Dominated splitting and zero volume for incompressible three flows

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Abstract

We prove that there exists an open and dense subset of the incompressible 3-flows of class $C^2$ such that, if a flow in this set has a positive volume regular invariant subset with dominated splitting for the linear Poincaré flow, then it must be an Anosov flow. With this result we are able to extend the dichotomies of Bochi–Mañé (see Bessa 2007 Ergod. Theory Dyn. Syst. 27 1445–72, Bochi 2002 Ergod. Theory Dyn. Syst. 22 1667–96, Mañé 1996 Int. Conf. on Dynamical Systems (Montevideo, Uruguay, 1995) (Harlow: Longman) pp 110–9) and of Newhouse (see Newhouse 1977 Am. J. Math. 99 1061–87, Bessa and Duarte 2007 Dyn. Syst. Int. J. submitted Preprint 0709.0700) for flows with singularities. That is, we obtain for a residual subset of the $C^1$ incompressible flows on 3-manifolds that: (i) either all Lyapunov exponents are zero or the flow is Anosov and (ii) either the flow is Anosov or else the elliptic periodic points are dense in the manifold.

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1. Introduction

Incompressible flows are a traditional subject from fluid mechanics, see, e.g., [20]. These flows are associated with divergence-free vector fields; they preserve a volume form on the ambient manifold and thus come equipped with a natural invariant measure. On compact manifolds this provides an invariant probability giving positive measure (volume) to all non-empty open subsets. Therefore for vector fields $X$ in this class we have $\Omega(X) = M$ by the Poincaré
recurrence theorem, where $\Omega(X)$ denotes the non-wandering set. In particular, such flows can have neither sinks nor sources, and in general do not admit Lyapunov stable sets, either for the flow itself or for the time reversed flow.

Let $X^r(M)$ be the space of $C^r$ vector fields, for any $r \geq 1$, and $X^r_{\mu}(M)$ the subset of divergence-free vector fields defining incompressible (or conservative) flows. It is natural to study these flows under the measure theoretic point of view, besides the geometrical one.

The device of Poincaré sections has been used extensively to reduce several problems arising naturally in the setting of flows to lower-dimensional questions about the behaviour of a transformation. Recent breakthroughs in the understanding of generic volume-preserving diffeomorphisms on surfaces have non-trivial consequences for the dynamics of generic incompressible flows on three-dimensional manifolds.

The Bochi–Mañé theorem [13] asserts that, for a $C^1$ residual subset of area-preserving diffeomorphisms, either the transformation is Anosov (i.e. globally hyperbolic) or the Lyapunov exponents are zero Lebesgue almost everywhere (i.e. there is no asymptotic growth of the length of vectors in any direction for almost all points). This was announced by Mañé in [24] but only a sketch of a proof was available in [26]. The complete proof presented by Bochi in [13] admits extensions to higher dimensions, obtained by Bochi and Viana in [15], stating in particular that either the Lyapunov exponents of a $C^1$ generic volume-preserving diffeomorphism are zero Lebesgue almost everywhere or else the system admits a dominated splitting for the tangent bundle dynamics. A survey of this theory can be found in [14].

Recently (see theorem 1.1) one of the coauthors was able to use, adapt and fully extend the ideas of the original proof by Bochi to the setting of generic conservative flows on three-dimensional compact boundaryless manifolds without singularities in [9]. The presence of singularities imposes some differences between discrete and continuous systems. The ideas from the Bochi–Ma proof were partially extended to a dense subset of $C^1$ incompressible flows (see theorem 1.2) admitting singularities but without a full dichotomy between zero exponents and global hyperbolicity in the same work [9].

There are related results from Arbieto–Matheus in [4], where it is proved that $C^1$ robustly transitive volume-preserving 3-flows must be Anosov, with the help of a new perturbation lemma for divergence-free vector fields, and also from Horita–Tahzibi in [22], where it is proved that robustly transitive symplectomorphisms must be partially hyperbolic. One of the coauthors together with Rocha proved in [11] that robustly transitive volume-preserving $n$-flows must have dominated splitting.

There are older $C^1$ dichotomy results for low-dimensional transformations. A result of fundamental importance in the theory of generic conservative diffeomorphisms on surfaces was obtained by Newhouse in [30]. Newhouse’s theorem states that $C^1$ generic area-preserving diffeomorphisms on surfaces either are Anosov or else the elliptical periodic points are dense. A refined version of these results was presented by Arnoux in [5] in the family of four-dimensional symplectomorphisms. Even more recently Saghin–Xia [38] generalized Arnoux’s result for the multidimensional symplectic case, and in [10] one of the coauthors together with Duarte obtained a similar dichotomy for $C^1$-generic incompressible flows without singularities on 3-manifolds: either the flow is Anosov or else the elliptic periodic orbits are dense in the manifold.

Here we complete the results of [9, 10] fully extending the dichotomy from generic non-singular vector fields to generic vector fields in the family of $C^1$ all incompressible flows on 3-manifolds.

The main step is our arguments is to show that if a $C^2$ incompressible flow on a 3-manifold admits a positive volume invariant subset (not necessarily closed) formed by regular orbits with a very weak form of hyperbolicity, known as dominated decomposition, then there cannot be
any singularity on the closure of this set, under a mild non-resonant condition on the possible eigenvalues at the singularities. This leads easily to the conclusion that the closure of this invariant subset is a positive volume hyperbolic subset.

Adapting arguments from Bochi–Viana [14] to the flow setting it is proved that incompressible $C^1$ flows with positive volume compact invariant hyperbolic sets must be globally hyperbolic. Finally using standard arguments from Bochi–Viana [9, 10, 14] these results imply the $C^1$ generic dichotomies mentioned above for incompressible flows without any extra condition on the singularities.

1.1. Definitions and statement of the results

In what follows $M$ will always be a $C^\infty$ compact connected boundaryless three-dimensional Riemannian manifold. We denote by $\mu$ a volume form on $M$ and by $\text{dist}$ the distance induced on $M$ by the Riemannian scalar product, denoted by $\langle \cdot, \cdot \rangle$.

We begin by recalling Oseledets’ theorem for measure preserving flows and the notion of linear Poincaré flow first introduced by Doering in [18].

Consider $X \in \mathfrak{X}^1_\mu(M)$ and the associated flow $X^t : M \to M$. Oseledets’ theorem [31] guarantees that we have, for $\mu$-a.e. point $x \in M$, a measurable splitting of the tangent bundle at $x$, $T_x M = E^u_x \oplus \cdots \oplus E^s_x(x)$, called the Oseledets splitting, and real numbers $\lambda_1(x) > \cdots > \lambda_k(x)$ called Lyapunov exponents such that $DX_t^x(E_i^u) = E_i^u(x)$ and

$$
\lim_{t \to \pm \infty} \frac{1}{t} \log \|DX_t^x \cdot v'\| = \lambda_i(x)
$$

for any $v' \in E_i^u \setminus 0$ and $i = 1, \ldots, k(x)$. Oseledets’ theorem allows us to conclude also that

$$
\lim_{t \to \pm \infty} \frac{1}{t} \log |\det(DX_t^x)| = \sum_{i=1}^{k(x)} \lambda_i(x) \cdot \dim(E_i^u),
$$

(1.1)

which is related to the sub-exponential decrease in the angle between any subspaces of the Oseledets splitting along $\mu$-a.e. orbits. Since $DX_t^x(X(x)) = X(X_t(x))$ the direction of the vector field is one of the Oseledets subspaces and it is associated with a zero Lyapunov exponent. The full $\mu$-measure subset of points where these exponents and directions are defined is referred to as the set of Oseledets points of $X$.

In the volume-preserving setting we have $|\det(DX_t^x)| = 1$. Hence on 3-manifolds by (1.1) either $\lambda_j(x) = -\lambda_3(x) > 0$ or both are zero. If $\lambda_1(x) > 0$, then we obtain two directions $E^u$ and $E^s$, respectively, associated with $\lambda_1(x)$ and $\lambda_3(x)$ which we denote by $\lambda_u(x)$ and $\lambda_s(x)$.

