Gravitational Yang-Lee Model.
Four Point Function

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Abstract
The four-point perturbative contribution to the spherical partition function of the gravitational Yang-Lee model is evaluated numerically. An effective integration procedure is due to a convenient elliptic parameterization of the moduli space. At certain values of the “spectator” parameter the Liouville four-point function involves a number of “discrete terms” which have to be taken into account separately. The classical limit, where only discrete terms contribute, is also discussed. In addition, we conjecture an explicit expression for this partition function at the “second solvable point” where the spectator matter is in fact another \( \mathcal{M}_{2/5} \) (Yang-Lee) minimal model.

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1. Introduction

This work is a direct continuation of the previous developments of refs. [2] and [3] where a kind of analytic-numeric extrapolation of the perturbative series has been applied to study the spherical partition function of a perturbed conformal matter immersed to the quantized 2D gravity. In these works the scaling Yang-Lee model (complemented with a conformal “spectator matter”) is taken as an example of the perturbed matter theory. Similar development for the massive free Majorana fermions (the scaling Ising model) coupled to 2D gravity is reported in ref. [4].

For the further convenience we summarize in this section few main concepts of [3]. This is mostly to facilitate subsequent references and introduce notations. It doesn’t make the present report self consistent, a minimum acquaintance with the previous one [3] is assumed.

**Gravitational Yang-Lee** model (GYL) can be conventionally defined through the Lagrangian density

\[ \mathcal{L}_{GYL} = \mathcal{L}_{\text{matter}} + \mathcal{L}_L + \mathcal{L}_{gh} \]  

(1.1)

where \( \mathcal{L}_L \) is referred to as the Liouville component of the theory

\[ \mathcal{L}_L = \frac{1}{4\pi} (\partial_\phi)^2 + \mu e^{2b\phi} \]  

(1.2)

and \( \mathcal{L}_{gh} \) is the Lagrangian of the \( c_{gh} = -26 \) conformal ghost \( BC \) system. As usual, \( \phi \) is the Liouville field, \( \mu \) is the cosmological constant and \( b \) is a parameter related to the Liouville central charge \( c_L = 1 + 6Q^2 \) through the “background charge” \( Q = b + b^{-1} \). The matter component consists of some unperturbed “spectator” CFT with central charge \( c_{sp} \) and the \( \mathcal{M}_{2/5} \) minimal CFT model (critical Yang-Lee model [6] with \( c_{YL} = -22/5 \)) perturbed by the only \( \mathcal{M}_{2/5} \) non-trivial primary field \( \varphi \), i.e., the basic Yang-Lee field of dimension \( \Delta = -1/5 \)

\[ \mathcal{L}_{\text{matter}} = \mathcal{L}_{sp} + \mathcal{L}_{YL} + \frac{i\lambda}{2\pi} \varphi(x) e^{2g\phi} \]  

(1.3)

Parameters \( b \) in (1.2) and \( g \) in the interaction term of eq.(1.3), are determined by \( c_{sp} \) and \( \Delta \) through the balance equations

\[ c_{sp} + c_{YL} + c_L + c_{gh} = 0 \]

\[ g(Q - g) + \Delta = 1 \]  

(1.4)

In the case of spherical geometry, which we only consider in the present study, any details of the spectator matter are not important except for the parameter \( c_{sp} \). This component is added to get a formal access to the parameter \( b^2 \) of the model, in particular, to have a link with the classical limit. The interaction term contains a dimensional coupling constant \( \lambda \sim \mu^{1/\rho} \) where

\[ \rho = bg^{-1} \]  

(1.5)
The **spherical partition function** $Z(\mu, \lambda)$ of the model is developed as a systematic perturbative series in the coupling constant $\lambda$

$$
\frac{Z(\mu, \lambda)}{Z(\mu, 0)} = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{(2\pi)^n n!} \langle \langle (\varphi e^{2g\phi})^n \rangle \rangle_{\text{LG}}
$$

(1.6)

where $\langle \langle (\varphi e^{2g\phi})^n \rangle \rangle_{\text{LG}}$ are (integrated and normalized) $n$-point functions in the Liouville gravity (i.e., at $\lambda = 0$).

The normalized correlation functions scale as

$$
\langle \langle (\varphi e^{2g\phi})^n \rangle \rangle_{\text{LG}} = (\pi \mu)^{-n/\rho} a_n
$$

(1.7)

with some dimensionless numbers $a_n$, i.e., the perturbative development is in fact a series in powers of the dimensionless scaling parameter $\lambda \mu^{-1/\rho}$. It is convenient to introduce also the unnormalized correlation functions

$$
\langle (\varphi e^{2g\phi})^n \rangle_{\text{LG}} = (\pi \mu)^{Q/b-n/\rho} G_n
$$

(1.8)

In the Liouville gravity these correlation functions are evaluated as the integrals of the products of the Liouville and $\mathcal{M}_{2/5}$ $n$-point functions

$$
\langle (\varphi e^{2g\phi})^n \rangle_{\text{LG}} = \langle C \bar{C}(x_1)C \bar{C}(x_2)C \bar{C}(x_3) \rangle_{\text{gh}} \times
\int \langle \varphi(x_1)\ldots\varphi(x_n) \rangle_{\text{YL}} \langle V_g(x_1)\ldots V_g(x_n) \rangle_L d^2x_1 \ldots d^2x_n
$$

(1.9)

where $\langle \ldots \rangle_{\text{gh}}$, $\langle \ldots \rangle_{\text{YL}}$ and $\langle \ldots \rangle_L$ are related to respectively ghost, $\mathcal{M}_{2/5}$ minimal model and (unnormalized) Liouville correlations. We also denote $V_g = \exp(2g\phi)$. Geometrically the $n - 3$ dimensional integral in (1.9) is the integral over the moduli space of a sphere with $n$ punctures.

As in [3], in this paper we are mainly interested in the fixed area partition function $Z_A(\lambda)$, which is related to $Z(\mu, \lambda)$ as

$$
Z(\mu, \lambda) = \int_{(0)}^{\infty} Z_A(\lambda) e^{-\mu A} \frac{dA}{A}
$$

(1.10)

The lower limit (0) here is simply a particular prescription how to regularize the divergency of the integral at small $A$ [3]. The fixed area partition function has the following scaling form

$$
Z_A(\lambda) = Z_A(0) z(h)
$$

(1.11)

where $Z_A(0)$ scales as $A^{-Q/b}$. The scaling function $z(h)$ is a regular expansion

$$
z(h) = \sum_{n=0}^{\infty} z_n (-h)^n
$$

(1.12)

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3Here notations differ slightly from those of [3], where the coefficients $a_n$ were dimensional and included the multiplier $(2\pi)^n n!$. 

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in the fixed area dimensionless scaling parameter

\[ h = \lambda \left( \frac{A}{\pi} \right)^{1/\rho} \] (1.13)

Apparently, coefficients \( z_n \) are related to the numbers \( G_n \) in (1.8) as

\[ z_n = \frac{a_n \Gamma(-1 - b^{-2})}{(2\pi)^n n! \Gamma(n\rho^{-1} - b^{-2} - 1)} \] (1.14)

Sometimes it is convenient to express these coefficients directly

\[ z_n = \frac{\mathcal{I}_n}{(2\pi)^n n! Z_L^{(A)}} \] (1.15)

through the non-normalized fixed area correlation functions

\[ \langle (\varphi e^{2\varphi})^n \rangle^{(A)} = \left( \frac{A}{\pi} \right)^{n/\rho - Q/b} \mathcal{I}_n = \frac{(A\pi^{-1})^{n/\rho - Q/b} G_n}{\Gamma(n\rho^{-1} - 1 - b^{-2})} \] (1.16)

and the fixed area Liouville partition function

\[ Z_L^{(A)} = \frac{\gamma^{1/2} b^2}{\pi^3 \Gamma(b^{-2} - 1)} \] (1.17)

It was argued in [3] that \( z(h) \) is an entire function of \( h \).

