ESTIMATES OF THE DERIVATIVES OF THE HEAT KERNEL
ON SYMMETRIC AND LOCALLY SYMMETRIC SPACES,
AND APPLICATIONS

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Abstract. We obtain pointwise upper bounds on the derivatives of the heat
kernel on symmetric and locally symmetric spaces of noncompact type. Applying
these estimates we prove the \( L^p \)-boundedness of Littlewood-Paley-Stein operators,
the Laplacian and the gradient of the heat operator, and the Riesz operator.

1. Introduction

In this article we prove estimates of the derivatives of the heat kernel on symmetric
and locally symmetric spaces of noncompact type.

A symmetric space is a homogeneous space that can be described as a coset
Riemannian manifold \( X = G/K \), where \( G \) is a semisimple Lie group and \( K \) is a
maximal compact subgroup (see \cite{12} for more details). From now on, \( X \) will denote
an \( n \)-dimensional symmetric space.

Let \( g \) and \( \mathfrak{k} \) be the Lie algebras of \( G \) and \( K \) respectively. Let also \( p \) be the subspace
of \( g \) which is orthogonal to \( \mathfrak{k} \) with respect to the Killing form. Let \( a \) be an abelian
maximal subspace of \( p \), \( a^* \) its dual and let \( \Sigma \subset a^* \) be the root system of \((g,a)\).
Choose a set \( \Sigma^+ \) of positives roots. Let \( \rho \) be the half-sum of positive roots counted
with multiplicity. Let \( a^+ \subset a \) be the corresponding positive Weyl chamber and let \( \overline{a^+} \)
be its closure. We have the Cartan decomposition \( G = K(\exp \overline{a^+})K \). If \( x \in G \), then it is written as \( x = k_1 (\exp H) k_2, k_1, k_2 \in K, H \in \overline{a^+} \).

Let \( \Delta \) be the Laplace-Beltrami operator on \( X \). Then, the heat kernel \( h_t \) of \( X \) is
the fundamental solution of the heat equation \( \partial_t h_t = \Delta h_t \). It is a \( K \)-bi-invariant
function i.e. if \( x = k_1 (\exp H) k_2 \in X \), then \( h_t(x) = h_t(\exp H) \).

One of the main objectives in the present article is to prove the following result,
which answers a question that was raised by Professor Michel Marias.

Theorem 1. If \( X \) is a symmetric space of noncompact type, then for all \( \epsilon > 0 \) and
\( i \in \mathbb{N} \) there is a constant \( c > 0 \) such that
\[
\left| \frac{\partial^i h_t}{\partial t^i} (\exp H) \right| \leq c t^{-(n/2) - i} e^{-(1-\epsilon)\left(\|\rho\|^2 t + \langle \rho, H \rangle + \| H \|^2 / (4t)\)} ,
\]
for all \( t > 0 \) and \( H \in \overline{a^+} \).

As a first application of Theorem 1 we obtain gradient estimates of the heat kernel.

Corollary 1. If \( X \) is a symmetric space of noncompact type, then for all \( \epsilon > 0 \) there
is a constant \( c > 0 \), such that
\[
\| \nabla h_t(\exp H) \| \leq c t^{-(n+1)/2} e^{-(1-\epsilon)\left(\|\rho\|^2 t + \langle \rho, H \rangle + \| H \|^2 / (4t)\)} ,
\]
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for all $t > 0$ and $H \in \overline{a_t}$.

Next, we obtain estimates of the heat kernel on a locally symmetric space. Let $\Gamma$ be a discrete torsion free subgroup of $G$. Then the locally symmetric space $M = \Gamma \setminus X$, equipped with the projection of the canonical Riemannian structure of $X$, becomes a Riemannian manifold. Denote by $\Delta_M$ the Laplacian on $M$ and by $d_M$ the Riemannian distance on $M$. We denote by $\pi : X \rightarrow M$ the canonical projection and for $x \in X$ we set $\tilde{x} = \pi(x)$. Mandouvalos and Davies in [8] obtained estimates of the heat kernel $h_t^M(\tilde{x}, \tilde{y})$ in the case when $X = \mathbb{H}^{n+1}$ and then Weber in [22] in a class of locally symmetric spaces $M = \Gamma \setminus X$. In Theorem 2 below, we generalize the estimates of Weber in [22]. Denote by

$$P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma y)}, \quad x, y \in X,$$

and

$$\delta(\Gamma) = \inf\{s \in (0, \infty) : P_s(x, y) < \infty\},$$

the Poincaré series and the critical exponent respectively.

Set also

$$\rho_m = \min_{H \in \mathfrak{H}, \|H\| = 1} \langle \rho, H \rangle.$$

Note that $\delta(\Gamma) \leq 2\|\rho\|$ and that $\rho_m \leq \|\rho\|$.

Let $\delta(\Gamma) < \rho + \rho_m$. Consider $\alpha_2 \in (\delta(\Gamma), \|\rho\| + \rho_m)$ and $\alpha_1, \alpha_3 \in [0, 1]$ such that $\alpha_1\alpha_3 \in \left((\alpha_2 - \rho_m) / \|\rho\|^2, 1\right]$.

**Theorem 2.** Suppose that $M = \Gamma \setminus X$, with $\delta(\Gamma) < \|\rho\| + \rho_m$. Then, for all $\epsilon > 0$ there is a constant $c > 0$, such that

$$\left| \frac{\partial^2 h_t^M(\tilde{x}, \tilde{y})}{\partial t^2} \right| \leq c t^{-n/2 - i} e^{-(1-\epsilon)\left(\|\rho\|^2 t + (\alpha_2 - \delta(\Gamma)) d_M(\tilde{x}, \tilde{y}) + (1-\alpha_3) d_y^2(\tilde{x}, \tilde{y}) / (4t)\right)}$$

$$\times P_{\epsilon + \delta(\Gamma)}(x, y),$$

for all $t > 0$ and $\tilde{x}, \tilde{y} \in M$.

Note that if $\delta(\Gamma) < \rho_m$, then $\rho_m \in (\delta(\Gamma), \|\rho\| + \rho_m)$. Thus, we can consider $\alpha_2 = \rho_m$ and $((\alpha_2 - \rho_m) / \|\rho\|^2) = 0$, consequently we can consider $\alpha_1 = \alpha_3 = 0$. So, we observe that the above result extends both the cases $\delta(\Gamma) < \rho_m$ and $\delta(\Gamma) \geq \rho_m$ covered by Weber in [22]

Let us now assume that $\Gamma$ is convex co-compact. As explained in Section 2.1 in this case the Poincaré series are uniformly bounded for all $s > \delta(\Gamma)$ and $x, y \in X$.

Combining this fact with Theorem 2 we obtain the following result.

**Corollary 2.** Assume that $\Gamma$ is convex co-compact and $M = \Gamma \setminus X$, with $\delta(\Gamma) < \|\rho\| + \rho_m$. Then, for all $\epsilon > 0$ there is a constant $c > 0$, such that

$$\left| \frac{\partial^2 h_t^M(\tilde{x}, \tilde{y})}{\partial t^2} \right| \leq c t^{-n/2 - i} e^{-(1-\epsilon)\left(\|\rho\|^2 t + (\alpha_2 - \delta(\Gamma)) d_M(\tilde{x}, \tilde{y}) + (1-\alpha_3) d_y^2(\tilde{x}, \tilde{y}) / (4t)\right)},$$

for all $t > 0$ and $\tilde{x}, \tilde{y} \in M$.

Next, we apply Theorem 1 and Corollary 1 in order to prove $L^p$-boundedness of various operators related to the heat semigroup $H_t^M = e^{-\Delta_M t}$.

We shall now assume that the locally symmetric space $M = \Gamma \setminus G/K$ belongs in the family $\mathcal{L}M$, i.e. if it is contained in the following three classes: (i). $\Gamma$ is a lattice,
(ii). $G$ possesses Kazhdan’s property (T), (iii). $\Gamma \backslash G$ is non-amenable. See [18] for more details.

If $M \in L\mathcal{M}$, then Lohoué and Marias proved in [18] an analogue of the Kunze-Stein phenomenon. More precisely, if $p \in (1, \infty)$, denote by $p'$ its conjugate and set

$$s(p) = 2 \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\}.$$

If $\kappa$ is a $K$-bi-invariant function and $\ast |\kappa|$ denotes the convolution operator whose kernel is $|\kappa|$, then in [18, Section 3] it is proved that

$$\| \ast |\kappa| \|_{L^p(M) \to L^p(M)} \leq c \int_G |\kappa(g)| \phi_{-\iota \eta_T}(g)^{s(p)} dg,$$

where $\phi_{\lambda}$ are the elementary spherical functions, $\eta_T$ is a vector of the euclidean sphere $S(0, (\|\rho\|^2 - \lambda_0)^{1/2})$ of $\mathfrak{a}^*$ and $\lambda_0$ is the bottom of the spectrum of the Laplacian $-\Delta_M$. Note that $\|\eta_T\| < \|\rho\|$, since $\lambda_0 > 0$ (see [23, Theorem 3.18] for more details).

