Statistical modelling of the cosmological dispersion measure

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ABSTRACT
We have investigated the basic statistics of the cosmological dispersion measure (DM)—such as its mean, variance, probability distribution, angular power spectrum and correlation function—using the state-of-the-art hydrodynamic simulations, IllustrisTNG300, for the fast radio burst (FRB) cosmology. To model the DM statistics, we first measured the free-electron abundance and the power spectrum of its spatial fluctuations. The free-electron power spectrum turns out to be consistent with the dark matter power spectrum at large scales, but it is strongly damped at small scales (≤ 1 Mpc) owing to the stellar and active galactic nucleus feedback. The free-electron power spectrum is well modelled using a scale-dependent bias factor (the ratio of its fluctuation amplitude to that of the dark matter). We provide analytical fitting functions for the free-electron abundance and its bias factor. We next constructed mock sky maps of the DM by performing standard ray-tracing simulations with the TNG300 data. The DM statistics are calculated analytically from the fitting functions of the free-electron distribution, which agree well with the simulation results measured from the mock maps. We have also obtained the probability distribution of source redshift for a given DM, which helps in identifying the host galaxies of FRBs from the measured DMs. The angular two-point correlation function of the DM is described by a simple power law, \( \xi(\theta) \approx 2400(\theta/\mathrm{deg})^{-6} \, \mathrm{pc}^2 \, \mathrm{cm}^{-6} \), which we anticipate will be confirmed by future observations when thousands of FRBs are available.

Key words: cosmology: large-scale structure of Universe – galaxies: intergalactic medium – methods: numerical – radio continuum: transients

1 INTRODUCTION
A fast radio burst (FRB) is a radio pulse (~ ms wide) coming from a cosmological distance (see reviews by Cordes & Chatterjee (2019) and Petroff et al. (2019)). After the first detection (Lorimer et al. 2007), more than hundreds of FRBs have been reported to date (Petroff et al. 2016). Ongoing and future surveys such as ASKAP2, CHIME3, UTMOST4, FAST5, STARE2 (Bochenek et al. 2020) and SKA6 will detect thousands of events per year (e.g., Connor et al. 2016; Hashimoto et al. 2020). Many FRB-progenitor models have been proposed, but the origins of FRBs are still obscure (e.g., Popov & Postnov 2010; Totani 2013; Kashiyama et al. 2013; Cordes & Wasserman 2016; Murase et al. 2016; Metzger et al. 2017; Kumar et al. 2017; Levin et al. 2020; Lyubarsky 2020; Ioka & Zhang 2020; Ioka 2020). More observations are needed to differentiate between them. From the frequency dependence of the arrival time from a FRB, the projected free-electron density along the line of light (i.e., the dispersion measure (DM)) can be measured. Similarly, from the frequency dependence of the polarisation angle, the line-of-sight component of the magnetic field (i.e., the rotation measure (RM)) can also be measured. Because FRBs are extragalactic sources, these DMs and RMs directly map the cosmological free-electron distribution (Ioka 2003; Inoue 2004) and the cosmic magnetic fields (e.g., Akahori et al. 2016; Michilli et al. 2018).

The primordial abundance of baryons is currently measured to sub-percent-level accuracy by the cosmic microwave background (CMB) and Big Bang nucleosynthesis (BBN) (Planck Collaboration et al. 2020; Cooke et al. 2018). However, in the late-time universe, the baryon abundance and its spatial distribution are still poorly constrained by observations (e.g., Fukugita & Peebles 2004; Shull et al. 2012). About one-third of the baryons are still missing (the so-called ‘missing baryons’), although they are likely to be low-density ionised gas in the intergalactic medium (IGM). The cosmological DM is a powerful tool to probe for the missing baryons (Ioka 2003; Inoue 2004). Very recently, Macquart et al. (2020) measured the baryon density from five host-galaxy-identified FRBs. Their result is independent of, but consistent with, the Planck and BBN results. Keane et al. (2016) provided a similar constraint from a single event.

FRB catalogue at http://frbcat.org
2 https://www.atnf.csiro.au/projects/askap/
3 https://chime-experiment.ca/
4 https://astronomy.swin.edu.au/research/utmost/
5 https://fast.bao.ac.cn/
6 https://www.skatelescope.org/
7 https://frbtheorycat.org/index.php/Main_Page

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Because FRBs and their DMs have unique cosmological properties, many cosmological applications have been proposed. The DMs of far-distant FRBs are a unique probe of cosmological reionisation (Ioka 2003; Inoue 2004; Caleb et al. 2019; Dai & Xia 2020). Gravitational lensing of FRBs also enables searches for intervening compact objects that may constitute the dark matter (e.g., Zheng et al. 2014; Muñoz et al. 2016; Oguri 2019; Jow et al. 2020; Liao et al. 2020). If the host galaxy is identified, the redshift–DM relation can constrain the dark energy models (e.g., Gao et al. 2014; Zhou et al. 2014). The angular auto-correlation of the DM directly maps free-electron clustering (e.g., Masui & Sigurdson 2015; Shirasaki et al. 2017), and its large-scale signal may contain primordial non-Gaussianity (Reischke et al. 2020). The cross-correlation of the DM and foreground galaxies provides the free-electron distribution around the galaxies (McQuinn 2014; Shirasaki et al. 2017; Mahhavacheril et al. 2019), as well as helping to constrain the redshift distribution of the host galaxies (Rafiei-Ravandi et al. 2019). The cross-correlation of the DM and the thermal Sunyaev–Zel’dovich signal (tSZ, Sunyaev & Zeldovich 1970) gives further information about ionised gas, because the tSZ effect measures the projected electron pressure (Muñoz & Loeb 2018).

Theoretical studies of the DM statistics have been based on the analytical halo model or hydrodynamic simulations because these are able to explore the non-linear free-electron distribution. McQuinn (2014) calculated the DM statistics (variance and probability distribution) from the halo model (e.g., Cooray & Sheth 2002) for given model ingredients such as spatial halo clustering, halo mass function and free-electron density profile in halos. Mahhavacheril et al. (2019) and Dai & Xia (2020) computed the angular power spectrum of the DM based on the halo model. Cosmological hydrodynamic simulations are the most reliable tools for investigating the free-electron distribution in the universe. Dolag et al. (2015) studied the DM probability distribution based on hydrodynamic simulations (the Magneticum Pathfinder; Dolag et al. 2016). Zhu et al. (2018) estimated the dispersion and scattering measures in the IGM using their cosmological hydrodynamic simulations. Pol et al. (2019) made a full-sky map of the DM using the MICE ONION simulation (Fosalba et al. 2008), and they computed the mean, variance and probability distribution of the DM. That was a dark-matter-only simulation, and they assumed that the free electrons exactly trace the dark matter. Shirasaki et al. (2017) performed a similar analysis using their own dark-matter simulation. Jaroszynski (2019) recently studied the cosmological DM (its mean, variance and probability distribution) using a public hydrodynamic simulation, the original Illustris (Vogelsberger et al. 2014).

The previous simulation studies did not compare their measurements with analytical predictions of the DM statistics (such as its variance and power spectrum), where the analytical solutions are useful for future data analyses. Previous analytical studies on the DM statistics assumed that free electrons exactly trace the underlying dark matter (Masui & Sigurdson 2015; Shirasaki et al. 2017; Rafiei-Ravandi et al. 2019), although this assumption breaks down at small scales (≤ 1 Mpc), as shown in subsection 3.3. The main purpose of the present work is to provide an analytical model for the DM statistics (such as its mean, variance, angular power spectrum and correlation function). The analytical model is based on a standard two-point statistics. Because the DM statistics are fully determined by the free-electron statistics, we first measure the free-electron distribution from the latest cosmological hydrodynamic simulations, IllustrisTNG, the successor to Illustris (e.g., Nelson et al. 2018). We use the largest-box run from these TNG simulations (named TNG300, for which the side length of the cubic box is \( L = 205 h^{-1} \) Mpc ≃ 300 Mpc), which is suitable for cosmological studies. We measure the free-electron abundance and the power spectrum of its spatial fluctuations over a wide range of redshifts (\( z = 0–5 \)) and scales (≈ 0.1–200 \( h^{-1} \) Mpc) in TNG300. We then make fitting functions for them to model the free-electron distribution. The DM statistics are calculated analytically using these fitting functions. We next construct mock sky maps of the DM using the TNG300 data and measure the DM statistics from them to check the accuracy of the analytical model. The three spatial-resolution runs in TNG300 are used to check the numerical convergence of the results. The presented model is applicable, in principle, for other cross-correlations, such as DM–galaxy, DM–weak lensing and DM–tSZ cross-correlations. As thousands of FRBs will be available in the relatively near future, we expect this kind of statistical study to be required. Throughout this paper, we mainly study the cosmological DM (i.e., excluding contributions from the Milky Way and host galaxies).

