MATRIX NONLINEAR SCHRÖDINGER EQUATIONS
AND MOMENT MAPS INTO LOOP ALGEBRAS†

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Abstract

It is shown how Darboux coordinates on a reduced symplectic vector space may be
used to parametrize the phase space on which the finite gap solutions of matrix nonlinear
Schrödinger equations are realized as isospectral Hamiltonian flows. The parametrization
follows from a moment map embedding of the symplectic vector space, reduced by suitable
group actions, into the dual ˜g+∗ of the algebra ˜g+ of positive frequency loops in a Lie
algebra g. The resulting phase space is identified with a Poisson subspace of ˜g+∗ consisting
of elements that are rational in the loop parameter. Reduced coordinates associated to the
various Hermitian symmetric Lie algebras (g, k) corresponding to the classical Lie algebras
are obtained.

1. Introduction

In a series of recent papers [AHP, AHH1, AHH2] a systematic method was
developed for parametrizing quasi-periodic solutions to integrable systems of PDE’s
in terms of Darboux coordinates on a finite dimensional phase space. The general
approach consists of reducing symplectic vector spaces under suitable continuous or
discrete Hamiltonian group actions, and embedding the reduced spaces as Poisson sub-
manifolds in the dual ˜g+∗ of the positive frequency part of a loop algebra ˜g. The image
space consists of loops in g extending as rational functions of the loop parameter, and
the PDE’s in question arise as the compatibility conditions for pairs of commutative
isospectral Hamiltonian flows induced by spectral invariants. A number of examples

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have been studied within this framework, including the cubically nonlinear Schrödinger (NLS) equation and the coupled, two-component nonlinear Schrödinger (CNLS) equation.

In [FK], Fordy and Kulish gave a list of generalized nonlinear Schrödinger equations related to the Hermitian symmetric spaces corresponding to classical Lie algebras, together with their associated matrix Lax pair commutative flows. The general form for such equations is

\[
\begin{align*}
\sqrt{-1} q_t &= q_{xx} + 2qpq \\
-\sqrt{-1} p_t &= p_{xx} + 2pqp,
\end{align*}
\]

(1.1)

where \(q\) and \(p^T\) are complex \(a \times b\) matrices. Specific cases are obtained by imposing further invariance conditions under involutive automorphisms.

In Hamiltonian terms, the corresponding Lax equations may be viewed as representing flows in the dual of the positive part \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^+\) of the loop algebra \(\tilde{\mathfrak{sl}}(r, \mathbb{C})\), or subalgebras obtained as fixed point sets under involutive automorphisms. Using the Lie Poisson bracket structure on \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^+\), the Adler-Kostant-Symes (AKS) theorem gives the Lax pair representations:

\[
\begin{align*}
\frac{d}{dx} \mathcal{L}(\lambda) &= [(\lambda \mathcal{L}(\lambda))_+, \mathcal{L}(\lambda)] \\
\frac{d}{dt} \mathcal{L}(\lambda) &= [(\lambda^2 \mathcal{L}(\lambda))_+, \mathcal{L}(\lambda)]
\end{align*}
\]

(1.2a,b)

for Hamilton’s equations corresponding to suitably chosen elements of the Poisson commuting ring \(\mathcal{F} \equiv I(\tilde{\mathfrak{sl}}(r, \mathbb{C})^+)\big|_{\tilde{\mathfrak{sl}}(r, \mathbb{C})^+}\) of \(Ad^\ast\)-invariant functions on \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^\ast\), restricted to the subspace \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^{++}\). Here \(\lambda\) is the loop parameter and \(\mathcal{L}(\lambda)\) is chosen as holomorphic outside a disc \(D\) in the complex \(\lambda\)-plane centered at 0. The + subscript in (1.2a,b) means projection to the positive part \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^+\) of the loop algebra (i.e. loops that extend holomorphically to the interior of the circle \(S^1 = \partial D\)). As usual, there is an identification understood between \(\tilde{\mathfrak{sl}}(r, \mathbb{C})\) and a dense subspace of \(\tilde{\mathfrak{sl}}(r, \mathbb{C})^\ast\), determined by the \(Ad^\ast\)-invariant metric

\[
<X, Y> = \frac{1}{2\pi i} \oint_{S^1} \text{tr}(X(\lambda)Y(\lambda)) \frac{d\lambda}{\lambda}, \ X, Y \in \tilde{\mathfrak{sl}}(r, \mathbb{C}).
\]

(1.3)

Putting appropriate invariant restrictions on the leading terms of the Laurent expansion of \(\mathcal{L}(\lambda)\), the equations (1.1) become the compatibility conditions for the \(x\)- and \(t\)-flows determined by (1.2a,b).

For a subalgebra \(\mathfrak{g} \subset \mathfrak{sl}(r, \mathbb{C})\) with loop algebra \(\tilde{\mathfrak{g}}\) we denote by \(\tilde{\mathfrak{g}}^0 \subset \tilde{\mathfrak{g}}\) the subalgebra of elements extending holomorphically outside \(S^1\), identified via \(<, >\) with
\[ \tilde{g}^\pm \text{. (No notational distinction will be made between elements of } \tilde{g} \text{ (resp. } \tilde{g}^0) \text{ and the corresponding element of } \tilde{g}^* \text{ (resp. } \tilde{g}^+). } \]

The specific form for \( L \) giving rise to equation (1.1) as compatibility conditions of (1.2a,b) is:

\[
L(\lambda) = L_0 + \lambda^{-1}L_1 + \lambda^{-2}L_2 + \cdots + \lambda^{-n+1}L_{n-1} \tag{1.4}
\]

with

\[
L_0 = \frac{\sqrt{-1}}{a+b} \begin{pmatrix} bI_a & 0 \\ 0 & -aI_b \end{pmatrix} \tag{1.5a}
\]

\[
L_1 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \tag{1.5b}
\]

\[
L_2 = \sqrt{-1} \begin{pmatrix} qp & -qx \\ px & -pq \end{pmatrix}. \tag{1.5c}
\]

A general framework for studying the “finite-gap” quasi-periodic solutions of such Lax pair AKS flows was developed in [AHP, AHH1, AHH2], using moment map embeddings of finite dimensional symplectic vector spaces, reduced by certain Hamiltonian group actions, into the duals of loop algebras. The image space is a finite dimensional Poisson submanifold of \( \tilde{sl}(r, \mathbb{C})^* \) consisting of a union of coadjoint orbits whose elements are rational in the loop parameter. The original space, which we refer to as the “generalized Moser space” (cf. [AHP, M]), may be viewed as consisting of pairs \((F, G)\) of maximal rank \((N \times r)\) matrices \((N > r)\) parametrizing rank-\(r\) perturbations of a fixed diagonal \(N \times N\) matrix \(A\), with eigenvalues \(\{\alpha_i\}_{i=1,...,n}\) of multiplicities \(\{k_i\}_{i=1,...,n}\) and \(k_i \leq r, \sum_{i=1}^{n} k_i = N\). The general form for such a moment map is:

\[
\tilde{J} : M^{N \times r} \times M^{N \times r} \longrightarrow \tilde{sl}(r, \mathbb{C})^{++} \tag{1.6}
\]

\[
(F, G) \longmapsto \mathcal{N}, \tag{1.6a}
\]

where

\[
\mathcal{N}(\lambda) = \lambda G^T (A - \lambda I) F, \tag{1.6b}
\]

\(F\) and \(G\) are \(N \times r\) matrices chosen so that \(\text{tr}(\mathcal{N}) = 0\), and the space of pairs \((F, G)\) is given the symplectic structure

\[
\omega = \text{tr}(dF^T \wedge dG). \tag{1.7}
\]

We denote by \(\mathcal{M}^k\), where \(k = (k_1, k_2, \ldots k_n)\), the space of pairs \((F, G)\) of such \(N \times r\) matrices, subject to the generic requirement that the \(k_i \times r\) dimensional blocks \(F_i\) and \(G_i\) corresponding to the eigenspaces of \(A\) with eigenvalue \(\alpha_i\) have maximal rank \(k_i\), and endowed with the symplectic structure (1.7). Assuming \(A\) to be diagonal, we obtain \(\mathcal{N}\) in the form:

\[
\mathcal{N}(\lambda) = \sum_{i=1}^{n} \frac{\lambda N_i}{\alpha_i - \lambda}, \tag{1.8}
\]
where
\[ N_i = G_i^T F_i. \]

