Coupling analysis of a 5-degree-of-freedom hybrid manipulator based on a global index

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Abstract
Parallel kinematic machines have been applied in aerospace and automotive manufacturing due to their potentials in high speed and high accuracy. However, there exists coupling in parallel kinematic machines, which makes dynamic analysis, rigidity enhancement, and control very complicated. In this article, coupling characteristics of a 5-degree-of-freedom (5-dof) hybrid manipulator are analyzed based on a local index and a global index. First, velocity analysis as well as acceleration analysis of the robot is conducted to provide essential information for dynamic modeling. Then the dynamic model is built based on the principle of virtual work. Whereas the mass matrix is off-diagonal, a local coupling index as well as a global index is defined, based on which coupling characteristics of the robot are analyzed. Results show that distributions of coupling indices are symmetric due to its structural features. And dimensional parameters, structural parameters, as well as mass parameters have a large influence on the system’s coupling characteristics. Research conducted in the article is of great help in optimal design and control. Meanwhile, the method proposed in the article can be applied to other types of parallel kinematic machines or hybrid manipulators.

Keywords
Parallel kinematic machine, rigid dynamics, principle of virtual work, coupling analysis

Introduction
Parallel kinematic machines (PKMs) have received significant attention both in terms of theoretical research and practical applications due to their merits of high
loads, high accuracy, and high speed when compared with serial kinematic machines (SKMs). The typical success can be verified by the application of the Tricept and Sprint Z3 in assembly and milling in aerospace and automotive industry.\textsuperscript{1–4} As is known, PKMs have large stiffness and large loading capability due to their closed-loop structural features. However, it is the closed-loop structural features that make limbs of a PKM coupled, which makes it very complicated for topological synthesis, dimensional design, performance enhancement, and control system design. Limbs of PKMs are coupled, which demonstrates that the driving force/torque of one certain limb is related to other limbs. Despite that there are many references focusing on kinematic analysis,\textsuperscript{5–7} stiffness modeling,\textsuperscript{8–10} dynamic analysis,\textsuperscript{11–13} and calibration\textsuperscript{14–16} of PKMs, references about coupling analysis are very scarce. As far as the authors are concerned, coupling analysis of PKMs involves two basic issues, one is how to establish a coupling model, and the other is how to quantify coupling strength.

In general, the dynamic model of a PKM is a highly nonlinear complex system with multiple inputs/outputs, which can be regarded as the basis of coupling analysis. There are mainly several methods of dynamic modeling, that is, Lagrange method,\textsuperscript{17,18} Newton-Euler method\textsuperscript{19,20} Kane’s method,\textsuperscript{21,22} the principle of virtual work.\textsuperscript{23,24} The Lagrange method is based on the system’s energy. Bonnemains et al.\textsuperscript{25} built a dynamic model of Tripteror X7, based on which inertia parameters were indentified. The Newton-Euler method describes inertia parameters, force, and acceleration of a PKM by establishing one Newton equation and one Euler equation of each part. Based on the Newton-Euler method, Wang et al.\textsuperscript{26} proposed a simplified dynamic model which was used to the optimal design of the driving force. Relatively speaking, the Kane’s method is complicated, which calculates the generalized driving force and generalized inertia force of a system by defining partial velocity and partial angular velocity. The principle of virtual work is the most commonly used method due to its easy principle. Kalani et al.\textsuperscript{27} presented an improved dynamic model of a 6-UPS PKM based on the principle of virtual work. In addition, the Gibbs-Appell method is usually applied to establish dynamic equations of PKMs.\textsuperscript{28} As for the coupling strength, there are no too much references talking about how to quantify. Bergerman et al.\textsuperscript{29} developed a dynamic coupling index for an underactuated manipulator, which can express the coupling degree between passive joints and active joints. Inspired by the work conducted in Bergerman et al.,\textsuperscript{29} this article attempts to analyze coupling in PKMs.

According to discussions above, this article deals with the coupling strength of a 5-dof hybrid manipulator. The remainder of the article is organized as follows. Kinematic analysis is described in Section 2, which can provide essential information for the following dynamic modeling. Based on the principle of virtual work, the rigid dynamic model is established in Section 3. Then a local coupling index and a global index are proposed, based on which coupling characteristics are analyzed in Section 4. Finally, main conclusions are drawn in Section 5.
Description of mechanical structures and inverse kinematic analysis

Figure 1 shows a 3D model of the Trimule robot proposed in Dong et al.,\textsuperscript{10} which contains a PKM module and a 2R module. In general, the PKM module is composed of one base, one platform, three active universal–prismatic–spherical (UPS) limbs and one passive universal–prismatic (UP) limb. The three active UPS limb assemblages have the same topological structures, which has an oscillating limb connected to the base by a universal joint and a stretchable limb connected to the platform by a spherical joint. And the passive UP limb assemblage mainly consists of a thin limb and a thick limb as shown in Figure 2. The thin limb is connected to the base by a universal joint while the thick limb is fixedly connected to the platform. The 2R module consists of two revolute joints which are defined as the A axis and the C axis, respectively, which can improve flexibility of the robot.

