General Structural Results for Potts Model Partition Functions on Lattice Strips

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Abstract

We present a set of general results on structural features of the $q$-state Potts model partition function $Z(G, q, v)$ for arbitrary $q$ and temperature Boltzmann variable $v$ for various lattice strips of arbitrarily great width $L_y$ vertices and length $L_x$ vertices, including (i) cyclic and Möbius strips of the square and triangular lattice, and (ii) self-dual cyclic strips of the square lattice. We also present an exact solution for the chromatic polynomial for the cyclic and Möbius strips of the square lattice with width $L_y = 5$ (the greatest width for which an exact solution has been obtained so far for these families). In the $L_x \to \infty$ limit, we calculate the ground-state degeneracy per site, $W(q)$ and determine the boundary $B$ across which $W(q)$ is singular in the complex $q$ plane.

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1 Introduction

The $q$-state Potts model has served as a valuable model for the study of phase transitions and critical phenomena [1, 2]. On a lattice, or, more generally, on a (connected) graph $G$, at temperature $T$, this model is defined by the partition function

$$Z(G, q, v) = \sum_{\{\sigma_i\}} e^{-\beta\mathcal{H}}$$  \hspace{1cm} (1.1)$$

with the (zero-field) Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$$  \hspace{1cm} (1.2)$$

where $\sigma_i = 1, ..., q$ are the spin variables on each vertex (site) $i \in G$; $\beta = (k_B T)^{-1}$; and $\langle ij \rangle$ denotes pairs of adjacent vertices. The graph $G = G(V, E)$ is defined by its vertex set $V$ and its edge set $E$; we denote the number of vertices of $G$ as $n = n(G) = |V|$ and the number of edges of $G$ as $e(G) = |E|$. We use the notation

$$K = \beta J, \quad a = e^K, \quad v = a - 1$$  \hspace{1cm} (1.3)$$

so that the physical ranges are (i) $a \geq 1$, i.e., $v \geq 0$ corresponding to $\infty \geq T \geq 0$ for the Potts ferromagnet, and (ii) $0 \leq a \leq 1$, i.e., $-1 \leq v \leq 0$, corresponding to $0 \leq T \leq \infty$ for the Potts antiferromagnet. One defines the (reduced) free energy per site $f = -\beta F$, where $F$ is the actual free energy, via

$$f(\{G\}, q, v) = \lim_{n \to \infty} \ln[Z(G, q, v)^{1/n}]$$  \hspace{1cm} (1.4)$$

where we use the symbol $\{G\}$ to denote $\lim_{n \to \infty} G$ for a given family of graphs. In the present context, this $n \to \infty$ limit corresponds to the limit of infinite length for a strip graph of the square lattice of fixed width and some prescribed boundary conditions.

Let $G' = (V, E')$ be a spanning subgraph of $G$, i.e. a subgraph having the same vertex set $V$ and an edge set $E' \subseteq E$. Then $Z(G, q, v)$ can be written as the sum

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')}$$  \hspace{1cm} (1.5)$$

where $k(G')$ denotes the number of connected components of $G'$. Since we only consider connected graphs $G$, we have $k(G) = 1$. The formula (1.3) enables one to generalize $q$ from $\mathbb{Z}_+$ to $\mathbb{R}_+$ (keeping $v$ in its physical range). The formula (1.5) shows that $Z(G, q, v)$ is a polynomial in $q$ and $v$. 

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The Potts model partition function is, up to a prefactor, equal to a quantity of major current mathematical interest, the Tutte (also called Tutte/Whitney) polynomial. The Tutte polynomial of a graph \( G \), \( T(G, x, y) \), is given by \([4]-[6]\)

\[
T(G, x, y) = \sum_{G' \subseteq G} (x - 1)^{k(G') - k(G)}(y - 1)^{c(G')}
\]

(1.6)

where \( k(G') \), \( e(G') \), and \( n(G') = n(G) \) denote the number of components, edges, and vertices of \( G' \), and \( c(G') = e(G') + k(G') - n(G') \) is the number of independent circuits in \( G' \). As stated in the text, \( k(G) = 1 \) for the graphs of interest here. Now let

\[
x = 1 + \frac{q}{v}
\]

(1.7)

and

\[
y = a = v + 1
\]

(1.8)

so that

\[
q = (x - 1)(y - 1) .
\]

(1.9)

Then

\[
Z(G, q, v) = (x - 1)^{k(G)}(y - 1)^{n(G)}T(G, x, y) .
\]

(1.10)

For a planar graph \( G \) the Tutte polynomial satisfies the duality relation

\[
T(G, x, y) = T(G^*, y, x)
\]

(1.11)

where \( G^* \) is the (planar) dual to \( G \). Some reviews of Tutte polynomials include \([3]-[4]\).

One interesting special case of the Potts model is the zero-temperature limit \((v = -1)\) of the Potts antiferromagnet, where the partition function is identical to the chromatic polynomial \( P(G, q) \) counting the number of ways of coloring the vertices of a graph with \( q \) colors subject to the condition that no adjacent vertices have the same color \([10, 7, 11, 12]\)

\[
Z(G, q, -1) = P(G, q) .
\]

(1.12)

The minimum number of colors necessary for this coloring is the chromatic number of \( G \), denoted \( \chi(G) \). For sufficiently large \( q \), on a given lattice or graph \( G \), the Potts antiferromagnet exhibits nonzero ground state entropy (without frustration). This is of interest as an exception to the third law of thermodynamics \([13, 14]\). A physical example of residual entropy at low temperatures is provided by ice \([15]\). Nonzero ground state entropy is equivalent to a ground state degeneracy per site (vertex), \( W > 1 \), since \( S_0 = k_B \ln W \). We have

\[
W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} .
\]

(1.13)
A subtlety in the definition (1.13) due to the noncommutativity
\[ \lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n} \quad (1.14) \]
at certain points \( q_s \) (typically, \( q = 0, 1, \ldots, \chi(G) \)) was discussed in [16].

Using the formula (1.5) for \( Z(G, q, v) \), one can generalize \( q \) from \( \mathbb{Z}_+ \) not just to \( \mathbb{R}_+ \) but to \( \mathbb{C} \) and \( a \) from its physical ferromagnetic and antiferromagnetic ranges \( 1 \leq a \leq \infty \) and \( 0 \leq a \leq 1 \) to \( a \in \mathbb{C} \). In particular, we shall be interested here in the case of the \( T = 0 \) antiferromagnet and the corresponding zeros of \( Z(G, q, -1) = P(G, q) \) (called chromatic zeros). A subset of these zeros can form a continuous accumulation set in the \( n \to \infty \) limit, denoted \( B \). This locus occurs where there is a non-analytic switching between different \( \lambda_{Z,G,j} \) of maximal magnitude and is thus determined as the solution to the equation of degeneracy in magnitude of these maximal or dominant \( \lambda_{Z,G,j} \)'s. As discussed earlier [17, 18, 19, 20], in the infinite-length limit where the locus \( B \) is defined, for a given width and transverse boundary condition (free or periodic) \( B \) depends on the longitudinal boundary conditions but is independent of whether they involve orientation reversal or not. Thus, in the present context, for a given \( L_y, B \) is the same for the cyclic and Möbius strips. Following the notation in [16], we denote the maximal region in the complex \( q \) plane to which one can analytically continue the function \( W(\{G\}, q) \) from physical values where there is nonzero ground state entropy as \( R_1 \). The maximal value of \( q \) where \( B_q \) intersects the (positive) real axis is denoted \( q_c(\{G\}) \). Thus, region \( R_1 \) includes the positive real axis for \( q > q_c(\{G\}) \).

In the present work we shall give a number of general results on structural features of the \( q \)-state Potts model partition function \( Z(G, q, v) \) for arbitrary \( q \) and temperature Boltzmann variable \( v \) for various lattice strips of arbitrarily great width \( L_y \) vertices and length \( L_x \) vertices. We shall also discuss an exact solution for the chromatic polynomial for the cyclic and Möbius strips of the square lattice with width \( L_y = 5 \) (the greatest width for which an exact solution has been obtained so far for these families). In the \( L_x \to \infty \) limit, we calculate the ground-state degeneracy per site, \( W(q) \) and determine the boundary \( B \) across which \( W(q) \) is singular in the complex \( q \) plane.

There are several motivations for this work. We have mentioned the basic importance of nonzero ground state entropy in statistical mechanics. From the point of view of rigorous statistical mechanics, exact solutions are always valuable for the insight that they give into the behavior of the given system under study. The results derived here give further insight into the general structure of the Potts model partition function for lattice strips. The study of \( W(\{G\}, q) \) with \( q \) generalized to complex values enables one to gain a deeper understanding of the behavior of this function for \( q \in \mathbb{Z}_+ \), in much the same way as the study of functions of a complex variable imparts a deeper understanding of functions of a real variable in
mathematics. In particular, one sees how the value $q_c$ on the real axis corresponds to an intersection of the locus $B$ with this axis. In addition to the papers [16, 17, 18, 19, 20], some relevant studies of Potts model partition functions and/or chromatic polynomials for strip graphs of regular lattices include [21]-[52]. Related mathematical papers, in addition to those already mentioned, include [53]-[58], and Ref. [59] contains further references.

2 Structural Results

2.1 General

In this section we briefly review some general structural results which will be used for our new results below. A general form for the Potts model partition function for the strip graphs $G_m$ considered here, or more generally, for recursively defined families of graphs comprised of $m$ repeated subunits (e.g. the columns of squares of height $L_y$ vertices that are repeated $L_x = m$ times to form an $L_x \times L_y$ strip of a regular lattice with some specified boundary conditions), is [19]

$$Z(G_m, q, v) = \frac{N_{Z,G,\lambda}}{c_{Z,G,j}(\lambda_{Z,G,j}(q, v))^m}$$

(2.1.1)

where $N_{Z,G,\lambda}$, $c_{Z,G,j}$, and $\lambda_{Z,G,j}$ depend on the type of recursive family $G$ (lattice structure and boundary conditions) but not on its length $m$. For strips with periodic longitudinal boundary conditions (e.g., cyclic and torus strips) eq. (2.1.1) follows from the property that $Z(G_m, q, v)$ can be expressed as the trace of a certain transfer matrix,

$$Z(G_m, q, v) = \text{Tr}(T_Z^m) .$$

(2.1.2)

Let

$$C_{Z,G} = \sum_{j=1}^{C_{Z,L_y}} c_{Z,G,j} .$$

(2.1.3)

For the strips with periodic longitudinal boundary conditions, $C_{Z,G}$ is thus the dimension of the matrix $T_Z$. In particular, for the cyclic strip graphs of the square and triangular lattices considered here, the repeated subgraph is the path graph consisting of $L_y$ vertices, and so, denoting $C_{Z,G} = C_{Z,L_y}$ as before, one has [16]

$$C_{Z,L_y} = q^{L_y} .$$

(2.1.4)

For strips with reversed-orientation periodic longitudinal boundary conditions (Möbius and Klein bottle boundary conditions), eq. (2.1.2) is modified by the insertion of the requisite
orientation-reversal operator in the trace, with the result that

$$C_{Z,Mb,L_y} = q^{\left\lfloor \frac{L_x}{2} \right\rfloor}$$

(2.1.5)

where \([\nu]\) denotes the integral part of \(\nu\).

Equivalently, for the Tutte polynomial of a recursive graph

$$T(G_m, x, y) = \sum_{j=1}^{N_{T,G,\lambda}} c_{T,G,j}(q)(\lambda_{T,G,j}(x, y))^m$$

$$= \frac{1}{x - 1} \sum_{j=1}^{N_{T,G,\lambda}} \bar{c}_{T,G,j}(q)(\lambda_{T,G,j}(x, y))^m$$

(2.1.6)

where

$$\bar{c}_{T,G,j} = c_{Z,G,j}.$$  

(2.1.7)

For the special case of the \(T = 0\) antiferromagnet, the partition function, or equivalently, the chromatic polynomial \(P(G_m, q)\) has the corresponding form

$$P(G_m, q) = \sum_{j=1}^{N_{P,G,\lambda}} c_{P,G,j}(\lambda_{P,G,j}(q))^m.$$  

(2.1.8)

Let

$$C_{P,G} = \sum_{j=1}^{N_{P,G,\lambda}} c_{P,G,j}.$$  

(2.1.9)

Here, the strip graph \(G\) is indexed by its width, \(L_y\). The analogue of (2.1.4) for the chromatic polynomial for cyclic strip graphs of the square and triangular lattices is

$$C_{P,L_y} = P(T_{L_y}, q) = q(q - 1)^{L_y - 1}$$

(2.1.10)

where \(T_n\) denotes the tree graph with \(n\) indices, and the analogue of (2.1.5) for the chromatic polynomial of the Möbius strip of the square lattice is

$$C_{P,Mb,L_y} = \begin{cases} 0 & \text{for even } L_y \\ P(T_{\frac{L_y+1}{2}}, q) & \text{for odd } L_y \end{cases}.$$  

(2.1.11)

### 2.2 Cyclic and Möbius Strips of the Square and Triangular Lattice

In \([46]\) it was shown that for cyclic and Möbius strips of the square lattice of fixed width \(L_y\) and arbitrary length \(L_x\) (and also for cyclic strips of the triangular lattice) the coefficients \(c_j(q)\) in the Potts model partition function (2.1.1), and hence, \textit{a fortiori}, in the chromatic...
polynomial \((2.1.8)\), are polynomials in \(q\) with the property that there is a unique polynomial, denoted \(c^{(d)}\), of degree \(d\) in \(q\). Further, this was shown to be a Chebyshev polynomial of the second kind:

\[
c^{(d)} = U_{2d}(q^{1/2}/2) = \sum_{j=0}^{d} (-1)^j \binom{2d - j}{j} q^{d-j} \tag{2.2.1}
\]

where \(U_n(x)\) is the Chebyshev polynomial of the second kind. A number of properties of these coefficients were derived in [46]; for our present work, we shall need the following special values:

\[
c^{(d)} = (-1)^d \quad \text{for} \quad q = 0 \tag{2.2.2}
\]

If \(q = 1\) then \(c^{(d)} = \begin{cases} 
1 & \text{if } d = 0 \mod 3 \\
0 & \text{if } d = 1 \mod 3 \\
-1 & \text{if } d = 2 \mod 3
\end{cases} \tag{2.2.3}
\]

If \(q = 2\) then \(c^{(d)} = \begin{cases} 
1 & \text{if } d = 0, 1 \mod 4 \\
-1 & \text{if } d = 2, 3 \mod 4
\end{cases} \tag{2.2.4}
\]

In contrast, for the M"{o}bius strips of the triangular lattice, the coefficients in the Potts model partition function and chromatic polynomial do not, in general, have the form \(c^{(d)}\) [36].

