WEAKLY–EXCEPTIONAL QUOTIENT SINGULARITIES

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Abstract. A singularity is said to be weakly–exceptional if it has a unique purely log terminal blow up. In dimension 2, V. Shokurov proved that weakly–exceptional quotient singularities are exactly those of types $D_n, E_6, E_7, E_8$. This paper classifies the weakly–exceptional quotient singularities in dimensions 3 and 4.

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1. Introduction

Let $G \in \text{SL}_N(\mathbb{C})$. Then $G$ has a natural linear action on $\mathbb{C}^N$, and by linearity it induces an action of a group $\bar{G}$ on $\mathbb{P}^{N-1}$, where $\bar{G}$ is the image of $G$ in the natural projection of $\text{SL}_N(\mathbb{C})$ onto $\text{PGL}_N(\mathbb{C})$. Since the two groups are closely related, I will move from considering $\bar{G}$ to considering $G$ and back at will, immediately assuming any results proven for $\bar{G}$ as proven for $G$ (up to scalar multiplication) and vice versa. In this article, I will assume that the action of $\bar{G}$ is faithful, since any non–faithful actions will have been looked at when considering a faithful action of a smaller group.

1.1. Notation. Here I will define some standard notation that I will be using throughout this paper:

- $\mathbb{Z}_n$ is the cyclic group of order $n$.
- $D_{2n}$ is the dihedral group of order $2n$.
- $A_n$ is the alternating group of degree $n$.
- $S_n$ is the permutation group of degree $n$.
• $\mathbb{Z}_n$, $\mathbb{D}_2n$, $\mathbb{A}_n$, $\mathbb{S}_n$ are the binary versions of the relevant groups, i.e. their central extensions by scalars (see for example [20]).
• $\mathbb{V}_4$ is the Klein group of size 4.
• $A \times B$ is the direct product of groups $A$ and $B$.
• $A \rtimes B$ is the semidirect product of groups $A$ and $B$.
• $\zeta_n$ is a primitive $n$–th root of unity. Whenever two such are used in defining generators of the same group, the choice is assumed to be consistent, i.e. $\zeta_{ab} = \zeta_b$ for $a, b \in \mathbb{Z}$.

Note that the groups $\mathbb{V}_4$, $\mathbb{D}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic. The notation used for this group will denote the context in which I am considering it: $\mathbb{V}_4$ will be used whenever I am considering it as a subgroup of $\mathbb{A}_4$ or $\mathbb{S}_4$, and one of the other two will be used whenever I am considering it on its own.

1.2. Type of group actions. Throughout this section, let $H$ be a group acting on $\mathbb{C}^N$. From the more algebraic point of view, one can distinguish between several possible types of action of $H$:

Definition 1.1. The action of $H$ is transitive, if for any non–zero vector $x \in \mathbb{C}^N$, the $H$–orbit of $x$ spans $\mathbb{C}^N$. Otherwise the action is intransitive.

Example 1.2. Put the coordinates of $\mathbb{C}^4$ into the form of a $2 \times 2$ matrix via

$$(x, y, u, v) \mapsto \begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

Now, given $A, B \in \text{SL}_2(\mathbb{C})$, can define a linear action on this space via

$$[A, B] : \begin{pmatrix} x & y \\ u & v \end{pmatrix} \mapsto A \begin{pmatrix} x & y \\ u & v \end{pmatrix} B^T$$

Note that this action preserves a quadric in $\mathbb{P}^3$ defined by the vanishing of the determinant of the matrix form.

For $m, n$ odd positive integers, consider the action of a central extension of the group $(\mathbb{Z}_n \times \mathbb{Z}_m) \times \mathbb{D}_4$ generated by:

$$\left[ \begin{pmatrix} \zeta_{4n}^2 & 0 \\ 0 & \zeta_{4n}^{-2} \end{pmatrix}, \text{id} \right], \left[ \text{id}, \begin{pmatrix} \zeta_{4m}^2 & 0 \\ 0 & \zeta_{4m}^{-2} \end{pmatrix} \right],$$

$$\left[ \begin{pmatrix} \zeta_{4n}^2 & 0 \\ 0 & \zeta_{4n}^{-2} \end{pmatrix}, \begin{pmatrix} \zeta_{4m}^m & 0 \\ 0 & \zeta_{4m}^{-m} \end{pmatrix} \right], \left[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]$$

(where $\zeta_k$ is a primitive $k$–th root of unity). This action is intransitive, since it preserves the linear subspaces defined by $x = v = 0$ and by $y = u = 0$. 
Example 1.3. In the notation of Example 1.2, consider the action of a central extension of the group $(\mathbb{Z}_n \times \mathbb{Z}_m) \rtimes D_4$ generated by:

\[
\begin{bmatrix}
(\zeta_{4n}^2 & 0 \\
0 & \zeta_{4n}^{-2}
\end{bmatrix}, \text{id}, \begin{bmatrix}
(\zeta_{4m}^2 & 0 \\
0 & \zeta_{4m}^{-2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
(\zeta_{4n}^n & 0 \\
0 & \zeta_{4n}^{-n}
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
\zeta_{4m}^m & 0 \\
0 & \zeta_{4m}^{-m}
\end{bmatrix}
\]

(cf. Lemma 4.3, 10). It can easily be checked that this action is transitive.

Although the groups in Examples 1.2 and 1.3 look very similar, both being central extensions of quotients of $D_{4n} \times D_{4m}$ by normal subgroups isomorphic to $D_4$, only one of them is transitive and hence interesting for the purposes of this paper.

Definition 1.4. The action of $H$ is primitive if there does not exist a decomposition of $\mathbb{C}^N$ into proper linear subspaces, such that the action of $H$ permutes the subspaces. If such a decomposition does exist, then the action is imprimitive.

Example 1.5. In the notation of Example 1.2, consider the action of a central extension of $(V_4 \times V_4) \rtimes S_3$ generated by:

\[
\begin{bmatrix}
(\zeta_8^2 & 0 \\
0 & -\zeta_8^2
\end{bmatrix}, \text{id}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \text{id}, \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & \zeta_8
\end{bmatrix}, \begin{bmatrix}
1 & 0 \\
0 & -\zeta_8
\end{bmatrix}, \begin{bmatrix}
\zeta_8^7 & \zeta_8 \\
\zeta_8^5 & \zeta_8
\end{bmatrix}, \begin{bmatrix}
\zeta_8^7 & \zeta_8 \\
\zeta_8^5 & \zeta_8
\end{bmatrix}, \begin{bmatrix}
0 & \zeta_8 \\
-\zeta_8^7 & 0
\end{bmatrix}, \begin{bmatrix}
0 & \zeta_8 \\
-\zeta_8^7 & 0
\end{bmatrix}
\]

(cf. Lemma 4.3, 7). This group is imprimitive, since its action permutes the four 1-dimensional linear subspaces spanned by $(1,0,0,1)$, $(1,0,0,-1)$, $(0,1,1,0)$ and $(0,1,-1,0)$ (in the basis used above).

Example 1.6. Take the group generated by the group in Example 1.5 and an additional element

\[
\begin{bmatrix}
1/\sqrt{2} & \zeta_8^7 & \zeta_8 \\
\zeta_8^3 & \zeta_8 & \zeta_8
\end{bmatrix}, \text{id}
\]

turning the group in the previous example into a central extension of $(A_4 \times A_4) \rtimes \mathbb{Z}_2$ (cf. Lemma 4.3, 2). Then it can easily be checked that the subspace decomposition given in 1.5 is no longer valid, and in fact this group action is primitive.
Definition 1.7. A transitive imprimitive group \( H \) is monomial, if any irreducible representation of \( H \) is induced from a linear representation of some subgroup \( H' \leq H \). Otherwise it is non–monomial.

Example 1.8. In the notation of Example 1.2 consider the action a central extension of the group \(((\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2\) generated by transposing the matrix form of \( C_4 \) and

\[
\begin{bmatrix}
\zeta_8^2 & 0 \\
0 & -\zeta_8^2
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad \text{id}, \\
\text{id}, \quad \begin{bmatrix}
\zeta_8^2 & 0 \\
0 & -\zeta_8^2
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \\
\frac{1}{\sqrt{2}} \begin{bmatrix}
\zeta_8^5 & \zeta_8^7 \\
\zeta_8 & \zeta_8^5
\end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix}
\zeta_8^5 & \zeta_8^7 \\
\zeta_8 & \zeta_8^5
\end{bmatrix}
\]

Take 4 distinct lines in \( \mathbb{C}^4 \), spanned by vectors \((1,0,0,1)\), \((1,0,0,-1)\), \((0,1,1,0)\) and \((0,1,-1,0)\) (in the basis used above). One can check directly that the group’s action is induced from the action of its subgroup that fixes these four subspaces. Therefore it is monomial.

The isomorphism class of this group can be given either as \((\mathbb{Z}_2)^2 \rtimes S_4\) (see Example 4.4) or as \(((\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2\) (see Lemma 4.3, 16).

