ON SOLVABLE SUBGROUPS OF THE CREMONA GROUP

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Abstract. The Cremona group Bir(\(\mathbb{P}^2\)) is the group of birational self-maps of \(\mathbb{P}^2\). Using the action of Bir(\(\mathbb{P}^2\)) on the Picard-Manin space of \(\mathbb{P}^2\), we characterize its solvable subgroups. If \(G \subset \text{Bir}(\mathbb{P}^2)\) is solvable, nonvirtually Abelian, and infinite, then up to finite index: either any element of \(G\) is of finite order or conjugate to an automorphism of \(\mathbb{P}^2\), or \(G\) preserves a unique fibration that is rational or elliptic, or \(G\) is, up to conjugacy, a subgroup of the group generated by one hyperbolic monomial map and the diagonal automorphisms.

We also give some corollaries.

1. Introduction

We know properties on finite subgroups ([16]), finitely generated subgroups ([6]), uncountable maximal Abelian subgroups ([13]), nilpotent subgroups ([14]) of the Cremona group. In this article, we focus on solvable subgroups of the Cremona group.

Let \(G\) be a group. Recall that \([g, h] = ghg^{-1}h^{-1}\) denotes the commutator of \(g\) and \(h\). If \(\Gamma_1\) and \(\Gamma_2\) are two subgroups of \(G\), then \([\Gamma_1, \Gamma_2]\) is the subgroup of \(G\) generated by the elements of the form \([g, h]\) with \(g \in \Gamma_1\) and \(h \in \Gamma_2\). We define the derived series of \(G\) by setting

\[G^{(0)} = G, \quad G^{(n+1)} = [G^{(n)}, G^{(n)}] \quad \forall n \geq 0.\]

The soluble length \(\ell(G)\) of \(G\) is defined by

\[\ell(G) = \min\{k \in \mathbb{N} \cup \{0\} | G^{(k)} = \{\text{id}\}\}\]

with the convention: \(\min \emptyset = \infty\). We say that \(G\) is soluble if \(\ell(G) < \infty\). The study of soluble groups started a long time ago, and any linear soluble...
subgroup is up to finite index triangularizable (Lie–Kolchin theorem, [23, Theorem 21.1.5]). The assumption “up to finite index” is essential: for instance, the subgroup of PGL(2, C) generated by \(
abla = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(
abla = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) is isomorphic to \(S_3\) so is solvable but is not triangularizable.

**Theorem A.** Let \(G\) be an infinite, solvable, non virtually Abelian subgroup of Bir\((\mathbb{P}_2^2)\). Then, up to finite index, one of the following holds:

1. any element of \(G\) is either of finite order, or conjugate to an automorphism of \(\mathbb{P}_2^2\);
2. \(G\) preserves a unique fibration that is rational, in particular \(G\) is, up to conjugacy, a subgroup of \(\text{PGL}(2, \mathbb{C}(y)) \rtimes \text{PGL}(2, \mathbb{C})\);
3. \(G\) preserves a unique fibration that is elliptic;
4. \(G\) is, up to birational conjugacy, contained in the group generated by \(\left\{(x^p y^q, x^r y^s), (\alpha x, \beta y)\mid \alpha, \beta \in \mathbb{C}^* \right\}\),

where \(M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\) denotes an element of \(\text{GL}(2, \mathbb{Z})\) with spectral radius \(> 1\).

The group \(G\) preserves the two holomorphic foliations defined by the 1-forms \(\alpha_1 x dy + \beta_1 y dx\) and \(\alpha_2 x dy + \beta_2 y dx\) where \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) denote the eigenvectors of \(^t M\).

Furthermore if \(G\) is uncountable, case 3. does not hold.

**Examples.**

- Denote by \(S_3\) the group generated by \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\). As we recall before \(S_3 \cong S_3\). Consider now the subgroup \(G\) of Bir\((\mathbb{P}_2^2)\) whose elements are the monomial maps \((x^p y^q, x^r y^s)\) with \(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\) \(\in S_3\). Then any element of \(G\) has finite order, and \(G\) is solvable; it gives an example of case 1.
- Other examples that illustrate case 1. are the following groups

\[
\{(\alpha x + \beta y + \gamma, \delta y + \varepsilon)\mid \alpha, \delta \in \mathbb{C}^*, \beta, \gamma, \varepsilon \in \mathbb{C} \} \subset \text{Aut}(\mathbb{P}_2^2),
\]

and

\[
E = \{(\alpha x + P(y), \beta y + \gamma)\mid \alpha, \beta \in \mathbb{C}^*, \gamma, P \in \mathbb{C}[y] \} \subset \text{Aut}(\mathbb{C}^2).
\]

- The centralizer of a birational map of \(\mathbb{P}_2^2\) that preserves a unique fibration that is rational is virtually solvable ([9, Corollary C]); this example falls in case 2 (see Section 3.2).
- In [10, Proposition 2.2] Cornulier proved that the group

\[
\langle (x + 1, y), (x, y + 1), (x, xy) \rangle
\]

is solvable of length 3, and is not linear over any field; this example falls in case 2. The invariant fibration is given by \(x = \text{cst}\).

