On the strongly coupled heterotic string

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Abstract

We analyze in detail the anomaly cancellation conditions for the strongly coupled \(E_8 \times E_8\) heterotic string introduced by Horava and Witten and find new features compared to the ten-dimensional Green-Schwarz mechanism. We project onto ten dimensions the corresponding Lagrangian of the zero-mode fields. We find that it has a simple interpretation provided by the conjectured heterotic string/fivebrane duality. The part which originates from eleven-dimensions is naturally described in fivebrane language. We discuss physical couplings and scales in four dimensions.
1 Introduction

It was recently proposed [1] that the strong coupling limit of the $E_8 \times E_8$ heterotic string is described by eleven-dimensional supergravity [2] compactified on $S_1/Z_2$, with the two $E_8$ gauge groups propagating on the two boundaries. This exciting conjecture allows to hope for a better understanding of the nonperturbative regime of the ten and four dimensional heterotic superstring and opens new perspectives for particle physics string phenomenology.

In Section 2, we analyze the analog of the ten-dimensional Green-Schwarz anomaly cancellation mechanism in this context. An important new feature appears, namely the three-form field $C_{ABC}$, $A, B, C = 1 \cdots 10$, which has no zero modes in ten dimensions generically transforms under gauge transformations, in addition to the expected gauge transformation of $C_{11AB}$, the analog of the antisymmetric tensor field of weakly coupled theory.

In Sections 3-4, we project onto ten dimensions the Lagrangian describing the strongly coupled $E_8 \times E_8$ heterotic string. The result can be interpreted in two different ways, by using the membrane origin of the eleven-dimensional theory (membrane quantization condition):

i) in ten-dimensional string units, the resulting Lagrangian is similar to a mixture of tree-level and string one-loop terms.

ii) in natural $M$-theory units, which we argue to be the ten-dimensional fivebrane units, the Lagrangian is similar to a mixture of tree-level and one-loop fivebrane terms, with all terms originating from eleven-dimensions being tree-level.

The natural $M$-theory units are dual to the string units in the sense of strong/weak coupling duality. By compactifying from ten to nine dimensions on a circle we interpret the Kaluza-Klein states and states describing the wrapping of the membrane around the torus in the fivebrane language. By compactification from ten to four dimensions, in natural fivebrane units, we identify in Section 5 the physical couplings and scales and discuss the issue of string unification.

Finally some conclusions are drawn together with some comments.

2 Anomaly cancellation in strongly coupled $E_8 \times E_8$ heterotic string

The issue of anomaly cancellation was already discussed in [1], [3] and [4]. Due to the new features we encounter, we discuss in some detail the way it works and compare it to the ten-dimensional Green-Schwarz mechanism (GSM).
In the following we use the differential forms notations in eleven dimensions. For a \( p \)-form \( A \) we define components by the usual definition
\[
A = \frac{1}{p!} A_{I_1 \ldots I_p} dx^{I_1} \wedge \cdots \wedge dx^{I_p}.
\]
We use the upstairs formulation of \[3\], e.g. we work on \( M_{11} \) directly and impose a \( Z_2 \) symmetry on the Lagrangian. In most computations, we use formulae of the type
\[
d[e(x^{11})A] = 2\delta(x^{11})dx^{11}A + \epsilon(x^{11})dA ,
\]
where \( \epsilon(x^{11}) = 1 \) for \( x^{11} > 0 \) and \( -1 \) for \( x^{11} < 0 \) and \( A \) is an arbitrary form. It was shown in \[3\] that the Bianchi identity for the field strength of the three-form \( C \) appearing in eleven-dimensional SUGRA is (we are only interested in one of the two ten-dimensional pieces, around, say, \( x^{11} = 0 \))
\[
dG = a\delta(x^{11})dx^{11}\hat{I}_4 ,
\]
where we defined \( a \equiv k_{11}^2/\sqrt{2}\lambda^2 \) as a function of the eleven-dimensional gravitational coupling \( k_{11} \) and the gauge coupling \( \lambda \) and \( \hat{I}_4 = 1/2trR^2 - trF^2 \). The solution of \( (3) \) is to modify the definition of the field strength
\[
G = 6dC - a\delta(x^{11})dx^{11}Q_3 ,
\]
where \( Q_3 = 1/2\omega_{3L} - \omega_{3Y} \). The value of the restriction of \( G \) around the ten-dimensional manifold \( M_{10} \) is changed accordingly
\[
G| = \frac{a}{2}\epsilon(x^{11})\hat{I}_4 + \cdots ,
\]
where \( \cdots \) denote terms which vanish for \( x^{11} = 0 \). Compatibility of \( (3) \) and \( (2) \) gives the value of the restriction of the three-form on the ten-dimensional boundary:
\[
C| = \frac{a}{12}\epsilon(x^{11})Q_3 + d\Omega ,
\]
where \( \Omega \) is a two-form field which can be written, near \( M_{10} \), by \( Z_2 \) symmetry as \( \Omega = \epsilon(x^{11}) \Omega' \), with \( \Omega' \) a two-form living on \( M_{10} \). The term responsible for the anomaly cancellation is the Chern-Simons term \[2\]
\[
W = -\frac{\sqrt{2}}{k_{11}^2} \int_{M_{11}} C \wedge G \wedge G .
\]
The variation \( \delta C \) is found by imposing \( \delta G = 0 \) in \( (3) \). By using \( \delta Q_3 = dQ_2^1 \), where \( Q_2^1 \) is a two-form linear in the gauge transformation parameter, we find that the result can be parametrized by a free parameter \( \alpha \) and reads
\[
\delta C = \alpha \delta C_1 + (1 - \alpha)\delta C_0 + d\Lambda_\alpha ,
\]
where \( \Lambda_\alpha \) is a two-form and we defined
\[
\delta C_1 \equiv \frac{a}{12}\epsilon(x^{11})dQ_2^1 , \quad \delta C_0 \equiv -\frac{a}{6}\delta(x^{11})dx^{11}Q_2^1 .
\]
Identifying the gauge transformation of (3) with the restriction of (7) to $M_{10}$ we find that
\[
\delta \Omega' = - \frac{a}{12} (1 - \alpha) Q_2^1 + \Lambda_\alpha \, .
\] (9)

