Symmetries and causes of the coincidence of the radiation spectra of mirrors and charges in 1 + 1 and 3 + 1 spaces

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Abstract

This paper discusses the symmetry of the wave field that lies to the right and left of a two-sided accelerated mirror in 1 + 1 space and satisfies a single condition on it. The symmetry is accumulated in the Bogolyubov matrix coefficients $\alpha$ and $\beta$ that connect the two complete sets of solutions of the wave equations. The amplitudes of the quantum processes in the right and left half-spaces are expressed in terms of $\alpha$ and $\beta$ and are related to each other by transformation (12). Coefficient $\beta^{*}_{\omega'}\omega$ plays the role of the source amplitude of a pair of particles that are directed to opposite sides with frequencies $\omega$ and $\omega'$ but that are in either the left or the right half-space as a consequence of the reflection of one of them. Such an interpretation makes $\beta^{*}_{\omega'}\omega$ observable and explains the equalities, given by Eq. (1) and found earlier by Nikishov and author [Zh. Eksp. Teor. Fiz. 108, 1121 (1995)] and by author [Zh. Eksp. Teor. Fiz. 110, 526 (1996)] that the radiation spectra of a mirror in 1+1 space coincide with those of charges in 3 + 1 space by the fact that the moment of the pair emitted by the mirror coincide with the spin of the single particle emitted by the charge.

1 Introduction

It was found in Refs.[1] and [2] that the spectra of bosons and fermions emitted by an accelerated mirror in 1 + 1 space coincide with the spectra of photons and scalar quanta emitted by electric and scalar charges in 3 + 1 space when the latter move along the same trajectory as does the mirror. Namely, the Bogolyubov coefficients $\beta_{\omega\omega}^{B,F}$ that describe the spectra of the Bose and Fermi radiations of an accelerated mirror and the Fourier transforms of the density of the 4-current $j_{\alpha}(k_+,k_-)$ and the scalar charge density $\rho(k_+,k_-)$ that describe the spectra of the photons and scalar quanta emitted by electric and scalar charges are connected by the relationships

$$|\beta_{\omega\omega}^{B}|^2 = \frac{1}{e^2}|j_{\alpha}(k_+,k_-)|^2, \quad |\beta_{\omega\omega}^{F}|^2 = \frac{1}{e^2}|\rho(k_+,k_-)|^2.$$  (1)
It is assumed here that the components \( k_\pm = k^0 \pm k^1 \) of the wave 4-vector \( k^\alpha \) of the quantum emitted by the charge are identified with the doubled frequencies \( \omega \) and \( \omega' \) of the quanta emitted by the mirror:

\[
2\omega = k_+, \quad 2\omega' = k_-,
\]

and \( e \) is the electrical or scalar charge in Heaviside units.

However, there is a substantial physical difference between the right-hand and left-hand quantities in Eqs. (1), i.e., between the radiation spectra of the charges and of the mirror. Whereas the former are the distribution of the mean number of radiated quanta over the two independent components \( k_+ \) and \( k_- \) of the wave vector of the quantum (as a consequence of the azimuthal symmetry of the radiation, there is no dependence on the third independent variable), the latter have a more complex interpretation. Actually, they will be the spectra of the mean number of quanta emitted by the mirror to the right only after integration over frequency \( \omega' \) [3]:

\[
d\bar{n}_\omega = \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega'}{2\pi} |\beta\omega'|^2.
\]

(3)

If the mirror is two-sided and infinitely thin, then, besides the quanta emitted to the right with the spectrum given by Eq. (3), it will (as we shall see) also emit quanta to the left with the spectrum

\[
d\bar{n}'_\omega = \frac{d\omega'}{2\pi} \int_0^\infty \frac{d\omega}{2\pi} |\beta\omega'|^2.
\]

(4)

The thought naturally arises whether it is not possible to regard the quantity

\[
|\beta\omega'|^2 \frac{d\omega d\omega'}{(2\pi)^2}
\]

as the mean number of pairs of quanta, one of which, with frequency \( \omega \) in the interval \( d\omega \), is emitted by the mirror to the right, while the other, with frequency \( \omega' \) in the interval \( d\omega' \), is emitted to the left. In this case, two frequencies \( \omega \) and \( \omega' \) would be observable, characterizing one event: the emission of a pair of quanta by the mirror, in the same way as two components \( k_+ \) and \( k_- \) also characterize one event: the emission of one quantum by a charge. As we can see, with certain nontrivial complications, such a treatment is actually valid. In any case, the mirror emits quanta in pairs.

It is elucidated that this circumstance helps to understand another difference between the coincident spectra of a charge and a mirror. While the bosons and fermions emitted by a mirror have a spin of 0 and 1/2, the photons and scalar quanta emitted by electrical and scalar charges have a spin of 1 and 0. Even though the quanta have different spin, the radiation spectra of the charges coincide with the boson and fermion spectra of the mirror.

This coincidence is explained by the fact that, unlike charges, the mirror emits particles in pairs, and a pair of spinless bosons can have a total moment of 1, while a pair of fermions can have a total moment of 0. Then the moment of the pair emitted by the mirror coincides with the spin of the particle emitted by the charge. The fact that, upon reflection, \( \beta^B_{\omega'\omega} \) behaves...
like a pseudoscalar while $\beta_{\omega}^P$ behaves like a scalar can serve as an indirect confirmation of this (see Sections 2 and 4).

It is shown in Section 2 that the system of Bogolyubov coefficients obtained for a right-sided mirror (i.e., for the field to the right of a mirror with a boundary condition on it), because of the properties of mirror symmetry, also describes the processes in the field to the left of a mirror with the same boundary condition. In other words, the same system of Bogolyubov coefficients characterizes the behavior of the field in all of space—both to the right and to the left of a two-sided mirror.

Section 3 describes the connection between the integral quantities that characterize the radiation of a two-sided mirror, their behavior under certain space–time transformations, and the symmetry (or asymmetry) of the space–time regions where they are formed.

The symmetry of the Bogolyubov coefficients reflects the symmetry of two inequivalent complete systems of solutions of wave equation, definite and smooth in all of $1 + 1$ space, satisfying inside it—on the trajectory of the mirror—a single condition and characterized by propagation of a monochromatic component of each solution toward the right in one system and toward the left in the other. When the field is quantized and when the usual comparison of monochromatic plane waves to particles is made, these two systems of solutions form in and out systems for the field to the right of the trajectory and out and in systems for the field to the left of it. Therefore, the quantum processes in the field to the right and to the left of the mirror are independent, even though they are described by a single system of Bogolyubov coefficients. In particular, the particle-production amplitudes to the right and to the left of the mirror, the single-particle scattering amplitudes in these regions, etc. are connected with transformation (12). Such amplitudes, certain frequency distributions, and also the distribution of pair-production probabilities over the number of pairs, which is invariant relative to transformation (12), are computed in Section 4. It is shown that $\beta_{\omega}^*\omega^\prime$ plays the role of the source amplitude of a pair of particles potentially emitted to the right and to the left with frequencies $\omega$ and $\omega^\prime$, with the spin of a boson pair equaling 1, while that of the fermion pair equals 0.

