AN INFINITELY DIFFERENTIABLE FUNCTION WITH
COMPACT SUPPORT: DEFINITION AND PROPERTIES

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1. INTRODUCTION.

Infinitely differentiable functions of compact support defined on \( \mathbb{R} \) play an important role in Analysis. Usually, one constructs examples using an idea of Cauchy. For this example the derivatives are cumbersome. This problem makes me search for a better example.

Looking at a rough plot of such a function and its derivative (see figure 1) I asked if it was possible that the derivative could be formed with two homothetic copies of the same function translated conveniently. So I posed the following question:

Does there exist a function \( \varphi \in \mathcal{D}(\mathbb{R}) \) such that:

(a) \( \text{supp}(\varphi) = [-1, 1] \),
(b) \( \varphi(t) > 0 \) for any \( t \in (-1, 1) \),
(c) \( \varphi(0) = 1 \),
(d) and there is a constant \( k > 0 \) such that

\[
\varphi'(t) = k(\varphi(2t + 1) - \varphi(2t - 1))
\]

We will prove that there is a unique solution \( \varphi \) satisfying the above conditions. For this unique solution the value of the constant \( k \) is 2. No other value of \( k \) gives a solution.

The function \( \varphi \) has many other properties. It can be interpreted as a probability (theorem 3), \( \varphi \) and some of its translates form a partition of unity (theorem 5), its derivatives can be computed easily (theorem 4), and the most notable, it is not a rational function but its values at dyadic points are rational numbers that are effectively computable. Since its derivatives are related to the same function, not only the values of \( \varphi \) but also those of its derivatives \( \varphi^{(k)}(t) \) are rational number at dyadic points.

The only reference that we know about this function is a paper [4] by Jessen and Wintner (1935) where the function \( \varphi \) is defined by means of its Fourier transform, as an example of an infinitely differentiable function, but Jessen and Wintner do not give any other property of this function.

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2. Existence and Unicity.

**Theorem 1.** There is a unique infinitely differentiable function with compact support \( \varphi : \mathbb{R} \to \mathbb{R} \) and such that:

(a) \( \text{supp}(\varphi) = [-1, 1] \).
(b) \( \varphi(t) > 0 \) for any \( t \) in the open set \((-1, 1)\).
(c) \( \varphi(0) = 1 \).
(d) There is a constant \( k > 0 \) such that for any \( t \in \mathbb{R} \)

\[
\varphi'(t) = k(\varphi(2t + 1) - \varphi(2t - 1))
\]

and the constant \( k \) appearing in (d) is necessarily equal to 2.

**Proof.** First, assuming that \( \varphi \) exists, we will prove the unicity of \( \varphi \) and that \( k = 2 \).

Since \( \varphi \in \mathcal{D}(\mathbb{R}) \) its Fourier transform is an entire function

\[
\hat{\varphi}(z) = \int_{\mathbb{R}} \varphi(t) e^{-2\pi izt} \, dt
\]

The Fourier transform of \( \varphi'(t) \), \( \varphi(2t + 1) \) and \( \varphi(2t - 1) \) are

\[
2\pi iz\hat{\varphi}(z), \quad e^{\pi iz}\hat{\varphi}(\frac{z}{2}), \quad e^{-\pi iz}\hat{\varphi}(\frac{z}{2})
\]

respectively. Condition (d) yields

\[
\hat{\varphi}(z) = k \frac{\sin \pi z}{\pi z} \hat{\varphi}(\frac{z}{2}).
\]

By induction, we obtain from (2) that

\[
\hat{\varphi}(z) = \left( \frac{k}{2} \right) \sum_{n=0}^{\infty} \sin \frac{\pi z}{2^{n+1}} \hat{\varphi}(\frac{z}{2^n}).
\]

Conditions (a) and (b) imply that \( \hat{\varphi}(0) = \int \varphi(t) \, dt > 0 \), so that taking limits for \( n \to \infty \) we obtain \( k = 2 \) and

\[
\hat{\varphi}(z) = \hat{\varphi}(0) \prod_{n=0}^{\infty} \frac{\sin \frac{\pi z}{2^{n+1}}}{\frac{\pi z}{2^{n+1}}}
\]

If there is a solution to our problem it is unique, because by the inversion formula

\[
\varphi(t) = \int_{\mathbb{R}} \hat{\varphi}(x) e^{2\pi itx} \, dx
\]

and condition (c) will fix the value of the constant \( \hat{\varphi}(0) \).

