EXACT CRITICAL EXPONENTS OF THE STAIRCASE MODEL

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Abstract

The staircase model is a recently discovered one-parameter family of integrable two-dimensional continuum field theories. We analyze the novel critical behavior of this model, seen as a perturbation of a minimal conformal theory $M_p$: the leading thermodynamic singularities are simultaneously governed by all fixed points $M_p, M_{p-1}, \ldots, M_3$. The exponents of the magnetic susceptibility and the specific heat are obtained exactly. Various corrections to scaling are discussed, among them a new type specific to crossover phenomena between critical fixed points.

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In the past few years, there has been renewed interest in exploring the consequences of integrability in two-dimensional statistical systems. At a critical point, an infinite number of integrals of motion appears if the theory is conformally invariant; one has even a partial classification of such universality classes [1]. For some perturbations away from criticality, a subset of these integrals survives and makes the theory solvable even at finite correlation length $\xi$. Two types of such systems with a generic $(p-1)$-critical point ($p = 3, 4, \ldots$) described by the minimal conformal theory $M_p$ have been known:

(a) lattice models [2], whose manifold of integrability is parametrized about the fixed point $M_p$ by a relevant temperature-like thermodynamic parameter as well as marginal and irrelevant parameters governing the lattice effects, and

(b) exact factorizable scattering matrices for certain relevant perturbations of the conformal fixed points $M_p$ [3]. These define massive continuum field theories that describe universal scaling behavior. The thermodynamic Bethe ansatz [4] is a way to calculate the universal ground state energy of the associated Hamiltonian on a circle of circumference $R$,

$$E_0(R, \xi) = \frac{2\pi}{R} f(R/\xi). \quad (1)$$

The scaling function $f(\rho)$ shows a simple crossover from the thermodynamic regime $R \gg \xi$ to the conformal regime $R \ll \xi$; its ultraviolet limit is determined by the central charge $c$ of the asymptotic conformal theory: $\lim_{\rho \to 0} f(\rho) = -c/12$.

The “staircase model” is a one-parameter family $M(\theta_0)$ ($\theta_0 > 0$) of factorizable scattering theories with a new and more intricate scaling behavior, discovered very recently by Al.B. Zamolodchikov [5]. These theories contain a single type of massive particles that are characterized by the purely elastic $S$-matrix

$$S(\theta, \theta_0) = (\sinh \theta - i \cosh 2\theta_0)/(\sinh \theta + i \cosh 2\theta_0),$$

written in terms of the Lorentz-invariant rapidity difference $\theta$. The scaling function $f(\rho, \theta_0)$ shows a staircase pattern that interpolates between all central charges $c_p$ (see fig. 1). Hence the theory
$M(\theta_0)$ is described by a renormalization group trajectory that comes close to each fixed point $M_p$ for a RG “time” interval $\theta_0$, whereafter it crosses over to the next lower fixed point $M_{p-1}$ [5].

In another paper [6], I argued that from a Lagrangian point of view, this model can be understood as any fixed point theory $M_p$, perturbed by a linear combination of its weakest relevant scaling field $\phi_{(1,3)}$ and its leading irrelevant scaling field $\phi_{(3,1)}$,

$$L = L^*_p + t_p \phi_{(1,3)} - \bar{t}_p \phi_{(3,1)},$$

(2)

$t_p > 0$ and $\bar{t}_p > 0$ are dimensionful coupling constants with renormalization group eigenvalues $y_p = 4/(p+1)$ and $\bar{y}_p = -4/p$, respectively. The purpose of this Letter is to show in detail that this leads to a unique critical behavior as a function of the relevant “temperature” $t_p$: for $t_p = 0$, the theory is in the universality class of $M_p$, but the thermodynamic singularities as $t_p \downarrow 0$ are determined not by $M_p$ alone, but simultaneously by all fixed points $M_p, M_{p-1}, \ldots, M_3$. Specifically, I obtain the exact exponents of the magnetic susceptibility

$$\chi(t_p) \sim t_p^{-\gamma_p}$$

(3)

and the specific heat

$$C(t_p) \sim t_p^{-\alpha_p} \log t_p$$

(4)

and discuss various corrections to scaling.

