The domination number of round digraphs

Abstract: The concept of the domination number plays an important role in both theory and applications of digraphs. Let $D = (V, A)$ be a digraph. A vertex subset $T \subseteq V(D)$ is called a dominating set of $D$, if there is a vertex $t \in T$ such that $\forall v \in A(D)$ for every vertex $v \in V(D) \setminus T$. The dominating number of $D$ is the cardinality of a smallest dominating set of $D$, denoted by $\gamma(D)$. In this paper, the domination number of round digraphs is characterized completely.

Keywords: domination number, round digraph, local tournament, purely local tournament

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1 Introduction

The domination theory of graphs was derived from a board game in ancient India. In 1962, Ore formally gave the definitions of the dominating set and the domination number in [1]. Due to the universality of its applications to both theoretical and practical problems, domination has become one of the important research topics in graph theory. A summary of most important results and applications can be found in [2]. Problems of resource allocations and scheduling in networks are frequently formulated as domination problems on underlying graphs (digraphs). By contrast, domination in digraphs has not yet gained the same amount of attention, although it has several useful applications as well. For example, it has been used in the study of answering skyline query in the database [3] and routing problems in networks [4]. The relationship among domination numbers of different orientations of a graph was studied in [5]. The relevant results about domination numbers of digraphs can be found in [6–10]. Recent studies on domination theory include [11–13].

We refer the reader to [14] for terminology and notation not defined in this paper. Let $D = (V, A)$ be a digraph, which means that $V$ and $A$ represent the vertex set and the arc set of $D$, respectively. The order of $D$ is the number of vertices in $D$, denoted by $|V(D)|$. If $uv$ is an arc, then we say that $u$ dominates $v$ (or $v$ is dominated by $u$) and use the notation $u \rightarrow v$ to denote this. For a vertex $v$ of a digraph $D$, we define the vertex set $N_D^+(v) = \{u \in V \mid vu \in A\}$, $N_D^-(v) = \{w \in V \mid wv \in A\}$. We also call the vertex set $N_D(v) = N_D^+(v) \cup N_D^-(v)$ the out-neighbourhood, the in-neighbourhood and the neighbourhood of the vertex $v$, respectively. $d_D^+(v)$ is the number of all arcs with tail $v$, and we call $d_D^-(v)$ the out-degree of $v$. $d_D^+(v)$ is the number of all arcs with head $v$, and we call $d_D^-(v)$ the in-degree of $v$. If each arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that $H$ is induced by $X = V(H)$ and denote it by $H = D[X]$. We call $H$ an induced subdigraph of $D$.

Let $D$ be a digraph. Let $v_1, v_2, \ldots, v_n$ be a vertex labelling of $D$. If there is always $i < j$ for every arc $v_iv_j$ in $D$, then we often refer to the vertex labelling as an acyclic ordering of $D$. A walk in $D$ is an alternating sequence
A digraph on \( n \) vertices is round if we can label its vertices \( v_1, v_2, \ldots, v_n \) so that for each \( i \), we have \( \mathcal{N}_D(v_i) = \{ v_{i+1}, \ldots, v_{d^{-1}(v_i)} \} \) and \( \mathcal{N}_D(v_i) = \{ v_{i-1}, \ldots, v_1 \} \) (all subscripts are taken modulo \( n \)). Let \( D = (V, A) \) be a digraph, and let \( T \) be a subset of the vertices of \( D \). If for every vertex \( v \in V(D) \setminus T \), there is a vertex \( t \in T \) such that \( tv \in A(D) \), then we say that \( T \) is a dominating set of \( D \) and denote it by \( T \to D \). The dominating number of \( D \) is the cardinality of a smallest dominating set of \( D \), denoted by \( \gamma(D) \).

We need the following lemma and theorem in order to prove the main theorems.

**Lemma 1.1.** [17] Every round digraph is locally semicomplete.

**Theorem 1.2.** [18] Let \( D \) be a strong tournament on \( n \geq 3 \) vertices. For every \( x \in V(T) \) and every integer \( k \in \{3, 4, \ldots, n\} \), there exists a \( k \)-cycle through \( x \) in \( D \). In particular, a tournament is Hamiltonian if and only if it is strong.

In this paper, the domination number of a round digraph is characterized by studying the round local tournament and the round non-local tournament, respectively.