We will see later that (c) implies \( \hat{\varphi}(0) = 1 \), so that in what follows we will use \( \hat{\varphi}(z) \) to denote the function defined in (4) assuming \( \hat{\varphi}(0) = 1 \).

Now we will show that the solution \( \varphi \) exists. We start from the function \( \hat{\varphi}(z) \) defined in (4). Since the infinite product converges uniformly in compact sets, the function \( \hat{\varphi}(z) \) is entire. Equation (2) may be used to expand it in power series

\[
\hat{\varphi}(z) = \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{(2k)!} (2\pi z)^{2k},
\]

where the \( c_k \) are rational numbers defined by the recurrence

\[
(2k + 1)2^{2k}c_k = \sum_{h=0}^{k} \binom{2k + 1}{2h} c_h.
\]
From equation (7) we obtain that the numbers $c_k$ are positive. Also we have

$$c_k = \frac{F_k}{(2k + 1)(2k - 1) \cdots 1} \prod_{n=1}^{k} (2^{2n} - 1)^{-1},$$  

where $F_k$ are natural numbers, $F_0 = 1, F_1 = 1, F_2 = 19, F_3 = 2915, F_4 = 2788989$. Using the known formulas

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \cos \frac{z}{2^n}, \quad \text{and} \quad \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

we obtain

$$\hat{\varphi}(z) = \prod_{m=1}^{\infty} \left(\cos \frac{\pi z}{2m}\right)^m = \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)^{1 + \nu_2(m)}$$

where $\nu_2(m)$ is the greatest exponent such that $2^{\nu_2(m)}$ divides $m$.

It is clear that $\hat{\varphi}$ restricted to $\mathbb{R}$ is infinitely differentiable. We will show also that it is a rapidly decreasing function.

Let $f(x) = (\sin x)/x$. For $x \in \mathbb{R}^*$, we have $|f(x)| \leq 1$ and $|\sin x| \leq 1$. For all $n$

$$|x^n \hat{\varphi}(x)| = \left|x^n \prod_{h=0}^{\infty} f(\pi x/2^h)\right| \leq |x^n \prod_{h=0}^{n-1} f(\pi x/2^h)| \leq 2^n (2\pi)^{-n}.$$

It is easy to see that there is a constant $M_r \geq 0$ for each $r \in \mathbb{N}$ such that

$$|\partial^r f(\pi x/2^h)| \leq \pi^{r-h} M_r.$$

Applying the rule to differentiate an infinite product and the same idea used above to bound $|x^n \hat{\varphi}(x)|$ we obtain

$$|x^n \partial^r \hat{\varphi}(x)| \leq \sum_{S} \frac{r!}{s_1! \cdots s_l!} \sum_{H} \prod_{i=1}^{t} \left|\partial^{s_i} f(\pi x/2^{h_i})\right| \left|x^n \prod_{h \neq h_i} f(\pi x/2^h)\right| \leq \sum_{S} \frac{r!}{s_1! \cdots s_l!} M_{s_1} \cdots M_{s_l} \left(\sum_{H} \pi^{r-s_{1}h_{1} + \cdots - s_{l}h_{l}}\right) 2^{(n+r)} \pi^{-n} < \infty$$

where the sum extended to $S$ refers to all sets $\{s_1, \ldots, s_l\}$ of natural numbers such that $s_1 + \cdots + s_l = r$ and $s_i \geq 1$ and the sum in $H$ to all sets $\{h_1, \ldots, h_l\}$ of $t$ distinct natural numbers.

Once we have proved that $\hat{\varphi}$ is a test function in Schwartz space we define $\varphi$ by means of equation (5). It follows that $\varphi$ is infinitely differentiable and rapidly decreasing. Since $\hat{\varphi}$ satisfies (2) with $k = 2$, we obtain that $\varphi$ satisfies condition (d) with $k = 2$. We will show that $\varphi$ also satisfies conditions (a), (b) and (c). Instead of using Paley-Wiener’s Theorem we prefer to use another method, which gives us some additional information.

