Some New Reverses and Refinements of Inequalities for Relative Operator Entropy

S.S. DRAGOMIR

1 Mathematics, College of Engineering & Science, Victoria University  
PO Box 14428, Melbourne City, MC 8001, Australia  
sever.dragomir@vu.edu.au, http://rgmia.org/dragomir

2 School of Computer Science & Applied Mathematics, University of the Witwatersrand  
Private Bag 3, Johannesburg 2050, South Africa

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Abstract: In this paper we obtain new inequalities for relative operator entropy $S(A|B)$ in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 \leq m < M$.

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1. Introduction

Kamei and Fujii [6, 7] defined the relative operator entropy $S(A|B)$, for positive invertible operators $A$ and $B$, by

$$S(A|B) := A^{\frac{1}{2}} \left( \ln \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \quad (1.1)$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [12].

In general, we can define for positive operators $A$, $B$

$$S(A|B) := s - \lim_{\varepsilon \to 0^+} S(A + \varepsilon 1_H|B)$$

if it exists, here $1_H$ is the identity operator.

For the entropy function $\eta(t) = -t \ln t$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A = S(A|1_H) \geq 0$$

for positive contraction $A$. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.
Following [8, pp. 149-155], we recall some important properties of relative operator entropy for \(A\) and \(B\) positive invertible operators:

(i) We have the equalities
\[
S(A|B) = -A^{1/2} \left( \ln A^{1/2} B^{-1/2} A^{1/2} \right) A^{1/2} \\
= B^{1/2} \eta \left( B^{-1/2} A B^{-1/2} \right) B^{1/2}.
\]

(ii) We have the inequalities
\[
S(A|B) \leq A \left( \ln \|B\| - \ln A \right) \quad \text{and} \quad S(A|B) \leq B - A.
\]

(iii) For any \(C\), \(D\) positive invertible operators we have that
\[
S(A + B|C + D) \geq S(A|C) + S(B|D).
\]

(iv) If \(B \leq C\) then
\[
S(A|B) \leq S(A|C).
\]

(v) If \(B_n \downarrow B\) then
\[
S(A|B_n) \downarrow S(A|B).
\]

(vi) For \(\alpha > 0\) we have
\[
S(\alpha A|\alpha B) = \alpha S(A|B).
\]

(vii) For every operator \(T\) we have
\[
T^* S(A|B) T \leq S(T^* A T|T^* B T).
\]

The relative operator entropy is \textit{jointly concave}, namely, for any positive invertible operators \(A, B, C, D\) we have
\[
S(t A + (1 - t) B | t C + (1 - t) D) \geq t S(A|C) + (1 - t) S(B|D)
\]
for any \(t \in [0, 1]\).

For other results on the relative operator entropy see [1, 4, 9, 10, 11, 13]. Observe that, if we replace in (1.2) \(B\) with \(A\), then we get
\[
S(B|A) = A^{1/2} \eta (A^{-1/2} B A^{-1/2}) A^{1/2} \\
= A^{1/2} \left( -A^{-1/2} B A^{-1/2} \ln (A^{-1/2} B A^{-1/2}) \right) A^{1/2},
\]
therefore we have

\[ A^{1/2}(A^{-1/2}BA^{-1/2} \ln (A^{-1/2}BA^{-1/2}))A^{1/2} = -S(B|A) \]  

(1.4)

for positive invertible operators \( A \) and \( B \).

It is well known that, in general \( S(A|B) \) is not equal to \( S(B|A) \).

In [15], A. Uhlmann has shown that the relative operator entropy \( S(A|B) \) can be represented as the strong limit

\[ S(A|B) = s - \lim_{t \to 0} \frac{A^{\nu}_t B - A}{t}, \]  

(1.5)

where

\[ A^{\nu}_t := A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}, \quad \nu \in [0, 1], \]

is the weighted geometric mean of positive invertible operators \( A \) and \( B \). For \( \nu = \frac{1}{2} \) we denote \( A_2 B \).

This definition of the weighted geometric mean can be extended for any real number \( \nu \) with \( \nu \neq 0 \).

For \( t > 0 \) and the positive invertible operators \( A, B \) we define the Tsallis relative operator entropy (see also [3]) by

\[ T_t(A|B) := \frac{A^{\nu}_t B - A}{t}. \]

The following result providing upper and lower bounds for relative operator entropy in terms of \( T_t(\cdot | \cdot) \) has been obtained in [6] for \( 0 < t \leq 1 \). However, it holds for any \( t > 0 \).

