SOLUTIONS WITH INTERSECTING P-BRANES RELATED TO TODA CHAINS

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Abstract

Solutions in multidimensional gravity with $m$ $p$-branes related to Toda-like systems (of general type) are obtained. These solutions are defined on a product of $n+1$ Ricci-flat spaces $M_0 \times M_1 \times \ldots \times M_n$ and are governed by one harmonic function on $M_0$. The solutions are defined up to the solutions of Laplace and Toda-type equations and correspond to null-geodesics of the (sigma-model) target-space metric. Special solutions relating to $A_m$ Toda chains (e.g. with $m = 1, 2$) are considered.

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I Introduction

At present there exists a special interest to the so-called $M$-theory (see, for example, [1]-[2]). This theory is “supermembrane” analogue of superstring models [3] in $D = 11$. The low-energy limit of $M$-theory after a dimensional reduction leads to models governed by a Lagrangian containing metric, fields of forms and scalar fields. These models contain a large variety of so-called $p$-brane solutions (see [1]-[12] and references therein).

In [25] it was shown that after dimensional reduction on the manifold $M_0 \times M_1 \times \ldots \times M_n$ and when the composite $p$-brane ansatz for fields of forms is considered the problem is reduced to the gravitating self-interacting $\sigma$-model with certain constraints imposed. (For electric $p$-branes see also [16, 17, 26].) This representation may be considered as a powerful tool for obtaining different solutions with intersecting $p$-branes (analogues of membranes). In [25] the Majumdar-Papapetrou type solutions (see [15]) were obtained (for non-composite case see [16, 17]). These solutions correspond to Ricci-flat $(M_i, g_i^f)$, $(g^f$ is metric on $M_i)$ $i = 1, \ldots, n$, and were also generalized to the case of Einstein internal spaces [23]. Earlier some special classes of these solutions were considered in [6, 7, 11, 19, 20, 21]. The obtained solutions take place, when certain (block-)orthogonality relations (on couplings parameters, dimensions of ”branes”, total dimension) are imposed. In this situation a class of cosmological and spherically-symmetric solutions was obtained [31, 40]. Special cases were also considered in [12, 27, 29, 30]. The solutions with the horizon were considered in details in [1, 22, 23, 31, 33].

In models under consideration there exists a large variety of Toda-chain solutions, when certain intersection rules are satisfied [31]. Cosmological and spherically symmetric solutions with $p$-branes and $n$ internal spaces related to $A_m$ Toda chains were previously considered in [12, 13] and [39, 42].

II The model

We consider a model governed by the action [25]

$$S = \int d^Dz \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \Delta} \theta_a ! \exp[2 \lambda_a(\varphi)] |F^a|^2 \right\}$$

where $g = g_{MN} dz^M \otimes dz^N$ is a metric, $\varphi = (\varphi^a) \in \mathbb{R}^l$ is a vector of scalar fields, $(h_{\alpha\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix $(l \in \mathbb{N})$, $\theta_a = \pm 1$, $F^a = dA^a$ is a
$n_a$-form ($n_a \geq 1$), $\lambda_a$ is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{aa}\varphi^a$, $a \in \Delta$, $\alpha = 1, \ldots, l$. Here $\Delta$ is some finite set.

We consider a manifold

\[ M = M_0 \times M_1 \times \ldots \times M_n, \tag{2.2} \]

with a metric

\[ g = e^{2\tau(x)} g^0 + \sum_{i=1}^{n} e^{2\phi_i(x)} g^i \tag{2.3} \]

where $g^0 = g^0_{\mu\nu}(x)dx^\mu \otimes dx^\nu$ is a metric on the manifold $M_0$, and $g^i = g^i_{m,n_i}(y_i)dy^m_i \otimes dy^{n_i}_i$ is an Einstein metric on $M_i$ satisfying the equation

\[ R_{m,n_i}[g^i] = \xi_i g^i_{m,n_i}, \tag{2.4} \]

$m_i, n_i = 1, \ldots, d_i$; $\xi_i = \text{const}$, $i = 1, \ldots, n$. (Here we identify notations for $g^i$ and $\hat{g}^i$, where $\hat{g}^i = p_i^*g^i$ is the pullback of the metric $g^i$ to the manifold $M$ by the canonical projection: $p_i : M \to M_i$, $i = 0, \ldots, n$. An analogous agreement will be also kept for volume forms etc.)

Any manifold $M_\nu$ is supposed to be oriented and connected and $d_\nu \equiv \dim M_\nu$, $\nu = 0, \ldots, n$. Let

\[ \tau_i \equiv \sqrt{|g^i(y_i)|dy^1_i \wedge \ldots \wedge dy^{d_i}_i}, \quad \varepsilon(i) \equiv \text{sign}(\det(g^i_{m,n_i})) = \pm 1 \tag{2.5} \]

denote the volume $d_i$-form and signature parameter respectively, $i = 1, \ldots, n$. Let $\Omega = \Omega_n$ be a set of all subsets of $\{1, \ldots, n\}$, $|\Omega| = 2^n$. For any $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote

\[ \tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}, \quad d(I) \equiv \sum_{i \in I} d_i, \quad \varepsilon(I) \equiv \prod_{i \in I} \varepsilon(i). \tag{2.6} \]

We also put $\tau(\emptyset) = \varepsilon(\emptyset) = 1$ and $d(\emptyset) = 0$.