Let $\mu_n$ be the Radon measure in $\mathbb{R}$ whose Fourier transform is

$$\mathcal{F}(\mu_m) = \prod_{k=1}^{m} \left(\cos \frac{\pi x}{2^k}\right)^k.$$

Since

$$\mathcal{F}\left(\frac{1}{2} \delta_{2^{-k-1}} + \frac{1}{2} \delta_{2^{-k-1}}\right) = \cos \frac{\pi x}{2^k},$$

we can use this result to prove that $\varphi$ satisfies condition (d).
\( \mu_m \) is the convolution product

\[ \mu_m = \sum_{k=1}^{\infty} \left( \frac{1}{2^k} \delta_{2^{-k-1}} + \frac{1}{2} \delta_{-2^{-k-1}} \right)^k \]

where the powers have also the meaning of convolution products.

It is clear that the total variation \( \| \mu_m \| = 1 \), \( \mu_m \geq 0 \) and \( \text{supp}(\mu_m) \subset [-1, 1] \).

The last assertion follows from

\[ \sum_{k=1}^{\infty} \frac{k}{2^k+1} = 1. \]

**Lemma 1.** Let \( (\mu_m) \) be the sequence of measures defined in (12). This sequence of measures converges in the weak-* topology \( \sigma(\mathcal{M}_b(\mathbb{R}), C^*(\mathbb{R})) \) towards the measure \( \varphi_\lambda \) with density \( \varphi \) with respect to Lebesgue measure \( \lambda \).

**Proof.** Denote by \( C^*(\mathbb{R}) \) the Banach space of complex valued bounded functions defined on \( \mathbb{R} \). Since the measures \( \mu_m \) are on the unit ball of the dual space, which is weakly compact, there is a measure \( \mu \) that is a weak cluster point to the sequence \( \mu_m \). Since \( F(\mu_m) \to F(\varphi_\lambda) \) pointwise, we have \( F(\mu) = F(\varphi_\lambda) \). Since \( F \) is injective in the space of bounded Radon measures, we obtain \( \mu = \varphi_\lambda \). Therefore \( \varphi_\lambda \) is the only weak cluster point, so that it is the weak limit of the sequence \( \mu_m \). \( \square \)

Since \( \mu_m \to \varphi_\lambda \) with weak convergence, it follows that \( \varphi \) satisfies condition (a) and, since \( \varphi \) is continuous it follows that \( \varphi(x) \geq 0 \) for all \( x \in \mathbb{R} \).

Now we know that \( \int \varphi(t) \, dt = \varphi(0) = 1 \). This fact, together with the fact that \( \text{supp}(\varphi) = [-1, 1] \) yields

\[
\varphi(0) = \int_{-1}^{0} \varphi'(t) \, dt = \int_{-1}^{0} 2(\varphi(2t + 1) - \varphi(2t - 1)) \, dt \\
= 2 \int \varphi(2t + 1) \, dt = \int \varphi(u) \, du = 1.
\]

and \( \varphi \) satisfies condition (c).

It remains to show that \( \varphi \) satisfies (b). By the same reasoning as above we have for every \( x \in (-1, 0) \)

\[ \varphi(x) = 2 \int_{-1}^{x} \varphi(2t + 1) \, dt. \]

Therefore \( \varphi(x) \) is not decreasing in \( (-1, 0) \) (since \( \varphi'(x) \geq 0 \)). Since \( \varphi \) is an even function, \( \varphi(x) > 0 \) implies \( \varphi(t) > 0 \) for all \( t \in (-x, x) \). If \( \varphi(x) > 0 \) we have \( \varphi((x - 1)/2) > 0 \), therefore \( \varphi(t) > 0 \) for \( t \in (-1, 1) \). \( \square \)

### 3. Other expressions for \( \varphi \).

We have seen two possible definitions of \( \varphi \): the expression (5) and that given in Lemma 1. We will give another two. One as the limit of a sequence of step functions and another by means of an integral. We need some previous notations and definitions.