In order to facilitate derivation, the schematic diagram of the hybrid manipulator and several coordinate systems are depicted in Figure 3. Herein, \( C \) is the end point of the machine tool. \( A \) denotes the intersection point of A axis and C axis. \( A_i \) (\( i = 1 \) to 3) denotes the geometrical center of the \( i \)th spherical joint. \( A_4 \) is the connection point on the interface of the platform and the passive limb. \( B_k \) (\( k = 1 \) to 4) denotes the geometrical center of the \( k \)th universal joint. \( C_k \) (\( k = 1 \) to 4) is the end point of the \( k \)th limb. The machine tool body-fixed frame \( C-x_Cy_Cz_C \) is attached to the end point of the tool \( C \), with the \( x_C \) axis parallel to the C axis’s rotation axis, the \( z_C \) axis parallel to the machine tool’s axis, and the \( y_C \) axis can be decided by the right-hand rule. The A axis body-fixed frame \( A-x_Ay_Az_A \) is built at the point \( A \),
where the $x_A$ axis parallel to the C axis’s rotation axis, the $z_A$ axis parallel to the axis of the A axis, and the $y_A$ axis can be decided by the right-hand rule. $B_4$-xyz is the global frame attached to the point $B_4$, where the $x$ axis points from $B_4$ to $B_2$ and the $y$ axis is vertically up to $B_3B_2$. Then direction of the $z$ axis can be determined by the right-hand rule. The platform body-fixed frame $P$-uvw is set at the midpoint of $A_3A_2$, with the $u$ axis parallel to $A_3A_2$ and the $w$ axis vertical to the
platform. Accordingly, the $v$ axis can be decided by the right-hand rule. The limb body-fixed frame $B_k$-$x_k$-$y_k$-$z_k$ is set at the point $B_k$, where the $z_k$ axis is along the $k$th limb, the $y_k$ axis is along one rotation axis of the $k$th universal joint, and the $x_k$ axis can be determined by the right-hand rule. To be specific, $B_1$-$x_1$-$y_1$-$z_1$ in the 1st UPS limb is shown in Figure 3.

Assume that the transformation matrix of the platform body-fixed frame $P$-$uvw$ with respect to the global frame $B_4$-$xyz$ can be defined as

$$B_4 R_P : \text{Trans}(P - uvw \rightarrow B_4 - xyz) \quad (1)$$

Similarly, transformation matrices of frames $B_k$-$x_k$-$y_k$-$z_k$ and $C$-$xCyCzC$ with respect to the global frame $B_4$-$xyz$ can be expressed, respectively, as

$$B_4 R_{Bk} : \text{Trans}(B_k - x_ky_kz_k \rightarrow B_4 - xyz) \quad (2)$$

$$B_4 R_C : \text{Trans}(C - xCyCzC \rightarrow B_4 - xyz) \quad (3)$$

For content limitation, detailed inverse kinematics is not described in the article. Since there is no coupling in the 2R module, the following will focus on kinematic and dynamic modeling of the PKM module.

**Velocity analysis**

Position of the point $P$ $r_P$ can be expressed in the global frame $B_4$-$xyz$ as

$$r_P = b_i + q_i w_i - a_i, (i = 1\sim 3) \quad (4)$$

$$r_P = q_4 w_4 \quad (5)$$

where $b_i$ is the vector pointing from $B_4$ to $B_i$; $q_i$ and $w_i$ denote the length and unit vector of the $i$th UPS limb, respectively; $a_i$ is the vector pointing from $P$ to $A_i$ expressed in the global frame $B_4$-$xyz$; $q_4$ and $w_4$ denote the length and unit vector of the UP limb, respectively.

Taking derivation of equations (4) and (5) yields

$$v_P = \dot{q}_i w_i + q_i (\omega_Li \times w_i) - \omega_p \times a_i, (i = 1\sim 3) \quad (6)$$

$$v_P = \dot{q}_4 w_4 + q_4 (\omega_p \times w_4) \quad (7)$$

where $v_P$ and $w_P$ are the linear velocity of the point $P$ and the angular velocity of the platform, respectively; $\dot{q}_i$ and $\omega_Li$ denote the velocity of the $i$th prismatic joint and the angular velocity of the $i$th UPS limb, respectively.

Taking inner dot of equation (6) by $w_i$ leads to

$$\dot{q}_i = w_i^T v_P + (a_i \times w_i)^T \omega_P, (i = 1\sim 3) \quad (8)$$

Taking inner dot of equation (7) by $u$ and $v$ ($u$ and $v$ are unit vectors of $u$ and $v$ axes), respectively, yields
\[ u^T v_P + q_4 (u \times w_4)^T \omega_P = 0 \quad (9) \]
\[ v^T v_P + q_4 (v \times w_4)^T \omega_P = 0 \quad (10) \]

It is noted that the rotation matrix \( B_4 R_P \) can be determined by rotating about the \( x \) axis with an angle \( \psi \) first and then rotating about the \( v \) axis with an angle \( \theta \). According to the principle of linear superposition, the angular velocity of the platform can be expressed as

\[ \omega_P = \dot{\psi} x + \dot{\theta} v \quad (11) \]

Taking inner dot of equation (11) by \( n = v \times x \) yields

\[ n^T \omega_P = 0 \quad (12) \]

