Bloch-Ogus theory for smooth and semi-stable schemes in mixed characteristic

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Abstract

We study Bloch-Ogus theory and the Gersten conjecture for homology theories with duality satisfying certain properties, in particular for étale cohomology with finite coefficients coprime to the residue characteristic of the base, for smooth and semi-stable schemes in mixed characteristic. We prove the Gersten conjecture in the smooth case and prove a special case in the semi-stable situation. As a corollary of the smooth case we obtain the surjectivity of the Galois symbol map for arbitrary local rings over an excellent discrete valuation ring.

1 Introduction

Let $S$ be the spectrum of an excellent discrete valuation ring $\mathcal{O}_K$ with local parameter $\pi$, field of fractions $K$ and residue field $k$. For any scheme $X$ of finite type over $S$, one can define the concept of a homology theory with duality $(\mathcal{H}^\ast, \mathcal{H}_\ast)$, that is a pair of a homology and a cohomology theory satisfying certain axioms, for example Poincaré duality (see Section 2). Let $\mathcal{H}^\ast(n)$ be the sheafification of the presheaf $U \mapsto \mathcal{H}^\ast(U, n)$ on $X$ in the Zariski topology. For the sheaf $\mathcal{H}^\ast(n)$, one can formulate the Gersten conjecture, which says that the sequence of sheaves

$$0 \rightarrow \mathcal{H}^q(n) \rightarrow \bigoplus_{x \in X^{(0)}} i_{x,*} \mathcal{H}^q(x, n) \rightarrow \bigoplus_{x \in X^{(1)}} i_{x,*} \mathcal{H}^{q-1}(x, n-1) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(d)}} i_{x,*} \mathcal{H}^{q-d}(x, n-d) \rightarrow 0$$

is exact for all $n, q \in \mathbb{N}$ if $X$ is regular of dimension $d$. If this is the case, then we say that the Gersten conjecture holds for $\mathcal{H}$ in the Zariski topology. Our main theorem reads as follows:

**Theorem 1.1.** (Thm. 3.5) Let $S$ and $X$ be as above and $(\mathcal{H}^\ast, \mathcal{H}_\ast)$ a homology theory with duality satisfying the principal triviality Property 2.7 and lifting Property 2.8. If $X$ is smooth over $S$, then the Gersten conjecture holds for $\mathcal{H}$ in the Zariski topology. Our main theorem in the semi-stable situation is the following:

**Theorem 1.3.** (Thm. 4.1) Let $\mathcal{O}_K$ be a henselian discrete valuation ring with local parameter $\pi$ and residue field $k$. Let $(\mathcal{H}^\ast, \mathcal{H}_\ast)$ be a homology theory with duality satisfying the blow-up Property 2.6, the principal triviality Property 2.7 and the lifting Property 2.8. Let

$$X = \text{Spec}(\mathcal{O}_K[t_1, t_2]/(t_1 t_2 - \pi)),$$

then the Galois symbol map

$$\hat{K}_q^M(R)/\ell \rightarrow H^q_{\text{ét}}(R, \mu^q_\ell)$$

is surjective. In particular, the graded ring $\bigoplus_{q \in \mathbb{N}} H^q_{\text{ét}}(R, \mu^q_\ell)$ is generated in degree 1.
Let $\mathcal{H}^q(n)$ be the sheafification of the presheaf $U \mapsto H^q(U, n)$ on $X$ in the Zariski topology. Assume that for every open subset $W \subset X$ containing $\text{Spec}(O_X[t_1, ..., t_d]/(t_1t_2 - \pi, t_1)) \cong A^1_k$ the morphism $\delta_k^1 \to W$ is homologically effaceable at the origin. Then the complex of sheaves

$$0 \to \mathcal{H}^q(n) \to \bigoplus_{x \in X^{(0)}} i_{*, x}H^q(x, n) \to \bigoplus_{x \in X^{(1)}} i_{*, x}H^{q-1}(x, n - 1) \to \bigoplus_{x \in X^{(2)}} i_{*, x}H^{q-2}(x, n - 2) \to 0$$

is exact.

Theorem 1.1 has a long history. In the case that $S$ is a field and $X$ is smooth over $S$, Theorem 1.1 is due to Bloch and Ogus [1] who use “Quillen’s trick” in their proof, an idea based on Noether normalisation which Quillen used in [17] to prove the Gersten conjecture for algebraic K-theory for smooth schemes over fields. The theorem of Bloch and Ogus has been studied further by Colliot-Thélène, Hoobler and Kahn in [2] where they give a different proof replacing “Quillen’s trick” by Gabber’s geometric presentation lemma which allows further applications. They therefore call the theorem the “Bloch-Ogus-Gabber theorem”. The case of the Gersten conjecture for étale cohomology with finite coefficients coprime to $\text{char}(k)$ for a regular scheme in equal characteristic can be reduced to the smooth case by a method developed by Panin (see [16, Sec. 5]). In mixed characteristic the case of $X$ smooth over $S$ has been proved in the Nisnevich topology, assuming additionally that $k$ is infinite, by Schmidt and Strunk for étale cohomology with finite coefficients coprime to $\text{char}(k)$ in [22], using their version of Gabber’s geometric presentation lemma in mixed characteristic from [21]. Theorem 1.1 improves their result to the Zariski topology and arbitrary residue fields. The main difference is that we make use of “Quillen’s trick” and follow ideas of Gillet and Levine [7]. The general case of the Gersten conjecture for a regular scheme in mixed characteristic is wide open for étale cohomology with finite coefficients coprime to $\text{char}(k)$ but also for any other theory for which one would expect it to hold; examples include motivic cohomology, K-theory and Milnor K-theory. This mixed characteristic case would, however, have many applications, for example to the (contravariant) functoriality of motivic cohomology in the Chow group and higher zero-cycle range (see [13] and Remark 1.4 below). Theorem 1.3 is a first step in this direction. The method we develop in the proof, and which so far only works in relative dimension one, is to use blow-ups and an induction on the number of irreducible components of the special fiber. This is inspired by [9, Sec. 3].

Remark 1.4 (Functoriality in the semi-stable case). The Gersten conjecture in the Zariski or Nisnevich topology for étale cohomology with finite coefficients coprime to $\text{char}(k)$ for a regular scheme $X$ flat over $S$ as above with special fiber $X_1$ a simple normal crossing divisor would imply the Gersten conjecture for the improved Milnor K-sheaf $\hat{K}^M_{q,X,Nis}/m$, with $m$ coprime to $\text{char}(k)$, in the Nisnevich topology. Indeed, the sheaf $\hat{K}^M_{q,X,Nis}/m$ identifies with the sheaf $H^q(\mu^m_1)$ for henselian local rings: Milnor K-theory is rigid by Proposition 3.7 and étale cohomology is rigid by proper base change; passing to a bigger residue field and using a standard norm argument, the two can be identified. The other terms in the respective Gersten resolutions are identified by the norm residue isomorphism. This allows to define a restriction map

$$CH^q(X)/m \to H^{2q}_{cdh}(X_1, \mathbb{Z}/m\mathbb{Z}(q))$$

through the diagram

$$\begin{array}{ccc}
H^q_{Nis}(X, \hat{K}^M_q/m) & \to & H^q_{Nis}(X_1, \hat{K}^M_q/m) \\
\cong & & \cong \\
CH^q(X)/m & - - - & H^{2q}_{cdh}(X_1, \mathbb{Z}/m\mathbb{Z}(q)),
\end{array}$$

in which the left vertical isomorphism, that is Bloch’s formula, comes from the Gersten conjecture. For the isomorphism on the right see [13, Sec. 8.2]. This restriction map is studied in loc. cit. for $q$ the relative dimension of $X$ over $S$.

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2 Homology theories with duality

We define homology theories with duality. The main references are [14], [1] and [10].

Definition 2.1. Let $S$ be the spectrum of a field or an excellent Dedekind domain. Let $\mathcal{C}_S$ be the category of schemes separated and of finite type over $S$. Let $\mathcal{C}_{S*}$ be the category with the same objects as $\mathcal{C}_S$ but where morphisms are just the proper maps in $\mathcal{C}_S$. Let $Ab$ be the category of abelian groups.