Thus, the terms \(\lambda_{Z,L_y,j}(q, v)\) (equivalently, \(\lambda_{T,L_y,j}(x, y)\)) and \(\lambda_{P,L_y,j}(q)\) that occur in \((2.1.1)\), \((2.1.6)\), and \((2.1.8)\) can be classified into sets, with the \(\lambda_{Z,L_y,j}(q, v)\), \(\lambda_{T,L_y,j}(x, y)\), and \(\lambda_{P,L_y,j}(q)\) in the \(d'\)th set being defined as those terms with respective coefficients \(c_{Z,L_y,j} = \bar{c}_{T,L_y,j}\) and \(c_{P,L_y,j}\) being equal to \(c^{(d)}\). In Ref. [46] the numbers of such terms, denoted \(n_{Z}(L_y, d)\) and \(n_{P}(L_y, d)\) respectively, were calculated. Note that

\[
n_{Z}(L_y, d) = n_{T}(L_y, d) . \tag{2.2.5}
\]

Labelling the eigenvalues with coefficient \(c^{(d)}\) as \(\lambda_{Z,L_y,d,j}(q, v)\) with \(1 \leq j \leq n_{Z}(L_y, d)\), the Potts model partition function can be written in the form

\[
Z(G[L_y \times m, cyc.], q, v) = \sum_{d=0}^{L_y} \sum_{j=1}^{n_{Z}(L_y, d)} (\lambda_{Z,L_y,d,j})^m \tag{2.2.6}
\]

or equivalently, for the Tutte polynomial,

\[
T(G[L_y \times m, cyc.], x, y) = \frac{1}{x - 1} \sum_{d=0}^{L_y} \sum_{j=1}^{n_{T}(L_y, d)} (\lambda_{T,L_y,d,j})^m \tag{2.2.7}
\]

where

\[
\lambda_{Z,L_y,d,j}(q, v) = v^{L_y} \lambda_{T,L_y,d,j}(x, y) \tag{2.2.8}
\]

with the relations \((1.7)\) and \((1.8)\).
With analogous labelling, the chromatic polynomial has the form

\[ P(G[L_y \times m, cyc.], q) = \sum_{d=0}^{L_y} \left[ c^{(d)} \sum_{j=1}^{n_P(L_y, d)} (\lambda_{P,L_y,d,j})^{m} \right]. \] (2.2.9)

Combining (2.2.6) with (2.1.4) and (2.2.9) with (2.1.10), one has the following relations for the cyclic strip graphs of the square and triangular lattice of width \( L_y \) and arbitrary length \([46]\)

\[ C_{Z,L_y} = \sum_{d=0}^{L_y} c^{(d)} n_{Z}(L_y, d) = q^{L_y} \] (2.2.10)

\[ C_{P,L_y} = \sum_{d=0}^{L_y} c^{(d)} n_{P}(L_y, d) = q(q - 1)^{L_y - 1}. \] (2.2.11)

The total numbers, \( N_{Z,L_y,\lambda} \) and \( N_{P,L_y,\lambda} \), of different terms \( \lambda_{Z,L_y,j} \) and \( \lambda_{P,L_y,j} \) in eqs. (2.1.1) and (2.1.8) are given by \([46]\)

\[ N_{Z,L_y,\lambda} = N_{T,L_y,\lambda} = \sum_{d=0}^{L_y} n_{Z}(L_y, d) \] (2.2.12)

and

\[ N_{P,L_y,\lambda} = \sum_{d=0}^{L_y} n_{P}(L_y, d). \] (2.2.13)

We shall recall here some results on the numbers \( n_{P}(L_y, d) \); these are nonzero for \( 0 \leq d \leq L_y \) and are given by

\[ n_{P}(L_y, L_y) = 1 \] (2.2.14)

\[ n_{P}(1,0) = 1 \] (2.2.15)

with all other numbers \( n_{P}(L_y, d) \) being determined by the two recursion relations \([46]\)

\[ n_{P}(L_y + 1, 0) = n_{P}(L_y, 1) \] (2.2.16)

\[ n_{P}(L_y + 1, d) = n_{P}(L_y, d - 1) + n_{P}(L_y, d) + n_{P}(L_y, d + 1) \] for \( L_y \geq 1 \) and \( 1 \leq d \leq L_y + 1 \). (2.2.17)

The solution to these relations yielded, in particular, the results

\[ n_{P}(L_y, 0) = M_{L_y - 1} \] (2.2.18)

where \( M_n \) is the Motzkin number,

\[ M_n = \sum_{j=0}^{n} (-1)^j C_{n+1-j} \binom{n}{j} \] (2.2.19)
where
\[ C_n = \frac{1}{n+1} \binom{2n}{n} \] (2.2.20)
is the Catalan number. Further,
\[ n_P(L_y, 1) = M_{L_y} \] (2.2.21)
and
\[ n_P(L_y, L_y - 1) = L_y . \] (2.2.22)
The total number of \( \lambda_{Z,L_y,j} \)'s in the Potts model partition function \( Z([L_y \times m, cyc.], q, v) \) or \( Z([L_y \times m, Mb], q, v) \) is
\[ N_{Z,L_y,\lambda} = \binom{2L_y}{L_y} \] (2.2.23)
and the total number of \( \lambda_{P,L_y,j} \)'s in the chromatic polynomial \( P([L_y \times m, cyc.], q) \) or \( P([L_y \times m, Mb], q) \) is
\[ N_{P,L_y,\lambda} = 2(L_y - 1)! \sum_{j=0}^{[L_y/2]} \frac{(L_y - j)}{(j!)^2 (L_y - 2j)!} . \] (2.2.24)

For arbitrary \( L_y \), eq. (2.2.14) shows that there is a unique \( \lambda_{P,L_y,d} \) corresponding to the coefficient \( c(d) \) of highest degree, \( d = L_y \), and this term is
\[ \lambda_{P,L_y,d} = (-1)^{L_y} \equiv \lambda_{P,L_y,L_y} . \] (2.2.25)

As indicated, since this eigenvalue is unique, it is not necessary to append a third index, as with the other \( \lambda \)'s, and we avoid this for simplicity. The result (2.2.25) corresponds to the analogous property that in the full Potts model partition function or equivalent Tutte polynomial, \( n_Z(L_y, L_y) = 1 \) and the unique \( \lambda_{Z,L_y,d=L_y} \) with coefficient \( c(L_y) \) in \( Z([L_y \times L_x, cyc.], q, v) \) or \( Z([L_y \times L_x, Mb], q, v) \) is
\[ \lambda_{Z,L_y,d=L_y} = v^{L_y} . \] (2.2.26)
Equivalently, the unique \( \lambda_{T,L_y,d=L_y} \) with reduced coefficient \( \bar{c}_{T,G,j} = c(L_y) \) in \( T([L_y \times L_x, cyc.], x, y) \) or \( T([L_y \times L_x, Mb], x, y) \) is
\[ \lambda_{T,L_y,d=L_y} = 1 . \] (2.2.27)
(Here, one should not confuse the arguments of the Tutte polynomial \( x, y \) with the longitudinal and transverse directions \( x, y \).)
For a strip of fixed width $L_y$ and arbitrary length $L_x = m$ with Möbius boundary conditions, $(FBC_y, TPBC_x)$, the general form of the Potts model partition function was determined in [46] to be

$$Z(G[L_y \times m, Mb], q, v) = \sum_{d=0}^{d_{\text{max, Mb}}} \left\{ c^{(d)} \sum_{\eta=\pm 1} \eta \left[ \sum_{j=1}^{n_{Z,Mb}(L_y,d,\eta)} (\lambda_{Z,L_y,d,\eta,j})^m \right] \right\}$$

(2.2.28)

where

$$d_{\text{max, Mb}} = \begin{cases} \frac{L_y^2}{2} & \text{for even } L_y \\ \frac{(L_y+1)^2}{2} & \text{for odd } L_y \end{cases}$$

(2.2.29)

Correspondingly,

$$T(G[L_y \times m, Mb], x, y) = \frac{1}{x-1} \sum_{d=0}^{d_{\text{max, Mb}}} \left\{ c^{(d)} \sum_{\eta=\pm 1} \eta \left[ \sum_{j=1}^{n_{T,Mb}(L_y,d,\eta)} (\lambda_{T,L_y,d,\eta,j})^m \right] \right\}$$

(2.2.30)

with $n_{T,Mb}(L_y,d,\eta) = n_{Z,Mb}(L_y,d,\eta)$. For the special case of the chromatic polynomial,

$$P(G[L_y \times m, Mb], q) = \sum_{d=0}^{d_{\text{max, Mb}}} \left\{ c^{(d)} \sum_{\eta=\pm 1} \eta \left[ \sum_{j=1}^{n_{P,Mb}(L_y,d,\eta)} (\lambda_{Z,L_y,d,\eta,j})^m \right] \right\}.$$  

(2.2.31)

Combining (2.2.28) and (2.2.31) with these formulas, one has the relations for the Möbius strip graphs of the square and triangular lattice of width $L_y$ and arbitrary length $L_x$ [46]

$$C_{Z,Mb,L_y} = \sum_{d=0}^{d_{\text{max, Mb}}} c^{(d)} \sum_{\eta=\pm 1} \eta n_{Z,Mb}(L_y,d,\eta) = q^{\frac{L_y+1}{2}}$$

(2.2.32)

$$C_{P,Mb,L_y} = \sum_{d=0}^{d_{\text{max, Mb}}} c^{(d)} \sum_{\eta=\pm 1} \eta n_{P,Mb}(L_y,d,\eta) = \begin{cases} 0 & \text{for even } L_y \\ P(T_{\frac{L_y+1}{2}}, q) & \text{for odd } L_y \end{cases}$$

(2.2.33)

In [46] the $n_{Z,Mb}(L_y,d,\eta)$ and $n_{P,Mb}(L_y,d,\eta)$ were determined and a set of transformations were given that relate the Potts model partition functions for the cyclic and Möbius strips.

### 2.3 Self-Dual Cyclic Strips of the Square Lattice

Consider a strip of the square lattice with (i) a fixed transverse width $L_y$, (ii) an arbitrarily great length $L_x$, (iii) periodic longitudinal boundary conditions, such that (iv) each vertex on one side of the strip, which we take to be the upper side (with the strip oriented so that the longitudinal, $x$ direction is horizontal) are joined by edges to a single external vertex. A strip graph of this type will be denoted generically as $G_D$ (where the subscript $D$ refers to the
self-duality) and, when its size is indicated, as $G_D(L_y \times L_x)$. Although this family of graphs differs from the simple recursive cyclic or Möbius lattice strips, owing to the feature that all of the vertices on the upper side are connected to a single external vertex, we showed \[50, 51\] that the Potts model partition function and its special case, the chromatic polynomial, for the $G_D(L_y \times L_x)$ family have a structure analogous to that for the cyclic and Möbius strips of the square lattice and cyclic strip of the triangular lattice, in the sense that there is a unique coefficient which is a polynomial of degree $d$ in $q$, denoted $\kappa^{(d)}$. There are $n_{Z, G_D}(L_y, d)$ terms $\lambda_{Z, G_D, L_y, d, j}$ in $Z(G_D(L_y \times L_x), q, v)$ having coefficient $\kappa^{(d)}$. The Potts model partition function, Tutte polynomial, and chromatic polynomial were shown to have the following forms \[50, 51\]

$$Z(G_D[L_y \times L_x], q, v) = \sum_{d=1}^{L_y+1} \left[ \kappa^{(d)} \sum_{j=1}^{n_{Z, G_D}(L_y, d)} (\lambda_{Z, G_D, L_y, d, j})^m \right]$$

(2.3.1)

or equivalently, for the Tutte polynomial,

$$T(G_D[L_y \times m], x, y) = \sum_{d=1}^{L_y+1} \left[ \bar{\kappa}^{(d)} \sum_{j=1}^{n_{T, G_D}(L_y, d)} (\lambda_{T, G_D, L_y, d, j})^m \right]$$

(2.3.2)

where

$$n_{T, G_D}(L_y, d) = n_{Z, G_D}(L_y, d)$$

(2.3.3)

and

$$\kappa^{(d)} = q \bar{\kappa}^{(d)}$$

$$= \sqrt{q} U_{2d-1} \left( \frac{\sqrt{q}}{2} \right)$$

$$= \sum_{j=0}^{d-1} (-1)^j \binom{2d - 1 - j}{j} q^{d-j}$$

$$= \kappa^{(1)} + \kappa^{(d-1)}$$

(2.3.4)

where $U_n(z)$ is the Chebyshev polynomial of the second kind. Thus, $\kappa^{(1)} = q$, $\kappa^{(2)} = q(q-2)$, $\kappa^{(3)} = q(q-1)(q-3)$, and so forth for higher values of $d$. From (1.10) it follows that

$$\lambda_{Z, G_D, L_y, d, j}(q, v) = v^{L_y} \lambda_{T, G_D, L_y, d, j}(x, y), \quad 1 \leq d \leq L_y + 1$$

(2.3.5)

with the relations (1.7) and (1.8). Clearly, for the total numbers, $N_{Z, G_D, L_y, \lambda} = N_{T, G_D, L_y, \lambda}$. 

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For the chromatic polynomial,

\[ P(G_D[L_y \times L_x], q) = \sum_{d=1}^{L_y+1} \left[ \kappa^{(d)} \sum_{j=1}^{n_{P,G_D}(L_y,d)} (\lambda_{P,G_D,L_y,d,j})^m \right]. \tag{2.3.6} \]

The analogues to (2.2.10) and (2.2.11) for this self-dual strip of the square lattice are

\[ \sum_{d=1}^{L_y+1} \kappa^{(d)} n_{Z,G_D}(L_y, d) = q^{L_y+1} \] \tag{2.3.7}

\[ \sum_{d=1}^{L_y+1} \kappa^{(d)} n_{P,G_D}(L_y, d) = P(T_{L_y+1}, q) = q(q - 1)^{L_y}. \tag{2.3.8} \]

A number of properties of the \( \kappa^{(d)} \) coefficients were derived in [50]; for our present work, we use the following special values

\[ \kappa^{(d)} = 0 \quad \text{for} \quad q = 0 \tag{2.3.9} \]

If \( q = 1 \) then \( \kappa^{(d)} = \begin{cases} 0 & \text{if } d = 0 \mod 3 \\ 1 & \text{if } d = 1 \mod 3 \\ -1 & \text{if } d = 2 \mod 3 \end{cases} \tag{2.3.10} \]

If \( q = 2 \) then \( \kappa^{(d)} = \begin{cases} 0 & \text{if } d \text{ is even} \\ 2(-1)^k & \text{if } d \text{ is odd and } d = 2k + 1 \end{cases} \tag{2.3.11} \)

The numbers \( n_{Z,G_D}(L_y, d) \) and \( n_{P,G_D}(L_y, d) \) were determined in [50] and related to \( n_Z(L_y, d) \) and \( n_P(L_y, d) \) for cyclic strips of the square and triangular lattices. For the total number of terms in the Potts model partition function,

\[ N_{Z,G_D,L_y,\lambda} = \left( \frac{2L_y + 1}{L_y + 1} \right). \tag{2.3.12} \]

The analogous total number of terms in the chromatic polynomial is given by

\[ \frac{1}{2} \left[ \left( \frac{1 + x}{1 - 3x} \right)^{1/2} - 1 \right] - x = \sum_{L_y=1}^{\infty} N_{P,G_D,L_y,\lambda} x^{L_y+1} \tag{2.3.13} \]

and satisfies

\[ N_{P,G_D,L_y,\lambda} = \frac{1}{2} N_{P,L_y+1,\lambda} \tag{2.3.14} \]

where the latter refers to cyclic or Möbius strips of the square or triangular lattice.
3 Structural Sum Rules

3.1 Relations Between Sums of $n_Z(L_y,d)$

In this section we derive new structural relations for the Potts model partition function and chromatic polynomial. These involve sums of terms and hence, following a traditional nomenclature in physics, we shall refer to them as sum rules.