## 2 The domination number of a round local tournament

### 2.1 The domination number of a round purely local tournament which is non-strong

Let \( D \) be a round purely local tournament which is non-strong. Let \( P_n = v_0v_1 \ldots v_{n-1} \) be a directed path of \( D \). If there is an arc \( v_iv_j \in A(D) \) satisfying \( j - i \geq 2 \) for \( i, j \in \{0, 1, \ldots, n-1\} \), then the arc \( v_iv_j \) is called a cross arc on \( P_n \). If there is no cross arc \( v_{i_0}v_{j_0} \) on \( P_n \) such that \( i_0 < i < j < j_0 \), then the cross arc \( v_{i_0}v_{j_0} \) is called a maximal cross arc on \( P_n \). We call the vertex set \( \{v_0, v_1, \ldots, v_i\} \) covered by the maximal cross arc \( v_{i_0}v_{j_0} \). We call the set \( G \) a maximal cross-arc chain on \( P_n \), if there is a maximal cross-arc set \( G = \{v_i, v_j\} \) is a maximal cross arc on \( P_n \), \( t \in \{0, 1, \ldots, k-1\} \) on \( P_n \) satisfying one of the following conditions:

1. For \( k = 1 \),
   (a) \( v_{i_0}v_{j_0} \) is a maximal cross arc on \( P_n \) and there is no set \( \{a, a', \beta, \beta'\} \subseteq \{0, 1, \ldots, n-1\} \) such that \( a < i_0 < a' \) and \( \beta < j_0 < \beta' \), where \( v_{a}v_{a'}, v_{\beta}v_{\beta'} \) are two cross arcs on \( P_n \); or
   (b) \( v_{i_0}v_{j_0} \) is a maximal cross arc on \( P_n \). There is a maximal cross arc \( v_{i_0}v_{j_0} \) on \( P_n \) such that \( y < i_0 < y' \) (or \( y < j_0 < y' \)) and \( i_0 - y = 1 \) (or \( y' - j_0 = 1 \)), and there is no cross arc \( v_yv_{y'} \) such that \( \tau < i_0 < \tau' \) and \( i_0 - \tau \geq 2 \) (or \( \tau < j_0 < \tau' \) and \( \tau' - j_0 \geq 2 \)) (Figure 1(a)).
For $k \geq k_1$, let $G_2 = (V_0, V_1)$ be a round purely local tournament which is non-strong. Let $G_2 = (V_0, V_1)$ be two maximal cross-arc chains on $P = v_0 v_1 \ldots v_{16}$. $P_1 = v_0 v_2, P_2 = v_2 v_4 v_6, P_3 = v_5 v_16$ are all maximal pure subpaths on $P$. (b) $k \geq 2$: $v_j v_k, v_k v_j$ are all maximal cross arcs on path $P = v_0 v_1 \ldots v_{16}$. $G = (v_k v_j | t = 0, 1, 2, 3)$ is an only maximal cross-arc chain on path $P = v_0 v_1 \ldots v_{16}$, in which $v_j v_k$ is an only invalid cross arc in $G$. $P_1 = v_0 v_2, P_2 = v_4 v_8 v_{16}$ are all maximal pure subpaths on $P$.

Figure 1: (a) $k = 1$: $G_1 = (V_0, V_1, \ldots, V_6)$ are two maximal cross-arc chains on $P = v_0 v_1 \ldots v_{16}$. $P_1 = v_0 v_2, P_2 = v_2 v_4 v_6, P_3 = v_5 v_{16}$ are all maximal pure subpaths on $P$; (b) $k \geq 2$: $v_j v_k, v_k v_j$ are all maximal cross arcs on path $P = v_0 v_1 \ldots v_{16}$. $G = (v_k v_j | t = 0, 1, 2, 3)$ is an only maximal cross-arc chain on path $P = v_0 v_1 \ldots v_{16}$, in which $v_j v_k$ is an only invalid cross arc in $G$. $P_1 = v_0 v_2, P_2 = v_4 v_8 v_{16}$ are all maximal pure subpaths on $P$.

Subsequently, we show the partition problem of the vertices of a round purely local tournament which is non-strong. Let $D$ be a round purely local tournament which is non-strong on $n$ vertices. Let $G = (V_0, V_1, \ldots, V_6)$ be a round purely local tournament which is non-strong. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. Since $D$ is a round purely local tournament which is non-strong, $P = v_0 v_1 \ldots v_{n-1}$ is the only Hamilton path in $D$. If there exists a cross arc on $P$, then it can only be forward arc (that is, from the vertex with a small subscript to the vertex with a large subscript). Thus, $V(D)$ can form a partition. It means $V(D) = \bigcup_{i=0}^{n-1} B_i$. $B_i \cap B_j = \emptyset$ for any $i \neq j$. Figure 1 illustrates these definitions.

