A PERTURBATIVE ANALYSIS OF SYNCHROTRON SPECTRAL INDEX VARIATION OVER THE MICROWAVE SKY

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ABSTRACT

In this paper, we implement a perturbative approach, first proposed by Bouchet & Gispert, to estimate the variation of the spectral index of galactic polarized synchrotron emission, using a linear combination of simulated Stokes $Q$ polarization maps of selected frequency bands from WMAP and Planck observations on a region of sky dominated by the synchrotron Stokes $Q$ signal. We find that a first order perturbative analysis recovers the input spectral index map well. Along with the spectral index variation map, our method provides a fixed reference index, $\beta_0$, over the sky portion being analyzed. Using Monte-Carlo simulations, we find that $\langle \beta_0 \rangle = -2.84 \pm 0.01$, which matches very closely with the position of a peak at $\beta(p) = -2.85$ of the empirical probability density function of input synchrotron indices obtained from the same sky region. For thermal dust, the mean recovered spectral index $\langle \beta_d \rangle = 2.00 \pm 0.004$ from simulations, matches very well with the spatially fixed input thermal dust spectral index $\beta_d = 2.00$. As accompanying results of the method, we reconstruct cosmic microwave background, thermal dust, and a synchrotron template map with fixed spectral indices over the entire sky region. We use, in our analysis, full pixel–pixel noise covariance matrices of all frequency bands, estimated from the sky region being analyzed. The perturbative technique of this work (1) can build a model with an arbitrary but sufficient degree of accuracy (and precession) as allowed by the data and (2) can produce maximum likelihood estimators for reference indices and templates asymptotically.

Key words: cosmic background radiation – cosmology: observations – diffuse radiation

1. INTRODUCTION

Observations of the microwave signal by WMAP and Planck satellite missions, in the frequency range of 23–857 GHz, can be used to obtain a wealth of precise cosmological information (e.g., precise values of basic cosmological parameters (Bennett et al. 2013; Hinshaw et al. 2013; Planck Collaboration et al. 2015c)), topology of space (Planck Collaboration et al. 2015f), laws and nature of physical conditions existing at different epoch marked by different phases of dynamical evolution of spacetime (Planck Collaboration et al. 2014, 2015d, 2015e, 2015g)) as well as precise information about various astrophysical mechanisms operating since recent times that cause diffuse emissions inside the Milky Way (Bennett et al. 2003; Dobler 2012; Planck Collaboration et al. 2013, 2015b). However, extracting this information accurately with precision requires a method that separates different components from their observed mixture using a model that is flexible enough to describe the data with an arbitrary but desired degree of minute closeness as permissible by the data. A completely accurate model may produce ambiguous results if the data is simply insufficient to constrain all model parameters correctly, e.g., consider the astrophysical problem of reconstructing various galactic emission components, which is a highly complex task due to the inherent degeneracy existing in the morphological pattern of these emissions, the degeneracy arising due to similarities in physical models of multiple emissions, or other reasons. Such uncertainties in reconstructed foreground components may cause cosmological inferences drawn from the analysis of reconstructed cosmic microwave background (CMB) maps to be biased (Tegmark et al. 2000) if these uncertainties are not known or if they are not properly taken into account.

One of the prime sources of uncertainties against properly extracting the primordial signal and other galactic foreground components from observed maps is the intrinsic variation of the spectral index of synchrotron emission (Tegmark 1998) over the sky due to variations in number density of cosmic-ray electrons in our galaxy. In this paper, we implement a novel method using simulated WMAP and Planck Stokes $Q$ frequency maps to perturbatively model the synchrotron spectral index variation over the sky following the approach proposed by Bouchet & Gispert (1999). The perturbative nature of the method is very desirable since it allows one to build a model with just enough accuracy that it is still meaningful within the available data. As associated results of the method, we reconstruct a best-fit map for the CMB and templates for thermal dust and synchrotron emission.

Usefulness of the method described in this paper can also be seen from a different perspective. There exist several methods in the literature for CMB or (and) foreground component reconstruction, which can be grouped broadly in the following three categories.

1. The so-called blind, internal-linear-combination approach (Tegmark & Efstathiou 1996; Maino et al. 2002; Tegmark et al. 2003; Delabrouille et al. 2009; Saha 2011; Bennett 2012; Planck Collaboration et al. 2015a), where one removes foregrounds based upon the fact that CMB follows the blackbody spectrum, whereas other components do not. In principle, this method can also be used to produce a cleaned map of any of the foreground components (Remazeilles et al. 2011), provided its spectral index is known a priori and is constant over the sky portion being analyzed. The method by definition is a non-maximal likelihood (non-ML)
approach since it incorporates minimum information about the physical components and no information about detector noise.

2. The second class (Eriksen et al. 2008a, 2008b; Planck Collaboration et al. 2015a, 2015b) is one in which as much prior information about all components is used as possible, along with the detector noise model, to perform an ML analysis of the data. The outcome of this method is ML estimates of all components.

3. The third one is an intermediate approach of the first and second categories. Although, here, one incorporates information about various components available from prior observations, (e.g., Maximum Entropy Method (Hobson et al. 1998; Bennett et al. 2003; Hinshaw et al. 2007; Gold et al. 2009, 2011) and the Wiener filtering approach (Bunn et al. 1994; Tegmark & Efstathiou 1996; Bouchet et al. 1999), it is a non-ML approach.

All the above methods have merit and demerit of their own. The ML approach has advantages. It is the optimal method in the sense that it retains all information regarding the data and provides maximum precision on the derived parameters. The ML method may, however, be biased if the model used for the data is not accurately known or if there are degeneracies between the parameters. The non-ML approaches are advantageous since data is processed with minimal assumptions. However, it is a sub-optimal method and can be biased. Therefore, a question arises: can we develop a method that requires minimal assumptions in terms of the data model (e.g., no requirement to use templates) and at the same time, a method that is an ML approach with respect to at least almost all of its parameters? So far, there has not been any attempt in the literature to answer this question. The novel perturbative approach for component separation and for the determination of the variation of the spectral index of synchrotron emission followed in this paper, on the one hand, is a linear-combination method, and on the other hand, is an asymptotically ML approach with respect to all of its component templates and best-fit spectral indices for synchrotron and thermal dust. The method incorporates information about spectral properties of components and the detector noise model, thus preserving properties of the first and second categories described above.

