The Liouville theorem as a problem of common eigenfunctions

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla, 72570 Puebla, Pue., México
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Abstract

It is shown that, by appropriately defining the eigenfunctions of a function defined on the extended phase space, the Liouville theorem on solutions of the Hamilton–Jacobi equation can be formulated as the problem of finding common eigenfunctions of \( n \) constants of motion in involution, where \( n \) is the number of degrees of freedom of the system.

1 Introduction

In the framework of the Hamiltonian formulation of classical mechanics, the Liouville theorem asserts that, for a mechanical system with \( n \) degrees of freedom, if we have \( n \) constants of motion in involution, \( F_1, F_2, \ldots, F_n \) (that is, \( \{ F_i, F_j \} = 0 \) for \( i, j = 1, 2, \ldots, n \), where \( \{ , \} \) is the Poisson bracket), then a complete solution of the Hamilton–Jacobi (HJ) equation can be found by quadrature \(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\).

More precisely, if \( F_1(q_i, p_i, t), \ldots, F_n(q_i, p_i, t) \) are \( n \) constants of motion in involution, that is,

\[
\frac{\partial F_i}{\partial t} + \{ F_i, H \} = 0, \quad i = 1, 2, \ldots, n
\]

and

\[
\{ F_i, F_j \} = 0, \quad i, j = 1, 2, \ldots, n,
\]

where \( \{ , \} \) denotes the Poisson bracket (with the convention \( q_i, p_j = \delta_{ij} \)), then, assuming that

\[
\det \left( \frac{\partial F_i}{\partial p_j} \right) \neq 0,
\]

so that, locally at least, we can express the \( p_i \) as functions of \( q_j, F_j, \) and \( t \), the differential form

\[
p_i(q_i, F_j, t) dq_i - H(q_i, p_i(q_i, F_j, t), t) dt
\]

is the differential of some function, \( S(q_i, t) \), which is a complete solution of the HJ equation (here and in what follows, there is summation over repeated indices).

The aim of this paper is to show that the Liouville theorem can be formulated in another form, closer to the standard formalism of quantum mechanics. Specifically, we shall show that if \( S(q_i, t) \) is a common eigenfunction of the functions \( F_1, F_2, \ldots, F_n \) (a concept to be defined below) then, by adding to \( S \) an appropriate function of \( t \) only, one obtains a complete solution of the HJ equation. In Section 2 we present the definition of the eigenfunctions of a function \( f(q_i, p_i, t) \) and we show that two functions, \( f(q_i, p_i, t) \) and \( g(q_i, p_i, t) \), have common eigenfunctions if and only if their Poisson bracket vanishes. Then, we prove that, if conditions \(^1\)\(^2\)\(^3\) hold, a common eigenfunction of \( F_1, F_2, \ldots, F_n \) is, up to an additive function of \( t \) only, a complete solution of the HJ equation. In Section 3 we give some illustrative examples, emphasizing the fact that we can make use of constants of motion that depend explicitly on the time.

The statement of the Liouville theorem presented here allows us to see that the Liouville theorem is analogous to one of the methods employed to solve the Schrödinger equation, where we look for the common eigenfunctions of a complete set of mutually commuting operators that also commute with the Hamiltonian (e.g., for a spherically symmetric Hamiltonian we consider the common eigenfunctions of \( L^2 \), \( L_z \) and \( H \)).
2 Eigenfunctions of a function and complete solutions of the Hamilton–Jacobi equation

We start by giving the definition of the eigenfunctions of a real-valued function \( f(q_i, p_i, t) \): We shall say that \( S(q_i, t) \) is an eigenfunction of \( f(q_i, p_i, t) \), with eigenvalue \( \lambda \), if \( S \) is a solution of the first-order partial differential equation

\[
f(q_i, \frac{\partial S}{\partial q_i}, t) = \lambda. \tag{5}\]

(It may be noticed that if \( f \) is a time-independent Hamiltonian, then (5) is the corresponding time-independent HJ equation.) We note that if \( S(q_i, t) \) is an eigenfunction of \( f(q_i, p_i, t) \) with eigenvalue \( \lambda \), then so is \( S(q_i, t) + \phi(t) \), for any function \( \phi(t) \) of \( t \) only, and that the solutions of (5) will depend parametrically on \( \lambda \). Of course, in order for (5) to be a differential equation, \( f \) must depend on one of the \( p_i \), at least.