**Theorem 1.** Let \( A, B \) be two positive invertible operators, then for any \( t > 0 \) we have

\[ T_t(A|B)(A^{\nu}_t B)^{-1} A \leq S(A|B) \leq T_t(A|B). \]  

(1.6)

In particular, we have for \( t = 1 \) that

\[ (1_H - AB^{-1}) A \leq S(A|B) \leq B - A, \]  

(6)

and for \( t = 2 \) that

\[ \frac{1}{2} (1_H - (AB^{-1})^2) A \leq S(A|B) \leq \frac{1}{2} (BA^{-1}B - A). \]  

(1.8)
The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}(A|B) := 2(A^* \circ B - A)$$

and

$$T_{1/2}(A|B)(A^*_{1/2}B)^{-1}A = 2(1_H - A(A^*B)^{-1})A,$$

hence by (1.6) we get

$$2(1_H - A(A^*B)^{-1})A \leq S(A|B) \leq 2(A^*B - A) \leq B - A. \quad (1.9)$$

Motivated by the above results, in this paper we obtain new inequalities for the relative operator entropy in the case of operators satisfying the condition $m_1 A \leq B \leq M_1 A$, with $0 < m < M$.

2. INEQUALITIES FOR LOG-FUNCTION

We have:

**THEOREM 2.** For any $a, b > 0$ we have the inequalities

$$\frac{1}{2b \min\{a, b\}}(b - a)^2 \geq \ln b - \ln a - \frac{b - a}{b} \geq \frac{1}{2b \max\{a, b\}}(b - a)^2 \quad (2.1)$$

and

$$\frac{1}{2a \min\{a, b\}}(b - a)^2 \geq \frac{b - a}{a} - \ln b + \ln a \geq \frac{1}{2a \max\{a, b\}}(b - a)^2. \quad (2.2)$$

**Proof.** We have

$$\int_a^b \frac{b - t}{t} dt = \int_a^b \frac{1}{t} dt - \int_a^b dt = b(\ln b - \ln a) - (b - a)$$

giving that

$$\ln b - \ln a - \frac{b - a}{b} = \frac{1}{b} \int_a^b \frac{b - t}{t} dt \quad (2.3)$$

for any $a, b > 0$.

Let $b > a > 0$, then

$$\frac{1}{a} \int_a^b (b - t) dt \geq \int_a^b \frac{b - t}{t} dt \geq \frac{1}{b} \int_a^b (b - t) dt$$
giving that
\[ \frac{1}{2a} (b - a)^2 \geq \int_a^b \frac{b - t}{t} \, dt \geq \frac{1}{2b} (b - a)^2. \] (2.4)

Let \( a > b > 0 \), then
\[ \frac{1}{b} \int_b^a (t - b) \, dt \geq \int_a^b \frac{b - t}{t} \, dt = \int_b^a \frac{t - b}{t} \, dt \geq \frac{1}{a} \int_b^a (t - b) \, dt \]

giving that
\[ \frac{1}{2b} (b - a)^2 \geq \int_a^b \frac{b - t}{t} \, dt \geq \frac{1}{2a} (b - a)^2. \] (2.5)

Therefore, by (2.4) and (2.5) we get
\[ \frac{1}{2 \min\{a, b\}} (b - a)^2 \geq \int_a^b \frac{b - t}{t} \, dt \geq \frac{1}{2 \max\{a, b\}} (b - a)^2, \]

for any \( a, b > 0 \).

By utilising the equality (2.3) we get the desired result (2.1). \( \blacksquare \)

**Corollary 1.** For any \( y > 0 \) we have
\[ \frac{1}{2y \min\{1, y\}} (y - 1)^2 \geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2y \max\{1, y\}} (y - 1)^2, \] (2.6)
\[ \frac{1}{2 \min\{1, y\}} (y - 1)^2 \geq y - 1 - \ln y \geq \frac{1}{2 \max\{1, y\}} (y - 1)^2. \] (2.7)

**Remark 1.** Since for any \( a, b > 0 \) we have \( \max\{a, b\} \min\{a, b\} = ab \), then (2.1) and (2.2) can also be written as
\[ \frac{1}{2a} \max\{a, b\} \left( \frac{b - a}{b} \right)^2 \geq \ln b - \ln a - \frac{b - a}{b} \]
\[ \geq \frac{1}{2a} \min\{a, b\} \left( \frac{b - a}{b} \right)^2 \] (2.8)

and
\[ \frac{1}{2b} \max\{a, b\} \left( \frac{b - a}{a} \right)^2 \geq \frac{b - a}{a} - \ln b + \ln a \]
\[ \geq \frac{1}{2b} \min\{a, b\} \left( \frac{b - a}{a} \right)^2 \] (2.9)
for any $a, b > 0$.

The inequalities can also be written as

$$\frac{1}{2} \max\{1, y\} \left(\frac{y - 1}{y}\right)^2 \geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2} \min\{1, y\} \left(\frac{y - 1}{y}\right)^2 \quad (2.10)$$

and

$$\frac{1}{2y} \max\{1, y\} (y - 1)^2 \geq y - 1 - \ln y \geq \frac{1}{2y} \min\{1, y\} (y - 1)^2, \quad (2.11)$$

for any $y > 0$.

In the recent paper [2] we obtained the following inequalities that provide upper and lower bounds for the quantity $\ln b - \ln a - \frac{b-a}{a}$:

$$\frac{1}{2} \frac{(b-a)^2}{\min^2\{a, b\}} \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2} \frac{(b-a)^2}{\max^2\{a, b\}}, \quad (2.12)$$

where $a, b > 0$ and

$$\frac{(b-a)^2}{ab} \geq \frac{b-a}{a} - \ln b + \ln a \quad (2.13)$$

for any $a, b > 0$.

