On Lightlike Geometry of Indefinite Sasakian Statistical Manifolds

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Abstract. In this study, we introduce indefinite sasakian statistical manifolds and lightlike hypersurfaces of an indefinite sasakian statistical manifold. Some relations among induced geometrical objects with respect to dual connections in a lightlike hypersurface of an indefinite statistical manifold are obtained. Some examples related to these concepts are also presented. Finally, we prove that an invariant lightlike submanifold of indefinite sasakian statistical manifold is an indefinite sasakian statistical manifold.

1 Introduction

Neural networks can be applied to solving numerous complex optimization problems in electromagnetic theory. Applied physicist B. Bartlett presented unsupervised machine learning model for computing approximate electromagnetic field solutions [2]. In April 2019, the Event Horizon Telescope (EHT) collaboration released the first image of the shadow of a black hole with the help of deep learning algorithms. This image is direct evidence of the existence of black holes and general theory of relativity [3]. This is also an indirect evidence of the existence of lightlike geometry in the universe.

A statistical manifold, a new branch of mathematics, is a generalization of the Riemannian manifold and is used to model the information; and also uses tools of differential geometry to study statistical inference, information loss and estimation [4]. Statistical manifolds also have many application areas such as neural networks, machine learning and artificial intelligence.

Lightlike geometry is one of the most important research areas in differential geometry and has many applications in physics and mathematics, such as general relativity, electromagnetism and black hole theory.

In 1975, Efron [5] first emphasized the role of differential geometry in statistics. Differential geometrical tools were used by Amari to develop this idea [6], [7]. In 1989, Vos [8] obtained fundamental equations of geometry of submanifolds of statistical manifolds. In 2009, hypersurfaces of a statistical manifold are studied by Furuhata [9]. Many studies have been done on both statistical manifolds and lightlike geometry over the last few decades [10]-[25]. However, no study combining these two notions has been done in the literature so far.

Motivated by these circumstances, in this study, we introduce the lightlike geometry of an indefinite sasakian statistical manifold. In Section 2, we present basic definitions and results about statistical manifolds and lightlike hypersurfaces. In Section 3, we show that the induced connections on a lightlike hypersurface of a statistical manifold need not be dual and a lightlike hypersurface need not be a statistical manifold. Moreover, we show that the
second fundamental forms are not degenerate. Finally, an example is given. In Section 4, we introduce indefinite sasakian statistical manifolds and we obtain the characterization theorem of indefinite sasakian statistical manifolds. This section is concluded with two examples. In Section 5, we consider lightlike hypersurfaces of indefinite sasakian statistical manifolds. We characterize the parallelness, totality geodeticity and integrability of some distributions. In this section we also give two examples. In Section 6, we prove that an invariant lightlike submanifold of indefinite sasakian statistical manifold is an indefinite sasakian statistical manifold.

2 Preliminaries

Let \((\bar{M}, \bar{g})\) be an \((m+2)\)-dimensional semi-Riemannian manifold with index(\(\bar{g}\)) = \(q \geq 1\). Let \((M, g)\) be a hypersurface of \((\bar{M}, \bar{g})\) with \(g = \bar{g}|_M\). If the induced metric \(g\) on \(M\) is degenerate, then \(M\) is called a lightlike (null or degenerate) hypersurface ([14], [15], [16]). In this case, there exists a null vector field \(\xi \neq 0\) on \(M\) such that
\[
g(\xi, X) = 0, \quad \forall X \in \Gamma(TM).
\]
(2.1)
The radical or the null space of \(T_xM\), at each point \(x \in M\), is a subspace \(\text{Rad } T_xM\) defined by
\[
\text{Rad } T_xM = \{\xi \in T_xM : g_x(\xi, X) = 0, \ X \in \Gamma(TM)\}.
\]
(2.2)
The dimension of \(\text{Rad } T_xM\) is called the nullity degree of \(g\). We recall that the nullity degree of \(g\) for a lightlike hypersurface of \((\bar{M}, \bar{g})\) is 1. Since \(g\) is degenerate and any null vector being orthogonal to itself, \(T_xM^\perp\) is also null and
\[
\text{Rad } T_xM = T_xM \cap T_xM^\perp.
\]
(2.3)
Since \(\dim T_xM^\perp = 1\) and \(\dim \text{Rad } T_xM = 1\), we have \(\text{Rad } T_xM = T_xM^\perp\). We call \(\text{Rad } TM\) a radical distribution and it is spanned by the null vector field \(\xi\). The complementary vector bundle \(S(TM)\) of \(\text{Rad } TM\) in \(TM\) is called the screen bundle of \(M\). We note that any screen bundle is non-degenerate. This means that
\[
TM = \text{Rad } TM \perp S(TM),
\]
(2.4)
with \(\perp\) denoting the orthogonal-direct sum. The complementary vector bundle \(S(TM)^\perp\) of \(S(TM)\) in \(TM\) is called screen transversal bundle and it has rank 2. Since \(\text{Rad } TM\) is a lightlike subbundle of \(S(TM)^\perp\) there exists a unique local section \(N\) of \(S(TM)^\perp\) such that
\[
\bar{g}(N, N) = 0, \quad \bar{g}(\xi, N) = 1.
\]
(2.5)
Note that \(N\) is transversal to \(M\) and \(\{\xi, N\}\) is a local frame field of \(S(TM)\) and there exists a line subbundle \(\text{ltr}(TM)\) of \(T\bar{M}\), and it is called the lightlike transversal bundle, locally spanned by \(N\). Hence we have the following decomposition:
\[
T\bar{M} = TM \oplus \text{ltr}(TM) = S(TM) \perp \text{Rad } TM \oplus \text{ltr}(TM),
\]
(2.6)
where \(\oplus\) is the direct sum but not orthogonal ([14], [15]). In view of the splitting (2.6), we have the following Gauss and Weingarten formulas, respectively,
\[
\nabla_XY = \nabla_XY + h(X, Y),
\]
(2.7)
\[ \nabla_X N = -A_N X + \nabla_X^t N \]  
(2.8)

for any \( X, Y \in \Gamma(TM) \), where \( \nabla_X Y, A_N X \in \Gamma(TM) \) and \( h(X, Y), \nabla_X^t N \in \Gamma(\text{tr}(TM)) \). If we set
\[
B(X, Y) = \bar{g}(h(X, Y), \xi) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla_X^t N, \xi),
\]
then (2.7) and (2.8) become
\[ \nabla_X Y = \nabla_X Y + B(X, Y)N, \]  
(2.9)

\[ \nabla_X N = -A_N X + \tau(X) N, \]  
(2.10)

respectively. Here, \( B \) and \( A \) are called the second fundamental form and the shape operator of the lightlike hypersurface \( M \), respectively [14]. Let \( P \) be the projection of \( S(TM) \) on \( M \). Then, for any \( X \in \Gamma(TM) \), we can write
\[ X = PX + \eta(X)\xi, \]  
(2.11)

where \( \eta \) is a 1-form given by
\[ \eta(X) = \bar{g}(X, N). \]  
(2.12)

