PARACANONICAL BASE LOCUS, ALBANESE MORPHISM, AND SEMI-ORTHOGONAL INDECOMPOSABILITY OF DERIVED CATEGORIES

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Abstract. Motivated by an indecomposability criterion of Xun Lin for the bounded derived category of coherent sheaves on a smooth projective variety $X$, we study the paracanonical base locus of $X$, that is the intersection of the base loci of $\omega_X \otimes P_\alpha$ for all $\alpha \in \text{Pic}^0 X$. We prove that this is equal to the relative base locus of $\omega_X$ with respect to the Albanese morphism of $X$. As an application, we get that bounded derived categories of Hilbert schemes of points on certain surfaces do not admit non-trivial semi-orthogonal decompositions. We also have a consequence on the indecomposability of bounded derived categories in families. Finally, our viewpoint allows to unify and extend some results recently appearing in the literature.

1. Introduction

In this note, we are interested in finding hypothesis ensuring the indecomposability of the bounded derived category of a variety $X$, that is the nonexistence of non-trivial semi-orthogonal decompositions of $D^b(X)$ (see §2.1 for definitions). In recent years, bounded derived categories of projective varieties – and their semi-orthogonal decompositions – have been much studied. The interest in their indecomposability mainly rests on its conjectural relation with the minimal model program (see [27] and §2.1):

Conjecture 1.1. Let $X$ be a smooth projective variety. If $D^b(X)$ has no non-trivial semi-orthogonal decompositions, then $X$ is minimal, i.e., the canonical line bundle $\omega_X$ is nef.

Examples of varieties whose derived categories admit no non-trivial semiorthogonal decompositions are, for instance, varieties with trivial, or, more generally, algebraically trivial canonical bundle ([6] and [27, Corollary 1.7], respectively), curves of genus $\geq 1$ ([33], varieties whose Albanese morphism is finite ([39, Theorem 1.4])].

It is well known that the converse direction in Conjecture 1.1 is false (a counterexample is furnished by Enriques surfaces ([44]), but we have the following folklore (see [5, Conjecture 1.6], or [2, Question E]):

Conjecture 1.2. Let $X$ be a smooth projective variety. If $\omega_X$ is nef and effective, then $D^b(X)$ has no non-trivial semi-orthogonal decompositions.

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To be precise, in ([39, Theorem 1.4], Pirozhkov proved that such varieties are noncommutatively stably semiorthogonally indecomposable, which is a stronger notion than indecomposability.
At the moment of writing, this conjecture (as even the above Conjecture 1.1) is widely open in general and there are just some (classes of) varieties for which it has been verified: see [27, 2], besides the references already quoted above. A particularly interesting case is given by symmetric products of curves. It has been conjectured by Belmans-Galk in-Mukhopadhyay [3, Conjecture 1.1] and, independently, by Biswas-Gómez-Lee [5, Conjecture 1.4], that the bounded derived category of the \( n \)-th symmetric product of a smooth projective curve of genus \( g \geq 2 \) has no non-trivial semi-orthogonal decompositions, if \( n \leq g - 1 \). Very recently, Lin [29, Theorem 1.9] nicely proved it by using a criterion that we will recall below (see Theorem 1.4). This complements some previous partial results in [5] and [2].

A result of Kawatani-Okawa is particularly inspiring for us. These authors established a relation between the base locus \( \text{Bs} |\omega_X| \) of the canonical line bundle of \( X \) and the non-existence of semi-orthogonal decompositions (SODs for short) of \( D^b(X) \). Among other things, they proved [27, Corollary 1.5]:

**Theorem 1.3** (Kawatani-Okawa). If \( \text{Bs} |\omega_X| \) is a finite set (possibly empty), then \( D^b(X) \) has no non-trivial SODs.

So, in particular, a weak version of Conjecture 1.2 is known to be true: the global generation of \( \omega_X \) implies the nonexistence of SODs of \( D^b(X) \). In [29], Lin focused instead on the paracanonical base locus of a variety \( X \), that is the closed subset

\[
\text{PBs} |\omega_X| := \bigcap_{\alpha \in \text{Pic}^0X} \text{Bs} |\omega_X \otimes P_\alpha|,
\]

where \( \omega_X \) is the canonical line bundle of \( X \), and \( P_\alpha \) denotes the topologically trivial line bundle on \( X \) corresponding to the (closed) point \( \alpha \) of the Picard variety \( \text{Pic}^0X \). Lin refined Kawatani-Okawa theorem as follows:

**Theorem 1.4** (Lin). If \( \text{P Bs} |\omega_X| \) is a finite set (possibly empty), then \( D^b(X) \) has no non-trivial SODs.

The study of paracanonical base loci has its own interest. For instance, their emptiness (and related properties) has been implicitly studied in, e.g., [37, 1, 21], where several results of quite different flavor are obtained (see also [36] and the references therein). Note that, since \( \omega_X \) is an invertible sheaf, by definition \( \text{P Bs} |\omega_X| \) equals the support of the cokernel of the sum of evaluation maps

\[
\bigoplus_{\alpha \in \text{Pic}^0X} H^0(X, \omega_X \otimes P_\alpha) \otimes P_\alpha^\vee \rightarrow \omega_X.
\]

Here we observe that, using some standard results of generic vanishing theory due to Chen, Jiang, Pareschi, Popa and Schnell (that will be recalled in §2.2 for reader’s convenience), it is possible, quite easily, to give a different account of this locus, introducing a “relative” point of view (see Theorem 1.5 below). This allows to generalize Theorem 1.3 to a relative setting where the Albanese morphism

\[
a_X : X \rightarrow \text{Alb} X
\]

of \( X \) appears. More precisely, Lin’s criterion can be read as a relative version of Kawatani and Okawa’s one (see Corollary 1.8 below). Before giving the statement, let us fix some other notations.
We denote as usual by \( a_X^* a_X \ast \omega_X \rightarrow \omega_X \)
the adjuction morphism. The corresponding relative base ideal is
\[ b_X := b(\omega_X, a_X) = \text{Im} \left[ (a_X^* a_X \ast \omega_X \otimes \omega_X^X \rightarrow \mathcal{O}_X) \right], \]
and the relative base locus of \( \omega_X \) with respect to the Albanese morphism \( a_X \) is, by definition, the closed subset cut out by the relative base ideal \( b_X \).

