Numerical solution of generalized fractional Volterra integro-differential equations via approximation the Bromwich integral

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Abstract. In this paper, we consider the generalized fractional Volterra integro-differential equations with the regularized Prabhakar derivative and represent the solution of this type of equations in the form of Bromwich integral in the complex plane. Then we select the hyperbolic contour as an optimal contour to approximate the Bromwich integral. Further, an example to show absolute errors for various parameters by using our numerical scheme on hyperbolic contour is given.

1. Introduction and Preliminaries

The fractional Volterra integro-differential equations (FVIDEs) have many applications in physics, engineering, economics, diffusion problems. As the exact solutions of FVIDEs are difficult in many cases, so numerical methods are proposed to obtain the solution of such types of [1, 6].

In this work, a numerical scheme based on Laplace transform is constructed to approximate the generalized FVIDEs with the regularized Prabhakar derivative. By applying Laplace transform, we get the solution of generalized FVIDEs in the sense of contour integral in the complex plane. Then we select the hyperbolic contour and use trapezoidal rule with equal step size to approximate this integral. Finally, the performance of the numerical method is tested for an example.

Definition 1 For \( f \in L^1[0, b] \), the Prabhakar integral operator with generalized Mittag-Leffler function [5]

\[
E^\gamma_{\rho,\mu}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)} \frac{x^k}{k!}, \quad \gamma, \rho, \mu \in \mathbb{C}, \quad \Re(\rho) > 0, \Re(\mu) > 0, \quad (1)
\]

in its kernel is defined as follows [3]

\[
E^\gamma_{\rho,\mu,\omega,0+}(x) = \int_0^x (x - u)^{\mu-1}E^\gamma_{\rho,\mu}(\omega(x - u)^\rho)f(u)du, \quad 0 < x < b \leq \infty. \quad (2)
\]
Definition 2. The regularized Prabhakar derivative for \( f \in AC^m[0, b] \) is defined by [3]
\[
^{cD}_{\rho, \mu, \omega, 0^+} f(x) = E_{\rho, m-\mu, \omega, 0^+} \frac{d^m}{dx^m} f(x), \quad \rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\rho), \Re(\mu) > 0.
\] (3)
The notation \( AC^m[0, b] \) is the space of real-valued functions \( f(x) \) with continuous derivatives up to order \( m-1 \) on \([0, b]\) such that \( f^{(m-1)}(x) \) belongs to the space of absolutely continuous functions \( AC[0, b] \):
\[
AC^m[0, b] = \{ f : [0, b] \rightarrow \mathbb{R} ; \frac{d^{m-1}}{dx^{m-1}} f(x) \in AC[0, b] \}.
\]

Lemma 3. For \( m-1 < \mu \leq m \), the Laplace transform of regularized Prabhakar fractional derivative (3) has the form [2]
\[
\mathcal{L}^{cD}_{\rho, \mu, \omega, 0^+} f(x); s = s^\mu (1-\omega s^{-\rho})^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^\gamma f^{(k)}(0),
\] (4)
where \( F(s) \) is the Laplace transform of \( f(x) \).

2. Numerical approximation of generalized FVIDEs

Consider the following generalized FVIDEs with the regularized Prabhakar derivative
\[
^{cD}_{\rho, \mu, \omega, 0^+} f(t) = g(t) + \int_0^t k(t, \tau)f(\tau)d\tau, \quad m-1 \leq \alpha < m, m \in \mathbb{N}.
\] (5)
For the convolution type of equation (5), the kernel will be of the form \( k(t, \tau) = k(t-\tau) \). Therefore, the equation (5) becomes
\[
^{cD}_{\rho, \mu, \omega, 0^+} f(t) = g(t) + \int_0^t k(t-\tau)f(\tau)d\tau.
\] (6)
Taking the Laplace transform on the both side of (6) with respect to \( t \), and using the Laplace transform of the regularized Prabhakar fractional derivative, we have
\[
s^\mu (1-\omega s^{-\rho})^\gamma F(s) - \sum_{k=0}^{m-1} s^{\mu-k-1}(1-\omega s^{-\rho})^\gamma f^{(k)}(0) = \mathcal{L}(g(t); s) + K(s)\mathcal{L}(f(t); s),
\]
where \( F(s) = \mathcal{L}(f(t); s) \), \( G(s) = \mathcal{L}(g(t); s) \) and \( K(s) = \mathcal{L}(k(t); s) \). Finally, we get
\[
F(s) = \frac{G(s)}{s^\mu(1-\omega s^{-\rho})^\gamma + K(s)} + \sum_{k=0}^{m-1} \frac{s^{\mu-k-1}(1-\omega s^{-\rho})^\gamma f^{(k)}(0)}{s^\mu(1-\omega s^{-\rho})^\gamma + K(s)},
\]
Taking inverse Laplace, the problem reduces to compute the following integral in the complex plane
\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds.
\] (7)
The numerical method for inversion of the Laplace transform is based on the approximation of the Bromwich complex contour integral. We select the contour of integration to approximate the path from $c - i\infty$ to $c + i\infty$. To this end we consider hyperbolic contour [7, 8]. The parametric equation of hyperbolic contour is given by [4, 7, 8]

