Seeing into a nearly black star

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A nearly black star of semi-transparent, spherical, massive shell containing a few point-like light sources inside would not be perceived like a three dimensional ball for a localized observer outside the shell in terms of the affine or parallax distance. As the radius of the spherical shell approaches the Schwarzschild radius, the perceived distance between the front and rear surfaces of the shell would go to zero while the images of most of the interior emitters would squeeze around the shell surfaces in terms of the affine or parallax distance. So the Schwarzschild black hole formed from the star would be thought of as a membrane for the observers who can only measure the parallax distance and/or affine distance. However, the depth information of a point source inside the nearly black star can still be resolved in terms of the radar or luminosity distance, which needs the knowledge about the radar signal or the standard candles input earlier by the observer outside the star.

I. INTRODUCTION

A field can be considered as a collection of oscillators, each oscillator is labeled with a space point, and each is interacting with the oscillators labeled as its spatial neighbors (and perhaps with itself, too) [1, 2]. Since the entropy in statistical physics is proportional to the dimension of the phase space of the total system, or the degrees of freedom of the system, it is natural to think that the entropy of a field in a region is proportional to the number of the oscillators, namely, the number of the space points, or the spatial volume of the region on which the field is defined. It is therefore intriguing that the entropy of a field on the background of a black hole geometry is found to be proportional to the area rather than the volume of the black hole [3, 4].

Much effort has been made in various perspectives to understand how the black hole entropy should be proportional to its area rather than its volume. The major direction is to count the degeneracy of microscopic states to each single macroscopic state of a stationary black hole [5]. The leading-order results of such counting in string theory [6] and loop quantum gravity [7] can match the area law of black hole entropy in many cases. Nevertheless, the calculations in this direction are usually started with an eternal black hole whose horizon has been existing. Having those interesting results, one may still ask: Is there a continuous transition from the entropy of a collapsing star, which is supposed to be proportional to the volume, to the entropy of the black hole formed from the star, proportional to its area?

A sound answer comes from the study on the systems of negative specific heat due to the presence of long-range attractive forces [8]. When the long-range interaction is gradually turned on, the entropy of a system can make a smooth transition from a volume-scaling to an area-scaling quantity [9]. The long-range gravitational interaction of the field/matter in a nearly black star may have been strong enough to constrain the individual behaviors of the elements and reduce the effective degrees of freedom of the whole system.

In this paper we would suggest an alternative thought, following the line in the opening paragraph. If every localized observer outside a star perceives the collection of space points of a star like a membrane rather than a ball, then the entropy of the field in the star should be proportional to the area of the membrane rather than the volume of a ball for the outside world. To show that such situation is possible, we are studying how the position information of the point-like emitters distributed inside a spherical massive shell, which is about to form a black hole, would be perceived by an observer localized outside the photon sphere of the star. We will see that, as the shell radius approaches the Schwarzschild radius, in terms of the affine and parallax distances the interior of the shell would be perceived like a 2D membrane rather than a 3D ball on which local measurements on the field can be positioned. However, the interior of the shell still looks like a 3D ball in terms of the radar or the luminosity distances whenever the nearly black star is not truly black. The key difference between these two kinds of distance measures suggests that, by including or ignoring the full knowledge about the signals, the field entropy inside the star can be switched between a volume-scaling and an area-scaling quantity for localized observers outside.

This paper is organized as follows. We review the null geodesic equations describing the light rays in a spherically symmetric spacetime in Section II. The light rays here would not only refer to some eikonal limit of realistic electromagnetic waves, which may be scattered by the interior material of a star, but also the ideal light rays in relativistic
physics to specify the causal structure [10], or any messenger fields or particles weakly interacting with matter. Then we focus our attention on a simple geometry produced by a semi-transparent spherical massive shell in Section III where we determine the affine distance from a point source of light inside or outside the spherical shell to an observer localized outside the star. In Section IV we introduce the parallax distances perceived by the observer with baselines in two orthogonal directions, as well as the trinocular distance to compare with the affine distances of the interior emitters. In Section V we further examine if other measures of distance such as the radar distance and the luminosity distance would give similar results. Then we summarize and discuss our results in Section VI.

II. NULL GEODESICS IN A SPHERICALLY SYMMETRIC (3+1)D SPACETIME

Suppose a spherical star collapses radially in a very slow rate, so slow that in the period of our interest the spacetime geometry can be approximately described by the static, spherically symmetric metric,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

(1)

A light ray in this background geometry satisfies the geodesic equations

$$\ddot{r} + \frac{A'}{A} \dot{r}^2 = 0,$$

(2)

$$\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \ddot{\varphi}^2 = 0,$$

(3)

$$\ddot{\varphi} + \frac{2}{r} \dot{r} \dot{\varphi} + 2 \cot \theta \dot{\varphi} \dot{\theta} = 0,$$

(4)

and the null condition

$$-A\dot{r}^2 + B\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 = 0,$$

(5)

where the dots and primes denote the derivatives with respect to some affine parameter \(\lambda\) and the \(r\)-coordinate, respectively. Eq. (5) implies \(\partial_\lambda (r^2 \sin^2 \theta \dot{\varphi}) = 0\), and so along the light ray one has \(r^2 \sin^2 \theta \dot{\varphi} = K\), which is a constant. If \(K\) is not zero, we can insert \(\dot{\varphi} = K/(r^2 \sin^2 \theta)\) into Eq. (4) and then multiply the equation by \(\dot{\theta}\) to obtain \(\partial_\lambda [r^4 \dot{\theta}^2 + (K^2/\sin^2 \theta)] = 0\), or

$$r^4 \ddot{\theta}^2 = C - \frac{K^2}{\sin^2 \theta},$$

(6)

where \(C\) is another constant.

Suppose an observer is localized around some point outside the star, and we choose the \(z\)-axis joining the center of the star (the origin in bookkeeper coordinates [11] given in 4) and the localized observer. Then in bookkeeper coordinates the observer would be localized around a point of \(\sin \theta = 0\) while \(r \neq 0\). Since we are looking into how the star light would be perceived by the observer, we are only interested in the light rays passing through the localized observer. However, for any finite \(C\) and nonzero \(K\) the right hand side of (7) diverges to negative infinity as \(\sin \theta \to 0\) while the left hand side is positive definite. Thus \(K\) has to be zero for the light rays seen by the observer, and these light rays must have \(\dot{\varphi} = 0\) off the \(z\)-axis from [5]. In other words, each light ray from a point-like emitter to the localized observer on the \(z\)-axis will be lying on a constant-\(\varphi\) hypersurface by symmetry.

Allowing that \(\theta\) can be negative while requiring \(\dot{\varphi} = 0\) in Eq. (4), we find \(\partial_\lambda (r^2 \dot{\theta}) = 0\) and so

$$r^2 \ddot{\theta} = b,$$

(8)

with a constant \(b\) for the light rays. Eliminating \(\dot{t}^2\) by (6) and then introducing (8), Eq. (3) becomes

$$\partial_\lambda \partial_\lambda r = -V'(r),$$

(9)

where \(\partial_\lambda \equiv \sqrt{AB}\partial_\lambda\) can be thought of as the effective time-derivative operator and

$$V(r) = \frac{b^2 A(r)}{2r^2}.$$

(10)
as the effective potential for radial motion of a particle of unit mass. Multiplying both sides of (9) by $\partial_A r$, one finds 
\[ \partial_A \left( \frac{1}{2} (\partial_A r)^2 + V \right) = 0, \]
and thus
\[ \frac{1}{2} \left( \sqrt{A(r)} B(r) \dot{r} \right)^2 + V(r) = E \]
with a constant of integration $E$, interpreted as the effective total energy of that particle.

Denote the radius of the star by $r_s$ and the position of the localized observer $O$ by $(r_o, \theta_o)$ with $r_o > r_s$ and $\theta_o = 0$ on the $r\theta$-plane of constant $\varphi$. To match the affine parameter $\lambda$ to the local radar distance (cf. Section V A) around the observer determined by the observer’s proper time $d\tau = A(r_o)dt^2$, we impose the normalization condition
\[ r^2 \bigg|_{r=r_o} = \frac{1}{B(r_o)} \left( 1 - r_o^2 \theta^2(r_o) \right) \]
from (6) for $\lambda$. In this normalization (11) is simply
\[ E = \frac{A(r_o)}{2} = \frac{A_o}{2}, \]
after (8) is inserted. Given the metric components $A(r)$ and $B(r)$, and the position of the localized observer $r_o$, the only free parameter for the null geodesics of our interest is $b$. Later in (19) we will see that $b$ determines the angle of arrival of the light ray for a localized observer at finite distance, and $b$ will be identified as the impact parameter for a localized observer at $r \to \infty$.

