AN INCOMPLETE EQUILIBRIUM WITH A STOCHASTIC ANNUITY

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Abstract. We prove the global existence of an incomplete, continuous-time finite-agent Radner equilibrium in which exponential agents optimize their expected utility over both running consumption and terminal wealth. The market consists of a traded annuity, and, along with unspanned income, the market is incomplete. Set in a Brownian framework, the income is driven by a multidimensional diffusion, and, in particular, includes mean-reverting dynamics.

The equilibrium is characterized by a system of fully coupled quadratic backward stochastic differential equations, a solution to which is proved to exist under Markovian assumptions.

1. Introduction

We prove the existence of a Radner equilibrium in an incomplete, continuous-time finite-agent market setting. The economic agents act as price takers in a fully competitive setting and maximize exponential utility from running consumption and terminal wealth. An annuity in one-net supply is traded on a financial market, and it pays a constant running and terminal dividend to its shareholders. The agents choose between consuming their income and dividend streams or investing in the annuity.

Although our setting and the income dynamics are quite general, our financial market looks relatively simple at first glance. The only available asset is the annuity, and the agents’ only choice at any given moment is how much to consume, keeping in mind that the only way to transfer wealth from one time to the next is through the annuity. This apparent simplicity is quite misleading, since the scarcity of the available traded assets leads to market incompleteness, a notorious difficulty in equilibrium analysis. Indeed, the fewer assets the agents have at their disposal, the
less efficient the market becomes and the harder it becomes to use the standard tools such as the representative agent approach. In our case, this lack of assets is pushed to its limit.

Admittedly, it would be more realistic to consider markets with several assets, both risky and riskless, where the incompleteness is derived from the constraints on each asset's ability to incorporate all the risk present in the environment. We believe that the exploration of such problems is one of the most interesting and important areas of future research in this area. Unfortunately, the formidable mathematical difficulties present in virtually all such problems leave them outside the scope of the techniques available to us today.

One of the advantages of our model is its ability to incorporate various income stream dynamics, including unspanned mean-reverting income streams (which have been studied extensively for their empirical relevance; see, e.g., [Wan04, Wan06, Coc14]). To the best of our knowledge, our model is the first with exponential agents to incorporate unspanned mean-reverting income in equilibrium and prove the existence of such an equilibrium. The general income streams we study lead to stochastic annuity dynamics, which prevent a money market account from being replicated by trading in the annuity in equilibrium.

Our approach crucially relies on the presence of a traded annuity. We also need utility functions of exponential type and a Markovian assumption on the dynamics of the income streams in order to obtain conveniently structured individual agent problems, amenable to a BSDE analysis. Even so, the analysis involves a non-standard Ansatz for the value function, as we need to formally treat the asset price $A$ as a quantity that, in standard models, plays the role of a money market account.

We are not the first to introduce a traded annuity into an equilibrium model (see, e.g., [VV99, Cal01, CLM12, CL14, Wes18]). Our contribution is to recognize the role of a traded annuity price in the individual agent value functions, even when general income streams render the annuity dynamics computationally intractable.

The backward stochastic differential equation (BSDE)/PDE-system approach to incomplete market equilibria dates back to [Zit12, Zha12, CL15, KXŽ15, XŽ18], with the early work relying on a smallness-type assumption on some ingredient of the model (the time-horizon, size of the endowment, etc.) The mathematical analysis of the present paper is quite involved and relies heavily on some recent results of [XZ18], which overcome smallness conditions and treat the existence and stability of solutions to quadratic systems of BSDE. Moreover, the applicability of those results in our setting is not at all immediate and is contingent on a number of a-priori estimates specific to our model.

Notation and conventions. For $J,d \in \mathbb{N}$, the set of $J \times d$-matrices is denoted by $\mathbb{R}^{J \times d}$. The Euclidean space $\mathbb{R}^J$ is identified with the set of $\mathbb{R}^{J \times 1}$, i.e., vectors in
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We work on a finite time horizon \([0, T]\) with \(T > 0\), where \(\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}\) is the usual augmentation of the filtration generated by a \(d\)-dimensional Brownian motion \(B\).

The stochastic integral with respect to \(B\) is taken for \(\mathbb{R}^{1 \times d}\)-valued (row) processes as if \(dB\) were a column of its components, i.e., \(\int \sigma(t) dB_t\) stands for \(\sum_{j=1}^{d} \int \sigma_j(t) dB^j\).

Similarly, for a process \(Z\) with values in \(\mathbb{R}^{J \times d}\), \(\int Z_t dB_t\) is an \(\mathbb{R}^J\)-valued process whose components are the stochastic integrals of the rows \(Z^i\) of \(Z\) with respect to \(dB_t\).

For a function defined on a domain in \(\mathbb{R}^d\), the derivative \(Du\) is always assumed to take row-vector values, i.e., \(Du(x) \in \mathbb{R}^{d \times 1}\). If \(u\) is \(\mathbb{R}^J\) valued, the Jacobian \(Du\) will, as usual, be interpreted as an element of \(\mathbb{R}^{J \times d}\). The Hessian, \(D^2u\) of a scalar-valued function takes values in \(\mathbb{R}^{d \times d}\), and we will have no need for Hessians of vector-valued maps in this paper.

To relieve the notation, we omit the time-index from many expressions involving stochastic processes but keep (and abuse) the notation \(dt\) for an integral with respect to the Lebesgue measure.

The set of all adapted, continuous and uniformly bounded processes is denoted by \(\mathcal{S}^\infty\), and the set of all processes of bounded mean oscillation by \(\text{BMO}\) (we refer the reader to [Kaz94] for all the necessary background on the \(\text{BMO}\) processes). The family of all \(B\)-integrable processes \(\sigma\) such that \(\int \sigma dB\) is in \(\text{BMO}\) is denoted by \(\text{bmo}\).

The set of all \(\mathbb{F}\)-progressively measurable process is denoted by \(\mathcal{P}\). \(\mathcal{P}^r\) denotes the set of all \(c \in \mathcal{P}\) with \(\int_0^T |c|^r\ dt < \infty\), a.s. The same notation is used for scalar, vector or matrix-valued processes - the distinction will always be clear from the context.

2. The problem

2.1. Model primitives. The model primitives can be divided into three groups. In the first one, we describe the uncertain environment underlying the entire economy. In the second, we postulate the form of the dynamics of the traded asset, and in the third we describe the characteristics of individual agents. A single real consumption good is taken as the numeraire throughout.

