Quantum versus Classical Online Algorithms with Advice and Logarithmic Space

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Abstract. In this paper, we consider online algorithms. Typically the model is investigated with respect to competitive ratio. We consider algorithms with restricted memory (space) and explore their power. We focus on quantum and classical online algorithms. We show that there are problems that can be better solved by quantum algorithms than classical ones in a case of logarithmic memory. Additionally, we show that quantum algorithm has an advantage, even if deterministic algorithm gets advice bits. We propose “Black Hats Method”. This method allows us to construct problems that can be effectively solved by quantum algorithms. At the same time, these problems are hard for classical algorithms. The separation between probabilistic and deterministic algorithms can be shown with a similar method.

Keywords: quantum computing, online algorithms, quantum vs classical, quantum models, computational complexity, automata, branching programs, obdd, decision diagrams

1 Introduction

Online algorithms are well-known as a computational model for solving optimization problems. The defining property of this model is that algorithm reads an input piece by piece and should return output variables after some of input variables immediately, even if the answer depends on whole input. An online algorithm should return an output for minimizing an objective function. There are different methods to define the effectiveness of algorithms \cite{19,20,22}. But the most standard is the competitive ratio \cite{29,39}. It is a ratio between output’s price for an online algorithm and optimal offline algorithm. We focus on a new model of online algorithms, \textit{quantum online algorithms} that use a power of quantum computing for solving online minimization problem. It was introduced in \cite{31,32}.

Typically, online algorithms has unlimited computational power and the main restriction is lack of knowledge on future input variables. And there are many

\* This work was supported by ERC Advanced Grant MQC. The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.
problems that can be formulated in this terms. At the same time it is quite
interesting to solve online minimization problem in a case of big input stream.
We consider big stream such that it cannot be stored in a memory. In that case we
can discuss online algorithms with restricted memory. We can find such problems,
for example on data streams processing problems. In the paper we allow online
algorithm to use only $s$ bits of memory, for given integer $s$. Another point of view
of the same model is streaming algorithms for an online minimization problem.
And such classical models were considered in $[10,26,21,31]$. Automata for online
minimization problems were considered in $[32]$. 

It is already known that quantum online algorithms can perform better than
classical ones in a case of logarithmic memory $[31]$. At the same time it is inter-
esting to consider logarithmic space (memory). In this case only quantum online
algorithms with repeated test was considered in $[41]$. 

We also interested in advice complexity measure $[34,17,18]$. In this model online
algorithm gets some bits of advice about an input. Trusted Adviser sending these
bits knows the whole input and has an unlimited computational power. The
question is “how many advice bits are enough to reduce competitive ratio or to
make the online algorithm as the same effective as the offline algorithm in the
worst case?”. This question has different interpretations. One of them is “How
many information an algorithm should know about a future for solving a problem
effectively?”. Another one is “If we have an expensive channel which can be used
for pre-processed information about the future, then how many bits we should
send by this channel to solve a problem effectively?”. Researchers pay attention
to deterministic and probabilistic or randomized online algorithms with advice
$[27,33,36,34]$. 

In the paper we present “Black Hats Method” for constructing hard online
minimization problems. Using this method we suggest problems that shows fol-
lowing separation between power of quantum and classical algorithms:

– There is a problem (BHR) that has quantum online algorithm that uses
$O(\log n)$ qubits with better competitive ratio than any classical (randomize
or deterministic) online algorithms that use $(\log_2 n)^{O(1)}$ bits of memory.
Additionally, we show that if deterministic online algorithm with $(\log_2 n)^{O(1)}$
bits of memory gets advice then quantum online algorithms still shows better
competitive ratio. The problem is based on $R$ function and results from $[38]$. 

– There is a problem (BHE) that has quantum and randomize online algo-
rithms that use $O(\log n)$ qubits with better competitive ratio than any de-
terministic online algorithms even if it has no restriction on memory. Addition-
ally, we show that if deterministic online algorithm with $(\log_2 n)^{O(1)}$
bits of memory gets advice then quantum or randomized online algorithms still
shows better competitive ratio. The problem is based on Equality function
and results from $[57,6,12,13,11,23]$. 

Online algorithms with restricted memory are similar to streaming algorithms
$[35,26]$, Branching programs $[40]$ and automata $[14,15,14]$. Researchers also compare
classical and quantum cases for these models $[15,12,13,8,30,14,15,35,2,28,25]$. 

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The paper is organized in the following way. We present definitions in Section 2. In Section 3 we suggest Black Hats Method. Discussion Quantum and Randomize vs Deterministic online algorithms in Section 4. Last Section 5 contains results on Quantum vs Classical Online Algorithms.

2 Preliminaries

Firstly, let us define the online optimization problem. All following definitions we give with respect to [34,31]. Also the definitions are agreed with [41].

**Definition 1 (Online Minimization Problem).** An online minimization problem consists of a set $I$ of inputs and a cost function. Every input $I \in I$ is a sequence of requests $I = (x_1, \ldots, x_n)$. Furthermore, a set of feasible outputs (or solutions) is associated with every $I$; every output is a sequence of answers $O = (y_1, \ldots, y_n)$. The cost function assigns a positive real value $\text{cost}(I, O)$ to every input $I$ and any feasible output $O$. For every input $I$, we call any feasible output $O$ for $I$ that has the smallest possible cost (i.e., that minimizes the cost function) an optimal solution for $I$.

Let us define an online algorithm for this problem as an algorithm which gets requests $I$ one by one and should return answers $O$ immediately, even if optimal solution can depend on future requests.

**Definition 2 (Deterministic online algorithm).** Consider an input $I$ of an online minimization problem. An online algorithm $A$ computes the output sequence $A(I) = (y_1, \ldots, y_n)$ such that $y_i$ is computed from $x_1, \ldots, x_i, y_1, \ldots, y_{i-1}$. We denote the cost of the computed output by $\text{cost}(A(I)) = \text{cost}(I, A(I))$.

This setting can also be regarded as a request-answer game: an adversary generates requests, and an online algorithm has to serve them one at a time [10].

We use competitive ratio as a main measure of quality of the online algorithm. It is the ratio of costs of the algorithm’s solution and a solution of an optimal offline algorithm, in the worst case.

**Definition 3 (Competitive Ratio).** Online algorithm $A$ is $c$-competitive if there exists a non-negative constant $\alpha$ such that, for every input $I$, we have: $\text{cost}(A(I)) \leq c \cdot \text{cost}(\text{Opt}(I)) + \alpha$, where $\text{Opt}$ is an optimal offline algorithm for the problem. We also call $c$ the competitive ratio of $A$. If $\alpha = 0$, then $A$ is called strictly $c$-competitive; $A$ is optimal if it is strictly $1$-competitive.

Let us define an Online Algorithm with Advice. We can save that advice is some information about future input.