Numerically, we obtain the light rays reaching the observer $O$ on the $r\theta$-plane by first solving $r(\lambda)$ from (11) with $13)$ and the initial conditions (12) and $r = r_o$ at $\lambda = 0$, then integrating (8) to get $\theta(\lambda) = b \int_0^\lambda r^{-2}(\bar{\lambda})d\bar{\lambda} + \theta_o$ with $\theta_o = 0$ here.

### III. EMITTERS IN A SPHERICAL MASSIVE SHELL

Consider a star of a spherical thin shell of radius $r_s$, total mass $M$, and centered at $C$ chosen as the origin of bookkeeper coordinates (1), containing a few point-like light sources of negligible masses inside the spherical shell. The mass of the star is concentrated on the shell of negligible thickness with a uniform surface density and the shell is semi-transparent so that light rays can go through the shell and a localized observer at a fixed radius outside the star can see the front and the rear surfaces of the shell, like viewing a dusty hollow glass sphere, as well as those light emitters inside. Suppose the interior of the star is otherwise empty and the bending of the light rays are purely due to the metric, which has $12)$ $13)$,
\[ A(r) = 1/B(r) = 1 - \frac{2M}{r} \quad \text{for} \quad r > r_s, \]
and
\[ A(r) = A_s = 1 - \frac{2M}{r_s}, \quad B(r) = 1 \quad \text{for} \quad r \leq r_s. \]

Note that $A_s \to 0$ as $r_s \to 2M$. Then (11) can be written as $E = T + V$, where
\[ T = \begin{cases} \dot{r}^2/2 & \text{for } r > r_s, \\ A_s \dot{\theta}^2/2 & \text{for } r \leq r_s, \end{cases} \quad V = \begin{cases} \frac{\dot{\theta}^2}{2A_s} (1 - \frac{2M}{r}) & \text{for } r > r_s, \\ \frac{\dot{\theta}^2}{2A_s} A_s & \text{for } r \leq r_s. \end{cases} \]

Note that light rays of different $b$ have different effective potentials $V$.

When $r_s \geq 3M$, $V(r)$ is monotonically decreasing as $r$ increases. If $r_s < 3M$, as shown in Figure 1, the Schwarzschild metric outside the shell forms a local maximum of $V$ at $r = 3M$, where the photon sphere is located. The light rays with $E < V(3M)$ started from the inside of the spherical shell cannot cross the barrier of the effective potential and will be trapped in the photon sphere. Similarly, a null geodesic started at an observer localized outside the photon sphere cannot reach the surface of the spherical shell unless $E > V(3M) = b^2/(54M^2)$. Once the null geodesic can reach and enter the shell from the outside, the closest radius $r$ to the shell center it can possibly reach is
\[ r_{\min} = \sqrt{\frac{b^2 A_s}{2E}} = |b| \sqrt{\frac{A_s}{A_o}}, \]
where $\dot{r} = 0$ and so the effective kinetic energy $T = 0$. One can see that $r_{\min} \to 0$ as $r_s \to 2M$.

To our purpose, below we only consider the simplest model where the whole star has sunk into its photon sphere ($2M < r_s < 3M$) and the observer is localized outside the photon sphere ($r_o > 3M$).
FIG. 1: The effective potential given in (10), with $b = 3$ (solid curve), $b = b_c = 4.5$ (dashed), and $b = 5.5$ (dotted). Here the total mass of the spherical shell is $M = 1$, the shell and the observer are situated at $r = r_s = 2.25M$ and $r_o = 8M$, respectively, and $b_c = 3\sqrt{3}M$ defined above Eq. (23) corresponds to the angle of arrival of the rays forming the boundary of the star’s image. The horizontal line represents the effective total energy $E = A_o/2 = 3/8$. The least values of the bookkeeper $r$-coordinate that the rays with $b = 3$ ($< b_c$) and $5.5$ ($> b_c$) can reach are $r_{\text{min}} = r_3 \approx 1.1547M < 2M$ and $r_{5.5} \approx 4.8783M > 3M$, respectively.

A. Angle of arrival and affine distance

The angle of arrival for the light ray reaching the localized observer at $r = r_o$ is

$$\theta_o = \tan^{-1} \frac{r_o\dot{\theta}(r_o)}{\dot{r}(r_o)} = \tan^{-1} \frac{b/r_o}{\sqrt{A_o [1 - (b/r_o)^2]}} \tag{18}$$

in the $r\theta$-plane in bookkeeper coordinates, assuming that the observer is always facing to the shell center $C$ at the origin. The perceived angle of arrival would then be

$$\hat{\theta}_o = \tan^{-1} \frac{r_o\dot{\theta}(r_o)}{\dot{r}(r_o)} = \tan^{-1} \frac{b/r_o}{\sqrt{1 - (b/r_o)^2}} = \sin^{-1} \frac{b}{r_o} \tag{19}$$

in terms of the radar coordinates, $ds^2 = dr^2/A_o + r_o^2 d\theta^2 \equiv dt^2 + r_o^2 d\theta^2$ for $dt = d\phi = 0$, around the observer at $r = r_o$. The above $\theta_o$, $\hat{\theta}_o$, and $b$ are also allowed to have negative values. Here one should choose $b^2 \leq r_o^2$ to make $\dot{r}(r_o)$ real, and $\theta_o, \hat{\theta}_o \to \pm \pi/2$ as $b \to \pm r_o$.

From [11], the affine distance $d_A$ of a null geodesic connecting an event or emitter $e$ at $(r_e, \theta_e)$ and the observer $O$ at $(r_o, \theta_o)$ is defined as

$$d_A = \left| \int_e^{r_o} d\lambda \right| = \left| \int_e^{r_o} \sqrt{2(E - V(r))} \, dr \right|. \tag{20}$$

The above integrand will be real if the connecting null geodesic exists classically. For the events and emitters outside the shell with the whole null geodesics being in the Schwarzschild geometry, one has

$$d_A = \left| \int_{r_e}^{r_o} \frac{r^2 dr}{\sqrt{A_o r^4 - b^2 r^2 + 2Mb^2 r}} \right|. \tag{21}$$

and for the events/emitters inside the shell,

$$d_A = \int_{r_e}^{r_o} \sqrt{\frac{A_s}{A_o - A_b b^2/r^2}} \, dr + \int_{r_e}^{r_o} \frac{r^2 dr}{\sqrt{A_o r^4 - b^2 r^2 + 2Mb^2 r}}$$

$$= \sqrt{\frac{A_s}{A_o}} \left( \sqrt{r_e^2 - r_{e,\text{min}}^2} + \sqrt{r_o^2 - r_{\text{min}}^2} \right) + \int_{r_e}^{r_o} \frac{r^2 dr}{\sqrt{A_o r^4 - b^2 r^2 + 2Mb^2 r}} \tag{22}$$

with “$-$” for $|eF| \leq |eR|$ and “$+$” for $|eF| > |eR|$ in Figures [3] and [4] where $F$ and $R$ are the points on the front and rear surfaces for the observer with respect to the affine distance.

Around $r = r_s$, since the effective potential $V$ in [16] is continuous and $E$ is a constant, the effective kinetic energy $T$ has to be continuous, too. This implies that $\dot{r}|_{r_s+\epsilon} = \sqrt{A_s}r|_{r_s+\epsilon}$, $\epsilon \to 0+$, and so for the light ray traveling for
FIG. 2: (Left) The light rays (red, black, and gray curves) around a spherical shell (orange) and received by the localized observer $O$ are represented in bookkeeper coordinates $r$ and $\theta$, plotted as polar coordinates on a 2D plane (of some constant $\varphi$). Here $M = 1$, $r_o = 8M$, and $r_s = 2.01M$. The photon sphere at $r = 3M$ and the Schwarzschild radius $r = 2M$ are represented in blue-dotted and red-dotted curves, respectively. The red ray has $\theta_o = \theta_s$, which is the minimum angle of arrival within which the localized observer can collect the signals from all the point sources inside the shell. (Right) The “perceived” spherical shell in the observer’s local frame in terms of the affine distance $d_A$ and the angle of arrival $\theta_o$ for the shell radii $r_s = 4M$ (gray, short-dashed curve), $r_s = 3.1M$ (gray, long-dashed), and $r_s = 2.01M$ (black) in a constant-$\varphi$ plane. Here $(\tilde{x}, \tilde{z}) \equiv (d_A \cos \theta_o, d_A \sin \theta_o)$. The black curves actually extend to infinity with the asymptotes represented in green. For comparison, the spherical shells of different radii are represented in orange short-dashed, long-dashed, and solid curves, together with the photon sphere (blue-dotted) and the Schwarzschild radius (red-dotted) in the $r\theta$-plane of bookkeeper coordinates, with the $z$-coordinate scaled by $1/\sqrt{A_o}$. One can see that the affine distance between the front and rear surfaces of the spherical shell decreases as $r_s \to 2M$.

