case $\text{Im} \phi'_1 = x_0\mathcal{O}_{L_1}(l - 2)$. Since $L_1 \cup L_2 \cup L'_2$ is arithmetically CM, the graded $S$-module $H^0(\mathcal{O}_{L_1 \cup L_2 \cup L'_2})$ is generated by $1 \in H^0(\mathcal{O}_{L_1 \cup L_2 \cup L'_2})$. One deduces an exact sequence:

$$0 \to S(L_1)(l - 2) \to H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}) \to H^0(\mathcal{O}_{L_1 \cup L_2 \cup L'_2}) \to 0$$

where the left morphism maps $1 \in S(L_1)$ to the element of $H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}(-l + 2))$ whose image into $H^0(\mathcal{O}_{X}(-l + 2)) \oplus H^0(\mathcal{O}_{L_2}(-l + 2)) \oplus H^0(\mathcal{O}_{L'_2}(-l + 2))$ is $(x_0x_1e_1, 0, 0)$.

(b) In this case $\text{Im} \phi_1 = \mathcal{O}_{L_1}(l - 1)$. One deduces an exact sequence:

$$0 \to S(L_1)(l - 1) \to H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}) \to H^0(\mathcal{O}_{L_1 \cup L_2 \cup L'_2}) \to 0$$

where the left morphism maps $1 \in S(L_1)$ to the element of $H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}(-l + 1))$ whose image into $H^0(\mathcal{O}_{X}(-l + 1)) \oplus H^0(\mathcal{O}_{L_2}(-l + 1)) \oplus H^0(\mathcal{O}_{L'_2}(-l + 1))$ is $(x_0e_1, 0, 0)$.

(c) In this case $\text{Im} \phi_1 = x_0\mathcal{O}_{L_1}(l - 1)$. One deduces an exact sequence:

$$0 \to S(L_1)(l - 1) \to H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}) \to H^0(\mathcal{O}_{L_1 \cup L_2 \cup L'_2}) \to 0$$

where the left morphism maps $1 \in S(L_1)$ to the element of $H^0(\mathcal{O}_{X \cup L_2 \cup L'_2}(-l))$ whose image into $H^0(\mathcal{O}_{X}(-l)) \oplus H^0(\mathcal{O}_{L_2}(-l)) \oplus H^0(\mathcal{O}_{L'_2}(-l))$ is $(e_1, 0, 0)$. \hfill $\Box$

**Lemma B.24.** The homogeneous ideal $I(W) \subset S$ of $W = L_1^{(1)} \cup L_2 \cup L_3$ is generated by $x_2x_3, x_1x_2^2, x_1x_3^2$ and admits the following graded free resolution:

$$0 \to 2S(-4) \xrightarrow{\begin{pmatrix} -x_1x_2 & -x_1x_3 \\ x_3 & 0 \\ x_2 \end{pmatrix}} S(-2) \oplus 2S(-3) \to I(W) \to 0.$$  

**Proof.** $I(W)$ is, by definition:

$$(x_2^2, x_2x_3, x_3^2) \cap (x_1, x_3) \cap (x_1, x_2) = (x_2x_3, x_3^2, x_1x_2^2) \cap (x_1, x_2) = (x_2x_3, x_1x_2, x_1x_3^2).$$

If $W'$ is the curve directly linked to $W$ by the complete intersection defined by $x_2x_3$ and $x_1(x_2^2 + x_3^2)$ then $I(W') = (x_2, x_3) (x_2 \cdot x_1x_2^2 = x_2 \cdot x_1(x_2^2 + x_3^2) - x_1x_3 \cdot x_2x_3)$, i.e., $W' = L_1$. One can apply, now, Ferrand’s result about liaison. \hfill $\Box$

**Proposition B.25.** Let $X$ be the double structure on the line $L_1$ considered at the beginning of subsection \textbf{B.1}. Then $I(X \cup L_2 \cup L_3) = Sx_1F_2 + I(L_1^{(1)} \cup L_2 \cup L_3)$. 

Proof. If $x_1 \mid b$ then one has an exact sequence (look at the beginning of this subsection):

$$0 \longrightarrow \mathcal{I}_{L_1}^{(1)} \oplus L_2 \oplus L_3 \longrightarrow \mathcal{I}_{X \cup L_2 \cup L_3} \xrightarrow{\psi_1} \mathcal{O}_{L_1}(-l - 2)$$

where $\psi_1$ is the composite morphism $\mathcal{I}_{X \cup L_2 \cup L_3} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2)$. Since $L_1 \cap L_3 = \{P_i\}$, Coker $\psi_1$ is an $\mathcal{O}_{\{P_i\}}$-module hence $x_1 \mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im} \psi_1 \subseteq \mathcal{O}_{L_1}(-l - 2)$.

Claim. Im $\psi_1 = x_1 \mathcal{O}_{L_1}(-l - 3)$.

Indeed, assume that Im $\psi_1 = \mathcal{O}_{L_1}(-l - 2)$. Since $H^1_0(\mathcal{I}_{L_1}^{(1)} \oplus L_2 \oplus L_3) = 0$ (by Lemma B.24) it follows that there exists $f \in H^0(\mathcal{I}_{X \cup L_2 \cup L_3}(l + 2))$ such that $\psi_1(f) = 1 \in H^0(\mathcal{O}_{L_1})$. But $\psi_1(f) = \psi(f)$ hence, by the proof of Prop. [B.11] $f$ must have the form:

$$f = F_2 + f_0 x_2 x_3 + f_1 x_3^2 + f_2 x_1 x_2^2.$$ 

Since $0 = f \mid L_3 = a(x_0, 0)x_3 + f_1(x_0, 0, 0, x_3)x_3^2$ it follows that $a(x_0, 0) = 0$ hence $x_1 \mid a$. But this contradicts the fact that $a$ and $b$ are coprime.

It follows from the Claim that $\psi_1$ factorizes as $\mathcal{I}_{X \cup L_2 \cup L_3} \xrightarrow{\psi_2} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2)$ with $\psi_2$ mapping $x_1 F_2 \in H^0(\mathcal{I}_{X \cup L_2 \cup L_3}(l + 3))$ to $1 \in H^0(\mathcal{O}_{L_1})$. One deduces an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup L_2 \cup L_3) \longrightarrow I(X \cup L_2 \cup L_3) \xrightarrow{H^0_*(\psi_2)} S(L_1)(-l - 3) \longrightarrow 0.$$ 

If $x_1 \nmid b$ then one has an exact sequence:

$$0 \longrightarrow \mathcal{I}_{L_1}^{(1)} \oplus L_2 \oplus L_3 \longrightarrow \mathcal{I}_{X \cup L_2 \cup L_3} \xrightarrow{\psi'_1} \mathcal{O}_{L_1}(-l - 3)$$

where $\psi'_1$ is the composite morphism $\mathcal{I}_{X \cup L_2 \cup L_3} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3)$. Since $x_1 F_2 \mid L_3 = 0$ it follows that $x_1 F_2 \in I(X \cup L_2 \cup L_3)$. But $\psi'_1(x_1 F_2) = \psi'(x_1 F_2) = 1 \in H^0(\mathcal{O}_{L_1})$. One deduces that $\psi'_1$ is an epimorphism and that, moreover, one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup L_2 \cup L_3) \longrightarrow I(X \cup L_2 \cup L_3) \xrightarrow{H^0_*(\psi'_1)} S(L_1)(-l - 3) \longrightarrow 0.$$ 

Notice that one can use this exact sequence (and the similar one for the case $x_1 \mid b$) to get a concrete graded free resolution of $I(X \cup L_2 \cup L_3)$. □

Proposition B.26. Let $X$ be the double structure on the line $L_1$ considered at the beginning of Subsection B.2. Then the graded $S$-module $H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3})$ admits the following graded free resolution:

$$0 \longrightarrow \begin{array}{c} 2S(-3) \oplus \delta_2 \oplus 3S(-2) \delta_1 \oplus S \oplus \delta_0 \end{array} \xrightarrow{H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3})} 0$$

where

$$\delta_2 : S(l - 3) \rightarrow 2S(l - 2) \delta_1 : 2S(l - 2) \rightarrow S(l - 1) \delta_0 : S(l - 1) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3})$$
with \( \delta_0 = (1, x_1 e_1) \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:
\[
\begin{pmatrix}
  x_1 x_2 & x_1 x_3 & x_2 x_3 & 0 & 0 \\
  -a & -b & 0 & x_2 & x_3 \\
\end{pmatrix},
\begin{pmatrix}
  -x_3 & 0 & 0 \\
  x_2 & -x_2 & 0 \\
  0 & x_1 & 0 \\
  b & -b & -x_3 \\
  -a & 0 & x_2 \\
\end{pmatrix}.
\]

**Proof.** \( I(L_1 \cup L_2 \cup L_3) = (x_2, x_3) \cap (x_1, x_3) \cap (x_1, x_2) = (x_1 x_2, x_1 x_3, x_2 x_3) \). One deduces a minimal graded free resolution:
\[
0 \rightarrow 2S(-3) \rightarrow 3S(-2) \rightarrow I(L_1 \cup L_2 \cup L_3) \rightarrow 0.
\]

If \( x_1 \mid b \), i.e., if \( b = x_1 b_1 \) we use the exact sequence (defined at the beginning of this subsection):
\[
0 \rightarrow \text{Im} \phi' \rightarrow \mathcal{O}_{X \cup L_2 \cup L_3} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_3} \rightarrow 0
\]
with \( \phi' \) the composite morphism \( \mathcal{I}_{L_1 \cup L_2 \cup L_3} \rightarrow \mathcal{I}_{L_1 \cup L_2} \rightarrow \mathcal{O}_{L_1}(l - 1) \). Recall, from the proof of Prop. 12 that \( \phi'(x_3) = b_1 \) and \( \phi'(x_1 x_2) = a \) hence:
\[
\phi'_1(x_1 x_2) = a, \quad \phi'_1(x_1 x_3) = x_1 b_1 = b, \quad \phi'_1(x_2 x_3) = 0.
\]
It follows that \( \phi'_1 \) is an epimorphism. Moreover, since \( L_1 \cup L_2 \cup L_3 \) is arithmetically CM the graded \( S \)-module \( H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_3}) \) is generated by \( 1 \in H^0(\mathcal{O}_{L_1 \cup L_2 \cup L_3}) \). One deduces an exact sequence:
\[
0 \rightarrow S(L_1)(l - 1) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3}) \rightarrow H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_3}) \rightarrow 0
\]
where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3}(-l + 1)) \) whose image into \( H^0_*(\mathcal{O}_X(-l + 1)) \oplus H^0_*(\mathcal{O}_{L_2}(-l + 1)) \oplus H^0_*(\mathcal{O}_{L_3}(-l + 1)) \) is \( (x_1 e_1, 0, 0) \).

If \( x_1 \nmid b \) we use the exact sequence:
\[
0 \rightarrow \text{Im} \phi_1 \rightarrow \mathcal{O}_{X \cup L_2 \cup L_3} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_3} \rightarrow 0
\]
with \( \phi_1 \) the composite morphism \( \mathcal{I}_{L_1 \cup L_2 \cup L_3} \rightarrow \mathcal{I}_{L_1 \cup L_2} \rightarrow \mathcal{O}_{L_1}(l) \). Recall, from the proof of Prop. 12 that \( \phi(x_3) = b \) and \( \phi(x_1 x_2) = x_1 a \) hence:
\[
\phi_1(x_1 x_2) = x_1 a, \quad \phi_1(x_1 x_3) = x_1 b, \quad \phi_1(x_2 x_3) = 0.
\]
One deduces that \( \text{Im} \phi_1 = x_1 \mathcal{O}_{L_1}(l - 1) \) and that one has an exact sequence:
\[
0 \rightarrow S(L_1)(l - 1) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3}) \rightarrow H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_3}) \rightarrow 0
\]
where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L_3}(-l + 1)) \) whose image into \( H^0_*(\mathcal{O}_X(-l + 1)) \oplus H^0_*(\mathcal{O}_{L_2}(-l + 1)) \oplus H^0_*(\mathcal{O}_{L_3}(-l + 1)) \) is \( (x_1 e_1, 0, 0) \). \( \square \)
Lemma B.27. The homogeneous ideal $I(W) \subset S$ of $W = L_1^{(1)} \cup L_2 \cup L_1'$ is generated by $x_0 x_2 x_3, x_0 x_3^2, x_1 x_2 x_3, x_1 x_3^2$ and admits the following graded free resolution:

$$0 \rightarrow S(-5) \xrightarrow{d_2} 5S(-4) \xrightarrow{d_1} 5S(-3) \xrightarrow{d_0} I(W) \rightarrow 0$$

where $d_1$ and $d_2$ are defined by the matrices:

$$
\begin{pmatrix}
-x_3 & 0 & 0 & -x_1 & 0 \\
x_2 & 0 & 0 & 0 & -x_1 \\
0 & -x_3 & 0 & 0 & 0 \\
0 & x_2 & -x_3 & x_0 & 0 \\
0 & 0 & x_2 & 0 & x_0
\end{pmatrix}
= 
\begin{pmatrix}
-x_1 \\
0 \\
x_3 \\
-x_2
\end{pmatrix}.
$$

Proof. By definition:

$$I(W) = (x_2^2, x_2 x_3, x_3^2) \cap (x_1, x_3) \cap (x_0, x_1) = (x_2 x_3, x_3^2, x_1 x_3^2) \cap (x_0, x_1) = (x_0 x_2 x_3, x_0 x_3^2, x_1 x_3^2, x_1 x_2 x_3, x_1 x_3^2).$$

Since $I(L_1 \cup L_1') = (x_0 x_2, x_0 x_3, x_1 x_2, x_1 x_3)$, it follows that:

$I(W) = S x_1 x_3^2 + I(L_1 \cup L_1') x_3$.

Moreover, by Lemma [B.1], $I(L_1 \cup L_1')$ admits a graded free resolution of the form:

$$0 \rightarrow S(-4) \rightarrow 4S(-3) \rightarrow 4S(-2) \rightarrow I(L_1 \cup L_1') \rightarrow 0.$$

One can apply, now, Remark [B.4]. Actually, by the main property of the basic double linkage, $L_1 \cup L_1'$ can be obtained from $W$ by two direct linkages. Concretely, these two linkages are the following ones: if $W'$ is the curve directly linked to $W$ by the complete intersection defined by $x_0 x_3^2$ and $x_1 x_2^2$ then, by Remark [B.3b],

$$I(W') = (x_3, x_1 x_2) \cap (x_2, x_0 x_3) \cap (x_0, x_2^3) = (x_0 x_3, x_0 x_1 x_2, x_1 x_2^3, x_1 x_3^2).$$

If $W''$ is the curve directly linked to $W'$ by the complete intersection defined by $x_0 x_3$ and $x_1 x_2^2$ then, by the same Remark, $I(W'') = (x_2, x_3) \cap (x_0, x_1)$ hence $W'' = L_1 \cup L_1'$. □

Proposition B.28. Let $X$ be the double structure on the line $L_1$ considered at the beginning of Subsection B.1.

(a) If $l = -1$ and $b = 0$ then $I(X \cup L_2 \cup L_1') = (x_0 x_3, x_1 x_3, x_1 x_3^2)$.

(b) If $l \geq 0$ and $x_1 \mid b$ then $I(X \cup L_2 \cup L_1') = S F_2 + I(L_1^{(1)} \cup L_2 \cup L_1')$.

(c) If $x_1 \nmid b$ then $I(X \cup L_2 \cup L_1') = S x_1 F_2 + I(L_1^{(1)} \cup L_2 \cup L_1')$.

Proof. (a) In this case $F_2 = x_3$ hence $I(X) = (x_3, x_3^2)$ hence $I(X \cup L_2 \cup L_1') = (x_3, x_3^2) \cap (x_1, x_3) \cap (x_0, x_1) = (x_0 x_3, x_1 x_3, x_1 x_3^2)$.

(b) We use the exact sequence:

$$0 \rightarrow J_{L_1^{(1)} \cup L_2 \cup L_1'} \rightarrow J_{X \cup L_2 \cup L_1'} \xrightarrow{\psi_1} \mathcal{O}_{L_1}(-l - 2)$$
defined at the beginning of this subsection. Since \( L_1 \cap L'_1 = \emptyset \) it follows that \( \psi_1 \) is an epimorphism. Moreover, since \( l \geq 0 \) it follows that \( F_2 | L'_1 = 0 \) hence \( F_2 \in I(X \cup L_2 \cup L'_1) \). Since \( \psi_1(F_2) = \psi(F_2) = 1 \in H^0(\mathcal{O}_{L_1}) \) one deduces an exact sequence:

\[
0 \rightarrow I(L_1^{(1)} \cup L_2 \cup L'_1) \rightarrow I(X \cup L_2 \cup L'_1) \xrightarrow{H^0_*(\psi_1)} S(L_1)(-l - 2) \rightarrow 0.
\]

(c) We use the exact sequence:

\[
0 \rightarrow \mathcal{I}_{L_1^{(1)} \cup L_2 \cup L'_1} \rightarrow \mathcal{I}_{X \cup L_2 \cup L'_1} \xrightarrow{\psi'_1} \mathcal{O}_{L_1}(-l - 3)
\]
defined at the beginning of this subsection. Since \( L_1 \cap L'_1 = \emptyset \) it follows that \( \psi'_1 \) is an epimorphism. Moreover, since \( x_1F_2 | L'_1 = 0 \) it follows that \( x_1F_2 \in I(X \cup L_2 \cup L'_1) \). Since \( \psi'_1(x_1F_2) = \psi'(x_1F_2) = 1 \in H^0(\mathcal{O}_{L_1}) \) one deduces an exact sequence:

\[
0 \rightarrow I(L_1^{(1)} \cup L_2 \cup L'_1) \rightarrow I(X \cup L_2 \cup L'_1) \xrightarrow{H^0_*(\psi'_1)} S(L_1)(-l - 3) \rightarrow 0.
\]

Notice that the exact sequences deduced in the above proof can be used to get a concrete graded free resolution of \( I(X \cup L_2 \cup L'_1) \).

**Proposition B.29.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection [B.1].

(a) If \( x_1 \mid b \), i.e., if \( b = x_1b_1 \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L'_1}) \) admits the following free resolution:

\[
0 \rightarrow 2S(-3) \xrightarrow{\delta_2} 3S(-2) \xrightarrow{\delta_1} S \oplus \delta_0 H^0_*(\mathcal{O}_{X \cup L_2 \cup L'_1}) \rightarrow 0
\]

with \( \delta_0 = (1, x_1e_1) \) and \( \delta_1, \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
  x_0x_3 & x_1x_2 & x_1x_3 & 0 & 0 \\
  -x_0b_1 & -a & -b & x_2 & x_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
  -x_1 & 0 & 0 \\
  0 & -x_3 & 0 \\
  x_0 & x_2 & 0 \\
  0 & b & -x_3 \\
  0 & -a & x_2
\end{pmatrix}.
\]

(b) If \( x_1 \nmid b \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup L_2 \cup L'_1}) \) admits the following free resolution:

\[
0 \rightarrow \oplus 2S(-3) \xrightarrow{\delta_2} 3S(-2) \xrightarrow{\delta_1} S \oplus \delta_0 H^0_*(\mathcal{O}_{X \cup L_2 \cup L'_1}) \rightarrow 0
\]

with \( \delta_0 = (1, e_1) \) and \( \delta_1, \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
  x_0x_3 & x_1x_2 & x_1x_3 & 0 & 0 \\
  -x_0b & -x_1a & -x_1b & x_2 & x_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
  -x_1 & 0 & 0 \\
  0 & -x_3 & 0 \\
  x_0 & x_2 & 0 \\
  0 & x_1b & -x_3 \\
  0 & -x_1a & x_2
\end{pmatrix}.
\]
Proof. The homogeneous ideal of \( L_1 \cup L_2 \cup L_1' \) is:
\[
(x_2, x_3) \cap (x_1, x_3) \cap (x_0, x_1) = (x_0x_3, x_1x_2, x_1x_3).
\]
One deduces that \( L_1 \cup L_2 \cup L_1' \) is directly linked by the complete intersection defined by \( x_0x_3 \) and \( x_1x_2 \) to the line \( L_2' \) of equations \( x_0 = x_2 = 0 \). Using Ferrand’s result about liaison, one gets the following graded free resolution of \( I(L_1 \cup L_2 \cup L_1') \):
\[
\begin{pmatrix}
-x_1 & 0 \\
0 & -x_3 \\
x_0 & x_2
\end{pmatrix}
\]
\[
0 \rightarrow 2S(-3) \rightarrow 3S(-2) \rightarrow I(L_1 \cup L_2 \cup L_1') \rightarrow 0.
\]
It results that \( L_1 \cup L_2 \cup L_1' \) is arithmetically CM, hence the graded \( S \)-module \( H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_1'}) \) is generated by \( 1 \in H^0(\mathcal{O}_{L_1 \cup L_2 \cup L_1'}) \).

