Quantum corrections for the phase diagram of systems with competing order

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Abstract

We use the effective potential method of quantum field theory to obtain the quantum corrections to the zero temperature phase diagram of systems with competing order parameters. We are particularly interested in two different scenarios: regions of the phase diagram where there is a bicritical point, at which both phases vanish continuously, and the case where both phases coexist homogeneously. We consider different types of couplings between the order parameters, including a bilinear one. This kind of coupling breaks time-reversal symmetry and it is only allowed if both order parameters transform according to the same irreducible representation. This occurs in many physical systems of actual interest like competing spin density waves, different types of orbital antiferromagnetism, elastic instabilities of crystal lattices, vortices in a multigap SC and also applies to describe the unusual magnetism of the heavy fermion compound URu2Si2. Our results show that quantum corrections have an important effect on the phase diagram of systems with competing orders.

Keywords: quantum phase transitions, strongly correlated systems, competing orders, quantum corrections

(Some figures may appear in colour only in the online journal)

1. Introduction

The study of strongly correlated electronic systems (SCES) constitute an exciting area of condensed matter physics. It deals with the interesting, emergent phenomena that arise from the interactions between the electrons. Among the different types of ground states presented by SCES we find, for example, ferromagnetism, spin density wave, Mott insulator, unconventional superconductivity, heavy Fermi liquid and charge density wave [1, 2]. In many cases, these systems can sustain different ground states for the same values of the external parameters. This leads to a competition between different orderings and eventually to coexistence. In general SCES are sensitive to small changes of a control parameter [2, 3], as external pressure, applied magnetic field or doping levels. This allows to study these materials as they transit through quantum phase transitions between the different ground states as the external parameter is varied.

Among the SCES materials a very interesting class comprises inter-metallic compounds containing unstable f-shell elements, as Ytterbium (Yb), Cerium (Ce) and Uranium (U). Since the f-ions are displayed on the sites of a lattice, these compounds have lattice translation invariance and ideally their resistivity should vanish as temperature approaches zero [2, 4, 5]. The physical properties of these systems is a direct result of the competition between the Kondo effect and RKKY interactions, with the former favoring the formation of a non-magnetic ground state [6, 7]. At zero temperature there is a quantum critical point (QCP) separating a magnetic phase from a heavy Fermi liquid state [3, 5]. At finite temperatures, in the magnetic side of the phase diagram, there is a line of magnetic transitions, in general antiferromagnetic (AF), that vanishes at the QCP [4–6]. In some heavy fermion compounds, at very low temperatures, experiments have shown that they can exhibit superconductivity (SC) near or in coexistence with the AF phase close to the magnetic QCP [8].
Competing orders, SC-AF-Structural, also appear in the iron arsenide SC [9, 10]. More recently, it was discovered that some pnictides, such as, LaFeAsO(1−x)F x, PrFeAsO(1−x)F x, (Sr1−Na)xFe2As2 and (Ba1−K)xFe2As2, exhibit competing AF-SC order separated by a first-order phase transition and can only coexist in phase-separated macroscopic regions of the sample [9, 11−13].

Thus, experimental results show that systems with strong correlations present many interesting phenomena and very rich phase diagrams, with competing orders and coexistence [13]. Non-Fermi liquid behavior points to the existence of underlying QCPs and there is also the possibility of zero temperature first order transitions [8, 14−16]. The study of the phase diagrams of SCES with competing orders is one of the most fundamental issues that have not yet been clarified.

Even in systems with a single type of order, instabilities in the phase diagram can arise due to the coupling of the order parameter to other excitations, not necessarily associated with a symmetry breaking. For example, materials as superconductors with charged excitations can couple to the fluctuations of the electromagnetic field [5]. Magnetic materials can couple to elastic excitations [17] and in metallic ferromagnets the magnetization can couple to electron-hole excitations of the Fermi liquid [14−16, 18]. In many cases the effect of these couplings is to change the order of the transition associated with the relevant order parameter and this may occur even for zero temperature [14−16, 18].

For the case of competing instabilities, different types of couplings between the order parameters are treated in the literature. The nature of these couplings is dictated by symmetry arguments. Time-reversal invariance, for example, leads to a quartic coupling in the different fields, such as, for quadrupolar [13, 19] or spin nematic orderings [20, 21]. In general symmetry precludes a bilinear coupling that, however, is allowed if the two order parameters transform according to an irreducible representation of the point group. We consider bilinear couplings, whenever it is allowed by symmetry.

In this paper, we investigate the effect of quantum corrections for the zero temperature phase diagram of systems with different types of coupling between the order parameters. We consider the case of a bicritical point where both phases vanish continuously, and that there is a region of coexistence between the different types of ordering in the ground state. The problem where different orderings exist in close proximity, but in different regions of the phase diagram has been studied before [8]. Our aim is to obtain the quantum corrections and verify how they modify the classical predictions. We consider the cases where the coupling between the different order parameters is a conventional quartic coupling, but also that of a bilinear interaction, whenever it is allowed by symmetry.

We use the effective potential method of quantum field theory [8, 29, 30] that provides the more direct and simple way to obtain the quantum corrections to the classical action. We work in a region of parameter space close to the QCPs of the competing ground states, such that, their order parameters are small and allow for a Landau type of expansion of the free energy. For simplicity, we describe the quantum dynamics of the critical modes of the competing phases, by propagators associated with a dynamic exponent $z = 1$ [8, 31−33].

The paper is organized as follows: in section 2 we use Landau theory for a system of coupled order parameters to classically analyze the phase diagram of the system for different couplings. In section 3 we present the method of the effective potential which will be used to take into account the first order quantum corrections to the phase diagram. Section 4 describes our results for the quantum effects near to the QCPs of the different orders. Finally, we discuss the consequences of the quantum corrections in section 5 and reserve the appendix to show mathematical details.

2. Landau theory—classical analysis

The simplest way to describe classically competing orders and their phase transitions is through Landau’s theory [13, 21]. The order parameters are a representation of a symmetry group of the system and have the role of describing the existence or not of a respective order or symmetry breaking. The system is assumed to be near its critical points, such that, both order parameters are small and the free energy can be expanded in terms of them. The presence of a given term in the expansion of the Landau free energy is dictated by the symmetries of the problem. The equilibrium state and its properties close to the phase transitions can be obtained by minimizing the free energy considered, in our case, as a function of the two competing order parameters.

