Double Descent in Random Feature Models: Precise Asymptotic Analysis for General Convex Regularization

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Abstract

We prove rigorous results on the double descent phenomenon in random features (RF) model by employing the powerful Convex Gaussian Min-Max Theorem (CGMT) in a novel multi-level manner. Using this technique, we provide precise asymptotic expressions for the generalization of RF regression under a broad class of convex regularization terms including arbitrary separable functions. We further compute our results for the combination of $\ell_1$ and $\ell_2$ regularization case, known as elastic net, and present numerical studies about it. We numerically demonstrate the predictive capacity of our framework, and show experimentally that the predicted test error is accurate even in the non-asymptotic regime.

1 Introduction

Common wisdom in statistics suggests that when modelling a data set, increasing the number of model parameters beyond the number of data points should result in overfitting, diminishing the ability of the model to generalize to unseen data. Neural Networks (NNs) have consistently shown a capacity to forgo this principle and generalize even when the model size grows well beyond the interpolation threshold. More surprisingly, many studies confirm that in this dimension regime, the performance of NNs consistently improves with increasing the model size. This “double descent” phenomenon has been recently studied and is well-understood for simple models under $\ell_2$ regularization, especially a partially trained shallow NN model known as random features (RF) \cite{Rahimi2007, Mei2019}. A comprehensive theory for more complex models is yet to be discovered and the burgeoning research points out to the key role of complex and implicit forms of regularization beyond $\ell_2$ regularization\cite{Smith2021}. Motivated by this observation and towards a better understanding of the double descent phenomenon, we introduce a novel framework for studying overparameterization in the RF model that enables an accurate prediction of the expected network behaviour under generic convex regularization functions.

In general, overparameterized systems of equations have a multitude of solutions. While many solutions may result in small or even zero training loss, few have the capacity to generalize effectively to unseen data. In simple models such as linear regression, it has previously been shown, e.g. in \cite{Bartlett2020, Tsigler2020} that “benign overfitting” may occur, i.e. overparameterization may not impact the testing performance, if the solution closely approximates a minimum $\ell_2$ norm solution. In this vein, linear regression models with Gaussian covariates have been extensively examined in \cite{Hastie2019, Advani2020}, and their asymptotic behavior is characterized when the number of model parameters grows unboundedly. A step towards the study of NNs was taken in \cite{Mei2019} by extending the above studies to linear regression of random features obtained from an untrained shallow NN with $\ell_2$ regularization, and correspondingly establishing double descent. More recently, \cite{dAscoli2020} has shown that under various conditions, the RF model with $\ell_2$ regularization undergoes phase transitions with respect to the formation of the double descent peak at the interpolation threshold. In particular, they predict that strongly regularized models do not exhibit double descent, but instead continuously decrease their testing error at larger model sizes.
Despite a considerable progress, these simple models fall short of capturing the full complexity of large scale NNs as they are in general restricted to $\ell_2$ regularization. Instead, the generalization behavior of NNs is widely attributed to an implicit form of regularization imposed by the training procedure, namely backpropagation $\text{Smith et al.}(2021)$; $\text{Arora et al.}(2019)$. For this reason, the study of an arbitrary regularization effect on the asymptotic model performance is instrumental for a fuller understanding of NNs. Similar to $\text{Mei & Montanari}(2019)$, we focus on the RF model. Our main goal in this paper is to introduce a novel framework for studying the RF model with strongly convex but otherwise arbitrary regularization functions. We show that our framework leads to precise and simple expressions for arbitrary separable convex regularization functions, e.g. the $\ell_p^p$ norms for $p \geq 0$. A summary of our contributions is given below.

1.1 Main results

- We propose a novel framework that allows us to characterize the expected asymptotic training and testing loss of RF linear regression under arbitrary regularization functions. The proposed framework is based on the following technical contributions:
  - In Theorem 4.1, we prove a general Gaussian replacement result showing that the behavior of generic random features is asymptotically equivalent to Gaussian features with a similar mean and covariance matrix, under milder conditions than previous work, such as $\text{Mei & Montanari}(2019)$.
  - In Theorem 4.2, we apply the celebrated convex Gaussian Min-Max Theorem (CGMT) to characterize the behavior of the equivalent Gaussian features. For this purpose, we overcome the notorious difficulty of the CGMT in treating correlated features by introducing a novel multi-level scheme. With this approach, we achieve simple expressions for generic regularization functions.

- We specialize our results to the case of elastic net regularization ($\ell_1$ and $\ell_2^2$) and give exact expressions for the expected asymptotic training and testing loss. Unlike the previous studies that consider untrained random features, the case of elastic net can be interpreted as training features by selection (pruning). We compare the results to the numerical experiments and verify their validity. Furthermore, we confirm and characterize the double descent behaviour in the elastic net setup.

2 Related Works

There has been a recent focus in the literature on the generalization behavior of machine learning models in the interpolation regime. Arguably this was begun by $\text{Zhang et al.}(2016)$, who demonstrated that deep NNs are capable of generalizing even when the network memorizes the entire data set. $\text{Belkin et al.}(2018)$ shortly thereafter showed similar results in kernel learning, with generalization capacity being shown in kernel regression $\text{Belkin et al.}(2019)$ and kernel ridgeless regression $\text{Liang & Rakhlin}(2020)$.

The double descent phenomenon has a long history $\text{Loog et al.}(2020)$, but was first discussed in its modern form by $\text{Belkin et al.}(2019a)$. The case of linear regression is studied in $\text{Belkin et al.}(2020)$ and $\text{Hastie et al.}(2019)$. The former investigates finite dimensional models with Gaussian covariates and the latter provides asymptotic expressions for the prediction error in the high dimensional case. In the case of linear regression, non asymptotic generalization bounds have also been determined by $\text{Bartlett et al.}(2020)$; $\text{Tsiger & Bartlett}(2020)$.

With respect to NNs, asymptotic prediction error for the RF model $\text{Hastie et al.}(2019)$ and $\ell_2^2$ regularized RF $\text{Mei & Montanari}(2019)$ has been previously determined. We extend these results by determining the asymptotic behavior of the RF model with more general regularization functions. Understanding the generalization behavior of linear models may hold relevance to more general NNs. Results from the study of the Neural Tangent Kernel regime, $\text{Jacot et al.}(2018)$; $\text{Allen-Zhu et al.}(2019)$; $\text{Du et al.}(2019)$, have shown that overparameterized models may be expressed in terms of linear dynamical systems with an implicit form of regularization.
For the derivation of our results concerning the RF model we make use of the Convex Gaussian Min Max theorem Thrampoulidis et al. (2014); Gordon (1988). This theorem has been used to study the universality of regularized least square solutions Panahi & Hassibi (2017) including LASSO regularization. LASSO and basis pursuit in the case of strong and weak features has also been examined by Bosch et al. (2021). A major difficulty with the CGMT is in the case of correlated covariates, as in the RF model. In the case of $\ell^2_2$ regularization rotational symmetry may be applied to study correlated models, such as in Chang et al. (2021), however we are not aware of any application of the CGMT considering NNs with more generic regularization. As a contribution of this paper, we resolved the issue of correlated covariates with a novel approach involving multiple applications of CGMT and extend the analysis of regularized least squares into RF features.

3 Random Features and Preliminaries

We consider the case of an RF model, consisting of a two-layer NN expressed by

$$f(z_i; \theta, W) = \sum_{j=1}^{m} \theta_j \sigma(\langle w_j, z_i \rangle / \sqrt{d}),$$  \hspace{1cm} (1)$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a nonlinear activation function. Each $z_i$ is an element of the data set $\{(z_i, y_i) \in \mathbb{R}^d \times \mathbb{R}\}_{i=0}^n$ and each $z_i \sim N(0, I_d)$. The weights of the first layer are given by the matrix $W$, where the rows $w_j$ are distributed according to $w_j \sim N(0, I_d)$. In the RF model, the weights $w_j$ are fixed at initialization and remain untrained, optimization is exclusive to the parameters $\theta = (\theta_j)$. We define the $(m \times n)$ matrix $X$ of features such that $X_{ij} = \sigma(\langle w_j, z_i \rangle / \sqrt{d})$. To train this NN, we consider a square loss function with generic regularization $g(\lambda, \theta)$, in which $\lambda \in \mathbb{R} \geq 0$ is a regularization parameter. We require $g$ to be thrice differentiable and strongly convex. Although our results assume a generic form of such regularization terms, we are particulary interested in a case where $g$ is separable in the sense that

$$g(\lambda, \theta) = \sum_{i=1}^{m} g_i(\lambda, \theta_i).$$  \hspace{1cm} (2)$$

Under these conditions, we may equivalently express training the model (1) as a regularized minimization problem of the following regression form

$$P(X) := \min_{\theta} \frac{1}{2n} ||y - X\theta||^2_2 + g(\lambda, \theta).$$  \hspace{1cm} (3)$$

We consider this problem in the regime that $n, m, d$ all grow large at constant ratios, we define these ratios by:

$$n \rightarrow \infty, \quad m \rightarrow \infty, \quad \eta := \frac{n}{\gamma}.$$

Finally we let the relationship between $X$ and $y$ be given by

$$y = \frac{1}{\sqrt{m}}X^T \theta^* + \nu,$$

where $\theta^* = (\theta_j^*)$ is the "true" solution and $\nu = (\nu_i)$ is an i.i.d noise, with $E[\nu_i] = 0$, $E[\nu_i^2] = \sigma^2_\nu < \infty$ and $E[\nu_i^4] < \infty$. We define the vector $e = \theta - \frac{1}{\sqrt{m}}\theta^*$ and note that we can express the optimization problem in (3) as

$$P(X) = \min_{e} \frac{1}{2n} ||\nu - X^T e||^2_2 + g \left( \lambda, e + \frac{1}{\sqrt{m}}\theta^* \right).$$  \hspace{1cm} (6)$$

