MARTINGALE INEQUALITIES IN NONCOMMUTATIVE SYMMETRIC SPACES

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Abstract. We provide generalizations of Burkholder’s inequalities involving conditioned square functions of martingales to the general context of martingales in noncommutative symmetric spaces. More precisely, we prove that Burkholder’s inequalities are valid for any martingale in noncommutative space constructed from a symmetric space defined on the interval \((0, \infty)\) with Fatou property and whose Boyd indices are strictly between 1 and 2. This answers positively a question raised by Jiao and may be viewed as a conditioned version of similar inequalities for square functions of noncommutative martingales. Using duality, we also recover the previously known case where the Boyd indices are finite and are strictly larger than 2.

1. Introduction

In classical martingale theory, a fundamental result due to Burkholder ([8, 9, 20]) can be described as follows: given a probability space \((\Omega, \mathcal{F}, P)\), let \(\{\mathcal{F}_n\}_{n \geq 1}\) be an increasing sequence of \(\sigma\)-fields of \(\mathcal{F}\) such that \(\mathcal{F} = \bigvee \mathcal{F}_n\). If \(2 \leq p < \infty\) and \(f = (f_n)_{n \geq 1}\) is a \(L_p\)-bounded martingale adapted to the filtration \(\{\mathcal{F}_n\}_{n \geq 1}\), then (using the convention that \(\mathcal{F}_0 = \mathcal{F}_1\)),

\[
\sup_{n \geq 1} [\mathbb{E}|f_n|^p]^{1/p} \simeq_p \left[ \mathbb{E}\left( \sum_{n \geq 1} \mathbb{E}[|df_n|^2|\mathcal{F}_{n-1}] \right)^{p/2} \right]^{1/p} + \left[ \sum_{n \geq 1} \mathbb{E}[|df_n|^p] \right]^{1/p},
\]

where \(\simeq_p\) means equivalence of norms up to constants depending only on \(p\). The random variable \(s(f) = \left( \sum_{n \geq 1} \mathbb{E}[|df_n|^2|\mathcal{F}_{n-1}] \right)^{1/2}\) is called the conditioned square function of the martingale \(f\) and the equivalence (1.1) is generally referred to as Burkholder’s inequalities. The equivalence (1.1) was established by Burkholder as the martingale difference sequence generalizations of Rosenthal’s inequalities which state that if \(2 \leq p < \infty\) and \((g_n)_{n \geq 1}\) is a sequence of independent mean-zero random variables in \(L_p(\Omega, \mathcal{F}, P)\) then

\[
\left( \mathbb{E}\left( \sum_{n \geq 1} |g_n|^p \right)^{1/p} \right)^{1/p} \simeq_p \left( \sum_{n \geq 1} \mathbb{E}|g_n|^2 \right)^{1/2} + \left( \sum_{n \geq 1} \mathbb{E}|g_n|^p \right)^{1/p}.
\]

Probabilistic inequalities involving independent random variables and martingales inequalities play important roles in many different areas of mathematics. Burkholder/Rosenthal inequalities in particular have many applications in probability theory and structures of symmetric spaces in Banach space theory. On the other hand, a recent trend in the general study of martingale inequalities is to find analogues of classical inequalities in the context of noncommutative \(L_p\)-spaces. We refer to [39, 26, 28, 41] for additional information on noncommutative martingales inequalities. Noncommutative analogues of (1.1) and (1.2) were extensively studied by Junge.

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and Xu in [29, 30]. They obtained that if $2 \leq p < \infty$ and $x = (x_n)_{n \geq 1}$ is a noncommutative martingale that is $L_p$-bounded then

$$\|x\|_p \simeq_p \max \left\{ \|s_c(x)\|_p, \|s_r(x)\|_p, \left( \sum_{n \geq 1} \|dx_n\|_p^p \right)^{1/p} \right\}$$

(1.3)

where $s_c(x)$ and $s_r(x)$ denote the column version and the row version of conditioned square functions which we refer to the next section for formal definitions. Moreover, they also treated the corresponding inequalities for the range $1 < p < 2$ which are dual versions of (1.3) and read as follows: if $x = (x_n)_{n \geq 1}$ is a noncommutative martingale in $L_2(M)$ then

$$\|x\|_p \simeq_p \inf \left\{ \|s_c(y)\|_p + \|s_r(z)\|_p + \left( \sum_{n \geq 1} \|dw_n\|_p^p \right)^{1/p} \right\}$$

(1.4)

where the infimum is taken over all $x = y + z + w$ with $y$, $z$, and $w$ are martingales. The differences between the two cases $1 < p < 2$ and $2 \leq p < \infty$ are now well-understood in the field. In [21], inequalities (1.3) and (1.4) were extended to the case of noncommutative Lorentz spaces $L_{p,q}(M)$ for $1 < p < \infty$ and $1 \leq q < \infty$. Motivated by this extension, it is natural to ask if some versions of noncommutative Burkholder’s inequalities remain valid in the general context of noncommutative symmetric spaces. This question was explicitly raised in [22, Problem 3.5].

Martingale inequalities for square functions which we refer to the next section for formal definitions. Moreover, they also treated extensions of Junge’s noncommutative Doob maximal inequalities in some symmetric spaces were recently established in [14, 22], and various interpolation techniques play significant roles in all the results stated above. The present paper solves the problem discussed above. Our main result can be summarized as follows: assume that $E$ is a rearrangement invariant function space on $(0, \infty)$ that satisfy some natural conditions and has nontrivial Boyd indices $1 < p_E \leq q_E < \infty$ and $M$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. We obtain generalizations of (1.3) and (1.4) that read as follow:

If $1 < p_E \leq q_E < 2$, then

$$\|x\|_{E(M)} \simeq_{E} \inf \left\{ \|s_c(y)\|_{E(M)} + \|s_r(z)\|_{E(M)} + \left( \sum_{n \geq 1} \|dx_n\|_{E(M)\overline{\otimes} \ell_\infty}\right)^{1/p} \right\}$$

(1.5)

where as in (1.4), the infimum is taken over all decompositions $x = y + z + w$ with $y$, $z$, and $w$ are martingales in $E(M, \tau)$.

If $2 < p_E \leq q_E < \infty$, then

$$\|x\|_{E(M)} \simeq_{E} \max \left\{ \|s_c(x)\|_{E(M)}, \|s_r(x)\|_{E(M)} + \left( \sum_{n \geq 1} \|dx_n\|_{E(M)\overline{\otimes} \ell_\infty}\right)^{1/p} \right\}$$

(1.6)

We note that (1.6) was recently established by Dirksen in [12]. His approach follows closely the original argument used in [29] taking advantage of the fact mentioned earlier that the noncommutative Burkholder-Gundy inequalities for square functions are valid for noncommutative martingales in some general symmetric spaces. Thus, our main motivation is primarily to establish the equivalence (1.5).
Our approach is based on another discovery made in the next section that, in some sense, one inequality in the equivalence in (1.3) can be achieved with a decomposition that works simultaneously for all $1 < p < 2$. We refer to Theorem 2.8 below for more information. This simultaneous decomposition allows us to efficiently apply results from interpolation theory. Namely, we use concrete realization of noncommutative symmetric spaces as interpolations of noncommutative $L_p$-spaces by means of $K$-functionals and $J$-functionals. The non-trivial inequality in the equivalence (1.6) will be deduced from (1.5) using duality. Unlike the $L_p$-cases, this duality technique does not seem to apply for the other direction. That is, at the time of this writing, we lack necessary ingredients to deduce (1.5) from (1.6).

The paper is organized as follows: in Section 2, we provide some preliminary results concerning noncommutative symmetric spaces, interpolation theory, and martingale inequalities. In particular, we establish a decomposition result from noncommutative martingales that sets up the use of interpolations. Section 3 is devoted entirely to the statement and proof of our main result. In the last section, we discuss some related results, provide examples, and point to related open questions concerning Burkholder’s inequalities.

Our notation and terminology are standard as may be found in the books [6, 34, 45].

2. Definitions and Preliminary results

2.1. Noncommutative spaces. In this subsection, we review some basic facts on rearrangement invariant spaces and their noncommutative counterparts that are relevant for our presentation.

For a semifinite von Neumann algebra $\mathcal{M}$ equipped with a faithful normal semifinite trace $\tau$, let $\tilde{\mathcal{M}}$ denote the topological $*$-algebra of all measurable operators with respect to $(\mathcal{M}, \tau)$ in the sense of [36]. For $x \in \tilde{\mathcal{M}}$, define its generalized singular number by

$$\mu_t(x) = \inf\{\lambda > 0; \tau(e^{[x]}(\lambda, \infty)) \leq t\}, \quad t > 0$$

where $e^{[x]}$ is the spectral measure of $|x|$. The function $t \mapsto \mu_t(x)$ from $(0, \infty)$ into $[0, \infty)$ is right-continuous and nonincreasing ([19]). For the case where $\mathcal{M}$ is the abelian von Neumann algebra $L_\infty(0, \infty)$ with the trace given by integration with respect to the Lebesgue measure, $\tilde{\mathcal{M}}$ becomes the linear space of all measurable functions $L_0(0, \infty)$ and $\mu(f)$ is the decreasing rearrangement of the function $|f|$ in the sense of [34].

We recall that a Banach function space $(E, \| \cdot \|_E)$ on $(0, \infty)$ is called symmetric if for any $g \in E$ and any measurable function $f$ with $\mu(f) \leq \mu(g)$, we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. The Köthe dual of a symmetric space $E$ is the function space defined by setting:

$$E^\times = \left\{ f \in L_0(0, \infty) : \int_0^\infty f(t)g(t)dt < \infty, \forall g \in E \right\}.$$

When equipped with the norm $\|f\|_{E^\times} := \sup\{\int_0^\infty f(t)g(t) \ dt : \|g\|_E \leq 1\}$, $E^\times$ is a symmetric Banach function space.

The symmetric Banach function space $E$ is said to have the Fatou property if, whenever $0 \leq f_\alpha \uparrow_\alpha \subseteq E$ is an upwards directed net with $\sup_\alpha \|f_\alpha\|_E < \infty$, it follows that $f = \sup_\alpha f_\alpha$ exists in $E$ and $\|f\|_E = \sup_\alpha \|f_\alpha\|_E$. It is well-known that $E$ has the Fatou property if and only if the natural embedding of $E$ into its Köthe bidual $E^{\times\times}$ is a surjective isometry. Examples of symmetric spaces with the Fatou property are separable symmetric spaces and duals of separable symmetric spaces.

Another concept that is central to the paper is the notion of Boyd indices which we now introduce. Let $E$ be a symmetric Banach space on $(0, \infty)$. For $s > 0$, the dilation operator
$D_s : E \to E$ is defined by setting

$$D_s f(t) = f(t/s), \quad t > 0, \quad f \in E.$$ 

The lower and upper Boyd indices of $E$ are defined by

$$p_E := \lim_{s \to \infty} \frac{\log s}{\log \| D_{1/s} \|} \quad \text{and} \quad q_E := \lim_{s \to 0^+} \frac{\log s}{\log \| D_{1/s} \|},$$

respectively. It is well-known that $1 \leq p_E \leq q_E \leq \infty$ and if $E = L^p$ for $1 \leq p \leq \infty$ then $p_E = q_E = p$. We shall say that $E$ has non-trivial Boyd indices whenever $1 < p_E \leq q_E < \infty$. We refer to [6, 34] for unexplained terminology from function space theory.

For a given symmetric Banach function space $(E, \| \cdot \|_E)$ on the interval $(0, \infty)$, we define the corresponding noncommutative space by setting:

$$E(\mathcal{M}, \tau) = \{ x \in \hat{\mathcal{M}} : \mu(x) \in E \}.$$ 

Equipped with the norm $\| x \|_{E(\mathcal{M}, \tau)} := \| \mu(x) \|_E$, the space $E(\mathcal{M}, \tau)$ is a complex Banach space ([32]) and is referred to as the noncommutative symmetric space corresponding to the function space $(E, \| \cdot \|_E)$. We remark that if $1 \leq p < \infty$ and $E = L^p(\mathbb{R})$ then $E(\mathcal{M}, \tau) = L^p(\mathcal{M}, \tau)$ is the usual noncommutative $L^p$-space associated with $(\mathcal{M}, \tau)$.

