Emptiness formation probability of the six-vertex model and the sixth Painlevé equation

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Abstract. We show that the emptiness formation probability of the six-vertex model with domain wall boundary conditions at its free-fermion point is a \( \tau \)-function of the sixth Painlevé equation. Using this fact we derive asymptotics of the emptiness formation probability in the thermodynamic limit.

Contents

1. Introduction 2
   1.1. The model 2
   1.2. Emptiness formation probability 4
   1.3. Main result: asymptotic expansions 5
2. EFP and integrable structures 6
   2.1. EFP as a Fredholm determinant 6
   2.2. The resolvent operator 10
   2.3. The Fuchs pair 12
3. EFP and P6 16
   3.1. The sixth Painlevé equation 16
   3.2. Riemann-Hilbert problem 17
   3.3. The Fredholm determinant as the \( \sigma \)-function 20
4. Asymptotic expansions of the EFP in the thermodynamic limit 22
   4.1. Two asymptotic regimes 23
   4.2. Asymptotic expansion in the disordered regime 25
   4.3. Asymptotic expansion in the ordered regime 32
5. Conclusion 35
Appendix A. The Wasow theorem and the \( \sigma \)-form of P6 38
Appendix B. Saddle-point approach to asymptotic expansion of the EFP in the ordered regime 41
Appendix C. Explicit form for the exponentially small correction in the ordered regime 45
References 46

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1. Introduction

One of the most intriguing facts about correlation functions of solvable models of statistical mechanics is that in many cases they can be described in terms of the Painlevé equations or their generalizations. Famous examples are provided by correlation functions of the two-dimensional Ising model, related to the third Painleve equation [1,2], and correlation functions of an impenetrable Bose gas, related to the fifth Painlevé equation [3,4]. At the same time this relationship carries mainly academic interest, whilst asymptotic analysis of the correlation functions is usually performed via the associate integrable structures rather than with the help of these ordinary differential equations (ODEs).

In this paper, we consider the six-vertex model with the domain wall boundary conditions and discuss a particular correlation function, called the emptiness formation probability (EFP). We show that, for the model with the Boltzmann weights satisfying the free-fermion condition, this correlation function appears to be a $\tau$-function of the sixth Painlevé equation (P6). Using this connection we derive asymptotic expansions of the EFP in the thermodynamic limit.

It is important to stress that here we consider asymptotics of P6 for the case where the coefficients of the equation are large while its argument is a finite parameter. Such type of asymptotics is not thoroughly studied for P6 yet. Therefore to cope with this problem, we initially intended to use one of the known asymptotic methods for the Painlevé equations, namely, the isomonodromy deformation technique by Jimbo-Its-Novokshenov [5,6] or Deift-Zhou [7] asymptotic analysis of the corresponding Riemann-Hilbert problem. However, we were quite surprised that we have been able to construct asymptotics (both the leading terms and corrections) without any sophisticated techniques—just with the help of the $\sigma$-form of P6. Indeed, finding asymptotics, especially the correction terms, of solutions of ODEs by substituting corresponding asymptotic expansions into the ODE is the standard asymptotic technique in the case where the asymptotics with respect to the argument of the corresponding ODE is constructed. On this way, one usually finds a recurrence relation for determination of the coefficients. Our case is different: we arrive at the recurrence relation for the first derivatives of the correction terms so that to find the explicit formulas we need the initial data. It turn out that for correction terms of arbitrary order these initial data can be obtained from representations of the EFP at the critical points of P6. It seems that the same technique may appear useful for other analogous problems.

1.1. The model. We recall that the six-vertex model (also known as the ice-type model) is a statistical mechanics model defined on a square lattice. The local states of the model are arrows placed on the edges of the lattice. The admissible configurations of the model are those in which there are equal numbers of incoming and outgoing arrows in each lattice vertex. The condition of local conservation of the number of incoming and outgoing arrows is known as the ice-rule, and it selects six possible arrow configurations around the vertex, see Fig. 1. For periodic boundary conditions the model was solved by Lieb [8,10] and Sutherland [11] (for a review, see, e.g., [12]).

Domain wall boundary conditions (DWBC) are defined as follows. Consider a finite square lattice formed by intersection of $N$ horizontal and $N$ vertical lines (the so-called $N \times N$ lattice). For such a lattice one may require that the local states on
Figure 1. The six types of arrow configurations around a vertex allowed in the six-vertex model and their Boltzmann weights.

Figure 2. An $N \times N$ lattice ($N = r + s + q$) with domain wall boundary conditions, with the additional condition that the vertices of an $s \times (s + q)$ lattice at the top left corner are all of type 2—the probability of this configuration is $F_{r,s,q}$ (shown the case $r = 5$, $s = 3$, and $q = 1$).

external edges are fixed in a special way consistent with the ice-rule, namely, using the description of the local states in terms of arrows on edges, that all arrows on the external vertical edges (i.e., at the top and bottom boundaries) are incoming, while those on the external horizontal edges (i.e., on the left and right boundaries) are outgoing, see Fig. 2. The six-vertex model on the $N \times N$ lattice with such fixed boundary conditions is called the six-vertex model with DWBC; it was considered for the first time by Korepin [13].

To define the partition function of this model, let us denote by $\Omega_N$ the set of all arrow configurations of the $N \times N$ lattice with DWBC and with the ice-rule obeyed in each lattice vertex. Assigning to the $i$th arrow configuration around a vertex the Boltzmann weight $w_i$, see Fig. 1 the partition function $Z_N$ is defined as

$$Z_N = \sum_{C \in \Omega_N} \prod_{i=1,\ldots,6} w_i^{n_i(C)}.$$

Here $n_i(C)$, $i = 1,\ldots,6$, is the number of vertices of type $i$ in the configuration $C$, $\sum_i n_i(C) = N^2$. The partition function was computed, for a more general inhomogeneous (with position-dependent weights) model, in terms of an $N \times N$ determinant by Izergin [14]; for the homogeneous model the determinant has the Hankel structure [15]. The free energy per site was derived for various regimes and by various methods by Korepin and Zinn-Justin [16,17]. A detailed analysis of the asymptotic expansion of the partition function in the thermodynamic limit,
using the connection of Hankel determinants and matrix models with the Riemann-Hilbert problem [18], was given in paper [19] and a series of papers by Bleher and collaborators [20–24].

1.2. Emptiness formation probability. To define the EFP, let us set \( N = r + s + q \), where \( r \), \( s \), and \( q \) are nonnegative integers, and consider configurations of six-vertex model on the \( N \times N \) lattice with DWBC in which the vertices belonging to the \( s \times (s + q) \) rectangle at the top left corner of the lattice are all of type 2, see Figs. 1 and 2. We denote the set of such configurations as \( \Omega_{r,s,q} \); obviously, \( \Omega_{r,s,q} \subseteq \Omega_N \). We denote the EFP by \( F_{r,s,q} \) and define it as a probability of observing a configuration belonging to \( \Omega_{r,s,q} \) for a given \( r, s, q \),

\[
F_{r,s,q} = Z_N^{-1} \sum_{C \in \Omega_{r,s,q}} \prod_{i=1,\ldots,6} w_i^{n_i(C)}.
\]

In what follows we always assume that \( s \leq r \), since otherwise \( F_{r,s,q} \) is just zero (for \( s > r \) the set \( \Omega_{r,s,q} \) is empty). A closed expression for the EFP in the terms of a multiple \((s\text{-fold})\) integral was obtained by Colomo and the second author in [25]. As it was demonstrated in [26], this result make it possible to address phase separation phenomena in the model, and in one of its special cases provides the limit shape of large altering-sign matrices [27].

In this paper we study the EFP of the six-vertex model with DWBC in the case of the weights obeying the so-called free-fermion condition, \( w_1w_2 + w_3w_4 = w_5w_6 \). Namely, we choose the weights in the form

\[
w_1 = w_2 = \sqrt{1 - \alpha}, \quad w_3 = w_4 = \sqrt{\alpha}, \quad w_5 = w_6 = 1,
\]

where \( \alpha \in (0, 1) \) is a parameter. We note that this case is related to enumeration of domino tilings, where the parameter \( \alpha \) plays the role of “bias” [28–30]. In the main text we prove the following result.

**Theorem 1.1.** Define

\[
\sigma = \alpha(\alpha - 1) \frac{d}{d\alpha} \log F_{r,s,q} = \frac{(r + q + s)^2}{4} \alpha + \frac{(r + q + s)q + 2rs}{4},
\]

then

\[
\alpha^2(\alpha - 1)^2 \sigma'(\sigma'')^2 + \left\{ (1 - 2\alpha)(\sigma')^2 + 2\sigma\sigma' + \nu_1\nu_2\nu_3\nu_4 \right\}^2
\]

\[
= (\sigma' + \nu_1) (\sigma' + \nu_2) (\sigma' + \nu_3) (\sigma' + \nu_4),
\]

\[
\sigma' = \frac{d}{d\alpha} \sigma, \quad \sigma = \frac{d}{d\alpha} \log F_{r,s,q}.
\]

**Remark 1.1.** Note that \( F_{r,s,q} \) is a polynomial in \( \alpha \), see for details representation (2.2) obtained in Sect. 2.1. Thus, (1.2) implies that \( \sigma = \sigma(\alpha) \) is a rational solution of (1.1). The corresponding solution of the canonical form of P6 is also rational. This solution defines isomonodromy deformations of the associated linear Fuchsian ODE with the monodromy matrices \( \pm I \) (see Sect. 3.3 for details).

**Remark 1.2.** The EFP can be expressed as

\[
F_{r,s,q} = C_{r,s,q} \alpha^{rs}(1 - \alpha)^{(r+q)_r},
\]

where quantity \( C_{r,s,q} \) is independent of \( \alpha \) and \( \tau = \tau(\alpha) \) is the Jimbo-Miwa \( \tau \)-function of P6 [4], see also Sect. 3.4. The pre-factor of the \( \tau \)-function in (1.4) is

\[
\frac{d}{d\alpha} \log F_{r,s,q} = \frac{(r + q + s)^2}{4} \alpha + \frac{(r + q + s)q + 2rs}{4}.
\]
a matter of the definition (normalization) of the $\tau$-function, therefore $F_{r,s,q}$ is, in fact, the $\tau$-function of P6.

We recall some basic fact about the theory of $P_6$ in Sect. 3.1. Our starting point in proving the theorem is the formula for the EFP expressing it as a Fredholm determinant of some linear integral operator of the so-called integrable type. This representation, among some others, was derived from the general multiple integral representation by specifying it to the case of the weights \((\ref{1.1})\) in \cite{31}. We review some of these results in Sect. 2.1. The theorem is proven in Sects. 2 and 3.

As a comment to this result, we mention that appearance of a Painlevé equation in the context of the six-vertex model with DWBC was already noticed in \cite{32} where it was shown that the partition function in the ferroelectric regime serves as a $\tau$-function of the fifth Painlevé equation. In turn, P6 appeared in the study of the EFP at zero-temperature for the Heisenberg XX0 spin chain \cite{33}, which can be seen as the six-vertex model at its free-fermion point with periodic boundary conditions.

In fact, the relation of $F_{r,s,q}$ with $P_6$ reported in Theorem 1.1 can be extracted from the works by Johansson, Baik and Rains, and Forrester and Witte \cite{34–36} (see a discussion at the end of Sect. 2.1). Here we present a different proof based on Fredholm determinants of integrable linear integral operators, developed in \cite{3, 37}. This method is also important for asymptotic study of $F_{r,s,q}$ as it can be used to provide initial conditions in our proof of Theorem 1.3 below. The connection of the EFP with $P_6$ allows us to study behavior of the EFP in the thermodynamic limit, in which the integers $r$, $s$, and $q$ are large, with their ratios fixed; the variable $\alpha$, defining the weights, is considered as a parameter. An important special case of such a limit is where the region of the vertices of type 2 at the top left corner has a macroscopically square shape, that corresponds to the ratios $q/s$ and $q/r$ vanishing in the limit. In this paper we compute the asymptotic expansion for $F_{r,s,q}$ in the case of $q = 0$.

### 1.3. Main result: asymptotic expansions

Define

$$v \equiv \frac{s}{r}, \quad u \equiv \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}, \quad v, u \in (0, 1).$$

As discussed in Sect. 4.1, the EFP has different asymptotic expansions for $u < v < 1$ (referred to as the EFP in the disordered regime) and for $0 < v < u$ (referred to as the EFP in the ordered regime). Our results about the EFP in the thermodynamic limit are formulated as follows.

**Theorem 1.2.** For $s/r = v \in (u, 1)$, as $s, r \to \infty$,

$$\log F_{r,s,0} = -\phi s^2 - \frac{1}{12} \log s + \frac{1}{8} \log \left(1 - \frac{(1 - u)^2 v^2}{v^2 - u^2}\right) - \frac{1}{12} \log \frac{1 - v^2}{2} + \zeta'(-1)$$

$$+ \sum_{k=1}^{n} a_{2k}s^{-2k} + O(s^{-2n-2}), \quad (1.5)$$

where

$$\phi = \log \frac{v}{u} - \frac{(1 - v^2)^2}{2v^2} \log \frac{1 - v}{1 - u} - \frac{(1 + v^2)^2}{2v^2} \log \frac{1 + v}{1 + u}, \quad (1.6)$$

$$\zeta'(-1) = -0.1654211437 \ldots \text{is the derivative of the Riemann zeta-function, and } a_{2k} = a_{2k}(u, v) \text{ are rational functions of } u^2 \text{ and } v^2 \text{ with poles at } v^2 = 1 \text{ and } v^2 = u^2.$$
In particular,

\[ a_2 = \frac{v^2(1-v^2)(2v^4 + 5u^2v^2 - u^4)}{64(v^2 - u^2)^3} - \frac{(1 + v^2)(v^2 - (1 - v^2)^2)}{120(1 - v^2)^3} - \frac{1}{64}, \]

\[ a_4 = -\frac{u^2(1 - v^2)}{256(v^2 - u^2)^6} \left[ 10v^{10}u^2 - 2v^{10} - 90v^8u^2 + 140v^8u^4 + 105v^6u^6 - 160v^6u^4 - 4v^4u^8 + 5v^4u^6 - 6v^2u^8 + v^2u^{10} + u^{10} \right] \]

\[ - \frac{v^6(4v^4 + 5v^2 - 10)}{504(1 - v^2)^4} + \frac{31}{16128}. \]

The estimate \( O(s^{-2n-2}) \) of the error term in (1.5) is uniform with respect to the parameter \( v \) on any compact subset of the interval \((u, 1)\).

**Theorem 1.3.** For \( s/r = v \in (0, u) \), as \( s, r \to \infty \),

\[
\log (1 - F_{r,s,0}) = -\chi s - \log s + \log \frac{v^2(1 - u^2)^2}{32\pi \sqrt{u}(u - v)(1 - uv)^{3/2}} + \sum_{k=1}^n b_k s^{-k} + O(s^{-n-1}), \quad (1.7)
\]

where

\[
\chi = \frac{4}{v} \log \left( \frac{\sqrt{1 - v^4} + \sqrt{u(u - v)}}{\sqrt{1 - u^2}} \right) - 4 \log \left( \frac{\sqrt{u(1 - uv)} + \sqrt{u - v}}{\sqrt{(1 - u^2)v}} \right) \quad (1.8)
\]

and \( b_k = b_k(u, v) \). In particular,

\[ b_1 = -\frac{9v^2(1 + u^4) - 36uv(1 + v^2)(1 + u^2) - (2 - 142v^2 + 2v^4)u^2}{48\sqrt{u}(u - v)(1 - uv)^{3/2}}, \]

\[ b_2 = \frac{v^2}{64uv(1 - v^2)(1 - uv)} \left[ 3v^2(u^8 + 1) - 8uv(v^2 + 1)(u^6 + 1) + 4u^2(10v^4 + v^2 + 10)(u^4 + 1) - 120uv^3(v^2 + 1)(u^2 + 1) - (16v^4 - 370v^2 + 16)u^4 \right]. \]

The estimate \( O(s^{-n-1}) \) of the error term in (1.7) is uniform with respect to the parameter \( v \) on any compact subset of the interval \((0, u)\).

**Remark 1.3.** Since \( s \) and \( r \) are integers \( v \) is a rational number, \( v = s_0/r_0 \), where \( 0 < s_0 < r_0 \) are co-prime natural numbers. Asymptotic expansions presented in Theorems 1.2 and 1.3 are understood in the natural way: put \( s \equiv s_0 \) and \( r \equiv r_0 \), where \( n \) is a positive integer and \( n \to \infty \). As a further comment, we note that one can modify the problem and ask about asymptotics for irrational values of \( v \). Our results also deliver the answer to this question. Consider rational approximation, \( v = s_n/r_n + \delta_n \), where \( s_n \) and \( r_n \) are co-prime natural numbers, and \( \delta_n \to 0 \) is the corresponding error. With the help of the Farey series \( \mathbb{F} \), we can find such approximations that \( \delta_n = \kappa_n/r_n^2 \), where \( |\kappa_n| < 1 \). For the irrational \( v \), Theorems 1.2 and 1.3 remain valid but with \( v \) substituted by \( v - \delta_n \). In particular, for the optimal rational approximations (obtained with the help of the continued fractions or the Farey series) it is convenient first to rewrite the asymptotics in terms of the large parameter \( r \) instead of \( s \) by using the relation \( s = vr \), then substitute \( v \) with \( v - \kappa_n/r_n^2 \), and, finally, re-expand asymptotics with respect to \( r = r_n \).
Remark 1.4. Existence of the asymptotic expansions (1.7) and (1.5) are proved in Sect. 4.2 and Sect. 4.3, respectively, and the coefficients $a_{2k}$’s and $b_k$’s can be obtained by recurrent procedures via substitution of these expansions into the $\sigma$-form of $P_6$, see Theorem 1.1. The coefficients are derived successively with the linear differential equations of the first order whose initial data are given in these sections.

The asymptotic formulas (1.5) and (1.7) confirms and extends the result of [30], where only the leading $O(s^2)$-term of the asymptotic expansion of $\log F_{r,s,0}$ was obtained. Formula (1.8) for $\chi$ reproduces the result obtained by Johansson [34], see, for details, Sect. 4.3.

As a comment to the asymptotic expansion of the EFP, we mention that the value $v = u$ is the critical value at which the third-order phase transition of the leading term takes place. This phase transition has a combinatorial interpretation in the context the of random domino tilings on Aztec diamonds with a cut-off corner [30]. Recently, a possible third-order phase transition in the thermal correlations of XX0 spin chain, although for a peculiar choice of the parameters, was pointed out in [39]. Similar phenomena can be seen when the EFPs of quantum spin chains and vertex models are treated by conformal field theory methods [40] and when the vector Chern-Simons theory is considered on $S^2 \times S^1$ [41]. A detailed discussion of the third order phase transition phenomena from the random matrix models point-of-view can be found in [42]. We discuss aspects of the third-order phase transition related to the EFP in terms of the associated Painlevé equations in Conclusion.

2. EFP and integrable structures

The aim of this section is to present an overview of the results which lead to formulation of the EFP as a special solution of an integrable system (in our case P6). We start with recalling presentation of $F_{r,s,q}$ as a Fredholm determinant of a linear integral operator, next we discuss the properties of the corresponding resolvent kernel, and finally we show how to convert the associated system of integral equations to Fuchs (a Lax-type) pair for P6.

2.1. EFP as a Fredholm determinant. We mention here three representations for the EFP, valid for the six-vertex model with the weights (1.1): in terms of a multiple integral, in terms of Hankel determinant, and in terms a Fredholm determinant of a linear integral operator acting on a closed contour in the complex plane. Others can be found in [31] where interrelations between these representations are proved. In our study of the EFP we mostly rely on the last form, in terms of the Fredholm determinant.