Example 1.9. In the notation of Example 1.2 consider the action of a central extension of the group \((\mathbb{Z}_n \times \mathbb{A}_4) \rtimes \mathbb{Z}_2\) generated by:

\[
\begin{bmatrix}
\text{id}, & \begin{bmatrix}
\zeta_8^2 & 0 \\
0 & -\zeta_8^2
\end{bmatrix} \\
\text{id}, & \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\end{bmatrix}, \\
\begin{bmatrix}
\text{id}, & \begin{bmatrix}
\zeta_8^2 & 0 \\
0 & -\zeta_8^2
\end{bmatrix} \\
\frac{1}{\sqrt{2}} \begin{bmatrix}
\zeta_8^5 & \zeta_8^7 \\
\zeta_8 & \zeta_8^5
\end{bmatrix}, \quad \begin{bmatrix}
\zeta_{2n} & 0 \\
0 & \zeta_{2n}^\ast
\end{bmatrix}, \quad \text{id}, \\
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & \zeta_8 \\
-\zeta_8^7 & 0
\end{bmatrix}
\]

(cf. Lemma 4.3, 4). The action of this group is not primitive, since it permutes the 2–dimensional subspaces \( x = y = 0 \) and \( u = v = 0 \), but neither is it monomial, since one cannot subdivide \( \mathbb{C}^4 \) into four 1–dimensional linear subspaces that the group permutes.

To summarise, the action of \( H \) must belong to exactly one of the following types:

- Intransitive
- Transitive
  - Primitive
  - Imprimitive
    - Monomial
    - Non–monomial
1.3. **Types of singularities.** From the more geometric point of view, it is more interesting to look at the variety $\mathbb{C}^n/G$. This variety has a singularity at $G(0)$, which can be categorised as follows:

**Definition 1.10** (see [6, Definition 1.14]). Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity. The singularity is said to be exceptional if for every effective $\mathbb{Q}$-divisor $D_V$ on the variety $V$ one of the following two conditions is satisfied:

- either the log pair $(V, D_V)$ is not log canonical,
- or for every resolution of singularities $\pi : U \rightarrow V$ there exists at most one $\pi$-exceptional divisor $E_1 \subset U$ such that

$$K_U + D_U \sim_{\mathbb{Q}} \pi^*(K_V + D_V) + a(V, D_V, E_1) E_1 + \sum_{i=2}^{r} a(V, D_V, E_i) E_i,$$

where $a(V, D_V, E_1) = -1$, while $E_2, \ldots, E_r$ are $\pi$-exceptional divisors different from $E_1$, and $D_U$ is the proper transform of the divisor $D_V$ on the variety $U$.

**Proposition 1.11** (see [16, Proposition 2.1]). Let $G < SL_N(\mathbb{C})$ be a finite subgroup that induces an exceptional singularity. Then $G$ is primitive.

**Corollary 1.12.** For any given $N$, only finitely many finite subgroups of $SL_N(\mathbb{C})$ induce exceptional singularities.

One can define a larger class of singularities by:

**Definition 1.13.** Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and let $\pi : W \rightarrow V$ be a birational morphism, such that the following hypotheses are satisfied:

- the exceptional locus of $\pi$ consists of one irreducible divisor $E$ such that $O \in \pi(E)$,
- the log pair $(W, E)$ has purely log terminal singularities.
- the divisor $-E$ is a $\pi$-ample $\mathbb{Q}$-Cartier divisor.

We then say that $\pi : W \rightarrow V$ is a plt blow up of the singularity $(V \ni O)$.

**Theorem 1.14** (see [6, Theorem 3.7]). The birational morphism $\pi : W \rightarrow V$ does exist.

**Definition 1.15** (see [6, Definition 3.8]). We say that $(V \ni O)$ is weakly-exceptional if it has a unique plt blow up.

**Corollary 1.16** (see [6, Corollary 3.13]). If $(V \ni O)$ is exceptional, then $(V \ni O)$ is weakly-exceptional.
Theorem 1.17 (see [6, Theorem 1.30]). Let $G < SL_N(\mathbb{C})$ be a finite subgroup that induces a weakly-exceptional singularity. Then $G$ is transitive.

This result will be used frequently throughout this paper, since for any finite group $G$, it allows to severely limit the number of actions worth checking.

Example 1.18 ($\mathbb{P}^1$). As an example, one can consider the case $N = 2$. It is a well-known fact that the only finite subgroups of $Aut(\mathbb{P}^1)$ are:

- Cyclic: $\mathbb{Z}_n$, $n \geq 1$.
- Dihedral: $D_{2n} = \langle a, b \mid a^n = b^2 = id, bab = a^{-1} \rangle$ ($n \geq 2$).
- Polyhedral groups $A_4, S_4, A_5$.

Lifting their action to $SL_2(\mathbb{C})$, get the actions of the following groups ($\zeta_n$ is taken to be a primitive $n$–th root of unity):

- Binary cyclic group
  $$\overline{\mathbb{Z}}_n = \langle a \mid a^{2n} = 1 \rangle$$
  All its faithful representations are 1–dimensional, and are of the form $a \mapsto \zeta_{2n}^l$, some $l \in \mathbb{Z}$. Thus a 2–dimensional representation has to be a direct sum of two such.
- Binary dihedral group
  $$\overline{D}_{2n} = \langle a, b \mid a^n = b^2 = b^4 = 1, aba^{-1} = a^{-1} \rangle$$
  The suitable 2–dimensional representations of this group are indexed by different choices of $\zeta_{2n}$. They are:
  $$a \mapsto \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
- Binary tetrahedral group
  $$\overline{A}_4 = \langle \zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123) \rangle$$
  (using standard notation for elements of the symmetric group).
  Similarly to above, the suitable 2–dimensional representations of this group are determined by the choice of $\zeta_8$. They are:
  $$\zeta_4(12)(34) \mapsto \begin{pmatrix} \zeta_8^2 & 0 \\ 0 & -\zeta_8^2 \end{pmatrix}, \quad \zeta_4(14)(23) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta_4(234) \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^7 & \zeta_8 \end{pmatrix}$$
• **Binary octahedral group**

\[
\mathbb{S}_4 = \langle \zeta_4(12)(34), \zeta_4(14)(23), \zeta_4(123), \zeta_4(34) \rangle
\]

This group only has 2 suitable representations, each having a subrepresentation isomorphic to the representation of \( \mathbb{A}_4 \) that uses the same value of \( \zeta_8 \). The extra generator acts as

\[
\zeta_4(34) \mapsto \begin{pmatrix} 0 & \zeta_8 \\ -\zeta_8^2 & 0 \end{pmatrix}
\]

• **Binary icosahedral group**

\[
\mathbb{A}_5 = \langle \zeta_4(12345), \zeta_4(12)(34) \rangle
\]

\[
\zeta_4(12345) \mapsto \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & -\zeta_5^2 \end{pmatrix},
\zeta_4(12)(34) \mapsto \frac{1}{\sqrt{5}} \begin{pmatrix} -\zeta_5 + \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}
\]

One can see that these group actions are of the following types:

- The cyclic groups act intransitively.
- \( \mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5 \) have primitive actions
- Dihedral groups have imprimitive monomial actions.

On the geometric side, \( \mathbb{A}_4, \mathbb{S}_4, \mathbb{A}_5 \) give rise to exceptional singularities, the dihedral groups give rise to weakly–exceptional, but not exceptional singularities. The cyclic groups give rise to singularities that are not weakly–exceptional. For the proofs of these results, see [19].

**Corollary 1.19.** In the 2–dimensional case, the exceptional singularities are those of types \( E_6, E_7, E_8 \). The weakly–exceptional singularities are those of types \( E_6, E_7, E_8, D_n \).

Since this example fully settles the \( N = 2 \) case, assume for the rest of this paper that \( N \geq 3 \). When the dimension \( N \leq 6 \), the exceptional singularities have been fully classified in [6], [7] and [14]. The aim of this paper is to obtain a list of group actions that induce weakly–exceptional singularities in dimension 3 and 4. This will be done using the exceptionality criteria obtained by Y. Prokhorov, I. Cheltsov and C. Shramov in [6] and [16], using the language of Tian’s alpha–invariant introduced in [21] and [22] by G. Tian and S-T. Yau.

For dimension \( N = 3 \), the classification of finite subgroups of \( \text{SL}_3(\mathbb{C}) \) is a classical result by G. Miller, H. Blichfeldt and L. Dickson (see [15]). A modern exposition of this result has been made by S. Yau and Y. Yu (see [23]). The first main result of this paper uses the classification to obtain an explicit list of groups inducing weakly–exceptional singularities:
Theorem 1.20. Let $G < SL_3(\mathbb{C})$ be a transitive finite subgroup. Then $G$ induces a weakly–exceptional singularity if and only if one of the following holds:

- $G$ is conjugate to a monomial group, that is not isomorphic to a central extension of $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3$ or $(\mathbb{Z}_2)^2 \rtimes S_3$.
- $G$ is conjugate to an irreducible faithful action of a central extension of one of:
  - $A_6$,
  - Klein’s simple group $K_{168}$ of order 168,
  - the Hessian group $H_{648}$ of order 648,
  - the normal subgroup $F_{216} \triangleleft H_{648}$ of order 216,
  - the normal subgroup $E_{108} \triangleleft F_{216}$ of order 108.

Remark 1.21. This is a full classification, since a group that induces a weakly–exceptional singularity must act transitively (see Theorem [1.17]).

Section 3 of this paper is devoted to proving this theorem. The result settles the 3–dimensional case, complimenting the earlier result by D. Markushevich and Y. Prokhorov:

Theorem 1.22 (see [14]). In the notation of Theorem [1.20], $G$ induces an exceptional singularity if and only if it is conjugate to an irreducible faithful action of a central extension of $A_6$, $K_{168}$, $H_{648}$ or $F_{216}$.

