ABSTRACT

Q-learning is a popular reinforcement learning algorithm. This algorithm has however been studied and analysed mainly in the infinite horizon setting. There are several important applications which can be modeled in the framework of finite horizon Markov decision processes. We develop a version of Q-learning algorithm for finite horizon Markov decision processes (MDP) and provide a full proof of its stability and convergence. Our analysis of stability and convergence of finite horizon Q-learning is based entirely on the ordinary differential equations (O.D.E) method. We also demonstrate the performance of our algorithm on a setting of random MDP.

1 Introduction

Markov decision process (MDP) is a popular framework to study sequential decision making under uncertainty. An MDP is generally defined via a 5-tuple \( \langle S, A, P, R, \beta \rangle \) where \( S \) is the set of states, \( A \) is the set of actions, \( P \) is the transition probability matrix, \( R \) is the reward function and \( \beta \) is a scalar that can take the values \( 0 < \beta \leq 1 \). In infinite horizon discounted reward problems, we require \( \beta < 1 \). Quite often, in real-life applications, \( P \) and \( R \) are not available to us, however, we have access to data in the form of several ‘state-action-reward-next state’ tuples. Reinforcement learning algorithms learn the optimal policies and value functions from such data samples. Amongst the model free reinforcement learning algorithms, Q-learning is a popular one that has been widely studied both theoretically and over a range of applications.

Now we give a brief survey of Q-learning-type algorithms in the literature. Q-learning has been extensively studied and many improvements have been proposed to the basic algorithm in the literature. In Double Q-learning [van Hasselt 2010], two estimators of Q-values are used to improve the empirical performance by reducing the over-estimations of Q-values. Speedy Q-learning [Ghavamzadeh et al. 2011] is another algorithm where bounds on the number of iterations needed for convergence have been improved. Generalized speedy Q-learning [John et al. 2020] improves further upon Speedy Q-learning by adopting the technique of successive relaxation. The improvement occurs because the contraction factor of the successive relaxation Bellman operator is lower in value than that of the standard Bellman operator. In Zap Q-learning [Devraj et al. 2019], a two-timescale update rule for matrix gain is used which results in an improvement over standard Q-learning.

Even though there are many applications that are finite horizon in nature, previous works have mainly dealt with the infinite horizon setting. Even if the number of stages is reasonably large, using the infinite horizon setting as an approximation introduces errors in the solutions. For instance, a stationary policy is invariably optimal in infinite horizon settings but rarely so when the setting is finite horizon in nature. Further, finite horizon settings allow for non-stationary transition probabilities and reward functions, unlike infinite horizon MDPs. Most of the development in reinforcement learning has been towards the design of algorithms for infinite horizon problems. Finite horizon temporal difference learning has been studied recently in [De Asis et al. 2020]. A finite horizon version of Q-learning has been studied in [Garcia and Ndiaye 1998]. However, a rigorous proof of convergence has not been shown in that
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reference. We present the finite horizon Q-learning algorithm and provide a rigorous proof of convergence along with an experimental study on random MDPs. The main contributions in this paper are as follows:

- We present a Finite Horizon Q-learning algorithm for the general case where the transition dynamics and reward structure are also stage-dependent in addition to them being dependent on states and actions.
- We provide a stochastic approximation based analysis for the stability and convergence of Finite Horizon Q-learning. We show that the Finite horizon Q-learning recursions are both stable and converge to the set of optimal Q-values almost surely.
- We show the results of experiments on a setting of random MDPs and observe that our results conform with the theoretical results.

The rest of the paper is arranged as follows. In the next section, we present basic results in finite horizon reinforcement learning. We present here the Finite Horizon Dynamic Programming (FHDP) algorithm and show that it gives the optimal Q-values. This is then followed in the next section by a description of our proposed algorithm – the Finite Horizon Q learning. In the subsequent section, we present the complete proof of stability and convergence of our finite horizon reinforcement learning algorithm. We then present the results of experiments conducted in the setting of random MDPs in the subsequent section. The final section presents conclusions and describes future work.