In [25], using the method of Yang-Baxter algebra (the algebra of the quantum monodromy matrix) [43] and the methods of orthogonal polynomials theory, the EFP was evaluated in the form of a multiple integral. In the case of the weights (1.1) the result can be formulated as follows.

**Proposition 2.1.** The EFP of the six-vertex model with DWBC with the weights (1.1) is represented in the form

$$F_{r,s,q} = \frac{(-1)^s(s+1)^s}{s!} \int_{C_0} \cdots \int_{C_0} \prod_{1 \leq j < k \leq s} (z_j - z_k)^2 \prod_{j=1}^{s} (\alpha z_j + 1 - \alpha)^{r+q} \cdot \frac{d^s z}{(2\pi i)^s},$$

(2.1)
where $C_0$ is a simple, closed, counter-clockwise oriented contour around the point $z = 0$ of the complex plane, and lying in its small vicinity.

The representation (2.1) was used in the study of the so-called arctic curve of the six-vertex model with DWBC, i.e., the curve which describes the spacial separation of order and disorder in the thermodynamic limit [26, 44]. However, the integral (2.1) can hardly be studied by the random matrix model methods, since the problem of finding of an equilibrium measure cannot be solved explicitly (see discussion in [44]). This fact stimulated to look for another, simple representations. In particular, evaluating the integrals it can be shown that the EFP is given in terms of a Hankel determinant.

**Proposition 2.2.** The EFP admits the representation:

$$ F_{r,s,q} = \frac{(q!)^s}{\prod_{k=0}^{s-1}(q+k)!} \frac{(1 - \alpha)^{s+q}}{\alpha^{s-1/2}} \det_{1 \leq j,k \leq s} \left[ \sum_{m=0}^{r-1} m^{j+k-2} \binom{m+q}{m} \alpha^m \right]. \quad (2.2) $$

Using a peculiar structure of the matrix in (2.2), the finite-size determinant here can converted into Fredholm determinants of integral operators of various types. Indeed, the matrix in (2.2) can be represented as a difference of two matrices, one involving the summation over $m \in \mathbb{Z}_{\geq 0}$ and second over $m \in \mathbb{Z}_{> r}$; the first matrix is related with the ensemble of Meixner polynomials and hence it can be explicitly inverted, while the second one gives rise to a Fredholm determinant structure of the finite-size determinant. This finite-size determinant can be further transformed by keeping the Fredholm structure of the determinant. In particular, it can be written as a Fredholm determinant of a linear integral operator acting on a closed contour in the complex plane. Specifically, here we deal with the integral operator $\hat{K}$ acting on trial functions by the formula

$$ \left( \hat{K}f \right)(\lambda) = \oint_{C_0} K(\lambda,\mu) f(\mu) \, d\mu. $$

The function $K(x, y)$ is the kernel of the operator, and $C_0$ is the contour given as a circle of small radius around the point $\lambda = 0$, with counter-clockwise orientation, i.e., the same contour appeared above in the multiple integral representation for the EFP. Besides this contour, the functions defining the kernel involve integrals over the contour $C_\infty$, which is a circle of large radius, counter-clockwise oriented around the origin (the circle of a small radius around the point $\lambda = \infty$, clockwise oriented around it).

We also recall that the Fredholm determinant of a linear integral operator is defined as

$$ \text{Det} \left( 1 - \hat{K} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{1 \leq i,j \leq n} \det_{1 \leq i,j \leq n} [K(\lambda_i, \lambda_j)] \, d^n \lambda. $$

Using that $\text{Det}(1 - \hat{K}) = \exp \{ \text{Tr} \log(1 - \hat{K}) \}$, and defining the function $\log(1 - \hat{K})$ by its power series expansion in powers of $\hat{K}$, we can also define the Fredholm determinant of the integral operator $\hat{K}$ by the formula

$$ \text{Det} \left( 1 - \hat{K} \right) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \int K(\lambda_1, \lambda_2) \cdots K(\lambda_n, \lambda_1) \, d^n \lambda \right\}. \quad (2.3) $$

We now formulate the Fredholm determinant representation for the EFP, derived in [31].
Proposition 2.3. The EFP admits representation in terms of the Fredholm determinant 
\[ F_{r,s,q} = \text{Det} \left( 1 - \hat{K} \right), \]  
where \( \hat{K} = \hat{K}_{r,s,q} \) is the integral operator acting on the contour \( C_0 \) and possessing the kernel 
\[ K(\lambda, \mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2\pi i(\lambda - \mu)}, \quad \lambda, \mu \in C_0, \]  
where the functions \( e_{\pm}(\lambda) \), which depend on the integers \( r, s, \) and \( q \), i.e., \( e_{\pm}(\lambda) = e_{\pm, r,s,q}(\lambda) \), are 
\[ e_-(\lambda) = \frac{(\lambda - \alpha)^{r/2}}{(\lambda - 1)^{(r+q)/2}\lambda^{s/2}} \]  
and 
\[ e_+(\lambda) = e_-(\lambda)E(\lambda), \]  
where the function \( E(\lambda) \) is 
\[ E(\lambda) = \frac{\nu}{(\nu - \alpha)^r(\nu - \lambda)} \]  
and \( C_0 \) and \( C_\infty \) are counter-clockwise oriented circles of small and large radii, respectively.

The details of the proof can be found in [31]. Here we just mention the idea of the proof: upon change of the variable \( \nu \mapsto 1/\nu \), which replaces the contour \( C_\infty \) by the contour \( C_0 \), the trace of the \( n \)th power of the operator \( \hat{K} \) reads 
\[ \text{Tr} \left( \hat{K}^n \right) = (-1)^n q \int_{C_0} \cdots \int_{C_0} \frac{d^n \nu \, d^n \lambda}{\lambda_{i+1}^{r+q}(1 - \lambda_i)(1 - \lambda_{i+1})} \]  
where \( \lambda_{i+1} := \lambda_1 \). Since all \( \lambda_i \)'s and \( \nu_i \) are integrated over the contours of small radii, the factors \( (1 - \lambda_i \nu_i)^{-1} \) and \( (1 - \lambda_{i+1} \nu_i)^{-1} \) can be expanded in the Taylor series. Re-ordering the summations and integrations here, various Fredholm-type determinant representations can be obtained, in particular, in terms of a finite-size matrix, which leads to the Hankel determinant representation (2.2).

It is necessary to mention that the representation (2.2), up to a redefinition of the discrete parameters and when rewritten in its random matrix model integral form (with a discrete measure), appears to be coinciding with certain quantity discussed in a different context by Johansson [34] (see Proposition 1.3 therein). In turn, Baik and Rains [35,45] showed that this quantity can be represented as certain average over circular unitary ensemble. Using the Okamoto theory [46], Forrester and Witte [36] next showed this average serves as \( \tau \)-function corresponding to a classical solution of P6. In what follows we show that this result, which can be summarized in the form of Theorem 1.1, can be established directly from the Fredholm determinant representation (2.4), using the notion of the integrable integral operators introduced by Its, Izergin, Korepin, and Slavnov [37].
2.2. The resolvent operator. Consider Fredholm integral equations

\[ f_{\pm}(\lambda) - \oint_{C_0} K(\lambda, \mu) f_{\pm}(\mu) \, d\mu = e_{\pm}(\lambda), \]

which can be written in operator form as

\[ \left[ (1 - \hat{K}) f_{\pm} \right](\lambda) = e_{\pm}(\lambda). \]  (2.9)

For brevity we omit dependence of all functions on the parameter \( \alpha \) and discrete parameters \( r, s, \) and \( q \). We consider solutions of (2.9) belonging to the class of analytic functions in \( \lambda \)-plane which can be presented in a vicinity of the point \( \lambda = 0 \) in the form

\[ f_{\pm}(\lambda) = h_1(\lambda) e^{-\lambda} + h_2(\lambda) e^{\lambda}, \]

where \( h_1(\lambda) \) and \( h_2(\lambda) \) are analytic functions at the origin. For given values of \( r, s, \) and \( q \) there exists a finite number of points \( \alpha = \alpha_1, \ldots, \alpha_p \), where

\[ p = r(s + q), \]

such that \( \dim \ker(1 - \hat{K}) = 0 \) for \( \alpha \in \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_p\} \), and \( \dim \ker(1 - \hat{K}) = 1 \) at \( \alpha \in \{\alpha_1, \ldots, \alpha_p\} \). It is important to note that \( \alpha_1, \ldots, \alpha_p \notin (0, 1) \) that is the operator \( 1 - \hat{K} \) is invertible for the case of our interest, \( \alpha \in (0, 1) \). Therefore, one can define the resolvent \( \hat{R} \) of the operator \( \hat{K} \),

\[ (1 - \hat{K}) \left(1 + \hat{R}\right) = 1. \]  (2.10)

As we show below, the kernel the operator \( \hat{R} \) can be written in terms of the functions \( f_{\pm}(\lambda) \), exactly in the same way as the kernel of the operator \( \hat{K} \) is given in terms of the functions \( e_{\pm}(\lambda) \).

It is easy to see that (2.9) can be rewritten in the form

\[ X(\lambda) \hat{e}(\lambda) = \hat{f}(\lambda), \]

where

\[ \hat{e}(\lambda) = \begin{pmatrix} e_+(\lambda) \\ e_-(-\lambda) \end{pmatrix}, \quad \hat{f}(\lambda) = \begin{pmatrix} f_+(\lambda) \\ f_-(-\lambda) \end{pmatrix}, \]

and

\[ X(\lambda) = \begin{pmatrix} 1 - H_{-+}(\lambda) & H_{++}(\lambda) \\ -H_{-+}(\lambda) & 1 + H_{++}(\lambda) \end{pmatrix}, \]  (2.11)

where the functions \( H_{mn}(\lambda) \) are

\[ H_{mn}(\lambda) = \oint_{C_0} \frac{e_m(\mu)f_n(\mu)}{\mu - \lambda} \, d\mu, \quad m, n = \{+, -\}. \]  (2.12)

It turns out that the matrix \( X(\lambda) \) can be easily inverted, due to the following remarkable property.

**Proposition 2.4.** The determinant of the matrix (2.11) is equal to one,

\[ \det X(\lambda) = 1, \quad \lambda \in \mathbb{C}. \]

**Proof.** To prove the result we use the same trick as in [43], Chap. XIV, App. C, but we will rely only the known structure of the kernel of the operator \( \hat{K} \),
instead of that for the resolvent. Note that the functions $H_{+-}(\lambda)$ and $H_{-+}(\lambda)$ can be written in the form

$$H_{+-}(\lambda) = \oint_{C_0} \left[ \frac{\left( 1 - \hat{K} \right) f_+(\mu) f_-(\mu)}{\mu - \lambda} \right] \frac{d\mu}{2\pi i}$$

and

$$H_{-+}(\lambda) = \oint_{C_0} \frac{f_+(\nu)}{\nu - \lambda} \left[ \frac{\left( 1 - \hat{K} \right) f_-(\nu)}{\nu - \lambda} \right] \frac{d\nu}{2\pi i},$$

respectively. Subtracting the second expression from the first one and using the fact that the kernel is symmetric, $\hat{K}(\mu, \nu) = \hat{K}(\nu, \mu)$, we have

$$H_{+-}(\lambda) - H_{-+}(\lambda) = \oint_{C_0} \frac{d\mu}{2\pi i} \oint_{C_0} f_-(\mu) K(\mu, \nu) f_+(\nu) \left( \frac{1}{\nu - \lambda} - \frac{1}{\mu - \lambda} \right) d\nu.$$

Since

$$\frac{1}{\nu - \lambda} - \frac{1}{\mu - \lambda} = \frac{\mu - \nu}{(\mu - \lambda)(\nu - \lambda)}$$

and

$$(\mu - \nu) K(\mu, \nu) = e_+(\mu) e_-(\nu) - e_-(\mu) e_+(\nu),$$

it is fairly easy to see that in fact we have the relation

$$H_{+-}(\lambda) - H_{-+}(\lambda) = H_{+-}(\lambda) H_{-+}(\lambda) - H_{--}(\lambda) H_{++}(\lambda).$$

Obviously, the last relation implies that $\det X(\lambda) = 1$. \hfill \Box

**Corollary 2.1.** The inverse matrix has the form

$$X^{-1}(\lambda) = \begin{pmatrix} 1 & H_{+-}(\lambda) \\ H_{-+}(\lambda) & 1 - H_{++}(\lambda) \end{pmatrix}. \quad (2.13)$$

**Corollary 2.2.** The functions $e_\pm(\lambda)$ solve the following integral equations

$$f_\pm(\lambda) = e_\pm(\lambda) + \oint_{C_0} R(\lambda, \mu) e_\pm(\mu) d\mu, \quad (2.14)$$

where

$$R(\lambda, \mu) = \frac{f_+(\lambda) f_-(\mu) - f_-(\lambda) f_+(\mu)}{2\pi i(\lambda - \mu)}. \quad (2.15)$$

**Proof.** Using (2.12) it is straightforward to rewrite the formula

$$X^{-1}(\lambda) \tilde{f}(\lambda) = \tilde{c}(\lambda)$$

in the form of the integral equations (2.14) with the kernel (2.15). \hfill \Box

Note that (2.14) can be written in the following operator form

$$f_\pm(\lambda) = \left[ 1 + \hat{R} \right] e_\pm(\lambda). \quad (2.16)$$

Comparing (2.16) with (2.9), we conclude that the operator $\hat{R}$ is the resolvent of the operator $\hat{K}$, defined by the formula (2.10). In other words, we have just showed that if the operator $\hat{K}$ possesses the kernel (2.5), then the functions $f_\pm(\lambda)$, defined as the solutions of the equations (2.9), determine the kernel of the resolvent by formula (2.15).

In a similar manner one can calculate action of the operator $(1 - \hat{K})^{-1} = 1 + \hat{R}$ on various functions. In what follows we often use the following result.
Lemma 2.1. Let the vector $g(\lambda; \nu)$ be

$$
\vec{g}(\lambda; \nu) = \left( \frac{g_+(\lambda; \nu)}{g_-(\lambda; \nu)} \right), \quad g_\pm(\lambda; \nu) = \frac{e_\pm(\lambda)}{\lambda - \nu},
$$

(2.17)

then

$$
\left[ \left( 1 + \hat{R} \right) \vec{g}(\cdot; \nu) \right](\lambda) = \frac{1}{\lambda - \nu} X^{-1}(\nu) \vec{f}(\lambda).
$$

(2.18)

Proof. A direct calculation gives us

$$
\left[ \left( 1 + \hat{R} \right) g_\pm(\cdot; \nu) \right](\lambda) = \frac{e_\pm(\lambda)}{\lambda - \nu} + \int_{C_0} R(\lambda, \mu) \frac{e_\pm(\mu)}{\mu - \nu} \, d\mu
$$

$$
= \frac{f_\pm(\lambda)}{\lambda - \nu} + \left( 1 + \frac{\lambda - \mu}{\mu - \nu} \right) \int_{C_0} R(\lambda, \mu) e_\pm(\mu) \, d\mu
$$

$$
= \frac{f_\pm(\lambda)}{\lambda - \nu} + \int_{C_0} R(\lambda, \mu)(\lambda - \mu) e_\pm(\mu) \, d\mu
$$

$$
= \frac{f_\pm(\lambda)}{\lambda - \nu} + H_{\pm}(\nu) \frac{f_\pm(\lambda)}{\lambda - \nu} - H_{\pm}(\nu) \frac{f_\pm(\lambda)}{\lambda - \nu},
$$

(2.19)

where relations (2.16) and (2.15) have been used. Comparing the result in (2.19) with (2.13), we arrive at (2.18). □

2.3. The Fuchs pair. One of the main properties of the functions $f_\pm(\lambda)$ is that they obey linear first-order differential equations in the variables $\alpha$ and $\lambda$. These equations appear to be a vector form of the famous Fuchs pair for $P6$.

Note that there exist also difference equations with respect to the discrete parameters $r$, $s$, and $q$, shifting them by $\pm 1$. These equations, together with the one with respect to $\alpha$, constitute the set of the Lax-type equations of our problem. They can be viewed as the so-called Schlesinger transformations for the Fuchs pair given below, or derived in a straightforward way using the method presented in the proof.

Proposition 2.5. The functions $f_\pm(\lambda)$ is the solution of the system of linear differential equations

$$
\frac{d}{d\lambda} \vec{f}(\lambda) = A(\lambda) \vec{f}(\lambda), \quad \frac{d}{d\alpha} \vec{f}(\lambda) = B(\lambda) \vec{f}(\lambda),
$$

with

$$
A(\lambda) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_\alpha}{\lambda - \alpha}, \quad B(\lambda) = -\frac{A_\alpha}{\lambda - \alpha},
$$

(2.20)

where the matrices $A_0$, $A_1$, and $A_\alpha$ are given by

$$
A_\nu = \frac{\theta_\nu}{2} X(\nu) S(\nu) X^{-1}(\nu) \quad (\nu = 0, 1, \alpha).
$$

(2.21)

Here

$$
\theta_0 = s, \quad \theta_1 = r + q, \quad \theta_\alpha = -r,
$$

(2.22)

and the matrix $X(\lambda)$ is given in (2.11) and the matrix $S(\lambda)$ is

$$
S(\lambda) = \begin{pmatrix} 1 & -2E(\lambda) \\ 0 & -1 \end{pmatrix}.
$$

(2.23)
Proof. Here we consider a proof based on the straightforward differentiation of the functions $f_{\pm}(\lambda)$; a simpler proof, based on analytical properties of the matrix $X(\lambda)$, is given in Sect. 3.1.

We first consider the derivatives of functions $f_{\pm}(\lambda)$ with respect to the parameter $\alpha$. We begin with calculating the derivatives of functions $e_{\pm}(\lambda)$. From (2.6), for the function $e_-(\lambda)$ we have

$$
\frac{d}{d\alpha}e_-(\lambda) = -\frac{r}{2(\lambda - \alpha)}e_-(\lambda). \quad (2.24)
$$

To obtain the similar relation for the function $e_+(\lambda)$, let us consider first the function $E(\lambda)$, see (2.7), for which we have

$$
\frac{d}{d\alpha}E(\lambda) = r \oint_{C_0} \frac{(\nu - 1)^{\gamma + q\nu}}{(\nu - \alpha)^{\gamma + 1}(\nu - \lambda)} d\nu
$$

Using

$$
\frac{1}{(\nu - \alpha)(\nu - \lambda)} = \frac{1}{\lambda - \alpha} \left( \frac{1}{\nu - \lambda} - \frac{1}{\nu - \alpha} \right),
$$

we can rewrite the relation above as

$$
\frac{d}{d\alpha}E(\lambda) = \frac{r}{\lambda - \alpha} (E(\lambda) - E(\alpha))
$$

and therefore

$$
\frac{d}{d\alpha}e_+(\lambda) = \frac{r}{2(\lambda - \alpha)}e_+(\lambda) - \frac{rE(\alpha)}{\lambda - \alpha} e_-(\lambda), \quad (2.25)
$$

where we have also used (2.24).