2.1.1. The state process. For \(d \in \mathbb{N}\), we start with an \(\mathbb{R}^d\)-valued state process \(\xi\) whose dynamics is given by

\[
 d\xi_t = \Lambda(t, \xi_t)\ dt + \Sigma(t, \xi_t)\ dB_t, \quad \xi_0 = x_0 \in \mathbb{R}^d
\]

(2.1)

where the measurable functions \(\Lambda : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) and \(\Sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}\) satisfy the following the regularity assumption:
Assumption 2.1 (Regularity of the state process). There exists a constant $K > 0$ such that for all $t, t' \in [0, T], x, x' \in \mathbb{R}^d$ and $z \in \mathbb{R}^{d \times 1}$ we have

1. $|\Lambda(t, x)| \leq K$ and $|\Lambda(t, x) - \Lambda(t, x')| \leq K|x - x'|,$
2. $|\Sigma(t, x)| \leq K,$ and $|\Sigma(t, x) - \Sigma(t, x')| \leq K(\sqrt{|t' - t|} + |x - x'|)$ and
3. $|\Sigma(t, x)z| \geq \frac{1}{K} |z|.$

Remark 2.2. Under Assumption 2.1 the SDE (2.1) admits a unique strong solution. The full significance of the assumptions above, however, will only be apparent in the later sections and is related to the ability to use certain existence results for systems of backward stochastic differential equations.

2.1.2. The traded asset. Our market consists of a single real asset $A$ in one-net supply, whose dynamics we postulate to be of the following form:

$$dA_t = (A_t \mu_t - 1) dt + A_t \sigma_t dB_t, \quad A_T = 1,$$  \hfill (2.2)

with the processes $\mu$ and $\sigma$ to be determined in the equilibrium. It can be interpreted as an annuity which pays a dividend at the rate 1 during $[0, T]$, as well as a unit lump sum payment at time $T$.

Let $\Gamma$, the coefficient space, denote the set of all pairs $\gamma = (\mu, \sigma)$, where $\mu$ is a scalar-valued process and $\sigma$ is an $\mathbb{R}^{1 \times d}$-valued bmo process. For simplicity, we often identify the market $A^\gamma$ with its coefficient pair $\gamma = (\mu, \sigma)$, and talk simply about the market $\gamma$. The set of all markets given by (2.2) is not bijectively parametrized by $\Gamma$ as not every $\gamma \in \Gamma$ defines a market. Indeed, the terminal condition $A_T = 1$ imposes a nontrivial relationship between $\mu$ and $\sigma$; for example, if $\mu$ is deterministic, $\sigma$ either has to vanish or one of its components has to be truly stochastic. The set of those $\gamma \in \Gamma$ that do define a market is denoted by $\Gamma_f$ and its elements are said to be feasible. If we need to stress that it comes with feasible coefficients $\gamma \in \Gamma_f$, we write $A^\gamma$ for the process given by (2.2).

2.1.3. Agents. There are a finite number $I \in \mathbb{N}$ of economic agents, each of which is characterized by the following four elements:

1. the risk-aversion coefficient $\alpha_i > 0$. It fully characterizes the agent’s utility function $U^i$ which is of exponential form
   $$U^i(c) = -\frac{1}{\alpha_i}e^{-\alpha_i c} \text{ for } c \in \mathbb{R}.$$
2. the random-endowment (stochastic income) rate. Each agent receives an endowment of the consumption good at the rate $e^i_t = e^i(t, \xi_t)$ and a lump sum $e^i_T = e^i(T, \xi_T)$ at time $T$, for some function $e^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}.$
3. the initial holding $\pi^i_0 \in \mathbb{R}$ is the initial number of shares of the annuity $A$ held by the agent.

With the cumulative endowment rate defined by

$$e = \sum_{i=1}^I e^i,$$

we impose the following regularity conditions:
Assumption 2.3 (Regularity of the endowment rates).

(1) Each $e^i$ is bounded and continuous, while its terminal section $e^i(T, \cdot)$ is $\alpha$-Hölder continuous for some $\alpha \in (0, 1]$.

(2) The cumulative endowment process $e_t = e(t, \xi_t)$ is a semimartingale with the decomposition

$$e(t, \xi_t) = e(0, x_0) + \int_0^t \mu_e(s, \xi_s) \, ds + \int_0^t \sigma_e(s, \xi_s) \, dB_s,$$

where the drift function $\mu_e : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous and $\sigma_e(s, \xi_s)$ is a bmo process.

We will often abuse notation and write $e^i$ both for the function $e^i : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and the stochastic process $e^i_t = e^i(t, \xi_t)$. The same applies to other functions applied to $(t, \xi_t)$ - such as $e$ or $\mu_e$.

Remark 2.4. It is worth stopping here to give a few examples of state processes $\xi$ and the functions $e^i$ which satisfy all the regularity conditions imposed so far. Once the coefficients $\Lambda$ and $\Sigma$ for $\xi$ are picked so as to satisfy Assumption 2.1, then Assumption 2.3 is easy to check for a sufficiently smooth $e^i$ by a simple application of Itô’s formula.

The more interesting observation is that there is room for improvement. It may seem that the boundedness imposed in Assumption 2.1 rules out some of the most important classes of state processes such as the classical mean-reverting (Ornstein-Uhlenbeck) processes. This is not the case, as we have the freedom to choose both the state process $\xi$, and the deterministic function $e^i$ applied to it, while only caring about the resulting composition. We illustrate what we mean by that with a simple example. The reader will easily add the required bells and whistles to it, and adapt it to other similar frameworks.

We assume that $d = 1$ and that we are interested in the random endowment rate $e^i(t, \eta_t)$ where $e^i$ is a bounded and appropriately smooth function, and $\eta_t$ is an Ornstein-Uhlenbeck process with the dynamics

$$d\eta_t = \theta(\bar{\eta} - \eta_t) \, dt + \sigma_\eta \, dB_t,$$

and parameters $\theta, \sigma_\eta > 0$ and $\eta_0, \bar{\eta} \in \mathbb{R}$. Since the drift function $x \mapsto \theta(\bar{\eta} - x)$ is not bounded, the process $\eta$ does not satisfy the conditions of Assumption 2.1. The process $\eta$ admits, however, an explicit expression in terms of a stochastic integral of a deterministic process with respect to the underlying Brownian motion:

$$\eta_t = \bar{\eta} + (\eta_0 - \bar{\eta})e^{-\theta t} + \sigma_\eta e^{-\theta t} \int_0^t e^{\theta s} \, dB_s \quad (2.3)$$

If we define the state process $\xi$ by

$$d\xi_t = e^{-\theta t} \, dB_t, \quad \xi_0 = 0,$$

i.e., if we set $\Lambda(t, x) = 0$ and $\Sigma(t, x) = e^{-\theta t}$, the boundedness of the time horizon $[0, T]$ allows use to conclude that $\Lambda$ and $\Sigma$ satisfy Assumption 2.1. Moreover, by
the choice \( f^i(t,x) = e^i(t, \eta + (\eta_0 - \eta)e^{-\theta t} + \sigma_\eta x) \) yields
\[
f^i(t, \xi_t) = e^i(t, \eta_t).
\]
This way, we can represent a function of an interesting, but not entirely compliant state process \( \eta \) as a (modified) function of a regular state process \( \xi \). The boundedness (and other regularity properties) of the function \( e^i \) are inherited by \( f^i \), thanks to the boundedness from above and away from zero of the function \( t \mapsto e^{-\theta t} \).

