Dirichlet-Poincaré profiles of graphs and groups

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Abstract

We define Poincaré profiles of Dirichlet type for graphs of bounded degree, in analogy with the Poincaré profiles (of Neumann type) defined in [HMT19]. The obvious first definition yields nothing of interest, but an alternative definition yields a spectrum of profiles which are quasi–isometry invariants and monotone with respect to subgroup inclusion. Moreover, in the extremal cases $p = 1$ and $p = \infty$, they detect the Følner function and the growth function respectively.

1 Introduction

In [HMT19], a spectrum of monotone coarse invariants for bounded degree graphs were introduced: these were called $L^p$-Poincaré profiles and are defined for $p \in [1, \infty]$. At the extremes $p = 1$ and $p = \infty$, the profiles detect the separation profile (see [BST12]) and the growth function of the graph respectively.

A more accurate name is $L^p$-Neumann–Poincaré profiles, since they are built from Poincaré constants of Neumann type.

The goal of this paper is to provide both a short and a long answer to the following question pointed out to us by Laurent Saloff–Coste:

**Question 1.** What happens if you replace Poincaré constants of Neumann type by Poincaré constants of Dirichlet type?

Let us fix some notation. Let $\Gamma$ be a finite graph and let $f : V\Gamma \to \mathbb{R}$. We define $\nabla f : V\Gamma \to \mathbb{R}$ by $\nabla f(v) = \max \{|f(v) - f(w)| : vw \in E\Gamma\}$.

Given a graph $X$ and a finite subgraph $\Gamma \leq X$ we define the $L^p$-Dirichlet–Poincaré constant of $\Gamma$ as follows:

$$Dh^p_X(\Gamma) = \inf \left\{ \frac{\|\nabla f\|_p}{\|f\|_p} : f : V\Gamma \to \mathbb{R}, f|_{\partial X\Gamma} \equiv 0, f \neq 0 \right\},$$

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where $\partial_X\Gamma = \{ v \in V \Gamma \mid d_X(v, VX \setminus V\Gamma) = 1 \}$. We make the convention that $Dh^p_X(\Gamma) = +\infty$ whenever $\Gamma = \partial_X\Gamma$. Notice that replacing $f$ by $|f|$ does not change the norm, preserves the property $f|_{\partial_X\Gamma} \equiv 0$ and does not increase $\|\nabla f\|_p$. Therefore we may always assume that all functions we consider are non-negative.

When the ambient graph $X$ is clear we will simply write $Dh^p(\Gamma)$ for $Dh^p_X(\Gamma)$.

Following [HMT19] we could define the $L^p$-Dirichlet-Poincaré profile of $X$ by

$$D^*\Lambda^p_X(n) = \sup \{ |\Gamma| Dh^p_X(\Gamma) \mid \Gamma \leq X, \partial_X\Gamma \neq VT, |VT| \leq n \}.$$ 

While $D^*\Lambda^p$ defines a monotone coarse invariant, it is sadly not a very interesting one, providing the short answer to the question.

**Proposition 2.** Let $X$ be a connected infinite graph with maximal vertex degree $d$. Then

$$D^*\Lambda^p_X(n) = \begin{cases} (d + 1)^{\frac{1}{p}}n & \text{if } p < \infty, \\ n & \text{if } p = \infty. \end{cases}$$

Instead, let us make the following alternative definition. Let $X$ be a graph of bounded degree. The $L^p$-Dirichlet-Poincaré profile of $X$ is given by

$$D\Lambda^p_X(n) = \inf \{ |\Gamma| Dh^p_X(\Gamma) : \Gamma \leq X, |VT| \geq n \}.$$ 

In general, these profiles are not monotone coarse invariants (nor even monotone under quasi-isometric embeddings), but they are quasi-isometry invariants.

**Theorem 3.** Let $X, Y$ be quasi-isometric graphs of bounded degree. Then, for every $p \in [1, \infty]$,

$$D\Lambda^p_X(n) \simeq D\Lambda^p_Y(n).$$

We consider functions using a standard partial order: given two functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f \lesssim g$ if there exists a constant $C$ such that $f(n) \leq Cg(Cn) + C$ for all $n \in \mathbb{N}$, and $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$. We write $f \lesssim_{u,v,\ldots} g$ to indicate that the constant $C$ depends on $u, v, \ldots$. Hence, Dirichlet-Poincaré profiles are well-defined for finitely generated groups. Moreover, they behave monotonically with respect to subgroups.

**Theorem 4.** Let $H$ be a finitely generated subgroup of a finitely generated group $G$. Then, for every $p \in [1, \infty]$,

$$D\Lambda^p_H(n) \lesssim D\Lambda^p_G(n).$$

The remainder of the paper is devoted to initial properties of these quasi-isometry invariants organised in direct analogy with the corresponding theory of Neumann-Poincaré profiles. We begin with the extremal cases $p = 1$ and $p = \infty$.

For $p = \infty$ the Dirichlet-Poincaré profile also depends only on growth.
Proposition 5. Let $X$ be a bounded degree graph.

\[ D \Lambda_{\infty}^X(n) \simeq \min \left\{ \frac{m}{\kappa(m)} \mid m \geq n \right\} \]

where $\kappa(m) = \max \{ k \mid \exists x : |B(x,k)| \leq m \}$ is the (lower) inverse growth function.