Equations (8)–(10) and (12) can be written in a matrix form as follows

\[
J \begin{bmatrix} v_P \\ \omega_P \end{bmatrix} = \dot{\theta}
\]

(13)

where \( \dot{\theta} = [\dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3 0 0 0]^T \); \( J = \begin{bmatrix} J_a \\ J_c \end{bmatrix} \) is the generalized Jacobian matrix of the PKM module, \( J_a \) and \( J_c \) are defined as the active Jacobian matrix and the constraint Jacobian matrix, respectively, which can be expressed as

\[
J_a = \begin{bmatrix} w_1 \\ a_1 \times w_1 \\ w_2 \\ a_2 \times w_2 \\ w_3 \\ a_3 \times w_3 \end{bmatrix}^T
\]

(14)
\[
J_c = \begin{bmatrix} u \\ v \\ q_4 (u \times w_4) \\ q_4 (v \times w_4) \\ n \end{bmatrix}^T
\]

(15)

Taking cross product of equation (6) by \( w_i \), the angular velocity of the \( i \)th UPS limb \( \omega_{Li} \) can be expressed as

\[ \omega_{Li} = J_{oli} \begin{bmatrix} v_P \\ \omega_P \end{bmatrix}, (i = 1\sim3) \]

(16)

where \( J_{oli} = 1/q_i [w_i \times] (E_{3 \times 3} - [a_i \times]), [w_i \times] \) and \([a_i \times]\) are skew matrices of vectors \( w_i \) and \( a_i \).

Supposing that the mass center of the \( i \)th UPS limb is defined as \( D_i \) and the vector pointing from \( B_i \) to \( D_i \) is \( r_i \) in the global frame \( B_4-xyz \), the velocity of the mass center \( D_i \) can be expressed in the frame \( B_4-xyz \) as

\[ v_{Li} = v_{Bi} + \omega_{Li} \times r_i, (i = 1\sim3) \]

(17)

where \( v_{Bi} \) denotes the velocity of point \( B_i \) and there exists \( v_{Bi} = 0 \).
Substituting equation (16) into equation (17), the velocity of $D_i$ can be calculated by

$$v_{Li} = J_{vi} \begin{bmatrix} v_p \\ \omega_p \end{bmatrix}, (i = 1\sim3)$$ (18)

where $J_{vi} = -1/q_i[r_i\times](E_{3\times3} - [a_i\times])$ and $[r_i\times]$ are the skew matrix of $r_i$.

Combining equations (16) with (18), the velocity of the mass center $D_i$ and the angular velocity of the $i$th UPS limb can be written in a matrix form as follows

$$\begin{bmatrix} v_{Li} \\ \omega_{Li} \end{bmatrix} = J_{Li} \begin{bmatrix} v_p \\ \omega_p \end{bmatrix}, (i = 1\sim3)$$ (19)

where $J_{Li} = \begin{bmatrix} J_{vi} \\ J_{oi} \end{bmatrix}$ can be defined as the transformation matrix of the platform’s velocity with respect to that of the $i$th UPS limb.

Similarly, define $r_4$ as the position vector of the UP limb’ mass center $D_4$, whose velocity can thus be expressed as

$$v_{L4} = \omega_p \times r_4$$ (20)

Thus, the velocity of the mass center $D_4$ and angular velocity of the UP limb can be written as

$$\begin{bmatrix} v_{L4} \\ \omega_{L4} \end{bmatrix} = J_{L4} \begin{bmatrix} v_p \\ \omega_p \end{bmatrix}$$ (21)

where $J_{L4} = \begin{bmatrix} 0_{3\times3} & -[r_4\times] \\ 0_{3\times3} & E_{3\times3} \end{bmatrix}$, $[r_4\times]$ denotes the skew matrix of vector $r_4$.

**Acceleration analysis**

Taking inner dot of equation (7) by $w_4$ yields

$$\dot{q}_4 = w_4^T v_p$$ (22)

Taking derivation of equation (13) yields

$$\dot{J} \begin{bmatrix} \dot{a}_p \\ \dot{\xi}_p \end{bmatrix} + J \begin{bmatrix} \dot{v}_p \\ \dot{\omega}_p \end{bmatrix} = \ddot{q}$$ (23)

where $\dot{J} = \begin{bmatrix} \dot{J}_a \\ \dot{J}_c \end{bmatrix}$ is the derivative matrix of $J$ with respect to time. And there exist

$$\dot{J}_a = \begin{bmatrix} \omega_{L1} \times w_1 \\ (\omega_p \times a_1) \times w_1 + a_1 \times (\omega_{L1} \times w_1) \\ \omega_{L2} \times w_2 \\ (\omega_p \times a_2) \times w_2 + a_2 \times (\omega_{L2} \times w_2) \\ \omega_{L3} \times w_3 \\ (\omega_p \times a_3) \times w_3 + a_3 \times (\omega_{L3} \times w_3) \end{bmatrix}^T$$ (24)
\[
\begin{bmatrix}
\dot{J} \\
\dot{\xi}_L
\end{bmatrix} = \begin{bmatrix}
J_L \quad \dot{J}_L \quad \dot{\xi}_L
\end{bmatrix} = \begin{bmatrix}
a_L \\
\dot{a}_p \\
\dot{v}_p
\end{bmatrix} + \begin{bmatrix}
\dot{v}_p \\
\dot{a}_p \\
\dot{\xi}_L
\end{bmatrix}, (i = 1 \sim 3)
\]

