Some Simple Predictions
from $E_{11}$ Symmetry

Peter West

Vienna, Preprint ESI 1505 (2004) 
August 12, 2004

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via anonymous ftp from FTP.ESI.AC.AT
or via WWW, URL: http://www.esi.ac.at
Some simple predictions from $E_{11}$ symmetry

Peter West
Department of Mathematics
King’s College, London WC2R 2LS, UK

and

Erwin Schrödinger International Institute for Mathematical Physics,
Boltzmanngasse 9,
1090 Wien, Austria

Abstract
The simplest consequences of the common $E_{11}$ symmetry of the eleven dimensional, IIA and IIB theories are derived and are shown to imply the known relations between these three theories.
1. Introduction

It has been argued that eleven dimensional supergravity [1] when suitably extended possess a non-linearly realised \(E_{11}\) symmetry [2]. Furthermore, both the IIA supergravity \([3]\) and IIB supergravity theories \([4,5]\), when suitably extended, should also possess a non-linearly realised \(E_{11}\) symmetry \([2,6]\). As explained below, the three different theories arise from the same underlying algebra due to the different possible embeddings in \(E_{11}\) of the sub-algebras that describe gravity \([2,6]\). Their common \(E_{11}\) symmetry can be exploited to find explicit relations between the eleven dimensional, IIA and IIB non-linearly realised theories \([7]\). Indeed, one can find a one to one correspondence between the fields that occur in any two of these three theories providing a very concrete idea of what M theory actually is \([7]\). In this paper we derive the simplest of these relations which are those that involve fields that are associated with the Cartan sub-algebra of \(E_{11}\). We recover the known relations between the eleven dimensional, IIA and IIB theories, when dimensionally reduced on a suitable torus, in a simple way. Originally these relations were found by using a mixture of string and solitonic properties \([8,9,10,11,12,13]\), but we will show that they follow from the way the sub-algebra associated with the gravity sectors of the different theories are embedded in \(E_{11}\). We also give an example of how the correspondence works for a field not associated with the Cartan sub-algebra and derive the effect of the Weyl transformations of the \(E_8^{+++}\) theory for the IIA and IIB theories.

A Kac-Moody algebra is specified by its Cartan matrix \(A_{ab}\) which by definition must satisfies the following properties:

\[
A_{aa} = 2, \quad (1.1)
\]

\[
A_{ab} \text{ for } a \neq b \text{ are negative integers or zero}, \quad (1.2)
\]

and

\[
A_{ab} = 0 \implies A_{ba} = 0 \quad (1.3)
\]

The index range \(a,b = 1,\ldots,r\) where \(r\) is the rank of the Kac-Moody algebra. The Kac-Moody algebra is formulated in terms of its Chevalley generators denoted by \(H_a, E_a, F_a, a = 1,\ldots,r\) which obey the Serre relations;

\[
[H_a, H_b] = 0, \quad [H_a, E_b] = A_{ab} E_b, \quad [H_a, F_b] = -A_{ab} F_b, \quad [E_a, F_b] = \delta_{ab} H_a, \quad (1.4)
\]

and

\[
[E_a, \ldots [E_a, E_b] \ldots] = 0, \quad [F_a, \ldots [F_a, F_b] \ldots] = 0 \quad (1.5)
\]

In equation (1.5) there are \(1 - A_{ab}\) \(E_a\)'s in the first equation and the same number of \(F_a\)'s in the second equation. Given the Cartan matrix \(A_{ab}\), one can construct the Kac-Moody algebra by taking the multiple commutators of the \(E_a\)'s, and separately the \(F_a\)'s, subject to the above Serre relations. We recognise the \(H_a\) generators as those of the Cartan sub-algebra and the \(E_a\)'s \((F_a\)'s) as the generators of the positive (negative) simple roots, hence, a Kac-Moody algebra is uniquely specified by its Cartan matrix \(A_{ab}\) which is also encoded in its Dynkin diagram. For the case when all the simple roots are of equal length, the Dynkin diagram consists of \(r\) dots labeled \(a = 1,\ldots,r\) with \(-A_{ab}\) lines between nodes labeled \(a\) and \(b\). The Cartan matrix can be expressed in terms of the simple roots.
$\alpha_a$ as $A_{ab} = 2\frac{\langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle}$. Although the construction of the Kac-Moody algebra is simple in theory, in practice it is difficult and the generators are not known explicitly for any Kac-Moody algebra except those that are affine, finite dimensional or for certain special algebras associated with string theory.

We invite the reader to draw the Dynkin diagram of $E_{11}$ by drawing ten nodes connected together by a single horizontal line. We label these nodes from left to right by the integers one to ten and then add a further node, labeled eleven, above node eight and attached by a single vertical line. This algebra is just one of a special class of algebras, called very extended algebras, which were studied in [14]. One begins with the Dynkin diagram of any finite dimensional semi-simple lie algebra $G$, and adds the affine node to find the affine algebra $G^+$. One then adds a further node (called the extended node) attached to the affine node by a single horizontal line and finally yet another node (called the very extended node) attached to the extended node by a single horizontal line. One denotes the very extended algebra by $G^{+++}$. Hence, we also denote $E_{11}$ as $E_{8}^{+++}$.

As the reader may readily verify, any Kac-Moody algebra is invariant under the Cartan involution which acts on the Chevalley generators as $E_a \mapsto -F_a$, $F_a \mapsto -E_a$, $H_a \mapsto -H_a$. The sub-algebra that is invariant under the Cartan involution is then generated by $E_a - F_a$.

We note that this sub-algebra contains none of the generators $H_a$ of the Cartan sub-algebra of the Kac-Moody algebra.