Let us consider initially the case of two one-component real order parameters $\phi_1$ and $\phi_2$. Thus, each order parameter transforms with an irreducible representation of the $Z_2$ group. We consider bilinear, we well as, quadratic couplings between them. The Landau free energy density of this system takes the form,

$$f = \frac{\alpha_1}{2} \phi_1^4 + \frac{\alpha_2}{4} \phi_2^4 + \frac{\alpha_{12}}{2} \phi_1^2 \phi_2^2 + \frac{\alpha_{13}}{2} \phi_1^2 \phi_3^2 + \gamma \phi_1 \phi_2. \quad (1)$$

For $\gamma = 0$, $f$ has two independent Ising symmetries ($Z_2 \times Z_2$), meaning that $f$ is invariant under the transformations $\phi_1 \rightarrow -\phi_1$ and $\phi_2 \rightarrow -\phi_2$, independently. However, the term $\gamma \neq 0$, breaks this symmetry to one single Ising symmetry ($Z_2 \times Z_2 \rightarrow Z_2$), corresponding to changing the sign of $\phi_1$ and $\phi_2$ simultaneously. We work at zero temperature, such that, the quantities $\alpha_i = a_i(P)$, $\alpha_m = a_m(P)$ are functions of a control parameter $P$, as pressure, for example. In the absence of the bilinear coupling $\gamma = 0$, these quantities vanish at the critical points for their respective order parameters, i.e. at critical pressures $P_{c1}$ and $P_{c2}$, such that, $a_i(P_{c1}) = 0$ and $a_m(P_{c2}) = 0$. Furthermore, we take $u_{13} > 0$ implying that when the couplings $u_i$ and $\gamma$ between the order parameters vanish, we have two independent second order phase transitions at $P_{c1}$ and $P_{c2}$. When the coupling between the order parameters, $u_i$ and $\gamma$ are positive we have competition between the different phases [8, 13, 22, 34].
Minimizing the free energy density, equation (1), with respect to both order parameters, $\phi_1$ and $\phi_2$, we obtain the following expressions for these quantities,

$$
\phi_1^2 = \frac{[a_i u_m - a_m u_i + \gamma (u_m - u_i)]}{u_i^2 - u_m u_i},
$$

$$
\phi_2^2 = \frac{[a_m u_i - a_i u_m + \gamma (u_i - u_m)]}{u_i^2 - u_m u_i}.
$$

Replacing these expressions into equation (1) we can rewrite the equilibrium free energy density as,

$$
f = f_s + f_m - \frac{2a_i a_m u_i}{4D_{ms}} + \frac{\gamma^2 (-u_i - u_m + 2u_0)}{4D_{ms}} + \frac{\gamma}{D_{ms}} [a_i u_m - a_m u_i + \gamma (u_m - u_i)]^{1/2} \\
\times [a_m u_i - a_i u_m + \gamma (u_i - u_m)]^{1/2},
$$

where, $f_s = \frac{\phi_i^2}{4M}$, $f_m = \frac{\phi_i^2}{4M}$ and $D_{ms} = u_i^2 - u_m u_i$. If we take $a_i = a_1 (P - P_c)$, $a_m = a_2 (P_c - P)$, the coefficients of the quartic terms $u_i, u_m$ and $\gamma$ as positive constants, the equations above describe a variety of zero temperature phase diagrams and phase transitions as a function of the control parameter $P$. We are mainly interested in the different phases and on the nature of the transitions that occur in the presence of the couplings between order parameters. We consider first the classical case and the effect of each type of interaction separately. Next, we study the effect of quantum corrections in the phase diagrams and quantum phase transitions.

2.1. Quartic coupling

Following the same procedures above, for the case of an exclusive quartic interaction, i.e. $\gamma = 0$, we get

$$
f = f_m + \frac{(a_i u_m - a_m u_i)^2}{4M D_{ms}} = f_s + \frac{(a_m u_i - a_i u_m)^2}{4M D_{ms}},
$$

where $D_{ms} = u_i^2 - u_m u_i$, as before. Notice, from equation (4), that phase coexistence is only possible if $D_{ms} < 0$, i.e. if $u_i < \sqrt{u_m u_i}$, so that the energy density of the two coexisting phases is smaller than the condensation energies, $f_m$ and $f_s$ of the individual phases. We call attention here for a particular point in the phase diagram where the two phases vanish at the same critical value of the control parameter, i.e. for $P_{c1} = P_{c2} = P_c$. It is easy to verify from the equations above that for $a_i = a_m = 0$, both order parameters vanish at this point, even for finite $u_i$ and $D_{ms} < 0$. On the other hand, this is not necessarily the case in the presence of a finite bilinear coupling $\gamma$. This can be seen from equations (2) and (3) and will be discussed in more detail below.

2.2. Bilinear coupling

We now study the case that $u_i = 0$ in equation (1), but there is a finite bilinear coupling between the order parameters. The equilibrium values of the order parameters is given by

$$
\phi_1^2 = \frac{(a_i + \gamma)}{u_i},
$$

$$
\phi_2^2 = \frac{(a_m + \gamma)}{u_m},
$$

which, when substituted in the expression for the free energy density yield,

$$
f_c = f_s + f_m + \gamma \left(\frac{(a_i + \gamma) (a_m + \gamma)}{u_i u_m}\right) + \frac{\gamma^2}{4} \left(\frac{u_i + u_m}{u_i u_m}\right).
$$

Equation (6) shows that phase coexistence is only possible for $\gamma < 0$, such that, the total free energy density of the system $f_c$ is less than the sum $f_s + f_m$. This condition is also implicit in equation (5).

Considering again the special point in the phase diagram, $a_i = a_m = 0$, we notice a quite different behavior from that of the previous case. Indeed from equation (5), we realize that at the classical level the system can have finite order parameters induced by the bilinear coupling $\gamma$, even for $a_i = a_m = 0$ (see figure 1). The phase diagrams shown in figure 1 are strongly affected by quantum fluctuations as will be discussed further on in the text.

2.3. Both couplings present

In this case, we can see from equation (3) that the condition for coexistence requires that $\gamma < 0$ and that $u_i < \sqrt{u_m u_i}$. For the special case $a_i = a_m = 0$, we notice from equation (2) that both order parameters are finite due to the bilinear coupling $\gamma$ as long as $\gamma$ remains negative and the conditions $u_m > u_i$ and $u_i > u_0$ are satisfied. Notice that the latter automatically imply $u_i < \sqrt{u_m u_i}$.

In the next section we are going to calculate the quantum corrections to the ground state energy of the system described by equation (1) in the one-loop approximation. It will turn out that in this approximation, the contribution from the bilinear term to these corrections is an infinite constant independent of the fields. This problem can be circumvented applying a linear transformation with $\det = 1$ (rotation) to the classical fields.
\[ \phi_1 = \alpha \varphi_1 + \beta \varphi_2 \]
\[ \phi_2 = -\beta \varphi_1 + \alpha \varphi_2 \] (7)

with \( \alpha^2 + \beta^2 = 1 \). We can parametrize this transformation with just one parameter \( \theta \) taking,
\[ \alpha = \cos \theta ; \quad \beta = \sin \theta. \] (8)

Replacing equation (7) into the expression of the ground state energy, we get to lowest order in the fields,
\[ f = r_1 \varphi_1^2 + r_2 \varphi_2^2 + \lambda_1 \varphi_1^4 + \lambda_2 \varphi_2^4 + \lambda_{12} \varphi_1^2 \varphi_2^2 + \delta_1 \varphi_1 \varphi_2 + \delta_{12} \varphi_1 \varphi_2 \] (9)

where,
\[ r_1 = \frac{a_s}{2} \cos^2 \theta + \frac{a_m}{2} \sin^2 \theta - \gamma \cos \theta \sin \theta \]
\[ r_2 = \frac{a_s}{2} \sin^2 \theta + \frac{a_m}{2} \cos^2 \theta + \gamma \cos \theta \sin \theta \] (10)

and \( \theta \) is such that,
\[ \tan 2\theta = \frac{2\gamma}{a_m - a_s}. \] (11)

The \( \lambda_i \) and \( \delta_i \) are new arbitrary constants. Notice that the bilinear coupling has been replaced by new terms of higher order in the fields (with coefficients \( \delta_i \)), but with the same symmetry properties. The analysis of the classical ground state energy in this new basis is similar to that developed before. It is important to notice that the main effect of the coupling \( \gamma \) at this level is to shift the QCPs as can be seen in equation (5) and is shown in figure 1. The analysis is more simple if carried out close to the new QCPs at \( r_1 = 0 \) and \( r_2 = 0 \).