In the last section, Sec. 5, a similar method is used to treat the emission by an accelerated mirror of pairs the particle and antiparticle of which are not identical.

A system of units in which $\hbar = c = 1$ is used in this article. To simplify the formulas in Sections 4 and 5, the frequencies are considered discrete, integration over $d\omega/2\pi$ is replaced by summation over $\omega$, and the delta function $2\pi\delta(\omega - \omega'')$ is replaced by the Kronecker symbol $\delta_{\omega\omega''}$.

### 2 Symmetry of the Bogolyubov coefficients and radiation of accelerated two-sided mirror

Let us consider the connection between radiation spectra and other quantities in two problems in which the mirror trajectories $x = \xi_1(t)$ and $x = \xi_2(t)$ differ by reflection: $\xi_1(t) = -\xi_2(t)$. Then, if the first trajectory is described on the plane of variables $u = t - x$, $v = t + x$ by the function $v = v_1 = f(u)$, the second trajectory will be described by the function inverse
Similarly, the Bogolyubov coefficients for a fermion field $f$ replaced by formations of Eqs. (8) and (11), i.e., the transition from trajectory $v$ to $g$, is absorbed. Then, according to Eqs. (8) and (11), these last are also equal to $\alpha_{\omega'\omega}[f]$ and $\beta_{\omega'\omega}[f]$ if the frequencies of the monochromatic waves propagating to the right and to the left are denoted as $\omega$ and $\omega'$, as was assumed for the field to the right of the trajectory.

Thus, for the field to the left of a mirror moving along trajectory $f(u)$, $\alpha_{\omega'\omega}[f]$ and $\beta_{\omega'\omega}[f]$ are the amplitudes of waves with frequencies $\omega'$ and $-\omega'$ contained in the reflected...
part of the \textit{in} wave with frequency $\omega$, while $\alpha^*_{\omega'}[f]$ and $\mp \beta_{\omega'}[f]$ are the amplitudes of waves with frequencies $\omega$ and $-\omega$ contained in the incident part of the \textit{out} wave with frequency $\omega'$. Therefore, the matrix that connects the \textit{in} and \textit{out} waves of the field to the left of the mirror differs from the analogous matrix for the field to the right of it by transformation (12).

So the transition from trajectory $f(u)$ to the mirror-symmetric $g(u)$ is equivalent to considering the field on the part of the Minkowski plane not to the right but to the left of trajectory $f(u)$ with the previous boundary condition on the mirror.

The mean number of particles formed by a two-sided infinitely thin mirror on the left part of the Minkowski plane is the same as on the right, since the integral

$$ N = \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} |\beta_{\omega'}|^2 $$

(13)

does not change when $\beta_{\omega'}$ is replaced by $\mp \beta_{\omega'}$. At the same time, the energy

$$ \mathcal{E}' = \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} \omega' |\beta_{\omega'}|^2 $$

(14)

emitted by the mirror to the left, generally speaking, is not equal to the energy

$$ \mathcal{E} = \int_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} \omega |\beta_{\omega'}|^2 $$

(15)

emitted to the right.

The equality of the mean numbers of particles emitted by a two-sided accelerated mirror to the right and to the left suggests that the particles are generated in pairs and fly off in opposite directions. Quantity (5) is usually considered as the mean number of actual quanta with frequency $\omega$ in the interval $d\omega$, emitted to the right when a quantum with frequency $\omega'$ in the interval $d\omega'$ is absorbed from the vacuum from the right. The question arises whether it is not possible to regard the same quantity as the mean number of pairs of quanta emitted to the right and to the left with frequencies $\omega$ and $\omega'$ in the intervals $d\omega$ and $d\omega'$, respectively. In other words, is $N^{-1}|\beta_{\omega'}|^2$ the two-dimensional probability distribution of frequencies $\omega$ and $\omega'$ of two quanta escaping to the right and to the left with momenta $\omega$ and $-\omega'$?

Such an interpretation of the frequency distribution of bosons (fermions) emitted by a mirror in $1+1$ space would make the coincidence of this distribution with the radiation spectrum of an electric (scalar) charge in $3+1$ space detected in Refs. [1] and [2] less formal. Although, in the case of mirror emission, the random quantities are the frequencies $\omega$ and $\omega'$ of two bosons (fermions) escaping in different directions, while in the case of charge emission, the random quantities are the components $k_+$ and $k_-$ of the wave vector of one vector (scalar) quantum emitted to the right or to the left, corresponding to the sign of $k_+ - k_- > 0$ or $< 0$.

Let us give two more evidences of left–right symmetry of the wave field of an accelerated mirror that are reflected by the Bogolyubov coefficients.
First, Eqs. (6) and (7) or (9) and (10), obtained for the field to the right of the mirror for the Bogolyubov coefficients, represent $|\beta_{\omega,\omega'}|^2$ by a double integral over the entire $uv$ plane, as shown by Ref. [2]. Thus

$$|\beta_{\omega,\omega'}^B|^2 = -\text{Re} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \exp [i\omega u + i\omega' f(u) - i\omega' v - i\omega g(v)],$$

while $|\beta_{\omega,\omega'}^F|^2$ differs from Eq. (16) by an additional factor of $-\sqrt{f'(u)g'(v)}$ under the integral. Similarly, in the double integral for the mean number of particles emitted to the right,

$$N_{B,F}^{B,F} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} du K_{B,F}^{B,F}(u),$$

the integration is carried out over the entire $uv$ plane, i.e., over all of Minkowski space, and not over the part of it lying to the right of the trajectory. The wave fields to the right and to the left of the trajectory that satisfy the same condition on it are described by a single analytical function and therefore are not independent. Therefore, the frequencies of the quanta emitted to the right and to the left are also not independent.

Second, the mean energies $\mathcal{E}$ and $\mathcal{E}'$ emitted to the right and to the left, according to Ref. [2], can be represented as integrals over the proper time $\tau$ of the mirror:

$$\mathcal{E}^B = \frac{1}{12\pi} \int_{-\infty}^{\infty} d\tau a^2 \sqrt{f'} - d(a\sqrt{f'}),$$

$$\mathcal{E}'^B = \frac{1}{12\pi} \int_{-\infty}^{\infty} d\tau a^2 \sqrt{f'} + d\left(\frac{a}{\sqrt{f'}}\right),$$

Here $a$ is the acceleration of the mirror in its proper system.