We now give a brief exposition of non-linear realisations so that the reader may appreciate the more general setting, although much of this is not required later on in the paper. A non-linear realisation is specified by an algebra together with a chosen local sub-algebra. In the case of interest in this paper these are $E_8^{+++}$ and, essentially, the Cartan invariant sub-algebra respectively. The non-linear realisation is then just a theory which is invariant under $g \mapsto g_0 gh$ where $g$ and $g_0$ belong to the group and $h$ to the local subgroup and also $g$ and $h$ are local. The meaning of local depends on how one encodes space-time and we refer the reader to reference [19] for a recent discussion of this point for the theories being considered here. However, in the conventional applications of non-linear realisations, and in essence also here, it means that $g$ and $h$, unlike $g_0$, depend on space-time. In general these invariances are not sufficient to determine the theory uniquely, but if the local sub-algebra is large enough it they do. The local transformations allow one to choose $g$ in the form $g = \exp(\sum A \cdot R)$ where the sum is over all generators of the algebra that are not in the local sub-algebra and their coefficients are the fields of the theory. In our case, $g$ then contains all the generators of the Borel sub-algebra of $E_8^{+++}$.

So little is known about Kac-Moody algebras that it is difficult to calculate the general properties of a non-linear realisation based upon them. However, by setting to zero all the fields of the non-linear realisation, except those associated with the Cartan sub-algebra, the group element takes the very simple form

$$g = \exp(\sum_a q_a H_a)$$

The fields $q_a$ are then the only fields of the theory. Provided one restricts ones attention to operations that preserve the Cartan sub-algebra it is then essentially trivial to examine the consequences. Such is the case for Weyl transformations. Indeed, these were considered in just such a setting for the $E_8^{+++}$ non-linear realisation appropriate to the eleven
dimensional theory and the Weyl transformations were shown [15] to be none other than the U-duality transformations [18].

The algebra $G^{+++}$ contains a $GL(D)$ sub-algebra, with generators $K^{a b}, a, b = 1, \ldots, D$ which leads in the non-linear realisation to the gravity sector of the resulting theory where $D$ is the dimension of the space-time of the theory. The $A_{D-1}$, or $SL(D)$, part of this sub-algebra is obtained by taking $D - 1$ dots of the Dynkin diagram of $G^{+++}$ which are connected to the very extended node, i.e. selecting an $A_{D-1}$ sub-Dynkin diagram which contains as an extreme node the very extended node. As we shall see, there is more than one way to do this in general and these lead to different physical theories. The part of the group element of $G^{+++}$ which contains the generators of the preferred sub-algebra is of the form $\exp(\sum_{a \leq b} h_a h_b K^{a b})$ and carrying out the non-linear realisation one finds that the vierbein $e_{\mu}^{a}$ is identified with $e_{\mu}^{a} = (\epsilon^{a})_{\mu}^{\alpha}$, where in this last equation, we treat $h$ as a matrix [2,16]. This $Gl(D)$, or in some cases the $SL(D)$, sub-algebra is referred to as the gravity sub-algebra and the $D - 1$ dots of the Dynkin diagram of $G^{+++}$ which belong to the $SL(D)$, or $A_{D-1}$, sub-algebra are referred to as the gravity line.

The eleven dimensional, IIA and IIB theories all have an underlying $E_{8}^{+++}$, but they are distinguished by their different gravity sub-algebras. The eleven dimensional theory must possess an $A_{10}$ gravity algebra and there is only one such algebra. We must choose the gravity algebra to be the $A_{10}$ sub-Dynkin diagram that consist of nodes labeled one to ten. That is it is found by deleting node eleven in the $E_{8}^{+++}$ Dynkin diagram [2].

The IIA and IIB theories are ten dimensional and so to find these theories we seek an $A_{9}$ gravity algebra. Looking at the $E_{8}^{+++}$ Dynkin diagram there are only two ways to do this. Starting from the very extended node we must choose a $A_{9}$ sub-Dynkin diagram, but once we get to the junction of $E_{8}^{+++}$ Dynkin diagram, situated at the node labeled 8, we can continue along the horizontal line with two further nodes taking only the first node to belong to the $A_{9}$, or we can find the final $A_{9}$ node by taking it to be the only node in the other choice of direction at the junction. These two ways correspond to the IIA and IIB theories respectively. Hence, in the IIA theory we take the gravity line to be nodes labeled one to nine inclusive while for the IIB theory the gravity line contains nodes one to eight and in addition node eleven [2,6].

The gravity sub-algebra is such that $K^{a}_{a}, a = 1, \ldots, D$ are part of the Cartan sub-algebra of $E_{8}^{+++}$. For the eleven dimensional theory, these eleven generators span the Cartan sub-algebra and so one can also write the group element of equation (1.6) in the form

$$g = \exp(\sum_{a=1}^{11} h^{a} K_{a}) = \exp(h^{T} K)$$  \hspace{1cm} (1.7)

In the second equation we have used matrix notation whose meaning should be clear. The relationship between the Chevalley generators $H_{a}$ and the physical generators $K^{a}_{a}$ can be written in matrix form as $K = \rho H$. It is given by [2]

$$H_{a} = K^{a}_{a} - K^{a+1}_{a+1}, a = 1, \ldots, 10, \ H_{11} = -\frac{1}{3}(K_{1}^{1} + \ldots + K_{8}^{8}) + \frac{2}{3}(K_{9}^{9} + K_{10}^{10} + K_{11}^{11})).$$ \hspace{1cm} (1.8)
We also record the relations

\[ E_a = K^{a+1}_a, \, a = 1, \ldots, 10, \, E_{11} = R^{91011}, \]  

(1.9)
between the Chevalley generators \( E_a \) and the simple root generators of SL(11) and the generator \( R^{a_1a_2a_3} \) which is responsible in the non-linear realisation for the introduction of the gauge field \( A_{a_1a_2a_3} \) of the eleven dimensional supergravity theory. Hence, keeping only fields associated with the Cartan sub-algebra implies keeping only the diagonal parts of the metric and, as we will see below for the IIA and IIB theories, also the dilaton field.