For fields of forms we consider the following composite electromagnetic ansatz

\[ F^a = \sum_{I \in \Omega_{a,e}} F^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} F^{(a,m,J)} \tag{2.7} \]

where

\[ F^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \tag{2.8} \]
\[ F^{(a,m,J)} = e^{-2\lambda_a(\varphi)} *(d\Phi^{(a,m,J)} \wedge \tau(J)) \tag{2.9} \]

are elementary forms of electric and magnetic types respectively, $a \in \Delta$, $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$ and $\Omega_{a,e}$, $\Omega_{a,m}$ are non-empty subsets of $\Omega$. In \[\text{\[2.9\]}\] $* = *[g]$ is the Hodge operator on $(M, g)$. For scalar functions we put

\[ \varphi^a = \varphi^a(x), \quad \Phi^a = \Phi^a(x), \tag{2.10} \]
Here and below
\[ S = S_e \cup S_m, \quad S_v = \bigcup_{a \in \triangle} \{a\} \times \{v\} \times \Omega_{a,v}, \quad (2.11) \]
\[ v = e, m. \] The set \( S \) consists of elements \( s = (a_s, v_s, I_s) \), where \( a_s \in \triangle \), \( v_s = e, m \) and \( I_s \in \Omega_{a_s,v_s} \).

Due to (2.8) and (2.9)
\[ d(I) = n_a - 1, \quad d(J) = D - n_a - 1, \quad (2.12) \]
for \( I \in \Omega_{a,e}, \quad J \in \Omega_{a,m} \).

II.1 The sigma model

Let \( d_0 \neq 2 \) and
\[ \gamma = \gamma_0(\phi) \equiv \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \phi^j, \quad (2.13) \]
i.e. the generalized harmonic gauge is used.

We impose the restriction on sets \( \Omega_{a,v} \). These restrictions guarantee the block-diagonal structure of a stress-energy tensor (like for the metric) and the existence of \( \sigma \)-model representation \([25]\).

We denote \( w_1 \equiv \{i|i \in \{1, \ldots, n\}, \quad d_i = 1\} \), and \( n_1 = |w_1| \) (i.e. \( n_1 \) is the number of 1-dimensional spaces among \( M_i, \quad i = 1, \ldots, n \)).

**Restriction 1.** Let 1a) \( n_1 \leq 1 \) or 1b) \( n_1 \geq 2 \) and for any \( a \in \triangle, \quad v \in \{e, m\}, \quad i, j \in w_1, \quad i < j \), there are no \( I, J \in \Omega_{a,v} \) such that \( i \in I, \quad j \in J \) and \( I \setminus \{i\} = J \setminus \{j\} \).

**Restriction 2** (only for \( d_0 = 1, 3 \)). Let 2a) \( n_1 = 0 \) or 2b) \( n_1 \geq 1 \) and for any \( a \in \triangle, \quad i \in w_1 \) there are no \( I \in \Omega_{a,m}, \quad J \in \Omega_{a,e} \) such that \( \bar{I} = \{i\} \cup J \) for \( d_0 = 1 \) and \( J = \{i\} \cup \bar{I} \) for \( d_0 = 3 \). Here and in what follows
\[ \bar{I} \equiv \{1, \ldots, n\} \setminus I. \quad (2.14) \]

These restrictions are satisfied in the non-composite case \([10, 17]\): \( |\Omega_{a,v}| = 1 \), (i.e when there are no two \( p \)-branes with the same color index \( a, \quad a \in \Delta \).) Restriction 1 and 2 forbid certain intersections of two \( p \)-branes with the same color index for \( n_1 \geq 2 \) and \( n_1 \geq 1 \) respectively.

It was proved in \([25]\) that equations of motion for the model (2.1) and the Bianchi identities: \( dF^s = 0, \quad s \in S_m \), for fields from (2.3)–(2.13), when Restrictions 1 and 2 are imposed, are equivalent to equations of motion for the \( \sigma \)-model governed by the action
\[ S_\sigma = \int d^{d_0}x \sqrt{|g^0|} \left\{ R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_{\mu} z^A \partial_{\nu} z^B - \sum_{s \in S} e^{-2U_{s}^{\alpha}} g^{0\mu\nu} \partial_{\mu} \Phi^{s} \partial_{\nu} \Phi^{s} - 2V \right\}, \quad (2.15) \]
where \((z^A) = (\phi^i, \varphi^a)\), the index set \(S\) is defined in (2.11),

\[
V = V(\phi) = -\frac{1}{2} \sum_{i=1}^{n} \xi_i d_i e^{-2\phi^i + 2\gamma_0(\phi)}
\]  

(2.16)
is the potential,

\[
(\hat{G}_{AB}) = \left( \begin{array}{cc}
G_{ij} & 0 \\
0 & h_{\alpha\beta}
\end{array} \right),
\]

(2.17)
is the target space metric with

\[
G_{ij} = d_i \delta_{ij} + \frac{d_i d_j}{d_0 - 2},
\]

(2.18)
are vectors, \(s = (a_s, v_s, I_s)\), \(\chi_e = +1\), \(\chi_m = -1\);

\[
\delta_i = \sum_{j \in I} \delta_{ij}
\]

(2.20)is the indicator of \(i\) belonging to \(I\): \(\delta_{II} = 1\) for \(i \in I\) and \(\delta_{II} = 0\) otherwise; and

\[
\varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta_{as},
\]

(2.21)s \(\in S\), \(\varepsilon[g] \equiv \text{sign det}(g_{MN})\). More explicitly (2.21) reads \(\varepsilon_s = \varepsilon(I_s) \theta_{as}\) for \(v_s = e\) and \(\varepsilon_s = -\varepsilon[g] \varepsilon(I_s) \theta_{as}\), for \(v_s = m\).