The rest of this paper is organised as follows: Section 2 introduces the theory of two-point DM statistics. Section 3 measures the free-electron abundance and its power spectrum in the TNG300 data and provides fitting functions for them. Section 4 describes a procedure for making mock sky maps of the DM. Section 5 presents our main results: comparisons between the simulation results measured from the mock maps and analytical predictions. Section 6 discusses the host-galaxy contribution and provides comparisons with other hydrodynamic simulations. Finally, section 7 summarises this work.

Throughout this paper, we adopt a cosmological model consistent with the Planck 2015 best-fit flat ΛCDM model (Planck Collaboration 2016): matter density \( \Omega_m = 1 - \Omega_\Lambda = 0.3089 \), baryon density \( \Omega_b = 0.0486 \), Hubble parameter \( h = 0.6774 \), spectral index \( n_s = 0.9667 \), and amplitude of the matter density fluctuations on the scale of \( 8 h^{-1} \) Mpc \( \sigma_8 = 0.8159 \). This model is the same as that adopted in the TNG simulations. All physical quantities (such as length, wavenumber and number density) will be given in comoving units.

### 2 THEORY OF THE COSMOLOGICAL DISPERSION MEASURE

This section presents the theoretical basics of the cosmological DM: the mean and fluctuations (subsection 2.1) and the two-point statistics (subsection 2.2).

#### 2.1 The mean and fluctuations

Three major components contribute to the observed DM: the Milky Way, the host galaxy and the intervening cosmological medium. The Milky Way contribution can be inferred from the Galactic free-electron distribution, which is modelled by pulsar measurements (e.g., the NE2001 model: Cordes & Lazio 2002). The host-galaxy contribution decreases for more distant sources in proportion to \((1 + z_h)^{-1}\), where \(z_h\) is the source redshift, due to cosmological time dilation and the Doppler frequency shift (if its intrinsic property in the rest frame does not evolve with time, e.g., Zhou et al. 2014). In contrast, the cosmological contribution increases roughly in proportion to \(z_h\) (e.g., Ioka 2003), and it exceeds the host-galaxy contribution for \(z_h \gtrsim 0.3\). Therefore, throughout this paper, we mainly consider the cosmological contribution, and hereafter, DM refers to that alone. The host-galaxy contribution will be briefly discussed in subsection 6.1.

We consider an FRB at an angular position \( \theta = (\theta_1, \theta_2) \) on the...
sky and redshift \( z_s \), as shown in Fig. 1. The vector \( r \) points to the intervening gas at \( z \), and its absolute value is the comoving distance

\[
r(z) = \int_0^z \frac{cdz'}{H(z')}.
\]  

(1)

where \( H(z) \) is the Hubble expansion rate. Denoting the number density of free electrons at \( r \) and \( z \) by \( n_e(r; z) \), the DM is obtained by integrating \( n_e \) along the line of sight (e.g., Inoue 2003; Inoue 2004):

\[
\Delta M(\theta; z_s) = \int_0^\infty \frac{cdz}{H(z)} n_e(r; z)(1 + z).
\]  

(2)

Note that the number density \( n_e \) is given in comoving units.

The number density \( n_e \) can be decomposed into its spatial mean \( \bar{n}_e \) and fluctuations \( \delta_e \):

\[
n_e(r; z) = \bar{n}_e(z) [1 + \delta_e(r; z)].
\]  

(3)

The spatial average of the second term vanishes: \( \langle \delta_e \rangle = 0 \). As for the first term, the total number density of electrons (including both free electrons and those bound to atoms) in the universe is

\[
\bar{n}_e = \left( X_p + \frac{1}{2} Y_p \right) \frac{\bar{\rho}_b}{m_p}.
\]  

(4)

where \( \bar{\rho}_b \) is the comoving cosmological baryon density, and \( m_p \) is the proton mass (e.g., Deng & Zhang 2014). The quantities \( X_p \) and \( Y_p \) denote the primordial mass fractions of hydrogen and helium, respectively, and are set to \( X_p = 1 - Y_p = 0.76 \) to be consistent with TNG. We ignore the time evolution of \( \bar{n}_e \) due to stellar nucleosynthesis, because it is negligibly small. Introducing the free-electron fraction at \( z \), \( f_e(z) \), the free-electron number density is written as

\[
\bar{n}_e(z) = f_e(z) \bar{n}_e(z).
\]  

(5)

where \( f_e = 1 \) corresponds to full ionisation. After hydrogen and helium were fully ionised at \( z \approx 3 \), \( f_e \) is assumed to be close to unity. However, current observational constraints on \( f_e \) still have a large variation (\( f_e = 0.7-1 \)), e.g., Fukugita & Peebles 2004; Shull et al. 2012; McQuinn 2016; Walters et al. 2019; Li et al. 2020). Note that \( f_e \) in Eq. (5) includes all free electrons, both inside and outside of intervening galaxies. In other words, \( f_e \) is the spatial mean fraction averaged over all galaxies and the IGM. In this paper, we do not introduce the free-electron fraction in IGM, \( f_{IGM} \). One reason is that \( f_{IGM} \) depends on the boundary between the galaxies and the IGM, and that boundary is ambiguous. Another reason is that some FRB signals may pass through an intervening galaxy; this probability may be low, but it gives a large DM. We will measure \( f_e \) from the TNG300 simulations in section 3.

Similarly to \( n_e \), the DM can be decomposed into two terms,

\[
\Delta M(\theta; z_s) = \bar{\Delta M}(z_s) + \sigma \Delta M(\theta; z_s).
\]  

(6)

The mean and fluctuations of the DM can be written from Eqs. (2)–(5) in the forms

\[
\bar{\Delta M}(z_s) = \int_0^\infty \frac{cdz}{H(z)} W(z),
\]  

(7)

\[
\sigma \Delta M(\theta; z_s) = \int_0^\infty \frac{cdz}{H(z)} W(z) \delta_e(r; z),
\]  

(8)

with a kernel

\[
W(z) = \frac{\bar{\rho}_b}{m_p} \left( X_p + \frac{1}{2} Y_p \right) f_e(z)(1 + z).
\]  

(9)

The mean baryon density is rewritten as \( \bar{\rho}_b = \Omega_b \rho_{cr} = 3H_0^2\Omega_b/(8\pi G) \), where \( \rho_{cr} \) is the cosmological critical density.

### 2.2 The two-point statistics

This subsection discusses the angular correlation function and its Fourier transform (i.e., the power spectrum) of the DM fluctuations. Previously, several authors have studied the angular power spectrum of the DM (e.g., Masui & Sigurdson 2013; Shirasaki et al. 2017; Madhavacheril et al. 2019; Dai & Xia 2020). Here, we simply summarise their results.\(^8\)

The angular correlation function of the DM between \( \theta_1 \) and \( \theta_2 \) at the same source redshift \( z_s \) is defined as

\[
\xi(\theta_{12}; z_s) = \langle \delta \Delta M(\theta_1; z_s) \delta \Delta M(\theta_2; z_s) \rangle.
\]  

(10)

Because of the isotropy of the universe, the correlation function is a function of the separation \( r_{12} = \theta_1 - \theta_2 \). Throughout this paper, we assume \( r_{12} \ll 1 \), i.e. the flat-sky approximation is valid. From Eqs. (8) and (10), under the Limber and the flat-sky approximations, the correlation function reduces to

\[
\xi(\theta_{12}; z_s) = \frac{1}{2\pi} \int_0^\infty \frac{cdz}{H(z)} W^2(z) \int_0^\infty dk k P_e(k; z) J_0(\theta_{12} k r(z)) \, ,
\]  

(11)

where \( J_0 \) is the zero-th order Bessel function, and \( k \) is the wavenumber of the density fluctuations. The power spectrum of the free-electron fluctuations is defined as

\[
P_e(k; z) = (2\pi)^3 \delta^2(k + k') \equiv \langle \delta_e(k; z) \delta_e(k'; z) \rangle,
\]  

(12)

where \( \delta_e(k; z) \) is the Fourier transform of \( \delta_e(r; z) \), and \( \delta^2_\text{Dirac} \) is the Dirac delta function.