There have been many attempts in the literature to measure the synchrotron spectral index of our Galaxy. Reich & Reich (1988) find a map of the synchrotron spectral index between 420 and 1420 MHz using radio continuum surveys. Following the method of the TT plot, Davies et al. (1996) find that the synchrotron temperature spectral index lies in the range of $-2.8$ to $-3.2$ from a strip at $\delta = 40^\circ$ latitude sections of 408 MHz and 1420 MHz maps. Platania et al. (1998) find a synchrotron spectral index between 420 MHz and 7.5 GHz as $-2.76 \pm 0.11$. Bennett et al. (2003) find a flatter synchrotron spectral index of approximately $-2.5$ near the galactic plane and a steeper index of approximately $-2.9$ toward the higher galactic latitude. Using WMAP nine-year data, Fuskeland et al. (2014) estimate a polarized synchrotron spectral index toward the galactic plane as $-2.98 \pm 0.01$, and a steeper spectral index, $-3.12 \pm 0.04$, toward the higher galactic latitude.

\section{2. BASIC PROBLEM}

Let us assume that we have observations of CMB and foreground polarizations at $n_f$ different frequency bands. Each frequency band contains a contribution from $n$ different physical components, namely, CMB and emissions from $n-1$ foreground components—each with a given spectral index all over the sky—plus detector noise.\footnote{Detector noise does not count as a physical component.} If the foreground component $j$ has a fixed spectral index, $\beta_j$, all over the sky, net polarized Stokes $Q$ signal, $Q(p)$, at a pixel $p$, due to CMB, $Q_c(p)$, all foregrounds, $Q_f(p)$, and detector noise, $Q_n(p)$ at a frequency $\nu_i$ in thermodynamic temperature unit is given by

$$Q(p) = Q_c(p) + \sum_{j=1}^{n-1} a_j \left( \frac{\nu_p}{\nu_{0j}} \right)^{\beta_j} Q_{0j}(p) + Q_n(p), \quad (1)$$

where $a_i \equiv a(\nu_i)$ denotes the conversion factor from the antenna to the thermodynamic temperature unit at frequency $\nu_i$ for all foreground components. CMB anisotropy at any given pixel $p$ is independent of frequency $\nu_i$ due to its blackbody nature. Each foreground template $Q_{0j}$ is defined with reference to frequency $\nu_{0j}$, which may be different for different foreground components. We note that, in general, observed maps for different frequencies are convolved by different instrument response functions. However, we implicitly assume that frequency maps, $Q(p)$, in Equation (1) have already been brought to a common resolution by properly deconvolving first by respective instrument response functions and then convolving again by the common response function. Since all frequency maps of our work are convolved by a common Gaussian beam function of FWHM = $20^\circ$, we omit references to a beam function in all equations of this paper containing frequency maps, CMB, or foreground templates.

For the polarized signal, we assume that there are only two foreground components at microwave frequencies, namely, synchrotron and thermal dust, since so far observationally only these two components are detected to be polarized. Moreover, the spectral index for synchrotron emission varies with sky position due to the variation of relativistic electron density over the sky. In the presence of a variation of the spectral index for synchrotron, but a fixed spectral index for the thermal dust

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\textit{The Astrophysical Journal, 829:113 (12pp), 2016 October 1}

\textbf{SAHA & ALURI}
component, Equation (1) reduces to

\[ Q_j(p) = Q_c(p) + a_i \left( \frac{\nu_i}{\nu_{0j}} \right)^{\delta_i} Q_{0i}(p) \]

\[ + a_i \left( \frac{\nu_i}{\nu_{0d}} \right)^{\delta_i} Q_{0d}(p) + Q_{0i}(p), \]

(2)

where \( \beta_i(p) \) represents the synchrotron spectral index at a pixel \( p \) and \( \beta_0 \) is the global thermal dust spectral index. The motivation for our work is to obtain a reliable estimate of \( \beta_i(p) \) using a perturbative analysis, without assuming any prior template models for foregrounds or CMB.

3. PERTURBATIVE DATA MODEL

Let us consider a given region of the sky. A synchrotron spectral index at a pixel \( p \) can be written as \( \beta_i(p) = \beta_{0i} + \Delta \beta_i(p) \), where \( \beta_{0i} \) represents some reference spectral index for the region and \( \Delta \beta_i(p) \) represents an actual variation of the spectral index over \( \beta_{0i} \) inside the region. Using the direction dependent spectral index, \( \beta_i(p) \), in the second term of Equation (2), one can show that (Bouchet & Gispert 1999),

\[ \left( \frac{\nu_i}{\nu_{0i}} \right)^{\delta_i} = \left( \frac{\nu_i}{\nu_{0i}} \right)^{\beta_{0i}} \left( 1 + \Delta \beta_i(p) \ln \left( \frac{\nu_i}{\nu_{0i}} \right) \right) \]

(3)

where the second line follows from the Taylor expansion on the left hand side of the above equation up to first order in \( \Delta \beta_i(p)/\beta_{0i} \), assuming \( \Delta \beta_i(p)/\beta_{0i} \ll 1 \). Using Equation (3) in Equation (2), we obtain

\[ Q_j(p) = Q_c(p) + a_i \left( \frac{\nu_i}{\nu_{0i}} \right)^{\beta_{0i}} Q_{0i}(p) + a_i \left( \frac{\nu_i}{\nu_{0d}} \right)^{\beta_i} \ln \left( \frac{\nu_i}{\nu_{0d}} \right) \]

\[ \times Q_{0d}(p) \Delta \beta_i(p) + a_i \left( \frac{\nu_i}{\nu_{0d}} \right)^{\delta_i} Q_{0d}(p) + Q_{0d}(p). \]

(4)

Apart from CMB and detector noise, our data model, given by the above equation, can now be interpreted nicely to consist of a total of three foreground emission components, namely, synchrotron with a constant spectral index of \( \beta_{0i} \), a second component \( Q_{0i}(p) \Delta \beta_i(p) \), with its frequency variation \( a_i \left( \frac{\nu_i}{\nu_{0d}} \right)^{\delta_i} \ln \left( \frac{\nu_i}{\nu_{0d}} \right) \), and finally a thermal dust component. As we can see from the above equation, our model renders the variation of the spectral index of the synchrotron component in terms of a new foreground component with rigid frequency scaling. We define a set of templates, \( T_j \), where \( j \in \{1, 2, 3, 4\} \) following

\[ T_1(p) = Q_c(p) \]

\[ T_2(p) = Q_{0d}(p) \]

\[ T_3(p) = Q_{0i}(p) \Delta \beta_i(p) \]

\[ T_4(p) = Q_{0d}(p). \]