3. Quantum corrections

The method of the effective potential provides the most simple and direct approach to obtain information about the effects of quantum corrections on classical phase diagrams. At the one-loop level, the effective potential \( V_{\text{eff}} \) is given by the expression
\[ V_{\text{eff}}(\varphi_i) = V_0(\varphi_i) + \hbar \Gamma^{(1)}(\varphi_i) + O(\hbar^2), \] (12)

where \( V_0(\varphi_i) \) is the classical potential (the Landau free energy density described in the previous section), and \( \Gamma^{(1)}(\varphi_i) \) codifies the quantum fluctuations at linear order in \( \hbar \). \( \varphi_i \) are the homogeneous order parameters. Then, the actual phases of the system are reached by computing \( \partial V_{\text{eff}} / \partial \varphi_i = 0 \).

Following [8, 29, 30], the quantum correction is given by
\[ \Gamma^{(1)}(\varphi_i) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{ln} (\text{det}[1 - M(k)]) + \text{counterterms}. \] (13)

The ‘counterterms’ are necessary to renormalize the theory, and \( M(k) \) is a matrix whose elements \( [M]_{lm} \) are given by
\[ [M]_{lm} = -G^0(\varphi_i) \left[ \frac{\partial^2 V_{\text{eff}}(\varphi_i)}{\partial \varphi_l \partial \varphi_m} \right]_{\varphi_i = \varphi_i}. \] (14)

Here, \( G^0(k) \) are the propagators of the \( \varphi_i \) fields and \( V_{\text{eff}}^0 \) is the interaction part of the classical potential (i.e. the Landau free energy density without the mass terms \( (m^2) \)). Dynamical effects are included through the frequency dependent propagators.

For simplicity, we will consider that the two phases have the same dynamics described by propagators associated with a dynamical exponent \( z = 1 \) [8, 31–33], that characterizes a non-dissipative behavior. It results from the relation between the \( n \)-th order time derivatives and the gradient terms in non-stationary Ginzburg–Landau equation, i.e. the frequency and wave-vector dependence of the propagators [8, 31–33]. However, we may have different behaviors for the propagators and consequently distinct dynamic critical exponent \( z \). For example, \( z = 2 \) is generally associated with a dissipative behavior [8, 32, 33], among others [32]. Preliminary results indicate that the choice of different dynamics, and of the dimensionality of the system can affect the final results [35].

The type of dynamics that we consider in this work is appropriate to magnetic phases with excitations with linear dispersion relations, superconductors [4, 5, 8, 32] or to superfluid liquid \(^3\)He [31]. The relevant propagators are given by,
\[ G^{(1,2)}(k) = G^0(\omega, q) = \frac{1}{k^2 + r_{1,2}}, \] (15)
where \( k^2 = \omega^2 + q^2 \) (Euclidean space).

4. Effects of quantum corrections—results

After the calculation of the derivatives in equation (14) we get the expression for the first order quantum corrections for the effective potential, equation (13), as (see appendix)
\[ \Gamma^{(1)} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{ln} \left[ \left( 1 + \frac{12\lambda_1 \varphi_1^2 + 2\lambda_2 \varphi_2^2 + 6\delta_1 \varphi_1 \varphi_2}{k^2 + r_1} \right)^{-1} \right] \times \left( 1 + \frac{12 \lambda_2 \varphi_2^2 + 2\lambda_2 \varphi_1^2 + 6\delta_1 \varphi_1 \varphi_2}{k^2 + r_2} \right)^{-1} - \frac{(3\delta_1 \varphi_1^2 + 3\delta_2 \varphi_2^2 + 4\lambda_1 \varphi_1 \varphi_2)^2}{(k^2 + r_1)(k^2 + r_2)} + \text{ct.} \] (16)

where \( \text{ct} \) stands for counterterms. In order to calculate the integrals in equation (16), we perform an expansion of the logarithm up to second order terms in \((3\delta_1 \varphi_1^2 + 3\delta_2 \varphi_2^2 + 4\lambda_1 \varphi_1 \varphi_2)\) and get,
\[ \Gamma^{(1)} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \text{ln} \left[ \left( 1 + \frac{b_1}{k^2 + r_1} \right) \left( 1 + \frac{b_2}{k^2 + r_2} \right) \right] - \frac{(3\delta_1 \varphi_1^2 + 3\delta_2 \varphi_2^2 + 4\lambda_1 \varphi_1 \varphi_2)^2}{2(B^2 - A^2)} \times \left( \frac{1}{k^2 + A^2} - \frac{1}{k^2 + B^2} \right) + \text{ct.} \] (17)

where,
\( b_1 = 12\lambda_1\varphi_1^2 + 2\lambda_{12}\varphi_2^2 + 6\delta_1\varphi_1\varphi_2; \ A^2 = r_1 + b_1 \)
\( b_2 = 12\lambda_2\varphi_2^2 + 2\lambda_{12}\varphi_1^2 + 6\delta_2\varphi_1\varphi_2; \ B^2 = r_2 + b_2. \) (18)

We can calculate the integrals in equation (17) noticing that \( d^d k = S_d k^{d-1} d^d k, \) where \( S_d = (2\pi)^d/\Gamma(d/2) \) and \( S_d = 2\pi^2. \) Therefore, the integral in the first line of equation (17) \([5, 8]\) and the integral in the second line of equation (17) can be easily calculated (see appendix A).

Since we are interested in the regime where both \( r_1 \) or \( r_2 \) are small, i.e. near to the QCP of both phases, we will use an expansion of the effective potential in terms of \( r_1 \) and \( r_2. \) Let us consider some particular cases.

4.1. Bicritical point, \( r_1 = r_2 = 0 \)

We start calculating the quantum corrections to the classical phase diagram when both QCPs coincide (Full lines (blue) in figure 1), such that, \( r_1 = r_2 = 0. \) The quantity \( \Gamma^{(1)} \) for \( r_1 = r_2 = 0 \) can be obtained from the previous calculations, i.e. summing equations (A.3) and (A.4) (see appendix A) and assuming \( \Lambda \gg 1. \) We get,

\[
\Gamma^{(1)} = \frac{\pi^2}{(2\pi)^2} \left[ \frac{\Lambda^2}{2} (b_1 + b_2) + \frac{b_2^2}{4} \ln \left( \frac{b_1}{\Lambda^2} \right) + \frac{b_1^2}{4} \ln \left( \frac{b_2}{\Lambda^2} \right) \right] - \frac{(b_1^2 + b_2^2)}{8} + \frac{3\delta_1\varphi_1^2 + 3\delta_2\varphi_2^2 + 4\lambda_{12}\varphi_1\varphi_2}{2} \right) \]  
\times \left[ \ln \left( \frac{1}{\Lambda^2} \right) + \frac{b_1 \ln (b_1) - b_2 \ln (b_2)}{b_1 - b_2} \right] 
+ ct. \]  