The vertex set partition of a round purely local tournament $D$ which is non-strong. $V(D) = \bigcup_{i=0}^{n-1} B_i$, where $B_0 = \{v_0\}$, $B_1 = \{v_0, v_2, v_3\}$, $B_2 = \{v_2, v_4, v_5\}$, $B_3 = \{v_4, v_6, v_7\}$, $B_4 = \{v_6\}$, $B_5 = \{v_0, v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$, $B_6 = \{v_{16})\}$.

Figure 2: The vertex set partition of a round purely local tournament $D$ which is non-strong. $V(D) = \bigcup_{i=0}^{n-1} B_i$, where $B_0 = \{v_0\}$, $B_1 = \{v_0, v_2, v_3\}$, $B_2 = \{v_2, v_4, v_5\}$, $B_3 = \{v_4, v_6, v_7\}$, $B_4 = \{v_6\}$, $B_5 = \{v_0, v_2, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}\}$, $B_6 = \{v_{16})\}$.

Lemma 2.1. Let $D$ be a round purely local tournament which is non-strong on $n$ vertices. Let $P = v_0 v_1 \ldots v_{n-1}$ be a Hamilton path in $D$. If $P_m = v_0 v_1 \ldots v_{l(m-1)}$ is a maximal pure subpath on $P$, then $y(P_m) = \left\lfloor \frac{m}{2} \right\rfloor$. 

According to the partition about the vertex set above, the following conclusions can be obtained.
Proof. Since $P_m$ is a directed path, $v_j \rightarrow v_{j+1}$ for $j \in \{i_0, i_0 + 1, \ldots, i_0 + (m - 2)\}$.

When $m$ is even, let $T = \{v_{i_0}, v_{i_0+1}, v_{i_0+2}, \ldots, v_{i_0+(m-2)}\}$. Since $v_j \rightarrow v_{j+1}$ for $j \in \{i_0, i_0 + 1, \ldots, i_0 + (m - 2)\}$, we have $T \rightarrow P_m$. Thus, $\gamma(P_m) \leq |T| = \frac{m}{2}$. Choosing any vertex set $M = \{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}\} \subseteq V(D)$, there must exist a vertex $v_i \in V(P_m)$ such that $v_i, v_{j+1} \notin M$. By maximal pure subpath $P_m$, we see that $v_i \rightarrow v_{j+1}$ for any vertex $v_i \in M$ since $M \cap \{v_{j+1}\} = \emptyset$. Therefore, $M$ is not a dominating set of $P_m$. For the arbitrariness of $M$, we have $\gamma(P_m) \geq \frac{m}{2}$. Thus, $\gamma(P_m) = \frac{m}{2} = \left\lceil \frac{m}{2} \right\rceil$.

When $m$ is odd, let $T = \{v_{i_0}, v_{i_0+1}, v_{i_0+3}, \ldots, v_{i_0+(m-2)}\}$. It is easy to see that $T \rightarrow P_m$ and $|T| = \left\lceil \frac{m}{2} \right\rceil$. So $\gamma(P_m) \leq \left\lceil \frac{m}{2} \right\rceil$. Choosing any vertex set $M' = \{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-3)}\} \subseteq V(D)$, there must exist a vertex $v_i \in V(P_m)$ such that $v_i, v_{j+1} \notin M'$. By maximal pure subpath $P_m$, we have $v_i \rightarrow v_{j+1}$ for any vertex $v_i \in M'$. Then $M'$ is not a dominating set of $P_m$. It implies $\gamma(P_m) \geq \frac{m}{2}$. Thus, $\gamma(P_m) = \frac{m}{2} = \left\lceil \frac{m}{2} \right\rceil$.

Lemma 2.2. Let $D$ be a round purely local tournament which is non-strong on $n$ vertices. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. $P = v_0v_1v_2 \ldots v_{n-1}$ is the Hamiltonian path in $D$. If $B = \bigcup_{k=0}^{n-1} \{v_0, v_{k+1}, \ldots, v_k\} \subseteq V(D)$ is covered by the maximal cross-arc chain $G = \{v_0v_{j+1}\} t = 0, 1, 2, \ldots, k - 1$, then $\gamma(D(B)) = \tau$, where $\tau$ is the number of all valid arcs in $G$, and all subscripts are ordered in the round ordering.

Proof. Now we distinguish two cases to prove this lemma.

Case 1. There is no invalid cross arc in $G$.

According to the definitions of the maximal cross arc and the round digraph, there must be $v_i \rightarrow v_l$ where $l \in \{i + 1, i + 2, \ldots, j\}$ for $t \in \{0, 1, 2, \ldots, k - 1\}$. Since $i + 1 [i + 2, \ldots, j - 1] \in \{0, 1, \ldots, k - 1\}$, we have $v_l \rightarrow v_{j+1} \rightarrow (D(B))$. Thus, $\gamma(D(B)) \leq k = \tau$.

