Slightly Superexponential Parameterized Problems

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Abstract
A central problem in parameterized algorithms is to obtain algorithms with running time \( f(k) \cdot n^{O(1)} \) such that \( f \) is as slow growing function of the parameter \( k \) as possible. In particular, the first natural goal is to make \( f(k) \) single-exponential, that is, \( c^k \) for some constant \( c \). This has led to the development of parameterized algorithms for various problems where \( f(k) \) appearing in their running time is of form \( 2^{O(k)} \). However there are still plenty of problems where the “slightly superexponential” \( f(k) \) appearing in the best known running time has remained non single-exponential even after a lot of attempts to bring it down. A natural question to ask is whether the \( f(k) \) appearing in the running time of the best-known algorithms is optimal for any of these problems.

In this paper, we examine parameterized problems where \( f(k) \) is \( k^{O(k)} = 2^{O(k \log k)} \) in the best known running time and for a number of such problems, we show that the dependence on \( k \) in the running time cannot be improved to single exponential. More precisely we prove following tight lower bounds, for three natural problems, arising from three different domains:

- The pattern matching problem Closest String is known to be solvable in time \( 2^{O(d \log d)} \cdot n^{O(1)} \) and \( 2^{O(d \log |Σ|)} \cdot n^{O(1)} \). We show that there is no \( 2^{o(d \log d)} \cdot n^{O(1)} \) time algorithm, unless Exponential Time Hypothesis (ETH) fails.
- The graph embedding problem Distortion, that is, deciding whether a graph \( G \) has a metric embedding into the integers with distortion at most \( d \) can be done in time \( 2^{O(d \log d)} \cdot n^{O(1)} \). We show that there is no \( 2^{o(d \log d)} \cdot n^{O(1)} \) time algorithm, unless ETH fails.
- The Disjoint Paths problem can be solved in time \( 2^{O(w \log w)} \cdot n^{O(1)} \) on graphs of treewidth at most \( w \). We show that there is no \( 2^{o(w \log w)} \cdot n^{O(1)} \) time algorithm, unless ETH fails.

To obtain our result we first prove the lower bound for variants of basic problems: finding cliques, independent sets, and hitting sets. These artificially constrained variants form a good starting point for proving lower bounds on natural problems without any technical restrictions and could be of independent interest. We believe that many further results of this form can be obtained by using the framework of the current paper.

1 Introduction
The goal of parameterized complexity is to find ways of solving NP-hard problems more efficiently than brute force: here aim is to restrict the combinatorial explosion to a parameter that is hopefully much smaller than the input size. Formally, a parameterization of a problem is assigning an integer \( k \) to each input instance and we say that a parameterized problem is fixed-parameter tractable (FPT) if there is an algorithm that solves the problem in time \( f(k) \cdot |I|^{O(1)} \), where \( |I| \) is the size of the input and \( f \) is an arbitrary computable function depending on the parameter \( k \) only. There is a long list of NP-hard problems that are FPT under various parameterizations: finding a vertex cover of size \( k \), finding a cycle of length \( k \), finding a maximum independent set in a graph of treewidth at most \( k \), etc. For more background, the reader is referred to the monographs [24, 28, 51].

The practical applicability of fixed-parameter tractability results depends very much on the form of the function \( f(k) \) in the running time. In some cases, for example in results obtained from Graph Minors theory, the function \( f(k) \) is truly horrendous (towers of exponentials), making the result purely of theoretical interest. On the other hand, in many cases \( f(k) \) is a moderately growing exponential function: for example, \( f(k) = 1.2738^k \) in the current fastest algorithm for finding a vertex cover of size \( k \) [10], which can be further improved to \( 1.1616^k \) in the special case of graphs with maximum degree 3 [57]. For some problems, \( f(k) \) can be even subexponential (e.g., \( c^{\sqrt{k}} \) ) [18, 17, 16, 1].

The implicit assumption in the research on fixed-parameter tractability is that whenever a reasonably natural problem turns out to be FPT, then we can improve \( f(k) \) to \( c^k \) with some small \( c \) (hopefully \( c < 2 \)) if we work on the problem hard enough. Indeed, for some basic problems, the current best running time was obtained after a long sequence of incremental

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improvements. However, it is very well possible that for some problems there is no algorithm with single-exponential \( f(k) \) in the running time.

In this paper, we examine parameterized problems where \( f(k) \) is “slightly superexponential” in the best known running time: \( f(k) \) is of the form \( k^{O(k)} = 2^{O(k \log k)} \). Algorithms with this running time naturally occur when a search tree of height at most \( k \) and branching factor at most \( k \) is explored, or when all possible permutations, partitions, or matchings of a \( k \) element set are enumerated. For a number of such problems, we show that the dependence on \( k \) in the running time cannot be improved to single exponential. More precisely, we show that a \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) time algorithm for these problems would violate the Exponential Time Hypothesis (ETH): the assumption that there is no \( 2^{o(n)} \)-time algorithm for \( n \)-variable 3SAT [37].

In the first part of the paper, we prove the lower bound for variants of basic problems: finding cliques, independent sets, and hitting sets. These variants are artificially constrained such that the search space is of size \( 2^{O(k \log k)} \) and we prove that a \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) time algorithm would violate ETH. The results in this section demonstrate that for some problems the natural \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) upper bound on the search space is actually a tight lower bound on the running time. More importantly, the results on these basic problems form a good starting point for proving lower bounds on natural problems without any technical restrictions.

In the second part of the paper, we use our results on the basic problems to prove tight lower bounds for three natural problems from three different domains:

- The pattern matching problem CLOSEST STRING is known to be solvable in time \( 2^{O(d \log d)} \cdot |I|^{O(1)} \) [33] and \( 2^{O(d \log |\Sigma|)} \cdot |I|^{O(1)} \) [46]. We show that there is no \( 2^{o(d \log d)} \cdot |I|^{O(1)} \) and \( 2^{o(d \log |\Sigma|)} \cdot |I|^{O(1)} \) time algorithm, unless ETH fails.

- The graph embedding problem DISTORTION, that is, deciding whether a \( n \) vertex graph \( G \) has a metric embedding into the integers with distortion at most \( d \) can be done in time \( 2^{O(d \log d)} \cdot n^{O(1)} \) [27]. We show that there is no \( 2^{o(d \log d)} \cdot n^{O(1)} \) time algorithm, unless ETH fails.

- The DISJOINT PATHS problem can be solved in time \( 2^{O(w \log w)} \cdot n^{O(1)} \) on \( n \) vertex graphs of treewidth at most \( w \) [54]. We show that there is no \( 2^{o(w \log w)} \cdot n^{O(1)} \) time algorithm, unless ETH fails.

We remark that the algorithm given in [54] does not mention the running time for DISJOINT PATHS as \( 2^{O(w \log w)} \cdot n^{O(1)} \) on graphs of bounded treewidth but a closer look reveals that it is indeed the case. We expect that many further results of this form can be obtained by using the framework of the current paper. Thus parameterized problems requiring “slightly superexponential” time \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) is not a shortcoming of algorithm design or pathological situations, but an unavoidable feature of the landscape of parameterized complexity.

It is important to point out that it is a real possibility that some \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) time algorithm can be improved to single-exponential dependence with some work. In fact, there are examples of well-studied problems where the running time was “stuck” at \( 2^{O(k \log k)} \cdot |I|^{O(1)} \) for several years before some new algorithmic idea arrived that made it possible to reduce the dependence to \( 2^{O(k)} \cdot |I|^{O(1)} \):  

- In 1985, Monien [48] gave a \( k! \cdot n^{O(1)} \) time algorithm for finding a cycle of length \( k \) in a graph on \( n \) vertices. Alon, Yuster, and Zwick [2] introduced the color coding technique in 1995 and used it to show that a cycle of length \( k \) can be found in time \( 2^{O(k)} \cdot n^{O(1)} \).

- In 1995, Eppstein [25] gave an \( O(k^k n) \) time algorithm for deciding if a \( k \)-vertex planar graph \( H \) is a subgraph of an \( n \)-vertex planar graph \( G \). Very recently, Dorn [21] gave an improved algorithm with running time \( 2^{O(k)} \cdot n \). One of the main technical tools in this result is the use of sphere cut decompositions of planar graphs, which was used earlier to speed up algorithms on planar graphs in a similar way [22].

- In 1995, Downey and Fellows [23] gave a \( kO(k) \cdot n^{O(1)} \) time algorithm for FEEDBACK VERTEX SET (given an undirected graph \( G \) on \( n \) vertices, delete \( k \) vertices to make it acyclic). A randomized \( 4k \cdot n^{O(1)} \) time algorithm was given in 1999 [7, 6]. The first deterministic \( 2^k \cdot n^{O(1)} \) time algorithms appeared only in 2005 [35, 15, 14], using the technique of iterative compression introduced by Reed et al. [52].

As we can see in the examples above, achieving single exponential running time often requires the invention of significant new techniques. Therefore, trying to improve the running time for a problem whose best known parameterized algorithm is slightly superexponential can lead to important new discoveries and developments. However, in this paper we identify problems for which such an improvement is very unlikely. The \( 2^{O(k \log k)} \) dependence on \( f(k) \) seems to be inherent to these problems and one should not waste too much time on trying to achieve single-exponential dependence.
There are some lower bound results on FPT problems in the parameterized complexity literature, but not of the form that we are proving here. Cai and Juedes [9] proved that if the parameterized version of a MAXSNP-complete problems (such as VERTEX COVER on graphs of maximum degree 3) can be solved in time $2^{o(k)} \cdot |I|^{O(1)}$, then ETH fails. Using parameterized reductions, this result can be transferred to other problems: for example, assuming ETH, there is a no $2^{o(\sqrt{k})} \cdot |I|^{O(1)}$ time algorithm for planar versions of VERTEX COVER, INDEPENDENT SET, and DOMINATING SET (and this bound is tight). However, no lower bound above $2^{O(k)}$ was obtained this way for any problem so far.

Flum, Grohe, and Weyer [29] tried to rebuild parameterized complexity by redefining fixed-parameter tractability as $2^{O(k)} \cdot |I|^{O(1)}$ time and introducing appropriate notions of reductions, completeness, and complexity classes. This theory could be potentially used to show that the problems treated in the current paper are hard for certain classes, and therefore they are unlikely to have single-exponential parameterized algorithms. However, we see no reason why these problems would be complete for any of those classes (for example, the only complete problem identified in [29] is actually FPT is a model checking on problem on words for which it was already known that $f(k)$ cannot even be elementary). Moreover, we are not only giving evidence against single-exponential time algorithms in this paper, but show that the $2^{O(k \log k)}$ dependence is actually tight.

2 Basic problems

In this section, we modify basic problems in such a way that they can be solved in time $2^{O(k \log k)} \cdot |I|^{O(1)}$ by brute force, and this is best possible assuming ETH. In all the problems of this section, the task is to select exactly one element from each row of a $k \times k$ table such that the selected elements satisfy certain constraints. This means that the search space is of size $k^k = 2^{O(k \log k)}$.