We say that $x \in M$ is a singularity of $X$ if $X(x) = 0$ and we denote by $S(X)$ the set of all singularities of $X$. The complement $M \setminus S(X)$ is the set of regular points of $X$. For a regular point $z$ of $X$ denote by $N_z = \{v \in T_z M : \langle v, X(z) \rangle = 0\}$ the orthogonal complement of the flow direction $[X]_z = [X(z)] := \mathbb{R} \cdot X(z)$ in $T_z M$. Denote by $O_z : T_z M \to N_z$ the orthogonal projection of $T_z M$ onto $N_z$. For every $t \in \mathbb{R}$ define

$$
P^t_X(z) : N_z \to N_{X(z)} \quad \text{by} \quad P^t_X(z) = O_{X(z)} \circ DX^t_z.
$$

It is easy to see that $P = \{P^t_X(z) : t \in \mathbb{R}, X(z) \neq 0\}$ satisfies the cocycle identity

$$
P^{t+s}_X(z) = P^t_X(X^s(z)) \circ P^s_X(z)
$$

for every $t, s \in \mathbb{R}$.

The family $P$ is called the linear Poincaré flow of $X$.

If we have an Oseledets point $x \notin S(X)$ and $\lambda_1(x) > 0$, the Oseledets splitting on $T_z M$ induces a $P^t_X$-invariant splitting on $N_z$, say $O_{X^s(z)} = N^s_x \circ \varphi = u, s$. If $\lambda_1(x) = 0$, then the
$P^1_p$-invariant splitting is trivial. Using (1.1) it is easy to see that the Lyapunov exponents of $P^1_p(x)$ associated with the subspaces $N^u$ and $N^s$ are, respectively, $\lambda_u(x) \geq 0$ and $\lambda_s(x) \leq 0$.

We now define dominated structures for the linear Poincaré flow. Given a regular invariant subbundle $A$ for $X \in \mathcal{X}^1(M)$, that is $A \cap S(X) = \emptyset$, an invariant splitting $N^1 \oplus N^2$ of the normal bundle $N_M$ for the linear Poincaré flow $P^1_x$ is said to be $m$-dominated, if there exists an integer $m$ such that for every $x \in \Lambda$ we have the domination relation

$$
\frac{\left\| P^1_p(x) \mid N^1 \right\|}{\left\| P^1_p(x) \mid N^2 \right\|} \leq \frac{1}{2},
$$

(1.2)

Dominated splittings are automatically continuous on the Grassmanian of plane subbundles of the tangent bundle, see, e.g., [16, 21] for an exposition of the theory. In particular, the dimensions of the subbundles are constant on each connected component of $\Lambda$.

As is traditional we say that a vector field is Anosov if the flow preserves a globally defined hyperbolic structure, that is, the tangent bundle $TM$ splits into three continuous $D_X^1$-invariant subbundles $E \oplus [X] \oplus G$ where $[X]$ is the flow direction, the sub-bundle $E \neq 0$ is uniformly contracted and the sub-bundle $G \neq 0$ is uniformly expanded by $DX_t$ for $t > 0$. Note that for an Anosov flow $X$ the entire manifold is $m$-dominated for some $m \in \mathbb{N}$. The fact that the dimensions of the subbundles are constant on the entire manifold implies that $S(X) = \emptyset$ for an Anosov vector field.

Denote by $\mathcal{X}^1(M)^{\ast}$ the subset of $\mathcal{X}^1(M)$ of $C^1$-incompressible flows but without singularities.

**Theorem 1.1 ([9], theorem 1).** There exists a residual set $\mathcal{R} \subset \mathcal{X}^1(M)^{\ast}$ such that for $X \in \mathcal{R}$, either $X$ is Anosov or else for Lebesgue almost every $p \in M$ all the Lyapunov exponents of $X^p$ are zero.

Developing the ideas of the proof of this result one can also obtain the following statement on denseness of dominated splitting, now admitting singularities.

**Theorem 1.2 ([9], theorem 2).** There exists a dense set $\mathcal{D} \subset \mathcal{X}^1(M)^{\ast}$ such that for $X \in \mathcal{D}$, there are invariant subsets $D$ and $Z$ whose union has full measure, such that

- for $p \in Z$ the flow has only zero Lyapunov exponents;
- $D$ is a countable increasing union $\Lambda_m$ of invariant sets admitting an $m$-dominated splitting for the linear Poincaré flow, where $m$ is a strictly increasing integer sequence.

We recall another $C^1$-type result for incompressible three-dimensional flows without fixed points. Preliminary versions for the discrete symplectic case were presented in [5, 30, 38], respectively, for surfaces, four-dimensional manifolds and $2n$-dimensional manifolds.

**Theorem 1.3 ([10], theorem 1.2).** Given $\varepsilon > 0$, any open subset $U$ of $M$ and a non-Anosov vector field $X \in \mathcal{X}^1(M)^{\ast}$, there exists $Y \in \mathcal{X}^1(M)^{\ast}$ such that $Y$ is $C^1$-$\varepsilon$-close to $X$ and $Y$ has an elliptic closed orbit intersecting $U$.

We are able to extend theorems 1.1 and 1.3 to the full family of incompressible $C^1$ flows. Here are our main results.

**Theorem A.** There exists a generic subset $\mathcal{R} \subset \mathcal{X}^1(M)$ such that for $X \in \mathcal{R}$

- either $X$ is Anosov,
- or else for Lebesgue almost every $p \in M$ all the Lyapunov exponents of $X^p$ are zero.

**Theorem B.** Let $\varepsilon > 0$, an open subset $U$ of $M$ and a non-Anosov vector field $X \in \mathcal{X}^1(M)$ be given. Then there exists $Y \in \mathcal{X}^1(M)$ such that $Y$ is $C^1$-$\varepsilon$-close to $X$ and $Y^p$ has an elliptic closed orbit intersecting $U$. 
From theorem B we can follow *ipsis verbis* the proof of theorem 1.3 of [10] to deduce the next generic result.

**Corollary 1.4.** There exists a $C^1$ residual set $\mathcal{R} \subset X_\mu^1(M)$ such that if $X \in \mathcal{R}$, then $X$ is Anosov or else the elliptic closed orbits of $X$ are dense in $M$.

It is well known that a $C^2$ dynamical system admitting a hyperbolic set with positive measure must be globally hyperbolic: see, e.g., Bowen–Ruelle [17] and Bochi–Viana [14]. Recently in [2] this was extended to transitive sets having a weaker form of hyperbolicity called *partial hyperbolicity* with the extra assumption of non-uniform expansion along the central direction. Also in [1] similar results were obtained for positive volume *singular-hyperbolic* sets for $C^2$ (not necessarily incompressible) flows.

We extend these results for an even weaker type of hyperbolicity, i.e. for sets with a dominated splitting. Both theorems A and B are deduced from the following result.

**Theorem C.** There exists an open and dense subset $G \subset X_\mu^2(M)$ such that for every $X \in G$ with a regular invariant set $\Lambda$ (not necessarily closed) satisfying:

1. the linear Poincaré flow over $\Lambda$ has a dominated decomposition; and
2. $\Lambda$ has positive volume: $\mu(\Lambda) > 0$;

then $X$ is Anosov and the closure of $\Lambda$ is the whole of $M$.

1.2. Overview of the arguments and organization of the paper

The proofs of theorems A and B follow standard arguments from Bochi [6, 9, 10] assuming theorem C together with the denseness of $C^2$ incompressible flows among $C^1$ incompressible ones given by Zuppa in [40]. We present these arguments in the following section 2.

We give now an outline of the proof of theorem C. Fix $X \in X_\mu^2(M)$ and assume that there exists an invariant subset $\Lambda$ for $X$ (not necessarily compact) without singularities (i.e. formed by regular orbits of $X$) and with positive volume: $\mu(\Lambda) > 0$. We show that

1. the closure $A$ of $\Lambda$ cannot contain singularities.
2. If $A$ is a compact invariant set without singularities and with dominated decomposition of the linear Poincaré flow, then $A$ is a uniformly hyperbolic set.
3. A uniformly hyperbolic set $A$ with positive volume for a $C^2$ incompressible flow must be the whole $M$.

For the last item above we adapt the arguments from [14, appendix B] to the flow setting.

2. Generic dichotomies for incompressible flows

Here we prove theorems A and B assuming theorem C.