Taking the AKS Hamiltonians \( \Phi_x, \Phi_t \in \mathcal{F} \) for the \( x \) and \( t \) flows as
\[
\Phi_x(X) = \frac{1}{2} \text{tr} \left( \frac{a(\lambda)}{\lambda^{n-1}} X(\lambda)^2 \right)_0, \quad (1.9a)
\]
\[
\Phi_t(X) = \frac{1}{2} \text{tr} \left( \frac{a(\lambda)}{\lambda^{n-2}} X(\lambda)^2 \right)_0, \quad (1.9b)
\]
where the subscript 0 means taking the \( \lambda^0 \) term in the Laurent expansion centered at \( \lambda = 0 \) and \( a(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i) \) is the minimal polynomial of \( A \), gives Hamilton’s equations in the Lax pair form
\[
\frac{d}{dx} N(\lambda) = \left[ \left( \frac{a(\lambda)}{\lambda^{n-1}} X(\lambda) \right)_+, N(\lambda) \right] \quad \text{(1.10a)}
\]
\[
\frac{d}{dt} N(\lambda) = \left[ \left( \frac{a(\lambda)}{\lambda^{n-2}} X(\lambda) \right)_+, N(\lambda) \right]. \quad \text{(1.10b)}
\]

Defining
\[
\mathcal{L} := -\frac{a(\lambda)}{\lambda^n} N(\lambda) \quad \text{(1.11)}
\]
gives the equations (1.2a,b), with \( \mathcal{L} \) a polynomial in \( \lambda^{-1} \) of degree \( n - 1 \). From equation (1.11), we obtain the leading terms \( \mathcal{L}_0, \mathcal{L}_1 \) and \( \mathcal{L}_2 \) in terms of the matrices \( (F_i, G_i) \) as
\[
\mathcal{L}_0 = \sum_{i=1}^{n} G_i^T F_i \quad \text{(1.12a)}
\]
\[
\mathcal{L}_1 = \sum_{i=1}^{n} \alpha_i (G_i^T F_i - \mathcal{L}_0) \quad \text{(1.12b)}
\]
\[
\mathcal{L}_2 = -\sum_{j \geq k} \alpha_j \alpha_k \mathcal{L}_0 + \sum_{i=1}^{n} \alpha_i (\alpha_i G_i^T F_i - \mathcal{L}_1). \quad \text{(1.12c)}
\]

The form (1.5a) for the leading term \( \mathcal{L}_0 \) determines an invariant manifold under such AKS flows, since \( \mathcal{L}_0 \) is just the moment map generating the conjugation action under \( \text{Sl}(r, C) \), and hence is conserved by the AKS flows. The other two conditions (1.5b,c) determining the form of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are also invariant under the flows.

In the following section we shall first briefly recall how to obtain the special structure (1.5a-c) for \( \mathcal{L} \) and then show how the moment map (1.6) gives coordinates parametrizing certain solutions of the nonlinear Schrödinger equations associated to the Hermitian symmetric Lie algebra \( (\mathfrak{sl}(a + b, C), \mathfrak{sl}(a, C) \oplus \mathfrak{sl}(b, C) \oplus C^*) \) and its real
forms \((\mathfrak{su}(a + r, s), \mathfrak{s}(u(a) \oplus u(r, s)))\). A Darboux coordinate atlas is obtained for the reduced space \(\mathcal{M}_{\mathfrak{sl}(a+b,C)}^k\), obtained by requiring \(\mathcal{N}(\lambda)\) to be traceless, as well as for the real forms obtained by choosing fixed points under antilinear involutions.

Section 3 deals with hermitian symmetric pairs of the type \((\mathfrak{so}(2l), u(r)), (\mathfrak{so}(2l + 2), \mathfrak{so}(2l) \oplus \mathfrak{so}(2))\), \((\mathfrak{so}(2l + 3), \mathfrak{so}(2l + 1) \oplus \mathfrak{so}(2))\) and \((\mathfrak{sp}(l), u(l))\) as well as their corresponding noncompact and complex forms. Global Darboux coordinates are obtained from the restriction that the moment map take its values in the total space of one of the classical Lie algebras. The detailed computations for these are omitted, the results being given in tabular form.

We emphasize that the purpose of this work is to give an intrinsic canonical parametrization of the invariant finite dimensional sector of the phase space underlying certain solutions of the matrix NLS equations; namely, those of “finite gap” type or more generally, quasi-periodic flows lying on the stationary manifolds of higher AKS invariants. The actual integration of these finite dimensional flows in terms of theta functions, obtained from linear flows of divisors on the associated invariant spectral curves, will be the subject of a subsequent work [W].

2. Darboux Coordinates for the \((\mathfrak{sl}(a + b, C), \mathfrak{sl}(a, C) \oplus \mathfrak{sl}(b, C) \oplus C)\) and \((\mathfrak{su}(a + l, s), \mathfrak{s}(u(a) \oplus u(l, s)))\) Matrix NLS Equations

In this section, we compute Darboux coordinates for matrix nonlinear Schrödinger equations of the type (1.1) and the real forms

\[ \sqrt{-1} u_t = u_{xx} + 2(uDu^T)u, \]

obtained by setting \(u = q = -Dq^T\) where \(u \in C^{a \times b}\) and \(D\) is a Hermitian matrix which, up to a base change can be taken of the form \(\begin{pmatrix} I_l & 0 \\ 0 & -I_s \end{pmatrix}\), \(l + s = b\).

First we derive the form for \(\mathcal{L}_0, \mathcal{L}_1\) and \(\mathcal{L}_2\) given in (1.5a-c). The co-adjoint action of the subgroup of constant loops \(Sl(r, C)\) on \(\tilde{\mathfrak{sl}}(a + b, C)^{+*} \sim \mathfrak{sl}(a + b, C)\), \(a + b = r\), given by

\[ \text{Ad}_g(X)(\lambda) = g(X(\lambda))g^{-1}, \]

with \(X(\lambda) = \sum_{i=0}^{0} X_i\lambda^i\), is generated by the moment map \(J(X) = X_0\). Since our Hamiltonians (1.9a,b) (and all other Hamiltonians in \(\mathcal{F}\)) are invariant under this action, \(\mathcal{L}_0\) is an invariant of the flows.

By a Hermitian symmetric Lie algebra (cf. [KN]), we understand a Lie algebra \(\mathfrak{g}\) with subalgebra \(\mathfrak{k}\) and an involutive automorphism \(\sigma : \mathfrak{g} \to \mathfrak{g}\) such that \(\mathfrak{k}\) is the \(+1\) eigenspace and, denoting by \(\mathfrak{m}\) the \(-1\) eigenspace, an element \(B \in \mathfrak{k}\) exists satisfying

\[ i) \ \mathfrak{k} = \ker(ad B) \]

\[ ii) \ (ad B|_m)^2 = -\text{id}_m. \]
The element $B$ has the same properties (2.3a,b) in the complexification of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{m}$. The $\mathfrak{k}$ and $\mathfrak{m}$ components of any element $X \in \mathfrak{g}$ will henceforth be denoted $X_k$ and $X_m$ respectively.

In particular, for $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$ define the involutive automorphism

$$\sigma_0 : \mathfrak{sl}(a + b, \mathbb{C}) \rightarrow \mathfrak{sl}(a + b, \mathbb{C})$$

$$X \rightarrow I_{a,b} X I_{a,b}$$

(2.4)

with $I_{a,b} = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$. This also determines an involutive automorphism $\tilde{\sigma}_0$ on the associated loop algebra $\tilde{\mathfrak{g}}$ through

$$(\tilde{\sigma}_0(X))(\lambda) = \sigma_0(X(\lambda)), \; X \in \tilde{\mathfrak{g}}.$$  

(2.5)

The decomposition of $\mathfrak{sl}(a + b, \mathbb{C})$ into the $+1$ eigenspace $\mathfrak{h} = \mathfrak{sl}(a, \mathbb{C}) \oplus \mathfrak{sl}(b, \mathbb{C}) \oplus \mathbb{C}$ consisting of the diagonal $a \times a$ and $b \times b$ blocks and the complementary $-1$ eigenspace $\mathfrak{m}$ can be interpreted as the complexification of the Hermitian symmetric Lie algebra $(\mathfrak{su}(a + b), \mathfrak{s}(\mathbb{u}(a) \oplus \mathbb{u}(b)))$, with properties (2.3a,b) in both cases satisfied by the same element $B$. Using the induced involution (2.5) we also have a splitting of the loop algebra $\tilde{\mathfrak{sl}}(a + b, \mathbb{C})$, with $+1$ eigenspace $\tilde{\mathfrak{h}}$ and $-1$ eigenspace $\tilde{\mathfrak{m}}$ and relations (2.3a,b) also hold when $\mathfrak{h}$, $\mathfrak{m}$ are replaced by $\tilde{\mathfrak{h}}$, $\tilde{\mathfrak{m}}$.