For instance, according to this definition, the eigenfunctions of the function

\[
F(q, p, t) = m\omega q \sin \omega t + p \cos \omega t, \tag{6}
\]

where \( m \) and \( \omega \) are constants, are the solutions of the differential equation

\[
m\omega q \sin \omega t + \frac{\partial S}{\partial q} \cos \omega t = \lambda,
\]

which can be readily integrated giving

\[
S = \lambda q \sec \omega t - \frac{m\omega}{2} q^2 \tan \omega t + \phi(t), \tag{7}
\]

where \( \phi(t) \) is an arbitrary function of \( t \) only. Note that \( \lambda \) may be a function of \( t \).

If \( S(q_i, t) \) is a common eigenfunction of \( f(q_i, p_i, t) \) and \( g(q_i, p_i, t) \), with eigenvalues \( \lambda \) and \( \mu \), respectively, that is, \( S \) satisfies (5) and

\[
g(q_i, \frac{\partial S}{\partial q_i}, t) = \mu,
\]

then, differentiating with respect to \( q_i \), making use of the chain rule, we obtain

\[
\frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} = 0 \quad \text{and} \quad \frac{\partial g}{\partial q_i} + \frac{\partial g}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} = 0,
\]

hence,

\[
\{ f, g \} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} = -\frac{\partial f}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} \frac{\partial q}{\partial p_i} + \frac{\partial g}{\partial p_j} \frac{\partial^2 S}{\partial q_j \partial q_i} \frac{\partial f}{\partial p_i} = 0.
\]

Thus, if \( f(q_i, p_i, t) \) and \( g(q_i, p_i, t) \) possess common eigenfunctions, then \( \{ f, g \} = 0 \).

In order to see that the converse is also true, we now assume that \( \{ f, g \} = 0 \). If \( f \) and \( g \) are functionally independent, then there exists, locally at least, a set of canonical coordinates, \( Q_i, P_i \), such that, \( P_1 = f \) and \( P_2 = g \). Then, the eigenvalue equations for \( f \) and \( g \) are \( \partial S/\partial Q_1 = \lambda \) and \( \partial S/\partial Q_2 = \mu \), which have the simultaneous solutions \( S = \lambda Q_1 + \mu Q_2 + \phi(Q_3, \ldots, Q_n, t) \), where \( \phi \) is an arbitrary function of \( n - 1 \) variables, thus showing that \( f \) and \( g \) have common eigenfunctions.

(Note that this does not mean that every eigenfunction of \( f \) is an eigenfunction of \( g \) (cf. [7], Sec. 2.9).) The expression for \( S \) in terms of the original coordinates, \( (q_i, t) \), is not given by the simple substitution of the \( Q_i \) as functions of \( (q_i, p_i, t) \) [8]; what is relevant here is the existence of common eigenfunctions for \( f \) and \( g \).

In the case where \( f \) and \( g \) are functionally dependent, the eigenvalue equations for \( f \) and \( g \) are equivalent to each other and, trivially, possess common solutions.
2.1 Alternative formulation of the Liouville theorem