Let \( p_n \) be the sequence of polynomials defined by the recurrence

\[ p_0 = 1; \quad p_n(x) = p_{n-1}(x^2)(1 + x)^n. \]
It is easy to see that

\[ p_n(x) = \prod_{k=1}^{n} \left( \frac{1 - x^{2^k}}{1 - x} \right) \]

The degree \( g_n \) of \( p_n \) is given by the equations

\[ g_0 = 0, \quad g_n = 2g_{n-1} + n. \]

Therefore

\[ \frac{g_n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n}. \]

Equations (12) and (14) show that \( \mu_n \) is the measure obtained when we substitute each power \( x^m \) by \( \delta_{g_{m-1}, \frac{m}{2^n}} \) in the polynomial

\[ 2^{-\left( \frac{m}{2^n} \right)} p_n(x). \]

For each \( n \in \mathbb{N} \), let \( \varphi_n \) be the step function obtained from the polynomial \( 2^{-\left( \frac{m}{2^n} \right)} p_n(x) \) substituting each power \( x^m \) by the characteristic function of the interval

\[ \left[ \frac{2m + 1 - g_n}{2^{n+1}}, \frac{2m + 1 - g_n}{2^{n+1}} \right] \]

multiplied by \( 2^n \). We have then:

**Theorem 2.** \( \varphi \) is the limit of the sequence of step functions \( \varphi_m \).

**Proof.** It suffices to observe that for a characteristic function \( f \) of an interval with dyadic extremes, we have

\[ \lim_{m \to \infty} \mu_m(f) = \lim_{m \to \infty} \int \varphi_m f = \int \varphi f, \]

and the fact, easily proved, that \( \varphi_m \) is monotonous non decreasing in \((-1, 0)\) and monotonous not increasing in \((0, 1)\), and that \( \varphi_m(0) = 1 \). \( \square \)

It is easy to see that

\[ p_{m+1}(x) = p_m(x)(1 + x + x^2 + \cdots + x^{2^m+1-1}) \]

This gives us an easy algorithm to obtain the \( \varphi_m \), and also shows that

\[ p_m(x) = (1 + x)(1 + x + x^2 + x^3) \cdots (1 + x + \cdots + x^{2m-1}). \]

Therefore we have a combinatorial interpretation of the coefficient of \( x^r \) in \( p_m(x) \):

The coefficient of \( x^r \) in \( p_m(x) \) is the number of partitions of \( r \), in \( m \) parts \( r = s_1 + s_2 + \cdots + s_m \) such that \( 0 \leq s_i \leq 2^i - 1 \).

**Theorem 3.** Let \( \sigma = \bigotimes_{k=1}^\infty \lambda_k \) be the measure defined on \([0, 1]^\mathbb{N}\), \( \lambda_k \) being the Lebesgue measure on \([0, 1]\). For \(-1 \leq x \leq 0 \) we have

\[ \varphi(x) = \sigma \left\{ (x_k) : 0 \leq \sum_{k=1}^\infty \frac{x_k}{2^k} \leq x + 1 \right\} \]

**Proof.** Let \( \nu_k \) be the measure in \([-1, 1]^\mathbb{N}\)

\[ \nu_k = \bigotimes_{m=1}^\infty \left( \frac{1}{2} \delta_{2^{-m} - k} + \frac{1}{2} \delta_{2^{-m} - k} \right) \]

\((k = 1, 2, \ldots)\) and let \( (t_{k,1}, t_{k,2}, \ldots) \) denote the variables in the space \([-1, 1]^\mathbb{N}\).
Let $\mu$ be the measure defined on $\{0,1\}^\mathbb{N}$ as the product of the measure assigning 0 and 1 measure 1/2.

Then $\nu_k = f_k(\mu)$ the image measure, with $f_k\{0,1\}^\mathbb{N} \to [-1,1]^\mathbb{N}$ given by $f_k(\varepsilon_1,\varepsilon_2,\ldots) = (t_{k,1},t_{k,2},\ldots)$ where

$$t_{k,m} = \begin{cases} 2^{-m-k} & \text{when } \varepsilon_m = 1, \\ -2^{-m-k} & \text{when } \varepsilon_m = 0. \end{cases}$$

$\mu$ is also the image measure of Lebesgue measure on $[0,1]$ by the application $g: [0,1] \to \{0,1\}^\mathbb{N}$ defined by $g(x) = (\varepsilon_1,\varepsilon_2,\ldots)$ if $x = \sum_{m=1}^\infty (\varepsilon_m/2^m)$ with $\varepsilon_m \in \{0,1\}$. The function $g$ is well defined only almost everywhere but this is no difficulty.