Returning to the general case, since $\mathcal{L}_0$ is an invariant of the flows, we may choose it equal to $B \in \mathfrak{h}$, as mentioned above ($\mathfrak{h}$ is considered here as representing the constant loops in $\tilde{\mathfrak{h}}$). For any Hamiltonian $\phi \in \mathcal{F}$ with flow parameter $\tau$ let $d\phi(\mathcal{L}) = A\lambda^{-1} + O(\lambda^{-2})$, with $\mathcal{L}$ given by (1.4). The AKS theorem implies

$$\frac{d}{d\tau} \mathcal{L}_1 = -[A, \mathcal{L}_0].$$

(2.6a)

It follows from (2.3a) that $\mathcal{L}_{1\mathfrak{h}}$ is an invariant of the flows, which we set equal to zero as an initial condition. In particular, from Hamilton’s equation (1.2a) we deduce

$$\frac{d}{dx} \mathcal{L}_1 = [\mathcal{L}_0, \mathcal{L}_2] \in \mathfrak{m}$$

(2.6b)

The equations of motion then also imply

$$\mathcal{L}_{2\mathfrak{m}} = \frac{d}{dx} \mathcal{L}_1$$

(2.6c)

due to (2.3b) and (2.6b). Finally, an easy computation shows that the condition

$$\mathcal{L}_{2\mathfrak{h}} = \frac{1}{2}[[\mathcal{L}_0, \mathcal{L}_1], \mathcal{L}_1]$$

(2.6d)
is invariant under the flows of all the Hamiltonians in the ring of functions $\mathcal{F}$, and hence may be consistently imposed. For the particular case $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$, we have $\mathcal{L}_0 = B$ given by equation (1.5a) and equations (2.6a-d) are equivalent to the special form (1.5b,c) for $\mathcal{L}_1, \mathcal{L}_2$.

The moment map $\tilde{\mathcal{J}}$ in (1.6) generally takes its values in $\tilde{\mathfrak{gl}}(r, \mathbb{C})^{++}$. In order that it be defined so as to take values in $\tilde{\mathfrak{sl}}(\alpha + b, \mathbb{C})^{++}$, we have to impose restrictions on its domain in $\mathcal{M}^k$. The corresponding submanifold of $\mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})} \subset \mathcal{M}^k$ is defined by

$$\mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})} = \{(F, G) \in \mathcal{M}^k \mid \text{tr}(G_i^T F_i) = 0, \ i = 1, \ldots, n\}.$$ 

This is not a symplectic manifold; to obtain one, we have to quotient by the $(\mathbb{C}^*)^n$ group-action on $\mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}$ defined by $(z(F, G))_i = (z_i F_i, z_i^{-1} G_i)$, $i = 1, \ldots, n$, where $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$, whose orbits are the null foliation of the restriction of the symplectic form $\omega$ to $\mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}$. This amounts to a zero moment map reduction with respect to the action of the central subgroup $\mathcal{D}$ of the loop group $\tilde{\mathfrak{gl}}(r, \mathbb{C})$ corresponding to multiples of the identity matrix. Let $\pi : \mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})} \to \mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}/(\mathbb{C}^*)^n$ be the associated projection. On $\mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}/(\mathbb{C}^*)^n$ we have symplectic charts $(\varphi^{\tilde{\mathcal{J}}}, \pi(U^{\tilde{\mathcal{J}}}))$ defined as follows:

$$\varphi^{\tilde{\mathcal{J}}} : \pi(U^{\tilde{\mathcal{J}}}) \longrightarrow \varphi^{\tilde{\mathcal{J}}}(\pi(U^{\tilde{\mathcal{J}}})) \subset \mathbb{C}^{N_{r-n}} \times \mathbb{C}^{N_{r-n}}$$

$$\pi(F, G) \longmapsto (G_i^{l_i c_i} F_i, (G_i^{l_i c_i})^{-1} \tilde{G}_i)_{i=1, \ldots, n}$$

(2.7)

where $\tilde{F}_i (\tilde{G}_i)$ means just $F_i (G_i)$ with the $F_i^{l_i c_i}$ ($G_i^{l_i c_i}$) component suppressed, considered as an element of $\mathbb{C}^{r k_i-1}$ and

$$U^{\tilde{\mathcal{J}}} = \{ (F, G) \in \mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}/(\mathbb{C}^*)^n \mid G_i^{l_i c_i} \neq 0, F_i^{l_i c_i} \neq 0 \}$$

$$\tilde{l} = (l_1, \ldots, l_n) \quad \tilde{c} = (c_1, \ldots, c_n)$$

$$l_i \in \{ K_i + 1, \ldots, K_i + k_i \} \quad K_i := \sum_{j=1}^{i-1} k_j \quad c_i \in \{ 1, \ldots, r \}$$

(2.8)

But $\pi(U^{\tilde{\mathcal{J}}})$ is diffeomorphic to the following symplectic submanifold of $\mathcal{M}^k$:

$$\pi(U^{\tilde{\mathcal{J}}}) \cong \{ (F, G) \in \mathcal{M}^k \mid G_i^{l_i c_i} = 1, F_i^{l_i c_i} = - \sum_{(\delta, \beta) \neq (l_i, c_i)} G_i^{\delta_i \beta_i} F_i^{\delta_i \beta_i} \neq 0, i = 1, \ldots, n \}$$

(2.9)

which defines a section of the quotient projection $\pi : \mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})} \to \mathcal{M}^k_{\tilde{\mathfrak{sl}}(r, \mathbb{C})}/(\mathbb{C}^*)^n$. The symplectic form (1.7) restricted to this manifold becomes

$$\omega_{\pi(U^{\tilde{\mathcal{J}}})} = \sum_{i=1}^{n} \sum_{\delta_i = K_i + 1}^{K_i + k_i} \sum_{\beta_i = 1}^{r} dG_i^{\delta_i \beta_i} \wedge dF_i^{\delta_i \beta_i}.$$ 

(2.10)
The moment map $\tilde{J}$ then is defined on the submanifolds (2.9) as follows:

$$
\tilde{J}_{\text{sl}(a+b, C)}(\pi(F, G)) = \lambda \sum_{i=1}^{n} \frac{G_i^T F_i}{\alpha_i - \lambda} = -\frac{\lambda^n}{a(\lambda)} \sum_{i=0}^{n-1} \lambda^{-i} L_i
$$

(2.11)

where the map may be viewed as defined either on the quotient space $M^k_{\text{sl}(r, C)}/(\mathbb{C}^*)^n$ or, equivalently, on the sections whose images are defined by the diffeomorphism (2.9).

Now, the condition (1.5a) imposed on $L_0$ pulls back through the moment map to the space $M^k$ and gives us the following level sets on our charts $\tilde{\pi}(\tilde{U})$ (2.9). Setting $F_i = (\hat{X}_i, \check{X}_i)$ and $G_i = (\hat{Y}_i, \check{Y}_i)$ with $\hat{X}_i, \hat{Y}_i \in \mathbb{C}^{k_i \times a}$ and $\check{X}_i, \check{Y}_i \in \mathbb{C}^{k_i \times b}$ we obtain from the diagonal blocks of the condition (1.5a)

$$
\sum_{i=1}^{n} \hat{Y}_i^T \hat{X}_i = \frac{b\sqrt{-1}}{a+b} I_a
$$

(2.12a)

$$
\sum_{i=1}^{n} \check{Y}_i^T \check{X}_i = -\frac{a\sqrt{-1}}{a+b} I_b,
$$

(2.12b)

and from its off-diagonal blocks

$$
\sum_{i=1}^{n} \hat{Y}_i^T \check{X}_i = 0
$$

(2.13a)

$$
\sum_{i=1}^{n} \check{Y}_i^T \hat{X}_i = 0.
$$

(2.13b)

The vanishing of the diagonal blocks in condition (1.5b) implies

$$
\sum_{i=1}^{n} \alpha_i (\hat{Y}_i^T \hat{X}_i - \frac{b\sqrt{-1}}{a+b} I_a) = 0
$$

(2.14a)

$$
\sum_{i=1}^{n} \alpha_i (\check{Y}_i^T \check{X}_i + \frac{a\sqrt{-1}}{a+b} I_b) = 0,
$$

(2.14b)

while the off-diagonal blocks give $q$ and $p$ in terms of the linear coordinates $(\hat{X}_i, \hat{Y}_i, \check{X}_i, \check{Y}_i)$ as

$$
q = \sum_{i=1}^{n} \alpha_i \hat{Y}_i^T \check{X}_i, \quad p = \sum_{i=1}^{n} \alpha_i \hat{Y}_i^T \hat{X}_i.
$$

(2.15)