2.2. Admissibility and equilibrium.

**Definition 2.5.** Given a feasible set of coefficients \( \gamma = (\mu, \sigma) \in \Gamma_f \), a pair \((\pi, c)\) of scalar processes is said to be a \( \gamma \)-admissible strategy for agent \( i \) if

1. \(|c| + |\pi(A^\gamma \mu - 1)| \in \mathcal{P}^1 \) and \( \pi A^\gamma \sigma \in \text{bmo} \).
2. the gains process \( X = X^{\pi, \gamma} = \pi A^\gamma \) is a semimartingale which satisfies the self-financing condition
\[
dx = \pi da^\gamma + (e^i - c + \pi) dt.
\]

The set of all \( \gamma \)-admissible strategies for agent \( i \) is denoted by \( \mathcal{A}^i \), and the subset of \( \mathcal{A}^i \) consisting of the strategies with \( \pi(0) = \pi_0 \) is given by \( \mathcal{A}^i(\pi_0) \).

**Definition 2.6.** We say that \( \gamma^* \in \Gamma_f \) is the set of equilibrium market coefficients (and \( A^{\gamma^*} \) an equilibrium market) if there exist \( \gamma^* \)-admissible strategies \((\hat{\gamma}^i, \hat{c}^i) \in \mathcal{A}^i(\pi^i_0), i = 1, \ldots, I\) such that the following two conditions hold:

1. **Single-agent optimality:** For each \( i \) and all \((\pi, c) \in \mathcal{A}^i(\pi^i_0)\) we have
\[
E[\int_0^T U^i(\hat{c}^i_t) \, dt] + E[U^i(\hat{\gamma}^i_t, \hat{c}^i_t + e^i_T)] \geq E[\int_0^T U^i(c_t) \, dt] + E[U^i(X^\pi, c + e^i_T)]
\]
2. **Market clearing:**
\[
\sum_{i=1}^I \hat{\gamma}^i = 1 \quad \text{and} \quad \sum_{i=1}^I \hat{c}^i = e + 1 \quad \text{on } [0, T), \quad \text{and} \quad \sum_{i=1}^I X^\pi_t, \hat{c}^i = 1, \ \text{a.s.}
\]

3. Results

3.1. A BSDE characterization. Our first result is a characterization of equilibria in terms of a system of backward stochastic differential equations (BSDE). These systems consist of \( 1 + I \) equations, with the first component generally playing a different role from the other \( I \). For that reason, it pays to depart slightly from the classical notation \( (Y, Z) \), where \( Y \) has as many components as there are equations, and the driver \( Z \) is a matrix process whose additional dimension reflects the number of driving Brownian motions. Instead, we use the notation \( ((a,Y), (\sigma, Z)) \) where \( a \) is a scalar and \( Y \) is \( \mathbb{R}^{I\times 1} \)-valued. Similarly, \( \sigma \) and \( Z \) are \( \mathbb{R}^{I\times d} \) and \( \mathbb{R}^{I\times d} \)-valued processes, respectively. As usual, we say that \( ((a,Y), (\sigma, Z)) \) is an \( (\mathcal{S}^\infty \times \text{bmo}) \)-solution if all the components of \( a \) and \( Y \) are in \( \mathcal{S}^\infty \), and all components of \( (\sigma, Z) \) are in \( \text{bmo} \). To simplify the notation, we also introduce the following, derived, quantities:
\[
\frac{1}{\alpha} := \sum_{i=1}^I \frac{1}{\alpha^i}, \quad \text{and} \quad \kappa^i := \frac{\alpha^i}{\alpha} > 0 \text{ so that } \sum_{i=1}^I \kappa^i = 1.
\]
Theorem 3.1 (A BSDE Characterization). Suppose that $\sum_{i=1}^{I} \pi^i_0 = 1$, that Assumption 2.3 holds, and that $(\{a, Y\}, (\sigma, Z))$ is an $\mathcal{S}^\infty \times \text{bmo}$-solution to
\begin{align*}
\frac{\partial}{\partial t} \pi^i = \left( \begin{array}{c}
- \mu^i 
\end{array} \right) + \frac{1}{2} \sum_{i=1}^{I} \kappa^i |Z^i|^2 - \exp(-a) 
dt, \quad a_T = 0 
\end{align*}
\begin{align*}
\frac{\partial}{\partial t} \sigma^i = \left( \begin{array}{c}
\sigma^i 
\end{array} \right) + \frac{1}{2} \sum_{i=1}^{I} \kappa^i |Z^i|^2 + \exp(-a)(1 + a + Y^i - \alpha^i e^i) 
dt, \quad Y^i_T = \alpha^i e^i, \quad 1 \leq i \leq I. \tag{3.2}
\end{align*}

Then $A = \exp(a)$ is an equilibrium annuity price with market coefficients $(\mu, \sigma) \in \Gamma_f$, where $\mu$ is given by
\begin{align*}
\mu = \alpha \mu_e + \frac{1}{2} |\sigma|^2 - \frac{1}{2} \sum_{i=1}^{I} \kappa^i |Z^i|^2. \tag{3.3}
\end{align*}

Remark 3.2. We note that the validity of Theorem 3.1 above does not depend on Assumption 2.3. In fact, no Markovian assumption is needed for it, at all. Moreover, the full force of Assumption 2.3 is not needed, either. It would be enough to assume that each $e^i$ is in bmo and that the cumulative endowment process $e$ is a semimartingale of the form $de = \mu_e dt + \sigma_e dB$, where $\mu_e$ and $\sigma_e$ are general bmo processes and not necessarily bounded functions of a state process.