Then, our main result is
\[ \textbf{Theorem 1.5}. \text{ Let } X \text{ be a smooth projective variety, or a compact Kähler manifold.} \text{ Then the paracanonical base locus } \text{PBs} | \omega_X | \text{ equals the relative base locus of } \omega_X \text{ with respect to the Albanese morphism of } X. \]
In particular, we get that \( \omega_X \) is \( a_X \text{-relatively globally generated} \) (this means that the natural morphism \( a_X^* a_X \ast \omega_X \rightarrow \omega_X \) is surjective, i.e., \( b_X = \mathcal{O}_X \)) if and only if \( \text{PBs} | \omega_X | \) is empty.

Since \( a_X \) finite implies \( a_X^* a_X \ast \omega_X \rightarrow \omega_X \), we have that

\[ \textbf{Corollary 1.6}. \text{ The paracanonical base locus of a variety with finite Albanese morphism is empty}. \]

\[ \textbf{Remark 1.7}. \text{ Similarly, if } X \text{ has maximal Albanese dimension (i.e., the Albanese morphism } a_X \text{ is generically finite onto its image), the paracanonical base locus of } X \text{ is contained in the exceptional locus of } a_X, \text{ which is defined as the inverse image, via } a_X, \text{ of the points in } a_X(X) \text{ having non-finite fibers. This gives a partial answer to [31, Question 8.7].} \]

Moreover, from Theorems 1.5 and 1.4 it follows
\[ \textbf{Corollary 1.8}. \text{ If } b_X \text{ defines a finite set (possibly empty), then } D^b(X) \text{ has no non-trivial SODs}. \]

This partially generalizes the above-mentioned result of Pirozhkov on varieties having finite Albanese morphism. It also allows to give an alternative proof of [29, Theorem 1.9] on indecomposability of derived categories of symmetric products of curves (see §4.2), and to put both these results under the same perspective.

On the other hand, Okawa recently informed us that he can prove the indecomposability of the derived category of a smooth projective surface \( S \) with nef \( \omega_S \) and irregularity \( h^0(S, \Omega^1_S) = \dim \text{Alb} S > 0 \). It would be interesting to know if the same holds true in higher dimensions, for instance in the case of minimal varieties of maximal Albanese dimension.

Our second result concerns indecomposability of derived categories of punctual Hilbert scheme on surfaces. Let \( S \) be a smooth projective surface and \( S^{[n]} \) the Hilbert scheme of points of length \( n \) on \( S \). In [2, Proposition 5.1], the authors proved that, if \( \text{Bs} | \omega_S | \) is empty, then, for all \( n \geq 1 \), \( \text{Bs} | \omega_S^{[n]} | \) is empty too. Hence, by Theorem 1.3 \( D^b(S^{[n]}) \) has no non-trivial SODs. It is natural to ask if something similar holds true taking into account the paracanonical base locus instead of the usual canonical base locus. Theorem 1.5 turns out to be useful in this case. Indeed, by using it, we prove

\[ \text{See the comment below Theorem 2.3}. \]
Theorem 1.9. Let \( n > 0 \) be a natural number. If
\[
\bigcap_{\alpha \in U} \text{Bs} |\omega_S \otimes P_\alpha| = \emptyset
\]
for all non-empty open subset \( U \subseteq \text{Pic}^0 S \) then \( \text{PBs} |\omega_S[n]| = \emptyset \). Therefore, \( \text{D}^b(S[n]) \) has no non-trivial SODs.

This should be seen as a counterpart to the symmetric product of curves situation mentioned above (see also §4.2 and Remark 4.4). It adds a new class of examples to the list of varieties appearing in the literature, whose derived categories are indecomposable. For instance, it applies to surfaces having finite Albanese morphism and Albanese image of general type thanks to [37, Proposition 5.5(iii)] (see also [22, Lemma 2.1]).

The proof of Theorem 1.9 will be given in §4.3. The last statement follows from Lin’s criterion (Theorem 1.4).

Lastly, we consider indecomposability in families and we give a variant of [2 Corollary B], where the authors proved that, if \( f: \mathcal{X} \to T \) is a smooth projective family of varieties, with \( T \) an irreducible algebraic variety over \( \mathbb{C} \), and there exists a point \( t_0 \in T \) such that \( \text{Bs} |\omega_{\mathcal{X}_{t_0}}| \) is finite (possibly empty), then \( \text{D}^b(\mathcal{X}_t) \) has no non-trivial SODs for all \( t \in T \). Here, as usual, \( \mathcal{X}_t \) denotes the variety \( f^{-1}(t) \). By assuming that the Albanese morphisms of the varieties appearing in the family \( f \) vary “smoothly”, we can prove that the same result is true, if the paracanonical base locus \( \text{PBs} |\omega_{\mathcal{X}_{t_0}}| \) is finite (possibly empty) for a point \( t_0 \in T \). Namely, in order to precisely state our result, let us recall that, given a smooth projective family \( f: \mathcal{X} \to T \), there exists a commutative diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\varphi} & \text{Alb} \mathcal{X}/T \\
\downarrow f & & \downarrow \eta \\
T & & \\
\end{array}
\]
where \( \eta \) is a smooth morphism such that, for all \( t \in T \), the fiber \( (\text{Alb} \mathcal{X}/T)_t \) is \( \text{Alb} \mathcal{X}_t \), and the restriction \( \varphi_t: \mathcal{X}_t \to (\text{Alb} \mathcal{X}/T)_t \) is the Albanese morphism \( a_{\mathcal{X}_t}: \mathcal{X}_t \to \text{Alb} \mathcal{X}_t \). The morphism \( \varphi \) is a universal morphism from the family \( f \) to families of abelian varieties over \( T' \), for any morphism \( T' \to T \) (see [16 n. 236], or [15, 8] and the references therein): \( \varphi \) is called the relative Albanese morphism for \( f \) and \( \text{Alb} \mathcal{X}/T \) is the relative Albanese variety for \( f \). Thanks to Theorem 1.5 we have the following result whose proof is the content of §5:

Theorem 1.10. Let \( f: \mathcal{X} \to T \) be a smooth projective family of varieties as above, and assume that its relative Albanese morphism \( \varphi \) is smooth. If \( \text{PBs} |\omega_{\mathcal{X}_{t_0}}| \) is finite (possibly empty) for a certain point \( t_0 \in T \), then \( \text{D}^b(\mathcal{X}_t) \) has no non-trivial SODs for all \( t \in T \).