A homology theory $H(n) = \{H_q(n)\}_{q \in \mathbb{Z}}$ on $\mathcal{C}_S$, for some fixed $n \in \mathbb{Z}$, is a sequence of covariant functors:

$$H_q(\dash, n): \mathcal{C}_{S*} \to Ab$$

satisfying the following conditions:

1. (Presheaf in the étale topology) For each étale morphism $j: V \to X$ in $\mathcal{C}_S$, there is a map $j^*: H_q(X, n) \to H_q(V, n)$, associated to $V$ in a functorial way.

2. (Localisation) If $i: Y \hookrightarrow X$ is a closed immersion with open complement $j: U \to X$, there is a long exact sequence (called localization sequence)

$$\cdots \to H_q(Y, n) \xrightarrow{i_*} H_q(X, n) \xrightarrow{j^*} H_q(U, n) \xrightarrow{\partial} H_{q-1}(Y, n) \to \cdots$$

This sequence is functorial with respect to proper maps and open immersions in an obvious way.

Remark 2.2. (Twist) The twist $n$ in the above definition is fixed and could therefore be dropped from the notation. We keep it to keep track of the twist in the corresponding cohomology theory.

Definition 2.3. A cohomology theory with supports is given by abelian groups $H^q_Y(X, n)$ for any closed immersion $Y \hookrightarrow X$ of schemes in $\mathcal{C}_S$ and for any $q, n \in \mathbb{Z}$, which satisfies the following conditions. For $Y = X$, $H^q_Y(X, n)$ is simply denoted by $H^q(X, n)$.

1. (Contravariant functoriality) The groups $H^q_Y(X, n)$ are contravariant functorial for Cartesian squares

$$\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow \scriptstyle{i'} & & \downarrow \scriptstyle{f} \\
Y & \xrightarrow{i} & X, 
\end{array}$$

i.e. there exists a pullback $f^*: H^q_Y(X, n) \to H^q_{Y'}(X', n)$.

2. (Localisation) For closed immersions $Z \hookrightarrow Y \hookrightarrow X$ there is a long exact sequence

$$\cdots \to H^q_Z(X, n) \to H^q_Y(X, n) \to H^q_{Y-Z}(X-Z, n) \xrightarrow{\partial} H^{q+1}_Z(X, n) \to \cdots$$

compatible with pullbacks. Note that this implies in particular that

$$H^q_Y(X, n) \simeq H^q_{Y_{red}}(X, n)$$

for a closed immersion $Y \hookrightarrow X$ and the reduced part $Y_{red}$ of $Y$.

3. (Excision) Let $Y \hookrightarrow X \in \mathcal{C}_S$ be a closed immersion and $U$ be an open subset of $X$ containing $Z$. Then there is an isomorphism

$$H^q_Z(X, n) \xrightarrow{\gamma} H^q_Z(U, n).$$

Definition 2.4. A homology theory with duality $(H_*, H^*)$ is a tuple of a homology theory $H_*$ in the sense of Definition 2.1 and a cohomology theory with supports $H^*$ in the sense of Definition 2.3 such that the following structural properties exist:
(1) (Cap product with supports) For any closed immersion $Y \hookrightarrow X$ in $\mathcal{C}_S$ and $p, q \in \mathbb{Z}$, there is a pairing

$$\cap : H^p_p(X, m) \times H^q_Y(X, n) \to H_{p-q}(Y, m-n).$$

(2) (Compatibility of cap product with restriction) Consider a Cartesian diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow f_Y & & \downarrow f_X \\
Y & \longrightarrow & X
\end{array}
$$

such that $f_X$ is étale. Then for $a \in H^q_Y(X, n)$ and $z \in H^p_p(X, m)$ we have that $f^*_X(a) \cap f^*_X(z) = f^*_Y(a \cap z)$.

(3) (Projection formula) Consider a Cartesian diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow f_Y & & \downarrow f_X \\
Y & \longrightarrow & X
\end{array}
$$

such that $f_X$ is proper. Let $a \in H^q_Y(X, n)$ and $z \in H^p_p(X', m)$. Then the diagram

$$
\begin{array}{ccc}
H^p_p(X', m) & \times & H^q_Y(X', n) \\
\downarrow f_{X*} & & \downarrow f_{X*} \\
H^p_p(X, m) & \times & H^q_Y(X, n)
\end{array}
\xrightarrow{\cap} 
\begin{array}{ccc}
H_{p-q}(Y', m-n) & \rightarrow & H_{p-q}(Y, m-n) \\
\downarrow f_{Y*} & & \downarrow f_{Y*} \\
H_{p-q}(Y, m-n)
\end{array}
$$

commutes in the sense that $f_{X*}(z) \cap a = f_{Y*}(z \cap f^*_X(a))$.

(4) (Fundamental class) If $X \in \mathcal{C}_S$ is of dimension $d$ with structural morphism $\pi : X \to S$, then there is a global section $\eta_X$ of $H^{2d}(X, d - \dim \pi(X))$ such that if $\alpha : X' \to X$ is étale, then $\alpha^* \eta_X = \eta_{X'}$.

(5) (Poincaré duality) If $\pi : X \to S \in \mathcal{C}_S$ is regular of dimension $d$ and $Y \hookrightarrow X$ a closed immersion, then the cap product with $\eta_X$ induces an isomorphism

$$\eta_X \cap - : H_Y^{2d-\eta}(X, d - \dim \pi(X) - n) \to H_q(Y, n).$$

**Lemma 2.5.** Let $(H_*, H^*)$ be a homology theory with duality. Suppose that the square

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow f_Y & & \downarrow f_X \\
Y & \longrightarrow & X
\end{array}
$$

is Cartesian and that $f_X$ is étale. Then the square

$$
\begin{array}{ccc}
H^q_Y(X', n) & \longrightarrow & H^{2d-q}(Y', d - \dim \pi(X) - n) \\
\downarrow f_{X*} & & \downarrow f_{Y*} \\
H^q_Y(X, n) & \longrightarrow & H^{2d-q}(Y, d - \dim \pi(X) - n)
\end{array}
$$

commutes.

**Proof.** This follows from (2) and (4).
We will need the following properties:

**Property 2.6.** (Descent for blow-ups) Let $X \in \text{Ob} \mathcal{C}_S$ and $\pi: \tilde{X} \to X$ be the blow-up of $X$ along a closed subscheme $Z$. Let $E \hookrightarrow \tilde{X}$ be the exceptional divisor. We say that a homology theory with duality $(H_*, H^*)$ satisfies the descent for blow-ups property if for any integer $q$ and any $X, Z$ as above, there is a split exact sequence

$$0 \to \tilde{H}_q(E, n) \to H_q(\tilde{X}, n) \overset{\mathcal{Z}}{\to} H_q(X, n) \to 0,$$

where

$$0 \to \tilde{H}_q(E, n) := \ker(\pi_*: H_q(E, n) \to H_q(Z, n)).$$

**Property 2.7.** (Principal triviality) Let $\pi: X \to S \in \mathcal{C}_S$. Let $i: W \hookrightarrow X$ be a smooth principal divisor and $\dim(W) = d$. Assume that $W$ is flat over $\pi(X)$. We say that a homology theory with duality $(H_*, H^*)$ satisfies the principal triviality property if the morphism $i_*: H_{2d}(W, d - \dim \pi(X)) \to H_{2d}(X, d - \dim \pi(X))$ is zero for all $X, W$ as above.

**Property 2.8.** (Lifting) Let $R$ be a regular semi-local ring and $Z$ be the set of closed points of $\text{Spec}(R)$. We say that a homology theory with duality $(H_*, H^*)$ satisfies the lifting property if the sequence

$$0 \to H^q(R, n) \to H^q(K(R), n) \to H^{q-1}(Z, n - 1) \to 0$$

is exact.

### 2.2 An example: étale cohomology

The following is the main example which satisfies the above definitions and properties.