The Fourier transform of the DM fluctuations is given by

\[
\delta \Delta M(\ell; z_s) = \int d^2 \theta \delta \Delta M(\theta; z_s) e^{-i\ell \cdot \theta}.
\]  

(13)

where \( \ell = (\ell_1, \ell_2) \) is the two-dimensional vector of multipole moments. Similarly to \( P_e(k; z) \), the angular power spectrum of the DM is defined as

\[
C_{\Delta M}(z_s) = (2\pi)^2 \delta^2_\text{Dirac}(\ell + \ell') \equiv \langle \delta \Delta M(\ell; z_s) \delta \Delta M(\ell'; z_s) \rangle.
\]  

(14)

\(^8\) A detailed discussion of the two-point statistics of projected random fields is found in, e.g., section 2.4 of Bartelmann & Schneider (2001) and section 9.1 of Dodelson (2003).
From the above equations (10)–(14), the angular power spectrum is obtained as
$$C_ℓ(\ell) = \int d^2 \theta \xi(\theta, z_\ell) e^{-i \ell \theta},$$
$$= \int_0^{2\pi} \, d\theta \frac{W^2(\theta)}{H(z)} P_e(\ell) \int_0^\infty \, d\theta \frac{\tilde{\chi}(z_\ell)}{r(z_\ell)}.$$ (15)

This equation relates the 3D power spectrum of the free electrons to the 2D power spectrum of the DM.

The variance of the DM is simply obtained by setting $θ_1 = θ_2$ in Eqs. (10) and (11):
$$\sigma_{DM}(z_\ell)^2 = \langle [\delta DM(\theta, z_\ell)]^2 \rangle,$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \, d\theta \frac{W^2(\theta)}{H(z)} \int_0^\infty \, d\theta \frac{\tilde{\chi}(z_\ell)}{r(z_\ell)}.$$ (16)

This is consistent with the analytical result in McQuinn (2014, their section 2). Theoretical models of the ionised fraction $f_e(z)$ and the power spectrum $P_e(k; z)$ are required to compute the above two-point statistics. We will calibrate these functions using TNG300 in the next section.

3 CALIBRATION WITH TNG300

This section briefly introduces the TNG simulations (subsection 3.1) and then measures the free-electron fraction $f_e(z)$ (subsection 3.2) and the power spectrum $P_e(k; z)$ (subsection 3.3).

3.1 The TNG simulations

We investigate the spatial distribution of free electrons in the universe using the TNG dataset9 (Marinacci et al. 2018; Naiman et al. 2018; Nelson et al. 2018; Pillepich et al. 2018b; Springel et al. 2018). The simulations follow the gravitational clustering of matter (dark matter and baryons) as well as astrophysical processes such as star and galaxy formation, gas cooling, and stellar and active galactic nucleus (AGN) feedback. The gravitational evolution and magneto-hydrodynamic processes were computed with the moving-mesh code AREPO (Springel 2010). The simulations incorporate astrophysical processes in a subgrid model, thereby enabling them to follow the processes of galaxy formation and evolution. The TNG project produced three sets of simulations in different-sized cubic boxes, with three mass resolutions for each box size. Here, we used the largest box (referred to as TNG300), with side length $L = 205 \ h^{-1} \ Mpc$ ($\approx 300 \ Mpc$), because our interest is the large-scale distribution of free electrons. To check the numerical convergence, we used the three resolutions from high to low (referred to as TNG300-1 to -3, respectively). This box contains the same number of dark-matter and baryon particles. The number of particles and the mass resolution are listed in Table 1. The TNG team also performed dark-matter-only runs, in which the number of dark-matter particles was the same as in TNG300. In this case, the N-body particles represent both components (baryons and dark matter), but the simulations follow the gravitational evolution only. Such simulations help to see the impact of dark matter on the free-electron clustering. Here we used the highest-resolution run (named TNG300-1-Dark). The TNG team have released the simulation data at 20 redshifts in the range $z = 0$–12 (named “full” snapshots).

In this paper, we used all the datasets up to $z = 8$, as listed in Table

2. The first column is the redshift $z$, the second is the comoving distance to $z$ and the third refers to the TNG snapshot number.

Each baryon particle has one of three forms: gas, star or supermassive black hole. Free electrons are contained only in the gas particles. At the initial redshift ($z = 127$), all the baryon particles are gas. As time evolves, the gas falls into the halos, and star formation begins in high-density regions (Pillepich et al. 2018a). Some gas particles then convert to stars or black holes. However, even at $z = 0$, most of the baryon particles are still gas (the gas mass fraction is $> 96\%$). The time evolution of each mass fraction measured in TNG300-1 is summarised in Table 3. The mass fraction is obtained from the total mass of each component in the box divided by the total baryonic mass ($\approx \rho_0 L^3$). The TNG team followed the time evolution of the atomic abundances of H, He and seven other species (C, N, Ne, Mg, Si and Fe). The public data contains the atomic abundance in each gas particle. For hydrogen, the data includes neutral (H$_0$) and ionised (H$^+$) fractions. In Table 3, the gas-mass fraction ($f_{gas}$) is further decomposed into neutral and ionised hydrogen ($f_{H_0}$ and $f_{H^+}$) and helium ($f_{He}$), where $f_{He}$ includes both neutral and ionised states. The mass fraction of elements heavier than H is negligible.

The hydrogen is ionised abruptly at the epoch of cosmic reionisation (between $z = 5$ and 6). We comment that the mass fraction of stars ($f_{star}$) reaches 3.2% at $z = 0$, which is slightly smaller than the observed values $f_{star} = 6.0 \pm 1.3\%$ (Fukugita & Peebles 2004) and 7 ± 2% (Shull et al. 2012; Nicastro et al. 2018). Therefore the gas fraction $f_{gas}$ and the resulting free-electron fraction $f_e$ in TNG300-1 may be overestimated by approximately a few percent.

Previous work (Jaroszynski 2019) on the DM used the lowest-resolution run of Illustris: the side length of the box is $75 \ h^{-1} \ Mpc$, and it contains $2 \times 455^3$ baryon and dark-matter particles. Therefore, TNG300 has better mass resolution and a larger simulation volume. Their work did not check the numerical convergence among the different resolutions. Illustris is known to predict AGN feedback that is too strong (e.g., Chisari et al. 2019), which may affect the free-electron distribution in the halos.

3.2 Free-electron abundance

The TNG team also provided the abundance of free electrons in each gas particle, which is the total free-electron abundance for all atoms (not only for hydrogen). By summing up all the gas particles in each snapshot, we obtained the number density of free electrons and its fraction, $\bar{n}_e(z)$ and $f_e(z)$, as defined in Eq. (5). Here we measured $f_e(z)$ at the 16 redshifts in the range $z = 0$–8 listed in Table 2. The result is plotted in Fig. 2. At high $z (\ge 6)$, the gas is still neutral (i.e., $f_e \approx 0$). The fraction $f_e$ rises abruptly at the epoch of hydrogen reionisation ($z \sim 6$) and increases further at the epoch of helium reionisation ($z \sim 4$). At relatively low $z (\lesssim 3)$, $f_e$ decreases slightly because some fraction of the electrons becomes confined in stars and black holes (see Table 3). A small fraction of the electrons is in neutral hydrogen (HI and H$_2$) in galaxies ($f_{He} \approx 1\%$ in Table 3; the cosmological HI distribution was recently studied using hydrodynamic simulations in, e.g., Villaescusa-Navarro et al. 2018; Ando et al. 2019).