Notice that $W$ is invariant under the usual three-form gauge transformations \( \delta C = d\Lambda_\alpha \) only if $\Lambda_\alpha = 0$ on $M_{10}$, which will be assumed in the following. It is interesting that, even by putting $\Lambda_\alpha = 0$, $\delta C$ depends on $\alpha$ as a total derivative
\[
\delta C = \delta C_0 + \frac{\alpha a}{12} \epsilon (x^{11}) Q_2^1 = \delta C_1 - \frac{a}{12} (1 - \alpha) d[\epsilon (x^{11}) Q_2^1] \, .
\] (10)

However, the two-form in the bracket does not vanish on $M_{10}$ and therefore, as (6) is not invariant under the 3-form gauge transformations, the value of the anomaly depends on $\alpha$. The gauge variation of $W$ can be split into two pieces:
\[
\delta W = \alpha \delta_1 W + (1 - \alpha) \delta_0 W \, .
\] (11)

By using (2), (4) and (8) we find immediately
\[
\delta_0 W = \frac{\sqrt{2} a^3}{24 k_{11}^2} \int_{M_{10}} Q_2^1 \wedge \hat{I}_4 \wedge \hat{I}_4 \, .
\] (12)

In order to find $\delta_1 W$ we insert (8) into (6), integrate by parts, use the formulae (1), (2) and (4). The result is
\[
\delta_1 W = \frac{\sqrt{2} a^3}{8 k_{11}^2} \int_{M_{10}} Q_2^1 \wedge \hat{I}_4 \wedge \hat{I}_4 \, .
\] (13)

On the other hand, the ten-dimensional $E_8 \times E_8$ anomaly is (see, for example, [4])
\[
\delta \Gamma = \frac{1}{48 (2\pi)^5} \int_{M_{10}} Q_2^1 \wedge \left[ \frac{1}{4} \hat{I}_4 \wedge \hat{I}_4 - X_8 \right] \, ,
\] (14)

where $X_8 \equiv - \frac{1}{8} tr R^4 + \frac{1}{32} (tr R^2)^2$. By comparison of (11) and (14) we find that the $\hat{I}_4 \wedge \hat{I}_4$ part of the anomaly is cancelled provided the following relation holds
\[
\frac{k_{11}^4}{\lambda^6} = - \frac{1}{4(1 + 2\alpha)(2\pi)^5} \, .
\] (15)