2 Preliminaries

Our basic setting involves a finite horizon MDP with horizon length $N < \infty$. Let $n = 0, 1, \ldots, N - 1$ denote the $N$ stages of decision making with $N$ as the termination instant. We consider here a setting involving cost minimization as opposed to reward maximization as it appears to be more natural for optimal control problems. Let $g_n(i, a, j)$ (resp. $p(i, a, j)$) denote the single-stage cost (resp. transition probability) when the state at instant $n$ is $i$ and action chosen is $a$, and the next state (i.e., the one at instant $n+1$) is $j$. Further, let $g_N(i)$ denote the terminal cost when the terminating state is $i \in S$. Let $S$ and $A$ respectively denote the state and action spaces of the MDP. In particular, we let $A(i) \subseteq A$ be the set of feasible actions in state $i$. In general, one may let $S$ and $A$ be time-dependent sets. We however select them to be time-invariant for simplicity.

Let $\pi = \{\pi_0, \pi_1, \ldots, \pi_{N-1}\}$ represent a policy where $\pi_k(i) \in A(i), \forall k = 0, 1, \ldots, N - 1$. The idea is that when following policy $\pi$, at instant $k$, the action is chosen according to the function $\pi_k$, $k = 0, 1, \ldots, N - 1$. Let $J^\pi(i)$, $i \in S$, denote the long-term expected cost:

$$J^\pi(i) = E_\pi \left[ \sum_{k=0}^{N-1} g_k(s_k, a_k, s_{k+1}) + g_N(s_N) \mid s_0 = i \right].$$

Let $\Pi$ denote the set of all policies as above. The goal then is to find a policy $\pi^* \in \Pi$ that gives the optimal long-term expected cost given by

$$J^*(i) \triangleq J^{\pi^*}(i) = \min_{\pi \in \Pi} J^\pi(i), \ i \in S.$$

Define Q-values $Q_\pi(i, a)$ as follows:

$$Q_\pi(i, a) = E_\pi \left[ \sum_{k=0}^{N-1} g_k(s_k, \pi_k(s_k), s_{k+1}) + g_N(s_N) \mid s_0 = i, a_0 = a \right].$$

Here the initial action chosen in the initial state $i$ is $a$. Subsequent actions are chosen according to the policy $\pi$ that now kicks in from instant 1 onwards. Let

$$Q^*(i, a) = \min_{\pi \in \Pi} Q_\pi(i, a),$$

where $\Pi$ is the set of policies starting from instant 0. Note however that since the initial action is $a$ (in state $s$), the initial action is not according to $\pi_0$ in general. Consider now the finite horizon Dynamic Programming (DP) algorithm [1]-[4] in terms of the Q-function.

$$Q_N(s_N, a_N) = g_N(s_N),$$

$$Q_k(s_k, a_k) = E_{s_{k+1}} \left[ g_k(s_k, a_k, s_{k+1}) + \min_{a_{k+1} \in A(s_{k+1})} \left( Q_{k+1}(s_{k+1}, a_{k+1}) \right) \right],$$

for $k = N - 1, N - 2, \ldots, 0$. [Equation 1 and 2]
Assume for some \( k \), thus, the second term on the RHS above can be written as we describe below. A wide range of learning rate choices can be explored that satisfy the standard stochastic approximation conditions that are however shown for the case of finite horizon state-value function (not the state-action value function or the Q-value function as here). We give the details below for completeness. Let \( Q_k^*(s_k,a_k) = E_{s_{k+1}}[g_k(s_k,a_k)] \), \( k = 0,1,\ldots,N-1 \) and

\[
Q^*(N)(s_N,a_N) = g_N(s_N).
\]

Assume for some \( k \) and all feasible \( (s_{k+1},a_{k+1}) \) tuples, \( Q_{k+1}^*(s_{k+1},a_{k+1}) = Q_{k+1}(s_{k+1},a_{k+1}) \). Then,

\[
Q_k^*(s_k,a_k) = E_{s_{k+1}}[g_k(s_k,a_k)] + \min_{\pi_k+1} E_{s_{k+2},\ldots} [g_N(s_{N+1})]
\]

\[
+ \sum_{i=k+2}^{N-1} g_i(s_i,\pi_i(s_i),s_{i+1})]
\]

