The relationship between costs for quantum error mitigation and non-Markovian measures

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Quantum error mitigation (QEM) has been proposed as an alternative method of quantum error correction (QEC) to compensate errors in quantum systems without qubit overhead. While Markovian gate errors on digital quantum computers are mainly considered previously, it is indispensable to discuss a relationship between QEM and non-Markovian errors because non-Markovian noise effects inevitably exist in most of the solid state systems. In this work, we investigate the QEM for non-Markovian noise, and show that there is a clear relationship between costs for QEM and non-Markovian measures. We exemplify several non-Markovian noise models to bridge a gap between our theoretical framework and concrete physical systems. This discovery may help designing better QEM strategies for realistic quantum devices with non-Markovian environments.

I. INTRODUCTION

It is now widely accepted that quantum computing will enable us to solve classically intractable tasks such as Shor’s algorithm for prime factorization [1], quantum simulation for quantum many-body systems [2], and HHL (Harrow-Hassidim-Lloyd) algorithm for solving linear equations [3]. However, the effect of decoherence does impose an inevitable impact on the reliability and efficiency of quantum computation, and hence suppressing physical errors is crucial to obtain reliable results [2, 4–7]. Although fault-tolerant quantum computing based on quantum error correction can resolve that difficulty, it is not likely to happen for a while because of requiring the large number of physical qubits per single logical qubit.

Quantum error mitigation (QEM) methods have been proposed to mitigate errors in digital quantum computing, which is compatible with near-term quantum computers with the restricted number of qubits and gate operations as it does not rely on encoding required in fault-tolerant quantum computing [8–12]. For example, probabilistic error cancellation can perfectly cancel the effect of noise if the complete description of the noise model is given [9, 10]. We apply recovery quantum operations \( \mathcal{E}_R \) to invert noise processes of gates \( N_G \) such that \( \mathcal{E}_R = N_G^{-1} \). Since inverse channel of noisy process is generally unphysical channel, we need to realize this by applying single-qubit operations with a classical post-processing of measurement outcomes. Also, the repetition of quantum circuits need to be \( C^2 \) times greater to achieve the same accuracy as before QEM, where \( C \) is an overhead factor determined by the noise model and operations used in the QEM procedure. We call this overhead factor as QEM costs throughout this paper. We can write QEM costs of the error mitigation of the quantum circuit

as \( C = \prod_{k=1}^{N_G} c_k \) where \( c_k \) is a QEM cost of \( k \)-th noisy gate and \( N_G \) is the number of gates. Therefore, in order to suppress costs of QEM, we need to investigate the property of \( c_k \) and optimize it.

Recently, Sun et al. [13] proposed a general quantum error mitigation scheme that can also be applied to continuous quantum systems such as analog quantum simulators. The continuous time evolution of a quantum system is described by

\[
\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] + \mathcal{L}[\rho(t)],
\]

where \( H \) is the Hamiltonian, \( \rho_N(t) \) denotes a density matrix under noisy dynamics, \( \mathcal{L}[\rho_N(t)] \) is the superoperator describing the effect of the environment, which should be mitigated. Even when \( \mathcal{L} \) consists of local Lindblad operators, the effects of them easily propagate to the entire system, resulting in highly correlated noise. Note that Eq. [1] can be rewritten as \( \rho_N(t + \delta t) = \mathcal{E}_N(\rho_N(t)) \) where \( \mathcal{E}_N \) denotes a superoperator of noisy dynamics for a small time interval \( \delta t \). Thus, similarly to probabilistic cancellation, denoting \( \mathcal{E}_I \) as the ideal process, we can apply the recovery channel \( \mathcal{E}_R \) such that \( \mathcal{E}_R\mathcal{E}_N = \mathcal{E}_I \) using additional single-qubit operations and classical post-processing of measurement results. In Ref. [13], the stochastic QEM method was introduced to implement recovery operations in the limit of \( \delta t \to 0 \) via Monte-Carlo sampling. Note that QEM costs corresponding to \( \mathcal{E}_R(t) \) can be described as \( c(t) \approx 1 + c'(t)\delta t \). Therefore, a QEM cost from \( t = 0 \) to \( t = T \) can be described as \( C(T) = \exp\left[\int_0^T dt c'(t)\right] \).

