Cotorsion pairs, Gorenstein dimensions and triangle-equivalences

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Abstract

Let \((A, B)\) be a complete hereditary cotorsion pair in \(\text{Mod}\ R\). Yang and Ding made a general study of \(B\) dimensions of complexes in [56]. In this paper, we define the notion of Gorenstein \(B\) dimensions for complexes by applying the model structure induced by \((A, B)\), which can be used to describe how Gorenstein dimensions of complexes should work for any complete hereditary cotorsion pair. Characterizations of the finiteness of Gorenstein \(B\) dimensions for complexes are given. As a consequence, we study relative cohomology groups for complexes with finite Gorenstein \(B\) dimensions. Moreover, the relationships between Gorenstein \(B\) dimensions and \(B\) dimensions for complexes are given. Next we get two triangle-equivalences between the homotopy category of a hereditary abelian model structure, the singularity category of an exact category and the stable category of a Forbenius category. As applications, some necessary and sufficient conditions for the validity of the Finitistic Dimension Conjecture are given. In particular, we show that the Finitistic Dimension Conjecture is true for an Artin algebra \(R\) if and only if the homotopy category \(H_0(M)\) of the hereditary abelian model structure \(M = (\mathcal{X}, \text{cores } \hat{\mathcal{Y}}^{<\infty}, \mathcal{G}(\mathcal{Y}))\) is triangle-equivalent to the stable category \(\mathcal{X} \cap \mathcal{G}(\mathcal{Y})\) if and only if there is a triangle-equivalence \(\mathbb{D}_{sg}(\mathcal{X}) \cong \mathcal{X} \cap \mathcal{G}(\mathcal{Y})\), where \((\mathcal{X}, \mathcal{Y})\) is the cotorsion pair cogenerated by the class of finitely generated modules with finite projective dimension in \(\text{Mod}R\) and \(\mathbb{D}_{sg}(\mathcal{X}) := \mathbb{D}^b(\mathcal{X})/\mathbb{K}^b(\mathcal{X} \cap \mathcal{Y})\) is the singularity category of \(\mathcal{X}\).

Key Words: cotorsion pair; Gorenstein dimension; model structure; singularity category; triangle-equivalence; finitistic dimension.

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1. Introduction

Avramov and Foxby [6] explored projective and injective dimensions arising from constructions of \(dg\)-projective and \(dg\)-injective resolutions of complexes. Using the notions of \(dg\)-projective and \(dg\)-injective resolutions, one can define Gorenstein projective and Gorenstein injective dimensions for complexes (see [43, 53]). Cotorsion pairs were invented by Salce [50] in the category of abelian groups, and rediscovered by Enochs and coauthors [13, 27, 28, 29, 30] in the 1900’s. The most

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obvious example of a cotorsion pair is \((\text{Mod}R, \mathcal{I})\). Let \((A, B)\) be a complete hereditary cotorsion pair in \(\text{Mod}R\). Then for any complex \(X\), [56, Theorem 2.4] provides a general construction of \(\text{dg-B} \) resolutions of \(X\). As an application, Yang and Ding make a general study of \(B\) dimensions of complexes. The main purpose of this paper is to describe how Gorenstein dimensions of complexes should work for any complete hereditary cotorsion pair.

The first step in studying Gorenstein dimension for complexes with respect to any complete hereditary cotorsion pair is to give an appropriate definition. By means of a complete hereditary cotorsion pair \((A, B)\), we define a class of left \(R\)-modules, denoted by \(\mathcal{G}(B)\). Such modules are called Gorenstein \(B\)-modules. Detailed definitions can be found in Definition 3.1 below. In [40] Hovey laid out a correspondence between abelian model structures on a complete and cocomplete abelian category \(\mathcal{D}\) and two complete cotorsion pairs on \(\mathcal{D}\). Motivated by this, Gillespie [31] showed that the flat cotorsion pair \((\mathcal{F}, C)\) can induce a flat model structure in the category \(\mathcal{C}(R)\) of chain complexes of \(R\)-modules. Furthermore, Yang and Liu showed that any complete hereditary cotorsion pair \((A, B)\) in \(\text{Mod}R\) can induce a model structure in \(\mathcal{C}(R)\) (see [57, Corollary 3.8]). By means of this model structure, we can give the definition of Gorenstein dimensions for complexes with respect to any complete hereditary cotorsion pair (see Definition 3.10 below). The next result is our first main theorem which characterizes the finiteness of Gorenstein \(B\) dimensions for complexes. See 3.15 for the proof.

**Theorem 1.1.** Let \((A, B)\) be a complete hereditary cotorsion pair in \(\text{Mod}R\) and \(M\) a complex. The following are equivalent for each integer \(n\):

1. \(\mathcal{GB}\)-dim\(M \leq n\);
2. \(-\inf M \leq n\) and there exists a fibrant-cofibrant resolution \(M \leftarrow QM \rightarrow RQM\) of \(M\) such that \(Z_{-n}(RQM) \in \mathcal{G}(B)\);
3. \(-\inf M \leq n\) and \(Z_{-n}(I) \in \mathcal{G}(B)\) for any \(M \sim I\) with \(I\) \(dg\)-injective;
4. \(-\inf M \leq n\) and there exists a quasi-isomorphism \(M \rightarrow B\) with \(B\) a \(dg\)-\(B\) complex such that \(Z_{-n}(B) \in \mathcal{G}(B)\);
5. \(-\inf M \leq n\) and \(Z_{-n}(B) \in \mathcal{G}(B)\) for any \(M \sim B\) with \(B\) a \(dg\)-\(B\) complex;
6. \(-\inf M \leq n\) and \(Z_{-n}(RQM) \in \mathcal{G}(B)\) for each fibrant-cofibrant resolution \(M \leftarrow QM \rightarrow RQM\) of \(M\);
7. For each fibrant-cofibrant resolution \(M \leftarrow QM \rightarrow RQM\) of \(M\), there exists a Tate \(B\) resolution \(M \leftarrow QM \rightarrow RQM \rightarrow^\tau T\) of \(M\) with each \(\tau_i\) a split monomorphism such that \(\tau_i = \text{id}_{(RQM)}\) for \(i \leq -n\).

Furthermore, if \(\mathcal{GB}\)-dim\(M < \infty\), then

\[
\mathcal{GB}\)-dim\(M = \sup \{-\inf \text{RHom}_R(X, M) \mid X \in A \cap B\}.
\]

If we set \((A, B) = (\text{Mod}R, \mathcal{I})\), the Gorenstein \(B\) dimension of a complex \(M\) defined here is exactly the Gorenstein injective dimension of \(M\) defined by Asadollahi and Salarian in [3, Definition 2.2].

The relative cohomology theory was initiated by Butler and Horrocks [16] and Eilenberg and Moore [26] and has been revitalized recently by a number of authors (see, for example [8, 28, 29, 37, 52]), notably, Avramov and Martsinkovsky [8] and Enochs and Jenda [29]. Based on the notions of proper \(X\)-coresolutions, one can define the relative cohomology functors \(\text{Ext}^i_X(\ldots, \ldots)\). Detailed definitions can be found in Definition 2.2 below.
Assume that $\mathcal{A} = (\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\text{Mod}_R$. Let $M$ and $N$ be complexes of $R$-modules. Then we have the following exact sequence of complexes:

$$0 \to \text{Hom}_R(\mathcal{Q}M, \mathcal{Q}N) \to \text{Hom}_R(\mathcal{Q}M, \mathcal{Q}N) \to \tilde{\text{Hom}}_R(\mathcal{Q}M, \mathcal{Q}N) \to 0.$$ 

By [42, Definition 3.7], the $n$th Tate-Vogel cohomology group, denoted by $\tilde{\text{ext}}_\mathcal{A}^n(M, N)$, is defined as

$$\tilde{\text{ext}}_\mathcal{A}^n(M, N) = H_{-n}(\tilde{\text{Hom}}_R(\mathcal{Q}M, \mathcal{Q}N)).$$

Thus we have a long exact sequence

$$\cdots \longrightarrow \tilde{\text{ext}}_\mathcal{A}^n(M, N) \longrightarrow \text{Ext}^n_R(M, N) \longrightarrow \tilde{\text{ext}}_\mathcal{A}^{n-1}(M, N) \longrightarrow \tilde{\text{ext}}_\mathcal{A}^{n+1}(M, N) \longrightarrow \cdots,$$

where $\tilde{\text{ext}}_\mathcal{A}^n(M, N) = H_{-n}(\tilde{\text{Hom}}_R(\mathcal{Q}M, \mathcal{Q}N))$ for each integer $n$ (see Fact 4.1 and Remark 4.2 below). So we have the following theorem. See 4.12 for the proof.

**Theorem 1.2.** Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\text{Mod}_R$ and $N$ an $R$-module with finite Gorenstein $\mathcal{B}$ dimension, and let $\mathcal{G}(\mathcal{B})$ be the class of Gorenstein $\mathcal{B}$-modules. For any $R$-module $M$ in $\mathcal{A}$, we have the following isomorphisms:

1. $\text{Ext}^1_{\mathcal{G}(\mathcal{B})}(M, N) \cong \ker(\tilde{\text{ext}}_\mathcal{A}^1(M, N))$;
2. $\text{Ext}^n_{\mathcal{G}(\mathcal{B})}(M, N) \cong \tilde{\text{ext}}_\mathcal{A}^n(M, N)$ for any integer $n > 1$.

We note that Theorem 1.2 is motivated by [53, Remark 6.7], where the author pointed out that there are some “obstacles” to define relative cohomology groups for complexes. Theorem 1.2(2) shows that $\tilde{\text{ext}}_\mathcal{A}^n(-, -)$ defined here extends the relative cohomology for modules with finite Gorenstein $\mathcal{B}$ dimension defined in Definition 2.2 whenever $n$ is an integer with $n > 1$.

By [56, Definition 3.1], the $\mathcal{B}$ dimension of a complex $N$, denoted by $\mathcal{B}\text{-dim}N$, is defined as

$$\mathcal{B}\text{-dim}N = \inf\{\sup\{i \mid B_{-i} \neq 0\} \mid M \simeq B \text{ with } B \in \mathcal{B}\},$$

where the symbol “$\simeq$” stands for quasi-isomorphism. Note that each module in $\mathcal{B}$ is a Gorenstein $\mathcal{B}$-module. It seems natural to investigate the relationships between Gorenstein $\mathcal{B}$ dimensions and $\mathcal{B}$ dimensions for complexes. Motivated by this, we have the following theorem. See 5.3 for the proof.

**Theorem 1.3.** Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\text{Mod}_R$ and $N$ a complex.

1. There is an inequality

$$\mathcal{G}\mathcal{B}\text{-dim}N \leq \mathcal{B}\text{-dim}N,$$

and the equality holds if $\mathcal{B}\text{-dim}N < \infty$.

2. If $\mathcal{G}\mathcal{B}\text{-dim}N < \infty$, then the following are equivalent:
   (a) $\mathcal{B}\text{-dim}N = \mathcal{G}\mathcal{B}\text{-dim}N$;
   (b) $\tilde{\text{ext}}_\mathcal{A}^i(X, N) = 0$ for all $i \in \mathbb{Z}$ and all $\mathcal{B}$-complexcxes $X$;
   (c) $\tilde{\text{ext}}_\mathcal{A}^i(X, N) = 0$ for some $i \in \mathbb{Z}$ and all $R$-modules $X \in \mathcal{A}$;
   (d) For each fibrant-cofibrant resolution $N \leftarrow \mathcal{Q}N \rightarrow \mathcal{Q}\mathcal{Q}N$ of $N$, there exist a small enough $n$ such that

$$\tilde{\text{ext}}^i(M, \mathcal{Q}n(\mathcal{Q}\mathcal{Q}N)) : \text{Ext}^i_{\mathcal{G}(\mathcal{B})}(M, \mathcal{Q}n(\mathcal{Q}\mathcal{Q}N)) \to \text{Ext}^i_R(M, \mathcal{Q}n(\mathcal{Q}\mathcal{Q}N))$$

is an isomorphism for all $i \in \mathbb{Z}$ and any $R$-module $M$ in $\mathcal{A}$.  


We note that the equality $B\text{-dim}N = GB\text{-dim}N$ in Theorem 1.3(2) does not hold in general. See Remark 5.4 below.

Let $X$ be a class of left $R$-modules and $n$ a non-negative integer. We let $\text{cores } \hat{X} \leq n$ ($\text{cores } \hat{X} < \infty$) be the class of left $R$-modules $M$ with $X\text{-id}(M) \leq n$ ($X\text{-id}(M) < \infty$). Detailed definitions can be found in Section 2. By [29, Theorem 12.3.1], a left and right Noetherian ring $R$ is Gorenstein if and only if $\text{Mod}R = \text{cores } \hat{G} \leq n$ for some non-negative integer $n$, where $\mathcal{G}\mathcal{I}$ is the class of Gorenstein injective $R$-modules. In [40] Hovey obtained the Gorenstein injective model structure on $\text{Mod}R$ whenever $R$ is Gorenstein. On the other hand, in order to study the representation theory of algebras, singularity categories are defined as the Verdier’s quotient triangulated categories $D_{sg}(R) := D^b(\text{mod}R)/\mathbb{K}^b(\mathcal{P})$, where $D^b(\text{mod}R)$ is the bounded derived category of finitely presented modules over a (left) coherent ring $R$ and $\mathbb{K}^b(\mathcal{P})$ is the bounded homotopy category of finitely generated projective $R$-modules [15]. Similar quotient triangulated categories were also studied by several people (see, for example, [10, 19, 20, 35]). Dualizing the proof of [19, Theorem 3.3], one can show that there is a triangle-equivalence $D^b(\text{Mod}R)/\mathbb{K}^b(\mathcal{I}) \cong \mathcal{G}\mathcal{I}$ whenever $R$ is a Gorenstein ring, where $\mathcal{G}\mathcal{I}$ is the stable category of $\mathcal{G}\mathcal{I}$ modulo $\mathcal{I}$.

In the general case, we have the following result; see Theorems 6.3, 6.4 and 7.14.

**Theorem 1.4.** Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\text{Mod}R$. Then $\mathcal{A} \cap \mathcal{G}(\mathcal{B})$ is a Frobenius category with $\mathcal{A} \cap \mathcal{B}$ the full subcategory of projectives and injectives. If $\mathcal{A} \cap \mathcal{G}(\mathcal{B})$ is the stable category of $\mathcal{A} \cap \mathcal{G}(\mathcal{B})$ modulo $\mathcal{A} \cap \mathcal{B}$, then the following are equivalent:

1. $\mathcal{GB}\text{-dim}M < \infty$ for any complex $M$ in $D^{-}(R)$;
2. $\mathcal{M} = (\mathcal{A}, \text{cores } \hat{\mathcal{B}} < \infty, \mathcal{G}(\mathcal{B}))$ is a hereditary abelian model structure and its homotopy category $H_0(\mathcal{M})$ is triangle-equivalent to $\mathcal{A} \cap \mathcal{G}(\mathcal{B})$;
3. There is a triangle-equivalence $D_{sg}(\mathcal{A}) \cong \mathcal{A} \cap \mathcal{G}(\mathcal{B})$, where $D_{sg}(\mathcal{A}) := D^b(\mathcal{A})/\mathbb{K}^b(\mathcal{A} \cap \mathcal{B})$ is the singularity category of $\mathcal{A}$ (see Definition 7.6 below).

Let $\mathcal{P}^{<\infty}$ be the class of finitely generated modules with finite projective dimension. The left little finitistic dimension of a ring $R$ is

$$\text{findim}(R) = \sup\{\text{pd}(P) \mid P \in \mathcal{P}^{<\infty}\}.$$ \hspace{1cm}

Recall that the Finitistic Dimension Conjecture states that the little finitistic dimension $\text{findim}(R)$ is finite for every Artin algebra $R$ (see [5, p.409] and [9]). This conjecture is also related to many other homological conjectures and attracts many algebraists, see for instance [1, 24, 55, 59]. The following result gives some criteria for the validity of this conjecture. See 8.7 for the proof.

**Theorem 1.5.** Let $R$ be an Artin algebra and $\mathcal{X} = (\mathcal{X}, \mathcal{Y})$ the cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$ in $\text{Mod}(R)$. Then the following are equivalent:

1. $\mathcal{GY}\text{-dim}M < \infty$ for any complex $M$ in $D^{-}(R)$;
2. $\mathcal{GY}\text{-dim}M < \infty$ for any complex $M$ in $D^{-}(R)$;
3. $\mathcal{GY}\text{-dim}M < \infty$;
4. $\mathcal{M} = (\mathcal{X}, \text{cores } \hat{\mathcal{Y}} < \infty, \mathcal{G}(\mathcal{Y}))$ is a hereditary abelian model structure and its homotopy category $H_0(\mathcal{M})$ is triangle-equivalent to the stable category $\mathcal{X} \cap \mathcal{G}(\mathcal{Y})$;
5. There is a triangle-equivalence $D_{sg}(\mathcal{X}) \cong \mathcal{X} \cap \mathcal{G}(\mathcal{Y})$, where $D_{sg}(\mathcal{X}) := D^b(\mathcal{X})/\mathbb{K}^b(\mathcal{X} \cap \mathcal{Y})$ is the singularity category of $\mathcal{X}$.
Furthermore, if $\text{findim}(R) < \infty$, then

$$\text{findim}R = \sup\{n \in \mathbb{Z} \mid \text{ext}^n_X(X, R) \neq 0 \text{ for some } X \in \mathcal{X} \cap \mathcal{Y}\}.$$ 

We conclude this section by summarizing the contents of this paper. Section 2 contains notations and definitions for use throughout this paper. In Section 3, we give definitions of Gorenstein dimensions for complexes with respect to any complete hereditary cotorsion pair and prove Theorem 1.1. Section 4 is devoted to studying relative cohomology groups for complexes with finite Gorenstein $\mathcal{B}$ dimensions and prove Theorem 1.2. In Section 5, the relationships between Gorenstein $\mathcal{B}$ dimensions and $\mathcal{B}$ dimensions for complexes are studied, including the proof of Theorem 1.3. In Section 6, we consider Frobenius categories and model structures, including the proofs of (1) $\Leftrightarrow$ (2) and the first claim in Theorem 1.4. Section 7 is devoted to studying singularity categories, including the proof of (1) $\Leftrightarrow$ (3) in Theorem 1.4. In Section 8, we characterize when the little finitistic dimension is finite and prove Theorem 1.5.

2. Preliminaries

Throughout this paper, $R$ is an associative ring and $\text{Mod}R$ is the class of left $R$-modules. All “$R$-modules” and “complexes” mean “left $R$-modules” and “chain complexes of left $R$-modules”, respectively. We use the term “subcategory” to mean a “full and additive subcategory that is closed under isomorphisms”. $\mathcal{E}$ is the class of exact complexes. $\mathcal{P}$ and $\mathcal{I}$ denote the classes of projective and injective $R$-modules, respectively.

Next we recall basic definitions and properties needed in the sequel. For more details the reader can consult [7], [23], [29], [32] or [34].

**Complexes.** Let $\mathcal{D}$ be an additive category. We denote by $\text{C}(\mathcal{D})$ the category of complexes in $\mathcal{D}$; the objects are complexes and morphisms are chain maps. We write the complexes homologically, so an object $X$ of $\text{C}(\mathcal{D})$ is of the following form

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots.$$ 

If $X_i = 0$ for $i \neq 0$ we identify $X$ with the object of $\mathcal{D}$ in degree 0, and object $M$ in $\mathcal{D}$ is thought of as the stalk complex concentrated in degree zero. Let $X_{\leq n}$ be the complex with $i$th component equal to $X_i$ for $i \leq n$ and to 0 for $i > n$, and $X_{\geq n}$ be the complex with $i$th component equal to $X_i$ for $i \geq n$ and to 0 for $i < n$. The $i$th shift of $X$ is the complex $X[i]$ with $n$th component $X_{n-i}$ and differential $\partial_{n}^{X[i]} = (-1)^i \partial_{n-i}^X$. The mapping cone of a morphism $\varphi : X \rightarrow Y$ is the complex $\text{Con}(\varphi)$ defined by $\text{Con}(\varphi)_n = Y_n \oplus X_{n-1}$ and $\partial_n^{\text{Con}(\varphi)} = \begin{pmatrix} \partial_n^Y & \varphi_{n-1} \\ 0 & -\partial_{n-1}^X \end{pmatrix}$.

