ON DEFORMATION WITH CONSTANT MILNOR NUMBER AND NEWTON POLYHEDRON

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Abstract. We show that every $\mu$-constant family of isolated hypersurface singularities satisfying a nondegeneracy condition in the sense of Kouchnirenko, is topologically trivial, also is equimultiple.

Let $f: (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be the germ of a holomorphic function with an isolated singularity. The Milnor number of a germ $f$, denoted by $\mu(f)$, is algebraically defined as the \text{dim} $\mathcal{O}_n/J(f)$, where $\mathcal{O}_n$ is the ring of complex analytic function germs : $(\mathbb{C}^n,0) \to (\mathbb{C},0)$ and $J(f)$ is the Jacobian ideal in $\mathcal{O}_n$ generated by the partial derivatives $\{\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\}$. We recall that the multiplicity $m(f)$ is defined as the lowest degree in the power series expansion of $f$ at $0 \in \mathbb{C}^n$. Let $F: (\mathbb{C}^n \times \mathbb{C},0) \to (\mathbb{C},0)$ be the deformation of $f$ given by $F(z,t) = f(z) + \sum c_\nu(t)z^\nu$, where $c_\nu: (\mathbb{C},0) \to (\mathbb{C},0)$ are germs of holomorphic functions. We use the notation $F_t(z) = F(z,t)$ when $t$ is fixed. Let $m_t$ denote the multiplicity and $\mu_t$ denote the Milnor number of $F_t$ at the origin. The deformation $F$ is equimultiple (resp. $\mu$-constant) if $m_0 = m_t$ (resp. $\mu_0 = \mu_t$) for small $t$. It is well-known by the result of Lê-Ramanujam [9], that for $n \neq 3$, the topological type of the family $F_t$ is constant under $\mu$-constant deformations. The question is still open for $n = 3$. However, under some additional assumption, positive answers have been given. For example, if $F_t$ is non-degenerate in the sense of Kouchnirenko [7] and the Newton boundary $\Gamma(F_t)$ of $F_t$ is independent of $t$, i.e., $\Gamma(F_t) = \Gamma(f)$, it follows that $\mu^*(F_t)$ is constant, and hence $F_t$ is topologically trivial (see [12, 16] for details). Motivated by the Briançon-Speder $\mu$-constant family $F_t(z) = z_1^5 + z_2z_3^2 + z_4^{15} + tz_1z_2^6$, which is topologically trivial but not $\mu^*$-constant, M. Oka [13] shows that any non-degenerate family of type $F(z,t) = f(z) + tz^A$ for $A = (A_1, \ldots, A_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of nonnegative integers and $z^A = z_1^{A_1}z_2^{A_2} \cdots z_n^{A_n}$ as usual, is topologically trivial, under the assumption of $\mu$-constancy. Our purpose of this paper is to generalize this result, more precisely, we show that every $\mu$-constant non-degenerate family $F_t$ with not necessarily Newton boundary $\Gamma(F_t)$ independent of $t$, is topologically trivial. Moreover, we show that $F$ is equimultiple, which gives a positive answer to a question of Zariski [19, 5, 14] for a non-degenerate family. To prove the main result (Theorem 4 below), we shall use the notion of $(c)$-regularity in stratification theory, introduced by K. Bekka in [3], which is weaker than Whitney regularity, nevertheless $(c)$-regularity implies topological triviality. First, we give a characterization of $(c)$-regularity (Theorem 2.1 below). By using it, we can show that the $\mu$-constancy condition for a non-degenerate family implies Bekkas $(c)$-regularity condition and then obtain the topological triviality as a corollary.

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Notation. To simplify the notation, we will adopt the following conventions: for a function $F(z,t)$ we denote by $\partial F$ the gradient of $F$ and by $\partial_z F$ the gradient of $F$ with respect to variables $z$.

Let $\varphi$, $\psi: (\mathbb{C}^n, 0) \to \mathbb{R}$ be two function germs. We say that $\varphi(x) \leq \psi(x)$ if there exists a positive constant $C > 0$ and an open neighborhood $U$ of the origin in $\mathbb{C}^n$ such that $\varphi(x) \leq C \psi(x)$, for all $x \in U$. We write $\varphi(x) \sim \psi(x)$ if $\varphi(x) \leq \psi(x)$ and $\psi(x) \leq \varphi(x)$.

