A LOCAL EXISTENCE RESULT FOR POINCARÉ-EINSTEIN METRICS

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Abstract. Given a closed Riemannian manifold \((M, g_M)\) of dimension \(n \geq 3\), we prove the existence of a conformally compact Einstein metric \(g_+\) defined on a collar neighborhood \(M \times (0, 1]\) whose conformal infinity is \([g_M]\).

Keywords: Einstein metric, conformally compact, local existence

1. INTRODUCTION

Let \(X\) be the interior of a compact manifold with boundary \(\bar{X}\) of dimension \(n + 1\), and let \(M = \partial X\) denote the boundary. A metric \(g_+\) defined on \(X\) is said to be conformally compact if there is a defining function \(\rho \in C^\infty(X)\) with \(\rho > 0\) and \(d\rho \neq 0\) on \(\partial X\), such that \(\rho^2 g\) extends to a metric \(\overline{g}\) on \(\bar{X}\). Since we can multiply \(\rho\) by any smooth positive function on \(X\), a conformally compact metric naturally defines a conformal class of metrics \([g]\) on \(M = \partial X\), called the conformal infinity of \((X, g)\).

If in addition \(g_+\) satisfies the Einstein condition, which we normalize by

\[
Ric(g_+) = -ng_+, \tag{1.1}
\]

then we say that \((X, g_+)\) is a Poincaré-Einstein (P-E) manifold. The motivating example of P-E manifolds is the Poincaré ball model of hyperbolic space \((\mathbb{B}^{n+1}, g_{BH})\), and in this case the conformal infinity is the conformal class of the round sphere \(S^n = \partial \mathbb{B}^{n+1}\). P-E manifolds play a fundamental role in the Fefferman-Graham theory of conformal invariants (see [7]), and in the AdS/CFT correspondence in quantum field theory (see, for example, [15]). Our main interest in this paper is the question of existence: given a conformal class \([g_M]\) on the closed manifold \(M = \partial X\), is there a Poincaré-Einstein metric \(g_+\) defined in \(X\) whose conformal infinity is \([g_M]\)?

A seminal existence result was proved by Graham-Lee in [8]: given a metric \(\gamma\) sufficiently close to the round metric \(\gamma_0\) on the sphere \(S^n\), there is a Poincaré-Einstein metric \(g_+\) on the ball \(B^{n+1}\) whose conformal infinity is \([\gamma]\). Later, Lee [14] extended this prove the existence of P-E metrics whose conformal infinity is sufficiently close to the conformal infinity of a given P-E metric, provided the linearized operator (suitably defined) is invertible. Anderson [1] proved a more general existence result on \(S^3\): any conformal class with positive Yamabe invariant is the conformal infinity of a P-E metric.

By contrast, in joint work with Q. Han [11] the first author proved a non-existence result for conformal classes on \(S^7\): there are infinitely many conformal classes (which can be taken in different components of the space of PSC metrics) which cannot be the conformal infinity of a P-E metric in the ball \(B^8\). The proof uses in a crucial way the work of Gromov-Lawson [9] on the space of PSC metrics.
on $S^7$, demonstrating that the existence is Poincaré-Einstein fillings is influenced by the topology of $X$ as well as the geometry of the conformal infinity.

Since there are obstructions to the global existence of Poincaré-Einstein fillings, in this paper we consider a local version: given a closed Riemannian manifold $(M, g_M)$, we find a conformally compact Einstein metric $g_+ \text{ defined on a collar neighborhood } M \times (0, 1]$ such that the conformal infinity of $g_+$ is $[g_M]$ (a more precise statement is given below). If $M$ is real analytic, then there is always a P-E metric defined on a collar neighborhood $M \times (0, 1]$; this was proved when $M$ is odd-dimensional by Fefferman-Graham in [7], and in the even-dimensional case by Kichenassamy in [12]. Also, LeBrun used twistor methods to construct an ASD Poincaré-Einstein metric in a collar neighborhood of any real analytic three-manifold, see [13]. Our interest in this paper is therefore in the $C^\infty$ category, and our main result is:

**Theorem 1.** Let $(M, g_M)$ be a smooth, connected, closed manifold of dimension $n \geq 3$. Then there is a metric $g_+$ defined on $X = M \times (0, 1]$ with the following properties:

(i) $(X, g_+)$ is a manifold with boundary $\partial X = M \times \{1\} \cong M$ satisfying the Einstein condition:

$$\text{Ric}(g_+) + ng_+ = 0.$$ 

(ii) $(X, g_+)$ is conformally compact with conformal infinity given by $(M, [g_M])$. More precisely, there is a defining function $\rho \in C^\infty(X)$ such that $\bar{g} = \rho^2 g_+$ defines a $C^0$-metric on the compact manifold with boundary $\bar{X} = M \times [0, 1]$ with

$$\bar{g} |_{M \times \{0\}} = g_M.$$ 

To give a sketch of our approach we begin by considering the model case. Let $dx^2$ denote the Euclidean metric on $\mathbb{R}^n$, and on $H^{n+1} = \mathbb{R} \times \mathbb{R}^n$ let $g_H$ denote the hyperbolic metric

$$g_H = dt^2 + e^{2t} dx^2.$$ 

We can recover the standard upper half-space model by letting $t = \log \frac{1}{y}$, so that

$$g_H = dy^2 + \frac{dx^2}{y^2}.$$ 

In particular, restricting to $\{(x, y) : x \in \mathbb{R}^n, y \in (0, 1]\}$ we obtain an Einstein metric $g_+ = g_H$ on the manifold with boundary $H_+^{n+1} = (0, \infty) \times \mathbb{R}^n$, whose compactification $\bar{g} = y^2 g_H$ gives the Euclidean metric on the boundary.

Given a compact manifold $(M, g_M)$ and $\epsilon > 0$ small, as a first approximation we define the metric

$$g_{\epsilon} = dt^2 + e^{2t} \epsilon^{-2} g_M$$

on $(0, \infty) \times M$. On a fixed compact set, when $\epsilon > 0$ is small the metric $g_{\epsilon}$ is close to the hyperbolic metric $g_H$. Our goal is to perturb $g_{\epsilon}$ to obtain a Poincaré-Einstein metric $g_+ = g_{\epsilon} + h$ on $M_+ = [0, \infty) \times M$. If we compactify by letting $\bar{g} = \epsilon^2 e^{-2t} g_+$, then assuming $h$ decays fast enough it follows that $\bar{g} |_{y=0} = g_M$ as required.

One advantage of rescaling $(M, g_M)$ and considering $g_{\epsilon}$ is that the linearized problem can be reduced, via a cutting and pasting method, to the linearized problem
on the model space $\mathbf{H}_1$, where Fourier transform methods can be used. This is somewhat reminiscent of “gluing” problems along submanifolds in the literature, such as Taubes [17] and more specifically Brendle [4] in the context of gauge theory. One key difference in our setting is that our model geometry is not a product.

It turns out that the metric $g_\epsilon$ is not a sufficiently good approximation. Roughly, $g_\epsilon$ is a solution up to an error of order $\epsilon^2$, but our estimates for the linearized operator require the error to be of order smaller than $\epsilon^4$ in order to use a fixed point argument. To remedy this we appeal to the formal solutions of Fefferman-Graham [7] to ‘correct’ $g_\epsilon$; see Lemma 2 below.

As in the global existence problem for Einstein metrics we also need to compensate for diffeomorphism invariance by introducing a ‘gauge-fixed’ version of the problem. We will consider a slight variant of the mapping defined by Graham-Lee in [9], but the essential idea is the same: we add a Lie derivative term à la DeTurck [6] in order to cancel out the degeneracies in the symbol of the linearized operator.

To prove that a zero of the gauge-fixed mapping is an Einstein metric, Graham-Lee used the Bianchi condition along with a maximum principle argument (see Lemma 2.2 of [9]). To prove the analogous result in our setting we need to impose an appropriate boundary condition on the ‘inner’ boundary. This introduces a number of technical issues that have no obvious counterpart in the work of Graham-Lee or Graham. For example, we will see that our (gauge-fixed) linear operator will in general have a finite dimensional cokernel, and we need to append the domain of the nonlinear mapping in order to get surjectivity. In addition our boundary condition is not elliptic, since it is underdetermined. One could attempt to add additional boundary conditions such as those introduced by Schlenker [16] and Anderson [2] to obtain an elliptic boundary value problem, however it seems difficult to identify the cokernels of these operators.

In this context we should also mention the work of Chruściel-Delay-Lee-Skinner on boundary regularity for Poincaré-Einstein metrics [5], in which they construct a harmonic map on a collar neighborhood of the boundary using a perturbation argument (see Theorem 4.5). However, they are imposing Dirichlet boundary conditions, and the invertibility of their linearized map follows from Theorem C of [14].

In the next section we will begin by introducing the nonlinear problem and the ‘inner’ boundary condition, and assuming the invertibility of the linearized problem we prove our main result. The remainder of the paper will be concerned with constructing a right inverse for the linearized operator.

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2. THE NONLINEAR PROBLEM

As in the Introduction, let $(M, g_M)$ be a compact $n$-manifold, and for $\epsilon > 0$ we define the metric $g_\epsilon$ on $M \times (0, \infty)$ by

$$g_\epsilon = dt^2 + e^{2t} \epsilon^{-2} g_M.$$
We want to find a symmetric 2-tensor $h$ with sufficient decay at infinity so that
\begin{equation}
\text{Ric}(g_\varepsilon + h) + n(g_\varepsilon + h) = 0,
\end{equation}
i.e., $g = g_\varepsilon + h$ is a Poincaré-Einstein metric. Before providing an outline of our argument, we begin with some preliminary remarks and definitions.

We will work in weighted Hölder spaces $C^{k,\alpha}_\delta = e^{-\delta t}C^{k,\alpha}$, with the norm
\[ \|f\|_{C^{k,\alpha}_\delta} = \|e^{\delta t}f\|_{C^{k,\alpha}} \]
in terms of the usual Hölder spaces (see Lee [14] Chapter 3). This norm extends to sections of the various tensor bundles; e.g. $\mathcal{C}^{k,\alpha}_\delta(S^2)$ will denote the space of symmetric two-tensors with respect to this norm. We will choose the weight $\delta = 1$; in practice any weight $\delta \in (0, n)$ would work, provided we start with a sufficiently good approximate solution. Constructing a better approximate solution than $g_\varepsilon$ is the point of our first technical lemma:

**Lemma 2.** Given $(M, g_M)$, there are symmetric 2-tensors $k^{(2)}, k^{(4)}$ defined on $M$ such that if
\begin{equation}
g'_\varepsilon = g_\varepsilon + k^{(2)} + e^{-2\varepsilon^2 k^{(4)}},
\end{equation}
then
\begin{equation}
\|\text{Ric}(g'_\varepsilon) + n g'_\varepsilon\|_{C^{0,\alpha}_1} = O(\varepsilon^6).
\end{equation}

**Proof.** In [7], Fefferman-Graham proved the existence of a one-parameter family of metrics $\gamma_r$ on $M$ such that the metric on $M \times (0, 1]$ given by
\[ g_+ = r^{-2}(dr^2 + \gamma_r) \]
satisfies
\begin{equation}
\text{Ric}(g_+) + n g_+ = O(r^\infty)
\end{equation}
when $n$ is odd, and
\begin{equation}
\text{Ric}(g_+) + n g_+ = O(r^{n-2})
\end{equation}
when $n$ is even. The metric $\gamma_r$ is given by a formal power series
\begin{equation}
\gamma_r = g_M + k^{(2)} r^2 + \cdots
\end{equation}
in even powers of $r$ up to order $n - 1$ when $n$ is odd, and up to order $n - 2$ when $n$ is even. Moreover, the coefficients in this range are determined by $g_M$, and obtained by differentiating (2.4) (or (2.5)) and evaluating at $r = 0$. Up to a diffeomorphism fixing $M$, when $n$ is odd there is in fact a unique formal power series solution of (2.4). When $n$ is even, formal power series exist but they are not unique (even modulo diffeomorphisms); see Theorem 2.3 of [7].

Applying the Fefferman-Graham result to our setting, we conclude the following: When the dimension $n$ is odd, there are tensors $k^{(2)}, k^{(4)}$ determined by $g_M$ such that the metric
\begin{equation}
\tilde{g} = r^{-2}(dr^2 + g_M + k^{(2)} r^2 + k^{(4)} r^4)
\end{equation}
satisfies
\begin{equation}
\text{Ric}(\tilde{g}) + n \tilde{g} = O(r^6).
\end{equation}
The same holds when $n \geq 6$ is even. When $n = 4$, the coefficient $k^{(4)}$ in (2.7) is not determined by $g_M$, but one can choose such a tensor so that (2.8) holds.
To complete the proof of the lemma, for $0 < r \leq \epsilon$ we let
\[ t = \log \frac{\epsilon}{r}. \]
Then we can rewrite the metric in (2.7) as
\[ \tilde{g} = dt^2 + \epsilon 2t^{-2} g_M + k^{(2)} + \epsilon^{-2} t^2 k^{(4)}, \]
which holds on $M \times [0, \infty)$. Also, by (2.8),
\[ \text{Ric}(\tilde{g}) + n\tilde{g} = O(\epsilon^6 e^{-6t}). \]
Taking $g'_\epsilon = \tilde{g}$, the estimate (2.3) follows. □

Remark 3. Since $g_\epsilon$ and $g'_\epsilon$ are uniformly equivalent, we can use either to measure norms defined above.

To slightly rephrase our goal in light of the preceding, we want to find a symmetric 2-tensor $h \in C^{2,\alpha}_1$ with sufficient decay at infinity so that
(2.9)
\[ \text{Ric}(g'_\epsilon + h) + n(g'_\epsilon + h) = 0. \]
The next issue we address is the well known lack of ellipticity of the linearization of this equation. We overcome this by using the standard technique of modifying by a ‘gauge-fixing’ term. To explain this we need to introduce some notation.

For metrics $g$ and $\tilde{g}$ define the mapping
(2.10)
\[ N_{\tilde{g},g}[h] = \text{Ric}(\tilde{g} + h) + n(\tilde{g} + h) + \delta^*_{\tilde{g} + h} \beta_g(h), \]
where
(2.11)
\[ \beta_g(h)_{ij} = \frac{1}{2} \left( \nabla_{\tilde{g} + h} \gamma_{ij} + \nabla_{\tilde{g} + h} \omega_{ij} \right) \]
is the Bianchi operator, and
(2.12)
\[ \delta^*_{\tilde{g} + h}(\omega)_{ij} = \frac{1}{2} \left( \nabla_{\tilde{g} + h} \omega_{ij} + \nabla_{\tilde{g} + h} \omega_{ij} \right) \]
is the $L^2$-adjoint of the divergence operator. We also let
(2.13)
\[ L_{\tilde{g},g}(h) = \frac{d}{ds} N_{\tilde{g},g}[sh]\bigg|_{s=0} \]
denote the linearization of $N$ at $h = 0$. It follows that
\[ L_{\tilde{g},g}(h) = (D \text{Ric}_{\tilde{g}} + n)h + \delta^*_{\tilde{g}} \beta_g(h), \]
where $D \text{Ric}$ denotes the linearization of the Ricci tensor. From standard formulas (see e.g. Besse [3]) we have
(2.14)
\[ L_{\tilde{g},g}(h) = -\frac{1}{2} \Delta_{\tilde{g}} h + \mathfrak{D}_{\tilde{g}}(h) + nh + \mathcal{R}_{\tilde{g}}(h), \]
where $\mathfrak{D}_{\tilde{g}}$ is given by
(2.15)
\[ \mathfrak{D}_{\tilde{g}}(h) = \delta^*_{\tilde{g}} \{ \beta_{\tilde{g}}(h) - \beta_g(h) \}, \]
and $\mathcal{R}_{\tilde{g}}$ is given by
(2.16)
\[ \mathcal{R}_{\tilde{g}}(h)_{jk} = -\check{R}^a_{\ jk} h_{ab} + \frac{1}{2} (\check{R}^a_{ k a j} + \check{R}^a_{ j a k}), \]
in terms of the curvature of $\tilde{g}$. Notice that if $\tilde{g} = g$ (or more generally, if $\tilde{g} - g$ is sufficiently small) then the linearized operator is elliptic.
Remark 4. Although it will slightly complicate the argument in certain parts, overall it is much easier to work with the Bianchi operator with respect to the metric \( g_\epsilon \) (instead of \( g'_\epsilon \)) when defining the gauge-fixing term. As we will see below (Lemma 5), the boundary condition will also be defined in terms of \( g_\epsilon \).