(26)

where \( J_{Li} \) is the derivative matrix of \( J_L \) with respect to time, which can be expressed as \( \dot{J}_{Li} = \begin{bmatrix}
\dot{J}_{vi} \\
\dot{J}_{wi}
\end{bmatrix} \). And there exist

\[
\dot{J}_{wi} = \frac{1}{q_i} \left\{ \left( \left[ (\omega_L \times w_i) \right] \times \right) \left( E_{3 \times 3} \quad -[a_i] \times \right) \right. + \left. [w_i] \times ([0_{3 \times 3} \quad [(a_i \times \omega_p) \times])] \right\}
\]

(27)

\[
\dot{J}_{vi} = \frac{1}{q_i} \left\{ \left( \frac{\dot{q}_i}{q_i} [r_i] \right) - \left[ (\omega_L \times r_i) \times \right] \right. \left. [w_i] \times ([E_{3 \times 3} \quad -[a_i] \times]) \right. \\
- \left. [r_i] \times ([\omega_L \times w_i] \times ([E_{3 \times 3} \quad -[a_i] \times]) + [w_i] \times ([0_{3 \times 3} \quad [(a_i \times \omega_p) \times])] \right\}
\]

(28)

Similarly, taking derivation of equation (21) yields

\[
\begin{bmatrix}
a_L \\\n\dot{\xi}_L \end{bmatrix} = \begin{bmatrix}
J_{L4} \quad \dot{J}_{L4} \quad \dot{\xi}_{L4}
\end{bmatrix} = \begin{bmatrix}
a_p \\
\dot{a}_p \\
\dot{v}_p
\end{bmatrix} + \begin{bmatrix}
\dot{v}_p \\
\dot{a}_p \\
\dot{\xi}_{L4}
\end{bmatrix}, \left[ \Delta_4 \times \right] \text{ denotes the skew matrix of vector } \Delta_4 = r_4 \times \omega_p.
\]

(29)

**Formulation of rigid dynamics**

In this section, the principle of virtual work is applied to formulate the rigid dynamic model of the PKM module. And the 2R module will be treated as the platform's load whose inertia matrix is position-dependent. In order to facilitate derivation, the force diagram of the PKM module is depicted in Figure 4.

**Virtual work conducted by the platform.** Assume that the external load \([F \quad M]^T\) is applied at the point \(P\) in the platform, of which the virtual work can be expressed as

\[
W^F_P = \begin{bmatrix}
v_p \\
\omega_p
\end{bmatrix}^T \begin{bmatrix}
F \\
M
\end{bmatrix}
\]

(30)

The inertial force and moment of the platform can be expressed as
Figure 4. Force diagram of the PKM module.

\[
\begin{bmatrix}
F_p \\
M_p
\end{bmatrix} =
\begin{bmatrix}
-m_P a_p \\
-I_P \xi_P - \omega_P \times I_P \omega_P
\end{bmatrix}
\]

(31)

where \( m_P \) and \( I_P \) are the equivalent mass and inertia matrix of the platform measured in the global frame \( B_{4-xyz} \), respectively. It is noted that mass of the 2R module can be regarded as load applied at the platform. Thus there exists

\[
m_P = m_{P_0} + m_A + m_C, \quad I_P = I_{P_0} + I_A + I_C
\]

(32)

where \( m_{P_0}, m_A, \) and \( m_C \) are the mass of the platform itself, the A axis, and the C axis, respectively; \( I_{P_0}, I_A, \) and \( I_C \) are the inertia matrices of the platform itself, the A axis and the C axis expressed in the global frame \( B_{4-xyz} \). It is noted that \( I_{P_0}, I_A, \) and \( I_C \) are calculated about the platform’s mass center \( P \).

Then the virtual work conducted by the platform’s inertial force and moment can be expressed as

\[
W^a_p = \begin{bmatrix}
v_p \\
\omega_p
\end{bmatrix}^T \begin{bmatrix}
F_p \\
M_p
\end{bmatrix}
\]

(33)

Substituting equation (31) into equation (33) yields

\[
W^a_p = -\begin{bmatrix}
v_p \\
\omega_p
\end{bmatrix}^T \left( \begin{bmatrix}
m_P e & 0 \\
0 & I_P
\end{bmatrix} \begin{bmatrix}
a_p \\
\xi_P
\end{bmatrix} + \begin{bmatrix}
0 \\
\omega_P \times I_P \omega_P
\end{bmatrix} \right)
\]

(34)

where \( e \) is an identity matrix in \( 3 \times 3 \).
The virtual work conducted by the platform’s gravity can be expressed as

\[ W_g^P = [v_P \omega_P]^T [m_P g] \]  

(35)

where \( g \) denotes the gravity’s acceleration.