The first terms under the integral in Eqs. (19) and (20) represent the energy irreversibly emitted by the mirror respectively to the right and to the left of the section $d\tau$ of the trajectory. In the mirror’s proper system, these portions of the energy are identical and equal $a^2 d\tau/12\pi$, whereas the portions of irreversibly emitted momentum equal $\pm a^2 d\tau/12\pi$. In the laboratory system, these portions of the energy, because of the opposite directions of their motion with respect to the velocity $\beta$ of the source, acquire Doppler factors $\sqrt{f'}$ and $1/\sqrt{f'}$. We recall that $\sqrt{f'} = \sqrt{(1+\beta)/(1-\beta)}$. The second, Schott terms under the integrals in Eqs. (19) and (20) “smear” the formation region of radiation, as a result of
which, for the formation of radiation of energy, such intervals $\Delta \tau$ on which the irreversibly emitted energy exceeds the change of the Schott energy are substantial; i.e.,

$$\Delta \tau a^2 \sqrt{f'} > |a \sqrt{f'}|, \quad \Delta \tau \frac{a^2}{\sqrt{f'}} > \left| \frac{a}{\sqrt{f'}} \right|,$$

or $\Delta \tau > a^{-1}$; the proper time interval must be greater than the inverse proper acceleration. The proper acceleration determines the characteristic frequency of the radiation in the proper system and its scatter: $\omega \sim \Delta \omega \sim a$. Therefore, the condition $\Delta \tau a > 1$ is equivalent to the indeterminacy relation $\Delta \tau \Delta \omega > 1$.

### 3 Symmetry and the relations of certain integral quantities

The following representations for the mean number $N$ of emitted particles and the mean emitted energies $E = E_+$ and $E' = E_-$ are convenient for explaining their properties with respect to certain space–time transformations:

$$N^B = \int_{-\infty}^{\infty} dudv S(u, v), \quad N^F = -\int_{-\infty}^{\infty} dudv \sqrt{f'(u)g'(v)} S(u, v),$$

$$E_\pm^B = \int_{-\infty}^{\infty} dudv A_\pm(u, v), \quad E_\pm^F = -\int_{-\infty}^{\infty} dudv \sqrt{f'(u)g'(v)} A_\pm(u, v),$$

where $S$ and $A_\pm$ are singular functions ($\varepsilon, \delta \to +0$):

$$S(u, v) = \frac{1}{8\pi^2} \left[ \frac{1}{(v - f(u) - i\varepsilon)(g(v) - u - i\delta)} + \text{c.c.} \right],$$

$$A_+(u, v) = \frac{1}{8\pi^2 i} \left[ \frac{1}{(v - f(u) - i\varepsilon)(g(v) - u - i\delta)^2} - \text{c.c.} \right],$$

$$A_-(u, v) = \frac{1}{8\pi^2 i} \left[ \frac{1}{(v - f(u) - i\varepsilon)^2(g(v) - u - i\delta)} - \text{c.c.} \right].$$

1. Lorentz transformations. The quantities $S(u, v)$, $\sqrt{f'(u)g'(u)}$, and $dudv$ are scalars with respect to the Lorentz transformations, while $A_\pm(u, v)$ transform as the $\pm$ components of a vector. Therefore, $N^{B,F}$ are Lorentz invariants, while $E^{B,F}_\pm$ are the $\pm$ components of a vector.

2. Mirror symmetry. When the trajectory is replaced by a mirror-symmetric trajectory, $f(u) \to g(u)$, $g(v) \to f(v)$, the integrals $N[f]$ and $E_\pm[f]$ transform, respectively, to

$$N[g] = N[f], \quad E_\pm[g] = E_\mp[f],$$

(27)
since, for such a replacement,

\[ S(u, v) \rightarrow S(v, u), \quad A_\pm(u, v) \rightarrow A_\mp(v, u), \quad \sqrt{f'(u)g'(v)} \rightarrow \sqrt{g'(u)f'(v)}, \]

after which the transformed integrals \( N \) and \( \mathcal{E}_\pm \) differ from the untransformed \( N \) and \( \mathcal{E}_\pm \) only in the designation of the variables of integration. Thus, the mean numbers of particles emitted from the same trajectory to the right and to the left are identical and do not change when the trajectory is replaced by the mirror-symmetric one, while the mean energies emitted to the right and to the left are different and transform into each other when such replacement is made.

3. Synchromirror transformation. This discrete transformation consists of replacing co-ordinates \( u \) and \( v \) with the coordinates

\[ \tilde{u} = g(v), \quad \tilde{v} = f(u), \quad \text{so that} \quad u = g(\tilde{v}), \quad v = f(\tilde{u}), \] (28)

Points \((u, v)\) and \((\tilde{u}, \tilde{v})\), which are related by transformation (28), lie on the Minkowski plane on different sides of the trajectory of the mirror on the intersection of the light cones whose vertices are found on the trajectory at points \( A(u, f(u)) \) and \( B(g(v), v) \). The points of any compact region lying to the right of the trajectory are mapped one-to-one into the points of a compact region lying to the left of the trajectory.

Functions \( S(u, v) \) and \( A_\pm(u, v) \) are form-invariant relative to transformation (28); i.e.,

\[ S(u, v) \equiv S(g(\tilde{v}), f(\tilde{u})) = S(\tilde{u}, \tilde{v}), \quad A_\pm(u, v) = A_\mp(\tilde{u}, \tilde{v}). \] (29)

Since the area element \( dudv\sqrt{f'(u)g'(v)} \) appearing in the Fermi integrals \( N^F \) and \( \mathcal{E}^F_\pm \) is also form-invariant relative to transformation (28); i.e.,

\[ dudv\sqrt{f'(u)g'(v)} = d\tilde{u}d\tilde{v}\sqrt{f'(\tilde{u})g'(\tilde{v})}, \] (30)

the contributions to the Fermi intervals from any two regions on the \( uv \) plane related by symmetry transformation (28) are identical. In particular, the contributions from the entire region to the right and the entire region to the left of the trajectory are identical.

In the Bose integrals \( N^B \) and \( \mathcal{E}^B_\pm \), the contributions of the right-hand and left-hand regions related by transformation (28) are, generally speaking, different, since the area element \( dudv \) that appears in these integrals, unlike the functions \( S \) and \( A_\pm \) being integrated, is mapped by transformation (28) into the unequal element \( d\tilde{u}d\tilde{v} \):

\[ dudv = d\tilde{u}d\tilde{v}f'(\tilde{u})g'(\tilde{v}), \quad d\tilde{u}d\tilde{v} = dudvf'(u)g'(v). \] (31)

Therefore, the contributions to the Bose integrals from these two elementary areas are proportional to their areas; i.e., their ratio equals the Jacobian of the transformation.

Transformation (28) of the variables of integration of course does not change the values of the integrals \( N \) and \( \mathcal{E}_\pm \). Its meaning is that the local contributions to \( N \) and \( \mathcal{E}_\pm \) from any pair of right-hand and left-hand regions associated by transformation (28) have a definite symmetry or asymmetry. Namely, for Fermi integrals, this symmetry consists of the equality of such contributions, whereas, for Bose integrals, it consists of left–right asymmetry of the contributions, determined by the Jacobian of the transformation.
Radiation of two-sided mirror, quantum approach

For a consistent description of the quantum wave field lying both to the right and to the left of the mirror and satisfying a single condition on the mirror, it is convenient to use the two complete sets \{φ_{\text{out }\omega}, φ_{\text{out }\omega}^*\} and \{φ_{\text{in }\omega}, φ_{\text{in }\omega}^*\} of solutions of the wave equation, given in Refs. [1] and [2]. Possessing in the right-hand Minkowski plane the physical meaning of the \textit{out} and \textit{in} sets and satisfying the boundary condition on the mirror, these solutions can be smoothly extended into the left half-plane with no change of their functional form. However, in the left half-plane, these sets acquire the physical meaning of the \textit{in} and \textit{out} sets, respectively, and they must be designated as \{φ_{\text{in }\omega}, φ_{\text{in }\omega}^*\} and \{φ_{\text{out }\omega}, φ_{\text{out }\omega}^*\}.