**Definition 4 (Online Algorithm with Advice).** Consider an input $I$ of an online minimization problem. An online algorithm $A$ with advice computes the output sequence $A^\phi(I) = (y_1, \ldots, y_n)$ such that $y_i$ is computed from $\phi, x_1, \ldots, x_i$, where $\phi$ is the message from Adviser, who knows the whole input. $A$ is $c$-competitive with advice complexity $b = b(n)$ if there exists a non-negative constant $\alpha$ such that, for every $n$ and for any input $I$ of length at most $n$, there exists some $\phi$ such that $\text{cost}(A^\phi(I)) \leq c \cdot \text{cost}(\text{Opt}(I)) + \alpha$ and length of $\phi$ is at most $b$ bits.
Next, let us define a randomized online algorithm.

**Definition 5 (Randomized Online Algorithm).** Consider an input $I$ of an online minimization problem. A randomized online algorithm $R$ computes the output sequence $R^I := R^I(I) = (y_1, \ldots, y_n)$ such that $y_i$ is computed from $\psi, x_1, \ldots, x_i$, where $\psi$ is the content of a random tape, i.e., an infinite binary sequence, where every bit is chosen uniformly at random and independently of all the others. By $\text{cost}(R^I(I))$ we denote the random variable expressing the cost of the solution computed by $R$ on $I$. $R$ is $c$-competitive in expectation if there exists a non-negative constant $\alpha$ such that, for every $I$, $E[\text{cost}(R^I(I))] \leq c \cdot \text{cost}(\text{Opt}(I)) + \alpha$, where $\text{Opt}$ is an optimal offline algorithm for the problem.

Now we are ready to define a quantum online algorithm. You can read more about quantum computation in [15].

**Definition 6 (Quantum Online Algorithm).** Consider an input $I$ of an online minimization problem. A quantum online algorithm $Q$ computes the output sequence $Q(I) = (y_1, \ldots, y_n)$ such that $y_i$ is computed from $x_1, \ldots, x_i$. The algorithm can measure qubits several times during computation. By $\text{cost}(Q(I))$ we denote the cost of the solution computed by $Q$ on $I$. Note that quantum computation is probabilistic process. $Q$ is $c$-competitive in expectation if there exists a non-negative constant $\alpha$ such that, for every $I$, $E[\text{cost}(Q(I))] \leq c \cdot \text{cost}(\text{Opt}(I)) + \alpha$, where $\text{Opt}$ is an optimal offline algorithm for the problem.

Let us describe computational process of quantum online algorithm that use $q$ qubits of memory. For a given $n > 0$, a quantum online algorithm $P$ with $q$ qubits defined on input $\{0, \ldots, d-1\}^n$ and outputs variables $\{0, \ldots, t-1\}^m$, is a 3-tuple $P = (T, |\psi\rangle_0, \text{Result})$, where $T = \{T_j : 1 \leq j \leq n\}$ and $T_j = (G_j^0, \ldots, G_j^{d-1})$ are (left) unitary matrices representing the transitions. Here $G_j^0, \ldots, G_j^{d-1}$ is applied on the $j$-th step. And a choice is determined by the input variable. $|\psi\rangle_0$ is a initial vector from $2^q$-dimensional Hilbert space over the field of complex numbers. $\text{Result} = \{\text{Result}_1, \ldots, \text{Result}_n\}$, where $\text{Result}_1 : \{0, \ldots, 2^q-1\} \to \{0, \ldots, t-1\}$ is function that converts result of measurement to output variable.

For any given input $\nu \in \{0, \ldots, d-1\}^n$, the computation of $A$ on $\nu$ can be traced by a $2^q$-dimensional vector from Hilbert space over the field of complex numbers. The initial one is $|\psi\rangle_0$. In each step $j$, $1 \leq j \leq n$, the input variable $x_j$ is tested and then the corresponding unitary operator is applied: $|\psi\rangle_j = G_j^{x_j}(|\psi\rangle_{j-1})$, where $|\psi\rangle_j$ represents the state of the system after the $j$-th step, for $1 \leq j \leq n$. We can measure one of qubits or more. Let the program was in state $|\psi\rangle = (v_1, \ldots, v_{2^q})$ before measurement and let us measure the $i$-th qubit. And let states with numbers $j_1^0, \ldots, j_{2^q-1}^0$ correspond to 0 value of the $i$-th qubit, and states with numbers $j_1^1, \ldots, j_{2^q-1}^1$ correspond to 1 value of the $i$-th qubit. The result of measurement of $i$-th qubit is 1 with probability $p_{r_1} = \sum_{u=1}^{w/2} |v_{ju}|^2$ and 0 with probability $p_{r_0} = 1 - p_{r_1}$. The algorithm can measure qubits on any step after unitary transformation. If $A$ measured $v$ qubits then as result it gets number $u \in \{0, \ldots, 2^v - 1\}$. In some step $j$, with respect to definition of problem, the algorithm returns $\text{Result}_j(v)$. 


Let us define online algorithms with restricted memory. Let deterministic online algorithm $A$ be an algorithm which uses at most $s$ bits of memory on processing any input $I$. We can define similar restrictions for randomized algorithms and algorithms with advice. Let quantum online algorithm $Q$ be an algorithm using at most $t$ quantum bits of memory on processing any input $I$.

Another point of view on online algorithms with restricted memory is considering them as streaming algorithms for online minimization problem. You can read more about streaming algorithm in literature \[37,9\]. Shortly it is algorithms that use small size of memory and read input variables one by one.

At the same time, there is OBDD models and automata models that are good abstractions for streaming algorithms. You can read more about classical and quantum OBDDs in \[40,38,3,4,5,1,30\]. Formal definition of OBDDs and automata in Abstract A. Following relations between automata, id-OBDDs (OBDD that reads input variables in natural order) and streaming algorithms is folklore:

**Lemma 1.** Following claims are right:

1. Let a quantum (probabilistic) id-OBDD $P$ with width $2^w$ computes Boolean function $f$ then there is a quantum (randomize) streaming algorithm computing $f$ that uses $w + \lceil \log_2 n \rceil$ qubits.
2. Let a quantum automaton $A$ with size $2^w$ recognizes a language $L$, then there is a quantum streaming algorithm recognizing $L$ that uses $w$ qubits.
3. If any deterministic (probabilistic) id-OBDD $P$ that computing a Boolean function $f$ has width at least $2^w$ then any deterministic (randomize) streaming algorithm computing $f$ uses at least $w + \lceil \log_2 n \rceil$ bits.
4. If any deterministic (probabilistic) automaton $A$ that recognizes a language $L$ has width at least $2^w$, then any deterministic (randomize) streaming algorithm recognizing $L$ uses at least $w$ bits.