(a) We use the exact sequence (look at the beginning of this subsection):
\[
0 \rightarrow \text{Im} \phi' \rightarrow \mathcal{O}_{X \cup L_2 \cup L_1'} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_1'} \rightarrow 0
\]
with \( \phi' \) the composite morphism \( \mathcal{I}_{L_1 \cup L_2 \cup L_1'} \rightarrow \mathcal{I}_{L_1 \cup L_2} \rightarrow \mathcal{O}_{L_1}(l - 1) \). Since \( L_1 \cap L_1' = \emptyset \), \( \phi' \) is an epimorphism. One deduces an exact sequence:
\[
0 \rightarrow S(L_1)(l - 1) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2 \cup L_1'}) \rightarrow H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_1'}) \rightarrow 0
\]
where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0(\mathcal{O}_{X \cup L_2 \cup L_1'}(-l + 1)) \) whose image into \( H^0(\mathcal{O}_X(-l + 1)) \oplus H^0(\mathcal{O}_{L_2}(-l + 1)) \oplus H^0(\mathcal{O}_{L_1'}(-l + 1)) \) is \( (x_1e_1, 0, 0) \).

(b) We use the exact sequence:
\[
0 \rightarrow \text{Im} \phi \rightarrow \mathcal{O}_{X \cup L_2 \cup L_1'} \rightarrow \mathcal{O}_{L_1 \cup L_2 \cup L_1'} \rightarrow 0
\]
with \( \phi \) the composite morphism \( \mathcal{I}_{L_1 \cup L_2 \cup L_1'} \rightarrow \mathcal{I}_{L_1 \cup L_2} \rightarrow \mathcal{O}_{L_1}(l) \). Since \( L_1 \cap L_1' = \emptyset \), \( \phi \) is an epimorphism. One deduces an exact sequence:
\[
0 \rightarrow S(L_1)(l) \rightarrow H^0_*(\mathcal{O}_{X \cup L_2 \cup L_1'}) \rightarrow H^0_*(\mathcal{O}_{L_1 \cup L_2 \cup L_1'}) \rightarrow 0
\]
where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0(\mathcal{O}_{X \cup L_2 \cup L_1'}(-l)) \) whose image into \( H^0(\mathcal{O}_X(-l)) \oplus H^0(\mathcal{O}_{L_2}(-l)) \oplus H^0(\mathcal{O}_{L_1'}(-l)) \) is \( (e_1, 0, 0) \).
\[\square\]

Lemma B.30. Let \( L \subset \mathbb{P}^3 \) be the line of equations \( \ell = x_3 = 0 \), where \( \ell = x_1 + cx_2, c \neq 0 \). Then:
\[
I(L_1^{(1)} \cup L \cup L_1') = (x_0, x_1)(x_2x_3, x_3^2, \ell x_2^2).
\]

Proof. According to Lemma B.10, \( L_1^{(1)} \cup L \) is arithmetically CM. It follows, now, from Lemma B.11 that \( I(L_1^{(1)} \cup L \cup L_1') = I(L_1^{(1)} \cup L)I(L_1') \).
\[\square\]

Proposition B.31. Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection B.1 and let \( L \subset \mathbb{P}^3 \) be the line of equations \( \ell = x_3 = 0 \), where \( \ell = x_1 + cx_2, c \neq 0 \).

(a) If \( x_1 \mid b \) and \( l = -1 \) then \( I(X \cup L \cup L_1') = (x_0, x_1)(x_3, \ell x_2^2) \).

(b) If \( x_1 \mid b \) and \( l = 0 \) then \( I(X \cup L \cup L_1') = (x_0, x_1)(-\ell x_2 + ax_3, x_2x_3, x_3^2) \).
(c) If \( x_1 \mid b \), i.e., if \( b = x_1 b_1 \), and \( l \geq 1 \) then \( I(X \cup L \cup L'_1) = S(F_2 - cb_1 x^2_2) + I(L_1^{(1)} \cup L \cup L'_1) \).

(d) If \( x_1 \nmid b \) and \( l = -1 \) then \( I(X \cup L \cup L'_1) = (x_0, x_1)(\ell F_2, x_2 x_3, x^2_3) \).

(e) If \( x_1 \nmid b \) and \( l \geq 0 \) then \( I(X \cup L \cup L'_1) = \ell F_2 + I(L_1^{(1)} \cup L \cup L'_1) \).

Proof. (a) If \( l = -1 \) the condition \( x_1 \mid b \) means that \( b = 0 \), hence \( F_2 = x_3 \), hence \( I(X) = (x_3, x^2_2) \). One deduces that \( I(X \cup L) = (x_3, \ell x^2_2) \) and one can apply, now, Lemma B.11

(b) If \( l = 0 \) and \( x_1 \mid b \) then one can assume that \( b = x_1 \), hence \( F_2 = -x_1 x_2 + ax_3 \). It follows, from Prop. B.11 that:

\[
I(X \cup L) = (-x_1 x_2 + ax_3 - cx^2_2, x_2 x_3, x^2_3, \ell x^2_2) = (-\ell x_2 + ax_3, x_2 x_3, x^2_3).
\]

One deduces that \( X \cup L \) is directly linked by the complete intersection defined by \(-\ell x_2 + ax_3 \) and \( x^2_3 \) to the line \( L \), hence it is arithmetically CM. One can apply, now, Lemma B.11

(c) If \( x_1 \mid b \) then one deduces, from the proof of Prop. B.11 the existence of an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(1)} \cup L} \longrightarrow \mathcal{I}_{X \cup L} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0
\]

from which one derives an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(1)} \cup L \cup L'_1} \longrightarrow \mathcal{I}_{X \cup L \cup L'_1} \xrightarrow{\psi_1} \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0.
\]

Since \( l \geq 1 \), the element \( F_2 - cb_1 x^2_2 \) of \( I(X \cup L) \) belongs to \( I(X \cup L \cup L'_1) \) and this implies that the sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L \cup L'_1) \longrightarrow I(X \cup L \cup L'_1) \xrightarrow{H^0(\psi)} S(L_1)(-l - 2) \longrightarrow 0
\]

is exact.

(d) If \( l = -1 \) the condition \( x_1 \nmid b \) is equivalent to \( b \neq 0 \). One can assume that \( b = -1 \), hence \( F_2 = x_2 + ax_3 \), hence \( I(X) = (x_2 + ax_3, x^2_3) \). It follows that \( I(X \cup L) = ((x_1 + cx_2)(x_2 + ax_3), x_2 x_3, x^2_3) \). One deduces that \( X \cup L \) is directly linked by the complete intersection defined by \((x_1 + cx_2)(x_2 + ax_3) \) and \( x^2_3 \) to \( L \), hence it is arithmetically CM. One can apply, now, Lemma B.11

(e) If \( x_1 \nmid b \) one deduces, as in the proof of (c), an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(1)} \cup L \cup L'_1} \longrightarrow \mathcal{I}_{X \cup L \cup L'_1} \xrightarrow{\psi'_1} \mathcal{O}_{L_1}(-l - 3) \longrightarrow 0.
\]

Since \( l \geq 0 \), the generator \( \ell F_2 \) of \( I(X \cup L) \) (see Prop. B.11) belongs to \( I(X \cup L \cup L'_1) \) and this implies that the sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L \cup L'_1) \longrightarrow I(X \cup L \cup L'_1) \xrightarrow{H^0(\psi'_1)} S(L_1)(-l - 3) \longrightarrow 0
\]

is exact.

Notice that using the exact sequences from the above proof one can get a concrete graded free resolution of \( I(X \cup L \cup L'_1) \). \( \square \)
Lemma B.32. Let $L''_1 \subset \mathbb{P}^3$ be the line of equations $\ell_0 = \ell_1 = 0$, where $\ell_0 = x_0 - x_2$ and $\ell_1 = x_1 - x_3$. $L_1, L'_1$ and $L''_1$ are mutually disjoint and are contained in the quadric $Q \subset \mathbb{P}^3$ of equation $x_0x_3 - x_1x_2 = 0$. Then the homogeneous ideal $I(Y)$ of $Y = L_1 \cup L'_1 \cup L''_1$ is generated by $x_0x_3 - x_1x_2$, $x_0\ell_0x_2$, $x_0\ell_0x_3$, $x_1\ell_1x_2$ and $x_1\ell_1x_3$ and admits the following graded free resolution:

$$0 \to 2S(-5) \xrightarrow{d_2} 6S(-4) \xrightarrow{d_1} S(-2) \oplus 4S(-3) \to I(Y) \to 0$$

where $d_1$ and $d_2$ are defined by the matrices:

$$
\begin{pmatrix}
-x_0\ell_0 & x_0x_1 & -x_1\ell_1 & 0 & \ell_0x_3 + x_1x_2 & 0 \\
-x_1 & 0 & 0 & -x_3 & 0 & 0 \\
x_0 & -x_1 & 0 & x_2 & -x_3 & 0 \\
0 & x_0 & -x_1 & 0 & x_2 & -x_3 \\
0 & 0 & x_0 & 0 & 0 & x_2
\end{pmatrix},
\begin{pmatrix}
-x_3 & 0 \\
x_2 & -x_3 \\
0 & x_2 \\
x_1 & 0 \\
x_0 & x_1 \\
0 & -x_0
\end{pmatrix}.
$$

Proof. Tensorizing by $\mathcal{I}_{L_1 \cup L'_1}$ the exact sequence:

$$0 \to \mathcal{O}_p(-2) \xrightarrow{(-\ell_1, \ell_0)} 2\mathcal{O}_p(-1) \xrightarrow{(\ell_0, \ell_1)} \mathcal{O}_p \xrightarrow{\mathcal{I}_{L''_1}} 0$$

one gets an exact sequence:

$$0 \to \mathcal{I}_{L_1 \cup L'_1}(-2) \to 2\mathcal{I}_{L_1 \cup L'_1}(-1) \to \mathcal{I}_Y \to 0.$$  

Since $H^1(\mathcal{I}_{L_1 \cup L'_1}(i)) = 0$ for $i \geq 1$ it follows that $I(Y)$ coincides with $I(L_1 \cup L'_1)I(L''_1)$ in degrees $\geq 3$. On the other hand, it is well known that $I(Y)_2 = k(x_0x_3 - x_1x_2)$. Consequenly:

$$I(Y) = S(x_0x_3 - x_1x_2) + (x_0x_2, x_0x_3, x_1x_2, x_1x_3)\ell_0, \ell_1).$$

Noticing that $q := x_0x_3 - x_1x_2 = \ell_0x_3 - \ell_1x_2$ one gets:

$$\ell_0x_1x_2 = \ell_0\ell_0x_3 - \ell_0q, \ell_0x_1x_3 = x_1\ell_1x_2 + x_1q,$$
$$x_0\ell_1x_2 = x_0\ell_0x_3 - x_0q, x_0\ell_1x_3 = x_1\ell_1x_2 + \ell_1q.$$

It follows that $I(Y)$ is (minimally) generated by the elements from the statement.

Let, now, $X'$ be the divisor $2L'_1$ on $Q$. Using the Segre isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q$ one gets exact sequences:

$$0 \to \mathcal{O}_p(-2) \xrightarrow{q} \mathcal{I}_Y \to \mathcal{O}_Q(-3,0) \to 0$$
$$0 \to \mathcal{O}_p(-2) \xrightarrow{q} \mathcal{I}_{L_1 \cup X'} \to \mathcal{O}_Q(-3,0) \to 0.$$  

One deduces that $I(Y)$ and $I(L_1 \cup X')$ admit graded free resolutions of the same numerical shape. Using Prop. B.9 the last exact sequence from its proof, and Remark A.8 it follows
that \( I(L_1 \cup X') \) is generated by \( x_0x_3 - x_1x_2, x_0^2x_2, x_0^2x_3, x_1^2x_2, x_1^2x_3 \) and admits the following graded free resolution:

\[
0 \rightarrow 2S(-5) \xrightarrow{d_1'} 6S(-4) \xrightarrow{d_1''} S(-2) \oplus 4S(-3) \rightarrow I(L_1 \cup X') \rightarrow 0
\]

where \( d_1' \) and \( d_1'' \) are defined by the matrices:

\[
\begin{pmatrix}
-x_0^2 & x_0x_1 & -x_1^2 & 0 & x_0x_3 + x_1x_2 & 0 \\
-x_1 & 0 & 0 & -x_3 & 0 & 0 \\
x_0 & -x_1 & 0 & x_2 & -x_3 & 0 \\
0 & x_0 & -x_1 & 0 & x_2 & -x_3 \\
0 & 0 & x_0 & 0 & 0 & x_2
\end{pmatrix},
\begin{pmatrix}
x_2 & -x_3 \\
x_0 & x_2 \\
x_1 & 0 \\
-x_0 & x_1 \\
0 & -x_0 & x_1 \\
-x_3 & 0 & 0 \\
x_2 & -x_3 & 0 \\
0 & x_2 & -x_3
\end{pmatrix}.
\]

One can easily guess, now, a similar graded free resolution of \( I(Y) \), which is the one from the statement. \( \square \)

**Lemma B.33.** Using the notation from the statement of Lemma B.32, the homogeneous ideal \( I(W) \subset S \) of \( W = L_1^{(1)} \cup L_1' \cup L_1'' \) is generated by:

\[
x_2(x_0x_3 - x_1x_2), x_3(x_0x_3 - x_1x_2), x_0\ell_0x_2^2, x_0\ell_0x_2x_3, x_0\ell_0x_3^2, x_1\ell_1x_2x_3, x_1\ell_1x_3^2
\]

and admits the following graded free resolution:

\[
0 \rightarrow 3S(-6) \xrightarrow{d_2} S(-4) \oplus 8S(-5) \xrightarrow{d_1} 2S(-3) \oplus 5S(-4) \rightarrow I(W) \rightarrow 0
\]

where \( d_1 \) and \( d_2 \) are defined by the matrices:

\[
\begin{pmatrix}
x_2 & 0 & 0 & 0 & 0 & 0 \\
x_0 & 0 & 0 & \ell_0x_3 + x_1x_2 & 0 & 0 \\
0 & -x_3 & 0 & 0 & -x_1 & 0 \\
0 & x_2 & -x_3 & 0 & 0 & x_0 \\
0 & 0 & x_2 & -x_3 & 0 & 0 \\
0 & 0 & 0 & x_2 & -x_3
\end{pmatrix},
\begin{pmatrix}
x_0\ell_0 & 0 & 0 \\
x_1 & 0 & 0 \\
-x_0 & x_1 \\
0 & -x_0 & x_1 \\
0 & 0 & -x_0 \\
x_2 & -x_3 & 0 \\
0 & x_2 & -x_3 \\
0 & 0 & x_2
\end{pmatrix}.
\]

**Proof.** Tensorizing by \( \mathcal{H}_{L_1' \cup L_1''} \) the exact sequence:

\[
0 \rightarrow \mathcal{O}_Y(-2) \xrightarrow{(x_2, x_3)} 2\mathcal{H}_{L_1}(-1) \xrightarrow{(x_2, x_3)} \mathcal{O}_Y \xrightarrow{\mathcal{H}_{L_1^{(1)}}} 0
\]

one gets an exact sequence:

\[
0 \rightarrow \mathcal{H}_{L_1' \cup L_1''}(-2) \xrightarrow{(x_2, x_3)} 2\mathcal{H}_{L_1 \cup L_1' \cup L_1''}(-1) \xrightarrow{(x_2, x_3)} \mathcal{H}_W \xrightarrow{0}.
\]
admits the following minimal graded free resolution:

Lemma B.34. The graded $S$-submodule $S(L_1)_{\geq 2} := \bigoplus_{i \geq 2} S(L_1)_i$ of $S(L_1) := H^0(\mathcal{O}_{L_1})$ admits the following minimal graded free resolution:

$$0 \to 2S(-5) \xrightarrow{d_3} 7S(-4) \xrightarrow{d_2} 8S(-3) \xrightarrow{d_1} 3S(-2) \xrightarrow{d_0} S(L_1)_{\geq 2} \to 0$$
with \( d_0, d_1, d_2, d_3 \) defined by the matrices:

\[
(x_0^2, x_0x_1, x_1^2), \begin{pmatrix}
-x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 \\
x_0 & -x_1 & 0 & 0 & x_2 & x_3 & 0 \\
0 & x_0 & 0 & 0 & 0 & x_2 & x_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
x_2 & x_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_2 & x_3 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & -x_3 & 0 & 0 \\
0 & x_1 & 0 & 0 & x_2 & 0 & 0 \\
-x_0 & 0 & x_1 & 0 & 0 & -x_3 & 0 \\
0 & -x_0 & 0 & x_1 & 0 & x_2 & 0 \\
0 & 0 & -x_0 & 0 & 0 & 0 & -x_3 \\
0 & 0 & 0 & -x_0 & 0 & 0 & x_2
\end{pmatrix}, \begin{pmatrix}
-x_3 & 0 \\
x_2 & 0 \\
0 & -x_3 \\
0 & x_2 \\
-x_1 & 0 \\
x_0 & -x_1 \\
0 & x_0
\end{pmatrix}.
\]

**Proof.** Using the exact sequence:

\[
0 \to 2S(L_1)(-3) \xrightarrow{(x_0^2, x_0x_1, x_1^2)} 3S(L_1)(-2) \to S(L_1)_{\geq 2} \to 0.
\]

one sees that the tensor product of the complexes:

\[
2S(-3) \xrightarrow{(x_0^2, x_0x_1, x_1^2)} 3S(-2), S(-2) \xrightarrow{(x_2)} 2S(-1) \xrightarrow{(x_2, x_3)} S
\]

is a minimal graded free resolution of \( S(L_1)_{\geq 2} \) over \( S \).

**Proposition B.35.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection B.1. Let \( L_1' \subset \mathbb{P}^3 \) be the line of equations \( \ell_0 = \ell_1 = 0 \), where \( \ell_0 = x_0 - x_2 \) and \( \ell_1 = x_1 - x_3 \). \( L_1, L_1' \) and \( L_1'' \) are mutually disjoint and are contained in the quadric \( Q \subset \mathbb{P}^3 \) of equation \( x_0x_3 - x_1x_2 = 0 \).

(a) If \( l = -1 \) then:

\[
I(X \cup L_1' \cup L_1'') = Sx_0\ell_0F_2 + Sx_1\ell_0F_2 + Sx_1\ell_1F_2 + I(L_1^{(1)} \cup L_1' \cup L_1'').
\]

(b) If \( l = 0 \) and \( \{F_2 = 0\} \neq Q \) then:

\[
I(X \cup L_1' \cup L_1'') = S\ell_0F_2 + S\ell_1F_2 + I(L_1^{(1)} \cup L_1' \cup L_1'').
\]

(c) If \( l = 0 \) and \( \{F_2 = 0\} = Q \) or if \( l \geq 1 \) then, writing \( F_2 = x_0F_2' + x_1F_2'' \), one has:

\[
I(X \cup L_1' \cup L_1'') = S(\ell_0F_2' + \ell_1F_2'') + I(L_1^{(1)} \cup L_1' \cup L_1'').
\]

**Proof.** Tensorizing by \( \mathcal{I}_{L_1', L_1''} \) the exact sequence:

\[
0 \to \mathcal{I}_{L_1'} \to \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2) \to 0
\]
where \( \eta \) maps \( F_2 \) to \( 1 \in H^0(\mathcal{O}_{L_1}) \) and \( x_2^2, x_2x_3, x_3^2 \) to 0, one gets an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1}^{(1)} \otimes L_2^{(1)} \longrightarrow \mathcal{I}_{X \cup L_1^1 \cup L_1^2} \psi \longrightarrow \mathcal{O}_{L_1}(-l-2) \longrightarrow 0.
\]

Since, by Lemma \[B.33\] \( H^1(\mathcal{I}_{L_1}^{(1)} \otimes L_2^{(1)}(i)) = 0 \) for \( i \geq 3 \) it follows that \( H^0(\psi(i)) \) is surjective for \( i \geq 3 \). On the other hand:

\[
H^0(\mathcal{I}_{X \cup L_1^1 \cup L_1^2}(2)) \subseteq H^0(\mathcal{I}_{L_1} \cup L_2^{(1)}(2)) = k(x_0x_3 - x_1x_2).
\]

(a) In this case, by Lemma \[B.38\] \( X \) is the divisor \( 2L_1 \) in the plane \( H \) of equation \( F_2 = 0 \). Since \( H \cap Q \) is the union of \( L_1 \) and of a different line intersecting it it follows that \( X \) is not contained in \( Q \) hence \( H^0(\mathcal{I}_{X \cup L_1^1 \cup L_1^2}(2)) = 0 \). One deduces an exact sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L_1' \cup L_1'') \longrightarrow I(X \cup L_1' \cup L_1'') \xrightarrow{H^0(\psi)} S(L_1 \geq 2(1)) \longrightarrow 0.
\]

Since \( x_0\ell_2F_2, x_1\ell_2F_2, x_1\ell_1F_2 \) belong to \( I(X \cup L_1' \cup L_1'') \) and are linearly independent one gets the assertion from the statement.

Notice that by using the last exact sequence and Lemma \[B.34\] one can get a non-minimal graded free resolution of \( I(X \cup L_1' \cup L_1'') \). This resolution has, actually, excessive length 3 but using Remark \[A.8\] one can get from it a minimal free resolution (of length 2).