For arbitrary $t \in \{1, 2, \ldots, k - 2\}, v_lv_{h+1}$ is a valid cross arc of $G$. By the definition of a valid cross arc, $|l - h| \geq 2$. Since $G$ is a maximal cross-arc chain, $|i - h| \geq 2$ and $|j - h| \geq 2$. Thus, there is at least a vertex $v_{i_0}$ in the vertex set $\{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}\}$ such that $v_{i_0}$ is only covered by the maximal cross arc $v_{i_0}v_{i_0+1}$ of $G$, where $t \in \{0, 1, 2, \ldots, k - 1\}$. Let $A_t = \{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}\}$, $t \in \{0, 1, \ldots, k - 1\}$. It is easy to see $A_t \cap A_j = \emptyset$, $i, j \in \{0, 1, \ldots, k - 1\}$. For any vertex set $M \subseteq B$ for $|M| = k - 1$, there exists $a \in \{0, 1, \ldots, k - 1\}$ such that $A_a \cap M = \emptyset$. By the round purely local tournament $D$ which is non-strong, we have $v_i \rightarrow v_{a_0}$ for any $v_i \in A_0$. For any vertex $v_i \in A_a$, there is only covered by the maximal cross arc $v_{i_0}v_{i_0+1}$ of $G$. We have $v_i \rightarrow v_{a_0}$ for $v_i \in \{i_0, i_0 + 1, \ldots, i_0 + (m - 2)\} \subseteq V(D)$, where $\tau$ is the number of all valid arcs in $G$. According to Case 1, $\gamma(D(B)) \geq \tau$. Thus, $\gamma(D(B)) = \tau = \gamma(D(B))$ (Figure 3(a)).

Case 2. There is at least one invalid cross arc in $G$.

Let $v_{i_0}v_h$ be any invalid cross arc of $G$. Accordingly to the definition of an invalid cross arc, $i_0 - h = |1| = 1$. It implies that $\{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}\}$ is covered by the maximal cross arc $v_{i_0}v_{i_0+1}$ and $\{v_{i_0+1}, v_{i_0+2}, \ldots, v_{i_0+(m-2)}\}$ is covered by the maximal cross arc $v_{i_0+1}v_{i_0+2}$. By the definitions of the cover and the round digraph, we obtain $\{v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}\} \rightarrow \{v_{i_0+1}, v_{i_0+2}, \ldots, v_{i_0+(m-2)}\}$.

For the arbitrariness of $v_{i_0}, v_{i_0+1}, \ldots, v_{i_0+(m-2)}$, all vertices covered by the invalid cross arcs in $G$ can be covered by the valid cross arcs in $G$. This means $\gamma(D(B)) \leq \tau$, where $\tau$ is the number of valid arcs in $G$. According to Case 1, $\gamma(D(B)) \geq \tau$ can be proved similarly. Then, $\gamma(D(B)) = \tau$ (Figure 3(b)).

Lemma 2.3. Let $D$ be a round purely local tournament which is non-strong on $n$ vertices. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. Let $P = v_0v_1v_2 \ldots v_{n-1}$ be the Hamiltonian path in $D$, $P = v_0v_1v_2 \ldots v_{n-1}$ be a maximal subpath on $P$, and $G = \{v_0v_{j+1}\} k = 1, 2, \ldots, t - 1 \subseteq V(D)$ be a maximal cross-arc chain on $P$ adjacent to $P(P)$. If $B$ and $C$ represent the set of vertices covered by $P(P)$ and $G(G)$, respectively, then $\gamma(D(B \cup C)) = \gamma(D(B)) + \gamma(D(C))$. 

