THE CUBIC SZEGŐ EQUATION WITH A LINEAR PERTURBATION

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Abstract. We consider the following Hamiltonian equation on the $L^2$ Hardy space on the circle $S^1$,
\[ i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1), \quad \alpha \in \mathbb{R}, \]
where $\Pi$ is the Szegő projector. The above equation with $\alpha = 0$ was introduced by Gérard and Grellier as an important mathematical model [5, 7, 3]. In this paper, we continue our studies started in [22], and prove our system is completely integrable in the Liouville sense. We study the motion of the singular values of the related Hankel operators and find a necessary condition of norm explosion. As a consequence, we prove that the trajectories of the solutions will stay in a compact subset, while more initial data will lead to norm explosion in the case $\alpha > 0$.

1. Introduction

The purpose of this paper is to study the following Hamiltonian system,
\[ i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1), \quad x \in S^1, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \]
(1.1)
where the operator $\Pi$ is defined as a projector onto the non-negative frequencies, which is called the Szegő projector. When $\alpha = 0$, the equation above turns out to be the cubic Szegő equation,
\[ i\partial_t u = \Pi(|u|^2 u), \]
(1.2)
which was introduced by P. Gérard and S. Grellier as an important mathematical model of the completely integrable systems and non-dispersive dynamics [5, 7]. For $\alpha \neq 0$, by changing variables as $u = \sqrt{|\alpha|} \tilde{u}(|\alpha|t)$, then $\tilde{u}$ satisfies
\[ i\partial_t \tilde{u} = \Pi(|\tilde{u}|^2 \tilde{u}) + \text{sgn}(\alpha)(\tilde{u}|1). \]
(1.3)
Thus our target equation with $\alpha \neq 0$ becomes
\[ i\partial_t u = \Pi(|u|^2 u) \pm (u|1). \]
(1.4)

1.1. Lax Pair structure. Thanks to the Lax pairs for the cubic Szegő equation (1.2) [7], we are able to find a Lax pair for (1.1). To introduce the Lax pair structure, let us first define some useful operators and notation. For $X \subset \mathcal{D}'(S^1)$, we denote
\[ X_+ (S^1) := \{ u(e^{i\theta}) \in X, \ u(e^{i\theta}) = \sum_{k \geq 0} \hat{u}(k) e^{ik\theta} \}. \]
(1.5)
For example, $L^2_+$ denotes the Hardy space of $L^2$ functions which extend to the unit disc $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ as holomorphic functions

$$u(z) = \sum_{k \geq 0} \hat{u}(k)z^k, \sum_{k \geq 0} |\hat{u}(k)|^2 < \infty .$$ \hfill (1.6)

Then the Szegő operator $\Pi$ is an orthogonal projector $L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$.

Now, we are to define a Hankel operator and a Toeplitz operator. By a Hankel operator we mean a bounded operator $\Gamma$ on the sequence space $\ell^2$ which has a Hankel matrix in the standard basis $\{e_j\}_{j \geq 0}$,

$$(\Gamma e_j, e_k) = \gamma_{j+k}, \ j, k \geq 0 ,$$ \hfill (1.7)

where $\{\gamma_j\}_{j \geq 0}$ is a sequence of complex numbers. More backgrounds on the Hankel operators can be found in [20].

Let $S$ be the shift operator on $\ell^2$,

$$Se_j = e_{j+1}, \ j \geq 0 .$$

It is easy to show that a bounded operator $\Gamma$ on $\ell^2$ is a Hankel operator if and only if

$$S^* \Gamma = \Gamma S .$$ \hfill (1.8)

\textbf{Definition 1.1.} For any given $u \in H^1_+(\mathbb{S}^1)$, $b \in L^\infty(\mathbb{S}^1)$, we define two operators $H_u$, $T_b : L^2_+ \to L^2_+$ as follows. For any $h \in L^2_+$,

$$H_u(h) = \Pi(\overline{uh}) ,$$

$$T_b(h) = \Pi(bh) .$$ \hfill (1.9)

$$T_b$$ is $\mathbb{C}$–linear and is self-adjoint if and only if $b$ is real-valued.

Moreover, $H_u$ is a Hankel operator. Indeed, it is given in terms of Fourier coefficients by

$$\hat{H_u(h)}(k) = \sum_{\ell \geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)} ,$$

then

$$S^* H_u(h) = \sum_{k,\ell \geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)}S^* e_k = \sum_{k,\ell \geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)}e_k ,$$

$$H_u S h = \sum_{k,\ell \geq 0} \hat{u}(k)e_k\overline{\hat{h}(\ell)e_{\ell+1}} = \sum_{k,\ell \geq 0} \hat{u}(k+\ell+1)\overline{\hat{h}(\ell)}e_k ,$$

which means $S^* H_u = H_u S$, thus $H_u$ is a Hankel operator. We may also represent $T_b$ in terms of Fourier coefficients,

$$\hat{T_b(h)}(k) = \sum_{\ell \geq 0} \hat{b}(k-\ell)\overline{\hat{h}(\ell)} ,$$

then its matrix representation, in the basis $e_k, k \geq 0$, has constant diagonals, $T_b$ is a Toeplitz operator.
We now define another operator $K_u := T^*_z H_u$. In fact $T^*_z$ is exactly the shift operator $S$ as above, we then call $K_u$ the shifted Hankel operator, which satisfying the following identity
\begin{equation}
K_u^2 = H_u^2 - (\cdot | u)u. \tag{1.11}
\end{equation}

Using the operators above, Gérard and Grellier found two Lax pairs for the Szegő equation (1.2).

\textbf{Theorem 1.1.} [5, Theorem 3.1] Let $u \in C(R, H^s_z(\mathbb{S}^1))$ for some $s > 1/2$. The cubic Szegő equation (1.2) has two Lax pairs $(H_u, B_u)$ and $(K_u, C_u)$, namely, if $u$ solves (1.2), then
\begin{equation}
\frac{dH_u}{dt} = [B_u, H_u], \quad \frac{dK_u}{dt} = [C_u, K_u], \tag{1.12}
\end{equation}
where
\begin{align*}
B_u &:= \frac{i}{2}H_u^2 - iT_{|u|^2}, \quad C_u = \frac{i}{2}K_u^2 - iT_{|u|^2}.
\end{align*}

For $\alpha \neq 0$, the perturbed Szegő equation (1.1) is globally well-posed and by simple calculus, we find that $(H_u, B_u)$ is no longer a Lax pair, in fact,
\begin{equation}
\frac{dH_u}{dt} = [B_u, H_u] - i\alpha(u(1))H_1. \tag{1.13}
\end{equation}
Fortunately, $(K_u, C_u)$ is still a Lax pair.

\textbf{Theorem 1.2.} [22] Given $u_0 \in H^\frac{i}{2}_z(\mathbb{S}^1)$, there exists a unique global solution $u \in C(R; H^\frac{i}{2}_z)$ of (1.1) with $u_0$ as the initial condition. Moreover, if $u_0 \in H^s_z(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C^\infty(R; H^\frac{i}{2}_z)$. Furthermore, the perturbed Szegő equation (1.1) has a Lax pair $(K_u, C_u)$, namely, if $u$ solves (1.1), then
\begin{equation}
\frac{dK_u}{dt} = [C_u, K_u]. \tag{1.14}
\end{equation}
An important consequence of this structure is that, if $u$ is a solution of (1.1), then $K_{u(t)}$ is unitarily equivalent to $K_{u_0}$. In particular, the spectrum of the $\mathbb{C}$-linear positive self-adjoint trace class operator $K_u^2$ is conserved by the evolution.

Denote
\begin{equation}
\mathcal{L}(N) := \{u : \text{rk}(K_u) = N, N \in \mathbb{N}^+\}. \tag{1.15}
\end{equation}
Thanks to the Lax pair structure, the manifolds $\mathcal{L}(N)$ are invariant under the flow of (1.1). Moreover, they turn out to be spaces of rational functions as in the following Kronecker type theorem.

\textbf{Theorem 1.3.} [22] $u \in \mathcal{L}(N)$ if and only if $u(z) = \frac{A(z)}{B(z)}$ is a rational function with
\begin{equation*}
A, B \in \mathbb{C}[z], A \land B = 1, \deg(A) = N \text{ or } \deg(B) = N, B^{-1}((0)) \cap \mathbb{D} = \emptyset,
\end{equation*}
where $A \land B = 1$ means $A$ and $B$ have no common factors.

Our main objective of the study on this mathematical model (1.1) is on the large time unboundedness of the solution. This general question of existence of unbounded Sobolev trajectories comes back to [1], and was addressed by several authors for various Hamiltonian PDEs, see e.g. [2, 6, 12, 13, 14, 11, 15, 16, 17, 19, 21]. We have already considered the case with initial data $u_0 \in \mathcal{L}(1)$ and found that
Theorem 1.4. [22] Let $u$ be a solution to the $\alpha$–Szegő equation,
\[
\begin{cases}
i\partial_t u = \Pi(|u|^2 u) + \alpha(u)1, \quad \alpha = \mathbb{R}, \\
u(0, x) = u_0(x) \in L^1(1).
\end{cases}
\tag{1.16}
\]

For $\alpha < 0$, the Sobolev norm of the solution will stay bounded, uniform if $u_0$ is in some compact subset of $L^1(1)$,
\[\|u(t)\|_{H^s} \leq C, \quad C \text{ does not depend on time } t, \quad s \geq 0.\]

For $\alpha > 0$, the solution $u$ of the $\alpha$–Szegő equation has an exponential-on-time Sobolev norm growth,
\[\|u(t)\|_{H^s} \simeq e^{C_\alpha|t|}, \quad s > \frac{1}{2}, \quad C_\alpha > 0, \quad |t| \to \infty,\]
if and only if
\[E_\alpha = \frac{1}{4}Q^2 + \frac{1}{2}Q,\]
with $E_\alpha$ and $Q$ as the two conserved quantities, energy and mass.

1.2. Main results. We continue our studies on the cubic Szegő equation with a linear perturbation (1.1) on the circle $S^1$ with more general initial data $u_0 \in L^N(1)$ for any $N \in \mathbb{N}^+$.

Firstly, the system is integrable since there are a large amount of conservation laws which comes from the Lax pair structure (1.14).

Theorem 1.5. Let $u(t, x)$ be a solution of (1.1). For every Borel function $f$ on $\mathbb{R}$, the following quantity
\[L_f(u) := (f(K^2_u)u) - \alpha(f(K^2_u)11),\]
is conserved.

