Asymptotically Flat Holography
and
Strings on the Horizon

by

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ABSTRACT

Recently, Klemm and others [hep-th/0104141] have successfully generalized
the Cardy-Verlinde formula for an asymptotically flat spacetime of arbitrary
dimensionality. And yet, from a holographic perspective, the interpretation
of this formula remains somewhat unclear. Nevertheless, in this paper, we
incorporate the implied flat space/CFT duality into a study on boundary
descriptions of a $d$-dimensional Schwarzschild-black hole spacetime. In par-
ticular, we demonstrate that the (presumably) dual CFT adopts a string-like
description and, moreover, is thermodynamically equivalent to a string that
lives on the stretched horizon of the bulk black hole. Significantly, a similar
equivalence has recently been established in both an asymptotically dS and
AdS context. On this basis, we argue that the asymptotically flat Cardy-
Verlinde formula does, indeed, have a holographic pedigree.
1 Introduction

In spite of much attention, a consistent theory of quantum gravity remains one of the most challenging problems in theoretical physics. Nonetheless, one can still hope to partially comprehend this elusive, fundamental theory by looking for imprints that it may leave on a semi-classical framework. One likely imprint is the “holographic principle” [1, 2], which indicates that a physical system of dimensionality \( d \) can be effectively described by a system that lives in one dimension fewer.

It is interesting to note that the holographic principle started out as an intuitive leap of faith from the Bekenstein entropy bound [3]. Since its literary inception, however, this principle has attained considerable support; in particular, by way of the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [4, 5, 6]. The essence of this correspondence is that a \( d+1 \)-dimensional theory of AdS gravity (i.e., Einstein gravity with a negative cosmological constant) is dually related to a \( d \)-dimensional CFT that lives on a timelike boundary of the AdS bulk. For instance, boundary values of bulk fields serve as sources for the corresponding CFT. This realization has lead to explicit calculations of boundary correlation functions in terms of strictly bulk formalism [7].

On the basis of the holographic principle, it has been argued that some sort of analogue to the AdS/CFT duality should exist for any bulk theory that includes gravity [8]. However, this is anything but a trivial extension, considering that gravitational theories with a non-negative cosmological constant have no accessible boundary at spatial infinity. At least naively, the existence of such a boundary is critically important to many relevant calculations on account of the infrared-ultraviolet connection [8]. That is to say, the high-energy regime of a boundary-induced CFT can only be accessed in the large radial limit of the dual bulk theory. Such accessibility is important because as one moves deeper and deeper into the bulk (i.e., towards smaller radii), the CFT probes lower and lower energies, and the fine-grained details of the holographic picture are lost.

In spite of the above discussion, an analogous de Sitter (i.e., positive cosmological constant) or dS/CFT correspondence has indeed been established [9]. The problem of a missing boundary at spatial infinity has nicely been

\(^1\)For further citations in support of the dS/CFT duality, see Ref. [10]. Also see Refs. [11].
circumvented with the assumption that the dual CFT is now a Euclidean one that lives on a spacelike boundary of the dS bulk. It should be noted, however, that the dS/CFT duality remains on weaker ground than its AdS analogue. This can primarily be attributed to dS spacetimes lacking a string-theoretical description and an objective observer [14].

Also of interest is the flat-space limit of the AdS/CFT correspondence, which translates into a bulk gravity theory with a vanishing cosmological constant. In principle, this should be a perfectly valid limiting procedure. In practice, however, rigorous calculations are inhibited by a non-localization of the holographic mapping as the flat-space limit is approached [13].

An interesting subplot in the AdS/CFT correspondence is the renowned Cardy-Verlinde formula [16]. In the cited paper (also see [17]), Verlinde exploited the relevant duality to demonstrate that, for an arbitrary-dimensional bulk, the thermodynamics of the dual CFT could be expressed in a form which mimics the Cardy entropic formula [18]. Also of interest, the sub-extensive contribution to the CFT entropy plays the role of the Cardy “central charge”. Paradoxically, the Cardy formula originated in a strictly two-dimensional (CFT) framework, and there is no a priori basis for its generalization to higher-dimensional theories. Furthermore, the Cardy central charge is a Virasoro algebraic parameter; with the Virasoro algebra describing the symmetries of strictly two-dimensional CFTs [19].

