SEGREGATED VECTOR SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS WITH ELECTROMAGNETIC POTENTIALS

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ABSTRACT. In this paper, we study the following nonlinear Schrödinger system in $\mathbb{R}^3$
\[
\begin{cases}
(\nabla - A(y))^2 u + \lambda_1(|y|)u = |u|^2u + \beta|v|^2u, & x \in \mathbb{R}^3, \\
(\nabla - A(y))^2 v + \lambda_2(|y|)v = |v|^2v + \beta|u|^2v, & x \in \mathbb{R}^3,
\end{cases}
\]
where $A(y) = A(|y|) \in C^1(\mathbb{R}^3, \mathbb{R})$ is bounded, $\lambda_1(|y|), \lambda_2(|y|)$ are continuous positive radial potentials, and $\beta \in \mathbb{R}$ is a coupling constant. We prove that if $A(y), \lambda_1(y), \lambda_2(y)$ satisfy some suitable conditions, the above problem has infinitely many non-radial segregated solutions.

1. Introduction. This paper concerns with the following nonlinear Schrödinger system
\[
\begin{cases}
(\nabla - A(y))^2 u + \lambda_1(|y|)u = |u|^2u + \beta|v|^2u, & x \in \mathbb{R}^3, \\
(\nabla - A(y))^2 v + \lambda_2(|y|)v = |v|^2v + \beta|u|^2v, & x \in \mathbb{R}^3,
\end{cases}
\]
(1.1)
where $A(y) = A(|y|) \in C^1(\mathbb{R}^3, \mathbb{R})$ is bounded, $\lambda_1(|y|) = \lambda_1(|y|)$ and $\lambda_2(|y|) = \lambda_2(|y|)$ are continuous positive radial functions, $\beta \in \mathbb{R}$ is a coupling constant.

This paper was motivated by some works that have appeared in recent years related to Schrödinger equation of this kind
\[
i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{-i}{\hbar} \nabla - A(y) \right)^2 \psi + G(y) \psi - f(y, |\psi|)\psi, \quad (t, y) \in \mathbb{R} \times \mathbb{R}^3,
\]
and Schrödinger system of this kind
\[
\begin{aligned}
-\Delta u + P(|y|)u &= \mu u^3 + \beta \frac{p}{p+4} |v|^q |u|^{p-2}u, & x \in \mathbb{R}^3, \\
-\Delta v + Q(|y|)v &= \nu v^3 + \beta \frac{p}{p+4} |u|^p |v|^{q-2}v, & x \in \mathbb{R}^3.
\end{aligned}
\]
(1.3)
Here $\psi(y, t)$ takes on complex values, $\hbar$ is the Planck constant, $i$ is the imaginary unit, $G: \mathbb{R}^3 \to \mathbb{R}$ denotes an electric potential, $A = (A_1, A_2, A_3): \mathbb{R}^3 \to \mathbb{R}^3$ is a

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vector potential with magnetic field, \( B = \text{curl} A \), \( 2 \leq p \leq 5 \), \( \mu, \nu > 0 \), \( 2 \leq q \leq 5 \) and \( p + q \leq 6 \). Moreover, the Schrödinger operator is defined by

\[
\left( \frac{\hbar}{i} \nabla - A(y) \right)^2 \psi = -\hbar^2 \Delta \psi - \frac{2\hbar}{i} A \nabla \psi + |A|^2 \psi - \frac{\hbar}{i} \psi \text{div} A.
\]

This type of (1.2) arises in different physical theories, e.g. the description of Bose-Einstein condensates, plasma physics and nonlinear optics (see [1, 30] and the references therein). When we are always focused on finding standing wave solutions for (1.2), namely, waves of the form \( \psi(y, t) = e^{-i \omega t} u(y) \) for some function \( u : \mathbb{R}^N \to \mathbb{C} \), one is led to solve the following complex equation in \( \mathbb{R}^3 \)

\[
\left( \frac{\hbar}{i} \nabla - A(y) \right)^2 u + (G(y) - E) u = f(y, |u|) u.
\]

For simplicity, letting \( V(y) = G(y) - E \), then the above problem can be rewritten as

\[
\left( \frac{\hbar}{i} \nabla - A(y) \right)^2 u + V(y) u = f(y, |u|) u,
\]

which received a lot of interest in recent years. For the existence of ground state solutions, multi-bump solutions, infinitely many solutions or asymptotic behavior of the solutions to (1.4), one can refer to [6, 10, 15–18, 20, 29] and the references therein.

The nonlinearly coupled Schrödinger equations (1.3) has also been studied extensively (see [3–5, 7, 8, 13, 14, 21–25, 28]), which is motivated by applications to nonlinear optic and Bose-Einstein condensation [2]. Recently, in [27], by using the finite reduction method, Peng and Wang obtained the existence of infinitely many non-radial positive and sign-changing segregated solutions for (1.3) with radial symmetric potentials \( P(|x|), Q(|x|) \) satisfying some algebra decay assumptions. Particularly, if \( p = q = 2 \), they did not only get the existence of infinitely many non-radial positive segregated solutions, but also constructed the existence of infinitely many non-radial positive synchronized solutions for (1.3). In [5], Ao and Wei obtained the existence of infinitely many solutions of (1.3) for nonsymmetric potentials \( P(|x|), Q(|x|) \) satisfying some exponential decay assumptions.

Motivated by works just described, more precisely by results found in [27], a natural question is whether the same phenomenon of multiplicity holds for (1.1). So in this article, we intend to investigate the existence of infinitely many segregated solutions for (1.1). In order to state our main result, we give the conditions imposed on \( A(|y|) \), \( \lambda_1(|y|) > 0 \) and \( \lambda_2(|y|) > 0 \) as follows:

(\( \lambda_1 \)) There exist constants \( a \in \mathbb{R}, m > 1 \) and \( \theta_1 > 0 \) such that as \( |y| \to +\infty \),

\[
\lambda_1(|y|) = 1 + \frac{a}{|y|^m} + O\left( \frac{1}{|y|^{m+\theta_1}} \right).
\]

(\( \lambda_2 \)) There exist constants \( b \in \mathbb{R}, n > 1 \) and \( \theta_2 > 0 \) such that as \( |y| \to +\infty \),

\[
\lambda_2(|y|) = 1 + \frac{b}{|y|^n} + O\left( \frac{1}{|y|^{n+\theta_2}} \right).
\]

(\( A \)) There exist constants \( \alpha \in \mathbb{R}, \ell > 1 \) and \( \theta_3 > 0 \) such that as \( |y| \to +\infty \),

\[
A(|y|) = A(r) + \frac{\alpha}{|y|^\ell} + O\left( \frac{1}{|y|^{\ell+\theta_3}} \right),
\]

where \( r = |x^1| \), which is defined later.

Now we state our main result as follows:
Remark 1.3. Our argument allows us to treat the following general system

\[ \ell > 0. \]

Theorem 1.1. Suppose that \( \lambda_1(|y|) \) satisfies (\( \lambda_1 \)), \( \lambda_2(|y|) \) satisfies (\( \lambda_2 \)), \( A(|y|) \) satisfies (\( A \)) and \( m = n = \ell, a > 0, b > 0, c > 0 \). Then there exists \( \beta^* > 0 \) such that for any \( \beta < \beta^* \), problem (1.1) has infinitely many non-radial segregated solutions \( (u_k, v_k) \), whose energy can be arbitrarily large. Furthermore, as \( k \to \infty \),

\[ \|u_k(\cdot) - v_k(T_k(\cdot))\|_{H^1(\mathbb{R}^3, \mathbb{C})} + \|u_k(\cdot) - v_k(T_k(\cdot))\|_{L^\infty(\mathbb{R}^3, \mathbb{C})} \to 0, \]

where \( T_k(\cdot) \) is the rotation on the \((x_1, x_2)\) plane of \( \frac{\pi}{k} \).

Remark 1.2. The radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinated system

\( m > 0, \theta_1 > 0 \) and \( p > 0 \).

\( (A1) \) \( A(y) = A(y', y_3) = A(|y'|, |y_3|) \), where \( y = (y', y_3) \in \mathbb{R}^2 \times \mathbb{R} \).

\( (A2) \) \( A(|y|) = A(r) + \frac{a}{|y|^m} + O\left(\frac{1}{|y|^{m+q}}\right) \) as \( |y| \to +\infty \), where constants \( a \in \mathbb{R} \), \( m > 1 \) and \( \theta_2 > 0 \) and \( q > 0 \).