The infinite volume specific energy \( \mathcal{E}_{\text{vac}}(b^2) = -\mu_c \) enters the leading \( A \to \infty \) asymptotic of \( Z_A(\lambda) \)

\[ Z_A(\lambda) \sim Z_{\infty} \left( \frac{A}{\pi} \right)^{-Q_{\text{IR}}/b_{\text{IR}}} \exp \left( -\mathcal{E}_{\text{vac}}(b) A \right) \] (1.18)

This quantity scales as

\[ \mu_c = f_0(b^2) \lambda^\rho \] (1.19)

Here we introduced dimensionless function \( f_0(b^2) \) which is an important universal characteristic of GYL. Numerical study of this quantity at different values of the “spectator” parameter \( b^2 \) is the main topic of ref. [3] and of the present article. Parameters \( Q_{\text{IR}} = b_{\text{IR}}^{-1} + b_{\text{IR}} \) and \( b_{\text{IR}} \) in (1.18) are fixed by the “IR central charge balance” [3]

\[ 1 + 6Q_{\text{IR}}^2 + c_{\text{sp}} = 26 \] (1.20)

In terms of the scaling function \( z(h) \) eq. (1.18) reads as the following asymptotic behavior at \( h \to \infty \)

\[ \log z(h) = \pi f_0(b) h^\rho + (\delta + 1/2) \log h + O(1) \] (1.21)
where
\[ \delta + 1/2 = \rho \left( Qb^{-1} - Q_{IR}b_{IR}^{-1} \right) \tag{1.22} \]

All the above considerations are either kinematical or based on natural physical assumptions. A less trivial observation is that in GYL (as well as in a number of other important gravitational models, see e.g., [4]) the asymptotic (1.21) holds in the whole complex plane away from the negative real axis and that all zeros of this entire function \( z(h) \) are real. This property has been established numerically in the classical limit \( b^2 = 0 \) [2] and in the special, exactly solvable case \( b^2 = 0.4 \). It is then extended, as a conjecture, to the whole region of the parameter. Combined with the asymptotic behavior (1.21) this feature leads to an effective analytic-numeric algorithm, which allows to restore the scaling function to an impressive accuracy starting from a few first perturbative coefficients \( z_n \) in the expansion (1.12) (see [3] for more details).

In the previous study we used these coefficients up to \( z_3 \), where the numerical evaluation of the matter and Liouville correlation functions doesn’t offer any technical difficulties. In the present article we develop a method of numerical integration over moduli, which allows to calculate the four-point function in (1.9) to a precision sufficient to improve the results of [3].

For completeness we quote here the expressions for \( z_2 \) and \( z_3 \) used in [3]
\[ z_2 = -\frac{\gamma^{1-2g/b}(b^2)\Gamma(b^{-2}-1)\gamma(2gb-b^2)}{8\Gamma(1+b^{-2}-2gb^{-1})} \tag{1.23} \]
\[ z_3 = \kappa \frac{\gamma^{1-3g/b}(b^2)\Gamma(b^{-2}-1)\gamma(3gb-b^2)\Upsilon_b(b)\Upsilon_b^3(2g)}{48b\Gamma(2+b^{-2}-3gb^{-1})\Upsilon_b(3g)\Upsilon_b^3(g)} \]
where
\[ \kappa = \frac{\gamma^{3/2}(1/5)}{5\gamma^{1/2}(3/5)} = 1.91131 \ldots \tag{1.24} \]
in the last expression is related to the basic structure constant \( C_{\phi\phi\phi} \) in the critical Yang-Lee model [6]. Special function \( \Upsilon_b(x) \) is the standard element of the Liouville field theory construction (see [7] or [8] for the definitions and properties).

2. Matter four point function

In the four-point function the integral (1.9) reduces to
\[ \left\langle (\varphi e^{2g\phi})^4 \right\rangle_{LG} = \int G_{YL}(x, \bar{x})G_L(x, \bar{x})d^2x \tag{2.1} \]
where both in the Yang-Lee four point function
\[ G_{YL}(x, \bar{x}) = \left\langle \varphi(0)\varphi(1)\varphi(\infty)\varphi(x) \right\rangle_{YL} \tag{2.2} \]
and the Liouville one
\[ G_L(x, \bar{x}) = \left\langle V_g(0)V_g(1)V_g(\infty)V_g(x) \right\rangle_L \tag{2.3} \]
we have used the projective invariance of the integrand to put \( x_1, x_2 \) and \( x_3 \) to 0, 1 and \( \infty \) respectively.

The \( M_{2/5} \) four point function reads explicitly [3] as

\[
G_{YL}(x, \bar{x}) = \mathcal{F}_I(x)\mathcal{F}_I(\bar{x}) - \kappa^2 \mathcal{F}_\varphi(x)\mathcal{F}_\varphi(\bar{x})
\]  

(2.4)

where \( \kappa \) is from (1.24) and the blocks \( \mathcal{F}_{I,\varphi}(x) \) are expressed through the hypergeometric functions

\[
\mathcal{F}_I(x) = x^{2/5}(1 - x)^{1/5} F_1(2/5, 3/5, 6/5, x)
\]

\[
\mathcal{F}_\varphi(x) = x^{1/5}(1 - x)^{1/5} F_1(1/5, 2/5, 4/5, x)
\]

(2.5)

3. Liouville four point function

The non-normalized Liouville four point function \([2,3]\) can be represented as an integral over the intermediate momentum \( P \) [7] (see also [3] from where the expressions below are read off). In our context it looks more convenient to start with the fixed area Liouville four-point function

\[
G_{L}^{(A)}(A) = \left( \frac{A}{\pi} \right)^{(4g - Q)/b} g_L(x, \bar{x})
\]

(3.1)

Function \( g_L(x, \bar{x}) \) in general can be evaluated through the following integral representation

\[
g_L(x, \bar{x}) = \frac{\mathcal{R}_g}{\Gamma(4g - 1 - b^2 - 1)} \int dP r_g(P) \mathcal{F}_P(x) \mathcal{F}_P(\bar{x})
\]

(3.2)

The prime near the integral sign denotes possible discrete terms (see below) and

\[
\mathcal{F}_P(x) = \mathcal{F}_P \left( \begin{array}{c} \Delta_g \\ \Delta_g \\ \Delta_g \\ \Delta_g \end{array} \right) x^0
\]

(3.3)

is the general four point conformal block with all four external dimensions \( \Delta_g = g(Q - g) = 6/5 \), the central charge \( c_L \) and the intermediate dimension \( Q^2/4 + P^2 \). This function was introduced in [5]. In Appendix A we recapitulate some details end explicit constructions concerning this object. As in the previous paper [3] we use the notations

\[
\mathcal{R}_g = \left( \gamma^2(b^2 - 2b^2) \right)^{(Q - 4g)/b} \frac{\mathcal{Y}_b^4(b) \mathcal{Y}_b^4(2g)}{\pi^2 \mathcal{Y}_b^4(2g - Q/2)}
\]