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Proof. Let \( T_B \) and \( T_C \) be the minimum dominating set of \( D(B) \) and \( D(C) \), respectively. It implies \( \gamma(D) = |T_B| = |T_C| \). It is easy to see \( T_B \cup T_C \rightarrow D(B \cup C) \). Thus, we have \( \gamma(D(B \cup C)) \leq |T_B| + |T_C| \). Without loss of generality, suppose that all the cross arcs in \( G \) are valid. Choose any vertex set \( M \subseteq V(D) \) satisfying \( |M| = |T_B| + |T_C| - 1 \). According to the proof of Lemma 2.2 and the structure of the round purely local tournament which is non-strong, the minimum dominating set of \( D(C) \) must be \( \gamma(D) = \{v_k|k = 1, 2, \ldots, t\} \) (where \( v_k \in \{v_0, v_{k+1}, \ldots, v_k \} \) is a vertex in \( C \) with the smallest subscript which is only covered by the valid cross arc \( v_k \). It is not difficult to see that \( |E| = |T_C| \). Thus, if \( |M \cap E| \leq |T_C| - 1 \), then \( M \) cannot dominate \( D(C) \), which means \( M \) is not a dominating set of \( D \). Otherwise, \( M \) contains \(|T_B| - 1\) vertices in \( B \) at most. As we know from Lemma 2.1, there exists \( v_k \in \{v_0, v_{k+1}, \ldots, v_k \} \) such that \( v_k \cap M \neq \emptyset \). By the structure of \( D \) and the pure subpath \( D(B) \), we have \( M \rightarrow \{v_0, v_{k+1}\} \), i.e. \( M \) is not a dominating set of \( D(B \cup C) \). Due to the arbitrariness of \( M \), \( \gamma(D(B \cup C)) \geq |T_B| + |T_C| \). Thus, \( \gamma(D(B \cup C)) = |T_B| + |T_C| = \gamma(D(B)) + \gamma(D(C)) \) (Figure 4). \( \square \)

Figure 4: An example for Lemma 2.3.

According to the aforementioned results, we can get the following result about the dominating number of a round purely local tournament which is non-strong.

**Theorem 2.4.** Let \( D \) be a round purely local tournament which is non-strong on \( n \) vertices. Let \( v_0, v_1, \ldots, v_n \) be a round labelling of \( D \). \( P = v_0v_1 \ldots v_{n-1} \) is the Hamiltonian path of \( D \). If there exist \( k \) maximal cross-arc chains \( G_i \) (\( i = 1, 2, \ldots, k \)) and \( l \) maximal pure subpaths \( P_j \) (\( j = 1, 2, \ldots, l \)) on \( P \), then \( \gamma(D) = \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \left\lceil \frac{n_j}{2} \right\rceil \), where \( l \in \{k - 1, k, k + 1\} \), \( \tau_i \) is the number of valid cross arcs in \( G_i \), and \( n_j \) is the number of vertices contained in \( P_j \).

**Proof.** According to the structure of \( D \), we obtain the vertices of \( D \) are either covered by a maximal cross-arc chain or covered by a maximal pure subpath. According to the definition of the maximal cross-arc chain and the maximal pure subpath, one can get \( \gamma(D) \leq \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \left\lceil \frac{n_j}{2} \right\rceil \) by Lemmas 2.1 and 2.2.
We give a proof for $\gamma(D) \geq \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \lceil \frac{n}{2} \rceil$ as follows. Choose any vertex set $M \subseteq V(D)$ for $|M| \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \lceil \frac{n}{2} \rceil$, then one of the following two cases holds at least:

**Case 1.** There is a maximal cross-arc chain $G_\alpha = \{v_\alpha v_\beta \mid i = 1, 2, \ldots, \tau_\alpha \}$ such that $|M \cap \{v_\beta \mid \beta = i, i + 1, \ldots, j_\alpha \}| \leq \tau_\alpha - 1$.

By the proof of Case 1 in Lemma 2.2, there must be $v_{\alpha_0} \in \{v_\beta \mid \beta = i, i + 1, \ldots, j_\alpha \}$ such that $M \not\rightarrow v_{\alpha_0}$. Then $M$ cannot dominate $D$.

**Case 2.** There is a maximal pure subpath $P_\beta$ such that $|M \cap V(P_\beta)| \leq \left\lceil \frac{n_{\beta}}{2} \right\rceil - 1$, where $n_{\beta} = |V(P_\beta)|$.

According to Lemma 2.1, there must be a vertex set $\{v_\gamma, v_{\gamma + 1}\} \subseteq V(P_\beta)$ such that $M \cap \{v_\gamma, v_{\gamma + 1}\} = \emptyset$. Due to the structure of $D$, it is obvious $N_D(v_{\gamma + 1}) = \{v_\beta\}$, which means $M \not\rightarrow v_{\gamma + 1}$. Then $M$ cannot dominate $D$.

Therefore, $M$ is not a dominating set of $D$ anyway. By the arbitrariness of $M$, $\gamma(D) \geq \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \lceil \frac{n}{2} \rceil$. Thus, $\gamma(D) = \sum_{i=1}^{k} \tau_i + \sum_{j=1}^{l} \left\lceil \frac{n}{2} \right\rceil$. $\square$

### 2.2 The domination number of a round purely local tournament which is strong

**Theorem 2.5.** Let $D$ be a directed cycle $v_0 v_1 \ldots v_{n-1} v_0$, then $\gamma(D) = \left\lceil \frac{n}{2} \right\rceil$, where all subscripts are taken modulo $n$.