We start with a sequence of simple lemmas. We say that the vector field $X$ is *aperiodic* if the volume of the set of all closed orbits for the corresponding flow is zero.
Lemma 2.1. There exists a $C^1$-dense set $\mathcal{D} \subseteq \mathcal{X}_\mu^1(M)$ such that if $X \in \mathcal{D}$, then

- $X$ is aperiodic;
- $X$ is of class $C^r$ for some $r \geq 2$; and
- every invariant $m$-dominated set $\Lambda$ has zero or full measure, for any $m \in \mathbb{N}$.

Proof. Let $\mathcal{KS}$ be the $C^r$ generic subset given in [36, theorem 1(i)], for some $r \geq 2$, so that $X \in \mathcal{KS}$ is $C^r$ and admits countably many closed orbits only, all of which are hyperbolic or elliptic. According to the results in [40], $\mathcal{X}_\mu^r(M)$ is also $C^1$-dense on $\mathcal{X}_\mu^1(M)$, for $r \geq 2$. Therefore, we can find a set $\mathcal{D}$ such that $X \in \mathcal{D}$ is aperiodic, of class at least $C^2$ and given any $m$-dominated invariant subset $\Lambda$ of $M$ for $X$, by theorem C we have that either $\Lambda$ has zero volume, or $X$ is Anosov, and so $\Lambda = M$. \hfill $\square$

We define as in [15] or [9], the integrated upper Lyapunov exponent

$$L(X) = \lim_{n \to +\infty} \frac{1}{n} \int_M \log \| P^n_X(x) \| d\mu(x),$$

which is an upper semicontinuous function $L : \mathcal{X}_\mu^1(M) \to \mathbb{R}$.

The proof of the next result follows [9, proposition 3.2] step by step, only replacing hyperbolic invariant subset with $m$-dominated invariant subset in the relevant places of the argument.

Proposition 2.2. Let $X \in \mathcal{X}_\mu^2(M)$ be an aperiodic vector field and assume that every $m$-dominated invariant subset has zero volume.

For every given $\epsilon, \delta > 0$ there exists an incompressible $C^1$ vector field $Y$ such that $Y$ is $\epsilon$-$C^1$-close to $X$ and $L(Y) < \delta$.

Proof of theorem A. Let $\mathcal{D}$ be given by lemma 2.1. Denote by $\mathcal{A}$ the $C^r$-stable subset of Anosov incompressible flows. By upper semicontinuity of $L$, for every $k \in \mathbb{N}$, the set $\mathcal{A}_k = \{ X \in \mathcal{X}_\mu^1(M) : L(X) < 1/k \}$ is open. Then proposition 2.2 implies that $\mathcal{A}_k$ dense in the complement $\mathcal{A}^c$ of $\mathcal{A}$ in $\mathcal{X}_\mu^1(M)$. We define a $C^1$ residual set by

$$\mathcal{R} = \bigcap_{k \in \mathbb{N}} (\mathcal{A} \cup \mathcal{A}_k).$$

It is straightforward to check that $\mathcal{R}$ satisfies the statement of theorem A. \hfill $\square$

Now we start the proof of theorem B. But first we recall a basic result which is a consequence of the persistence of dominated splittings, see, e.g., [16].

Lemma 2.3. Given a subset $\Lambda$ with $m$-dominated splitting for a vector field $X$, there exists a neighbourhood $U$ of $\Lambda$ and $\delta > 0$ such that the set $\Lambda_Y(U) := \cap_{t \in \mathbb{R}} Y^t(U)$ has a $(m+1)$-dominated splitting for any vector field $Y$ which is $\delta$-$C^1$-close to $X$.

This means that perturbing the original flow $X$ to $Y$ around an invariant $m$-dominated set, we can in (1.2) switch from $1/2$ to $1/2 + \varepsilon$ for a very small $\varepsilon$ and for every regular orbit of $Y$ which remains nearby $\Lambda$.

The following perturbation lemmas from [10] are the main tools in our arguments to prove theorem B.

Lemma 2.4 (Small angle perturbation ([10], proposition 3.8)). Let $X \in \mathcal{X}_\mu^1(M)$ and $\varepsilon > 0$ be given. There exists $\theta = \theta(\varepsilon, X) > 0$ such that if a hyperbolic periodic orbit $\mathcal{O}$ for $X$ has angle between its stable and unstable directions smaller than $\theta$, then we can find an $\varepsilon$-$C^1$-close volume-preserving vector field $Y$ such that $\mathcal{O}$ is an elliptic periodic orbit for $Y'$. 

Another setting where one can create a nearby elliptic periodic orbit is the following.

**Lemma 2.5 (Large angle perturbation ([10], proposition 3.13)).** Let \( X \in \mathcal{X}^1_{\mu}(M) \) and \( \varepsilon, \theta > 0 \) be given. There exists \( m = m(\varepsilon, \theta) \in \mathbb{N} \) and \( T(m) > 0 \) such that if \( \mathcal{O} \) is a hyperbolic periodic orbit for \( X \) with

- angle between its stable and unstable directions bounded from below by \( \theta \);
- period larger than \( T(m) \), and
- the linear Poincaré flow along \( \mathcal{O} \) is not \( m \)-dominated,

then we can find a \( \varepsilon \)-\( C^1 \)-close vector field \( Y \) such that \( \mathcal{O} \) is an elliptic periodic orbit for \( Y \).

Conversely the absence of elliptic periodic orbits for all nearby perturbations implies uniform bounds on hyperbolic orbits with big enough period. This is an easy consequence of the two previous lemmas 2.4 and 2.5 which we state for future reference.

**Lemma 2.6.** Let \( X \in \mathcal{X}^1_{\mu}(M) \) and \( \varepsilon > 0 \) be given and set \( \theta = \theta(\varepsilon, X) \), \( m = m(\varepsilon, \theta) \) and \( T = T(m) \) given by lemmas 2.4 and 2.5.

Assume that all divergence-free vector fields \( Y \) which are \( \varepsilon \)-\( C^1 \)-close to \( X \) do not admit elliptic closed orbits. Then for every such \( Y \) all closed orbits with period larger than \( T \) are hyperbolic, \( m \)-dominated and with angle between its stable and unstable directions bounded from below by \( \theta \).

**Proof of theorem B.** Let \( \mathcal{P} \) be the residual set given by Pugh’s general density theorem in [35], that is, \( \mathcal{P} \) is the family of all divergence-free vector fields \( X \) such that \( \Omega(X) \) is the closure of the set of periodic orbits, all of them hyperbolic or elliptic, and \( \Omega(X) = M \) by the Poincaré recurrence theorem.

We take any \( X \in \mathcal{X}^1_{\mu}(M) \) which is not approximated by an Anosov flow. Then by a small \( C^1 \) perturbation we can assume that \( X \) belongs to \( \mathcal{P} \) and that \( X \) is still not approximated by an Anosov flow. We fix some open set \( U \) and \( \varepsilon > 0 \).

If some elliptic closed orbit of \( X \) intersects \( U \) there is nothing to prove, just set \( Y = X \). Otherwise we fix \( \varepsilon > 0 \) small and consider three cases:

(A) All closed orbits of \( X \) which intersect \( U \) are hyperbolic, and some of them have a small angle, less than \( \theta = \theta(\varepsilon, X) \), provided by lemma 2.4, between the stable and unstable directions.

(B) All closed orbits of \( X \) which intersect \( U \) are hyperbolic, with angle between stable and unstable directions bounded from below by \( \theta \), but some of them, with period larger than \( T \), do not admit any \( m \)-dominated splitting for the linear Poincaré flow, where \( m = m(\varepsilon, \theta) \) and \( T = T(m) \) are given by lemma 2.5, and \( \theta = \theta(\varepsilon, X) \) is given as before by lemma 2.4.

(C) All closed orbits of \( X \) which intersect \( U \) and have period larger than \( T \) are hyperbolic, with \( m \)-dominated splitting, and with the angle between the stable and unstable directions bounded from below by \( \theta \), where \( m = m(\varepsilon, \theta) \) and \( T = T(m) \) are given by lemma 2.5, and \( \theta = \theta(\varepsilon, X) \) is given as before by lemma 2.4.

Case (A) implies the desired conclusion for some zero-divergence vector field \( Y \) \( \varepsilon \)-\( C^1 \)-close to \( X \) by lemma 2.4. Analogously for case (B) by the choice of the bounds \( m, T \) and by lemma 2.5.

