On the remainder in the weighted length spectrum for strictly hyperbolic Fuchsian groups

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Abstract. In this paper, we consider the remainder in a weighted form of the length spectrum for compact Riemann surfaces of genus greater than or equal to two. Earlier, we conducted a similar research where we applied the Cauchy residue theorem over two different square boundaries, one of which intersected the corresponding critical line, and some, quite complex estimates for the logarithmic derivative of the associated zeta functions of Selberg and Ruelle. The main goal of this paper is to achieve the same length spectrum with the same remainder as in our previous study, but in a much simpler way.

1. Introduction
Our notation will be based of [7] and [12].

Let $F$ be a compact Riemann surface of genus $g \geq 2$. We can therefore represent $F$ as a quotient space $\Gamma \backslash H$, where $\Gamma$ is a strictly hyperbolic Fuchsian group and $H$ is the upper half-plane.

Thus, $\Gamma \subseteq PSL(2, \mathbb{R})$.

If $I$ is the fundamental polygon of $F$, then $\partial I = \alpha_1^+ \beta_1^- \alpha_1^- \beta_1^+ \ldots \alpha_g^+ \beta_g^- \alpha_g^- \beta_g^+$, where the sides $\alpha_k^-$, $\alpha_k^+$, $\beta_k^-$, $\beta_k^+$ are identified in pairs.

We assume that the sides of $I$ are piecewise smooth.

As it is known, the upper half-plane $H$ comes with the Poincare metric $ds = \frac{|dz|}{y}$, whose corresponding area element is $d\mu(z) = y^{-2} dxdy$.

Note that the Poincare metric has Gaussian curvature $K = -1$ (see, e.g., [15]).

Suppose that $\pi : H \to F$ is the universal covering map.

By projecting the Poincare metric onto $F$ via $\pi$, $F$ becomes a compact Riemannian manifold.

Furthermore, by the Gauss-Bonnet theorem (see, e.g., [14])

$$A = Area(F) = \mu(I) = 4\pi (g - 1).$$

Note that the automorphic group $\Gamma$ is determined up to a conjugation in $PSL(2, \mathbb{R})$. However, the Poincare metric on $F$ is not affected by such conjugations because of the invariance properties of the Poincare metric on $H$ (we assume through the rest of the paper that $F$ carries the Poincare metric). Furthermore, the Gaussian curvature is still $K = -1$. 


In [8], the author concluded that the asymptotic distribution of the closed geodesics is highly influenced by the eigenvalues of the Laplace operator $\Delta$ on $F$, i.e., by the eigenvalues for the problem $\Delta f + \lambda f = 0$ on $F$.

The Selberg zeta function $Sel (s)$ for the group $\Gamma$ is an entire function of order 2, having a sequence of zeros at $0, -1, -2, \ldots$, with the zero at $s = 1$ simple, and having additional zeros in the critical strip $0 < \Re (s) < 1$. The zeros in the critical strip are located at points which are solutions of the equations $s (s - 1) = \lambda_n$, where $\lambda_n$ ranges through the sequence of eigenvalues, omitting $\lambda_0 = 0$, for the problem $\Delta f + \lambda f = 0$ on $F$.

The multiplicity of such a zero is the same as the multiplicity of the corresponding eigenvalue (see, e.g., [10], [13]).

Note that the detailed description of the locations and the orders of the zeros of $Sel (s)$ will be given in the sequel.

In [12, p. 245, Th. 2] (see also, [9]), the author derived the following length spectrum.

If $Sel (s)$ has zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\left( \frac{3}{4}, 1 \right)$, then, there exist constants $c_1, c_2, \ldots, c_n$, such that

$$\varphi_0 (x) = x + c_1 x^{\alpha_1} + \cdots + c_n x^{\alpha_n} + O \left( x^{\frac{3}{4}} \right)$$

as $x \to +\infty$ (the functions $\varphi_n (x)$, $n \in \mathbb{N} \cup \{ 0 \}$ are introduced below).