The form of the \( H_a \) of equation (1.8) can essentially be determined given that they must obey equation (1.4) with the Cartan matrix of \( E_8^{+++} \), together with the knowledge of the simple roots generators of equation (1.10) and that tensors, such as \( R^{a_1a_2a_3} \), transform in the obvious way under GL(11), i.e. \([K_{cd}, R^{a_1a_2a_3}] = \delta_d^{a_1} R^{a_2a_3} + \delta_d^{a_2} R^{a_3a_1} + \delta_d^{a_3} R^{a_1a_2} \).

We denote quantities in the IIA and IIB theories with a tilde and hat respectively. For these theories the Cartan sub-algebra of the gravity sub-algebra, i.e. the \( \tilde{K}_{a_a}, \, a_1, \ldots, 10 \) for the IIA theory and the \( \hat{K}_{a'_a}, \, a = 1, \ldots, 10 \) for the IIB theory, account for only ten of the eleven generators of the Cartan sub-algebra of \( E_8^{+++} \). The final commuting generator is associated with the dilaton which appears in the IIA and IIB theories. We denote this generator by the symbol \( R \) and the dilaton by \( A \) with appropriate tildes or hats. As such, for the IIA theory the \( E_8^{+++} \) group element of equation (1.6) can be written in terms of the physical generators in the form

\[ \tilde{g} = \exp(\sum_{a=1}^{10} \tilde{h}_a \tilde{K}_a) \exp(\tilde{A} \tilde{R}) = \exp(\tilde{h} \tilde{K}) \]  

(1.10)

In the second equation we have used matrix notation for which \( \tilde{h} \) is a column vector whose first ten components are \( \tilde{h}_a, \, a = 1, \ldots, 10 \) and whose eleventh component is \( \tilde{A} \), similarly \( \tilde{K} \) has its first ten components as \( \tilde{K}_a \) and eleventh component \( \tilde{R} \). The Cartan sub-algebra generators \( H_a \) of \( E_{11} \) and the physical generators \( \tilde{K}_{a_a}, \, a = 1, \ldots, 10 \) and \( \tilde{R} \) are related by \( H = \tilde{\delta} \tilde{K} \) which is given by [2]

\[ H_a = \tilde{K}_{a_a} - \tilde{K}_{a+1_a}, \, a = 1, \ldots, 9, \, H_{10} = -\frac{1}{8}(\tilde{K}_{11} + \ldots + \tilde{K}_{99}) + \frac{7}{8} \tilde{K}_{10}^{10} - \frac{3}{2} \tilde{R}, \]  

\[ H_{11} = -\frac{1}{4}(\tilde{K}_{11} + \ldots + \tilde{K}_{88}) + \frac{3}{4}(\tilde{K}_{99} + \tilde{K}_{1010}) + \tilde{R}. \]  

(1.11)

While the \( E_a \) Chevalley generators of \( E_8^{+++} \) are given in terms of IIA generators by [2]

\[ E_a = \tilde{K}_{a+1_a}, \, a = 1, \ldots, 9, \, E_{10} = \tilde{K}_{10}^{10}, \, E_{11} = \tilde{R}^{910}. \]  

(1.12)

The fields associated with the generators \( \tilde{R}^a \) and \( R^{ab} \) in the non-linear realisation are the one form and two form fields of the IIA supergravity theory.

Equating the Chevalley generators \( H_a \) of equations (1.8) and (1.11) we find that the generators in the physical basis of the eleven dimensional and IIA theory are related by [2]

\[ K^a_a = \tilde{K}_{a_a}, \, a = 1, \ldots, 10, \, K^{11}_{11} = \frac{1}{8} \sum_{a=1}^{10} \tilde{K}_a^a + \frac{3}{2} \tilde{R}. \]  

(1.13)
For the IIB theory, the generators $K^a_a$, $a = 1, \ldots, 10$ and $\hat{R}$ span the Cartan sub-algebra of $E_{8}^{++}$ and so the group element of equation (1.6) can be expressed as

$$\hat{g} = \exp\left(\sum_{a=1}^{10} \hat{h}_a^a \hat{K}^a_a\right) \exp(\hat{A}^R) = \exp(\hat{h}^T \hat{K})$$  \hspace{1cm} (1.14)

In the second equation we have used matrix notation for which $\hat{h}$ is a column vector whose first ten components are $\hat{h}_a^a$, $a = 1, \ldots, 10$ and whose eleventh component is $\hat{A}$ and similarly $\hat{K}$ has its first ten components as $\hat{K}^a_a$ and eleventh component $\hat{R}$. The relationship between the Cartan sub-algebra generators $H_a$ of $E_{8}^{++}$ and the physical generators $\hat{K}^a_a$, $a = 1, \ldots, 10$ and $\hat{R}$ can be written in the form $H = \hat{\rho} \hat{K}$ and it is explicitly given by [6]

$$H_a = \hat{K}_a^a - \hat{K}_{a+1}^{a+1}, a = 1, \ldots, 8, \quad H_9 = \hat{K}_9^9 + \hat{K}_{10}^{10} + \hat{R} - \frac{1}{4} \sum_{a=1}^{10} \hat{K}_a^a,$$

$$H_{10} = -2\hat{R}, \quad H_{11} = \hat{K}_9^9 - \hat{K}_{10}^{10}$$  \hspace{1cm} (1.15)

The Chevalley generators $E_a$ of $E_{8}^{++}$, as they appears in IIB theory are given by [6]

$$E_a = \hat{K}_a^a_{a+1}, a = 1, \ldots, 8, \quad E_9 = \hat{K}_9^9, \quad E_{10} = \hat{R}_2, \quad E_{11} = \hat{K}_9^{10}. \hspace{1cm} (1.16)$$

The fields associated with the generators $\hat{R}_1^{ab}$ and $\hat{R}_2$ are the NS-NS two form and the axion, $\hat{\chi}$ of the IIB theory. The last equation reflects the fact that the node labeled eleven is the last node in the IIB gravity line.