The measure $\varphi(t) dt$ is the limit of the $\mu_m$, therefore for all integrable $f$,

$$\int f(t)\varphi(t) dt = \int f\left(\sum_{k=1}^\infty t_{k,m}\right) d\bigotimes_{k=1}^\infty \nu_k.$$ 

Since each $\nu_k$ is an image measure the last integral can be transformed in an integral on $[0,1]^\mathbb{N}$ with respect to the measure $\sigma = \otimes_{k=1}^\infty A$.

The relation $f_k \circ g(x_k) = (t_{k,1},t_{k,2},\ldots)$ implies $x_k = \sum_{m=1}^\infty (\varepsilon_m/2^m)$ with $\varepsilon_m \in \{0,1\}$, $t_{k,m} = 2^{-m-k}$ if $\varepsilon_m = 1$ and $t_{k,m} = -2^{-m-k}$ when $\varepsilon_m = 0$. Therefore

$$\sum_{m} t_{k,m} = \sum_{m=1}^\infty \varepsilon_m 2^{-m-k} - \left(\sum_{m=1}^\infty 2^{-m-k} - \sum_{m=1}^\infty \varepsilon_m 2^{-m-k}\right) = x_k 2^{-k+1} - 2^{-k}$$

From this we get

$$\int f(t)\varphi(t) dt = \int f\left(\sum_{k=1}^\infty x_k 2^{-k+1} - 1\right) d\sigma.$$ 

Taking $f(t) = \chi_{[-1,2x+1]}(t)$ with $-1 \leq x \leq 0$,

$$\varphi(x) = \int_{-1 \leq \sum_{k=1}^\infty x_k 2^{-k+1} - 1 \leq 2x+1} d\sigma = \int_{0 \leq \sum_{k=1}^\infty x_k 2^{-k} \leq x+1} d\sigma$$

$$= \sigma\left\{(x_k): 0 \leq \sum_{k=1}^\infty x_k 2^{-k} \leq x + 1\right\}$$

In other words we have proved the Proposition: Let $x_k$ be independent random variables uniformly distributed in $[0,1]$, $\varphi(x)$ (with $-1 \leq x \leq 0$) is equal to the probability that the sum $\sum x_k 2^{-k}$ be $\leq x + 1$. \hfill \Box

4. Properties.

**Theorem 4.** Let

$$\theta(t) = \sum_{k=0}^\infty (-1)^{s(k)} \varphi(t - 2k - 1)$$

where $s(k)$ denotes the sum of the digits of $k$ when written in base 2. Then

(a) $\theta$ is an infinitely differentiable function.

(b) $\theta'(t) = 2\theta(2t)$.

(c) For $t \in [-1,1]$, $\varphi^{(k)}(t) = 2^{(k+1) t} \theta(2^k t + 2^k)$. 

Proof. The sum in the definition of $\theta(t)$ is locally finite, therefore $\theta$ is infinitely differentiable and its derivative is

$$
\theta'(t) = \sum_{k=0}^{\infty} (-1)^{s(k)}2(\varphi(2t - 4k - 2 + 1) - \varphi(2t - 4k - 2 - 1))
$$

$$
= 2 \sum_{k=0}^{\infty} ((-1)^{s(k)} \varphi(2t - 2(2k + 1) - 1) - (-1)^{s(k)} \varphi(2t - 2(2k + 1) - 1))
$$

using the definition of $s(k)$ this yields

(21) $\theta'(t) = 2\theta(2t)$.

By repeated differentiation of (21) we obtain

(22) $\theta^{(k)}(t) = 2^{(k+1)/2} \theta(2^{k+1}t)$.