Finally, the diagonal blocks of conditions (1.5c) imply the constraints

$$
\sum_{i=1}^{n} \alpha_i^2 \hat{Y}_i^T \hat{X}_i - \sqrt{-1}(qp + \sum_{j \geq k} \alpha_j \alpha_k \frac{b}{a+b} I_a) = 0
$$

(2.16a)

$$
\sum_{i=1}^{n} \alpha_i^2 \check{Y}_i^T \check{X}_i + \sqrt{-1}(pq + \sum_{j \geq k} \alpha_j \alpha_k \frac{a}{a+b} I_b) = 0.
$$

(2.16b)
Now let us turn to the general real form (2.1) of the nonlinear Schrödinger equation. The equations of motion and the choice of $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are obtained in the same way as for the complex case, with the understanding that we are now working in a reduction to the Hermitian symmetric Lie algebra ($\mathfrak{g} = \mathfrak{su}(a+l, s), \mathfrak{k} = \mathfrak{su}(a) \oplus \mathfrak{u}(l, s)$). This space is obtained by requiring invariance of the image of the moment map under the antilinear involutive automorphism

$$
\tilde{\rho} : \tilde{\mathfrak{sl}}(a+l+s, \mathbb{C}) \to \tilde{\mathfrak{sl}}(a+l+s, \mathbb{C})
$$

$$
\tilde{\rho}(X) = -tX^T(\tilde{\lambda})t^{-1},
$$

where $t = \begin{pmatrix} I_{a+l} & 0 \\ 0 & -I_s \end{pmatrix}$, as well as under the involution (2.5). Requiring that the moment map $\tilde{J}_{\mathfrak{sl}(a+b, \mathbb{C})}$ take its values in the fixed point set $\tilde{\mathfrak{su}}(a+l, s)^* = \{X \in \tilde{\mathfrak{sl}}(a+l+s, \mathbb{C})^* \mid \tilde{\rho}(X) = X\}$ forces further restrictions on the space $\mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}$. The method used in computing these restrictions will apply equally to the reductions considered in Section 3. (See also [AHP] for details on the general approach to reductions under involutive automorphisms.) To implement such a discrete reduction we must impose the following condition, reducing the domain of $\tilde{J}$ within the space $\mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}$:

$$
\tilde{W}_{\mathfrak{su}(a+l, s)} = \{\pi(F, G) \in \mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}/(\mathbb{C}^*)^n \mid \tilde{\rho}(\lambda \sum_{i=1}^n G_i^TF_i) = \lambda \sum_{i=1}^n G_i^TF_i, \forall \lambda\}.
$$

(2.18)

This implies that the $\alpha_i$ are either real or come in complex conjugate pairs. We may reorder the $\alpha_i$ such that $\alpha_{2i} = \overline{\alpha_{2i-1}}, \ i = 1, \ldots, m$ and $\alpha_{j} = \overline{\alpha_{j}}, \ j = 2m + 1, \ldots, n$. Thus (2.18) is equivalent to the constraints

$$
G_{2i}^TF_{2i} = -tF_{2i-1}^TG_{2i-1}t^{-1}, \ i = 1, \ldots, m
$$

(2.19a)

$$
G_j^TF_j = -tF_{j-1}^TG_jt^{-1}, \ j = 2m + 1, \ldots, n
$$

(2.19b)

on the pairs $(F_i, G_i)$. The reduced space $\mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}/(\mathbb{C}^*)^n$ splits into several connected components satisfying these constraints, one of which is

$$
W_{\mathfrak{su}(a+l, s)} = \{\pi(F, G) \in \mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}/(\mathbb{C}^*)^n \mid G_{2i-1} = -\overline{F}_{2i}t, G_{2i} = \overline{F}_{2i-1}t, \ i = 1, \ldots, m, \ G_j = \sqrt{-1}F_jt, j = 2m + 1, \ldots, n \}.
$$

(2.20)

Let $\text{pr}_F : \mathbb{C}^{Nr-n} \times \mathbb{C}^{Nr-n} \to \mathbb{C}^{Nr-n}$ be the projection to the first factor. The charts $(\varphi_{\mathfrak{su}}, \pi(U^{\tilde{i}c}))$, defined on $\mathcal{M}^k_{\mathfrak{sl}(r, \mathbb{C})}/(\mathbb{C}^*)^n$ give rise to charts $(\varphi_{\mathfrak{su}}^{\tilde{i}c}, U_{\mathfrak{su}}^{\tilde{i}c})$ on $W_{\mathfrak{su}(a+l, s)}$ defined by

$$
U_{\mathfrak{su}}^{\tilde{i}c} = \pi(U^{\tilde{i}c}) \cap W_{\mathfrak{su}(a+l, s)}
$$

$$
\varphi_{\mathfrak{su}}^{\tilde{i}c} = \text{pr}_F \circ \varphi_{\mathfrak{su}}^{\tilde{i}c} |_{U_{\mathfrak{su}}^{\tilde{i}c}}
$$

(2.21)
which, in coordinates, gives:

\[
\varphi_{su}^E(\pi(F, G)) = (G_1^{l_1, c_1} \tilde{F}_1, \ldots, G_n^{l_n, c_n} \tilde{F}_n) \in C^{N_r-n}.
\] (2.22)

Restricting the symplectic form (2.10) to these charts we get the following symplectic form on our new charts

\[
\omega_{U_i} = \sum_{i=1}^{m} \left( \sum_{\beta_i = K_i + 1}^{K_i + k_i} \sum_{\delta_i = 1}^{a+l} (dF_{2i}^\beta_i \delta_i \wedge dF_{2i-1}^\beta_i \delta_i + dF_{2i}^{\beta_i \delta_i} \wedge dF_{2i-1}^{\beta_i \delta_i}) \right)
\]

\[
\omega_{U_i} = \sum_{i=1}^{m} \left( \sum_{\beta_i = K_i + 1}^{K_i + k_i} \sum_{\delta_i = a+l+1}^{a+l} (dF_{2i}^\beta_i \delta_i \wedge dF_{2i-1}^\beta_i \delta_i + dF_{2i}^{\beta_i \delta_i} \wedge dF_{2i-1}^{\beta_i \delta_i}) \right)
\]

\[
+\sqrt{-1} \sum_{j=2m+1}^{n} \sum_{\beta_i = K_i + 1}^{K_i + k_i} \sum_{\delta_i = 1}^{a+l} (dF_j^\beta_i \delta_j \wedge dF_j^{\beta_i \delta_j})
\]

\[
-\sqrt{-1} \sum_{j=2m+1}^{n} \sum_{\beta_i = K_i + 1}^{K_i + k_i} \sum_{\delta_i = a+l+1}^{a+l+1} (dF_j^\beta_i \delta_j \wedge dF_j^{\beta_i \delta_j}).
\] (2.23)

Note that the terms with the minus sign are a consequence of the choice of the matrix \(t\) in (2.17), (2.19a,b) and (2.20).

Expressing the restriction of the moment map \(\tilde{J}_{sl(a+b, C)}\) to \(W_{su(a+l, s)}\) as \(\tilde{J}_{su(a+l, s)}\)
we have:

\[
J_{su(a+l,s)}(\pi(F,G)) = \lambda \sum_{i=1}^{m} \left( \begin{array}{cc}
-\bar{X}_{2i}^T \bar{X}_{2i-1} & -\bar{X}_{2i}^T \bar{X}_{2i-1} \\
-\bar{D} \bar{X}_{2i}^T \bar{X}_{2i-1} & -\bar{D} \bar{X}_{2i}^T \bar{X}_{2i-1}
\end{array} \right) \frac{\alpha_{2i-1} - \lambda}{\bar{\alpha}_{2i-1} - \lambda} + \lambda \sum_{i=1}^{m} \left( \begin{array}{cc}
\bar{X}_{2i-1}^T \bar{X}_{2i} & \bar{X}_{2i-1}^T \bar{X}_{2i} \\
\bar{D} \bar{X}_{2i-1}^T \bar{X}_{2i} & \bar{D} \bar{X}_{2i-1}^T \bar{X}_{2i}
\end{array} \right) + \sqrt{-1} \lambda \sum_{j=2m+1}^{n} \left( \begin{array}{cc}
\bar{X}_j^T \bar{X}_j & \bar{X}_j^T \bar{X}_j \\
\bar{D} \bar{X}_j^T \bar{X}_j & \bar{D} \bar{X}_j^T \bar{X}_j
\end{array} \right) \frac{\alpha_j - \lambda}{\bar{\alpha}_j - \lambda}.
\]

