Lower bounds for the isoperimetric numbers of random regular graphs

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Abstract
The vertex isoperimetric number of a graph $G = (V, E)$ is the minimum of the ratio $|\partial_V U|/|U|$ where $U$ ranges over all non-empty subsets of $V$ with $|U|/|V| \leq u$ and $\partial_V U$ is the set of all vertices adjacent to $U$ but not in $U$. The analogously defined edge isoperimetric number — with $\partial_V U$ replaced by $\partial_E U$, the set of all edges with exactly one endpoint in $U$ — has been studied extensively. Here we study random regular graphs. For the case $u = 1/2$, we give asymptotically almost sure lower bounds for the vertex isoperimetric number for all $d \geq 3$. Moreover, we obtain a lower bound on the asymptotics as $d \to \infty$. We also provide asymptotically almost sure lower bounds on $|\partial_E U|/|U|$ in terms of an upper bound on the size of $U$, and analyse the bounds as $d \to \infty$.

1 Introduction
In this paper we consider versions of the isoperimetric number of random regular graphs. These are explicit indicators of the notion generally called expansion (see below for the relevant definitions). Random regular graphs give nondeterministic examples of expander graphs, and as mentioned in [12, Section 4.6], there is great interest in the edge and vertex expansion of sets of varying sizes. Here we obtain explicit bounds on the expansion of sets with given size in random regular graphs. We concentrate on the vertex version, which is more difficult and less well studied than the edge version.

Let $G$ be a graph on $n$ vertices. For a subset $U$ of its vertex set $V = V(G)$, let $\partial_V(U)$ denote the set of all vertices adjacent to a vertex in $U$ but not in $U$. Similarly, let $\partial_E(U)$ denote the set of all edges with exactly one end in $U$. Note that $|\partial_V(U)| \leq |\partial_E(U)|$. For any $0 < u \leq 1/2$ the $u$-edge isoperimetric number is defined as

\[ i_{E,u}(G) = \min_{|U| \leq un} \frac{|\partial_E(U)|}{|U|}, \]

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and likewise for any $0 < u \leq 1$ the $u$-vertex isoperimetric number of $G$ as
\[
    iv_u(G) = \min_{|U| \leq un} |\partial_V(U)|/|U|.
\]

The 1/2-edge [1/2-vertex] isoperimetric number is often referred to as the edge [vertex] isoperimetric number, and in this case, we simplify notation as $iE(G)$ and $iv(G)$. For the edge version, this makes immediate sense since a lower bound on the number of edges joining $S \subseteq V$ to $V \setminus S$ is obtained as $iE(G)\rho$ where $\rho = \min\{|S|, |V \setminus S|\}$. For the vertex isoperimetric number, the situation is not quite so symmetrical since $|\partial_V(U)| \neq |\partial_V(V \setminus U)|$ in general. However, $u = 1/2$ has some uses. Note that the iterated neighbourhoods of any vertex in $G$ expand by (at least) a factor $\alpha = 1 + iv(G)$ until they reach size $n/2$ (or more). Hence an easy upper bound on the diameter of $G$ is $2\log_\alpha(n/2)$. To give another example, Sauerwald and Stauffer [16] recently showed that if a certain rumour spreading process takes place on a regular graph, where informed vertices randomly select a neighbour to inform (i.e. the push model), then all vertices are informed, w.h.p., after $O((1/iv) \cdot \log^5 n)$ steps of the process. Hence a lower bound on $iv$ gives an upper bound (holding w.h.p.) for the time at which all vertices are aware of the rumour.

A graph is $d$-regular if all its vertices are of degree $d$. Let $G_{n,d}$ denote the uniform probability space on the set of all $d$-regular graphs on $n$ vertices that are simple (i.e., have no loops or multiple edges). A property holds asymptotically almost surely (a.a.s.) in a sequence of probability spaces on $\{\Omega_n\}$ if the probability that an element of $\Omega_n$ has the given property converges to 1 as $n \to \infty$. Define
\[
    iv_u(d) = \sup\{\ell : iv_u(G_{n,d}) \geq \ell \text{ a.a.s.}\}
\]
and define $iE_u(d)$ similarly. In the case $u = 1/2$, we simply write $iv(d)$ and $iE(d)$.

We can now describe our results more explicitly. Our main purpose is to provide asymptotically almost sure lower bounds for the vertex expansion of random regular graphs.

In Section 1 we highlight some results in the literature that relate to vertex and edge expansion of regular graphs.

In Section 4 the pairing model, as used by Bollobás [5] to investigate $iE(d)$, i.e., for the case of edge expansion, is discussed. As we shall see, this model is also helpful for studying $iv_u(d)$.

In Section 2 we introduce the method we use. In short, we obtain lower bounds on vertex expansion using the first moment method: for a sequence of non-negative, integer-valued random variables $\{X_n\}$, provided that $E(X_n) \to 0$ as $n \to \infty$, it follows that $X_n = 0$ a.a.s. For an introduction to probabilistic techniques in discrete mathematics, refer to [3] for instance. In [5], the first moment method easily provides lower bounds on $iE(d)$. However, in bounding $iv_u(d)$, the method yields bounds which are initially quite opaque. The main complication is that both $\partial_V$ and $\partial_E$ need to be considered. For a sequence $u(n) \to u_0 \in [0,1/2]$ and numbers $s$ and $y$, we define a random variable $X$ that counts the number of subsets $U$ with $|U| = un$, $|\partial_V| = sn$ and $|\partial_E| = yn$ in a graph from $G_{n,d}$. For such a sequence, we use the first moment method to determine the range of $s$ for which $X = 0$ a.a.s. This leads us to define a $u$-vertex expansion number $Iv_u(d)$, which can be thought of as an asymptotic profile for the vertex expansion of subsets $U \subset V(G_{n,d})$ with $|U| \sim un$. (See Section 1 for the precise definition.)

In Section 5 we obtain lower bounds $A_d(u)$ on $iv_u(d)$, and at Table 4 we provide approximate values for $A_d(u)$ for several values of $d$ and $u$. Since one expects the isoperimetric
number of a graph to be obtained by larger sets, it is reasonable to conjecture that the \( u \)-isoperimetric and expansion numbers coincide, i.e. \( i_{V,u}(d) = I_{V,u}(d) \), and hence, for all \( d \geq 3 \) and \( u \in (0, 1/2) \), \( i_{V,u}(d) \geq A_d(u) \). Unfortunately, the formulae do not seem to have a convenient explicit form, and so, this is not straightforward to show for the cases \( u < 1/2 \). We plan to address \( i_{V,u}(d) \), for \( u < 1/2 \), at a later time.

In the present work, we deal with the case \( u = 1/2 \). In Section 5 we obtain explicit lower bounds on the vertex isoperimetric number \( i_{V}(d) \) for all \( d \geq 3 \).

**Theorem 1.** For \( d \geq 3 \),
\[
i_{V}(d) \geq A_d(1/2) = 4\tilde{y}_d(1-1/2^d)/d
\]
where \( \tilde{y}_d \) is the smallest positive zero of the function
\[
F_d(y) = \log \left( \frac{(2^d-1)^s}{2^{d/2+s-1}(1-2s)^{1/2-s}s^s} \right) \bigg| s=2y(1-1/2^d)/d.
\]

Table 2 provides approximate values for \( A_d(1/2) \) for several values of \( d \).

In Section 6 we apply Theorem 1 to obtain a lower bound on the asymptotics of \( i_{V}(d) \) as \( d \to \infty \).

**Corollary 2.** As \( d \to \infty \),
\[
i_{V}(d) \geq 1 - 2/d + O((\log d)/d^2).
\]

Corollary 2 improves upon the information that is otherwise available from spectral results. See Section 2 for a discussion on this.

In Section 7 we switch attention to the edge isoperimetric number. Bollobás [5] computed lower bounds with the first moment method for \( i_{E}(d) \), i.e. for the case of edge expansion at \( u = 1/2 \), for all \( d \geq 3 \). Therein it is shown that for sufficiently large \( d \),
\[
i_{E}(d) \geq d/2 - \sqrt{d\log 2}
\]
and so
\[
\lim_{d \to \infty} i_{E}(d)/d = 1/2.
\]

Further, it is noted that for \( n \geq d+1 \), if \( G \) is a \( d \)-regular graph in \( n \) vertices and \( U \) is selected uniformly from \( \{1, 2, \ldots, n\} \) such that \( |U| = [n/2] \), then
\[
E(\partial U) = d[n/2][n/2]/(n-1).
\]

Hence, the lower bound (1) is, as stated in [5], ‘essentially best possible’ for large \( d \). Then, more generally, it is claimed in [5] that
\[
\lim_{d \to \infty} i_{E,u}(d)/d = 1 - u
\]
for all \( 0 < u < 1/2 \), however no explicit lower bounds are given. Our contribution is to find lower bounds on \( i_{E,u}(d) \), for all \( d \geq 3 \) and \( 0 < u \leq 1/2 \). The argument is similar to that of [5] in that we compute the expected number of sets in a random regular graph with an edge boundary of a given size, and then apply the first moment method. However, our use of the \( u \)-edge expansion number \( I_{E,u}(d) \) (analogous to \( I_{V,u}(d) \)) yields a simple method for computing ‘best possible’ lower bounds on \( i_{E,u}(d) \) for all \( d \geq 3 \) and \( 0 < u \leq 1/2 \). The following result is proved in Section 7.
Theorem 3. For $d \geq 3$ and $u \in (0, 1/2]$,

$$i_{E,u}(d) \geq \hat{A}_d(u) = \hat{y}_{d,u}/u$$

where $\hat{y}_{d,u}$ is the smallest positive zero of the function

$$\hat{f}_d(u, y) = \log \left( \frac{(du)^{du}(d(1-u))^{(1-u)}}{(du-y)(du-y)/2(d-1y)^{d-1y}/2d^{d/2}u^u(1-u)^{1-u}y^y} \right).$$

Table 3 provides approximate values for $\hat{A}_d(u)$ for several values of $d$ and $u$.