Recall that a linear operator $T : X \to Y$ is called a semi-embedding if $T$ is one to one and $T(B_X)$ is a closed subset of $Y$ where $B_X = \{ x \in X : \| x \| \leq 1 \}$. As in the commutative case, if $1 \leq p < p_E \leq q_E < q \leq \infty$ then the space $E(\mathcal{M}, \tau)$ is intermediate to the spaces $L^p(\mathcal{M}, \tau)$ and $L^q(\mathcal{M}, \tau)$ in the sense that

$$L^p(\mathcal{M}, \tau) \cap L^q(\mathcal{M}, \tau) \subseteq E(\mathcal{M}, \tau) \subseteq L^p(\mathcal{M}, \tau) + L^q(\mathcal{M}, \tau)$$

with the inclusion maps being continuous. Moreover, if $E$ satisfies the Fatou property, one can readily verify that the second inclusion map $E(\mathcal{M}, \tau) \hookrightarrow L^p(\mathcal{M}, \tau) + L^q(\mathcal{M}, \tau)$ is a semi-embedding. These facts will be used in the sequel.

We end this subsection with the following elementary lemma. It will be used in the proof of our main result. We include a proof for completeness.

**Lemma 2.1.** Assume that $1 < p < q < 2$ and let $u \in L_p(\mathcal{M}) \cap L_q(\mathcal{M})$. There exists a sequence $(u_m)_{m \geq 1}$ in $L_1(\mathcal{M}) \cap L_2(\mathcal{M})$ with $\lim_{m \to \infty} \| u_m - u \|_{L_p(\mathcal{M}) \cap L_q(\mathcal{M})} = 0$ and both sequences $(\| u_m \|_p)_{m \geq 1}$ and $(\| u_m \|_q)_{m \geq 1}$ are increasing and converge to $\| u \|_p$ and $\| u \|_q$, respectively.

**Proof.** We may assume without loss of generality that $\mathcal{M}$ is semifinite and $\sigma$-finite. Let $\varphi$ be a normal faithful state in $\mathcal{M}_e$ ([15, Proposition II-3.19]). Below, $1$ denotes the identity of $\mathcal{M}$. Inductively, we construct two sequences of projections $(e_k)_k$ and $(f_k)_k$ with the following properties:

1. $\{ f_k ; k \geq 1 \}$ is a family of mutually disjoint projections with $\tau(f_k) < \infty$ for all $k \geq 1$;
2. $f_1 \leq e_1$ and for every $k \geq 2$, $f_k \leq e_k \leq 1 - \sum_{i=1}^{k-1} f_i$;
3. $\tau(1 - \sum_{i=1}^{k-1} f_i - e_k) < 2^{-k}$;
4. $\varphi(e_k - f_k) < 2^{-k}$;
5. $uf_k$ and $u^*f_k$ belong to $\mathcal{M}$.

The inductive proof uses the basic definition of $\tau$-measurability of operators. Since $u$ and $u^*$ are $\tau$-measurable operators, there exists a projection $e_1$ with $\tau(1 - e_1) < 2^{-1}$ and such that $ue_1$ and $u^*e_1$ belong to $\mathcal{M}$. As $e_1$ is semifinite, we may choose a projection $f_1 \leq e_1$ with $\tau(f_1) < \infty$ and $\varphi(e_1 - f_1) < 2^{-1}$. Assume that the construction is done for $\{ e_1, \ldots, e_k \}$ and $\{ f_1, \ldots, f_k \}$ satisfying the properties listed in the items above. We consider $(1 - \sum_{i=1}^{k} f_i)u(1 - \sum_{i=1}^{k} f_i)$ and $(1 - \sum_{i=1}^{k} f_i)u^*(1 - \sum_{i=1}^{k} f_i)$ as $\tau$-measurable operators affiliated with the von Neumann algebra
We may choose \( e_{k+1} \leq 1 - \sum_{i=1}^{k} f_i \) with \( \tau((1 - \sum_{i=1}^{k} f_i) - e_{k+1}) < 2^{-(k+1)} \) and such that both operators \((1 - \sum_{i=1}^{k} f_i)u_{e_{k+1}} \) and \((1 - \sum_{i=1}^{k} f_i)u_{e_{k+1}}^* \) already belong to \( \mathcal{M} \) by assumption, we have that both \( u_{e_{k+1}} \) and \( u_{e_{k+1}}^* \) belong to \( \mathcal{M} \). Next, by semifiniteness, we choose \( f_{k+1} \leq e_{k+1} \) with \( \tau(f_{k+1}) < \infty \) and \( \varphi(e_{k+1} - f_{k+1}) < 2^{-(k+1)}. \) All required items are satisfied by \( e_{k+1} \) and \( f_{k+1}. \) The induction process is complete.

Next, it readily follows from item (iii) and item (iv) that both sequences of projections \( (1 - \sum_{i=1}^{k-1} f_i) \) and \( (e_{k+1} - f_{k+1}) \) converge to zero for the strong operator topology (see for instance [45 Proposition 5.3, p. 148]). For \( m \geq 1, \) set \( v_m = \sum_{i=1}^{m} f_i. \) Then \( (v_m)_m \) is an increasing sequence of projections that converges to 1 for the strong operator topology and such that for every \( m \geq 1, \) \( \tau(v_m) < \infty \) and \( uv_m \in \mathcal{M}. \) For \( m \geq 1, \) let \( u_m := uv_m. \) We claim that \( (u_m)_m \) provides the desired sequence.

Indeed, since \( u_m \in L_q(\mathcal{M}) \cap \mathcal{M} \) and \( 1 < q < 2, \) we have \( u_m \in L_2(\mathcal{M}). \) Moreover, by Hölder’s inequality,

\[
\|u_m\|_1 \leq \tau(v_m)^{1/2}\|u_m\|_2 < \infty.
\]

This shows that \( u_m \in L_1(\mathcal{M}) \cap L_2(\mathcal{M}). \) Furthermore, from the identities

\[
\max \left\{ \|u - u_m\|_p, \|u - v_m\|_q \right\} = \max \left\{ \|u(1 - v_m)\|_p, \|u(1 - v_m)\|_q \right\} = \max \left\{ \|1 - v_m\|_p^2 \|1 - v_m\|_q^2 \right\},
\]

we get that \( \lim_{n \to \infty} \|u - u_m\|_{L_p(\mathcal{M}) \cap L_q(\mathcal{M})} = 0. \) On the other hand, if \( s \) is equal to either \( p \) or \( q, \) it follows from the identity \( \|u_m\|_s = \|u|v_m|u\|_s^{1/2} \) that \( (\|u_m\|_s)_m \) forms an increasing sequence that converges to \( \|u\|_s. \)

We refer to [10] for extensive discussions on various properties of noncommutative symmetric spaces.

2.2. Function spaces and interpolations. In this subsection, we will discuss concrete description of certain classes of noncommutative symmetric spaces as interpolations of noncommutative \( L_p \)-spaces that are relevant for our method of proof in the next section. We begin by recalling that for a given compatible Banach couple \((X_0, X_1), \) a Banach space \( Z \) is called an interpolation space if \( X_0 \cap X_1 \subset Z \subset X_0 + X_1 \) and whenever a bounded linear operator \( T : X_0 + X_1 \to X_0 + X_1 \) is such that \( T(X_0) \subset X_0 \) and \( T(X_1) \subset X_1 \) we have \( T(Z) \subset Z \) and \( \|T\| \leq C \max\{\|T : X_0 \to X_0\|, \|T : X_1 \to X_1\|\} \) for some constant \( C. \) In this case, we write \( Z \in \text{Int}(X_0, X_1). \) We refer to [3, 7, 31] for more on interpolations.

In this paper we rely heavily on the notions of \( K \)-functionals and \( J \)-functionals which we now review:

For a compatible Banach couple \((X_0, X_1), \) we define the \( J \)-functional by setting for any \( x \in X_0 \cap X_1 \) and \( t > 0, \)

\[
J(x, t; X_0, X_1) = \max \left\{ \|x\|_{X_0}, t\|x\|_{X_1} \right\}.
\]

As a dual notion, the \( K \)-functional is defined by setting for any \( x \in E_0 + E_1 \) and \( t > 0, \)

\[
K(x, t; X_0, X_1) = \inf \left\{ \|x_1\|_{X_0} + t\|x_2\|_{X_1}; x = x_1 + x_2 \right\}.
\]

If the compatible couple \((X_0, X_1), \) is clear from the context, then we will simply write \( J(x, t) \) and \( K(x, t) \) in place of \( J(x, t; X_0, X_1) \) and \( K(x, t; X_0, X_1), \) respectively. It is now quite well-known that any symmetric Banach function space with the Fatou property that belongs to \( \text{Int}(L_p, L_q) \) is given by a \( K \)-method. More precisely, we have the following result due to Brudnyi and Krugliak (see for instance [31 Theorem 6.3]).
Theorem 2.2. Let $E$ be a symmetric Banach function space on $(0, \infty)$ with the Fatou property. If $E \in \text{Int}(L_p(0, \infty), L_q(0, \infty))$ for $1 \leq p < q \leq \infty$, then there exists a function space $F$ on $(0, \infty)$ such that $f \in E$ if and only if $K(f, \cdot, L_p, L_q) \in F$ and there exists a constant $C$ such that

$$C^{-1} \|K(f, \cdot)\|_F \leq \|f\|_E \leq C \|K(f, \cdot)\|_F.$$

We will use the corresponding $J$-method of the above theorem. This was studied in [4, 5]. We review the basic construction of this method and introduce a discrete version that is quite essential in the next section.

Suppose that an element $x \in X_0 + X_1$ admits a representation

$$x = \int_0^\infty u(t) \frac{dt}{t},$$

where $u(\cdot)$ is measurable function that takes its values in $X_0 \cap X_1$ and the integral is convergent in $X_0 + X_1$. For any given representation $u(\cdot)$, we set for $s > 0$,

$$j(u, s) = \int_s^\infty \frac{1}{t} J(u(t), t) \frac{dt}{t}.$$

Given a symmetric Banach function space $F$ defined on $(0, \infty)$, the interpolation space $(X_0, X_1)_{F,j}$ consists of elements $x \in X_0 + X_1$ which admits a representation as in (2.1) and are such that

$$\|x\|_{F,j} = \inf \left\{ \|j(u, \cdot)\|_F \right\} < \infty,$$

where the infimum is taken over all representation $u$ of $x$ as in (2.1). We refer to [4, 5] for a comprehensive study of this interpolation method along with some other equivalent methods. As noted above, we will need a discrete version of this method. This is standard but we could not find any reference in the literature for this particular method so we provide the details.

We define the interpolation space $(X_0, X_1)_{F,j}$ to be the space of elements $x \in X_0 + X_1$ which admits a representation

$$x = \sum_{\nu \in \mathbb{Z}} u_{\nu} \quad \text{(convergence in } X_0 + X_1)$$

with $u_{\nu} \in X_0 \cap X_1$ and are such that

$$\|x\|_{F,j} = \inf \left\{ \|j(\{u_{\nu}\}, \cdot)\|_F \right\} < \infty,$$

where the decreasing function $j(\{u_{\nu}\}, \cdot)$ is defined by

$$j(\{u_{\nu}\}, t) = \sum_{\gamma \geq \nu + 1} 2^{-\gamma} J(u_{\gamma}, 2^\gamma) \quad \text{for } t \in [2^\nu, 2^{\nu+1}),$$

and the infimum is taken over all representations of $x$ as in (2.4). Clearly, the function $j(\{u_{\nu}\}, t)$ takes only countably many values. Thus, we may call $(X_0, X_1)_{F,j}$ as a discrete interpolation method.

Our purpose is to show that as in the case of real interpolation methods, this discrete version is equivalent to the continuous version described earlier. More precisely, we have:

Lemma 2.3. Let $x \in X_0 + X_1$. Then $x \in (X_0, X_1)_{F,j}$ if and only if $x \in (X_0, X_1)_{F,j}$. More precisely, the following inequalities hold:

$$\frac{1}{4} \|x\|_{F,j} \leq \|x\|_{F,j} \leq 4 \|x\|_{F,j}.$$
Proof. Assume that $x \in (X_0, X_1)_{F,j}$ and fix a representation $x = \int_0^\infty u(t) \, dt/t$ such that

(2.5) \quad \|j(u, \cdot)\|_F \leq 2\|x\|_{F,j}.