A homomorphism $\varphi : X \rightarrow Y$ of degree $n$ is a family of $(\varphi_i)_{i \in \mathbb{Z}}$ of homomorphisms $\varphi_i : X_i \rightarrow Y_{i+n}$ in $\mathcal{D}$. In this case, we set $|\varphi| = n$. All such homomorphisms form an abelian group, denoted by $\text{Hom}_\mathcal{D}(X, Y)_n$; it is clearly isomorphic to $\prod_{i \in \mathbb{Z}} \text{Hom}_\mathcal{D}(X_i, Y_{i+n})$. We let $\text{Hom}_\mathcal{D}(X, Y)$ be the complex of $\mathbb{Z}$-modules with $n$th component $\text{Hom}_\mathcal{D}(X, Y)_n$ and differential

$$\partial(\varphi) = \partial^Y \varphi - (-1)^{|\varphi|} \varphi \partial^X.$$ 

If $\mathcal{D} = \text{Mod}R$ is the category of (left) $R$-modules, we write $\text{Hom}_R(X, Y)$ for $\text{Hom}_{\text{Mod}R}(X, Y)$ for all complexes $X$ and $Y$.  

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For any \( i \in \mathbb{Z} \), the cycles in \( \text{Hom}_D(X, Y)_i \) are the chain maps \( X \to Y \) of degree \( i \). A chain map of degree 0 is a morphism. Two morphisms \( \beta \) and \( \beta' \) in \( \text{Hom}_D(X, Y)_0 \) are called chain homotopic, denoted by \( \beta \sim \beta' \), if there exists a degree 1 homomorphism \( \nu \) such that \( \partial(\nu) = \beta - \beta' \). A chain homotopy equivalence is a morphism \( \varphi : X \to Y \) for which there exists a morphism \( \psi : Y \to X \) such that \( \varphi \psi \sim \text{id}_Y \) and \( \psi \varphi \sim \text{id}_X \).

The chain homotopy category of \( D \) will be denoted by \( \mathbb{K}(D) \). Its objects are the same as \( C(D) \) and morphisms are the chain homotopy classes of morphisms of complexes.

If \( D = \text{Mod} R \) is the category of (left) \( R \)-modules, we write \( C(R) \) (resp., \( \mathbb{K}(R) \)) for \( C(\text{Mod} R) \) (resp., \( \mathbb{K}(\text{Mod} R) \)). It is known that \( D \) is additive (resp., abelian) then so is \( C(D) \). In particular, \( C(R) \) is an abelian category and \( \mathbb{K}(R) \) is an additive category. We use subscripts +, −, b to denote boundedness conditions. For example, \( C^+(R) \) is the full subcategory of \( C(R) \) of left bounded (or bounded above) complexes. To every complex

\[
X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \cdots
\]

in \( C(R) \), the \textit{nth homology module} of \( X \) is the module \( H_n(X) = \ker(\partial_n^X)/\text{im}(\partial_{n+1}^X) \). We also set \( Z_n(X) = \ker(\partial_n^X) \), \( B_n(X) = \text{im}(\partial_{n+1}^X) \) and \( C_n(X) = \text{coker}(\partial_{n+1}^X) \).

A \textit{quasi-isomorphism} \( \varphi : X \to Y \) with \( X \) and \( Y \) in \( C(R) \) is a morphism such that the induced map \( H_n(\varphi) : H_n(X) \to H_n(Y) \) is an isomorphism for all \( n \in \mathbb{Z} \). The morphism \( \varphi \) is a quasi-isomorphism if and only if \( \text{Con}(\varphi) \) is exact. Two complexes \( X \) and \( Y \) are \textit{equivalent} \cite[1.1.11, p.164]{Enochs}, and denoted by \( X \simeq Y \), if they can be linked by a sequence of quasi-isomorphisms with arrows in the alternating directions.

Let \( \mathcal{H} \) be a subcategory of \( \text{Mod} R \). Then a complex \( L \) is \textit{Hom}_R(\mathcal{H}, -) \textit{exact} (resp., \( \text{Hom}_R(-, \mathcal{H}) \textit{exact} \)) if the complex \( \text{Hom}_R(M, L) \) (resp., \( \text{Hom}_R(L, M) \)) is exact for each \( M \in \mathcal{H} \).

\textbf{Cotorsion pairs.} Let \( D \) be an abelian category and \( \mathcal{X} \) a subcategory of \( D \). For an object \( M \in D \), write \( M \in ^{\perp} \mathcal{X} \) (resp., \( M \in ^{\perp} \mathcal{X} \)) if \( \text{Ext}^{\perp}_D(M, X) = 0 \) (resp., \( \text{Ext}^D(M, X) = 0 \)) for each \( X \in \mathcal{X} \). Dually, we can define \( M \in \mathcal{X}^{\perp} \) and \( M \in \mathcal{X}^{\perp} \).

Following Enochs \cite{Enochs}, Hovey \cite{Hovey} and Salce \cite{Salce}, a \textit{cotorsion pair} is a pair of classes \((A, B)\) in \( D \) such that \( A^{\perp} = B \) and \( ^{\perp} B = A \). A cotorsion pair \((A, B)\) is said to be \textit{hereditary} \cite{Hovey} if \( \text{Ext}^{i}_D(A, B) = 0 \) for all \( i \geq 1 \) and all \( A \in A \) and \( B \in B \). If we restrict the abelian category \( D \) to the category of chain complexes of \( R \)-modules or the category of \( R \)-modules, then by \cite{Hovey}, the condition that \((A, B)\) is hereditary is equivalent to that if whenever \( 0 \to L' \to L \to L'' \to 0 \) is exact with \( L, L'' \in A \), then \( L' \in A \), or equivalently, if whenever \( 0 \to B' \to B \to B'' \to 0 \) is exact with \( B', B'' \in B \), then \( B'' \in B \).

Let \( M \) be an object in \( D \). A morphism \( \phi : M \to X \) with \( X \in \mathcal{X} \) is called an \( \mathcal{X} \) \textit{preenvelope} of \( M \) if for any morphism \( f : M \to X' \) with \( X' \in \mathcal{X} \), there is a morphism \( g : X \to X' \) such that \( g\phi = f \). A monomorphism \( \phi : M \to B \) with \( B \in \mathcal{X} \) is said to be a \textit{special} \( \mathcal{X} \) \textit{preenvelope} of \( M \) if \( \text{coker}(\phi) \in ^{\perp} \mathcal{X} \). Dually we have the definitions of an \( \mathcal{X} \) precover and a special \( \mathcal{X} \) precover.

A cotorsion pair \((A, B)\) in \( D \) is called \textit{complete} if every object \( M \) of \( D \) has a special \( B \) preenvelope and a special \( A \) precover. If we choose \( D = \text{Mod} R \) for some ring \( R \), the most obvious example of a complete hereditary cotorsion pair is \((\text{Mod} R, \mathcal{T})\).
Model structures. In [40] Hovey laid out a correspondence between (nice enough) abelian model structures on a bicomplete abelian category $\mathcal{D}$ and cotorsion pairs on $\mathcal{D}$. Essentially, a model structure on $\mathcal{D}$ is two complete cotorsion pairs $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$, where $\mathcal{Q}$ is the class of cofibrant objects, $\mathcal{R}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects. And a model structure on $\mathcal{D}$ is determined by the above cotorsion pairs in the following way: the (trivial) cofibrations are the monomorphisms with (trivially) cofibrant cokernel, the (trivial) fibrations are the epimorphisms with (trivially) fibrant kernel and the weak equivalences are the maps that can be factored as a trivial cofibration followed by a trivial fibration.

Hovey’s correspondence makes it clear that an abelian model structure can be succinctly represented by a triple $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$. By a slight abuse of language we often refer to such a triple as an abelian model structure. Moreover, we also call an abelian model structure $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ hereditary [32] when the two cotorsion pairs $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ in Hovey’s correspondence are hereditary. Denote by $H_0(\mathcal{M})$ the homotopy category of an abelian model structure $\mathcal{M}$. An important feature of hereditary abelian model structures is that $H_0(\mathcal{M})$ is always a triangulated category, in the sense of Verdier, in the case that $\mathcal{M}$ is hereditary. We refer to [32], [39] and [41] and for a more detailed discussion on this matter.

Derived Categories. The derived category of the category of chain complexes of $R$-modules, denoted by $\mathbb{D}(R)$, is the category of chain complexes of $R$-modules localized at the class of quasi-isomorphisms (see [36, 54]). The symbol $\simeq$ is used to designate isomorphisms in $\mathbb{D}(R)$. The homological position and size of a complex $X$ are captured by the numbers supremum and infimum defined by

$$\text{sup} X = \sup \{i \in \mathbb{Z} \mid H_i(X) \neq 0\}, \quad \text{inf} X = \inf \{i \in \mathbb{Z} \mid H_i(X) \neq 0\}.$$ 

By convention $\text{sup} X = -\infty$ and $\text{inf} X = \infty$ if $X \simeq 0$. The full subcategories $\mathbb{D}^+(R), \mathbb{D}^-(R)$ and $\mathbb{D}^b(R)$ consist of complexes $X$ with $H_i(X) = 0$ for, respectively, $i > 0$, $i < 0$ and $|i| > 0$.

Denote by $\text{RHom}_R(-,-)$ the right derived functor of the homomorphism functor of complexes; by [6] and [51] no boundedness conditions are needed on the arguments. That is, for $X, Y \in \mathbb{D}(R)$, the complexes $\text{RHom}_R(X,Y)$ are uniquely determined up to isomorphism in $\mathbb{D}(R)$, and they have the usual functorial properties. We set $\text{Ext}_R^i(X,Y) = H_{-i}(\text{RHom}_R(X,Y))$ for $i \in \mathbb{Z}$. For modules $X$ and $Y$ this agrees with the notation of classical homological algebra.

**Definition 2.1.** ([52]) For every complex $X$ in $\mathbb{C}(R)$ with $X_n = 0 = H_{-n}(X)$ for all $n > 0$, the natural morphism $M = H_0(X) \to X$ is a quasi-isomorphism. In this event, $X$ is an $\mathcal{X}$-coresolution of $M$ if each $X_n \in \mathcal{X}$, and the associated exact sequence

$$X^+ = 0 \to M \to X_0 \to X_{-1} \to \cdots$$ 

is the augmented $\mathcal{X}$-coresolution of $M$ associated to $X$. An $\mathcal{X}$-coresolution $X$ of $M$ is proper if $X^+$ is $\text{Hom}_R(-, \mathcal{X})$ exact. The $\mathcal{X}$-injective dimension of $M$ is the quantity

$$\mathcal{X}\text{-id}(M) = \inf \{\sup \{n \geq 0 \mid X_{-n} \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$ 

Let $n$ be a non-negative integer. For convenience, we set

- $\hat{\mathcal{X}}^{\leq n}$ = the class of $R$-modules $M$ with $\mathcal{X}\text{-id}(M) \leq n$;
- $\hat{\mathcal{X}}^{< \infty}$ = the class of $R$-modules $M$ with $\mathcal{X}\text{-id}(M) < \infty$. 

Definition 2.2. ([52]) Let $\mathcal{X}$ be a class of $R$-modules and $N$ an $R$-module. If $N$ has a proper $\mathcal{X}$-coresolution $N \to X$, then for each integer $n$ and each $R$-module $M$, the $n$th relative cohomology group, denoted by $\text{Ext}^n_{\mathcal{X}}(M, N)$, is defined as

$$\text{Ext}^n_{\mathcal{X}}(M, N) = \text{H}_{-n}(\text{Hom}_R(M, X)).$$

We refer to [29, Section 8.2], [37, 2.4] and [52, Section 4] for a detailed discussion on this matter.

In the following sections, we always assume that $\mathbf{A} = (\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\text{Mod}R$.

3. Gorenstein $\mathcal{B}$ dimensions for complexes

In this section, we study some properties of Gorenstein $\mathcal{B}$-modules. Furthermore, we investigate Gorenstein $\mathcal{B}$ dimensions for complexes. We start with the following definition.

Definition 3.1. Let $R$ be a ring.

(1) An $R$-module $M$ is a Gorenstein $\mathcal{B}$-module if $M \in (\mathcal{A} \cap \mathcal{B})^\perp$ and there is a $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact exact sequence $\cdots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in \mathcal{A} \cap \mathcal{B}$.

(2) An exact complex $X$ is called totally $\mathcal{B}$-acyclic if each entry of $X$ belongs to $\mathcal{A} \cap \mathcal{B}$ and $Z_i(X)$ is a Gorenstein $\mathcal{B}$-module for every $i \in \mathbb{Z}$.

In what follows, we write $\mathcal{G}(\mathcal{B})$ for the class of Gorenstein $\mathcal{B}$-modules.

Remark 3.2. We note that if we set $(\mathcal{A}, \mathcal{B}) = (\text{Mod}R, \mathcal{I})$, the class of Gorenstein $\mathcal{B}$-modules defined here is exactly the class of Gorenstein injective modules (see [28, 38]).

Lemma 3.3. Let $R$ be a ring and $M$ an $R$-module. Then the following are equivalent:

(1) $M$ is a Gorenstein $\mathcal{B}$-module;

(2) $M \in (\mathcal{A} \cap \mathcal{B})^\perp$ and there is a $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact exact sequence $\cdots \to B_1 \to B_0 \to M \to 0$ with each $B_i \in \mathcal{B}$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). We assume that $\cdots \to B_1 \to B_0 \to M \to 0$ is a $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact exact sequence of $R$-modules with each $B_i \in \mathcal{B}$. Let $K_1 = \ker(B_0 \to M)$. Then $0 \to K_1 \to B_0 \to M \to 0$ is $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact. Since $M \in (\mathcal{A} \cap \mathcal{B})^\perp$, so is $K_1$. Note that $(\mathcal{A}, \mathcal{B})$ is complete. Then there exists an exact sequence $0 \to L_1 \to X_0 \to B_0 \to 0$ with $X_0 \in \mathcal{A} \cap \mathcal{B}$ and $L_1 \in \mathcal{B}$. Consider the following pullback diagram:

$$
\begin{array}{ccccccc}
0 & \to & 0 \\
\uparrow & & \uparrow \\
L_1 & \to & L_1 \\
\uparrow & & \uparrow \\
0 & \to & T_1 & \to & X_0 & \to & M & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & K_1 & \to & B_0 & \to & M & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0.
\end{array}
$$
The exactness of $0 \to L_1 \to T_1 \to K_1 \to 0$ implies $T_1 \in (A \cap B) \perp$. Let $K_2 = \ker(B_1 \to B_0)$. Then $0 \to K_2 \to B_1 \to K_1 \to 0$ is $\text{Hom}_R(A \cap B, -)$ exact. Thus we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
L_1 & L_1 & & & & & & 0 \\
0 & K_2 & Y_0 & T_1 & 0 \\
0 & K_2 & B_1 & K_1 & 0 \\
0 & 0 & & & & & & 0 \\
\end{array}
\]

Since $L_1$ and $B_1$ are in $B$, so is $Y_0$. Note that $K_2 \in (A \cap B) \perp$, and so $0 \to K_2 \to Y_0 \to T_1 \to 0$ is $\text{Hom}_R(A \cap B, -)$ exact. Since $\cdots \to B_3 \to B_2 \to K_2 \to 0$ is $\text{Hom}_R(A \cap B, -)$ exact, we get a $\text{Hom}_R(A \cap B, -)$ exact exact sequence $\cdots \to Y_2 \to Y_1 \to Y_0 \to T_1 \to 0$, where $Y_0 \in B$ and $Y_i = B_{i+1}$ for $i = 1, 2, \cdots$. By proceeding in this manner, we get a $\text{Hom}_R(A \cap B, -)$ exact exact sequence $\cdots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in A \cap B$.

A class $\mathcal{X}$ of $R$-modules is called \textit{injectively resolving} \cite{38} if $I \subseteq \mathcal{X}$ and for every exact sequence $0 \to X' \to X \to X'' \to 0$ of $R$-modules with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

\textbf{Proposition 3.4.} The following are true for any ring $R$.

1. The class $\mathcal{G}(B)$ is injectively resolving. Furthermore, $\mathcal{G}(B)$ is closed under direct products and direct summands.

2. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of $R$-modules. If $Y \in \mathcal{G}(B)$ and $Z \in \mathcal{G}(B)$, then $X \in \mathcal{G}(B)$ if and only if $X \in (A \cap B) \perp$.

\textbf{Proof.} (1) One easily checks that $\mathcal{G}(B)$ is closed under direct products by Lemma 3.3. To prove that $\mathcal{G}(B)$ is injectively resolving, we consider any exact sequence $0 \to X \to Y \to Z \to 0$ of $R$-modules with $X \in \mathcal{G}(B)$. First assume that $Z \in \mathcal{G}(B)$. Then there exist $\text{Hom}_R(A \cap B, -)$ exact exact sequences:

\[
\cdots \to L_1 \to L_0 \to X \to 0 \quad \text{and} \quad \cdots \to K_1 \to K_0 \to Z \to 0
\]

with all $L_i, K_i \in A \cap B$. By the proof of \cite[Lemma 8.2.1]{29}, we can construct a $\text{Hom}_R(A \cap B, -)$ exact exact sequence $\cdots \to X_1 \to X_0 \to Y \to 0$ with all $X_i \in A \cap B$. By assumption, $X \in (A \cap B) \perp$ and $Z \in (A \cap B) \perp$. So $Y \in (A \cap B) \perp$ and $Y \in \mathcal{G}(B)$, as desired.
Next we assume that $Y \in \mathcal{G}(\mathcal{B})$. Then there exists an exact sequence $0 \to H_0 \to B_0 \to Y \to 0$ of $R$-modules with $B_0 \in \mathcal{B}$ and $H_0 \in \mathcal{G}(\mathcal{B})$. Consider the following pullback diagram:

$$
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
H_0 & H_0 \\
\downarrow & \downarrow \\
0 & U_0 & B_0 & Z & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & X & Y & Z & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0.
\end{array}
$$

Since $H_0 \in \mathcal{G}(\mathcal{B})$ and $X \in \mathcal{G}(\mathcal{B})$, $U_0 \in \mathcal{G}(\mathcal{B})$ by the proof above. Thus there exists a $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact sequence $\cdots \to B_2 \to B_1 \to U_0 \to 0$ with all $B_i \in \mathcal{B}$. Assembling the sequence above and $0 \to U_0 \to B_0 \to Z \to 0$, we get a $\text{Hom}_R(\mathcal{A} \cap \mathcal{B}, -)$ exact sequence $\cdots \to B_2 \to B_1 \to B_0 \to Z \to 0$ with all $B_i \in \mathcal{B}$. Since $Y \in (\mathcal{A} \cap \mathcal{B})^\perp$ and $X \in (\mathcal{A} \cap \mathcal{B})^\perp$ by hypothesis, $Z$ belongs to $(\mathcal{A} \cap \mathcal{B})^\perp$. So $Z \in \mathcal{G}(\mathcal{B})$ by Lemma 3.3, as desired.

Finally we have to show that the class $\mathcal{G}(\mathcal{B})$ is closed under direct summands. Since $\mathcal{G}(\mathcal{B})$ is injectively resolving and closed under direct products, $\mathcal{G}(\mathcal{B})$ is closed under direct summands by [38, Proposition 1.4].

