We prove a conjecture of Callan in [1] that OEIS sequence A006012 counts a certain kind of permutation. Call this sequence \( (a_n)_{n=1}^{\infty} \); then \( a_n \) is defined by \( a_1 = 1, a_2 = 2, \) and \( a_n = 4a_{n-1} - 2a_{n-2} \) (the actual sequence in the OEIS is offset by one, so \( a_0 = 1, a_1 = 2, \) and the recursion is the same). The conjecture states that \( a_n \) is equal to the number of permutations of length \( n \) for which no subsequence \( abcd \) has the following two properties: \( c = b + 1 \) and \( \max\{a, c\} < \min\{b, d\} \).

We can rewrite this conjecture in the language of pattern avoidance, in particular, using the dashed notation for generalized pattern avoidance introduced in [2]. Therefore, we define a pattern to be a permutation \( \pi \) to be a permutation if it does not contain any occurrence of the pattern, and a permutation avoids a pattern if it does not contain any occurrence of any of them. If \( A \) is a set of patterns, we will write \( \text{Av}(A) \) for the set of permutations which avoid them, and \( \text{Av}_n(A) \) for the set of length-\( n \) permutations which avoid them. The following two examples should help clarify these definitions.

**Example:** The permutation 251346 contains the subsequence 5146 which is an occurrence of the pattern 3-1-24 because the elements of the subsequence occur in the same relative order as 3124, and the 4 and 6 are consecutive in the original permutation (the 5 and 1 are also consecutive - that is allowed but not necessary).

**Example:** The permutation 251346 avoids 32-1-4 (i.e. \( 251346 \in \text{Av}_n(\{32-1-4\}) \subseteq \text{Av}(\{32-1-4\}) \)).

Using this notation, we rewrite the conjecture as \( a_n = |\text{Av}_n(\{1-32-4, 1-42-3, 2-31-4, 2-41-3\})| \).

We will prove two propositions. The first is that if \( A = \{1-32-4, 1-42-3, 2-31-4, 2-41-3\} \) and \( B = \{1-3-2-4, 1-4-2-3, 2-3-1-4, 2-4-1-3\} \), then \( \text{Av}(A) \) and \( \text{Av}(B) \) are the same set. The second proposition is that \( |\text{Av}_n(B)| \) follows the defining recurrence of \( a_n \), i.e. \( |\text{Av}_1(B)| = 1, |\text{Av}_2(B)| = 2, |\text{Av}_n(B)| = 4|\text{Av}_{n-1}(B)| - 2|\text{Av}_{n-2}(B)| \).

**Proposition 1:** Let \( A = \{1-32-4, 1-42-3, 2-31-4, 2-41-3\} \) and \( B = \{1-3-2-4, 1-4-2-3, 2-3-1-4, 2-4-1-3\} \). The sets \( \text{Av}(A) \) and \( \text{Av}(B) \) are the same.