Consider first a crossover between two critical fixed points which is characterized by a renormalization group trajectory $u(\tau)$. The two-point correlation functions $G_{ij}(r,u) \equiv \langle \phi_i(0,u)\phi_j(r,u) \rangle$ of local fields $\phi_i$ are path-ordered integrals

$$G_{ij}(e^{r_2-\tau_1}r, u(\tau_1)) =$$

$$G_{kl}(r, u(\tau_2)) \left( P\exp(-\int_{\tau_1}^{\tau_2} d\tau x(u(\tau))) \right)^k_i \left( P\exp(-\int_{\tau_1}^{\tau_2} d\tau x(u(\tau))) \right)^l_j$$

(5)

over the matrix of anomalous dimensions $x(u)$, which is given in perturbation theory about the ultraviolet fixed point by

$$x_i^j(u) = x_i^j(0) + C_{i,k}^j u^k + O(u^2)$$

(6)
in terms of the asymptotic anomalous dimensions and operator product coefficients $C^i_{jk}$. These equations imply the *mixing of fields* under the renormalization group: the infrared asymptotic scaling fields, i.e. the eigenvectors of the matrix $x(u^*)$, are linear combinations of the ultraviolet scaling fields, i.e. the eigenvectors of $x(0)$. If e.g. the local magnetization $\sigma(r)$ is expanded in the infrared scaling basis, the lowest appearing eigenvalue $x$ gives its scaling dimension and the other eigenvalues $x', \ldots$ contribute corrections to scaling. Even if $\sigma(r)$ is chosen a pure scaling field in the ultraviolet, the mixing of fields dictates that the infrared asymptotics of its two-point function $G_\sigma(r)$ is $G_\sigma(r) \sim r^{-2x}(1 + Ar^{-2(x'-x)} + \ldots)$. Besides the usual corrections to scaling due to irrelevant variables and the nonlinearity of scaling variables, this new type of corrections appears in the crossover scaling functions between two critical points since the matrix of anomalous dimensions cannot be diagonalized simultaneously in the ultraviolet and the infrared. The mixing is, of course, restricted by the symmetries that are preserved under the crossover, such as spin-reversal symmetry or self-duality.

A multiple crossover involving the fixed points $M_p, M_{p-1}, \ldots, M_3$ is characterized by the appearance of $p - 1$ different length scales $\xi_{p+1,p}, \xi_{p,p-1}, \ldots, \xi_{3,2} \equiv \xi$. At distances $\xi_{p',p'} < r < \xi_{p',p'}$ (corresponding to RG times $\tau_{p',p'} < \tau < \tau_{p',p'} - 1$), the behavior of correlation functions is governed by the fixed point $M_{p'}$; for $r \gtrsim \xi$, they decay exponentially. At any such fixed point, the the expansion of the local magnetization contains the most relevant scaling field $\phi_{(2,2)}$ of dimension $x_{p'} = 3/(2p'(p'+1))$ (the flow of subleading operators is more subtle due to operator mixing). Hence, in order to extract the leading thermodynamic singularity, one may integrate in Eq. (3) over the piecewise constant function $x(\tau) = x_{p'}$ for $\tau_{p',p'} < \tau < \tau_{p',p'} - 1$; this suppresses in addition the familiar corrections to scaling of the form $(1 + Br^{y_i})$ due to an irrelevant coupling of dimension $y_i$. The singular part of the susceptibility
then consists of a sum of terms

\[
\int_{\xi_{p' + 1, p'}}^{\xi_{p', p'} - 1} 2\pi r dr G_\sigma (r) = G_\sigma (\xi_{p' + 1, p'}) \left( \frac{\xi_{p' + 1, p'} - 1}{\xi_{p', p'} - 1} \right)^{-2x_{p'}} \xi_{p', p'}^2 \xi_{p', p'} - 1 = \\
\xi_{p, p'}^{2 - 2x_{p'}} \left( \frac{\xi_{p' - 1, p} - 2}{\xi_{p, p'} - 1} \right)^{2 - 2x_{p' - 1}} \cdots \left( \frac{\xi_{p' - 1, p'} - 2}{\xi_{p', p'} - 1} \right)^{2 - 2x_{p'}}.
\]

(7)

Since \( x_{p'} \leq 1 \) for all \( p' \), the leading singularity comes from the integration region \( \xi_{4,3} < r < \xi \); the other terms are a third source of corrections to scaling.