Each such solution is actually unambiguously characterized by the frequency \(\omega\) or \(\omega'\) of its monochromatic component travelling to the right or to the left and by the condition on the mirror. For a Lorentzian transformation with velocity \(\beta\) along the \(x\) axis, the frequencies \(\omega\) and \(\omega'\) transform into \(\tilde{\omega}\) and \(\tilde{\omega}'\) according to the mutually inverse laws

\[
\tilde{\omega} = D^{-1}(\beta) \omega, \quad \tilde{\omega}' = D(\beta) \omega', \quad D(\beta) = \sqrt{\frac{1+\beta}{1-\beta}},
\]

where \(D(\beta)\) is the Doppler factor. \(\omega\) and \(\omega'\) thus possess the opposite covariance. Below, frequencies that transform like \(\omega\) will be equipped with an even number of primes, while those that transform like \(\omega'\) will have an odd number. Then the subscript \textit{in} or \textit{out}, in addition to the frequency, will simply indicate the side of the Minkowski plane on which the solution is considered.

The expansion of the solutions of the first set in the solutions of the second set and the inverse expansion have been written by us (in the right-hand half-plane) in the form

\[
φ_{\text{out }\omega} = α_\omega φ_{\text{in }\omega} + β_\omega φ_{\text{in }\omega}^*, \quad φ_{\text{in }\omega}' = α_\omega^* φ_{\text{out }\omega} + β_\omega^* φ_{\text{out }\omega}^*,
\]

or, if matrix notation is used,

\[
\begin{pmatrix}
φ_{\text{out}} \\
φ_{\text{out}}^*
\end{pmatrix} = \begin{pmatrix}
α & β \\
β^* & α^*
\end{pmatrix} \begin{pmatrix}
φ_{\text{in}} \\
φ_{\text{in}}^*
\end{pmatrix}, \quad \begin{pmatrix}
φ_{\text{in}} \\
φ_{\text{in}}^*
\end{pmatrix} = \begin{pmatrix}
α^* & β \\
β^* & α
\end{pmatrix} \begin{pmatrix}
φ_{\text{out}} \\
φ_{\text{out}}^*
\end{pmatrix}.
\]

As a consequence of the orthogonality and normalization of the solutions in both sets, the matrices that appear in Eqs. (35) are mutually inverse. This means that the Bogolyubov coefficients satisfy four independent matrix relations:

\[
α^+ α \mp β^+ β = 1, \quad β^+ α^* \mp α^+ β^* = 0,
\]

\[
αα^* \mp β^* β = 1, \quad αβ^+ \mp β* α = 0.
\]

On the left-hand half-plane, Eqs. (33)–(35) are conserved, but a new physical meaning requires the interchange of the subscripts \textit{in} ⇀ \textit{out} in the functions, which is equivalent to transformation (12).
For a quantized field in the right half-plane, the connection of the \( \text{in} \) and \( \text{out} \) absorption and creation operators \( a \) and \( a^+ \) is given by the Bogolyubov transformations

\[
\begin{pmatrix}
    a_{\text{in}} \\
    a_{\text{in}}^+
\end{pmatrix} = \begin{pmatrix}
    \alpha & \beta^* \\
    \beta & \alpha^*
\end{pmatrix}
\begin{pmatrix}
    a_{\text{out}} \\
    a_{\text{out}}^+
\end{pmatrix},
\quad
\begin{pmatrix}
    a_{\text{out}} \\
    a_{\text{out}}^+
\end{pmatrix} = \begin{pmatrix}
    \alpha^+ & \mp \beta^+ \\
    \mp \beta & \bar{\alpha}
\end{pmatrix}
\begin{pmatrix}
    a_{\text{in}} \\
    a_{\text{in}}^+
\end{pmatrix}.
\]

(37)

For a field in the left-hand half-plane, an interchange of the subscripts \( \text{in} \leftrightarrow \text{out} \) is required on operators \( a \) and \( a^+ \) in transformations (37). This again is equivalent to transformation (12).

Following DeWitt’s paper [3] and its notation, we represent the vector of the vacuum state of the field in the distant past in the form of an expansion in the vectors of the \( n \)-particle states of the field in the distant future:

\[
|\text{in}\rangle = e^{iW} \sum_{n=0}^{\infty} \frac{i^n/2}{n!} \sum_{\text{i}_{12}...\text{i}_n} V_{\text{i}_{12}...\text{i}_n} |\text{i}_1\text{i}_2...\text{i}_n\text{out}\rangle.
\]

(38)

In our case, by the quantum numbers \( i_1i_2...i_n \) of the \( \text{out} \) states of the individual particles should be understood frequencies that transform like \( \omega \) or like \( \omega' \) if one is dealing with the field, respectively, to the right or to the left of the mirror.

Using the equation \( a_{\text{in}} |\text{in}\rangle = 0 \), transformations (37), and the expansion given by Eq. (38), it is easy to show [3,4] that the relative production amplitudes \( V_{i_1i_2...i_n} \) of \( n \) particles equal zero for odd \( n \), whereas, for even \( n \), they are expressed in terms of the production amplitude of a pair of particles:

\[
V_{i_1i_2...i_n} = \sum_p \delta_p V_{i_1i_2} V_{i_3i_4}...V_{i_{n-1}i_n}.
\]

(39)

Here \( \sum_p \) denotes summation over \( n!/2^{n/2}(n/2)! \) different pairings of subscripts \( i_1i_2...i_n \), while \( \delta_p = 1 \) for bosons and \( \delta_p = \pm 1 \) for fermions when the permutation leading to the given pairing is, respectively, even or odd. The production amplitudes of a pair of particles with frequencies \( \omega'' \) and \( \omega \) in the right-hand region and frequencies \( \omega''' \) and \( \omega' \) in the left-hand region equal

\[
V_{\omega''\omega} = i(\alpha^{-1}\beta^*)_{\omega''\omega}, \quad V_{\omega'''\omega'} = -i(\beta\alpha^{-1})^*_{\omega'''\omega'}.
\]

(40)

They are related to each other by transformation (12), which is symmetric for a Bose field and antisymmetric for a Fermi field, as follows from Eqs. (36).

The indicated number of terms in the amplitude of Eq. (39) appears in connection with its symmetrization (antisymmetrization) and equals the number \( n! \) of permutations of its subscripts, reduced by a factor of \( 2^{n/2} \) because of the already existing symmetry (antisymmetry) of the two-particle amplitudes and by a factor of \( (n/2)! \) because of the inessentiality of permutations of these amplitudes.