(3.4)

and

\[
r_g(P) = \frac{\pi^2 \mathcal{Y}_b(2iP) \mathcal{Y}_b(-2iP) \mathcal{Y}_b^4(2g - Q/2)}{\mathcal{Y}_b^2(b) \mathcal{Y}_b^2(2g - Q/2 - iP) \mathcal{Y}_b^2(2g - Q/2 + iP) \mathcal{Y}_b^2(Q/2 - iP) \mathcal{Y}_b^2(Q/2 + iP)}
\]

(3.5)
The last function enters the integrand in $P$ in (3.2) and therefore a quick numerical evaluation is important. Expression (3.5) admits the following integral representation

$$r_g(P) = \sinh(2\pi b^{-1} P) \sinh(2\pi b P)$$

$$\times \exp \left( -8 \int_0^\infty \frac{dt}{t} \frac{t^2 \sin^2 \left( Q - 2g \right) t - e^{-Qt} \cos^2 Pt}{\sinh bt \sinh b^{-1} t} \right)$$

(convergent at $g > Q/4$). The integral here is convergent if $g > Q/4$, i.e., in our example, at $b^2 > b_0^2$ (where $b_0^2 = (11 - 4\sqrt{6})/5 = 0.2404\ldots$, see below). At smaller values of $b^2$ a slightly more complicated expression applies

$$r_g(P) = \sinh(2\pi b^{-1} P) \sinh(2\pi b P) \frac{\gamma^4(3/2 + b^{-2}/2 - 2g b^{-1})}{\gamma^2(3/2 + b^{-2}/2 - 2g b^{-1} + ib^{-1} P) \gamma^2(3/2 - b^{-2}/2 - 2g b^{-1} - ib^{-1} P)}$$

$$\times \exp \left( -8 \int_0^\infty \frac{dt}{t} \frac{t^2 \sin^2 \left( b - 2g \right) t - e^{-Qt} \cos^2 Pt}{\sinh bt \sinh b^{-1} t} \right)$$

This integral representation converges in the interval $0 < b^2 < (23 - 4\sqrt{19})/15 = 0.371\ldots$, which complements the region of convergence of (3.4) (and has essential overlap with it).

The integral in (3.2) is understood literally, i.e., it goes along the real axis, only if $b^2 > b_0^2 = 0.2404\ldots$. At $b^2 = b_0^2$ two double poles of $r_g(P)$ at $iP = \pm(Q/2 - 2g)$ cross the integration contour and must be picked up explicitly as the discrete terms. Then, at $b^2 = b_1^2 = (4\sqrt{139} - 43)/25 \approx 0.1664\ldots$ the same happens with the poles at $iP = \pm(Q/2 - 2g - b)$ and so on. In general the pair of double poles $iP = \pm(Q/2 - 2g - nb)$ with $n = 0, 1, 2, \ldots$ shows up in the form of the discrete term at $b^2 < b_n^2$ where

$$b_n^2 = \frac{4\sqrt{25n^2 + 60n + 54} - 10n - 33}{5(4n^2 + 4n - 3)}$$

Below we will use the notation

$$P_n = i(2g + nb - Q/2)$$

In the presence of the discrete terms expression (3.2) reads

$$g_L(x, \bar{x}) = \sum_{n=0}^{N_d-1} D_n(x, \bar{x}) + \frac{\mathcal{R}_g}{\Gamma(4gb^{-1} - b^{-2} - 1)} \int \frac{dP}{4\pi} r_g(P) \mathcal{F}_P(x) \mathcal{F}_P(\bar{x})$$

where $N_d$ is the actual number of discrete terms

$$N_d = \text{Floor} \left[ \sqrt{1 + b^{-4} - 14b^{-2}/5 - b^{-2}/2 + 1/2} \right]$$
while in the last “integral” term the ordinary integration over real $P$ is implied. The discrete terms are in a sense “logarithmic”

$$D_n(x, \bar{x}) = \mathcal{N}_n \mathcal{F}_{P_n}(x) \mathcal{F}_{P_n}(\bar{x})(2 \Re f_{P_n}(x) + U_n) \quad (3.12)$$

where, as in [4], we introduced the logarithmic derivative in $P$ of the general Liouville block

$$f_P(x) = i \frac{d}{dP} \log \mathcal{F}_P(x) \quad (3.13)$$

The logarithms appearing in this derivative are due to a kind of degeneracy which occur at equal external dimensions and leads to the double poles in the integrand of (3.2). Apparently this effect hides nothing conceptually new and there is no point to talk about the Liouville field theory as of “logarithmic CFT”\(^4\). In the four point function with different external dimensions there are no logarithms. The constants $\mathcal{N}_n$ and $U_n$ in (3.12) read explicitly

$$\mathcal{N}_n = \frac{\gamma(Q-4g/b)(b^2)\gamma(4gb - b^2 + 2nb^2)}{b^2 \Gamma(2 + b^{-2} - 4gb^{-1})} \times \prod_{k=0}^{n-1} \frac{(4g - b^{-1} - b + kb)^2}{(4g - b^{-1} + (k + n - 1)b)^2} \frac{\gamma^2(4gb + (k + n - 1)b^2)}{\gamma^2(2gb + kb^2)\gamma^2(1 + (k + 1)b^2)}$$

and

$$U_n = 2\nu(b)(4g - Q + nb) - 2\nu(4g - Q + 2nb) + 4\nu(2g + nb) - 2\nu(4g + 2nb) - 2\nu(b) - 4(n + 1)b \log b - 4bC_E + 2b \sum_{k=1}^{n} (\psi(-kb^2) + \psi(1 + kb^2)) \quad (3.15)$$

In the last expression we have, as in ref. [4], introduced the notation

$$\nu_b(x) = \frac{d}{dx} \log \Upsilon_b(x) \quad (3.16)$$

for the logarithmic derivative of the $\Upsilon_b$-function. This special function can be evaluated through the integral representation

$$\nu_b(x) = \int_0^\infty dt \left( \frac{\sinh(Q - 2x)t}{\sinh bt \sinh b^{-1} t} - \frac{Q - 2x}{t} e^{-2t} \right) \quad (3.17)$$

convergent in the strip $0 < \Re x < Q$. Outside this region one of the following relations, whichever more convenient, can be used to render the argument to the strip of convergence

$$\nu_b(x + b) - \nu_b(x) = b(-2 \log b + \psi(bx) + \psi(1 - bx)) \quad (3.18)$$

$$\nu_b(x + b^{-1}) - \nu_b(x) = b^{-1}(2 \log b + \psi(b^{-1} x) + \psi(1 - b^{-1} x))$$

\(^4\)Whatever this last term means.
4. Elliptic modular parameter

After all these preliminaries we can turn to the integral (2.1). For the fixed area four point function it reads

$$I_4 = \left\langle (\varphi e^{2q\phi})^4 \right\rangle_{L_G}^{(A=\pi)} = 6 \int_G G_{Y\ell L}(x, \bar{x}) g_{\ell L}(x, \bar{x}) d^2x$$  \hspace{1cm} (4.1)

As in ref. [3], in eq. (4.1) we used the symmetry of the integral under the six element modular subgroup of projective group, generated by the transformations $x \to 1/x$ and $x \to 1 - x$. This group divides the complex plane of $x$ in 6 regions, the fundamental region $G = \{ \text{Re} x < 1/2; |1 - x| < 1 \}$ and its 5 images. The integral in (4.1) is reduced to $G$ while the factor 6 in front of the integral takes into account the equivalent images.

