Numerical indications on the semiclassical limit of the flipped vertex

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Abstract
We introduce a technique for testing the semiclassical limit of a quantum gravity vertex amplitude. The technique is based on the propagation of a semiclassical wave packet. We apply this technique to the newly introduced ‘flipped’ vertex in loop quantum gravity, in order to test the intertwiner dependence of the vertex. Under some drastic simplifications, we find very preliminary, but surprisingly good numerical evidence for the correct classical limit.

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Suppose you are explicitly given the propagation kernel $W_t(x, y)$ of a one-dimensional nonrelativistic quantum system defined by a Hamiltonian operator $H$

$$W_t(x, y) = \langle x | e^{-\frac{i}{\hbar} H t} | y \rangle$$ (1)

and you want to study whether the classical ($\hbar \to 0$) limit of this quantum theory yields a certain given classical evolution. One of the (many) ways of doing so is to propagate a wave packet $\psi_{x,p}(x)$ with $W_t(x, y)$.

Suppose that in the time interval $t$ the classical theory evolves the initial position and momentum $x_i, p_i$ to the final values $x_f, p_f$. Then you can consider a semiclassical wave packet $\psi_{x_i,p_i}(y)$ centered on the initial values $x_i, p_i$, compute its evolution under the kernel

$$\phi(x) := \int dy W_t(x, y) \psi_{x_i,p_i}(y)$$ (2)

and ask whether or not this state is a semiclassical wave packet centered on the correct final values $x_f, p_f$. In this paper, we consider the possibility of using this method for exploring the semiclassical limit of the dynamics of nonperturbative quantum gravity.

An explicit expression for the vertex amplitude in loop quantum gravity has been recently proposed in [1]; we call it here the ‘flipped’ vertex, following [2]. This expression corrects

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certain difficulties that have emerged with the Barrett–Crane vertex amplitude [3]. In particular, the Barrett–Crane vertex was shown to have a good semiclassical behavior as far as its dependence on spin variables was concerned, but not in its intertwiner sector. The flipped vertex \( W(j_{nm}, i_n) \) is a function of 10 spin variables \( j_{nm} \) where \( n, m = 1, \ldots, 5 \) and five intertwiner variables \( i_n \) and it is hoped that it will correct the intertwiner dependence of the Barrett–Crane vertex. Here we investigate this intertwiner dependence. A number of variants of the flipped vertex have appeared in the literature [4], but we do not consider these variants here.

The derivation of the vertex amplitude presented in [2] indicates that the process described by one vertex can be seen as the dynamics of a single cell in a Regge triangulation of general relativity. This is a fortunate situation, because it allows us to give a simple and direct geometrical interpretation of the dynamical variables entering the vertex amplitude, and a simple formulation of the dynamical equations. The boundary of a Regge cell is formed by five tetrahedra joined along all their faces, thus forming a closed space with the topology of a 3-sphere. Denote by \( A_{nm} \) the area of the triangle \((nm)\) that separates the tetrahedra \(n\) and \(m\). Denote by \( \alpha_{(mp,qr)}^n \) the angle between the triangles \((mp)\) and \((qr)\) in the tetrahedron \(n\). And denote by \( \Theta_{1nm} \) the angle between the normals to the tetrahedra \(n\) and \(m\). These quantities determine entirely the intrinsic \((A_{nm}, \alpha_{(mp,qr)}^n)\) and extrinsic \((\Theta_{1nm})\) classical geometry of the boundary surface.

The 10 spins \( j_{nm} \) are the quantum numbers of the areas \( A_{nm} \) [2, 5]. The five intertwiners \( i_n \) are the quantum numbers associated with the angles \( \alpha_{(mp,qr)}^n \). More precisely, they are the eigenvalues of the quantity
\[
i_n^{(mp,qr)} = A_{mp}^2 + A_{qr}^2 + 2A_{mp}A_{qr} \cos \alpha_{(mp,qr)}^n.
\]
Each tetrahedron has six such angles, of which only two are independent (at given values of the areas); but the two corresponding quantum operators do not commute [6] and a basis of the Hilbert space on which they act can be obtained by diagonalizing just a single arbitrary one among these angles. Therefore the intrinsic geometry of the boundary of a classical Regge cell is determined by 20 numbers, but the corresponding quantum numbers are only 15: the 15 quantities \( j_{nm}, i_n \). These are the fifteen arguments of the vertex. When using the intertwiners \( i_n \), we have of course to specify to which pairing \( i_n^{(mp,qr)} \) we are referring.