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

as provided by (2.2), (2.12) and (2.13) is better?

It has been shown in [2] that neither of the upper bounds in (2.12) and (2.13) is always best.

Consider now the difference

$$D_1(a, b) := \frac{1}{2a \min\{a, b\}} (b-a)^2 - \frac{1}{2} \frac{(b-a)^2}{\min^2\{a, b\}}$$

$$= \frac{1}{2} \frac{(b-a)^2}{a \min^2\{a, b\}} (\min\{a, b\} - a) \leq 0,$$

which shows that upper bound in (2.2) is always better than the upper bound in (2.12).
Consider the difference

\[ D_2(a, b) := \frac{1}{2a \min\{a, b\}} (b - a)^2 - \frac{(b - a)^2}{ab} \]

\[ = \frac{1}{2ab \min\{a, b\}} (b - a)^2 (b - 2 \min\{a, b\}) , \]

which can take both positive and negative values for \( a, b > 0 \), showing that neither of the bounds (2.2) and (2.13) is always best.

Now, consider the difference

\[ d(a, b) := \frac{1}{2a \max\{a, b\}} (b - a)^2 - \frac{1}{2 \max^2\{a, b\}} (b - a)^2 \]

\[ = \frac{1}{2a \max^2\{a, b\}} (b - a)^2 (\max\{a, b\} - a) \geq 0 , \]

which shows that lower bound in (2.2) is always better than the lower bound in (2.12).

**Corollary 2.** If \( y \in [k, K] \subset (0, \infty) \), then we have the local inequalities

\[
\frac{1}{2 \min\{1, k\}} \frac{(y - 1)^2}{y} \geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2 \max\{1, K\}} \frac{(y - 1)^2}{y} , \tag{2.14}
\]

\[
\frac{1}{2 \min\{1, k\}} (y - 1)^2 \geq y - 1 - \ln y \geq \frac{1}{2 \max\{1, K\}} (y - 1)^2 , \tag{2.15}
\]

\[
\frac{1}{2} \max\{1, K\} \left( \frac{y - 1}{y} \right)^2 \geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2} \min\{1, k\} \left( \frac{y - 1}{y} \right)^2 , \tag{2.16}
\]

\[
\frac{1}{2} \max\{1, K\} \frac{(y - 1)^2}{y} \geq y - 1 - \ln y \geq \frac{1}{2} \min\{1, k\} \frac{(y - 1)^2}{y} . \tag{2.17}
\]

**Proof.** If \( y \in [k, K] \subset (0, \infty) \), then by analyzing all possible locations of the interval \([k, K]\) and 1 we have

\[
\min\{1, k\} \leq \min\{1, y\} \leq \min\{1, K\} ,
\]

\[
\max\{1, k\} \leq \max\{1, y\} \leq \max\{1, K\} .
\]
By using the inequalities (2.6) and (2.7) we have

\[
\frac{1}{2y \min\{1, y\}} (y - 1)^2 \geq \frac{1}{2y \min\{1, y\}} (y - 1)^2 \\
\geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2y \max\{1, y\}} (y - 1)^2 \\
\geq \frac{1}{2y \max\{1, K\}} (y - 1)^2
\]

and

\[
\frac{1}{2 \min\{1, k\}} (y - 1)^2 \geq \frac{1}{2 \min\{1, y\}} (y - 1)^2 \\
\geq y - 1 - \ln y \geq \frac{1}{2 \max\{1, y\}} (y - 1)^2 \\
\geq \frac{1}{2 \max\{1, K\}} (y - 1)^2
\]

for any \( y \in [k, K] \), that prove (2.14) and (2.15).

The inequalities (2.16) and (2.17) follows by (2.16) and (2.17).

If we consider the function \( f(y) = \frac{(y - 1)^2}{y} \), \( y > 0 \), then we observe that

\[
f'(y) = \frac{y^2 - 1}{y^2} \quad \text{and} \quad f''(y) = \frac{2}{y^3},
\]

which shows that \( f \) is strictly decreasing on \((0, 1)\), strictly increasing on \([1, \infty)\) and strictly convex for \( y > 0 \). We also have \( f\left(\frac{1}{2}\right) = f(y) \) for \( y > 0 \).