It turns remarkably convenient to introduce the “elliptic” modular parameter through the standard map

$$\tau = i \frac{K(1 - x)}{K(x)}$$ \hspace{1cm} (4.2)

where

$$K(x) = \frac{1}{2} \int_0^1 \frac{dt}{[t(1 - t)(1 - xt)]^{1/2}}$$  \hspace{1cm} (4.3)

is the complete elliptic integral of the first kind. The integral (4.1) becomes

$$I_4 = 6\pi^2 \int_F |x(1 - x)\theta_3^4(q)|^2 G_{Y\ell L}(q, \bar{q}) g_{\ell L}(q, \bar{q}) d^2\tau$$  \hspace{1cm} (4.4)

where $F = \{ |\tau| > 1; |\text{Re} \tau| < 1/2 \}$ is now the standard fundamental region of the modular group and

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$ \hspace{1cm} (4.5)

is the usual $\theta$-series in

$$q = e^{i\pi \tau}$$ \hspace{1cm} (4.6)

There are two important advantages in the form (4.4). As it has been argued in refs. [10], a general four point conformal block admits a convenient recursive representation, which looks particularly simple in terms of the elliptic parameter $q$ and the so called “elliptic” four point block. Also, while the two blocks (2.5) are known in closed form, the Liouville block (3.3) at general $P$ is only computed as a power series in $x$ or $q$. The recursive algorithm in the elliptic representation gives directly the power series in $q$, which is argued to have much better convergence then that in $x$. In particular, in the elliptic fundamental region $\max_F |q|^2 = \exp(-\pi \sqrt{3}) = 0.00433 \ldots$ while in its $x$ image $\max_G |x|^2 = 1$ near the cusps of the fundamental region, where the convergence of the $x$-series is questionable.
Elliptic representations of the four point conformal blocks is recapitulated in the Appendix. It is also shown there that the products of the Yang-Lee and Liouville blocks read in the elliptic parametrization as

\[ x(1-x)\theta_3^2(q)\mathcal{F}_\alpha(x)\mathcal{F}_\varphi(x) = (16q)^{\Delta_\alpha + P^2 + \lambda_1^2} H_\alpha(q)H_\varphi(q)H_{sp}(q) \] (4.7)

Here and below the index \( \alpha = I, \varphi \) specifies the intermediate representation in the Yang-Lee block, in particular \( \Delta_I = 0 \) and \( \Delta_\varphi = -1/5 \). The corresponding elliptic blocks \( H_\alpha(q) \) can be found as a series in \( q \) either through the explicit expressions (2.5)

\[ \mathcal{F}_I(x) = (16q)^{9/40}[x(1-x)]^{7/40}\theta_3^{1/2}(q)H_I(q) \] (4.8)

\[ \mathcal{F}_\varphi(x) = (16q)^{1/40}[x(1-x)]^{7/40}\theta_3^{1/2}(q)H_\varphi(q) \]

or via the recursive relation (A.8). In both cases we have

\[
\begin{align*}
H_I(q) &= 1 - \frac{21}{22}q^2 - \frac{51}{88}q^4 + \frac{1989}{5456}q^6 - \frac{111489}{3579136}q^8 + \frac{1612779}{3579136}q^{10} - \ldots \\
H_\varphi(q) &= 1 - \frac{5}{6}q^2 - \frac{91}{152}q^4 + \frac{1967}{8816}q^6 - \frac{8211}{70528}q^8 + \frac{405647}{987392}q^{10} - \ldots 
\end{align*}
\] (4.9)

The “central charge deficit” part

\[
H_{sp}(q) = (1 - q^2 - q^4 + \ldots)^{-c_{sp}/2}
\] (4.10)

is carried out in the Appendix. Finally, the \( q \)-expansion of the Liouville elliptic block is easily generated through recursive relation (A.8). Although the calculation is straightforward, the result is somewhat cumbersome at higher orders, so that here we quote it explicitly only up to \( O(q^2) \) (the notation \( \lambda_{m,n} \) is explained in [A.2])

\[
H_P(q) = 1 + \frac{(16p^2 - b^2)^2 q^2}{4(1 - b^4)(P^2 + \lambda_{1,2}^2)} + \frac{(16p^2 - b^{-2})^2 q^2}{4(1 - b^4)(P^2 + \lambda_{2,1}^2)} + O(q^4)
\] (4.11)

where \( p = Q/2 - g \). In numerical calculations below we used the expansion of \( H_P(q) \) up to the order \( q^8 \). In our symmetric case (all external dimensions are equal) all the elliptic blocks are series in \( q^2 \).

5. Modular integral

The Liouville correlation function can be separated into the “integral part” and, sometimes, a number of discrete terms (3.10). Consequently

\[
\mathcal{I}_4 = \sum_{n=0}^{N_4-1} \mathcal{I}_{\text{disc}, n} + \mathcal{I}_{\text{int}}
\] (5.1)
For numerical integration it is convenient to further separate each term in two parts, corresponding to two matter blocks in (2.4)

\[ I_{\text{int}} = J_{\text{int}}^{(f)} - \kappa^2 J_{\text{int}}^{(v)} \]  
\[ I_{\text{disc, } n} = J_{\text{disc, } n}^{(f)} - \kappa^2 J_{\text{disc, } n}^{(v)} \]  

Consider first the integral part

\[ J_{\text{int}}^{(f)} = \frac{3\pi R_g}{\Gamma(4gb^{-1} - Qb^{-1})} \int_0^\infty r_g(P) dP \int_{F} \left| (16q)^{P^2+Q^2/4-1+\Delta_{\alpha}} H_P^{(\alpha)}(q) \right|^2 d^2\tau \]  

The product of the elliptic blocks

\[ H_P^{(\alpha)}(q) = H_{\alpha}(q) H_{sp}(q) H_{P}(q) \]  

is developed in a power series in \( q \)

\[ H_{\alpha}(q) H_{sp}(q) H_{P}(q) = \sum_{r=0}^\infty H_r^{(\alpha)}(P) q^r \]  

so that the integrand in (5.3) as a double power series in \( q \) and \( \bar{q} \). In each term the integration in \( \tau_2 = \text{Im} \tau \) can be carried out explicitly. The result is in terms of the following function

\[ \Phi(A, r, l) = \int_{F} d^2\tau \left| 16q \right|^{2A} q^r \bar{q}^l = \frac{(16)^{2A}}{\pi(2A + r + l)} \int_{-1/2}^{1/2} \cos(\pi(r - l)x)e^{-\pi\sqrt{1-x^2}(2A+r+l)} dx \]  

Notice, that if the integral is divergent at \( \tau_2 \to \infty \), this reduction automatically takes care of the divergency (in the sense of analytic continuation). We arrive at the series

\[ J_{\text{int}}^{(f)} = \frac{3\pi R_g}{\Gamma(4gb^{-1} - Qb^{-1})} \sum_{L=0}^\infty A_{\text{int, } L}^{(\alpha)} \]  

where in the last sum

\[ A_L^{(\alpha)} = \int_0^\infty r_g(P) dP \sum_{k=0}^L H_k^{(\alpha)}(P) H_{L-k}^{(\alpha)}(P) \Phi(P^2 + Q^2/4 + \Delta_{\alpha} - 1, k, L - k) \]  

In our symmetric case only even \( L \) contribute. Each term in (5.8) is suppressed by a factor \( \max_F |q|^{2L} \) and in practice the series in \( L \) converges very fast. Below we found it sufficient to sum up to \( L = 6 \) to reach the 8 – 9 digit precision (see table 1).