We now assume that \( F_1, \ldots, F_n \) are \( n \) functions satisfying (1)–(3), and we consider a common eigenfunction \( S(q_i, t) \) of \( F_1, \ldots, F_n \), with eigenvalues \( \lambda_1, \ldots, \lambda_n \), respectively, then, assuming that the eigenvalues are constant, differentiating with respect to \( t \) both sides of the equation

\[
F_i(q_j, \frac{\partial S}{\partial q_j}, t) = \lambda_i, \tag{8}
\]

making use of the chain rule, the Hamilton equations and (1), we have

\[
0 = \frac{\partial F_i}{\partial q_j} \dot{q}_j + \frac{\partial F_i}{\partial p_j} \dot{p}_j + \frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial q_k} \frac{\partial^2 S}{\partial q_k \partial q_j} \dot{q}_k = \frac{\partial F_i}{\partial p_j} \frac{\partial H}{\partial p_j} + \frac{\partial F_i}{\partial q_j} \frac{\partial H}{\partial q_j}, \quad i = 1, \ldots, n.
\]

By virtue of (3), the last equations are equivalent to

\[
0 = \frac{\partial^2 S}{\partial t \partial q_j} + \frac{\partial^2 S}{\partial q_k \partial q_j} \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial q_j}, \quad j = 1, \ldots, n,
\]

which implies that the expression inside the brackets is a function of \( t \) only,

\[
\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = \chi(t).
\]

Thus,

\[
\tilde{S} = S - \int_0^t \chi(u) \, du
\]

is a solution of the HJ equation. We can verify that this solution is complete by differentiating (8) with respect to \( \lambda_j \), which gives

\[
\frac{\partial F_i}{\partial p_k} \frac{\partial^2 S}{\partial \lambda_j \partial q_k} = \delta_{ij}.
\]

Taking into account (3), this last equation shows that \( \det(\partial^2 S/\partial \lambda_j \partial q_k) \neq 0 \).

3 Examples

In this section we give some examples of the method presented above.

3.1 One-dimensional harmonic oscillator

The function

\[
F(q, p, t) = m \omega q \sin \omega t + p \cos \omega t
\]

already considered above [see (6)], is a constant of motion if the Hamiltonian is given by

\[
H = \frac{p^2}{2m} + \frac{m \omega^2}{2} q^2, \tag{9}
\]

where \( \omega \) is a constant. According to the results of the preceding section, if \( \lambda \) is a constant,

\[
S = \lambda q \sec \omega t - \frac{m \omega}{2} q^2 \tan \omega t + \phi(t) \tag{10}
\]
must be a solution of the HJ equation for the Hamiltonian \( H \), if the function \( \phi \) is appropriately chosen. A direct computation yields

\[
\frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + \frac{m\omega^2}{2} q^2 + \frac{\partial S}{\partial t} = \frac{\lambda^2}{2m} \sec^2 \omega t + \phi'(t),
\]

and, therefore, choosing \( \phi(t) = -\frac{\lambda^2}{2m} \tan \frac{\omega t}{2m} \), we obtain the complete solution of the HJ equation

\[
S = \lambda q \sec \omega t - \left( \frac{\lambda^2}{2m} + \frac{m\omega^2}{2} q^2 \right) \tan \frac{\omega t}{\omega}.
\]

Note that, in this case, \( H \) is also a constant of motion and, as pointed out above, the equation that determines the eigenfunctions of \( H \) is just the time-independent HJ equation. However, the constant of motion \( \text{eq}(5) \) leads to simpler expressions.

### 3.2 Particle in a time-dependent force field

As a second example we consider the time-dependent Hamiltonian

\[
H = \frac{p^2}{2m} - ktq,
\]

where \( k \) is a constant. One can readily verify that

\[
F = p - \frac{kt^2}{2}
\]

is a constant of motion and that the eigenfunctions of \( F \), i.e., the solutions of

\[
\frac{\partial S}{\partial q} - \frac{kt^2}{2} = \lambda,
\]

are given by

\[
S = \lambda q + \frac{kt^2}{2} q + \phi(t),
\]

where \( \phi(t) \) is an arbitrary function of \( t \) only. Then

\[
\frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 - \frac{kt^2}{2} + \frac{\partial S}{\partial t} = \frac{\lambda^2}{2m} + \frac{\lambda kt^2}{2m} + \frac{k^2 t^4}{8m} + \phi'(t),
\]

hence, choosing

\[
\phi(t) = -\frac{\lambda^2 t}{2m} - \frac{\lambda kt^3}{6m} - \frac{k^2 t^5}{40m},
\]

(11) is a complete solution of the HJ equation (which is not separable).

