Singularity-Free Two-Body Equation with Confining Interactions in Momentum Space

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Abstract We are developing a covariant model for all mesons that can be described as quark-antiquark bound states in the framework of the Covariant Spectator Theory (CST) in Minkowski space. The kernel of the bound-state equation contains a relativistic generalization of a linear confining potential which is singular in momentum space and makes its numerical solution more difficult. The same type of singularity is present in the momentum-space Schrödinger equation, which is obtained in the nonrelativistic limit. We present an alternative, singularity-free form of the momentum-space Schrödinger equation which is much easier to solve numerically and which yields accurate and stable results. The same method will be applied to the numerical solution of the CST bound-state equations.

1 Introduction

This work is part of an effort to develop a manifestly covariant model for a unified description of all mesons that can be understood as \( q\bar{q} \) states bound by a confining interaction. Mesons containing only heavy quarks are essentially nonrelativistic systems, but relativity is an essential ingredient in a theory that includes light quarks. In addition, the requirements of chiral symmetry have to be respected for a realistic description of the pion. Our theoretical framework is the Covariant Spectator Theory (CST), which is based on Relativistic Quantum Field Theory and can be viewed as a reorganization of the Bethe-Salpeter equation (for a recent brief review see Ref. [1]). By incorporating the quark self-interaction through the same relativistic kernel that describes the interaction between two different quarks, we improve on previous work by Gross et al. [2–5] and make the dynamics self-consistent [6].

Figure 1 shows how the \( q\bar{q} \) CST–Bethe-Salpeter vertex function with both external quark momenta off mass shell is related to four distinct CST vertex functions. It corresponds to the usual Bethe-Salpeter equation, except that only propagator pole contributions are included in the loop integration. To determine the four CST vertex functions, each of the two external quark momenta is, one at a time, placed on its positive- or negative-energy mass shell. The four possible choices then yield a closed system of four equations, referred to as the “CST four-channel equations”. It can be shown that they are charge-conjugation symmetric as required for the description of equal-mass \( q\bar{q} \) systems [6].

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The CST bound-state equations are homogeneous integral equations formulated in momentum space. A relativistic generalization of a linear potential is used as confining interaction, to which constant or color-Coulomb interactions can be added. Our goal is to construct a model that provides an accurate description of both the energy spectrum and the meson structure, the latter of which is probed in elastic and transition form factors. However, in order to determine the solutions of the CST equations, reliable numerical methods for dealing with confining interactions in momentum space are required.

To develop such a method we study the nonrelativistic limit in which the CST equation becomes the Schrödinger equation. This is very convenient because exact solutions of the Schrödinger equation with a linear potential for S-waves are known in coordinate space in terms of Airy functions. The validity of our numerical techniques can therefore be tested by comparison with the exact results.

\section{Linear Confinement in Momentum Space}

It is a priori not obvious how best to treat the linear coordinate-space potential \( W_L(\mathbf{r}) = \sigma r \) in momentum space. One way is to start from a screened version of the form

\[ W_{L,\varepsilon}(\mathbf{r}) = \frac{\sigma}{\varepsilon} (1 - e^{-\varepsilon r}) = W_{A,\varepsilon}(\mathbf{r}) - W_{A,\varepsilon}(0), \]  

where

\[ W_{A,\varepsilon}(\mathbf{r}) = -\frac{\sigma}{\varepsilon} e^{-\varepsilon r}, \]  

and \( W_L(\mathbf{r}) = \lim_{\varepsilon \to 0} W_{L,\varepsilon}(\mathbf{r}) \). After calculating the Fourier transform of \( W_{A,\varepsilon}(\mathbf{r}) \),

\[ V_{A,\varepsilon}(\mathbf{q}) = \int d^3 r W_{A,\varepsilon}(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} = -\frac{8\pi \sigma}{(q^2 + \varepsilon^2)^2}, \]

and observing that

\[ \int \frac{d^3 q}{(2\pi)^3} V_{A,\varepsilon}(\mathbf{q}) = -\frac{\sigma}{\varepsilon}, \]

one can write the screened linear potential in momentum space as

\[ V_{L,\varepsilon}(\mathbf{q}) = \int d^3 r W_{L,\varepsilon}(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} = V_{A,\varepsilon}(\mathbf{q}) - (2\pi)^3 \delta^{(3)}(\mathbf{q}) \int \frac{d^3 q'}{(2\pi)^3} V_{A,\varepsilon}(\mathbf{q}'). \]

The unscreened limit becomes

\[ V_L(\mathbf{q}) = \lim_{\varepsilon \to 0} V_{L,\varepsilon}(\mathbf{q}) = \left[ V_A(\mathbf{q}) - (2\pi)^3 \delta^{(3)}(\mathbf{q}) \int \frac{d^3 q'}{(2\pi)^3} V_A(\mathbf{q}') \right], \]

where

\[ V_A(\mathbf{q}) = -\frac{8\pi \sigma}{q^4}. \]

The form (7) suggests that a covariant generalization of the linear potential in momentum space is obtained by the replacement \( q^2 \to -q^2 \), where \( q \) is the four-vector of the transferred momentum. This generalization guarantees that the correct linear potential is reproduced in the nonrelativistic limit. That it is also strictly confining when used in the kernel of the CST bound-state equation has been demonstrated in Ref. [5].