Finally, we use theorem A to show that if \( X \) is in case (C) and we assume that every \( C^1 \)-nearby vector field \( Y \) does not admit elliptic periodic orbits through \( U \), then we get a contradiction. This proves the statement of theorem B.
If $X$ is in case (C), then from Lemma 2.6 we know that every periodic orbit intersecting $U$, for every vector field $Y$ $\epsilon$-$C^1$-close to $X$, with period larger than $T$, is hyperbolic with uniform bounds on $m$ and $\theta$.

From Theorem A, since $X$ is not approximated by an Anosov flow, there exists an incompressible vector field $Y$, which is $\epsilon/3$-$C^1$-close to $X$, admitting a full $\mu$-measure subset $Z$ where all Lyapunov exponents for $Y$ are zero. Moreover we can assume that $Y$ is aperiodic, that is, the set of all periodic orbits has volume zero.

Let $\hat{U} \subset U$ be a measurable set with positive measure. Let $R \subset \hat{U}$ be the set given by the Poincaré recurrence theorem (see, e.g., [25]) with respect to $Y$. Then every $x \in R$ returns to $\hat{U}$ infinitely many times under the flow $Y^t$ and is not a periodic point. Denote by $T$ the set of positive return times to $\hat{U}$ under $Y^t$.

Given $x \in Z \cap R$ and $0 < \delta < \log 2/m$, there exists $t_k \in \mathbb{R}$ such that

$$e^{-\delta t} < \|P_t(x)\| < e^{\delta t} \quad \text{for every } t \geq t_k.$$

Let us choose $\tau \in T$ such that $\tau > \max\{t_k, T\}$.

The $Y^\tau$-orbit of $x$ can be approximated for a very long time $\tau > 0$ by a periodic orbit of a $C^1$-close flow $Z$: given $r, \tau > 0$ we can find a $\epsilon/3$-$C^1$-neighbourhood $U$ of $Y$ in $\mathcal{X}^1_\epsilon(M)$, a vector field $Z \in \mathcal{U}$, a periodic orbit $p$ of $Z$ with period $\ell$ and a map $g : [0, \tau] \to [0, \ell]$ close to the identity such that

- $\text{dist}(Y^t(x), Z^{g(t)}(p)) < r$ for all $0 \leq t \leq \tau$;
- $Z = Y$ over $M \setminus \bigcup_{0 \leq t \leq \ell} (B(p, r) \cap B(Z^t(p), r))$.

This is Pugh’s $C^1$ closing lemma adapted to the setting of conservative flows, see [35]. Letting $r > 0$ be small enough we obtain also that

$$e^{-\delta t} < \|P_t^\tau(x)\| < e^{\delta t} \quad \text{with } \tau > T. \quad (1.1)$$

Now it is easy to see that $Z$ is $\epsilon$-$C^1$-close to $X$, so that the orbit of $p$ under $Z$ satisfies the conclusion of Lemma 2.6. In particular, we have that

$$\frac{\|DP^\ell_p\|}{\|DP^\ell_p N^u\|} \leq \frac{1}{2} \quad \text{for all } x \in O_Z(p),$$

otherwise we would use Lemma 2.5 and produce an elliptic periodic orbit for a flow $\epsilon$-$C^1$-close to $X$. Since the subbundles $N^{c,u}$ are one-dimensional we write $p_i := Z^m(p)$ for $i = 0, \ldots, [\ell/m]$ with $[r] := \max\{k \in \mathbb{Z} : k \leq r\}$ and

$$\frac{\|DP^\ell_p\|}{\|DP^\ell_p N^u\|} \leq \frac{\|DP^\ell_{Z^m} N^v\|}{\|DP^\ell_{Z^m} N^u\|} \leq C(p, Z) \cdot \left(1 + \frac{1}{2}\right)^{\lfloor \ell/m \rfloor}, \quad (2.2)$$

where $C(p, Z) = \sup_{0 \leq t \leq \ell} \left(\|DP^\ell_{Z^t} N^v\| \cdot \|DP^\ell_{Z^t} N^u\|^{-1}\right)$ depends continuously on $Z$ in the $C^1$ topology. There exists then a uniform bound on $C(p, Z)$ for all vector fields $Z$ which are $C^1$-close to $X$.

We note that we can take $\ell > T$ arbitrarily big by letting $r > 0$ be small enough in the above arguments. Therefore (2.2) ensures that $\|DP^\ell_p(p)\| = \|DP^\ell_p N^u_p\|$ and also

$$\frac{1}{\ell} \log \|DP^\ell_p\| \leq \frac{1}{\ell} \log C(p, Z) + \frac{[\ell/m]}{\ell} \log \frac{1}{2} + \frac{1}{\ell} \log \|DP^\ell_p\|.$$

Moreover since $Z$ is volume preserving we have that the sum of the Lyapunov exponents along $O_Z(p)$ is zero, that is (we recall that $\ell$ is the period of $p$)

$$\frac{1}{\ell} \log \|DP^\ell_p N^u_p\| = -\frac{1}{\ell} \log \|DP^\ell_p N^u_p\|.$$
The constants in (2.2) do not depend on $\ell$ so taking the period very big we deduce that
\[
\frac{1}{\ell} \log \|DP_\ell(p)\| \geq \frac{1}{m} \log 2 > \delta.
\]
This contradicts (2.1) and completes the proof of theorem B. \hfill \Box

3. Dominated splitting and regularity

Here we prove that positive volume regular invariant subsets with dominated splitting cannot admit singularities in their closure and thus are essentially uniformly hyperbolic sets. This result is used to prove theorem C.

We denote by $\mathcal{X}^{1+}(M)$ the set of all $C^1$ vector fields $X$ whose derivative $DX$ is Hölder continuous with respect to the given Riemannian norm, and we say that $X \in \mathcal{X}^{1+}(M)$ is of class $C^1$. We clearly have
\[
\mathcal{X}^1(M) \supset \mathcal{X}^{1+}(M) \supset \mathcal{X}^r(M), \text{ for every } r \geq 2.
\]

**Proposition 3.1.** Let $X \in \mathcal{X}^{1+}_\mu(M)$ be given. Assume that $\Lambda$ is a regular $X^t$-invariant subset of $M$ with positive volume and admitting a dominated splitting. Then the closure $A$ of the set of Lebesgue density points of $\Lambda$ does not contain singularities.

We recall that a compact invariant subset $\Lambda$ of $X \in \mathcal{X}^1_\mu(M)$ is (uniformly) hyperbolic if
\[
T\Lambda = E \oplus [X] \oplus G
\]
is a continuous $DX^t$-invariant splitting with the sub-bundle $E \neq 0$ uniformly contracted and the sub-bundle $G \neq 0$ uniformly expanded by $DX^t$ for $t > 0$.

According to [9, lemma 2.4] a compact invariant set without singularities of a $C^1$ three-dimensional vector field admitting a dominated splitting for the linear Poincaré flow is a uniformly hyperbolic set. Then we obtain the following.

**Corollary 3.2.** Let $X \in \mathcal{X}^{1+}_\mu(M)$ and $\Lambda$ be a regular $X^t$-invariant subset of $M$ with positive volume and admitting a dominated splitting. Then the closure $A$ of the set of Lebesgue density points of $\Lambda$ is a hyperbolic set.

This implies in particular that there are neither singular-hyperbolic sets (e.g. Lorenz-like sets or singular-horseshoes) nor partially hyperbolic sets (see, e.g., [16] or [28] for the definitions) with positive volume for $C^1$ incompressible flows on three-dimensional manifolds. A similar conclusion for singular-hyperbolic sets was obtained by Arbieto–Matheus in [4] but assuming that the invariant compact subset is robustly transitive.

The proof of proposition 3.1 is divided into several steps, which we state and prove as a sequence of lemmas in the following subsections.

3.1. Bounded angles, eigenvalues and Lorenz-like singularities

Denote by $D(\Lambda)$ the subset of the Lebesgue density points of $\Lambda$, that is, $x \in D(\Lambda)$ if $x \in \Lambda$ and
\[
\lim_{r \to 0^+} \frac{\mu(\Lambda \cap B(x, r))}{\mu(B(x, r))} = 1.
\]
It is well known (see, e.g., [37] or [29]) that almost every point of a measurable set is a Lebesgue density point, that is $\mu(\Lambda \setminus D(\Lambda)) = 0$. Moreover since every non-empty open subset of $M$ has positive $\mu$-measure, we see that $D(\Lambda)$ is contained in the closure of $\Lambda$.