We use property of Boolean functions called "subfunctions". Let us define it.

Let $\theta_u = (\{1, \ldots, u\}, \{u+1, \ldots, n\}) = (X_A, X_B)$ be a partition of set $X$ into two parts, for $1 \leq u \leq n$. Below we will use equivalent notations for Boolean functions $f(X)$ and $f(X_A, X_B)$. Let $f_\sigma$ be a subfunction of Boolean function $f$ for some partition $\theta_u$, where $\sigma \in \{0, 1\}^u$. Function $f_\sigma$ is obtained from $f$ by fixing $x_1 = \sigma_1, \ldots, x_{|X_A|} = \sigma_{|X_A|}$, then $f_\sigma(X_B) = f(\sigma, X_B)$. Let $N^u(f)$ be number of different subfunctions with respect to partition $\theta_u$.

### 3 Black Hats Method

Let us define a method which allows to construct hard online minimization problems. We call it “Black Hats Method”. It is generalization of $PNH$ problem from \[31\].

Let we have a Boolean function $f : \{0, 1\}^m \rightarrow \{0, 1\}$, then online minimization problem $BH^k_{r, w, t}(f)$, for integers $k, r, w, t$, where $k \mod t = 0$, is following:

We have $k$ guardians and $k$ prisoners. They stay one by one in a line like $GPGP \ldots$, where $G$ is guardian, $P$ is prisoner. Prisoner $P_i$ has input $X_i$ of
length $m$ and computes function $f(X_i)$. If result is 1 then it paints his hat in black, otherwise he paints it in white. Each guardian wants to know a parity for number of following black hats. We separate sequential guardians into $t$ blocks. The cost of a block is $r$ if all guardians of the clock are right, and $w$ otherwise. We want to minimize the cost of output.

Let us define the problem formally:

**Definition 7 (Black Hats Method).** We have a Boolean function $f_m : \{0, 1\}^m \rightarrow \{0, 1\}$, then online minimization problem $BH^I_{k,r,w}(f)$, for integers $k, r, w, t$, where $k \mod t = 0$, is following:

Suppose that the input $I = (x_1, \ldots, x_n)$ of length $n$ and $k$ positive integers $m_1, \ldots, m_k$ such that $n = \sum_{i=1}^{k}(m_i + 1)$. Let $I$ is always such that $I = 2, X_1, 2, X_2, 2, X_3, 2, \ldots, 2, X_k$, where $X_i \in \{0, 1\}^{m_i}$, for $i \in \{1, \ldots, k\}$. Let $O$ be an output and $O' = y_1, \ldots, y_k$ be an output bits corresponding to input variables with value 2 (another words output variables $I$ for guardians). Output $y_i$ corresponds to input variable $x_i$, where $i = j + \sum_{r=1}^{j-1}(m_r)$. Let $z_j(I) = \bigoplus_{r=1}^{k} f_m(X_i)$

Let us separate all output variables $y_i$ to $t$ blocks of length $z = k/t$. The cost of $i$-th block is $c_i$. Where $c_i = r$, if $y_i = z_j$ for $j \in \{(i-1)z + 1, \ldots, i \cdot z\}$; and $c_i = w$ otherwise.

Cost of the whole output is $cost'(I, O) = c_1 + \cdots + c_t$.

We also consider two specific cases of the problem: $BH^I_{k,r,w}(f) = BH^I_{k,t,r}(f)$ and $BH^I_{k,r,w}(f) = BH^I_{k,r,w}(f)$

We have following complexity results for $BH^I_{k,r,w}(f)$ problem:

**Theorem 1.** Let Boolean function $f$ is such that there are no deterministic streaming algorithms that compute $f$ using space less than $s$ bits. Then for any deterministic online algorithm $A$ using space less than $s$ bits and solving $BH^I_{k,r,w}(f)$, there is input $I_A$ such that $cost'(I_A, A(I_A)) = tw$ for any $t \in \{1, \ldots, k\}$. And there are no deterministic online algorithm that $c$-competitive, where $c < w/r$.

**Proof.** Let us consider any online algorithm $A$ for $BH^I_{k,r,w}(f)$ problem. Let $A$ returns value $y_1$ for the first input bit $x_1$. Note that by the definition of the problem $x_1 = 2$.

Let us prove that there is part of input $X_1^0 \in \{0, 1\}^{m_1}$ such that $A$ returns $y_2 = 0$ and $X_1^1 \in \{0, 1\}^{m_1}$ such that $A$ returns $y_2 = 1$, but at the same time $f(X_1^0) = f(X_1^1) = \sigma_1$, for some $\sigma_1 \in \{0, 1\}$.

Assume that there is no such triple $(\sigma_1, X_1^0, X_1^1)$. Then it means that we can construct offline algorithm $A'$ that uses space less than $s$ and such that $A'(X_1^i) = A'(X_1^i)$ iff $f(X_1^i) = f(X_1^i)$. Algorithm $A'$ emulates work of algorithm $A$. It means that $A$ computes $f$ or $-f$. It is contradiction with the claim of the theorem.

Let us choose $X_1 = X_1^{y_1 \oplus \sigma_1}$. In that case $A$ returns $y_2 = y_1 \oplus \sigma_1$ on request of the second guardian.

Let us prove that for $i \in \{2, \ldots, k\}$ there is part of input $X_i^0 \in \{0, 1\}^{m_i}$ such that $A$ returns $y_{i+1} = 0$ and $X_i^1 \in \{0, 1\}^{m_i}$ such that $A$ returns $y_{i+1} = 1$, but at the same time $f(X_i^0) = f(X_i^1) = \sigma_i$, for some $\sigma_i \in \{0, 1\}$.
Assume that there are no such triple \((\sigma_i, X^0_i, X^1_i)\). Then it means that we can construct streaming algorithm \(A'\) that uses space less than \(s\) and such that \(A'(X^0_i) = A'(X^1_i)\) iff \(f(X^0_i) = f(X^1_i)\). Algorithm \(A'\) emulates work of algorithm \(A\). It means that \(A'\) computes \(f\) or \(-f\). It is contradiction with the claim of the theorem.

Let us show that we can suggest an input \(\sigma\).

Let us consider an input \(x_1 = X^0_1 \oplus \sigma_1\). In that case \(A\) returns \(y_t = y_i \oplus \sigma_i\) on the request of the \((i + 1)\)-th guardian.

Let us consider an input \(x_t = 2X_12X_12 \ldots 2X_k\), where \(X_k\) is such that \(\sigma_k = f(X_k) \neq y_k\), or \(\sigma_k = y_k \oplus 1\).

Optimal offline solution is \(z_1, \ldots, z_k\) where \(z_j = \bigoplus_{i=j}^k \sigma_i\)

Let us prove that \(z_j \neq y_j\) for each \(j \in \{1, \ldots, k\}\) by induction. \(y_k \neq z_k\) by definition of \(X_k\). Let us show that if \(y_j = y_j'\) then \(y_{j-1} = y_{j-1}'\).