(5)

With these definitions, total polarized emission of Equation (4), at a frequency of \( \nu_i \) is then given by

\[ Q_j(p) = \sum_{j=1}^{4} T_j(p) s_j + Q_{0d}(p), \]

(6)

where we have denoted emission from \( j \)th component at channel \( i \) (in thermodynamic temperature unit) by \( T_j(p) s_j \). \( T_j(p) \) denotes a template for the \( j \)th component based on reference frequency \( \nu_{0j} \) and \( s_j \), for a given \( j \), denotes elements of the so-called shape vector, \( \{s_j\} \), for component \( j \). For foregrounds with emissions following usual power-law behavior of the shape vector are given by

\[ s_j = \left( \frac{\nu_{0j}}{\nu_{0i}} \right)^{\delta_i} a(\nu_i), \]

(7)

where \( \delta_i \) represents a spatially fixed spectral index of the \( j \)th (foreground) component. Following the method described in Section 5, we reconstruct each template defined by Equation (5). Due to the presence of detector noise, the reconstructed templates, \( \hat{T}_j(p) \), are, however, different from \( T_j(p) \). Once all template components are reconstructed, the synchrotron spectral index variation map is obtained following

\[ \Delta \hat{\beta}_i(p) = \hat{T}_i(p)/\hat{T}_2(p). \]

(8)

4. FORMALISM

4.1. Component Reconstruction and Weights

From a mixture of \( n \) physical components present in different frequency bands, our aim is to isolate a best-fit map of each one of them by removing the rest. This can be achieved by reconstructing any one component first and then repeating the method for all other components. Let us denote the component to be reconstructed by a Greek letter subscript, e.g., \( \alpha \) and other components to be removed by any Roman letter subscript taking values from the difference set \( \{1, 2, \ldots, n\} - \{\alpha\} \). We denote a shape vector for the component to be reconstructed by \( \{s_j\} \), while the \( \alpha \)th component from remaining \( n - 1 \) components has a shape vector of \( \{s_j\} \). To reconstruct the cleaned template, \( \hat{T}_\alpha(p) \), for component \( \alpha \), we form a linear combination of available maps, following

\[ \hat{T}_\alpha(p) = \sum_{i=1}^{n} w_{\alpha i} Q_i(p), \]

(9)

where the linear weights, \( w_{\alpha i} \), for \( \alpha \) component is obtained by minimizing the total pixel–pixel variance, \( \sigma^2_{\alpha} \), of the cleaned template, \( \hat{T}_\alpha(p) \). Pixel variance of this template is defined as

\[ \sigma^2_{\alpha} = \frac{N_{\text{pix}}}{\sum_{p=1}^{N_{\text{pix}}} \left( \hat{T}_\alpha(p) - \sum_{p'} \hat{T}_\alpha(p') \right)^2}, \]

(10)

where \( N_{\text{pix}} \) denotes the total number of pixels available for analysis in a single frequency map after any suitable mask has been applied. Using the matrix notation above, the equation

\[ \sum_{p=1}^{N_{\text{pix}}} \begin{pmatrix} \hat{T}_\alpha(p) \\ \hat{T}_\beta(p) \\ \vdots \\ \hat{T}_\gamma(p) \end{pmatrix} = \mathbf{W} \cdot \mathbf{Q}, \]

(11)

where \( \mathbf{W} \) is a matrix of weights and \( \mathbf{Q} \) is a matrix of maps.

5 For the CMB component, elements of the shape vector are unity for all frequency bands and the choice of \( \nu_{0i} \) is irrelevant.

The Astrophysical Journal, 829:113 (12pp), 2016 October 1

Saha & Aluri
The term describes the where, The Astrophysical Journal, column space of since we use low-resolution maps for our analysis. However, as described in Section 5.2, we are also, at the same time, to be determined from the data. (components that are to be removed. vector of this component, while simultaneously being ortho-
desired component with known spectral index see that noise, Due to the underlying linearity in our method, one can clearly noise in the (Equation 15), noise in the has a contribution from noise of all input frequency maps. Specifically, 
\[ \hat{Q}_i(p, \nu_i) = s_i^n \sum_{j=1}^{n_b} w_{ij} Q_i^n(p). \] 
Hence the total noise, \( \hat{Q}_i^n(p) \), due to all recovered components at frequency \( \nu_i \), is given by, 
\[ \hat{Q}_i^n(p) = \sum_{k=1}^{n} \hat{Q}_i^n(p, \nu_i) = \sum_{k=1}^{n} s_{i}^k \sum_{j=1}^{n_b} w_{ij} Q_i^n(p) \]
where we have defined 
\[ \tilde{w}_{ij} = \sum_{k=1}^{n} s_{i}^k w_{jk}. \] 
We form a difference map at each frequency, \( \nu_i \), by subtracting the recovered and input frequency map at the same frequency, \( \nu_i \). Using Equation (18), noise in the \( i \)th difference map is given by 
\[ Q_i^n(p) - \hat{Q}_i^n(p) = (1 - \tilde{w}_{ij}) Q_i^n(p) - \sum_{j \neq i} \tilde{w}_{ij} Q_j^n(p). \] 
Assuming that the noise of different frequency bands has zero mean and that noise properties of different detectors are uncorrelated with one another, noise covariance of the \( i \)th

\[ \sigma_{i}^2 = [W]_i \mathbb{V} [W]_i^T, \] 
where \( [W]_i \) is a \( 1 \times n_b \) weight matrix, \( (w_0, w_1, \ldots, w_{n_b}) \), for component \( \alpha \). \( [V] \) represents the \( n_b \times n_b \) covariance matrix, with elements \( \mathbb{V}_{i'j'} \) representing the covariance between frequency maps \( \mathbb{Q}(p) \) and \( \mathbb{Q}_i(p') \). As described in detail in Saha et al. (2008), the choice of weights that minimizes \( \sigma_{i}^2 \) is given by 
\[ [W]_i = \frac{[s]_i [V]^+ [s]_i^T}{[s]_i [V]^+ [s]_i^T}, \] 
where \( [V]^+ \) represents the Moore–Penrose generalized inverse (Moore 1920; Penrose 1955) of \( [V] \). We note that, in practice, \( [s]_i \) is not known a priori since the spectral indices for the components to be cleaned (e.g., synchrotron, thermal dust etc.) are also, at the same time, to be determined from the data. However, as described in Section 5.2, we find these weights for different components by varying their spectral indices in different pre-defined intervals. Each set of weights thus obtained then can be used to obtain a cleaned template for a given component.