We believe (and hope) that Theorem 1.5 could also be useful in some other context, possibly far from those considered in this note.

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3This means that \( \omega_S \) is continuously globally generated with respect to the Albanese morphism \( a_S \) (see §2.2 below). This notion, in general, neither implies nor it is implied by the global generation of \( \omega_S \).

4Recall that such properties are inherited by taking finite ramified coverings.
Notations. All varieties are assumed to be smooth, projective, irreducible and defined over \( \mathbb{C} \). A coherent sheaf on a variety is simply called a sheaf. Let \( \mathcal{P} \) be the normalized Poincaré line bundle on \( \text{Alb} X \times \text{Pic}^0 X \). Given a closed point \( \alpha \in \text{Pic}^0 X \), we denote by \( P_\alpha := \mathcal{P}|_{\text{Alb} X \times \{ \alpha \}} \) the corresponding line bundle on \( \text{Alb} X \). Recall that \( \text{Pic}^0(\text{Alb} X) \cong \text{Pic}^0 X \) via the pullback along the Albanese morphism \( a_X: X \to \text{Alb} X \) of \( X \). So, by a slight abuse of notation, the line bundle \( a_X^* P_\alpha \) on \( X \) corresponding to \( \alpha \in \text{Pic}^0 X \) is simply denoted by \( P_\alpha \).

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2. Prerequisites

Here we recall the definition of semi-orthogonal decomposition of a derived category, and some notions and results concerning the generic vanishing theory.

2.1. Semi-orthogonal decompositions. Let \( D^b(X) \) be the bounded derived category of a smooth projective variety \( X \). If there exist full triangulated subcategories \( A \) and \( B \) of \( D^b(X) \) such that
\[
D^b(X) = \left\langle A, B \right\rangle,
\]
(i.e., \( A \) and \( B \) generate the triangulated category \( D^b(X) \), and
\[
\text{Hom}(b, a) = 0
\]
for any object \( a \in A \) and \( b \in B \), then we say that \( (2.1) \) is a semi-orthogonal decomposition (SOD for short) of \( D^b(X) \).

Definition 2.1. A semi-orthogonal decomposition of \( D^b(X) \) is said to be non-trivial if \( A \) and \( B \) are both non-zero. If \( D^b(X) \) has no non-trivial semi-orthogonal decompositions, it is said to be indecomposable.

We refer the reader to [28] for an overview on SODs in algebraic geometry. The indecomposability of the derived category of a variety is (conjecturally) strongly related to its birational geometry. Indeed, the DK-hypothesis of [25] (see also [26]) predicts that, if there exists a \( K \)-inequality between two smooth projective varieties \( X \) and \( Y \), say \( X \leq_K Y \)\(^5\) then there is a fully faithful functor of triangulated categories \( D^b(X) \hookrightarrow D^b(Y) \). This would imply the existence of a semi-orthogonal decomposition of \( D^b(Y) \)
\[
D^b(Y) = \left\langle D^b(X)^\perp, D^b(X) \right\rangle,
\]
where \( D^b(X)^\perp \) is the right orthogonal complement of \( D^b(X) \) inside \( D^b(Y) \). See [26], and the references therein, for more information about this (and related) ideas.

\(^5\)This means that there exists a third smooth projective variety \( Z \) with birational morphisms \( f: Z \to X \) and \( g: Z \to Y \) such that \( g^* \omega_Y \otimes f^* \omega_X \) is effective.
2.2. **Generic vanishing.** Let $A$ be an abelian variety, and $\mathcal{F}$ be a sheaf on $A$. We denote by

$$V^i(A, \mathcal{F}) := \{ \alpha \in \text{Pic}^0 A \mid h^i(A, \mathcal{F} \otimes P_{\alpha}) > 0 \}$$

the $i$-th cohomological support loci of $\mathcal{F}$. These are closed algebraic subset of Pic$^0 A$ by semicontinuity.

**Definition 2.2.** The sheaf $\mathcal{F}$ is said to be:

i) $GV$, if codim$_{\text{Pic}^0 A} V^i(A, \mathcal{F}) \geq i$, for all $i \geq 0$;

ii) $M$-regular, if codim$_{\text{Pic}^0 A} V^i(A, \mathcal{F}) > i$, for all $i > 0$.

We refer the reader to the surveys [34] and [36] (and the references therein) for the main results surrounding such notions.

For us, the central property is the following:

**Proposition 2.3** ([35], Proposition 2.13). Let $\mathcal{F}$ be an $M$-regular sheaf on $A$. Then, for any non-empty open subset $U \subseteq \text{Pic}^0 A$, the sum of evaluation maps

$$\bigoplus_{\alpha \in U} H^0(A, \mathcal{F} \otimes P_{\alpha}) \otimes P_{\alpha}^\vee \to \mathcal{F}$$

is surjective.

A sheaf satisfying the condition (2.2) is said to be **continuously globally generated**. More generally, a sheaf $\mathcal{G}$ on a projective variety $Y$ admitting a non-trivial morphism $g: Y \to B$ to an abelian variety $B$, is said to be continuously globally generated with respect to $g$, if

$$\bigoplus_{\beta \in V} H^0(Y, \mathcal{G} \otimes g^* P_{\beta}) \otimes g^* P_{\beta}^\vee \to \mathcal{G}$$

is surjective for all non-empty open subset $V \subseteq \text{Pic}^0 B$. This notion was introduced by Pareschi and Popa in [35], and it has to be considered as a “continuous” variant of the usual notion of generation by global sections.