For every $\nu \in \mathbb{Z}$, let $u_\nu = \int_{2^\nu}^{2^{\nu+1}} u(t) \, dt/t$. Then, we have $x = \sum_{\nu \in \mathbb{Z}} u_\nu$ with convergence in $X_0 + X_1$. We claim that for every $s > 0$,

(2.6) \quad j(\{u_\nu\}_\nu, s) \leq 2j(u, s).

To verify this claim, assume that $2^\nu \leq s < 2^{\nu+1}$ for some $\nu \in \mathbb{Z}$. Then,

$$j(\{u_\nu\}_\nu, s) = \sum_{\gamma \geq \nu+1} 2^{-\gamma} J(u_\gamma, 2^\gamma)$$

$$\leq \sum_{\gamma \geq \nu+1} 2^{-\gamma} \int_{2^\gamma}^{2^{\gamma+1}} J(u(t), 2^\gamma) \, dt/t$$

$$\leq 2 \sum_{\gamma \geq \nu+1} \int_{2^\gamma}^{2^{\gamma+1}} t^{-1} J(u(t), 2^\gamma) \, dt/t$$

$$\leq 2 \int_{s}^{\infty} t^{-1} J(u(t), t) \, dt/t = 2j(u, s)$$

which proves (2.6). Taking the norms on the symmetric space $F$, we deduce that

$$\|x\|_{F,j} \leq \|j(\{u_\nu\}_\nu, \cdot)\|_F \leq 2\|j(u, \cdot)\|_F \leq 4\|x\|_{F,j}$$

where the last inequality is from (2.5).

Conversely, let $x \in (X_0, X_1)_{F,j}$. We may choose a representation $x = \sum_{\nu \in \mathbb{Z}} u_\nu$ satisfying

(2.7) \quad \|j(\{u_\nu\}_\nu, \cdot)\|_F \leq 2\|x\|_{F,j}.

We define $u(t) = u_\nu / \log 2$ for $2^\nu \leq t < 2^{\nu+1}$. Clearly, $u(\cdot)$ takes its values in $X_0 \cap X_1$ and is measurable. Moreover, we have

$$x = \sum_{\nu \in \mathbb{Z}} u_\nu = \sum_{\nu \in \mathbb{Z}} \int_{2^\nu}^{2^{\nu+1}} u_\nu / \log 2 \, dt/t = \int_0^\infty u(t) \, dt/t$$

with the convergence of the integral taken in $X_0 + X_1$. That is, the integral $\int_0^\infty u(t) \, dt/t$ is a representation of $x$. Now we estimate the function $j(u, \cdot)$.

Assume that $2^\nu \leq s < 2^{\nu+1}$ for some $\nu \in \mathbb{Z}$. Then we have the following estimates:

$$j(u, s) = \int_{s}^{\infty} t^{-1} J(u(t), t) \, dt/t$$

$$\leq \sum_{\gamma \geq \nu} \int_{2^\gamma}^{2^{\gamma+1}} t^{-1} J(u(t), t) \, dt/t$$

$$\leq (\log 2)^{-1} \sum_{\gamma \geq \nu} \int_{2^\gamma}^{2^{\gamma+1}} t^{-1} J(u_\gamma, t) \, dt/t.$$
Using the well-known inequality that $J(x, s) \leq \max(1, s/t)J(x, t)$ for every $x \in X_0 \cap X_1$ and $s, t > 0$, we further get

$$j(u, s) \leq (\log 2)^{-1} \sum_{\gamma \geq \nu} \int_{2^\gamma}^{2^{\gamma+1}} 2^{-\gamma} J(u_\gamma, 2^{\gamma}) \, dt/t$$

$$\leq \sum_{\gamma \geq \nu} 2^{-\gamma} J(u_\gamma, 2^{\gamma}) = j(\{u_\nu\}_\nu, s/2).$$

We observe that since $j(\{u_\nu\}_\nu, \cdot)$ is a decreasing function on $(0, \infty)$ and $F$ is a symmetric space, we have $\|j(\{u_\nu\}_\nu, \cdot)\|_F \leq 2\|j(\{u_\nu\}_\nu, \cdot)\|_F$. We may conclude that

$$\|x\|_{F,j} \leq \|j(u, \cdot)\|_F \leq 2\|j(\{u_\nu\}_\nu, \cdot)\|_F \leq 4\|x\|_{F,j}$$

where the last inequality follows from (2.7). The proof of the lemma is complete. \hfill \Box

Combining [4] Theorem 9.3, [5] Theorem 3.5, and Lemma 2.3 we may state the following result which is one of the decisive tools we use in our proof.

**Theorem 2.4.** If $E$ is a symmetric Banach function space on $(0, \infty)$ with the Fatou property then the following are equivalent:

(i) $1 < p < p_E \leq q_E < q < \infty$.

(ii) There exists a symmetric Banach function space $F$ on $(0, \infty)$ with nontrivial Boyd indices such that:

$$E = (L_p(0, \infty), L_q(0, \infty))_{\hat{F}, j} \quad \text{(with equivalent norms)}.$$

As is now well-known, the preceding interpolation result automatically lifts to the noncommutative setting (see [40] Corollary 2.2):

**Corollary 2.5.** Let $E$ be a symmetric Banach function space on $(0, \infty)$ with the Fatou property. Then the following are equivalent:

(i) $1 < p < p_E \leq q_E < q < \infty$.

(ii) There exists a symmetric Banach function space $F$ on $(0, \infty)$ with nontrivial Boyd indices such that for every semifinite von Neumann algebra $(N, \sigma)$,

$$E(N, \sigma) = (L_p(N, \sigma), L_q(N, \sigma))_{\hat{F}, j},$$

with equivalent norms depending only on $E$, $p$, and $q$.

We record a general fact about interpolations of linear operators between two noncommutative spaces for further use.

**Proposition 2.6 ([17]).** Assume that $E \in \text{Int}(L_p, L_q)$ and $M$ and $N$ are semifinite von Neumann algebras. Let $T : L_p(M) + L_q(M) \to L_p(N) + L_q(N)$ be a linear operator such that $T : L_p(M) \to L_p(N)$ and $T : L_q(M) \to L_q(N)$ are bounded. Then $T$ maps $E(M)$ into $E(N)$ and the resulting operator $T : E(M) \to E(N)$ is bounded. Moreover, we have the following estimate:

$$\|T : E(M) \to E(N)\| \leq C \max \{\|T : L_p(M) \to L_p(N)\|, \|T : L_q(M) \to L_q(N)\|\}$$

for some absolute constant $C$. 

2.3. Noncommutative martingales. Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by \((\mathcal{M}_n)_{n \geq 1}\) an increasing sequence of von Neumann subalgebras of \(\mathcal{M}\) whose union is weak*-dense in \(\mathcal{M}\). For \(n \geq 1\), we assume that there exists a trace preserving conditional expectation \(\mathcal{E}_n\) from \(\mathcal{M}\) onto \(\mathcal{M}_n\). It is well-known that \(\mathcal{E}_n\) extends to a contractive projection from \(L_p(\mathcal{M}, \tau)\) onto \(L_p(\mathcal{M}_n, \tau_n)\) for all \(1 \leq p \leq \infty\). More generally, if \(E\) is a symmetric Banach function space on \((0, \infty)\) then \(\mathcal{E}_n\) is a contraction from \(E(\mathcal{M}, \tau)\) onto \(E(\mathcal{M}_n, \tau_n)\), where \(\tau_n\) denotes the restriction of \(\tau\) on \(\mathcal{M}_n\).

**Definition 2.7.** A sequence \(x = (x_n)_{n \geq 1}\) in \(L_1(\mathcal{M})\) is called a *noncommutative martingale* with respect to \((\mathcal{M}_n)_{n \geq 1}\) if \(\mathcal{E}_n(x_{n+1}) = x_n\) for every \(n \geq 1\).

If in addition, all \(x_n\)’s belong to \(E(\mathcal{M})\) then \(x\) is called an \(E(\mathcal{M})\)-martingale. In this case we set
\[
\|x\|_{E(\mathcal{M})} = \sup_{n \geq 1} \|x_n\|_{E(\mathcal{M})}.
\]

If \(\|x\|_{E(\mathcal{M})} < \infty\), then \(x\) is called a bounded \(E(\mathcal{M})\)-martingale.

Let \(x = (x_n)_{n \geq 1}\) be a noncommutative martingale with respect to \((\mathcal{M}_n)_{n \geq 1}\). Define \(dx_n = x_n - x_{n-1}\) for \(n \geq 1\) with the usual convention that \(x_0 = 0\). The sequence \(dx = (dx_n)_{n \geq 1}\) is called the *martingale difference sequence* of \(x\). A martingale \(x\) is called a *finite martingale* if there exists \(N\) such that \(dx_n = 0\) for all \(n \geq N\). In the sequel, for any operator \(x \in E(\mathcal{M})\), we denote \(x_n = \mathcal{E}_n(x)\) for \(n \geq 1\).

Let us now review the definitions of the square functions and Hardy spaces of noncommutative martingales. Following \([39]\), we consider the column and row versions of square functions relative to a (finite) martingale \(x = (x_n)_{n \geq 1}\) as follows:
\[
S_c(x) = \left(\sum_{k \geq 1} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_r(x) = \left(\sum_{k \geq 1} |dx_k^*|^2\right)^{1/2}.
\]

Define \(\mathcal{H}_E^c(\mathcal{M})\) (resp. \(\mathcal{H}_E^r(\mathcal{M})\)) as the completion of all finite \(L_1(\mathcal{M}) \cap \mathcal{M}\)-martingales under the norm \(\|x\|_{\mathcal{H}_E^c} = \|S_c(x)\|_{E(\mathcal{M})}\) (resp. \(\|x\|_{\mathcal{H}_E^r} = \|S_r(x)\|_{E(\mathcal{M})}\)). The mixture Hardy space of noncommutative martingales is defined as follows. For \(1 \leq p_E \leq q_E < 2\),
\[
\mathcal{H}_E(\mathcal{M}) = \mathcal{H}_E^c(\mathcal{M}) + \mathcal{H}_E^r(\mathcal{M})
\]
equipped with the norm
\[
\|x\|_{\mathcal{H}_E} = \inf \left\{ \|y\|_{\mathcal{H}_E^c} + \|z\|_{\mathcal{H}_E^r} \right\},
\]
where the infimum is taken over all \(y \in \mathcal{H}_E^c(\mathcal{M})\) and \(z \in \mathcal{H}_E^r(\mathcal{M})\) such that \(x = y + z\). For \(2 \leq p_E \leq q_E < \infty\),
\[
\mathcal{H}_E(\mathcal{M}) = \mathcal{H}_E^c(\mathcal{M}) \cap \mathcal{H}_E^r(\mathcal{M})
\]
equipped with the norm
\[
\|x\|_{\mathcal{H}_E} = \max \left\{ \|x\|_{\mathcal{H}_E^c}, \|x\|_{\mathcal{H}_E^r} \right\}.
\]
These definitions mirror the well-documented difference between the two cases \(1 \leq p < 2\) and \(2 \leq p < \infty\) for the special case where \(E = L_p(0, \infty)\).

We now consider the conditioned versions of the above definitions. Our approach is based on the conditioned spaces introduced by Junge in \([29]\). Since this is very crucial in the sequel, we review the basic setup. Below, we use the convention that \(\mathcal{E}_0 = \mathcal{E}_1\).