The equations of motion of any dynamical system can be expressed as constraints on the set formed by the initial, final and (if it is the case) boundary variables. For instance, in the case of the evolution of a free particle in the time interval \( t \), the equations of motion can be expressed as constraints on the set \((x_i, p_i, x_f, p_f)\). These constraints are of course \( m(x_f - x_i)/t = p_i = p_f \). (For the general logic of this approach to dynamics, see [8].)

In general relativity, the Einstein equations can be seen as constraints on boundary variables \( A_{nm}, \alpha_{(mp,qr)}^n \) and \( \Theta_{1nm} \). These, in fact, can be viewed as the ensemble of the initial, boundary and final data for a process happening inside the boundary 3-sphere. Such constraints are a bit difficult to write explicitly, but one solution is easy: the one that corresponds to flat space and to the boundary of a regular 4-simplex. This is given by all equal areas \( A_{nm} = j_0 \), all equal angles \( i_n = i_0 \), and \( \Theta_{1nm} = \Theta \), where elementary geometry gives
\[
i_0 = \frac{2}{\sqrt{3}} j_0, \quad \cos \Theta = -\frac{1}{4}.
\]

It follows immediately from these considerations that a boundary wave packet centered on these values must be correctly propagated by the vertex amplitude, if the vertex amplitude is to give the Einstein equations in the classical limit.
The simplest wave packet we may consider is a diagonal Gaussian wave packet

$$\psi(j_{nm}, i_n) = \prod_{nm, n} \left[ \psi(j_{nm}) \prod_n \psi(i_n) \right],$$

(5)

where

$$\psi(j_{nm}) = e^{-r(j_{nm} - j_{0})^2 + \Theta_{1_{nm}}},$$

(6)

and

$$\psi(i) = N \sqrt{d_i} e^{-\sigma(i - i_{0})^2 + \Theta_{1} i}.$$

(7)

The normalization factors \(\sqrt{d_i} = \sqrt{2i + 1}\), here and below, are required by the conventions we use in this paper, where the 3\(j\)-symbols are normalized. For details on the conventions, see [3]. The constants \(\sigma\) and \(\theta\) are fixed by the requirement that the state is peaked on the value \(i_n = i_0\) also when we change the pairing at the vertex. That is, by the requirement that all angles of the tetrahedron are equally peaked on \(i_n = i_0\). It was shown in [7] that this requirement fixes these constants to the values

$$\sigma = \frac{3}{4j_{0}}, \quad \theta = \frac{\pi}{2}.$$

(8)

In other words, the state considered is formed by a Gaussian state on the spins, with \(\Theta\)-phases given by the extrinsic curvature and by a ‘coherent tetrahedron’ state (see [7])

$$\psi(i) = N \sqrt{d_i} e^{-\sigma(i - i_{0})^2 + \Theta_{1} i}$$

(9)

for each tetrahedron.

Let us write the wave packet (5) as an ‘initial state’ times a ‘final state’ by viewing the process represented by the spacetime region described by the Regge cell as a process evolving four tetrahedra into one. That is, let us write this state in the form

$$\psi(j_{nm}, i_n) = \psi_\text{init}(j_{nm}, i'_n) \psi(i),$$

(10)

where \(i'_n = (i_1, \ldots, i_4)\). Then we can test the classical limit of the vertex amplitude by computing the evolution of the four ‘incoming’ tetrahedra generated by the vertex amplitude

$$\phi(i) = \sum_{j_{nm}, i'_n} W(j_{nm}, i'_n, i) \psi_\text{init}(j_{nm}, i'_n),$$

(11)

where \(i\) is \(i_5\), and comparing \(\phi(i)\) with \(\psi(i)\). If the vertex amplitude has general relativity as its classical limit, then we expect that in the semiclassical limit, namely for large \(j_{0}\), the evolution should evolve the ‘initial’ boundary state \(\psi_\text{init}(j_{nm}, i'_n)\) into a final state \(\phi(i)\) which is still a wave packet centered on the same classical tetrahedron as the state \(\psi(i)\) given in (9). That is, \(\phi(i)\) must be a state ‘similar’ to \(\psi(i)\), plus perhaps quantum corrections representing the quantum spread of the wave packet.