The remaining anomaly cancellation can be achieved from another term in the $M$ theory Lagrangian [3], which can be viewed as a term cancelling fivebrane worldvolume anomalies [4] or, by compactifying one dimension, as a one-loop term in the type $IIA$ superstring [8]. In our conventions, it reads
\[
W_5 = - \frac{T_3}{2\sqrt{2}(2\pi)^4} \int_{M_{11}} C \wedge X_8 \, ,
\] (16)
where $T_3$ is the membrane tension related to $\alpha'$ by $T_2 = 2\pi \alpha'^{1/2} T_3 = \frac{1}{2\pi \alpha'}$. We use the membrane quantization condition \cite{7, 4, 2}:

$$\frac{(2\pi)^2}{k_{11}^2 T_3} = 2m, \quad m = \text{integer or halfinteger}.$$ (17)

The gauge and Lorentz variation of $W_5$ is then easily computed to be

$$\delta W_5 = \frac{a\sqrt{2}}{24(2\pi)^{10/3}(2mk_{11}^2)^{1/3}} \int_{M_{10}} Q_2^1 \wedge X_8.$$ (18)

Notice that, due to (10) and $dtrR^2 = dtrR^4 = 0$, $\delta W_5$ is independent of $\alpha$. The result is that $\delta W + \delta W_5 + \delta \Gamma = 0$ if the gauge and the eleven-dimensional gravitational couplings are related by the relation

$$m^{1/3} \lambda^2 = 2\pi (4\pi k_{11}^2)^{2/3}.$$ (19)

The final step is to compare (15) with (19). We find that the gauge anomaly is completely cancelled if the membrane quantization integer (or half-integer) parameter $m$ is related to $\alpha$ by

$$\alpha = -\frac{1}{2}(1 + \frac{1}{m}).$$ (20)

For example, $m = 1$ implies $\alpha = -1$. The relation (19) has a physical meaning only for $m > 0$. A puzzling result is that the value $\alpha = 0$, which is the most natural value from a weak coupling ten-dimensional string point of view is only obtained for the unphysical value $m = -1$. To be more precise, we rewrite the gauge transformations (7) in component fields (with $Q_2^1 = -tr\epsilon F$, $\epsilon$ being the gauge transformation parameter):

$$\delta C_{ABC} = -\alpha \frac{a}{12} \epsilon^{(x^{11})} [\partial_A (tr\epsilon F_{BC}) \pm 2 \text{perm.}],$$ (21)

$$\delta C_{11AB} = (1 - \alpha) \frac{a}{6} \delta(x^{11}) tr\epsilon F_{AB},$$ (22)

where $A, B, C$ are ten-dimensional indices. The contribution of $\delta C_{ABC}$ to the gauge anomaly has no weak coupling heterotic string interpretation and would disappear only for $\alpha = 0$. Precisely in this case we could identify $C_{11AB}$ with the ten-dimensional string antisymmetric tensor field $B_{AB}$. However, this corresponds, by using (19) to an imaginary gauge coupling.

A more general framework for comparing with the weakly coupled GSM consists of using a more general modification of the four-form field strength compatible with the Bianchi identity (3)

$$G = 6dC - a\beta \delta(x^{11}) dx^{11} Q_3 + \frac{a}{2} (1 - \beta) \epsilon(x^{11}) \hat{I}_4,$$ (23)
where $\beta$ is an arbitrary real parameter. In this case, the eqs. (19) and (20) become
\[ m^{1/3} \lambda^2 = 2\pi \beta (4\pi k_{11}^2)^{2/3}, \quad \alpha = \frac{1}{2} \left( 1 + \frac{\beta^2}{m} \right). \] (24)

The case corresponding to the weakly coupled GSM corresponds to the set of values $\alpha = 0, \beta = -1, m = -1$. This more general approach induces additional corrections to the SUSY transformation law for $C$:
\[ \tilde{\delta}C = -\frac{a}{12} \beta \alpha \epsilon(x) d(tr A \tilde{A}) + \frac{a}{6} \beta (1 - \alpha) \delta(x) dx^{11} tr A \tilde{A}, \] (25)
which imply an additional correction to the SUSY transformation law for $G$ compared to \[3\]. It reads
\[ \tilde{\delta}G = a \left[ 2\beta \delta(x) dx^{11} tr (F \tilde{A}) - (1 - \beta) \epsilon(x) tr (F \tilde{F}) \right], \] (26)
and this asks for additional terms in the fermionic part of the Lagrangian in \[3\] in order to preserve supersymmetry. This is a viable alternative, but we will not pursue further this observation here. Therefore, it seems that even for $\beta = 1$ the anomaly cancellation is essentially different in strong coupling regime of the heterotic string, which is a puzzle in our usual understanding of anomalies.