### III Exact solutions with one harmonic function

#### III.1 Toda-like Lagrangian

Action (2.13) may be also written in the form

\[
S_\sigma = \int d^d x \sqrt{|g^0|} \left\{ R[g^0] - \mathcal{G}_{\bar{A}\bar{B}}(X) g^{0\mu\nu} \partial_\mu X^{\bar{A}} \partial_\nu X^{\bar{B}} - 2V \right\}
\]

(3.1)where \(X = (X^\bar{A}) = (\phi^i, \varphi^a, \Phi^s) \in \mathbb{R}^N\), and minisupermetric

\[
\mathcal{G} = \mathcal{G}_{\bar{A}\bar{B}}(X) dX^{\bar{A}} \otimes dX^{\bar{B}}
\]

(3.2)on minisuperspace

\[
\mathcal{M} = \mathbb{R}^N, \quad N = n + l + |S|
\]

(3.3)(\(|S|\) is the number of elements in \(S\)) is defined by the relation

\[
(\mathcal{G}_{\bar{A}\bar{B}}(X)) = \left( \begin{array}{ccc}
G_{ij} & 0 & 0 \\
0 & h_{\alpha\beta} & 0 \\
0 & 0 & \varepsilon_s \exp(-2U^s(X)) \delta_{ss'}
\end{array} \right).
\]

(3.4)
Here we consider exact solutions to field equations corresponding to the action (3.1)

\[ R_{\mu\nu}[g^0] = G_{\dot{A}\dot{B}}(X)\partial_{\mu}X^{\dot{A}}\partial_{\nu}X^{\dot{B}} + \frac{2V}{d_0 - 2g^0_{\mu\nu}}, \]  

(3.5)

\[ \frac{1}{\sqrt{|g^0|}}\partial_{\mu}[\sqrt{|g^0|}G_{\dot{C}\dot{B}}(X)g^{0\mu\nu}\partial_{\nu}X^{\dot{B}}] - \frac{1}{2}G_{\dot{A}\dot{B},\dot{C}}(X)g^{0\mu\nu}\partial_{\mu}X^{\dot{A}}\partial_{\nu}X^{\dot{B}} = V, \]  

(3.6)

\[ s \in S. \]  

Here \( \Delta[g^0] \) is the Laplace-Beltrami operator corresponding to \( g^0 \) and \( V_{\dot{C}} = \partial V/\partial X^{\dot{C}}. \)

We put

\[ X^{\dot{A}}(x) = F^{\dot{A}}(H(x)), \]  

(3.7)

where \( F : (u_-, u_+) \to \mathbb{R}^N \) is a smooth function, \( H : M_0 \to \mathbb{R} \) is a harmonic function on \( M_0 \), i.e.

\[ \Delta[g^0]H = 0, \]  

(3.8)

satisfying \( u_- < H(x) < u_+ \) for all \( x \in M_0 \).

The substitution of (3.7) into eqs. (3.5) and (3.6) leads us to the relations

\[ R_{\mu\nu}[g^0] = G_{\dot{A}\dot{B}}(F(u))\dot{F}^{\dot{A}}\dot{F}^{\dot{B}}\partial_{\mu}H\partial_{\nu}H + \frac{2V}{d_0 - 2g^0_{\mu\nu}}, \]  

(3.9)

\[ \left[ \frac{d}{du} (G_{\dot{C}\dot{B}}(F(u))\dot{F}^{\dot{B}}) - \frac{1}{2}G_{\dot{A}\dot{B},\dot{C}}(F(u))\dot{F}^{\dot{A}}\dot{F}^{\dot{B}} \right]g^{0\mu\nu}\partial_{\mu}H\partial_{\nu}H = V, \]  

(3.10)

where \( u = H(x) \) and \( \dot{f} = df/du \).

Let all spaces \((M_i, g^i)\) be Ricci-flat, i.e.

\[ R_{\mu\nu}[g^i] = 0, \]  

(3.11)

\[ i = 0, \ldots, n. \]  

In this case the potential is zero : \( V = 0 \) and the field equations (3.9) and (3.10) are satisfied identically if \( F = F(u) \) obey the Lagrange equations for the Lagrangian

\[ L = \frac{1}{2}G_{\dot{A}\dot{B}}(F)\dot{F}^{\dot{A}}\dot{F}^{\dot{B}} \]  

(3.12)

with the zero-energy constraint

\[ E = \frac{1}{2}G_{\dot{A}\dot{B}}(F)\dot{F}^{\dot{A}}\dot{F}^{\dot{B}} = 0. \]  

(3.13)

This means that \( F : (u_-, u_+) \to \mathbb{R}^N \) is a null-geodesic map for the minisupermetric (3.2).

Thus, we are led to the Lagrange system (3.12) with the minisupermetric \( G \) defined in (3.4).

The problem of integrability may be simplified if we integrate the Lagrange equations corresponding to \( \Phi^s \) (i.e. the Maxwell equations for \( s \in S_e \) and Bianchi identities for \( s \in S_m \)):

\[ \frac{d}{du} \left( \exp(-2U^s(z))\dot{\Phi^s} \right) = 0 \iff \dot{\Phi^s} = Q_s\exp(2U^s(z)), \]  

(3.14)
where \( Q_s \) are constants, \( s \in S \). Here \( (F^A) = (z^A, \Phi^s) \). We put
\[
Q_s \neq 0,
\]
(3.15)
for all \( s \in S \).

For fixed \( Q = (Q_s, s \in S) \) the Lagrange equations for the Lagrangian (3.12) corresponding to \( (z^A) = (\phi^s, \varphi^a) \), when equations (3.14) are substituted, are equivalent to the Lagrange equations for the Lagrangian
\[
L_Q = \frac{1}{2} \hat{G}_{AB} \dot{z}^A \dot{z}^B - V_Q,
\]
where
\[
V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp[2U_s(z)],
\]
(3.17)
\((\hat{G}_{AB})\) are defined in (2.17) respectively. The zero-energy constraint (3.13) reads
\[
E_Q = \frac{1}{2} \hat{G}_{AB} \dot{z}^A \dot{z}^B + V_Q = 0.
\]
(3.18)

### III.2 Toda-type solutions

Let us define the scalar product as follows
\[
(U, U') = \hat{G}^{AB} U_A U'_B,
\]
(3.19)
for \( U, U' \in R^{n+l} \), where \( (\hat{G}^{AB}) = (\hat{G}_{AB})^{-1} \). The scalar products (3.19) for vectors \( U^s \) were calculated in [25]
\[
(U^s, U'^s) = d(I_s \cap I'_s) + \frac{d(I_s)d(I'_s)}{2-D} + \chi_s \chi_{s'} \lambda_{aa_s} \lambda_{ab_s} h^{\alpha\beta},
\]
(3.20)
where \( (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1} ; s = (a_s, v_s, I_s) \) and \( s' = (a'_s, v'_s, I'_s) \) belong to \( S \).