Denote by $H_t = e^{-\Delta t}$ the heat semigroup on $X$. Fix $i \in \mathbb{N}$. Then, for all $\sigma \geq 0$ we consider as in [2, 15], the $\sigma$-maximal operator

$$H_{\sigma}^{\max}(f) = \sup_{t > 0} e^{\sigma t} t^i \frac{\partial^i}{\partial t^i} H_t f,$$

and the Littlewood-Paley-Stein operator

$$H_{\sigma}(f)(x) = \left( \int_0^\infty e^{2\lambda t} \left( \left| t^i \frac{\partial^i}{\partial t^i} H_t f(x) \right|^2 + |\nabla_x H_t f(x)|^2 \right) \frac{dt}{t} \right)^{1/2},$$

where $\nabla$ denotes the Riemannian gradient on $X$.

We prove the following theorem.

**Theorem 3.** Suppose that $M \in L\mathcal{M}$. Then, the operators $H_{\sigma}^{\max}$ and $H_{\sigma}$ are bounded on $L^p(M), p \in (1, \infty)$, provided that

$$\sigma < s(p) (\|\rho\| - \|\eta_T\|) (2 \|\rho\| - s(p) (\|\rho\| - \|\eta_T\|)).$$

The Littlewood-Paley-Stein operator was first introduced and studied by Lohoué [16], in the case of Riemannian manifolds of non-positive curvature. In a variety of geometric settings it has been proved that $H_{\sigma}^{\max}$ and $H_{\sigma}$ are bounded on $L^p, p \in (1, \infty)$, under some conditions on $\sigma$ (see for example [16]). In particular, in the case of a symmetric space $X$, Anker in [2] has shown that $H_{\sigma}^{\max}$ and $H_{\sigma}$ are bounded in $L^p(X)$, provided $\sigma < 4 \|\rho\|^2 / pp'$. In the case of symmetric spaces, where $\eta_T = 0$ (see [18] for more details), we observe that the condition on $\sigma$ becomes $\sigma < \|\rho\|^2 s(p)(2 - s(p)) = 4 \|\rho\|^2 / pp'$, thus we extend the result of Anker in [2].

Next we shall prove $L^p$-boundedness for the Riesz transform $\nabla(-\Delta_M)^{-1/2}$. We prove the following theorem.

**Theorem 4.** Suppose that $M \in L\mathcal{M}$. Then, the Riesz transform $\nabla(-\Delta_M)^{-1/2}$ is bounded on $L^p(M)$, for all $p \in (1, \infty)$.

This operator has been studied in the case of symmetric spaces by Anker in [2] and in the case of locally symmetric by Mandouvalos and Marias in [20] where they prove that the Riesz operator $\nabla(-\Delta_M)^{-1/2}$ is bounded on $L^p(M)$ for an interval around 2. In the class $L\mathcal{M}$ of locally symmetric spaces we are considering, we improve the result in [20] and we prove that the Riesz operator is bounded on $L^p(M)$, for all
$p \in (1, \infty)$. This result has been first proved for locally symmetric spaces by Li, Kunstmann and Weber in \cite{13}.

Finally, we shall obtain $L^p$-estimates for the operators $\Delta_M e^{-t \Delta_M}$ and $\nabla_x e^{-t \Delta_M}$. We prove the following theorem.

**Theorem 5.** Suppose that $M \in \mathcal{LM}$. Then, for all $p \in (1, \infty)$ there exists $\epsilon > 0$ such that

\begin{equation}
\|\Delta_M e^{-t \Delta_M}\|_{L^p(M) \to L^p(M)} \leq \begin{cases} 
ct^{-1}, & \text{if } 0 \leq t < 1, \\
\epsilon^{-1}, & \text{if } t \geq 1,
\end{cases}
\end{equation}

and

\begin{equation}
\|\nabla_x e^{-t \Delta_M}\|_{L^p(M) \to L^p(M)} \leq \begin{cases} 
ct^{-1/2}, & \text{if } 0 \leq t < 1, \\
\epsilon^{-1}, & \text{if } t \geq 1.
\end{cases}
\end{equation}

These operators have been studied for complete Riemannian manifolds with bounded geometry by Davies in \cite{7}. In the class $\mathcal{LM}$ of locally symmetric spaces we are considering, we improve the result of \cite{7}. More precisely, in \cite{7} the $L^p$-norm of the operators $\Delta_M e^{-t \Delta_M}$ and $\nabla_x e^{-t \Delta_M}$ is proved to be bounded by a constant for $t \geq 1$.

In our case, we prove that it decays exponentially as $t \to \infty$, thus we extend the result of Davies.

There is a very rich and long literature concerning heat kernel estimates, marked by the works of Li and Yau \cite{14}, Varopoulos, Saloff-Coste and Coulhon \cite{21}, Alexopoulos \cite{1}, and Grigor’yan \cite{9}, to name a few. Davies and Mandouvalos in \cite{8}, proved estimates of the heat kernel in hyperbolic spaces and Kleinian groups. Anker and Ji in \cite{3} and Anker and Ostellari in \cite{5}, and Weber in \cite{22}, obtained estimates of the heat kernel in the case of symmetric and locally symmetric spaces, respectively. The time derivative of the heat kernel has been estimated in \cite{19} for hyperbolic spaces, in \cite{15} for some manifolds of exponential volume growth, and in \cite{7} and \cite{10} in a more general setting. More precisely, in \cite{15}, Li and Sjögren study the Littlewood-Paley-Stein operator in the hyperbolic space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ where the Laplacian is given a drift in the $\mathbb{R}^+$ direction and they prove that it is of weak type $(1,1)$.

Let us now outline the organization of the paper. In Section 2 we recall some basic definitions about symmetric spaces and the heat kernel. In addition, we recall some results providing estimates of the heat kernel and estimates of its derivatives. In Section 3 we prove Theorem 1. Next, as a direct application, we prove gradient estimates of the heat kernel. In Section 4 we prove estimates of the derivatives of the heat kernel in the setting of locally symmetric spaces. In Section 5 we study Littlewood-Paley-Stein operators, the Riesz operator and the Laplacian and the gradient of the heat operator. We prove that they are bounded on $L^p(M)$, where $M$ is a locally symmetric space in the family $\mathcal{LM}$.

Throughout this article the different constants will always be denoted by the same letter $c$. When their dependence or independence is significant, it will be clearly stated.

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2. Preliminaries

2.1. Symmetric and locally symmetric spaces. We shall recall some basic facts on symmetric and locally symmetric spaces. For details, see [2, 4, 12, 18].

As it is already mentioned in the Introduction, a symmetric space $X$ is the quotient $G/K$, where $G$ is a semisimple Lie group and $K$ is a maximal compact subgroup.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{k}$ the Lie algebra of $K$ respectively. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is the Cartan decomposition at the Lie algebra level. Let us choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. For any $\alpha \in \mathfrak{a}^*$, let $ad(X)(Y) = [X,Y]$ for all $X,Y \in \mathfrak{g}$ and set

$$ \mathfrak{g}_\alpha := \{ Y \in \mathfrak{g} : ad(H)(Y) = \alpha(H)Y, \text{ for all } H \in \mathfrak{a} \}. $$

If $\mathfrak{g}_\alpha \neq \{0\}$, then $\alpha \in \mathfrak{a}^* \setminus \{0\}$ is called a root of the pair $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ is called the root space. Denote by $\Sigma$ the set of all roots. If $\alpha$ is a root, then the only multiples of $\alpha$ that can also be roots are $\pm \frac{1}{2} \alpha, \pm 2 \alpha$, and $-\alpha$. A positive root $\alpha$ is called indivisible if $\frac{1}{2} \alpha$ is not a root. We call an $H \in \mathfrak{a}$ regular if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$.

The set of every regular elements of $\mathfrak{a}^*$ is the complement of a union of finitely many hyperplanes and the connected components are called Weyl chambers. Let us fix a Weyl chamber $\mathfrak{a}^+$. With respect to this Weyl chamber a root $\alpha$ is said to be positive if $\alpha(H) > 0$ for all $H \in \mathfrak{a}^+$. We denote by $\Sigma^+$ the set of positive roots and by $\Sigma^+_0$ the set of indivisible positive roots. If $\overline{\mathfrak{a}^+}$ is the closure of $\mathfrak{a}^+$ then we denote by $\mathfrak{a}_{\mathfrak{a}^+}$ and $\overline{\mathfrak{a}_{\mathfrak{a}^+}}$ the cones corresponding to $\mathfrak{a}^+$ and $\overline{\mathfrak{a}^+}$ in $\mathfrak{a}^*$ (see [23] for more details).