**Proposition 1** For the strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$ with $(FBC_y, PBC_x)$ (i.e., cyclic) boundary conditions,

$$
\sum_{0 \leq d \leq L_y} n_Z(L_y,d) = \sum_{0 \leq d \leq L_y} n_Z(L_y,d) = \frac{1}{2} \left( \frac{2L_y}{L_y} \right) .
$$

(3.1.1)

**Proof** Evaluating (2.2.10) for $q = 0$ and using (2.2.2) yields the first equality. From this equality and the fact that the sum of the number of eigenvalues $\lambda_{Z,L_y,d,j}$ with coefficients of even and odd $d$ is equal to (2.2.23), the last equality in (3.1.1) follows. □

Clearly, eq. (3.1.1) also applies with $n_Z(L_y,d)$ replaced by $n_T(L_y,d)$. Note that the right-hand side of the last equality in (3.1.1) is equal to the number of directed animals of length $n = L_y$ on the triangular lattice [46]. The first few values of this quantity for $1 \leq L_y \leq 5$ are 1, 3, 10, 35, and 126, respectively.

It is of interest to observe a related structural feature for cyclic strips of the square or triangular lattice. We start by noting that for any graph $G$, $Z(G,q,v) = 0$ if $q = 0$, as is clear from (1.5). We apply this to the case where $G$ is the cyclic strip of the square or triangular lattice, denoted for short simply as $[L_y \times L_x, cyc.]$ with $L_x = m$, and use (2.2.2) to obtain

$$
Z([L_y \times m, cyc.], q = 0, v) = 0
$$

$$
= \sum_{0 \leq d \leq L_y} \sum_{d \text{ even}} (\lambda_{Z,L_y,d,j}(0,v))^m - \sum_{0 \leq d \leq L_y} \sum_{d \text{ odd}} (\lambda_{Z,L_y,d,j}(0,v))^m .
$$

(3.1.2)

We observe from our explicit calculations that the terms $\lambda_{Z,L_y,d,j}(q,v)$ do not, in general, vanish at $q = 0$. Since the cancellation between the various $m$'th powers of the eigenvalues must occur for arbitrary $m$, they must, in fact, cancel in pairs. Note that this provides a second proof of the result (3.1.1), since the cancellation requires that the number of eigenvalues $\lambda_{Z,L_y,d,j}$ with even $d$ must be equal to the number of eigenvalues $\lambda_{Z,L_y,d,j}$ with odd $d$.

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Next, we have

**Prop. 2** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq L_y} n_Z(L_y, d) - \sum_{d=0 \mod 3}^{d=2 \mod 3} n_Z(L_y, d) = 1 .
$$

**(Proof)** To prove this, we evaluate (2.2.10) for $q = 1$ and use (2.2.3). \(\square\)

**Prop. 3** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq L_y} n_Z(L_y, d) - \sum_{d=0, 1 \mod 4}^{d=2, 3 \mod 4} n_Z(L_y, d) = 2^{L_y} .
$$

**(Proof)** To prove this, we evaluate (2.2.10) for $q = 2$ and use (2.2.4). \(\square\)

**Prop. 4** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq L_y} n_Z(L_y, d) = \frac{1}{2} \binom{2L_y}{L_y} + 2^{L_y-1}.
$$

$$
\sum_{d=0, 1 \mod 4}^{d=2, 3 \mod 4} n_Z(L_y, d) = \frac{1}{2} \binom{2L_y}{L_y} - 2^{L_y-1}.
$$

**(Proof)** This result follows from (3.1.4) and the fact that the sum $\sum_{0 \leq d \leq L_y} n_Z(L_y, d) = N_{Z,L_y,\lambda}$, for which one has the expression given in eq. (2.2.23). \(\square\)

### 3.2 Relations Between Sums of $n_P(L_y, d)$

**Prop. 5** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq L_y} n_P(L_y, d) = \sum_{0 \leq d \leq L_y} n_P(L_y, d)
$$

$$
= (L_y - 1)! \sum_{j=0}^{[\frac{L_y}{2}]} \frac{(L_y - j)}{(j!)^2(L_y - 2j)!} .
$$

(3.2.1)
Proof Evaluating (2.2.11) for \( q = 0 \) and using (2.2.2) yields the first equality. From this equality and the fact that the sum of the number of eigenvalues \( \lambda_{P,L,y,d,j} \) with coefficients of even and odd \( d \) is equal to (2.2.24), the last equality in (3.2.1) follows. \( \square \)

Note that the right-hand side of (3.2.1) is equal to the number of directed animals of length \( L_y \) on the square lattice \([46]\). The first few values of this quantity for \( 1 \leq L_y \leq 5 \) are 1, 2, 5, 13, and 35, respectively.

Again, one can remark on a related structural feature for cyclic strips of the square or triangular lattice. Clearly, \( P(G,q) = 0 \) if \( q = 0 \). We apply this to the case where \( G \) is the cyclic strip of the square or triangular lattice, denoted for short simply as \([L_y \times L_x, cyc.]\) with \( L_x = m \), and use (2.2.2) to obtain

\[
P([L_y \times m, cyc.], q = 0) = 0
= \sum_{0 \leq d \leq L_y \atop d \text{ even}} n_P(L_y,d) \sum_{j=1}^{\lambda_{P,L_y,d,j}(q = 0)} m - \sum_{0 \leq d \leq L_y \atop d \text{ odd}} n_P(L_y,d) \sum_{j=1}^{\lambda_{P,L_y,d,j}(q = 0)} m. \tag{3.2.2}
\]

We observe from our calculations that the \( \lambda_{P,L_y,d,j} \)'s are not, in general, zero for \( q = 0 \). Since the cancellation between the various \( m \)'th powers of the eigenvalues must occur for arbitrary \( m \), they must, in fact, cancel in pairs. Hence, the number of eigenvalues \( \lambda_{P,L_y,d,j} \) with even \( d \) must be equal to the number of eigenvalues \( \lambda_{P,L_y,d,j} \) with odd \( d \). This yields another way of seeing the result (3.2.1).

Next,

Prop. 6 For the cyclic strip of the square or triangular lattice of width \( L_y \geq 2 \) and arbitrary length \( L_x \)

\[
\sum_{0 \leq d \leq L_y \atop d = 0 \text{ mod } 3} n_P(L_y,d) = \sum_{0 \leq d \leq L_y \atop d = 2 \text{ mod } 3} n_P(L_y,d)
= \frac{1}{2} \left[ N_{P,L_y,d} - N_{P,L_y-1,d} \right]. \tag{3.2.3}
\]

where the expression for \( N_{P,L_y,d} \) was given in eq. (2.2.24).

Proof To prove the first equality, we evaluate (2.2.11) for \( q = 1 \) and use (2.2.3). For the second equality, we observe that from the results of [46],

\[
\sum_{0 \leq d \leq L_y \atop d = 1 \text{ mod } 3} n_P(L_y,d) = N_{P,L_y-1,\lambda}. \tag{3.2.4}
\]
where the expression for $N_{P,L_y,\lambda}$ was given in (2.2.24). Together with the first equality in (3.2.3) and the formula from [46] for the total sum, (2.2.24), this yields the second equality.

For $2 \leq L_y \leq 6$ the values of the right-hand side of the second equality in (3.2.3) are 1, 3, 8, 22, 61. The case $L_y = 1$ is special since for this case $C_{P,L_y=1} = 1$ rather than 0 at $q = 1$, so that the analogue to eq. (3.2.3) reads simply $n_p(1,0) = 1$.

**Prop. 7** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

\[
\sum_{0 \leq d \leq L_y \atop d = 0,1 \mod 4} n_p(L_y, d) - \sum_{0 \leq d \leq L_y \atop d = 2,3 \mod 4} n_p(L_y, d) = 2 .
\]  

**Proof** To prove this, we evaluate (2.2.11) for $q = 2$ and use (2.2.4).

**Prop. 8** For the cyclic strip of the square or triangular lattice of width $L_y$ and arbitrary length $L_x$

\[
\sum_{0 \leq d \leq L_y \atop d = 0,1 \mod 4} n_p(L_y, d) = \frac{1}{2} N_{P,L_y,\lambda} + 1
\]

\[
\sum_{0 \leq d \leq L_y \atop d = 2,3 \mod 4} n_p(L_y, d) = \frac{1}{2} N_{P,L_y,\lambda} - 1 .
\]  

**Proof** This result follows from (3.2.5) and the fact that the sum $\sum_{0 \leq d \leq L_y} n_p(L_y, d) = N_{P,L_y,\lambda}$, for which one has the expression given in eq. (2.2.24).

For an arbitrary connected graph with at least two vertices, $P(G, q) = 0$ for $q = 1$. It follows, in particular, that

\[
P([L_y \times m, cyc.], q = 1) = 0
\]

\[
= \sum_{0 \leq d \leq L_y \atop d = 0 \mod 3} n_p(L_y, d) \sum_{j=1} \lambda_{P,L_y,d,j}(q = 1)^m - \sum_{0 \leq d \leq L_y \atop d = 2 \mod 3} n_p(L_y, d) \sum_{j=1} \lambda_{P,L_y,d,j}(q = 1)^m .
\]  

However, in contrast to the case for $q = 0$, the cancellation here is not pairwise because some of the $\lambda$’s vanish. Specifically, as will be shown below, the term $\lambda_{P,L_y,d,j}$ for $d = L_y - 1$ and $j = 1$ is $q - 1$, which vanishes at $q = 1$.  

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4 Sum Rules Applicable to Möbius Strips of the Square Lattice

4.1 Relations Between Sums of $n_{Z,Mb}(L_y, d)$

Prop. 9 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq d_{max,Mb}} \left[ n_{Z,Mb}(L_y, d, +) - n_{Z,Mb}(L_y, d, -) \right]_{d \text{ even}} = \sum_{0 \leq d \leq d_{max,Mb}} \left[ n_{Z,Mb}(L_y, d, +) - n_{Z,Mb}(L_y, d, -) \right]_{d \text{ odd}}.
$$

(4.1.1)

**Proof** To prove this, we evaluate (2.2.32) for $q = 0$ and use (2.2.2). □

Applying the general fact that for any graph $G$, $Z(G, q, v) = 0$ for $q = 0$, to the Möbius strip of the square lattice and using (2.2.2), we obtain

$$
Z(sq[Mb, L_y \times m], q = 0, v) = 0 = \sum_{0 \leq d \leq d_{max,Mb}} \left( \lambda_{Z,L,d,\eta,j}(q = 0) \right)^m - \sum_{0 \leq d \leq d_{max,Mb}} \left( \lambda_{Z,L,d,\eta,j}(q = 0) \right)^m
$$

where

$$
C = \eta(-1)^d.
$$

(4.1.2)

(4.1.3)

Since the cancellation between the various $m$’th powers of the eigenvalues must occur for arbitrary $m$, they must, in fact, cancel in pairs. This provides another way of understanding the result in (4.1.1).

Prop. 10 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq d_{max,Mb}} \left[ n_{Z,Mb}(L_y, d, +) - n_{Z,Mb}(L_y, d, -) \right]_{d \equiv 0 \text{ mod } 3} + \sum_{0 \leq d \leq d_{max,Mb}} \left[ -n_{Z,Mb}(L_y, d, +) + n_{Z,Mb}(L_y, d, -) \right]_{d \equiv 2 \text{ mod } 3} = 1.
$$

(4.1.4)
Proof To prove this, we evaluate (2.2.32) for $q = 1$ and use (2.2.3). □

Prop. 11 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop d = 0, 1 \mod 4} \left[ n_{Z,\text{Mb}}(L_y, d, +) - n_{Z,\text{Mb}}(L_y, d, -) \right] \\
+ \sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop d = 2, 3 \mod 4} \left[ -n_{Z,\text{Mb}}(L_y, d, +) + n_{Z,\text{Mb}}(L_y, d, -) \right] \\
= 2^\left\lfloor \frac{L_y + 1}{2} \right\rfloor .
$$

(4.1.5)

Proof To prove this, we evaluate (2.2.32) for $q = 2$ and use (2.2.4). □

4.2 Relations Between Sums of $n_{P,\text{Mb}}(L_y, d)$

Prop. 12 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$

$$
\sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop d \text{ even}} \left[ n_{P,\text{Mb}}(L_y, d, +) - n_{P,\text{Mb}}(L_y, d, -) \right] \\
= \sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop d \text{ odd}} \left[ n_{P,\text{Mb}}(L_y, d, +) - n_{P,\text{Mb}}(L_y, d, -) \right].
$$

(4.2.1)

Proof To prove this, we evaluate (2.2.33) for $q = 0$ and use (2.2.2). □

Applying the general fact that for any graph $G$, $P(G, q) = 0$ for $q = 0$, to the Möbius strip of the square lattice and using (2.2.2), we obtain the following relation, where the $\lambda$'s are evaluated at $q = 0$:

$$
P(sq[L_y \times m, Mb], q = 0) = 0
= \sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop \eta = \pm 1; \ C = +1} (\lambda_{P,L,d,\eta,j}(q = 0))^m \\
- \sum_{0 \leq d \leq d_{\text{max},\text{Mb}} \atop \eta = \pm 1; \ C = -1} (\lambda_{P,L,d,\eta,j}(q = 0))^m.
$$

(4.2.2)

where $C$ was defined in (4.1.3). Since the cancellation between the various $m$'th powers of the eigenvalues must occur for arbitrary $m$, they must, in fact, cancel in pairs. This is another way of understanding the result (4.2.1).
Prop. 13 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$,

$$\sum_{0 \leq d \leq d_{\text{max},Mb}} [n_{P,Mb}(L_y, d, +) - n_{P,Mb}(L_y, d, -)]$$

$$= \sum_{0 \leq d \leq d_{\text{max},Mb}} [n_{P,Mb}(L_y, d, +) - n_{P,Mb}(L_y, d, -)].$$

(4.2.3)

Proof To prove this, we evaluate (2.2.33) for $q = 1$ and use (2.2.3). □

Prop. 14 For the Möbius strip of the square lattice of width $L_y$ and arbitrary length $L_x$,

$$\sum_{0 \leq d \leq d_{\text{max},Mb}} [n_{P,Mb}(L_y, d, +) - n_{P,Mb}(L_y, d, -)]$$

$$+ \sum_{0 \leq d \leq d_{\text{max},Mb}} [-n_{P,Mb}(L_y, d, +) + n_{P,Mb}(L_y, d, -)]$$

$$= 2.$$

(4.2.4)

Proof To prove this, we evaluate (2.2.33) for $q = 2$ and use (2.2.4). □

5 Sum Rules for Self-Dual Strips of the Square Lattice

5.1 Relations Between Sums of $n_{Z,G_D}(L_y, d)$

We first note that for the self-dual strips of the square lattice $G_D(L_y \times L_x)$, there is no analogue of the sum rules (3.1.1) and (3.2.1) because all of the coefficients $\kappa^{(d)}$ have $q$ as a factor and hence vanish at $q = 0$. We do, however, find several sum rules.