Our conclusions for the minimal choice $\beta = 1$ in (24) differ from that in \[3\]. We find that $C_{ABC}, A, B, C = 1 \cdots 10$ must transform under gauge transformations for a real gauge coupling. Correspondingly, the GSM does not directly reduce to the ten-dimensional case. Comparing with the downstair (manifold with boundary) approach, it is interesting to note that the gauge transformations (7), imposed by the modification of the corresponding field strength, are different from those of \[4\]. The downstair gauge transformations of \[4\] are obtained for $\Omega = 0$ in \[3\], but this choice is incompatible with \[3\]. This conclusion holds in the more general case described in (23).

### 3 Strongly coupled $E_8 \times E_8$ heterotic string Lagrangian viewed from ten dimensions

We try now to interpret different terms of the heterotic Lagrangian in the strongly coupled regime in a heterotic weak coupling language, by using the membrane quantization condition (17), which allows us to relate the string slope $\alpha'$ to $k_{11}$ by \[3\]
\[ \alpha' = \left[ \frac{2mk_{11}^2}{(2\pi)^8} \right]^{\frac{1}{3}}. \] (27)

\footnote{We can recover our results in the downstair approach by using $C| = \frac{1}{12} Q_3 + d\Omega$, (4), and using the integration by part formula $\int_{M_{11}} d\omega = \int_{x^{11}=\pi} \omega - \int_{x^{11}=0} \omega$.}
We certainly do not want to propose to reduce the \( M \)-theory physics to a ten-dimensional physics, just to show that projecting the Horava-Witten Lagrangian onto ten-dimensions we can naturally interpret the result in terms of heterotic weak coupling variables. In the following we only consider zero modes of the fields in a Kaluza-Klein decomposition, which are interpreted as the usual ten-dimensional heterotic fields in the weak coupling description. For this, we develop eleven-dimensional fields in a Fourier expansion, for example

\[
C_{11AB}(x^{11}, x) = \sum_{n=0}^{\infty} \frac{\cos nx^{11}}{\alpha'^{-1/2}} C^{(n)}_{11AB}(x) ,
\]

where \( x = x_1 \cdots x_{10} \) and keep into account only \( C^{(0)} \). Similarly, we expand

\[
\delta(x^{11}) = \frac{1}{2\pi\alpha'^{1/2}} + \frac{1}{\pi\alpha'^{1/2}} \sum_{n=1}^{\infty} \frac{\cos nx^{11}}{\alpha'^{1/2}} .
\]

Then, by using (19), (27), (28), (29) we can write the zero-mode part of eq. (3) as (we set \( \omega_{3L} = 0 \) in the following, as all our considerations below can be trivially generalized by the inclusion of higher-order gravitational terms)

\[
G^{(0)} = 6dC^{(0)} + \frac{\alpha'}{2\sqrt{2}} dx^{11} \omega_{3Y} ,
\]

which is similar to the ten-dimensional relation \( H = dB - \frac{\alpha'}{2} \omega_{3Y} \). Notice that the membrane quantization parameter \( m \) cancelled out in the final expression.

The bosonic part of the strongly coupled \( E_8 \times E_8 \) heterotic string Lagrangian is

\[
L = \frac{1}{\kappa_{11}^2} \int_{M^{11}} d^{11}x \sqrt{g} \left( -\frac{1}{2} R^{(11)} - \frac{1}{48} G_{IJKL} G^{IJKL} \right) - \frac{\sqrt{2}}{3456\pi_2} \int_{M^{11}} d^{11}x \epsilon^{I_1 \cdots I_{11}} C_{I_1 I_2 I_3} G_{I_4 \cdots I_7} G_{I_8 \cdots I_11} - \frac{1}{X^2} \int_{M^{10}} d^{10}x \sqrt{g} \frac{1}{4} tr F_{AB} F^{AB} ,
\]

where \( I, J, K, L = 1 \ldots 11 \).

We will be mainly interested in the projection of this Lagrangian onto one of its two boundaries, let’s say \( x^{11} = 0 \). The physical ten-dimensional bosonic fields, which are invariant under the \( Z_2 \) orbifold projection are \( g_{11,11}, g_{AB}, A_B \) and \( C_{11AD} \). Moreover, by considerations explained in [10], in the ten-dimensional string metric \( g_{st,AB} = e^{\frac{2\phi}{3}} g_{AB} \) we have \( g_{11,11} = e^{4\phi/3} \), where \( \phi \) is the heterotic string dilaton. In the following, all the fields with an eleven-dimensional origin mean implicitly the corresponding zero-modes part. For the zero mode of \( G_{ABCD} \) we use (4). We neglect in the following the Green-Schwarz gravitational anomaly terms which are irrelevant for our purposes.
The result in the string metric is