Particle production in pairs is explained by the linearity of the Bogolyubov transformations in the operators \( a \) and \( a^+ \). Operator \( a_{\text{in}} \), when it acts on the \( n \)-particle \( \text{out} \) state, transforms it into a superposition of \( n - 1 \)-particle and \( n + 1 \)-particle \( \text{out} \) states. Therefore, in the expansion of the null vector \( a_{\text{in}} |\text{in}\rangle \) in the \( n \)-particle \( \text{out} \) states, the expansion coefficients equal to zero represent the linear relation between the amplitudes of the \( n + 1 \)-particle and \( n - 1 \)-particle creations. Since \( n \geq 0 \), the amplitude of the single-particle production
$V_{i_1}$ is equal to zero, and, along with it, all the formation amplitudes of an odd number of particles.

The absolute amplitudes of the $n$-particle production are determined and are related to the relative amplitudes by

$$\langle \text{out } i_1 i_2 \ldots i_n | \text{in} \rangle \equiv \langle \text{out } a_{\text{out } i_n} \ldots a_{\text{out } i_2} a_{\text{out } i_1} | \text{in} \rangle = e^{iW} i^{n/2} V_{i_1 i_2 \ldots i_n}. \tag{41}$$

The vacuum-conservation amplitude $\langle \text{out} | \text{in} \rangle = e^{iW}$ is determined to within a phase factor by the fact that the total probability of the transition from the initial vacuum state is equal to one:

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \ldots i_n} |\langle \text{out } i_1 i_2 \ldots i_n | \text{in} \rangle|^2 = e^{-2 \text{Im } W} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \ldots i_n} |V_{i_1 i_2 \ldots i_n}|^2. \tag{42}$$

The sum of the relative probabilities

$$q_n = \frac{1}{n!} \sum_{i_1 i_2 \ldots i_n} |V_{i_1 i_2 \ldots i_n}|^2 \tag{43}$$

of the production of $n$ particles (or of $n/2$ pairs) on the right-hand side of Eq. (42) we shall call the statsum. It can be shown that, in the case considered here, in which pairs of identical particles and antiparticles are formed, the statsum equals

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 i_2 \ldots i_n} |V_{i_1 i_2 \ldots i_n}|^2 = \text{det} (1 \mp M) = \exp \left( \mp \frac{1}{2} \text{tr} \ln (1 \mp M) \right), \tag{44}$$

where $M = VV^+$ is a Hermitian positive-semidefinite matrix formed from the matrices in Eqs. (40). In particular, the first three terms of the statsum, determined by the relative amplitudes

$$1, \quad V_{i_1 i_2}, \quad V_{i_1 i_2} V_{i_3 i_4} \pm V_{i_1 i_3} V_{i_2 i_4} + V_{i_1 i_4} V_{i_2 i_3}, \tag{45}$$

and by Eq. (43), are equal, respectively, to

$$q_0 = 1, \quad q_2 = \frac{1}{2} \text{tr } M, \quad q_4 = \frac{1}{8} \left( \text{tr } M \right)^2 \pm \frac{1}{4} \text{tr } M^2. \tag{46}$$

The absolute probabilities of forming $n$ pairs are equal to $p_{2n} = p_0 q_{2n}$, where $p_0$ is the vacuum-conservation probability:

$$p_0 = e^{-2 \text{Im } W}, \quad 2 \text{Im } W = \mp \frac{1}{2} \text{tr} \ln (1 \mp M). \tag{47}$$

Since the relative probabilities $q_{2n}(M)$ of producing $n$ pairs are homogeneous functions of degree $n$, $q_{2n}(\lambda M) = \lambda^n q_{2n}(M)$, it is convenient to compute the mean number of pairs from

$$\bar{n} = \sum_{n=0}^{\infty} n p_{2n} = p_0 \lambda \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \lambda^n q_{2n}(M)|_{\lambda=1} = \lambda \frac{\partial}{\partial \lambda} 2 \text{Im } W(\lambda M)|_{\lambda=1} = \frac{1}{2} \text{tr} \frac{M}{1 \mp M}. \tag{48}$$
Matrices $M$ are different for the right-hand and left-hand regions:

$$M = VV^+ = \beta^+\beta(1 \pm \beta^+\beta)^{-1},$$  \hspace{1cm} (49)$$
$$M = VV^+ = \beta^*\bar{\beta}(1 \pm \beta^*\bar{\beta})^{-1},$$  \hspace{1cm} (50)$$
but are related to each other by transformation (12). However, the positive-definite quantities $\text{tr}M^n$, $n = 1, 2, \ldots$, are invariants of this transformation. Therefore, the total probabilities given above for conservation of the vacuum, $p_0$, and of the production of $n$ pairs, $p_{2n}$, and the mean number of pairs, $\bar{n}$, are identical for the right-hand and left-hand regions. In particular, the quantities

$$p_0 = e^{-2\text{Im}W}, \quad 2\text{Im}W = \frac{1}{2}\text{tr} \ln(1 \pm \beta^+\beta),$$  \hspace{1cm} (51)$$
$$p_2 = e^{-2\text{Im}W} \frac{1}{2}\text{tr} \beta^+\beta(1 \pm \beta^+\beta)^{-1},$$  \hspace{1cm} (52)$$
$$\bar{n} = \frac{1}{2}\text{tr} \beta^+\beta$$  \hspace{1cm} (53)$$
do not change under transformation (12) or $\beta^+\beta \rightarrow \beta\beta^+$.

Nevertheless, the frequency distributions of the probabilities and of the mean number of particles possess no left–right symmetry. Thus, the production probability of one pair, one particle of which has a definite frequency while the other has any frequency, equals

$$p_{2\omega} = e^{-2\text{Im}W} \left( \frac{\beta^+\beta}{1 \pm \beta^+\beta} \right)_{\omega\omega},$$  \hspace{1cm} (54)$$
for the right-hand region and equals

$$p'_{2\omega'} = e^{-2\text{Im}W} \left( \frac{\beta\beta^+}{1 \pm \beta\beta^+} \right)_{\omega'\omega'},$$  \hspace{1cm} (55)$$
for the left-hand region. The frequency distributions of the mean number of particles emitted by the mirror to the right and to the left are also functionally different from each other:

$$N_\omega = (\beta^+\beta)_{\omega\omega}, \quad N'_{\omega'} = (\beta\beta^+)_{\omega'\omega'}. $$  \hspace{1cm} (56)$$