**Remark.** In case 1 if there exists an integer \(d\) such that \(\deg \phi \leq d\) for any \(\phi \in G\), then there exist a smooth projective variety \(M\) and a birational map \(\psi : M \to \mathbb{P}_2^2\) such that \(\psi^{-1} G \psi\) is a solvable subgroup of Aut\((M)\) (see Section 3.3). But there is some solvable subgroups \(G\) with only elliptic ele-
ments that do not satisfy this property: the group $E$ introduced in Examples. Let us mention another example: Wright constructs Abelian subgroups $H$ of $\text{Aut}(\mathbb{C}^2)$ such that any element of $H$ is of finite order, $H$ is unbounded and does not preserve any fibration ([28]).

In Section 3, we prove Theorem A: we first assume that our solvable, infinite and non virtually Abelian subgroup $G$ contains a hyperbolic map, then that it contains a twist and no hyperbolic map, and finally that all elements of $G$ are elliptic. In the last section (Section 4), we also

- recover the following fact: if $G$ is an infinite nilpotent subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$, then $G$ does not contain a hyperbolic map;
- remark that we can bound the soluble length of a nilpotent subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ by the dimension of $\mathbb{P}^2_\mathbb{C}$ as Epstein and Thurston did in the context of Lie algebras of rational vector fields on a connected complex manifold;
- give a negative answer to the following question of Favre: does any solvable and finitely generated subgroup $G$ of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ contain a subgroup of finite index whose commutator subgroup is nilpotent? if we assume that $[G,G]$ is not a torsion group;
- give a description of the embeddings of the solvable Baumslag–Solitar groups into the Cremona group.

### 2. Some properties of the birational maps

**First definitions.** Let $S$ be a projective surface. We will denote by $\text{Bir}(S)$ the group of birational self-maps of $S$; in the particular case of the complex projective plane the group $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ is called *Cremona group*. Take $\phi$ in $\text{Bir}(S)$, we will denote by $\text{Ind}_\phi$ the set of points of indeterminacy of $\phi$; the codimension of $\text{Ind}_\phi$ is $\geq 2$.

A birational map from $\mathbb{P}^2_\mathbb{C}$ into itself can be written

$$(x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),$$

where the $\phi_i$’s denote some homogeneous polynomials of the same degree and without common factors of positive degree. The *degree* of $\phi$ is equal to the degree of the $\phi_i$’s. Let $\phi$ be a birational map of $\mathbb{P}^2_\mathbb{C}$. One can define the *dynamical degree* of $\phi$ as

$$\lambda(\phi) = \lim_{n \to +\infty} \left(\deg \phi^n \right)^{1/n}.$$ 

More generally, let $S$ be a projective surface, and $\phi : S \to S$ be a birational map. Take any norm $\|\cdot\|$ on the Néron–Severi real vector space $\text{N}^1(S)$. If $\phi^*$ is the induced action by $\phi$ on $\text{N}^1(S)$, we can define

$$\lambda(\phi) = \lim_{n \to +\infty} \| (\phi^n)^* \|^{1/n}.$$ 

Remark that this quantity is a birational invariant: if $\psi : S \to S'$ is a birational map, then $\lambda(\psi \phi \psi^{-1}) = \lambda(\phi)$. 
Classification of birational maps. The algebraic degree is not a birational invariant, but the first dynamical degree is; more precisely one has a classification of birational maps based on the degree growth. Before stating it, let us first introduce the following definitions. Let $\phi$ be an element of $\text{Bir}(\mathbb{P}^2_{\mathbb{C}})$. If
- $(\deg \phi^n)_{n \in \mathbb{N}}$ is bounded, we say that $\phi$ is an elliptic map,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows linearly, we say that $\phi$ is a Jonquières twist,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows quadratically, we say that $\phi$ is a Halphen twist,
- $(\deg \phi^n)_{n \in \mathbb{N}}$ grows exponentially, we say that $\phi$ is a hyperbolic map.

Theorem 2.1 ([15], [20], [3]). Let $\phi$ be an element of $\text{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Then one and only one of the following cases holds
- $\phi$ is elliptic, furthermore if $\phi$ is of infinite order, then $\phi$ is up to birational conjugacy an automorphism of $\mathbb{P}^2_{\mathbb{C}}$,
- $\phi$ is a Jonquières twist, $\phi$ preserves a unique fibration that is rational and every conjugate of $\phi$ is not an automorphism of a projective surface,
- $\phi$ is a Halphen twist, $\phi$ preserves a unique fibration that is elliptic and $\phi$ is conjugate to an automorphism of a projective surface,
- $\phi$ is a hyperbolic map.

In the three first cases, $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$.