1More generally, we require local stability around the optimal point. Despite not being strongly convex, the $\ell_1$ norm satisfies these stability conditions and is hence supported by our analysis. See for example Candes et al. (2006).
Let \( \hat{e} \) be the value that minimizes this problem and hence \( \hat{\theta} = \hat{e} + \frac{1}{\sqrt{m}} \theta^* \) is the solution to (3). Finally, let \( \tilde{f} = f(z; \hat{\theta}, W) \) denote the network in which the weights of the second layer are given by \( \hat{\theta} \). Suppose that a new data point \((z, y)\) is similarly generated by

\[
y = \frac{1}{\sqrt{m}} \mathbf{x}^T \theta^* + \nu,
\]

where \( \nu \) is an i.i.d copy of \( \nu_i \) and \( x_j = (\sigma(\langle w_j, z \rangle / \sqrt{d}) \). Then, the average generalization risk, \( R_{RF} \), is expressed as

\[
R_{RF}(\hat{\theta}; \theta^*) := \mathbb{E}_z[\tilde{f}(z) - y]^2 = \mathcal{B}^2 + \mathcal{V}^2 + \nu^2,
\]

where \( \mathcal{B} \) is the bias and \( \mathcal{V} \) is the variance of the model with respect to generalization on new data:

\[
\mathcal{B} = \mathbb{E}_z[\tilde{f}(z) - f^*(z)]
\]

\[
\mathcal{V} = \mathbb{E}_z \left\| \tilde{f}(z) - \mathbb{E}_z \hat{f}(z) \right\|_2^2,
\]

where \( f^* = f(z; \theta^*, W) \) is the network with true weights \( \theta^* \) in the second layer.

### 4 Main results

In this paper, we obtain exact asymptotic values of generalization risk, bias and variance in (8) and (9) in the limit of \( m \). Our strategy is summarized in two main results: In the first one, we show that replacing \( \mathbf{X} \) by a suitable Gaussian matrix will not change statistical properties of the solution. In the second one, we provide a framework for analyzing the solution of the Gaussian feature matrix.

#### 4.1 Universality by Gaussian Replacement

We state this step by the following theorem:

**Theorem 4.1 (Gaussian Replacement).** Let \( \mathbf{X} \) be given by \( \mathbf{X}_{ij} = \sigma(\langle \mathbf{w}_j, \mathbf{z}_i \rangle / \sqrt{d}) \), where \( \mathbf{w}_j \)s are fixed and \( \mathbf{z}_i \sim \mathcal{N}(0, \mathbf{I}_d) \) are i.i.d.. Denote respectively by \( \mathbf{x}_i \) and \( \mathbf{R}(\mathbf{W}) = \mathbb{E}_{\mathbf{x}_i}[\mathbf{x}_i^T \mathbf{x}_i^T] \) the \( i \)th row (treated as a column vector) of \( \mathbf{X} \) and their common covariance conditioned on the weights \( \mathbf{W} = \{\mathbf{w}_j\} \). Let \( \tilde{\mathbf{X}} \) be defined such that the rows \( \tilde{\mathbf{x}}_i \sim \mathcal{N}(0, \mathbf{R}(\mathbf{W})) \) are i.i.d.. Furthermore, let the elements of \( \theta^* \) be distributed i.i.d according to some distribution \( \zeta \).

Then, if the problem \( P(\mathbf{X}) \) as given in (4) converges to a constant \( C \) with optimal value \( \theta(\tilde{\mathbf{X}}) \) in the limit of \( n, m, d \to \infty \) then \( P(\mathbf{X}) \) will converge to the same constant \( C \) and the solution \( \theta(\mathbf{X}) \) will converge to the solution \( \theta(\tilde{\mathbf{X}}) \) in the weak topology of the smooth functions \( h(e, \theta^*) \) with absolutely bounded first and second derivatives.

As such, as long as the number of data points is sufficiently large, the statistical model behaviour will be indistinguishable from a model trained on a Gaussian data set with a similar covariance.

#### 4.1.1 Proof Sketch

While the proof may be found in the appendix [3] we give a brief overview of the main steps here. In the case that the features are i.i.d and centered, a proof of Gaussian replacement exists in [Panahi & Hassibi 2017]. In the current case with the correlation \( \mathbf{R}(\mathbf{W}) \) present, this proof requires modification in certain steps. The proof relies on the so called Lindeberg’s argument. We consider the difference in the expectation value of a family of test functions applied to the optimal value \( h(P(\mathbf{X})) \). Then, we sequentially replace each row of \( \mathbf{X} \) with its Gaussian equivalent row of \( \tilde{\mathbf{X}} \) and show that at each replacement, the amount of change between the expected values of all the test functions is bounded by \( O(m^{-5/4}) \). Applying the triangle inequality and performing the sum over all row replacements, we conclude that the total difference in the expectation of the test functions between \( h(\mathbf{X}) \) and \( h(\tilde{\mathbf{X}}) \) is \( O(m^{-1/4}) \). In the limit of \( m \) growing to infinity, the difference becomes negligible. A similar methodology is applied to show the convergence of the solution \( \theta(\mathbf{X}) \) to that of \( \theta(\tilde{\mathbf{X}}) \) under a family of test functions, resulting to the desired weak convergence.
4.2 Gaussian Feature Matrix

We now examine the solutions space of the problem \( P(\mathbf{X}) \) in which the row of \( \mathbf{X} \) are Gaussian and distributed with covariance \( \mathbf{R}(\mathbf{W}) \), that is a generated from of the random features model. We note that by Theorem 4.1, these solutions will be asymptotically identical to those of the non-Gaussian case. We first make an observation about the structure of \( \mathbf{R}(\mathbf{W}) \). As shown by [Mei & Montanari 2019], we note that the activation function in the asymptotic limit is equivalent to the following Gaussian covariate model

\[
\sigma((\mathbf{z}_i, \mathbf{w}_j)/\sqrt{d}) \to \rho_0 + \rho_1 (\mathbf{z}_i, \mathbf{w}_j) / \sqrt{d} + \rho_2 \mathbf{v}_{ij},
\]

where \( \mathbf{v}_{ij} \sim \mathcal{N}(0, 1) \) and \( \rho_i \) are constants based on the choice of activation function given by

\[
\rho_0 = \mathbb{E}_a[\sigma(a)], \quad \rho_1 = \mathbb{E}_a[a\sigma(a)], \quad \rho_2 = \mathbb{E}_a[\sigma^2(a)] - \rho_1^2 - \rho_0^2.
\]

The expectation is taken with respect to \( a \sim \mathcal{N}(0, 1) \). The covariance matrix conditioned on the weights \( \mathbf{R}(\mathbf{W}) \) may now be expressed in the following form

\[
\mathbf{R}(\mathbf{W}) = \mathbb{E}_a[\mathbf{x}_i \mathbf{x}_j^T] = \rho_0^2 11^T + \frac{\rho_1^2}{d} \mathbf{W} \mathbf{W}^T + \rho_2^2 \mathbf{I},
\]

in the asymptotic limit. We now state our results about the solution space of the problem \( P(\mathbf{X}) \) when \( \mathbf{X} \) is normally distributed.

**Theorem 4.2.** Consider the problem \( P(\mathbf{X}) \) given in (1), such that each row of \( \mathbf{X} \) is distributed according to \( \mathbf{x}_i \sim \mathcal{N}(0, \mathbf{R}(\mathbf{W})) \), with \( \mathbf{R}(\mathbf{W}) \) given in (14). Further, assume that the rows \( \mathbf{w}_j \sim \mathcal{N}(0, 1) \) are i.i.d.

Let \( \phi_1, \phi_2 \) be standard Gaussians of degree \( m \), and \( \phi_0 \) a standard Gaussian of degree 1. For any choice of positive real numbers \( \beta, q, \xi, \) and \( r \), let

\[
c_1 = \frac{\beta^2 \rho_2^2 \xi}{2q^2 r} + \frac{\beta \rho_1^2}{2q}, \quad c_2 = \left( \frac{\beta^2 \rho_2^2 \xi^2}{q^2} + \beta^2 \rho_1^2 \right)^{1/2}
\]

and take \( \varphi \) as a standard Gaussian vector satisfying:

\[
\frac{c_2}{\sqrt{n}} \varphi = \frac{\beta \rho_2 \xi}{q \sqrt{d}} \phi_1 + \frac{\beta \rho_1}{\sqrt{n}} \phi_2.
\]

Consider the solution \( \mathbf{e} \) to the following optimization problem

\[
\max_{\mu \in \mathbb{R}} \min_{s \in \mathbb{R}} \min_{\mathbf{c} \in \mathbb{R}} c_1 ||\mathbf{c}||^2_2 + \frac{c_2}{\sqrt{n}} (\varphi, \mathbf{c}) + g\left(\lambda, \mathbf{e} + \frac{1}{\sqrt{m}} \theta^*\right) + \frac{\beta n \rho_0}{2qm} s^2 - \frac{\beta \rho_0 \phi_0}{\sqrt{n}} s + \mu(s - 1^T \mathbf{e}).
\]