From (2.9), we have
\[ (\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \]  
(2.13)

for all \( X, Y, Z \in \Gamma(TM) \), where the induced connection \( \nabla \) is a non-metric connection on \( M \). From (2.4), we have
\[ \nabla_X W = \nabla_X^* W + h^*(X, W) = \nabla_X^* W + C(X, W)\xi, \]  
(2.14)

\[ \nabla_X \xi = -A^*_\xi X - \tau(X)\xi \]  
(2.15)

for all \( X \in \Gamma(TM) \), \( W \in \Gamma(S(TM)) \), where \( \nabla_X^* W \) and \( A^*_\xi X \) belong to \( \Gamma(S(TM)) \). Here \( C \), \( A^*_\xi \) and \( \nabla^* \) are called the local second fundamental form, the local shape operator and the induced connection on \( S(TM) \), respectively. We also have
\[ g(A^*_\xi X, W) = B(X, W), \ g(A^*_\xi X, N) = 0, \ B(X, \xi) = 0, \ g(A_N X, N) = 0. \]  
(2.16)

Moreover, from the first and third equations of (2.16), we have
\[ A^*_\xi \xi = 0. \]  
(2.17)

Now we define some statistical basic concepts

**Definition 2.1** [9] Let \( \widetilde{M} \) be a smooth manifold. Let \( \widetilde{D} \) be an affine connection with the torsion tensor \( T^{\widetilde{D}} \) and \( \widetilde{g} \) a semi-Riemannian metric on \( \widetilde{M} \). Then the pair \( (\widetilde{D}, \widetilde{g}) \) is called a statistical structure on \( \widetilde{M} \) if

1. \( (\widetilde{D}_X \widetilde{g})(Y, Z) = (\widetilde{D}_Y \widetilde{g})(X, Z) = \bar{g}(T^{\widetilde{D}}(X, Y), Z) \)
   for all \( X, Y, Z \in \Gamma(TM) \), and

2. \( T^{\widetilde{D}} = 0. \)
Definition 2.2 Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold. Two affine connections $\tilde{D}$ and $\tilde{D}^*$ on $\tilde{M}$ are said to be dual with respect to the metric $\tilde{g}$, if
\[
Z\tilde{g}(X,Y) = \tilde{g}(\tilde{D}_ZX,Y) + \tilde{g}(X,\tilde{D}_Z^*Y) \tag{2.18}
\]
for all $X,Y,Z \in \Gamma(T\tilde{M})$.

A statistical manifold will be represented by $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$. If $\tilde{\nabla}$ is Levi-Civita connection of $\tilde{g}$, then
\[
\tilde{\nabla} = \frac{1}{2}(\tilde{D} + \tilde{D}^*). \tag{2.19}
\]

In (2.18), if we choose $\tilde{D}^* = \tilde{D}$ then Levi-Civita connection is obtained.

Lemma 2.3 For statistical manifold $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$, we set
\[
\mathbb{K} = \tilde{D} - \tilde{\nabla}. \tag{2.20}
\]
Then we have
\[
\mathbb{K}(X,Y) = \mathbb{K}(Y,X), \quad \tilde{g}(\mathbb{K}(X,Y),Z) = \tilde{g}(\mathbb{K}(X,Z),Y), \tag{2.21}
\]
for any $X,Y,Z \in \Gamma(TM)$.

Conversely, for a Riemannian metric $g$, if $\mathbb{K}$ satisfies (2.21), the pair $(\tilde{D} = \tilde{\nabla} + \mathbb{K}, \tilde{g})$ is a statistical structure on $\tilde{M}$ [19].

Let $(M, g)$ be a submanifold of $(\tilde{M}, \tilde{g})$. If $(M, g, D, D^*)$ is a statistical manifold, then $(M, g, \tilde{D}, \tilde{D}^*)$ is called a statistical submanifold of $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$, where $D, D^*$ are affine dual connections on $M$ and $\tilde{D}, \tilde{D}^*$ are affine dual connections on $\tilde{M}$ (see [7], [9],[8]).

3 Lightlike hypersurface of a statistical manifold

Let $(M, g)$ be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$. Then, Gauss and Weingarten formulas with respect to dual connections are given by [9]
\[
\tilde{D}_X Y = D_X Y + B(X,Y)N, \tag{3.1}
\]
\[
\tilde{D}_X N = -A_N X + \tau(X)N \tag{3.2}
\]
\[
\tilde{D}_X^* Y = D_X^* Y + B^*(X,Y)N, \tag{3.3}
\]
\[
\tilde{D}_X^* N = -A_N^* X + \tau^*(X)N, \tag{3.4}
\]
for all $X,Y \in \Gamma(TM), \quad N \in \Gamma(ltrTM)$, where $D_X Y$, $D_X^* Y$, $A_N X$, $A_N^* X \in \Gamma(TM)$ and
\[
B(X,Y) = \tilde{g}(\tilde{D}_X Y, \xi), \quad \tau(X) = \tilde{g}(\tilde{D}_X N, \xi),
\]
\[
B^*(X,Y) = \tilde{g}(\tilde{D}_X^* Y, \xi), \quad \tau^*(X) = \tilde{g}(\tilde{D}_X^* N, \xi).
\]
Here, $D$, $D^*$, $B$, $B^*$, $A_N$ and $A_N^*$ are called the induced connections on $M$, the second fundamental forms and the Weingarten mappings with respect to $\bar{D}$ and $\bar{D}^*$, respectively. Using Gauss formulas and the equation (2.18), we obtain

$$Xg(Y, Z) = g(\bar{D}X Y, Z) + g(Y, \bar{D}X^* Z) + B(X, Y)\eta(Z) + B^*(X, Z)\eta(Y). \quad (3.5)$$

From the equation (3.5), we have the following result.