Since 4 is not a prime number, $SL_4(\mathbb{C})$ contains significantly more finite subgroups than $SL_3(\mathbb{C})$ does. The list of finite primitive subgroups is a classical result, that can be found in H. Blichfeldt’s book (see [2]). The list of transitive imprimitive finite subgroups can be obtained from papers by D. Flannery (see [9]) and B. Höflich (see [10]). In order to avoid reproducing the tables of groups from the works above, I will instead produce a list of transitive group actions that give rise to singularities that are not weakly exceptional. Therefore, the second main result of this paper is:

Theorem 1.23. If $G < SL_4(\mathbb{C})$ is a finite transitive subgroup whose action induces a singularity that is not weakly–exceptional, then $G$ must be conjugate to a central extension of one of (using the notation of Section 4.2 where appropriate):

- A primitive group that is one of:
  - $S_5$, whose action preserves the diagonal cubic surface.
  - $\mathbb{Z}_2 \rtimes A_5$, acting as the third symmetric power of its irreducible faithful 2–dimensional representation.
  - Twisted diagonal group $(A_5, 1, A_5, 1)_{\alpha} \cong A_5$. 

- 9 groups \((H_1, H_1, H_2, H_2) \cong H_1 \times H_2\), for different choices of \(H_1, H_2 \in \{A_4, S_4, A_5\}\)
- 3 families of groups isomorphic to one of: \((A_4 \times A_4) \times \mathbb{Z}_2\), \((S_4 \times S_4) \times \mathbb{Z}_2\), \((A_5 \times A_5) \times \mathbb{Z}_2\).
- Groups isomorphic to \(A_5 \times \mathbb{Z}_2\).
- \(\frac{1}{2}[S_3 \times S_1] \cong (A_4 \times A_4) \times \mathbb{Z}_2 \cong (S_4, A_4, S_4, A_4)\).
- \(((A_4 \times A_4) \times \mathbb{Z}_2) \times \mathbb{Z}_2 \cong (S_4, A_4, S_4, A_4) \times \mathbb{Z}_2\).

- An imprimitive non-monomial group that is one of:
  - 3 families of groups \(D_{2m} \times H_2\), where \(H_2 \in \{A_4, S_4, A_5\}\)
  - \(\frac{1}{2}[D_{2m} \times S_1] \cong (Z_m \times A_4) \times \mathbb{Z}_2 \cong (D_{2m}, Z_m, S_4, A_4)\).
  - \(\frac{1}{2}[D_{4m} \times S_1] \cong (D_{2m} \times A_4) \times \mathbb{Z}_2 \cong (D_{4m}, D_{2m}, S_4, A_4)\) for \(m \geq 2\).
  - \(\frac{1}{6}[D_{6m} \times S_4] \cong (Z_m \times V_4) \times S_3 \cong (D_{6m}, Z_m, S_4, V_4)\).

- A monomial group that is one of:
  - \(((Z_3)^3 \times \mathbb{Z}_2) \times \mathbb{Z}_2\), acting as shown in Lemma 4.2
  - \((Z_3)^3 \times S_4\), acting as shown in Lemma 4.2
  - Twisted diagonal groups \((H_1, 1, H_1)_\alpha \cong H_1\) isomorphic to \(A_4\) or \(S_4\).
  - 1 family of groups \(D_{2m} \times D_{2n}\)
  - \(D_{2m} \times D_{2n} \times \mathbb{Z}_2\)
  - Groups isomorphic to \(D_{4n} \times \mathbb{Z}_2\).
  - Groups isomorphic to \(A_4 \times \mathbb{Z}_2\) or \(S_4 \times \mathbb{Z}_2\).
  - \(\frac{1}{3}[A_4 \times A_4] \cong (V_4 \times V_4) \times S_3 \cong (S_4, V_4, A_4, V_4)\).
  - \(\frac{1}{2}[D_{2m} \times D_{4n}] \cong (Z_m \times D_{2n}) \times \mathbb{Z}_2 \cong (D_{2m}, Z_m, D_{4n}, D_{2n})\) \(m, n \geq 2\).
  - \(\frac{1}{4}[D_{4m} \times D_{4n}]_\alpha \cong (Z_m \times Z_n) \times D_4 \cong (D_{4m}, Z_m, D_{4n}, Z_n)_\alpha\) \((\text{where } \alpha(b) = a_{2m}^m, \alpha(a_{2m}^m) = b)\).
  - \(\frac{1}{2}[D_{4m} \times D_{4n}] \cong (D_{2m} \times D_{2n}) \times \mathbb{Z}_2 \cong (D_{4m}, D_{2m}, D_{4n}, D_{2n})\) \(m, n \geq 2\).
  - \(((V_4 \times V_4) \times S_3) \times \mathbb{Z}_2 \cong (S_4, V_4, S_4, V_4) \times \mathbb{Z}_2\).
  - \(((V_4 \times V_4) \times Z_3) \times \mathbb{Z}_2 \cong (A_4, V_4, A_4, V_4) \times \mathbb{Z}_2\).
  - \(((Z_m \times Z_m) \times D_4) \times \mathbb{Z}_2 \cong (D_{4m}, Z_m, D_{4m}, Z_m)_\alpha \times \mathbb{Z}_2\) \((\text{where } \alpha(b) = a_{2m}^m, \alpha(a_{2m}^m) = b)\).
  - \(((D_{2m} \times D_{2m}) \times \mathbb{Z}_2) \times \mathbb{Z}_2 \cong (D_{4m}, D_{2m}, D_{4m}, D_{2m}) \times \mathbb{Z}_2\) for \(m \geq 2\).

**Remark 1.24.** As in the 3–dimensional case, this directly implies a full classification, since any group inducing a weakly–exceptional singularity must act transitively (by Theorem 1.17).
Section 4 is devoted to proving this result. This fully settles the 4–dimensional case, since the list of finite subgroups of $SL_4(\mathbb{C})$ inducing exceptional singularities is known and can be found in a paper by I. Cheltsov and C. Shramov (see [6]).

2. Preliminaries

2.1. The short exact sequence. Let $\overline{G} < PGL_N(\mathbb{C})$ be a finite subgroup, and $G$ be its lift to $SL_N(\mathbb{C})$. Assume that $S \subset \mathbb{P}^{N-1}$ is a proper $\overline{G}$–invariant subvariety. Then there exists a natural homomorphism $\pi_S : G \to Aut(S)$, so get a short exact sequence

$$0 \to G_0 \to G \to G_S \to 0$$

where $G_S = \pi_S(G)$, $G_0 = \ker \pi_S$.

Proposition 2.1. Let $G_0 < SL_N(\mathbb{C})$ be a finite subgroup, and $\overline{G}_0$ the image of its natural embedding into $PGL_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a subvariety that is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{N-1}$, and assume $\overline{G}_0$ fixes $S$ pointwise. Then $\overline{G}_0$ is trivial.

Proof: Pick $g \in G_0$. Then $<g>$ is a finite abelian group, and so (in some basis for $\mathbb{C}^4$) consists of diagonal matrices. Let $\overline{g}$ be the image of $g$ under the natural projection into $PGL_4(\mathbb{C})$.

Let $e_1, \ldots, e_N \in S$ be distinct points, such that their lifts $e_1, \ldots, e_n$ to $\mathbb{C}^N$ span all of $\mathbb{C}^N$ (these exist, since $S$ is not contained in a proper linear subspace of $\mathbb{P}^{N-1}$). Then $e_i$ are eigenvectors of $g$, and let $\lambda_i$ be the corresponding eigenvalues. Reordering $e_i$–s if necessary, let $1 \leq m \in \mathbb{Z}$ be such that $\lambda_1 = \ldots = \lambda_m$ and $\lambda_n \neq \lambda_m \forall m < n \leq N$.

Assume $m < N$. Then take $A \subset \mathbb{C}^N$ to be the linear subspace spanned by $e_1, \ldots, e_m$ and $B \subset \mathbb{C}^N$ to be the linear subspace spanned by $e_{m+1}, \ldots, e_N$. Let $\overline{A}, \overline{B}$ be their natural projections into $\mathbb{P}^{N-1}$. These are proper linear subspaces, so $\exists \overline{p} \in S \setminus (\overline{A} \cup \overline{B})$.

This means there are at least $1 + N - m$ distinct linear eigenspaces for $g$ not contained in $A$, at least one of which is not contained in $B$ either. Therefore must have $\lambda_n = \lambda_m$ for some $m < n \leq N$, contradicting the choice of $m$.

This means $m = N$, and so $g$ is a scalar matrix. \qed

Corollary 2.2. Let $G < SL_N(\mathbb{C})$ and let $\overline{G}$ be its natural projection into $PGL_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a $\overline{G}$–invariant subvariety, that is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{N-1}$. Let $\pi_s : G \to Aut(S)$ be the natural homomorphism. Then $\ker \pi_S = 0$. 
Remark 2.3. If $S \subset \mathbb{P}^{n-1}$ is an irreducible surface, then either it is contained in an $(N - 2)$–dimensional linear subspace of $\mathbb{P}^{n-1}$ or it is not contained in the union of any two proper linear subspaces of $\mathbb{P}^{n-1}$.