There exists a particular value of \( \beta, q, \xi, \) and \( r \) satisfying the following system of equations

\[
\frac{\beta(n-d)}{2n} q^2 - \frac{\beta^2 \rho_2^2 \xi}{2q} ||\mathbf{c}||^2_2 = \frac{\beta \rho_2^2}{2q} ||\mathbf{c}||^2_2 + \frac{\beta \rho_1}{\sqrt{d}} \langle \phi_1, \mathbf{c} \rangle + \frac{\rho_0^2}{2m} s^2 - \frac{\beta^2 \rho_1^2}{2} + \frac{\beta \rho_0^2}{2}
\]

\[
\left( \frac{\rho_2^2}{q^2 r} ||\mathbf{c}||^2_2 + \frac{\xi d}{rn} - 1 \right) \beta = -\frac{\rho_2^2}{2q} ||\mathbf{c}||^2_2 - \frac{\rho_1}{q \sqrt{d}} \langle \phi_1, \mathbf{e} \rangle - \frac{\rho_2}{\sqrt{d}} \langle \phi_2, \mathbf{e} \rangle + \frac{\rho_0}{\sqrt{n}} \phi_0 s + \frac{\xi d}{2q} + \frac{\beta \rho_0^2}{2} - \frac{\beta \rho_1^2}{2} + \frac{\beta \rho_0^2}{2} + \frac{\beta \rho_1^2}{2} + \frac{\beta \rho_0^2}{2}
\]

\[
r^2 = \frac{\beta \rho_2^2}{q^2} ||\mathbf{c}||^2_2 + \frac{\beta^2 \rho_1^2}{\eta}
\]

\[
\xi = \frac{q}{\beta} + \frac{\rho_1}{\sqrt{d}} \langle \mathbf{e}, \phi_2 \rangle.
\]
Then, the joint empirical distribution of \((\sqrt{m}e, \theta^*)\) converges pointwise to the empirical distribution of \((\sqrt{m}e, \theta^*)\) in the asymptotic limit of \(n, m, d \rightarrow \infty\) at constant ratios. Furthermore, let \(f = f(\cdot; W, \theta + \theta^*)\). Then,

\[
B(f) \rightarrow B(\hat{f}), \quad V(f) \rightarrow V(\hat{f}),
\]

pointwise.

**Remark 4.3.** Theorem 4.2 defines an optimization problem of \(\bar{e}\), that when solved for constant values of \(\beta, q, \xi, r\) produces a solution \(\bar{e}(\beta, q, \xi, r)\) depending on them. The values of \(\beta, q, \xi, r\) must then be ascertained by solving the system of equations given by equations (18) - (21). We denote these values by \(\beta^*, q^*, \xi^*, r^*\). Then, the particular solution \(\bar{e}^* = \bar{e}(\beta^*, q^*, \xi^*, r^*)\) is associated with the trained RF model. The expected asymptotic training error is obtained by plugging \(\bar{e}^*\) into (17). The test error may be obtained by noting that \(\bar{e}(\beta^*, q^*, \xi^*, r^*) = \theta(\beta^*, q^*, \xi^*, r^*) - \frac{1}{\sqrt{m}}\theta^*\), in conjunction with the average risk for unseen points in (8) and (9).

**Remark 4.4.** There exists an appealing approach for solving the system of equations in (18) - (21). This system corresponds to the optimal point of a 4-dimensional characteristic optimization problem stated in the supplement equation (57). Accordingly, we solve the system of equations by applying a standard gradient-free optimization technique to the characteristic optimization. For more details, see [4, 1] in the supplement.

**Remark 4.5.** We note that asymptotically, \(\bar{e}\) has a similar joint empirical distribution with \(\theta^*\) to the actual error vector and hence \(\theta := \bar{e} + \theta^*/\sqrt{m}\) has a similar joint distribution to the solution \(\theta\). Moreover, as long as the terms \(\|\bar{e}\|_2^2, \langle \phi_1, \bar{e} \rangle\) and \(\langle \phi_2, \bar{e} \rangle\) asymptotically concentrate for any choice of \(\beta, q, \xi\) and \(r\), the equivalent solution vector \(\theta\) is simply obtained by first adding a Gaussian noise \(\sqrt{\frac{2}{m}}\varphi\), a uniform bias \(\mu \mathbf{1}\) and then applying the proximal operator \(\text{prox}_{c_1g}\) of \(g\) with step size \(c_2\):

\[
\bar{e} = \text{prox}_{c_1g} \left( \frac{\theta^*}{\sqrt{m}} + \frac{c_2}{\sqrt{m}} + \mu \mathbf{1} \right)
\]

**Remark 4.6.** For separable regularization functions in (2), the above procedure is simplified, because the optimization in (17) is separable, the proximal operator is element-wise and \(\bar{e}\) has independent entries. Then, the terms \(\|\bar{e}\|_2^2, \langle \phi_1, \bar{e} \rangle\) and \(\langle \phi_2, \bar{e} \rangle\) concentrate on their expected values, which can be easily computed.

**Remark 4.7.** We note that in the case that the activation function \(\sigma\) is centered, such that \(\rho_0 = \mathbb{E}_0[\sigma(\alpha)] = 0\), where \(\alpha\) is a standard normal, that the optimization function given in (17) is simplified substantially, as the optimization over \(\mu\) and \(s\) becomes no longer necessary. The optimization then takes the form

\[
\min_{\bar{e}} c_1\|\bar{e}\|_2^2 + \frac{c_2}{\sqrt{m}}\phi + g \left( \lambda, \bar{e} + \frac{1}{\sqrt{m}}\theta^* \right).
\]

In effect, the optimization over \(s\) and \(\mu\) introduces a bias that accounts for the asymmetry in the distribution of the activation function around 0.

**4.2.1 Proof sketch**

We make use of the convex Gaussian min max theorem [Thrampoulidis et al. (2014)] which shows that there exists an asymptotic equivalence between problems of the following type:

\[
\min_{x \in S_x} \max_{y \in S_y} y^T G x + \psi(x, y) \quad (26)
\]

\[
\min_{x \in S_x} \max_{y \in S_y} \|x\|_2 g^T y + \|y\|_2 h^T x + \psi(x, y), \quad (27)
\]

where the elements of \(G, g\) and \(h\) are drawn from i.i.d \(\mathcal{N}(0, 1)\), \(\psi\) is an arbitrary concave-convex function and \(S_x\) and \(S_y\) are convex sets. To prove Theorem 1.2, we first fix \(W\) and translate the minimization problem given in (6) into a min-max problem of the form in (26) by suitable
transformations and change of variables. Then, we invoke the CGMT which eliminates the randomness (in $G$) due the data set $z$ and reexpress the problem in terms of (27). Unfortunately, the resulting expression is well-known to be intractable as it depends on a complex form of covariance matrix $R(W)$. Here, we introduce a key novel step. We show that assuming random weights $W$, under further suitable transformations, the resulting equivalent form in (27) can itself be transformed into the form of (26) with a new random matrix $G$ representing the randomness of the weights. This allows us to apply the CGMT again which eliminates the randomness of the weights as well. Finally, we simplify the expressions obtained by the second CGMT application, which leads to the results in Theorem 4.2. The full proof is given in the Appendix A.

5 Regularized Random Features Examples

In this section we derive expressions for the expected asymptotic behavior for some choices of the regularization function $g$. We focus specifically on $\ell_2^2$ regularization, i.e. ridge regression, as well as $\ell_1 + \ell_2^2$ regularization, known as elastic net. The behavior of some additional regularization functions, such as pure $\ell_1$ may be found in the supplementary material. In the cases below, we assume, for simplicity, that the activation function is centered such that $\rho_0 = 0$. We also assume the "true" solution $\theta^*$ to be distributed according to some distribution $\zeta$, and consider a new testing data point $z$ drawn from a similar distribution to $z_i$. Then, we let $(x)_i = \sigma((z, w_i) / \sqrt{d})$.

5.1 $\ell_2^2$ Regularization

We first consider the case of ridge regression, in which the optimization problem is expressed as

$$
\min_{\theta} ||y - X^T \theta||_2^2 + \lambda ||\theta||_2^2. \quad (28)
$$

Here, $\lambda$ is the regularization parameter, and is tuneable. The case of $\ell_2^2$ regularization has been considered previously, for instance in Mei & Montanari (2019) and d’Ascoli et al. (2020). However, in both cases, the labels $y_i$ are a linear function of the input data $z_i$, i.e. $y_i = \langle \psi, z_i \rangle + \epsilon_i$ for a fixed vector $\psi$ and a random noise or perturbation term $\epsilon$. As seen in (7), we define $y$ as a linear function of the features $x$. To the best of the authors’ knowledge, this setup has not been previously discussed. Note that unlike the previous studies, our setup covers all nonlinear relations $y = f^*(x)$ which is well-approximated in the linear span of the neural features $x_i(z) = \sigma((w_i, x))$ basis (basis expansion model).

For the random features model described above, we find asymptotic expressions for the solution $\hat{\theta}$. According to Theorem 4.2, this solution has a similar statistics to an equivalent solution of (17):

$$
\hat{\theta} = \frac{c_1}{\sqrt{m}} \theta^* - \frac{c_2}{2\sqrt{m}(c_1 + \lambda)} \varphi. \quad (29)
$$

The bias and the variance of the found solutions, with respect to a new unseen feature $x$ is given by

$$
B = \frac{\lambda}{\sqrt{m}(c_1 + \lambda)} \mathbb{E}_{x, \theta^*}||x^T \theta^*||_2^2 \quad (30)
$$

$$
V = \frac{c_2}{2\sqrt{m}(c_1 + \lambda)} \mathbb{E}_{x}||x^T \varphi||_2^2, \quad (31)
$$

where $\varphi \sim N(0, I_m)$. To compute the statistics of the solution we need to calculate $c_1, c_2$ by solving (18) - (21), which is explained in appendix A.1. We note that as expected, the presence of the $\ell_2^2$ regularization introduces a bias with respect to $\lambda$.