Remark 1.3. Our argument allows us to treat the following general system

\[ \begin{cases} (\nabla \cdot - A(y))u^2 + \lambda_1(|y|)u = \mu u^3 + \beta \frac{p}{p+q}|v|^q |u|^{p-2}u, & x \in \mathbb{R}^3, \\ (\nabla \cdot - A(y))v^2 + \lambda_2(|y|)v = \nu v^3 + \beta \frac{q}{p+q} |u|^p |v|^{q-2}v, & x \in \mathbb{R}^3, \end{cases} \]

where \( 2 \leq p \leq 5, 2 \leq q \leq 5 \) and \( p + q \leq 6 \). Proceeded as done in proof of Theorem 1.1, we can get the same result as Theorem 1.1.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

For any function \( K(y) > 0 \), the sobolev space \( H^1_K(\mathbb{R}^3, \mathbb{C}) \) is endowed with the standard norm

\[ \|u\|_K = \left( \int_{\mathbb{R}^3} \left( \left| \frac{\nabla u}{i} - A(y)u \right|^2 + K(y)|u|^2 \right) \right)^{\frac{1}{2}}, \]

which is induced by the inner product

\[ \langle u, v \rangle_K = Re \int_{\mathbb{R}^3} \left( \frac{\nabla u}{i} - A(y)u \right) \left( \frac{\nabla v}{i} - A(y)v \right) + Re \int_{\mathbb{R}^3} K(y) uv. \]

Denote \( H \) to be the product space \( H^1_{\lambda_1}(\mathbb{R}^3, \mathbb{C}) \times H^1_{\lambda_2}(\mathbb{R}^3, \mathbb{C}) \) with the norm

\[ \|(u, v)\| = \|u\|_{\lambda_1} + \|v\|_{\lambda_2}. \]

It is well known that the following problem

\[ \begin{cases} -\Delta u + u = u^3, & u > 0 \ in \ \mathbb{R}^3, \\ u(0) = \max_{y \in \mathbb{R}^3} u(y), & u(y) \in H^1(\mathbb{R}^3) \end{cases} \] \hspace{2cm} (1.5) \]

has a unique solution, denoted by \( U \), which is non-degenerate and satisfies for some \( c > 0 \) (see [19]),

\[ U(r)re^r \to c, \ U'(r)re^r \to -1, \ as \ r = |y| \to \infty. \]
Let $U_c : \mathbb{R}^3 \rightarrow \mathbb{C}$ be a least-energy solution of the following equation
\[ -\Delta u + u = |u|^2 u, \quad u(y) \in H^1(\mathbb{R}^3, \mathbb{C}). \] (1.6)

Then by energy comparison (see [18]), one has
\[ U_c(y) = e^{i\sigma} U(y) \]
for some choice of $\sigma \in [0, 2\pi]$. Moreover, form [11, 12], we know that $U_c$ is non-degenerate.

In this paper, we will use the least energy solutions of problem
\[ \left( \frac{\nabla}{i} - A_0 \right)^2 u + u = |u|^2 u, \quad u \in H^1(\mathbb{R}^3, \mathbb{C}), \] (1.7)
where $A_0 = (A_{0,1}, A_{0,2}, A_{0,3})$ is a constant vector, to build up the approximate solutions for (1.1). It is easy to find that $u$ is a solutions of (1.7) if and only if $e^{-iA_0 \cdot y} u(y)$ is a solution of (1.6). From the non-degeneracy of $U_c$, we can infer that $e^{iA_0 \cdot y} U_c$ is non-degenerate.

Define
\[ x^i = (r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0), \quad i = 1, \cdots, k, \]
and
\[ y^i = (r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0), \quad i = 1, \cdots, k, \]
where $r \in [r_0 k \ln k, r_1 k \ln k]$ for some constant $r_1 > r_0 > 0$.

Furthermore, we let
\[ \mathcal{H}_{\lambda_1} = \left\{ u : u \in H^1(\mathbb{R}^3, \mathbb{C}), u \text{ is even in } y_j, j = 2, 3, \right. \]
\[ \left. u(r \cos \theta, r \sin \theta, y_3) = u(r \cos(\theta + \frac{2i\pi}{k}), r \sin(\theta + \frac{2i\pi}{k}), y_3) \right\}. \]

and we can define $\mathcal{H}_{\lambda_2}$ similarly.

Set
\[ \eta_i(y) = i\sigma + iA(x^i) \cdot (y - x^i), \quad \eta^*_i(y) = i\sigma + iA(y^i) \cdot (y - y^i) \]
and
\[ Z_r(y) = \sum_{i=1}^k \eta_i(y) U_{x^i}(y) := \sum_{i=1}^k Z_{x^i}, \quad Z^*_r(y) = \sum_{i=1}^k \eta^*_i(y) U_{y^i}(y) := \sum_{i=1}^k Z^*_{y^i}, \]
where $U_\xi = U(y - \xi)$ for some $\xi \in \mathbb{R}^3$.

To prove Theorem 1.1, it suffices to verify the following result:

**Theorem 1.4.** Under the assumption of Theorem (1.1), there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.1) has a solution $(u_k, v_k)$ of the form
\[ (u_k, v_k) = (Z_{r_k}(y) + \varphi_k, Z^*_{r_k}(y) + \psi_k), \]
where $(\varphi_k, \psi_k) \in \mathcal{H}_{\lambda_1} \times \mathcal{H}_{\lambda_2}$, $r_k \in [r_0 k \ln k, r_1 k \ln k]$, and as $k \to +\infty$, \[ \| (\varphi_k, \psi_k) \|_H \to 0. \]

The rest of the paper is organized as follows. In Section 2, we will carry out a reduction procedure. We prove our main result in Section 3. Finally, in Appendix, some basic estimates and an energy expansion for the functional corresponding to problem (1.1) will be established.
2. The finite-dimensional reduction. In this section, we carry out a finite-
dimensional reduction.

Write

\[ Y_{i,1} = \frac{\partial Z_{x_i}}{\partial r}, \quad Y_{i,2} = \frac{\partial Z_{x_i}}{\partial \sigma}, \]

and

\[ Y_{i,1}^* = \frac{\partial Z_{y_i}^*}{\partial r}, \quad Y_{i,2}^* = \frac{\partial Z_{y_i}^*}{\partial \sigma}. \]

Define

\[ E = \left\{ (u, v) \in \mathcal{H}_A \times \mathcal{H}_\lambda, \left\langle \sum_{i=1}^k |Z_{x_i}^*|^2 Y_{i,t}, u \right\rangle = 0, \left\langle \sum_{i=1}^k |Z_{y_i}^*|^2 Y_{i,t}^*, v \right\rangle = 0, \ t = 1, 2 \right\}, \]

where \( \langle u, v \rangle = \text{Re} \int_{\mathbb{R}^3} u \bar{v} \). In this section, we suppose

\[ r \in \mathbb{D}_k = \left( \frac{m}{2\pi} - \mu \right) k \ln k, M \ln k \],

where \( \mu > 0 \) is a small and \( M \) is a large constant depending on \( a, b, m, A_1, A_2 \) (seen in the Appendix).

Let

\[ J(\varphi, \psi) = I(Z_r + \varphi, Z_r^* + \psi), \quad (\varphi, \psi) \in E, \]

where

\[ I(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \left| \frac{\nabla u}{i} - A(|y|)u \right|^2 + \lambda_1(|y|)|u|^2 + \left| \frac{\nabla v}{i} - A(|y|)v \right|^2 + \lambda_2(|y|)|v|^2 \right) \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} (|u|^4 + |v|^4) - \frac{\beta}{2} \int_{\mathbb{R}^3} |u|^2|v|^2. \]  

(2.1)

Now we can expand

\[ J(\varphi, \psi) = J(0, 0) + I(\varphi, \psi) + \frac{1}{2} L(\varphi, \psi) + R(\varphi, \psi), \quad (\varphi, \psi) \in E, \]  

(2.2)

where

\[ l(\varphi, \psi) = \text{Re} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|y|) \right) Z_r (A(|y|) - A(|y|)) \varphi + (A(|y|) - A(|y|)) Z_r \left( \frac{\nabla}{i} - A(|y|) \right) \varphi \right. \]

\[ + (A(|y|) - A(|y|))^2 Z_r \varphi \]  

\[ + \sum_{i=1}^k \int_{\mathbb{R}^3} (\lambda_1(|y|) - 1) \text{Re}(Z_{x_i} \bar{\varphi}) + \int_{\mathbb{R}^3} \left[ \sum_{i=1}^k |Z_{x_i}|^2 \text{Re}(Z_{x_i} \bar{\varphi}) - |Z_r|^2 \text{Re}(Z_r \bar{\varphi}) \right. \]

\[ + \text{Re} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|y|) \right) Z_r (A(|y|) - A(|y|)) \bar{\psi} (A(|y|) - A(|y|)) Z_r \left( \frac{\nabla}{i} - A(|y|) \right) \psi \right. \]

\[ + (A(|y|) - A(|y|))^2 Z_r \bar{\psi} \]  

\[ + \sum_{i=1}^k \int_{\mathbb{R}^3} (\lambda_2(|y|) - 1) \text{Re}(Z_{y_i} \bar{\psi}) + \int_{\mathbb{R}^3} \left[ \sum_{i=1}^k |Z_{y_i}|^2 \text{Re}(Z_{y_i} \bar{\psi}) - |Z_r|^2 \text{Re}(Z_r \bar{\psi}) \right. \]

\[ - \beta \int_{\mathbb{R}^3} |Z_r|^2 \text{Re}(Z_r \bar{\psi}) + |Z_r|^2 \text{Re}(Z_r \bar{\psi}) \], \]

\[ L(\varphi, \psi) = \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|y|) \right) \varphi + \lambda_1(|y|)|\varphi|^2 - |Z_r|^2 |\varphi|^2 - 2 |\text{Re}(Z_r \bar{\psi})|^2 \right] \]
There exists a constant $\lambda_2 > 0$, such that for any $\varphi, \psi \in \mathbb{E}$,

$$\|L(u, v)\| \leq C\|\langle \varphi, \psi \rangle\|, \quad (u, v) \in \mathbb{E}.$$

Now we consider the invertibility of $L$.