Finally, $|\varphi(x)| \ll |\psi(x)|$ (when $x$ tends to $x_0$) means $\lim_{x \to x_0} \frac{\varphi(x)}{\psi(x)} = 0$.

1. Newton polyhedron, main results

First we recall some basic notions about the Newton polyhedron (see [7, 12] for details), and state the main result.

Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function defined by a convergent power series $\sum c_\nu x^\nu$, we define $\text{supp}(f) = \{ \nu \in \mathbb{N}^n : c_\nu \neq 0 \}$. Also, let $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, \text{ each } x_i \geq 0, i = 1, \ldots, n\}$. The Newton polyhedron of $f$, denoted by $\Gamma(f) \subset \mathbb{R}^n$, is defined by the convex hull of $\{k + v \mid k \in \text{supp}(f), \ v \in \mathbb{R}_+^n\}$, and let $\Gamma(f)$ be the Newton boundary, i.e., the union of the compact faces of $\Gamma(f)$. For a face $\gamma$ of $\Gamma(f)$, we write $f_\gamma(z) := \sum_{\nu \in \gamma} c_\nu x^\nu$. We say that $f$ is non-degenerate if, for any face $\gamma$ of $\Gamma(f)$, the equations $\frac{\partial f_\gamma}{\partial x_1} = \cdots = \frac{\partial f_\gamma}{\partial x_n} = 0$ have no common solution on $x_1 = \cdots = x_n \neq 0$. The power series $f$ is said to be convenient if $\Gamma_+(f)$ meets each of the coordinate axes. We let $\Gamma_-(f)$ denote the compact polyhedron which is the cone over $\Gamma_+(f)$ with the origin as a vertex. When $f$ is convenient, the Newton number $\nu(f)$ is defined as $\nu(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n$, where the $V_n$ are the $n$-dimensional volumes of $\Gamma_-(f)$ and for $1 \leq k \leq n-1$, $V_k$ is the sum of the $k$-dimensional volumes of the intersection of $\Gamma_-(f)$ with the coordinate planes of dimension $k$. The Newton number may also be defined for a non-convenient analytic function (see [6]). Finally, we define the Newton vertices of $f$ as $\text{ver}(f) = \{ \alpha : \alpha \text{ is a vertex of } \Gamma(f) \}$.

Now we can state the main result

Theorem 1.1. Let $F: (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that $F_t$ is non-degenerate. Then $F_t$ is topologically trivial, and moreover, $F$ is equimultiple.

Remark 1.2. In the above theorem, we do not require the independence of $t$ for the Newton boundary $\Gamma(F_t)$

2. Criterion for (c)-regularity

By way of notation, we let $G(k,n)$ denote the set of $k$-dimensional linear subspace of the vector space $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Let $M$ be a smooth manifold, and let $X$, $Y$ be smooth submanifolds of $M$ such that $Y \subset X$ and $X \cap Y = \emptyset$. 
(i) (Whitney (a)-regularity)

\((X,Y)\) is (a)-regular at \(y_0 \in Y\) if:

for each sequence of points \(\{x_i\}\) which tends to \(y_0\) such that the sequence of tangent spaces \(\{T_{x_i}X\}\) tends to \(\tau\) in the grassmannian \(G(\dim X, \dim M)\), one has \(T_{y_0}Y \subset \tau\). We say \((X,Y)\) is (a)-regular if it is (a)-regular at any point \(y_0 \in Y\).

(ii) (Bekka (c)-regularity)

Let \(\rho\) be a smooth non-negative function such that \(\rho^{-1}(0) = Y\). \((X,Y)\) is (c)-regular at \(y_0 \in Y\) for the control function \(\rho\) if:

for each sequence of points \(\{x_i\}\) which tends to \(y_0\) such that the sequence of tangent spaces \(\{\text{Ker}\rho(x_i) \cap T_{x_i}X\}\) tends to \(\tau\) in the grassmannian \(G(\dim X - 1, \dim M)\), one has \(T_{y_0}Y \subset \tau\). \((X,Y)\) is (c)-regular at \(y_0\) if it is (c)-regular for some control function \(\rho\). We say \((X,Y)\) is (c)-regular if it is (c)-regular at any point \(y_0 \in Y\).