For $t \in [-1, 1]$ we have $\varphi(t) = \theta(t + 1)$ so that

(23) $\varphi^{(k)}(t) = 2^{(k+1)/2} \theta(2^{k+1}t + 2^k)$, if $t \in [-1, 1]$.

□

This proves that on any dyadic point $t = q/2^n$ the Taylor expansion is a polynomial

(24) $T(t, x) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(t)}{k!} x^k$

and for $q$ odd the degree of $T(t, x)$ is $n$.

Corollary. The function $\varphi$ is not analytic on any point of the interval $[-1, 1]$.

Theorem 5. For $u > 0$ and $t \in \mathbb{R}$ we have

(25) $\sum_{k \in \mathbb{Z}} \varphi(t + uk) = \sum_{k \in \mathbb{Z}} \frac{1}{u} \varphi \left( \frac{k}{u} \right) e^{2\pi i k \frac{t}{u}}$.

Proof. The left hand side of (25) is locally finite, therefore the sum is infinitely differentiable. It is a periodic function of $t$ with period $u$. Therefore it has a Fourier series expansion

$$
\sum_{k \in \mathbb{Z}} \varphi(t + uk) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k \frac{t}{u}}
$$
where
\[ a_n = \frac{1}{u} \int_0^u \sum_{k \in \mathbb{Z}} \varphi(t + uk) e^{-2\pi i n \frac{t}{u}} dt = \sum_{k \in \mathbb{Z}} \frac{1}{u} \int_0^u \varphi(t + uk) e^{-2\pi i n \frac{t}{u}} dt = \frac{1}{u} \int_{uk}^{u(k+1)} \varphi(v) e^{-2\pi i n \frac{v}{u}} dv = \frac{1}{u} \int \varphi(v) e^{-2\pi i n \frac{v}{u}} dv = \frac{1}{u} \hat{\varphi}(\frac{n}{u}). \]

\[ \square \]

Some particular cases of (25) are interesting:

(26) \[ \sum_{k \in \mathbb{Z}} \varphi\left(t + \frac{k}{n}\right) = n \quad \text{for} \quad n \in \mathbb{N}. \]

Furthermore

(27) \[ \sum_{k \in \mathbb{Z}} \varphi(t + k) = 1. \]

which is equivalent to

(28) \[ \varphi(t) + \varphi(t - 1) = 1, \quad \text{for} \quad t \in [0, 1]. \]

Also, from (25) it follows that

(29) \[ \sum_{k \in \mathbb{Z}} \varphi(t + 2k) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \hat{\varphi}\left(\frac{k}{2}\right) e^{\pi i kt}, \]

which is no more than the Fourier expansion

(30) \[ \varphi(t) = \frac{1}{2} + \sum_{k=0}^{\infty} \hat{\varphi}\left(\frac{2k+1}{2}\right) \cos(2k+1)\pi t, \]

valid for \( t \in [-1, 1] \) and which has good convergence properties.

The product (9) implies that the sign of the coefficient \( \hat{\varphi}\left((2k+1)/2\right) \) is the parity of \( 1 + v_2(1) + 1 + v_2(2) + \cdots + 1 + v_2(k) = k + v_2(k!) = s(k) \), therefore also equal to the sign of \( \theta(k) \).

Equation (25) is not only a Fourier expansion, it is also Poisson’s formula applied to \( \varphi(t + x) \). For \( t = 0 \) it yields

(31) \[ \sum_{m \in \mathbb{Z}} \varphi(ma) = \sum_{m \in \mathbb{Z}} \frac{1}{a} \hat{\varphi}\left(\frac{m}{a}\right), \]

and using the knowledge about the support of \( \varphi \), this implies

(32) \[ a + 2a \varphi(a) = \sum_{m \in \mathbb{Z}} \frac{1}{a} \hat{\varphi}\left(\frac{m}{a}\right), \quad \text{for} \quad \frac{1}{2} \leq a \leq 1. \]

5. VALUES AT DYADIC POINTS.

Theorem 6. For each natural number \( n \) we have

(33) \[ \int_0^1 t^{n-1} \varphi(t) dt = (n - 1)! 2^{\left(\frac{n}{2}\right)} \varphi(1 - 2^{-n}). \]

(34) \[ \int_0^1 t^{2n} \varphi(t) dt = \frac{c_n}{2}. \]
where \( c_n \) are the rational numbers that appear in the expansion (6) of \( \varphi \).