The TNG300-1 result can be fitted by
$$f_{e}(z) = a (z + b)^{0.02} \left[1 - \tanh (c (z - z_0)) \right],$$ (17)
with $a = 0.475, b = 0.703, c = 3.19$ and $z_0 = 5.42$. Here, $z_0$ corresponds to the epoch of hydrogen reionisation. For $z \gg z_0$, $f_e$ approaches zero. On the other hand, for $z \to 0$, $f_e \to 2a \approx 0.95$. This fit agrees with the TNG300-1 results to within a deviation $\Delta f_e = 0.012$ in the range $z = 0$–8. There are few-percent deviations.
Table 1. Summary of the TNG300 simulations used in this paper: the numbers of baryon and dark-matter particles \(N_{\text{baryon}}, N_{\text{dark}}\), the average masses of baryon and dark-matter particles \(m_{\text{baryon}}, m_{\text{dark}}\), the minimum gravitational softening length of the gas cells \(\epsilon_{\text{gas, min}}\), and the mean size of the gas cells \(r_{\text{gas}} \equiv L / N_{\text{gas}}^{1/3}\). The upper three runs follow both the gravitational evolution and astrophysical processes, while the bottom one follows only the former. The side length of the simulation box is \(L = 205 \, h^{-1}\) Mpc in all runs.

|          | \(N_{\text{baryon}}\) | \(N_{\text{dark}}\) | \(m_{\text{baryon}}(h^{-1}\text{M}_\odot)\) | \(m_{\text{dark}}(h^{-1}\text{M}_\odot)\) | \(\epsilon_{\text{gas, min}}(h^{-1}\text{kpc})\) | \(r_{\text{gas}}(h^{-1}\text{kpc})\) |
|----------|-----------------------|----------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| TNG300-1 | 2500^3                | 2500^3               | 7.4 \times 10^8                          | 4.0 \times 10^7                          | 0.25                                     | 82                                       |
| TNG300-2 | 1250^3                | 1250^3               | 6.0 \times 10^7                          | 3.2 \times 10^6                          | 0.5                                      | 164                                      |
| TNG300-3 | 625^3                 | 625^3                | 4.8 \times 10^8                          | 2.5 \times 10^6                          | 1.0                                      | 328                                      |

| TNG300-1-Dark | – | 2500^3 | – | 4.7 \times 10^7 | – | – |

Table 2. Output redshift \(z\), comoving distance \(r(z)\) and snapshot number in the TNG dataset.

| \(z\) | \(r(z)\) \((h^{-1}\text{Mpc})\) | snapshot |
|-------|--------------------------------|----------|
| 0     | 0                              | 99       |
| 0.1   | 293                            | 91       |
| 0.2   | 571                            | 84       |
| 0.3   | 834                            | 78       |
| 0.4   | 1083                           | 72       |
| 0.5   | 1318                           | 67       |
| 0.7   | 1747                           | 59       |
| 1     | 2301                           | 50       |
| 1.5   | 3034                           | 40       |
| 2     | 3599                           | 33       |
| 3     | 4411                           | 25       |
| 4     | 4973                           | 21       |
| 5     | 5390                           | 17       |
| 6     | 5716                           | 13       |
| 7     | 5978                           | 11       |
| 8     | 6196                           | 8        |

Table 3. Mass fractions of gas, stars and super-massive black holes to the total baryons measured in TNG300-1. The values are given in percentages \((i.e., f_{\text{gas}} + f_{\text{star}} + f_{\text{bh}} = 100\%)\). The gas is further decomposed into neutral and ionised hydrogen \((\text{H}_0\) and \(\text{H}^+\)) and helium \((\text{He})\), which satisfy \(f_{\text{gas}} = f_{\text{H}_0} + f_{\text{H}^+} + f_{\text{He}}\).

| \(z\) | \(f_{\text{gas}}\) | \(f_{\text{H}_0}\) | \(f_{\text{H}^+}\) | \(f_{\text{He}}\) | \(f_{\text{star}}\) | \(f_{\text{bh}}\) |
|-------|------------------|------------------|------------------|------------------|------------------|------------------|
| 0     | 96.8             | 1.4              | 71.9             | 23.3             | 3.2              | 0.02             |
| 1     | 97.7             | 0.9              | 73.2             | 23.5             | 3.7              | 0.01             |
| 3     | 99.3             | 0.9              | 74.6             | 23.8             | 0.7              | 5 \times 10^{-3} |
| 5     | 99.8             | 1.3              | 74.5             | 24.0             | 0.2              | 1 \times 10^{-3} |
| 6     | 99.9             | 7.4              | 1.9              | 24.0             | 0.1              | < 10^{-3}        |
| 7     | 100.0            | 75.0             | 1.0              | 24.0             | 0.06             | < 10^{-3}        |
| 8     | 100.0            | 75.5             | 0.4              | 24.0             | 0.03             | < 10^{-3}        |

Figure 2. Time evolution of the free-electron fraction measured in TNG300. The purple, blue and green symbols represent the results from TNG300-1, -2 and -3 (from high- to low-resolution runs), respectively. The red curve is the fit to TNG300-1 given in Eq. (17).

Table 3. Mass fractions of gas, stars and super-massive black holes to the total baryons measured in TNG300-1. The values are given in percentages \((i.e., f_{\text{gas}} + f_{\text{star}} + f_{\text{bh}} = 100\%)\). The gas is further decomposed into neutral and ionised hydrogen \((\text{H}_0\) and \(\text{H}^+\)) and helium \((\text{He})\), which satisfy \(f_{\text{gas}} = f_{\text{H}_0} + f_{\text{H}^+} + f_{\text{He}}\).

3.3 Free-electron power spectrum

We next measured the power spectrum of free electrons in TNG300 following the standard procedure (see, e.g., Springel et al. 2018, their section 2.2). The TNG team provided the mass of free electrons in each gas particle. To measure the density contrast, we assigned the free-electron mass to 1024\(^3\) regular grid cells in the box using the cloud-in-cell (CIC) interpolation with the interfacing scheme (e.g., Jing 2005; Sefusatti et al. 2016). The Fourier transform of the density field \(\delta(k)\) was then obtained with fast Fourier transform (FFT)\(^1\). To explore smaller scales, we also employed the folding method (Jenkins et al. 1998), which folds the particle positions \(\text{x}\) into a smaller box of side length \(L/10\) by replacing \(\text{x}\) with \(\text{x} \% (L/10)\) (where \(\%\) denotes the reminder of \(a/b\)). This procedure effectively increases the spatial resolution by 10 times. The minimum and maximum wavenumbers in the 1024\(^3\) cells are \(k_{\text{min}} = 2\pi/L = 0.025 \, h\, \text{Mpc}^{-1}\) (where \(L = 205 \, h^{-1}\) Mpc) and \(k_{\text{max}} = 512 \, k_{\text{min}} = 12.9 \, h\, \text{Mpc}^{-1}\), respectively. The folding

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\(^1\) FFTW (Fast Fourier Transform in the West) at \url{http://www.fftw.org}. 
scheme enlarges $k_{\text{max}}$ by 10 times. The power spectrum is reliable up to the particle Nyquist wavenumber, which is determined by the mean separation of the gas particles $r_{\text{gas}}$ in Table 1: $k_{\text{Nyq}} = \pi/r_{\text{gas}}$. The values of $k_{\text{Nyq}}$ are 38.3 (19.1 and 9.6) $h$ Mpc$^{-1}$ for TNG300-1 (-2 and -3).

The power spectrum is measured as

\[ P_e(k) = \frac{1}{N_{\text{mode}}} \sum_{k' | k} |\delta_e(k')|^2, \tag{18} \]

where the summation is performed in the spherical shell $k - \Delta k/2 < |k' | < k + \Delta k/2$ and $N_{\text{mode}}$ is the number of Fourier modes in the shell with bin-width ($\Delta \log_{10} k = 0.1$). The spectrum $P_e(k)$ in Eq. (18) contains the shot noise contribution

\[ P_{e,\text{shot}} = \frac{L^3}{N_{\text{eff}}}, \tag{19} \]

where $N_{\text{eff}}$ is the effective number of gas particles in the box. Denoting $m_i$ as the free-electron mass of the $i$-th gas particle, we have $N_{\text{eff}} = \sum_i m_i^2/\langle m_i^2 \rangle$. If all gas particles have equal mass, then $N_{\text{eff}} = N_{\text{gas}}$ (where $N_{\text{gas}}$ is the number of gas particles). The shot noise was subtracted from the measured $P_e(k)$. Here we measured $P_e(k)$ up to $z = 5$, because $P_e(k)$ is noisy for $z \geq 6$ owing to the low free-electron abundance.