**Example 2.9.** Let $S = \text{Spec} \mathcal{O}_K$ be the spectrum of a discrete valuation ring $\mathcal{O}_K$ with residue field $k$. Let $\Lambda = \mathbb{Z}/\ell^n \mathbb{Z}$ and $\ell$ be a prime number coprime to char($k$). Let $\Lambda(n) = \mu_{\ell^n}$ for $n \in \mathbb{N}$ and $\Lambda(n) = \text{Hom}(\Lambda(-n), \Lambda)$ for $-n \in \mathbb{N}$ denote the Tate twists. One can define an étale homology theory with duality as follows: define the homology of a morphism $\pi: X \to S \in \mathcal{C}_S$ to be

$$H^q_\pi(X, n) = H^{2d-q}(X_{\text{et}}, R \pi^1 \Lambda(-n))$$

and the cohomology with support in $Y \hookrightarrow X$ to be

$$H^q_\pi(X, n) = H^q(Y, \Lambda(n)).$$

Indeed, it is shown in [10, Ex. 2.2] and [18, Prop. 1.5] that $H^q_\pi(\cdot)$ defines a homology theory. The fact that $H^q_\pi(X, n)$ is a cohomology theory with supports is well known and it is shown in [18, Lem. 1.8] that these two form a homology theory with duality. In particular, if $\dim X = d$ and $X$ is regular, then there is an isomorphism

$$\tau_{X,Y}: H^{2d-q}_\pi(X, \Lambda(d - \dim(\pi(X)) - n)) \overset{\cong}{\to} H_q(Y, \Lambda(n)).$$

For the convenience of the reader, we sketch the proof of this isomorphism if $\pi$ is flat, following the proof of the existence of this isomorphism of loc. cit.. For the case in which $\pi$ factorises through the closed point of $S$, i.e. is not flat, we refer to [1, Sec. 2] or also [18, Lem. 1.8]. Let

$$T_{\tau_\pi}: R\pi_!\Lambda(d - 1 + n)[2d - 2] \to R\Lambda_S(n)$$

be the trace morphism due to Deligne [8, XVIII, Thm. 2.9] and

$$T_\pi: \Lambda_X(d - 1 + n)[2d - 2] \to R\pi^1 \Lambda_S(n)$$

its adjoint. Applying $H^{2d-q}_\pi(X, -)$ to $T_\pi$ gives the morphism $\tau_{X,Y}$. It therefore suffices to show that $T_{\tau_\pi}$ is an isomorphism. Fix a locally closed embedding $\gamma: X \hookrightarrow \mathbb{P} := \mathbb{P}_S^N$. Let $\rho: \mathbb{P} \to S$ be the projection. Then $T_{\tau_\pi}$ factors as

$$\Lambda_X(d - 1 + n)[2d - 2] \xrightarrow{\text{Gys}_\gamma} R\gamma^1 \Lambda(N + n)|_\mathbb{P}[2N] \xrightarrow{T_{\tau_\pi}} R\gamma^1 \rho^1 \Lambda_S(n) = R\rho^1 \Lambda_S(n)$$

by [4, Ch. 4, Thm. 2.3.8(i)]. Since the Gysin morphism $\text{Gys}_\gamma$ is an isomorphism by absolute purity [5] and $T_{\tau_\pi}$ is an isomorphism by [8, XVIII, Thm. 3.2.5] (i.e. Poincaré duality for $\mathbb{P}$), we get that $T_{\tau_\pi}$ is an isomorphism.
We now show that the étale homology theory with duality, or short étale cohomology, of Example 2.9 has Properties 2.6, 2.7 and 2.8.

**Lemma 2.10** (See also [18, Lem. 6.1]). Let $X \in C_S$ be regular of dimension $d$. Let $\pi: \tilde{\cal{X}} \to X$ be the blow-up of $X$ along a closed subscheme $V \subset X$. Let $W$ be a closed subscheme of $X$ not contained in $V$ and $W$ its strict transform. Let $E_W \hookrightarrow W$ be the exceptional divisor. Then there is a split exact sequence

$$0 \to \tilde{H}_q(E_W, n) \to H_q(W, n) \to H_q(W - E_W, n) \to 0,$$

where

$$\tilde{H}_q(E_W, n) := \ker(\pi_*: H_q(E_W, n) \to H_q(V \cap W, n)).$$

I.o.w. the étale cohomology theory of Example 2.9 satisfies the blow-up Property 2.6.

**Proof.** There is a commutative diagram with exact sequences

$$\begin{array}{cccccc}
H_{q+1}(\tilde{W} - E_W, n) & \longrightarrow & H_q(E_W, n) & \longrightarrow & H_q(\tilde{W}, n) & \longrightarrow & H_q(W - E_W, n) \\
\cong & & & & & & \\
H_{q+1}(W - V \cap W, n) & \longrightarrow & H_q(V \cap W, n) & \longrightarrow & H_q(W, n) & \longrightarrow & H_q(W - V \cap W, n).
\end{array}$$

We need to show that the map $\pi_*: H_q(\tilde{W}, n) \to H_q(W, n)$ is split surjective. We define a split $\pi^*: H_q(W, n) \to H_q(\tilde{W}, n)$ of $\pi_*$ to be the composition

$$\pi^*: H_q(W, n) \cong H^{2d-q}_W(X, d - \dim(\pi(X) - n)) \overset{\pi^*}{\longrightarrow} H^{2d-q}_W(\tilde{X}, d - \dim(\pi(X) - n)) \cong H_q(\tilde{W}, n)$$

and show that $\pi_*\pi^* = \text{Id}$. By the projection formula we have a commutative diagram of cap-product pairings

$$\begin{array}{cccccc}
H^0(\tilde{X}, 0) \times H^{2d-q}_W(\tilde{X}, d - \dim(\pi(X) - n)) & \longrightarrow & H^{2d-q}_W(\tilde{X}, d - \dim(\pi(X) - n)) \\
\downarrow & & & & & \downarrow \\
H^0(X, 0) \times H^{2d-q}(X, d - \dim(\pi(X) - n)) & \longrightarrow & H^{2d-q}(X, d - \dim(\pi(X) - n)).
\end{array}$$

This implies that $\pi_*\pi^*(a) = \pi_*(1 \cap \pi^*(a)) = \pi_*(1) \cap a = a$ since $\pi_*(1) = 1$ as $\pi$ is an isomorphism outside of $E$. □

**Lemma 2.11.** The étale cohomology theory of Example 2.9 satisfies the principal triviality Property 2.7.

**Proof.** Let $\pi: X \to S \in C_S$. Let $i: W \hookrightarrow X$ be a smooth principal divisor and $\dim(W) = d$. Assume that $W$ is flat over $\pi(X)$. Consider the commutative diagram

$$\begin{array}{ccc}
H_{2d}(W, d - \dim(\pi(X)) & \longrightarrow & H_{2d}(X, d - \dim(\pi(X)) \\
\cong & & \cong \\
H^0(W, \Lambda) & \longrightarrow & H^2(X, \Lambda(1)).
\end{array}$$

The lower horizontal map sends the fundamental class of $W$ to zero since $[\cal{O}_W] = 0 \in \text{Pic}(X)$ as $W$ is principle which implies the statement. Indeed, we have a commutative diagram

$$\begin{array}{cccc}
H^0(W, \Lambda) & \longrightarrow & H^2(X, \Lambda(1)) & \longrightarrow & \text{Pic}(X) \\
& & & & \\
& & & & \text{Pic}(X)
\end{array}$$

and $1 \in H^0(W, \Lambda)$ and $1 \in H^1_W(\cal{X}, \cal{O}_X^\times)$ have the same image in $H^2(X, \Lambda(1))$. □
The following lemma is proved in [2, Thm. B.2.1]. We give a sketch for the sake of completeness. Note that the case of discrete valuation rings is sufficient for applications to the Gersten conjecture in the smooth case (see the end of the proof of Corollary 3.5), while we need the semi-local case in Section 4 for the semi-stable case.