So far, QEM for Markovian noise was mainly considered and non-Markovian noise was not investigated well. The development of the QEM for non-Markov noise is practically important, because non-Markovian noise is relevant in most of the solid state systems such as superconducting qubits, nitrogen vacancy centers in diamond, and spin qubits in quantum dots [14–19].

In addition to the practical motivation for non-Markovian noise, the concept of quantum non-
Markovianity has been extensively studied from a fundamental interest of the characterization and the quantification of the backflow of the information from an environment [26, 27]. There are some definitions of non-Markovianity such as semigroup definition [28], divisibility definition [29, 30], and BLP (proposed by Breuer, Laine and Piilo) definition [31] (Ref. 20 discusses a hierarchical relation between these definitions). Throughout this paper, we adopt divisible maps as Markovian processes and we will show that precise definition in later section.

Moreover, there are many applications to utilize non-Markovianity in a positive way for quantum information processing, including quantum Zeno effects [26, 27], dynamical decoupling [28], Loschmidt echo and criticality [29], continuous-variable quantum key distribution [30], time-invariant discord [31], quantum chaos [32], dynamical decoupling [28], Loschmidt echo and criticality [29, 33], and quantum metrology [34, 35]. These motivate the researchers to investigate the properties of non-Markovianity.

In this paper, we investigate QEM costs for the case of non-Markovian noise. The stochastic QEM can be naturally applied to time-dependent non-Markovian noise to fully compensate for physical errors. We show that QEM costs reduce in the non-Markovian region. We also find a clear relationship between costs of QEM and previously reported non-Markovian measures, decay rate measure [36] and RHP (Rivas, Huelga, and Plenio) measure [24, 36]. We calculate QEM costs for two experimental setups showing non-Markovianity as examples: The one is a controllable open quantum system, which consists of a long-lived qubit coupled with a short-lived qubit. This system has been realized in the NMR (nuclear magnetic resonance) experiments [37, 38]. The other example is a qubit dispersively coupled with a dissipative resonator [39, 41]. This discovery may illuminate how to construct efficient QEM procedures.

The rest of this paper is organized as follows. In Sec. II, we review the stochastic QEM proposed in Ref. 13. In Sec. III, we review the definition and the measure of non-Markovianity. In Sec. IV, we discuss the relation between QEM costs and the measure of non-Markovianity, and study QEM costs for specific models. Finally, we summarize and discuss our results in Sec. V.

II. STOCHASTIC QUANTUM ERROR MITIGATION

In this section, we review the stochastic QEM [13]. Suppose that the dynamics of the system of interest can be described by Eq. (1). Here, we assume that the local noise and the coupling to the environment is sufficiently weak and the continuous dynamics of the system can be described by the time-dependent Lindblad master equation. Now we express the evolution of the state from $t$ to $t + \delta t$ as $\rho(t + \delta t) = \mathcal{E}_N(t)(\rho(t))$ and $\rho(t + \delta t) = \mathcal{E}_f(t)(\rho(t))$, corresponding to the noisy and the ideal process, respectively. The ideal process represents the unitary dynamics without any noisy operators in Eq. (1). We hope to emulate the ideal evolution $\mathcal{E}_f$ by mitigating errors of the process $\mathcal{E}_N$. When the evolution is affected by local noise operators, i.e., $\mathcal{L}$ can be decomposed as a linear combination of local noise operators, by using a recovery operation $\mathcal{E}_Q(t)$, we can efficiently find a decomposition:

$$\mathcal{E}_f(t) = \mathcal{E}_Q(t)\mathcal{E}_N(t)$$

$$\mathcal{E}_Q(t) = \sum_i \mu_i \mathcal{R}_i = \langle c(t) \sum_i \text{sgn}(\mu_i) p_i \mathcal{R}_i \rangle.$$