We have the Cartan decomposition of $G$:

$$ G = K \exp \overline{\mathfrak{a}^+}K. $$

Define $m_\alpha := \dim \mathfrak{g}_\alpha$ to be the multiplicity of a root $\alpha$ and let

$$ \rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha $$

be half the sum of the positive roots counted according to their multiplicity.

Let $x, y \in X$ and consider a base point $x_0 \in X$. Then, there are $g, h \in G$ such that $x = g(x_0)$ and $y = h(x_0)$. Because of the Cartan decomposition, there are $k_1, k_2 \in K, H \in \mathfrak{a}^+$ such that $g^{-1}h = k_1 \exp Hk_2$. Then, the distance $d(x,y)$ of $x, y \in X$ is given by

$$ d(x,y) = d(x_0, \exp Hx_0) = \|H\|. $$

If $\Gamma \subset G$ is a discrete torsion free subgroup of isometries of $X$, then the quotient space $M = \Gamma \backslash X$ equipped with the projection of the Riemannian metric of $X$ is a Riemannian manifold and is called locally symmetric space. If $\pi : X \rightarrow M$ is the canonical projection, we write $\tilde{x} = \pi(x)$. The distance $d_M$ on $M$ is given by

$$ d_M(\tilde{x}, \tilde{y}) = \min_{\gamma \in \Gamma} d(x, \gamma y). $$

2.2. Estimates of the Poincaré series in the convex co-compact case. Denote by $L(\Gamma) \subseteq \partial X$ the limit set of $\Gamma$ and by $C(L(\Gamma))$ its convex hull. Recall that $\Gamma$ is said to be convex co-compact if $\Gamma \backslash C(L(\Gamma))$ is compact.

The counting function $N(x,y,R), x,y \in X, R > 0, \Gamma$ is defined by

$$ N(x,y,R) = \sharp \{ \gamma \in \Gamma : d(x, \gamma y) \leq R \}, $$

where $\sharp(A)$ is the cardinal of the set $A$. 

In [6, Theorem 7.2], it is proved that if \( \Gamma \) is convex co-compact, then there is a constant \( C_1 > 1 \) such that
\[
C_1^{-1} e^{R\delta(\Gamma)} \leq N(x, x, R) \leq C_1 e^{R\delta(\Gamma)}.
\]

In [24, Theorem 6.2.4] it is proved that there is a positive function \( c(x), x \in X \) and a constant \( C_2 > 1 \), such that
\[
C_2^{-1} c(x) c(y) e^{R\delta(\Gamma)} \leq N(x, y, R) \leq C_2 c(x) c(y) e^{R\delta(\Gamma)}.
\]

Putting \( x = y \) in (2.5) and combining (2.4) with (2.5), we get that
\[
(C_1^{-1} C_2^{-1/2} x) \leq (C_1 C_2)^{1/2}, \quad x \in X.
\]

Thus, if \( \Gamma \) is convex co-compact then we obtain the following uniform estimate of \( N(x, y, R) \): there exists \( c > 0 \), such that for all \( x, y \in X \),
\[
c^{-1} e^{R\delta(\Gamma)} \leq N(x, y, R) \leq c e^{R\delta(\Gamma)}.
\]

Then one can write
\[
N(x, y, k) = \# \Gamma_k(x, y).
\]

Using (2.6) and (2.7) we have that
\[
P_s(x, y) \leq \sum_{k=0}^{\infty} e^{-ks} N(x, y, k + 1) \leq c \sum_{k=0}^{\infty} e^{-ks} e^{(k+1)\delta(\Gamma)} = \frac{ce^{\delta(\Gamma)+s}}{e^s - e^{\delta(\Gamma)}},
\]
for all \( s > \delta(\Gamma) \) and \( x, y \in X \).

2.3. The heat kernel on symmetric and locally symmetric spaces. Denote by \( h_t \) heat kernel on the symmetric space \( X \). Then the heat kernel \( h_t^M(\bar{x}, \bar{y}) \) on \( M \) is given by the formula
\[
h_t^M(\bar{x}, \bar{y}) = \sum_{\gamma \in \Gamma} h_t(x, \gamma y), \quad \text{for all } x, y \in X, \quad t > 0.
\]

The heat kernel on symmetric spaces has been extensively studied [3, 5]. Sharp estimates of the heat kernel have been real hyperbolic space have been obtained in [8] while in [3] Anker and Ji and Anker and Ostellari in [5], generalized results of [8] to all symmetric spaces of noncompact type. They proved the following sharp estimate
\[
h_t(\exp H) \asymp t^{-n/2} \left( \prod_{\alpha \in \Sigma^+} (1 + \langle \alpha, H \rangle)(1 + t + \langle \alpha, H \rangle)^{(m_\alpha + (m_\alpha)/2) - 1} \right) \\
\times e^{-\|\rho\|^2 - \langle \rho, H \rangle - \|H\|^2/(4t)},
\]
for all \( H \in \mathfrak{a}^\perp \) and all \( t > 0 \). Recall that we write \( f \asymp h \) for functions \( f \) and \( h \) if there is a positive constant \( c > 0 \) such that \( \frac{1}{c} h \leq f \leq ch \).
Note that (2.9) implies the following estimate
\[ h_t^M(x, y) \leq c t^{-n/2} (1 + t)^m e^{-\|\rho\|^2 t - d^2_M(x, y)/(4t)} P_s(x, y), \]
for all \( x, y \in X \) and all \( t > 0 \).

If \( \rho_m \leq \delta(\Gamma) \leq \rho_m + \|\rho\|, \) then for all \( \epsilon > 0, \) there is a constant \( c > 0 \) such that
\[ h_t^M(x, y) \leq c t^{-n/2} e^{-\|\rho\|^2 t - (\delta(\Gamma) - \rho_m + \epsilon)^2 t} P_{\epsilon/2 + \delta(\Gamma)}(x, y), \]
for all \( x, y \in X \) and all \( t > 0 \).

Estimates of the derivatives of the heat kernel are obtained by Mandouvalos and Tselepidis in [19] for the case of real hyperbolic spaces and Grigory’an in [10] derived Gaussian upper bounds for all the time derivatives of the heat kernel, under some assumptions on the on-diagonal upper bound for \( h_t \) on an arbitrary complete non-compact Riemannian manifold \( X \). More precisely, he proves that if there exists an increasing continuous function \( f(t) > 0, t > 0, \) such that
\[ h_t(x, x) \leq \frac{1}{f(t)}, \]
then,
\[ \left| \frac{\partial^i h_t}{\partial t^i} \right|(x, y) \leq \frac{1}{\sqrt{f(t)} f_{2i}(t)}, \]
for all \( i \in \mathbb{N}, t > 0, x, y \in X, \)
where the sequence of functions \( f_i = f_i(t), \) is defined by
\[ f_0(t) = f(t) \]
and \( f_i(t) = \int_0^t f_{i-1}(s) ds, i \geq 1. \)

3. Estimates of the derivatives of the heat kernel

In this section we give the proof of Theorem [1]. More precisely we shall prove the following estimate: for all \( \epsilon > 0 \) and \( i \in \mathbb{N}, \) there is a \( c > 0 \) such that
\[ \left| \frac{\partial^i h_t}{\partial t^i} (\exp H) \right| \leq c t^{-(n/2) - i} e^{-(1-\epsilon)(\|\rho\|^2 t + \langle \rho, H \rangle + \|H\|^2/(4t))}, \]
for all \( H \in \mathfrak{a}^X \) and all \( t > 0. \)

For the proof of (3.1) we need several lemmata. The following lemma is technical but important for the proof of Theorem [1].
Lemma 1. Let
\[
\alpha > \beta \geq 0, \quad D \geq D_*, \quad B \geq B_*, \quad C \geq C_*,
\]
and assume that for fixed $H \in \mathbb{R}^n$ the function $f_H : (0, +\infty) \to \mathbb{R}$, satisfies
\[
|f_H(t)| \leq ct^{-\alpha}(1 + t)^\beta(1 + \|H\|)^\gamma e^{-Dt - B\langle\rho, H\rangle - C\|H\|^2/(4t)}
\]
and
\[
\left|\frac{d^2 f_H}{dt^2}(t)\right| \leq ct^{-\alpha-2}(1 + t)^\beta(1 + \|H\|)^\gamma e^{-D_*t - B_\ast \langle\rho, H\rangle - C_*\|H\|^2/(4t)}.
\]
Then, for all $\epsilon > 0$, there is a constant $c > 0$, such that for all $H \in \mathbb{R}^n$,
\[
\left|\frac{df_H}{dt}(t)\right| \leq ct^{-\alpha-1}(1 + t)^\beta(1 + \|H\|)^\gamma e^{-((D_* + D)t/2 + (B_* + B)\langle\rho, H\rangle/2 + (C_* + C_\ast)\|H\|^2/8t)}
\]
where $\lambda_\epsilon = \frac{1-\epsilon}{1+\epsilon}$.