In the same way that $\Lambda^1$ can be expressed in terms of the Cheeger constants of finite graphs, $D \Lambda^1$ is related to the Cheeger constants of infinite graphs.

We recall that the Cheeger constant of an infinite graph is given by

\[ h(X) = \inf \left\{ \frac{|\partial X A|}{|A|} \mid A \subset V X, |A| < \infty \right\} \]

Theorem 6. Let $X$ be a connected graph of bounded degree. Then the following are equivalent

- $h(X) > 0$,
- $D \Lambda_{X}^1(n) \simeq n$,
- $D \Lambda_{X}^p(n) \simeq n$ for every $p \in [1, \infty)$.

When $X$ is the Cayley graph of a finitely generated group $G$, $h(X) > 0$ if and only if $G$ is non-amenable (this is commonly known as Følner’s criterion). From this we may easily deduce that $D \Lambda_{X}^1$ is not monotone under quasi-isometric embeddings, since solvable groups of exponential growth admit undistorted free sub-semigroups [CT08], so there is a quasi-isometric embedding of a 4-regular tree (a Cayley graph of the non-amenable free group on two generators) into a solvable (and hence amenable) group.

Moreover, one can completely express $D \Lambda_{X}^1$ in terms of the Følner function of $X$:

\[ F_X(n) = \min \left\{ |\Gamma| : \Gamma \leq X, \frac{|\partial X \Gamma|}{|\Gamma|} \leq \frac{1}{n} \right\}. \]

Theorem 7. Let $X$ be a bounded degree graph. Then, for $p \in (1, \infty)$,

\[ D \Lambda_{X}^p(n) \lesssim \min \left\{ \frac{m}{(F_X(m))^{\frac{1}{p}}} \mid m \geq n \right\} \]

where $F_X(n)$ is the inverse Følner function $F_X(n) = \max \{ k \mid F(k) \leq n \}$. In the case $p = 1$ we have the stronger result

\[ D \Lambda_{X}^{p}(n) \simeq \min \left\{ \frac{m}{F_X(m)} \mid m \geq n \right\}. \]

Since Følner functions of amenable groups have been extensively studied there are a number of immediate consequences of this result, some of which we list in §3.3.1. Grigorchuk and Pansu conjecture that the Følner function grows either polynomially, or at least exponentially [Gri14 Conjecture 5(ii)]. Reinterpreted in terms of Dirichlet-Poincaré profiles this is equivalent to the following:
Conjecture 8. Let $G$ be a finitely generated group. Either there is some $d$ such that $DA_G^p(n) \asymp n^{1-\frac{1}{d}}$ for all $p \in [1, \infty)$, or $DA_G^p(n) \gtrsim \frac{n}{\log(n)}$ for all $p \in [1, \infty)$.

Continuing the comparison with Neumann-Poincaré profiles, we see that Dirichlet-Poincaré profiles are also monotonic with respect to $p \in [1, \infty)$, however the relationship with $DA^\infty$ is very different:

Theorem 9. Let $X$ be a connected graph of bounded degree. Then for all $1 \leq p \leq q < \infty$:

$$D\Lambda_X^p(n) \lesssim D\Lambda_X^q(n).$$

Theorem 10. Let $G$ be a finitely generated group. Then

$$DA_G^\infty(n) \lesssim DA_G^1(n).$$

Theorem 10 is a direct consequence of Varopoulos’ inequality (cf. [CSC93, Théorème 1]).

We finish with some upper bounds coming from geometric properties of groups. From [HMT19 Proposition 9.5] we see that groups with finite linearly-controlled asymptotic dimension (cf. [As82]) satisfy $\Lambda_G^p(n) \lesssim \Lambda_G^\infty(n)$ for all $p \in [1, \infty)$. For Dirichlet-Poincaré profiles this follows from Theorem 7 in the case $p = 1$ via work of Nowak.

Corollary 11. [Now07, Theorem 7.1] Let $G$ be a finitely generated amenable group with finite linearly-controlled asymptotic dimension. Then

$$DA_G^1(n) \asymp \frac{n}{\kappa(n)}$$

where $\kappa(n)$ is the inverse growth function $\kappa(n) = \min \{ |k| \mid |B(1,k)| > n \}$.

The key examples of groups satisfying the above hypotheses are polycyclic groups and wreath products $F \wr \mathbb{Z}$ with $F$ finite. These groups also have controlled Følner pairs in the sense of [Tes11]. For these groups we have the following:

Theorem 12. Let $G$ be a finitely generated amenable group with controlled Følner pairs. Then, for all $p \in [1, \infty)$

$$DA_G^p(n) \asymp \frac{n}{\kappa(n)}$$

where $\kappa(n)$ is the inverse growth function $\kappa(n) = \min \{ k \mid |B(1,k)| > n \}$.

1.1 Questions

It is natural to ask for which groups $\Lambda_G^p \simeq DA_G^p$. For non-amenable groups this will be exceptionally rare, as $\Lambda_G^p(n) \simeq n$ if and only if $G$ contains an expander $(\Gamma_n)_n$ where the $|\Gamma_n|$ grow at most exponentially in $n$. 


Amongst amenable groups this equality appears to be much more common. We know from Theorem 12 [HMT19] and [HMT] that these profiles are equal for all virtually polycyclic groups.