Let \(F : (\mathbb{C}^n \times \mathbb{C}, \{0\} \times \mathbb{C}) \to (\mathbb{C}, 0)\) be a deformation of an analytic function \(f\). We denote by \(\Sigma(V_F) = \{F^{-1}(0) - \{0\} \times \mathbb{C}, \{0\} \times \mathbb{C}\}\) the canonical stratification of the germ variety \(V_F\) of the zero locus of \(F\). We may assume that \(f\) is convenient, this is not a restriction when it defines an isolated singularity, in fact, by adding \(z^N\) for a sufficiently large \(N\) for which the isomorphism class of \(F_t\) does not change. Hereafter, we will assume that \(f\) is convenient,

\[ X = F^{-1}(0) - \{0\} \times \mathbb{C}, \ Y = \{0\} \times \mathbb{C} \text{ and } \rho(z) = \sum_{\alpha \in \text{ver}(F_t)} z^\alpha. \]

Here \(\text{ver}(F_t)\) denotes the Newton vertices of \(F_t\) when \(t \neq 0\).

Note that by the convenience assumption on \(f\), \(\rho^{-1}(0) = Y\).

We also let

\[ \partial \rho = \sum_{i=1}^n \frac{\partial \rho}{\partial z_i} \frac{\partial}{\partial z_i} + \frac{\partial \rho}{\partial \bar{z}_i} \frac{\partial}{\partial \bar{z}_i} = \partial_z \rho + \partial_{\bar{z}} \rho \]

and

\[ \partial F = \sum_{i=1}^n \frac{\partial F}{\partial z_i} \frac{\partial}{\partial z_i} + \frac{\partial F}{\partial t} = \partial_z F + \partial_t F. \]

**Calculation of the map \(\partial_z \rho|_X\)**

First of all we remark that \(\partial_z \rho = \partial_z \rho|_X + \partial_z \rho|_N\) (where \(N\) denotes the normal space to \(X\)). Since \(N\) is generated by the gradient of \(F\), we have that \(\partial_z \rho = \partial_z \rho|_X + \eta \partial F\). On the other hand, \((\partial_z \rho|_X, \partial F) = 0\), so we get \(\eta = \frac{\langle \partial_z \rho, \partial F \rangle}{\|\partial F\|^2}\). It follows that

\[ \tag{2.1} \partial_z \rho|_X = \partial_z \rho - \frac{\langle \partial_z \rho, \partial F \rangle}{\|\partial F\|^2} \partial F = \partial_z \rho|_X + \partial_z \rho|_X t, \]

where

\[ (\partial_z \rho|_X)_z = \partial_z \rho - \frac{\langle \partial_z \rho, \partial F \rangle}{\|\partial F\|^2} \partial F, \quad (\partial_z \rho|_X)_t = -\frac{\langle \partial_z \rho, \partial F \rangle}{\|\partial F\|^2} \partial_t F \]

and

\[ |\partial_z \rho|_X|^2 = \frac{|\partial F|^2 |\partial_z \rho|^2 - |\langle \partial_z \rho, \partial F \rangle|^2}{|\partial F|^2} = \frac{\|\partial F \wedge \partial_z \rho\|^2}{|\partial F|^2}. \]

Then we can characterise the (c)-regularity as follows:
Theorem 2.1. Consider $X$ and $Y$ as above. The following conditions are equivalent

(i) $(X,Y)$ is $(c)$-regular for the the control function $\rho$.
(ii) $(X,Y)$ is $(a)$-regular and $|\partial_z \rho| |X| \ll |\partial_z \rho| |X|$ as $(z,t) \in X$ and $(z,t) \to Y$.
(iii) $|\partial_t F| \ll \frac{\|\partial F \land \partial_z \rho\|}{|\partial_z \rho|}$ as $(z,t) \in X$ and $(z,t) \to Y$.

Proof. Since (i) $\iff$ (ii) is proved in [1], Theorem 1, and (iii) $\Rightarrow$ (ii) is trivial, it is enough to see (ii) $\Rightarrow$ (iii).