By the properties of \( f \) we then have that

\[
\max_{y \in [k, K]} \frac{(y - 1)^2}{y} = \begin{cases} 
\frac{(k - 1)^2}{k} & \text{if } K < 1, \\
\max\left\{ \frac{(k - 1)^2}{k}, \frac{(K - 1)^2}{K} \right\} & \text{if } k \leq 1 \leq K, \\
\frac{(K - 1)^2}{K} & \text{if } 1 < k,
\end{cases}
\]

\[=: U(k, K)\]
and

\[
\min_{y \in [k, K]} \frac{(y - 1)^2}{y} = \begin{cases} 
\frac{(1-K)^2}{K} & \text{if } K < 1, \\
0 & \text{if } k \leq 1 \leq K, \\
\frac{(k-1)^2}{k} & \text{if } 1 < k,
\end{cases} \tag{2.19}
\]

\[=: u(k, K). \]

We can provide now some global bounds as follows. From (2.14) we then get for any \( y \in [k, K] \) that

\[
\frac{1}{2 \min\{1, k\}} U(k, K) \geq \ln y - \frac{y - 1}{y} \geq \frac{1}{2 \max\{1, K\}} u(k, K), \tag{2.20}
\]

while from (2.17) we get for any \( y \in [k, K] \) that

\[
\frac{1}{2} \max\{1, K\} U(k, K) \geq y - 1 - \ln y \geq \frac{1}{2} \min\{1, k\} u(k, K). \tag{2.21}
\]

Consider

\[
Z(k, K) := \max_{y \in [k, K]} (y - 1)^2 \tag{2.22}
\]

\[
= \begin{cases} 
(1-k)^2 & \text{if } K < 1, \\
\max\{(1-k)^2, (K-1)^2\} & \text{if } k \leq 1 \leq K, \\
(K-1)^2 & \text{if } 1 < k,
\end{cases}
\]

and

\[
z(k, K) := \min_{y \in [k, K]} (y - 1)^2 = \begin{cases} 
(1-K)^2 & \text{if } K < 1, \\
0 & \text{if } k \leq 1 \leq K, \\
(k-1)^2 & \text{if } 1 < k.
\end{cases} \tag{2.23}
\]

By making use of (2.15) we get

\[
\frac{1}{2 \min\{1, k\}} Z(k, K) \geq y - 1 - \ln y \geq \frac{1}{2 \max\{1, K\}} z(k, K), \tag{2.24}
\]

for any \( y \in [k, K] \).
Consider the function \( g(y) = \left( \frac{y-1}{y} \right)^2 \), \( y > 0 \), then we observe that
\[
g'(y) = \frac{2(y-1)}{y^2} \quad \text{and} \quad g''(y) = \frac{2(3-2y)}{y^4},
\]
which shows that \( g \) is strictly decreasing on \((0, 1)\), strictly increasing on \([1, \infty)\) strictly convex for \( y \in (0, 3/2) \) and strictly concave on \((3/2, \infty)\).

Consider
\[
W(k, K) := \max_{y \in [k, K]} \left( \frac{y-1}{y} \right)^2 \quad \text{(2.25)}
\]
\[
= \begin{cases} 
\left( \frac{1-k}{k} \right)^2 & \text{if } K < 1, \\
\max \left\{ \left( \frac{1-k}{k} \right)^2, \left( \frac{K-1}{K} \right)^2 \right\} & \text{if } k \leq 1 \leq K, \\
\left( \frac{K-1}{K} \right)^2 & \text{if } 1 < k,
\end{cases}
\]

and
\[
w(k, K) := \min_{y \in [k, K]} \left( \frac{y-1}{y} \right)^2 = \begin{cases} 
\left( \frac{1-K}{K} \right)^2 & \text{if } K < 1, \\
0 & \text{if } k \leq 1 \leq K, \\
\left( \frac{k-1}{k} \right)^2 & \text{if } 1 < k.
\end{cases} \quad \text{(2.26)}
\]

Then by (2.16) we get
\[
\frac{1}{2} \max \{1, K\} W(k, K) \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2} \min \{1, k\} w(k, K) \quad \text{(2.27)}
\]
for any \( y \in [k, K] \).

3. Operator inequalities

We have the following:

**Lemma 1.** Let \( x \in [k, K] \) and \( t > 0 \), then we have
\[
\frac{1}{2 \min \{1, k^t\}} \left( \frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right) \geq \ln x - \frac{1 - x^{-t}}{t} \geq \frac{1}{2 \max \{1, K^t\}} \left( \frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right) \geq 0.
\]
and
\[
\frac{1}{2} \max \{1, K^t\} t \left( \frac{1 - x^{-t}}{t} \right)^2 \geq \ln x - \frac{1 - x^{-t}}{t} \geq \frac{1}{2} \min \{1, k^t\} t \left( \frac{1 - x^{-t}}{t} \right)^2 \geq 0.
\] (3.2)

**Proof.** Let \( y = x^t \in [k^t, K^t] \). By using the inequality (2.14) we have
\[
\frac{1}{2} \min \{1, k^t\}(x^t + x^{-t} - 2) \geq t \ln x - \frac{x^t - 1}{x^t}
\]
that is equivalent to (3.1).