Proof. Having fixed an $(\mathcal{S}^\infty, \text{bmo})$-solution $(\{a, Y\}, (\sigma, Z))$, we set $A = \exp(a)$ and define $\mu$ as in (3.3), so that $A$ satisfies (2.2). With the market coefficients $\gamma = (\mu, \sigma)$ fixed, we pick an agent $i \in \{1, \ldots, I\}$ and a pair $(\pi, c) \in \mathcal{A}_i(\pi^i_0)$, and define processes $X^i, \hat{V}^i$ and $V^i$ by
\begin{align*}
X^i = \pi A, \quad \hat{V}^i = -\exp(-\alpha^i X^i/A - Y^i) \quad \text{and} \quad V^i = \hat{V}^i + \int_0^T -\exp(-\alpha^i c_i) \, dt.
\end{align*}

The self-financing property of $(\pi, c)$ implies that the semimartingale decomposition of $V^i$ is given by $dV^i = \mu_V \, dt + \sigma_V \, dB$, where
\begin{align*}
\mu_V = -\exp(-\alpha^i c) + \frac{\gamma^i}{\alpha^i} (1 - \log(-\gamma^i/\alpha^i)) - (\alpha^i c) \frac{\hat{V}^i}{\alpha^i} \quad \text{and} \quad \sigma_V = -\hat{V}^i Z^i.
\end{align*}

Young's inequality implies that $\mu_V \leq 0$ and that the coefficients $\mu_V$ and $\sigma_V$ are regular enough to conclude that $V^i$ is a supermartingale for all admissible $(\pi, c)$. Therefore,
\begin{align*}
\mathbb{E}[\int_0^T U^i(c_s) \, ds] + \mathbb{E}[U^i(X^i_T + e^i_T)] &= \frac{1}{\alpha^i} \mathbb{E}[\int_0^T -\exp(-\alpha^i c_s) \, ds] + \frac{1}{\alpha^i} \mathbb{E}[\exp(-\alpha^i (X^i_T + e^i_T))] \\
&= \frac{1}{\alpha^i} \left( \mathbb{E}[\int_0^T -\exp(-\alpha^i c_s) \, ds] + \mathbb{E}[\exp(-\alpha^i X^i_T/A - Y^i_T)] \right) \\
&= \frac{1}{\alpha^i} \mathbb{E}[V^i_T] \leq \frac{1}{\alpha^i} V^i_0 = -\frac{1}{\alpha^i} \exp(-\alpha^i \pi^i_0 - Y^i_0).
\end{align*}

Next, in order to characterize the optimizer, we construct a consumption process for which $\mu_V = 0$. More precisely, we let the process $\tilde{X}^i$ be the unique solution of the following linear SDE:
\begin{align*}
\tilde{X}^i_0 = \pi^i_0 A_0, \quad d\tilde{X}^i = \left( \mu \tilde{X}^i + (e^i - \frac{1}{\alpha^i} (a + Y^i) - \frac{\tilde{X}^i}{\alpha^i}) \right) \, dt + \tilde{X}^i \sigma \, dB, \quad (3.4)
\end{align*}
and set
\[ \hat{c}^i = \frac{1}{\alpha^i}(a + Y^i) + \frac{\hat{X}^i}{\alpha^i}, \quad \hat{\pi}^i = \frac{\hat{X}^i}{\alpha^i}. \]

It follows immediately that \((\hat{\pi}^i, \hat{c}^i) \in \mathcal{A}^i(\pi^i_0)\) and that the process \(\hat{X}\) is the associated gains process. The choice of \(\hat{c}^i\), through \(\hat{X}^i\), makes the process \(V^i\) a martingale and the pair \((\hat{\pi}^i, \hat{c}^i)\) optimal for agent \(i\).

Turning to market clearing, we consider the process
\[ F = a + \sum_{i=1}^I \kappa^i Y^i - \bar{\alpha} e, \]
whose dynamics are given by
\[ dF = (\sigma + \sum_{i=1}^I \kappa^i Z^i - \bar{\alpha} \sigma_e) dB + \exp(-a) F dt, \quad F_T = 0. \quad (3.5) \]
In other words, the pair \((Y, \zeta) = (F, \sigma + \sum_{i=1}^I \kappa^i Z^i - \bar{\alpha} \sigma_e)\) is an \(S^\infty \times \text{bmo}\)-solution to the linear BSDE
\[ dY = \zeta dB + \exp(-a) Y dt, \quad Y_T = 0. \]

Since \(a\) is bounded, the coefficients of this BSDE are globally Lipschitz, and, therefore, by the uniqueness theorem (see [Zha17, Theorem 4.3.1, p. 84]), we can conclude that \(F = 0\). That implies that
\[ a + \sum_{i=1}^I \kappa^i Y^i = \bar{\alpha} e \text{ on } [0, T], \]
and so,
\[ \sum_{i=1}^I \hat{c}^i = e + \frac{1}{\bar{\alpha}} \sum_{i=1}^I \hat{X}^i. \]
The form of the dynamics (3.4) of each \(\hat{X}^i\) leads to the following dynamics for \(\hat{X} = \sum_{i=1}^I \hat{X}^i\):
\[ d\hat{X} = (\mu \hat{X} - \frac{1}{\bar{\alpha}} \hat{X}) dt + \hat{X} \sigma dt. \quad (3.6) \]
The assumption that \(\sum \pi^i_0 = 1\) implies that \(\hat{X}_0 = A_0\), which, in turn, implies that the process \(A\) is also a solution to (3.6). By uniqueness, we must have \(\hat{X} = A\) and conclude that the clearing conditions are satisfied. \(\square\)

3.2. Existence of an equilibrium. Next, we show that under additional assumptions on the problem ingredients - most notably that of a Markovian structure - the characterization of Theorem 3.1 can be used to establish the existence of an equilibrium market.

**Theorem 3.3.** Under Assumptions 2.1 and 2.3, the system (3.2) admits an \(S^\infty \times \text{bmo}\)-solution.

The BSDE characterization of Theorem 3.1 immediately implies the main result of the paper:

**Corollary 3.4.** Under Assumptions 2.1 and 2.3 there exists a set \(\gamma^* = (\mu^*, \sigma^*)\) of feasible market coefficients such that \(A^{\gamma^*}\) is an equilibrium market.

**Proof of Theorem 3.3.** In certain situations it will be convenient to standardize the notation, so we also write \(Y^0\) for \(a\), \(Z^0\) for \(\sigma\), and set
\[ g^i(x) = \begin{cases} 0, & i = 0, \\ \alpha^i e^i(T, x), & 1 \leq i \leq I. \end{cases} \]
The $dt$-terms in (3.2) define the driver $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{I+1} \times \mathbb{R}^{(I+1)\times d} \to \mathbb{R}^{I+1}$ in the usual way:

$$
\begin{align*}
    f^0(t, x, y, z) &= \bar{\alpha}\mu_c(t, x) - \frac{1}{2} \sum_{i=1}^{I} \kappa^i |z^i|^2 - \exp(-y^0), \\
    f^i(t, x, y, z) &= \frac{1}{2} |z^i|^2 + \exp(-y^0) \left(1 + y^0 + y^i - \alpha^i e^i(t, x)\right), \quad \text{for } i = 1, \ldots, I.
\end{align*}
$$

The system (3.2), written in the new notation, becomes

$$
dY^i_t = f^i(t, \xi_t, Y_t, Z_t) dt + Z^i_t dB_t, \quad Y^i_T = g^i(\xi_T), \quad i = 0, \ldots, I. \tag{3.7}
$$