2.2. Exceptionality criteria.

Definition 2.4 (see [6]). Let $X$ be a smooth Fano variety (see [12]) of dimension $n$, and let $g = g_{ij}$ be a Kähler metric, such that
\[
\omega = \frac{-1}{2\pi} \sum g_{ij} d\bar{z}_i \wedge dz_j \in c_1 (X)
\]
Let $G \leq \text{Aut} (X)$ be a compact subgroup, such that $g$ is $G$–invariant. Let $P_G (X, g)$ be the set of $C^2$ smooth $G$–invariant functions, such that $\forall \phi \in P_G (X, g)$,
\[
\omega + \frac{-1}{2\pi} \partial \bar{\partial} \phi > 0
\]
and $\sup_X \phi = 0$. Then the $G$–invariant $\alpha$–invariant of $X$ is
\[
\alpha_G (X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \exists C \in \mathbb{R}, \text{ such that } \forall \phi \in P_G (X, g), \int_X e^{-\lambda \phi} \omega^n \leq C \right\}
\]
where $n$ is the dimension of $X$.

The number $\alpha_G (X)$ was introduced in [21] and [22].

Definition 2.5 (see [6, Definition 2.1]). Let $X$ be a variety with at most Kawamato log terminal singularities (see [13, Definition 3.5]) and $D$ an effective $\mathbb{Q}$–divisor on $X$. Let $Z \subseteq X$ be a closed non–empty subvariety. Then the log canonical threshold of $D$ along $Z$ is
\[
c_Z (X, D) = \sup \{ \lambda \in \mathbb{Q} \mid \text{the pair } (X, \lambda D) \text{ is log canonical along } Z \}
\]
To simplify notation, write $c_X (X, D) = c (X, D)$.

There exists an equivalent complex analytic definition of the log canonical threshold:

Proposition 2.6 (see [13, Proposition 8.2]). Let $X$ be a smooth complex variety, $Z$ a closed non–empty subscheme of $X$, and $f$ a non–zero regular function on $X$. Then
\[
c_Z (X, f) = \sup \{ \lambda \in \mathbb{Q} \mid |f|^{-\lambda} \text{ is locally } L^2 \text{ near } Z \}
\]

Remark 2.7. There exists a very similar definition for real varieties. For more details, see [1, Section 7.2.2] and [17].
Definition 2.8 (see [6, Definition 3.2]). Let $G$ be a finite subgroup in $GL_N(\mathbb{C})$, where $N \geq 2$, and let $G$ be its image under the natural projection into $PGL_N(\mathbb{C})$. Then the global $\bar{G}$–invariant log canonical threshold of $\mathbb{P}^{N-1}$ is:

$$lct(\mathbb{P}^{N-1}, \bar{G}) = \inf \left\{ c(\mathbb{P}^{N-1}, D) \mid D \text{ is a } \bar{G} \text{–invariant } \mathbb{Q} \text{–Cartier effective } \mathbb{Q} \text{–divisor on } \mathbb{P}^{N-1}, \right. \left. \text{such that } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^{N-1}} \right\}$$

Remark 2.9 (see [5, Theorem A.3]). $lct(\mathbb{P}^{N-1}, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^{N-1})$

Theorem 2.10 (see [6, Theorem 3.15]). The singularity $(V \ni O)$ is weakly–exceptional $\iff lct(\mathbb{P}^{N-1}, \bar{G}) \geq 1$, where $N = \dim V$.

Remark 2.11. A similar condition is often necessary in order to compute groups of birational automorphisms of direct products of varieties. For example, see [4, Theorem 6.5].

Theorem 2.12 (see [6, Theorem 3.18]). Suppose that $G < SL_3(\mathbb{C})$ is a finite group and $G$ is its image under the natural projection into $PGL_3(\mathbb{C})$. Then the following are equivalent:

- the inequality $lct(\mathbb{P}^2, \bar{G}) \geq 1$ holds,
- the group $G$ does not have semi-invariants of degree at most 2.

Theorem 2.13 (see [6, Theorem 4.1]). Suppose that $G < SL_4(\mathbb{C})$ is a finite group and $G$ is its image under the natural projection into $PGL_4(\mathbb{C})$. Then the inequality $lct(\mathbb{P}^3, \bar{G}) \geq 1$ holds if and only if the following conditions are satisfied:

- the group $G$ is transitive,
- the group $G$ does not have semi-invariants of degree at most 3,
- there is no $\bar{G}$–invariant smooth rational cubic curve in $\mathbb{P}^3$.

Equivalently, the same criterion can be stated as:

Theorem 2.14. Suppose that $G < SL_4(\mathbb{C})$ is a finite group and $\bar{G}$ is its image under the natural projection into $PGL_4(\mathbb{C})$. Then the inequality $lct(\mathbb{P}^3, \bar{G}) \geq 1$ holds if and only if the following conditions are satisfied:

- the group $G$ is transitive,
- the group $G$ does not have semi-invariants of degree at most 3,
- $G$ is not the central extensions of $A_5$ acting as the third symmetric power of its irreducible 2–dimensional representation.

Proof: Assume $\bar{G} < PGL_4(\mathbb{C})$ is a finite transitive subgroup that does not fix any quadric or cubic surface, and $C \subset \mathbb{P}^3$ is a smooth rational $\bar{G}$–invariant cubic curve.
It is a well-known fact that any smooth rational cubic curve is projectively isomorphic to the rational normal curve, which is the image of

\[ C : (x : y) \in \mathbb{P}^1 \mapsto (x^3 : x^2y : xy^2 : y^3) \in \mathbb{P}^3 \]

This means that \( \bar{G} \) must be isomorphic to one of the finite automorphism groups of \( \mathbb{P}^1 \), with its action induced by the action of \( \mathbb{P}^1 \) via \( C \). This means \( \bar{G} \) must be one of the following:

- Cyclic or dihedral group: neither these groups nor their central extensions by scalar matrices in \( \text{SL}_4(\mathbb{C}) \) have an irreducible 4-dimensional representation. Therefore such a \( G \) would be intransitive.
- \( A_4 \) or \( S_4 \): The induced action of these two groups preserves \( \{ (x : y : u : v) \in \mathbb{P}^3 \mid xy - uv = 0 \} \), which is a smooth quadric surface.
- \( A_5 \): This action is primitive. From the embedding of \( C \) into \( \mathbb{P}^3 \), it is easy to see that the action on \( \mathbb{C}^4 \) must correspond to the third symmetric power of the irreducible 2-dimensional representation of \( \mathfrak{A}_5 \), or its central extension.

\[ \Box \]

**Remark 2.15.** The last condition is necessary, since this action is transitive and does not preserve any projective surfaces of degree 2 or 3 (can be checked directly or seen in [6, Proof of Lemma 4.9]).

**Example 2.16** (cf. Example 4.5). Let

\[ S = \{ (x : y : u : v) \in \mathbb{P}^3 \mid x^m + y^m + u^m + w^m = 0 \}, \]

where \( m \geq 5 \). Let \( \bar{G} = \text{Aut}(S) \), and \( G \) be the lift of \( \bar{G} \) to \( \text{SL}_4(\mathbb{C}) \). Then \( G \) induces a weakly-exceptional singularity.

By Theorem 2.14, it is enough to prove that \( G \) is transitive, has no semiinvariants of degree at most 3, and \( \bar{G} \) does not leave a smooth rational cubic curve invariant. Let \( d \) be the minimal degree of a semiinvariant of \( G \).

It can be observed that \( G \) is a central extension of \( (\mathbb{Z}_m)^3 \rtimes \mathfrak{S}_4 \), acting by permuting the basis and multiplying the coordinates by \( m \)-th roots of unity. This means \( G \) is transitive, i.e. \( d > 1 \).

Consider \( \alpha : (x, y, u, v) \mapsto (\zeta_m x, \zeta_m^{-1} y, u, v) \). By applying \( \alpha \) and the element of \( G \) switching coordinates \( x \) and \( y \) to the general quadric or cubic polynomial in 4 variables, it is easy to see that any semiinvariant of degree 2 or 3 must be of the form \( xyp_0(u, v) + p_2(u, v) \) or \( xyp_1(u, v) + p_3(u, v) \) respectively, where \( p_n \) are polynomials of degree \( n \).
in two variables. Applying other elements of $G$ that permute the basis, one can easily deduce that neither of these can be semiinvariant of $G$. (By similar considerations, one can see that any semiinvariant of $G$ of degree 4 must be a scalar multiple of $xyuv$ which in fact is a semiinvariant of $G$).

It remains to prove that $\bar{G}$ does not leave any smooth rational cubic curve invariant. Assume it does, and call this curve $C$. Since $C$ is rational, its automorphism group is isomorphic to the automorphism groups of $\mathbb{P}^1$. Then let $\pi : \bar{G} \rightarrow \text{Aut}(C)$ be the natural homomorphism with image $\bar{G}_C$ and kernel $H \leq \bar{G}$. Then $\bar{G}_C \cong \bar{G}/H$. Checking all possible normal subgroups of $\bar{G}$ against the list of finite automorphism groups of $\mathbb{P}^1$, it is easy to see that $\bar{G}_C$ must be one of $S_4$, $S_3$ and $Z_2$. But that would mean that $\alpha$ (as defined above) leaves $C$ pointwise invariant, and hence $C$ must be contained in a proper linear subspace of $\mathbb{P}^3$, which $G$ would have to preserve, contradicting its transitivity. Therefore no such $C$ exists, and $G$ induces a weakly–exceptional singularity.

2.3. Effect of transitivity on an action’s possible invariants. Let $\bar{G} < \text{PGL}_{N-1}(\mathbb{C})$, $N > 2$, be a finite transitive group and $G$ its lift to $\text{SL}_N(\mathbb{C})$. Let $S \subset \mathbb{P}^{N-1}$ be a subvariety left invariant by all of $\bar{G}$.