Let \(\mathcal{E} : \mathcal{M} \to \mathcal{N}\) be a normal faithful conditional expectation, where \(\mathcal{N}\) is a von Neumann subalgebra of \(\mathcal{M}\). For \(0 < p \leq \infty\), we define \(L_p^0(\mathcal{M}, \mathcal{E})\) to be the completion of \(\mathcal{M} \cap L_p(\mathcal{M})\) with respect to the quasi-norm
\[
\|x\|_{L_p(\mathcal{M}, \mathcal{E})} = \|\mathcal{E}(x^* x)\|^{1/2}_{p/2}.
\]
It was shown in [26] that for every \( n \) and \( 0 < p \leq \infty \) there exists an isometric right \( \mathcal{M}_n \)-module map \( u_{n,p} : L^p_n(\mathcal{M}, \mathcal{E}_n) \rightarrow L^p_n(\mathcal{M}; \ell_2^p) \) such that

\[
(2.8) \quad u_{n,p}(x)^* u_{n,q}(y) = \mathcal{E}_n(x^* y),
\]

for all \( x \in L^p_n(\mathcal{M}; \mathcal{E}_n) \) and \( y \in L^q_n(\mathcal{M}; \mathcal{E}_n) \). More generally, for \( 0 < p \leq \infty \), we consider the space \( L^\text{cond}_p(\mathcal{M}; \ell_2^p) \) as the completion of the finite sequence \( (a_n)_{n \geq 1} \) in \( L_1(\mathcal{M}) \cap \mathcal{M} \) with respect to the norm:

\[
(2.9) \quad \| (a_n) \|_{L^\text{cond}_p(\mathcal{M}; \ell_2^p)} = \| \left( \sum_{n \geq 1} \mathcal{E}_{n-1} |a_n|^2 \right)^{1/2} \|_p.
\]

The space \( L^\text{cond}_p(\mathcal{M}; \ell_2^p) \) can be isometrically embedded into an \( L^p \)-space associated to a semifinite von Neumann algebra by means of the following map:

\[
U_p : L^\text{cond}_p(\mathcal{M}; \ell_2^p) \rightarrow L^p_p(\mathcal{M} \overline{\otimes} B(\ell_2^2))
\]

defined by setting

\[
U_p ((a_n)_{n \geq 1}) = \sum_{n \geq 1} e_{n,1} \otimes u_{n-1,p}(a_n),
\]

where \( (e_{i,j})_{i,j \geq 1} \) is the family of unit matrices in \( B(\ell_2^2) \). From [28], it follows that if \( (a_n)_{n \geq 1} \in L^\text{cond}_p(\mathcal{M}; \ell_2^p) \) and \( (b_n)_{n \geq 1} \in L^\text{cond}_q(\mathcal{M}; \ell_2^p) \) for \( 1/p + 1/q \leq 1 \) then

\[
U_p ((a_n))^* U_q ((b_n)) = \sum_{n \geq 1} \mathcal{E}_{n-1} (a_n^* b_n).
\]

In particular, \( \| (a_n) \|_{L^\text{cond}_p(\mathcal{M}; \ell_2^p)} = \| U_p ((a_n)) \|_p \) and hence \( U_p \) is indeed an isometry. Below, we will simply write \( U \) for \( U_p \). We refer the reader to [26] and [28] for more details on the preceding construction.

Now, we generalize the notion of conditioned spaces to the setting of symmetric spaces. We consider the algebraic linear map \( U \) restricted to the linear space of finite sequences in \( L_1(\mathcal{M}) \cap \mathcal{M} \) taking its values in \( L_1(\mathcal{M} \overline{\otimes} B(\ell_2^2)) \cap \mathcal{M} \overline{\otimes} B(\ell_2^2) \). For a given finite sequence \( (a_n)_{n \geq 1} \) in \( L_1(\mathcal{M}) \cap \mathcal{M} \), we set:

\[
\| (a_n) \|_{E^\text{cond}(\mathcal{M}; \ell_2^p)} = \left\| \left( \sum_{n \geq 1} \mathcal{E}_{n-1} |a_n|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} = \| U((a_n)) \|_{E(\mathcal{M} \overline{\otimes} B(\ell_2^2))}.
\]

This is well-defined and induces a norm on the linear space of finite sequence in \( L_1(\mathcal{M}) \cap \mathcal{M} \). If we define \( E^\text{cond}(\mathcal{M}; \ell_2^p) \) to be the completion of the set of finite sequences of elements of \( L_1(\mathcal{M}) \cap \mathcal{M} \) with respect to the above norm, then \( U \) extends to an isometry from \( E^\text{cond}(\mathcal{M}, \ell_2^p) \) into \( E(\mathcal{M} \overline{\otimes} B(\ell_2^2)) \) which we will still denote by \( U \). Moreover, if \( E^\text{c} \) denotes the Köthe dual of \( E \) then for any \( (a_n)_{n \geq 1} \in E^\text{cond}(\mathcal{M}; \ell_2^p) \) and \( (b_n)_{n \geq 1} \in (E^\text{c})^\text{cond}(\mathcal{M}; \ell_2^p) \), then the following identity holds:

\[
(2.10) \quad U((a_n))^* U((b_n)) = \sum_{n \geq 1} \mathcal{E}_{n-1} (a_n^* b_n).
\]

Similarly, we may define the corresponding row version \( E^\text{cond}(\mathcal{M}; \ell_2^p) \) which can also be viewed as a subspace of \( E(\mathcal{M} \overline{\otimes} B(\ell_2^2)) \) as row vectors. This is done by simply considering adjoint operators.

Now, let \( x = (x_n)_{n \geq 1} \) be a finite martingale in \( L_2(\mathcal{M}) \). We set

\[
\begin{align*}
&s_c(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1} |dx_k|^2 \right)^{1/2} \quad \text{and} \quad s_r(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1} |dx_k^*|^2 \right)^{1/2}.
\end{align*}
\]
These are called the column and row conditioned square functions, respectively. Define \( h^c_E(\mathcal{M}) \) (resp. \( h^r_E(\mathcal{M}) \)) as the completion of all finite \( \mathcal{M} \cap L_1(\mathcal{M}) \)-martingales under the norm \( \| x \|_{h^c_E} = \| s_r(x) \|_{E(\mathcal{M})} \) (resp. \( \| x \|_{h^r_E} = \| s_c(x) \|_{E(\mathcal{M})} \)). We observe that \( \| x \|_{h^c_E} = \| (dx_n) \|_{E^{\text{cond}}(\mathcal{M}; E^2)} \). Therefore \( h^c_E(\mathcal{M}) \) may be viewed as the subspace of \( E^{\text{cond}}(\mathcal{M}; E^2) \) consisting of martingale difference sequences. A fortiori, there is an isometric embedding

\[
    h^c_E(\mathcal{M}) \subset E(\mathcal{M}) \otimes B(\ell_2(\mathbb{N}^2)).
\]

We can make similar assertion for \( h^r_E(\mathcal{M}) \). These isometric embedding will be heavily used in the sequel.

We also need the diagonal Hardy space \( h^d_E(\mathcal{M}) \) as the subspace of \( E(\mathcal{M} \otimes \ell_\infty) \) consisting of all martingale difference sequences. As above, we define the conditioned version of martingale Hardy spaces as follows. If \( 1 \leq p_E \leq q_E < 2 \), then

\[
    h^c_E(\mathcal{M}) = h^d_E(\mathcal{M}) + h^c_E(\mathcal{M}) + h^r_E(\mathcal{M})
\]

equipped with the norm

\[
    \| x \|_{h^c_E} = \inf \left\{ \| w \|_{h^d_E} + \| y \|_{h^c_E} + \| z \|_{h^r_E} \right\},
\]

where the infimum is taken over all \( w \in h^d_E(\mathcal{M}), y \in h^c_E(\mathcal{M}), \) and \( z \in h^r_E(\mathcal{M}) \) such that \( x = w + y + z \). If \( 2 \leq p_E \leq q_E < \infty \), then

\[
    h^c_E(\mathcal{M}) = h^d_E(\mathcal{M}) \cap h^c_E(\mathcal{M}) \cap h^r_E(\mathcal{M})
\]

equipped with the norm

\[
    \| x \|_{h^c_E} = \max \left\{ \| x \|_{h^d_E}, \| x \|_{h^c_E}, \| x \|_{h^r_E} \right\}.
\]

For the case where \( E = L_p(0, \infty) \), we will simply write \( \mathcal{H}_p(\mathcal{M}), h_p(\mathcal{M}) \), ect... in place of \( \mathcal{H}_{L_p}(\mathcal{M}), h_{L_p}(\mathcal{M}) \), ect... From the noncommutative Burkholder-Gundy inequalities and Burkholder inequalities in [20, 30], we have

\[
    \mathcal{H}_p(\mathcal{M}) = h_p(\mathcal{M}) = L_p(\mathcal{M})
\]

with equivalent norms for all \( 1 < p < \infty \). The latter equality constitutes the primary topic of this paper.

The next theorem is the main result of this section. Its main feature is that it gives a decomposition that provides norms estimates simultaneously for all \( p \in (1, 2) \). This fact is very crucial in our approach in the next section.

**Theorem 2.8.** There exists a family \( \{ \kappa_p : 1 < p < 2 \} \subset \mathbb{R}_+ \) satisfying the following: if \( x \in L_1(\mathcal{M}) \cap L_2(\mathcal{M}) \), then there exist \( a, b, c \in \cap_{1 < p < 2} L_p(\mathcal{M}) \) such that

1. \( x = a + b + c \),
2. for every \( 1 < p < 2 \), the following inequality holds:

\[
    \| a \|_{h^d_E} + \| b \|_{h^c_E} + \| c \|_{h^r_E} \leq \kappa_p \| x \|_p.
\]

**Proof.** **Case 1.** Assume that \( \mathcal{M} \) is finite and \( \tau \) is a normalized trace. The proof uses a weak-type decomposition from [43]. We consider the interpolation couple \( (L_1(\mathcal{M}), L_2(\mathcal{M})) \).

Let \( x \in L_2(\mathcal{M}) \). According to [7, Lemma 3.3.2], there is a representation \( x = \sum_{\nu \in \mathbb{Z}} u_\nu \) (convergent in \( L_1(\mathcal{M}) \)) of \( x \) satisfying, for every \( \nu \in \mathbb{Z} \),

\[
    J(u_\nu, 2^\nu) \leq 4K(x, 2^\nu).
\]
Since $\tau(1) = 1$, we may apply [33, Theorem 3.1]. There exists an absolute constant $\kappa > 0$ such that, for each $\nu \in \mathbb{Z}$, we can find three adapted sequences $\alpha^{(\nu)}$, $\beta^{(\nu)}$, and $\gamma^{(\nu)}$ in $L_2(M)$ such that:

\begin{align}
\text{(2.12)} & \quad d_n(u_\nu) = \alpha_n^{(\nu)} + \beta_n^{(\nu)} + \gamma_n^{(\nu)} \text{ for all } n \geq 1, \\
\text{(2.13)} & \quad J\left(\sum_{n \geq 1} \alpha_n^{(\nu)} \otimes e_n, t\right) \leq \kappa J(u_\nu, t), \quad t > 0, \\
\text{(2.14)} & \quad J\left(\sum_{n \geq 1} \mathcal{E}_{n-1}(\|\beta_n^{(\nu)}\|^2)^{1/2}, t\right) \leq \kappa J(u_\nu, t), \quad t > 0, \\
\text{(2.15)} & \quad J\left(\sum_{n \geq 1} \mathcal{E}_{n-1}(\|\gamma_n^{(\nu)}\|^2)^{1/2}, t\right) \leq \kappa J(u_\nu, t), \quad t > 0,
\end{align}

where the $J$-functional in the left hand side of the inequality in (2.13) are taken relative to the interpolation couple $(L_{1,\infty}(M), L_2(M))$ and those from the left hand sides of (2.14) and (2.15) are taken with respect to the interpolation couple $(L_{1,\infty}(M), L_2(M))$. We set

$$\alpha_n = \sum_{\nu \in \mathbb{Z}} \alpha_n^{(\nu)}, \quad \beta_n = \sum_{\nu \in \mathbb{Z}} \beta_n^{(\nu)}, \quad \text{and} \quad \gamma_n = \sum_{\nu \in \mathbb{Z}} \gamma_n^{(\nu)}.$$ 

Then we obtain three adapted sequence $\alpha = (\alpha_n)_n$, $\beta = (\beta_n)_n$, and $\gamma = (\gamma_n)_n$. Define

$$da_n = \alpha_n - \mathcal{E}_{n-1}(\alpha_n), \quad db_n = \beta_n - \mathcal{E}_{n-1}(\beta_n), \quad \text{and} \quad dc_n = \gamma_n - \mathcal{E}_{n-1}(\gamma_n).$$

We claim that the resulting operators $a$, $b$, and $c$ satisfy the conclusion of the theorem. Indeed, it is clear from the construction that $x = a + b + c$. For the second item, we will verify the statement separately for $a$, $b$, and $c$. We begin with the operator $a$. This will be deduced from the next lemma. For $0 < \theta < 1$ and $1 < p < 2$, $(\cdot, \cdot)_{\theta,p,K}$ denote the discrete real interpolation methods using the $J$-functionals and $K$-functionals, respectively. We refer to [7] for definitions.