**Lemma 2.1.** There exists a constant $C > 0$, independent of $k$, such that for any $r \in \mathbb{D}_k$,

$$\|L(u, v)\| \leq C\|\langle \varphi, \psi \rangle\|, \quad (u, v) \in \mathbb{E}.$$

**Lemma 2.2.** There exist $\beta^* > 0, \rho > 0$, independent of $k$, such that for any $r \in \mathbb{D}_k$,

$$\|L(u, v)\| \geq \rho\|\langle \varphi, \psi \rangle\|, \quad (u, v) \in \mathbb{E}.$$

**Proof.** Arguing by contradiction, we suppose that there are $k \to \infty, r_k \in \mathbb{D}_k$ and $(u_k, v_k) \in \mathbb{E}$ with $\|(u_k, v_k)\|^2 = k$ and

$$\langle L(u_k, v_k), (\varphi, \psi) \rangle = o(1)\|\langle u_k, v_k \rangle\|\|\langle \varphi, \psi \rangle\|, \quad \forall (\varphi, \psi) \in \mathbb{E}. \quad (2.3)$$

and

$$R(\varphi, \psi) = \int_{\mathbb{R}^3} \left[ Re(\nabla i - A(|y|))u \nabla A(|y|) \psi + \lambda_1(|y|)Re(u \overline{\varphi}) - Re|Z_r|^2u \overline{\varphi} - 2Re(Z_r \overline{\varphi})Re(Z_r \overline{\psi}) \right]$$

$$+ \int_{\mathbb{R}^3} \left[ Re(\nabla i - A(|y|))v \nabla A(|y|) \psi + \lambda_2(|y|)Re(v \overline{\varphi}) - Re|Z_r|^2v \overline{\varphi} - 2Re(Z_r \overline{\varphi})Re(Z_r \overline{\psi}) \right]$$

is a bounded bi-linear functional in $\mathbb{E}$. Thus, there exists a bounded linear operator $L$ from $\mathbb{E}$ to $\mathbb{E}$ such that for any $(u, v), (\varphi, \psi) \in \mathbb{E}$,

$$\langle L(u, v), (\varphi, \psi) \rangle = \int_{\mathbb{R}^3} \left[ Re(\nabla i - A(|y|))u \nabla A(|y|) \psi + \lambda_1(|y|)Re(u \overline{\varphi}) - Re|Z_r|^2u \overline{\varphi} - 2Re(Z_r \overline{\varphi})Re(Z_r \overline{\psi}) \right]$$

$$+ \int_{\mathbb{R}^3} \left[ Re(\nabla i - A(|y|))v \nabla A(|y|) \psi + \lambda_2(|y|)Re(v \overline{\varphi}) - Re|Z_r|^2v \overline{\varphi} - 2Re(Z_r \overline{\varphi})Re(Z_r \overline{\psi}) \right]$$

Using the above discussion, we can get the following results.

**Lemma 2.1.** There exists a constant $C > 0$, independent of $k$, such that for any $r \in \mathbb{D}_k$,

$$\|L(u, v)\| \leq C\|\langle \varphi, \psi \rangle\|, \quad (u, v) \in \mathbb{E}.$$

Now we consider the invertibility of $L$.

**Lemma 2.2.** There exist $\beta^* > 0, \rho > 0$, independent of $k$, such that for any $r \in \mathbb{D}_k$, if $\beta < \beta^*$, then

$$\|L(u, v)\| \geq \rho\|\langle \varphi, \psi \rangle\|, \quad (u, v) \in \mathbb{E}.$$

**Proof.** Arguing by contradiction, we suppose that there are $k \to \infty, r_k \in \mathbb{D}_k$ and $(u_k, v_k) \in \mathbb{E}$ with $\|(u_k, v_k)\|^2 = k$ and

$$\langle L(u_k, v_k), (\varphi, \psi) \rangle = o(1)\|\langle u_k, v_k \rangle\|\|\langle \varphi, \psi \rangle\|, \quad \forall (\varphi, \psi) \in \mathbb{E}. \quad (2.3)$$
For \( i = 1, \cdots, k \), define
\[
\Omega_i = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R} : \frac{\langle x', (x')' \rangle}{\|x'\|^2} \geq \cos \frac{\pi}{k} \right\},
\]
and
\[
\overline{\Omega}_i = \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R} : \frac{\langle x', (y')' \rangle}{\|x'\|^2} \geq \cos \frac{\pi}{k} \right\}.
\]
Next, for simplicity, we will use \( r \) replace \( r_k \). By symmetry, we see from (2.3) that
\[
\begin{align*}
\int_{\Omega_i} \left[ \text{Re} \left( \frac{\nabla}{i} - A(|y|) \right) u_k \left( \frac{\nabla}{i} - A(|y|) \right) \varphi + \lambda_1(|y|) \text{Re}(u_k \overline{\varphi}) - \overline{\text{Re}(Z_r^2 u_k \varphi)} \right]
&- 2 \text{Re}(Z_r \overline{u_k}) \text{Re}(Z_r \overline{\varphi}) \\
+ \int_{\Omega_i} \left[ \text{Re} \left( \frac{\nabla}{i} - A(|y|) \right) v_k \left( \frac{\nabla}{i} - A(|y|) \right) \psi + \lambda_2(|y|) \text{Re}(v_k \overline{\psi}) - \overline{\text{Re}(Z_r^2 v_k \psi)} \right]
&- 2 \text{Re}(Z_r \overline{v_k}) \text{Re}(Z_r \overline{\psi}) \\
&- \beta \int_{\Omega_i} \left[ |Z_r|^2 \text{Re}(v_k \overline{\varphi}) + |Z_r|^2 \text{Re}(u_k \overline{\varphi}) + 2 \text{Re}(Z_r \overline{u_k}) \text{Re}(Z_r \overline{\varphi}) \right]
&+ 2 \text{Re}(Z_r \overline{\varphi}) \text{Re}(Z_r \overline{\varphi}) \\
&= \frac{1}{K} \left< L(u_k, v_k), (\varphi, \psi) \right> = o(1) \left( \frac{1}{\sqrt{K}} \right) \| (\varphi, \psi) \|, \ \forall \ (\varphi, \psi) \in \mathcal{E}.
\end{align*}
\] (2.4)

In particular,
\[
\begin{align*}
\int_{\Omega_i} \left[ |(\frac{\nabla}{i} - A(|y|)) u_k|^2 + \lambda_1(|y|) |u_k|^2 - |Z_r|^2 |u_k|^2 - 2 |\text{Re}(Z_r \overline{u_k})|^2 \right]
&+ \int_{\Omega_i} \left[ |(\frac{\nabla}{i} - A(|y|)) v_k|^2 + \lambda_2(|y|) |v_k|^2 - |Z_r|^2 |v_k|^2 - 2 |\text{Re}(Z_r \overline{v_k})|^2 \right] \\
&- \beta \int_{\Omega_i} \left[ |Z_r|^2 |v_k|^2 + |Z_r|^2 |u_k|^2 + 4 \text{Re}(Z_r \overline{u_k}) \text{Re}(Z_r \overline{v_k}) \right] = o(1),
\end{align*}
\] (2.5)

and
\[
\begin{align*}
\int_{\Omega_i} \left[ |(\frac{\nabla}{i} - A(|y|)) u_k|^2 + \lambda_1(|y|) |u_k|^2 + |(\frac{\nabla}{i} - A(|y|)) v_k|^2 + \lambda_2(|y|) |v_k|^2 \right]
&= 1.
\end{align*}
\] (2.6)

Obviously, these estimates (2.4), (2.5) and (2.6) are also true in \( \overline{\Omega}_1 \).

Let
\[
u_k^* = u_k(y - x^1), \quad v_k^* = v_k(y - y^1).
\]
Now we consider \( u_k(y) \) in detail. For any \( R > 0 \), as \( |x^i - x^1| = 2r \sin \left( \frac{(i-1)\pi}{k} \right) \geq c \ln k \) is contained in \( B_R(x^1) \subset \Omega_1 \). So (2.6) gives
\[
\int_{B_R(0)} \left[ |(\frac{\nabla}{i} - A(r)) u_k^*|^2 + |u_k^*|^2 + |(\frac{\nabla}{i} - A(r)) v_k^*|^2 + |v_k^*|^2 \right] \leq C.
\]
Thus, we may assume that there is \( u^* \) such that as \( k \to \infty \),
\[
\begin{align*}
u_k^* &\to u^* \text{ in } H^1_{loc}(\mathbb{R}^3; \mathbb{C}), \quad u_k^* \to u^* \text{ in } L^2_{loc}(\mathbb{R}^3; \mathbb{C}).
\end{align*}
\]
Since \( u_k \) is even in \( y_j, j = 2, 3 \), \( u^* \) is even in \( y_j, j = 2, 3 \) and satisfies
\[
\operatorname{Re} \int_{\mathbb{R}^3} \eta(y)|U|^2 U^\ast = 0, \quad \operatorname{Re} \int_{\mathbb{R}^3} \eta(y)|U|^2 \frac{\partial U}{\partial y_1} U^\ast = 0, \tag{2.7}
\]
where and in this sequel we always denote \( \eta(y) = e^{i\sigma+iA(x^1)}y_1 \).