Let $\sigma^2_k$ be an eigenvalue of $K^2_u$, and $f$ be the characteristic function of the singleton $\{\sigma^2_k\}$, then
\[\ell_k(u) := ||u'_k||^2 - \alpha||v'_k||^2\]
is conserved, where $u'_k$, $v'_k$ are the projections of $u$ and $1$ onto ker$(K^2_u - \sigma^2_k)$, and $|| \cdot ||$ denotes the $L^2$–norm on the circle. Generically, on the $2N + 1$–dimensional complex manifold $L(N)$, we have $2N + 1$ linearly independent and in involution conservation laws, which are $\sigma_k$, $1 \leq k \leq N$ and $\ell_m$, $0 \leq m \leq N$. Thus, the system (1.1) can be approximated by a sequence of systems of finite dimension which are completely integrable in the Liouville sense.

Secondly, we prove the existence of unbounded trajectories for data in $L^N(1)$ for any arbitrary $N \in \mathbb{N}^+$. One way to capture the unbounded trajectories of solutions is via the motion of singular values of $H^2_u$ and $K^2_u$. In the case with $\alpha = 0$, all the eigenvalues of $H^2_u$ and $K^2_u$ are constants, but the eigenvalues of $H^2_u$ are no longer constants for $\alpha \neq 0$, which makes the system more complicated.

By studying the motion of singular values of $H_u$ and $K_u$, we gain that the necessary condition and existence of crossing which means the two closest eigenvalues of $H_u$ touch some eigenvalue of $K_u$ at some finite time. A remarkable observation is that the Blaschke products of $K_u$ never change their $S^1$ orbits as time goes.

The main result on the large time behaviour of solutions is as below.
Theorem 1.6. Let $u_0 \in \mathcal{L}(N)$ for any $N \in \mathbb{N}^+$. If $\alpha < 0$, the trajectory of the solution $u(t)$ of the $\alpha$–Szegő (1.1) stays in a compact subset of $\mathcal{L}(N)$. In other words, the Sobolev norm of the solution $u(t)$ will stay bounded,

$$\|u(t)\|_{H^s} \leq C, \ C \text{ does not depend on time } t, \ s \geq 0.$$  

While for $\alpha > 0$, there exists $u_0 \in \mathcal{L}(N)$ which leads to a solution with norm explosion at infinity. More precisely,

$$\|u(t)\|_{H^s} \simeq e^{C_\alpha (2s-1)|t|}, \ t \to \infty, \ \forall \ s \geq \frac{1}{2}.$$  

Remark 1.1.

1. In the case $\alpha = 0$, there are two Lax pairs, the conserved quantities are much simpler, which are the eigenvalues of $H^2_n$ and $K^2_n$. While in the case $\alpha \neq 0$, the eigenvalues of $H^2_n$ are no longer conserved, which makes our system more complicated.

2. For the cubic Szegő equation with $\alpha = 0$, Gérard and Grellier [4] have proved there exists a $G_\delta$ dense set $g$ of initial data in $C^\infty_+ := \cap_s H^s$, such that for any $v_0 \in g$, there exist sequences of time $t_n$ and $t_n$ such that the corresponding solution $v$ of the cubic Szegő equation

$$i\partial_t v = \Pi_+(|v|^2 v), \ v(0) = v_0,$$  

satisfies

$$\forall r > \frac{1}{2}, \ \forall M \geq 1, \ \frac{\|v(t_n)\|_{H^r}}{|t_n|^M} \to \infty, \ n \to \infty,$$  

while

$$v(t_n) \to v_0 \text{ in } C^{\infty}_+, \ n \to \infty.$$  

Here, by considering the rational data in the case $\alpha \neq 0$, we proved the existence of solutions with exponential growth in time rather than $\limsup$.

There is another non dispersive example with norm growth by Oana Pocovnicu [21], who studied the cubic Szegő equation on the line $\mathbb{R}$, and found there exist solutions with Sobolev norms growing polynomially in time as $|t|^{s-1}$ with $s \geq 1/2$.

3. For the case $\alpha > 0$, we now have solutions of (1.1) with different growths, uniformly bounded, growing in fluctuations with a $\limsup$ super-polynomial in time growth, and exponential in time growth. Indeed, it is easy to show that $zu(t, z^2)$ is a solution to the $\alpha$–Szegő equation if $u(t, z)$ solves the cubic Szegő equation (1.19). Thus, for the cubic Szegő equation with a linear perturbation (1.1), there also exist solutions with such an energy cascade as in (1.20) and (1.21).

4. In this paper, we consider data in $\mathcal{L}(N)$ for any arbitrary $N \in \mathbb{N}^+$. The data we find which lead to a large time norm explosion are very special. An interesting observation is that the equations on $u_k'$ and $v_k'$ look similar to the original $\alpha$–Szegő equation,

$$\frac{\partial}{\partial t} \begin{pmatrix} u_k' \\ v_k' \end{pmatrix} = -i \begin{pmatrix} T_{|u|^2} & \alpha(u|1) \\ -(1|u) & T_{|u|^2} - \sigma_k^2 \end{pmatrix} \begin{pmatrix} u_k' \\ v_k' \end{pmatrix},$$  

which gives us some hope to extend our results to general rational data.
1.3. Organization of this chapter. In section 2, we recall the results about the singular values of $H_u$ and $K_u$ [9]. In section 3, we introduce the conservation laws and prove the integrability. In section 4, we study the motion of the singular values of the Hankel operators $H_u$ and $K_u$, the eigenvalues of $H_u$ move and may touch some eigenvalue of $K_u$ at finite time while the eigenvalues of $K_u$ stay fixed with the corresponding Blaschke products stay in the same orbits. In section 5, we present a necessary condition of the norm explosion, and as a direct consequence, we know that for $\alpha < 0$, the trajectories of the solutions stay in a compact subset. In section 6, we study the norm explosion with $\alpha > 0$ for data in $L(N)$ with any $N \in \mathbb{N}^*$. We present some open problems in the last section.

2. Spectral analysis of the operators $H_u$ and $K_u$

In this section, let us introduce some notation which will be used frequently and some useful results by Gérard and Grellier in their recent work [9]. We consider $u \in H^s_+(\mathbb{S}^1)$ with $s > \frac{1}{4}$. The Hankel operator $H_u$ is compact by the theorem due to Hartman [18]. Let us introduce the spectral analysis of operators $H^2_u$ and $K^2_u$. For any $\tau \geq 0$, we set

$$E_u(\tau) := \ker(H^2_u - \tau^2 I), \quad F_u(\tau) := \ker(K^2_u - \tau^2 I).$$

If $\tau > 0$, the $E_u(\tau)$ and $F_u(\tau)$ are finite dimensional with the following properties.

Proposition 2.1. [9] Let $u \in H^s_+(\mathbb{S}^1) \setminus \{0\}$ with $s > 1/2$, and $\tau > 0$ such that

$$E_u(\tau) \neq \{0\} \quad \text{or} \quad F_u(\tau) \neq \{0\}.$$

Then one of the following properties holds.

1. $\dim E_u(\tau) = \dim F_u(\tau) + 1$, $u \notin E_u(\tau)$, and $F_u(\tau) = E_u(\tau) \cap u^\perp$.
2. $\dim F_u(\tau) = \dim E_u(\tau) + 1$, $u \notin F_u(\tau)$, and $E_u(\tau) = F_u(\tau) \cap u^\perp$.

Moreover, if $u_\rho$ and $u_\rho'$ denote respectively the orthogonal projections of $u$ onto $E_u(\rho)$, $\rho \in \Sigma_H(u)$, and onto $F_u(\sigma)$, $\sigma \in \Sigma_K(u)$ with

$$\Sigma_H(u) := \{\tau > 0 : u \notin E_u(\tau)\}, \quad \Sigma_K(u) := \{\tau \geq 0 : u \notin F_u(\tau)\}.$$

Then

1. $\Sigma_H(u)$ and $\Sigma_K(u)$ are disjoint, with the same cardinality;
2. if $\rho \in \Sigma_H(u)$,

$$u_\rho = ||u_\rho||^2 \sum_{\sigma \in \Sigma_K(u)} \frac{u_\sigma'}{\rho^2 - \sigma^2},$$

(2.2)

3. if $\sigma \in \Sigma_K(u)$,

$$u_\sigma' = ||u_\sigma'||^2 \sum_{\rho \in \Sigma_H(u)} \frac{u_\rho}{\rho^2 - \sigma^2},$$

(2.3)

4. A non-negative number $\sigma$ belongs to $\Sigma_K(u)$ if and only if it does not belong to $\Sigma_H(u)$ and

$$\sum_{\rho \in \Sigma_H(u)} ||u_\rho||^2 \frac{1}{\rho^2 - \sigma^2} = 1.$$
By the spectral theorem for $H^2_u$ and $K^2_u$, which are self-adjoint and compact, we have the following orthogonal decomposition

$$L^2_u = \oplus_{\tau \geq 0} E_u(\tau) = \oplus_{\tau > 0} F_u(\tau).$$

Then we can write $u$ as

$$u = \sum_{\rho \in \Sigma_\mu(u)} u_\rho = \sum_{\sigma \in \Sigma_\kappa(u)} u'_\sigma.$$  \hfill (2.6)

In fact, we are able to describe these two sets $E_u(\tau)$ and $F_u(\tau)$ more explicitly. Recall that a finite Blaschke product of degree $k$ is a rational function of the form

$$\Psi(z) = e^{-i\psi} \frac{P(z)}{D(z)},$$

where $\psi \in \mathbb{S}^1$ is called the angle of $\Psi$ and $P$ is a monic polynomial of degree $k$ with all its roots in $\mathbb{D}$, $D(z) = e^{kF}(\frac{1}{z})$ as the normalized denominator of $\Psi$. Here a monic polynomial is a univariate polynomial in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1. We denote by $B_k$ the set of all the Blaschke functions of degree $k$.

**Proposition 2.2.** [9] Let $\tau > 0$ and $u \in H^2_u(\mathbb{S}^1)$ with $s > \frac{1}{s}$.

1. Assume $\tau \in \Sigma_\mu(u)$ and $\ell := \dim E_u(\tau) = \dim F_u(\tau) + 1$. Denote by $u_\tau$ the orthogonal projection of $u$ onto $E_u(\tau)$. There exists a Blaschke function $\Psi_\tau \in B_{\ell - 1}$ such that

$$\tau u_\tau = \Psi_\tau H_u(\tau),$$

and if $D$ denotes the normalized denominator of $\Psi_\tau$,

$$E_u(\tau) = \left\{ \frac{f}{D(z)} H_u(\tau), \ f \in \mathbb{C}_{\ell - 1}[z] \right\},$$

$$F_u(\tau) = \left\{ \frac{g}{D(z)} H_u(\tau), \ g \in \mathbb{C}_{\ell - 2}[z] \right\},$$

and for $a = 0, \ldots, \ell - 1$, $b = 0, \ldots, \ell - 2$,

$$H_a \left( \frac{e^a}{D(z)} H_u(\tau) \right) = \tau e^{-i\psi_\tau} \frac{e^{a - 1}}{D(z)} H_u(\tau),$$

$$K_a \left( \frac{e^b}{D(z)} H_u(\tau) \right) = \tau e^{-i\psi_\tau} \frac{e^{b - 2}}{D(z)} H_u(\tau),$$

where $\psi_\tau$ denotes the angle of $\Psi_\tau$.