**Proof.** Firstly, for all $\delta > 0$, we note that by applying twice the mean value theorem, one can prove that
\[
\left|\frac{df_H}{dt}(t)\right| \leq \frac{1}{\delta} (|f_H(t)| + |f_H(t + \delta)|) + \delta \sup_{\tau \in (t, t+\delta)} \left|\frac{d^2 f_H}{dt^2}(\tau)\right|.
\]

Note that $t \to t^{-\alpha}(1 + t)^\beta$ is a decreasing function of $t$, since $\alpha > \beta$, therefore
\[(t + \delta)^{-\alpha}(1 + t + \delta)^\beta \leq t^{-\alpha}(1 + t)^\beta.
\]
Thus (3.5) and the estimates (3.3) and (3.4) imply that
\[
\left|\frac{df_H}{dt}(t)\right| \leq \frac{1}{\delta} t^{-\alpha}(1 + t)^\beta(1 + d)^\gamma e^{-Dt - B\langle\rho, H\rangle - C\|H\|^2/(4t)} + \\
+ c\delta t^{-\alpha-2}(1 + t)^\beta(1 + d)^\gamma e^{-(D_*t - B_\ast \rho, H\rangle - C_*\|H\|^2/(4t)}.
\]
Choose now
\[
\delta = \epsilon t e^{-(D_* - D)t/2 - (B_* - B)\langle\rho, H\rangle/2 - (C_* - C_\ast)\|H\|^2/8t)}.
\]
Thus,
\[
\left|\frac{df_H}{dt}(t)\right| \leq \frac{1}{\epsilon} t^{-\alpha-1}(1 + t)^\beta(1 + d)^\gamma e^{-(D_* - D)t/2 - (B_* - B)\langle\rho, H\rangle/2 - (C_* - C_\ast)\|H\|^2/8t)} + \\
+ c\epsilon t^{-\alpha-1}(1 + t)^\beta(1 + d)^\gamma e^{-(D_* - D)t/2 - (B_* - B)\langle\rho, H\rangle/2 - (C_* - C_\ast)\|H\|^2/8t)} - C_*\|H\|^2/(4t+\delta)
\]
From (3.2) it follows that $\delta \leq \epsilon t$. Thus
\[
\frac{1}{2t} - \frac{1}{t + \delta} \leq -\frac{1 - \epsilon}{2t(1 + \epsilon)} = -\frac{\lambda_\epsilon}{2t}.
\]
Consequently,
\[
\frac{C - C_*\|H\|^2}{2} - \frac{\|H\|^2}{4(t + \delta)} \leq -\frac{\|H\|^2}{4t} C\lambda_\epsilon + C_\ast
\]
and similarly
\[
\frac{C - C_*}{2} \frac{\|H\|^2}{4t} + \frac{C_* \|H\|^2}{4(t + \delta)} \geq \frac{\|H\|^2}{2} \frac{C_* \lambda + C}{4t} \geq \frac{\|H\|^2}{2} \frac{C_\lambda + C_*}{4t}.
\]

Thus, from (3.6), (3.7) and (3.8) it follows that
\[
\left| \frac{df}{dt} (H) (t) \right| \leq ct^{\alpha - 1} (1 + t)^\beta (1 + \|H\|)^\gamma e^{-(D_* + D)/2 + (B_* + B) \langle \rho, H \rangle/2 + (C_* + C\lambda) \|H\|^2/8t)}.
\]

We shall now apply the estimate (2.12) of Grigoryan in the case of symmetric spaces of noncompact type.

**Lemma 2.** Suppose that \( X \) is a symmetric space of noncompact type. For all \( i \in \mathbb{N} \) there is a constant \( c > 0 \) such that
\[
\left| \frac{\partial^i h_t}{\partial t^i} (\exp H) \right| \leq ct^{\alpha - 1} (1 + t)^\beta (1 + \|H\|)^\gamma e^{-(D_* + D)/2 + (B_* + B) \langle \rho, H \rangle/2 + (C_* + C\lambda) \|H\|^2/8t)}.
\]

**Proof.** We note that according to (2.11),
\[
h_t (\exp H) \leq t^{\alpha - 1} (1 + t)^\beta (1 + \|H\|)^\gamma e^{-(D_* + D)/2 + (B_* + B) \langle \rho, H \rangle/2 + (C_* + C\lambda) \|H\|^2/8t)}.
\]

Thus, we can apply (2.12) with
\[
f(t) = t^{\alpha - 1} (1 + t)^\beta (1 + \|H\|)^\gamma e^{-(D_* + D)/2 + (B_* + B) \langle \rho, H \rangle/2 + (C_* + C\lambda) \|H\|^2/8t)}.
\]

Next, we use Lemma 1 and an inductive argument in order to obtain a sequence of estimates for the time derivatives.

**Lemma 3.** Suppose that \( X \) is a symmetric space of noncompact type. Let us fix \( \epsilon > 0 \) and set \( \lambda_\epsilon = \frac{1 - \epsilon}{1 + \epsilon} \). Then, for all \( i, \ell \in \mathbb{N} \), there are constants \( c, \beta_\ell, \gamma_\ell > 0 \) such that
\[
\left| \frac{\partial^i h_t}{\partial t^i} (\exp H) \right| \leq ct^{\alpha - 1} (1 + t)^\beta (1 + \|H\|)^\gamma e^{-(D_* + D)/2 + (B_* + B) \langle \rho, H \rangle/2 + (C_* + C\lambda_\epsilon) \|H\|^2/8t)}.
\]
for all $t > 0$ and $H \in \overline{a^+}$, where $m, A$ are defined in (2.10) and $c$ is a constant that depends on $\epsilon, i, k$. Furthermore the sequences $\beta^i_k, \gamma^i_k$ satisfy the iteration formulas

$$\beta^i_k = \frac{1}{2}(\beta^i_{k-1} + \beta^i_{k+1}),$$

(3.13)

$$\gamma^i_k = \frac{1}{2}(\gamma^i_{k-1} + \gamma^i_{k+1}),$$

and the initial conditions

$$\beta^0_0 = 0, \gamma^0_0 = 0, \text{ for all } i \geq 1, \beta^0_i = 1, \gamma^0_i = 1, \text{ for all } \ell \geq 0.$$

Proof. For every $\ell \in \mathbb{N}$ consider the following statement $L(\ell)$: for all $i \in \mathbb{N}$, $\frac{d^\alpha h^i_\ell}{dt}(\exp H)$ satisfies the estimate (3.12) and the constants $\beta^i_\ell, \gamma^i_\ell$ appearing in (3.12) satisfy the iteration formulas (3.13) and the initial conditions (3.14). We shall then prove by induction, that $L(\ell)$ holds for every $\ell \in \mathbb{N}$.

For $\ell = 0$ we have to prove that for all $i \in \mathbb{N}$, $\frac{d^\alpha h}{dt}(\exp H)$ satisfies the estimate (3.12) and that the constants $\beta^0_0, \gamma^0_0 > 0$ satisfy $\beta^0_0 = \gamma^0_0 = 0$, for all $i \geq 1$, and $\beta^0_0 = \gamma^0_0 = 1$.

Indeed, from Lemma 2 we get that for $i \geq 1$

$$|\frac{\partial^ih_t}{\partial t^i}(\exp H)| \leq ct^{-(n/2) - i}(1 + t)^m, \text{ for all } t > 0, H \in \overline{a^+}.$$

But, $1 \leq (1 + \|H\|)^A$. Thus

$$\frac{|\partial^ih_t}{\partial t^i}(\exp H)| \leq ct^{-(n/2) - i}(1 + t)^m(1 + \|H\|)^A, \text{ for all } t > 0, H \in \overline{a^+},$$

i.e. (3.12) holds true for all $i \geq 1$, with $\beta^0_0 = \gamma^0_0 = 0$. Furthermore, from the estimates of the heat kernel in (2.11), we obtain that

$$|h_t(\exp H)| \leq ct^{-n/2}(1 + t)^m(1 + \|H\|)^A e^{-\langle \|\rho\|^{2t} + (\rho, H) + \|H\|^2/4t \rangle}.$$ 

Thus (3.12) holds true also for $i = 0$ and $\beta^0_0 = \gamma^0_0 = 1$. Therefore the statement $L(0)$ holds true.

Let us assume now that $L(\ell - 1)$ holds true. Thus, for all $i \in \mathbb{N}$, there are constants $c, \beta^0_{\ell - 1}, \gamma^0_{\ell - 1} > 0$ such that $\frac{d^\alpha h^i_{\ell - 1}}{dt}(\exp H)$ satisfies the estimate (3.12).

We shall prove that $L(\ell)$ holds true. Indeed, from the estimates of the heat kernel in (2.11), we have that

$$|h_t(\exp H)| \leq ct^{-n/2}(1 + t)^m(1 + \|H\|)^A e^{-\langle \|\rho\|^{2t} + (\rho, H) + \|H\|^2/4t \rangle}.$$ 

Thus (3.12) holds true for $i = 0$ with $\beta^0_\ell = \gamma^0_\ell = 1$.