(19)

Proceeding as usual with the renormalization process (see appendix B), we determine the counterterms, with the condition that they eliminate the dependence of the effective potential on the cut-off \( \Lambda. \) Using equation (12), where from now on we put \( b = 1, \) we finally get,

\[
V_{\text{eff}}(\varphi_1, \varphi_2) = \tilde{\lambda}_1\varphi_1^4 + \tilde{\lambda}_2\varphi_2^4 + \tilde{\lambda}_{12}\varphi_1^2\varphi_2^2 + \tilde{\delta}_1\varphi_1^2\varphi_2^2 + \tilde{\delta}_2\varphi_1^2\varphi_2^2 + O(\varphi^6). \]  

(20)

The tilde quantities represent effective couplings renormalized by quantum corrections and are given together with the terms of higher orders in the fields \( O(\varphi^6) \) in the appendix C. Analogously to the analysis of the classical case, let us consider the quantum corrections for some particular cases of equation (20).

For an exclusive quartic coupling, i.e. taking \( \delta_1 = 0 \) and finite (positive) \( \lambda_{12} \) on equation (20), we can verify that the first order quantum corrections do not introduce any qualitative change in the phase diagram. In other words, even taking into account quantum corrections, the minimum of the effective potential remains at the origin as can be seen in figure 2. This implies that the bicritical point is stable in the presence of quantum corrections due to a quartic coupling. Summarizing, quantum corrections due to an exclusive quartic coupling do not affect the phase diagram in figure 1 (full lines (blue line)) and the bicritical point survives these interactions at the classical and quantum level.

Next, we are interested in the case that \( \lambda_{12} = 0 \) and a finite (positive) bilinear coupling is allowed. For simplicity, we can, without loss of generality, take the limit when \( \delta_1 \) tends to \( \delta_2 \) and \( \lambda_2 \) tends to \( \lambda_1. \) In other words, the terms of \( O(\varphi^8) \) in equation (20) can be neglected. Notice also that in the presence of the bilinear coupling, the physical region of the phase diagram is that where the fields have the same sign, such that, time reversal symmetry is preserved. Thus, in order to verify the behavior of the system in this region of parameters, we make a cut in the 3D phase diagram taking the fields \( \varphi_1 = \varphi_2 \) in equation (20).

After all these simplifications equation (20) can be rewritten in the form,

\[
V_{\text{eff}} = \frac{1}{64\pi^2} \varphi_1^4 \left[ \frac{128\pi^2(\lambda_1 + \delta_2)}{\text{classical term}} 
+ 18\delta_2^2 \left[ \ln \left( \frac{6(\varphi_1^2(2\lambda_1 + \delta_2))^2}{\delta_2 \langle \varphi \rangle^2} \right) \right] 
+ 4\ln \left( \frac{\varphi_1^2(2\lambda_1 + \delta_2)}{\delta_2 \langle \varphi \rangle^2} \right) - \frac{85}{6} \right] + 288\lambda_1(\lambda_1 + \delta_2) 
\times \left[ \ln \left( \frac{\varphi_1^2(2\lambda_1 + \delta_2)}{2\delta_2 \langle \varphi \rangle^2} \right) - \frac{25}{6} \right] \]  

(21)

where \( \langle \varphi \rangle \) correspond to the minima of the effective potential (see appendix B). We can identify from equation (21) two Coleman–Weinberg-like terms that give rise to minima in the effective potential, outside the origin \([30]\). The plot of the 3D effective potential in figure 3 shows that quantum corrections due to an exclusive bilinear coupling has a radically different effect from that of purely quartic interactions. The quantum corrections due to the former break the symmetry of the fields and induce coexistence between these orders even for \( \delta_1 \) positive.

For completeness we consider now the stability of the bicritical point when both couplings are present. Due to the presence of the bilinear coupling we can use the same arguments of the exclusive bilinear case to simplify the effective potential given in equation (20). Therefore, following the same previous procedures for the exclusive bilinear coupling, but now for \( \lambda_{12} \) finite (positive) we get,
and positive. When we take into account quantum corrections of a finite bilinear coupling, with and without quantum corrections the effective potential, equation (22), to obtain the classical occurrence for smaller values of the symmetry breaking field. The plot for equation (22) is very similar to that of figure 3 with the quantum corrections (see text). In this figure the phase effect of the coupling with is identified with a superconducting phase and that with is with an antiferromagnet. The QCP of these phases are labelled as SC-QCP and AF-QCP, respectively.

4.2. Coexistence phase diagram

Let us consider now the more general case where both and are different from zero. Figure 4 shows a possible schematic phase diagram for the particular case of a superconductor and an antiferromagnet where both phases coexist in a region of the phase diagram. Classically, this occurs whenever and are negative, even in the absence of a coupling between these phases. In the presence of the quartic coupling and for and , the required condition for coexistence, in the case of positive and , is that . This guarantees that the coexistence of phases lowers the energy of the system. For and , we can check from equation (9) that coexistence is always possible. In the general case that and are finite, we find that these quantities compete, the former favoring coexistence between phases. The condition in this case is that with . When these are zero we recover the previous condition for biquadratic interactions only.

We now obtain the quantum corrections for this case of coexistence of phases. The problem where and is mathematically more intricate than that of the previous section. However, we can still make analytical progress whenever the system is deep in one of the phases, but at the QCP of the other, as shown in figure 4. Then, we treat here the problem where the material is in an ordered phase, such that, say and , but at the QCP of the other, i.e. with and , small, although allowed to be finite. The symmetric case corresponding to small and negative, such that, the system is in the ordered phase, but is tuned to the QCP of , i.e. to .
can be treated in the same way and yields equivalent results since the dynamics of the propagators are considered identical in this paper. For the former conditions, the effective potential is obtained expanding $\Gamma^{(1)}(\varphi_1, \varphi_2)$ for $r_2$ small but $r_1 = 0$. We get,

$$\Gamma^{(1)} = \frac{\pi^2}{(2\pi)^2} \left[ \frac{\Lambda^2}{2} (b_1 + b_2) + \frac{b_1^2}{4} \ln \left( \frac{b_1}{\Lambda^2} \right) \right.$$

$$- \left( \frac{b_1^2 + b_2^2}{4} + (r_2 + b_2)^2 \right) \ln \left( \frac{r_2 + b_2}{\Lambda^2} \right) - \frac{r_2^2}{4} \ln \left( \frac{r_2}{\Lambda^2} \right)$$

$$\left. + \frac{(3b_1\varphi_1^2 + 3b_2\varphi_2^2 + 4\lambda_{12}\varphi_1\varphi_2)^2}{4} \times \left( \ln \left( \frac{1}{\Lambda^2} \right) + \frac{B^2\ln(B^2) - b_1\ln(b_1)}{B^2 - b_1} \right) \right] + ct$$

(23)

where $B^2 = r_2 + b_2$, $b_1$ and $b_2$ are given in equation (18). Notice that both fields $\varphi_1$ and $\varphi_2$ are kept finite in this expansion even taking $r_1 = 0$ because of the possibility of $\varphi_1$ being induced by the couplings.