Prop. 15 For the self-dual strip of the square lattice $G_D(L_y \times L_x)$

$$\sum_{1 \leq d \leq L_y + 1 \atop d = 1 \text{ mod } 3} n_{Z,G_D}(L_y, d) - \sum_{1 \leq d \leq L_y + 1 \atop d = 2 \text{ mod } 3} n_{Z,G_D}(L_y, d) = 1.$$

(5.1.1)

Proof To prove this, we evaluate (2.3.7) at $q = 1$ and use (2.3.10). □
Prop. 16  For the self-dual strip of the square lattice, $G_D(L_y \times L_x)$,

$$
\sum_{\substack{1 \leq d \leq L_y+1 \\ d=2k+1}} (-1)^k n_{Z,G_D}(L_y, d) = 2^{L_y} .
$$
\hfill (5.1.2)

Proof  To prove this, we evaluate \(2.3.4\) at \(q = 2\) and use \(2.3.11\). \(\square\)

5.2 Relations Between Sums of \(n_{P,G_D}(L_y, d)\)

Prop. 17  For the self-dual strip of the square lattice $G_D(L_y \times L_x)$,

$$
\sum_{\substack{1 \leq d \leq L_y+1 \\ d=1 \mod 3}} n_{P,G_D}(L_y, d) = \sum_{\substack{1 \leq d \leq L_y+1 \\ d=2 \mod 3}} n_{P,G_D}(L_y, d) .
$$
\hfill (5.2.1)

Proof  To prove this, we evaluate \(2.3.8\) at \(q = 1\) and use \(2.3.10\). \(\square\)

Prop. 18  For the self-dual strip of the square lattice $G_D(L_y \times L_x)$,

$$
\sum_{\substack{1 \leq d \leq L_y+1 \\ d=2k+1}} (-1)^k n_{P,G_D}(L_y, d) = 1 .
$$
\hfill (5.2.2)

Proof  To prove this, we evaluate \(2.3.8\) at \(q = 2\) and use \(2.3.11\). \(\square\)

As before for other strip graphs, one may analyze how cancellations occur in the chromatic polynomial at values of \(q\) where it vanishes. Thus, let us consider the value \(q = 1\) and use the fact that for any connected graph \(G\) with at least two vertices, \(P(G, q) = 0\) if \(q = 1\). Applying this to \(G = G_D(L_y \times L_x)\) and using the property \(2.3.10\), we obtain the following relation, where the \(\lambda\)'s are evaluated at \(q = 1\):

$$
P(G_D(L_y \times m, q = 1) = 0
\begin{align*}
= & \sum_{1 \leq d \leq L_y+1} \sum_{d=1 \mod 3} n_{P,G_D}(L_y, d) (\lambda_{P,G_D,L_y,d,j})^m - \sum_{1 \leq d \leq L_y+1} \sum_{d=2 \mod 3} n_{P,G_D}(L_y, d) (\lambda_{P,G_D,L_y,d,j})^m . \hfill (5.2.3)
\end{align*}
$$

For the cases where we have carried out explicit calculations, we find that the \(\lambda\)'s do not vanish at \(q = 1\) for this family of graphs. Since the cancellation in \(5.2.3\) between the various \(m\)'th powers of the eigenvalues must occur for arbitrary \(m\), they must, in fact, cancel in pairs.
Since the family $G_D(L_y \times L_x)$ contains triangles, $P(G_D[L_y \times L_x], q) = 0$ for $q = 2$. One may investigate what is implied by this property. Using (2.3.11), one finds

$$P(G_D[L_y \times m], q = 2) = 0$$

$$= 2 \sum_{1 \leq d \leq L_y} (-1)^k \sum_{j=1}^{n_P(G_D, L_y, d)} (\lambda_{P,G_D, L_y, d, j}(q = 2))^m .$$

(5.2.4)

However, this cancellation does not, in general, occur in a pairwise manner, because some of the $\lambda_{P,G_D, L_y, 2k+1,j}$ vanish. For example, for $L_y = 2$, $n_P(G_D, 1) = 2$ while $n_P(G_D, 3) = 1$, but the $\lambda_{P,G_D, 2,1,j}$, given by eq. (6.3) in [50],

$$\lambda_{P,G_D, 2,1,j} = \frac{1}{2} \left[ q^2 - 5q + 7 \pm (q^4 - 6q^3 + 15q^2 - 22q + 17)^{1/2} \right] j = 1, 2$$

(5.2.5)

vanishes at $q = 2$.

### 6 Sum Rules for Other Recursive Families of Graphs

Proceeding in a manner similar to that above, one can obtain analogous sum rules for other recursive families of graphs. We comment briefly on one such family. In [59] we presented theorems determining the structure of the Potts model partition function for a cyclic clan graph (CG). We recall some relevant definitions. A complete graph $K_r$ is a graph containing $r$ vertices with the property that each vertex is connected by edges to every other vertex. The join of two graphs $H_1$ and $H_2$, denoted $H_1 + H_2$, is the graph formed by connecting each vertex of $H_1$ to all of the vertices of $H_2$ with edges. Then a (homogeneous) cyclic clan graph, denoted $G[(K_r)_m, jn]$ is a recursive graph composed of a set of $m$ complete graphs $K_r$, such that the linkage between two adjacent pairs of $K_r$’s is a join (abbreviated $jn$). The Potts model partition function has the form [59]

$$Z(G[(K_r)_m, jn], q, v) = \sum_{d=0}^{r} \mu_d \sum_{j=1}^{n_{Z,CG}(r,d)} (\lambda_{Z,CG,r,d,j})^m$$

(6.1)

where $\mu_0 = 1$ and

$$\mu_d = \binom{q}{d} - \binom{q}{d-1} = \frac{q(q-1)(q-2d+1)}{d!} \text{ for } 1 \leq d \leq r$$

(6.2)

where

$$q(r) = \prod_{s=0}^{r-1} (q - s)$$

(6.3)
is the falling factorial in combinatorics. The numbers \( n_{Z,CG}(r,d) \) were determined in [59].

Let us apply the relation
\[ Z(G, q = 0, v) = 0 \]
for this family. All of the \( \mu_d \) coefficients for \( d \geq 2 \) contain \( q \) as a factor and hence vanish identically for \( q = 0 \). Using \( \mu_1 = q - 1 \), the condition \( Z(G, q = 0, v) = 0 \) reduces to
\[ n_{Z,CG}(r,0) \sum_{j=1}^{n_{Z,CG}(r,1)} (\lambda_{Z,CG,r,0,j})^m = \sum_{j=1}^{n_{Z,CG}(r,1)} (\lambda_{Z,CG,r,1,j})^m . \] (6.4)

Since this must hold for arbitrary \( m \) and since, in general, the relevant \( \lambda \)'s are nonvanishing, it follows that the cancellation must occur in a pairwise manner, and hence that \( n_{Z,CG}(r,0) = n_{Z,CG}(r,1) \). This is in agreement with a result that we already derived in [59], viz., that \( n_{Z,CG}(r,0) = n_{Z,CG}(r,0) = 2^{r-1} \). So here one does not obtain any new sum rule. A similar comment applies for the chromatic polynomial \( P(G[\{K_r\}_m, jn], q) \), where \( n_{P,CG}(r,d) = 1 \) for all \( d \) (with \( 0 \leq d \leq r \) [25].

7 Determination of \( \lambda_{P,L_y,d=L_y-1} \) for Families of Lattice Strip Graphs

7.1 Cyclic and Möbius Strips of the Square Lattice

We have succeeded in determining the terms \( \lambda_{P,L_y,d=L_y-1} \) with coefficient \( c(L_y-1) \) in the chromatic polynomial for the cyclic and Möbius strips of the square lattice. There are \( n_P(L_y, L_y - 1) = L_y \) of these terms, by eq. (2.2.22). Since all of the results in this section and the rest of the paper (including the appendix) refer to the chromatic polynomial, we shall use an abbreviated notation omitting the subscript \( P \) in the terms \( \lambda_{P,L_y,d} \). For our calculation, we use the sieve method of [37] to calculate the \( L_y \times L_y \) transfer matrix for these \( \lambda \)'s with coefficient \( c(L_y-1) \). This transfer matrix is denoted \( T^{(L_y,d)} \). To avoid awkward notation, we leave the type of lattice implicit. In subsequent sections, when the same symbol is used for strips of other lattices, it will be understood that its meaning is specific to those sections. It is convenient to extract a prefactor and define a reduced transfer matrix:
\[ T^{(L_y,L_y-1)} = (-1)^{L_y+1} T^{(L_y,L_y-1)} . \] (7.1.1)

**Theorem 1** For the cyclic or Möbius strip of the square lattice with arbitrary width \( L_y \) and arbitrary length \( L_x = m \), the eigenvalues \( \lambda_{L_y,d=1,j} \) of for coefficient degree \( d = L_y - 1 \) are
\[ \lambda_{L_y,d=L_y-1,j} = (-1)^{L_y+1}(q - a_{sq,L_y,j}) , \quad 1 \leq j \leq L_y \] (7.1.2)
where

\[ a_{sq,L_yj} = 1 + 4 \cos^2 \left( \frac{(L_y + 1 - j)\pi}{2L_y} \right), \quad 1 \leq j \leq L_y. \]  

(7.1.3)

**Proof**

We begin with the following lemma:

**Lemma 1** For the cyclic strip of the square lattice with arbitrary width \( L_y \) and length \( L_x = m \), denoted \( sq[L_y \times L_x, \text{cyc.}] \), the matrix \( \tilde{T}^{(L_y,L_y-1)} \) for \( L_y \geq 2 \) is given by

\[ \tilde{T}^{(L_y,L_y-1)}_{11} = \tilde{T}^{(L_y,L_y-1)}_{L_y,L_y} = q - 2 \]  

(7.1.4)

\[ \tilde{T}^{(L_y,L_y-1)}_{jj} = q - 3 \quad \text{for} \quad 2 \leq j \leq L_y - 1 \]  

(7.1.5)

\[ \tilde{T}^{(L_y,L_y-1)}_{jj+1} = \tilde{T}^{(L_y,L_y-1)}_{j+1,j} = -1 \quad \text{for} \quad 1 \leq j \leq L_y - 1 \]  

(7.1.6)

with other elements equal to zero. For \( L_y = 1 \), \( \tilde{T}^{(1,0)} = \lambda_{1,0} = q - 1 \).

Thus, for example, for \( L_y = 5 \),

\[ \tilde{T}^{(5,4)} = \begin{pmatrix} q - 2 & -1 & 0 & 0 & 0 \\ -1 & q - 3 & -1 & 0 & 0 \\ 0 & -1 & q - 3 & -1 & 0 \\ 0 & 0 & -1 & q - 3 & -1 \\ 0 & 0 & 0 & -1 & q - 2 \end{pmatrix}. \]  

(7.1.7)

We note that \( \tilde{T}^{(L_y,L_y-1)} \) is a special case of a Jacobi matrix, in the terminology of linear algebra (where a Jacobi matrix \( A \) is defined as a matrix with the property that \( A_{ij} = 0 \) if \( |i - j| \geq 2 \) [60].

**Proof** One can construct the cyclic strip of the square lattice via the repetition of a subgraph which is a transverse slice consisting of the path graph with \( L_y \) vertices such that the linkage \( L \) between two successive such path graphs is the identity linkage, so that vertex 1 of the first path graph is connected by an edge to vertex 1 of the next path graph, and so forth for all \( L_y \) vertices. We denote this linkage as \( L = \{(1,1), (2,2), \ldots, (L_y, L_y)\} \). Now applying the sieve method of [37], we consider the functions \( \alpha: V \rightarrow \{1, 2, \ldots, q\} \) that are proper \( q \)-coloring of the path (or tree) graph with \( L_y \) vertices, so that \( V \) is the vertex set \( V = \{1, 2, \ldots, L_y\} \). Let us choose the basis so that

\[ [\alpha](\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}. \]  

(7.1.8)
Define the compatibility matrix as follows,

\[(T)_{\alpha\beta} = \begin{cases} 
1 & \text{if } (\alpha, \beta) \text{ is compatible with } L \\
0 & \text{otherwise}
\end{cases} \quad (7.1.9)\]

Denote \([1|h]\) as the function that takes the value 1 for the coloring that assign color \(h\) to vertex 1, and takes the value 0 otherwise. Similarly let \([1, 2, ..., h_1, h_2, ...]\) refer to the coloring choice such that color \(h_1\) is assigned to vertex 1, color \(h_2\) is assigned to vertex 2 and so forth. The coloring without restriction is denoted \(u\). We have

\[T[\alpha] = \sum_{X \subseteq V} (-1)^{|X|}[X|\alpha] \quad (7.1.10)\]

where \(X\) refers to all subsets of the \(L_y\) vertices of the path graph, and \(|X|\) is the number of vertices in \(X\). For example, for \(L_y = 3\), we have

\[T[\alpha] = u - [1|\alpha_1] - [2|\alpha_2] - [3|\alpha_3] + [1, 2|\alpha_1, \alpha_2] + [1, 3|\alpha_1, \alpha_3] + [2, 3|\alpha_2, \alpha_3] - [1, 2, 3|\alpha_1, \alpha_2, \alpha_3]. \quad (7.1.11)\]

Concentrating now on the case \(d = L_y - 1\), we observe that the invariant subspace of the matrix \(T\) is spanned by

\[[1, 2, ..., j - 1, j + 1, ..., L_y|h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}], \quad 1 \leq j \leq L_y. \quad (7.1.12)\]

Therefore,

\[T[1, 2, ..., j - 1, j + 1, ..., L_y|h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}] = T \left( \sum_H [\alpha] \right) = \sum_H T[\alpha] \quad (7.1.13)\]

where \(H\) means \(\alpha_1 = h_1, \alpha_2 = h_2, ..., \alpha_{j-1} = h_{j-1}, \alpha_j = h_{j+1}, ..., \alpha_{L_y} = h_{L_y}\), and \(h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}\) are different from each other.