Let us briefly point out that the Cardy-Verlinde formula [16] has since been generalized to a myriad of scenarios; including asymptotically AdS (for instance, [20]), asymptotically dS (for instance, [21]) and, even, asymptotically flat [22, 23] spacetimes. Moreover, the status of the “Casimir entropy” (i.e., the sub-extensive contribution to the CFT entropy) as a legitimate analogue to the Cardy central charge has since been substantiated [22, 23, 25]. In particular, these studies have identified the Casimir entropy with the “C-function” of a renormalization-group flow (when the conformal symmetry of the boundary theory is broken), in perfect analogy to the central charge of a two-dimensional broken CFT [27].

Let us return our attention to the paradox of CFT entropies consistently adopting Cardy-like descriptions, regardless of the dimensionality. In a sense, this is not particularly surprising in view of some prior studies by Carlip and

\[ \text{For an up-to-date list of citations, see Ref. [24].} \]
Solodukhin [28, 29]. These authors have shown that the two-dimensional Cardy formula is relevant to the entropic calculation of virtually any space-time (of any dimensionality) that admits a black hole (or black hole-like) event horizon. In spite of this universality, a lack of clarity at the interpretative level remains conspicuously absent. Recently, however, Halyo was able to shed some light on this matter [31]. In the context of a $d+1$-dimensional asymptotically dS bulk, this author has shown that the dually related ($d$-dimensional) CFT behaves, thermodynamically, just like a string. In particular, a linear relationship between the entropy and extensive energy of the boundary theory is in evidence. In view of this stringy description, the $d$-dimensional CFT can be interpreted as having only two relevant dimensions.

Halyo went on to demonstrate [31] that the CFT thermodynamic properties, including an effective string tension and energy, are directly related to those of a string that lives on the “stretched” (cosmological) horizon of the asymptotically dS bulk. (This latter picture follows from the “black hole-string correspondence”, as originally advocated by Susskind [32].) Furthermore, when the properties of the stretched horizon have properly been renormalized (see the end of this section), then these two stringy descriptions are, in fact, thermodynamically equivalent. That is, the CFT can alternatively be viewed as a horizon-based string having a renormalized tension. Notably, this program was later generalized for the case of an AdS-black hole bulk [34]. Given that a string is fundamentally a two-dimensional object, a Cardy-like formulation for the entropy of any CFT or, by duality, any black hole begins to make sense.

The focus of the current paper is a further generalization of the Halyo treatment [31] (also see [34]) to the case of an asymptotically flat black hole spacetime. Such a study has importance for the following two reasons. Firstly, considering the priorly discussed work of Carlip and Solodukhin [28, 29], one might anticipate that a stringy description of a duality-induced CFT is a universal feature of any bulk theory that admits an event horizon. Thus, for completeness sake, it is important to verify that this description persists in the case of an asymptotically flat bulk.

Secondly, let us first point out that a definitively dual CFT for an asymp-

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See Ref. [30] for relevant discussion.

A stretched black hole (or cosmological) horizon [32, 33] refers to a timelike surface that lies at a distance of the fundamental string scale ($l_s$) above the event horizon.
totically flat spacetime remains a relatively open question. Nonetheless, Klemm et al. have managed to generalize the Cardy-Verlinde formula for just such a flat-space scenario. However, it remains unclear, at least to the current author, if this generalized Cardy-Verlinde formula should be interpreted as an intriguing coincidence or as a true manifestation of the holographic principle. (In this regard, we note that the formula in question, although certainly valid, did not follow directly from holographic arguments.) We further note that the associated Casimir energy lacks the monotonicity properties, as relevant to a renormalization-group flow, which are evident in AdS-inspired dualities. We believe that this issue can be somewhat clarified by way of the “CFT-stretched horizon correspondence”. That is to say, if such a duality is in evidence for an asymptotically flat bulk, as it is for asymptotically AdS and dS scenarios, it would strongly support the holographic viewpoint.

The remainder of this paper is organized as follows. In Section 2, we begin by introducing the bulk solution of interest; namely, an arbitrary-dimensional Schwarzschild (i.e., asymptotically flat) black hole. The relevant black hole thermodynamics are presented, and these are used to formulate the thermodynamics of a (presumably) dual CFT. It is then shown that the CFT entropy satisfies Klemm et al.’s generalization of the Cardy-Verlinde formula. We also demonstrate the anticipated string-like behavior of the CFT and accordingly identify an effective string tension and energy.