The restrictions (1.5a-c) on \( L_0, L_1 \) and \( L_2 \) required to obtain the CNLS equation may be expressed in terms of the reduced coordinates over the space \( W_{su(a+l,s)} \). The constraints implied by the choice (1.5a) of \( L_0 \) or equivalently (2.12a,b), (2.13a,b) are

\[
\sum_{i=1}^{m} (-\bar{X}_{2i} \bar{X}_{2i-1} + \bar{X}_{2i-1} \bar{X}_{2i}) + \sqrt{-1} \sum_{j=2m+1}^{n} \bar{X}_j \bar{X}_j = \frac{b \sqrt{-1}}{a + b} I_a \tag{2.25a}
\]

\[
\sum_{i=1}^{m} (-\bar{X}_{2i} \bar{X}_{2i-1} + \bar{X}_{2i-1} \bar{X}_{2i}) + \sqrt{-1} \sum_{j=2m+1}^{n} \bar{X}_j \bar{X}_j = \frac{-a \sqrt{-1}}{a + b} D \tag{2.25b}
\]

\[
\sum_{i=1}^{m} (-\bar{X}_{2i} \bar{X}_{2i-1} + \bar{X}_{2i-1} \bar{X}_{2i}) + \sqrt{-1} \sum_{j=2m+1}^{n} \bar{X}_j \bar{X}_j = 0. \tag{2.25c}
\]

From the equations (2.15), together with the reality conditions (2.19a,b), we find \( u \) in terms of the reduced coordinates \( \bar{X}_i, \bar{X}_i \):

\[
u = \sum_{i=1}^{m} (\alpha_{2i-1} \bar{X}_{2i-1} \bar{X}_{2i} - \alpha_{2i-1} \bar{X}_{2i} \bar{X}_{2i-1}) + \sqrt{-1} \sum_{j=2m+1}^{n} \alpha_j \bar{X}_j \bar{X}_j. \tag{2.26}
\]

From the constraints (2.14a,b), together with the reality conditions (2.19a,b) we have

\[
\sum_{i=1}^{m} (-\alpha_{2i-1} \bar{X}_{2i} \bar{X}_{2i-1} + \bar{\alpha}_{2i-1} \bar{X}_{2i-1} \bar{X}_{2i}) + \sqrt{-1} (\sum_{j=2m+1}^{n} \bar{X}_j \bar{X}_j - \sum_{i=1}^{n} \alpha_i \frac{b}{a + b} I_a) = 0 \tag{2.27a}
\]

\[
\sum_{i=1}^{m} (-\alpha_{2i-1} \bar{X}_{2i} \bar{X}_{2i-1} + \bar{\alpha}_{2i-1} \bar{X}_{2i-1} \bar{X}_{2i}) + \sqrt{-1} (\sum_{j=2m+1}^{n} \bar{X}_j \bar{X}_j + \sum_{i=1}^{n} \alpha_i \frac{a}{a + b} D) = 0. \tag{2.27b}
\]
while from the constraints (2.16a,b), we have

$$\sqrt{-1} \left( \sum_{j=2m+1}^{n} \alpha_j^2 \hat{X}_j \hat{X}_j - \sum_{j \geq k} \alpha_j \alpha_k \frac{b}{a+b} I_a + u D \hat{u}^T \right) + \sum_{i=1}^{m} (- \alpha_{2i-1}^2 \hat{X}_{2i} \hat{X}_{2i} - \hat{u}^T \hat{u}) = 0 \quad (2.28a)$$

$$\sqrt{-1} \left( \sum_{j=2m+1}^{n} \alpha_j^2 \hat{X}_j \hat{X}_j + \sum_{j \geq k} \alpha_j \alpha_k \frac{a}{a+b} D - \hat{u}^T \hat{u} \right) + \sum_{i=1}^{m} (- \alpha_{2i-1}^2 \hat{X}_{2i} \hat{X}_{2i-1} + \hat{u}^T \hat{u}) = 0 \quad (2.28b)$$

Thus, under the Hamiltonian flow induced by the collective Hamiltonians $\tilde{J}_{su(a+l,s)}^x \Phi_x$ and $\tilde{J}_{su(a+l,s)}^t \Phi_t$ on $W_{su(a+l,s)}$, with $\Phi_x, \Phi_t$ given in equations (1.9a,b), subject to the invariant constraints (2.25a-c), (2.28a,b), the resulting function $u(x,t)$ given in equation (2.26) satisfies the CNLS equation (2.1).

### 3. Symmetric Space Nonlinear Schrödinger Equations

The preceding section explained in detail how the Lax pair flows determined by (1.10a,b) on $\tilde{\mathfrak{sl}}(r, \mathbb{C})^{+\ast}$ give the reduced Darboux coordinates and the invariant level sets on the reduced space give rise to solutions of the coupled nonlinear Schrödinger (CNLS) equation corresponding to the Hermitian symmetric Lie algebra $(\mathfrak{su}(a+l, s), \mathfrak{s}(u(a) \oplus u(l, s)))$ and its complexification $(\mathfrak{sl}(a+b, \mathbb{C}), \mathfrak{sl}(a, \mathbb{C}) \oplus \mathfrak{sl}(b, \mathbb{C}) \oplus \mathbb{C})$. In this section we construct reduced coordinates for CNLS equations corresponding to the other Hermitian symmetric Lie algebras considered by Fordy and Kulish [FK]. Referring to the classification of symmetric spaces in [H], the Hermitian symmetric Lie algebras $(\mathfrak{so}(2l), u(l))$ ($DIII$) and $(\mathfrak{sp}(l), u(l))$ ($CI$) as well as their corresponding noncompact and complex forms can be obtained as fixed points of involutive automorphisms on $(\mathfrak{su}(2l), \mathfrak{s}(u(l) \oplus u(l)))$ ($AIII$) and on its corresponding noncompact and complex forms. On the other hand, the complex structure for the $AIII$-cases is contained in the $\mathfrak{u}(1)$-subalgebra, which is not invariant under the involution leading to the Hermitian symmetric Lie algebras associated with $(\mathfrak{so}(2+k), \mathfrak{so}(2) \oplus \mathfrak{so}(k))$ ($BDI$). Therefore the $BDI$-Hermitian symmetric Lie algebras cannot be realized as subcases of the $AIII$-cases. Nevertheless, we can obtain them by reduction of the corresponding generic systems in $\mathfrak{gl}(r, \mathbb{C})$.

For all these cases the resulting Darboux coordinates are global and directly obtained by a reduction of $\mathfrak{gl}(r, \mathbb{C})$ to the classical complex Lie algebras $\mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{o}_n$ and
to their corresponding real forms. We could have obtained the coordinates for the CI- and DIII-cases by imposing further constraints on the coordinates associated with the AIII-cases, but the reduction from $\mathfrak{gl}(r, \mathbb{C})$ avoids the need for local charts. We emphasize that only the constraints on the Moser space induced by reductions to the classical Lie algebras are solved, and not the additional invariant constraints arising from the splitting $g = \mathfrak{t} \oplus \mathfrak{m}$ of the Hermitian symmetric algebras underlying equations (1.5a-c). The latter are simply imposed, as in the previous section, upon the initial data, and continue to hold valid under the flows.

The following are the involutions to be applied to the AIII-Hermitian symmetric Lie algebras in order to get the CI- and DIII-Hermitian symmetric Lie algebras.

$$\sigma_{\text{CI}}(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (3.1a)
$$\sigma_{\text{DIII}}(X) = -\begin{pmatrix} 0 & 1 \\ I & 0 \end{pmatrix} X^T \begin{pmatrix} 0 & 1 \\ I & 0 \end{pmatrix}$$  \hspace{1cm} (3.1b)

The corresponding additional restrictions on the functions $q, p$ and $u$ defined by equations (2.15) and (2.26) and the equations (1.1) and (2.1) are given in Table I. Note that in all these equations, the matrices $p, q$ and $u$ are of dimension $l \times l$. The restrictions given in rows 2 and 3 should be interpreted as invariant quadratic constraints satisfied by the matrices $\{F_i, G_i\}$ on a suitably defined reduced manifold $W$. The general form of the matrices $L_0, L_1$ and $L_2$ entering in the Lax equations (1.2a,b) is still given by equations (1.5a-c).

| Involution: $\sigma_{\text{CI}}$ | $\text{sp}(2l, \mathbb{C})/\text{gl}(l, \mathbb{C})$ | $\text{sp}(2l), \mathfrak{s}(u(l) \oplus u(l))$ | $\text{sp}(l, \mathfrak{s}(u(l) \oplus u(l)))$ |
|----------------------------------|------------------------------------------------|-------------------------------------|---------------------------------|
| Corresponding CNLS equation      | $\sqrt{-1}q_t = q_{xx} - 2pq$                     | $\sqrt{-1}u_t = u_{xx} + 2uu^T u$   | $\sqrt{-1}u_t = u_{xx} - 2uu^T u$ |
| Involutions: $\sigma_{\text{CI}}$ | $q = q^T$                                        | $u = u^T$                           | $u = u^T$                       |
|                                  | $p = p^T$                                        |                                     |                                 |

| Involution: $\sigma_{\text{DIII}}$ | $\text{so}(2l, \mathbb{C})/\text{gl}(l, \mathbb{C})$ | $\text{so}(2l)$                      | $\text{so}^*(2l)$ |
|-----------------------------------|------------------------------------------------|-------------------------------------|-----------------|
| Involutions: $\sigma_{\text{DIII}}$ | $q = -q^T$                                    | $u = -u^T$                         | $u = -u^T$      |
|                                  | $p = -p^T$                                    |                                     |                  |