\(z_{j-1} = \sigma_{j-1} \oplus z_j\) and \(y_j = \sigma_{j-1} \oplus y_{j-1}\) or \(y_{j-1} = \sigma_{j-1} \oplus y_j\). \(y_j \neq z_j\) means that \(z_{j-1} = \sigma_{j-1} \oplus z_j \neq \sigma_{j-1} \oplus y_j = y_{j-1}\).

Therefore, all answers are wrong and \(cost^t(I, A(I)) = tw\), for any \(t \in \{1, \ldots, k\}\). So competitive ratio \(c\) is at least \(c \geq tw/(tr) = w/r\).

**Theorem 2.** Let Boolean function \(f\) is such that there are no randomized streaming algorithms that compute \(f\) with bounded error using space less than \(s\) bits. Then for any randomized online algorithm \(A\) using space less than \(s\) bits and solving \(BH^w_{k, r, w}(f)\), there is input \(I, A\) such that for any \(t \in \{1, \ldots, k\}\) and \(z = k/t\) we have: \(cost^t(I, A(I)) = t \cdot ((1 - 2^{-w})w + 2^{r-2}r)\). And there are no randomized online algorithm that \(c\)-competitive, where \(c < 2^{-w} + (1 - 2^{-w})w/r\).

**Proof.** By the way similar to proof of Theorem 1 we can show that algorithm \(A\) cannot compute any \(y_i\) with bounded error. It means that only way to answer is guessing \(y_i\) with probability 1/2. So, the cost of \(i\)-th block \(c_i = (1 - 2^{-w})w + 2^{-2}r\), because the algorithm should guess all output bits for getting cost \(r\). Therefore, \(cost^t(I, A(I)) = t \cdot ((1 - 2^{-w})w + 2^{r-2}r)\). And competitive ratio \(c\) is \(c \geq t \cdot ((1 - 2^{-w})w + 2^{r-2}r)/(tr) = 2^{-w} + (1 - 2^{-w})w/r\).

We also have bound for competitive ratio in case when the algorithm has unlimited computational power.

**Theorem 3.** There is no deterministic online algorithm \(A\) even without restricted memory computing \(BH^w_{k, r, w}(f)\) that is \(c\)-competitive, for \(c < [(t + 1)/2] \cdot w + [(t - [(t + 1)/2]) \cdot r]/(tr)\). Particularly, there is no algorithm for \(BH^1_{k, r, w}(f)\) that is \(c\)-competitive, for \(c < w/r\); and no algorithm for \(BH^2_{k, r, w}(f)\) that is \(c\)-competitive, for \(c < w/r\), odd \(k\).

**Proof.** Let us show that we can suggest an input \(I\) such that at least \([(k + 1)/2] \cdot w + [(t - [(t + 1)/2]) \cdot r]/(tr)\) guardians returns wrong answers.

Let the algorithm \(A\) receives the input \(I = (x_1, \ldots, x_n) = (2, X_1, 2, X_2, 2, \ldots, 2, X_k)\), such that \(X_i, \in \{0, 1\}^m\). Let \(X_1, \ldots, X_k\) be such that \(f(X_i) = 0\) for \(i \in \{1, \ldots, k - 1\}\).

Then \(A\) receives part \((2, X_1, 2, X_2, 2, \ldots, 2, X_{k-1}, 2)\) of the input and returns \(y_1, \ldots, y_{k-1}, y_k\). Let \(b = 1\), if \(y_1 + \ldots + y_k \geq [(k + 1)/2]\); and \(b = 0\), otherwise.
Then we choose $X_k$ such that $f(X_k) \neq b$. In that case $z_1 = \cdots = z_k = f(X_k) \neq b$. Therefore at least $|\left((k + 1)/2 \right)|$ guardians return wrong answers.

Worst case is first $\left((t + 1)/2 \right)$ guardians returns wrong answer. So, $\left((t + 1)/2 \right)$ blocks will be ”wrong”. And competitive ratio $c \geq \left((t + 1)/2 \right) \cdot w + \left((t - (t + 1)/2) \right) \cdot r)/(tr)$. Hence $\text{cost}^1(I, A(I)) = w$ and $A$ is $(w/r)$-competitive in case of $BH^1_{k,r,w}(f)$. And $\text{cost}^2(I, A(I)) = w$, for odd $k$, and $A$ is $(w/r)$-competitive in case of $BH^2_{k,r,w}(f)$.

**Theorem 4.** Let Boolean function $f$ such that there is randomize (quantum) offline streaming algorithm $R$ that is compute $f$ with bounded error $\varepsilon$ using space $s$ bits, where $0 \leq \varepsilon < 0.5$. Then there is probabilistic (quantum) online algorithm $A$ using space at most $s + 1$ bits (qubits) solving $BH^r_{k,r,w}(f)$ such that for any input $I_A$ and $v = (2\varepsilon - 1)^2, z = k/t$ we have

$$\mathbb{E} \left[ \text{cost}^t(I, O) \right] = \frac{(1 - \varepsilon)^{z-1}}{4} \left( (1 + (-1)^z) z + (1 - (-1)^z) \frac{v^{2\lfloor t/2 \rfloor + 2} - 1}{v^3 - v} \right) (v - w) + tw$$

**Proof.** We will use following online algorithm $A$:

**Step 1** $A$ guesses first bit of output with equal probability. The algorithm stores current result in bit $p$. For randomize case: $p = 1$ or $p = 0$ with probability 0.5; for quantum case we use Hadamar-Walsh transformation. Then $A$ returns $y_1 = p$, in quantum case measure $p$.

**Step 2** Then the Algorithm reads $X^1$ and computes $p = p \oplus R(X^1)$, where $R(X^1)$ is the result of computation for $R$ on input $X^1$. Then the algorithm returns $y_2 = p$ or result of measurement in quantum case.

**Step i** The Algorithm reads $X^{i - 1}$ and computes $p = p \oplus R(X^{i - 1})$. Then $A$ returns $y_i = p$ or result of measurement in quantum case.

Let us compute cost of output for this algorithm. Let us consider new cost function $\text{cost}'(I, O)$. For this function, a “right” block costs 1 and a “wrong” block costs 0. In that case $\text{cost}'(I, O) = (r - w) \cdot \text{cost}'(I, O) + tw$. Therefore in following proof we can consider only $\text{cost}'(I, O)$ function.

Firstly let us compute $p_1$ the probability that block $i$ is a “right” block or costs 1.

Let $i = 1$. Then, if $i$-th block is “right” then all $z - 1$ prisoners inside block returns right answer and guess of the first guardian is right. A probability of this event is $p_1 = 0.5 \cdot (1 - \varepsilon)^{z-1}$.