In Section 3, we consider a much different effective theory; with this one living on a boundary close to the horizon of the Schwarzschild-black hole. Following Susskind’s proposal of a black hole-string correspondence, we demonstrate that the near-horizon geometry can be described as a single fundamental string that lives on this so-called stretched horizon. Furthermore, the associated thermodynamic properties of the string are appropriately identified.

In Section 4, we go on to compare these two stringy descriptions (of the Schwarzschild bulk) and ultimately confirm their thermodynamic equivalence. Essential to this identification is a coordinate rescaling of the properties of the string at the stretched horizon. This process can be viewed as

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5 Jing has also considered such a generalization.
6 This is contrary to the original Cardy-Verlinde expression and many of its subsequent generalizations.
a renormalization that accounts for the gravitational red shift occurring between the horizon and a cosmological observer (see below). Finally, Section 5 contains a summary and an overview.

Before proceeding to the main body of the paper, we present a brief discussion on the black hole-string correspondence, as this duality plays a prominent role in the analysis to follow.

Roughly a decade ago, Susskind proposed [32] a one-to-one correspondence between sufficiently massive black holes and highly excited fundamental strings. Let us now review the logistics of this proposal. As string coupling increases, the size of a string state must ultimately fall below its Schwarzschild radius and, hence, the string state should naturally evolve into a black hole [35]. Conversely, as string coupling decreases, the size of a black hole must eventually fall below the string scale and, hence, the near-horizon geometry should no longer be interpreted as a black hole per se. Thus, a black hole and some sort of string configuration can be regarded as strongly and weakly coupled manifestations of the same entity. Now consider that, at large enough mass, a typical string state consists of a small number of highly excited strings. However, as such a configuration approaches the high-temperature conditions of the horizon (i.e., the Hagedorn temperature), a single string will become the entropically preferred state [36]. Alternatively, an external observer would view the various strings as having melted together into a single string that fills up the stretched horizon [37]. On the basis of these arguments, a one-to-one correspondence between black holes and strings does indeed follow.

Contrary to the above perspective, there is, of course, an observed discrepancy between black hole and string thermodynamics. In particular, the entropy of a string has a linear energy dependence, whereas the entropy of (for instance) a four-dimensional Schwarzschild black hole depends quadratically on its mass. However, Susskind has conjectured [32] that this discrepancy can be accounted for with an appropriate renormalization of the string energy. Such a renormalization is, indeed, necessary by virtue of the large gravitational red shift occurring between the stretched horizon and an external observer. Exploiting the Rindler-like near-horizon geometry of a Schwarzschild black hole, Susskind was able to demonstrate the feasibility of this conjecture.

The validity of the black hole-string correspondence has since been rigorously substantiated for various black hole (as well as black hole-like) scenarios
Generally speaking, the counting of string states yields an entropy that coincides with the anticipated Bekenstein-Hawking form up to some ambiguity in the numerical coefficient. Let us further note that the same philosophy has also been applied to special classes of supersymmetric extremal and near-extremal black holes. (For instance, Ref.[45] and, for a review, Ref.[46].) Remarkably, in these cases, the Bekenstein-Hawking form has been reproduced with no ambiguity. Significant to this precision is a special property of such models; namely, supersymmetry protects the mass against renormalization [47].

2 A CFT Description of Schwarzschild Black Holes

In this section, we will review a proposal by Klemm et al. [22] of a generalized Cardy-Verlinde formula [18, 16] for an asymptotically flat black hole. We will also re-interpret the dual CFT (as implied by the generalized formula) as an effective stringy theory in analogy to Refs.[31, 34].

Let us begin here by considering the bulk model of interest; namely, a spherically symmetric black hole in an \( n+2 \)-dimensional spacetime with vanishing cosmological constant.\(^7\) In a static gauge, such a black hole can be uniquely described by the well-known Schwarzschild solution (generalized for arbitrary dimensionality). That is:

\[
ds_{n+2}^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2d\Omega_n^2,
\]

(1)

where:

\[
h(r) = 1 - \frac{\omega_n M}{r^{n-1}},
\]

(2)

\[
\omega_n = \frac{16\pi G}{nV_n}.
\]

(3)

Here, \(d\Omega_n^2\) denotes the line element of a unit \( n \)-sphere with volume \( V_n \), \( G \) is the \( n+2 \)-dimensional Newton constant, and \( M \) (a constant of integration) is the conserved mass of the associated black hole.