2. Assume $\tau \in \Sigma_\kappa(u)$ and $m := \dim F_u(\tau) = \dim E_u(\tau) + 1$. Denote by $u'_\tau$ the orthogonal projection of $u$ onto $F_u(\tau)$. There exists an inner function $\Psi_\tau \in B_{m - 1}$ such that

$$K_u(u'_\tau) = \tau \Psi_\tau u'_\tau,$$

and if $D$ denotes the normalized denominator of $\Psi_\tau$,

$$E_u(\tau) = \left\{ \frac{f}{D(z)} u'_\tau, \ f \in \mathbb{C}_{m - 1}[z] \right\},$$

$$F_u(\tau) = \left\{ \frac{g}{D(z)} u'_\tau, \ g \in \mathbb{C}_{m - 2}[z] \right\},$$

\hfill (2.11) \hfill (2.12)
and, for \( a = 0, \ldots, m - 1 \), \( b = 0, \ldots, m - 2 \),

\[
K_u \left( \frac{z^a}{D(z)} u'_r \right) = re^{-i\psi_r} \frac{z^{m-a-1}}{D(z)} u'_r, \quad (2.13)
\]

\[
H_u \left( \frac{z^{b+1}}{D(z)} u'_r \right) = re^{-i\psi_r} \frac{z^{m-b-1}}{D(z)} u'_r, \quad (2.14)
\]

where \( \psi_r \) denotes the angle of \( \Psi_r \).

We call the elements \( \rho_j \in \Sigma_H(u) \) and \( \sigma_k \in \Sigma_K(u) \) as the dominant eigenvalues of \( H_u \) and \( K_u \) respectively. Due to the above achievements, they are in a finite or infinite sequence

\[
\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots \to 0,
\]

we denote by \( \ell_j \) and \( m_k \) as the multiplicities of \( \rho_j \) and \( \sigma_k \) respectively. In other words,

\[
\dim E_u(\rho_j) = \ell_j,
\]

\[
\dim F_u(\sigma_k) = m_k.
\]

Therefore, we may define the dominant ranks of the operators as

\[
\text{rk}_d(H_u) := \sum_j \ell_j,
\]

\[
\text{rk}_d(K_u) := \sum_k m_k,
\]

while the ranks of the operators are

\[
\text{rk}(H_u) = \sum_j \ell_j + \sum_k (m_k - 1),
\]

\[
\text{rk}(K_u) = \sum_j (\ell_j - 1) + \sum_k m_k.
\]

In this paper, \( u_j \) and \( u'_k \) denote the orthogonal projections of \( u \) onto \( E_u(\rho_j) \) and \( F_u(\sigma_k) \) respectively, while \( v_j \) and \( v'_k \) denote the orthogonal projections of 1 onto \( E_u(\rho_j) \) and \( F_u(\sigma_k) \). The \( L^2 \)-norms of \( u_j \) and \( u'_k \) can be represented in terms of \( \rho_j \)'s and \( \sigma_k \)'s, which was already observed in [8].

**Lemma 2.1.** Let \( u \in H^1(S^1) \), \( \Sigma_H(u) = \{\rho_j\} \) and \( \Sigma_K(u) = \{\sigma_k\} \) with

\[
\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0.
\]

Then

\[
\|u_j\|^2 = \frac{\prod (\rho_j^2 - \sigma_l^2)}{\prod (\rho_j^2 - \rho_l^2)} \cdot \|u'_k\|^2 = \frac{\prod (\rho^2 - \sigma_k^2)}{\prod (\sigma^2 - \sigma_k^2)}.
\]

**Proof.** First, we have

\[
((I - xH^2)| 1) = \prod_l \frac{1 - x\sigma_l^2}{1 - x\rho_l^2}.
\]
In fact, we can rewrite the left hand side as
\[
((I - xH^2_u)^{-1} | 1) = \sum_{\ell} \frac{\|v_\ell\|^2}{1 - xp_\ell^2} + 1 - \sum_{\ell} \|v_\ell\|^2.
\]

From Proposition 2.2,
\[
v_j = \left(1, \frac{H_u(u_j)}{\|H_u(u_j)\|}\right) \frac{H_u(u_j)}{\|H_u(u_j)\|},
\]
combined with \(\Psi_j H_u(u_j) = \rho_j u_j\), we get
\[
\|v_j\|^2 = \frac{\|(1, H_u(u_j))\|^2}{\|H_u(u_j)\|^2} = \frac{\|(H_u(1), u_j)\|^2}{\rho_j^2 \|u_j\|^2} = \frac{\|u_j\|^2}{\rho_j^2}.
\]

Thus
\[
\prod_{\ell} \frac{1 - x\sigma_\ell^2}{1 - x\rho_\ell^2} = \sum_{\ell} \frac{||u_\ell||^2}{\rho_\ell^2(1 - x\rho_\ell^2)} + 1 - \sum_{\ell} \frac{||u_\ell||^2}{\rho_\ell^2}.
\]

We get, identifying the residues at \(x = 1/\rho_j^2\),
\[
\|u_j\|^2 = \frac{\prod_{\ell} (\rho_j^2 - \sigma_\ell^2)}{\prod_{\ell \neq j} (\rho_j^2 - \rho_\ell^2)}.
\]

On the other hand, since
\[
1 - x((I - xK^2_u)^{-1} u | u) = \frac{1}{\|(I - xH^2_u)^{-1} | 1\|},
\]
then
\[
1 - x\left(\sum_k \frac{||u'_k||^2}{1 - x\sigma_k^2} + ||u||^2 - \sum_k ||u'_k||^2\right) = \prod_{\ell} \frac{1 - x\sigma_\ell^2}{1 - x\sigma_k^2},
\]
we get, identifying the residues at \(x = 1/\sigma_k^2\),
\[
||u'_k||^2 = \frac{\prod_{\ell} (\sigma_k^2 - \sigma_\ell^2)}{\prod_{\ell < k} (\sigma_k^2 - \sigma_\ell^2)}.
\]

3. Conservation laws and the \(\alpha\)--Szego hierarchy

We endow \(L^2_\alpha(S^1)\) with the symplectic form
\[
\omega(u, v) = 4\text{Im}(u | v).
\]

Then (1.1) can be rewritten as
\[
\partial_t u = X_{E_\alpha}(u),
\]
with \(X_{E_\alpha}\) as the Hamiltonian vector field associated to the Hamiltonian function given by
\[
E_\alpha(u) := \frac{1}{4} \int_{S^1} |u|^4 \frac{d\theta}{2\pi} + \frac{\alpha}{2} |u(1)|^2.
\]
The invariance by translation and by multiplication by complex numbers of modulus 1 gives two other formal conservation laws

mass: $Q(u) := \int_{S^1} |u|^2 \frac{d\theta}{2\pi} = \|u\|_{L^2}^2,$
momentum: $M(u) := (Du|u), D := -i\partial_\theta = z\partial_z.$

Moreover, the Lax pair structure leads to the conservation of the eigenvalues of $K_n^2.$ So it is obvious the system is completely integrable for the data in the 3-dimensional complex manifold $\mathcal{L}(1).$ Then what about the general case, for example in $\mathcal{L}(N)$ with arbitrary $N \in \mathbb{N}^+?$ Fortunately, we are able to find many more conservation laws by its Lax pair structure (1.14). We will then show our system is still completely integrable with data in $\mathcal{L}(N)$ in the Liouville sense.

3.1. Conservation laws. Thanks to the Lax pair structure, we are able to find an infinite sequence of conservation laws.

**Theorem 3.1.** For every Borel function $f$ on $\mathbb{R},$ the following quantity

$L_f(u) := \left( f(K_n^2)u|u\right) - \alpha \left( f(K_n^2)(1)|1 \right)$

is a conservation law.

**Proof.** From the Lax pair identity

$$\frac{dK_n}{dt} = [C_n, K_n],$$
we infer

$$\frac{d}{dt}K_n^2 = [-iT_{|u|^2}, K_n^2],$$
and consequently, for every Borel function $f$ on $\mathbb{R},$

$$\frac{d}{dt}f(K_n^2) = [-iT_{|u|^2}, f(K_n^2)].$$

On the other hand, the equation reads

$$\frac{d}{dt}u = -iT_{|u|^2}u - i\alpha(u|1).$$

Therefore we obtain

$$\frac{d}{dt}\left( f(K_n^2)u|u\right) = \left( [-iT_{|u|^2}, f(K_n^2)]u|u\right) - i\left( f(K_n^2)T_{|u|^2}u|u\right) + i\left( u|f(K_n^2)T_{|u|^2}u\right)$$

$$-i\alpha(u|1)\left( f(K_n^2)(1)|1\right) + i\alpha(1|u)\left( f(K_n^2)(1)u\right)$$

$$= -i\alpha\left( f(K_n^2)(1)|1\right) - (1|u)f(K_n^2(1)) \right).$$

Now observe that

$$(1|u)u = H_n^2(1) - K_n^2(1) = T_{|u|^2}(1) - K_n^2(1).$$
We obtain
\[
\frac{d}{dt}(f(K_u^2)u|u) = -i\alpha \left[ (f(K_u^2)(1)|T_{u|u}^2(1)) - (T_{u|u}^2(1)|f(K_u^2)(1)) \right]
\]
\[
= \alpha \left( -iT_{u|u}^2, f(K_u^2) \right)(1)|1 \right)
\]
\[
= \alpha \frac{d}{dt} \left( f(K_u^2)(1)|1 \right).
\]

\[\square\]

3.2. The $\alpha$–Szegő hierarchy. By the theorem above, for any $n \in \mathbb{N},$
\[
L_n(u) := (K_u^{2n}(u) | u) - \alpha (K_u^{2n}(1) | 1)
\]
is conserved. Then the manifold $\mathcal{L}(N)$ is of $2N + 1$– complex dimension and admits $2N + 1$ conservation laws, which are
\[
\sigma_k, k = 1, \cdots, N \text{ and } L_n(u), n = 0, 1, \cdots, N.
\]
We are to show that all these conservation laws are in involve. Since the $\sigma_k$’s are constants, it is sufficient to show that all these $L_n$ satisfy the Poisson commutation relations
\[
\{L_n, L_m\} = 0. \quad (3.2)
\]
Let us begin with the following lemma which helps us better understand the conserved quantities.