With this notation we can now reformulate our goal: to find a solution of

\[
N_{g'_\epsilon, g_\epsilon} [h] = 0.
\]

(2.17)

In contrast to (2.9) the linearized operator \( L_{g'_\epsilon, g_\epsilon} \) is now elliptic, since \( g'_\epsilon - g_\epsilon \) is small when \( \epsilon > 0 \) is small. Unfortunately, it is not necessarily surjective, so we need to allow additional variations of the metric \( g'_\epsilon \). We therefore consider the following modification of (2.17):

\[
(r, h) \mapsto N_{g'_\epsilon + r, g_\epsilon} [h] = \text{Ric}(g'_\epsilon + r + h) + n(g'_\epsilon + r + h) + \delta^*_{g'_\epsilon + r + h} \beta_{g_\epsilon} (h),
\]

where \( r \) will be chosen in a suitable finite-dimensional space to compensate for the lack of surjectivity of \( L_{g'_\epsilon, g_\epsilon} \).

We also need to verify that a zero of the mapping in (2.18) defines an Einstein metric. The following result is a boundary-value version of Lemma 2.2 of [9], and as a byproduct it also specifies the boundary condition we will impose:

**Lemma 5.** Suppose that \((r, h)\) is a zero of the mapping in (2.18) with \( r, h \in C^2_{1, \alpha} \) small enough so that \( g_+ = g'_\epsilon + h + r \) defines a Riemannian metric in \( M \times [0, \infty) \).

Assume

(i) On the boundary \( \{t = 0\} \), we have

\[
\beta_{g_\epsilon} (h) = 0.
\]

(2.19)

(ii) For some \( K < 0 \), \( \text{Ric}(g_+) \leq Kg_+ \).

Then \( \beta_{g_\epsilon} (h) = 0 \) on \( M \times [0, \infty) \), and hence (by (2.18)) \( g_+ \) is a Poincaré-Einstein metric.

**Proof.** We let \( \omega = \beta_{g_\epsilon} (h) \). Applying the Bianchi identity to (2.18), we obtain

\[
\beta_{g_\epsilon} (\delta^*_{g_+} \omega) = 0.
\]

As in [9], this implies

\[
\Delta_{g_+} |\omega|^2 \geq -K |\omega|^2.
\]

Since \( \omega = 0 \) on the boundary \( \{t = 0\} \), and \( \omega \to 0 \) as \( t \to \infty \), the maximum principle implies that \( \omega = 0 \) everywhere. \( \square \)

We are thus led to studying the linearization of the mapping in (2.18), subject to the boundary condition \( \beta_{g_\epsilon} (h)|_{t=0} = 0 \). Using (2.14), the linearization of (2.18) is given by

\[
\bar{L}_{g'_\epsilon, g_\epsilon} : E \times (C^2_{1, \alpha})_\beta \to C^1_{0, \alpha}
\]

\[
(r, h) \mapsto (D\text{Ric}g'_\epsilon + n)r + L_{g'_\epsilon, g_\epsilon} (h),
\]

where \( E \) is a certain finite dimensional subspace of \( C^2_{1, \alpha} \), to be determined later, and \((C^2_{1, \alpha})_\beta\) denotes the space of symmetric two tensors \( h \in C^2_{1, \alpha} \) satisfying the boundary condition \( \beta_{g_\epsilon} (h)|_{t=0} = 0 \).

Most of our work in the paper will be constructing a right inverse for this linearized operator, leading to the following, proved in Section 4.4.
**Theorem 6.** Let $\epsilon, \alpha > 0$ be sufficiently small. If the metric $g_M$ is chosen generically in its conformal class, then for a suitable finite dimensional subspace $E$ the linearized operator $L_{\nabla \epsilon}$ has a right inverse $R$, satisfying $\|R\| \leq C\epsilon^{-2-\alpha}$ for a constant $C$ independent of $\epsilon$.

Using this result together with Lemma 2 a standard contraction mapping argument can be used to solve Equation 2.18 as follows. Let us define the operator $Q$ by

$$\text{Ric}(g'_\epsilon + r + h) + n(g'_\epsilon + r + h) + \delta^*_g + \kappa \beta_g(h) = \text{Ric}(g'_\epsilon) + n g'_\epsilon + \nabla\nabla\epsilon, g_\epsilon, (r, h) + Q(r, h),$$

and define $F$ by

$$\mathcal{F} : E \times (C^2_1) \beta \to E \times (C^2_1) \beta$$

$$(r, h) \mapsto -R \left[ \text{Ric}(g'_\epsilon) + n g'_\epsilon + Q(r, h) \right].$$

A fixed point of $\mathcal{F}$ then necessarily satisfies Equation (2.18).

Define the set

$$\mathcal{U} = \{ (r, h) \in E \times (C^2_1) : \| (r, h) \| \leq \epsilon^3 \},$$

using the norm

$$\| (r, h) \| = \| r \|_{C^2_1} + \| h \|_{C^2_1}.$$

**Proposition 7.** For sufficiently small $\epsilon, \alpha$ the map $F$ defines a contraction $\mathcal{F} : \mathcal{U} \to \mathcal{U}$, and so it has a fixed point.

**Proof.** First note that by differentiating Equation (2.20) with respect to $g'_\epsilon$ and applying the mean value theorem (or alternatively expanding $Q$ as a power series), we find that as long as $\| (r_1, h_1) \|, \| (r_2, h_2) \| < \kappa < c_0$ for a fixed constant $c_0$, we have

$$\| Q(r_1, h_1) - Q(r_2, h_2) \|_{C^2_1} \leq C\kappa \| (r_2 - r_1, h_2 - h_1) \|.$$

Using our bound for the right inverse $R$, it follows that as long as $(r_i, h_i) \in \mathcal{U}$, and $\epsilon$ is sufficiently small, we have

$$\| F(r_1, h_1) - F(r_2, h_2) \| \leq C\epsilon^{1-\alpha} \| (r_1 - r_2, h_1 - h_2) \|,$$

and so $F$ is a contraction.

Finally to check that $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$ we let $(r, h) \in \mathcal{U}$. Then

$$\| F(r, h) \| \leq \| F(r, h) - F(0, 0) \| + \| F(0, 0) \|$$

$$\leq C\epsilon^{1-\alpha} \| (r, h) \| + C\epsilon^{1-\alpha} \leq \epsilon^3$$

for sufficiently small $\epsilon$. Here we used that by Lemma 2 and the bound for $R$ we have $\| F(0, 0) \| \leq C\epsilon^{-2-\alpha} \epsilon^6$. \hfill $\square$

The existence of a fixed point of $F$ together with Proposition 5 then completes the proof of Theorem 1. In the remainder of this section we give a brief outline of the proof of Theorem 6.

The first step, in Section 3 is to carefully analyze the linearized operator $L_H = L_{g_H, g_H}$ in the model case when $g_H$ is the hyperbolic metric, i.e. $M = \mathbb{R}^n$ and $g_M$ is the Euclidean metric. The main result here is Theorem 3 below, which roughly speaking says the following: given a 2-tensor $u$ supported inside the unit ball in the
spatial direction, and satisfying an additional "orthogonality condition" \( I(u) = 0 \), we can solve \( L_h(u) = 0 \) with \( h \) satisfying the Bianchi boundary condition (with respect to \( g_H \)), such that \( h \) is localized in the sense that it has good decay in the spatial directions. Here \( I(u) \) is a one-form on \( M \), see (3.2) for its definition.

To illustrate this, consider the following simple analogous result. Let \( \Delta \) be the Laplacian on the product space \( \mathbb{R}^n \times X \) for a compact Riemannian manifold \( X \), and let \( u \) be a function supported in \( B_1 \times X \). We can then construct a solution of \( \Delta h = u \) with \( h \) satisfying the Bianchi boundary condition (with respect to \( g_H \)), such that \( h \) is localized in the sense that it has good decay in the spatial directions. Here \( I(u) \) is a one-form on \( M \), see (3.2) for its definition.

The next step is to globalize this result to the case when \( M \) is a compact manifold. The idea is that when \( \epsilon \) is sufficiently small, then locally \( (M, \epsilon^{-2}g_M) \) is well approximated by Euclidean space. We can then solve the equation \( L_{g'(\epsilon), g} h = u \) on \( M \times [0, \infty) \) as long as \( u \) satisfies the orthogonality condition \( I(u) = 0 \), by chopping \( u \) up into pieces supported in approximately Euclidean balls, and combining the “local” inverses constructed in the model space. The decay of the corresponding local solutions ensures that we get a good estimate for the error obtained from combining these local solutions. We need some additional steps to ensure that after this cutting and pasting procedure we can still impose the Bianchi condition.

It remains to deal with the case when \( I(u) \neq 0 \). Since \( I(u) \) is a one-form on \( M \), we are able to reduce this to inverting a suitable linear operator on \( M \). More precisely, we consider the operator

\[
T : C^{2, \alpha}(\Omega^1(M)) \to C^{0, \alpha}(\Omega^1(M))
\]

\[
\omega \mapsto I \circ L_{g'(\epsilon), g} (\epsilon^{-2}e^{-nt} \omega \otimes dt)
\]

It turns out that \( T \), which depends on \( \epsilon \), converges to an elliptic operator \( T_0 \) as \( \epsilon \to 0 \), but \( T_0 \) is not necessarily surjective. It is this issue that we overcome by incorporating an additional finite dimensional space \( E \) of symmetric 2-tensors on \( M \times [0, \infty) \) in the problem, and instead we consider the operator

\[
\overline{T} : E \times C^{2, \alpha}(\Omega^1(M)) \to C^{0, \alpha}(\Omega^1(M))
\]

\[
(r, \omega) \mapsto I \left[ (DRic_{g'_\epsilon} + n)r + L_{g'(\epsilon), g} (\epsilon^{-2}e^{-nt} \omega \otimes dt) \right]
\]

Although this operator is not elliptic in \( r \), we only need a finite dimensional space \( E \) since the cokernel of \( T \) is finite dimensional. It turns out that as long as \( g_M \) admits no Killing vector fields, we can choose a finite dimensional space \( E \) such that \( \overline{T} \) is surjective. This is then enough to construct the right inverse required in Theorem 6.

3. The linearized operator on Hyperbolic space

In this section we study the linearized operator \( L_{g^H} = L_{g_H, g^H} \) in (2.14) on hyperbolic space \( H^{n+1} \) with the hyperbolic metric \( g^H \). A standard calculation gives

\[
L_{g^H} h = -\frac{1}{2} \Delta g^H h - h + (\text{tr} g^H) g^H.
\]

A basic result (see [14], Theorem 5.9) is the following:
Theorem 8. On hyperbolic space $H^{n+1}$, the linearized operator $L = L_{gH}$ at the hyperbolic metric is an isomorphism $L : C^k_\delta \to C^{k-2, \alpha}_\delta$, as long as $|\delta - n/2| < n/2$. In particular, this holds for our choice of weight.

The main technical result we will need is a variant Theorem 8 solving a boundary value problem. As above let $H^{n+1}_+ = R^n \times [0, \infty)$, a subset of hyperbolic space equipped with the hyperbolic metric

$$g_{H} = dt^2 + e^{2t}(dx^i)^2.$$ 

We will sometimes write $x^0 = t$. Indices $i, j, k, l, \ldots$ run from 1 to $n$, while indices $a, b, c, \ldots$ run from 0 to $n$.

For a symmetric 2-tensor $u$ on $H^{n+1}_+$ define the one-form $I : T R^n \to R$ on $R^n$ by

$$(3.2) I(u)(V) = \int_0^\infty u(V, \partial_t)e^{-2t}dt$$

where $V \in TR^n$. More generally, given a manifold $M$ and a symmetric 2-tensor $u \in C_{\delta}^{0, \alpha}(M \times [0, \infty))$, then (3.2) defines a one-form $I(u)$ on $M$ as long as $\delta > -2$.

Theorem 9. Suppose that $u \in C_{1}^{0, \alpha}$ is a symmetric two-tensor on $H^{n+1}_+$ supported in $B_1 \times [0, \infty)$, with $I(u) = 0$. Then there exists a symmetric two-tensor $h \in C_{1}^{2, \alpha}$ on $H^{n+1}_+$ satisfying

1. $Lh = u$, and $||h||_{C^2, \alpha} \leq C||u||_{C^0, \alpha}$ for a uniform constant $C$.
2. $\beta_{gH}(h) = 0$ along the boundary $\{t = 0\}$.
3. For any $\delta \in (0, 1)$, $h$ decays in the $x^i$ directions, at a rate of at least $|x|^{-n-1+\delta}$. More precisely, let $A_{R-1, R} = (B_R \setminus B_{R-1}) \times [0, \infty)$. We have $||h||_{C^{0, \alpha}_1(A_{R-1, R})} \leq CR^{-n-1+\delta}||u||_{C^{0, \alpha}_1}$ for all $R > 1$, for a uniform constant $C$.

We define the linear operator $P_H$ by setting $P_H(u) = h$.

The rest of this section will be devoted to the proof of Theorem 9. Since it is rather involved, we begin with a sketch.

Given a symmetric 2-tensor $u$ as in the statement of the theorem, the first step is to construct a solution $h_0$ of

$$(3.3) Lh_0 = u$$

on $[0, \infty) \times M$, using the Green’s function of $L$. Note that this solution will not in general satisfy the Bianchi condition $\beta_{gH}h_0 = 0$ on the boundary $\{t = 0\}$. Therefore, we need to ‘correct’ our solution by solving the homogeneous boundary-value problem

$$(3.4) \begin{cases} Lh_1 = 0 \text{ in } H^{n+1}_+, \\ \beta_{gH}(h_1) = \beta_{gH}(h_0) \text{ on } \partial H^{n+1}_+ = \{t = 0\} \times R^n, \end{cases}$$

where $h_0$ solves (3.3). Then taking $h = h_0 - h_1$, we arrive at a solution of the original problem. We will solve the homogeneous problem using the Fourier transform, and analyzing the resulting ODEs. The required decay in Theorem 9 will be obtained by controlling the singularity of the Fourier transform at the origin, and the orthogonality condition $I(u) = 0$ is used to ensure that the terms with the worst singularity vanish, thereby improving the decay of the solution.
The Fourier transform of the homogeneous problem. We begin by writing down explicit formulas for the components of \( Lh \) for a symmetric 2-tensor \( h \) with respect to the coordinates \( x^i, t \). We will write \( x^0 = t \), and use the convention that indices \( i, j, k, \ldots \) run from 1 to \( n \), while \( a, b, c, \ldots \) run from 0 to \( n \).

Lemma 10. With respect to the basis \( \{ \partial_{x^0}, \ldots, \partial_{x^n} \} \), the only nonzero Christoffel symbols are

\[
\Gamma^0_{jk} = -e^{2t} \delta_{jk},
\]

\[
\Gamma^i_{0k} = \Gamma^i_{k0} = \delta^i_k,
\]

where \( \nabla \) is the Riemannian connection.

More generally, if \((M, g_M)\) is a Riemannian manifold, \( g = dt^2 + e^{2t} g_M \) is a warped product metric, and \( \{ x^i \} \) are local coordinates on \( M \), then the only nonzero Christoffel symbols with respect to the coordinate system \( \{ x^1, \ldots, x^n, x^0 = t \} \) on \( M \times [0, \infty) \) are

\[
\Gamma^m_{jk} = (\Gamma^m_M)_{jk},
\]

\[
\Gamma^0_{jk} = -e^{2t} (g_M)_{jk},
\]

\[
\Gamma^i_{0k} = \Gamma^i_{k0} = \delta^i_k,
\]

where \( \Gamma_M \) are the Christoffel symbols with respect to \( g_M \).