\(^7\)In this study, we will limit considerations to \( n \geq 2 \), as it is well established that an asymptotically flat, three-dimensional spacetime does not admit black hole solutions. We are also assuming no conserved charges other than the mass.
For a non-vanishing (positive) $M$, there will be a single black hole horizon, which corresponds to the positive root of $h(r)$. Denoting this horizon by $r = R$, we thus obtain:

$$R^{n-1} = \omega_n M.$$  \hfill (4)

The associated black hole thermodynamics (energy, temperature, entropy) can readily be identified via standard techniques \cite{48}:

$$E_{Sch} = M,$$  \hfill (5)

$$T_{Sch} = \left. \frac{1}{4\pi} \frac{dh}{dr} \right|_{r=R} = \frac{n-1}{4\pi R},$$  \hfill (6)

$$S_{Sch} = \frac{\nu_n R^n}{4G} = \frac{4\pi}{n\omega_n} R^n.$$  \hfill (7)

Following Klemm et al. \cite{22}, let us assume that Schwarzschild black holes, like their AdS-Schwarzschild counterparts, have the following dual description: an $n+1$-dimensional CFT that lives on a timelike boundary of the bulk spacetime. The CFT thermodynamics should then be related to those of a bulk up to a simple red-shift factor (with this ambiguity being a direct consequence of the conformal symmetry of the boundary theory) \cite{4, 5, 6}. Because of the relative simplicity of the bulk theory, we will assume that, in this case, the red-shift factor is equal to unity. Significant to this choice is the existence of a single length scale, $R$, for the bulk geometry. (Actually, this choice is difficult to justify; except that, because of the conformal symmetry at the boundary, we are free to do so. One might say that we started with the generalized Cardy-Verlinde formula \cite{22} and then defined the CFT thermodynamics so as to obtain a suitable match. The significance of this distinction will be elaborated on in the final section.)

Given the above considerations, we have:

$$E_{CFT} = E_{Sch} = \frac{1}{\omega_n} R^{n-1},$$  \hfill (8)

$$T_{CFT} = T_{Sch} = \frac{n-1}{4\pi R},$$  \hfill (9)

$$S_{CFT} = S_{Sch} = \frac{4\pi}{n\omega_n} R^n.$$  \hfill (10)
where we have applied Eq.(4) in obtaining the first result. Note that the CFT entropy is, in fact, uniquely defined, as entropy is never affected by the choice of scale [6].

Now following Verlinde [16], we will define the Casimir contribution ($E_C$) to the CFT energy as the violation of $E(\lambda S, \lambda V) = \lambda E(S, V)$; i.e., the violation in the Euler identity. This definition leads to the following relation [16] (now suppressing the subscript CFT on relevant thermodynamic quantities):

$$E_C = n [E + pV - TS],$$

where $V = R^n V_n$ is the volume enclosed by the CFT boundary and $p = E/nV$ is the pressure. Here, we have assumed the equation of state for radiative matter, as is appropriate for a theory with a traceless stress tensor. Straightforward evaluation of the above yields:

$$E_C = 2 \frac{R^{n-1}}{\omega_n} = 2E.$$  

(12)

Let us recall the Cardy-Verlinde formula for the CFT entropy as relevant to an asymptotically AdS bulk [16]:

$$S = 2\pi R_n \sqrt{E_C [2E - E_C]}.$$  

(13)

Naively applying this formula to the asymptotically flat case, we obtain the nonsensical result of $S = 0$. However, this unpleasant situation can be dealt with by first noting that the standard Cardy formula for a two-dimensional CFT [18]:

$$S = 2\pi \sqrt{\frac{c}{6} \left[ L_o - \frac{c}{24} \right]}.$$  

(14)

takes on a simplified form when the conformal weight of the ground state is zero. This simplification being:

$$S = 2\pi \sqrt{\frac{cL_o}{6}}.$$  

(15)

Note that $c$ is the central charge and $L_o$ is the eigenvalue of the zero-mode generator of the corresponding Virasoro algebra, which can be used to describe the symmetries of a two-dimensional CFT [18].
By direct analogy, this consideration implies that, under “appropriate circumstances”, the Cardy-Verlinde should be modified as follows:

\[ S = \frac{2\pi R}{n} \sqrt{E_C(2E)}. \]  