### 3.3 Particle in two dimensions

As a final example we consider the Hamiltonian

\[
H = \frac{1}{2m} \left[ \left( p_x + \frac{eB}{2c} y \right)^2 + \left( p_y - \frac{eB}{2c} x \right)^2 \right],
\]

which corresponds to a charged particle of mass \( m \) and electric charge \( e \) in a uniform magnetic field \( B \). The functions

\[
F_1 = \frac{1}{2} (1 + \cos \omega t) p_x - \frac{1}{2} p_y \sin \omega t + \frac{m\omega}{4} x \sin \omega t - \frac{m\omega}{4} (1 - \cos \omega t) y,
\]

\[
F_2 = \frac{1}{2} (1 + \cos \omega t) p_y + \frac{1}{2} p_x \sin \omega t + \frac{m\omega}{4} y \sin \omega t + \frac{m\omega}{4} (1 - \cos \omega t) x,
\]

where \( \omega \equiv eB/mc \), are constants of motion in involution, which correspond to the values of the canonical momenta \( p_x \) and \( p_y \) respectively, at \( t = 0 \).
From (13) and (14) one finds that the common eigenfunctions of $F_1$ and $F_2$, with eigenvalues $\lambda_1$ and $\lambda_2$, respectively, are

$$S = \lambda_1 x + \lambda_2 y + \tan \frac{1}{2} \omega t \left[ \lambda_2 x - \lambda_1 y - \frac{m\omega}{4} (x^2 + y^2) \right] + \phi(t),$$

where $\phi(t)$ is an arbitrary function of $t$ only. Substituting this expression into the HJ equation one finds that $S$ is a solution of this equation if and only if

$$\frac{\lambda_1^2 + \lambda_2^2}{2m} \sec^2 \frac{1}{2} \omega t + \phi'(t) = 0,$$

hence,

$$S = \lambda_1 x + \lambda_2 y - \left[ \left( \lambda_1 + \frac{m\omega}{2} y \right)^2 + \left( \lambda_2 - \frac{m\omega}{2} x \right)^2 \right] \frac{\tan \frac{1}{2} \omega t}{m\omega}$$

is a complete solution of the HJ equation.

4 Concluding remarks

As pointed out above, the formulation of the Liouville theorem given here makes use of terms analogous to those employed in the standard formalism of quantum mechanics, thus providing another example of the parallelism between both theories. Another advantage of the version of the Liouville theorem given above is that its proof is shorter than those usually presented in the textbooks.

References

[1] Whittaker, E.T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 4th ed. Cambridge University Press, Cambridge (1993).

[2] Vilasi, G.: Hamiltonian Dynamics. World Scientific, Singapore (2001).

[3] Babelon, O., Bernard, D., Talon, M.: Introduction to Classical Integrable Systems. Cambridge University Press, Cambridge (2003).

[4] Fasano, A., Marmi, S.: Analytical Mechanics. Oxford University Press, Oxford (2006).

[5] DiBenedetto, E.: Classical Mechanics. Birkhäuser, New York (2011).

[6] Torres del Castillo, G.F.: Applications and extensions of the Liouville theorem on constants of motion, Rev. Mex. Fis. 57, 245-249 (2011).

[7] Sneddon, I.N.: Elements of Partial Differential Equations. Dover, New York (2006).

[8] Torres del Castillo, G.F., Cruz Domínguez, H.H., de Yta Hernández, A., Herrera Flores, J.E., Sierra Martínez, A.: Mapping of solutions of the Hamilton–Jacobi equation by an arbitrary canonical transformation, Rev. Mex. Fis. 60, 301-304 (2014).