Define
\[
\tilde{E} = \left\{ \varphi \in H^1(\mathbb{R}^3, \mathbb{C}), \quad \operatorname{Re} \int_{\mathbb{R}^3} \eta(y)|U|^2 U^\ast = 0, \quad \operatorname{Re} \int_{\mathbb{R}^3} \eta(y)|U|^2 \frac{\partial U}{\partial y_1} \varphi = 0 \right\}.
\]
For any \( R > 0 \), let \( \varphi \in C_0^\infty(B_R(0), \mathbb{C}) \cap \tilde{E} \) and be even in \( y_j, j = 2, 3 \). Define \( \varphi_k = \varphi(y-x_1) \in C_0^\infty(B_R(x_1), \mathbb{C}) \). Choosing \( (\varphi, \psi) = (\varphi_k, 0) \) and inserting \( (\varphi, \psi) \) into \( (2.4) \), we find that as \( k \to \infty \),
\[
\operatorname{Re} \int_{\Omega_1} \left( \frac{\nabla}{i} - A(|y|) \right) u_k \left( \frac{\nabla}{i} - A(|y|) \right) \varphi_k + \operatorname{Re} \int_{\Omega_1} \lambda_1(|y|) u_k \varphi_k \\
\to \operatorname{Re} \int_{\mathbb{R}^3} \left( \frac{\nabla}{i} - A(|x_1|) \right) u^* \left( \frac{\nabla}{i} - A(|x_1|) \right) \varphi + \operatorname{Re} \int_{\mathbb{R}^3} u^* \varphi.
\]

\( \operatorname{Re} \int_{\Omega_1} \left[ |Z_r|^2 u_k \varphi_k + 2(\varphi, \psi) \operatorname{Re}(Z_r \varphi) \right] \to \operatorname{Re} \int_{\mathbb{R}^3} \left[ U^2 u^* \varphi + 2 \eta(y) U \varphi \operatorname{Re}(\eta(y) U \varphi) \right] \),
and
\[
\beta \int_{\Omega_1} \left[ |Z_r|^2 \operatorname{Re}(u_k \varphi_k) + 2 \operatorname{Re}(Z_r \varphi) \operatorname{Re}(Z_r \varphi) \right] \to 0.
\]
Thus we have
\[
\operatorname{Re} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|x_1|) \right) u^* \left( \frac{\nabla}{i} - A(|x_1|) \right) \varphi + u^* \varphi - U^2 u^* \varphi \\
-2\eta(y) U \varphi \operatorname{Re}(\eta(y) U \varphi) \right] = 0. \tag{2.8}
\]

Since \( u^* \) is even in \( y_j, j = 2, 3 \), \( (2.8) \) holds for any function \( \varphi \in C_0^\infty(B_R(0), \mathbb{C}) \cap \tilde{E} \),
which is odd in \( y_j, j = 2, 3 \). Hence, \( (2.8) \) holds for any function \( \varphi \in C_0^\infty(B_R(0), \mathbb{C}) \cap \tilde{E} \). By the density of \( C_0^\infty(B_R(0), \mathbb{C}) \) in \( H^1(\mathbb{R}^3, \mathbb{C}) \), one can find for any \( \varphi \in \tilde{E} \),
\[
\operatorname{Re} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|x_1|) \right) u^* \left( \frac{\nabla}{i} - A(|x_1|) \right) \varphi + u^* \varphi - U^2 u^* \varphi \\
-2\eta(y) U \varphi \operatorname{Re}(\eta(y) U \varphi) \right] = 0. \tag{2.9}
\]
But (2.9) holds for \( \varphi = c_0 \eta U + c_1 \frac{\partial (U^2)}{\partial y_1} \). Thus, (2.9) is true for any \( \varphi \in H^1(\mathbb{R}^3, \mathbb{C}) \),
which means that \( u^* = c_0 \eta U + c_1 \frac{\partial (U^2)}{\partial y_1} \) for some constants \( c_0, c_1 \). From (2.7), it is easy to check that \( c_0 = c_1 = 0 \) and then \( u^* = 0 \).

Applying the same argument on \( \Omega_1 \), we can prove that as \( k \to \infty \),
\[
v_k \to 0 \quad \text{in} \ H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{C}), \quad v_k^* \to 0 \quad \text{in} \ L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{C}).
\]
As a result,
\[
\int_{B_R(x^1)} |u_k|^2 = o(1), \quad \int_{B_R(y^*)} |v_k|^2 = o(1).
\]
Note that by Lemma A.2, we get
\[
|Z_r(y)| \leq Ce^{-\frac{|y-x_1|}{2}}, \quad y \in \Omega_1, \quad |Z_r^*(y)| \leq Ce^{-\frac{|y-y^1|}{2}}, \quad y \in \tilde{\Omega}_1.
\]
Thus, it follows from (2.3) that
\[
o(1)k = \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) u_k \right]^2 + \lambda_1(\rho)|u_k|^2 - |Z_r|^2|u_k|^2 - 2|\text{Re}(Z_r \overline{u}_k)|^2 \\
+ \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) v_k \right]^2 + \lambda_2(\rho)|v_k|^2 - |Z_r|^2|v_k|^2 - 2|\text{Re}(Z_r \overline{v}_k)|^2 \\
- \beta \int_{\mathbb{R}^3} \left[ |Z_r|^2|u_k|^2 + |Z_r|^2|v_k|^2 + 4\text{Re}(Z_r \overline{u}_k)\text{Re}(Z_r \overline{v}_k) \right] \\
= \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) u_k \right]^2 + \lambda_1(\rho)|u_k|^2 - |Z_r|^2|u_k|^2 - 2|\text{Re}(Z_r \overline{u}_k)|^2 \\
- \beta \int_{\mathbb{R}^3} \left[ |Z_r|^2|u_k|^2 + |Z_r|^2|v_k|^2 + 4\text{Re}(Z_r \overline{u}_k)\text{Re}(Z_r \overline{v}_k) \right] \\
- k \int_{\Omega_1} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) u_k \right]^2 + 2\text{Re}(Z_r \overline{u}_k)|^2 \right] \\
- k \int_{\Omega_1} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) v_k \right]^2 + 2|\text{Re}(Z_r \overline{v}_k)|^2 \right] \\
\geq \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(\rho) \right) u_k \right]^2 + \lambda_1(\rho)|u_k|^2 - |Z_r|^2|u_k|^2 - 2|\text{Re}(Z_r \overline{u}_k)|^2 \\
- \beta \int_{\mathbb{R}^3} \left[ |Z_r|^2|u_k|^2 + |Z_r|^2|v_k|^2 \right] - 4\beta k \int_{\Omega_1} \left[ |Z_r|u_k \right| Z_r^*|v_k| \\
- 3k \int_{\Omega_1} \left[ |Z_r|^2|u_k|^2 + |Z_r|^2|v_k|^2 \right] \\
\geq k - C\beta k + O(e^{-\frac{\pi}{k}})k + (o(1) + O(e^{-R}))k,
\]
(2.10)
where we used the fact that
\[
|x^1 - y^1| = 2r_k \sin \frac{\pi}{2k},
\]
and
\[
\int_{\mathbb{R}^3} \left[ |Z_r|^2|u_k|^2 + |Z_r|^2|v_k|^2 \right] \leq C \int_{\mathbb{R}^3} \left[ |u_k|^2 + |v_k|^2 \right] \leq Ck
\]
for some constant \(C > 0\) independent of \(k\).

If we choose \(\beta < \beta^* = \frac{1}{C}\), then (2.10) implies a contradiction for large \(R\) and \(k\).