(16)

It is not difficult to verify that this revised form of the Cardy-Verlinde entropy (16) is, indeed, satisfied by the thermodynamic properties (8,10,12) of our conjectured CFT. Furthermore, it is straightforward to substantiate the following relation:

\[ S = \frac{4\pi R}{n} E. \]  

(17)

The significance of the above expression is the direct proportionality that exists between the CFT entropy and energy. (It is perhaps a subtle distinction, but, to a boundary observer, \( R \) represents a fixed ultraviolet cutoff \[8\] rather than a mass-dependent horizon radius. That is to say, this string-like behavior is not evident to a bulk observer; even though, strictly speaking, the thermodynamic relations are equivalent.) Notably, such linearity is a characteristic behavior of strings \[46\]. With this in mind, we can re-interpret Eq.(17) as the equation of state for an “effective string” having an energy:

\[ \mathcal{E} = \frac{2C}{n} E \]  

(18)

and having a tension:

\[ \mathcal{T} = \frac{C^2}{2\pi R^2}. \]  

(19)

Here, we have introduced a yet-to-be-determined parameter, \( C \). On the basis of prior studies \[31,34\], it can be anticipated that, in general, \( C = C(R; n, k) \).

The above string-like behavior is not a trivial outcome, inasmuch as the CFT typically has at least three spacetime dimensions, whereas a string is fundamentally a two-dimensional entity. This outcome is, however, not a complete surprise, considering that the relevant degrees of freedom for any black hole can evidently be described by a two-dimensional CFT \[28,24\]. That the Cardy-Verlinde description itself \[16\] has its genesis in two dimensions \[18\] also supports this notion. It would appear that some universal but as-of-yet-unknown principle underlies these various approaches.
3 A Stringy Description of Schwarzschild Black Holes

In the preceding section, we have shown that the CFT of interest (i.e., a CFT that is possibly dual to an asymptotically flat spacetime) exhibits some characteristics of a string-like object. With this mind, we will next consider a stringy description of a Schwarzschild black hole as based on a one-to-one correspondence that appears to exist between black holes and strings [32]. More specifically, Susskind has proposed that a massive black hole can be identified with a highly excited string which is located at the so-called stretched horizon [33, 37]. Furthermore, Susskind has argued that the apparent thermodynamic differences (between these two pictures) can be attributed to the long-range effects of the gravitational field; that is, the immense red shift occurring between the stretched horizon and a distant observer. (See Section 1 for supplementary discussion.)

To exploit the proposed correspondence, it is first necessary (or, at least, convenient) to transform the near-horizon black hole geometry into a Rindler-like form. We can accomplish this task by introducing a new radial coordinate, \( y \), in accordance with \( r = R + y \) (with \( y << R \) being assumed). Up to first order in \( y/R \), the metric function \( h(r) \) (2) can now be written as:

\[
 h(y) \approx \frac{n-1}{R} y. \tag{20}
\]

Incorporating the above into Eq. (1), we find the following near-horizon form for the line element:

\[
 ds^2_{NH} = -\left(\frac{n-1}{R}\right)^2 dt^2 + \frac{1}{y} \frac{R}{n-1} dy^2 + R^2 d\Omega^2_n. \tag{21}
\]

It is useful if \( y \) is replaced with a coordinate that directly measures the proper radial distance from the horizon. Denoting the proper distance as \( \rho \), we obtain (up to the first perturbative order):

\[
 \rho \approx \sqrt{\frac{R}{n-1}} \int_y^\infty \frac{dy}{\sqrt{y}} = 2 \sqrt{\frac{Ry}{n-1}}. \tag{22}
\]

The near-horizon line element (21) can now be expressed in the following Rindler-like form:

\[
 ds^2_{NH} = \left(\frac{n-1}{4R^2}\right) \rho^2 dt^2 + d\rho^2 + R^2 d\Omega^2_n. \tag{23}
\]
The near-horizon geometry can be identically formulated as a Rindler spacetime:

\[ ds^2_{\text{NH}} = -\rho^2 d\tau^2 + d\rho^2 + R^2 d\Omega^2_n, \]  

provided that the associated temporal coordinate is defined as follows:

\[ \tau \equiv \frac{n-1}{2R} t. \]  

Note that the Rindler time, \( \tau \), is a dimensionless quantity.