**Lemma 2.9.** For every $0 < \theta < 1$ and every $1 < p < 2$,

$$\left\| (\alpha_n)_{n \geq 1} \right\|_{L_{1,\infty}(M), L_2(M)} \leq 4\kappa \|x\|_{\theta,p,K}.$$  

For $\nu \in \mathbb{Z}$, let $[\alpha]^{(\nu)} = \sum_n \alpha_n^{(\nu)} \otimes e_n$. The series $\sum_n [\alpha]^{(\nu)}$ is a representation of $\sum_n \alpha_n \otimes e_n$. Then the lemma follows immediately from combining (2.11) and (2.13).

Fix $1 < p < 2$ and $1/p = (1 - \theta) + \theta/2$. We appeal to the facts that for any semifinite von Neumann algebra $\mathcal{N}$,

$$L_p(\mathcal{N}) = [L_{1,\infty}(\mathcal{N}), L_2(\mathcal{N})]_{\theta,p,K} \quad \text{and} \quad L_p(\mathcal{N}) = [L_{1}(\mathcal{N}), L_2(\mathcal{N})]_{\theta,p,K}.$$ 

The above lemma yields a constant $c_p$ such that

$$\left(\sum_n \|\alpha_n\|_p\right)^{1/p} = \| (\alpha_n)_{n \geq 1} \|_{L_p(M)} \leq c_p \|x\|_{\theta,p,K}.$$ 

Applying the fact that expectations are contractive projections in $L_p(M)$ gives

$$\|a\|_{\nu} \leq 2c_p \|x\|_p.$$
Therefore, \((2.14)\) becomes,
\[
\left(\sum_{n \geq 1} E_{n-1}(|\beta_n|^2)^{1/2}, 2^\nu\right) = J\left(\beta(\nu), 2^\nu; L^{1,\infty}(M \otimes B(\ell^2(\mathbb{N}^2))), L^2(M \otimes B(\ell^2(\mathbb{N}^2)))\right).
\]

Therefore, \((2.14)\) becomes,
\[
(2.16) \quad J\left(\beta(\nu), 2^\nu; L^{1,\infty}(M \otimes B(\ell^2(\mathbb{N}^2))), L^2(M \otimes B(\ell^2(\mathbb{N}^2)))\right) \leq \kappa J(u_\nu, 2^\nu).
\]

Using similar argument as in the estimate of \(a\) with \(M \otimes \ell_\infty\) replaced by \(M \otimes B(\ell^2(\mathbb{N}^2))\), we get as in Lemma \(2.9\) that for every \(0 < \theta < 1\) and \(1 < p < 2\),
\[
\|\beta\|_{L^p(M \otimes B(\ell^2(\mathbb{N}^2)))} \leq c_p \|x\|_p.
\]
This is equivalent to
\[
\left\|\left(\sum_{n \geq 1} E_{n-1}(|\beta_n|^2)^{1/2}\right)\right\|_p \leq c_p \|x\|_p.
\]
Using Kadison’s inequality \(E_{n-1}(\beta_n)^*E_{n-1}(\beta_n) \leq E_{n-1}|\beta_n|^2\) for all \(n \geq 1\), we deduce that
\[
\|b\|_{h_\theta^p} \leq c_p \|x\|_p.
\]
A similar argument can be applied to get the estimate
\[
\|c\|_{h_\theta^p} \leq c_p \|x\|_p.
\]
Combining the above three estimates clearly provides the second item in the statement of the theorem. This completes the proof for the finite case.

**Case 2.** Assume now that \(M\) is infinite. Without loss of generality, we may assume that \(M\) is \(\sigma\)-finite. We note first that Case 1 extends easily to any finite case with the trace \(\tau\) being not necessarily normalized (with the same constants as the case of normalized trace). Since there is a trace preserving conditional expectation \(E_1 : M \rightarrow M_1\), it is known that \(\tau|_{M_1}\) remains semifinite.

Fix an increasing sequence of projections \((e_k)_{k \geq 1} \subset M_1\) with \(\tau(e_k) < \infty\) for all \(k \geq 1\) and such that \((e_k)_{k \geq 1}\) converges to \(1\) for the strong operator topology. For each \(k\), consider the finite von Neumann algebra \((e_k M e_k, \tau|_{e_k M e_k})\) with the filtration \((e_k M_n e_k)_{n \geq 1}\). If we denote by \(E_n^{(k)}\) the trace preserving conditional expectation from \(e_k M e_k\) onto \(e_k M_n e_k\) then \(E_n^{(k)}\) is just the restriction of \(E_n\) on \(e_k M e_k\). This is the case since the \(e_k\)’s were chosen from the smallest subalgebra \(M_1\). Therefore, if \(y \in e_k M e_k\) then one can easily verify that
\[
\|y\|_{h_\theta^p(e_k M e_k)} = \|y\|_{h_\theta^p(M_1)}, \quad \|y\|_{h_\theta^p(e_k M e_k)} = \|y\|_{h_\theta^p(M)}, \quad \text{and} \quad \|y\|_{h_\theta^p(e_k M e_k)} = \|y\|_{h_\theta^p(M)}.
\]

Let \(x \in L_2(M) \cap L_1(M)\). For each \(k \geq 1\), \(e_k x e_k \in L_2(e_k M e_k)\). From Case 1., there exists a decomposition \(e_k x e_k = a^{(k)} + b^{(k)} + c^{(k)}\) with the property that for every \(1 < p < 2\),
\[
\|a^{(k)}\|_{h_\theta^p} + \|b^{(k)}\|_{h_\theta^p} + \|c^{(k)}\|_{h_\theta^p} \leq \kappa_p \|e_k x e_k\|_p.
\]
Fix an ultrafilter \(\mathcal{U}\) on \(\mathbb{N}\) containing the Fréchet filter. For any given \(1 < p < 2\), the weak-limit along the ultrafilter \(\mathcal{U}\) of the sequence \((a^{(k)})_{k \geq 1}\) exists in \(h_\theta^p(M)\). It is crucial here to observe that such weak-limits are independent of \(p\) (since they are automatically weak-limits of the same
sequence in $L_1(M\otimes\ell_\infty) + L_2(M\otimes\ell_\infty)$. Similar observations can be made with the sequences $(b^{(k)})_{k \geq 1}$ and $(c^{(k)})_{k \geq 1}$. Set

$$a = \operatorname{w-lim}_{k \uparrow \ell} a^{(k)}, \quad b = \operatorname{w-lim}_{k \uparrow \ell} b^{(k)}, \quad \text{and} \quad c = \operatorname{w-lim}_{k \uparrow \ell} c^{(k)}$$

in $h^p_p(M)$, $h^p_p(M)$, and $h^p_p(M)$, respectively. We also observe that for every $1 < p < 2$, it is easy to verify that $\lim_{k \to \infty} \|e_k x e_k - x\|_p = 0$. A fortiori, $\lim_{k \uparrow \ell} \|e_k x e_k - x\|_p = 0$. All these facts lead to the decomposition:

$$x = a + b + c.$$ 

Furthermore, for every $1 < p < 2$, we have

$$\|a\|_{h^p_p} + \|b\|_{h^p_p} + \|c\|_{h^p_p} \leq \sup_k \left\{ \|a^{(k)}\|_{h^p_p} + \|b^{(k)}\|_{h^p_p} + \|c^{(k)}\|_{h^p_p} \right\}$$

$$\leq \kappa_p \sup_k \|e_k x e_k\|_p \leq \kappa_p \|x\|_p$$

where $\kappa_p$ is the constant from Case 1. The proof is complete. \hfill $\square$

Remarks 2.10. 1) Since the noncommutative Burkholder inequalities do not hold for $p = 1$, the validity of our simultaneous decomposition can not include the left endpoint of the interval $(1, 2)$. On the other hand, using known estimates from real interpolation $(\theta, p, K)$ and $(\theta, p, J)$ methods of classical Lebesgue spaces, we can derive that there is an absolute constant $C$ such that for $1/p = (1 - \theta) + \theta/2$, we have $\kappa_p \leq C\theta^{-2}(1 - \theta)^{-1/2-1/p}$. It follows that $\kappa_p$ is of order $(p - 1)^{-2}$ when $p \to 1$ and of order $(2 - p)^{-1}$ when $p \to 2$. In particular, our method of proof does not allow extension of the decomposition to any of the endpoints of the interval $[1, 2]$. As we only get that $\kappa_p = O((p - 1)^{-2})$ when $p \to 1$, our argument does not yield the optimal order for the constants for the Burkholder inequalities from $[13]$.

2) In the argument for the Case 2, we were forced to use ultrafilter since we needed the weak-limits to exist simultaneously for uncountably many values of $p$.

3) Junge and Perrin also considered simultaneous type decomposition for conditioned Hardy spaces in $[28]$. Our Theorem 2.8 above should be compared with $[28]$ Theorem 5.9]. See also Corollary 4.4 below for similar type simultaneous decomposition for the case of martingale Hardy space norms.

3. Burkholder’s inequalities in symmetric spaces

The following is the principal result of this article. It provides extensions of noncommutative Burkholder’s inequalities for martingales in general noncommutative symmetric spaces.

**Theorem 3.1.** Let $E$ be a symmetric Banach function on $(0, \infty)$ satisfying the Fatou property. Assume that either $1 < p_E \leq q_E < 2$ or $2 < p_E \leq q_E < \infty$. Then

$$E(M) = h_E(M).$$

That is, a martingale $x = (x_n)_{n \geq 1}$ is bounded in $E(M)$ if and only if it belongs to $h_E(M)$ and

$$\|x\|_{E(M)} \simeq_E \|x\|_{h_E}.$$  

As noted in the introduction, the preceding theorem solves positively a question raised in $[22]$. The new result here is the case where $1 < p_E \leq q_E < 2$. The case $2 < p_E \leq q_E < \infty$ was established by Dirksen in $[12]$ Theorem 6.2] but we will also provide an alternative approach for this range. We remark that under the assumption of Theorem 3.1, the Banach function space $E$ is fully symmetric in the sense of $[17]$ but this extra property will not be needed in the proof.
We divide the proof into four separate parts according to \(1 < p_E \leq q_E < 2\) or \(2 < p_E < q_E < \infty\), each case involving two inequalities. The main difficulty in the proof is Part II below. Part III will be deduced from Part II via duality. The other two parts will be derived from standard use of interpolations of linear operators.

3.1. **The case \(1 < p_E \leq q_E < 2\).** Let \(E\) be a symmetric Banach function space on \((0, \infty)\) with the Fatou property and satisfying \(1 < p_E \leq q_E < 2\). Throughout the proof, we fix \(p\) and \(q\) so that \(1 < p < p_E \leq q_E < q < 2\). In this case, \(E \in \text{Int}(L_p, L_q)\).

**Part I.** We will verify that there exists a constant \(c_E\) such that for every \(x \in h_E(M)\), we have

\[
\|x\|_{E(M)} \leq c_E \|x\|_{h_E}.
\]

Let \(\Theta : \ell_p(L_p(M)) + \ell_q(L_q(M)) \to L_p(M) + L_q(M)\) be the linear operator defined by:

\[
\Theta((a_n)_{n \geq 1}) = \sum_{n \geq 1} \varepsilon_n(a_n) - \varepsilon_{n-1}(a_n).
\]

Then \(\Theta : \ell_p(L_p(M)) \to L_p(M)\) and \(\Theta : \ell_q(L_q(M)) \to L_q(M)\) are bounded operators. Indeed, if \(s\) is either \(p\) or \(q\), then the operator \(\Theta\) may be viewed as the composition of \(\Theta_1 : \ell_s(L_s(M)) \to h^d_s(M)\) defined by

\[
\Theta_1((a_n)_{n \geq 1}) = (\varepsilon_n(a_n) - \varepsilon_{n-1}(a_n))_{n \geq 1}
\]

and the canonical map \(\Theta_2 : h^d_s(M) \to L_s(M)\) defined by:

\[
\Theta_2((d_n)_{n \geq 1}) = \sum_{n \geq 1} d_n.
\]

Both operators are well-defined for finite sequences. Since conditional expectations are bounded on \(L_s(M)\), \(\Theta_1\) is clearly bounded. The boundedness of \(\Theta_2\) follows directly from the noncommutative Burkholder’s inequality for \(L_s\)-bounded martingales \((1 < s < 2)\). Applying the interpolation from Proposition 2.6, we conclude that \(\Theta : E(M \otimes \ell_\infty) \to E(M)\) is bounded.