**Remark 2.17.** In the notation above, the following hold:

- $\bar{G}$ cannot be cyclic, as in that case, in some basis for $\mathbb{C}^N$, $G$ must consist of diagonal matrices (as all irreducible representations of abelian groups are 1–dimensional), and hence would not be transitive.
- $S$ cannot be contained in a proper linear subspace of $\mathbb{P}^{N-1}$, since in that case $G$ would have to fix a non–empty proper linear subspace of $\mathbb{C}^N$ and so would not be acting transitively. In particular, $\bar{G}$ cannot fix any point, line (if $N \geq 3$) or degree 1 surface (if $N \geq 4$) in $\mathbb{P}^{N-1}$.

**Remark 2.18.** In the notation above, if $p \in S$ is an isolated singularity, then $\forall \bar{g} \in \bar{G}$, $\bar{g}(p) \in S$ is an isolated singularity of the same type. For example, if $S$ is a cubic surface in $\mathbb{P}^3$ and $p \in S$ is an $A_2$ singularity, then so is $\bar{g}(p)$ for every $\bar{g} \in \bar{G}$. Similarly, a line on $S$ must be mapped into another line on $S$.

**Lemma 2.19.** In the notation above, let $S$ have only isolated singularities. Then for any given singularity type, either $S$ has no singularities of this type or is has at least $N$ of them.
**Proof:** Let \( S \) have \( k \) (\( 1 \leq k < N \)) singularities of some given type, positioned at \( p_1, \ldots, p_k \). Since \( S \) is \( \bar{G} \)-invariant,
\[
\bar{G}\{p_1, \ldots, p_k\} = \{p_1, \ldots, p_k\}
\]
Therefore, \( p_i \) span a non-empty invariant linear subspace of \( \mathbb{P}^N \), which is proper since \( k < N \). This (by Remark 2.17) contradicts the transitivity of \( G \). \( \square \)

### 3. Three-Dimensional Case

Assume \( N = 3 \) throughout this section. This section is devoted to proving Theorem 1.20.

The list of all finite subgroups of \( \text{SL}_3(\mathbb{C}) \) have been given in [23]. It is:

**Proposition 3.1** (see [23, Theorem A]). *Up to conjugacy, any finite subgroup of \( \text{SL}_3(\mathbb{C}) \) belongs to one of the following types:*

**(A)** Diagonal abelian group.

**(B)** Group isomorphic to a transitive finite subgroups of \( \text{GL}_2(\mathbb{C}) \) and not conjugate to a group of type \( [A] \).

**(C)** Group generated by the group in \( [A] \) and \( T \) and not conjugate to a group of types \( [A] \) or \( [B] \).

**(D)** Group generated by the group in \( [C] \) and \( Q \) and not conjugate to a group of types \( [A] \)–\( [C] \).

**(E)** Group of order 108 generated by \( S \), \( T \) and \( V \).

**(F)** Group of order 216 generated by the group in \( [E] \) and an element \( P := UVU^{-1} \).

**(G)** Hessian group of order 648 generated by the group in \( [E] \) and \( U \).

**(H)** Simple group of order 60 isomorphic to alternating group \( A_5 \).

**(I)** Simple group of order 168 isomorphic to permutation group generated by \( (1234567) \), \( (142)(356) \), \( (12)(35) \).

**(J)** Group of order 180 generated by the group in \( [H] \) and \( W \).

**(K)** Group of order 504 generated by the group in \( [I] \) and \( W \).

**(L)** Group \( G \) of order 1080 with its quotient \( G/ <W> \) isomorphic to the alternating group \( A_6 \).

Where the matrices used are given in Figure 1.

These groups can be put into the form of the algebraic classification of groups given in the earlier sections as follows:

**Remark 3.2.** The list of groups in Proposition 3.1 can be subdivided as follows:

- Groups of types \( [A] \) and \( [B] \) are intransitive.
- Groups of types \( [E] \)–\( [L] \) are primitive.
Figure 1. Matrices used in the finite subgroup classification

\[ S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \omega \end{pmatrix}, \quad Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad P = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \quad W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \]

Where \( \omega = e^{2\pi i/3} \), \( \epsilon^3 = \omega^2 \) and \( abc = -1 \) in such a way that \( Q \) generates a finite group.

- Groups of types (C) and (D) are monomial.

Proof: Immediate from the lists of generators given in [23]. □

Let \( G < \text{SL}_3(\mathbb{C}) \) be a transitive finite subgroup with image \( \bar{G} \) under the natural projection into \( \text{PGL}_3(\mathbb{C}) \).

Lemma 3.3. Assume the singularity the action of \( G \) induces is not weakly–exceptional. Then:

- \( \bar{G} \) leaves a smooth conic \( C \subset \mathbb{P}^2 \) invariant.
- \( G \) is isomorphic to one of \( D_{2n}, A_4, S_4, A_5 \) (some \( n \geq 2 \)) or to one of their central extensions by scalar matrices.

Proof: By Theorem 2.12, \( \bar{G} \) must preserve a conic \( C \subset \mathbb{P}^2 \). If \( C \) is singular, then it must have exactly one isolated singularity, which is impossible by Lemma 2.19. Therefore, \( C \) must be smooth.

Any smooth conic in \( \mathbb{P}^2 \) is rational and is not contained in any proper linear subspace of \( \mathbb{P}^2 \), so by Proposition 2.1, \( \bar{G} \) must be one of the finite transitive subgroups of \( \text{Aut}(\mathbb{P}^1) \), and \( G \) must be its central extension. □

Proof of Theorem 1.20: Groups of types (A) and (B) can be excluded immediately, since they are not transitive. Assume the singularity \( G \) induces is not weakly–exceptional. Then, comparing the lists in Proposition 3.1 and Lemma 3.3, \( G \) must be conjugate to one of:

- A central extension of \( (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_3 \cong A_4 \), where both \( \mathbb{Z}_2 \)-s act diagonally and \( \mathbb{Z}_3 \) permutes the basis (such a group is of type (C)).
• A central extension of \((\mathbb{Z}_2)^2 \rtimes S_3 \cong S_4\), where both \(\mathbb{Z}_2\)-s act diagonally and \(S_3\) permutes the basis (such a group is of type (D)).

• A central extension of \(A_5\) (such a group is of type (H) or (J)).

Let the \(\mathbb{P}^2\) that \(G\) is acting on have coordinates \((x : y : z)\). In each of the cases, fix \(G\) to be the member of its conjugacy class, whose generators are given in [23]. Looking at the generators, it is immediate that the groups of types (C) and (D) in this list leave the smooth conic defined by \(x^2 + y^2 + z^2 = 0\) invariant. Similarly, the groups of types (H) or (J) leave the smooth conic defined by \(x^2 - yz = 0\) invariant.

The statement of Theorem 1.20 follows by renaming the remaining groups to fit with the more widely used notation. \(\square\)

4. Four-dimensional case

The aim of this section is to prove Theorem 1.23. Throughout the section, let \(G < SL_4(\mathbb{C})\) be a finite transitive subgroup, and \(\tilde{G}\) its image under the natural projection \(SL_4(\mathbb{C}) \rightarrow PGL_4(\mathbb{C})\).

In view of Theorem 2.14, the only transitive \(\tilde{G}\) that are not weakly exceptional are the finite subgroups of automorphism groups of surfaces of degree 2 or 3 and \(A_5\) with one specific action. When considering automorphisms of a surface \(S\), without loss of generality one can assume that there is no \(\tilde{G}\)-invariant surface \(S'\) of smaller degree.

Lemma 4.1. Let \(S \subset \mathbb{P}^3\) be a \(\tilde{G}\)-invariant surface of minimal degree. Then either \(\deg S \geq 4\) or \(S\) is smooth.

Proof: Since \(G\) is transitive, \(\deg S \geq 2\). If \(\deg S = 2\) and \(S\) is singular, then \(S\) has exactly 1 isolated singularity. This is impossible (by Lemma 2.19), so if \(\deg S = 2\) then \(S\) must be smooth.

If \(\deg S = 3\) and \(S\) has a singular curve \(C\), then, since the surface \(S\) is a \(\tilde{G}\)-invariant surface of minimal degree and hence irreducible, by [3, Case E] \(C\) is a line. Since \(\tilde{G}(S) = S\), \(\tilde{G}(C) = C\), and so there exists a \(\tilde{G}\)-invariant line, contradicting transitivity of \(G\). Therefore if \(\deg S = 3\) then \(S\) must have at worst isolated singularities.

If \(\deg S = 3\) and \(S\) is singular with only isolated singularities, then by [3], the singularity types form one of the following collections: \((A_1), (2A_1), (A_1, A_2), (3A_1), (A_1, A_3), (2A_1, A_2), (4A_1), (A_1, A_4), (2A_1, A_3), (A_1, 2A_2), (A_1, A_5)\). Since by Lemma 2.19 \(S\) cannot have exactly 1, 2 or 3 singularities of any given type, \(S\) has to have 4 \(A_1\) singularities. Since there is only one such surface (see, for example, [3]), \(S\) must be the Cayley cubic, defined by

\[
S = \{(x : y : u : v) \in \mathbb{P}^3 \mid xyu + xyv + xuv + yuv = 0\}
\]
But in this case $S$ contains exactly 9 lines, 6 of which are going through pairs of singular points and the other 3 defined by

$$x + y = 0 = u + v, \ x + u = 0 = y + v\ \text{and} \ x + v = 0 = y + u$$

These last three lines are coplanar and must be mapped to each other by all of $\bar{G}$. Therefore, $G$ preserves a plane, contradicting the transitivity assumption for $G$. Thus if $S$ is a cubic surface, then it must be smooth. \(\square\)

Summarising, if $\bar{G}$ is transitive but the singularity it induces is not weakly–exceptional, any $\bar{G}$–invariant surface $S$ must be a smooth surface of degree 2 or 3. These cases will be considered separately in the next two sections.