Next, let us consider the thermodynamics as measured by a hypothetical Rindler observer at the stretched horizon. The dimensionless horizon temperature is known to be [49]:

\[ T_R = \frac{1}{2\pi}. \]  

Meanwhile, the Rindler entropy should still be given by the Bekenstein-Hawking area law [50, 51] (or its \( n+2 \)-dimensional analogue), and so:

\[ S_R = S_{\text{Sch}}. \]

To determine the dimensionless Rindler energy, we simply apply the first law of thermodynamics; that is, \( dE_R = T_R dS_R \). This process yields:

\[ E_R = \frac{1}{2\pi} S_R = \frac{2R^n}{n\omega_n}, \]  

with the right-most relation following from Eqs.(4,27).

Significantly, the linear relation between Rindler entropy and energy is (once again) indicative of string-like behavior. By way of Susskind’s conjecture [32], we can attribute this behavior to the state of a single string that lives on the stretched horizon. Note that the associated string tension, in dimensionless Rindler coordinates, can readily be identified as:

\[ T_R = \frac{1}{2\pi}. \]  

Meanwhile, the dimensionless string energy is trivially \( \mathcal{E}_R = E_R \).

For a hypothetical observer located near the horizon, the fundamental length scale will be the string length, \( l_s \), as it is this quantity that, by definition, determines the radial extent of the stretched horizon [33, 34]. Hence,
we can obtain the dimensional thermodynamics of the stretched horizon by rescaling the Rindler relations in terms of this length. In particular, quantities with units of energy (or inverse length) should be divided by \( l_s \). That is:

\[
T_{SH} = \frac{T_R}{l_s} = \frac{1}{2\pi l_s},
\]

\[
S_{SH} = S_R = S_{Sch},
\]

\[
\mathcal{E}_{SH} = \frac{\mathcal{E}_R}{l_s} = \frac{2R^n}{n\omega_n l_s},
\]

\[
T_{SH} = \frac{T_R}{l_s^2} = \frac{1}{2\pi l_s^2}.
\]

It is interesting to note that \( T_{SH} \) agrees with the Hagedorn temperature of a string [10].

Let us re-emphasize that, in accordance with the black hole-string correspondence [32], the apparent discrepancy in thermodynamic properties (between those of the stretched horizon and those of the black hole) should be attributable to the long-range effects of the gravitational field. Notably, this conjectured duality has, indeed, been verified for a number of models (for instance, [39]) up to some ambiguity in the numerical coefficients. Furthermore, it has been demonstrated that the same arguments lead to a thermodynamic identification between the string at the stretched horizon and the (effective) string on the CFT boundary [31, 34]. This last point is of particular relevance to the analysis of the following section.

4 A CFT/String Correspondence?

In the preceding analysis, we have observed two different string-like descriptions of an asymptotically flat (black hole) spacetime; with the strings in question being located on the “CFT boundary” and the stretched black hole horizon. The purpose of the current section is to ascertain if these stringy theories are thermodynamically equivalent. Given that such an equivalence has been found for both asymptotically AdS [34] and dS [31] bulk geometries, we will view a similar outcome here as strong support for the proposed duality between asymptotically flat spacetimes and CFTs [15].
The stringy theories of interest can only be judiciously compared if the pertinent measurements are being carried out by a common observer. Hence, it is necessary that the thermodynamics of the stretched horizon be suitably rescaled for a cosmological observer (i.e., one who is far away from the horizon in terms of all relevant length scales). We can accomplish this task by rescaling the (dimensionless) Rindler thermodynamic quantities (26-29) in terms of the “cosmological” time coordinate, $t$. Utilizing the precise relation between $t$ and the Rindler time coordinate (viz Eq.(25)), we thus obtain the following rescaled quantities:

$$T'_{SH} = \frac{d\tau}{dt} T_R = \frac{n-1}{4\pi R}.$$  \hfill (34)

$$S'_{SH} = S_R = S_{Sch},$$  \hfill (35)

$$E'_{SH} = \frac{d\tau}{dt} E_R = \frac{n-1}{n\omega} \frac{R^{n-1}}{R^2},$$  \hfill (36)

$$\mathcal{T}'_{SH} = \left(\frac{d\tau}{dt}\right)^2 \mathcal{T}_R = \frac{(n-1)^2}{8\pi R^2}.$$  \hfill (37)

Note that a prime indicates that a rescaling has taken place.