Below, we derive an analytical expression for weights (i.e., Equation (15)) that minimizes variance due to physical components by giving an outline of the detailed mathematical procedure. The basic idea behind this equation is that under the assumption of negligible detector noise\(^6\), reconstruction of any desired component (with known spectral index) can be achieved if the weight vector becomes “parallel” to the shape vector of this component, while simultaneously being orthogonal to the shape vector of each one of the rest of the components that are to be removed. Using variance and covariance of different components, we can write 
\[ [V] = \sigma_{i}^2 [s]_i^T [s]_i + [r]_i^T [s]_i + [s]_i^T [r]_i + [F], \] 
where \( [r]_i \) is a \( 1 \times n_b \) matrix describing random chance correlation between the component to be reconstructed and all of the other components at each of the \( n_b \) frequency bands. \( [F] \) describes the \( n_b \times n_b \) covariance matrix of \( n-1 \) components, hence it has a rank of \( n-1 \). The above equation can be written as 
\[ [V] = \sigma_{i}^2 [s]_i^T [s]_i + [A]_i, \] 
where \( [A]_i = [r]_i^T [s]_i + [A]_2 \) and \( [A]_2 = [s]_i^T [r]_i + [F] \). Using the above equation, we can apply the generalized Shermann–Morisson formula (Meyer 1973; Baksalary et al. 2003) successively to obtain both the numerator and the denominator of Equation (12) (e.g., as in Saha et al. 2008). After a long set of algebra and neglecting all terms containing \( [r]_i \), Equation (12) reduces to 
\[ [W]_i = \frac{[s]_i ([I] - [F][F]^+)[s]_i^T}{[s]_i ([I] - [F][F]^+)[s]_i^T}. \] 
The term \( [F][F]^+ \) can be interpreted as the projector on the column space of \( [F] \). Hence \( ([I] - [F][F]^+) \) is the orthogonal

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\(^6\) The assumption of limiting detector noise is justified for our current work, since we use low-resolution maps for our analysis.
difference map is given by

\[ N^{ij,D}_{pp'} = (1 - \tilde{w}^i_0)^2 \langle \tilde{Q}^n_i(p) \tilde{Q}^n_i(p') \rangle + \sum_{j=1}^{n_0} (\tilde{w}^j_0)^2 \langle \tilde{Q}^n_j(p) \tilde{Q}^n_j(p') \rangle \]

\[
= (1 - \tilde{w}^i_0)^2 N^{i,i}_{pp'} + \sum_{j=1}^{n_0} (\tilde{w}^j_0)^2 N^{j,j}_{pp'}, \tag{21}
\]

where \( N^{i,j}_{pp'} \) is the noise covariance matrix of the \( i \)(\( j \))th input frequency map.

4.3. Data-model Mismatch Statistic

Using the noise covariance matrix obtained in the previous section, we define the mismatch between the data and the model by means of the usual \( \chi^2 \) statistic, following

\[ \chi^2 = \frac{1}{N} \sum_{i,p,p'} (Q_i(p) - \tilde{Q}_i(p)) (N^{i,j}_{pp'}(Q_i(p) - \tilde{Q}_i(p'))) , \tag{22} \]

where \( (N^{i,j}_{pp'})^{-1} \) represents the Moore–Penrose generalized inverse of \( N^{i,j}_{pp'} \) and \( N = n_b \times N_{\text{pix}} \) denotes the total number of surviving pixels counting all frequency bands after any suitable mask has been applied on the maps. Since \( \tilde{Q}_i(p) \) depends upon \( \beta_0, \beta_d \) through Equation (16) and \( N^{i,j}_{pp'} \) depends upon \( \beta_0, \beta_d \) and frequency band noise covariance matrices through Equation (21), we have \( \chi^2 = \chi^2(\beta_0, \beta_d) \). Clearly, \( \beta_0 \) and \( \beta_d \) enter as the minimizing variables for the \( \chi^2 \) statistic. Our best-fit values for these spectral indices are the ones that together minimize the \( \chi^2 \) given by the above equation.

4.4. Relationship with ML Approaches

Although Equation (22) may resemble an ML estimation of spectral indices (assuming a uniform prior) for a given noise model, we note that our method is not an ML approach, since we solve only the first order perturbative expansion, e.g., as given in Equation (4), which is not an exact equation, instead of using a full and completely accurate power-law model as given in Equation (2). However, the perturbative expansion will tend to an ML approach asymptotically. Moreover, our method can be extended to more than merely first order in perturbation expansion, whenever it is allowed by the data, i.e., when the instrument’s noise level is relatively small and the total number of free parameters to be estimated is less than or equal to the total number of frequency maps available. At some maximum order of perturbative expansion (given the small variation of the synchrotron spectral index parameter), our results will practically be indistinguishable from the true ML results within the noise limit of instruments. So sufficiently higher order perturbative analysis will possess the desirable property that its results will be identical in terms of both parameter values and error bars derived on them with the ML results, within the instrument’s noise uncertainty.