**Step 1 (truncation).** We start by truncating the driver $f$ to obtain a sequence of well-behaved, Lipschitz problems. More precisely, given $N > 0$ we define

$$
\iota_N(x) = \max(\min(x, N), -N) \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad q_N(z) = |z| \iota_N(|z|), \quad \text{for } z \in \mathbb{R}^{1 \times d},
$$

so that $\iota_N$ and $q_N$ are Lipschitz functions with Lipschitz constants 1 and $N$, respectively. Moreover,

$$
|\iota_N(x)| \leq N \quad \text{and} \quad |q_N(z)| \leq N |z|.
$$

Using the functions defined above, for each $N \in \mathbb{N}$ we pose a truncated version of (3.2):

$$
\begin{align*}
da &= \sigma dB + \left(\bar{\alpha}\mu_c - \frac{1}{2} \sum_{i=1}^{I} \kappa^i q_N(Z^i) - \exp(-\iota_N(a))\right) dt, \\
dY^i &= Z^i dB + \left(\frac{1}{2} q_N(Z^i) + \exp(-\iota_N(a))(1 + \iota_N(a) + \iota_N(Y^i) - \alpha^i e^i)\right) dt
\end{align*}
$$

(BSDE$_N$) with the terminal conditions $Y^i_T = \iota_N(g^i_T)$ and $a_T = 0$. We define the driver $(t, x, y, z) \mapsto f^{(N)}(t, x, y, z)$ from the $dt$-terms in the standard way.

For each $N \in \mathbb{N}$, $f^{(N)}$ is continuous in all of its variables, uniformly Lipschitz in both $z$ and $y$, and $f^{(N)}(t, x, 0, 0)$ is bounded. Assumption 2.1 guarantees that the same is true for the function $F^{(N)}(t, x, y, z) = -f^{(N)}(t, x, y, z \Sigma^{-1}(t, x))$. Therefore, we can apply Proposition 4.1 in the Appendix to conclude that there exists a solution $(Y^{(N)}, Z^{(N)})$ to (BSDE$_N$) of the form

$$
Y^i_t = v^{(N)}(t, \xi_t), \quad Z^i_t = w^{(N)}(t, \xi_t),
$$

with $v^{(N)} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{I+1}$ bounded and $w^{(N)} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{(I+1)\times d}$ such that $Z^{(N)}$ is a bmo process. We note that existence for (BSDE$_N$) is also guaranteed by the classical result [PP90, Theorem 3.1, p. 58], but only in the class $S^2 \times \mathcal{H}^2$, which is too big for our purposes.

**Step 2 (uniform estimates).** The bounds guaranteed by Proposition 4.1 all depend on the truncation constant $N$, so our next task is to explore the special structure of our system and establish bounds in terms of universal quantities. A universal constant, in this proof, will be a quantity that depends on the constants $\alpha^i$, the time-horizon $T$ and the $S^\infty$-bounds on $e^i$ and $\mu_c$, but not on $N$. We denote such a constant by $C$, and allow it to change from line to line.
Let \(((a^{i(N)}, Y^{i(N)}, \sigma^{(N)}, Z^{(N)})\) be the solution to the truncated system from Step 1 above. It follows from the dynamics of \(a^{(N)}\) and the fact that \(q_N(z) \geq 0\) for all \(z \in \mathbb{R}^{1 \times d}\) that \(a^{(N)} - \int_0^t \bar{a}\mu_e dt\) is a supermartingale, so that for all \(t \in [0, T]\),

\[
a_t^{i(N)} \geq \mathbb{E}[a_T^{i(N)} - \int_t^T \bar{a}\mu_e dt] \geq -(T-t)\|\bar{a}\mu_e\|_{S^\infty}, \text{ i.e., } a_t^{i(N)} \geq -C.
\]

Next, we turn to \(Y^{(N)}\) and use the fact that the components of \(Y\) are coupled only through \(a\). This way, we can get uniform bounds on \(Y^{i(N)}\) if we manage to produce a uniform bound on the function of \(a\) appearing on the right-hand side. We start by using the following easy-to-check inequality

\[
\exp(-x)(1 + |x|) \leq \exp(2x^-), \text{ for all } x \in \mathbb{R},
\]

and the fact that \((t_N(x))^{-} \leq (x)^{-}\) for all \(x\), to obtain that for all \(t \in [0, T]\),

\[
\exp(-t_N(a_t^{i(N)})) \left(1 + |t_N(a_t^{i(N)})|\right) \leq C.
\]

It is readily checked that there exist a bounded measurable function \(\delta^{(N)} : \mathbb{R}^{1 \times d} \to \mathbb{R}^d\), such that

\[
q_N(z) = z \delta^{(N)}(z) = \sum_{j=1}^d z_j \delta^{i_j(N)}(z), \text{ for } z = (z_1, \ldots, z_d) \in \mathbb{R}^{1 \times d}.
\]

Therefore, for each \(i = 1, \ldots, I\), there exists a probability measure \(\mathbb{P}^i = \mathbb{P}^{i,N} \sim \mathbb{P}\) under which the process \(\bar{B} = B + \int_0^t \delta^{(N)}(Z_t^{i,(N)}) dt\) is a Brownian motion on \([0, T]\). Since \(Z^{i,(N)}\) is guaranteed to be in bmo, it remains in bmo under the measure \(\mathbb{P}^i\) (see [Kaz94] Theorem 3.3, p. 57]. Therefore, the process \(\int Z^{i,(N)} dB^i\) is a \(\mathbb{P}^i\)-martingale and we can take the expectation of the \(i\)-th equation with respect to \(\mathbb{P}^i\) to obtain

\[
\|Y_t^{i,(N)}\| \leq \mathbb{E}^i[|t_N(a_t^{i,(N)}(T))| |\mathcal{F}_t|] + \int_t^T \mathbb{E}^i \left[\exp(-t_N(a_s^{i,(N)})) \left(1 + |t_N(a_s^{i,(N)})|\right) |\mathcal{F}_t\right] ds
\]

\[
+ \int_t^T \mathbb{E}^i \left[\exp(-t_N(a_s^{i,(N)})) |t_N(Y_s^{i,(N)}) - \alpha^i e_s^i| |\mathcal{F}_t\left.\right| ds
\]

\[
\leq C \left(1 + \int_t^T \mathbb{E}^i[|y_t^{i,(N)}| |\mathcal{F}_t| ds\right) \leq C \left(1 + \int_t^T y_t^{i}(s) ds \right),
\]

where \(y^i(t) = ||Y_t^{i,(N)}||_{S^\infty}\). Thus, \(y^i\) satisfies

\[
y^i(t) \leq C \left(1 + \int_t^T y^i(s) ds \right), \text{ for all } t \in [0, T],
\]

for some universal constant \(C\). Gronwall’s inequality implies that \(y^i(0) = ||Y_0^{i,(N)}||_{S^\infty}\) is bounded by another universal constant, so we conclude that there exists a universal \(S^\infty\)-bound on all \(Y^{i,(N)}\).

