Adiabatic Ground-State Properties of Spin Chains with Twisted Boundary Conditions

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We study the Heisenberg spin chain with twisted boundary conditions, focusing on the adiabatic flow of the energy spectrum as a function of the twist angle. In terms of effective field theory for the nearest-neighbor model, we show that the period 2 (in unit $2\pi$) obtained by Sutherland and Shastry arises from irrelevant perturbations around the massless fixed point, and that this period may be rather general for one-dimensional interacting lattice models at half filling. In contrast, the period for the Haldane-Shastry spin model with $1/r^2$ interaction has a different and unique origin for the period, namely, it reflects fractional statistics in Haldane’s sense.

I. INTRODUCTION

In recent rapid progress in one-dimensional (1D) quantum systems, exactly solvable models have been playing a vital role to understand low-energy critical properties. For example, the idea of twisted boundary conditions has been successfully combined with the exact solution to reveal a rich structure of the excitation spectrum. In this context, Sutherland and Shastry examined the spectral flow in the $XXX$ Heisenberg model with twisted boundary conditions using the Bethe ansatz method. They found a remarkable feature in the spectral flow, which is explained briefly here. Note that the Heisenberg $XXX$ model with twisted boundary conditions is equivalent to the interacting fermion model on a ring threaded by a magnetic flux. Since each fermion on a ring feels a gauge potential, it acquires a non-trivial phase factor when it moves along the ring and returns to the previous position. Let us denote this phase as $e^{2\pi i \phi}$. Recall first that the full spectrum at $\phi = 1$ should be equivalent to that at $\phi = 0$. However, if we increase $\phi$ gradually from 0 and follow each eigenstate adiabatically, each state does not necessarily come back to the original state at $\phi = 1$ since the spectral flow occurs and levels cross each other. Sutherland and Shastry found that the ground state at $\phi = 0$ becomes the first excited state at $\phi = 1$, and returns to the ground state again at $\phi = 2$ for the antiferromagnetic $XXX$ model with zero magnetic field. The resulting period 2 shows a sharp contrast to that for non-interacting systems which have the period $N$ for $N$-site systems. In this connection, ferromagnetic spin models were discussed in [5], paying attention to the motion of string solutions. Also, Berry’s phase was calculated in [6]. The spectral flow in the Hubbard model and the $t$-$J$ model was investigated in [8].

Stimulated by the above studies, we are naturally lead to fundamental questions, i.e. what is essential to determine the period in the spectral flow, and how the spectral flow occurs for more generic cases. In this paper we examine this problem. We show that the presence of irrelevant perturbations among the massless fixed point is essential to determine the period of 2, implying that the period 2 may be rather general for 1D interacting systems with massless excitations at half-filling (or spin systems with zero magnetic field). We find, however, that this is not the case for a special class of models, i.e. the 1D models with $1/r^2$ interaction. One can see a different origin for the periodicity in this model, namely, the period is controlled by Haldane’s fractional exclusion statistics.

This paper is organized as follows. In the next section, we first reexamine how the spectral flow occurs for the antiferromagnetic Heisenberg $XXX$ chain by using low-energy effective theory. In §3, we study the Haldane-Shastry model with $1/r^2$ long-range interaction, and find that the period of the ground state is determined by the statistical interaction. We discuss the results in terms of the motif picture in fractional exclusion statistics. A brief summary is given in §4.