**Proof.** We can check, by differentiation, that in the sequence of functions

\[
\begin{align*}
    f_0(t) &= \varphi(t), \\
    f_1(t) &= \varphi\left(\frac{t}{2} - \frac{1}{2}\right), \\
    f_2(t) &= 2\varphi\left(\frac{t}{4} - \frac{1}{4} - \frac{1}{2}\right), \\
    f_k(t) &= 2^{(\frac{k}{2})} \varphi\left(\frac{t}{2^k} - \frac{1}{2^k} - \frac{1}{2^{k-1}} - \cdots - \frac{1}{2}\right)
\end{align*}
\]

each function is a primitive in \([-1, 1]\) of the previous one and all vanish at the point \( t = -1 \). So integrating by parts

\[
\int_0^1 t^n \varphi(t) \, dt = (-1)^n \int_{-1}^0 t^n \varphi(t) \, dt = (-1)^n \int_{-1}^0 t^n f_0(t) \, dt
\]

\[
= (-1)^n n \int_{-1}^0 t^{n-1} f_1(t) \, dt = (-1)^n (-1)^n n! \int_{-1}^0 f_n(t) \, dt
\]

\[
= n! f_{n+1}(0) = n! 2^{\left(\frac{n+1}{2}\right)} \varphi(1 - 2^{-n-1}).
\]

Moreover

\[
\sum_{n=0}^\infty \frac{x^n}{n!} \int_{-1}^1 t^n \varphi(t) \, dt = \int_{-1}^1 e^{xt} \varphi(t) \, dt = \varphi\left(\frac{ix}{2\pi}\right) = \sum_{k=0}^\infty \frac{c_k}{(2k)!} x^{2k},
\]

and this proves (34). \( \square \)

From the two formulas we obtain

\[
\varphi(1 - 2^{-2n-1}) = \frac{2^{\left(\frac{2n+1}{2}\right)}}{2^{(2n)!}} \frac{F_n}{(2n+1)(2n-1) \cdots 1} \prod_{k=1}^n (2^k - 1)^{-1},
\]

where \( F_k \) are the integers defined in (8).

We may compute in a similar way all the numbers \( \varphi(1 - 2^{-n}) \). With this objective notice that

\[
\int_0^1 \varphi(t)e^{-2\pi ixt} \, dt = \frac{1}{2\pi i x} + \int_0^1 \varphi'(t) e^{-2\pi ixt} \, dt = \frac{1}{2\pi i x} \left( 1 - e^{-\pi x} \varphi\left(\frac{ix}{2}\right) \right).
\]

Therefore

\[
\int_0^1 e^{xt} \varphi(t) \, dt = \sum_{n=0}^\infty \frac{x^n}{n!} \int_0^1 t^n \varphi(t) \, dt = -\frac{1}{x} \left( 1 - e^{\frac{ix}{4\pi}} \varphi\left(\frac{ix}{4\pi}\right) \right)
\]

from which we obtain \( \varphi(1 - 2^{-n}) \). Another way to compute these numbers is to use

\[
f(x) = 1 + x \int_0^1 e^{xt} \varphi(t) \, dt = e^x \varphi\left(\frac{ix}{4\pi}\right),
\]

together with the fact that

\[
f(2x) = \frac{e^x - 1}{x} f(x).
\]

Therefore

\[
f(x) = \sum_{n=0}^\infty \frac{d_n}{n!} x^n,
\]
where \( d_0 = 1 \) and we have the recurrence
\[
(n + 1)(2^n - 1)d_n = \sum_{k=0}^{n-1} \binom{n + 1}{k} d_k.
\]
(40)

It follows that there are integers \( G_n \) such that
\[
d_n = \frac{G_n}{(n + 1)!} \prod_{k=1}^{n} (2^k - 1)^{-1}.
\]
(41)

The numbers \( d_n \), equation (33) and
\[
d_n = n \int_{0}^{1} t^{n-1} \varphi(t) \, dt
\]
determine the values of \( \varphi(1 - 2^{-n}) \).