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**Table 1**

| Frequency, \( \nu \) (GHz) | WMAP/Planck Map Name | \( \alpha(\nu) \) | \( \sigma_0(\mu K) \) Ref. | \( \sigma_1(\mu K) \) Ref. |
|----------------------------|-------------------|-----------------|-----------------|-----------------|
| 23                         | WMAP K            | 1.014           | 40.86           | 1435.0          |
| 30                         | Planck, LFI       | 1.023           | 36.56           |                 |
| 33                         | WMAP Ka           | 1.029           | 41.19           | 1472.0          |
| 41                         | WMAP Q1           | 1.044           | 56.30           | 2254.0          |
| 41                         | WMAP Q2           | 1.044           | 53.41           | 2140.0          |
| 44                         | Planck, LFI       | 1.051           | 37.03           |                 |
| 61                         | WMAP V1           | 1.099           | 70.86           | 3234.0          |
| 61                         | WMAP V2           | 1.099           | 63.05           | 2958.0          |
| 71                         | Planck, LFI       | 1.136           | 37.11           |                 |
| 100                        | Planck, HFI       | 1.283           | 15.83           |                 |
| 143                        | Planck, HFI       | 1.646           | 11.80           |                 |
| 217                        | Planck, HFI       | 2.961           | 19.39           |                 |

Note. List of WMAP and Planck frequency maps used in the analysis. The third column denotes antenna to thermodynamic conversion factors at different frequencies. In the last column, \( \sigma_0 \) values represent the polarized noise level of WMAP detectors for a single observation. In the fourth column, \( \sigma_0 \) values describing Planck polarized noise levels are consistent with the Planck blue book (Consortium 2005) values and scaled to \( N_{\text{side}} = 512 \). For WMAP \( \sigma_0 \) values represent \( \sigma_c/\sqrt{N_{\text{side}}(p)} \) where \( (p) \) denotes the all-sky average and \( N_{\text{side}}(p) \) represents the effective number of observations at a given pixel \( p \) for a given \( N_{\text{side}} = 512 \) map. The fourth column clearly shows a lower noise level of Planck compared to WMAP. Wherever multiple differencing assemblies are available for WMAP, a polarized noise level for each differencing assembly is also mentioned in order. For WMAP, the actual noise level per pixel at \( N_{\text{side}} = 512 \) is given by \( \sigma(p) = \sigma_c/\sqrt{N_{\text{side}}(p)} \). For a Planck frequency, the band noise level per pixel at \( N_{\text{side}} = 512 \) is assumed to be uniform with \( \sigma(p) = \sigma_c \) as the specified values at \( N_{\text{side}} = 512 \). \( ^a \) Stokes \( (Q, U) \) polarization noise level in \( \mu K \) thermodynamic temperature unit at \( N_{\text{side}} = 512 \).

5. METHODOLOGY

5.1. Input Data

5.1.1. Frequency bands

We include simulated Stokes \( Q \) polarization maps of WMAP K, Ka, Q, V bands, all three Planck LFI bands, and three lowest frequency Planck HFI bands. We do not include the WMAP W band in our analysis since it contains a somewhat larger noise level compared to other WMAP bands. We do not include the two highest frequency Planck HFI maps since they only measure temperature. Moreover, with the inclusion of the HFI 353 GHz map in our analysis, \( \chi^2 \) values become increasingly more sensitive to the thermal dust model, giving lesser weightage to the synchrotron component. Hence we also exclude this frequency band from our analysis. Detailed specifications of the frequency bands used in this work are given in Table 1.

Since pixel values of spectral index variation map, \( \Delta \beta(p) \), are small compared to the foreground emission, any significant residual noise in the reconstructed template maps results in its potentially noisy reconstruction following Equation (8). Owing to the delicate noise sensitivity of the reconstructed spectral index map, we chose to minimize the detector noise at the beginning of the analysis. For this purpose, we use Stokes \( Q \) frequency maps at \( N_{\text{side}} = 8 \) smoothed by a Gaussian beam function of FWHM = \( 20' \).
5.1.2. Mask

We see from Equation (8) that the $\Delta \tilde{\beta}_s(p)$ map may contain noisy pixels if corresponding pixels in the reconstructed synchrotron template, $\tilde{T}_s(p)$, or $\tilde{T}_b(p)$, are also noise dominated. To avoid noisy reconstruction of the spectral index map, we remove those pixels from all 10 input frequency maps, which contain weak synchrotron Stokes $Q$ polarization signals. For this purpose, based upon the smoothed Stokes $Q$ synchrotron template at the K band, we make a mask that excises all pixels with absolute values less than 10 $\mu$K in the antenna temperature unit. We call this mask the pm10 mask and apply it to all frequency maps for the reconstruction of the spectral index map.

5.1.3. Physical Components

We use WMAP “base fit” MCMC (Markov Chain Monte-Carlo) synchrotron and thermal dust polarization maps for our simulations. These maps are provided at $N_{\text{side}} = 64$ on the LAMBDA website. As mentioned on the LAMBDA website, for some of the pixels of these maps, MCMC fitting resulted in errors. For both Q and U polarized synchrotron maps at K band, using WMAP’s bit-coded error flag file, we first identify all pixels for which WMAP MCMC fit resulted in error. The process identifies a total of 1315 pixels at $N_{\text{side}} = 64$. Pixel value for each of these bad pixels is then replaced by the average pixel value of its nearest neighbors. If a bad pixel has one or more bad neighbors, we average only over the good ones. Following a similar procedure, we obtain a polarized thermal dust template at W band at $N_{\text{side}} = 64$. We downgrade each of these two templates to $N_{\text{side}} = 8$. We generate polarized CMB maps compatible to WMAP LCDM spectrum directly at $N_{\text{side}} = 8$. We smooth both foreground templates and CMB maps by the Gaussian beam of FWHM = 20°.

5.1.4. Spectral Index Map

For spectral index, we use the WMAP “best-fit” MCMC spectral index map. This map is provided by the WMAP science team at $N_{\text{side}} = 64$. We downgrade this map at $N_{\text{side}} = 8$ and subsequently smooth by a Gaussian beam of FWHM = 20°.

This smoothed spectral index map defines our primary spectral index map, $\beta_s(p)$, for this work with which we compare the reconstructed spectral index map obtained by our method.

5.1.5. Detector Noise Maps and Covariance Matrix

Generating noise maps and thus modeling the noise properties in the low-resolution maps are somewhat more involved than the simple smoothing operations that needed to be performed on foreground templates and CMB maps at $N_{\text{side}} = 8$ in Section 5.1.3. We take into account WMAP’s scan-modulated detector noise pattern by using $N_{\text{noise}}(p)$ information associated with each nine-year Difference Assembly (DA) map available on the LAMBDA website at $N_{\text{side}} = 512$. We first compute noise variance at each pixel following $\sigma^2(p) = \sigma^2_{\text{noise}}(p)$ at $N_{\text{side}} = 512$ for all DA maps between the K to V bands. Since for each of the Q and V bands two DA maps are available, we simply weigh down the average variance of these two DA maps at each pixel, by a factor of two, to obtain noise variance maps corresponding to an average DA map within the frequency bands. For Planck LFI and HFI maps, we use a simple uniform white noise model. Pixel white noise levels, $\sigma(p)$, for Planck maps are obtained from Planck blue book specifications, $\sigma_{\text{blue}}$, which represents noise in the beam area of the corresponding map in units of $\mu$K/K. Pixel noise levels for Planck maps at $N_{\text{side}} = 512$ are then given by, $\sigma(p) = \sigma_{\text{blue}} T_{\text{CMB}} B_{\text{wthm}} / \Delta \theta$, where $B_{\text{wthm}}$ represents the FWHM of Planck beams in arcmin, $\Delta \theta = 69°$, represents the pixel size of the $N_{\text{side}} = 512$ map and $T_{\text{CMB}} = 2.73$ K.