Let $i > 1$. If $i$-th block is “right” then two conditions should be true: (i) All $z - 1$ prisoners inside block should return right answers. (ii) A number of preceding guardians that return wrong answer should be even.

Probability of the first condition is $(1 - \varepsilon)^{z-1}$. Let us compute probability of the second condition. If the guess of the first guardian is right then even number of $u = (i - 1)z$ prisoners should have error in computation. A probability of exactly $2j$ errors among $(i - 1)z$ prisoners is $\varepsilon^{2j}(1 - \varepsilon)^{u - 2j} \cdot \binom{u}{2j} = (1 - \varepsilon)^u \cdot \left(\frac{1}{1 - \varepsilon}\right)^{2j} \cdot \binom{n}{2j}$.
So, probability of event that even number of $u = (i - 1)z$ prisoners have error is following:

$$
\sum_{j=0}^{u/2} \left(1 - \varepsilon\right)^u \cdot \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j} \cdot \left(\frac{u}{2j}\right) = (1 - \varepsilon)^u \sum_{j=0}^{u/2} \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j} \cdot \left(\frac{u}{2j}\right)
$$

$$
= (1 - \varepsilon)^u \cdot 0.5 \cdot \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u - \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u
$$

If guess of the first guardian is wrong then odd number of $u = (i - 1)z$ prisoners should have error in computation. Probability of such event is:

- if $u$ is even then

$$
\sum_{j=0}^{u/2} \left(1 - \varepsilon\right)^u \cdot \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j+1} \cdot \left(\frac{u}{2j+1}\right) = (1 - \varepsilon)^u \sum_{j=0}^{u/2} \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j+1} \cdot \left(\frac{u}{2j+1}\right)
$$

$$
= (1 - \varepsilon)^u \cdot 0.5 \cdot \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u - \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u
$$

- if $u$ is odd then

$$
\sum_{j=0}^{(u-1)/2} \left(1 - \varepsilon\right)^u \cdot \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j+1} \cdot \left(\frac{u}{2j+1}\right) = (1 - \varepsilon)^u \cdot \sum_{j=0}^{(u-1)/2} \left(\frac{\varepsilon}{1 - \varepsilon}\right)^{2j+1} \cdot \left(\frac{u}{2j+1}\right)
$$

$$
= (1 - \varepsilon)^u \cdot 0.5 \cdot \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u + \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u
$$

The probability of the event is

$$
p_i = (1 - \varepsilon)^{z-1} \cdot 0.5 \cdot (1 - \varepsilon)^u \cdot 0.5 \cdot \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u - \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u +
$$

$$
+ 0.5 \cdot (1 - \varepsilon)^u \cdot 0.5 \cdot \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u + (-1)^{u+1} \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u =
$$

$$
= (1 - \varepsilon)^{z-1} \cdot 0.25 \cdot (1 - \varepsilon)^u \left(2 \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^u + (1 + (-1)^u+1) \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^u\right) =
$$

$$
= (1 - \varepsilon)^{z-1} \cdot 0.25 \cdot (1 - \varepsilon)^{i-1}z \left(2 \left(\frac{\varepsilon}{1 - \varepsilon} + 1\right)^{(i-1)z} + (1 + (-1)^{(i-1)z+1}) \left(\frac{\varepsilon}{1 - \varepsilon} - 1\right)^{(i-1)z}\right) =
$$

$$
= (1 - \varepsilon)^{z-1} \cdot 0.25 \left(2 + (1 + (-1)^{(i-1)z+1}) (2\varepsilon - 1)^{(i-1)z}\right)
$$
If \( z \) is even then \( p_i = 0.5(1 - \varepsilon)^{z-1} \).

If \( z \) is odd then \( p_i = 0.25(1 + (-1)^i)(1 - \varepsilon)^{z-1}(2\varepsilon - 1)^{(i-1)z} \).

So expected cost is \( \mathbb{E}[cost'(I, O)] = \sum_{i=1}^{t} (p_i \cdot 1 + (1 - p_i) \cdot 0) = p_1 + \sum_{i=2}^{t} p_i \)

if \( z \) is even then \( \mathbb{E}[cost'(I, O)] = 0.5(1 - \varepsilon)^{z-1}t \) if \( z \) is odd then

\[
\mathbb{E}[cost'(I, O)] = 0.5(1-\varepsilon)^{z-1}\left( 1 + \sum_{r=1}^{\lfloor t/2 \rfloor} ((2\varepsilon - 1)^z)^{2r-1} \right) = 0.5(1-\varepsilon)^{z-1}\left( 1 + \frac{v^{2\lfloor t/2 \rfloor+2} - 1}{v^3 - v} \right),
\]

for \( v = (2\varepsilon - 1)^z \).

Therefore we get final formula for \( cost'(I, O) \):

\[
\mathbb{E}[cost'(I, O)] = 0.25(1-\varepsilon)^{z-1}\left( (1 + (-1)^z)t + (1 - (-1)^z) \left( 1 + \frac{v^{2\lfloor t/2 \rfloor+2} - 1}{v^3 - v} \right) \right)
\]

And final formula as in claim of the theorem:

\[
\mathbb{E}[cost'(I, O)] = \frac{(1 - \varepsilon)^{z-1}}{4} \left( (1 + (-1)^z)t + (1 - (-1)^z) \left( 1 + \frac{v^{2\lfloor t/2 \rfloor+2} - 1}{v^3 - v} \right) \right) (r - w) + tw
\]

### 3.1 Advice complexity

Let us consider the model with advice. The \( BH_{k,r,w}^s(f) \) problem has following properties with respect to online algorithms with advice.

**Theorem 5.** Let Boolean function \( f \) is such that there are no deterministic offline streaming algorithms that computes \( f \) using space less than \( s \) bits. Then for any deterministic online algorithm \( A \) using space less than \( s - 2b \) bits, \( b \) advice bits and solving \( BH_{k,r,w}^s(f) \), there is input \( I_A \) such that \( cost'(I_A, A(I_A)) \geq hr + (t - h)w \), for \( h = \lfloor b/z \rfloor, z = k/t \). And competitive ratio \( c \) of the algorithm is \( c \geq \frac{hr + (t - h)w}{t} \).

**Proof.** Let us proof that if the algorithm gets \( b \) advice bits then there is an input such that at least \( k - b \) prisoners returns wrong answer. We prove it by induction.

Firstly, let us prove for \( b = k \). Then Adviser can send all results of guardians and the algorithm returns all answers right.

Secondly, let us prove for \( b = 0 \). It means that we have not any advice and we get situation from Theorem[1]

Thirdly, let us prove for other cases. Assume that for any pair \( (b', k') \) such that \( b' < b \) or \( k' < k \) we already proved the claim. We focus on the first prisoner.