The Picard–Manin and bubble spaces. Let $S$, and $S_i$ be complex projective surfaces. If $\pi: S_1 \to S$ is a birational morphism, one gets $\pi^*: N^1(S) \to N^1(S_1)$ an embedding of Néron–Severi groups. Take two birational morphisms $\pi_1: S_1 \to S$ and $\pi_2: S_2 \to S$; the morphism $\pi_2$ is above $\pi_1$ if $\pi_1^{-1}\pi_2$ is regular. Starting with two birational morphisms one can always find a third one that covers the two first. Therefore, the inductive limit of all groups $N^1(S_i)$ for all surfaces $S_i$ above $S$ is well-defined; it is the Picard-Manin space $Z_S$ of $S$. For any birational map $\pi$, $\pi^*$ preserves the intersection form and maps nef classes to nef classes hence the limit space $Z_S$ is endowed with an intersection form of signature $(1, \infty)$ and a nef cone.

Let $S$ be a complex projective surface. Consider all complex and projective surfaces $S_i$ above $S$, that is all birational morphisms $\pi_i: S_i \to S$. If $p$ (resp. $q$) is a point of a complex projective surface $S_1$ (resp. $S_2$), and if $\pi_1: S_1 \to S$ (resp. $\pi_2: S_2 \to S$) is a birational morphism, then $p$ is identified with $q$ if $\pi_1^{-1}\pi_2$ is a local isomorphism in a neighborhood of $q$ that maps $q$ onto $p$. The bubble space $B(S)$ is the union of all points of all surfaces above $S$ modulo the equivalence relation induced by this identification. If $p$ belongs to $B(S)$ represented by a point $p$ on a surface $S_i \to S$, denote by $E_p$ the exceptional divisor of the blow-up of $p$ and by $e_p$ its divisor class viewed as a point in $Z_S$. The following properties are satisfied

\[
\begin{align*}
  e_p \cdot e_q &= 0 & &\text{if } p \neq q, \\
  e_p \cdot e_p &= -1.
\end{align*}
\]
Hyperbolic space $\mathbb{H}_S$. Embed $N^1(S)$ as a subgroup of $Z_S$; this finite dimensional lattice is orthogonal to $e_p$ for any $p \in B(S)$, and

$$Z_S = \left\{ D + \sum_{p \in B(S)} a_p e_p \middle| D \in N^1(S), a_p \in \mathbb{R} \right\}.$$ 

The completed Picard-Manin space $\overline{Z}_S$ of $S$ is the $L^2$-completion of $Z_S$; in other words

$$\overline{Z}_S = \left\{ D + \sum_{p \in B(S)} a_p e_p \middle| D \in N^1(S), a_p \in \mathbb{R}, \sum a^2_p < +\infty \right\}.$$ 

The intersection form extends as an intersection form with signature $(1, \infty)$ on $\overline{Z}_S$. Let

$$\overline{Z}_S^+ = \{ d \in \overline{Z}_S | d \cdot c \geq 0 \ \forall c \in \overline{Z}_S \}$$ 

be the nef cone of $\overline{Z}_S$ and

$$\mathcal{L}\overline{Z}_S = \{ d \in \overline{Z}_S | d \cdot d = 0 \}$$ 

be the light cone of $\overline{Z}_S$.

The hyperbolic space $\mathbb{H}_S$ of $S$ is then defined by

$$\mathbb{H}_S = \{ d \in \overline{Z}_S^+ | d \cdot d = 1 \}.$$ 

Let us remark that $\mathbb{H}_S$ is an infinite dimensional analogue of the classical hyperbolic space $\mathbb{H}^n$. The distance on $\mathbb{H}_S$ is defined by

$$\cosh(\text{dist}(d, d')) = d \cdot d' \ \forall d, d' \in \mathbb{H}_S.$$ 

The geodesics are intersections of $\mathbb{H}_S$ with planes. The projection of $\mathbb{H}_S$ onto $\mathbb{P}(\overline{Z}_S)$ is one-to-one, and the boundary of its image is the projection of the cone of isotropic vectors of $\overline{Z}_S$. Hence

$$\partial \mathbb{H}_S = \{ \mathbb{R}^+ d | d \in \overline{Z}_S^+, d \cdot d = 0 \}.$$ 

Isometries of $\mathbb{H}_S$. If $\pi: S' \to S$ is a birational morphism, we get a canonical isometry $\pi^*$ (and not only an embedding) between $\mathbb{H}_S$ and $\mathbb{H}_{S'}$. This allows to define an action of $\text{Bir}(S)$ on $\mathbb{H}_S$. Consider a birational map $\phi$ on a complex projective surface $S$. There exists a surface $S'$, and $\pi_1: S' \to S$, $\pi_2: S' \to S$ two morphisms such that $\phi = \pi_2 \pi_1^{-1}$. One can define $\phi_*$ by

$$\phi_* = (\pi^*_{2})^{-1} \pi^*_1;$$

in fact, one gets a faithful representation of $\text{Bir}(S)$ into the group of isometries of $\mathbb{H}_S$ (see [6]).