To show that (ii) $\Rightarrow$ (iii), it suffices to show this on any analytic curve $\lambda(s) = (z(s), t(s)) \in X$ and $\lambda(s) \to Y$. Indeed, we have to distinguish two cases:

First case, we suppose that along $\lambda$, $|\langle \partial_z \rho, \partial F \rangle| \sim |\partial_z \rho| |\partial F|$, hence by (2.1) and (ii), we have

$$|\langle \partial_z \rho |X| \rangle| = \frac{|\langle \partial_z \rho, \partial F \rangle|}{|\partial F|^2} \ll \frac{\|\partial F \land \partial_z \rho\|}{|\partial F|}.$$ But this clearly implies

$$|\partial_t F| \ll \frac{\|\partial F \land \partial_z \rho\|}{|\partial_z \rho|}$$ along the curve $\lambda(s)$,

where $|\langle \partial_z \rho, \partial F \rangle| \sim |\partial_z \rho| |\partial F|$.

Second case, we suppose that along $\lambda$, $|\langle \partial_z \rho, \partial F \rangle| \ll |\partial_z \rho| |\partial F|$, thus

$$\|\partial F \land \partial_z \rho\| \sim |\partial_z \rho| |\partial F|$$ along the curve $\lambda(s)$.

On the other hand, by the Whitney $(a)$-regularity in (ii) we get

$$|\partial_t F| \ll |\partial F|.$$ Therefore, $|\partial_t F| \ll |\partial F| \sim \frac{\|\partial F \land \partial_z \rho\|}{|\partial_z \rho|}$ along the curve $\lambda(s)$. The Theorem 2.1 is proved.

\[\Box\]

3. Proof of the Theorem 1.1

Before starting the proofs, we will recall some important results on the Newton number and the geometric characterization of $\mu$-constancy.

Theorem 3.1 (A. G. Kouchnirenko [7]). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity, then the Milnor number $\mu(f) \geq \nu(f)$. Moreover, the equality holds if $f$ is non-degenerate.

As an immediate corollary we have

Corollary 3.2 (M. Furuya [8]). Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two germs of holomorphic functions with $\Gamma_+(g) \subset \Gamma_+(f)$. Then $\nu(g) \geq \nu(f)$.

On the other hand, concerning the $\mu$-constancy, we have

Theorem 3.3 (Greuel [5], Lê-Saito [10], Teissier [10]). Let $F : (\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be the deformation of a holomorphic $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent.

1. $F$ is a $\mu$-constant deformation of $f$.
2. $\frac{\partial F}{\partial t}$ $\in J(F_t)$, where $J(F_t)$ denotes the integral closure of the Jacobian ideal of $F_t$ generated by the partial derivatives of $F$ with respect to the variables $z_1, \ldots, z_n$. 

\[\Box\]
(3) The deformation $F(z, t) = F_t(z)$ is a Thom map, that is,
\[ \sum_{j=1}^{m} \frac{\partial F}{\partial t_j} \ll \| \partial F \| \text{ as } (z, t) \to (0, 0). \]

(4) The polar curve of $F$ with respect to $\{t = 0\}$ does not split, that is,
\[ \{(z, t) \in \mathbb{C}^n \times \mathbb{C}^m \mid \partial_z F(z, t) = 0\} = \{0\} \times \mathbb{C}^m \text{ near } (0, 0). \]

We now want to prove theorem 1.1. In fact, let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a deformation of a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that $F_t$ is non-degenerate. Then, by theorem 3.1, we have

\[ \mu(f) = \nu(f) = \mu(F_t) = \nu(F_t). \]

Consider the deformation $\tilde{F}$ of $f$ given by
\[ \tilde{F}(z, t, \lambda) = F_t(z) + \sum_{\alpha \in \text{ver}(F_t)} \lambda^\alpha z^\alpha. \]

From the upper semi-continuity of Milnor number \[11\], we obtain

\[ \mu(f) \geq \mu(\tilde{F}_t, \lambda) \text{ for } (t, \lambda) \text{ near } (0, 0). \]

By Theorem 3.1 and Corollary 3.2, therefore
\[ \mu(\tilde{F}_t, \lambda) \geq \nu(\tilde{F}_t, \lambda) \geq \nu(f). \]

It follows from (3.1) and (3.2) that the deformation $\tilde{F}$ is $\mu$-constant, and hence, by Theorem 3.3 we get

\[ |\partial_t F| + \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \ll |\partial_z F| + \sum_{\alpha \in \text{ver}(F_t)} \lambda^\alpha z^\alpha | \text{ as } (z, t, \lambda) \to (0, 0, 0). \]

Therefore, for all $\alpha \in \text{ver}(F_t)$ we have $|z^\alpha| \ll |\partial f|$, and so $m(z^\alpha) \geq m(f)$. Hence the equality $m(F_t) = m(f)$ follows. In other words, $F$ is equimultiple.