(b) Since, by Prop. \[B.9\] \( H^0(\mathcal{I}_{X \cup L_1^1}(2)) = kF_2 \), our hypothesis implies that one has \( H^0(\mathcal{I}_{X \cup L_1^1 \cup L_1^2}(2)) = 0 \). One deduces an exact sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L_1' \cup L_1'') \longrightarrow I(X \cup L_1' \cup L_1'') \xrightarrow{H^0(\psi)} S(L_1 \geq 2(-2)) \longrightarrow 0.
\]

It remains to notice that \( \ell_2F_2 \) and \( \ell_1F_2 \) belong to \( H^0(\mathcal{I}_{X \cup L_1' \cup L_1''}(3)) \) and that \( \psi(\ell_1F_2) = \eta(\ell_iF_2) = x_i \in H^0(\mathcal{O}_{L_1}(1)), i = 0, 1 \).

Notice, also, that using the last exact sequence and the resolution of \( S(L_1) \) from the discussion following Prop. \[A.12\] one can get a non-minimal graded free resolution of \( I(X \cup L_1' \cup L_1'') \) of length 3. Using Remark \[A.8\] one can get from it a minimal free resolution (of length 2).

(c) In this case one gets an exact sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L_1' \cup L_1'') \longrightarrow I(X \cup L_1' \cup L_1'') \xrightarrow{H^0(\psi)} S(L_1(-l-2)) \longrightarrow 0
\]

and it suffices to notice that \( \ell_0F_2 + \ell_1F_2'' \) belongs to \( H^0(\mathcal{I}_{X \cup L_1^1 \cup L_1^2}(l+2)) \) and that \( \psi(\ell_0F_2 + \ell_1F_2'') = 1 \in H^0(\mathcal{O}_{L_1}) \). Notice, also, that using the last exact sequence one can get a graded free resolution of \( I(X \cup L_1 \cup L_1'') \).

\[ \square \]

B.4. **A double line union a conic.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection \[B.1\] and let \( C \) be a (nonsingular) conic in \( \mathbb{P}^3 \).

One has to consider one of the following four possibilities:

(i) \( L_1 \cap C = \emptyset \);
(ii) \( L_1 \cap C \neq \emptyset \) and \( L_1 \) is not contained in the plane of \( C \);
(iii) \( L_1 \) tangent to \( C \);
(iv) \( L_1 \) secant to \( C \).
(iv) \( L_1 \cap C \) consists of two points.

We recall the following elementary:

**Lemma B.36.** Let \( Q_0, Q_1, Q_2 \) be the coordinate points \((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\) of \( \mathbb{P}^2 \). Consider a (nonsingular) conic \( C \subseteq \mathbb{P}^2 \), containing \( Q_0 \) and \( Q_1 \) and such that \( T_{Q_0} C = Q_0 Q_2 = \{ x_1 = 0 \} \) and \( T_{Q_1} C = Q_1 Q_2 = \{ x_0 = 0 \} \). Then the equation of \( C \) is of the form \( c_0 x_0 x_1 + c_1 x_2^2 = 0 \) with \( c_0, c_1 \in k \setminus \{ 0 \} \).

In case (i) one can assume, up to a linear change of coordinates in \( \mathbb{P}^3 \), that \( C \) is contained in the plane \( \{ x_1 = 0 \} \) spanned by \( P_0, P_2, P_3 \), that \( P_2, P_3 \in C \) and that \( T_{P_2} C = T_{P_3} C = L_2, T_{P_2} C = T_{P_3} C = L_3 \). Lemma B.36 implies that \( C \) is defined by equations of the form \( x_1 = x_0^2 + c x_2 x_3 = 0 \) with \( c \in k \setminus \{ 0 \} \).

In case (ii) one can assume, up to a linear change of coordinates in \( \mathbb{P}^3 \), that \( C \) is contained in the plane \( \{ x_2 = 0 \} \) spanned by \( P_0, P_1, P_2 \), that \( P_0, P_2 \in C \) and that \( T_{P_0} C = T_{P_2} C = L_1, T_{P_0} C = T_{P_2} C = L_3 \). Lemma B.36 implies that \( C \) is defined by equations of the form \( x_2 = x_0^2 + c x_2 x_3 = 0 \) with \( c \in k \setminus \{ 0 \} \).

In case (iii) one can assume, up to a linear change of coordinates in \( \mathbb{P}^3 \), that \( C \) is contained in the plane \( \{ x_3 = 0 \} \) spanned by \( P_0, P_1, P_2 \), that \( P_0, P_1 \in C \) and that \( T_{P_0} C = T_{P_1} C = L_2, T_{P_0} C = T_{P_1} C = L_3 \). Lemma B.36 implies that \( C \) is defined by equations of the form \( x_3 = x_0^2 + c x_2 x_3 = 0 \) with \( c \in k \setminus \{ 0 \} \).

In case (iv) one can assume, up to a linear change of coordinates in \( \mathbb{P}^3 \), that \( C \) is contained in the plane \( \{ x_3 = 0 \} \) spanned by \( P_0, P_1, P_2 \), that \( P_0, P_1 \in C \) and that \( T_{P_0} C = T_{P_1} C = L_2, T_{P_0} C = T_{P_1} C = L_3 \). Lemma B.36 implies that \( C \) is defined by equations of the form \( x_3 = x_0^2 + c x_2 x_3 = 0 \) with \( c \in k \setminus \{ 0 \} \).

The following lemma follows immediately from Lemma B.1.

**Lemma B.37.** Let \( C \subseteq \mathbb{P}^3 \) be the conic of equations \( x_1 = x_0^2 + c x_2 x_3 = 0, c \in k \setminus \{ 0 \} \). Put \( q_1 := x_2^2 + c x_2 x_3 \). Then \( I(L_1^{(1)} \cup C) \) is generated by \( x_1 x_2, x_1 x_2 x_3, x_1 x_3, x_2^2 q_1, x_2 x_3 q_1, x_2^2 q_1 \) and admits the following minimal graded free resolution:

\[
0 \rightarrow 2S(-6) \xrightarrow{d_2} 2S(-4) \oplus 3S(-3) \xrightarrow{d_1} 5S(-5) \oplus 3S(-4) \xrightarrow{d_0} I(L_1^{(1)} \cup C) \rightarrow 0
\]

with \( d_1 \) and \( d_2 \) defined by the matrices:

\[
\begin{pmatrix}
-x_3 & 0 & 0 & 0 & -q_1 & 0 & 0 \\
0 & -x_3 & 0 & 0 & 0 & -q_1 & 0 \\
0 & 0 & x_2 & -x_3 & 0 & x_1 & 0 \\
0 & 0 & x_2 & -x_3 & 0 & x_1 & 0 \\
0 & 0 & x_2 & -x_3 & 0 & x_1 & 0 \\
0 & 0 & x_2 & -x_3 & 0 & x_1 & 0
\end{pmatrix},
\begin{pmatrix}
q_1 & 0 & 0 & -x_1 & 0 & -x_3 & 0 \\
0 & q_1 & 0 & 0 & -x_1 & 0 & -x_3 \\
-x_1 & 0 & 0 & -x_1 & 0 & -x_3 & 0 \\
x_2 & -x_3 & 0 & x_2 & 0 & 0 & 0
\end{pmatrix}
\]
Lemma B.38. Let $X$ be the double structure on the line $L_1$ considered at the beginning of Subsection B.1 and let $C$ be the conic from Lemma B.37.

(a) If $l = -1$ then $I(X \cup C) = I(X)I(C)$.
(b) If $l = 0$, writing $F_2 = x_0F'_2 + x_1F''_2$, one has $I(X \cup C) = S(q_1F'_2 + x_0x_1F''_2) + Sx_1F_2 + I(L_1^{(1)} \cup C)$.
(c) If $l \geq 1$, writing $F_2 = x_0^2F'_2 + x_1F''_2$, one has $I(X \cup C) = S(q_1F'_2 + x_1F''_2) + I(L_1^{(1)} \cup C)$.

Proof. Tensorizing by $\mathcal{S}$ the exact sequence:

$$0 \to \mathcal{I}_{L_1^{(1)}} \to \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(−l−2) \to 0$$

one gets an exact sequence:

$$0 \to \mathcal{I}_{L_1^{(1)} \cup C} \to \mathcal{I}_{X \cup C} \xrightarrow{\psi} \mathcal{O}_{L_1}(−l−2) \to 0.$$ 

It follows from Lemma B.37 that $H^1(\mathcal{I}_{L_1^{(1)} \cup C}(i)) = 0$ for $i \geq 3$ hence $H^0(\psi(i))$ is surjective for $i \geq 3$.

(a) In this case, by Lemma B.38, $X$ is the divisor $2L_1$ on the plane $H \subset \mathbb{P}^3$ of equation $F_2 = 0$ and the result follows from Lemma B.1. By the same lemma, one can get a graded free resolution of $I(X \cup C)$.

(b) In this case $H^0(\mathcal{I}_{X \cup C}(2)) = 0$. Indeed, assume that there exists a non-zero $f \in H^0(\mathcal{I}_{X \cup C}(2))$. Since $H^0(\mathcal{I}_X(1)) = 0$, $f$ cannot vanish identically on the plane $\{x_1 = 0\}$. On the other hand, $f$ vanishes in $P_0 \in \{x_1 = 0\} \setminus C$ hence it cannot vanish on $C$ which is a contradiction.

One deduces, now, an exact sequence:

$$0 \to I(L_1^{(1)} \cup C) \to I(X \cup C) \xrightarrow{H^0(\psi)} S(L_1)^+(-2) \to 0.$$ 

It remains to notice that $q_1F'_2 + x_0x_1F''_2$ and $x_1F_2$ belong to $H^0(\mathcal{I}_{X \cup C}(3))$ and that they are mapped by $\psi$ to $x_0 \in H^0(\mathcal{O}_{L_1}(1))$ and $x_1 \in H^0(\mathcal{O}_{L_1}(1))$, respectively. Notice, also, that using the above exact sequence and the resolution of $S(L_1)^+$ that can be found in the discussion following Prop. A.12 one can get a graded free resolution of $I(X \cup C)$.

(c) In this case one has an exact sequence:

$$0 \to I(L_1^{(1)} \cup C) \to I(X \cup C) \xrightarrow{H^0(\psi)} S(L_1)(-l-2) \to 0.$$ 

It remains to notice that $q_1F'_2 + x_1F''_2$ belongs to $H^0(\mathcal{I}_{X \cup C}(l + 2))$ and that it is mapped by $\psi$ to $1 \in H^0(\mathcal{O}_{L_1})$. Notice, also, that using the above exact sequence one can get a graded free resolution of $I(X \cup C)$.

Lemma B.39. Let $C \subset \mathbb{P}^3$ be the conic of equations $x_1 = x_0x_3 + cx_2^2 = 0$, $c \in k \setminus \{0\}$. Put $q_2 := x_0x_3 + cx_2^2$. Then $I(L_1^{(1)} \cup C)$ is generated by $x_2q_2, x_3q_2, x_1x_2^2, x_1x_2x_3, x_1x_3^2$ and admits the following minimal graded free resolution:

$$0 \to S(-5) \xrightarrow{d_2} 5S(-4) \xrightarrow{d_1} 5S(-3) \xrightarrow{d_0} I(L_1^{(1)} \cup C) \to 0.$$
where \( d_1 \) and \( d_2 \) are defined by the matrices:

\[
\begin{pmatrix}
-x_3 & 0 & 0 & -x_1 & 0 \\
x_2 & 0 & 0 & 0 & -x_1 \\
0 & -x_3 & 0 & cx_2 & 0 \\
0 & x_2 & -x_3 & x_0 & cx_2 \\
0 & 0 & x_2 & 0 & x_0
\end{pmatrix}
\begin{pmatrix}
-x_1 \\
cx_2 \\
x_0 \\
x_3 \\
-x_2
\end{pmatrix}.
\]

**Proof.** We notice, firstly, that \( q_2 \in I(L_1 \cup C) \) and \( x_1 \notin I(L_1) \) hence:

\[ I(L_1 \cup C) = S q_2 + x_1 I(L_1) = (q_2, x_1 x_2, x_1 x_3). \]

One has an exact sequence:

\[ 0 \rightarrow \mathcal{I}_{L_1 \cup C} \rightarrow \mathcal{I}_{L_1 \cup C} \rightarrow \mathcal{I}_{L_1} / \mathcal{I}_{L_1}^2 \rightarrow 0 \]

with \( \rho(q_2) = (0, x_0) \in H^0(2\mathcal{O}_{L_1}(1)) \), \( \rho(x_1 x_2) = (x_1, 0) \) and \( \rho(x_1 x_3) = (0, x_1) \). Any homogeneous element \( f \) of \( I(L_1 \cup C) \) can be written as:

\[ f = f_1 q_2 + f_2 x_1 x_2 + f_3 x_1 x_3. \]

If \( f \) belongs to \( I(L_1^{(1)} \cup C) \) then:

\[ 0 = \rho(f) = (x_1(f_2 \mid L_1), x_0(f_1 \mid L_1) + x_1(f_3 \mid L_1)). \]

One deduces that \( f_2 \in (x_2, x_3) \) and that there exist polynomials \( g \in k[x_0, x_1] \) and \( h_1, h_3 \in (x_2, x_3) \) such that:

\[ f_1 = -x_1 g + h_1, \quad f_3 = x_0 g + h_3. \]

It follows that \( f = g(-x_1 q_2 + x_0 x_1 x_3) + h_1 q_2 + f_2 x_1 x_2 + h_3 x_1 x_3 \). Since \(-x_1 q_2 + x_0 x_1 x_3 = -cx_1 x_2^2\) one deduces that \( I(L_1^{(1)} \cup C) \) is generated by the elements from the statement.

Now, let \( W_1 \) be the curve directly linked to \( L_1^{(1)} \cup C \) by the complete intersection defined by \( x_3 q_2 \) and \( x_1 x_2^2 \). We assert that:

\[ I(W_1) = (x_0 x_3, x_0 x_1 x_2, x_1 x_2^2, x_2^2 x_3). \]

Indeed, let \( J \) denote the ideal from the righthand side of the equality we want to prove. It is easy to check that:

\[ J \cdot I(L_1^{(1)} \cup C) \subseteq (x_3 q_2, x_1 x_2^2) \]

hence \( J \subseteq I(W_1) \). On the other hand, as we noticed in the last part of the proof of Lemma \[B.27\], \( J = I(W') \) where \( W' \) is the curve directly linked to \( L_1^{(1)} \cup L_2 \cup L_1' \) by the complete intersection defined by \( x_0 x_3^2 \) and \( x_1 x_2^2 \). Since \( \deg W_1 = 4 = \deg W' \) and since \( W_1 \subseteq W' \) it follows that \( W_1 = W' \) hence \( I(W_1) = J \).

As we saw in the last part of the proof of Lemma \[B.27\], \( W' \) (hence \( W_1 \)) is directly linked to \( L_1 \cup L_1' \) by the complete intersection defined by \( x_0 x_3 \) and \( x_1 x_2^2 \). Since \( \mathcal{I}_{L_1 \cup L_1'} \) admits a resolution of the form:

\[ 0 \rightarrow \mathcal{O}_P(-4) \rightarrow 4\mathcal{O}_P(-3) \rightarrow 4\mathcal{O}_P(-2) \rightarrow \mathcal{I}_{L_1 \cup L_1'} \rightarrow 0 \]
one deduces, from Ferrand’s result about liaison, that \( \mathcal{I}_W \) has a monad of the form:

\[
0 \longrightarrow 3\mathcal{O}_\mathbb{P}(-3) \longrightarrow 5\mathcal{O}_\mathbb{P}(-2) \longrightarrow \mathcal{O}_\mathbb{P}(-1) \longrightarrow 0
\]

and, then, that \( \mathcal{I}_W \) admits a resolution of the form:

\[
0 \longrightarrow \mathcal{O}_\mathbb{P}(-5) \longrightarrow 5\mathcal{O}_\mathbb{P}(-4) \longrightarrow 5\mathcal{O}_\mathbb{P}(-3) \longrightarrow \mathcal{I}_W \longrightarrow 0.
\]

One can easily determine the differentials of this resolution. \( \square \)

**Proposition B.40.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection [B.3] and let \( C \subset \mathbb{P}^3 \) be the conic from Lemma [B.39].

(a) If \( x_1 | b \) and \( l = -1 \) then \( I(X \cup C) = (x_0x_3 + cx_2^2, x_1x_3) \).

(b) If \( x_1 | b \) and \( l \geq 0 \) then \( I(X \cup C) = S \left( F_2 + c\frac{a(x_0,0)}{x_0}x_2^2 \right) + I(L_{1,(1)}^{(1)} \cup C) \).

(c) If \( x_1 \not| b \) then \( I(X \cup C) = Sx_1F_2 + I(L_{1,(1)}^{(1)} \cup C) \).

**Proof.** Using Lemma [B.3], one deduces from the exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_1^{(1)}} \longrightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0
\]

an exact sequence:

\[
0 \longrightarrow \mathcal{I}_{L_{1,(1)}^{(1)} \cup C} \longrightarrow \mathcal{I}_{X \cup C} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2)
\]

where \( \psi \) is the composite morphism:

\[
\mathcal{I}_{X \cup C} \longrightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2).
\]

Moreover, \( \text{Coker } \psi \) is an \( \mathcal{O}_{L_1 \cup C} = \mathcal{O}_{(R_1)} \)-module hence \( x_1\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im } \psi \subseteq \mathcal{O}_{L_1}(-l - 2) \).

(a) If \( l = -1 \) then the condition \( x_1 | b \) means that \( b = 0 \), hence \( F_2 = x_3 \), hence \( I(X) = (x_3, x_2^2) \). It follows that:

\[
I(X \cup C) = (x_3, x_2^2) \cap (x_1, x_0x_3 + cx_2^2) = (x_3, x_0x_3 + cx_2^2) \cap (x_1, x_0x_3 + cx_2^2) = (x_0x_3 + cx_2^2, x_1x_3).
\]

(b) If \( x_1 | b \) and \( l \geq 0 \) then \( F_2 + c\frac{a(x_0,0)}{x_0}x_2^2 \in I(X \cup C) \). Since \( \psi \left( F_2 + c\frac{a(x_0,0)}{x_0}x_2^2 \right) = \eta \left( F_2 + c\frac{a(x_0,0)}{x_0}x_2^2 \right) = 1 \in H^0(\mathcal{O}_{L_1}) \), one deduces that \( \psi \) is an epimorphism and that the sequence:

\[
0 \longrightarrow I(L_{1,(1)}^{(1)} \cup C) \longrightarrow I(X \cup C) \xrightarrow{H^0(\psi)} S(L_1)(-l - 2) \longrightarrow 0
\]

is exact. Notice that using this exact sequence one can get a graded free resolution of \( I(X \cup C) \).

(c) We show, firstly, that:

**Claim.** \( \text{Im } \psi = x_1\mathcal{O}_{L_1}(-l - 3) \).
Indeed, assume that $\text{Im} \psi = \mathcal{O}_{L_1}(-l-2)$. Then, for $m > 0$, there exists $f \in H^0(\mathcal{I}_{X \cup C}(l+2+m))$ such that $\psi(f) = x_0^m \in H^0(\mathcal{O}_{L_1}(m))$. Since $\psi(f) = \eta(f)$ it follows that:

$$f = x_0^m F_2 + f_0 x_2^2 + f_1 x_2 x_3 + f_2 x_3^2$$

with $f_0, f_1, f_2 \in S_{t+m}$. On the other hand, since $f \in H^0(\mathcal{I}_C(l+2+m))$ it follows that:

$$f = g_1 x_1 + g_2 (x_0 x_3 + c x_2^2)$$

with $g_1 \in S_{t+m+1}$ and $g_2 \in S_{t+m}$. Restricting the two different formulae of $f$ to the line $L'_2$ of equations $x_1 = x_3 = 0$ one gets:

$$-x_0^m b(x_0, 0) x_2 + (f_0 | L'_2) x_2^2 = c(g_2 | L'_2) x_2^2.$$

Since $x_1 \nmid b$ one has $b(x_0, 0) \neq 0$ and this leads to a contradiction.