II. ADIABATIC FLOW IN THE $XXZ$ HEISENBERG CHAIN

We first study the Heisenberg $XXX$ spin chain model with nearest neighbor interactions. The corresponding Hamiltonian is given for the system with even $N$ sites,

$$H = \sum_{j=1}^{N} \left[ -\frac{1}{2} J (S_j^+ S_{j+1}^- + h.c.) + J_z S_j^z S_{j+1}^z \right]$$

$$= H_0 + H_{\text{int}}, \quad (2.1)$$

where $S_j^z$ is a spin-1/2 generator at $j$th site, and $J$ and $J_z$ are antiferromagnetic coupling constants with $J_z \geq 0$. We are concerned with the massless phase of this model ($J_z/J \leq 1$) with twisted boundary conditions.
\( S_j^{\pm} = e^{\pm 2\pi i \phi} S_j^\pm \), \( S_j^{\pm+N} = S_j^{\pm} \) for any \( j \).

\[ (2.2) \]

After the work of Sutherland and Shastry, it was found in the numerical studies of the Bethe-ansatz solution that the gap formation due to the finite-size effect may be relevant for the period of the spectral flow. To see what kind of the interaction is actually relevant, and moreover to address the problem in more general cases including non-integrable models, we here analyze the spectral flow by means of effective field theory in the continuum limit. According to the Jordan-Wigner transformation, \( S_j^z = 1/2 - n_j \), and \( S_j^z = \psi_j e^{i \sum n(k) n_k} \), where \( n_j = \psi_j^+ \psi_j \), the Hamiltonian (2.1) with the boundary condition (2.2) is rewritten in terms of the fermion field \( \psi_j \),

\[ H_0 = \frac{J}{2} \sum_{j=1}^{N} \left( \psi_j^+ \psi_{j+1} + \text{h.c.} \right), \]

\[ H_{\text{int}} = J_s \sum_{j=1}^{N} \left( n_j - \frac{1}{2} \right) \left( n_{j+1} - \frac{1}{2} \right). \]

\[ (2.3) \]
\[ (2.4) \]

Henceforth, we restrict ourselves to the half-filling case (zero magnetic field) with even \( N \). In order to correctly describe the spectral flow for the finite system, one should seriously cope with the parity effect on the ground state, i.e. the effect depending on whether the number of particles is even or odd. This is briefly summarized in Appendix A. Passing to the continuum limit, the fermion operators on the lattice \( \psi_j / \sqrt{a/2\pi} \) can be expressed in terms of the continuum field operators, \( \psi(x) = e^{-ipF} \psi_L(x) + e^{ipF} \psi_R(x) \) with \( \psi_{L(R)}(x) = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} e^{2\pi i (k + \phi)/l} \psi_{L(R), k+\phi} \), where \( l \) is the linear size of the system, related with the lattice spacing \( a \) by \( l = aN \), \( p_F = \pi/2a \) is the Fermi momentum defined by \( p_F = \pi/2a \) and \( \phi = \phi + 1/2 \) (see Appendix A). The one-particle dispersion is given by a linear form \( \epsilon(p) \sim \pm v_F p = \pm \frac{2\pi}{l} v_F (k + \phi) \) around the Fermi point \( p = \pm p_F \), where \( v_F \) is the Fermi velocity defined by \( v_F \approx Ja \). Then, there appears infinitely deep Dirac sea which have to be regularized. In the presence of twisted boundaries, this regularization is somehow subtle. First, we define the vacuum state as the lowest energy state at a fixed fermion number for a given \( \phi \), which is written as

\[ |0\rangle = \prod_{k+\phi > 0} \prod_{k'+\phi \leq 0} \psi_{L,k+\phi}^\dagger \psi_{R,k'+\phi}^\dagger |0\rangle, \]

\[ (2.5) \]

where \( |0\rangle \) denotes the reference state with no particles. Accordingly, the Hamiltonian (2.3) is rewritten as

\[ H_0 = \sum_{k \in \mathbb{Z}} v_F p \left( \psi_{L,k+\phi}^\dagger \psi_{R,k+\phi} - \psi_{R,k+\phi}^\dagger \psi_{L,k+\phi} - \psi_{L,k+\phi}^\dagger \psi_{L,k+\phi} - \psi_{R,k+\phi}^\dagger \psi_{R,k+\phi} \right), \]

\[ (2.6) \]

where normal ordering \( : \) is defined according to the sign of \( k + \phi \), not that of \( k \). The Casimir energy is easily obtained as \( E_\phi = \frac{2\pi v_F}{l} (\phi^2 - \frac{1}{4}) \) by the zeta-function regularization, where \( \Phi = \phi \) for \( 0 \leq \phi \leq 1/2 \) and \( \Phi = 1 - \phi \) for \( 1/2 \leq \phi \leq 1 \).