Applying Theorem 3 in Section 7 we obtain lower bounds on the asymptotics of $i_{V,u}(d)$, as $d \to \infty$, for all $0 < u \leq 1/2$.

Corollary 4. Fix $u \in (0, 1/2]$. As $d \to \infty$,

$$\hat{A}_d(u) = d(1-u) - 2(1-u)\sqrt{d \log(u^u(1-u)^{u-1})} + o(\sqrt{d}).$$

Hence

$$i_{E,u}(d) \geq d(1-u + o(1)) \quad (as \ d \to \infty).$$

2 Background

A $u$-edge [$u$-vertex] $\alpha$-expander, $\alpha > 0$, is a graph that has a $u$-edge [$u$-vertex] isoperimetric number at least as large as $\alpha$. The $u$-isoperimetric numbers of a graph may be viewed as indicators for its level of connectivity. Expander graphs provide a wealth of theoretical interest and have many applications. For a thorough exposition of the theory and applications of expander graphs, see [12] and the references therein.

Regular graphs are known to be good expanders with high probability, however, determining the isoperimetric number with precision is a difficult task. For instance, as shown by Golovach [11], even the problem of determining whether $i_E(G) \leq q$ provided $q \in \mathbb{Q}$ and $G$ has degree sequence bounded by 3, is $\text{NP}$-complete. Thus, bounds for the isoperimetric numbers of regular graphs are of interest.

Buser [6] showed that for all $n \geq 4$ there exists a cubic (3-regular) graph on $n$ vertices that has an edge isoperimetric number of at least $1/128$. To quote Bollobás [5]:

Buser’s proof . . . was very unorthodox in combinatorics and very exciting: it used the spectral geometry of the Laplace operator on Riemann surfaces, Kloosterman sums and the Jacquet-Langlands theory. As Buser wrote: the proof ‘is rather complicated and it would be more satisfactory to have an elementary proof.’

Using the standard first moment method Bollobás [5] provided a simple proof that much more is true. In [5] it is shown that in fact

$$i_E(d) \geq d/2 - \sqrt{d\log 2} \quad (as \ d \to \infty).$$

Several bounds for small $d$ are provided in [5]; such as $i_E(3) > 2/11$, $i_E(4) > 11/25$ and $i_E(5) > 18/25$. Other bounds have been found by analysing the spectral gap of the adjacency matrix of regular graphs. Let $\lambda(G)$ denote the second largest (i.e. largest other than $d$)
eigenvalue in the adjacency matrix of $G$. Alon and Millman [2] proved that if $G$ is $d$-regular then

$$i_E(G) \geq (d - \lambda)/2.$$  

Further, Alon [1] showed if $n > 40d^9$ and $G$ is $d$-regular, then

$$i_E(G) \leq d/2 - 3\sqrt{d}/16\sqrt{2}.$$  

The best lower bound to date for cubic graphs, $i_E(3) \geq 1/4.95$, was found by Kostochka and Melnikov [13] using an edge weighting argument. Upper bounds for $i_E(d)$ are available via results for the bisection width of regular graphs. Note that the (edge) bisection width of a graph $G = (V, E)$ is defined as

$$b(G) = \min_{(n-1)/2 \leq |U| \leq n/2} |\partial E(U)|/|U|,$$

and so clearly for any graph $G$,

$$i_E(G) \leq 2b(G).$$

The best upper bound for cubic graphs at present is that of Monien and Preis [14]. Therein it is proved $b(G) \leq 1/6 + \epsilon$ for all $\epsilon > 0$ for all sufficiently large connected cubic graphs $G$. (The required lower bound on the size of $G$ depends on $\epsilon$.) Hence $i_E(3) \leq 1/3$. This bound was found by way of an algorithmic procedure which attempts to find a small cut. The best upper bounds to date for $d > 3$ have been found by Díaz, Do, Serna and Wormald [8, 9] via randomized greedy algorithms analysed using the differential equation method. From the results of [8] it follows that $i_E(4) \leq 2/3$, and from the numerical values in [9] we obtain further values such as $i_E(5) \leq 1.0056$, $i_E(6) \leq 1.3348$ and $i_E(7) \leq 1.7004$.

In comparison, less information about the vertex isoperimetric numbers of regular graphs is available. Of course if $G$ is $d$-regular then

$$i_{V,u}(G) \leq i_{E,u}(G) \leq d \cdot i_{V,u}(G)$$

for all $0 < u \leq 1/2$; but these bounds are far from sharp for most values of $u$. Some interesting results are as follows. Tanner [17] proved that if $G \in \mathcal{G}_{n,d}$ and $\lambda(G) \leq \alpha d$, then

$$i_{V,u}(G) \geq 1/(u(1 - \alpha^2) + \alpha^2) - 1.$$  

Friedman [10] showed by a detailed examination of numbers of closed walks that

$$\lambda(G) \leq 2\sqrt{d - 1} + \epsilon$$

a.a.s. in $\mathcal{G}_{n,d}$ for any $\epsilon > 0$. Thus it follows for any $d \geq 3$ and $0 < u \leq 1/2$,

$$i_{V,u}(d) \geq 1/(u(1 - 4(d - 1)/d^2) + 4(d - 1)/d^2) - 1.$$  

So, in particular,

$$i_V(d) \geq 1 - 8/d + O(1/d^2). \quad (2)$$

Finally, one other result of interest concerns the expansion of small sets, see [12, Theorem 4.16.1]. For any $d \geq 3$ and $\delta > 0$, there exists an $\epsilon_\delta > 0$ for which

$$i_{E,\epsilon_\delta}(G) \geq d - 2 - \delta \quad (3)$$
a.a.s. in $\mathcal{G}_{n,d}$. In fact, the same is true of vertex expansion. In the sequel, we refer to the above as the small sets property. In both cases, the expansion parameter $d - 2$ is best possible. In particular, for any $k$,

$$\min_{|U| = k} |\partial_V U|/|U| \leq \min_{|U| = k} |\partial_E U|/|U| \leq d - 2 + 2/k.$$  \hfill (4)

For details see [12, Subsection 5.1.1].

### 3 Model for analysis

To analyse $\mathcal{G}_{n,d}$ we use of the pairing model $\mathcal{P}_{n,d}$, described as follows. Suppose there are $n$ cells, each containing $d$ points, where $dn$ is even. $\mathcal{P}_{n,d}$ is the uniform probability space on the set of all perfect matchings of the $dn$ points. By collapsing each cell of a given $H \in \mathcal{P}_{n,d}$ into a single vertex a $d$-regular multigraph $\pi_H$ on $n$ vertices is obtained. The pairing model is due to Bollobas, who was the first to suggest directly deducing properties of random graphs from the model, though similar models appear in earlier works (see [18] for details).

It is known that $\mathbb{P}(\pi_H$ is simple) is bounded away from 0 as $n$ tends to infinity, and that all $d$-regular simple graphs are selected with equal probability through the process of choosing an $H \in \mathcal{P}_{n,d}$ uniformly at random and then constructing $\pi_H$. Thus, to prove that a property occurs a.a.s in $\mathcal{G}_{n,d}$, it is enough to prove that the pairings in $\mathcal{P}_{n,d}$ a.a.s. have the corresponding property. A survey of properties of random $d$-regular graphs proved using this model is in [18].

We make isoperimetric definitions for pairings to coincide with the same parameters for the corresponding (multi)graphs. For a pairing $H \in \mathcal{P}_{n,d}$ and a subset $U$ of its $n$ cells, let $\partial_V(H)(U)$ denote the set of all cells adjacent to a cell in $U$ but not in $U$. Similarly, let $\partial_E(H)(U)$ denote the set of all edges with exactly one endpoint in a cell of $U$. Note that $|\partial_V(H)(U)| \leq |\partial_E(H)(U)|$. For any $0 < u \leq 1$ the $u$-vertex isoperimetric number for a pairing $H \in \mathcal{P}_{n,d}$ is defined as

$$i_{V,u}(H) = \min_{|U| \leq un} |\partial_V(H)(U)|/|U|,$$

and likewise for any $0 < u \leq 1/2$ the $u$-edge isoperimetric number is defined as

$$i_{E,u}(H) = \min_{|U| \leq un} |\partial_E(H)(U)|/|U|.$$

Furthermore, put

$$i_{V,u}(\mathcal{P}, d) = \sup \{ \ell : i_{V,u}(\mathcal{P}_{n,d}) \geq \ell \text{ a.a.s.} \}$$

and define $i_{E,u}(\mathcal{P}, d)$ analogously.