It remains that $\psi$ factorizes as:

$$\mathcal{I}_{X \cup C} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l-3) \xrightarrow{\delta_2} \mathcal{O}_{L_1}(-l-2)$$

with $\psi'$ an epimorphism. Since $\psi'(x_1 F_2) = 1 \in H^0(\mathcal{O}_{L_1})$ one deduces that the sequence:

$$0 \longrightarrow I(L_1^{(l)} \cup C) \longrightarrow I(X \cup C) \xrightarrow{H^0(\psi')} S(L_1)(-l-3) \longrightarrow 0$$

is exact from which the description of $I(X \cup C)$ from the statement follows. Notice, also, that using this exact sequence one can get a graded free resolution of $I(X \cup C)$. \(\Box\)

**Proposition B.41.** Under the hypothesis of Prop. B.40:

(a) If $x_1 | b$, i.e., if $b = x_1 b_1$ then the graded $S$-module $H^0_*(\mathcal{O}_{X \cup C})$ admits the following free resolution:

$$0 \longrightarrow 2S(-3) \oplus 3S(-2) \oplus S \longrightarrow \oplus_{S(l-3)} \oplus_{2S(l-2)} \oplus_{S(l-1)} H^0_*(\mathcal{O}_{X \cup C}) \longrightarrow 0$$

where $\delta_0 = (1, x_1 e_1)$ and $\delta_1, \delta_2$ are defined by the matrices:

$$\begin{pmatrix} x_0 x_3 + c x_2^2 & x_1 x_2 & x_1 x_3 & 0 & 0 \\ -x_0 b_1 & -a & -b & x_2 & x_3 \end{pmatrix}, \begin{pmatrix} -x_1 & 0 & 0 \\ c x_2 & -x_3 & 0 \\ x_0 & x_2 & 0 \\ ca & b & -x_3 \\ 0 & -a & x_2 \end{pmatrix}.$$

(b) If $x_1 \nmid b$ then the graded $S$-module $H^0_*(\mathcal{O}_{X \cup C})$ admits the following free resolution:

$$0 \longrightarrow 2S(-3) \oplus 3S(-2) \oplus S \longrightarrow \oplus_{2S(l-2)} \oplus_{2S(l-1)} \oplus_{S(l)} H^0_*(\mathcal{O}_{X \cup C}) \longrightarrow 0$$
Lemma B.42. Let \( C \subseteq \mathbb{P}^3 \) be the conic of equations \( y_3 = x_1^2 + cx_0x_2 = 0, \) \( c \in k \setminus \{0\} \). Put \( q_3 := x_1^2 + cx_0x_2 \). Then \( I(L_1^{(1)} \cup C) = (x_2x_3, x_3^2, x_2q_3) \) and admits the following minimal

graded free resolution:

\[
0 \rightarrow \bigoplus_{S(-5)} S(-3) \left(\begin{array}{cc}
-x_3 & -x_2q_3 \\
x_2 & 0 \\
0 & x_3
\end{array}\right) \rightarrow \bigoplus_{S(-4)} 2S(-2) \rightarrow I(L_1^{(1)} \cup C) \rightarrow 0.
\]

Proof. Let \( f = f_0x_2^2 + f_1x_2x_3 + f_2x_3^2 \) be a homogeneous element of \( I(L_1^{(1)}) \). Since \( x_3 \) vanishes on \( C \) it follows that if \( f \) vanishes on \( C \) then \( f_0x_2^2 \) vanishes on \( C \) hence \( f_0 \) vanishes on \( C \) hence \( f_0 \in (x_3, q_3) \). One deduces that \( I(L_1^{(1)} \cup C) \) is generated by the elements from the statement.

Now, if \( W \) is the curve directly linked to \( L_1^{(1)} \cup C \) by the complete intersection defined by \( x_2^2 \) and \( x_2^2q_3 \) then \( I(W) = (x_3, x_2q_3) \) hence \( W = L_1 \cup C \). Using Ferrand’s result about liaison one gets the graded free resolution from the statement. \( \square \)

**Proposition B.43.** Let \( X \) be the double structure on the line \( L_1 \) considered at the beginning of Subsection [B.1] and let \( C \subset \mathbb{P}^3 \) be the conic from Lemma [B.42]

(a) If \( x_1^2 \mid b \), i.e., if \( b = x_1^2b_2 \) then:

\[
I(X \cup C) = S(-q_3b_2x_2 + ax_3) + I(L_1^{(1)} \cup C).
\]

(b) If \( x_1 \mid b \) but \( x_1^2 \notmid b \), i.e., if \( b = x_1b_2 \) with \( x_1 \notmid b_1 \) then:

\[
I(X \cup C) = S(-q_3b_1x_2 + ax_1x_3) + I(L_1^{(1)} \cup C).
\]

(c) If \( x_1 \notmid b \) then \( I(X \cup C) = Sb_3F_2 + I(L_1^{(1)} \cup C) \).

Proof. According to Lemma [B.5] there is an exact sequence:

\[
0 \rightarrow \mathcal{I}_{L_1^{(1)} \cup C} \rightarrow \mathcal{I}_{X \cup C} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2)
\]

where \( \psi \) is the composite morphism:

\[
\mathcal{I}_{X \cup C} \rightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2).
\]

Moreover, \( \text{Coker } \psi \) is an \( \mathcal{O}_{L_1 \cup C} \)-module, hence \( x_2^2\mathcal{O}_{L_1}(-l - 4) \subseteq \text{Im } \psi \subseteq \mathcal{O}_{L_1}(-l - 2) \).

**Claim 1.** If \( \text{Im } \psi = \mathcal{O}_{L_1}(-l - 2) \) then \( x_1^2 \mid b \).

Indeed, since \( H^1(\mathcal{I}_{L_1^{(1)} \cup C}) = 0 \) (by Lemma [B.42]) there exists \( f \in H^0(\mathcal{I}_{X \cup C}(l + 2)) \) such that \( \psi(f) = 1 \in H^0(\mathcal{O}_{L_1}) \). Since \( \psi(f) = \eta(f) \) it follows that:

\[
f = F_2 + f_0x_2^2 + f_1x_2x_3 + f_2x_3^2.
\]

On the other hand, \( f \in H^0(\mathcal{O}_C(l + 2)) \) hence:

\[
f = g_1x_3 + g_2(x_1^2 + cx_0x_2).
\]

Restricting the two different expressions of \( f \) to the plane \( H \) of equation \( x_3 = 0 \) one gets:

\[-bx_2 + (f_0 \mid H)x_2^2 = (g_2 \mid H)(x_1^2 + cx_0x_2).
\]
One deduces that $g_2 \mid H = x_2g'_2$ with $g'_2 \in k[x_0, x_1, x_2]$ hence:
\[-b + (f_0 \mid H)x_2 = g'_2(x_1^2 + cx_0x_2)\] Restricting this relation to the line $L_1$ of equations $x_2 = x_3 = 0$ one gets $-b = (g'_2 \mid L_1)x_1^2$ whence the Claim.

Claim 2. If $x_1\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im} \psi$ then $x_1 \mid b$.

Indeed, since $\text{H}^1(\mathcal{I}_{L_1^{(1)})}) = 0$ (by Lemma [B.42]) there exists $f \in \text{H}^0(\mathcal{I}_{X \cup C}(l + 3))$ such that $\psi(f) = x_1 \in \text{H}^0(\mathcal{O}_{L_1}(1))$. Since $\psi(f) = \eta(f)$ it follows that:
\[f = x_1F_2 + f_0x_2^2 + f_1x_2x_3 + f_2x_3^2\]

On the other hand, $f \in \text{H}^0(\mathcal{O}_C(l + 2))$ hence:
\[f = g_1x_3 + g_2(x_1^2 + cx_0x_2)\]

Restricting the two different expressions of $f$ to the plane $H$ of equation $x_3 = 0$ one gets:
\[-bx_1x_2 + (f_0 \mid H)x_2 = (g_2 \mid H)(x_1^2 + cx_0x_2)\]

One deduces that $g_2 \mid H = x_2g'_2$ with $g'_2 \in k[x_0, x_1, x_2]$ hence:
\[-bx_1 + (f_0 \mid H)x_2 = g'_2(x_1^2 + cx_0x_2)\]

Restricting this relation to the line $L_1$ of equations $x_2 = x_3 = 0$ one gets $-b = (g'_2 \mid L_1)x_1$ whence the Claim.

(a) In this case $-q_3b_2x_2 + ax_3$ belongs to $\text{H}^0(\mathcal{I}_{X \cup C}(l + 2))$ and $\psi(-q_3b_2x_2 + ax_3) = \eta(-q_3b_2x_2 + ax_3) = x_1 \in \text{H}^0(\mathcal{O}_{L_1}(1))$. It follows that $\psi$ is an epimorphism and that one has an exact sequence:
\[0 \longrightarrow I(L_1^{(1)}) + C \longrightarrow I(X \cup C) \xrightarrow{\text{H}^0(\psi)} \text{S}(L_1)(-l - 2) \longrightarrow 0\]
from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$.

(b) In this case, by Claim 1, $\text{Im} \psi \subseteq x_1\mathcal{O}_{L_1}(-l - 3)$. On the other hand, $-q_3b_1x_2 + ax_3$ belongs to $\text{H}^0(\mathcal{I}_{X \cup C}(l + 3))$ and $\psi(-q_3b_1x_2 + ax_3) = \eta(-q_3b_1x_2 + ax_3) = x_1 \in \text{H}^0(\mathcal{O}_{L_1}(1))$. It follows that $\psi$ factorizes as:
\[\mathcal{I}_{X \cup C} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2)\]
with $\psi'$ an epimorphism and that one has an exact sequence:
\[0 \longrightarrow I(L_1^{(1)}) + C \longrightarrow I(X \cup C) \xrightarrow{\text{H}^0(\psi')} \text{S}(L_1)(-l - 3) \longrightarrow 0\]
from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$. 
(c) In this case, by Claim 2, $\text{Im } \psi = x_1^2 \mathcal{O}_{L_1}(-l - 4)$. $q_3F_2$ belongs to $H^0(\mathcal{I}_{X \cup C}(l + 4))$ and $\psi(q_3F_2) = \eta(q_3F_2) = x_1^2 \in H^0(\mathcal{O}_{L_1}(2))$. It follows that $\psi$ factorizes as:

$$ \mathcal{I}_{X \cup C} \xrightarrow{\psi''} \mathcal{O}_{L_1}(-l - 4) \xrightarrow{x_1^2} \mathcal{O}_{L_1}(-l - 2) $$

with $\psi''$ an epimorphism and that one has an exact sequence:

$$ 0 \rightarrow I(L_1^{(i)} \cup C) \rightarrow I(X \cup C) \xrightarrow{H^0(\psi'')} S(L_1)(-l - 4) \rightarrow 0 $$

from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$. \qed

**Proposition B.44.** Under the hypothesis of Prop. [B.43]:

(a) If $x_2 \mid b$, i.e., if $b = x_1^2 b_2$ then the graded $S$-module $H_*^0(\mathcal{O}_{X \cup C})$ admits the following free resolution:

$$ 0 \rightarrow S(-4) \xrightarrow{\oplus} S(-1) \xrightarrow{\delta_2} S(-3) \xrightarrow{\oplus} S(-1) \xrightarrow{\delta_0} H_*^0(\mathcal{O}_{X \cup C}) \rightarrow 0 $$

with $\delta_0 = (1, x_1^2 e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$ \begin{pmatrix} x_3 & x_2q_3 & 0 & 0 \\ -b_2 & -a & x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} -x_2q_3 & 0 \\ x_3 & 0 \\ -q_3b_2 & -x_3 \\ a & x_2 \end{pmatrix}. $$

(b) If $x_1 \mid b$ but $x_2 \nmid b$, i.e., if $b = x_1 b_1$ with $x_1 \nmid b_1$ then the graded $S$-module $H_*^0(\mathcal{O}_{X \cup C})$ admits the following free resolution:

$$ 0 \rightarrow S(-4) \xrightarrow{\oplus} S(-1) \xrightarrow{\delta_2} S(-3) \xrightarrow{\oplus} S(-1) \xrightarrow{\delta_0} H_*^0(\mathcal{O}_{X \cup C}) \rightarrow 0 $$

with $\delta_0 = (1, x_1 e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$ \begin{pmatrix} x_3 & x_2q_3 & 0 & 0 \\ -b_1 & -x_1 a & x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} -x_2q_3 & 0 \\ x_3 & 0 \\ -q_3b_1 & -x_3 \\ x_1 a & x_2 \end{pmatrix}. $$
(c) If \( x_1 \nmid b \) then the graded \( S \)-module \( H^0_*(O_{X \cup C}) \) admits the following free resolution:

\[
0 \rightarrow S(-4) \oplus S(-3) \oplus S(-1) \oplus S(l) \oplus H^0_*(O_{X \cup C}) \rightarrow 0
\]

with \( \delta_0 = (1, e_1) \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
  x_3 & x_2 q_3 & 0 & 0 \\
  -b & -x_1^2 & x_2 & x_3
\end{pmatrix},
\begin{pmatrix}
  -x_2 q_3 & 0 \\
  x_3 & 0 \\
  -q_3 b & -x_3 \\
  x_1^2 & x_2
\end{pmatrix}.
\]

Proof. One has \( L(L_1 \cup C) = (x_3, x_2 q_3) \) whence an exact sequence:

\[
0 \rightarrow S(-4) \oplus S(-1) \oplus (x_3, x_2 q_3) \rightarrow S \rightarrow H^0_*(O_{L_1 \cup C}) \rightarrow 0.
\]

Now, by Lemma [3.3] one has an exact sequence:

\[
0 \rightarrow \text{Im} \phi \rightarrow O_{X \cup C} \rightarrow O_{L_1 \cup C} \rightarrow 0
\]

where \( \phi \) is the composite morphism:

\[
\mathcal{I}_{L_1 \cup C} \rightarrow \mathcal{I}_L \rightarrow \mathcal{I}_{L_1} / \mathcal{I}_{L_1}^2 \simeq 2\mathcal{O}_{L_1}(-1) \xrightarrow{(a, b)} \mathcal{O}_{L_1}(l).
\]

Moreover, \( \text{Coker} \phi \) is an \( O_{L_1 \cup C} \)-module hence \( \mathcal{I}_1 = \mathcal{O}_{L_1}(l-2) \subseteq \text{Im} \phi \subseteq \mathcal{O}_{L_1}(l) \). Notice that:

\[
\phi(x_3) = b, \quad \phi(x_2 q_3) = x_1^2 a.
\]

(a) In this case \( \text{Im} \phi = x_1^2 \mathcal{O}_{L_1}(l-2) \) and one has an exact sequence:

\[
0 \rightarrow S(L_1)(l-2) \rightarrow H^0_*(O_{X \cup C}) \rightarrow H^0_*(O_{L_1 \cup C}) \rightarrow 0
\]

where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0_*(O_{X \cup C}(-l + 2)) \) whose image into \( H^0_*(O_X(-l + 2)) \oplus H^0_*(O_C(-l + 2)) \) is \((x_1^2 e_1, 0)\).

(b) In this case \( \text{Im} \phi = x_1 \mathcal{O}_{L_1}(l-1) \) and one has an exact sequence:

\[
0 \rightarrow S(L_1)(l-1) \rightarrow H^0_*(O_{X \cup C}) \rightarrow H^0_*(O_{L_1 \cup C}) \rightarrow 0
\]

where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0_*(O_{X \cup C}(-l + 1)) \) whose image into \( H^0_*(O_X(-l + 1)) \oplus H^0_*(O_C(-l + 1)) \) is \((x_1 e_1, 0)\).

(c) In this case \( \text{Im} \phi = \mathcal{O}_{L_1}(l) \) and one has an exact sequence:

\[
0 \rightarrow S(L_1)(l) \rightarrow H^0_*(O_{X \cup C}) \rightarrow H^0_*(O_{L_1 \cup C}) \rightarrow 0
\]

where the left morphism maps \( 1 \in S(L_1) \) to the element of \( H^0_*(O_{X \cup C}(-l)) \) whose image into \( H^0_*(O_X(-l)) \oplus H^0_*(O_C(-l)) \) is \((e_1, 0)\). \( \square \)
Lemma B.45. Let $C \subseteq \mathbb{P}^3$ be the conic of equations $x_3 = x_0x_1 + cx_2^2 = 0$, $c \in k \setminus \{0\}$. Put $q_4 := x_1^2 + cx_0x_2$. Then $I(L_1^{(1)} \cup C) = (x_2x_3, x_3^2, x_2^2q_4)$ and admits the following minimal graded free resolution:

$$0 \longrightarrow S(-3) \oplus S(-5) \longrightarrow 2S(-2) \oplus S(-4) \longrightarrow I(L_1^{(1)} \cup C) \longrightarrow 0.$$ 

Proof. Let $f = f_0x_2^2 + f_1x_2x_3 + f_2x_3^2$ be a homogeneous element of $I(L_1^{(1)})$. Since $x_3$ vanishes on $C$ it follows that if $f$ vanishes on $C$ then $f_0x_2^2$ vanishes on $C$ hence $f_0 \in (x_3, q_4)$. One deduces that $I(L_1^{(1)} \cup C)$ is generated by the elements from the statement.

Now, if $W$ is the curve directly linked to $L_1^{(1)} \cup C$ by the complete intersection defined by $x_3^2$ and $x_2^2q_4$ then $I(W) = (x_3, x_2q_4)$ hence $W = L_1 \cup C$. Using Ferrand’s result about liaison one gets the graded free resolution from the statement. \hfill \Box 

Proposition B.46. Let $X$ be the double structure on the line $L_1$ considered at the beginning of Subsection B.1 and let $C \subseteq \mathbb{P}^3$ be the conic from Lemma B.45.

(a) If $x_0x_1 \mid b$, i.e., if $b = x_0x_1b_2$ then:

$$I(X \cup C) = S(-q_4b_2x_2 + ax_3) + I(L_1^{(1)} \cup C).$$

(b) If $x_0 \mid b$ but $x_1 \nmid b$, i.e., if $b = x_0b_0$ with $x_1 \nmid b_0$ then:

$$I(X \cup C) = S(-q_4b_0x_2 + ax_1x_3) + I(L_1^{(1)} \cup C).$$

(c) If $x_1 \mid b$ but $x_0 \nmid b$, i.e., if $b = x_1b_1$ with $x_0 \nmid b_1$ then:

$$I(X \cup C) = S(-q_4b_1x_2 + ax_0x_3) + I(L_1^{(1)} \cup C).$$

(d) If $x_0 \nmid b$ and $x_1 \nmid b$ then $I(X \cup C) = Sq_4F_2 + I(L_1^{(1)} \cup C)$.

Proof. According to Lemma B.5 there is an exact sequence:

$$0 \longrightarrow J_{L_1^{(1)} \cup C} \longrightarrow J_{X \cup C} \xrightarrow{\psi} O_{L_1}(-l - 2)$$

where $\psi$ is the composite morphism:

$$J_{X \cup C} \longrightarrow J_X \xrightarrow{\eta} O_{L_1}(-l - 2).$$

Moreover, $\text{Coker } \psi$ is an $O_{L_1 \cap C} = O_{\{P_b, P_3\}}$-module, hence $x_0x_1O_{L_1}(-l - 4) \subseteq \text{Im } \psi \subseteq O_{L_1}(-l - 2)$.

Claim 1. If $\text{Im } \psi = O_{L_1}(-l - 2)$ then $x_0x_1 \mid b$.

Indeed, since $H^2_*(J_{L_1^{(1)} \cup C}) = 0$ (by Lemma B.45) there exists $f \in H^0(J_{X \cup C}(l + 2))$ such that $\psi(f) = 1 \in H^0(O_{L_1})$. Since $\psi(f) = \eta(f)$ it follows that:

$$f = F_2 + f_0x_2^2 + f_1x_2x_3 + f_2x_3^2.$$
On the other hand, \( f \in H^0(\mathcal{O}_C(l + 2)) \) hence:
\[
f = g_1x_3 + g_2(x_0x_1 + cx_2^2).
\]
Restricting the two different expressions of \( f \) to the plane \( H \) of equation \( x_3 = 0 \) one gets:
\[
-bx_2 + (f_0 \mid H)x_2^2 = (g_2 \mid H)(x_0x_1 + cx_2^2).
\]
One deduces that \( g_2 \mid H = x_2g'_2 \) with \( g'_2 \in k[x_0, x_1, x_2] \) hence:
\[
-b + (f_0 \mid H)x_2 = g'_2(x_0x_1 + cx_2^2).
\]
Restricting this relation to the line \( L_1 \) of equations \( x_2 = x_3 = 0 \) one gets \(-b = (g'_2 \mid L_1)x_0x_1 \) whence the Claim.

**Claim 2.** If \( x_1\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im } \psi \) then \( x_0 \mid b \).

Indeed, since \( H^1_*(\mathcal{I}^{(l)\cup C}) = 0 \) (by Lemma [B.45]) there exists \( f \in H^0(\mathcal{I}_X \cup C(l + 3)) \) such that \( \psi(f) = x_1 \in H^0(\mathcal{O}_L(1)) \). Since \( \psi(f) = \eta(f) \) it follows that:
\[
f = x_1F_2 + f_0x_2^2 + f_1x_2x_3 + f_2x_3^2.
\]
On the other hand, \( f \in H^0(\mathcal{O}_C(l + 2)) \) hence:
\[
f = g_1x_3 + g_2(x_0x_1 + cx_2^2).
\]
Restricting the two different expressions of \( f \) to the plane \( H \) of equation \( x_3 = 0 \) one gets:
\[
-bx_1x_2 + (f_0 \mid H)x_2^2 = (g_2 \mid H)(x_0x_1 + cx_2^2).
\]
One deduces that \( g_2 \mid H = x_2g'_2 \) with \( g'_2 \in k[x_0, x_1, x_2] \) hence:
\[
-bx_1 + (f_0 \mid H)x_2 = g'_2(x_0x_1 + cx_2^2).
\]
Restricting this relation to the line \( L_1 \) of equations \( x_2 = x_3 = 0 \) one gets \(-b = (g'_2 \mid L_1)x_0 \) whence the Claim.