**Theorem 3.1** [1] Let $(M, g)$ be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$. Then:

(i) Induced connections $D$ and $D^*$ need not be dual.

(ii) A lightlike hypersurface of a statistical manifold need not be a statistical manifold with respect to the dual connections.

Using Gauss and Weingarten formulas in (3.5), we get

$$(D_X g)(Y, Z) + (D_X^* g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) + B^*(X, Y)\eta(Z) + B^*(X, Z)\eta(Y). \quad (3.6)$$

**Proposition 3.2** [1] Let $(M, g)$ be a lightlike hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$. Then the following assertions are true:

(i) Induced connections $D$ and $D^*$ are symmetric connection.

(ii) The second fundamental forms $B$ and $B^*$ are symmetric.

**Proof.** We know that $T^{\tilde{D}} = 0$. Moreover,

$$T^D(X, Y) = \tilde{D}_X Y - \tilde{D}_Y X - [X, Y] = d_X Y - d_Y X - [X, Y] + B(X, Y)\eta - B(Y, X)\eta = 0. \quad (3.7)$$

Comparing the tangent and transversal components of (3.7), we obtain

$$B(X, Y) = B(Y, X), \quad T^D = 0,$$

where $T^D$ is the torsion tensor field of $D$. Thus, second fundamental form $B$ is symmetric and induced connection $D$ is symmetric connection.

Similarly, it can be shown that the second fundamental form $B^*$ is symmetric and the induced connection $D^*$ is a symmetric connection. ■

Let $P$ denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.4). Then, we have

$$D_X PY = \nabla_X PY + \bar{h}(X, PY), \quad (3.8)$$

$$D_X \xi = -\bar{A}_\xi X + \bar{\nabla}_X \xi = 0 \quad (3.9)$$
for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\nabla_XPY$ and $\overline{\nabla}_\xi X$ belong to $\Gamma(S(TM))$, $\nabla$ and $\overline{\nabla}$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. Here, $\overline{\nabla}$ and $\overline{\nabla}$ are called screen second fundamental form and screen shape operator of $S(TM)$, respectively. If we define

$$C(X, PY) = g(\overline{\nabla}_X(PY), N), \quad (3.10)$$

$$\varepsilon(X) = g(\overline{\nabla}_X N, \forall X, Y \in \Gamma(TM). \quad (3.11)$$

One can show that

$$\varepsilon(X) = -\tau(X).$$

Therefore, we have

$$D_X PY = \nabla_X PY + C(X, PY)\xi, \quad (3.12)$$

$$D_X\xi = -\overline{\nabla}_\xi X - \tau(X)\xi = 0, \forall X, Y \in \Gamma(TM). \quad (3.13)$$

Here $C(X, PY)$ is called the local screen fundamental form of $S(TM)$.

Similarly, the relations of induced dual objects on $S(TM)$ are given by

$$D_X^* PY = \nabla_X^* PY + C^*(X, PY)\xi, \quad (3.14)$$

$$D_X^*\xi = -\overline{\nabla}_\xi X - \tau^*(X)\xi = 0, \forall X, Y \in \Gamma(TM). \quad (3.15)$$

Using (3.5), (3.12), (3.14) and Gauss-Weingarten formulas, the relationship between induced geometric objects are given by

$$B(X, \xi) + B^*(X, \xi) = 0, g(A_N X + A_N^* X, N) = 0, \quad (3.16)$$

$$C(X, PY) = g(A_N^* X, PY), \quad C^*(X, PY) = g(A_N X, PY). \quad (3.17)$$

Now, using the equation (3.16) we can state the following result.

**Proposition 3.3** [1] Let $(M, g)$ be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then second fundamental forms $B$ and $B^*$ are not degenerate.

Additionally, due to $\widetilde{D}$ and $\widetilde{D}^*$ are dual connections we obtain

$$B(X, Y) = g(\overline{A}_\xi X, Y) + B^*(X, \xi)\eta(Y), \quad (3.18)$$

$$B^*(X, Y) = g(\overline{A}_\xi X, Y) + B(X, \xi)\eta(Y). \quad (3.19)$$

Using (3.18) and (3.19) we get

$$\overline{A}_\xi \xi + \overline{A}_\xi \xi = 0.$$

**Example 3.4** Let $(R^4_1, \tilde{g})$ be a 4-dimensional semi-Euclidean space with signature $(-, -, +, +)$ of the canonical basis $(\partial_0, \ldots, \partial_3)$. Consider a hypersurface $M$ of $R^4_2$ given by

$$x_0 = x_1 + \sqrt{2}\sqrt{x_2^2 + x_3^2}.$$

For simplicity, we set $f = \sqrt{x_2^2 + x_3^2}$. It is easy to check that $M$ is a lightlike hypersurface whose radical distribution $RadTM$ is spanned by

$$\xi = f(\partial_0 - \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3).$$
Then the lightlike transversal vector bundle is given by

\[ ltr(TM) = \text{Span} \left\{ N = \frac{1}{4f^2} \left\{ f(-\partial_0 + \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3) \right\} \right\}. \]

It follows that the corresponding screen distribution \( S(TM) \) is spanned by

\[ \{W_1 = \partial_0 + \partial_1, W_2 = -x_3\partial_2 + x_2\partial_3\}. \]

Then, by direct calculations we obtain

\[ \tilde{\nabla}_X W_1 = \tilde{\nabla}_{W_1} X = 0, \]
\[ \tilde{\nabla}_{W_2} W_2 = -x_2\partial_2 - x_3\partial_3, \]
\[ \tilde{\nabla}_\xi \xi = \sqrt{2}\xi, \tilde{\nabla}_{W_2} \xi = \tilde{\nabla}_\xi W_2 = \sqrt{2}W_2, \]

for any \( X \in \Gamma(TM) \) [16].