Assume that $\Lambda$ is a $X^t$-invariant set without singularities such that $\mu(\Lambda) > 0$ and write $A$ for the closure of $D(\Lambda)$ in what follows. Note that $A$ is contained in the closure of $\Lambda$. 
Lemma 3.3. Suppose that the linear Poincaré flow over $\Lambda$ has a dominated splitting for $X$. Then there exist a neighbourhood $V$ of $\Lambda$, a neighbourhood $U$ of $X$ in $X^1(M)$ (not necessarily contained in the space of conservative flows) and $\eta > 0$ such that for every $Y \in U$, every periodic orbit contained in $U$ is hyperbolic of saddle type and its eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $\lambda_1 < -\eta$ and $\lambda_2 > \eta$. Moreover the angle between the unstable and stable directions of these periodic orbits is greater than $\eta$.

Proof. The dominated splitting for the linear Poincaré flow extends by continuity to every regular orbit $O$ which remains close to $\Lambda$ for a $C^1$ nearby flow $Y$, this is lemma 2.3. The domination implies that the eigenvalues $\lambda_1 \leq \lambda_2$ of $O$ satisfy $\lambda_1 + 2\kappa \leq \lambda_2$ for some $\kappa > 0$ which only depends on the domination constant of $\Lambda$. Since the flow $Y$ is close to being conservative, we have $|\lambda_1 + \lambda_2| \leq \epsilon$, where we can take $\epsilon < \kappa/2$ just by letting $Y$ be in a small $C^1$-neighbourhood of $X$.

Thus we have $-\lambda_2 - \epsilon \leq \lambda_1$ which implies $-\lambda_2 - \epsilon + 2\kappa \leq \lambda_1 + 2\kappa \leq \lambda_2$ and so $2\lambda_2 \geq 2\kappa - \epsilon > 0$ on the one hand. On the other hand $\lambda_1 \leq \epsilon - \lambda_2$ implies $\lambda_1 \leq \epsilon - (\kappa - \epsilon/2) = 3\epsilon/2 - \kappa < 0$.

Hence there exists $\eta > 0$, independent of $Y$ in a $C^1$ neighbourhood of $X$, and independent of the periodic orbit $O$ of $Y$ in a neighbourhood of $\Lambda$, such that $\lambda_1 < -\eta$ and $\lambda_2 > \eta$, as stated.

For the angle bound we argue by contradiction as in [28]: assume there exists a sequence of flows $Y_n \rightrightarrows_{n \to \infty} X$ and of periodic orbits $O_n = Y_n$ contained in the neighbourhood $V$ of $\Lambda$ such that the angle $\alpha_n$ between the unstable subspace and the stable direction satisfies $\alpha_n \rightrightarrows_{n \to \infty} 0$.

Then as in the proof of [28, theorem 3.6] (or [3, theorem 3.31]) we can find (through a flow version of Frank’s lemma, see [19] and [3, appendix]) an arbitrarily small $C^1$ perturbation $Z_n$ of $Y_n$, for all big enough $n \geq 1$, sending the stable direction close to the unstable direction along the periodic orbit, such that the orbit of $O_n$ becomes a sink or a source for $Z_n$. This contradicts the first part of the statement of the lemma. \qed

We say that a singularity $\sigma$ is Lorenz-like for $X$ if $DX(\sigma)$ has three real eigenvalues $\lambda_2 \leq \lambda_1 \leq \lambda_3$ satisfying $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

Lemma 3.4. Assume that $X \in X^1_\mu(M)$ is such that all singularities are hyperbolic with no resonances (real eigenvalues are all distinct). Then the singularities $S(X) \cap \Lambda$ are all Lorenz-like for $X$ or for $-X$.

Remark 3.5. The assumptions of the lemma above hold true for an open and dense subset of all $C^r$ vector fields, both volume preserving or not.

Proof. Fix $\sigma$ in $S(X) \cap \Lambda$ if this set is non-empty (otherwise there is nothing to prove). By assumption on $X$ we know that $\sigma$ is hyperbolic. As in [28] we show first that $\sigma$ has only real eigenvalues. For otherwise we would get a conjugate pair of complex eigenvalues $\omega$, $\overline{\omega}$ and a real one $\lambda$ and, by reversing time if needed, we can assume that $\lambda < 0 < \text{Re}(\omega)$.

Since $\mu(\Lambda) > 0$ there are infinitely many distinct orbits of $\Lambda$ passing through every given neighbourhood of $\sigma$, for each regular orbit of a flow is a regular curve, and so does not fill volume in a three-dimensional manifold.

Using the connecting lemma of Hayashi adapted to conservative flows (see, e.g., [39]) we can find a $C^1$-close flow $Y$ preserving the same measure $\mu$ with a saddle-focus connection associated with the continuation $\sigma_Y$ of the singularity $\sigma$. By a small perturbation of the vector field we can assume that $Y$ is of class $C^\infty$ and still $C^1$-close to $X$ (see, e.g., [40]).
We can now unfold the saddle-focus connection as in [12] to obtain a periodic orbit with all Lyapunov exponents equal to zero (an elliptic closed orbit) for a $C^1$-close flow and near $A$. This contradicts lemma 3.3, since such an orbit is contained in a neighbourhood of $\Lambda$. This shows that complex eigenvalues are not allowed for any singularity in $A$.

Let then $\lambda_2 \leq \lambda_3 \leq \lambda_1$ be the eigenvalues of $\sigma$. We have $\lambda_2 < 0 < \lambda_3$ because $\sigma$ is hyperbolic. The preservation of volume implies that $\lambda_2 = -(\lambda_1 + \lambda_3) < 0$ so that $-\lambda_3 < \lambda_1$.

We have now two cases:

$\lambda_3 < 0$: this implies $\lambda_2 < -\lambda_3 < 0 < -\lambda_1 < \lambda_1$ by the non-resonance assumption, and $\sigma$ is Lorenz-like for $X$;

$\lambda_3 > 0$: since $\lambda_1 = -(\lambda_2 + \lambda_3) > 0$ the non-resonance assumption ensures that $\lambda_2 < -\lambda_3 < 0 < \lambda_3 < \lambda_1$, so $\sigma$ is Lorenz-like for $-X$.

The proof is complete. $\square$

3.2. Invariant manifolds of a positive volume set with dominated splitting for the linear Poincaré flow

3.2.1. Invariant manifolds and (non-uniform) hyperbolicity. An embedded disc $\gamma \subset M$ is a (local) strong-unstable manifold, or a strong-unstable disc, if $\text{dist}(X^{-t}(x), X^{-t}(y))$ tends to zero exponentially fast as $t \to +\infty$, for every $x, y \in \gamma$. In the same way $\gamma$ is called a (local) strong-stable manifold, or a strong-stable disc, if $\text{dist}(X^t(x), X^t(y)) \to 0$ exponentially fast as $n \to +\infty$, for every $x, y \in \gamma$. It is well known that every point in a uniformly hyperbolic set possesses a local strong-stable manifold $W_{ss}^{loc}(x)$ and a local strong-unstable manifold $W_{uu}^{loc}(x)$ which are discs tangent to $E_s$ and $G_s$ at $x$, respectively, with topological dimensions $d_E = \dim(E)$ and $d_G = \dim(G)$, respectively. Considering the action of the flow we get the (global) strong-stable manifold

$$W^{ss}(x) = \bigcup_{t>0} X^{-t}\left(W_{ss}^{loc}(X^t(x))\right)$$

and the (global) strong-unstable manifold

$$W^{uu}(x) = \bigcup_{t>0} X^t\left(W_{uu}^{loc}(X^{-t}(x))\right)$$

for every point $x$ of a uniformly hyperbolic set. Similar notions are defined in a straightforward way for diffeomorphisms. These are immersed submanifolds with the same differentiability of the flow or the diffeomorphism. In the case of a flow we also consider the stable manifold $W^s(x) = \bigcup_{t \in \mathbb{R}} X^t(W^{ss}(x))$ and unstable manifold $W^u(x) = \bigcup_{t \in \mathbb{R}} X^t(W^{uu}(x))$ for $x$ in a uniformly hyperbolic set, which are flow invariant.