**Proof:** We will show that any permutation containing an occurrence of an element of \( B \) must also contain an occurrence of an element of \( A \) (the converse is immediately clear). Let \( \pi \) be a permutation. First, note that a subpermutation \( \pi_a\pi_b\pi_c\pi_d \) of \( \pi \) is an occurrence of a pattern in \( A \) if and only if \( c = b + 1 \) and \( \max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\} \) (in fact, this is the definition Callan provides in the OEIS). Similarly, a subpermutation \( \pi_a\pi_b\pi_c\pi_d \) of \( \pi \) is an occurrence of a pattern in \( B \) if and only if \( \max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\} \).
Choose any element of $B$, and suppose that $\pi$ contains an occurrence of this element. As noted above, this means that we can find $a < b < c < d$ such that $\max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\}$. Let $e$ be the largest index less than $c$ such that $\pi_e > \max\{\pi_a, \pi_c\}$, i.e. $e = \max\{i: i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$. Because $b$ is an element of $\{i: i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$, it follows that $e$ exists and $a < b \leq e < e + 1 \leq c < d$. Now, we claim that $\pi_a \pi_e \pi_{e+1} \pi_d$ is an occurrence of a pattern in $A$. Obviously $e + 1 = e + 1$, and so it remains to check that $\max\{\pi_a, \pi_{e+1}\} < \min\{\pi_e, \pi_d\}$. Because $\max\{\pi_a, \pi_e\} < \min\{\pi_b, \pi_d\}$, we conclude that $\pi_a < \pi_d$ and by the choice of $e$, we also have $\pi_a < \pi_e$. Now, either $e + 1 = c$, in which case $\pi_{e+1} = \pi_c$, or else $\pi_{e+1} < \max\{\pi_a, \pi_c\}$ because otherwise we would have chosen $e + 1$ as the $\max\{i: i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$ instead of $e$. It follows that $\pi_{e+1} \leq \max\{\pi_a, \pi_e\} < \pi_d, \pi_e$ for the same reasons as $\pi_a$. Therefore, $\max\{\pi_a, \pi_{e+1}\} < \min\{\pi_e, \pi_d\}$ and $\pi_a \pi_e \pi_{e+1} \pi_d$ is an occurrence of a pattern in $A$. We conclude that the permutations avoiding the patterns of $A$ are the same as the permutations avoiding the patterns of $B$.

**Proposition 2:** The number of permutations of length $n$ avoiding all patterns in $B$ (and hence in $A$) satisfies the recurrence $a_1 = 1, a_2 = 2, a_n = 4a_{n-1} - 2a_{n-2}$.

**Proof:** Since $Av_1(B) = \{1\}$ and $Av_2(B) = \{12, 21\}$, the initial conditions hold. Our strategy will be as follows: given $Av_n(B)$, define four maps which, when all of them are applied to all the permutations of $Av_{n-1}(B)$, will generate all of the permutations of $Av_n(B)$. Then, we will count how many permutations of $Av_n(B)$ are double counted in this way, and find that there are two for every element of $Av_{n-2}(B)$, thereby establishing the recurrence.

Note that, for a permutation to avoid all patterns of $A$, it must be the case that either 1 and 2 occur consecutively (not necessarily in that order) or either 1 or 2 is the last element of the permutation. This observation motivates the following definitions of the four maps $f_{\text{before}}, f_{\text{after}}, f_{\text{end}}, f_{\text{bump}}$. Let $f_{\text{before}}$ be the function that inputs a permutation and outputs that permutation with all elements increased by 1 and a 1 inserted immediately before the new 2. Let $f_{\text{after}}$ be the function that also inputs a permutation and outputs that permutation with all the elements increased by 1 and a 1 inserted immediately after the new 2. Similarly, let $f_{\text{end}}$ be the function that inputs a permutation, increases all its elements by 1 and puts a 1 at the end of it, and let $f_{\text{bump}}$ be the function that inputs a permutation, increases all its elements by 1, replaces the new 2 with a one and puts a 2 at the end. The following example gives a concrete illustration of the four functions.

**Example:** Let $\pi = 31542$. Then $f_{\text{before}}(\pi) = 412653, f_{\text{after}}(\pi) = 421653, f_{\text{end}}(\pi) = 426531,$ and $f_{\text{bump}}(\pi) = 416532$. Note that $\pi \in Av(B)$, and so are all its images.