**Lemma 3.1.** Let $u \in H^\downarrow(\mathbb{S}^1), \Sigma_H(u) = \{\rho\}$ and $\Sigma_{K}(u) = \{\sigma\}$ with
\[
\rho_1 > \sigma_1 > \rho_2 > \cdots \geq 0.
\]
Denote
\[
J_x(u) := ((1 - xH_u^2)^{-1}(1) | 1),
\]
\[
Z_x(u) := (1 | (1 - xH_u^2)^{-1}(u)),
\]
\[
F_x(u) := ((1 - xK_u^{2})^{-1}(u) | u),
\]
\[
E_x(u) := ((1 - xK_u^{2})^{-1}(1) | 1).
\]
Then
\[
F_x(u) = \frac{J_x(u) - 1}{xJ_x(u)}, \quad (3.3)
\]
\[
E_x(u) = J_x(u) - \frac{|Z_x(u)|^2}{J_x(u)}. \quad (3.4)
\]
**Proof.** Recall (1.11), for any $f \in H^\downarrow$, we have
\[
K_u^{2}f = H_u^{2}f - (f | u)u.
\]
Denote
\[
w(f) = (1 - xH_u^{2})^{-1}(f) - (1 - xK_u^{2})^{-1}(f), \quad (3.5)
\]
then
\[
    w(f) = x(f \mid (1 - xK_u^2)^{-1}(u))(1 - xH_u^2)^{-1}(u) = x(f \mid (1 - xH_u^2)^{-1}(u))(1 - xK_u^2)^{-1}(u) .
\]

We may observe the two vectors \((1 - xH_u^2)^{-1}(u)\) and \((1 - xK_u^2)^{-1}(u)\) are co-linear,
\[
    (1 - xK_u^2)^{-1}(u) = A(1 - xH_u^2)^{-1}(u), \quad A \in \mathbb{R} .
\]  

(3.6)

Let us choose \(f = u\), then
\[
    \left( w(u) \mid u \right) = (1 - A)((1 - xH_u^2)^{-1}(u) \mid u) = Ax((1 - xH_u^2)^{-1}(u)^2 .
\]  

(3.7)

We are to calculate the factor \(A\). Since
\[
    x(u \mid (1 - xH_u^2)^{-1}(u)) = x(1 \mid (1 - xH_u^2)^{-1}H_u(1)) = \sum_{n \geq 0} x^{n+1}(H_u^{n+1}(1) \mid 1) = \sum_{n \geq 0} x^n(H_u^n(1) \mid 1) - 1 = J_x - 1 ,
\]

thus (3.7) yields
\[
    1 - A = (J_x - 1)A ,
\]

which means
\[
    A = \frac{1}{J_x} .
\]

So (3.6) turns out to be
\[
    (1 - xK_u^2)^{-1}(u) = \frac{1}{J_x}(1 - xH_u^2)^{-1}(u) ,
\]  

(3.8)

then combined with the definition of \(w(f)\), we have
\[
    (1 - xH_u^2)^{-1}(f) - (1 - xK_u^2)^{-1}(f) = \frac{x}{J_x}\left( f \mid (1 - xH_u^2)^{-1}(u) \right)(1 - xH_u^2)^{-1}(u) .
\]  

(3.9)

Using the equality (3.8),
\[
    F_x = \left( (1 - xK_u^2)^{-1}(u) \mid u \right) = \frac{1}{J(x)}((1 - xH_u^2)^{-1}(u) \mid u)
\]
\[
    = \frac{1}{J(x)}((1 - xH_u^2)^{-1}H_u(1) \mid 1) = \frac{J_x - 1}{xJ_x} .
\]

Now, we turn to prove (3.4). Use again (3.5) with \(f = 1\),
\[
    \left( w(1) \mid 1 \right) = \left( (1 - xH_u^2)^{-1}(1) - (1 - xK_u^2)^{-1}(1) \mid 1 \right) = J_x - E_x
\]
\[
    = x(1 \mid 1 - xH_u^2)^{-1}(1))(1 - xK_u^2)^{-1}(1)\mid 1 = xZ_{f_x}((1 - xK_u^2)^{-1}(u)\mid 1) ,
\]

plugging (3.6),
\[
    \left( (1 - xK_u^2)^{-1}(u) \mid 1 \right) = \frac{1}{J_x}(1 - xH_u^2)^{-1}(u)\mid 1 = \frac{Z_x}{J_x} ,
\]

then
\[
    J_x - E_x = xZ_x \frac{Z_x}{J_x} = x\frac{|Z_x|^2}{J_x} ,
\]  

(3.10)

which leads to (3.4). \(\square\)
Now, we are ready to show the following cancellation for the Poisson brackets of the conservation laws.

**Theorem 3.2.** For any $x \in \mathbb{R}$, we set

$$L_x(u) = ((1 - xK^2_x)^{-1}(u) \mid u) - \alpha((1 - xK^2_x)^{-1}(1) \mid 1),$$

Then $L_x(u(t))$ is conserved, and for every $x, y$,

$$\{L_x, L_y\} = 0.$$ (3.11)

**Proof.** Using the previous Lemma, we may rewrite

$$L_x = \frac{1}{\lambda}(1 - \frac{1}{J_x}) - \alpha E_x,$$ (3.12)

with

$$J_x(u) := ((1 - xH^2_x)^{-1}(1) \mid 1) = 1 + x((1 - xH^2_x)^{-1}(u) \mid u),$$

$$E_x(u) := ((1 - xK^2_x)^{-1}(1) \mid 1) = J_x(u) - x\frac{|Z_x(u)|^2}{J_x(u)},$$

$$Z_x(u) := (1 \mid (1 - xH^2_x)^{-1}(u)).$$

Recall that the identity

$$\{J_x, J_y\} = 0$$ (3.13)

which was obtained in [5, section 8]. We then have

$$\{L_x, L_y\} = \alpha\left(\frac{y}{xJ_x} \{J_x, |Z_x|^2\} - \frac{x}{yJ_y} \{J_y, |Z_x|^2\}\right) + \alpha^2\{E_x, E_y\}.$$ (3.14)

Let us first prove that $\{E_x, E_y\} = 0$. Notice that

$$E_x(u) = J_x(S^*u),$$ (3.15)

therefore

$$dE_x(u) \cdot h = dJ_x(S^*u) \cdot (S^*h) = \omega(S^*h, X_{J_x}(S^*u)) = \omega(h, SX_{J_x}(S^*u)).$$

We conclude

$$X_{E_x}(u) = SX_{J_x}(S^*U),$$

thus

$$\{E_x, E_y\}(u) = dE_x(u) \cdot X_{E_y}(u) = dJ_x(S^*u) \cdot S^*SX_{J_x}(S^*u)$$

$$= dJ_x(S^*u) \cdot X_{J_x}(S^*u) = \{J_x, J_x\}(S^*u) = 0.$$ (3.16)

We now show that the coefficient of $\alpha$ in (3.14) vanishes identically. It is enough to work on the generic states of $\mathcal{L}(N)$, so we can use the coordinates

$$(\rho_1, \ldots, \rho_{N+1}, \sigma_1, \ldots, \sigma_N, \varphi_1, \ldots, \varphi_{N+1}, \theta_1, \ldots, \theta_N)$$

for which we recall that

$$\omega = \sum_{j=1}^{N+1} d\left(\frac{\rho_j^2}{2}\right) \wedge d\varphi_j + \sum_{k=1}^{N} d\left(\frac{\sigma_k^2}{2}\right) \wedge d\theta_k.$$ (3.17)

Moreover, we have

$$\rho_1 \mu_j = e^{-i\varphi_j} H_u(u_j),$$ (13)
therefore,
\[ Z_x(u) = \sum_{j=1}^{N+1} \frac{||u_j||^2}{\rho_j(1 - x\rho_j^2)} e^{i\varphi_j}. \]

Since
\[ J_x(u) = \prod_{j=1}^{N+1} (1 - x\sigma_j^2) \prod_{j=N+1}^{N+1} (1 - x\rho_j^2), \]
we know that
\[ \{ J_x, \varphi_j \} = \frac{2xJ_x}{1 - x\rho_j^2}, \]
and we infer
\[ \{ J_x, Z_y \} = 2iJ_x \sum_{j=1}^{N+1} \frac{||u_j||^2}{\rho_j(1 - x\rho_j^2)(1 - y\rho_j^2)} e^{i\varphi_j} = \frac{2ixJ_x}{x - y} (xZ_x - yZ_y). \] (3.16)
Consequently,
\[ \{ J_x, |Z_y|^2 \} = 2Re(\overline{Z}_y J_x, Z_y) = -\frac{4x^2J_x}{x - y} \text{Im}(\overline{Z}_y Z_x). \] (3.17)

We conclude that
\[ \frac{y}{xJ_x} J_y \{ J_x, |Z_y|^2 \} - \frac{x}{yJ_y} J_x \{ J_y, |Z_y|^2 \} = -\frac{4xy}{(x - y)J_x J_y} (\text{Im}(\overline{Z}_y Z_x) + \text{Im}(\overline{Z}_x Z_y)) = 0. \] (3.18)

This completes the proof. \(\Box\)

The last part of this section is devoted to proving that functions \((L_n(u))_{0 \leq n \leq N}\) are generically independent on \(L(N)\). Actually, it is sufficient to discuss the case \(|\alpha| \ll 1\). For \(\alpha\) small enough, we may consider the term \(\alpha(K^{2n}(u)|1)\) as a perturbation, then we only need to study the independence of \(F_n := (K^{2n}(u)|u)\). Using the formula (3.12), for any \(0 \leq n \leq N\),
\[ F_n = J_{n+1} - \sum_{k+j=n} F_k J_j, \]
with \(J_n = (H^{2n}_u|1|1)\). Assume there exists a sequence \(c_n\) such that
\[ \sum_{n\geq0} c_n F_n = 0, \]
we are to prove that \(c_n \equiv 0\). Indeed,
\[ \sum_{n\geq0} c_n J_{n+1} - \sum_{n\geq0} \sum_{k+j=n} c_n F_k J_j = \sum_n (c_n - \sum_{0 \leq k \leq (n-1)} c_{n+k+1} F_k) J_{n+1} = 0, \]
since all the \(J_{n+1}\) are independent in the complement of a closed subset of measure 0 of \(L(N)\) [5], then for every \(n\),
\[ c_n - \sum_{0 \leq k \leq (n-1)} c_{n+k+1} F_k = 0. \]
Thus \(c_N = c_{N-1} = \cdots = c_0 = 0\).
Finally, we now have $2N + 1$ linearly independent and in involution conservation laws on a dense open subset of $2N + 1$ dimensional complex manifold $\mathcal{L}(N)$, thus our system is completely integrable in the Liouville sense.