Here, $c(t) = \sum_i |\mu_i|$, $p_i = |\mu_i|/c(t)$, and $\{\mathcal{R}_i\}$ is a set of polynomial number of physical operations applied for QEM. For a given decomposition, the ideal process $\mathcal{U}_f$ from $t = 0$ to $t = T$ can be decomposed as

$$\mathcal{U}_f \approx \prod_{n=0}^{N_d-1} \mathcal{E}_f(n\delta t) = C(T) \sum_{\bar{i}} \prod_{n=0}^{N_d-1} \mathcal{R}_{\bar{i}} \mathcal{E}_N(n\delta t) + O(T\delta t),$$

where $N_d = T/\delta t$, $\bar{i} = (i_1, i_2, ..., i_{N_d})$, $p_{\bar{i}} = \prod_{n=0}^{N_d-1} p_{i_n}$, $s_{\bar{i}} = \prod_{n=0}^{N_d-1} \text{sgn}(\mu_{i_n})$ and a QEM cost can be described as $C(T) = \sum_{\bar{i}} c(n\delta t)$. Suppose that the initial state for the quantum circuit is $\rho_{1n}$, we have

$$\rho_{1}(T) = C(T) \sum_{\bar{i}} p_{\bar{i}} s_{\bar{i}} \rho_{1n} + O(T\delta t),$$

where $\rho_{1}(T)$ is the density operator after the ideal process $\rho_{1}(T) = \mathcal{U}_f(\rho_{1n})$ and $\rho_{1n} = (\prod_{n=0}^{N_d-1} \mathcal{R}_{i_n} \mathcal{E}_N(n\delta t))(\rho_{1n})$. When measuring a observable $M$, since the expectation value for the state $\rho$ equals to $\langle M \rangle_{\rho} = \text{Tr}[\rho M]$, we obtain

$$\langle M \rangle_{\rho_{1}} = C(T) \sum_{\bar{i}} p_{\bar{i}} s_{\bar{i}} \langle M \rangle_{\rho_{1n}} + O(T\delta t).$$

We can obtain the $\langle M \rangle_{\rho_{1}}$ of the Eq. (6) from the actual experiment as follows: Firstly, we generate the recovery operation $\mathcal{R}_i$ with a probability $p_i$, with a time interval $\delta t$ until time $T$, and measure the observable $M$, and we record the measurement outcome after multiplying the factor of $s_{\bar{i}}$. Secondly, we repeat the same procedure to reduce the statistical uncertainty. Finally, we estimate the value of $C(T) \sum_{\bar{i}} p_{\bar{i}} s_{\bar{i}} \langle M \rangle_{\rho_{1n}}$ from the measurement results, and this approximates the error-free expectation value.

Since $\mathcal{E}_N \approx \mathcal{E}_f$ for a small $\delta t$ and the recovery operation becomes an identity operation in almost all the cases, we can use the Monte Carlo method to stochastically realize continuous recovery operations $\mathcal{R}_i$ corresponding to $\delta t \to +0$ to eliminate a discretization error $O(T\delta t)$. This procedure is similar to the one employed in the simulation of stochastic Schrödinger equation. Refer to Ref. 13.
In this paper, to quantify non-Markovianity, we adopt the previous studies (for example, see review papers [20, 21]). Eq. (9) has a similar form to the Lindblad Markovian and other words, the sign of \( \gamma \) dependent decoherence operator satisfying \( \text{Tr}[L_k^\dagger(t)L_k(t)] = 0 \) and \( \text{Tr}[L_k^\dagger(t)L_k(t)] = 1 \). Here, \( \{A, B\} = AB + BA \) denote anticommutator. It is worth mentioning that the Eq. (9) has a similar form to the Lindblad Markovian master equation, except that the decay rate \( \gamma_k \) can be negative in some time interval. In fact, if and only if all the decay rates \( \gamma_k \) are non-negative for all the time \( t \), the dynamical maps are CP-divisible [23]. In other words, the sign of \( \gamma_k \) characterizes whether the dynamical maps are Markovian or non-Markovian.