The isometries of $\mathbb{H}_S$ are classified in three types ([4], [19]). The translation length of an isometry $\phi_*$ of $\mathbb{H}_S$ is defined by

$$L(\phi_*) = \inf \{ \text{dist}(p, \phi_*(p)) | p \in \mathbb{H}_S \}.$$ 

If the infimum is a minimum, then
either it is equal to 0 and \( \phi \) has a fixed point in \( \mathbb{H}_S \), \( \phi \) is thus elliptic,
— or it is positive and \( \phi \) is hyperbolic. Hence, the set of points \( p \in \mathbb{H}_S \) such that \( \text{dist}(p, \phi(p)) \) is equal to \( L(\phi) \) is a geodesic line \( \text{Ax}(\phi) \subset \mathbb{H}_S \). Its boundary points are represented by isotropic vectors \( \omega(\phi) \) and \( \alpha(\phi) \) in \( \mathbb{Z}_S \) such that
\[
\phi(\omega(\phi)) = \lambda(\phi) \omega(\phi), \quad \phi(\alpha(\phi)) = \frac{1}{\lambda(\phi)} \alpha(\phi).
\]
The axis \( \text{Ax}(\phi) \) of \( \phi \) is the intersection of \( \mathbb{H}_S \) with the plane containing \( \omega(\phi) \) and \( \alpha(\phi) \); furthermore, \( \phi \) acts as a translation of length \( L(\phi) = \log \lambda(\phi) \) along \( \text{Ax}(\phi) \) (see [8, Remark 4.5]). For all \( p \in \mathbb{H}_S \) one has
\[
\lim_{k \to +\infty} \frac{\phi^{-k}(p)}{\lambda(\phi)} = \alpha(\phi), \quad \lim_{k \to +\infty} \frac{\phi^k(p)}{\lambda(\phi)} = \omega(\phi).
\]
When the infimum is not realized, \( L(\phi) = 0 \) and \( \phi \) is parabolic: \( \phi \) fixes a unique line in \( \mathcal{LZ}_S \); this line is fixed pointwise, and all orbits \( \phi^n(p) \) in \( \mathbb{H}_S \) accumulate to the corresponding boundary point when \( n \) goes to \( \pm \infty \).

There is a strong relationship between this classification and the classification of birational maps of the complex projective plane ([6, Theorem 3.6]): if \( \phi \) is an element of Bir(\( \mathbb{P}_2^C \)), then
- \( \phi \) is an elliptic isometry if and only if \( \phi \) is an elliptic map;
- \( \phi \) is a parabolic isometry if and only if \( \phi \) is a twist;
- \( \phi \) is a hyperbolic isometry if and only if \( \phi \) is a hyperbolic map.

**Tits alternative.** Cantat proved the Tits alternative for the Cremona group ([6, Theorem C]): let \( G \) be a finitely generated subgroup of Bir(\( \mathbb{P}_2^C \)), then
- either \( G \) contains a free non-Abelian subgroup,
- or \( G \) contains a subgroup of finite index that is solvable.

As a consequence, he studied finitely generated and solvable subgroups of Bir(\( \mathbb{P}_2^C \)) without torsion ([6, Theorem 7.3]): let \( G \) be such a group, there exists a subgroup \( G_0 \) of \( G \) of finite index such that
- either \( G_0 \) is Abelian,
- or \( G_0 \) preserves a foliation.

### 3. Proof of Theorem A

#### 3.1. Solvable groups of birational maps containing a hyperbolic map.**

Let us recall the following criterion (for its proof see, for example, [11]) used on many occasions by Klein, and also by Tits ([26]) known as Ping-Pong Lemma: let \( H \) be a group acting on a set \( X \), let \( \Gamma_1, \Gamma_2 \) be two subgroups of \( H \), and let \( \Gamma \) be the subgroup generated by \( \Gamma_1 \) and \( \Gamma_2 \). Assume that \( \Gamma_1 \) contains at least three elements, and \( \Gamma_2 \) at least two elements. Suppose that there exist two non-empty subsets \( X_1, X_2 \) of \( X \) such that
\[
X_2 \not\subseteq X_1, \quad \gamma(X_2) \subset X_1 \quad \forall \gamma \in \Gamma_1 \setminus \{\text{id}\}, \quad \gamma'(X_1) \subset X_2 \quad \forall \gamma' \in \Gamma_2 \setminus \{\text{id}\}.
\]

Then \( \Gamma \) is isomorphic to the free product \( \Gamma_1 \ast \Gamma_2 \). The Ping-Pong argument allows us to prove the following.

**Lemma 3.1.** A solvable, non-Abelian subgroup of \( \text{Bir}(\mathbb{P}_C^2) \) cannot contain two hyperbolic maps \( \phi \) and \( \psi \) such that \( \{ \omega(\phi \cdot \cdot), \alpha(\phi \cdot \cdot) \} \neq \{ \omega(\psi \cdot \cdot), \alpha(\psi \cdot \cdot) \} \).