Equating the Chevalley generators $H_a$ of equations (1.8) and (1.15) we find that the generators in the physical basis of the eleven dimensional and IIB theory are related by [7]

$$K_a^a = \hat{K}_a^a, \quad a = 1, \ldots, 9, \quad \hat{K}_{10}^{10} = \frac{1}{3} \sum_{a=1}^{9} K_a^a - \frac{2}{3}(K_{10}^{10} + K_{11}^{11}), \quad \hat{R} = -\frac{1}{2}(K_{10}^{10} - K_{11}^{11})$$  \hspace{1cm} (1.17)

For completeness we note the relationship between the IIA and IIB physical generators;

$$\hat{K}_a^a = \hat{K}_a^a, \quad a = 1, \ldots, 9, \quad \hat{K}_{10}^{10} = \frac{1}{4} \sum_{a=1}^{9} \hat{K}_a^a - \frac{3}{4} \hat{K}_{10}^{10} - \hat{R},$$

$$\hat{R} = \frac{1}{16} \sum_{a=1}^{9} \hat{K}_a^a - \frac{7}{16} \hat{K}_{10}^{10} + \frac{3}{4} \hat{R}.$$  \hspace{1cm} (1.18)

We note that the generator corresponding to the node labeled ten in the eleven dimensional theory is $K_{10}^{10}$ and so is associated with the exchange of the ten and eleven space-time coordinates, while in the IIB theory it is $\hat{R}_2$ which is the non-perturbative part of the $SL(2,\mathbb{Z})$ symmetry of the IIB theory.
2. Relations between the eleven dimensional, IIA and IIB theories

As explained in reference [7], the common $E_{8}^{+++}$ origin of these three theories implies a one to one correspondence between the fields of the three theories. In particular, any field in the non-linearly realised IIB theory arises in the group element as the coefficient of a particular generator which is in the Borel sub-algebra of $E_{8}^{+++}$, however, the generators of $E_{8}^{+++}$ are essentially unique and so we can identify this generator from the viewpoint of the eleven dimensional theory. For example, the component graviton field $h_{910}$ of the IIB theory is associated with the generator $K_{910}$ which is equal to the Chevalley generator $E_{11}$ of $E_{8}^{+++}$.

However, from the eleven dimensional perspective this Chevalley generator is equal to the generator $R_{91011}$ that is associated with the field $A_{91011}$ which is one component of the third rank anti-symmetric field of the eleven dimensional supergravity theory. In this section, we will find these correspondences at the simplest possible level.

2.1 The correspondence between the eleven dimensional and IIA theories

To find the correspondence for the Cartan sub-algebra we simply equate the two group elements in the eleven dimensional and IIA theories of equations (1.7) and (1.10) respectively:

$$g = \tilde{g} \text{ or } \exp\left(\sum_{a=1}^{11} h_{a} a K_{a}\right) = \exp\left(\sum_{a=1}^{10} \tilde{h}_{a} a \tilde{K}_{a}\right) \exp(\tilde{A} \tilde{R}) \quad (2.1)$$

Using equation (1.13), we conclude that

$$\tilde{h}_{a} a = h_{a} a + \frac{1}{8} h_{11} a_{11}, \ a = 1, \ldots, 10, \ \tilde{A} = \frac{3}{2} h_{11} a_{11} \quad (2.2)$$

We expect these relations to hold even if one does not carries out dimensional reduction of the theory on a torus, but then one must also carry out a corresponding exchange of the generalised coordinates [17]. However, if we do dimensionally reduce some of the dimensions on a torus then it is useful to change to the variables

$$h_{a} a = \ln \frac{R_{a}}{l_{p}}, \ a = 1, \ldots, 11 \quad (2.3)$$

where $l_{p}$ is the eleven dimensional Planck scale. We note that in the group elements used to construct the non-linear realisation the fields are dimensionless and so the resulting part of the action in $D$ space-time dimensions that has two space-time derivatives is multiplied by $l_{p}^{-(D-2)}$. In particular, we will apply the change of variable to the constant background part of the fields. For a rectangular torus, the coordinate and parameterisation invariant length of its cycle in the $a$ direction is $l_{p} \int c_{a} a dx^{a} = R_{a}$.

Similarly we introduce the analogous IIA variables by

$$\tilde{h}_{a} a = \ln \frac{\tilde{R}_{a}}{\tilde{l}_{p}}, \ a = 1, \ldots, 10, \ \tilde{A} = \ln \tilde{g}_{s} \quad (2.4)$$
where $\tilde{l}_p$ is the ten dimensional Planck scale of the IIA theory. Comparing the low energy action with that calculated from string scattering allows us to identify the string scale $l_s$ by $(\tilde{l}_p)^8 = \tilde{g}_s^2 (\tilde{l}_p)^8$ and $\tilde{g}_s$ in equation (2.4) with the string coupling constant in the usual way.

The last relation in equation (2.2) implies that

$$\tilde{g}_s^2 = \left(\frac{R_{11}}{l_p}\right)^3$$

Since the eleven dimensional theory after reduction on a circle coincides with the IIA theory we may take $\tilde{R}_a = R_a$, $a = 1,\ldots, 10$ and then we find that

$$\left(\frac{l_p}{\tilde{l}_p}\right)^{12} = \tilde{g}_s \quad \text{or} \quad l_p^3 = (\tilde{l}_p)^3 \tilde{g}_s.$$ (2.6)

The first relation in the above equation together with equation (2.5) implies that $\frac{\tilde{R}_{11}}{l_p} = \frac{1}{\tilde{g}_s}$. Equations (2.5) and (2.6) are the known relations between the IIA theory and the so called eleven dimensional M theory. They encouraged the idea that eleven dimensional M theory is the strong coupling limit of the IIA string theory [10,11].