From the inequality (2.16) we have for \( y = x^t \)
\[
\frac{1}{2} \max \{1, K^t\}(1 - 2x^{-t} + x^{-2t}) \geq t \ln x - \frac{x^t - 1}{x^t}
\]
that is equivalent to (3.2). }

We have:

**Theorem 3.** Let \( A, B \) be two positive invertible operators and the constants \( M > m > 0 \) with the property that
\[
mA \leq B \leq MA.
\] (3.3)

Then for any \( t > 0 \) we have
\[
\frac{1}{2 \min \{1, m^t\}} T_t(A|B)(A^{-1} - (A^*tB)^{-1})A
\]
\[
\geq S(A|B) - T_t(A|B)(A^*tB)^{-1}A
\]
\[
\geq \frac{1}{2 \max \{1, M^t\}} T_t(A|B)(A^{-1} - (A^*tB)^{-1})A \geq 0
\] (3.4)
\[
1 \frac{1}{2} \max \{1, M^l\} t (T_t(A|B)(A_{st}B)^{-1})^2 A \\
\geq S(A|B) - T_t(A|B)(A_{st}B)^{-1} A \\
\geq 1 \frac{1}{2} \min \{1, m^l\} t (T_t(A|B)(A_{st}B)^{-1})^2 A \geq 0.
\]

**Proof.** Since \(mA \leq B \leq MA\) and \(A\) is invertible, then by multiplying both sides with \(A^{-1/2}\) we get \(m1_H \leq A^{-1/2}BA^{-1/2} \leq M\). Denote \(X = A^{-1/2}BA^{-1/2}\) and by using the functional calculus for \(X\) that has its spectrum contained in the interval \([m, M]\) and the inequality (3.1), we get

\[
\geq \frac{1}{2} \min \{1, m^l\} \left( \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^t - 1_H - \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^{-t} \right)
\]

\(\geq 0\)

for any \(t > 0\).

Now, if we multiply both sides of (3.6) by \(A^{1/2}\), then we get

\[
\geq A^{1/2} \left( \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^t - 1_H - \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^{-t} \right) A^{1/2}
\]

\(\geq A^{1/2} (\ln (A^{-1/2}BA^{-1/2})) A^{1/2} - A^{1/2} \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} A^{1/2}
\]

\(\geq \frac{1}{2} \max \{1, M^l\} A^{1/2} \left( \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^t - 1_H - \frac{1}{t} \left( A^{-1/2}BA^{-1/2} \right)^{-t} \right) A^{1/2} \geq 0
\]

for any \(t > 0\).

Observe that

\[A^{1/2} \ln (A^{-1/2}BA^{-1/2}) A^{1/2} = S(A|B),\]
Some new reverses and refinements of inequalities

\[ A^{1/2} \left( \frac{A^{-1/2}BA^{-1/2}}{t} \right)^t - 1 A^{1/2} = \frac{A_{\alpha t}B - A}{t} = T_t(A|B), \]

\[ A^{1/2} \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right) A^{1/2} \]

\[ = A^{1/2} \left( \frac{A^{-1/2}BA^{-1/2}}{t} \right)^t (A^{-1/2}BA^{-1/2})^{-t} - (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \]

\[ = A^{1/2} (A^{-1/2}BA^{-1/2})^{-t} - 1_H (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \]

\[ = A^{1/2} (A^{-1/2}BA^{-1/2})^{-t} - 1_H A^{1/2} A^{-1/2} (A^{-1/2}BA^{-1/2})^{-t} A^{-1/2} \]

\[ = T_t(A|B)(A_{\alpha t}B)^{-1}A \]

and then by (3.7) we get

\[ \frac{1}{2 \min\{1, m^t\}} T_t(A|B)(1_H - (A_{\alpha t}B)^{-1}A) \]

\[ \geq S(A|B) - T_t(A|B)(A_{\alpha t}B)^{-1}A \]

\[ \geq \frac{1}{2 \max\{1, M^t\}} T_t(A|B)(1_H - (A_{\alpha t}B)^{-1}A) \geq 0 \]

that is equivalent to (3.4).

From the inequality (3.2) we also have

\[ \frac{1}{2 \max\{1, M^t\}} t \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)^2 \]

\[ \geq \ln (A^{-1/2}BA^{-1/2}) - \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \]

\[ \geq \frac{1}{2 \min\{1, m^t\}} t \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)^2 \geq 0. \]
Now, if we multiply both sides of (3.9) by $A^{1/2}$, then we get

$$
\frac{1}{2} \max\{1, M^t\} t A^{1/2} \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)^2 A^{1/2}
$$

(3.10)

$$
\geq A^{1/2} (\ln (A^{-1/2}BA^{-1/2})) A^{1/2} - A^{1/2} \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} A^{1/2}
$$

$$
\geq \frac{1}{2} \min\{1, m^t\} t A^{1/2} \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)^2 A^{1/2} \geq 0.
$$

From (3.8) we have, by multiplying both sides by $A^{-1/2}$, that

$$
\frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} = A^{-1/2} T_i(A|B)(A_{\#} B)^{-1} A^{1/2}.
$$

Then

$$
A^{1/2} \left( \frac{1_H - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)^2 A^{1/2}
$$

$$
= A^{1/2} \left( A^{-1/2} T_i(A|B)(A_{\#} B)^{-1} A^{1/2} \right)^2 A^{1/2}
$$

$$
= A^{1/2} A^{-1/2} T_i(A|B)(A_{\#} B)^{-1} A^{1/2} A^{-1/2} T_i(A|B)(A_{\#} B)^{-1} A^{1/2} A^{1/2}
$$

$$
= T_i(A|B)(A_{\#} B)^{-1} T_i(A|B)(A_{\#} B)^{-1} A
$$

$$
= (T_i(A|B)(A_{\#} B)^{-1})^2 A,
$$

which together with (3.10) produces the desired result (3.5).