**Proof.** Assume by contradiction that \( \{ \omega(\phi \cdot \cdot), \alpha(\phi \cdot \cdot) \} \neq \{ \omega(\psi \cdot \cdot), \alpha(\psi \cdot \cdot) \} \). Then the Ping-Pong argument implies that there exist two integers \( n \) and \( m \) such that \( \psi^n \) and \( \phi^m \) generate a subgroup of \( G \) isomorphic to the free group \( F_2 \) (see [6]). But \( \langle \phi, \psi \rangle \) is a solvable group: contradiction. \( \square \)

Let \( G \) be an infinite solvable, non-virtually Abelian, subgroup of \( \text{Bir}(\mathbb{P}_C^2) \). Assume that \( G \) contains a hyperbolic map \( \phi \). Let \( \alpha(\phi \cdot \cdot) \) and \( \omega(\phi \cdot \cdot) \) be the two fixed points of \( \phi \cdot \cdot \) on \( \partial \mathbb{H}_{\mathbb{P}_C^2} \), and \( \text{Ax}(\phi \cdot \cdot) \) be the geodesic passing through these two points. As \( G \) is solvable there exists a subgroup of \( G \) of index \( \leq 2 \) that preserves \( \alpha(\phi \cdot \cdot), \omega(\phi \cdot \cdot), \) and \( \text{Ax}(\phi \cdot \cdot) \) (see [6, Theorem 6.4]); let us still denote by \( G \) this subgroup. Note that there is no twist in \( G \) since a parabolic isometry has a unique fixed point on \( \partial \mathbb{H}_{\mathbb{P}_C^2} \). One has a morphism \( \kappa: G \to \mathbb{R}_{>0} \) such that

\[
\psi_\cdot(\ell) = \kappa(\psi) \ell
\]

for any \( \ell \) in \( \mathbb{Z}_{\mathbb{P}_C^2} \) lying on \( \text{Ax}(\phi \cdot \cdot) \).

The kernel of \( \kappa \) is an infinite subgroup that contains only elliptic maps. Indeed the set of elliptic elements of \( G \) coincides with \( \ker \kappa \); and \( [G, G] \subset \ker \kappa \) so if \( \ker \kappa \) is finite, \( G \) is Abelian up to finite index which is by assumption impossible.

**Gap property.** If \( \psi \) is a hyperbolic birational map of \( G \), then \( \kappa(\psi) = L(\psi \cdot \cdot) = \log \lambda(\psi) \). Recall that \( \lambda(\phi) \) is an algebraic integer with all Galois conjugates in the unit disk, that is a Salem number, or a Pisot number. The smallest known number is the Lehmer number \( \lambda_L \approx 1.176 \) which is a root of \( X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1 \). Blanc and Cantat prove in [2, Corollary 2.7] that there is a gap in the dynamical spectrum \( \Lambda = \{ \lambda(\phi) | \phi \in \text{Bir}(\mathbb{P}_C^2) \} \): there is no dynamical degree in \( [1, \lambda_L] \).

The gap property implies that in fact \( \kappa \) is a morphism from \( G \) to a subgroup of \( \mathbb{R}_{>0} \) isomorphic to \( \mathbb{Z} \).

**Elliptic subgroups of the Cremona group with a large normalizer.** Consider in \( \mathbb{P}_C^2 \) the complement of the union of the three lines \( \{ x = 0 \}, \{ y = 0 \} \) and \( \{ z = 0 \} \). Denote by \( U \) this open set isomorphic to \( \mathbb{C}^* \times \mathbb{C}^* \). One has an action of \( \mathbb{C}^* \times \mathbb{C}^* \) on \( U \) by translation. Furthermore \( \text{GL}(2, \mathbb{Z}) \) acts on \( U \) by monomial maps

\[
\begin{bmatrix} p & q \\ r & s \end{bmatrix} \mapsto ((x, y) \mapsto (x^{p}y^{q}, x^{r}y^{s})).
\]

One thus has an injective morphism from \( (\mathbb{C}^* \times \mathbb{C}^*) \rtimes \text{GL}(2, \mathbb{Z}) \) into \( \text{Bir}(\mathbb{P}_C^2) \). Let \( G_{\text{toric}} \) be its image.
One can now apply [12, Theorem 4] that says that if there exists a short exact sequence
\[ 1 \to A \to N \to B \to 1, \]
where \( N \subset \text{Bir}(\mathbb{P}^2_C) \) contains at least one hyperbolic element, and \( A \subset \text{Bir}(\mathbb{P}^2_C) \) is an infinite and elliptic\(^1\) group, then \( N \) is up to conjugacy a subgroup of \( G_{\text{toric}} \). Hence, up to birational conjugacy \( G \subset G_{\text{toric}} \). Recall now that if \( \psi \) is a hyperbolic map of the form \( (x^a y^b, x^c y^d) \), then to preserve \( \alpha(\psi) \) and \( \omega(\psi) \) is equivalent to preserve the eigenvectors of the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). We can now thus state:

**Proposition 3.2.** Let \( G \) be an infinite solvable, non-virtually Abelian, subgroup of \( \text{Bir}(\mathbb{P}^2_C) \). If \( G \) contains a hyperbolic birational map, then \( G \) is, up to conjugacy and finite index, a subgroup of the group generated by
\[ \{ (x^p y^q, x^r y^s), (\alpha x, \beta y)|\alpha, \beta \in \mathbb{C}^* \}, \]
where \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) denotes an element of \( \text{GL}(2, \mathbb{Z}) \) with spectral radius \( > 1 \).