(2) The “only if” part is clear by definition. For the “if” part, since $Z \in \mathcal{G}(\mathcal{B})$, there exists an exact sequence $0 \to K \to L \to Z \to 0$ with $L \in \mathcal{A} \cap \mathcal{B}$ and $K \in \mathcal{G}(\mathcal{B})$. Consider the following pullback diagram:

$$
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
K & K \\
\downarrow & \downarrow \\
0 & X & H & L & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & X & Y & Z & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0.
\end{array}
$$

By (1) and the exactness of the middle column in the above diagram, $H \in \mathcal{G}(\mathcal{B})$. Note that $X \in (\mathcal{A} \cap \mathcal{B})^{\perp_1}$ and $L \in \mathcal{A} \cap \mathcal{B}$. It follows that the middle row in the above diagram is split. So $X \in \mathcal{G}(\mathcal{B})$ by (1). \qed

**Definition 3.5.** ([31, Definition 3.3]) Let $R$ be ring and $X$ a complex.

(1) $X$ is called an $\mathcal{A}$ complex if it is exact and $Z_nX \in \mathcal{A}$ for all $n$. 

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(2) $X$ is called a $\mathcal{B}$ complex if it is exact and $Z_nX \in \mathcal{B}$ for all $n$.
(3) $X$ is called a $dg\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for each $n$, and $\text{Hom}_R(X, B)$ is exact whenever $B$ is a $\mathcal{B}$ complex.
(4) $X$ is called a $dg\mathcal{B}$ complex if $X_n \in \mathcal{B}$ for each $n$, and $\text{Hom}_R(A, X)$ is exact whenever $A$ is an $\mathcal{A}$ complex.

In what follows, we denote the class of $\mathcal{A}$ (resp., $\mathcal{B}$) complexes by $\tilde{\mathcal{A}}$ (resp., $\tilde{\mathcal{B}}$) and the class of $dg\mathcal{A}$ (resp., $dg\mathcal{B}$) complexes by $dg\tilde{\mathcal{A}}$ (resp., $dg\tilde{\mathcal{B}}$). By [31, Theorem 3.12], we have $dg\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$ and $dg\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$ whenever $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\text{Mod}R$. In particular, if $(\mathcal{A}, \mathcal{B}) = (\text{Mod}R, \mathcal{I})$ then $dg\mathcal{I}$ complexes are exactly $dg$-injective complexes (see [6, 23]). We refer to [6, 23] and [31] for a more detailed discussion on this matter.

**Lemma 3.6.** ([56, Corollaries 2.7 and 2.8]) If $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\text{Mod}R$, then the induced cotorsion pairs $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ in $\mathcal{C}(R)$ are both complete and hereditary. Furthermore, $dg\tilde{\mathcal{A}} \cap \mathcal{E} = \tilde{\mathcal{A}}$ and $dg\tilde{\mathcal{B}} \cap \mathcal{E} = \tilde{\mathcal{B}}$, where $\mathcal{E}$ is the class of exact complexes. So there exists a model structure on $\mathcal{C}(R)$, denoted by $\mathcal{C}(R)_{\mathcal{M}\mathcal{B}}$, satisfying that:

1. the weak equivalences are the quasi-isomorphisms;
2. the cofibrations (resp., trivial cofibrations) are the monomorphisms whose cokernels are in $dg\tilde{\mathcal{A}}$ (resp., $\tilde{\mathcal{A}}$);
3. the fibrations (resp., trivial fibrations) are the epimorphisms whose kernels are in $dg\tilde{\mathcal{B}}$ (resp., $\tilde{\mathcal{B}}$).

In particular, $dg\tilde{\mathcal{A}}$ is the class of cofibrant objects and $dg\tilde{\mathcal{B}}$ is the class of fibrant objects.

Some nice introductions to the basic ideal of a model category can be found in [25, 39].

**Fact 3.7.** Let $M$ be a complex. Then $M$ has a cofibrant replacement $p_M : QM \to M$ in $\mathcal{C}(R)_{\mathcal{M}\mathcal{A}}$ and a fibrant replacement $i_M : M \to RM$ in $\mathcal{C}(R)_{\mathcal{M}\mathcal{B}}$, where $QM$ is cofibrant and $p_M$ is a trivial fibration, and $RM$ is fibrant and $i_M$ is a trivial cofibration. We can insist these exist functorially if one wishes. We refer to [25, Sections 4 and 5] and [33, Section 4] for a detailed discussion on this matter.

**Definition 3.8.** ([42, Definition 3.1]) Let $R$ be a ring and $M$ a complex. A **fibrant-cofibrant resolution** of $M$ is a diagram $M \xrightarrow{p_M} QM \xrightarrow{i_M} RM$ of morphisms of complexes with $p_M$ a cofibrant replacement in $\mathcal{C}(R)_{\mathcal{M}\mathcal{A}}$ and $i_M$ a fibrant replacement in $\mathcal{C}(R)_{\mathcal{M}\mathcal{B}}$.

**Remark 3.9.** We note that $RM$ in the above definition is in $dg\tilde{\mathcal{A}} \cap dg\tilde{\mathcal{B}}$. As far as the notions of special $dg\tilde{\mathcal{A}}$ precovers and special $dg\tilde{\mathcal{B}}$ preenvelopes are concerned, a cofibrant replacement $p_M$ is exactly a special $dg\tilde{\mathcal{A}}$ precover of $M$ and a fibrant replacement $i_M$ is exactly a special $dg\tilde{\mathcal{B}}$ preenvelope of $M$.

**Definition 3.10.** Let $R$ be a ring and $M$ a complex.

1. A **Tate $\mathcal{B}$ resolution** of $M$ is a diagram $M \leftarrow QM \to \mathcal{R}QM \to T$ of morphisms of complexes with $M \leftarrow QM \to \mathcal{R}QM$ a fibrant-cofibrant resolution of $M$ such that $T$ is a totally $\mathcal{B}$-acyclic complex and $\tau_i$ is bijective for all $i \ll 0$. A Tate $\mathcal{B}$ resolution is **split** if $\tau_i$ is a split monomorphism for all $i \in \mathbb{Z}$.
The Gorenstein $B$ dimension, $\mathcal{GB}$-$\dim M$, of $M$ is defined as
\[
\mathcal{GB}$-$\dim M = \inf \left\{ -n \in \mathbb{Z} \mid M \leftarrow QM \rightarrow RQM \xrightarrow{T} T \text{ is a Tate } B \text{ resolution of } M \text{ such that } \tau_i \text{ is bijective for each } i \leq n \right\}.
\]

**Remark 3.11.** (1) Let $M$ be an $R$-module. By [6, 1.4.I, p.133] and (1) $\iff$ (3) of Theorem 1.1 from the introduction, we note that the Gorenstein $B$ dimension of $M$ defined here is exactly the $\mathcal{GB}$-injective dimension of $M$ defined in Section 2.

(2) If we replace the cotorsion pair $(A, B)$ with the cotorsion pair $(\text{Mod} R, I)$ in the definition above, then, for each complex $M$, one easily checks that the Tate $I$ resolution of $M$ defined here is the complete coresolution of $M$ given by Asadollahi and Salarian in [3, Definition 2.1]. By (i) $\iff$ (iii) of Theorem 2.3 in [3] and (1) $\iff$ (3) of Theorem 1.1 from the introduction, we get that the Gorenstein $I$ dimension of $M$ defined here is exactly the Gorenstein injective dimension of $M$ defined by Asadollahi and Salarian in [3, Definition 2.2].

**Lemma 3.12.** ([42, Lemma 3.5]) If $F \in \text{dg} \tilde{A}$ and $X \in \text{dg} \tilde{B}$, then $R\text{Hom}_R(F, X)$ can be represented by $\text{Hom}_R(F, X)$.

**Lemma 3.13.** If $X \simeq I$ in $D^{-}(R)$ with $I$ a dg-injective complex and $X$ a dg-$B$ complex, then
\[
\text{Ext}^1_R(A, Z_n(X)) \cong \text{Ext}^1_R(A, Z_n(I))
\]
for any $A \in \mathcal{A}$ and any integer $n$ with $\text{inf } I \geq n$.

**Proof.** By assumption, we get that $\text{inf } X = \text{inf } I \geq n > -\infty$. For each $A \in \mathcal{A}$, we have
\[
\text{Ext}^1_R(A, Z_n(X)) = H_{-1}(R\text{Hom}_R(A, Z_n(X)))
\]
\[
\cong H_{-1}(R\text{Hom}_R(A, X_{\leq n}[-n]))
\]
\[
= H_{n-1}(R\text{Hom}_R(A, X_{\leq n}))
\]
\[
\cong H_{n-1}(\text{Hom}_R(A, X_{\leq n}))
\]
\[
= H_{n-1}(\text{Hom}_R(A, X))
\]
\[
\cong H_{n-1}(R\text{Hom}_R(A, X))
\]
\[
\cong H_{n-1}(\text{Hom}_R(A, I))
\]
\[
\cong H_{-1}(R\text{Hom}_R(A, I_{\leq n}[-n]))
\]
\[
\cong H_{-1}(\text{Hom}_R(A, \text{Hom}_R(A, I)))
\]
\[
= \text{Ext}^1_R(A, Z_n(I))
\]
where the second and third isomorphisms follow from [31, Lemma 3.4] and Lemma 3.12. This completes the proof.

**Lemma 3.14.** Let $M \leftarrow QM \rightarrow RQM \xrightarrow{T} T$ be a Tate $B$ resolution of $M$. If $n$ is an integer such that $\tau_i$ is bijective for all $i \leq n$, then there exist a Tate $B$ resolution $M \leftarrow QM \rightarrow RQM \xrightarrow{T'} T'$ with each $\tau'_i$ a split monomorphism such that $\tau'_i$ is bijective for all $i \leq n$ and a homotopy equivalence $\alpha : T' \to T$ such that $\tau = \alpha \tau'$ and $\alpha_i = \text{id}_{T'_i}$ for all $i \leq n$.

**Proof.** The proof is dual to that of [8, Construction 3.7].

Now we can give the proof of Theorem 1.1.
3.15. Proof of Theorem 1.1. (1) $\Rightarrow$ (2). By hypothesis, there is a Tate $B$ resolution $M \leftarrow QM \rightarrow \mathcal{R}QM \xrightarrow{\sim} T$ such that $\tau_{n<} : T_{n<} \rightarrow (\mathcal{R}QM)_{n<}$ is an isomorphism of complexes. Hence $Z_{n}(\mathcal{R}QM) \cong Z_{n}(T)$ and $H_{i}(T) \cong H_{i}(\mathcal{R}QM) = 0$ for all $i \leq -n$. So $Z_{n}(\mathcal{R}QM) \in \mathcal{G}(B)$ and $\inf M = \inf \mathcal{R}QM \geq -n$.

(2) $\Rightarrow$ (3). By hypothesis, there is a $dg-B$ complex $B$ with $M \cong B$ such that $Z_{n}(B) \in \mathcal{G}(B)$. Let $I$ be a $dg$-injective complex with $M \cong I$. Thus $B \cong I$, and hence there is a quasi-isomorphism $\varphi : B \rightarrow I$ by [6, 1.4.1, p.133].

If $\varphi$ is injective, then there is an exact sequence $0 \rightarrow B \xrightarrow{\varphi} I \rightarrow K \rightarrow 0$ of complexes with $K$ an exact complex. Since both $B$ and $I$ are $dg$-$B$ complexes, $K$ is a $dg$-$B$ complex by Lemma 3.6. Note that $K$ is an exact complex. Then $K \in \mathcal{B}$ and $Z_{n}(K) \in \mathcal{B}$. Thus there is an exact sequence $0 \rightarrow Z_{-n}(B) \rightarrow Z_{-n}(I) \rightarrow Z_{-n}(K) \rightarrow 0$ with $Z_{-n}(K) \in \mathcal{B}$ and $Z_{-n}(B) \in \mathcal{G}(B)$, and so $Z_{-n}(I) \in \mathcal{G}(B)$ by Proposition 3.4(1).

Suppose that $\varphi$ is not injective. By Lemma 3.6, there is a special $\mathcal{B}$ preenvelope $B \rightarrow G$ of $B$. Thus $B \rightarrow I \oplus G$ is an injective quasi-isomorphism with $I \oplus G \in dg\mathcal{B}$ such that $Z_{-n}(I \oplus G) \cong Z_{-n}(I) \oplus Z_{-n}(G)$. Note that $Z_{-n}(I \oplus G) \in \mathcal{G}(B)$ by the proof above, and so $Z_{-n}(I) \in \mathcal{G}(B)$ by Proposition 3.4(1).

(3) $\Rightarrow$ (4) follows from Lemma 3.6 and [6, 1.4.1, p.133].

(4) $\Rightarrow$ (3) The proof is similar to that of (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (5). Let $B$ be a $dg$-$B$ complex with $M \cong B$. By hypothesis, there exists a $dg$-$B$ complex $I$ with $M \cong I$ such that $Z_{-n}(I) \in \mathcal{G}(B)$. Let $X$ be any $R$-module in $A \cap B$. Then $\text{Ext}^{1}_{R}(X, Z_{-n}(I)) = 0$. By Lemma 3.13, we get that $\text{Ext}^{1}_{R}(X, Z_{-n}(B)) = 0$.

Note that there exists an exact sequence $0 \rightarrow B \rightarrow L \rightarrow K \rightarrow 0$ in $\mathcal{C}(R)$ with $L$ a $dg$-injective complex and $K$ an exact complex. Since $L$ and $B$ are $dg$-$B$ complexes, $K$ is a $dg$-$B$ complex by Lemma 3.6. Thus $K \in \mathcal{B}$ and $Z_{-n}(K) \in \mathcal{B}$. One easily checks that the sequence $0 \rightarrow Z_{-n}(B) \rightarrow Z_{-n}(L) \rightarrow Z_{-n}(K) \rightarrow 0$ is exact. Since $Z_{-n}(L) \in \mathcal{G}(B)$ by (3), $Z_{-n}(B) \in \mathcal{G}(B)$ by Proposition 3.4(2).

(5) $\Rightarrow$ (6) and (7) $\Rightarrow$ (1) are trivial.

(6) $\Rightarrow$ (7). By Lemma 3.14, it suffices to show that there exists a Tate $B$ resolution $M \leftarrow QM \rightarrow \mathcal{R}QM \xrightarrow{\sim} T$ of $M$ with $\tau_{i} = \text{id}_{(\mathcal{R}QM)_{i}}$ for $i \leq -n$. By hypothesis, there exists a fibrant-cofibrant resolution $M \leftarrow QM \rightarrow \mathcal{R}QM$ of $M$ such that $Z_{-n}(\mathcal{R}QM) \in \mathcal{G}(B)$. Then there exists a $\text{Hom}_{R}(A \cap B, -)$ exact exact sequence $\cdots \rightarrow B_{1} \rightarrow B_{0} \rightarrow Z_{-n}(\mathcal{R}QM) \rightarrow 0$ with each $B_{i} \in A \cap B$. Let $\widehat{X} = \Sigma^{1-n}X$ where $X$ is the complex $\cdots \rightarrow B_{2} \rightarrow B_{1} \rightarrow B_{0} \rightarrow 0$. Note that $\mathcal{R}QM \in dgA \cap dg\mathcal{B}$. Thus there exists a morphism $\gamma : (\mathcal{R}QM)_{\geq 1-n} \rightarrow \widehat{X}$ such that the following diagram

\[
\begin{array}{c}
\cdots \\
\gamma_{2-n} \\
\gamma_{1-n} \\
\cdots \\
\widehat{X}_{2-n} \\
\widehat{X}_{1-n} \\
Z_{n}(\mathcal{R}QM) \\
0
\end{array}
\xrightarrow{\gamma_{2-n}}
\begin{array}{c}
(\mathcal{R}QM)_{2-n} \\
(\mathcal{R}QM)_{1-n} \\
Z_{n}(\mathcal{R}QM) \\
0
\end{array}
\xrightarrow{\gamma_{1-n}}
\begin{array}{c}
Z_{n}(\mathcal{R}QM) \\
0
\end{array}
\]

commutes. Let $T$ be the complex obtained by splicing $\widehat{X}$ and $(\mathcal{R}QM)_{\leq -n}$ along $Z_{-n}(\mathcal{R}QM)$. One easily checks that $T$ is an exact complex with each entry in $A \cap B$ and $Z_{i}(T)$ is a Gorenstein $B$ module for every $i \in \mathbb{Z}$. Set

\[
\tau_{i} = \begin{cases} 
\gamma_{i} & \text{for } i > -n, \\
\text{id}_{(\mathcal{R}QM)_{i}} & \text{for } i \leq -n.
\end{cases}
\]

Thus $\tau : \mathcal{R}QM \rightarrow T$ is a morphism, as desired.
To show the last claim, \( GB\text{-dim}M = \sup \{-\inf R\text{Hom}_R(X, M) | X \in A \cap B\} \) holds when \( GB\text{-dim}M = -\infty \) by noting that \( GB\text{-dim}M = -\infty \) if and only if \( M \) is exact. By assumption, we assume that \( GB\text{-dim}M = g < \infty \) is an integer. First we need to show that

\[
\sup \{-\inf R\text{Hom}_R(X, M) | X \in A \cap B\} \leq g.
\]

By (3) and Proposition 3.4(1), there exists a dg-injective complex \( I \) such that \( M \cong I \) and \( Z_{-n}(I) \in G(\mathcal{B}) \) for all \( n \geq g \). For each \( i \geq 1 \) and every \( X \in A \cap B \), we get that

\[
\begin{align*}
H_{-g-i}(R\text{Hom}_R(X, M)) &= H_{-g-i}(\text{Hom}_R(X, I)) \\
&= H_{-1}(\text{Hom}_R(X, I_{\leq g-i+1}[g+i-1])) \\
&= H_{-1}(R\text{Hom}_R(X, Z_{-g-i+1}(I))) \\
&= \text{Ext}^{1}_{R}(X, Z_{-g-i+1}(I)) \\
&= 0.
\end{align*}
\]

This implies that \(-\ inf R\text{Hom}_R(X, M) \leq g \) for all \( X \in A \cap B \), as desired.

Next we need to show that \( \sup \{-\inf R\text{Hom}_R(X, M) | X \in A \cap B\} \geq g \). Suppose on the contrary that \( \sup \{-\inf R\text{Hom}_R(X, M) | X \in A \cap B\} < g \). Since \( GB\text{-dim}M = g \), there exists a Tate \( B \) resolution \( M \leftarrow Q_{\mathcal{M}} \rightarrow RQ_{\mathcal{M}} \rightarrow T \) with each \( T_i \in A \cap B \) such that \( Z_{-g}(RQ_{\mathcal{M}}) = Z_{-g}(T) \) and \( \tau_i = \text{id}_{(RQ_{\mathcal{M}})} \), for all \( i \leq -g \). Let \( j : Z_{-g}(RQ_{\mathcal{M}}) \rightarrow (RQ_{\mathcal{M}})_{-g} \) be the inclusion. Then there exists an epimorphism \( q : T_{1-g} \rightarrow Z_{-g}(RQ_{\mathcal{M}}) \) such that \( \partial_{n-g} = jq \) and a morphism \( s : (RQ_{\mathcal{M}})_{1-g} \rightarrow Z_{-g}(RQ_{\mathcal{M}}) \) such that \( \partial_{n-g}^{RQ_{\mathcal{M}}} = js \). Note that \( H_{-g}(R\text{Hom}_R(T_{1-g}, RQ_{\mathcal{M}})) = 0 \) by hypothesis. Then \( H_{-g}(\text{Hom}_R(T_{1-g}, RQ_{\mathcal{M}})) = 0 \) by Lemma 3.12. Hence we have the following exact sequence

\[
\text{Hom}_R(T_{1-g}, (RQ_{\mathcal{M}})_{1-g}) \rightarrow \text{Hom}_R(T_{1-g}, (RQ_{\mathcal{M}})_{-g}) \rightarrow \text{Hom}_R(T_{1-g}, (RQ_{\mathcal{M}})_{-1}).
\]

It is easy to check that \( \text{Hom}_R(T_{1-g}, s) : \text{Hom}_R(T_{1-g}, (RQ_{\mathcal{M}})_{1-g}) \rightarrow \text{Hom}_R(T_{1-g}, Z_{-g}(RQ_{\mathcal{M}})) \) is an epimorphism. Then there exists a map \( \beta : T_{1-g} \rightarrow (RQ_{\mathcal{M}})_{1-g} \) such that \( q = s\beta \). Since \( q \) is an epimorphism, so is \( s \). Hence \( -\inf M \leq g - 1 \). Note that \( q : T_{1-g} \rightarrow Z_{-g}(RQ_{\mathcal{M}}) \) is an epic \( A \cap B \) precover of \( Z_{-g}(RQ_{\mathcal{M}}) \). Then \( s : (RQ_{\mathcal{M}})_{1-g} \rightarrow Z_{-g}(RQ_{\mathcal{M}}) \) is an epic \( A \cap B \) precover of \( Z_{-g}(RQ_{\mathcal{M}}) \). Thus \( Z_{1-g}(RQ_{\mathcal{M}}) \in (A \cap B)^{+1} \), and so \( Z_{1-g}(RQ_{\mathcal{M}}) \) is a Gorenstein \( B \) module by Proposition 3.4(2). This is a contradiction. So \( GB\text{-dim}M = \sup \{-\inf R\text{Hom}_R(X, M) | X \in A \cap B\} \). This completes the proof.

\[\square\]

**Corollary 3.16.** Let \( 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \) be an exact sequence of complexes. If two complexes have finite Gorenstein \( B \) dimension, then so does the third.