We also measured the matter power spectrum $P_{\text{dm}}(k)$ in the dark-matter-only (dmo) run (TNG300-1-Dark in Table 1). This spectrum $P_{\text{dm}}(k)$ can be used to clarify the difference in clustering between free electrons and dark matter.

Figure 3 plots the measured power spectra at several redshifts ($z = 0–3$). The purple, blue and green symbols are $P_e(k)$, while the grey circles are $P_{\text{dm}}(k)$. The dotted lines denote the shot-noise contribution. The vertical axis is $k^3P_e(k)$, which represents the contribution to the DM variance per $\ln k$ from Eq. (16). The figure shows that density fluctuations at $k = 1–10$ $h$ Mpc$^{-1}$ (corresponding to a scale of $2\pi/k \approx 1$ Mpc) contribute most to the DM variance. The shot noise is negligibly small around this peak. The spectrum $P_e(k)$ agrees with $P_{\text{dm}}(k)$ at large scales ($k < 1$ $h$ Mpc$^{-1}$) but is strongly suppressed at intermediate and small scales ($k \geq 1$ $h$ Mpc$^{-1}$). Springel et al. (2018) previously measured the power spectrum of the gas in the TNG simulations and gave a physical explanation for this suppression: the stellar and AGN feedback expels gas from the halos and suppresses the gas clustering, especially at low $z$, but gas cooling enhances clustering at very small scales, $k > 10$ $h$ Mpc$^{-1}$. In fact, $k^2P_e(k)$ rises slightly for $k > 10$ $h$ Mpc$^{-1}$, especially for the higher-resolution run. The results for $P_e(k)$ at small scales ($k \geq 10$ $h$ Mpc$^{-1}$) do not converge among the different simulations owing to the lack of spatial resolution. The dashed-orange curves are Halofit results from a fitting formula for non-linear $P_{\text{dm}}(k)$ (Smith et al. 2003;
Takahashi et al. 2012. These curves agree with the dark-matter-only simulation results very well.

To model $P_{d}(k; z)$, we introduce the bias factor $b_{e}(k; z)$ defined by

$$b_{e}^{2}(k; z) = \frac{P_{e}(k; z)}{P_{dmo}(k; z)}.$$  \hspace{1cm} (20)

Figure 4 plots the measured bias. The bias approaches unity in the small-$k$ limit, but it is suppressed at large $k$ ($\gtrsim 1$ h Mpc$^{-1}$). At the largest scales (i.e., the smallest $k$), the bias is very close to unity, although it is slightly smaller than unity (by approximately a few percent), especially at high $z$.\footnote{Shaw et al. (2012) previously measured $b_{e}^{2}(k)$ from their hydrodynamic simulations. Their result is somewhat smaller than ours in the low-$k$ limit: $b_{e}^{2}(k) = 0.6-1$ and varies with $z$ (see the right panel of their Fig. 2). However, according to the cosmological perturbation theory of mixed components (baryons and dark matter), the baryon-fluctuation amplitude is only slightly smaller (< 4%) than the dark matter one for $k \leq 0.1$ h Mpc$^{-1}$ and $z \leq 3$ (e.g., Somogyi & Smith 2010, their Fig.1).} This is because the baryon-density fluctuations gradually catch up to the dark matter fluctuations after the epoch of decoupling (at $z \approx 1100$). The red curves are our fits to TNG300-1, where the bias is calibrated at 10 redshifts in the range $z = 0-5$ ($z = 0, 0.2, 0.4, 0.7, 1, 1.5, 2, 3, 4$ and $5$). The range of $k$ included in the fit is determined such that the TNG300-1 and -2 results agree to within 20%. The bias factor is fitted by the function

$$b_{e}^{2}(k; z) = \frac{b_{e}^{2}(z)}{1 + (k/k_{s}(z))^{\gamma(z)}},$$ \hspace{1cm} (21)

with

$$b_{e}^{2}(z) = 0.971 - 0.013 z,$$

$$\gamma(z) = 1.91 - 0.59 z + 0.10 z^{2},$$

$$k_{s}(z) = 4.36 - 3.24 z + 3.10 z^{2} - 0.42 z^{3},$$

where $k_{s}$ has units of h Mpc$^{-1}$. This function agrees with the simulation results for $P_{d}(k; z)/P_{dmo}(k; z)$ to within 3.5 (10.8\%) for $k < 2$ (10) h Mpc$^{-1}$ in the range $z = 0-5$.

The user can compute $P_{e}(k)$ from the bias factor (21) and the $P_{dmo}(k)$ model. Accurate fitting formulas for non-linear $P_{dmo}(k)$ have been presented, such as Halofit (Smith et al. 2003; Takahashi et al. 2012), HMcode (Mead et al. 2015) and the Mira-Titan emulator (Lawrence et al. 2017). These formulas agree with the latest dark-matter simulations to within 5\% up to $k = 10$ h Mpc$^{-1}$ (e.g., Smith & Angulo 2019, their Fig. 6). Halofit and HMcode are implemented in public codes such as CAMB\footnote{https://camb.info/} and CLASS\footnote{http://class-code.net/}.

4 MAKING MOCK SKY MAPS OF THE DM

This section describes our procedure for making mock maps of the DM. We placed the simulation boxes along the line-of-sight direction using periodic boundary conditions, as shown in Fig. 5. The observer is placed at a corner of the box at $z = 0$. The field of view was set to be a square of $6 \times 6$ deg$^2$. To avoid repeating the same structure along the line of sight, we tilted the main axis (denoted by the dotted line) of the line of sight by 5 deg from the box axis. The lower-$z$ box in Table 2 was placed closer to the observer.
For a given comoving distance \( r \), we used the box nearest to \( r \). For instance, from Table 2, the lowest-\( z \) box (at \( z = 0 \)) was used for \( r/(h^{-1} \text{ Mpc}) \leq 293/2 \), the second-lowest box (at \( z = 0.1 \)) was used for \( 293/2 < r/(h^{-1} \text{ Mpc}) \leq (293 + 571)/2 \) and so on. Note that, due to the periodicity of the box, for \( r > 205 h^{-1} \text{ Mpc}/(6 \text{ deg}) \approx 2.0 h^{-1} \text{ Gpc} \), the same structure may appear more than once in the field of view.

Free electrons are included in the gas particles. For each gas particle, the TNG team provided the spatial position, gas mass \( m_{\text{gas}} \), density \( \rho_{\text{gas}} \), and free-electron number density \( n_{e,\text{gas}} \). We assume that each gas particle is described by a sphere of constant density with the radius \( r_{\text{gas}} \) determined via \( m_{\text{gas}} = (4\pi r_{\text{gas}}^3/3)\rho_{\text{gas}} \).

The DM is rewritten from Eq. (2) as

\[
\text{DM}(\theta; z_s) = \int_0^\infty dr_n e(r,z) (1+z(r)),
\]

(22)

where \( r_n = r(z_s) \) is the comoving distance to the source, and \( \theta \) denotes the angular position in the field of view, i.e., \( \theta = (\theta_1, \theta_2) \) with \( |\theta_1, \theta_2| \leq 3 \text{ deg} \). Light rays are emitted from the observer and propagate along straight lines in the field. The DM is computed by summing the contributions from all gas particles intersecting the light-ray path:

\[
\text{DM}(\theta; z_s) = \sum_{i} n_{e,i}^{2D}(b_i)(1+z_i),
\]

(23)

where \( n_{e,i}^{2D} \) is the free-electron column density of the \( i \)-th gas particle and \( b_i \) is the impact parameter (i.e., the minimum separation between the ray path and the position of the \( i \)-th particle). The redshift \( z_i \) is calculated from the comoving distance using Eq. (1). Assuming that the \( i \)-th gas particle has a constant density \( \rho_{\text{gas}}^{\text{const}} \) and radius \( r_{\text{gas},i} \), its column-density profile is given by

\[
 n_{e,i}^{2D}(b_i) = 2 n_{e,i}^{\text{gas}} \rho_{\text{gas}}^{\text{const}} \sqrt{r_{\text{gas},i}^2 - b_i^2},
\]

(24)

for \( b_i < r_{\text{gas},i} \), and \( n_{e,i}^{2D}(b_i) = 0 \) otherwise. The DM in Eq. (23) is computed along the straight-line ray-path up to \( z_i = 3 \). We comment that each light ray passes through a sufficient number of gas spheres. For instance, in TNG300-1, there are \( 1.0 \times 10^4, 2.2 \times 10^4 \) and \( 3.7 \times 10^4 \) gas spheres intersecting a single light ray up to \( z_s = 0.4, 1 \) and 2, respectively. For TNG300-2 (-3), this number simply decreases by a factor of 2 (4). If the number of intersecting gas spheres follows a Poisson distribution, the accuracy of the DM in Eq. (23) is roughly given by \((\text{the number})^{-1/2}\).