Assume that there is input \( X^1 \in \{0, 1\}^{m_1} \) for the first prisoner that it returns wrong answer, then we use it and get situation for \( k - 1 \) prisoners and \( b \) advice bits. In that case \( k - b - 1 \) prisoners is wrong, plus first one also wrong.

Assume that the algorithm always returns right answer for the first prisoner.

So we can describe the process of communication with Adviser in the following way: Adviser separates all possible inputs in \( 2^b \) non-overlapping groups \( G_1, \ldots, G_{2^b} \) and sends number of the group that contains current input to the
algorithm. After that $A$ processes the input with knowledge that an input can be only from this group.

Let us consider three sets of groups: $I_0 = \{G_i : \forall \sigma \in \{0, 1\}^m \text{ such that } \sigma \text{ is an input for the first prisoner and } f(\sigma) = 0\}$, $I_1 = \{G_i : \forall \sigma \in \{0, 1\}^m \text{ such that } \sigma \text{ is an input for the first prisoner and } f(\sigma) = 1\}$, $I_{10} = \{G_1, \ldots, G_{2^k}\} \setminus (I_1 \cup I_0)$.

Let $|I_0| \neq 0$, for $a \in \{0, 1\}$. If $|I_0| \leq 2^{b-1}$ then as $X^1$ we take any input from any group $G \in I_a$. Hence we have at most $2^{b-1}$ possible groups for Adviser. We can say that Adviser can encode it in $b-1$ bits string. Therefore we get situation for $b-1$ advice bits and $k-1$ prisoners, where the claim is true. If $|I_0| > 2^{b-1}$ then as $X^1$ we take any input from any group $G \notin I_a$. Hence we have at most $2^{b-1}$ possible groups for Adviser and same situation, where the claim is true.

Let $|I_0| = |I_1| = 0$. Suppose that the algorithm $A$ such that it skips $v$ bits and then using advice continue computation. Let $N^v(f_{m_i}) = 2^v > 1$. The algorithm knows about previous input only number of a group that comes from Adviser. Let us see two inputs $\sigma, \sigma' \in \{0, 1\}^v$ such that $\sigma \neq \sigma'$ and $f_\sigma \neq f_{\sigma'}$. Hence there is input $\gamma \in \{0, 1\}^{m_i-v}$ such that $f_\sigma(\gamma) \neq f_{\sigma'}(\gamma)$. Inputs $\sigma \gamma$ and $\sigma' \gamma$ cannot be at the same group of inputs that sends Adviser. Suppose that these two inputs are in a same group. Then the Algorithm process only $\gamma$ and returns same result for both inputs. At the same time $f_{\sigma}(\gamma) \neq f_{\sigma'}(\gamma)$. Let sets $J, J' \subset \{G_1, \ldots, G_{2^k}\}$ be such that all sets from $J$ contain $\sigma \gamma$ as the first prisoner input and all sets from $J'$ contain $\sigma' \gamma$ as the first prisoner input. Let us choose smallest one and choose as $X^1$ this input. So, size of this set is less or equal to $2^{b-1}$. Therefore we have situation for $k-1$ prisoner and $b-1$ advice bits.

Suppose, that the algorithm does not skip any bits. Then let us focus on position $u$ such that $N^u(f_{m_i}) = 2^u$. Let us consider the set of inputs $T \subset \{0, 1\}^u$ such that for any $\sigma, \sigma' \in T, \sigma \neq \sigma'$ we have $f_\sigma \neq f_{\sigma'}$. Hence $|T| = 2^u$. Let the algorithm use $s' < s - 2b$ bits of memory. Therefore after $u$ steps algorithm using $s'$ bits of memory and $b$ bits of advice can separate all inputs from the part before $u$-th variable to $2^{s'+b}$ groups. And inside group following behavior of algorithm will be same. By the Pigeonhole principle there is at least two inputs $\sigma, \sigma' \in T, \sigma \neq \sigma'$ that belongs to one group that obtained by the algorithm. We have $f_\sigma \neq f_{\sigma'}$, it means that there is $\gamma \in \{0, 1\}^{m_i-u}$ such that $f_\sigma(\gamma) \neq f_{\sigma'}(\gamma)$ and $f(\sigma \gamma) \neq f(\sigma' \gamma)$. There is at least one of group such that it contains inputs $\sigma \gamma$ and $\sigma' \gamma$. Therefore, the algorithm will return same result on $\sigma \gamma$ and $\sigma' \gamma$. Hence return wrong result on one of the two inputs. We get contradiction with assumption that the first prisoner return right answer.

The previous discussion means that better strategy for adviser is sending right answers for $b$ guardians. If the algorithm knows answer for $i$-th guardian then it can just ignore result of $(i-1)$-th prisoner.

We can show that guardian that do not get answer from Adviser (“unknown” guardians) will return wrong answers. We use the proof technique as in Theorem 8. We use exactly the same approach for all segments between “known” guardians.

So, because of properties of cost function, better strategy is send information about all guardians of a block. So it means that the Algorithm can get $[b/u]$
full blocks and cost for each of them will be \( r \), for \( u = k/t \). And other blocks have at least one “wrong” guardians and these blocks have cost \( w \). Therefore, we can construct input such that it’s coast is \( \lfloor b/u \rfloor \cdot r + (t - \lfloor b/u \rfloor)w \). And the competitive ratio \( c \) of the algorithm is \( c \geq \frac{\lfloor b/u \rfloor \cdot r + (t - \lfloor b/u \rfloor)w}{tr} \).

In next Corollary we show when advice bit does not help us at all.

**Corollary 1.** Let Boolean function \( f \) such that there are no deterministic offline streaming algorithms that computes \( f \) using space less than \( s \) bits. Then for any deterministic online algorithm \( A \) using space less than \( s - 2b \) bits, \( b \) advice bits and solving \( BH_{k,r,w}^t(f) \), where \( b < z, z = k/t \), there is an input \( I_A \) such that \( \text{cost}^t(I_A, A(I_A)) = tw \). And competitive ratio \( c \) of the algorithm is \( c = w/r \).

### 4 Quantum and Probabilistic vs Deterministic Algorithms

In this section we will use results for OBDDs. Let us apply the Black Hats Method from Section 3 to \( EQ_m : \{0,1\}^n \rightarrow \{0,1\} \) Boolean function from [6]. Let us define the function. Boolean function \( EQ_n : \{0,1\}^n \rightarrow \{0,1\} \) is such that

\[
EQ(x_1, \ldots, x_{\lfloor n/2 \rfloor}, x_{\lfloor n/2 \rfloor + 1}, \ldots, x_n) = 1, \text{ if } (x_1, \ldots, x_{\lfloor n/2 \rfloor}) = (x_{\lfloor n/2 \rfloor + 1}, \ldots, x_n),
\]

and 0 otherwise. It is known from [6, 4] that there is a quantum and randomized OBDDs that compute \( EQ_m \) using linear width.