The second term on the RHS above can be written as

\[
\min_{\pi_{k+1}} E_{s_{k+2}}[g_{k+1}(s_{k+1+1},\pi_{k+1}(s_{k+1}),s_{k+2})]
\]

\[
+ \sum_{i=k+2}^{N-1} g_i(s_i,\pi_i(s_i),s_{i+1})]
\]

Thus,

\[
Q_k^*(s_k,a_k) = E_{s_{k+1}}[g_k(s_k,a_k)] + \min_{a_{k+1}} Q_{k+1}(s_{k+1},a_{k+1})
\]

\[
= E_{s_{k+1}}[g_k(s_k,a_k)] + \min_{a_{k+1}} Q_{k+1}(s_{k+1},a_{k+1})
\]

\[
= Q_k(s_k,a_k),
\]

\( \forall s_k \in S, a_k \in A(s_k) \). This completes the induction step.

3 Finite Horizon Q-learning Algorithm

Note that the DP algorithm is a numerical procedure that computes the optimal Q-values (as shown by Proposition 1). This however relies on the fact that knowledge of the transition probabilities is available. In most real-life situations, however, such information is not available and one only has available (as described earlier) data samples of 'state-action-reward-next state' tuples. As with the regular infinite horizon Q-learning algorithm of [Watkins and Dayan 1992], we present a stochastic approximation version of the finite horizon DP algorithm in Q-values. We provide below the full details of the finite horizon Q-learning algorithm. In the algorithm below, we let the learning rate (or step-size) be \( \alpha(m) = \frac{1}{(m+1)^{1/10}} \). This choice was seen to perform well in experiments. Nonetheless a wide range of learning rate choices can be explored that satisfy the standard stochastic approximation conditions that we describe below.

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The text above is a combination of the natural language representation of the mathematical content and the LaTeX formatting. The goal is to ensure that the text is readable and accessible to a human reader.
Algorithm 1 Finite Horizon Q-Learning

Notation:
- $Q^n_i(a)$: Q-value at state $i$, action $a$, stage $n$, recursion $m$.
- $a(m)$: step-size at recursion index $m$.
- $Q_N(i, a)$: Q-value for state $i$ and action $a$ at terminal stage ($N$).
- $g_n(i, a, j)$: Single stage reward for stage $n$ where current state is $i$, action is $a$ and next state is $j$.
- $g_N(i)$: Terminal reward at the $N^{th}$ stage when terminal state is $i$.
- $A(j)$: Set of feasible actions in state $j$.
- $\eta(i, a)$: Sampling function taking input $(i, a)$ as state-action pair and returns the next state.

Input: Samples of the form $(i, a, r, j)$.
Output: Updated Q-value $Q^{n+1}_i(a)$ estimated after $m$ iterations of the algorithm.

Initialization: $Q^0_i(a) = 0, \forall (i, a), n = 0, \ldots, N - 1$, and $Q^0_N(i, a) = g_N(i), \forall (i, a)$

1: procedure FINITE HORIZON Q-LEARNING:
2:   $a(m) = \frac{1}{(m+1)^{1/10}}$
3:   $j = \eta(i, a)$ (from samples)
4:   $Q^{n+1}_i(a) = (1 - a(m))(Q^n_i(a)) + a(m)$
5:   $\times \min_{b \in A(j)} Q^{n+1}_b(j, b)$, $n = 0, 1, \ldots, N - 1$,
6:   $Q^{N+1}_i(a) = g_N(i)$, $\forall (i, a)$ tuples.
7:   return $Q^{N+1}_i(a)$

4 Proof of Stability and Convergence

In this section, we give a proof of the stability and convergence of our finite horizon Q-learning algorithm. Our proof relies on verifying the Borkar and Meyn conditions for stability and convergence of general stochastic approximation algorithms, see Borkar and Meyn [2000]. We first describe the conditions for stability and convergence as described in Borkar and Meyn [2000] as well as the main results there. Later we shall show that our algorithm meets these conditions.