To obtain derivatives of the functions $f_{\pm}(\lambda)$ with respect to $\alpha$, we differentiate the defining relation (2.9), that gives

$$
\frac{d}{d\alpha}f_{\pm}(\lambda) = \oint_{C_0} \left[ \frac{d}{d\alpha}K(\lambda, \mu) \right] f_{\pm}(\mu) d\mu - \oint_{C_0} K(\lambda, \mu) \frac{d}{d\alpha} f_{\pm}(\mu) d\mu
$$

$$
= \frac{d}{d\alpha}e_{\pm}(\lambda). \quad (2.26)
$$

Using (2.24) and (2.25), the derivative of $K(\lambda, \mu)$ can be computed, with the result

$$
\frac{d}{d\alpha}K(\lambda, \mu) = -\frac{e_+(\lambda)e_-(\mu) + e_-(\lambda)e_+(\mu) - 2E(\alpha)e_-(\lambda)e_-(\mu)}{4\pi i(\lambda - \alpha)(\mu - \alpha)}. \quad (2.27)
$$

Thus, moving the second term in the left-hand side of (2.26) to the right, we can rewrite (2.26) as follows:

$$
\left[ (1 - \hat{K}) \frac{d}{d\alpha}f_{\pm} \right](\lambda) = \frac{d}{d\alpha}e_{\pm}(\lambda)
$$

$$
- \frac{r}{2(\lambda - \alpha)} [H_{\pm}(\alpha)e_+(\lambda) + H_+(\alpha)e_-(\lambda) - 2E(\alpha)H_{\pm}(\alpha)e_{\pm}(\lambda)].
$$

In the case of the plus-sign subscript, using (2.28), we thus obtain

$$
\left[ (1 - \hat{K}) \frac{d}{d\alpha}f_+ \right](\lambda) = (1 - H_+(\alpha)) \frac{re_+(\lambda)}{2(\lambda - \alpha)}
$$

$$
- \left[ H_{++}(\alpha) + 2E(\alpha)(1 - H_+(\alpha)) \right] \frac{re_-(\lambda)}{2(\lambda - \alpha)}. \quad (2.28)
$$
Similarly, in the case of the minus-sign subscript, using (2.24), we get

$$\left[ (1 - \hat{K}) \frac{d}{d\alpha} f_{-} \right] (\lambda) = -H_{-}(\alpha) \frac{r e_{-}(\lambda)}{2(\lambda - \alpha)}$$

$$- \left[ 1 + H_{+}(\alpha) + 2E(\alpha)H_{-}(\alpha) \right] \frac{r e_{-}(\lambda)}{2(\lambda - \alpha)}. \quad (2.29)$$

Now the desired result can be obtained by acting with the operator \((1 - \hat{K})^{-1} = 1 + \hat{R}\) on these two relations.

Comparing (2.28) and (2.29) with (2.11), one can note that these two relations can also be written as

$$\left[ (1 - \hat{K}) \frac{d}{d\alpha} f \right] (\lambda) = \frac{r}{2(\lambda - \alpha)} X(\alpha)S(\alpha)\hat{g}(\lambda; \alpha),$$

where the matrix \(S(\lambda)\) is defined in (2.23). Specifying \(\nu = \alpha\) in (2.18), we obtain

$$\frac{d}{d\alpha} f(\lambda) = \frac{r}{2(\lambda - \alpha)} X(\alpha)S(\alpha)\hat{f}(\lambda). \quad (2.30)$$

Clearly, the matrix standing in the right-hand side is exactly the matrix \(B(\lambda)\) appearing in (2.20).

Let us now find the derivatives of the functions \(f_{\pm}(\lambda)\) with respect to the variable \(\lambda\). We first obtain those of the functions \(e_{\pm}(\lambda)\). It is straightforward for \(e_{-}(\lambda)\),

$$\frac{d}{d\lambda} e_{-}(\lambda) = -\frac{1}{2} \left( \frac{s}{\lambda} + \frac{r + q}{\lambda - 1} - \frac{r}{\lambda - \alpha} \right) e_{-}(\lambda). \quad (2.31)$$

To obtain that of \(e_{+}(\lambda)\), we note that the derivative of the function \(E(\lambda)\), after integration by parts, reads

$$\frac{d}{d\lambda} E(\lambda) = \int_{C_{0}} \frac{\nu^s(\nu - 1)^{r+q}}{(\nu - \lambda)(\nu - \lambda)^{r+q}} \frac{d\nu}{2\pi i}$$

$$= \int_{C_{0}} \left( \frac{d}{d\nu} \frac{\nu^s(\nu - 1)^{r+q}}{(\nu - \lambda)^r} \right) \frac{1}{\nu - \lambda} d\nu$$

$$= \left( \frac{s}{\lambda}(E(\lambda) - E(0)) + \frac{r + q}{\lambda - 1}(E(\lambda) - E(1)) \right) - \frac{r}{\lambda - \alpha} (E(\lambda) - E(\alpha)).$$

Hence,

$$\frac{d}{d\lambda} e_{+}(\lambda) = \frac{1}{2} \left( \frac{s}{\lambda} + \frac{r + q}{\lambda - 1} - \frac{r}{\lambda - \alpha} \right) e_{+}(\lambda)$$

$$- \left( \frac{sE(0)}{\lambda} + \frac{(r + q)E(1)}{\lambda - 1} - \frac{rE(\alpha)}{\lambda - \alpha} \right) e_{-}(\lambda), \quad (2.32)$$

where (2.31) have been used.

Now we are prepared for calculations with the functions \(f_{\pm}(\lambda)\). Differentiate the defining relation (2.9) to obtain

$$\frac{d}{d\lambda} f_{\pm}(\lambda) = \int_{C_{0}} \left[ \frac{d}{d\lambda} + \frac{d}{d\mu} \right] K(\lambda, \mu) f_{\pm}(\mu) d\mu - \int_{C_{0}} K(\lambda, \mu) \frac{d}{d\mu} f_{\pm}(\mu) d\mu$$

$$= \frac{d}{d\lambda} e_{\pm}(\lambda). \quad (2.33)$$
Here the second term can be expressed in terms of the functions $e_{\pm}(\lambda)$. Indeed, since
\[
\left(\frac{d}{d\lambda} + \frac{d}{d\mu}\right)\frac{1}{\lambda - \mu} = 0,
\]
differentiation of the kernel affects only the numerator,
\[
\left(\frac{d}{d\lambda} + \frac{d}{d\mu}\right) K(\lambda, \mu) = \frac{e_-'(\lambda)e_-(\mu) - e'_+(\lambda)e_+(\mu) + e_+(\lambda)e'_-(\mu) - e_-(\lambda)e'_+(\mu)}{2\pi i (\lambda - \mu)},
\]
where prime denotes the derivative with respect to the argument of the function. Using (2.31) and (2.32), we obtain
\[
\left(\frac{d}{d\lambda} + \frac{d}{d\mu}\right) K(\lambda, \mu)
= \frac{1}{2} \left[ s \lambda \mu + \frac{r + q}{(\lambda - 1)(\mu - 1)} - \frac{r}{(\lambda - \alpha)(\mu - \alpha)} \right] e_+(\lambda)e_-(\mu) + \frac{r + q}{(\lambda - 1)(\mu - 1)} - \frac{r}{(\lambda - \alpha)(\mu - \alpha)}\right] e_-(\lambda)e_+(\mu) + \frac{r + q}{(\lambda - 1)(\mu - 1)} - \frac{r}{(\lambda - \alpha)(\mu - \alpha)}\right] e_+(\lambda)e_+(\mu)
\]
Thus, moving the second term in the left-hand side of (2.33) to the right, we arrive at the following relation:
\[
\left(1 - \hat{K}\right) f_+ = \frac{d}{d\lambda} e_+(\lambda) + \frac{s}{2\lambda} \left( H_{-\pm}(0)e_+(\lambda) + (H_{\pm}(0) - 2E(0)H_{-\pm}(0))e_-(\lambda) \right)
- \frac{r + q}{2(\lambda - 1)} \left( H_{-\pm}(1)e_+(\lambda) + (H_{\pm}(1) - 2E(1)H_{-\pm}(1))e_-(\lambda) \right)
+ \frac{r}{2(\lambda - \alpha)} \left( H_{-\pm}(\alpha)e_+(\lambda) + (H_{\pm}(\alpha) - 2E(\alpha)H_{-\pm}(\alpha))e_-(\lambda) \right)
\]
Using (2.31) and (2.32) and switching to the vector form, one can see that (2.31) is just the relation
\[
\left(1 - \hat{K}\right) \vec{f} = \frac{s}{2} X(0)S(0)\vec{g}(\lambda; 0) + \frac{r + q}{2} X(1)S(1)\vec{g}(\lambda; 1) - \frac{r}{2} X(\alpha)S(\alpha)\vec{g}(\lambda; \alpha),
\]
where functions $g_{\pm}(\lambda; \nu)$ are defined in (2.17).

Finally, acting with the operator $(1 - \hat{K})^{-1} = 1 + \hat{R}$ on the last relation by making use of (2.19), we obtain
\[
\frac{d}{d\lambda} \vec{f}(\lambda) = \left\{ \frac{s}{2\lambda} X(0)S(0)X^{-1}(0) + \frac{r + q}{2(\lambda - 1)} X(1)S(1)X^{-1}(1)
- \frac{r}{2(\lambda - \alpha)} X(\alpha)S(\alpha)X^{-1}(\alpha) \right\} \vec{e}(\lambda).
\]
Clearly, the matrix appearing in the right-hand side of (2.35) coincides with the matrix $A(\lambda)$ given in (2.20). □
3. EFP and P6

The main goal of this section is to prove Theorem 1.1. We begin with recalling some basic facts of the theory of P6. Next, we discuss analytic properties of the matrix $X(\lambda)$ in terms of the related objects of the theory of integrable systems. Finally, we use all these ingredients to show that the Fredholm determinant of the operator $\hat{K}$ serves as a $\tau$-function of P6.

3.1. The sixth Painlevé equation. The canonical form of P6 (cf. [47]) reads,

$$
\frac{d^2 y}{d\alpha^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - \alpha} \right) \left( \frac{dy}{d\alpha} \right)^2 - \left( \frac{1}{\alpha} + \frac{1}{\alpha - 1} + \frac{1}{y - \alpha} \right) \frac{dy}{d\alpha}
$$

$$
+ \frac{y(y - 1)(y - \alpha)}{\alpha^2(\alpha^2 - 1)} \left( a + b\frac{\alpha}{y^2} + c\frac{\alpha - 1}{(y - 1)^2} + d\frac{\alpha(\alpha - 1)}{(y - \alpha)^2} \right). \tag{3.1}
$$

For generic choice of the coefficients $a$, $b$, $c$, and $d$ solutions $y = y(\alpha)$ are transcendental functions called the sixth Painlevé transcendents. These functions cannot be expressed by a finite number of operations through the known transcendental (hypergeometric-type, elliptic, etc), algebraic, and elementary functions (for a more precise formulation, see [48, 49]).

At the same time, it is known (cf. [46]) that for some particular choices of coefficients there exist special, and even general, solutions which can be expressed in terms of Gauss hypergeometric, algebraic, elementary functions (and composition of elliptic and hypergeometric functions). In Sect. 3.3 we show that the EFP can be expressed in terms of a rational solution of P6. Classification of rational solutions of P6 via its representation as the isomonodromy deformation of a linear $2 \times 2$ matrix Fuchsian ODE of the first order is given in [50]. We do not use this classification result, however, we need the above mentioned isomonodromy representation of P6.

Consider the $2 \times 2$ matrix linear Fuchsian ODE

$$
\frac{d}{d\lambda} Y(\lambda) = A(\lambda)Y(\lambda), \quad A(\lambda) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_\alpha}{\lambda - \alpha}, \tag{3.2}
$$

where matrices $A_0$, $A_1$, and $A_\alpha$ are independent of $\lambda$. In generic situation the isomonodromy deformations coincide with the Schlesinger deformations, and they are governed by the following equation with respect to $\alpha$:

$$
\frac{d}{d\alpha} Y(\lambda) = B(\lambda)Y(\lambda), \quad B(\lambda) = -\frac{A_\alpha}{\lambda - \alpha}. \tag{3.3}
$$

The compatibility condition of (3.2) and (3.3), which is called the Schlesinger system, is equivalent, according to Jimbo and Miwa [4], to P6 equation (3.1). More precisely, choose normalization of (3.2) such that the matrices $A_0$, $A_1$, and $A_\alpha$ are traceless, and the matrix $A_\infty := -(A_0 + A_1 + A_\alpha)$ is diagonal

$$
A_\infty = \begin{pmatrix}
\theta_\infty/2 & 0 \\
0 & -\theta_\infty/2
\end{pmatrix}, \quad \theta_\infty \neq 0.
$$

Denote the eigenvalues of the matrices $A_k$ as $\pm\theta_k/2$, $k = 0, 1, \alpha, \infty$. Actually, while matrices $A_k$ are functions of $\alpha$, their eigenvalues $\theta_k$ are in fact independent of $\alpha$, since they are the first integrals of the Schlesinger system.
Denoting by \((A_k)_{ij}\), \(i,j = 1,2\), the matrix elements of \(A_k\), and noting the relation
\[
(A_0)_{12} + (A_1)_{12} + (A_\alpha)_{12} = 0,
\]
one can verify that the equation
\[
\frac{(A_0)_{12}}{y} + \frac{(A_1)_{12}}{y - 1} + \frac{(A_\alpha)_{12}}{y - \alpha} = 0
\]
has in the generic situation (that is, for \((A_1)_{ij} + \alpha (A_\alpha)_{ij} \neq 0\)) unique solution \(y\), provided that the coefficients in (3.1) are chosen as follows:
\[
a = \frac{(\theta_\infty - 1)^2}{2}, \quad b = -\frac{\theta_0^2}{2}, \quad c = \frac{\theta_1^2}{2}, \quad d = \frac{1 - \theta_2^2}{2}.
\]
One can associate with isomodromy deformations of (3.2) the so-called \(\tau\)-function, which is defined modulo a constant factor by the relation
\[
\frac{d}{d\alpha} \log \tau \equiv \text{res} \, \text{tr} \left( \frac{A_0 + \theta_0/2}{\lambda} + \frac{A_1 + \theta_1/2}{\lambda - 1} + \frac{A_\alpha + \theta_\alpha/2}{\lambda - \alpha} \right)^2
\]
\[
= \text{tr} \left[ A_\alpha \left( \frac{A_0}{\alpha} + \frac{A_1}{\alpha - 1} \right) \right] + \frac{\theta_0}{\alpha} \left( \frac{\theta_0}{\alpha} + \frac{\theta_1}{\alpha - 1} \right).
\]
We note that our defining relation for the \(\tau\)-function (3.4) (with the right-hand side being the Hamiltonian of P6) exactly reproduce that of Jimbo and Miwa [4], since we use the traceless normalization of the matrices \(A_k\), \(k = 0,1,\alpha\).

The function
\[
|\sigma| = \alpha(\alpha - 1) \frac{d}{d\alpha} \log \tau
\]
\[
+ (\nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3) \left( \frac{\nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3 + \nu_1 \nu_4 + \nu_2 \nu_4 + \nu_3 \nu_4}{2} \right)
\]
where
\[
\nu_1 = \frac{\theta_\alpha + \theta_\infty}{2}, \quad \nu_2 = \frac{\theta_\alpha - \theta_\infty}{2}, \quad \nu_3 = -\frac{\theta_0 + \theta_1}{2}, \quad \nu_4 = -\frac{\theta_0 - \theta_1}{2},
\]
satisfies the equation (the so-called \(\sigma\)-form of P6):
\[
\alpha^2(\alpha - 1)^2 \sigma'(\sigma'')^2 + \{(1 - 2\alpha)(\sigma')^2 + 2\sigma' + \nu_1 \nu_2 \nu_3 \nu_4\}^2
\]
\[
= \left( \sigma' + \nu_2^2 \right) \left( \sigma' + \nu_3^2 \right) \left( \sigma' + \nu_4^2 \right).
\]
The prime denotes differentiation with respect to \(\alpha\). We note the \(\sigma\)-function can also be written in the form
\[
\sigma = \alpha(\alpha - 1) \frac{d}{d\alpha} \text{tr} \left[ A_\alpha \left( \frac{A_0}{\alpha} + \frac{A_1}{\alpha - 1} \right) \right] + \nu_1 \nu_2 \alpha - \frac{\nu_1 \nu_2 + \nu_3 \nu_4}{2},
\]
which will be convenient for our purposes below.

3.2. Riemann-Hilbert problem. Our result stated in Proposition [2.5] yields a special vector solution of the system of ODEs (3.2), (3.3) in terms of the solution of the integral equation (2.9), where the matrices \(A_0\), \(A_1\), and \(A_\alpha\) are given by (2.21)–(2.23), (2.11), and (2.12). Here we construct a matrix fundamental solution corresponding to the vector one, calculate its monodromy data and, finally, reformulate the problem of solving of the Fredholm integral equation as the solution of a proper Riemann-Hilbert problem or as a solution of the inverse monodromy problem for ODE (3.2). As a byproduct of our considerations here we give another
proof of Proposition 2.5 and obtain a formula for $X'(\lambda)$ which will appear important in what follows. It should be stressed that this formula, just like the statement of Proposition 2.5, can be proved by the direct differentiation, while their derivations based on the analytic properties of $X(\lambda)$ are easier modulo the considerations given below.

First, we discuss the structure of the matrix $Y(\lambda)$ in our case. Looking at (2.21) and taking into account that the matrix $S(\lambda)$ admits factorization

$$S(\lambda) = \begin{pmatrix} 1 & E(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -E(\lambda) \\ 0 & 1 \end{pmatrix},$$

(3.8)

we observe that the matrices $A_0$, $A_1$, and $A_s$ are in fact residues of the logarithmic derivative $[\partial_\lambda Y(\lambda)]Y(\lambda)^{-1}$ of the matrix

$$Y(\lambda) = X(\lambda) \begin{pmatrix} 1 & E(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/e_-(\lambda) & 0 \\ 0 & e_-(\lambda) \end{pmatrix},$$

(3.9)

where the functions $E(\lambda)$ and $e_-(\lambda)$ are given by (2.8) and (2.6), respectively. Below we show that $Y(\lambda)$ is a fundamental solution of the Fuchs pair (3.2), (3.3). Note that the second column of $Y(\lambda)$ is nothing but our vector solution $\tilde{f}(\lambda)$.

The standard approach of proving that $Y(\lambda)$ is the solution of the Fuchs pair (3.2), (3.3) consists in inspecting its analytic properties. Equation (3.9) defines the matrix $Y(\lambda)$ as analytic function on $\mathbb{C} \setminus \{C_0 \cup C_\infty\}$, i.e., piecewise analytic in $\mathbb{C}$, with singular points at $\lambda = 0, \alpha, 1, \infty$. In case some numbers $r, s, \text{or} q$ are odd one has to make one or two cuts in $\mathbb{C}$ connecting the singular points. If so, then one should consider $\mathbb{C}$ with the cuts, that makes no substantial changes in the following.