5.2 Elastic net model

We further consider the case of a mixture of an $\ell_2^2$ and an $\ell_1$ regularization terms. In the appendix, we also derived expressions for the pure $\ell_1$ regularization known as Least Absolute Shrinkage and Selection Operator (LASSO). In the case of elastic net, the training optimization is expressed as

$$
\min_{\theta} ||y - X^T \theta||_2^2 + \lambda_1 ||\theta||_2^2 + \lambda_2 \sqrt{m} ||\theta||_1. \quad (32)
$$
The $\ell_1$ regularization term promotes sparse solutions, which is widely used in feature selection applications. In the RF model, the weights $W$ from the input to the hidden layer are randomly initialized and frozen. Note that sparsity in $\theta$ results in some of the nodes of the hidden layer being discarded as their corresponding coefficients $\beta_j$ are set to zero. As such, finding the optimal solution is in part achieved by finding the correct subset of the weights $W$ that allow for a small training error. From this perspective, $\ell_1$ regularization may be considered a rough proxy for training the NN, in which the weights are not updated but are selected. For a similar application see for example Gallicchio & Scardapane (2020). Merely applying an $\ell_1$ regularization (LASSO) has well-known limitations. For example, the number of selected features never exceeds the number of the data points and the technique is sensitive to correlated features. An additional $\ell_2^2$ term is incorporated in the elastic net formalism to alleviate these difficulties.

Now, we present an application of Theorem 4.2 for analyzing the elastic net solution with RFs. We obtain the values of $(\hat{\theta})_i$ for each $i = 1, \ldots, m$ as follows:

$$
(\hat{\theta})_i = \begin{cases} 
0 & -\alpha_{1i} \leq \varphi_i \leq \alpha_{1i} \\
\frac{c_2 \varphi_i}{\sqrt{m}} - \frac{\lambda_i}{2 \sqrt{m} (c_1 + \lambda_i)} & \alpha_{1i} < -\varphi_i \\
\frac{c_2 \varphi_i}{\sqrt{m}} - \frac{\lambda_i}{2 \sqrt{m} (c_1 + \lambda_i)} + \frac{\lambda_i}{2 \sqrt{m} (c_1 + \lambda_i)} & \varphi_i > \alpha_{2i} \\
\end{cases}
$$

where each term $\varphi_i$ is a standard normal random variable with unit variance, and where $\alpha_{1i}$ and $\alpha_{2i}$ are given by

$$
\alpha_{1i} = \frac{(-\lambda_2 + 2 c_1 \theta_i^*)}{c_2} \quad \alpha_{2i} = \frac{(\lambda_2 + 2 c_1 \theta_i^*)}{c_2}.
$$

The solution for $(\hat{\theta})_i$ may be more succinctly written in the form a soft thresholding operator. The exact expressions for the bias and the variance are unwieldy, and as such they are given in full in the appendix C.3. We also postpone the details of solving the system of equations in (18) - (21) and numerically obtaining the values of $c_1, c_2$ to Appendix A.1.

According to Theorem 4.2 different statistics of the solution $\hat{\theta}$ can be computed from $\theta$. For example, the sparsity of the solution is determined by noting that each element $(\hat{\theta})_i$ is zero with a non-zero probability. Letting $s$ be equal to the expected number of non-zero elements of $\theta$, then value of the sparsity may be determined to be

$$
s = \sum_i Q(\alpha_{1i}) + Q(\alpha_{2i}),
$$

where $Q(\cdot)$ is the Q-function. In the asymptotic limit of $m \to \infty$, $s$ will also grow to infinity. As such, it is more relevant to consider the ratio $s/m$ the fractional proportion of the number of nonzero elements of $\theta$. In C.3.1 we show that

$$
\frac{s}{m} = \frac{2(c_1 + \lambda_1)}{c_2^2} \mathbb{E}[F] \quad \frac{s}{m} = \frac{\gamma s}{n},
$$

where $F$ is the value that $\frac{\mathbb{E}[\ell_1(\theta, \phi)]}{\sqrt{n}}$ asymptotically concentrates on. A full expression for $F$ may be found in equation (111) in the supplement.

In the case of LASSO regularization ($\lambda_1 = 0$), the expression for $s$ and the ratio the ratio $\frac{s}{n}$, can still be computed by (34) and (35), by setting $\lambda_1 = 0$.

6 Experiments

6.1 Experimental setup

Using the expressions obtained in the previous section, we examine the cases of $\ell_2^2$ and $\ell_1 + \ell_2^2$ regularization experimentally, by varying the strength of the regularization parameters $\lambda$ between the experiments. We choose the tanh activation function as it is an anti-symmetric function, resulting in $\rho_0 = 0$. We further set $\delta = 1$ and plot the results with respect to $\frac{1}{s} = \frac{m}{n}$, the ratio between the number of model parameters to the number of data points. The empirical results presented are averages over 100 training data sets.
Figure 1: Theoretically predicted (solid line) and numerically determined (markers) values of the (a) training error and (b) generalization error for the random features model with $\ell^2_2$ regularization as a function of $\gamma = \frac{n}{m}$, for varying values of regularization strengths of $\lambda_1$ and $\lambda_2$.

6.2 $\ell^2_2$ Regularization

We plot the numerical and theoretically determined values of the generalization error for the random features model with $\ell^2_2$ regularization, we show these results in figure 1.

As seen in the figure, our mathematical technique is capable of predicting the expected generalization error with a high degree of accuracy. We also observe that increasing the strength of the regularization parameter diminishes the peak present at the interpolation threshold of $n = m$.

As we have pointed out, previous works reported a similar behavior in other setups concerning $\ell^2_2$ regularization d’Ascoli et al. (2020).

6.3 Elastic net model

6.3.1 Generalization Error

We compare the experimental and theoretically derived generalization error for a range of the regularization parameter $\lambda_1, \lambda_2$ in the elastic net model. In figure 2(a) and 2(b), we respectively plot the training error and the generalization error of the RF elastic net model as a function of $\gamma$, for several values of the the regularization parameters. We plot both the theoretical prediction, as the solid curve, and the corresponding averaged empirical values. We note that our expressions accurately predict the expected behavior of a network, the small deviation explained by the fact that $n, m$ are finite. In our experiments we set $n + m = 1000$. However, the discrepancy is only notable in a small range near the interpolation peak, suggesting the validity of our expressions in a wide range of networks of a non asymptotic size. We observe, in both plots, that small values of the regularization parameters leads to a spike in the generalization error at the interpolation threshold, which in this particular model, is slightly less than $\gamma = 1$. We further note that as the regularization parameter increases in strength, the interpolation peak diminishes. This is consistent with other results on the study of the double descent phenomenon d’Ascoli et al. (2020).

6.3.2 Sparsity

Our theory allows for an asymptotic prediction of the sparsity of the regularized RF solutions. We continue our focus on the elastic net model and plot $s$, the number of non-zero elements in the solution $\theta$, as a ratio with respect to $m$ and to $n$. We consider a variety of values of $\gamma = \frac{n}{m}$, and at various regularization strengths. The result is depicted in Figure 3(a) and 3(b), respectively.

As noted in the underparameterized regime $n >> m$, the ratio $s \frac{m}{n}$ is 1, implying a non-sparse solution. Once the interpolation threshold $\gamma \approx 1$ is reached the relative sparsity $s \frac{m}{n}$ drops, asymptotically approaching zero in the limit of $m >> n$. These results occurred for all values of $\lambda_2$, corresponding to the strength of the $\ell_1$ regularization. However, the strength of the regularization
Figure 2: Theoretically predicted (solid line) and numerically determined (markers) values of the training error (a) and generalization error (b) for the random features model with $\ell_1 + \ell_2$ regularization as a function of $1/\gamma = \frac{m}{n}$, for varying values of regularization strengths of $\lambda_1$ and $\lambda_2$.

Figure 3: Theoretically predicted value of the sparsity $s$ as a ratio to (a) $m$ the number of model parameters or (b) $n$ the number of data points for the random features model with $\ell_1 + \ell_2$ regularization as a function of $1/\gamma = \frac{m}{n}$, for varying values of regularization strengths of $\lambda_1$ and $\lambda_2$. 

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parameter $\lambda_2$ did affect the value of $\gamma$ at which this transition occurred. At larger values of $\lambda_2$, sparse solutions may be found even before the interpolation threshold.

With respect to $\frac{1}{n}$, the ratio of sparsity to the number of data points $\frac{s}{n}$ experiences a phase transition at around the interpolation threshold. The exact position depends on the regularization strength $\lambda_2$.

7 Conclusions

We propose a mathematical framework for the study of the random features (RF) model with generic regularization functions. We prove that RF models and Gaussian models with the same covariance in the feature matrix, converge to the same optimal solution and optimal value under mild conditions. We make use of the celebrated convex Gaussian min max theorem (CGMT) to show an equivalence between those Gaussian models and an alternative constrained minimization problem that is more readily solvable for a wide range of regularization functions. We consider two of separable regularization functions, namely $\ell_2^2$, ridge regression, as well as $\ell_1 + \ell_2^2$, elastic net regression, and give expected asymptotic expressions for the bias and variance of such problems. We further show numerically that our framework correctly predicts both training and testing error and show that these predictions have a high degree of accuracy even in the case of RF models of non asymptotic size.
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Zhang, C., Bengio, S., Hardt, M., Recht, B., and Vinyals, O. Understanding deep learning requires rethinking generalization. 2016. URL http://arxiv.org/abs/1611.03530 arxiv:1611.03530 Published in ICLR 2017.
A Proof of Theorem 4.2

We examine the problem as given in [3]
\[
\min_\nu \frac{1}{2n} ||\nu - X^T e||_2^2 + g \left( \lambda, e + \frac{\theta^*}{\sqrt{m}} \right).
\]  

In the subsequent analysis we will drop the regularization term \( g \left( \lambda, e + \frac{\theta^*}{\sqrt{m}} \right) \) for the sake of notational clarity as this term is not relevant for the primary derivation. We first introduce another maximization over \( a \)
\[
\min_\nu \max_\nu - \frac{1}{n} \langle a, \nu - X^T e \rangle - \frac{1}{2} ||a||_2^2.
\]  