We may prove now the following theorem:

**Theorem 7.** The function \( \varphi \) takes rational values at each dyadic point.

**Proof.** Let \( t = q/2^n \) with \( |q| < 2^n \). We compute \( \varphi(q2^{-n}) \). Since \( \varphi \) and all its derivatives vanish at the point \(-1\), Taylor’s theorem with the rest in integral form gives us
\[
\varphi(q2^{-n}) = \int_{-1}^{t} \frac{(t-x)^n}{n!} \varphi^{(n+1)}(x) \, dx.
\]

Applying our formula for the derivatives of \( \varphi \) we obtain
\[
\varphi(t) = \frac{1}{n!} 2^{(n+1)/2} \int_{-1}^{t} (t-x)^n \theta(2^{n+1}(1+x)) \, dx.
\]

Since for \( 2h \leq 2^{n+1}(1+x) \leq 2(h+1) \) we have
\[
\theta(2^{n+1}(1+x)) = (-1)^{s(h)} \varphi(2^{n+1}(1+x) - 2h - 1)
\]
and putting \( 2^{n+1}(1+x) - 2h - 1 = u \) we obtain
\[
\varphi(t) = \frac{1}{n!} 2^{(n+1)/2} \sum_{h=0}^{q+2n-1} (-1)^{s(h)} \int_{-1}^{1} (t - \frac{u}{2n+1} - \frac{2h + 1}{2n+1} + 1)^n \varphi(u) \, du
\]
\[
= \frac{1}{n!} 2^{(n+1)/2} \sum_{h=0}^{q+2n-1} (-1)^{s(h)} \int_{-1}^{1} (2(q-h) + 2^{n+1} - 1 - u)^n \varphi(u) \, du
\]
\[
= \frac{1}{n!} 2^{(n+1)/2} \sum_{h=0}^{q+2n-1} (-1)^{s(h)} \sum_{k=0}^{n} \binom{n}{k} (2(q-h) + 2^{n+1} - 1 - k)^n \int_{\mathbb{R}} u^k \varphi(u) \, du.
\]

This formula, together with equality
\[
\int_{-1}^{1} u^n \varphi(u) \, du = (1 + (-1)^n) \int_{0}^{1} u^n \varphi(u) \, du
\]
and (34) proves our theorem, and we obtain
\[
\varphi(q2^{-n}) = 2 \sum_{h=0}^{q+2n-1 \lfloor n/2 \rfloor} (-1)^{s(h)} 2^{(2^{n+1})-\left(\frac{n+1}{2}\right)} (2(q-h) + 2^{n+1} - 1)^{n-2k} \varphi(1 - 2^{-2k-1})
\]
\[ \square \]
For the computation we may first obtain the common denominator of $\varphi(q^{2^{-n}})$ for a fixed $n$, and using (30) it is possible then to compute the exact value of $\varphi(q^{2^{-n}})$. For $n = 5$ the common denominator is $33177600 = 2^{14}3^45^2$ and we obtain

| $q$ | $33177600 \varphi(q/32)$ | $q$ | $33177600 \varphi(q/32)$ | $q$ | $33177600 \varphi(q/32)$ |
|-----|------------------------|-----|------------------------|-----|------------------------|
| 0   | 33177600               | 11  | 26622019               | 22  | 4893712                |
| 1   | 33177581               | 12  | 24768000               | 23  | 3470381                |
| 2   | 33175312               | 13  | 22784381               | 24  | 2304000                |
| 3   | 33152381               | 14  | 20733712               | 25  | 1396781                |
| 4   | 33062400               | 15  | 18662381               | 26  | 746512                 |
| 5   | 32842819               | 16  | 16588800               | 27  | 334781                 |
| 6   | 32431088               | 17  | 14515219               | 28  | 115200                 |
| 7   | 31780819               | 18  | 12443888               | 29  | 25219                  |
| 8   | 30873600               | 19  | 10393219               | 30  | 2288                   |
| 9   | 29707219               | 20  | 8409600                | 31  | 19                     |
| 10  | 28283888               | 21  | 6555581                | 32  | 0                      |

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