**Claim 3.** If \( x_0\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im } \psi \) then \( x_1 \mid b \).

(a) In this case \(-q_4b_2x_2 + ax_3 \) belongs to \( H^0(\mathcal{I}_X \cup C(l + 2)) \) and \( \psi(-q_4b_2x_2 + ax_3) = \eta(-q_4b_2x_2 + ax_3) = 1 \in H^0(\mathcal{O}_{L_1}) \). It follows that \( \psi \) is an epimorphism and that one has an exact sequence:
\[
0 \rightarrow I(L_1^{(l)} \cup C) \rightarrow I(X \cup C) \xrightarrow{H^0(\psi)} S(L_1)(-l - 2) \rightarrow 0
\]
from which one gets the generators of \( I(X \cup C) \) from the statement. Using this exact sequence one can also get a graded free resolution of \( I(X \cup C) \).

(b) In this case, by Claim 1, \( \text{Im } \psi \neq \mathcal{O}_{L_1}(-l - 2) \). On the other hand, \(-q_4b_0x_2 + ax_3 \) belongs to \( H^0(\mathcal{I}_X \cup C(l + 3)) \) and \( \psi(-q_4b_0x_2 + ax_3) = \eta(-q_4b_0x_2 + ax_3) = x_1 \in H^0(\mathcal{O}_{L_1}(1)) \). It follows that \( \psi \) factorizes as:
\[
\mathcal{I}_X \cup C \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2)
\]
with $\psi'_1$ an epimorphism and that one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup C) \longrightarrow I(X \cup C) \xrightarrow{H_0^s(\psi'_1)} S(L_1)(-l - 3) \longrightarrow 0$$

from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$.

(c) In this case, by Claim 1, $\text{Im} \psi \neq \mathcal{O}_{L_1}(-l - 2)$. On the other hand, $-q_4b_1x_2 + ax_0x_3$ belongs to $H^0(\mathcal{I}_{X\cup C}(l + 3))$ and $\psi(-q_4b_1x_2 + ax_0x_3) = \eta(-q_4b_1x_2 + ax_0x_3) = x_0 \in H^0(\mathcal{O}_{L_1}(1))$. It follows that $\psi$ factorizes as:

$$\mathcal{I}_{X\cup C} \xrightarrow{\psi'_0} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_0} \mathcal{O}_{L_1}(-l - 2)$$

with $\psi'_0$ an epimorphism and that one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup C) \longrightarrow I(X \cup C) \xrightarrow{H_0^s(\psi'_0)} S(L_1)(-l - 3) \longrightarrow 0$$

from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$.

(d) In this case, by Claims 2 and 3, $\text{Im} \psi = x_0x_1\mathcal{O}_{L_1}(-l - 4)$. $q_4F_2$ belongs to $H^0(\mathcal{I}_{X\cup C}(l + 4))$ and $\psi(q_4F_2) = \eta(q_4F_2) = x_0x_1 \in H^0(\mathcal{O}_{L_1}(2))$. It follows that $\psi$ factorizes as:

$$\mathcal{I}_{X\cup C} \xrightarrow{\psi''} \mathcal{O}_{L_1}(-l - 4) \xrightarrow{x_0^2x_1} \mathcal{O}_{L_1}(-l - 2)$$

with $\psi''$ an epimorphism and that one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup C) \longrightarrow I(X \cup C) \xrightarrow{H_0^s(\psi'')} S(L_1)(-l - 4) \longrightarrow 0$$

from which one gets the generators of $I(X \cup C)$ from the statement. Using this exact sequence one can also get a graded free resolution of $I(X \cup C)$. ■

**Proposition B.47.** Under the hypothesis of Prop. B.46:

(a) If $x_0x_1 | b$, i.e., if $b = x_0x_1b_2$ then the graded $S$-module $H_0^s(\mathcal{O}_{X\cup C})$ admits the following free resolution:

$$0 \longrightarrow S(-3) \xrightarrow{\delta_1} S(-1) \oplus S(l - 3) \xrightarrow{\delta_2} S \oplus S(l - 4) \xrightarrow{\delta_0} H_0^s(\mathcal{O}_{X\cup C}) \xrightarrow{} 0$$

with $\delta_0 = (1, x_0x_1e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$\begin{pmatrix} x_3 & x_2q_4 & 0 & 0 \\ -b_2 & -a & x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} -x_2q_4 & 0 \\ x_3 & 0 \\ -q_4b_2 & -x_3 \\ a & x_2 \end{pmatrix}.$$
(b) If $x_0 \mid b$ but $x_1 \nmid b$, i.e., if $b = x_0b_0$ with $x_1 \nmid b_0$ then the graded $S$-module $H_*(\mathcal{O}_{X\cup C})$ admits the following free resolution:

$$0 \rightarrow \bigoplus_{S(-4)} \xrightarrow{\delta_2} \bigoplus_{S(-3)} \xrightarrow{\delta_1} \bigoplus_{S(l-1)} \xrightarrow{\delta_0} H_*(\mathcal{O}_{X\cup C}) \rightarrow 0$$

with $\delta_0 = (1, x_0e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$\begin{pmatrix} x_3 & x_2q_4 & 0 & 0 \\ -b_0 & -x_1a & x_2 & x_3 \end{pmatrix}, \begin{pmatrix} -x_2q_4 & 0 \\ x_3 & 0 \\ -q_4b_0 & -x_3 \\ x_1a & x_2 \end{pmatrix}.$$

(c) If $x_1 \mid b$ but $x_0 \nmid b$, i.e., if $b = x_1b_1$ with $x_0 \nmid b_1$ then the graded $S$-module $H_*(\mathcal{O}_{X\cup C})$ admits the following free resolution:

$$0 \rightarrow \bigoplus_{S(-4)} \xrightarrow{\delta_2} \bigoplus_{S(-3)} \xrightarrow{\delta_1} \bigoplus_{S(l-1)} \xrightarrow{\delta_0} H_*(\mathcal{O}_{X\cup C}) \rightarrow 0$$

with $\delta_0 = (1, x_1e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$\begin{pmatrix} x_3 & x_2q_4 & 0 & 0 \\ -b_1 & -x_0a & x_2 & x_3 \end{pmatrix}, \begin{pmatrix} -x_2q_4 & 0 \\ x_3 & 0 \\ -q_4b_1 & -x_3 \\ x_0a & x_2 \end{pmatrix}.$$

(d) If $x_0 \nmid b$ and $x_1 \nmid b$ then the graded $S$-module $H_*(\mathcal{O}_{X\cup C})$ admits the following free resolution:

$$0 \rightarrow \bigoplus_{S(-4)} \xrightarrow{\delta_2} \bigoplus_{S(-3)} \xrightarrow{\delta_1} \bigoplus_{S(l-1)} \xrightarrow{\delta_0} H_*(\mathcal{O}_{X\cup C}) \rightarrow 0$$

with $\delta_0 = (1, e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$\begin{pmatrix} x_3 & x_2q_4 & 0 & 0 \\ -b & -x_0x_1a & x_2 & x_3 \end{pmatrix}, \begin{pmatrix} -x_2q_4 & 0 \\ x_3 & 0 \\ -q_4b_1 & -x_3 \\ x_0x_1a & x_2 \end{pmatrix}.$$
Proof. One has $L(L_1 \cup C) = (x_3, x_2q_4)$ whence an exact sequence:

$$
0 \rightarrow S(-4) \xrightarrow{\begin{pmatrix} -x_2q_4 \\ x_3 \end{pmatrix}} S(-1) \oplus (x_3, x_2q_4) \xrightarrow{S(-3)} S \rightarrow H^0(\mathcal{O}_{L_1 \cup C}) \rightarrow 0.
$$

Now, by Lemma B.5 one has an exact sequence:

$$
0 \rightarrow \text{Im } \phi \rightarrow \mathcal{O}_{X \cup C} \rightarrow \mathcal{O}_{L_1 \cup C} \rightarrow 0
$$

where $\phi$ is the composite morphism:

$$
\mathcal{J}_{L_1 \cup C} \rightarrow \mathcal{J}_{L_1} \rightarrow \mathcal{J}_{L_1}/\mathcal{J}_{L_1}^2 \simeq 2\mathcal{O}_{L_1}(-1) \xrightarrow{(a, b)} \mathcal{O}_{L_1}(l).
$$

Moreover, $\text{Coker } \phi$ is an $\mathcal{O}_{L_1 \cup C} = \mathcal{O}_{(P_b, P)}$-module hence $x_0x_1\mathcal{O}_{L_1}(l - 2) \subseteq \text{Im } \phi \subseteq \mathcal{O}_{L_1}(l)$.

Notice that:

- (a) In this case $\text{Im } \phi = x_0x_1\mathcal{O}_{L_1}(l - 2)$ and one has an exact sequence:
  $$
  0 \rightarrow S(L_1)(l - 2) \rightarrow H^0(\mathcal{O}_{X \cup C}) \rightarrow H^0(\mathcal{O}_{L_1 \cup C}) \rightarrow 0
  $$

- (b) In this case $\text{Im } \phi = x_0\mathcal{O}_{L_1}(l - 1)$ and one has an exact sequence:
  $$
  0 \rightarrow S(L_1)(l - 1) \rightarrow H^0(\mathcal{O}_{X \cup C}) \rightarrow H^0(\mathcal{O}_{L_1 \cup C}) \rightarrow 0
  $$

- (c) In this case $\text{Im } \phi = x_1\mathcal{O}_{L_1}(l - 1)$ and one has an exact sequence:
  $$
  0 \rightarrow S(L_1)(l - 1) \rightarrow H^0(\mathcal{O}_{X \cup C}) \rightarrow H^0(\mathcal{O}_{L_1 \cup C}) \rightarrow 0
  $$

- (d) In this case $\text{Im } \phi = \mathcal{O}_{L_1}(l)$ and one has an exact sequence:
  $$
  0 \rightarrow S(L_1)(l) \rightarrow H^0(\mathcal{O}_{X \cup C}) \rightarrow H^0(\mathcal{O}_{L_1 \cup C}) \rightarrow 0.
  $$

\[\square\]

B.5. Union of two double lines. Let $X, X'$ and $X''$ be double structures on the lines $L_1, L_1'$ and $L_2$, respectively, defined by exact sequences:

$$
0 \rightarrow \mathcal{J}_X \rightarrow \mathcal{J}_{L_1} \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(l) \rightarrow 0,
$$

$$
0 \rightarrow \mathcal{J}_X' \rightarrow \mathcal{J}_{L_1'} \xrightarrow{\varepsilon'} \mathcal{O}_{L_1'}(l') \rightarrow 0,
$$

$$
0 \rightarrow \mathcal{J}_X'' \rightarrow \mathcal{J}_{L_2} \xrightarrow{\varepsilon''} \mathcal{O}_{L_2}(l'') \rightarrow 0.
$$
where \( \varepsilon, \varepsilon', \varepsilon'' \) are composite morphisms:

\[
\mathcal{I}_L \to \mathcal{I}_L / \mathcal{I}^2_L \cong 2\mathcal{O}_L (-1) \xrightarrow{(a, b)} \mathcal{O}_L (l),
\]

\[
\mathcal{I}'_L \to \mathcal{I}'_L / \mathcal{I}^2'_L \cong 2\mathcal{O}_L (-1) \xrightarrow{(a', b')} \mathcal{O}_L (l'),
\]

\[
\mathcal{I}_L^2 \to \mathcal{I}_L^2 / \mathcal{I}^2_L \cong 2\mathcal{O}_L (l') \xrightarrow{(a'', b'')} \mathcal{O}_L (l'').
\]

Putting \( F_2 := -b(x_0, x_1)x_2 + a(x_0, x_1)x_3, F'_2 := -b'(x_2, x_3)x_0 + a'(x_2, x_3)x_1 \) and \( F''_2 := -b''(x_0, x_2)x_1 + a''(x_0, x_2)x_3 \) one has:

\[
I(X) = (F_2, x_2^2, x_2x_3, x_3^2), \quad I(X') = (F'_2, x_0^2, x_0x_1, x_1^2), \quad I(X'') = (F''_2, x_1^2, x_1x_3, x_3^2).
\]

Recall, also, the exact sequences:

\[
0 \to \mathcal{I}'_L \to \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_L (-l - 2) \to 0,
\]

\[
0 \to \mathcal{I}'_L \to \mathcal{I}'_X \xrightarrow{\eta'} \mathcal{O}'_L (-l' - 2) \to 0,
\]

\[
0 \to \mathcal{I}''_L \to \mathcal{I}''_X \xrightarrow{\eta''} \mathcal{O}''_L (-l'' - 2) \to 0.
\]

The following result is an immediate consequence of Lemma B.1:

**Lemma B.48.** \( I(L_1^{(1)} \cup L_2^{(1)}) = (x_0^2, x_0x_1, x_1^2)(x_2^2, x_2x_3, x_3^2) \) and the tensor product of the complexes:

\[
2S(-3) \xrightarrow{\begin{pmatrix} -x_3 & 0 \\ x_2 & -x_3 \\ 0 & x_2 \end{pmatrix}} 3S(-2), \quad 2S(-3) \xrightarrow{\begin{pmatrix} -x_1 & 0 \\ x_0 & -x_1 \\ 0 & x_0 \end{pmatrix}} 3S(-2)
\]

is a minimal graded free resolution of this ideal. \( \square \)

**Lemma B.49.** With the notation introduced at the beginning of this subsection:

(a) If \( l = -1 \) then \( I(X \cup L_1^{(1)}) = Sx_0^2F_2 + Sx_0x_1F_2 + Sx_1^2F_2 + I(L_1^{(1)} \cup L_1^{(1)})) \).

(b) If \( l = 0 \) then \( I(X \cup L_1^{(1)}) = Sx_0F_2 + Sx_1F_2 + I(L_1^{(1)} \cup L_1^{(1)}) \).

(c) If \( l \geq 1 \) then \( I(X \cup L_1^{(1)}) = SF_2 + I(L_1^{(1)} \cup L_1^{(1)}) \).

**Proof.** Tensorizing by \( \mathcal{I}'_L \) the exact sequence:

\[
0 \to \mathcal{I}'_L \to \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_L (-l - 2) \to 0
\]

one gets an exact sequence:

\[
0 \to \mathcal{I}'_L \to \mathcal{I}_X \xrightarrow{\psi} \mathcal{O}_L (-l - 2) \to 0.
\]
Lemma B.48 implies that $H^1(\mathcal{I}_{L_1^{(1)\cup}L_1^{(1)}}(i)) = 0$ for $i \geq 3$. On the other hand, one has $H^0(\mathcal{I}_{L_1\cup L_1^{(1)}}(2)) = 0$ hence $H^0(\mathcal{O}_{X\cup L_1^{(1)}}(2)) = 0$. One deduces that:

$$\text{Im} H^0_*(\psi) = \bigoplus_{i \geq 3} H^0(\mathcal{O}_{L_1}(-l - 2 + i)).$$

(a) In this case $X$ is the divisor $2L_1$ on the plane $H \supset L_1$ of equation $F_2 = 0$ hence a complete intersection. The assertion from the statement follows, now, from Lemma B.1. Notice that, using the same lemma, one can get a minimal graded free resolution of $I(X \cup L_1^{(1)})$.

(b) In this case one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup L_1^{(1)}) \longrightarrow I(X \cup L_1^{(1)}) \xrightarrow{H^0_*(\psi)} S(L_1)_+(-l - 2) \longrightarrow 0.$$ 

The assertion from the statement follows noticing that $x_0F_2$ and $x_1F_2$ belong to $I(X \cup L_1^{(1)})$. Notice, also, that using the above exact sequence one can get a graded free resolution of $I(X \cup L_1^{(1)})$ (a minimal free resolution of the graded $S$-module $S(L_1)_+$ can be found in the discussion following Prop. A.12).

(c) In this case one has an exact sequence:

$$0 \longrightarrow I(L_1^{(1)} \cup L_1^{(1)}) \longrightarrow I(X \cup L_1^{(1)}) \xrightarrow{H^0_*(\psi)} S(L_1)(-l - 2) \longrightarrow 0.$$ 

The assertion from the statement follows noticing that $F_2$ belongs to $I(X \cup L_1^{(1)})$. Notice, also, that using the above exact sequence one can get a graded free resolution of $I(X \cup L_1^{(1)})$. \hfill \Box

**Proposition B.50.** With the notation from the beginning of this subsection, assume that $l \geq l'$.

(a) If $l = -1$ (hence $l' = -1$) then $I(X \cup X') = S x_0^2F_2 + S x_0x_1F_2 + S x_1^2F_2 + I(L_1^{(1)} \cup X')$.

(b) If $l = 0$ and $l' = -1$ then $I(X \cup X') = S x_0F_2 + S x_1F_2 + I(L_1^{(1)} \cup X')$.

(c) If $l = 0$, $l' = 0$ and $kF_2 \neq kF_2'$ then $I(X \cup X') = S x_0F_2 + S x_1F_2 + I(L_1^{(1)} \cup X')$.

(d) If $l = 0$, $l' = 0$ and $kF_2 = kF_2'$ then $I(X \cup X') = SF_2 + I(L_1^{(1)} \cup L_1^{(1)})$, i.e., $X \cup X'$ is the divisor $2L_1 + 2L_1'$ on the quadric surface $Q \subset \mathbb{P}^3$ of equation $F_2 = 0$.

(e) If $l \geq 1$ then $I(X \cup X') = SF_2 + I(L_1^{(1)} \cup X')$.

Notice that, by Lemma B.49

$$I(L_1^{(1)} \cup X') = \begin{cases} 
S x_0^2F_2' + S x_0x_3F_2' + S x_3^2F_2' + I(L_1^{(1)} \cup L_1^{(1)}), & \text{if } l' = -1 ; \\
S x_0F_2' + S x_3F_2' + I(L_1^{(1)} \cup L_1^{(1)}), & \text{if } l' = 0 ; \\
SF_2' + I(L_1^{(1)} \cup L_1^{(1)}), & \text{if } l' \geq 1 .
\end{cases}$$

**Proof.** Tensorizing by $\mathcal{I}_{X'}$, the exact sequence:

$$0 \longrightarrow \mathcal{I}_{L_1^{(1)}} \longrightarrow \mathcal{I}_{X} \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2) \longrightarrow 0$$


one gets an exact sequence:

\[ 0 \rightarrow \mathcal{J}_{L_1^{(1)} \cup X'} \rightarrow \mathcal{J}_{X \cup X'} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2) \rightarrow 0. \]

Recall, from the proof of Lemma B.49, that one has an exact sequence:

\[ 0 \rightarrow \mathcal{J}_{L_1^{(1)} \cup L_1^{(i)}} \rightarrow \mathcal{J}_{L_1^{(1)} \cup X'} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l' - 2) \rightarrow 0. \]

Using Lemma B.48, one deduces that \( H^1(\mathcal{J}_{L_1^{(1)} \cup X'}, i) = 0 \) for \( i \geq \max(3, l' + 1) \).

(a) In this case \( X \) and \( X' \) are complete intersections hence, by Lemma B.1, \( (X \cup X') = I(X)I(X') \). From the same lemma one can get a minimal free resolution of \( I(X \cup X') \).

(b) In this case one has an exact sequence:

\[ 0 \rightarrow \mathcal{J}_{L_1^{(1)} \cup X'} \rightarrow \mathcal{J}_{X \cup X'} \xrightarrow{\psi} \mathcal{O}_{L_1}(-2) \rightarrow 0 \]

and \( H^1(\mathcal{J}_{L_1^{(1)} \cup X'}, i) = 0 \) for \( i \geq 3 \). We assert that \( H^0(\mathcal{J}_{X \cup X'}(2)) = 0 \).

Indeed, all the quadric surfaces containing \( X \) are nonsingular. \( X' \) is the divisor 2\( L_1' \) on the plane \( H \supset L_1' \) of equation \( F_0 = 0 \). The intersection of \( H \) with a nonsingular quadric containing \( L_1' \) is the union of \( L_1' \) and of another (different) line. It follows that no quadric surface containing \( X \) can contain \( X' \).

One deduces, now, that one has an exact sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X') \rightarrow I(X \cup X') \xrightarrow{H^0(\psi)} S(L_1)(-2) \rightarrow 0. \]

The assertion from the statement follows noticing that \( x_0F_2 \) and \( x_1F_2 \) belong to \( I(X \cup X') \).

Notice, also, that using the above exact sequence one can get a graded free resolution of \( I(X \cup X') \) (a minimal free resolution of the graded \( S \)-module \( S(L_1)_+ \) can be found in the discussion following Prop. A.12).

(c) It follows, from Prop. B.9, that \( H^0(\mathcal{J}_{X \cup L_1}(2)) = kF_2 \) and that \( H^0(\mathcal{J}_{X \cup L_1}(2)) = kF_2' \).

One deduces, from the hypothesis, that \( H^0(\mathcal{J}_{X \cup X'}(2)) = 0 \). One can use, now, the same argument as in case (b).

(d) The argument used in case (c) shows that \( H^0(\mathcal{J}_{X \cup X'}(2)) = kF_2 = kF_2' \). One deduces the existence of an exact sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X') \rightarrow I(X \cup X') \xrightarrow{H^0(\psi)} S(L_1)(-2) \rightarrow 0 \]

from which the assertion from the statement follows. Notice, also, that using this exact sequence one can get a graded free resolution of \( I(X \cup X') \).