We define an affine connection \( \tilde{D} \) as follows

\[ \tilde{D}_X W_1 = \tilde{D}_{W_1} X = 0, \tilde{D}_{W_2} W_2 = -2x_2\partial_2 \]
\[ \tilde{D}_\xi \xi = \sqrt{2}\xi, \]
\[ \tilde{D}_{W_2} \xi = \tilde{D}_\xi W_2 = \sqrt{2}W_2. \] (3.20)

Then using (2.19) we obtain

\[ \tilde{D}^* X W_1 = \tilde{D}^*_{W_1} X = 0, \tilde{D}^*_{W_2} W_2 = -2x_3\partial_3 \]
\[ \tilde{D}^*_\xi \xi = \sqrt{2}\xi, \]
\[ \tilde{D}^*_{W_2} \xi = \tilde{D}^*_\xi W_2 = \sqrt{2}W_2. \] (3.21)

Then \( \tilde{D} \) and \( \tilde{D}^* \) are dual connections. Here, one can easily see that \( T\tilde{D} = 0 \) and \( \tilde{D}\tilde{g} = 0 \).

From Definition 2.1, we say that \((R^4, \tilde{g}, \tilde{D}, \tilde{D}^*)\) is a statistical manifold.

## 4 Indefinite sasakian statistical manifolds

In order to call a differentiable semi-Riemannian manifold \((\tilde{M}, \tilde{g})\) of dimension \( n = 2m + 1 \) as practically contact metric one, a \((1, 1)\) tensor field \( \tilde{\varphi} \), a contravariant vector field \( \nu \), a 1–form \( \eta \) and a Riemannian metric \( \tilde{g} \) should be admitted, which satisfy

\[ \tilde{\varphi}\nu = 0, \eta(\tilde{\varphi}X) = 0, \eta(\nu) = \epsilon, \]
\[ \tilde{\varphi}^2(X) = -X + \eta(X)\nu, \tilde{g}(X, \nu) = \epsilon\eta(X), \]
\[ \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \epsilon = \mp 1 \]

for all the vector fields \( X, Y \) on \( \tilde{M} \). When a practically contact metric manifold performs

\[ (\tilde{\nabla}_X \tilde{\varphi})Y = \tilde{g}(X, Y)\nu - \epsilon\eta(Y)X, \]
\[ \tilde{\nabla}_X \nu = -\tilde{\varphi}X, \]

\( \tilde{M} \) is regarded as an indefinite sasakian manifold. In this study, we assume that the vector field \( \nu \) is spacelike.
Definition 4.1 Let $(\tilde{g}, \tilde{\phi}, \nu)$ be an indefinite sasakian structure on $\tilde{M}$. A quadruplet $(\tilde{D} = \tilde{\nabla} + \mathbb{K}, \tilde{g}, \tilde{\phi}, \nu)$ is called a indefinite sasakian statistical structure on $\tilde{M}$ if $(\tilde{D}, \tilde{g})$ is a statistical structure on $\tilde{M}$ and the formula

$$K(X, \tilde{\phi}Y) = -\tilde{\phi}K(X, Y)$$ (4.6)

holds for any $X, Y \in \Gamma(TM)$. Then $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\phi}, \nu)$ is said to be an indefinite sasakian statistical manifold.

An indefinite sasakian statistical manifold will be represented by $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\phi}, \nu)$. We remark that if $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\phi}, \nu)$ is an indefinite sasakian statistical manifold, so is $(\tilde{M}, \tilde{D}^*, \tilde{g}, \varphi, \nu)$ [18], [19].

Theorem 4.2 Let $(\tilde{M}, \tilde{D}, \tilde{g})$ be a statistical manifold and $(\tilde{g}, \tilde{\phi}, \nu)$ an almost contact metric structure on $\tilde{M}$. $(\tilde{D}, \tilde{g}, \tilde{\phi}, \nu)$ is an indefinite sasakian statistical structure if and only if the following conditions hold:

$$\tilde{D}_X \varphi Y - \tilde{\phi} \tilde{D}_X Y = g(Y, X)\nu - \tilde{g}(Y, \nu)X,$$ (4.7)

$$\tilde{D}_X \nu = -\tilde{\phi} X + g(\tilde{D}_X \nu, \nu),$$ (4.8)

for all the vector fields $X, Y$ on $\tilde{M}$.

Proof. Using (2.20) we get

$$\tilde{D}_X \varphi Y - \tilde{\phi} \tilde{D}_X Y = (\tilde{\nabla}_X \varphi)Y + \mathbb{K}(X, \varphi Y) + \varphi \mathbb{K}(X, Y)$$ (4.9)

for all the vector fields $X, Y$ on $\tilde{M}$. If we consider Definition 4.1 and the equation (4.4), we have the formula (4.7). If we write $\tilde{D}^*$ instead of $\tilde{D}$ in (4.7), we have

$$\tilde{D}^*_X \varphi Y - \tilde{\phi} \tilde{D}^*_X Y = g(Y, X)\nu - \tilde{g}(Y, \nu)X,$$ (4.10)

Substituting $\nu$ for $Y$ in (4.10), we have the equation (4.8).

Conversely using (4.7), we obtain

$$\tilde{\phi} \{ \tilde{D}_X \varphi^2 Y - \varphi \tilde{D}^*_X \varphi Y \} = 0.$$ (4.11)

Assume (4.2) and (4.8) as well, we get

$$0 = -\tilde{\phi} \tilde{D}_X Y + \tilde{g}(Y, \nu)X - \tilde{g}(X, \nu)\tilde{g}(Y, \nu)\nu + \tilde{D}^*_X \varphi Y - \tilde{g}(\tilde{D}^*_X \varphi, \varphi Y)\nu,$$

From (4.3), this equation gives us (4.10).

Now, we will prove that (4.4) and (4.6) by using (4.7) and (4.10). Using (4.7) and (4.10), respectively, we have the following equations

$$(\tilde{\nabla}_X \varphi)Y - g(Y, X)\nu + \tilde{g}(Y, \nu)X = \mathbb{K}(X, \varphi Y) + \varphi \mathbb{K}(X, Y),$$

and

$$(\tilde{\nabla}_X \varphi)Y - g(Y, X)\nu + \tilde{g}(Y, \nu)X = -\mathbb{K}(X, \varphi Y) - \varphi \mathbb{K}(X, Y).$$

This last two equations verifies (4.4) and (4.6).
Example 4.3 Let \( \tilde{M} = (R^5, \tilde{g}) \) be a semi-Euclidean space, where \( \tilde{g} \) is of the signature \((-,-,+,+,+)\) with respect to canonical basis \( \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z} \} \). Defining
\[
\eta = dz, \quad \nu = \frac{\partial}{\partial z}, \\
\tilde{\varphi}(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i}, \quad \tilde{\varphi}(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}, \quad \tilde{\varphi}(\frac{\partial}{\partial z}) = 0,
\]
where \( i = 1, 2 \). It can easily see that \( (\tilde{\varphi}, \nu, \eta, \tilde{g}) \) is an indefinite Sasakian structure on \( R^5 \). If we choose \( \mathbb{K}(X, Y) = \tilde{g}(Y, \nu)\tilde{g}(X, \nu) \), then \( \tilde{D} = \tilde{\nabla} + \mathbb{K}, \tilde{g}, \tilde{\varphi}, \nu \) is an indefinite Sasakian statistical structure on \( \tilde{M} \).