**Proof.** The proof is similar to that of [53, Theorem 3.9(1)]. \[\square\]

By [56, Definition 3.1], the \( B \) dimension of a complex \( M \), denoted by \( B\text{-dim}M \), is defined as \( B\text{-dim}M = \inf \{\sup \{i | B_{-i} = 0\} | M \cong B \} \), where the symbol “\( \cong \)” stands for quasi-isomorphism.

**Corollary 3.17.** Let \( R \) be a ring and \( M \) a complex. Then there is an inequality

\[
GB\text{-dim}M \leq B\text{-dim}M,
\]

and the equality holds if \( B\text{-dim}M < \infty \).

**Proof.** Set \( B\text{-dim}M = m \) and \( GB\text{-dim}M = n \). There is nothing to prove if \( m = \infty \). We may assume that \( m \) is finite. It follows from Theorem 1.1 and [56, Theorem 3.4] that \( n \leq m \).
Suppose $n < m$. Choose a dg-injective complex $I$ such that $M \cong I$. Note that $-\inf M \leq n$ and $Z_{-n}(I) \in \mathcal{G}(\mathcal{B})$ by Theorem 1.1. Then there exists an exact sequence of $R$-modules

$$0 \to Z_{-n}(I) \to I_{-n} \to I_{-n-1} \to \cdots \to I_{1-m} \to Z_{-m}(I) \to 0.$$ 

Applying [56, Theorem 3.4] again, we get that $Z_{-m}(I) \in \mathcal{B}$. Thus there exists an exact sequence of $R$-modules with $K \in \mathcal{B}$ and $L \in \mathcal{A} \cap \mathcal{B}$. Consider the following pullback diagram:

$$\begin{array}{ccc}
Z_{1-m}(I) & \to & Z_{1-m}(I) \\
\downarrow & & \downarrow \\
I_{1-m} & \to & 0 \\
\downarrow & & \downarrow \\
K & \to & L \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}$$

Since $K$ and $I_{1-m}$ belong to $\mathcal{B}$, so is $U$. Note that $Z_{-n}(I) \in \mathcal{G}(\mathcal{B})$ by the proof above. It follows from Proposition 3.4(1) that $Z_{1-m}(I) \in \mathcal{G}(\mathcal{B})$. Thus the middle column in the above diagram is split. So $Z_{1-m}(I)$ is in $\mathcal{B}$. We proceed in this manner to get that $Z_{-n}(I)$ is in $\mathcal{B}$. Now [56, Theorem 3.4] implies that $m \leq n$, which contradicts our assumption. Hence $m = n$, as desired.

**Corollary 3.18.** For every family of complexes $\{M_i\}_{i \in I}$ one has

$$\mathcal{GB} \text{-dim } \prod_{i \in I} M_i = \sup_{i \in I} \{ \mathcal{GB} \text{-dim } M_i \}.$$

**Proof.** The proof is similar to that of [53, Corollay 3.5].

4. Relative cohomology groups for complexes with finite Gorenstein $\mathcal{B}$ dimensions

The goal of this section is to study relative cohomology groups for complexes with finite Gorenstein $\mathcal{B}$ dimensions and prove Theorem 1.2. To this end, we start with the following fact from Hu and Ding [42].

**Fact 4.1.** Let $M$ and $N$ be two complexes. According to Lemma 3.6, there are two fibrant-cofibrant resolutions $M \leftarrow QM \to RQM$ and $N \leftarrow QN \to RN$ of $M$ and $N$, respectively. A homomorphism $\beta \in \text{Hom}_R(RQM, RN)$ is bounded below (or right bounded) if $\beta_i = 0$ for all $i \ll 0$. The subset $\text{Hom}_R^\leftarrow(RQM, RN)$ of $\text{Hom}_R(RQM, RN)$, consisting of all bounded below homomorphisms, is a subcomplex with components

$$\text{Hom}_R^\leftarrow(RQM, RN)_n = \{ (\varphi_i) \in \text{Hom}_R(RQM, RN)_n \mid \varphi_i = 0 \text{ for all } i \ll 0 \}.$$ 

We set

$$\widehat{\text{Hom}}_R(RQM, RN) = \text{Hom}_R(RQM, RN)/\text{Hom}_R^\leftarrow(RQM, RN).$$
By [42, Definition 3.7], the $n$th *Tate-Vogel cohomology group*, denoted by $\widetilde{\text{ext}}^n_{A}(M, N)$, is defined as

$$\widetilde{\text{ext}}^n_{A}(M, N) = H_{-n}(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N})).$$

**Remark 4.2.** Let $M$ and $N$ be two complexes. According to Fact 4.1, there exists an exact sequence of complexes

$$0 \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to \widehat{\text{Hom}}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to 0.$$  

It follows from Lemma 3.12 that $\text{Ext}^n_R(M, N) = H_{-n}(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}))$ for each integer $n$. Let $\text{ext}^n_{A}(M, N) = H_{-n}(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}))$ for each integer $n$. Then we have a long exact sequence

$$\cdots \to \text{ext}^n_{A}(M, N) \to \text{Ext}^n_R(M, N) \to \text{ext}^n_{A}(M, N) \to \text{ext}^{n+1}_{A}(M, N) \to \cdots.$$  

By [42, Lemma 3.4], one can see that $\text{ext}^n_{A}(-,-)$ is a cohomological functor for each integer $n$, independent of the choice of cofibrant replacements and fibrant replacements.

Next we give general techniques for computing cohomologies $\text{ext}^n_{A}(M, N)$ and $\text{ext}^n_{A}(M, N)$ whenever $N$ is a complex with finite Gorenstein $B$ dimension.

**Theorem 4.3.** Let $R$ be a ring and $N$ a complex of finite Gorenstein $B$ dimension. For each complex $M$ and each integer $n$, there exist isomorphisms

$$\text{ext}^n_{A}(M, N) \cong H_{-n}(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, T)) \text{ and } \text{ext}^{n+1}_{A}(M, N) \cong H_{-n}(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, L_N)),$$  

where $M \leftarrow \mathcal{Q}M \to \mathcal{R}Q\mathcal{M}$ is a fibrant-cofibrant resolution of $M$ and $N \leftarrow \mathcal{Q}N \to \mathcal{R}Q\mathcal{N} \xrightarrow{T} T$ is a split Tate $B$ resolution of $N$ with $L_N = \text{coker}(\tau_i)$ such that each $\tau_i$ is a split monomorphism and $\tau_i = \text{id}_{(\mathcal{R}Q\mathcal{N})}$, for all $i \ll 0$. In the case when $M$ is a left bounded dg-$A$ complex, we have

$$\text{ext}^n_{A}(M, N) \cong H_{-n}(\text{Hom}_R(M, T)) \text{ and } \text{ext}^{n+1}_{A}(M, N) \cong H_{-n}(\text{Hom}_R(M, L_N)).$$

**Proof.** By Theorem 1.1, there exists a split Tate $B$ resolution $N \leftarrow \mathcal{Q}N \to \mathcal{R}Q\mathcal{N} \xrightarrow{T} T$ of $N$ such that each $\tau_i$ is a split monomorphism and $\tau_i = \text{id}_{(\mathcal{R}Q\mathcal{N})}$, for all $i \ll 0$. Let $L_N = \text{coker}(\tau)$. Then the exact sequence $0 \to \mathcal{R}Q\mathcal{N} \to T \to L_N \to 0$ of complexes is split in each degree. Let $M \leftarrow \mathcal{Q}M \to \mathcal{R}Q\mathcal{M}$ be a fibrant-cofibrant resolution of $M$. Applying the functor $\text{Hom}_R(\mathcal{R}Q\mathcal{M}, -)$ to the exact sequence above, we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to \widehat{\text{Hom}}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}) \to 0 \\
\downarrow \\
0 \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, T) \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, T) \to \widehat{\text{Hom}}_R(\mathcal{R}Q\mathcal{M}, T) \to 0 \\
\downarrow \\
0 \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, L_N) \to \text{Hom}_R(\mathcal{R}Q\mathcal{M}, L_N) \to \widehat{\text{Hom}}_R(\mathcal{R}Q\mathcal{M}, L_N) \to 0 \\
\downarrow \\
0 \quad 0 \quad 0.
\end{array}
\]

Since $L_N \in \mathcal{C}^-(R)$, $\text{Hom}_R(\mathcal{R}Q\mathcal{M}, L_N) = \text{Hom}_R(\mathcal{R}Q\mathcal{M}, L_N)$. For each integer $n \in \mathbb{Z}$, we have $H_n(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N})) \cong H_n(\text{Hom}_R(\mathcal{R}Q\mathcal{M}, T))$. Note that $\mathcal{R}Q\mathcal{M} \in \text{dg}A \cap \text{dg}B$ and $T$ is an
exact complex such that $\text{Hom}_R(X,T)$ is exact for all $X \in A \cap B$. Then $\text{Hom}_R(RQ, M, T)$ is exact. Thus $H_n(\text{Hom}_R(RQ, M, T)) \cong H_n(\text{Hom}_R(RQ, M, T))$ for each integer $n \in \mathbb{Z}$, and hence

$$H_n(\text{Hom}_R(RQ, M, T)) \cong H_n(\text{Hom}_R(RQ, M, T)) \cong H_n(\text{Hom}_R(RQ, M, T))$$

for each integer $n \in \mathbb{Z}$. So $\text{ext}^n_A(M, N) \cong H_{-n}(\text{Hom}_R(RQ, M, T))$ for each integer $n \in \mathbb{Z}$.

By the exactness of the left column in the above diagram, we have the following exact sequence

$$\cdots \longrightarrow H_n(\text{Hom}_R(RQ, M, T)) \longrightarrow H_n(\text{Hom}_R(RQ, M, L_N)) \longrightarrow H_n(\text{Hom}_R(RQ, M, L_N))$$

Since $\text{Hom}_R(RQ, M, T)$ is exact by the proof above, we get that

$$H_{n-1}(\text{Hom}_R(RQ, M, T)) \cong H_n(\text{Hom}_R(RQ, M, L_N)) = H_n(\text{Hom}_R(RQ, M, L_N))$$

for all integers $n \in \mathbb{Z}$. So $\text{ext}^{n+1}_A(M, N) \cong H_{-n}(\text{Hom}_R(RQ, M, L_N))$ for all integers $n \in \mathbb{Z}$.

Now assume that $M$ is a left bounded $dg$-$A$ complex. Note that there is an exact sequence $0 \to M \to RM \to KM \to 0$ of complexes with $RM \in dg\tilde{B} \cap \mathbb{C}_C(R)$ and $KM \in \tilde{A} \cap \mathbb{C}_C(R)$ by [42, Lemma 3.11]. Hence both $\text{Hom}_R(KM, T)$ and $\text{Hom}_R(KM, L_N)$ are exact by [22, Lemma 2.5]. It is easy to check that the sequences

$$0 \to \text{Hom}_R(KM, T) \to \text{Hom}_R(RM, T) \to \text{Hom}_R(M, T) \to 0$$

and

$$0 \to \text{Hom}_R(KM, L_N) \to \text{Hom}_R(RM, L_N) \to \text{Hom}_R(M, L_N) \to 0$$

are exact. Thus for any integer $n$, we have

$$H_n(\text{Hom}_R(RM, T)) \cong H_n(\text{Hom}_R(M, T)), \ H_n(\text{Hom}_R(RM, L_N)) \cong H_n(\text{Hom}_R(M, L_N)).$$

So $\text{ext}^n_A(M, N) \cong H_{-n}(\text{Hom}_R(RM, T))$ and $\text{ext}^{n+1}_A(M, N) \cong H_{-n}(\text{Hom}_R(M, L_N))$ for all integers $n \in \mathbb{Z}$. This completes the proof. \hfill \Box

**Corollary 4.4.** If $M$ is a complex of finite Gorenstein $B$ dimension, then we have

$$\mathcal{G}B\text{-dim} M = \sup\{n \in \mathbb{Z} \mid \text{ext}^n_A(X, M) \neq 0 \text{ for some } X \in A \cap B\}.$$  

**Proof.** Let $X \in A \cap B$. Then $\text{ext}^n_A(X, M) = 0$ for all $n \in \mathbb{Z}$ by Theorem 4.3. It follows from Remark 4.2 that $\text{ext}^n_A(X, M) \cong \text{Ext}^n_R(X, M)$ for all $n \in \mathbb{Z}$ and all $X \in A \cap B$. So the result is clear by Theorem 1.1. \hfill \Box

**Corollary 4.5.** Let $M$ be a complex with $\mathcal{G}B\text{-dim} M < \infty$. Then the following are equivalent:

1. $\mathcal{B}\text{-dim} M = \mathcal{G}B\text{-dim} M$;
2. $\text{ext}^i_A(X, M) = 0$ for all $i \in \mathbb{Z}$ and all $dg$-$A$ complexes $X$;
3. $\text{ext}^i_A(X, M) = 0$ for some $i \in \mathbb{Z}$ and all $R$-modules $X \in A$.

**Proof.** (1) $\Rightarrow$ (2) holds by [42, Theorem 1.1(2)].

(2) $\Rightarrow$ (3) is trivial.
(3) ⇒ (1). By Theorem 1.1, there exists a Tate \( \mathcal{B} \) resolution \( M \leftarrow \mathcal{Q}M \rightarrow \mathcal{RQ}M \rightarrow T \) of \( M \). Let \( X \) be any \( R \)-module in \( \mathcal{A} \). Then there exists an integer \( i \) such that \( H_{-i}(\text{Hom}_R(X,T)) = 0 \) by (3) and Theorem 4.3. Thus we have the following exact sequence

\[
\text{Hom}_R(X,T_{i-1}) \xrightarrow{\text{Hom}_R(X,\partial^n_{T_{i-1}})} \text{Hom}_R(X,T_{i-1}) \xrightarrow{\text{Hom}_R(X,\partial^n_{T_{i}})} \text{Hom}_R(X,T_{i-1-1}).
\]

Let \( t : Z_{-i}(T) \rightarrow T_{-i} \) be the canonical injection. Then there exists an epimorphism \( s : T_{1-i} \rightarrow Z_{-i}(T) \) such that \( \partial^n_{T_{-i}} = ts \). Thus \( \text{Hom}_R(X,s) : \text{Hom}_R(X,T_{1-i}) \rightarrow \text{Hom}_R(X,Z_{-i}(T)) \) is epic. Note that \( 0 \rightarrow Z_{1-i}(T) \rightarrow T_{1-i} \xrightarrow{s} Z_{-i}(T) \rightarrow 0 \) is an exact sequence of \( R \)-modules. Then we have an exact sequence

\[
\text{Hom}_R(X,T_{1-i}) \rightarrow \text{Hom}_R(X,Z_{-i}(T)) \rightarrow \text{Ext}^1_R(X,Z_{1-i}(T)) \rightarrow \text{Ext}^1_R(X,T_{1-i}) = 0.
\]

Thus we have \( \text{Ext}^1_R(X,Z_{1-i}(T)) = 0 \) for any \( X \in \mathcal{A} \), and hence \( Z_{1-i}(T) \in \mathcal{B} \). Consequently, \( Z_{-j}(T) \in \mathcal{B} \) for all \( j > i \). Let \( j \) be an integer such that \( j > \max\{i, \mathcal{GB}\text{-dim} M\} \). Then \( Z_{-j}(\mathcal{Q}M) \in \mathcal{B} \). Thus \( \mathcal{B}\text{-dim} M < \infty \) by [56, Theorem 3.4], and so \( \mathcal{GB}\text{-dim} M = \mathcal{B}\text{-dim} M < \infty \) by Corollary 3.17. This completes the proof.

**Lemma 4.6.** Let \( 0 \rightarrow B \rightarrow A \rightarrow N \rightarrow 0 \) be an exact sequence of \( R \)-modules with \( B \in \mathcal{B} \) and \( A \in \mathcal{A} \). Then \( \mathcal{B}\text{-dim} N = \mathcal{B}\text{-dim} A \) and \( \mathcal{GB}\text{-dim} N = \mathcal{GB}\text{-dim} A \).

**Proof.** The proof is straightforward. \( \square \)

**Lemma 4.7.** Let \( M \) be an \( R \)-module with finite Gorenstein \( \mathcal{B} \) dimension and \( f : A \rightarrow M \) a special \( \mathcal{A} \) precover of \( M \). Then \( A \) has a split Tate \( \mathcal{B} \) resolution \( A \xrightarrow{id} A \rightarrow \mathcal{R}A \xrightarrow{\tau} T \). Hence there exists a degreewise split exact sequence of complexes

\[
0 \rightarrow \mathcal{R}A \rightarrow \tilde{T} \rightarrow \Sigma X \rightarrow 0
\]

with \( \tilde{T} = (T_{\leq 0})^+ \) such that \( X \) is a proper \( \mathcal{G}(\mathcal{B}) \)-coresolution of \( A \).

**Proof.** By Theorem 1.1 and Lemma 4.6, \( A \) has a split Tate \( \mathcal{B} \) resolution \( A \xrightarrow{id} A \rightarrow \mathcal{R}A \xrightarrow{\tau} T \). Thus there is a non-negative integer \( n \) such that \( \tau_i \) is bijective for all \( i \leq -n \). We set \( \tilde{T} = (T_{\leq 0})^+ \), that is

\[
\tilde{T}_i = \begin{cases} T_i & \text{if } i \leq 0; \\ Z_0(T) & \text{if } i = 1; \\ 0 & \text{if } i > 1,
\end{cases}
\]

and \( \partial^n_{\tilde{T}} = \begin{cases} \partial^n_{T} & \text{if } i \leq 0; \\ q & \text{if } i = 1; \\ 0 & \text{if } i > 0,
\end{cases} \)

where \( q : Z_0(T) \rightarrow T_0 \) is the canonical injection. Let \( \tilde{\beta} : \mathcal{R}A \rightarrow \tilde{T} \) be a morphism such that \( \tilde{\beta}_i = \tau_i \) for all \( i \leq 0 \) and \( \tilde{\beta}_i = 0 \) for all \( i > 0 \). Let \( X = \Sigma^{-1}\text{coker}(\tilde{\beta}) \). Note that \( \text{coker}(\tau) \) is a complex with each entry in \( \mathcal{A} \cap \mathcal{B} \). Thus \( X_0 = Z_0(T) \in \mathcal{G}(\mathcal{B}) \), \( X_{-1} \in \mathcal{A} \cap \mathcal{B} \) for \( 1 \leq i \leq n \), and \( X_{-i} = 0 \) for \( i \geq n+1 \) and \( i \leq -1 \). One easily checks that \( A \cong H_0(X) \) and \( H_{-i}(X) = 0 \) for \( 1 \leq i \leq n-1 \). Note that \( \text{Ext}^1_R(Z_{-i}(X),G) \cong \text{Ext}^{n+1-i}_R(X_{-n},G) \) for \( 1 \leq i \leq n-1 \) and any \( G \in \mathcal{G}(\mathcal{B}) \). Then \( \text{Ext}^1_R(Z_{-i}(X),G) = 0 \) for \( 1 \leq i \leq n-1 \) and any \( G \in \mathcal{G}(\mathcal{B}) \) by Definition 3.1. Hence \( X \) is a proper \( \mathcal{G}(\mathcal{B}) \)-coresolution of \( A \). So we have the following exact sequence of complexes:

\[
0 \rightarrow \mathcal{R}A \rightarrow \tilde{T} \rightarrow \Sigma X \rightarrow 0
\]

with \( \tilde{T} = (T_{\leq 0})^+ \). This completes the proof. \( \square \)

**Proposition 4.8.** Let \( A \) be an \( R \)-module in \( \mathcal{A} \) with finite Gorenstein \( \mathcal{B} \) dimension. For any \( R \)-module \( M \) in \( \mathcal{A} \), we have the following isomorphisms:
By the long exact sequence theorem, we have the following exact sequence:

\[ L : 0 \to RA \to \tilde{T} \to \Sigma X \to 0 \]

with \( \tilde{T} = (T_{<0})^+ \) such that \( X \) is a proper \( G(B) \)-coresolution of \( A \).