What makes this equation useful is the fact that any two subsequent crossover length scales have the same ratio

\[
\frac{\xi_{p', p'} - 1}{\xi_{p', p' + 1}} = e^{\theta_0}.
\]

(8)

This feature of the exact solution is consistent with the Lagrangian formulation (2), as can be inferred from renormalization group and scaling arguments (for details see [6]). Under the flow between \( M_p \) and \( M_{p - 1} \), the irrelevant running coupling \( \bar{u}_p \) and the relevant coupling \( u_{p - 1} \) mix, which can be expressed as the dimensionless equation

\[
t_{p - 1}^{-1} t_{p - 1}^{-y_{p - 1}} / \bar{y}_{p - 1} = t_{p}^{-1} \bar{y}_{p} / y_{p}
\]

(9)

Expressing these dimensionful couplings in terms of the crossover length scales

\[
t_{p'}^{-1} / y_{p'} \sim \xi_{p', p' - 1} \sim t_{p' - 1}^{-1} / \bar{y}_{p'}
\]

(10)

and using the exponent relation \( y_{p - 1} = 4/p = -\bar{y}_p \) then shows that the ratio \( (\xi_{p+1, p}/\xi_{p, p-1})/(\xi_{p, p-1}/\xi_{p-1, p-2}) \) is asymptotically constant.

From (7), (8) and (10), one obtains

\[
\gamma_p = \frac{1}{y_p} (2 - 2x_p + 2 - 2x_{p - 1} + \ldots + 2 - 2x_3) = \frac{(p - 2)(2p + 1)}{4}.
\]

(11)

The local energy density contains at any fixed point the most relevant field that is even under spin reversal, i.e. \( \phi_{(3,3)} \) of dimension \( \tilde{x}_{p'} = 4/(p'(p' + 1)) \) for \( p' > 3 \) and
\( \phi_{(1,3)} \) of dimension \( \bar{x}_3 = 1 \) for \( p = 3 \). Hence for the specific heat, one obtains in a similar way

\[
\alpha_p = \frac{1}{y_p} \left( 2 - 2\bar{x}_p + 2 - 2\bar{x}_{p-1} + \ldots + 2 - 2\bar{x}_3 \right) = \frac{p(p - 3)}{2}; \tag{12}
\]

the additional logarithmic singularity in Eq. (4) comes from the integration region \( \xi_{4,3} < r < \xi \), which is governed by the Ising fixed point \( M_3 \). Zamolodchikov’s S-matrix has also been related to multiple crossovers in the \( D \)-series of minimal models \[7\], based on a thermodynamic Bethe ansatz analysis for antiperiodic boundary conditions \[8\]. A \( D \)-series staircase model has the same Lagrangian description (2) and the same flow of scaling fields that are even under spin reversal, but Eq. (11) is replaced by a similar sum over scaling dimensions of odd \( D \)-series fields.

From a renormalization group point of view, two (related) aspects of this system are remarkable.

(a) The irrelevant parameter \( \bar{t}_p \) in (2) affects the leading exponents associated to the relevant perturbation \( t_p \). This effect, encountered also \[9\] in regime IV of the integrable lattice models of Andrews, Baxter and Forrester \[2\], is due to mixing under the renormalization group \[6\].

(b) Several fixed points democratically share the responsibility for the exponents (11) and (12). Generically, the ratio of different crossover “times” \( \tau_{p',p'-1} - \tau_{p'+1,p'} \) would be singular as \( t_p \to 0 \). That this is not the case in the staircase model is yet another of the amazing fine-tunings occurring in integrable systems.
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Fig. 1. The ground state scaling function $f(\rho, \theta_0)$ for the staircase model $M(\theta_0)$. All steps have the same logarithmic width $\theta_0$. 