**Lemma 2.3.** There exists \(C > 0\), independent of \(k\), such that
\[
\|R^{(i)}(\varphi, \psi)\| \leq C\| (\varphi, \psi) \|^3, \quad i = 0, 1, 2.
\]

**Proof.** Note that for any \(a \in \mathbb{C}\), \(|\text{Re}a| \leq |a|\). Then
\[
|R(\varphi, \psi)| = \left| \int_{\mathbb{R}^3} \left[ \text{Re}(Z_r \varphi) |\varphi|^2 + \text{Re}(Z_r \overline{\psi}) |\psi|^2 + \frac{1}{4} |\varphi|^4 + \frac{1}{4} |\psi|^4 \right] \\
- \beta \int_{\mathbb{R}^3} \left[ |Z_r + \varphi|^2 |Z_r^* + \psi|^2 - |Z_r|^2 |Z_r^*|^2 - 2|Z_r|^2 \text{Re}(Z_r \overline{\psi}) \right] \right|
\]
Thus, finding a critical point for $J$ and there is a constant $C$ such that
\[
1794 \quad \text{Hence we get our conclusion.}
\]

**Proposition 2.4.** There exists an integer $k_0 > 0$, such that for any $k \geq k_0$, there is a $C^1$ map from $(\mathbb{D}_k \times [0, 2\pi]) \times (\mathbb{D}_k \times [0, 2\pi])$ to $\mathcal{H}_k \times \mathcal{H}_k : (\varphi, \psi) = (\varphi_{r, \sigma}, \psi_{r, \sigma})$, $r = |x^1|$ satisfying $(\varphi, \psi) \in \mathcal{E}$ and
\[
\left\langle \frac{\partial J(\varphi, \psi)}{\partial (\varphi, \psi)}, (\varphi, \psi) \right\rangle = 0, \quad \forall (\varphi, \psi) \in \mathcal{E},
\]
and there is a constant $C$ such that
\[
\| (\varphi, \psi) \| \leq Ck \left( \frac{1}{r^m} + \frac{1}{r^n} \right) + C|\beta|k^\frac{1}{2} \frac{k}{r} e^{-r/2}. \tag{2.11}
\]
Moreover, $(\varphi_{r, \sigma}, \psi_{r, \sigma}) = e^{i\sigma} (\varphi_{r, 0}, \psi_{r, 0})$ for any $\sigma \in [0, 2\pi]$.

**Proof.** Noting that the Lemma 2.5 below, $l(\varphi, \psi)$ is a bounded linear functional in $\mathcal{E}$. Thus, there is an $l_k \in \mathcal{E}$ such that
\[
l(\varphi, \psi) = (l_k, (\varphi, \psi)).
\]
Thus, finding a critical point for $J(\varphi, \psi)$ is equivalent to solving
\[
l_k + L(\varphi, \psi) + R(\varphi, \psi) = 0. \tag{2.12}
\]
By Lemma 2.2, $L$ is invertible and then (2.12) can be rewritten as
\[
(\varphi, \psi) = A(\varphi, \psi) = -L^{-1}l_k - L^{-1}R(\varphi, \psi). \tag{2.13}
\]
Let 
\[ \mathcal{N} = \{ (\varphi, \psi) : (\varphi, \psi) \in \mathbb{E}, \| (\varphi, \psi) \| \leq k^{1+\tau} \left( \frac{1}{r^m} + \frac{1}{r^n} + \frac{1}{r^t} \right) + |\beta|k^{\frac{1+\tau}{r}} e^{-r\frac{\pi}{\tau}} \} , \]
where \( \tau > 0 \) is small.

From Lemma 2.3 and Lemma 2.5 below, for \( k \) large, we find
\[
\| A(\varphi, \psi) \| \leq C\| l_k \| + C\| (\varphi, \psi) \|^2
\]
\[
\leq C \left( \frac{k}{r^m} + \frac{k}{r^n} + \frac{k}{r^t} \right) + C k^{1+\tau} \left( \frac{1}{r^m} + \frac{1}{r^n} + \frac{1}{r^t} \right) + k^{\frac{1+\tau}{r}} e^{-r\frac{\pi}{\tau}} \tag{2.14}
\]
and
\[
\| A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2) \| = \| L^{-1} R'(\varphi_1, \psi_1) - L^{-1} R'(\varphi_2, \psi_2) \|
\]
\[
\leq C \| R'(\varphi_1, \psi_1) - R'(\varphi_2, \psi_2) \|
\]
\[
\leq C \left( \| (\varphi_1, \psi_1) \| + \| (\varphi_2, \psi_2) \| \right) \| (\varphi_1, \psi_1) - (\varphi_2, \psi_2) \|
\]
\[
\leq \frac{1}{2} \| (\varphi_1, \psi_1) - (\varphi_2, \psi_2) \|.
\]

Hence, \( A \) maps from \( \mathcal{N} \) to \( \mathcal{N} \) and is a contraction map. So applying the contraction mapping theorem, for any \((r, \sigma) \in \mathbb{D}_k \times [0, 2\pi]\), we can find a unique \((\varphi_{r, \sigma}, \psi_{r, \sigma}) \in \mathbb{E}\) such that \((\varphi_{r, \sigma}, \psi_{r, \sigma}) = A(\varphi_{r, \sigma}, \psi_{r, \sigma})\). Moreover, it follows from (2.13) that
\[
\| (\varphi, \psi) \| \leq C k \left( \frac{1}{r^m} + \frac{1}{r^n} + \frac{1}{r^t} \right) + C |\beta|k^{\frac{1}{r}} e^{-r\frac{\pi}{\tau}}.
\]

Finally, to finish the proof, we only need to prove that \((\varphi_{r, \sigma}, \psi_{r, \sigma}) = e^{i\sigma}(\varphi_{r, 0}, \psi_{r, 0})\) for any \( \sigma \in \mathbb{R} \). It is easy to see that \((e^{i\sigma} \varphi_{r, 0}, e^{i\sigma} \psi_{r, 0})\) satisfies (2.12) since \((\varphi_{r, 0}, \psi_{r, 0})\) satisfies (2.12). So by the uniqueness of \((\varphi_{r, \sigma}, \psi_{r, \sigma})\), we see that \((\varphi_{r, \sigma}, \psi_{r, \sigma}) = e^{i\sigma}(\varphi_{r, 0}, \psi_{r, 0})\). This completes our proof.

**Lemma 2.5.** There exists \( C > 0 \), independent of \( k \), such that
\[
\| l_k \| \leq C k \left( \frac{1}{r^m} + \frac{1}{r^n} + \frac{1}{r^t} \right) + C |\beta|k^{\frac{1}{r}} e^{-r\frac{\pi}{\tau}}.
\]

**Proof.** Recall that
\[
I(\varphi, \psi) = \text{Re} \int_{\mathbb{R}^3} \left[ (\nabla_i - A(r)) Z_r(A(r) - A(|y|))\varphi + (A(r) - A(|y|))^2 Z_r \nabla \right]
\]
\[
+ \sum_{i=1}^k \int_{\mathbb{R}^3} (\lambda_i(|y|) - 1) \text{Re}(Z_{x_i} \nabla) + \sum_{i=1}^k |Z_{x_i}|^2 \text{Re}(Z_{x_i} \nabla) - |Z_r|^2 \text{Re}(Z_r \nabla)
\]
\[
+ \text{Re} \int_{\mathbb{R}^3} \left[ (\nabla_i - A(r)) Z_{r^2} Z_r^2 (A(r) - A(|y|))\varphi + (A(r) - A(|y|))^2 Z_r \nabla \right]
\]
\[
+ \sum_{i=1}^k \int_{\mathbb{R}^3} (\lambda_2(|y|) - 1) \text{Re}(Z_{y_i} \nabla) + \sum_{i=1}^k |Z_{y_i}|^2 \text{Re}(Z_{y_i} \nabla) - |Z_r|^2 \text{Re}(Z_r \nabla)
\]
\[- \beta \int_{\mathbb{R}^3} \left[ |Z_r|^2 \text{Re}(Z_r^* \overline{\psi}) + |Z_r|^2 \text{Re}(Z_r \overline{\varphi}) \right].\]

Firstly, since
\[
\left( \frac{\nabla}{r} - A(r) \right)^2 Z_r + Z_r = \sum_{i=1}^{k} |Z_{x_i}|^2 Z_{x_i},
\]

\[
\text{Re} \int_{\mathbb{R}^3} \sum_{i,j=1}^{k} |Z_{x_i}|^2 Z_{x_i} Z_{x_j}^* = \int_{\mathbb{R}^3} \sum_{i=1}^{k} |Z_{x_i}|^4 + \text{Re} \int_{\mathbb{R}^3} \sum_{i \neq j} |Z_{x_i}|^2 Z_{x_i} Z_{x_j}^*
\]
\[
= k \int_{\mathbb{R}^3} U^4 + k \text{Re} \int_{\mathbb{R}^3} \sum_{j=2}^{k} |Z_{x_j}|^2 Z_{x_j} Z_{x_j}^*
\]
\[
= k \int_{\mathbb{R}^3} U^4 + k \sum_{j=2}^{k} \cos(A(r)(x^j - x^1)) \int_{\mathbb{R}^3} U_{x_i}^4 U_{x_j}
\]
\[
= k \int_{\mathbb{R}^3} U^4 + O\left( k^2 e^{-\frac{2\pi}{r}} \right),
\]

by Hölder inequality and condition \(A\), we have
\[
\text{Re} \int_{\mathbb{R}^3} \left( \frac{\nabla}{r} - A(r) \right) Z_r (A(r) - A(|y|)) \overline{\varphi}
\]
\[
\leq \left( \int_{\mathbb{R}^3} \left( \frac{\nabla}{r} - A(r) \right) |Z_r|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |\varphi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \left( \text{Re} \int_{\mathbb{R}^3} \sum_{i,j=1}^{k} |Z_{x_i}|^2 Z_{x_i} Z_{x_j}^* \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |\varphi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \frac{k}{r^d} + k \frac{k}{r} e^{-\frac{2\pi}{r}} \right) \| (\varphi, \psi) \|,
\]

and similarly
\[
\text{Re} \int_{\mathbb{R}^3} \left[ (A(r) - A(|y|)) Z_r \left( \frac{\nabla}{r} - A(r) \right) \varphi + (A(r) - A(|y|))^2 Z_r \overline{\varphi} \right]
\]
\[
\leq C \left( \frac{k}{r^d} + k \frac{k}{r} e^{-\frac{2\pi}{r}} \right) \| (\varphi, \psi) \|,
\]
\[
\text{Re} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{r} - A(r) \right) Z_r^* (A(r) - A(|y|)) \overline{\psi} (A(r) - A(|y|)) Z_r^* \left( \frac{\nabla}{r} - A(r) \right) \psi \right]
\]
\[
+ (A(r) - A(|y|))^2 Z_r^* \overline{\psi} \right] \leq C \left( \frac{k}{r^d} + k \frac{k}{r} e^{-\frac{2\pi}{r}} \right) \| (\varphi, \psi) \|.
\]