Since low-energy properties of the present system can be described by the shifted U(1) Kac-Moody algebra for left- and right-going currents,

\[ J_{L(R)}(x) = \psi_{L(R)}^\dagger \psi_{L(R)}; (x) + \frac{2\pi}{l} \phi, \]

\[ (2.7) \]

the spectrum of the Hamiltonian (2.6) is given by its representation. As is well known, there exist infinite number of the primary states labeled by integers \( Q_L \) and \( Q_R \), each of which forms a conformal tower with descendant states labeled by integers \( N_L \) and \( N_R \). By taking into account the effect of twisting, we have the spectrum for the eigenstate denoted by \( |Q_L,Q_R; N_L, N_R; \phi\rangle \),

\[ E_\phi(Q_L, Q_R; N_L, N_R; \phi) = \frac{1}{2} (Q_L - \phi^2) + \frac{1}{2} (Q_R + \phi)^2 + N_L + N_R - \frac{1}{12}, \]

\[ (2.8) \]
in unit \( 2\pi v_F / l \). Since we restrict ourselves to the half-filling, we have the constraint for quantum numbers, \( Q_L + Q_R = 0 \).

We start by examining how the spectral flow of (2.8) occurs as a function of \( \phi \), paying special attention to the motion of the ground state for \( \phi = 0 \) which is denoted by \( |a\rangle \equiv |0,0;0,0;\phi\rangle \). This state remains as the ground state up to \( \phi = 1/2 \) as seen from eq.(2.8) and also from Fig.3. At this point there occurs a level-crossing between the state \( |a\rangle \) and the primary state \( |1,-1;0,0;\phi\rangle \) which is the first excited state at \( \phi = 0 \). In other words, we can say that there occurs a pair-creation from the absolute ground state (lowest energy state) at this level-crossing point, as drawn in Fig.1. Note that this pair creation is related to the fact that one of the rapidities on the real axis jumps to the \( i \pi \) line with the twist angle being increased in the Bethe ansatz description. As \( \phi \) is further increased, the initial ground state \( |a\rangle \) becomes the first excited state at \( \phi = 1 \), where there occurs four-fold degeneracy among the state \( |a\rangle \), the other primary state \( |b\rangle \equiv |2,-2;0,0;\phi\rangle \), and two descendant states \( |c\rangle \equiv |1,-1;1,0;\phi\rangle \), \( |d\rangle \equiv |1,-1;0,1;\phi\rangle \). At this point, the whole spectrum recovers the original structure at \( \phi = 0 \), though the initial ground state \( |a\rangle \) is raised up to the excited state. If we further increase \( \phi \), the state \( |a\rangle \) is continuously raised to highly excited state with smoothly crossing with other levels. This smooth rise of the ground state implies that the period of the spectral flow becomes macroscopic, which is consistent with the known fact that the period of the spectral flow for non-interacting systems is given by \( N \) for \( N \)-site systems.
end, we bosonize the interaction term in a standard way. Expressing the interaction Hamiltonian in terms of the field operator, we write down the Hamiltonian as

\[ H = H_0 + H_J + H_b, \]

\[ H_J = \frac{4g}{2\pi} \int_0^l dx J_L J_R, \quad (2.9) \]

\[ H_b = -\frac{g}{2\pi} \int_0^l dx \left[ (\psi_L^\dagger \psi_R)^2 + (\psi_R^\dagger \psi_L)^2 \right], \quad (2.10) \]