We denote by $[k] \times [k]$ the set of elements in a $k \times k$ table, where $(i, j)$ is the element in row $i$ and column $j$. Thus selecting exactly one element from each row gives a set $(1, \rho(1)), \ldots, (k, \rho(k))$ for some mapping $\rho : [k] \to [k]$. In some of the variants, we not only require that exactly one element is selected from each row, but we also require that exactly one element is selected from each column, that is, $\rho$ has to be a permutation.

The lower bounds for such permutation problems will be essential for proving hardness results on CLOSEST STRING (Section 3) and DISTORTION (Section 4). The key step in obtaining the lower bounds for permutation problems is the randomized reordering argument of Theorem 2.3. The analysis and derandomization of this step is reminiscent of the color coding [2] and chromatic coding [1] techniques.

To prove that a too fast algorithm for a certain problem $P$ contradicts the Exponential Time Hypothesis, we will reduce 3-COLORING on $n$-vertex graphs to problem $P$ and argue that the algorithm would solve 3-COLORING in time $2^{o(n)}$. It is a well-known fact that such an algorithm for 3-COLORING would violate ETH.

**PROPOSITION 2.1.** ([37]) Assuming ETH, there is no $2^{o(n)}$ time algorithm for deciding whether an $n$-vertex graph is 3-colorable; and there is no $2^{o(m)}$ time algorithm for $m$-clause 3SAT.

2.1 $k \times k$ Clique The first problem we investigate is the variant of the standard clique problem where the vertices are the elements of a $k \times k$ table, and the clique we are looking for has to contain exactly one element from each row.

| $k \times k$ CLIQUE |
|----------------------|
| **Input:** A graph $G$ over the vertex set $[k] \times [k]$ |
| **Parameter:** $k$ |
| **Question:** Is there a $k$-clique in $G$ with exactly one element from each row? |

Note that the graph $G$ in the $k \times k$ CLIQUE instance has $O(k^2)$ vertices at most $O(k^4)$ edges, thus the size of the instance is $O(k^4)$.

**THEOREM 2.1.** Assuming ETH, there is no $2^{o(k \log k)}$ time algorithm for $k \times k$ CLIQUE.

**Proof.** Suppose that there is an algorithm $A$ that solves $k \times k$ CLIQUE in $2^{o(k \log k)}$ time. We show that this implies that 3-COLORING on a graph with $n$ vertices can be solved in time $2^{o(n)}$, which contradicts ETH by Proposition 2.1.

Let $H$ be a graph with $n$ vertices. Let $k$ be the smallest integer such that $3^n/k+1 \leq k$, or equivalently, $n \leq k \log_3 k - k$. Note that such a finite $k$ exists for every $n$ and it is easy to see that $k \log k = O(n)$ for the smallest such $k$. Intuitively, it will be useful to think of $k$ as a value somewhat larger than $n / \log n$ (and hence $n/k$ is somewhat less than $\log n$).

Let us partition the vertices of $H$ into $k$ groups $X_1, \ldots, X_k$, each of size at most $[n/k]$. For every $1 \leq i \leq k$, let us fix an enumeration of all the proper 3-colorings of $H[X_i]$. Note that there are most $3^{[n/k]} \leq 3^{n/k+1} \leq k$ such 3-colorings for every $i$. We say that a proper 3-coloring $c_i$ of $H[X_i]$ and a proper 3-coloring $c_j$ of $H[X_j]$ are compatible if together they form a proper coloring of
$H[X_i \cup X_j]$; for every edge $uv$ with $u \in X_i$ and $v \in X_j$, we have $c_i(u) \neq c_j(v)$. Let us construct a graph $G$ over the vertex set $[k] \times [k]$ where vertices $(i_1, j_1)$ and $(i_2, j_2)$ with $i_1 \neq i_2$ are adjacent if and only if the $j_1$-th proper coloring of $H[X_i]$ and the $j_2$-th proper coloring of $H[X_i]$ are compatible (this means that if, say, $H[X_i]$ has less than $j_1$ proper colorings, then $(i_1, j_1)$ is an isolated vertex).

We claim that $G$ has a $k$-clique having exactly one vertex from each row if and only if $H$ is 3-colorable. Indeed, a proper 3-coloring of $H$ induces a proper 3-coloring for each of $H[X_i], \ldots, H[X_k]$. Let us select vertex $(i, j)$ and only if the proper coloring of $H[X_i]$ induced by $c$ is the $j$-th proper coloring of $H[X_i]$. It is clear that we select exactly one vertex from each row and they form a clique: the proper colorings of $H[X_i]$ and $H[X_j]$ induced by $c$ are clearly compatible. For the other direction, suppose that $(1, \rho(1)), \ldots, (k, \rho(k))$ form a $k$-clique for some mapping $\rho : [k] \rightarrow [k]$. Let $c_i$ be the $\rho(i)$-th proper 3-coloring of $H[X_i]$. The colorings $c_1, \ldots, c_k$ together define a coloring $c$ of $H$. This coloring $c$ is a proper 3-coloring: for every edge $uv$ with $u \in X_i$, and $v \in X_j$, the fact that $(i_1, \rho(i_1))$ and $(i_2, \rho(i_2))$ are adjacent means that $c_i$ and $c_j$ are compatible, and hence $c_i(u) \neq c_j(v)$.

Running the assumed algorithm $A$ on $G$ decides the 3-colorability of $H$. Let us estimate the running time of constructing $G$ and running algorithm $A$ on $G$. The graph $G$ has $k^2$ vertices and the time required to construct $G$ is polynomial in $k$: for each $X_i$, we need to enumerate at most $k$ proper 3-colorings of $G[X_i]$. Therefore, the total running time is $2^{o(k \log k)} \cdot O(1) = 2^{o(n)}$ (using that $k \log k = O(n)$). It follows that we have a $2^{o(n)}$ time algorithm for 3-COLORING, contradicting ETH. □

$k \times k$ PERMUTATION CLIQUE is a more restricted version of $k \times k$ CLIQUE: in addition to requiring that the clique contains exactly one vertex from each row, we also require that it contains exactly one vertex from each column. In other words, the vertices selected in the solution are $(1, \rho(1)), \ldots, (k, \rho(k))$ for some permutation $\rho$ of $[k]$. Given an instance $I$ of $k \times k$ CLIQUE having a solution $S$, if we randomly reorder the vertices in each row, then with some probability the reordered version of solution $S$ contains exactly one vertex from each row and each column of the reordered instance. In Theorem 2.2, we use this argument to show that a $2^{o(k \log k)}$ time algorithm for $k \times k$ PERMUTATION CLIQUE gives a randomized $2^{o(k \log k)}$ time algorithm for $k \times k$ CLIQUE. Section 2.2 shows how the proof of Theorem 2.2 can be derandomized.

**Theorem 2.2.** If there is a $2^{o(k \log k)}$ time algorithm for $k \times k$ PERMUTATION CLIQUE, then there is a randomized $2^{o(m)}$ time algorithm for $m$-clause 3SAT.

**Proof.** We show how to transform an instance $I$ of $k \times k$ CLIQUE into an instance $I'$ of $k \times k$ PERMUTATION CLIQUE with the following properties: if $I$ is a no-instance, then $I'$ is a no-instance, and if $I$ is a yes-instance, then $I'$ is a yes-instance with probability at least $2^{-O(k)}$. This means that if we perform this transformation $2^{O(k)}$ times and accept $I$ as a yes-instance if and only if at least one of the $2^{O(k)}$ constructed instances is a yes-instance, then the probability of incorrectly rejecting a yes-instance can be reduced to an arbitrary small constant. Therefore, a $2^{o(k \log k)}$ time algorithm for $k \times k$ PERMUTATION CLIQUE implies a randomized $2^{O(k)} \cdot 2^{o(k \log k)} = 2^{o(k \log k)}$ time algorithm for $k \times k$ CLIQUE.

Let $(c(i, j)) : [k] \times [k] \rightarrow [k]$ be a mapping chosen uniform at random; we can imagine $c$ as a coloring of the $k \times k$ vertices. Let $c'(i, j) = \bigstar$ if there is a $j' \neq j$ such that $c(i, j) = c(i, j')$ and let $c'(i, j) = c(i, j)$ otherwise (i.e., if $c(i, j) = x \neq \bigstar$, then no other vertex has color $x$ in row $i$). The instance $I'$ of $k \times k$ PERMUTATION CLIQUE is constructed the following way: if there is an edge between $(i_1, j_1)$ and $(i_2, j_2)$ in instance $I$ and $c'(i_1, j_1), c'(i_2, j_2) \neq \bigstar$, then we add an edge between $(i_1, c'(i_1, j_1))$ and $(i_2, c'(i_2, j_2))$ in instance $I'$. That is, we use mapping $c$ to rearrange the vertices in each row. If vertex $(i, j)$ clashes with some other vertex in the same row (that is, $c(i, j) = \bigstar$), then all the edges incident to $(i, j)$ are thrown away.

Suppose that $I'$ has a $k$-clique $(1, \rho(1)), \ldots, (k, \rho(k))$ for some permutation $\rho$ of $[k]$. For every $i$, there is a unique $\delta(i)$ such that $c(i, \delta(i)) = \rho(i)$: otherwise $(i, \rho(i))$ is an isolated vertex in $I'$. It is easy to see that $(1, \delta(i)), \ldots, (k, \delta(k))$ is a clique in $I$: vertices $(i_1, \delta(i_1))$ and $(i_2, \delta(i_2))$ have to be adjacent, otherwise there would be no edge between $(i_1, \rho(i_1))$ and $(i_2, \rho(i_2))$ in $I'$. Therefore, if $I$ is a no-instance, then $I'$ is a no-instance as well.

Suppose now that $I$ is a yes-instance: there is a clique $(1, \delta(i)), \ldots, (k, \delta(k))$ in $I$. Let us estimate the probability that the following two events occur:

1. For every $1 \leq i_1 < i_2 \leq k$, $c(i_1, \delta(i_1)) \neq c(i_2, \delta(i_2))$.
2. For every $1 \leq i \leq k$ and $1 \leq j \leq k$ with $j \neq \delta(i)$,
$c(i, \delta(i)) \neq c(i, j)$.

Event (1) means that $c(1, \delta(1)), \ldots, c(k, \delta(k))$ is a permutation of $[k]$. Therefore, the probability of (1) is $k! / k^k = e^{-\theta(k)}$ (using Stirling’s Formula). For a particular $i$, event (2) holds if $k - 1$ randomly chosen values are all different from $c(i, \delta(i))$. Thus the probability that (2) holds for a particular $i$ is
(1−1/k)(k−1) ≥ e−1 and the probability that (2) holds for every i is at least e−k. Furthermore, events (1) and (2) are independent: we can imagine the random choice of the mapping c as first choosing the values c(1, δ(1)), . . . , c(k, δ(k)) and then choosing the remaining k² − k values. Event (1) depends only on the first k choices, and for any fixed result of the first k choices, the probability of event (2) is the same. Therefore, the probability that (1) and (2) both hold is e−O(k).

