Quantization of the canonical tensor model and an exact wave function

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Abstract. Tensor models in various forms are being studied as models of quantum gravity. Among them the canonical tensor model has a canonical pair of rank-three tensors as dynamical variables, and is a pure constraint system with first-class constraints. The Poisson algebra of the first-class constraints provides an algebraically consistent way of discretizing the Dirac algebra for general relativity. This paper successfully formulates the Wheeler-DeWitt quantization of the canonical tensor model. Formally one can obtain wave functions of the “universe” by solving the partial differential equations representing the constraints. For the simplest non-trivial case, the unique wave function is exactly and globally obtained. Although this case is far from being realistic, the wave function is physically interesting; locality is favored, and there exists a locus of configurations with features of the beginning of the universe.

1. Introduction

Tensor models [1] were first proposed to describe simplicial quantum gravity in dimensions higher than two, with the hope to extend the success of matrix models for the two dimensional case. However, because of some unfavored features, these original tensor models have been unsuccessful so far. On the other hand, the difficulties have been overcome by the advent of the colored tensor models [2], and various interesting analytical results have been derived [3]. Yet, since the dominant simplicial manifolds in colored tensor models are branched polymers [4], a serious open question remains on how wide-spread spaces like our universe can be realized in tensor models. A resolution may be obtained by considering time direction more seriously. In fact, the Causal Dynamical Triangulation (CDT), which is simplicial quantum gravity with a causal time direction, has numerically been shown to generate wide-spread simplicial spaces [5], while the above developments of colored tensor models are in Euclidean signature.

Being motivated by the above consideration, the author of the present paper has proposed a tensor model in the canonical formalism, dubbed canonical tensor model [6, 7, 8]. It has a canonical pair of rank-three tensors as its dynamical variables, and is defined as a pure first-class constraint system. The first-class constraint Poisson algebra guarantees the consistency of “many-fingered time” evolutions, and the fact that it has structure functions makes the dynamics non-trivial. In view of these features, the canonical tensor model is very similar to general relativity, and in fact it is possible to derive the Dirac constraint algebra [9] for general relativity from the constraint algebra of the canonical tensor model by taking a formal locality limit [6]. Since the Dirac constraint algebra plays a major role in geometrodynamics, it would be a highly interesting possibility that the canonical tensor model reproduces general relativity.
in a certain, yet unknown, infrared limit. For this to be seen, first of all, a space-like object must be dynamically generated from the canonical tensor model, and more specifically the aforesaid formal locality limit must be derived from dynamics. The purpose of the present paper is to take a first step in this direction by formulating the quantization of the canonical tensor model, and obtaining a wave function of the "universe" for the simplest case [10].

2. Quantization of the canonical tensor model

This paper deals with the Wheeler-DeWitt quantization of the minimal canonical tensor model [10], the classical form of which has been introduced in [8]. The dynamical variables of the model are a canonical pair of rank-three tensors, $M_{abc}, P_{abc}$ $(a, b, c = 1, 2, \ldots, N)$. In the quantization, they are lifted to Heisenberg operators as

$$[\hat{M}_{abc}, \hat{P}_{def}] = i \sum_\sigma \delta_{a\sigma(d)} \delta_{b\sigma(e)} \delta_{c\sigma(f)}, \quad [\hat{M}_{abc}, \hat{M}_{def}] = [\hat{P}_{abc}, \hat{P}_{def}] = 0, \quad (1)$$

where the summation over $\sigma$ is for all the permutations of $def$. The operators are assumed to satisfy the properties corresponding to the classical case,

$$\hat{M}_{abc} = \hat{M}_{abc}^\dagger = \hat{M}_{cba} = \hat{M}_{bca} = \hat{M}_{cab} = \hat{M}_{bac} = \hat{M}_{acb} = \hat{M}_{cba}, \quad (2)$$

$$\hat{P}_{abc} = \hat{P}_{abc}^\dagger = \hat{P}_{bca} = \hat{P}_{cab} = \hat{P}_{abc} = \hat{P}_{acb} = \hat{P}_{cba}, \quad (3)$$

where $\dagger$ denotes the Hermitian conjugate of operators.

The model has three kinds of constraints, $\hat{J}_{[ab]}$, $\hat{D}$, $\hat{H}_a$. The operator ordering in the constraints can be determined by imposing the Hermiticity and the first-class closure of the constraint algebra [10]. The results are

$$\hat{J}_{[ab]} = \frac{1}{4} \left( \hat{M}_{bcd} \hat{P}_{acd} - \hat{M}_{acd} \hat{P}_{bcd} \right), \quad (4)$$

$$\hat{D} = -\frac{1}{6} \left( \hat{M}_{abc} \hat{P}_{abc} - i \frac{N(N+1)(N+2)}{2} \right), \quad (5)$$

$$\hat{H}_a = \frac{1}{2} \left( \hat{M}_{abc} \hat{M}_{bde} \hat{P}_{cde} - i \frac{(N+2)(N+3)}{2} \hat{M}_{abb} \right). \quad (6)$$