We note that solving for \( a \) returns the minimization problem above. This optimization problem satisfies the conditions for the application of the Convex Gaussian Min Max Theorem (CGMT) [Thrampoulidis et al. 2014]. As shown in equation (14) the covariance of \( X^T \) is given by
\[
R(W) = E[X^T X] = \rho_0^2 11^T + \rho_1^2 I_m + \frac{\rho_2^2}{d} WW^T.
\]  

We let \( X = RU \) in which \( U \) is a matrix consisting entirely of i.i.d standard Gaussians, the application of the CGMT allow us to reexpress the problem as
\[
\min_\nu \max_\nu - \frac{1}{n} \langle a, \nu - R \rangle - ||R^{1/2} e||_2 \langle h, a \rangle - \frac{1}{2} ||a||_2^2,
\]  

in which \( g \) and \( h \) are standard Gaussian vectors of dimension \( m \) and \( n \) respectively. We note that by the properties of the CGMT that the two problems (36) and (39) will have asymptotically equivalent solutions. We let \( ||a||_2 = \beta \sqrt{n} \). We note that in this case the maximization may equivalently expressed as
\[
\max_a \equiv \max_{\beta > 0} \max_{a ||a||_2 = \beta \sqrt{n}}
\]  

Solving the inner maximization we obtain
\[
\min_\nu \max_{\beta > 0} - \frac{1}{\sqrt{n}} \left( -\beta \langle w, R^{1/2} g \rangle + \|\nu - ||R^{1/2} e||_2 \beta \|_2^2 \right) - \frac{\beta^2}{2}.
\]  

We now note that asymptotically,
\[
\|\nu - ||R^{1/2} e||_2 \beta \|_2 \rightarrow \frac{n}{\sqrt{2}} \sqrt{\sigma^2 + eRe}.
\]  

We further note that for any \( x \), the \( \sqrt{x} \) may be given by
\[
\sqrt{x} = \min_{q > 0} \frac{x}{2q} + \frac{q}{2}.
\]  

We plug this to the expression above, and obtain:
\[
\min_\nu \max_{\beta > 0} \min_{q > 0} - \frac{\beta}{\sqrt{n}} \langle e, R^{1/2} g \rangle + \frac{\beta}{2q} e^T Re + \frac{\beta \sigma^2}{2q} + \frac{\beta q}{2} - \frac{\beta^2}{2}.
\]  

We now substitute in the definition of \( R \), and we make use of the properties of normally distributed variables to obtain
\[
R^{1/2} g = \tilde{g} \sim N(0, R) = N(0, \rho_0^2 11^T + \rho_1^2 I + \rho_1^2/d WW^T)
\equiv \frac{\rho_0}{\sqrt{m}} 11^T \phi_0 + \rho_2 \phi_2 + \frac{\rho_1}{\sqrt{d}} W \phi_1
\]  

in which \( \phi_0 \) and \( \phi_2 \) are \( m \)-dimensional standard normals and \( \phi_1 \) is a \( d \) dimensional standard normal. Substituting we obtain:
\[
\min_\nu \max_{\beta > 0} \min_{q > 0} - \frac{\beta}{\sqrt{mn}} \langle 1^T e, 1^T \phi_0 \rangle - \frac{\beta \rho_2}{\sqrt{n}} \langle e, \phi_2 \rangle - \frac{\beta \rho_1}{\sqrt{md}} \langle W^T e, \phi_1 \rangle + \frac{\beta \rho_1^2}{2q} \|e\|_2^2
\]  
\[
+ \frac{\beta \rho_0^2}{2qm} (1^T e)^2 + \frac{\beta \rho_1^2}{2qd} \|W^T e\|_2^2 + \frac{\beta \sigma^2}{2q} + \frac{\beta q}{2} - \frac{\beta^2}{2}.
\]  

\[ \]
The majority of these terms are directly separable with respect to \( \textbf{e} \). In the following we focus only on the terms containing \( W^T \textbf{e} \), ie
\[
\frac{\beta \rho_1}{2 q d} \| W^T \textbf{e} \|_2^2 - \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, W^T \textbf{e} \rangle .
\]
(48)

First we complete the square to obtain
\[
\frac{\beta \rho_1^2}{2 q d} \left\| W^T \textbf{e} - \frac{q \sqrt{d}}{\rho_1 \sqrt{n}} \phi_1 \right\|_2^2 - \frac{\beta \rho_1}{2 n} \| \phi_1 \|_2^2.
\]
(49)

We note that in the large \( d \) limit the value of \( \| \phi_1 \|_2^2 \) approaches \( d \). We further introduce another maximization over \( p \)
\[
\min_{\textbf{p}} \max_{e} \frac{\beta \rho_1^2}{q d} \| \textbf{p} \|_2 \langle \phi_3, \textbf{e} \rangle + \frac{\beta \rho_1^2}{q d} \| \textbf{e} \|_2 \langle \phi_4, \textbf{p} \rangle - \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, \textbf{p} \rangle - \frac{\beta \rho_1^2}{2 q d} \| \textbf{p} \|_2^2 - \frac{\beta \rho_2}{2 n}.
\]
(50)

This optimization is once again in the form in which we might apply the CGMT. We note that \( W \) has a covariance matrix that is the identity matrix. Applying the CGMT we obtain
\[
\min_{\textbf{e}} \max_{\textbf{p}} \frac{\beta \rho_1^2}{q d} \| \textbf{p} \|_2 \langle \phi_3, \textbf{e} \rangle + \frac{\beta \rho_1^2}{q d} \| \textbf{e} \|_2 \langle \phi_4, \textbf{p} \rangle - \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, \textbf{p} \rangle - \frac{\beta \rho_1^2}{2 q d} \| \textbf{p} \|_2^2 - \frac{\beta \rho_2}{2 n},
\]
in which \( \phi_3, \phi_4 \) are \( m \)-dimensional and \( d \)-dimensional standard normals respectively. Once again we may split the optimization over \( p \) in the following manner
\[
\max_{\xi > 0} \max_{p, \|e\|_2 = \sqrt{\xi / \rho_1}} \langle \phi_3, \textbf{e} \rangle + \frac{\beta \rho_1^2}{q d} \| \textbf{e} \|_2 \langle \phi_4, \textbf{p} \rangle - \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, \textbf{p} \rangle - \frac{\beta \rho_1^2}{2 q d} \| \textbf{p} \|_2^2 - \frac{\beta \rho_2}{2 n}.
\]
(52)

The inner optimization may be solved to obtain:
\[
\min_{\xi > 0} \frac{\beta \rho_1}{q \sqrt{d}} \langle \phi_3, \textbf{e} \rangle + \left\| \frac{\beta \rho_1}{q \sqrt{d}} \| \textbf{e} \|_2 \langle \phi_4, \textbf{p} \rangle - \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, \textbf{p} \rangle \right\|_2 \xi - \frac{\beta \rho_1^2}{2 q} \| \textbf{e} \|_2^2 - \frac{\beta \rho_2}{2 n}.
\]
(53)

We note that in the limit of large \( d \) the 2-norm takes the following form
\[
\lim_{d \to \infty} \sqrt{\frac{\beta \rho_1^2}{2 q d} \| \textbf{e} \|_2^2 + \frac{\beta \rho_1}{\sqrt{d n}} \langle \phi_1, \textbf{p} \rangle \| \textbf{e} \|_2}
\]
(54)

We once again we replace this square root with another minimization with respect to \( r > 0 \) and return the terms that we ignored earlier to obtain:
\[
\min_{\xi > 0} \min_{\beta > 0} \min_{q > 0} \min_{r > 0} \left( \frac{\beta \rho_1 \xi}{2 q \sqrt{d}} + \frac{\beta \rho_1^2}{2 q} \right) \| \textbf{e} \|_2^2 + \frac{\beta \rho_1 \xi}{q \sqrt{d}} \langle \phi_3, \textbf{e} \rangle - \frac{\beta \rho_2}{\sqrt{d n}} \langle \phi_2, \textbf{e} \rangle
\] + \frac{\beta \rho_2}{2 q m} (1^T \textbf{e})^2 - \frac{\beta \rho_0}{\sqrt{m n}} (1^T \textbf{e}, 1^T \phi_0) + \frac{\beta \xi d}{2 r n} - \frac{\beta \xi}{2} - \frac{\xi r}{2} - \frac{\beta q d}{2 n} + \frac{\beta q}{2} - \frac{\beta^2}{2} + g \left( \lambda, \textbf{e} + \frac{\theta}{\sqrt{m}} \right).
\]
(55)

Finally we note that \( 1^T \phi_0 \) is a sum of \( m \) standard normals with unit variance. As such, making use of the properties of the sums of Gaussian random variables we represent it as \( \sqrt{m} \phi_0 \) where \( \phi_0 \sim \mathcal{N}(0, 1) \). We further note that \( (1^T \textbf{e})^2 \) is not separable, to deal with this we introduce a constraint of the form \( s \) such that
\[
\min_{\xi > 0} \min_{\beta > 0} \min_{q > 0} \min_{r > 0} \min_{s} \left( \frac{\beta \rho_1 \xi}{2 q \sqrt{d}} + \frac{\beta \rho_1^2}{2 q} \right) \| \textbf{e} \|_2^2 + \frac{\beta \rho_1 \xi}{q \sqrt{d}} \langle \phi_3, \textbf{e} \rangle - \frac{\beta \rho_2}{\sqrt{d n}} \langle \phi_2, \textbf{e} \rangle
\] + \frac{\beta \rho_2}{2 q m} s^2 - \frac{\beta \rho_0}{\sqrt{m}} s + \frac{\beta \xi d}{2 r n} - \frac{\beta \xi}{2} - \frac{\xi r}{2} - \frac{\beta q d}{2 n} + \frac{\beta q}{2} - \frac{\beta^2}{2} + g \left( \lambda, \textbf{e} + \frac{\theta}{\sqrt{m}} \right)
\] s.t.
\[
s = 1^T \textbf{e}.
\]
(56)
We may express this optimization in dual form with the Lagrange multiplier $\mu$ as

$$\max_{\beta > 0} \min_{q > 0} \min_{\xi > 0} \min_{r > 0} \min_{\mu, s, \rho} \max_{e, s, \rho} \left( \frac{\beta^2 \rho_1^2 \xi}{2q^2 r} + \frac{\beta \rho_2^2}{2q} \right) ||e||_2^2 + \frac{\beta \rho_1 \xi}{q \sqrt{d}} \langle \phi_1, e \rangle - \frac{\beta \rho_2}{\sqrt{n}} \langle e, \phi_2 \rangle + \frac{\beta \rho_0^2}{2qm} s^2 - \frac{\beta \rho_0 \phi_0}{\sqrt{n}} s + \frac{\beta^2 \xi d}{2q} - \frac{\xi d}{2q} + \frac{\xi d}{2q} + \frac{\beta \rho_1^2}{2q} + \frac{\beta \rho_2^2}{2q} - \frac{\beta^2}{2} + \mu (s - 1^T e) + g \left( \lambda, e + \frac{\theta^*}{\sqrt{m}} \right). \tag{57}$$