On the other hand,
\[
\left| \sum_{i=1}^{k} \int_{\mathbb{R}^3} (\lambda_1(|y|) - 1) \text{Re}(Z_{x_i} \overline{\varphi}) \right| \leq \sum_{i=1}^{k} \int_{\mathbb{R}^3} \lambda_1(|y|) - 1 |U_{x_i}| |\varphi|
\]
\[
\leq k \left( \int_{\mathbb{R}^3} \lambda_1(|y| - x^1|) - 1 |U^2| \right)^{\frac{1}{2}} |\varphi| |\lambda_1|
\]
\[
\leq C k \frac{1}{p_m} \| (\varphi, \psi) \|,
\]

where we used the fact that
\[
\int_{\mathbb{R}^3} |\lambda_1(|y| - x^1|) - 1 |U^2|
\]
Similarly,
\[
\left| \sum_{i=1}^{k} \int_{\mathbb{R}^3} (\lambda_2(|y|) - 1) Re(Z_{y^i}^{*} \overline{\psi}) \right| \leq C k \frac{1}{r^n} ||(\varphi, \psi)||,
\]

Now we see from Lemma A.1 that
\[
\left| \sum_{i=1}^{k} \int_{\mathbb{R}^3} |Z_{x^i}|^2 Re(Z_{x^i} \overline{\varphi}) - |Z_{x}^i|^2 Re(Z_{x} \overline{\varphi}) \right|
\]
\[
\leq \int_{\mathbb{R}^3} \left| |Z_{x}|^2 Z_{x} - \sum_{i=1}^{k} |Z_{x^i}|^2 Z_{x^i} \right| |\varphi|
\]
\[
\leq \int_{\mathbb{R}^3} \left( |Z_{x}|^2 Z_{x} - \sum_{i=1}^{k} |Z_{x^i}|^2 Z_{x^i} \right)^{\frac{3}{4}} \left( \int_{\mathbb{R}^3} |\varphi|^4 \right)^{\frac{1}{4}}
\]
\[
\leq C \int_{\mathbb{R}^3} \left( \sum_{i \neq j} |Z_{x^i}|^3 |Z_{x^j}| \right)^{\frac{3}{4}} ||\varphi||_{\lambda_1}
\]
\[
\leq C k \frac{3}{2} \int_{\mathbb{R}^3} \left( \sum_{i=2}^{k} |U_{x^i}|^3 U_{x^i} \right)^{\frac{3}{4}} ||\varphi||_{\lambda_1}
\]
\[
\leq C k \frac{3}{2} \int_{\mathbb{R}^3} \left( \sum_{i=2}^{k} \frac{e^{-|x^i - x^j|^2}}{|x^i - x^j|^2} \right)^{\frac{3}{4}} ||\varphi||_{\lambda_1}
\]
\[
\leq C k \frac{3}{2} \frac{k}{r} e^{-r^2 \varphi} ||(\varphi, \psi)||,
\]
and
\[
\left| \sum_{i=1}^{k} \int_{\mathbb{R}^3} |Z_{y^i}|^2 Re(Z_{y^i}^{*} \overline{\psi}) - |Z_{y}^i|^2 Re(Z_{y}^{*} \overline{\psi}) \right| \leq C k \frac{3}{2} k \frac{e^{-r^2 \varphi}}{r} ||(\varphi, \psi)||.
\]

Finally, by Lemma A.2, we find
\[
|\beta \int_{\mathbb{R}^3} |Z_x|^2 Re(Z_x^{*} \overline{\varphi})| \leq |\beta| k \int_{\Omega_1} |Z_x|^2 \left( U_{y^1} + \sum_{i=2}^{k} U_{y^i} \right) |\psi|
\]
\[
\leq C |\beta| k \int_{\Omega_1} U_{x^1}^2 \left( U_{y^1} + \sum_{i=2}^{k} U_{y^i} \right) |\psi|
\]
\[
\leq C |\beta| k \int_{\Omega_1} \left( U_{x^1} e^{-|x^1 - y^1|^2} + U_{x^1} \sum_{i=2}^{k} e^{-|x^1 - y^1|^2} \right) |\psi|
\]
\[
\leq C |\beta| k \frac{3}{4} k \frac{e^{-r \varphi}}{r} ||(\varphi, \psi)||,
\]
and
\[
|\beta \int_{\mathbb{R}^3} |Z_y|^2 Re(Z_y^{*} \overline{\varphi})| \leq C |\beta| k \frac{3}{4} k \frac{e^{-r \varphi}}{r} ||(\varphi, \psi)||.
\]
Therefore, from the above estimates, we have proved our result.

3. **Proof of the main result.** In this section, we come to prove the main theorem.

**Proof of Theorem 1.4.** Let \((\varphi_{r,\sigma}, \psi_{r,\sigma})\) be the map obtained in Proposition 2.4. Define

\[
F(r) = I(Z_r + \varphi_{r,\sigma}, Z_r^* + \psi_{r,\sigma}), \quad \forall \, r \in \mathbb{D}_k.
\]

Applying the same argument used in [9, 26], we can easily check for \(r \) sufficiently large, if \( r \) is a critical point of \( F(r) \), then \((Z_r + \varphi_{r,\sigma}, Z_r^* + \psi_{r,\sigma})\) is a critical point of \( I(u, v) \).

It follows from Lemmas 2.3, 2.5 and A.3 that

\[
F(r) = I(Z_r, Z_r^*) + \frac{1}{2} \langle L(\varphi_{r,\sigma}, \psi_{r,\sigma}), (\varphi_{r,\sigma}, \psi_{r,\sigma}) \rangle + R(\varphi_{r,\sigma}, \psi_{r,\sigma})
\]

\[
= I(Z_r, Z_r^*) + O\left( \|r_k\| \|L(\varphi_{r,\sigma}, \psi_{r,\sigma})\| + \|L(\varphi_{r,\sigma}, \psi_{r,\sigma})\|^2 \right)
\]

\[
= I(Z_r, Z_r^*) + O\left( \frac{\beta^2}{r} e^{-2r} + O\left( \frac{1}{r^{m-1+\epsilon}} + \frac{1}{r^{n-1+\epsilon}} + \frac{1}{r^{\beta-1+\epsilon}} \right) \right)
\]

\[
= k\left( A + \frac{a + b}{r^m A_1} - A_2 \frac{k}{r} e^{-2r} - \left( o(1) + O\left( \frac{\beta^2}{\ln k} \right) \right) \frac{k}{r} e^{-2r} + O\left( \frac{1}{r^{m+\epsilon}} \right) \right).
\]

For any \( \beta < \beta^* \), where \( \beta^* \) is defined in Lemma 2.2, we can take \( k_0 > 0 \) such that

\( A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) > 0 \) for \( k \geq k_0 \). Now consider the maximization problem

\[
\max_{r \in \mathbb{D}_k} F(r).
\]

Assume that (3.1) is achieved by some \( r_k \in \mathbb{D}_k \). We will prove that \( r_k \) is an interior point of \( \mathbb{D}_k \).

When \( o(1) + O\left( \frac{\beta^2}{\ln k} \right) > 0 \), we let

\[
g(t) = \frac{a + b}{t^m k^m A_1} - \left( A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) \right) \frac{1}{t} e^{-2\pi t}.
\]

Then

\[
g'(t) = - \frac{m(a + b)A_1}{k^m t^{m+1}} + \frac{2\pi \left( A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) \right) e^{-2\pi t}}{t} 
\]

\[+ \left( A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) \right) \frac{e^{-2\pi t}}{t^2}.
\]

But we can easily check \( g(t) \) has a maximum point \( t_k \) satisfying

\[
m(a + b)A_1 = \frac{A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) e^{-2\pi t}}{t} (2\pi + \frac{1}{t}).
\]

Thus

\[
t_k = \left( \frac{m}{2\pi} + o(1) \right) \ln k,
\]

and

\[
g(t_k) = \frac{a + b}{t_k^m k^m A_1} \left( 1 + O\left( \frac{1}{t_k} \right) \right) .
\]

As a result, the function

\[
g(r) = \frac{a + b}{r^m} A_1 - \left( A_2 + o(1) + O\left( \frac{\beta^2}{\ln k} \right) \right) \frac{k}{r} e^{-2r} + O\left( \frac{1}{r^{m+\epsilon}} \right).
\]
has a maximum point
\[ r_k = k t_k = \left( \frac{m}{2\pi} + o(1) \right) k \ln k \]
with
\[ \hat{g}(r_k) = g(k t_k) = \frac{c_1 + o(1)}{|\ln k|^m k^m} \]
for some constant \( c_1 > 0 \) depending only on \( a, b, A_1, m \).