\[
L^{(10)} = \frac{2\pi \alpha^\prime}{\kappa^2} \int_{M^{10}} d^{10}x \sqrt{g_{st}} e^{-2\phi} \left[ -\frac{1}{2} R^{(10)} - 4(\partial \phi)^2 - \frac{1}{12} G_{11ABC}^2 \right. \\
- \frac{3 \kappa^{4/3}}{8(4 \pi)^10/3} e^{2\phi} (F^2)^2 - \frac{\sqrt{2} \kappa^{4/3} m^{2/3}}{64 (4 \pi)^10/3} e^{2\phi} C_{11} F^2 F^2 \\
- \frac{m^{1/3}}{2 \pi (4 \pi \kappa^2)^{2/3}} \int_{M^{10}} d^{10}x \sqrt{g_{st}} \left( \frac{1}{4} e^{-2\phi} trF_{AB} F^{AB} \right),
\]

(32)

where we defined \((F^2)^2 \equiv F_{[AB} F^a_{CD]} F^{b,[AB} F^{b,CD]}\) and \(C_{11} F^2 F^2 \equiv \epsilon^{A_1 \ldots A_{10}} C_{11A_1 A_2} F^a_{[A_3 A_4} F^a_{A_5 A_6]} F^b_{[A_7 A_8} F^b_{A_9 A_{10}]}\). The Lagrangian \(L^{(10)}\) can be written in a more transparent way by using heterotic string notations. We define the ten-dimensional Newton constant by \(k_{11}^2 = 2\pi \alpha^\prime / k_{10}^2\) and use (27). Then \(L^{(10)}\) reads

\[
L^{(10)} = \frac{1}{2 \kappa^2} \int_{M^{10}} d^{10}x \sqrt{g_{st}} e^{-2\phi} \left[ -R^{(10)} - 4(\partial \phi)^2 - \frac{1}{6} G_{11ABC}^2 - \frac{\alpha^\prime}{4} trF_{AB} F^{AB} \\
- \frac{3 \kappa^2}{32(2 \pi)^5} e^{2\phi} (F^2)^2 \right] - \frac{m}{2569 \sqrt{2}(2 \pi)^5} \kappa^{1/3} \int_{M^{10}} d^{10}x C_{11} F^2 F^2.
\]

(33)

This expression is very similar to a weakly-coupled heterotic string Lagrangian containing a mixture of tree-level and one-loop terms, where the one-loop terms in (33) have an additional factor of \(e^{2\phi}\) compared to the tree-level terms. By the identification \(6 \sqrt{2} C_{11}^{(0)} = B_{AB}, \sqrt{2} C_{11}^{(0)} = H_{ABC}\), needed in order to identify (30) with the string relation \(H = dB - \frac{\alpha^\prime}{2} \omega_{3Y}\), we get the usual Green-Schwarz term for \(m = -1\), in agreement with the conclusions of the preceding paragraph.

The analogy with a weakly-coupled heterotic Lagrangian becomes stronger by looking more detail at the one-loop terms of the type \(F^4\) in the ten-dimensional \(E_8 \times E_8\) heterotic string. They were computed long-time ago in [11] and were found to be proportional to \(\frac{1}{(2\pi)^{10}} t^{ABCDEFGH} trF_{AB} F_{CD} F_{EF} F_{GH}\), where the tensor \(t^{ABCDEFGH}\) is defined by the expression

\[
t^{ABCDEFGH} = \frac{1}{2} (g^{AC} g^{BD} - g^{AD} g^{BC}) (g^{EG} g^{FH} - g^{EH} g^{FG}) \\
+ \frac{1}{2} (g^{CE} g^{DF} - g^{CF} g^{DE}) (g^{GA} g^{HB} - g^{GB} g^{HA}) - \frac{1}{2} (g^{AE} g^{BF} - g^{AF} g^{BE}) (g^{CG} g^{DH} - g^{CH} g^{DG}) \\
+ \frac{1}{2} (g^{BC} g^{DE} g^{GF} g^{HA} + g^{BE} g^{FC} g^{DG} g^{HA} + g^{BE} g^{FG} g^{CH} g^{DA} + \text{perm.}).
\]

(34)