Now, let $a: X \to A$ be a morphism from a smooth projective variety $X$ to an abelian variety $A$. The higher direct images $R^j a_* \omega_X$ give non-trivial examples of $GV$ sheaves (see [17]). Recently, it has been observed that indeed much more is true:

**Theorem 2.4** (Chen-Jiang decomposition). Given a morphism $a: X \to A$ and an integer $j \geq 0$, there exists a canonical decomposition

$$R^j a_* \omega_X = \bigoplus_i \pi_i^* \mathcal{F}_i \otimes P_{\alpha_i},$$

(2.3)

where $\pi_i: A \to A_i$ are quotient of abelian varieties with connected fibers, the $\mathcal{F}_i$'s are $M$-regular sheaves on $A_i$, and $\alpha_i \in \text{Pic}^0 A$ are torsion points.

This powerful result was proved by Chen-Jiang [11], assuming $a$ generically finite, and soon after by Pareschi-Popa-Schnell [38] in complete generality and even in the Kähler setting. More precisely, in [38 Theorem A], the authors proved that the Chen-Jiang decomposition (2.3) holds true for a holomorphic mapping $a: X \to T$ from a compact Kähler manifold $X$ to a compact complex torus $T$,
and, moreover, the $M$-regular sheaves $F_i$, that are defined on the corresponding compact complex tori $T_i$, have \textit{projective} support for all $i$. This allows to say that the $F_i$’s are continuously globally generated (see \textit{op.cit.}, §20), that is precisely what we need here. Indeed, this fact – along with the decomposition (2.3) itself– is more than sufficient to prove Theorem 1.5.

See also [43] for a simplified proof of Theorem 2.4 in the projective case, following the lines of [11]. The Chen-Jiang decomposition also holds true for the pushforwards of pluricanonical bundles $a_X^{\omega_X^{\otimes m}}$, $m \geq 2$, of a smooth projective variety $X$ by Lombardi-Popa-Schnell [30], and it was recently applied to the study of derived invariants in [9] and [10].

3. Proof of Theorem 1.5

We are now ready to give the proof of our first result. We may assume that $X$ is smooth projective, given that the proof goes completely analogous in the compact Kähler case (see the comment below Theorem 2.4). Let $a_X : X \to \text{Alb} X$ be the Albanese morphism of $X$, and

$$a_X^* \omega_X = \bigoplus_i \pi_i^* F_i \otimes P_{\alpha_i} \quad (3.1)$$

be the Chen-Jiang decomposition of $a_X^* \omega_X$ (see §2.2). Hence, for each index $i$, $\pi_i : \text{Alb} X \to A_i$ is a quotient of abelian varieties with connected fibers, $F_i$ is an $M$-regular sheaf on $A_i$, and $\alpha_i \in \text{Pic}^0 X$ is a torsion point. The sheaves $F_i$ are continously globally generated by [35, Proposition 2.13]. This means, in particular, that the sum of evaluation maps

$$\bigoplus_{\alpha \in \text{Pic}^0 A_i} H^0(A_i, F_i \otimes P_{\alpha}) \otimes P_{\alpha}^\vee \to F_i \quad (3.2)$$

is surjective. Note that the dual morphism $\hat{\pi}_i : \text{Pic}^0 A_i \hookrightarrow \text{Pic}^0 X$ is injective, hence we simply denote by $\alpha = \hat{\pi}_i(\alpha) \in \text{Pic}^0 X$ the image of an element $\alpha \in \text{Pic}^0 A_i$. Moreover, by the projection formula,

$$H^0(\text{Alb} X, \pi_i^* F_i \otimes P_{\alpha}) = H^0(A_i, F_i \otimes P_{\alpha})$$

if $\alpha \in \text{Pic}^0 A_i$. Hence, applying $\pi_i^*$ to (3.2) and tensoring by $P_{\alpha_i}$, we get that, a fortiori, the morphism

$$\bigoplus_{\alpha \in \text{Pic}^0 X} H^0(\text{Alb} X, \pi_i^* F_i \otimes P_{\alpha}) \otimes P_{\alpha_i}^\vee \otimes P_{\alpha_i} \to \pi_i^* F_i \otimes P_{\alpha_i} \quad (3.3)$$

is surjective. Now we change notation as follows: we call $\alpha_i = -\beta \in \text{Pic}^0 X$. So we have that the map

$$\bigoplus_{\beta \in \text{Pic}^0 X} H^0(\text{Alb} X, \pi_i^* F_i \otimes P_{\alpha_i} \otimes P_{\alpha_i} \otimes P_{\alpha_i}) \otimes P_{\alpha_i}^\vee \otimes P_{\alpha_i} \to \pi_i^* F_i \otimes P_{\alpha_i} \quad (3.3)$$

is surjective, and this holds true for any index $i$ appearing in the Chen-Jiang decomposition of $a_X^* \omega_X$. Therefore, taking the sum over $i$ in (3.3) – and using (3.1) –, we get the following

\textbf{Lemma 3.1.} Let $X$ be a smooth projective variety, or a compact Kähler manifold. Then, the sum of evaluation maps

$$\bigoplus_{\beta \in \text{Pic}^0 X} H^0(\text{Alb} X, a_X^* \omega_X \otimes P_{\beta}) \otimes P_{\beta}^\vee \to a_X^* \omega_X \quad (3.4)$$

is surjective.
Now, if we apply $a^*_X$ to (3.4) and post-compose with the natural morphism $a^*_X a_X^* \omega_X \to \omega_X$, we recover the evaluation morphism

$$\bigoplus_{\beta \in \text{Pic}^0 X} H^0(X, \omega_X \otimes P_\beta) \otimes P_\beta^\vee \to a^*_X a_X^* \omega_X \to \omega_X,$$