$$s = \beta(1 - \sin(\lambda + c)\cosh(\zeta)) + i\beta\cos(\lambda + c)\sinh(\zeta), \quad -\infty < \zeta < +\infty,$$

where $\beta$, $\lambda$ and $c$ are parameters and need to be optimized for better accuracy. More details about the hyperbolic contour are given in [4, 8]. The numerical solution can be represented in the following form

$$f(t) = L^{-1}\{F(s)\}; s \to t = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s(\zeta)) e^{s(\zeta)t} s'(\zeta) ds.$$

If we use equal weight quadrature rule, i.e trapezoidal rule with step size $h$, then the equation (9) can be approximated as

$$f_N(t) = \frac{h}{2\pi i} \sum_{j=-N}^{N} F(s(\zeta_j)) e^{s(\zeta_j)t} s'(\zeta_j), \quad 1 < \alpha < 2, \quad \zeta_j = jh.$$

3. Example

Consider the following generalized FVIDE

$$\mathcal{C}_D^{\gamma \rho, \mu, \omega, 0} f(t) = t^{\mu - 1} E_{\rho, \mu}^{\gamma} (\omega t^\rho) + \int_0^t (t - \tau)^{\mu - 1} E_{\rho, \mu}^{\gamma} (\omega(t - \tau)^\rho) f(\tau) d\tau, \quad f(0) = 0, \quad \rho, \omega, \gamma \in \mathbb{R}, \quad \mu \in (0, 1).$$

The exact solution of (1) is

$$f(t) = \sum_{i=0}^{\infty} t^{2\mu(i+1)-1} E_{\rho, 2\mu(i+1)}^{2\gamma(i+1)} (\omega t^\rho).$$

To give the approximate solution of (11), we use the presented numerical scheme. Applying the Laplace transform to equation (11) with respect to $t$, we have

$$s^{\mu}(1 - \omega s^{-\rho})^{\gamma} F(s) - s^{\mu-1}(1 - \omega s^{-\rho})^{\gamma} f(0) = s^{-\mu}(1 - \omega s^{-\rho})^{-\gamma} + s^{-\mu}(1 - \omega s^{-\rho})^{-\gamma} F(s).$$

Since $f(0) = 0$, so we get

$$F(s) = \frac{s^{2\mu(1-\omega s^{-\rho})^{-2\gamma}}}{1-s^{2\mu(1-\omega s^{-\rho})^{-2\gamma}}}, \quad \rho, \omega, \gamma \in \mathbb{R}, \quad 0 < \mu < 1.$$

From the above equation and equation (10), we get the approximate solution as

$$f_N(t) = \frac{h}{2\pi i} \sum_{j=-N}^{N} \frac{(s(\zeta_j))^{-2\mu(1-\omega s(\zeta_j))^{-\rho})^{-2\gamma}}{1-(s(\zeta_j))^{-2\mu(1-\omega s(\zeta_j))^{-\rho})^{-2\gamma}} e^{s(\zeta_j)t} s'(\zeta_j), \quad \zeta_j = jh.$$

For our numerical experiments, we chose the parameters such that the absolute error has the least value. So, we take the parameters as follows
\[
\begin{align*}
  c &= 0.1, \\
  \lambda &= 0.8, \\
  h &= \frac{1}{N} \phi_\lambda, \\
  \beta &= \frac{1}{\phi_\lambda ((4\pi\lambda - \pi^2)N)},
\end{align*}
\]

where \( \phi_\lambda = \cosh^{-1}\left(\frac{2\lambda}{(4\lambda\pi - \pi\sin(\lambda))}\right) \). Table 1 and Figures 1-2 show the absolute errors for approximate solution of generalized FVIDE (11). Figure 1 shows the absolute errors for \( t = 0.1, \gamma = 5, \rho = 0.1, \mu = 0.99, \omega = -20 \) and various \( N \). Figure 2 shows the absolute errors for \( N = 100, \gamma = 5, \rho = 0.1, \omega = -20 \) and different \( t \) and \( \mu \).