We also show that condition (3.3), in fact, implies Bekka’s (c)-regularity, hence, this deformation is topologically trivial. For this purpose, we need the following lemma (see [15]).

Lemma 3.4. Suppose $F_t$ is a deformation as above, then we have

\[ \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \ll \inf_{\eta \in \mathbb{C}} \{ |\partial F + \sum_{\alpha \in \text{ver}(F_t)} \eta^\alpha \partial_z z^\alpha| \} \text{ as } (z, t) \to (0, 0), F(z, t) = 0. \]

Proof. Suppose (3.4) does not hold. Then by the curve selection lemma, there exists an analytic curve $p(s) = (z(s), t(s))$ and an analytic function $\eta(s)$, $s \in [0, \epsilon)$, such that:
(a) $p(0) = 0$;
(b) $F(p(s)) \equiv 0$, and hence $dF(p(s)) \frac{dp}{ds} \equiv 0$;
(c) along the curve $p(s)$ we have
\[ \sum_{\alpha \in \text{ver}(F_t)} |z^\alpha| \geq |\partial F + \sum_{\alpha \in \text{ver}(F_t)} \eta(s)^\alpha \partial_z z^\alpha|. \]
Set
\begin{equation}
(3.5) \quad g(z, \overline{z}) = \left( \sum_{\alpha \in \text{ver}(F_1)} \overline{\varepsilon}^\alpha z^\alpha \right)^{\frac{1}{2}} \quad \text{and} \quad \gamma(s) = \eta(s)g(z(s), \overline{z}(s))
\end{equation}

First suppose that \( \gamma(s) \rightarrow 0 \). Since \( |\overline{\varepsilon}| \leq g \), we have,
\[
\lambda_\alpha = \frac{\gamma(s)\overline{\varepsilon}^\alpha(s)}{g(z(s), \overline{z}(s))} \rightarrow 0, \quad \forall \alpha \in \text{ver}(F_1)
\]
Next, using (3.3) and (3.5) it follows
\[
\sum_{\alpha \in \text{ver}(F_1)} |z^\alpha(s)| \ll |\partial F(p(s))| + \sum_{\alpha \in \text{ver}(F_1)} \eta(s)\overline{\varepsilon}^\alpha(s) \partial_z z^\alpha(s) \quad \text{as} \quad s \rightarrow 0,
\]
which contradicts (c).

Suppose now that the limit of \( \gamma(s) \) is not zero (i.e., \( |\gamma(s)| \gtrsim 1 \)). Since \( p(0) = 0 \) and \( g(z(0), \overline{z}(0)) = 0 \), we have, asymptotically as \( s \rightarrow 0 \),
\begin{equation}
(3.6) \quad s\frac{dp}{ds}(s) \sim |p(s)| \quad \text{and} \quad s\frac{d}{ds}g(z(s), \overline{z}(s)) \sim g(z(s), \overline{z}(s)).
\end{equation}

But
\begin{equation}
(3.7) \quad \frac{d}{ds}g(z(s), \overline{z}(s)) = \sum_{\alpha \in \text{ver}(F_1)} \frac{1}{2g(z(s), \overline{z}(s))} \left( \overline{\varepsilon}^\alpha d\overline{z}^\alpha + z^\alpha d\overline{\varepsilon}^\alpha \right).
\end{equation}

We have \( \overline{\varepsilon}^\alpha d\overline{z}^\alpha \frac{d}{ds} = \overline{z}^\alpha d\overline{\varepsilon}^\alpha \frac{d}{ds} \) and \( 1 \lesssim |\gamma(s)| \). Thus,
\begin{equation}
(3.8) \quad \left| \frac{d}{ds}g(z(s), \overline{z}(s)) \right| \lesssim \sum_{\alpha \in \text{ver}(F_1)} \frac{\gamma(s)}{g(z(s), \overline{z}(s))} \overline{\varepsilon}^\alpha d\overline{z}^\alpha \frac{d}{ds}.
\end{equation}

This together with (3.6), (3.5) and (b) gives
\[ g(z(s), \overline{z}(s)) \sim \left| s\frac{d}{ds}g(z(s), \overline{z}(s)) \right| \lesssim s \sum_{\alpha \in \text{ver}(F_1)} \eta(s)\overline{\varepsilon}^\alpha \partial z^\alpha + \partial F(p(s)) \frac{dp}{ds}. \]

Hence
\[ g(z(s), \overline{z}(s)) \lesssim s \left| \frac{dp}{ds}(s) \right| \sum_{\alpha \in \text{ver}(F_1)} \eta(s)\overline{\varepsilon}^\alpha \partial z^\alpha + \partial F(p(s)) \right|,
\]
which contradicts (c). This ends the proof of Lemma.