**Lemma 2.12.** The étale cohomology theory of Example 2.9 satisfies the lifting Property 2.8.

**Proof.** We recall the notation: let $R$ be a regular semi-local ring and $Z$ be the set of closed points of $\text{Spec}(R)$. We need to show that the sequence

$$0 \to H^q(R, n) \to H^q(K(R), n) \to H^{q-1}(Z, n-1) \to 0$$

is exact. Let $R^h$ be the henselisation of $R$ along $Z$. Let $\alpha \in H^{q-1}_{et}(Z, \Lambda(n-1))$. By Gabber’s affine analogue of proper base change we have that $H^{q-1}_{et}(Z, \Lambda(n-1)) \cong H^{q-1}_{et}(R^h, \Lambda(n-1))$. This implies that $\alpha$ is in the image of some $\alpha' \in H^{q-1}_{et}(A, \Lambda(n-1))$ where $A$ is an étale neighbourhood of $Z$ and quasi-finite over $R$. Let $R'$ be the integral closure of $R$ in $F' = \text{Frac}(A)$. Let $f: \text{Spec}(R') \to \text{Spec}(R)$ be the natural map and recall that there is a trace map on cohomology induced by the counit map $f_*f^* \to \text{Id}$ (see [8, exposé XVIII]). Then $T' := f^{-1}(Z)$. We consider the commutative diagram

\[
\begin{array}{cccccc}
H^q_{et}(R', \Lambda(n)) & \rightarrow & H^q_{et}(F', \Lambda(n)) & \rightarrow & H^q_{et}(R', \Lambda(n)) & \rightarrow & \bigoplus_p H^q_{et}(R'/p, \Lambda(n-1)) \\
\downarrow\text{tr} & & \downarrow\text{tr} & & \downarrow\text{tr} & & \\
H^q_{et}(R, \Lambda(n)) & \rightarrow & H^q_{et}(\text{Frac}(R), \Lambda(n)) & \rightarrow & H^q_{et}(Z, \Lambda(n-1)) & \rightarrow & \bigoplus_m H^q_{et}(R/m, \Lambda(n-1)), \\
\end{array}
\]

where the $p$ are the prime ideals of $R'$ lying over $Z$ and the $p$ are the maximal ideals of $R$. By the Chinese remainder theorem we can choose an element $g \in F''$ which is congruent to $0$ at a copy of $Z$ in $T''$ and congruent to $1$ otherwise. Then $\partial \circ \text{tr}(\alpha' \cup g) = \text{tr} \circ (\oplus \partial)(\alpha' \cup g) = \alpha$. \qed

**Remark 2.13.** Let $S = \text{Spec} O_K$ be the spectrum of a discrete valuation ring $O_K$ with perfect residue field $k$ of characteristic $p > 0$. Let $X$ be a regular semi-stable scheme over $S$ and let $\mathcal{T}_r(n)_X$ denote the $p$-adic étale Tate twists defined in [20]. One can define a $p$-adic étale homology theory in weight $-1$ as follows (see [11, Sec. 4.10]): define the homology of a morphism $\pi: X \to S \in C_S$ to be

$$H_q(X, \mathcal{Z}/p^n(-1)) = H^{2-q}(X_{et}, R_\pi^p \mathcal{T}_r(1)_S)$$

and the cohomology with support in $Y \hookrightarrow X$ to be

$$H^q_Y(X, n) = H^q_Y(X_{et}, \mathcal{T}_r(n)_X).$$

There is a Poincaré duality isomorphism

$$H^{2d-q}_r(X, \mathcal{T}_r(d)_X) \cong H^{2-q}(X_{et}, \mathcal{T}_r(d)[2d-2]) \cong H^{2-q}(X_{et}, Rf^p \mathcal{T}_r(1)_S) \cong H_q(Y, -1),$$

the relative duality theorem of Sato [20, Thm. 7.3.1] giving the isomorphism in the middle. However, the theory not being defined for arbitrary weights, it does not seem to be possible to define cap products with supports.

### 3 The Gersten Conjecture: The Smooth Case

Let $S$ be the spectrum of an excellent discrete valuation ring $O_K$ and $X$ be a regular and flat scheme over $S$. Let $(H_*, H^*)$ be a homology theory with duality and let $\mathcal{H}^*(n)$ be the sheafification of the presheaf...
In this section let \( U \mapsto H^*(U, n) \) for the Zariski topology on \( X \). We begin by recalling some general reductions which imply that the sequence of sheaves in the Zariski topology on \( X \)

\[
0 \to H^0(n) \to \bigoplus_{x \in X^{(0)}} i_{*,x}H^q(x, n) \to \bigoplus_{x \in X^{(1)}} i_{*,x}H^{q-1}(x, n-1) \to \cdots \to \bigoplus_{x \in X^{(d)}} i_{*,x}H^{q-d}(x, n-d) \to 0 \quad (\ast)
\]

is exact (Section 3.1). Then in Section 3.3 we prove the smooth case and in Section 4 a part of the semi-stable case.

### 3.1 General reductions

In this section let \( (H_*, H^*) \) be a homology theory with duality and let \( H^*(n) \) be the sheafification of the presheaf \( U \mapsto H^*(U, n) \) for the Zariski topology on \( X \).

**Definition 3.1.**  
(1) Let \( X \) be a scheme separated and of finite type over \( S \). We let

\[
Z^p(X) := \{ Z \subset X | Z \text{ closed, codim}_X(Z) \geq p \}.
\]

If \( X \) is clear from the context, then we also write \( Z^p \) instead of \( Z^p(X) \). Let \( Z^p/Z^{p+1} \) denote the ordered set of pairs \( (Z, Z') \in Z^p \times Z^{p+1} \) such that \( Z' \subset Z \), with the ordering \( (Z, Z') \geq (Z_1, Z'_1) \) if \( Z \supset Z_1 \) and \( Z' \supset Z'_1 \).

(2) For the Zariski topology on \( X \), we define \( H^0_{Z^p}(n) \) to be the sheaf associated to the presheaf

\[
U \mapsto \lim_{\to} H^0_{Z^p/U}(U, n)
\]

and \( H^0_{Z^p/Z^{p+1}}(n) \) to be the sheaf associated to the presheaf

\[
U \mapsto \lim_{\to} H^0_{Z^p/Z^{p+1}/U}(U, n).
\]

(3) Let

\[
H_q(x, n) := \lim_{\to} H_q(U, n),
\]

\[
H^q(x, n) := \lim_{\to} H^q(U, n)
\]

and

\[
H^q_{\infty}(X, n) := \lim_{\to} H^q_{[x] \cap U}(U, n),
\]

where the colimit is taken over all open subschemes \( U \) in \( \{x\} \) and all open subschemes \( U \) in \( X \) containing \( x \) respectively.

The complex in \((\ast)\) is exact if the natural map

\[
(\ast\ast): H^q_{Z^{p+1}}(n) \to H^q_{Z^p}(n)
\]

is zero for all \( p, q, n \in \mathbb{N} \). Indeed, assuming \((\ast\ast) = 0\) for all \( p, q, n \in \mathbb{N} \), the exact sequences

\[
\ldots \to H^q_{Z^p/Z^{p+1}}(n) \to H^q_{Z^p}(n) \to H^q_{Z^p/Z^{p+1}}(n) \to \ldots
\]

and

\[
\ldots \to H^q_{Z^{p+1}/Z^{p+2}}(n) \to H^q_{Z^{p+2}}(n) \to H^q_{Z^{p+1}/Z^{p+2}}(n) \to \ldots
\]

splice together as follows:

\[
\ldots \to H^q_{Z^p/Z^{p+1}}(n) \to H^q_{Z^{p+1}}(n) \to H^q_{Z^{p+1}/Z^{p+2}}(n) \to H^q_{Z^{p+2}}(n) \to \ldots
\]
Assuming that $X$ is regular, the Poicaré duality isomorphism (Definition 2.4(5)) implies that

$$H^q_x(X, n) \cong H^{q-2p}(x, n - p)$$

for $\text{codim}_X(x) = p$. This in turn implies\(^1\) that there are isomorphisms

$$H^{q-1}_{Z^p/Z^{p+1}}(n) \cong \bigoplus_{x \in Z^p/Z^{p+1}} i_x H^{q-2p-1}(x, n - p)$$

and

$$H^q_{Z^{p+1}/Z^{p+2}}(n) \cong \bigoplus_{x \in Z^{p+1}/Z^{p+2}} i_x H^{q-2p-2}(x, n - p - 1).$$

The following definition is used in applications to show that $(**)$ is zero.

**Definition 3.2.** Let $f : Z_1 \to Z_2$ be a morphism in $C_{S*}$ and $T \subseteq Z_2$ a finite set. We say that $f$ is **homologically effaceable** at $T$, iff there is an open $U \subseteq Z_2$ containing $T$ such that the composition

$$H_*(Z_1) \to H_*(Z_2) \to H_*(U)$$

is zero.