Here we are interested in exact solutions for a special case when the vectors \( U^s \in R^{n+l} \) satisfy the following conditions
\[
K_s = (U^s, U^s) \neq 0,
\]
(3.21)
for all \( s \in S \), and a ("quasi-Cartan") matrix
\[
(A_{ss'}) = \begin{pmatrix}
2(U^s, U'^s) \\
(U'^s, U'^s)
\end{pmatrix}
\]
(3.22)
is a non-degenerate one. Here some ordering in \( S \) is assumed. It follows from (3.21) and the non-degeneracy of the matrix (3.22) that the vectors \( U^s, s \in S \), are linearly independent. Hence, the number of the vectors \( U^s \) should not exceed the dimension of \( R^{n+l} \), i.e.
\[
|S| \leq n + l.
\]
(3.23)
From (3.20)-(3.22) we get the following intersection rules

\[
d(I_s \cap I_{s'}) = \frac{d(I_s)d(I_{s'})}{D - 2} - \chi_s \chi_{s'} \lambda_{a_s} \cdot \lambda_{a_{s'}} + \frac{1}{2} K_{s'} A_{ss'},
\]

(3.24)
s \neq s'.

The exact solutions to Lagrange equations corresponding to (3.16) with the potential (3.17) could be readily obtained using the relations from Appendix. The solutions read

\[
z^A = \sum_{s \in S} \frac{U^s_A}{(U^s, U^s)} q^s + c^A u + \bar{c}^A,
\]

(3.25)

where \( q^s \) are solutions to Toda-type equations

\[
\ddot{q}^s = -B_s \exp\left( \sum_{s' \in S} A_{ss'} q^{s'} \right),
\]

(3.26)

with

\[
B_s = 2K_s A_s, \quad A_s = \frac{1}{2} \varepsilon_s Q_s^2,
\]

(3.27)
s \in S. These equations correspond to the Lagrangian

\[
L_{TL} = \frac{1}{4} \sum_{s, s' \in S} K^{-1}_{ss'} q^{s'} q^{s'} - \sum_{s \in S} A_s \exp\left( \sum_{s' \in S} A_{ss'} q^{s'} \right).
\]

(3.28)

Vectors \( c = (c^A) \) and \( \bar{c} = (\bar{c}^A) \) satisfy the linear constraint relations (see (6.11) in Appendix)

\[
U^s(c) = U^s_A c^A = \sum_{i \in I_s} d_i c^i - \chi_s \lambda_{a_s \alpha} c^\alpha = 0,
\]

(3.29)

\[
U^s(\bar{c}) = U^s_A \bar{c}^A = \sum_{i \in I_s} d_i \bar{c}^i - \chi_s \lambda_{a_s \alpha} \bar{c}^\alpha = 0,
\]

(3.30)
s \in S.

The contravariant components \( U^{sA} = \hat{G}^{AB} U^s_B \) are [25, 31]

\[
U^{si} = G^{ij} U^s_j = \delta_{iI_s} - \frac{d(I_s)}{D - 2}, \quad U^{s\alpha} = -\chi_s \lambda^\alpha_{a_s},
\]

(3.31)

Here (as in [19])

\[
G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D},
\]

(3.32)
i, j = 1, \ldots, n, are the components of the matrix inverse to \((G_{ij})\) from (2.18).

Using (3.23) and (3.31) we obtain

\[
\phi^i = \sum_{s \in S} h_s \left( \delta_{iI_s} - \frac{d(I_s)}{D - 2} \right) q^s + c^i u + \bar{c}^i,
\]

(3.33)

and

\[
\varphi^\alpha = -\sum_{s \in S} h_s \chi_s \lambda^\alpha_{a_s} q^s + c^\alpha u + \bar{c}^\alpha,
\]

(3.34)
\[ \gamma_0(\phi) = -\sum_{s \in S} \frac{d(I_s)}{D - 2} h_s q^s + c^0 u + \bar{c}^0, \]  

(3.35)

where we denote

\[ h_s = K_s^{-1} \]  

(3.36)

and

\[ c^0 = \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j c^j, \quad \bar{c}^0 = \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \bar{c}^j. \]  

(3.37)

The zero-energy constraint reads (see Appendix)

\[ 2E = 2E_{TL} + \hat{G}_{ABC} c^A c^B = \]  

(3.38)

where

\[ E_{TL} = \frac{1}{4} \sum_{s,s' \in S} K_s^{-1} A_{ss'} q^s q^{s'} + \sum_{s \in S} A_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}) \]  

(3.39)

is an integration constant (energy) for the solutions from (3.26).

From the relation

\[ \exp(2U^s(z)) = \prod_{s' \in S} f_{s'}^{-A_{ss'}} \]  

(3.40)

following from (3.22), (3.25), (3.29) and (3.30) we get for electric-type forms (2.8)

\[ F^s = Q_s \left( \prod_{s' \in S} f_{s'}^{-A_{ss'}} \right) dH \wedge \tau(I_s), \]  

(3.41)

\[ s \in S_e, \]  

and for magnetic-type forms (2.9)

\[ F^s = \exp[-2\lambda_\alpha(\varphi)] \left[ Q_s \left( \prod_{s' \in S} f_{s'}^{-A_{ss'}} \right) dH \wedge \tau(I_s) \right] = \check{Q}_s(*_0 dH) \wedge \tau(I_s), \]  

(3.42)

\[ s \in S_m, \]  

where \( \check{Q}_s = Q_s \varepsilon(I_s) \mu(I_s) \) and \( \mu(I) = \pm 1 \) is defined by the relation \( \mu(I)dH \wedge \tau(I_0) = \tau(\bar{I}) \wedge dH \wedge \tau(I), I_0 = \{1, \ldots, n\} \) (see eq. (2.26) in [25]). Here \( *_0 = *[g^0] \) is the Hodge operator on \( (M_0, g^0) \).