In principle, one can use all the noise variance maps at $N_{\text{side}} = 512$ corresponding to all frequency bands of Table 1 to obtain representations of noise maps at $N_{\text{side}} = 512$ and then downgrade them to $N_{\text{side}} = 8$ for use in our simulation. However, one needs to simulate many such maps at $N_{\text{side}} = 512$, since we need to obtain pixel noise covariance at $N_{\text{side}} = 8$ by downgrading the high-resolution noise maps. Since generating noise maps at $N_{\text{side}} = 512$ and subsequently downgrading them many times is a computationally expensive process, which requires $N_{\text{pix}} = 12 \times 512 \times 512$ of floating point operations per simulation, we devise a completely equivalent approach that is computationally significantly fast compared to the former method. Instead of downgrading $N_{\text{side}} = 512$ noise maps, we obtain noise variance maps at $N_{\text{side}} = 8$ by simply multiplying the total noise variance of all $N_{\text{side}} = 512$ pixels lying inside each $N_{\text{side}} = 8$ pixel by the ratio of pixel numbers of a HEALPix map at $N_{\text{side}} = 8$ and $N_{\text{side}} = 512$. Such a method is always feasible for our method and does not introduce any pixel–pixel correlation at $N_{\text{side}} = 8$ since noise in the original $N_{\text{side}} = 512$ maps is uncorrelated from pixel to pixel. Using these noise variance maps, we simulate noise maps at $N_{\text{side}} = 8$. These noise maps are then smoothed by a Gaussian beam of FWHM = 20°, to generate realistic noise maps at $N_{\text{side}} = 8$ for each frequency band. We obtain 10000 Monte-Carlo simulations of smoothed noise maps at $N_{\text{side}} = 8$. Since the smoothing operations introduce pixel to pixel noise covariance, we obtain a full pixel to pixel noise covariance matrix over surviving pixels (after application of pm10 mask) at $N_{\text{side}} = 8$ corresponding to each frequency band. We have shown a noise covariance matrix for the V-band

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9 We apply the pm10 mask only for reconstruction of the spectral index map. We present results for reconstruction of CMB and foreground components (e.g., Section 6.3) over the entire sky.
masked map, in Figure 1. The non-diagonal nature of this matrix indicates non-trivial noise properties in the maps.

Using the $N_{\text{side}} = 8$ smoothed CMB Stokes $Q$ polarized map, $Q_i(p)$, synchrotron and thermal dust templates ($Q_{0s}$ and $Q_{0d}$ respectively), spectral index map, $\beta_s(p)$, and detector noise map, $Q^n_i(p)$ $(i = 1, 2, \ldots, n_0)$ as obtained above, we generate simulated frequency maps for WMAP and Planck frequency bands, using Equation (2) and knowing the values of $a_i$. We use $\beta_d = 2.0$, $\nu_{0s} = 23$ GHz and $\nu_{0d} = 94$ GHz in this work.

5.2. Analysis

Since spectral indices of neither synchrotron, nor thermal dust components are known a priori, we perform a reconstruction of all components defined by Equation (5) for a suitable range of both of these indices. We vary $\beta_{0s}$ between $-2.65$ to $-3.05$ with a step of $-0.01$. For each of these values $\beta_d$ is varied between $1.80$ and $2.20$ with a step size of $0.01$. For each pair $(\beta_{0s}, \beta_d)$, such generated, we compute shape vectors for all foreground template components. Combining these with the shape vector of CMB, we estimate a set of weights using Equation (15) to reconstruct all template components including CMB. For each pair $(\beta_{0s}, \beta_d)$, we form reconstructed frequency maps following Equation (16) using all the reconstructed template components. We then form the difference map at each frequency by subtracting a reconstructed frequency map from the corresponding input frequency map. The noise covariance of the $i$th difference map over the surviving pixels for a given pair of indices is defined by the matrix, $N_{ij}^{\text{pp}}$, defined by Equation (21). Once the difference map is formed, we obtain $\chi^2(\beta_{0s}, \beta_d)$ for all indices following Equation (22). Our best-fit values for indices are the ones that minimize $\chi^2$ over all indices. Using these values, we obtain a spectral index variation map following Equation (8). We validate our method by performing 1000 Monte-Carlo simulations of the entire procedure described above. We note that since CMB follows a blackbody distribution its shape vector remains identical for all foreground indices and for all simulations. Since the weights, as given by Equation (15), depend only upon foreground and CMB shape vectors (without any dependence on random detector noise, which varies from simulation to simulation) for the purpose of reconstruction of all components, we calculate these weights only once for the entire range of spectral indices and store them on the disk. We then use these weights to reconstruct all components for all Monte-Carlo simulations for all indices. Moreover, we also precompute Moore–Penrose generalized inverses of detector noise covariance matrices of all difference maps for all pairs of spectral indices $(\beta_{0s}, \beta_d)$ and store them on the disk. In effect, a Singular Value Decomposition algorithm with a cutoff of $1.0 \times 10^{-7}$ was used to implement Moore–Penrose inverses. An alternative approach would have been to invert these covariance matrices after adding a small regularization noise to each one of them.