Moreover, the same $\Delta r$ in the bookkeeper $r$-coordinate, the affine distance $\Delta \lambda$ just inside the shell is shrunk from the one just outside the shell by a factor $\sqrt{A_o} < 1$. As $r_s$ approaches the Schwarzschild radius $2M$ and so $\sqrt{A_o} \to 0$, the affine distance from the front to the rear surfaces of the shell for the observer, $2\sqrt{(A_o/A_s)(r_s^2 - r_o^2)}$, goes to zero at every angle of arrival, and the emitters inside the shell along the same light ray with the affine distance in between becomes more and more difficult to be resolved by the observer. In other words, our nearly black spherical star “perceived” by the localized observer is not a ball in terms of $(d_A, \theta_o)$. Rather, it would look like a pancake or a contact lens, and will get into a membrane as its radius $r$ goes down to $2M$, as shown in Figure 2 (right). The information of the depth of the interior emitters from the shell surface in terms of the affine distance will be lost right before the birth of the black hole.

Also in Figure 2 (right), one can see that as $r_s$ is going down to the vicinity of $3M$, which is the radius of the photon sphere, the edge of the 3D map of the shell in terms of the affine distance and the perceived angle of arrival starts to stretch backward along an asymptotic cone for the observer. When $2M < r_s < 3M$, the affine distance of the edge of the 3D “image” goes to infinity, and the angle of arrival of the boundary of the shell’s image $\tilde{\theta}_a$ for the localized observer at $r_o$ corresponds to the rays coming exactly from the photon sphere $r = 3M$ with $E = V(3M)$, which requires $b = b_c$ with $E = A_o/2 = b_c^2/(2r_s^2)[1 - (2M/r)]|_{r=3M}$, i.e. $b_c = 3M/\sqrt{6E} = 3\sqrt{3A_o}/M$. Thus

$$\tilde{\theta}_a^c = \sin^{-1} \frac{b_c}{r_o} = \sin^{-1} \frac{3\sqrt{3A_o}}{r_o} M.$$

For the observer, the received rays with $|b| > b_c$ or $|\tilde{\theta}_a| > \tilde{\theta}_a^c$ must have never been inside the spherical shell.

B. Polar angles and impact parameter

Combining (8) and (11), the polar-angle difference of the event/emitter $e$ and the observer $O$ connected by a null geodesic is

$$\theta_e - \theta_o = \int_{\theta_o}^{\theta_e} d\theta = \int_{\varphi}^{e} \frac{b}{r^2} (-d\lambda) = \int_{r_e}^{r_o} b \frac{A(r)B(r)}{r^2 \sqrt{2(E - V(r))}} dr$$

(24)
in bookkeeper coordinates. For \( r < r_s \) the geometry is similar to Minkowski space, and so the section of a light ray inside the shell is a straight line. For the cases with both \( r \) and \( O \) outside or on the shell, after (16) is inserted,

\[
\theta_e - \theta_o = \int_{r_o}^{r_s} \frac{b \, dr}{\sqrt{\Lambda_0 r^3 - b^2 r^2 + 2 M b^2 r}}
\]  

(25)
can be calculated numerically.

For the observer very far from the shell, \( A_o \rightarrow 1 \) as \( r_o \rightarrow \infty \), and \( \theta_o = 0 \) for the observer situated on the \( z \)-axis. Then \( b \) can be interpreted as the impact parameter for the incident photons from \( r \rightarrow \infty \), since for \( r > b \), (25) implies \( \theta(r) = 0 \approx \int_{r^o}^{\infty} \frac{b \, dr}{r^2} = b/r \) asymptotically, such that the constant \( b \approx r \theta(r) \approx r \sin \theta = \rho \), which is the distance from the point \((r, \theta, \varphi)\) in the light ray to the \( z \)-axis joining the shell center \( C \) and the localized observer \( O \) in bookkeeper coordinates.

By letting \( \theta_e = \theta_s \equiv \theta(r_s) \) (put \( e \) at \( F \) in Figure [3]), one can use the integral in (25) to fit the value of \( b \) numerically for the light ray from that \( \theta_s \) to the observer at \((r_o, \theta_o)\) on the same \( r\theta \)-plane. For example, as \( r_s \rightarrow 2M \) and the shell is becoming a Schwarzschild black hole, the observer at \( \theta_o = 0 \) and \( r_o \rightarrow \infty \) would receive the light rays of \( b = b_1 \approx 2.8477M \) for \( \theta_s = \pi/2 \), and \( b \approx 4.4573M \) for \( \theta_s = \pi \). For the observers at finite \( r_o \), the values of \( b \) for the same \( \theta_s \) would be larger than those for \( r_o \rightarrow \infty \).

C. Direct images

In Figure 2 (left) one can see that the light rays going through the shell roughly rotate about the shell center \( C \) and scan the whole interior of the shell as a whole along \( b \) and so \( \theta_e \) in (18) increases from zero. The red curve with the section inside the shell perpendicular to the \( z \)-axis in Figure 2 (left) represents the light ray with \( \theta_n = \theta_1 \), which is the minimum angle of arrival within which the localized observer at \( r_o > 3M \) could collect the signals from all the possible point-sources inside the shell. Indeed, a rotation of the \( r\theta \)-plane about the \( z \)-axis from \( \varphi = 0 \) to \( 2\pi \) will make the shaded region in Figure 2 (left) passing through every interior point of the spherical shell at least once. In the angular interval \( |\theta_n| \leq \theta_1 \) for the observer, most of the point-like emitters inside the shell have only single point-like images. The only exceptions are those emitters located in the small region \(-r_{\min}(\theta_1) < z < 0\) around the \( z \)-axis (between the vertical red curve and the shell center \( C \) in Figure 2 (left)), where the point-like emitters right on the \( z \)-axis would produce Einstein rings, and the ones slightly off the \( z \)-axis may produce double images asymmetric to the center of the image of the whole star. All of them are direct images. Around the center of the spherical shell \( C \) there is a core region of radius \( r < r_{\min}(\theta_1) \) in which the null geodesics of higher \( b \) from the observer \( O \) will never reach, and so the emitters in that core region can only be seen in \( |\theta_n| \leq \theta_1 \) for the observer.

The polar angle \( \theta_n = \theta(r_s) \) where the red ray is passing through the upper shell surface in Figure 2 (left) is greater than \( \pi/2 \) because \( r_{\min} \) is still significant there. As \( r_s \rightarrow 2M \) and so \( r_{\min} \rightarrow 0 \), the \( \theta_s \) corresponding to \( \theta_1 \) will go to \( \pi/2 \) and the core region mentioned above will shrink to the point \( C \). For the cases with \( 2M < r_s < 3M \) and \( r_o > 3M \), as one keeps increasing \( \theta_n \) from \( \theta_1 \) towards \( \theta_n^* \) (corresponding to \( \theta_n^* \)), each point-like emitter inside the shell but outside the core region starts to be seen repeatedly, such as first, and second, and perhaps infinitely many higher-order indirect images in the picture of the whole star, if the whole setup is stable for an infinitely long period and the observer has an infinite angular resolution.

The direct images of two emitters inside the spherical shell may overlap (along the same ray of some value of \( b \)) while their higher-order indirect images split (along two rays of different \( b \)), and vice versa, since \( r_{\min} \) varies for different \( b \) in (17). From this the observer may be able to extract the information of the relative positions of different interior emitters. As \( r_s \rightarrow 2M \), however, image-overlapping of two emitters will become uniform in all orders. Thus the observer could not resolve the depth information of the emitters in this way from the picture of the whole star taken right before the star becomes a black hole.

IV. PARALLAX DISTANCES

In previous section we have learned that during the gravitational collapse a 3D spherical star would be perceived more and more like a 2D membrane by a localized observer in terms of the affine distance. One may argue that the affine distance is a mathematical construction, which is not measurable directly by physical means. Below we examine whether the observation would be similar in terms of physical measurables such as the parallax distance.

To determine the parallax distance our localized observer should set a baseline of non-zero length. We assume the baseline is infinitesimal to suppress the ambiguity. In a spherically symmetric spacetime, the parallax distance to a point source of light may be well determined using a baseline either parallel or perpendicular to the polar direction.
with respect to the $z$-axis joining the origin at $C$ and the localized observer $O$ or the emitter $e$, though the values of the parallax distances determined by these two orthogonal baselines are different in general for the localized observer. Of these two directions, however, the parallax distance could not be determined straightforwardly, and the observer may need a more sophisticated way to obtain a reasonable measure of distance, e.g. the trinocular distance [14].