This is a straightforward calculation, and we will omit the proof. Using these formulas, we have the following identities for the components of the covariant derivatives of a symmetric two-tensor:

\[
\nabla_i h_{jk} = \partial_i h_{jk} + e^{2t} \delta_{ij} h_{0k} + e^{2t} \delta_{ik} h_{0j} - h_{ij}
\]

\[
\nabla_0 h_{jk} = \partial_0 h_{jk} - 2h_{jk}
\]

\[
\nabla_i h_{j0} = \partial_i h_{j0} + e^{2t} \delta_{ij} h_{00} - h_{ij}
\]

\[
\nabla_0 h_{j0} = \partial_0 h_{j0} - h_{j0}
\]

\[
\nabla_j h_{00} = \partial_j h_{00} - 2h_{00}
\]

\[
\nabla_0 h_{00} = \partial_0 h_{00}.
\]

Using these formulas we can compute the Bianchi operator:

\[
\beta(h)_a = g^{bc} \nabla_k h_{ac} - \frac{1}{2} \nabla_a (g^{bc} h_{bc}).
\]

Its components are

\[
\beta(h)_i = e^{-2t} \partial_i h_{1i} + \partial_0 h_{i0} + h_{i0} - \frac{1}{2} e^{-2t} \partial_i h_{kk} - \frac{1}{2} \partial_i h_{00}
\]

\[
\beta(h)_0 = e^{-2t} \partial_0 h_{00} + nh_{00} + \frac{1}{2} \partial_0 h_{00} - \frac{1}{2} e^{-2t} \partial_0 h_{ii}.
\]
We can also take another covariant derivative and compute the components of the rough laplacian acting on symmetric 2-tensors $\Delta = g^{ab}_{ij} \nabla_a \nabla_b$:

$$
\Delta h_{jk} = e^{-2t} \partial_t \delta_{ij} h_{jk} + \partial_t^2 h_{jk} + (n - 4) \partial_t h_{jk} + (2 - 2n) h_{jk}
+ 2 e^{2t} h_{00} \delta_{jk} + 2(\partial_t h_{0k} + \partial_k h_{0j}),
$$

$$
\Delta h_{j0} = e^{-2t} \partial_t \delta_{i} h_{j0} + \partial_t^2 h_{j0} + (n - 2) \partial_t h_{j0} - 2(n + 1) h_{j0}
+ 2 \partial_j h_{00} - 2 e^{-2t} \partial_i h_{i j}
$$

$$
\Delta h_{00} = e^{-2t} \partial_t \delta_{i} h_{00} + \partial_t^2 h_{00} + n \partial_t h_{00} - 2n h_{00}
- 4 e^{-2t} \partial_i h_{0i} + 2 e^{-2t} h_{kk}.
$$

Combining the above, we can write the equation $L h = u$ as a system of equations in the components of $h$ and $u$:

$$
u_{jk} = e^{-2t} \partial_t \partial_t h_{jk} + \partial_t^2 h_{jk} + (n - 4) \partial_t h_{jk} + (4 - 2n) h_{jk}
- 2 h_{i j} \delta_{jk} + 2(\partial_t h_{0k} + \partial_k h_{0j})
$$

$$
u_{j0} = e^{-2t} \partial_t \partial_t h_{j0} + \partial_t^2 h_{j0} + (n - 2) \partial_t h_{j0} - 2(n + 1) h_{j0}
+ 2 \partial_j h_{00} - 2 e^{-2t} \partial_i h_{i j}
$$

$$
u_{00} = e^{-2t} \partial_t \partial_t h_{00} + \partial_t^2 h_{00} + n \partial_t h_{00} - 2n h_{00}
- 4 e^{-2t} \partial_i h_{0i}.
$$

(3.6)

In the following, we will use upper-case letters to denote the Fourier transforms of components of $h$, scaled by additional powers of $e^{-t}$. This amounts to writing our tensor $h$ in terms of an orthonormal frame, and it leads to an ODE system which is easier to analyze. With this in mind we define

$$
H_{ij}(t, \xi) = e^{-2t} \hat{h}_{ij}(t, \xi) = e^{-2t} \int_{\mathbb{R}^n} e^{-\sqrt{-1} \xi \cdot x} h_{ij}(t, x) dx,
$$

(3.7)

$$
H_{j0}(t, \xi) = e^{-t} \hat{h}_{j0}(t, \xi) = e^{-t} \int_{\mathbb{R}^n} e^{-\sqrt{-1} \xi \cdot x} h_{j0}(t, x) dx,
$$

$$
H_{00}(t, \xi) = \hat{h}_{00}(t, \xi) = \int_{\mathbb{R}^n} e^{-\sqrt{-1} \xi \cdot x} h_{00}(t, x) dx,
$$

and similarly we will write $U_{ij} = e^{-2t} \hat{u}_{ij}$, etc. After applying the Fourier transform to the system (3.6), we obtain the following system of ODEs:

$$
H''_{jk} + n H'_{jk} - 2 \delta_{jk} H_{pp} - 2 \sqrt{-1} e^{-t}(\xi_j H_{0k} + \xi_k H_{0j}) - e^{-2t} |\xi|^2 H_{jk} = U_{jk}
$$

$$
H''_{j0} + n H'_{j0} - (n + 1) H_{j0} - 2 e^{-t} \sqrt{-1}(\xi_j H_{00} - \xi_i H_{ij}) - e^{-2t} |\xi|^2 H_{j0} = U_{j0}
$$

$$
H''_{00} + n H'_{00} - 2n H_{00} + 4 \sqrt{-1} e^{-t} \xi_i H_{0i} - e^{-2t} |\xi|^2 H_{00} = U_{00},
$$

(3.8)

and we are for now interested in the case when $U = 0$. 

Applying the Fourier transform to the components of the Bianchi operator in (3.5) gives

\[ B_\xi(H)_i = e^{-t\hat{\beta}(h)}_i = H'_0 + (n + 1)H_{i0} - e^{-t}\sqrt{-1}\xi_{ij}H_{ij} + \frac{1}{2}e^{-t}\sqrt{-1}\xi_{ii}H_{pp} + \frac{1}{2}e^{-t}\sqrt{-1}\xi_{ii}H_{00} \]

(3.9)

\[ B_\xi(H)_0 = \hat{\beta}(h)_0 = nH_{00} + \frac{1}{2}H'_{i0} - \frac{1}{2}H''_{pp} - H_{pp} - e^{-t}\sqrt{-1}\xi_{ii}H_{00}. \]

3.2. Solutions for small \( \xi \). We will assume that \( |\xi| \) is small, and find solutions of the system of ODEs as perturbations of solutions to the simpler system when \( \xi = 0 \), as a power series in \( \xi, \bar{\xi} \). Let us write the ODEs (3.5) with \( U = 0 \) as \( L_\xi(H) = 0 \). If we write

\[ H(\xi, t) = H(0, t) + \xi_0 \partial_{\xi,0}H(0, t) + \xi_1 \partial_{\xi,1}H(0, t) + \ldots, \]

then we can obtain equations satisfied by \( H(0, t) \) and \( \partial_{\xi,0}H(0, t) \) by differentiating the equation \( L_\xi(H) = 0 \) and setting \( \xi = 0 \). In particular, \( H(0, t) \) satisfies

\[ H''_{jk} + nH'_{jk} - 2\delta_{jk}H_{pp} = 0 \]

(3.11)

\[ H''_{0j} + nH'_{0j} - (n + 1)H_{0j} = 0 \]

\[ H''_{00} + nH'_{00} - 2nH_{00} = 0. \]

\( \partial_{\xi,0}H(0, t) \) also satisfies the same equations, while \( \partial_{\xi,1}H(0, t) \) satisfies

\[ \left[ \partial_{\xi,0}H''_{jk} + n\partial_{\xi,1}H'_{jk} - 2\delta_{jk}\partial_{\xi,0}H_{pp} \right] - 2e^{-t}\sqrt{-1}(\delta_{ij}H_{00} + \delta_{jk}H_{0j}) = 0 \]

(3.12)

\[ \left[ \partial_{\xi,0}H''_{0j} + n\partial_{\xi,1}H'_{0j} - (n + 1)\partial_{\xi,0}H_{0j} \right] - 2e^{-t}\sqrt{-1}(\delta_{ij}H_{00} - \delta_{ij}H_{0i}) = 0 \]

\[ \left[ \partial_{\xi,0}H''_{00} + n\partial_{\xi,1}H'_{00} - 2n\partial_{\xi,0}H_{00} \right] + 4\sqrt{-1}e^{-t}\delta_{0i}H_{00} = 0. \]

For higher order derivatives \( \partial_{\xi,k}^2 \), we will have a system that we write schematically as

\[ L(\partial_{\xi,k}^2 H) + e^{-t} \ast \partial_{\xi,k-1}^2 H + e^{-2t} \ast \partial_{\xi,k-2}^2 H = 0, \]

where \( L \) is the homogeneous 2nd order operator appearing in square brackets above.

The solutions of the system for \( H, \partial_{\xi}H \) that we write down below will all be of order \( e^{-nt} \) and \( e^{-(n+1)t} \) respectively, or smaller. Because of the additional factors of \( e^{-t} \), it follows that the inhomogeneous equations (3.13) for \( \partial_{\xi,0}^2 H \) has a solution of order \( e^{-(n+k)t} \), or smaller, and so the solution \( H \) given by the series (3.10) satisfies \( H = O(e^{-nt}) \) as \( t \to \infty \).

3.2.1. Solutions of type I. Let \( a_{ij} \) be any trace free symmetric matrix. Define

\[ H_{ij} = \sqrt{-1}a_{ij}e^{-nt}, \]

\[ H_{i0} = 0, \]

\[ H_{00} = 0. \]

This solves the equations (3.11). We can set \( \partial_{\xi,0}H = 0 \), and also in (3.12) only the equations involving \( \partial_{\xi,0}H_{0j} \) are inhomogeneous. So we can let \( \partial_{\xi,0}H_{ij}, \partial_{\xi,0}H_{00} = 0 \), while \( \partial_{\xi,0}H_{0j} \) satisfies

\[ \left[ \partial_{\xi,0}H''_{0j} + n\partial_{\xi,1}H'_{0j} - (n + 1)\partial_{\xi,0}H_{0j} \right] - 2e^{-t}a_{ij}e^{-nt} = 0. \]
A solution of this ODE is
\[ \partial_\xi H_{j0} = -\frac{2}{n+2} a_{ij} t e^{-(n+1)t}. \]

We can similarly obtain solutions of the equations obtained by differentiating \( L_\xi (H) = 0 \) more than once, and solve them inductively. The inhomogeneous terms in these equations will all be of order \(|\xi|^2 e^{-(n+2)t} \) or smaller. It follows that we can find a solution \( H^1 \) of our system (3.8) such that
\[
\begin{align*}
H^1_{ij} &= \sqrt{-1} a_{ij} e^{-nt} + O(|\xi|^2 e^{-(n+2)t}), \\
H^1_{i0} &= b_{ij} \xi_j e^{-(n+1)t} + O(|\xi|^2 e^{-(n+2)t}), \\
H^1_{00} &= O(|\xi|^2 e^{-(n+2)t}).
\end{align*}

Let \( B_\xi \) denote the Fourier transform of the Bianchi operator, i.e., the operator appearing on the RHS of (3.9). Then
\[
\begin{align*}
B_\xi (H^1)_i &= e^{-(n+1)t} \xi_j a_{ij} \left[ \frac{-2}{n+2} + 1 \right] + O(|\xi|^2 e^{-(n+2)t}), \\
B_\xi (H^1)_0 &= O(|\xi|^2 e^{-(n+2)t}).
\end{align*}
\]

Evaluating at \( t = 0 \) we have
\[
\begin{align*}
B_\xi (H^1)_i |_{t=0} &= \xi_j a_{ij} \frac{n}{n+2} + O(1), \\
B_\xi (H^1)_0 |_{t=0} &= O(|\xi|^2).
\end{align*}
\]

### 3.2.2. Solutions of type II, III.

We now let
\[ \lambda = \frac{n + \sqrt{n^2 + 8n}}{2}, \]
and note that \( n + 1 < \lambda < n + 2 \). For constants \( a, b \), let us set
\[
\begin{align*}
H_{jk} &= a e^{-\lambda t} \delta_{jk}, \\
H_{j0} &= 0, \\
H_{00} &= b e^{-\lambda t}.
\end{align*}
\]

These give a solution of (3.11). Again, from (3.12) only the equations for \( \partial_\xi H_{j0} \) have a nonzero inhomogeneous term:
\[
\left[ \partial_\xi H''_{j0} + n \partial_\xi H'_{j0} - (n + 1) \partial_\xi H_{j0} \right] - 2 \sqrt{-1} e^{-t} (b - a) \delta_{ij} e^{-\lambda t} = 0.
\]

A solution of this equation is
\[ \partial_\xi H_{j0} = K (b - a) \delta_{ij} e^{-(\lambda+1)t}, \]
where
\[ K = \frac{2 \sqrt{-1}}{\lambda^2 + (2-n)\lambda - 2n}. \]

As before, it follows that we can find a solution \( \tilde{H} \) of (3.8) satisfying
\[
\begin{align*}
\tilde{H}_{jk} &= a e^{-\lambda t} \delta_{jk} + O(|\xi|^2 e^{-(\lambda+2)t}), \\
\tilde{H}_{j0} &= (b - a) \xi_j e^{-(\lambda+1)t} + O(|\xi|^2 e^{-(\lambda+2)t}), \\
\tilde{H}_{00} &= b e^{-\lambda t} + O(|\xi|^2 e^{-(\lambda+2)t}).
\end{align*}
\]
Lemma 12. For each $H$, this solution $a$ is a two-dimensional space of solutions of type IV, satisfying

$$\left. B_\xi(H)_{ij}\right|_{t=0} = \left[ \left( (\lambda - n)K + \frac{n-2}{2}\sqrt{-1} \right) a + \left( (n - \lambda)K + \frac{1}{2}\sqrt{-1} \right) b \right] \xi_i + O(\|\xi\|^2),$$

$$\left. B_\xi(H)_{00}\right|_{t=0} = \left( n - \frac{\lambda}{2} \right) b - \left( n - \frac{\lambda}{2} \right) a + O(\|\xi\|^2).$$

Choosing $a, b$ suitably, we obtain two different solutions, $H^2, H^3$ of (3.17), satisfying

$$\left. B_\xi(H^2)_{ij}\right|_{t=0} = \xi_i + O(\|\xi\|^2),$$

$$\left. B_\xi(H^2)_{00}\right|_{t=0} = O(\|\xi\|^2),$$

and

$$\left. B_\xi(H^3)_{ij}\right|_{t=0} = O(\|\xi\|^2),$$

$$\left. B_\xi(H^3)_{00}\right|_{t=0} = 1 + O(\|\xi\|^2).$$

3.2.3. Solutions of type IV. With the same choice of $\lambda$ as above, set

$$H_{jk} = e^{-\lambda t}\delta_{jk},$$

$$H_{00} = e^{-\lambda t},$$

$$H_{0i} = b_i e^{-(n+1)t},$$

for arbitrary $b_1, \ldots, b_n$. This tensor satisfies (3.11) and as above, we can iteratively solve inhomogeneous ODEs for $\partial_\xi H$ to find a solution $H^4 = O(e^{-nt})$. We will not need to know the value of the Bianchi operator for these solutions.

**Lemma 11.** The solutions of types I, II, III and IV together form an $(n+1)(n+2)/2$-dimensional space of solutions of $L_\xi(H) = 0$. Moreover, all of them decay at a rate of at least $e^{-nt}$ as $n \to \infty$.

**Proof.** To explain the dimension count: the solutions of type I are in one-to-one correspondence with trace-free symmetric $n \times n$ matrices; hence the dimension of this space of solutions is $n(n+1)/2 - 1$. The solutions of type II and III depend on two different choices of the parameter $a$, hence there is a two-dimensional space of these kinds of solutions. Finally, the set of solutions of type IV is obviously $n$-dimensional, since we can choose the vector $(b_1, \ldots, b_n)$ arbitrarily. Summing, we have $[n(n+1)/2 - 1] + 2 + n = (n+1)(n+2)/2$. It is clear from the leading terms in (3.14), (3.16), and (3.17) that this family of solutions is linearly independent. □

3.2.4. Prescribing the boundary condition for small $\xi$. We can now combine the solutions $H^1, H^2, H^3$ that we obtained above, to find that for any symmetric matrix $a_{ij}$ (not necessarily trace free), and constant $a$, there is a solution of $L_\xi(H) = 0$ satisfying

$$\left. B_\xi(H)_{ij}\right|_{t=0} = a_{ij}\xi_j + O(\|\xi\|^2)$$

$$\left. B_\xi(H)_{00}\right|_{t=0} = a + O(\|\xi\|^2).$$

This solution $H$ is a smooth function of $\xi$, $a_{ij}$, $a$, and in addition $H = O(e^{-nt})$.

**Lemma 12.** For each $1 \leq a \leq n + 1$ and $\xi \neq 0$, we can find a solution $H^a$ (with the same decay properties) satisfying

$$\left. B_\xi(H^a)_{ij}\right|_{t=0} = e_a,$$
where $e_a \in \mathbb{R}^{n+1}$ is a standard basis vector.

Proof. Define $a_{ij}$ to be the symmetric matrix such that $a_{1i} = |\xi|^{-2}\xi_i$ for all $i$, and $a_{ii} = -|\xi|^{-2}\xi_i$ for $i = 2, \ldots, n$, and $a_{ij} = 0$ for the other entries. Also, let $a = 0$. The corresponding solution $H$ satisfies

$$B_\xi(H)|_{t=0} = 1 + O(|\xi|),$$

$$B_\xi(H)|_{t=0} = O(|\xi|), \text{ for } i > 1$$

$$B_\xi(H)|_{t=0} = O(|\xi|^2).$$

We can repeat this construction replacing the index 1 with any $j > 1$, and finally we can also set $a_{ij} = 0, a = 1$. In this way, for any standard basis vector $e_a \in \mathbb{R}^{n+1}$ we can obtain a solution $\tilde{H}$ satisfying

$$B_\xi(\tilde{H})|_{t=0} = e_a + O(|\xi|).$$

For sufficiently small $\xi$, say $|\xi| < \kappa$, we can then take linear combinations

$$H^a = \lambda_a \tilde{H}^a + \sum_{b \neq a} \lambda_b \tilde{H}^b,$$

where $\lambda_a = 1 + O(|\xi|)$ and $\lambda_b = O(|\xi|)$ for $b \neq a$, and $H^a$ will satisfy

$$B_\xi(H^a)|_{t=0} = e_a.$$

\[\square\]

The key question for us is the nature of the singularity of these solutions $H^a$ at $\xi = 0$. From the preceding discussion we see that the components of each $H^a$ have the form

$$H^a_{bc} = |\xi|^{-2}\Phi^a_{bc}(\xi, t),$$

where the $\Phi^a_{bc}$ are smooth functions of $\xi, t$ satisfying $\Phi^a_{bc}(0, t) = 0$ and $\Phi^a_{bc}(\xi, t) = O(e^{-nt})$.