**Proof.** Since $D$ is a directed cycle, we have $v_i \rightarrow v_{i+1}$ for $0 \leq i \leq n - 1$.

When $n$ is even, let $T = \{v_0, v_2, v_4, \ldots, v_{n-2}\}$. By the condition $v_i \rightarrow v_{i+1}$ for $0 \leq i \leq n - 1$, we obtain $T \rightarrow D$, and then $\gamma(D) \leq \frac{n}{2}$. Choose arbitrary vertex set $M = \{v_i, v_{i+1}, \ldots, v_{i+\frac{n}{2}-1}\}$, there must exist $\alpha \in \{0, 1, 2, \ldots, n - 1\}$ such that $\{v_\alpha, v_{\alpha + 1}\} \not\subseteq M$. Since $D$ is a cycle, we have $M \not\rightarrow v_{\alpha + 1}$. So $M$ is not a dominating set of $D$. By the arbitrariness of $M$, we have $\gamma(D) \geq \frac{n}{2}$. Therefore, $\gamma(D) = \frac{n}{2}$.

When $n$ is odd, it is easy to see the vertex set $T = \{v_0, v_1, v_3, \ldots, v_{n-1}\}$ is a dominating set of $D$ because $D$ is a directed cycle. Thus, $\gamma(D) \leq \left\lceil \frac{n}{2} \right\rceil$. Choose any vertex set $M = \{v_j, v_{j+1}, \ldots, v_{j+\frac{n-1}{2}}\}$, there must exist $\beta \in \{0, 1, 2, \ldots, n - 1\}$ such that $\{v_\beta, v_{\beta + 1}\} \not\subseteq M$. Since $D$ is a cycle, $M \not\rightarrow v_{\beta + 1}$. Therefore, $M$ is not a dominating set of $D$. For the arbitrariness of $M$, we get $\gamma(D) \geq \left\lceil \frac{n}{2} \right\rceil$. Thus, $\gamma(D) = \left\lceil \frac{n}{2} \right\rceil$. $\square$

For the convenience of proof, we define as follows similar to that of Section 2.1. Let $D$ be a round purely local tournament which is strong. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. Let $C_n = v_0 v_1 v_2 \ldots v_{n-1} v_0$ be the Hamilton cycle in $D$. The arc connecting any two non-adjacent vertices on the cycle is called a **cross arc** on $C_n$. Let $v_i \rightarrow v_j$ be a cross arc on $C_n$. If there is no cross arc $v_i \rightarrow v_j$ on $C_n$ such that $i < j < j_0 (j_0 < j_0 < i < i_0)$, then the cross arc $v_i \rightarrow v_j$ is called a **maximal cross arc** on $C_n$. We call the vertex set $\{v_i, v_{i+1}, \ldots, v_{j-1}\}$ covered by the maximal cross arc $v_i \rightarrow v_j$. We call the set $G$ a **maximal cross-arc chain** on $C_n$ and the vertex set $\bigcup_{k=0}^{l-1}\{v_i, v_{i+1}, \ldots, v_{j_k}\}$ covered by the maximal cross-arc chain $G$, if there is a maximal cross-arc set $\{v_i v_{j_0}, v_{j_1}, \ldots, v_{j_l}\}$ maximal cross arc on the cycle $C_n, \tau \in \{0, 1, \ldots, k - 1\}$ on $C_n$ satisfied:

1. For $k = 1$,
   a) $v_i v_j$ is a maximal cross arc on $C_n$ and there is no set $\{a, a', \beta, \beta'\} \subseteq \{0, 1, \ldots, n - 1\}$ such that $a < i_0 < a'$ and $\beta < j_0 < \beta'$, where $v_i v_{a'}, v_j v_{\beta'}$ are a cross arcs on $C_n$; or
   b) $v_i v_j$ is a maximal cross arc on $C_n$. There is a maximal cross arc $v_i v_{r'}$ on $C_n$ such that $y < i_0 < y'$ (or $y < i_0 < y'$) and $i_0 - y = 1$ (or $y' - j_0 = 1$), and there is no cross arc $v_i v_{r'}$ such that $\tau < i_0 < \tau'$ and $i_0 - \tau \geq 2$ (or $\tau < j_0 < \tau'$ and $\tau' - j_0 \geq 2$) (Figure 5(a)).
If there is a maximal cross arc $v_i v_j \in G$ satisfying $|i_{t+1} - j_{t-1}| = 1$, then the maximal cross arc $v_i v_j$ is an invalid cross arc of $G$, where $1 \leq t \leq k - 2$. In addition to the invalid cross arcs in $G$, we call the remaining maximal cross arcs the valid cross arcs of $G$. Let $P_m$ be a subpath on $C_n$. If all the vertices on $P_m$ are not covered by any maximal cross-arc chain, then call $P_m$ a pure subpath on $C_n$. If there is no vertex $v_k \in V(C_n)$ such that $D(V(P_{m}) \cup \{v_k\})$ is a pure subpath on $C_n$, then $P_m$ is a maximal pure subpath of $C_n$ and $V(P_{m})$ is covered by $P_m$. Figure 5 illustrates these definitions.