By a rather straightforward algebra one can prove the following formula

\[
t^{ABCDEFGH} trF_{AB} F_{CD} F_{EF} F_{GH} = \frac{1}{100} t^{ABCDEFGH} (trF_{AB} F_{CD})(trF_{EF} F_{GH})
\]
\[ \frac{1}{25} \left[ -(TrF^A F^C D)(TrF_A F_C D) + 2(TrF^A B F^C D)(TrF_B C F_D A) \right] \]
\[ + \frac{1}{50} \left[ (-TrF^A B F_A B)(TrF^C D F_C D) + 2(TrF^A B F_B C)(TrF^C D F_D A) \right], \]
(35)

where \( Tr \) is the trace in the adjoint representation (which is also the fundamental representation) of \( E_8 \). On the other hand, the \( F^4 \) term appearing in (33) can be rewritten as

\[ 9(F^2)^2 = 3(TrF_A B F_C D)(TrF^A B F^C D) + 6(TrF_A B F_C D)TrF^A D F^B C. \]
(36)

Comparing (35) and (36) we conclude that the \( F^4 \) term appearing in the strongly coupled Horava-Witten heterotic Lagrangian projected onto ten-dimensions is indeed similar to a one-loop effect in the weakly-coupled heterotic string, but the numerical factor in (33) is (for \( m = 1 \)) three times bigger. Still, it is interesting that the combination in eq. (36) already appears in (35). Actually, a certainly better way to define a ten-dimensional Lagrangian is to integrate over the Kaluza-Klein modes. This could perhaps reproduce correctly the one-loop result, but this is beyond the goal of this letter (see for ex. [21] for an integration from five to four dimensions in the same context). Consequently, from a weak-coupling point of view, the projected Lagrangian is similar to a mixture of tree-level and one-loop terms, but with value of the couplings renormalized compared to their weak-coupling values. Notice, however, that all this picture is rather formal because we are in a strongly coupled regime for the heterotic string where a perturbative Lagrangian in the string coupling does not have too much sense. In particular, we believe that it is not justified to further compactify to four-dimensions in string units in order to define four-dimensional couplings and scales. A better picture is obtained by going to the dual fivebrane picture to be analyzed in the next paragraph.

4 Fivebrane picture

It is well known that the eleven-dimensional supergravity can be obtained from the worldvolume action of the eleven-dimensional supermembrane by imposing the Kappa symmetry [13], giving a possible membrane origin of M-theory. On the other hand, there are arguments by Duff et al. [4] in favor of a membrane/fivebrane duality in eleven dimensions. In this section we argue that the part of \( L^{(10)} \) originating from eleven-dimensions is naturally described in ten-dimensional fivebrane units by using arguments very similar to that used by Duff and coll. [4] in order to support the heterotic / fivebrane duality in ten dimensions. Indeed, our form for \( L^{(10)} \) is very close to the low-energy fivebrane

\[ ^2 \text{This would be important for the proposal made in [2]. We thank S.P. de Alwis for this remark} \]
Lagrangian of Duff and coll., except that the $F^4$ terms appearing in (33) are only a part of the terms conjectured in [14]. We therefore follow closely their analysis trivially adapted to our case. The fivebrane tension $\beta'$ is related to the string tension by the relation

$$ m(2\pi)^5\alpha'\beta' = 2k_{10}^2. $$

By using the membrane quantization condition (17) we get the analog of (27) for the fivebrane tension

$$ \beta' = \left[ \frac{2k_{11}^2}{(2\pi)^5m^2} \right]^{\frac{1}{3}}. $$

By performing a Weyl transformation $g_{st} = e^{2\phi}g_5$, where $g_5$ is the ten-dimensional fivebrane metric and performing the Hodge duality $K = \sqrt{2}e^{-\phi}G_{11}$, with $(G_{11})_{ABC} \equiv G_{11ABC}$, we get the Lagrangian

$$ L^{(10)} = \frac{1}{2\kappa_{10}^2} \int_{M^{10}} d^{10}x \sqrt{g_5} e^{2\phi} \left[ -R^{(10)} - \frac{1}{12}K^2 - \frac{3m^2\beta'}{64}(F^2)^2 - \frac{k_{10}^2}{2(2\pi)^5m\beta'} e^{-\frac{2\phi}{3}} trF_{AB}F^{AB} \right]. $$