Along with the amplitudes given by Eq. (41) for particle production from vacuum by the mirror, it is necessary to consider the amplitudes of single-particle scattering by the mirror

$$\langle \text{out} \omega' | \omega | \text{in} \rangle = \langle \text{out} | a_{\text{out}} \omega a_{\text{in}}^+ \omega | \text{in} \rangle = e^{iW} a_{\omega\omega'},$$  \hspace{1cm} (57)$$
$$\langle \text{out} \omega' | \omega | \text{in} \rangle = \langle \text{out} | a_{\text{out}} \omega a_{\text{in}}^+ \omega | \text{in} \rangle = e^{iW} a_{\omega\omega'}^{-1*},$$  \hspace{1cm} (58)$$
for the right-hand and left-hand regions, respectively. These amplitudes differ only in their phases. Of course, they are related to each other by transformation (12), but we shall be interested in their relation to the corresponding pair-production amplitudes:

$$\langle \text{out} \omega'' | \omega' | \text{in} \rangle = -e^{iW} (\alpha^{-1}\beta^*)_{\omega''\omega} = -\sum_{\omega'} \langle \text{out} \omega'' | \omega' | \text{in} \rangle \beta'_{\omega''\omega},$$  \hspace{1cm} (59)$$
\[
\langle \text{out } \omega' \omega''|\text{in} \rangle = e^{iW(\beta \alpha^{-1})} \omega' \omega'' = \sum_{\omega} \bar{\beta}_{\omega'}^{*} \langle \text{out } \omega''|\text{in} \rangle.
\]

(60)

Since the pair-production amplitudes and the single-particle scattering amplitudes are quantities that can in principle be experimentally measured from the corresponding probabilities, Eqs. (59) and (60) make it possible to experimentally measure \(\beta_{\omega'}^{*}\). Moreover, these relationships make it possible to regard \(\beta_{\omega'}^{*}\) as the amplitude of the source of a pair of particles potentially emitted to the right and to the left with frequencies \(\omega\) and \(\omega'\), respectively. In this case, if a particle with frequency \(\omega\) actually escaped to the right, a particle with frequency \(\omega'\) does not escape to the left, but experiences an internal reflection and is actually emitted to the right with altered frequency \(\omega''\). Conversely, if a particle with frequency \(\omega'\) actually escaped to the left, a particle with frequency \(\omega\) cannot escape to the right, but, after internal reflection, is actually emitted to the left with another frequency \(\omega''\).

For fermions, amplitude \(\beta_{\omega'}^{F}\) is diagonal in the projection of the spin of the in and out waves (see Ref. [2]). But one of the waves forming \(\beta_{\omega'}^{F}\) has a negative frequency and therefore describes an antiparticle with frequency and spin projection opposite in sign to the frequency and spin projection of this wave (see §26 in Ref. [5] or §9 of chap. 2 in Ref. [6]). Thus, the spin of a pair of generated fermions equals zero. This is confirmed by the scalar nature of the identically equal integrals in Eqs. (9) and (10), in which \(du \sqrt{f'(u)}\) and \(dv \sqrt{g'(v)}\) are elements of proper time \(d\tau\), and by their coincidence,

\[
\beta_{\omega'}^{F} = \frac{1}{e} \rho(k_+, k_-)
\]

(61)

with the Fourier component of the scalar-charge density in 3 + 1 space.

Amplitude \(\beta_{\omega'}^{B}\) of the source of a boson pair, according to Eqs. (6) and (7), is linearly expressed in terms of the Fourier components \(j_{\pm}(k)\) of the current density of an electric charge in 3 + 1 space:

\[
\beta_{\omega'}^{B} = \sqrt{\frac{k_+ j_-}{k_- e}} = \sqrt{\frac{k_- j_+}{k_+ e}},
\]

(62)

\[
j_- = e \int_{-\infty}^{\infty} du \exp \left[ i \frac{1}{2} (k_+ u + k_- f(u)) \right], \quad j_+ = e \int_{-\infty}^{\infty} dv \exp \left[ i \frac{1}{2} (k_- v + k_+ g(v)) \right],
\]

(63)

see also Eqs. (1) and (2) in this paper and Eqs. (43) and (44) in Ref. [1]. The last equality in Eq. (62) is none other than the current-transverseness condition, \(k_+ j_- + k_- j_+ = 0\). It can also be seen from Eq. (62) that \(\beta_{\omega'}^{B}\) is a pseudoscalar, since, at the reflection \(k_\pm \rightarrow k_\mp\), \(j_\pm \rightarrow j_\mp\), and \(\beta^{B}\) changes sign. Vector \(j_\alpha(k)\) is spacelike and, in a system where \(k_+ = k_-\) (or \(\omega = \omega'\)), has only a spatial component, precisely equal to \(e \beta_{\omega'}^{B}\). In covariant form,

\[
e \beta_{\omega'}^{B} = \epsilon_{\alpha\beta} k^\alpha j^\beta / \sqrt{k_+ k_-}.
\]

Thus, the source of a boson pair is the conserved current vector given by Eqs. (63), and this means that the spin of a pair equals 1, see [7].

The fact that the spin of a boson pair equals 1 while that of a fermion pair equals 0 is essential for understanding the coincidence of the spectra of a mirror and of a charge.
If $\beta_{\omega,\omega}^*$ is small, i.e., if the mean number of emitted quanta is small, then, as is easy to obtain from Eqs. (6) and (9),

$$\alpha_{\omega,\omega} \approx 2\pi \delta(\tilde{\omega}' - \tilde{\omega}), \quad \alpha_{\omega,\omega}^{-1} \approx 2\pi \delta(\tilde{\omega} - \tilde{\omega}')$$

where $\tilde{\omega}$ and $\tilde{\omega}'$ are related to $\omega$ and $\omega'$ by transformation (32), in which $\beta$ is the effective velocity of the mirror on the emission section. In this approximation, the emission amplitudes given by Eqs. (59) and (60) for pairs of particles with frequencies $\omega$ and $\omega''$ to the right and pairs of particles with frequencies of $\omega'$ and $\omega'''$ to the left equal, respectively,

$$\langle \text{out} \omega'' | \text{in} \rangle \approx -e^{iW} D^{-1}(\beta) \beta_{\omega,\omega}^*, \quad \omega' = D^{-2}(\beta) \omega''$$

(65)

$$\langle \text{out} \omega' | \omega'' | \text{in} \rangle \approx e^{iW} D(\beta) \beta_{\omega,\omega}^*, \quad \omega = D^2(\beta) \omega''$$

(66)

These formulas, including the connection between the frequencies of the waves incident on the mirror and reflected from it, confirm the interpretation of $\beta_{\omega,\omega}^*$ given above.