As an aside, let us compare the above outcomes with those of Eqs.(30-33); that is, the thermodynamic properties of the string at the stretched horizon as measured by a local observer. In going from a horizon to a cosmological perspective, we observe an effective renormalization of roughly $l_s^{-1} \rightarrow R^{-1}$. Since the black hole-string correspondence [32] presumes that $R >> l_s$, this renormalization implies a significant reduction in the energy, string tension and temperature as measured by an external observer. Such a renormalization is, however, expected by virtue of the large gravitational red shift occurring between the horizon and a cosmological vantage point [32].

In the prior related works [31, 34], a further rescaling was necessary to account for an observer who is specifically located at the CFT boundary. We happily note that this is not the case here, since the relevant red-shift factor (between the bulk and the CFT boundary) was originally taken to be unity. Thus, we are now in a position to directly compare the thermodynamic properties of the stringy CFT [11-13,18,19] with the renormalized properties of the stretched horizon (34-37). This comparison goes as follows.

We begin here by noting that the entropies $S$ and $S'_{SH}$ trivially agree, as this dimensionless quantity is unaffected by any coordinate rescalings.
On a less trivial note, we observe the following thermal coincidence (cf. Eqs. (19, 34)):

$$T = \frac{n - 1}{4\pi R} = T_{SH}'.$$  

(38)

Next, let us compare the pair of expressions for the string tension. Recalling Eq. (19), we see that the CFT-inspired tension depends on an arbitrary parameter, \( C \). Our approach will be to identify \( T \) with \( T_{SH}' \) (37) and use this relation to define \( C \). (This \textit{ad hoc} identification will be suitably be tested via the string-energy comparison to follow.) On this basis, we find:

$$C = \frac{n - 1}{2}. \quad (39)$$

It is interesting that \( C \) is strictly a (dimensional-dependent) constant. This is in contrast to studies in an AdS and a dS bulk context, where the analogous parameter was found to depend directly on the horizon radius [31, 34].

Finally, let us consider a comparison of the string energies. Substituting the above identity for \( C \) into Eq. (18) for the CFT-inspired string energy, we obtain:

$$E = \frac{n - 1}{n\omega_n} R^{n-1}, \quad (40)$$

where we have also incorporated Eq. (8). By way of the above and Eq. (36), it is now clear to see that:

$$E = E_{SH}' \quad (41)$$

thus establishing the thermodynamic equivalence of the two stringy theories.

To briefly summarize, we have demonstrated a precise correspondence between the renormalized thermodynamics of the stretched horizon and the thermodynamics of the (conjectured) conformal boundary theory. This duality suggests that the string which appears to live on the CFT boundary (cf. Eqs. (18, 19)) is really just the string at the stretched horizon as viewed by a cosmological observer. Moreover, the observed correspondence adds support to the notion of a conformal boundary description of an asymptotically flat spacetime (given that analogous outcomes were obtained [31, 34] with bulk spacetimes for which a CFT duality is well-established). The relative simplicity of the prior calculations may obscure the non-triviality of this outcome. Hence, let us emphasize that this pair of boundary theories has no \textit{a priori} relationship.
As a final note in this section, we point out an agreement (up to an insignificant constant factor) between the “stringy energy” ($E$) and the total energy ($E$) of the CFT. This may seem a trivial outcome; however, no such agreement was found in the prior, related studies in an AdS and dS context [31, 34]. For instance, given an AdS-Schwarzschild bulk spacetime, these two energies were found to differ by $\approx E_E - 2E_C$ (where $E_E$ is the purely extensive contribution to the energy of the CFT). Considering that this energy discrepancy was attributed to an ambiguity in defining conserved charges in an asymptotically AdS or dS background [52, 53], it is quite encouraging that no such discrepancy arises in asymptotically flat spacetime (where conserved charges are unambiguously defined).

5 Conclusion

In summary, we have been studying some holographic aspects of an asymptotically flat spacetime of arbitrary dimensionality. In particular, we have focused on static, spherically symmetric bulk solutions admitting a single black hole horizon; that is, $n+2$-dimensional Schwarzschild black holes. The analysis formally began with a review of the pertinent bulk formalism, including the black hole thermodynamic expressions. After these preliminaries, we considered a pair of effective boundary theories; both of which have a conjectured duality with the black hole bulk spacetime. (We again point out that our treatment follows from a related paper by Halyo [31]. Also see Ref.[34].)