### 2.2 The correspondence between the eleven dimensional and IIB theories

We now find the analogous relations between the fields, which are associated with their Cartan sub-algebra, of the the eleven dimensional and IIB theories. Equating the eleven dimensional and IIB group elements of equation (1.6) and equation (1.14) we find that

$$g = \tilde{g} \quad \text{or} \quad \exp\left(\sum_{a=1}^{11} h_a K_a\right) = \exp\left(\sum_{a=1}^{10} \tilde{h}_a \tilde{K}_a\right) \exp(\tilde{A} \tilde{R})$$

which using the identifications of equations (1.17) implies that

$$h_a^a = \tilde{h}_a^a + \frac{1}{3} h_{10}^{10}, \quad h_{10}^{10} = -\frac{2}{3} h_{10}^{10} - \frac{1}{2} \tilde{A}, \quad h_{11}^{11} = -\frac{2}{3} h_{10}^{10} + \frac{1}{2} \tilde{A}.$$ (2.8)

These relations hold without compactifications, but for a torus compactification it is appropriate to adopt the variables

$$\tilde{h}_a^a = \ln \frac{\tilde{R}_a}{l_p}, \quad a = 1,\ldots, 10, \quad \tilde{A} = \ln \tilde{g}_s$$

where $l_p$ is the Planck length in the IIB theory and $\tilde{g}_s$ its string coupling. Introducing the IIB string scale by $(\tilde{l}_p)^8 = \tilde{g}_s^2 (\tilde{l}_p)^8$ the relations given in equation (2.8) become

$$\frac{l_p^4 \tilde{g}_s}{\tilde{l}_p^3} = \tilde{R}_{10}, \quad \frac{\tilde{R}_{10}^6}{l_p^3} = \frac{l_p^4 \tilde{g}_s^4}{\tilde{l}_p^3}, \quad \frac{R_{10}^6}{l_p^3} = \frac{l_p^4 \tilde{g}_s^4}{\tilde{l}_p^3}.$$ (2.10)
respectively. These are equivalent to the more familiar relations

\[
\frac{\hat{R}}{R_{10}} = \frac{R_{11}}{R_{10}}, \quad \hat{R} = \frac{l_p}{R_{10}}, \quad \frac{l_p}{R_{10}R_{11}}
\]

which relate the eleven dimensional theory reduced on rectangular torus with radii \(R_{10}\) and \(R_{11}\) to the IIB theory reduced on a circle of radius \(\hat{R}_{10}\) [9,12,13].

As explained in reference [7], there is a one to one map between all the fields of the IIB and the eleven dimensional non-linearly realised theories and not just those associated with the Cartan sub-algebra. We close this section by giving a simple illustration of how this map works for a field outside the Cartan sub-algebra. Equations (1.9) and (1.16) state that \(E_{10} = K^{0}_{11} = \hat{R}_2\) and as explained at the beginning of this section this implies that the eleven dimensional field \(h_{10}^{11}\) corresponds to the axion field \(\hat{\chi}\) of the IIB theory. We now enlarge the fields which are non-zero by including these fields in addition to those associated with the Cartan sub-algebra. As a result, the eleven dimensional group element takes the form

\[
g = \exp\left(\sum_{a=1}^{11} h_a\, a^{K_a} \right) \exp(h_{10}^{11} K_{10}^{11}). \tag{2.12}
\]

Putting only the Cartan sub-algebra elements in the first exponential will allow us to perform the computation more easily, but it is not quite the form given in the non-linear realisation of references [2,16] and used to find the eleven dimensional supergravity theory. As a result, we must use the form of the vierbein that follows from the \(g\) of equation (2.12); its non-vanishing components are given by

\[
e^\mu_a = \delta^\mu_\mu, \quad a, \mu = 1, \ldots, 9, \quad e^\mu_a = \left( \begin{array}{cc} e^{h_{10}^{10}} & e^{h_{10}^{10} h_{11}^{11}} \\ 0 & e^{h_{11}^{11}} \end{array} \right)^\mu_a, \quad a, \mu = 10, 11 \tag{2.13}
\]

On the other hand, the IIB group element can be written as

\[
\hat{g} = \exp\left(\sum_{a=1}^{10} \hat{h}_a\, a^{K_a} \right) \exp(\hat{\chi}\hat{R}_2) \exp(\hat{\chi}\hat{R}_2) = \exp\left(\sum_{a=1}^{10} \hat{h}_a\, a^{K_a} \right) \exp(\hat{\chi}\hat{R}_2) \exp(e^{\hat{\chi}} \hat{R}_2) \tag{2.14}
\]

The first form of \(\hat{g}\) is the one used to construct the non-linear realisation of IIB supergravity in [6] while the second form is suitable for our comparison with eleven dimensional group element. To change from one form to the other we used the relation \([\hat{R}, \hat{R}_2] = -\frac{1}{2}[R_{10}, E_{10}] = -E_{10} = -\hat{R}_2\).

Setting \(\hat{g} = \hat{g}\) and using equations (1.17), (1.9) and (1.16) we find the same relations of equation (2.8) as well as

\[
e^{\hat{\chi}} \hat{\chi} = h_{10}^{11} \tag{2.15}
\]

Let us now suppose that the ten and eleven directions of the eleven dimensional theory are a torus with lengths \(R_{10}\) and \(R_{11}\). To discuss the properties of the torus it is simplest to make a rigid coordinate transformation from the coordinates \(x^T = (x^{10}, x^{11})\) to the coordinates \(y^T = (y^{10}, y^{11})\) that diagonalises the metric in these directions. In particular,
we will diagonalise the vierbein in the ten and eleven directions. We denoted the latter by the matrix $e$ which can be read off from the last relation in equation (2.13). The transformation $e \to \Lambda e$ given by
\[
\Lambda = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},
\]
where $m = -\frac{e_{10}^{11}}{e_{11}^{11}}$, has the desired result. The new vierbein has the same diagonal components as the old one. Using equation (2.13) we find that $m = -h_{10}^{11}\exp(h_{10}^{10} - h_{11}^{11})$. In the diagonal coordinates $y$ we take the cycles of the torus to be given by $y^{11} = u$, $y^{10} = 0$, $0 \leq u < 1$ and $y^{10} = v$, $y^{11} = 0$, $0 \leq v < 1$. The coordinate and parameterisation invariant length of the first cycle is $\int_0^1 e_{11}^{11} \frac{du}{u} = e_{11}^{11} = \Omega_{11}$ and similarly with the invariant length of the second cycle is given by $e_{10}^{10} = \Omega_{10}$. Hence, we still have the relation $\frac{\Omega_{11}}{\Omega_{10}} = \exp(\Lambda \hat{A}) = \hat{g}_s$ of equation (2.11).

