A proof of the polycirculant conjecture

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Abstract

This paper presents a solution of the polycirculant conjecture which states that every vertex-transitive graph $G$ has an automorphism that permutes the vertices in cycles of the same length. This is done by identifying vertex-transitive graphs as coset graphs. For a coset graph $H$, an equivalence relation $\sim$ is defined on the vertices of cosets with classes as double cosets of the stabiliser and any other proper subgroup $A'$ of a transitive group $A$ of $G$. Induced left translations of elements of the subgroup $A'$ are semi-regular since they preserve these double cosets and acts regularly on each of them. The coset graph is equivalent to $G$ by a theorem of Sabidussi.

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1 Introduction

This paper sets out to prove the polycirculant conjecture. The polycirculant conjecture states that every vertex-transitive finite graph is a polycirculant.

A graph $G = (V, E)$ is a set $V$ together with an irreflexive and symmetric binary relation $E$ defined on $V$. The two arcs $(x, y), (y, x) \in E$ are identified
into an edge \([x, y]\). All graphs discussed in this paper are loopless, connected and finite.

An automorphism of a graph is a permutation of its vertices which preserves edges. The set of all permutations of a graph \(G\) is denoted by \(\text{Aut} G\) and constitutes a group under composition. A graph \(G\) is vertex-transitive if given any two vertices \(x, y \in V(G)\) there is an automorphism which maps \(x\) to \(y\). Let \(r \geq 1\) and \(s \geq 2\) be integers. An automorphism \(\alpha\) of a graph is said to \((r, s)\)-semiregular if it has \(r\) orbits of length \(s\) and semiregular if it is \((r, s)\)-semiregular for some \(r\) and \(s\). A graph \(G\) of order \(n\) is called a circulant if it has a \((1, n)\)-semiregular automorphism.

It is easy to see that a graph is a circulant if and only if it is Cayley. A natural relaxation of the concept of a graph being a circulant is the concept of it being polycirculant. In 1981, Dragan Marušić asked if every vertex-transitive graph is a polycirculant. The question was also independently asked by D. Jordan.

The main thrust of this note is the following result.

1 Theorem Every vertex-transitive graph is a polycirculant.

To prove the theorem, we first represent vertex-transitive graphs as coset graphs. It is a classical Sabidussi Representation Theorem that vertex-transitive graphs are coset graphs.

For convenience and to fix notation, we will first present the Sabidussi Representation Theorem and then we give the proof of our main result.

2 Main Results

2.1 Sabidussi’s Representation Theorem

In this section we give a presentation of Sabidussi Representation Theorem that is amenable to proving the polycirculant theorem.

Let \(G\) be a vertex-transitive graph and \(A\) a subgroup of \(\text{Aut} G\) which acts
transitively on \( V(G) \).

Fix \( u \in V(G) \) (base point) and consider the stabilizer of \( u \) in \( A \):

\[
A_u := \{ \alpha \in A : \alpha(u) = u \}.
\]

Let \( A/A_u \) be the left cosets of \( A_u \) in \( A \) and \( B := \{ \alpha \in A : [u, \alpha(u)] \in E(G) \} \).

Note that \( B \) is a Cayley set in \( A \), i.e., the identity is not in \( B \) and if \( b \in B \) then \( b^{-1} \in B \).

We define a graph \( H \) as follows:

\[
V(H) := A/A_u,
\]

\[
[\alpha A_u, \beta A_u] \in E(H) \iff \text{there exists } v \in \beta A_u \text{ such that } v \in \alpha A_u \sigma, \sigma \in B.
\]

2 Lemma With the notation as above, \( H \) is isomorphic to \( G \).

The constructed graph \( H \) is called a coset graph. In view of Lemma 2 we have that

3 Theorem (Sabidussi [5]) Every vertex-transitive graph is a coset graph.

2.2 Proof of the theorem

Here the notation is as in Section 2.1. Our proof of the theorem utilizes the relationship between Cayley graphs and vertex-transitive graphs. The most used characterization of Cayley graph is that given by Sabidussi. It reads as follows:

4 Proposition (Sabidussi [5]) A graph \( G \) is Cayley if and only if \( \text{Aut } G \) contains a subgroup \( A \) which acts regularly on \( V(G) \).