Now, let \(x \in h^d_E(M)\). Then \((dx_n)_{n \geq 1} \in E(M \otimes \ell_\infty)\) with \(\|x\|_{h^d_E} = \|(dx_n)_{n \geq 1}\|_{E(M \otimes \ell_\infty)}\) and \(\Theta((dx_n)_{n \geq 1}) = x\). It follows that

\[
(3.1) \quad \|x\|_{E(M)} \leq \|\Theta : E(M \otimes \ell_\infty) \to E(M)\| \cdot \|x\|_{h^d_E}.
\]

In a similar way, we may consider the linear map

\[
\Lambda : L_p(M \otimes B(\ell_2(N^2))) + L_q(M \otimes B(\ell_2(N^2))) \to L_p(M) + L_q(M)
\]

as the composition of the natural projection from \(L_p(M \otimes B(\ell_2(N^2))) + L_q(M \otimes B(\ell_2(N^2)))\) onto \(h^d_p(M) + h^d_q(M)\) described in [26] and the canonical operator from \(h^d_p(M) + h^d_q(M)\) into \(L_p(M) + L_q(M)\). Then for \(s = p, q\), the operator \(\Lambda : L_s(M \otimes B(\ell_2(N^2))) \to L_s(M)\) is bounded. As above, we obtain by interpolation that

\[
\Lambda : E(M \otimes B(\ell_2(N^2))) \to E(M)
\]

is bounded. Now, if \(x \in h^c_E(M)\) then \((dx_n)_{n \geq 1}\) may be viewed as an element of \(E(M \otimes B(\ell_2(N^2)))\) and \(\Lambda((dx_n)_{n \geq 1}) = x\). We may deduce that

\[
(3.2) \quad \|x\|_{E(M)} \leq \|\Lambda : E(M \otimes B(\ell_2(N^2))) \to E(M)\| \cdot \|x\|_{h^c_E}.
\]

The corresponding row version can be deduced by considering adjoint operators.
To conclude the proof, assume that \( x = w + y + z \) where \( w \in h^a_E(\mathcal{M}) \), \( y \in h^r_E(\mathcal{M}) \), and \( z \in h_E^r(\mathcal{M}) \). Then the above inequalities leads to

\[
\|x\|_{E(\mathcal{M})} \leq c_E \left( \|w\|_{h_E^a} + \|y\|_{h_E^r} + \|z\|_{h_E^r} \right)
\]

where \( c_E \) depends only on \( E \). Taking the infimum over all decompositions \( x = w + y + z \), we deduce that \( \|x\|_{E(\mathcal{M})} \leq c_E \|x\|_{h_E^r} \).

**Part II.** We consider now the reverse inequalities. That is, there exists a constant \( \beta_E \) such that for every \( x \in E(\mathcal{M}) \),

\[
\|x\|_{h_E^r} \leq \beta_E \|x\|_{E(\mathcal{M})^*}
\]

The proof is much more involved and requires several steps. Our approach relies on two essential facts. As stated in Corollary 2.5, noncommutative symmetric spaces have concrete representations as interpolation spaces. The second fact is the simultaneous decomposition obtained in the previous section.

According to Corollary 2.5 we may fix a symmetric Banach function space \( F \) on \((0, \infty)\) with nontrivial Boyd indices and such that for any semifinite von Neumann algebra \( \mathcal{N} \), we have:

\[
E(\mathcal{N}) = \left[ L_p(\mathcal{N}), L_q(\mathcal{N}) \right]_{F,j},
\]

where \([\cdot, \cdot]_{F,j}\) is the interpolation method introduced in the previous section. We begin with the following intermediate result.

**Lemma 3.2.** Let \( x \in L_p(\mathcal{M}) \cap L_q(\mathcal{M}) \). For every \( \varepsilon > 0 \), there exist \( x_\varepsilon \in L_p(\mathcal{M}) \cap L_q(\mathcal{M}) \), \( a_\varepsilon \in h^a_E(\mathcal{M}) \), \( b_\varepsilon \in h^r_E(\mathcal{M}) \), and \( c_\varepsilon \in h_E^r(\mathcal{M}) \) with:

1. \( \|x - x_\varepsilon\|_{L_p(\mathcal{M}) \cap L_q(\mathcal{M})} < \varepsilon \);
2. \( x_\varepsilon = a_\varepsilon + b_\varepsilon + c_\varepsilon \);
3. \( \|a_\varepsilon\|_{h_E^a} + \|b_\varepsilon\|_{h_E^r} + \|c_\varepsilon\|_{h_E^r} \leq \eta \|x\|_{E(\mathcal{M})^*} \).

**Proof.** Let \( x \in L_p(\mathcal{M}) \cap L_q(\mathcal{M}) \) and \( \varepsilon > 0 \). Using the interpolation couple \( (L_p(\mathcal{M}), L_q(\mathcal{M})) \), fix a representation \( x = \sum_{\nu \in \mathbb{Z}} u_\nu \) (convergent in \( L_p(\mathcal{M}) + L_q(\mathcal{M}) \)) such that

\[
(3.3) \quad \left\| j(\{u_\nu\}_{\nu \in \mathbb{Z}}) \right\|_F \leq 2 \|x\|_{E(\mathcal{M})^*}.
\]

Note that the \( u_\nu \)'s belong to \( L_p(\mathcal{M}) \cap L_q(\mathcal{M}) \). Using Lemma 2.1 for each \( \nu \in \mathbb{Z} \), we may choose \( u^{(m_\nu)}_\nu \in L_1(\mathcal{M}) \cap L_2(\mathcal{M}) \) satisfying the following properties:

1. \( \|u^{(m_\nu)}_\nu - u_\nu\|_{L_p(\mathcal{M}) \cap L_q(\mathcal{M})} \leq \frac{\varepsilon}{4|\nu| + 1} \);
2. \( \|u^{(m_\nu)}_\nu\|_q \leq \|u_\nu\|_q \);
3. \( \|u^{(m_\nu)}_\nu\|_p \leq \|u_\nu\|_p \).

The last two conditions imply that for every \( \nu \in \mathbb{Z} \) and every \( t > 0 \),

\[
J(u^{(m_\nu)}_\nu, t) \leq J(u_\nu, t),
\]

which furthermore leads to the following inequality:

\[
(3.4) \quad j(\{u^{(m_\nu)}_\nu\}_{\nu \in \mathbb{Z}}) \leq j(\{u_\nu\}_{\nu \in \mathbb{Z}}).
\]

We define the operator \( x_\varepsilon \) by setting:

\[
x_\varepsilon := \sum_{\nu \in \mathbb{Z}} u^{(m_\nu)}_\nu.
\]
Then it satisfies the following norm estimates:
\[
\|x_\varepsilon - x\|_{L_p(M) \cap L_q(M)} \leq \sum_{\nu \in \mathbb{Z}} \|u^{(m)}_{\nu} - u_\nu\|_{L_p(M) \cap L_q(M)} \\
\leq \sum_{|\nu|+1}^\nu \varepsilon \leq \varepsilon.
\]
In particular, \(x_\varepsilon \in L_p(M) \cap L_q(M)\) and the first item in the statement of Lemma 3.2 is satisfied. The crucial fact here is that all \(u^{(m)}_{\nu}\)s in the representation of \(x_\varepsilon\) belong to \(L_1(M) \cap L_2(M)\) so that Theorem 2.8 can be applied to each of the \(u^{(m)}_{\nu}\)s. That is, for every \(\nu \in \mathbb{Z}\), there exist \(a_\nu, b_\nu, c_\nu\) in \(\cap_{1 \leq s \leq 2} L_s(M)\) satisfying:
\[
u\]
and if \(s\) is equal to either \(p\) or \(q\), then
\[
u
where \(\kappa(p, q) = \max\{\kappa_p, \kappa_q\}\).

Below, for each \(\nu \in \mathbb{Z}\), \(d_\nu\) denotes the sequence \((d_\nu(a_\nu))_{n \geq 1}\) in \(L_p(M \otimes \ell_\infty) \cap L_q(M \otimes \ell_\infty)\) and \(d_\nu\) and \(d_\nu^*\) denote the operators \(U[(d_\nu(b_\nu))_{n \geq 1}]\) and \(U[(d_\nu(c_\nu^*))_{n \geq 1}]\) in \(L_p(M \otimes B(\ell_2(\mathbb{N})) \cap L_q(M \otimes B(\ell_2(\mathbb{N}^2)))\), respectively.

First, we observe that (3.6) can be reinterpreted using the \(J\)-functionals as follows:
\[
J(da_\nu, t) \leq \kappa(p, q) J(u^{(m)}_{\nu}, t), \ \ t > 0,
\]
\[
J(db_\nu, t) \leq \kappa(p, q) J(u^{(m)}_{\nu}, t), \ \ t > 0,
\]
\[
J(d\nu^*, t) \leq \kappa(p, q) J(u^{(m)}_{\nu}, t), \ \ t > 0
\]
where the \(J\)-functional on the left side of (3.7) is taken using the couple \([L_p(M \otimes \ell_\infty), L_q(M \otimes \ell_\infty)]\) and those on the left sides of inequalities (3.8) and (3.9) were computed using the couple \([L_p(M \otimes B(\ell_2(\mathbb{N}))), L_q(M \otimes B(\ell_2(\mathbb{N}^2)))\]).

We need the following propositions of the sequences \((da_\nu)_{\nu \in \mathbb{Z}}, (db_\nu)_{\nu \in \mathbb{Z}},\) and \((dc_\nu)_{\nu \in \mathbb{Z}}\). We refer to [11] for definition and criterion for unconditionally Cauchy series in Banach spaces.

**Sublemma 3.3.** (1) \(\sum_{\nu \in \mathbb{Z}} da_\nu\) is a weakly unconditionally Cauchy series in \(E(M \otimes \ell_\infty)\). (2) \(\sum_{\nu \in \mathbb{Z}} db_\nu\) and \(\sum_{\nu \in \mathbb{Z}} dc_\nu^*\) are weakly unconditionally Cauchy series in \(E(M \otimes B(\ell_2(\mathbb{N}^2)))\).

Moreover, there exist a constant \(\kappa_E\) such that:
\[
\max \left\{ \sup_{N \geq 1} \|S_N(a)\|_{E(M \otimes \ell_\infty)}, \sup_{N \geq 1} \|S_N(b)\|_{E(M \otimes B(\ell_2(\mathbb{N}^2))), \sup_{N \geq 1} \|S_N(c^*)\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))} \right\} \\
\leq \kappa_E \|x\|_{E(M)},
\]
where for each \(N \geq 1\), \(S_N(a) = \sum_{|\nu| \leq N} da_\nu, S_N(b) = \sum_{|\nu| \leq N} db_\nu,\) and \(S_N(c^*) = \sum_{|\nu| \leq N} dc_\nu^*\).

To verify the first item in Sublemma 3.3 we note that if \(S\) if a finite subset of \(\mathbb{Z}\) then it follows from (3.4) and (3.7) that
\[
\hat{j} \left(\{da_\nu\}_{\nu \in S}, \cdot \right) \leq \kappa(p, q) \hat{j} \left(\{u^{(m)}_{\nu}\}_{\nu \in S}, \cdot \right) \leq \kappa(p, q) \hat{j} \left(\{u_\nu\}_{\nu \in S}, \cdot \right).
\]

By the definition of \(\langle \cdot, \cdot \rangle_{F,\hat{j}}\), for every finite sequence of scalars \((\theta_\nu)_{\nu \in S}\) with \(|\theta_\nu| = 1\), we have
\[
\|\sum_{\nu \in S} \theta_\nu da_\nu\|_{L_p(M \otimes \ell_\infty), L_q(M \otimes \ell_\infty)} \leq \kappa(p, q) \|\hat{j} \left(\{u_\nu\}_{\nu \in S}, \cdot \right)\|_E \leq 2\kappa(p, q) \|x\|_{F,\hat{j}}.
\]
where the last inequality is from (3.3). Now we use the facts that
\[ E(\mathcal{M}\otimes \ell_\infty) = [L_p(\mathcal{M}\otimes \ell_\infty), L_q(\mathcal{M}\otimes \ell_\infty)]_{F,j} \] and \( E(\mathcal{M}) = [L_p(\mathcal{M}), L_q(\mathcal{M})]_{F,j} \)
to deduce that there exists a constant \( \kappa_E \) such that:
\[ (3.10) \quad \left\| \sum_{\nu \in S} \theta_\nu da_\nu \right\|_{E(\mathcal{M}\otimes \ell_\infty)} \leq \kappa_E \| x \|_{E(\mathcal{M})}. \]

Since this is the case for any arbitrary finite subset of \( Z \), it proves that the series \( \sum_{\nu \in Z} da_\nu \) is weakly unconditionally Cauchy in \( E(\mathcal{M}\otimes \ell_\infty) \).