We note that these notions are well defined for a hyperbolic periodic orbit, since this compact set is itself a hyperbolic set.

Now we observe that since $A$ has positive volume, the dominated splitting of the linear Poincaré flow implies that the Lebesgue measure $\mu_A$ normalized and restricted to $A$ is a (non-uniformly) hyperbolic invariant probability measure, see, e.g., [7]: every Lyapunov exponent of $\mu_A$ is non-zero, except along the direction of the flow. Indeed (recall the arguments in the proof of lemma 3.3), the Lyapunov exponents $\lambda_1 \leq 0 \leq \lambda_2$ along every Oseledec’s regular orbit satisfy $\lambda_1 + \lambda_2 = 0$ since the flow is incompressible, and for every Oseledec’s regular orbit in $A$ (a non-empty set because $A$ has positive volume) the exponents also satisfy $\lambda_1 + 2\kappa \leq \lambda_2$ for some $\kappa > 0$ depending only on the domination strength—in particular $\kappa$ does not depend on the orbit chosen inside $\Lambda$. Thus there exists $\eta > 0$ such that $\lambda_2 = -\lambda_1 > \eta$ along every Oseledec’s regular orbit inside $\Lambda$. 


Assuming from now on that \( X \in X^{1+}(M) \) we have, according to the non-uniform hyperbolic theory (see [7, 32, 33]), that there are smooth strong-stable and strong-unstable discs tangent to the directions corresponding to negative and positive Lyapunov exponents, respectively, at \( \mu_A \) almost every point. The sizes of these discs depend measurably on the point as well as the rates of exponential contraction and expansion. We can define as before the strong-stable, strong-unstable, stable and unstable manifolds at \( \mu_A \) almost all points.

In addition, since \( \mu \) is a smooth invariant measure, we can use [8, theorem 11.3] and conclude that there are at most countably many ergodic components of \( \mu_A \). Therefore we assume from now on that \( \mu_A \) is ergodic without loss of generality.

In addition, hyperbolic smooth ergodic invariant probability measures for a \( C^{1+} \) dynamics are in the setting of Katok’s closing lemma, see [23] or [8, section 15]. In particular we have that the support of \( \mu_A \) is contained in the closure of the closed orbits inside \( A \)

\[
\text{supp}(\mu_A) \subset \overline{\text{Per}(X)} \cap A,
\]

where the periodic points in our setting are all hyperbolic by lemma 3.3.

### 3.2.2. Almost all invariant manifolds are contained in \( A \)

Now we adapt the arguments in [14] to our setting to deduce the following. Let \( \mu_s \) and \( \mu_u \) denote the measure induced on (strong-)unstable and (strong-)stable manifolds by the Lebesgue volume form \( \mu \).

**Lemma 3.6.** For \( \mu_A \) almost every \( x \) the corresponding invariant manifolds satisfy

\[
\mu_s(W^{ss}(x) \setminus A) = 0 \quad \text{and} \quad \mu_u(W^{uu}(x) \setminus A) = 0
\]

that is, the invariant manifolds are \( \mu_{s,u} \) mod 0 contained in \( A \).

In addition, since \( A \) is closed and every open subset of either \( W^{ss}(x) \) or \( W^{uu}(x) \) has positive \( \mu_s \) or \( \mu_u \) measure, respectively, then we see that in fact

\[
W^{ss}(x) \subset A \quad \text{and} \quad W^{uu}(x) \subset A \quad \text{for } \mu \text{-almost every } x.
\]

To prove lemma 3.6 we need a bounded distortion property along invariant manifolds which is provided by [8, theorems 11.1 and 11.2]. To state this properly we need the notion of hyperbolic block for a hyperbolic invariant probability measure.

### 3.2.3. Hyperbolic blocks and bounded distortion along invariant manifolds

The measurable dependence of the invariant manifolds on the base point means that for each \( \kappa \in \mathbb{N} \) we can find a compact hyperbolic block \( \mathcal{H}(\kappa) \) and positive numbers \( C_\kappa \) satisfying

- \( \text{dist}(X'(y), X'(x)) \leq C_\kappa e^{-\tau_y} \cdot \text{dist}(y, x) \) for all \( t > 0 \) and \( y \in W^{ss}_\text{loc}(x) \), and analogously for \( y \in W^{uu}_\text{loc}(x) \) exchanging the sign of \( t \);
- \( C_\kappa \leq \kappa \) and \( \tau_x \geq \kappa^{-1} \) for every \( x \in \mathcal{H}(\kappa) \);
- \( \mathcal{H}(\kappa) \subset \mathcal{H}(\kappa + 1) \) for all \( \kappa \geq 1 \) and \( \mu_A(\mathcal{H}(\kappa)) \to 1 \) as \( \kappa \to +\infty \);
- the \( C^1 \) strong-stable and strong-unstable discs \( W^{ss}_\text{loc}(x) \) and \( W^{uu}_\text{loc}(x) \) vary continuously with \( x \in \mathcal{H}(\kappa) \) (in particular the sizes of these discs and the angle between them are uniformly bounded from zero for \( x \) in \( \mathcal{H}(\kappa) \)).

Now we have the bounded distortion property.

**Theorem 3.7** ([8], theorems 11.1 and 11.2). Fix \( \kappa \in \mathbb{N} \) such that \( \mu_A(\mathcal{H}(\kappa)) > 0 \). Then the function

\[
h'(x, y) := \prod_{i \geq 0} \left| \frac{\det Df \mid E^s(f^i(x))}{\det Df \mid E^u(f^i(y))} \right|
\]
is Hölder-continuous for every \( x \in \mathcal{H}(\kappa) \) and \( y \in W^u_{\text{loc}}(x) \), where \( f := X^1 \) is the time-1 map of the flow \( X^t \) and \( E^i \) is the direction corresponding to negative Lyapunov exponents.

An analogous statement is true for a function \( h^u \) on the unstable discs in \( \mathcal{H}(\kappa) \) exchanging \( E^s \) with the direction corresponding to positive Lyapunov exponents and reversing the sign of \( i \) in the product above.

Note that since \( \mathcal{H}(\kappa) \) is compact, there exists \( 0 < h_\kappa < \infty \) such that \( \max\{h^u, h^s\} \leq h_\kappa \) on \( \mathcal{H}(\kappa) \).

### 3.2.4. Recurrent and Lebesgue density points

We are now ready to start the proof of lemma 3.6. Let us take a strong-unstable disc \( W^{uu}(x) \) satisfying simultaneously

- \( x \in \mathcal{H}(\kappa) \),
- \( \mu_u(W^{uu}(x) \cap A) > 0 \) and
- \( x \) is a \( \mu_u \) density point of \( W^{uu}(x) \cap A \).

For this it is enough to take \( \kappa \) big enough since by the absolute continuity of the foliation of strong-unstable discs a positive volume subset, as \( \mathcal{H}(\kappa) \), must intersect almost all strong-stable discs on a subset of \( \mu_u \) positive measure, see, e.g., [34].

Using the recurrence theorem we can also assume without loss of generality that \( x \) is recurrent inside \( \mathcal{H}(\kappa) \), that is, there exists a strictly increasing sequence of integers \( n_1 < n_2 < \cdots \) such that

\[
  x_k := f^{n_k}(x) \in \mathcal{H}(\kappa) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad x_k \to x.
\]

Therefore we can consider the disc \( W_k = f^{-n_k}(W^{uu}_{\text{loc}}(x_k)) \). Observe that \( W_k \subset W^{uu}(x) \) is a neighbourhood of \( x \) and since the sizes of the strong-unstable discs on \( \mathcal{H}(\kappa) \) are uniformly bounded we see that \( \text{diam}(W_k) \to 0 \) exponentially fast as \( k \to +\infty \).