### A. Definition of non-Markovianity

There are some definitions of Markovianity in quantum dynamics, but throughout this paper, we adopt a definition introduced in [24]. Here, a Markovian map is defined as a CP-divisible map: a dynamical map \( \mathcal{E}_{(t,0)} \) from 0 to \( t \) is CP-divisible if the map \( \mathcal{E}_{(t,s)} (0 \leq s \leq t) \) defined by

\[
\mathcal{E}_{(t,s)} = \mathcal{E}_{(t,0)} \mathcal{E}_{(s,0)}^{-1}
\]

is completely positive for all time \( s \). Otherwise, dynamical maps are non-Markovian. Particularly when the inverse of dynamical maps \( \mathcal{E}_{(t,0)} \) exists, even though the dynamical maps are non-Markovian, the equations of the dynamics can be written in the canonical form of the time-local master equation [36].

\[
\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)] + \sum_k \gamma_k(t) \left[ L_k(t) \rho(t) L_k^\dagger(t) - \frac{1}{2} \{ L_k^\dagger(t)L_k(t), \rho(t) \} \right],
\]

where \( \gamma_k(t) \) is a decay rate and \( L_k(t) \) is a time-dependent decoherence operator satisfying \( \text{Tr}[L_k(t)] = 0 \) and \( \text{Tr}[L_k^\dagger(t)L_k(t)] = 1 \). Here, \( \{A, B\} = AB + BA \) denote anticommutator. It is worth mentioning that the Eq. (9) has a similar form to the Lindblad Markovian master equation, except that the decay rate \( \gamma_k \) can be negative in some time interval. In fact, if and only if all the decay rates \( \gamma_k \) are non-negative for all the time \( t \), the dynamical maps are CP-divisible [23]. In other words, the sign of \( \gamma_k \) characterizes whether the dynamical maps are Markovian or non-Markovian.

### B. Measure of non-Markovianity

There are several non-Markovian measures proposed in previous studies (for example, see review papers [20, 21]). In this paper, to quantify non-Markovianity, we adopt the decay rate measure [30]:

\[
F(t', t) = \sum_k \int_t^{t'} ds \frac{\gamma_k(s) - \gamma_k(s)}{2}.
\]

Since Markovian dynamical maps give \( F(t, t') = 0 \), this measure can be interpreted as the total amount of non-Markovianity. Moreover, it is shown that this measure is equivalent to the RHP (Rivas, Huelga, and Plenio) measure [24, 36], which quantifies the degree of non-completeness of the map \( \mathcal{E}_{(t,s)} \) based on Choi-Jamiołkowski isomorphism [43, 44].

### III. PROPERTIES OF NON-MARKOVIANITY

In this section, we review typical properties of non-Markovianity [30, 31].

#### A. Definition of non-Markovianity

We derive QEM costs for non-Markovian dynamics and discuss the direct relation between costs and the measure of non-Markovianity. Note that, since the time-local master equation Eq. (9) is derived from a given Hamiltonian, modifications of the Hamiltonian for applying recovery operations could affect the form of time-local master equation. However, to derive a relation between QEM costs and non-Markovian measures, we assume that recovery operations do not change the equation Eq. (9).