where \( g \equiv J_z a/2\pi, H_0 \) is a free Hamiltonian with a renormalized velocity \( v_F' = u_F (1 + J_z g/\pi) \), and \( H_J \) is a marginal current-current interaction. Note that \( H_b \) can be incorporated into the free part,

\[ \mathcal{L} = \frac{1}{2} \left( \psi_L^\dagger \partial_t \psi_L + \psi_R^\dagger \partial_t \psi_R \right) + \frac{4g}{2\pi} J_L J_R \rightarrow \frac{1}{2\pi r^2} \partial \varphi \partial \varphi, \]

\[ (2.11) \]

with dimensionless coupling \( r^2 = 1/(1 + 4g) \sim 1 - 2J_z/\pi J \).

Let us first note that neither \( H_J \) nor \( H_b \) lifts the twofold degeneracy at \( \phi = 1/2 \), hence causing no effects on the spectral flow. We thus concentrate on the behavior around \( \phi = 1 \). Recall that the effect due to \( H_J \) can be incorporated in the free form \( E_{0,0}(Q_L, Q_R; N_L, N_R; \phi) = E_0(rQ_L, rQ_R; N_L, N_R; r\phi) \) in unit \( 2\pi v_F'/l \). So, \( H_J \) only separates the primary states \( |a\rangle \) and \( |b\rangle \) from the descendant states \( |c\rangle \) and \( |d\rangle \), so that the state \( |a\rangle \) can be raised up with smooth crossing with \( |b\rangle \) at \( \phi = 1 \). This implies that the marginal operator \( H_J \) does not affect the periodicity of the spectral flow, i.e. the period is still \( N \).

We now look at the effect of \( H_b \), which is an irrelevant operator for \( J_z/J \leq 1 \) and vanishes at the massless fixed point after the renormalization. Nevertheless, it can still affect the spectral flow for the finite-size system. By recalling the bosonization rules for fermion operators, \( \psi_L \rightarrow V_L \equiv e^{i\varphi L} \) and \( \psi_R \rightarrow V_{R-r} \equiv e^{-i\varphi R} \), the interaction \( H_b \) is replaced by the sine-Gordon form,

\[ H_b \rightarrow -\frac{g}{\pi} \int_0^l dx \cos 2r \varphi, \]

\[ (2.12) \]

in a standard way, where \( \varphi_L (\varphi_R) \) is a left- (right-) moving component of \( \varphi \), i.e., \( \varphi = \varphi_L + \varphi_R \). We now wish to evaluate the effect of \( H_b \) on the spectrum at \( \phi = 1 \) in the first order perturbation for the finite-size system with length \( l \). The two-fold degenerate primary states \( |a\rangle \) and \( |b\rangle \) can be written as

\[ |a\rangle = |V_L(r)|V_{R-r}, \quad |b\rangle = |V_{L-r}|V_{R-r}, \]

\[ (2.13) \]

where \( |V_{L(R)}(r)| \) denotes the primary state with dimension \( r \). From the asymptotic behavior of the two- and three-point functions of the vertex operators, we can see \( \langle r|V_{L(R)}(x)|V_{L(R)}(x) - r \rangle = (2\pi^2)^{2r^2}. \) The matrix element between these states is thus calculated as

\[ \langle a|H_b|b\rangle = -\frac{g}{2\pi} \int_0^l dx \langle V_{L(r)}V_{L(r)}|V_{R-r}V_{R-r} \rangle \]

\[ = -g \left( \frac{2\pi}{r} \right) \frac{4r^2-1}{4}. \]