Now \( W^{uu}_{\text{loc}}(x) \) is one-dimensional in our setting and thus the shrinking of \( W_k \) to \( x \) together with the \( f \)-invariance of \( A \) are enough to ensure

\[
  \frac{\mu_u(f^{-n_k}(W^{uu}_{\text{loc}}(x_k) \setminus A))}{\mu_u(f^{-n_k}(W^{uu}_{\text{loc}}(x_k)))} = \frac{\mu_u(W_k \setminus A)}{\mu_u(W_k)} \to 0.
\]

Finally the bounded distortion given by theorem 3.7 implies

\[
  \frac{\mu_u(f^{-n_k}(W^{uu}_{\text{loc}}(x_k) \setminus A))}{\mu_u(f^{-n_k}(W^{uu}_{\text{loc}}(x_k)))} = \frac{\int_{W^{uu}_{\text{loc}}(x_k) \setminus A} |\det Df^{-n_k}| |E^u(z)| d\mu_u(z)}{\int_{W^{uu}_{\text{loc}}(x_k)} |\det Df^{-n_k}| |E^u(z)| d\mu_u(z)} \geq 1 - \frac{\mu_u(W^{uu}_{\text{loc}}(x_k) \setminus A)}{\mu_u(W^{uu}_{\text{loc}}(x_k))},
\]

which means that

\[
  \frac{\mu_u(W^{uu}_{\text{loc}}(x_k) \setminus A)}{\mu_u(W^{uu}_{\text{loc}}(x_k))} \leq h_\kappa \cdot \frac{\mu_u(W_k \setminus A)}{\mu_u(W_k)}
\]

for all \( k \geq 1 \). Hence we get \( \mu_u(W^{uu}_{\text{loc}}(x) \setminus A) = 0 \) by the choice of \( x_k \) and the continuous dependence of the strong-unstable discs on the points of the hyperbolic block \( \mathcal{H}(\kappa) \). The argument for the stable direction is the same. Since the points of a full \( \mu_A \) measure subset have all the properties we used, this concludes the proof of lemma 3.6 and of the property (3.2).
3.2.5. Dense invariant manifolds of a periodic orbit. Now we use the density of periodic points in $A$ (property (3.1)). Consider again a hyperbolic block $H(\kappa) \in A$ such that $\mu_A(H(\kappa)) > 0$. For any given $x \in H(\kappa)$ and $\delta > 0$ there exists a hyperbolic periodic orbit $O(p)$ intersecting $B(x, \delta)$. Because the sizes and angles of the stable and unstable discs of points in $H(\kappa)$ are uniformly bounded away from zero, we can ensure that we have the following transversal intersections $^4$

$$W^u(p) \cap W^s(x) \neq \emptyset \neq W^s(p) \cap W^u(x).$$

This together with the inclination lemma implies that

$$\overline{W^u(p)} = \overline{W^s(x)} \subset A \quad \text{and} \quad W^s(p) = \overline{W^u(x)} \subset A.$$

(3.3)

Moreover since we can pick any $x \in H(\kappa)$ we can assume without loss that $x$ has a dense orbit in $A$ (since we took $\mu_A$ to be ergodic) and then we can strengthen (3.3): there exists a periodic orbit $O(p)$ inside $A$ such that

$$W^u(p) = A \quad \text{and} \quad W^s(p) = A.$$ (3.4)

3.2.6. Absence of singularities in $A$. We recall that the alpha-limit set of a point $p \in M$ with respect to the flow $X$ is the set $\alpha(p)$ of all limit points of $X^{-t}(p)$ as $t \rightarrow +\infty$. Likewise the omega-limit set is the set $\omega(p)$ of limit points of $X^t(p)$ when $t \rightarrow +\infty$. Both these sets are flow-invariant.

Using property (3.4) we consider, on the other hand, the invariant compact subset of $A$ given by

$$L = \alpha_X(W^{ss}(p))$$

the closure of the accumulation points of backward orbits of points in the strong-stable manifold of the periodic orbit $O(p)$. By (3.4) we have $L = A$. On the other hand, considering $N = \omega_X(W^{uu}(p))$ we likewise obtain that $N = A$.

Let us assume that $\sigma$ is a singularity contained in $A$. By lemma 3.4 $\sigma$ is either Lorenz-like for $X$ or Lorenz-like for $-X$.

In the former case, we would get $W^{ss}(\sigma) \subset A$ because any compact part of the strong-stable manifold of $\sigma$ is accumulated by backward iterates of a small neighbourhood $\gamma$ inside $W^{ss}(x)$. Here we are using that the contraction along the strong-stable manifold, which becomes an expansion for negative time, is uniform. In the latter case we would get $W^{uu}(\sigma) \subset A$ by a similar argument reversing the time direction.

We now explain that each one of these possibilities leads to a contradiction with the dominated splitting of the linear Poincaré flow on the regular orbits of $A$, following an argument in [28]. It is enough to deduce a contradiction for a Lorenz-like singularity for $X$, since the other case reduces to this one through a time inversion.

If $W^{ss}(\sigma) \cap A \setminus \{\sigma\} \supset \{y\}$ for some point $y \in A$ and for some singularity $\sigma \in A$, then we have countably distinct regular orbits of $A$ accumulating on $y \in W^{ss}(x)$ (by the definition of $A$) and on a point $q \in W^{ss}(\sigma)$ (by the dynamics of the flow near $\sigma$).

Applying the connecting lemma, we obtain a saddle-connection associated with the continuation of $\sigma$ for a $C^1$-close vector field $Y$, known as ‘orbit-flip’ connection, that is, there is a homoclinic orbit $\Gamma$ associated with $\sigma_Y$ such that $W^{ss}(\sigma_Y)$ intersects $W^s(\sigma_Y)$ transversely along $\Gamma$, i.e., $\Gamma = W^{ss}(\sigma_Y) \cap W^s(\sigma_Y)$, and also $\Gamma \cap W^{uu}(\sigma_Y) \neq \emptyset$.

These connections can be $C^1$ approximated by ‘inclination-flip’ connections for another $C^1$ nearby vector field $Z$, not necessarily conservative, see, e.g., [3, 27]. This means that the

$^4$ Recall the difference between $W^{uu}(p)$ and $W^u(p)$ etc. in the flow setting.
continuation $\sigma_Z$ of the singularity has an associated homoclinic orbit $\gamma$ such that $W^u(\sigma_Z)$ intersects $W^s(\sigma_Z)$ along $\gamma$ but not transversely, and $\gamma \cap W^{st}(\sigma_Z) = \emptyset$.

However, the presence of ‘inclination-flip’ connections is an obstruction to the dominated decomposition of the linear Poincaré flow for nearby regular orbits. This contradicts lemma 2.3 and concludes the proof of proposition 3.1.

4. Uniform hyperbolicity

Here we conclude the proof of theorem C, showing that proper invariant hyperbolic subsets of a $C^{1+}$ incompressible flow cannot have positive volume.

**Proposition 4.1.** Let $A$ be a compact invariant hyperbolic subset for $X \in \mathcal{X}^{1+}_\mu(M)$. Then either $\mu(A) = 0$ or else $X$ is an Anosov flow and $A = M$.

The proof of proposition 4.1 is given as a sequence of intermediate results along the rest of this section. Assuming this result we easily have the following.

**Proof of theorem C.** From corollary 3.2 we have that a regular invariant subset with positive volume with dominated splitting for the linear Poincaré flow admits a positive volume subset which is hyperbolic. Therefore the flow of $X$ is Anosov from proposition 4.1. □

4.1. Positive volume hyperbolic sets and conservative Anosov flows

We start the proof by recalling the notion of partial hyperbolicity.

Let $\Lambda$ be a compact invariant subset for a $C^1$ flow on a compact boundaryless manifold $M$ with dimension at least 3. We say that $\Lambda$ is **partially hyperbolic** if there are a continuous invariant tangent bundle decomposition $T\mathcal{X}M = E^s \oplus E^c$ and constants $\lambda, K > 0$ such that for all $x \in \Lambda$ and for all $t \geq 0$

- $E^c$ dominates $E^s$: $\|DX^c(x) \cdot E^s_x\| \cdot \|DX^{-t} \cdot E^c_{\alpha(x)}\| \leq Ke^{-\lambda t}$
- $E^s$ is uniformly contracting: $\|DX^s\cdot E^s_x\| \leq Ke^{-\lambda t}$.

We note that for a partially hyperbolic set of a flow the flow direction must be contained in the central bundle.

Now we recall the following result.

**Theorem 4.2 ([1], theorem 2.2).** Let $f : M \to M$ be a $C^{1+}$ diffeomorphism and let $\Lambda \subset M$ be a partially hyperbolic set with positive volume. Then $\Lambda$ contains a strong-stable disc.