If for every $x \in X$ and every open subset $x \in W \subseteq X$ and every $x \in Z_1 \subseteq Z_2(W)$ there exists a $Z_2 \in Z^p(W)$ and a homologically effaceable morphism $f : Z_1 \to Z_2$ at $x$, then $(**)$ is zero. In cohomological terms, this is equivalent to saying that the composition

$$H^*_Z(W) \to H^*_{ZU}(U) \to H^*_U(U)$$

(which factorises through $H^*_U(U)$) is zero.

### 3.2 A key lemma due to Bloch and Ogus

In this section we transfer two lemmas of Bloch and Ogus to the relative situation. The proofs go through the same way but we give them for the convenience of the reader - all that is needed are the smoothness assumptions on certain morphisms, which are relative notions.

**Lemma 3.3.** [1, Lem. 5.1] Let

$$\begin{array}{ccc}
Z_2 & \xrightarrow{i_2} & X_2 \\
\downarrow{g} & & \downarrow{f} \\
Z_1 & \xrightarrow{i_1} & X_1
\end{array}$$

be a commutative square in $C_S$ with $i_1, i_2$ closed immersions and $f, g$ smooth. Let $T \subseteq X_2$ be a finite set of points contained in an affine scheme. Then after replacing $Z_2$ and $X_2$ by Zariski neighbourhoods of $T$, there exists a closed subscheme $X'_2 \subseteq X_2$ containing $Z_2$ such that the induced morphism $f : X'_2 \to X_1$ is still smooth and the square

$$\begin{array}{ccc}
Z'_2 & \xrightarrow{i_2} & X'_2 \\
\downarrow{g} & & \downarrow{f'} \\
Z'_1 & \xrightarrow{i_1} & X_1
\end{array}$$

is Cartesian.

---

\(^1\)We use the fact that if $T_1, \ldots, T_r$ are pairwise disjoint closed subsets of $X$, then $\bigoplus_i H^q_{T_i}(X, A) \cong H^q_{U/T_i}(X, A)$ (see [2, Lemma 1.2.1]).
**Proof.** Let $Y = f^{-1}(Z_1) \subset X_2$. Let $\mathcal{I}$ be the ideal defining $Z_2$ as a subscheme of $X_2$ and $\bar{\mathcal{I}}$ be the ideal defining $Z_2$ as a subscheme of $Y$.

\[ \begin{array}{ccc}
\bar{\mathcal{I}} & \subset & \mathcal{I} \\
\downarrow & & \downarrow \\
Y & \subset & X_2 \\
\downarrow & & \downarrow \\
Z_1' & \subset & X_1
\end{array} \]

Then since $Y$ and $Z_2$ are smooth over $Z_1$ there is a short exact sequence

$$0 \to \bar{\mathcal{I}}/\mathcal{I}^2 \to \Omega^1_{Y/Z_1} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Z_2} \to \Omega^1_{Z_2/Z_1} \to 0$$

of locally free sheaves on $Z_2$. Replacing $X_2$ if necessary by an open neighbourhood of $T$, we can find sections $f_1, \ldots, f_r$ of $\mathcal{I}$ whose images $\bar{f}_1, \ldots, \bar{f}_r$ in $\bar{\mathcal{I}}$ form a basis of $\bar{\mathcal{I}}/\mathcal{I}^2$. Let $X_2' \subset X_2$ be the subscheme defined by $f_1 = \ldots = f_r = 0$.

By base change we have that $\Omega^1_{Y/Z_1} \cong \Omega^1_{X_2/Z_1} \otimes_{\mathcal{O}_{X_2}} \mathcal{O}_Y$, which implies that the differentials $df_1, \ldots, df_r \in \Omega^1_{X_2/Z_1}$ are independent in some neighbourhood of $T$. This implies that after shrinking, $X_2'$ is smooth over $X_1$ since $\dim \Omega^1_{X_2/Z_1} = \dim \Omega^1_{X_2/Z_1}/(df_1, \ldots, df_r) \geq \dim X_2 - \dim X_1$. Now it follows from Nakayama’s lemma that the $f_1, \ldots, f_r$ generate $\bar{\mathcal{I}}$ in some neighbourhood of $T$ and therefore $\mathcal{I}$ is generated by $f_1, \ldots, f_r$ and the sheaf of ideals $\mathcal{I}_Y$ defining $Y$ in $X_2$. This means that the lower square in the statement of the lemma is Cartesian, indeed taking a quotient by $f_1, \ldots, f_r$ identifies $Z_2$ and $Y$ in $X_2'$.

**Key lemma 3.4.** [1, Prop. 4.5] Let $S$ be the spectrum of an excellent discrete valuation ring $\mathcal{O}_K$. Let $(H_*, H^*)$ be a homology theory with duality on $\mathcal{C}_S$ satisfying the principal triviality Property 2.7. Let $i: Z_1 \hookrightarrow Z_2$ be a closed immersion of affine schemes in $\mathcal{C}_S$ and $\pi: Z_2 \to Z_1$ be a section of $i$ which is smooth of relative dimension 1 at a finite set $T$ of points in $Z_2$. Then $i$ is homologically effaceable at $T$.

**Proof.** We can assume that $Z_1$ is flat over $S$. Otherwise we are in the situation of [1, Prop. 4.5]. Since $Z_1$ and $Z_2$ are affine we can find a commutative diagram

\[ \begin{array}{ccc}
Z_2' & \xrightarrow{i_2} & X_2 \\
\downarrow & \downarrow & \downarrow \\
Z_1' & \xrightarrow{i_1} & X_1
\end{array} \]

with $X_1$ and $X_2$ smooth over $S$, $f$ smooth and $i_1$ and $i_2$ closed immersions. By Lemma 3.3 we can find a neighbourhood $U$ of $T$ in $X_2$ and a closed subscheme $X_2' \subset U \cap X_2$ containing $Z_2' = Z_2 \cap U$ such that $f': X_2' \to X_1$ is smooth of relative dimension 1, $T \subset Z_2'$ and such that the square

\[ \begin{array}{ccc}
Z_2' & \xrightarrow{i_2} & X_2' \\
\downarrow & \downarrow & \downarrow \\
Z_1' & \xrightarrow{i_1} & X_1
\end{array} \]

is Cartesian. Let $Z_1' = i^{-1}(Z_2')$. Then the map $\alpha: Z_1' \to Z_1$ induced by $\pi'$ is an open immersion and we get a commutative square

\[ \begin{array}{ccc}
Z_1' & \xrightarrow{i_2} & Z_2' \\
\downarrow & \downarrow & \downarrow \\
Z_1' & \xrightarrow{i_1} & X_1
\end{array} \]

(1)
Applying Lemma 3.3 to the outer square of the diagram (1), we get a commutative diagram

\[ \begin{array}{ccc}
Z'_1 & \xrightarrow{h} & X'_2 \\
\alpha \downarrow & & \downarrow g \\
Z'_1 & \xrightarrow{i} & X_1
\end{array} \]

in which the square on the left is Cartesian. Since \( f' \) and \( g \) are smooth, \( h \) is a regular immersion and since \( g \) is of relative dimension 0, this implies that \( W \), also smooth over \( S \), is a smooth Cartier divisor on \( X'_2 \). Shrinking \( X'_2 \) further around \( T \), we may assume that \( W \) is a principle Cartier divisor.

Putting these diagrams together, we get a commutative diagram

\[ \begin{array}{ccc}
Z'_1 & \xrightarrow{h} & W \\
\alpha \downarrow & & \downarrow g \\
Z'_1 & \xrightarrow{i} & X_1
\end{array} \]

in which \( \alpha \) is an open immersion and \( g \) is étale.

Let \( d = \dim X_1 = \dim W \) (and therefore \( \dim X'_2 = d + 1 \)). Consider the commutative diagram

\[ \begin{array}{ccc}
H^q_{Z_1}(X_1, n) & \xrightarrow{f^*} & H^q_{Z'_2}(X'_2, n) \\
\cap \eta_{Z_1} \cong & & \cap h^* \eta_{W} \\
H_{2d-q}(Z_1, d - 1 - n) & \xrightarrow{g^*} & H_{2d-q}(Z'_2, d - 1 - n) \\
\alpha^* \cong & & \cap h_{*}, n \\
H_{2d-q}(Z'_1, d - 1 - n)
\end{array} \]

in which the bottom left square commutes by Lemma 2.5 and the triangle at the top commutes since \( g = h \circ f \). The bottom right square commutes by the projection formula.