We now turn our attention to interference effects in the production of Bose and Fermi particles. They become most substantial when matrices $M$ for bosons and fermions satisfy the conditions

$$\mp \frac{1}{2} \text{tr} \ln(1 \mp M) = \mp \ln \left(1 \mp \frac{1}{2} \text{tr} M\right),$$

i.e.,

$$\frac{1}{2} \text{tr} M^n = \left(\frac{1}{2} \text{tr} M\right)^n, \quad n = 2, 3, \ldots.$$  

(67)

Then the statsum given by Eq. (44) for Bose and Fermi particles reduces, respectively, to

$$\left(1 - \frac{1}{2} \text{tr} M\right)^{-1} \quad \text{and} \quad 1 + \frac{1}{2} \text{tr} M.$$  

(68)

This means that the probabilities of producing $n$ pairs of bosons form the geometrical progression

$$p_{2n}^B = p_0^B q_2^n, \quad p_0^B = 1 - \frac{1}{2} \text{tr} M, \quad q_2^B = \frac{1}{2} \text{tr} M,$$

(69)

while the probabilities of emitting two or more pairs of fermions disappear; i.e., only the production of one fermion pair is possible:

$$p_0^F = \left(1 + \frac{1}{2} \text{tr} M\right)^{-1}, \quad p_2^F = p_0^F \frac{1}{2} \text{tr} M, \quad p_{2n}^F = 0, \quad n \geq 2.$$  

(70)

In other words, the conditions given by Eqs. (67) denote the most constructive interference of bosons and the most destructive interference of fermions. In these cases, the mean-square fluctuation of number of boson pairs is always greater than $\bar{n}_B$, while that of the fermion pairs is less than $\bar{n}_F$, being equal to $\bar{n}(1 \pm \bar{n})$, where

$$0 < \bar{n}_B = \frac{\frac{1}{2} \text{tr} M}{1 - \frac{1}{2} \text{tr} M} < \infty, \quad 0 < \bar{n}_F = \frac{\frac{1}{2} \text{tr} M}{1 + \frac{1}{2} \text{tr} M} < 1.$$  

(71)
We are less interested in the case in which interference effects can be neglected. In this case,\[ \text{tr } M^k \ll \text{tr } M, \quad k \geq 2, \quad (72) \]
and the probability distribution over the number of generated pairs coincides with the Poisson distribution:
\[ p_{2n} = e^{-\bar{n}} \frac{(\bar{n})^n}{n!}, \quad \bar{n} = \frac{1}{2} \text{tr } \beta^+ \beta. \quad (73) \]

5 Emission of pairs consisting of nonidentical particles and antiparticles

In the case of pair production of nonidentical particles and antiparticles (ab pairs), the direct and inverse Bogolyubov transformations (37) are replaced by
\[ \left( \begin{array}{c} a_{in} \\ b_{in}^+ \end{array} \right) = \left( \begin{array}{cc} \alpha_{aa} & \beta_{ab}^* \\ \beta_{ba} & \alpha_{bb}^* \end{array} \right) \left( \begin{array}{c} a_{out} \\ b_{out}^+ \end{array} \right), \quad \left( \begin{array}{c} a_{out} \\ b_{out}^+ \end{array} \right) = \left( \begin{array}{cc} \alpha_{aa}^+ & \pm \beta_{ba}^+ \\ \mp \beta_{ab}^+ & \tilde{\alpha}_{bb} \end{array} \right) \left( \begin{array}{c} a_{in} \\ b_{in}^+ \end{array} \right). \quad (74) \]

These transformations contain not two but four matrices \( \alpha_{aa}, \alpha_{bb}, \beta_{ab}, \) and \( \beta_{ba} \), which satisfy not the four Eqs.(36) but the six equations
\[ \begin{align*}
\alpha_{aa} \alpha_{aa}^+ \mp \beta_{ba} \beta_{ba} &= 1, \\
\beta_{ba} \alpha_{bb}^+ \mp \alpha_{aa} \beta_{ab}^+ &= 0, \\
\alpha_{bb} \alpha_{bb}^+ \mp \beta_{ab} \beta_{ab} &= 1, \\
\alpha_{aa} \beta_{bb}^+ \mp \beta_{ab} \tilde{\alpha}_{bb} &= 0.
\end{align*} \quad (75) \]

However, these relationships can be written in the form of Eqs.(36) if \( \alpha \) and \( \beta \) stand for the 2×2 matrices consisting of the indicated quarters:
\[ \alpha = \left( \begin{array}{cc} \alpha_{aa} & 0 \\ 0 & \alpha_{bb} \end{array} \right), \quad \beta = \left( \begin{array}{cc} 0 & \beta_{ab} \\ \beta_{ba} & 0 \end{array} \right). \quad (76) \]

As can be seen from Eqs. (74), the interchange \( \text{in} \leftrightarrow \text{out} \) is now equivalent to the interchange
\[ \alpha_{aa} \rightarrow \alpha_{aa}^+, \quad \alpha_{bb} \rightarrow \alpha_{bb}^+, \quad \beta_{ab} \rightarrow \mp \tilde{\beta}_{ab}, \quad \beta_{ba} \rightarrow \mp \beta_{ba}, \quad (77) \]
which can be represented in the form of the transformation (12) if \( \alpha \) and \( \beta \) stand for the matrices of Eqs.(76).

Using for the in-vacuum state an expansion of the type of Eq.(38) and the equations \( a_{in}|in\rangle = b_{in}|in\rangle = 0 \), it can be shown that all the emission amplitudes of an odd number of particles equal zero, while the production amplitudes of an even number of particles are products of the production amplitudes of ab pairs:
\[ V_{\omega'\omega}^{ab} = i(\alpha_{aa}^{-1} \beta_{ab}^*)_{\omega'\omega}, \quad V_{\omega''\omega'}^{ab} = -i(\beta_{ab} \beta_{bb}^{-1})_{\omega''\omega'}, \quad (78) \]
respectively for the right-hand and the left-hand regions. As follows from Eqs. (75), the amplitudes given by Eqs. (78) possess intrinsic Bose symmetry or Fermi antisymmetry:
\[ V_{\omega'\omega}^{ab} = \pm V_{\omega'\omega}^{ba}, \quad \equiv \pm i(\alpha_{bb}^{-1} \beta_{ab})_{\omega'\omega}, \quad V_{\omega''\omega'}^{ab} = \pm V_{\omega''\omega'}^{ba}, \quad \equiv \mp i(\beta_{ba} \alpha_{aa}^{-1})_{\omega''\omega'}. \quad (79) \]
Thus, the formation amplitude of an \(ab\) pair can be denoted via \(V_{i_1i_2}\), where the subscript \(i_1\) characterizes the state of the particle and \(i_2\) that of the antiparticle. The production of two \(ab\) pairs is described by the amplitude

\[
V_{i_1i_2i_3i_4} = V_{i_1i_2} V_{i_3i_4} \pm V_{i_3i_2} V_{i_1i_4}, \tag{80}
\]

symmetric (antisymmetric) separately with respect to states \(i_1\) and \(i_3\) of the particles and separately with respect to states \(i_2\) and \(i_4\) of the antiparticles. We also write the production amplitude of three pairs:

\[
V_{i_1i_2i_3i_4i_5i_6} = V_{i_1i_2} V_{i_3i_4} V_{i_5i_6} \pm V_{i_3i_2} V_{i_1i_4} V_{i_5i_6} \pm V_{i_1i_2} V_{i_3i_4} V_{i_5i_6} \pm V_{i_3i_2} V_{i_1i_4} V_{i_5i_6}, \tag{81}
\]

In the general case, the production amplitude of \(n/2\) pairs has the form

\[
V_{i_1i_2\ldots i_n} = \sum_p \delta_p V_{i_1i_2} V_{i_3i_4} \ldots V_{i_{n-1}i_n}, \tag{82}
\]

where the sum is taken over all \((n/2)!\) terms that differ by a permutation of the odd subscripts (or, what is the same thing, by a permutation of the even subscripts), with \(\delta_p = \pm 1\) in the case of fermions for an even or odd permutation, respectively, while \(\delta_p = 1\) in the case of bosons. Then amplitude \(V_{i_1i_2\ldots i_n}\) will be symmetric (antisymmetric) both over particle states \(i_1i_2\ldots i_{n-1}\) and over antiparticle states \(i_2i_4\ldots i_n\).