We shall complete the proof of Theorem 1.1. Since \( \Gamma_+(\partial_t F) \subset \Gamma_+(F_1) \). Then, by an argument, based again on the curve selection lemma, we get the following inequality
\begin{equation}
(3.9) \quad |\partial_t F| \lesssim \sum_{\alpha \in \text{ver}(F_1)} |z^\alpha|.
\end{equation}

Then, by the above Lemma 3.4 we obtain
\[ |\partial_t F| \ll \inf_{\eta \in \mathcal{C}} \{|\partial F + \eta \partial_z \rho|\} \text{ as } (z, t) \rightarrow (0, 0), \quad F(z, t) = 0,
\]
we recall that
\[ \rho(z) = \sum_{\alpha \in \text{ver}(F_1)} z^\alpha \overline{\varepsilon}^\alpha.
\]
But
\[
\inf_{\eta \in \mathbb{C}} \left\{ \left| \partial F + \eta \partial F \right|^2 \right\}^2 = \frac{\left| \partial F \right|^2 \left| \partial F \right|^2 - \left| \left( \partial F, \partial F \right) \right|^2}{\left| \partial F \right|^2} = \frac{\left| \partial F \wedge \partial F \right|^2}{\left| \partial F \right|^2}.
\]
Therefore, by Theorem 2.1, we see that the canonical stratification \( \Sigma(V_F) \) is \((c)-regular\) for the control function \( \rho \), then \( F \) is a topologically trivial deformation (see 3).

This completes the proof of Theorem 1.1.

**Remark 3.5.** We should mention that our arguments still hold for any \( \mu \)-constant deformation \( F \) of a weighted homogeneous polynomial \( f \) with isolated singularity. Indeed, we can find from Varchenko’s theorem [18] that \( \mu(f) = \nu(f) = \mu(F_t) = \nu(F_t) \). Thus, the above proof can be applied.

Unfortunately this approach does not work, if we only suppose that \( f \) is non-degenerate.

For consider the example of Altman [2] defined by
\[
F_t(x, y, z) = x^5 + y^6 + z^5 + y^3z^2 + 2tx^2y^2z + t^2x^4y,
\]
which is a \( \mu \)-constant degenerate deformation of the non-degenerate polynomial \( f(x, y, z) = x^5 + y^6 + z^5 + y^3z^2 \). He showed that this family has a weak simultaneous resolution. Thus, by Laufer’s theorem [8], \( F \) is a topologically trivial deformation. But we cannot apply the above proof because \( \mu(f) = \nu(f) = \mu(F_t) = 68 \) and \( \nu(F_t) = 67 \) for \( t \neq 0 \).

We conclude with several examples.

**Example 3.6.** Consider the family given by
\[
F_t(x, y, z) = x^{13} + y^{20} + z^6y^5 + tx^6y^8 + t^2x^{10}y^3 + z^l, \ l \geq 7.
\]
It is not hard to see that this family is non-degenerate. Moreover, by using the formula for the computation of Newton number we get \( \mu(F_t) = \nu(F_t) = 153l + 32 \). Thus, by theorem [14] we have that \( F_t \) is topologically trivial. We remark that this deformation is not \( \mu^* \)-constant, in fact, the Milnor numbers of the generic hyperplane sections \( \{z = 0\} \) of \( F_0 \) and \( F_t \) for \( t \neq 0 \) are 260 and 189 respectively.

**Example 3.7.** Let
\[
F_t(x, y, z) = x^{10} + x^3y^4z + y^l + z^l + t^3x^4y^5 + t^5x^4y^5
\]
where \( l \geq 6 \). Since \( \mu(F_t) = 2l^2 + 32l + 9 \) and \( F_t \) is a non-degenerate family, it follows from Theorem 1.1 that \( F \) is a topologically trivial deformation.

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