However, it is certainly not the case that \( \Lambda^1_G(n) \simeq D\Lambda^1_G(n) \) holds for amenable groups. By [HM19] there are elementary amenable groups \( G_d \) such that

\[
\Lambda^1_{G_d}(n) \lesssim \log^d(n) \lesssim \frac{n}{\log(n)} \lesssim D\Lambda^1_{G}(n).
\]

where \( \log^d \) denotes the \( d \)th iterate of \( \log \). Both of the following questions appear to be open.

**Question 1.1.** Does \( \Lambda^p_{G}(n) \lesssim D\Lambda^p_{G}(n) \) hold for every \( p \in [1, \infty] \) and every finitely generated group \( G \)?

**Question 1.2.** Does there exist a finitely generated group \( G \) and \( 1 \leq p < q < \infty \) such that \( D\Lambda^p_{G}(n) \not\simeq D\Lambda^q_{G}(n) \)?

### 1.2 Disclaimer

Some results in this note are likely to be well-known to experts in the area. The goal here is to represent them as profiles in the style of [HMT19], and hopefully to provoke new questions.

## 2 The short answer

We recall that

\[
D^*\Lambda^p_X(n) := \sup \{ |\Gamma| D h^p_X(\Gamma) : \Gamma \leq X, \partial_X \Gamma \neq VT, |VT| \leq n \}.
\]

**Proposition 2.1.** Let \( X \) be an infinite graph of maximal degree \( d \). Then, for all \( n \geq d + 1 \),

\[
D^*\Lambda^p_X(n) = \begin{cases} 
(d + 1)^{\frac{p}{p}} n & \text{if } p < \infty, \\
n & \text{if } p = \infty.
\end{cases}
\]

**Proof.** Let \( v \in VX \) have degree exactly \( d \) and let \( \Gamma_n \leq X \) satisfy \( |\Gamma| = n \) and \( \partial_X \Gamma = \{v\} \). Let \( f : \Gamma_n \to \mathbb{R} \) be a function such that \( f|_{\partial_X \Gamma_n} = 0 \). Firstly suppose \( p < \infty \).

\[
\|f\|_p = |f(v)| \quad \text{and} \quad \|\nabla f\|_p = (d + 1)^{\frac{p}{p}} |f(v)|. \tag{1}
\]

Hence \( Dh^p(\Gamma_n) \geq (d + 1)^{\frac{p}{p}} \). Next we prove an upper bound on \( Dh^p_X(\Gamma) \) for any finite \( \Gamma \leq X \) with \( VT \setminus \partial_X \Gamma \neq \emptyset \). Fix a vertex \( v \in VT \setminus \partial_X \Gamma \) and define \( f = 1_{\{v\}} \). Then

\[
Dh^p_X(\Gamma) \leq \frac{\|\nabla f\|_p}{\|f\|_p} \leq (d + 1)^{\frac{p}{p}} \leq d + 1. \tag{2}
\]

If \( p = \infty \), then following the above strategy \( \Box \) can be replaced by \( \|f\|_\infty = |f(v)| = \|\nabla f\|_\infty \). Hence \( Dh^\infty_X(\Gamma_n) \geq 1 \). Moreover, taking the same function \( f = 1_{\{v\}} \) \( \Box \) can be replaced with \( Dh^\infty_X(\Gamma) \leq 1 \) and thus the result holds when \( p = \infty \). \( \Box \)
3 The long answer

We now pass to the definition of Dirichlet-Poincaré profile we will consider for the remainder of the paper.

**Definition 3.1.** Let $X$ be a graph of bounded degree. The $L^p$-Dirichlet-Poincaré profile of $X$ is given by

$$D\Lambda_X^p(n) = \inf \{ |\Gamma| \, D\h_x^p(\Gamma) : \Gamma \leq X, |V\Gamma| \geq n \}.$$ 

3.1 Elementary observations

We begin with two elementary but useful observations.

**Lemma 3.2.** Let $X$ be a graph, let $B$ be a finite subgraph of $X$ and $A$ a subgraph of $B$. Then for all $p$, $D\h^p(B) \leq D\h^p(A)$.

**Proof.** For each $f : V A \to [0, \infty)$ satisfying $f|_{\partial X A} \equiv 0$, define $f'(b) = \begin{cases} f(b) & \text{if } b \in V A, \\ 0 & \text{otherwise.} \end{cases}$ It is clear that for every $p$, $\|f\|_p = \|f\|_p$ and $\|\nabla f\|_p = \|\nabla f\|_p$. The result follows. \hfill \Box

Given a finite subgraph $\Gamma \leq X$ and some $a \geq 1$, we define

$$D\h^p_a(\Gamma) = \inf \left\{ \frac{\|\nabla a f\|}{\|f\|} : f|_{\partial X \Gamma} \equiv 0 \right\},$$

where $\nabla a f(x) = \sup \{ |f(y) - f(y')| : y, y' \in B_\Gamma(x, a) \}$.