The proof of the second item follows the same pattern. As above, if \( S \) is a finite subset of \( Z \), then it follows from (3.1) and (3.8) that:
\[ j\left( \{ db_\nu \}_{\nu \in S} \right) \leq \kappa(p,q) j\left( \{ u_\nu^{mu} \}_{\nu \in S} \right) \leq \kappa(p,q) j\left( \{ u_\nu \}, \cdot \right). \]

Using similar argument as above, we may deduce that for every finite subset \( S \) of \( Z \) and for every sequence of scalars \( (\theta_\nu)_{\nu \in S} \) with \( |\theta_\nu| = 1 \),
\[ (3.11) \quad \left\| \sum_{\nu \in S} \theta_\nu db_\nu \right\|_{E(\mathcal{M}\otimes B(\ell_2(N^2)))} \leq \kappa_E \| x \|_{E(\mathcal{M})}. \]

This again shows that the series \( \sum_{\nu \in Z} db_\nu \) is weakly unconditionally Cauchy in \( E(\mathcal{M}\otimes B(\ell_2(N^2))) \). The proof for the series \( \sum_{\nu \in Z} dc_\nu \) is identical. The inequality stated in Sublemma 3.3 follows from (3.10), (3.11), and the corresponding inequality for \( \sum_{\nu \in Z} dc_\nu \). Sublemma 3.3 is verified.

Next, we note that since \( L_p(\mathcal{M}\otimes \ell_\infty) + L_q(\mathcal{M}\otimes \ell_\infty) \) is a reflexive space, the series \( \sum_{\nu \in Z} da_\nu \) is unconditionally convergent in \( L_p(\mathcal{M}\otimes \ell_\infty) + L_q(\mathcal{M}\otimes \ell_\infty) \). Similarly, both series \( \sum_{\nu \in Z} db_\nu \) and \( \sum_{\nu \in Z} dc_\nu \) are unconditionally convergent in \( L_p(\mathcal{M}\otimes B(\ell_2(N^2))) + L_q(\mathcal{M}\otimes B(\ell_2(N^2))) \). Now we set:
\[ da_\varepsilon := \sum_{\nu \in Z} da_\nu, \quad db_\varepsilon := \sum_{\nu \in Z} db_\nu, \quad \text{and} \quad dc_\varepsilon := \sum_{\nu \in Z} dc_\nu. \]

We claim that
\[ (3.12) \quad \max \left\{ \| d(a_\varepsilon) \|_{E(\mathcal{M}\otimes \ell_\infty)}, \| d(b_\varepsilon) \|_{E(\mathcal{M}\otimes B(\ell_2(N^2))}), \| d(c_\varepsilon) \|_{E(\mathcal{M}\otimes B(\ell_2(N^2))}) \right\} \leq \kappa_E \| x \|_{E(\mathcal{M})}. \]

To verify this claim, we use the fact mentioned in the previous section that for every semifinite von Neumann algebra \( \mathcal{N} \), the inclusion map from \( E(\mathcal{N}) \) into \( L_p(\mathcal{N}) + L_q(\mathcal{N}) \) is a semi-embedding. Indeed, if \( \rho = \kappa_E \| x \|_{E(\mathcal{M})} \), then from Sublemma 3.3 \( (\mathcal{N}(a))_{N \geq 1} \) is a sequence in the \( \rho \)-ball of \( E(\mathcal{M}\otimes \ell_\infty) \) that converges to \( da_\varepsilon \) for the norm topology of \( L_p(\mathcal{M}\otimes \ell_\infty) + L_q(\mathcal{M}\otimes \ell_\infty) \). By semi-embedding, we have \( da_\varepsilon \in E(\mathcal{M}\otimes \ell_\infty) \) with \( \| da_\varepsilon \|_{E(\mathcal{M}\otimes \ell_\infty)} \leq \rho \). Identical argument can be applied to \( db_\varepsilon \) and \( dc_\varepsilon \) to deduce that \( \| d(b_\varepsilon) \|_{E(\mathcal{M}\otimes B(\ell_2(N^2))}) \leq \rho \) and \( \| d(c_\varepsilon) \|_{E(\mathcal{M}\otimes B(\ell_2(N^2))}) \leq \rho \).

We have verified (3.12).

To complete the proof, we identify \( da_\varepsilon, db_\varepsilon, \) and \( dc_\varepsilon \) as martingales \( a_\varepsilon \in \mathcal{H}_E^d(\mathcal{M}), b_\varepsilon \in \mathcal{H}_E^c(\mathcal{M}), \) and \( c_\varepsilon \in \mathcal{H}_E^c(\mathcal{M}) \), respectively. Then it is clear from the construction that
\[ x_\varepsilon = a_\varepsilon + b_\varepsilon + c_\varepsilon. \]

Moreover, (3.12) can be restated as:
\[ \max \left\{ \| a_\varepsilon \|_{\mathcal{H}_E^d}, \| b_\varepsilon \|_{\mathcal{H}_E^c}, \| c_\varepsilon \|_{\mathcal{H}_E^c} \right\} \leq \kappa_E \| x \|_{E(\mathcal{M})}. \]

This clearly implies the last item in Lemma 3.2. The proof is complete. \( \square \)

The next step provides the desired decomposition for all \( x \in L_p(\mathcal{M}) \cap L_q(\mathcal{M}). \)
Lemma 3.4. There exists a constant $\beta_E$ such that every $x \in L_p(M) \cap L_q(M)$ admits a decomposition $x = a + b + c$ satisfying:

$$\|a\|_E + \|b\|_{\overline{E}} + \|c\|_{\overline{E}} \leq \beta_E \|x\|_{E(M)}.$$

**Proof.** We use semi-embedding technique. Using Lemma 3.2, we construct sequences of operators $(x_m), (a_m), (b_m),$ and $(c_m)$ such that:

1. $\lim_{m \to \infty} \|x_m - x\|_{L_p(M) \cap L_q(M)} = 0;$
2. $x_m = a_m + b_m + c_m$ for all $m \geq 1;$
3. $\|a_m\|_{h_E} + \|b_m\|_{\overline{h}_E} + \|c_m\|_{\overline{h}_E} \leq \eta_E \|x\|_{E(M)}.$

Let $\rho = \eta_E \|x\|_{E(M)}$ and for $m \geq 1$, set

$$\tilde{a}_m := (d_n(a_m))_{n \geq 1} \in E(M \otimes \ell_\infty),$$
$$\tilde{b}_m := U((d_n(a_m))_{n \geq 1}) \in E(M \otimes B(\ell_2(\mathbb{N}^2))),$$
$$\tilde{c}_m := U((d_n(c_m))_{n \geq 1}) \in E(M \otimes B(\ell_2(\mathbb{N}^2))).$$

We observe that the sequence $(\tilde{a}_m)$ belongs to the $\rho$-ball of $E(M \otimes \ell_\infty)$. Similarly, $(\tilde{b}_m)_m$ and $(\tilde{c}_m)_m$ belong to the $\rho$-ball of $E(M \otimes B(\ell_2(\mathbb{N}^2)))$.

Since the spaces $L_p(M \otimes \ell_\infty) + L_q(M \otimes \ell_\infty)$ and $L_p(M \otimes B(\ell_2(\mathbb{N}^2))) + L_q(M \otimes B(\ell_2(\mathbb{N}^2)))$ are reflexive, we may assume (after taking subsequences if necessary) that $(\tilde{a}_m)_m$ converges to $\tilde{a}$ for weak topology in $L_p(M \otimes \ell_\infty) + L_q(M \otimes \ell_\infty)$ and both $(\tilde{b}_m)_m$ and $(\tilde{c}_m)_m$ converge (for the weak topology of $L_p(M \otimes B(\ell_2(\mathbb{N}^2)))$) to $\tilde{b}$ and $\tilde{c}$, respectively.

As a consequence of the fact that the inclusion mappings are semi-embeddings, it is clear that these limits satisfy:

$$\max \left\{ \|\tilde{a}\|_{E(M \otimes \ell_\infty)}, \|\tilde{b}\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))}, \|\tilde{c}\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))} \right\} \leq \rho.$$

Next, we note that if $\tilde{a} = (\tilde{a}^{(n)})_{n \geq 1}$ then for each $n \geq 1$, $\tilde{a}^{(n)} = \varliminf_{m \to \infty} d_n(a_m)$ in $L_p(M) + L_q(M)$. In particular, $\tilde{a}^{(n)} \in L_p(M_n) + L_q(M_n)$ and $\mathcal{E}_{n-1}(\tilde{a}^{(n)}) = 0$. That is, $(\tilde{a}^{(n)})_{n \geq 1}$ is a martingale difference sequence. If we set $a := \sum_n \tilde{a}^{(n)}$, then

$$\|a\|_{h_E} = \|\tilde{a}\|_{E(M \otimes \ell_\infty)} \leq \rho.$$

Similarly, one can easily check that there exist martingales $b$ and $c$ such that $\tilde{b} = U((d_n(b))_{n \geq 1})$ and $\tilde{c} = U((d_n(c))_{n \geq 1})$ which further satisfy:

$$\|b\|_{\overline{h}_E} = \|\tilde{b}\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))} \leq \rho \quad \text{and} \quad \|c\|_{\overline{h}_E} = \|\tilde{c}\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))} \leq \rho.$$

It is now clear from the construction that $x = a + b + c$ and $\|a\|_{h_E} + \|b\|_{\overline{h}_E} + \|c\|_{\overline{h}_E} \leq 3\rho$. Thus, we have verified Lemma 3.4.

To conclude the proof of Part II, it is enough to note that since $L_p(M) \cap L_q(M)$ is dense in $E(M)$. The assertion that $\|x\|_{h_E} \leq \beta_E \|x\|_{E(M)}$ for all $x \in E(M)$ then follows immediately from Lemma 3.4.

3.2. The case $2 < p_E \leq q_E < \infty$. Assume now that $E$ is a symmetric Banach function space on $(0, \infty)$ satisfying the Fatou property and $2 < p_E \leq q_E < \infty$.

**Part III.** We will verify that for every $x \in h_E(M)$,

$$\|x\|_{E(M)} \lesssim \|x\|_{h_E}.$$
This will be deduced from Part II using duality. Let $E^\times$ be the Köthe dual of $E$. The noncommutative symmetric space $E^\times(M)$ is the Köthe dual of $E(M)$ in the sense of [15]. Since $E$ has the Fatou property, it follows that for every $x \in E(M)$, we have $\|x\|_{E(M)} = \|x\|_{E^\times(M)}$. In particular, the closed unit ball of $E^\times(M)$ is a norming set for $E(M)$.