The first boundary theory of interest was a conformal field theory that lives on a timelike hypersurface of the asymptotically flat spacetime. Such a CFT is expected to be dually related to the bulk theory, presuming a well-defined flat-space limit of the AdS/CFT correspondence [4, 5, 6, 15]. On the basis of this proposed duality and a relevant study by Klemm et al. [22], we were able to identify various thermodynamic properties of the CFT and show that they satisfy a Cardy-Verlinde-like form [18, 19]. Moreover, we were able to demonstrate a linear relationship between the entropy and energy of the CFT. Significantly, this outcome implies that the CFT can also be interpreted as a string that lives on the associated cosmological boundary. On the basis of this stringy description, we identified an effective string tension and energy for the CFT.

The second boundary theory under consideration was based on one-to-one
correspondence that has been conjectured between massive black holes and highly excited strings [32]. This duality is most evident when one transforms the near-horizon geometry of a black hole into a Rindler-like form. Following this program, we are able to deduce the thermodynamic properties of a string that lives on the (so-called) stretched horizon of a Schwarzschild black hole. Although the stringy thermodynamic relations appear to differ from those of the associated black hole, it has convincingly been argued (for instance, [39]) that such discrepancies can be attributed to the long-range effects of the gravitational field.

In the final portion of this analysis, we compared the thermodynamic relations for this pair of effective boundary theories. For such a comparison, it was first necessary to rescale the thermodynamic properties of the stretched horizon. In particular, we invoked a coordinate transformation from Rindler time to “cosmological” time. In this way, both boundary theories could be viewed from the perspective of a common cosmological observer. It is interesting to note that the net effect of this rescaling is a significant renormalization of the tension, temperature and energy of the string at the stretched horizon. Such a renormalization can be attributed to the substantial gravitational red shift that arises between the horizon and a cosmological observation point [32].

Ultimately, we were able to demonstrate a precise equivalence between the renormalized thermodynamics of the stretched horizon and the thermodynamics of the “stringy CFT” (provided that an arbitrary parameter had been suitably fixed). These identifications included the temperature, string energy, string tension and, trivially, the entropy. This remarkable equivalence implies that the CFT thermodynamics can be re-interpreted as the renormalized thermodynamics of a string that lives nears the black hole horizon. Given that the two boundary theories are a priori unrelated, we find this identification to be considerably non-trivial.

In conclusion, the results of this paper appear to be particularly significant on two distinct levels; let us highlight these in turn. Firstly, we again point out the success of this general treatment for asymptotically dS [31], asymptotically AdS [34] and, now, asymptotically flat bulk scenarios. Hence, we are suitably positioned to extrapolate a universal link between CFTs on asymptotic boundaries and the thermodynamics of horizon-based strings. Ultimately, this link may help us understand why any relevant CFT entropy consistently conforms to a Cardy-like description [18, 19]. In this
regard, it is interesting to take note of some prior studies by Carlip and Solo-
dukhin [28, 29]. These authors demonstrated that a Cardy-like description of
entropy is likely a universal characteristic of black hole (and black hole-like)
geometries. Since all paths lead to the Cardy formula, one would expect
that these various approaches are probing the same fundamental principle.
Unfortunately, this principle may remain obscured (at least in the short run)
by our semi-classical limitations in understanding gravity.

Secondly, let us consider our results in the context of the proposed flat
space/CFT correspondence. Here, we again point out that our CFT ther-
modynamic framework was based, in large part, on a generalization of the
Cardy-Verlinde formula for an asymptotically flat spacetime (viz. Klemm et
al. [22]). That is to say, we first assumed the formula and then utilized it in
defining the CFT thermodynamic properties. This is converse to the typical
practice whereby one exploits the relevant duality (AdS/CFT or dS/CFT)
in deriving the corresponding CFT thermodynamics. Thus, the holographic
interpretation of the flat-space Cardy-Verlinde formula is somewhat unclear.
However, the results of this paper would appear to support a holographic
pedigree for this formula, given that similar outcomes have previously been
achieved in scenarios having well-established dualities [31, 34]. In spite of our
“success”, it would still be preferable to have this holographic connection es-
tablished on a more rigorous level. Hopefully, this issue will be addressed in
a future investigation.

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