There are some particular inequalities of interest as follows.

For $t = 1$ we get from (3.4) and (3.5) that

$$
\frac{1}{2} \min\{1, m\} (B - A)(A^{-1} - B^{-1}) A
$$

$$
\geq S(A|B) - (1_H - AB^{-1}) A
$$

(3.11)

$$
\geq \frac{1}{2} \max\{1, M\} (B - A)(A^{-1} - B^{-1}) A \geq 0
$$
\[
\frac{1}{2} \max\{1, M\}(1_H - AB^{-1})^2 A \\
\geq S(A|B) - (1_H - AB^{-1})A \\
\geq \frac{1}{2} \min\{1, m\}(1_H - AB^{-1})^2 A \geq 0. \tag{3.12}
\]

For \( t = 1/2 \) we get from (3.4) and (3.5) that
\[
\frac{1}{\min\{1, \sqrt{m}\}}(A^*B - A)(A^{-1} - (A^*B)^{-1})A \\
\geq S(A|B) - 2(1_H - A(A^*B)^{-1})A \\
\geq \frac{1}{\max\{1, \sqrt{M}\}}(A^*B - A)(A^{-1} - (A^*B)^{-1})A \geq 0
\]
and
\[
\max\{1, \sqrt{M}\}(1_H - A(A^*B)^{-1})^2 A \\
\geq S(A|B) - 2(1_H - A(A^*B)^{-1})A \\
\geq \min\{1, \sqrt{m}\}(1_H - A(A^*B)^{-1})^2 A \geq 0. \tag{3.14}
\]

For \( t = 2 \) we get from (3.4) and (3.5) that
\[
\frac{1}{4\min\{1, m^2\}}(BA^{-1}B - A)(A^{-1} - B^{-1}AB^{-1})A \\
\geq S(A|B) - \frac{1}{2}(1_H - (AB^{-1})^2)A \\
\geq \frac{1}{4\max\{1, M^2\}}(BA^{-1}B - A)(A^{-1} - B^{-1}AB^{-1})A \geq 0.
\tag{3.15}
\]
and
\[
\frac{1}{4}\max\{1, M^2\}(1_H - (AB^{-1})^2)^2 A \\
\geq S(A|B) - \frac{1}{2}(1_H - (AB^{-1})^2)A \\
\geq \frac{1}{4}\min\{1, m^2\}(1_H - (AB^{-1})^2)^2 A \geq 0.
\tag{3.16}
\]

We have the following:
**Lemma 2.** Let \( x \in [m, M] \) and \( t > 0 \), then we have

\[
\frac{1}{2 \min \{1, m^t\}} t \left( \frac{x^t - 1}{t} \right)^2 \geq \frac{x^t - 1}{t} - \ln x
\]

\[
\geq \frac{1}{2 \max \{1, M^t\}} t \left( \frac{x^t - 1}{t} \right)^2
\]

(3.17)

and

\[
\frac{1}{2 \max \{1, M^t\}} \left( \frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right) \geq \frac{x^t - 1}{t} - \ln x
\]

\[
\geq \frac{1}{2 \min \{1, m^t\}} \left( \frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right).
\]

(3.18)

**Proof.** Let \( y = x^t \in [m^t, M^t] \). By using the inequality (2.15) we have (3.17) and by (2.17) we have (3.18). \( \square \)

We also have:

**Theorem 4.** Let \( A, B \) be two positive invertible operators and the constants \( M > m > 0 \) with the property (3.3). Then for any \( t > 0 \) we have

\[
\frac{1}{2 \min \{1, m^t\}} t T_t(A|B) A^{-1} T_t(A|B)
\]

\[
\geq T_t(A|B) - S(A|B)
\]

(3.19)

and

\[
\frac{1}{2 \max \{1, M^t\}} T_t(A|B)(1_H - (A^*_t B)^{-1} A)
\]

\[
\geq T_t(A|B) - S(A|B)
\]

(3.20)

\[
\geq \frac{1}{2} \min \{1, m^t\} T_t(A|B)(1_H - (A^*_t B)^{-1} A) \geq 0.
\]

**Proof.** If we use the inequality (3.17) for the selfadjoint operator \( X = A^{-1/2} B A^{-1/2} \) that has its spectrum contained in the interval \([m, M]\), then we...
get

\[
\frac{1}{2 \min\{1, m^t\}} t \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right)^2 \geq \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \ln (A^{-1/2}BA^{-1/2})
\]

\[
\geq \frac{1}{2 \max\{1, M^t\}} t \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right)^2 \geq 0
\]

for any \( t > 0 \).