Next, we show the partition problem of the vertices of a round purely local tournament which is strong.

Let $D$ be a round purely local tournament which is strong. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. Then $C = v_0 v_1 \ldots v_{n-1}$ is the only Hamilton cycle in $D$. Similar to the round purely local tournament which is non-strong, $V(D)$ can form a partition, which means $V(D) = \bigcup_{i=0}^{m} B_i$, $B_i \cap B_j = \emptyset$ for any $i \neq j \in \{0, 1, \ldots, m\}$, where $B_i$ is covered by either some maximal pure subpath or some maximal cross-arc chain (Figure 5).

**Theorem 2.6.** Let $D$ be a round purely local tournament which is strong on $n$ vertices. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. $C = v_0 v_1 \ldots v_{n-1}$ is a Hamilton cycle in $D$. If there are $k$ maximal cross-arc chains $G_i$ on $C$, then $\gamma(D) = \sum_{i=0}^{k-1} \left( \pi_i + \lceil \frac{n}{2} \rceil \right)$, where $\pi_i$ indicates the number of valid cross arcs in $G_i$, $n_i$ indicates the number of vertices contained in the maximal pure subpath $P_i$ on $C$ for $i \in \{0, 1, \ldots, k-1\}$.

**Proof.** Obviously, there are $k$ maximal pure subpaths on $C$. Without loss of generality, let $P_0 = v_0 v_1 \ldots v_{10-1}$, $G_0 = \{v_{10} v_0 \mid t = 1, 2, \ldots, n_0\}$, $P_1 = v_1 v_{10} v_1 v_{10} v_2 \ldots v_{n-2}, G_1 = \{v_1 v_{10} \mid t = 1, 2, \ldots, n_1\}$, $P_{k-1} = v_{k-1} v_{k-1} v_{k-2} \ldots v_{10}, G_{k-1} = \{v_{n-2} v_{n-1} v_{k-1} \mid t = 1, 2, \ldots, n_{k-1}\}$, where $n_{k-1} = n - 1$ (Figure 6). Let $D' = D - v_{n-1} v_0$. By Theorem 2.4, we have $\gamma(D') = \sum_{t=0}^{k-1} \left( \pi_i + \lceil \frac{n}{2} \rceil \right)$. Since $N_{D'}(v_0) = \emptyset$, $v_0$ must be contained in any dominating set of $D'$. Also, owing to $N_{D'}(v_0) = \{v_{n-1}\}$ and the proof of Lemma 2.2, one can obtain that $v_{n-1}$ has been dominated by $v_{n-1} v_{n-1}$. Thus, $\gamma(D) = \gamma(D') = \sum_{t=0}^{k-1} \left( \pi_i + \lceil \frac{n}{2} \rceil \right)$. \qed
2.3 The domination number of a round tournament

**Theorem 2.7.** Let $D$ be a strong round tournament with $n$ vertices. Let $v_0, v_1, \ldots, v_n$ be a round labelling of $D$. Then $\gamma(D) = 2$, where all subscripts are taken modulo $n$.

**Proof.** Since $D$ is a strong tournament, $d^+(v) \leq n - 2$ for any vertex $v \in V(D)$. It implies any vertex in $D$ cannot form a dominating set of $D$, i.e. $\gamma(D) \geq 2$. Furthermore, we have $V(D) \geq 3$ and $v_0$ is adjacent to $v_n$, since $D$ is a strong tournament. Then the following two cases will be considered.

**Case 1.** $v_0 \rightarrow v_{n-2}$.

According to the definition of the round digraph, we have $v_0 \rightarrow v_t$ as well as $v_{n-2} \rightarrow v_{n-1}$, where $t \in \{1, 2, \ldots, n - 2\}$. Therefore, $\{v_0, v_{n-2}\} \rightarrow D$, which means $\gamma(D) \leq 2$. Thus, we have $\gamma(D) = 2$.