#### A. General form of QEM costs

Here, we derive the general form of QEM costs for the time-local quantum master equation Eq. (9). The key idea is to represent the decoherence operators \( L_k(t) \) using the process matrix form [15]:

\[
L_k(t) = \sum_{i=1}^{d^2-1} \frac{1}{d^2} \text{Tr}[L_k(t)G_i] G_i,
\]

where the operators \( G_i \) satisfy the conditions \( G_i = I \otimes N \), \( G_i G_j \) and \( (G_i)^2 = I \otimes N \), and \( d = 2^N \) is the dimension of the state vector of \( N \) qubits. An example of \( \{G_i\}_i \) is a set of Pauli products, i.e., \( G_i \in \{I, X, Y, Z\} \otimes N \). Then, Eq. (9) can be rewritten as

\[
\frac{d}{dt} \rho_N(t) = \sum_{i,j=0}^{d^2-1} M_{ij}(t) G_i \rho_N(t) G_j,
\]

where \( M(t) \) is an \( d^2 \times d^2 \) Hermitian matrix defined by

\[
M_{ij}(t) = \begin{cases}
\frac{1}{2d^2} \sum_k \gamma_k(t) \text{Tr}[L_k^\dagger(t)G_i] \text{Tr}[L_k(t)G_j] & (i, j \geq 1) \\
-\frac{1}{2d^2} \sum_k \gamma_k(t) \text{Tr}[L_k^\dagger(t)L_k(t)G_i] & (i \geq 1, j = 0) \\
-\frac{1}{2d^2} \sum_k \gamma_k(t) \text{Tr}[L_k^\dagger(t)L_k(t)G_j] & (i = 0, j \geq 1) \\
-\sum_k \gamma_k(t) & (i = j = 0)
\end{cases}
\]

and \( M(t) \) can be diagonalized using an unitary matrix \( u \)

\[
M_{ij}(t) = \sum_{l=0}^{d^2-1} u_{il}(t) q_l(t) u_{jl}^*(t).
\]
Therefore, we obtain
\[ \frac{d}{dt}\rho_N(t) = \sum_{i=0}^{d^2-1} q_i(t) B_i(t) \rho_N(t) B_i^\dagger(t), \]

where \( B_i(t) \) is an operator \( B_i(t) = \sum_{i=\geq 0}^{d^2-1} u_i(t) G_i \).

Using Eq. [14], we can derive QEM costs. By choosing the recovery operation at time \( t \)

\[ \mathcal{E}_Q(t) = c(t) \left( p_0(t) \mathcal{I} + \sum_{l \geq 1} \text{sgn}(-q_l(t)) p_l(t) B_l(t) \right), \]

where \( c(t) = 1 + (-q_0(t) + \sum_{l \geq 1} |q_l(t)|) dt, p_l(t) = |q_l(t)| dt \) (\( l \geq 1 \)), \( p_0(t) = 1 - \sum_{l \geq 1} p_l(t) \), and \( B_l(t) \rho = B_l(t) \rho B_l^\dagger(t) \).

Hence, the general form of QEM costs is given by

\[ C(T) = \exp \left[ \int_{0}^{T} \left( -q_0(t) + \sum_{l \geq 1} |q_l(t)| \right) dt \right]. \]

**B. The effect of non-Markovianity on QEM costs**

We assume \( L_k^1(t)L_k(t) = L_k(t)L_k^1(t) = \mathcal{I} \otimes N \) for all \( k \) in Eq. [10], such as Pauli operators. In this case, the matrix \( M \) can be easily diagonalized because \( M_{ii} = 0 \) and \( M_{ii} = 0 \) for all \( i (i \geq 1) \) and the unitary matrix \( u \) is determined by \( u_k = \frac{1}{\sigma^2} \text{Tr}[L_k(t) G_i] \) \( (i, k \geq 1) \), \( u_{ii} = u_{ii} = 0 \) \( (i, k \geq 1) \), \( u_{00} = 1 \), and the eigenvalues of \( M \) are \( q_k = \gamma_k(t) \) \( (k \geq 1) \) and \( q_0 = -\sum_{k \geq 1} \gamma_k(t) \). We can derive QEM costs as

\[ C(T) = \exp \left[ \sum_{k} \int_{0}^{T} \left( |\gamma_k(t)| + \gamma_k(t) \right) dt \right]. \]