The discrete terms, if any, are treated similarly. It is again convenient to single out the “logarithmic” part in each of the integrals \( J_{\text{disc, } n}^{(f)} \)

\[ J_{\text{disc, } n}^{(f)} = 6\pi^2 N_n \left( L_n^{(\alpha)} + K_n^{(\alpha)} \right) \]  

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where \((P_n)\) is from eq. (3.9)
\[
L_{n}^{(a)} = 4\pi(Q/2 - 2g - nb) \int \left| (16q)^{P_n^2 + \Delta_\alpha + Q^2/4 - 1} H_{P_n}^{(a)}(q) \right|^2 \Im \tau \, d^2\tau \]  
(5.10)
\[
K_{n}^{(a)} = \int \left| (16q)^{P_n^2 + \Delta_\alpha + Q^2/4 - 1} H_{P_n}^{(a)}(q) \right|^2 \left( -i \frac{d}{dP} \log H_{P_n} \right|_{P=P_n} + S_n \) \, d^2\tau \]  
(5.11)

(only the Liouville elliptic blocks \(H_{P}(q)\) depend on \(P\) and therefore appear in the logarithmic derivative) and
\[
S_n = U_n - 4(Q/2 - 2g - nb) \log 16 \tag{5.11}
\]

As in the case of the integral part, the expansion in \(q\) and \(\bar{q}\) induces the “level” series
\[
L_{n}^{(a)} = \sum_{L=0}^{\infty} B_{n,L}^{(a)} \]  
(5.12)
\[
M_{n}^{(a)} = \sum_{L=0}^{\infty} C_{n,L}^{(a)} \]  
(5.13)

The integrals are then evaluated in the same way as (5.11)
\[
B_{n,L}^{(a)} = 4\pi(Q/2 - 2g - nb) \sum_{k=0}^{L} H_k^{(a)}(P_n) H_{L-k}^{(a)}(P_n) \Phi'(P_n^2 + Q^2/4 + \Delta_\alpha - 1, k, L-k) \tag{5.13}
\]
\[
C_{n,L}^{(a)} = \sum_{k=0}^{L} \left( S_n H_{L-k}^{(a)}(P_n) - 2H_{L-k}^{(a)}(P_n) \right) H_k^{(a)}(P_n) \Phi(P_n^2 + Q^2/4 + \Delta_\alpha - 1, k, L-k) \]

where
\[
H_k^{(a)}(P) = i \frac{d}{dP} H_k^{(a)}(P) \]  
(5.14)

A new function
\[
\Phi'(A, r, l) = \int d^2\tau \left| 16q \right|^{2A} q^r \bar{q}^l \Im \tau \]  
(5.15)
\[
= \frac{(16)^{2A}}{(2A + r + l)^2 \pi^2} \int_{-1/2}^{1/2} \cos(\pi(r - l)x)e^{-\pi\sqrt{1-x^2}(2A+r+l)} \left( 1 + \pi(2A + r + l)\sqrt{1-x^2} \right) dx
\]

has been introduced to treat the logarithmic part. The “level” series also converges very fast (see next section).

6. Numerical results

In table II we present the numerical values of the fourth perturbative coefficient
\[
z_4 = \frac{I_4}{24(2\pi)^4 Z_L^{(A)}} \]  
(6.1)
at different values of $b^2$ together with the second and third ones already evaluated in ref. [3]. The forth column contains the preliminary estimates $z_4^{(\text{est})}$ of $z_4$ on the basis of $z_2$, $z_3$ and analytic properties of $z(h)$, as explained in [3]. Also two “exact” values are produced at the “solvable” points $b^2 = 0.4$ and $b^2 = 0.3$ (see sect.8 for details).

| $b^2$ | $z_2 \times 10^2$ | $z_3 \times 10^3$ | $z_4 \times 10^4$ | $z_4^{(\text{est})} \times 10^4$ | $z_4^{(\text{exact})} \times 10^4$ |
|-------|----------------|----------------|----------------|-----------------|-----------------|
| 0.00  | −8.92857       | 22.9899        | −31.891804     | −31.8938        |
| 0.01  | −8.83599       | 22.5977        | −31.114313     | −31.1164        |
| 0.05  | −8.43801       | 20.9364        | −27.881220     | −27.8839        |
| 0.10  | −7.86500       | 18.6240        | −23.557311     | −23.5604        |
| 0.15  | −7.18331       | 16.0092        | −18.942809     | −18.9461        |
| 0.20  | −6.35922       | 13.0616        | −14.131054     | −14.1345        |
| 0.25  | −5.34942       | 9.78831        | −9.3340267     | −9.33745        |
| 0.30  | −4.09998       | 6.28732        | −4.94949021    | −4.95242        |
| 0.35  | −2.55378       | 2.87061        | −1.61778168    | −1.61953        |
| 0.40  | −0.71440                   | 0.35675        | −0.085061507   | −0.085303       |

Table 1: Numerical values for the second, third and forth order perturbative coefficients in the fixed area scaling function $z(h)$. In the forth column we place the estimate of the four-point coefficient from the sum rules, as explained in ref. [3]. Exact values, where available, are presented for comparison.

At $b^2 < b^2_0$ the fourth coefficient contains the contributions from the integral part and discrete terms in (3.10)

$$z_4 = \sum_{n=0}^{N_4} z_4^{(d)}(n) + z_4^{(\text{int})}$$

(6.2)

In table 2 the structure of the four point integral as a sum of discrete and integral contributions is illustrated numerically. At small $b^2$ the integral part becomes negligible while more and more discrete terms appear. In order, only few first of these discrete terms really contribute at sufficiently small $b^2$. E.g., at $b^2 = 0.01$ in fact there are as many as 49 discrete terms and at $b^2 = 0$ (see next section) their number is infinite. In the table we quote only those which count at the precision level chosen (approximately 9 decimal digits).

Finally, our results for $z_4$ are used in the analytic-numeric procedure described in the first article [3]. This allows to correct the previous numerical results for the scaling function $z(h)$. In particular we improve the numerical approximations for the specific vacuum energy parameter $f_0 (b^2)$ (see eq. (1.19) or [3]). The new numbers $f_0^{(4)}$, which take into account the perturbative coefficients up to $z_4$ are presented in table 3 and compared with the previous approximations as well as with the exact values $f_0^{(\text{exact})}$ where the last are available.
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
b^2 & 0.00 & 0.01 & 0.05 & 0.10 & 0.15 & 0.20 \\
\hline
z_4^{(\text{int})} \times 10^4 & 0.00 & 8.6 \times 10^{-26} & 0.006087 & 1.7311 & 2.318 & 0.799 \\
z_4^{(d)}(0) \times 10^4 & -11.4450 & -10.5160 & -6.88429 & -4.187 & -6.397 & -14.93 \\
z_4^{(d)}(1) \times 10^4 & -15.7357 & -15.5090 & -13.9653 & -11.51 & -14.864 & \\
z_4^{(d)}(2) \times 10^4 & -3.94363 & -4.16729 & -4.97850 & -5.770 & \\
z_4^{(d)}(3) \times 10^4 & -0.662455 & -0.77641 & -1.43314 & -3.829 & \\
z_4^{(d)}(4) \times 10^4 & -0.092114 & -0.12415 & -0.41407 & \\
z_4^{(d)}(5) \times 10^4 & -0.011464 & -0.01846 & -0.13010 & \\
z_4^{(d)}(6) \times 10^4 & -0.001324 & -0.00265 & -0.04666 & \\
z_4^{(d)}(7) \times 10^4 & -1.44668 \times 10^{-4} & -3.76 \times 10^{-4} & -0.02083 & \\
z_4^{(d)}(8) \times 10^4 & -1.51392 \times 10^{-5} & -5.34 \times 10^{-5} & -0.01443 & \\
z_4^{(d)}(9) \times 10^4 & -1.52956 \times 10^{-6} & -7.65 \times 10^{-6} & \\
z_4^{(d)}(10) \times 10^4 & -1.50085 \times 10^{-7} & -1.11 \times 10^{-6} & \\
z_4^{(d)}(11) \times 10^4 & -1.43677 \times 10^{-8} & -1.66 \times 10^{-7} & \\
z_4^{(d)}(12) \times 10^4 & -1.34666 \times 10^{-9} & -2.53 \times 10^{-8} & \\
\hline
\end{array}
\]

Table 2: Relative contributions of the integral and discrete terms in the sum \((6.2)\) at different values of \(b^2\).