**Table 1**: Absolute errors of equation (11) using our numerical scheme and hyperbolic contour.

| \( N \) | \( t \) | \( \gamma \) | \( \rho \) | \( \mu \) | \( \omega \) | Absolute error |
|---|---|---|---|---|---|---|
| 20 | 0.1 | 0.2 | 0.1 | 0.99 | -1 | 8.9582e-05 |
| 30 | 0.1 | 0.2 | 0.1 | 0.99 | -1 | 6349e-08 |
| 40 | 0.1 | 0.2 | 0.1 | 0.99 | -1 | 13994e-12 |
| 50 | 0.1 | 0.2 | 0.1 | 0.99 | -1 | 4.0246e-16 |
| 100 | 0.1 | 0.2 | 0.1 | 0.99 | -1 | 3892e-17 |
| 100 | 1 | 0.2 | 0.1 | 0.99 | -1 | 4.2388e-06 |
| 100 | 0.01 | 0.2 | 0.1 | 0.99 | -1 | 8.6744e-18 |
| 100 | 0.01 | 1.5 | 0.1 | 0.99 | -1 | 8.6737e-19 |
| 100 | 0.01 | 9 | 0.1 | 0.99 | -1 | 8.7352e-22 |
| 100 | 0.01 | 9 | 0.2 | 0.99 | -1 | 1.5439e-16 |
| 100 | 0.01 | 9 | 0.4 | 0.99 | -1 | 7419e-17 |
| 100 | 0.01 | 9 | 0.1 | 0.95 | -1 | 4.3376e-19 |
| 100 | 0.01 | 9 | 0.1 | 0.75 | -1 | 1276e-16 |
| 100 | 0.01 | 9 | 0.1 | 0.75 | -0.1 | 2.8866e-15 |
| 100 | 0.01 | 9 | 0.1 | 0.75 | -10 | 4400e-20 |
| 100 | 0.01 | 9 | 0.1 | 0.75 | -45 | 7.4447e-24 |
Figure 1: Absolute errors for approximate solution of generalized FVIDE (11) for $t = 0.1$, $\gamma = 5$, $\rho = 0.1$, $\mu = 0.99$, $\omega = -20$ and various $N$.

Figure 2: Absolute errors for approximate solution of generalized FVIDE (11) for $N = 100$, $\gamma = 5$, $\rho = 0.1$, $\omega = -20$ and different $t$ and $\mu$.

4. Conclusion

In this work we constructed a numerical method for approximating the generalized FVIDEs
including the regularized Prabhakar derivative. The proposed numerical method is based on Laplace transform and quadrature rule. Then we gave an example and showed the absolute value errors for approximating the solution of generalized FVIDE.

References

[1] Akbar M, Nawaz R, Ahsan S, Nisar K S, Abdel Aty A H and Eleuch H 2020 New approach to approximate the solution for the system of fractional order Volterra integro-differential equations Results Phys. 19 103453.
[2] Eshaghi S, Ansari A, Khoshsiar Ghaziani R and Ahmadi Darani M 2017 Fractional Black-Scholes model with regularized Prabhakar derivative Publ. de l’Institut Math. 102 116 121-132.
[3] Garra R, Gorenflo R, Polito F and Tomovski Z 2014 Hilfer-Prabhakar derivatives and some applications Appl. Math. Comput. 242 576-589.
[4] McLean W and Thomée V 2010 Numerical solution via Laplace transforms of a fractional order evolution equation J. Integral Equ. Appl. 22 57-94.
[5] Prabhakar T R 1971 A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 7-15.
[6] Rajagopala N, Balajia S, Seethalakshmi R and Balajib V S 2020 A new numerical method for fractional order Volterra integro-differential equations Ain Shams Eng. J. 11 171-177.
[7] Uddin M and Uddin M 2020 On the numerical approximation of Volterra integro-differential equation using Laplace transform Comput. Methods Differ. Equ. 8 305-313.
[8] J. Weideman and L. Trefethen, Parabolic and hyperbolic contours for computing the Bromwich integral, Math. Comput., 76 (2007), 1341–1356.