The boundary values of the functions $X(\lambda)$ and $E(\lambda)$ suffer jumps on the oriented contours $C_0$ and $C_\infty$ (recall that both contours are counter-clockwise oriented with respect to the origin). These jumps can be found with the help of the Sokhotski-Plemelj formulas. Namely, let $\Gamma$ be an oriented contour; denoting $Y_{\Gamma}^\pm(\lambda) = Y(\lambda \pm \epsilon)$ as $\epsilon \to 0$ with $\lambda \in \Gamma$, such that the point $\lambda + \epsilon$ (respectively, $\lambda - \epsilon$) lies in the domain on the left-hand (right-hand) side of the oriented contour $\Gamma$, we have

$$Y_{\Gamma}^+(\lambda) = Y_{\Gamma}^-(\lambda) G_\Gamma, \quad \lambda \in \Gamma, \quad \Gamma = \{C_0, C_\infty\},$$

(3.10)

where

$$G_{C_0} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad G_{C_\infty} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

(3.11)

Using equations (3.9), (3.10), (3.12), and (2.8) one confirms the following asymptotic behavior at the singular points:

$$Y(\lambda) \approx_{\lambda \to \nu} \begin{cases} \Phi_0(\nu) + \sum_{k=1}^{\infty} \Phi_k(\nu)(\lambda - \nu)^k, & \nu = 0, \alpha, 1, \\ \frac{(\lambda - \nu)^{\theta_\infty/2}}{(\lambda - \nu)^{-\theta_\infty/2}}. & \nu = \infty \end{cases}$$

(3.12)

where $\det \Phi_0(\nu) = 1$ and the normalization condition at the point $\lambda = \infty$ is

$$Y(\lambda) \approx_{\lambda \to \infty} \begin{cases} I + \sum_{k=1}^{\infty} \Phi_k(\infty) \left(\frac{1}{\lambda}\right)^k, & \theta_\infty = -(s + q), \\ \frac{(\lambda^{-\theta_\infty/2}}{\lambda^{\theta_\infty/2}}. & \theta_\infty = -(-s + q). \end{cases}$$

(3.13)

The series in (3.12) and (3.13) are convergent in some neighborhoods of the corresponding singular points, and $\theta_\nu$'s are given by (2.22).
We can summarize the analytic properties of $Y(\lambda)$ as the following singular Riemann-Hilbert problem:

1. The $2 \times 2$ matrix function $Y(\lambda)$ is analytic in $\mathbb{C} \setminus \{C_0 \cup C_\infty\}$, where circles $C_0$ and $C_\infty$ are centered at 0 and $\infty$, respectively;
2. The boundary values on the contours $C_0$ and $C_\infty$ satisfy the jump conditions (3.10), (3.11);
3. The function $Y(\lambda)$ has singular points at $\lambda = 0, \alpha, 1, \infty$ where it has asymptotic expansions (3.12) and (3.13);
4. The points $\lambda = \alpha$ and $\lambda = 1$ belong to the annulus domain between the circles $C_0$ and $C_\infty$.

The existence of the solution of the Riemann-Hilbert problem can be established by the reference to the integral equation (2.9) and representation (3.9) for $Y(\lambda)$; the uniqueness can be proven by standard arguments based on the Liouville theorem. In fact, the Riemann-Hilbert problem is a reformulation of the analytic properties of the matrix-function $Y(\lambda)$, which thus can be seen as its implicit solution. Note that this Riemann-Hilbert problem can solved explicitly with the help of the theory Schlesinger transformations.

The solution of the Riemann-Hilbert problem delivers the generalized solution of the Fuchs pair (3.2), (3.3). This can be established in the standard way by noticing that the logarithmic derivatives $\partial_\lambda Y(\lambda) Y(\lambda)^{-1}$ and $\partial_\nu Y(\lambda) Y(\lambda)^{-1}$ are rational functions on $\mathbb{C}$ with the first order poles at 0, $\alpha$, 1, and $\infty$. Calculation of the residues at these points with the help of representation (3.9) provides us an alternative proof of Proposition 2.5, since our vector solution $\vec{f}(\lambda)$ is nothing but the second column of $Y(\lambda)$. Note that, if one does not assume representation (3.9), then the residue-matrices $A_\nu$, $\nu = 0, \alpha, 1$, can be calculated in terms of the entries the matrix $\Phi_1(\infty)$.

Using the solution of the Riemann-Hilbert problem we can construct the classical solution to the pair (3.2), (3.3) just by making analytic continuation on the whole complex plane of any piece of the function $Y(\lambda)$; since we have three pieces we obtain three fundamental solutions related to each other by constant matrices. We take the canonical solution, i.e., the one obtained by the analytic continuation from the disk centered at infinity with the asymptotics (3.13). Our purpose is to characterize this solution by its monodromy data. If we denote this solution as $Y_\infty(\lambda)$ then it can be characterized by the following asymptotic expansions.

**Proposition 3.1.** The matrix-function $Y_\infty(\lambda)$ is the unique fundamental solution of (3.2) with the residue matrices $A_\nu$ defined by (2.21) and (2.22) with the following expansions at the regular singular points:

$$
Y_\infty(\lambda) \to \nu = \frac{\Phi_0(\nu) + \sum_{k=1}^\infty \Phi_k(\nu) (\lambda - \nu)^k}{\left(\begin{array}{cc}
(\lambda - \nu)^{\theta_\nu/2} & 0 \\
0 & (\lambda - \nu)^{-\theta_\nu/2}
\end{array}\right) C_\nu, \\
\nu = 0, \alpha, 1, \tag{3.14}
$$

with the normalization condition

$$
Y_\infty(\lambda) \to \lambda \to \infty \left( I + \sum_{k=1}^\infty \Phi_k(\infty) \left(\frac{1}{\lambda}\right)^k \right) \left(\begin{array}{cc}
\lambda^{-\theta_\infty/2} & 0 \\
0 & \lambda^{\theta_\infty/2}
\end{array}\right),
\right)$$

19
where \( \theta_\infty = -(s + q) \), \( \det \Phi_0(\nu) = 1 \), and the matrices \( C_\nu, \nu = 0, 1, \alpha \), are

\[
C_0 = G_{C_0}^{-1}G_{C_\infty}^{-1}, \quad C_\alpha = C_1 = G_{C_\infty}^{-1}.
\]

**Remark 3.1.** It is clear that the monodromy matrices

\[
M_\nu = e^{i \pi \theta_\nu} I, \quad \nu = 0, \alpha, 1, \infty,
\]
do not uniquely define \( Y_\infty(\lambda) \). So, in this specific case we call the monodromy data \( C_\nu \). Note that because integers \( \theta_0 > 0 \), \( \theta_1 > 0 \), and \( \theta_\alpha < 0 \), the matrices \( C_\nu \) are defined by asymptotic expansions \( (3.14) \) up to the left multiplication:

\[
C_0 \to \begin{pmatrix} a_0 & b_0 \\ 0 & 1/a_0 \end{pmatrix} C_0, \quad C_1 \to \begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix} C_1, \quad C_\alpha \to \begin{pmatrix} a_\alpha & 0 \\ b_\alpha & 1/a_\alpha \end{pmatrix} C_\alpha,
\]

\( (3.15) \)

where \( a_\nu, b_\nu \in \mathbb{C}, \nu = 0, 1, \alpha \).

Note that our singular matrix Riemann-Hilbert problem on two contours can be reformulated in various ways as different singular and regular Riemann-Hilbert problems. In particular, one can formulate the Riemann-Hilbert problem on an oriented simple closed contour \( \gamma \) on \( \mathbb{C} \), which surrounds a domain containing the points \( \lambda = 0 \) and \( \lambda = 1 \) but excluding the point \( \lambda = 1 \), and with the jump matrix given by \( \gamma = G_{C_0} G_{C_\infty} \). This can be done in two ways: first, by performing analytic continuation of the solutions \( Y_\infty(\lambda) \) and \( Y_0(\lambda) \) (the solution which is analytic inside the disc surrounded by the contour \( C_0 \)), and, second, just by observing that in the Fredholm kernel \( \hat{K} \) one can merge the contours \( C_0 \) and \( C_\infty \) into one contour \( \gamma \) and then repeat the whole construction of this section literally. The equivalence of these two procedures can be established by using \( (3.15) \).

As a byproduct of our discussion above, one immediately obtains, with the help of \( (3.9) \), the following formula

\[
\frac{d}{d\lambda} X(\lambda) = A(\lambda)X(\lambda) - X(\lambda) \left( \frac{s}{2\lambda} S(0) + \frac{r + q}{2(\lambda - 1)} S(1) - \frac{r}{2(\lambda - \alpha)} S(\alpha) \right).
\]

\( (3.16) \)

This formula is crucial in derivation of relation \( (3.19) \) below. Note that \( (3.16) \) also be derived directly, just by using the expressions for the derivatives of functions \( e_k(\lambda) \) and \( f_k(\lambda) \) entering the definition of the matrix \( X(\lambda) \). Then expression \( (3.16) \) yields an alternative prove of the system \( (3.2), (3.3) \), by noticing that the second term would not appear if instead of \( X(\lambda) \) there would stand the matrix \( X(\lambda) \) times some proper matrix factor, multiplied from the right. Taking into account that the matrix \( S(\lambda) \) given by \( (2.23) \) admits factorization \( (3.8) \), it is easy to see that such a matrix factor can be chosen as in the expression \( (3.9) \) for the matrix \( Y(\lambda) \).

### 3.3. The Fredholm determinant as the \( \sigma \)-function.

To identify connection of the Fredholm determinant \( \sigma \) with \( \text{P}6 \), we consider the logarithmic derivative of the determinant with respect to the parameter \( \alpha \),

\[
\frac{d}{d\alpha} \log \det \left( 1 - \hat{K} \right) = -\text{Tr} \left[ \left( 1 - \hat{K} \right)^{-1} \frac{d}{d\alpha} \hat{K} \right]
\]

\[
= -\text{Tr} \left[ \left( 1 + \hat{R} \right) \frac{d}{d\alpha} \hat{K} \right]
\]

\[
= -\oint_{\bar{C}_0} \left( 1 + \hat{R} \right) \frac{d}{d\alpha} \hat{K}(\cdot, \mu) \right) (\mu) \, d\mu.
\]

\( (3.17) \)
To evaluate these integrals, we first note that the $α$-derivative of the kernel $K(λ, µ)$, see (2.27), can be written in terms of the functions $g_±(λ; α)$, defined in (2.17), as follows

$$\frac{d}{dα} K(λ, µ) = -\frac{r}{4πi} [g_+(λ; α)g_-(µ; α) + g_-(λ; α)g_+(µ; α) - 2E(α)g_-(λ; α)g_-(µ; α)].$$

(3.17)

Introducing a notation (see also (2.17))

$$\bar{g}_*(λ; ν) = \begin{pmatrix} g_-(λ; ν) \\ -g_+(λ; ν) \end{pmatrix},$$

we note that the linear combination of the functions standing in the brackets in (3.17) can also be written as

$$\frac{d}{dα} K(λ, µ) = -\frac{r}{4πi} \bar{g}_*^T(µ; α)S(α)\bar{g}(λ; α) = -\frac{r}{4πi} \text{tr} \left[ \bar{g}(λ; α)\bar{g}_*^T(µ; α)S(α) \right],$$

where the matrix $S(λ)$ is given in (2.29). Then, using the result of action of the operator $1 + \hat{R}$ on the functions $g_±(λ; ν)$ in its vector form (2.15), we have

$$\left[(1 + \hat{R}) \frac{d}{dα} K(·, µ) \right](λ) = -\frac{r}{4πi(µ - α)} \text{tr} \left[ X^{-1}(α)\bar{f}(λ; α)\bar{g}_*^T(µ; α)S(α) \right].$$

Setting here $λ = µ$ and integrating over $µ$ gives

$$\oint_{C_α} \frac{1}{µ - α} \bar{f}(µ; α)\bar{g}_*^T(µ; α) \frac{dµ}{2πi} = \oint_{C_α} \frac{1}{(µ - α)^2} \begin{pmatrix} f_+(µ)e_-(µ) & -f_+(µ)e_+(µ) \\ f_-(µ)e_-(µ) & -f_-(µ)e_+(µ) \end{pmatrix} \frac{dµ}{2πi}$$

$$= \frac{d}{dλ} \begin{pmatrix} H_{++}(λ) & -H_{++}(λ) \\ H_{−−}(λ) & -H_{−−}(λ) \end{pmatrix} \bigg|_{λ=α}$$

$$= \frac{d}{dλ} X(λ) \bigg|_{λ=α}.$$
Setting $\lambda = \alpha$ for the remaining two terms and writing $\theta_{\alpha}/2$ for the pre-factor in (3.18), we have

\[
\frac{d}{d\alpha} \log \text{Det} \left( 1 - \hat{K} \right) = \frac{\theta_{\alpha}}{2 \alpha} \text{tr} \left[ S(\alpha) X^{-1}(\alpha) \left( \frac{A_0 X(\alpha)}{\alpha} - \frac{\theta_0 X(\alpha) S(0)}{2\alpha} + \frac{A_1 X(\alpha)}{\alpha - 1} - \frac{\theta_1 X(\alpha) S(1)}{2(\alpha - 1)} \right) \right]
\]

\[
= \text{tr} \left[ A_{\alpha} \left( \frac{A_0}{\alpha} + \frac{A_1}{\alpha - 1} \right) \right] - \frac{\theta_{\alpha}}{4} \text{tr} \left[ S(\alpha) \left( \frac{\theta_0 S(0)}{\alpha} + \frac{\theta_1 S(1)}{\alpha - 1} \right) \right].
\]

Taking into account that the matrix $S(\lambda)$ is upper triangular, see (2.23), we obtain

\[
\frac{d}{d\alpha} \log \text{Det} \left( 1 - \hat{K} \right) = \text{tr} \left[ A_{\alpha} \left( \frac{A_0}{\alpha} + \frac{A_1}{\alpha - 1} \right) \right] - \frac{\theta_{\alpha}}{2} \left( \frac{\theta_0}{\alpha} + \frac{\theta_1}{\alpha - 1} \right). \tag{3.19}
\]

Using (3.19), thus we have for the trace

\[
\text{tr} \left[ A_{\alpha} \left( \frac{A_0}{\alpha} + \frac{A_1}{\alpha - 1} \right) \right] = \frac{d}{d\alpha} \log \text{Det} \left( 1 - \hat{K} \right) + \frac{(\nu_1 + \nu_2)(-2\nu_3 + \nu_3 + \nu_4)}{2\alpha(\alpha - 1)},
\]

where $\nu_k$’s are defined in (3.3).

Substituting the expression for the trace above into (3.7), we arrive at the following expression for the $\sigma$-function

\[
\sigma = \alpha(\alpha - 1) \frac{d}{d\alpha} \log \text{Det} \left( 1 - \hat{K} \right) + (\nu_1 \nu_2 - \nu_1 \nu_3 - \nu_2 \nu_4)\alpha - \frac{\nu_1 \nu_2 + \nu_3 \nu_4 - (\nu_1 + \nu_2)(\nu_3 + \nu_4)}{2}.
\]

Taking into account that from our previous discussion it follows that

\[
\theta_0 = s, \quad \theta_1 = r + q, \quad \theta_{\alpha} = -r, \quad \theta_{\infty} = -(\theta_0 + \theta_1 + \theta_{\alpha}) = -(s + q),
\]

the $\nu_k$’s are therefore identified to be

\[
\nu_1 = \nu_3 = -\frac{r + q + s}{2}, \quad \nu_2 = -\frac{r - q - s}{2}, \quad \nu_4 = \frac{r + q - s}{2}. \tag{3.20}
\]

Using these values, for the $\sigma$-function we obtain the following expression

\[
\sigma = \alpha(\alpha - 1) \frac{d}{d\alpha} \log \text{Det} \left( 1 - \hat{K} \right) - \frac{(r + q + s)^2}{4} \alpha + \frac{(r + q + s)q + 2rs}{4}.
\]

It satisfies (3.10) with the $\nu_k$’s as specified in (3.20). Since the EFP is represented in as the Fredholm determinant, $F_{r,s,q} = \text{Det}(1 - \hat{K})$, we conclude that we arrive at the statement of Theorem 1.11 which is thus proved.

4. Asymptotic expansions of the EFP in the thermodynamic limit

Under the thermodynamic limit of a physical quantity given on a lattice one usually understands behavior of this quantity as the size of the lattice becomes large. Specifically, in the case of the EFP besides the size of the lattice, $N = r + s + q$, there are geometric parameters, $s$ and $q$, which describe the frozen rectangle at the top left corner of the lattice. Therefore it is interesting to consider the thermodynamic limit $N \to \infty$ preserving the geometry of the problem, i.e., keeping the ratios $s/N$ and $q/N$ fixed. Here we consider solution of this problem in the important particular case where $q = 0$; it will be convenient for us to fix the ratio $s/r$ and perform the limit $s \to \infty$. We show that this problem can be solved in a systematic
way using the $\sigma$-form of P6 and the asymptotic behavior of $F_{r,s,q}$ at the singular points of this equation.

We begin with discussion of physical interpretation of leading terms of asymptotic expansions in two different regimes. The details of calculations of the asymptotic behavior of $F_{r,s,q}$ in each regime follow next. In Appendices we give an alternative derivation of the asymptotics in one of the regimes.

4.1. Two asymptotic regimes. Recall that the EFP is defined as a probability of observing the certain arrow configuration in the six-vertex model with DWBC. It can be seen as a ratio of two partition functions:

$$F_{r,s,q} = \frac{Z_{r,s,q}}{Z_N}$$

where $Z_{r,s,q}$ and $Z_N$ are the partition functions of the model on the lattice with the frozen corner, and on the original, unmodified lattice, respectively. Since both partition functions in the thermodynamic limit behave as $\exp(-Vf)$ where $V$ is the volume (the number of lattice sites) and $f$ is the free-energy per site, we conclude that in the limit, where $r,s,q \to \infty$, we should have

$$F_{r,s,q} = \exp\{-\phi s^2 + o(s^2)\}.$$  \hspace{1cm} (4.1)

The quantity $\varphi$ is the function of the ratios $r/s$ and $q/s$, and the parameter $\alpha$; moreover, $\varphi \geq 0$ since $Z_{r,s,q} \leq Z_N$. This function has the meaning of the change of the free-energy per site due to the freezing of a macroscopically large rectangle at the corner of the lattice. It is clear that if $s > r$, i.e., the bottom-right corner of the frozen rectangle lies below the counter-diagonal of the square, then, as follows from the ice-rule, $F_{r,s,q} = 0$ and hence $\varphi = \infty$. At $s = r$, it can be shown that for generic weights $F_{r,s,q} = (w_1w_2)^{s(s+q)}Z_rZ_{r+q}/Z_N$; in our case of the free-fermion weights (1.1) $Z_n = 1$ for all $n \in \mathbb{N}$, that follows from the results of papers [14,28], hence $F_{r,s,q} = (1-\alpha)^{s(s-1)}$. At $s = 0$, obviously, $F_{r,s,q} = 1$, hence $\varphi = 0$. Thus the problem consists in studying the EFP in the thermodynamic limit for $0 < s < r$.

It is clear that $\varphi$ is monotonically non-decreasing function for $s \in (0,r)$. Thus, there may exist a region of values of $s$, adjacent to zero, where $\varphi \equiv 0$. In this case the quantity $o(s^2)$ in (4.1) decays much faster than $s^2$, namely, it is exponentially small as $s \to \infty$. In this region the EFP is close to 1, and we have the following behavior

$$F_{r,s,q} = 1 - \exp\{-\chi s + o(s)\}, \hspace{1cm} \chi \geq 0.$$  \hspace{1cm} (4.2)

Note that this kind of behavior can be expected if we recall that the EFP can be treated as an $s$-point quantum correlation function [44] and that ferroelectric correlations in the six-vertex model are exponentially decaying.

It is to be also mentioned that our EFP has the meaning of a cumulative distribution function in the context of the random growth model, considered by Johansson in [34], who proved the existence of the two asymptotic regimes (4.1) and (4.2). The functions $\varphi$ and $\chi$ are called there as the lower and upper tail deviation functions, respectively.

The existence of the two asymptotic regimes (4.1) and (4.2) is a manifestation of the spatial separation of phases of the six-vertex model with DWBC in the thermodynamic limit, the so-called arctic ellipse phenomenon. Exactly the same phenomenon is often mentioned in the literature in relation to the dimer problem on an Aztec diamond graph (or domino tilings of an Aztec diamond) [28,51].