7. Classical limit

In ref. [2] the classical limit of the Liouville four point function has been considered. In particular, the “symmetric” function with four equal dimensions \(\sigma\) admits the following integral representation

\[
g_{\text{cl}}(x, \bar{x}) = \pi^3 \int_1 \frac{(2s - 1) ds \Gamma^2(2\sigma + s - 1) \Gamma^2(2\sigma - s) \Gamma^4(s)}{2\pi i \Gamma^4(2\sigma) \Gamma^2(2s)} \mathcal{F}^{(\text{cl})}(\sigma, s, y) \mathcal{F}^{(\text{cl})}(\sigma, s, \bar{y}) \tag{7.1}
\]

The “classical block” \(\mathcal{F}^{(\text{cl})}(\sigma, s, y)\) is expressed explicitly through the hypergeometric function

\[
\mathcal{F}^{(\text{cl})}(\sigma, s, y) = y^{s-2\sigma} \binom{2}{s} F_1(s, s, 2s, y) \tag{7.2}
\]

while the integration contour \(\uparrow\) goes up along the imaginary axis to the left from the poles of \(\Gamma^2(2\sigma - s)\) and to the right from all other singularities of the integrand. We are going to argue that this expression is consistent with the classical limit of the fixed area four-point function \((3.2)\) provided we identify \(\Delta_y = \sigma\) and \((7.1)\) with the normalized correlation function

\[
\lim_{b^2 \to 0} \frac{g_{\text{cl}}(x, \bar{x})}{Z_L^{(A)}} = g_{\text{cl}}(x, \bar{x}) \tag{7.3}
\]
Table 3: Specific energy $f_0$ determined with the use of first two, three and four perturbative coefficients. When available, the exact values are quoted for comparison.

To this order, let us evaluate the integral through the residues at the infinite sequence of the “right” poles at $s = s_n$, where

$$s_n = 2\sigma + n$$

This is a rightful procedure, as the large $s$ asymptotic of the integrand shows. Thus

$$g_c(x, \bar{x}) = \sum_{n=0}^{\infty} D_n^{(cl)}(x, \bar{x})$$

where

$$D_n^{(cl)}(x, \bar{x}) = N_n^{(cl)} \left| \mathcal{F}^{(cl)}(\sigma, s_n, x) \right|^2 \left( 2 \text{Re} \frac{d}{ds} \log \mathcal{F}^{(cl)}(\sigma, s, x) \bigg|_{s=s_n} + U_n^{(cl)} \right)$$

We introduced the notations

$$N_n^{(cl)} = \frac{\pi^3}{(4\sigma + 2n - 1)(n!)^2} \prod_{k=0}^{n-1} \frac{(2\sigma + k)^4}{(4\sigma + n + k - 1)^2}$$

and

$$U_n^{(cl)} = 4\psi(4\sigma + 2n) - 2\psi(4\sigma + n - 1) - 4\psi(2\sigma + n) + 2\psi(1 + n) - \frac{2}{4\sigma + 2n - 1}$$

Comparing (7.7) and (7.8) with (3.14) and (3.15) respectively, it is easy to see that

$$\lim_{b \to 0} \frac{N_n}{b Z_L} = N_n^{(cl)}$$

$$\lim_{b \to 0} b U_n = U_n^{(cl)}$$
Also, it is well known (see e.g. [10]) that
\[
\lim_{b^2 \to 0} F_P \left( \begin{array}{ccc} \sigma & \sigma & \sigma \\ \sigma & \sigma & \sigma \end{array} \right) | x \right) = F^{(\text{cl})}(\sigma, s, x) \tag{7.10}
\]
provided \( \sigma \) and \( s = P^2 + Q^2/4 \) are kept finite in the limit. Thus, the classical limit of the normalized discrete term (3.12) coincides with (7.6)
\[
\lim_{b^2 \to 0} \frac{D_n(x, \bar{x})}{Z_L^{(A)}} = D_n^{(\text{cl})}(x, \bar{x}) \tag{7.11}
\]
In the classical limit the integral in eq.(3.10) is saturated by the infinite sequence of discrete terms, the integral one vanishing. This proves (7.3).

The classical block can be rendered to the form (cp. eq.(A.16) in the Appendix)
\[
F^\alpha F^{(\text{cl})}(\sigma, s, x) = \left( 16q^4 \right) \Delta^\alpha + s - 1 \frac{H^\alpha(q) H^{(\text{cl})}(q)}{x(1-x)} \tag{7.12}
\]
In the general expression (A.16) only the product \( H^{(\text{cl})}(q) = H_{sp}H_L \) allows the classical limit.

The classical elliptic block reads
\[
H^{(\text{cl})}(q) = \eta^{sM/2 - 25/2} q^{16\sigma}(q^2) h_s(q) \tag{7.13}
\]
In this expression
\[
h_s(q) = \left( \frac{x}{16q} \right)^s 2F_1(s, s, 2s, x) \tag{7.14}
\]
\[
= 1 - \frac{4s(2s-3)}{2s+1} q^2 + \frac{2s (8s^2 - 14s + 9)}{2s+3} q^4 + \ldots
\]
while
\[
\eta(q^2) = 1 - q^2 - q^4 + q^{10} + \ldots \tag{7.15}
\]
is the standard Dedekind product (A.18) and
\[
\theta_0(q^2) = 1 - 2q^2 + 2q^8 - 2q^{18} + \ldots = \sum_{n=-\infty}^{\infty} (-)^n q^{2n^2} \tag{7.16}
\]
the usual theta series. It is also straightforward to verify directly that \( H^{(\text{cl})}(q) \) is the limit of \( H_{sp}H_P \) as \( b^2 \to 0 \) and \( P^2 \to s - Q^2/4 \).

For our particular application in the GYL model the classical elliptic block is evaluated through the explicit formula
\[
H^{(\text{cl})}(q) = \eta^{-147/10} \theta_0^{96/5} h_s(q) \tag{7.17}
\]
while the matter elliptic blocks $H_\alpha$ remains the same as in eq.(4.9). As usual, we are going to use the $q$-expansions. Denote

$$H_\alpha(q)H^{(cl)}(q) = \sum_{k=0}^{\infty} h_k^{(\alpha)} q^k$$

$$H_\alpha(q) \frac{d}{ds} H^{(cl)}(q) = \sum_{k=0}^{\infty} h'_k^{(\alpha)} q^k$$

In the classical case the integral is the sum of the infinite number of discrete terms

$$I_{n}^{(cl)} = \sum_{n=0}^{\infty} I_n^{(cl)}$$

where

$$I_n^{(cl)} = 6 \int_G D_n^{(cl)}(x, \bar{x}) G_{Y\Lambda}(x, \bar{x}) d^2 x = j_n^{(l)} - \kappa^2 j_n^{(e)}$$