Now \( \cap \eta_{X_1} \) is an isomorphism by Poincaré Duality and \( \cap h_{*}, n \cap W = \cap 0 = 0 \). Indeed, the right vertical map is given by \( H^q_{Z_2}(X'_2, n) \times H_{2d}(X'_2, d - 1 - n) \xrightarrow{- \cap h_{*}, n} H_{2d-q}(Z'_2, d - 1 - n) \) and the image of \( \eta_{W} \) under the map \( h_{*} : H_{2d}(W, d - 1) \to H_{2d}(X'_2, d - 1) \) is zero by Property 2.7 since \( W \) is a smooth and locally principle divisor. This and the commutativity imply that \( \iota^* \alpha^* = 0 \). But now the commutative diagram

\[ \begin{array}{ccc}
Z_1 \cap Z'_2 = Z'_1 & \xrightarrow{\alpha} & Z_1 \\
\iota' \downarrow & & \downarrow i \\
Z_2 \cap U = Z'_2 & \xrightarrow{\beta} & Z_2,
\end{array} \]

in which \( \alpha \) and \( \beta \) are open, implies that the diagram

\[ \begin{array}{ccc}
H_{2d-q}(Z'_1, d - n) & \xrightarrow{\alpha^*} & H_{2d-q}(Z_1, d - n) \\
\iota' \downarrow & & \downarrow i^* \\
H_{2d-q}(Z'_2, d - n) & \xrightarrow{\beta^*} & H_{2d-q}(Z_2, d - n)
\end{array} \]

commutes and therefore that the composition

\[ H_{2d-q}(Z_1, d - n) \xrightarrow{i^*} H_{2d-q}(Z_2, d - n) \xrightarrow{\beta^*} H_{2d-q}(Z'_2, d - n) \]

is zero which means that \( i \) is homologically effaceable at \( T \).

\[ \square \]
3.3 Proof of the smooth case

Corollary 3.5. Let $S = \text{Spec} \mathcal{O}_K$ be the spectrum of an excellent discrete valuation ring $\mathcal{O}_K$ and $X$ be a smooth scheme over $S$. Let $(H_\ast, H^\ast)$ be a homology theory with duality on $C_2$ satisfying the principal triviality Property 2.7 and the lifting Property 2.8. Let $H^\ast(n)$ be the sheafification of the presheaf $U \mapsto H^\ast(U, n)$. Then the sequence of sheaves

$$0 \to H^0(n) \to \bigoplus_{x \in X^{(0)}} i_{x, x}H^q(x, n) \to \bigoplus_{x \in X^{(1)}} i_{x, x}H^{q-1}(x, n-1) \to \cdots \to \bigoplus_{x \in X^{(d)}} i_{x, x}H^{q-d}(x, n-d) \to 0$$

is exact.

Proof. Let $x \in X$. We need to show that the map $(\ast \ast)$ is zero at $x$. In order to simplify notation let $X$ be an open neighbourhood of $x$. Let $Z' \in Z^{p+1}(X), x \in Z'$, be flat over $\mathcal{O}_K$ or $p > 0$. Then up to shrinking $X$ by [7, Lem. 1] there exists a smooth morphism of relative dimension 1

$$g: X \to A$$

such that the restriction $f := g|_{Z'}: Z' \to A$ is quasi-finite. Note that if $Z'$ is not flat over $\mathcal{O}_K$ but of codimension $> 1$, then [7, Lem. 1] still applies since $Z'$ is contained in a flat divisor of $X$. Let $Z'' := Z' \times_A X$ and $i: Z' \to Z''$ be the induced closed immersion. By Zariski’s main theorem $f'$ factorises into an open immersion $Z'' \to Z''$ and a finite map $\tilde{f}': Z'' \to X$. There is commutative diagram

$$\begin{array}{ccc}
Z' & \xrightarrow{i} & Z'' \\
\downarrow{g'} & \downarrow{g} & \downarrow{\tilde{g}} \\
Z & \xrightarrow{f} & A
\end{array}$$

Since $g$ is smooth at $x$ and of relative dimension 1, $g'$ is smooth of relative dimension 1 at the points of $S' = f'^{-1}(x)$. Furthermore, since $f'$ is finite, $Z = f'(Z'') \in Z^q(X)$ and $Z' \subset Z$. By Lemma 3.4 $i$ is homologically effaceable at $x$, that is the there is an open neighbourhood $U''$ of $S'$ in $Z''$ such that the composition

$$H_q(Z', n) \to H_q(Z'', n) \to H_q(U'', n)$$

is zero. This implies that the composition

$$H_q(Z', n) \to H_q(Z'', n) \to H_q(U'', n)$$

is zero. Since $\tilde{f}'$ is finite, we can find a neighbourhood $U$ of $x$ in $X$ such that $U' = \tilde{f}'^{-1}(U) \subset U''$. The commutative diagram

$$\begin{array}{cccc}
H_q(Z', n) & \xrightarrow{f'_*} & H_q(Z, n) & \xrightarrow{f'_*} \ H_q(U, n) \\
\downarrow{0} & & \downarrow{0} & \downarrow{0} \\
H_q(Z', n) & \xrightarrow{f'_*} & H_q(Z'', n) & \xrightarrow{f'_*} H_q(U'', n) & H_q(U', n)
\end{array}$$

implies the effacibility of the map $Z' \to Z$ for subschemes $Z'$ which are flat over $\mathcal{O}_K$ or of codimension $> 1$. This implies that the map

$$(\ast \ast): \mathcal{H}^q_{Z^{p+1}}(n) \to \mathcal{H}^q_{Z^p}(n)$$

is zero for all $p > 0$ and arbitrary $q$. 

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It remains to show that
\[(**): \mathcal{H}_q^0(Z,1)(n) \to \mathcal{H}_q^0(Z,2)(n)\]
is zero. This map fits into the exact sequence \(\mathcal{H}_q^0(Z,1)(n) \to \mathcal{H}_q^0(Z,2)(n) \to \mathcal{H}_q^0(Z,2)(n)\) (see Section 3.1). By definition \(\mathcal{H}_q^0(Z,2)(n) = H^q(\mathcal{O}_{X,x},n)\) and \(\mathcal{H}_q^0(Z,2)(n) = H^q(K(X),n)\). We can therefore also show that the composition \(H^q(\mathcal{O}_{X,x},n) \to H^q(\mathcal{O}_{X,x},n) \to H^q(K(X),n)\) is injective, where \(\eta\) is the generic point of the special fiber of \(X\). The first map is obtained by localising the exact sequence \(\mathcal{H}_q^0(Z,1)(n) \to \mathcal{H}_q^0(Z,2)(n) \to \mathcal{H}_q^0(Z,2)(n)\) at \(x\) and by what we showed above the first map in this sequence is zero. The map \(H^q(\mathcal{O}_{X,x},n) \to H^q(K(X),n)\) fits into a long exact sequence
\[\ldots \to H^q(K(X),n+1) \xrightarrow{\partial} H^q(k(\eta),n) \to H^q(\mathcal{O}_{X,x},n) \to H^q(K(X),n) \xrightarrow{\partial} H^{q+1}(k(\eta),n-1) \to \ldots\]
But \(\partial\) is surjective by the lifting Property 2.8.

3.4 An application to the Galois symbol map

Corollary 3.6. Let \(\mathcal{O}_K\) be an excellent discrete valuation ring with residue field \(k\) and \(R\) be a local \(\mathcal{O}_K\)-algebra. Let \(\ell \in \mathbb{N}_{>0}\) be coprime to \(\text{char}(k)\). Then the Galois symbol map
\[\tilde{K}_q^M(R)/\ell \to H^q_\text{ét}(R,\mu^\otimes_q)\]
is surjective. In particular, the graded ring
\[\bigoplus_{q \in \mathbb{N}} H^q_\text{ét}(R,\mu^\otimes_q)\]
is generated in degree 1.