To obtain the matrix for level \(L_y - 1\), we only have to consider the terms in the summation of eq. (7.1.10) with \(|X| = L_y - 1\) and \(|X| = L_y\). Now the first row of \(T^{(L_y,L_y-1)}\) is obtained by applying \(T\) on \([1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}]\). The nonzero elements are

\[(-1)^{L_y-1}((q - 1)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + ([1, 2, ..., L_y - 2, L_y|h_1, h_2, ..., h_{L_y-2}, h_{L_y}]))
\]

\[+(-1)^{L_y}([1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}])\]

\[= (-1)^{L_y-1}((q - 2)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}])
\]

\[-[1, 2, ..., L_y - 2, L_y|h_1, h_2, ..., h_{L_y-2}, h_{L_y}]) \quad (7.1.14)\]
and we have the corresponding result for the last row. Applying $T$ on $[1, 2, ..., j - 1, j + 1, ..., L_y]h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}$ for $2 \leq j \leq L_y - 1$, we find that the nonzero elements are

\[
(-1)^{L_y-1}[-[1, 2, ..., j, j + 2, ..., L_y]h_1, h_2, ..., h_j, h_{j+2}, ..., h_{L_y}] \\
+(q - 2)[1, 2, ..., j - 1, j + 1, ..., L_y]h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}] \\
+(-[1, 2, ..., j - 2, j, ..., L_y]h_1, h_2, ..., h_{j-2}, h_j, ..., h_{L_y}) \\
+(-1)^{L_y}[1, 2, ..., j - 1, j + 1, ..., L_y]h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}] \\
= (-1)^{L_y-1}[-[1, 2, ..., j, j + 2, ..., L_y]h_1, h_2, ..., h_j, h_{j+2}, ..., h_{L_y}] \\
+(q - 3)[1, 2, ..., j - 1, j + 1, ..., L_y]h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}] \\
-\left[-1, 2, ..., j - 2, j, ..., L_y|h_1, h_2, ..., h_{j-2}, h_j, ..., h_{L_y}\right] \\
\] (7.1.15)

and, taking into account the relation (7.1.1), the lemma follows for the case of the cyclic strip. It was proved in [20] that the $\lambda$'s for the cyclic and Möbius strips of the square lattice are identical to each other, and similarly for the triangular lattice. Finally, for the $L_y = 1$ case, $T^{(1,0)} = \lambda_{1,0} = q - 1$ is obtained by an explicit elementary calculation. This completes the proof of the lemma. \quad \square

Next, we have

**Lemma 2** The eigenvalues of the (reduced) transfer matrix $\bar{T}^{(L_y,L_y-1)}$ are $q - a_{L_y,j}$ for $1 \leq j \leq L_y$.

**Proof** For the case $L_y = 1$, an elementary explicit calculation yields the result. For $L_y \geq 2$, consider the $L_y \times L_y$ matrix $A(L_y)$ given by

\[
A(L_y)_{j \ j+1} = A(L_y)_{j+1 \ j} = -1 \quad \text{for} \quad 1 \leq j \leq L_y - 1 \quad (7.1.16)
\]

with all other elements equal to zero, and $A(L_y) = 0$ for $L_y = 1$. Denote the characteristic polynomial of $A(L_y)$ in terms of variable $\omega$ as $C[A(L_y)] = \det(\omega I - A(L_y))$, where $I$ is the $L_y \times L_y$ identity matrix. This polynomial satisfies the recursion relation (for $L_y \geq 2$)

\[
C[A(L_y)] = \omega C[A(L_y - 1)] - C[A(L_y - 2)] \quad (7.1.17)
\]

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where, for $L_y = 2$ we formally set $C[A(0)] \equiv 1$. As in our earlier related work \cite{16, 50}, we observe that this is the same as the recursion relation satisfied by the Chebyshev polynomial of the second kind, $U_n(x)$, for $x = \omega/2$, i.e.,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) . \quad (7.1.18)$$

Solving the recursion relation (7.1.17), we obtain

$$C[A(L_y)] = U_{L_y}(\omega/2) = \sum_{j=0}^{[L_y]} (-1)^j \binom{L_y-j}{j} (\omega)^{L_y-2j} \quad (7.1.19)$$

where we again use the notation $[\nu]$ to denote the integral part of $\nu$. Using the relation

$$U_n(\cos \phi) = \sin((n+1)\phi) \sin \phi \quad (7.1.20)$$

we express this equivalently as

$$C[A(L_y)] = \frac{\sin((L_y+1)\arccos(\omega/2))}{\sin[\arccos(\omega/2)]} \quad (7.1.21)$$

It follows that the roots of the characteristic polynomial $C[A(L_y)]$, i.e., the eigenvalues of the matrix $A(L_y)$, are

$$\omega_j = 2 \cos\left(\frac{j\pi}{L_y+1}\right) \quad \text{for } 1 \leq j \leq L_y . \quad (7.1.22)$$

Now the reduced transfer matrix $\tilde{T}^{(L_y, L_y-1)}$ can be written for $L_y \geq 2$ as

$$\tilde{T}^{(L_y, L_y-1)} = (q-3)I + \tilde{T}^{(L_y, L_y-1)} \quad (7.1.23)$$

and the matrix $\tilde{T}^{(L_y, L_y-1)}$ is given by

$$\tilde{T}^{(L_y, L_y-1)}_{11} = \tilde{T}^{(L_y, L_y-1)}_{L_y, L_y} = 1 \quad (7.1.24)$$

$$\tilde{T}^{(L_y, L_y-1)}_{j, j+1} = \tilde{T}^{(L_y, L_y-1)}_{j+1, j} = -1 \quad \text{for } 1 \leq j \leq L_y - 1 \quad (7.1.25)$$

with all other elements 0. Let us denote the characteristic polynomial of $\tilde{T}^{(L_y, L_y-1)}$ in terms of the variable $\omega$ as $C[\tilde{T}^{(L_y, L_y-1)}]$. We find that this characteristic polynomial satisfies the following recursion relation

$$C[\tilde{T}^{(L_y, L_y-1)}] = (\omega - 2)C[A(L_y - 1)] . \quad (7.1.26)$$

Using eq. (7.1.22), we find that the roots of $C[\tilde{T}^{(L_y, L_y-1)}]$ are given by

$$\omega_j = 2 \cos\left(\frac{j\pi}{L_y}\right) \quad \text{for } 0 \leq j \leq L_y - 1 . \quad (7.1.27)$$
Combining eqs. (7.1.23) and (7.1.27) (and relabelling the roots so that the $a_{L_y,j}$ increase monotonically as a function of $j$) then yields the result in the lemma. □

Finally, we use the result from [17, 20] that the $\lambda_{P,G,j}$’s for the cyclic and Möbius strips graphs $G$ of a given lattice and width $L_y$ are the same. The theorem then follows. □

We remark that eqs. (5.2) and (5.3) in [41] are in accord with this structure of the transfer matrix.

Several corollaries of Theorem 1 and the associated lemmas are of interest. Each of these applies to the cyclic or Möbius strips of the square lattice; we leave this implicit in the notation. Since the proofs are straightforward, we omit them.

**Corollary 1**  
The trace and determinant of $\bar{T}^{(L_y,L_y-1)}$ are

\[
\text{Tr}(\bar{T}^{(L_y,L_y-1)}) = 2 + L_y(q - 3) \tag{7.1.28}
\]

and

\[
\det(\bar{T}^{(L_y,L_y-1)}) = \prod_{j=1}^{L_y}(q - a_{sq,L_y,j}) \tag{7.1.29}
\]

where the $a_{sq,L_y,j}$ were given in (7.1.3).

**Corollary 2**  
For arbitrary $L_y$, $q - 1$ is an eigenvalue of $\bar{T}^{(L_y,L_y-1)}$ with multiplicity 1, given by

\[
\lambda_{L_y,j=1} = q - 1 . \tag{7.1.30}
\]

**Corollary 3**  
If $L_y = 0 \mod 2$, i.e., $L_y \geq 2$ is even, then $q - 3$ is an eigenvalue of $\bar{T}^{(L_y,L_y-1)}$ with multiplicity 1 given by

\[
\lambda_{L_y,j=1+(L_y/2)} = q - 3 . \tag{7.1.31}
\]

**Corollary 4**  
If $L_y = 0 \mod 3$, then $q - 2$ and $q - 4$ are eigenvalues of $\bar{T}^{(L_y,L_y-1)}$, each with multiplicity 1, given by

\[
\lambda_{L_y,j=1+(L_y/3)} = q - 2 \tag{7.1.32}
\]

\[
\lambda_{L_y,j=1+(2L_y/3)} = q - 4 . \tag{7.1.33}
\]

**Corollary 5**  
The roots $a_{sq,L_y,j}$ monotonically increases from 1, for $j = 1$, to $1 + 4 \cos^2(\pi/(2L_y))$ for $j = L_y$. As $L_y \to \infty$, these roots become dense on the interval $[1, 5]$. 

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7.2 Self-Dual Cyclic Strips of the Square Lattice

**Theorem 2**  For the self-dual cyclic strip of the square lattice with arbitrarily width \(L_y\) and length \(L_x = m\), the eigenvalues \(\lambda_{P,L_y,d,j}\) for coefficient degree \(d = L_y - 1\) are

\[
\lambda_{L_y,d=1,1,j} = (-1)^{L_y+1} \left( q - a_{G,D,L_y,j} \right) , \quad 1 \leq j \leq L_y \tag{7.2.1}
\]

where

\[
a_{G,D,L_y,j} = 1 + 4 \cos^2 \left( \frac{(2j - 1)\pi}{2L_y + 1} \right) , \quad 1 \leq j \leq L_y . \tag{7.2.2}
\]

**Proof**  We begin with the following lemma:

**Lemma 3**  For the self-dual cyclic strip of the square lattice with arbitrarily width \(L_y\) and length \(L_x = m\), the reduced transfer matrix \(\bar{T}^{(L_y,L_y-1)}\) for \(L_y \geq 2\) is given by

\[
\begin{align*}
\bar{T}^{(L_y,L_y-1)}_{11} &= q - 2 \\
\bar{T}^{(L_y,L_y-1)}_{jj} &= q - 3 \quad \text{for} \quad 2 \leq j \leq L_y \\
\bar{T}^{(L_y,L_y-1)}_{j,j+1} &= \bar{T}^{(L_y,L_y-1)}_{j+1,j} = -1 \quad \text{for} \quad 1 \leq j \leq L_y - 1
\end{align*} \tag{7.2.3}
\]

with other elements equal to zero. For \(L_y = 1\), \(\bar{T}^{(1,0)} = \lambda_{1,0} = q - 2\).

**Proof**  This proceeds in a manner similar to the proof given above, with the difference that the coloring is further constrained by the feature that all of the vertices on the upper side of the strip are connected by edges to a single external vertex. This difference has the effect of replacing the \(q - 2\) by \(q - 3\) in the entry \(\bar{T}^{(L_y,L_y-1)}_{L_y,L_y}\). \(\square\)

An example for \(L_y = 5\) is

\[
\bar{T}^{(5,4)} = \begin{pmatrix}
q - 2 & -1 & 0 & 0 & 0 \\
-1 & q - 3 & -1 & 0 & 0 \\
0 & -1 & q - 3 & -1 & 0 \\
0 & 0 & -1 & q - 3 & -1 \\
0 & 0 & 0 & -1 & q - 3
\end{pmatrix} . \tag{7.2.6}
\]

Next,

**Lemma 4**  The eigenvalues of \(\bar{T}^{(L_y,L_y-1)}\) are given by

\[
q - a_{G,D,L_y,j} \quad \text{for} \quad 1 \leq j \leq L_y . \tag{7.2.7}
\]
Proof} For the case $L_y = 1$, an explicit calculation yields the result. For $L_y \geq 2$, consider the $L_y \times L_y$ matrix $A(L_y)$ given by

$$\tilde{T}^{(L_y, L_y^{-1})} = (q - 3)I + \tilde{T}^{(L_y, L_y^{-1})}$$  \hspace{1cm} (7.2.8)

where again $I$ is the $L_y \times L_y$ identity matrix, and the matrix $\tilde{T}^{(L_y, L_y^{-1})}$ is given by

$$\tilde{T}^{(L_y, L_y^{-1})}_{11} = 1$$  \hspace{1cm} (7.2.9)

with all other elements 0. Let us denote the characteristic polynomial of $\tilde{T}^{(L_y, L_y^{-1})}$ in terms of the variable $\omega$ as $C[\tilde{T}^{(L_y, L_y^{-1})}]$. We observe that this characteristic polynomial satisfies the recursion relations for $L_y \geq 2$

$$C[\tilde{T}^{(L_y, L_y^{-1})}] = (\omega - 1)C[A(L_y - 1)] - C[A(L_y - 2)]$$

$$= C[A(L_y)] - C[A(L_y - 1)]$$  \hspace{1cm} (7.2.11)

where the matrix $A(L_y)$ and its characteristic polynomial $C[A(L_y)]$ were given in Lemma 2, and we continue to use the formal definition $C[A(0)] = 1$. Using eq. (7.1.21), we find that $C[\tilde{T}^{(L_y, L_y^{-1})}]$ can be written as

$$C[\tilde{T}^{(L_y, L_y^{-1})}] = \frac{\sin[(L_y + 1)\arccos(\omega/2)]}{\sin[\arccos(\omega/2)]} - \frac{\sin[L_y\arccos(\omega/2)]}{\sin[\arccos(\omega/2)]}$$

$$= \frac{2\cos[(L_y + 1/2)\arccos(\omega/2)]\sin[\arccos(\omega/2)/2]}{\sin[\arccos(\omega/2)]}$$

$$= \frac{\cos[(L_y + 1/2)\arccos(\omega/2)]}{\cos[\arccos(\omega/2)/2]}.$$  \hspace{1cm} (7.2.12)

Therefore, the roots of the characteristic polynomial $C[\tilde{T}^{(L_y, L_y^{-1})}]$ are given by

$$\omega_j = 2 \cos\left(\frac{(2j - 1)\pi}{2L_y + 1}\right) \text{ for } 1 \leq j \leq L_y. $$  \hspace{1cm} (7.2.13)

Combining eqs. (7.2.8) and (7.2.13) then yields the result in the lemma. □

Finally, combining Lemmas 3 and 4 yields the theorem. □

As is the case with $a_{sq,L_y,j}$, the roots $a_{GD,L_y,j}$ become dense on the interval $[1, 5]$ as $L_y \to \infty$. It is also straightforward to derive the following corollary:

**Corollary 6** If $L_y = 1 \mod 3$, then $q - 2$ is a root of the reduced transfer matrix $\tilde{T}^{(L_y, L_y^{-1})}$ for the self-dual cyclic strip of the square lattice of width $L_y$. 

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7.3 Cyclic and Möbius Strips of the Triangular Lattice

We have also determined the corresponding $L_y \times L_y$ transfer matrix for the triangular lattice. As noted before, to avoid awkward notation, we shall use the same symbol for the transfer matrix $T^{(L_y,d)}$, but understand that it is different for this lattice strip than for the previous strips. We have the following theorem for $\bar{T}^{(L_y, L_y^{-1})}$.