3.3. Solutions for large $\xi$. Consider again the ODEs (3.8), satisfied by the Fourier transform $H$ of a solution of $Lh = 0$. We now study solutions of this system for large $\xi$, with the aim of prescribing $\beta(h)$ at $t = 0$. The following simple observation shows that this is equivalent to studying solutions of the system with $|\xi| = 1$, but $t \to -\infty$.

Lemma 13. Suppose that $H(\xi, t)$ is a solution of the system (3.8). Then for any $T \in \mathbb{R}$ another solution is given by $\tilde{H}(\xi, t) = H(e^T \xi, t + T)$. In addition, applying the Fourier transform of the Bianchi operator, we have

$$B_\xi(\tilde{H})|_{t=0} = B_{e^T \xi}(H)|_{t=T}.$$

For $\xi$ with $|\xi| = 1$, the system (3.8) is of the form

$$H'' + nH' + Q_0H + e^{-t}Q_1(\xi)H - e^{-2t}H = 0,$$

for suitable matrices $Q_0, Q_1$, where only $Q_1$ depends on $\xi$. After a change of variables $s = e^{-t}$, we obtain

$$s^2 \frac{d^2}{ds^2}H - (n - 1)s \frac{d}{ds}H + Q_0H + sQ_1(\xi)H - s^2 H = 0.$$
Writing \( J = \frac{d}{ds} H \) we have the equivalent first order system

\[
\frac{d}{ds} H = J, \\
\frac{d}{ds} J = H + (n - 1)s^{-1}J - Q_1(\xi)s^{-1}H - Q_0s^{-2}H.
\]

The leading coefficients are given by the matrix

\[
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix},
\]

which has eigenvalues 1, \(-1\) with multiplicity \((n + 1)(n + 2)/2\) each. The system has an irregular singularity of rank 1 as \( s \to \infty \), and so there will be \((n + 1)(n + 2)/2\) linearly independent solutions which as \( s \to \infty \) have leading order term \( s^{-r}e^s \) for suitable \( r \), and \((n + 1)(n + 2)/2\) solutions which decay like \( s^{-r}e^s \). We are interested in the solutions that blow up as \( s \to \infty \), and for these each component of \( H \) has an asymptotic expansion of the form

\[
(3.20) \quad H_{ab} \sim s^{-r}e^s(c_{ab} + c_{ab}^{(1)}s^{-1} + c_{ab}^{(2)}s^{-2} + \ldots).
\]

If we substitute this asymptotic power series into our system, then the leading terms are of order \( s^{2-r}e^s \), and these cancel in each equation. The vanishing of the next order term, \( s^{1-r}e^s \) gives rise to a system of linear equations for the coefficients \( c = c_{ab} \):

\[
-2rc - (n - 1)c + Q_1(\xi)c = 0,
\]

so \( c \) is an eigenvector of the matrix \( Q_1(\xi) \), with eigenvalue \( 2r + n - 1 \).

**Lemma 14.** The matrix \( Q_1(\xi) \) is diagonalizable, with real eigenvalues.

**Proof.** This follows from the fact that \( Q_1(\xi) \) is self adjoint in a suitable basis. More precisely, let us write \( A_{jk} = H_{jk} \) for \( j \neq k \), \( A_{j0} = H_{j0} \), \( A_{jj} = \frac{1}{\sqrt{2}}H_{jj} \), and \( A_{00} = \frac{1}{\sqrt{2}}H_{00} \). In this basis we have

\[
(Q_1(\xi)A)_{jk} = -2\sqrt{-1}(\xi_jA_{k0} + \xi_kA_{j0}), \quad \text{for } j \neq k
\]

\[
(Q_1(\xi)A)_{j0} = -2\sqrt{-2}\xi_jA_{00} + 2\sqrt{-1}\sum_{k \neq j} \xi_kA_{jk} + 2\sqrt{-2}\xi_jA_{jj}
\]

\[
\frac{1}{\sqrt{2}}(Q_1(\xi)A)_{jj} = -2\sqrt{-2}\xi_jA_{j0}
\]

\[
\frac{1}{\sqrt{2}}(Q_1(\xi)A)_{00} = 2\sqrt{-2}\sum_j \xi_jA_{j0},
\]

so that \( Q_1(\xi) \) is self adjoint. \( \Box \)

From this lemma we obtain that there are \((n + 1)(n + 2)/2\) linearly independent solutions of our system with asymptotic expansion \((3.20)\), where the value of \( r \) may depend on the solution. The type I, II, III solutions that we found in the previous subsections cannot decay as \( s \to \infty \) by the maximum principle. This can be viewed as an instance of the argument in the proof of Lemma 5 or more precisely its linearization around the hyperbolic metric. To see this note that for any fixed \( \xi \) these ODE solutions define periodic elements \( h \) in the kernel of \( L_{\mathbf{H}} \) on \( \mathbb{R}^n \times (-\infty, \infty) \). Letting \( \omega = \beta_{\mathbf{H}}(h) \) as in the proof of Lemma 5 we find that \( |\omega| \) cannot admit an interior maximum. But \( \omega \to 0 \) as \( t \to \infty \), so \( h \) cannot decay as
$t \to -\infty$ (i.e. $s \to \infty$) as well. It follows that the type I, II or III solutions have asymptotics of the form \((3.20)\) as $s \to \infty$. Translating back to the $t$-variable, the conclusion is the following.

**Proposition 15.** For any eigenvector $c = c_{ab}$ of the matrix $Q_1(\xi)$ we obtain a solution $H$ of the system $L_\xi(H) = 0$. As $t \to -\infty$ each component $H_{ab}$ has asymptotic expansion

$$H_{ab}(t) \sim e^{rt} e^{-t}(c_{ab} + c_{ab}^{(1)} e^t + c_{ab}^{(2)} e^{2t} + \ldots),$$

while as $t \to \infty$, we have $|H| = O(e^{-nt})$.

Let us now look at the boundary condition. Substituting $H_{ab}$ into \((3.9)\), the leading terms are

\[(3.21)\]

$$B_\xi(H)_i = \left[-c_{i0} - \sqrt{-1} \xi_i c_{00} + \frac{1}{2} \sqrt{-1} \xi_i c_{pp} + \frac{1}{2} \sqrt{-1} \xi_i c_{00}\right] e^{(r-1) t} e^{-t} + O(e^{rt} e^{-t})$$

$$B_\xi(H)_0 = \left[-\frac{1}{2} c_{00} + \frac{1}{2} c_{pp} - \sqrt{-1} \xi_i c_{00}\right] e^{(r-1) t} e^{-t} + O(e^{rt} e^{-t}),$$

and more precisely $B_\xi(H)$ has an asymptotic expansion in powers of $e^t$. Note that the leading coefficients do not depend on $r$. Let us write

$$B_\xi(H) \sim e^{(r-1)t} e^{-t} \left(R(\xi) c + R^{(1)}(\xi, r) c e^t + \ldots\right),$$

where $R(\xi)$ is independent of $r$. We have the following

**Lemma 16.** The matrix $R(\xi)$ has a right inverse for all $\xi$ with $|\xi| = 1$.

**Proof.** Since our problem is rotationally invariant in $\mathbb{R}^n$, it is enough to check this for a single unit vector $\xi$, for instance $\xi = (1, 0, 0, \ldots, 0)$, in which case it is straightforward. \qed

Let us fix an eigenvector $c$ of $Q_1(\xi)$, and define

$$\tilde{H}(t) = H(t - T)e^{rT}e^{-T}.$$

Then, using Lemma 13 we have

$$L_{e^T \xi}(\tilde{H}) = 0,$$

$$\tilde{H}_{ab}(0) \sim c_{ab} + c_{ab}^{(1)} e^{-T} + \ldots,$$

$$B_{e^T \xi}(\tilde{H})|_{t=0} \sim e^T R(\xi) c + R^{(1)}(\xi, r) c + \ldots.$$

Writing $\zeta = e^T \xi$, and recalling that $|\xi| = 1$, we have

$$L_\zeta(\tilde{H}) = 0,$$

$$\tilde{H}_{ab}(0) \sim c_{ab} + c_{ab}^{(1)} |\zeta|^{-1} + \ldots,$$

$$B_\zeta(\tilde{H})|_{t=0} \sim |\zeta| R(|\zeta|^{-1} \zeta) c + R^{(1)}(|\zeta|^{-1} \zeta, r) c + \ldots.$$

From this and Lemma 16 it follows that as long as $|\zeta|$ is sufficiently large, say $|\zeta| > \kappa^{-1}$, we can take suitable linear combinations of our solutions $\tilde{H}$ (for different eigenvectors $c$) with coefficients that have an asymptotic expansion in powers of $|\zeta|^{-1}$, and obtain $H^a$ satisfying $L_\zeta(H^a) = 0$, $B_\zeta(H^a)|_{t=0} = e_a$, and

$$H^a(0) \sim \Psi^{(-1)}_a(\zeta) + \Psi^{(-2)}_a(\zeta) + \Psi^{(-3)}_a(\zeta) + \ldots,$$
where for each $i$, $\Psi_a^{(i)}$ is homogeneous of degree $i$, and smooth on the unit sphere. In addition we have $H^a(t) = O(e^{-nt})$ as $t \to \infty$. More precisely we have the following estimate.

**Proposition 17.** For $|\zeta| > \kappa^{-1}$ the solutions $H^a$ satisfy

$$|\partial_1^i H^a(t)| \leq C_i |\zeta|^{-(n-1)} e^{-n t}$$

for all $i$, and $t \geq 0$, with suitable constants $C_i$.

**Proof.** Let $|\xi| = 1$, fix an eigenvector $c$ of $Q_1(\xi)$ as above, and let $H(t)$ be the corresponding solution of $L_\xi(H) = 0$. From the asymptotic behavior of $H$ as $t \to -\infty$ we have that for suitable $c, C > 0$

$$(\log |H|)' < -ce^{-t} < -(n-1)$$

for $t < -C$, while the behavior as $t \to \infty$ implies that

$$(\log |H|)' < -(n-1)$$

for $t > C$. It follows from this that for any $s \in \mathbb{R}$ and $t \geq 0$, we have

$$\log |H|(s + t) < \log |H|(s) + C - (n-1)t,$$

i.e.

$$|H|(s + t) \leq Ce^{-(n-1)t} |H|(s)$$

for a different constant $C$. Since the derivatives $\partial_1^i H$ have analogous asymptotics to $H$, they also satisfy estimates of the form (3.22).

For large $T$, let

$$\tilde{H}_T(t) = H(t - T)e^{\tau T}e^{-\tau T}$$

as above. By the asymptotics of $H$, we have that $e^{\tau T}e^{-\tau T} |H|(-T)$ is bounded for large $T$, and so using (3.22) we have, for $t \geq 0$, that

$$|\tilde{H}_T(t)| \leq Ce^{-(n-1)t},$$

with $C$ independent of $T$. Computing a derivative

$$e^{-T}\partial_T \tilde{H}_T(t) = \left[ -e^{-T}H'(t - T) - H(t - T) + re^{-T}H(t - T) \right]e^{\tau T}e^{-\tau T},$$

and so using the analogous estimate to (3.22) for $H'$ together with a bound on $e^{(r-1)T}e^{-\tau T} |H'|(-T)$ for large $T$, we obtain

$$|e^{-T}\partial_T \tilde{H}_T(t)| \leq Ce^{-(n-1)t},$$

for $t \geq 0$. We can bound further derivatives $(e^{-T}\partial_T)^i \tilde{H}_T$ in a similar way.

Using the substitution $T = \log |\zeta|$, this implies that

$$|\partial_1^i \log |\zeta|(t)| \leq Ce^{-(n-1)t}$$

for $t \geq 0$. The solutions $H^a$ are obtained by taking linear combinations of such $\tilde{H}$, with coefficients that are of order $|\zeta|^{-1}$, and have an asymptotic expansion in powers of $|\zeta|^{-1}$. The required estimates follow from this. $\square$
3.4. Prescribing the Bianchi operator for all $\xi$. We have seen in section 3.2.4 that for sufficiently small $|\xi|$ we can find solutions $\tilde{H}^a$ of (3.8), such that $B_\xi(\tilde{H}^a)|_{t=0} = e_a$. Applying Lemma 13 this means that if we fix $\xi$ with $|\xi| = 1$, then we have solutions $H^a$ of $L_\xi(H^a) = 0$ with $B_\xi(H^a)|_{t=T} = e_a$ for some large $T$. A crucial result is the following.

**Proposition 18.** The vectors $B_\xi(H^a)(t) \in C_n+1$ are linearly independent for all $t \in \mathbb{R}$.

**Proof.** This follows from the maximum principle, analogously to Lemma 5. Indeed, if there was a value of $t$ at which the vectors were not linearly independent, then we could form a linear combination and take the inverse Fourier transform to obtain a periodic element $h$ in the kernel of $L_{\gamma h}$ for which $\omega = \beta_{\gamma h}(h)$ vanishes at some value of $t$. This contradicts that $|\omega|$ cannot admit an interior maximum. □

Applying Lemma 13 again, it follows that for all $\xi$ we can find suitable solutions $H^a$ of $L_\xi(H^a) = 0$, satisfying $B_\xi(H^a)|_{t=0} = e_a$. In the previous two subsections we have constructed special collections of such $H^a$ for sufficiently small, and for sufficiently large $|\xi|$ respectively. Combining these with suitable cutoff functions, we obtain the following.

**Proposition 19.** For all $\xi \neq 0$ we have solutions $H^a(\xi, t)$ of $L_\xi(H^a) = 0$, $B_\xi(H^a)|_{t=0} = e_a$, which depend smoothly on $\xi$ such that in addition we have

1. For small $\xi$

\[
H^a(\xi, 0) = |\xi|^{-2} \Phi^a(\xi)
\]

for smooth $\Phi^a$ with $\Phi^a(0) = 0$.

2. For large $\xi$ we have an asymptotic expansion

\[
H^a(\xi, 0) \sim \Psi^{(-1)}_a(\xi) + \Psi^{(-2)}_a(\xi) + \Psi^{(-3)}_a(\xi) + \ldots,
\]

where each $\Psi^{(i)}_a$ is homogeneous of degree $i$, and smooth on the unit sphere.

3. For $t \geq 0$ and all $\xi \neq 0$ we have

\[
|\partial_\xi H^a(\xi, t)| \leq C_i |\xi|^{-1-i} e^{-\gamma (n-1)t},
\]

for constants $C_i$.

We can now state the main result of this subsection.

**Proposition 20.** Suppose that $\eta \in T^*\mathbb{H}^{n+1}|_{t=0}$ is a one-form, satisfying the following estimates:

1. $\|\eta\|_{C^{1,\alpha}} \leq C$,
2. $\|\eta\|_{C^{1,\alpha}(B_R \setminus B_{R-\delta})} \leq CR^{-2n+1/2}$ for $R > 1$,

and in addition for all $i = 1, \ldots, n$, and each component $\eta_i$, we have

\[
\int_{\mathbb{R}^n} \eta_i \, dx = \int_{\mathbb{R}^n} x_i \eta_i \, dx = 0.
\]

Then there exists a symmetric two tensor $h \in C^{2,\alpha}_\delta(\mathbb{H}^{n+1})$ satisfying $L h = 0$ in $\mathbb{H}^{n+1}$, such that $h$ has the boundary condition $\eta$, i.e. $\beta(h)|_{t=0} = \eta$ and in addition $h$ satisfies the following decay estimate, for any $\delta \in (0, 1)$:

\[
\|h\|_{C^{2,\alpha}_\delta(A_{R-1/2}, R)} < C'(1 + R)^{-n+1/2}
\]

for all $R > 0$, where $C'$ depends on the constant $C$ and on $\delta$. 

Proof. We use the solutions $H^a$ of $L_\xi(H^a) = 0$ from Proposition 19 to define

$$H(\xi, t) = \sum a \hat{\eta}_a(\xi) H^a(\xi, t).$$

Then the inverse Fourier transform $h(x, t)$ of $H$ will satisfy $L h = 0$, and by construction $\beta(h)|_{t=0} = \eta$ will hold. What remains is to verify that $h$ satisfies the required estimates. We will first focus on the relevant estimates at $t = 0$.

Let us define a cutoff function $\rho$ such that $\rho(s) = 1$ for $s < 1/2$, and $\rho(s) = 0$ for $s > 1$. Let us write $h = h_1 + h_2$ where $h_1$ is the inverse Fourier transform of $H_1 = \rho(|\xi|) H$, and $h_2$ is the inverse transform of $H_2 = (1 - \rho(|\xi|)) H$. I.e. we collect the small Fourier modes in $h_1$, and the large ones in $h_2$. We prove the required estimates for $h_1, h_2$ separately.