**Virtual work conducted by the \( i \)th UPS limb**

Assume that the external force \( f_i \) (actually caused by the driving torque) is applied at the \( i \)th UPS limb, whose virtual work can be expressed as

\[ W_{fi}^{Li} = \dot{q}_i f_i \]  

(36)

The inertial force and moment of the \( i \)th UPS limb can be expressed as

\[
\begin{bmatrix}
F_{Li}^a \\
M_{Li}^a
\end{bmatrix}
= 
\begin{bmatrix}
-m_{Li} a_{Li} \\
-I_{Li} \xi_{Li} - \omega_{Li} \times I_{Li} \omega_{Li}
\end{bmatrix}
\]  

(37)

where \( m_{Li} \) and \( I_{Li} \) are the equivalent mass and inertia matrix of the \( i \)th UPS limb measured in the global frame \( B_4-xyz \). It is noted that \( m_{Li} \) and \( I_{Li} \) are contributed by the stretchable limb and oscillating limb. And the inertia matrix \( I_{Li} \) is position-dependent.

Then the virtual work conducted by the \( i \)th UPS limb’s inertial force and moment can be expressed as

\[ W_{Li}^a = [v_{Li} \omega_{Li}]^T [F_{Li}^a M_{Li}^a] \]  

(38)

Substituting equation (37) into equation (38) yields

\[ W_{Li}^a = -[v_{Li} \omega_{Li}]^T \left( \begin{bmatrix}
m_{Li} e \\
0
\end{bmatrix} \begin{bmatrix}
a_{Li} \\
\xi_{Li}
\end{bmatrix} + \begin{bmatrix}
0 \\
0
\end{bmatrix} \begin{bmatrix}
\omega_{Li} \times I_{Li} \omega_{Li}
\end{bmatrix} \right) \]  

(39)

The virtual work conducted by the \( i \)th UPS’s gravity can be expressed as

\[ W_{Li}^g = [v_{Li} \omega_{Li}]^T [m_{Li} g] \]  

(40)

**Virtual work conducted by the UP limb**

The inertial force and moment of the UP limb can be expressed as

\[
\begin{bmatrix}
F_{LA}^a \\
M_{LA}^a
\end{bmatrix}
= 
\begin{bmatrix}
-m_{LA} a_{LA} \\
-I_{LA} \xi_{LA} - \omega_{LA} \times I_{LA} \omega_{LA}
\end{bmatrix}
\]  

(41)
where \( m_{L4} \) and \( I_{L4} \) are the equivalent mass and inertia matrix of the UP limb measured in the global frame \( B_4-xyz \).

Then the virtual work conducted by the UP limb’s inertial force and moment can be expressed as

\[
W_{aL4} = \begin{bmatrix} v_{L4} \\ \omega_{L4} \end{bmatrix}^T \begin{bmatrix} F_{L4}^a \\ M_{L4}^a \end{bmatrix}
\]  

(42)

Substituting equation (41) into equation (42) yields

\[
W_{aL4} = - \begin{bmatrix} v_{L4} \\ \omega_{L4} \end{bmatrix}^T \left( \begin{bmatrix} m_{L4} e & 0 \\ 0 & I_{L4} \end{bmatrix} \begin{bmatrix} a_{L4} \\ \xi_{L4} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_{L4} \times I_{L4} \omega_{L4} \end{bmatrix} \right)
\]  

(43)

The virtual work conducted by the UP limb’s gravity can be expressed as

\[
W_{gL4} = \begin{bmatrix} v_{L4} \\ \omega_{L4} \end{bmatrix}^T \begin{bmatrix} m_{L4}g \\ 0 \end{bmatrix}
\]  

(44)

**Dynamic equation of the PKM module**

Based on the principle of virtual work, there exists

\[
W_P + W_{pL} + W_{pL}^F + \sum_{i=1}^{3} (W_{Li} + W_{Li}^a + W_{Li}^g) + W_{L4}^a + W_{L4}^g = 0
\]  

(45)

Substituting equations (13), (19), (21), (23), (26), (29), (30), (34)–(36), (39), (40), (43) and (44) into equation (45) yields

\[
M\ddot{U} + C\dot{U} + G + Q - \begin{bmatrix} F \\ M \end{bmatrix} = J^Tf
\]  

(46)

where \( U \) is the displacement of the platform, \( C \) is the coefficient matrix of \( \dot{U} \), \( G \) is the system’s gravity, \( Q \) is the Coriolis force which can be ignored if the PKM works in a low speed, \( f = [f_1 \ f_2 \ f_3 \ f_1p \ f_2p \ f_3p]^T \), \( M \) is the mass matrix of the PKM module, which can be expressed as

\[
M = M_P + \sum_{i=1}^{3} M_{Li} + M_{L4}, \quad C = \sum_{i=1}^{3} C_{Li} + C_{L4}
\]  

(47)

\[
G = -G_P - \sum_{i=1}^{3} G_{Li} - G_{L4}, \quad Q = Q_P + \sum_{i=1}^{3} Q_{Li} + Q_{L4}
\]  

