The Landau-Lifshitz equation of the ferromagnetic spin chain and Oseen-Frank flow

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Abstract

In this paper, we consider the Landau-Lifshitz equation of the ferromagnetic spin chain from $\mathbb{R}^2$ to the unit sphere $S^2$ under the general Oseen-Frank energy. We obtain global existence and uniqueness of weak solutions for large energy data; moreover, the number of singular points is finite.

1 Introduction

The $d$-dimensional classical system for the isotropic Heisenberg chain with spin vector $\mathbf{n} = (n_1, n_2, n_3)$ is described by the Hamiltonian density (without external magnetic field) $H = |\nabla \mathbf{n}|^2/2$. The spin equation of motion with the Gilbert damping term (without the external magnetic field) has the form

$$\partial_t \mathbf{n} = \alpha \mathbf{n} \times (\mathbf{n} \times \frac{\delta H}{\delta \mathbf{n}}) - \beta \mathbf{n} \times \frac{\delta H}{\delta \mathbf{n}}, \quad (1.1)$$

where $\alpha \geq 0$ is the Gilbert damping constant and $\beta$ is the exchange constant satisfying $\alpha^2 + \beta^2 = 1$ and $H$ is the Hamiltonian density. Explicitly, this gives the following classical Landau-Lifshitz equation

$$\partial_t \mathbf{n} = \beta \mathbf{n} \times \Delta \mathbf{n} - \alpha \mathbf{n} \times (\mathbf{n} \times \Delta \mathbf{n}). \quad (1.2)$$

The above system (1.1) or (1.2) is called the Landau-Lifshitz equation or the Landau-Lifshitz-Gilbert equation, which was first derived on phenomenological grounds by Landau-Lifshitz in [22]. It gives rise to a continuum spin wave theory. Note that the above system (1.2) reduces to the heat flow of harmonic maps when $\alpha = 1, \beta = 0$ and to the Schrödinger flow when $\alpha = 0, \beta = 1$.

Motivated by the study on the heat flow of harmonic maps (see [17, 31, 32, 33, 9] and so on) and Schrödinger flow (see [13, 14, 34] and so on), much progress has been made recently in the analysis of the Landau-Lifshitz-Gilbert Equation (1.2). For example, see [2] for the existence of global weak solutions of (1.2) under the Neumann boundary condition in any dimensions, and see [10, 27, 12, 25, 28, 4, 15, 19] and the references therein for partial regularity and the analysis of singularity of the system (1.2). More recently, the existence of partially smooth, global weak solutions of (1.2) similar to [32], has been obtained by Guo-Hong [18] for $d = 2$, Melcher [26] for $d = 3$, and Wang [35] for $d = 4$ with Dirichlet
boundary conditions. More recently, the first author and Guo [29, 30] studied the fractional generalization of the Landau-Lifshitz equation and obtained local well-posedness and global existence of weak solutions.

In this paper, we shall consider the case when the energy density is replaced by the Oseen-Frank energy density. The Oseen-Frank energy density expresses the free energy density of a nematic liquid crystal in terms of its optic axis, and is a measure of the increase in the Helmholtz free energy per unit volume due to deviations in the orientational ordering away from a uniformly aligned nematic director configuration. See [20] for the analysis for the minimizers of the Oseen-Frank energy. Let 

\[ W = W(n, \nabla n) \]

be the Oseen-Frank density of the form

\[ W(n, \nabla n) = k_1(\text{div} n)^2 + k_2|n \times (\nabla \times n)|^2 + k_3|n \cdot (\nabla \times n)|^2 + (k_2 + k_4)(\text{tr}(\nabla n)^2 - (\text{div} n)^2), \]

where \( k_1, k_2, k_3, k_4 \) are elastic constants depending on the materials and temperature.

Replacing \( H \) in (1.1) with \( W \), we obtain the Landau-Lifshitz equation of Oseen-Frank energy as follows:

\[ \partial_t n = -\alpha n \times (n \times h) + \beta n \times h, \quad (1.3) \]

where the vector field \( h \) is given by

\[ h = -\frac{\delta W}{\delta n} = (\nabla_i W_{p_i} - W_{n_i}), \]

where \( p_i = \nabla_i n_i \) and we adopt the standard summation convention. Throughout this paper, we denote

\[ W_{n_i} = \frac{\partial W(n, p)}{\partial n_i}, \quad W_{p_i} = \frac{\partial W(n, p)}{\partial p_i}. \]

In what follows, we give explicit form of the vector field \( h \). For this, we rewrite \( W(n, \nabla n) \) as in [20]

\[ W(n, \nabla n) = a|\nabla n|^2 + V(n, \nabla n), \]

where \( a = \min\{k_1, k_2, k_3\} \) and

\[ V(n, \nabla n) = (k_1 - a)(\text{div} n)^2 + (k_2 - a)|n \times (\nabla \times n)|^2 + (k_3 - a)|n \cdot (\nabla \times n)|^2. \]

In this way, we have the following (see [36])

**Lemma 1.1** It holds that

\[
(\nabla_a W_{p_i}) = 2a\Delta n + 2(k_1 - a)\nabla \text{div} n - 2(k_2 - a)\text{curl}(n \times (\text{curl} n \times n)) - 2(k_3 - a)\text{curl}(\text{curl} n \cdot n),
\]

\[ (W_{n_i}) = 2(k_3 - k_2)(\text{curl} n \cdot n)(\text{curl} n), \]

In particular, we have

\[
h = 2a\Delta n + 2(k_1 - a)\nabla \text{div} n - 2(k_2 - a)\text{curl}(\text{curl} n)
- 2(k_3 - k_2)\text{curl}(\text{curl} n \cdot n) - 2(k_3 - k_2)(\text{curl} n \cdot n)(\text{curl} n). \quad (1.4)
\]
In particular, when \( k_1 = k_2 = k_3 = 1 \), (1.3) with (1.4) reduces to the classical Landau-Lifshitz equation (1.2).

In [21], Hong-Xin proved that global existence of weak solution for the Oseen-Frank flow in 2D (i.e. \( \alpha = 1, \beta = 0 \) in (1.3)) whose singular points are finite and the uniqueness of weak solution was obtained by the later two authors of the present paper in [37] (see also [23] for different assumptions).

We are aimed to generalize the above results to the general Landau-Lifshitz equation (1.3) with \( \alpha, \beta > 0 \). Note that \( \partial_3 n = 0 \) in the 2-D case. Let \( b \in S^2 \) be a constant vector and we define
\[
H^1_b(\mathbb{R}^2; S^2) = \left\{ u : u - b \in H^1(\mathbb{R}^2; \mathbb{R}^3), |u| = 1 \text{ a.e. in } \mathbb{R}^2 \right\}.
\]

Our main results state as follows.

**Theorem 1.2** Assume that the initial data \( n_0 \in H^1_b(\mathbb{R}^2; S^2) \). Then there exists a unique global weak solution \( n \) of the system (1.3), which is smooth in \( \mathbb{R}^2 \times ((0, +\infty) \setminus \{T_i\}_{i=1}^L) \) with a finite number of singular points \( (x^l_i, T_i) \), \( 1 \leq l \leq L_i \). Moreover, there are two constants \( \epsilon_0, R_0 > 0 \) such that each singular point \( (x^l_i, T_i) \) is characterized by the condition
\[
\limsup_{t \uparrow T_i} \int_{B_R(x^l_i)} |\nabla n|^2(\cdot, t)dx > \epsilon_0
\]
for any \( R > 0 \) with \( R \leq R_0 \).

**Remark 1.3** The above theorem generalizes the existence and uniqueness results of the equation (1.2) in [18], and also generalize the existence result in [21]. The main difference is the introduced Oseen-Frank energy, which makes the system (1.3) does not keep the parabolic property. By constructing strong solutions of a new approximate system, we obtain the local well-posedness and global weak solutions of (1.3). Different with [32, 18], it’s not easy to obtain the uniqueness as said in [21], since the positivity of the diffusion term \( \delta_h \times n \) under the metric of \( L^2 \) norm is unknown. Instead, we introduce a type of weak Oseen-Frank metric as in [37]. Our goal is to combine the work of Oseen-Frank energy and the Schrödinger part \( n \times h \) together.

The rest of the paper is organized as follows. In Section 2, we obtain global existence of weak solution for the system (1.3) by using the local well-posedness and blow-up results in the Appendix. In Section 3, we prove that the weak solution obtained in Section 2 is unique indeed. At last, the local well-posedness and blow-up results for the Landau-Lifshitz system (1.3) with general Oseen-Frank energy are obtained in the Appendix.

**2 Global existence of weak solutions in \( \mathbb{R}^2 \)**

Let \( E(t) = \int_{\mathbb{R}^2} W(n, \nabla n)(x, t)dx \) for \( t \geq 0 \) and \( E_0 = E(0) = \int_{\mathbb{R}^2} W(n_0, \nabla n_0)(x)dx \). Moreover,
\[
E_R(n(\cdot, t); x) = \int_{B_R(x)} |\nabla n(y, t)|^2dy.
\]
For two constants $\tau$ and $T$ with $0 \leq \tau < T$, we denote
\[
V(\tau, T) : = \{ n : \mathbb{R}^2 \times [\tau, T] \to S^2 | n \text{ is measurable and satisfies } \}
\]
\[
es\sup_{\tau \leq t \leq T} \int_{\mathbb{R}^2} |n(t)|^2 dx + \int_{\tau}^{T} \int_{\mathbb{R}^2} |\nabla^2 n|^2 + |\partial_t n|^2 dx dt < \infty \}.
\]

2.1 A priori estimates

The following technical lemma can be found in [32].

Lemma 2.1 There are constants $C$ and $R_0$ such that for any $u \in V(0, T)$ and any $R \in (0, R_0]$, we have
\[
\int_{\mathbb{R}^2 \times [0, T]} |\nabla u|^4 dx dt \leq C \esssup_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla u|^2 dx \cdot \left( \int_{\mathbb{R}^2 \times [0, T]} |\nabla^2 u|^2 + R^{-2} \int_{\mathbb{R}^2 \times [0, T]} |\nabla u|^2 dx dt \right). \tag{2.1}
\]

First of all, we have the following basic energy estimates.