If, however, $Sel (s)$ has no zeros in $\left( \frac{3}{4}, 1 \right)$, then

$$\varphi_0 (x) = x + O \left( x^{\frac{3}{4}} \right)$$

as $x \to +\infty$.

The main goal of this research is to derive an analogous result for the function $\frac{\varphi_1 (x)}{x}$, i.e., a weighted form of the corresponding length spectrum.

In our earlier research [1, p. 466, Th. 1], we derived one such result applying quite complex mathematical apparatus. The obtained remainder $O \left( x^{\frac{3}{4}} \log x \right)$, however, was much better than the classical one $O \left( x^{\frac{3}{4}} \right)$.

Motivated by this fact, that a $\varphi_1$ analogue of the classical length spectrum yields a better result, we give yet another proof of Theorem 1 [1] based on application of much simpler mathematical techniques.

2. Preliminaries

We adopt the functions $\varphi_0 (x)$ and $\varphi_n (x)$ from [12, p. 245].

The first one is given as a sum indexed over $\Gamma_{hyp}$ with $\text{Norm} \left( \gamma_0 \right) \leq x$, where $\Gamma_{hyp}$ denotes the set of $\Gamma$-conjugacy classes of hyperbolic elements in $\Gamma$. $\text{Norm} \left( \gamma_0 \right) = \exp \left( \text{len} \left( \text{Geo}_{\gamma_0} \right) \right)$, $\text{len} \left( \text{Geo}_{\gamma_0} \right)$ is the length of the prime geodesic $\text{Geo}_{\gamma_0}$ associated to the conjugacy class $\gamma_0$ (it is known fact that prime geodesics over $F$ correspond to the conjugacy classes of primitive hyperbolic elements in $\Gamma$). More precisely, if $\gamma \in \Gamma_{hyp}$, then $\gamma$ is an exponent of some primitive $\gamma_0$, with the degree $j \left( \gamma \right) \in \mathbb{N}$.

For such $\gamma$, $\Phi \left( \gamma \right)$ is defined by $\Phi \left( \gamma \right) = \text{len} \left( \text{Geo}_{\gamma_0} \right)$.

The functions $\varphi_n (x)$, $n \in \mathbb{N}$ are defined inductively.

Through the rest of the paper, we shall assume that $l$ is a number, $l \in \mathbb{N}$, and that $d$ and $T$ are constants which will be fixed later.

For a function $f \left( x \right)$, we define $D_{l-1,+}$ by
\[ \mathcal{D}_{l-1, +} f (x) = f (x + (l - 1) d) - (l - 1) f (x + (l - 2) d) + \frac{(l - 1) (l - 2)}{2} f (x + (l - 3) d) - \ldots + (-1)^{l-1} f (x). \]

By [3, p. 315, (22)], \( \mathcal{D}_{l-1, +} f (x) \) can be represented as an iterated integral (if \( f \) is a differentiable function of appropriate order).

Thus, for some \( x \leq \tilde{x} \leq x + (l - 1) d \), an analogue of the estimate (23) in [3] holds also true. The number of zeros of \( \text{Sel} (s) \) on the critical line will be denoted by \( n (t) \).

We shall apply the fact that \( n (t) \) can be estimated as \( \frac{A}{\pi^2} t^2 \).

3. Main result

We shall prove the following theorem.

Theorem 1. Let \( F \) be a compact Riemann surface of genus \( g \geq 2 \). Then,

\[
\varphi_l (x) = \frac{1}{x} + \sum_{s_0 \in S_{0, \mathbb{R}}, \frac{1}{2} < s_0 < 1} \text{ord} (s_0, 0) s_0^{-1} (s_0 + 1)^{-1} x^{s_0} + O \left( x^{\frac{1}{2}} \log x \right)
\]

as \( x \to +\infty \), where \( S_{0, \mathbb{R}} \) is the set of zeros of \( \text{Sel} (s) \) such that \( s_0 \in (0, 1) \), and \( \text{ord} (s_0, 0) \) is the multiplicity of \( s_0 \).