For $i \geq 1$, consider the function

$$f_H(t) = \frac{\partial^{i-1}h_t}{\partial t^{i-1}}(\exp H).$$

From the validity of $L(\ell - 1)$, we get that for $i - 1$ and $i + 1$ we have that

$$|f_H(t)| = \frac{|\partial^{i-1}h_t}{\partial t^{i-1}}(\exp H)| \leq ct^{-\alpha}(1 + t)^\beta (1 + d)^e^{-D\|t - B(\rho, H) + C\|\|H\|^2/(4t)}}$$

$$|\frac{d^2f_H(t)}{dt^2}| = \frac{|\partial^{i+1}h_t}{\partial t^{i+1}}(\exp H)| \leq ct^{-\alpha - 2}(1 + t)^\beta (1 + d)^e^{-D\|t - B(\rho, H) + C\|\|H\|^2/(4t)},$$

with $\alpha = (n/2) + i - 1, \beta = m, \gamma = A, D = B = \beta^i_{\ell - 1}, C = \gamma^i_{\ell - 1}$ and $D_* = B_* = \beta^{i+1}_{\ell - 1}, C_* = \gamma^{i+1}_{\ell - 1}$. 

Thus, by Lemma 4 applied for the function $f_H(t)$, it follows that
\[
\left| \frac{df_H}{dt}(t) \right| = \left| \frac{\partial^i h_t}{\partial t^i} (\exp H) \right| \leq c t^{-(n/2)-i} (1+t)^m (1+\|H\|^2) A e^{-\beta_i(\|\rho\|^2 t+\langle\rho,H\rangle)} e^{-\gamma_i\|H\|^2/(4t)},
\]
for all $i \geq 1$, where $\beta_i^i$ and $\gamma_i^i$ satisfy (3.13). Thus, the statement $L(\ell)$ is valid and the proof of the lemma is complete. \hfill \square

**Remark 1.** The constant $c = c(i, \ell, \epsilon)$ in relation (3.12) of Lemma 3 depends on $i, \ell$ and $\epsilon$ and it increased to infinity (when either $i \to \infty$ or $\ell \to \infty$ or $\epsilon \to 0$), but we only need the fact that it is finite for fixed $i, \ell, \epsilon$.

**Lemma 4.** For any $i \in \mathbb{N}$,
\[
\lim_{\ell \to \infty} \left( \frac{1 - \sqrt{1 - \lambda_i}}{\ell} \right)^i \quad \text{and} \quad \lim_{\ell \to \infty} \beta_i^\ell = 1.
\]

**Proof.** We shall deal only with $\gamma_i^\ell$. The proof that $\lim_{\ell \to \infty} \beta_i^\ell = 1$ is similar, and we shall omit it.

**Claim 1.** For every $\ell \in \mathbb{N}$ consider the following statement $L(\ell)$: for all $i \in \mathbb{N}$,
\[
(3.16) \quad \gamma_i^\ell \leq 1.
\]
We shall prove by induction that $L(\ell)$ is valid for all $\ell \in \mathbb{N}$.

For $\ell = 0$ we have to prove that for all $i \in \mathbb{N}$, we have that $\gamma_i^0 \leq 1$. Indeed, this is a consequence of the initial conditions $\gamma_0^0 = 0$ and $\gamma_0^0 = 1$. Thus $L(0)$ holds true.

Let us assume now that $L(\ell - 1)$ holds true. Thus, for all $i \in \mathbb{N}$, we have that $\gamma_i^{\ell-1} \leq 1$.

We shall prove that $L(\ell)$ holds true. Recall that by the induction assumption, for all $i \in \mathbb{N}$, for $i - 1$ and $i + 1$ we have that $\gamma_i^{i-1} \leq 1$ and $\gamma_i^{i+1} \leq 1$. Thus, from (3.13) it follows that
\[
\gamma_i^\ell = \frac{\lambda_i}{2} \gamma_i^{\ell-1} + \frac{1}{2} \gamma_i^{\ell+1} \leq \frac{\lambda_i}{2} + \frac{1}{2} \leq 1,
\]
thus the statement $L(\ell)$ is valid and this completes the proof of Claim 1.

**Claim 2.** For every $\ell \in \mathbb{N}$ consider the following statement $L(\ell)$: for all $i \in \mathbb{N}$,
\[
(3.18) \quad \gamma_i^\ell \leq \gamma_i^{\ell+1}.
\]
We shall prove that $L(\ell)$ is valid for all $\ell \in \mathbb{N}$. We proceed once again by induction in $\ell \in \mathbb{N}$.

For $\ell = 0$ we have to prove that for all $i \in \mathbb{N}$, $\gamma_i^0 \leq \gamma_i^1$. Indeed, from (3.14) it follows that $\gamma_0^i = 0 \leq \gamma_0^i$, for all $i > 0$ and $\gamma_0^0 = 1 = \gamma_0^1$. Therefore the statement $L(0)$ holds true.

Let us assume now that $L(\ell - 1)$ holds true, i.e. that for all $i \in \mathbb{N}$, $\gamma_i^{\ell-1} \leq \gamma_i^\ell$.

We shall prove that $L(\ell)$ holds true, i.e. that for all $i \in \mathbb{N}$, $\gamma_i^\ell \leq \gamma_i^{\ell+1}$. Recall that by (3.13) we have that
\[
(3.19) \quad \gamma_i^\ell = \frac{1}{2} (\lambda_i \gamma_i^{\ell-1} + \gamma_i^{\ell+1}).
\]
Then by the induction assumption for $i - 1$ and $i + 1$ we have that $\gamma_i^{\ell-1} \leq \gamma_i^{i-1}$ and $\gamma_i^{\ell+1} \leq \gamma_i^{i+1}$. Hence, from (3.19) we get that
\[
\gamma_i^\ell \leq \frac{1}{2} (\lambda_i \gamma_i^{i-1} + \gamma_i^{i+1}) = \gamma_i^{i+1}.
\]
Thus the statement $L(\ell)$ is valid and the proof of Claim 2 is complete.
Claim 3. For all $i \in \mathbb{N}$,

$$
\lim_{\ell \to \infty} \gamma_i^\ell = \left(1 - \sqrt{1 - \lambda_{\epsilon_1}}\right)^i.
$$

Note that by Claim 2, the sequence $\gamma_i^\ell$ is increasing in $\ell$ and by Claim 1, $\gamma_i^\ell$ is bounded above. Thus $\lim_{\ell \to \infty} \gamma_i^\ell$ exists and since $0 \leq \gamma_i^\ell \leq 1$, then

$$
\lim_{\ell \to \infty} \gamma_i^\ell = \gamma_i \leq 1.
$$

Note that $\gamma_0^0 = 1$, for all $\ell \in \mathbb{N}$, thus $\gamma_0 = 1$.

Now, taking limits in the iteration formula (3.19) we obtain that

$$
\gamma_i = \frac{1}{2} (\lambda_{\epsilon_1} \gamma_{i-1} + \gamma_{i+1}),
$$

thus

$$
\gamma_{i+1} - 2\gamma_i + \lambda_{\epsilon} \gamma_{i-1} = 0.
$$

This is a homogeneous linear recurrence relation with constant coefficients and the solutions of this equation are given by

$$
\gamma_i = C_1 r_1^i + C_2 r_2^i, \quad C_1, C_2 \in \mathbb{R},
$$

where $r_1, r_2$ are the roots of the equation

$$
r^2 - 2r + \lambda_{\epsilon} = 0.
$$

Thus, we conclude that

$$
\gamma_i = C_1 \left(1 - \sqrt{1 - \lambda_{\epsilon_1}}\right)^i + C_2 \left(1 + \sqrt{1 - \lambda_{\epsilon_1}}\right)^i,
$$

for some $C_1, C_2 \in \mathbb{R}$.

Since $0 \leq \gamma_i \leq 1$, we get $C_2 = 0$, otherwise $\lim_{i \to \infty} \gamma_i = \infty$. Also, since $\gamma_0 = 1$, we get $C_1 = 1$. Thus, from (3.21) for $C_1 = 1, C_2 = 0$, we get (3.20) and the proof is complete. \hfill $\square$

End of the proof of Theorem [1] To complete the proof of Theorem [1] notice that $\lim_{\epsilon_1 \to 0} \left(1 - \sqrt{1 - \lambda_{\epsilon_1}}\right)^i = 1$. Thus, taking $\ell \in \mathbb{N}$ sufficiently large and $\epsilon_1$ sufficiently close to zero, one has $\gamma_i^\ell \geq 1 - \epsilon$ and $\beta_i^\ell \geq 1 - \epsilon$. Thus, from (3.12) and (3.16) it follows that

$$
\left| \frac{\partial^k h_t}{\partial t^k} (x, y) \right| \leq c t^{-(n/2)-i} (1 + t)^\alpha (1 + ||H||) A e^{-\epsilon (||\rho||^2 t + ||\rho, H|| + ||H||^2/4t)}.
$$

Taking now into account that if $\alpha, \beta > 0$ then there exists a constant $c = c(\alpha, \beta)$ such that $x^\alpha \leq c e^{\beta x}$ for all $x > 0$, we conclude that for every $\epsilon > 0$, there exists a constant $c > 0$ such that

$$
\left| \frac{\partial^k h_t}{\partial t^k} (x, y) \right| \leq c t^{-(n/2)-i} e^{-\epsilon (||\rho||^2 t + ||\rho, H|| + ||H||^2/4t)},
$$

and the proof of Theorem [1] is complete.

