Ultrasonic cavity solitons

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Abstract – We report on a new type of localized structure, an ultrasonic cavity soliton, supported by large aspect-ratio acoustic resonators containing viscous media. These states of the acoustic and thermal fields are robust structures, existing whenever a spatially uniform solution and a periodic pattern coexist. Direct proof of their existence is given both through the numerical integration of the model and through the analysis and numerical integration of a generalized Swift-Hohenberg equation, derived from the microscopic equations under conditions close to nascent bistability. An analytical solution for the ultrasonic cavity soliton is given.

Many systems in nature, when driven far from equilibrium, can self-organize giving rise to a large variety of patterns or structures. Although studied intensively for most of the last century, it has only been during the past thirty years that pattern formation has emerged as a specific branch of science [1]. One of the most relevant features of pattern formation is its universality: systems with different microscopic descriptions frequently exhibit similar patterns on a macroscopic level. This universal character of pattern formation is evidenced when the microscopic models are reduced, under given assumptions, to simpler equations describing the evolution of a single variable, the so-called order parameter [2]. Order parameter equations are usually obtained near some critical point, and belong to few and well-known classes, such as the Ginzburg-Landau or the Swift-Hohenberg equations and their variants. They are often based on system symmetries and are independent of the microscopic differences among systems, providing a theoretical framework to understand the origins of non-equilibrium pattern formation [1].

This approach to pattern formation has been extensively applied to nonlinear optical cavities [3–5], such as lasers, optical parametric oscillators or Kerr (cubically nonlinear) resonators, where light in the transverse plane of the cavity has been shown to develop patterns with different symmetries (rolls, hexagons, and also quasi-patterns) as well as cavity solitons (CS). The latter correspond to localized solutions often resulting from a bistability between two stationary, spatially extended states of the system.

However, despite the existing analogies between optics and acoustics, acoustical resonators in the nonlinear regime have been much less explored than their optical counterparts. Furthermore, pattern formation studies in acoustics are almost lacking. The main reason lies in the weak dispersion of sound in common homogeneous media, which is responsible for the growth of higher harmonics, leading to wave distortion and shock formation. These effects are absent in optics, which is dispersive in nature. However, in some special cases it is still possible to avoid the nonlinear distortion and recover the analogies [6]. It is, for example, the case of sound beams propagating in viscous media characterized by a strong absorption (e.g. glycerine), where sound velocity depends on fluid temperature, resulting in an additional nonlinearity mechanism of thermal origin. For the majority of fluids, temperature variations induced by an intense acoustic field result in a decrease of sound velocity, leading to a self-focusing of the beam. In the case of viscous fluids the characteristic length of self-focusing effects is much shorter than the effect corresponding to the development of shock waves [7]. Also, high-frequency components are strongly absorbed in such media, so in practice the use of a quasimonochromatic (optical) description for wave propagation is justified.

In a viscous medium sound propagates with a speed \( c \) that depends significantly on temperature, \( c = c_0(1 - \sigma T) \), where \( c_0 \) is the speed of sound at some
equilibrium (ambient) temperature, $T'$ denotes the variation of the medium temperature from that equilibrium due to the intense acoustic wave, and $\sigma$ is the parameter of thermal nonlinearity. The propagation of sound in such a medium has been shown [7,8] to be described in terms of two coupled equations for pressure, $p'$, and temperature, $T'$, deviations. These equations have been used to address problems such as self-focusing and self-transparency of sound [7]. They have been also the basis for the analysis of temporal dynamic phenomena in acoustic resonators [9]. In this case, a viscous fluid is bounded by two flat and parallel reflecting surfaces. One of the surfaces, vibrating at a frequency $f$, is an ultrasonic source providing the external forcing. Previous studies on this system [9,10] have reported, in the frame of the plane-wave approximation, bistability and complex temporal dynamics, in good agreement with the corresponding experiments. In this letter we extend the previous model by considering the effects of sound diffraction and temperature diffusion in a large aperture resonator. These effects, which are responsible of the spatial coupling, can play an important role when the Fresnel number of the resonator $F = l^2/\lambda L \gg 1$ (being $l$ and $L$ its transverse and longitudinal dimensions, respectively).