(3.5)

where we used that $H^0(\text{Alb} X, a_X^* \omega_X \otimes P_\beta) = H^0(X, \omega_X \otimes P_\beta)$ for all $\beta \in \text{Pic}^0 X$. The proof of Theorem 1.5 follows from (3.5). Indeed, if $x \in X$ is not contained in the relative base locus of $\omega_X$ with respect to $a_X$, then the sum of evaluation maps

$$\bigoplus_{\beta \in \text{Pic}^0 X} H^0(X, \omega_X \otimes P_\beta) \otimes P_\beta^\vee \to \omega_X,$$

is surjective at $x$ by (3.5). Therefore, by definition, $x \notin \text{PBs} |\omega_X|$. The other inclusion is completely similar (and indeed easier, since it does not need Lemma 3.1). □

**Remark 3.2.** Let $a: X \to A$ be a morphism from a smooth projective variety $X$ to an abelian variety $A$. The properties of $a_X^* \omega_X$ we used in the proof of Theorem 1.5 hold also true for $a^* \omega_X$ (see §2.2). Therefore, if we define the $a$-paracanonical base locus as

$$\text{PBs}_a|\omega_X| := \bigcap_{\alpha \in \text{Pic}^0 A} \text{Bs} |\omega_X \otimes a^* P_\alpha|,$$

we have that this is equal to the relative base locus of $\omega_X$ with respect to the morphism $a: X \to A$. Note that $\text{PBs} |\omega_X| \subseteq \text{PBs}_a|\omega_X|$, for any $a$.

More generally, the same happens for a holomorphic map $a: X \to T$ from a compact Kähler manifold $X$ to a compact complex torus $T$, and, in the projective setting, for $a^* \omega_X^{\otimes m}$ with $m \geq 2$ (see the comment below Theorem 2.4), once one defines the $a$-parapluricanonical base locus $\text{PBs}_a|\omega_X^{\otimes m}|$ likewise.

4. Examples

In this section, we present some non-trivial examples where the hypothesis of Corollary 1.8 can be (quickly) checked.

4.1. Varieties with finite Albanese morphism. As already observed, it is well known that, if a variety $X$ has finite Albanese morphism $a_X$, then its canonical bundle $\omega_X$ is $a_X$-relatively globally generated. Thus, $D^0(X)$ is indecomposable if $a_X$ is finite. We refer the reader to the paper [24] for some fundamental structural results on varieties with finite Albanese morphism, and to [39] for a stronger theorem on indecomposability, in this case.

4.2. Symmetric products of curves. Let $C$ be a smooth projective curve of genus $g \geq 2$. Let $J(C)$ be its Jacobian variety. We denote by $C^{(n)}$ the $n$-th symmetric product of $C$. Let

$$a_n := a_{C^{(n)}}: C^{(n)} \to J(C), \quad D \mapsto O_C(D - np_0)$$

(4.1)

be the Albanese morphism of $C^{(n)}$, where $p_0 \in C$ is a fixed point.
In [29], by algebraically moving techniques, Lin proved that PBs $|\omega_{\mathbb{C}^g}|$ is empty if $n \leq g - 1$. In particular, $D^k(\mathbb{C}^g)$ has no non-trivial SODs in that range. By making use of Theorem 1.5, we see how this fact also follows from a classical description of the symmetric product of curves appearing in [11] Namely, we have

**Proposition 4.1.** Let $n \leq g - 1$. Then the canonical bundle $\omega_{\mathbb{C}^g}$ is $a_n$-relatively globally generated.

**Proof.** Let 

$$i_{n-1} : \mathbb{C}^{(n-1)} \to \mathbb{C}^g, \quad D \mapsto D + p_0,$$

and denote its image by $x = i_{n-1}(\mathbb{C}^{(n-1)}) \subseteq \mathbb{C}^g$. Let $\Theta$ be the theta divisor on $J(\mathbb{C})$, and $\theta \in \text{NS}(\mathbb{C}^g)$ the class of the pullback of $\Theta$ via $a_n$. We recall that algebraic and numerical equivalence of divisors coincide on $\mathbb{C}^g$, and that

$$\omega_{\mathbb{C}^g} = \theta + (g - n - 1)x$$

holds in $\text{NS}(\mathbb{C}^g)$ (see, e.g., [5, Lemma 2.1]). We prove that, for any $m \geq 0$, the map

$$a_n^*a_n^*\mathcal{O}_{\mathbb{C}^g}(\theta + mx) \to \mathcal{O}_{\mathbb{C}^g}(\theta + mx)$$

is surjective. The case $m = g - n - 1$ is the desired result. Since $\theta$ is a pullback line bundle via $a_n$, it is enough to prove this without the twist by $\theta$. For $m = 0$ this is evident, while for $m = 1$ this holds by [11, Proposition 10]. Assume $m \geq 2$. Then, we have that $a_n^*a_n^*\mathcal{O}_{\mathbb{C}^g}(mx) \to \mathcal{O}_{\mathbb{C}^g}(mx)$ is surjective by induction on $m$, thanks to the following commutative diagram

$$
\begin{array}{ccc}
a_n^*a_n^*\mathcal{O}_{\mathbb{C}^g}((m-1)x) & \xrightarrow{ev} & \mathcal{O}_{\mathbb{C}^g}(\theta + mx) \\
\downarrow & & \downarrow \\
a_n^*a_n^*\mathcal{O}_{\mathbb{C}^g}(mx) & \xrightarrow{ev} & \mathcal{O}_{\mathbb{C}^g}(mx).
\end{array}
$$

\[ \square \]

### 4.3. Hilbert schemes of points on surfaces

Here we prove Theorem 1.9.