**Lemma 2.** For arbitrary \( \epsilon \in (0,1) \) the function \( EQ_n \) can be computed with one-sided error \( \epsilon \) by a quantum OBDD of width \( O(n) \), where \( n \) is the length of the input.

**Lemma 3.** For arbitrary \( \epsilon \in (0,1) \) the function \( EQ_n \) can be computed with one-sided error \( \epsilon \) by a randomized OBDD of width \( O(n) \), where \( n \) is the length of the input.

**Lemma 4.** There is no deterministic OBDD of width \( 2^{o(n)} \) computing \( EQ_m \).

The next corollary follows from these lemmas and Lemma 1.

**Corollary 2.** Following claims are right:

- there are quantum and randomized streaming algorithms that computes \( EQ_n \) using \( O(n) \) qubits with one-sided error \( \epsilon \);
- there is no deterministic streaming algorithm that computes \( EQ_m \) using \( o(n) \) bits.

Let us consider \( BHE_{k,r,w}^t = BH_{k,r,w}^t(EQ_m) \) problem. Let us discuss properties of the problem:

**Theorem 6.** Following claims are right for \( P^t = BHE_{k,r,w}^t \) and \( t \in \{1, \ldots, k\} \), \( z = k/t \).
1. There are quantum and randomized online algorithm $Q$ that uses $O(\log n)$ qubits and solves $P^t$ such that $Q$ is $c$ expected competitive for $d = (2\varepsilon - 1)^z$ and

$$c \geq \frac{(1 - \varepsilon)^z - 1}{4} \left((1 + (-1)^z) t + (1 - (-1)^z) \left(1 + \frac{d^{[1/2]}+2-1}{d^3 - d}\right)\right) (r-w)/(rt+w/r)$$

2. There is no deterministic online algorithm $A$ using $o(n)$ bits of memory and solving $P^t$ that is $c$-competitive for $c < w/r$.

3. There is no deterministic online algorithm $A$ using $o(n) - 2b$ bits of memory, $b = o(n)$ advice bits and solving $P^t$ that is $c$-competitive for $c < qhr + (t-h)w$, $h = \lfloor b/z \rfloor$.

4. There is no deterministic online algorithm $A$ using $o(n) - 2b$ bits of memory, $b = o(n), b < z$ advice bits and solving $P^t$ that is $c$-competitive for $c < w/r$.

Proof. The Claims follows from Corollary 2 and following properties of Black Hats Method: Theorems 4, 1, 2, 5 and Corollary 1.

This Theorem gives us following important results:

1. Quantum and randomize online algorithms with logarithmic space for $BHE^{1}_{k,r,w}$ have better competitive ratio than
   - any deterministic online algorithm with logarithmic space and $o(n)$ advice bits.
   - any deterministic online algorithm without restriction on memory.

2. Increasing advice bits for deterministic algorithm for $BHE^{t}_{k,r,w}$ gives us better competitive ratio in a case of logarithmic space, for $t > 1, t \leq k/2$.
   But competitive ratio still worse than for quantum and randomized online algorithms.

5 Quantum vs Classical Algorithms

In this section as in the previous one we will use result for OBDDs. Let us apply the Black Hats Method from Section 3 to $R_{\nu,l,m,n}$ Boolean function from [38]:

Let $\ket{1}, \ldots, \ket{m}$ be the standard basis of $\mathbb{C}^n$. Let $V_0$ and $V_1$ denote the subspaces spanned by the first and last $n/2$ of these basis vectors. Let $0 < \nu < 1/\sqrt{2}$.

The input for the function $R_{\nu,l,m,n}$ consists of $3l(m + 1)$ boolean variables $a_{i,j}, b_{i,j}, c_{i,j}, 1 \leq i \leq l, 1 \leq j \leq m + 1$, which are interpreted as universal $(\varepsilon,l,m)$-codes for three unitary $n \times n$-matrices $A, B, C$, where $\varepsilon = 1/(3n)$. The function takes the value $z \in \{0,1\}$ if the Euclidean distance between $CBA\ket{1}$ and $V_z$ is at most $\nu$. Otherwise the function is undefined.

It is known from [38] that there is quantum OBDD that computes $R_{\nu,l,m,n}$ using polynomial width.

Lemma 5. Let $0 < \nu < 1/\sqrt{2}$. The function $R_{\nu,3g,9g^2,n}$ with an input size of $N = 81g^2b + 9g = O(n^3 \log^2 n)$ has QOBDDs with error at most $\nu^2$ and width $O(N^{4/5} / \log^{8/5} N)$. 

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Lemma 6. Let \( 0 < \nu < 1/\sqrt{2} \). Each randomized OBDD with bounded error for the function \( R_{\nu, 3g, 9b, n} \) on \( N = 81g^2b + 9g = O(n^5 \log^2 n) \) variables has size \( 2^{\Omega(N^{1/10} / \log^{1/5} N) / N} \).

The next corollary follows from two previous lemmas and Lemma 1.

Corollary 3. Following claims are right:

- there is a quantum streaming algorithm that computes \( R_{\nu, 3g, 9b, n} \) using \( O(\log n) \) qubits with bounded error \( \nu^2 \);
- there is no probabilistic streaming algorithm that computes \( R_{\nu, 3g, 9b, n} \) using \( o(n^{1/25}) \) bits with bounded error.

Let us consider \( BHR_{k, r, w, n, g, b, \nu}^t = BH_{k, r, w}(R_{\nu, 3g, 9b, n}) \) problem. Let us discuss properties of the problem:

Theorem 7. Following claims are right for \( P_t = BHR_{k, r, w, n, g, b, \nu}^t, t \in \{1, \ldots, k\} \), \( z = k/t \).

1. There is quantum online algorithm \( Q \) using \( O(\log n) \) qubits that solving \( P_t \) that is \( c \) expected competitive, for \( d = (2\nu^2 - 1)^t \) and
   \[
   c \geq \frac{(1 - \nu^2)^{z-1}}{4} \left( (1 + (-1)^z) t + (1 - (-1)^z) \left( 1 + \frac{d^{2\lfloor d/2 \rfloor} + 1 - 1}{d^3 - d} \right) \right) (r-w)/(rt)+w/r
   \]
2. There is no deterministic online algorithm \( A \) using \( o(n^{1/25}) \) bits of memory and solving \( P_t \) that is \( c \)-competitive, for \( c < w/r \).
3. There is no randomize online algorithm \( A \) using \( o(n^{1/25}) \) bits of memory and solving \( P_t \) that is \( c \)-competitive, for \( c < 2^{-z} + (1 - 2^{-z})w/r \).
4. There is no deterministic online algorithm \( A \) using \( o(n^{1/25}) - 2b \) bits of memory, \( b = o(n^{1/25}) \) advice bits and solving \( P_t \) that is \( c \)-competitive, for \( c \geq h + (t-h)w, \) where \( h = \lfloor b/z \rfloor \).
5. There is no randomize online algorithm \( A \) using \( o(n^{1/25}) - 2b \) bits of memory, \( b = o(n^{1/25}) \), \( b < z \) advice bits and solving \( P_t \) that is \( c \)-competitive, for \( c < w/r \).