Proof. By [12, p. 245, Th. 1.'], \( \varphi_l (x) \) can be represented as the sum \( \sum_{p=0}^{l} (-1)^p W_p \), where \( W_p \) is the sum \( \sum_{z \in A_{p,l}} a_{x,p,l} \).

Here, \( A_{p,l} \) denotes the set of poles of the corresponding function, and \( a_{x,p,l} \)'s are the attached residues.

In general, we may point out that the Selberg zeta function \( \text{Sel} (s + p) \) has a zero at \( 1 - p \) of order 1, a zero at \( 0 - p \) of order \( 2g - 1 \), zeros at \( -k - p, k \in \{1, 2, \ldots \} \), whose orders are \( (2g - 2)(2k + 1) \).

The non-trivial zeros of \( \text{Sel} (s + p) \) are all contained in the union of the interval \( (0 - p, 1 - p) \) with the vertical line \( \frac{1}{2} - p + i \mathbb{R} \), i.e., in \( (-p, 1 - p) \cup \left( \frac{1}{2} - p + i \mathbb{R} \right) \).

As we already noted, the values \( 0, -1, \ldots, -l \) are zeros of \( \text{Sel} (s + p) \). Hence, these values are poles of order two of the corresponding function.

The values \( -(l + 1), -(l + 2), \ldots \) are then the simple poles of the same function.

Finally, the value 1 is a simple pole if \( p = 0 \).

The set of zeros \( s_p \) of \( \text{Sel} (s + p) \) such that \( s_p \notin \mathbb{Z} \) will be denoted by \( S_p \).

In other words, \( S_p \subseteq (-p, 1 - p) \cup \left( \frac{1}{2} - p + i \mathbb{R} \right) \).

Note that the values \( s_p \), where \( s_p \in S_p \) are also simple poles of the function in the case at hand.

Now, we determine the residues \( a_{x,p,l} \)'s for \( z \in A_{p,l} \).

Let \( z \) be a zero of \( \text{Sel} (s + p) \) of multiplicity \( \text{ord} (z, p) \).

Furthermore, let \( q_i (z, p) \)'s be the corresponding coefficients in the expansion of the logarithmic derivative of \( \text{Sel} (s + p) \) near \( z \).

If \( z \in A_{p,l} \), and \( z = s_p \in S_p \), then \( a_{x,p,l} \)'s are calculated in the same way as in [3, p. 314, (13)].

Suppose that \( z \in A_{p,l} \), and \( z = -j \in \{0, -1, \ldots, -l\} \).

For the final form of the corresponding residue \( a_{x,j,p,l} \) in this case, we refer to [3, (14)].
Now, suppose that \( z \in A_{p,l}, z = -j \in \{- (l + 1), -(l + 2), \ldots \}. \)

In this case, \( a_{-j,p,l} \) is given by \( \text{ord} (-j, p) \prod_{q=0}^{l} (-j + q)^{-1} x^{-j+l}. \)

Furthermore, if \( z \in A_{0,l}, z = 1, \) then \( a_{1,0,l} = \frac{x^{1+i}}{(1+i)!}. \)

We shall consider the following subsets of the set \( S_p: S_{p,R} = S_p \cap \mathbb{R}, \) and \( S_{p,\frac{1}{2} - p} = S_p \setminus S_{p,R} \) for \( p \in \{0, 1\}. \)

Thus, \( S_{p,R} \subseteq (-p, 1 - p) \) and \( S_{p,\frac{1}{2} - p} \subseteq \frac{1}{2} - p + i \mathbb{R}. \)

Assume that \( z \in A_{p,l}, \) and \( z \in S_{p,\frac{1}{2} - p}. \)

By the very definition of the operator \( D_{l-1,+}, \) it immediately follows that \( d^{-(l-1)} D_{l-1,+} a_{z,p,l} \) can be estimated by \( O \left( d^{-(l-1)} |z|^{-l-1} x^{\frac{1}{2}+l} \right). \)