We have $h_1 = \sum a h_1^a$, where

$$h_1^a(x, 0) = F^{-1} \left[ \rho(|\xi|) \hat{\eta}_a(\xi) |\xi|^{-2} \Phi^a(\xi) \right],$$

in terms of the formula (3.23) for $H^a$, and the inverse Fourier transform $F^{-1}$. In terms of convolutions we have

$$h_1^a(x, 0) = \eta_a * F^{-1} \left[ \rho(|\xi|) \Phi^a(\xi) \right] * F^{-1}(|\xi|^{-2}).$$

Since $\rho(|\xi|)$ is smooth and compactly supported, the inverse Fourier transform of $\rho(|\xi|) \Phi^a(\xi, t)$ is a Schwarz function. It follows that the function

$$N(x) = \eta_a * F^{-1} \left[ \rho(|\xi|) \Phi^a(\xi) \right]$$

satisfies the same decay estimates as $\eta_a$, but for all derivatives rather than just the $C^{1,\alpha}$ norm. In addition we have $\Phi^a(0) = 0$, and the assumption 3.26 implies $\hat{\eta}(0) = \partial_\xi \hat{\eta}(0) = 0$ for all $i$. For $N$ this implies

$$\int_{\mathbb{R}^n} N(x) \, dx = \int_{\mathbb{R}^n} x_i N(x) \, dx = \int_{\mathbb{R}^n} x_i x_j N(x) \, dx = 0$$

for all $i, j$.

At the same time in the sense of distributions we have

$$F^{-1}(|\xi|^{-2}) = c|x|^{2-n},$$

for a dimensional constant $c$, so

$$h_1^a(x, t) = c N(x) * |x|^{2-n}.$$

The required decay estimate for $h_1^a$ follows from this.

Let us now consider $h_2$. Then $h_2$ is a sum of terms $h_2^a$, where

$$h_2^a(x, 0) = F^{-1} \left[ (1 - \rho(|\xi|)) \hat{\eta}_a(\xi) H^a(\xi) \right]$$

$$= \eta_a * F^{-1} \left[ (1 - \rho(|\xi|)) H^a(\xi) \right].$$

Using the asymptotic expansion (3.24) for $H^a(\xi)$, we have

$$H^a(\xi) \sim \Psi_a^{(-1)}(\xi) + \ldots + \Psi_a^{(-n-2)}(\xi) + \Theta_a(\xi),$$

where for large $\xi$ we have $\nabla_\xi \Theta_a(\xi) = O(|\xi|^{-n-3-k})$. It follows that

$$K \Theta_a(x) = F^{-1} \left[ (1 - \rho(|\xi|)) \Theta_a(\xi) \right]$$

for large $\xi$.
is smooth on \( \mathbb{R}^n \), and \( K_{\Theta_n}, \nabla_x K_{\Theta_n}, \nabla_x^2 K_{\Theta_n} \) decay exponentially fast. In particular \( \eta_* K_{\Theta_n} \) satisfies the required estimates.

Let us write
\[
K^{(-i)}_a = \mathcal{F}^{-1} \left[ (1 - \rho(|\xi|))\Psi^{(-i)}_a \right].
\]

Then the distribution \( \nabla_x^i K^{(-i)}_a \) is the Fourier transform of a function which for large \( \xi \) is homogeneous of degree zero. The decay of the derivatives of \( \Psi^{(-i)}_a \) implies that \( K^{(-i)}_a \) has singular support at the origin, and all of its derivatives decay exponentially fast away from the origin. It follows from these properties (as in Gilbarg-Trudinger, Section 4.3 for the Poisson equation) that for each \( i \),
\[
\eta_* \nabla_x^i K^{(-i)}_a
\]
decays in \( C^{1,\alpha} \) (or in any other Hölder space) at the same rate as \( \eta_a \). Since \( i \geq 1 \), we obtain the required \( C^{2,\alpha} \) estimates for \( h_\xi(\cdot, 0) \).

We now consider \( h(x, t) \) for \( t \geq 0 \). Our goal is to show that \( e^t|h(x, t)| \leq C'(1 + |x|)^{-n-1+\delta} \), and then Schauder estimates together with our estimate for the boundary values of \( h \) imply the required \( C^{2,\alpha} \) estimates. As above, for each \( t \), \( h(x, t) \) is obtained as a convolution of components of \( \eta \) with the Fourier transforms of the solutions \( H^\alpha(\xi, t) \) of Proposition \ref{p:convolution}. The property \eqref{e:partially_supported} together with Lemma \ref{lem:estimate_2} and Lemma \ref{lem:estimate_3} below implies the result. \(\square\)

**Lemma 21.** Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) satisfies the estimates
\[
|\partial_x^j f(x)| \leq C_j |x|^{-1-i} \tag{3.28}
\]
for \( x \neq 0 \). For any \( \delta \in (0, 1) \), the Fourier transform \( \hat{f} \) of \( f \) in the sense of distributions then satisfies
\[
|\partial_\xi^j \hat{f}(\xi)| \leq \begin{cases} 
C_j' |\xi|^{1-n-j-\delta}, & \text{for } |\xi| < 2 \\
C_j' |\xi|^{1-n-j+\delta}, & \text{for } |\xi| > 1.
\end{cases} \tag{3.29}
\]

**Proof.** We will first prove the \( j = 0 \) case of the required inequality, i.e. we prove that assuming the estimate \eqref{e:partially_supported}, we have
\[
|\hat{f}(\xi)| \leq \begin{cases} 
C' |\xi|^{1-n-\delta}, & \text{for } |\xi| < 2 \\
C' |\xi|^{1-n+\delta}, & \text{for } |\xi| > 1.
\end{cases}
\]

Let us write \( f = f_1 + f_2 \), where \( f_1 \) is supported in \( B_2(0) \), and \( f_2 \) is supported on \( \mathbb{R}^n \setminus B_1(0) \).

Consider \( f_1 \) first. Let us write
\[
\frac{n-1-\delta}{2} = k + s,
\]
where \( k \in \mathbb{Z} \) and \( s \in (0, 1) \). The estimates \eqref{e:partially_supported} imply that
\[
|\Delta_x^{k+s} f_1(x)| \leq \begin{cases} 
C |x|^{-n+\delta}, & \text{for } 0 < |x| < 2 \\
C |x|^{-n-2s}, & \text{for } |x| > 1,
\end{cases}
\]
where we are using the fractional Laplacian \( \Delta^s \). To see this, note that from our assumptions we have
\[
|\partial_x^i \Delta_x^k f_1(x)| \leq C |x|^{-1-k-i},
\]
and the required estimate then follows from the integral formula
\[
\Delta_x^{k+s} f_1(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy,
\]
where \(u(x) = \Delta_x^k f_1(x)\). In particular we find that \(\Delta_x^{k+s} f_1 \in L^1\), and so on the Fourier transform side we obtain that \(|\xi|^{2(k+s)} \hat{f}_1\) is bounded, i.e.
\[
|\hat{f}_1(\xi)| < C|\xi|^{-n+\delta},
\]
for all \(\xi\). At the same time the fact that \(f_1\) is compactly supported implies that \(\hat{f}_1\)
is actually smooth and in particular it is bounded near \(\xi = 0\).

We can deal with \(f_2\) in a similar way, letting
\[
\frac{n - 1 + \delta}{2} = k + s
\]
this time. Since \(\partial_x^N f_2 \in L^1\) for all \(N > n - 1\), it follows that \(\hat{f}_2\) decays at infinity faster than any polynomial, while a similar argument to the above, using the fractional Laplacian, shows that
\[
|\hat{f}_2(\xi)| < C|\xi|^{-n-\delta}
\]
for \(0 < |\xi| < 1\), say. Combining these estimates for \(\hat{f}_1, \hat{f}_2\), we obtain the required bound for \(\hat{f}\).

Given the estimate (5.29) for \(j = 0\), we can obtain the general case if we replace \(f\) by \(P_j(x)f(x)\) for degree \(j\) monomials \(P_j\).

**Lemma 22.** Suppose that \(f: \mathbb{R}^n \to \mathbb{R}\) satisfies \(|f(x)| \leq C(1 + |x|)^{-2n+1/2}\), and
\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} x_i f(x) \, dx = 0
\]
for each \(i\). Let \(K: \mathbb{R}^n \to \mathbb{R}\) be such that for some \(\delta \in (0,1)\),
\[
|\partial_i^2 K(x)| \leq \begin{cases} C_i |x|^{-n-\delta}, & \text{for } |x| < 2 \\ C_i |x|^{-n+\delta}, & \text{for } |x| > 1. \end{cases}
\]
Then the convolution \(g = f \ast K\) satisfies \(|g(x)| \leq C'(1 + |x|)^{-1-n+\delta}\).

**Proof.** We can expand \(K(x-y)\) in a Taylor series around \(y = 0\), and the series will converge on the region \(|y| < |x|/2\), say:
\[
K(x-y) = K(x) - y_i \partial_i K(x) + O(|y|^2|x|^{-1-n+\delta}).
\]
We then have
\[
g(x) = \int_{|y| < |x|/2} f(y) K(x-y) \, dy + \int_{|x|/2 \leq |y| \leq 2|x|} f(y) K(x-y) \, dy \\
+ \int_{|y| > 2|x|} f(y) K(x-y) \, dy.
\]
In estimating the first integral we use the Taylor expansion of \(K(x-y)\), while the other two integrals can be estimated directly. \(\square\)
3.5. The proof of Theorem 9. In this section we will give the proof of Theorem 9. The first step is to solve the inhomogeneous problem in (3.3):

**Proposition 23.** Suppose that \( u \in \mathcal{C}^{0,\alpha}_1 \) is a symmetric two-tensor on \( \mathbf{H}^{n+1}_+ \) supported in \( \mathcal{B}_1 \times [0, \infty) \). Then there exists a symmetric two-tensor \( h_0 \) on \( \mathbf{H}^{n+1}_+ \) satisfying \( Lh_0 = u \), and \( h_0 \) satisfies the estimate

\[
\|h_0\|_{\mathcal{C}^{2,\alpha}\mathcal{L}} \leq C(1 + R)^{-2n+1/2}\|u\|_{\mathcal{C}^{0,\alpha}_1},
\]

for all \( R > 0 \).

**Proof.** First by reflecting \( u \) across the boundary of \( \mathbf{H}^{n+1}_+ \), and multiplying by a cutoff function, we extend \( u \) to a tensor \( \tilde{u} \) on all of hyperbolic space \( \mathbf{H}^{n+1}_+ \), with

\[
\|\tilde{u}\|_{\mathcal{C}^{2,\alpha}\mathcal{L}} \leq C\|u\|_{\mathcal{C}^{0,\alpha}_1}.
\]

We can then apply Theorem 8 to obtain the required tensor \( h_0 \) on all of \( \mathbf{H}^{n+1}_+ \), and we simply restrict it to \( \mathbf{H}^{n+1}_+ \). The required decay of \( h_0 \) in the \( x_j \)-directions follows from the decay result [13, Proposition 5.2]. \( \square \)

The next step in the proof of Theorem 9 is to let \( \eta = \beta_\mathbf{H}(h_0)|_{t=0} \), and try using Proposition 20 to find \( h_1 \) such that \( Lh_1 = 0 \), and \( \beta_\mathbf{H}(h_1)|_{t=0} = \eta \). For this we need to check the integral conditions (3.26), which are equivalent to \( \hat{\eta}(0) = \partial_t \hat{\eta}(0) = 0 \). This is where the condition \( I(u) = 0 \) enters, but we will need to further adjust \( h_0 \) before these conditions hold. We first have the following.

**Proposition 24.** Suppose that \( u \in \mathcal{C}^{0,\alpha}_1 \) satisfies \( I(u) = 0 \), and that \( h_0 \) satisfies \( Lh_0 = u \). Let \( \eta = \beta_\mathbf{H}(h_0)|_{t=0} \). Then for small \( \xi \) the components of the Fourier transform \( \hat{\eta} \) satisfy

\[
\hat{\eta}(\xi) = \xi_j A_{ij} + O(|\xi|^2),
\]

for a symmetric matrix \( A_{ij} \).

**Proof.** We need to show that \( \hat{\eta}(0) = 0 \), and that the skew-symmetric part of the first derivative of \( \hat{\eta} \) vanishes at the origin, i.e.

\[
\partial_\xi \hat{\eta}(0) - \partial_\xi \hat{\eta}(0) = 0.
\]

Let us denote by \( H_0(\xi, t) \) the Fourier transform of \( h_0 \) with additional exponential factors as before in Equation (3.7). Similarly \( U(\xi, t) \) is the Fourier transform of \( u \) with additional exponential factors. The equation \( Lh_0 = u \) then implies that \( L_\xi H_0 = U \), where \( L_\xi \) is the operator given by the left hand side of (3.28). In particular, the components \( H_{0,j0}(0, t) \) for \( \xi = 0 \) satisfy the ODEs

\[
H_{0,j0}' + nH_{0,j0} - (n + 1)H_{0,j0} = U_{j0}.
\]

The condition \( I(u) = 0 \) says that for all \( x \) we have

\[
\int_0^\infty u_{j0}(x, t)e^{-2t} dt = 0,
\]

and so taking the Fourier transform, and letting \( \xi = 0 \) we get

\[
\int_0^\infty U_{j0}(0, t)e^{-t} dt = 0,
\]

recalling that \( U_{j0} = e^{-t}u_{j0} \). Applying Lemma 24 we find that for each \( j \),

\[
H_{0,j0}'(0, 0) + (n + 1)H_{0,j0}(0, 0) = 0.
\]
Using the formula \((3.9)\) for the Fourier transform of the Bianchi operator, we then have
\[
\hat{\eta}_{ij}(0) = H'_{0,j0}(0,0) + (n+1)H_{0,i0}(0,0) = 0,
\]
as required.

We next look at the first derivative of \(\hat{\eta}\), and for this we differentiate the equation \(L_\xi H_0 = U\) with respect to \(\xi\). We only need certain components of the derivative, so let us define
\[
S_{ij}(t) = \partial_{\xi_i} H_0,j0 - \partial_{\xi_j} H_0,i0 \bigg|_{\xi=0}.
\]
Differentiating the equation \(L_\xi H_0 = U\) with respect to \(\xi\) and then setting \(\xi = 0\), we obtain
\[
S''_{ij} + nS'_{ij} - (n+1)S_{ij} = g_{ij}(t),
\]
where
\[
g_{ij}(t) = \partial_{\xi_i} U_{j0} - \partial_{\xi_j} U_{i0} \bigg|_{\xi=0}.
\]
From the properties of the Fourier transform we have
\[
g_{ij}(t) = -\sqrt{-1} \int_{\mathbb{R}^n} x_j e^{-t u_{i0}(x,t)} - x_i e^{-t u_{j0}(x,t)} dx,
\]
and so \((3.31)\) for all \(x\) implies
\[
\int_0^\infty g_{ij}(t) e^{-t} dt = 0.
\]
Just as above, Lemma \(\text{25}\) then implies that \(S'_{ij}(0) + (n+1)S_{ij}(0) = 0\).

At the same time \((3.9)\) implies that
\[
\partial_{\xi} \hat{\eta}_i - \partial_{\xi} \hat{\eta}_j \bigg|_{\xi=0} = S'_{ij}(0) + (n+1)S_{ij}(0) = 0,
\]
which is what we wanted to prove. \(\Box\)

We used the following in the previous argument.

**Lemma 25.** Suppose that \(f : [0, \infty) \to \mathbb{R}\) is a decaying solution of
\[
f'' + nf' - (n+1)f = g.
\]
Then \(f'(0) + (n+1)f(0) = 0\) if and only if
\[
\int_0^\infty g(s)e^{-s} ds = 0.
\]

**Proof.** The solutions of the homogeneous equation are \(e^t, e^{-(n+1)t}\). Note that the decaying homogeneous solution \(\phi(t) = e^{-(n+1)t}\) satisfies \(\phi'(0) + (n+1)\phi(0) = 0\), and so it is enough to check the statement of the lemma for one particular solution.

A decaying fundamental solution of the ODE is
\[
\Gamma(t) = \begin{cases} ae^t & \text{if } t < 0 \\ ae^{-(n+1)t} & \text{if } t > 0, \end{cases}
\]
for a suitable constant \(a\), and so a decaying solution of the ODE is
\[
f(t) = \int_{-\infty}^\infty g(s)\Gamma(t-s) ds.
\]
It follows that
\[
f'(0) + (n + 1)f(0) = \int_{-\infty}^{\infty} g(s) \left[ \Gamma'(-s) + (n + 1)\Gamma(-s) \right] ds
\]
\[
= \int_0^{\infty} g(s) \left[ ae^{-s} + (n + 1)ae^{-s} \right] ds
\]
\[
= a(n + 2) \int_0^{\infty} g(s)e^{-s} ds.
\]
The result follows.