Proceeding with the renormalization process to eliminate the cut-off, we obtain the counterterms and the following expression for the effective potential,

$$V_{\text{eff}}(\varphi_1, \varphi_2) = r_2\varphi_2^2 + \lambda_1\varphi_2^4 + \lambda_2\varphi_1^2 + \lambda_{12}\varphi_1\varphi_2^2$$

$$+ \delta_1'\varphi_1\varphi_2 + \delta_2'\varphi_1\varphi_2^2 + \bar{\rho}(\varphi_{1,2})$$

$$+ \frac{1}{(4\pi)^2} \eta \left[ \ln \left( \frac{B^2}{r_2^2} \right) - \frac{1}{2} \right]$$

(24)

where $\bar{\rho}(\varphi_{1,2})$ contains terms of higher orders in the fields and,

$$\eta = (3r_2\delta_2\varphi_1\varphi_2 + 6b_2\varphi_1^2 + r_2\lambda_{12}\varphi_2^2).$$

(25)

Again the prime quantities represent effective couplings renormalized by quantum corrections and are given together with the higher orders terms $\bar{\rho}(\varphi_{1,2})$ in the appendix D. The terms $B^2 = r_2 + b_2$, $b_1$ and $b_2$ are given by equation (18) above.

The effective potential, equation (24), specifically in the $\eta$ contribution, contains a $\varphi_1^2$ term with a small coefficient whose effect is just to renormalize the classical potential and does not produce qualitative changes in the classical phase diagram. However, the presence of a $\varphi_1^2$ term in $\eta$ gives rise to interesting physical consequences that we now analyze in detail.

First notice that for $\delta_1 = 0$ in the effective potential, equation (24), the term in brackets $\left[ \cdots \right]$ multiplied by $\eta$ and involving $\varphi_1^2$ is always negative. This term in turn couples to the product $r_2\lambda_{12}$ that is also negative since the system is in the ordered $\varphi_2$ phase, i.e. $r_2 < 0$. This implies that the coefficient of the $\varphi_1^2$ term due to the quantum correction, $-1/(4\pi)^2 r_2\lambda_{12}$, is always positive. The physical significance of this positive $\varphi_1^2$ term is that the QCP of the $\varphi_1$ phase has been pushed away towards the QCP of the $\varphi_2$ phase due to the competition introduced by $\lambda_{12}$ coupling between these phases. The system that was at the QCP of the $\varphi_1$ phase is now in its disordered phase. The deeper the system is in the $\varphi_2$ phase, the larger is this effect since $|r_2|$ is larger. The effect also increases with the intensity of the interaction $\lambda_{12}$ clearly manifesting the competitive nature of this coupling that acts in the sense of reducing the region of coexistence in the phase diagram.

![Figure 5. Plot of the complete effective potential for $r_1$ small, with both couplings finite (schematic), i.e. of the full equation (24). The minima occur for finite values of both order parameters, $\varphi_1, \varphi_2$, such that, there is coexistence of phases (see text). For $\delta_1$ finite, the physical region of the phase diagram is that where $\varphi_1$ and $\varphi_2$ have the same sign. The plot shows a cut in a 3D graphic for $\varphi_1 = \varphi_2$ (2D graphic) in order to visualize the region of interest.](image-url)

If the same analysis is carried out at the new QCP, $r'_1$ (see figure 4), the same effect occurs until $r'_1 = r'_2$ and we arrive at the stable situation studied previously of a quantum bicritical point. For completeness, we point out the rather trivial case of a negative $\lambda_{12}$ in which case the opposite effect is observed and coexistence is enhanced by this coupling. As shown in figure 4 the new QCP of the $\varphi_1$ phase has moved to $r'_1$ due to the negative interaction.

Next we consider the case the quartic interaction $\lambda_{12} = 0$, but we turn on the couplings $\delta_{1,2}$. The terms multiplied by $\eta$ that arise in the quantum corrections due to these couplings are proportional to $r_2\varphi_2$ and in particular there are no terms in $\varphi_1^2$ coupled to $\delta_{1,2}$ as can be seen from equation (24). However, the coefficient of the term $\varphi_1\varphi_2^2$ has an opposite behavior to that obtained for the coupling $\lambda_{12}$, since both $r_2$ and $b_2$ are negative, with the negative sign coming from the terms in brackets of equation (24). We can then state that the quantum corrections arising from these couplings unlike the case of the previous coupling $\lambda_{12}$ favor an increase of the region of coexistence.

Finally, we discuss the stability of the coexistence phase described by the complete effective potential in the presence of both kinds of couplings, i.e. the full equation (24). As shown before, at the classical level coexistence is always possible whenever $\delta_{1,2} < 0$, even when $r_2 = 0$, since this lowers the total condensation energy. This is also valid with quantum corrections. The complete effective potential assumes a Mexican hat shape as shown in figure 5. The minima occur for finite values of both order parameters, $\varphi_1, \varphi_2$, and consequently there is a coexistence of phases. Nevertheless, as discussed previously, the $\lambda_{12}$ parameter couples with $\varphi_1^2$ term and we
have competition between the different orderings depending on the quantities $r_2$ and $\lambda_{12}$. In addition the biquadratic and bilinear couplings also compete producing different trends for the coexistence of phases in the global phase diagram.

5. Conclusions

We have investigated in this work the effect of quantum corrections to the phase diagram of systems with competing order parameters. We considered the case of a bicritical point in the classical phase diagram, where two phases vanish simultaneously and continuously, and also the region of this diagram where there is coexistence between the two.

We studied these problems in the presence of two types of couplings between the order parameters. One is a conventional quartic coupling with a positive sign that describes competition between the different orderings [13, 19–21]. The other is a bilinear interaction that is allowed only in special cases where the order parameters have the same symmetry [22, 23]. These couplings have different effects in the phase diagram.

We have obtained that a classical bicritical point is stable to quartic interactions even when quantum fluctuations are taken into account. However, this is not the case in the presence of a bilinear coupling. We have shown that any finite positive bilinear interaction breaks the symmetry of the bicritical point and gives rise to phase coexistence. This is a purely quantum effect and resembles the physics of the Coleman–Weinberg mechanism [30] where coupling to a gauge field gives rise to symmetry breaking.

In the region of coexistence, we obtained that the effect of quantum fluctuations in the presence of quartic interactions is to reduce the region of phase coexistence in the phase diagram. While in the classical case coexistence is allowed in the presence of a quartic coupling, if the condition $\lambda_{12}^2 < 4\lambda_1\lambda_2$ is satisfied, when quantum corrections are included no coexistence is possible in the presence of these quartic interactions. This is not the case for a bilinear coupling that favors coexistence even when quantum fluctuations are included. When both interactions are present, in fact for any finite bilinear coupling, the different phases can coexist at zero temperature.

For completeness, we point out that the problem where different phases are in close proximity, but do not coexist, has been studied before [4, 5, 8]. There, critical fluctuations of one phase interfere with those of the other changing the nature of the quantum phase transitions from continuous to first order [4, 5, 8].

In this paper we describe the dynamics of the different orderings by propagators associated with the same dynamic exponent $z = 1$ [8, 31–33]. Preliminary results indicate that the choice of different dynamics and the dimensionality of the system can affect the final results [35].