Remark 2. If $X$ is a Cartan-Hadamard manifold then, [11], the heat kernel $h_t$ of $X$ satisfies some pointwise bounds of the type

$$
h_t(x, y) \leq \frac{c}{\min\{1, t^\alpha\}} e^{-At - Bd(x,y) -Cd^2(x,y)/t}, \quad t > 0, \quad x, y \in X,
$$
for some positive constants $c, A, B, C$ and $\alpha$. Proceeding as in Section 3, one can prove the following estimate: for all $\epsilon > 0$ and $i \in \mathbb{N}$, there is a constant $c > 0$ such that

$$\left| \frac{\partial^i h_t}{\partial t^i}(x, y) \right| \leq \frac{c}{\min\{1, t^{\alpha+i}\}} e^{-(1-\epsilon)(At+Bd(x,y)+C\rho^2(x,y)/t)}.$$  

Proof of Corollary 3. Let us recall that if $X$ is a complete, non-compact, $n$-dimensional Riemannian manifold, with Ricci curvature bounded from below by $-R^2$, then by [14] for $\gamma > 1$, we have that

$$\frac{\|\nabla h_t(x, y)\|^2}{h_t^2(x, y)} - \frac{1}{h_t(x, y)} \frac{\partial h_t(x, y)}{\partial t} \leq \frac{nR^2\gamma^2}{\sqrt{2(\gamma-1)}} + \frac{n\gamma^2}{2t} \quad \text{for all } t > 0, x, y \in X.$$  

Using Theorem 1 and (3.22) we shall prove Corollary 1. Indeed, by (3.22), it follows that

$$\|\nabla h_t(x, y)\|^2 \leq h_t^2(x, y) \left( \frac{nR^2\gamma^2}{\sqrt{2(\gamma-1)}} + \frac{n\gamma^2}{2t} \right) + h_t(x, y) \frac{\partial h_t(x, y)}{\partial t}.$$  

Next, note that there is a constant $c > 0$ such that

$$\frac{nR^2\gamma^2}{\sqrt{2(\gamma-1)}} + \frac{n\gamma^2}{2t} \leq c \frac{1+t}{t}.$$  

Thus, using the heat kernel estimate (2.11) as well as Theorem 1 and inequalities (3.23) and (3.24), we obtain

$$\|\nabla h_t(\exp H)\|^2 \leq \left( c_1 t^{-n/2} (1+t)^m (1 + \|H\|)^A e^{-\|\rho\|^2 t} e^{-(\rho, H)} e^{-\|H\|^2/(4t)} \right)^2 \left( \frac{1+t}{t} \right) + c_2 \left( t^{-n/2} (1+t)^m (1 + \|H\|)^A e^{-\|\rho\|^2 t} e^{-(\rho, H)} e^{-\|H\|^2/(4t)} \right) \times \left( t^{-(n/2)} e^{-(1-\epsilon)\|\rho\|^2 t} e^{-(1-\epsilon)(\rho, H)} e^{-(1-\epsilon)\|H\|^2/(4t)} \right).$$

But, using that for $\alpha, \beta > 0, x^\alpha \leq e^{\beta x}$ for all $x > 0$, we conclude that for every $\epsilon > 0$ there exists a constant $c > 0$, such that

$$\|\nabla h_t(\exp H)\| \leq ct^{-(n+1)/2} e^{-(1-\epsilon)(\|\rho\|^2 t+(\rho, H)+\|H\|^2/(4t))},$$

and the proof of Corollary 1 is complete.

4. Estimates of the time derivatives of the heat kernel on locally symmetric spaces

In this section we obtain estimates of the heat kernel time derivatives in the case of a locally symmetric space $M = \Gamma \backslash X$. Our results extend the estimates of Weber, [22, 23], we presented in Section 2.3.

Suppose that $\delta(\Gamma) < \|\rho\| + \rho_m$. Recall that

$$h_t^M(\tilde{x}, \tilde{y}) = \sum_{\gamma \in \Gamma} h_t(x, \gamma y), \quad \text{for all } x, y \in X, t > 0.$$  

and

$$d_M(\tilde{x}, \tilde{y}) = \inf_{\gamma \in \Gamma} d(x, \gamma y).$$
Consider $\alpha_2 \in (\delta(\Gamma), \|\rho\| + \rho_m)$ and $\alpha_1, \alpha_3 \in (0, 1]$ such that $\alpha_1 \alpha_3 \in \left(\frac{\alpha_2 - \rho_m}{\|\rho\|}, 1\right]$. We shall prove that for all $\epsilon > 0$ there is a constant $c > 0$, such that

$$
(4.2) \quad \left| \frac{\partial^2 h^M}{\partial t^2} (\bar{x}, \bar{y}) \right| \leq c t^{-\alpha_1 \|\rho\| - \alpha_2 \|\rho\| + \alpha_3 \frac{\|\rho\|^2}{4t}} (x, y).
$$

According to Theorem 1,

$$
\left| \frac{\partial^2 h^M}{\partial t^2} (\exp H) \right| \leq c t^{-\alpha_1 \|\rho\|^2 t + \alpha_2 \|\rho\| + \alpha_3 \frac{\|\rho\|^2}{4t}} (x, y).
$$

Thus, taking into account that $d(x, y) = \|H\|$, we obtain the estimate

$$
(4.3) \quad \left| \frac{\partial^2 h^M}{\partial t^2} (x, y) \right| \leq c t^{-\alpha_1 \|\rho\| - \alpha_2 \|\rho\| + \alpha_3 \frac{\|\rho\|^2}{4t}} (x, y).
$$

Note that

$$
(4.4) \quad \alpha_1 \|\rho\|^2 t + (\rho_m - \alpha_2) d_M (\bar{x}, \bar{y}) + \alpha_3 \frac{d_M^2 (\bar{x}, \bar{y})}{4t} \geq 0,
$$

since $\alpha_1 \alpha_3 \in \left(\frac{\alpha_2 - \rho_m}{\|\rho\|}, 1\right]$ , with equality when $\alpha_2 = \rho_m > \delta(\Gamma)$ and $\alpha_1 = \alpha_3 = 0$.

Thus, from (4.3) and (4.4) it follows that

$$
(4.5) \quad \left| \frac{\partial^2 h^M}{\partial t^2} (x, y) \right| \leq c t^{-\alpha_1 \|\rho\|^2 t + \alpha_2 d_M (x, y) + \alpha_3 \frac{d_M^2 (x, y)}{4t}} (x, y).
$$

Thus, a summation argument implies that for every $\epsilon > 0$ we have

$$
\left| \frac{\partial^2 h^M}{\partial t^2} (\bar{x}, \bar{y}) \right| \leq c t^{-\alpha_1 \|\rho\|^2 t + \alpha_2 d_M (\bar{x}, \bar{y}) + \alpha_3 \frac{d_M^2 (\bar{x}, \bar{y})}{4t}} (x, y),
$$

and the proof is complete.

5. Applications

In this section we apply the estimates of the derivatives of the heat kernel and we obtain the $L^p$-boundedness of some operators related to the heat semigroup. More precisely we prove Theorem 3, Theorem 4 and Theorem 5.

5.1. Proof of Theorem 3. We shall prove that the operators $H^{\max}_\sigma$ and $H_\sigma$ defined in Section 1 are bounded on $L^p(M), 1 < p < \infty$, provided that

$$
\sigma < s(p) \left( \|\rho\| - \|\eta\| \right) \left( 2 \|\rho\| - s(p) \left( \|\rho\| - \|\eta\| \right) \right),
$$

where

$$
(5.1) \quad s(p) = \min \left\{ \frac{1}{p}, \frac{1}{p'} \right\} \leq 1.
$$
This is the analogue of Anker’s result \cite{2} in the case of a locally symmetric space $M$, with $M$ in the class $\mathcal{LM}$. We shall give only the proof for $H^\text{max}_\sigma$. The proof for $H_\sigma$ is similar and then omitted.