**Proof of Theorem 1.23**: By Theorem 2.14 and the discussion above, the theorem is an immediate consequence of Lemmas 4.2 and 4.3. \(\square\)

### 4.1. If $S$ is a smooth $\bar{G}$–invariant cubic surface

This section is devoted to proving the following lemma:

**Lemma 4.2.** If $G < \text{PGL}_4(\mathbb{C})$ is a finite transitive subgroup that does not fix any quadric surface, $G$ its lift to $\text{SL}_4(\mathbb{C})$ and $S \subset \mathbb{P}^3$ a smooth $\bar{G}$–invariant cubic surface, then $G$ must be isomorphic to a central extension of one of:

- $((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. This produces a monomial action.
- $(\mathbb{Z}_3)^3 \rtimes \text{S}_4$. This produces a monomial action.
- $\text{S}_5$. This produces a primitive action.

by scalar matrices, acting as described below.

As stated before, $\bar{G} < \text{Aut}(S)$ is a finite subgroup, so by [11], $\bar{G}$ must belong to one of the following isomorphism classes:

1. $\{e\}, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8$.
2. $(\mathbb{Z}_2)^2$
3. $\text{S}_3$, the symmetric group of degree 3.
4. $\text{S}_3 \times \mathbb{Z}_2$
5. $\text{S}_4$
6. $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$
7. $((\mathbb{Z}_3)^3 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
8. $\text{S}_5$
9. $(\mathbb{Z}_3)^3 \rtimes \text{S}_4$

4.1.1. Case $\bar{G}$ cyclic. The groups in (11) are all cyclic, so have a degree 1 invariant surface.
4.1.2. Case $\bar{G} \cong (\mathbb{Z}_2)^2$. This is a dihedral group of order 4, and neither it nor its central extension by a group of scalar matrices in $\text{SL}_4(\mathbb{C})$ have an irreducible 4–dimensional representation. Therefore this group does not act transitively.

4.1.3. Case $\bar{G} \cong S_3$. This is the dihedral group of order 6, and neither it nor its central extension by a group of scalar matrices in $\text{SL}_4(\mathbb{C})$ have a 4–dimensional irreducible representation. Therefore the action of this group is intransitive.

4.1.4. Case $\bar{G} \cong S_3 \times \mathbb{Z}_2$. This is the dihedral group of order 12, so, similarly to the case above, the corresponding action is intransitive.

4.1.5. Case $\bar{G} \cong S_4$. This group by itself has no 4–dimensional irreducible representations, while its central extension has (up to a choice of a root of unity) only one such, which preserves a quadric surface (see the twisted diagonal actions).

4.1.6. Cases $\bar{G} \cong (\mathbb{Z}_3)^2 \times \mathbb{Z}_2$ and $\bar{G} \cong ((\mathbb{Z}_3)^3 \times \mathbb{Z}_2) \times \mathbb{Z}_2$. Write $G'$ for the lift of $\bar{G}$ to $\text{SL}_4(\mathbb{C})$. As stated in [11], these two cases correspond to groups $\bar{G} = G'^{0}_{54}/C(\bar{G}'^{0}_{54})$ and $\bar{G}' = G'^{0}_{54} \times \mathbb{Z}_2$. This means there exist elements $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\epsilon} \in \bar{G}'$, such that $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ generate $G'^{0}_{54}$, with $\bar{\alpha}$ generating its centre, $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ of order 3, $\bar{\delta}, \bar{\epsilon}$ of order 2. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be lifts of $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ (respectively) to $\text{SL}_4(\mathbb{C})$.

Let $h_1 := \alpha^3, h_2 := \beta^3$. $\bar{\alpha}, \bar{\beta}$ commute, so say $\beta \alpha = \alpha \beta h_3$. By the structure of the lift, $h_i$ are scalar matrices of order 1, 2 or 4. Then

$$h_1^3 h_2 = (\alpha^2 \beta \alpha)^3 = (\beta h_1 h_3)^3 = h_2 h_1^3 h_3^3$$

and so $h_3 = \text{id}$. Similarly, get $\alpha, \beta, \gamma$ all commuting. Hence the corresponding matrices can all be taken to be diagonal (by choosing a suitable basis). It is then easy to see that $\delta$ and $\epsilon$ must act as elements of a central extension of $S_4$ permuting the basis.

Since $\bar{G}'$ has only one normal subgroup of index 2, and $\bar{G}'$ has no centre (otherwise $\bar{G}'/C(\bar{G}')$ would be on the list of groups acting on a cubic surface), $\bar{\delta} \bar{\epsilon} \neq \bar{\epsilon} \bar{\delta}$. Therefore, up to conjugation, $\delta$ interchanges the first and the second basis vectors, and $\epsilon$ interchanges the first basis vector with the third one and the second basis vector with the fourth one.
This means that $G$ is not transitive, while $G'$ is transitive and (up to conjugation) is generated by

$$
\begin{bmatrix}
\zeta_j & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \zeta_j^{-1}
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \zeta_j & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \zeta_j^{-1}
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \zeta_j & 0 \\
0 & 0 & 0 & \zeta_j^{-1}
\end{bmatrix}
$$

This group clearly leaves the cubic polynomial $x^3 + y^3 + z^3 + w^3$ (in coordinates $(x, y, z, w)$ for $\mathbb{C}^3$) semi-invariant. This action of $G'$ is monomial.

4.1.7. Case $\bar{G} \cong S_5$. According to [18], this is the automorphism group of the irreducible diagonal cubic surface

$$\begin{align*}
S &= \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4 \left| \begin{array}{l}
x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0, \\
x_0 + x_1 + x_2 + x_3 + x_4 = 0
\end{array} \right. \right\}
\end{align*}
$$

which is embeddable into $\mathbb{P}^3$ via

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 - x_1 : x_1 - x_2 : x_2 - x_3 : x_3 - x_4)$$

In this case $G$ acts by permuting the basis of the $\mathbb{C}^5$ containing the $\mathbb{P}^4$. This action descends to the relevant $\mathbb{C}^4$ as

$$
(12) \mapsto \zeta_8 \begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad
(2345) \mapsto \zeta_8^7 \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1
\end{bmatrix}
$$

where $\zeta_8$ is a primitive 8-th root of unity. One should note that this action is primitive and irreducible. Furthermore, the only other 4-dimensional irreducible representation of $S_5$ has a semi-invariant quadric (see the section on quadric surfaces).

4.1.8. Case $\bar{G} \cong (\mathbb{Z}_3)^3 \rtimes S_4$. As stated in [18], this group acts by permuting the basis vectors of $\mathbb{C}^4$ arbitrarily and multiplying them by arbitrary cube roots of unity. Hence (up to conjugation) $G$ is a central extension of such a group by scalar matrices.

This group clearly leaves the cubic polynomial $x^3 + y^3 + z^3 + w^3$ (in coordinates $(x, y, z, w)$) semi-invariant. The action of this group is monomial.
4.2. If $S$ is a smooth $\bar{G}$–invariant quadric surface. Let the group $G \subset \text{PGL}_4(\mathbb{C})$ be a finite irreducible transitive subgroup, $\bar{G}$ its lift to $\text{SL}_4(\mathbb{C})$ and $S \subset \mathbb{P}^3$ a smooth $\bar{G}$–invariant quadric surface. This section is devoted to creating a list of the possible values that $G$ (equivalently, $\bar{G}$) can take in this situation. The final list is presented in Lemma 4.3.

In this case there exists a basis for $\mathbb{P}^3$, in which $S$ is the image of the Veronese embedding $\mathcal{V} : ((a : b), (c : d)) \mapsto (ac : ad : bc : bd)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. The image of this embedding is

$$S = \{ (x : y : u : v) \in \mathbb{P}^3 \mid xv - yu = 0 \}$$

It is possible to present $\mathbb{P}^3$ as the set of non–zero $2 \times 2$ matrices. From now on, consider the “matrix form” of $\mathbb{P}^3$ to be equivalent to its standard presentation. Name the coordinates as follows:

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \equiv (x : y : u : v)$$

Then $S$ takes the form of the set of matrices with zero determinant, and the automorphism group of $S$ is the group of operations on the $2 \times 2$ matrix leaving the determinant semiinvariant. $\bar{G}$ must then be conjugate to the image of a finite subgroup $\hat{H}$ of this group.

$\mathbb{P}^1 \times \mathbb{P}^1$ has exactly two rulings, which $H$ can either preserve or interchange. Consider the short exact sequence

$$1 \longrightarrow H \longrightarrow \hat{H} \longrightarrow \mathbb{S}_2 \longrightarrow 1$$

where $\mathbb{S}_2$ permutes the two rulings. Then $H \leq \hat{H}$ is the maximal subgroup that preserves the ruling. Let $\tau \in \hat{H} \setminus H$ if this set is non–empty. Note that if $\tau$ exists, it can be chosen to be an involution. Let $\pi_i : H \rightarrow H_i$ be the projections of $H$ on the two components of $\mathbb{P}^1 \times \mathbb{P}^1$. Then have two more short exact sequences

$$1 \longrightarrow K_2 \longrightarrow H \longrightarrow H_2 \longrightarrow 1$$

$$1 \longrightarrow K_1 \longrightarrow H \longrightarrow H_1 \longrightarrow 1$$

It is clear that $K_1 \cap K_2 = \{1\}$. Therefore, for $i \neq j$ ($i, j \in \{1, 2\}$),

$$K_i \cong \hat{K}_i := \{ kK_j \mid k \in K_i \} \leq H/K_j = H_i$$

In this notation, $H_1/\hat{K}_1 \cong H/(K_1K_2) \cong H_2/\hat{K}_2$, so the group can be defined completely by the $5$–tuple $(H_1, K_1, H_2, K_2, \alpha)$, where $\alpha$ is an isomorphism $H_1/\hat{K}_1 \rightarrow H_2/\hat{K}_2$. In return, if $H$ is known, one can reconstruct $\alpha$ by making it map $\pi_1(a)\hat{K}_1 \mapsto \pi_2(a)\hat{K}_2$.