### 3.2. Solvable groups with a twist.
Consider a solvable, non-Abelian subgroup \( G \) of \( \text{Bir}(\mathbb{P}^2_C) \). Let us assume that \( G \) contains a twist \( \phi \); the map \( \phi \) preserves a unique fibration \( \mathcal{F} \) that is rational or elliptic. Let us prove that any element of \( G \) preserves \( \mathcal{F} \). Denote by \( \alpha(\phi) \in \partial \mathbb{H}_{\mathbb{P}^2_C} \) the fixed point of \( \phi \). Take one element in \( L \mathbb{Z}_{\mathbb{P}^2_C} \) still denoted \( \alpha(\phi) \) that represents \( \alpha(\phi) \). Take \( \varphi \in G \) such that \( \varphi(\alpha(\phi)) \neq \alpha(\phi) \). Then \( \psi = \varphi \phi \varphi^{-1} \) is parabolic and fixes the unique element \( \alpha(\psi) \) of \( L \mathbb{Z}_{\mathbb{P}^2_C} \) proportional to \( \varphi(\alpha(\phi)) \). Take \( \varepsilon > 0 \) such that \( U(\alpha(\phi), \varepsilon) \cap U(\alpha(\psi), \varepsilon) = \emptyset \) where
\[ U(\alpha, \varepsilon) = \{ \ell \in L \mathbb{Z}_{\mathbb{P}^2_C} | \alpha \cdot \ell < \varepsilon \}. \]
Since \( \psi \) is parabolic, then for \( n \) large enough \( \psi^n(U(\alpha(\phi), \varepsilon)) \) is included in a \( U(\alpha(\psi), \varepsilon) \). For \( m \) sufficiently large \( \phi^m \psi^n(U(\alpha(\phi), \varepsilon)) \subset (U(\alpha(\phi), \varepsilon/2)) \subset U(\alpha(\psi), \varepsilon) \); hence \( \phi^m \psi^n \) is hyperbolic. You can by this way build two hyperbolic maps whose sets of fixed points are distinct: this gives a contradiction with Lemma 3.1. So for any \( \varphi \in G \) one has: \( \alpha(\phi) = \alpha(\varphi) \); one can thus state the following result.

**Proposition 3.3.** Let \( G \) be a solvable, non-Abelian subgroup of \( \text{Bir}(\mathbb{P}^2_C) \) that contains a twist \( \phi \). Then
- if \( \phi \) is a Jonquières twist, then \( G \) preserves a rational fibration, that is up to birational conjugacy \( G \) is a subgroup of \( \text{PGL}(2, \mathbb{C}(y)) \times \text{PGL}(2, \mathbb{C}) \),
- if \( \phi \) is a Halphen twist, then \( G \) preserves an elliptic fibration.

**Remark 3.4.** Both cases are mutually exclusive.

\(^1\) A subgroup of \( \text{Bir}(\mathbb{P}^2_C) \) is elliptic if it fixes a point in \( \mathbb{H}_{\mathbb{P}^2_C} \).
Note that if $G$ is a subgroup of $\text{Bir}(\mathbb{P}^2_{\mathbb{C}})$ that preserves an elliptic fibration, then $G$ is countable ([5]). Let us explain briefly why. A smooth rational projective surface $S$ is a Halphen surface if there exists an integer $m > 0$ such that the linear system $|-mK_S|$ is of dimension 1, has no fixed component, and has no base point. The smallest positive integer for which $S$ satisfies such a property is the index of $S$. If $S$ is a Halphen surface of index $m$, then $K_S^2 = 0$ and, by the genus formula, the linear system $|-mK_S|$ defines a genus 1 fibration $\pi : S \to \mathbb{P}^1_{\mathbb{C}}$. This fibration is relatively minimal in the sense that there is no $(-1)$-curve contained in a fiber. The following properties are equivalent:

- $S$ is a Halphen surface of index $m$,
- there exists an irreducible pencil of curves of degree $3m$ with 9 base points of multiplicity $m$ in $\mathbb{P}^2_{\mathbb{C}}$ such that $S$ is the blow-up of the 9 base points and $|-mK_S|$ is the proper transform of this pencil (the base points set may contain infinitely near points).

As a corollary of the classification of relatively minimal elliptic surfaces the relative minimal model of a rational elliptic surface is a Halphen surface of index $m$ ([22, Chapter 2, Section 10]). Up to conjugacy $G$ is a subgroup of $\text{Aut}(S)$ where $S$ denotes a Halphen surface of index $m$. The action of $G$ on $\text{NS}(S)$ is almost faithful, and $G$ is a discrete (it preserves the integral structure of $\text{NS}(S)$) and virtually Abelian (it preserves the intersection form and the class of the elliptic fibration) subgroup of $\text{Aut}(S)$. So one has the following.

**Corollary 3.5.** If $G$ is an uncountable, solvable, non-Abelian subgroup of $\text{Bir}(\mathbb{P}^2_{\mathbb{C}})$, then $G$ doesn’t contain a Halphen twist.