Now, as in the general case, we single out the “logarithmic” integral

$$j_n^{(\alpha)} = 6 \pi^2 N_n^{(cl)} (L_n^{(\alpha)} + M_n^{(\alpha)} + K_n^{(\alpha)})$$

where,

$$L_n^{(\alpha)} = 2\pi \int_F |(16q)^{\Delta_\alpha+2\sigma+n-1} h_\alpha(q)|^2 \text{Im} \tau \ d^2 \tau$$

$$M_n^{(\alpha)} = \int_F |(16q)^{\Delta_\alpha+2\sigma+n-1} h_\alpha(q)|^2 (U_n^{(cl)} - 2 \log 16) \ d^2 \tau$$

$$K_n^{(\alpha)} = -2 \int_F |(16q)^{\Delta_\alpha+2\sigma+n-1}|^2 \frac{d}{ds} (h_\alpha(q) h_\alpha(\bar{q})) \ d^2 \tau$$

Then, each term is obtained as a “level by level” sum in $L$ through the double $q$ and $\bar{q}$-expansions.

$$L_n^{(\alpha)} = 2\pi \sum_{L=0}^{\infty} \sum_{k=0}^{L} h_k^{(\alpha)} (s_n) h_{L-k}^{(\alpha)} (s_n) \Phi'(\Delta_\alpha + s_n - 1, k, L - k)$$

$$M_n^{(\alpha)} = (U_n^{(cl)} - 2 \log 16) \sum_{L=0}^{\infty} \sum_{k=0}^{L} h_k^{(\alpha)} (s_n) h_{L-k}^{(\alpha)} (s_n) \Phi(\Delta_\alpha + s_n - 1, k, L - k)$$

$$K_n^{(\alpha)} = -2 \sum_{L=0}^{\infty} \sum_{k=0}^{L} h_k^{(\alpha)} (s_n) h'_{L-k}^{(\alpha)} (s_n) \Phi(\Delta_\alpha + s_n - 1, k, L - k)$$

Every component is straightforwardly evaluated with the use of the integrals and . In practical calculations (given the required precision of 9 digits) we found it sufficient to sum up to $L = 10$ in eq. (7.23).
Through all these calculations we arrive at the contribution of $n$-th discrete term to the fixed area perturbative coefficient

$$z_{4}^{(cl)}(n) = \frac{I_{n}^{(cl)}}{24(2\pi)^{4}}$$

(7.24)

Several first contributions are presented in table 2 to manifest the convergence. The discrete terms sum up to the number

$$z_{4}^{(cl)} = \sum_{n=0}^{\infty} z_{4}^{(cl)}(n) = -31.8918039 \times 10^{-4}$$

(7.25)

already obtained earlier [2] by means of a different numerical approach. Notice, that we had to take into account as many as 10 discrete terms to achieve this 9 digit precision quoted.

8. **Integrable points** $b^{2} = 0.3$ and $b^{2} = 0.4$

In this section we comment about the two exactly solvable points $b^{2} = 0.4$ and $b^{2} = 0.3$ in the family of GYL models. The solution at the “pure Yang-Lee” point $b^{2} = 2/5$ through the matrix model approach has been already discussed in the previous article [3]. Here we recapitulate the essence very briefly. This case of the “minimal gravity” is related to the flow from the tricritical to the critical points in the generic one matrix model [13]. The corresponding scaling function $Z_{YL}(x,t)$, which is interpreted (up to an overall scale) as the spherical partition function of the continuous gravity, is determined explicitly as

$$\frac{\partial^{2}}{\partial x^{2}} Z_{YL}(x,t) = u(x,t)$$

(8.1)

through a solution $u(x,t)$ of the following simple algebraic equation

$$x = u^{3} - tu$$

(8.2)

Parameters $t$ and $x$ are, again up to some normalization constants, the cosmological constant and $\varphi$-perturbation coupling respectively. Comparing the series expansion generated by (8.1) and (8.2)

$$Z_{YL}(x,t) = Z_{YL}(0,t) \left( 1 - \frac{105}{16} \left( \frac{x}{t^{3/2}} \right) + \frac{105}{8} \left( \frac{x}{t^{3/2}} \right)^{2} - \frac{35}{16} \left( \frac{x}{t^{3/2}} \right)^{3} + \ldots \right)$$

(8.3)

with the perturbative coefficients evaluated in the field theoretic approach, it is easy to relate

$$\frac{x}{t^{3/2}} = \frac{\lambda l_{YL}}{(\pi \mu)^{3/2}}$$

(8.4)

where

$$l_{YL} = \frac{\gamma^{1/2}(4/5)}{4\gamma(2/5)} = 0.0845223 \ldots$$

(8.5)
Then, the fixed area scaling function (1.12) at \( b^2 = 0.4 \) reads explicitly

\[
z(h) = -2\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(l_{YL}h)^n}{n!\Gamma(n/2 - 1/2)}
\] (8.6)

In particular

\[
z_2 = -l_{YL}^2 = -0.00714401
\]
\[
z_3 = \sqrt{\pi} l_{YL}^3 = 0.000356752
\] (8.7)
\[
z_4 = -\frac{1}{6} l_{YL}^4 = -8.50615 \times 10^{-6}
\]

The last number and

\[
f_0(0.4) = \frac{3}{\pi} \left( \frac{l_{YL}}{2} \right)^{2/3} = 0.11585962159187
\] (8.8)

are quoted in tables I and II as the corresponding exact values for this point.

In ref. [3] it has been argued that at \( b^2 = 0.3 \) our GYL model is also a version of minimal gravity. It arises as a particular perturbation of the minimal \( M_{3/10} \) model coupled to the Liouville gravity. Therefore there are serious reasons to believe that at this point the model is again exactly solvable. Unfortunately yet no exact solution in the framework of the matrix model approach is known. However, it is very natural to expect the existence of a closed analytic expression, e.g., for the scaling function (1.12). We conjecture the following explicit form, apparently motivated by more general structures discovered in [14] (see also [8])

\[
z_{3/10}(h) = \Gamma(-1/3) \sum_{n=0}^{\infty} \frac{(l_{eg}h)^n}{n!\Gamma(n/3 - 1/3)}
\] (8.9)

where the scale factor \( l_{eg} \) is easily figured out through the comparison with the perturbative coefficients of sect.1

\[
l_{eg} = \frac{\gamma(1/3)}{2^{1/3}3^{5/6}(3/10)} = 0.23254 \ldots
\] (8.10)

This explicit expression gives rise to the following values of the first perturbative coefficients

\[
z_2 = -0.0409998231725
\]
\[
z_3 = 0.00628731873887
\]
\[
z_4 = -0.000494949020548
\]
\[
z_5 = 0.0000257778956523
\] (8.11)
We consider the comparison of these numbers with those in the corresponding row of the table as a convincing support in favor of our conjecture. Analyzing the asymptotic of \((8.9)\) one finds, in addition, that at \(b^2 = 0.3\)

\[
f_0 = \frac{4}{\pi} \left( \frac{l_{eg}}{3} \right)^{3/4} = 0.187044 \ldots \tag{8.12}
\]

This number is produced in table 2 as the corresponding “exact” value.

\[
\Gamma(4/3) = 3/10 \quad (x, t) = \frac{13}{3} - 81/3640 \quad (t x^{-4/3})^4
\]

Figure 1: Integration contour in the representation \((8.17)\). Dashed are the wedges where the integrand decreases at large \(|u|\).