We model the resonator as usual [9,10]: the intracavity pressure field is decomposed into two counterpropagating traveling waves, $p' = p_+ e^{i(\omega t-kz)} + p_- e^{i(\omega t+kz)} + \text{c.c.}$, where $t$ is time, $z$ is the axial (propagation) coordinate, $\omega = 2\pi f$, and $k = \omega/c_0$, whose complex amplitudes $p_\pm$ are related through their reflections at the boundaries, and the temperature field is decomposed into a homogeneous component and a grating component, $T' = T_h + T_g e^{i2kz} + T_g^* e^{-i2kz}$. All these amplitudes are slowly varying functions of space and time as the fast (acoustical) variations are explicitly taken into account through the complex exponentials. Under the assumption of highly reflecting plates we can adopt a mean-field model, where the slowly varying amplitudes do not depend on the axial coordinate $z$ and $p_- = p_+ \equiv p$, ending up with the following dimensionless equations [11]:

\[
\begin{align*}
\tau_p \partial_t P &= -P + P_{in} + i\nabla^2 P + i (H + G - \Delta) P, \\
\partial_t H &= -H + D \nabla^2 H + 2 |P|^2, \\
\partial_t G &= -\tau_g G + D \nabla^2 G + |P|^2. 
\end{align*}
\]

(1)

Here $P = \left(\frac{\sigma \tau_p t_h a_0}{2\sqrt{\epsilon_0 \sigma_0 c_p}}\right)^{1/2} p$, $H = \omega t_p \sigma T_h$, and $G = \omega t_p \sigma T_g$, are new normalized variables, $\tau = t/t_h$ is time measured in units of the relaxation time $t_h$ of the temperature field homogeneous component, and $\tau_p = t_p/t_h$ and $\tau_g = t_g/t_h$ are the normalized relaxation times of the intracavity pressure field and the temperature grating component, respectively. Their original values are given by $t_p^{-1} = c_0 T/2L + c_0 a_0$, and $t_g^{-1} = 4k^2 \chi$, where $\chi = \kappa / \rho_0 c_p$ is the coefficient of thermal diffusivity, $\rho_0$ is the equilibrium density of the medium and $\kappa$ and $c_p$ are the thermal conductivity and the specific heat of the fluid at constant pressure, respectively. Other parameters are the detuning $\Delta = (\omega - \omega_c) / \gamma_p$, with $\omega_c$ the longitudinal cavity mode frequency that lies nearest to the driving frequency $\omega$, and $P_{in} = \frac{c_0 t_p}{2F} \left(\frac{\sigma \tau_p t_h a_0}{2\sqrt{\epsilon_0 \sigma_0 c_p}}\right)^{1/2} p_{in}$, $p_{in}$ being the injected pressure plane-wave amplitude, which we take as real without loss of generality. Finally $\nabla^2$ is the transverse Laplacian operator, where the dimensionless transverse coordinates ($x, y$) are measured in units of the diffraction length $l_d = c_0 \sqrt{\lambda / 2\pi}$, and the normalized diffusion coefficient $D = \chi t_h / l_d^2$. We note that when spatial derivatives are ignored and the complex fields are written in terms of moduli and phases, the model of [9,10] is retrieved. This kind of generalization to consider transverse effects is well known in optics (see, e.g., [12]).

The model parameters can be estimated for a typical experimental situation [9]. We consider a resonator with high-quality plates ($T = 0.1$), separated by $L = 5$ cm, driven at a frequency $f = 2$ MHz, and containing glycerine at $10^3$ C. Under these conditions the medium parameters are $c_0 = 2 \times 10^3$ m s$^{-1}$, $\sigma_0 = 10$ m$^{-1}$, $\rho_0 = 1.2 \times 10^3$ kg m$^{-3}$, $c_p = 4 \times 10^3$ J kg$^{-1}$ K$^{-1}$, $\sigma = 10^2$ K$^{-1}$, and $\kappa = 0.5$ W m$^{-1}$ K$^{-1}$ ($\chi = 10^3$ m$^{-2}$ s$^{-1}$). In this case $t_p = 2 \times 10^{-5}$ s, $t_g = 6 \times 10^{-2}$ s, and our length unit $l_d = 2$ mm. For a resonator with a large Fresnel number, the relaxation of the homogeneous component of the temperature is mainly due to the heat flux through the boundaries, and can be estimated from the Newton’s cooling law as $t_h \sim 10^5$ s. (Remind that this is our time unit.) Then the diffusion constant $D \sim 10^6$, and the normalized decay times $\tau_p \sim 10^{-6}$, and $\tau_g \sim 10^{-2}$ under usual conditions. We see that the problem is typically very stiff: $0 < \tau_p \ll \tau_g \ll 1$. In the following the results will be given to the lowest nontrivial order in these smallest decay times in order to not overburden the expressions.