**Lemma 3.3.** Let $\Gamma$ be a finite graph with maximal degree $d$, let $a \geq 1$. There exists a constant $c = c(a, d) > 0$ such that

$$cD\h^p_a(\Gamma) \leq D\h^p(\Gamma) \leq D\h^p_a(\Gamma).$$

**Proof.** The right-hand inequality is obvious. Fix $a \geq 1$ and let $B_a$ be the maximal cardinality of a closed ball of radius $a$ in $\Gamma$. For every $x \in \{ \nabla a f \geq t \}$ choose $y, y' \in B_\Gamma(x, a)$ such that $|f(y) - f(y')| \geq t$. Consider geodesics from $y$ to $x$ and from $x$ to $y'$. By the triangle inequality there is an edge $uv$ on one of these geodesics such that $|f(u) - f(v)| \geq \frac{t}{2a}$. It follows that there is some $u \in B_\Gamma(x, a)$ contained in $\{ \nabla f \geq \frac{t}{2a} \}$. Hence

$$|\{ \nabla f \geq t \}| \leq B_a \left( \frac{t}{2a} \right) \leq d^{a+1} \left| \{ \nabla f \geq \frac{t}{2a} \} \right|.$$ 

Using the co-area formula:

$$\sum_{x \in V\Gamma} |g(x)| = \int_{\mathbb{R}^+} \{ g \geq t \} \, dt$$
and \(4\) we have
\[
\sum_{x \in V} |\nabla_a f(x)|^p \leq d^{a+1} \sum_{x \in V} |2a \nabla f(x)|^p \leq (2a)^p d^{a+1} \sum_{x \in V} |\nabla f(x)|^p.
\]
Hence \(\|\nabla_a f\|_p \leq 2ad^{a+1} \|\nabla f\|_p\), and \(4\) follows. \(\square\)

### 3.2 Quasi-isometry invariance

These Dirichlet-Poincaré profiles are quasi-isometry invariants.

**Theorem 3.4.** Let \(X, Y\) be infinite connected bounded degree graphs and let \(q : X \to Y\) be a quasi-isometry. Then for any \(p \in [1, \infty)\), \(DA^p_X(n) \simeq DA^p_Y(n)\).

**Proof of Theorem 3.4.** Let \(q : X \to Y\) be a \((K, C)\)-quasi-isometry, so for all \(x, x' \in VX\),
\[
K^{-1} d_X(x, x') - C \leq d_Y(q(x), q(x')) \leq K d_X(x, x') + C,
\]
and for every \(y \in VY\) there is some \(x\) so that \(d_Y(q(x), y) \leq C\). Let \(d, d'\) be the maximal vertex degrees of \(X, Y\) respectively. We will prove \(DA^p_X(n) \simeq DA^p_Y(n)\) by constructing for each finite \(\Gamma \leq X\) a graph \(\Gamma' \leq Y\) with a comparable number of vertices and \(Dh^p_{\Gamma}(\Gamma')\) bounded from above by a fixed multiple of \(Dh^p_{\Gamma}(\Gamma)\).

For each finite \(\Gamma \leq X\) we define \(\Gamma'\) to be the full subgraph of \(Y\) whose vertex set is the closed \(C\)-neighbourhood of \(q(\Gamma)\). Given any \(f : VT \to [0, \infty)\) with \(f|_{\partial_X \Gamma} \equiv 0\) we define a comparison function \(f' : \Gamma' \to [0, \infty)\). If \(y \in \partial_Y \Gamma'\) define \(f'(y) = 0\), otherwise define \(f'(y) = \max\{|f(x)| : d_Y(y, q(x)) \leq C\}\).

We first find a lower bound for \(\|f'\|_p\) in terms of \(\|f\|_p\). For every \(x \in V\), \(f'(q(x)) \geq f(x)\), and the pre-image of a vertex in \(Y\) has diameter at most \(KC\). Hence
\[
\|f'\|_p^p = \sum_{y \in \Gamma'} f'(y)^p \geq (d + 1)^{-KC} \sum_{x \in \Gamma} f(x)^p = (d + 1)^{-KC} \|f\|_p^p. \tag{5}
\]

We next find an upper bound for \(\|\nabla f\|_p\) in terms of \(\|\nabla f\|_p\). For \(a \geq 1\) define \(\Gamma_a\) to be the full subgraph of \(X\) whose vertex set is the closed \(a\)-neighbourhood of \(VT\) in \(X\). Define \(f_a : VT_a \to [0, \infty)\) by \(f_a(x) = f(x)\) if \(x \in VT\) and \(0\) otherwise. By direct calculation, \(\|\nabla f_a\|_p = \|\nabla f\|_p\), so by Lemma 2.3 for any \(a\) there is a constant \(L = L(a, d) > 0\) such that
\[
\|\nabla_a f_a\|_p \leq L \|\nabla f\|_p.
\]