### 4 Lower bounds for vertex expansion

For a sequence $u = u(n)$ with $0 < u \leq 1$ for all $n$, we define the $u$-vertex expansion number to be

$$I_{V,u}(d) = \sup \left\{ \ell : \min_{U \subset V, |U| = un} \frac{|\partial_V U|}{un} \geq \ell \text{ a.a.s. in } \mathcal{G}_{n,d} \right\}.$$

The motivation to study the vertex expansion number is the following relation to the isoperimetric number.
Lemma 5. Fix $0 < u_0 \leq 1$. Then

$$i_{V,u_0}(d) \geq \inf_{0 \leq u \leq u_0} \inf_{w \to u} I_{V,w}(d),$$

where the second infimum is over sequences $w(n)$ with $0 < w \leq 1$.

Proof. Set $L$ to be right hand side of the inequality, and assume by way of contradiction that, for some $\epsilon > 0$, $i_{V,u_0}(d)$ is not at least $L - \epsilon$ a.a.s. Then for all $n$ in some infinite set $S$ of positive integers, and some $\epsilon' > 0$, we have $\mathbb{P}(i_{V,u_0}(G_{n,d}) < L - \epsilon) > \epsilon'$. Thus, there exists a function $w(n) > 0$ with a limit point in $[0, u_0]$ such that in $G_{n,d}$

$$\mathbb{P}\left( \min_{|U| = u(n)n} \frac{\partial_V(U)}{|U|} < L - \epsilon \right) > \epsilon'$$

for all $n \in S$. Since $w(n)$ is bounded, restricted to $n \in S$ it has a convergent subsequence with some limit $u$ where $0 \leq u \leq u_0$. For any sequence $w'(n) \to u$ with $w'(n) = w(n)$ for all $n \in S$, we have $I_{V,w'}(d) \leq L - \epsilon$, giving the desired contradiction. \(\square\)

Of course we expect $I_{V,u}(d) \geq i_{V,u_0}(d)$ if $u \to u_0 > 0$ since it seems natural to expect that large sets will cause the most problems. Equality will often hold but is not always straightforward to prove, so in some cases we will have to be satisfied with explicit results in the form of a fixed continuous function $f$ such that $I_{V,u} \geq f(c)$ when $u(n) \sim c$. From our argument, we will be able to conclude that for fixed $u$, $I_{V,u} \geq \min \{f(c) : 0 \leq c \leq u\}$.

To analyse $I_{V,u}(d)$ it will be useful to first look at the analogously defined quantity $I_{V,u}(\mathcal{P}, d)$ for pairings.

The main complication in bounding $I_{V,u}(\mathcal{P}, d)$ via the first moment method is that both $\partial_V$ and $\partial_E$ must be considered to compute the expected number of elements of $\mathcal{P}_{n,d}$ with $i_{V,u}$ equal to some specified value. Consequently, bounding $I_{V,u}$ is more involved than $I_{E,u}$, as in the latter case we need only take $\partial_E$ into account.

For a randomly selected element of $\mathcal{P}_{n,d}$, let $X_{u,s,y,d}^{(n)}$ denote the number of subsets of $V$ of size $un$ that have $|\partial_V| = sn$ and $|\partial_E| = yn$. Here $s$, $u$ and $y$ will be functions of $n$. Let $C_{n,s,y}$ denote the coefficient of $x^{yn}$ in the polynomial

$$\left( \sum_{j=1}^{d} \binom{d}{j} x^j \right)^{sn} = ((x + 1)^d - 1)^{sn},$$

so that $C_{n,s,y}$ is the number of ways to choose $yn$ elements of $sdn$ items partitioned into $sn$ groups of cardinality $d$ each, such that at least one item is chosen from each group. Note also that $M(2m) = (2m)!/m!2^m$ is the number of perfect matchings of $2m$ points, so for instance $|\mathcal{P}_{n,d}| = M(dn)$. Then in $\mathcal{P}_{n,d}$,

$$\mathbb{E}(X_{u,s,y,d}^{(n)}) = C_{n,s,y} \frac{n}{un} \frac{n - un}{sn} \frac{dun}{yn} \frac{(yn)!M((du - y)n)M((d - du - y)n)}{M(dn)}.$$

where the binomial coefficients choose a set $U$ of $un$ vertices, their $sn$ neighbours, and the $yn$ points inside them that join to points outside $U$, and the other factors count choices of the pairs with the obvious restrictions. For any $x > 0$,

$$C_{n,s,y} \leq x^{-yn} ((x + 1)^d - 1)^{sn}.$$
We will use this upper bound for various $x > 0$ as our estimate for $C_{n,s,y}$.

By Stirling’s approximation, for any $x > 0$,

$$
\left(\mathbb{E} X_{u,s,y,d}^{(n)}\right)^{1/n} \leq \frac{\left((d^u d - d u - y)^{(d^u d - d u - y)/2}((x + 1)^d - 1)^s \phi(n)\right)}{x^u u^s (1 - u - s)(1 - u - s)(y - y)^{(d^u d - d u - y)/2 d^u d/2}},
$$

where $\phi(n) = n^{O(1/n)}$ contains factors which are of polynomial size before taking the $n$th root. Hence for $x > 0$, we have

$$
\log \mathbb{E} \left(X_{u,s,y,d}^{(n)}\right) \leq n \left(f_d(u, s, y, x) + o(1)\right)
$$

where

$$
f_d(u, s, y, x) = du \log(d u) + (d - d u - y)(\log(d - d u - y))/2 + s \log \left((x + 1)^d - 1\right)
- y \log x - u \log u - s \log s - (1 - u - s) \log(1 - u - s)
- (d u - y)(\log(d u - y))/2 - (d \log d)/2.
$$

One particular value we will use is $x = x_0$, defined as the value at which $\partial f/\partial x = 0$, or equivalently

$$
\frac{sd(x + 1)^d - 1}{(x + 1)^d - 1} = \frac{y}{x}.
$$

This choice of $x$ is important, since if $(y - s)n \to \infty$ and $(s d - y)n \to \infty$ as $n \to \infty$ and $x_0 > 0$ is the unique solution to (7), then (for reasons explained below)

$$
x_0^{-y}((x_0 + 1)^d - 1)^s \sim C_{n,s,y}^{1/n} \quad \text{(as } n \to \infty).\n$$

Since the relevant range of $y$ is $s \leq y \leq ds$ (as for any $U \subset V$, $|\partial_U U| \leq |\partial E U| \leq d \cdot |\partial_V U|$), it can be observed by [8] that using $x = x_0$ to bound $C_{n,s,y}$ in our argument leads to just as good final results as using the precise formula.

We briefly outline two arguments for the asymptotics at [8]. One simple option is to use the limit theorems of Bender [4, Theorem 3 and Theorem 4] (cf. [15] and [7]). Another more transparent way to obtain [9] is as follows: Let $Y_i$ be i.i.d. random variables with support $\{0, 1, \ldots, d - 1\}$ and taking values $j$ w.p. $(d^u d - d u - y)/2^d - 1$. Put $Y(sn) = \sum_{i=1}^{sn} Y_i$, and observe

$$
C_{n,s,y} = \mathbb{P}(Y(sn) = y n).
$$

Log-concave sequences are unimodal. Let $y^* n$ denote an exponent associated with the coefficient in the $sn$th convolution of $p(x)$ attaining (as close as possible to) the centre of mass. By the Berry-Esseen inequality, $Y(sn)$ is asymptotically normal. Hence the asymptotics of $C_{n,s,y}^{y*}$ can be established. Moreover, the asymptotics of an arbitrary coefficient, $C_{n,s,y}$ say, may be obtained as follows. For any $r \in \mathbb{R}$, we have

$$
\left[ \sum_{i=1}^{sn} Y_i \right]^r = \left[ x^m \right] p(x)^n
$$

where $[x^m] g(x)$ denotes the coefficient of $x^m$ in the polynomial $g$. Observe that since $p(x)$ has log-concave coefficients, then so does $p(xr)$ and it then follows by a well known property that the same holds for $p(xr)^n$. Hence, selecting $r$ such that the centre of mass in $p(xr)^n$ is located at the coefficient of $x^m$, the asymptotics of $[x^m] p(x)^n$ can be found by [4, Lemma 1].
Note that \( f \) is continuous on the natural domain in question, with the convention \( 0 \log 0 = 0 \). Define
\[
M_d(u, s, y) = \min_{x \geq 0} f_d(u, s, y, x),
\]
\[
h_d(u, s) = \max_{s \leq y \leq \min\{ds, du\}} M_d(u, s, y)
\]
and
\[
H_d(u) = \min \{ s : h_d(u, s) \geq 0 \}.
\]
A little examination of \( f \) shows that the various min’s and max’s exist and are continuous. Recalling that every relevant \( x \) leads to an upper bound in \( \{0\} \), we may now deduce the following.

**Lemma 6.** \( H_d \) has the following properties.

(a) \( H_d(u) = 0 \) if and only if \( u \in \{0, 1\} \).