Lemma 2.2 (The basic energy estimates) Assume that $n$ is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $n_0 \in H^1_b(\mathbb{R}^2)$. Then, for all $0 < t < T$ there holds
\[
\int_{\mathbb{R}^2} W(n, \nabla n)(x, t) dx + \alpha \int_{0}^{t} \int_{\mathbb{R}^2} |\partial_t n|^2 dx ds \leq E_0.
\]

Proof: Multiply $\partial_t n$ on both sides of the equation (1.3) and integrate on $\mathbb{R}^2$, then the property $|n| = 1$ implies that
\[
\int_{\mathbb{R}^2} |\partial_t n|^2 dx = \alpha \int_{\mathbb{R}^2} \partial_t n \cdot (n \times (h \times n)) dx + \beta \int_{\mathbb{R}^2} \partial_t n \cdot (n \times h) dx
\]
\[
\quad = \alpha \int_{\mathbb{R}^2} \partial_t n \cdot h dx + \beta \int_{\mathbb{R}^2} \partial_t n \cdot (n \times h) dx
\]
Noting that the definition of the molecular field $h$, we get
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W(n, \nabla n)(x, t) dx = \int_{\mathbb{R}^2} (-h) \cdot \partial_t n dx.
\]
It follows that
\[
\int_{\mathbb{R}^2} |\partial_t n|^2 dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(n, \nabla n)(x, t) dx = \beta \int_{\mathbb{R}^2} \partial_t n \cdot (n \times h) dx. \tag{2.2}
\]
Now we estimate the term $\partial_t n \cdot (n \times h)$ as in [18]. The equation (1.3) show that
\[
\partial_t n = \alpha n \times (h \times n) + \beta n \times h,
\]
then we have
\[
n \times \partial_t n = \alpha n \times h + \beta n \times (n \times h),
\]
hence using $\alpha^2 + \beta^2 = 1$ we arrive at
\[
\mathbf{n} \times \partial_t \mathbf{n} + \frac{\beta}{\alpha} \partial_t \mathbf{n} = \frac{1}{\alpha} \mathbf{n} \times \mathbf{h},
\]
which yields that
\[
\partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) = \beta |\partial_t \mathbf{n}|^2. \tag{2.3}
\]
Combining the estimates (2.2) and (2.3), we have
\[
\alpha^2 \int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 \, dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \, dx = 0,
\]
and the proof is completed by integrating with respect to time. \qed

As in [32, 18], the key ingredient for global existence of weak solution is a local monotonicity inequality, and our results state as follows.

**Lemma 2.3 (The local monotonicity inequality)** Assume that $\mathbf{n}$ is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H^1_b(\mathbb{R}^2)$. Then, for all $0 < t < T$ and $x_0 \in \mathbb{R}^2$ there holds
\[
E_R(\mathbf{n}(\cdot, t); x_0) \leq E_{2R}(\mathbf{n}_0(\cdot); x_0) + C_0 \frac{t}{R^2} E_0,
\]
where $C_0$ is an absolute constant independent of $t, R$ and $\mathbf{n}$.

**Proof:** Let $\phi(x)$ be a smooth cut-off function satisfying $\phi(x) = 1$ for $x \in B_R(x_0)$ and $\phi(x) = 0$ when $|x - x_0| > 2R$. Multiply $\partial_t \mathbf{n} \phi^2$ on both sides of (1.3), then we have
\[
\int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 \phi^2 \, dx = \alpha \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot \mathbf{h} \phi^2 \, dx + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) \phi^2 \, dx,
\]
and using the following relation
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) \, dx = \int_{\mathbb{R}^2} (-\mathbf{h}) \cdot \partial_t \mathbf{n} \phi^2 \, dx - 2 \int_{\mathbb{R}^2} \mathbf{W}_{ij}(\mathbf{n}, \nabla \mathbf{n}) \partial_t \mathbf{n}^i \partial_t \phi \phi \, dx,
\]
\[
\int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 \phi^2 \, dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) \, dx \leq 2\alpha \int_{\mathbb{R}^2} \mathbf{W}_{ij}(\mathbf{n}, \nabla \mathbf{n}) \partial_t \mathbf{n}^i \partial_t \phi \phi \, dx + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) \phi^2 \, dx,
\]
and using the equality of (2.3) for the term $\partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h})$ again, there holds
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) \, dx \leq C_0 \int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 |\nabla \phi|^2 \, dx \leq C_0 \frac{1}{R^2} E_0.
\]
Then the proof is complete. \qed
Lemma 2.4 (The positive diffusion) Assume that $n$ is a smooth solution of the Landau-Lifshitz equation (1.3) in $\{0, T\} \times \mathbb{R}^2$ and the initial data $n_0 \in H^1_0(\mathbb{R}^2)$. Then there exists $\epsilon_1 > 0$, such that for all $R \in (0, R_0)$ with $R_0 > 0$, if

$$\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla n(\cdot, t)|^2 dx < \epsilon_1,$$

then there hold

$$\int_{\mathbb{R}^2 \times [0, T]} |\nabla^2 n|^2 dxdt \leq C(1 + TR^{-2})E_0, \quad (2.4)$$

and

$$\int_{\mathbb{R}^2 \times [0, T]} |\nabla n|^4 dxdt \leq C\epsilon_1(1 + TR^{-2})E_0. \quad (2.5)$$

Proof: Due to the embedding inequality (2.1), it suffices to prove the first inequality (2.4). Since

$$\frac{d}{dt} \int_{\mathbb{R}^2} W(n, \nabla n)dx = \int_{\mathbb{R}^2} (W_{n^i} - \nabla_i W_p^j) \cdot n^i dx = -\int_{\mathbb{R}^2} h \cdot n dx,$$

using the equation of (1.3) we have

$$\int_{\mathbb{R}^2} W(n, \nabla n)(\cdot, t) dx + \alpha \int_{0}^{t} \int_{\mathbb{R}^2} (n \times (h \times n)) \cdot h dx ds \leq E_0.$$

Next we prove the positivity of the diffusion term. Using Lemma 1.1 and $n \cdot \Delta n = -|\nabla n|^2$, we derive that

$$\int_{\mathbb{R}^2 \times [0, T]} (n \times (h \times n)) \cdot h dx dt$$

$$\geq \int_{\mathbb{R}^2 \times [0, T]} (n \times (\nabla_i W_p^j \times n))\nabla_i W_p^j dx dt - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla n|^2(|\nabla^2 n| + |\nabla n|^2) dx dt$$

$$\geq 2a \int_{\mathbb{R}^2 \times [0, T]} \nabla_i W_p^j \cdot \Delta n dx dt + 2a \int_{\mathbb{R}^2 \times [0, T]} \Delta n \cdot (\nabla_i W_p^j - 2a \Delta n) dx dt$$

$$+ \int_{\mathbb{R}^2 \times [0, T]} (n \times ((\nabla_i W_p^j - 2a \Delta n) \times n)) \cdot (\nabla_i W_p^j - 2a \Delta n) dx dt$$

$$- C \int_{\mathbb{R}^2 \times [0, T]} |\nabla n|^2(|\nabla^2 n| + |\nabla n|^2) dx dt$$

$$\geq 4a \int_{\mathbb{R}^2 \times [0, T]} [a|\Delta n|^2 + 2(k_1 - a)|\nabla \text{div} n|^2 + 2(k_2 - a)|\nabla (\nabla \times n \times n)|^2$$

$$+ 2(k_3 - a)|\nabla (\nabla \times n \cdot n)|^2] dx dt - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla n|^2(|\nabla^2 n| + |\nabla n|^2) dx dt,$$

and the first estimate (2.4) follows from the embedding inequality (2.1) by choosing a small $\epsilon_1$. \hfill \Box
Concluding the above local monotonicity inequality in Lemma 2.3 and the positive diffusion in Lemma 2.4, we have the following corollary.

**Corollary 2.5** Assume that \( \mathbf{n} \) is a smooth solution of the Landau-Lifshitz equation (1.3) in \((0, T) \times \mathbb{R}^2\) and the initial data \( \mathbf{n}_0 \in H^1_0(\mathbb{R}^2) \). Then, there exists \( R > 0 \) such that \( \sup_{x \in \mathbb{R}^2} E_2 R(\mathbf{n}_0(\cdot); x) \leq \frac{\epsilon_1}{2} \), and

\[
\int_{\mathbb{R}^2 \times [0,t]} |\nabla \mathbf{n}|^2 dxdt + \int_{\mathbb{R}^2 \times [0,t]} |\nabla^2 \mathbf{n}|^2 dxdt \leq C(E_0 + \epsilon_1),
\]

hold for \( t < \frac{\epsilon R^2}{2C_0 E_0} \), where \( C_0 \) is given in Lemma 2.3.

Next, we use the idea of Lemma 2.4 and the estimates in Corollary 2.5 to obtain a higher interior regularity of the solution.

**Lemma 2.6** Assume that \( \mathbf{n} \) is a smooth solution of the Landau-Lifshitz equation (1.3) in \((0, T) \times \mathbb{R}^2\) and the initial data \( \mathbf{n}_0 \in H^1_0(\mathbb{R}^2) \). Then there is a constant \( \epsilon_1 \) such that for all \( R \in (0, R_0] \), if

\[
\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}(\cdot, t)|^2 dx < \epsilon_1,
\]

then, for all \( t \in (\tau, T) \) with \( \tau \in (0, T) \), it holds that

\[
\int_{\mathbb{R}^2} |\nabla^2 \mathbf{n}(\cdot, t)|^2 dx + \int_{\tau}^t \int_{\mathbb{R}^2} |\nabla^3 \mathbf{n}(\cdot, s)|^2 dxds \leq C(\epsilon_1, E_0, \tau, T, \frac{T}{R^2}).
\]

**Proof:** First, we can differentiate \( \nabla_\beta \) to (1.3), multiply it by \( \nabla_i \mathbf{h}(i = 1, 2) \), and we get

\[
\frac{d}{dt} \int_{\mathbb{R}^2} a |\Delta \mathbf{n}|^2 + (k_1 - a) |\nabla \text{div} \mathbf{n}|^2 + (k_2 - a) |\nabla (\mathbf{n} \times (\nabla \times \mathbf{n}))|^2 dx
\]

\[+ \frac{d}{dt} \int_{\mathbb{R}^2} (k_3 - a) |\nabla (\mathbf{n} \cdot (\nabla \times \mathbf{n}))|^2 dx \leq -\alpha \int_{\mathbb{R}^2} \nabla_i (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx - \beta \int_{\mathbb{R}^2} \nabla_i (\mathbf{n} \times \mathbf{h}) \cdot \nabla_i \mathbf{h} dx
\]

\[+ C \int_{\mathbb{R}^2} |\nabla \mathbf{n}_i| |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}| + |\mathbf{n}_i| |\nabla^2 \mathbf{n}|^2 dx \leq -\alpha \int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx
\]

\[+ C \int_{\mathbb{R}^2} |\nabla \mathbf{n}_{\mathbf{i}}| |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}| + |\mathbf{n}_{\mathbf{i}}| |\nabla^2 \mathbf{n}| |\nabla \mathbf{n}| + |\nabla \mathbf{n}| |\nabla^3 \mathbf{n}| |\nabla^2 \mathbf{n}| dx \leq -\alpha \int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx + C \int_{\mathbb{R}^2} |\nabla^2 \mathbf{n}|^2 |\nabla \mathbf{n}|^2 + |\nabla \mathbf{n}| |\nabla^3 \mathbf{n}| |\nabla^2 \mathbf{n}| dx \]

Note the fact that \( |\mathbf{n} \cdot \nabla_i \Delta \mathbf{n}| \leq C |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}| \), and similar estimates as in Lemma 2.4 imply

\[
\int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx \geq 3a^2 \int_{\mathbb{R}^2} |\nabla^3 \mathbf{n}|^2 - C \int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 |\nabla^2 \mathbf{n}|^2 dx.
\]
Due to the interpolation inequality
\[
\|\nabla^2 n\|_4 \leq C \|\nabla^2 n\|_2^{1/2} \|\nabla^3 n\|_2^{1/2},
\]
we have
\[
\int_{\mathbb{R}^2} |\nabla n|^2 |\nabla^2 n|^2\,dx \leq \delta \|\nabla^3 n\|_2^2 + C(\delta) \|\nabla n\|_4^4 \|\nabla^2 n\|_2^2,
\]
thus Gronwall’s inequality and Corollary 2.5 imply the required estimates. □

Indeed, using the above idea by induction, one can prove the smooth property of \(n\), and we omit the proof (similar arguments for Ericksen-Leslie system, see [36, Corollary 4.6]).