On the other side, an application of the mean value theorem, yields that

\[
|\text{ord} (z, p)| |z|^{-1} |z + 1|^{-1} \left( x + \sum_{i=2}^{l} q_i \right)^{\frac{1}{2}-p+1} \quad \text{dominates} \quad d^{-(l-1)} D_{l-1,+} a_{z,p,l} \quad \text{for some} \quad 0 \leq q_i \leq d, \quad i \in \{2, 3, \ldots, l\}.
\]

Thus, \( O \left( |z|^{-2} x^{\frac{3}{2}} \right) \) is the second estimate for \( d^{-(l-1)} D_{l-1,+} a_{z,p,l}. \)

Having in mind these two estimates, one easily concludes that the sum

\[
\sum_{z \in S_{p,\frac{1}{2} - p}} d^{-(l-1)} D_{l-1,+} a_{z,p,l},
\]

and hence the sum

\[
\frac{1}{p} \sum_{z \in S_{p,\frac{1}{2} - p}} d^{-(l-1)} D_{l-1,+} a_{z,p,l} \quad \text{are} \quad O \left( x^{\frac{3}{2}} \log T \right) + O \left( d^{-(l-1)} x^{\frac{1}{2}+l} T^{-l+1} \right).
\]

Furthermore, \( \frac{1}{2} x^2 + O (dx) + O \left( d^2 \right) \) dominates \( d^{-(l-1)} D_{l-1,+} a_{1,0,l}. \)

Suppose that \( d = x^\alpha (\log x)^\beta \) and \( T = x^\gamma (\log x)^\delta. \)

Then,

\[
d^2 = x^{2\alpha} (\log x)^{2\beta}, \\
x^{\frac{3}{2}} \log T = \gamma x^{\frac{3}{4}} \log x + \delta x^{\frac{3}{2}} \log \log x, \\
d^{-l+1} x^{\frac{1}{2}+l} T^{-l+1} = x^{- (l+1) \alpha + \frac{1}{2} + l + (l+1) \gamma} (\log x)^{-(l+1) \beta + (l+1) \delta}.
\]

Since \( \log \log x \) is dominated by \( \log x, \) we want that

\[
2\alpha = \frac{3}{2} = (-l + 1) \alpha + \frac{1}{2} + l + (-l + 1) \gamma, \\
2\beta = 1 = (-l + 1) \beta + (-l + 1) \delta.
\]

It is not so hard to conclude that \( \alpha = \frac{3}{4}, \beta = \frac{1}{2}, \gamma = \frac{1}{4} \) and \( \delta = \frac{1+l}{2(1+l)}, \) i.e., \( d = x^{\frac{3}{4}} (\log x)^{\frac{1}{2}} \) and \( T = x^{\frac{1}{2}} (\log x)^{\frac{1+l}{2(1+l)}}. \)

Since, in this case, \( O (dx) = O \left( x^{\frac{3}{4}} (\log x)^{\frac{1}{2}} \right), \) it follows that the aforementioned remainders are all \( O \left( x^{\frac{7}{4}} (\log x)^{\frac{1}{2}} \right). \)
Next, we consider
\[
dx = x^{\alpha+1} (\log x)^\beta, \\
x^{d/2} \log T = \gamma x^{d/2} \log x + \delta x^{d/2} \log \log x, \\
d^{-l+1} x^{1/2} + l T^{-l+1} = x^{(-l+1)\alpha + \frac{1}{2} + l + (-l+1)\gamma} (\log x)^{(-l+1)\beta + (-l+1)\delta}. \]

Reasoning in the same way as in the previous case, we are interested in
\[
\alpha + 1 = \frac{3}{2} = (-l + 1) \alpha + \frac{1}{2} + l + (-l + 1) \gamma, \\
\beta = 1 = (-l + 1) \beta + (-l + 1) \delta.
\]

Thus, \( \alpha = \frac{1}{2}, \beta = 1, \gamma = \frac{1}{2} \) and \( \delta = \frac{l}{1-l}, \) that is, \( d = x^{\frac{1}{2}} (\log x) \) and \( T = x^{\frac{1}{2}} (\log x)^{\frac{l}{1-l}}. \)

Also, in this case, the error term \( O(d^2) \) is \( O(x (\log x)^2). \)

Consequently, the error terms given above are all \( O\left(x^{\frac{d}{2}} \log x\right). \)

Since the equality \( O(dx) = O(d^2) \) yields at least the error term \( O\left(x^2\right), \) the remaining two cases are not interesting for our research.