4. Multiplicity and Blaschke product

Recall the notation in section 2, there are two kinds of eigenvalues of $K_\mu$, some are the dominant eigenvalues of $K_\mu$, which are denoted as $\sigma_k \in \Sigma_k(u)$, while the others are the dominant eigenvalues of $H_\mu$ with multiplicities larger than 1. Let us denote $u(t)$ as the solution of the $\alpha-$Szegő equation with $\alpha \neq 0$. Fortunately, we are able to show that for almost all $t \in \mathbb{R}$, the Hankel operator $H_{u(t)}$ has single dominant eigenvalues with multiplicities equal to 1. In other words, for almost every time $t \in \mathbb{R}$,

$$\text{rk}_d K_{u(t)} = \text{rk}_d K_{u(t)} = \text{rk}_d K_{u_0}.$$ 

We call the phenomenon that $H_{u(t)}$ has some eigenvalue $\sigma$ with multiplicity $m \geq 2$ as crossing at $\sigma$ at $t_0$.

4.1. The motion of singular values. Let us first introduce the following Kato-type lemma.

**Lemma 4.1** (Kato). Let $P(t)$ be a projector on a Hilbert space $\mathcal{H}$ which is smooth in $t \in I$, then there exists a smooth unitary operator $U(t)$, such that

$$P(t) = U(t)P(0)U^*(t),$$

and

$$\frac{d}{dt} U(t) = Q(t)U(t), \quad U(0) = \text{Id}, \quad (4.1)$$

with $Q(t) = [P'(t), P(t)]$.

**Proof.** By simple calculus, we can prove $Q^* = -Q$. Since $P(t)$ is smooth in time, then by the Cauchy theorem for linear ordinary equations, $U(t)$ is well defined. The unitary property of $U(t)$ for every $t$ is a consequence of the anti self-adjointness of $Q$.

$$\frac{d}{dt} U(t)U(t)^* = \frac{d}{dt} U^*U + U^* \frac{d}{dt} U = U^*Q^*U + U^*QU = 0,$$

thus $U(t)^*U(t) = \text{Id}$. On the other hand,

$$\frac{d}{dt} (U(t)U(t)^*) = \frac{d}{dt} UU^* + U^* \frac{d}{dt} U^* = QUU^* - UU^*Q.$$

It is obvious that $\text{Id}$ is a solution to the linear equation $\frac{d}{dt} A = QA - AQ$ with $A(0) = \text{Id}$, using the uniqueness of solutions, we have $U(t)U^*(t) = \text{Id}$. We now prove that $U^*(t)P(t)U(t)$ does not depend on $t$.

$$\frac{d}{dt} (U^*(t)P(t)U(t)) = \frac{d}{dt} U^*(t)P(t)U(t) + U^*(t) \frac{d}{dt} P(t)U(t) + U^*(t)P(t) \frac{d}{dt} U(t)$$

$$= U^*QPU + U^*P'U + U^*PQU$$

$$= U^*(P' + [P, Q])U$$

$$= U^*(P' - PP' - P'P)U = 0$$

where we have used $P^2 = P$. This completes the proof. \qed
If $u_0 \in H^s_+$ with $s > 1$, then the solution $u(t)$ of the $\alpha$–Szegő equation (1.1) is real analytic in $t$ valued in $H^s_+$. By the Lax pair for $K_u$, we know that the singular values of $K_u$ are fixed, with constant multiplicities.

**Proposition 4.1.** Given any initial data $u_0 \in H^s_+$ with $s > 1$, let $u$ be the corresponding solution to the $\alpha$–Szegő equation. Let $\sigma > 0$ be a singular eigenvalue of $K_u$ with multiplicity $m$, and write

$$\sigma_+ > \sigma > \sigma_-$$

where $\sigma_+$, $\sigma_-$ are the closest singular values of $K_u$, possibly, $\sigma_+ = +\infty$ or $\sigma_- = 0$. Then one of the following two possibilities occurs.

1. $\sigma$ is a singular value of $H_{u(t)}$ with multiplicity $m + 1$ for every time $t$, and $u$ is a solution of the cubic Szegő equation (1.2).

2. There exists a discrete subset $T_\varepsilon$ of times outside of which the singular values of $H_{u(t)}$ in the interval $(\sigma_-, \sigma_+)$ are $\rho_1$, $\rho_2$ of multiplicity 1, and $\sigma$ of multiplicity $m - 1$ if $m \geq 2$, with

$$\rho_1 > \sigma > \rho_2,$$

and $\rho_1$, $\rho_2$ are analytic on every interval contained into the complement of $T_\varepsilon$.

**Proof.** Let us assume that $\sigma$ is a singular value of multiplicity $m + 1$ of $H_{u(t)}$ for some time $t_0$. Then we may select $\delta > 0$ and $\varepsilon > 0$ such that

$$\sigma_+ > \sigma + \varepsilon > \sigma > \sigma - \varepsilon > \sigma_-$$

such that, for every $t \in [t_0 - \delta, t_0 + \delta]$, $\sigma^2 - \varepsilon$ and $\sigma^2 + \varepsilon$ are not eigenvalues of $H^2_{u(t)}$. Then we know that $H^2_{u(t)}$ has either $\sigma^2$ as an eigenvalue of multiplicity $m + 1$, or admits in $(\sigma^2 - \varepsilon, \sigma^2 + \varepsilon)$ two eigenvalues of multiplicity 1, $\rho_1, \rho_2$ on both sides of $\sigma$. Set

$$P(t) := (2i\pi)^{-1} \int_{C(\sigma^2, \varepsilon)} (zI - H^2_{u(t)})^{-1} dz . 

(4.2)$$

We know that $P(t)$ is an orthogonal projector, depending analytically of $t \in (t_0 - \delta, t_0 + \delta)$, and that $P(t_0)$ is just the projector onto

$$E(t_0) := \ker(H^2_{u(t_0)} - \sigma^2 I) .$$

Consider the selfadjoint operator

$$A(t) := H^2_{u(t)} P(t)$$

acting on the $(m + 1)$-dimensional space $E(t) = \text{Ran} P(t)$. Then its characteristic polynomial is

$$P(\lambda, t) = (\sigma^2 - \lambda)^{m-1} (\lambda^2 + a(t)\lambda + b(t)) ,$$

where $a$, $b$ are real analytic, real valued functions, such that

$$a^2 - 4b \geq 0 .$$

Notice that the condition $a(t)^2 - 4b(t) = 0$ is precisely equivalent to the fact that $H^2_{u(t)}$ has $\sigma^2$ as an eigenvalue of multiplicity $m + 1$. Since this function is analytic, it is either identically 0, or different from 0 for $0 < |t - t_0| < \delta$ and $\delta > 0$ small enough. Moreover, by the following perturbation analysis, the first condition only occurs if

$$|u(t)) = 0$$
for every \( t \in (t_0 - \delta, t_0 + \delta) \). Since \((1|u)\) is a real analytic function of \( t \), this would imply that it is identically 0, whence \( u \) is a solution of the cubic Szegő equation. We now come back to the perturbation analysis, let \( U(t) \) be a unitary operator given as in the Kato-type lemma above, denote
\[
B(t) = U^*(t)A(t)U(t) ,
\]
then
\[
B(t_0) = \sigma^2 \text{Id}P(t_0) .
\]
Let us calculate the derivative of \( B \), we find
\[
\frac{d}{dt} B(t) = \frac{d}{dt} \left( U^*(t)H^2_{w(t)}U(t)U^*(t)P(t)U(t) \right) = \frac{d}{dt} \left( U^*(t)H^2_{w(t)}U(t)P(t_0) \right) .
\]
Since \( \frac{d}{dt} U(t) = Q(t)U(t) \) with \( Q(t) = [P'(t), P(t)] \), then
\[
\frac{d}{dt} B(t) = U^* \left( \frac{d}{dt} H^2_{w(t)} + [H^2_{w(t)}, Q(t)] \right) U P(t_0) ,
\]
using \((1.13)\),
\[
\frac{d}{dt} H^2_{w(t)} = [B_w, H^2_w] - i\alpha(u|1)H_1 H_u + i\alpha(1|u)H_u H_1 .
\]
For any \( h_1, h_2 \in E(t_0) \),
\[
([B_w, H^2_w]h_1, h_2) + ([H^2_w, Q]h_1, h_2) = 0 ,
\]
then
\[
\left( \frac{d}{dt} B(t_0) h_1, h_2 \right) = -i\alpha[(u(t_0)|1)(h_1|u(t_0))(1|h_2) - (1|u(t_0))(u(t_0)|h_2)(1|1)] .
\]
Denote by \( v, w \) as the projections onto \( E(t_0) \) of \( 1 \) and \( u \) respectively. If \((u(t_0)|1) \neq 0\), then the corresponding matrix under the base \((v, w)\) turns out to be
\[
\begin{pmatrix}
-i\alpha(u|1)(v|w) & i\alpha(1|u)||w|^2 \\
-i\alpha(u|1)||v|^2 & i\alpha(1|u)(w|v)
\end{pmatrix}
\]
which has a negative determinant if \((u(t_0)|1) \neq 0\). For the case \((u(t_0)|1) = 0\) with \( \frac{d}{dt^m}(u|1)(t_0) \neq 0 \) for some \( n \in \mathbb{N} \), we only need to consider \( \frac{d^{m+1}}{dt^{m+1}}(B(t))(t_0) \),
\[
\left( \frac{d^{m+1}}{dt^{m+1}} B(t_0) h_1, h_2 \right) = -i\alpha \left[ \left( \frac{d^m}{dt^m}(u|1)(t_0)(1|u(t_0))(h_1|1) - \frac{d^m}{dt^m}(1|u)(t_0)(u(t_0)|h_2)(h_1|1) \right) \right] ,
\]
with any \( h_1, h_2 \in E(t_0) \). It is similar as the case \( n = 0 \). This completes the proof.