A. Baseline in polar direction

Let us choose a new $z$-axis joining the origin at the shell center $C$ and the point-like emitter $e$ in the spatial part of bookkeeper coordinates in (1), then specify the location of the observer as $O(r_o, \theta_o, \varphi_o)$ (Figure 3). As we argued in Section II, each light ray connecting the emitter and the observer is always in a $r\theta$-plane of constant $\varphi$ by symmetry, and here the constant is $\varphi_o$. Suppose the baseline of the localized observer around $O$ is going in the $\theta$-direction about the $z$-axis in this $r\theta$-plane. A light ray of the impact parameter $b$ emitted by a point source $e$ and received by the observer $O$ at the angle of arrival $\theta_a$ has the tangent line $r(s) = [r_o \cos \theta_o + s \cos(\theta_o - \theta_a + \pi), r_o \sin \theta_o + s \sin(\theta_o - \theta_a + \pi)]$ at $O$ $(s \in \mathbb{R}$ and $r(0) = (r_o, \theta_o))$, in terms of the rectangular coordinates $[z, x] = [r \cos \theta, r \sin \theta]$ on the $r\theta$-plane of $\varphi = \varphi_o$. Suppose another light ray from the same emitter $e$ but with a slightly different value of the impact parameter $b_e$, where $\varepsilon$ is a small parameter to be chosen later and $b_e \rightarrow b$ as $\varepsilon \rightarrow 0$, is lying on the same $r\theta$-plane of $\varphi = \varphi_o$ and reaches a slightly different point $(r_o, \theta_e)$ on the baseline from $O$. The tangent line of the second light ray at $(r_o, \theta_e)$ is $r_e(s_e) = [r_o \cos \theta_e^o + s_e \cos(\theta_e^o - \theta_o^o + \pi), r_o \sin \theta_e^o + s_e \sin(\theta_e^o - \theta_o^o + \pi)]$, $s_e \in \mathbb{R}$. These two tangent lines intersect at

$$s = \frac{r_o \left[ (\theta_e^o - \theta_o^o) - \sin \left( \theta_e^o - \theta_o^o \right) \right] - \sin \left[ \theta_o^o - \left( \theta_e^o - \theta_o^o \right) \right]}{\sin \left[ \left( \theta_e^o - \theta_o^o \right) - \left( \theta_o^o - \theta_a^o \right) \right]} = \frac{r_o \theta_e^o \cos \theta_a^o}{\theta_e^o - \theta_a^o} + O(\varepsilon),$$

(26)

where $\theta_e^o = \lim_{\varepsilon \to 0} (\theta_e^o - \theta_o^o)/\varepsilon$ and $\theta_a^o = \lim_{\varepsilon \to 0} (\theta_a^o - \theta_o^o)/\varepsilon$. Thus the parallax distance determined by the infinitesimal baseline in the $\theta$-direction for the observer $O$ would be

$$d_\parallel \equiv N(\theta_o)r_o \cos \theta_a^o \frac{\theta_e^o}{\theta_e^o - \theta_a^o},$$

(27)

where a normalization factor $N(\theta_o) \equiv \sqrt{A_o^{-1} \cos^2 \theta_a^o + \sin^2 \theta_a^o}$ is introduced to match the radar distance $\tilde{d}$ around the observer $O$ (cf. Section V). Choose $\varepsilon$ as the angle between the two light rays of $b$ and $b_e$ in the vicinity of the emitter $e$ (Figure 3). Given $\theta_a$ in (15), and notice that $\theta_a^o$ has the same form as $\theta_a$ except $b$ is replaced by $b_e = \sqrt{(A_o/A_s) r_{\min}^o} = b + \varepsilon b' + O(\varepsilon^2)$, one has

$$\theta_e^o = \frac{b' \sqrt{A_o r_o^2}}{\sqrt{r_o^2 - b^2 \left[ A_o(r_o^2 - b^2)^2 + b^4 \right]}}.$$

(28)

where $b'$ may depend on the location of $e$ on the $r\theta$-plane.
When the whole light ray from the source $e$ to the vicinity of the observer $O$ is outside of the spherical shell, given $\theta_o$ in (25) and $\theta_o^\prime$ with $b$ in $\theta_o$ replaced by $b_e$, one has

$$\theta_o^\prime = \int_{r_e}^{r_o} \frac{b'_A r_o^4 dr}{(A_o r_o^4 - b_o^2 r_o^2 + 2 M b_o^2 r_o^2)^{3/2}},$$

since $\theta_e$ is fixed. Substitute (29) and (28) into (27), one finds that the factor $b'$ cancels and so the value of $b'$ is irrelevant in this case.

Consider a light ray started at $R$ and received by the observer $O$ represented as the black path in Figure 3, where $R$ is on the rear surface of the shell with respect to the affine distance for the observer. If we put a point source $e$ at another point than $R$ on the same black path, then one of the light rays emitted by $e$ can go along the black path to reach the observer $O$. Let us put an emitter $e$ at $O$, and then slowly bring $e$ away from $O$ towards $R$ along the black path in Figure 3, while the $z$-axis from $C$ pointing to $e$ varies following the move of $e$. When the emitter $e$ is still outside the shell, as the emitter is brought away from $O$ along the black path of a light ray, the parallax distance $d_\parallel$ is a monotonically increasing function of the affine distance $d_A = \Delta \lambda$ for the observer, as shown in Figure 3 (left). When $d_A$ is small, we find $d_\parallel \approx d_A$. As the path of the light ray from $e$ to $O$ starts to bend on the $r\theta$-plane, the value of $d_\parallel$ deviates from $d_A$ and becomes less than $d_A$. As the emitter $e$ goes towards the spherical shell further, $d_\parallel$ tends to saturate but still increasing until $e$ touches $F$ on the front surface of the shell with respect to $d_A$ for the observer. Indeed, as the emitter $e$ is brought away from the observer $O$ along the black path in Figure 3, $\theta_o^\prime/b'$ is a positive constant from (28), and the integrand of $\theta_o^\prime/b'$ is positive definite from (29), so $(-\theta_o^\prime/b')$ is increasing as $r_o - r_e$ increases from zero. Rewrite (27) into $d_\parallel = N(\theta_a) r_o \cos \theta_a \times (-\theta_o^\prime/b') / [(-\theta_o^\prime/b') + (\theta_o/b')]$, the behavior of $d_\parallel$ described above is obvious.

For the cases with the emitter $e$ inside the spherical shell, as in Figure 4, one can see that $r_{\min}^\prime$ (blue-dotted) : $r_{\min}$(black-dotted) = $\sin \angle CF_e e : \sin \angle CF_e e = \sin \angle CeF_e : \sin \angle CeF_e$, so one has $\sin(\delta \pm (\epsilon - \epsilon)) = \sin \delta \sin(\gamma \pm \epsilon)/\sin \gamma$, where $\delta \equiv \angle CR e = \angle CF e = \sin^{-1}(r_{\min}/r_o)$, $\gamma \equiv \angle CeF_e$, and $"+"$ and "$-"$ work for the cases of $|eF| > |eR|$ (Figure 4 (left)) and $|eF| \leq |eR|$ (Figure 4 (right)), respectively. This implies $\epsilon = \delta [1 + (\tan \delta/\tan \gamma)] + O(\epsilon^2)$ as $\epsilon \to 0$. Then from (24) and Figure 3, one has $b = \sqrt{A_o/A_s r_o \sin \delta}$ and $b_e = \sqrt{A_o/A_s r_o \sin(\delta \pm (\epsilon - \epsilon))}$, and so

$$b' = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (b_e - b) = \sqrt{\frac{A_o}{A_s} r_o \sin \delta} \frac{\tan \delta}{\tan \gamma}.$$  

Further, from (24) and Figure 4 one can see that

$$\theta_o = \theta_s - \int_{r_o}^{r_e} \frac{b dr}{\sqrt{A_o r_o^4 - b^2 r_o^2 + 2 M b_o^2 r_o^2}},$$

$$\theta_o^\prime = \theta_s \pm \epsilon - \int_{r_o}^{r_e} \frac{b_e dr}{\sqrt{A_o r_o^4 - b_e^2 r_o^2 + 2 M b_o^2 r_o^2}}.$$
for the light rays emitted by \(e\) and passing through the points \(F(r_s, \theta_s)\) and \(F_e(r_s, \theta_s \pm \epsilon)\), respectively. Therefore,

\[
\theta'_e = 1 + \frac{\tan \delta}{\tan \gamma} - \frac{\int_{r_s}^{r_o} b'_A e_A r^4 dr}{\frac{b'_A e_A r^4}{(A_o r^4 - b^2 r^2 + 2M b^2 r^3)^{3/2}} - \theta'_o |F_e - \theta'_o |F_e} \tag{33}
\]

and \(b'\) will not be canceled in \(\Box\) when the emitter is inside the spherical shell \((\delta < \gamma < \pi - \delta)\). Now \(d_{\|}\) can be arranged in the form

\[
d_{\|} = N(\theta_a) r_o \cos \theta_a \frac{\kappa(\gamma) - \theta'_o |F_e}{\kappa(\gamma) - [\theta'_o - \theta'_o |F_e]} \tag{34}
\]