The discussion conducted above, yields that \( O\left(x^{\frac{d}{2}} \log x\right) \) is the remainder we are looking for, and is established for \( d = x^{\frac{1}{2}} \log x \) and \( T = x^{\frac{1}{2}} (\log x)^{\frac{l}{1-l}}. \)

The sum over \( s_p \in S_{p,R}, \frac{1}{2} < s_p < 1, p \in \{0, 1\}, \) is obviously equal to the sum over \( s_0 \in S_{0,R}, \frac{1}{2} < s_0 < 1. \)

Thus, for \( d = x^{\frac{1}{2}} \log x, \) the sum is equal to
\[
\sum_{s_0 \in S_{0,R}} \text{ord}(s_0) (s_0)^{-1} (s_0 + 1)^{-1} x^{s_0+1} + O\left(x^{\frac{d}{2}} \log x\right).
\]

A we noted earlier, \( Sel(s+p) \) has zeros at \( -k - p, k \in \{1, 2, \ldots\} \) of orders \( (2g - 2) (2k + 1). \)

In particular, \( Sel(s) \) has zeros at \( -k, k \in \{1, 2, \ldots\} \) of orders \( (2g - 2) (2k + 1), \) while \( Sel(s+1) \) has zeros at \( -k - 1, k \in \{1, 2, \ldots\} \) of orders \( (2g - 2) (2k + 1). \)

Moreover, we denoted by \( \text{ord}(z, 0) \) the multiplicity of the zero \( z \) of \( Sel(s). \)

In particular, \( \text{ord}(z, 1) \) is the multiplicity of the zero \( z \) of \( Sel(s+1). \)

Thus, \( \text{ord}(-k, 0) = (2g - 2) (2k + 1), k \in \{1, 2, \ldots\}, \) and \( \text{ord}(-k - 1, 1) = (2g - 2) (2k + 1), k \in \{1, 2, \ldots\}. \)

Hence, \( \text{ord}(-j, 0) = (2g - 2) (2j + 1), \text{ord}(-j, 1) = (2g - 2) (2j + 1), (2g - 2) \)

It follows that,
\[\sum_{p=0}^{1} (-1)^p \sum_{j=l+1}^{+\infty} a_{-j,p,l} = (2g - 2) \sum_{j=l+1}^{+\infty} \frac{2j + 1}{(-j)(-j+1) \cdots (-j+l)} x^{-j+l} \]

\[= (-1)^{l+1} (4g - 4) \sum_{j=l+1}^{+\infty} \frac{1}{j(j+1) \cdots (j-l)} x^{-j+l}.\]

Thus,\[\left|\sum_{p=0}^{1} (-1)^p \sum_{j=l+1}^{+\infty} a_{-j,p,l}\right| \leq (4g - 4) x^{-l} \sum_{j=l+1}^{+\infty} \frac{1}{j(j+1) \cdots (j-l)}.\]

Since the last series converges, we conclude that the sum of \(a_{-j,p,l}\)'s over \(j \in \{l+1, l+2, \ldots\}, p \in \{0, 1\}\) is \(O(x^{-1})\).