**Corollary 2.7** Assume that \(n\) is a smooth solution of the Landau-Lifshitz equation (1.3) in \((0, T) \times \mathbb{R}^2\) and the initial data \(n_0 \in H^1_b(\mathbb{R}^2)\). Then there is a constant \(\epsilon_1 > 0\) such that for all \(R \in (0, R_0]\), if
\[
\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla n(\cdot, t)|^2\,dx < \epsilon_1,
\]
then, for all \(t \in (\tau, T)\) with \(\tau \in (0, T)\), for any \(l \geq 1\) it holds that
\[
\int_{\mathbb{R}^2} |\nabla^{l+1} n(\cdot, t)|^2\,dx + \int_\tau^t \int_{\mathbb{R}^2} |\nabla^{l+2} n(\cdot, s)|^2\,dx\,ds \\
\leq C(l, \epsilon_1, E_0, \tau, T, \frac{T}{R^2}). \tag{2.8}
\]
Moreover, \(n\) is regular for all \(t \in (0, T)\).

### 2.2 Existence of global weak solution

Now we complete the proof of the existence part in Theorem 1.2. Similar to [32, 24, 36], we sketch its step for completeness.

For any data \(n_0 \in H_b^1(\mathbb{R}^2; S^2)\), one can approximate it by a sequence of smooth maps \(n^k_0\) in \(H_b^1(\mathbb{R}^2; S^2)\), and we can assume that \(\nabla n^k_0 \in H_b^1(\mathbb{R}^2; S^2)\) (see [31]). Due to the absolute continuity property of the integral, for any \(\epsilon_1 > 0\), there exists \(R_0 \geq R_1 > 0\) such that
\[
\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla n_0|^2\,dx \leq \epsilon_1,
\]
and by the strong convergence of \(n^k_0\),
\[
\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla n^k_0|^2\,dx \leq 2\epsilon_1
\]
for a sufficient large \(k\). Without loss of generality, we assume that it holds for all \(k \geq 1\).

For the data \(n^k_0\), by Theorem A.1 there exists a time \(T^k\) and a strong solution \(n^k\) such that
\[
\nabla n^k \in C \left([0, T^k]; H^4(\mathbb{R}^2)\right).
\]
Hence there exists $T^k_0 \leq T^k$ such that
\[
\sup_{0 < t < T^k_0} \int_{B_R(x)} |\nabla n^k(y, t)|^2 dy \leq (8 + \frac{1}{a})\epsilon_1,
\]
where $R \leq R_1/2$. However, by the local monotonic inequality in Lemma 2.3, we have $T^k_0 \geq \frac{\epsilon_1 R^2}{C_0 E_0} = T_0 > 0$ uniformly. For any $0 < \tau < T_0$, by the estimates in Corollary 2.7 for any $l \geq 1$ we get
\[
\sup_{\tau < t < T_0} \int_{\mathbb{R}^2} |\nabla^{l+1} n^k|^2(\cdot, t) dx + \int_\tau^{T_0} \int_{\mathbb{R}^2} |\nabla^{l+2} n^k|^2(\cdot, s) dx ds \leq C(l, \epsilon_1, E_0, \tau, T_0, \frac{T_0}{R^2}). \tag{2.9}
\]
Moreover, the energy inequality in Lemma 2.2 a priori estimates in Lemma 2.4 and the equation (1.3) yield that
\[
E(n^k)(t) \leq E_0, \quad 0 < t < T^k, \tag{2.10}
\]
and
\[
\int_{\mathbb{R}^2 \times [0, T^k]} (|\nabla^2 n^k|^2 + |\partial_t n^k|^2 + |\nabla n^k|^4) dx dt \leq C(\epsilon_1, C_0, E_0). \tag{2.11}
\]
Hence the above estimates (2.9), (2.11) and Aubin-Lions Lemma yield that there exists a solution $n - b \in W^{2,1}_2(\mathbb{R}^2 \times [0, T_0]; \mathbb{R}^3)$ such that (at most up to a subsequence)
\[
n^k - b \rightarrow n - b, \quad \text{locally in } W^{3,1}_2(\mathbb{R}^2 \times (0, T_0); \mathbb{R}^3).
\]
By (2.10), $\nabla n(t) \rightharpoonup \nabla n_0$ weakly in $L^2(\mathbb{R}^2)$, thus $E(n_0) \leq \liminf_{t \to 0} E(n(t))$. On the other hand, by the energy estimates of $(n^k)$, we have
\[
E(n_0) \geq \limsup_{t \to 0} E(n(t)).
\]
Hence, $\nabla n(t) \to \nabla n_0$ strongly in $L^2(\mathbb{R}^2)$ and $n$ is the solution of the equation (1.3) with the initial data $n_0$. From the weak limit of regular estimates (2.9), we know that $n \in C^\infty(\mathbb{R}^2 \times (0, T_0))$ and $\nabla^{l+1} n(\cdot, T_0) \in L^2(\mathbb{R}^2)$ for any $l \geq 1$. By Theorem A.1 there exists a unique smooth solution of (1.3) with the initial data $n(\cdot, T_0)$, which is still written as $n$, and blow-up criterion yields that if $n$ blows up at finite time $T^*$, then
\[
\|\nabla n\|_{L^\infty(\mathbb{R}^2)}(t) \to \infty, \quad \text{as } t \to T^*.
\]
As a result, we have
\[
|\nabla^4 n|(x, t) \not\in L^\infty_1 L^2_2((T_0, T^*) \times \mathbb{R}^2) \tag{2.12}
\]
We assume that $T_1$ is the first singular time of $n$, then we have
\[
n \in C^\infty(\mathbb{R}^2 \times (0, T_1); S^2) \quad \text{and} \quad n \not\in C^\infty(\mathbb{R}^2 \times (0, T_1]; S^2);
\]
and by Corollary 2.7 and (2.12), there exists $\epsilon_0 > 0$ such that
\[
\limsup_{t \uparrow T_1} \sup_{x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla n|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.
\]
Finally, since \( n - b \in C^0([0, T_1], L^2(\mathbb{R}^2)) \) by the interpolation inequality (similarly see P330, [24]), we can define
\[
n(T_1) - b = \lim_{t \uparrow T_1} n(t) - b \quad \text{in} \quad L^2(\mathbb{R}^2).
\]
On the other hand, by the energy inequality \( \nabla n \in L^\infty(0, T_1; L^2(\mathbb{R}^2)) \), hence \( \nabla n(t) \rightharpoonup \nabla n(T_1) \).
Similarly we can extend \( T_1 \) to \( T_2 \) and so on. It’s easy to check that the energy loss at every singular time \( T_i \) for \( i \geq 1 \) is at least \( \epsilon_1 \), thus the number \( L \) of the singular time is finite. Moreover, singular points at every singular time are finite by similar arguments as in [32], since \( \partial_t u \in L^2_{x,t} \) in Lemma 2.2 and the local monotonicity inequality in Lemma 2.3 hold. Assume that singular points are \( (x_j, T_i) \) with \( 1 \leq j \leq L_i \) and \( i \leq L \), and we have
\[
\limsup_{t \uparrow T_i} \int_{B_R(x_j)} |\nabla n|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.
\]
The proof is complete. \( \square \)

### 3 Uniqueness of weak solution

In this section, we follow the same route as in [37] and prove the following uniqueness theorem. The main difference is to deal with the Schrödinger part \( n \times h \).

**Theorem 3.1** Let \( n^1 \) and \( n^2 \) be two weak solutions of the Landau-Lifshitz equation \((1.3)\) in \( \mathbb{R}^2 \) obtained in Theorem 1.2 with the same initial data \( n_0 \). Then we have
\[
n^1(t) = n^2(t)
\]
for any \( t \in [0, +\infty) \).

Let \( n^1 \) and \( n^2 \) be two weak solutions of the Landau-Lifshitz equation \((1.3)\) in \( \mathbb{R}^2 \) obtained in Theorem 1.2 with the same initial data \( n_0 \). Let
\[
\delta_n = n^2 - n^1,
\]
then we infer that
\[
\partial_t \delta_n = \alpha \delta_{n \times (h \times n)} + \beta \delta_{n \times h}.
\] \( \text{(3.1)} \)

Here and in what follows, we denote \( f^i = f(n^i) \) for \( i = 1, 2 \) and \( \delta_f = f^2 - f^1 \) if \( f \) is a function of \( n \).