**Example 3.6.** Let us come back to the example given in Section 1. If $\phi \in \text{Bir}(\mathbb{P}^2_{\mathbb{C}})$ preserves a unique fibration that is rational then one can assume that up to birational conjugacy this fibration is given, in the affine chart $z = 1$, by $y = \text{cst}$. If $\phi$ preserves $y = \text{cst}$ fiberwise, then

- $\phi$ is contained in a maximal Abelian subgroup denoted $\text{Ab}(\phi)$ that preserves $y = \text{cst}$ fiberwise ([13]),
- the centralizer of $\phi$ is a finite extension of $\text{Ab}(\phi)$ (see [9, Theorem B]).

This allows us to establish that if $\phi$ preserves a fibration not fiberwise, then the centralizer of $\phi$ is virtually solvable. For instance, if $\phi = (x + a(y), y + 1)$ (resp. $(b(y)x, \beta y)$ or $(x + a(y), \beta y)$ with $\beta \in \mathbb{C}^*$ of infinite order) preserves a unique fibration, then the centralizer of $\phi$ is solvable and metabelian ([9, Propositions 5.1 and 5.2]).

3.3. **Solvable groups with no hyperbolic map, and no twist.** Let $M$ be a smooth, irreducible, complex, projective variety of dimension $n$. Fix a Kähler form $\kappa$ on $M$. If $\ell$ is a positive integer, denote by $\pi_i : M^\ell \to M$ the projection onto the $i$th factor. The manifold $M^\ell$ is then endowed with the Kähler form $\sum_{i=1}^\ell \pi_i^* \kappa$ which induces a Kähler metric. To any $\phi \in \text{Bir}(M)$
one can associate its graph $\Gamma_\phi \subset M \times M$ defined as the Zariski closure of
$$\{(z,\phi(z)) \in M \times M | z \in M \setminus \text{Ind} \phi\}.$$ 
By construction $\Gamma_\phi$ is an irreducible subvariety of $M \times M$ of dimension $n$.
Both projections $\pi_1, \pi_2: M \times M \to M$ restrict to a birational morphism $\pi_1, \pi_2: \Gamma_\phi \to M$.

The total degree $\text{tdeg} \phi$ of $\phi \in \text{Bir}(M)$ is defined as the volume of $\Gamma_\phi$ with respect to the fixed metric on $M \times M$:
$$\text{tdeg} \phi = \int_{\Gamma_\phi} (\pi_1^* \kappa + \pi_2^* \kappa)^n = \int_{M \setminus \text{Ind} \phi} (\kappa + \phi^* \kappa)^n.$$

Let $d \geq 1$ be a natural integer, and set
$$\text{Bir}_d(M) = \{ \phi \in \text{Bir}(M) \mid \text{tdeg} \phi \leq d \}.$$ 
A subgroup $G$ of $\text{Bir}(M)$ has bounded degree if it is contained in $\text{Bir}_d(M)$ for some $d \in \mathbb{N}^*$.

Any subgroup $G$ of $\text{Bir}(M)$ that has bounded degree can be regularized, that is up to birational conjugacy all indeterminacy points of all elements of $G$ disappear simultaneously.

**Theorem 3.7 ([27]).** Let $M$ be a complex projective variety, and let $G$ be a subgroup of $\text{Bir}(M)$. If $G$ has bounded degree, there exists a smooth, complex, projective variety $M'$, and a birational map $\psi: M' \dashrightarrow M$ such that $\psi^{-1}G\psi$ is a subgroup of $\text{Aut}(M')$.

The proof of this result can be found in [21], [29]; an heuristic idea appears in [7].

4. Applications

4.1. Nilpotent subgroups of $\text{Bir}(\mathbb{P}^2_C)$. Let us recall that if $G$ is a group, the descending central series of $G$ is defined by
$$C^0G = G, \quad C^{n+1}G = [G, C^nG] \quad \forall n \geq 0.$$ 
We say that $G$ is nilpotent if there exists $j \geq 0$ such that $C^jG = \{\text{id}\}$. If $j$ is the minimum non-negative number with such a property, we say that $G$ is of nilpotent class $j$. Nilpotent subgroups of the Cremona group have been described:

**Theorem 4.1 ([14]).** Let $G$ be a nilpotent subgroup of $\text{Bir}(\mathbb{P}^2_C)$. Then
- either $G$ is up to finite index metabelian,
- or $G$ is a torsion group.

We find an alternative proof of [14, Lemma 4.2] for $G$ infinite:

**Lemma 4.2.** Let $G$ be an infinite, nilpotent, non-virtually Abelian subgroup of $\text{Bir}(\mathbb{P}^2_C)$. Then $G$ does not contain a hyperbolic map.
Proof. The group $G$ is also solvable. Assume by contradiction that $G$ contains a hyperbolic map; then according to Theorem A up to birational conjugacy and finite index there exists $\mathcal{Y} \subset \mathbb{C}^* \times \mathbb{C}^*$ infinite such that $G$ is generated by $\phi = (x^p y^q, x^r y^s)$ and
\[
\{(\alpha x, \beta y) | (\alpha, \beta) \in \mathcal{Y}\}.
\]
The group $C^1 G$ contains
\[
\{[\phi, (\alpha x, \beta y)] | (\alpha, \beta) \in \mathcal{Y}\} = \{(\alpha^{p-1} \beta^q x, \alpha^r \beta^s - 1 y) | (\alpha, \beta) \in \mathcal{Y}\}
\]
that is infinite since $\mathcal{Y}$ is infinite. Suppose that $C^i G$ contains the infinite set
\[
\{(\alpha^{\ell_i} \beta^{n_i} x, \alpha^{k_i} \beta^{m_i} y) | (\alpha, \beta) \in \mathcal{Y}\}
\]
($\ell_i$, $n_i$, $k_i$ and $m_i$ are some functions in $p, q, r$ and $s$); then $C^{i+1} G$ contains
\[
\{(\alpha^{(p-1)\ell_i + q n_i} \beta^{(p-1)k_i + q m_i} x, \alpha^{r \ell_i + q m_i} \beta^{r k_i + (s-1) n_i} y) | (\alpha, \beta) \in \mathcal{Y}\}
\]
that is still infinite. □