In the matrix form described above, $H_1$ acts on $\mathbb{P}^3$ by left matrix multiplication, and $H_2$ acts by transposed right matrix multiplication.
The involution switching the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to transposing the matrix. If $h_1 \in H_1$ acts on the first component of $\mathbb{P}^1 \times \mathbb{P}^1$ and $h_2 \in H_2$ acts on the second one, then write $[h_1, h_2]$ to denote this action. Explicitly, for any $2 \times 2$ matrices $A$ and $B$,

$$[A, B] \left( \begin{pmatrix} x & y \\ u & v \end{pmatrix} \right) = A \begin{pmatrix} x & y \\ u & v \end{pmatrix} B^T$$

It is worth noting that if $\tau$ exists, then conjugation by $\tau$ provides isomorphisms $H_1 \cong H_2$ and $K_1 \cong K_2$.

4.2.1. **Types of group action.** The actions of the finite automorphism groups of the quadric surface on $\mathbb{C}^4$ will need to be put into one of the following four categories:

- Intransitive.
- Transitive monomial.
- Transitive non–monomial imprimitive.
- Primitive (hence transitive).

This can be done directly by looking at the representations used to build the action. To make the explanations more simple, it will be assumed that $H_1$ and $H_2$ both contain a non–scalar diagonal matrix. This will mean that any proper invariant subspace must have a basis, which is a subset of the chosen basis for $\mathbb{C}^4$. The representations used are all listed in Example 1.18, and clearly each of them contains such a matrix. Now fix this basis for the remainder of this section.

4.2.2. **Transitivity:** Assume first that $\tau$ (an element of $\hat{H}$ interchanging the rulings) does not exist. Then it can be seen that in order for $G$ to be transitive, both $H_1$ and $H_2$ need to be transitive: If $H_1$ is intransitive, then (in the matrix presentation of $\mathbb{P}^3$) the rows of $\mathbb{P}^3$ give invariant lines, i.e.

$$\{x = y = 0\}, \{u = v = 0\} \subset \{(x : y : u : v) \in \mathbb{P}^3\}$$

are invariant subspaces. Similarly, if $H_2$ is intransitive, then there exist $H$–invariant lines corresponding to the columns of the matrix, i.e.

$$\{x = u = 0\}, \{y = v = 0\} \subset \{(x : y : u : v) \in \mathbb{P}^3\}$$

are invariant subspaces. Even when both $H_i$ are transitive, $G$ can still be intransitive, as illustrated in Examples 1.2 and 1.3. However, this is essentially the only configuration where this happens.

This means that for the action of $G$ to be transitive, need (for any $i \neq j \in \{1, 2\}$) one of the following to hold:

- $H_i, H_j \in \{A_4, S_4, A_5\}$
- $H_i \in \{A_4, S_4, A_5\}, H_j = D_{2n}$
• $H_1 = \mathbb{D}_{4m}$, $H_2 = \mathbb{D}_{4n}$ and the action contains elements of the form $[a^l, b]$ or $[b, a^k]$, not just $[a^l b, a^k b]$ ($k, l \in \mathbb{Z}$).

If an involution $\tau$ exists, then it must keep one of the diagonals of the matrix form of $\mathbb{P}^3$ invariant, i.e.

$$\{x = v = 0\}, \{y = u = 0\} \subset \{(x : y : u : v) \in \mathbb{P}^3\}$$

are left invariant by $\tau$. However, if $\tau$ exists, then $H_1 \cong H_2$, so the only two intransitive cases where it can potentially make the action transitive are when $H_1 \cong H_2 \cong \mathbb{D}_{2n}$ (the action remains intransitive, as $[a^l b, a^k b]$ leaves the same two proper subspaces invariant as $\tau$ does) or when $H_1 \cong H_2 \cong \mathbb{Z}_n$ (the action remains intransitive, as $\tau$ only permutes pairs of coordinates, so the group has a pair of distinct invariant proper subspaces).

The discussion above has been summarised in Table 1.

| $H_1$ | $H_2$ | $A_4, S_4, A_5$ | $D_{2m}$ | $\mathbb{Z}_m$ |
|-------|-------|------------------|----------|--------------|
| $A_4, S_4, A_5$ | Transitive | Transitive | Intransitive |
| $D_{2n}$ | Transitive | Depends on action | Intransitive |
| $\mathbb{Z}_n$ | Intransitive | Intransitive | Intransitive |

4.2.3. **Primitivity**: Assume the action of $G$ is transitive. Then the question of the action being monomial, imprimitive nonmonomial or primitive becomes of interest.

By direct computation, it is easy to see that the place of $G$ in this classification depends on the matrices in $H_i$ that have 3 or more non-zero entries, and how these matrices are combined in $G$, i.e. on the isomorphism $\alpha: H_1/K_1 \to H_2/K_2$ (as defined above).

Throughout the remainder of this section, say elements of a given type are “coupled”, if $\forall [g, g'] \in G$ with $g$ an element of this type, then $g'$ must be either be an element of the same type or a product of such an element and an element of a different type. Otherwise say that elements of this type are “separated”. For example, if $H_1 = H_2 = \mathbb{D}_{2n}$, then elements $b$ (see generators for $\mathbb{D}_{2n}$ given in Example 1.18) are separated in $G$ if $G$ contains an element $[a^k, ba^l]$ for some $k, l \in \mathbb{Z}$. Otherwise they are coupled.

With this in mind, direct computation provides the following criteria (putting $i \neq j \in \{1, 2\}$):

- If $H_1, H_2$ dihedral and $G$ transitive, then $G$ acts monomially.
• If $H_i \in \{A_4, S_4, A_5\}$ and $H_j = D_{2n}$, then the action of $G$ is non-monomial imprimitive.
• If $H_1 \cong H_2 \cong A_5$ then the action of $G$ is primitive.
• If $H_1, H_2 \in \{A_4, S_4\}$ and the 3-cycles are separated, then the action of $G$ is primitive.
• If $H_1 \cong H_2 \cong A_4$ and the 3-cycles are coupled then the action of $G$ is monomial.
• If $H_i \cong S_4, H_j \cong A_4$ and the 3-cycles are coupled, then the action is imprimitive non-monomial.
• If $H_i \cong S_4, H_j \cong S_4$, the 3-cycles are coupled, but the odd permutations are separated, then the action is primitive.

This list is clearly not exhaustive, but it is sufficient for determining the nature of all the groups below. The (non-)existence of $\tau$ does not influence this classification (can be checked by direct computation).

4.2.4. Possible isomorphism classes of $\bar{G}$. Since $\bar{G}$ is a finite group leaving a smooth quadric $S$ invariant, its action must be equal (as shown in section 2.1) to a suitable of one of the finite automorphism groups of a smooth 2–dimensional quadric. Thus $\bar{G}$ must be conjugate to the image of one of the finite groups given in [8], Theorem 4.9.

In order to make the structure of each of the groups slightly more explicit, the group structure will also be given in the notation

$$(H_1, K_1, H_2, K_2)_\alpha,$$

where $H_i, K_i$ are as before, and $\alpha$ is the gluing isomorphism between $H_1/K_1$ and $H_2/K_2$. Where only one such isomorphism exists, $\alpha$ will be omitted. For each isomorphism class, several representations of the group can be chosen. However, looking at the list of possible choices of representations, it is clear that the different representations will differ by at most an outer automorphism, so all the properties that are of interest in this discussion will be the same for all of them. Therefore, for each isomorphism class, any faithful representation of $H_i$ can be chosen. For any group $(H_1, K_1, H_2, K_2)_\alpha$, there also exists a group $(H_2, K_2, H_1, K_1)_{\alpha^{-1}}$, which corresponds to the same group with the components of the ruling of the quadric swapped. These two groups are conjugate to each other.

**Lemma 4.3.** If $\bar{G} < PGL_4(\mathbb{C})$ is a finite irreducible transitive subgroup, $G$ its lift to $SL_4(\mathbb{C})$ and $S \subset \mathbb{P}^3$ a smooth $\bar{G}$–invariant quadric surface, and $\tau \in G$ and element interchanging the ruling of $S$ (if such exists), then $\bar{G}$ must be conjugate to one of the groups in the list below.
First assume \( \tau \) does not exist, i.e. \( \bar{G} \) leaves the ruling of \( \mathbb{P}^1 \times \mathbb{P}^1 \) invariant. Bearing the analysis above in mind, for \( G \) to be transitive, it must be conjugate to one of the following:

1. Product subgroups \((H_1, H_1, H_2, H_2) \cong H_1 \times H_2\) for some finite groups \( H_i \in \text{Aut} (\mathbb{P}^1)\). Taking different choices for \( H_1, H_2 \), get the following groups of the form \( H_1 \times H_2\):
   
   - (a) 9 primitive groups when \( H_1, H_2 \in \{ A_4, S_4, A_5 \} \)
   
   - (b) 3 families of nonmonomial imprimitive groups \( \mathbb{D}_{2m} \times H_2\), where \( H_2 \in \{ A_4, S_4, A_5 \} \)
   
   - (c) 1 family of monomial groups \( \mathbb{D}_{2m} \times \mathbb{D}_{2n} \)

2. Twisted diagonal subgroups \((H_1, 1, H_1, 1)_{\alpha} \cong H_1\) for some finite group \( H_1 \in \text{Aut} (\mathbb{P}^1)\). This gives 3 families of groups, indexed by the choice of isomorphism \( \alpha \). They are:
   
   - (a) Monomial groups isomorphic to \( A_4 \) or \( S_4 \).
   