Let \(y_0 \in \Gamma'\) and choose \(y_1\) so that \(y_0 y_1 \in E\Gamma'\) and \(\nabla f'(y_0) = |f'(y_0) - f'(y_1)|\). Now we may choose \(x_0, x_1 \in V\) such that \(f'(y_j) = f(x_j)\) and \(d_Y(q(x_j), y_j) \leq C\). We have \(d_Y(q(x_0), q(x_1)) \leq 2C + 1\) so
\[
d_X(x_0, x_1) \leq Kd_Y(q(x_0), q(x_1)) + KC = 3KC + K.
\]
Choosing \( a \geq 3KC + K \) we see that
\[
\nabla_a f_a(x_0) \geq |f_a(x_0) - f_a(x_1)| = |f'(y_0) - f'(y_1)| = \nabla f'(y_0).
\]
Now the set of \( y_0 \in VT' \) for which we may choose a fixed \( x_0 \in VT \) is contained in the closed ball of radius \( C \) around \( q(x_0) \). Thus
\[
\|\nabla f'\|_p \leq (d' + 1)^C \|\nabla_a f_a\|_p \leq L(d' + 1)^C \|\nabla f\|_p.
\]
Combining this with (5), we have \( Dh^p(\Gamma') \leq (d + 1)^{2KC} L(d' + 1)^C Dh^p(\Gamma) \). If \( |\Gamma| \geq n \), then \( |\Gamma'| \geq (d + 1)^{-KC} n \), hence
\[
DA^n_X(p(n) \leq (d + 1)^{KC} L(d' + 1)^C DA^p_X((d + 1)^{KC} n),
\]
so \( DA^n_X(p(n) \lesssim_{K,C,d,d'} DA^p_X(n) \). The opposite inequality is obtained by considering a quasi-inverse \( r : Y \to X \) of \( q \).

### 3.3 The extremal cases \( p = 1 \) and \( p = \infty \)

We next explore the extremal cases, starting with \( p = \infty \).

**Proposition 3.5.** Let \( X \) be an infinite graph and let \( \Gamma \) be a subgraph of \( X \). Define \( l_\Gamma \) to be the radius of the largest ball in \( X \) which is contained in \( \Gamma \). Then \( Dh^\infty(\Gamma) = l_\Gamma^{-1} \) and

\[
DA^\infty_X(n) \simeq \inf \left\{ \frac{m}{\underline{\kappa}(m)} \mid m \geq n \right\}
\]

where \( \underline{\kappa}(m) \) is the maximal \( k \) such that there is a ball of radius \( k \) in \( X \) containing at most \( m \) vertices.

**Proof.** Let \( \Gamma \) be a finite subgraph of \( X \) with \( VT \neq \partial X \Gamma \) and let \( f : VT \to [0, \infty) \) satisfy \( f|_{\partial X \Gamma} \equiv 0 \). Pick \( x \in VT \) such that \( f(x) = \|f\|_\infty \). Let \( P \) be a path from \( x \) to a vertex in \( \partial X \Gamma \) which has length at most \( l_\Gamma \geq 1 \). It follows from the triangle inequality that \( \|\nabla f\|_\infty \geq \frac{1}{l_\Gamma} \|f\|_\infty \), hence
\[
Dh^\infty(\Gamma) \geq \frac{1}{l_\Gamma}.
\]

Now fix \( x \) so that \( B_X(x, l_\Gamma) \subseteq \Gamma \) and define \( f : VT \to [0, \infty) \) by \( f(y) = \max \{0, l_\Gamma - d_X(x, y)\} \). It is clear that \( \|f\|_\infty = l_\Gamma \) and \( \|\nabla f\|_\infty \leq 1 \). Hence \( Dh^\infty(\Gamma) \leq \frac{1}{l_\Gamma} \). By definition
\[
DA^\infty_X(n) \simeq \inf \left\{ \frac{\Gamma}{l_\Gamma} \mid \Gamma \leq X, l_\Gamma \geq 1, |\Gamma| \geq n \right\} \simeq \inf \left\{ \frac{m}{\underline{\kappa}(m)} \mid m \geq n \right\}.
\]

Recall that \( \Lambda^\infty_X(n) \simeq \sup \left\{ \frac{m}{\bar{\kappa}(m)} \mid m \leq n \right\} \) where \( \bar{\kappa} \) is the (upper) inverse growth function \( \bar{\kappa}(m) = \max \{k \mid \forall x \ B(x; k) \leq m \} \). For most groups (certainly those with polynomial or exponential growth), \( DA^\infty_X(n) \simeq \Lambda^\infty_X(n) \).
As with Poincaré profiles of Neumann type, the $L^1$-Dirichlet-Poincaré profile is determined by a combinatorial connectivity constant. We recall that the Cheeger constant of an infinite graph with bounded degree is given by

$$h(X) = \inf \left\{ \frac{|\partial X A|}{|A|} \mid A \subset V X, |A| < \infty \right\}$$

**Theorem 3.6.** Let $X$ be a connected graph of bounded degree. The following are equivalent

1. $\Lambda^p_X(n) \not\simeq n$ for every $p \in [1, \infty)$,
2. $\Lambda^1_X(n) \not\simeq n$,
3. $h(X) = 0$.

**Proof.** $(i) \Rightarrow (ii)$ is immediate. Fix $d$ to be the maximal degree of a vertex in $X$. We start with $(ii) \Rightarrow (iii)$. Suppose $\Lambda^1_X(n) \not\simeq n$. From (2) we have $Dh^1(\Gamma) \leq d + 1$ for every finite subgraph $\Gamma$ of $X$. Therefore, there must be a sequence of finite subgraphs $\Gamma_n$ of $X$ such that $Dh^1(\Gamma_n) \leq \frac{1}{n}$.