Different with [32] [18], it’s not easy to obtain the positivity of the diffusion term \( \nabla \delta n \) under the metric of \( L^2 \) norm, since we can’t use the property of \( \nabla n \cdot n = -|\nabla n|^2 \) from \( |n| = 1 \). Instead, we introduce a type of weak Oseen-Frank metric
\[
W(t) = \sup_{j \geq 0} 2^{-2js} \int_{\mathbb{R}^2} W^j(t, x) dx + \|\Delta_{-1}\delta_n\|_2^2
\]
with \( s \in (0, 1) \) and
\[
W^j(x, t) = a|\nabla \Delta_j \delta_n|^2 + (k_1 - a)|\text{div} \Delta_j \delta_n|^2
\]
\[
+ (k_2 - a)|n^2 \times (\nabla \times \Delta_j \delta_n)|^2 + (k_3 - a)|n^2 \cdot (\nabla \times \Delta_j \delta_n)|^2.
\]

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The proof of Theorem 3.1 is based on the following two propositions. To state them, we introduce
\[ \tilde{h}(t) \overset{\text{def}}{=} 1 + \| (\nabla n^1, \nabla n^2) \|_4^4 + \| (\partial_t n^1, \partial_t n^2) \|_2^2 + \| (\nabla n^1, \nabla n^2) \|_{H^1}^2. \]

**Proposition 3.2** It holds that
\[ \frac{d}{dt} \| \Delta_1 \delta n \|_2^2 \leq C \tilde{h}(t) W(t). \]

**Proposition 3.3** For any \( j \geq 0 \) and \( \epsilon > 0 \), it holds that
\[ \frac{d}{dt} \int_{\mathbb{R}^2} W^j(x, t) dx + 3\alpha^2 \| \Delta_j \nabla^2 \delta n \|_2^2 \leq C 2^{2j} \tilde{h}(t) W(t) + \epsilon \sum_{l=j-9}^{j+9} 2^{4l} \| \Delta_l \delta n \|_2^2. \]

For the moment, let us assume that these propositions are correct and complete the proof of Theorem 3.1. Assume that \( T_1^i \) is the first blow-up time of \( n^i \) with \( i = 1, 2 \). We know from Lemma 2.4 that
\[ \int_{\mathbb{R}^2 \times [0, T_1 - \theta]} |\nabla^2 n|^2 + |\nabla n|^4 dx dt < +\infty, \]  
where \( \theta > 0 \) and \( T_1 = \min\{T_1^1, T_1^2\} \). And using the equation (1.3), we get
\[ \partial_t n^i \in L^2((0, T_1 - \theta) \times \mathbb{R}^2). \]  
(3.3)

Proposition 3.2 and Proposition 3.3 ensure that
\[ \frac{d}{dt} \left( \int_{\mathbb{R}^2} W^j dx + \| \Delta_1 \delta n \|_2^2 \right) + ca^{2j} \| \Delta_j \delta n \|_2^2 \leq C 2^{2j} \tilde{h}(t) W(t) + \epsilon \sum_{l=j-9}^{j+9} 2^{4l} \| \Delta_l \delta n \|_2^2. \]
Noting that \( \int_{\mathbb{R}^2} W^j dx + \| \Delta_1 \delta n \|_2^2 \geq c 2^{j} \| \Delta_j \delta n \|_2^2 \), we deduce by taking \( \epsilon \) small enough that
\[ W(t) \leq C \int_0^t \tilde{h}(\tau) W(\tau) d\tau. \]

By (3.2) and (3.3), \( \tilde{h}(t) \in L^1(0, T_1 - \theta) \). Then by Gronwall’s inequality, we get \( W(t) = 0 \) for \( t \in [0, T_1 - \theta] \) for any \( \theta > 0 \). Hence, \( n^1(t) = n^2(t) \) on \([0, T_1]\) with \( T_1 > 0 \) the first singular time of the solution \( n^1 \) or \( n^2 \). Since \( n^i \in C_w([0, +\infty); H^1_w) \), \( n^1(T_1) = n^2(T_1) \). Then the same arguments show that there exists a \( T_2 > T_1 \) such that \( n^1(t) = n^2(t) \) on \([T_1, T_2]\), where \( T_2 \) is the second singular time of the solution \( n^1 \) or \( n^2 \). Since the number of singular time is finite, we can conclude that \( n^1(t) = n^2(t) \) for \( t \in [0, +\infty) \). \( \square \)
3.1 Littlewood-Paley theory and nonlinear estimates

Let us recall some basic facts on Littlewood-Paley theory (see [3] for more details). Choose two nonnegative radial functions $\chi, \phi \in \mathcal{S}(\mathbb{R}^n)$ supported respectively in $\{\xi \in \mathbb{R}^n, |\xi| \leq \frac{3}{4}\}$ and $\{\xi \in \mathbb{R}^n, \frac{1}{4} \leq |\xi| \leq \frac{3}{4}\}$ such that for any $\xi \in \mathbb{R}^n$,

$$\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j}\xi) = 1.$$  

The frequency localization operator $\Delta_j$ and $S_j$ are defined by

$$\Delta_j f = \phi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y) f(x-y) dy, \quad \text{for} \quad j \geq 0,$$

$$S_j f = \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x-y) dy,$$

$$\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \quad \text{for} \quad j \leq -2,$$

where $h = \mathcal{F}^{-1} \phi$, $\tilde{h} = \mathcal{F}^{-1} \chi$. With this choice of $\phi$, it is easy to verify that

$$\Delta_j \Delta_k f = 0, \quad \text{if} \quad |j-k| \geq 2; \quad \Delta_j (S_{k} f \Delta_k f) = 0, \quad \text{if} \quad |j-k| \geq 5. \quad (3.4)$$

In terms of $\Delta_j$, the norm of the inhomogeneous Besov space $B^{s}_{p,q}$ for $s \in \mathbb{R}$, and $p, q \geq 1$ is defined by

$$\|f\|_{B^{s}_{p,q}} \overset{def}{=} \left\| \{2^{js} \| \Delta_j f \|_p \}_{j \geq -1} \right\|_{\ell^q},$$

and

$$\|f\|_{B^{s}_{p,\infty}} \overset{def}{=} \sup_{j \geq -1} \{2^{js} \| \Delta_j f \|_p \}.$$

We will constantly use the following Bernstein’s inequality [3].

**Lemma 3.4** Let $c \in (0,1)$ and $R > 0$. Assume that $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Then

$$\text{supp} \hat{f} \subset \{|\xi| \leq R\} \Rightarrow \| \partial^\alpha f \|_q \leq CR^{[\alpha] + n(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$

$$\text{supp} \hat{f} \subset \{cR \leq |\xi| \leq R\} \Rightarrow \|f\|_p \leq CR^{-[\alpha]} \sup_{|\beta| = [\alpha]} \| \partial^\beta f \|_p,$$

where the constant $C$ is independent of $f$ and $R$.

We need the following nonlinear estimates, see [37] for more details.

**Lemma 3.5** Let $s \in (0,1)$. For any $j \geq -1$, we have

$$\| \Delta_j (fgh) \|_2 \leq C 2^{js} (\|f\|_\infty + \|f \|_2) \|g\|_{B^{1-s}_{2,\infty}} \|h\|_2.$$

**Lemma 3.6** Let $s \in (0,1)$. For any $j \geq -1$, we have

$$\| \Delta_j (f \nabla gh) \|_2 \leq C 2^{js} \|g\|_{B^{1-s}_{2,\infty}} (\|f\|_\infty \|h\|_{H^1} + \|\nabla f\|_4 \|h\|_4 + \|\nabla^2 f\|_2 \|h\|_2)$$

$$+ C 2^{j+\frac{9}{2} + \sum_{l=j-9}^{j+9}} \| \Delta_l \nabla g \|_{\frac{3}{2}}.$$

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Lemma 3.7 Let \( s \in (0,1) \). For any \( j \geq -1 \), it holds that
\[
\| [\Delta_j, f] \nabla g \|_2 \leq C 2^j \| \nabla f \|_4 \| g \|_{B^{-s}_{2,\infty}}^\frac{1}{2} + \sum_{|j' - j| \leq 4} 2^{j'} \| \Delta_{j'} g \|_2^\frac{3}{2} + C 2^{j/2} \| g \|_{B^{-s}_{2,\infty}} \left( \| f \|_\infty + \| \nabla^2 f \|_2 \right).
\]

3.2 Proof of Proposition 3.2

Using the equation (3.1), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_{-1} \delta_n \|_2^2 = \alpha \langle \Delta_{-1} \delta_n \times \Delta_{-1} \delta_n \rangle + \beta \langle \Delta_{-1} \delta_n \times h, \Delta_{-1} \delta_n \rangle \triangleq I.
\]

Recall that the formula of \( h \) in (1.4), and we could write \( I \) as
\[
I = \langle \Delta_{-1} (M \nabla^2 \delta_n), \Delta_{-1} \delta_n \rangle + \sum_{i=1,2} \langle \Delta_{-1} (M \nabla^2 n^i \delta_n), \Delta_{-1} \delta_n \rangle + \sum_{i,k=1,2} \langle \Delta_{-1} (M \nabla n^i \nabla n^k \delta_n), \Delta_{-1} \delta_n \rangle.
\]

Here and in what follows, we denote by \( M \) a polynomial function of \((n^1, n^2)\) with degree no greater than 4, which may be different from line to line. Then by Lemma 3.5 (for \( I_2, I_4 \)), Lemma 3.6 and Lemma 3.4 (for \( I_1, I_3 \)), we get
\[
|I| \leq C \bar{h}(t) W(t).
\]

Thus the proof is complete. \( \square \)

3.3 Proof of Proposition 3.3

Let us first derive the following evolution inequality for the Oseen-Frank density.

Lemma 3.8 For any \( j \geq 0 \), it holds that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W^j(t, x) dx + 3aa^2 \| \Delta_j \nabla^2 \delta_n \|_2^2 \leq B_1 + \cdots + B_6,
\]
where \( B_i \) will be given in the proof.

The key part of the above lemma is the positivity of the diffusion term \( n \times (\Delta_j \delta_h \times n) \cdot \Delta_j \delta_h \). It’s important to analyse the main parts of \( \Delta_j \delta_h \) and \( \Delta_j \delta_{\nabla \times (h \times n)} \) (the second derivative terms). Using \( \text{curl}(fu) = f \text{curl} u + \nabla f \times u \), \( h \) in (1.4) can be rewritten as
\[
h = 2a \Delta n + 2(k_1 - a) \nabla \text{div} n - 2(k_2 - a) \text{curl} \text{curl} n - 2(k_3 - k_2) (\nabla \text{curl} n \cdot n) \times n
\]
\[\quad - 2(k_3 - k_2)(2(n \cdot \text{curl} n) \text{curl} n + (\nabla n \cdot \text{curl} n) \times n), \tag{3.5}\]

hence the main parts of \( \Delta_j \delta_h \) is
\[
W_1 = 2a \Delta_j \delta_n + 2(k_1 - a) \nabla \text{div} \Delta_j \delta_n - 2(k_2 - a) \text{curl} \text{curl} \Delta_j \delta_n
\]
\[\quad - 2(k_3 - k_2)(\nabla \text{curl} \Delta_j \delta_n \cdot n^2) \times n^2. \tag{3.6}\]
Note that by (3.5)
\[ (h \cdot n)n \]
\[ = (-2a|\nabla n|^2 + 2(k_1 - a)n \cdot \nabla \text{div}n - 2(k_2 - a)n \cdot \text{curl}\text{curl}n - 4(k_3 - k_2)(n \cdot \text{curl}n)^2)n \]
\[ = 2(k_1 - a)(n \cdot \nabla \text{div}n)n - 2(k_2 - a)(n \cdot \text{curl}\text{curl}n)n \]
\[ - 2a|\nabla n|^2n - 4(k_3 - k_2)(n \cdot \text{curl}n)^2n, \]
and \( n \times (h \times n) = h - (h \cdot n)n. \) We deduce
\[ \delta_{n \times (h \times n)} = 2a\Delta \delta_n + 2(k_1 - a)\nabla \text{div}\delta_n - 2(k_2 - a)\text{curl}\text{curl}\delta_n - 2(k_3 - k_2)(\nabla \text{curl}\delta_n \cdot n^2) \times n^2 \]
\[ - 2(k_1 - a)(n^2 \cdot \nabla \text{div}\delta_n)n^2 + 2(k_2 - a)(n^2 \cdot \text{curl}\text{curl}\delta_n)n^2 \]
\[ + \sum_{i=1,2} (M\delta_n \nabla^2 n^i + M\delta_{\nabla n} \nabla n^i) + \sum_{i,k=1,2} M\nabla n^i \nabla n^k \delta_n. \quad (3.7) \]