Since \( u(t) \) satisfying \((1|u(t)) = 0\) would be a solution of the cubic Szegő equation, which is well studied by Gérard and Grellier [5, 7, 6, 10]. We assume \((1|u)\) is not identically zero in the rest of this article. From the discussion above, we have

**Corollary 4.1.** The dominant eigenvalues of \( H_{w(t)} \) are of multiplicity 1 for almost all \( t \in \mathbb{R} \).

Recall the notation in section 2, by rewriting the conservation laws in Theorem 3.1 as
\[
L_n := K^{2n}_u(u|1) - \alpha K^{2n}_u(1|1) = \sum_k \sigma^{2n}_k \left( ||u_k||^2 - \alpha ||v_k||^2 \right) ,
\]
we get the following conserved quantities

\[ \ell_k := ||u_k'||^2 - \alpha ||v_k'||^2. \]  

(4.4)

**Lemma 4.2.** Let \( \alpha > 0 \). If there exists a crossing at \( \sigma_k \) at time \( t = t_0 \), then \( \ell_k < 0 \).

**Proof.** Since there is a crossing at \( \sigma_k \), then \( \sigma_k \in \Sigma_H(u(t_0)) \) with multiplicity \( m \geq 2 \). Then

\[ F_u(\sigma_k) = E_u(\sigma_k) \cap u^\perp = \left\{ \frac{g}{D} H_u(uk) : g \in \mathbb{C}_{m-2}[z] \right\}. \]

Hence, \( u_k' = 0 \) while \( v_k' \neq 0 \), since

\[ ||v_k'|| = \frac{(1, H_u(uk))}{||H_u(uk)||} = \frac{||u_k||}{\sigma_k} \neq 0. \]  

(4.5)

Thus \( \ell_k = ||u_k'||^2 - \alpha ||v_k'||^2 < 0 \) for \( \alpha > 0 \). \( \square \)

Here, we present an example to show the existence of crossing.

**Example 4.1** (Existence of crossing). Let \( u_0(z) = \frac{z-p}{1-pz} \) with \( p \neq 0 \) and \( |p| < 1 \), and \( u \) be the corresponding solution to the equation

\[ i\partial_t u = \Pi(|u|^2 u) + (u|1). \]  

(4.6)

It is obvious that \( u_0 \in L(1) \) and \( 1 \in \Sigma_H(u_0) \) with multiplicity 2, and

\[ L_i(u) = \left( K_u^2(u) | u \right) - \left( K_u^2(1) | 1 \right) = -(1 - |p|^2) < 0. \]

Let us represent the Hamiltonian function \( E = \frac{1}{4}||u||_{L^4}^4 + \frac{1}{2}(|u|1)^2 \) under the coordinates \( \rho_1, \rho_2, \sigma, \varphi_1, \varphi_2, \theta \),

\[
E = \frac{1}{4}(\rho_1^4 + \rho_2^4 - \sigma^4) + \frac{1}{2} \rho_1^2(\rho_1^2 - \sigma^2)^2 + \rho_2^2(\sigma^2 - \rho_2^2)^2 + 2\rho_1\rho_2(\rho_1^2 - \rho_2^2)(\sigma^2 - \rho_2^2) \cos(\varphi_1 - \varphi_2)
\]

\[ \left( \rho_1^2 - \rho_2^2 \right)^2 \]

\[ = \frac{1}{4} + \frac{1}{2}|p|^2. \]

Notice that \( \sigma = 1 \) and \( \rho_1^2 + \rho_2^2 - \sigma^2 = ||u||_{L^2}^2 = 1 \), then \( \rho_1^2 + \rho_2^2 = 2 \). Set \( I = \frac{\rho_1^2 - \rho_2^2}{2}, \varphi = \varphi_1 - \varphi_2, \) then \( \rho_1^2 = 1 + I \) and \( \rho_2^2 = 1 - I \), thus we can rewrite \( E \) as

\[ E = \frac{1}{4}(1 + 2I^2) + \frac{1}{4}(1 + \sqrt{1 - I^2} \cos(\varphi)) . \]

Thus

\[
\frac{dI}{dt} = -2 \frac{\partial E}{\partial \varphi} = \frac{1}{2} \sqrt{1 - I^2} \sin(\varphi) = \pm \frac{1}{2} \sqrt{4I^2 + (8|p|^2 - 5)I^2 + 4|p|^2(1 - |p|^2)} = \pm \sqrt{(a - I^2)(b + I^2)},
\]
with \( a, b \) satisfy
\[
\begin{aligned}
& a > 0, b > 0, \\
& ab = |p|^2(1 - |p|^2), \\
& a - b = 2|p|^2 - 5/4.
\end{aligned}
\]

Recall the definition of Jacobi elliptic functions. The incomplete elliptic integral of the first kind \( F \) is defined as
\[
F(\varphi, k) \equiv \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},
\]
then the Jacobi elliptic function \( sn \) and \( cn \) are defined as follows,
\[
\begin{aligned}
& sn(F(\varphi, k), k) = \sin \varphi, \\
& cn(F(\varphi, k), k) = \cos \varphi.
\end{aligned}
\]

Then we may solve the above equation,
\[
I(t) = \sqrt{\text{acn}} \left( \sqrt{a + b}(t - t_0) + F(\pi/2, \sqrt{a + b}), \sqrt{a + b} \right).
\]
Therefore, there exists a discrete set of time \( t_0 \in T_c \), such that \( I(t) = 0 \) for every \( t \in T_c \). In other words, crossing happens at \( t \in T_c \).

4.2. Blaschke product. We aim to show that the Blaschke products \( \Psi(t) \) of \( K_{u(t)} \) do not change their \( S^1 \)-orbits as times grows even before or after crossings.

**Proposition 4.2.** For any open interval \( \Omega \) contained into the complement of \( T_c \), for any \( \sigma_k \in \Sigma_{K(u(t))} \) with \( t \in \Omega \),
\[
K_{u(t)}u'_k(t) = \sigma_k \Psi_k(t)u'_k(t).
\]
Then there exists a function \( \psi_k(t) : \Omega \to S^1 \), such that
\[
\Psi_k(t) = e^{i\psi_k(t)}\Psi_k(0), \quad t \in \Omega.
\]

**Proof.** Differentiating the above equation (4.7) and using the Lax pair structure (1.14), one obtains
\[
[C_u, K_u](u'_k) + K_u \frac{du'_k}{dt} = \sigma_k \Psi_k u'_k + \sigma_k \Psi_k \frac{du'_k}{dt}.
\]
Recall \( u'_k = P_k(u) \), where \( P_k \) as (4.2) by replacing \( H_u \) with \( K_u \), then
\[
\frac{d}{dt}P_k(t) = [C_u, P_k].
\]
Rewriting \( \Pi(|u|^2 u) = T_{\mu_2}(u) = (iC_u + \frac{i}{2}K^2_k)u \), then the \( \alpha \)-Szegő equation (1.1) turns out to be
\[
\frac{du}{dt} = (C_u - \frac{i}{2}K^2_n)u - i\alpha(u|1),
\]
then
\[
\frac{du'_k}{dt} = \left( \frac{d}{dt}P_k(u) + P_k \frac{du}{dt} \right) = [C_u, P_k]u + P_kC_uu - \frac{i}{2}K^2_{n}P_k(u) - i\alpha(u|1)P_k(1).
\]
thus
\[
\frac{du'_k}{dt} = -i T_{|\omega|^2} u'_k - i\alpha(u | 1) \frac{(1 | u'_k)}{(u'_k | u'_k)} u'_k .
\] (4.10)

Then (4.9) and (4.10) obtained above lead to
\[
\left( \Psi_k - i \left( \sigma_k^2 + 2\alpha \text{Re} \left[ \frac{(u | 1) (1 | u'_k)}{(u'_k | u'_k)} \right] \right) \Psi_k \right) u'_k = -i [T_{|\omega|^2}, \Psi_k] (u'_k) .
\]
We claim that
\[
[T_{|\omega|^2}, \Psi_k] (u'_k) = 0 .
\]
Therefore
\[
\Psi_k(t) = e^{i(\sigma_k^2 t + \gamma_k(t))} \Psi_k(0) ,
\]
where
\[
\gamma_k(t) = 2\alpha \int_0^t \text{Re} [(u(t') | 1)(1 | u'_k(t'))] \left| u'_k(t') \right|^2 dt' .
\]
It remains to prove the claim (one can also refer to [9, Theorem 8] for the proof). We first prove that, for any \( \chi_p(z) = \frac{r}{1 - \overline{p} z} \) with \(|p| < 1\),
\[
[T_{|\omega|^2}, \chi_p] f = 0
\]
for any \( f \in F_u(\sigma_k) \) such that \( \chi_p f \in F_u(\sigma_k) \). For any \( L^2 \) function \( g \),
\[
[\Pi, \chi_p] g = (1 - |p|^2) H_{1/(1 - \overline{p} z)}(h) ,
\]
where \((\text{Id} - \Pi) g = S h\). Consequently, the range of \([\Pi, \chi_p]\) is one dimensional, directed by \( \frac{1}{1 - \overline{p} z} \). In particular, \([T_{|\omega|^2}, \chi_p] f \) is proportional to \( \frac{1}{1 - \overline{p} z} \). Since
\[
([T_{|\omega|^2}, \chi_p] f | 1) = (T_{|\omega|^2} \chi_p f - \chi_p T_{|\omega|^2} f | 1)
= (\chi_p f | H_u^1(1)) - (\chi_p | 1)(H_u^1 f | 1)
= (H_u^2 (\chi_p f) | 1) - (\chi_p | 1)(H_u^2 f | 1)
= (\chi_p f - (\chi_p | 1)f)u(1) ,
\]
We used (3.6) to gain the last equality. Since \( \chi_p f - (\chi_p | 1)f \in F_u(\sigma_k) \) is orthogonal to 1, by Proposition 2.2, \( \chi_p f - (\chi_p | 1)f \in E_u(\sigma_k) \), hence \( \chi_p f - (\chi_p | 1)f \in F_u(\sigma_k) \) is orthogonal to \( u \). This proves that \([T_{|\omega|^2}, \chi_p] f = 0\). \( \square \)

Therefore, we have

**Corollary 4.2.**

\[
\text{rk} K_{\omega}(t) = \text{rk}_{\omega} K_{\omega}(t) = \text{rk} K_{\omega}, \ a.e. \ t < \infty .
\]

We know that \( \Psi_k(t) \) is defined for every \( t \) in an open subset \( \Omega \) of \( \mathbb{R} \) consisting of the complement of a discrete closed subset, corresponding to crossings at \( \sigma_k^2 \). Furthermore, by Proposition 4.2, on each connected component of \( \Omega \), the zeroes of \( \Psi_k(t) \) are constant. Together with the following property, \( \Psi_k(t) \) never changes its orbit even after the crossings.