For each $n$, let $f : V \Gamma_n \to \mathbb{R}$ satisfy $f|\partial X \Gamma_n \equiv 0$, $f \geq 0$, and $\|\nabla f\|_1 \leq 2n$. Using the co-area formula (cf. [HMT19, Proposition 6.6])

$$\|\nabla f\|_1 = \int_{\mathbb{R}^+} |\partial X \{ f > t \}| dt,$$

and $\|f\|_1 = \int_{\mathbb{R}^+} |\{ f > t \}| dt$, we see that there is some $t > 0$ such that $S_t = \{ f > t \} \subset V \Gamma$ satisfies

$$\frac{|\partial X S_t|}{|S_t|} \leq \frac{2}{n}.$$

Thus $h(X) = 0$.

Finally, we show $(iii) \Rightarrow (i)$. Suppose $h(X) = 0$. If $X$ is finite there is nothing to prove, so assume it is infinite. There is a family of finite subgraphs $\Gamma_n$ ($n \geq 2$) of $X$ such that

$$\frac{|\partial X \Gamma_n|}{|\Gamma_n|} \leq \frac{1}{n}. \quad (6)$$

Since $X$ is infinite and connected, $|\partial X \Gamma_n| \geq 1$, so $|\Gamma_n| \geq n$. Let $f_n$ be the characteristic function of the set $V \Gamma_n \setminus \partial X \Gamma_n$. By construction

$$\|f_n\|_p = (|V \Gamma_n \setminus \partial X \Gamma_n|)^\frac{1}{p} \geq \left( \frac{n - 1}{n} \frac{1}{|\Gamma_n|} \right)^\frac{1}{p},$$

and $\nabla f_n$ is the characteristic function of the set of vertices in $\Gamma_n$ at distance $\leq 1$ from $\partial X \Gamma_n$. Hence

$$Dh^p(\Gamma_n) \leq \frac{\|\nabla f_n\|_p^p}{\|f_n\|_p^p} \leq \left( (d + 1) \frac{n}{n - 1} \frac{1}{|\Gamma_n|} \right)^\frac{1}{p} \leq \left( \frac{d + 1}{n - 1} \right)^\frac{1}{p}.$$
where the last step uses (6). It follows that \( DA_X^1(n) \not\equiv n \), since for every \( n \),

\[
DA_X^p(\Gamma_n) \leq |\Gamma_n| Dh^p(\Gamma_n) \leq |\Gamma_n| \left( \frac{d + 1}{n - 1} \right)^{\frac{1}{p}}.
\]  

(7)

\[ \square \]

**Corollary 3.7.** Let \( X \) be a graph of bounded degree. Then \( DA_X^p(n) \not\equiv n \) for every \( p \in [1, \infty) \) if and only if \( h(X) > 0 \).

**Proof.** The forward implication is immediate from Theorem 3.6, the reverse implication follows from monotonicity (Proposition 3.12) and Theorem 3.6. \( \square \)

Recall that for a graph \( X \) satisfying \( h(X) = 0 \) the **Følner function** is:

\[
F(n) = \min \left\{ k \mid \frac{|\partial X \Gamma|}{|\Gamma|} \leq \frac{1}{n} \text{ for some } \Gamma \leq X \text{ with } |\Gamma| = k \right\}.
\]

**Corollary 3.8.** Let \( X \) be a graph of bounded degree such that \( h(X) = 0 \). Then for \( p \in (1, \infty) \)

\[
DA_X^p(n) \lesssim \inf \left\{ \frac{m}{F_X(m)} \mid m \geq n \right\}
\]

where \( F_X(m) = \max \{ k \mid F(k) \leq m \} \). In the case \( p = 1 \) we have

\[
DA_X^1(n) \simeq \inf \left\{ \frac{m}{F_X(m)} \mid m \geq n \right\}.
\]

**Proof.** For the upper bound fix \( m \), and let \( \Gamma' \) be a subgraph of \( X \) satisfying

\[
|\Gamma'| \leq m \quad \text{and} \quad \frac{1}{F_X(m) + 1} < \frac{|\partial X \Gamma|}{|\Gamma|} \leq \frac{1}{F_X(m)}
\]

From the proof of Theorem 3.6 (iii) \( \Rightarrow (i) \), we have that

\[
Dh_X^p(\Gamma') \leq \left( \frac{d + 1}{F_X(m) - 1} \right)^{\frac{1}{p}}
\]

Hence, for any \( m \)-vertex subgraph \( \Gamma \) of \( X \) containing \( \Gamma \) we have

\[
Dh_X^p(\Gamma) \leq \left( \frac{d + 1}{F_X(m) - 1} \right)^{\frac{1}{p}}
\]

by Lemma 3.2. Thus

\[
DA_X^p(m) \leq m \left( \frac{d + 1}{F_X(m) - 1} \right)^{\frac{1}{p}}
\]

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as required. For the lower bound, let $\Gamma$ be a finite subgraph of $X$. Fix $k$ maximal such that $Dh^1(\Gamma) \leq \frac{1}{k}$, so $|\Gamma| \geq F(k)$. If $F(k) < n$ there is nothing to prove. If $F(k) \geq n$, then by assumption

$$|\Gamma| Dh^1(\Gamma) \geq \frac{F(k)}{k+1} \geq \frac{F(k)}{2k} \geq \inf \left\{ \frac{m}{F_X(m)} \mid m \geq n \right\},$$

as required.