(e) In this case one has \( H^1(\mathcal{J}_{L_1^{(1)} \cup X'}(l + 2)) = 0 \) (one takes into account that \( l \geq l' \)). One deduces the existence of an exact sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X') \rightarrow I(X \cup X') \xrightarrow{H^0(\psi)} S(L_1)(-l - 2) \rightarrow 0 \]
from which the assertion from the statement follows. Notice, also, that using this exact sequence one can get a graded free resolution of $I(X \cup X')$. □

**Lemma B.51.** $I(L_1^{(1)} \cup L_2^{(1)}) = (x_3, x_1x_2)^2 = (x_3^2, x_1x_2x_3, x_1x_2^2)$ and admits the following minimal graded free resolution:

\[
0 \rightarrow S(-4) \bigoplus S(-5) \rightarrow S(-4) \oplus S(-3) \rightarrow I(L_1^{(1)} \cup L_2^{(1)}) \rightarrow 0.
\]

**Proof.** $I(L_1^{(1)} \cup L_2^{(1)}) = (x_3^2, x_2x_3, x_3^2) \cap (x_3^2, x_1x_3, x_3^2) = (x_3^2, x_1x_2x_3, x_1x_2^2).$ $L_1^{(1)} \cup L_2^{(1)}$ is directly linked to $L_1 \cup L_2$ by the complete intersection defined by $x_3^2$ and $x_1x_2^2$. Applying Ferrand’s result about liaison one gets the minimal free resolution from the statement. □

**Lemma B.52.** With the notation introduced at the beginning of this subsection:

(a) If $x_1 \mid b$ then $I(X \cup L_2^{(1)}) = Sx_1F_2 + I(L_1^{(1)} \cup L_2^{(1)})$.

(b) If $x_1 \nmid b$ then $I(X \cup L_2^{(1)}) = Sx_1^2F_2 + I(L_1^{(1)} \cup L_2^{(1)})$.

**Proof.** According to Lemma [B.5] one has an exact sequence:

\[
0 \rightarrow \mathcal{I}_{L_1^{(1)} \cup L_2^{(1)}} \rightarrow \mathcal{I}_{X \cup L_2^{(1)}} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2)
\]

where $\psi$ is the composite morphism:

\[
\mathcal{I}_{X \cup L_2^{(1)}} \rightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2).
\]

Moreover, Coker $\psi$ is an $\mathcal{O}_{L_1 \cap L_2^{(1)}}$-module. Since $I(L_1 \cap L_2^{(1)}) = (x_1^2, x_2, x_3)$ it follows that $x_1^2 \mathcal{O}_{L_1}(-l - 4) \subseteq \text{Im} \psi \subseteq \mathcal{O}_{L_1}(-l - 2)$.

**Claim 1.** $\text{Im} \psi \subseteq x_1 \mathcal{O}_{L_1}(-l - 3)$.

Indeed, assume that $\text{Im} \psi = \mathcal{O}_{L_1}(-l - 2)$. Since, by Lemma [B.5] $\text{H}^1_s(\mathcal{I}_{L_1^{(1)} \cup L_2^{(1)}}) = 0$ there exists $f \in \text{H}^0(\mathcal{I}_{X \cup L_2^{(1)}}(l + 2))$ such that $\psi(f) = 1 \in \text{H}^0(\mathcal{O}_{L_1})$. Since $\psi(f) = \eta(f)$ it follows that:

\[
f = F_2 + f_0x_2^2 + f_1x_2x_3 + f_2x_3^2.
\]

On the other hand, since $f \in I(L_2^{(1)})$ one has:

\[
f = g_0x_1^2 + g_1x_1x_3 + g_2x_3^2.
\]

Comparing the two different expressions of $f$ one deduces that each monomial appearing in $F_2 := -b(x_0, x_1)x_2 + a(x_0, x_1)x_3$ must be divisible by one of the monomials $x_1^2, x_1x_3, x_2^2, x_2x_3, x_3^2$. It follows that $x_1^2 \mid b$ and $x_1 \mid a$ which is not possible because $a$ and $b$ are coprime.
Using this exact sequence one sees that $I$ factorizes as:

\[ \mathcal{I}_{X \cup L_2^{(1)}} \xrightarrow{\psi'} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2) \]

with $\psi'$ and epimorphism and that one has an exact sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup L_2^{(1)}) \rightarrow I(X \cup L_2^{(1)}) \xrightarrow{H^0(\psi')} S(L_1)(-l - 3) \rightarrow 0. \]

Using this exact sequence one sees that $I(X \cup L_2^{(1)})$ is generated by the elements from the statement. One can also get, from this sequence, a graded free resolution of this ideal.

(b) Using the same kind of argument as in the proof of Claim 1 one shows that:

**Claim 2.** If $x_1 \nmid b$ then $\text{Im} \psi = x_1^2 \mathcal{O}_{L_1}(-l - 4)$ (that is, it is not equal to $x_1 \mathcal{O}_{L_1}(-l - 3)$).

Since $x_1^2 F_2 \in I(X \cup L_2^{(1)})$ and $\psi(x_1^2 F_2) = \eta(x_1^2 F_2) = x_1^2 \in H^0(\mathcal{O}_{L_1}(2))$ one deduces, taking into account Claim 2, that $\psi$ factorizes as:

\[ \mathcal{I}_{X \cup L_2^{(1)}} \xrightarrow{\psi''} \mathcal{O}_{L_1}(-l - 4) \xrightarrow{x_1^2} \mathcal{O}_{L_1}(-l - 2) \]

with $\psi''$ and epimorphism and that one has an exact sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup L_2^{(1)}) \rightarrow I(X \cup L_2^{(1)}) \xrightarrow{H^0(\psi'')} S(L_1)(-l - 4) \rightarrow 0. \]

Using this exact sequence one sees that $I(X \cup L_2^{(1)})$ is generated by the elements from the statement. One can also get, from this sequence, a graded free resolution of this ideal. □

**Proposition B.53.** With the notation introduced at the beginning of this subsection, assume that $l \geq l''$.

(a) If $x_1 \mid b$ (i.e., $b = x_1 b_1$), $x_2 \mid b''$ (i.e., $b'' = x_2 b_1''$) and $b_1(x_0,0) \neq b_1''(x_0,0)$ then:

\[ I(X \cup X'') = S \left( F_2 + b(x_0,0)x_1x_2 - a(x_0,0)x_3 + \frac{a(x_0)}{a''(x_0,0)} F_2'' \right) + S x_2 F_2'' + I(L_1^{(1)} \cup L_2^{(1)}) \]

(b) If $x_1 \mid b$ (i.e., $b = x_1 b_1$), $x_2 \mid b''$ (i.e., $b'' = x_2 b_1''$) and $b_1(x_0,0) \neq b_1''(x_0,0)$ then:

\[ I(X \cup X'') = S x_1 F_2 + S x_2 F_2'' + I(L_1^{(1)} \cup L_2^{(1)}) \]

(c) If $x_1 \mid b$ and $x_2 \nmid b''$ then:

\[ I(X \cup X'') = S x_1 F_2 + S x_2 F_2'' + I(L_1^{(1)} \cup L_2^{(1)}) \]

(d) If $x_1 \nmid b$ and $x_2 \mid b''$ then:

\[ I(X \cup X'') = S x_1^2 F_2 + S x_2 F_2'' + I(L_1^{(1)} \cup L_2^{(1)}) \]

(e) If $x_1 \nmid b$ and $x_2 \nmid b''$ then:

\[ I(X \cup X'') = S \left( x_1 F_2 + b(x_0,0)x_1x_2 + \frac{b(x_0)}{\psi'(x_0,0)} x_2 F_2'' \right) + S x_2 F_2'' + I(L_1^{(1)} \cup L_2^{(1)}) \]
Proof. From Lemma [B.5] one has an exact sequence:

\[ 0 \longrightarrow \mathcal{I}_{L_1^{(1)}}^{(1)} \longrightarrow \mathcal{I}_{X \cup X''} \xrightarrow{\rho} \mathcal{O}_{L_1}(-l - 2) \]

with \( \rho \) the composite morphism:

\[ \mathcal{I}_{X \cup X''} \longrightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2). \]

By Lemma [B.52]:

\[ I(L_1^{(1)} \cup X'') = \begin{cases} 
Sx_2F''_2 + I(L_1^{(1)} \cup L_2^{(1)}), & \text{if } x_2 \parallel b''; \\
Sx_2^2F''_2 + I(L_1^{(1)} \cup L_2^{(1)}), & \text{if } x_2 \parallel b''. 
\end{cases} \]

Now, \( \rho \) can be also written as a composite morphism:

\[ \mathcal{I}_{X \cup X''} \longrightarrow \mathcal{I}_{X \cup L_2} \xrightarrow{\psi} \mathcal{O}_{L_1}(-l - 2) \]

with \( \psi \) the composite morphism \( \mathcal{I}_{X \cup L_2} \rightarrow \mathcal{I}_X \xrightarrow{\eta} \mathcal{O}_{L_1}(-l - 2) \). By Lemma [B.5] again, one has an exact sequence:

\[ 0 \longrightarrow \mathcal{I}_{X \cup X''} \longrightarrow \mathcal{I}_{X \cup L_2} \xrightarrow{\phi''} \mathcal{O}_{L_2}([l'']) \]

where \( \phi'' \) is the composite morphism:

\[ \mathcal{I}_{X \cup L_2} \longrightarrow \mathcal{I}_{L_2} \xrightarrow{\varepsilon''} \mathcal{O}_{L_2}([l'']). \]

By Prop. [B.11] one has:

\[ I(X \cup L_2) = \begin{cases} 
(F_2, x_2x_2, x_3^2, x_1x_2^2), & \text{if } x_1 \parallel b; \\
(x_2F_2, x_2x_2, x_3^2, x_1x_2^2), & \text{if } x_1 \parallel b. 
\end{cases} \]

Recalling, from the beginning of the subsection, the definition of \( \varepsilon'' \) one gets:

\[ \phi''(F_2) = -b_1(x_0, 0)x_2a'' + a(x_0, 0)b'', \quad \text{if } b = x_1b_1, \]

\[ \phi''(x_1F_2) = -b(x_0, 0)x_2a'', \quad \text{if } x_1 \parallel b, \]

\[ \phi''(x_2x_3) = x_2b'', \quad \phi''(x_3^2) = 0, \quad \phi''(x_1x_2^2) = x_1^2a''. \]

Claim 1. (a) If \( x_1 \parallel b \) then \( x_1\mathcal{O}_{L_1}(-l - 3) \subseteq \text{Im} \rho \subseteq \mathcal{O}_{L_1}(-l - 2). \)

(b) If \( x_1 \notparallel b \) then \( x_1^2\mathcal{O}_{L_1}(-l - 4) \subseteq \text{Im} \rho \subseteq x_1\mathcal{O}_{L_1}(-l - 3). \)

Indeed, in case (a) \( b = x_1b_1 \) hence \( x_1F_2 \) belongs to \( I(X \cup L_2^{(1)}) \subseteq I(X \cup X'') \) and \( \rho(x_1F_2) = \eta(x_1F_2) = x_1 \in H^0(\mathcal{O}_{L_1}(1)) \).

In case (b) it follows from the proof of Prop. [B.11] that \( \text{Im} \psi = x_1\mathcal{O}_{L_1}(-l - 3) \) hence \( \text{Im} \rho \subseteq x_1\mathcal{O}_{L_1}(-l - 3) \). On the other hand, by Lemma [B.5] \( \text{Coker} \rho \) is an \( \mathcal{O}_{L_1 \cap X''} \)-module. Since \( I(X'') = (F''_2, x_1^2, x_1x_3, x_3^2) \) one deduces that \( x_1^2\mathcal{O}_{L_1}(-l - 4) \subseteq \text{Im} \rho. \)

Claim 2. If \( x_1 \parallel b \) (i.e., if \( b = x_1b_1 \)) and \( \text{Im} \rho = \mathcal{O}_{L_1}(-l - 2) \) then \( x_2 \parallel b'' \) (i.e., \( b'' = x_2b''_1 \)) and \( \frac{b''_1(x_0, 0)}{a(x_0, 0)} = \frac{b''_1(x_0, 0)}{a(x_0, 0)}. \)
Indeed, if $\text{Im } \rho = \mathcal{O} L_1(-l - 2)$ then, for $m \gg 0$, there exists $f \in \mathcal{H}^0(\mathcal{I}_{X \cup X'}(l + 2 + m))$ such that $\rho(f) = x_0^m \in \mathcal{H}^0(\mathcal{O} L_1(m))$. Since $f \in \mathcal{H}^0(\mathcal{I}_{X \cup L_2}(l + 2 + m))$ and $\rho(f) = \psi(f)$ it follows that:

$$f = x_0^m F_2 + f_0 x_2 x_3 + f_1 x_3^2 + f_2 x_1 x_2^2.$$ 

On the other hand, $\phi''(f) = 0$ hence:

$$-x_0^m b_1(x_0, 0)x_2 a'' + x_0^m a(x_0, 0)b'' + (f_0 | L_2)x_2 b'' + (f_2 | L_2)x_2^2 a'' = 0.$$ 

Since $x_1 \mid b$ and $a$ and $b$ are coprime it follows that $x_1 \nmid a$ hence $a(x_0, 0) \neq 0$. One deduces, now, from the last relation, that $x_2 \mid b''$, i.e., $b'' = x_2 b''_1$ and then that:

$$(x_0^m a(x_0, 0) + (f_0 | L_2)x_2)b''_1 = (x_0^m b_1(x_0, 0) - (f_2 | L_2)x_2)a''.$$ 

Since $a''$ and $b''_1$ are coprime it follows that there exists $g \in k[x_0, x_2]$ such that:

$$x_0^m a(x_0, 0) + (f_0 | L_2)x_2 = ga''$$

and

$$x_0^m b_1(x_0, 0) - (f_2 | L_2)x_2 = gb''_1.$$ 

Restricting these two relations to the line $L_1$ of equations $x_2 = x_3 = 0$ and then dividing the second relation by the first one one gets $\frac{b_1(x_0, 0)}{a(x_0, 0)} = \frac{b''_1(x_0, 0)}{a''(x_0, 0)}$.

**Claim 3.** If $x_1 \nmid b$ and $\text{Im } \rho = x_1 \mathcal{O} L_1(-l - 3)$ then $x_2 \nmid b''$.

Indeed, if $\text{Im } \rho = x_1 \mathcal{O} L_1(-l - 3)$ then, for $m \gg 0$, there exists $f \in \mathcal{H}^0(\mathcal{I}_{X \cup X'}(l + 3 + m))$ such that $\rho(f) = x_0^m x_1 \in \mathcal{H}^0(\mathcal{O} L_1(m+1))$. Since $f \in \mathcal{H}^0(\mathcal{I}_{X \cup L_2}(l+3+m))$ and $\rho(f) = \psi(f)$ it follows that:

$$f = x_0^m x_1 F_2 + f_0 x_2 x_3 + f_1 x_3^2 + f_2 x_1 x_2^2.$$ 

On the other hand, $\phi''(f) = 0$ hence:

$$-x_0^m b(x_0, 0)x_2 a'' + (f_0 | L_2)x_2 b'' + (f_2 | L_2)x_2^2 a'' = 0$$ 

from which one deduces that:

$$(f_0 | L_2)b'' = (x_0^m b(x_0, 0) - (f_2 | L_2)x_2)a''.$$ 

Restricting the last relation to the line $L_1$ of equations $x_2 = x_3 = 0$ one gets:

$$x_0^m b(x_0, 0)a''(x_0, 0) = f_0(x_0, 0, 0, 0)b''(x_0, 0).$$ 

Since $x_1 \nmid b$ one has $b(x_0, 0) \neq 0$. If one would have $x_2 \mid b''$, which is equivalent to $b''(x_0, 0) = 0$, it would follow that $a''(x_0, 0) = 0$, i.e., that $x_2 \mid a''$ which would contradict the fact that $a''$ and $b''$ are coprime. It thus remains that $x_2 \nmid b''$.

(a) Let us put:

$$\Phi := F_2 + b_1(x_0, 0)x_1 x_2 - a(x_0, 0)x_3 + \frac{a(x_0, 0)}{a''(x_0, 0)}F_2''.$$
One has:

\[ F_2 + b_1(x_0, 0)x_1x_2 - a(x_0, 0)x_3 \in (x_1 x_3, x_1^2x_2) \subseteq I(L_1 \cup L_2^{(1)}), \]

\[ b_1(x_0, 0)x_1x_2 - a(x_0, 0)x_3 + \frac{a(x_0, 0)}{a'(x_0, 0)}F_2'' = \]

\[ = \frac{a(x_0, 0)}{a'(x_0, 0)}(b_1(x_0, 0)x_1x_2 - a''(x_0, 0)x_3 + F_2'') \in (x_2 x_3, x_1 x_2^2) \subseteq I(L_1^{(1)} \cup L_2). \]

It follows that \( \Phi \in I(X \cup X'') \) and that \( \rho(\Phi) = \eta(\Phi) = \eta(F_2) = 1 \in H^0(\mathcal{O}_{L_1}). \) One deduces the exactness of the sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X'') \rightarrow I(X \cup X'') \xrightarrow{H^0(\rho)} S(L_1)(-l - 2) \rightarrow 0. \]

This implies that \( I(X \cup X'') \) is generated by the elements from the statement. Using this exact sequence one can also get a graded free resolution of \( I(X \cup X''). \)

Notice, also, that, as a consequence of the above relations:

\[ x_1 \left( b_1(x_0, 0)x_1x_2 - a(x_0, 0)x_3 + \frac{a(x_0, 0)}{a'(x_0, 0)}F_2'' \right) \in (x_1 x_2 x_2, x_1^2 x_2^2) \subseteq I(L_1^{(1)} \cup L_2^{(1)}) \]

hence \( x_1 F_2 \equiv x_1 \Phi \left( \text{mod } I(L_1^{(1)} \cup L_2^{(1)}) \right). \)

(b) It follows, from Claim 1(a) and Claim 2, that \( \text{Im } \rho = x_1 \mathcal{O}_{L_1}(-l - 3) \) hence \( \rho \) factorizes as:

\[ \mathcal{I}_{X \cup X''} \xrightarrow{\rho'} \mathcal{O}_{L_1}(-l - 3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l - 2) \]

with \( \rho' \) an epimorphism. Since:

\[ x_1 F_2 = -b_1 x_1^2 x_2 + a x_1 x_3 \in I(X \cup L_2^{(1)}) \subseteq I(X \cup X'') \]

and since \( \rho(x_1 F_2) = \eta(x_1 F_2) = x_1 \in H^0(\mathcal{O}_{L_1}(1)) \) it follows that \( \rho'(x_1 F_2) = 1 \in H^0(\mathcal{O}_{L_1}) \) hence the sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X'') \rightarrow I(X \cup X'') \xrightarrow{H^0(\rho')} S(L_1)(-l - 3) \rightarrow 0. \]

is exact. One deduces that \( I(X \cup X'') \) is generated by the elements from the statement. One can also get, using this exact sequence, a graded free resolution of \( I(X \cup X''). \)

(c) The argument for case (b) works verbatim for case (c), too.

(d) It follows, from Claim 1(b) and Claim 3, that \( \text{Im } \rho = x_1^2 \mathcal{O}_{L_1}(-l - 4) \) hence \( \rho \) factorizes as:

\[ \mathcal{I}_{X \cup X''} \xrightarrow{\rho''} \mathcal{O}_{L_1}(-l - 4) \xrightarrow{x_1^2} \mathcal{O}_{L_1}(-l - 2) \]

with \( \rho'' \) an epimorphism. Since \( x_1^2 F_2 \in I(X) I(L_2^{(1)}) \subseteq I(X \cup X'') \) and since \( \rho(x_1^2 F_2) = \eta(x_1^2 F_2) = x_1^2 \in H^0(\mathcal{O}_{L_1}(2)) \) it follows that \( \rho''(x_1^2 F_2) = 1 \in H^0(\mathcal{O}_{L_1}) \) hence the sequence:

\[ 0 \rightarrow I(L_1^{(1)} \cup X'') \rightarrow I(X \cup X'') \xrightarrow{H^0(\rho'')} S(L_1)(-l - 4) \rightarrow 0. \]

is exact. One deduces that \( I(X \cup X'') \) is generated by the elements from the statement. One can also get, using this exact sequence, a graded free resolution of \( I(X \cup X''). \)
(e) Put \( \Psi := x_1F_2 + b(x_0,0)x_1x_2 + \frac{b(x_0,0)}{b'(x_0,0)} x_2F''_2 \). One has:

\[
x_1F_2 + b(x_0,0)x_1x_2 \in (x_1x_3, x_1^2x_2) \subset I(L_1 \cup L_2^{(1)}),
\]

\[
b(x_0,0)x_1x_2 + \frac{b(x_0,0)}{b'(x_0,0)} x_2F''_2 = \frac{b(x_0,0)}{b'(x_0,0)} (b'(x_0,0)x_1x_2 + x_2F''_2) \in (x_2x_3, x_1^2x_2) \subset I(L_1^{(1)} \cup L_2).
\]

It follows that \( \Psi \in I(X \cup X'') \) and that \( \rho(\Psi) = \eta(\Psi) = \eta(x_1F_2) = x_1 \in H^0(\mathcal{O}_{L_1}(1)) \). Taking into account Claim 1(b), one deduces that \( \rho \) factorizes as:

\[
\mathcal{I}_{X \cup X''} \xrightarrow{\rho'} \mathcal{O}_{L_1}(-l-3) \xrightarrow{x_1} \mathcal{O}_{L_1}(-l-2)
\]

with \( \rho' \) an epimorphism and that the sequence:

\[
0 \longrightarrow I(L_1^{(1)} \cup L_2^{(1)}) \xrightarrow{H^0(\rho')} \mathcal{O}_{L_1}(-l-3) \longrightarrow 0.
\]

is exact. It follows that \( I(X \cup X'') \) is generated by the elements from the statement. One can also get, using this exact sequence, a graded free resolution of \( I(X \cup X'') \).