Consider the following stochastic approximation recursion in $\mathbb{R}^d$:

$$X(m + 1) = X(m) + a(m)(h(X(m)) + M(m + 1)), \quad (3)$$

where $m \geq 0$, $X(m) = (X_1(m), \ldots, X_d(m))^T$, $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $a(m)$ is a sequence of positive real numbers, and $M(n), n \geq 0$ is a zero-mean noise sequence.

Consider now the O.D.E:

$$\dot{x}(t) = h(x(t)), \quad (4)$$

The limit points of the stochastic recursion (3) can be seen to be limit points of the O.D.E (4), see Benaïm [1996]. The O.D.E approach to stochastic approximation (originally due to Ljung [1977] and Kushner and Clark [1978]) is thus useful as one can understand the limit set of the stochastic recursion by analysing the limit set of the corresponding O.D.E. Let $x^*$ denote the unique globally asymptotically stable (UGAS) attractor for the O.D.E (4). Notice that while such an attractor may not always exist, in our setting, since we are analysing the finite horizon Q-learning algorithm, it will be seen that the UGAS attractor exists and just corresponds to the vector of optimal Q-values.

We now list the assumptions of Borkar and Meyn [2000] followed by the main results from that paper summarized as Theorem 1 – Theorem 2 below.

**Assumption 1**
1. The function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous.
2. The sequence of functions $h_r(x) \triangleq \frac{h(rx)}{r}$ satisfy that $h_r(x) \rightarrow h_\infty(x)$, for some function $h_\infty : \mathbb{R}^d \rightarrow \mathbb{R}^d$ uniformly on compacts.
We now recall the update rule of the finite horizon Q-learning algorithm:

\[ \dot{x}(t) = h_\infty(x(t)) \]  

(5)

has the origin in \( \mathbb{R}^d \) as it’s UGAS attractor.

**Assumption 2**

1. The sequence \( \{M(n), \mathcal{F}_n, n \geq 0\} \) is a square-integrable martingale difference sequence where \( \mathcal{F}_n = \sigma(X(i), M(i), i \leq n) \) is an associated sequence of sigma fields.

2. For some constant \( C_0 < \infty \) and any initial condition \( X(0) \in \mathbb{R}^d, n \geq 0, \)

\[ E[\|M(n+1)\|_2 | \mathcal{F}_n] \leq C_0(1+ \|X(n)\|^2), \]  

(6)

for all \( n \geq 0. \)

**Assumption 3** The sequence \( \{a(n)\} \) is a deterministic sequence of step-sizes that satisfy the following conditions:

1. \( a(n) > 0, \forall n \geq 0, \)
2. \( \sum_n a(n) = \infty, \)
3. \( \sum_n a(n)^2 < \infty \)

**Theorem 1 (Theorem 2.1 of Borkar and Meyn [2000])** Under Assumptions 1–3 for any initial condition \( X(0) \in \mathbb{R}^d, \)

\[ \sup_n \|X(n)\| < \infty \text{ almost surely (a.s).} \]

**Theorem 2 (Theorem 2.2 of Borkar and Meyn [2000])** Assume Assumptions 1–3 hold and that the ODE (4) has a unique globally asymptotically stable equilibrium \( x^* \). Then for any initial condition \( X(0) \in \mathbb{R}^d, \)

\[ \lim_{n \to \infty} X(n) = x^*. \]

We now recall the update rule of the finite horizon Q-learning algorithm:

\[ Q^{m+1}_n(i, a) = Q^m_n(i, a) + a(m)(g_n(i, a, \eta^m_n(i, a))) \]

\[ + \min_{b \in A(\eta^m_n(i, a))} Q^m_{n+1}(\eta^m_n(i, a), b) - Q^m_n(i, a)), \]

(7)

\[ n = 0, 1, \ldots, N - 1, m \geq 0, \]

\[ Q^{m+1}_N(i, a) = g_N(i), \forall m \geq 0, \]

(8)

\( \forall(i, a). \) In the above \( \eta^m_n(i, a) \) are i.i.d random variables having the distribution \( p_n(i, a, \cdot). \) Let \( G_m = \sigma(Q^m_n(i, a), \eta^m_n(i, a), n = 0, 1, \ldots, N - 1), k \leq m, l < m, m \geq 0 \) denote the sequence of associated sigma fields. Here we let \( \eta^m_n(i, a) = 0, \forall(i, a) \) tuples.