Denote the main parts of \( \Delta_j \delta_{n \times (h \times n)} \) as follows.
\[ H_1 = 2a\Delta \Delta_j \delta_n + 2(k_1 - a)\nabla \text{div}\Delta_j \delta_n - 2(k_2 - a)\text{curl}\text{curl}\Delta_j \delta_n \]
\[ - 2(k_3 - k_2)(\nabla \text{curl}\Delta_j \delta_n \cdot n^2) \times n^2 - 2(k_1 - a)(n^2 \cdot \nabla \text{div}\Delta_j \delta_n)n^2 \]
\[ + 2(k_2 - a)(n^2 \cdot \text{curl}\text{curl}\Delta_j \delta_n)n^2. \quad (3.8) \]

**Lemma 3.9** Assume that \( W_1, H_1 \) state as above, then we have
\[ \frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \| \Delta \Delta_j \delta_n \|^2_2 - B_1. \]
where
\[ B_1 = |\langle M\nabla n^2 \Delta_j \nabla^2 \delta_n, \Delta_j \nabla \delta_n \rangle|. \]

**Proof of Lemma 3.9** Let \( S_1 = \Delta_j \Delta \delta_n \), and
\[ H_2 = (k_1 - a)\nabla \text{div}\Delta_j \delta_n - (k_2 - a)\text{curl}\text{curl}\Delta_j \delta_n - (k_3 - k_2)(\nabla \text{curl}\Delta_j \delta_n \cdot n^2) \times n^2. \]
Then we find
\[ \frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx = \int_{\mathbb{R}^2} (aS_1 + H_2) \cdot (aS_1 + H_2 - (n^2 \cdot H_2)n^2) dx \]
\[ = a^2 \| S_1 \|^2_2 + a \langle H_2, S_1 \rangle + \| H_2 \times n^2 \|^2_2 + a \langle S_1, n^2 \times (H_2 \times n^2) \rangle \]
\[ + \frac{a^2}{4} n^2 \times S_1 \|^2_2 - \frac{a^2}{4} n^2 \times S_1 \|^2_2 \]
\[ \geq \frac{3}{4} a^2 \| \Delta_j \Delta \delta_n \|^2_2 + a \langle H_2, S_1 \rangle. \]
Furthermore, by Lemma 3.3 we have
\[ \langle H_2, S_1 \rangle = (k_1 - a)\| \nabla \text{div}\Delta_j \delta_n \|^2_2 + (k_2 - a)\| \text{curl}\text{curl}\Delta_j \delta_n \|^2_2 \]
\[ + (k_3 - k_2) \langle \nabla \Delta_j \text{curl}\delta_n \cdot n^2, \nabla \Delta_j \text{curl}\delta_n \cdot n^2 \rangle - B_1 \geq -B_1. \]
Hence we get
\[ \frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \| \Delta \delta_n \|_2^2 - B_1. \]

The proof is complete. \( \square \)

**Proof of Lemma 3.8.** Due to the definition of \( W^j \), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W^j(t,x) dx = \int_{\mathbb{R}^2} -\nabla_i W^j \partial_t \Delta_j \delta_n + W^j_p (n^j_\ell)_t dx
\]
\[ \triangleq - \int_{\mathbb{R}^2} \nabla_i W^j \partial_t \Delta_j \delta_n dx + B_1. \]

Using the equation (1.3), we get
\[
- \int_{\mathbb{R}^2} (\nabla_i W^j) \partial_t \Delta_j \delta_n dx = - \alpha \int_{\mathbb{R}^2} (\nabla_i W^j) \Delta_j (n^2 \times (h^2 \times n^2) - n^1 \times (h^1 \times n^1)) dx
\]
\[ - \beta \int_{\mathbb{R}^2} (\nabla_i W^j) \Delta_j (n^2 \times h^2 - n^1 \times h^1) dx
\]
\[ \triangleq \alpha I' + \beta I''. \]

So, we conclude that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} W^j(t,x) dx \leq \alpha I' + \beta I'' + B_2. \tag{3.9}
\]

As in Lemma 1.1, we have
\[
\nabla_a W^j_{p_\alpha} = 2a \Delta_j \delta_n + 2(k_1 - a) \nabla \text{div} \Delta_j \delta_n - 2(k_2 - a) \text{curl} \text{curl} \Delta_j \delta_n
\]
\[ - 2(k_3 - k_2) \text{curl}(n^2 \cdot \text{curl} \Delta_j \delta_n) n^2, \]
where \( p_{\alpha} = \nabla_a \Delta_j (n^2 - n^1)_t \), and we get
\[
\nabla_a W^j_{p_\alpha} = W_1 + \mathcal{M} \nabla n^2 \nabla \Delta_j \delta_n, \tag{3.10}
\]
that is, \( W_1 \) is also the main part of \( \nabla_a W^j_{p_\alpha} \). Then we have
\[
I'' = - \int_{\mathbb{R}^2} (\nabla_i W^j_{p_\alpha}) \cdot (n^2 \times \Delta_j \delta_n) dx - \int_{\mathbb{R}^2} (\nabla_i W^j_{p_\alpha}) \cdot ([\Delta_j, n^2 \times \delta_n]) dx
\]
\[ - \int_{\mathbb{R}^2} (\nabla_i W^j_{p_\alpha}) \Delta_j (\delta_n \times h^1) dx
\]
\[ \triangleq B_5 + B_3 + B_4, \]
where \( B_5 \) can be further decomposed into
\[
B_5 = - \int_{\mathbb{R}^2} (\nabla_i W^j_{p_\alpha}) \cdot (n^2 \times (\Delta_j \delta_n - W_1)) dx - \int_{\mathbb{R}^2} (\nabla_i W^j_{p_\alpha} - W_1) \cdot (n^2 \times W_1). \]
On the other hand, for the estimate of $I'$, we have
\[
I' = - \int_{\mathbb{R}^2} W_1 \cdot H_1 dx - \int_{\mathbb{R}^2} (\nabla_i W_j^{\text{ij}} - W_1) \cdot H_1 dx
- \int_{\mathbb{R}^2} \nabla_i W_j^{\text{ij}} \cdot (\Delta_j \delta_n \cdot n - H_1) dx
\]
\[
\triangleq - \int_{\mathbb{R}^2} W_1 \cdot H_1 dx + B_1 + B_6.
\]

Due to Lemma 3.9,
\[
\frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \| \Delta \Delta_j \delta_n \|_2^2 - B_1,
\]
which along with (3.9) gives the lemma. \qed

Now we follow the same route as in [37] and begin with the estimates of $B_i$.

- **Estimate of $B_1$.**
  By (3.10) and the definition of $W_1$ and $H_1$, we have
  \[
  B_1 \leq C \| \nabla n^2 \|_4 \| \Delta_j \nabla \delta_n \|_4 \| \Delta_j \nabla^2 \delta_n \|_2 \leq C \| \nabla n^2 \|_4 \| \Delta_j \nabla \delta_n \|_2^2 \| \Delta_j \nabla^2 \delta_n \|_2^2
  \]
  \[
  \leq C \| \nabla n^2 \|_4 \| \Delta_j \nabla \delta_n \|_2^2 + C^2 \| \Delta_j \delta_n \|_2^2
  \]
  \[
  \leq C \| \nabla n^2 \|_4 \| \Delta_j \nabla \delta_n \|_2^2 + C^2 \| \Delta_j \delta_n \|_2^2 + C^2 \| \Delta_j \delta_n \|_2^2 W(t).
  \]

- **Estimate of $B_2$.** Recall that
  \[
  W_j^{\text{ij}} = 2(k_3 - k_2) (n^2 \cdot \text{curl} \Delta_j \delta_n) \text{curl} \Delta_j \delta_n,
  \]
  then Lemma 3.4 yields that
  \[
  B_1 \leq C \| \Delta_j \delta_n \|_2 \| \partial n^2 \|_2 \| \Delta_j \delta_n \|_{\infty} \leq C \| \nabla n^2 \|_4 \| \Delta_j \delta_n \|_2^2 \| \partial n^2 \|_2
  \]
  \[
  \leq C \| \nabla n^2 \|_4 \| \Delta_j \delta_n \|_2^2 + C \| \Delta_j \delta_n \|_2^2 \| \Delta_j \delta_n \|_2^2
  \]
  \[
  \leq C \| \nabla n^2 \|_4 \| \Delta_j \delta_n \|_2^2 + C \| \Delta_j \delta_n \|_2^2 + C \| \Delta_j \delta_n \|_2^2 W(t).
  \]

- **Estimate of $B_6$, $B_3$, $B_4$, $B_5$.**
  By (3.10) and Lemma 3.3 for $j \geq 0$ we have
  \[
  \| \nabla_i W_j^{\text{ij}} \|_2 \leq C (\| \nabla^2 \Delta_j \delta_n \|_2 + \| \nabla n^2 \|_2 \| \Delta_j \delta_n \|_2) \leq C \| \Delta_j \delta_n \|_2.
  \]

Denote $\mathcal{B}$ the following form
\[
\sum_{i=1,2} \Delta_j (\mathcal{M} \nabla^2 n^i \delta_n) + \Delta_j (\mathcal{M} \nabla n^i \nabla \delta_n) + \sum_{i,k=1,2} \Delta_j (\mathcal{M} \nabla n^i \nabla n^k \delta_n) + [\Delta_j, \mathcal{M}] \nabla^2 \delta_n.
\]
Then by (3.7) and (3.12), we have
\[
B_6 \leq C \| \Delta_j \delta_n \|_2 \| \mathcal{B} \|_2,
\]

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where $\|\mathcal{B}\|_2$ is bounded by

$$
\sum_{i=1,2} \|\Delta_j (\mathcal{M} \nabla^2 \mathbf{n}^i \delta_n)\|_2 + \|\Delta_j (\mathcal{M} \nabla \mathbf{n}^i \nabla \delta_n)\|_2 + \sum_{i,k=1,2} \|\Delta_j (\mathcal{M} \nabla \mathbf{n}^i \nabla \mathbf{n}^k \delta_n)\|_2 + \|[\Delta_j, \mathcal{M}] \nabla^2 \delta_n\|_2.
$$

Then it follows from Lemma 3.5–Lemma 3.7 that

$$
B_6 \leq \epsilon \sum_{l=j-9}^{j+9} 2^d \|\Delta_l \delta_n\|_2^2 + C2^{2j} \tilde{h}(t) W(t).
$$

Moreover,

$$
|B_3| + |B_4| + |B_5| \leq C2^{2j} \|\Delta_j \delta_n\|_2 \|\mathcal{B}\|_2,
$$

and

$$
|B_3| + |B_4| + |B_5| + |B_6| \leq 4\epsilon \sum_{l=j-9}^{j+9} 2^d \|\Delta_l \delta_n\|_2^2 + C2^{2j} \tilde{h}(t) W(t).
$$

Thus, Proposition 3.3 follows from Lemma 3.8 and the estimates for $B_i$. \hfill \Box

### A Local well-posedness results in $\mathbb{R}^d$ with $d = 2, 3$

The symbol $\langle \cdot, \cdot \rangle$ denotes the integral in $\mathbb{R}^d$ with $d = 2, 3$. Moreover, $\mathcal{P}(\cdot, \cdot, \cdot, \cdot)$ denotes a polynomial depending on the arguments in the parentheses whose order, for example, is less than 10.