One can thus see from this expression that the twofold degenerate primary states are lifted by this irrelevant perturbation, making the new states \( |\pm\rangle \equiv (|a\rangle \pm |b\rangle)/\sqrt{2} \). This level repulsion produces a gap between them, \( \Delta = 2g \left( \frac{2\pi}{r} \right)^{4r^2-1} \). Hence, with the increase of \( \phi \) the initial ground state is continuously raised up to the excited states till \( \phi = 1 \), then follows the lower branch \( |+\rangle \) smoothly, and finally returns to the absolute ground state at \( \phi = 2 \). We thus end up with the conclusion that the period of the spectral flow reduces from \( N \) to \( 2 \) due to irrelevant interaction \( H_b \), which reproduces the results of Sutherland and Shastry. An important point in the present analysis is that this periodicity is determined by the existence of irrelevant perturbations. So, we can say that the period \( 2 \) for the spectral flow is rather common to general (integrable and non-integrable) spin chain models as well as to interacting lattice fermion models in massless phase, because these irrelevant interactions are naturally involved in ordinary lattice models. Before closing this section, we wish to mention the following point. So far we have restricted ourselves to the antiferromagnetic cases. In the case of the ferromagnetic interaction \( (J_z < 0) \) with massless excitations, it may not be easy to treat the spectral flow directly by our approach, because bound states are formed during the process of the spectral flow, and their finite-size effects play a quite specific role for the period of the ground state. In particular, for special values of the coupling \( J_z \) the period of the flow is completely modified, which may not be treated by the present approach naively. Field theoretic description of the ferromagnetic case is still open and to be solved in the future study.

### III. HALDANE-SHASTRY SPIN CHAIN WITH 1/R^2 INTERACTION

It is now interesting to ask what happens for the spectral flow if we sweep away all the irrelevant interactions from the model. As already mentioned, the free fermion model is a typical example without such irrelevant perturbations, which has the period \( N \) for \( N \)-site systems.

We have another interesting model which is completely free from such irrelevant interactions, i.e., the Haldane-Shastry (HS) spin model with \( 1/r^2 \) interaction. This Hamiltonian is known as the fixed-point Hamiltonian which describes “free particles” obeying fractional exclusion statistics with statistical interaction parameter \( g \). One would naively expect that the period of the spectral flow may be \( N \), because there is no level-repulsion among the energy levels due to irrelevant interactions. This question was actually raised in, but has remained...
still open. We wish to address this problem in the remainder of the paper. It will be shown that the period of the spectral flow in this model is naturally interpreted as \(g\), which directly reflects fractional exclusion statistics.

Let us introduce the model Hamiltonian describing the spin chain with \(1/r^2\) exchange:

\[
H = \frac{1}{2} \sum_{n=1}^{N} \sum_{n' = 1}^{N-1} \sum_{l=-\infty}^{\infty} \frac{1}{(n + lN)^2} \times \left(S_n^z S_{n+n'+lN}^z + S_n^y S_{n+n'+lN}^y + \Delta S_n^x S_{n+n'+lN}^x\right),
\]

(3.1)

where we take the anisotropic parameter \(\Delta = \frac{1}{2}g(g - 1)\) with an even integer \(g\). It is known that \(g\) is regarded as statistical interaction in ideal exclusion statistics. Instead of periodic boundary conditions for the ordinary Haldane-Shastry model, we impose twisted boundary condition (2.3) in order to study the spectral flow. At \(\phi = 0\), the model is reduced to the ordinary Haldane-Shastry model.

As has been shown in Ref.\[14\], the exact eigenstate of the twisted model can be obtained in the well-known Jastrow form, and its energy spectrum is correctly reproduced by the asymptotic Bethe ansatz. The resulting expression is quite simple, which is written down here. Namely, a certain series of the exact spectrum for the sector of \(S_z = N/2 - M\) is given by \(E_{\text{total}} = (\epsilon(k))^2(E(\phi) + \epsilon)\), where \(\epsilon = \frac{1}{6}(N^2 - 1)\left\{\frac{1}{4}N\Delta + M(1 - \Delta)\right\}\) and

\[
E(\phi) = \sum_{i=1}^{M} \varepsilon(\bar{k}_i).
\]

(3.2)

The single-particle dispersion relation in this expression is explicitly given by

\[
\varepsilon(\bar{k}) = \left(2|\bar{k}| - N + 1\right)\bar{k} - |\bar{k}|(|\bar{k}| + 1),
\]