(1) Let \( M \) be an \( R \)-module in \( A \). Applying \( \text{Hom}_R(M, -) \) to the exact sequence \( L \) gives rise to the following exact sequence of complexes:

\[ 0 \to \text{Hom}_R(M, RA) \to \text{Hom}_R(M, \tilde{T}) \to \text{Hom}_R(M, \Sigma X) \to 0. \]

By the long exact sequence theorem, we have the following exact sequence:

\[ H_0(\text{Hom}_R(M, \tilde{T})) \to H_0(\text{Hom}_R(M, \Sigma X)) \to H_{-1}(\text{Hom}_R(M, RA)) \to H_{-1}(\text{Hom}_R(M, \tilde{T})). \]

It follows from Lemma 3.12 and Theorem 4.3 that the following sequence

\[ 0 \to H_0(\text{Hom}_R(M, \Sigma X)) \to \text{Ext}^1_R(M, A) \to \text{Ext}^1_R(M, A) \]

is exact. Since \( X \) is a proper \( G(B) \)-coresolution of \( A \), \( \text{Ext}^1_{G(B)}(M, A) \cong H_{-1}(\text{Hom}_R(M, X)) \) by Definition 2.2. Note that \( H_{-1}(\text{Hom}_R(M, X)) \cong H_0(\text{Hom}_R(M, \Sigma X)) \). So \( \text{Ext}^1_{G(B)}(M, A) \cong \ker(\tilde{\varepsilon}^1_R(M, A)) \), as desired.

(2) Let \( n \) be an integer with \( n > 1 \). Note that \( \text{Ext}^n_A(M, A) \cong H_{1-n}(\text{Hom}_R(M, LA)) \) by Theorem 4.3. By Lemma 4.7, we have

\[ H_{1-n}(\text{Hom}_R(M, LA)) = H_{1-n}(\text{Hom}_R(M, \Sigma X)). \]

Then \( \text{Ext}^n_A(M, A) \cong H_{1-n}(\text{Hom}_R(M, \Sigma X)) \). It follows from Definition 2.2 that \( \text{Ext}^n_{G(B)}(M, A) \cong H_{-n}(\text{Hom}_R(M, X)) \) since \( X \) is a proper \( G(B) \)-coresolution of \( N \). Note that \( H_{-n}(\text{Hom}_R(M, X)) \cong H_{1-n}(\text{Hom}_R(M, \Sigma X)). \) So \( \text{Ext}^n_{G(B)}(M, A) \cong \text{ext}^n_A(M, A). \) This completes the proof. \( \Box \)

**Lemma 4.9.** Let \( 0 \to B \to A \to N \to 0 \) be an exact sequence of \( R \)-modules with \( B \in B \) and \( A \in A \). For each \( R \)-module \( M \) in \( A \), we have the following commutative diagram with exact rows

\[ \cdots \to \text{ext}^i_A(M, A) \to \text{Ext}^i_R(M, A) \to \text{ext}^i_A(M, A) \to \cdots \]

\[ \cdots \to \text{ext}^i_A(M, N) \to \text{Ext}^i_R(M, N) \to \text{ext}^i_A(M, N) \to \cdots \]

satisfying that

(1) \( g^i : \text{Ext}^i_R(M, A) \to \text{Ext}^i_R(M, N) \) is an isomorphism for any \( i \geq 1 \).

(2) \( h^i : \text{ext}^i_A(M, A) \to \text{ext}^i_A(M, N) \) is an isomorphism for each \( i \in \mathbb{Z} \).

**Proof.** Since \( B \in B \), there is an exact sequence \( \cdots \to X_1 \to X_0 \to B \to 0 \) of \( R \)-modules such that \( X_i \in A \cap B \) for \( i \geq 0 \) and \( \text{coker}(X_i \to X_{i-1}) \in B \) for \( i \geq 1 \). Let \( QB \) be the complex \( \cdots \to X_1 \to X_0 \to 0 \). Then \( QB \to B \) is a cofibrant replacement with \( QB \) in \( dg\mathcal{A} \cap dg\mathcal{B} \). Hence
$\mathcal{R} QB = QB$. Dualizing the proof of [42, Lemma 3.12], we have the following commutative diagram with exact rows such that the columns are cofibrant replacements:

$$
\begin{array}{ccccccccc}
0 & \to & QB & \to & QA & \to & QN & \to & 0 \\
0 & \to & B & \to & A & \to & N & \to & 0.
\end{array}
$$

Note that there exists an exact sequence $0 \to QA \to \mathcal{R} QA \to L \to 0$ of complexes with $L$ an $A$ complex. Consider the following pushout diagram:

$$
\begin{array}{ccccccccc}
0 & \to & QB & \to & QA & \to & QN & \to & 0 \\
0 & \to & QB & \to & \mathcal{R} QA & \to & W & \to & 0 \\
0 & \to & L & \to & L & \to & 0 & \to & 0.
\end{array}
$$

Since $QB$ and $\mathcal{R} QA$ are in $d g \tilde{B}$, so is $W$ by Lemma 3.6. Let $\mathcal{R} Q N = W$. Then $QN \to \mathcal{R} Q N$ is a fibrant replacement. Thus we have the following commutative diagram with exact rows such that the columns are fibrant-cofibrant replacements:

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{R} QB & \to & \mathcal{R} QA & \to & \mathcal{R} Q N & \to & 0 \\
0 & \to & QB & \to & QA & \to & QN & \to & 0 \\
0 & \to & B & \to & A & \to & B & \to & 0.
\end{array}
$$

Let $M$ be an $R$-module in $A$. Choose any fibrant-cofibrant resolution $M \leftarrow QM \to \mathcal{R} QM$ of $M$. Then we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
0 & \to & Hom_R(\mathcal{R} QM, QB) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QB) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QB) & \to & 0 \\
0 & \to & Hom_R(\mathcal{R} QM, QA) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QA) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QA) & \to & 0 \\
0 & \to & Hom_R(\mathcal{R} QM, QN) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QN) & \to & Hom_R(\mathcal{R} QM, \mathcal{R} QN) & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0.
\end{array}
$$
Since $\mathcal{R}Q\mathcal{B} \in \mathcal{C}^-(R), \Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{B}) = \Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{B})$. For each integer $n \in \mathbb{Z}$, we have $H_n(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{A})) \cong H_n(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}))$. It follows from Lemma 3.12 that $\Ext^i_R(M, B) \cong H_{-i}(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{B}))$ for each integer $i \in \mathbb{Z}$. Since $M \in \mathcal{A}$ and $B \in \mathcal{B}$, $H_i(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{B})) = 0$ for $i \leq -1$. Thus $H_i(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{A})) \cong H_i(\Hom_R(\mathcal{R}Q\mathcal{M}, \mathcal{R}Q\mathcal{N}))$ for $i \leq -1$. Applying the long exact sequence theorem to the commutative diagram above, we have the desired commutative diagram in Lemma 4.9. This completes the proof. □

**Fact 4.10.** Let $N$ be an $R$-module in $\mathcal{A}$ with finite Gorenstein $\mathcal{B}$ dimension. Then $N$ has a proper $\mathcal{G}(\mathcal{B})$-coresolution $N \to X$ by Lemma 4.7. Note that the class $\mathcal{B}$ is preenveloping by hypothesis. Choose a proper $\mathcal{B}$-coresolution $N \to L$ and a morphism $\gamma : X \to L$ lifting the identity on $N$. For each $R$-module $M$ in $\mathcal{A}$, we have $\Hom_R(M, L) \cong R\Hom_R(M, N)$ by Lemma 3.12 and the morphism of complexes

$$\Hom_R(M, \gamma) : \Hom_R(M, X) \to \Hom_R(M, L)$$

induces a natural homomorphism of abelian groups

$$e^n(M, N) : \Ext^n_{\mathcal{G}(\mathcal{B})}(M, N) \to \Ext^n_R(M, N)$$

for every $n \in \mathbb{Z}$. The groups and the maps defined above do not depend on the choices of coresolutions and liftings by [29, Exercise 3, p.169].

**Lemma 4.11.** Let $M$ be an $R$-module in $\mathcal{A}$ and $N$ an $R$-module with $\mathcal{G}\mathcal{B}$-$\dim N < \infty$. For each integer $i$ with $i \geq 1$, we have the following commutative diagram such that the columns are isomorphic:

$$\begin{array}{ccc}
\Ext^i_{\mathcal{G}(\mathcal{B})}(M, A) & \xrightarrow{e^i(M,A)} & \Ext^i_R(M, A) \\
\downarrow \cong & & \downarrow \cong \\
\Ext^i_{\mathcal{G}(\mathcal{B})}(M, N) & \xrightarrow{e^i(M,N)} & \Ext^i_R(M, N),
\end{array}$$

where $A \to N$ is a special $\mathcal{A}$ precover of $N$.

**Proof.** Let $f : A \to N$ be a special $\mathcal{A}$ precover of $N$. Then we have an exact sequence $0 \to B \to A \to N \to 0$ of $R$-modules with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Since $\mathcal{G}\mathcal{B}$-$\dim N < \infty$, there is a non-negative integer $n$ such that $\mathcal{G}\mathcal{B}$-$\dim N = \mathcal{G}\mathcal{B}$-$\dim A \leq n$. By the proof of Lemma 4.7, $A$ has a proper $\mathcal{G}(\mathcal{B})$-coresolution $\beta : A \to X$ such that $X_0 \in \mathcal{G}(\mathcal{B}), X_{-1} \in \mathcal{A} \cap \mathcal{B}$ for $1 \leq i \leq n-1$ and $X_{-i} = 0$ for $i \geq n$. Consider the following pushout diagram:

$$\begin{array}{cccccccc}
0 & 0 \\
0 & \arrow{d} & B & \arrow{d} & A & \arrow{d} & N & \arrow{d} & 0 \\
& \downarrow{\beta_0} & \arrow{r} & \arrow{d}{\alpha} & & \arrow{l}{\lambda} & \arrow{d} & L & \arrow{d} & 0 \\
0 & \arrow{d} & B & \arrow{d}{\lambda} & X_0 & \arrow{d} & Z_{-1}(X) & \arrow{d} & 0 \\
& \downarrow{Z_{-1}(X)} & \arrow{r} & \arrow{d} & Z_{-1}(X) & \arrow{d} & 0 & \arrow{d} & 0.
\end{array}$$
Note that $B \in \mathcal{B}$ and $X_0 \in \mathcal{G}(\mathcal{B})$. Then $L \in \mathcal{G}(\mathcal{B})$ by Proposition 3.4(1). Assume that $Y$ is a complex such that $Y_0 = L$, $Y_{-i} = X_{-i}$ for $1 \leq i \leq n-1$ and $Y_{-i} = 0$ for $i \geq n$. Thus $\eta : N \rightarrow Y$ is a proper $\mathcal{G}(\mathcal{B})$-coresolution of $N$, where $\eta_0 = \alpha$ and $\eta_i = 0$ for $i \neq 0$. Let $K = \cdots \rightarrow B \rightarrow 0 \rightarrow \cdots$ with $B$ in the 0th position and 0 in the other positions. It is easy to see that $0 \rightarrow K \rightarrow X \rightarrow Y$ is an exact sequence of complexes, where $\gamma_0 = \lambda$ and $\gamma_i = \text{id}_{X_i}$ for $i \leq -1$.

Note that $A \in \mathcal{A}$. Then $A$ has a proper $\mathcal{B}$-coresolution $\beta' : A \rightarrow X'$ such that $X'_i \in \mathcal{A} \cap \mathcal{B}$ and $\mathcal{Z}_d(X') \in \mathcal{A}$ for $i \leq 0$. By the foregoing proof, $N$ has a proper $\mathcal{B}$-coresolution $\eta' : N \rightarrow Y'$ with $Y'_i = X'_{-i}$ for $i \geq 1$ such that the following diagram

$\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow & & \downarrow \beta' \\
A & \longrightarrow & N \\
\downarrow & & \downarrow \eta' \\
0 & \longrightarrow & X'_0 \\
0 & \longrightarrow & \gamma \\
\downarrow & & \downarrow \varphi \\
Y' & \longrightarrow & 0
\end{array}$

is commute. Thus $0 \rightarrow K \rightarrow X' \rightarrow Y'$ is an exact sequence of complexes, where $\gamma_0 = \mu$ and $\gamma_i = \text{id}_{X_i}$ for $i \leq -1$. Since each $R$-module in $\mathcal{B}$ belongs to $\mathcal{G}(\mathcal{B})$, there exists $\varphi : X \rightarrow X'$ such that $\varphi \beta = \beta'$. Note that $\eta'_0 \varphi = \mu \beta_0' = \mu \varphi_0 \beta_0$. Using the pushout of homomorphisms $f$ and $\beta_0$, we have a morphism $\rho : Y_0 \rightarrow Y'_0$ such that $\rho \gamma_0 = \gamma_0' \varphi_0$. Let $\psi : Y \rightarrow Y'$ be a morphism such that $\psi_0 = \rho$ and $\psi_i = \varphi_i$ for $i \leq -1$. Thus we have the following diagram of complexes with exact rows:

$\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow & & \downarrow \beta \\
X & \longrightarrow & Y \\
\downarrow & & \downarrow \varphi \\
0 & \longrightarrow & Y'
\end{array}$

Let $M$ be an $R$-module in $\mathcal{A}$. Thus we have the following diagram of complexes:

$\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_R(M,K) \\
\downarrow & & \downarrow \\
\text{Hom}_R(M,X) & \longrightarrow & \text{Hom}_R(M,Y) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_R(M,K) \\
0 & \longrightarrow & \text{Hom}_R(M,X') \\
\downarrow & & \downarrow \\
\text{Hom}_R(M,Y') & \longrightarrow & \text{Hom}_R(M,Y')
\end{array}$

Note that $H_i(\text{Hom}_R(M,K)) = 0$ for $i \leq -1$. Applying the long exact sequence theorem to the commutative diagram above, we have the desired commutative diagram in Lemma 4.11. This completes the proof.

We now finish this section by giving the proof of Theorem 1.2 as follows.

4.12. **Proof of Theorem 1.2.** Since $N$ is an $R$-module with $\mathcal{G}\mathcal{B}\text{-dim}N < \infty$, there exists an exact sequence $0 \rightarrow B \rightarrow A \rightarrow N \rightarrow 0$ of $R$-modules with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Thus $\mathcal{G}\mathcal{B}\text{-dim}N = \mathcal{G}\mathcal{B}\text{-dim}A < \infty$. Let $M$ be an $R$-module in $\mathcal{A}$. By Proposition 4.8, Lemma 4.9 and Lemma 4.11, we have $\text{Ext}^1_{\mathcal{G}(\mathcal{B})}(M,N) \cong \text{Ext}^1_{\mathcal{G}(\mathcal{B})}(M,A) \cong \ker(\tilde{\varepsilon}_R(M,A)) \cong \ker(\tilde{\varepsilon}_R(M,N))$ and $\text{Ext}^n_{\mathcal{G}(\mathcal{B})}(M,N) \cong \text{Ext}^n_{\mathcal{G}(\mathcal{B})}(M,A) \cong \text{ext}_A^n(M,N)$ for any integer $n > 1$. This completes the proof.

5. **Comparisons between Gorenstein $\mathcal{B}$ dimensions and $\mathcal{B}$ dimensions**

Our goal of this section is to investigate the relationships between Gorenstein $\mathcal{B}$ dimensions and $\mathcal{B}$ dimensions for complexes. To this end, we start with the following exact sequence for modules with finite Gorenstein $\mathcal{B}$ dimension, connecting relative and Tate-Vogel cohomologies via a long exact
sequence. We will refer to a sequence of this form as an Avramov-Martsinkovsky exact sequence. The sequence is similar to [2, Theorem 3.10] and [8, Theorem 7.1].

**Theorem 5.1.** Assume that $N$ is an $R$-module such that $\mathcal{GB}$-$\dim N \leq g$ with $g \geq 1$ an integer. For each $R$-module $M$ in $\mathcal{A}$, there is a long exact sequence

$$
0 \to \text{Ext}_1^{\mathcal{G}(B)}(M, N) \to \text{Ext}_1^R(M, N) \to \text{ext}_A^1(M, N) \to \text{ext}_A^1(M, N) \to 0.
$$

**Proof.** Let $A \to N$ be a special $\mathcal{A}$ precover of $N$. By Theorem 1.1 and Lemma 4.6, there exists a split Tate $\mathcal{B}$ resolution $A \xleftarrow{id} A \to \mathcal{RA} \xrightarrow{\tau} T$ of $A$ such that $\tau_i = id_{(\mathcal{RA})_i}$ for all $i \leq -g$. Thus $\text{coker}(\tau)$ is a complex such that $(\text{coker}(\tau))_i = 0$ for all $i \leq -g$, and hence $H_{-g}(\text{Hom}_R(M, coker(\tau))) = 0$. Note that $\text{ext}_A^{g+1}(M, A) \cong H_{-g}(\text{Hom}_R(M, coker(\tau)))$ by Theorem 4.3. It follows that $\text{ext}_A^{g+1}(M, A) = 0$.

By Remark 4.2, we have the following exact sequence:

$$
\mathcal{X} : \cdots \to \text{ext}_A^1(M, A) \xrightarrow{\text{ext}_A^1(M, A)} \text{Ext}_1^R(M, A) \xrightarrow{\text{ext}_A^1(M, A)} \text{ext}_A^1(M, A) \to \cdots.
$$

Applying Theorem 4.8 and Fact 4.10 to the exact sequence $\mathcal{X}$ above, we have the following exact sequence:

$$
0 \to \text{Ext}_1^{\mathcal{G}(B)}(M, A) \xrightarrow{\text{ext}_A^1(M, A)} \text{Ext}_1^R(M, A) \xrightarrow{\text{ext}_A^1(M, A)} \text{ext}_A^1(M, A) \to 0.
$$

By Lemmas 4.9 and 4.11, we have the desired commutative diagram in Theorem 5.1. This completes the proof. \hfill $\square$

**Corollary 5.2.** Let $N$ be an $R$-module with $\mathcal{GB}$-$\dim N < \infty$. Then the following are equivalent:

1. $\mathcal{B}$-$\dim N = \mathcal{GB}$-$\dim N$;
2. $\varepsilon^n(M, N) : \text{Ext}_n^{\mathcal{G}(B)}(M, N) \to \text{Ext}_n^R(M, N)$ is an isomorphism for all $n \in \mathbb{Z}$ and all $R$-modules $M$ in $\mathcal{A}$.