Suppose that (1) and (2) both hold. Event (2) implies that c(i, δ(i)) = c′(i, δ(i)) ≠ ⋄ for every 1 ≤ i ≤ k. Event (1) implies that if we set ρ(i) := c(i, δ(i)), then ρ is a permutation of [k]. Therefore, the clique (1, ρ(1)), . . . , (k, ρ(k)) is a solution of I′, as required. □

In the next section, we show that instead of random colorings, we can use a certain deterministic family of colorings. This will imply:

**Corollary 2.1.** **Assuming ETH, there is no 2O(k log k) time algorithm for k × k PERMUTATION CLIQUE.**

### 2.2 Derandomization

In this section, we give a coloring family that can be used instead of the random coloring in the proof of Theorem 2.2. We call a graph G to be a cactus-grid graph if the vertices are elements of a k×k table and the graph precisely consists of a clique containing exactly one vertex from each row and each column. This together with the fact that the size of G is bounded by 2O(k log k). There is also another explicit construction of perfect hash functions. There are some known constructions for set H. We summarize them below.

**Proposition 2.2.** [2, 50] **There exists explicit construction H of (n, k)-family of perfect hash functions of size O(n log n).**

Proof. Our idea for deterministic coloring family F for cactus-grid graphs is to keep k functions f₁, . . . , fₖ where each fᵢ is an element of a (k, k′)-family of perfect hash functions for some k′ and use it to map the elements of {i} × k (the column i). We will ensure that the colors used in each column is private to the column and it is not used on the vertices of any other columns. This will ensure that we get the desired coloring family.

We make our intuitive idea more precise below. A description of a function f ∈ F consists of a tuple having

- a set S ⊆ [k];
- a tuple (k₁, k₂, . . . , kₗ) where kᵢ ≥ 1, ℓ = |S| and ∑ᵢ=₁ ℓ kᵢ = k;
- ℓ functions f₁, . . . , fₖ where fᵢ ∈ Hᵢ and Hᵢ is a (kᵢ, k′ᵢ)-family of perfect hash functions.

The set S tells us which all columns the clique intersects. Let the elements of S = {s₁, . . . , sₖ} be sorted in increasing order, say s₁ < s₂ . . . < sₖ. Then the tuple (k₁, k₂, . . . , kₖ) tells us that the column sᵢ, 1 ≤ j ≤ ℓ, contains kᵢ vertices from the clique. Hence with this interpretation, given a tuple (S, (k₁, . . . , kₖ), f₁, . . . , fₖ) we define the coloring function g : [k] × [k] → [k] as follows. Every element in [k] × {1, . . . , k} \ S is mapped to k+1. Now for vertices in [k] × {sᵢ} (vertices in column sᵢ), we define g(i, j) = fⱼ(i) + ∑₁≤ₗ≤ₖ kⱼ. We do this for every j between 1 and ℓ. This concludes the description. Now we show that it is indeed a coloring family for
cactus-grid graphs. Given a cactus grid graph $G$, we first look at the columns it intersects and that forms our set $S$ and then the number of vertices it intersects in each column makes the the tuple $(k_1, k_2, \ldots, k_l)$. Finally for each of the columns there exists a function $h$ in the perfect $(k, k_i)$-hash family that maps the elements of clique in this column one to one with $[k_i]$; we store this function corresponding to this column. Now we show that the function $g$ corresponding to this tuple properly colors $G$. The function $g$ assigns different values from $[k]$ to the columns in $S$ and hence we have that the vertices of clique gets distinct colors as in each column we have a function $f_i$ that is one to one on the vertices of $S$. Now we look at the edge with both end-points in the same row. If any of the end-point occurs in column that is not in $S$ then we know that it has been assigned $k+1$ while the vertex from the clique has been assigned color from $[k]$. If both end-points are from $S$ then the off-set we use to give different colors to vertices in these columns ensures that these end-points get different colors. This shows that $g$ is indeed a proper coloring of $G$. This shows that for every cactus-grid graph we have a function $g \in F$. Finally, the bound on the size of $F$ is as follows,

$$2^k \cdot 4^k \prod_{i=1}^{l} (11^k \log k) \leq 2^{O(k)(\log k)} \leq 2^{O(k \log \log k)}.$$ 

This concludes the proof. □

The bound achieved in the last lemma on the size of $F$ is sufficient for our purpose but it is not as small as $2^{O(k)}$ that one can obtain using a simple application of probabilistic methods. In fact one can obtain a family $\mathcal{F}$ of size $2^{O(k)}$. This will appear in the full version of the paper.

2.3 $k \times k$ Independent Set The lower bounds in Section 2.1 for $k \times k$ (PERMUTATION) CLIQUE obviously hold for the analogous $k \times k$ (PERMUTATION) INDEPENDENT SET problem: by taking the complement of the graph, we can reduce one problem to the other. We state here a version of the independent set problem that will be a convenient starting point for reductions in later sections:

| 2$\times$2 Bipartite Permutation Independent Set |
|-----------------------------------------------|
| **Input:** A graph $G$ over the vertex set $[2k] \times [2k]$ where every edge is between $I_1 = \{(i, j) \mid i, j \leq k\}$ and $I_2 = \{(i, j) \mid i, j \geq k + 1\}$. |
| **Parameter:** $k$ |
| **Question:** Is there an independent set $(1, \rho(1)), \ldots, (2k, \rho(2k)) \subseteq I_1 \cup I_2$ in $G$ for some permutation $\rho$ of $[2k]$? |

That is, the upper left quadrant $I_1$ and the lower right quadrant $I_2$ induce independent sets, and every edge is between these two independent sets. The requirement that the solution is a subset of $I_1 \cup I_2$ means that $\rho(i) \leq k$ for $1 \leq i \leq k$ and $\rho(i) \geq k + 1$ for $k + 1 \leq i \leq 2k$.

**Theorem 2.3.** Assuming ETH, there is no $2^{o(k \log k)}$ time algorithm for $2k \times 2k$ Bipartite Permutation Independent Set.

**Proof.** Given an instance $I$ of $k \times k$ PERMUTATION INDEPENDENT SET, we construct an equivalent instance $I'$ of $2k \times 2k$ Bipartite Permutation Independent Set the following way. For every $1 \leq i \leq k$ and $1 \leq j, j' \leq k, j \neq j'$, we add an edge between $(i, j)$ and $(i + k, j' + k)$ in $I'$. If there is an edge between $(i_1, j_1)$ and $(i_2, j_2)$ in $I$, then we add an edge between $(i_1, j_1)$ and $(i_2 + k, j_2 + k)$ in $I'$. This completes the description of $I'$.

Suppose that $I$ has a solution $(1, \delta(1)), \ldots, (k, \delta(k))$ for some permutation $\delta$ of $[2k]$. Then it is obvious from the construction of $I'$ that $(1, \delta(1)), \ldots, (k, \delta(k)), (1 + k, \delta(1) + k), \ldots, (2k, \delta(k) + k)$ is an independent set of $I$ and $\delta(1), \ldots, \delta(k), \delta(1) + k, \ldots, \delta(k) + k$ is clearly a permutation of $[2k]$. Suppose that $(1, \rho(1)), \ldots, (2k, \rho(2k))$ is solution of $I'$ for some permutation $\rho$ of $[2k]$. By definition, $\rho(i) \leq k$ for $1 \leq i \leq k$. We claim that $(1, \rho(k)), \ldots, (k, \rho(k))$ is an independent set of $I$. Observe first that $\rho(i + k) = \rho(i) + k$ for every $1 \leq i \leq k$; otherwise there is an edge between $(i, \rho(i))$ and $(i + k, \rho(i + k))$ in $I'$. If there is an edge between $(i_1, \rho(i_1))$ and $(i_2, \rho(i_2))$ in $I$, then by construction there is an edge between $(i_1, \rho(i_1))$ and $(i_2 + k, \rho(i_2) + k) = (i_2 + k, \rho(i_2 + k))$ in $I'$, contradicting the assumption that $(1, \rho(k)), \ldots, (2k, \rho(2k))$ is an independent set in $I'$. □

2.4 $k \times k$ Hitting Set Hitting Set is a $W[2]$-complete problem, but if we restrict the universe to a $k \times k$ table where only one element can be selected from each row, then it can be solved in time $O^*(k^k)$ by brute force.
We say that the mapping $\rho$ hits a set $S \subseteq [k] \times [k]$, if $(i, \rho(i)) \in S$ for some $1 \leq i \leq S$. Note that unlike for $k \times k$ CLIQUE and $k \times k$ INDEPENDENT SET, the size of the $k \times k$ HITTING SET cannot be bounded by a function of $k$.

It is quite easy to reduce $k \times k$ INDEPENDENT SET to $k \times k$ HITTING SET: for every pair $(i_1, j_1), (i_2, j_2)$ of adjacent vertices, we need to ensure that they are not selected simultaneously, which can be forced by a set that contains every element of rows $i_1$ and $i_2$, except $(i_1, j_1)$ and $(i_2, j_2)$. However, in Section 3.1 we prove the lower bound for CLOSEST STRING by reduction from a restricted form of $k \times k$ HITTING SET where each set contains at most one element from each row. The following theorem proves the lower bound for this variant of $k \times k$ HITTING SET. The basic idea is that an instance of $2k \times 2k$ BIPARTITE PERMUTATION INDEPENDENT SET can be transformed in an easy way into an instance of HITTING SET where each set contains at most one element from each column and we want to select exactly one element from each row and each column. By adding each row as a new set, we can forget about the restriction that we want to select exactly one element from each row: this restriction will be automatically satisfied by any solution. Therefore, we have a HITTING SET instance where we have to select exactly one element from each column and each set contains at most one element from each column. By changing the role of rows and columns, we arrive to a problem of the required form.

**Theorem 2.4.** Assuming ETH, there is no $2^{o(k \log k)} \cdot n^{O(1)}$ time algorithm for $k \times k$ HITTING SET, even in the special case when each set contains at most one element from each row.

**Proof.** To make the notation in the proof less confusing, we introduce a transposed variant of the problem (denote by $k \times k$ HITTING SET$^T$), where exactly one element has to be selected from each column. We prove the lower bound for $k \times k$ HITTING SET$^T$ with the additional restriction that each set contains at most one element from each column; this obviously implies the theorem.