We are now ready to complete the proof of Theorem 9. Consider again \( \eta = \beta_H(h_0)|_{t=0} \) for the \( h_0 \) given by Proposition 23. Using solutions of (3.8) for small \( \xi \) satisfying (3.18) we can find a solution \( \tilde{H}(\xi, t) \) of \( L_{\xi}(\tilde{H}) = 0 \), vanishing for \( |\xi| > 0 \), depending smoothly on \( \xi \), and such that
\[
\hat{\eta} - B_\xi(\tilde{H})|_{t=0} = O(|\xi|^2).
\]
The inverse Fourier transform \( \tilde{h} \) of \( \tilde{H} \) decays exponentially fast (it is in the Schwarz space). We can then apply Proposition 20 to find \( h_1 \) satisfying
\[
\beta_H(h_1)|_{t=0} = \eta - \beta_H(\tilde{h})|_{t=0},
\]
as well as the decay estimates (3.27). We finally let
\[
h = h_0 - \tilde{h} - h_1.
\]
This satisfies \( Lh = Lh_0 = u \), the boundary condition \( \beta_H(h) = 0 \), and the required decay estimates.

We also have the following improvement over Proposition 20 when the only nonzero component of \( \eta \) is \( \eta_0 \).

Proposition 26. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is a \( C^{1,\alpha} \) function supported in the unit ball \( B_1 \). There exists a symmetric two tensor \( h \in C^2_{1,\alpha}(\mathbb{H}^{n+1}_1) \) satisfying
\[
\begin{align*}
(1) & \quad Lh = 0, \\
(2) & \quad \beta(h)_i|_{t=0} = 0 \text{ for all } i \text{ and } \beta(h)_0|_{t=0} = f, \\
(3) & \quad h \text{ satisfies the decay estimate}
\end{align*}
\]
\[
\|h\|_{C^2_{1,\alpha}(A_{R-1,R})} < C_k'(1 + R)^{-k}\|f\|_{C^{1,\alpha}},
\]
for any \( k > 0 \), and \( C_k' \) depending on \( k \).

Proof. The solution \( h \) is constructed using the Fourier transform just like before, but for small \( \xi \) only the solutions \( H^a \) do not have a \( |\xi|^{-1} \) singularity this time, as can be seen in (3.18). This translates to better decay properties of \( h \) without the need for a condition like (3.26). \qed

4. The linearized problem on \([0, \infty) \times M\)

In this section we use Theorem 9 to invert the linearized operator on \( M \times [0, \infty) \), at first modulo a finite dimensional space. In this and subsequent sections we will need to do some local calculations with respect to the warped product metric
\[
g_\epsilon = dt^2 + e^{2\epsilon}e^{-2}g_M.
\]
In particular, \( \{x^i\} \) will denote local coordinates on \( M \), and \( \{x^1, \ldots, x^n, x^0 = t\} \) the corresponding coordinate system on \( M \times [0, \infty) \). We will use \( a, b, c, \ldots \) for indices ranging from 0 to \( n \), and \( i, j, k, \ldots \) for those ranging from 1 to \( n \), as before. We will
also use the obvious identifications between vector fields on \( M \) and \([0, \infty)\) and their lifts to vector fields on the product manifold, usually without comment. Recall the improved approximate solution
\[
g'_t = g_t + k^{(2)} + e^{-2t}e^2k^{(4)},
\]
where \( k^{(2)}, k^{(4)} \) are fixed tensors on \( M \) expressed in terms of \( g_M \). Since \( g_t \) is uniformly equivalent to \( g'_t \), we can use either of them to measure norms.

We now compute the Bianchi operator and the variation of the Ricci curvature with respect to the metric \( g_t \). The nonzero Christoffel symbols are given by
\[
\Gamma^i_{jk} = \Gamma^i_{M,jk},
\]
\[
\Gamma^0_{jk} = -\epsilon \gamma_{2t} g_M^M_{jk} = -g_{jk},
\]
\[
\Gamma^i_{0k} = \Gamma^i_{M0k} = \delta_k^i,
\]
where \( \Gamma^i_{M,jk} \) denote the Christoffel symbols of \( g_M \).

The general formula for the variation of the Ricci curvature is
\[
(D\text{Ric})_{g_t}(h) = -\frac{1}{2}\nabla^a \nabla_a h + \delta_{g_t} \beta_{g_t}(h) + \mathcal{R}(h),
\]
where
\[
\beta_{g_t}(h)_a = \nabla^b h_{ab} - \frac{1}{2} \nabla_a (g_k^k h_{bc}),
\]
\[
\delta_{g_t} \omega = \frac{1}{2} (\nabla_a \omega_b + \nabla_b \omega_a),
\]
and
\[
\mathcal{R}(h)_{cd} = -R^{a}_{e,d} h_{ab} + \frac{1}{2} (R^a_{c,d} h_{ab} + R^{a}_{e,d} h_{ac}),
\]
in terms of the curvature tensor of \( g_t \).

We are particularly interested in the \( j0 \)-component of the variation of the Ricci curvature. For this we have the following formulas:
\[
\nabla^a \nabla_a h_{j0} = \epsilon^2 e^{-2t}g_M^M \nabla_i \nabla_j h_{0i} + \partial^2_0 h_{j0} + (n - 2) \partial_0 h_{j0} - (4n + 2) h_{j0} + 2 \partial_j h_{00} - 2 \epsilon^2 e^{-2t}g_M^M \nabla_k h_{ij},
\]
\[
2\delta_{g_t} \beta_{g_t}(h)_{j0} = \epsilon^2 e^{-2t}g_M^M \nabla_i \nabla_j h_{0i} + \partial^2_0 h_{j0} + (n - 2) \partial_0 h_{j0} - 2nh_{j0} - \epsilon^2 e^{-2t} \partial_j (g_M^M \nabla_k h_{0i}) + \epsilon^2 e^{-2t} \partial_0 (g_M^M \nabla_k h_{ji}) + 2 \epsilon^2 e^{-2t} \partial_j (g_M^M h_{ik}) - 4 \epsilon^2 e^{-2t} g_M^M \nabla_k h_{ij} + (n + 1) \partial_j h_{00}.
\]

The curvature of \( g \) satisfies
\[
R^0_{j0k} = -\epsilon^2 e^{2t} g_M^M_{jk},
\]
\[
R_{00} = -n,
\]
\[
R_{jk} = R^M_{jk} - ng_{jk},
\]
where \( R^M \) is the Ricci curvature of \( g_M \).

\[
\mathcal{R}(h)_{j0} = -(1 + n) h_{j0} + \frac{1}{2} \epsilon^2 e^{-2t}g_M^M R^M_{kij} h_{ij}.
\]
Combining all of these we obtain

\[-2D(\text{Ric}_g + n)h_{j0} = \epsilon^2 e^{-2t} \Delta_M h_{j0} - \epsilon^2 e^{-2t} g_M^{ik} \nabla^M_j \nabla^M_k h_{i0} + \epsilon^2 e^{-2t} \partial_0 \partial_j (g_M^{ik} h_{ik}) - 2 \epsilon^2 e^{-2t} \partial_j (g_M^{ik} h_{ik}) - \epsilon^2 e^{-2t} \partial_0 g_M^{ik} \nabla^M_k h_{ji} + 2 \epsilon^2 e^{-2t} g_M^{ik} \nabla^M_k h_{ji} - (n - 1) \partial_j h_{00},\]

(4.2)

where \(\Delta_M\) denotes the Hodge Laplacian on 1-form on \((M, g_M)\).

Finally, for the Bianchi operator we have

\[
\begin{align*}
\beta_{g_\epsilon}(h)_j &= \epsilon^2 e^{-2t} g_M^{ik} \nabla^M_k h_{ij} + nh_{j0} + \partial_0 h_{j0} - \frac{1}{2} \epsilon^2 e^{-2t} \partial_j (g_M^{ik} h_{ik}) - \frac{1}{2} \partial_j h_{00} \\
\beta_{g_\epsilon}(h)_0 &= \epsilon^2 e^{-2t} g_M^{ik} \nabla^M_k h_{i0} + \frac{1}{2} \partial_0 h_{00} + nh_{00} - \frac{1}{2} \epsilon^2 e^{-2t} \partial_0 (g_M^{ik} h_{ik}).
\end{align*}
\]

(4.3)

4.1. Preliminary results on fixing the boundary values. The results in this section will allow us to make sure that our solutions of the linearized problem satisfy the Bianchi condition on the boundary.

**Proposition 27.** Suppose that \(\eta\) is a section of \(T^*(M \times [0, \infty])|_{t=0}\), in \(C^{1,\alpha}\). We can find a symmetric 2-tensor \(h \in C^{2,\alpha}\) on \(M \times [0, \infty)\), supported in \(M \times [0, 1]\), such that

\[\beta_{g_\epsilon}(h)|_{t=0} = \eta,\]

and in addition \(\|h\|_{C^{2,\alpha}} \leq C\|\eta\|_{C^{1,\alpha}_{\epsilon^{-2}g_M}}\) for a constant \(C\) independent of \(\epsilon\), once \(\epsilon\) is sufficiently small.

**Proof.** The form \(\eta\) decomposes as \(\eta = \eta_1 dx^i + \eta_0 dt\), where \(\eta_1 dx^i\) is a one form on \(M\) and \(\eta_0\) is a function on \(M\). Using Lemma 28 below, we can find a one-form \(\omega_i\) and a function \(f\) on \(M\) such that

\[
\begin{align*}
\Delta_{\epsilon^{-2}g_M} \omega_j - \omega_j &= \eta_j \\
\Delta_{\epsilon^{-2}g_M} f - f &= \eta_0,
\end{align*}
\]

where \(\Delta_{\epsilon^{-2}g_M} = \epsilon^2 g_M^{ik} \nabla^M_i \nabla^M_k\) denotes the rough laplacian. In addition, we have the estimates

\[
\|\omega\|_{C^{3,\alpha}_{\epsilon^{-2}g_M}} + \|f\|_{C^{3,\alpha}_{\epsilon^{-2}g_M}} \leq C \|\eta\|_{C^{1,\alpha}_{\epsilon^{-2}g_M}}.
\]

(4.4)

Define the 2-tensor \(\tilde{h}\) on \(M \times [0, \infty)\) by setting

\[
\begin{align*}
\tilde{h}_{ij} &= \nabla^M_i \omega_j + \nabla^M_j \omega_i, \\
\tilde{h}_{i0} &= \nabla^M_i f, \\
\tilde{h}_{00} &= 0.
\end{align*}
\]
We have \( \|\tilde{h}\|_{C^{2,\alpha}} \leq C\|\eta\|_{C^{1,\alpha}} \), and applying the Bianchi operator of \( g_\epsilon \) according to (4.3),

\[
\beta_{g_\epsilon} (\tilde{h})_{i=0} = \epsilon^2 g_M^{ik} \nabla_k^M \tilde{h}_i + \eta \tilde{h}_j + \tilde{h}_j 0 - \frac{1}{2} \epsilon^2 g_M^{ik} \nabla_j^M \tilde{h}_{ik} \\
= \epsilon^2 g_M^{ik} \nabla_k^M (\nabla_i^M \omega_j + \nabla_j^M \omega_i) - \epsilon^2 g_M^{ik} \nabla_i^M \nabla_j^M \omega_i + n \nabla_j^M f \\
= \eta_j + \omega_j + \epsilon^2 g_M^{ik} \Ric^{M}_{k} \omega_i + n \nabla_j^M f \\
\beta_{g_\epsilon} (\tilde{h})_{b=0} = \epsilon^2 g_M^{ik} \nabla_k^M \tilde{h}_b \\
= \epsilon^2 g_M^{ik} \nabla_k^M \nabla_i^M f \\
= \eta_0 + f.
\]

It follows that if we write \( \tau = \beta_{g_\epsilon} (\tilde{h})_{i=0} - \eta \), then by (4.4)

\[
\|\tau\|_{C^{2,\alpha}} \leq C\|\eta\|_{C^{1,\alpha}}.
\]

We now define the 2-tensor \( k \) on \( M \times [0, \infty) \) by

\[
k_{j0} = \tau_{j0} \\
k_{00} = 2t \tau_0.
\]

We have \( \|k\|_{C^{2,\alpha}} \leq C\|\eta\|_{C^{1,\alpha}} \) and in addition \( \beta_{g_\epsilon} (k)|_{t=0} = \tau \) (note that \( \tau \) vanishes when \( t = 0 \), and so only the terms involving a \( t \)-derivative survive).

Finally we define

\[
h = \chi \cdot (\tilde{h} - k),
\]

where \( \chi = \chi(t) \) is a cutoff function such that \( \chi(t) = 1 \) for \( t < 1/2 \), and \( \chi(t) = 0 \) for \( t > 1 \). Then \( h \) is supported in \( M \times [0, 1] \), it satisfies the required \( C^{2,\alpha} \) estimate since \( \tilde{h} \) and \( k \) do, and by construction it satisfies \( \beta_{g_\epsilon} (h)|_{t=0} = \eta \).

We have used the following result in the previous argument.

**Lemma 28.** Let \( (M, g_M) \) be compact. For sufficiently small \( \epsilon > 0 \), and any \( i \), the linear map \( D : C^{3,\alpha}_{\epsilon^{-2}g_M} (\Omega^i (M)) \to C^{1,\alpha}_{\epsilon^{-2}g_M} (\Omega^i (M)) \) given by

\[
D : \alpha \mapsto \Delta_{g^{-2}g_M} \alpha - \alpha,
\]

where \( \Delta_{g^{-2}g_M} = \epsilon^2 g_M^{ik} \nabla_j \nabla_k \) is the rough laplacian, is invertible. Moreover, the inverse is bounded independently of \( \epsilon \).

**Proof.** We will write down an approximate inverse for \( D \). We cover \( M \) with unit balls with respect to the metric \( \epsilon^{-2}g_M \), and let \( \gamma_1, \ldots, \gamma_N \) be a partition of unity subordinate to this cover. We have \( N_\epsilon = O(\epsilon^{-n}) \), and we can assume that all derivatives of the \( \gamma_i \) are uniformly bounded. Given \( u \in C^{1,\alpha}_{\epsilon^{-2}g_M} (\Omega^i (M)) \), we write

\[
u = \sum_{i=1}^{N_\epsilon} \gamma_i u.
\]

Using normal coordinates in each ball, we view \( \gamma_i u \) as a tensor on \( \mathbb{R}^n \) supported in the unit ball. On \( \mathbb{R}^n \) we can solve the equation

\[
\Delta_0 h_i - h_i = \gamma_i u,
\]
where $\Delta_0$ denotes the Euclidean Laplacian. Moreover, the solution decays in $C^\infty$ faster than any polynomial: for $|x| > 1$ and any $k, d$, we have
\[ |\partial_x^k h_i(x)| \leq C_{k,d}|x|^{-d}\|u\|_{C^{1,\alpha}_{r-2g_M}}, \]
since the Green’s function of the operator $\Delta_0 - 1$ on $\mathbb{R}^n$ decays exponentially fast (as can be seen using the Fourier transform for instance).

We can now reassemble these local solutions $h_i$ as follows. We fix a radius $R > 2$, and let $\chi_R$ denote a cutoff function supported in $B_R(0)$, and equal to 1 in $B_{R-1}(0)$. By the decay of $h_i$ we have
\[ \|\Delta_0(h_i - \chi_R h_i)\|_{C^{1,\alpha}_{r-2g_M}} \leq C_dR^{-d}\|u\|_{C^{1,\alpha}_{r-2g_M}} \]
for any $d > 0$.

Once $\epsilon$ is sufficiently small, we can use normal coordinates to view each $\chi_R h_i$ as a tensor on $M$, supported in an $R$-ball. On such an $R$-ball, if we compare $\epsilon^{-2}g_M$ with the Euclidean metric $\delta_{ij}$ in normal coordinates, we have
\[ \|\epsilon^{-2}g_{M,ij} - \delta_{ij}\|_{C^k} = O(\epsilon^2 R^2), \]
and so
\[ \|(\Delta_{e^{-2}g_M} - \Delta_0)(\chi_R h_i)\|_{C^{1,\alpha}_{e^{-2}g_M}} \leq C\epsilon^2 R^2\|u\|_{C^{1,\alpha}_{e^{-2}g_M}}. \]
Combining this with (4.5) we obtain
\[ \|\Delta_{e^{-2}g_M} (\chi_R h_i) - \chi_R h_i - \gamma_i u\|_{C^{1,\alpha}_{e^{-2}g_M}} \leq C_d(\epsilon^2 R^2 + R^{-d})\|u\|_{C^{1,\alpha}_{e^{-2}g_M}}. \]
We now define
\[ h = \sum_{i=1}^{N} \chi_R h_i, \]
and estimate the error
\[ \|\Delta_{e^{-2}g_M} h - h - u\|_{C^{1,\alpha}_{e^{-2}g_M}} \leq R^nC_d(\epsilon^2 R^2 + R^{-d})\|u\|_{C^{1,\alpha}_{e^{-2}g_M}}. \]
In this estimate we used the fact that each $\chi_R h_i$ is supported on an $R$-ball, and so at each point of $M$, the number of terms that contribute is of order $R^n$. It is now clear that if we choose $d > n$, then $R$ sufficiently large, and finally $\epsilon$ sufficiently small, we can ensure that
\[ \|\Delta_{e^{-2}g_M} h - h - u\|_{C^{1,\alpha}_{e^{-2}g_M}} \leq \frac{1}{2}\|u\|_{C^{1,\alpha}_{e^{-2}g_M}}, \]
and so the map $F : u \mapsto h$ that we defined is an approximate inverse for the linear operator $D$. In particular $DF$ is invertible, and $F(DF)^{-1}$ is the required inverse for $D$. \qed

We will also need the following, which allows us to correct the boundary values when they only contain a $dt$ component.