Proof. The Claims follows from Corollary 3 and following properties of Black Hats Method: Theorems 4, 1, 2, 5 and Corollary 1.

This Theorem gives us following important results:

1. Quantum online algorithm with logarithmic space for \( BHR_{k, r, w, n, k, b, \nu}^t \) have better competitive ratio than:
   - any randomize online algorithm with logarithmic space; and any deterministic online algorithm with logarithmic space even if it use \( b = o(n^{1/25}) \) advice bits.
   - any deterministic online algorithm without restriction on memory.
2. Increasing advice bits for deterministic and randomize algorithms for \( BHR_{k, r, w, n, k, b, \nu}^t \) gives us better competitive ratio in case of logarithmic space, for \( t > 1, t \leq k/2 \). But competitive ratio still worse than for quantum online algorithm.

Acknowledgements.
We thank Andris Ambainis, Alexanders Belovs and Abuzer Yakarilmaz from University of Latvia for helpful discussions.
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A Definition of OBDD and Automaton

OBDD is a restricted version of a branching program (BP). BP over a set $X$ of $n$ Boolean variables is a directed acyclic graph with two distinguished nodes $s$ (a source node) and $t$ (a sink node). We denote it $P_{s,t}$ or just $P$. Each inner node $v$ of $P$ is associated with a variable $x \in X$. A deterministic $P$ has exactly two outgoing edges labeled $x = 0$ and $x = 1$ respectively for that node $v$. The program $P$ computes Boolean function $f(X) (f : \{0,1\}^n \rightarrow \{0,1\})$ as follows: for each $\sigma \in \{0,1\}^n$ we let $f(\sigma) = 1$ iff there exists at least one $s - t$ path (called accepting path for $\sigma$) such that all edges along this path are consistent with $\sigma$. A size of branching program $P$ is a number of nodes. Ordered Binary Decision Diagram (OBDD) is a BP with following restrictions: (i) Nodes can be partitioned into levels $V_1, \ldots, V_{\ell + 1}$ such that $s$ belongs to the first level $V_1$ and sink node $t$ belongs to the last level $V_{\ell + 1}$. Nodes from level $V_j$ have outgoing edges only to nodes of level $V_{j+1}$, for $j \leq \ell$. (ii) All inner nodes of one level are labeled by the same variable. (iii) Each variable is tested on each path only once.

A width $w(P)$ of a program $P$ is $w(P) = \max_{1 \leq j \leq \ell} |V_j|$. OBDD $P$ reads variables in its individual order $\theta(P) = (j_1, \ldots, j_n)$. We consider only natural order of width. In this case we denote model as id-OBDD.

Probabilistic OBDD (POBDD) can have more than two edges for node, and choose one of them using probabilistic mechanism. POBDD $P$ computes Boolean function $f$ with bounded error $0.5 - \varepsilon$ if probability of right answer is at least $0.5 + \varepsilon$.

Let us define a quantum OBDD. That is given in different terms, but you can see that they are equivalent, see §3 for more details. For a given $n > 0$, a quantum OBDD $P$ of width $w$ defined on $\{0,1\}^n$, is a 4-tuple $P = (T, |\psi\rangle_0, \text{Accept}, \pi)$, where $T = \{T_j : 1 \leq j \leq n \text{ and } T_j = (G_j^0, G_j^1)\}$ are ordered pairs of (left) unitary matrices representing the transitions. Here $G_j^0$ or $G_j^1$ is applied on the $j$-th step. And a choice is determined by the input bit. $|\psi\rangle_0$ is a initial vector from $w$-dimensional Hilbert space over the field of complex numbers. $|\psi\rangle_0 = |q_0\rangle$ where $q_0$ corresponds to the initial node. Accept $\subset \{1, \ldots, w\}$ is a set of accepting nodes. $\pi$ is a permutation of $\{1, \ldots, n\}$ defines the order of input bits.

For any given input $\nu \in \{0,1\}^n$, the computation of $P$ on $\nu$ can be traced by a $w$-dimensional vector from Hilbert space over the field of complex numbers. The initial one is $|\psi\rangle_0$. In each step $j$, $1 \leq j \leq n$, the input bit $x_{\theta(j)}$ is tested and then the corresponding unitary operator is applied: $|\psi\rangle_j = G_j^{x_{\theta(j)}}(|\psi\rangle_{j-1})$, where $|\psi\rangle_j$ represents the state of the system after the $j$-th step, for $1 \leq j \leq n$. We can measure one of qubits. Let the program was in state $|\psi\rangle = (v_1, \ldots, v_w)$ before measurement and let us measure the $i$-th qubit. And let states with numbers $j_1^0, \ldots, j_{w/2}^0$ correspond to 0 value of the $i$-th qubit, and states with numbers $j_1^1, \ldots, j_{w/2}^1$ correspond to 1 value of the $i$-th qubit. The result of measurement of $i$-th qubit is 1 with probability $pr_1 = \sum_{u=1}^{w/2} |v_{j_u}^1|^2$ and 0 with probability $pr_0 = 1 - pr_1$. In the end of computation program $P$ measures all qubits. The accepting (return 1) probability $Pr_{\text{accept}}(\sigma)$ of $P_{\nu}$ on input $\sigma$ is $Pr_{\text{accept}}(\nu) = \sum_{\nu \in \text{Accept}} v_i^2$, for $|\psi\rangle_n = (v_1, \ldots, v_w)$.
Let $P_\varepsilon(\nu) = 1$ if $P$ accepts input $\nu \in \{0, 1\}^n$ with probability at least $0.5 + \varepsilon$, and $P_\varepsilon(\nu) = 0$ if $P$ accepts input $\nu \in \{0, 1\}^n$ with probability at most $0.5 - \varepsilon$, for $\varepsilon \in (0, 0.5]$. We say that a function $f$ is computed by $P$ with bounded error if there exists an $\varepsilon \in (0, 0.5]$ such that $P_\varepsilon(\nu) = f(\nu)$ for any $\nu \in \{0, 1\}^n$. We can say that $P$ computes $f$ with bounded error $0.5 - \varepsilon$.

**Automata.** We can say that automaton is id-OBDD such that transition function for each level is same.