### 3.3.1 Consequences for the Følner function

We give three consequences of Corollary 3.8 for finitely generated groups.

**Theorem 3.9.** [Now07, Theorem 7.1] Let $G$ be a finitely generated amenable group with finite linearly controlled asymptotic dimension. Then

$$DA^1_G(n) \approx \inf \left\{ \frac{m}{\kappa(m)} \mid m \geq n \right\},$$

where $\kappa$ is the inverse growth function of $G$.

**Theorem 3.10.** [Ers03] We have the following:

1. For $G = \mathbb{Z} \wr \mathbb{Z}$, $DA^1_G(n) \approx \frac{n \log \log n}{\log n}$.
2. For $G = \mathbb{Z}_2 \wr \mathbb{Z}^d$, $DA^1_G(n) \approx \frac{n}{\log n^{1/d}}$.
3. For $G = \mathbb{Z} \wr (\mathbb{Z} \wr (\mathbb{Z} \wr \ldots \wr \mathbb{Z})))$ where $\mathbb{Z}$ occurs $k$ times, we have $DA^1_G(n) \approx n \left( \frac{\log \log n}{\log n} \right)^{1/k}$.
4. For $G = ((\ldots ((\mathbb{Z} \wr \mathbb{Z}) \wr \ldots \wr \mathbb{Z})))$ where $\mathbb{Z}$ occurs $k$ times, we have $DA^1_G(n) \approx \frac{n}{\phi^k n}$, where $\phi$ is the $k$-1-fold iteration of log divided by the $k$-fold iteration of log.

**Theorem 3.11.** [Ers06] For every function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} \frac{f(n)}{n} = 0$, there is a finitely generated group of intermediate growth such that $DA^1_G(n) \gtrsim f(n)$.

### 3.4 Dependence on $p$

Dirichlet-Poincaré profiles satisfy many of the same properties as the Poincaré profile $A^p_X$ such as monotonicity:

**Proposition 3.12.** Let $X$ be a graph of bounded degree. For every $1 \leq p \leq q < \infty$ there is a constant $C = C(p, q)$ such that

$$DA^p_X(n) \leq CDA^q_X(n).$$
Proof. Let $d$ be the maximal vertex degree of $X$. Choose $\Gamma \leq X$ with $n \leq |\Gamma| < \infty$ and $g : VT \to [0, \infty)$ such that $g|_{\partial X \Gamma} = 0$ and

$$|\Gamma| \frac{\|\nabla g\|_q}{\|g\|_q} \leq 2DA_X^\infty(n).$$

Define $f : VT \to [0, \infty)$ by $f(v) = g(v)^{q/p}$. Now $\|f\|^p_p = \|g\|^q_q$. By $\star$ we need only consider functions $g$ such that $\|\nabla_a g\|_q \leq (d + 1)^{\frac{q}{p}} \|g\|_q$.

By the mean value theorem (see e.g. Matoušek [Mat97, Lemma 4]), for every $s, t \in \mathbb{R}$ and $\alpha \geq 1,$

$$\{|s|^\alpha - |t|^\alpha\} \leq \alpha(|s|^{\alpha-1} + |t|^{\alpha-1})|s - t|.$$ 

For each $v \in VT$ we apply this to $s = g(v)$, $t = g(u)$, $\alpha = \frac{q}{p}$ for each edge $vw \in E\Gamma$ to see that

$$\nabla f(v) \leq \left(\frac{2q}{p}\right)^p g_1(v)^{-\frac{q}{p}} \nabla g(v)$$

where $g_1(v) = \max \{ |g(u)| : d_{\Gamma}(v, u) \leq 1 \}$. By definition $g_1(v) \leq g(v) + \nabla g(v)$.

Now

$$\|g\|^q_q Dh^p_a(\Gamma)^p = \|f\|^p_p Dh^p_a(\Gamma)^p$$

$$\leq \sum_{v \in VT} \nabla f(v)^p$$

$$\leq \left(\frac{2q}{p}\right)^p \sum_{v \in VT} (|g(v)| + \nabla g(v))^{q-p} \nabla g(v)^p$$

$$\leq \left(\frac{2q}{p}\right)^p \sum_{v \in VT} |g(v)|^{q-p} \nabla g(v)^p + \|\nabla g\|^q_q$$

$$\leq \left(\frac{2q}{p}\right)^p \left(\|g\|^{q-p}_q \|\nabla g\|_q^p + (d + 1)^{\frac{q}{p}} \|g\|^{q-p}_q \|\nabla g\|^q_q\right)$$

$$\leq_{p, q} \|g\|^{q-p}_q \|\nabla g\|_q^p,$$

where $(\star)$ follows from $(s + t)^{\alpha} \leq 2^{\alpha}(s^{\alpha} + t^{\alpha})$ for any $s, t, \alpha > 0$, and $(\dagger)$ follows from Hölder’s inequality and $\|\nabla_a g\|_q \leq (d + 1)^{\frac{q}{p}} \|g\|_q$. Rearranging and taking $p$th roots, we see that

$$Dh^p(\Gamma) \leq_{p, q} \frac{\|\nabla g\|_q}{\|g\|_q}.$$ 

The relationship between $DA^\infty$ and $DA^1$ is a well-known inequality.