In terms of the original $x$ coordinates which are related by $x = A^T y$ the cycles of the torus are $x^{10} = 0$, $x^{11} = u$; $0 \leq u < 1$ and $x^{10} = v$, $x^{11} = mv$; $0 \leq v < 1$. If we define the complex coordinate $z = x^{11} - ix^{10}$ then the periods corresponding to the first and second cycles are $ \tau_1 = 1$ and $ \tau_2 = \pi$ respectively where $\tau = \tau_1 + i\tau_2$ with $\tau_1 = 1$ and $\tau_2 = -m$. Hence, the modulus parameter of the torus is given by
\[
\frac{\tau_1}{\tau_2} = m = \hat{\chi}
\]
This agrees with the identification of references [12] and [13] after one takes into account that one must make the field redefinition $\hat{\chi} \to \exp(-\hat{A})\hat{\chi}$ to find the $\hat{\chi}$ of [6] from that of [12] in order to gain agreement between the field equations of the two references. By a judicious choice of coordinates we can, as in [12], arrange for $\tau_2$ to be $\exp(-\hat{A})$, but the physically relevant quantity $\frac{\tau_1}{\tau_2}$ remains the same.

3. Weyl transformations in the IIB and IIA theories

The Weyl reflection $S_a$ corresponding to the simple root $\alpha_a$ on any weight $\beta$ is given by $S_a \beta = \beta - 2 \frac{(\beta, \alpha_a)}{(\alpha_a, \alpha_a)} \alpha_a$. For the simple roots this becomes
\[
S_a \alpha_b = \alpha_b - 2 \frac{(\alpha_b, \alpha_a)}{(\alpha_a, \alpha_a)} \alpha_a = (s_a)^\circ \alpha_c
\]
(3.1)
The action of the Weyl transformation $S_a$ on the Cartan sub-algebra of a Kac-Moody algebra is given by
\[
H'_b = S_a H_b = (s_a)^\circ H_c
\]
(3.2)
Since the Weyl group acts on Cartan sub-algebra generators to give Cartan sub-algebra generators it makes sense to consider their action on elements restricted to be of the form of equation (1.6). Writing the group element in matrix form $g = \exp(q^T H)$, we conclude that the Weyl group acts on the fields $q$ as $q'^T = q^T s$, or $q' = s^T q$ as $s^2 = I$. Clearly, these transformations hold for the eleven dimensional theory and the IIA and IIB theories.

To find the physical effects of the Weyl transformations we need to find their action on the physical variables $h_a \tilde{a}$ and also the dilaton field, for the cases of the IIA and IIB

10
Theories, however, the relationship between the Chevalley generators $H_a$ and the physical generators depends upon which theory we are considering and so the effect of the Weyl transformations on the physical generators and fields is different for each theory. Using matrix notation, in the eleven dimensional theory we may write $H = \rho K$ and then the effect of the Weyl transformation is $S_a K = K' = \rho^{-1} S_a \rho K = \rho_{a} K$ and so the physical fields $h$ transform as $h' = r_a^T h$. However, for the IIB theory, $H = \rho K$ and so we have the equations

$$S_a \dot{K} = \dot{K}' = \rho^{-1} S_a \rho \dot{K} = \dot{r}_a \dot{K}, \quad \dot{h}' = r_a^T \dot{h}$$

(3.3)

The equation for IIA being found by replacing $\ddot{\cdot}$ by $\dddot{\cdot}$'s. Using equations (2.3), (2.4) and (2.9) the effects of the Weyl transformations can then be readily deduced on the radii of any compactified directions and the appropriate length scales and coupling constants.

This calculation was carried out in reference [15] for the eleven dimensional theory and we briefly summarize the result. The Weyl transformations $S_a$, $a = 1, \ldots, 10$ implies that $R_a \leftrightarrow R_{a+1}$, $l_p \rightarrow l_p$. However, $S_{11}$ induces the transformations $h_{a} = h_{a} + \frac{1}{2}(h_9 + h_{10} + h_{11})$, $a = 1, \ldots, 8$ and $h_{a} = h_{a} - \frac{1}{2}(h_9 + h_{10} + h_{11})$, $a = 9, 10, 11$ which in turn implies that

$$R'_{a} = \frac{\tilde{R}_{a}}{R_{a+1} R_{11}}, \quad R'_{10} = \frac{\tilde{R}_{10}}{R_{11} R_{9}}, \quad R'_{11} = \frac{\tilde{R}_{11}}{R_{9} R_{10}}, \quad (l'_p)^3 = \frac{r_{9}}{R_{9} R_{10} R_{11}}$$

(3.4)

For the IIB theory, the Weyl transformations $S_a$, $a = 1, \ldots, 8$ correspond $K_{a} \leftrightarrow K_{a+1}$, for $a = 1, \ldots, 8$ as well as $R \rightarrow \tilde{R}$. The effect on the variables of equation (2.9) is $\tilde{R}_a \rightarrow \tilde{R}_{a+1}$ for $a = 1, \ldots, 7$ as well as $\tilde{y}_s \rightarrow \tilde{y}_s$. The Weyl transformation $S_{11}$ leaves $\tilde{R}$ and all the $K_{a}$ inert except for $K_{9} \leftrightarrow K_{10}$. The effect is to take $\tilde{R}_9 \rightarrow \tilde{R}_{10}$ with all other variables being inert. This is consistent with the node labeled eleven being the last on the gravity line of the IIB theory and one finds that all the Weyl transformations corresponding to all points on the gravity line just exchanges the corresponding radii.