From this we obtain the inner optimization given in theorem 4.2. The values of $\beta, q, \xi, r$ must satisfy the relations given in theorem 4.2, which maybe obtained by taking the derivative of equation (57) with respect to the relevant parameter and setting it equal to zero. From this we obtain equations (18)-(21).

### A.1 Numerical Techniques

In the case that $g \left( \lambda, e + \frac{\theta^*}{\sqrt{m}} \right)$ is separable such that

$$g \left( \lambda, e + \frac{\theta^*}{\sqrt{m}} \right) = \sum_{i} g_i \left( \lambda, e_i + \frac{\theta^*}{\sqrt{m}} \right), \tag{59}$$

Then the inner optimization over $e, s, \mu$ may be solved explicitly, and substituted into the expression (57). The optimization over $\beta, q, \xi$ and $r$ may be done numerically. By noting that the original problem was convex and that each application of the CGMT maintained convex-concave relationships, each parameter $\beta, q, \xi, r$ will have a single minimum or maximum value. We may therefore use a method similar to Newtons method. For each parameter, we pick and upper and lower bound and substitute in those values. Whichever is smaller (minimization) or larger (maximization) remains fixed while the other bound is increased or decreased accordingly. After a number of iterations the program converges to the optimal solution. We solve the problem sequentially, first picking values of $\beta, q, \xi$ as constants, and then optimizing over $r$. Then using the results from the inner optimization to update the bounds for the outer optimization until we reach convergence.

### B Proof of Theorem 4.1

We note that the proof of Gaussian replacement in the case in which the elements of $X$ are i.i.d and centered has already been proven in Panahi & Hassibi (2017). In the present case, in which the elements of $X$ are correlated requires some modification. More specifically the majority of the proof may remain identical but lemma 3 of the supplementary material of Panahi & Hassibi (2017) needs to be changed. Lemma 3, is dependent on a definition for regular and scaled regular vector, given by Panahi & Hassibi (2017, Supplement, Definition 1), which we reproduce here

**Definition B.1.** We call a random matrix regular if it consists of independent and identical distributed entries with vanishing first, third, and fifth moments, unit variance, and finite fourth and sixth moments. We call an $m \times n$ matrix $A$ scaled regular if it can be written as $A = A' / \sqrt{m}$, where $A'$ is regular.

The lemma Panahi & Hassibi (2017, Supplement, Lemma 3) makes the following claim
Lemma B.2. Suppose that the $m$-dimensional vector $\sqrt{m}h$ has i.i.d regular entries. For any $m \times m$ matrix $S$, vector $p$ and $a = 1, 2, 3$

$$\mathbb{E}[|p^T h|^{2a}] \leq \frac{(2a)!}{a!} \frac{||p||^{2a}}{m} \mathbb{E}[|h|^{2a}]$$

(60)

$$\mathbb{E}[h^T S h] = \frac{T_r(S)}{m}$$

(61)

$$\mathbb{E}[||h^T S h||^2] \leq \left( \frac{T_r(S^2)}{m^2} \right)^2 + \frac{2T_r(S^4)}{m^4} \mathbb{E}[|h|^4]$$

(62)

We consider the feature matrix $X$ and focus on its scaled counterpart $X = \sqrt{m}X'$. We consider $h$ to given by $x$ in the present case, with $x$ being defined to be a scaled row of $X'$, such that

$$x_i = \frac{1}{\sqrt{m}}\sigma((z, w_i)) \equiv \frac{\rho_0}{\sqrt{m}} + \frac{\rho_1}{\sqrt{dm}} (z, w_i) + \frac{\rho_2}{\sqrt{m}} \hat{w}_i \rightarrow \frac{1}{\sqrt{m}} (x_i^0 + x_i^1 + x_i^2)$$

(63)

$$x_i^0 = \rho_0, \quad x_i^1 = \frac{\rho_1}{\sqrt{d}} (z, w_i), \quad x_i^2 = \rho_2 \hat{w}_i$$

(64)

in which $z, w_i, \hat{w}_i$ are all standard Gaussian Normals of dimension $d, d$ and 1 respectively. This Gaussian covariate model is equivalent in the asymptotic limit, as was show by Mei & Montanari (2019). We note that in this case $x_i^0$ for $i = 1, 2$ has vanishing first, third and fifth moments, and finite second, fourth and sixth moments. More specifically we note $x_i^2$ is regular except for its non unit variance, and that

$$\mathbb{E}[x_i^1 x_i^1^T] = \frac{\rho_1^2}{d} \mathbb{E}_w \mathbb{E}_z W z z^T W = \rho_1^2 I$$

(65)

$$\mathbb{E}[x_i^2 x_i^2^T] = \rho_2^2 I$$

(66)

We may now restate the lemma as follows

Lemma B.3. Let $x$ be an $m$-dimensional vector given in [65]. For any matrix $S$, vector $p$ and $a = 1, 2, 3$

$$\mathbb{E}[|p^T x|^{2a}] \leq \frac{(2a)!}{a!} \frac{||p||^{2a}}{m} \mathbb{E}[(x_1^2)^{2a}] \mathbb{E}[(x_1^2)^{2a}] + \frac{(2a)!}{a!} \frac{||p||^{2a}}{m} \left( \mathbb{E}[|x_1^2|^{2a}] + \mathbb{E}[|x_1^4|^{2a}] \right)$$

(67)

$$\mathbb{E}[h^T S h] \leq \frac{\rho_1^2 + \rho_2^2}{m} \frac{T_r(S)}{m} + \rho_2^2 ||S||_F$$

(68)

$$\mathbb{E}[||h^T S h||^2] \leq \rho_0^2 ||S||_F + \left( \frac{T_r(S^2)}{m^2} \right)^2 (\rho_1^2 + \rho_2^2 + 6\rho_1^2 \rho_2^2) + 2 \frac{T_r(S^4)}{m^4} (\mathbb{E}[|x_1^4|^4] + \mathbb{E}[|x_1^2|^4] + 6\rho_1^2 \rho_2^2) + 6 \frac{T_r(S^4)}{m^4} \mathbb{E}||S||_F^4 \rho_0^2 (\rho_1^2 + \rho_2^2)$$

(69)

Proof. For the first part we first multinomially expand

$$\mathbb{E}[|p^T x|^{2a}] = \frac{1}{m^a} \sum_{i+j=2a} \binom{2a}{i,j} \mathbb{E}(p^T x)^i (p^T x)^j (p^T x)^{2a-i-j}$$

(70)

$$= \frac{1}{m^a} \sum_{i, \text{even}} \binom{2a}{i} \mathbb{E}(p^T x)^i (p^T x)^{2a-i} + \mathbb{E}(p^T x)^{2a-i} + \frac{1}{m^a} \sum_{i, \text{even}} \binom{2a}{i} \mathbb{E}(p^T x)^i (p^T x)^{2a-i}$$

where we have made use of the fact that $x_1$ and $x_2$ are independent, and that odd powers of $(p^T x_1)$ and $(p^T x_2)$ have zero expectation.

The proof of the lemma above only relies on the vanishing first, third, and fifth moments, as
such for the second term we may upper bound it using the lemma above.

\[
\frac{1}{m^a} \sum_{i, \text{even}} 2a \binom{2a}{i} E(p^T x^i)(p^T x^{2a-i}) \leq \frac{1}{m^a} \sum_{i, \text{even}} 2a \binom{2a}{i} \frac{|p|^2}{m} E(|x_i|^4) + \frac{(2a - i)!}{((2a - i)/2)! \times 2^{2a-i}} |p|^2 \rho E(|x_i^{2a-i}|) \leq \frac{(2a)!}{a!} |p|^2 \rho E(|x_i^{2a}|) \rho E(|x_i^{2a}|) (71)
\]

For the first part we note that \( p^T x^0 = 1^T p \leq |p| \leq |p| \sqrt{m} \). The first term may therefore be upper bounded by

\[
\frac{1}{m^a} \sum_{i, \text{even}} 2a \binom{2a}{i} (p^T x^0)^i (p^T x^{2a-i}) \leq \sum_{i, \text{even}} 2a \binom{2a}{i} |p|^2 i \frac{(i)!}{(i/2)! \times 2^i} |p|^2 \rho E(|x_i^{2a-i}|) + \frac{(2a - i)!}{((2a - i)/2)! \times 2^{2a-i}} |p|^2 \rho E(|x_i^{2a-i}|) \leq \frac{(2a)!}{a!} |p|^2 \rho E(|x_i^{2a}|) E(|x_i^{2a}|) (72)
\]

for the second part we note that

\[
E[\text{Tr}(S^2) = \text{Tr}(SE[xx^T])] = (\frac{\rho_1^2 + \rho_2^2}{m}) + \frac{\rho_0^2}{m} \text{Tr}(11^T S) \leq (\frac{\rho_1^2 + \rho_2^2}{m}) + \rho_0^2 \|S\|_F (73)
\]