The tree-level field equations for the three-form, written in form language

$$ d^*G = -\frac{1}{\sqrt{2}} G \wedge G, $$

after projection onto ten-dimensions and use of (14), (19) and (38) read, in fivebrane units

$$ dK = -m^2\beta' trF^4 $$

and become the Bianchi identity for $K$. Notice that all the terms with an eleven-dimensional origin in (33) are tree-level if the fivebrane coupling is $\lambda_5 = e^{-\frac{\phi}{3}}$. A notable exception is the usual Yang-Mills term $F^2$ (of a ten dimensional origin), which is now one-loop (whatever that means for a five-brane). Also, $g_5$ is actually the natural metric for the M-theory. More precisely, the fivebrane units in which the dilaton has no kinetic term are obtained from the eleven dimensional supergravity in the natural units

$$ g_{11} = e^{4\phi}, g_{AB} = g_{AB}^{(11)}. $$

An additional argument for heterotic/fivebrane duality can be obtained by compactifying one additional coordinate, $x_{10}$. As argued in [4], in the M-theory metric the masses of states in the compactified theory are given by

$$ M^2 = \frac{l^2}{R_{11}^2} + \frac{m^2}{R_{10}^2} + n^2 R_{10}^2 R_{11}^2, $$

where $l, m$ are Kaluza-Klein modes and $n$ describes the wrapping of the membrane around the two-torus. These states should have an interpretation in the heterotic
and in the fivebrane picture. In the string metric, they become

$$M_h^2 = \frac{l^2}{\lambda_{E_8}^2} + \frac{m^2}{R_{E_8}^2} + n^2 R_{E_8}^2,$$

(43)

where $R_{11} = \lambda_{E_8}^{2/3}$ and $R_{10} = \frac{R_{E_8}}{\lambda_{E_8}^{1/3}}$. In the fivebrane metric, we get analogously

$$M_5^2 = l^2 \lambda_5^4 + \frac{m^2}{R_5^2} + n^2 R_5^2 \lambda_5^2,$$

(44)

where $R_5 = R_{10} = R_{E_8}/\lambda_{E_8}^{1/3}$ and $\lambda_5 = \lambda_{E_8}^{-1/3}$. The natural interpretation of these expressions is that nonperturbative string states labeled by $l$ become perturbative in fivebrane picture. The string winding states $n$ become windings of the string solitonic solution around the compactified torus and Kaluza-Klein $m$ states keep their interpretation in the dual picture.

The Lagrangian (39) is in the perturbative regime when the heterotic string is strongly coupled and can eventually be compactified to four dimensions.

5 Couplings and scales in four dimensions

We now compactify to four dimensions in order to define physical quantities relevant for phenomenology. A special attention will be paid for writing four-dimensional relations in the appropriate metric for the weak and strong coupling respectively and for using four-dimensional fields $S$ and $T$ defined in the Witten truncation [15]. We follow the conventions and notations of [16]. For the weakly-coupled heterotic string, the ten-dimensional terms we are interested in then read

$$L_h^{10} = - \int d^{10}x \sqrt{g_{st}} e^{-2\phi} \left[ \frac{4}{\alpha' R} + \frac{1}{\alpha' \delta} tr F^2 + \cdots \right].$$

(45)

In the following, six-dimensional compact indices will be denoted by latin letters $i, j$ and four-dimensional indices by greek letters $\mu, \nu$. Four dimensional string units are obtained by setting

$$g_{ij}^{10} = e^\sigma \delta_{ij}, \quad g_{\mu\nu}^{10} = g_{\mu\nu}.$$

(46)

By defining the grand unification scale $M_{GUT}$ by $V_6^{-1} = M_{GUT}^6$, where $V_6$ is the volume of the compactified space, we get for the Newton constant $G_N$ and GUT coupling $\alpha_{GUT}$ in four dimensions:

$$16\pi G_N = \frac{t^3}{4s} \alpha'^4 M_{GUT}^6, \quad 16\pi \alpha_{GUT} = \frac{t^3}{s} \alpha'^3 M_{GUT}^6.$$

(47)

\footnotetext[3]{The following considerations apply in the lowest order in $l_{11}^{2/3}$, in which case we can approximate the compactified space by $K \times S_1/Z_2$, where $K$ is a Calabi-Yau manifold.}
where \( s = e^{-2\phi}e^{3\sigma}, t = e^{\sigma}. \) By combining the two equations, we get the well known relation

\[
G_N = \frac{1}{4} \alpha' \alpha_{GUT}.
\]  

(48)

In the weak-coupling regime \( e^{2\phi} < 1 \) which, by using (47) and (48) reads

\[
G_N > \left( \frac{\pi}{4} \right)^{1/3} \frac{\alpha_{GUT}^{4/3}}{M_{GUT}^2},
\]

(49)

which is too large and gives rise to the so-called string unification problem [17], [18].