Proof. Assume first that \(R\) is smooth. Then the first assertion follows from the commutative diagram with exact rows
\[
\begin{array}{cccccc}
\tilde{K}_q^M(R)/\ell & \longrightarrow & K_q^M(\text{Frac}(R))/\ell & \longrightarrow & \bigoplus_{x \in \text{Spec}(R)^{\text{reg}}} K_q^{M-1}(R)/\ell & \longrightarrow \ldots \\
0 & \longrightarrow & H^q_\text{ét}(R,\mu^\otimes_q) & \longrightarrow & H^q_\text{ét}(\text{Frac}(R),\mu^\otimes_q) & \longrightarrow \bigoplus_{x \in \text{Spec}(R)^{\text{reg}}} H^q_\text{ét}^{M-1}(k(x),\mu^\otimes_{q-1}) & \longrightarrow \ldots \\
\end{array}
\]
The first row is exact by [15, Thm. 1.1]. Indeed, the proofs of loc. cit. go through in exactly the same way with finite coefficients. The second row is exact by Corollary 3.5. The vertical arrows from the second one onwards are isomorphisms by the Bloch-Kato conjecture proven by Rost and Voevodsky. This implies that the first vertical map is surjective.

Let now \(R\) be an arbitrary local \(\mathcal{O}_K\)-algebra and let \(A \to R\) be a henselian surjection from an ind-smooth local \(\mathcal{O}_K\)-algebra \(A\). Consider the commutative diagram
\[
\begin{array}{cccc}
\tilde{K}_q^M(A)/\ell & \longrightarrow & \tilde{K}_q^M(R)/\ell \\
0 & \longrightarrow & H^q_\text{ét}(A,\mu^\otimes_q) & \longrightarrow & H^q_\text{ét}(R,\mu^\otimes_q) \\
\end{array}
\]
The left vertical map is surjective by the last paragraph and since both étale cohomology and improved Milnor K-theory commute with filtered colimits of rings. The bottom map is an isomorphism by Gabber’s affine analogue of the proper base change theorem [6]. In sum, we get that the map \(\tilde{K}_q^M(R)/\ell \to \tilde{K}_q^M(R)/\ell\) is surjective.

Since the map \(K_q^M(R)/\ell \to \tilde{K}_q^M(R)/\ell\) is surjective by [12, Thm. B], the composition \(K_q^M(R)/\ell \to \tilde{K}_q^M(R)/\ell \to H^q_\text{ét}(R,\mu^\otimes_q)\) is surjective implying the second assertion.

\[\square\]
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We include the following proposition, which is stronger than the surjectivity needed in the previous proof, for the convenience of the reader. It says that Milnor K-theory with finite invertible coefficients is rigid (for the precise meaning of rigid see the proposition). This is probably well-known to the expert and generalises [3, Lem. 3.2], whose proof we follow closely.

**Proposition 3.7.** Let $k$ be a commutative ring. Let $R$ be an arbitrary local $k$-algebra and let $A \to R$ be a henselian surjection from an ind-smooth local $k$-algebra $A$. Then the natural map

$$K^M_q(A)/\ell \to K^M_q(R)/\ell$$

is an isomorphism.

**Proof.** The statement is trivial for $q = 0$. For $q = 1$ it follows from Hensel's lemma that $A^x/\ell \cong R^x/\ell$. Let $q \geq 2$. There is a commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{StR}_q A^x & \longrightarrow & T_q A^x & \longrightarrow & K^M_q(A) & \longrightarrow & 0 \\
0 & \longrightarrow & \text{StR}_q R^x & \longrightarrow & T_q R^x & \longrightarrow & K^M_q(R) & \longrightarrow & 0,
\end{array}
$$

where $T_q A^x = (A^x)^{\otimes q}$ and $T_q R^x = (R^x)^{\otimes q}$. $\text{StR}_q A^x = \langle x_1 \otimes \cdots \otimes x_q \in T_q A^x \mid \exists i \neq j : x_i + x_j = 1 \rangle$ and $\text{StR}_q R^x = \langle x_1 \otimes \cdots \otimes x_q \in T_q R^x \mid \exists i \neq j : x_i + x_j = 1 \rangle$ are the Steinberg groups. Tensoring with $\mathbb{Z}/\ell$ we get the commutative diagram with exact rows

$$
\begin{array}{cccc}
\text{StR}_q A^x/\ell & \longrightarrow & T_q A^x/\ell & \longrightarrow & K^M_q(A)/\ell & \longrightarrow & 0 \\
\text{StR}_q R^x/\ell & \longrightarrow & T_q R^x/\ell & \longrightarrow & K^M_q(R)/\ell & \longrightarrow & 0,
\end{array}
$$

in which the middle vertical map is an isomorphism since $T_q A^x/\ell \cong T_q (A^x/\ell) \cong T_q (R^x/\ell) \cong T_q R^x/\ell$. Therefore, by the five-lemma, it suffices to show that $\alpha$ is surjective. Let $r_1 \otimes \cdots \otimes r_q \in \text{StR}_q R^x$. Without loss of generality we may assume that $q = 2$ and $r_1 + r_2 = 1$. Let $a_1, a_2 \in A^x$ map to $r_1$ and $r_2$ respectively. Then $a_1 + a_2 = : u \in 1 + I$, where $I = \text{Ker}(A \to R)$. Note that $I \subset \text{Jac}(A)$ implies that $1 + I \subset A^x$. Therefore $u^{-1}(a_1 + a_2) = u^{-1}a_1 + u^{-1}a_2 = 1$ and $u^{-1}a_1 \otimes u^{-1}a_2 \in \text{StR}_2 A^x$ maps to $r_1 \otimes r_2 \in \text{StR}_2 R^x$ under $\alpha$. \qed

4 **The Gersten conjecture: the semi-stable case**

In this section we study the Gersten conjecture in the semi-stable case étale locally. Étale locally a semi-stable scheme is of the form

$$\text{Spec}(\mathcal{O}_K[t_1, \ldots, t_d]/(t_1 \cdot \ldots \cdot t_e - \pi))$$

for $e \in \mathbb{Z}$ and $1 \leq e \leq d$.

We will use the following notation:

- If $X$ is a Noetherian scheme and $V$ a closed subscheme, then we denote the blow-up of $X$ along $V$ by $\tilde{X}$.

- If $W$ is a closed subscheme of $X$, then we denote the strict transform of $W$ by $\tilde{W}$.

- We denote the exceptional divisors of $\pi_X : \tilde{X} \to X$ and $\pi_W : \tilde{W} \to W$ by $E_X$ and $E_W$ respectively.

---

\[2\text{That } A \to R \text{ is a henselian surjection means that it is surjective and that } (A, I = \text{Ker}(A \to R)) \text{ is a henselian pair. The property defining such a pair, which we use here, is the following: Given a polynomial } f(x) \in A[x] \text{ and a root } \alpha \in A/I \text{ of } f \in (A/I)[x] \text{ with } f'(\alpha) \in (A/I)^{\times}, \text{ then } \alpha \text{ lifts to a root } \alpha' \in A \text{ of } f. \]
Bloch-Ogus theory in mixed characteristic

In this section we drop the twist in the homology theory from notation. Our main theorem is the following:

**Theorem 4.1.** Let \( \mathcal{O}_K \) be a henselian discrete valuation ring with local parameter \( \pi \) and residue field \( k \). Let \( (H^1, H^0) \) be a homology theory with duality satisfying the blow-up Property 2.6, the principal triviality Property 2.7 and the lifting Property 2.8. Let \( 1 \leq e \leq d, d, e \in \mathbb{N}, \) and

\[
X = \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_d]/(t_1 \cdot \ldots \cdot t_e - \pi)).
\]

Let \( \mathcal{H}^*(n) \) be the sheafification of the presheaf \( U \mapsto H^*(U, n) \) on \( X \) in the Zariski topology. Assume that \( d = 2 \) and that for every open subset \( W \subset X \) containing \( \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_d]/(t_1t_2 - \pi, t_1)) \cong \mathbb{A}^1_k \) the morphism \( \mathbb{A}^1_k \rightarrow W \) is homologically effaceable at the origin. Then the complex of sheaves

\[
0 \rightarrow \mathcal{H}^0(n) \rightarrow \bigoplus_{x \in X^{(0)}} i_* x^* H^q(x, n) \xrightarrow{\partial_0} \bigoplus_{x \in X^{(1)}} i_* x^* H^{q-1}(x, n - 1) \xrightarrow{\partial_1} \bigoplus_{x \in X^{(2)}} i_* x^* H^{q-2}(x, n - 2) \rightarrow 0 \tag{2}
\]

is exact.