Let \( Q^n = (Q^m_n(i, a), i \in S, a \in A(i), n = 0, 1, \ldots, N)^T. \) We rewrite (7)-(8) in the following unified form:

\[ Q^{m+1}_n(i, a) = Q^m_n(i, a) + a(m)(h_n(i, a, Q^m) + M^{m+1}_n(i, a)), n = 0, 1, \ldots, N, \]

where for \( n = 0, 1, \ldots, N - 1, \)

\[ h_n(i, a, Q^m) = \sum_j p_n(i, a, j)(g_n(i, a, j) \]

\[ + \min_{b \in A(j)} Q^m_{n+1}(j, b) - Q^m_n(i, a), \]

\[ M^{m+1}_n(i, a) = \]

\[ g_n(i, a, \eta^m_n(i, a)) + \min_{b \in A(\eta^m_n(i, a))} Q^m_{n+1}(\eta^m_n(i, a), b) \]

\[ - \sum_j p_n(i, a, j)(g_n(i, a, j) + \min_{b \in A(j)} Q^m_{n+1}(j, b)). \]
Also, for \( n = N \), we have
\[
h_n^m(i, a) = 0 \text{ and } M_n^{m+1}(i, a) = 0.
\]

Let \( M_n^m \equiv (m^m(i, a), n = 0, 1, \ldots, N, a \in A(i), i \in S)^T \). Also, let \( h(Q) = (h_n(i, a, Q), a \in A(i), i \in S, n = 0, 1, \ldots, N)^T \). The recursions (7)-(8) can thus together be written as
\[
Q^{m+1} = Q^m + a(m)(h(Q^m) + M^{m+1}), \quad m \geq 0.
\]

In what follows, we shall use \( \| \cdot \| \) to denote the sup or the max norm. We now proceed by verifying Assumptions 1-3.

**Proposition 2** The functions \( h, h_r \) (defined as in Assumption 7), \( \forall r \geq 1 \), and \( h_\infty \) are Lipschitz continuous.

**Proof 2** We show first that for two functions \( Q \) and \( Q' \),
\[
| \min_{a \in A(i)} Q(i, a) - \min_{a \in A(i)} Q'(i, a) | \leq \max_{a \in A(i)} | Q(i, a) - Q'(i, a) |,
\]
for all \((i, a)\) tuples. Note that given functions \( f \) and \( g \) and a set \( A \),
\[
\inf_{x \in A} (f(x) + g(x)) = \inf_{x \in A, x = y} (f(x) + g(y)) \\
\geq \inf_{x, y \in A} (f(x) + g(y)) = \inf_{x \in A} f(x) + \inf_{y \in B} g(y).
\]

Using \( f - g \) in place of \( f \), one obtains
\[
\inf_{x \in A} ((f - g)(x) + g(x)) \geq \inf_{x \in A} (f - g)(x) + \inf_{x \in A} g(x), \quad \text{or}
\]
\[
\inf_{x \in A} (f(x) - g(x)) \leq \inf_{x \in A} f(x) - \inf_{x \in A} g(x).
\]

Let \( e(x) = -g(x), \forall x \). Then
\[
\inf_{x \in A} (f(x) + e(x)) \leq \inf_{x \in A} f(x) + \sup_{x \in A} e(x), \quad \text{or}
\]
\[
\inf_{x \in A} (f(x) + e(x)) - \inf_{x \in A} f(x) \leq \sup_{x \in A} e(x).
\]

Again with \( e(x) = g(x) - f(x) \), we have
\[
\inf_{x \in A} g(x) - \inf_{x \in A} f(x) \leq | \sup_{x \in A} (g(x) - f(x)) |.
\]

We now claim that
\[
| \sup_{x \in A} (g(x) - f(x)) | \leq \sup_{x \in A} | g(x) - f(x) |.
\]