(3.3)

where \(|\bar{k}|\) denotes the Gauss symbol. Here \(\bar{k}\) is a pseudo-momentum including the effects of the twist, which is to be determined. Note that the effect of \(1/r^2\) interaction is incorporated via the renormalization of \(\bar{k}\) in eq. (3.2), which is given by a solution to the Bethe equation,

\[
\bar{k}_i = I_i + \phi + \frac{1}{2}(g - 1) \sum_{j \neq i} \text{sgn}(\bar{k}_i - \bar{k}_j),
\]

(3.4)

where \(I_i\) is an integer (or half integer) which specifies the energy spectrum. We note that the key quantities to correctly follow the flow of the spectrum are the set of quantum numbers \(\{I_i\}\), and also the crystal momentum,

\[
K(\phi) = \frac{2\pi M}{N} \sum_{i=1}^{M} \bar{k}_i.
\]

(3.5)

\[\text{A. Isotropic case}\]

By taking a suitable set of quantum numbers \(\{I_i\}\), we can determine the exact flow of the spectrum. Let us first consider the isotropic case \(\Delta = 1\) \((g = 2)\) by assuming that \(N\) is a multiple of 4.\[14\] We see from (3.2) that the spectral flow of the ground state occurs continuously up to \(\phi = 1\) with the adiabatic increase of \(\phi\), by setting \(I_i = (M - 1)/2 + i\) with \(i = 1, 2, \cdots, M\). Although we encounter a cusp structure at \(\phi = 1\) in (3.2), we may pass through it by keeping the quantum numbers \(\{I_i\}\) unchanged modulo the periodicity of them in (3.4). Based on these observations, we obtain the natural spectral flow of the ground state for the system with the statistical interaction \(g = 2\), resulting in the expression,

\[
E(\phi) = \begin{cases} 
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 + 2), & \text{for } 0 \leq \phi \leq 1 \\
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 - 10), & \text{for } 1 \leq \phi \leq 2 
\end{cases}
\]

(3.6)

Therefore the period of the spectral flow is 2, which may have a different origin from that for the nearest-neighbor Heisenberg model. We will indeed explain below that the obtained spectral flow actually realizes the motif picture of fractional exclusion statistics. Since the above exact solution produces only a certain series of the exact spectrum, we have complementarily calculated the exact spectral flow for the finite system numerically. In Fig. 3, the results for the \(N = 8\) system with \(\Delta = 1\) \((g = 2)\) are shown. It is seen that there are not level repulsions at any level crossing points, which we have discussed for the nearest-neighbor model in the previous section. They are replaced by characteristic cusp structures, which are indeed observed in the exact result of (3.6). One can easily trace the spectral flow of the ground state in Fig. 2 according to (3.6).

\[\text{B. Anisotropic case}\]

In contrast to the isotropic case, it may not be straightforward to discuss the spectral flow in the anisotropic case. It is indeed not trivial to figure out what kind of the ground state is realized in thermodynamic limit. Nevertheless, we find that the present approach can be still applied to discuss properties of the finite system. For example, the spectrum shown in Fig. 3 which is numerically calculated for the \(N = 8\) system with \(\Delta = 6\) \((g = 4)\), can be well described by (3.4). Namely, similar analyses to the isotropic case enable us to write down the analytic expression for the natural spectral flow as

\[
E(\phi) = \begin{cases} 
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 + 8), & \text{for } 0 \leq \phi \leq 1 \\
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 + 20), & \text{for } 1 \leq \phi \leq 2 \\
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 - 58), & \text{for } 2 \leq \phi \leq 3 \\
\frac{1}{4}N\phi - \frac{1}{12}N(N^2 - 16), & \text{for } 3 \leq \phi \leq 4 
\end{cases}
\]

(3.7)
For example, it is seen from the momentum conservation (3.3) that starting from the ground state at $\phi = 0$, we should follow the upper branch of the flow (ii) after the level crossing point at $\phi = 1$ in Fig.3. The expression (3.7) is obtained for the sector with the total spin $S_z = 2$, which is confirmed to be indeed the ground state for the $N = 8$ system. Thus, the period of the spectral flow for the ground state is regarded as 4, reflecting the statistical interaction $g = 4$.