Let $S$ be a smooth projective surface. Given an integer $n > 0$, we denote by $S^{[n]}$ the Hilbert scheme of points of length $n$ on $S$, and by $S^{(n)}$ the $n$-fold symmetric product. It is well-known (see [14]) that $S^{[n]}$ is smooth and that the Hilbert-Chow morphism $\pi : S^{[n]} \to S^{(n)}$ is a canonical resolution of singularities of $S^{(n)}$. Moreover,

$$a_{S^{[n]} : S^{[n]} \to S^{(n)}} \xrightarrow{f_n} \text{Alb} S$$

(4.2)

is the Albanese morphisms of $S^{[n]}$, where $f_n$ is the morphism induced by $a_S$ by addition on $\text{Alb} S$. In particular, $\text{Alb} (S^{[n]}) \cong \text{Alb} S$ (see [14]). Let us point out that (4.2) is a natural generalization of the morphism (1.1).

The Hilbert-Chow morphism $\pi$ is a crepant resolution, i.e., $\omega_{S^{[n]}} = \pi^*\omega_{S^{(n)}}$, and $S^{(n)}$ is a Gorenstein variety, so that $\omega_{S^{(n)}}$ is a line bundle. It is easy to see that $\omega_{S^{[n]}}$ is $a_{S^{[n]}}$-relatively globally generated.

---

6Interestingly enough, the main result of [11] was also used by Belmans and Krug (see Remark 1.4 below) to give a simpler proof of a result of Toda constructing a non-trivial semi-orthogonal decomposition of $D^b(\mathbb{C}^g)$, when $n \geq g \geq 2$.

7We also refer to [13] [19] for a detailed account on these morphisms.
globally generated, if $\omega_{S(n)}$ is $f_n$-relatively globally generated. Indeed, since $\pi_* O_{S[n]} = O_{S(n)}$, by the projection formula we have that

$$a^*_{S[n]} a_{S[n]} \omega_{S[n]} = \pi^* f_n^* f_n \pi_* \pi^* \omega_{S(n)} = \pi^* f_n^* f_n \omega_{S(n)}.$$  

Hence, if

$$f_n^* f_n \omega_{S(n)} \to \omega_{S(n)}$$

is surjective, by applying $\pi^*$ to it we get that

$$a^*_{S[n]} a_{S[n]} \omega_{S[n]} \to \omega_{S[n]}$$

is surjective, too.

So we reduced to prove

**Proposition 4.2.** If $\omega_S$ is continuously globally generated with respect to $a_S$, then

$$f_n^* f_n \omega_{S(n)} \to \omega_{S(n)}$$

is surjective.

**Proof.** Let $S^n$ be the usual product variety. We consider the following commutative diagram

$$
\begin{array}{ccc}
S^n & \xrightarrow{q} & S^{(n)} \\
\downarrow a_{S^n} & & \downarrow f_n \\
(\text{Alb } S)^n & \xrightarrow{\sigma} & \text{Alb } S
\end{array}
$$

where $q$ is the quotient morphism by the action of the symmetric group $\mathfrak{S}_n$ on $S^n$, $a_{S^n} = (a_S)^n$ is the Albanese morphism of $S^n$, and $\sigma$ is the addition morphism.

We make use of the following technical fact:

**Proposition 4.3.** Let $X$ be a smooth projective variety such that

$$\bigcap_{\alpha \in U} \text{Bs } |\omega_X \otimes P_\alpha| = \emptyset,$$

for any non-empty open subset $U \subseteq \text{Pic}^0 X$. Let $n \geq 1$ be a natural number. If $\sigma_n : \text{Alb } X^n \to \text{Alb } X$ denotes the addition morphism, then the sum of evaluation maps

$$ev_{n, U} : \bigoplus_{\alpha \in U} H^0(X^n, \omega_X \otimes \sigma_n^* P_\alpha) \otimes \sigma_n^* \nu^\vee \to \omega_X^n$$

is surjective, for all non-empty open subset $U \subseteq \text{Pic}^0 X$.

Granting it for the moment, we apply this to the surface $X = S$. So, in particular,

$$\bigcap_{\alpha \in \text{Pic}^0 S} \text{Bs } |\omega_{S^n} \otimes \sigma_n^* P_\alpha| = \emptyset.$$

---

8 Otherwise said: if $\omega_X$ is continuously globally generated with respect to $a_X$, then $\omega_{X^n}$ is continuously globally generated with respect to the composition $a_n := \sigma_n \circ a_{X^n} : X^n \to \text{Alb } X$.
Since $S^n$ is quasi-compact, there exists a finite number $N$ such that, for some $\alpha_1, \ldots, \alpha_N \in \text{Pic}^0 S$, the intersection $\bigcap_{k=1}^N B_n |\omega_{S^n} \otimes \sigma^* P_{\alpha_k}|$ is empty, i.e., the sum of evaluation morphisms

$$\bigoplus_{k=1}^N H^0(S^n, \omega_{S^n} \otimes \sigma^* P_{\alpha_k}) \otimes \sigma^* P_{\alpha_k}^\vee \rightarrow \omega_{S^n}$$

is surjective. Looking at the diagram (4.3), note that $q_* \sigma^* P_{\alpha_k}^\vee = q_* \alpha_n^* \sigma^* P_{\alpha_k}^\vee = q_* q^* f_n^* P_{\alpha_k}^\vee = f_n^* P_{\alpha_k}^\vee \otimes q_* \mathcal{O}_{S^n}$ for all $k$, by the projection formula. Moreover, since $\omega_{S^n} = q^* \omega_{S(n)}$, we have that $q_* \omega_{S^n} = \omega_{S(n)}(\alpha) \otimes q_* \mathcal{O}_{S^n}$ as well. Since $q$ is finite, we see, by taking the pushforward $q_*$ in (4.4), that the morphism

$$\bigoplus_{k=1}^N H^0(S^n, \omega_{S^n} \otimes \sigma^* P_{\alpha_k}) \otimes f_n^* P_{\alpha_k}^\vee \otimes q_* \mathcal{O}_{S^n} \rightarrow \omega_{S(n)} \otimes q_* \mathcal{O}_{S^n}$$

is surjective, too. Now, since $\mathcal{O}_{S(n)}$ is contained in $q_* \mathcal{O}_{S^n}$ as a direct summand, there exists a surjective morphism $q_* \mathcal{O}_{S^n} \rightarrow \mathcal{O}_{S(n)}$ and we get the following commutative diagram

$$\bigoplus_{k=1}^N H^0(S^n, \omega_{S^n} \otimes \sigma^* P_{\alpha_k}) \otimes f_n^* P_{\alpha_k}^\vee \otimes q_* \mathcal{O}_{S^n} \rightarrow \omega_{S(n)} \otimes q_* \mathcal{O}_{S^n}$$