(48)
where \( M_P = \begin{bmatrix} m_P e & 0 \\ 0 & I_P \end{bmatrix} \), \( M_{Li} = J_{Li}^T m_{Li} e 0 \\ 0 & I_{Li} \end{bmatrix} J_{Li} \), \( M_{L4} = J_{L4}^T m_{L4} e 0 \\ 0 & I_{L4} \end{bmatrix} J_{L4} \),
\( C_{Li} = J_{Li}^T \begin{bmatrix} m_{Li} e & 0 \\ 0 & I_{Li} \end{bmatrix} J_{Li} \), \( C_{L4} = J_{L4}^T \begin{bmatrix} m_{L4} e & 0 \\ 0 & I_{L4} \end{bmatrix} J_{L4} \), \( G_P = \begin{bmatrix} m_P g \\ 0 \end{bmatrix}, \ G_{Li} = J_{Li}^T \begin{bmatrix} m_{Li} g \\ 0 \end{bmatrix}, \ G_{L4} = J_{L4}^T \begin{bmatrix} m_{L4} g \\ 0 \end{bmatrix} \),
\( Q_P = \begin{bmatrix} 0 \\ \omega P \times I_P \omega_P \end{bmatrix}, \ Q_{Li} = J_{Li}^T \begin{bmatrix} 0 \\ \omega L_i \times I_{Li} \omega_{Li} \end{bmatrix}, \) and
\( Q_{L4} = J_{L4}^T \begin{bmatrix} 0 \\ \omega L_4 \times I_{L4} \omega_{L4} \end{bmatrix} \).

If the generalized Jacobian matrix \( J \) is invertible, the dynamic model can be expressed in the joint space as
\[
M^* \ddot{q} + C^* \dot{q} + G^* + Q^* - J^{-T} \begin{bmatrix} F \\ M \end{bmatrix} = f
\]  

(49)

where \( M^* \) denotes the mass matrix of the PKM module in the joint space, which can be expressed as
\[
M^* = J^{-T} M J^{-1}, \quad C^* = J^{-T} C J^{-1} - M^* J J^{-1} \quad (50)
\]
\[
G^* = J^{-T} G, \quad Q^* = J^{-T} Q
\]  

(51)

**Coupling analysis**

In general, the driving force caused by \( C^* \dot{q} \) and \( Q^* \) is very small and the term \( G^* \) can be compensated by feedforward control. Therefore, if the external force is zero, the driving force can be mostly determined by the inertial force as follows
\[
f \approx M^* \ddot{q}
\]  

(52)

In order to extract the three driving force of the three UPS limbs, the following is given
\[
\tilde{f} = \tilde{M} \ddot{q}
\]  

(53)

where \( \tilde{f} = [f_1 \quad f_2 \quad f_3]^T, \tilde{M} = EM^* E^T, E = \begin{bmatrix} e \\ 0 \end{bmatrix}, \tilde{q} = [q_1 \quad q_2 \quad q_3]^T \).

It is noted that the mass matrix \( \tilde{M} \) is off-diagonal, which can be regarded as the root of the PKM’s coupling. In general, the mass matrix \( \tilde{M} \) can be expressed as
\[
\tilde{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}
\]  

(54)

If the PKM is not coupling, off-diagonal elements should be zero. Therefore, \( \kappa_i \) can be regarded as the local coupling index, which can be expressed as
$k_i = P_{3j = 1, j 
eq i} \frac{m_{ij}}{m_{ii}}$ (55)

It can be seen from equation (55) that the larger the local coupling index $k_i$ is, the more coupled the system is.

**Parameters of the hybrid manipulator**

Main geometrical parameters and mass parameters are listed in Tables 1 and 2, respectively. It is noted that inertia of all parts listed in Table 2 is measured about their mass centers in their body-fixed frames.

**Coupling analysis at a typical configuration**

Taking a typical configuration as an example, where $x = y = 0$ m, $z = 1.35$ m, $\alpha = 90^\circ$, $\beta = 0^\circ$, the calculated mass matrix is shown in equation (56). Herein, $\alpha$ and $\beta$ are Euler angles of the machine tool frame $C-x_Cy_Cz_C$ with respect to the global frame $B_4-x_yz$.

$$
\tilde{M} = \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix} = \begin{bmatrix}
559.1358 & -267.7605 & -267.7605 \\
-267.7605 & 470.5595 & -162.5702 \\
-267.7605 & -162.5702 & 470.5595
\end{bmatrix}
$$

(56)

As can be seen obviously, the mass matrix is not diagonal. Based on equation (55), three local coupling indices are calculated, that is, $k_1 = 0.9578$, $k_2 = k_3 = 0.9145$. It can be found that $k_1$ is larger than $k_2$ and $k_3$, demonstrating that the 1st UPS limb has a larger coupling strength with the other two UPS limbs. And $k_2$ is equal to $k_3$, which is caused by that the 2nd UPS limb and the 3rd UPS limb are distributed symmetrically at this configuration.
Distributions of local coupling indices over a working plane

Distributions of three local coupling indices over the working plane of $z = 1.35 \text{ m}$ are demonstrated in Figure 5. Herein, both of $x$ and $y$ vary from $-0.2 \text{ m}$ to $0.2 \text{ m}$.