The spatially uniform steady state can be obtained by neglecting the derivatives in eqs. (1). Introducing the notation $W = |P|^2$, $W_{in} = |P_{in}|^2$, one has $G = \tau_g W, H = 2W$, and

\[
W_{in} = W + (\Delta - 2W)^2 W.
\]

(2)

The characteristic curve $W$ vs. $W_{in}$ can display an S-shape, typical of the optical bistability of coherently driven optical Kerr cavities, as we show next. The two turning points of the characteristic and its inflection point are given by $W_{\pm} = \frac{2\Delta \pm \sqrt{4\Delta^2 - 4\Delta^2}}{2}$ and $W_1 = \frac{\Delta}{4}$, and the corresponding values of the input pressure follow from eq. (2). Note that bistability requires $\Delta > \Delta_0 = \sqrt{3}$, the nascent bistability (NB) occurring at $\Delta = \Delta_0$, in which case $W_{\pm} = W_1 = W_0 = \frac{\Delta}{4}$ (see footnote 1). This NB inflection point occurs at $P_{in,0} = \frac{2}{\sqrt{\pi} t_h \gamma}$, where $P_0 = \frac{\sqrt{3} - 1}{2 \sqrt{\pi}}$.  

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1 Bistability actually requires simultaneous stability of two solutions, what can occur in our model as we show below. Nevertheless, here the important feature is the multiplicity of the characteristic around the inflection point.
Inspired by previous works on passive optical cavities [13,14] we shall concentrate our study on conditions close to nascent bistability (\( \Delta \approx \Delta_0 \)) and to the vicinity of the inflection point, \((P, H, G, P_n) \approx (P_0, H_0, G_0, P_{n,0})\), where interesting pattern formation properties can be envisaged. We advance that, by focusing on the above conditions we can describe in a consistent way the S-shape of the characteristic curve by using standard multiple scale methods [15].

We consider next the stability of the spatially uniform steady state. We consider perturbations of the form \( \exp(\mathbf{i}k \cdot \mathbf{r} + \alpha t) \) and linearize the model eqs. (1) with respect to them. Further considering the relevant case \( 0 < \tau_p \ll \tau_g \ll 1 \), see typical values above, one obtains that there are five branches of characteristic exponents, \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \), which are continuous functions of \( k = |\mathbf{k}| \) and that, to the leading order, read

\[
\lambda_1 = \frac{a}{\tau_p}, \quad \lambda_2 = -\frac{a}{\tau_p} - 1, \quad \lambda_3 = -\frac{a}{\tau_p} - Dk^2, \quad \lambda_4 = -\frac{a}{\tau_p} - Dk^2 + 2W, \quad \lambda_5 = -1 - Dk^2 + 4\alpha W, \quad (3)
\]

\[
\alpha = \frac{\Delta + k^2 - 2W}{\Delta - 2W}, \quad \beta = a + bk^2 + O(k^4), \quad a = 4W - \frac{a}{\tau_g} - 1, \quad b = 4W - \frac{a}{\tau_g} - D, \quad \beta_0 = \Delta - 2W. \quad (4)
\]

\[
\lambda_5 = -1 - Dk^2 + 4\alpha W \quad (3)
\]

The homogeneous steady solution is then implicitly given by \( 4\mu = (3h_0^2 - 2\sqrt{3}\delta)h_0 \), being multi-valued when \( \delta > 0 \). In the following we focus on the bistable region, extending between the turning points \( h_{0,\pm} = \pm\sqrt{28/3}h_0^{3/2} \). A linear stability analysis shows that, for \( d > 0 \), only the upper branch can be modulationally unstable [19], giving rise to a pattern with wave number \( k_s = (15h_0 - 4d)/12 \). These results are confirmed by the numerical integration of eq. (4), performed in a one-dimensional case, \( \nabla^2 = \partial_x^2 \), which describes, \( e.g. \), a physical situation in which the resonator has a slab waveguide configuration confining the sound in the \( y \)-direction. Figure 1(a) shows the spatiotemporal

![Figure 1](image.png)
evolution (from bottom to top) of an initially homogeneous distribution, where the development of a modulational instability is observed, in agreement with the previous analysis.