6. RESULTS

6.1. Spectral Index Map

For each simulation described in Section 5, we construct a spectral index variation map, $\Delta \beta_s(p)$, over the reference index, $\beta_{0s}$, following Equation (5). The results for reconstruction of spectral index map are shown in Figure 2. The colored area of the Mollweide projection of all the three panels shows the region on the sky that survives after applying the $\text{pm10}$ mask. The top panel shows the mean reconstructed synchrotron spectral index map, $\langle \beta_s(p) \rangle$, defined as, $\langle \beta_s(p) \rangle = \langle \beta_{0s, \text{min}} \rangle + \langle \Delta \beta_s(p) \rangle$, where $\beta_{0s, \text{min}}$ denotes a value of $\beta_{0s}$ (e.g., see Equation (4)) corresponding to $\chi^2_{\text{min}}$ and the average is performed over all 1000 simulations. The middle panel is a simple standard deviation map for any one of the reconstructed spectral index maps. The lowermost panel shows the difference between the average reconstructed index map and the input spectral index map, $\langle \beta_s(p) \rangle - \beta_s(p)$. Clearly, the difference map shows that our method can accurately reconstruct the spectral index map inside the $\text{pm10}$ mask. The reconstruction difference toward higher (lower) galactic coordinates becomes slightly larger due to weak polarized signal in these regions.

For any given Monte-Carlo simulation, $\chi^2(\beta_{0s}, \beta_d)$ values show a clear minimum, $\chi^2_{\text{min}}$, at some values of $(\beta_{0s}, \beta_d) = (\beta_{0s, \text{min}}, \beta_{d, \text{min}})$. Although a topographical plot of $\chi^2(\beta_{0s}, \beta_d)$ depends upon the particular random simulation
under consideration, the features of such plots that arise on average due to foregrounds, CMB, and detector noise, can be studied by making a plot of $\chi^2(b_0, b_0)$ (where the average is performed over all 1000 simulations for each pair $(b_0, b_0)$) with respect to both indices. We have shown this at the top panel of Figure 3. The elongated contours of this plot show the relatively higher sensitivity of the data with the variation of synchrotron spectral index compared to the same variation of the dust spectral index. In the bottom panel of this figure, we show a standard deviation map of $\chi^2(b_0, b_0)$ for any one of the 1000 simulations.

The probability density function of this figure follows a very symmetric distribution for $\chi^2_{\text{min}}$. This can be expected due to a very large number of degrees of freedom (≈4260) available in the analysis. In the bottom panel of this figure, we have shown the probability distribution of 20° smoothed $N_{\text{side}} = 8$ input spectral index map, from the region that survives after the application of the $pm10$ mask. A bimodal structure of the distribution is readily identifiable corresponding to spectral indices of $\sim-2.85$ and $-3.16$ respectively. Comparing these values with the mean reconstructed spectral index $-2.84 \pm 0.01$, it is interesting to note that our method picks out the peak corresponding to a flatter index from the available pair of doubly degenerate ones.

6.2. Weight

One very important component of this work is the weights. We show weights for reconstruction of $Q_{\text{h}}(p)$ and $Q_{\text{h}}(p)\Delta\beta(p)$ templates for the entire range of spectral indices, respectively, in the top and bottom panels of Figure 5. At the low-frequency side, where synchrotron emission is dominant, weights, for reconstruction of the synchrotron template, tend to be positive for all indices. We see this for the K, LFI 30 GHz, and Ka bands. For all other bands except
the HFI 217 GHz band, weights are negative for all indices. HFI 217 GHz band weights become positive for all indices to cancel out signals from CMB, thermal dust, $Q_0(p)$, and at the same time to preserve $Q_0$. For the reconstruction of $Q_0(p)\Delta\beta_s(p)$, template weights must completely project out its shape vector from the data, at the same time, the weight vector needs to be orthogonal to shape vectors of synchrotron and other components. Because of the close analytical similarity in the shape of vectors $Q_0(p)$ and $Q_0(p)\Delta\beta_s(p)$, the orthogonality condition leads to a somewhat larger magnitude of weights for $Q_0(p)\Delta\beta_s(p)$ than the $Q_0(p)$ template. The relatively larger magnitude of weights can more easily be seen from the second panel than the first one. For the second panel and for the K band, weights take the largest negative value, since, from the expression of the shape vector for the $Q_0(p)\Delta\beta_s(p)$ component, $(a(\nu)/(\nu/\nu_0))^{1+(3\nu)}\ln(\nu/\nu_0))$, the shape vector has a zero component at the K band. For low-frequency LFI 30 GHz, Ka, Q, and LFI 44 GHz frequency bands, all weights are positive, indicating a larger shape vector at lower frequency. For V, LFI 71 GHz, HFI 100 GHz, and HFI 143 GHz frequency bands weights become negative, since shape vector decreases in this frequency range. Finally, for HFI 217 GHz, frequency band weights again become positive to project out the $Q_0(p)\Delta\beta_s(p)$ component completely and to cancel out the rest of the components.

Figure 5. Top panel: weights for the reconstruction of $Q_0(p)$. Bottom panel: weights for the reconstruction of $Q_0(p)\Delta\beta_s(p)$. For any given panel, plots in consecutive rows from left to right indicate, respectively, weights for K, LFI 30 GHz, Ka, Q, LFI 44 GHz, V, LFI 71 GHz, HFI 100 GHz, HFI 143 GHz, and HFI 217 GHz frequency bands.
6.3. CMB and Foreground

As associated results of our method, we reconstruct asymptotically ML templates for all components using the best-fit spectral index values $\beta_{\text{tr}, \min}$, $\beta_{\text{d}, \min}$ corresponding to $\chi^2_{\min}$ obtained in Section 6.1. We show the results in Figure 6. Unlike the case for reconstructed synchrotron spectral map, we do not apply a mask on the recovered templates, since CMB and foreground signals are expected to be stronger than the spectral index variations. Top, middle, and bottom panels of this figure show results for CMB, thermal dust, and synchrotron template reconstruction, respectively, obtained from a randomly chosen Monte-Carlo simulation with seed = 100. All the color maps of this figure are in $\mu$K units (thermodynamic for CMB, antenna for foreground templates). Following the usual row, column convention of a matrix, we represent any image of this figure by the pair $(i, j)$. $(1, 1)$ image represents the reconstructed CMB map against the input CMB map for the same simulation, shown as the $(1, 2)$ image. As one can easily visually identify, the large-scale features of these two maps are very nearly identical. The $(1, 3)$ image shows the difference between the reconstructed and the input CMB map and hence is dominated by residual detector noise. The noisy nature of the difference map is also indicated by the symmetric color table plotted below the $(1, 3)$ image. The $(1, 4)$ image represents the standard deviation map estimated from the set of all $(1, 3)$ maps resulting from 1000 Monte-Carlo simulations of the component separation algorithm. As expected, the standard deviation map clearly indicates a scan-modulated noise pattern as applicable for WMAP observations. There exist some foreground induced errors on the galactic plane. Such errors can be reduced by extending our method to a second-order perturbative analysis.