The Weyl transformation $S_{10}$ acts on the Cartan sub-algebra as $H'_{10} = -H_{10}$, $H'_{9} = H_{9} + H_{10}$ all other elements being inert. Using equation (1.15) we find that these transformations imply that

$$\tilde{R}' = -\tilde{R}, \quad K'_{a} = \tilde{K}_{a}$$

(3.5)

Using equations (2.9) and (3.3), the effect on the physical variables is given by

$$\tilde{A}' = -\tilde{A}, \quad h'_{a} = h_{a}$$

(3.6)

Which in turn implies that

$$\tilde{y}'_{s} = \frac{1}{\tilde{y}_s}, \quad \tilde{R}'_{a} = \tilde{R}_{a}, \quad \tilde{h}'_{a} = \tilde{h}_{a}$$

(3.7)

This is just the non-perturbative S-duality transformations of the IIB theory which holds if the theory is compactified or not. This is to be expected as the node labeled ten just leads to an $SL(2,\mathbb{R})$ transformation of the supergravity theory. We note that in the eleven
dimensional theory, node eleven is the last node in the gravity line of this theory and the corresponding Weyl transformation swops the eleventh and tenth coordinates.

Finally, we consider the Weyl transformation $S_9$ which induces the transformations $H'_9 = -H_9$, $H'_1 = H_1 + H_9$, $H'_8 = H_8 + H_9$ with all other elements of the Cartan sub-algebra being inert. The transformation on the physical generators is given by

$$\hat{K}'^a_a = \hat{K}^a_a, \ a = 1, \ldots, 8, \ \hat{K}'^a_9 = \hat{K}^a_9 + \frac{1}{4}(\hat{K}^1_1 + \ldots + \hat{K}^8_8) - \frac{3}{4}(\hat{K}^9_9 + \ldots + \hat{K}^{10}_{10}) - \hat{R}, \ a = 9, 10,$$

$$\hat{R}' = \hat{R} + \frac{1}{8}(\hat{K}^1_1 + \ldots + \hat{K}^8_8) - \frac{3}{8}(\hat{K}^9_9 + \ldots + \hat{K}^{10}_{10}) - \frac{1}{2}\hat{R} \quad (3.8)$$

The corresponding effect on the fields of the IIB theory is

$$\hat{h}'^a_a = \hat{h}^a_a + \frac{1}{4}(h^9_9 + \hat{h}^{10}_{10} + \frac{1}{2}A), \ a = 1, \ldots, 8$$

$$\hat{h}'^a_9 = \hat{h}^a_9 - \frac{3}{4}(h^9_9 + \hat{h}^{10}_{10} + \frac{1}{2}A), \ a = 9, 10, \ \hat{A}' = \hat{A} - (h^9_9 + \hat{h}^{10}_{10} + \frac{1}{2}A) \quad (3.9)$$

As a result the variables of equation (2.9) transform as

$$\hat{R}' = \hat{R}_a, \ a = 1, \ldots, 8, \ \frac{\hat{R}'_9}{\hat{R}_9} = \frac{\hat{R}^2}{\hat{R}_9}, \ a = 9, 10, \ \frac{\hat{g}'_s}{g_s} = \frac{\hat{g}^2}{g_s} \quad (3.10)$$

and $\hat{\ell}' = \hat{\ell}$. We recognise this as a double T duality seen from the IIB viewpoint.

We now briefly discuss the effect on the Weyl transformations of $E_8^{+++}$ for the IIA theory. The Weyl transformations $S_a, \ a = 1, \ldots, 9$ takes $K'^a_a \leftrightarrow K^{a+1}_a$ and so $R_a \leftrightarrow R_{a+1}$. The Weyl transformation $S_{11}$ leads to the double T duality

$$R'_a = R_a, \ a = 1, \ldots, 8; \ \frac{\hat{R}'_9}{\hat{R}_9} = \frac{\hat{R}^2}{\hat{R}_9}; \ \frac{\hat{R}'_{10}}{\hat{R}_{10}} = \frac{\hat{R}^2}{\hat{R}_9}; \ \frac{\hat{g}'_s}{g_s} = \frac{\hat{g}^2}{g_s}; \ \hat{\ell}' = \hat{\ell} \quad (3.11)$$

Finally, the Weyl transformation $S_{10}$ induces the changes

$$\hat{h}'^a_a = \hat{h}^a_a + \frac{1}{8}\hat{h}^{10}_{10} + \frac{3}{32}A, \ a = 1, \ldots, 9$$

$$\hat{h}'^{10}_{10} = \hat{h}^{10}_{10} - \frac{7}{8}\hat{h}^{10}_{10} + \frac{21}{32}A, \ a = 9, 10, \ \hat{A}' = \hat{A} + \frac{3}{2}\hat{h}^{10}_{10} - \frac{9}{8}A \quad (3.12)$$

which leads to

$$R'_a = R_a, \ a = 1, \ldots, 9; \ \frac{\hat{R}'_{10}}{\hat{R}_{10}} = \hat{g}^2_s; \ \hat{g}'_s = \frac{\hat{g}^2}{g_s}; \ \hat{\ell}' = \hat{\ell} \quad (3.13)$$

Clearly, this is a non-perturbative relation which is in some sense the IIA analogue of the SL(2,Z) symmetry of the IIB theory.
3. Discussion

One could use the same techniques as used in this paper to identify the relations between other $G^{+++}$ non-linearly realised theories where there is a choice of gravity sub-algebra.