If we multiply both sides of this inequality by \( A^{1/2} \) we get

\[
\frac{1}{2 \min\{1, m^t\}} t A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right)^2 A^{1/2}
\]

\[
\geq A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right) - A^{1/2} - A^{1/2} \left( \ln (A^{-1/2}BA^{-1/2}) \right) \ A^{1/2}
\]

\[
\geq \frac{1}{2 \max\{1, M^t\}} t A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right)^2 A^{1/2} \geq 0
\]

for any \( t > 0 \).

Since

\[
A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right) A^{1/2} = T_t(A|B),
\]

then

\[
\left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right) = A^{-1/2}T_t(A|B)A^{-1/2}
\]

and

\[
A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right)^2 A^{1/2}
\]

\[
= A^{1/2}A^{-1/2}T_t(A|B)A^{-1/2}A^{-1/2}T_t(A|B)A^{-1/2}A^{1/2}
\]

\[
= T_t(A|B)A^{-1/2}T_t(A|B)
\]

for any \( t > 0 \).
By making use of (3.21) we then get (3.19).
By using inequality (3.18) we have

\[
\frac{1}{2} \max\{1, M^t\} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \frac{1 - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right)
\]

\[
\geq \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \ln(A^{-1/2}BA^{-1/2})
\]

\[
\geq \frac{1}{2} \min\{1, m^t\} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \frac{1 - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right) \geq 0,
\]

for any \( t > 0 \).

If we multiply both sides of this inequality by \( A^{1/2} \) we get

\[
\frac{1}{2} \max\{1, M^t\} A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \frac{1 - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right) A^{1/2}
\]

\[
\geq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} A^{1/2} - A^{1/2} \left( \ln(A^{-1/2}BA^{-1/2}) \right) A^{1/2}
\]

\[
\geq \frac{1}{2} \min\{1, m^t\} A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} - \frac{1 - (A^{-1/2}BA^{-1/2})^{-t}}{t} \right) A^{1/2}
\]

\[
\geq 0
\]

for any \( t > 0 \), and the inequality (3.20) is obtained.

For \( t = 1 \) we get from (3.19) and (3.20) that

\[
\frac{1}{2\min\{1, m\}} (B - A)A^{-1}(B - A)
\]

\[
\geq B - A - S(A|B)
\]

\[
\geq \frac{1}{2} \max\{1, M\} (B - A)A^{-1}(B - A) \geq 0
\]

and

\[
\frac{1}{2} \max\{1, M\} (B - A)(1_H - B^{-1}A)
\]

\[
\geq B - A - S(A|B)
\]

\[
\geq \frac{1}{2} \min\{1, m\} (B - A)(1_H - B^{-1}A) \geq 0.
\]
For $t = 1/2$ we get from (3.19) and (3.20) that
\[
\frac{1}{\min \{1, \sqrt{m}\}} (A^*B - A) A^{-1} (A^*B - A)
\geq 2(A^*B - A) - S(A|B) \quad (3.24)
\geq \frac{1}{\max \{1, \sqrt{M}\}} (A^*B - A) A^{-1} (A^*B - A) \geq 0
\]
and
\[
\max \{1, \sqrt{M}\} (A^*B - A)(1_H - (A^*B)^{-1}A)
\geq 2(A^*B - A) - S(A|B) \quad (3.25)
\geq \min \{1, \sqrt{m}\} (A^*B - A)(1_H - (A^*B)^{-1}A) \geq 0.
\]

For $t = 2$ we get from (3.19) and (3.20) that
\[
\frac{1}{4 \min \{1, m^2\}} (BA^{-1}B - A) A^{-1} (BA^{-1}B - A)
\geq \frac{1}{2} (BA^{-1}B - A) - S(A|B) \quad (3.26)
\geq \frac{1}{4 \max \{1, M^2\}} (BA^{-1}B - A) A^{-1} (BA^{-1}B - A) \geq 0
\]
and
\[
\frac{1}{4} \max \{1, M^2\} (BA^{-1}B - A) \left(1_H - (B^{-1}A)^2\right)
\geq \frac{1}{2} (BA^{-1}B - A) - S(A|B) \quad (3.27)
\geq \frac{1}{4} \min \{1, m^2\} (BA^{-1}B - A) \left(1_H - (B^{-1}A)^2\right) \geq 0.
\]

4. Some global bounds

For $[m, M] \subset (0, \infty)$ and $t > 0$ and by the use of (2.18) we define
\[
U_t(m, M) := U(m^t, M^t)
= \begin{cases} 
\frac{(m^t-1)^2}{m^t} & \text{if } M < 1, \\
\max \left\{ \frac{(m^t-1)^2}{m^t}, \frac{(M^t-1)^2}{M^t} \right\} & \text{if } m \leq 1 \leq M, \\
\frac{(M^t-1)^2}{M^t} & \text{if } 1 < m,
\end{cases}
\]
and by (2.19)

\[
\begin{align*}
  u_t(m, M) := u(m^t, M^t) &= \begin{cases} 
    \frac{(1-M^t)^2}{M^t} & \text{if } M < 1, \\
    0 & \text{if } m \leq 1 \leq M, \\
    \frac{(m^t-1)^2}{m^t} & \text{if } 1 < m.
  \end{cases}
\end{align*}
\]