Here, we define the quantity \( D(t', t) = \sum_{k} \int_{0}^{t'} ds |\gamma_k(s)| \), which is equivalent to the QEM costs \( e^{D(t', t)} \) for the Markovian case with decay rates \( |\gamma_k(t)| \). By using \( D(t', t) \) and \( F(t', t) \), we can rewrite the QEM costs as

\[ C(T) = \exp \left[ 2 \left( D(T, 0) - F(T, 0) \right) \right]. \]

From this equation, we can understand that as the amount of non-Markovianity in Eq. [10] increases, QEM costs are reduced. More specifically, in a time region with \( \gamma_k(t) < 0 \) for all \( k \), QEM costs do not increase at all.

**C. Study of specific models**

Here, we study QEM costs for specific models. Although the implementation of recovery operations could change the form of time local master equation, we discuss the case it is invariant, i.e., recovery operations commute with both the system Hamiltonian and the noise operators. In this case, since we can perform the recovery operations at the end of the dynamics, the time local master equation is not affected by recovery processes.

We consider a two-qubit system where a long-lived qubit is coupled with a short-lived qubit. (Another example of a qubit dispersively coupled with a dissipative resonator is illustrated in Appendix A.) Importantly, this system has been realized with nuclear magnetic resonance, and the non-Markovian noise has been controlled by implementation of the pulse [37] [38]. The equation of the two-qubit model is given by

\[ \frac{d\rho_N^{(1+2)}(t)}{dt} = i \left[ \rho_N^{(1+2)}(t), \frac{J}{4} Z \otimes Z \right] + \mathcal{L}[\rho_N^{(1+2)}(t)], \]

\[ \mathcal{L}[\rho] = \sum_{k=1}^{2} \frac{\gamma_k}{4} \left( 2 L_k \rho L_k^\dagger - \{ L_k^2, \rho \} \right), \]

where \( \rho_N^{(1+2)} \) denotes the density operator of the two-qubit system, \( J \) denotes a coupling strength between the qubits, \( \gamma_1 = 2 \gamma s \) denotes a thermalization rate associated with a Lindblad operator \( L_1 = I \otimes \sigma \), \( \gamma_2 = 2 \gamma (1 - s) \) denotes an energy relaxation rate associated with a Lindblad operator \( L_2 = I \otimes \sigma \), \( \sigma_+ = (X + iY)/2 \) \( (\sigma_- = (X - iY)/2) \) denotes a raising (lowering) operator, \( s \) denotes a control parameter determined by the environmental temperature \( (0 \leq s \leq 1/2) \). Only the second qubit is directly connected to the Markovian environment and the first qubit is influenced by the environment through the second qubit. Here, \( 1/J \) denotes a time scale of the exchange of the information between the first qubit and the second qubit, while \( 1/\gamma \) denotes the decoherence time for the second qubit. In the regime of \( J/\gamma > 1 \) where the information exchange between qubits occurs in a faster time scale than the environmental decoherence of the second qubit, the dynamics of the reduced density operator of the first qubit could be non-Markovian. This means that the first qubit receives the backflow of the information from the second qubit before losing the coherent information to the environment.

Here, we choose an initial state as \( \rho_N^{(1+2)}(0) = |+ \rangle \langle +| \otimes \rho_{\text{Gibbs}} \), where \( |+ \rangle = (|0 \rangle + |1 \rangle)/2 \) is the superposition of the computational basis \( |0 \rangle \) and \( |1 \rangle \), and \( \rho_{\text{Gibbs}} = s|0 \rangle \langle 0| + (1-s) |1 \rangle \langle 1| \) is the Gibbs state corresponding to the Lindbladian in Eq. [19]. In this case, the equation of its dynamics is given by