If we apply \(\mathcal{D}_{l-1,+}\) to this sum, the remainder \(O(x^{-1})\) will not be changed. Consequently, for \(d = x^{3/2} \log x\), the error term \(O(x^{-1})\) multiplied by \(d^{-(l-1)}\) gives us\[O\left(\left(x^{3/2} \log x\right)^{-l+1} x^{-1}\right) = O\left(x^{-3/2 + 1} (\log x)^{-l+1}\right) = O\left(x^{-3/2} (\log x)^{-1}\right). \tag{1}\]

Since\[\left|\sum_{p=0}^{1} (-1)^p \sum_{j=2}^{l} d^{-(l-1)} \mathcal{D}_{l-1,+}a_{-j,p,l}\right| \leq x^{-1} \sum_{p=0}^{1} \sum_{j=2}^{l} |o(-j,p,l)|,

where the constant \(o(-j,p,l)\) is given by\[o(-j,p,l) = (-1)^{l-2} or d(-j,p)(l-j)! (j-2)! \prod_{q=0}^{l} (-j + q)^{-1},\]
it follows that the corresponding sum is estimated by \(O(x^{-1})\).

Reasoning in the same way as above (applying the main properties of \(\mathcal{D}_{l-1,+}\)), we obtain that the sum corresponding to \(a_{-1,p,l}, p \in \{0, 1\}\) resp. the sum corresponding to \(a_{0,p,l}, p \in \{0, 1\}\), is \(O(\log x)\) resp. \(O(x \log x)\).

Note that the sum over \(s_p \in S_{p,R}, -1 < s_p < 0, p \in \{0, 1\}\) resp. the sum over \(s_p \in S_{p,R}, 0 < s_p \leq \frac{1}{2}, p \in \{0, 1\}\), is actually the sum over \(s_1 \in S_{1,R}, -1 < s_1 < 0\) resp. the sum over \(s_0 \in S_{0,R}, 0 < s_0 \leq \frac{1}{2}\).

These two sums, however, are \(O(x)\) and \(O(x^{3/2})\), respectively.

Having in mind the subsets of \(A_{p,l}\) we have considered above, we may write
In order to derive (1), we first proved that the sum of 

\[ a_{j,p,l} \]

is

\[ \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{j+p+l} a_{j,p,l} = 0. \]

Thus,

\[ d^{-l-1} D_{l-1,-} \varphi_l (x) \]

is not larger than \( \varphi_1 (x) \), and the fact that \( \varphi_1 (x) \) is not larger than \( d^{-l-1} D_{l-1,-} \varphi_l (x) \) \( D_{l-1,-} \) is defined in [11] in a similar way, it follows that for \( d = x^{\frac{3}{2}} \log x \) and \( T = x^{\frac{3}{2}} (\log x)^{\frac{3}{4}} \)

\[ \varphi_1 (x) = \frac{1}{2} x^2 + \sum_{s_0 \in S_{0,0}} \text{ord} (s_0,0) s_0^{-1} (s_0 + 1)^{-1} x^{s_0+1} + O \left( x^{\frac{3}{2}} \log x \right). \]

Thus,

\[ \frac{\varphi_1 (x)}{x} = \frac{1}{2} x + \sum_{s_0 \in S_{0,0}} \text{ord} (s_0,0) s_0^{-1} (s_0 + 1)^{-1} x^{s_0} + O \left( x^{\frac{1}{2}} \log x \right). \]

This completes the proof. \( \square \)

4. Remarks
In order to derive (1), we first proved that the sum of \( a_{-j,p,l} \)'s over \( j \in \{ l+1, l+2, \ldots \} \), \( p \in \{ 0, 1 \} \), is \( O (x^{-1}) \).

Note that one could also use the fact that \( d^{-l-1} D_{l-1,+} a_{-j,p,l} \) is

\[ \text{ord} (-j, p) (-j)^{-1} (-j+1)^{-1} x^{-j+1} \]

for some \( x \leq x_{-j,p,l} \leq x + (l-1) d \), to directly estimate the sum

\[ \sum_{p=0}^{1} (-1)^{p} \sum_{j=l+1}^{+\infty} d^{-l-1} D_{l-1,+} a_{-j,p,l}. \]
In [1], the authors also derived a weighted length spectrum for real hyperbolic manifolds with cusps. Such research represents a generalization of the researches conducted in [11], [2].

A weighted form of the result obtained in [3] (see also, [4]) is derived in [5].

For yet another proof of the main result in [12], we refer to [6].

Note that the author in [16] also considered a weighted length spectrum. His object of research were the automorphic $L$-functions.

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