**Proposition 3.13.** [CSC93, Théorème 1] Let $X$ be a graph of bounded degree satisfying the pseudo-Poincaré inequality (for example Cayley graphs of finitely generated groups)

$$\|f - f_r\|_1 \leq Cr \|\nabla f\|_1$$

where $f_r(x) = |B(x, r)|^{-1} \sum_{v \in B(x, r)} f(v)$. Then $DA_X^\infty(n) \lesssim DA_X^1(n)$. 

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3.5 Monotonicity with respect to subgroups

Theorem 3.14. Let $H$ be a finitely generated subgroup of a finitely generated group $G$. Then, for every $p \in [1, \infty]$, 

$$D\Lambda^p_H(n) \lesssim D\Lambda^p_G(n).$$

Proof. For $p = \infty$, this follows from Proposition 3.5. The rest of the proof is adapted from [DK18, Theorem 18.100].

Let $S \subset T$ be finite symmetric generating sets for $H$ and $G$ respectively. Let $X = \text{Cay}(H, S)$ and $Y = \text{Cay}(G, T)$.

Let $\Gamma$ be a finite subgraph of $Y$ with $m$ vertices. Now $\Gamma$ intersects finitely many cosets of $H$ which we label $g_1 H, \ldots, g_k H$ (note $k$ will depend on $\Gamma$). Denote $\Gamma_i = g_i^{-1}(\Gamma \cap g_i Y)$ considered as a subgraph of $Y$. For each function $f : \Gamma \to [0, \infty)$ let $f_i : \Gamma_i \to [0, \infty)$ be defined by $f_i(x) = f(g_i(x))$. Now

$$\sum_{i=1}^{k} \|\nabla^Y f_i\|_p \leq \|\nabla^X f\|_p = \epsilon_{f,p} \|f\|_p = \epsilon_{f,p} \sum_{i=1}^{k} \|f_i\|_p,$$

for some $\epsilon_{f,p}$. Therefore, there is some $i$ such that

$$\|\nabla^Y f_i\|_p \leq \epsilon_{f,p} \|f_i\|_p.$$

Now if $f|_{\partial \Gamma} \equiv 0$ then $f_i|_{\partial \Gamma} \equiv 0$, so $Dh^p_X(\Gamma_i) \leq \epsilon_{f,p}$. It is immediate that $|\Gamma_i| \leq m$, by Lemma 3.2 we see that $Dh^p_X(\Gamma') \leq \epsilon_{f,p}$ for any $\Gamma' \leq Y$ which contains $\Gamma_i$.

Since, by definition, $Dh^p_Y(\Gamma) = \inf \{ \epsilon_{f,p} : f : V \Gamma \to [0, \infty), f|_{\partial \Gamma} \equiv 0 \}$ we have that for every $\Gamma \leq Y$ with $|\Gamma| = m$ there is some $\Gamma' \leq X$ with $|\Gamma'| = m$ and $Dh^p_X(\Gamma') \leq Dh^p_Y(\Gamma)$. Thus

$$D\Lambda^p_X(n) \leq D\Lambda^p_Y(n). \qedhere$$

3.6 Controlled Følner pairs

Let us recall the definition.

Definition 3.15. [Tes11, Definition 4.8] Let $X$ be a graph of bounded degree. We say a family of pairs of finite subsets of $VX$ $(H_m, H'_m)_{m \in \mathbb{N}}$ is a controlled sequence of Følner pairs if there exists a constant $C \geq 1$ such that

- $N_m(H_m) = \{ x \in VX : d(x, H_m) \leq m \} \subseteq H'_m$,
- $|H'_m| \leq C |H_m|$,
- $\text{diam}(H'_m) \leq C m$.

Proposition 3.16. Let $G$ be a finitely generated group which admits a controlled sequence of Følner pairs. Then for all $p \in [1, \infty]$, 

$$D\Lambda^p_G(n) \lesssim \frac{n}{\kappa(n)}.$$ (8)
Proof. For each $m$ consider the function $f : H_m' \to [0, \infty)$ given by $f(v) = \max \{0, m - d_X(v, H_m)\}$. It is clear that $f|_{\partial X H_m} \equiv 0$, and (for $p \in [1, \infty)$)

$$Dh_p(H_m') \leq \frac{\|\nabla f\|_p}{\|f\|_p} \leq \frac{2 \|H_m'\|^\frac{1}{p}}{\|H_m\|^\frac{1}{p} m} \leq \frac{2C_1^\frac{1}{p}}{m},$$

while for $p = \infty$ we immediately have $Dh_\infty(H_m') \leq \frac{2}{m}$.

Now for each $n$ choose $m$ maximal so that $|H_m'| \leq n$ and let $\Gamma$ be any $n$-vertex subgraph of $X$ such that $H_m' \subseteq \text{VT}$. By Lemma 3.2 $Dh_p(\Gamma) \leq \frac{2C_1^\frac{1}{p}}{m}$, hence

$$D\Lambda^p_X(n) \leq \frac{2C_1^\frac{1}{p} n}{m} \lesssim \frac{n}{\kappa(n)}.$$

Let us justify the final inequality. Define $b_m = |B(1, m)|$. Now $b_m \leq |H_m'| \leq n < |H_{m+1}'| \leq b_{C(m+1)}$. Hence $m \leq \kappa(n) \leq C(m+1)$.

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