**Proof.** (1) $\Rightarrow$ (2). By hypothesis, there is a non-negative integer $g$ such that $\mathcal{B}$-$\dim M = \mathcal{GB}$-$\dim M \leq g$. Let $M$ be an $R$-module in $\mathcal{A}$. One easily checks that $\text{Ext}_0^{\mathcal{G}(B)}(M, N) \cong \text{Hom}_R(M, N) \cong \text{Ext}_0^R(M, N)$ and $\text{Ext}_n^{\mathcal{G}(B)}(M, N) = 0 = \text{Ext}_n^R(M, N)$ for $n < 0$ or $n > g$. For $1 \leq n \leq g$, $\varepsilon^n(M, N) : \text{Ext}_n^{\mathcal{G}(B)}(M, N) \to \text{Ext}_n^R(M, N)$ is an isomorphism by Corollary 4.5 and Theorem 5.1. So (2) follows.

(2) $\Rightarrow$ (1) holds by Corollary 4.5 and Theorem 5.1. \hfill $\square$

We are now in a position to prove Theorem 1.3 from the introduction.

**5.3. Proof of Theorem 1.3.** (1). The result follows from Corollary 3.17.

(2). $(a) \iff (b) \iff (c)$ hold by Corollary 4.5.

$(a) \Rightarrow (d)$. Let $N \leftarrow QN \to \mathcal{RN}N$ be a fibrant-cofibrant resolution of $N$. Then there is an integer $n$ such that $Z_n(\mathcal{RN}N) \in \mathcal{B}$ by [56, Theorem 3.4]. So (d) follows from Corollary 5.2.

$(d) \Rightarrow (a)$. Let $N \leftarrow QN \to \mathcal{RN}N$ be a fibrant-cofibrant resolution of $N$. Then there is an integer $s$ such that $H_i(\mathcal{RN}N) = 0$ and $Z_i(\mathcal{RN}N) \in \mathcal{G}(\mathcal{B})$ for all $i < s$ by Theorem 1.1. By (d),
there exist an integer \( n \) with \( n < s \) such that
\[
\varepsilon^i(M, Z_n(RQN)) : \text{Ext}^i_G(M, Z_n(RQN)) \to \text{Ext}^i_R(M, Z_n(RQN))
\]
is an isomorphism for all \( i \in \mathbb{Z} \) and any \( R \)-module \( M \) in \( A \). It follows from Corollary 5.2 that 
\( B \)-dim\( Z_n(RQN) \) < \( \infty \). Thus \( B \)-dim\( N \) < \( \infty \) by [56, Theorem 3.4], and so (a) holds by Corollary 4.5. This completes the proof.

\[\square\]

Remark 5.4. Let \( R \) be a commutative Artin local ring. Then \( \text{findim}(R) < \infty \) by [4, Proposition 1.3 and Theorem 1.7]. If we let the cotorsion pair \((A, B)\) in Theorem 1.3 be the cotorsion pair \((X, \mathcal{P})\) cogenerated by \( \mathcal{P}^{<\infty} \) in Theorem 1.5, then \( B \)-dim\( N \) = \( GB \)-dim\( N \) holds for any complex \( N \) by Theorem 1.5 and Corollary 3.17. On the other hand, the equality \( B \)-dim\( N \) = \( GB \)-dim\( N \) in Theorem 1.3(2) does not hold in general. For example, let \( R = \mathbb{Z}/4\mathbb{Z} \) and \( (A, B) = (\text{Mod}R, \mathcal{I}) \), then \( N = 2R \) is a Gorenstein \( B \)-module by [29, Theorem 12.3.1] and [49, Corollary 4.6], i.e., \( GB \)-dim\( N \) = 0, but \( B \)-dim\( N \) = \( \infty \).

6. Frobenius categories and model structures

We start this section with the following definition which is cited from [17], see also [45, 48].

**Definition 6.1.** Let \( \mathcal{B} \) be an additive category. A kernel-cokernel pair \((i, p)\) in \( \mathcal{B} \) is a pair of composable morphisms
\[
L \xrightarrow{i} M \xrightarrow{p} N
\]
such that \( i \) is a kernel of \( p \) and \( p \) is a cokernel of \( i \). Let \( \varepsilon \) be a class of kernel-cokernel pairs on \( \mathcal{B} \) closed under isomorphisms, a kernel-cokernel pair \((i, p)\) is called a conflation if \((i, p) \in \varepsilon \), and we denote it by
\[
L \xrightarrow{i} M \xrightarrow{p} N.
\]
We call \( i \) an inflation and \( p \) a deflation.

The pair \((\mathcal{B}, \varepsilon)\) (or simply \( \mathcal{B} \)) is called an exact category if it satisfies the following conditions.

- [E0] For any object \( B \) in \( \mathcal{B} \), the identity morphism \( \text{id}_B \) is both an inflation and a deflation.
- [E1] The class of inflations is closed under compositions.
- [E1]\(^0\) The class of deflations is closed under compositions.
- [E2] The pull-out of an inflation along an arbitrary morphism exists and yields an inflation.
- [E2]\(^0\) The push-out of a deflation along an arbitrary morphism exists and yields a deflation.

Recall that an object \( P \) in \( \mathcal{B} \) is projective provided that the functor \( \text{Hom}_{\mathcal{B}}(P, -) \) sends conflations to short exact sequences; this is equivalent to that any deflation ending at \( P \) splits. The exact category \( \mathcal{B} \) is said to have enough projective objects provided that each object \( X \) fits into a deflation \( d : P \to X \) with \( P \) projective. Dually one has the notions of injective objects and having enough injective objects.

An exact category \( \mathcal{B} \) is said to be Frobenius provided that it has enough projective and enough injective objects, and the class of projective objects coincides with the class of injective objects. The importance of Frobenius categories lies in that they give rise naturally to triangulated categories; see [34].

Denote by \( \varepsilon \) the class of all exact sequences of the form
\[
0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0
\]
with all terms in \( \mathcal{A} \). We have the following observation.

**Lemma 6.2.** \((\mathcal{A}, \varepsilon)\) is an exact category with enough injective objects. In particular, \( \mathcal{A} \cap \mathcal{B} \) is the full subcategory of all injective objects.

**Proof.** The assertion that \((\mathcal{A}, \varepsilon)\) is an exact category follows from [45, 4.1]. Now let \( M \in \mathcal{A} \). Since \((\mathcal{A}, \mathcal{B})\) is a complete hereditary cotorsion pair in \( \text{Mod} \mathcal{R} \), we have an exact sequence \( 0 \to M \to H \to M' \to 0 \) with \( H \in \mathcal{A} \cap \mathcal{B} \) and \( M' \in \mathcal{A} \). So \((\mathcal{A}, \varepsilon)\) has enough injective objects and \( \mathcal{A} \cap \mathcal{B} \) is the full subcategory of all injective objects.

Now consider the full subcategory \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) of \( \mathcal{A} \), it is easy to check that \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) is closed under extensions in sense that, for every conflation \( 0 \to X' \to X \to X'' \to 0 \) in \( \varepsilon \), \( X', X'' \in \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) implies \( X \in \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \). Then it follows from [17, 13.3] that \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) equipped with the exact structure of \((\mathcal{A}, \varepsilon)\) is an exact subcategory of \((\mathcal{A}, \varepsilon)\). Moreover, we have the following theorem which is the first claim of Theorem 1.4 from the introduction.

**Theorem 6.3.** \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) is a Frobenius category with \( \mathcal{A} \cap \mathcal{B} \) the full subcategory of projective and injective objects.

**Proof.** Note that \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \subseteq (\mathcal{A} \cap \mathcal{B})^\perp \cap (\mathcal{A} \cap \mathcal{B})^\perp \). So \( \mathcal{A} \cap \mathcal{B} \) is the full subcategory of projective and injective objects of \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \). Next we will show that \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) has enough projective and injective objects.

Let \( M \in \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \). Since \((\mathcal{A}, \mathcal{B})\) is a complete hereditary cotorsion pair, we have an exact sequence \( 0 \to M \to X \to M' \to 0 \) with \( X \in \mathcal{A} \cap \mathcal{B} \) and \( M' \in \mathcal{A} \). Since \( M, X \in \mathcal{G}(\mathcal{B}) \), it follows from Proposition 3.4(1) that \( M' \in \mathcal{G}(\mathcal{B}) \), and then \( M' \in \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \). So \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) has enough injective objects. Note that \( M \in \mathcal{G}(\mathcal{B}) \). Then \( M \in (\mathcal{A} \cap \mathcal{B})^\perp \) and there is a \( \text{Hom}_\mathcal{R}(\mathcal{A} \cap \mathcal{B},-) \) exact exact sequence \( \cdots \to X_1 \to X_0 \to M \to 0 \) with each \( X_i \in \mathcal{A} \cap \mathcal{B} \). Put \( K_1 := \ker(X_0 \to M) \), then there is a \( \text{Hom}_\mathcal{R}(\mathcal{A} \cap \mathcal{B},-) \) exact exact sequence \( 0 \to K_1 \to X_0 \to M \to 0 \) and then \( K_1 \in (\mathcal{A} \cap \mathcal{B})^\perp \).

Thus \( K_1 \in \mathcal{G}(\mathcal{B}) \) by Proposition 3.4(2). Because both \( X_0 \) and \( M \) lie in \( \mathcal{A} \), \( K_1 \in \mathcal{A} \). Therefore \( K_1 \in \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \), and then \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) has enough projective objects. \( \square \)

By Theorem 6.3, we denote by \( \widetilde{\mathcal{A} \cap \mathcal{G}(\mathcal{B})} \) the stable category of \( \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \) modulo \( \mathcal{A} \cap \mathcal{B} \). Then \( \widetilde{\mathcal{A} \cap \mathcal{G}(\mathcal{B})} \) is a triangulated category.

Now we can give the following theorem which contains (1) \( \iff \) (2) of Theorem 1.4 from the introduction.

**Theorem 6.4.** The following are equivalent for any ring \( \mathcal{R} \):

1. \( \text{GB-dim} M < \infty \) for any complex \( M \) in \( \text{D}^- (\mathcal{R}) \);
2. \( \text{Mod} \mathcal{R} = \text{cores} \big( \mathcal{G}(\mathcal{B}) \big)_{\leq n} \);
3. \( \text{Mod} \mathcal{R} = \text{cores} \big( \mathcal{G}(\mathcal{B}) \big) \) for some non-negative integer \( n \);
4. \( \mathcal{A} \subseteq \text{cores} \big( \mathcal{G}(\mathcal{B}) \big)_{\leq n} \);
5. \( \mathcal{M} = (\mathcal{A}, \text{cores} \big( \mathcal{G}(\mathcal{B}) \big)_{\leq \infty}) \) is a hereditary abelian model structure and its homotopy category \( H_0(\mathcal{M}) \) is triangle-equivalent to \( \widetilde{\mathcal{A} \cap \mathcal{G}(\mathcal{B})} \).

**Proof.** (1) \( \implies \) (2) and (3) \( \implies \) (4) are trivial.

(2) \( \implies \) (3). To prove (3), it suffices to show that \( \sup \{ \text{GB-dim} M \mid M \text{ is an } \mathcal{R} \text{-module} \} < \infty \) byRemark 3.11(1). If \( \sup \{ \text{GB-dim} M \mid M \text{ is an } \mathcal{R} \text{-module} \} = \infty \), then for any positive integer \( n \),
we have an $R$-module $M_n$ with $\mathcal{G}B\text{-dim}M_n > n$. Note that there exists a positive integer $k$ such that $\mathcal{G}B\text{-dim}\prod_{n \geq 1}M_n < k$ by (2). It follows from Corollary 3.18 that $\mathcal{G}B\text{-dim}M_n < k$ for any integer $n \geq 1$. This is a contradiction, as desired.

(3) $\Rightarrow$ (1). Let $M$ be a complex in $\mathbb{D}^-(R)$. Then $-\inf M \leq m$ for some positive integer $m$. Let $I$ be a dg-injective complex with $M \simeq I$. Note that $\mathcal{G}B\text{-dim}Z_{-m}I \leq n$ for some non-negative integer $n$ by (3). So $Z_{-n-m}(I)$ is a Gorenstein $B$-module and $\mathcal{G}B\text{-dim}M \leq n + m$ by Theorem 1.1, as desired.

(4) $\Rightarrow$ (3) holds by Lemma 4.6.

(2) $\Rightarrow$ (5). It is easy to check that cores $\hat{\mathcal{B}}^{<\infty}$ is closed under retracts and if two out of three terms in a short exact sequence are in cores $\hat{\mathcal{B}}^{<\infty}$ then so is the third. This means that cores $\hat{\mathcal{B}}^{<\infty}$ is a thick subcategory of $\text{Mod}R$ by [41, Definition 2.3]. By [40, Theorem 2.5] and [32, Theorem 4.3], we only need to show that $(\mathcal{A}, \mathcal{G}(B) \cap \text{cores} \hat{\mathcal{B}}^{<\infty})$ and $(\mathcal{A} \cap \text{cores} \hat{\mathcal{B}}^{<\infty}, \mathcal{G}(B))$ are complete hereditary cotorsion pairs.

For the first one, we only need to show that $\mathcal{G}(B) \cap \text{cores} \hat{\mathcal{B}}^{<\infty} = B$. It is easy to check that $\mathcal{B} \subseteq \mathcal{G}(B)$ and $\mathcal{B} \subseteq \text{cores} \hat{\mathcal{B}}^{<\infty}$. For the reverse containment, let $M \in \mathcal{G}(B) \cap \text{cores} \hat{\mathcal{B}}^{<\infty}$. Put $\mathcal{B}\text{-id}(M) \leq n < \infty$. We proceed by induction on $n$. The case $n = 0$ is trivial.

For $n = 1$, assume that $M$ is an $R$-module in $\mathcal{G}(B)$ with $\mathcal{B}\text{-id}(M) \leq 1$. Then there exists an exact sequence $0 \to M \to X \to Y \to 0$ of $R$-modules with $X, Y \in \mathcal{B}$. Since $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair, there exists an exact sequence $0 \to K \to L \to Y \to 0$ of $R$-modules with $L \in \mathcal{A} \cap \mathcal{B}$ and $K \in \mathcal{B}$. Consider the following pullback diagram:

Since $K$ and $X$ belong to $\mathcal{B}$, so is $G$. Note that $M \in \mathcal{G}(B)$ and $L \in \mathcal{A} \cap \mathcal{B}$. It follows that the middle row in the above diagram is split. So $M \in \mathcal{B}$, as desired.

For $n \geq 2$, assume that $M$ is an $R$-module in $\mathcal{G}(B)$ with $\mathcal{B}\text{-id}(M) \leq n$. Then there exists an exact sequence $0 \to M \to B_0 \to B_{-1} \to \cdots \to B_{-n} \to 0$ of $R$-modules with $B_{-i} \in \mathcal{B}$ for $i = 0, 1, 2, \cdots, n$. Let $K = \ker(B_{-1} \to B_{-2})$. Then $\mathcal{B}\text{-id}(K) \leq n - 1$. It follows from Proposition 3.4(1) that $K \in \mathcal{G}(B)$. Thus $K \in \mathcal{B}$ by induction hypothesis, and hence $\mathcal{B}\text{-id}(M) \leq 1$. So $M \in \mathcal{B}$ by the proof above.

For the second one, we assume that $\text{Mod}R = \hat{\text{cores} \mathcal{G}(B)^{<\infty}}$. Let $M$ be an $R$-module. Dualizing the proof of [58, Lemma 3.1], we can construct two exact sequences $0 \to B \to A \to M \to 0$ and $0 \to M \to W \to A' \to 0$ of $R$-modules with $A, A' \in \hat{\text{cores} \mathcal{A} \cap \mathcal{B}^{<\infty}}$ and $B, W \in \mathcal{G}(B)$. Hence $(\text{cores} \mathcal{A} \cap \mathcal{B}^{<\infty}, \mathcal{G}(B))$ is a complete cotorsion pair by [11, Definition 3.1 and Remark 3.2, p.88-89].
We want to show that cores $\overline{A \cap B}^{<\infty} = \text{cores } \hat{B}^{<\infty} \cap A$. Then containment cores $\overline{A \cap B}^{<\infty} \subseteq \text{cores } \hat{B}^{<\infty} \cap A$ is straightforward. For the reverse containment, let $M \in \text{cores } \hat{B}^{<\infty} \cap A$. Then there exists an integer $n$ such that $\text{B-id}(M) \leq n$. Since $(A, B)$ is a complete hereditary cotorsion pair and $M \in A$, there exists an exact sequence $0 \to M \to B_0 \to B_{-1} \to \cdots$ of $R$-modules with each $B_i \in A \cap B$ and $\ker(B_i \to B_{i-1}) \in A$ for all $i \leq 0$. Hence $\ker(B_{-n} \to B_{-n-1}) \in B$ and the proof is dual to that of [21, Theorem 1.2.7]. Thus $\ker(B_{-n} \to B_{-n-1}) \in A \cap B$ and $M \in \text{cores } \overline{A \cap B}^{<\infty}$. It follows that $(A \cap \text{cores } \hat{B}^{<\infty}, G(B))$ is a complete cotorsion pair. So $(A \cap \text{cores } \hat{B}^{<\infty}, G(B))$ is a hereditary cotorsion pair by Proposition 3.4(1), as desired.

$(5) \Rightarrow (2)$. Let $M$ be an $R$-module. Note that $(A \cap \text{cores } \hat{B}^{<\infty}, G(B))$ is a complete cotorsion pair by $(5)$. Then there exist an exact sequence $0 \to M \to K \to L \to 0$ of $R$-modules with $K \in G(B)$ and $L \in \text{cores } \hat{B}^{<\infty}$. It follows from Corollary 3.17 that $L$ belongs to cores $\hat{G(B)}$. So $M \in \text{cores } \hat{B}^{<\infty}$. This completes the proof. □

**Corollary 6.5.** Assume that every $R$-module has finite Gorenstein injective dimension. Then there is a model structure on $\text{Mod}_R$, the Gorenstein injective model structure, in which the cofibrant objects are the modules in $\text{Mod}_R$, the fibrant objects are the Gorenstein injective modules and trivial objects are the modules with finite injective dimension.

**Proof.** The result follows from Theorem 1.4 and Corollary 6.4. □

**Remark 6.6.** We note that Bravo, Gillespie and Hovey [14] generalizes [40, Theorem 8.6] to the case that $R$ is a left Noetherian ring. By [12, Example 2.8], there exists a non-Noetherian ring $R$ such that every $R$-module has finite Gorenstein injective dimension. Thus Corollary 6.5 also generalizes [40, Theorem 8.6]. Moreover, our method here is different from that in [14].

### 7. Singularity categories

In this section, we will denote the exact category $(\mathcal{A}, \varepsilon)$ in Lemma 6.2 by $\mathcal{A}$ for short and denote by $\mathcal{C}^*(\mathcal{A})$ the complex category of $\mathcal{A}$ and by $\mathcal{K}^*(\mathcal{A})$ the homotopy category of $\mathcal{A}$, where $* \in \{\text{blank}, +, -, b\}$.

For an $\mathcal{A}$ complex $X \in \mathcal{C}^*(\mathcal{A})$, the $i$th differential factors as $X_i \to Z_{i-1}(X) \to X_{i-1}$ with $0 \to Z_i(X) \to X_i \to Z_{i-1}(X) \to 0$ lies in $\varepsilon$ for every integer $i$. Then it is an acyclic complex in the sense of [17]. We call a morphism in $\mathcal{K}^*(\mathcal{A})$ an $\mathcal{A}$-quasi-isomorphism if its mapping cone is homotopy equivalent to an $\mathcal{A}$ complex. It is clear to check that an $\mathcal{A}$-quasi-isomorphism is a quasi-isomorphism.

Denote by $\mathcal{K}^*_{\text{ac}}(\mathcal{A})$ the full subcategory of $\mathcal{K}^*(\mathcal{A})$ consisting of $\mathcal{A}$-acyclic complexes. It follows from [17, section 10] that $\mathcal{K}^*_{\text{ac}}(\mathcal{A})$ is a thick subcategory of $\mathcal{K}^*(\mathcal{A})$.