**Proposition 29.** Let $f \in C^{1,\alpha}(M, \mathbb{R})$. We can find $h \in C^{2,\alpha}(S^2)$ satisfying
\[ (1) \|h\|_{C^{2,\alpha}_1} \leq C\|f\|_{C^{1,\alpha}_{e^{-2}g_M}}, \]
\[ (2) \text{We have the boundary condition} \]
\[ \beta_{\gamma_j}(h)_{|t=0} = 0, \text{ for all } j \]
\[ \beta_{\gamma_0}(h)_{|t=0} = f \]

(3) $h$ is approximately in the kernel of $L_{g', g}$, in the sense that for any $\delta > 0$ there is a $C_\delta$ such that
\begin{equation}
\|L_{g', g} h\|_{C^{0, \alpha}_1} \leq C_\delta \epsilon^{2-\delta} \|f\|_{C^{1, \alpha}_{\epsilon^2 g_M}}.
\end{equation}

**Proof.** This result should be compared with Proposition 27, where arbitrary boundary values are allowed, but this comes at the cost of a worse estimate for $L_{g', g} h$. The proof is similar to the preceding proof, using the local result Proposition 26. As in the previous proof we write
\[ f = \sum_{i=1}^{N_\epsilon} \gamma_i f, \]
and apply Proposition 26 to each $\gamma_i f$. We obtain $h_i$ satisfying $L_0(h_i) = 0$, and $\beta_0(h_i)|_{t=0} = f$, emphasizing that we are using the Euclidean operators $L_0$ and $\beta_0$ here. We define
\[ \tilde{h} = \sum_{i=1}^{N_\epsilon} \chi_{R_\epsilon} h_i \]
as above, but we now allow the radius $R_\epsilon$ to depend on $\epsilon$. Let us write $b$ for the tensor given by $b_0 = 0$, and $b_0 = f$ on the slice $\{ t = 0 \}$. Estimating the errors as in (4.6), we will have
\[ \|\beta_{g_\epsilon}(\tilde{h})|_{t=0} - b\|_{C^{1, \alpha}_{g_M}} \leq R_\epsilon^2 C_\delta (\epsilon^2 R_\epsilon^2 + R_\epsilon^{-4}) \|f\|_{C^{1, \alpha}_{\epsilon^2 g_M}}. \]
Choosing $R_\epsilon = \epsilon^{-\tau}$ for some small $\tau > 0$, this implies
\[ \|\beta_{g_\epsilon}(\tilde{h})|_{t=0} - b\|_{C^{1, \alpha}_{g_M}} \leq C_\delta (\epsilon ^{2-2n}\tau + \epsilon \tau) \|f\|_{C^{1, \alpha}_{\epsilon^2 g_M}} \leq C_\delta \epsilon^{2-\delta} \|f\|_{C^{1, \alpha}_{\epsilon^2 g_M}}, \]
if $\tau = (2+n)^{-1}\delta$ and $d$ is sufficiently large. Similarly we have
\[ \|L_{g', g_\epsilon} \tilde{h}\|_{C^{0, \alpha}_1} \leq C_\delta \epsilon^{2-\delta} \|f\|_{C^{1, \alpha}_{\epsilon^2 g_M}}. \]
We can now apply Proposition 27 to perturb $\tilde{h}$ to $h$ satisfying $\beta_{g_\epsilon}(h)|_{t=0} = b$, while still satisfying the required estimate (4.7) for $L_{g', g_\epsilon} h$. \qed

4.2. **Inverting the linearized operator on the kernel of $I$.** We now move on to inverting the linearized operator, on the kernel of $I$. Recall that given any symmetric 2-tensor $u$ on $M \times [0, \infty)$, we defined the 1-form $I(u)$ on $M$ as in (4.2). We then have the following.

**Proposition 30.** We have a linear map
\[ P_\epsilon : C^{0, \alpha}_1(\ker I) \rightarrow C^{2, \alpha}_1(S^2), \]
satisfying the following.

1. There is a uniform bound $\|P_\epsilon u\|_{C^{2, \alpha}_1} \leq C \|u\|_{C^{0, \alpha}_1}$,
2. $P_\epsilon$ is an approximate inverse to $L_{g_\epsilon}$ in the sense that
\[ \|L_{g', g_\epsilon} P_\epsilon u - u\|_{C^{0, \alpha}_1} \leq C \epsilon^p \|u\|_{C^{0, \alpha}_1}, \]
where we can take any $1 < p < 4/3$.
3. $P_\epsilon(u)$ satisfies the Bianchi boundary condition, i.e. $\beta_{g_\epsilon}(P_\epsilon u)|_{t=0} = 0$. 

Proof: We construct an approximate inverse \( \tilde{P}_\epsilon \) for \( L_{g', g} \) in a very similar way to the proofs of Lemma \( \text{(29)} \) and Proposition \( \text{(29)} \). The difference is that the local result used here, Theorem \( \text{(2)} \) does not give rise to solutions with decay properties as strong as the local results used above. As a result the estimates required to obtain an approximate inverse are more delicate.

As before, let us cover \( M \) with unit balls \( \{B^i\}_{1 \leq i \leq N} \) with respect to the metric \( \epsilon^{-2}g_M \), and let \( \gamma_1, \ldots, \gamma_N \) be a partition of unity subordinate to this cover. We have \( N_\epsilon = O(\epsilon^{-n}) \), and we can assume that all derivatives of the \( \gamma_i \) are bounded uniformly. Given \( u \in C^{0, \alpha}(\ker I) \), we can express \( u \) as

\[
u = \sum_{i=1}^{N_\epsilon} \gamma_i u.
\]

Let \( B^i = B^i_{\epsilon^{-2}g_M}(p_i, 1) \) be one such ball of the covering, and let \( \{x^\mu\} \) be coordinates centered at \( p_i \) that are normal with respect to \( \epsilon^{-2}g_M \). We may assume these coordinates are defined on all of \( B^i \). In particular, if \( \{y^\mu\} \) are coordinates centered at \( p_i \) that are normal with respect to \( g_M \), then we can just take \( x^\mu = \epsilon^{-1}y^\mu \) to be the dilated coordinates, and it is clear that the \( x \)-coordinates are defined on \( B^i \) once \( \epsilon > 0 \) is small enough.

We can use the \( x \)-coordinates on \( B^i \) to view \( \gamma_i u \) as a 2-tensor on \( H^{n+1}_x \), supported in \( B_1 \times [0, \infty) \subset H^{n+1} \), where \( B_1 \) is a (Euclidean) unit ball. Also, \( \gamma_i u \) satisfies \( I(\gamma_i u) = 0 \), so we can apply Theorem \( \text{(2)} \). Letting \( P_{H} \) denote the inverse of the linearized operator \( L_{g_H} \) on the model space, we obtain solutions \( h_i = P_{H}(\gamma_i u) \) of \( L_{g_H}(h_i) = \gamma_i u \). By the estimates of Theorem \( \text{(2)} \) we have

\[
\|h_i\|_{C^{2, \alpha}(AR_{\epsilon^{-1}R})} \leq C\|u\|_{C^{0, \alpha}_1} R^{-n-1+\delta},
\]

where \( \delta \in (0, 1) \) and \( R > 1 \). Let \( R_\epsilon = \epsilon^{-2/3} \), and let \( \chi_{R_\epsilon} \) be a cutoff function supported in the Euclidean ball \( B_{R_\epsilon} \), equal to 1 in \( B_{R_{\epsilon}^{-1}} \). Using the \( x \)-coordinates we can identify the balls \( B_{\epsilon^{-2}g_M}(p_i, R_\epsilon) \) with Euclidean balls \( B(0, R_\epsilon) \), and view \( \chi_{R_{\epsilon}} h_i \) as a 2-tensor on \( M \times [0, \infty) \). We define

\[
\tilde{P}_\epsilon(u) = \sum_{i=1}^{N_\epsilon} \chi_{R_{\epsilon}} h_i.
\]

In order to estimate the norm \( \|\tilde{P}_\epsilon u\|_{C^2_{1, \alpha}} \), note that at each point \( (p, t) \in M \times [0, \infty) \), there will be contributions to \( \tilde{P}_\epsilon u \) from those \( h_i \), for which the center of the corresponding ball in our covering of \( M \) is of distance \( k < R_\epsilon + 1 \) from \( p \). There will be approximately \( k^{n-1} \) balls whose distance from \( p \) is in the interval \( [k-1, k) \), and the corresponding functions \( \chi_{R_{\epsilon}h_i} \) will contribute \( k^{-n+1+\delta} \|u\|_{C^{0, \alpha}_1} \) to the norm of \( \tilde{P}_\epsilon u \) at \( p \), because of the decay of \( h_i \). Adding up these contributions we have

\[
\|\tilde{P}_\epsilon u\|_{C^2_{1, \alpha}} \leq C \sum_{k=1}^{[R_\epsilon]} k^{n-1} k^{-n+1+\delta} \|u\|_{C^{0, \alpha}_1}
\]

\[
\leq C\|u\|_{C^{0, \alpha}_1},
\]

since \( \delta < 1 \). This gives the required bound on \( \tilde{P}_\epsilon \).

Next we need to estimate the error

\[
\|L_{g', g} \tilde{P}_\epsilon(u) - u\|_{C^{0, \alpha}_1}.
\]
In the same way we can express the operator
\[\omega = \gamma_i u_i + E,\]
where \(E\) is supported in \(B_{R_e} \setminus B_{R_{e-1}}\), and it is bounded by the \(C^{2,\alpha}\) norm of \(h_i\) there. From the decay of \(h_i\) we then get
\[
\|L_{g_i, g_i}(\chi_{R_i} h_i) - \chi_{R_i} L_{g_i} h_i\|_{C^{0,\alpha}_1(A_{R_{e-1}, R_{e}})} \leq C \|h_i\|_{C^{2,\alpha}_1(B_{R_e} \setminus B_{R_{e-1}})} \leq CR_{e}^{-n+\delta} \|h\|_{C^{0,\alpha}_1},
\]
and the error vanishes outside the annular region \(A_{R_{e-1}, R_{e}}\).

Next we consider the error arising from the difference between \(L'_{g_i, g_i}\) and \(L_{g_i}\). To do this we first observe that on the set \(C_R = B_R \times [0, \infty)\), where \(1 < R < R_e\), we have
\[
\|g'_{r_i} - g_{iH}\|_{C^{2}(C_R)} = O(\epsilon^2 R^2).
\]
Since \(g'_{r_i}\) and \(g_{iH}\) are close in \(C^2\), we want to show the corresponding linear operators are close (in a sense that will be made precise below).

Recall the formula for \(L'_{g_i, g_i}\) in (2.10):
\[
L'_{g_i, g_i}(h) = -\frac{1}{2} \Delta_{g_i} h + \mathfrak{D}_{g_i}(h) + nh + \mathcal{R}_{g_i}(h),
\]
where
\[
\mathfrak{D}_{g_i}(h) = \delta^g_{g_i}\{\beta_{g_i}(h) - \beta_{g_i}(h)\},
\]
and \(\mathcal{R}_{g_i}\) is given by (2.10), with \(\tilde{g} = g'_{r_i}\). Also,
\[
L_{g_i}(h) = -\frac{1}{2} \Delta_{g_{iH}} h + nh + \mathcal{R}_{g_{iH}}(h),
\]
where \(\mathcal{R}_{g_{iH}}\) is now computed with respect to the curvature of \(g_{iH}\). Subtracting, we have
\[
(L'_{g_i, g_i} - L_{g_{iH}})(h) = -\frac{1}{2} (\Delta_{g_i} - \Delta_{g_{iH}}) h + \mathfrak{D}_{g_i}(h) + (\mathcal{R}_{g_i} - \mathcal{R}_{g_{iH}})(h).
\]
Therefore, we need to estimate each of the terms on the right-hand side.

To estimate the term with \(\mathfrak{D}\), we use the fact that the Bianchi operator \(\beta_{g}h\) with respect to a local coordinate system can be schematically written as
\[
\beta_{g}h = g^{-1} * \partial h + g^{-1} * g^{-1} * \partial g * h,
\]
where * denotes the operation of tensor products and contractions. It follows that given two metrics \(g, g'\),
\[
\beta_{g'}h - \beta_{g}h = ((g')^{-1} - g^{-1}) * \partial h + ((g')^{-1} - g^{-1}) * (g')^{-1} * \partial g' * h + \cdots + g^{-1} * g^{-1} * \partial (g' - g) * h.
\]
In the same way we can express the operator \(\delta^g_{\omega}\) as
\[
\delta^g_{\omega} = \partial \omega + g^{-1} * \partial g * \omega.
\]
If we take \(g = g_e\) and \(g' = g'_{r_i}\), then combining (4.11) and (4.12) we have
\[
\mathfrak{D}_{g'_{r_i}}(h) = ((g'_{r_i})^{-1} - g_e^{-1}) * \partial^2 h + \cdots + g^{-1} * g^{-1} * \partial^2 (g'_{r_i} - g_e) * h.
\]
On the set \( C_R = B_R \times [0, \infty) \), where \( 1 < R < R_* \), we have

\[
\|g'_\epsilon - g_\epsilon\|_{C^2(C_R)} + \|(g'_\epsilon)^{-1} - g_\epsilon^{-1}\|_{C^2(C_R)} = O(\epsilon^2 R^2).
\]

Consequently,

\[
\|\mathcal{D} g'_\epsilon(h)\|_{C^{0,\alpha}(AR_{1-R})} \leq C\epsilon^2 R^2\|h\|_{C^{2,\alpha}(AR_{1-R})},
\]

for \( 1 < R < R_* \).

To estimate the the remaining terms in (4.10), we argue in a similar way. For a metric \( g \), we can schematically write the rough laplacian with respect to a local coordinate system as

\[
\Delta g = g^{-1} * \partial^2 h + g^{-1} * \partial g * \partial h + g^{-1} * g^{-1} * \partial g * \partial h + g^{-1} * g^{-1} * \partial^2 g * h.
\]

We can then estimate the difference \((\Delta g'_\epsilon - \Delta g_\epsilon)h\) using (4.14) and (4.17) to get

\[
\|\Delta g'_\epsilon - \Delta g_\epsilon\|_{C^{0,\alpha}(AR_{1-R})} \leq C\epsilon^2 R^2\|h\|_{C^{2,\alpha}(AR_{1-R})},
\]

We can estimate the difference of the curvature terms in a similar manner, and combining all of these estimates we conclude

\[
\|L g'_\epsilon, g_\epsilon - L g_\epsilon(h)\|_{C^{0,\alpha}(AR_{1-R})} \leq C\epsilon^2 R^2\|h\|_{C^{2,\alpha}(AR_{1-R})},
\]

for \( 1 < R < R_* \). Combining this with the errors introduced by the cut-off functions estimated in (4.13), we obtain

\[
\|L g'_\epsilon, g_\epsilon - L g_\epsilon\|_{C^{0,\alpha}(AR_{1-R})} \leq C\epsilon^2 R^2\|h\|_{C^{2,\alpha}(AR_{1-R})} \leq C\epsilon^2 R^{-n+1+\delta}\|u\|_{C^{0,\alpha}},
\]

for \( 1 < R < R_* + 1 \), and the error vanishes for larger \( R \).

To estimate the difference \( L g'_\epsilon, g_\epsilon - P_\epsilon(u) - u \), we need to sum up all the contributions from (4.13) and (4.17).