We homogeneously emitted 5400\(^2\) rays through the \( 6 \times 6 \text{ deg}^2 \) field and computed their DMs using Eq. (23). The resulting angular resolution is 4 arcsec (= 6 deg/5400). We stored the DM data up to \( z_s = 3 \) at every \( \Delta z = 0.02 \) step. To see the statistical variation
among the maps, we prepared 10 such maps by recycling the same simulation data. Here the recycling procedure is as follows: (i) swap the coordinates (e.g., \(x \leftrightarrow y\)) for all particles in the box, (ii) shift the coordinates (e.g., \(x \rightarrow x + x_0\) with an arbitrary constant \(x_0\) where the coordinate origin can be freely chosen under the periodic boundary conditions) for all particles and (iii) finally place these boxes as in Fig. 5 and perform the same ray-tracing calculation. The swapped coordinates (i) and the coordinate shift (ii) were randomly chosen for each map. We prepared the 10 maps for each of the three resolution runs. We checked that the observer does not belong to any halo (the TNG also provides halo catalogues containing halo positions and radii), and thus, the measured DM does not contain the observer’s halo contribution.

Figure 6 is a contour map of the DM from TNG300-1 at \(z_s = 1\). This is one of the 10 maps. The red (blue) regions correspond to foreground clusters or galaxies (voids). We present an analysis of the 10 maps in the following section.

5 RESULTS

This section presents measurements of the DM statistics from the mock maps: the mean and variance (subsection 5.1), probability distribution of the DM (subsection 5.2), probability distribution of \(z_s\) for a given DM (subsection 5.3), angular power spectrum (subsection 5.4) and angular correlation function (subsection 5.5). Comparisons with the analytical results using the fitting functions (given in section 3) are also presented.

5.1 Mean and variance of the DM

We measured the mean and variance of the DM from the 10 mock maps. As there are 5400\(^2\) data points in each map, the total number of rays (= 10\(\times 5400^2 \approx 2.9 \times 10^8\)) is sufficient for statistical analysis. Figure 7 plots the mean with the standard deviation as a function of \(z_s\). The simulation results are measured at several values of \(z_s\) (= 0.2, 0.4, 0.7, 1, 1.5 and 2). The mean and variance are obtained by using all the rays in the 10 maps. The solid and dashed red curves are the theoretical mean and standard deviation given in Eqs. (7) and (16), respectively. Here and hereafter, the fitting formula for \(f_e(z)\) and \(P_e(k;z)\) given in section 3 are used to compute the analytical predictions, and the theory refers to the analytical model (in section 2) using these fitting formulas. The figure shows that the theory agrees with the simulations very well. The mean DM is approximately proportional to \(z_s\), \(\sigma_{\text{DM}}(z_s) = 1000 \times z_s \text{ pc cm}^{-3}\), which is consistent with previous work (e.g., Ioka 2003). The dash-dotted orange curve is the same as the solid one, but assuming \(f_e = 1\). In this and following figures, the theory refers to the analytical formula in (section 2) using the fitting formulas for \(f_e(z)\) and \(P_e(k;z)\) (in section 3).

Figure 8. Standard deviation of the DM as a function of \(z_s\). The purple, blue and green symbols are the TNG300-1, -2 and -3 results. The solid red curve denotes the analytical prediction given in Eq. (16). The dash-dotted orange curve is the same as the solid one, but assuming \(f_e = 1\).

fluctuations with \(k = 1-10 \text{ h Mpc}^{-1}\) contribute most to \(\sigma_{\text{DM}}\) (see also subsection 3.3), and these runs resolve this scale sufficiently. The TNG300-3 results give a slightly smaller result because of the lowest resolution. From this figure, the standard deviation is roughly given by \(\sigma_{\text{DM}}(z_s) \approx 230 \times z_s^{0.5} \text{ pc cm}^{-3}\), which is consistent with the halo-model prediction (McQuinn 2014; Macquart et al. 2020, see also Kumar & Linder 2019). The previous ray-tracing simulation (Jaroszynski 2019, see the dashed line in their Fig. 1) gave
\( \sigma_{\text{DM}} \approx 200 \) and 300 pc cm\(^{-3} \) at \( z_s = 1 \) and 2, respectively, which are also consistent with our results.

Previously, Shirasaki et al. (2017) and Pol et al. (2019) studied the DM statistics using dark-matter only simulations, assuming the free-electron distribution to be the same as the dark matter distribution (i.e., \( b_e = 1 \)). The dash-dotted orange curve in Fig. 8 corresponds to this case. This assumption may overestimate \( \sigma_{\text{DM}} \) by a factor of two.

We comment that although the TNG300 simulations do not contain density fluctuations larger than the box size \( L (= 205 \, \text{k}^{-1} \, \text{Mpc}) \), this does not affect the results for \( \sigma_{\text{DM}} \). If we set the large-scale cutoff \( P_c(k) = 0 \) for \( k < 2\pi/L = 0.03 \, \text{h} \, \text{Mpc}^{-1} \) in Eq. (16), the variance \( \sigma_{\text{DM}}^2 \) is underestimated by \( < 0.5 \% \) in the range \( z_s = 0–2 \), because \( k^2P_c(k) \) at such a large scale is too small to give a contribution.

### 5.2 Probability distribution of the DM

Figure 9 plots the probability distribution function (PDF) of the DM for several source redshifts \( (z_s = 0.4, 0.7, 1 \) and 1.5). The coloured histograms correspond to the different TNG resolutions, which are consistent with each other. The PDF is highly skewed, especially at low \( z_s \), owing to the strong non-Gaussianity of the density fluctuations. The red curves are log-normal distributions with the mean and variance given by the DM-map measurements from TNG300-1. At higher \( z_s \), the simulations approach log-normal distributions. This model is roughly consistent with the simulations, but it has broader tails around the peak and is less skewed than the simulations.

Figure 10 shows a comparison of the PDF with previous fitting formulas. Das & Ostriker (2006) measured the PDF of the projected matter density using dark-matter N-body simulations. Their purpose was to investigate the PDF of the weak-lensing convergence field. They proposed a modified log-normal distribution (given in their Eq. (11)). Dolag et al. (2015) performed cosmological hydrodynamic simulations and measured the PDF. Their formula depends only on \( z_s \) (given in their Eq. (6)). Macquart et al. (2020) proposed a skewed Gaussian PDF calibrated by the halo-model prediction (McQuinn 2014). Here we used their best-fit model (their Eq. (4) with \( \alpha = \beta = 3 \)). In Das & Ostriker (2006) and Macquart et al. (2020), the formulas contain free parameters, but they are fully determined by the mean and variance of the DM measured from TNG300-1. The figure shows that all the formulas show better agreement with the simulation than the simple log-normal model.

### 5.3 Probability distribution of the source redshift for a given DM

Figure 11 plots the mean and 1\( \sigma \) standard deviations for the source redshift \( z_s \) inferred from the measured DM. The coloured symbols are the simulation results, while the dash-dotted orange curve is the analytical DM–\( z_s \) relation (7). As clearly shown in the figure, the DM–\( z_s \) relation underestimates \( z_s \), especially at low \( z_s \), owing to the highly skewed distribution of the DM (shown in Figs. 9 and 10). As the peak of the DM is lower than the analytical mean for a given \( z_s \) in Fig. 9, the inferred \( z_s \) is higher than the analytical mean for a given DM. This trend is consistent with a previous finding (Pol et al. 2019, their Fig. 3). The figure shows that the standard deviation of \( z_s \) is approximately 20% for DM > 500 pc cm\(^{-3} \) but becomes larger for a nearer source. As the statistics of \( z_s \) are useful in searching for the host galaxy of an FRB from the measured DM, we fitted the mean \( \bar{z}_s \) and standard deviation \( \sigma_{z_s} \) from TNG300-1 in the range DM = 100–2000 pc cm\(^{-3} \):

\[
\bar{z}_s(\text{DM}) = 0.015 \, \text{DM}^{0.26} + 9.4 \times 10^{-4} \, \text{DM},
\]

\[
\sigma_{z_s}(\text{DM}) = 0.0024 \, \text{DM}^{0.61} + 1.2 \times 10^{-5} \, \text{DM},
\]

where DM is in units of pc cm\(^{-3} \). These formulas are plotted as the solid and dashed red curves in Fig. 11. Though the relation (25) was derived from TNG300-1 for a specific \( f_c(z) \) model, it may be used for any \( f_c(z) \) model, so long as \( f_c(z) \) does not evolve strongly with time (which is valid in our case at \( z < 2 \), where \( f_c = 0.95 \), as shown in Fig. 2). In this case, the relation (25) may be used by replacing DM with DM × (0.95/\( f_c \)) for an arbitrary \( f_c \).