The problem investigated in this paper is extremely relevant for many systems in condensed matter physics. They range from high temperature superconductors where different phases compete inside the superconducting dome to heavy fermion materials [36–42]. In the latter antiferromagnetism and superconductivity have clearly been observed in coexistence close to an AF QCP. Our results can be useful to identify the form of the interactions between the order parameters in this system.

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Appendix A. Calculation of the integrals of equation (17)

In this section we will perform the calculus of the integrals from equation (17) above in text, which is given by,

$$\Gamma^{(1)} = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left[ 1 + \frac{b_1}{k^2 + r_1} \right] \ln \left[ 1 + \frac{b_2}{k^2 + r_2} \right] - \frac{(3\delta_1\varphi_1^2 + 3\delta_2\varphi_2^2 + 4\lambda_{12}\varphi_1\varphi_2)^2}{2(B^2 - A^2)} \times \int \frac{d^d k}{(2\pi)^d} \left[ \frac{1}{k^2 + A^2} - \frac{1}{k^2 + B^2} \right] + ct. \tag{A.1}$$

where,

$$b_1 = 12\lambda_1\varphi_1^2 + 2\lambda_{12}\varphi_2^2 + 6\delta_1\varphi_1\varphi_2; A^2 = r_1 + b_1$$

$$b_2 = 12\lambda_2\varphi_2^2 + 2\lambda_{12}\varphi_1^2 + 6\delta_2\varphi_1\varphi_2; B^2 = r_2 + b_2. \tag{A.2}$$

We can calculate the integrals in equation (17) noticing that $d^d k = S_d d^{d-1} d k$, where $S_d = (2\pi)^{d/2}/\Gamma(d/2)$ and $S_2 = 2\pi^2$. Therefore, the integral in the first line of equation (A.1) is easily calculated [5, 8] and yields,

$$I_a = I_1 + I_2$$

where,

$$I_{1,2} = \frac{\pi^2}{(2\pi)^d} \left[ \frac{\Lambda^2}{2} b_{1,2} + \frac{(b_{1,2} + r_{1,2})^2}{4} \ln \left( \frac{b_{1,2} + r_{1,2}}{\Lambda^2} \right) - \frac{b_{1,2}^2}{8} - \frac{r_{1,2}^2}{4} \ln \left( \frac{r_{1,2}}{\Lambda^2} \right) \right] \tag{A.3}$$

with $\Lambda$ an ultraviolet cut-off.

The integral in the second line of equation (17) has the following solution,

$$I_b = I_A + I_B,$$

where,

$$I_{A,B} = \frac{\pi^2}{(2\pi)^d} \left[ \frac{1}{2} \Lambda^2 + \frac{(A,B)^2}{2} \ln \left( (A,B)^2 \right) - \frac{(A,B)^2}{2} \ln \left( (A,B)^2 + \Lambda^2 \right) \right]. \tag{A.4}$$

The quantities $A$ and $B$ are defined in equation (18).

Appendix B. Renormalization

In this section we will apply the renormalization process to renormalize the effective potential in the presence of both, bilinear and biquadratic couplings in the region of coexistence. We can get the particular cases easily from this calculation.
Expanding the effective potential, equation (23), for \( r_1 \) small yields,

\[
\Gamma^{(1)} = \frac{\pi^2}{(2\pi)^4} \left[ \frac{\Lambda^2}{2} (b_1 + b_2) + \frac{b_1^2}{4} \ln \left( \frac{b_1}{\Lambda^2} \right) - \frac{b_1^2}{8} \right] + \frac{1}{4} \left[ \frac{\Lambda^2}{2} (b_1 + b_2) + \frac{b_1^2}{4} \ln \left( \frac{b_1}{\Lambda^2} \right) - \frac{b_1^2}{8} \right]
\]

\[
+ \frac{1}{4} \ln \left( \frac{\Lambda^2}{2\Lambda^2} \right) + \frac{\Theta_1}{4} \ln \left( \frac{\Lambda^2}{2\Lambda^2} \right) - \frac{\Theta_1^2}{8}
\]

\[ \times \left( \ln \left( \frac{1}{\Lambda^2} \right) + B^2 \ln(B^2) - A^2 \ln(A^2) \right) + \text{ct.} \] (B.1)

The counterterms necessary for renormalization with additional quantum corrections must have terms of the same form as those from the classical free energy, equation (9). Thus, we have to calculate terms of the type,

\[
\frac{1}{2} C_1 \varphi_1^2, \quad \frac{1}{2} C_2 \varphi_2^2, \quad C_3 \varphi_1 \varphi_2, \quad \frac{1}{4!} D_1 \varphi_1^3.
\]

The calculation of these terms requires that we define, as usual [5, 8, 29, 30], the following conditions,

\[
\left[ \frac{d^2 \Gamma^{(1)}}{d \varphi_1^2} \right]_{\varphi_{12}=0} = r_1 = 0; \quad \left[ \frac{d^4 \Gamma^{(1)}}{d \varphi_1^4} \right]_{\varphi_{12}=0} = \lambda_1
\]

\[
\left[ \frac{d^2 \Gamma^{(1)}}{d \varphi_2^2} \right]_{\varphi_{12}=0} = r_2; \quad \left[ \frac{d^4 \Gamma^{(1)}}{d \varphi_2^4} \right]_{\varphi_{12}=0} = \lambda_2
\]

\[
\left[ \frac{d^2 \Gamma^{(1)}}{d \varphi_1^2 d \varphi_2^2} \right]_{\varphi_{12}=0} = \lambda_1 \lambda_2; \quad \left[ \frac{d^4 \Gamma^{(1)}}{d \varphi_1^2 d \varphi_2^2} \right]_{\varphi_{12}=0} = \gamma
\]

\[
\left[ \frac{d^2 \Gamma^{(1)}}{d \varphi_1^2 d \varphi_2^2} \right]_{\varphi_{12}=0} = \delta_1; \quad \left[ \frac{d^4 \Gamma^{(1)}}{d \varphi_1^2 d \varphi_2^2} \right]_{\varphi_{12}=0} = \delta_2
\] (B.3)

where \( \langle \varphi_{12} \rangle \) correspond to the minima of the effective potential. Proceeding with the calculation of the derivatives in equation (B.3), we obtain the counterterms,

\[
\frac{1}{2} C_1 \varphi_1^2 = -\frac{1}{2} \frac{\pi^2}{(2\pi)^4} \left[ \Lambda^2 (12 \lambda_1 + 2 \lambda_{12}) + 2 r_2 \lambda_2 \left( 1 + 2 \ln \left( \frac{|r_2|}{\Lambda^2} \right) \right) \right] \varphi_1^2
\] (B.4)

\[
\frac{1}{2} C_2 \varphi_2^2 = -\frac{1}{2} \frac{\pi^2}{(2\pi)^4} \left[ \Lambda^2 (12 \lambda_2 + 2 \lambda_{12}) + 6 r_2 \lambda_2 \left( 1 + 2 \ln \left( \frac{|r_2|}{\Lambda^2} \right) \right) \right] \varphi_2^2
\] (B.5)