So, the bottom horizontal map is surjective, and, by adjunction, it factorizes as

$$\bigoplus_{k=1}^N H^0(S^n, \omega_{S^n} \otimes \sigma^* P_{\alpha_k}) \otimes f_n^* P_{\alpha_k}^\vee \rightarrow f_n^* f_n \omega_{S(n)} \rightarrow \omega_{S(n)}.$$  

In order to conclude the proof of Proposition 4.2 and hence of Theorem 1.9, it only remains to give the

**Proof of Proposition 4.3.** The proof is essentially the same of [35, Proposition 2.12] and it uses the method of “reducible sections”. We proceed by induction on $n$. For $n = 1$ there is nothing to prove. Let us denote by $\pi_i: X^n \rightarrow X$ the $i$-th projection. Since $\alpha \in \text{Pic}^0 X$, we have

$$\sigma^*_n P_{\alpha} = \pi^*_1 P_{\alpha} \otimes \cdots \otimes \pi^*_n P_{\alpha} =: P_{\alpha}^{\boxtimes n},$$

where $\sigma_n: \text{Alb} X^n \rightarrow \text{Alb} X$ is the addition morphism. This can be easily proved from the see-saw principle (see, e.g., [32, pp. 74-75]) and induction on $n$.

Let $\bar{x} = (x_1, \ldots, x_n) \in X^n$ be an arbitrary point. Since we are assuming that $\omega_X$ is continuously globally generated with respect to $\alpha_X$, there exists a non-empty open subset $U_{\bar{x}_n} \subseteq \text{Pic}^0 X$ such that $x_n \notin B_1 |\omega_X \otimes P_{\alpha}|$ for all $\alpha \in U_{\bar{x}_n}$.

---

9Proof: Let $\mathcal{L}_{n}$ be the ideal sheaf of $x_n$ in $X$. By upper semicontinuity, there exist open subsets $U_1, U_2 \subseteq \text{Pic}^0 X$ such that $h^0(X, \omega_X \otimes P_{\alpha})$ (resp. $h^0(X, \omega_X \otimes \mathcal{L}_{n} \otimes P_{\alpha})$) takes its minimal value, when $\alpha \in U_1$ (resp. $\alpha \in U_2$). Let $U_{\bar{x}_n} := U_1 \cap U_2$. Then, since by the hypothesis the morphism

$$\bigoplus_{\alpha \in U_{\bar{x}_n}} H^0(X, \omega_X \otimes P_{\alpha} ) \otimes P_{\alpha}^\vee \rightarrow \omega_X$$

is surjective at $x_n$, we must have that $h^0(X, \omega_X \otimes \mathcal{L}_{n} \otimes P_{\alpha}) < h^0(X, \omega_X \otimes P_{\alpha})$ for all $\alpha \in U_{\bar{x}_n}$. This implies that the evaluation morphism $H^0(X, \omega_X \otimes P_{\alpha} ) \otimes \mathcal{O}_X \rightarrow \omega_X \otimes P_{\alpha}$ is surjective at $x_n$, for all $\alpha \in U_{\bar{x}_n}$. 


Let $U \subseteq \text{Pic}^0 X$ be a non-empty open subset, and denote $V_x := U \cap U_x$. Let $p: X^n = X^{n-1} \times X \to X^{n-1}$ be the first projection. We consider the following commutative diagram where maps are given by evaluation:\footnote{Note that $H^0(p^* \omega_X^{\leq n-1} \otimes p^* P_{\alpha}^{\leq n-1}) \neq 0$ if and only if $H^0(\pi^*_n \omega_X \otimes \pi^*_n P_{\alpha}) \neq 0$.}

\[
\begin{array}{c}
\bigoplus_{\alpha \in V_x} H^0(p^* \omega_X^{\leq n-1} \otimes p^* P_{\alpha}^{\leq n-1}) \otimes H^0(\pi^*_n \omega_X \otimes \pi^*_n P_{\alpha}) \otimes \pi^*_n P_{\alpha}^\vee \\
\bigoplus_{\alpha \in V_x} H^0(p^* \omega_X^{\leq n-1} \otimes p^* P_{\alpha}^{\leq n-1}) \otimes p^*(P_{\alpha}^\vee)^{\leq n-1} \otimes \pi^*_n \omega_X \otimes C(\tilde{x}) \ar{r} & \omega_X \otimes C(x) \ar{r}
\end{array}
\]

By the inductive hypothesis, the bottom horizontal morphism is surjective. Moreover, since $V_x \subseteq U_{x}$, the left vertical morphism is surjective as well. Therefore, by Nakayama’s lemma,

$$x \notin \text{Coker}(ev_{n, V_x}) = \bigcap_{\alpha \in V_x} \text{Bs} | \omega_X \otimes \pi^*_n P_{\alpha}|.$$  

Since $V_x \subseteq U$, one has $\bigcap_{\alpha \in V_x} \text{Bs} | \omega_X \otimes \pi^*_n P_{\alpha}| \subseteq \bigcap_{\alpha \in V_x} \text{Bs} | \omega_X \otimes \pi^*_n P_{\alpha}|$, and, therefore, $x \notin \text{Coker}(ev_{n, U})$. Since the point $x \in X^n$ was arbitrary, we finally have that $ev_{n, U}$ is surjective. \footnote{Indeed, the projection formula gives $H^0(p^* \omega_X^{\leq n-1} \otimes p^* P_{\alpha}^{\leq n-1}) = H^0(p_* \mathcal{O}_{X^n} \otimes \omega_X^{\leq n-1} \otimes P_{\alpha}^{\leq n-1}) = H^0(\omega_X^{\leq n-1} \otimes \pi^*_n P_{\alpha})$.}

This concludes the proof of Theorem 1.9. \footnote{Indeed, the projection formula gives $H^0(p^* \omega_X^{\leq n-1} \otimes p^* P_{\alpha}^{\leq n-1}) = H^0(p_* \mathcal{O}_{X^n} \otimes \omega_X^{\leq n-1} \otimes P_{\alpha}^{\leq n-1}) = H^0(\omega_X^{\leq n-1} \otimes \pi^*_n P_{\alpha})$.}

**Remark 4.4.** As observed, the case of surfaces treated in §4.3 is the natural generalization of the one of curves of §4.2. However, comparing Proposition 4.1 and Theorem 1.9, it appears a visible difference on the index $n$ in the two cases. The reason behind that has already been properly explained in [2, Remark 5.2].