**Proof of the $L^p$-boundedness of $H^\text{max}_\sigma$.** Denote by $k_\sigma(\exp H)$ the kernel of the operator $H^\text{max}_\sigma$. Recall that if $f \in C_0^\infty(M)$, then

\begin{equation}
(5.2) \quad H^\text{max}_\sigma f(g) = \int_G k_\sigma(x^{-1}g)f(x)dx = \sup_{t>0} e^{\sigma t} \frac{\partial^i}{\partial t^i} \int_G h_t(x^{-1}g)f(x)dx.
\end{equation}

Using that $h_t$ is $K$-bi-invariant, we obtain that if $f \in C_0^\infty(M)$, then $H^\text{max}_\sigma f$ is right $K$-invariant and left $\Gamma$-invariant i.e. a function on $M$. Thus (5.2) define $H^\text{max}_\sigma$ as an operator on $M$.

Recall that

$$k_\sigma(\exp H) = \sup_{t>0} e^{\sigma t} \frac{\partial^i}{\partial t^i} h_t(\exp H), \quad t > 0, \quad H \in \mathbf{a}^r.$$ 

Write

$$k_\sigma = \sup_{0<t<1} e^{\sigma t} \frac{\partial^i}{\partial t^i} h_t + \sup_{t \geq 1} e^{\sigma t} \frac{\partial^i}{\partial t^i} h_t = k_\sigma^0 + k_\sigma^\infty.$$ 

Denote by $H^0_\sigma$, $H^\infty_\sigma$ the convolution operators whose kernels are $k_\sigma^0$ and $k_\sigma^\infty$ respectively.

We shall split $k_\sigma^0$ into two parts

\begin{equation}
(5.3) \quad k_\sigma^0 = \psi k_\sigma^0 + (1-\psi) k_\sigma^0 = k_\sigma^{0,0} + k_\sigma^{0,\infty},
\end{equation}

where $\psi \in C^\infty_c(K\backslash G/K)$ with $\psi \equiv 1$ near the origin and $\psi \equiv 0$ in $B(0, 2)^c$.

The splitting (5.3) of $k_\sigma^0$ implies the following splitting of $H^0_\sigma$:

$$H^0_\sigma(f)(x) = (f * k_\sigma^{0,0})(x) \quad \text{and} \quad H^\infty_\sigma(f)(x) = (f * k_\sigma^{0,\infty})(x).$$

For the treatment of the operators $H^\infty_\sigma$ and $H^{0,\infty}_\sigma$ we need the following lemma.

**Lemma 5.** For all $\epsilon > 0$ there exists $c > 0$ such that

$$|k_\sigma^{0,\infty}(\exp H)|, \quad |k_\sigma^{0,\infty}(\exp H)| \leq ce^{-\epsilon <\rho, H>_\Delta} e^{-(1-\epsilon)\|H\|/\sqrt{\|\rho^2 - \frac{\sigma}{4\pi} t/\epsilon}},$$

for all $H \in \mathbf{a}^r$.

**Proof.** From the estimates (1.1) of the derivative $\frac{\partial^i}{\partial t^i} h_t$ provided by Theorem 1 we get that

$$|k_\sigma^{\infty}(\exp H)| = \sup_{t \geq 1} e^{\sigma t} \left| \frac{\partial^i}{\partial t^i} h_t(\exp(H)) \right|$$

$$\leq \sup_{t \geq 1} e^{-\epsilon <\rho, H>_\Delta} \left( \|\rho\|^2 t + <\rho, H>_\Delta + \|H\|^2/(4t) \right)$$

$$\leq \sup_{t \geq 1} e^{-\epsilon <\rho, H>_\Delta} e^{-(1-\epsilon)\|H\|^2/(4t)}$$

$$\leq ce^{-(1-\epsilon) <\rho, H>_\Delta} e^{-(1-\epsilon)\|H\|/\sqrt{\|\rho^2 - \frac{\sigma}{4\pi} t/\epsilon}}.$$
Similarly, from (11), we get that for \( \|H\| \geq 2 \),
\[
|k_{\sigma}^{0,0}(\exp H)| \leq \operatorname{csup}_{t \leq 1} -n/2 e^{-(1-\epsilon)\|H\|^2/(4t)} e^{\sigma t} e^{-(1-\epsilon)(\|\rho\|^2 t + \langle \rho, H \rangle)}
\leq \operatorname{csup}_{t \leq 1} -n/2 \|H\|^n/2 e^{-(1-2\epsilon)\|H\|^2/(4t)} e^{\sigma t} e^{-(1-\epsilon)(\|\rho\|^2 t + \langle \rho, H \rangle)}
\leq \operatorname{csup}_{t \leq 1} e^{-(1-2\epsilon)\|H\|^2/(4t)} e^{\sigma t} e^{-(1-2\epsilon)(\|\rho\|^2 t + \langle \rho, H \rangle)}
\leq ce^{-(1-2\epsilon)\|H\|^2/(4t)} e^{-(1-2\epsilon)\|H\|^2/2} e^{-\sigma t}.
\]
where we have used the inequality
\[
(1 - 2\epsilon) \|H\|^2/(4t) + t (1 - 2\epsilon) \|\rho\|^2 - \sigma \geq (1 - 2\epsilon) \sqrt{\|\rho\|^2 - \frac{\sigma}{1 - 2\epsilon}}.
\]

\[ \square \]

**Proof of Theorem** \( \Box \) Recall that we have the following splitting of the maximal operator \( H_\sigma \):
\[
H_\sigma = H_{\sigma}^{0,0} + H_{\sigma}^{0,\infty} + H_{\sigma}^{\infty}.
\]

To prove the theorem, we shall prove the \( L^p \)-boundedness of the above operators separately.

**Step 1. Continuity of** \( H_{\sigma}^{0,0} \) **on** \( L^p(M) \). First, we note that Anker \[2\] p. 278] proves that \( H_{\sigma}^{0,0} \) is bounded on \( L^p(X) \) for all \( p \in (1, \infty) \), but in fact the same arguments give the continuity of \( H_{\sigma}^{0,0} \) on \( L^p(G) \). Next, proceed as in \[14\] Sections 4.3 and 5.1 where it is proved that for \( u, v \in C_0^\infty (\Gamma \backslash G) \), there are functions \( u_j \in L^p(G) \) and \( v_j \in L^{p'}(G) \), \( j \in \mathbb{N} \), such that
\[
\sum_j \|u_j\|_{L^p(G)} \|u_j\|_{L^{p'}(G)} \leq \|u\|_{L^p(\Gamma \backslash G)} \|u\|_{L^{p'}(\Gamma \backslash G)}
\]
and
\[
I(u, v) := \int_{\Gamma \backslash G} H_{\sigma}^{0,0} u (\tilde{x}) v (\tilde{x}) d\tilde{x}
\]
\[
= \sum_j \int_G (k_{\sigma}^{0,0} * u_j) (x) \tilde{v}_j (x) dx,
\]
where \( \tilde{u} (x) = u (x^{-1}) \).

But, as it is already mentioned, the operator \( f \rightarrow k_{\sigma}^{0,0} * f \) is bounded on \( L^p(G) \), \( p \in (1, \infty) \), \[2\] p. 278]. So, from (5.4) and (5.5), it follows that
\[
|I(u, v)| \leq \sum_j \|k_{\sigma}^{0,0} * u_j\|_{L^p(G)} \|v_j\|_{L^{p'}(G)}
\]
\[
\leq \sum_j c \|u_j\|_{L^p(G)} \|v_j\|_{L^{p'}(G)}
\]
\[
\leq c \|u\|_{L^p(\Gamma \backslash G)} \|v\|_{L^{p'}(\Gamma \backslash G)} ,
\]
which implies that \( H_{\sigma}^{0,0} \) is bounded on \( L^p(M) \) for all \( p \in (1, \infty) \).

**Step 2. Continuity of** \( H_{\sigma}^{\infty} \) **on** \( L^p(M) \). According to Kunze-Stein phenomenon for locally symmetric spaces we recalled in Section 1,
where \( s(p) \) is given in (5.1) and \( \eta_H \) is a vector on the euclidean sphere \( S(0, (\|\rho\|^2 - \lambda_0)^{1/2}) \) of \( \mathfrak{a}^* \). Thus, to prove Theorem 3 it is enough to show that the integral in (5.6) converges.

Recall that by the Cartan decomposition, the Haar measure on \( G \) is written as

\[
\int_{G} f(g) \, dg = \int_{K} dk \int_{\mathfrak{a}^+} \delta(H) \, dH \int_{K} f(k_1 (\exp H) k_2) \, dk_2,
\]

where, \( dk \) is the normalised Haar measure of \( K \) and the modular function \( \delta(H) \) satisfies:

\[
\delta(H) \leq c^{2(\rho,H)}, \quad H \in \mathfrak{a}^+.
\]

From (5.7) it follows that if \( f \) is \( K \)-bi-invariant, then

\[
\int_{G} f(g) \, dg = \int_{\mathfrak{a}^+} f(\exp H) \, \delta(H) \, dH.
\]

First, recall that there are positive constants \( c \) and \( \alpha \) such that

\[
0 < \phi_{-i\lambda}(\exp H) \leq c(1 + \|H\|)^\alpha \exp(\lambda(H) - \rho(H)), \quad H, \lambda \in \mathfrak{a}^+,
\]

see [18] for more details.