These constraint operators form a first-class constraint algebra given by

$$[\hat{H}(T_1), \hat{H}(T_2)] = i \hat{J}(\{\hat{T}_1, \hat{T}_2\}), \quad (7)$$

$$[\hat{J}(V), \hat{H}(T)] = i \hat{H}(VT), \quad (8)$$

$$[\hat{J}(V_1), \hat{J}(V_2)] = i \hat{J}(\{V_1, V_2\}), \quad (9)$$

$$[\hat{D}, \hat{H}(T)] = i \hat{H}(T), \quad (10)$$

$$[\hat{D}, \hat{J}(V)] = 0, \quad (11)$$

where

$$\hat{H}(T) = T_a \hat{H}_a, \quad (12)$$

$$\hat{J}(V) = V_{[ab]} \hat{J}_{[ab]}, \quad (13)$$

$$\hat{J}(\hat{V}) = \hat{V}_{[ab]} \hat{J}_{[ab]}, \quad (14)$$

$$\hat{T}_{bc} = T_a \hat{M}_{abc}. \quad (15)$$
Here, operators are distinguished from classical variables by \( \hat{\cdot} \).

The above first-class algebraic closure of the quantum constraints guarantees the existence (at least locally) of the Wheeler-DeWitt wave functions \( \Psi \), which are solutions to the Wheeler-DeWitt equations,

\[
\hat{\mathcal{H}}_a \Psi = \hat{\mathcal{J}}_{[ab]} \Psi = \hat{\mathcal{D}} \Psi = 0,
\]

in appropriate representations of operators.

3. The exact wave function for \( N = 2 \)

This section solves the Wheeler-DeWitt equations (16) explicitly for the simplest non-trivial case. Consider a representation of the operators in terms of \( M \) as

\[
\Psi = \Psi(M),
\]

\[
\hat{M}_{abc} = M_{abc},
\]

\[
\hat{P}_{abc} = -i\Delta(abc) \frac{\partial}{\partial M_{abc}},
\]

where \( \Delta(abc) \) is a multiplicity factor defined by

\[
\Delta(abc) = \begin{cases} 
6 & \text{for } a = b = c, \\
2 & \text{for } a = b \neq c, b = c \neq a, c = a \neq b, \\
1 & \text{for } a \neq b, b \neq c, c \neq a.
\end{cases}
\]

This factor is needed to consistently take account of the properties (1) and (2). Then the Wheeler-DeWitt equations (16) are a set of first-order partial differential equations for \( \Psi(M) \).

The total number of the first-order partial differential equations in (16) is given by

\[ N + N(N - 1)/2 + 1 = (N^2 + N + 2)/2, \]

while the number of degrees of freedom of \( M_{abc} \) is \( N(N + 1)(N + 2)/6 \). Thus, for \( N = 2 \), the Wheeler-DeWitt equations have a unique solution, granted that it is also globally consistent. It is elementary to obtain the solution, and the result is [10]

\[
\Psi(M) = c_0 \sqrt{\varepsilon_{abc} \varepsilon_{def} h^e f^g h^f g^i h^h g^i h^j h^k M_{aef} M_{bgh} M_{ceg} M_{dfh}} M_{aef} M_{bgh} M_{cde} M_{def} M_{aef} M_{bfc}
\]

with a numerical constant \( c_0 \).

Since the wave function (21) has infinite peaks where the denominator vanishes, there exists a kind of preference for such configurations. One can prove that, by using the kinematical \( SO(N) \) transformations generated by \( \hat{J}_{[ab]} \), such configurations can always be transformed to the diagonal form,

\[
M_{abc} = m_a \delta_{ab} \delta_{ac},
\]

with real \( m_a \).

The canonical tensor model can be interpreted as describing the Hamiltonian dynamics of fuzzy spaces [11]. In this interpretation, \( M_{abc} \) corresponds to the structure constants defining a fuzzy space characterized by an algebra,

\[
f_a * f_b = M_{abc} f_c,
\]
among functions $f_a$ ($a = 1, 2, \cdots, N$) on a fuzzy space. Physically speaking, functions can be regarded as fuzzy “points” (in an appropriate basis of functions), and in the case of the diagonal form (22), the “points” are mutually independent, because

$$f_a \ast f_b = m_a \delta_{ab} f_a.$$  

(24)

Thus, the divergent configurations of the wave function (21) correspond to fuzzy spaces in which locality is maximized.

The configurations where the wave function vanishes can also be characterized as follows. When the numerator of (21) vanishes, one can show that there exists a real vector $v_a$ such that $M_{abc} v_b v_c$ vanishes. In this sense, the configuration $M$ is degenerate at the locus, which would agree with the interpretation that the locus is the beginning of the “universe”, since our universe should have started from a point-like state (or a very small state).

4. Summary and discussion

In this paper, the Wheeler-DeWitt quantization of the minimal canonical tensor model has successfully been formulated. The classical constraints have been lifted to quantum ones, and their commutation algebra has been shown to be first-class. These constraints form a consistent set of Wheeler-DeWitt equations, and the wave function of the “universe” can be obtained by solving them. Indeed, the unique wave function for $N = 2$ has explicitly been obtained and its physical interpretation has been discussed. Although the case $N = 2$ is far from being realistic, the wave function shows physically interesting features such as that locality is favored, and that there exists a locus of configurations which have characteristics of the beginning of the universe.

An obvious question for future study is whether the physically interesting properties of the wave function found for $N = 2$ can be generalized for $N \geq 3$ or not. In particular, the emergence of locality is essentially important for the constraint algebra of the canonical tensor model to be identified with the Dirac or the hypersurface deformation algebra of general relativity.

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