**Definition 7.1.** [47, 17] The derived category $\mathbb{D}^*(\mathcal{A})$ of $\mathcal{A}$ is defined to be the Verdier quotient $\mathbb{D}^*(\mathcal{A}) := \mathcal{K}^*(\mathcal{A})/\mathcal{K}^*_{\text{ac}}(\mathcal{A})$.

**Remark 7.2.** Each morphism $X \to Y$ in $\mathbb{D}^*(\mathcal{A})$ is given by an equivalence class of right fractions $f/s$ or left fractions $s/f$ as presented by $X \xleftarrow{\sim} Z \xrightarrow{f} Y$ or $X \xrightarrow{f} Z \xleftarrow{\sim} Y$, where the doubled arrow means $s$ is an $\mathcal{A}$-quasi-isomorphism.

**Lemma 7.3.** Let $X$ be a complex of $C(A)$. Then $X \in \mathbb{D}^b(\mathcal{A})$ if and only if up to isomorphism, the $i$th differential $\partial_i^X$ of $X$ factors as $X_i \xrightarrow{s_i} Z_{i-1}(X) \xrightarrow{t_{i-1}} X_{i-1}$ with $0 \to Z_i(X) \xrightarrow{v_i} X_i \xrightarrow{v_{i-1}} Z_{i-1}(X) \to 0$ lies in $\varepsilon$ for $|i| \geq 0$. 27
Proof. The “only if” part is trivial, so we will show the “if” part.

Now let $X$ be a complex and assume that the $i$th differential $\partial_i^X$ of $X$ factors as $X_i \xrightarrow{v_i} Z_{i-1}(X) \xrightarrow{u_i} X_{i-1}$ with $0 \to Z_i(X) \xrightarrow{u_i} X_i \xrightarrow{v_i} Z_{i-1}(X) \to 0$ lies in $\varepsilon$ for $|i| \gg 0$. Then there exists some $n_0 > 0$ such that $i$th differential $\partial_i^X$ of $X$ factors as $X_i \xrightarrow{v_i} Z_{i-1}(X) \xrightarrow{u_i} X_{i-1}$ with $0 \to Z_i(X) \xrightarrow{u_i} X_i \xrightarrow{v_i} Z_{i-1}(X) \to 0$ lies in $\varepsilon$ for $|i| \geq n_0$. Consider the truncation:

$$X' := \cdots \to X_{-n_0+1} \to X_{-n_0} \to Z_{-n_0-1}(X) \to 0$$

of $X$. We claim that the morphism:

$$\begin{array}{c}
X' : & \cdots & \to & X_{-n_0+1} & \to & X_{-n_0} & \to & Z_{-n_0-1}(X) & \to & 0 & \to & \cdots \\
\downarrow f & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X : & \cdots & \to & X_{-n_0+1} & \to & X_{-n_0} & \to & X_{-n_0-1} & \to & X_{-n_0-2} & \to & \cdots
\end{array}$$

is an $A$-quasi-isomorphism or equivalently, $\text{Con}(f)$ is an $A$ complex.

By assumption, $f$ is a quasi-isomorphism, then

$$\text{Con}(f) = \cdots \to X_{-n_0+1} \oplus X_{-n_0} \to X_{-n_0} \oplus Z_{-n_0-1} \to X_{-n_0-1} \to X_{-n_0-2} \to \cdots$$

is exact. Since $Z_i(\text{Con}(f)) \cong Z_i(X)$ lies in $A$ for $i \leq -n_0 - 1$, $\text{Con}(f)$ is an $A$ complex and then $X \cong X'$ in $D^b(A)$.

Now consider the truncation:

$$X'' := 0 \to Z_{n_0}(X) \to X_{n_0} \to \cdots \to X_{-n_0} \to Z_{-n_0-1}(X) \to 0$$

of $X'$. By the same argument as above we get that the morphism:

$$\begin{array}{c}
X' : & \cdots & \to & X_{n_0+2} & \to & X_{n_0+1} & \to & X_n & \to & \cdots & \to & X_{-n_0} & \to & Z_{-n_0-1}(X) & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X'' : & \cdots & \to & 0 & \to & Z_n(X) & \to & X_n & \to & \cdots & \to & X_{-n_0} & \to & Z_{-n_0-1}(X) & \to & 0
\end{array}$$

is an $A$-quasi-isomorphism and hence $X' \cong X''$ in $D^b(A)$. Therefore, $X \cong X' \cong X''$ in $D^b(A)$ and then $X \in D^b(A)$. \qed

Denote by $K^{+, b}(A \cap B)$ the full subcategory of $K^+(A \cap B)$ consisting of complexes $X$ with the $i$th differential $X_i \to X_{i-1}$ factors as $X_i \to Z_{i-1}(X) \to X_{i-1}$ and $0 \to Z_i(X) \to X_i \to Z_{i-1}(X) \to 0$ lies in $\varepsilon$, for $i \ll 0$.

**Proposition 7.4.** $K^{+, b}(A \cap B)$ is a triangulated subcategory of $K^-(A \cap B)$.

**Proof.** It is easy to check that $K^{+, b}(A \cap B)$ is closed under shift functors $[1]$ and $[-1]$. Let $f : X \to Y$ be an isomorphism in $K^+(A \cap B)$ with $X \in K^{+, b}(A \cap B)$. Then $\text{Con}(f)$ is a null-homotopic complex and then an $A$ complex. It follows from $X \in K^{+, b}(A \cap B)$ that $H_i(Y) \cong H_i(X) = 0$ for $i \ll 0$. Note that $0 \to Y \to \text{Con}(f) \to X[1] \to 0$ is an exact sequence of complexes. We have the following
with exact rows and columns. It follows from \(Z_{i-1}(X), Z_i(\text{Con}(f)) \in \mathcal{A}\) that \(Z_i(Y) \in \mathcal{A}\) for \(i \leq 0\) and then \(Y \in \mathcal{K}^{+, b}(A \cap B)\). Thus \(\mathcal{K}^{+, b}(A \cap B)\) is closed under isomorphisms.

Now let \(f : X \to Y\) be a morphism in \(\mathcal{K}^{+, b}(A \cap B)\). By a similar argument as above we have that \(\text{Con}(f) \in \mathcal{K}^{+, b}(A \cap B)\) and we complete the proof. \(\square\)

**Proposition 7.5.** There exist triangle-equivalences \(\mathcal{K}^+(A \cap B) \simeq \mathcal{D}^+(A)\) and \(\mathcal{K}^{+, b}(A \cap B) \simeq \mathcal{D}^b(A)\).

**Proof.** Let \(G : \mathcal{K}^+(A \cap B) \to \mathcal{D}^+(A)\) be the composition functor \(\mathcal{K}^+(A \cap B) \xrightarrow{i} \mathcal{K}^+(A) \xrightarrow{Q} \mathcal{D}^+(A)\), where \(i, Q\) are the canonical functors. It follows from [18, Theorem 3.3.1(1)] that \(G\) is a triangle-equivalence. Now by Lemma 7.3 that the restriction of \(G\) to the full subcategory \(\mathcal{K}^{+, b}(A \cap B)\) induces a triangle-equivalence \(\mathcal{K}^{+, b}(A \cap B) \simeq \mathcal{D}^b(A)\). \(\square\)

It is easy to check that \(\mathcal{K}^b(A \cap B)\) is a thick subcategory of \(\mathcal{D}^b(A)\), now we introduce the following definition.

**Definition 7.6.** The Verdier quotient \(\mathcal{D}_{sg}(A) := \mathcal{D}^b(A) / \mathcal{K}^b(A \cap B)\) is called the singularity category of \(A\).

Inspired by [19, Theorem 3.3], we wonder whether or not \(\mathcal{D}_{sg}(A)\) is algebraically closed, i.e., if there exists some Frobenius category such that \(\mathcal{D}_{sg}(A)\) is triangle-equivalent to the stable category of this Frobenius category.

Note that \(A \cap \mathcal{G}(B)\) is a Frobenius category with \(A \cap B\) the full subcategory of projectives and injectives by Theorem 6.3. Let \(0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0\) be a conflation of \(A \cap \mathcal{G}(B)\). We have a commutative diagram of conflations in \(A \cap \mathcal{G}(B)\):

\[
\begin{array}{ccc}
0 & \xrightarrow{u} & X & \xrightarrow{v} & Y & \xrightarrow{w} & Z & \to 0 \\
0 & \xrightarrow{u} & X & \xrightarrow{v} & H & \xrightarrow{w} & T(X) & \to 0,
\end{array}
\]

where \(H \in A \cap B\). Then \(X \xrightarrow{H} Y \xrightarrow{Z} Z \xrightarrow{T(X)} T(X)\) is a triangle in \(A \cap \mathcal{G}(B)\) with \(T : A \cap \mathcal{G}(B) \to A \cap \mathcal{G}(B)\) the shift functor. For more details, we refer to [34, Chapter I].

In the following denote the composition functor \(\mathcal{A} \cap \mathcal{G}(B) \hookrightarrow \mathcal{A} \xrightarrow{\mathcal{D}^b(A)} \mathcal{D}_{sg}(A)\) with all canonical ones by \(F\), then it is clear to see \(F\) induces a functor \(F' : \mathcal{A} \cap \mathcal{G}(B) \to \mathcal{D}_{sg}(A)\).
Proposition 7.7. $F' : \mathcal{A} \cap \mathcal{G}(\mathcal{B}) \to \mathbb{D}_{sg}(\mathcal{A})$ is fully faithful.

The proof of the proposition involves the following two lemmas.

Lemma 7.8. The canonical functor $\theta : \mathcal{A} \to \mathbb{D}^{b}(\mathcal{A})$ which sends every $X \in \mathcal{A}$ to the stalk complex concentrated in degree zero and sends every morphism $f : X \to Y$ in $\mathcal{A}$ to $\text{id}_{Y} \setminus f : X \xrightarrow{f} Y \xleftarrow{\text{id}_{Y}} Y$ is fully faithful.

Proof. It suffices to show that for any $X, Y \in \mathcal{A}$, the map $\theta : \text{Hom}_{R}(X, Y) \to \text{Hom}_{\mathbb{D}^{b}(\mathcal{A})}(X, Y)$ is an isomorphism.

Let $f \in \text{Hom}_{R}(X, Y)$. If $F(f) = 0$, then there exists an $\mathcal{A}$-quasi-isomorphism and hence a quasi-isomorphism $s : Y \to Z$ such that $sf \sim 0$, and then $H_{0}(s)H_{0}(f) = 0$. Since $H_{0}(s)$ is an isomorphism, $f = H_{0}(f) = 0$. On the other hand, let $s \setminus f : X \xrightarrow{f} U \xleftarrow{s} Y$ be a morphism of $\mathbb{D}^{b}(\mathcal{A})$, where $s$ is an $\mathcal{A}$-quasi-isomorphism. Consider the truncation:

$$U' := 0 \to C_{0}(U) \to U_{-1} \xrightarrow{\partial_{U}} U_{-2} \to \cdots$$

of $U$ and the canonical map $p : U \to U'$. It follows that $ps : Y \to U'$ is a quasi-isomorphism and then $\text{Con}(ps) = 0 \to Y \to C_{0}(U) \to U_{-1} \xrightarrow{\partial_{U}} U_{-2} \to \cdots$ is exact. Note that $\text{Con}(ps)$ is right bounded and $Y \in \mathcal{A}$. Hence $C_{0}(U) \in \mathcal{A}$ and then $ps$ is an $\mathcal{A}$-quasi-isomorphism. Let $g := H_{0}(s)^{-1}H_{0}(f) \in \text{Hom}_{R}(X, Y)$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{H_{0}(s)} & H_{0}(U) \\
\downarrow{s} & & \downarrow{p} \\
U & \xrightarrow{p} & U',
\end{array}$$

where $H_{0}(U) \to U'$ is the canonical map, so $psg = psH_{0}(s)^{-1}H_{0}(f) = pf$. Thus we get the following commutative diagram of complexes:

$$\begin{array}{ccc}
X & \xrightarrow{p} & U' \\
\downarrow{g} & \xleftarrow{s} & \downarrow{ps} \\
Y & \xleftarrow{\text{id}_{Y}} & Y
\end{array}$$

which implies $\theta(g) = \text{id}_{Y} \setminus g = s \setminus f$. \hfill \Box

Lemma 7.9. The following are true for any ring $R$:

1. If $X \in \mathcal{A} \cap \mathcal{G}(\mathcal{B})$ and $H \in \mathbb{K}^{b}(\mathcal{A} \cap \mathcal{B})$ with $H_{i} = 0$ for any $i \leq 0$, then $\text{Hom}_{\mathbb{D}^{b}(\mathcal{A})}(X, H) = 0$.
2. If $Y \in \mathcal{A} \cap \mathcal{G}(\mathcal{B})$ and $H \in \mathbb{K}^{b}(\mathcal{A} \cap \mathcal{B})$ with $H_{i} = 0$ for any $i \geq 0$, then $\text{Hom}_{\mathbb{D}_{sg}(\mathcal{A})}(H, Y) = 0$.

Proof. (1). Note that $H \in \mathbb{K}^{b}(\mathcal{A} \cap \mathcal{B})$ with $H_{i} = 0$ for any $i \leq 0$. One easily checks that $\text{Hom}_{\mathbb{K}^{b}(\mathcal{A})}(X, H) = 0$. It follows from [18, Corollary 3.3.4] that $\text{Hom}_{\mathbb{D}^{b}(\mathcal{A})}(X, H) \cong \text{Hom}_{\mathbb{K}^{b}(\mathcal{A})}(X, H)$. So $\text{Hom}_{\mathbb{D}_{sg}(\mathcal{A})}(X, H) = 0$, as desired.

(2). Since $Y \in \mathcal{A} \cap \mathcal{G}(\mathcal{B})$, there is an exact sequence $0 \to Y \to L_{0} \to L_{-1} \to \cdots \to$ of $R$-modules with $L_{i} \in \mathcal{A} \cap \mathcal{B}$ and $\ker(L_{i} \to L_{i-1}) \in \mathcal{A} \cap \mathcal{G}(\mathcal{B})$ for any $i \leq 0$ by Theorem 6.3. Hence $Y \cong L$ in $\mathbb{D}^{b}(\mathcal{A})$, where $L$ is the complex $0 \to L_{0} \to L_{-1} \to \cdots \to$. It follows from [18, Corollary 3.3.4] that
\[ \text{Hom}_{\mathbb{D}^b(A)}(H, Y) \cong \text{Hom}_{\mathbb{K}(A)}(H, L). \] Note that \( L \) is a complex with each \( Z_i(L) \in A \cap \mathcal{G}(\mathcal{B}). \) Then the sequence
\[ \text{Hom}_A(X, L_{i+1}) \to \text{Hom}_A(X, L_i) \to \text{Hom}_A(X, L_{i-1}) \]
is exact for any \( i \leq -1 \) and any \( X \in A \cap \mathcal{B}. \) By hypothesis, there is a non-negative \( m \) such that \( H_i = 0 \) for \( i < -m. \) Let \( f : H \to L \) be a morphism in \( \mathbb{K}(A). \) It follows that \( \partial^L_m f_m = 0. \) Note that \( \text{Hom}_A(H_{-m}, L_{-m+1}) \to \text{Hom}_A(H_{-m}, L_{-m}) \to \text{Hom}_A(H_{-m}, L_{-m-1}) \) is exact. Then there is a morphism \( s_m : H_{-m} \to L_{-m-1} \) such that \( f_m = \partial^L_{m+1} s_m. \) Since \( \partial^L_{m+1}(f_{m+1} - s_m \partial^H_{m+1}) = \partial^L_{m+1} f_{m+1} - \partial^L_{m+1} s_m \partial^H_{m+1} = \partial^L_{m+1} f_{m+1} - f_m \partial^H_{m+1} = 0, \) there is a morphism \( s_{m+1} : H_{-m+1} \to L_{-m} \) such that \( f_{m+1} - s_m \partial^H_{m+1} = \partial^L_{m+2} s_m, \) by noting that the sequence \( \text{Hom}_A(H_{-m+1}, L_{-m+2}) \to \text{Hom}_A(H_{-m+1}, L_{-m+1}) \to \text{Hom}_A(H_{-m+1}, L_{-m}) \) is exact. Hence \( f_{m+1} = \partial^L_{m+2} s_{m+1} + s_m \partial^H_{m+1}. \) By proceeding in this manner, we get a morphism \( s_i : H_{-i} \to L_{-i-1} \) such that \( f_{-i} = \partial^L_{i+1} s_{-i} + s_{i-1} \partial^H_{-i} \) for \( 1 \leq i \leq m - 2. \) This implies that \( f : H \to L \) is null-homotopic. So \( \text{Hom}_{\mathbb{D}^b(A)}(H, Y) = \text{Hom}_{\mathbb{K}(A)}(H, L) = 0. \) This completes the proof. \[ \square \]

7.10. Proof of Proposition 7.7. In the following, we will use a doubled arrow to denote a morphism lying in the saturated multiplicative system determined by the thick subcategory \( \mathbb{K}^b(A \cap \mathcal{B}) \) of \( \mathbb{D}^b(A). \) A morphism in \( \mathbb{D}_{\mathcal{G}^b}(A) \) from \( X \) to \( Y \) is denoted by right fraction \( f/s : X \xleftarrow{s} Z \xrightarrow{f} Y. \)

If \( F(f) = 0 \) for some \( f : X \to Y \) in \( A \cap \mathcal{G}(\mathcal{B}). \) Then there exists \( s : Z \to X \) in \( \mathbb{D}^b(A) \) with \( \text{Con}(s) \in \mathbb{K}^b(A \cap \mathcal{B}) \) such that \( fs = 0. \) We have a triangle \( Z \xrightarrow{s} X \xrightarrow{h} \text{Con}(s) \to Z[1] \) in \( \mathbb{D}^b(A). \)

Since \( fs = 0, \) \( f = f'h \) for some \( f' : \text{Con}(s) \to Y. \) Let \( H = \text{Con}(s). \) Then \( H \in \mathbb{K}^b(A \cap \mathcal{B}). \) Consider the following triangle in \( \mathbb{D}^b(A): \)
\[ H_{\leq-1} \to H \to H_{\geq0} \to H_{\leq-1}[1]. \] (7.1)

By applying the functor \((- , Y) : = \text{Hom}_{\mathbb{D}^b(A)}(- , Y) \) to (7.1) we get an exact sequence of abelian groups:
\[ (H_{\geq0}, Y) \to (H, Y) \to (H_{\leq-1}, Y) \]
It follows from Lemma 7.9(2) that \( (H_{\leq-1}, Y) = 0. \) So \( f' \) and hence \( f \) factors through \( H_{\geq0}. \) We may suppose \( H_i = 0 \) for \( i \leq -1. \) Consider the triangle: \( H_{\geq1}[1] \to H_0 \to H \to H_{\geq1}. \) Note that \( \text{Hom}_{\mathbb{D}^b(A)}(X, H_{\geq1}) = 0 \) by Lemma 7.9(1). Thus \( h \) factors through \( H_0 \in A \cap \mathcal{B}, \) and hence \( f \) also factors through \( H_0 \in A \cap \mathcal{B} \) in \( \mathbb{D}^b(A). \) So \( f : X \to Y \) factors through \( H_0 \in A \cap \mathcal{B} \) by Lemma 7.8. This implies that \( F' \) is faithful.