\[
C_3 \varphi_1 \varphi_2 = -\frac{\pi^2}{(2\pi)^4} \left[ \Lambda^2 (3 \delta_1 + 6 \delta_2) + 3 \delta_2 \left( 1 + 2 \ln \left( \frac{|r_2|}{\Lambda^2} \right) \right) \right] \varphi_1 \varphi_2
\] (B.6)

\[
\frac{1}{4!} D_1 \varphi_1^4 = -\frac{1}{4!} \frac{\pi^2}{(2\pi)^4} \left[ 16 \lambda_1^2 \left( 198 + 54 \ln \left( \frac{12 \lambda_1 \langle \varphi_1 \rangle^2}{\Lambda^2} \right) \right) + 8 \lambda_1^2 \left( 11 + 3 \ln \left( \frac{2 \lambda_1 \langle \varphi_1 \rangle^2}{\Lambda^2} \right) \right) \right] \varphi_1^4
\] (B.7)

\[
\frac{1}{4!} D_2 \varphi_2^4 = -\frac{1}{4!} \frac{\pi^2}{(2\pi)^4} \left[ 16 \lambda_2^2 \left( 198 + 54 \ln \left( \frac{12 \lambda_2 \langle \varphi_2 \rangle^2}{\Lambda^2} \right) \right) + 8 \lambda_2^2 \left( 11 + 3 \ln \left( \frac{2 \lambda_2 \langle \varphi_2 \rangle^2}{\Lambda^2} \right) \right) \right] \varphi_2^4
\] (B.8)

\[
\frac{1}{4!} D_3 \varphi_1 \varphi_2^3 = -\frac{1}{4!} \frac{\pi^2}{(2\pi)^4} \left[ 90 + 36 \ln \left( \frac{6 \delta_1 \langle \varphi_1 \rangle \langle \varphi_2 \rangle}{\Lambda^2} \right) \right] \varphi_1 \varphi_2^3
\] (B.9)

\[
\frac{1}{6} D_4 \varphi_1^3 \varphi_2 = -\frac{1}{6} \frac{\pi^2}{(2\pi)^4} \left[ 8 \lambda_1 \delta_1 \left( 99 + 27 \ln \left( \frac{12 \lambda_1 \langle \varphi_1 \rangle^3}{\Lambda^2} \right) \right) + 4 \delta_2 \lambda_{12} \left( 27 + 9 \ln \left( \frac{2 \lambda_{12} \langle \varphi_2 \rangle^2}{\Lambda^2} \right) \right) \right] \varphi_1^3 \varphi_2
\] (B.10)

\[
\frac{1}{6} D_5 \varphi_1 \varphi_2^4 = -\frac{1}{6} \frac{\pi^2}{(2\pi)^4} \left[ 8 \lambda_2 \delta_1 \left( 99 + 27 \ln \left( \frac{12 \lambda_2 \langle \varphi_2 \rangle^3}{\Lambda^2} \right) \right) + 4 \delta_1 \lambda_{12} \left( 27 + 9 \ln \left( \frac{2 \lambda_{12} \langle \varphi_2 \rangle^2}{\Lambda^2} \right) \right) \right] \varphi_1 \varphi_2^3
\] (B.11)
Replacing the counterterms into equation (B.1) we can write the renormalized effective potential in the form,

\[ V_{\text{eff}} = r_2 \varphi_2^4 + \lambda_1 \varphi_1^4 + \lambda_2 \varphi_2^4 + \lambda_{12} \varphi_1^2 \varphi_2^2 + \delta_1 \varphi_1^3 \varphi_2 + \delta_2 \varphi_1 \varphi_2^3 + \frac{\pi^2}{(2\pi)^4} \left[ 36 \lambda_1^2 \ln \left( \frac{b_1}{12 \lambda_1 \langle \varphi_1 \rangle^2} \right) \right. \\
\left. + 12 \lambda_1 \lambda_{12} \ln \left( \frac{b_1}{12 \lambda_1 \langle \varphi_1 \rangle^2} \right) \right] \phi_1^4 \]

\[ + \left[ 9 \delta_1^2 \ln \left( \frac{b_1}{6 \lambda_1 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) \right] \phi_2^4 + \left[ 9 \delta_2^2 \ln \left( \frac{b_1}{6 \lambda_2 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) \right] \phi_1^2 \phi_2^2 \]

\[ + \left[ 6 \lambda_1 \lambda_{12} \ln \left( \frac{b_1}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) \right] \phi_1^2 \phi_2^2 \]

\[ + \left[ 6 \lambda_2 \lambda_{12} \ln \left( \frac{b_1}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) \right] \phi_1^2 \phi_2^2 \]

\[ + 6 \lambda_1 \lambda_{12} \ln \left( \frac{b_1}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) \phi_1^2 \phi_2^2 \]

\[ + 6 \lambda_2 \lambda_{12} \ln \left( \frac{b_1}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) \phi_1^2 \phi_2^2 \]

\[ + \left( 3 \delta_1 \delta_2 \phi_1 \phi_2 + 6 \delta_2 \phi_1 \phi_2^2 + 6 \delta_2 \phi_1^2 \phi_2 + 6 \delta_2 \phi_1 \phi_2^2 \right) \ln \left( \frac{b_1 + b_2}{b_1 - b_2} \right) \]

\[ - \left( 150 \lambda_1^2 \phi_1^4 + \frac{25}{6} \lambda_1 \phi_1^4 + \frac{150 \lambda_2 \phi_2^4 + \frac{25}{6} \lambda_2 \phi_2^4}{6} \right) \]

\[ + 27 \lambda_1 \lambda_{12} \phi_1^2 \phi_2^2 + 27 \lambda_2 \lambda_{12} \phi_1^2 \phi_2^2 \]

\[ + 42 \lambda_1 \lambda_{12} \phi_1^2 \phi_2 + 42 \lambda_1 \lambda_{12} \phi_1^2 \phi_2^3 + 150 \delta_1 \phi_1^3 \phi_2 + 21 \lambda_2 \lambda_{12} \phi_1 \phi_2^3 + 21 \lambda_2 \lambda_{12} \phi_1^2 \phi_2^3 + 21 \lambda_2 \lambda_{12} \phi_1 \phi_2^3 \]

\[ + \left( \frac{3}{2} \delta_2 \phi_1 \phi_2 + 3 \delta_2 \phi_1 \phi_2^2 + \frac{1}{2} \delta_2 \phi_1 \phi_2^2 \right) \]

\[ + \left( \delta_1 \phi_1^3 + 3 \delta_2 \phi_1^2 + 4 \lambda_{12} \phi_1 \phi_2 \right) \]

\[ \times \left( B^2 \ln(B^2) - b_1 \ln(b_1) \right) \]  \hspace{1cm} \text{(B.12)}

where,

\[ b_1 = 12 \lambda_1 \phi_1^2 + 2 \lambda_2 \phi_2^2 + 6 \lambda_1 \phi_1 \phi_2 \]

\[ b_2 = 12 \lambda_2 \phi_2^2 + 2 \lambda_1 \phi_2^2 + 6 \lambda_2 \phi_1 \phi_2 \]