**Theorem 3** The general matrix $\bar{T}^{(L_y, L_y^{-1})}$ for the cyclic strip of the triangular lattice with $L_y \geq 2$ is given by

\[
\bar{T}_{11}^{(L_y, L_y^{-1})} = q - 3 \tag{7.3.1}
\]

\[
\bar{T}_{ij}^{(L_y, L_y^{-1})} = q - 4 \text{ for } 2 \leq i \leq L_y - 1, \ 1 \leq j \leq i \tag{7.3.2}
\]

\[
\bar{T}_{L_y j}^{(L_y, L_y^{-1})} = q - 2 \text{ for } 1 \leq j \leq L_y \tag{7.3.3}
\]

\[
\bar{T}_{j, j+1}^{(L_y, L_y^{-1})} = -1 \text{ for } 1 \leq j \leq L_y - 1 . \tag{7.3.4}
\]

**Proof** We use the sieve formula again. Consider the strip of the triangular lattice of width $L_y$ and arbitrary length $L_x$ to be constructed by starting with the corresponding cyclic strip of the square lattice and adding edges connecting the upper right and lower left vertices of the squares on the strip, so that the edge set is $L = \{(1,1), (2,1), (2,2), (3,2), \ldots (L_y, L_y)\}$. In this case, there is more than one edge connecting a vertex on one path graph forming a transverse slice to the adjacent path graph, so eq. (7.1.10) has to be written in a more general form,

\[
T[\alpha] = \sum_{X \subseteq V} (-1)^{|X|} \sum_{\ell} |X| [X|\alpha_{\ell}] \tag{7.3.5}
\]

where we sum over all different sets of edges connecting the vertices of the right-hand path graph, $X$, to the vertices of the left-hand path graph for each such pair. As an example, for the $L_y = 3$ case, this is

\[
T[\alpha] = u - [1|\alpha_1] - [1|\alpha_2] - [2|\alpha_2] - [2|\alpha_3] - [3|\alpha_3] + [1, 2|\alpha_1, \alpha_2] + [1, 2|\alpha_1, \alpha_3] + [1, 2|\alpha_2, \alpha_3]
\]

\[
+ [1, 3|\alpha_1, \alpha_3] + [1, 3|\alpha_2, \alpha_3] + [2, 3|\alpha_2, \alpha_3] - [1, 2, 3|\alpha_1, \alpha_2, \alpha_3] . \tag{7.3.6}
\]

For $d = L_y - 1$, the invariant subspace of $T$ is again spanned by the terms given in eq. (7.1.12). The nonzero elements in the first row of $T^{(L_y, L_y^{-1})}$ are

\[
(-1)^{L_y-1}((q - 1)[1, 2, \ldots, L_y - 1|h_1, h_2, \ldots, h_{L_y-1}] + (-[1, 2, \ldots, L_y - 1|h_1, h_2, \ldots, h_{L_y-1}])
\]

\[
+ (-[1, 2, \ldots, L_y - 2, L_y|h_1, h_2, \ldots, h_{L_y-2}, h_{L_y}]) + (-1)^{L_y}[1, 2, \ldots, L_y - 1|h_1, h_2, \ldots, h_{L_y-1}]
\]

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\[(7.3.7)\]

For row \(j\) from 2 to \(L_y - 1\), the nonzero elements are

\[
(-1)^{L_y-1}(-[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + (q - 2)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}]
\]

\[+(-[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}]) + ...\]

\[+(q - 2)[1, 2, ..., j, j + 2, ..., L_y|h_1, h_2, ..., h_j, h_{j+2}, ..., h_{L_y}]\]

\[+(-[1, 2, ..., j, j + 2, ..., L_y|h_1, h_2, ..., h_j, h_{j+2}, ..., h_{L_y}])\]

\[+(q - 2)[1, 2, ..., j - 1, j + 1, ..., L_y|h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}]\]

\[+(-[1, 2, ..., j - 1, j + 1, ..., L_y|h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}])\]

\[+(q - 2)[1, 2, ..., j - 2, j, ..., L_y|h_1, h_2, ..., h_{j-2}, h_j, ..., h_{L_y}]\]

\[+(-1)^{L_y}[1, 2, ..., j - 2, j, ..., L_y|h_1, h_2, ..., h_{j-2}, h_j, ..., h_{L_y}])\]

\[= (-1)^{L_y-1}((q - 4)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + ...\]

\[+(q - 4)[1, 2, ..., j, j + 2, ..., L_y|h_1, h_2, ..., h_j, h_{j+2}, ..., h_{L_y}]\]

\[+(q - 4)[1, 2, ..., j - 1, j + 1, ..., L_y|h_1, h_2, ..., h_{j-1}, h_{j+1}, ..., h_{L_y}]\]

\[+[1, 2, ..., j - 2, j, ..., L_y|h_1, h_2, ..., h_{j-2}, h_j, ..., h_{L_y}])\]  \( (7.3.8)\)

The nonzero elements for the last row are

\[
(-1)^{L_y-1}(-[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + (q - 1)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + ...\]

\[+(-[1, 3, ..., L_y|h_1, h_3, ..., h_{L_y}]) + (q - 1)[1, 3, ..., L_y|h_1, h_3, ..., h_{L_y}]\]
\[ + (q - 1)[2, 3, ..., L_y|h_2, h_3, ..., h_{L_y}] + (-1)^{L_y}[2, 3, ..., L_y|h_2, h_3, ..., h_{L_y}] \]
\[ = (-1)^{L_y-1}((q - 2)[1, 2, ..., L_y - 1|h_1, h_2, ..., h_{L_y-1}] + ... + (q - 2)[2, 3, ..., L_y|h_2, h_3, ..., h_{L_y}] . \]

(7.3.9)

The theorem then follows. \( \blacksquare \)

Thus, for example, for \( L_y = 5 \),
\[ \bar{T}^{(5,4)} = \begin{pmatrix} q - 3 & -1 & 0 & 0 & 0 \\ q - 4 & q - 4 & -1 & 0 & 0 \\ q - 4 & q - 4 & q - 4 & -1 & 0 \\ q - 4 & q - 4 & q - 4 & q - 4 & -1 \\ q - 2 & q - 2 & q - 2 & q - 2 & q - 2 \end{pmatrix} . \]  

(7.3.10)

We also recall the result from [17, 20] that the \( \lambda_{P,G,j} \)'s for the cyclic and Möbius strips graphs \( G \) of a given lattice and width \( L_y \) are the same.

Here the eigenvalues of the cyclic triangular lattice do not have a simple linear form \( q - a_{L_y,j} \) as was the case for the eigenvalues for this degree \( d = L_y - 1 \) in the case of the cyclic and Möbius strips of the square lattice given in eqs. (7.1.2) and (7.1.3) or the eigenvalues for the self-dual strip of the square lattice given in eq. (7.2.1) with (7.2.2).

However, we do find the following corollary, which is easily derived from the theorem.

**Corollary 7**  For the cyclic strip of the triangular lattice
\[ \det(\bar{T}^{(L_y,L_y-1)}) = (q - 2)^2(q - 3)^{L_y-2} \]  
\[ \text{Tr}(\bar{T}^{(L_y,L_y-1)}) = 3 + L_y(q - 4) . \]  

(7.3.11)  
(7.3.12)

For example, for \( L_y = 2 \), we have, for the eigenvalue of the reduced transfer matrix,
\[ \lambda_{2,1,j} = \frac{1}{2} \left[ 2q - 5 \pm \sqrt{9 - 4q} \right] \text{ for } j = 1, 2 \]  

(7.3.13)  
in agreement with eqs. (5.12) and (5.13) of [31] (taking account of eq. (7.1.1)). For \( L_y = 3 \), we have
\[ \lambda_{3,2,1} = q - 2 \]  
\[ \lambda_{3,2,j} = \frac{1}{2} \left[ 2q - 7 \pm \sqrt{25 - 8q} \right] \text{ for } j = 2, 3 \]  

(7.3.14)  
(7.3.15)  
in agreement our eqs. (2.16) and (2.17) in [14].
8 Chromatic Polynomial for the Cyclic Strip of the Square Lattice of Width $L_y = 5$

8.1 General

In this section we give our solution for the chromatic polynomial of the $L_y \times L_x$ cyclic strip of the square lattice with width $L_y = 5$. For $L_x$ beyond the first few degenerate cases, this has $n = L_x L_y$ vertices and $e = L_x(2L_y - 1)$ edges. The chromatic number is $\chi = 2$ for $L_x$ even and $\chi = 3$ for $L_x$ odd, independent of $L_y$.

We list below the specific $c^{(d)}$'s that will appear in our results.

\[
\begin{align*}
  c^{(0)} &= 1, & c^{(1)} &= q - 1, & c^{(2)} &= q^2 - 3q + 1, & (8.1.1) \\
  c^{(3)} &= q^3 - 5q^2 + 6q - 1, & (8.1.2) \\
  c^{(4)} &= (q - 1)(q^3 - 6q^2 + 9q - 1), & (8.1.3) \\
\end{align*}
\]

and

\[
\begin{align*}
  c^{(5)} &= q^5 - 9q^4 + 28q^3 - 35q^2 + 15q - 1. & (8.1.4)
\end{align*}
\]

For this family, using the general structural formulas derived in [46], we know that

\[
\begin{align*}
  n_P(5,0) = M_4 = 9, & \quad n_P(5,1) = n_P(5,2) = 21, & \quad n_P(5,3) = 13, & \quad n_P(5,4) = 5 & (8.1.5)
\end{align*}
\]

as well as $n_P(5,5) = 1$. Summing these or equivalently evaluating the general formula (2.2.24) for the case $L_y = 5$, one has, for the total number of $\lambda_{5,d,j}$'s, $N_{P,5,5} = 70$. We have calculated the chromatic polynomials by first computing the transfer matrices $T_{L_y,d}$ for $0 \leq d \leq L_y$.

We find

\[
P(sq[5 \times m, cyc.], q) = c^{(0)} \sum_{j=1}^{9} (\lambda_{5,0,j})^m + c^{(1)} \sum_{j=1}^{21} (\lambda_{5,1,j})^m + c^{(2)} \sum_{j=1}^{21} (\lambda_{5,2,j})^m + c^{(3)} \sum_{j=1}^{13} (\lambda_{5,3,j})^m + c^{(4)} \sum_{j=1}^{5} (\lambda_{5,4,j})^m + c^{(5)} (\lambda_{5,5})^m & (8.1.6)
\]

where $\lambda_{5,0,j}$ for $j = 1, 2$ and $3 \leq j \leq 9$ are roots of equations of degree 2 and 7 in $q$, $\lambda_{5,1,j}$ for $1 \leq j \leq 8$ and $9 \leq j \leq 21$ are roots of equations of respective degrees 8 and 13, $\lambda_{5,2,j}$ for $1 \leq j \leq 9$ and $10 \leq j \leq 21$ are roots of equations of respective degrees 9 and 12, and $\lambda_{5,3,j}$ for $1 \leq j \leq 5$ and $6 \leq j \leq 13$ are roots of equations of respective degrees 5 and 8. We discuss these $\lambda$'s next.
8.2 \( \lambda_{5,0,j} \)

The quadratic equations for \( \lambda_{5,0,j} \) factorize over the field \( \mathbb{Q}[\sqrt{5}] \), so that these eigenvalues are polynomials in \( q \) and the elements of this field:

\[
\lambda_{5,0,j} = \frac{1}{2} \left[ 2q^3 - 13q^2 + 28q - 19 \pm (q - 1)(q - 3)\sqrt{5} \right] \quad \text{for} \quad j = 1, 2 .
\]  

(8.2.1)

The \( \lambda_{5,0,j} \), \( 3 \leq j \leq 9 \) are identical to the \( \lambda \)'s for the \( L_y = 5 \) strip with free boundary conditions, and the degree-7 equation for these is

\[
\xi^7 + \sum_{j=1}^{7} b_{sq5FF,j} \xi^{7-j} = 0
\]

(8.2.2)

where the expressions \( b_{sq5FF,j} \) were given in eqs. (A.20)-(A.26) of [41].

8.3 \( \lambda_{5,d,j} \), \( d = 1, 2 \)

The \( \lambda_{5,d,j} \) with \( d = 1 \) and \( d = 2 \) and \( 1 \leq j \leq 21 \) are too complicated to present here. The computer files are available on request from the authors.

8.4 \( \lambda_{5,3,j} \)

The \( \lambda_{5,3,j} \) are the eigenvalues of the transfer matrix

\[
T^{(5,3)} = \begin{pmatrix}
    t_{11} & q - 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    q - 2 & t_{22} & q - 3 & q - 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    -1 & q - 3 & t_{33} & -1 & q - 3 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
    0 & q - 2 & -1 & t_{44} & q - 3 & q - 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
    0 & -1 & q - 3 & q - 3 & t_{55} & -1 & q - 3 & q - 3 & -1 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & q - 2 & -1 & t_{66} & q - 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -1 & q - 3 & q - 2 & t_{77} & -1 & q - 2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & q - 3 & 0 & -1 & t_{88} & q - 3 & -1 & 0 & 1 & 1 \\
    0 & 0 & 0 & 0 & -1 & 0 & q - 2 & q - 3 & t_{99} & q - 2 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 0 & -1 & q - 2 & t_{1010} & 0 & 0 & 1 & 1 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & q - 2 & 0 & 0 & 0 & 0 \\
    2 - q & 1 & 2 - q & 0 & 0 & 0 & 0 & 0 & -1 & q - 2 & t_{1010} & 0 & 0 & 1 \\
    0 & 0 & 2 - q & 0 & 1 & 0 & 0 & 0 & 2 - q & 0 & 0 & 0 & q - 2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 - q & 1 & 2 - q & 0 & 0 & 0 & q - 2 \\
\end{pmatrix}
\]

(8.4.1)

where

\[
t_{11} = -q^2 + 4q - 5, \quad t_{22} = -(q - 2)(q - 3), \quad t_{33} = -q^2 + 5q - 8 \]

8.4

\[
t_{44} = -(q - 2)(q - 3), \quad t_{55} = -(q - 3)^2, \quad t_{66} = -(q - 2)^2
\]
As noted above, the characteristic polynomial of this matrix factorizes into degree-8 and degree-5 polynomials.

8.5 \( \lambda_{5,4,j} \) and \( \lambda_{5,5} \)

As a special case of our general formula (7.1.2) with (7.1.3) for \( \lambda_{L_y, L_y-1,j} \), of which there are \( L_y \) eigenvalues, we have

\[
\begin{align*}
\lambda_{5,4,1} &= q - a_{sq,5,1} = q - 1 \\
\lambda_{5,4,2} &= q - a_{sq,5,2} = \left( q - \frac{5 - \sqrt{5}}{2} \right) = q - 1.38196... \\
\lambda_{5,4,3} &= q - a_{sq,5,3} = \left( q - \frac{7 - \sqrt{5}}{2} \right) = q - 2.38196... \\
\lambda_{5,4,4} &= q - a_{sq,5,4} = \left( q - \frac{5 + \sqrt{5}}{2} \right) = q - 3.61803... \\
\lambda_{5,4,5} &= q - a_{sq,5,5} = \left( q - \frac{7 + \sqrt{5}}{2} \right) = q - 4.61803...
\end{align*}
\]

We note the connection between the appearance of the algebraic numbers \( a + b\sqrt{5} \), i.e., elements of the field \( \mathbb{Q}[\sqrt{5}] \) in both the \( \lambda_{5,4,j} \) for \( 2 \leq j \leq 5 \) and in \( \lambda_{5,0,\ell} \) for \( \ell = 1,2 \). Indeed, the term involving \( \sqrt{5} \) in \( \lambda_{5,0,1} \) and \( \lambda_{5,0,2} \) is proportional to \( (q - 1)(q - 3) \), which is \( \lambda_{4,0,1} \).