Notice, also, that, as a consequence of the above relations:

\[
x_1 \left( b(x_0,0)x_1x_2 + \frac{b(x_0,0)}{b'(x_0,0)} x_2F''_2 \right) \in (x_1x_2x_3, x_1^2x_2) \subset I(L_1^{(1)} \cup L_2^{(1)})
\]

hence \( x_1^2F_2 \equiv x_1 \Psi \pmod{I(L_1^{(1)} \cup L_2^{(1)})} \).

\[\square\]

**Corollary B.54.** Under the hypothesis of Prop. [B.53] \( X \cup X'' \) is locally complete intersection in the cases (a) and (e) and it is not locally complete intersection in the other cases.

**Proof.** We cut \( X \cup X'' \) with the plane \( H \subset \mathbb{P}^3 \) of equation \( x_1 - x_2 = 0 \). The composite map \( k[x_0, x_1, x_3] \hookrightarrow S \twoheadrightarrow S(H) \) is bijective and allows one to identify \( S(H) \) with \( k[x_0, x_1, x_3] \).

Under this identification, the restrictions to \( H \) of the generators of \( I(L_1^{(1)} \cup L_2^{(1)}) \) are:

\[
x_3^2 \mid H = x_2^2, \quad x_1x_2x_3 \mid H = x_1^2x_3, \quad x_1^2x_2 \mid H = x_1^3.
\]

(a) Using the notation from the proof of Prop. [B.53] one has \( \Phi \mid H \equiv F_2 \mid H \equiv a(x_0,0)x_3 \pmod{(x_1, x_3)^2} \). It follows that the 1-dimensional scheme \( H \cap \{\Phi = 0\} \) is nonsingular at \( P_0 \). Since \( H \cap (X \cup X'') \) is a subscheme of \( H \cap \{\Phi = 0\} \) one deduces that it is locally complete intersection in \( H \) hence \( X \cup X'' \) is locally complete intersection at \( P_0 \).

(b) One has \( x_1F_2 \mid H \equiv a(x_0,0)x_1x_3 \pmod{(x_1, x_3)^3} \) and \( x_2F''_2 \mid H \equiv a''(x_0,0)x_1x_3 \pmod{(x_1, x_3)^3} \). Using [16, Chap. I, Ex. 5.4(a)] (in that exercise one has equality if and only if \( Y \) and \( Z \) have no common tangent direction at \( P \)) one deduces that \( H \cap (X \cup X'') \) is not locally complete intersection in \( H \).

(c) \( x_1F_2 \mid H \equiv a(x_0,0)x_1x_3 \pmod{(x_1, x_3)^3} \) and \( x_2^2F''_2 \mid H \equiv (x_1, x_3)^3 \).

(d) \( x_1^2F_2 \in (x_1, x_3)^3 \) and \( x_2F''_2 \mid H \equiv a''(x_0,0)x_1x_3 \pmod{(x_1, x_3)^3} \).

(e) Using the notation from the proof of Prop. [B.53] one has:

\[
\Psi \mid H \equiv -b(x_0,0) \left( x_1^2 - \left( \frac{a(x_0,0)}{b(x_0,0)} + \frac{a(x_0,0)}{b'(x_0,0)} \right) x_1x_3 \right) \pmod{(x_1, x_3)^3}.
\]
If \( Y \subset H \) (resp., \( Z \subset H \)) is the effective divisor of equation \( (\Psi \mid H) = 0 \) (resp., \( x_3^2 = 0 \)) it follows, from the above mentioned exercise from Hartshorne’s book, that the intersection multiplicity of \( Y \) and \( Z \) at \( P_0 \) is 4. Since \( H \cap (X \cup X'') \) is a subscheme of \( Y \cap Z \) of degree 4 concentrated at \( P_0 \) one deduces that \( H \cap (X \cup X'') \) is locally complete intersection in \( H \) hence \( X \cup X'' \) is locally complete intersection at \( P_0 \). \( \square \)

**Remark B.55.** Under the hypothesis of Prop. B.53 recall the exact sequences:

\[
0 \to \mathcal{I}_{X \cup L_2} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\phi} \mathcal{O}_{L_1}(l) , \quad 0 \to \mathcal{I}_{L_1 \cup X''} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\phi''} \mathcal{O}_{L_2}(l'') ,
\]

where \( \phi \) and \( \phi'' \) are the composite morphisms:

\[
\mathcal{I}_{L_1 \cup L_2} \to \mathcal{I}_{L_1} \xrightarrow{\varepsilon} \mathcal{O}_{L_1}(l) , \quad \mathcal{I}_{L_1 \cup L_2} \to \mathcal{I}_{L_2} \xrightarrow{\varepsilon''} \mathcal{O}_{L_2}(l'') .
\]

One deduces an exact sequence:

\[
0 \to \mathcal{I}_{X \cup X''} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{(\phi, \phi'')} \mathcal{O}_{L_1}(l) \oplus \mathcal{O}_{L_2}(l'') .
\]

Moreover, since \( I(L_1 \cup L_2) = (x_3, x_1x_2) \) and since:

\[
\phi(x_3) = b, \quad \phi(x_1x_2) = x_1a, \quad \phi''(x_3) = b'', \quad \phi''(x_1x_2) = x_2a'',
\]

it follows that if \( x_1 \mid b \) (resp., \( x_2 \mid b'' \)) then \( \phi \) (resp., \( \phi'' \)) factorizes as:

\[
\mathcal{I}_{L_1 \cup L_2} \xrightarrow{\phi_1} \mathcal{O}_{L_1}(l - 1) \xrightarrow{x_1} \mathcal{O}_{L_1}(l) \quad \text{(resp.,) \quad \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\phi''} \mathcal{O}_{L_2}(l'' - 1) \xrightarrow{x_2} \mathcal{O}_{L_2}(l'').}
\]

(a) If \( x_1 \mid b \) (i.e., \( b = x_1b_1 \)), \( x_2 \mid b'' \) (i.e., \( b'' = x_2b_1'' \)) and \( \frac{b_1(x_0, 0)}{a(x_0, 0)} = \frac{b_1'(x_0, 0)}{a'(x_0, 0)} \) then one has an exact sequence:

\[
0 \to \mathcal{I}_{X \cup X''} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\left(\begin{smallmatrix} \phi_1 \\ \phi''_1 \end{smallmatrix} \right)} \mathcal{O}_{L_1}(l - 1) \oplus \mathcal{O}_{L_2}(l'' - 1) \xrightarrow{(a''(x_0, 0), -a(x_0, 0))} \mathcal{O}_{P_0}(l + l'') \to 0 .
\]

(b) If \( x_1 \mid b \) (i.e., \( b = x_1b_1 \)), \( x_2 \mid b'' \) (i.e., \( b'' = x_2b_1'' \)) and \( \frac{b_1(x_0, 0)}{a(x_0, 0)} \neq \frac{b_1'(x_0, 0)}{a'(x_0, 0)} \) then one has an exact sequence:

\[
0 \to \mathcal{I}_{X \cup X''} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\left(\begin{smallmatrix} \phi_1 \\ \phi''_1 \end{smallmatrix} \right)} \mathcal{O}_{L_1}(l - 1) \oplus \mathcal{O}_{L_2}(l'' - 1) \to 0 .
\]

(c) If \( x_1 \mid b \) and \( x_2 \nmid b'' \) then one has an exact sequence:

\[
0 \to \mathcal{I}_{X \cup X''} \to \mathcal{I}_{L_1 \cup L_2} \xrightarrow{\left(\begin{smallmatrix} \phi_1 \\ \phi'' \end{smallmatrix} \right)} \mathcal{O}_{L_1}(l - 1) \oplus \mathcal{O}_{L_2}(l'') \to 0 .
\]
(d) If $x_1 \nmid b$ and $x_2 \mid b''$ then one has an exact sequence:

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow 0.$$ 

(e) If $x_1 \nmid b$ and $x_2 \nmid b''$ then one has an exact sequence:

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow \mathcal{J} \rightarrow 0.$$ 

Indeed, in case (a) let $H \subset \mathbb{P}^3$ be the plane of equation $x_3 = 0$. Coker $(\phi_1, \phi_1'')$ is concentrated in $P_0$ and coincides with the cokernel of the morphism:

$$\mathcal{O}_H^{-1} \oplus \mathcal{O}_H^{-2} \rightarrow \mathcal{O}_L^{-1} \oplus \mathcal{O}_L^{-2}$$

hence with the cokernel of the morphism:

$$\mathcal{O}_H^{-1} \oplus \mathcal{O}_H^{-2} \oplus \mathcal{O}_H^{-2} \oplus \mathcal{O}_H^{-2} \rightarrow \mathcal{O}_H^{-1} \oplus \mathcal{O}_H^{-1} \oplus \mathcal{O}_H^{-1} \oplus \mathcal{O}_H^{-1}$$

The cokernel of the last morphism is annihilated by $x_1b_1$, $x_1a$, $x_2b''$, $x_2a''$. Since $b_1$ and $a$ (resp., $b''_1$ and $a''$) are coprime one deduces that Coker $(\phi_1, \phi_1'')$ is annihilated by $x_1$ and $x_2$ hence it is an $\mathcal{O}_{P_0}$-module. It follows that the sequence:

$$\mathcal{O}_{P_0}^{-1} \oplus \mathcal{O}_{P_0}^{-2} \oplus \mathcal{O}_{P_0}^{-2} \rightarrow \mathcal{O}_{P_0}^{-1} \oplus \mathcal{O}_{P_0}^{-1} \rightarrow \text{Coker} (\phi_1, \phi_1'') \rightarrow 0$$

is exact and assertion (a) follows.

The assertions (b)-(e) can be proven similarly.

Recall, now, that the graded $S$-module $H^0_x(\mathcal{O}_X)$ (resp., $H^0_x(\mathcal{O}_{X'}$)) is generated by 1 and by an element $e_1 \in H^0(\mathcal{O}_X(-l))$ (resp., $e''_1 \in H^0(\mathcal{O}_{X'}(-l''))$).

Proposition B.56. Using the notation introduced at the beginning of this subsection, assume that $l \leq l''$. Define $a_1, b_1, b_2, a''_1, b''_1$ and $b''_2$ by the relations:

$$a = a(x_0, 0) + a_1x_1, \ b = b(x_0, 0) + b_1x_1, \ b_1 = b_1(x_0, 0) + b_2x_1,$$
$$a'' = a''(x_0, 0) + a''_1x_2, \ b'' = b''(x_0, 0) + b''_1x_2, \ b''_1 = b''_1(x_0, 0) + b''_2x_2.$$
(a) If \( x_1 \mid b, \ x_2 \mid b'' \) and \( \frac{b_1(x_0,0)}{a(x_0,0)} = \frac{b_1''(x_0,0)}{a''(x_0,0)} \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup X''}) \) admits a free resolution of the form:

\[
0 \to S(l - 3) \xrightarrow{\delta_2} S(-2) \oplus S(-1) \oplus S \oplus 2S(l - 2) \oplus 2S(l'' - 3) \oplus S(l'' - 2) \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup X''}) \to 0
\]

with \( \delta_0 = (1, x_1e_1 + \frac{a''(x_0,0)}{a(x_0,0)}x_2e_1'', x_2e_1'') \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
  x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
  -b_1 & -a & x_2 & x_3 & 0 & 0 \\
  -b''_1 & -a'' & 0 & 0 & x_1 & x_3
\end{pmatrix}
\begin{pmatrix}
  -x_1x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
  -x_1b_1 & -x_3 & 0 \\
  a & x_2 & 0 \\
  -x_2b''_2 & \frac{a''(x_0,0)}{a(x_0,0)}b_1 & 0 & -x_3 \\
  a'' & 0 & -x_1
\end{pmatrix}
\]

(b) If \( x_1 \mid b, \ x_2 \mid b'' \) and \( \frac{b_1(x_0,0)}{a(x_0,0)} \neq \frac{b''_1(x_0,0)}{a''(x_0,0)} \) then the graded \( S \)-module \( H^0_*(\mathcal{O}_{X \cup X''}) \) admits a free resolution of the form:

\[
0 \to S(l - 3) \xrightarrow{\delta_2} S(-2) \oplus S(-1) \oplus S \oplus 2S(l - 2) \oplus 2S(l'' - 3) \oplus S(l'' - 2) \xrightarrow{\delta_0} H^0_*(\mathcal{O}_{X \cup X''}) \to 0
\]

with \( \delta_0 = (1, x_1e_1, x_2e_1'') \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
  x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
  -b_1 & -a & x_2 & x_3 & 0 & 0 \\
  -b''_1 & -a'' & 0 & 0 & x_1 & x_3
\end{pmatrix}
\begin{pmatrix}
  -x_1x_2 & 0 & 0 \\
  x_3 & 0 & 0 \\
  -x_1b_1 & -x_3 & 0 \\
  a & x_2 & 0 \\
  -x_2b''_2 & 0 & -x_3 \\
  a'' & 0 & x_1
\end{pmatrix}
\]
(c) If $x_1 \mid b$ and $x_2 \nmid b''$ then the graded $S$-module $H^0_+(\mathcal{O}_{X \cup X''})$ admits a free resolution of the form:

$$
\begin{array}{cccccccc}
0 & \to & S(l - 3) & \xrightarrow{\delta_2} & S(-2) & \oplus & S(l - 1) & \xrightarrow{\delta_0} & H^0_+(\mathcal{O}_{X \cup X''}) & \to & 0 \\
& & & & & & & & \\
& & & & & & & & 2S(l - 1) & \oplus & S(l'' - 1) \\
\end{array}
$$

with $\delta_0 = (1, x_1e_1, e''_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
 x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
 -b_1 & -a & x_2 & x_3 & 0 & 0 \\
 -b'' & -x_2a'' & 0 & 0 & x_1 & x_3
\end{pmatrix},
\begin{pmatrix}
 -x_1x_2 & 0 & 0 \\
 x_3 & 0 & 0 \\
 -x_1b_1 & -x_3 & 0 \\
 a & x_2 & 0 \\
 -x_2b'' & 0 & -x_3 \\
 x_2a'' & 0 & x_1
\end{pmatrix}.
$$

(d) If $x_1 \nmid b$ then the graded $S$-module $H^0_+(\mathcal{O}_{X \cup X''})$ admits a free resolution of the form:

$$
\begin{array}{cccccccc}
0 & \to & S(l - 2) & \xrightarrow{\delta_2} & S(-2) & \oplus & S(l) & \xrightarrow{\delta_0} & H^0_+(\mathcal{O}_{X \cup X''}) & \to & 0 \\
& & & & & & & & \\
& & & & & & & & 2S(l - 1) & \oplus & S(l'' - 2) \\
\end{array}
$$

with $\delta_0 = (1, e_1 + \frac{b''(x_0, 0)}{b(x_0, 0)}e''_1, x_2e''_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
 x_3 & x_1x_2 & 0 & 0 & 0 & 0 \\
 -b & -x_1a & x_2 & x_3 & 0 & 0 \\
 -b'' & -a'' & -\frac{b''(x_0, 0)}{b(x_0, 0)} & 0 & x_1 & x_3
\end{pmatrix},
\begin{pmatrix}
 -x_1x_2 & 0 & 0 \\
 x_3 & 0 & 0 \\
 -x_1b & -x_3 & 0 \\
 x_1a & x_2 & 0 \\
 -x_2b'' & 0 & -x_3 \\
 a'' & -\frac{b''(x_0, 0)}{b(x_0, 0)} & x_1
\end{pmatrix}.
$$

Proof. By Lemma B.5 one has an exact sequence:

$$
0 \to \text{Im} \phi'' \to \mathcal{O}_{X \cup X''} \to \mathcal{O}_{X \cup L_2} \to 0
$$

where $\phi''$ is the composite morphism:

$$
\mathcal{I}_{X \cup L_2} \to \mathcal{I}_{L_2} \xrightarrow{\varepsilon''} \mathcal{O}_{L_2}(l'').
$$
According to Prop. 3.11 one has:

\[ I(X \cup L_2) = \begin{cases} 
(F_2, x_2x_3, x_3^2, x_1x_2^2), & \text{if } x_1 \mid b; \\
(x_1F_2, x_2x_3, x_3^2, x_1x_2^2), & \text{if } x_1 \nmid b.
\end{cases} \]

Taking into account the definition of \( \varepsilon'' \) one gets:

\[
\phi''(F_2) = -b_1(x_0, 0)x_2a'' + a(x_0, 0)b'' = \\
= a(x_0, 0)b''(x_0, 0) + (-b_1(x_0, 0)a''(x_0, 0) + a(x_0, 0)b''_1(x_0, 0))x_2 + \\
+ (-b_1(x_0, 0)a'_1 + a(x_0, 0)b''_2)x_2^2, \text{ if } x_1 \mid b,
\]

\[
\phi''(x_1F_2) = -b(x_0, 0)x_2a'' = -b(x_0, 0)a''(x_0, 0)x_2 - b(x_0, 0)a_1'x_2^2, \text{ if } x_1 \nmid b,
\]

\[
\phi''(x_2x_3) = x_2b'' \quad \phi''(x_3^2) = 0 \quad \phi''(x_1x_2^2) = x_2^2a''.
\]

We shall identify the element \( e_1 \) of \( H_0^0(\mathcal{O}_X) \) (resp., \( e''_1 \) of \( H_0^0(\mathcal{O}_X'') \)) to the element \( (e_1, 0) \) (resp., \( (0, e''_1) \)) of \( H_0^0(\mathcal{O}_X) \oplus H_0^0(\mathcal{O}_X'') \). Using the exact sequence:

\[
0 \longrightarrow H_0^0(\mathcal{O}_X \cup X'') \longrightarrow H_0^0(\mathcal{O}_X) \oplus H_0^0(\mathcal{O}_X'') \longrightarrow H_0^0(\mathcal{O}_{X \cap X''})
\]

and taking into account that \( X \cap X'' \) is concentrated at \( P_0 \) one deduces that if an element \( \xi \) of \( H_0^0(\mathcal{O}_X) \oplus H_0^0(\mathcal{O}_X'') \) has the property that \( x_0^m\xi \) belongs to \( H_0^0(\mathcal{O}_{X \cup X''}) \) for some \( m \geq 0 \) then \( \xi \) itself belongs to \( H_0^0(\mathcal{O}_{X \cup X''}) \).

Finally, let us recall that Prop. 3.12 provides a graded free resolution for \( H_0^0(\mathcal{O}_{X \cup L_2}) \).

(a) In this case \( \text{Im } \phi'' = x_2^2\mathcal{O}_{L_2}(l'' - 2) \). By Prop. 3.12(a), the graded \( S \)-module \( H_0^0(\mathcal{O}_{X \cup L_2}) \) is generated by 1 and by \( x_1e_1 \in H^0(\mathcal{O}_{X \cup L_2}(-l+1)) \). Since \( l \leq l'' \) it follows that \( H^1((\text{Im } \phi'')(-l + 1)) \cong H^1(\mathcal{O}_{L_2}(l'' - l - 1)) = 0 \). One deduces an exact sequence:

\[
0 \longrightarrow S(L_2)(l'' - 2) \longrightarrow H_0^0(\mathcal{O}_{X \cup X''}) \longrightarrow H_0^0(\mathcal{O}_{X \cup L_2}) \longrightarrow 0
\]

where the left morphism maps 1 \( \in S(L_2) \) to \( x_2^2e''_1 \in H^0(\mathcal{O}_{X \cup X''}(-l'' + 2)) \) (identified, as we assumed above, to the element \( (0, x_2^2e''_1) \) of \( H_0^0(\mathcal{O}_X) \oplus H_0^0(\mathcal{O}_X'') \)).

Claim 1. \( x_1^2e_1 \) belongs to \( H_0^0(\mathcal{O}_{X \cup X''}) \).

Indeed, since \( x_1e_1 \) belongs to \( H_0^0(\mathcal{O}_{X \cup L_2}) \), there exists an element of \( H_0^0(\mathcal{O}_{X \cup X''}) \) of the form \( x_1e_1 + fe''_1 \). Since \( x_1 \) annihilates \( e''_1 \) the claim follows.

Claim 2. \( x_1e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)}x_2e''_1 \) belongs to \( H_0^0(\mathcal{O}_{X \cup X''}) \).