\[ \frac{d\rho_N^{(1)}(t)}{dt} = i \left[ \rho_N^{(1)}(t), \frac{S(t)}{2} Z \right] + \frac{\gamma(t)}{2} \left( Z \rho_N^{(1)}(t) Z - \rho_N^{(1)}(t) \right), \]

where \( \rho_N^{(1)}(t) \) is the reduced density operator of the first qubit and \( Z \) is a Pauli \( Z \) matrix. In this case, the decay rate \( \gamma(t) \) and \( S(t) \) are given by

\[ \gamma(t) - i S(t) = -\frac{1}{f(t)} \frac{d}{dt} f(t), \]

where \( f(t) \) is the fidelity of the state. This equation allows us to extract the relaxation rate \( \gamma(t) \) from the decay rate \( S(t) \) and vice versa.

\[ C(T) = \exp \left[ 2 \left( D(T, 0) - F(T, 0) \right) \right]. \]
\[ \gamma(t) = \frac{J^2}{2} \left( e^{\lambda_+ t} - e^{\lambda_- t} \right) + \frac{J^2}{2} \left( e^{\lambda_- t} - e^{\lambda_+ t} \right) \] and \( \lambda_{\pm} \) are the two solutions of an equation \( \lambda^2 + \gamma \lambda + (2iJ\gamma(1-2s) + J^2)/4 = 0 \). From Eq. (17), QEM costs \( C(T) \) can be derived as

\[ C(T) = \exp \left( \int_0^T dt \frac{\gamma(t) + \gamma(t)}{2} \right). \] (22)

Fig. 1 shows the numerical results of \( \gamma(t) \) and \( C(t) \) at time \( t \) with experimental parameters in Ref. [37]. In Fig. 1, \( \gamma(t) \) becomes negative in some time interval and therefore the dynamics is actually non-Markovian. Moreover, the area where \( \gamma(t) \geq 0 \) holds only contributes to QEM costs \( C(t) \), as shown in Fig. 1. In other words, QEM costs \( C(t) \) do not increase at all in the region satisfying \( \gamma(t) < 0 \), and the area of its region is equivalent to the non-Markovian measure in Eq. (10).

V. DISCUSSIONS

In this work, we discuss a relationship between the QEM and non-Markov measures. Non-Markovianity is characterized by a negative decay rate of the dissipator. Interestingly, QEM costs do not increase at all when the all decay rates are negative. This demonstrates that non-Markovianity can contribute to reduce cost of the QEM. We show specific physical systems as examples that support our theoretical analysis. We focus on the case where the decoherence operators can be described by a set of orthogonal operators such as Pauli operators, and leave more general cases for a future work. Our work helps understanding properties of QEM and may lead to sophisticated construction of QEM for realistic quantum systems with non-Markovian noise.

ACKNOWLEDGMENTS

This work was supported by Leading Initiative for Excellent Young Researchers MEXT Japan and JST presto (Grant No. JPMJPR1919) Japan. This paper is partly based on results obtained from a project, JPNP16007, commissioned by the New Energy and Industrial Technology Development Organization (NEDO), Japan. This work was supported by MEXT Quantum Leap Flagship Program (MEXT Q-LEAP) (Grant No. JPMXS0120319794, JPMXS0118068682) and JST ERATO (Grant No. JPMJER1601).

Appendix A: A qubit disperively coupled with a dissipative resonator

Here, we also consider a qubit-resonator system where a qubit is disperively coupled with a lossy resonator. The qubit is affected by a dephasing induced from the interaction with the resonator, and this dynamics has been studied in Refs. [39–41]. When the frequency of the qubit is significantly detuned from that of the resonator, the Hamiltonian is given by

\[ H_{q+r} = \chi Z a^\dagger a, \] (A1)

where \( \chi/\pi \) is the dispersive frequency shift of the qubit per photon and \( a^\dagger a \) is the creation (amnihilation) operator of the photon in the resonator. For simplicity, we assume that we are in a rotating frame, and we only consider the interaction Hamiltonian in Eq. (A1). The dynamics of the system can be described by the Lindblad master equation