Given an instance $I$ of $2k \times 2k$ BIPARTITE PERMUTATION INDEPENDENT SET, we construct an equivalent $2k \times 2k$ HITTING SET$^T$ instance $I'$ on the universe $[2k] \times [2k]$. For $1 \leq i \leq k$, let set $S_i$ contain the first $k$ elements of row $i$ and for $k + 1 \leq i \leq 2k$, let set $S_i$ contain the last $k$ elements of row $i$. For every edge $e$ in instance $I$, we construct a set $S_e$ the following way. By the way $2k \times 2k$ BIPARTITE PERMUTATION INDEPENDENT SET is defined, we need to consider only edges connecting some $(i_1, j_1)$ and $(i_2, j_2)$ with $i_1, j_1 \leq k$ and $i_2, j_2 \geq k + 1$. For such an edge $e$, let us define

$$S_e = \{(i_1, j') \mid 1 \leq j' \leq k, j' \neq j_1\} \cup \{(i_2, j') \mid k + 1 \leq j' \leq 2k, j' \neq j_2\}.$$

Suppose that $(1, \delta(1)), \ldots, (2k, \delta(2k))$ is a solution of $I$ for some permutation $\rho$ of $[2k]$. We claim that it is a solution of $I'$. As $\rho$ is a permutation, the set satisfies the requirement that it contains exactly one element from each column. As $\delta(i) \leq k$ if and only if $i \leq k$, the set $S_e$ is hit for every $1 \leq i \leq 2k$. Suppose that there is an edge $e$ connecting $(i_1, j_1)$ and $(i_2, j_2)$ such that set $S_e$ of $I'$ is not hit by this solution. Elements $(i_1, \delta(i_1))$ and $(i_2, \delta(i_2))$ are selected and we have $1 \leq \delta(i_1) \leq k$ and $k + 1 \leq \delta(i_2) \leq 2k$. Thus if these two elements do not hit $S_e$, then this is only possible if $\delta(i_1) = j_1$ and $\delta(i_2) = j_2$. However, this means that the solution for $I$ contains the two adjacent vertices $(i_1, j_1)$ and $(i_2, j_2)$, a contradiction.

Suppose now that $(\rho(1), 1), \ldots, (\rho(2k), 1)$ is a solution for $I'$. Because of the sets $S_i, 1 \leq i \leq 2k$, the solution contains exactly one element from each row, i.e., $\rho$ is a permutation of $2k$. Moreover, the sets $S_1, \ldots, S_k$ have to be hit by the $k$ elements in the first $k$ columns. This means that $\rho(i) \leq k$ if $i \leq k$ and consequently $\rho(i) > k$ if $i > k$. We claim that $(\rho(1), 1), \ldots, (\rho(2k), 1)$ is also a solution of $I$. It is clear that the only thing that has to be verified is that these $2k$ vertices form an independent set. Suppose that $(\rho(j_1), j_1)$ and $(\rho(j_2), j_2)$ are connected by an edge $e$. We can assume that $\rho(j_1) \leq k$ and $\rho(j_2) > k$, which implies $j_1 \leq k$ and $j_2 > k$. The solution for $I'$ hits set $S_e$, which means that either the solution selects an element $(\rho(j_1), j')$ or an element $(\rho(j_2), j')$. Elements $(\rho(j_1), j_1)$ and $(\rho(j_2), j_2)$ are the only elements of this form in the solution, but neither of them appears in $S_e$. Thus $(\rho(1), 1), \ldots, (\rho(2k), 2k)$ is indeed a solution of $I$. □

3 Closest String

Computational biology applications often involve long sequences that have to be analyzed in a certain way. One core problem is finding a “consensus” of a given set of strings: a string that is close to every string in the input. The CLOSEST STRING problem defined below formalizes this task.
We denote by $d(s, s_i)$ the Hamming distance of the strings $s$ and $s_i$, that is, the number of positions where they have different characters. The solution $s$ will be called the center string. 

**Closest String** and its generalizations (**Closest Substring, Distinguishing (Sub)string Selection, Consensus Patterns**) have been thoroughly explored both from the viewpoint of approximation algorithms and fixed-parameter tractability [46, 56, 47, 33, 43, 12, 26, 32, 41, 20]. In particular, Gramm et al. [33] showed that **Closest String** is fixed-parameter tractable parameterized by $d$: they gave an algorithm with running time $O(d^k \cdot |I|)$. The algorithm works over an arbitrary alphabet $\Sigma$ (i.e., the size of the alphabet is part of the input). It is an obvious question whether the dependence on $d$ can be reduced to single exponential, i.e., whether the running time can be improved to $2^{O(d)} \cdot |I|^{O(1)}$. For small fixed alphabets, Ma and Sun [46] achieved single-exponential dependence on $d$: the running time of their algorithm is $|\Sigma|^{O(d)} \cdot |I|^{O(1)}$. Improved algorithms with running time of this form, but with better constants in the exponent were given in [56, 12]. We show here that the $d^k$ and $|\Sigma|^d$ dependence are best possible (assuming ETH): the dependence cannot be improved to $2^{o(d \log d)}$ or to $2^{o(d \log |\Sigma|)}$. In particular, single exponential dependence on $d$ cannot be achieved if the alphabet size is unbounded.

**Theorem 3.1.** Assuming ETH, there is no $2^{o(d \log d)} \cdot |I|^{O(1)}$ or $2^{o(d \log |\Sigma|)} \cdot |I|^{O(1)}$ time algorithm for **Closest String**.

**Proof.** We prove the theorem by a reduction from the Hitting Set problem considered in Theorem 2.4. Let $I$ be an instance of $k \times k$ Hitting Set with sets $S_1, \ldots, S_m$; each set contains at most one element from each row. We construct an instance $I'$ of **Closest String** as follows. Let $\Sigma = [2k + 1], L = k$, and $d = k - 1$ (this means that the center string has to have at least one character common with every input string). Instance $I'$ contains $(k + 1)m$ input strings $s_{x,y}$ $(1 \leq x \leq m, 1 \leq y \leq k + 1)$. If set $S_x$ contains element $(i, j)$ from row $i$, then the $i$-th character of $s_{x,y}$ is $j$; if $S_x$ contains no element of row $i$, then the $i$-th character of $s_{x,y}$ is $y + k$. Thus set $s_{x,y}$ describes the elements of set $S_x$, with a dummy value between $k + 1$ and $2k + 1$ marking the rows disjoint from $S_x$. The strings $s_{x,1}, \ldots, s_{x,k+1}$ differ only in the choice of the dummy values.

We claim that $I'$ has a solution if and only if $I$ has. Suppose that $(1, \rho(1)), \ldots, (k, \rho(k))$ is a solution of $I$ for some mapping $\rho : [k] \rightarrow [k]$. Then the center string $s = \rho(1) \ldots \rho(k)$ is a solution of $I'$: if element $(i, \rho(i))$ of the solution hits set $S_x$ of $I$, then both $s$ and $s_{x,y}$ have character $\rho(i)$ at the $i$-th position. For the other direction, suppose that center string $s$ is a solution of $I'$. As the length of $s$ is $k$, there is a $k + 1 \leq y \leq 2k + 1$ that does not appear in $s$. If the $i$-th character of $s$ is some $1 \leq c \leq k$, then let $\rho(i) = c$; otherwise, let $\rho(i) = 1$ (or any other arbitrary value). We claim that $(1, \rho(1)), \ldots, (k, \rho(k))$ is a solution of $I$, i.e., it hits every set $S_x$ of $I$. To see this, consider the string $s_{x,y}$, which has at least one character common with $s$. Suppose that character $c$ appears at the $i$-th position in both $s$ and $s_{x,y}$. It is not possible that $c > k$: character $y$ is the only character larger than $k$ that appears in $s_{x,y}$, but $y$ does not appear in $s$. Therefore, we have $1 \leq c \leq k$ and $\rho(i) = c$, which means that element $(i, \rho(i)) = (i, c)$ of the solution hits $S_x$.

The claim in the previous paragraph shows that solving instance $I'$ using an algorithm for **Closest String** solves the $k \times k$ Hitting Set instance $I$. Note that the size $n$ of the instance $I'$ is polynomial in $k$ and $m$. Therefore, a $2^{o(d \log d)} \cdot |I|^{O(1)}$ or a $2^{o(d \log |\Sigma|)} \cdot |I|^{O(1)}$ algorithm for **Closest String** would give a $2^{o(k \log k)} \cdot (km)^{O(1)}$ time algorithm for $k \times k$ Hitting Set, violating ETH (by Theorem 2.4).

### 4 Distortion

Given an undirected graph $G$ with the vertex set $V(G)$ and the edge set $E(G)$, a metric associated with $G$ is $M(G) = (V(G), D)$, where the distance function $D$ is the shortest path distance between $u$ and $v$ for each pair of vertices $u, v \in V(G)$. We refer to $M(G)$ as to the **graph metric of $G$**. Given a graph metric $M$ and another metric space $M'$ with distance functions $D$ and $D'$, a mapping $f : M \rightarrow M'$ is called an **embedding of $M$ into $M'$**. The mapping $f$ has **contraction** $c_f$ and **expansion** $e_f$ if for every pair of points $p, q$ in $M$, $D(p,q) \leq D'(f(p), f(q)) \cdot c_f$ and $D(p,q) \cdot e_f \geq D'(f(p), f(q))$ respectively. We say that $f$ is **non-contracting** if $c_f$ is at most 1. A non-contracting mapping $f$ has **distortion** $d$ if $d_f$ is at most $d$. One of the most well studied case of graph embedding is when the host metric $M'$ is $\mathbb{R}^1$ and $D'$ is the Euclidean distance. This is also called embedding the graph into integers or line. Formally, the problem of **Distortion** is defined as follows.
The problem of finding embedding with good distortion between metric spaces is a fundamental mathematical problem [38, 44] that has been studied intensively [3, 4, 5, 40]. Embedding a graph metric into a simple low-dimensional metric space like the real line has proved to be a useful algorithmic tool in various fields (for an example see [36] for a long list of applications). Bădoiu et al. [4] studied Distortion from the viewpoint of approximation algorithms and exact algorithms. They showed that there is a constant $a > 1$, such that $a$-approximation of the minimum distortion embedding into the line, is NP-hard and provided an exact algorithm computing embedding of a $n$ vertex graph into line with distortion $d$ in time $n^{O(d)}$. Subsequently, Fellows et al. [27] improved the running time of their algorithm to $d^{O(d)} \cdot n$ and thus proved Distortion to be fixed parameter tractable parameterized by $d$. We show here that the $d^{O(d)}$ dependence in the running time of Distortion algorithm is optimal (assuming ETH). To achieve this we first obtain a lower bound on an intermediate problem called Constrained Permutation, then give a reduction that transfers the lower bound from Constrained Permutation to Distortion. The superexponential dependence on $d$ is particularly interesting, as $c^d$ time algorithms for finding a minimum distortion embedding of a graph on $n$ vertices into line have been given by Fomin et al. [30] and Cygan and Pilipczuk [13].