The phenomenon consists in a peculiar form of statistically dominating configurations of the six-vertex model with DWBC on the $N \times N$ lattice as $N \to \infty$. The arctic ellipse arises when $N \times N$ lattice is scaled to a square of unit side length,
and it divides this square on two types of domains: the interior of the ellipse and the four corner domains, see Fig. 3. The local states (arrows) are disordered in the interior of the ellipse, while they are ordered in the each of the four corner domains due to strong influence of the boundary conditions. Recall that the EFP describes the probability that all arrows are ordered (frozen) in the rectangle at the top left corner of the lattice. There are two principal cases, whether the rectangle intersects the arctic ellipse or not. When the rectangle intersects the arctic ellipse one may expect that the EFP behaves as given by (4.1). Since in this case the rectangle contains a portion of the domain of the disorder, we will say that then the EFP is in the disordered regime. If the rectangle does not intersects the arctic ellipse then one may expect that the EFP behaves as given by (4.2). Since in this case the rectangle encloses only a region of order we will say that the EFP is in the ordered regime. Indeed, when the rectangle does not intersects the arctic ellipse it does not essentially influence on the number of the states of the system; thus the EFP is close to 1 for large $N$, and, in fact, equals to 1 in the thermodynamic limit. Contrary, when it intersects the arctic ellipse it starts to order the disordered domain and reduces significantly the number of states, so that the EFP should be small for large $N$, and, in fact, vanishes in the limit. In Fig. 3 these two situations are shown in the case $q = 0$, i.e., when the rectangle is just square.

In calculations we limit ourselves here to the case $q = 0$. In this case the analytic results appear considerably simpler without altering the main ingredients of the method. We take the two remaining integer parameters, $s$ and $r$, $0 < s < r$ to be large in the limit, with their ratio being a non-vanishing parameter

$$v \equiv \frac{s}{r}, \quad v \in (0, 1).$$
The condition of intersection of the square with the arctic ellipse reads

\[ v = u, \quad u \equiv \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}. \quad (4.3) \]

For \( v \in (u, 1) \) the EFP is in the disordered regime, and for \( v(0, u) \) it is in the ordered one. Sometimes instead of natural variables \( v \) and \( u \), it is useful to use the variables \( \alpha \) and \( \beta \), in which the condition of intersection of the ellipse reads

\[ \alpha = \beta, \quad \beta \equiv \left( \frac{1 - v}{1 + v} \right)^2. \quad (4.4) \]

For \( \alpha \in (\beta, 1) \) the EFP is in the disordered regime, and for \( \alpha \in (0, \beta) \) it is in the ordered one.

### 4.2. Asymptotic expansion in the disordered regime.

We begin with the study of the EFP in the disordered regime, which corresponds to \( v \in (u, 1) \) or \( \alpha \in (\beta, 1) \), by considering behavior of \( F_{r,s,q} \) at the critical point \( \alpha = 1 \) of P6.

It was noticed in [30] that \( F_{r,s,0} \sim C_{r,s}(1 - \alpha)^{s^2} \) where the constant \( C_{r,s} \) is essentially given by the value of the Hankel determinant in (2.2) at \( \alpha = 1 \). However, this asymptotics does not uniquely characterize the corresponding solution of the \( \sigma \)-form of P6, and therefore we need a more elaborated result.

**Proposition 4.1.** As \( \alpha \to 1 \), the EFP behaves as follows

\[ F_{r,s,0} = C_{r,s}(1 - \alpha)^{s^2} \left( 1 - \frac{s(r - s)}{2}(1 - \alpha) + \frac{s(r - s)(2s^3r - 2s^4 - 3s^2 + 1)}{4(4s^2 - 1)}(1 - \alpha)^2 + O((1 - \alpha)^3) \right), \quad (4.5) \]

where the constant \( C_{r,s} \) has the explicit form

\[ C_{r,s} = \prod_{j=0}^{s-1} \frac{(j + r)!/(j!)^2}{(r - j - 1)!(2j)!(2j + 1)!}. \quad (4.6) \]

**Proof.** At \( \alpha = 1 \) the Hankel determinant in (2.2) appears to be related in the canonical way with an ensemble of Hahn polynomials [30, 52]. At \( q = 0 \) we deal with the special case of the Hahn polynomials \( Q_n(x; 0, 0, r - 1) \); for the notation and basic properties see, e.g., [53]. We will use the normalized version of these polynomials with the coefficient of the highest-order term equal to 1,

\[ p_n(x) = \frac{(1)_n(-r + 1)_n}{(n + 1)_n} Q_n(x; 0, 0, r - 1) = \frac{(1)_n(-r + 1)_n}{(n + 1)_n} \binom{3}{-n, n + 1, x} F_2 \left( -n, n + 1, -x \right| 1, 1, -r + 1 \right), \]

where \( (a)_n = a(a + 1) \cdots (a + n - 1) \) is the Pochhammer symbol. These polynomials satisfy the orthogonality condition

\[ \sum_{x=0}^{r-1} p_n(x)p_{n'}(x) = h_n \delta_{nn'}, \quad h_n = \frac{(n!)^4(r + n)!}{(2n)!(2n + 1)!(r - n - 1)!} \quad (4.7) \]

and the recurrence relation

\[ xp_n(x) = p_{n+1}(x) + \frac{(r - 1)}{2} p_n(x) + \frac{h_n}{h_{n-1}} p_{n-1}(x), \quad \frac{h_n}{h_{n-1}} = \frac{n^2(r^2 - n^2)}{4(4n^2 - 1)} \quad (4.8) \]

Let us denote the Hankel matrix in the representation (2.2) by \( H(\alpha) \); we also write for brevity \( H \equiv H(1) \). At \( q = 0 \) (for the case of \( q \neq 0 \), see [52]) the entries of
the matrix $H$ appear to be the moments of the orthogonality measure in (4.7) and therefore
\[
\det H = \det \left[ \sum_{m=0}^{r-1} m_j^{j+k-2} \right] = \prod_{j=0}^{s-1} h_j = \prod_{j=0}^{s-1} \frac{(j!)^4(r+j)!}{(2j)!(2j+1)!(r-j-1)!}.
\]
Taking into account the value of the prefactor in (2.2), we obtain the expression (4.0) for the constant $C_{r,s}$.

Let us now focus on the terms of the $(1 - \alpha)$-expansion in (4.5). We denote $H' \equiv \frac{d}{d\alpha} H(\alpha)|_{\alpha=1}$ and $H'' \equiv \frac{d^2}{d\alpha^2} H(\alpha)|_{\alpha=1}$. Using the fact that $\det H(\alpha) = \exp(\text{tr} \log H(\alpha))$ and differentiating, one can show that the coefficients of the Taylor expansion
\[
\det H(\alpha) = \det H \cdot \left(1 + c_1(1 - \alpha) + \frac{c_2}{2}(1 - \alpha)^2 + O \left((1 - \alpha)^3\right)\right),
\]
can be written as
\[
c_1 = -\text{tr}(H'H^{-1}), \quad c_2 = (\text{tr}(H'H^{-1}))^2 - \text{tr}(H'H^{-1})^2 + \text{tr}(H''H^{-1}).
\]
Now we use the fact that
\[
H'_{jk} = \left. \frac{1}{(j-1)!\partial x^{j-1}} \frac{1}{(k-1)!\partial y^{k-1}} \sum_{n=0}^{r-1} \frac{p_n(x)p_n(y)}{h_n} \right|_{x=y=0}.
\]
Taking into account that $H'_{jk} = \sum_{m=0}^{r-1} m_j^{j+k-1}$ and $H''_{jk} = \sum_{m=0}^{r-1} m_j^{j+k-1}(m-1)$, one can reduce evaluation of the traces in the expressions above to the standard sums involving the related polynomials, namely
\[
\sum_{m=0}^{r-1} \frac{mp_n^2(m)}{h_n} = \frac{r-1}{2}, \quad \sum_{m=0}^{r-1} \frac{m^2p_n^2(m)}{h_n} = \frac{h_{n+1}}{h_n} + \left(\frac{r-1}{2}\right)^2 + \frac{h_n}{h_{n-1}},
\]
which can be easily derived from the orthogonality condition (4.7) and the recurrence relation (4.8). In this way, for the coefficient $c_1$ we get
\[
c_1 = -\sum_{n=0}^{s-1} \frac{\sum_{m=0}^{r-1} mp_n^2(m)}{h_n} = -\frac{s(r-1)}{2}.
\]
For the coefficient $c_2$, after the straightforward but more involved calculations, we obtain
\[
c_2 = -\frac{s(r-1)}{2} + \frac{s^2(1 - 2r + s^2)}{4} + \frac{s^4(r^2 - s^2)}{4s^2 - 1}.
\]
Finally, taking into account that
\[
\frac{1}{\alpha^{s(s-1)/2}} = 1 + \frac{s(s-1)}{2}(1 - \alpha) + \frac{s(s-1)(s^2 - s + 2)}{8}(1 - \alpha)^2 + O \left((1 - \alpha)^3\right),
\]
we arrive at (4.9).

As a direct consequence of (4.5), we have the following.

**Corollary 4.1.** As $\alpha \to 1$,
\[
\log F_{r,s,0} = \log C_{r,s} + s^2 \log(1 - \alpha) - \frac{s(r-s)}{2}(1 - \alpha)
\]
\[
- \frac{s(r-s)(7s^2 - 8s - 2)}{8(4s^2 - 1)}(1 - \alpha)^2 + O \left((1 - \alpha)^3\right).
\]
To construct the asymptotics of $\log F_{r,s,0}$ for large $s$ and $r = s/v$, we need the corresponding behavior of the coefficients of (14.9). A nontrivial calculation concerns only logarithmic terms in (4.9). A nontrivial calculation concerns only even powers of $1/s$. The same property appears to be valid for the explicit terms in (4.9).

As established by Barnes:

$$B(r) = \frac{\pi^{s+1/2} G^2(1/2) G(r + s + 1) G(r - s + 1)}{2^{2s-1} G^2(r + 1) G(s + 1/2) G(s + 3/2)}. \quad (4.10)$$

The Barnes G-function satisfy the relations

$$G(z + 1) = \Gamma(z) G(z), \quad G(1) = G(2) = G(3).$$

As established by Barnes:

$$\log G(z + 1) = \frac{z^2}{2} \log z - \frac{3}{4} z^2 + \frac{\log 2\pi}{2} z - \frac{1}{12} \log z + \zeta'(-1) + \sum_{k=1}^{n} \frac{B_{2k+2}}{4k(k+1)} z^{-2k} + O(z^{-2n-2}), \quad (4.11)$$

where $\zeta'(-1) = -0.165142...$ is the derivative of the Riemann function $\zeta(z)$ at $z = -1$, and $B_{2k}$’s are the Bernoulli numbers:

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad \ldots \quad (4.12)$$

Therefore, taking the logarithm of (4.10) and using (4.11), we arrive at the following result.

**Proposition 4.2.** As $s, r \to \infty$, with $s/r = v$ fixed, $v \in [0, 1]$, we have

$$\log C_{r,s} = -s^2 \left( \log 4v - \frac{(1 + v)^2}{2v^2} \log(1 + v) - \frac{(1 - v)^2}{2v^2} \log(1 - v) \right) - \frac{1}{12} \log s - \frac{1}{12} \log \left( \frac{1 - v^2}{2} \right) + \zeta'(-1) + \sum_{k=1}^{n} \frac{B_{2k}}{s^2} + O(s^{-2n-2}), \quad (4.13)$$

where the coefficients $B_{2k} = B_{2k}(v)$ are

$$B_{2k} = \frac{B_{2k+2}}{4k(k+1)} \left( \frac{v^{2k}}{(1 + v)^{2k}} + \frac{v^{2k}}{(1 - v)^{2k}} - 2v^{2k} \right) - \sum_{m=0}^{k-1} \frac{B_{2(k-m)+2}}{2^{2m+1} (k-m)(k-m+1)} \left( \frac{2k-1}{2m} \right) \frac{4k^2 + 6k - 1}{2^{2k+3} k(k+1)(2k+1)},$$

where $\binom{2k-1}{2m}$ are Binomial coefficients. For example,

$$B_2 = -\frac{1}{8} \left( \frac{1}{15} - \frac{1 + v^2}{15(1 - v^2)^2} \right), \quad B_4 = -\frac{v^6}{504(1 - v^2)^4} - \frac{31}{16128}.$$

Note that, besides single logarithmic term, the expansion for $\log C_{r,s}$ involves only even powers of $1/s$. The same property appears to be valid for the explicit terms in (4.9).
To derive expansions of the EFP in the thermodynamic limit, we start with analyzing the $\sigma$-form of P6 in the large $s$ limit. We note that for $q = 0$ the $\sigma$-function in (1.2) reads

$$\sigma = \left( -\frac{(1 + v)^2}{4v^2} - \frac{1}{2v} \right) s^2 + \alpha(\alpha - 1) \frac{d}{d\alpha} \log F_{r,s,0}. \quad (4.14)$$

Absence of the odd powers of $s$ in the $\sigma$-form of P6 (1.3) together with the expansions (4.9) and (4.13) suggests that the $\sigma$-function may be searched in the form of the following asymptotic ansatz in the decaying powers of $s^2$:

$$\sigma = \sigma_2 s^2 + \sigma_0 + \sigma_{-2}s^{-2} + \cdots, \quad s \to \infty. \quad (4.15)$$

Note that if we succeed in the construction of the expansion (4.15) with the coefficients $\sigma_{2k} = \sigma_{2k}(\alpha)$, which are analytical functions of $\alpha$, then the Wazow theorem, see Th. 36.1 in Chap. IX of [55], implies that there exists a genuine solution with asymptotics (4.15). To justify that this solution actually coincides with the solution given by (4.14) one can verify that both solutions have the same asymptotics as $\alpha \to 1$. Here we present the general scheme of derivation of asymptotics; more details are given in Remark 4.3 below.

The expansion (4.15) can be constructed in a standard way by substituting it in (1.3). On this way we first obtain the leading term, by requiring that it reproduces the asymptotic condition at $\alpha \to 1$, and next prove that the further terms can be obtained recursively.

It happens that the leading term $\sigma_2$ can be found as a solution of the first-order ODE which is obtained by noticing that the first term in the left-hand side of (1.3), which contains second-order derivatives, is sub-leading as $s$ large. Besides the overall factor $\sigma_2$, the resulting first-order ODE splits on two equations

$$\frac{1 + v^2}{4v^2} + \sigma_2 + (1 - \alpha)\sigma'_2 = 0, \quad \frac{(1 - v^2)^2}{16v^4} + \sigma'_2 \left( \frac{1 + v^2}{4v^2} - \sigma_2 + \alpha\sigma'_2 \right) = 0. \quad (4.16)$$

These equations have the following general solutions

$$\sigma_2 = C_1 (\alpha - 1) - \frac{1 + v^2}{4v^2}, \quad \sigma_2 = C_{11} \alpha + \frac{1 + v^2}{4v^2} + \frac{(1 - v^2)^2}{16C_{11} v^4}, \quad (4.17)$$

respectively. Here $C_1$ and $C_{11}$ are integration constants, which may depend on $v$. Moreover, the second equation has two partial solutions

$$\sigma_2 = \frac{1 + v^2}{4v^2} \pm \frac{1 - v^2}{2v^2} \sqrt{\alpha}. \quad (4.18)$$

Consider now the function $\varphi$, defined by (4.11). Using (4.14) and (4.15) we find that

$$\varphi = \int \left( \sigma_2 + \frac{1 + v^2}{4v^2} \alpha - \frac{1}{2v} \right) - \alpha(1 - \alpha) + C,$$

where $C = C(\alpha)$ is an integration constant. The function $\varphi$ should obey the following asymptotic condition as $\alpha \to 1$:

$$\varphi = - \log(1 - \alpha) + \log 4v - \frac{(1 + v^2)^2}{2v^2} \log(1 + v) - \frac{(1 - v)^2}{2v^2} \log(1 - v) + \frac{1 - \nu}{2v} - \frac{(1 - \nu)(1 - 7v)}{32v^2} \log((1 - \alpha)^2 + O((1 - \alpha)^3). \quad (4.19)$$
The function \((\varphi)_I\) which corresponds to solution \((\sigma_2)_I\) reads

\[
(\varphi)_I = - \left( C_I + \frac{(1 + v)^2}{4v^2} \right) \log \alpha + C.
\]

This function does not have asymptotics (4.19), because of the absence the leading term \(- \log(1 - \alpha)\).

In the case of the function \((\varphi)_II\), which corresponds to \((\sigma_2)_II\), the leading term \(- \log(1 - \alpha)\) of the expansion (4.19) dictates the choice \(C_{II} = - \frac{1 - v^2}{4v^2}\), then we get

\[
(\varphi)_II = - \log(1 - \alpha) - \frac{1 - v}{2v} \log \alpha + C.
\]

As \(\alpha \to 1\),

\[
(\varphi)_II = - \log(1 - \alpha) + C + \frac{1 - v}{2v}(1 - \alpha) + \frac{1 - v}{4v}(1 - \alpha)^2 + O\left((1 - \alpha)^3\right).
\]

Therefore, because of the coefficient of the term \((1 - \alpha)^2\), the asymptotics (4.19) cannot be satisfied for any choice of \(C\).

Now consider the functions \((\varphi)_\pm\) corresponding to partial solutions (4.18),

\[
(\varphi)_\pm = - \log \left( 1 \pm \sqrt{\alpha} \right) - \frac{1}{v^2} \log \left( 1 \mp \sqrt{\alpha} \right) + \frac{(1 - v)^2}{4v^2} \log \alpha + C.
\]

Asymptotics \(\alpha \to 1\) of the function \((\varphi)_+\) has the wrong coefficient of the leading term, namely, \(-1/v^2\), compare with (4.19). For the function \((\varphi)_-\), as \(\alpha \to 1\), we get

\[
(\varphi)_- = - \log(1 - \alpha) + C - \frac{1 - v^2}{v^2} \log 2
\]

\[
+ \frac{1 - v}{2v}(1 - \alpha) - \frac{(1 - v)(1 - 7v)}{32v^2}(1 - \alpha)^2 + O\left((1 - \alpha)^3\right).
\]

Therefore, comparing the last asymptotics with (4.19), we observe that \(\varphi\) is given by the function \((\varphi)_-\) with the proper choice of the integration constant \(C\),

\[
\varphi = - \log \left( \frac{1 - \sqrt{\alpha}}{2} \right) - \frac{1}{v^2} \log \left( \frac{1 + \sqrt{\alpha}}{2} \right) + \frac{(1 - v)^2}{4v^2} \log \alpha
\]

\[
+ \log v - \frac{(1 + v)^2}{2v^2} \log(1 + v) - \frac{(1 - v)^2}{2v^2} \log(1 - v). \quad (4.20)
\]

Note that \(\varphi\) is positive for \(\alpha \in (\beta, 1)\), because it can be presented in the form

\[
\varphi = \int_{\beta}^{\alpha} \left( \frac{1 - v}{2v} - \frac{1 + v}{2v} \sqrt{\alpha} \right)^2 \frac{d\alpha}{\alpha(1 - \alpha)}, \quad (4.21)
\]

where \(\beta\) is given by (4.4).

Formula (4.20), when written in terms of the parameter \(u\), is exactly the expression (1.16) and coincides with the leading term of the asymptotic expansion of the EFP in the disordered regime obtained in [30].

Thus we have just constructed the term \(\sigma_2\) of the expansion (4.15); the further terms \(\sigma_{-2k}, k = 0, 1, \ldots,\) satisfy the recurrence relation of the following form

\[
(\nu^2(1 + \sqrt{\alpha})^2 - (1 - \sqrt{\alpha})^2)^{3k+1} \sigma_{-2k} = P(v, \alpha; \sigma_2, \ldots, \sigma_{-2k+2}), \quad (4.22)
\]
where $P$ is a polynomial in all its variables. Thus all coefficients $\sigma_{-2k}$ can be recursively constructed as analytical functions of $\alpha \in \mathbb{C}$ defined in the disc centered at $\alpha = 1$ and of radius $1 - \beta$.