Now for any \( X, Y \in A \cap \mathcal{G}(\mathcal{B}) \) and any morphism \( f/s : X \xleftarrow{s} Z \xrightarrow{f} Y \) in \( \mathbb{D}_{\mathcal{G}^b}(A). \) We have a triangle
\[ Z \xrightarrow{s} X \xrightarrow{h} H \to Z[1] \]
in \( \mathbb{D}^b(A) \) with \( H \in \mathbb{K}^b(A \cap \mathcal{B}). \) Consider the following triangle in \( \mathbb{D}^b(A): \)
\[ H_{\leq0} \xrightarrow{i} H \xrightarrow{j} H_{\geq1} \to H_{\leq0}[1]. \]
Since \( \text{Hom}_{\mathbb{D}^b(A)}(X, H_{\geq 1}) = 0 \) by Lemma 7.9(1), \( h = ih' \) for some \( h' : X \to H_{\leq 0} \). By the octahedral axiom, we have the following commutative diagram

\[
\begin{array}{c}
Y' \xrightarrow{a} X \xrightarrow{h'} H_{\leq 0} \xrightarrow{H'_{[1]}} Y'_{[1]} \\
\downarrow t \downarrow \downarrow \downarrow \downarrow \\
Z \xrightarrow{s} X \xrightarrow{h} H \xrightarrow{Z_{[1]}} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
H_{\geq 1}[{-1}] \xrightarrow{i} H \xrightarrow{H_{[1]}} \\
\end{array}
\]

with all rows and the leftmost column be triangles in \( \mathbb{D}^b(A) \). Since \( \text{Hom}_{\mathbb{D}^b(A)}(H_{\leq 0}[{-1}], Y) = 0 \) by Lemma 7.9(2), \( ft = f'a = f's't \) for some \( f' : X \to Y \). Thus \( f/s = F(f') \) and \( F' \) is full. This completes the proof.

\[ \blacksquare \]

**Proposition 7.11.** \( F' : A \cap G(B) \to \mathbb{D}_{sg}(A) \) is a triangle functor.

**Proof.** Let \( X \to Y \to Z \to T(X) \) be a triangle in \( A \cap G(B) \), then it comes from the following commutative diagram of conflations in \( A \cap G(B) \):

\[
\begin{array}{c}
0 \xrightarrow{} X \xrightarrow{i} Y \xrightarrow{i} Z \xrightarrow{i} 0 \\
0 \xrightarrow{} X \xrightarrow{i} H \xrightarrow{i} T(X) \xrightarrow{i} 0, \\
\end{array}
\]

where \( H \in A \cap B \). This induces a commutative diagram of triangles in \( \mathbb{D}^b(A) \)

\[
\begin{array}{c}
X \xrightarrow{} Y \xrightarrow{} Z \xrightarrow{} X[1] \\
X \xrightarrow{} H \xrightarrow{} T(X) \xrightarrow{} X[1]. \\
\end{array}
\]

It is sent to a commutative diagram of triangles in \( \mathbb{D}_{sg}(A) \). Since \( H \in A \cap B \) is zero, \( T(X) \cong X[1] \) in \( \mathbb{D}_{sg}(A) \). Furthermore, it is easy to check that isomorphism \( T(X) \cong X[1] \) is functorial in \( X \). Thus \( F' \) is a triangle functor.

\[ \blacksquare \]

Put

\[ \mathbb{D}^b(A)_{fGB} := \{ M \in \mathbb{D}^b(A) : GB\text{-dim} M < \infty \}. \]

**Lemma 7.12.** \( \mathbb{D}^b(A)_{fGB} \) is a triangulated subcategory of \( \mathbb{D}^b(A) \).

**Proof.** Clearly \( \mathbb{D}^b(A)_{fGB} \) is closed under shift functors \([1], [-1]\). Let \( M \) and \( N \) be two complexes with \( M \cong N \) in \( \mathbb{D}^b(A) \). Then \( M \cong N \) in \( \mathbb{D}(R) \). By Theorem 1.1, we have \( GB\text{-dim} M < \infty \) if and only if \( GB\text{-dim} N < \infty \). Hence \( \mathbb{D}^b(A)_{fGB} \) is closed under isomorphisms.

Assume that \( M \to N \to L \to M[1] \) is a triangle in \( \mathbb{D}^b(A) \) such that \( M \) and \( N \) are in \( \mathbb{D}^b(A)_{fGB} \). Then there exists some triangle \( X \xrightarrow{f} Y \to \text{Con}(f) \to X[1] \) in \( \mathbb{D}^b(A) \) such that \( M \to N \to L \to \]
$M[1]$ is its image under the canonical functor. Thus we have an isomorphism of triangles:

\[
\begin{array}{ccccccc}
M & \rightarrow & N & \rightarrow & L & \rightarrow & M[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \rightarrow & Y & \rightarrow & \text{Con}(f) & \rightarrow & X[1]
\end{array}
\]

in $\mathbb{D}^b(A)$, and then $L \cong \text{Con}(f)$. Since $\mathbb{D}^b(A)_{GB}$ is closed under isomorphisms and $M, N \in \mathbb{D}^b(A)_{GB}$, $X, Y \in \mathbb{D}^b(A)_{GB}$. By the exactness of the sequence of complexes $0 \rightarrow Y \rightarrow \text{Con}(f) \rightarrow X[1] \rightarrow 0$, we have that $\text{Con}(f) \in \mathbb{D}^b(A)_{GB}$ by Corollary 3.16. Hence $L \in \mathbb{D}^b(A)_{GB}$ and we complete the proof.

**Proposition 7.13.** $F' : A \cap \mathcal{G}(B) \rightarrow \mathbb{D}_{sg}(A)$ is dense if and only if $\mathcal{G}B$-$\dim M < \infty$ for any complex $M$ in $\mathbb{D}^-(R)$.

**Proof.** For the “if” part, assume that $\mathcal{G}B$-$\dim M < \infty$ for any complex $M$ in $\mathbb{D}^-(R)$ and any $X \in \mathbb{D}_{sg}(A)$. It follows from Proposition 7.5 that $X \cong H$ in $\mathbb{D}^b(A)$ for some $H \in \mathbb{K}^+(A \cap B)$ and then there exists some $n_0 \leq 0$ such that $H$ is isomorphic to the complex:

\[
\cdots \rightarrow H_{n_0+1} \rightarrow H_{n_0} \rightarrow Z_{n_0-1}(H) \rightarrow 0.
\]

The above complex induces a triangle in $\mathbb{D}^b(A)$ and hence a triangle in $\mathbb{D}_{sg}(A)$:

\[
H_{\geq n_0}[1] \rightarrow Z_{n_0-1}(H)[n_0-1] \rightarrow H \rightarrow H_{\geq n_0}.
\]

Since $H_{\geq n_0} \in K^b(A \cap B)$, $H \cong Z_{n_0-1}(H)[n_0-1]$ in $\mathbb{D}_{sg}(A)$. By assumption, we may assume $\mathcal{G}B$-$\dim Z_{n_0-1}(H) = m_0 < \infty$. Let $0 \rightarrow Z_{n_0-1}(H) \rightarrow I_0 \rightarrow I_{-1} \rightarrow \cdots \rightarrow I_{-m_0+1} \rightarrow I_{-m_0} \rightarrow \cdots$ be an exact complex of $A$ with each $I_i \in A \cap B$. Denote by $I$ the complex $0 \rightarrow I_0 \rightarrow I_{-1} \rightarrow \cdots \rightarrow I_{-m_0+1} \rightarrow I_{-m_0} \rightarrow \cdots$. It follows from Theorem 1.1 that $Z_{-m_0}(I) \in A \cap \mathcal{G}(B)$. Thus $Z_{n_0-1}(H) \cong Z_{-m_0}(I)[-m_0]$ in $\mathbb{D}_{sg}(A)$, and hence $X \cong H \cong Z_{n_0-1}(H)[n_0-1] \cong Z_{-m_0}(I)[-m_0]$ in $\mathbb{D}_{sg}(A)$, where $I_0 = -m_0+n_0-1$. Because $Z_{-m_0}(I) \in A \cap \mathcal{G}(B)$ and $A \cap \mathcal{G}(B)$ is a Frobenius category, there is an exact complex $\cdots \rightarrow J_{-l_0} \rightarrow J_{-l_0-1} \rightarrow \cdots \rightarrow J_1 \rightarrow J_0 \rightarrow Z_{-m_0}(I) \rightarrow 0$ of $A$ with each $J_i \in A \cap B$ and each cycle $Z_i(J) := \ker(J_i \rightarrow J_{i-1}) \in A \cap \mathcal{G}(B)$. Thus $Z_{-m_0}(I) \cong Z_{-l_0-1}(J)[-l_0]$ in $\mathbb{D}_{sg}(A)$ and hence $X \cong Z_{-l_0-1}(J)$ in $\mathbb{D}_{sg}(A)$. Therefore $X \cong F'(Z_{-l_0-1}(J))$.

For the “only if” part, assume that $F' : A \cap \mathcal{G}(B) \rightarrow \mathbb{D}_{sg}(A)$ is dense and any $M \in A$. It follows that $M \cong F(G)$ in $\mathbb{D}_{sg}(A)$ for some $G \in A \cap \mathcal{G}(B)$. Let $s \circ f : M \rightarrow Z \cong G$ be an isomorphism in $\mathbb{D}_{sg}(A)$ with $\text{Con}(s) \in \mathbb{K}^b(A \cap B)$, then $\text{Con}(f) \in \mathbb{K}^b(A \cap B)$. Consider the triangle:

\[
G \cong Z \rightarrow \text{Con}(s) \rightarrow G[1]
\]

in $\mathbb{D}^b(A)$. Since both $G$ and $\text{Con}(s)$ lie in $\mathbb{D}^b(A)_{GB}$, we have $Z \in \mathbb{D}^b(A)_{GB}$ by Lemma 7.12. Then it follows from $Z \in \mathbb{D}^b(A)_{GB}$ and $\text{Con}(f) \in \mathbb{D}^b(A)_{GB}$ that $M \in \mathbb{D}^b(A)_{GB}$. Therefore $\mathcal{G}B$-$\dim M < \infty$. So $\mathcal{G}B$-$\dim M < \infty$ for any complex $M$ in $\mathbb{D}^-(R)$ by Theorem 6.4. This completes the proof.

We end this section with the following theorem which contains (1) $\Leftrightarrow$ (3) of Theorem 1.4 from the introduction

**Theorem 7.14.** $F' : A \cap \mathcal{G}(B) \rightarrow \mathbb{D}_{sg}(A)$ is a triangle-equivalence if and only if $\mathcal{G}B$-$\dim M < \infty$ for any complex $M$ in $\mathbb{D}^-(R)$.

**Proof.** The result follows directly from Propositions 7.7, 7.11 and 7.13. \(\square\)
8. Applications to the finitistic dimension

Our goal in this section is to characterize when the little finitistic dimension is finite. To this end, we will let \((X, Y)\) be the cotorsion pair cogenerated by \(P^{<\infty}\), that is, \(Y = (\mathcal{P}^{<\infty})^\perp_1\) and \(X = \perp_1 Y\), where \(\mathcal{P}^{<\infty}\) is the class of finitely generated modules with finite projective dimension.

**Lemma 8.1.** ([42, Lemma 4.1]) The cotorsion pair \((X, Y)\) is both complete and hereditary in \(\text{Mod} R\). Moreover, the induced cotorsion pairs \((\tilde{X}, \text{dg} \tilde{Y})\) and \((\text{dg} \tilde{X}, \tilde{Y})\) of \((X, Y)\) are both complete and hereditary in \(\mathcal{C}(R)\).

**Lemma 8.2.** ([42, Proposition 4.3]) The following are equivalent for each non-negative integer \(n\):

1. \(\text{findim}(R) \leq n\);
2. \(Y\)-\text{dim}\(M\) \(\leq n\) for any \(R\)-module \(M\);
3. \(Y\)-\text{dim}\(M\) \(\leq n - \inf M\) for any complex \(M\)
   Moreover if \(R\) is a left coherent ring, then the above conditions are also equivalent to:
4. \(Y\)-\text{dim}\(R\) \(\leq n\)

**Lemma 8.3.** Let \(M\) be an \(R\)-module and \(n\) a non-negative integer. Then \(G\)\(Y\)-\text{dim}\(M\) \(\leq n\) if and only if there exists an exact sequence \(0 \rightarrow M \rightarrow G \rightarrow L \rightarrow 0\) of \(R\)-modules with \(G \in \mathcal{G}(Y)\) and \(Y\)-\text{dim}\(L\) \(\leq n - 1\) (if \(n = 0\), this should be interpreted as \(L = 0\)).

**Proof.** The “if” part is clear. For the “only if” part, we assume that \(G\)\(Y\)-\text{dim}\(M\) \(\leq n\). We proceed by induction on \(n\). The case \(n = 0\) is trivial.

For \(n = 1\), there exists an exact sequence \(0 \rightarrow M \rightarrow Y \rightarrow K \rightarrow 0\) of \(R\)-modules with \(Y \in \mathcal{Y}\) and \(K \in \mathcal{G}(\mathcal{Y})\) by Theorem 1.1. Note that \(K \in \mathcal{G}(\mathcal{Y})\). Then there exists an exact sequence \(0 \rightarrow U \rightarrow L \rightarrow K \rightarrow 0\) of \(R\)-modules with \(U \in \mathcal{G}(\mathcal{Y})\) and \(L \in \mathcal{Y}\) by Definition 3.1. Consider the following pullback diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
U & \rightarrow & U \\
\downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & G & \rightarrow & L & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & Y & \rightarrow & K & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0.
\end{array}
\]

Note that \(U \in \mathcal{G}(\mathcal{Y})\) and \(Y \in \mathcal{Y}\). So \(G \in \mathcal{G}(\mathcal{Y})\) by Proposition 3.4(1), as desired.

For \(n \geq 2\), there exists an exact sequence \(0 \rightarrow K \rightarrow T \rightarrow U \rightarrow 0\) of \(R\)-modules with \(Y \in \mathcal{Y}\) and \(G\)\(Y\)-\text{dim}\(K\) \(\leq n - 1\) by Theorem 1.1. By induction hypothesis, we have an exact sequence \(0 \rightarrow K \rightarrow T \rightarrow U \rightarrow 0\) of \(R\)-modules with \(T \in \mathcal{G}(\mathcal{Y})\) and \(Y\)-\text{dim}\(U\) \(\leq n - 2\). Thus there exists an exact sequence \(0 \rightarrow W \rightarrow V \rightarrow T \rightarrow 0\) of \(R\)-modules with \(W \in \mathcal{G}(\mathcal{Y})\) and \(V \in \mathcal{Y}\) by Definition 3.1.
Hence we have the following pullback diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
W & W & \downarrow & \downarrow & \downarrow & \downarrow & W \\
0 & L & V & U & 0 \\
0 & K & T & U & 0 \\
0 & 0 & . & . & . & . & 0
\end{array}
\]

Note that \( V \in \mathcal{Y} \) and \( \mathcal{Y}\text{-dim}\, U \leq n - 2 \). It follows that \( \mathcal{Y}\text{-dim}\, L \leq n - 1 \). Consider the following pullback diagram:

\[
\begin{array}{ccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
W & W & \downarrow & \downarrow & \downarrow & \downarrow & W \\
0 & M & G & L & 0 \\
0 & M & Y & K & 0 \\
0 & 0 & . & . & . & . & 0
\end{array}
\]

Note that \( W \in G(\mathcal{Y}) \) and \( Y \in \mathcal{Y} \). So \( G \in G(\mathcal{Y}) \) by Proposition 3.4(1). This completes the proof. □

**Lemma 8.4.** If \( M \) is an \( R \)-module in \( \perp_1 \mathcal{G}(\mathcal{Y}) \), then \( \mathcal{G}\mathcal{Y}\text{-dim}\, M = \mathcal{Y}\text{-dim}\, M \).

**Proof.** Let \( M \) be an \( R \)-module in \( \perp_1 \mathcal{G}(\mathcal{Y}) \). It follows from Corollary 3.17 that \( \mathcal{G}\mathcal{Y}\text{-dim}\, M \leq \mathcal{Y}\text{-dim}\, M \). We need show that \( \mathcal{Y}\text{-dim}\, M \leq \mathcal{G}\mathcal{Y}\text{-dim}\, M \). Assume that \( \mathcal{G}\mathcal{Y}\text{-dim}\, M = n \) is a non-negative integer. If \( n = 0 \), there exists an exact sequence \( 0 \to W \to B \to M \to 0 \) of \( R \)-modules with \( W \in G(\mathcal{Y}) \) and \( B \in \mathcal{Y} \) by Definition 3.1. Since \( M \in \perp_1 \mathcal{G}(\mathcal{Y}) \), the exact sequence \( 0 \to W \to B \to M \to 0 \) is split. Hence \( M \in \mathcal{Y} \), as desired. For \( n \geq 1 \), there exists an exact sequence \( 0 \to M \to G \to L \to 0 \) of \( R \)-modules with \( G \in G(\mathcal{Y}) \) and \( \mathcal{Y}\text{-dim}\, L \leq n - 1 \) by Lemma 8.3. Note that there exists an exact sequence \( 0 \to X \to Y \to G \to 0 \) of \( R \)-modules with \( X \in G(\mathcal{Y}) \) and \( Y \in \mathcal{Y} \) by Definition 3.1.
Consider the following pullback diagram:

\[
\begin{array}{ccccccc}
0 & 0 \\
\downarrow & \downarrow \\
X & X \\
\downarrow & \downarrow \\
0 & T & Y & L & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & M & G & L & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0.
\end{array}
\]

Since \( Y \in \mathcal{Y} \) and \( \mathcal{Y}\text{-dim}L \leq n-1 \), \( \mathcal{Y}\text{-dim}T \leq n \). Note that \( M \in \perp_{1} \mathcal{G}(\mathcal{Y}) \) and \( X \in \mathcal{G}(\mathcal{Y}) \). Then the left column \( 0 \to X \to T \to M \to 0 \) in the above diagram is split. Thus \( T \cong X \oplus M \), and hence \( \mathcal{Y}\text{-dim}M \leq n \). So \( \mathcal{Y}\text{-dim}M \leq \mathcal{G}\mathcal{Y}\text{-dim}M \). This completes the proof. \( \square \)

**Proposition 8.5.** Let \( R \) be an Artin algebra and \( X = (X, Y) \) the cotorsion pair cogenerated by \( \mathcal{P}^{<\infty} \) in \( \text{Mod}(R) \). Then the following are equivalent:

1. \( \text{findim}(R) \leq n \);
2. \( \mathcal{G}\mathcal{Y}\text{-dim}M \leq n \) for any \( R \)-module \( M \);
3. \( \mathcal{G}\mathcal{Y}\text{-dim}M \leq n - \inf M \) for any complex \( M \);
4. \( \mathcal{G}\mathcal{Y}\text{-dim}R \leq n \).

**Proof.** (1) \( \Rightarrow \) (2) holds by Corollary 3.17 and Lemma 8.2.

(2) \( \Rightarrow \) (3). There is nothing to prove \( \inf M = \infty \) or \( \inf M = -\infty \). For \( \inf M = s \) with \( s \) an integer. Let \( I \) be a dg-injective complex with \( M \cong I \). Then \( \mathcal{G}\mathcal{Y}\text{-dim}Z_{s}(I) \leq n \) by (2). Hence \( Z_{s-n}(I) \in \mathcal{G}\mathcal{Y} \). So \( \mathcal{G}\mathcal{Y}\text{-dim}M \leq n - s \), as desired.

(3) \( \Rightarrow \) (4) is trivial.

(4) \( \Rightarrow \) (1) follows from Lemmas 8.2 and 8.4. \( \square \)

**Corollary 8.6.** Let \( R \) be an Artin algebra. Then \( \mathcal{G}(\mathcal{Y}) = \mathcal{Y} \) whenever \( \text{findim}(R) < \infty \).

**Proof.** The result holds by Proposition 8.5 and Corollary 3.17. \( \square \)

We now finish this paper by giving the proof of Theorem 1.5 as follows.

**8.7. Proof of Theorem 1.5.** (1) \( \iff \) (2) \( \iff \) (3) hold by Proposition 8.5.

(2) \( \iff \) (4) \( \iff \) (5) follow from Theorem 1.4.

The last claim follows from Corollary 4.4. This completes the proof. \( \square \)

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