Figure 12 plots the PDFs of the \( z_s \) for given DMs. The histograms are the simulation results, while the vertical dashed-orange lines are the values of \( z_s \) inferred from the DM–\( z_s \) relation (7). The analytical expectation of \( z_s \) is thus systematically lower than the true value by 10–20%, especially for a low DM. The red curves are Gaussian distributions with mean and variance given by Eq. (25). The PDF of the \( z_s \) is well described by a Gaussian.

Walker et al. (2020) recently derived a PDF of the \( z_s \) by a different approach. Their PDF is based on Bayes’ theorem and uses their DM probability distribution and a given FRB redshift distribution. Their PDF (in their Fig. 5) seems consistent with ours, but their result depends on the prior FRB redshift distribution. Hackstein et al. (2020) performed a similar analysis using the same approach.

### 5.4 Angular power spectrum of the DM

We measured the angular power spectrum of the DM in the same way as discussed for the free-electron power spectrum in subsection 3.3. The Fourier transform of the DM fluctuations in the \( i \)-th map (\( i = 1, 2, \cdots, 10 \)) is denoted as \( \delta\text{DM}(\ell; z_s) \) from Eq. (13). Then the power spectrum for this map is obtained as

\[
C_{\ell,i}(z_s) = \frac{1}{N_{\text{mode}, \ell}} \sum_{I' \in \ell} \left| \delta\text{DM}(I'; z_s) \right|^2,
\]

where the summation is performed in the annulus \( \ell - \Delta \ell/2 < |I'| < \ell + \Delta \ell/2 \) and \( N_{\text{mode}, \ell} \) is the number of Fourier modes in the annulus with bin-width \( \Delta \) (\( \ell \log_{10} \ell = 0.1 \)). We measured \( C_{\ell,i}(z_s) \) for the 10 maps to calculate its mean and variance among the maps.

Figure 13 shows the angular power spectra of the DM at \( z_s = 0.4 \) and 1. The symbols with error bars denote the simulation results for the mean and standard deviation among the maps. Here, the minimum multipole is determined by the side length of the map: \( \ell = 2 \pi/(6 \, \text{deg}) = 60 \). The angular resolution of the maps (\( \approx 4 \) arcsec = 6 deg/5400) is good enough to resolve the signal up to \( \ell = 10^3 \). If \( C_{\ell,i} \) obeys a Gaussian distribution, the error bars scale in proportion to \( (\text{survey area})^{-1/2} \), where \( \Delta \ell \) is the bin-width. The solid red curves are the theory (15). The dashed red curves include the effect of the finite size of the simulation box, where we simply set \( P_c(k) = 0 \) for \( k < 2\pi/L \) in Eq. (15). This effect only slightly suppresses \( C_{\ell,i} \) at large scales \( \ell < 100 \). The theory and simulations agree well over a wide range of \( \ell \), but the simulations are slightly suppressed at small \( \ell < 10^3 \). This may be caused by the sample variance of the 10 maps (in other words, 10 maps may not be a sufficient number to measure the mean of \( C_{\ell,i} \) precisely). The peak of \( C_{\ell,i} \) in this figure roughly corresponds to the peak of \( k^2P_c(k) \) in Fig. 3 via \( \ell_{\text{peak}} \approx k_{\text{peak}} \ell = 2000 \, [k_{\text{peak}}/(2 \, \text{h} \, \text{Mpc}^{-1})] \, [\pi/(1 \times 10^{-3} \, \text{Gpc})] \) from Eq. (15).

In actual measurements of \( C_{\ell,i} \), as indicated by several authors.
Figure 9. Probability distributions of the DM for several source redshifts $z_s$, as measured from the mock maps. The purple, blue and green histograms are the TNG300-1, -2 and -3 results. The vertical dashed line denotes the mean DM measured from TNG300-1. The red curves are log-normal distributions. The vertical axis is in arbitrary units.

Figure 10. Comparison of probability distributions of the DM at $z_s = 0.5$. The purple histogram is the TNG300-1 result, while the curves denote previous fitting formulas: Das & Ostriker (2006, dashed red), Dolag et al. (2015, dotted brown) and Macquart et al. (2020, dash-dotted orange) in subsection 5.2.

Figure 11. Source redshift inferred from the measured DM. The symbols with $1\sigma$ errorbars are the simulation results. The dash-dotted orange curve is the analytical DM–$z_s$ relation (7). The solid (dashed) red curve is the fit to the mean ($1\sigma$ error bars), given in Eq. (25).

Figure 14 plots the contribution from different redshifts to $C_\ell$. At smaller (larger) multipoles $\ell$, nearby (distant) structures mainly contribute to $C_\ell$ because they appear larger (smaller) in the sky. Especially for $\ell \gtrsim 10$, local structures at $z \lesssim 0.01$ (corresponding to $r \lesssim 30 \ h^{-1} \text{ Mpc}$) mainly determine the signal.

We comment that the analytical prediction of $C_\ell$ is less accurate for very small $\ell (< 10)$, because it was derived under the flat-sky approximation. The accuracy of the Limber and flat-sky approximations for projected galaxy clustering and weak-lensing statistics has been discussed in, e.g., Kilbinger et al. (2017) and Fang et al. (2020). Further studies are necessary to estimate the accuracy of these approximations in the DM statistics.
5.5 Angular correlation function of the DM

From Eq. (10), the correlation function in the i-th map is given by

$$\xi_i(\theta; z_s) = \frac{1}{N_{\text{pair}}(\theta_i, \theta_j) \Delta \theta} \sum_{\theta_j \in \theta_i} \frac{\sum_{j=1}^{N_{\text{pair}}} (\mathbf{DM}(\theta_j; z_s) - \overline{DM}(z_s))^2}{\sigma^2_{DM,host}(n_{host})}$$

where the summation is done in the range $\theta - \Delta \theta/2 < |\theta_1 - \theta_2| < \theta + \Delta \theta/2$, and $N_{\text{pair}}$ is the number of DM pairs in the bin-width $\Delta \log_{10} \theta = 0.1$. The mean DM is estimated from the 10 maps. Similarly to $C_F$, we measured $\xi_i(\theta)$ for each of the 10 maps to estimate its mean and variance among the maps. We comment that
$C_T$ and $\xi(\theta)$ are not independent but rather are related via the Fourier transform.

Figure 15 plots the angular correlation functions at $z_s = 0.4$ and 1. The simulation results for the mean and standard deviations are obtained from the 10 mock maps. The standard deviations increase near the scale of the survey area ($\approx 6$ deg) because the number of independent DM pairs decreases. The solid red curves are the analytical mean (11). The dashed red curves include the effect of the finite simulation box, discussed in subsection 5.4. The theory agrees well with the simulations but slightly overestimates them at $\theta \approx 1$ deg and $z_s = 0.4$. This discrepancy may be caused by the small survey area, because such a large-scale signal is mainly determined by the finite-simulation-box effect discussed in subsection 5.4.

Finally, Figure 16 plots the theoretical correlation functions for a full-sky measurement. The solid red curves are the theory, and the orange-dashed lines are a power-law fit:

$$\xi(\theta; z_s) \approx 2400 (\frac{\theta}{\text{deg}})^{-1} \text{pc}^2 \text{cm}^{-6} \quad \text{for } \theta \geq 1 \text{ deg}. \quad (29)$$

We checked that this fit works well at $z_s \geq 0.3$ (i.e., it is insensitive to $z_s$, because such a large-scale signal is mainly determined by nearby structures, as shown in Fig. 14). Note that Eq. (29) simply scales as $\propto (f_e/0.95)^2$ for an arbitrary $f_e$. The shaded grey regions represent the standard deviation under the assumption of Gaussian density fluctuations (which is valid in the large-scale limit).