The relative probability

\[
q_n = \frac{1}{(n/2)!(n/2)!} \sum_{i_1i_2\ldots i_n} |V_{i_1i_2\ldots i_n}|^2 \tag{83}
\]

of producing \(n/2\) pairs consisting of nonidentical particles and antiparticles contains the factor \(1/(n/2)!(n/2)!\), which, along with the symmetry (antisymmetry) of amplitude \(V_{i_1i_2\ldots i_n}\) separately for even and separately for odd subscripts, makes it possible to sum over the particle and antiparticle states, considering the ranges of variation of the quantum numbers of these states to be independent. Without this factor, the sum over \(i_1i_2\ldots i_{n-1}\) would have had to contain only physically different states. In our case, for example, it would be unambiguous that the frequencies of the particles must satisfy the condition \(\omega_1 \geq \omega_3 \geq \ldots \geq \omega_{n-1}\), while the frequencies of the antiparticles must satisfy the condition \(\omega_2 \geq \omega_4 \geq \ldots \geq \omega_n\).

It is easy to construct the first four terms of the statsum in terms of the relative amplitudes shown above:

\[
q_0 = 1, \quad q_2 = \text{tr } M, \quad q_4 = \frac{1}{2} \text{tr } M^2 \pm \frac{1}{2} \text{tr } M^2, \quad q_6 = \frac{1}{6} \text{tr } M^3 \pm \frac{1}{2} \text{tr } M \text{tr } M^2 + \frac{1}{3} \text{tr } M^3. \tag{84}
\]

For the statsum as a whole, we get

\[
\sum_{n=0}^{\infty} \frac{1}{(n/2)!(n/2)!} \sum_{i_1i_2\ldots i_n} |V_{i_1i_2\ldots i_n}|^2 = \text{det}(1 \mp M)^{\mp 1} = \exp(\pm \text{tr } \ln(1 \mp M)). \tag{85}
\]
Here, as in Eq. (44), $M = VV^+$ is a Hermitian positive-semidefinite matrix. It is given by Eqs. (49) and (50), in which by $\beta$ is meant, respectively, $\beta_{ba}$ and $\beta_{ab}$.

Just as above, the absolute probabilities of the formation of $n$ pairs of nonidentical particles and antiparticles equal $p_{2n} = p_0q_{2n}$, where $p_0$ is the vacuum-conservation probability:

$$p_0 = e^{-2\Im W}, \quad 2\Im W = \mp \text{tr} \ln(1 \mp M).$$

(86)

The mean number of pairs, computed according to the rule given in Eq. (48), equals

$$\bar{n} = \text{tr} \frac{M}{1 \mp M}.$$  

(87)

It can be seen that these formulas differ from the corresponding Eqs. (47) and (48) for pair production of identical particles by replacing $(1/2)\text{tr}$ by $\text{tr}$ in the latter equations. Because of the $a \leftrightarrow b$ symmetry in the matrices, under the tr sign, by $\beta$ can be understood both $\beta_{ba}$ and $\beta_{ab}$.

It is easy to see that this rule connects all the formulas for the integral characteristics of pair production of identical particles with the formulas of the corresponding characteristics of $ab$-pair production. Thus, in order to obtain from Eqs. (51)–(53) and (67)–(73) the analogous expressions for $ab$-pair production, it is sufficient to replace $(1/2)\text{tr}$ in these formulas with $\text{tr}$ and by $\beta$ to understand $\beta_{ba}$ or $\beta_{ab}$.

As far as the spectral characteristics shown, for example, in Eqs. (54)–(56) are concerned, they undergo no changes when the transition is made to the case under consideration, if by $\beta$ is understood $\beta_{ba}$ ($\beta_{ab}$) for the spectrum of particles (antiparticles) emitted to the right and $\beta_{ab}$ ($\beta_{ba}$) for the spectrum of particles (antiparticles) emitted to the left.

In fact, for the differential probability $p_{2\omega}$ shown in Eq. (54), the original integral

$$p_{2\omega} = \int_0^\infty \frac{d\omega''}{2\pi}|\langle\text{out} \omega\omega''|\text{in}\rangle|^2$$  

(88)

represents it as the sum of the probabilities of physically different events regardless of whether the particles are identical or not. However, the total pair-formation probability $p_2$ as a sum of probabilities of physically different events for identical particles is represented by the integral

$$p_2 = \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega''}{2\pi}|\langle\text{out} \omega\omega''|\text{in}\rangle|^2 = \frac{1}{2} \int_0^\infty \frac{d\omega}{2\pi} p_{2\omega}.$$  

(89)

since the states in this case differ only by the values of the large frequency $\omega$ and the small frequency $\omega''$ of two identical particles. At the same time, for an $ab$ pair,

$$p_2 = \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega''}{2\pi}|\langle\text{out} \omega\omega''|\text{in}\rangle|^2 = \int_0^\infty \frac{d\omega}{2\pi} p_{2\omega}.$$  

(90)

since the states differ in the frequencies $\omega''$ and $\omega$ of the particle and antiparticle independently of each other, while the particle and antiparticle differ in turn in that they interact differently with the counters.
Turning to the amplitude of $ab$-pair production,

$$\langle \text{out} | b_{\omega''} a_{\omega'} | \text{in} \rangle \equiv \langle \text{out} \omega'' | \omega' \text{in} \rangle = -e^{iW} (\alpha_{aa}^{-1} \beta_{ab}^*) \omega'' = \mp e^{iW} (\alpha_{bb}^{-1} \beta_{ba}^*) \omega'' \omega, \quad (91)$$

we note that it reduces to a product of the source amplitude $\beta_{ab}^*$ or $\beta_{ba}^*$ of oppositely directed $a$ and $b$ particles and the backscattering amplitude $\alpha_{aa}^{-1}$ or $\alpha_{bb}^{-1}$ of one of them, as a result of which both particles of the pair move in the same direction. The symmetry of Eqs. (79) makes it impossible to establish which of the particles of the $ab$ pair experiences backscattering.

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**References**

[1] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **108**, 1121 (1995) [JETP **81**, 615 (1995)].
[2] V. I. Ritus, Zh. Eksp. Teor. Fiz. **110**, 526 (1996) [JETP **83**, 282 (1996)].
[3] B. S. DeWitt, Phys. Rep. C **19**, 295 (1975).
[4] R. M. Wald, Comm. Math. Phys. **45**, 9 (1975).
[5] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Nauka, Moscow, 1989; Pergamon Press, Oxford, 1971).
[6] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Nauka, Moscow, 1969; Wiley, New York, 1965).
[7] J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass, 1970; Mir, Moscow, 1976).