When \( o(1) \beta + O\left( \frac{A^2}{|m k|} \right) \leq 0 \), we define
\[ \hat{g}_1(r) = \frac{a + b}{r^m} A_1 - A_2 \frac{k}{r} e^{-2\pi \frac{m}{2\pi}}. \]
Then we can still verify that the maximum of \( \hat{g}_1(r) \) is
\[ \hat{g}_1(r_k) = \frac{c_2 + o(1)}{|\ln k|^m k^m} \]
for some constant \( c_2 > 0 \).

Furthermore, if \( o(1) \beta + O\left( \frac{A^2}{|m k|} \right) > 0 \), we have
\[ F(r_k) \geq k \left( A + \hat{g}(r_k) + \frac{o(1)}{|\ln k|^m k^m} \right) = k \left( A + \frac{c_1 + o(1)}{|\ln k|^m k^m} \right), \tag{3.3} \]
and if \( o(1) \beta + O\left( \frac{A^2}{|m k|} \right) \leq 0 \),
\[ F(r_k) \geq k \left( A + \hat{g}_1(r_k) + \frac{o(1)}{|\ln k|^m k^m} \right) = k \left( A + \frac{c_2 + o(1)}{|\ln k|^m k^m} \right). \tag{3.4} \]

Next we show that the maximum cannot be on the boundary of \( D_k \). Assume that \( r_k = (\frac{m}{2\pi} - \mu) k \ln k \) and \( o(1) \beta + O\left( \frac{A^2}{|m k|} \right) > 0 \). Then
\[ F(r_k) \leq k \left( A + \frac{c}{|\ln k|^m k^m} - \frac{A_2 e^{-2\pi (\frac{m}{2\pi} - \mu) k \ln k}}{(\frac{m}{2\pi} - \mu) k \ln k} + \frac{o(1)}{|\ln k|^m k^m} \right) \]
\[ \leq k \left( A + \frac{c}{|\ln k|^m k^m} - \frac{A_2}{|\ln k|^m k^m} \right) \left( \frac{m}{2\pi} - \mu \right) k^{2\pi (\frac{m}{2\pi} - \mu) k \ln k} \right) + \frac{o(1)}{|\ln k|^m k^m} \right) \]
\[ < k \left( A + \frac{o(1)}{|\ln k|^m k^m} \right), \]
which contradicts to (3.3). If \( o(1) \beta + O\left( \frac{A^2}{|m k|} \right) \leq 0 \),
\[ F(r_k) \leq k \left( A + \frac{c}{|\ln k|^m k^m} - \frac{A_2 + o(1) \beta + \beta^2 (\frac{m}{2\pi} - \mu) k \ln k}{(\frac{m}{2\pi} - \mu) k \ln k} + \frac{o(1)}{|\ln k|^m k^m} \right) \]
\[ \leq k \left( A + \frac{c}{|\ln k|^m k^m} - \frac{A_2 + o(1) \beta + \beta^2 (\frac{m}{2\pi} - \mu) k \ln k}{(\frac{m}{2\pi} - \mu) k \ln k} + \frac{o(1)}{|\ln k|^m k^m} \right) \]
\[ < k \left( A + \frac{o(1)}{|\ln k|^m k^m} \right), \]
which yields a contradiction to (3.4).

Assume that \( r_k = M k \ln k \). Then
\[ F(r_k) \leq k \left( A + \frac{(a + b) A_1}{M k \ln k} + \frac{o(1)}{|\ln k|^m k^m} \right) - k \left( A + \frac{c_1 + o(1)}{|\ln k|^m k^m} \right), \]
if \( M \) is large enough. This is also a contradiction to (3.3).

So we have proved that \( r_k \) is an interior point of \( D_k \) for large \( k \) and then \( r_k \) is a critical point of \( F(r) \).
Appendix A. Energy expansion. In this section, we will expand the energy $I(Z_r, Z^*_r)$, where
\begin{align*}
I(u, v) &= \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\nabla u}{i} - A(y)u \right)^2 + \lambda_1(y)|u|^2 + \frac{\nabla v}{i} - A(y)v \right)^2 + \lambda_2(y)|v|^2 \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} (|u|^4 + |v|^4) - \frac{\beta}{2} \int_{\mathbb{R}^3} |u|^2|v|^2.
\end{align*}

Firstly, we give one elementary inequality which is applied in the previous sections (see [20]).

**Lemma A.1.** For $p \geq 2$ and $k \in \mathbb{N}$, there is $C > 0$ such that for any $a_j \in \mathbb{C}$, $j = 1, 2, \ldots, k$
\begin{align*}
\left\| \sum_{j=1}^{k} a_j |z|^{p-2} \left( \sum_{j=1}^{k} a_j - \sum_{j=1}^{k} |a_j|^{p-2}a_j \right) \right\|_{p} &\leq C \sum_{i \neq j} |a_i|^{p-2}|a_j|.
\end{align*}

Now, we consider $Z_r$ and $Z^*_r$ and similar to Lemma A.1 in [27], we have

**Lemma A.2.** For any $\alpha \geq 1$, there is a constant $\epsilon$, such that
\begin{align*}
|Z_r(y)|^\alpha &= U^\alpha_x(y) + O\left( \frac{1}{r^\alpha} e^{-\frac{1}{2}|y-x|^2} \right), \quad \forall y \in \Omega_i, \\
|Z_r^*(y)|^\alpha &= U^\alpha_y(y) + O\left( \frac{1}{r^\alpha} e^{-\frac{1}{2}|y-y'|^2} \right), \quad \forall y \in \Omega^*_i,
\end{align*}
where
\begin{align*}
\Omega_i &= \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R} : \left( \frac{z'}{|z'|} \frac{(x_i)^i}{|x_i|^i} \right) \geq \cos \left( \frac{\pi}{k} \right) \right\}, \quad i = 1, 2, \ldots, k, \\
\overline{\Omega}_i &= \left\{ z = (z', z'') \in \mathbb{R}^2 \times \mathbb{R} : \left( \frac{z'}{|z'|} \frac{(y_i')'}{|y_i'|} \right) \geq \cos \left( \frac{\pi}{k} \right) \right\}, \quad i = 1, 2, \ldots, k.
\end{align*}

**Lemma A.3.** There is a small constant $\epsilon > 0$, such that
\begin{align*}
I(Z_r, Z^*_r) &= k \left( A + \left( \frac{a}{r^m} + \frac{b}{\gamma^m} \right) A_1 - A_2 \frac{k}{r} e^{-2r \bar{z}} - o(1) \frac{\beta}{r} e^{-2r \bar{z}} + \left( \frac{1}{r^{m+\epsilon}} + \frac{1}{r^{m+\epsilon}} \right) \right),
\end{align*}
where
\begin{align*}
A &= \frac{1}{2} \int_{\mathbb{R}^3} U^4, \quad A_1 = \frac{1}{2} \int_{\mathbb{R}^3} U^2, \\
\text{and } A_2 \text{ is a positive constant independent of } k.
\end{align*}

**Proof.** Note that
\begin{align*}
\int_{\mathbb{R}^3} \left| \left( \frac{\nabla}{i} - A(|y|) \right) Z_r \right|^2 &= \int_{\mathbb{R}^3} \left| \left( \frac{\nabla}{i} - A(r) \right) Z_r \right|^2 + \int_{\mathbb{R}^3} |A(r) - A(|y|)||Z_r|^2 \\
&\quad + 2\text{Re} \int_{\mathbb{R}^3} \left( \frac{\nabla}{i} - A(r) \right) Z_r (A(r) - A(|y|)) Z_r^*, \\
\left( \frac{\nabla}{i} - A(r) \right)^2 Z_{x^i} + Z_{x^i} &= |Z_{x^i}|^2 Z_{x^i},
\end{align*}
and
\begin{align*}
\int_{\mathbb{R}^3} |Z_r|^4 &= \int_{\mathbb{R}^3} \sum_{i=1}^{k} |Z_{x^i}|^2 + \text{Re} \sum_{i \neq j} |Z_{x^i} Z_{x^j}|^2
\end{align*}
Next, we estimate each term in (A.1).