In this section, we are aimed to prove the local existence and blow-up criterion for strong solutions of the system (1.3) in $\mathbb{R}^d$ with $d = 2, 3$. Firstly, we use the classical Friedrich’s method to construct the approximate solutions of (1.3) as in [38, 36]. The main difficulty lies in the Schrödinger term $\mathbf{n} \times \mathbf{h}$, which can’t be controlled by the term $\mathbf{n} \times (\mathbf{n} \times \mathbf{h})$ when $|\mathbf{n}| \neq 1$. Hence, we introduce an equivalent system of (1.3) as follows

$$
\partial_t \mathbf{n} = \alpha \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) + \beta \mathbf{n} \times [(\mathbf{n} \times \mathbf{h}) \times \mathbf{n}],
$$

(A.1)

Secondly, blow-up criterion is similar to [36]. We’ll use a better representation formula of $\mathbf{h} \cdot \mathbf{n}$ and the vertical property of $\mathbf{n} \times \mathbf{h}$ with respect to $\mathbf{n}$.

Our main theorem states as follows.

**Theorem A.1** Let $s \geq 2$ be an integer, and the initial data $\nabla \mathbf{n}_0 \in H^{2s} (\mathbb{R}^d)$ for $d = 2, 3$. Then there exist $T > 0$ and a solution $\mathbf{n}$ of the system (1.3) such that

$$
\nabla \mathbf{n} \in C ([0, T^*); H^{2s} (\mathbb{R}^d)).
$$

Moreover, if $T^*$ is the maximal existence time of the solution, then $T^* < +\infty$ implies that

$$
\int_0^{T^*} \|\nabla \mathbf{n}(t)\|_{L^\infty}^2 dt = +\infty.
$$

The following lemma will be frequently used for the commutator; for example see [5].
Lemma A.2 For $\alpha, \beta \in N^3$ or $N^2$, it holds that
\[
\|D^{\alpha}(fg)\|_{L^2} \leq C \sum_{|\gamma|=|\alpha|} (\|f\|_{L^\infty}\|D^{\gamma}g\|_{L^2} + \|g\|_{L^\infty}\|D^{\gamma}f\|_{L^2}),
\]
and
\[
\|[D^{\alpha}, f]D^{\beta}g\|_{L^2} \leq C \left( \sum_{|\gamma|=|\alpha|+|\beta|} \|D^{\gamma}f\|_{L^2}\|g\|_{L^\infty} + \sum_{|\gamma|=|\alpha|+|\beta|-1} \|\nabla f\|_{L^\infty}\|D^{\gamma}g\|_{L^2} \right).
\]

Let $a, k_1, k_2, k_3$ be the parameters of $h$, then we have the following inequality.

Lemma A.3 For any vector $f \in L^2(\mathbb{R}^d)$, there holds
\[
(k_2 - a)\|f\|_2^2 + (k_3 - k_2)\|\mathbf{n} \cdot f\|_2^2 \geq 0.
\]

In fact, on one hand
- if $a = k_1$, then either $k_3 \geq k_2$ or $|k_3 - k_2| \leq |k_2 - a|;$
- if $a = k_2$, then $k_3 \geq k_2$;
- if $a = k_3$, then $|k_3 - k_2| = |k_2 - a|,$
on the other hand, $|\mathbf{n} \times \text{curl}|^2 + |\mathbf{n} \cdot (\text{curl } \mathbf{n})|^2 = |\text{curl } \mathbf{n}|^2$ implies the above inequality.

Proof of Theorem A.1 It’s divided into three steps.

Step 1. Construction of the approximated solutions: Let $b \in S^2$ be a constant vector, $\mathbf{n}_0 : \mathbb{R}^d \to S^2$ such that $\mathbf{n}_0 - b \in H^k(\mathbb{R}^d)$ with $k > 0$. Let
\[
\mathcal{J}_\epsilon f = \mathcal{F}^{-1}(\phi(\frac{\xi}{\epsilon})\mathcal{F}f),
\]
where $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix \xi}dx$ is usual Fourier transform and $\phi(\xi)$ is a smooth cut-off function with $\phi = 1$ in $B_1$ and $\phi = 0$ outside of $B_2$. We construct the approximate system of (A.1),
\[
\begin{cases}
\partial_t \mathbf{n}_\epsilon = \alpha \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) + \beta \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \\
\mathbf{n}_\epsilon|_{t=0} = \mathcal{J}_\epsilon \mathbf{n}_0,
\end{cases}
\]
(A.2)
where
\[
\mathcal{J}_\epsilon h_\epsilon = 2a\Delta \mathcal{J}_\epsilon \mathbf{n}_\epsilon + 2(k_1 - a)\nabla \text{div } \mathcal{J}_\epsilon \mathbf{n}_\epsilon - 2(k_2 - a)\text{curl}(\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\text{curl } \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) - 2(k_3 - k_2)(\text{curl } \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \mathcal{J}_\epsilon \mathbf{n}_\epsilon) \cdot \text{curl } \mathcal{J}_\epsilon \mathbf{n}_\epsilon.
\]

By the Cauchy-Lipschitz theorem (for example, see [8]), we know that there exists a strictly maximal time $T_\epsilon$ and a unique solution $\mathbf{n}_\epsilon - \mathbf{n}_0 \in C([0, T_\epsilon); H^k(\mathbb{R}^d))$ for any $k > 0$.

Step 2. Uniform energy estimates: We consider the evolution of the following energy norm
\[
E_\alpha(\mathbf{n}_\epsilon) = \|\mathbf{n}_\epsilon - \mathbf{n}_0\|_2^2 + \int_{\mathbb{R}^d} W(\mathbf{n}_\epsilon, \nabla \mathbf{n}_\epsilon)dx + a\|\Delta^s \nabla \mathbf{n}_\epsilon\|_2^2 + (k_1 - a)\|\Delta^s \text{div } \mathbf{n}_\epsilon\|_2^2
\]
\[(k_2 - a)\|\Delta^s \text{curl } \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_2^2 + (k_3 - a)\|\Delta^s \text{curl } \mathbf{n}_\epsilon \cdot \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_2^2,
\]

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and it’s sufficient to prove that

\[
\frac{d}{dt} E_s(n_\epsilon) \leq C \mathcal{P}(\|n_\epsilon\|_{L^\infty}, \|\nabla n_\epsilon\|_{L^\infty}, \|\nabla^2 n_\epsilon\|_{L^\infty}) E_s(n_\epsilon) \leq \mathcal{F}(E_s(n_\epsilon)),
\]

(\ref{A.3})

where we used the embedding equality with \( s \geq 2 \) and \( \mathcal{F} \) is an increasing function with \( \mathcal{F}(0) = 0 \). Indeed, it means that there exists a \( T > 0 \) depending only on \( E_s(n_0) \) such that for all \( t \in [0, \min(T, T_\epsilon)] \),

\[
E_s(n_\epsilon) \leq 2E_s(n_0),
\]

which implies that \( T_\epsilon \geq T \) by a continuous argument. Then the uniform estimates for the solutions \( n_\epsilon \) on \([0, T]\) hold which yield that there exists a local solution \( n \) of \( \text{(A.1)} \) by the standard compactness arguments. Also, if \( |n_0| = 1 \), multiply \( n \) on both sides of \( \text{(A.1)} \) and we can obtain \( |n| = 1 \).

Next, we come to prove the estimate \( \text{(A.3)} \).

\section*{2.1. Lower order terms:} In fact, using the equation \( \text{(A.2)} \) we have

\[
\frac{1}{2} \frac{d}{dt} \|n_\epsilon - n_0\|_2^2 = (\partial_t n_\epsilon, n_\epsilon - n_0)
\]

\[
\leq C(1 + \|n_\epsilon\|_{L^\infty} + \|\nabla n_\epsilon\|_{L^\infty})^4(\|\nabla n_\epsilon\|_2 + \|\Delta n_\epsilon\|_2)\|n_\epsilon - n_0\|_2
\]

\[
\leq C(1 + \|n_\epsilon\|_{L^\infty} + \|\nabla n_\epsilon\|_{L^\infty})^4 E_s(n_\epsilon)
\]

and on the other hand

\[
\frac{d}{dt} \int_{\mathbb{R}^d} W(n_\epsilon, \nabla n_\epsilon)(\cdot, t)dx
\]

\[
= \int_{\mathbb{R}^d} \left(W_{n_\epsilon} - \nabla W_{p_\epsilon}\right)(n_\epsilon, \nabla n_\epsilon)
\]

\[
\cdot \mathcal{J}_r \left(\alpha(\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon)) + \beta ((\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon))\right) dx
\]

\[
\leq C \mathcal{P}(\|n_\epsilon\|_{L^\infty}, \|\nabla n_\epsilon\|_{L^\infty}) \|\nabla n_\epsilon\|_{H^1}^2,
\]

which are the required estimates.