with \(\kappa(\gamma) \equiv (\tan \gamma / \tan \delta) \mp 1\). That \(\gamma = \pi - \delta\) at \(F\) implies \(\theta'_o |F < 0, \theta'_o |F > 0, \text{ and } \theta'_o |F < 0\) from \(\Box\), \(\Box\) and \(\Box\). Thus when \(e\) is located at \(F\), \(\kappa = 0\) and both the denominator and the numerator of \(\Box\) are negative. Moving the emitter \(e\) from \(F\) towards \(R\) along the black path of light ray in Figure \(\Box\) or \(\Box\) the factor \(\kappa\) runs from \(0\) down to \(-\infty\) as \(e\) approaches the middle point between \(F\) and \(R\), where \(\gamma = \pi/2\) and \(d_o = N(\theta_o) r_o \cos \theta_o\). Then, as \(e\) goes from the middle point to \(R\), \(\kappa\) runs from \(+\infty\) down to \(2\). If \(\theta'_o |F > 2\), which is true when \(r_o\) is sufficiently large and the shell radius \(r_s\) is close enough to \(2M\) so that \(A_o\) is small enough by \(\Box\) and \(\Box\), on the way that \(\kappa\) drops from positive infinity down, \(\kappa\) first meets the value of \([\theta'_o - \theta'_o |F]\) causing the divergence of \(d_{\|}\) at that point. Then \(d_{\|}\) increases from negative infinity up as \(e\) keeps going, until \(\kappa\) passes through the value of \(\theta'_o |F\) where \(d_{\|} = 0\). After that point \(d_{\|}\) becomes positive and still increasing, all the way to the point that \(e\) arrives at \(R\) where \(\gamma = \delta, \kappa = 2, b'|R > 0, \theta'_o |R > 0\), and remarkably, \(d_{\|}|R \leq d_{\|}|F\).

The angle \(\gamma\) where \(d_{\|}\) diverges may look closer to \(\delta\) (for the emitter \(e\) at \(R\)) than it is to \(\pi/2\) (for \(e\) right at the middle point of \(F\) and \(R\)) in value. However, when considering \(d_{\|}\) as a function of the depth \(\ell \equiv \int_{F}^{R} d\lambda [r^2 + r^2 \delta^2]^{1/2} = \sqrt{A_o/A_s} (\lambda_e - \lambda_F)\) of the emitter \(e\) from \(F\) on the front surface, which is related to the angle \(\gamma\) by

\[
\tan \delta = \frac{\ell}{r_s \cos \delta} - 1 \tag{35}
\]

according to Figure \(\Box\), the singularity of \(d_{\|}(\ell)\) would look much closer to the middle point \((\ell = r_s \cos \delta = \sqrt{r^2 - r^2_{\min}})\) than it is to \(R\) \((\ell = 2r_s \cos \delta)\) when \(r_s\) is sufficiently close to \(2M\), e.g., Figure \(\Box\) (right). In the same plot one can also see that the slope of \(d_{\|}(\ell)\) is approximately zero in almost the whole domain \((0 \leq \ell \leq 2r_s \cos \delta)\) inside the shell except the neighborhood of the singularity, and so it would become more and more difficult to resolve the depth \(\ell\) of the emitter \(e\) from \(d_{\|}\) for the observer as \(r_s \rightarrow 2M\), unless \(e\) happens to be located around the singularity of \(d_{\|}(\ell)\).

An emitter \(e\) inside the shell and located on the straight line joining the shell center \(C\) and the observer \(O\) on the \(r\theta\)-plane has \(b = \theta_a = \theta_o = 0\), and the parallax distance of its direct image can be expressed in closed form,

\[
d_{\|} = \frac{1}{\sqrt{A_o}} \left[ r_o - \frac{r_s (r_s - \ell)}{r_s - (1 - \sqrt{A_o} \ell)} \right] , \tag{36}
\]
where $0 < \ell < 2r_s$. The behavior of $d_\parallel(\ell)$ in this case is similar to the one in Figure 5 (right). As $\ell$ increases from zero, the value of $d_\parallel(\ell)$ in (36) is close to the affine distance $d_A = (r_o - r_s + \sqrt{\lambda_o} \cdot \ell) / \sqrt{\lambda_o}$ for small $\ell$, then it deviates from $d_A$ around $\ell = r_s/(1 - \sqrt{\lambda_o})$ where it diverges.

Around the boundary of the image of the whole star, as $b \rightarrow b_c$ and $\theta_a \rightarrow \theta_a^c$, Eq. (33) and the statement above (23) imply $\theta_{\alpha}'|_{P}$ diverges to positive infinity due to the singularity of the integrand at $r = 3M$, while $\theta_{\alpha}|_{F}$, $\kappa|_{F}$, and $\kappa|_{R}$ are finite, and so the parallax distances $d_\parallel$ in (34) for the front and rear surfaces go to the same value $N(\theta_{\alpha}^o)r_o \cos \theta_{\alpha}^o$. Otherwise for $\theta_a < \theta_a^c$, $d_\parallel|_{R}$ is less than $d_\parallel|_{F}$, so the front surface of the shell with respect to the affine distance would be positioned behind the rear surface in terms of $d_\parallel$ by the observer. The images of the interior emitters between the front surface and roughly the shell center in bookkeeper coordinates would be located behind the perceived front surface, as indicated in Figure 5 (right), while the apparent locations of the interior emitters at even deeper $\ell$ would be somewhere in front of the perceived rear surface, and so the star looks inside out. Most of those emitter images look very close to the shell surfaces, as shown in Figure 7 (lower-right). As the shell radius $r_s$ goes to the Schwarzschild radius $2M$, the images of the shell surfaces and almost all the interior emitters squeeze to a membrane in terms of $d_\parallel$, except the emitters around the shell center $C$.

The zero value of $d_\parallel$ for an emitter around the shell center do not imply that the observer would see the image of the emitter in the vicinity of the observer. In fact, the parallax distance $d_\parallel$ at a fixed angle of arrival $\theta_a = \theta_a^o$ is ill-defined around the zero of $[26]$ or (34) where $\gamma = \gamma_0 \approx \kappa^{-1}(\theta_{\alpha}'|_{F})$ from (34). The observer at $(r_o, \theta_a)$ is a focal point of the light rays around the angle of arrival $\theta_a$ from the interior emitter at $\gamma_0$ or depth $\ell_0 \equiv \ell(\gamma_0)$, and the vanishing value of $d_\parallel(\ell_0)$ is associated with a zero length of baseline, which is beyond our assumption of infinitesimal baseline. Suppose we introduce a small but finite baseline in the polar direction with two telescopes situated at its two ends and the observer $O$ sitting in the middle. Then as the depth $\ell$ of an emitter is approaching $\ell_0$ along the interior section of the null geodesic, one of the telescope would start to miss all the light rays in the neighborhood of the angle of arrival $\theta_a$, and the parallax distance $d_\parallel$ cannot be determined in the conventional binocular way, until $\ell$ gets to be sufficiently away from $\ell_0$ and the missing image re-appears. According to such kind of image missing in one of the telescopes, anyway, the observer could still know that the location of the emitter is around the shell center $\ell \approx 2M$ rather than at other depths when $r_s$ is close to $2M$.

### B. Baseline perpendicular to polar direction

Suppose the baseline of the localized observer is instead going in the direction perpendicular to the plane of the light ray $\varphi = \varphi_o$ with a very small length. Given a light ray connecting the emitter $e$ and the observer $O$ in the plane of $\varphi_o$ such as the black path in Figure 3, a small rotation $\Delta \varphi$ about the $z$-axis joining $e$ and the shell center $C$ will give another light ray starting at the emitter $e$ and arriving the sphere of $r = r_o$ (that the observer is situated) at $\varphi_o + \Delta \varphi$. Then the new light ray can approximately meet one of the two ends of the baseline, with the mismatch in the $\theta$-direction significant only for $|\theta_o| \leq |\pi - \theta_o| < 2\Delta \varphi$ (otherwise the mismatch is mostly $\Delta \theta \sim (\Delta \varphi)^2$, and in particular $\Delta \theta = 0$ for $\theta_o = \pi/2$.) In Figure 3 one can see that the tangent lines of the light rays in the two ends of the baseline, which can be written as $r(s) = [r_o \cos \theta_o + s \cos(\theta_o - \theta_o + \pi), r_o \sin \theta_o + s \sin(\theta_o - \theta_o + \pi)]$ ($s \in \mathbb{R}$) in terms of the rectangular coordinates on the $r\theta$-plane of each light ray ($\varphi = \varphi_o \pm \Delta \varphi$), will intersect at $X$ on the axis of rotation. Therefore $s \sin \angle OXC = r_o \sin \angle O\varphi C$, or

$$
s \sin[\pi - (\theta_e - \theta_o - \pi) - \theta_a)] = r_o \sin(\theta_e - \theta_o - \pi)
$$

(37)

equals the distance between $O$ and the $z$-axis ($\theta_e = 0$ with our choice of the $z$-axis), and the parallax distance determined by the infinitesimal baseline in the $\varphi$-direction for the localized observer $O$ would be

$$
d_\perp = N(\theta_o) \frac{r_o \sin(\theta_e - \theta_o)}{\sin(\theta_e - \theta_o + \pi)}.
$$

(38)

Here $\theta_o$ and $\theta_e - \theta_o$ on the $r\theta$-plane have been given in (18) and (24), respectively.