C. Motif picture

We now discuss the origin of the period for the Haldane-Shastry model. Let us recall again that the Haldane-Shastry model describes the fixed-point Hamiltonian without irrelevant perturbations, which is indeed consistent with the above results that there is no level repulsions at any level-crossing points. So, in this ideal situation, it may be expected that one can see characteristic properties of fractional (exclusion) statistics in the spectral flow (3.8). To explain this in an intuitive way, it is convenient to use the notion of motif which is a fermionic occupation-number classification in the momentum space. We describe the behavior of the spectral flow in this language. The statistical interaction $g > 1$ means that the Pauli-principle acts stronger than the fermionic case due to the $1/r^2$ interaction, and as a consequence the spacing of the occupied states should be enlarged $g$-times larger than that of free fermions. This naturally leads us to the description by motif. To see the motif description clearly, we first describe the free system in this language. In this case, the Pauli principle controls the occupation of levels in the momentum space. Let us denote an occupied (unoccupied) state by 1 (0). Since there is no interaction among particles, the ground state motif is given by 000 · · · 00011 · · · 1100 · · · 000 which consists of $N/2$ 0’s and $N/2$ 1’s. Note that adding a unit flux $\phi$ shifts all 1’s uniformly to the right by one step. So, it is easily seen that this motif returns to the original one after adding the $N$ flux. Therefore, we can see that the period for the free fermion system is $N$, being consistent with the exact results mentioned before.

Let us now consider the present $1/r^2$ model with $g = 2$. In this case, the corresponding motif is constructed by all sequences of 0 and 1 not containing one or more consecutive 1’s, which reflect the repulsive interaction with $g = 2$. In the same way above, since adding a unit $\phi$ shifts 1’s to the right by one step, the ground state motif behaves as

$$01010 \cdots 01010 \xrightarrow{\delta\phi=1} 00101 \cdots 10101 \xrightarrow{\delta\phi=1} 01010 \cdots 01010. \quad (3.8)$$

Observing this, we see that the period of the spectral flow is indeed 2, which exactly reproduces (3.6). On the other hand, the motif for the ground state of the finite system shown in Fig.3 ($g = 4$, $N = 8$) is 001000100 · · · 001000100. Similarly to the $g = 2$ case, one easily finds that the motif has a period 4, reflecting the statistical interaction $g = 4$. So, we can naturally interpret the period observed for the Haldane-Shastry model in terms of the fractional exclusion statistics.

IV. SUMMARY

We have studied the adiabatic flow of the energy spectrum for spin chains. By using effective field theory for the XXZ Heisenberg model, we have clarified that the period 2 obtained by Sutherland and Shastry arises from leading irrelevant perturbations in the model. This implies that the period 2 is quite common to many 1D interacting quantum systems at half-filling (or spin systems with zero magnetic field), because ordinary quantum models naturally involve such perturbations. On the other hand, the quantum models with $1/r^2$ interaction has a nature of the fixed point Hamiltonian which is free from such irrelevant interactions. We have demonstrated that in this ideal system we can naturally see the statistical interaction in the period of the spectral flow.

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APPENDIX A: PARITY EFFECT ON JORDAN-WIGNER FERMIONS

In this appendix, we briefly discuss the parity effect on the ground state properties for Jordan-Wigner fermions. First let us recall the well-known fact that the ground state for spinless fermion systems at $\phi = 0$ is unique (doubly degenerate) according to whether the number of fermions is odd (even). So, if we introduce the external gauge field or alternatively twist boundary conditions, the system shows diamagnetic (paramagnetic) response depending on the parity with respect to the number of particles, i.e. the ground state energy first increases (decreases) with the increase of $\phi$. In contrast to this, we show below that the ground state at $\phi = 0$ for the present system is always unique and the response to the gauge field is diamagnetic.