So any nilpotent and infinite subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ falls in case (1), (2), (3) of Theorem A. If it falls in case (2) or (3) then $G$ is virtually metabelian ([14, Proof of Theorem 1.1]). Finally if $G$ falls in case (1), we can prove as in [14] that either $G$ is a torsion group, or $G$ is virtually metabelian.

4.2. Soluble length of a nilpotent subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$. Let us recall the following statement due to Epstein and Thurston ([17]): let $M$ be a connected complex manifold. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of the complex vector space of rational vector fields on $M$. Then $\mathfrak{h}^{(n)} = \{0\}$ if $n \geq \dim M$; hence, the soluble length of $\mathfrak{h}$ is bounded by the dimension of $M$. We have a similar statement in the context of birational maps; indeed a direct consequence of Theorem 4.1 is the following property: let $G \subset \text{Bir}(\mathbb{P}^2_\mathbb{C})$ be a nilpotent subgroup of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ that is not a torsion group, then the soluble length of $G$ is bounded by the dimension of $\mathbb{P}^2_\mathbb{C}$.

4.3. Favre’s question. In [18], Favre asked few questions; among them there is the following: does any solvable, finitely generated subgroup $G$ of $\text{Bir}(\mathbb{P}^2_\mathbb{C})$ contain a subgroup $H$ of finite index such that $[H, H]$ is nilpotent? We will prove that the answer is no if $[G, G]$ is not a torsion group.

Take $G$ a solvable and finitely generated subgroup of the Cremona group; besides suppose that $[G, G]$ is not a torsion group. Assume that the answer of Favre’s question is yes. Up to finite index one can assume that $[G, G]$ is nilpotent. According to Theorem 4.1 the group $G^{(1)} = [G, G]$ is up to finite index metabelian; in other words up to finite index $G^{(2)} = [G^{(1)}, G^{(1)}]$ is Abelian and so $G^{(3)} = [G^{(2)}, G^{(2)}] = \{\text{id}\}$, that is, the soluble length of $G$ is bounded by $3$ up to finite index. Consider the subgroup
\[
\langle (x + y^2, y), (x(1 + y), y), (x, \frac{y}{1 + y}), (x, 2y) \rangle
\]
of Bir($\mathbb{P}^2_C$). It is solvable of length 4 (see [24]): contradiction.

4.4. Baumslag–Solitar groups. For any integers $m, n$ such that $mn \neq 0$, the Baumslag–Solitar group $BS(m;n)$ is defined by the following presentation

$$BS(m;n) = \langle r, s | rs^m r^{-1} = s^n \rangle.$$ 

In [3], we prove that there is no embedding of $BS(m;n)$ into Bir($\mathbb{P}^2_C$) as soon as $|n|, |m|$, and 1 are distinct; it corresponds exactly to the case $BS(m;n)$ is not solvable. Indeed $BS(m;n)$ is solvable if and only if $|m| = 1$ or $|n| = 1$ (see [25, Proposition A.6]).

Proposition 4.3. Let $\rho$ be an embedding of $BS(1;n) = \langle r, s | rsr^{-1} = s^n \rangle$, with $n \neq 1$, into the Cremona group. Then

- the image of $\rho$ doesn’t contain a hyperbolic map,
- and

$$\rho(s) = (x, y + 1), \quad \rho(r) = (\nu(x), n(y + a(x)))$$

with $\nu \in \text{PGL}(2, \mathbb{C})$ and $a \in \mathbb{C}(x)$.

Proof. According to [3, Proposition 6.2, Lemma 6.3] one gets that $\rho(s) = (x, y + 1)$ and $\rho(r) = (\nu(x), n(y + a(x)))$ for some $\nu \in \text{PGL}(2, \mathbb{C})$ and $a \in \mathbb{C}(x)$.

Furthermore, $\rho(s)$ can neither be conjugate to an automorphism of the form $(ax, \beta y)$ (see [1]), nor to a hyperbolic birational map of the form $(\gamma x^p y^q, \delta x^r y^s)$ with $\begin{bmatrix} p & q \\ s & r \end{bmatrix}$$ \in \text{GL}(2, \mathbb{Z})$ of spectral radius $> 1$. As a consequence, Proposition 3.2 implies that $\rho(BS(1;n))$ does not contain a hyperbolic birational map. \hfill $\Box$

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