Example 4.4 In a 5- dimensional real space \( \tilde{M} = R^5 \), let \( \{x_i, y_i, z\}_{1 \leq i \leq 2} \) be cartesian coordinates on \( \tilde{M} \) and \( \{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z} \}_{1 \leq i \leq 2} \) be the natural field of frames. If we define 1- form \( \eta \), a vector field \( \nu \) and a tensor field \( \tilde{\varphi} \) as follows:
\[
\eta = dz - y_1dx_1 - x_1dy_1, \quad \nu = \frac{\partial}{\partial z}, \\
\tilde{\varphi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial y_1}, \quad \tilde{\varphi}(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial x_1}, \quad \tilde{\varphi}(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial z}.
\]

It is easy to check (4.1) and (4.2) and thus \( (\tilde{\varphi}, \nu, \eta) \) is an almost contact structure on \( R^5 \). Now, we define metric \( \tilde{g} \) on \( R^5 \) by
\[
\tilde{g} = (y_1^2 - 1)dx_1^2 - dx_2^2 + (x_1^2 + 1)dy_1^2 + dy_2^2 + dz^2 - y_1dx_1 \otimes dz - y_1dz \otimes dx_1 + x_1y_1dy_1 \otimes dx_1 - x_1dy_1 \otimes dz - x_1dz \otimes dy_1
\]
with respect to the natural field of frames. Then we can easily see that \( (\tilde{\varphi}, \nu, \eta, \tilde{g}) \) is an indefinite Sasakian structure on \( R^5 \). We set the difference tensor field \( \mathbb{K} \) as
\[
\mathbb{K}(X, Y) = \lambda \tilde{g}(Y, \nu)\tilde{g}(X, \nu) \nu,
\]
where \( \lambda \in C^\infty(\tilde{M}) \). Then, \( \tilde{D} = \tilde{\nabla} + \mathbb{K}, \tilde{g}, \tilde{\varphi}, \nu \) is an indefinite Sasakian statistical structure on \( \tilde{M} \).

5 Lightlike hypersurfaces of indefinite sasakian statistical manifolds

Definition 5.1 Let \( (M, g, D, D^*) \) be a hypersurface of indefinite Sasakian statistical manifold \( (\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu) \). The quadruplet \( (M, g, D, D^*) \) is called lightlike hypersurface of indefinite Sasakian statistical manifold \( (\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu) \) if the induced metric \( g \) is degenerate.

Let \( (\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu) \) be a \((2m+1)-\) dimensional Sasakian statistical manifold and \( (M, g) \) be a lightlike hypersurface of \( \tilde{M} \), such that the structure vector field \( \nu \) is tangent to \( M \). For any \( \xi \in \Gamma(RadTM) \) and \( N \in \Gamma(ltrTM) \), in view of (4.1)-(4.3), we have
\[
\tilde{g}(\xi, \nu) = 0, \quad \tilde{g}(N, \nu) = 0, \\
\tilde{\varphi}^2 \xi = -\xi, \quad \tilde{\varphi}^2 N = -N.
\]
Also using (3.1) and (4.8) we obtain
\[ B(\xi, \nu) = 0, \quad B(\nu, \nu) = 0, \quad (5.3) \]
\[ B^*(\xi, \nu) = 0, \quad B^*(\nu, \nu) = 0. \quad (5.4) \]

**Proposition 5.2** Let \((\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)\) be a \((2m + 1)\)-dimensional Sasakian statistical manifold and \((M, g, D, D^*)\) be its lightlike hypersurface such that the structure vector field \(\nu\) is tangent to \(M\). Then we have
\[ g(\widetilde{\varphi} \xi, \xi) = 0, \quad (5.5) \]
\[ g(\widetilde{\varphi} \xi, N) = -g(\xi, \widetilde{\varphi} N) = -g(A^*_N \xi, \nu), \quad (5.6) \]
\[ g(\widetilde{\varphi} \xi, \widetilde{\varphi} N) = 1, \quad (5.7) \]
where \(\xi\) is a local section of \(\text{Rad}TM\) and \(N\) is a local section of \(\text{ltr}TM\).

**Proof.** Using (4.8) and (3.1), we have
\[ g(\widetilde{\varphi} \xi, \xi) = g(-\widetilde{D}_\xi \nu + g(\widetilde{D}_\xi \nu, \nu) \nu, \xi) \]
\[ = g(-D_\xi \nu - B(\xi, \nu) N, \xi), \]
\[ = 0 \]

and
\[ g(\widetilde{\varphi} \xi, N) = g(-\widetilde{D}_\xi \nu + g(\widetilde{D}_\xi \nu, \nu) \nu, N) \]
\[ = g(\nu, \widetilde{D}^*_\xi N), \]
\[ = -g(A^*_N \xi, \nu). \]

From (4.3) and (5.1), we have (5.7).

Proposition 5.2 makes it possible to make the following decompositions:
\[ S(TM) = \{\widetilde{\varphi} \text{Rad}TM \oplus \widetilde{\varphi} \text{ltr}(TM)\} \perp L_0 \perp \langle \nu \rangle, \quad (5.8) \]
where \(L_0\) is non-degenerate and \(\widetilde{\varphi}\) - invariant distribution of rank \(2m - 4\) on \(M\). If we denote the following distributions on \(M\)
\[ L = \text{Rad}TM \perp \widetilde{\varphi} \text{Rad}TM \perp L_0, \quad L' = \widetilde{\varphi} \text{ltr}(TM), \quad (5.9) \]
then \(L\) is invariant and \(L'\) is anti-invariant distributions under \(\widetilde{\varphi}\). Also we have
\[ TM = L \oplus L' \perp \langle \nu \rangle. \quad (5.10) \]

Now, we consider two null vector field \(U\) and \(W\) and their 1-forms \(u\) and \(w\) as follows:
\[ U = -\widetilde{\varphi} N, \quad u(X) = \widetilde{g}(X, W), \quad (5.11) \]
\[ W = -\widetilde{\varphi} \xi, \quad w(X) = \widetilde{g}(X, U). \quad (5.12) \]

Then, for any \(X \in \Gamma(\widetilde{T}M)\), we have
\[ X = SX + u(X)U, \quad (5.13) \]
where $S$ projection morphism of $T\tilde{M}$ on the distribution $L$. Applying $\tilde{\varphi}$ to last equation, we obtain

\[
\tilde{\varphi}X = \tilde{\varphi}SX + u(X)\tilde{\varphi}U,
\]
\[
\tilde{\varphi}X = \varphi X + u(X)N,
\]

where $\varphi$ is a tensor field of type $(1, 1)$ defined on $M$ by $\varphi X = \tilde{\varphi}SX$.