\section*{2.2. Higher order term:} Direct calculation shows that

\[
\frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s n_\epsilon, \nabla \Delta^s n_\epsilon \rangle = -\alpha \langle \Delta^s(\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon)), \mathcal{J}_r \Delta^{s+1} n_\epsilon \rangle
\]

\[
- \beta \langle \Delta^s(\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon)), \mathcal{J}_r \Delta^{s+1} n_\epsilon \rangle
\]

\[
:= I_1 + I_2,
\]

\[
\frac{1}{2} \frac{d}{dt} \langle div \Delta^s n_\epsilon, div \Delta^s n_\epsilon \rangle = -\alpha \langle \Delta^s(\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon)), \mathcal{J}_r div \Delta^{s} n_\epsilon \rangle
\]

\[
- \beta \langle \Delta^s(\mathcal{J}_r n_\epsilon \times (\mathcal{J}_r h_\epsilon \times \mathcal{J}_r n_\epsilon)), \mathcal{J}_r div \Delta^{s} n_\epsilon \rangle
\]

\[
:= I_1' + I_2',
\]
\[
\frac{1}{2} \frac{d}{dt} \langle J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle \\
= \alpha \langle J_{t} n_{e} \times \Delta^{s} \text{curl} J_{t} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})), J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle \\
+ \beta \langle J_{t} n_{e} \times \Delta^{s} \text{curl} J_{t} ((J_{t} n_{e} \times J_{t} n_{e}) \times (J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle) \\
+ \alpha \langle J_{t}^{2} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})) \times \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle \\
+ \beta \langle J_{t}^{2} (J_{t} n_{e} \times ((J_{t} n_{e} \times J_{t} h_{e}) \times J_{t} n_{e})) \times \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle \\
:= I_{1}' + I_{2}' + I_{3}' + I_{4}' ,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \langle J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e} \rangle \\
= \alpha \langle J_{t} n_{e} \cdot \Delta^{s} \text{curl} J_{t} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})), J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e} \rangle \\
+ \beta \langle J_{t} n_{e} \cdot \Delta^{s} \text{curl} J_{t} ((J_{t} n_{e} \times J_{t} n_{e}) \times (J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e} \rangle) \\
+ \alpha \langle J_{t}^{2} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})) \cdot \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e} \rangle \\
+ \beta \langle J_{t}^{2} (J_{t} n_{e} \times ((J_{t} n_{e} \times J_{t} h_{e}) \times J_{t} n_{e})) \cdot \Delta^{s} \text{curl} n_{e}, J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e} \rangle \\
:= I_{1}'' + I_{2}'' + I_{3}'' + I_{4}'' .
\]

Then we have

\[
2(k_{2} - a)I_{2}' + 2(k_{2} - a)I_{1}'' + 2(k_{3} - a)I_{3}'' + 2(k_{3} - a)I_{4}'' \\
\leq CP\left(||n_{e}||_{L^{\infty}}, ||\nabla n_{e}||_{L^{\infty}}, ||\nabla^{2} n_{e}||_{L^{\infty}}\right)||\nabla n_{e}||_{H^{2s}}^{2}.
\]

By the formula of \( J_{t} n_{e} \) and commutator estimates in Lemma A.2 we get

\[
||\Delta^{s} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e}))||_{L^{2}} \\
\leq ||[\Delta^{s}, J_{t} n_{e}] (J_{t} h_{e} \times J_{t} n_{e})||_{L^{2}} + ||J_{t} n_{e} \times \Delta^{s} (J_{t} h_{e} \times J_{t} n_{e})||_{L^{2}} \\
\leq P\left(||n_{e}||_{L^{\infty}}, ||\nabla n_{e}||_{L^{\infty}}, ||\nabla^{2} n_{e}||_{L^{\infty}}\right) (||\nabla n_{e}||_{H^{2s}} + ||\Delta^{s} J_{t} h_{e} \times J_{t} n_{e}||_{2}) ,
\]

(A.4)

Recall the commutator estimates of \([J_{t}, f] \) in [36],

\[
||[J_{t}, f] \nabla g||_{L^{p}} \leq C(1 + ||\nabla f||_{L^{\infty}})||g||_{L^{p}},
\]

therefore we have

\[
2aI_{1} + 2(k_{1} - a)I_{1}' + 2(k_{2} - a)I_{1}'' + 2(k_{3} - a)I_{1}''' \\
= \alpha (\Delta^{s} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})), -2a J_{t} \Delta^{s+1} n_{e} - 2(k_{1} - a) J_{t} \nabla \text{div} \Delta^{s} n_{e} \\
+ 2(k_{2} - a) J_{t} \text{curl} ((J_{t} n_{e} \times \Delta^{s} \text{curl} n_{e}) \times J_{t} n_{e}) \\
+ 2(k_{3} - a) J_{t} \text{curl} ((J_{t} n_{e} \cdot \Delta^{s} \text{curl} n_{e}) \cdot J_{t} n_{e}) \\
\leq - \alpha (\Delta^{s} (J_{t} n_{e} \times (J_{t} h_{e} \times J_{t} n_{e})), \Delta^{s} J_{t} h_{e}) \\
+ C(\delta) P\left(||n_{e}||_{L^{\infty}}, ||\nabla n_{e}||_{L^{\infty}}, ||\nabla^{2} n_{e}||_{L^{\infty}}\right)||\nabla n_{e}||_{H^{2s}} + \delta ||\Delta^{s} J_{t} h_{e} \times J_{t} n_{e}||_{L^{2}}^{2} \\
\leq - \frac{\alpha}{2} (\Delta^{s} J_{t} h_{e} \times J_{t} n_{e}, \Delta^{s} J_{t} h_{e} \times J_{t} n_{e}) + CP\left(||n_{e}||_{L^{\infty}}, ||\nabla n_{e}||_{L^{\infty}}, ||\nabla^{2} n_{e}||_{L^{\infty}}\right)||\nabla n_{e}||_{H^{2s}}^{2}
\]

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Similarly,
\[
2aI_2 + 2(k_1 - a)I_3' + 2(k_2 - a)I_3'' + 2(k_3 - a)I_3'''
\leq -\beta \langle \Delta \varphi, (\mathcal{J}_n \times (\mathcal{J}_n \times \mathcal{J}_h) \times \mathcal{J}_h), \Delta^2 \mathcal{J}_h \rangle
+ C(\delta)\mathcal{P}(|n|_{L^\infty}, |\nabla n|_{L^\infty}, |\nabla^2 n|_{L^\infty})|\nabla n|_{L^2}^2 + \delta |\nabla^2 \mathcal{J}_h \times \mathcal{J}_h|^2_{L^2}
\leq C(\delta)\mathcal{P}(|n|_{L^\infty}, |\nabla n|_{L^\infty}, |\nabla^2 n|_{L^\infty})|\nabla n|_{L^2}^2 + 2\delta |\Delta^2 \mathcal{J}_h |^2_{L^2},
\]
where we used the relation
\[
\langle (\mathcal{J}_n \times (\mathcal{J}_n \times \Delta^2 \mathcal{J}_h \times \mathcal{J}_h)), \Delta^2 \mathcal{J}_h \rangle = 0,
\]
and thus
\[
\langle \Delta \varphi, (\mathcal{J}_n \times (\mathcal{J}_n \times \Delta^2 \mathcal{J}_h \times \mathcal{J}_h) \times \mathcal{J}_h), \Delta^2 \mathcal{J}_h \rangle
= \langle [\Delta^2 \mathcal{J}_h \times \mathcal{J}_h] \times (\mathcal{J}_n \times \mathcal{J}_h), \Delta^2 \mathcal{J}_h \rangle
\leq \langle [\Delta^2 \mathcal{J}_h \times \mathcal{J}_h] \times (\mathcal{J}_n \times \mathcal{J}_h), \nabla \Delta \mathcal{J}_h \rangle
+ C(\delta)\mathcal{P}(|n|_{L^\infty}, |\nabla n|_{L^\infty}, |\nabla^2 n|_{L^\infty})|\nabla n|_{L^2}^2 + 2\delta |\Delta^2 \mathcal{J}_h |^2_{L^2}.
\]
Combining the above estimates, the inequality (A.3) is satisfied by choosing \(\delta\) is sufficiently small, thus the proof of the local existence is complete.

**Step 3. Blow-up criterion.**

Let \(T^* < \infty\) be the maximal existence time of the solution. Then it is sufficient to prove that
\[
\frac{d}{dt}E_s(n) \leq C(1 + |\nabla n|_{L^\infty}^2)E_s(n), \tag{A.5}
\]
where
\[
E_s(n) = |n - n_0|_{L^2}^2 + \int_{\mathbb{R}^d} W(n, \nabla n) dx + a|\Delta^2 \nabla n|_{L^2}^2 + (k_1 - a)|\Delta \varphi|_{L^2}^2
+ (k_2 - a)|n \times \Delta^2 (\nabla \times n)|_{L^2}^2 + (k_3 - a)|n \cdot \Delta^2 (\nabla \times n)|_{L^2}^2.
\]
The proof of (A.5) is more subtle with respect to the existence, since we can't use the bound of \(|\nabla^2 n|_{L^\infty}\). However, at this time we have \(|n| = 1\), and \(n \cdot \Delta n = -|\nabla n|^2\).

**3.1. Lower order terms:** It is easy to see that
\[
\frac{d}{dt} |n - n_0|_{L^2}^2 = \langle \partial_t n, n - n_0 \rangle
= 2\langle \alpha n \times (h \times n) + \beta n \times h, n - n_0 \rangle
\leq C(\|\Delta n\|_2 + |\nabla n|_2^2)\|n - n_0\|_2 \leq CE_s(n),
\]
and
\[
\frac{d}{dt} \int_{\mathbb{R}^d} W(n, \nabla n)(\cdot, t) dt = \int_{\mathbb{R}^d} \left( W_{nt} - \nabla_i W_{hi} \right) \partial_t n^i dx = -\alpha \int_{\mathbb{R}^d} |n \times h|^2 dx.
\]
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**Step 3.2. Higher order term:** Direct calculation shows that

\[
\frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s \mathbf{n}, \nabla \Delta^s \mathbf{n} \rangle
\]

\[
= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^{s+1} \mathbf{n} \rangle - \beta \langle \Delta^s (\mathbf{n} \times \mathbf{h}), \Delta^{s+1} \mathbf{n} \rangle := I_1 + I_2,
\]

\[
\frac{1}{2} \frac{d}{dt} \langle \text{div} \Delta^s \mathbf{n}, \text{div} \Delta^s \mathbf{n} \rangle
\]

\[
= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \nabla \text{div} \Delta^s \mathbf{n} \rangle - \beta \langle \Delta^s (\mathbf{n} \times \mathbf{h}), \nabla \text{div} \Delta^s \mathbf{n} \rangle := I'_1 + I'_2,
\]

\[
= \alpha \langle \mathbf{n} \times \Delta^s \text{curl} \mathbf{n}, \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \beta \langle \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \times \mathbf{h}, \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \alpha \langle (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s \text{curl} \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \beta \langle (\mathbf{n} \times \mathbf{h}) \times \Delta^s \text{curl} \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
:= I''_1 + I''_2 + I'''_3 + I'''_4,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \langle \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n}, \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
= \alpha \langle \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n} \times (\mathbf{h} \times \mathbf{n}), \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \beta \langle \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n} \times \mathbf{h}, \mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \alpha \langle (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s \text{curl} \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
+ \beta \langle (\mathbf{n} \times \mathbf{h}) \cdot \Delta^s \text{curl} \mathbf{n} \times \Delta^s \text{curl} \mathbf{n} \rangle
\]

\[
:= I'''_1 + I'''_2 + I'''_3 + I'''_4.
\]