Finally for the third part we make use of the independence of each \( x_i \), and make use of the lemma above to obtain

\[
E[\|x^T S x\|^2] = E[(x^0^T S x^0)^2 + (x^1^T S x^1)^2] + 2(x^0^T S x^0) (x^1^T S x^1) + 2(x^0^T S x^2) + 2(x^1^T S x^2) + 4(x^0^T S x^1) + 4(x^1^T S x^1) + 4(x^0^T S x^2)^2 + (x^1^T S x^2)^2] \\
= \rho_0^2 \|S\|_F^2 + \left( \frac{\text{Tr}(S^2)}{m^2} \right)^2 \rho_0^4 + \rho_0^2 + 6 \rho_0^2 \rho_2^2 + 2 \left( \frac{\text{Tr}(S^4)}{m^4} \right) (E[|x_i|^4] + E[(x_i^2)^4] + 6 E[(x_i^1)^2] E[(x_i^2)^2]) + 6 \left( \frac{\text{Tr}(S^2)}{m^2} \right) \rho_0^2 (\rho_1^2 + \rho_2^2) (74)
\]

The change from the lemma \([B.2]\) to lemma \([B.3]\) is twofold. First each of the terms picks up a bias term proportional to \( \rho_0 \), and secondarily the constants with respect to the terms proportional to \( O(\frac{1}{m^2}) \) have changed. This second change does not change the efficacy of the original proof as the equivalence is taken in the asymptotic limit.

With respect to the bias terms, the proof provided by [Panahi & Hassibi (2017)] relies comparing \( X \) to \( \bar{X} \), in which \( \bar{X} \) is a gaussian matrix with the same covariance as \( X \). Noting that these expressions will have identical bias term results in these terms proportional to \( \rho_0 \) being eliminated during the comparison. More specifically, these terms are eliminated in equations 58 and 59, in Lemma 1 of the supplementary material of [Panahi & Hassibi (2017)].

### C Example cases

Here we derive the asymptotic behavior for a number of choices for the regularization function \( g(\lambda, \theta) \), we compute the expected solution \( \theta \) and expressions for the Bias and Variance. Total generalization error may easily be constructed from the solution, noting that generalization error may be decomposed into bias, variance and noise terms. In the following we assume that the activation function is centered around zero such that \( \rho_0 = 0 \), as is the case in functions like \( \tanh(x) \). For non zero \( \rho_0 \) additional bias terms must be computed.
C.1 L2 regularization

In this case the inner optimization takes the form
\[
\min_{\mathbf{e}} c_1 \|\mathbf{e}\|_2^2 + \frac{c_2}{\sqrt{m}} \langle \varphi, \mathbf{e} \rangle + \lambda \left\| \mathbf{e} + \frac{\theta_i^*}{\sqrt{m}} \right\|_2.
\] (75)

This optimization is separable such that
\[
\sum_{i=1}^{m} \min_{\mathbf{e}_i} c_1 \mathbf{e}_i^2 + \frac{c_2}{\sqrt{m}} \varphi_i \mathbf{e}_i + \lambda \left( \mathbf{e}_i + \frac{\theta_i^*}{\sqrt{m}} \right)^2
= \sum_{i=1}^{m} \min_{\mathbf{e}_i} (c_1 + \lambda) \mathbf{e}_i^2 + \frac{c_2 \varphi_i \sqrt{m} + 2 \lambda \theta_i^* \sqrt{m}}{\sqrt{nm}} \mathbf{e}_i + \frac{\lambda \theta_i^{*2}}{m}.
\] (76)

Completing the square we obtain
\[
\sum_{i=1}^{m} \min_{\theta_i} (c_1 + \lambda) \left( \mathbf{e}_i + \frac{c_2 \varphi_i \sqrt{m} + 2 \lambda \theta_i^* \sqrt{m}}{2 \sqrt{n}(c_1 + \lambda)} \right)^2 + \frac{\lambda \theta_i^{*2}}{m} - \frac{(c_2 \varphi_i \sqrt{m} + 2 \lambda \theta_i^* \sqrt{m})^2}{4(c_1 + \lambda)nm}.
\] (77)

Substituting in the definition of \( \mathbf{e}_i = \theta_i^* - \frac{\theta_i^*}{\sqrt{m}} \). We note that the optimization is minimized when
\[
\theta_i = \frac{c_1 \theta_i^*}{\sqrt{m}(c_1 + \lambda)} - \frac{c_2 \varphi_i}{2 \sqrt{n}(c_1 + \lambda)}.
\] (78)

From this we can determine the Bias and Variance to be
\[
\mathcal{B} = \frac{\lambda}{\sqrt{m}(c_1 + \lambda)} \mathbb{E}_{\mathbf{x}, \theta^*} \left[ \left\| \mathbf{x}^T \theta_i^* \right\|^2 \right]
\] (79)
\[
\mathcal{V} = \frac{c_2}{2 \sqrt{n}(c_1 + \lambda)} \mathbb{E}_{\mathbf{x}} \left[ \left\| \mathbf{x}^T \varphi \right\|^2 \right]
\] (80)

C.2 L1 Regularization

The inner optimization takes the form of
\[
\min_{\mathbf{e}} c_1 \|\mathbf{e}\|_2^2 + \frac{c_2}{\sqrt{m}} \langle \varphi, \mathbf{e} \rangle + \lambda \left\| \mathbf{e} + \frac{\theta_i^*}{\sqrt{m}} \right\|_1.
\] (81)

Note that the regularization factor has been normalized in this case. This optimization is once again separable such that
\[
\sum_{i=1}^{m} \min_{\mathbf{e}_i} c_1 \mathbf{e}_i^2 + \frac{c_2}{\sqrt{m}} \varphi_i \mathbf{e}_i + \lambda \left\| \mathbf{e}_i + \frac{\theta_i^*}{\sqrt{m}} \right\|_1.
\] (82)

We once again complete the square and replace \( \mathbf{e}_i = \theta_i^* - \frac{\theta_i^*}{\sqrt{m}} \)
\[
\sum_{i=1}^{m} \min_{\theta_i} c_1 \left( \theta_i^* - \frac{\theta_i^*}{\sqrt{m}} + \frac{c_2 \varphi_i}{2 c_1 \sqrt{n}} \right)^2 - \frac{c_2 \varphi_i^2}{4 c_1 n} + \frac{\lambda \theta_i^* |\theta_i^*|}{\sqrt{m}}.
\] (83)

This equation may be solved using a soft thresholding operator resulting in the solution
\[
\theta_i = \begin{cases} 
0 & \frac{\lambda}{\sqrt{m}} \theta_i^* - \frac{c_2 \varphi_i}{2 c_1 \sqrt{n}} - \frac{\lambda}{2 c_1 \sqrt{m}} \theta_i^* < \frac{\lambda}{2 c_1 \sqrt{m}} \\
\frac{\lambda}{\sqrt{m}} \theta_i^* - \frac{c_2 \varphi_i}{2 c_1 \sqrt{n}} - \frac{\lambda}{2 c_1 \sqrt{m}} \theta_i^* > \frac{\lambda}{2 c_1 \sqrt{m}} \\
\theta_i^* \theta_i^* > \frac{\lambda}{2 c_1 \sqrt{m}} \\
\theta_i^* \theta_i^* \theta_i^* \theta_i^* < \frac{\lambda}{2 c_1 \sqrt{m}}
\end{cases}
\] (84)

we define the following values:
\[
\alpha_{i1} = \frac{\lambda - 2 c_1 \theta_i^*}{c_2} \sqrt{\gamma} \quad \alpha_{i2} = \frac{\lambda + 2 c_1 \theta_i^*}{c_2} \sqrt{\gamma}
\] (85)
and note that this allows us to express the solution as

\[
\theta_i = \begin{cases} 
0 & -\alpha_{11} \leq \varphi_i \leq \alpha_{12} \\
\frac{\alpha_{21}}{\sqrt{m}} - e^{2\varphi_i} - \frac{\alpha_{22}}{\sqrt{m}} & \varphi_i < -\alpha_{11} \\
\frac{\alpha_{21}}{\sqrt{m}} + e^{2\varphi_i} + \frac{\alpha_{22}}{\sqrt{m}} & \varphi_i > \alpha_{21} 
\end{cases}
\]  

(86)

Or written more succinctly,

\[
\theta_i = \mathcal{T}_{\frac{\alpha_{21}}{\sqrt{m}}} \left( \frac{\theta_i}{\sqrt{m}} - e^{2\varphi_i} \right),
\]

(87)

in which \( \mathcal{T} \) is the soft thresholding operator. We now examine the \( E\theta \) we note that

\[
E\theta_i = E_{\varphi_i} \left[ \left( \frac{\theta_i}{\sqrt{m}} - e^{2\varphi_i} \right) \right] = \frac{\lambda}{2c_1 \sqrt{m}} \left( 1 - Q(\alpha_{11}) - Q(\alpha_{12}) \right) + \frac{\lambda}{2c_1 \sqrt{m}} (Q(\alpha_{11}) - Q(\alpha_{12})) + \frac{c_2}{2c_1 \sqrt{n}} \frac{1}{\sqrt{2 \pi}} e^{-\alpha_{11}^2/2 - \alpha_{12}^2/2} ,
\]

(88)