(b) Fix \( 0 < u_0 < 1 \). If \( u = u(n) \to u_0 \) as \( n \to \infty \), then
\[
I_{V,u}(d) \geq \frac{H_d(u_0)}{u_0}.
\]
In the case that \( u \to 0^+ \),
\[
I_{V,u}(d) \geq d - 2.
\]

(c) For any \( 0 < u_0 < 1 \), we have
\[
i_{V,u_0}(d) \geq \inf_{0 < u \leq u_0} \frac{H_d(u)}{u}.
\]

**Proof.** For part (a), note that \( H_d(u) = 0 \) if and only if \( u \in \{0, 1\} \). Observe that \( f_d(0, 0, 0, \cdot) = 0 = f_d(1, 0, 0, \cdot) \) (noting that when \( s = 0 \), the only possible value \( y \) in the max function in the definition of \( h_d(u, s) \) is 0, and then \( x \) does not appear in \( f \)). So \( h_d(0, 0) = h_d(1, 0) \geq 0 \) and hence \( H_d(0) = H_d(1) = 0 \). Conversely, suppose \( H_d(u) = 0 \). Then \( h_d(u, 0) \geq 0 \), and so we have \( f_d(u, 0, 0, \cdot) \geq 0 \). Thus, since
\[
\frac{d^2 f_d(u, 0, 0, \cdot)}{du^2} = \frac{d - 2}{2u(1 - u)} > 0
\]
for \( 0 < u < 1 \), \( u \) is either 0 or 1.

For part (b), consider \( 0 < u_0 < 1 \) and \( u(n) \to u_0 \). We have \( H_d(u_0) > 0 \) by (a), so we can let \( s_0 \) be positive with \( s_0 < H_d(u_0) \). Then \( h_d(u_0, s_0) < 0 \). By definition, \( h_d(u, s) \) is monotonic nondecreasing in \( s \). So, by continuity, for \( n \) sufficiently large, \( h_d(u, s) < h_d(u_0, s_0)/2 \) for all \( s \leq s_0 \). Then \( \{0\} \) tells us that in \( \mathcal{P}_{n,d} \) the expected number of sets \( U \) with \( |U| = un \), \( |\partial y U| = sn \leq s_0 n \) and \( |\partial E U| = yn \) is exponentially small for every relevant \( y \) (noting that \( y \geq s \) can be assumed because every boundary vertex has at least one boundary edge). Summing over all \( O(n^2) \) relevant values of \( s \) and \( y \), we deduce by the union bound that a.a.s. \( |\partial y U| \geq s_0 n \) for all \( U \) with \( |U| = un \). Hence, \( I_{V,u}(\mathcal{P}, d) \geq H_d(u_0)/u_0 \).

Thus, the first inequality in (b) follows in view of the relation between \( \mathcal{G}_{n,d} \) and \( \mathcal{P}_{n,d} \) discussed in Section \( \Box \).
In the case that \( u \to 0^+ \), we use the small sets property discussed at (3): for all \( \delta > 0 \) there is some \( N_\delta \in \mathbb{N} \) so that for \( n > N_\delta \), \( u(n) < \epsilon_\delta \) (\( \epsilon_\delta \) as guaranteed by the property), and hence

\[
\min_{|U|=u(n)n} |\partial_V U| \geq \min_{|U| \leq \epsilon_\delta n} |\partial_V U| \geq d - 2 - \delta
\]
a.a.s. in \( G_{n,d} \). The above holds for any \( \delta > 0 \), so we have

\[
I_{V,u}(d) \geq d - 2,
\]
which finishes the proof of (b).

For part (c), note that Lemma 6 implies

\[
i_{V,u_0}(d) \geq \min \left\{ \inf_{w \to 0} I_{V,w}(d), \inf_{0 < u \leq u_0} \frac{H_d(u)}{n} \right\}
\]
and the first of these is at least \( d - 2 \) by (b). On the other hand, by inequality (4), \( i_{V,u}(d) \leq d - 2 \) for any \( u > 0 \), and (c) follows.

Of course, this result is ‘best possible’ for the first moment method, in the sense that, from (3) and the earlier discussion, the expected number of sets of size \( s \) with a boundary which is slightly larger than \( H_d(u_0) \) cannot be exponentially small.

Now we need to discuss the behaviour of \( f_d(u,s,y,x) \), which we often abbreviate to \( f \). Similarly, since \( d \) and \( u \) are fixed for the whole discussion we often refrain from explicitly mentioning them as parameters of other functions.

A ‘direct attack’, solving for the minimum \( x \) and maximum \( y \) in the definition of \( h_d(u,s) \), and then analysing its behaviour as a function of \( s \), leads to calculations that seem too complicated. Instead we take an indirect approach: for each suitable \( y \) we will compute a value of \( s \) such that \( y \) is the maximiser. Even with this, we do not restrict ourselves to using the minimising \( x \), which, considering \( \partial f / \partial x \), turns out to be \( x_0 \). For an upper bound, we are free to choose any \( x \). To simplify the argument we will, for part of it, use a different choice of \( x \), which happens to coincide with \( x_0 \) everywhere that matters.

The relevant partial derivatives are

\[
\frac{\partial f}{\partial x} = \frac{sd(x+1)^{d-1} - y}{x} - 1
\]
and

\[
\frac{\partial f}{\partial y} = \log \hat{x}(y) - \log x
\]
where

\[
\hat{x}(y) = \sqrt{\frac{d-u-y}{d-du-y}}
\]
for all \( y < du \).

For any \( x \) and \( y \), define

\[
S(y, x) = \frac{y((x+1)^d-1)}{xd(x+1)^{d-1}}
\]
(9)
so that $s = S(y, x)$ satisfies (7), and set
\[ \hat{s}(y) = S(y, \hat{x}(y)) \] (10)
and
\[ F(y) = F_{d,u}(y) = f_d(u, \hat{s}(y), y, \hat{x}(y)). \] (11)

**Lemma 7.** Fix $0 < u \leq 1/2$. We have $d\hat{s}(y)/dy > 0$.

**Proof.** This follows easily by checking that $\hat{x}(y)$ is a nonincreasing function of $y$, and that $\partial S(x, y)/\partial x < 0$ (using $1 + dx - (x + 1)d < 0$).

We are now ready to state our main results for this section. From now on, we restrict attention to $u \leq 1/2$, since the result in Lemma 7 does not always extend for much larger $u$.

We say that a real-valued function $g$ with a real domain $D$ is unimodal with mode $\tilde{y}$ if $g(y)$ is strictly increasing for $y < \tilde{y}$ ($y \in D$) and strictly decreasing for $y > \tilde{y}$ ($y \in D$).

**Proposition 8.** Let $d \geq 3$ and $0 < u \leq 1/2$ and let $\hat{s}$ and $F$ be defined as in (10) and (11). Then
(a) $F$ is unimodal with mode $\tilde{y} = du(1 - u)$;
(b) $F$ has a unique zero $\bar{y} \in (0, \tilde{y})$, and we have that $H_d(u) \geq \hat{s}(\bar{y})$.

**Proof.** To analyse $F(y)$, we define
\[ g(s, y) = f_d(u, s, y, \hat{x}(y)), \]
and note that by definition $F(y) = g(\hat{s}(y), y)$. We have
\[ \frac{dF}{dy} = \left. \frac{\partial g}{\partial y} \right|_{s=\hat{s}(y)} + \left. \frac{\partial g}{\partial s} \right|_{s=\hat{s}(y)} \frac{d\hat{s}}{dy}. \]
Now
\[ \frac{\partial g}{\partial y} = \left. \frac{\partial f_d(u, s, y, x)}{\partial y} \right|_{x=\hat{x}(y)} + \left. \frac{\partial f_d(u, s, y, x)}{\partial x} \right|_{x=\hat{x}(y)} \frac{d\hat{x}}{dy}, \]
where the first partial derivative is 0 by the way we defined $\hat{x}$, and the second one, evaluated at $s = \hat{s}(y)$, is 0 by (10) and the comment above it. Hence, the first partial derivative in the above formula for $dF/dy$ is 0. Furthermore, by Lemma 7 $d\hat{s}(y)/dy > 0$. Thus, $dF/dy$ has the same sign as
\[ \left. \frac{\partial g}{\partial s} \right|_{s=\hat{s}(y)} = \left. \frac{\partial f_d(u, s, y, \hat{x}(y))}{\partial s} \right|_{s=\hat{s}(y)} = -\log \hat{s}(y) + \log (1 - u - \hat{s}(y)) + \log ((\hat{x}(y) + 1)d - 1). \]

Put $\tilde{y} = du(1 - u)$. Observe that
\[ \hat{x}(\tilde{y}) = \frac{u}{1 - u} \] (12)
and hence
\[ \hat{s}(\tilde{y}) = S(\tilde{y}, \hat{x}(\tilde{y})) = (1 - u)(1 - (1 - u)^d). \] (13)
Thus, after some simplifications, we find that
\[
\frac{\partial g}{\partial s} \bigg|_{s=\hat{s}(\tilde{y})} = 0.
\]
The partial derivative of \(-\log s + \log (1 - u - s)\) with respect to \(s\) is negative, \(d\hat{s}(y)/dy > 0\), and \(d\hat{x}(y)/dy < 0\) as can be verified directly. So \(\frac{\partial g}{\partial s} \bigg|_{s=\hat{s}(y)}\), and consequently also \(dF(y)/dy\), takes the value 0 at \(y = \tilde{y}\), and it is positive for \(y < \tilde{y}\) and negative for \(y > \tilde{y}\). This gives the unimodality claim in (a), so we may turn to (b).

We first address the existence of \(\tilde{y}\). Performing some straightforward manipulations we see that
\[
\lim_{y \to 0^+} F(y) = \frac{d-2}{2}(u \log u + (1-u) \log(1-u)) < 0.
\]
Then using the simplified expressions at (12) and (13), we observe that
\[
F(\tilde{y}) = -u \log u - (1-u) \log(1-u) > 0.
\]
Hence, there is a point \(\tilde{y} \in (0, \tilde{y})\) which satisfies \(F(\tilde{y}) = 0\). By the unimodality of \(F\), it is unique.