As can be seen clearly from Figure 5, the three local coupling indices are strongly position-dependent. To be specific, $k_1$ varies from $0.9403$ to $0.9736$; $k_2$ and $k_3$ change from $0.9104$ to $0.9304$. Further observation can be found that distributions of $k_1$ are symmetric about the plane of $x = 0$ while distributions of $k_2$ are symmetric to that of $k_3$ about the plane of $x = 0$. To make it clear, distributions of $k_2$ and $k_3$ over the $x$-$y$ plane are shown in Figure 6. The reason for this phenomenon is that the system is distributed symmetrically. To be specific, the 2nd UPS limb and the 3rd UPS limb are distributed symmetrically about the plane of $x = 0$. It can be found from Figure 4 that $k_1$ increases monotonously with the increment of $y$ and the three limbs demonstrate the most coupling strength when the system works at the configuration where $x = \pm 0.2 \text{ m}$, $y = 0.2 \text{ m}$.

Table 2. Mass parameters of the Trimule.

| Parameter                                      | Value          |
|------------------------------------------------|----------------|
| Mass of the oscillating limb $m_o$ (kg)        | 34.7504        |
| Inertia of the oscillating limb in the $x$ axis $I_{ox0}$ (kg·m$^2$) | 4.8210         |
| Inertia of the oscillating limb in the $y$ axis $I_{oy0}$ (kg·m$^2$) | 4.8186         |
| Inertia of the oscillating limb in the $z$ axis $I_{oz0}$ (kg·m$^2$) | 0.0496         |
| Mass of the stretchable limb $m_s$ (kg)        | 13.1696        |
| Inertia of the stretchable limb in the $x$ axis $I_{sx0}$ (kg·m$^2$) | 1.2023         |
| Inertia of the stretchable limb in the $y$ axis $I_{sy0}$ (kg·m$^2$) | 1.2023         |
| Inertia of the stretchable limb in the $z$ axis $I_{sz0}$ (kg·m$^2$) | 0.0081         |
| Mass of the thick limb $m_{tk}$ (kg)           | 18.2863        |
| Inertia of the thick limb in the $x$ axis $I_{tx0}$ (kg·m$^2$) | 0.3203         |
| Inertia of the thick limb in the $y$ axis $I_{ty0}$ (kg·m$^2$) | 0.3203         |
| Inertia of the thick limb in the $z$ axis $I_{tz0}$ (kg·m$^2$) | 0.1604         |
| Mass of the thin limb $m_{tn}$ (kg)            | 33.4241        |
| Inertia of the thin limb in the $x$ axis $I_{tx0}$ (kg·m$^2$) | 2.3216         |
| Inertia of the thin limb in the $y$ axis $I_{ty0}$ (kg·m$^2$) | 2.3216         |
| Inertia of the thin limb in the $z$ axis $I_{tz0}$ (kg·m$^2$) | 2.3380         |
| Mass of the platform $m_p$ (kg)                | 27.0335        |
| Inertia of the platform in the $u$ axis $I_{pu0}$ (kg·m$^2$) | 0.2725         |
| Inertia of the platform in the $v$ axis $I_{pv0}$ (kg·m$^2$) | 0.2345         |
| Inertia of the platform in the $w$ axis $I_{pw0}$ (kg·m$^2$) | 0.3244         |
| Mass of the A axis $m_A$ (kg)                  | 28.9460        |
| Inertia of the A axis in the $x$ axis $I_{Ax0}$ (kg·m$^2$) | 1.4970         |
| Inertia of the A axis in the $y$ axis $I_{Ay0}$ (kg·m$^2$) | 1.6407         |
| Inertia of the A axis in the $z$ axis $I_{Az0}$ (kg·m$^2$) | 0.3441         |
| Mass of the C axis $m_C$ (kg)                  | 40.6882        |
| Inertia of the C axis in the $x$ axis $I_{Cx0}$ (kg·m$^2$) | 1.0051         |
| Inertia of the C axis in the $y$ axis $I_{Cy0}$ (kg·m$^2$) | 0.5346         |
| Inertia of the C axis in the $z$ axis $I_{Cz0}$ (kg·m$^2$) | 0.6243         |

Distributions of local coupling indices over a working plane

Distributions of three local coupling indices over the working plane of $z = 1.35 \text{ m}$ are demonstrated in Figure 5. Herein, both of $x$ and $y$ vary from $-0.2 \text{ m}$ to $0.2 \text{ m}$.

As can be seen clearly from Figure 5, the three local coupling indices are strongly position-dependent. To be specific, $k_1$ varies from $0.9403$ to $0.9736$; $k_2$ and $k_3$ change from $0.9104$ to $0.9304$. Further observation can be found that distributions of $k_1$ are symmetric about the plane of $x = 0$ while distributions of $k_2$ are symmetric to that of $k_3$ about the plane of $x = 0$. To make it clear, distributions of $k_2$ and $k_3$ over the $x$-$y$ plane are shown in Figure 6. The reason for this phenomenon is that the system is distributed symmetrically. To be specific, the 2nd UPS limb and the 3rd UPS limb are distributed symmetrically about the plane of $x = 0$. It can be found from Figure 4 that $k_1$ increases monotonously with the increment of $y$ and the three limbs demonstrate the most coupling strength when the system works at the configuration where $x = \pm 0.2 \text{ m}$, $y = 0.2 \text{ m}$. 
Parametric analysis

As can be seen from Figure 5, three local coupling indices are strongly position-dependent. For the convenience of analysis, a global coupling index which demonstrates the average maximum local coupling index throughout the whole workspace is defined as

\[ \kappa = \frac{\int \kappa_{\text{max}, \rho} dV}{V} \]  

(57)

where \( \kappa_{\text{max}, \rho} \) is the largest value among \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) at a typical configuration \( \rho \); \( V \) denotes the volume of the entire workspace. Workspace of the manipulator is the combination of a cylinder portion and a spherical portion as depicted in Dong et al.\textsuperscript{10} For the convenience of analysis, the followings only analyze coupling strength in a prescribed task workspace where \( z = 1.15 \) to \( 1.35 \) m, \( x = -0.2 \) to \( 0.2 \) m, \( y = -0.2 \) to \( 0.2 \) m.