Again, we apply $\tilde{\varphi}$ to (5.14) and using (4.1)-(4.3) we have

\[
\tilde{\varphi}^2 X = \tilde{\varphi}\varphi X + u(X)\tilde{\varphi}N, \\
-X + g(X, \nu)\nu = \varphi^2 X - u(X)U.
\]

which means that

\[
\varphi^2 X = -X + g(X, \nu)\nu + u(X)U. \tag{5.15}
\]

Now applying $\varphi$ to the equation (5.15) and since $\varphi U = 0$, we have $\varphi^3 + \varphi = 0$ which gives that $\varphi$ is an $f$–structure.

**Definition 5.3** Let $(M, g, D, D^*)$ be a hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. The quadruplet $(M, g, D, D^*)$ is called screen semi-invariant lightlike hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$ if

\[
\tilde{\varphi}(\text{ltr}TM) \subset S(TM), \\
\tilde{\varphi}(\text{Rad}TM) \subset S(TM).
\]

We remark that a hypersurface of indefinite Sasakian statistical manifold is screen semi-invariant lightlike hypersurface.

**Example 5.4** Let us recall the example 4.3, Suppose that $M$ is a hypersurface of $R^5_2$ defined by

\[
x_1 = y_2.
\]

Then $\text{Rad}TM$ and $\text{ltr}(TM)$ are spanned by $\xi = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}$ and $N = \frac{1}{2}\{-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}\}$, respectively. Applying $\tilde{\varphi}$ to this vector fields, we have

\[
\tilde{\varphi}\xi = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1}, \tilde{\varphi}N = \frac{1}{2}\left\{\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}\right\}.
\]

Thus $M$ is a screen semi invariant lightlike hyperurface of indefinite sasakian statistical manifold $R^5_2$.

**Example 5.5** Let $M$ be a hypersurface of $(\tilde{\varphi}, \nu, \eta, \tilde{g})$ on $\tilde{M} = R^5$ in Example 4.4, Suppose that $M$ is a hypersurface of $R^5_2$ defined by

\[
x_2 = y_2,
\]
Then the tangent space $T\ M$ is spanned by $\{U_i\}_{1\leq i\leq 4}$, where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2}$, $U_3 = \frac{\partial}{\partial y_1}$, $U_4 = \nu$. $\mathrm{Rad}T\ M$ and $\mathrm{ltr}(T\ M)$ are spanned by $\xi = U_2$ and $N = \frac{1}{2}\{-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}\}$, respectively. Applying $\phi$ to this vector fields, we have
\[
\phi \xi = U_1 + U_3 + (x_1 + y_1)U_4, \quad \phi N = \frac{1}{2}\{-U_1 + U_3 + (x_1 - y_1)U_4\}.
\]

Thus $M$ is a screen semi invariant lightlike hypersurface of indefinite sasakian statistical manifold $\tilde{M}$. In view of (5.11) and (5.12), we have
\[
\tilde{g}(U, W) = 1.
\]
Thus $\langle U \rangle \oplus \langle W \rangle$ is non-degenerate vector bundle of $S(T\ M)$ with rank 2. If we consider (5.8) and (5.9), we get
\[
S(T\ M) = \{U \oplus W\} \perp L_0 \perp \langle \nu \rangle, \quad (5.16)
\]
and
\[
L = \mathrm{Rad}T\ M \perp \langle W \rangle \perp L_0, \quad L' = \langle U \rangle. \quad (5.17)
\]
Thus, for any $X \in \Gamma(T\ M)$, we can write
\[
X = PX + QX + g(X, \nu)\nu, \quad (5.18)
\]
where $P$ and $Q$ are projections of $T\ M$ into $L$ and $L'$. Thus, we can write $QX = u(X)U$. Using (4.1), (4.2), (4.3), (5.14) and (5.18), we have
\[
\varphi^2 X = -X + g(X, \nu)\nu + u(X)U.
\]
where $\varphi PX = \varphi X$. We can easily see that
\[
g(\varphi X, \varphi Y) = g(X, Y) - g(X, \nu)g(Y, \nu) - u(X)w(Y) - u(Y)w(X), \quad (5.19)
\]
for any $X, Y \in \Gamma(T\ M)$. Also we have the following identities:
\[
g(\varphi X, Y) = g(X, \varphi Y) - u(X)\eta(Y) - u(Y)\eta(X), \quad (5.20)
\]
\[
\varphi \nu = 0, \quad g(\varphi X, \nu) = 0. \quad (5.21)
\]
Thus, we have the following proposition

**Proposition 5.6** Let $(M, g, D, D^*)$ be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. Then $\varphi$ need not be a almost contact structure.

**Lemma 5.7** Let $(M, g, D, D^*)$ be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(T\ M)$, we have the following identities:
\[
D_X\varphi Y - \varphi D_X^* Y = u(Y)A_N X - B^*(X, Y)U + g(X, Y)\nu - g(\nu, Y)X, \quad (5.22)
\]
\[
D_X(u(Y)) - u(D_X^* Y) = -B(X, \varphi Y) - u(Y)\tau(X) \quad (5.23)
\]

**Proof.** Using Gauss and Weingarten formulas in (4.7) we obtain
\[
D_X\varphi Y + B(X, \varphi Y) + D_X(u(Y))N - u(Y)A_N X + u(Y)\tau(X)N - \varphi \nabla_X^* Y + B^*(X, Y)U = g(X, Y)\nu - g(\nu, Y)X \quad (5.24)
\]
If we take tangential and transversal parts of this last equation we have (5.22) and (5.23).
Similarly, we have the following lemma

Lemma 5.8 Let \((M, g, D, D^*)\) be a lightlike hypersurface of indefinite Sasakian statistical manifold \((\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)\). For any \(X, Y \in \Gamma(TM)\), we have the following identities:

\[
D^*_X \varphi Y - \varphi D^*_X Y = u(Y) A^*_N X - B(X, Y) U + g(X, Y) \nu - g(\nu, Y) X,
\]

\[
D^*_X (u(Y)) - u(D_X Y) = -B^*(X, \varphi Y) - u(Y) \tau^*(X)
\]

Lemma(5.7) and Lemma(5.8) are give us the following theorem.