We next show that, essentially, when computing \(H_d\), for the maximum in the definition of \(h_d\), we can restrict ourselves to points \((s, y)\) of the form \((\hat{s}(y), y)\). Fix \(y_0 < du\). We claim that for all \(y < du\),
\[
M_d(u, \hat{s}(y_0), y) \leq F(y_0) = f_d(u, \hat{s}(y_0), y_0, \hat{x}(y_0)).
\]
To see this, note that
\[
M_d(u, \hat{s}(y_0), y) \leq f_d(u, \hat{s}(y_0), y, \hat{x}(y_0))
\]
and from above
\[
\partial f_d(u, \hat{s}(y_0), y, \hat{x}(y_0))/\partial y = \log \hat{x}(y) - \log \hat{x}(y_0).
\]
Since \(d\hat{x}(y)^2/dy = d(2u - 1)/(d - du - y)^2\), \(\hat{x}(y)\) is a nonincreasing function of \(y\). This implies that for fixed \(y_0\), \(f_d(u, \hat{s}(y_0), y, \hat{x}(y_0))\) is maximised at \(y = y_0\), which gives (14). From this, it follows that
\[
h_d(u, \hat{s}(y_0)) = \max_{\hat{s}(y_0) \leq y \leq \hat{s}(y_0)} M_d(u, \hat{s}(y_0), y) \leq F(y_0).
\]
Recalling that \(\hat{s}\) is a continuous increasing function of \(y\) which tends to 0 from above with \(y\), it follows that if \(s < \hat{s}(\tilde{y})\) then \(s = \hat{s}(y)\) for some \(y < \tilde{y}\) and
\[
h_d(u, s) = h_d(u, \hat{s}(y)) \leq F(y) < F(\tilde{y}) = 0,
\]
where the last inequality follows since \(dF/dy > 0\) on \((0, \tilde{y})\). Thus \(H_d(u) \geq \hat{s}(\tilde{y})\), as required for (b).

Since \(\tilde{y}\) as in Proposition 8(b) depends on \(d\) and \(u\), we denote it by \(\bar{y}_{d,u}\). To be clear, let us emphasise that \(F(\bar{y}_{d,u}) = 0\) means
\[
f_d(u, \hat{s}(\bar{y}_{d,u}), \bar{y}_{d,u}, \hat{x}(\bar{y}_{d,u})) = 0.
\]
Next, we define
\[
A_d(u) = \hat{s}(\bar{y}_{d,u})/u.
\]
Combining the proposition with Lemma 4 we obtain the following immediately.
Corollary 9. If $u \to u_0$ where $0 < u_0 \leq 1/2$ is fixed, then

$$I_{V,u}(d) \geq A_d(u_0).$$

By this corollary and Lemma 6(c), for any $u_0 \in (0, 1/2]$, we have

$$i_{V,u_0}(d) \geq \inf_{0 < u \leq u_0} A_d(u). \quad (18)$$

Approximate values of $A_d(u)$ for various $d$ and $u$ are provided in Table 1. These were found by searching for the first zero of $F = F_{d,u}(y)$ and finding strictly positive and negative values of $F$ on either side of it. Recall that finding such a zero of $F$ as a function of $y$ means we have found $\hat{y}_{d,u}$ and hence $A_d(u)$ via (17). The entries in the table are monotonically decreasing in the columns, and this seems likely to hold for all $d$. If this is true, it would follow from (18) that $i_{V,u}(d) \geq A_d(u)$ in general.

| $u$  | 3    | 4    | 5    | 10   | 25   | 50   | 100  |
|------|------|------|------|------|------|------|------|
| 0.01 | 0.55822 | 1.24636 | 1.97397 | 5.71086 | 16.16640 | 30.80253 | 52.21931 |
| 0.05 | 0.43552 | 0.97129 | 1.52478 | 4.12128 | 9.57894 | 14.12199 | 17.14034 |
| 0.10 | 0.36513 | 0.80589 | 1.24807 | 3.13558 | 6.15315 | 7.78467 | 8.52607 |
| 0.15 | 0.31790 | 0.69369 | 1.06039 | 2.50085 | 4.35286 | 5.11942 | 5.43785 |
| 0.20 | 0.28136 | 0.60687 | 0.91620 | 2.04298 | 3.25720 | 3.68551 | 3.86267 |
| 0.25 | 0.25110 | 0.53536 | 0.79862 | 1.69322 | 2.52784 | 2.79584 | 2.90837 |
| 0.30 | 0.22503 | 0.47421 | 0.69923 | 1.41621 | 2.01058 | 2.19121 | 2.26836 |
| 0.35 | 0.20194 | 0.42060 | 0.61319 | 1.19112 | 1.62589 | 1.75381 | 1.80926 |
| 0.40 | 0.18108 | 0.37272 | 0.53737 | 1.00461 | 1.32904 | 1.42271 | 1.46383 |
| 0.45 | 0.16196 | 0.32936 | 0.46968 | 0.84761 | 1.09323 | 1.16336 | 1.19447 |
| 0.50 | 0.14420 | 0.28966 | 0.40859 | 0.71371 | 0.90142 | 0.95467 | 0.97850 |

Table 1: Approximate values for $A_d(u)$. By Corollary 9 these are approximate lower bounds for the $u$-vertex expansion number $I_{V,u}(d)$.

5 Lower bounds for the vertex isoperimetric number

For $i_{V,u_0}$, as opposed to $I_{V,u_0}$ (for a sequence $u(n) \to u_0$), we must consider vertex sets of cardinality less than or equal to $u_0 n$. As noted above at (18), the minimum of $A_d(u)$ over all $0 < u \leq u_0$ gives a lower bound for $i_{V,u_0}$. However, this turns out to be not so amenable to theoretical analysis.

In this section, for the case $u_0 = 1/2$, we use a less direct argument to show that $i_V(d) \geq A_d(1/2)$. This bound is ‘best possible’ using the first moment method, since it was best possible for $I_{V,u}(d)$ over all $u \to 1/2$. (See the comment after Lemma 6.)

The situation is manageable at $u_0 = 1/2$ largely due to the fact that in this case

$$\hat{x}(y) = \sqrt{\frac{d/2 - y}{d - d/2 - y}} = 1$$
for all relevant \( y \), and we get the simplified expression
\[
\hat{s}(y) = S(y, 1) = 2y(1 - 1/2^d)/d,
\]
(19)
which is useful for computing \( F(y) \) for \( u_0 = 1/2 \), and hence \( A_d(1/2) \). Corollary [9] provides us with the lower bound \( I_{V,u}(d) \geq A_d(1/2) \) when \( u \to 1/2 \). However, to get a lower bound on \( i_V(d) \) with Lemma [8] we need to also consider \( H_d(u)/u \) for \( 0 < u < 1/2 \). For such \( u \), we use inequality [5] to show that the case \( u_0 = 1/2 \) is critical, in the sense that \( A_d(1/2) \leq I_{V,u}(d) \) for any sequence \( u(n) \to u \in (0,1/2] \).

Let \( \hat{A} \) be the set of all \((u, s)\) with \( s/u < A_d(1/2) \) and \( 0 < u \leq 1/2 \). We will show that \( h_d(u, s) < 0 \) on \( \hat{A} \). It then follows that \( H_d(u) > s \) for all \((u, s) \in \hat{A} \), and then, by Lemma [6] we may conclude \( i_V(d) \geq A_d(1/2) \).

To this end, define
\[
\hat{h}_d(u, s) = \max_{s \leq y \leq \min\{ds, du\}} f_d(u, s, y, 1). \tag{20}
\]
Since \( f_d(u, s, y, 1) \geq M_d(u, s, y) \), we have \( \hat{h}_d(u, s) \geq h_d(u, s) \), and it suffices to show \( \hat{h}_d(u, s) < 0 \) on \( \hat{A} \). Our job will be made easier after we show that \( \partial f_d(u, s, y, 1)/\partial y < 0 \).

First of all, since we do not have a closed form for \( A_d(1/2) \), we obtain an initial estimate of the location of \( \hat{A} \) with the following lemma. Define
\[
C(d) = (d - 2)/(d - 1).
\]
In the case \( u = 1/2 \), let us write \( F_d = F_{d,1/2} \) as it appears in statement of Theorem [1]

**Lemma 10.** For any \( d \geq 3 \), \( 0 < A_d(1/2) < C(d) \), and for all \( s < A_d(1/2)/2 \), \( \hat{h}_d(1/2, s) < 0 \).

**Proof.** Fix \( d \). Put \( y_d = d(d-2)2^{d-2}/(d-1)(2^d-1) \). (We follow the convention \( a/bc = a/(bc) \), that is, multiplication by juxtaposition takes precedence over ‘/’. ) We have
\[
F_d(y_d) = -((d^2 - 3d + 2) \log 2 + \log(1/(d-1)))/2(d-1) \\
+ (d-2) \log ((d-1)(2^d-1)/(d-2))/2(d-1) \\
> -(d^2 - 3d + 2) \log 2 - (d-2) \log(2^d-1))/2(d-1) \\
= (d-2)(\log(2^d-1) - (d-1) \log 2)/2(d-1) \\
= (d-2) \log((2^d-1)/2^{d-1})/2(d-1) \\
> 0.
\]
Hence
\[
A_d(1/2) = 2\hat{s}(\bar{y}_{d,1/2})
\]
where \( 0 < \bar{y}_{d,1/2} < y_d \). Since \( \hat{s} \) is nonnegative and monotonically increasing by [19],
\[
0 < A_d(1/2) < 2\hat{s}(y_d) = 4y_d(1 - 1/2^d)/d = (d - 2)/(d - 1) = C(d).
\]
This establishes the first inequalities in the lemma.