In fact, by [42, Corollary 5.11], there exists a non-trivial semi-orthogonal decomposition of $D^b(C(n))$, where $C$ is a curve of genus $g \geq 2$, if $n \geq g$ (see also [41 Theorem D] and [23 Corollary 3.10]). Note that this is what is expected by Conjecture 1.1, because $C(n)$ is not minimal in the range $n \geq g \geq 2$.

5. **Indecomposability in families: proof of Theorem 1.10**

In this section we consider indecomposability of derived categories in certain families of varieties. Namely, taking notations as in the Introduction, we assume that $f: \mathcal{X} \to T$ is a smooth projective family of varieties, parametrized by a complex algebraic variety $T$, such that the relative Albanese morphism for $f$, denoted by

$$\varphi: \mathcal{X} \to \text{Alb} \mathcal{X}/T,$$

is a smooth morphism over $T$.

What we need to prove is the following
Proposition 5.1. Under the above assumption, the function
\[ u : T \to \mathbb{N} \cup \{-\infty\}, \quad t \mapsto u(t) := \dim \text{PBs} |\omega_X| \]
is upper semi-continuous.

Proof. The Proposition can be proved quite similarly to [2, Proposition 2.5]. We give a sketch of it. Let us consider the exact sequence
\[ \phi^* \varphi_* \omega_f \otimes \omega_f^\vee \to \mathcal{O}_X \to \mathcal{O}_{\text{PB}} \to 0, \]  
(5.1)
where \( \omega_f := \det \Omega_{X/T} \) is the relative canonical line bundle of the smooth morphism \( f : X \to T \), and \( \mathcal{PB} \subseteq X \) is the closed subscheme defined by the ideal sheaf \( \text{Im} [\phi^* \varphi_* \omega_f \otimes \omega_f^\vee \to \mathcal{O}_X] \). Given any \( t \in T \), we can restrict (5.1) to the fiber \( X_t \) of \( f \) over \( t \), obtaining the exact sequence
\[ a^*_X a_{X_t} \omega_X \otimes \omega_{X_t}^\vee \to \mathcal{O}_{X_t} \to \mathcal{O}_{\text{PB}_t} \to 0. \]  
(5.2)
Indeed, from the cartesian diagram
\[ \xymatrix{ X_t \ar[r]^\varphi \ar[d]^{\phi} & X \ar[d]^\varphi \ar[r] & (\text{Alb} X/T) \ar[d]^\iota \ar[r]^t & \text{Alb} X/T \ar[d]^\varphi \ar[r] & \mathcal{PB}_t \ar[r] & 0 } \]
we get that
\[ (\varphi^* \varphi_* \omega_f)|_{X_t} = \varphi_t^* \varphi_* \omega_f = \varphi_t^* \varphi_{\omega_{X_t}} = a^*_X a_{X_t} \omega_{X_t}, \]
where the first isomorphism follows by the commutativity of the diagram, and the last one by the definition of the relative Albanese morphism for \( f \). The central isomorphism comes from the base change of the smooth morphism \( \varphi \) (see [7, Lemma 1.3], or [10, p. 276]). More precisely, from the formula in op.cit. we have
\[ L^t \omega_{\varphi_* \omega_f} = R^t \omega_{t^* \varphi_* \omega_f}, \]
where \( L \) and \( R \) denote the left and right derived functors, respectively. By taking cohomology in degree 0, we get \( L^0 \omega_{t^* \varphi_* \omega_f} = \varphi_{t_* \omega_{X_t}} \). So we need to prove that
\[ L^0 \omega_{t^* \varphi_* \omega_f} = t^* \varphi_* \omega_f. \]  
(5.3)
Note that, by definition and the projection formula, \( R^q \varphi_* \omega_f = R^q (\omega_f \otimes \varphi^* \omega_\eta) = R^q (\varphi_* \omega_f \otimes \omega_\eta) \) (notations as in the diagram (1.1)). Since \( \varphi \) is a smooth morphism, the higher direct images \( R^q \varphi_* \omega_f \) are locally free sheaves for all \( q \) by Hodge theory (see, e.g., [12, Theorem 10.21]). Hence, \( R^q \varphi_* \omega_f \) are locally free too, and \( L^p \omega_{t^* (R^q \varphi_* \omega_f)} = 0 \) for \( p > 0 \). So the spectral sequence (see [20, (3.10), p. 81])
\[ E_2^{p,q} = L^p \omega_{t^* (R^q \varphi_* \omega_f)} \Rightarrow L^{p+q} (R^q \varphi_* \omega_f) \]
degenerates, and we get (5.3).

Therefore, from (5.2) and Theorem 1.5, we have that \( \mathcal{PB}_t = \text{PBs} |\omega_X| \) for any \( t \in T \), and, since the fiber dimension of the proper morphism \( \mathcal{PB} \to T \) is upper semi-continuous, this concludes the proof.

Proof of Theorem 1.10. At this point, the proof of Theorem 1.10 is exactly the same of [2, Theorem B], where one uses Proposition 5.1 instead of [2, Proposition 2.5] and Theorem 1.4 instead of [2, Theorem 1.1].
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