**Proof.** We begin the proof in the more general setting of arbitrary dimension \( d \) which is why the theorem is not directly formulated for \( d = 2 \). At some point, which we will highlight, we will have to assume that \( d = 2 \). We hope that the method of the proof, and its limitations in arbitrary dimension, will nevertheless be be illuminating.

The surjectivity of \( \partial_0 \) follows from the fact that for every \( x \in X^{(d)} \) we can choose a \( y \in X^{(d-1)} \) such that \( \{y\} \) is regular and contains \( x \). The boundary map \( H^{q-d+1}(y, n - d + 1) \xrightarrow{\partial_0} H^{q-d}(x, n - d) \) is then surjective by the lifting Property 2.8. This implies in particular that the map \( (\ast) : \mathcal{H}^q_{\mathcal{O}_K}(n) \rightarrow \mathcal{H}^q_{\mathcal{O}_K'}(n) \) is zero.

In order to show the exactness in the middle we fix \( d \) and proceed by induction on \( e \). The case \( e = 1 \) was proved in Corollary 3.5. We wish to show that if the theorem holds for \( e - 1 \), then it also holds for \( e \). We change the notation slightly. From now on let \( R = \mathcal{O}_K[t_1, \ldots, t_d]/(t_1 \cdot \ldots \cdot t_{e-1} - \pi) \) and \( X = \text{Spec}(R) \). Let \( \pi_X : \bar{X} \rightarrow X \) be the blow-up of \( X \) along \( V(I) \) for \( I = t_{e-1}R + t_e R \). Then \( \bar{X} \) can be covered by the open sets

\[
U_1 = \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_d, V]/(t_1 \cdot \ldots \cdot t_{e-1} - \pi, t_e V - t_{e-1})),
\]

and

\[
U_2 = \text{Spec}(\mathcal{O}_K[t_1, \ldots, t_d, W]/(t_1 \cdot \ldots \cdot t_{e-1} - \pi, t_e - t_{e-1}W)).
\]

Note that \( U_1 \) is exactly the object we want to study. Therefore if we show that the complex of sheaves \( (\ast) \) on \( \bar{X} \) is exact at all points \( x \in U_1 \), then we are done.

We fix a point \( x \in U_1 \). Let \( Z' \) be a closed subscheme of \( \bar{X} \) of codimension \( d - 1 \) which contains the point \( x \in U_1 \). Assume that \( Z' \) is flat over \( \mathcal{O}_K \). We need to show that there is a closed subscheme \( Z \subset \bar{X} \) of codimension \( \geq d - 2 \) containing \( Z' \) and an open subset \( U' \subset Z \) containing \( x \) such that the composition

\[
H_q(Z') \rightarrow H_q(Z) \rightarrow H_q(U')
\]

is zero. By our induction assumption we know that there is some \( Y \) containing \( \pi_X(Z') \) and some \( U \subset Y \) such that the composition \( H_q(\pi_X(Z')) \rightarrow H_q(Y) \rightarrow H_q(U) \) is zero. Note that \( Z' = \pi_X(Z') \). This
in which the right vertical sequence is exact by Property 2.6.

From here on we assume that $d = 2$. Therefore $E_U = E_X$ and $E_U \cap U_1 \cong E_X \cap U_1 \cong \mathbb{A}^1_k$. We choose a divisor $D \hookrightarrow X = \tilde{Y}$ such that $E_U - D \cong \mathbb{A}^1_k$ and consider the commutative diagram

$$
\begin{array}{c}
H_q(Z') & \to & H_q(\tilde{X}) & \to & H_q(\tilde{U} - D) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

in which the lower right vertical map is zero by assumption up to enlarging $D$. In sum, this shows that the morphism $Z' \hookrightarrow \tilde{X}$ is homologically effaceable at $x$. Next we show the same effaceability for any open subscheme $U$ of $\tilde{X}$ containing $x$ and any closed subscheme $Z' \subset U$ of dimension 1 which is flat over Spec $\mathcal{O}_K$. The diagram

$$
\begin{array}{c}
H_q(Z') & \to & H_q(\tilde{X}) & \to & H_q(\tilde{U} - D) \\
\downarrow & & \downarrow & & \downarrow \\
H_q(Z') & \to & H_q(U) & \to & H_q(U \cap (\tilde{U} - D))
\end{array}
$$

commutes. The equality of the left vertical morphism comes from the fact that the closure of $Z'$ in $\tilde{X}$ is contained in $U$ since $Z'$ is flat over Spec $\mathcal{O}_K$ and $\mathcal{O}_K$ is henselian.\(^3\) The composition in the first row is zero by the above, implying the claim. This proves that the map

$$
\mathcal{H}^q_{Z^{d-1},f} \to \mathcal{H}^q_{Z^{d-2}},
$$

where

$$
Z^{d-1,f} := \{Z \subset X \mid Z \text{ closed, } \text{codim}_X(Z) \geq d - 1, \dim(Z) = \dim(\im(Z) \subset \text{Spec}(\mathcal{O}_K))\},
$$

is zero. We note that this implies that the complex

$$
\mathcal{H}^q_{Z^{d-2}/Z^{d-1},f} \to \bigoplus_{x \in X^{(d-1),f}} i_{*,x}H^{q-d+1}(x, n - d + 1) \xrightarrow{\partial_0} \bigoplus_{x \in X^{(d)}} i_{*,x}H^{q-d}(x, n - d) \to 0,
$$

\(^3\)In cohomological terms the equality on the left is the excision isomorphism $H^q_{Z'}(\tilde{X}, n) \cong H^q_{Z'}(U, n)$. 


where $X^{(d-1)f}$ denotes the set of flat 1-cycles, is exact. We have already seen that the map

$$
\mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}(n)
$$

is zero. It remains to show that the map

$$
\mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}(n)
$$

are zero. This map fits into the exact sequence

$$
\cdots \to \mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}/Z_1 \to \cdots
$$

We show that the second map is injective on stalks. Let $x \in X$ be the singular point of the special fiber. Otherwise we are in the situation of Theorem 3.5. First note that $\mathcal{H}_Z^{d}(n)_x = H^q(O_{X,x}, n)$ and $\mathcal{H}_Z^{d}/Z_1(n)_x = H^q(K(X), n)$. Let $\eta_1$ and $\eta_2$ be the two generic points of the special fiber. The first map in the exact sequence

$$
\cdots \to \mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}(n) \to \mathcal{H}_Z^{d}/Z_1 \to \cdots
$$

is zero by the above. It therefore suffices to show that the map $\mathcal{H}_Z^{d}(Z_1 - \{\eta_1, \eta_2\})_x \to H^q(K(X), n)$ is injective. It is clear that $\mathcal{H}_Z^{d}(Z_1 - \{\eta_1, \eta_2\})_x \cong H^q(R(n), n)$, where $R$ is the semi-localisation at the special fiber, and in particular $R_{\ell_1, \ell_2} = K(X)$. Finally, the sequence

$$
H^q(R(n) = \mathcal{H}_Z^{d}(Z_1 - \{\eta_1, \eta_2\})_x \to H^q(Z_1, n) \to H^q(Z_1, n) = H^q(K(X), n),
$$

in which the middle term is isomorphic to $H^q(R|_{\mathcal{R}_{\mathcal{R}}}, n)$, is a sequence of injective maps by Property 2.8.

\begin{remark}
The assumption on the effaceability of $\mathcal{A}_1 \to W$ is often satisfied due to the $\mathcal{A}_1$-invariance of étale cohomology combined with the cohomological dimension of $k$.
\end{remark}

\begin{remark}
In [19, Thm. 9] Sakagaito, using a different method, proves that if $R$ is a mixed characteristic two-dimensional excellent regular henselian local ring and $\ell$ a positive integer which is invertible in $R$, then the Gersten conjecture holds for étale cohomology with $\mu_\ell^n$ coefficients. Our proof of Theorem 4.1 should be thought of as carrying the reduction to the dvr case in the proof of Theorem 3.5 from the smooth to the semi-stable situation as far as possible. Finally, we would like to raise the question if an analogue of Gabber’s representation lemma exists for regular semi-stable schemes in mixed characteristic.
\end{remark}

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