For the terms \( I_1, I'_1, I''_1, I'''_1 \), we have

\[
2aI_1 + 2(k_1 - a)I'_1 + 2(k_2 - a)I''_1 + 2(k_3 - a)I'''_1
\]

\[
= \alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), -2\Delta^{s+1} \mathbf{n} - 2(k_1 - a)\nabla \text{div} \Delta^s \mathbf{n} + 2(k_2 - a)\text{curl} \text{curl} \Delta^s \mathbf{n} \rangle
\]

\[
+ \alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), 2(k_3 - k_2)\langle (\mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n}) \text{curl} \nabla (\mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n}) \rangle \times \mathbf{n} \rangle
\]

\[
= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s \nabla \mathbf{n} \rangle
\]

\[
+ 2(k_3 - k_2)\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), (\mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n}) \text{curl} - \Delta^s (\mathbf{n} \cdot \text{curl} \mathbf{n}) \rangle
\]

\[
+ 2(k_3 - k_2)\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \nabla (\mathbf{n} \cdot \Delta^s \text{curl} \mathbf{n}) \times \mathbf{n} - \nabla \Delta^s (\mathbf{n} \cdot \text{curl} \mathbf{n}) \times \mathbf{n} \rangle
\]

\[
+ 2(k_3 - k_2)\alpha \langle \nabla \Delta^s (\mathbf{n} \cdot \text{curl} \mathbf{n}) \times \mathbf{n} - \Delta^s (\mathbf{n} \cdot \text{curl} \mathbf{n}) \times \mathbf{n} \rangle
\]  
(A.6)

where we have used the following relation, for a function \( f \) and a vector field \( \mathbf{u} \), there holds

\[
\text{curl}(f \mathbf{u}) = f \text{curl} \mathbf{u} + \nabla f \times \mathbf{u}.
\]

We will use the following Gagliardo-Sobolev inequality on \( \mathbb{R}^d \) (for example, see [1]). Let \( \tau \in \mathbb{N} \), and \( \tau \geq 2s - 1 \), then for \( 1 \leq j \leq \lfloor \tau/2 \rfloor \), \( \lfloor \tau/2 \rfloor + 1 \leq k \leq \tau \), and \( f \in H^{\tau+1}(\mathbb{R}^d) \), we have

\[
\| \nabla^j f \|_{L^\infty} \leq C \| \nabla f \|_{H^{\tau+1-\lfloor \tau/2 \rfloor}} \| f \|_{L^\infty}^{1-\frac{\lfloor \tau/2 \rfloor}{\tau+1-\lfloor \tau/2 \rfloor}},
\]

\[
\| \nabla^k f \|_{L^2} \leq C \| \nabla f \|_{H^{\tau+1-\lfloor \tau/2 \rfloor}} \| f \|_{L^\infty}^{1-\frac{\lfloor \tau/2 \rfloor}{\tau+1-\lfloor \tau/2 \rfloor}}.
\]
At last, we estimate
\[ \|\nabla^{r+1}\mathbf{n}\|_{L^2} \leq \|\nabla^r\mathbf{n}\|_{L^2} + \|\nabla^r\mathbf{n}\|_{L^\infty} \leq C\|\nabla\mathbf{n}\|_{W^{r+1}}. \tag{A.7} \]
Hence, for \( r \geq 2s - 1 \) with \( s \geq 2 \), the following inequalities hold,
\[ \begin{align*}
\|\nabla^{r+1}\mathbf{n}\|_{L^2} &\leq \|\nabla^r\mathbf{n}\|_{L^2} + \|\nabla^r\mathbf{n}\|_{L^\infty} \leq C\|\nabla\mathbf{n}\|_{W^{r+1}}, \\
\|\nabla^{r+1}\mathbf{n}\|_{L^2} &\leq \|\nabla^r\mathbf{n}\|_{L^2} + \|\nabla^r\mathbf{n}\|_{L^\infty} \leq C\|\nabla\mathbf{n}\|_{W^{r+1}}.
\end{align*} \tag{A.7} \]

By Lemma [A.2] and Gagliardo-Sobolev inequality [A.7], we have
\[ \begin{align*}
2aI_1 &+ 2(k_1 - a)I_1' + 2(k_2 - a)I_2' + 2(k_3 - a)I_3'
\leq -\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla\alpha W_{p^0}\rangle \\
&+ C(\|\Delta^s\mathbf{n}\|_{L^2} + \|\Delta^s\mathbf{n}\|_{L^2} (\|\nabla\mathbf{n}\|_{L^\infty} \geq \|\nabla^2\mathbf{n}\|_{L^\infty}) + \|\Delta^s\mathbf{h} \times \mathbf{n}\|_{L^2}) \\
&\cdot (\|\Delta^s\mathbf{n}\|_{L^2} \|\nabla\mathbf{n}\|_{L^\infty} + \|\nabla\mathbf{n}\|_{L^2} \|\Delta^s\nabla\mathbf{n}\|_{L^2} + \|\Delta^s\mathbf{n}\|_{L^2} \|\nabla^2\mathbf{n}\|_{L^\infty}) \\
&\leq -\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla\alpha W_{p^0}\rangle + C_\delta\|\nabla\mathbf{n}\|_{L^\infty} \|\Delta^s\nabla\mathbf{n}\|_{L^2} + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2. \tag{A.8}
\end{align*} \]

Note that
\[ \begin{align*}
&-\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla\alpha W_{p^0}\rangle \\
&= -\alpha\langle\Delta^s\left(\mathbf{n} \times \left((\mathbf{h} - \nabla\alpha W_{p^0}) \times \mathbf{n}\right)\right), \Delta^s\nabla\alpha W_{p^0}\rangle \\
&- \alpha\langle\Delta^s\left(\mathbf{n} \times (\nabla\alpha W_{p^0} \times \mathbf{n})\right), \Delta^s(2a\Delta\mathbf{n})\rangle \\
&- \alpha\langle\Delta^s\left(\mathbf{n} \times ((\nabla\alpha W_{p^0} - 2a\Delta\mathbf{n}) \times \mathbf{n})\right), \Delta^s(\nabla\alpha W_{p^0} - 2a\Delta\mathbf{n})\rangle \\
&- 2a\alpha\langle\Delta^s(\mathbf{n} \times (\Delta\mathbf{n} \times \mathbf{n})), \Delta^s(\nabla\alpha W_{p^0} - 2a\Delta\mathbf{n})\rangle \\
&= I_{11} + I_{12} + I_{13} + I_{14}. \tag{A.9}
\end{align*} \]

Note that \( \mathbf{h} - \nabla\alpha W_{p^0} = -W_{\mathbf{n}} = -2(k_3 - k_2)(\mathbf{n} \cdot \text{curl}\mathbf{n})\text{curl}\mathbf{n} \), we have
\[ I_{11} \leq C_\delta(1 + \|\nabla\mathbf{n}\|_{L^\infty}^2)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2, \]
and similar estimates hold for the term \( I_{13} \), since \( I_{13} \) can be written as the sum of a nonnegative term and a commutator term. As to \( I_{14} \), by Lemma [A.2] and [A.3] we have
\[ \begin{align*}
I_{14} &\leq -4a(k_1 - a)\alpha\langle\nabla\Delta^s\text{div}\mathbf{n}, \nabla\Delta^s\text{div}\mathbf{n}\rangle - 4a(k_2 - a)\alpha\langle\nabla\Delta^s\text{curl}\mathbf{n}, \nabla\Delta^s\text{curl}\mathbf{n}\rangle \\
&- 4a(k_3 - k_2)\alpha\langle\mathbf{n} \cdot \nabla\Delta^s\text{curl}\mathbf{n}, \nabla\Delta^s\mathbf{n}\rangle \\
&+ C_\delta(\|\nabla\mathbf{n}\|_{L^\infty}^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2 \\
&\leq C_\delta(\|\nabla\mathbf{n}\|_{L^\infty}^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2.
\end{align*} \]

At last, we estimate \( I_{12} \). Direct calculation shows that
\[ \begin{align*}
\nabla\alpha W_{p^0} \cdot \mathbf{n}' \\
&= -2k_2\|\nabla\mathbf{n}\|^2 - 2(k_3 - k_2)(\mathbf{n} \cdot \text{curl}\mathbf{n})^2 - 2(k_1 - k_2)(\text{div}\mathbf{n})^2 + 2(k_1 - k_2)\nabla(\mathbf{n}' \cdot \text{div}\mathbf{n}). \tag{A.10}
\end{align*} \]
Thus, by Lemma \[\text{(A.2)}\] and \[\text{(A.1)}\] we infer that

\[I_{12} = -2a\alpha \langle \Delta^s \nabla_l n^t \nabla_l \Delta^s \nabla_l, \Delta^{s+1} n \rangle + 2a\alpha \langle \Delta^s (\nabla^3 \nabla_l \Delta^s \nabla_l, \Delta^{s+1} n \rangle \\
= -4a^2 \alpha (\Delta^{s+1} n, \Delta^{s+1} n) - 4a(k_1 - a)\alpha \langle \Delta^s \nabla_l \Delta^s \nabla_l, \Delta^{s+1} n \rangle \\
- 4a(k_2 - a)\alpha \langle \Delta^s \nabla_l \Delta^s \nabla_l, \Delta^{s+1} n \rangle \\
- 4a(k_3 - k_2)\alpha \langle \Delta^s \nabla_l \Delta^s \nabla_l, \Delta^{s+1} n \rangle \\
+ 2a\alpha \langle \Delta^s ((\nabla^3 \nabla_l \Delta^s \nabla_l) \Delta^{s+1} n \rangle \\
\leq -4a^2 \alpha (\Delta^{s+1} n, \Delta^{s+1} n) + C_\delta (\|\nabla n\|_{L^\infty}^2 + 1)\|\Delta^s \nabla n\|_{L^2}^2 + 2\delta\|\Delta^{s+1} n\|_{L^2}^2,
\]

where we used Lemma \[\text{(A.2)}\] and \[\text{(A.3)}\] for the last term of the second equality, we have the following observation with the help of \[\text{(A.10)}\):

\[\langle \Delta^s \nabla_l (n^t \nabla_l) \Delta^{s+1} n \rangle = \langle \Delta^s \nabla_l (n^t \nabla_l) \Delta^{s+1} n \rangle - \langle \Delta^s \nabla_l (n^t \nabla_l) \Delta^{s+1} n \rangle + \langle \Delta^s \nabla_l (n^t \nabla_l) \Delta^{s+1} n \rangle.
\]

Similarly, we have

\[2aI_2 + 2(k_1 - a)I_2'' + 2(k_2 - a)I_2'' + 2(k_3 - a)I_2'''' \leq C_\delta (\|\nabla n\|_{L^\infty}^2 + 1)\|\Delta^s \nabla n\|_{L^2}^2 + \delta\|\Delta^{s+1} n\|_{L^2}^2,
\]

and

\[|I_3''| + |I_3''| + |I_4''| \leq C_\delta (\|\nabla n\|_{L^\infty}^2 + 1)\|\Delta^s \nabla n\|_{L^2}^2 + \delta\|\Delta^{s+1} n\|_{L^2}^2.
\]

Thus, the above arguments show that \[\text{(A.5)}\] is true. \[\square\]

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