For the Hermitian symmetric Lie algebras corresponding to BDI, the Lie-algebras $\text{so}(2 + 2l, \mathbb{C})$ and $\text{so}(2 + 2l + 1, \mathbb{C})$ are obtained as fixed point sets of the involutions

$$\sigma_{\text{DI}}(X) = -\begin{pmatrix} 0 & I_{l+1} \\ I_{l+1} & 0 \end{pmatrix} X^T \begin{pmatrix} 0 & I_{l+1} \\ I_{l+1} & 0 \end{pmatrix}, \ X \in \mathfrak{sl}(2 + 2l, \mathbb{C})$$  \hspace{1cm} (3.2a)
$$\sigma_{\text{BI}}(X) = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{l+1} \\ 0 & I_{l+1} & 0 \end{pmatrix} X^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{l+1} \\ 0 & I_{l+1} & 0 \end{pmatrix}, \ X \in \mathfrak{sl}(2 + 2l + 1, \mathbb{C})$$  \hspace{1cm} (3.2b)
(The separation into “$BI$” and “$DI$” is necessary due to our choice of a basis that makes the relation between the Lax matrices in (1.5a-c) and the corresponding nonlinear Schrödinger equations more transparent.) The involutive automorphisms determining the Hermitian symmetric Lie algebra decomposition are

$$
\tilde{\sigma}_{DI}(X) = \begin{pmatrix} I_{1,i} & 0 & 0 \\ 0 & I_{1,i} & 0 \\ 0 & 0 & I_{1,i} \end{pmatrix} X \begin{pmatrix} I_{1,i} & 0 & 0 \\ 0 & I_{1,i} & 0 \\ 0 & 0 & I_{1,i} \end{pmatrix}, \quad X \in \mathfrak{so}(2 + 2l, \mathbb{C})
$$

(3.3a)

$$
\tilde{\sigma}_{BI}(X) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & I_{1,i} & 0 \\ 0 & 0 & I_{1,i} \end{pmatrix} X \begin{pmatrix} -1 & 0 & 0 \\ 0 & I_{1,i} & 0 \\ 0 & 0 & I_{1,i} \end{pmatrix}, \quad X \in \mathfrak{so}(2 + 2l + 1, \mathbb{C}).
$$

(3.3b)

This gives the splitting $\mathfrak{so}(2+2l, \mathbb{C}) = \mathfrak{t}^1 \oplus \mathfrak{m}^1$ and $\mathfrak{so}(2+2l+1, \mathbb{C}) = \mathfrak{t}^2 \oplus \mathfrak{m}^2$ respectively, where $\mathfrak{t}^1 := \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(2l, \mathbb{C})$ and $\mathfrak{t}^2 := \mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(2l+1, \mathbb{C})$ are the $+1$ eigenspaces. The corresponding elements $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ given by equations (1.12a-c) will be denoted $\mathcal{L}_0^s$, $\mathcal{L}_1^{s,1}$ and $\mathcal{L}_2^{s,2}$, $s = 1, 2$ respectively, for the two cases. In the two cases the complex structures $I^1 := \text{ad}\mathcal{L}_0^{1|\mathfrak{m}^1}$, (resp. $I^2 : \text{ad}\mathcal{L}_0^{2|\mathfrak{m}^2}$) underlying the Hermitian symmetric Lie algebra is given by an element $\mathcal{L}_0^1$ (resp. $\mathcal{L}_0^2$) of the $\mathfrak{so}(2)$-subalgebra of the $+1$ eigenspace of the involutions (3.3a,b). Denoting by $E_{i,j}$ the matrix with 1 in the $ij$th position and zero elsewhere, we set

$$
\mathcal{L}_0^1 = \sqrt{-1}E_{1,1} - \sqrt{-1}E_{l+1,l+1} \in \mathfrak{so}(2 + 2l, \mathbb{C})
$$

(3.4a)

$$
\mathcal{L}_0^2 = \sqrt{-1}E_{2,2} - \sqrt{-1}E_{l+2,l+2} \in \mathfrak{so}(2 + 2l + 1, \mathbb{C}).
$$

(3.4b)

Using the same techniques as in the generic case we see that, with $\mathcal{L}_0^1, \mathcal{L}_0^2$ chosen as above, the $\mathfrak{t}^1$ (resp. $\mathfrak{t}^2$) components of $\mathcal{L}_1^1, \mathcal{L}_1^2$ are invariants of the flows which we choose equal to zero. This gives the general form:

$$
\mathcal{L}_1^1 = \begin{pmatrix} 0 & q^T & 0 & \tilde{q}^T \\ p & 0 & -\tilde{q} & 0 \\ 0 & -p^T & 0 & -p^T \\ \tilde{p} & 0 & -q & 0 \end{pmatrix} \in \mathfrak{so}(2 + 2l, \mathbb{C})
$$

(3.5a)

$$
\mathcal{L}_1^2 = \begin{pmatrix} 0 & -p^1 & 0 & q^1 & 0 \\ -q^1 & 0 & q^T & 0 & \tilde{q}^T \\ 0 & p & 0 & -\tilde{q} & 0 \\ p^1 & 0 & -\tilde{p}^T & 0 & -p^T \\ 0 & \tilde{p} & 0 & -q & 0 \end{pmatrix} \in \mathfrak{so}(2 + 2l + 1, \mathbb{C})
$$

(3.5b)

where $q, \tilde{q}, p, \tilde{p} \in \mathbb{C}^l$ and $q^1, p^1 \in \mathbb{C}$. Then $\mathcal{L}_2^1, \mathcal{L}_2^2$ are obtained by substituting (3.5a,b) into equations (2.6c,d).
From the compatibility conditions for equations (1.2a,b) with $L^3_0$, $L^1_1$ and $L^2_s$, $s = 1, 2$ chosen as above, we get the coupled nonlinear Schrödinger equations
\begin{align*}
\sqrt{-1}q_t &= q_{xx} - 2q(q^T p + q^T \tilde{p}) + 2q^T \tilde{q}p \\
\sqrt{-1}\tilde{q}_t &= \tilde{q}_{xx} - 2\tilde{q}(q^T p + q^T \tilde{p}) + 2q^T \tilde{q}p \\
-\sqrt{-1}p_t &= p_{xx} - 2p(q^T p + q^T \tilde{p}) + 2p^T \tilde{p}q \\
-\sqrt{-1}\tilde{p}_t &= \tilde{p}_{xx} - 2\tilde{p}(q^T p + q^T \tilde{p}) + 2p^T \tilde{p}q
\end{align*}
(3.6a)

for the Hermitian symmetric Lie algebra $(\mathfrak{so}(2 + 2l, C), \mathfrak{so}(2, C) \oplus \mathfrak{so}(2l, C))$, and
\begin{align*}
\sqrt{-1}q^1_t &= q^{1x} - 2q^1(q^T p + q^T \tilde{p} + q^1 p^1) + p^1(q^1 q^1 + 2q^T \tilde{q}) \\
-\sqrt{-1}p^1_t &= p^{1x} - 2p^1(q^T p + q^T \tilde{p} + q^1 p^1) + q^1(p^1 p^1 + 2p^T \tilde{p}) \\
\sqrt{-1}q_t &= q_{xx} - 2q(q^T p + q^T \tilde{p} + q^1 p^1) + \tilde{p}(q^1 q^1 + 2q^T \tilde{q}) \\
-\sqrt{-1}\tilde{p}_t &= \tilde{p}_{xx} - 2\tilde{p}(q^T p + q^T \tilde{p} + q^1 p^1) + q(p^1 p^1 + 2p^T \tilde{p})
\end{align*}
(3.6b)

for the Hermitian symmetric Lie algebra $(\mathfrak{so}(2 + 2l + 1, C), \mathfrak{so}(2, C) \oplus \mathfrak{so}(2l + 1, C))$.