\[ \frac{d\rho_{N}^{(q+r)}(t)}{dt} = i \left[ \rho_{N}^{(q+r)}(t), H_{q+r} \right] + \kappa/2 \left( 2\alpha \rho_{N}^{(q+r)}(t) a^\dagger - (a^\dagger a) \rho_{N}^{(q+r)}(t) \right), \] (A2)

where \( \kappa \) is a decay rate. We set the initial state to \( |\alpha\rangle \otimes |\alpha\rangle \), where \( |\alpha\rangle \) is a coherent state for \( \alpha \in \mathbb{C} \). This equation can be easily solved as \( \rho_{N}^{(q+r)}(t) = \sum_{i,j=0}^1 c_{ij}(t) |i\rangle \langle j| \otimes |\alpha_i(t)\rangle \langle \alpha_j(t)| \), where \( c_{00}(t) = c_{11}(t) = \frac{1}{2} \) and \( c_{10}(t) = c_{01}(t)^* \) and

\[ c_{01}(t) = \frac{c_{01}(0)}{|\alpha_1(t)|^2} \exp \left[ -|\alpha|^2 \frac{1 - e^{2(\chi - \kappa)t}}{1 - i\kappa/2\chi} \right]. \] (A3)

Here, \( \alpha_0(t) = e^{(i\chi - \kappa/2)t}\alpha \) and \( \alpha_1(t) = e^{(-i\chi - \kappa/2)t}\alpha \). The reduced density operator of the qubit is given by

\[ \rho_{N}^{(q)}(t) = \sum_{i,j=0}^1 c_{ij}(t) |i\rangle \langle j|, \] where \( c_{00}(t) = c_{11}(t) = \frac{1}{2} \) and \( c_{10}(t) = c_{01}(t)^* \) and \( c_{01}(t) = c_{01}(t) e^{2(\chi - \kappa)t} = \frac{1}{2} e^{-|\alpha|^2(x + iy)}. \) From this, the time-local master equation

![Graph](image-url)
From Eq. (A6), $\gamma(t)$ behaves as a damped oscillation with the time constant $\kappa$ and the angular frequency $2\chi$. QEM costs $C(T)$ are the same form as that in Section IV C and therefore $C(T)$ can decrease in the case of $\kappa < \chi$.

In Figs. 2 and 3, we show the numerical results of $\gamma(\kappa t)$ and $C(\kappa t)$. As is the same in Section IV C, the dynamics is actually non-Markovian because of some negative regions of $\gamma(\kappa t)$ in Fig. 2 and QEM costs $C(\kappa t)$ do not increase at all for those regions.

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\[
\frac{d\rho_N^{(q)}(t)}{dt} = i\left[\rho_N^{(q)}(t), S(t)\right] + \gamma(t) \left( Z\rho_N^{(q)}(t)Z - \rho_N^{(q)}(t) \right),
\]

(A4)

where $S(t) = |\alpha|^2 \frac{dx}{dt}$ and the decay rate of the qubit $\gamma(t) = |\alpha|^2 \frac{d\alpha}{dt}$ are given by

\[
S(t) = |\alpha|^2 e^{-\kappa t} (|\kappa(1 - \cos 2\chi t) - 2\chi \sin 2\chi t) + \frac{|\alpha|^2 e^{-\kappa t}}{1 + (\kappa/2\chi)^2} \left(2\kappa \cos 2\chi t + \left(2\chi - \frac{\kappa^2}{2\chi}\right) \sin 2\chi t\right),
\]

(A5)

\[
\gamma(t) = \frac{|\alpha|^2 \kappa e^{-\kappa t}}{1 + (\kappa/2\chi)^2} \left(\frac{\kappa}{\chi} \cos 2\chi t + \left(1 - \left(\frac{\kappa}{2\chi}\right)^2\right) \sin 2\chi t\right).
\]

(A6)
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