However, for our purpose we present yet another characterization which identifies and uses the fact that Cayley graphs should also be considered as coset graphs.
5 Proposition A graph $G$ is Cayley if and only if there is a transitive subgroup $A$ of $\text{Aut} \, G$ such that $A_u$ (the stabilizer of $u$ in $A$) is normal in $A$ for any $u \in V(G)$.

Proof
By Proposition 4, for a given Cayley graph $G$, $\text{Aut} \, G$ contains a subgroup $A$ which acts regularly on $V(G)$. Moreover, $A_u$ contains only the identity element and therefore is trivially normal in $A$.

Suppose that $G$ has it that $\text{Aut} \, G$ contains a transitive subgroup $A$ such that $A_u$ is normal in $A$. Then the quotient group $A/A_u$ together with the set

$$C := \{\alpha A_u \in A/A_u : \alpha \in A, [u, \alpha u] \in E(G)\}$$

forms a Cayley graph $\text{Cay}(A/A_u, C)$ isomorphic to $G$.

To complete the proof of Theorem 1 we need the following lemma.

6 Lemma Let $G$ be a graph and $u \in V(G)$. Let $A$ be a transitive subgroup of an automorphism group of a graph $G$ and $B := \{\alpha \in A : [u, \alpha u] \in E(G)\}$. If $A$ contains a proper subgroup $A'$ generated by $B' \subset B$, then it contains a semi-regular element.

Proof
For the vertex-transitive graph $G$, let $H$ be the quotient graph defined as in Section 2.1 by

$$V(H) = A/A_u,$$

$$[\alpha A_u, \alpha' A_u] \in E(H) \iff \text{for some } \tau \in \alpha A_u, \tau' \in \alpha A_u \text{ s.t. } \tau \beta = \tau', \beta \in B.$$  

We define a relation $\sim$ on $V(H)$ by

$$\alpha A_u \sim \alpha' A_u \iff \alpha, \alpha' \in A' \tau A_u,$$

i.e. $\alpha A_u$ and $\alpha' A_u$ are related if their union in contained in some double coset $A' \tau A_u$.

It is clear that $\sim$ is an equivalence relation.

We have that for any $\beta \in A'$ the map $\tilde{\lambda}_\beta : V(H) \rightarrow V(H)$ given by

$$\tilde{\lambda}_\beta(\alpha A_u) = \beta \alpha A_u$$
is an automorphism of $H$ that preserves equivalence classes of $\sim$. Moreover, $\Lambda_{A'} := \{\tilde{\lambda}_\beta, \beta \in A'\}$ is regular on each double coset. (We can consider the double coset $A'\tau A_u$ as a set of left cosets in the form

$$A'\tau A_u = \{\beta \alpha A_u : \beta \in A'\}$$

and so for each $\beta \in A'$ we have

$$\tilde{\lambda}_\beta (A'\tau A_u) = \beta A'\tau A_u = A'\tau A_u$$

for each double coset.) Hence the map $\tilde{\lambda}_\beta : V(H) \to V(H)$ is semi-regular. ■

**Proof of Theorem 1**

Let $G$ be a vertex-transitive graph, $u \in V(G)$ and $B := \{\alpha \in \text{Aut } A : [u, \alpha u] \in E(G)\}$.

If there is an element $\beta \in B$ such that $< \beta > = \text{Aut } G$ then the stabilizer of $u$ is normal in $\text{Aut } G$, hence by Proposition 5, $G$ is Cayley (Since $\text{Aut } G$ is Abelian). We therefore have that $G$ is polycirculant.

Otherwise, $< \beta >, \beta \in B$ is a proper subgroup of $\text{Aut } G$ and by Lemma 6, $\text{Aut } G$ contains a semi-regular element. (By considering the equivalence relation $\sim$ defining double cosets in the form $< \beta > \alpha A_u, \alpha \in \text{Aut } G$ as equivalence classes.) ■

**References**

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