Using the estimates of \( k_{\sigma}^\infty \) obtained in Lemma 5 and (5.9), we get that

\[
\int_{G} |k_{\sigma}^\infty(g)| \phi_{-i\lambda}(g) s(p) \, dg \leq c \int_{\mathfrak{a}^+} e^{- (1 - \epsilon) \left( |\rho,H| + \sqrt{\|\rho\|^2 - \frac{\sigma}{1 - \epsilon} \|H\|} \right)} e^{s(p)(|\eta_H,H| - |\rho,H|)} e^{2|\rho,H|} \, dH
\]

\[
\leq c \int_{\mathfrak{a}^+} e^{(1 + \epsilon - s(p))\|\rho\|\|H\| + s(p)\|\eta_H\|\|H\|} e^{- (1 - \epsilon)\|H\|\sqrt{\|\rho\|^2 - \frac{\sigma}{1 - \epsilon}}} \, dH.
\]

The integral above converges provided that

\[
(1 + \epsilon - s(p)) \|\rho\| + s(p) \|\eta_H\| - (1 - \epsilon) \sqrt{\|\rho\|^2 - \frac{\sigma}{1 - \epsilon}} < 0.
\]

Choosing \( \epsilon \) small enough, it follows from (5.11) that the integral in (5.10) converges when

\[
\sigma < s(p)(\|\rho\| - \|\eta_H\|)(2 \|\rho\| - s(p)(\|\rho\| - \|\eta_H\|)).
\]

Thus, \( H_{\sigma}^\infty \) is bounded on \( L^p(M) \), \( p \in (1, \infty) \), if (5.12) holds true.

**Step 3. Continuity of \( H_{\sigma}^{0,\infty} \) on \( L^p(M) \).** Similarly to the step 1, we have that

\[
\|H_{\sigma}^{0,\infty}\|_{L^p(M) \to L^p(M)} \leq c \int_{G} |k_{\sigma}^{0,\infty}(g)| \phi_{-i\lambda}(g) s(p) \, dg.
\]

Also, by Lemma 3 \( k_{\sigma}^{0,\infty} \) and \( k_{\sigma}^\infty \) satisfy the same estimates. Thus, exactly as in step 1, we conclude that \( H_{\sigma}^{0,\infty} \) is bounded on \( L^p(M) \), \( p \in (1, \infty) \), if (5.12) holds true. \( \square \)

**Remark 3.** Consider the Poisson operator \( P_t = e^{-t(-\Delta)^{1/2}} \), whose kernel is given by

\[
p_t = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} u^{-3/2} e^{-t^2/4u} h_u \, du,
\]

(see [2] for more details).
Define the corresponding $\sigma-$maximal operator and Littlewood-Paley-Stein operators. Then, in a similar way, one can prove that these operators are bounded on $L^p(M)$, provided that

$$\sigma < \sqrt{s(p) (\|\rho\| - \|\eta\|) (2 \|\rho\| - s(p) (\|\rho\| - \|\eta\|))}.$$  \hfill (5.14)

If $\eta = 0$ then the condition (5.14) on $\sigma$ becomes $\sigma < 2 \|\rho\| / \sqrt{pp'}$, thus we recover the result of Anker in [2].

5.2. Sketch of the proof of Theorem 4. In this section we deal with the Riesz operator $\nabla(-\Delta_M)^{-1/2}$ which is a convolution operator with kernel

$$k = \int_0^\infty \nabla h_t \frac{dt}{\sqrt{t}}.$$  

Note that $k$ is $K$-bi-invariant. As in the case of the operator $H_\sigma$, we split $k$ as follows:

$$k = \int_0^1 \nabla h_t \frac{dt}{\sqrt{t}} + \int_1^\infty \nabla h_t \frac{dt}{\sqrt{t}} = k^0 + k^\infty.$$  

We also split $k^0$ by writing

$$k^0 = \psi k^0 + (1 - \psi) k^0 = k^{0,0} + k^{0,\infty},$$  

where

$$\psi \in C_c^\infty(K \setminus G/K) \text{ with } \psi \equiv 1 \text{ near the origin and } \psi \equiv 0 \text{ in } B(0,2).$$

From Corollary 1 it follows that for all $\epsilon > 0$ there exists $c > 0$ such that

$$\|k^\infty\|, \|k^{0,\infty}\| \leq ce^{-(1-\epsilon)(\rho,H) + \|\rho\|/\|H\|}, \text{ for all } H \in \mathfrak{a}^+,$$  

Using the above estimate and proceeding exactly as in the proof of Theorem 3 one can prove the $L^p$-boundedness of $\nabla(-\Delta_M)^{-1/2}$.

5.3. Sketch of the proof of Theorem 5. In this section we prove Theorem 5 which gives estimates of the norms

$$\|\Delta_M e^{-t\Delta_M}\|_{L^p(M) \to L^p(M)} \text{ and } \|\nabla e^{-t\Delta_M}\|_{L^p(M) \to L^p(M)}.$$  

The operators $\Delta_M e^{-t\Delta_M}$ and $\nabla e^{-t\Delta_M}$ are convolution operators with kernel $\frac{\partial h_t}{\partial t}$ and $\nabla h_t$, respectively, which are $K$-bi-invariant. We shall deal only with the operator $S_t := \Delta_M e^{-t\Delta_M}$. The treatment of the operator $\nabla e^{-t\Delta_M}$ is similar and thus omitted.

Recall that from Theorem 11 if follows that for all $\epsilon > 0$ there exists $c > 0$ such that

$$\left| \frac{\partial h_t}{\partial t} \right| (\exp H) \leq ce^{-\epsilon} e^{-(1-\epsilon)(\rho,H) + \sqrt{\|\rho\|^2 - \epsilon}} \text{, for } t \geq 1, \text{ } H \in \mathfrak{a}^+.$$  

For $t \geq 1$, proceeding as in the proof of Theorem 3 the above estimate and the Kunze-Stein phenomenon, imply that

$$\|S_t\|_{L^p(M) \to L^p(M)} \leq ce^{-\epsilon t} \int_{\mathfrak{a}^+} e^{(1-s(p)+\epsilon)\|\rho\| - (1-\epsilon)\sqrt{\|\rho\|^2 - \epsilon}} dH.$$  \hfill (5.19)

The integral above converges whenever

$$s(p)(\|\eta\| - \|\rho\|) + \epsilon(\|\rho\| + \sqrt{\|\rho\|^2 - \epsilon}) + (\|\rho\| - \sqrt{\|\rho\|^2 - \epsilon}) < 0,$$

which holds true if sufficiently small $\epsilon > 0$ since $\|\eta\| < \|\rho\|$.

Furthermore, from (5.19) we get that

$$\|S_t\|_{L^p(M) \to L^p(M)} \leq ce^{-\epsilon t}, \text{ for all } t \geq 1.$$
For $t < 1$, as in the proof of Theorem 3, we split the kernel $\frac{\partial h_t}{\partial t}$ of the operator $S_t$ as follows:

\begin{equation}
\frac{\partial h_t}{\partial t} = \psi \frac{\partial h_t}{\partial t} + (1 - \psi) \frac{\partial h_t}{\partial t} (\exp H) = k^0_t + k^\infty_t,
\end{equation}

where $\psi$ is defined in (5.16).

The splitting (5.20) of $\frac{\partial h_t}{\partial t}$ implies the following splitting of $S_t$:

$$S_0f = (f * k^0_t)(x)$$

and

$$S^\infty_t(f)(x) = (f * k^\infty_t)(x).$$

For the treatment of the operators $S^\infty_t$ we proceed as in the case $t \geq 1$, i.e. first, we observe that from Theorem 1, if follows that for all $\epsilon > 0$ there exists $c > 0$ such that

\begin{equation}
|k^\infty_t (\exp H)| \leq ce^{-ct} e^{-(1-\epsilon)\langle\rho,H\rangle} + \sqrt{\|\rho\|^2 - \epsilon\|H\|}, \text{ for } t < 1, \ H \in \mathfrak{a}.
\end{equation}

Then, using the above estimate and the Kunze-Stein phenomenon we prove that

$$\|S^\infty_t\|_{L^p(M) \rightarrow L^p(M)} \leq ce^{-ct}, \text{ for all } t < 1.$$ 

For the continuity of $S^0_t$ on $L^p(M)$ we observe first that since the group $G$ is a complete Riemannian manifold with bounded geometry, the result of Davies [7, p. 376], can be applied and we deduce that

$$\|S^0_t\|_{L^p(G) \rightarrow L^p(G)} \leq \frac{c}{t}, \text{ for all } t < 1, \ p \in [1, \infty].$$

Next, the same approach as in the proof of Theorem 3 Step 1, allow us to deduce that

$$\|S^0_t\|_{L^p(M) \rightarrow L^p(M)} \leq \frac{c}{t}, \text{ for all } t < 1, \ p \in [1, \infty].$$

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