Our next goal is to produce universal bmo bounds on the processes \(Z^{i,(N)}\). This will follow by using the universal boundedness of the \(Z\)-free terms in the driver of \(Y^{i,(N)}\) obtained above. Since the \(i\)-th component of the driver \(f^{(N)}\) depends on \(Z^{(N)}\)
only through $Z^{i(N)}$, for $1 \leq i \leq I$, we can apply standard exponential-transform estimates. We adapt the argument in [EB13, Proposition 2.1, p. 2925] and define
\[ \phi(x) := \frac{\exp(2|x|) - 1 - 2|x|}{4} \text{ for } x \in \mathbb{R}, \]
noting that both $\phi$ and $\phi'$ are nonnegative and increasing, while $\phi \in C^2(\mathbb{R})$ with $\phi'' - 2|\phi'| = 1$. Thus, for any stopping time $\tau$ in $[0, T]$, Itô’s Lemma gives us that
\[
0 \leq \phi(Y^{i(N)}_\tau) \leq \mathbb{E}[\phi(Y^{i(N)}_\tau)|\mathcal{F}_\tau] + \mathbb{E} \left[ \int_\tau^T \phi'(Y^{i(N)}_s) \left( C \left( 1 + \|Y^{i(N)}_s\|_{S^\infty} \right) \right) ds \right] |\mathcal{F}_\tau | \\
\leq \phi(\|Y^{i(N)}\|_{S^\infty}) + C \int_0^T \phi'(\|Y^{i(N)}\|_{S^\infty})(1 + \|Y^{i(N)}\|_{S^\infty}) ds - \mathbb{E} \left[ \int_\tau^T \|Z^{i(N)}_s\|^2 ds \right] |\mathcal{F}_\tau |.
\]
Rearranging terms yields
\[
\mathbb{E} \left[ \int_\tau^T \|Z^{i(N)}_s\|^2 ds \right] |\mathcal{F}_\tau | \leq \phi(\|Y^{i(N)}\|_{S^\infty}) + C \int_0^T \phi'(\|Y^{i(N)}\|_{S^\infty})(1 + \|Y^{i(N)}\|_{S^\infty}) ds.
\]
The right-hand side admits a universal bound (independent of $N$ and $\tau$), and, hence, so does the left-hand side.

Finally, we go back to the equation satisfied by $a^{(N)}$ and note that the term $\exp(-\iota_N(a^{(N)}))$ is bounded because $(a^{(N)})^-$ is. We can bound $a^{(N)}$ from above in an $N$-independent manner, by a combination of the bmo-bounds on $Z^{(N)}$ and the sup norm of $\mu_\epsilon$. By taking expectations and using universal boundedness/bmo-property of all the other terms, we conclude that $\sigma^{(N)}$ also admits a universal bmo-bound.

Having the universal bounds on $Y^{(N)}$ and $a^{(N)}$, we can remove some of the truncations introduced in $\text{BSDE}^{(N)}_\epsilon$. Indeed, for $N$ larger than the largest of the $S^\infty$-bounds on $Y^{(N)}$ and $a^{(N)}$, we have
\[
\iota_N(Y^{i(N)}_\tau) = Y^{i(N)}_\tau \quad \text{and} \quad \iota_N(a^{(N)}_\tau) = a^{(N)}_\tau.
\]
Therefore, there exists a constant $N_0$ such that for $N \geq N_0$ the processes $(Y^{(N)}, a^{(N)})$ together with $(Z^{(N)}, \sigma^{(N)})$ solve the intermediate system
\[
da = \sigma dB + \left( \hat{\omega} \mu_\epsilon - \frac{1}{2} \sum_j \kappa^j q_N(Z^j) - \exp(-\iota_{N_0}(a)) \right) dt,
\]
\[
dY^i = Z^i dB + \left( \frac{1}{2} q_N(Z^i) + \exp(-\iota_{N_0}(a)) \iota_{N_0}(Y^i) + \iota_{N_0}(a) - \alpha^i \epsilon^i + 1 \right) dt
\]
(\text{BSDE}^{(N)}_\epsilon)
with the same terminal conditions as (3.2).
Step 3 (Bensoussan-Frehse conditions and the existence of a Lyapunov function). Mere boundedness in $S^{\infty} \times \text{sbo}$ is not sufficient to guarantee subsequential convergence of the solution $(Y^{(N)}, a^{(N)})$ of the truncated system to a limit which solves 
(BSDE$_N$). It has been shown, however, in [XZ18, Theorem 2.8, p. 501], that an additional property - namely the existence of a uniform Lyapunov function - will guarantee such a convergence. The existence of such a function can be deduced from another result of the same paper, [XZ18, Proposition 2.11, p. 503], once its conditions are checked. This proposition states that a uniformly bounded sequence of solutions of a sequence of BSDE such as (BSDE$_N$) admits a common Lyapunov function if the structure of its drivers satisfies the so called Bensoussan-Frehse conditions uniformly in $N$ (see [XZ18, Definition 2.10, p. 502] for the definition). It applies here because our system is of upper-triangular form when it comes to its quadratic dependence on $z$. More precisely, the driver of the system (BSDE$_N$) can be represented as a sum of two functions $f_1^{(N)}$ and $f_2^{(N)}$ given by

$$
(f_1^{(N)})^i(t, x, y, z) = \begin{cases} 
\bar{\alpha} \mu_z(t, x) - \exp(-t_{iN_0}(y)) & i = 0 \\
\exp(-t_{iN_0}(y)) (\iota_{N_0}(y^i) + \iota_{N_0}(y^0) - \alpha^i e^i(t, x) + 1) & 1 \leq i \leq I 
\end{cases}
$$

$$
(f_2^{(N)})^i(t, x, y, z) = \begin{cases} 
-\frac{\alpha}{2} \sum_{l=1}^{l=N} \kappa q N(z^l) & i = 0 \\
\frac{C}{N} q N(z^i) & 1 \leq i \leq I 
\end{cases}
$$

where the convention that $a = Y^0$ and $\sigma = Z^0$ is used. Therefore, there exists a universal constant $C$ such that, for all $1 \leq i \leq I + 1$, we have

$$
|f_1^{(N)}|^i(t, x, y, z)| \leq C
$$

as well as

$$
|f_2^{(N)}|^i(t, x, y, z)| \leq C(1 + \sum_{j=1}^{i} q N(z^j)) \leq C(1 + \sum_{j=1}^{i} z^j^2).
$$

Therefore, $f^{(N)}$ can be split into a subquadratic (in fact bounded) and an upper triangular component, allowing us to conclude that a uniform Lyapunov function for $(f^{(N)})_{N \geq N_0}$ can be constructed.