Consider the same example in Section IV A where we put an emitter $e$ at $O$ and bring $e$ slowly away from $O$ along the black path in Figure 3. One can see in Figure 5 (left) that initially the emitter would be perceived like a virtual image at a slowly growing parallax distance $d_\perp$ (blue curve), whose value is very close to the affine distance $d_A$ from $O$ to $e$. As the emitter $e$ is approaching $P_\infty$, the value of $d_\perp$ starts to deviate from the affine distance $d_A$, and then diverges at $P_\infty$ where the $z$-axis joining the emitter $e$ and the shell center $C$ is parallel to the tangent line of the light ray at $O$ such that $\theta_e - \theta_o + \theta_o = n\pi$, $n \in \mathbb{Z}$, and the incoming light rays from the emitter $e$ are parallel to each other around the two ends of the observer’s baseline. As the emitter $e$ is brought further from $P_\infty$ towards $R$, $d_\perp$ becomes negative and increases from negative infinity, while the focal point of the rays emitted from $e$ at different $\varphi$
is behind the observer and the tangents of the received rays will not intersect in front of the observer as the observer is facing to the center of the shell. When the emitter $e$ is sufficiently close to $P_0$, which is exactly on the straight line joining the shell center $C$ and the observer $O$, our definition (38) of $d_{\perp}$ breaks down. For $e$ at $P_0$, while $\theta_e - \theta_o = n\pi$ such that $d_{\perp} = 0$, the image of the emitter observed exactly at $O$ is an Einstein ring, rather than a point image at zero distance from the observer $O$. Each telescope at the two ends of the non-zero baseline of the observer $O$ would see double images of the emitter $e$ in this case, and the parallax distance for each image measured by the observer would rather be $d_{\parallel}$ introduced in Section IV A since $O$ is now on the $z$-axis and every baseline extended from the $z$-axis is in the polar direction.

As the emitter is moved further away from $P_0$ towards $R$, the distance $d_{\perp}$ grows from zero to some positive value where a real image of the emitter would be perceived by the observer. Before the emitter $e$ arrives at $F$ on the front surface of the shell with respect to the observer, depending on the value of the impact parameter $b$ and thus the angle of arrival $\theta_e$, the light rays outside the shell from $e$ to $O$ may contain zero (small $b$) to many ($b \rightarrow b_c$) intervals of negative $d_{\perp}$ similar to $(P_\infty, P_0)$.

Inside the shell, the black path in Figure 3 cross the straight lines parallel to $OX$ (gray dashed) at $P_\infty'$ and the straight line joining $C$ and $O$ (black) at $P_0$. The behavior of the parallax distance $d_{\perp}$ of the emitter $e$ around the interval $(P_\infty', P_0)$ is similar to those in $(P_\infty, P_0)$ for the observer $O$, as shown in Figure 3 (right). For every null geodesic with $0 < b < b_c$ started at the observer $O$ at finite $r_o$, going through $F$ on the front surface of the shell and arriving at $R$ on the rear surface, the section inside the shell must contain at least one of $P_\infty'$ and $P_0'$ if the shell radius $r_s$ is sufficiently close to the Schwarzschild radius $2M$. In Figure 5, $r_s = 2.05M$ is not close enough to $2M$ to see this. As $r_s$ goes to $2M$ further, the black path with $b$ fixed would still pass through the vicinity of the point $F$, while the straight-line section inside the shell will approximately be rotated about $F$ towards the shell center $C$ as $r_{\min} \rightarrow 0$ (also see Figure 6 (left)). In this limit, as we bring the emitter $e$ along the section of the black line from $F$ to $R$, namely, increasing the depth $\ell$, normally the angle $\theta_e - \theta_o$ will roughly be a constant, say, $\Delta$, until $e$ arrives $P_\infty'$ and/or $P_0'$ around the middle point with $\ell = \sqrt{r_s^2 - r_{\min}^2}$, where $\theta_e - \theta_o$ will suddenly jump from $\Delta$ to about $\Delta - \pi$, and then roughly keep this constant value as $e$ is brought to $R$. Thus in these normal cases (e.g. the blue dashed curve in Figure 6 (middle)), the observer $O$ would not be able to resolve the depth $\ell$ of the emitter $e$ inside the shell from the parallax distance $d_{\perp}$ if $e$ is outside the core region in which $P_\infty'$ and/or $P_0'$ as well as the sudden jump of $\theta_e - \theta_o$ by about $\pi$ occur.

Anomalous behavior happens in the cases with $\theta_e - \theta_o + \theta_o \approx n\pi$, $n \epsilon \mathbb{Z}$, when the smallest change in $\theta_e - \theta_o$ as one varies $\ell$ can be amplified by the denominator in (38), and so the observer appears to be able to resolve $d_{\perp}(\ell)$. In Figure 6 (middle) we show an example where $d_{\perp}$ is a monotonically increasing function of $\ell$, in contrast to the normal behavior mentioned above. As $r_s \rightarrow 2M$, the slope $\partial_\ell d_{\perp}$ increases and the depths of the emitters along these rays appear to be even more easily to be resolved around each window in the parameter range of $b$ where $\theta_e - \theta_o + \theta_o \approx n\pi$.  

FIG. 6: Along the black, gray, or the very-light-gray curve in the left plot, $\theta_e - \theta_o + \theta_o \approx \pi$ for the emitter inside the spherical shell whose $d_{\perp}(\ell)$ (blue curves in the middle plot) for the observer becomes monotonically increasing with its depth $\ell$. This behavior is different from the normal cases (dashed curves and the example in Figure 2). Here $r_s = 2.1M, b = 3.98548$ (very-light-colored curves), $r_s = 2.01M, b = 3.94182$ (light-colored), $r_s = 2.001M, b = 3.895$ (black, blue and red), $r_s = 2.001M, b = -1.37543$ (dashed curves in all plots, normal case), with other parameters the same as those in Figure 2. One can see that as $r_s \rightarrow 2M$ (from very-light blue to blue), the slope of $d_{\perp}(\ell)$ increases and $\ell$ appears to be more resolvable. However, what the observer is measuring locally is the difference of the angles of arrival at the two ends of the baseline, which is proportional to $d_{\perp}^{-1}$. In the right plot one can see that as $r_s \rightarrow 2M$, $d_{\perp}^{-1}$ goes to zero in most of $\ell$ inside the shell, and the resolution of $\ell$ away from the singular point slightly ahead of the middle point is also suppressed in this limit. We put $d_{\parallel}$ and $d_{\perp}^{-1}$ (red curves, see Section IV A) in each cases for comparison.
One may be tempted to conclude that the observer can use these windows and change the location \((\theta_o, \varphi_o)\) to scan the whole interior of the shell, or compare with the data from other observers to get the depth information inside the shell. Nevertheless, this idea would not be practical because a localized observer does not directly measure \(d\perp\). Rather, the observer infers \(d\perp = W/\delta\varphi\) with the length of the baseline \(W\) after directly measuring the difference between the angles of arrival \(\delta\varphi\) of the light rays from the emitter to the two ends of the baseline. As \(r_s \to 2M\), while \(d\perp(\ell)\) would be running from negative infinity to positive infinity as \(\ell\) goes from 0 to \(2\sqrt{r_s^2 - r_{\min}^2(\ell)}\) in this window of \(\theta_o\) or \(b\), the angular difference \(\delta\varphi \propto 1/d\perp\) is virtually zero for almost all \(\ell\) except those of the rays coming from the region giving \(d\perp \sim 0\) where \(\delta\varphi\) breaks down and the parallax distance would be given by \(\delta\varphi\). Thus, as \(r_s\) goes to 2\(M\), the observer outside the photon sphere eventually cannot resolve the depth \(\ell\) of the emitters from \(d\perp\) except those located around the shell center \(C\). In fact, what is happening in a normal case is similar: In the same limit \(r_s \to 2M\), the distance \(d\perp\) is almost constant for most of \(\ell\) except the point around the shell center \(C\), meaning that \(\delta\varphi \propto 1/d\perp\) is not resolvable for most of the values of depth \(\ell\), either.