Relations for the metric and scalar fields follows from (3.33)-(3.35)

\[ g = \left( \prod_{s \in S} f_s^{2d(I_s)h_s/(D-2)} \right) \left\{ \exp(2c^0 H + 2\bar{c}^0) g^0 \right\} \]  

\[ + \sum_{i=1}^{n} \left( \prod_{s \in S} f_s^{-2h_i h_s} \right) \exp(2\bar{c}^i H + 2\bar{c}^i) g^i, \]  

(3.44)

\[ \exp(\varphi^\alpha) = \left( \prod_{s \in S} h_s \chi_s \lambda_s^\alpha \right) \exp(c^\alpha H + \bar{c}^\alpha), \]  

(3.45)
\[ \alpha = 1, \ldots, l. \] Here

\[ f_s = f_s(H) = \exp(-q^s(H)), \]  

(3.46)

where \( q^s(u) \) is a solution to Toda-like equations (3.20) and \( H = H(x) \ (x \in M_0) \) is a harmonic function on \( (M_0, g^0) \) (see (3.8)).

The solution is presented by relations (3.41)-(3.46) with the functions \( q^s \) defined in (3.26) and the relations on the parameters of solutions \( c^A, \bar{c}^A \ (A = i, \alpha, 0), Q_s, h_s \) imposed in (3.29), (3.30), (3.37) and (3.36), respectively.

This solution describes a set of charged (by forms) overlapping \( p \)-branes \( (p_s = d(I_s) - 1, \ s \in S) \) “living” on submanifolds of \( M_1 \times \ldots \times M_n \). The solution is valid if the dimensions of \( p \)-branes and dilatonic coupling vector satisfy the relations (2.12).

### IV Solutions corresponding to \( A_m \) Toda chain

Here we consider exact solutions to equations of motion of a Toda-chain corresponding to the Lie algebra \( A_m = sl(m + 1, C) \) [50, 51],

\[ \dot{q}^s = -B_s \exp \left( \sum_{s' = 1}^{m} A_{ss'}q^{s'} \right), \]  

(4.1)

where

\[ (A_{ss'}) = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix} \]  

(4.2)

is the Cartan matrix of the Lie algebra \( A_m \) and \( B_s > 0, \ s, s' = 1, \ldots, m \). Here we put \( S = \{1, \ldots, m\} \).

The equations of motion (4.1) correspond to the Lagrangian

\[ L_T = \frac{1}{2} \sum_{s, s' = 1}^{m} A_{ss'} q^s q^{s'} - \sum_{s = 1}^{m} B_s \exp \left( \sum_{s' = 1}^{m} A_{ss'}q^{s'} \right). \]  

(4.3)

This Lagrangian may be obtained from the standard one [50] by separating a coordinate describing the motion of the center of mass.

Using the result of A. Anderson [51] we present the solution to eqs. (4.1) in the following form

\[ C_s \exp(-q^s(u)) = \sum_{r_1 < \ldots < r_s} v_{r_1} \cdots v_{r_s} \Delta^2(w_{r_1}, \ldots, w_{r_s}) \exp[(w_{r_1} + \ldots + w_{r_s})u], \]  

(4.4)

\( s = 1, \ldots, m \), where

\[ \Delta(w_{r_1}, \ldots, w_{r_s}) = \prod_{i < j}^s (w_{r_i} - w_{r_j}); \quad \Delta(w_{r_1}) \equiv 1, \]  

(4.5)
denotes the Vandermonde determinant. The real constants $v_r$ and $w_r$, $r = 1, \ldots, m + 1$, obey the relations
\begin{equation}
\prod_{r=1}^{m+1} v_r = \Delta^{-2}(w_1, \ldots, w_{m+1}), \quad \sum_{r=1}^{m+1} w_r = 0. \tag{4.6}
\end{equation}

In (4.4)
\begin{equation}
C_s = \prod_{s'=1}^{m} B_{ss'}^{-A_{ss'}}, \tag{4.7}
\end{equation}
where
\begin{equation}
A_{ss'} = \frac{1}{m+1} \min(s, s')[m+1 - \max(s, s')], \tag{4.8}
\end{equation}
s, $s' = 1, \ldots, m$, are components of a matrix inverse to the Cartan one, i.e. $(A_{ss'})^{-1}$ (see Ch.7 in [43]). Here
\begin{equation}
v_r \neq 0, \quad w_r \neq w_{r'}, \quad r \neq r', \tag{4.9}
\end{equation}
r, $r' = 1, \ldots, m + 1$. We note that the solution with $B_s > 0$ may be obtained from the solution with $B_s = 1$ (see [51]) by a certain shift $q^s \mapsto q^s + \delta^s$.

The energy reads [51]
\begin{equation}
E_T = \frac{1}{2} \sum_{s, s'=1}^{m} A_{ss'} q^s q^{s'} + \sum_{s=1}^{m} B_s \exp \left( \sum_{s'=1}^{m} A_{ss'} q^{s'} \right) = \frac{1}{2} \sum_{r=1}^{m+1} w_r^2. \tag{4.10}
\end{equation}

If $B_s > 0$, $s \in S$, then all $w_r, v_r$ are real and, moreover, all $v_r > 0$, $r = 1, \ldots, m + 1$. In a general case $B_s \neq 0$, $s \in S$, relations (4.4)-(4.7) also describe real solutions to eqs. (4.1) for suitably chosen complex parameters $v_r$ and $w_r$. These parameters are either real or belong to pairs of complex conjugate (non-equal) numbers, i.e., for example, $w_1 = \bar{w}_2$, $v_1 = \bar{v}_2$. When some of $B_s$ are negative, there are also some special (degenerate) solutions to eqs. (4.1) that are not described by relations (4.4)-(4.7), but may be obtained from the latter by certain limits of parameters $w_i$ (see example in the next section).