In most of the multivalued domain, the extended patterns emerging at the modulational instability are unstable, being a transient state. As shown in fig. 1(a), neighbor maxima collide and merge, the long-term evolution resulting in a number of localized structures or CSs. In fig. 1(b) the background peak amplitude vs. the control parameter \( \mu \) on the control parameter \( \mu \) are shown with dots. The background amplitude corresponds to the stable uniform state, while the peak value is very close to that of the underlying roll pattern which, as stated in the introduction, is a signature of CSs. Note that CSs are stable and robust structures whenever a pattern and a uniform state coexist [20,21]. Other properties which characterize CSs [22], in particular that they can be located freely across the transverse plane and that they can be written (erased) independently, have been successfully checked in eq. (4).

It is remarkable from fig. 1 that CSs can form spontaneously from the homogeneous steady state, differently from other systems which require a hard local excitation of the medium. Such circumstance is very relevant in the particular case considered here, where the nature of the driving source (a plane, vibrating rigid surface) makes it difficult implementing experimentally a localized excitation. Certainly, the proposal of techniques to address CSs in the acoustic resonator will be most relevant.

We have also obtained an analytical CS of eq. (4),

\[
h(x) = h_0 + h_sech^2(x/x_s),
\]

where the background \( h_0 \) corresponds to the stable homogeneous solution of lower amplitude. The expressions of the peak amplitude \( h_s \) and width \( x_s \) in (5) are quite involved but can be readily obtained substituting eq. (5) into eq. (4). The solution given by eq. (5) corresponds to an ultrasonic CS but also applies to optical CSs in those systems described by eq. (4). It is a singular solution in the sense that it exists only for a special value of the injected amplitude \( \mu \) determined by the rest of parameters \( (d, \delta) \), and corresponds to one case of the CS family found in the numerical study and illustrated in fig. 1(b).

The order parameter description developed above is a powerful tool to relate the dynamics of our system to those of other systems of different physical origins, evidences the universality of the reported phenomenon and implies that the thermoacoustic resonator must support CSs in a limit. However, one pays the price of a restricted validity, basically to work close to the NB point. In order to verify the robustness of the result, we have numerically solved the full microscopic model given by eqs. (1) for the realistic parameter values already evaluated. Figure 2(a) represents the S-shaped characteristic, eq. (2), and the result of the linear stability analysis of the homogeneous steady state (see caption). In fig. 2(b) the stationary transverse profiles of the acoustic and thermal fields are shown. This result evidences that CSs are robust solutions of the thermoacoustic model. As it occurs with CSs supported by the modified Swift-Hohenberg equation, eq. (4), CSs develop spontaneously from the homogeneous stationary state in the full microscopic model for the thermoacoustic resonator, giving rise to spatiotemporal evolution similar to the one shown in fig. 1(a). The signatures corresponding to CSs—that they can be located freely across the transverse plane and that they can be written (erased) independently—have been also checked for the CSs exhibited by the microscopic model, eqs. (1). Further characterization of these structures is in progress.

Concluding, we have studied the spatiotemporal dynamics of ultrasound in a resonator containing a viscous medium—a thermo-acoustic resonator. The microscopic equations, eqs. (1), have been reduced, close to the nascent bistability point, to a single-order-parameter equation, eq. (4), previously obtained in other contexts. The reduced model contains bistability and modulational instabilities. As a consequence, the system is shown to support cavity solitons, corresponding to states where ultrasound is highly localized in the transverse plane of the resonator. Analytical solutions of such ultrasonic CSs have been obtained, eq. (5). Finally, ultrasonic CSs have been also numerically demonstrated to exist in the microscopic thermo-acoustic model, thus evidencing the robustness of the phenomenon.

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