Indeed, one has in \( H_0^0(\mathcal{O}_X) \oplus H_0^0(\mathcal{O}_X'') \):

\[
x_1x_2 \cdot 1 = ax_1e_1 + a''x_2e''_1 = a(x_0, 0)x_1e_1 + a''(x_0, 0)x_2e_2 + a_1x_1^2e_1 + a''_1x_2e''_1.
\]

Since \( x_1^2e_1 \) and \( x_2^2e''_1 \) belong to \( H_0^0(\mathcal{O}_{X \cup X''}) \) it follows that \( a(x_0, 0)x_1e_1 + a''(x_0, 0)x_2e''_1 \) belongs to \( H_0^0(\mathcal{O}_{X \cup X''}) \) hence:

\[
a(x_0, 0) \left( x_1e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)}x_2e''_1 \right) \in H_0^0(\mathcal{O}_{X \cup X''}).
\]

The claim follows, now, using an observation from the beginning of the proof.
The assertion from case (a) of the statement is, now, a consequence of the exact sequence (B.6), of Prop. B.12(a) and of the following relations in $H^0(\mathcal{O}_{X_{\cup}X'}) \subset H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{X'})$:

$$
\begin{align*}
    x_3 \cdot 1 &= b e_1 + b'' e''_1 = b_1 x_1 e_1 + b''_1 x_1 e''_1 = b_1 \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) - b_1 \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1 + b''_1 x_1 e''_1 \\
    &= b_1 \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) - b_1 (x_0, 0) \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1 + b''_1 x_1 e''_1 = \\
    &= b_1 \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) - b''_1 (x_1, 0) x_2 e''_1 + b''_1 x_1 e''_1 = b_1 \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) + b''_1 x_1 e''_1 ; \\
    x_1 x_2 \cdot 1 &= a x_1 e_1 + a'' x_2 e''_1 = a \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) - a \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1 + a'' x_2 e''_1 = \\
    &= a \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) - a (x_0, 0) \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1 + a'' x_2 e''_1 = a \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) + a'' x_2 e''_1 ; \\
    x_2 \cdot \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) &= \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1 ; \quad x_3 \cdot \left(x_1 e_1 + \frac{a''(x_0, 0)}{a(x_0, 0)} x_2 e''_1\right) = 0 .
\end{align*}
$$

(b) In this case $\text{Im} \phi'' = x_2 \mathcal{O}_{L_2}(l'' - 1)$. By Prop. B.12(a) the graded $S$-module $H^0(\mathcal{O}_{X_{\cup}L_2})$ is generated by $1 \in H^0(\mathcal{O}_{X_{\cup}L_2})$ and by $x_1 e_1 \in H^0(\mathcal{O}_{X_{\cup}L_2}(-l + 1))$. One deduces, as in case (a), the existence of an exact sequence:

$$
(B.7) \quad 0 \rightarrow S(L_2)(l'' - 1) \rightarrow H^0(\mathcal{O}_{X_{\cup}X'}) \rightarrow H^0(\mathcal{O}_{X_{\cup}L_2}) \rightarrow 0
$$

where the left morphism maps $1 \in S(L_2)$ to $x_2 e''_1 \in H^0(\mathcal{O}_{X_{\cup}X'}(-l'' + 1))$. One shows, as in Claim 1, that $x_1 e_1$ belongs to $H^0(\mathcal{O}_{X_{\cup}X'})$ and then, as in Claim 2, that $x_1 e_1$ belongs to $H^0(\mathcal{O}_{X_{\cup}X'})$. The assertion from case (b) of the statement is, now, a consequence of the exact sequence (B.7) and of Prop. B.12(a).

(c) In this case $\text{Im} \phi'' = \mathcal{O}_{L_2}(l'')$. One deduces, as in the previous cases, the existence of an exact sequence:

$$
(B.8) \quad 0 \rightarrow S(L_2)(l'') \rightarrow H^0(\mathcal{O}_{X_{\cup}X'}) \rightarrow H^0(\mathcal{O}_{X_{\cup}L_2}) \rightarrow 0 ,
$$

where the left morphism maps $1 \in S(L_2)$ to $e''_1 \in H^0(\mathcal{O}_{X_{\cup}X'}(-l''))$, and the fact that $x_1 e_1$ belongs to $H^0(\mathcal{O}_{X_{\cup}X'})$. The assertion from case (c) of the statement is, now, a consequence of the exact sequence (B.8) and of Prop. B.12(a).

(d) In this case $\text{Im} \phi'' = x_2 \mathcal{O}_{L_2}(l'' - 1)$. By Prop. B.12(b) the graded $S$-module $H^0(\mathcal{O}_{X_{\cup}L_2})$ is generated by $1 \in H^0(\mathcal{O}_{X_{\cup}L_2})$ and by $e_1 \in H^0(\mathcal{O}_{X_{\cup}L_2}(-l))$. One deduces, as in case (a), the existence of an exact sequence:

$$
(B.9) \quad 0 \rightarrow S(L_2)(l'' - 1) \rightarrow H^0(\mathcal{O}_{X_{\cup}X'}) \rightarrow H^0(\mathcal{O}_{X_{\cup}L_2}) \rightarrow 0
$$

where the left morphism maps $1 \in S(L_2)$ to $x_2 e''_1 \in H^0(\mathcal{O}_{X_{\cup}X'}(-l'' + 1))$. One shows, as in Claim 1, that $x_1 e_1$ belongs to $H^0(\mathcal{O}_{X_{\cup}X'})$.

**Claim 3.** $e_1 + \frac{b''(x_0, 0)}{a(x_0, 0)} e''_1$ belongs to $H^0(\mathcal{O}_{X_{\cup}X'})$.

**Indeed,** one has in $H^0(\mathcal{O}_X) \oplus H^0(\mathcal{O}_{X'})$:

$$
\begin{align*}
    x_3 \cdot 1 &= b e_1 + b'' e''_1 = b(x_0, 0) e_1 + b_1 x_1 e_1 + b''(x_0, 0) e''_1 + b''_1 x_1 e''_1 .
\end{align*}
$$
According to Ferrand [15] there exists an exact sequence:

\[ \to H^0(B.9), \text{of Prop. B.12(b) and of the following relations in } H^0(\mathcal{O}_{X\cup X'}) \]

The assertion from case (d) of the statement is, now, a consequence of the exact sequence

\[ L \to x \to \to 0 \]

and where

\[ L \to x \to \to 0 \]

is a line bundle on \( X \) hence:

\[ (C \cup \nu) \in H^0(\mathcal{O}_{X\cup X'}) \]

such that, if \( t_0, t_1 \) is the canonical basis of \( H^0(\mathcal{O}_{\mathbb{P}^1}) \) then:

\[ x_0 | C = t_0^2, \quad x_1 | C = t_0 t_1, \quad x_2 | C = t_1^2, \quad x_3 | C = 0. \]

One has to consider two cases:

(I) \( \mathcal{L} \simeq \mathcal{O}_C(l) \) with \( l \geq -2 \);  
(II) \( \mathcal{L} \simeq \mathcal{O}_C(l) \otimes \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \) with \( l \geq -1 \).
In case (I) one has an exact sequence:

\[ 0 \rightarrow \mathcal{O}_C(l) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_C \rightarrow 0 \]

and one denotes by \( e \in H^0(\mathcal{O}_D(-l)) \) the image of \( 1 \in H^0(\mathcal{O}_C) \), while in case (II) one has an exact sequence:

\[ 0 \rightarrow \mathcal{O}_C(l) \otimes \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_C \rightarrow 0 \]

and one denotes by \( e_i \in H^0(\mathcal{O}_D(-l)) \) the image of \( t_i \in H^0(\nu_* \mathcal{O}_{\mathbb{P}^1}(1)) \), \( i = 0, 1 \).

**Lemma C.1.** \( I(C^{(1)}) \) admits the following minimal graded free resolution:

\[
0 \rightarrow S(-4) \oplus S(-5) \rightarrow S(-2) \oplus S(-3) \oplus I(C^{(1)}) \rightarrow 0.
\]

**Proof.** Let \( H \subset \mathbb{P}^3 \) be the plane of equation \( x_3 = 0 \). One has an exact sequence:

\[
0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow 4\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0.
\]

**Lemma C.2.** \( \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \) admits the following resolution:

\[
0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow 4\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0.
\]

**Proof.** Let \( H \subset \mathbb{P}^3 \) be the plane of equation \( x_3 = 0 \). One has an exact sequence:

\[
0 \rightarrow 2\mathcal{O}_H(-1) \rightarrow 4\mathcal{O}_H \rightarrow 2\mathcal{O}_H \rightarrow \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0.
\]

One deduces that the tensor product of the complexes:

\[
2\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3}, \quad \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3}
\]

is a resolution of \( \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \).

We treat, firstly, case (I). In this case, there exist \( a \in k[x_0, x_1, x_2]_{l+1} \), \( b \in k[x_0, x_1, x_2]_{l+2} \) such that \( \alpha = a \mid C \) and \( \beta = b \mid C \).

**Lemma C.3.** The morphism \( \varepsilon : \mathcal{I}_C \rightarrow \mathcal{O}_C(l) \) defined by \( \alpha = a \mid C \) and \( \beta = b \mid C \) is an epimorphism if and only if \( a \) and \( b \) have no common zero on \( C \).
Proposition C.4. Keeping the previously introduced notation, the homogeneous ideal $I(D)$ of the closed subscheme $D$ of $\mathbb{P}^3$ defined by the kernel of an epimorphism $\varepsilon$ as in Lemma C.3 is generated by:

$$F := \begin{vmatrix} a & b \\ x_3 & q \end{vmatrix}, \ x_3^2, \ x_3q, \ q^2$$

and admits the following graded free resolution:

$$0 \to S(-l - 6) \xrightarrow{d_2} S(-l - 5) \oplus S(-l - 4) \xrightarrow{d_1} S(-l - 3) \oplus S(-l - 5) \xrightarrow{d_0} I(D) \to 0$$

with $d_1$ and $d_2$ defined by the matrices:

$$d_1 = \begin{pmatrix} x_3 & q & 0 & 0 \\ b & 0 & -q & 0 \\ -a & b & x_3 & -q \\ 0 & -a & 0 & x_3 \end{pmatrix}, \ d_2 = \begin{pmatrix} -q \\ x_3 \\ -b \\ a \end{pmatrix}.$$ 

This resolution is minimal for $l \geq 0$ but not for $l = -2$ where $I(D) = (F, q^2)$ and for $l = -1$ where $I(D) = (F, x_3^2)$.

Proof. The image of $F \in H^0(\mathcal{I}_C(l + 3))$ by the map:

$$\mathcal{I}_C(l + 3) \to (\mathcal{I}_C/\mathcal{I}_C^2)(l + 3) \simeq \mathcal{O}_C(l + 2) \oplus \mathcal{O}_C(l + 1)$$

is $(-\beta, \alpha)$. It follows that $F \in H^0(\mathcal{I}_D(l + 3))$ and that $\eta(F) = 1 \in H^0(\mathcal{O}_C)$. One deduces the exactness of the sequence:

$$0 \to I(C^{(1)}) \to I(D) \xrightarrow{H^0(\eta)} S(C)(-l - 3) \to 0$$

and then one uses Lemma C.1. \qed

Proposition C.5. Keeping the previously introduced notation, let $D$ be the closed subscheme of $\mathbb{P}^3$ defined by the kernel of an epimorphism $\varepsilon$ as in Lemma C.3. Then the graded $S$-module $H^0_*(\mathcal{O}_D)$ admits the following free resolution:

$$0 \to S(-3) \oplus S(-l - 3) \xrightarrow{\delta_2} S(-2) \oplus S(l - 1) \xrightarrow{\delta_1} S(-1) \oplus S(l) \xrightarrow{\delta_0} H^0_*(\mathcal{O}_D) \to 0$$

with $\delta_1$ and $\delta_2$ defined by the matrices:

$$\delta_1 = \begin{pmatrix} x_3 & q & 0 & 0 \\ b & 0 & -q & 0 \\ -a & b & x_3 & -q \\ 0 & -a & 0 & x_3 \end{pmatrix}, \ \delta_2 = \begin{pmatrix} -q \\ x_3 \\ -b \\ a \end{pmatrix}.$$ 

This resolution is minimal for $l \geq 0$ but not for $l = -2$ where $H^0_*(\mathcal{O}_D) = (F, q^2)$ and for $l = -1$ where $H^0_*(\mathcal{O}_D) = (F, x_3^2)$.
with \( \delta_0 = (1, e) \) and with \( \delta_1 \) and \( \delta_2 \) defined by the matrices:

\[
\begin{pmatrix}
x_3 & q & 0 & 0 \\
-\alpha & -b & x_3 & q
\end{pmatrix}, \quad \begin{pmatrix}
-q & 0 \\
x_3 & 0 \\
b & -q
\end{pmatrix}.
\]

This resolution is minimal for \( l \geq 0 \) but not for the special cases \( l = -2 \) and \( l = -1 \) where \( D \) is a complete intersection.

**Proof.** Since the graded \( S \)-module \( H^0_*(\mathcal{O}_C) \) is generated by \( 1 \in H^0_*(\mathcal{O}_C) \) one deduces the exactness of the sequence:

\[
0 \longrightarrow S(C)(l) \longrightarrow H^0_*(\mathcal{O}_D) \longrightarrow H^0_*(\mathcal{O}_C) \longrightarrow 0.
\]

It suffices, now, to use the following relations in \( H^0_*(\mathcal{O}_D) \):

\[
x_3 \cdot 1 = \varepsilon(x_3) = \alpha = ae, \quad q \cdot 1 = \varepsilon(q) = \beta = be.
\]

We treat, finally, case (II). In this case there exist \( a_0, a_1 \in k[x_0, x_1, x_2]_{l+1}, b_0, b_1 \in k[x_0, x_1, x_2]_{l+2} \) such that:

\[
\alpha = (a_0 | C)t_0 + (a_1 | C)t_1, \quad \beta = (b_0 | C)t_0 + (b_1 | C)t_1.
\]

**Lemma C.6.** The morphism \( \varepsilon : \mathcal{I}_C \to \mathcal{O}_C(l) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \) defined by \( \alpha = (a_0 | C)t_0 + (a_1 | C)t_1 \) and \( \beta = (b_0 | C)t_0 + (b_1 | C)t_1 \) is an epimorphism if and only if

\[
a_0x_0 + a_1x_1, \quad a_0x_1 + a_1x_2, \quad b_0x_0 + b_1x_1, \quad b_0x_1 + b_1x_2
\]

have no common zero on \( C \).

**Proof.** One uses the fact that \( \alpha \) and \( \beta \) have no common zero on \( C \) and is only if \( \alpha t_0, \alpha t_1, \beta t_0, \beta t_1 \) have no common zero on \( C \).\( \square \)

**Proposition C.7.** Keeping the previously introduced notation, the homogeneous ideal \( I(D) \) of the closed subscheme \( D \) of \( \mathbb{P}^3 \) defined by the kernel of an epimorphism \( \varepsilon \) as in Lemma C.6 is generated by:

\[
F_0 := \begin{vmatrix}
a_0x_0 + a_1x_1 & b_0x_0 + b_1x_1 \\
x_3 & q
\end{vmatrix}, \quad F_1 := \begin{vmatrix}
a_0x_1 + a_1x_2 & b_0x_1 + b_1x_2 \\
x_3 & q
\end{vmatrix}, \quad x_3^2, x_3q, q^2
\]

and admits the following graded free resolution:

\[
0 \to 2S(-l - 6) \xrightarrow{d_2} S(-4) \xrightarrow{d_1} I(D) \to 0,
\]

where

\[
\begin{align*}
4S(-l - 5) & \oplus 2S(-l - 4) \\
S(-5) & \oplus S(-4) \oplus S(-3) \oplus S(-2) \oplus d_0 I(D)
\end{align*}
\]
with \( d_1 \) and \( d_2 \) defined by the matrices:

\[
\begin{pmatrix}
-x_2 & -x_1 & x_3 & 0 & 0 & 0 \\
x_1 & x_0 & 0 & x_3 & 0 & 0 \\
0 & 0 & b_0x_0 + b_1x_1 & b_0x_1 + b_1x_2 & -q & 0 \\
-b_0 & b_1 & -a_0x_0 - a_1x_1 & -a_0x_1 - a_1x_2 & x_3 & -q \\
a_0 & -a_1 & 0 & 0 & 0 & x_3
\end{pmatrix},
\begin{pmatrix}
x_3 & 0 \\
0 & -x_3 \\
-x_2 & -x_1 \\
x_1 & x_0 \\
-b_0 & b_1 \\
a_0 & -a_1
\end{pmatrix}.
\]

This resolution is minimal for \( l \geq 0 \) but not for \( l = -1 \) where the minimal resolution has the form \( 0 \to S(-5) \to 4S(-4) \to S(-2) \oplus 3S(-3) \to I(D) \to 0 \).

**Proof.** Recall the exact sequence:

\[
0 \to \mathcal{O}_C(-l - 4) \otimes \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2) \to \mathcal{O}_C(l) \otimes \nu_* \mathcal{O}_{\mathbb{P}^1}(1) \to 0.
\]

The image of \( F_i \in H^0(\mathcal{I}_C(l + 4)) \) by the map:

\[
\mathcal{I}_C(l + 4) \to (\mathcal{I}_C/\mathcal{I}_C^2)(l + 4) \cong \mathcal{O}_C(l + 3) \oplus \mathcal{O}_C(l + 2)
\]

is \((-\beta t_i, \alpha t_i), i = 0, 1\). It follows that \( F_i \in H^0(\mathcal{I}_D(l + 4)) \) and that \( \eta(F_i) = t_i \in H^0(\nu_* \mathcal{O}_{\mathbb{P}^1}(1)), i = 0, 1\). One deduces the exactness of the sequence:

\[
0 \to I(C(1)) \to I(D) \xrightarrow{H^0(\eta)} H^0(\nu_* \mathcal{O}_{\mathbb{P}^1}(1))(-l - 4) \to 0
\]

and one uses Lemma C.1, Lemma C.2 and the following relations:

\[
-x_2F_0 + x_1F_1 = b_0x_0q - a_0q^2, 
-x_1F_0 + x_0F_1 = -b_1x_3q + a_1q^2, 
-x_3F_0 = -(b_0x_0 + b_1x_1)x_3 + (a_0x_0 + a_1x_1)x_3q, 
-x_3F_1 = -(b_0x_1 + b_1x_2)x_3 + (a_0x_1 + a_1x_2)x_3q.
\]

**Proposition C.8.** Keeping the previously introduced notation, let \( D \) be the closed subscheme of \( \mathbb{P}^3 \) defined by the kernel of an epimorphism \( \varepsilon \) as in Lemma C.4. Then the graded \( S \)-module \( H^0(\mathcal{O}_D) \) admits the following free resolution:

\[
\begin{align*}
0 & \to S(3) \\
& \oplus \to 2S(l - 2) \\
& \delta_2 \to S(-2) \\
& \oplus \to 2S(l) \\
& S(-1) \to \delta_1 \to S(0) \\
& \oplus \to 4S(l - 1)
\end{align*}
\]

\( \delta_0 : H^0(\mathcal{O}_D) \to 0 \)
with $\delta_0 = (1, e_0, e_1)$ and with $\delta_1$ and $\delta_2$ defined by the matrices:

$$
\begin{pmatrix}
  x_3 & q & 0 & 0 & 0 & 0 \\
- a_0 & - b_0 & - x_2 & - x_1 & x_3 & 0 \\
- a_1 & - b_1 & x_1 & x_0 & 0 & x_3
\end{pmatrix}
= \begin{pmatrix}
  -q & 0 & 0 \\
  x_3 & 0 & 0 \\
  a_0 x_0 + a_1 x_1 & - x_3 & 0 \\
- a_0 x_1 - a_1 x_2 & 0 & - x_3 \\
  b_0 & - x_2 & - x_1 \\
  b_1 & x_1 & x_0
\end{pmatrix}.
$$

This resolution is minimal for $l \geq 0$ but not for $l = -1$ where the minimal free resolution has the form $0 \to 3S(-3) \to 5S(-2) \to S \oplus S(-1) \to H^0_*(\mathscr{O}_D) \to 0$.

**Proof.** One uses the exact sequence:

$$0 \to H^0_*(\nu^*\mathscr{O}_{\mathbb{P}^1}(l)) \to H^0_*(\mathscr{O}_D) \to H^0_*(\mathscr{O}_C) \to 0$$

and Lemma [C.2] In order to determine the remaining entries of the matrix of $\delta_1$ one uses the following relations in $H^0_*(\mathscr{O}_D)$:

$$x_3 \cdot 1 = \varepsilon(x_3) = \alpha = a_0 e_0 + a_1 e_1, \quad q \cdot 1 = \varepsilon(q) = \beta = b_0 e_0 + b_1 e_1,$$

and in order to determine the remaining entries of $\delta_2$ one uses the matrix relations:

$$\begin{pmatrix}
  - x_2 & - x_1 \\
  x_1 & x_0
\end{pmatrix}
\begin{pmatrix}
  - x_0 & - x_1 \\
  x_1 & x_2
\end{pmatrix}
\begin{pmatrix}
  a_0 \\
  a_1
\end{pmatrix}
= \begin{pmatrix}
  q \\
  0
\end{pmatrix}
\begin{pmatrix}
  a_0 \\
  a_1
\end{pmatrix}
= \begin{pmatrix}
  qa_0 \\
  qa_1
\end{pmatrix}.
$$

\[ \square \]

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