Consider first the case when \( \sup_{x \in A} (g(x) - f(x)) \geq 0 \). Then
\[
\sup_{x \in A} (g(x) - f(x)) \leq \sup_{x \in A} | g(x) - f(x) |.
\]

Now consider the case when \( \sup_{x \in A} (g(x) - f(x)) < 0 \). In this case, \( | g(x) - f(x) | = -(g(x) - f(x)), \forall x \). Then,
\[
| \sup_{x \in A} (g(x) - f(x)) | = - \sup_{x \in A} (g(x) - f(x))
\]
\[
= \inf_{x \in A} (-g(x) - f(x)) = \inf_{x \in A} | g(x) - f(x) | \\
\leq \sup_{x \in A} | g(x) - f(x) |.
\]

It follows that
\[
\inf_{x \in A} g(x) - \inf_{x \in A} f(x) \leq \sup_{x \in A} | g(x) - f(x) |.
\]

One may similarly show that
\[
\inf_{x \in A} f(x) - \inf_{x \in A} g(x) \leq \sup_{x \in A} | g(x) - f(x) |.
\]
The two inequalities above then imply
\[ |\inf_{x \in A} g(x) - \inf_{x \in A} f(x)| \leq \sup_{x \in A} |g(x) - f(x)|. \]

The claim follows upon substituting \( g(x) \) with \( Q(i, a) \), \( f(x) \) with \( Q^*(i, a) \), \( A \) with \( A(i) \) and noting that the sets \( S \) and \( A(i) \) for all \( i \) are finite, hence one replaces the inf with min and sup with max operators. It is now easy to see that \( h \) and \( h_r \) are Lipschitz continuous for all \( r \geq 1 \). Now define \( h_{\infty}(Q) = (h_{n,\infty}(i, a, Q), i \in S, a \in A(i), n = 0, 1, \ldots, N)^T \), where
\[ h_{n,\infty}(i, a, Q) = \lim_{r \to \infty} \frac{h_n(i, a, rQ)}{r} = \sum_j p_n(i, a, j) \min_{b \in A(j)} Q(j, b) - Q(i, a). \]

It is again clear from the foregoing that \( h_{\infty}(Q) \) is Lipschitz continuous as well.

Let \( Q^* = (Q^*_n(i, a), i \in S, a \in A(i), n = 0, 1, \ldots, N)^T \) denote the vector of optimal Q-values. By Proposition 1 we observe the DP algorithm (1)-(2) gives the optimal \( Q^* \).

**Lemma 1** The following hold:

1. The ODE \( \dot{Q} = h(Q) \) has \( Q^* \) as its unique globally asymptotically stable equilibrium.

2. The ODE \( \dot{Q} = h_{\infty}(Q) \) has the origin as its unique globally asymptotically stable equilibrium.

**Proof 3**

1. Note that the equilibria of the ODE \( \dot{Q} = h(Q) \) correspond to \( H \triangleq \{ Q \mid h(Q) = 0 \} \). However, it is easy to see that the system of equations that we obtain from \( h(Q) = 0 \) correspond to the solution of the DP algorithm (1)-(2) and which is precisely \( Q^* \). Moreover, \( Q^* \) is unique. A similar argument as in Abounadi et al. [2001] shows that \( Q^* \) is an asymptotically stable attractor of \( \dot{Q} = h(Q) \).

2. It is again easy to see that \( Q = 0 \) (the vector of all zeros or the origin) is an equilibrium of the ODE \( \dot{Q} = h_{\infty}(Q) \). The fact that it is unique can be seen again from the DP algorithm (1)-(2) for the special case when all single-stage costs \( g_k(i, a, j), k = 0, 1, \ldots, N-1 \) and \( g_N(i) \) are zero (for all \( i, j \in S, a \in A(i) \)). Again a similar argument as in Abounadi et al. [2001] shows that the origin is an asymptotically stable attractor of \( \dot{Q} = h_{\infty}(Q) \).

We now have the following result on the noise sequence \( M_{m+1}, m \geq 0 \).