We claim that (i) these four functions all map elements of $Av_{n-1}(B)$ to elements of $Av_n(B)$ and (ii) $f_{\text{before}}(Av_{n-1}(B)) \cup f_{\text{after}}(Av_{n-1}(B)) \cup f_{\text{end}}(Av_{n-1}(B)) \cup f_{\text{bump}}(Av_{n-1}(B)) \supseteq Av_n(B)$ (by claim (i), we could replace the ‘$\supseteq$’ in claim (ii) with ‘$=$’). To verify the first claim, choose some $\sigma \in Av_{n-1}(B)$, and consider each function in turn. If $f_{\text{before}}(\sigma)$ or $f_{\text{after}}(\sigma)$ contains an occurrence of a pattern in $A$, then this occurrence must use no more than 1 of the elements 1 and 2 (because they are consecutive in both $f_{\text{before}}(\sigma)$ and $f_{\text{after}}(\sigma)$ but can’t be in any pattern in $A$). Therefore, either this occurrence fails to use 1 and would have already been an occurrence of the pattern in $\sigma$, or else it fails to use 2, in which case it could have used
2 instead of 1 and been an occurrence of the pattern in $\sigma$. Thus, no such occurrence is possible in $f_{\text{before}}(\sigma)$ or $f_{\text{after}}(\sigma)$. In addition, if $f_{\text{end}}(\sigma)$ or $f_{\text{bump}}(\sigma)$ contains an occurrence of a pattern in $A$, then this occurrence cannot use the last element because that element is either a 1 or a 2, and patterns in $A$ only end with 3 or 4. So, this occurrence would already be an occurrence of the pattern in $\sigma$, and therefore cannot exist.

To verify the second claim, chose some $\pi \in \text{Av}_n(B)$. As previously noted, either 1 and 2 occur consecutively in $\pi$, or else either 1 or 2 is the final element of $\pi$. Let $\pi'$ be $\pi$ with the 1 removed and each element decreased by 1. We have introduced no new patterns, and so $\pi' \in \text{Av}_{n-1}(B)$. Suppose that 1 occurs immediately before 2 in $\pi$, then $f_{\text{before}}(\pi') = \pi$. If the 1 occurs immediately after 2 in $\pi$, then $f_{\text{after}}(\pi') = \pi$. If the 1 occurs at the end of $\pi$, then $f_{\text{end}}(\pi') = \pi$. If the 2 occurs at the end of $\pi$, then we will need to define $\pi''$ which is $\pi$ with the 1 removed, the 2 moved the position where the 1 used to be, and each element decreased by 1. Again, we have introduced no new patterns, and so $\pi'' \in \text{Av}_{n-1}(B)$, and $f_{\text{bump}}(\pi'') = \pi$.

If these four functions all had disjoint ranges, we could conclude that $a_n = 4a_{n-1}$. Unfortunately, some permutations are counted twice. Each $f$ outputs a certain kind of permutation: $f_{\text{before}}$ outputs permutations where 1 immediately precedes 2, $f_{\text{after}}$ outputs permutations where 2 immediately precedes 1, $f_{\text{end}}$ outputs permutations where 1 occurs at the end, and $f_{\text{bump}}$ outputs permutations where 2 occurs at the end. If a permutation fulfills two of these criteria it will be double-counted. Such permutations must be counted once by either $f_{\text{before}}$ or $f_{\text{after}}$ and again by either $f_{\text{end}}$ or $f_{\text{bump}}$ because no permutation can be counted by both $f_{\text{before}}$ and $f_{\text{after}}$ or both $f_{\text{end}}$ and $f_{\text{bump}}$. Thus, the final two elements of such permutations are 1 and 2 (not necessarily in that order). Let $g : \text{Av}_n(B) \to \text{Av}_{n-2}(B)$ be defined as the function which takes a permutation, removes from it the elements 1 and 2, and reduces all other elements by 2. If we restrict $g$ to those permutations which end in either 12 or 21, $g$ becomes a 2-to-1 map from the double-counted permutations of $\text{Av}_n(B)$ to the permutations of $\text{Av}_{n-2}(B)$, and so the number of double-counted permutations is twice $a_{n-2}$. It follows that $a_n = 4a_{n-1} - 2a_{n-2}$.

The author would like to thank his (intended) advisor Doron Zeilberger for introducing him to the conjecture and fixing typos in the original draft.

**References**

[1] OEIS Foundation Inc. (2017), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A006012

[2] E. Babson and E. Steingrímsson (2000). “Generalized permutation patterns and a classification of the Mahonian statistics,” *Sém. Lothar. Combin.* 44, Art: B44b