**Constrained Permutation**

**Input:** Subsets $S_1, \ldots, S_m$ of $[k]$

**Parameter:** $k$

**Question:** A permutation $\rho$ of $[k]$ such that for every $1 \leq i \leq m$, there is a $1 \leq j < k$ such that $\rho(j), \rho(j + 1) \in S_i$.

Given a permutation $\rho$ of $[k]$, we say that $x$ and $y$ are neighbors if $\{x, y\} = \{\rho(i), \rho(i + 1)\}$ for some $1 \leq i < k$. In the Constrained Permutation problem the task is to find a permutation that hits every set $S_i$ in the sense that there is a pair $x, y \in S_i$ that are neighbors in $\rho$.

**Theorem 4.1.** Assuming ETH, there is no $2^{o(k \log k)} m^{O(1)}$ time algorithm for Constrained Permutation.

**Proof.** Given an instance $I$ of $2k \times 2k$ Bipartite Permutation Independent Set, we construct an equivalent instance $I'$ of Constrained Permutation. Let $k' = 24k$ and for ease of notation, let us identify the numbers in $[k']$ with the elements $r_i^1, r_i^2, c_j^1, c_j^2$ for $1 \leq i \leq 3, 1 \leq i, j \leq 2k$. The values $r_i^1$ represent the rows and the values $c_j^1$ represent the columns. If $r_i^1$ and $c_j^1$ are neighbors in $\rho$, then we interpret it as selecting element $j$ from row $i$. More precisely, we want to construct the sets $S_1, \ldots, S_m$ in such a way that if $(1, \delta(1)), \ldots, (2k, \delta(2k))$ is a solution of $I$, then the following permutation $\rho$ of $[k']$ is a solution of $I'$:

$$r_1^1, \bar{r}_1^1, 1, c_3^1(1), \bar{c}_3^1(1), r_2^1, \bar{r}_2^1, 1, c_3^1(2), \bar{c}_3^1(2), \ldots, r_{2k}^1, \bar{r}_{2k}^1, 1, c_3^1(2k), \bar{c}_3^1(2k),$$

$$r_1^2, r_2^2, c_3^2(1), \bar{c}_3^2(1), r_2^2, r_3^2, c_3^2(2), \bar{c}_3^2(2), \ldots, r_{2k}^2, r_{2k}^2, c_3^2(2k), \bar{c}_3^2(2k),$$

$$r_1^3, r_3^3, c_3^3(1), c_3^3(1), r_3^3, r_4^3, c_3^3(3), c_3^3(3), \ldots, r_{2k}^3, r_{2k}^3, c_3^3(2k), c_3^3(2k).$$

The first property that we want to ensure is that every solution of $I'$ looks roughly like $\rho$ above: pairs $r_i^1r_j^1$ and pairs $c_i^1c_j^1$ alternate in some order. Then we can define a permutation $\delta$ such that $\delta(i) = j$ if $r_i^1\bar{r}_j^1$, is followed by the pair $c_i^1c_j^1$. The sets in instance $I'$ will ensure that this permutation $\delta$ is a solution of $I$. Let instance $I'$ contain the following groups of sets:

1. For every $1 \leq i \leq 3$ and $1 \leq i \leq 2k$, there is a set $\{r_i^1, r_i^2\}$,
2. For every $1 \leq i \leq 3$ and $1 \leq j \leq 2k$, there is a set $\{c_i^1, c_i^2\}$,
3. For every $1 \leq \ell < \ell'' \leq 3, 1 \leq i \leq 2k, X \subseteq [2k]$, there is a set $\{r_i^1, r_i^2\} \cup \{c_j^1 | j \in X\} \cup \{c_j^2 | j \notin X\}$,
4. For every $1 \leq i \leq k$, there is a set $\{r_i^1\} \cup \{c_j^1 | 1 \leq j \leq k\}$,
5. For every $k + 1 \leq i \leq 2k$, there is a set $\{r_i^1\} \cup \{c_j^1 | k + 1 \leq j \leq 2k\}$,
6. For every two adjacent vertices $(i_1, j_1) \in I_1$ and $(i_2, j_2) \in I_2$, there is a set $\{r_i^1, r_i^2\} \cup \{c_j^1 | 1 \leq j \leq k, j \neq j_1\} \cup \{c_j^2 | k + 1 \leq j \leq 2k, j \neq j_2\}$.

Recall that every edge of instance $I$ goes between the independent sets $I_1 = \{(i, j) | i, j \leq k\}$ and $I_2 = \{(i, j) | i, j \geq k + 1\}$. Let us verify first that if $\delta$ is a solution of $I$, then the permutation $\rho$ described above satisfies every set. It is clear that sets in the first two groups are satisfied. To see that every set in group 3 is satisfied, consider a set corresponding to a particular $1 \leq \ell < \ell'' \leq 3, 1 \leq i \leq 2k, X \subseteq [2k]$. If $\delta(i) \in X$, then
\( r_i^\ell \) and \( c_{\delta(i)}^\ell \) are neighbors and both appear in the set; if \( \delta(i) \notin X \), then \( r_i^\ell \) and \( c_{\delta(i)}^\ell \) are neighbors and both appear in the set. Sets in group 4 and 5 are satisfied because \( \delta(i) \leq k \) for \( 1 \leq i \leq k \) and \( \delta(i) \geq k + 1 \) for \( k+1 \leq i \leq 2k \). Finally, let \((i_1,j_1) \in V_1 \) and \((i_2,j_2) \in V_2 \) be two adjacent vertices and consider the corresponding set in group 6. As the solution of \( I \) is an independent set, either \( \delta(i_1) \neq j_1 \) or \( \delta(i_2) \neq j_2 \). In the first case, \( r_{i_1}^1 \) and \( c_{\delta(i_1)}^1 \) are neighbors and both appear in the set; in the second case, \( r_{i_2}^1 \) and \( c_{\delta(i_2)}^1 \) are neighbors and both appear in the set.

Next we show that if \( \rho \) is a solution of \( I' \), then a solution for \( I \) exists. We say that an element \( r_i^\ell \) is \textit{good} if its neighbors are \( r_i^\ell \) and \( c_j^\ell \) for some \( 1 \leq \ell' \leq 3 \) and \( 1 \leq j \leq 2k \). Similarly, an element \( c_j^\ell \) is \textit{good} if its neighbors are \( c_j^\ell \) and \( r_i^\ell \) for some \( 1 \leq \ell' \leq 3 \) and \( 1 \leq i \leq 2k \). Our first goal is to show that every \( r_i^\ell \) and \( c_j^\ell \) is good. The sets in group 1 and 2 ensure that \( r_i^\ell \) and \( r_i^{\ell'} \) are neighbors, and \( c_j^\ell \) and \( c_j^{\ell'} \) are neighbors.

We claim that for every \( 1 \leq \ell < \ell'' \leq 3 \), and \( 1 \leq i \leq 2k \), if elements \( r_i^\ell \) and \( r_i^{\ell''} \) are not neighbors, then both of them are good. Let us build a 4k-vertex graph \( B \) whose vertices are \( c_j^\ell \), \( c_j^{\ell''} \) (\( 1 \leq j \leq 2k \)). Let us connect by an edge those vertices that are neighbors in \( \rho \). Moreover, let us make \( c_j^\ell \) and \( c_j^{\ell''} \) adjacent for every \( 1 \leq j \leq 2k \). Observe that the degree of every vertex is at most 2 (as \( c_j^\ell \) has only one neighbor besides \( c_j^\ell \)). Moreover, \( B \) is bipartite: in every cycle, edges of the form \( c_j^\ell c_j^{\ell''} \) alternate with edges not of this form. Therefore, there is a partition \( (Y, \bar{Y}) \) of \( B \) such that the set \( Y \) (and hence \( \bar{Y} \)) contains exactly one of \( c_j^\ell \) and \( c_j^{\ell''} \) for every \( 1 \leq j \leq 2k \). Group 3 contains a set \( S_Y = \{ r_i^\ell, r_i^{\ell''} \} \cup Y \) and a set \( S_{\bar{Y}} = \{ r_i^\ell, r_i^{\ell''} \} \cup \bar{Y} \): as \( Y \) contains exactly one of \( c_j^\ell \) and \( c_j^{\ell''} \), there is a choice of \( X \) that yields these sets. Permutation \( \rho \) satisfies \( S_Y \) and \( S_{\bar{Y}} \), thus each of \( S_Y \) and \( S_{\bar{Y}} \) contains a pair of neighboring elements. By assumption, this pair cannot be \( r_i^\ell \) and \( r_i^{\ell''} \). As \( Y \) induces an independent set of \( B \), this pair cannot be contained in \( Y \) either. Thus the only possibility is that one of \( r_i^\ell \) and \( r_i^{\ell''} \) is the neighbor of an element of \( Y \). If, say, \( r_i^\ell \) is a neighbor of an element \( y \in Y \), then \( r_i^\ell \) is good. In this case, \( r_i^\ell \) is not the neighbor of any element of \( \bar{Y} \), which means that the only way two members of \( S_{\bar{Y}} \) are neighbors if \( r_i^{\ell''} \) is a neighbor of a member of \( \bar{Y} \), i.e., \( r_i^{\ell''} \) is also good.

At most one of \( r_i^\ell \) and \( r_i^{\ell''} \) can be the neighbor of \( r_i^1 \), thus we can assume that \( r_i^\ell \) and \( r_i^{\ell''} \) are not neighbors for some \( \ell \in \{2,3\} \). By the claim in the previous paragraph, \( r_i^1 \) and \( r_i^2 \) are both good. In particular, this means that \( r_i^1 \) is not the neighbor of \( r_i^2 \) and \( r_i^3 \), hence applying again the claim, it follows that \( r_i^2 \) and \( r_i^3 \) are both good. Thus \( r_i^1 \) is good for every \( 1 \leq \ell \leq 3 \) and \( 1 \leq i \leq 2k \), and the pigeonhole principle implies that \( c_j^1 \) is good for every \( 1 \leq \ell \leq 3 \) and \( 1 \leq i \leq 2k \).

As every \( c_j^1 \) is good, the sets in groups 4 and 5 can be satisfied only if every \( r_i^1 \) has a neighbor \( c_j^1 \). Let \( \delta(i) = j \) if \( c_j^1 \) is the neighbor of \( r_i^1 \); clearly \( \delta \) is a permutation of \( [2k] \). We claim that \( \delta \) is a solution of \( I \). The sets in group 4 and 5 ensure that \( \delta(i) \leq k \) for every \( 1 \leq i \leq k \) and \( \delta(i) \geq k + 1 \) if \( k + 1 \leq i \leq 2k \). To see that \( (1, \delta(1)), \ldots, (2k, \delta(2k)) \) is an independent set, consider two adjacent vertices \((i_1,j_1) \in I_1 \) and \((i_2,j_2) \in I_2 \). We show that it is not possible that \( \delta(i_1) = j_1 \) and \( \delta(i_2) = j_2 \). Consider the set \( S \) in group 6 corresponding to the edge connecting \((i_1,j_1) \) and \((i_2,j_2) \). As \( r_{i_1}^1, r_{i_2}^1, \) and \( c_j^1 \) is good, then only way \( S \) is can be satisfied is that \( r_{i_1}^1 \) or \( r_{i_2}^1 \) is the neighbor of some \( c_j^1 \) appearing in \( S \). If \( \delta(i_1) = j_1 \) and \( \delta(i_2) = j_2 \), then the \( c_j^1 \) and \( c_j^1 \) are the neighbors of \( r_{i_1}^1 \) and \( r_{i_2}^1 \), respectively, but \( c_j^1 \) and \( c_j^1 \) do not appear in \( S \). This shows that if there is a solution for \( I' \), then there is a solution for \( I \) as well.