**Proposition 3**

1. The sequence \( (M_{m+1}, G_m), m \geq 0 \) is a square-integrable martingale difference sequence.

2. For some constant \( C_1 > 0 \), we have
\[ E[\| M_{m+1} \|^2 | G_m] \leq C_1(1 + \| Q^m \|^2) \]

**Proof 4**

1. Recall that \( M_{m+1} = (M_{m+1}^n(i, a), n = 0, 1, \ldots, N-1, a \in A(i), i \in S)^T \), where for \( n = 0, 1, \ldots, N-1, \)
\[ M_{m+1}^n(i, a) = g_n(i, a, \eta_m^n(i, a)) + \min_{b \in A(\eta_m^n(i, a))} Q_{n+1}^m(\eta_m^n(i, a), b) \]
\[ - \sum_j p_n(i, a, j)(g_n(i, a, j) + \min_{b \in A(j)} Q_{n+1}^m(j, b)). \]

and \( M_{m+1}^0 = 0, \forall m \geq 0 \). It is easy to see that \( M_{m+1} \) is \( G_m \)-measurable \( \forall m \geq 0 \). Also, \( M_{m+1} \) are integrable random variables since \( g_n, n = 0, 1, \ldots, N \) are all uniformly bounded. Also, since \( Q_{n+1}^m \) are updated according to (7)-(8) from a given \( Q^0 \in \mathbb{R}_d \), it can be seen that one can find a constant \( K_n^m < \infty \) (uniformly over all sample paths), \( \| Q_{n+1}^m \| \leq K_n^m < \infty, \forall m \geq 0, n = 0, 1, \ldots, N \). Thus, \( M_{m+1} \) are all individually integrable. It is also easy to see that \( M_{m+1} \) are square-integrable random variables. Further, \[ E[g_n(i, a, \eta_m^n(i, a)) \]
\[ + \min_{b \in A(\eta_m^n(i, a))} Q_{n+1}^m(\eta_m^n(i, a), b) | G_m] \]
\[ = \sum_j p_n(i, a, j)(g_n(i, a, j) + \min_{b \in A(j)} Q_{n+1}^m(j, b)). \]
Thus, \( E[M_{n+1}^m \mid G_m] = 0, \forall m = 0, 1, \ldots, N - 1 \). Further, \( E[M_{N+1}^m \mid G_m] = 0 \) since \( M_{N+1}^m = 0, \forall m \geq 0 \).

2. Note that

\[
E[|g_n(i, a, \eta_{nm}(i, a)) + \min_{b \in A(\eta_{nm}(i, a))} Q_{nm+1}^m(\eta_{nm}(i, a), b)|^2 \mid G_m] \\
\leq 2(|g_n(i, a, \eta_{nm}(i, a)|^2 + |Q_{nm+1}^m(\eta_{nm}(i, a), b)|^2, \\
\forall b \in A(\eta_{nm}(i, a)). \]

The claim follows from the fact that the single-stage costs \( g_n, n = 0, 1, \ldots, N \) are all uniformly bounded.

We finally have the main result on stability and convergence.

**Theorem 3** The iterates \( Q^m, m \geq 0 \), given by the Q-learning algorithm (7)-(8) satisfy (a) \( \sup_m \parallel Q^m \parallel < \infty \) almost surely, and (b) \( Q^m \to Q^* \) as \( m \to \infty \) almost surely.

**Proof 5** Assumption 2 is shown as a consequence of Proposition 2 and Lemma 1. Proposition 3 shows that Assumption 2 holds. Finally, Assumption 3 can be seen to easily hold for our choice of step-sizes, viz., \( \alpha(m) = \left[ \frac{1}{(m + 1)/10} \right] \). The claim now follows from Theorem 2.

5 **Experiments and Results**

We implemented our Finite Horizon Q-learning algorithm on the setting of random MDPs for different combinations of the following triplet \((N, |S|, |A|)\) of number of stages, number of states and number of actions, respectively. The algorithm was terminated in each experiment for the following termination condition: \( \parallel Q_{prev} - Q_{curr} \parallel \leq \epsilon \), where

- \( Q_{prev} \): Vector of Q-values of size \( N \times |S| \times |A| \) obtained at the previous iteration
- \( Q_{curr} \): Vector of Q-values of size \( N \times |S| \times |A| \) obtained at the current iteration
- \( \epsilon \): a small tolerance or threshold level (set to 0.05 in the experiments)

The various quantities used in the following are explained below.