Now we can use an argument similar to the one presented in section 3.2.6.

**Lemma 4.3.** Let $X \in \mathcal{X}^{1+}_\mu(M)$ and $\Lambda$ be a compact invariant partially hyperbolic subset containing a strong-stable disc $\gamma$. Then $L = \alpha_X(\gamma) = \{\alpha(z) : z \in \gamma\}$ contains all stable discs through its points.

**Proof.** The partial hyperbolic assumption on $A$ ensures that every one of its points has a strong-stable manifold. Moreover

$$W^{ss}(z) \subset \Lambda \quad \text{for every } z \in \alpha(\gamma),$$  \hspace{1cm} (4.1)

since any compact part of the strong-stable manifold of $z$ is accumulated by backward iterates of any small neighbourhood of $x \in \gamma$ inside $W^{ss}(x)$. Here we are using that the contraction along the strong-stable manifold, which becomes an expansion for negative time, is uniform. □
Proof of proposition 4.1. Let $A$ be a hyperbolic subset for $X \in \mathcal{X}^1_h(M)$ with $\mu(A) > 0$. From lemma 4.3 we have that $L = \omega(\gamma)$ satisfies $W^{ss}(L) = \{W^{ss}(z) : z \in L\} \subset L$. This implies $W^s(L) = L$ by invariance.

Consider now $W^u(L) = \{W^u(z) : z \in L = W^s(L)\}$. This collection of unstable leaves crossing the stable leaves of $L$ forms a neighbourhood of $L$. But $U = W^u(L)$ being a neighbourhood of $L$ means that $L$ is a repeller: for $w \in U$ we have $\text{dist}(X^{-t}(w), L) \to 0$ as $t \to +\infty$.

This contradicts the preservation of the volume form $\mu$, unless $L$ is the whole of $M$. Thus $M = L \subset A$ and $X$ is Anosov.

\hfill $\square$

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References

[1] Alves J F, Araújo V, Pacífico M J and Pinheiro V 2007 On the volume of singular-hyperbolic sets Dyn. Syst. Int. J. 22 249–67
[2] Alves J F and Pinheiro V 2008 Topological structure of (partially) hyperbolic sets with positive volume Trans. Am. Math. Soc. at press
[3] Araújo V and Pacífico M J 2007 Three dimensional flows 25th Brazilian Mathematical Colloquium (Rio de Janeiro, Brazil) (Rio de Janeiro, Brazil: IMPA)
[4] Arbieto A and Matheus C 2007 A pasting lemma and some applications for conservative systems Ergod. Theory Dyn. Syst. 27 1399–417
[5] Arnaud M-C 2002 The generic symplectic $C^1$-diffeomorphisms of four-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point Ergod. Theory Dyn. Syst. 22 1621–39
[6] Avila A and Bochi J 2006 A generic $C^1$ map has no absolutely continuous invariant probability measure Nonlinearity 19 2717–25
[7] Barreira L and Pesin Y 2001 Lectures on Lyapunov exponents and smooth ergodic theory Proc. Symp. Pure Math. 69 3–106
[8] Barreira L and Pesin Y 2006 Smooth ergodic theory and nonuniformly hyperbolic dynamics Handbook of Dynamical Systems vol 1B (Amsterdam: Elsevier) pp 57–263 (With an appendix by Omri Sarig)
[9] Bessa M 2007 The Lyapunov exponents of generic zero divergence 3-dimensional vector fields Ergod. Theory Dyn. Syst. 27 1445–72
[10] Bessa M and Duarte P 2007 Abundance of elliptic dynamics on conservative 3-flows Dyn. Syst. Int. J. submitted Preprint 0709.1070v1
[11] Bessa M and Rocha J 2007 On $C^1$-robust transitivity of volume-preserving flows J. Diff. Equa Preprint 0707.2554v1
[12] Biragov V S and Shil’nikov L P 1992 On the bifurcation of a saddle-focus separatix loop in a three-dimensional conservative dynamical system Sel. Math. Sov. 11 333–40
[13] Bochi J 2002 Genericity of zero Lyapunov exponents Ergod. Theory Dyn. Syst. 22 1667–96
[14] Bochi J and Viana M 2004 Lyapunov exponents: how frequently are dynamical systems hyperbolic? Advances in Dynamical Systems (Cambridge: Cambridge University Press)
[15] Bochi J and Viana M 2005 The Lyapunov exponents of generic volume-preserving and symplectic maps Ann. Math. (2) 161 1423–85
[16] Bonatti C, Díaz J J and Viana M 2005 A global geometric and probabilistic perspective Mathematical Physics, III Dynamics Beyond Uniform Hyperbolicity (Encyclopedia of Mathematical Sciences vol 102) (Berlin: Springer)
Dominated splitting and zero volume 1653

[17] Bowen R and Ruelle D 1975 The ergodic theory of Axiom A flows Invent. Math. 29 181–202
[18] Doering C J 1987 Persistently transitive vector fields on three-dimensional manifolds Proc. Dynamical Systems and Bifurcation Theory (Rio de Janeiro, Brazil) vol 160 (Boston, MA: Pitman) pp 59–89
[19] Franks J 1971 Necessary conditions for stability of diffeomorphisms Trans. Am. Math. Soc. 158 301–8
[20] Gallavotti G 2002 Foundations of Fluid Dynamics (Berlin: Springer)
[21] Hirsch M, Pugh C and Shub M 1977 Invariant Manifolds (Lecture Notes in Mathematics vol 583) (New York: Springer)
[22] Hontz V and Tahzibi A 2006 Partial hyperbolicity for symplectic diffeomorphisms Ann. Inst. Henri Poincaré 23 641–61
[23] Katok A 1980 Lyapunov exponents, entropy and periodic orbits for diffeomorphisms Inst. Hautes Études Sci. Publ. Math. 51 137–73
[24] Mañé R 1984 Oseledec’s theorem from the generic viewpoint Proc. Int. Congress of Mathematicians (Warsaw, 1983) vol 1 and 2 (Warsaw: PWN) pp 1269–76
[25] Mañé R 1987 Ergodic Theory and Differentiable Dynamics (New York: Springer)
[26] Mañé R 1996 The Lyapunov exponents of generic area preserving diffeomorphisms Int. Conf. on Dynamical Systems (Montevideo, Uruguay 1995) (Pitman Research Notes in Mathematics Series vol 362) (Harlow: Longman) pp 110–9
[27] Morales C A and Pacifico M J 2001 Inclination-flip homoclinic orbits arising from orbit-flip Nonlinearity 14 379–93
[28] Morales C A, Pacifico M J and Pujals E R 2004 Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellors Ann. Math. (2) 160 375–432
[29] Munroe M E 1953 Introduction to Measure and Integration (Cambridge, MA: Addison-Wesley)
[30] Newhouse S E 1977 Quasi-elliptic periodic points in conservative dynamical systems Am. J. Math. 99 1061–87
[31] Oseledec V I 1968 A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems Trans. Moscow Math. Soc. 19 197–231
[32] Pesin Y 1976 Families of invariant manifolds corresponding to non-zero characteristic exponents Math. USSR. Izv. 10 1261–302
[33] Pesin Y B 1977 Characteristic Lyapunov exponents and smooth ergodic theory Russ. Math. Surv. 324 55–114
[34] Pugh C and Shub M 1989 Ergodic attractors Trans. Am. Math. Soc. 312 1–54
[35] Pugh C C and Robinson C 1983 The C 1 closing lemma, including Hamiltonians Ergod. Theory Dyn. Syst. 3 261–313
[36] Robinson R C 1970 Generic properties of conservative systems Am. J. Math. 92 562–603
[37] Rudin W 1987 Real and Complex Analysis 3rd edn (New York: McGraw-Hill)
[38] Saghin R and Xia Z 2006 Partial hyperbolicity or dense elliptic periodic points for C 1 -generic symplectic diffeomorphisms Trans. Am. Math. Soc. 358 5119–38
[39] Wen L and Xia Z 2000 C 1 connecting lemmas Trans. Am. Math. Soc. 352 5213–30
[40] Zuppa C 1979 Regularisation C∞ des champs vectoriels qui prèservent l’élément de volume Bull. Braz. Math. Soc. 10 51–6