For the energy (3.39) we get
\begin{equation}
E_{TL} = \frac{1}{2K} E_T = \frac{h}{4} \sum_{r=1}^{m+1} w_r^2. \tag{4.11}
\end{equation}

Here
\begin{equation}
K_s = K, \quad h_s = h = K^{-1}, \tag{4.12}
\end{equation}
$s \in S$.

Thus, in the $A_m$ Toda chain case (4.1) eqs. (4.4)-(4.12) should be substituted into relations (3.38) and (3.41)-(3.46) for the general solution.
IV.1 Examples for $d_0 > 2$

Here we consider the case $d_0 > 2$. Let matrix $(h_{\alpha \beta})$ be positively defined and $K = K_s > 0$. Then from the energy constraint (3.38) we get

$$E_{TL} \leq 0 \implies E_T \leq 0.$$  \hfill (4.13)

In this case

$$c^\alpha = c^i = 0 \iff E_{TL} = E_T = 0,$$  \hfill (4.14)

$i = 1, \ldots, n$, and $\alpha = 1, \ldots, l$. When $(h_{\alpha \beta})$ is negatively defined (as it takes place in 12-dimensional theory from [24]) there exist solutions with $E_{TL} > 0$.

IV.1.1 $A_1$-case.

Here we consider the case of one “brane”, i.e. $S = \{s\}$. Solving the Liouville equation

$$\ddot{q}^s = -B_s \exp(2q^s),$$  \hfill (4.15)

with $B_s = \varepsilon_s K_s Q_s^2$, we get

$$f_s(H) = |K_s|^{1/2}|Q_s|\tilde{f}_s(H),$$  \hfill (4.16)

where $H = H(x) > 0$ and

$$\tilde{f}_s(H) = \frac{1}{\sqrt{E_T}} \ch(\sqrt{E_T}H), \varepsilon_s K_s > 0, E_T > 0;$$  \hfill (4.17)

$$\frac{1}{\sqrt{E_T}} \sh(\sqrt{E_T}H), \varepsilon_s K_s < 0, E_T > 0;$$  \hfill (4.18)

$$\frac{1}{\sqrt{-E_T}} \sin(\sqrt{-E_T}H), \varepsilon_s K_s < 0, E_T < 0;$$  \hfill (4.19)

$$H, \varepsilon_s K_s < 0, E_T = 0.$$  \hfill (4.20)

Here

$$E_T = (\dot{q}^s)^2 + B_s \exp(2q^s).$$  \hfill (4.21)

In a special case $E_T = 0$ this solution agrees with those from [25, 36] if the following redefinition of the harmonic function is performed: $H \mapsto \hat{H}$,

$$\hat{H} = |K_s|^{1/2}|Q_s|H.$$  \hfill (4.22)

IV.1.2 $A_2$-case.

Now we consider the case $m = 2$ (for perfect fluid case see also [52]). Here we put $S = \{1, 2\}$. The solution reads

$$C_1 \exp(-q^1) = v_1 \exp(w_1 u) + v_2 \exp(w_2 u) + v_3 \exp(w_3 u),$$  \hfill (4.23)

$$C_2 \exp(-q^2) = v_1 v_2 (w_1 - w_2)^2 \exp(-w_3 u) + v_2 v_3 (w_2 - w_3)^2 \exp(-w_1 u) + v_3 v_1 (w_3 - w_1)^2 \exp(-w_2 u),$$  \hfill (4.24)
where
\[ w_1 + w_2 + w_3 = 0, \quad (4.25) \]
\[ v_1 v_2 v_3 = (w_1 - w_2)^2 (w_2 - w_3)^2 (w_3 - w_1)^2. \quad (4.26) \]

and
\[ C_1 = (B_1^2 B_2)^{-1/3}, \quad C_2 = (B_2 B_1)^{-1/3}. \quad (4.27) \]

Let \( K > 0 \). Then \( E_T \leq 0 \) and hence some of \( B_i \) should be negative.

Let \( B_1 < 0 \) and \( B_2 < 0 \). In the pseudo-Euclidean case, when \( \epsilon[g] = -1 \) and all \( \theta_a = 1 \), this means that \( \epsilon_s = \epsilon(I_s) = -1, i = 1, 2 \), (see (2.21)), i.e. all \( p \)-branes should contain an odd number of time submanifolds.

Let us consider solutions with a negative energy \( E_T = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2) < 0 \). (4.28)

In this case two of parameters \( w_i \) should be complex. Without loss of generality we put
\[ w_1 = -2\alpha, \quad w_2 = \alpha + i\beta, \quad w_3 = \alpha - i\beta, \quad (4.29) \]
\[ v_2 = ve^{i\theta}, \quad v_3 = ve^{-i\theta}, \quad (4.30) \]

where parameters \( \beta \neq 0 \), \( \alpha \) and \( v > 0 \) are real and
\[ v_1 v^2 = -\frac{1}{4} \beta^{-2}(9\alpha^2 + \beta^2)^{-2}. \quad (4.31) \]

Then relations (4.23) and (4.24) read
\[ |C_1| \exp(-q^1) = -v_1 \exp(-2\alpha u) + 2v \exp(\alpha u) \cos(\beta u + \theta), \quad (4.32) \]
\[ |C_2| \exp(-q^2) = 4\beta^2 v^2 \exp(2\alpha u) - 2vv_1 (9\alpha^2 + \beta^2) \exp(-\alpha u) \cos(\beta u + \theta + 2\varphi), \quad (4.33) \]

where \( |C_1| = (B_1^2 B_2)^{-1/3}, |C_2| = (B_2^2 B_1)^{-1/3}, |B_j| = Q_j^2, j = 1, 2 \), and
\[ 3\alpha + i\beta = (9\alpha^2 + \beta^2)^{1/2} e^{i\varphi}. \quad (4.34) \]

There exists also a degenerate solution with \( E_T = 0 \)
\[ |C_1| \exp(-q^1) = |C_2| \exp(-q^2) = \frac{1}{2} (u - u_0)^2, \quad (4.35) \]
\[ u_0 = \text{const}, \text{ that may be obtained from the solution (4.32),(4.33) with } \alpha = 0, v_1 = -2v, \]
\[ v = 1/2 \beta^2, \theta = -\beta u_0 \]
\[ |C_1| \exp(-q^1) = |C_2| \exp(-q^2) = \frac{2}{\beta^2} \sin^2[\beta (u - u_0)/2], \quad (4.36) \]
in the limit \( \beta \to 0 \).