(4.2)

By (2.20) and (2.21) we have for \( y = x^t \in [m^t, M^t] \) and \( t > 0 \) that

\[
\frac{1}{2t \min\{1, m^t\}} U_t(m, M) \geq \ln x - \frac{1 - x^{-t}}{t}
\]

\[
\geq \frac{1}{2t \max\{1, M^t\}} u_t(m, M)
\]

(4.3)

and

\[
\frac{1}{2t} \max\{1, m^t\} U_t(m, M) \geq \frac{x^t - 1}{t} - \ln x
\]

\[
\geq \frac{1}{2t} \min\{1, m^t\} u_t(m, M)
\]

(4.4)

where \( x \in [m, M] \) and \( t > 0 \).

Using (2.22) and (2.23) we define

\[
Z_t(m, M) := Z(m^t, M^t)
\]

\[
= \begin{cases} 
    (1 - m^t)^2 & \text{if } M < 1, \\
    \max\{(1 - m^t)^2, (M^t - 1)^2\} & \text{if } m \leq 1 \leq M, \\
    (M^t - 1)^2 & \text{if } 1 < m.
  \end{cases}
\]

(4.5)

and

\[
z_t(m, M) := z(m^t, M^t) = \begin{cases} 
    (1 - M^t)^2 & \text{if } M < 1, \\
    0 & \text{if } m \leq 1 \leq M, \\
    (m^t - 1)^2 & \text{if } 1 < m.
  \end{cases}
\]

(4.6)

By (2.24) we have for \( y = x^t \in [m^t, M^t] \) and \( t > 0 \) that

\[
\frac{1}{2t \min\{1, m^t\}} Z_t(m, M) \geq \frac{x^t - 1}{t} - \ln x
\]

\[
\geq \frac{1}{2t \max\{1, M^t\}} z_t(m, M)
\]

(4.7)
where \( x \in [m, M] \) and \( t > 0 \).

Utilising (2.25) and (2.26) we can define

\[
W_t(m, M) := W(m^t, M^t)
\]

\[
= \begin{cases} 
(\frac{1-m^t}{m^t})^2 & \text{if } M < 1, \\
\max \left\{ (\frac{1-m^t}{m^t})^2, (\frac{M^t-1}{m^t})^2 \right\} & \text{if } m \leq 1 \leq M, \\
(\frac{M^t-1}{M^t})^2 & \text{if } 1 < m,
\end{cases}
\]

and

\[
w_t(m, M) := W(m^t, M^t) = \begin{cases} 
(\frac{1-M^t}{M^t})^2 & \text{if } M < 1, \\
0 & \text{if } m \leq 1 \leq M, \\
(\frac{m^t-1}{m^t})^2 & \text{if } 1 < m.
\end{cases}
\]

By (2.24) we have for \( y = x^t \in [m^t, M^t] \) and \( t > 0 \) that

\[
\frac{1}{2t} \min \{1, M^t\} W_t(m, M) \geq \ln x - \frac{1 - x^{-t}}{t} \geq \frac{1}{2t} \min \{1, m^t\} w_t(m, M),
\]

where \( x \in [m, M] \) and \( t > 0 \).

**Theorem 5.** Let \( A, B \) be two positive invertible operators and the constants \( M > m > 0 \) with the property (3.3). Then for any \( t > 0 \) we have

\[
\frac{1}{2t} \min \{1, m^t\} U_t(m, M)A \geq S(A|B) - T_t(A|B)(A^*_tB)^{-1}A
\]

\[
\geq \frac{1}{2t} \max \{1, M^t\} u_t(m, M)A,
\]

\[
\frac{1}{2t} \max \{1, M^t\} W_t(m, M)A \geq S(A|B) - T_t(A|B)(A^*_tB)^{-1}A
\]

\[
\geq \frac{1}{2t} \min \{1, m^t\} w_t(m, M)A,
\]

\[
\frac{1}{2t} \min \{1, m^t\} Z_t(m, M)A \geq T_t(A|B) - S(A|B)
\]

\[
\geq \frac{1}{2t} \max \{1, M^t\} z_t(m, M)A.
\]
and

\[ \frac{1}{2t} \max \{1, M^t\} U_t(m, M)A \geq T_t(A\mid B) - S(A\mid B) \]

\[ \geq \frac{1}{2t} \min \{1, m^t\} u_t(m, M)A. \]

The proof follows by the inequalities (4.4), (4.5), (4.7) and (4.10) in a similar way as the one from the proof of Theorem 3 and we omit the details.

For \( t = 1, t = 1/2 \) and \( t = 2 \) one can obtain some particular inequalities of interest, however the details are not provided here.

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