The $(2, 1)$ image represents the reconstructed thermal dust template at $217 \text{GHz}$ for seed = 100. The reconstructed template matches very well with the input thermal dust template at the same frequency, viz., the $(2, 2)$ image. The difference between the reconstructed and input templates $(2, 3)$ image is dominated by noise. The standard deviation map for reconstructed thermal dust template is shown as the $(2, 4)$ image. Clearly, WMAP’s scan-modulated noise pattern is visible in this figure. There also exist some residual foreground errors in this map, which again can be eliminated by performing a higher order perturbative analysis.

The $(3, 1)$ image shows the reconstructed synchrotron template at $23 \text{GHz}$ for the same random seed. The input synchrotron template at $23 \text{GHz}$ is shown as the $(3, 2)$ image. Both these maps match with each other very well. The difference of the first and second maps is shown as the $(3, 3)$ image. The standard deviation map, image $(3, 4)$, shows the scan-modulated detector noise pattern, without any visible signature of residual foreground error.

The normalized reconstruction error in each of the above reconstructed templates can be quantified by dividing the difference of the output reconstructed template and the input one by the corresponding standard deviation map (i.e., dividing each map of column 3 by the corresponding map of column 4 of Figure 6). The dimensionless pixel values of these error maps should have zero mean and unity standard deviation for reliable template reconstruction. In Figure 7, we show these reconstruction error maps for CMB (top panel), thermal dust (middle panel), and the synchrotron template (bottom panel). We have shown the mean, $\mu$, and standard deviation, $\sigma$, values for CMB, thermal dust, and synchrotron templates in Table 2 for the entire sky, pm10 mask, and from the complementary region of the pm10 mask. Comparing the mean values, we conclude that, among all of these three regions, the reconstruction of CMB and thermal dust template components are most accurate for the pm10 sky region. The synchrotron template also has a small mean-error on the same sky region, compared to the maximum and minimum of the input template, indicating
cosmologists. This can be achieved relatively easily if (1) the observed sky signal consists of only those components that have rigid frequency scaling all over the sky, (2) the detector noise is negligible, and (3) the total number of these components is less than or equal to the total number of observed frequency bands. However, the task is complicated when the spectral index of any one of the components varies with sky positions. The stronger the variation, effectively one gets more components with rigid frequency scaling, which, if not taken into account in the analysis, may cause the results to be biased, or at the least difficult to interpret. The scenario becomes even worse in the presence of non-negligible detector noise. We have shown that in the presence of foregrounds following the power-law model with the spatially varying index and the small amount of detector noise one can recover the spectral index variation map reliably from the WMAP and Planck Stokes Q observations without any need to model foregrounds and or CMB in terms of any model templates. The only assumption we have made in this work is a power-law model for the foregrounds. However, this is not a necessary condition for our method to work, since one can directly fit for the shape vectors for all components by expanding the parameter space. Then with the help of perturbative expansion of the spectral index variation one can reconstruct the index variation map. Since the spectral index variation for the synchrotron component is small compared to other emissions (specifically, compared to synchrotron emission amplitude itself), and can be masked by the presence of a comparable level of detector noise contamination. We chose to work with maps containing low levels of detector noise, by smoothing the frequency maps by a $20^\circ$ beam window function. We use a set of full pixel-pixel detector noise covariance matrices to accurately model the smoothed detector noise pattern, and thus for better reconstruction of spectral index map. The condition for the large beam window can be relaxed easily for experiments or for sky regions with negligible detector noise. A very interesting future project will be to apply the method on the Planck polarized observations combined with WMAP, and with polarization sensitive, low noise, future CMB observations like CMBPol. For very low noise experiments in the future, it would be very useful to extend our method up to second-order perturbative analysis for an even more accurate understanding of cosmological and astrophysical information contained in the data.

We summarize the main results and conclusions of this work as follows.

1. We present and implement a first order perturbative method to estimate variation of the synchrotron spectral index over the microwave sky using simulated Stokes Q observations of WMAP, K, Ka, Q, V, Planck LFI, and three lowest frequency Planck HFI frequency bands. We validate the method by performing 1000 Monte-Carlo simulations over regions of the sky dominated by the synchrotron Stokes Q signal. Our results suggest that, following the first order analysis synchrotron spectral index variation map can be reconstructed reliably by our method with a narrow error margin.

2. As accompanying results of this work, we reconstruct CMB, thermal dust, and synchrotron template components. An error estimation on the reconstructed templates suggest that the template components can be reconstructed reliably by our method.

Table 2: Mean $\tilde{\mu}$, and Standard Deviation, $\tilde{s}_\mu$, Statistics of Normalized Error Maps

| Component | Fullsky | $pm10$ Mask | $pm10c$ Mask |
|-----------|---------|-------------|--------------|
| $\tilde{\mu}$ | $\tilde{s}_\mu$ | $\tilde{\mu}$ | $\tilde{s}_\mu$ |
| CMB      | 0.169   | 0.995       | 0.017        | 0.913        | 0.359   | 1.060 |
| Thermal Dust | −0.087  | 1.016       | 0.036        | 0.953        | −0.240  | 1.070 |
| Synchrotron | −0.210  | 0.947       | −0.294       | 0.918        | −0.106  | 0.974 |

Note. Mean and standard deviation values estimated from the normalized error maps of different template components. The $pm10c$ mask is complementary to the $pm10$ mask.

Figure 7. Normalized reconstruction error maps for the template reconstruction for CMB (top panel), thermal dust (middle panel) and synchrotron (bottom panel). Pixel values of all these maps are dimensionless.
3. Our reconstructed templates and best-fit spectral indices are asymptotically ML estimates.

4. The perturbative method of this work possesses an interesting property that it can be tuned to the data with an accuracy, as allowed by the data. This could be particularly very useful for the CMB and foreground component separation problem if the data is insufficient to constrain an otherwise completely accurate model, due to underlying degeneracies in the foreground components.

The work presented in this paper will allow an improved estimation of cosmological signal due to its detailed foreground modeling in terms of the variation of spectral indices. It will be useful to extract astrophysical information about galactic synchrotron emission, in particular, information about the galactic cosmic-ray electrons and the magnetic field. Finally, the method can be used to jointly extract all foreground components along with CMB signal from CMB observations.

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