Our general formula (2.2.25) yields

\[
\lambda_{5,5} = -1.
\]

9 Chromatic Polynomial for the Möbius Strip of the Square Lattice of Width \( L_y = 5 \)

We obtain this by applying the transformation rules derived in [16] to go from the chromatic polynomial for the strip with width \( L_y \) and arbitrary length \( L_x \) with \( (FBC_y, PBC_x) = \) cyclic boundary conditions to the corresponding strip with \( (FBC_y, TPBC_x) = \) Möbius boundary conditions. For the chromatic polynomial of the Möbius strip with width \( L_y = 5 \), we find

\[
P(sq[5 \times m, Mb], q) = c^{(0)} \left[ \sum_{j=1}^{7} (\lambda_{5,0,j+2})^m + \sum_{j=1}^{9} (\lambda_{5,2,j})^m - \sum_{j=1}^{2} (\lambda_{5,0,j})^m - \sum_{j=1}^{12} (\lambda_{5,2,j+9})^m \right]
\]
Expressing this in the general form (2.2.31), we have

\[
P(sq[5 \times m, Mb], q) = c^{(0)} \left( \sum_{j=1}^{16} (\lambda_{5,0,+j})^m - \sum_{j=1}^{14} (\lambda_{5,0,-j})^m \right) + c^{(1)} \left( \sum_{j=1}^{15} (\lambda_{5,1,+j})^m - \sum_{j=1}^{11} (\lambda_{5,1,-j})^m \right) + c^{(2)} \left( \sum_{j=1}^{8} (\lambda_{5,2,+j})^m - \sum_{j=1}^{5} (\lambda_{5,2,-j})^m \right) + c^{(3)} (\lambda_{5,5})^m \tag{9.2}
\]

where

\[
\lambda_{5,0,+j} = \lambda_{5,0,j+2} \quad \text{for} \quad 1 \leq j \leq 7 \tag{9.3}
\]

\[
\lambda_{5,0,+j} = \lambda_{5,2,j-7} \quad \text{for} \quad 8 \leq j \leq 16 \tag{9.4}
\]

\[
\lambda_{5,0,-j} = \lambda_{5,0,j} \quad \text{for} \quad 1 \leq j \leq 2 \tag{9.5}
\]

\[
\lambda_{5,0,-j} = \lambda_{5,2,j+7} \quad \text{for} \quad 3 \leq j \leq 14 \tag{9.6}
\]

\[
\lambda_{5,1,+j} = \lambda_{5,1,j+8} \quad \text{for} \quad 1 \leq j \leq 13 \tag{9.7}
\]

\[
\lambda_{5,1,+14} = \lambda_{5,4,2}, \quad \lambda_{5,1,+15} = \lambda_{5,4,4} \tag{9.8}
\]

\[
\lambda_{5,1,-j} = \lambda_{5,1,j} \quad \text{for} \quad 1 \leq j \leq 8 \tag{9.9}
\]

\[
\lambda_{5,1,-9} = \lambda_{5,4,1}, \quad \lambda_{5,1,-10} = \lambda_{5,4,3}, \quad \lambda_{5,1,-11} = \lambda_{5,4,5} \tag{9.10}
\]

\[
\lambda_{5,2,+j} = \lambda_{5,3,j+5} \quad \text{for} \quad 1 \leq j \leq 8 \tag{9.11}
\]

\[
\lambda_{5,2,-j} = \lambda_{5,3,j} \quad \text{for} \quad 1 \leq j \leq 5 \tag{9.12}
\]

\[
\lambda_{5,3,+j} = \lambda_{5,5} \tag{9.13}
\]

10 Locus \( B \) for \( L_y = 5 \) Cyclic/Möbius Strips of the Square Lattice

In this section we present the singular boundary \( B \) across which \( W(q) \) is singular in the complex \( q \) plane for the \( L_x \rightarrow \infty \) limit of the cyclic strip of the square lattice with width \( L_y = 5 \). We also recall that that \( B \) is the same for cyclic and Möbius strips \[16, 35, 36, 17, 41\], as was proved in \[20\]. A plot of \( B \) is shown in Fig. \[35\].
Figure 1: Locus $B$ for the $L_x \to \infty$ limit of the strip of the square lattice of width $L_y = 5$ strip with $(FBC_y, (T)PBC_x) = \text{cyclic (equiv. Möbius)}$ boundary conditions. For comparison, chromatic zeros calculated for the strip length $L_x = 20$ (i.e., $n = 100$ vertices) are shown.
The locus $\mathcal{B}$ is comprised of closed curves that separate the $q$ plane into several regions. This locus crosses the real axis at $q = 0$, $q = 2$, and a maximal value, $q_c$, which is

$$q_c = 2.582385...$$

The region $R_1$ includes the real axis for $q \geq q_c$ and $q \leq 0$ and extends outward to complex infinity from the outer envelope of $\mathcal{B}$. The region $R_2$ includes the real segment $2 \leq q \leq q_c$, while region $R_3$ includes the real segment $0 \leq q \leq 2$. In region $R_1$, the dominant $\lambda$ is the root of the degree-7 equation obtained from the appendix of [41] with the largest magnitude, which we label $\lambda_{R1}$. In region $R_2$, the dominant $\lambda_j$ is the root of the degree-12 equation with the largest magnitude, which we label $\lambda_{R2}$. Hence, $q_c$ is the solution of the equation of the degeneracy $|\lambda_{R1}| = |\lambda_{R2}|$. The value of $q_c$ that we have obtained for the present strip may be compared with the values that we obtained previously for the cyclic (Möbius) $L_y \times \infty$ strips of the square lattice with smaller widths, namely, $q_c = 2$ for $L_y = 1, 2$ [16], $q_c = 2.33654...$ for $L_y = 3$ [33, 36], and $q_c = 2.492845...$ for $L_y = 4$ [11]. We calculate that $W(\text{sq}[5 \times \infty, NBC_y, (T)PBC_x], q) = 1.317594$ at $q_c$, as given in eq. (10.1). In region $R_3$, the dominant $\lambda_j$ is the root of the degree-13 equation with the largest magnitude, which we label $\lambda_{R3}$. There are also complex-conjugate regions centered at approximately $q = 2.7 \pm 0.8i$; we denote these as $R_4, R_4^*$. The dominant $\lambda_j$ in these regions is the root of the degree-13 equation of maximal magnitude here; this is denoted $\lambda_{R4}$. We have

$$W = (\lambda_{R1})^{1/5}, \quad \text{for} \quad q \in R_1$$

$$|W| = (\lambda_{Rj})^{1/5}, \quad \text{for} \quad q \in R_j, \quad j = 2, 3, 4.$$  

(10.2)

(10.3)

(In regions other than $R_1$, only the magnitude $|W|$ can be determined unambiguously [17].)

Our previous results such as [33, 36] showed that there can also be very small sliver regions in the complex $q$ plane; we have not made an exhaustive search for these. As we found previously for $L_y = 3$ [33, 36] and $L_y = 4$ [11], the locus $\mathcal{B}$ has support for $Re(q) < 0$ as well as $Re(q) \geq 0$.

Fig. 1 also shows a comparison of the chromatic zeros calculated for a long finite cyclic strip with $L_x = 20$, i.e., $n = 100$ vertices, versus the asymptotic locus $\mathcal{B}$. One sees that, aside from the isolated real zero at $q = 1$, these chromatic zeros lie reasonably close to the locus $\mathcal{B}$. On the curve forming $\mathcal{B}$ passing through $q = 0$ and $q = q_c$, the density is highest on the right-hand side and somewhat lower in the region of $q = 0$. The density of chromatic zeros on the part of $\mathcal{B}$ passing through $q = 2$ and ending at upper and lower triple points is very low, while one observes an intermediate density on the complex-conjugate pair of bulb-like curves protruding to the right. These features are qualitatively the same as we
found for $L_y = 3$ (Fig. 1 of [35]) and for $L_y = 4$ (Fig. 1 of [11]). The differences in densities on different portions of $B$ were less pronounced for $L_y = 2$ (Fig. 1 of [16]). The elementary case $L_y = 1$ is special in two respects: (i) the complex chromatic zeros lie exactly $B$ for finite as well as infinite $L_x$, and (ii) their density is constant on $B$ [16].

11 Discussion

Here we make several further comments on these results for $B$

1. As in our earlier work, we characterize a strip graph as containing a global circuit if it contains a cycle along the longitudinal direction whose length goes to infinity as the length $L_x \to \infty$; in practice, this is equivalent to the property that the lattice strip graph has periodic longitudinal boundary conditions. For all of the strips of the square lattice containing global circuits that we have studied, the locus $B$ encloses regions of the $q$ plane including certain intervals on the real axis and passes through $q = 0$ and $q = 2$ as well as other possible points, depending on the family. Note that the presence of global circuits is a sufficient, but not necessary, condition for $B$ to enclose regions, as was shown in [31] (see Fig. 4 of that work). Our present results for the square lattice are in accord with, and strengthen the evidence for, the inference (conjecture) [18, 19] that

$$B \supset \{q = 0, 2\} \text{ for } sq[L_y, FBC_y, (T)PBC_x] \forall L_y \geq 1 \text{ . } (11.1)$$

(Of course $L_y = 1$ graphs with $(FBC_y, TPBC_x)$ and $(FBC_y, PBC_x)$ boundary conditions are identical.)

2. The crossing of $B$ at the point $q = 2$ for the (infinite-length limit of) strips with global circuits nicely signals the existence of a zero-temperature critical point in the Ising antiferromagnet (equivalent to the Ising ferromagnet on bipartite graphs). This has been discussed in [14] in the context of exact solutions for finite-temperature Potts model partition functions on the $L_y = 2$ cyclic and Möbius strips of the square lattice. In contrast, this connection is not, in general, present for strips with free longitudinal boundary conditions since $B$ does not, in general, pass through $q = 2$. Furthermore, for the strips without global circuits, there is no indication of any motion of the respective loci $B$ toward $q = 2$ as $L_y$ increases.

3. For cyclic strips, we note a correlation between the coefficient $c_{G_s,j}$ of the respective dominant $\lambda_{G_s,j}$’s in regions that include intervals of the real axis. Before, it was shown [18] that the $c_{G_s,j}$ of the dominant $\lambda_{G_s,j}$ in region $R_1$ including the real intervals
q > q_c(\{G\}) and q < 0 is c^{(0)} = 1, where the c^{(d)} were given in eqs. (2.2.1). We observe further that the c_{G,s,j} that multiplies the dominant \lambda_{G,s,j} in the region containing the intervals 0 < q < 2 is c^{(1)}. For the cyclic \(L_y = 3\) and \(L_y = 4\) strips, there is also another region containing an interval \(2 \leq q \leq q_c\) on the real axis, where \(q_c \simeq 2.34\) and \(2.49\) for \(L_y = 3, 4\); in this region, we find that the c_{G,s,j} multiplying the dominant \lambda_{G,s,j} is c^{(2)}.

4. For the \(L_x \to \infty\) limit of all of the strips of the square lattice containing global circuits, a \(q_c\) is defined, and our results for the cyclic and Möbius strips with widths from \(L_y = 1\) through \(L_y = 4\) indicate that \(q_c\) is a non-decreasing function of \(L_y\) in these cases. The same behavior was found for the strips of the triangular lattice with \(L_y = 2\) \([36]\) (and subsequently also \(L_y = 3, 4\) \([44]\)). This motivated the inference (conjecture) that \(q_c\) is a non-decreasing function of \(L_y\) for strips of regular lattices with \((FBC_y, (T)PBC_x)\) boundary conditions \([18]\), and our present results strengthen the support for this inference. Given that, as \(L_y \to \infty\), \(q_c\) reaches a limit, which is the \(q_c\) for the 2D lattice of the specified type (square, triangular, etc.), this inference leads to the following inequality:

\[
q_c(\Lambda, \infty, BC_y, (T)PBC_x) \leq q_c(\Lambda) .
\]

This is a non-decreasing function of \(L_y\). Our current results are in agreement with this conjecture.

5. One interesting feature of our new results is that the outermost envelope of the curves on \(B\) for the \(L_y = 5\) strip of the square lattice do not lie outside the envelope for \(B\) for the \(L_y = 4\) strip. This is evident from the comparison shown in Fig. \(2\). For reference, we also show the analogous pairwise comparisons for widths \(L_y = 2, 3\) and \(L_y = 3, 4\). For each of these, it was true that as \(L_y\) increases, the outer envelope of the locus \(B\) moves outward in a manner such that \(B\) for width \(L_y + 1\) encloses the locus \(B\) for \(L_y\), allowing for equality at some points, in particular, \(q = 0\) \([18]\). This is also true for the pairwise comparison of the loci \(B\) for \(L_y = 1\) and \(L_y = 2\), as is clear from Fig. \(3\) and the fact that the locus for \(L_y = 1\) is the circle \(|q - 1| = 1\). However, our new results show that this behavior does not persist in the comparison of \(L_y = 4\) and \(L_y = 5\) and hence is not general.

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Figure 2: Comparison of loci $B$ for the $L_z \to \infty$ limits of the strips of the square lattice of widths $L_y = 4$ (light curves) and $L_y = 5$ (heavy curves) with $(FBC_y, (T)PBC_x) = \text{cyclic (equiv. Möbius)}$ boundary conditions.
Figure 3: Comparison of loci $B$ for the $L_z \to \infty$ limits of the strips of the square lattice of widths $L_y = 2$ (light curves) and $L_y = 3$ (heavy curves) with $(FBC_y, (T)PBC_x) = $ cyclic (equiv. Möbius) boundary conditions.
Figure 4: Comparison of loci $\mathcal{B}$ for the $L_z \rightarrow \infty$ limits of the strips of the square lattice of widths $L_y = 3$ (light curves) and $L_y = 4$ (heavy curves) with $(FBC_y, (T)PBC_x) = \text{cyclic (equiv. Möbius)}$ boundary conditions.
12 Appendix: Structure of Chromatic Chromatic Polynomials for Cyclic and Möbius Strips of the Square Lattice

12.1 \( L_y = 2 \) Cyclic Strip

We list some known results to illustrate our general notation, for comparison with our new calculation of the \( L_y = 5 \) case. The chromatic polynomial of the cyclic strip with width \( L_y = 2 \) is \[22\]

\[
P(2 \times m, \text{cyc.}, q) = c^{(0)}(\lambda_{2,0,1})^m + c^{(1)} \sum_{j=1}^{2} (\lambda_{2,1,j})^m + c^{(2)}(\lambda_{2,2})^m
\]

\[
= (q^2 - 3q + 3)^m + (q - 1)\left[(1 - q)^m + (3 - q)^m\right] + (q^2 - 3q + 1)
\]

(12.1.1)

where

\[
\lambda_{2,0,1} = q^2 - 3q + 3
\]

(12.1.2)

\[
\lambda_{2,1,1} = -(q - a_{2,1}) = 1 - q
\]

(12.1.3)

\[
\lambda_{2,1,2} = -(q - a_{2,2}) = 3 - q
\]

(12.1.4)

\[
\lambda_{2,2} = 1.
\]

(12.1.5)

12.2 \( L_y = 2 \) Möbius Strip

The chromatic polynomial of the Möbius (\( Mb \)) strip with width \( L_y = 2 \) is \[22\]

\[
P(2 \times m, Mb, q) = c^{(0)} \left[(\lambda_{2,0,1})^m - (\lambda_{2,2})^m\right] + c^{(1)} \sum_{j=1}^{2} (-1)^j (\lambda_{2,1,j})^m
\]

\[
= \left[(q^2 - 3q + 3)^m - 1\right] + (q - 1)\left[-(1 - q)^m + (3 - q)^m\right]
\]