The size of the constructed instance \( I' \) is polynomial in \( 2k \). Thus if \( I' \) can be solved in time \( 2^{o(k \log k)} \cdot |I'| = 2^{o(k \log k)} \cdot 2^{O(k)} = 2^{o(k \log k)} \), then this gives a \( 2^{o(k \log k)} \) time algorithm for \( 2k \times 2k \) Bipartite Permutation Independent Set. \( \square \)

**Theorem 4.2.** Assuming ETH, there is no \( 2^{o(d \log d)} \) time algorithm for Distortion.

**Proof.** We prove the theorem by a reduction from the Constrained Permutation. Let \( I \) be an instance of Constrained Permutation consisting of subsets \( S_1, \ldots, S_m \) of \( [k] \). Now we show how to construct the graph \( G \), an input to Distortion corresponding to \( I \). For an ease of presentation we identify \( [k] \) with vertices \( u_1, \ldots, u_k \). We also set \( U = \{ u_1, \ldots, u_k \} \) and \( d = 2k \). The vertex set of \( G \) consists of all the vertices of sets of vertices.

- For every \( 1 \leq i \leq m \) and \( 1 \leq j \leq k \), \( u_i^j \). We also denote the set \( \{ u_1^1, \ldots, u_k^j \} \) by \( U_i \).
- A vertex \( s_i \) for each set \( S_i \).
- Two cliques \( C_a \) and \( C_b \) of size \( d + 1 \) consisting of vertices \( c_{a_1}, \ldots, c_{a_{d+1}} \) and \( c_{b_1}, \ldots, c_{b_{d+1}} \) respectively.
- A path \( P \) of length \( m \) (number of edges) consisting of vertices \( v_1, \ldots, v_{m+1} \).

We add the following more edges among these vertices. We add edges from all the vertices in clique \( C_a \) but \( c_{a_1} \) to \( v_1 \) and add edges from all the vertices in clique \( C_b \) but \( c_{b_1} \) to \( v_{m+1} \). For all \( 1 \leq i < m \) and \( 1 \leq j < k \), make
is that the distance between two consecutive vertices in $U_i$ we can “squeeze” in $s_i$ between those two vertices without disturbing the rest of the construction.

In the reverse direction, assume that we start with a distortion $d$ pushing embedding of $G$. Consider the layout of the graph induced on $C_a$ and the vertex $v_1$. This is a clique of size $d + 2$ minus an edge and hence $C_a \cup \{v_1\}$ can be layed out in two ways: $c^1_a, C_a \setminus \{c^1_a\}, v_1$ or $v_1, C_a \setminus \{c^1_a\}, c^1_a$. Since we can reverse the layout, we can assume without loss of generality that it is $c^1_a, C_a \setminus \{c^1_a\}, v_1$. Without loss of generality we can also assume that $v_1$ is placed on position 0. Since every vertex in $U_1$ is adjacent to $v_1$ and the negative positions are taken by the vertices in $C_a$, the $k = d/2$ vertices of $U_1$ must lie on the positions $\{1, \ldots , d\}$. We first argue that no vertex of $U_1$ occupies the position $d$. Suppose it does. Then the rightmost vertex of $U_2$ (to the right of $v_1$ in the embedding) must be on position at least $2d$. Simultaneously $v_2$ must be on position at most $d – 1$ since $d$ is already occupied and $v_2$ is adjacent to $v_1$. But $v_2$ is adjacent to the rightmost vertex of $U_2$ and hence the distance on the line between them becomes at least $d + 1$, a contradiction. So $U_1$ must use only positions in $\{1, \ldots , d – 1\}$. Since the distance between two consecutive vertices in $U_1$ is 2 together with the fact that we started with a pushing embedding imply that the vertices of $U_1$ occupy all odd positions of $\{1, \ldots , d – 1\}$. Now, $U_2$ must be on the positions in $\{d + 1, \ldots , 2d\}$ with the rightmost vertex in $U_2$ being on at least $2d – 1$. Since $d – 1$ is occupied by someone in $U_1$ and $v_2$ is adjacent to both $v_1$ and the rightmost vertex of $U_2$ it follows that $v_2$ must be on position $d$.

We can now argue similarly to the previous paragraph that $U_3$ does not use position $2d$, and hence $v_3$ is on position $2d$ while $U_2$ must use the odd positions of $\{d + 1, \ldots , 2d – 1\}$. We can repeat this argument for all $i$ and position the vertex $v_i$ of the path at $d(i – 1)$ and place the vertices of $U_i$ at odd positions between $d(i – 1)$ and $di$. Of course, all the vertices of the clique $C_b$ will come after $v_{m+1}$.

Consider the order in which the embedding puts the vertices of $U_1$. We claim that it must put the vertices of $U_2$ in the same order. Look at the embedding of $U_1$ and $U_2$ from left to right and let $j$ be the first index where $u_j^2$ of $U_1$ is placed between 0 and $d$ while $u_j^2$ of $U_2$ is placed between $d$ and $2d$ and $\alpha \neq \beta$. This implies that $u_i^2$ appears further back in the permutation of $U_2$ and hence the distance between the positions of $u_i^2$ in $U_1$ is more than $d$ while $u_i^2$ and $u_{i+1}^2$ are adjacent to each other in the graph. By repeating this argument for all $i$ and $i + 1$ we can show that order of all $U_i$’s is the same. Consider $s_i$. It must be put on some even position, with some vertices of $U_j$ coming before and

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Figure 1: A construction used in Theorem 4.2

$u_j^i$ adjacent to $v_i, v_{i+1}$ and $u_j^{i+1}$. For $1 \leq j \leq k$, make $u_j^m$ adjacent to $v_m, v_{m+1}$. Finally make $s_i$ adjacent to $u_j^i$ if $u_j^i \in S_i$. This concludes the construction. A figure corresponding to the construction can be found in Figure 4.

For our proof of correctness we also need the following known facts about distortion $d$ embedding of a graph into integers. For an embedding $g$, let $v_1, v_2, \ldots , v_n$ be an ordering of the vertices such that $g(v_1) < g(v_2) < \cdots < g(v_n)$. If $g$ is such that for all $1 \leq i < q, D(v_i, v_{i+1}) = |g(v_i) - g(v_{i+1})|$, then the mapping $g$ is called pushing embedding. It is known that pushing embeddings are always non-contracting and if $G$ can be embedded into integers with distortion $d$, then there is a pushing embedding of $G$ into integers with distortion $d$ [27].

Let a permutation $\rho$ of $[k] = U$ be a solution to $I$, an instance of CONSTRAINED PERMUTATION. This automatically leads to a permutation on $U$ that we represent by $\rho(U)$. There is a natural bijection between $U$ and $U_i$ with $u_j^i \in U_i$. So when we write $\rho(U_i)$ then this means that the vertices of $U_i$ are permutated with respect to $\rho$ and being identified with its counterpart in $U_i$. Now we give a pushing embedding for the vertices in $G$ with $c_a^i$ being placed at 0. All the vertices except the set vertices $s_i$ appear in the following order

$$c_a^1, \ldots , c_a^{d+1}, v_1, \rho(U_1), v_2, \rho(U_2), v_3, \ldots , v_m, \rho(U_m), v_{m+1}, c_b^{d+1}, \ldots , c_b^i.$$  

Since $\rho$ is a solution to $I$ we know that for every $S_i$ there exists a $1 \leq j < k$ such that $\rho(j) \rho(j+1) \in S_i$. We place $s_i$ between $\rho(v_i^j)$ and $\rho(v_{i+1}^j)$. By our construction the given embedding is pushing and hence non-contracting. To show that for every pair of vertices $u, v \in V(G)$, $D(u, v) \leq d \cdot |g(u) - g(v)|$, we only have to show that for every edge $uv \in E(G)$, $D(u, v) \leq d$. This can be readily checked from the construction. The crucial observation is that the distance between two consecutive vertices from $U_i$ is 2, and hence it must be at least distance 2 apart on the line.
after $s_i$. But then, because we started with pushing embedding we have that $s_i$ is adjacent to both those vertices, and hence $i = j$ as $s_i$ is adjacent to only the vertices in $U_i$.

Now we take the permutation $\rho$ for $[k]$, imposed by the ordering of $U_1$, as a solution to the instance $I$ of CONSTRAINED PERMUTATION. For every set $S_i$ we need to show that there exists a $1 \leq j < k$ such that $\rho(j), \rho(j + 1) \in S_i$. Consider the corresponding $s_i$ in the embedding and look at the vertices that are placed left and right of it. Let these be $u_\alpha^i$ and $u_\beta^i$. Then by construction $\alpha$ and $\beta$ are neighbors to $s_i$ in $G$ and hence $\alpha$ and $\beta$ belong to $S_i$. Now since the ordering of $U_i$’s are same we have that they are consecutive in the permutation $\rho$. This concludes the proof in the reverse direction.

The claim in the previous paragraph shows that an algorithm finding a distortion $d$ embedding of $G$ into line solves the instance $I$ of CONSTRAINED PERMUTATION. Note the number of vertices in $G$ is bounded by a polynomial in $k$ and $m$. Therefore a $2^{O(d \log d)}|V(G)|^{O(1)}$ algorithm for DISTORTION would give a $2^{O(k \log k)} \cdot (km)^{O(1)}$ algorithm for CONSTRAINED PERMUTATION, violating ETH by Theorem 4.1. $\square$

5 Disjoint Paths

There are many natural graph problems that are fixed-parameter tractable parameterized by the treewidth of the input graph. In most cases, these results can be obtained by well-understood dynamic programming techniques. In fact, Courcelle’s Theorem provide a clean way of obtaining such results. If the dynamic programming needs to keep track of a permutation, partition, or a matching at each node, then running time of such an algorithm is typically of the form $w^{O(w)} \cdot n^{O(1)}$ on graphs with treewidth $w$. We demonstrate a problem where this form of running time is necessary for the solution and it cannot be improved to $2^{O(w \log w)} \cdot n^{O(1)}$. We refer to Appendix ?? for the definitions of treewidth and pathwidth.