- \( Q^* \): Final value of the Q-value function (viewed as a vector over \( N \times S \times A \)) as obtained from the Finite Horizon Q-learning Algorithm.
- \( Q^{DP} \): Final value of the Q-value function (viewed as a vector over \( N \times S \times A \)) as computed by the finite horizon dynamic programming algorithm (1)-(2).
- Error = \( \parallel Q^* - Q^{DP} \parallel \) is the norm difference between \( Q^* \) and \( Q^{DP} \).
- \( J^*_n \): Value function at stage \( n \) (having \( |S| \) components) as computed by the algorithm.
- \( \pi_n^* \): Optimal policy function at stage \( n \), having \( |S| \) components.

We mention here that since the experimental setting was of random MDPs, we could make use of the transition probability information for running the DP algorithm (1)-(2) and estimate the error across various runs when using our Finite Horizon Q-learning algorithm.

| \((N, |S|, |A|)\) | \( \epsilon \) | Error | Number of Iterations |
|----------------|--------|-------|---------------------|
| (20, 50, 10)   | 0.05   | 6.8720| 82589               |
| (20, 20, 10)   | 0.05   | 3.5319| 38268               |
| (10, 20, 10)   | 0.05   | 2.1433| 28603               |
| (10, 5, 5)     | 0.05   | 1.2491| 7507                |

Table 1: Algorithm performance in different settings.

For the four different settings of \((N, |S|, |A|)\) as shown in Table 1, seven plots are provided for each of the settings. These individually correspond to the value function and policy obtained upon termination of the algorithm for the
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first stage, the middle stage and penultimate stage as well as error as a function of number of iterations (see above) as obtained when using the finite horizon Q-learning algorithm. Table I summarizes the performance in terms of the Error metric and the number of iterations needed in each of the four settings for the algorithm in order for norm-difference between the Q-values in the current and previous iterates of the algorithm to fall below $\epsilon = 0.05$. From the plots we see that (a) the error in each case diminishes as the number of iterates increases, (b) the value function as obtained from the Finite Horizon Q-learning algorithm closely resembles the value function obtained using the DP algorithm (1)-(2) and (c) the plot of the policy function at different stages indicates that in general the best actions for the various states are quite different.

6 Conclusions and Future Work

We presented the Q-learning algorithm for finite horizon MDPs and gave a complete proof of stability and convergence to the set of optimal Q-values. Experiments indicate that the algorithm achieves almost the same Q-values as the dynamic programming algorithm that works with full model information. It will be interesting to develop RL algorithms with function approximation in the future. Experiments on more detailed settings should also be performed in the future.

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(a) Estimated Value Function

(b) Estimated Policy

Figure 3: Performance for Stage 9 of (10, 5, 5) Setting

(a) Estimated Value Function

(b) Estimated Policy

Figure 4: Performance for Stage 0 of (10, 20, 10) Setting

(a) Estimated Value Function

(b) Estimated Policy

Figure 5: Performance for Stage 5 of (10, 20, 10) Setting
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Figure 6: Performance for Stage 9 of (10, 20, 10) Setting

Figure 7: Error Plots for Settings (10, 5, 5) and (10, 20, 10)

Figure 8: Performance for Stage 0 of (20, 20, 10)
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(a) Estimated Value Function

(b) Estimated Policy

Figure 9: Performance for Stage 10 of (20, 20, 10) Setting

(a) Estimated Value Function

(b) Estimated Policy

Figure 10: Performance for Stage 19 of (20, 20, 10)

(a) Estimated Value Function

(b) Estimated Policy

Figure 11: Performance for Stage 0 of (20, 50, 10)
Figure 12: Performance for Stage 10 of (20, 50, 10) Setting

Figure 13: Performance for Stage 19 of (20, 50, 10)

Figure 14: Error Plots for Settings (20, 20, 10) and (20, 50, 10)
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