**Proposition 4.3.** For every time \( t \) such that \( \Psi_k(t) \) is defined, the zeroes of \( \Psi_k(t) \) are the same.

**Proof.** The proposition is a consequence of the following lemma. \( \square \)
Lemma 4.3. There exists an analytic function $\Psi_k\beta$ defined in a neighborhood $\Omega'$ of $\Omega^c$ and valued into rational functions, and, for every $t \in \Omega \cap \Omega'$, there exists $\beta(t) \in \mathbb{T}$ such that

$$\Psi_k(t, z) = e^{\beta_k(t)}\Psi_k^{\beta}(t, z).$$

Proof. Since $\sigma_k^2$ is an eigenvalue of constant multiplicity $m$ of $K_{u(t)}^{2}$, the orthogonal projector $P_k(t)$ onto $F_{u(t)}(\sigma_k)$ is an analytic function of $t \in \mathbb{R}$. Consequently, the vector

$$v'_k(t) := P_k(t)(1)$$

depends analytically on $t$. Furthermore, $v'_k(t)$ is not 0 if $t \notin \Omega$. Indeed, from the description of $F_{u}(\tau)$ provided by Proposition 2.2 when $\tau$ is a singular value associated to the pair $(H_u, K_u)$, we observe that, if $\tau$ is $H$ dominant, the space $F_{u}(\tau)$ is not orthogonal to 1. Consequently, we can define, for $t$ in a neighborhood $\Omega'$ of $\Omega^c$,

$$\Psi_k^{\beta}(t, z) := \frac{K_{u(t)}(v'_k(t))(z)}{\sigma_k v'_k(t, z)}$$

as an analytic function of $t$ valued into rational functions of $z$. On the other hand, if $t \in \Omega$, Proposition 2.2 shows that

$$F_{u(t)}(\sigma_k) \cap u(t) = E_{u(t)}(\sigma_k) = F_{u(t)}(\sigma_k) \cap 1^\perp,$$

therefore $v'_k(t)$ is collinear to $u'_k(t)$,

$$v'_k(t) = (1 | u'_k(t)) \frac{u'_k(t)}{|u'_k(t)|^2}.$$

Since, from the definition of $\Psi_k(t)$,

$$K_{u(t)}(u'_k(t)) = \sigma_k \Psi_k(t) u'_k(t),$$

we infer that there exists an analytic $\beta_k$ on $\Omega \cap \Omega'$ valued into $\mathbb{T}$ such that

$$K_{u(t)}(v'_k(t)) = \sigma_k e^{-\beta_k(t)} \Psi_k(t) v'_k(t).$$

This completes the proof. \qed

5. Necessary condition of norm explosion

In this section, let $u(t)$ be the solution of $\alpha$–Szegö equation (1.1) with initial data $u_0 \in \mathcal{L}(N)$, $N \in \mathbb{N}^+$, $u^{\infty} = \lim u(t_n)$ for the weak * topology of $H^{1/2}$, for some sequence $t_n$ going to infinity. To study the large time behavior of solutions, it is equivalent to study the rank of the shifted Hankel operator $K_u$.

Lemma 5.1. The solution $u(t)$ to the $\alpha$–Szegö equation will stay in a compact subset of $\mathcal{L}(N)$ if and only if for all the adherent values $u^{\infty}$ of $u(t)$ at infinity,

$$\text{rk} K_{u^{\infty}} = \text{rk} K_{u_0}.$$

Proof. By the explicit formula of functions in $\mathcal{L}(N) \subset H^s$ for every $s$ in Theorem 1.3, $\text{rk} u(t) = N$ if and only if

$$u(z) = \frac{A(z)}{B(z)}$$

with $A, B \in \mathbb{C}[z], A \land B = 1, \deg(A) = N$ or $\deg(B) = N, B^{-1}(0) \ni \overline{\mathbb{D}} = \emptyset$. 

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Then a sequence of \((u_n)_n\) is in a relatively compact subset of \(\mathcal{L}(N)\) unless one of the poles of \(u_n\) approaches the unit disk \(\mathbb{D}\), then the corresponding limit \(u(z)\) will be in some \(\mathcal{L}(N')\) with \(N' < N\). \(\square\)

We first present a necessary condition of the norm explosion for any \(\alpha \in \mathbb{R} \setminus \{0\}\).

**Theorem 5.1.** If \(\text{rk} K_u < \text{rk} K_{u_0}\), then there exists some \(k\) such that \(\ell_k(u_0) = 0\).

**Corollary 5.1.** If \(\alpha < 0\), for any \(N \in \mathbb{N}^+\), given initial data \(u_0 \in \mathcal{L}(N)\), then the solution to the \(\alpha\)-Szegő equation stays in a compact subset of \(\mathcal{L}(N)\).

**Proof of Corollary 5.1.** Since \(\alpha < 0\), then \(\ell_k := ||u'_k||^2 - \alpha ||v_k'||^2 > 0\), due to Theorem 5.1, \(\text{rk} K_{u_0} \equiv \text{rk} K_{u_0}\). \(\square\)

**Proof of Theorem 5.1.** Assume \(\text{rk} K_{u_0} < \text{rk} K_{u_0}\), then there exists some \(k\) such that \(\dim F_{u_0}(\sigma_k) < \dim F_{u_0}(\sigma_k) = m\). We are to prove \(||u_{k}^\infty||^2 = 0\) and \(||v_{k}^\infty||^2 = 0\).

- \(||u_{k}^\infty||^2 = 0\).

There exists a time dependent Blaschke product \(\Psi_k\) of degree \(m - 1\) such that
\[
K^2_{u(t_n)}(u_k'(t_n)) = \sigma_k^2 u_k'(t_n), \quad K_{u(t_n)}(u_k'(t_n)) = \sigma_k \Psi_k(t_n) u_k'(t_n),
\]
(5.2)

By Proposition 4.3, any limit point of \(\Psi_k(t)\) as \(t\) goes to \(\infty\) is of degree \(m - 1\) as well. Since \(u_k'(t_n)\) is bounded in \(L^2\), up to a subsequence it converges weakly to some \(u_k^\infty \in L^2\). Passing to the limit in the identities (5.2), we get
\[
K^2_{u_k^\infty}(u_k^\infty) = \sigma_k^2 u_k^\infty, \quad K_{u_k^\infty}(u_k^\infty) = \sigma_k \Psi_k^\infty u_k^\infty,
\]
(5.3)

where \(\Psi_k^\infty\) is a Blaschke product of degree \(m - 1\). The latter identities (5.3) show that \(u_k^\infty\) and \(\Psi_k^\infty u_k^\infty\) belong to \(F_{u_k}(\sigma_k)\), hence, if \(u_k^\infty\) is not zero, the dimension of \(F_{u_k}(\sigma_k)\) is at least \(m\). Indeed, if we write \(\Psi_k^\infty = e^{-i\theta} \frac{P(z)}{D(z)}\), then
\[
F_{u_k}(\sigma_k) = \left\{ \frac{f}{D(z)} u_k^\infty, \quad f \in \mathbb{C}_{m-1}[z] \right\},
\]
(5.4)

- \(||v_{k}^\infty||^2 = 0\).

Recall the structure of \(F_{u_k}(\sigma_k)\) with \(\sigma_k \in \Sigma_K(u)\) in Proposition 2.2, the orthogonal projection of \(1\) onto the space \(F_{u_k}(\sigma_k), v_k'\) can be represented as
\[
v_k' = (1 \mid \frac{u_k'}{||u_k'||}) \frac{u_k'}{||u_k'||}.\]

If \(v_k^\infty \neq 0\), since \(||v_k'|| = \left|(1 \mid \frac{u_k'}{||u_k'||}\right|\), thus \(\frac{u_k'}{||u_k'||} \rightharpoonup v\) in \(L^2\) with \(v \neq 0\). Using the strategy in the first step above by replacing \(u_k'\) by \(\frac{u_k'}{||u_k'||}\), we have \(\dim F_{u_k}(\sigma_k) = m\). \(\square\)

6. LARGE TIME BEHAVIOR OF THE SOLUTION FOR THE CASE \(\alpha > 0\)

In this section, we prove for any \(N\), there exist solutions in \(\mathcal{L}(N)\) which admit an exponential on time norm explosion.
Theorem 6.1. For $\alpha > 0$, $u_0 \in H^s_+$ such that $\Sigma_{K}(u_0) = \{\sigma\}$ with multiplicity $k = \text{rk}_K u_0$. Then $\|u(t)\|_{H^s}$ grows exponentially on time,

$$\|u(t)\|_{H^s} \approx e^{C_s(2s-1)|t|} ,$$

if and only if

$$L_1(u) := (K_u^2(u)|u| - \alpha(K_u^2(1)|1| = 0 . \quad (6.1)$$

Let $u_0$ as in the theorem above. If $u_0$ is not a Blaschke product, we have

$$\Sigma_{H}(u_0) = \{\rho_1, \rho_2\} , \rho_1 > \sigma > \rho_2 .$$

Using the results by Gérard and Grellier [9], we have the explicit formula for the solution $u$ as

$$u(t, z) = \frac{\Delta_{11} - \Delta_{21}}{\text{det}(C(z))} e^{-i\varphi_1} + \frac{\Delta_{22} - \Delta_{12}}{\text{det}(C(z))} e^{-i\varphi_2} , \quad (6.2)$$

with $\Delta_{jk}$ as the minor determinant of $C(z)$ corresponding to line $k$ and column $j$, and

$$C(z) = \left( \begin{array}{cc} \frac{p_1 - \alpha z e^{-i\varphi_1}}{p_1^2 - \alpha^2} & \frac{p_1 - \alpha z e^{-i\varphi_2}}{p_2^2 - \alpha^2} \\
1 & 1 \end{array} \right) .$$

Then

$$u(t, z) = \frac{1}{p_2} - \frac{p_2 - \alpha z e^{-i\varphi_2}}{p_2^2 - \alpha^2} e^{-i\varphi_1} + \frac{\frac{p_2 - \alpha z e^{-i\varphi_1}}{p_2^2 - \alpha^2} - \frac{1}{p_1}}{p_1} e^{-i\varphi_2} .$$

An interesting fact is that $u$ is under the form

$$u(t, z) = b(t) + \frac{c'(t)z\Psi(t, z)}{1 - p'(t)z\Psi(t, z)},$$

where $b , p', c' \in \mathbb{C}$. Since $\Psi(t, z) = e^{i\psi(t)}z^\chi(t)$ with $\chi$ as a time independent Blaschke product, we then rewrite

$$u(t, z) = b(t) + \frac{c(t)z\chi'(t)}{1 - p(t)z\chi(t)} . \quad (6.3)$$

Lemma 6.1. Let $\chi$ be a time-independent Blaschke product. A function $u \in C^\infty(\mathbb{R}, H^s_+)$ with $s > \frac{1}{2}$ is a solution of the $\alpha$–Szegö equation,

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1) ,$$

if and only if

$$\tilde{u}(t, z) := u(t, z\chi(z))$$

satisfies the $\alpha$–Szegö equation.