Given an undirected graph $G$ and $p$ vertex pairs $(s_i, t_i)$, the DISJOINT PATHS problem asks whether there exists $p$ mutually vertex disjoint paths in $G$ linking these pairs. This is one of the classic problems in combinatorial optimization and algorithmic graph theory, and has many applications, for example in transportation networks, VLSI layout, and virtual circuits routing in high-speed networks. The problem is NP-complete if $p$ is part of the input and remains so even if restrict the input graph to be planar [39, 45]. However if $p$ is fixed then the problem is famously fixed-parameter tractable as a consequence of the seminal Graph Minors theory of Robertson and Seymour [53]. A basic building block in their algorithm for DISJOINT PATHS is an algorithm for DISJOINT PATHS on graphs of bounded treewidth. To our interest is the following parameterization of DISJOINT PATHS.

**Theorem 5.1.** Assuming ETH, there is no $2^{O(w \log w)} \cdot n^{O(1)}$ time algorithm for DIRECTED DISJOINT PATHS.

**Proof.** The key tool in the reduction from $k \times k$ HITTING SET to DIRECTED DISJOINT PATHS is the following gadget. For every $k \geq 1$ and set $S \subseteq [k] \times [k]$, we construct the gadget $G_{a,S}$ the following way.

- For every $1 \leq i \leq k$, it contains vertices $a_i$, $b_i$.
- For every $1 \leq i,j \leq k$, it contains a vertex $v_{i,j}$ and edges $a_i v_{i,j}$, $v_{i,j} b_j$.
- For every $1 \leq i \leq k$, it contains a directed path $P_i = c_{i,0}d_{i,1}v_{i,1}^1c_{i,1} \ldots d_{i,k}v_{i,k}^k c_{i,k}$.
- For every $1 \leq i,j \leq k$, it contains vertices $f_{i,j}$, $f_{i,j}^1$, $f_{i,j}^2$, and edges $b_j f_{i,j}$, $f_{i,j} c_{i,j}$, $f_{i,j}^1 f_{i,j}$, $f_{i,j}^2 f_{i,j}$.
- It contains two vertices $s$ and $t$, and for every $(i,j) \in S$, there are two edges $sd_{i,j}$, $d_{i,j} t$.

The demand pairs in the gadget are as follows:

- For every $1 \leq i \leq k$, there is a demand $(a_i, c_{i,k})$.
- For every $1 \leq i,j \leq k$, there is a demand $(f_{i,j}^1, f_{i,j}^2)$.
- There is a demand $(s, t)$. 

The best known algorithm for this problem runs in time $2^{O(w \log w)} \cdot n^{O(1)}$ [54] and here we show that this is indeed optimal. To get this lower bound first we give a linear parameter reduction from $k \times k$ HITTING SET to DIRECTED DISJOINT PATHS, a variant of DISJOINT PATHS where the input is a directed graph, parameterized by pathwidth of the underlying undirected graph. Finally we obtain a lower bound of $2^{O(k \log k)} |V(G)|^{O(1)}$ on DIRECTED DISJOINT PATHS parameterized by treewidth under ETH, by giving a linear parameter reduction from DIRECTED DISJOINT PATHS parameterized by pathwidth to DISJOINT PATHS parameterized by pathwidth.
This completes the description of the gadget. Observe that if a collection of paths form a solution for the gadget, then for every \( 1 \leq i \leq k \), exactly one of the vertices \( v_{i,1}, \ldots, v_{i,k} \) is used by the paths. We say that a solution represents the mapping \( \rho : [k] \rightarrow [k] \) if for every \( 1 \leq i \leq k \), vertex \( v_{i,\rho(i)} \) is used by the paths in the solution.

The following claim summarizes the most important properties of the gadget.

**Claim 1.** For every \( k \geq 1 \) and \( S \subseteq [k] \times [k] \), gadget \( G_{k,S} \) has the following properties:

1. For every \( \rho : [k] \rightarrow [k] \) that hits \( S \), gadget \( G_{k,S} \) has a solution that represents \( \rho \), and \( v^*_i,\rho(i) \) is not used by the paths in the solution for any \( 1 \leq i \leq k \).

2. If \( G_{k,S} \) has a solution that represents \( \rho \), then \( \rho \) hits \( S \) and vertex \( v^*_i,\rho(i) \) is used by the paths in the solution for every \( 1 \leq i \leq k \) and \( j \neq \rho(i) \).

**Proof.** To prove the first statement, we construct a solution in the following way. Demand \( (a_i,c_i,j) \) is satisfied by the path \( a_i v_{i,\rho(i)} b_{i,\rho(i)} f_{i,\rho(i)} c_i,\rho(i) \ldots c_i,k \), where we use a subpath of \( P_i \) to go from \( c_i,\rho(i) \) to \( c_i,k \). For every \( 1 \leq i,j \leq k \), if \( j \neq \rho(i) \), then the demand \( (f^1_{i,j}, f^2_{i,j}) \) is satisfied by the path \( f^1_{i,j} f_{i,j} f^2_{i,j} \). If \( j = \rho(i) \), then vertex \( f_{i,j} \) is already used by the demand \( (a_i,c_i,j) \). In the case demand \( (f^1_{i,j}, f^2_{i,j}) \) is satisfied by the path \( f^1_{i,j} c_i,0 \ldots c_i,j_1 f^2_{i,j} \). Finally, as \( \rho \) hits \( S \), there is a 1 \( \leq i \leq k \) such that \( (i,\rho(i)) \in S \) and hence the edges \( d_{i,\rho(i)} \) and \( d_{i,\rho(i)} \) exist. Therefore, we can satisfy the demand \( (s,t) \) via \( d_{i,\rho(i)} \). Note that this vertex is not used by the other paths: the path satisfying demand \( (a_i,c_i,k) \) uses \( P_i \) only from \( c_{i,0} \) to \( c_{i,\rho(i)} \), the path satisfying demand \( (f^1_{i,j}, f^2_{i,j}) \) uses \( P_i \) from \( c_{i,0} \) to \( c_{i,\rho(i)} \), and no other path reaches \( P_i \). This also implies that \( v^*_i,\rho(i) \) is used by none of the paths, as required.

For the second part, consider a solution of \( G_{k,S} \) representing some mapping \( \rho \). This means that the path of demand \( (a_i,c_i,k) \) uses vertex \( v^*_i,\rho(i) \) and hence \( b_{i,\rho(i)} \). The only way to reach \( c_i,k \) from \( b_{i,\rho(i)} \) without going through any other terminal vertex is using the path \( f_{i,\rho(i)} c_i,\rho(i) \ldots c_i,k \). This means that demand \( (f^1_{i,\rho(i)}, f^2_{i,\rho(i)}) \) cannot use vertex \( f^*_{i,\rho(i)} \), hence it has to use the path \( f^1_{i,\rho(i)} c_{i,0} \ldots c_{i,\rho(i)-1} f^2_{i,\rho(i)} \). It follows that for every \( 1 \leq i \leq k \) and \( j \neq \rho(i) \), vertices \( d_{i,j} \) and \( v^*_i,j \) are used by the paths satisfying demands \( (a_i,c_i,k) \) and \( (f^1_{i,\rho(i)}, f^2_{i,\rho(i)}) \). This shows that every \( v^*_i,j \) with \( j \neq \rho(i) \) is used by the paths in the solution. Moreover, the path satisfying \( (s,t) \) has to go through vertex \( d_{i,\rho(i)} \) for some \( i \). By the way the edges incident to \( s \) and \( t \) are defined, this is only possible if \( \rho(i) \in S \), that is, \( \rho \) hits \( S \).

Let \( S_1, \ldots, S_m \) be the sets appearing in the \( k \times k \) hitting set instance \( I \). We construct an instance \( \tilde{I} \) of directed disjoint paths consisting of \( m \) gadgets \( G_1, \ldots, G_m \), where gadget \( G_t \) \((1 \leq t \leq m)\) is a copy of the gadget \( G_{k,S_t} \), defined above. For every \( 1 \leq t < m \) and every \( 1 \leq i,j \leq k \), we identify vertex \( v^*_i,j \) of \( G_t \) and vertex \( v^*_i,j \) of \( G_{t+1} \). This completes the description of the instance \( \tilde{I} \) of directed disjoint paths.

We have to show that the pathwidth of the constructed graph \( \tilde{G} \) of \( \tilde{I} \) is \( O(k) \) and that \( \tilde{I} \) has a solution if and only if \( I \) has. To bound the pathwidth of \( \tilde{G} \), for every \( 0 \leq t \leq m, 1 \leq i,j \leq k \), let us define the bag \( B_{i,j} \) such that it contains vertices \( a_1, \ldots, a_k, b_1, \ldots, b_k, s, t, f, i,j, f^1_{i,j}, f^2_{i,j} \), and the path \( P_t \) of gadget \( G_t \) \((t = 0) \), and vertices \( a_1, \ldots, a_k, b_1, \ldots, b_k \) of gadget \( G_{t+1} \) \((t = m) \). It can be easily verified that the size of each bag is \( O(k) \) and if two vertices are adjacent, then they appear together in the same bag. Furthermore, if we order the bags lexicographically according to \((i,j)\), then each vertex appears precisely in an interval of the bags. This shows that the pathwidth of \( \tilde{G} \) is \( O(k) \).

Next we show that if \( I \) has a solution \( \rho : [k] \rightarrow [k] \), then \( \tilde{I} \) also has a solution. As \( \rho \) hits every \( S_t \), by the first part of the Claim, each gadget \( G_t \) has a solution representing \( \rho \). To combine these solutions into a solution for \( \tilde{I} \), we have to make sure that the vertices \( v^*_i,j \) that were identified are used only in one gadget. Since the solution for gadget \( G_t \) represents \( \rho \), it uses vertices \( v^*_i,\rho(i) \), \ldots, \( v^*_k,\rho(k) \), but no other \( v^*_i,j \) vertex. As vertex \( v^*_i,j \) of gadget \( G_t \) was identified with vertex \( v^*_i,j \) of gadget \( G_{t+1} \), these vertices might be used by the solution of \( G_{t+1} \) as well. However, the solution of \( G_{t-1} \) also represents \( \rho \) and as claimed in the first part of the Claim, the solution does not use vertices \( v^*_i,\rho(1), \ldots, v^*_k,\rho(k) \). Therefore, no conflict arises between the
solutions of $G_t$ and $G_{t-1}$.