So we have

\[ I(Z_r, Z_r^*) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ \left( \frac{\nabla}{i} - A(|y|) \right) Z_r |^2 + \lambda_1(|y|) |Z_r|^2 + \left| \left( \frac{\nabla}{i} - A(|y|) \right) Z_r^* |^2 + \lambda_2(|y|) |Z_r^*|^2 \right] - \frac{1}{4} \int_{\mathbb{R}^3} \left( |Z_r|^4 + |Z_r^*|^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^3} |Z_r|^2 |Z_r^*|^2 \]

\[ = \frac{k}{2} \int_{\mathbb{R}^3} U^4 + \frac{1}{2} \sum_{i \neq j} Re \int_{\mathbb{R}^3} |Z_{x_i}|^2 |Z_{x_j}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |Z_r|^2 \]

\[ + \frac{1}{2} \sum_{i \neq j} Re \int_{\mathbb{R}^3} |Z_{y_i}|^2 |Z_{y_j}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |Z_r^*|^2 \]

\[ + \frac{1}{2} \sum_{i \neq j} Re \int_{\mathbb{R}^3} |Z_{x_i}^*| |Z_{x_j}^*| |Z_{x_i}^*| |Z_{x_j}^*| + \frac{1}{2} \int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |Z_r^*|^2 \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} \left[ \sum_{i \neq j} |Z_{x_i}|^2 |Z_{x_j}|^2 + \left| \left( \sum_{i \neq j} Z_{x_i} Z_{x_j}^* \right) \right|^2 + 2Re \sum_{i \neq j} |Z_{x_i}|^2 Z_{x_i} Z_{x_j} \]
Moreover, similar to the above two equalities, it is easy to check
\[
\int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |Z_r|^2 = kO\left(\frac{1}{r^{t+\epsilon}}\right),
\]
and
\[
\int_{\mathbb{R}^3} |A(r) - A(|y|)|^2 |Z_r^*|^2 = kO\left(\frac{1}{r^{t+\epsilon}}\right).
\]
Secondly, noticing that
\[
\nabla Z_r = iA(r)Z_r + \sum_{j=1}^k e^{\eta_j} \nabla U_{x_j},
\]
where \(\eta_j = i\sigma + iA(x^j)(y - x^j)\), we find
\[
Re \int_{\mathbb{R}^3} \left(\frac{\nabla}{i} - A(r)\right) Z_r (A(r) - A(|y|))Z_r^* = k \int_{\mathbb{R}^3} U^2 + O\left(\frac{1}{r^{t+\epsilon}}\right),
\]
and
\[
Re \int_{\mathbb{R}^3} \left(\frac{\nabla}{i} - A(r)\right) Z_r^* (A(r) - A(|y|))Z_r = kO\left(\frac{1}{r^{t+\epsilon}}\right).
\]
But applying the same argument as before, we calculate
\[
\int_{\mathbb{R}^3} \sum_{i \neq j} |Z_{x^i}^2|Z_{x^j}|^2 = k \int_{\mathbb{R}^3} U^2 + O\left(\frac{k}{r} e^{-3r\frac{r}{2}}\right),
\]
\[
\int_{\mathbb{R}^3} \sum_{i \neq j} \left|Re Z_{x^i}Z_{x^j}^*\right|^2 = k \int_{\mathbb{R}^3} U^2 + O\left(\frac{k}{r} e^{-3r\frac{r}{2}}\right),
\]
\[
\sum_{i \neq j} Re \int_{\mathbb{R}^3} |Z_{x^i}^2 Z_{x^j}| = k \sum_{j=2}^k \cos(A(x^j)(x^j - x^{j-1})) \int_{\mathbb{R}^3} U^3 + O\left(\frac{k}{r} e^{-3r\frac{r}{2}}\right),
\]
and
\[
\int_{\mathbb{R}^3} \sum_{i \neq j} |Z_{x^i}^*|Z_{x^j}^*|^2 = k \int_{\mathbb{R}^3} U^2 + O\left(\frac{k}{r} e^{-3r\frac{r}{2}}\right),
\]
\[
\int_{\mathbb{R}^3} \left| \text{Re} \sum_{i \neq j} Z_{z_i}^* \overrightarrow{Z}_{y_i} \right|^2 = k \sum_{j=2}^{k} \int_{\Omega_i} U_{y_j}^2 U_{y_j}^2 + kO \left( \frac{k}{r} e^{-3r \bar{z}} \right),
\]

\[
\sum_{i \neq j} \text{Re} \int_{\mathbb{R}^3} |Z_{z_i}^*|^2 Z_{z_i}^* \overrightarrow{Z}_{y_i} = k \sum_{j=2}^{k} \cos(A(y^1)(y^j - y^1)) \int_{\Omega_i} U_{y_j}^3 U_{y_j} + kO \left( \frac{k}{r} e^{-3r \bar{z}} \right).
\]

Finally, from the Lemma A.4 in [27], one has as \( k \to \infty \),

\[
\int_{\Omega_1} U_{z_i}^2 U_{y_i}^2 = o(1) \frac{k}{r} e^{-2r \bar{z}}.
\]

So,

\[
\int_{\mathbb{R}^3} |Z_r|^2 |Z_r|^2 = k \int_{\Omega_1} |Z_{x_1} + Z_{x_2} + Z_{x_3} + \sum_{j=3}^{k-1} Z_{x_j}|^2 |Z_{y_1} + Z_{y_2} + Z_{y_3} + \sum_{j=3}^{k-1} Z_{y_j}|^2 \\
= k \int_{\Omega_1} \left[ |Z_{x_1}|^2 |Z_{y_1}|^2 + |Z_{x_2}|^2 |Z_{y_2}|^2 + |Z_{x_3}|^2 |Z_{y_3}|^2 + |Z_{x_1}|^2 |Z_{y_1}|^2 \right] + kO \left( \frac{k}{r} e^{-3r \bar{z}} \right) \\
= o(1) \frac{k^2}{r} e^{-2r \bar{z}} + kO \left( \frac{k}{r} e^{-3r \bar{z}} \right).
\]

As a result, combing the above estimates, we get

\[
I(Z_r, Z_r^*) = \frac{k}{2} \int_{\mathbb{R}^3} U^4 + \left( \frac{\alpha}{\pi} \int_{\mathbb{R}^3} U^2 + \frac{b}{\pi} \int_{\mathbb{R}^3} U^2 \right) + \frac{k}{2} \sum_{j=2}^{k} \left( \cos(A(x^1)(x^j - x^1)) \right) \\
- \frac{k}{4} \sum_{j=2}^{k} \left( 2 \int_{\Omega_1} U_{y_j}^3 U_{y_j} + 2 \cos(A(x^1)(x^j - x^1)) \right) \int_{\Omega_1} U_{z_j}^2 U_{z_j} \\
+ 2 \int_{\Omega_1} U_{y_j}^3 U_{y_j} + 2 \cos(A(y^1)(y^j - y^1)) \right) \int_{\Omega_1} U_{z_j}^3 U_{z_j} \\
- o(1) \frac{\beta k^2}{r} e^{-2r \bar{z}} + kO \left( \frac{1}{\pi r_{m+\epsilon}} + \frac{1}{\pi r_{p+\epsilon}} + \frac{1}{\pi r_{q+\epsilon}} + \frac{k}{r} e^{-3r \bar{z}} \right) \\
= \frac{k}{2} \int_{\mathbb{R}^3} U^4 + \left( \frac{\alpha}{\pi} \int_{\mathbb{R}^3} U^2 + \frac{b}{\pi} \int_{\mathbb{R}^3} U^2 \right) \\
+ \frac{k}{2} \left( \int_{\mathbb{R}^3} U_{z_1}^2 \sum_{j=2}^{k} \cos(A(x^1)(x^j - x^1)) \right) \frac{e^{-|x^1 - x^j|}}{|x^1 - x^j|} \\
+ \int_{\mathbb{R}^3} U_{y_j}^2 \sum_{j=2}^{k} \cos(A(y^1)(y^j - y^1)) \frac{e^{-|y^1 - y^j|}}{|y^1 - y^j|} \\
- \frac{k}{4} \left( 2 \int_{\Omega_1} U_{z_1}^2 \sum_{j=2}^{k} \frac{e^{-|x^1 - x^j|}}{|x^1 - x^j|} + 2 \int_{\Omega_1} U_{z_1}^2 \sum_{j=2}^{k} \cos(A(x^1)(x^j - x^1)) \frac{e^{-|x^1 - x^j|}}{|x^1 - x^j|} \right).
\]
\[+ 2 \int_{\Omega} U^2 \sum_{j=2}^{k} \frac{e^{-|y_j^1-y_j^2|}}{|y_j^1-y_j^2|} + 2 \int_{\Pi} U^2 \sum_{j=2}^{k} \cos(A(y_j^1)(y_j^1-y_j^2)) e^{-|y_j^1-y_j^2|} |y_j^1-y_j^2|^{\beta_k} \times \frac{k^2}{r} e^{-3r_j^{\frac{n}{m}}} + O(1) \frac{1}{r^{\nu_n+\epsilon}} + \frac{1}{r^{m+\epsilon}} + \frac{k^2}{r} e^{-3r_j^{\frac{n}{m}}} \]

where we used the fact that \(y \in \Omega_1, |y-x| \geq \frac{1}{2} |x^1-x^2|\) and \(y \in \Pi_1, |y-y^1| \geq \frac{1}{2} |y^1-y^2|\).

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