Proof. First of all, $z\chi(z) \in C^\infty(S^1)$, then $(z\chi(z))^n \in C^\infty(S^1)$ for any $n$, so that $u \in H^s_+$ implies $\tilde{u} \in H^s_+$. Assume $u$ is a solution of the $\alpha$–Szegö equation, it is equivalent to

$$i\partial_t \tilde{u}(t, n) = \sum_{p-q+r=n} \tilde{u}(t, p) \tilde{u}(t, q) \tilde{u}(t, r) + \alpha \tilde{u}(t, 0) \delta_{n0} , \forall n \geq 0 . \quad (6.4)$$

Since

$$\Pi(|u(z\chi(z))|^2 u(z\chi(z))) = \sum_{p-q+r \geq 0} \tilde{u}(p) \tilde{u}(q) \tilde{u}(r)(z\chi(z))^{p-q+r} ,$$

we obtain that $\tilde{u}$ satisfies the $\alpha$–Szegö equation.
Conversely, assume \( \tilde{u} \) satisfies the \( \alpha \)-Szegő equation, then we have

\[
i\partial_t \tilde{u}(n) (z \chi(z))^n = \sum_{p-q+r \geq 0} \tilde{u}(p) \tilde{u}(q) \tilde{u}(r) (z \chi(z))^{p-q+r} + \tilde{u}(0) .
\] (6.5)

Identifying the Fourier coefficients of 0 mode of both sides, we get equation (6.4) with \( n = 0 \). Then withdraw this quantity from both sides of (6.5) and simplify by \( z \chi(z) \). Continuing this process, we get all the equations (6.4) for every \( n \).

**Lemma 6.2.** Let \( \Psi \) be a Blaschke product of finite degree \( d \) and \( s \in [0, 1) \). There exists \( C_{\Psi,s} > 0 \) such that, for every \( p \in \mathbb{D} \),

\[
\left\| \frac{1}{1 - p\Psi} \right\|_{H^s(S^1)} \geq \frac{C_{\Psi,s}}{(1 - |p|)^{s+\frac{1}{2}}} .
\]

**Proof.** It is a classical fact that, for every \( u \in H^s(S^1) \), for every \( s \in [0, 1) \),

\[
\left\| u \right\|^2_{H^s(S^1)} \sim \int_{\mathbb{D}} |u'(z)|^2 (1 - |z|^2)^{1-2s} dL(z) ,
\]

where \( L \) denotes the bi-dimensional Lebesgue measure.

Let \( p \in \mathbb{D} \) close to the unit circle and \( \omega := \frac{p}{|p|} \).

Since \( \Psi \) is a Blaschke product of finite degree \( d \), the equation

\[
\omega \Psi(z) = 1
\]

admits \( d \) solutions on the circle. Moreover, these solutions are simple. Indeed, writing

\[
\Psi(z) = e^{-2\phi} \prod_{j=1}^{d} \frac{z - p_j}{1 - \overline{p}_j \overline{z}} , \quad |p_j| < 1 ,
\]

we have, for every \( z \in S^1 \),

\[
\frac{\Psi'(z)}{\Psi(z)} = \frac{1}{z} \sum_{j=1}^{d} \frac{|p_j|^2}{|z - p_j|^2} \neq 0 .
\]

Let \( \alpha \) be such a solution. For every \( z \) such that

\[
|z - \alpha| \leq (1 - |p|),
\]

we have, if \( 1 - |p| \) is small enough,

\[
|1 - p\Psi(z)| = |1 - p\Psi(\alpha) - p\Psi'(\alpha)(z - \alpha) + O(|z - \alpha|^2)| \leq C(1 - |p|).
\]
Therefore
\[
\left\| \frac{1}{1 - p \Psi} \right\|_{H^s(\mathbb{S}^1)}^2 \geq A_s \int_{\mathbb{D} \cap |z| \leq (1-|p|)} \left\| \frac{\Psi'(z)}{(1 - p \Psi(z))^2} \right\|^2 (1 - |z|^2)^{1-2s} \ dL(z)
\]
\[
\geq B_{\Psi,s} (1 - |p|)^{-4} \int_{\mathbb{D} \cap |z| \leq (1-|p|)} (1 - |z|^2)^{1-2s} \ dL(z)
\]
\[
\geq \frac{C_{\Psi,s}^2}{(1 - |p|)^{2s+1}}.
\]

Let us turn back to prove the theorem.

Proof of Theorem 6.1. Recall that
\[
L_1(u) = (K_0^2(u) | u) - \alpha (K_0^2(1) | 1)
\]
\[
= \frac{1}{2} (||u||_{L^4}^4 - ||u||_{L^4}^4) - \alpha (||u||_{L^4}^2 - |(u | 1)^2).
\]
Since $\chi(z)$ is an inner function, we have
\[
(\tilde{u} \mid \tilde{v}) = (u \mid v), \forall u, v,
\]
thus
\[
(\tilde{u}) = (u | 1), \ ||\tilde{u}||_{L^2} = ||u||_{L^2},
\]
and since
\[
\tilde{u}^2 = (\tilde{u})^2,
\]
then
\[
||\tilde{u}||_{L^4} = ||u||_{L^4}.
\]
As a consequence, $L_1(u) = L_1(\tilde{u}) = 0$.

The solution $\tilde{u}$ is under the form (6.3),
\[
u(t, z) = b(t) + \frac{c(t)\chi(z)}{1 - p(t)\chi(z)} = b - \frac{c}{p} + \frac{1}{p} - p\chi(z),
\]
thus
\[
||u||_{H^s} \simeq |c||\frac{1}{1 - p\chi(z)}||_{H^s},
\]
\[
\geq C_{\chi,s} \frac{|c|}{(1 - |p|)^{s+1/2}},
\]
where we used Lemma 6.2. Using the result in [22, Theorem 3.1] and its proof, we have
\[
\frac{|c|}{(1 - |p|)^{s+1/2}} \simeq (1 - |p|)^{-s+1/2} \simeq e^{C_s(2s-1)|\beta|}.
\]
Therefore, $\tilde{u}$ admit an exponential on time growth of the Sobolev norm $H^s$ with $s > \frac{1}{2}$. The proof is complete.
7. Perspectives

The main purpose of this work is to study the dynamics of the general solutions of the $\alpha$–Szegő equation (1.1). We have already observed the weak turbulence by considering some special rational data. We proved the existence of data with exponential in time growth, a natural question is about the genericity of data with such a high growth. Besides, an important open problem is to gain new informations on the solutions with infinite rank.

Another interesting question is about the cubic Szegő equation with other perturbations, for example, consider a Hamiltonian function

$$E(u) = \frac{1}{4}\|u\|^4_{L^4} + \frac{1}{2}F(||u||^2),$$

with a non linear function $F$. In this case, we still have one Lax pair $(K_u, C_u)$ while the conservation laws we found no longer exist. The question is to study the integrability and also the existence of turbulent solutions of this new Hamiltonian system.

References

[1] J. Bourgain. Problems in Hamiltonian PDE’s. Geom. Funct. Anal., (Special Volume, Part I):32–56, 2000.
[2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. Invent. Math., 181(1):39–113, 2010.
[3] P. Gérard and S. Grellier. L’équation de Szegő cubique. Séminaire: Équations aux Dérivées Partielles, 2008–2009, Exp. No. II, 19 p., online at slsedp.cedram.org/.
[4] P. Gérard and S. Grellier. On the growth of Sobolev norms for the cubic Szegő equation. Séminaire Laurent Schwartz, 2014–2015, EDP et applications, Exp. No. II, 20 p., online at slsedp.cedram.org/.
[5] P. Gérard and S. Grellier. The cubic Szegő equation. Ann. Sci. Éc. Norm. Supérieure (4), 43(5):761–810, 2010.
[6] P. Gérard and S. Grellier. Effective integrable dynamics for a certain nonlinear wave equation. Anal. PDE, 5(5):1139–1155, 2012.
[7] P. Gérard and S. Grellier. Invariant tori for the cubic Szegő equation. Invent. Math., 187(3):707–754, 2012.
[8] P. Gérard and S. Grellier. Inverse spectral problems for compact Hankel operators. J. Inst. Math. Jussieu, 13(2):273–301, 2014.
[9] P. Gérard and S. Grellier. Multiple singular values of Hankel operators. Preprint arXiv:1402.1716, 2014.
[10] P. Gérard and S. Grellier. An explicit formula for the cubic Szegő equation. Trans. Amer. Math. Soc., 367(4):2979–2995, 2015.
[11] B. Grébert and L. Thomann. Resonant dynamics for the quintic nonlinear Schrödinger equation. Ann. Inst. H. Poincaré Anal. Non Linéaire, 29(3):455–477, 2012.
[12] M. Guardia. Growth of Sobolev norms in the cubic nonlinear Schrödinger equation with a convolution potential. Comm. Math. Phys., 329(1):405–434, 2014.
[13] M. Guardia, E. Haus, and M. Procesi. Growth of Sobolev norms for the defocusing analytic NLS on $T^2$.
[14] M. Guardia and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation. J. Eur. Math. Soc. (JEMS), 17(1):71–149, 2015.
[15] Z. Hani. Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations. Arch. Ration. Mech. Anal., 211(3):929–964, 2014.
[16] Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia. Modified scattering for the cubic Schrödinger equation on product spaces and applications. Preprint arXiv:1311.2275v3.
[17] Z. Hani and L. Thomann. Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping. Preprint arXiv:1408.6213.
[18] P. Hartman. On completely continuous Hankel matrices. Proc. Amer. Math. Soc., 9:862–866, 1958.
[19] E. Haus and M. Procesi. Growth of Sobolev norms for the quintic NLS on $T^2$. Preprint arXiv:1405.1538.

[20] V. Peller. Hankel Operators and their applications. Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[21] O. Pocovnicu. Explicit formula for the solution of the Szegő equation on the real line and applications. Discrete Contin. Dyn. Syst., 31(3):607–649, 2011.

[22] H. Xu. Large-time blowup for a perturbation of the cubic Szegő equation. Anal. PDE, 7(3):717–731, 2014.

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