   - (b) Primitive groups isomorphic to \( A_5 \).
   
   The twisted diagonal groups isomorphic to the dihedral groups do not act transitively, as the relevant central extensions do not have any 4-dimensional irreducible representations.

3. \( \frac{1}{2} [S_4 \times S_4] \cong (A_4 \times A_4) \rtimes Z_2 \cong (S_4, A_4, S_4, A_4) \), a primitive group generated by elements corresponding to \([\text{id}, (12)(34), \text{id}]\), \([\text{id}, (12)(34)], [123], \text{id}\), \([\text{id}, (123)]\) and \([12], (12)]\).

4. \( \frac{1}{2} [D_{4m} \times S_4] \cong (Z_m \times A_4) \rtimes Z_2 \cong (D_{2m}, Z_m, S_4, A_4) \), a family of imprimitive nonmonomial groups generated by \([a_m, \text{id}]\), \([\text{id}, (12)(34)], [\text{id}, (123)]\) and \([b, (12)]\).

5. \( \frac{1}{2} [D_{4m} \times S_4] \cong (D_{2m} \times A_4) \rtimes Z_2 \cong (D_{4m}, D_{2m}, S_4, A_4) \) \((m \geq 2)\), a family of imprimitive nonmonomial groups generated by the action of \([a_{2m}^2, \text{id}]\), \([b, \text{id}]\), \([\text{id}, (12)(34)], [\text{id}, (123)]\) and \([a_{2m}^n, (12)]\).

6. \( \frac{1}{6} [D_{6m} \times S_4] \cong (Z_m \rtimes V_4) \rtimes S_3 \cong (D_{6m}, Z_m, S_4, V_4) \), a family of imprimitive non–monomial groups generated by \([a_{3m}, \text{id}]\), \([\text{id}, (12)(34)], [\text{id}, (13)(24)], [a_{3m}^n, (123)]\) and \([b, (12)]\).

7. \( \frac{1}{6} [S_4 \times S_4] \cong (V_4 \rtimes V_4) \rtimes S_3 \cong (S_4, V_4, S_4, V_4) \), a monomial group generated by \([12](34), \text{id}\), \([13](24), \text{id}\), \([\text{id}, (12)(34)], [123], (123)]\) and \([12], (12)]\).

8. \( \frac{1}{3} [A_4 \times A_4] \cong (V_4 \rtimes V_4) \rtimes Z_3 \cong (A_4, V_4, A_4, V_4) \), a monomial group generated by \([12](34), \text{id}\), \([13](24), \text{id}\), \([\text{id}, (12)(34)], [123], (123)]\) and \([123], (123)]\).

9. \( \frac{1}{2} [D_{2m} \times D_{4n}] \cong (Z_m \rtimes D_{2n}) \rtimes Z_2 \cong (D_{2m}, Z_m, D_{4n}, D_{2n}) \) \((m, n \geq 2)\), a family of monomial groups generated by \([a_m, \text{id}]\), \([\text{id}, a_{2m}^2], [\text{id}, b]\) and \([b, a_{2m}^n]\).
(10) $\frac{1}{3} [D_{4m} \times D_{4n}]_\alpha \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \times D_4 \cong (D_{4m}, \mathbb{Z}_m, D_{4n}, \mathbb{Z}_n)_\alpha$ (where $\alpha(b) = a_{2n}^m$, $\alpha(a_{2m}^m) = b$), a family of monomial groups generated by $[a_{2n}^m, \text{id}]$, $[\text{id}, a_{2n}^m]$, $[a_{2m}^m, b]$ and $[b, a_{2n}^m]$.

(11) $\frac{1}{2} [D_{4m} \times D_{4n}] \cong (D_{2m} \times D_{2n}) \times \mathbb{Z}_2 \cong (D_{4m}, D_{2m}, D_{4n}, D_{2n})$ (for $m, n \geq 2$), a family of monomial groups generated by $[a_{2m}^2, \text{id}]$, $[\text{id}, a_{2n}^2]$, $[\text{id}, b]$ and $[a_{2m}^m, a_{2n}^n]$.

Now assume that $\tau \in G$ exists. Then the normal subgroup of $G$ fixing the ruling of the quadric must also be a group of automorphisms of $S$. Furthermore, it must have $H_1 \cong H_2$ and $K_1 \cong K_2$, since conjugation by $\tau$ provides the two isomorphisms. That means that $G$ can be isomorphic to one of the following groups:

(12) $(H_1 \times H_1) \times 2 \cong (H_1, H_1, H_1, H_1) \times \mathbb{Z}_2$ ({$H_1 \in \text{Aut}(\mathbb{P}^1)$}). Taking different choices for $H_1$ and bearing in mind that choosing $H_1$ to be $\mathbb{Z}_n$ produces an intransitive group (see discussion above), get 1 family of monomial groups

(a) $(D_{2n} \times D_{2n}) \times \mathbb{Z}_2$
and 3 families of primitive groups, all of them indexed by the possible involutions acting on $H_1$:

(b) $(A_4 \times A_4) \times \mathbb{Z}_2$
(c) $(S_4 \times S_4) \times \mathbb{Z}_2$
(d) $(A_5 \times A_5) \times \mathbb{Z}_2$

(13) $H_1 \times \mathbb{Z}_2 \cong (H_1, 1, H_1, 1)\alpha$ ($H_1 \in \text{Aut}(\mathbb{P}^1)$). This gives 3 families of groups, indexed by the choice of isomorphism $\alpha$. They are:

(a) Monomial groups isomorphic to $D_{4n} \times \mathbb{Z}_2$.
(b) Monomial groups isomorphic to $A_4 \times \mathbb{Z}_2$ or $S_4 \times \mathbb{Z}_2$.
(c) Primitive groups isomorphic to $A_5 \times \mathbb{Z}_2$.

(14) $((A_4 \times A_4) \times \mathbb{Z}_2) \times \mathbb{Z}_2 \cong (S_4, A_4, S_4, A_4) \times \mathbb{Z}_2$, a family of primitive groups.

(15) $((V_4 \times V_4) \times S_3) \times \mathbb{Z}_2 \cong (S_4, V_4, S_4, V_4) \times \mathbb{Z}_2$, a family of monomial groups.

(16) $((V_4 \times V_4) \times Z_3) \times \mathbb{Z}_2 \cong (A_4, V_4, A_4, V_4) \times \mathbb{Z}_2$, a family of monomial groups.

(17) $((Z_m \times Z_m) \times D_4) \times \mathbb{Z}_2 \cong (D_{4m}, Z_m, D_{4m}, Z_m)_\alpha \times \mathbb{Z}_2$ (where $\alpha(b) = a_{2m}^m$, $\alpha(a_{2m}^m) = b$), a family of monomial groups.

(18) $((D_{2m} \times D_{2m}) \times \mathbb{Z}_2) \times \mathbb{Z}_2 \cong (D_{4m}, D_{2m}, D_{4m}, D_{2m}) \times \mathbb{Z}_2$ ({$m \geq 2$}), a family of monomial groups.

**Example 4.4.** When one thinks about finite groups in $\text{SL}_4(\mathbb{C})$ that leave a smooth quadric invariant, one of the first examples that come to mind is the monomial group $G = (\mathbb{Z}_2)^3 \rtimes S_4$, where the normal subgroup $H = (\mathbb{Z}_2)^3$ acts diagonally, and the binary symmetric group permutes the basis. Here, one of the size two subgroups of $H$ acts by scalar
matrices, so the action on the projective quadric surface is isomorphic to \((\mathbb{Z}_2)^2 \rtimes S_4 \cong ((\mathbb{V}_4 \times \mathbb{V}_4) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2\), which is in position (10) in the list above.

**Example 4.5** (cf. Example 2.16). Let

\[ S = \{(x : y : u : v) \in \mathbb{P}^3 \mid x^m + y^m + u^m + w^m = 0 \}, \]

where \(m \geq 5\). Let \(G = \text{Aut}(S)\), and \(\bar{G}\) be the lift of \(G\) to \(\text{SL}_4(\mathbb{C})\). Then \(G\) induces a weakly–exceptional singularity.

**Proof:** \(G\) is a central extension of \((\mathbb{Z}_m)^3 \rtimes \mathbb{S}_4\), acting by permuting the basis and multiplying the coordinates by arbitrary \(m\)–th roots of unity. Thus \(G\) is transitive. \(G\) is not isomorphic to any of the groups in Lemmas 4.2 or 4.3, so it does not have an invariant quadric or cubic surface. Furthermore, \(\bar{G} \not\cong \mathbb{A}_5\). Therefore, \(G\) induces a weakly–exceptional singularity.

\[\square\]

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