Since \( \hat{s}(y) \equiv 1 \) at \( u = 1/2 \) and so \( \hat{h}_d(1/2, s) = h_d(1/2, s) \), we have \( \hat{h}_d(1/2, s) < 0 \) for all \( s < A_d(1/2)/2 \) by [16] where, in this instance, \( \bar{y} = \bar{y}_{d,1/2} \).

We are ready to prove the main result for this section.
Fix $d \geq 3$. It will be convenient to parameterize $s$ in terms of $u$ and a new variable $r$. Set $s = ru$ and note that if $r = 0$ then necessarily $y = 0$ (since $\partial_V = 0$ if and only if $\partial_E = 0$). Recall that, by the discussion just before Lemma 10, it is enough to show that $\tilde{h}_d(u,s) < 0$ on $\mathcal{A}$. Observe that for relevant $s$ and $u$,

$$\frac{df_d(u,s,y,1)}{dy} = \log \sqrt{\frac{du - y}{d - du - y}} < 0.$$ 

Hence $\tilde{h}_d(u,ru) = f_d(u,ru,ru,1)$. Moreover, the final inequality in Lemma 10 takes care of the case $u = 1/2$. Thus, it suffices to show that

$$g_d(u,r) := f_d(u,ru,ru,1) < 0$$

on $\mathcal{A}$, where

$$\mathcal{A} = \{(u,r) : 0 < u < 1/2, 0 \leq r < A_d(1/2)\}.$$ 

For all $0 \leq r < C(d)$, it is easy to check that

$$\lim_{u \to 0^+} g_d(u,r) = 0.$$ 

Again, by the final inequality in Lemma 10, $g_d(1/2,r) < 0$ for all $0 < r < A_d(1/2)$. Hence it suffices to show that for each $0 \leq r < C(d)$, $g_d(u,r)$ is either strictly convex or strictly increasing in $u$ at every $0 < u < 1/2$. We partition the region $\mathcal{A}$ as follows, with a view to showing that $g_d(u,r)$ is either increasing or convex in each region. Define

$$\mathcal{A}_0 = \{(u,r) : r = 0, 0 < u < 1/2\}$$

$$\mathcal{A}_1 = \{(u,r) : 0 < r \leq \min\{c(d), A_d(1/2)\}, 0 < u < 1/2\}$$

$$\mathcal{A}_2 = \{(u,r) : c(d) < r < A_d(1/2), 0 < u < U(r,d)\}$$

$$\mathcal{A}_3 = \{(u,r) : c(d) < r < A_d(1/2), U(r,d) \leq u < 1/2\}$$

with

$$U(r,d) = \min \left\{ \frac{1}{2}, \frac{d(d - 2 - r)}{(r + 1)d^2 + (d - 2)r^2 - 2r - d(r + 2)} \right\}$$

and $c(d) = (d - 2)/(d + 2)$.

We analyse $g_d$ in the domain corresponding to each portion of $\mathcal{A}$, starting with $\mathcal{A}_0$. Since

$$\frac{d^2 g_d(u,0)}{du^2} = (d - 2)/2u(1 - u) > 0,$$

$g_d(u,r)$ is strictly convex in $u$ for $0 < u < 1/2$, as required for $\mathcal{A}_0$.

Assume hereafter $r > 0$. Note that

$$\frac{d^2 g_d(u,r)}{du^2} = \frac{\zeta(r,u,d)}{\eta(r,u,d)},$$

with

$$\zeta(r,u,d) = (1 - u - ru)d^2 + (-2 + ru - r^2u + 2u - r)d + 2ru(r + 1)$$

$$\eta(r,u,d) = 2u(1 - u - ru)(d - du - ru).$$
As $u \leq 1/2$ and $r \leq A_d(1/2) < C(d) < 1$, we have $\eta(r, u, d) > 0$. Further,
\[
\frac{d\zeta(r, u, d)}{du} = -(1 + r)(d - 2)(d + r) < 0.
\]
Hence to show $d^2 g_d(u, r)/du^2 > 0$, it is enough to determine that
\[
\zeta(r, 1/2, d) = (1 - d/2)r^2 + (1 - d/2 - d^2/2)r + d(d/2 - 1) > 0.
\]
The coefficient of $r$ and $r^2$ above are both negative since $d \geq 3$. Therefore $\zeta(r, 1/2, d) > 0$ in $A_1$ since
\[
\zeta(c(d), 1/2, d) = 4d(d - 2)/(d + 2)^2 > 0
\]
for $d \geq 3$. So $g_d(u, r)$ is strictly convex in $u$ for fixed $r$ such that $(u, r) \in A_1$, as required.

As seen above, $\zeta(r, u, d)$ is decreasing in $u$. Observe that whenever $U(r, d) \leq 1/2$, $\zeta(r, U(r, d), d) = 0$. Thus for any $r \in (c(d), C(d))$, $g_d(u, r)$ is strictly convex in $u$ for $0 < u < U(r, d)$. That is, $g_d(u, r)$ is strictly convex in $u$ for fixed $r$ such that $(u, r) \in A_2$. On the other hand, if $r \in (c(d), C(d))$ and $U(r, d) \leq u < 1/2$, then the situation is as follows. By the definition of $U(r, d)$, and the fact that $\zeta$ is linear in $u$, we deduce that $dg_d(u, r)/du$ is decreasing in $u$. Hence, to show that $g_d(u, r)$ is increasing in $u$, it suffices to show that
\[
\frac{dg_d(u, r)}{du}|_{u=1/2} = r \log((1 - r)/r) + r \log(2^d - 1) + d \log(d/(d - r)) + \log(1 - r)
\]
\[
= \log \left( \frac{(1 - r)^{1+r} (2^d - 1)^r d^d}{r^r (d - r)^d} \right)
\]
\[
> 0
\]
for $r \in [c(d), C(d)]$. To see that the above holds for $d \geq 8$, observe that for this range of $d$ we have
\[
(1 - r)^{1+r} < C(d)(1 - C(d))^{-14C(d)/d(c(d))} < (2^d - 1) \left( \frac{d}{d - c(d)} \right)^{d/C(d)} < (2^d - 1) \left( \frac{d}{d - r} \right)^{d/r}.
\]
For $d \leq 7$, put $\theta_d(r) = \frac{dg_d(u, r)}{du}|_{u=1/2}$. We have
\[
\frac{d^2 \theta_d(r)}{dr^2} = -\frac{2}{1 - r} - \frac{1 + r}{(1 - r)^2} - \frac{1}{r} + \frac{d}{(d - r)^2} < 0
\]
for $d \geq 3$ and $r < 1$, since each of the first three terms is less than $-1$ and the last is less than $1$. By direct calculation, all the endpoints $\theta_d(c(d))$ and $\theta_d(C(d))$ are positive for $3 \leq d \leq 7$. So by the concavity of $\theta_d(r)$ in $r$, we have that $\theta_d(r) > 0$ for all $d \leq 7$ and relevant $r$. Therefore $g_d(u, r)$ is strictly increasing in $u$ for fixed $r$ such that $(u, r) \in A_3$. Putting the above together, we conclude that $g_d < 0$ on $A_2 \cup A_3$.

Altogether we have shown $g_d < 0$ on $A = \bigcup_{i=0}^3 A_i$. As noted earlier, this implies $i_V(d) \geq A_d(1/2)$.

To supplement the values in Table 1, additional approximate values for $A_d(1/2)$ for various $d$ were generated by the same method, as shown in Table 2.
\[
\begin{array}{cccccccc}
\hline
d & \approx A_d & d & \approx A_d & d & \approx A_d & d & \approx A_d \\
3 & 0.14420 & 11 & 0.74355 & 19 & 0.90972 & 35 & 0.93269 \\
4 & 0.28966 & 12 & 0.76827 & 20 & 0.91338 & 40 & 0.94201 \\
5 & 0.40859 & 13 & 0.78897 & 21 & 0.91677 & 50 & 0.95467 \\
6 & 0.50190 & 14 & 0.80652 & 22 & 0.91991 & 60 & 0.96284 \\
7 & 0.57466 & 15 & 0.82155 & 23 & 0.92283 & 70 & 0.96854 \\
8 & 0.63178 & 16 & 0.83455 & 24 & 0.92555 & 80 & 0.97274 \\
9 & 0.67716 & 17 & 0.84587 & 25 & 0.92809 & 90 & 0.97596 \\
10 & 0.71371 & 18 & 0.85582 & 26 & 0.93046 & 100 & 0.97850 \\
\hline
\end{array}
\]

Table 2: Approximate values for \( A_d = A_d(1/2) \). By Theorem 1, these are approximate lower bounds for the vertex isoperimetric number \( i_V(d) \).

6 Asymptotic bounds for \( i_V(d) \)

The asymptotics of \( A_d(1/2) \), as \( d \to \infty \), can be computed as follows. For the case \( u = 1/2 \), we have, as stated above at (19), \( \hat{x}(y) = 1 \) and \( \hat{s}(y) = 2y(1 - 1/2^d)/d \).