Step 4 (Passage to a limit). It remains to use [XZ18, Theorem 2.8, p. 501] to conclude that a subsequence of $v^{(N)}$ converges towards a continuous function $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{I+1}$ such that $Y_t = v(t, \xi_t)$ and $Z_t = Dv(t, \xi_t)$ solves the limiting system

$$
da = \sigma dB + \left( \bar{\alpha} \mu_z - \frac{1}{2} \sum_{l} \kappa l |Z^l|^2 - \exp(-\iota_{N_0}(a)) \right) dt.
$$

$$
dY^i = Z^i dB + \left( \frac{1}{2} |Z^i|^2 + \exp(-\iota_{N_0}(a)) (\iota_{N_0}(Y^i) + \iota_{N_0}(a) - \alpha^i e^i + 1) \right) dt,
$$

(BSDE$^i$)

for $i = 1, \ldots, I$, where, as above $a = Y^0$ and $\sigma = Z^0$. As far as the conditions of Theorem 2.8 in [XZ18] are concerned, the most difficult one, the existence of a Lyapunov function, has been settled in Step 3. above. The other conditions - the uniform Hölder boundedness of the terminal conditions, and a-priori boundedness - are easily seen to be implied by our standing assumptions. Finally, since $Y$ is a
pointwise limit of a sequence of functions bounded by $N_0$, the same processes $(Y, Z)$ also solve the original BSDE (3.2) (without truncation at $N_0$).

4. Bounded solutions of Lipschitz quasilinear systems

The main result of this section, Proposition 4.1, collects some results on systems of heat equations with Lipschitz nonlinearities on derivatives up to the first order. We suspect that these results may be well-known to PDE specialists, but we were unable to find a precise reference under the same set of assumptions in the literature, and, therefore, decided to include a fairly self-contained proof.

In the sequel, $D$ denotes the derivative operator with respect to all spatial variables, i.e., all variables except $t$. For $d, J \in \mathbb{N}$ and $\beta \geq 0$, we define the following three Banach spaces:

1. $\mathbb{L}_\infty = \mathbb{L}_\infty^d(\mathbb{R}^d, \mathbb{R})$ or $\mathbb{L}_\infty = \mathbb{L}_\infty^{J \times d}$, depending on the context,
2. $W^{1, \infty} = W^{1, \infty}(\mathbb{R}^d, \mathbb{R}^J)$, with the norm $\|U\|_{W^{1, \infty}} = \|U\|_{L_\infty} + \|DU\|_{L_\infty}$,
3. $\mathbb{L}^{\beta}_J = \mathbb{L}^{\beta}_J([0, T); W^{1, \infty})$ - the Banach space of measurable functions $u : [0, T] \to W^{1, \infty}$, endowed with the exponentially weighted norm

$$\|u\|_{L^{\beta}_J} = \int_0^T e^{-\beta(t-T)} \|u(t, \cdot)\|_{W^{1, \infty}} dt.$$ 

The infinitesimal generator of the state process $\xi$ is given by

$$Au(t, x) = Du(t, x) \Lambda(t, x) + \frac{1}{2} \text{Tr} \left(D^2u(t, x)\Sigma(t, x)\Sigma^T(t, x)\right)$$

for $(t, x) \in [0, T] \times \mathbb{R}^d$.

**Proposition 4.1.** Suppose that $g : \mathbb{R}^d \to \mathbb{R}^J$ and $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^J \times \mathbb{R}^{J \times d} \to \mathbb{R}^J$ are measurable functions such that

- $|g(x)| \leq M$,
- $|F(t, x, 0, 0)| \leq M$, and
- $|F(t, x, y_2, z_2) - F(t, x, y_1, z_1)| \leq M(|y_2 - y_1| + |z_2 - z_1|),$

for some $M$ and all $t, x, y_1, y_2, z_1, z_2$, and that the functions $\Lambda$ and $\Sigma$ satisfy the conditions of Assumption 2.1 (with the constant $K$). Then the following statements hold:

1. The PDE system

$$u_t + Au + F(\cdot, \cdot, u, Du) = 0, \quad u(T, \cdot) = g$$

admits a weak solution $u$ on $[0, T]$. Moreover $u(t, \cdot) \in W^{1, \infty}$ for all $t \in [0, T]$ and there exists a constant $C = C(J, d, M, T, K) \in [0, \infty)$ such that

$$\|u(t, \cdot)\|_{L_\infty} \leq C \text{ for all } t \in [0, T] \text{ and } \int_0^T \|Du(t, \cdot)\|_{L_\infty} dt \leq C$$

2. Let $u$ denote a solution of (4.1) as in (1) above and let $\{\xi_t\}_{t \in [0, T]}$ be a strong solution of the SDE

$$d\xi_t = \Lambda(t, \xi_t) dt + \Sigma(t, \xi_t) dB_t.$$
The pair \((Y_t, Z_t)\), where \(Y_t = u(t, \xi_t)\) and \(Z_t = Du(t, \xi_t)\Sigma(t, \xi_t)\) is an \(S^\infty \times bmo\)-solution to the system

\[
dY_t^i = -F^i(t, \xi_t, Y_t, Z_t, \Sigma^{-1}(t, \xi_t)) \, dt + Z_t^i \, dB_t, \quad Y_t^i = g^i(\xi_T), \quad i = 1, \ldots, I
\]  

(4.2)

Proof. Throughout the proof, \(C\) will denote a constant which may depend on \(J, d, M, T\) or \(K\), but not on \(\beta, t, s\) or \(x\), and can change from line to line; we will call such a constant universal. The assumptions on \(F\) imply that uniformly in \(t\), and for all \(U, V \in W^{1, \infty}\), we have

\[
||F(t, \cdot, U, DU) - F(t, \cdot, V, DV)||_{L^\infty} \leq C||U - V||_{W^{1, \infty}}, \quad \text{and} \quad ||F(t, \cdot, U, DU)||_{L^\infty} \leq C(1 + ||U||_{W^{1, \infty}}).
\]  

(4.3) and (4.4)

Let \(p(t, x; s, x')\) denote a fundamental solution associated to the operator \(u_t + Au\), i.e., \((t, x) \mapsto p(t, x, s, x')\) solves

\[
p_t + Ap = 0 \quad \text{for} \quad t, x \in [0, s) \times \mathbb{R}^d
\]

classically and satisfies the boundary condition \(\lim_{t \to s^-} \int_{\mathbb{R}^d} \psi(x)p(t, x, s, x') \, dx = \psi(x')\) for each bounded and continuous \(\psi\). We refer the reader to [Fri64, Theorem 10, p. 23] and the discussion preceding it for existence of a positive fundamental solution under the conditions of Assumption 2.1. Moreover, the equations (6.12) and (6.13) on p. 24 of [Fri64] state that there exist universal constants \(C, \lambda > 0\) such that for \(t < s\) and all \(x, x'\) we have

\[
|p(t, x, s, x')| \leq C \varphi_\lambda(t, x, s, x') \quad \text{and} \quad |\partial_{x_k} p(t, x, s, x')| \leq C \frac{1}{\sqrt{\lambda(s-t)}} \varphi_\lambda(t, x, s, x'),
\]  

(4.5)

for all \(k = 1, \ldots, d\), where

\[
\varphi_\lambda(t, x; s, x') = \exp \left( - \frac{1}{2\lambda(s-t)} |x' - x|^2 \right)
\]

is the scaled heat kernel (which is, itself, a fundamental solution associated to \(u_t + \frac{1}{2} \lambda^2 \Delta u_\)).