First, we rewrite the boundary condition (2.2) in terms of Jordan-Wigner fermions,

$$\psi_{N+1} = \psi_1 e^{2\pi i (\phi + 1/2 - \mathcal{N})/2} \equiv \psi_1 e^{2\pi i (\phi + \phi_0)}, \quad (A1)$$

where $\mathcal{N} \equiv \sum_{j=1}^N n_j$ and $\phi_0 \equiv (M - 1)/2$ with $M$ being the number of fermions. Note that the extra phase
\(\phi_0\), originating from the minus sign in \(\psi_N^\dagger e^{-i\pi N} \psi_1 = -\psi_N^\dagger \psi_2 e^{-i\pi N}\), reflects the fact that the spin system is equivalent to the hard-core bosons. For even \(N\), we can see that \(\phi_0 = 1/2\) (0) for \(N/2\) = even (odd) for the half filling \(M = N/2\). This phase factor \(\phi_0\) distinguishes the present system from the ordinary fermion system, and plays a role to determine the ground state uniquely.

To confirm the above statement explicitly, we express the free Hamiltonian \(\{2,3\}\) in the diagonal form in Fourier space. The corresponding one-particle dispersion is given by \(\epsilon(k) \equiv -J \cos 2\pi(n + \phi + \phi_0)/N \equiv -J \cos(ap)\). Here the quantized momentum \(p\) is defined by \(p = 2\pi(k + \phi + \phi_0)/l \equiv 2\pi n/l\). Note that the twist angle shifts the momentum and plays a role of gauge potential. At \(\phi = 0\), \(n \in Z + \phi_0\), namely, \(n \in Z + 1/2\) (\(Z\)) for \(N/2\) = even (odd). The ground state configuration is then \(n = -(M - 1)/2\), \(-(M - 1)/2 + 1, \cdots, (M - 1)/2\) both for cases \(N/2\) = even and odd. Therefore, we can see that the ground state is always unique irrespective of the parity of the particle number. This is due to the extra phase \(\phi_0\) in eq. (A1).

Low energy excitations near the ground state are classified by the momentum \(ap = a(\pm p_F + p')\), where \(ap' = 2\pi(k' + \phi + 1/2)/N\) describes the excitation near the Fermi points. Note that the extra factor 1/2 is always accompanied in this expression, so that it is convenient to introduce \(\phi \equiv \phi + 1/2\) to simplify the expressions in the text.

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FIG. 1. Schematic diagram for spectral flow in the non-interacting case. It is described how the ground state configuration at \(\phi = 0\) evolves to the corresponding configuration at \(\phi = 1\) (occupied states for left- and right-going sectors are denoted by dots). It can be seen that \(|0; 0; 0; \phi = 1\rangle = |i; -1; 0; 0; \phi = 0\rangle\). Therefore, if we measure the fermion numbers \(Q_L\) and \(Q_R\) with respect to the vacuum state defined in the text, there occurs a pair creation at \(\phi = 1/2\). Namely, \(Q_R - Q_L\) changes by 2 during this process with keeping \(Q_L + Q_R = 0\).

FIG. 2. Exact spectral flow for the \(S^z = 0\) sector of the \(N = 8\) system with \(\Delta = 1\) (\(g = 2\)). Lower 10 levels are described.

FIG. 3. Exact spectral flow for the \(S^z = 2\) sector of the \(N = 8\) system with \(\Delta = 6\) (\(g = 4\)). Lower 10 levels are described. The spectral flow of the ground state occurs like \(i \rightarrow ii \rightarrow iii \rightarrow iv\). Note that we have folded the picture for \(0 \leq \phi \leq 4\) with respect to \(\phi = 2\) for simplicity.