For the direct images within \(0 < \theta_o \leq \theta_1\), where \(\theta_1\) corresponds to the red ray in Figure 2 (Left), there is no interval of negative \(d\perp\) for the emitter \(e\) outside the shell as it is brought away from \(O\) along the ray, and the parallax distance of the emitter \(e\) at \(F\) on the front surface, \(d\perp(\ell) = \frac{1}{\sin(\theta_s - \theta_o)}\) is positive for the observer. Here \(\theta_s - \theta_o\) is obtained from \(25\) with \(r_e = r_s\). In contrast, there must be one and only one \((P_\infty, P_0)\) interval for the emitter \(e\) inside the shell in this parameter range of \(\theta_o\), and the parallax distance of the emitter \(e\) at \(R\) on the rear surface for the observer, \(d\perp(\ell) = \frac{1}{\sin(\theta_s - \theta_o + \varphi)}\) is also positive but slightly less than \(d\perp(\ell)\) for sufficiently small \(\delta = \angle CFR = \sin^{-1}(r_{\min}/r_s) > 0\) in Figure 2. Thus, similar to the observation in terms of \(d\perp\), the direct image of the “front” surface (with respect to the affine distance \(d_A\)) of the shell for \(0 < \theta_o \leq \theta_1\) will be perceived by the observer like something located behind the image of the “rear” surface in terms of the parallax distance \(d\perp\). The interior emitters situated between \(F\) and \(P_\infty\) will be perceived further behind the “front” surface, and the real image of the emitters between \(P_0\) and \(R\) will be perceived like sitting in front of the “rear” surface of the shell, and so the star looks inside out in \(d\perp\), too. As \(r_s \to 2M\), \(\delta\) or \(\angle CFR\) in Figure 2 goes to zero, so the images of the rear and front surfaces eventually overlap in \(d\perp\) and the images of the interior emitters squeeze to the image of the shell surfaces. The only exceptions are those emitters around the shell center \(C\), whose direct images within \(0 < \theta_o \leq \theta_1\) may have parallax distances \(d\perp\) ranged from \(-\infty\) to \(\infty\) when perceived by the observer, as shown in Figure 7 (lower-left).

### C. baselines in other directions and trinocular distance

For the observer with the baseline not exactly parallel or perpendicular to the polar direction, the tangent lines of the rays arriving at the two ends of the baselines would not intersect in general. In this case the observer could simply rotate the baseline to where \(d||\) and \(d\perp\) can be defined. As we have learned earlier (e.g. Figure 7), the value of \(d\perp\) is generally different from \(d||\) of the same emitter. Since there is no rule to judge which one is better, the observer could further average these two distances to get a trinocular distance such as \(d_3\)

\[
d_3 = \frac{2}{(1/d||) + (1/d\perp)}.
\]

In Figure 5 we also compare the trinocular distance \(d_3\) (green dotted) with the parallax (binocular) distances \(d\perp\) and \(d||\), as well as the affine distance \(d_A\) of the emitter for the observer. As the emitter \(e\) is brought away from the observer \(O\) along the black path in Figure 5, one can see that the trinocular distance \(d_3\) can be very close to the affine distance \(d_A\) before \(d\perp\) approaches its first divergence at \(P_\infty\). After \(P_\infty\) the behavior of the trinocular distance \(d_3\) becomes totally different from \(d_A\). It diverges wherever \(d\perp = -d||\) and vanishes wherever \(d\perp\) or \(d||\) vanish.

In Figure 7 we show how a spherical shell with an array of the emitters inside would be perceived by an observer localized outside the photon sphere in terms of \(d_3\), \(d\perp\), and \(d||\). For small angle of arrival \(\theta_o\), once it is far enough from the direction of the first divergence of \(d\perp\), the properties of the images in terms of \(d_3\) are similar to \(d\perp\) and \(d||\). The spherical shell with interior emitters looks inside out for the observer. In particular, for \(\theta_o < \theta_1\), the direct image of the front surface of the shell in terms of \(d_3\) is very close to the one in the affine distance \(d_A\) (Figure 7 (upper-right).) As \(r_s \to 2M\), all the perceived distances \(d\perp\), \(d||\), and \(d_3\) between the front and rear surfaces of the spherical shell go to zero and the information of the depths of the interior emitters is not resolvable except those around the shell center. A light ray emitted by a point source in this core region can get a sufficiently low impact parameter \(|b| < b_c\) whenever \(r_s > 2M\) (see 17) and notice that \(r_{\min} \to 0\) for this light ray, which allows it to easily go beyond the potential barrier peaked at the photon sphere (proportional to \(b^2\) in 16) to reach the outside world while other light rays not sourced from this region would be mostly bound in the photon sphere when the shell radius \(r_s\) is sufficiently close to the Schwarzschild radius \(2M\). Nevertheless, the size of this core region would go below the scale of any emitter in the same limit \(r_s \to 2M\) so that, without introducing quantum effect, the final signals from this region right before...
the star becomes a black hole may not contain any structure interesting at the usual scale for the localized observer outside the photon sphere, though the range of the parallax distance $d_\perp$, $d_\parallel$, or $d_3$ of this region can be large when perceived by the observer.

V. OTHER MEASURES OF DISTANCE

So far we have seen that a nearly black spherical shell would be perceived like a membrane rather than a ball, with the information of the event/emitter depths inside the shell very hard to be resolved in terms of the affine and parallax distances by a localized observer outside the photon sphere. One may wonder if there is some way to extract the depth information of the interior emitters or event more easily as the nearly black shell is not truly black. It turns out that, at least, the radar distance and the luminosity distance can work.
A. Radar distance

Suppose the observer $O$ send a radar signal into the spherical shell at some moment, assuming the energy input by the radar signal would not turn the nearly black star to a black hole. The wavelength of the radar signal will be blue-shifted as it drops into the shell. Suppose the mass of the point-like object to be observed inside the shell is much greater than the energy of the blue-shifted photons of the radar signal, and the energy loss in the scattering event $e$ of the radar signal and the point-like object is negligible. Then the echoes climbing out the gravitational potential would be approximately at the same wavelength as the original radar signal when received by the observer or some other antenna at the same $r = r_o$ in bookkeeper coordinates.

To obtain the radar distance of an event inside the shell, we start with Eq. (2), which implies $\partial_\lambda (A(r) \dot{t}) = 0$ and so

$$\dot{t} = \frac{a}{A(r)}$$

(40)

with a constant of integration $a$. The value of $a$ cannot be chosen freely because of the constraint (6). Inserting (40), (3), and (11) into (6), one can see that $a = \sqrt{2E} = \sqrt{A_o}$ in our normalization for the affine parameter $\lambda$. Thus, for our localized observer at $r = r_o$, the radar distance of a scattering event $e$ is half of the proper-time difference of the moments of emitting and receiving the radar signals for the localized observer [15–17], namely, by (20) and $\Delta r_o = \sqrt{A_o} \Delta t$,

$$d_R = \frac{\Delta r_o}{2} = \left| \sqrt{A_o} \int_O^e \frac{a}{A(r)} d\lambda \right| = A_o \int_O^e \sqrt{\frac{B(r)}{2A(r)(E-V(r))}} dr,$$

(41)

where $\int_O^e dr = \int_{r_o}^{r_e} dr$ for $|eF| \leq |eR|$ and $\int_O^e dr = (\int_{r_{min}}^{r_e} + \int_{r_{min}}^{r_o}) dr$ for $|eF| > |eR|$ in Figures 3 and 4 with $r_{min}$ given in [17]. The section of the null geodesic of the radar signal inside the shell contributes

$$d_{R}^{p} = A_o \int_{F}^{c} dr \sqrt{\frac{1}{2A_s(E-V(r))}} = A_o \sqrt{A_s} \int_{F}^{c} dr \sqrt{\frac{1}{A_o - (A_o b^2/r^2)}} = \ell \sqrt{\frac{A_o}{A_s}}$$

(42)

with the depth $\ell = \sqrt{A_o/A_s}(\lambda_e - \lambda_F)$ defined earlier in Section [IVB]. The echoes from two events at different $\ell$ along the same null geodesic inside the shell could be easily distinguished by the observer’s clock even if the shell radius is close to the Schwarzschild radius $2M$, since the perceived time interval $\Delta d_{R}^{p}/c = \sqrt{A_o/(cA_s)} \Delta \ell$ will be dilated by a factor $1/\sqrt{A_s}$ as $A_s \rightarrow 0$ and become significant for the observer. Therefore, the depth information of an event inside the nearly-black star is in principle resolvable if the observer describes it in terms of the radar distance $d_R$, together with the angle of arrival $\theta_s$ of the echo of the radar signal.

B. Luminosity distance

If the emitters are some standard candles well known or even sent by the observer into the shell, then the apparent luminosity of the emitters can also reveal the information of their depths inside the shell to the observer.