**Case 2.** $v_{n-2} \rightarrow v_0$.

Since $D$ is a strong tournament, at this time we get $V(D) \geq 4$ for $v_{n-2} \rightarrow v_0$. If $v_0 \rightarrow v_{n-3}$, then $\{v_0, v_{n-2}\}$ is a dominating set of $D$ according to $v_0 \rightarrow \{v_1, \ldots, v_{n-3}\}$ and $v_{n-2} \rightarrow v_{n-1}$, i.e. $\gamma(D) \leq 2$. It implies $\gamma(D) = 2$. Conversely, if $v_{n-3} \rightarrow v_0$, then we consider $v_{n-4}$. Proceeding in this manner, if there exists a vertex $v_0 \in \{v_2, \ldots, v_{n-2}\}$ such that $v_0 \rightarrow v_{n-1}$, then $\{v_0, v_{n-3}\}$ is a dominating set of $D$. Otherwise, there will be $v_0 \rightarrow v_0, v_0 \in \{v_2, \ldots, v_{n-2}\}$. If $v_2 \rightarrow v_0$, then $v_2 \rightarrow v_{n-2}$ for $u_0 \in \{v_3, v_4, \ldots, v_{n-1}\}$ according to $v_2 \rightarrow v_1$ and the definition of a round digraph. From $v_0 \rightarrow v_1$, we know $\{v_0, v_2\}$ is a dominating set of $D$, which means $\gamma(D) \leq 2$. Then $\gamma(D) = 2$ (Figure 7).

**Theorem 2.8.** Let $D$ be a non-strong round tournament with $n$ vertices. $v_0, v_1, \ldots, v_n$ is a round labelling of $D$. Then $\gamma(D) = 1$, where all subscripts are taken modulo $n$.

**Proof.** Since $D$ is a round tournament which is non-strong, $v_0 \rightarrow v_i$ for $i \in \{1, 2, \ldots, n - 1\}$. Therefore, $\gamma(D) = 1$. 

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**Figure 6:** A round purely local tournament $D$ which is strong with $k$ maximal cross-arc chains and $k$ maximal pure subpaths.

**Figure 7:** An example for Theorem 2.7.
The domination number of a round non-local tournament

If a round digraph $D$ is a non-local tournament, then there is a 2-cycle in $D$ by Lemma 1.1. It implies $D$ is a semicomplete digraph or a purely local semicomplete digraph. According to the definition of the round digraph, one can know that $D$ is strongly connected.

**Theorem 3.1.** Let $D$ be a round non-local tournament with $n$ vertices. Let $v_0, v_1, \ldots, v_{n-1}$ be a round labelling of $D$. Then

$$
\gamma(D) = \begin{cases} 
1, & \text{if there exists } v_i \in V(D) \text{ such that } v_{i+1} \rightarrow v_i, \\
2, & \text{otherwise.}
\end{cases}
$$

**Proof.** Since $D$ is a strong round digraph, we have $v_0v_1\ldots v_{n-1}v_0$ is a Hamilton cycle of $D$. If there exists $v_i \in V(D)$ such that $v_{i+1} \rightarrow v_i$, then $v_{i+1} \rightarrow v_j$ where $j \in \{i + 2, i + 3, \ldots, i - 1, i\}$ according to the definition of the round digraph. Therefore, $\{v_{i+1}\}$ is a dominating set of $D$. Thus, $\gamma(D) = 1$ (Figure 8(a)).

Otherwise, there is a vertex set $\{v_\alpha, v_\beta\} \subseteq V(D)$ such that $\{v_\alpha v_\beta, v_\beta v_\alpha\} \subseteq A(D)$ satisfying $|\alpha - \beta| \neq 1$ according to the round non-locally tournament $D$ which is strong. By $v_\beta \rightarrow v_\alpha$, we have $v_\alpha \rightarrow v_\tau$ for $\tau \in \{\alpha + 1, \alpha + 2, \ldots, \beta - 1, \beta\}$. Similarly, we have $v_\beta \rightarrow v_\nu$ because $v_\nu \rightarrow v_\alpha$ for $\nu \in \{\beta + 1, \beta + 2, \ldots, \alpha - 1, \alpha\}$. Thus, $\{v_\alpha, v_\beta\}$ is a dominating set for $D$. Then $\gamma(D) \leq 2$. Choosing any vertex $v_\gamma \in V(D)$, we can see that $v_\gamma \rightarrow v_{i+1}$ by the known conditions, which means that $\{v_\gamma\}$ cannot form a dominating set of $D$. For the arbitrariness of $v_\gamma$, $\gamma(D) \geq 2$. Therefore, $\gamma(D) = 2$ (Figure 8(b)).

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