**Remark 4.1.** Similarly, one can construct expansions of the form (4.15) which correspond to remaining functions in (4.14) and (4.15), namely, to $(\varphi_2)_1$, $(\varphi_2)_1$, and $(\varphi_2)_2$. According to the Wazow theorem they correspond to different (more general) solutions of the $\sigma$-form of $P_6$.

**Remark 4.2.** The general solution of the $\sigma$-form of $P_6$ (1.3) with $q = 0$ is determined by two parameters in the expansion of the form (4.14), namely, by the coefficients of the terms $\log(1 - \alpha)$ and $(1 - \alpha)^2$. Our construction shows that if the coefficient of $\log(1 - \alpha)$ is fixed as $s^2$, then all solutions whose coefficients of $(1 - \alpha)^2$, as $s \to \infty$ and $s/r = \text{const}$., are of the form $(r - s)(7s - r)/32 + o(s^2)$ have the same asymptotic expansion (4.15) with the leading term given by $(\sigma_2)_2$. Therefore their asymptotics as $s \to \infty$ differ by exponentially small corrections which are defined by the function of $o(s^2)$. To find these corrections seems an interesting problem.

From (4.14) and (4.15) it follows that

$$
\log F_{r,s,0} = -s^2 \varphi - \frac{1}{12} \log s + \sum_{k=0}^{n} \frac{a_{2k}}{s^{2k}} + O(s^{-2n-2}),
$$

where the coefficients $a_{2k} = a_{2k}(\alpha, v)$ can be obtained by integration of the corresponding coefficients $\sigma_{-2k}$ and the integration constants are fixed by expansion (4.19). For $a_0$ we find

$$
a_0 = \frac{1}{8} \log \left( \frac{4v^2 \sqrt{\alpha}}{2(1+v^2)\sqrt{\alpha} - (1-v^2)(1+\alpha)} \right) - \frac{1}{12} \log \frac{1-v^2}{2} + \zeta(-1).
$$

Similarly, one can obtain further terms; for example, we find

$$
a_2 = \frac{(1-v^2)(1-v^{\alpha})^2 \left[ 2(1+\sqrt{\alpha})^4 v^4 + 5(1-\alpha)^2 v^2 - (1-\sqrt{\alpha})^4 \right]}{64 \left[ 2(1+v^2)\sqrt{\alpha} - (1-v^2)(1+\alpha) \right]^3}
- \frac{1}{8} \left( \frac{1}{8} - \frac{1+v^2}{15} + \frac{v^2(1+v^2)}{15(1-v^2)^2} \right)
$$

and

$$
a_4 = -\frac{(1-v^2)(1-v^{\alpha})^2}{256 \left[ 2(1+v^2)\sqrt{\alpha} - (1-v^2)(1+\alpha) \right]^5}
\left[ 8(1+\sqrt{\alpha})^8 (1-3\sqrt{\alpha} + \alpha) v^{10}
+ 10 (1-v^{\alpha})^2 (1+\sqrt{\alpha})^6 (5-46\sqrt{\alpha} + 5\alpha) v^8
- 5(1-\alpha)^2 (11+106\sqrt{\alpha} + 11\alpha) v^6
+ (1-\sqrt{\alpha})^4 (1+\alpha)^2 (1+18\sqrt{\alpha} + \alpha) v^4
- (1-\sqrt{\alpha})^8 (5+14\sqrt{\alpha} + 5\alpha) v^2 + (1-\sqrt{\alpha})^{10}
- \frac{v^6(v^6-4v^4+5v^2-10)}{504(1-v^2)^3} + \frac{31}{16128}. \right]
$$

Calculations show that the functions $a_{2k}$ become more and more cumbersome as $k$ increases. Finally, rewriting the above formulas in terms of the parameter $u$ instead of $\alpha$, see (4.3), we arrive at Theorem 1.2.
Remark 4.3. Our justification of ansatz (4.15) in fact includes the following steps:

1. Appearance of the formal expansion (4.15) as the simplest possible analytic (in $\alpha$) generalization of the behavior of $F_{r,s,0}$ for large $r,s$ ($s/r = \text{const.} \equiv v$) at $\alpha = 1$.

2. Substitution of the ansatz into the $\sigma$-form of $P_6$ equation to prove that this ansatz solves the equation.

3. Application of the Wasow theorem ([55], Th. 36.1) to prove that there exists at least one genuine solution of $P_6$ that has asymptotics (4.15).

4. Proof that the solution obtained via the Wasow theorem coincides with the function $\sigma$ related to $F_{r,s,0}$, see (4.14), based on matching of asymptotics expansions as $\alpha \to 1$ using the double character of asymptotics (4.15) with respect to $s \to \infty$ and $\alpha \to 1$.

Some of the items of Remark 4.3 require further comments.

Concerning item (2), using mathematical induction and formulas (4.18) and (4.22) one can prove that the coefficients $\sigma_{-2k}$, $k = 0, 1, \ldots$, are rational functions of $\sqrt{\alpha}$ with the poles only at the point $\alpha = \beta$. Obviously, these functions are holomorphic with respect to $\alpha$ in any simply-connected domain in $\mathbb{C} \setminus \{0, \beta\}$.

Concerning item (3), the Wasow theorem does not apply to the $\sigma$-form of $P_6$ directly, because it deals with the first-order vector ODEs resolved with respect to the derivatives, with holomorphic right-hand sides, and a small parameter $\epsilon$. This theorem applies instead to the corresponding representation of $P_6$ as a Hamiltonian system. The $\sigma$-function is intimately related to the Hamiltonian, and is given as a quintic polynomial in terms of the canonical variables [46]. Conversely, the canonical variables can be written as rational functions of $\sigma$, $\partial_\alpha \sigma$, $\partial^2_\alpha \sigma$ and $\alpha$ (see, e.g., Sect. 2.1 in [46]). Therefore, for the canonical variables of the Hamiltonian system there exist formal power-like expansions in $\epsilon$, where $\epsilon = 1/s$. The Wasow theorem also deals with two assumptions, (A) and (B). The assumption (A) concerns basically the definition of the domain (in the variable $\alpha$) of validity of the asymptotics; in our case it is evident that it is nonempty, and contains any segment in the interval $(\beta, 1)$. The assumption (B) concerns nongeneracy of the matrix of the vector ODE, after, possibly, a proper transformation of the canonical variables. In our case it follows from the fact that our $\epsilon$-expansions of the canonical variables are defined uniquely as long as the leading term is given. An example of such transformation for the first Painlevé equation is given by Wasow (see [55], p. 225). Our case is considered in Appendix A.

Concerning item (4), the matching of asymptotics is made by using the known asymptotics as $\alpha \to 1$ up to the terms of $O((1 - \alpha)^3)$. It is known [55] that this asymptotics defines uniquely the solution of the $\sigma$-form of $P_6$. Therefore, by comparing the expansions as $\alpha \to 1$, one can verify that the solutions, a priori defined in a different way, actually coincide. Strictly speaking, the Wasow theorem guarantees existence of at least one solution with the asymptotics (4.15). The other possible solutions may differ by terms decaying faster than any integer power of $\epsilon$ (e.g., exponentially small terms). In our case, because of exact coincidence of the coefficients (no free parameters) of the terms $(1 - \alpha)^k$, $k = 0, 1, 2$, the exponential

\[1\text{There is a misprint after Eq. (36.40) in [55]: the assumption (B) is meant rather than (A).}\]
small corrections (if any) does not influence on the matching procedure. In other words, the corresponding monodromy data of the solutions coincide.

4.3. Asymptotic expansion in the ordered regime. To fix integration constants for the terms of the asymptotic expansion of the EFP in the ordered region, \( v \in (0, u) \) (or, equivalently, \( \alpha \in (0, \beta) \)), we use the following result about the EFP in the limit \( \alpha \to 0 \).

**Proposition 4.3.** As \( \alpha \to 0 \), the EFP behaves as

\[
F_{r,s,q} = 1 - \left( \frac{r}{s} \right) \left( \frac{r + q}{s + q - 1} \right) \alpha^{r-s+1} + O \left( \alpha^{r-s+2} \right),
\]

where the standard notation for the binomial coefficients have been used, \( \binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \).

**Proof.** Consider the \( n=1 \) term in the sum (2.3), which is just the trace of the operator \( \hat{K} \), see Proposition 2.3,

\[
\text{Tr}(\hat{K}) = \int_{C_0} \frac{(\lambda - \alpha)^r}{(\lambda - 1)^{r+q}\lambda^s} \frac{d\lambda}{2\pi i} \int_{C_\infty} \frac{(w - 1)^{r+q}w^s}{(w - \alpha)^r(w^\lambda)^2} \frac{dw}{2\pi i}.
\]

Change variables:

\[
\lambda = \alpha \nu, \quad w = 1/\mu.
\]

Note that since \( 0 < \alpha < 1 \) contour \( C_0 \to C_0 \), while \( C_\infty \to -C_0 \), thus

\[
\text{Tr}(\hat{K}) = \alpha^{r-s+1}(-1)^q \int_{C_0} \frac{(1-\nu)^r}{(1-\nu^r\nu^s)^r} \frac{d\nu}{2\pi i} \int_{C_0} \frac{(1-\mu)^{r+q}}{(1-\alpha^r\mu^{r+q})(1-\alpha^r\mu)^2} \frac{d\mu}{2\pi i}.
\]

Therefore as \( \alpha \to 0 \) we find

\[
\text{Tr}(\hat{K}) = \alpha^{r-s+1}(-1)^q \int_{C_0} \frac{(1-\nu)^r}{\nu^s} \frac{d\nu}{2\pi i} \int_{C_0} \frac{(1-\mu)^{r+q}}{\mu^{r+q}} \frac{d\mu}{2\pi i} (1 + O(\alpha)).
\]

Now we calculate the above integrals via residues, that gives \((-1)^{s-1} \binom{r}{s-1}\) and \((-1)^{s+q-1} \binom{r+q}{s+q-1}\), respectively, and so we get the second term in the right-hand side of (4.23). Analogous consideration shows that \( \text{Tr}(\hat{K}^k) = O(\alpha^{k(r-s+1)}) \) as \( \alpha \to 0 \) and thus the further terms in (2.3) do not contribute to the leading order of the Taylor series of \( F_{r,s,q} \) at \( \alpha = 0 \).

Setting \( q = 0 \), we have the following result.

**Corollary 4.2.** As \( \alpha \to 0 \),

\[
\log(1 - F_{r,s,0}) = (r - s + 1) \log \alpha + 2 \log \left( \frac{r}{s-1} \right) + O(\alpha)
\]

and

\[
\log F_{r,s,0} = -\exp \left\{ (r - s + 1) \log \alpha + 2 \log \left( \frac{r}{s-1} \right) + O(\alpha) \right\}.
\]
Our next task is to get asymptotics of the logarithm of the binomial coefficient in (4.24) as $s \to \infty$, $r = s/v$. We recall the known asymptotics of logarithm of the Gamma-function as $z \to \infty$ (see, e.g., [56]),

$$\log \Gamma(z + a) = \left( z + a - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{n=1}^{m} \frac{(-1)^{n+1}B_{n+1}(a)}{n(n+1)z^n} + O(z^{-m-1}), \quad |\arg z| < \pi,$$

where $B_n(a)$ are the Bernoulli polynomials,

$$B_n(a) = \sum_{k=0}^{n} \binom{n}{k} B_k a^{n-k}.$$ 

Here $B_k$ are the Bernoulli numbers, $B_1 = -1/2$, $B_{2k+1} = 0$, $k = 1, 2, \ldots$, and $B_{2k}$ are given by (4.12).

Using the above formula for asymptotics of logarithm of the Gamma-function we arrive at the following result.

**Proposition 4.4.** As $s \to \infty$ and $r = s/v$,

$$2\log \left( \frac{r}{s-1} \right) = -2 \left( \log v + \frac{1-v}{v} \log(1-v) \right) s - \log s + \log \frac{v^2}{2\pi(1-v)^3} + \sum_{n=1}^{m} \frac{B_{2n}v^{2n-1} - (B_{2n} + 2n)(\frac{1}{v-1})^{2n-1} - B_{2n}}{n(2n-1)s^{2n-1}} + \sum_{k=1}^{m} \frac{v^{2k}}{k(1-v)^{2k}s^{2k}} + O(s^{-2m-1}). \quad (4.26)$$

Corollary 4.2 and Proposition 4.4 imply that the simplest possible asymptotic ansatz for the logarithm of $F_{r,s,0}$ as $s \to \infty$, $r = s/v$, reads

$$\log F_{r,s,0} = -\exp \left\{ -\chi s - \log s + \sum_{k=0}^{m} \frac{b_k}{s^k} + O(s^{-m-1}) \right\}. \quad (4.27)$$

In view of (4.14), the asymptotic ansatz for the $\sigma$-function, which corresponds to (4.27), is

$$\sigma = \left( -\frac{(1+v)^2}{4v^2} \alpha + \frac{1}{2v} \right) s^2 + e^{-s\omega} + O(e^{-2s\omega_0}), \quad (4.28)$$

where the function $\omega$ admits an asymptotic expansion in the inverse powers of $s$,

$$\omega = \omega_0 + \omega_1 s^{-1} + \omega_2 s^{-2} + \cdots. \quad (4.29)$$

At first glance, the asymptotic ansatz (4.28) is considerably different in comparison with the ansatz for the $\sigma$-function in the disordered regime, see (4.15). However, it can be viewed as the same ansatz but with the function $\varphi \equiv 0$, $\alpha \in (0, \beta)$. Indeed, as it is mentioned in Remark 4.2, an asymptotic expansion for the $\sigma$-function may contain exponentially small corrections, which are not indicated in (4.15). At the same time, in case $\varphi \equiv 0$, the corresponding function $\sigma_2$ is just given by the coefficient of the first term in (4.14). From the point of view of the solutions of (4.16), which govern $\sigma_2$, this expansion corresponds to the unique case where both general solutions (4.17) coincide, $C_1 = C_{11} = -(1+v^2)/4v^2$. It happens that in this
very case $\sigma_{-2k} \equiv 0$, $k = 0, 1, \ldots$. Comparing (1.14) and (4.28) with (1.2) we see that the function $\chi$ is equal to $\omega_0$, moreover it should be positive for $\alpha \in (0, \beta)$.

Justification of asymptotic ansatz (4.28), (4.29) can be obtained analogously as it was done for ansatz (1.15). However, here we have an independent proof for ansatz (4.27) based on the saddle-point method applied to the Fredholm determinant, presented in Appendix B. Therefore in what follows we consider an explicit construction of the coefficients in the series (4.29).

From the $\sigma$-form of P6 it follows that the $\omega$-function in all orders in $1/s$ obeys the following ODE:

$$
\alpha^2 (\alpha - 1)^2 (s(\omega')^2 - \omega'')^2 = [(\alpha - 1)s\omega' + 1] \left[ \frac{(1 + v)^2 \alpha - (1 - v)^2}{v^2} s\omega' + \frac{(1 + v)^2}{v^2} \right].
$$

(4.30)

For example, keeping the leading order in this equation, we find that the function $\chi \equiv \omega_0$ satisfies equation

$$
\alpha^2 (\alpha - 1)(\chi')^2 = \frac{(1 + v)^2 \alpha - (1 - v)^2}{v^2}.
$$

(4.31)

Since $\alpha \in (0, 1)$ and $\chi'$ is real, the right-hand side of (4.31) is negative, therefore $\alpha \in (0, \beta)$, where $\beta < 1$ is given in (4.4). Using (4.24) and (4.27), we note that $\chi$ has the following asymptotic behavior as $\alpha \to 0$,

$$
\chi = -\frac{1 - v}{v} \log \alpha + 2 \left( \log v + \frac{1 - v}{v} \log(1 - v) \right) + O(\alpha).
$$

(4.32)

Solving (4.31) for $\chi'$, we obtain

$$
\chi' = -\frac{1}{v \alpha} \sqrt{\frac{(1 - v)^2 - (1 + v)^2 \alpha}{1 - \alpha}} = -\frac{2}{(1 - \sqrt{\beta}) \alpha} \sqrt{\frac{\beta - \alpha}{1 - \alpha}},
$$

(4.33)

where sign minus is chosen to ensure the correct coefficient of the log $\alpha$ term in (4.32). Integrating (4.33) and fixing the integration constant according to the second term in (4.32), we obtain

$$
\chi = \frac{4}{1 - \sqrt{\beta}} \left\{ \sqrt{\beta} \log \left( \frac{\sqrt{\beta} + \alpha + (1 - \alpha)(\beta - \alpha)}{(1 + \sqrt{\beta}) \sqrt{\alpha}} \right) - \log \left( \frac{\sqrt{1 - \alpha} + \sqrt{\beta - \alpha}}{\sqrt{1 - \beta}} \right) \right\}.
$$

Note that this expression can also be rewritten as

$$
\chi = 2 \frac{1 + \sqrt{\beta}}{1 - \sqrt{\beta}} \log \left( \frac{\sqrt{\beta} + \alpha + (1 - \alpha)(\beta - \alpha)}{(1 + \sqrt{\beta}) \sqrt{\alpha}} \right)
$$

$$
- 2 \log \left( \frac{\sqrt{\beta} - \alpha + (1 - \alpha)(\beta - \alpha)}{(1 - \sqrt{\beta}) \sqrt{\alpha}} \right),
$$

(4.34)

or, noticing that the numerators of arguments of the logarithms are perfect squares, as

$$
\chi = 4 \frac{1 + \sqrt{\beta}}{1 - \sqrt{\beta}} \log \left( \frac{\sqrt{(1 + \sqrt{\alpha})(\sqrt{\beta} + \sqrt{\alpha})} + \sqrt{(1 - \sqrt{\alpha})(\sqrt{\beta} - \sqrt{\alpha})}}{\sqrt{2(1 + \sqrt{\beta}) \sqrt{\alpha}}} \right)
$$

$$
- 4 \log \left( \frac{\sqrt{(1 - \sqrt{\alpha})(\sqrt{\beta} + \sqrt{\alpha})} + \sqrt{(1 + \sqrt{\alpha})(\sqrt{\beta} - \sqrt{\alpha})}}{\sqrt{2(1 - \sqrt{\beta}) \sqrt{\alpha}}} \right).
$$

(4.35)
These expressions imply that $\chi|_{\alpha=\beta} = 0$, and furthermore $\chi > 0$ for $\alpha \in (0, \beta)$, since it can be written, similarly to (4.21), as the integral

$$\chi = \frac{2}{1 - \sqrt{\beta}} \int_{\alpha}^{\beta} \frac{1}{\alpha} \sqrt{\frac{\beta - \alpha}{1 - \alpha}} \, d\alpha.$$  

Using (4.34) it can be shown that $\chi = 2J(t + 1)$, where $J(t)$ is the upper tail rate function (for $\gamma = 1$) obtained by Johansson, see Eqs. (2.21) and (2.25) in [34], and $t = v^{-1} - 1$. Rewriting (4.35) in terms of $v$ and $u$, see (4.3), we arrive at (1.8).