Let us consider an example of the \( A_2 \)-solution in \( D = 11 \) supergravity [44]. We put \( n = 3, g^3 = -dt \otimes dt, d_1 = 2, d_2 = 5, d_0 = 3 \) (metrics \( g^0, g^1, g^2 \) are Ricci-flat). The
A_2\)-solution, describing a dyon configuration with electric 2-brane and magnetic 5-brane, corresponding to 4-form \( F \) and intersecting in 1-dimensional time manifold reads:

\[
g = c^{2/9} (\hat{H}^2 g^0 - \hat{H}^{-2} dt \otimes dt) + c^{8/9} g^1 + c^{-4/9} g^2,
F = cv_1 d\hat{H}^{-1} \wedge dt \wedge \tau_1 + cv_2 (\ast_0 d\hat{H}) \wedge \tau_1,
\]

where \( \hat{H} \) is the harmonic function on \((M_0, g^0)\) and \( \nu_1^2 = \nu_2^2 = 1 \). This solution corresponds to the degenerate solution (4.37). Here the following notations \( \hat{H} = H(\mid Q_1 \mid \mid Q_2 \mid)^{1/2} \), \( c = (\mid Q_2 \mid / \mid Q_1 \mid)^{1/3} \) are adopted. For \( c = 1 \) this solution coincides with that of [36].

V  Cosmological-type solutions

V.1 Solutions with Ricci-flat spaces

Let us consider a “cosmological” case: \( d_0 = 1 \) and

\[
M_0 = \mathbb{R}, \quad g^0 = wdu \otimes du,
\]

where \( w = \pm 1 \). Since \( H(u) = u \) is a harmonic function on \((M_0, g^0)\) we get for the metric and scalar fields from (3.43), (3.45)

\[
g = \left( \prod_{s \in S} f_s^{2d(r_s)h_s/(D-2)} \right) \left\{ \exp(2c^0 u + 2\bar{c}^0) wdu \otimes du \\
+ \sum_{i=1}^n \left( \prod_{s \in S} f_s^{-2h_s\delta_{is}} \right) \exp(2c^i u + 2\bar{c}^i) g^i \right\},
\]

\[
\exp(\varphi^\alpha) = \left( \prod_{s \in S} f_s^{h_s \chi_s^\alpha \lambda^\alpha_s} \right) \exp(c^\alpha u + \bar{c}^\alpha),
\]

\( \alpha = 1, \ldots, l \), where \( f_s = f_s(u) = \exp(\ast q^s(u)) \) and \( q^s(u) \) obey Toda-like equations (3.26).

Relations (3.37) and (3.38) take the form

\[
c^0 = \sum_{j=1}^n d_j c^j, \quad \bar{c}^0 = \sum_{j=1}^n d_j \bar{c}^j, \quad \bar{c}^0 = \sum_{j=1}^n d_j c^j, \quad 2E = 2E_{TL} + h_{\alpha\beta} c^\alpha c^\beta + \sum_{i=1}^n d_i (c^i)^2 - \left( \sum_{i=1}^n d_i c^i \right)^2 = 0,
\]

with \( E_{TL} \) from (3.39) and all other relations (e.g. constraints (3.29), (3.30)) and relations for forms (3.41) and (3.42) with \( H = u \) are unchanged. In a special \( A_m \) Toda chain case this solution was considered previously in [32].

V.2 Solutions with one curved space

The cosmological solution with Ricci-flat spaces may be also modified to the following case:

\[
\xi_1 \neq 0, \quad \xi_2 = \ldots = \xi_n = 0,
\]

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i.e. one space is curved and others are Ricci-flat and
\[ 1 \notin I_s, \quad \forall s \in S, \quad (5.7) \]
i.e. all “brane” submanifolds do not contain \( M_1 \). Relation (5.6) modifies the potential (3.17) by adding the term
\[ \delta V = \frac{1}{2} w \xi_1 d_1 \exp(2U^1(z)), \quad (5.8) \]
where \((d_1 > 1)\)
\[ U^1(z) = U^1_A z^A = -\phi^1 + \gamma_0(\phi), \quad (5.9) \]
\[ (U_1^A) = (-d_i^1 + d_i, 0). \quad (5.10) \]
For the scalar products we get \[31\]
\[ (U^1, U^1) = \frac{1}{d_1} - 1 < 0, \quad (5.11) \]
\[ (U^1, U^s) = 0 \quad (5.12) \]
for all \( s \in S \).
The solutions in the case under consideration may be obtained by a little modification of the solution (3.25) (see Appendix)
\[ z^A(u) = -\frac{U^1_A}{(U^1, U^1)} \ln |f_1(u - u_1)| - \sum_{s \in S} \frac{U^s_A}{(U^s, U^s)} \ln(f_s(u)) + c^A u + \bar{c}^A, \quad (5.13) \]
where
\[ f_1(\tau) = R \text{sh}(\sqrt{C^1_1} \tau), \quad C_1 > 0, \quad \xi_1 w > 0; \quad (5.14) \]
\[ R \sin(\sqrt{|C^1_1|} \tau), \quad C_1 < 0, \quad \xi_1 w > 0; \quad (5.15) \]
\[ R \text{ch}(\sqrt{C^1_1} \tau), \quad C_1 > 0, \quad \xi_1 w < 0; \quad (5.16) \]
\[ |\xi_1(d_1 - 1)|^{1/2} \tau, \quad C_1 = 0, \quad \xi_1 w > 0, \quad (5.17) \]
\( u_1 \) and \( C_1 \) are constants and \( R = |\xi_1(d_1 - 1)/C^1_1|^{1/2} \).
Vectors \( c = (c^A) \) and \( \bar{c} = (\bar{c}^A) \) satisfy the linear constraints (3.29), (3.30) and also additional constraints
\[ U^1(c) = U^1_A c^A = -c^1 + \sum_{j=1}^n d_j c^j = 0, \quad (5.18) \]
\[ U^1(\bar{c}) = U^1_A \bar{c}^A = -\bar{c}^1 + \sum_{j=1}^n d_j \bar{c}^j = 0. \quad (5.19) \]
The zero-energy constraint reads
\[ E = E_1 + E_{TL} + \frac{1}{2} \hat{g}_{AB} c^A c^B = 0, \quad (5.20) \]
where \( C_1 = 2E_1(U^1, U^1) \) or, equivalently,
\[
C_1 \frac{d_1}{d_1 - 1} = 2E_{TL} + h_\alpha \beta c^\alpha c^\beta + \sum_{i=2}^{n} d_i(c^i)^2 + \frac{1}{d_1 - 1} \left( \sum_{i=2}^{n} d_i c^i \right)^2 .
\] (5.21)