The reality conditions are
\begin{align*}
q &= -\bar{p} & \tilde{q} &= -\bar{\tilde{p}} \, ,
\end{align*}
(3.8a)

for both cases. The resulting reality conditions for $L^1_1$ and $L^2_2$ which reduce equations (3.6a) are
\begin{align*}
q &= -\bar{p} & \tilde{q} &= -\bar{\tilde{p}} \\
q^1 &= -\bar{p^1} \, .
\end{align*}
(3.8b)

The noncompact real forms $(\mathfrak{so}(2, 2l), \mathfrak{so}(2) \oplus \mathfrak{so}(2l))$ and $(\mathfrak{so}(2, 2l + 1), \mathfrak{so}(2) \oplus \mathfrak{so}(2l + 1))$ are obtained through the involutive automorphisms
\begin{align*}
\rho_{DI}(X) &= -\left( I_{l,t} 0 \atop 0 I_{l,t} \right) \overline{X}^T \left( I_{l,t} 0 \atop 0 I_{l,t} \right) , X \in \mathfrak{so}(2 + 2l, C) \, ,
\rho_{BI}(X) &= -\left( -1 0 \atop 0 I_{l,t} \right) \overline{X}^T \left( -1 0 \atop 0 I_{l,t} \right) , X \in \mathfrak{so}(2 + 2l + 1, C)
\end{align*}
(3.9a)
The resulting reduced form of $L_1^1$, $L_2^1$ and equations (3.6a) are given by
\begin{equation}
q = \bar{p} \quad \tilde{q} = \tilde{\bar{p}}. \quad (3.10a)
\end{equation}

The reality conditions giving the reduced form of $L_1^2$, $L_2^2$ and equations (3.6b) are
\begin{equation}
q = \bar{p} \quad \tilde{q} = \tilde{\bar{p}} \quad q^1 = \tilde{\bar{p}}. \quad (3.10b)
\end{equation}

Tables II(a − c) give a list of reduced coordinates, corresponding symplectic forms and Lax matrices $\mathcal{N}(\lambda)$ for all the Hermitian symmetric Lie algebras listed above. The reductions are given relative to the generic case by expressing the pairs $(F, G) \in \mathcal{M}^k$ of $N \times r$-complex matrices in terms of reduced coordinates, denoted $(X, Y)$ or $(w, X, Y)$, where $X$ and $Y$ are reduced rectangular blocks and $w \in \mathbb{C}^N$, subject to the remaining invariant constraints. The second column gives the restrictions and reality conditions on the $k_i \times r$-blocks $(F_i, G_i)$ as well as constraints on the eigenvalues of $A$ required in order that the moment map take its values in the subalgebras $\tilde{g}^{+*} \subset \tilde{\mathfrak{sl}}(r, \mathbb{C})^{+*}$ corresponding to the various Hermitian symmetric Lie algebras $(g, k)$. The independant variables entering in the $k_i \times r$ blocks $(F_i, G_i)$ are denoted $(X_i, Y_i)$ or $(w_i, X_i, Y_i)$ for the various cases as defined in the table. The invariant constraints to which these reduced coordinates are subject are obtained by substituting the expressions for $(F, G)$ given in the second column of Tables II(a − c) into equations (1.12a-c) and imposing the conditions (1.5a-c) for the cases in Table I and (3.4a,b), together with the vanishing terms in (3.5a,b) and the conditons (2.6d) imposed on $L_s^k$ for the remaining cases $BI$ and $DI$. The third column gives the corresponding symplectic forms in terms of the reduced coordinates. The fourth column expresses the matrices $q, \tilde{q}, q^1, p, \bar{p}, p^1$ or $u, u^1, v$ satisfying the matrix CNLS equations in terms of these reduced coordinates. The last column expresses, in terms of the reduced coordinates, the image $\mathcal{N}(\lambda)$ of the moment map whose isospectral flows, determined by equations (1.2a,b), imply that $q, \tilde{q}, q^1, p, \bar{p}, p^1, u, u^1, v$ satisfy the CNLS equations, given either in the first row of Table I or in equations (3.6a) or (3.6b).
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Table IIa
### Darboux Coordinates and CNLS Equations

#### 1. $\text{sp}(r, C)/\text{gl}(r, C)$

- **Reduced coords. & reality cons.**
  - $F = (X, Y)$
  - $G = (-Y, X)$
  - $F_i = (X_i, Y_i)$
  - $G_i = (-Y_i, X_i)$
  - $X, Y \in \mathbb{C}^{N \times r}$
  - $X_i, Y_i \in \mathbb{C}^{k_i \times r}$

- **Symplectic form**
  - $2tr(dY \wedge dX^T)$

- **$p, q; u(X, Y)$**
  - $q = -\sum_{i=1}^{n} \alpha_i Y_i^T Y_i$
  - $p = \sum_{i=1}^{n} \alpha_i X_i^T X_i$

#### 2. $\text{sp}(r)/u(r)$

- **Reduced coords. & reality cons.**
  - $\alpha_{2i-1} = \overline{\alpha_{2i}}$
  - $X_{2i-1} = -\overline{Y_{2i}}$
  - $Y_{2i-1} = \overline{X_{2i}}$
  - $i = 1, \ldots, m = n/2$

- **Symplectic form**
  - $2 \sum_{i=1}^{m} tr(dY_{2i} \wedge dX_{2i}^T + dY_{2i} \wedge dX_{2i}^T)$

- **$u = -\sum_{j=1}^{m} (\alpha_{2i, 2i}^T X_{2i} + \alpha_{2i, 2i} Y_{2i})$**

#### 3. $\text{so}(2r, C)/\text{gl}(r, C)$

- **Reduced coords. & reality cons.**
  - $G_i = (X_i, Y_i)$
  - $F_i = \gamma_{\kappa_i}(Y_i, X_i)$
  - $X_i, Y_i \in \mathbb{C}^{2\kappa \times r}$
  - $i = 1, \ldots, n$

- **Symplectic form**
  - $2 \sum_{i=1}^{n} tr(\gamma_{\kappa_i}(dX_i \wedge dY_i^T))$

- **$p, q; u(X, Y)$**
  - $q = \sum_{i=1}^{n} \alpha_i X_i^T \gamma_{\kappa_i} X_i$
  - $p = \sum_{i=1}^{n} \alpha_i Y_i^T \gamma_{\kappa_i} Y_i$

#### 4. $\text{so}(2r)/u(r)$

- **Reduced coords. & reality cons.**
  - $\alpha_{2i-1} = \overline{\alpha_{2i}}$
  - $Y_{2i-1} = -\gamma_{\kappa_{2i}} \overline{X_{2i}}$
  - $X_{2i-1} = -\gamma_{\kappa_{2i}} \overline{Y_{2i}}$
  - $i = 1, \ldots, m$

- **Symplectic form**
  - $2 \sum_{i=1}^{m} tr(\gamma_{\kappa_{2i}}(dX_{2i} \wedge dY_{2i}^T + dY_{2i} \wedge dX_{2i}^T))$
  - $+ 2\sqrt{-1} \sum_{j=2m+1}^{n} tr(\rho_j \overline{dX_j} \wedge dX_j^T)$

- **$u = \sum_{i=1}^{m} (\alpha_{2i} X_{2i}^T \gamma_{\kappa_{2i}} X_{2i} + \overline{\alpha_{2i}} Y_{2i}^T)$**
  - $+ \sum_{j=2m+1}^{n} \alpha_j X_j^T \gamma_{\kappa_j} X_j$

#### 5. $\text{so}^*(2r)/u(r)$

- **Reduced coords. & reality cons.**
  - $\alpha_j = \overline{\alpha_j}$
  - $Y_j = -\sqrt{-1} \gamma_{\kappa_j} \rho_j \overline{X_j}$
  - $j = 2m + 1, \ldots, n$

- **Symplectic form**
  - $2 \sqrt{-1} \sum_{j=1}^{n} tr(\overline{dX_j} \wedge dX_j^T)$

- **$u = \sum_{j=1}^{n} \alpha_j X_j^T \gamma_{\kappa_j} X_j$**
Table IIb
### DARBOUX COORDINATES AND CNLS EQUATIONS