Hence, as discussed after the proof of Proposition 8,

\[
A_d(1/2) = 4\bar{y}(1 - 1/2^d)/d
\]

where \( \bar{y} \) uniquely satisfies

\[
f_{d}(1/2, 2\bar{y}(1 - 1/2^d)/d, \bar{y}, 1) = 0.
\]

However, observe that

\[
f_{d}(1/2, s, y, 1) = s \log(2^d - 1) + (\log 2)/2 - s \log s - (1/2 - s) \log(1/2 - s) - (d \log 2)/2.
\]

Hence, when \( u = 1/2 \) and \( x = 1 \), \( y \) does not appear in \( f_d \). Thus, we investigate the asymptotics of \( s_d = 2\bar{y}(1 - 1/2^d)/d \) satisfying

\[
f_{d}(1/2, s_d, \cdot, 1) = 0.
\]

Moreover, note that by Lemma 10, \( A_d(1/2) > 0 \) for all \( d \geq 3 \). Hence, for all \( d \geq 3 \), \( s_d > 0 \). In fact, we can show that \( s_d \to 1/2 \). Writing \( \log(2^d - 1) = d \log 2 + \log(1 - 1/2^d) \) and manipulating, we obtain

\[
f_{d}(1/2, s, \cdot, 1) = s(- \log s - \log 2 + \log(1 - 2s) + d \log 2 + \log(1 - 1/2^d))
- (d \log 2)/2 + \log 2 - \log(1 - 2s)/2
= (s - 1/2) d \log 2 - s \log s + (s - 1/2) \log(1 - 2s) + (1 - s) \log 2 + O(s/2^d)
= (s - 1/2) d \log 2 + O(1).
\]

Setting this equal to 0, we conclude \( s_d = 1/2 + O(1/d) \) as \( d \to \infty \). With this in mind, we make the change of variables \( t = 1/2 - s_d \) in the above expression, obtaining

\[
0 = -td \log 2 - (1/2) \log(1/2) + O(1/d) + O(\log d/d) + (1/2) \log 2
\]
and hence
\[ t = 1/d + O(\log d/d^2). \]
Therefore
\[ A_d(1/2) = 1 - 2/d + O(\log d/d^2). \] (22)
By Theorem 1 and (22), we deduce Corollary 2. Observe that Corollary 2 provides a stronger bound on the asymptotics of \( i_V(d) \) as \( d \to \infty \) than (2).

7 A note on the edge isoperimetric number

In this section we show how the above arguments can be modified to obtain a.a.s. lower bounds for the edge isoperimetric number of random regular graphs. As discussed in Section 4, Bollobás in [5] computed lower bounds on \( i_E(d) \) for all \( d \geq 3 \) with the first moment method. Also, the asymptotics of \( i_E(d) \), as \( d \to \infty \) are investigated. Moreover, it is claimed that the arguments could be modified for \( 0 < u < 1/2 \). However no explicit lower bounds on \( i_{E,u}(d) \) are given, nor for the asymptotics of \( i_{E,u}(d) \) as \( d \to \infty \), for the cases \( 0 < u < 1/2 \). In this section we provide lower bounds on \( i_{E,u}(d) \) which result from direct application of the first moment method, for all \( d \geq 3 \) and \( 0 < u \leq 1/2 \). The bounds are analysed asymptotically as \( d \to \infty \) for fixed \( u \).

For a randomly selected element of \( \mathcal{P}_{n,d} \), let \( X_{u,y,d}^{(n)} \) denote the number of subsets of \( V \) of size \( un \) that have \( \partial V = yn \). Then
\[
\mathbb{E} \left( X_{u,y,d}^{(n)} \right) = \binom{n}{un} \binom{dn - dun}{yn} (yn)! M(dun - yn) M(dn - dun - yn) / M(dn),
\]
where \( M(2m) = (2m)! / m! 2^m \) counts the number of perfect matchings of \( 2m \) points, the binomial coefficients choose a set \( U \) consisting of \( un \) vertices and \( yn \) boundary edges, and the other factors count choices of the pairs with the obvious restrictions. Therefore, via Stirling’s approximation,
\[
\left( \mathbb{E} X_{u,y,d}^{(n)} \right)^{1/n} = \frac{(du)^{du} (d - du)^{d - du} \phi(n)}{u^n (1 - u) (du - y)^{(du - y)/2 (d - du - y)/2} d^{d/2}},
\]
where \( \phi(n) = n^{O(1/n)} \) contains the factors of polynomial size before taking the \( n \)th root. Hence
\[
\log \mathbb{E} \left( X_{u,y,d}^{(n)} \right) \leq n(\hat{f}_d(u,y) + o(1)) \] (23)
where
\[
\hat{f}_d(u,y) = du \log(du) + (d - du) \log(d - du) - u \log u - (1 - u) \log(1 - u) - y \log y - (du - y)(\log(du - y)) / 2 - (d - du - y)(\log(d - du - y)) / 2 - (d \log d) / 2.
\]
Up to this point, these facts are essentially contained in [5]. To get lower bounds on \( i_{E,u}(d) \) we find where \( \hat{f}_d < 0 \) and use (23). The argument is comparable to that of the preceding sections, however the situation is much simpler since in the current case we analyse a function with only two parameters. (Recall that in Section 4 there was a function \( f_d \), defined at (5), with parameters corresponding to the edge and vertex boundary sizes and also used to estimate a polynomial coefficient.)
Before stating the main lemmas and theorem, we collect some facts about \( \hat{f}_d \). Note that if \( G \in \mathcal{G}_{n,d} \) and \( |U| = un \) with \( u \leq 1/2 \) then \( |\partial E U| \leq dun \).

Fix \( d \geq 3 \) and \( 0 < u \leq 1/2 \). Note that \( \hat{f}_d(u, y) \) is strictly concave in \( y \) for \( 0 \leq y < du \). Indeed, so long as \( 0 \leq y < du \), we have

\[
\frac{d^2 \hat{f}_d(u, y)}{dy^2} = \frac{d(y - 2du(1-u))}{2y(du - y)(d - du - y)} < 0.
\]

We define

\[
\hat{A}_d(u) = \frac{1}{u} \min \{ y : \hat{f}_d(u, y) \geq 0 \}.
\]

To see that \( 0 < \hat{A}_d(u) \leq d(1 - u) \) observe that

\[
\lim_{y \to 0^+} \hat{f}_d(u, y) = \frac{d - 2}{2} (u \log u + (1 - u) \log(1 - u)) < 0,
\]

and that, after some simplifications,

\[
\hat{f}_d(u, du(1 - u)) = -u \log u - (1 - u) \log(1 - u) > 0.
\]

As we did for the case of vertex expansion, we will define a pointwise measure of edge expansion. For a sequence \( u = u(n) \) with \( 0 < u \leq 1/2 \) for all \( n \), we define the \( u \)-edge expansion number to be

\[
I_{E, u}(d) = \sup \left\{ \ell : \min_{U \subset V, |U| = un} \frac{|\partial E U|}{un} \geq \ell \text{ a.a.s. in } \mathcal{G}_{n,d} \right\}.
\]

We state here analogues of Lemmas 5 and 6 for the case of edge expansion. We do not provide the proofs since they are very similar.

**Lemma 11.** Fix \( 0 < u_0 \leq 1/2 \). Then

\[
i_{E, u_0}(d) \geq \inf_{0 \leq u \leq u_0} \inf_{w \to u} I_{E, w}(d),
\]

where the second infimum is over sequences \( w(n) \) with \( 0 < w \leq 1/2 \).

**Proof.** The proof is analogous to that of Lemma 5. \( \square \)

**Lemma 12.** \( \hat{A}_d \) has the following properties.

(a) Fix \( 0 < u_0 \leq 1/2 \). If \( u = u(n) \to u_0 \) as \( n \to \infty \), then

\[
I_{E, u}(d) \geq \hat{A}_d(u).
\]

In the case that \( u \to 0^+ \),

\[
I_{E, u}(d) \geq d - 2.
\]

(b) For any \( 0 < u_0 \leq 1/2 \), we have

\[
i_{E, u_0}(d) \geq \inf_{0 < u \leq u_0} \hat{A}_d(u).
\]
Proof. The proof is analogous to that of Lemma 6. (Note that the small sets property, as discussed at (3) and (4), also holds for edge expansion.)

As we now prove, \( \hat{A}_d(u) \) is, in fact, a lower bound for \( i_{E,u}(d) \) for all \( d \geq 3 \) and \( 0 < u \leq 1/2 \).

Proof of Theorem 3. Fix \( d \geq 3 \) and \( 0 < u \leq 1/2 \). As in the proof of Proposition 8, we will consider \( 0 < w \leq u \), and parameterize the variable \( y \) as \( rw \), where \( r \) is a new variable. Put

\[
B = \{(r, w) : 0 \leq r < \hat{A}_d(u), 0 < w \leq u \}.
\]

Once we show \( g_d(r, w) = \hat{f}_d(w, rw) < 0 \) over \( B \) the theorem will follow by Lemma 12 and inequality (23). Partition \( B \) as follows:

\[
B_1 = B \cap \{(r, w) : w \leq W_{r,d} \text{ or } r \leq R_{w,d} \}
\]

\[
B_2 = B \setminus B_1
\]

where

\[
W_{r,d} = \frac{d(d - 2 - r)}{(d - 2)(d + r)}
\]

\[
R_{w,d} = \frac{d(d - 2)(1 - w)}{d + (d - 2)w}.
\]

We have

\[
\frac{d^2 g_d(r, w)}{dw^2} = \frac{\eta(d, r, w)}{\zeta(d, r, w)}
\]

where

\[
\eta(d, r, w) = (1 - w)d^2 - (2(1 - w) + r(1 + w))d + 2rw
\]

\[
\zeta(d, r, w) = 2w(d - dw - rw)(1 - w).
\]

As \( w \leq u \leq 1/2 \) and \( r < du \), \( \zeta(d, r, w) > 0 \). Also, we have

\[
\eta(d, r, W_{r,d}) = \eta(d, R_{w,d}, w) = 0
\]

and if \( w < W_{r,d} \) or \( r < R_{w,d} \), then \( \eta(d, r, w) > 0 \). Hence, over \( B_1 \), \( g_d(r, w) \) is convex in \( w \) for any fixed \( r \).