\[ B^2 = |r_2 + b_2| \]  \hspace{1cm} \text{(B.13)}

**Appendix C. Effective couplings parameters renormalized—r_1 and r_2 expansion**

In this section, we give details of the calculation of the renormalized effective couplings parameters and the higher order field terms for the biquadratic point case and that leads to equation (20) in text. The calculation of the effective potential for this case proceeds in the same way as in the previous section, but expanding both in r_1 and r_2 conveniently, i.e. starting from equation (19). Thus, following the procedure of the previous section step by step we get the renormalized effective couplings parameters in the form,

\[ \hat{\lambda}_1 = \lambda_1 + \frac{\pi^2}{(2\pi)^4} \left[ 36 \lambda_1^2 \left( \ln \left( \frac{b_1}{12 \lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{25}{6} \right) \right. \\
\left. + \lambda_{12} \left( \ln \left( \frac{b_1}{2 \lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{25}{6} \right) \right] \]

\[ + \left[ \frac{9}{4} \delta_1^2 \left( \ln \left( \frac{b_1}{6 \lambda_1 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) - \frac{25}{6} \right) \right] \]  \hspace{1cm} \text{(C.1)}

\[ \hat{\lambda}_2 = \lambda_2 + \frac{\pi^2}{(2\pi)^4} \left[ 36 \lambda_2^2 \left( \ln \left( \frac{b_2}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{25}{6} \right) \right. \\
\left. + \lambda_{12} \left( \ln \left( \frac{b_2}{2 \lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{25}{6} \right) \right] \]

\[ + \left[ \frac{9}{4} \delta_2^2 \left( \ln \left( \frac{b_2}{6 \lambda_2 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) - \frac{25}{6} \right) \right] \]  \hspace{1cm} \text{(C.2)}

\[ \hat{\lambda}_{12} = \lambda_{12} + \frac{\pi^2}{(2\pi)^4} \left[ 9 \delta_1^2 \left( \ln \left( \frac{b_1}{6 \lambda_1 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) - 3 \right) \right. \\
\left. + 9 \delta_2^2 \left( \ln \left( \frac{b_2}{6 \lambda_2 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) - 3 \right) \right] \]

\[ + \lambda_{12} \left( \ln \left( \frac{b_1}{12 \lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \]

\[ + \lambda_{12} \left( \ln \left( \frac{b_2}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{7}{2} \right) \]  \hspace{1cm} \text{(C.3)}

\[ \hat{\delta}_1 = \delta_1 + \frac{\pi^2}{(2\pi)^4} \left[ 36 \delta_1 \lambda_1 \left( \ln \left( \frac{b_1}{12 \lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{25}{6} \right) \right. \\
\left. + 6 \lambda_{12} \left( \ln \left( \frac{b_1}{2 \lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \right] \]  \hspace{1cm} \text{(C.4)}

\[ \hat{\delta}_2 = \delta_2 + \frac{\pi^2}{(2\pi)^4} \left[ 36 \delta_2 \lambda_2 \left( \ln \left( \frac{b_2}{12 \lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{25}{6} \right) \right. \\
\left. + 6 \lambda_{12} \left( \ln \left( \frac{b_2}{2 \lambda_2 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \right] \]  \hspace{1cm} \text{(C.5)}
\[ O(\phi_{1,2}) = \frac{\pi^2}{(2\pi)^4} \left[ \frac{(3\delta_1\varphi_1^2 + 3\delta_2\varphi_2^2 + 4\lambda_1\varphi_1\varphi_2)^2}{4} \times \left( \frac{b_1 \ln(b_1) - b_2 \ln(b_2)}{b_1 - b_2} \right) \right] \quad \text{(C.6)} \]

where,
\[ b_1 = 12\lambda_1\varphi_1^2 + 2\lambda_2\varphi_2^2 + 6\delta_1\varphi_1\varphi_2; \]
\[ b_2 = 12\lambda_2\varphi_2^2 + 2\lambda_2\varphi_1^2 + 6\delta_2\varphi_1\varphi_2; \quad \text{(C.7)} \]

### Appendix D. Effective couplings parameters renormalized—\( r_1 \) expansion

Analogously to the previous section of this appendix, after obtaining the effective potential, we can write the renormalized effective coupling parameters and the higher order field terms in the coexistence phase, equation (24), as,

\[ \lambda'_1 = \lambda_1 + \frac{\pi^2}{(2\pi)^4} \left[ 36\lambda_1^2 \left( \ln \left( \frac{r_2 + b_2}{2\lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{25}{6} \right) \right] \quad \text{(D.1)} \]

\[ \lambda'_2 = \lambda_2 + \frac{\pi^2}{(2\pi)^4} \left[ 36\lambda_2^2 \left( \ln \left( \frac{r_2 + b_2}{2\lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{25}{6} \right) \right] \quad \text{(D.2)} \]

\[ \lambda'_{12} = \lambda_{12} + \frac{\pi^2}{(2\pi)^4} \left[ 9\delta_1^2 \left( \ln \left( \frac{b_1}{6\delta_1 \langle \varphi_1 \rangle \langle \varphi_2 \rangle} \right) - 3 \right) \right] \]

\[ + \frac{9\delta_2^2}{2\lambda_1 \langle \varphi_1 \rangle^2} \left( \ln \left( \frac{r_2 + b_2}{2\lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \]

\[ + \frac{12\lambda_1\lambda_1}{12\lambda_1 \langle \varphi_1 \rangle^2} \left( \ln \left( \frac{b_1}{12\lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \]

\[ + \frac{12\lambda_1\lambda_2}{12\lambda_2 \langle \varphi_2 \rangle^2} \left( \ln \left( \frac{r_2 + b_2}{12\lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{7}{2} \right) \quad \text{(D.3)} \]

\[ \delta'_1 = \delta_1 + \frac{\pi^2}{(2\pi)^4} \left[ 36\delta_1 \lambda_1 \left( \ln \left( \frac{r_2 + b_2}{12\lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{25}{6} \right) \right] \]

\[ + 6\delta_2 \lambda_1 \left( \ln \left( \frac{r_2 + b_2}{2\lambda_1 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \quad \text{(D.4)} \]

\[ \delta'_2 = \delta_2 + \frac{\pi^2}{(2\pi)^4} \left[ 36\delta_2 \lambda_2 \left( \ln \left( \frac{r_2 + b_2}{12\lambda_2 \langle \varphi_2 \rangle^2} \right) - \frac{25}{6} \right) \right] \]

\[ + 6\delta_1 \lambda_2 \left( \ln \left( \frac{b_1}{2\lambda_2 \langle \varphi_1 \rangle^2} \right) - \frac{7}{2} \right) \quad \text{(D.5)} \]

\[ \hat{\rho}(\phi_{1,2}) = \frac{\pi^2}{(2\pi)^4} \left[ \frac{(3\delta_1\varphi_1^2 + 3\delta_2\varphi_2^2 + 4\lambda_1\varphi_1\varphi_2)^2}{4} \times \left( \frac{B^2 \ln(B^2) - b_1 \ln(b_1)}{B^2 - b_1} \right) \right] \quad \text{(D.6)} \]

where,
\[ b_1 = 12\lambda_1\varphi_1^2 + 2\lambda_2\varphi_2^2 + 6\delta_1\varphi_1\varphi_2; \]
\[ b_2 = 12\lambda_2\varphi_2^2 + 2\lambda_2\varphi_1^2 + 6\delta_2\varphi_1\varphi_2; \quad \text{(D.7)} \]

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