Finally, we have to show that a solution for $\bar{I}$ implies that a solution for $I$ exists. We say that a solution for $\bar{I}$ is normal with respect to $G_t$ if the paths satisfying the demands in $G_t$ do not leave $G_t$ (the vertices $u_{i,j}$, $v_{i,j}^*$ that were identified are considered as part of both gadgets, so we allow the paths to go through these vertices). We show by induction that the solution for $\bar{I}$ is normal for every $G_t$. Suppose that this is true for $G_{t-1}$. If some path $P$ satisfying a demand in $G_t$ leaves $G_t$, then it has to enter either $G_{t-1}$ or $G_{t+1}$. If $P$ enters a vertex of $G_{t+1}$ that is not in $G_t$, then it cannot go back to $G_t$; the only way to reach a vertex $v_{i,j}$ of $G_{t+1}$ is from vertex $a_j$, which has indegree 0. Therefore, let us suppose that $P$ enters $G_{t-1}$ at some vertex $v_{i,j}^*$ of $G_{t-1}$. The only way the path can return to $G_t$ is via some vertex $v_{i,j}'$ of $G_{t-1}$ with $j' \geq j$. By the induction hypothesis, the solution is normal with respect to $G_{t-1}$, thus the second part of the Claim implies that there is a unique $v_{i,j}$ such that $v_{i,j}'$ is not used by the paths satisfying the demands in $G_{t-1}$. As $P$ can use only this vertex $v_{i,j}'$, it follows that $j' = j$ and hence path $P$ does not use any vertex of $G_{t-1}$ not in $G_t$. In other words, $P$ does not leave $G_t$.

Suppose now that the solution is normal with respect to every $G_t$, which means that it induces a solution for every gadget. Suppose that the solution of gadget $G_t$ represents mapping $\rho_t$. We claim that every $\rho_t$ is the same. Indeed, if $\rho_t(i) = j$, then the solution of $G_t$ uses vertex $u_{i,j}$ of $G_t$, which is identical to vertex $v_{i,j}^*$ of $G_{t-1}$. This means that the solution of $G_{t-1}$ does not use $v_{i,j}^*$, and by the second part of the Claim, this is only possible if $\rho_{t-1}(i) = j$. Thus $\rho_{t-1} = \rho_t$ for every $1 \leq i \leq m$, let $\rho$ be this mapping. Again by the claim, $\rho$ hits every set $S_t$ in instance $I$, thus $\rho$ is a solution of $I$. \(\Box\)

For our proof we will also need the following lemma.

**Lemma 5.1.** ([8]) Let $G$ be a graph (possibly with parallel edges) having pathwidth at most $w$. Let $G'$ be obtained from $G$ by subdividing some of the edges. Then the pathwidth of $G'$ is at most $w + 1$.

**Theorem 5.2.** Assuming ETH, there is no $2^{O(w \log w)} \cdot n^{O(1)}$ time algorithm for Disjoint Paths.

**Proof.** Let $\bar{I}$ be a instance of Directed Disjoint Paths on a directed graph $D$ having pathwidth $w$. We transform $D$ into an undirected graph $G$, where two adjacent vertices $\bar{v}_{in}$, $\bar{v}_{out}$ correspond to each vertex $v$ of $D$, and if $\bar{w}$ is an edge of $D$, then we introduce a new vertex $e_{uv}$ that is adjacent to both $\bar{u}_{out}$ and $\bar{v}_{in}$. It is not difficult to see that the pathwidth of $G$ is at most $2w + 1 = O(w)$: $G$ can be obtained from the underlying graph of $D$ by duplicating vertices (which at most doubles the size of each bag) and subdividing edges (which does not increase pathwidth).

Let $I$ be an instance of Disjoint Paths on $G$ where there is a demand $(\bar{v}_{out}, \bar{u}_{in})$ corresponding to every demand of $(v, u)$ of $\bar{I}$. It is clear that if $\bar{I}$ has a solution, then $I$ has a solution as well: every directed path from $u$ to $v$ in $D$ can be turned into a path connecting $\bar{u}_{out}$ and $\bar{v}_{in}$ in $G$. However, the converse is not true: it is possible that an undirected path $P$ in $G$ reaches $\bar{v}_{in}$ from $e_{uv}$ and instead of continuing to $\bar{v}_{out}$, it continues to some $e_{uw}$. In this case, there is no directed path corresponding to $P$ in $D$. We add further edges and demands to forbid such paths.

Let $B_1$, $\ldots$, $B_n$ be a path decomposition of $G$ having width $w' = O(w)$. For every vertex $x$ of $G$, let $\ell(x)$ and $r(x)$ be the index of the first and last bags, respectively, were $x$ appears. It is well-known that the decomposition can be chosen such that $r(x) \neq r(y)$ for any two vertices $x$ and $y$.

We modify $G$ to obtain a graph $G'$ the following way. If vertex $v$ has $d$ inneighbors $u_1, \ldots, u_d$ in $D$, then $\bar{v}_{in}$ has $d + 1$ neighbors in $G$: $\bar{v}_{out}$ and $d$ vertices $e_{u_1,v}$, $\ldots$, $e_{u_d,v}$. Suppose that the neighbors of $v$ are ordered such that $r(e_{u_i,v}) < \cdots < r(e_{u_d,v})$. We introduce $2d - 2$ new vertices $v_1, \ldots, v_{d-1}, v_1', \ldots, v_{d-1}'$ such that $v_i$ and $v_i'$ are both adjacent to $e_{u_i,v}$ and $e_{u_i,v}$. For every $1 \leq i \leq d - 1$, we introduce a new demand $(v_i^*, v_i'^*)$. Repeating this procedure for every vertex $v$ of $D$ creates an instance $I'$ of undirected Disjoint Paths on a graph $G'$.

We show that these new vertices and edges increase the pathwidth at most by a constant factor. Observe that $G'$ can be obtained from $G$ by adding two parallel edges between $e_{u_i,v}$ and $e_{u_i,v}'$ and subdividing them. Thus by Lemma 5.1, all we need to show is that adding these new edges increases pathwidth only by a constant factor. If $r(e_{u_i,v}) \geq \ell(e_{u_i+1,v})$, then the parallel edges between $e_{u_i,v}$ and $e_{u_i+1,v}$ can be added without changing the path decomposition: bag $B_{\ell(e_{u_i+1,v})}$ contains both vertices. If $r(e_{u_i,v}) < \ell(e_{u_i+1,v})$, then let us insert vertex $e_{u_i,v}$ into every bag $B_j$ for $r(e_{u_i,v}) < j \leq \ell(e_{u_i+1,v})$. Now bag $B_{\ell(e_{u_i+1,v})}$ contains both $e_{u_i,v}$ and $e_{u_i+1,v}$, thus we can add two parallel edges between them. Note that vertex $\bar{v}_{in}$ appears in every bag where $e_{u_i,v}$ is inserted: if not, then either $\bar{v}_{in}$ does not appear in bags with index at most $r(e_{u_i,v})$, or it does not appear in bags with index at least $\ell(e_{u_i+1,v})$, contradicting the fact that $\bar{v}_{in}$ is adjacent to both $e_{u_i,v}$ and $e_{u_i+1,v}$. Furthermore, vertices $e_{u_i,v}$ and $e_{u_i,v}'$ are not inserted into the same bag for any $i \neq j$: if $j > i$, then $r(e_{u_j,v}) \geq \ell(e_{u_i+1,v}) \geq \ell(e_{u_i+1,v})$. Therefore, the number of new vertices in each bag is at most the original size of the bag, i.e., the size of each bag increases.
by at most a factor of 2.

We claim that $I'$ has a solution if and only if $I$ has. If $I$ has a solution, then the directed path satisfying demand $(u, v)$ gives in a natural way an undirected path in $G'$ that satisfies demand $(v_{\text{out}}, v_{\text{in}})$. Thus we can obtain a pairwise disjoint collection of paths that satisfy the demands of the form $(v_{\text{out}}, v_{\text{in}})$. Note that if $v_{\text{out}}, e_{u_1v}, \ldots, e_{u_dv}$ are the neighbors of $v_{\text{in}}$ in $G'$, then the paths in this collection use at most one of the vertices $e_{u_1v}, \ldots, e_{u_dv}$, say, $e_{u_1v}$. Now we can satisfy the demands $(v_i^e, v_i^f)$ for every $1 \leq i \leq d - 1$: for $i < j$, we can use the path $v_i^e e_{u_i v} v_i^f$, for $i \geq j$, we can use the path $v_i^e e_{u_{i+1} v} v_i^f$. Thus instance $I'$ has a solution.

For the other direction, suppose that $I'$ has a solution. Let us call a path of this solution a main path if it satisfies a demand of the form $(v_{\text{out}}, v_{\text{in}})$. We claim that if $v_{\text{in}}$ is an internal vertex of a main path $P$, then $P$ contains $v_{\text{out}}$ as well. Otherwise, $P$ has to contain at least two of the neighbors $e_{u_1v}, \ldots, e_{u_dv}$ of $v_{\text{out}}$. In this case, less than $d - 1$ vertices out of $e_{u_1v}, \ldots, e_{u_dv}$ remain available for the $d - 1$ demands $(v_i^e, v_i^f), \ldots, (v_d^e, v_d^f)$, a contradiction.

Consider a main path $P$ that satisfies a demand $(v_{\text{out}}, v_{\text{in}})$ of $I'$. Clearly, $P$ cannot go through any terminal vertex other than $v_{\text{out}}$ and $v_{\text{in}}$. As $u$ has indegree 0 in $D$, path $P$ has to go to some $v_{\text{out}}$ and then to $v_{\text{in}}$ after starting from $v_{\text{out}}$. By our claim in the previous paragraph, the next vertex has to be $v_{\text{out}}$, then again some $e_{uw}$ and $z_{\text{in}}$ and so on. Thus there is a directed path in $D$ that corresponds to $P$ in $G'$. This means that directed paths corresponding to the main paths of the solution for $I'$ form a solution for $I$.

6 Conclusion

In this paper we showed that several parameterized problems have slightly superexponential running time unless ETH fails. In particular we showed for three well-studied problems arising in three different domains that the known superexponential algorithms are optimal: assuming ETH, there is no $2^{o(d \log d)} \cdot |I|^{O(1)}$ or $2^{o(d \log (|\Sigma|))} \cdot |I|^{O(1)}$ time algorithm for CLOSEST STRING, $2^{o(d \log d)} \cdot |I|^{O(1)}$ time algorithm for DISTORTION, and $2^{o(w \log w)} \cdot |I|^{O(1)}$ time algorithm for DISJOINT PATHS parameterized by treewidth. We believe that many further results of this form can be obtained by using the framework of the current paper. Some concrete problems that might be amenable to our framework are:

- Are the known parameterized algorithms for Point Line Cover [42, 34], Directed Feedback Vertex Set [11] and INTERVAL COMPLETION [55], parameterized by the solution size, running in time $2^{O(k \log k)} \cdot |I|^{O(1)}$ optimal?
- Are the known parameterized algorithms for HAMILTONIAN PATH [28], CONNECTED VERTEX COVER [49] and CONNECTED DOMINATING SET [19], parameterized by the treewidth $w$ of the input graph, running in time $2^{O(w \log w)} \cdot |I|^{O(1)}$ optimal?

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