Regarding \( B_2 \), we know that

\[
\frac{d^2 g_d(r, w)}{dw^2} < 0
\]

and so \( g_d(r, w)/dw \) is decreasing in \( w \). Thus, to show that \( g_d(r, w) \) for a fixed \( r \) is increasing in \( w \) over \( B_2 \), it suffices to show that

\[
\theta_u(r) = \left. \frac{dg_d(r, w)}{dw} \right|_{w=u} > 0
\]

for all \( R_{u,d} \leq r \leq d(1 - u) \), since \( R_{w,d} \) is decreasing in \( w \) and \( \hat{A}_{d,u} < d(1 - u) \). First of all, observe that

\[
\frac{d^2 \theta_u(r)}{dr^2} = \frac{d^2 (1 - 2u^2)r - 2d^3(1 - 2u + u^2)}{2r(d - r)(d - du - ru)^2} < 0
\]
for $0 < r < 2d(1 - 2u + u^2)/(1 - 2u^2)$. So as

$$\frac{2d(1 - 2u + u^2)}{(1 - 2u^2)} - d(1 - u) = \frac{d(1 - u)(2u^2 - 2u + 1)}{1 - 2u^2} > 0,$$

$\theta_u$ is strictly concave in $r$ over the interval in question. Thus we check that $\theta_u$ is positive at the endpoints. The right endpoint is positive since, after some simple manipulations, we see that

$$\frac{d\theta_u(d(1 - u))}{du} = -\frac{2}{2u(1 - u)} < 0$$

and

$$\theta_{1/2}(d(1 - 1/2)) = 0.$$

For the left endpoint, observe that

$$\theta_u(R_{u,d}) = (d - 1) \log \left( \frac{d - u}{1 - u} \right) + \frac{R_{u,d}}{2} \log(\varphi_u(d)) + \frac{d}{2} \log(\psi_u(d))$$

where

$$\varphi_u(d) = \frac{(d - du - R_{u,d}u)(du - R_{u,d}u)}{(R_{u,d}u)^2} = \frac{2d((d - 2)u + 1)}{u(1 - u)(d - 2)^2}$$

$$\psi_u(d) = \frac{d - du - R_{u,d}u}{du - R_{u,d}u} = \frac{d(1 - u)}{2u((d - 2)u + 1)}$$

Hence, after some simplifications, we find that

$$\frac{d^2\theta_u(R_{u,d})}{dd^2} = \frac{\delta_u(d) \log(\varphi_u(d)) + \gamma_u(d)}{d(d - 2)((d - 2)u + 1)((d - 2)u + d)^3}$$

where

$$\delta_u(d) = -4du(1 - u)(d - 2)((d - 2)u + 1)$$

$$\gamma_u(d) = ((d - 2)u + d)((4u^2 - u - 1)d^2 - 2(6u^2 - 4u + 1)d - 4u(1 - 2u))$$

Note that the coefficients of the second term in $\gamma_u(d)$ are non-positive for all $0 < u \leq 1/2$. Hence, observe that $\delta_u(d) < 0$ and $\gamma_u(d) < 0$ for all $d \geq 3$ and $0 < u \leq 1/2$. Note that since

$$\frac{d\varphi_u(d)}{dd} = \frac{-2((1 + 2u)d + 2(1 - 2u))}{u(1 - u)(d - 2)^3} < 0$$

and

$$\lim_{d \to \infty} \varphi_u(d) = \frac{2}{1 - u} \geq 2$$

we have that $\varphi_u(d) \geq 2$ for all $d \geq 3$ and $0 < u \leq 1/2$. We conclude that $d\theta_u(R_{u,d})/dd$ is decreasing in $d$ for any fixed $0 < u \leq 1/2$. Observe that

$$\lim_{d \to \infty} \frac{d\theta_u(R_{u,d})}{dd} = -\frac{\log(1 - u) + u \log 2}{1 + u} > 0$$
for all $0 < u \leq 1/2$. Therefore $\theta_u(R_{u,d})$ is increasing in $d$, and thus, since

$$\theta_u(R_{u,3}) \geq \theta_{1/2}(R_{1/2,3}) \approx 0.768$$

for all $0 < u \leq 1/2$, we conclude that $\theta_u(R_{u,d}) \geq 0$ for all relevant $d$ and $u$. Altogether, $g_d(r,w)$ is increasing in $w$ over $B_2$ for any fixed $r$.

Finally, for any $0 \leq r \leq d(1-u)$, it is easily seen that

$$\lim_{w \to 0^+} g_d(r,w) = 0.$$ 

Altogether, $i_{E,u}(d) \geq \hat{A}_d(u)$.

### Approximate values for $\hat{A}_d(u)$

Approximate values for $\hat{A}_d(u)$ are listed in Table 3.

| $u$   | 3    | 4    | 5    | 10   | 25   | 50   | 100  |
|-------|------|------|------|------|------|------|------|
| 0.01  | 0.57080 | 1.29152 | 2.07102 | 6.31585 | 20.00259 | 43.58306 | 91.53259 |
| 0.05  | 0.46150 | 1.06879 | 1.73912 | 5.49362 | 17.96765 | 39.83142 | 84.74426 |
| 0.10  | 0.39850 | 0.93300 | 1.52904 | 4.91775 | 16.36950 | 36.65008 | 78.56994 |
| 0.15  | 0.35544 | 0.83739 | 1.37806 | 4.48034 | 15.07700 | 33.96870 | 73.17830 |
| 0.20  | 0.32140 | 0.76038 | 1.25487 | 4.11019 | 13.93791 | 31.54266 | 68.19080 |
| 0.25  | 0.29262 | 0.69435 | 1.14821 | 3.78107 | 12.89392 | 29.72612 | 63.45361 |
| 0.30  | 0.26728 | 0.63557 | 1.05254 | 3.47967 | 11.91538 | 27.12097 | 58.89526 |
| 0.35  | 0.24435 | 0.58192 | 0.96469 | 3.19830 | 10.98467 | 25.04690 | 54.46794 |
| 0.40  | 0.22318 | 0.53205 | 0.88263 | 2.93177 | 10.08979 | 23.03451 | 50.13946 |
| 0.45  | 0.20332 | 0.48501 | 0.80492 | 2.67658 | 9.22247  | 21.06947 | 45.88731 |
| 0.50  | 0.18447 | 0.44011 | 0.73051 | 2.43002 | 8.37615  | 19.14025 | 41.69360 |

Table 3: Approximate values for $\hat{A}_d(u)$. By Theorem 3, these are approximate lower bounds for the $u$-edge isoperimetric number $i_{E,u}(d)$.

With Theorem 3 at hand, we can derive lower bounds on the asymptotics of $i_{E,u}(d)$ as $d \to \infty$.

**Proof of Corollary 4.** Put

$$\psi(u) = 2(1-u)\sqrt{\log(u^{-u}(1-u)^{-u-1})}.$$ 

For any $c > 0$ we have

$$\tilde{f}_d(u, u(d(1-u) - c\sqrt{d})) - \log u^{-u}(1-u)^{-u-1} = \mu_d(u,c) - \nu_d(u,c)$$

where

$$\mu_d(u,c) = \frac{cu\sqrt{d}}{2} \cdot \log \left( \frac{(du(1-u) - cu\sqrt{d})^2}{(du^2 + cu\sqrt{d})(d(1-u)^2 + cu\sqrt{d})} \right)$$

$$\nu_d(u,c) = \frac{d}{2} \cdot \log \left( \frac{1 - \frac{c}{(1-u)\sqrt{d}}}{(1-u)\sqrt{d}} \right)^{2u(1-u)} \left( 1 + \frac{c}{u\sqrt{d}} \right)^{u^2} \left( 1 + \frac{cu}{(1-u)^2\sqrt{d}} \right)^{(1-u)^2}. $$


Observe that

\[
\lim_{d \to \infty} \mu_d(u, c) = -\frac{c^2}{2(1-u)^2}
\]

\[
\lim_{d \to \infty} \nu_d(u, c) = -\frac{c^2}{4(1-u)^2}.
\]

Hence

\[
\lim_{d \to \infty} \hat{f}_d(u, u(d(1-u) - c\sqrt{d})) = -\frac{1}{4(1-u)^2}(c - \psi(u))(c + \psi(u)),
\]

and thus, for any \(\epsilon > 0\) and sufficiently large \(d\),

\[
\hat{A}_d(u) \geq d(1-u) - (\psi(u) + \epsilon)\sqrt{d}.
\]

Applying Theorem 3, the result is obtained.

Note that in the case \(u = 1/2\) the lower bound of Corollary 4 agrees with the lower bound at (1).

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