These properties, in particular, allow us to define the function \(\Phi[u] : [0, T] \times \mathbb{R}^d \to \mathbb{R}^J\) by

\[
\Phi[u](t, x) = \int_{\mathbb{R}^d} \int_t^T F \left( s, x', u(s, x'), Du(s, x') \right) p(t, x; s, x') \, ds \, dx',
\]  

(4.6)

for each \(u \in L^1_{\beta}\). The equation (4.4) guarantees that \(\Phi[u]\) is well-defined with \(\Phi[u](t, \cdot) \in L^\infty\).

The Gaussian bounds in (4.5) imply that one can pass the derivative under the integral sign to obtain

\[
\partial_{x_k} \Phi[u](t, x) = \int_{\mathbb{R}^d} \int_t^T F \left( s, x', u(s, x'), Du(s, x') \right) \partial_{x_k} p(t, x; s, x') \, ds \, dx',
\]  

(4.7)

Consequently \(t \mapsto \Phi[u](t, \cdot)\) is an a.e-defined measurable \([0, T] \to W^{1, \infty}\), for each \(u \in L^1_{\beta}\). To bound the norm of \(\Phi[u]\) we start with the following estimate,
A similar computation also yields
\[ ||\Phi[u](t, \cdot)||_{W^{1,\infty}} \leq C \int_t^T \left( 1 + ||u(s, \cdot)||_{W^{1,\infty}} \right) \int_{\mathbb{R}^d} \left( p(t, x; s, x') + \sum_{k=1}^{J} |\partial_{x_k} p(t, x; s, x')| \right) dx' \, ds \]
\[ \leq C \int_t^T \frac{1}{\sqrt{s-t}} (1 + ||u(s, \cdot)||_{W^{1,\infty}}) \, ds. \]
Furthermore, \((4.6)\) and \((4.7)\) imply that and, so,
\[ \Gamma \in L^\infty \text{ maps } u \text{ representations in } (4.6) \text{ and } (4.7) \text{ allow us to conclude that } \Gamma \text{ belongs to } L^\infty. \]
\[ \text{Next, for } g \in L^\infty, \text{ we define } \Psi[g](t, x) = \int_{\mathbb{R}^d} g(x') p(t, x; T, x') \, dx'. \]
so that, as above,
\[ ||\Psi[g](t, \cdot)||_{W^{1,\infty}} \leq C \sqrt{T-t} ||g||_{L^\infty} \text{ and } ||\Psi[g]||_{L^1_{\beta}} \leq \frac{C}{\sqrt{\beta}} ||g||_{L^\infty}, \]
and \(\Psi[g] \in L^1_{\beta}\) for each \(g \in L^\infty\). Therefore, the function
\[ \Gamma[u] = \Phi[u] + \Psi[g], \]
maps \(L^1_{\beta}\) into \(L^1_{\beta}\) and \((4.8)\) implies that it is Lipschitz, with constant \(C/\sqrt{\beta}\). Since \(C\) does not depend on \(\beta\), we can turn \(\Gamma\) into a contraction by choosing a large-enough \(\beta\), and conclude that \(\Gamma\) admits a unique fixed point \(u \in L^1_{\beta}\). The integral representations in \((4.8)\) and \((4.7)\) allow us to conclude that \(u\) and \(Du\) are continuous functions on \([0, T) \times \mathbb{R}^d\). Moreover, thanks to the Markov property of \(\xi\), we have
\[ u(t, \xi_t) = \mathbb{E}[g(\xi_T)] + \int_t^T f(s, \xi_s) \, ds |\mathcal{F}_t|, \ a.s. \]
where
\[ f(s, x) = F(s, x, u(s, x), Du(s, x)) \in \mathbb{R}^J. \]
Since \(||f(t, \cdot)||_{L^\infty} \leq C(1 + ||u(t, \cdot)||_{W^{1,\infty}})\) for all \(t\), the map \(t \mapsto ||u(t, \cdot)||_{W^{1,\infty}}\) belongs to \(L^1_{\beta}\), and (stripped of its norm) the space \(L^1_{\beta}\) does not depend on the choice of \(\beta\). Therefore,
\[ \left| |u(t, \xi_t) - \mathbb{E}[g(\xi_T)|\mathcal{F}_t]| \right|_{L^\infty} \leq \int_t^T ||f(s, \cdot)||_{L^\infty} \, ds \to 0 \text{ as } t \to T. \]
Since \( g \) is bounded, we have \( ||u(t, \cdot)||_{L^\infty} \leq C \) for all \( t \). Moreover, the martingale \( E[g(\xi_T)|\mathcal{F}_t] \) admits a continuous modification, so the process \( Y \), defined by

\[
Y_t = \begin{cases} 
  u(t, \xi_t), & t < T \\
  g(\xi_T), & t = T,
\end{cases}
\]

is a.s.-continuous. This allows us to conclude, furthermore, that \( Y_t + \int_0^t f(s, \xi_s) \) is a continuous modification of the martingale

\[
M_t = E[g(\xi_T) + \int_0^T f(s, \xi_s) \, ds|\mathcal{F}_t],
\]

making \( Y \) a semimartingale. To show that \((Y, Z)\) as in the statement indeed solves (1.2), we need to argue that the martingale \( M_t - M_0 \) must be of the form \( \int_0^t Du(s, \xi_s) \Sigma(s, \xi_s) \, dB_s \). This can be proven by approximation as in the proof of [XZ18, Lemma 4.4, p. 516]).

The last step is to argue that \((Y, Z)\) is an \( S^\infty \times \text{bmo}\)-solution. The function \( u \) is uniformly bounded, so it suffices to establish the bmo-property of \( Z \). This can be bootstrapped from the boundedness of \( Y \) by applying Itô’s formula to the bounded processes \( \exp(cY^i) \), \( i = 1, \ldots, J \), for large-enough constant \( c \). A similar argument is already presented on page 11 in the proof of Theorem 3.3, so we skip the details. \( \square \)

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