Based on the global coupling index, the followings are analyzed.
Dimensional parameters effects. Variations of $\kappa$ with respect to the size of the platform and the base are shown in Figure 7. Herein, $a$ denotes the size of the platform, which varies from 0.08 m to 0.16 m. And $b$ denotes the size of the base, varying from 0.28 m to 0.36 m.

As shown in Figure 7, the global coupling index $\kappa$ increases monotonously with the increment of $a$ while decreases with that of $b$. Further observation shows that the size of the base has a larger effect on the coupling strength than that of the platform. Therefore, increasing the size of the base is suggested preferentially to have a low coupled system so that the real control performance can be improved.

Structural parameters effects. Variations of $\kappa$ with respect to the length of active UPS and passive UP limbs are depicted in Figure 8. Herein, $l_{a1}$ and $l_{a2}$ are length of the stretchable limb and the oscillating limb while $l_{p1}$ and $l_{p2}$ are that of the thick limb and the thin limb as shown in Figure 2, respectively.

It can be seen from Figure 8 that the global coupling index $\kappa$ increases monotonously with the increment of the oscillating limb’s length $l_{a2}$ while decreases with that of the stretchable limb’s length $l_{a1}$. And it seems that the length of the oscillating limb $l_{a2}$ has a larger effect on $\kappa$ than that of the stretchable limb $l_{a1}$. Further observation demonstrates that the global coupling index $\kappa$ decreases monotonously with the increment of the thick limb’s length $l_{p1}$. And the global coupling index $\kappa$ seems to keep unchanged with the increment of the length of the thin limb $l_{p2}$. By comparing, it can be found that length of the active limbs has a larger effect on $\kappa$ than that of the passive limb. Therefore, it is suggested to decrease length of the oscillating limb preferentially to decrease the coupling strength of the system.

Figure 9 shows variations of $\kappa$ with respect to the depth of active UPS and passive UP limbs. Herein, $d_{a1}$ and $d_{a2}$ are depth of the stretchable limb and oscillating limb while $d_{p1}$ and $d_{p2}$ are that of the thick limb and thin limb, respectively. Depth of the active limbs and passive limb varies from 2 mm to 14 mm. For the
convenience of analysis, external diameters of the active limbs and passive limb keep unchanged.

It can be seen from Figure 9 that the global coupling index $k$ increases monotonously with the increment of the UPS limbs’ depth. By comparing, the global coupling index $k$ is more sensitive to the depth of the stretchable limb $\delta_{a1}$. And the effect of the oscillating limb’s depth $\delta_{a2}$ on $k$ is very small. On the contrary, the global coupling index $k$ decreases monotonously with the increment of the UP limb’s depth. And depth of the thick limb and thin limb seems to have the same effect on the system’s coupling strength.

In general, effects of the structural parameters on the system’s coupling strength are very weak, which can be found from Figures 8 and 9.
Mass parameters effects. Variations of $\kappa$ with respect to mass of the 2R module are depicted in Figure 10. Herein, $m_A$ and $m_C$ are mass of the A axis and C axis. As can be seen, the global coupling index $\kappa$ decreases monotonously with the increment of mass of the 2R module. Therefore, increasing mass of the 2R module properly is advised to reduce coupling strength of the system.

Combining Figures 7–10, it can be concluded that sizes of the platform and the base have an obvious effect on the global coupling strength. And effects of structural parameters of the active and passive limbs as well as mass of the 2R module are relatively low. Despite that, coupling may affect the real control performance or other indices in a large degree. Therefore, optimal design of the parallel/hybrid manipulator should be synthesized with kinematic, stiffness, and dynamic performance together.

Conclusion

Based on the rigid dynamic model, this article analyzes coupling characteristics of the Trimule robot. The main contributions of the article are drawn as follows.

1. A local coupling index is defined based on the off-diagonal mass matrix. Distributions of three local coupling indices are strongly position-dependent and are symmetric to the plane of $x = 0$ due to the mechanical features of the Trimule robot. And the three UPS limbs are coupled most when the robot works at the workspace boundaries.

2. A global coupling index is proposed which indicates the average maximum coupling index throughout the whole workspace. Parametric analysis demonstrates that dimensional parameters and structural parameters as well as mass parameters can affect the coupling strength to an extent. And the dimensional parameters of the PKM seem to have the largest influence.

Figure 10. Variations of $\kappa$ with respect to mass of the 2R module.
3. It is worth noting that optimal design of the Trimule should be conducted by considering kinematic performance, stiffness, dynamic characteristics, and coupling together, which will be published in the near future.

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