We have tested this hypothesis numerically, under a drastic approximation: replacing the Gaussian dependence on the spins with a state concentrated on \(j_{nm} = j_{0}\). That is, we have tested the hypothesis in the \(\tau \rightarrow \infty\) limit. We are not sure this approximation makes sense, because in this limit the \(\Theta\) phase dependence drops, and this may jeopardize the semiclassical limit. This approximation is therefore only a first tentative hypothesis, that allows us to perform numerical calculations. Numerical calculations have proven useful in quantum gravity [9], and the numerical investigation of the new vertex has already begun [10]; here we consider a different angle from which the problem can be explored.
The hypothesis we want to test is thus the following. We want to compare the evolved state
\[
\phi(i) = \sum_{i_1,\ldots,i_4} W(i_1,\ldots,i_4,i) \prod_{n=1}^{4} \psi(i_n)
\]
with the coherent tetrahedron state \(\psi(i)\), where
\[
W(i_n) := W(j_{nm},i_n)|_{j_{nm}=j_0}.
\]
If the function \(\phi(i)\) turns out to be sufficiently close to the coherent tetrahedron state \(\psi(i)\), we can say that, under the hypotheses given, the flipped vertex amplitude appears to evolve four coherent tetrahedra into one coherent tetrahedron, consistently with the flat solution of the classical Einstein equations.

From [2] the flipped vertex reads in the present case
\[
W(i_n) = \sum_{\ell^+_{i_n},\ell^-_{i_n}} 15 \left( \frac{j_0}{2},i_n^+ \right) 15 \left( \frac{j_0}{2},i_n^- \right) \prod_{n=1}^{4} f_{i_n}^{\ell^+_n,\ell^-_n},
\]
where
\[
f_{i_n}^{\ell^+_n,\ell^-_n} = \sqrt{d_{i_n^-} d_{i_n^+}} \prod_{n=1}^{4} \left( \frac{j_0}{2}, \frac{j_n}{2}, \frac{j_n}{2} \right)
\]
The \(15j\)-symbol is the contraction of five 4-valent intertwiners, each constructed with two Wigner \(3j\)-symbols and each normalized with a \(\sqrt{\Delta}\) factor. The matrix indicates the Wigner \(3j\)-symbol. The indices \(\ell_n\) are in the representation \(j_0\) and the indices \(b_n\) and \(c_n\) in the representation \(j_0/2\). The quantities \(i_{a_1-a_4}\) are the components of the invariant tensors that define the intertwiner \(i\). For technical details, see [3].

We have compared the two functions \(\psi(i)\) (coherent tetrahedron) and \(\phi(i)\) (evolved state) for the cases \(j_0 = 2\) and \(j_0 = 4\). The numerical results are shown in the figures below. The overall relative amplitude of \(\psi(i)\) and \(\phi(i)\) is freely adjusted by fixing the normalization constant \(N\) and therefore is not significant. The quantity \(i_{\text{mean}}\) is the mean value of \(i\). It gives the position of the wave packet. The quantity \(\sigma/2\) is the corresponding variance. It gives the (half) width of the wave packet. In figures 1 and 3 we compare the modulus square of the wavefunction (for the two values of \(j_0\)). In figures 2 and 4 we compare the modulus square of the discrete Fourier transform of the wavefunction: \(n\) stands for the \(n\)th multiple of the fundamental frequency \(2\pi/j_0\).
Figure 2. $j_0 = 2$. Modulus square of the (discrete) Fourier transform of the amplitude. Left: coherent tetrahedron ($n_{\text{mean}} \pm \sigma/2 = 1.25 \pm 0.27$). Right: evolved state ($n_{\text{mean}} \pm \sigma/2 = 1.15 \pm 0.31$).

Figure 3. $j_0 = 4$. Modulus square of the amplitude. Left: coherent tetrahedron ($i_{\text{mean}} \pm \sigma/2 = 4.88 \pm 0.56$). Right: evolved state ($i_{\text{mean}} \pm \sigma/2 = 4.85 \pm 0.96$).

Figure 4. $j_0 = 4$. Modulus square of the (discrete) Fourier transform of the amplitude. Left: coherent tetrahedron ($n_{\text{mean}} \pm \sigma/2 = 2.25 \pm 0.32$). Right: evolved state ($n_{\text{mean}} \pm \sigma/2 = 2.08 \pm 0.59$).

Case $j_0 = 2$. The agreement between the evolved state and the coherent tetrahedron state is quite good. Besides the overall shape of the state, note the concordance of the mean values and the widths of the wave packet. Considering the small value of $j_0$, which is far from the large scale limit, and the $\tau \to \infty$ limit we have taken, we find this quite surprising.

The same pattern repeats in the $j_0 = 4$ case:

The technique developed here is complementary to the one developed in [11], which is based on computing $n$-point functions. We expect that this technique could be better developed.

The numerical results presented above are preliminary and tentative, but they appear to support the expectation that the flipped vertex might in fact give general relativity in the classical limit.
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