From (5.11), (5.13) and
\[
U_{1\alpha} = - \delta_{1\alpha} d_1 ,
\] (5.22)
we get a relation for the metric
\[
g = \left( \prod_{s \in S} [f_s(u)]^{2d_s/(D-2)} \right) \left\{ [f_1(u - u_1)]^{2d_1/(1 - d_1)} \exp(2c^1 u + 2\bar{c}^1) \right\}
\times [wdu \otimes du + f_1^2(u - u_1)g^1] + \sum_{i=2}^{n} \left( \prod_{s \in S} [f_s(u)]^{-2h_s \delta_{i\alpha}} \right) \exp(2c^i u + 2\bar{c}^i)g^i \right\}. (5.23)
\]

All other relations are unchanged. (Here \( H(u) = u \) and \( \ast_0 dH = w \) in (3.42).) This solution in a special case of \( A_m \) Toda chain was obtained earlier in [39].

VI Appendix: Solutions for Toda-like system

Let
\[
L = \frac{1}{2} < \dot{x}, \dot{x}> - \sum_{s \in S} A_s \exp(2 < b_s, x >)
\] (6.1)
be a Lagrangian, defined on \( V \times V \), where \( V \) is \( n \)-dimensional vector space over \( \mathbf{R} \), \( A_s \neq 0 \), \( s \in S \); \( S \neq \emptyset \), and \( < \cdot, \cdot > \) is non-degenerate real-valued quadratic form on \( V \).

Let
\[
K_s = < b_s, b_s > \neq 0,
\] (6.2)
for all \( s \in S \).

Then, the Euler-Lagrange equations for the Lagrangian (6.1)
\[
\ddot{x} + \sum_{s \in S} 2A_s b_s \exp(2 < b_s, x >) = 0,
\] (6.3)
have the following solutions
\[
x(u) = \sum_{s \in S} \frac{q^s(u)b_s}{< b_s, b_s >} + \alpha u + \beta,
\] (6.4)
where \( \alpha, \beta \in V \),
\[
< \alpha, b_s > = < \beta, b_s > = 0,
\] (6.5)
s \( \in S \), and functions \( q^s(u) \) satisfy the Toda-like equations
\[
\ddot{q}^s = -2A_s K_s \exp(\sum_{s' \in S} A_{ss'} q^{s'}),(6.6)
\]
with
\[ A_{ss'} = \frac{2 < b_s, b_{s'} >}{< b_{s'}, b_{s'} >}, \quad (6.7) \]
s, s' \in S. Let the matrix \( A_{ss'} \) be a non-degenerate one. In this case vectors \( b_s, s \in S \), are linearly independent. Then eqs. (6.6) are field equations corresponding to the Lagrangian
\[ L_{TL} = \frac{1}{4} \sum_{s,s' \in S} K_s^{-1} A_{ss'} q_s \dot{q}_{s'} - \sum_{s \in S} A_s \exp\left( \sum_{s' \in S} A_{ss'} q_{s'} \right). \quad (6.8) \]

For the energy corresponding to the solution (6.4) we get
\[ E = \frac{1}{2} < \dot{x}, \dot{x} > + \sum_{s \in S} \exp(2 < b_s, x >) = E_{TL} + \frac{1}{2} < \alpha, \alpha >, \quad (6.9) \]
where
\[ E_{TL} = \frac{1}{4} \sum_{s,s' \in S} K_s^{-1} A_{ss'} q_s \dot{q}_{s'} + \sum_{s \in S} A_s \exp\left( \sum_{s' \in S} A_{ss'} q_{s'} \right), \quad (6.10) \]
is the energy function corresponding to the Lagrangian (6.9).

For dual vectors \( u^s \in V^* \) defined as \( u^s(x) = < b_s, x >, \forall x \in V \), we have \( < u^s, u^{l'} > = < b_s, b_{l'} > \), where \( < \cdot, \cdot > \) is dual form on \( V^* \). The orthogonality conditions (6.5) read
\[ u^s(\alpha) = u^s(\beta) = 0, \quad (6.11) \]
s \( \in S \).

VII Conclusions

Here we obtained a family of solutions in multidimensional gravity with \( p \)-branes generalizing Majumdar-Papapetrou type solutions \( \psi \) from \([25, 36]\) in a special case of one harmonic function. These solutions are related to Toda-like systems (of general type) and are defined up to the solutions of Laplace and Toda-type equations. We considered special solutions corresponding to \( A_m \) Toda lattices (written in a parametrization of A. Andersen \([51]\)). The general solutions may be also used for other open Toda lattices, e.g. corresponding to \( B_m, C_m, D_m \) series. The solutions also contain a class of “cosmological” and spherically symmetric solutions and may be used for investigation of possible black hole and wormhole configurations.

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