After we have determined $\omega_0 \equiv \chi$, the first derivatives of the further coefficients in the expansion (4.29), namely $\omega_0'$'s, can be derived uniquely from (4.30). Equations (4.14), (4.27), and (4.28) implies the following equation for the coefficients $b_k$:

$$\log \alpha + \log(1 - \alpha) + \log(-\chi') + \log \left( 1 + \sum_{k=0}^{\infty} \frac{b_k'}{(-\chi')^{s+1}} \right) + \sum_{k=0}^{\infty} \frac{b_k}{s^k} = -\sum_{k=0}^{\infty} \frac{\omega_{k+1}}{s^k}.$$  

For example, the first three coefficients $b_k$ are

$$b_0 = -\omega_1 - \log(-\chi') - \log(1 - \alpha),$$

$$b_1 = -\omega_2 + \frac{b_0'}{\chi'},$$

$$b_2 = -\omega_3 + \frac{b_1'}{\chi'} + \frac{(b_0')^2}{2(-\chi')^2}.$$  

The undermined constants in $\omega_k$ are fixed with the help of the expansions (4.25) and (4.26). Using this algorithm, we obtain the following expressions:

$$b_0 = \log \left( \frac{\alpha v^2}{2\pi\sqrt{1 - \alpha}[(1 - v)^2 - \alpha(1 + v^2)]^{3/2}} \right),$$

$$b_1 = \frac{(1 + v^4)(1 - \alpha)^2 + 9(v - v^3)(1 - \alpha^2) - 2v^2(10\alpha^2 - 11\alpha + 10)}{6\sqrt{1 - \alpha}[(1 - v)^2 - \alpha(1 + v^2)]^{3/2}},$$

$$b_2 = \frac{v^2}{(1 - \alpha)[(1 - v)^2 - (1 + v^2)\alpha]^3} \left[ (\alpha^2 + 8\alpha + 1)(1 - \alpha)^2(v^4 + 1) - 4(1 - \alpha)(\alpha^3 + 1)(v^3 + v) + 6(1 - \alpha)^3v^2 + 4\alpha(\alpha^2 + \alpha + 1)v^2 \right].$$  

Clearly,

$$\log(1 - F_{r,s,0}) = -\chi s - \log s + \sum_{k=0}^{m} \frac{b_k}{s^k} + O(s^{-m-1}),$$

that finishes the proof of Theorem 1.3.

It is interesting to note that (4.30) can be solved explicitly in terms of the hypergeometric function, see Appendix C.

5. Conclusion

The goal of this paper is twofold: to elaborate an algorithm for construction of full asymptotic expansions for a physically interesting quantity for an integrable model, and to show that the theory of P6 can serve as an effective tool for this purpose.
The situation is not absolutely standard since we used an ODE with respect to a continuous variable to find asymptotic expansions with respect to discrete parameters. At first glance, Toda or discrete Painlevé equations might be more suitable for this problem. Indeed these equations can be used in obtaining the leading terms of our asymptotic expansions \( [30] \), however, a perspective of obtaining higher order terms (HOTs) of the expansions with the help of the difference or differential-difference equations seems unclear.

Our problem looks like a standard object for asymptotic methods based on the Riemann-Hilbert conjugation problem or isomonodromy deformations (see Section 3.2). These methods however deliver the leading terms of the corresponding asymptotic expansions, while the explicit formulas for HOTs require much more deliberate and technically complicated efforts, so that anyway HOTs are easier to derive via substitution of the asymptotic expansions into the corresponding integrable (differential, difference, etc) equation.

In this paper we show that in our case not only HOTs but also the leading terms can be obtained via the substitution of asymptotic expansions into \( P_6 \). Since we use the ODE that plays an auxiliary role with respect to discrete parameters we need initial conditions for determination of all terms of the asymptotic expansion rather than just for the leading terms, as it usually happens in the standard approach. To get these initial conditions one needs to know asymptotic behavior of the \( \tau \)-function at critical points of \( P_6 \). In our case it is asymptotic behavior of \( F_{r,s,0} \) as \( \alpha \to 1 \) and \( \alpha \to 0 \), Propositions 4.1 and 4.3 respectively. The results stated in these propositions are, in fact, equivalent to the connection formula for the \( P_6 \) \( \tau \)-function. We have found no corresponding result in the literature, probably because our solution is too special from the point of view of the general theory of \( P_6 \).

Asymptotics as \( \alpha \to 0 \) is obtained in Proposition 4.3 with the the help of the Fredholm kernel \( \hat{K} \), which is the principle manifest of integrability. At the same time the proof of asymptotic expansion as \( \alpha \to 1 \) in Proposition 4.1 is obtained by an ad hoc observation, which may not occur for some other analogous problem. We used this observation because it simplifies our derivation, however we could employ a more deliberate scheme, we are going to outline below, where \( \alpha \to 1 \) asymptotics is not required. In principle, the connection problem for \( \tau \)-function of \( P_6 \) is a different and external problem in comparison to the one we consider here, so that we can just refer to it. However, there is a way to avoid this problem, or more precisely, to change it to another one: the connection problem for the “corresponding” \( \tau \)-function of the second Painlevé equation.

The key point of this scheme is the representation of \( F_{r,s,q} \) as a determinant of the integrable Fredholm operator \( \hat{K} \). The next step is the derivation of \( \sigma \)-form of \( P_6 \). Our problem is to find asymptotics of \( F_{r,s,q} \) for large variables \( r, s, \) and \( q \) in the thermodynamic limit \( (s/r = v = \text{const}) \) for all values of the parameter \( \alpha \), \( 0 < \alpha < 1 \). As shown in the paper, it is easy to find asymptotics in the ordered regime, i.e., in the interval \( 0 < v < u = (1 - \sqrt{\alpha})/(1 + \sqrt{\alpha}) \). This asymptotics requires \( \alpha \to 0 \) asymptotics and the \( \sigma \)-form of \( P_6 \) for \( \text{Det} \hat{K} \). In Appendix B we show that in this case we actually can find asymptotics directly, without any reference to \( P_6 \), by the help of the saddle-point method for the \( \text{Tr} \hat{K} \).

Then we observe that in the neighborhood \( u \sim v \) our asymptotic expansion is destroyed (see Theorem 1.3) and the so-called “boundary” variable \( z \) can be introduced, which, as is easy to guess from the structure of HOTs, reads \( z = \)
Substituting it into the \( \sigma \)-form of \( P_6 \) instead of \( \alpha \) one obviously finds that \( \sigma(\alpha) \rightarrow \sigma_2(z) \), where \( \sigma_2(z) = \int y^2(\tilde{z}) \, d\tilde{z} \) is a solution of the \( \sigma \)-form of the second Painlevé equation (P2). It is well-known that \( \sigma_2(z) = -\int y^2(\tilde{z}) \, d\tilde{z} \), where \( y(\tilde{z}) \) is the corresponding solution of the canonical form of P2,

\[
y'' = 2y^3 + \tilde{z}y + C. \tag{5.1}
\]

Let us explain why the appearance of P2 is an obvious thing here. As follows from the calculations presented in Appendix B there is a coalescence of two simple saddle-points in \( \text{Tr } K \) at \( u = v \). This is equivalent to coalescence of two simple saddle-points in the steepest-descent method for the corresponding RH problem or coalescence of four turning points in the isomonodromy deformation method. This process is described by the solution of the model RH-problem corresponding to P2 (see details, say, in [57]). Surely, one can deduce it directly from the corresponding differential equations with the help of the above-mentioned substitution and obtain one of the limits \( P_6 \rightarrow P_2 \) (see, say, discussion in introduction to [58]). How to choose the proper solution of equation (5.1) in the framework of this approach? The solution of P2 should match with the asymptotics in the ordered case: this means that in terms of variable \( z \) it should be exponentially decaying; thus the constant \( C \) in (5.1) vanishes. It is well-known that there exists only one-parameter family of solutions of P2 regular on the positive real semi-axis and exponentially decaying there. The value of this parameter should be determined via the matching procedure in the domain \( z \sim +s^\epsilon, \epsilon > 0 \), with the leading term of asymptotics in the ordered regime, \( v < u \).

On the negative semi-axis solutions of this family can be oscillating or possess infinite number of poles, depending on the value of the parameter of the family. There exists also the unique solution that separate these two types of asymptotic behavior, it can be treated as the solution with the only pole at the point at infinity, it is known as the Hastings-McLeod solution [59]. This solution grows as \( \sqrt{-z}/2 \) for \( z \rightarrow -\infty \).

Since we already know asymptotics of \( F_{r,s,q} \) in the disordered regime, which is regular and non-oscillating, it is obvious that the only suitable solution of P2 is the Hastings-McLeod solution. However from the formal point of view this fact should be established by applying the matching procedure described above.

Asymptotics of the Hastings-McLeod solution for \( z \sim -s^\epsilon, \epsilon > 0 \), serves as the “initial data” for asymptotics of \( F_{r,s,q} \) in the region \( v > u \). Obviously, to use the “P2-asymptotics” as the “initial data” we have to develop this asymptotic expansion up to all decaying orders of \( z \). From our point of view, this expansion is an important ingredient of the asymptotic scheme based on matching of asymptotic expansions. In certain cases, the scheme which can serve as an effective alternative to other asymptotic methods.

At this stage we just explain that the leading terms of the “P2-asymptotics” and the expansion constructed in Theorem 1.2 do match. For the leading term we have

\[
\log F_{r,s,q} \sim -\int z (z - \tilde{z})y^2(\tilde{z}) \, d\tilde{z} \sim -(v - u)^3s^2/12, \tag{5.2}
\]

since \( y \sim \sqrt{-z}/2 \). This shows that the “Painlevé asymptotics” in the transition domain matches at least with the leading order of asymptotics obtained in Theorem 1.2. Above we have given an explanation of appearance of P2 purely in
terms of the theory of differential equations and nonlinear special functions. For
the corresponding matrix model the result is well-known via the appearance of the
“Airy-kernel” of the associated integrable integral Fredholm operator $\hat{K}$.

Simultaneously, the last asymptotic relation in (5.2) explains the physically
interesting phenomenon of the third-order phase transition mentioned in Intro-
duction: the leading term of the asymptotic expansion of $\log F_{r,s,q}$ (denoted as $\varphi$
in Theorem 1.2) vanishes at $v = u$, together with the first two derivatives (with
respect to $v$ or $u$), whilst the third derivative does not. In other words, inter-
preting $\varphi$ as the change of the free energy per volume in the thermodynamic lim-
it, $-v^2 \varphi/(1+v)^2 \equiv \lim_{s,r \to \infty} \log F_{r,s,q}/(r+s)^2$, one concludes that (5.2) implies $\varphi \equiv 0$
for $v < u$, thus reproducing the result of [30] about the leading term of asymptotics
described by the function $\varphi$.

Appendix A. The Wasow theorem and the $\sigma$-form of P6

The $\sigma$-forms of Painlevé equations does not fit the setup of the Wasow theorem
[Th. 36.1 in Chap. IX of [55]], because they are quadratic with respect to the
second derivative of the $\sigma$-function. Nevertheless, in many cases this theorem can
be applied to these equations due to the presence of the Hamiltonian structure.

Here we give some more details to the general comments given at the end
of Sect. 4.2 that support item (3) of Remark 4.3. Namely, we show here that
the Wasow theorem applies to the representation of P6 as a Hamiltonian system,
and due to the one-to-one correspondence between the canonical variables and $\sigma$-
function (as proven by Okamoto [46]), it implies the validity of the similar result
for the $\sigma$-form of P6. We first consider the P6 as a Hamiltonian system in a general
setup, and next proceed to our case.

Since here we deal with P6 as a Hamiltonian system, we will rely upon results
obtained by Okamoto [46]. To keep contact with our discussion in the main text,
we make the following identification between parameters $\nu_1, \ldots, \nu_4$ of Jimbo-Miwa
and those $b_1, \ldots, b_4$ of Okamoto:

$$
\begin{align*}
\nu_1 &= \frac{1}{2}(\theta_\alpha + \theta_\infty) = b_3, \\
\nu_2 &= \frac{1}{2}(\theta_\alpha - \theta_\infty) = b_4, \\
\nu_3 &= -\frac{1}{2}(\theta_0 + \theta_1) = -b_1, \\
\nu_4 &= -\frac{1}{2}(\theta_0 - \theta_1) = -b_2.
\end{align*}
$$

(A.1)

Note that the parameters $b_1, b_2, b_3, b_4$ in (A.1) and the $\sigma$-function by Jimbo-Miwa,
which we use all over the paper, are in fact those denoted in [46] as $b_1^+, b_2^+, b_3^+, b_4^+$ and $h^+$, respectively (see Eqs. (C.55)-(C.61) of [3], and Eq. (1.13) and Props. 1.6 and
1.8 of [46]). This means that the canonical variables $q$ and $p$ appearing below are
related to those of P6 given in Sect. 3.1 [4] by a birational canonical transformation
(see Eq. (1.12) of [46]).

---

2 The parameters of P6 used by Jimbo-Miwa in [4] and by Okamoto in [46] are related as
$\theta_0 = \kappa_0$, $\theta_1 = \kappa_1$, $\theta_\alpha = \theta$, and $\theta_\infty = \kappa_\infty + 1$. 38
Given the set of parameters $b_1, \ldots, b_4$ and the $\sigma$-function, the canonical coordinate $q$ and momentum $p$ are given by the expressions
\[
q = \frac{1}{2A}[(b_3 + b_4)B + (\sigma' - b_3 b_4)C],
\]
\[
p = \frac{1}{2Aq(q - 1)}[-(\sigma' - w_2)B + (w_1 \sigma' - w_3)C],
\]
where
\[
A = (\sigma' + b_3^2)(\sigma' + b_4^2), \quad B = \alpha(\alpha - 1)\sigma'' + s_1 \sigma' - s_3, \quad C = 2(\alpha \sigma' - \sigma) - s_2. \tag{A.3}
\]

Here, $s_k$ and $w_k$, $k = 1, 2, 3$, are the $k$-th fundamental symmetric polynomials of $\{b_1, b_2, b_3, b_4\}$ and $\{b_1, b_3, b_4\}$, respectively. The corresponding system of Hamiltonian equations reads
\[
q' = \frac{q(q - 1)(q - \alpha)}{\alpha(\alpha - 1)} \left(2p - \frac{\theta_0}{q} - \frac{\theta_1}{q - 1} - \frac{\theta_2}{q - \alpha}\right),
\]
\[
p' = \frac{1}{\alpha(\alpha - 1)} \left\{ -3q^2 + 2(\alpha + 1)q - \alpha \right\} p^2
+ [\theta_0(2q - 1 - \alpha) + \theta_1(2q - \alpha) + \theta_2(2q - 1)]p - \frac{(\theta_0 + \theta_1 + \theta_2)^2 - \theta_2^2}{4}. \tag{A.4}
\]

Note, that eliminating $p$, one finds that $q$ solves the canonical form of $P6$ with $a = \frac{1}{4}\theta_2^2$, $b = -\frac{1}{4}\theta_0^2$, $c = \frac{1}{4}\theta_1^2$, and $d = -\frac{1}{4}\theta_0(\theta_1 + 2)$, in agreement with the birational canonical transformation (which can be identified as the change $\theta_\alpha \mapsto \theta_\alpha + 1$, $\theta_\infty \mapsto \theta_\infty + 1$ in the set of monodromy parameters of $P6$ given in Sect. 3.1).

Now we are ready to consider our case. In Sect. 4.2 we impose the following relations on the parameters:
\[
\nu_1 = -\frac{r + s}{2}, \quad \nu_2 = -\frac{r - s}{2}.
\]

Hence, $b_3 = -b_1 = -\nu_1$ and $b_4 = b_2 = \nu_2$, and so we have
\[
s_1 = 2\nu_2, \quad s_2 = -\nu_1^2 + \nu_2^2, \quad s_3 = -2\nu_1^2 \nu_2, \quad w_1 = \nu_2, \quad w_2 = -\nu_1^2, \quad w_3 = -\nu_1^2 \nu_2.
\]

We find it convenient to set
\[
\nu_1 = sv_1, \quad \nu_2 = sv_2,
\]
where
\[
v_1 \equiv -\frac{1 + v}{2v}, \quad v_2 \equiv -\frac{1 - v}{2v}, \quad v = \frac{s}{r}.
\]

We recall that $s$ is a large parameter, while $v_1$ and $v_2$ are finite.

Let us map the system (A.3) into the form considered by Wasow. Consider the formal asymptotic expansion for the $\sigma$-function (4.11), which is uniquely defined by its first term,
\[
\sigma = s^2 \left(\frac{v_1^2 + v_2^2}{2} - 2v_1 v_2 \sqrt{\alpha}\right) + O(1), \quad s \to \infty.
\]
Using it and equations (A.2) and (A.3), one finds the corresponding formal expansions for \( q \) and \( p \) in \( 1/s \). The first terms reads
\[
q = -\sqrt{\alpha} + s^{-1} \frac{(\alpha - 1)\sqrt{\alpha}(v_1 + v_2)}{4v_1\sqrt{\alpha - v_2}(v_2\sqrt{\alpha - v_1})} + O(s^{-2}),
\]
\[
p = s\frac{v_1\sqrt{\alpha - v_2}}{\sqrt{\alpha}(\sqrt{\alpha + 1})} + \frac{(\sqrt{\alpha - 1})[(v_1 + 2v_2)\sqrt{\alpha + v_2}]}{4\sqrt{\alpha}(\sqrt{\alpha + 1})(v_2\sqrt{\alpha - v_1})} + O(s^{-1}).
\] (A.5)

Introduce a 2-component vector function \( \vec{g} = \vec{g}(\alpha) \),
\[
\vec{g} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad q = y_1, \quad p = sy_2,
\]
where this transformation is motivated by the formal expansion (A.5). Taking into account that in our case \( \theta_0 = -\theta_\infty = -\nu_1 + \nu_2 \) and \( \theta_1 = -\theta_\nu = -\nu_1 - \nu_2 \), we can write the Hamiltonian system (A.4) in the following form:
\[
\epsilon \frac{d}{d\alpha} \vec{g} = \vec{f}(\alpha; \vec{g}), \quad \vec{f}(\alpha; \vec{g}) = \begin{pmatrix} f_1(\alpha; \vec{g}) \\ f_2(\alpha; \vec{g}) \end{pmatrix},
\] (A.6)
where \( \epsilon \equiv 1/s \) and
\[
f_1(\alpha; \vec{g}) = \frac{y_1(y_1 - 1)(y_1 - \alpha)}{\alpha(\alpha - 1)} \left( 2y_2 + \frac{v_1 - v_2}{y_1} - \frac{(v_1 + v_2)(\alpha - 1)}{(y_1 - 1)(y_1 - \alpha)} \right),
\]
\[
f_2(\alpha; \vec{g}) = \frac{y_2}{\alpha(\alpha - 1)} \left\{ \left[ -3y_1^2 + 2(\alpha + 1)y_1 - \alpha \right] y_2 - 2(v_1 - v_2)y_1 + 2v_1\alpha - 2v_2 \right\}.
\]

The Wasow theorem deals with the system of the form (A.6) and contains two additional conditions, called in [55] assumption (A) and assumption (B). We are going to verify that these conditions are valid in our case.

To verify assumption (A), see p. 218 in [55], we first assume that \( \alpha \in \overline{D} \) where \( \overline{D} \) is any closed disk in \( \mathbb{C} \setminus \{0, \beta, 1, \infty\} \). The case of our particular interest is where \( \alpha \) belongs to any closed segment in the interval \((\beta, 1)\). Assumption (A) requires also existence of a holomorphic vector function \( \vec{\phi} \) on \( \overline{D} \) such that
\[
\vec{f}(\alpha; \vec{\phi}) = 0.
\]
In our case
\[
\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \phi_1 = -\sqrt{\alpha}, \quad \phi_2 = \frac{v_1\sqrt{\alpha - v_2}}{\sqrt{\alpha}(\sqrt{\alpha + 1})}.
\] (A.7)
Note that \( \vec{\phi} \) is holomorphic for \( \alpha \in \overline{D} \). Thus, assumption (A) is verified.