The eleven dimensional, IIA and IIB theories are all expected to possess a non-linearly realised $E_8^{+++}$ symmetry [2,6]. Although, their differences arise from the way their gravity sub-algebras are embedded, their common symmetry allows one to establish a one to one correspondence between the fields of these theories [7]. In this paper, we have found the simplest consequences of this correspondence which are those for the fields associated with the Cartan sub-algebra of $E_8^{+++}$. We have recover the known relations [8,9,10,11] between the three theories. We also gave one example of the correspondence for a field outside the Cartan sub-algebra and recovered the fact [12,13] that the axion field of the IIB theory dimensionally reduced on a circle can be identified with the modulus of the two dimensional torus used to dimensionally reduce the eleven dimensional theory.

The correspondence between the three theories resulting from their common $E_8^{+++}$ symmetry implies many more results, such as the eleven dimensional origin of the massive IIA theory and the IIB space-filling brane [7]. However, the purpose of this paper is to demonstrate that the underlying $E_8^{+++}$ symmetry can be used to find results central to string theory in a very simple way.

As we noted above, the identifications of the fields of the three theories should hold even if one does not perform a dimensional reduction. In this case one is the fields which depend on the generalised coordinates [17] of the theory and, as explained in reference [7], one must then also swap the generalised coordinates of the theory. However, these includes central charge coordinates as well as the usual coordinates of space-time and their interchange will have far reaching effects on the theory.

We also computed the effect of the Weyl transformations of the IIA and IIB $E_8^{+++}$ theories on the diagonal components of the metric and dilaton to recover the expected U-duality symmetries of these theories. It would be interesting to compare these results with the different perturbative sub-algebras of the $E_8^{+++}$ algebra for the IIA and IIB theories found in [20].

Acknowledgments I wish to thank Matthias Gaberdiel and Dominic Clancy for useful discussions. I wish also to thank the Erwin Schrödinger International Institute for Mathematical Physics at Wien and the Department of Physics at Heraklion for their hospitality. This research was supported by a PPARC senior fellowship PPA/Y/S/2002/001/44 and in part by the PPARC grants PPA/G/O/2000/00451, PPA/G/S4/1998/00613 and the EU Marie Curie, research training network grant HPRN-CT-2000-00122.

References

[1] E. Cremmer, B. Julia and J. Scherk, Supergravity theory in eleven dimensions, Phys. Lett. 76B (1978) 409.
[2] P. West, $E_{11}$ and M Theory, Class. Quant. Grav. 18 (2001) 4443, hep-th/0104081
[3] I.C.G. Campbell and P. West, N=2 d=10 nonchiral supergravity and its spontaneous compactifications, Nucl. Phys. B243 (1984), 112; M. Huq, M. Namanzie, Kaluza-
Klein supergravity in ten dimensions, Class. Quant. Grav. 2 (1985); F. Giani, M. Pernici, \( N=2 \) supergravity in ten dimensions, Phys. Rev. D30 (1984), 325

[4] J. Schwarz and P. West, Symmetries and Transformations of chiral \( N=2, D=10 \) supergravity, Phys. Lett. B126 (1983), 301.

[5] J. Schwarz, Covariant field equations of chiral \( N=2, D=10 \) supergravity, Nucl. Phys. B226 (1983), 269; P. Howe and P. West, The complete \( N=2, d=10 \) supergravity, Nucl. Phys. B238 (1984), 181.

[6] I. Schnakenburg and P. West, Kac-Moody Symmetries of IIB supergravity, Phys. Lett. B 517 (2001) 137-145, hep-th/0107181

[7] P. West, The IIA, IIB and eleven dimensional theories and their common \( E_{11} \) origin, hep-th/0402140.

[8] J. Dai, R. Leigh and J. Polchinski, New connections between string theories, Mod. Phys. Lett. A4 (1989) 2073.

[9] M. Dine, P. Huet and N. Seiberg, Large and small radius in string theory, Nucl. Phys. B322 (1989), 301.

[10] P. Townsend, The eleven-dimensional supermembrane revisited, Phys. Lett. B 350 (1995) 184, hep-th/9501068.

[11] E. Witten, String theory dynamics in various dimensions, Nucl. Phys. B443 (19995) 85, hep-th/9503124.

[12] J. Schwarz, The Power of M Theory, Phys.Lett. B367 (1996) 97-103, hep-th/9510086; An \( SL(2,Z) \) Multiplet of Type IIB Superstrings, Phys.Lett. B360 (1995) 13-18; Erratum, B364 (1995) 252, hep-th/9508143.

[13] P. Aspinwall, Some relations between dualities in string theories, Nucl. Phys. Proc Suppl. 46 (1996) 30, hep-th/9508154.

[14] M. R. Gaberdiel, D. I. Olive and P. West, A class of Lorentzian Kac-Moody algebras, Nucl. Phys. B 645 (2002) 403-437, hep-th/0205068.

[15] F. Englert, L. Houart, A. Taormina and P. West, The Symmetry of M-theories, JHEP 0309 (2003) 020, hep-th/0304206.

[16] P. C. West, Hidden superconformal symmetry in M theory, JHEP 08 (2000) 007, hep-th/0005270

[17] P. West, \( E_{11}, SL(32) \) and Central Charges, Phys. Lett. B 575 (2003) 333-342, hep-th/0307098

[18] C.M. Hull and P.K. Townsend, Unity of superstring dualities, Nucl. Phys. B 438 (1995) 109, hep-th/9410167.

[19] A. Kleinschmidt and P. West, Representations of \( G^{+++} \) and the role of space-time, JHEP 0402 (2004) 033, hep-th/0312247.

[20] M. Gaberdiel and P. West, Kac-Moody algebras in perturbative string theory, JHEP 0208 (2002) 049, hep-th/0207032