The situation is different for the strongly-coupled regime. It is true that, viewed from ten dimensions the Lagrangian (33) is formally similar to a perturbative string one. However, due to the strong coupling it is more natural to work in the dual fivebrane coordinates. For notational convenience, we still use \( \alpha' \) instead of \( \beta' \) in our expressions. The relevant terms in the ten-dimensional Lagrangian are

\[
L_{10}^5 = -\int d^{10}x \sqrt{g_5} e^{\frac{2\phi}{3}} \left[ \frac{4}{\alpha'} R + e^{-\frac{2\phi}{3}} \frac{1}{\alpha'^2} tr F^2 + \cdots \right].
\]

(50)

The four-dimensional fivebrane units are obtained by setting [19]

\[
g_{5,ij}^{10} = e^\sigma \delta ij , \ g_{5,\mu\nu}^{10} = e^{-2\sigma} g_{\mu\nu},
\]

(51)

and the four-dimensional \( Re S \) and \( Re T \) fields are defined by

\[
s = e^{3\sigma} , \ t = e^\sigma e^{\frac{2\phi}{3}}.
\]

(52)

In this case, by compactifying (50) to four dimensions we get

\[
16\pi G_N = \frac{s}{4t} \alpha'^4 M_{GUT}^6 , \ 16\pi \alpha_{GUT} = \alpha'^3 M_{GUT}^6.
\]

(53)

By combining them, we get the analog of (48) and (49)

\[
G_N = \frac{s}{4t} \alpha' \alpha_{GUT} = \left( \frac{\pi}{4} \right)^{1/3} \frac{s^{4/3} \alpha_{GUT}^{1/3}}{t M_{GUT}^2}.
\]

(54)

The four-dimensional fivebrane coupling is \( t^{-1/2} \) which means \( t \) has large values in fivebrane weak coupling / string strong coupling regime. Clearly this is exactly what we need in (54) in order to accommodate the small observed value of the Newton constant \( G_N \). Notice that the threshold associated with the eleventh dimension

\[
M_{11} \equiv \frac{1}{R_{11}} = \frac{s^{1/3}}{t} << M_{GUT} = e^{\frac{\phi}{3}} = \frac{1}{s^{1/6}}
\]

(55)
is indeed lower than the unification scale $M_{GUT}$. At that point we would like to comment on different relations obtained in the literature. Witten’s relation \cite{Witten} reads

$$G_N = \left( \frac{\pi}{4} \right)^{1/3} e^{\frac{24}{\alpha} \frac{\alpha_{GUT}^{4/3}}{M_{GUT}^2}}$$  \hspace{1cm} (56)$$

and is obtained in our case by using ten-dimensional fivebrane units, which are the natural units for $M$-theory, as we argued in the preceding paragraph. However, the four-dimensional fivebrane units obtained through the Weyl rescaling (51) are probably more natural to use and lead to (54). The analysis performed in \cite{work} uses a string metric. The corresponding ten-dimensional Lagrangian is strongly coupled and cannot be, in our opinion, used to derive four-dimensional relations.

Different approaches attempting to define four-dimensional physical quantities have been discussed in detail in the literature and can be found in \cite{20}, \cite{21}.

### 6 Conclusions

The goal of this letter is to give a weak-coupling interpretation of different terms in the strongly coupled $E_8 \times E_8$ heterotic Lagrangian by projecting onto ten-dimensions and keeping only zero modes. This is possible by trading $M$-theory variables in terms of heterotic string variables by using the membrane quantization condition. The result is similar to a mixture of string tree-level and one-loop terms. There are two distinct one-loop terms. The first is the usual Green-Schwarz term, provided the anomaly cancellation mechanism at strong coupling reduces to the usual one at weak coupling. This is possible only with a non-minimal modification of the field strength of the three-form $C$ of the eleven-dimensional SUGRA ($\alpha = 0, \beta = m = -1$ with the notations of Section 2). The second term has algebraically the required structure to be interpreted as part of the heterotic one-loop $F^4$ corrections.

In strong coupling, the Lagrangian is better expressed in dual, fivebrane units. We argue that these units are more convenient for further compactification and for defining four-dimensional couplings and scales.

There are certainly many issues which deserve further investigation. One of them is to consistently integrate out the massive Kaluza-Klein modes, which give certainly additional contributions in the Lagrangian describing the zero-modes. Another issue is the clarification of the role played by the ten-dimensional antisymmetric tensor field in strong coupling regime.

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