In this case, the covariance between $\xi(\theta_1; z_s)$ and $\xi(\theta_2; z_s)$ is given by (Joachimi et al. 2008)

$$\text{Cov} [\xi(\theta_1; z_s)\xi(\theta_2; z_s)] = \frac{1}{S_W} \int_0^\infty d\ell \frac{\ell}{\pi} J_0(\ell \theta_1) J_0(\ell \theta_2) \left[ C_T(z_s) + C_{\text{shot}} \right]^2, \quad (30)$$

where $S_W$ is the survey area in steradians. The covariance is determined by the survey area at all scales and by the shot noise at small scales. The diagonal element (i.e., $\theta_1 = \theta_2$) corresponds to the variance. In this plot, the shot noise of the host galaxies is considered for the same three cases as in Fig. 13. The dotted black curves denote the standard deviation without the shot noise. This figure suggests that the density $n_{\text{host}} \approx 10 \text{deg}^{-2}$ is high enough to neglect the shot noise in the plotted range ($\theta > 0.1$ deg). For $n_{\text{host}} > 0.1 \text{deg}^{-2}$, the shot noise affects the standard deviation even at very large scales ($\theta \geq 10$ deg). At small scales, non-Gaussian fluctuations become important, and thus, the analytical prediction (30) underestimates the results. On larger scales, the standard deviation increases because there are fewer independent DM pairs in the full sky (i.e., the large-scale signal is limited by the cosmic variance). The down arrows in the lower panel denote the average angular separation of the host galaxies (i.e., $n_{\text{host}}^{-1}$) for $n_{\text{host}} = 0.1, 1$ and $10 \text{deg}^{-2}$ from right to left.
3(b), is orders of magnitude larger than our analytical expectation (29), although their error bars are still large. More FRB samples are required to determine whether their signal is of cosmological origin or not.

6 DISCUSSION

6.1 Host-galaxy contribution

So far we have not discussed the host-galaxy contribution, because there are two uncertainties in modelling it. First, the host-galaxy properties show significant diversity among the ~10 currently identified host galaxies (e.g. Tendulkar et al. 2017; Prochaska et al. 2019; Macquart et al. 2020). For instance, the repeating source FRB 121102 is located in a dwarf galaxy, while four other sources identified by ASKAP are in massive galaxies (Chatterjee et al. 2017; Bhandari et al. 2020). The spatial positions of the FRBs in their host galaxies also show variations from the centre to the outskirts. Second, the resolution of TNG300 is not fine enough to resolve the inner structure of a host galaxy. These large-box simulations are suitable for studying the cosmological distribution of free electrons, but to study the interiors of galaxies, finer-resolution (but smaller-box) runs—such as TNG50 and TNG100—are more suitable. Zhang et al. (2020) and Jaroszynski (2020) recently studied the host-galaxy contribution using TNG100.

When Pol et al. (2019) distributed the sources at a given $z_s$ in their DM simulation, they compared two cases: (i) the sources are distributed randomly, and (ii) its distribution is proportional to the local density contrast. They found that the latter significantly decreased the variance of the cosmological contribution, $\sigma^2_{\text{DM}}$. Their results suggest that $\sigma^2_{\text{DM}}$ also depends on host-galaxy properties such as its type (elliptical or spiral), mass or galaxy bias. More studies are needed on this topic, and we leave this for future work.

6.2 Comparison with other hydrodynamic simulations

Hydrodynamic simulations are the most reliable theoretical tool for studying the free-electron distribution. The cosmological DM has been studied using several simulations, such as Magneticum (Dolag et al. 2015), Illustris (Jaroszynski 2019) and TNG300 (this work). Although these previous results are fairly consistent with ours (see section 5), a more detailed quantitative comparison among various hydrodynamic simulations is desirable. The free-electron distribution in halos depends strongly on the stellar and AGN feedback model that expels internal gas to the outside of a galaxy.

Lim et al. (2020) recently studied the number-density profile of free electrons in halos with masses of $10^{12–14.5} M_\odot$ using three hydrodynamic simulations of Illustris, TNG300 and EAGLE (Schaye et al. 2015). These simulations show a discrepancy of ~30% at the halo radius $R_{500}$, as shown in their Figs. 5 and 6 (where $R_{500}$ is the radius within which the mean density is 500 times larger than the mean cosmological background density). The discrepancy is larger for a lower-mass halo, especially at smaller radius, because such halos are more sensitive to the feedback model. For instance, Illustris predicts a low inner profile due to strong feedback. In the halo model, $P_e(k)$ at small scales ($k \gtrsim 1 \text{ h Mpc}^{-1}$) is determined by the halo mass function and the free-electron density profile in the halos. Therefore, a similar level of discrepancy is probably present in $P_e(k)$.

Because the DM variance is sensitive to $k^2 P_e(k)$ around the peak ($k \approx 1–10 \text{ h Mpc}^{-1}$), the uncertainty in the feedback model may affect the variance. The angular power spectrum of the DM at small scales ($\ell > 10^3$) is also sensitive to the feedback. However, because its small-scale signal is strongly contaminated by the shot noise (see Fig. 13), the feedback effect will be difficult to observe in $C_\ell$.

7 CONCLUSIONS

We have investigated the basic statistics of the cosmological dispersion measure (DM) using the state-of-the-art hydrodynamic simulations, IllustrisTNG300. Our main purpose is to provide an analytical model for data analysis on the DM statistics.

First, we measured the free-electron fraction $f_e(z)$ and its power spectrum $P_e(k; z)$ from TNG300, which are key ingredients in the DM statistics. It turns out that $P_e(k; z)$ is consistent with the dark-matter-only power spectrum $P_{\text{dmo}}(k; z)$ at large scales ($k \lesssim 1 \text{ h Mpc}^{-1}$), but it is strongly suppressed at small scales ($k \gtrsim 1 \text{ h Mpc}^{-1}$) owing to stellar and AGN feedback. As a result, the free-electron fluctuations on scales $\approx 1 \text{ Mpc}$ contribute most to the DM variance (because $k^2 P_e(k; z)$ has a peak around that scale). To model $P_e(k; z)$, we introduced the free-electron bias factor defined by $b^2(k; z) = P_e(k; z)/P_{\text{dmo}}(k; z)$. We then provided simple fitting functions calibrated over a wide range of scales and epochs: $f_e(z)$ for $z = 0–8$ in Eq. (17) and $b_e(k; z)$ for $k < 10 \text{ h Mpc}^{-1}$ and $z = 0–5$ in Eq. (21). These fitting functions will be useful for future statistical analyses of the free-electron distribution.

Next, we prepared 10 mock sky maps of the DM using the TNG300 data, based on standard ray-tracing techniques. We then measured various DM statistics, such as its mean and variance, PDF of the DM, PDF of the source redshift $z_s$ for a given DM, angular power spectrum and angular correlation function. We calculated the analytical predictions using the fitting formulas for $f_e(z)$ and $P_e(k; z)$ and then validated them against the mock DM measurements. Basic statistics such as the mean, variance and PDF of the DM were consistent with previous work. The PDF of the DM is highly skewed, while the PDF of the $z_s$ is well approximated by a Gaussian. We provided a source redshift–DM relation—$z_s = z_{\text{DM}}(\text{DM})$ in Eq. (25)—which helps in identifying the host galaxies of FRBs from the measured DMs. The angular correlation function was also computed in subsection 5.5, and we expect it to be detected when thousands of FRBs are available in the coming years.

Throughout this paper, we compared the TNG300 results with three resolution runs to see the numerical convergence. We confirmed that our conclusions do not depend on the resolution, because all the runs resolve the dominant length scale of the free-electron fluctuations ($\approx 1 \text{ Mpc}$) sufficiently. Even so, because the gas distribution in halos is sensitive to the feedback model, quantitative comparisons with other hydrodynamic simulations are required for further systematic checks. The presented analytical model for the DM statistics will be updated easily by re-calibrating the fitting functions for $f_e(z)$ and $P_e(k; z)$ using more accurate future hydrodynamic simulations.

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