In which \( Q(\cdot) \) is the Q-function. Let

\[
\tilde{\theta}_i = \theta_i - E\theta_i = \frac{\theta_i}{\sqrt{m}} - E\theta_i = \frac{\lambda}{2c_1 \sqrt{m}} (1 - Q(\alpha_{11}) - Q(\alpha_{12})) + \frac{\lambda}{2c_1 \sqrt{m}} (Q(\alpha_{11}) - Q(\alpha_{12})) + \frac{c_2}{2c_1 \sqrt{n}} \frac{1}{\sqrt{2 \pi}} e^{-\alpha_{11}^2/2 - \alpha_{12}^2/2} ,
\]

(90)

and furthermore let

\[
\tilde{\tilde{\theta}}_i = \tilde{\theta}_i - E\tilde{\theta}_i = \frac{\lambda}{2c_1 \sqrt{m}} (1 - Q(\alpha_{11}) - Q(\alpha_{12})) + \frac{\lambda}{2c_1 \sqrt{m}} (Q(\alpha_{11}) - Q(\alpha_{12})) + \frac{c_2}{2c_1 \sqrt{n}} \frac{1}{\sqrt{2 \pi}} e^{-\alpha_{11}^2/2 - \alpha_{12}^2/2} .
\]

(91)

The Bias and Variance are now given by

\[
\mathcal{B} = E_{\mathbf{x},\theta^*} \| x^T \theta \|_2^2 \\
\mathcal{V} = E_{\mathbf{x},\theta^*} \| x^T \tilde{\theta} \|_2^2
\]

(92)

(93)

**C.2.1 Sparsity**

The effect of \( \ell_1 \) regularization is to promote sparse solution. Let \( s \) denote the number of elements of \( \theta \) that are non-zero, we note then that:

\[
s = E \sum_i 1_{\theta_i \neq 0} = \sum_i 1 - 1 = \sum_i 1 - 1 = \sum_i 1 - 1 = \sum_i 1 - 1 = \sum_i Q(\alpha_{11}) + Q(\alpha_{12})
\]

(94)

We further examine the term \( \frac{c_2}{\sqrt{m}} (\bar{e}, \varphi) \) we note that its expectation value is given by

\[
\frac{c_2}{\sqrt{m}} E[(\bar{e}, \varphi)] = \frac{c_2}{\sqrt{m}} \sum_i E_{\varphi_i} \theta_i = \frac{c_2}{\sqrt{m}} \sum_i E_{\varphi_i} \theta_i = \frac{c_2}{2c_1 \sqrt{n}} \sum_i E_{\mathcal{T}_i}
\]

(95)
where in the final equality we have made use of stein’s lemma, and where \( T' \) is the derivative of the soft thresholding operator, given in [87], with respect to its argument. We now note that the derivative of the soft thresholding operator is the same as our definition for sparsity above, symbolically

\[
1_{\{a \neq 0\}} T_i = T'_i.
\] (96)

As such we obtain that

\[
E[F] \equiv \frac{c_2}{\sqrt{n}} E(e, \varphi) = -\frac{c_2 s}{2c_1 n}.
\] (97)

We also note that \( F \) may equivalently be expressed as

\[
F = \sum_{i} F_i, \quad F_i = \begin{cases} 
\frac{c_2}{\sqrt{m}} \theta_i^* \varphi_i & -\alpha_i \leq \varphi_i \leq \alpha_i \\
\frac{c_2}{2mc_1} - \frac{c_2 \lambda \varphi_i}{2mc_1} & \varphi_i < -\alpha_i \\
\frac{c_2}{2mc_1} + \frac{c_2 \lambda \varphi_i}{2mc_1} & \varphi_i > \alpha_i,
\end{cases}
\] (98)

Which may be computed numerically. As such we may express sparsity as

\[
\frac{s}{n} = -\frac{2c_1}{c_2^2} E[F] \quad \frac{s}{m} = \frac{\gamma s}{n},
\] (99)

\section{C.3 L1 + L2 Regularization}

The inner optimization takes the form of

\[
\min_{\mathbf{e}} c_1 ||\mathbf{e}||^2_2 + \frac{c_2}{\sqrt{n}} \langle \varphi, \mathbf{e} \rangle + \lambda_1 \left||\mathbf{e} + \frac{\theta^*}{\sqrt{m}}\right||^2_2 + \lambda_2 \left||\mathbf{e} + \frac{\theta^*}{\sqrt{m}}\right||^2_1.
\] (100)

Note that the regularization factor with respect to the \( \ell_1 \) norm has been normalized in this case. This optimization is once again separable

\[
\sum_{i} \min_{\mathbf{e}_i} c_1 ||\mathbf{e}_i||^2_2 + \frac{c_2}{\sqrt{n}} \varphi_i \mathbf{e}_i + \lambda_1 \left||\mathbf{e}_i + \frac{\theta^*_i}{\sqrt{m}}\right||^2_2 + \lambda_2 \left||\mathbf{e}_i + \frac{\theta^*_i}{\sqrt{m}}\right||^2_1.
\] (101)

Completing the square and substituting the definition of the error vector we obtain

\[
\sum_{i} \min_{\boldsymbol{\theta}_i} \left( c_1 + \lambda_1 \right) \left( \boldsymbol{\theta}_i - \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} + \frac{c_2 \varphi_i}{2\sqrt{m}(c_1 + \lambda_1)} \right)^2 + \lambda_1 \frac{\theta^*_i}{m} + \lambda_2 \frac{\theta^*_i}{m}.
\] (102)

We may solve this for \( \theta_i \) to obtain

\[
\theta_i = \begin{cases} 
0 & \frac{c_2 \varphi_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \leq \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \\
\frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{c_2 \varphi_i}{2\sqrt{m}(c_1 + \lambda_1)} & \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \leq \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \leq \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \\
\lambda_2 & \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \leq \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \leq \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)}
\end{cases}
\] (103)

We make use of the definitions of \( \alpha_{11} \) and \( \alpha_{12} \) above. From this we determine the expectation value of \( \theta \). We find that

\[
E[\theta] = E \left[ \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{c_2 \varphi_i}{2\sqrt{m}(c_1 + \lambda_1)} - \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right) 1_{\{\varphi > \alpha_{11}\}} \right. \\
\left. + \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{c_2 \varphi_i}{2\sqrt{m}(c_1 + \lambda_1)} + \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right) 1_{\{\varphi < \alpha_{12}\}} \right] + Q(\alpha_{11}) \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} + \lambda_2 \right) + Q(\alpha_{12}) \left( -\frac{c_2 \varphi_i}{2\sqrt{m}(c_1 + \lambda_1)} + \lambda_2 \right)
\] (104)

\[
= Q(\alpha_{11}) \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right) + Q(\alpha_{12}) \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} + \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right)
\] (105)
Where we have made use of Steins lemma. We may once again define $\theta = \theta^* - \mathbb{E}\theta$ and $\bar{\theta} = \theta - \mathbb{E}\theta$. The exact expressions are given by:

$$
\bar{\theta}_i = \frac{\theta^*_i}{\sqrt{m}} \left( 1 - \frac{c_1 Q(\alpha_{i1})}{c_1 + \lambda_1} - \frac{c_1 Q(\alpha_{i2})}{c_1 + \lambda_1} \right) + \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} (Q(\alpha_{i2}) - Q(\alpha_{i1})) - \frac{c_2}{2\sqrt{m}(c_1 + \lambda_1)} \frac{1}{\sqrt{2\pi}} \left[ e^{\alpha_{i1}^2/2} - e^{\alpha_{i2}^2/2} \right]
$$

(106)

$$
\bar{\theta}_i = \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} - \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right) (1_{\varphi_i > \alpha_{i1}} - Q(\alpha_{i1})) + \left( \frac{c_1 \theta^*_i}{\sqrt{m}(c_1 + \lambda_1)} + \frac{\lambda_2}{2\sqrt{m}(c_1 + \lambda_1)} \right) (1_{\varphi_i < -\alpha_{i2}} - Q(\alpha_{i2})) - \frac{c_2}{2\sqrt{m}(c_1 + \lambda_1)} (\varphi_i + \frac{1}{\sqrt{2\pi}} \left[ e^{\alpha_{i1}^2/2} - e^{\alpha_{i2}^2/2} \right])
$$

(107)

the bias and variance then once again given by

$$
B = \mathbb{E}_{x, \theta^*} ||x^T \bar{\theta}||^2_2
$$

(108)

$$
V = \mathbb{E}_{x, \theta^*} ||x^T \bar{\theta}||^2_2
$$

(109)

C.3.1 Sparsity

The value of $s$, the number of nonzero elements in $\theta$ is given by equation (94) as in the case of $L1$ regularization and the ratios $s/n$ and $s/m$ are obtained in the same manner as discussed in C.2.1, we obtain

$$
\frac{s}{n} = - \frac{2(c_1 + \lambda_1)}{c_2^2} \mathbb{E}[F] \quad \frac{s}{m} = \gamma \frac{s}{n}
$$

(110)

Where in this case $F$ is given by

$$
F = \sum_{i} F_i = \begin{cases} 
- \frac{c_2}{\sqrt{mn}} \theta^*_i \varphi_i & -\alpha_{i1} \leq \varphi_i \leq \alpha_{i2} \\
- \frac{c_2^2 \varphi_i^2}{2n(c_1 + \lambda_1)} - \frac{c_2 \lambda_1 \theta_i \varphi_i}{\sqrt{nm(c_1 + \lambda_1)}} - \frac{c_2 \lambda_2 \varphi_i}{2\sqrt{nm(c_1 + \lambda_1)}} & \varphi_i < -\alpha_{i1} \\
- \frac{c_2^2 \varphi_i^2}{2n(c_1 + \lambda_1)} + \frac{c_2 \lambda_1 \theta_i \varphi_i}{\sqrt{nm(c_1 + \lambda_1)}} + \frac{c_2 \lambda_2 \varphi_i}{2\sqrt{nm(c_1 + \lambda_1)}} & \varphi_i > \alpha_{i2}
\end{cases}
$$

(111)