1. For each \( i \), the error coming from (4.13) appears only in an annulus \( B_{R_*} \setminus B_{R_*-1} \). Each point in \( M \) will be covered by roughly \( R_*^{-1} \) such annuli, and so the total contribution of this type of error at each point will be bounded by

   \[
   CR_*^{-1} R_*^{n-1+\delta}\|u\|_{C^{0,\alpha}} = CR_*^{-2+\delta}\|u\|_{C^{0,\alpha}}.
   \]

2. For each \( i \), the error coming from (4.17) appears on an \( R_* \)-ball, but it decays as we approach the boundary of the \( R_* \)-ball. We estimate this in a similar way to the way we bounded \( P_\epsilon u \) above. When \( \epsilon \) is sufficiently small, then on an \( R_* \) ball our cover of \((M, \epsilon^{-2}gM)\) with unit balls has centers that are roughly on the grid \( \mathbb{Z}^n \subset \mathbb{R}^n \) (in normal coordinates). We can sum up the contributions of these errors at the origin. If a unit ball has distance in the interval \([k-1, k)\) from the origin, then according to (4.17) it contributes an error of \( C\epsilon^2 k^{-n+1+\delta}\|u\|_{C^{0,\alpha}} \). There will be roughly \( k^{n-1} \) such balls, and \( k \) can range from 1 to \([R_*] \). The sum of errors will therefore be bounded by

   \[
   Ce^2 \sum_{k=1}^{[R_*]} k^{n-1} k^{-n+1+\delta}\|u\|_{C^{0,\alpha}} \leq Ce^2 R_*^{1+\delta}\|u\|_{C^{0,\alpha}}.
   \]

Adding up all of these contributions we have

\[
\|L g'_\epsilon, g_\epsilon - P_\epsilon(u) - u\|_{C^{0,\alpha}} \leq C(R_*^{-2+\delta} + \epsilon^2 R_*^{1+\delta})\|u\|_{C^{0,\alpha}} \leq Ce^{4+\frac{4}{\delta}}\|u\|_{C^{0,\alpha}},
\]
since $R_\epsilon = \epsilon^{-2/3}$. As $\delta \in (0, 1)$, we conclude
\[
\| L_{g'_\epsilon, g_\epsilon} \tilde{P}_\epsilon (u) - u \|_{C^{0, \alpha}_1} \leq C \epsilon^p \| u \|_{C^{0, \alpha}_1},
\]
for any $1 < p < 4/3$.

We still need to consider the boundary condition. Each $h_i = P(\gamma_i u)$ satisfies the boundary condition with respect to the hyperbolic metric, but we introduce an error when we multiply with the cutoff function $\chi_R$, and also when we use the metric $g_\epsilon$ instead of the hyperbolic metric. Accounting for the errors exactly as above, we have
\[
\| \beta_{g_\epsilon}(\tilde{P}_\epsilon u) \|_{C^{1, \alpha}_1} \leq C \epsilon^p \| u \|_{C^{0, \alpha}_1},
\]
where $1 < p < 4/3$. We can now use Proposition 27 to find a 2-tensor $k$ supported in $M \times [0, 1]$, satisfying $\| k \|_{C^{2, \alpha}_1} \leq C \epsilon^p \| u \|_{C^{0, \alpha}_1}$ and $\beta_{g_\epsilon}(k)|_{t=0} = \beta_{g_\epsilon}(\tilde{P}_\epsilon u)|_{t=0}$. Then we define $P_\epsilon u = \tilde{P}_\epsilon u - k$ and this will satisfy all of our requirements. \qed

4.3. The induced operator on $\Omega^1(M)$. In the previous section we considered the equation $L_{g', g_\epsilon}(h) = u$ for $u \in \ker I$. We now consider the complementary problem of solving $I \circ L_{g', g_\epsilon}(h) = \omega$ for a one-form $\omega$ on $M$. Let us define the operator
\[
T : C^{2, \alpha}_g(\Omega^1(M)) \to C^{0, \alpha}_g(\Omega^1(M)),
\]
\[
\omega \mapsto I\left(L_{g', g_\epsilon}(\epsilon^{-2} e^{-nt} \omega \otimes dt)\right),
\]
where we emphasize that we will now measure norms using the metric $g_M$ on $M$ instead of $\epsilon^{-2} g_M$.

The dependence of $T$ on $\epsilon$ is described by the following result.

**Proposition 31.** There is an elliptic operator
\[
T_0 : C^{2, \alpha}_g(\Omega^1(M)) \to C^{0, \alpha}_g(\Omega^1(M))
\]
such that
\[
\| T - T_0 \| \leq C \epsilon^{1-\alpha},
\]
for a constant $C$ independent of $\epsilon$.

**Proof.** To begin, we want to view $L_{g', g_\epsilon}$ as a perturbation of the hyperbolic model operator $L_{g_M}$. This will require us to use normal coordinates to identify the one-form $\omega$ on $M$ with a one-form in $H$. To this end, as in the proof of Proposition 30, we let $\{x^\mu\}$ denote normal coordinates with respect to $\epsilon^{-2} g_M$ defined on a ball in $M$. With respect to these coordinates, on the region $M \times [0, 1]$ we have
\[
\epsilon^{-2} g_M = \delta_{ij} + O(\epsilon^2),
\]
(see (4.19)). In addition, if we use these coordinates to define the hyperbolic metric $g_H = dt^2 + \epsilon^2 dx^2$, then $k^{(2)}, k^{(4)} = O(\epsilon^2)$, so
\[
g_\epsilon = g_H + O(\epsilon^2),
\]
\[
g'_\epsilon = g_H + O(\epsilon^2).
\]

Using the estimates in the proof of Proposition 30, we can write
\[
L_{g', g_\epsilon} = L_{g_H} + \epsilon^2 P + O(\epsilon^3),
\]
where $\mathcal{P}$ is a linear operator independent of $\epsilon$, determined by the terms of order $\epsilon^2$ in the difference (4.11). Note also that if $\|\omega\|_{C^2_{g'}} \leq 1$, then in the $x$-coordinates we have

$$
|\omega_j| \leq C\epsilon, \quad |\partial\omega_j| \leq C\epsilon^2, \quad |\partial^2\omega_j|_{C^0} \leq C\epsilon^3.
$$

It follows that

$$L_{g', g} e^{-2}\epsilon^{-nt} \odot dt = L_{g'} e^{-2}\epsilon^{-nt} \odot dt + \mathcal{P}(\epsilon^{-nt} \odot dt) + O(\epsilon^2).
$$

For tensors of the form $h = e^{-2}\epsilon^{-nt} \odot dt$, it follows from (4.21) that

$$L_{g'}(e^{-2}\epsilon^{-nt} \odot dt)_{j0} = e^{-2t}e^{-2}\epsilon^{-2}\partial_t \partial_j \omega_j.
$$

Let us write

$$\mathcal{P}(\epsilon^{-nt} \odot dt)_{j0} = A_2(\omega)_{j0} + A_1(\omega)_{j0} + A_0(\omega)_{j0},
$$

where $A_m$ denotes the degree $m$ part of the operator. The estimates (4.21) imply that

$$L_{g', g} e^{-2}\epsilon^{-nt} \odot dt)_{j0} = e^{-2t} e^{-2}\epsilon^{-2}\partial_t \partial_j \omega_j + A(\omega)_{j0} + O(\epsilon^2),
$$

where again we emphasize the components are with respect to the $x$-coordinates.

The same calculation also applies on the regions $M \times [T, T + 1]$ for all $T$, using normal coordinates for $e^{2T}\epsilon^{-2}g_M$. Applying the operator $I$ (i.e. integrating out the $t$ variable), we find that at least at the center of our coordinate system we have

$$I(L_{g', g} e^{-2}\epsilon^{-nt} \odot dt)_{j} = ce^{-2t}\partial_t \partial_j \omega_j + A(\omega)_{j} + O(\epsilon^2),
$$

where $c$ is a fixed constant arising from integrating the exponential term in $t$, and $A$ is a zeroth order operator on one-forms. Note that up to zeroth order terms, at the origin of our normal coordinate system $\partial_t \partial_j \omega_j$ is simply the rough Laplacian of $\omega$ with respect to the metric $g_M$. When we measure the $O(\epsilon^2)$ error term in (4.22) with respect to $g_M$ instead of $\epsilon^{-2}g_M$, then in the $C^{0, \alpha}$-norm we lose a factor of $\epsilon^{1+\alpha}$. In sum we have

$$
\|T \omega - c\Delta g_M \omega - \hat{A} \omega\|_{C^{0, \alpha}} \leq C\epsilon^{1-\alpha},
$$

where $\hat{A}$ is a zeroth order operator and $\Delta_{g_M}$ is the rough Laplacian on one-forms.

The specific form of $T_0$ is not important, but note that for instance if instead of $g'$ we use the metric $g$, then $T_0$ is the Hodge Laplacian on one forms. In particular $T_0$ is not necessarily surjective, already in this simple case. It is for this reason that we introduce a further finite dimensional space $E \subset C^{2, \alpha}_1(S^2)$, and consider the linear operator

$$
\mathcal{T} : E \times C^{2, \alpha}_{g_M}(\Omega^1(M)) \to C^{0, \alpha}_{g_M}(\Omega^1(M))
$$

$$(r, \omega) \mapsto I[(D\text{Ric}_{g'} + n)r + L_{g', g}(e^{-2t} \odot dt)].
$$

We then have the following.

**Proposition 32.** For a suitable finite dimensional subspace $E \subset C^{2, \alpha}_1$, the operator $\mathcal{T}$ has a right inverse with bound independent of $\epsilon$, as long as $\epsilon$ is sufficiently small. Here the norm on $E$ is the $C^{2, \alpha}_1$ norm.
Proof. Suppose that $r$ is a tensor of the form
\[ r = e^{-2}e^{-nt}h + f e^{-nt}dt \otimes dt, \]
where $h$ is a symmetric two-tensor on $M$, and $f$ is a function on $M$. Suppose in addition that $\text{tr}_{g_M}h = 0$. We then have the following formula (see (4.21)):
\[ -2D(\text{Ric}_{g_*} + n)r_{j0} = (n + 2)e^{-(n+2)t}g_M^{ij} \nabla^k_M h_{ij} - (n - 1)e^{-nt} \partial_j f. \]
It follows that for some nonzero dimensional constants $c_1, c_2$ we have
\[ I \circ D(\text{Ric}_{g_*} + n)r = c_1 \delta_{g_M}h + c_2 df. \]
A calculation shows that replacing $g_*$ by $g'_*$ only introduces lower order terms. More precisely
\begin{equation}
(4.23) \quad \| I \circ D(\text{Ric}_{g'_*} + n)r - c_1 \delta_{g_M}h - c_2 df \|_{C^\infty_{g_M}} \leq C \epsilon^2 (\|h\|_{C^2_{g_M}} + \|f\|_{C^2_{g_M}}). \end{equation}
We now observe that on the space of symmetric 2-tensors on $M$, the operator $\delta_{g_M}$ is underdetermined elliptic, and so its image is the orthogonal complement of $\ker \delta_{g_M}$, which can be identified with the space of Killing vector fields. A generic metric in the conformal class of $g_M$ has no Killing fields, and so $\delta_{g_M}$ is surjective. We assume from now that this is the case. Given any one-form $\eta$ on $M$, we can then find $h \in S^2(M)$ such that $\delta h = \eta$, and so
\[ \eta = \delta \left\{ h - \frac{1}{n} (\text{tr}_{g_M} h) g_M \right\} + \frac{1}{n} \text{tr}_{g_M} h. \]
It follows that for any $\eta \in \text{coker} T_0$ we can find $r$ as above, such that
\[ I \circ D(\text{Ric}_{g_*} + n)r = \eta. \]
Moreover since $\text{coker} T_0$ is finite dimensional, we have
\[ \|h\|_{C^2_{g_M}} + \|f\|_{C^2_{g_M}} \leq C \|\eta\|_{C^2_{g_M}}. \]
It follows that
\[ \|r\|_{C^2_{g_*}} \leq C \|\eta\|_{C^2_{g_M}}. \]
We can then use this, together with (4.19) and (4.23) to show the invertibility of $T$ for sufficiently small $\epsilon$. \hfill \square

4.4. Inverting the full linearized operator. We now combine the pieces developed in the previous sections. We consider the linearized operator
\[ T_{g'_*} : E \times (C^2_{g_*})_{g'_*} \rightarrow C^0_{g_*} \]
\[ (r, h) \mapsto (D\text{Ric}_{g'_*} + n)r + L_{g'_*, g_*}(h), \]
where $E$ is a finite dimensional subspace of $C^2_{g_*}$ as above. We can now prove Theorem 4 on finding a right inverse for $T_{g'_*}$. We state the result here again.

Theorem 33. Suppose that $(M, g_M)$ admits no Killing vector fields. Then for sufficiently small $\epsilon$ and $\alpha$ the operator $T_{g'_*}$ has a right inverse $R$, satisfying $\|R\| \leq C \epsilon^{-2-\alpha}$.

Proof. We construct an approximate inverse. Let $u \in C^0_{g_*}$, with $\|u\|_{C^1_{g_*}} \leq 1$. Then $I(u)$ is a one-form on $M$ satisfying the estimate
\[ \|I(u)\|_{C^0_{g_M}} \leq C \epsilon^{-1-\alpha}. \]
From Proposition 32 we have \( r \in E \) and \( \omega \in C^2_{gs}(\Omega^1 M) \) such that
\[
\overline{T}_{g_\epsilon}(r, \omega) = I(u),
\]
and
\[
\|r\|_{C^2_{\alpha}} \leq C\epsilon^{-1-\alpha}, \quad \|\omega\|_{C^2_{\alpha}} \leq C\epsilon^{-1-\alpha}.
\]
We let
\[
u = u - (D\text{Ric}_{g_\epsilon} + n) r - L_{g_\epsilon,g_\epsilon}(\epsilon^{-2}e^{-nt}\omega \otimes dt),
\]
so that by construction, \( u_1 \in \ker I \). At the same time
\[
\|D(\text{Ric}_{g_\epsilon} + n)r\|_{C^{0,\alpha}} \leq C\|r\|_{C^{2,\alpha}} \leq C\epsilon^{-1-\alpha},
\]
and we also have
\[
(4.24) \quad \|L_{g_\epsilon,g_\epsilon}(\epsilon^{-2}e^{-nt}\omega \otimes dt)\|_{C^{0,\alpha}} \leq C\epsilon^{-1-\alpha}.
\]
For the latter estimate note that by the formulas (3.6), in the model hyperbolic space, for a tensor of the form \( v = e^{-nt}\omega \otimes dt \) with an \( n \)-form \( \omega \) on \( \mathbb{R}^n \), we have
\[
\|L_{gh}v\|_{C^{0,\alpha}} \leq C\|\nabla \omega\|_{C^{1,\alpha}}
\]
since the terms that involve only \( t \)-derivatives of \( h \) cancel. Arguing similarly to (4.16) we then find that
\[
\|L_{g_\epsilon,g_\epsilon}v\|_{C^{0,\alpha}} \leq C\left[\|\nabla \omega\|_{C^{1,\alpha}} + \epsilon^2\|\omega\|_{C^{2,\alpha}}\right] \leq C\epsilon^2\|\omega\|_{C^{2,\alpha}}.
\]
The estimate (4.24) then follows, and as a consequence we have
\[
\|u_1\|_{C^{0,\alpha}} \leq C\epsilon^{-1-\alpha}.
\]
We now invoke Proposition 30 to find \( h_1 = P_r u_1 \), satisfying
\[
(4.25) \quad \|L_{g_\epsilon,g_\epsilon}h_1 - u_1\|_{C^{0,\alpha}} \leq C\epsilon^{-1-\alpha},
\]
where note that we can choose any \( p \in (1,4/3) \).

To construct our approximate solution what remains is to take care of the Bianchi boundary condition for the term \( \epsilon^{-2}e^{-nt}\omega \otimes dt \). Note that by (4.3) we have
\[
\beta_{g_\epsilon}(\epsilon^{-2}e^{-nt}\omega \otimes dt)_{t=0} = 0,
\]
and the estimates
\[
\|k\|_{C^{2,\alpha}} \leq C\|\omega\|_{C^{2,\alpha}} \leq C\epsilon^{-1-\alpha},
\]
\[
\|L_{g_\epsilon,g_\epsilon}k\|_{C^{0,\alpha}} \leq C\epsilon^{-1-\alpha} \epsilon^2 - \delta \leq C\epsilon^{1-\alpha-\delta},
\]
for any \( \delta > 0 \).

We now set
\[
h = h_1 + \epsilon^{-2}e^{-nt}\omega \otimes dt - k.
\]
By construction we have
\[ \beta(h)|_{t=0} = 0 \]
as required, and
\[ \|\omega\|_{C^{2,\alpha}_{t=-1/2,\pm M}} \leq C\epsilon\|\omega\|_{C^{2,\alpha}_{t=0,\pm M}} \leq C\epsilon^{-\alpha} \]
implies
\[ \|h\|_{C^{2,\alpha}_{t=0,\pm M}} \leq C\epsilon^{-2-\alpha}. \]
In addition we have \( \|r\|_{C^{2,\alpha}_{t=1}} \leq C\epsilon^{-1-\alpha} \), and
\[
\left\| u - L_{g'_\epsilon}(r, h) \right\|_{C^{0,\alpha}_1} \\
= \left\| u - (\mathcal{D} Ric_{g'_\epsilon} + n)r - L_{g'_\epsilon, g_n}(e^{-2\epsilon - n\epsilon t} \omega \otimes dt) - L_{g'_\epsilon, g_n} h_1 + L_{g'_\epsilon, g_n} k \right\|_{C^{0,\alpha}_1} \\
\leq C(\epsilon^{p-1-\alpha} + \epsilon^{1-\alpha-\delta}).
\]
If \( \alpha \) is sufficiently small, \( p > 1 \) and \( \delta \) is small, then for sufficiently small \( \epsilon \) the map \( u \mapsto (r, h) \) is then an approximate inverse for \( L_{g'_\epsilon} \), and we can perturb it to a genuine inverse. \( \square \)

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