To verify assumption (B), see p. 219 in [55], we define a new vector function
\[
\vec{g}_* = \vec{g} - \vec{\phi},
\]
and rewrite (A.6) as follows:
\[
\epsilon \frac{d}{d\alpha} \vec{g}_* = -\epsilon \frac{d}{d\alpha} \vec{\phi} + A\vec{g}_* + \vec{g}(\alpha; \vec{g}_*). \tag{A.8}
\]
Equation (A.8) exhibits the constant (with respect to \( \vec{g}_* \)), linear and nonlinear terms of the system of equations (A.6) about the zero-order (in \( \epsilon \)) solution \( \vec{\phi} \); the

\[\text{Note, that here we meet a special case of the Wasow theorem, where } \vec{f}(\alpha; \vec{g}) \text{ is independent of } \epsilon.\]
vector function \( \vec{g}(\alpha; \vec{y}_\ast) \) satisfies \( \partial_{\vec{y}_\ast k} \vec{g}(\alpha; \vec{y}_\ast)|_{\vec{y}_\ast = 0} = 0, \ k = 1, 2 \). Equation (A.8) is a special case of Eqn. (36.9) in [55]. The matrix \( \mathbf{A} \) reads:

\[
\mathbf{A} = \left( \begin{array}{cc}
-2\frac{v_1 \sqrt{\alpha} - v_2}{\alpha(1 - \sqrt{\alpha})} & 2\frac{1 + \sqrt{\alpha}}{1 - \sqrt{\alpha}} \\
-2\left[\frac{v_1 \alpha(2 + \sqrt{\alpha}) - v_2(1 + 2\sqrt{\alpha})}{\alpha^2(1 - \alpha)(1 + \sqrt{\alpha})^2}\right] & 2\frac{v_1 \sqrt{\alpha} - v_2}{\alpha(1 - \sqrt{\alpha})}
\end{array} \right).
\]

The determinant evaluates as

\[
\det \mathbf{A} = 4\left(\frac{v_1 \sqrt{\alpha} - v_2}{\alpha^{3/2}(1 - \alpha)^2}\right).
\]

Recall that the disordered regime which we consider in Sect. 4.2 corresponds to \( \alpha \in (\beta, 1] \), where \( \beta = \left(\frac{1}{1 + v_1^2}\right)^2 = v_2^2/v_1^2 \). Thus, \( \det \mathbf{A} \neq 0 \) for all values of \( \alpha \) of this regime, and assumption (B) of the Wasow theorem is fulfilled.

Hence, the Wasow theorem applies to the system (A.6), since both assumptions (A) and (B) of the Wasow theorem are satisfied in the disordered regime. Therefore, from the Wasow theorem it follows that there exists a genuine solution \( q, p \) of the system (A.4) with the asymptotic expansions (A.5). These expansions are uniform on any closed segment of \( (\beta, 1) \) and all power-terms in \( \epsilon \) can be constructed uniquely via the explicitly given zero-order solution (A.7). Due to the one-to-one correspondence between the canonical variables \( q, p \) and the \( \sigma \)-function, see Eqn. (1.6) and Prop. 1.2 in [46], there exist genuine solution \( \sigma \)-function with the formal expansion (4.15).

Note that the switch to the Hamiltonian system \( q \) and \( p \) has mostly theoretical rather than practical sense since from the construction above it is not obvious that the resulting expansion for the \( \sigma \)-function contains only even powers of \( s \). However, this fact is obvious due to the \( \sigma \)-form of P6 (1.3).

Appendix B. Saddle-point approach to asymptotic expansion of the EFP in the ordered regime

As in the main text we assume here that \( q = 0 \), which is not crucial for the approach discussed below. As mentioned in the main part of the paper the complete asymptotic expansion of \( F_{r,s,q} \) can be presented as transseries with a finite number (actually, of \( s \)) series. The transseries expansion of \( \log F_{r,s,q} \) consists of infinite number of series; the \( n \)-th series in this expansion is generated by the term \( \text{Tr}(\hat{K}^n) \).

Here we consider only construction of the first series of the complete transseries expansion, i.e., the series generated by \( \text{Tr}(\hat{K}) \); the other series can be studied analogously, however the corresponding calculations are more involved as they are related with the saddle-point method for integrals with \( n + 1 \) variables.

The main object of our studies here is the following integral:

\[
\text{Tr}(\hat{K}) = -\frac{\alpha^{r-s+1}}{(2\pi)^2} \int_{C_0} \int_{C_0} \frac{1}{(1 - \nu \mu)^2 (1 - \alpha \nu \mu)^r (1 - \alpha \mu)^r} d\nu d\mu \left( (1 - \nu)^r (1 - \mu)^r \nu^s \mu^s \right). \quad (B.1)
\]

We rewrite it in the following way

\[
\text{Tr}(\hat{K}) = -\frac{\alpha^{r-s+1}}{(2\pi)^2} \int_{C_0} \int_{C_0} e^{sS(\nu, \mu)} f(\nu, \mu) d\nu d\mu,
\]
where
\[ S(\nu, \mu) = S(\nu) + S(\mu), \quad S(\nu) = \frac{1}{\nu} \log \frac{1 - \nu}{1 - \alpha \nu} - \log(-\nu), \] (B.2)
and
\[ f(\nu, \mu) \equiv \frac{1}{(1 - \alpha \nu \mu)^2}, \]
Note that in the definition of \( S(\nu, \mu) \) we used the fact that \( s \) is an integer and we recall that \( v = s/r \).
Since the variables \( \nu \) and \( \mu \) are separated in the function \( S(\nu, \mu) \), one can successively apply the usual saddle-point method for a single variable with respect to each of these two variables. However, it is easier to employ the saddle-point method for multiple integrals. We follow below the version of this method developed by Fedoryuk [61].
Saddle points are defined by the system of equations \( \frac{\partial}{\partial \nu} S = \frac{\partial}{\partial \mu} S = 0 \). There are four saddle points with the coordinates \((\nu - \mu, \mu - \mu), (\nu - \mu, \mu + \mu), (\nu + \mu, \mu - \mu), \) and \((\nu + \mu, \mu + \mu), \)
where
\[ \nu_{\pm} = \mu_{\pm} = -1 - \frac{\alpha}{4\nu v} \left( \sqrt{1 - v \frac{1 + \sqrt{\alpha}}{1 - \sqrt{\alpha}}} \pm \sqrt{1 - v \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}} \right)^2. \]
In the following we assume that
\[ 0 < v < u \equiv \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}} \] (B.3)
This inequality, together with the condition \( 0 < \alpha < 1 \), implies that \( \nu_+ < -1/\alpha^{1/2} < \nu_- < 0 \).
To prove it one has to notice that \( \nu_+ \nu_- = 1/\alpha \) where both factors in the left-hand side are negative. Using this relation we find that
\[ S(\nu_+) + S(\nu_-) = -\frac{1 - v}{\nu} \log \alpha > 0. \] (B.4)
If we denote \( y = -\nu \) and consider \( \frac{d}{dy} S(-y) \) for \( y > 0 \), then it is easy to notice that \( \frac{d}{dy} S(-y) < 0 \), for \( 0 < y < y_- \) and \( y > y_+ \), and \( \frac{d}{dy} S(-y) > 0 \), for \( y_- < y < y_+ \), where \( y_{\pm} = -\nu_{\pm} \). Therefore,
\[ S(\nu_+, \mu_+) > S(\nu_+, \mu_-) = S(\nu_-, \mu_+) > S(\nu_-, \mu_-). \]
In other words, the saddle point surface should not contain the points \((\nu_+, \mu_+), (\nu_-, \mu_+), \) and \((\nu_+, \mu_-). \)
Thus, asymptotic expansion consists of the contribution related with the saddle point \((\nu_-, \mu_-) \). We use the general formula for this contribution obtained by Fedoryuk [61], which in our case can be written as follows,
\[ \text{Tr}(K) = \frac{\alpha^{r+s+1}}{2\pi} \frac{e^{2sS(\nu_-)}}{s S_{\nu \nu}(\nu_-)} \sum_{k=0}^{\infty} R_k(s; \alpha, v), \] (B.5)
where
\[ R_k(s; \alpha, v) = \frac{1}{k!} \left( \frac{\Delta}{2s S_{\nu \nu}(\nu_-)} \right)^k \left( f(\nu, \mu)e^{sS(\nu_+ \mu_+ \mu_-)} \right) \bigg|_{(\nu, \mu) = (\nu_- \mu_-)}, \]
and \( \Delta \) is the Laplacian,
\[
\Delta = \frac{\partial^2}{\partial \nu^2} + \frac{\partial^2}{\partial \mu^2}.
\]
The function \( S_1(\nu, \mu; \nu_-, \mu_-) \) is
\[
S_1(\nu, \mu; \nu_-, \mu_-) = S(\nu, \mu) - S(\nu_-, \mu_-) - \frac{1}{2} S''_{\nu\nu}(\nu_-) \left( (\nu - \nu_-)^2 + (\mu - \mu_-)^2 \right),
\]
where
\[
S''_{\nu\nu}(\nu_-) = \frac{(1 - v)\nu_- + 1 + v}{\nu_-^2 (1 - \nu_-)}.
\]

It is important to mention that (B.5) produces exponential decay for large \( s \). To prove this, we rewrite identity (B.4) as
\[
S(\nu_-) + S(\nu_+) + \frac{1 - v}{v} \log \alpha = 0
\]
and employ the inequality \( S(\nu_-) < S(\nu_+) \), that yields
\[
2S(\nu_-) + \frac{1 - v}{v} \log \alpha < 0.
\]

It is interesting to note that \( S(\nu_-) > 0 \). To show this let us consider \( S(-y) \) for positive \( y \). Since for \( 0 < y < 1 \) both terms (see the second equation in (B.2)) are positive, \( S(-y) > 0 \) for these values of \( y \). At the same time
\[
S(-y) = -\log y - \frac{1}{v} \log \alpha + O\left(\frac{1}{y}\right) < 0, \quad y \to \infty,
\]
so that all positive zeros of \( S(-y) \) should be greater than 1. On the other hand, it is easy to see that \( S(\frac{1}{\sqrt{\alpha}}) > 0 \), therefore one zero is greater than \( y_+ \) and the others (if any) should be located in the interval \((1, \frac{1}{\sqrt{\alpha}}]\). To prove that actually \( S(-y) > 0 \) for \( y \in (1, \frac{1}{\sqrt{\alpha}}] \) consider the following change of variables, \( y = \frac{1}{\alpha^\epsilon} \), where \( 0 < \epsilon \leq 1/2 \). In this notation \( S(-y) \) reads
\[
S(-y) = \frac{1}{v} \log \frac{1 + \alpha^\epsilon}{1 + \alpha^{1-\epsilon} v} + \frac{1 - v}{v} \log \frac{1}{\alpha^\epsilon} > 0.
\]

Now we study expansion (B.5) in more detail. Let us note that (B.5) does not represent an asymptotic series because of differentiation of the function \( \exp(sS_1) \) with respect to the large parameter \( s \). To get the asymptotic series one has to rearrange summation in a proper way, such that the series would represent an expansion in the decaying powers of \( s \). Let us write,
\[
R_k(s; \alpha, v) = \sum_{l=0}^{\left[ \frac{k}{3} \right]} \frac{R_{k,l}(\alpha, v)}{s^{k-l}},
\]
where \( \left[ \frac{k}{3} \right] \) is the entire part of \( \frac{k}{3} \). Note that
\[
R_0(s; \alpha, v) = R_{0,0}(\alpha, v) = f(\nu_-, \mu_-).
\]

Thus the asymptotic expansion of \( \text{Tr}(\hat{K}) \) reads
\[
\text{Tr}(\hat{K}) = \frac{\alpha f(\nu_-, \mu_-)}{2\pi S_{\nu\nu}(\nu_-)} \exp \left\{ s \left( 2S(\nu_-) + \frac{1 - v}{v} \log \alpha \right) - \log s \right\} \sum_{p=0}^{\infty} \frac{b_p}{s^p},
\]

43
where
\[
\hat{b}_p = \frac{1}{f(v_-, \mu_-)} \sum_{i=0}^{2p} R_{p+i} f(v_-, \mu_-).
\]

In particular,
\[
\begin{align*}
\hat{b}_0 &= 1, \\
\hat{b}_1 &= \frac{R_{1,0}(\alpha, v) + R_{2,1}(\alpha, v) + R_{3,2}(\alpha, v)}{f(v_-, \mu_-)}, \\
\hat{b}_2 &= \frac{R_{2,0}(\alpha, v) + R_{3,1}(\alpha, v) + R_{4,2}(\alpha, v) + R_{5,3}(\alpha, v) + R_{6,4}(\alpha, v)}{f(v_-, \mu_-)}.
\end{align*}
\]

To evaluate \( \hat{b}_1 \), we note that
\[
\begin{align*}
R_{1,0}(\alpha, v) &= \frac{\Delta f(v_-, \mu_-)}{2S'(v_-)}, \\
R_{2,1}(\alpha, v) &= -\frac{1}{4S'(v_-)^2} \left( S^{(4)}(v_-) + 2S^{(3)}(v_-)D \right) f(v_-, \mu_-), \\
R_{3,2}(\alpha, v) &= \frac{5}{12} \frac{S^{(6)}(v_-) f(v_-, \mu_-)}{S'(v_-)^3},
\end{align*}
\]

where
\[
D = \frac{\partial}{\partial v} + \frac{\partial}{\partial \mu},
\]
and we use the convention that \( Df(v_-, \mu_-) = Df(v, \mu)\big|_{v=\nu_- \mu=\mu_-} \). Using the above formulas, we obtain
\[
\hat{b}_1 = -\frac{(1 - \alpha)^2(v^4 + 1) + 9(1 - \alpha^2)(v^3 + v) - 2(10\alpha^2 - 11\alpha + 10)v^2}{6\sqrt{1 - \alpha}(1 - v)^2 - (1 + v)^2\alpha^{3/2}}.
\]

Similarly, to compute \( \hat{b}_2 \), we note that
\[
\begin{align*}
R_{2,0}(\alpha, v) &= \frac{\Delta^2 f(v_-, \mu_-)}{8S''(v_-)^2}, \\
R_{3,1}(\alpha, v) &= -\frac{1}{24S''(v_-)^3} \left( S^{(6)}(v_-) + 3S^{(5)}(v_-)D + 9S^{(4)}(v_-)\Delta \\
&\quad + 2S^{(3)}(v_-)(6\Delta D - D^2) \right) f(v_-, \mu_-), \\
R_{4,2}(\alpha, v) &= \frac{1}{96S''(v_-)^4} \left( 19S^{(4)}(v_-)^2 + 28S^{(3)}(v_-)S^{(5)}(v_-) \\
&\quad + 76S^{(3)}(v_-)S^{(4)}(v_-)D + 4S^{(3)}(v_-)^2(17\Delta + 3D^2) \right) f(v_-, \mu_-), \\
R_{5,3}(\alpha, v) &= -\frac{5}{48} \frac{S^{(6)}(v_-) f(v_-, \mu_-)}{S''(v_-)^3}, \\
R_{6,4}(\alpha, v) &= \frac{205 S^{(6)}(v_-) f(v_-, \mu_-)}{288 S''(v_-)^6}.
\end{align*}
\]
Using these formulas we obtain

\[
\hat{b}_2 = \frac{1}{72(1 - \alpha)(1 - \alpha^2 - (1 + \alpha^2)\alpha^3)[(1 - \alpha)^4(v^8 + 1) + 18(1 + \alpha)(1 - \alpha)^3(v^7 + v) + (113\alpha^2 + 782\alpha + 113)(1 - \alpha)^2(v^6 + v^2) - 18(1 - \alpha^2)(35\alpha^2 - 36\alpha + 35)(v^5 + v^3) + 12(83\alpha^4 + 1) - 194(\alpha^3 + \alpha) + 321\alpha^2)v^4].
\]

This result can be also written as follows:

\[
\hat{b}_1 = b_1, \quad \hat{b}_2 = \frac{\hat{b}_1^2}{2} + b_2.
\]

Here \(b_1\) and \(b_2\) are given by (4.36), where they are rewritten in terms of \(u\) (see Equation (B.3)).

**Appendix C. Explicit form for the exponentially small correction in the ordered regime**

As explained in Appendix B the complete asymptotic expansion in the ordered regime can be written in terms of transseries. The first series is defined by \(\text{Tr} \hat{K}\) and given by the double integral (B.1). At the same time our construction of asymptotics presented in Section 4.3 makes it possible to obtain another representation for this series. We recall that in terms of the \(\sigma\)-function the corresponding series is written in (4.28) as \(e^{-s\omega}\), where the function \(\omega\) is a solution of ODE (4.30). The general solution of (4.30) can be found explicitly, namely, it has the following form

\[
s\omega' = \frac{z^2 - \tilde{a}}{z^2 - \tilde{b} - (z^2 - \tilde{a})\alpha}, \quad \tilde{a} = \left(\frac{1 + v}{v}\right)^2, \quad \tilde{b} = \left(\frac{1 - v}{v}\right)^2,
\]

where the function \(z\) reads

\[
z = \pm \frac{2\alpha}{s} \frac{d}{d\alpha} \log \psi,
\]

with

\[
\psi = C_1 \alpha(s(1 - v)/2 + v) \left( s/v, -s / 1 + s(1 - v)/v \right)_{\alpha} + C_2 \alpha^{-s(1 - v)/2v} \left( s, -s/v / 1 - s(1 - v)/v \right)_{\alpha}.
\]

Here \(C_1\) and \(C_2\) are the constants of integration, which, in fact, are functions of \(s\) and \(v\).

Now we have to choose the solution which corresponds to \(F_{r,s,0}\). As follows from (4.14) and (4.28),

\[
\log F_{r,s,0} = -\int_0^\alpha \frac{e^{-s\omega} d\alpha}{\alpha(1 - \alpha)} + O(e^{-2s\omega}).
\]

Using asymptotics as \(\alpha \to 0\) of the Gauss hypergeometric function and comparing it with the expansion (4.28) we find that \(C_2 = 0\) and can fix the constant of integration while finding \(\omega\) from the first equation in (C.1). Before presenting our final result
we give an intermediate formula for $s\omega'$ which can be obtained by manipulation with (C.1) and (C.2) at $C_2 = 0$,

$$-s\omega' = \frac{d}{d\alpha} \log \left[ \alpha s^{1-2s} \left( z^2 - b - (z^2 - \tilde{a})\alpha \right) \right. \left. \frac{1}{1 + s(1-v)/v} \right] _{\alpha} 2F_1 \left( \frac{s/v, -s}{1 + s(1-v)/v} \right).$$

After integration and substitution of $s\omega'$ into (C.3) with yet indeterminate constant of integration, application of the formulas for differentiation of the hypergeometric function [56], determination of the integration constant as explained above, and some further simplifications, we arrive at the following representation of $\log F_{r,s,0}$ in the ordered regime,

$$\log F_{r,s,0} = -s^2 \left( \frac{r}{s} \right)^2 \int_0^\alpha \left\{ \frac{1}{1-\alpha} \left[ 2F_1 \left( \frac{s/v, -s}{1 + s(1-v)/v} \right) \right]^2 \right.$$ \left. - \frac{s(1-v)}{2s(1-v) + v} \frac{1}{2} \right\} \right. \left. \frac{d\alpha}{\left( \frac{1}{1-\alpha} \right) \frac{1}{2} \left[ 2F_1 \left( \frac{s/v, -s}{1 + s(1-v)/v} \right) \right]^2} \right.$$ \left. + O \left( e^{-2s\omega} \right). \quad (C.4) \right.$$

To see that this integral, in fact, is exponentially decaying as $s \to \infty$, one has to apply the saddle-point method to the corresponding integral representation of the hypergeometric function. Contrary to considerations of Appendix B, the asymptotic analysis of the representation (C.4) requires just the saddle-point method applied to a single integral. Note that the correction term in (C.4) has the order of the square of the first term.

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