**Darboux Coordinates for CNLS Equations (DI)**

| Algebra | Reduced coords. & reality coords. | Symplectic form | \(q, \tilde{q}, p, \tilde{p}; u, v\) |
|---------|-----------------------------------|----------------|------------------------------|
| \(\mathfrak{so}(2 + 2l, \mathbb{C})\) | \(\mathfrak{so}(2, \mathbb{C}) \oplus \mathfrak{so}(2l, \mathbb{C})\) | \(2 \sum_{i=1}^{n} tr(\gamma_{\kappa_i}(dX_i \wedge dY_i^T))\) | \(q = -\sum_{i=1}^{n} \alpha_i(\tilde{Y}_i)\) |
| \(\mathfrak{so}(2 + 2l)\) | \(\mathfrak{so}(2) \oplus \mathfrak{so}(2l)\) | \(2 \sum_{i=1}^{m} tr(\gamma_{\kappa_{2i}}(dX_{2i} \wedge dY_{2i}^T)\) | \(\tilde{q} = -\sum_{i=1}^{n} \alpha_i(\tilde{Y}_i)\) |
| \(\mathfrak{so}(2l)\) | \(\mathfrak{so}(2) \oplus \mathfrak{so}(2l)\) | \(-2 \sqrt{-1} \sum_{j=2m+1}^{n} tr(dX_j \wedge d(I_{1,l}X_j^T)\rho_j)\) | \(p = \sum_{i=1}^{n} \alpha_i(\tilde{X}_{2i})\) |
|         |                                    | \(-\sqrt{-1} \sum_{j=2m+1}^{n} tr(dX_{2i} \wedge dY_{2i}^T)\) | \(\tilde{p} = \sum_{i=1}^{n} \alpha_i(\tilde{Y}_{2i})\) |
|         |                                    | \(-\sum_{j=2m+1}^{n} \alpha_i(\tilde{Y}_{2i})\) |                                               |

- \(G_i = (X_i, Y_i)\)
- \(F_i = \gamma_{\kappa_i}(Y_i, X_i)\)
- \(X_i, Y_i \in \mathbb{C}^{2\kappa_i \times r}\)
- \(i = 1, \ldots, n\)

- \(\alpha_{2i-1} = \alpha_{2i}\)
- \(Y_{2i-1} = -\gamma_{\kappa_{2i}}X_{2i}\)
- \(X_{2i-1} = -\gamma_{\kappa_{2i}}Y_{2i}\)
- \(i = 1, \ldots, m\)
- \(\alpha_j = \overline{\alpha}_j\)
- \(Y_j = -\sqrt{-1} \gamma_{\kappa_j} \rho_j X_j\)
- \(j = 2m + 1, \ldots, n\)
Table IIc
| Algebra | Reduced coord. & reality cond. | Symplectic form | $q, \tilde{q}, p, \tilde{p}, q^1, p^1; u, v, u^1; j, j$ |
|---|---|---|---|
| $\frac{so(2+(2l+1),C)}{so(2,C) \oplus so(2l+1,C)}$ | $G_i = (w_i, X_i, Y_i)$ $F_i = \gamma_{\kappa_i}(w_i, Y_i, X_i)$ $w_i \in \mathbb{C}^{2\kappa_i}$ $X_i, Y_i \in \mathbb{C}^{2\kappa_i \times (l+1)}$ $i = 1, \ldots, n$ | $\sum_{i=1}^{n} tr(\gamma_{\kappa_i}(2dX_i \wedge dY_i^T + dw_i \wedge dw_i^T))$ | $q = -\sum_{i=1}^{n} \alpha_i \tilde{Y}_i^T \gamma_{\kappa_i}$ $\tilde{q} = -\sum_{i=1}^{n} \alpha_i \tilde{X}_i^T \gamma_{\kappa_i}$ $p = \sum_{i=1}^{n} \alpha_i \tilde{X}_i^T \gamma_{\kappa_i} Y_i$ $\tilde{p} = \sum_{i=1}^{n} \alpha_i \tilde{Y}_i^T \gamma_{\kappa_i} Y_i$ $q^1 = \sum_{i=1}^{n} \alpha_i w_i^T \gamma_{\kappa_i} X_i$ $p^1 = -\sum_{i=1}^{n} \alpha_i w_i^T \gamma_{\kappa_i} Y_i$ |
| $\frac{so(2+(2l+1))}{so(2) \oplus so(2l+1)}$ | $\alpha_{2i-1} = \overline{\alpha}_2$ $Y_{2i-1} = -\gamma_{\kappa_2i} \overline{X}_{2i}$ $X_{2i-1} = -\gamma_{\kappa_2i} \overline{Y}_{2i}$ $w_{2i-1} = -\gamma_{\kappa_2i} \overline{w}_{2i}$ $i = 1, \ldots, m$ | $\sum_{i=1}^{m} tr(\gamma_{\kappa_2i}(2dX_{2i} \wedge dY_{2i}^T + dw_{2i} \wedge dw_{2i}^T))$ $+ \sqrt{-1} \sum_{j=2m+1}^{n} tr(2d\overline{X}_j \wedge dX_j^T \rho_j + d\overline{w}_j \wedge dw_j^T)$ | $u = -\sum_{i=1}^{m} (\alpha_{2i} \tilde{Y}_{2i}^T \gamma_{\kappa_2i} X_{2i}^1 + \overline{\alpha}_{2i} \overline{X}_{2i} \gamma_{\kappa_2i} \overline{Y}_{2i})$ $+ \sqrt{-1} \sum_{j=2m+1}^{n} \alpha_j \tilde{X}_j^T \gamma_{\kappa_j}$ $v = -\sum_{i=1}^{m} (\alpha_{2i} \tilde{X}_{2i}^T \gamma_{\kappa_2i} X_{2i}^1 + \overline{\alpha}_{2i} \overline{Y}_{2i} \gamma_{\kappa_2i} \overline{Y}_{2i})$ $+ \sum_{j=2m+1}^{n} \alpha_j \tilde{Y}_j^T \gamma_{\kappa_j}$ $u^1 = \sum_{i=1}^{m} (\overline{\alpha}_{2i} w_{2i}^T \gamma_{\kappa_2i} X_{2i}^1 + \alpha_{2i} \overline{w}_{2i}^T \gamma_{\kappa_2i} \overline{Y}_{2i})$ $+ \sum_{j=2m+1}^{n} \alpha_j w_j^T \gamma_{\kappa_j} X_j^1$ |
\[ \begin{array}{ccc|c}
\text{Algebra} & \text{Reduced coord.} & \text{Symplectic form} & u, v, u^1(X, Y) \\
+1 \text{ eigenspace} & \text{& reality cond.} & & \\
\text{so}(2,2l+1) & \text{so}(2) \oplus \text{so}(2l+1) & & \\
\hline
\alpha_{2i-1} &=& \bar{\alpha}_{2i} & \sum_{i=1}^{m} \frac{\lambda N_i^{(1)}}{\alpha_{2i} - \lambda} \\
Y_{2i-1} &=& -\gamma_{2i} \bar{X}_{2i} I_{1,l} & + \frac{\lambda N_i^{(2)}}{\alpha_{2i} - \lambda} \\
X_{2i-1} &=& -\gamma_{2i} \bar{Y}_{2i} I_{1,l} & + \sum_{j=2m+1}^{n} \frac{\lambda N_j}{\alpha_j - \lambda} \\
w_{2i-1} &=& \gamma_{2i} \bar{w}_{2i} & + \sum_{j=2m+1}^{n} \frac{\lambda N_j}{\alpha_j - \lambda} \\
i = 1, \ldots, m & & & \\
Y_j &=& -\sqrt{-1} \gamma_{j} \rho_j \bar{X}_j I_{1,l} & + \sum_{j=2m+1}^{n} \frac{\lambda N_j}{\alpha_j - \lambda} \\
w_j &=& \sqrt{-1} \gamma_{j} \rho_j \bar{w}_j & + \sum_{j=2m+1}^{n} \frac{\lambda N_j}{\alpha_j - \lambda} \\
j = 2m+1, \ldots, n & & & \\
\hline
\end{array} \]

Notation:

1. For the complex Hermitian symmetric Lie algebras: \( \mathcal{N}(\lambda) = \sum_{i=1}^{n} \frac{\lambda N_i}{\alpha_i - \lambda} \).

2. For their real forms: \( \mathcal{N}(\lambda) = \sum_{i=1}^{m} \left( \frac{\lambda N_i^{(1)}}{\alpha_{2i} - \lambda} + \frac{\lambda N_i^{(2)}}{\alpha_{2i} - \lambda} \right) + \sum_{j=2m+1}^{n} \frac{\lambda N_j}{\alpha_j - \lambda} \).

3. By \( X_i^1 \) (resp. \( Y_i^1 \)) we denote the first column of \( X_i \) (resp. \( Y_i \)), and by \( \hat{X}_i \) (resp. \( \hat{Y}_i \)) the matrix \( X_i \) (resp. \( Y_i \)).

4. \( 2\kappa_i = k_i, i = 1, \ldots, n \)

5. \( \gamma_{\kappa} = \begin{pmatrix} 0 \\ -I_{\kappa} \\ 0 \end{pmatrix} \)

6. \( I_{k,l} = \begin{pmatrix} I_k \\ 0 \\ -I_l \end{pmatrix} \)

7. \( \rho_j = I_{\kappa_j,\kappa_j} \)