Theorem 5.9 A lightlike hypersurface \(M\) of an indefinite Sasakian statistical manifold \(\tilde{M}\) need not be a statistical manifold.

Proposition 5.10 Let \((M, g, D, D^*)\) be a lightlike hypersurface of indefinite Sasakian statistical manifold \((\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)\). For any \(X, Y \in \Gamma(TM)\), we have the following expressions:

(i) If the vector field \(U\) is parallel with respect to \(\nabla^*\), then

\[
A_N X = u(A_N X) U + \tau(A_N X) \nu, \quad \tau(X) = 0
\]

(ii) If the vector field \(U\) is parallel with respect to \(\nabla\), then

\[
A^*_N X = u(A^*_N X) U + \tau(A^*_N X) \nu, \quad \tau^*(X) = 0
\]

Proof. Replacing \(Y\) in (5.22) by \(U\), we obtain

\[-\varphi D^*_X Y = A_N X - B^*(X, U) U + g(X, U) \nu.\]

Applying \(\varphi\) to this equation and using (5.15), we get

\[D^*_X U - g(D^*_X U, \nu) - u(D^*_X U) U = \varphi A_N X.\]

If \(U\) is parallel with respect to \(\nabla^*\) then \(\varphi A_N X = 0\). From (5.14) we have \(\tilde{\varphi}(A_N X) = u(A_N X) N\). Applying \(\tilde{\varphi}\) this and using (4.2) we obtain \(A_N X = u(A_N X) U + \tau(A_N X) \nu\). Also, if we write \(U\) instead of \(Y\) in the equation (5.23), we have \(\tau(X) = 0\).

(5.28) can also be easily obtained by similar method.

Proposition 5.11 Let \((M, g, D, D^*)\) be a lightlike hypersurface of indefinite Sasakian statistical manifold \((\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)\). For any \(X, Y \in \Gamma(TM)\), we have the following expressions:

(i) If the vector field \(W\) is parallel with respect to \(\nabla^*\), then

\[\overline{A}_\xi X = g(\overline{A}_\xi X, \nu) + u(\overline{A}_\xi X) U, \quad \tau^*(X) = 0.\]

(ii) If the vector field \(W\) is parallel with respect to \(\nabla\), then

\[\overline{A}_\xi X = g(\overline{A}_\xi X, \nu) + u(\overline{A}_\xi X) U, \quad \tau(X) = 0.\]

Proof. If we write \(\xi\) instead of \(Y\) in the equation (5.22), we obtain

\[D_X \varphi \xi - \varphi D^*_X \xi = -B^*(X, \xi) U.\]
If $W$ is parallel with respect to $D$, using (3.15) and (5.12) in this equation, we obtain
\[
\varphi A^*_\xi X - \tau^*(X)W = -B^*(X, \xi)U.
\]
Applying $\tilde{\varphi}$ this and using (5.15) we have
\[
-\tilde{A}^*_\xi X + g(\tilde{A}^*_\xi X, \nu)\nu + u(\tilde{A}^*_\xi X)U = \tau^*(X)\xi
\]
If we take screen and radical parts of this last equation we have (5.29).
Similarly, we can easily see the equation (5.30).

**Definition 5.12** ([18], [22]) Let $(M, g)$ be a hypersurface of a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{D}, \tilde{D}^*)$.

(i) $M$ is called **totally geodesic with respect to** $\tilde{D}$ if $B = 0$.

(ii) $M$ is called **totally geodesic with respect to** $\tilde{D}^*$ if $B^* = 0$.

**Theorem 5.13** Let $(M, g, D, D^*)$ be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$.

(i) $M$ is totally geodesic with respect to $\tilde{D}$ if and only if
\[
D_X \varphi Y - \varphi D^*_X Y = g(X, Y)\nu, \forall X \in \Gamma(TM), Y \in \Gamma(L),
\]
\[
A^*_X X = -\varphi D^*_X U - g(X, U)\nu, \forall X \in \Gamma(TM).
\]

(ii) $M$ is totally geodesic with respect to $\tilde{D}^*$ if and only if
\[
D^*_X \varphi Y - \varphi D_X Y = g(X, Y)\nu, \forall X \in \Gamma(TM), Y \in \Gamma(L),
\]
\[
A^*_X X = -\varphi D_X U - g(X, U)\nu, \forall X \in \Gamma(TM).
\]

**Proof.** For any $X \in \Gamma(TM)$ we know that $u(Y) = 0$. Then the equations (5.22) and (5.25) are reduced to the equations, respectively
\[
D_X \varphi Y - \varphi D^*_X Y = -B^*(X, Y)U + g(X, Y)\nu,
\]
\[
D^*_X \varphi Y - \varphi D_X Y = -B(X, Y)U + g(X, Y)\nu.
\]
On the other hand, replacing $Y$ by $U$ in (5.22) and (5.25), respectively, we also have
\[
A^*_X X = -\varphi D^*_X U + B^*(X, U)U - g(X, U)\nu,
\]
\[
A^*_X X = -\varphi D_X U + B(X, U)U - g(X, U)\nu.
\]
If taking into account (5.35), (5.36), (5.37) and (5.38), we can easily obtain our assertion.

The following two theorems give a characterization of the integrability of distributions $L \perp \langle \nu \rangle$ and $L^\perp \langle \nu \rangle$, respectively.

**Theorem 5.14** Let $(M, g, D, D^*)$ be a screen semi-invariant hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. The following assertions are equivalent:

(i) The distribution $L \perp \langle \nu \rangle$ is integrable.
(ii) $B^*(X, \varphi Y) = B^*(\varphi X, Y)$, for all $X, Y \in \Gamma(L \perp \nu)$,
(iii) $B(X, \varphi Y) = B(\varphi X, Y)$, for all $X, Y \in \Gamma(L \perp \nu)$.