The luminosity $L$ of an isotropic emitter observed by an infinitesimal antenna of area $dA$ is proportional to the solid angle $d\Omega$ that all the emitted light rays hit the antenna around some angle of arrival have gone through earlier on the unit sphere surrounding the emitter. In (3+1)D Minkowski space, $dA = r_L^2 d\Omega$ where $r_L$ is the distance from the emitter to the localized observer. For an emitter as the standard candle of unit radiated power inside the spherical shell, the apparent luminosity that the radiated power experienced by the antenna would be $(\sqrt{A_s/A_o})^2 L$ due to gravitational redshift [18], thus the luminosity distance $d_L$ for an infinitesimal antenna can be determined as

$$d_L = \sqrt{dA/[(A_s/A_o)d\Omega]}$$

$$= \frac{A_o}{A_s} \sqrt{\frac{r_o \sin(\theta_o - \theta_s + \gamma + \delta)}{\sin \gamma}} |r_o \theta_o'|$$

(43)

where $r_o \sin(\theta_o - \theta_s + \gamma + \delta)$ is the bookkeeper distance of $O$ to the $z$-axis joining $e$ and $C$ (see Section [IVB] and Figures 3 and 4), and $\theta_o'$ has been given in Eq. (33). Since the factors

$$\frac{\sin(\theta_o - \theta_s + \gamma + \delta)}{\sin \gamma} = \frac{\ell \sin(\theta_o - \theta_s + \delta) - r_s \sin(\theta_o - \theta_s)}{r_s \sin \delta}$$

(44)
and \((\tan \delta / \tan \gamma)\) in \((33)\) are both linear in depth \(\ell\) from \((35)\), the luminosity distance \(d_L(\ell)\) is nearly linear in \(\ell\) in a large portion of its range. Such a dependence or contrast would be amplified by \(1/\sqrt{A_s}\) as \(r_s \to 2M\), though the overall apparent luminosity \((\sqrt{A_s/A_o})^2L\) goes to zero in the limit.

VI. SUMMARY AND DISCUSSION

We have studied how a nearly black star would be perceived by a localized observer outside the photon sphere of the star. By looking into a simple model of a semi-transparent spherical shell with a few point-like light emitters distributed inside, we find that, in terms of the affine distance and the parallax distance determined from outgoing light rays of the star, the observer would see that almost all the images of the interior emitters would squeeze around the shell surfaces while the distance between the images of the front and rear surfaces would go to zero as the shell radius \(r_s\) in bookkeeper coordinates is approaching the Schwarzschild radius \(2M\). The resolution of the depth information of the emitters inside the star is decreasing and the star perceived more and more like a 2D membrane rather than a 3D ball as the star is turning to a black hole.

However, using other measures such as the radar distance or the luminosity distance, the depth information of the interior emitters is always resolvable whenever the star is not a black hole. One may be tempted to conclude that the radar and luminosity distances simply do better than the parallax and affine distances, and the entropy of the nearly black star simply jumps from a volume-scaling quantity to some area-scaling quantity with a totally different nature when the star becomes a black hole. Such conclusion might be somewhat naive, anyway. Recall that, to determine the radar distance from the observer to an event, the event must have both the past and the future causal connection with the observer, and the observer must use the historical information of a received signal (echo) put earlier by her/himself (e.g. the emission time and the properties of the original radar signals). To determine the luminosity distance, the properties of the standard candles also have to be known by the observer, or even the candles were sent into the shell earlier by the outside observer herself/himself. Actually, if the emitters inside the shell are smart enough to measure the relative locations of each other by themselves and report to the observer, then the observer certainly will know the full information of the depth of the interior of the shell – if she or he can receive and understand the report, which requires two-way causal connections earlier between the smart emitters and the observer. In contrast, the pre-existing knowledge about a received signal is not needed in determining the parallax or affine distance (some physical assumptions on the signal source inside the shell would still be needed, anyway). Only the future causal connection from the emitting event to the receiving observer is sufficient, as if a past horizon were present behind the spherical shell for the observer. Thus, the observer’s pre-existing knowledge about the signal coming out of the star is crucial for the amount or scaling of the entropy of a nearly black star. Our result suggests that if an observer has that knowledge, by including or ignoring it, the observer would be able to switch between a volume law and a quasi-area law for the entropy of the nearly black star. Since more knowledge about the received signal held by the observer implies less information entropy in the signal as well as more star entropy (volume law) for the observer, the sum of the information entropy of the received signals and the star entropy seems to be conserved for the localized observer in our setup.

When the collapsing star is so close to a black hole that each signal from the interior is too weak or too red-shifted to be resolved by any observer, all those distances to the interior emitters or events will not be measurable for the outside world. With about Before this happens, however, the radar and luminosity distances would have been harder to be determined than the parallax distance because of the need of the historical information. For example, a radar signal may spend a time for a round trip much longer than the observer’s lifetime in her/his clock, and the standard candles sent by the observer may take a similar time scale in the observer’s clock to spread inside the star in order to explore the interior volume. Further, when the star is about to become a black hole, sending energy such as radar signals or standard candles into the star may turn the star to a black hole and the echoes of the radar signals or the light emitted by the standard candles would never reach the outside world. Thus for the observers witnessing the last stage of the black hole formation, more likely they would perceive the star like something closer to a 2D membrane than a 3D ball, and the area law of entropy eventually dominates after the thickness of the membrane is not resolvable (e.g. below the Planck scale).

For an observer localized far outside the photon sphere, the observational data on the interior emitters at the angle of arrival \(\theta_a \geq \theta_1\) are redundant (all are indirect images, see Figure 2 and Section III.C), and so they should not count in the entropy of the star. When the shell radius \(r_s\) is sufficiently close to the Schwarzschild radius \(2M\), all the direct images observed at the angles of arrival from \(\theta_a = 0\) to \(\theta_1\) can be mapped back to the region of the shell surface with \(\theta\) between 0 and \(\pi/2\) in bookkeeper coordinates, namely, the half sphere facing to the observer. Since the depth information of the emitters cannot be resolved, the entropy of the star would be proportional to the area of the half sphere \(A/2\) with \(A = 4\pi r_s^2\). In \([4]\) Bekenstein argued that the minimum increase in area of a Kerr black hole by dropping a particle into it is \(2\hbar\). Thus the entropy of the black hole evolved from our spherical star, to some
extent, could be roughly estimated as $S \approx (A/2)/(2\hbar) = A/(4\hbar)$, which is the Bekenstein-Hawking entropy.

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[1] H. Goldstein, *Classical Mechanics*, 2nd Ed (Addison-Wesley, Reading, 1980).
[2] S.-Y. Lin, *Instantaneous spatially local projective measurements are consistent in a relativistic quantum field*, Ann. Phys. 327 (2012) 3102-3115.
[3] J. M. Bardeen, B. Carter, and S. W. Hawking, *The Four Laws of Black Hole Mechanics*, Commun. Math. Phys. 31 (1973) 161.
[4] J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. D 7 (1973) 2333.
[5] J. D. Bekenstein, *Statistical black hole thermodynamics*, Phys. Rev. D 12 (1975) 2077; J. D. Bekenstein and A. Meisels, *Einstein A and B coefficients for a black hole*, Phys. Rev. D 15 (1977) 2775.
[6] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. B 379 (1996) 99; C. G. Callan and J. M. Maldacena, *D-brane approach to black hole quantum mechanics*, Nucl. Phys. B 472 (1996) 591; G. Compère, *The Kerr/CFT correspondence and its extensions*, Living Rev. Relativ. 20, 1 (2017) [Online Article]; cited [09/28/2019], and the references therein.
[7] C. Rovelli, *Black hole entropy from loop quantum gravity*, Phys. Rev. Lett. 77 (1996) 3288; K. Krasnov, *Geometrical entropy from loop quantum gravity*, Phys. Rev. D 55 (1997) 3505; A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, *Quantum geometry and black hole entropy*, Phys. Rev. Lett. 80 (1998) 904; A. Perez, *Black Holes in Loop Quantum Gravity*, Rep. Prog. Phys. 80 (2017) 126901, and the references therein.
[8] W. Thirring, *Systems with Negative Specific Heat*, Z. Physik 235 (1970), 339.
[9] J. Oppenheim, *Area scaling entropies for gravitating systems*, Phys. Rev. D 65 (2001) 024020; J. Oppenheim, *Thermodynamics with long-range interactions: From Ising models to black holes*, 68 (2003) 016108.
[10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, Cambridge, 1973); S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. 43 (1975) 212.
[11] E. F. Taylor and J. A. Wheeler, *Exploring Black Holes – Introduction to General Relativity*, (Addison Wesley Longman, San Francisco, 2000).
[12] W. Israel, *Singular Hypersurfaces and Thin Shells in General Relativity*, Nuovo Cimento 44B (1966) 1; erratum 48B (1967) 463.
[13] E. Poisson, *A Relativist’s Toolkit: The Mathematics of Black Hole Mechanics* (Cambridge University Press, 2004).
[14] A. J. S. Hamilton and G. Polhemus, *Stereoscopic visualization in curved spacetime: seeing deep inside a black hole*, New J. Phys. 12 123027 (2010).
[15] R. d’Inverno, *Introducing Einstein’s Relativity*, (Oxford University Press, Oxford, 1992).
[16] C. E. Dolby and S. F. Gull, *On radar time and the twin “paradox”*, Am. J. Phys. 69 (2001) 1257.
[17] S.-Y. Lin, *Notes on observational and radar coordinates for localized observers*, in preparation.
[18] S. Weinberg, *Gravitation and cosmology: Principles and applications of the general theory of relativity*, (Wiley, New York, 1972).