It seems also suggestive that the conjectured fixed area scaling function \((8.9)\) follows from the “matrix like” algebraic equation

\[
u^4 - xu = -t \tag{8.13}
\]

for the third derivative

\[
u = \frac{\partial^3}{\partial t^3} Z_{3/10}(x, t) \tag{8.14}
\]

of the “grand” partition function

\[
Z_{3/10}(x, t) = -\frac{x^{13/3}}{3} \sum_{n=0}^{\infty} \frac{\Gamma(4n/3 - 13/3)(tx^{-4/3})^n}{n! \Gamma(n/3 - 1/3)} \tag{8.15}
\]

\[
= -\frac{x^{13/3}}{3} \left( \frac{81}{3640} - \frac{tx^{-4/3}}{24} + \frac{9 (tx^{-4/3})^2}{20} - \frac{(tx^{-4/3})^3}{24} + \frac{(tx^{-4/3})^4}{90} + O(t^6) \right)
\]

20
Here the cosmological constant $x$ related to $\mu$ through

$$\frac{\lambda_{\text{eg}}}{(\pi \mu)^{4/3}} = \frac{t}{x^{4/3}}$$

(8.16)

Finally, let us mention a convenient integral representation for the scaling function

$$z_{3/10}(h) = 3 \int_C \exp \left(u^3 + h_{\text{eg}}u^{-1}\right) u^3 du$$

(8.17)

where the integration contour $C$ goes as it is shown in fig. 4

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**A. Elliptic four point block**

Here we summarize the results of refs. [10]. Consider a conformal theory with central charge $c$. It will prove convenient to introduce the “Liouville like” parameterization in terms of $b$

$$\frac{c - 1}{6} = (b^{-1} + b)^2$$

(A.1)

This is not a restriction for the value of $c$ since we allow $b$ to be complex if needed. It is also convenient to introduce the notation

$$\lambda_{m,n} = \frac{mb^{-1} + nb}{2}$$

(A.2)

Let $\Delta_i$, $i = 1, 2, 3, 4$ and $\Delta$ be the external and intermediate dimensions in the four point block [5], as it is illustrated in the picture below

$$\mathcal{F}_\Delta \left( \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \right| \begin{array}{c} \Delta_j \\ \Delta \end{array} \right) =$$

$$\begin{array}{c} x_j (= x) \\ x_2 (= 0) \\ x_3 (= I) \\ x_4 (= \infty) \end{array}$$
According to [10] it can be written as

\[
F_\Delta \left( \begin{array}{c|c} \Delta_1 & \Delta_3 \\ \Delta_2 & \Delta_4 \end{array} \right|x \right) = (16q)^{(\Delta_1-\Delta_2)/24} x^{(c-1)/24-\Delta_1-\Delta_2} (1-x)^{(c-1)/24-\Delta_3-\Delta_4} \theta_3^{(c-1)/24-\sum \Delta_i} (q) H_{\Delta} (q)
\]

where \( q \) is related to \( x \) as in eqs. (4.2), (4.6) and \( \theta_3(q) \) is defined in (4.5). In the function

\[
H_{\Delta}(q) = H_{\Delta} \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right| q \right)
\]

we suppressed the dependence on the external dimensions, which are conveniently parameterized in terms of \( \lambda_i, i = 1, 2, 3, 4 \) as

\[
\Delta_i = \frac{c-1}{24} + \lambda_i^2
\]

The \( H \)-function (the elliptic four point block) is a power series in \( q \)

\[
H_{\Delta} \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right) = \sum_{L=0}^{\infty} H_L \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right| \Delta \right) q^L
\]

which is believed to converge at \( |q| < 1 \). It can be effectively calculated through the recursive relation [10]

\[
H_{\Delta} \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right) = 1 + \sum_{(m,n)} \frac{q^{mn}}{\Delta - \Delta_{m,n}} R_{m,n} \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right) H_{\Delta_{m,n+mn}} \left( \begin{array}{c|c} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{array} \right| q \right)
\]

Here the sum is over all pairs \((m,n)\) of positive integers and

\[
\Delta_{m,n} = \lambda_{1,1}^2 - \lambda_{m,n}^2
\]

are the dimensions of degenerate representations of the Virasoro algebra with the central charge \( c \). The multipliers \( R_{m,n} \) in (A.8) read explicitly

\[
R_{m,n}(\lambda_i) = 2 \prod_{r,s}(\lambda_1 + \lambda_2 - \lambda_{r,s})(\lambda_1 - \lambda_2 - \lambda_{r,s})(\lambda_3 + \lambda_4 - \lambda_{r,s})(\lambda_3 - \lambda_4 - \lambda_{r,s}) \prod_{k,l} \lambda_{k,l}
\]

The products in (A.10) are over the following sets of integers \((r,s)\) and \((k,l)\)

\[
r = -m + 1, -m + 3, \ldots, m - 3, m - 1
\]

\[
s = -n + 1, -n + 3, \ldots, n - 3, n - 1
\]

and

\[
k = -m + 1, -m + 2, \ldots, m - 1, m
\]

\[
l = -n + 1, -n + 2, \ldots, n - 1, n
\]
while the prime sign near the last product symbol $\prod_{k,l}$ means that the two pairs $(k, l) = (0, 0)$ and $(m, n)$ are missing.

Relation (A.8) leads, in particular, to a recursive algorithm for the coefficients in the “level” expansion (A.7)

\[
H_0(\Delta) = 1 \\
H_L(\Delta) = \sum_{mn < L} \frac{R_{m,n}}{\Delta - \Delta_{m,n}} H_{L-mn}(\Delta_{m,n} + mn)
\]

where we have again suppressed the dependence on $\lambda_i$.

Now for the purposes of quantum gravity we want to combine two blocks of different conformal field theories, conventionally be the “matter” and the “Liouville” one, with central charges respectively $c_M$ and $c_L$. We do not necessarily require the “complementarity” of these quantities, introducing the “spectator” central charge

\[
c_{sp} = 26 - c_M - c_L
\]

to take care of the deficit. On the contrary we do require the complementarity of the “matter” and “Liouville” external dimensions $\Delta_i$ and $\tilde{\Delta}_i$

\[
\Delta_i + \tilde{\Delta}_i = 1
\]

for $i = 1, 2, 3, 4$. The “matter” and “Liouville” blocks are combined to

\[
\mathcal{F}^{(M)}_\Delta \left( \begin{array}{ccc} \Delta_1 & \Delta_3 \\ \Delta_2 & \Delta_4 \end{array} \right | x \right) \mathcal{F}^{(L)}_\Delta \left( \begin{array}{ccc} \tilde{\Delta}_1 & \tilde{\Delta}_3 \\ \tilde{\Delta}_2 & \tilde{\Delta}_4 \end{array} \right | x \right) = (16q)^{\Delta + \tilde{\Delta} - 1} \frac{H^{(M)}_{\Delta}(q)H^{(L)}_{\tilde{\Delta}}(q)H_{sp}(q)}{x(1-x)\theta_3^4(q)}
\]

where we conventionally denoted

\[
H_{sp}(q) = \eta^{-c_{sp}/2}(q^2)
\]

with

\[
\eta(q^2) = \prod_{k=1}^{\infty} (1 - q^{2k}) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)}
\]

the Dedekind function.

Notice, that we do not demand the intermediate dimensions $\Delta$ and $\tilde{\Delta}$ to be complementary. To avoid misunderstanding, let us stress that $H_{sp}(q)$ is simply a convenient notation and hardly can be interpreted as a “contribution of the spectator matter” to the block and the four point function. Some additional simplifications of the product of the elliptic blocks, which occur if the “matter” and “Liouville” CFT’s are indeed complementary (i.e., $c_{sp} = 0$) will be discussed in [12].
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