Proof. We know that $X \in \Gamma(L \perp \nu)$ if and only if $u(X) = g(X, W) = 0$. For any $X, Y \in \Gamma(L \perp \nu)$, using (3.1) and (5.14), we obtain

$$u[X, Y] = -u(D_XY) + u(D_YX).$$

From (5.23), we have

$$u[X, Y] = B^*(Y, \varphi X) - B^*(\varphi Y, X).$$

This gives the equivalence between (i) and (ii). Similarly we can easily see that the relation (i) and (iii).

Theorem 5.15 Let $(M, g, D, D^*)$ be a screen semi-invariant hypersurface of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. The following assertions are equivalent:
(i) The distribution $L' \perp \nu$ is integrable.
(ii) $A^*_X Y - A^*_Y X = g(X, \nu)Y - g(Y, \nu)X$, for all $X, Y \in \Gamma(L' \perp \nu)$.
(iii) $A_{XY}^* - A_{YX}^* = g(X, \nu)Y - g(Y, \nu)X$, for all $X, Y \in \Gamma(L' \perp \nu)$.

Proof. $X \in \Gamma(L' \perp \nu)$ if and only if $\varphi X = 0$. For any $X, Y \in \Gamma(L \perp \nu)$, using (3.2), (3.3) and (5.14) in (4.7), we have

$$\varphi D^*_X Y = -g(X, Y)\nu + \tilde{g}(Y, \nu)X - A_{\varphi Y} X + B^*(X, Y)U.$$

Therefore, we can get

$$\varphi[X, Y] = -A_{\varphi Y} X + A_{\varphi X} Y + \tilde{g}(Y, \nu)X - \tilde{g}(X, \nu)Y.$$

This gives the equivalence between (i) and (ii). Similarly we can easily see that the relation (i) and (iii).

6 Invariant submanifolds

Let $(M, g, D, D^*)$ be an invariant lightlike submanifold of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$. If $M$ is tangent to the structure vector field $\nu$, then $\nu$ belongs to $S(TM)$ (see [16]). For invariant lightlike submanifold, we have the following expressions:

$$\tilde{\varphi}(S(TM)) = S(TM), \tilde{\varphi}(\text{RadTM}) = \text{RadTM} \quad (6.1)$$

Proposition 6.1 Let $(M, g, D, D^*)$ be an invariant lightlike submanifold of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$ such that the structure vector field $\nu$ is tangent to $M$. For any $X, Y \in \Gamma(TM)$, we have the following identities:

$$D_X \varphi Y - \varphi D_X^* Y = g(X, Y)\nu - g(\nu, Y)X, \quad (6.2)$$

$$h(X, \tilde{\varphi} Y) = \tilde{\varphi} h^*(X, Y), \quad (6.3)$$

where $h$ and $h^*$ are second fundemental forms for affine dual connections $\tilde{D}$ and $\tilde{D}^*$, respectively.

Proof. Using (5.14) and Gauss formula in (4.7), we obtain

$$D_X \varphi Y + h(X, \tilde{\varphi} Y) - \varphi D_X^* Y - \tilde{\varphi} h^*(X, Y) = g(X, Y)\nu - g(\nu, Y)X.$$

If we take tangential and transversal parts of this last equation, our claim is proven.
Similar to the above proposition, the following proposition is given for dual connection $D^\ast$.

**Proposition 6.2** Let $(M, g, D, D^\ast)$ be an invariant lightlike submanifold of indefinite Sasakian statistical manifold $(\tilde{M}, \tilde{D}, \tilde{g}, \tilde{\varphi}, \nu)$ such that the structure vector field $\nu$ is tangent to $M$. For any $X, Y \in \Gamma(TM)$, we have the following identities:

\[
D^\ast_X \varphi Y - \varphi D_X Y = g(X, Y)\nu - g(\nu, Y)X; \quad (6.4)
\]

\[
h^\ast(X, \tilde{\varphi}Y) = \tilde{\varphi} h(X, Y), \quad (6.5)
\]

where $h$ and $h^\ast$ are second fundamental forms for affine dual connections $\tilde{D}$ and $\tilde{D}^\ast$, respectively.

From the equations (6.3) and (6.5), we have

\[
h(X, \nu) = 0, \quad h^\ast(X, \nu) = 0, \quad (6.6)
\]

A lightlike submanifold may not be an indefinite sasakian statistical manifold. The following theorem gives a case where this can happen.

**Theorem 6.3** An invariant lightlike submanifold of indefinite Sasakian statistical manifold is an indefinite sasakian statistical manifold.

**Proof.** In a invariant lightlike submanifold, $u(X) = 0$, for any $X \in \Gamma(TM)$. Then from (5.14) we have

\[
\varphi^2 X = -X + g(X, \nu)\nu.
\]

Since $\tilde{\varphi}X = \varphi X$, using (4.1), (4.2) and (4.3), we obtain

\[
\varphi \nu = 0, \quad \eta(\varphi X) = 0, \quad (6.7)
\]

\[
\tilde{g}(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (6.8)
\]

Then $(g, \varphi, \nu)$ is an almost contact metric structure.

Using (3.5), we get

\[
X g(\varphi Y, \varphi Z) = g(D_X \varphi Y, \varphi Z) + g(\varphi Y, D^\ast_X \varphi Z). \quad (6.9)
\]

This equation says that $D$ and $D^\ast$ are dual connections. Moreover torsion tensor of the connection $D$ is equal zero. Then, the equations (3.5) and (3.6) tell us that $(D, g)$ is a statistical structure.

If we consider Gauss formula and (4.8) we obtain

\[
D_X \nu = -\varphi X + g(D_X \nu, \nu)\nu. \quad (6.10)
\]

If we consider (6.2) and (6.10) in the theorem 4.2, our assertion are proven.

### 4. Conclusion and future work

In the present paper, firstly we have studied lightlike geometry of statistical manifolds, Later, we have introduced lightlike geometry of an indefinite sasakian statistical manifold which is a new classification of statistical manifolds and we have given some results for its induced geometrical objects. Some examples related to these concepts are also presented. Finally, we prove that an invariant lightlike submanifold of indefinite sasakian statistical manifold is an indefinite sasakian statistical manifold.

We hope that, this introductory study will bring a new perspective for researchers and researchers will further work on it focusing on new results not available so far on lightlike geometry.
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