(12.2.1)

Expressing this in the general form (2.2.31), we have

\[
P(2 \times m, Mb, q) = c^{(0)} \left[(\lambda_{2,0,+,1})^m - (\lambda_{2,0,-,1})^m\right] + c^{(1)} \left[(\lambda_{2,1,+,1})^m - (\lambda_{2,1,-,1})^m\right]
\]

(12.2.2)

where

\[
\lambda_{2,0,+,1} = \lambda_{2,0,1}
\]

(12.2.3)
$$\lambda_{2,0,-1} = \lambda_{2,2}$$  \hspace{1cm} (12.2.4)  \\
$$\lambda_{2,1,+1} = \lambda_{2,1,2}$$  \hspace{1cm} (12.2.5)  \\
$$\lambda_{2,1,-1} = \lambda_{2,1,1}$$  \hspace{1cm} (12.2.6)

12.3 \hspace{0.1cm} L_y = 3 Cyclic Strip

The chromatic polynomial for the cyclic strip of width $L_y = 3$ is \cite{32, 30}

$$P(3 \times m, cyc., q) = \sum_{j=1}^{2} (\lambda_{3,0,j})^m + \sum_{j=1}^{4} (\lambda_{3,1,j})^m + \sum_{j=1}^{3} (\lambda_{3,2,j})^m + \sum_{j=1}^{3} (\lambda_{3,3,j})^m$$  \hspace{1cm} (12.3.1)

where $\lambda_{3,0,j}$ for $j = 1, 2$ are the solutions to a quadratic equation given in \cite{30},

$$\lambda_{3,1,1} = -(q - 2)^2$$  \hspace{1cm} (12.3.2)

the $\lambda_{3,1,j}$ for $j = 4, 5, 6$ are roots of a cubic equation given in \cite{30}, and $\lambda_{3,2,j} = q - a_{3,j}$, with the specific values

$$\lambda_{3,2,1} = q - a_{3,1} = q - 1$$  \hspace{1cm} (12.3.3)  \\
$$\lambda_{3,2,2} = q - a_{3,2} = q - 2$$  \hspace{1cm} (12.3.4)  \\
$$\lambda_{3,2,3} = q - a_{3,3} = q - 4$$  \hspace{1cm} (12.3.5)

Finally,

$$\lambda_{3,3} = -1$$  \hspace{1cm} (12.3.6)

12.4 \hspace{0.1cm} L_y = 3 Möbius Strip

The chromatic polynomial for the Möbius strip of width $L_y = 3$ is \cite{17}

$$P(3 \times m, Mb, q) = \sum_{j=1}^{2} (\lambda_{3,0,j})^m + \sum_{j=1}^{3} (-1)^j (\lambda_{3,2,j})^m + \sum_{j=1}^{3} (\lambda_{3,1,j+1})^m - (\lambda_{3,1,1})^m + \sum_{j=1}^{3} (\lambda_{3,3,j})^m$$  \hspace{1cm} (12.4.1)

Expressing this in the general form \cite{2.2.34},

$$P(3 \times m, Mb, q) = \sum_{j=1}^{3} (\lambda_{3,0,j-1})^m - \sum_{j=1}^{2} (\lambda_{3,0,1})^m$$

44
\[
+ c^{(1)} \left[ \sum_{j=1}^{3} (\lambda_{3,1,+j})^m - (\lambda_{3,1,-1})^m \right] + c^{(2)} (\lambda_{3,3})^m \tag{12.4.2}
\]

where

\[
\lambda_{3,0,+j} = \lambda_{3,0,j} \quad \text{for} \quad j = 1, 2 \tag{12.4.3}
\]

\[
\lambda_{3,0,+3} = \lambda_{3,2,2} \tag{12.4.4}
\]

\[
\lambda_{3,0,-1} = \lambda_{3,2,1} \tag{12.4.5}
\]

\[
\lambda_{3,0,-2} = \lambda_{3,2,3} \tag{12.4.6}
\]

\[
\lambda_{3,1,+j} = \lambda_{3,1,j+1} \quad \text{for} \quad j = 1, 2, 3 \tag{12.4.7}
\]

\[
\lambda_{3,1,-1} = \lambda_{3,1,1} \tag{12.4.8}
\]

\[
\lambda_{3,2,+1} = \lambda_{3,3} \tag{12.4.9}
\]

12.5 \quad L_y = 4 \text{ Cyclic Strip}

The chromatic polynomial of the cyclic strip of width \( L_y = 4 \) is \[11\]

\[
P(4 \times m, \text{cyc.}) = c^{(0)} \sum_{j=1}^{4} (\lambda_{4,0,j})^m + c^{(1)} \sum_{j=1}^{9} (\lambda_{4,1,j})^m + c^{(2)} \sum_{j=1}^{8} (\lambda_{4,2,j})^m
\]

\[
+ c^{(3)} \sum_{j=1}^{4} (\lambda_{4,3,j})^m + c^{(4)} (\lambda_{4,4})^m \tag{12.5.1}
\]

where

\[
\lambda_{4,0,1} = (q - 1)(q - 3) \tag{12.5.2}
\]

\( \lambda_{4,0,j} \) for \( j = 2, 3, 4 \) are roots of a cubic equation, \( \lambda_{4,1,j} \) for \( 1 \leq j \leq 4 \) and \( 5 \leq j \leq 9 \) are roots of equations of respective degrees 4 and 5, \( \lambda_{4,2,j} \) for \( 1 \leq j \leq 3 \) and \( 4 \leq j \leq 8 \) are roots of equations of respective degrees 3 and 5, all given in \[11\], and \( \lambda_{4,3,j} = a_{4,j} - q \), with specific values

\[
\lambda_{4,3,1} = -(q - a_{4,1}) = 1 - q \tag{12.5.3}
\]

\[
\lambda_{4,3,2} = -(q - a_{4,2}) = 3 - \sqrt{2} - q \tag{12.5.4}
\]

\[
\lambda_{3,1,3} = -(q - a_{4,3}) = 3 - q \tag{12.5.5}
\]

\[
\lambda_{4,3,4} = -(q - a_{4,4}) = 3 + \sqrt{2} - q \tag{12.5.6}
\]

Finally,

\[
\lambda_{4,4} = 1 \tag{12.5.7}
\]
12.6 \( L_y = 4 \) Möbius Strip

The chromatic polynomial for the Möbius strip of width \( L_y = 4 \) is

\[
P(4 \times m, Mb) = c^{(0)} \left[ \sum_{j=1}^{3} (\lambda_{4,0,j+1})^m + \sum_{j=1}^{3} (\lambda_{4,2,j})^m - (\lambda_{4,0,1})^m - \sum_{j=1}^{5} (\lambda_{4,2,j+3})^m \right]
+ c^{(1)} \left[ \sum_{j=1}^{5} (\lambda_{4,1,j+4})^m - \sum_{j=1}^{4} (\lambda_{4,1,j})^m - (\lambda_{4,4})^m \right]
+ c^{(2)} \sum_{j=1}^{4} (-1)^j (\lambda_{4,3,j})^m .
\]

(12.6.1)

Expressing this in the general form (2.2.31), we have

\[
P(4 \times m, Mb) = c^{(0)} \left[ \sum_{j=1}^{6} (\lambda_{4,0,+,j})^m - \sum_{j=1}^{6} (\lambda_{4,0,-,j})^m \right]
+ c^{(1)} \left[ \sum_{j=1}^{5} (\lambda_{4,1,+,j})^m - \sum_{j=1}^{5} (\lambda_{4,1,-,j})^m \right]
+ c^{(2)} \left[ \sum_{j=1}^{2} (\lambda_{4,2,+,j})^m - \sum_{j=1}^{2} (\lambda_{4,2,-,j})^m \right]
\]

(12.6.2)

where

\( \lambda_{4,0,+,j} = \lambda_{4,0,j+1} \) for \( 1 \leq j \leq 3 \)
\( \lambda_{4,0,+,j} = \lambda_{4,2,j-3} \) for \( 4 \leq j \leq 6 \)
\( \lambda_{4,0,-,1} = \lambda_{4,0,1} \)
\( \lambda_{4,0,-,j} = \lambda_{4,2,j+2} \) for \( 2 \leq j \leq 6 \)
\( \lambda_{4,1,+,j} = \lambda_{4,1,j+4} \) for \( 1 \leq j \leq 5 \)
\( \lambda_{4,1,-,j} = \lambda_{4,1,j} \) for \( 1 \leq j \leq 4 \)
\( \lambda_{4,1,-,5} = \lambda_{4,4} \)
\( \lambda_{4,2,+,1} = \lambda_{4,3,2} \) , \( \lambda_{4,2,+,2} = \lambda_{4,3,4} \)
\( \lambda_{4,2,-,1} = \lambda_{4,3,1} \) , \( \lambda_{4,2,-,2} = \lambda_{4,3,3} \).

(12.6.3-12.6.11)
References

[1] R. B. Potts, Proc. Camb. Phil. Soc. 48 (1952) 106.

[2] F. Y. Wu, Rev. Mod. Phys. 54 (1982) 235.

[3] C. M. Fortuin and P. W. Kasteleyn, Physica 57 (1972) 536.

[4] W. T. Tutte, Canad. J. Math. 6 (1954) 80.

[5] W. T. Tutte, J. Combin. Theory 2 (1967) 301.

[6] W. T. Tutte Graph Theory, vol. 21 of Encyclopedia of Mathematics and its Applications, ed. Rota, G. C. (Addison-Wesley, New York, 1984).

[7] N. L. Biggs, Algebraic Graph Theory, Second Edition (Cambridge Univ. Press, Cambridge, 1993).

[8] D. J. A. Welsh, Complexity: Knots, Colourings, and Counting, London Math. Soc. Lect. Note Ser. 186 (Cambridge University Press, Cambridge, 1993).

[9] B. Bollobás, Modern Graph Theory (Springer, New York, 1998).

[10] G. D. Birkhoff, Ann. of Math. 14 (1912) 42.

[11] R. C. Read, J. Combin. Theory 4 (1968) 52.

[12] R. C. Read and W. T. Tutte, “Chromatic Polynomials”, in Selected Topics in Graph Theory, 3, (Academic Press, New York, 1988), p. 15.

[13] M. Aizenman and E. H. Lieb, J. Stat. Phys. 24 (1981) 279.

[14] Y. Chow and F. Y. Wu, Phys. Rev. B36 (1987) 285.

[15] L. Pauling, The Nature of the Chemical Bond (Cornell Univ. Press, Ithaca, 1960), p. 466.

[16] R. Shrock and S.-H. Tsai, Phys. Rev. E55 (1997) 5165.

[17] R. Shrock, Phys. Lett. A261 (1999) 57.

[18] R. Shrock, in the Proceedings of the 1999 British Combinatorial Conference, BCC99, Discrete Math. 231 (2001) 421.
[19] R. Shrock, Physica A 283 (2000) 388.
[20] S.-C. Chang and R. Shrock, Physica A 292 (2001) 307.
[21] E. H. Lieb, Phys. Rev. 162 (1967) 162.
[22] N. L. Biggs, R. M. Damerell, and D. A. Sands, J. Combin. Theory B 12 (1972) 123.
[23] S. Beraha, J. Kahane, and N. Weiss, J. Combin. Theory B 28 (1980) 52.
[24] N. L. Biggs, Bull. London Math. Soc. 9 (1977) 54.
[25] R. C. Read, A large family of chromatic polynomials, in Proc. 3rd Caribbean Conference on Combinatorics and Computing (1981) 23.
[26] R. J. Baxter, Chromatic polynomials of large triangular lattices, J. Phys. A 20 (1987) 5241.
[27] R. C. Read, Recent advances in chromatic polynomial theory, in Proc. 5th Caribbean Conf. on Combin. and Computing (1988).
[28] R. Shrock and S.-H. Tsai, Phys. Rev. E56 (1997) 4111.
[29] R. Shrock and S.-H. Tsai, Phys. Rev. E56 (1997) 3935.
[30] M. Roˇcek, R. Shrock, and S.-H. Tsai, Physica A252 (1998) 505.
[31] M. Roˇcek, R. Shrock, and S.-H. Tsai, Physica A259 (1998) 367.
[32] R. Shrock and S.-H. Tsai, Physica A259 (1998) 315.
[33] R. Shrock and S.-H. Tsai, Phys. Rev. E58 (1998) 4332; cond-mat/9808057.
[34] R. Shrock and S.-H. Tsai, J. Phys. A Letts. 32 (1999) L195.
[35] R. Shrock and S.-H. Tsai, Phys. Rev. E60 (1999) 3512.
[36] R. Shrock and S.-H. Tsai, Physica A 275 (2000) 429.
[37] N. L. Biggs, J. Combin. Theory B 82 (2001) 19.
[38] N. L. Biggs, LSE report LSE-CDAM-99-08 (1999), Bull. London Math. Soc., in press.
[39] N. L. Biggs and R. Shrock, J. Phys. A (Letts) 32 (1999) L489.
[40] S.-C. Chang and R. Shrock, Physica A 286 (2000) 189.
[41] S.-C. Chang and R. Shrock, Physica A 290 (2001) 402.
[42] S.-C. Chang and R. Shrock, Physica A 296 (2001) 183.
[43] S.-C. Chang and R. Shrock, Physica A 296 (2001) 234.
[44] S.-C. Chang and R. Shrock, Ann. Phys. 290 (2001) 124.
[45] J. Salas and A. Sokal, J. Stat. Phys., 104 (2001) 611.
[46] S.-C. Chang and R. Shrock, Physica A 296 (2001) 131.
[47] S.-Y. Kim and R. Creswick, E63 (2001) 066107.
[48] J. L. Jacobsen and J. Salas, J. Stat. Phys. 104 (2001) 701.
[49] J. Salas and R. Shrock, Phys. Rev. E 64 (2001) 011111.
[50] S.-C. Chang and R. Shrock, Physica A 301 (2001) 301.
[51] S.-C. Chang and R. Shrock, Phys. Rev. E 64 (2001) 066116.
[52] S.-C. Chang, J. Salas, and R. Shrock, J. Stat. Phys., in press (cond-mat/0108144).
[53] R. C. Read and G. F. Royle, in Graph Theory, Combinatorics, and Applications, Y. Alavi et al., eds. (Wiley, NY, 1991), vol. 2, p. 1009.
[54] D. Woodall, Discrete Math. 101 (1992) 333.
[55] B. Jackson, Combin. Probab. Comput. 2 (1993) 325.
[56] F. Brenti, G. Royle, and D. Wagner, Canadian J. Math. 46 (1994) 55.
[57] C. Thomassen, Combin. Probab. Comput. 6 (1997) 4555.
[58] J. Brown, J. Combin. Theory B 72 (1998) 251.
[59] S.-C. Chang and R. Shrock, Tutte polynomials for recursive families of graphs and their asymptotic limiting functions, paper for the CRM Workshop on Tutte Polynomials, Universitat Autònoma de Barcelona, Sept. 2001, math-ph/0112061.
[60] M. Marcus and H. Mine, A Survey of Matrix Theory and Matrix Inequalities (Dover, New York, 1964).