Fix $x \in E(M)$. For $\varepsilon > 0$, choose $y \in E^\times(M)$, with $\|y\|_{E^\times(M)} = 1$, and such that

$$\|x\|_{E(M)} \leq \varepsilon + \tau(xy^\ast).$$

From [34] Proposition 2.b.2, the Boyd indices of $E^\times$ satisfy $1 < p_{E^\times} \leq q_{E^\times} < 2$. Thus, using Part II, it follows that $y \in h_{E^\times}$. We may choose a decomposition $y = a + b + c$ satisfying:

$$\|a\|_{h_{E^\times}^c} + \|b\|_{h_{E^\times}^c} + \|c\|_{h_{E^\times}^c} \leq \kappa_{E^\times} + \varepsilon.$$

Now, $\tau(xy^\ast) = \tau(xa^\ast) + \tau(xb^\ast) + \tau(xc^\ast) = I + II + III$. We estimate $I$, $II$, and $III$ separately. Below, we denote by $\gamma$ and $\tau$ the usual traces on $\ell_\infty$ and $B(\ell_2(\mathbb{N}^2))$, respectively. For $I$, we have the following estimates:

$$I = \sum_{n \geq 1} \tau(dx_n da_n^\ast)$$

$$= \tau \otimes \gamma((dx_n)_{n \geq 1},(da_n^\ast)_{n \geq 1})$$

$$\leq \||dx_n)_{n \geq 1}\|_{E(M \otimes \ell_\infty)} \|\|da_n^\ast)_{n \geq 1}\|_{E^\times(M \otimes \ell_\infty)}$$

$$= \|x\|_{h_{E^\times}^c} \|a\|_{h_{E^\times}^c}.$$

For $II$, we use the identification of $h_{E}^c(M)$ and $h_{E^\times}^c(M)$ as subspaces of $E(M \otimes B(\ell_2(\mathbb{N}^2)))$ and $E^\times(M \otimes B(\ell_2(\mathbb{N}^2)))$, respectively. First, we write $II = \sum_{n \geq 1} \tau(dx_n db_n^\ast)$. Since the expectations $\mathcal{E}_k$'s are trace invariant, we have:

$$II = \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(db_n^\ast dx_n))$$

$$= \tau \left[ \sum_{n \geq 1} \mathcal{E}_{n-1}(db_n^\ast dx_n) \right].$$

From (2.10), we may write

$$II = \tau \otimes \text{tr} [U((db_n)_{n \geq 1})^*U((dx_n)_{n \geq 1})]$$

$$\leq \|U((db_n)_{n \geq 1})\|_{E^\times(M \otimes B(\ell_2(\mathbb{N}^2)))} \|U((dx_n)_{n \geq 1})\|_{E(M \otimes B(\ell_2(\mathbb{N}^2)))}$$

$$= \|b\|_{h_{E^\times}^c} \|x\|_{h_{E^\times}^c}.$$

The proof that $III \leq \|c\|_{h_{E^\times}^c} \|x\|_{h_{E^\times}^c}$ is identical so we omit the details. Combining the above estimates on $I$, $II$, and $III$, we derive that

$$\|x\|_{E(M)} \leq \varepsilon + (\kappa_{E^\times} + \varepsilon)\|x\|_{h_{E^\times}^c}.$$

Taking infimum over $\varepsilon$ gives the desired inequality.

**Part IV.** The remaining case is an easy interpolation of the noncommutative Burkholder inequalities for $L_s$-bounded martingales when $2 < s < \infty$. We include the details for completeness. Define

$$J^d : L_s(M) \to h^d_{b}(M).$$
to be the canonical inclusion \( x \mapsto (dx_n)_{n \geq 1} \). By the noncommutative Burkholder inequalities, \( J^d \) is bounded whenever \( 2 < s < \infty \). We view \( J^d \) as a bounded linear operator from \( L_s(M) \) into \( L_s(M \bar{} \ell_\infty) \). By interpolation, \( J^d : E(M) \to E(M \bar{} \ell_\infty) \) is bounded and clearly takes its values in \( h^s_E(M) \). This shows that for every \( x \in E(M) \),

\[
\|x\|_{h^s_E} \leq \|J^d : E(M) \to E(M \bar{} \ell_\infty)\| \|x\|_{E(M)}.
\]

Similarly, if we define \( J^c : L_s(M) \to h^s_c(M) \) to be the canonical inclusion as above then it may be viewed as a bounded linear operator from \( L_s(M) \) into \( L_s(M \bar{} B(\ell_2(\mathbb{N}^2))) \). By interpolation, \( J^c \) is bounded from \( E(M) \) into \( E(M \bar{} B(\ell_2(\mathbb{N}^2))) \) and takes its values in \( h^s_c(M) \). Therefore, for every \( x \in E(M) \),

\[
\|x\|_{h^s_c} \leq \|J^c : E(M) \to E(M \bar{} B(\ell_2(\mathbb{N}^2)))\| \|x\|_{E(M)}.
\]

Taking adjoints, we may also state that

\[
\|x\|_{h^s_c} = \|x^*\|_{h^s_c} \leq \|J^c : E(M) \to E(M \bar{} B(\ell_2(\mathbb{N}^2)))\| \|x\|_{E(M)}.
\]

The desired conclusion follows from combining the preceding three inequalities.

For the case where \( M \) is a finite von Neumann algebra equipped with a normal tracial state \( \tau \), it is more natural to consider symmetric Banach function spaces defined on the interval \([0, 1]\). However, the definition of the diagonal Hardy space uses the infinite von Neumann algebra \( M \bar{} \ell_\infty \). In this case, we may consider an extension of symmetric Banach function space \( E \) on \([0, 1]\) into a symmetric Banach function space on \((0, \infty)\) introduced in [23] (see also [1, 34] for more details).

Let \( Z^2_E \) be the symmetric space on \((0, \infty)\) of all measurable function \( f \) for which \( \mu(f)\chi_{(0,1]} \in E \) and \( \mu(f)\chi_{(1,\infty)} \in L_2(0, \infty), \) endowed with the norm

\[
\|f\|_{Z^2_E} = \max \left\{ \|\mu(f)\chi_{(0,1]}\|_{E}, \left( \sum_{n=0}^{\infty} \left( \int_n^{n+1} \mu_u(f) \, du \right)^2 \right)^{1/2} \right\}.
\]

It was shown in [34] Theorem 2.f.1 that if \( E \) has nontrivial Boyd indices then \( Z^2_E \) is isomorphic to \( E \). Using the symmetric Banach function space \( Z^2_E \) on the diagonal Hardy space, we may state the following variant of Theorem 3.1.

**Theorem 3.5.** Assume that \((M, \tau)\) is a finite von Neumann algebra with \( \tau \) being a normal tracial state and \( E \) is a symmetric Banach function space on \([0, 1]\) satisfying the Fatou property. Let \( x = (x_n)_{n \geq 1} \) be a bounded \( E(M) \)-martingale.

1. If \( 1 < p_E \leq q_E < 2 \), then

\[
\|x\|_{E(M)} \lesssim E \inf \left\{ \|w\|_{h^d_E}, \|y\|_{h^c_E}, \|z\|_{h^c_E} \right\}
\]

where the infimum runs over all decompositions \( x = w + y + z \) with \( w, y, \) and \( z \) are martingales.

2. If \( 2 < p_E \leq q_E < \infty \), then

\[
\|x\|_{E(M)} \lesssim E \max \left\{ \|x\|_{h^d_E}, \|x\|_{h^c_E} \right\}.
\]

The assumptions of Theorem 3.1 and Theorem 3.5 are equivalent to \( E \in \text{Int}(L_p, L_q) \) with \( 1 < p < q < 2 \) or \( 2 < p < q < \infty \). Indeed, if \( E \in \text{Int}(L_p, L_q) \) then \( p \leq p_E \leq q_E \leq q \). We do not know if our results extend to the case where \( E \in \text{Int}(L_p, L_q) \) for \( 1 < p < 2 \) or \( E \in \text{Int}(L_2, L_q) \) for
$q > 2$. On the other hand, since $h_1(M) \subseteq L_1(M)$, the argument used in Part I of the proof of Theorem [4.1] can be readily adjusted to prove the following:

If $E \in \text{Int}(L_1, L_q)$ where $q < 2$, then there exist a constant $c_E$ such that for every martingale $x \in h_E(M)$,

$$\|x\|_{E(M)} \leq c_E \|x\|_{h_E}.$$ 

Similarly, the argument of Part IV also gives:

If $E \in \text{Int}(L_2, L_q)$ where $2 < q < \infty$, then there exist a constant $c_E$ such that for every martingale $x \in E(M)$,

$$\|x\|_{h_E} \leq c_E \|x\|_{E(M)}.$$ 

We conclude this section by pointing out that as with the case of noncommutative Burkholder-Gundy inequalities, no equivalence of norms is known for the case where $1 < p_E < 2$ and $2 < q_E < \infty$.

### 4. Further remarks

We begin this section with a short discussion about the comparison between martingales Hardy spaces and conditioned martingale Hardy spaces associated with general symmetric spaces. Combining Theorem [4.1] with the main result of [22], we may state:

**Corollary 4.1.** Let $E$ be a symmetric Banach function on $(0, \infty)$ satisfying the Fatou property. Assume that either $1 < p_E \leq q_E < 2$ or $2 < p_E \leq q_E < \infty$. Then

$$H_E(M) = h_E(M)$$

with equivalent norms.

We recall the noncommutative Davis’ decomposition established in [27, 38] which states that $H_1(M) = h_1(M)$. In view of this equivalence, it seems reasonable to expect that the assumption $p_E > 1$ is not needed in the statement of Corollary [4.1]. Unfortunately, our interpolation technique is not efficient enough to apply to the case $p_E = 1$. We leave this as an open problem.

**Problem 4.2.** Assume that $p_E = 1$ and $q_E < 2$. Do we have $H_E(M) = h_E(M)$?

Our next result shows that one direction is always valid.

**Proposition 4.3.** Assume that $q_E < 2$. Then

1. $\max \{ \|a\|_{H_E}, \|a\|_{H_E} \} \leq c_E \|a\|_{h_E}$,
2. $\|a\|_{H_E} \leq c_E \|a\|_{h_E}$ and $\|a\|_{H_E} \leq c_E \|a\|_{h_E}$.

In particular, $h_E(M) \subseteq H_E(M)$ whenever $q_E < 2$.

**Proof.** Let $1 \leq s < 2$. It is an immediate consequence of the space $L_{s/2}(M)$ being a $s/2$-normed space that if $a \in h^d_s(M)$ that max $\{ \|a\|_{H^c_s}, \|a\|_{H^c_s} \} \leq \|a\|_{h^c_s}$. The fact that $\|a\|_{H^c_s} \leq 2^{1/s} \|a\|_{h^c_s}$ for every $a \in h^c_s(M)$ follows from [29, Theorem 7.1].

The general case can be achieved by interpolations. We include the argument for completeness. When $1 < s < 2$, it follows from above that the formal identity $\iota : h^d_s(M) \to H^c_s(M)$ is a contraction. Moreover, since conditional expectations are bounded on $L_s(M)$, the linear operator $\pi : L_s(M \otimes \ell_\infty) \to h^d_s(M)$ defined by

$$\pi((a_n)_{n \geq 1}) = (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n \geq 1}, \quad (a_n)_{n \geq 1} \in L_s(M \otimes \ell_\infty)$$

is bounded. Furthermore, if we denote by $j$ the natural isometry of $H^c_s(M)$ into $L_s(M \otimes B(\ell_2))$ then $j \pi : L_s(M \otimes \ell_\infty) \to L_s(M \otimes B(\ell_2))$ is bounded.
interpolation, where $L_2$, then $q < □$

Corollary 4.4. The above decomposition is related to another type of simultaneous decomposition considered by functions from [42] (see also [37]) and then follow the argument used in the proof of Theorem 2.8.

Next, we appeal to a result from [2, Theorem 2 and Remark 4] which asserts that if $q_E < q < 2$, then $E \in \text{Int}(L_1, L_q)$. Thus, by Proposition 2.6, $\mu \pi$ is bounded from $E(\mathcal{M} \overline{\otimes} \ell_\infty)$ into $E(\mathcal{M} \overline{\otimes} B(\ell_2))$. If $a \in h_E^c(\mathcal{M})$, then

$$\mu \pi((da_n)_{n \geq 1}) = \sum_{n \geq 1} da_n \otimes e_{n,1} \in E(\mathcal{M} \overline{\otimes} B(\ell_2))$$

where $(e_{i,j})_{i,j \geq 1}$ is the family of unit matrices in $B(\ell_2)$. We can conclude that

$$\|a\|_{\mathcal{H}_E^c} = \|\sum_{n \geq 1} da_n \otimes e_{n,1}\|_{E(\mathcal{M} \overline{\otimes} B(\ell_2))} \leq c_E \|(da_n)_{n \geq 1}\|_{E(\mathcal{M} \overline{\otimes} \ell_\infty)} = c_E \|a\|_{h_E^c}$$

where $c_E = \|\mu \pi : E(\mathcal{M} \overline{\otimes} \ell_\infty) \to E(\mathcal{M} \overline{\otimes} B(\ell_2))\|$.

The proof for the second item follows the same pattern. For $1 \leq s < 2$, the formal identity $\ell' : h_c^s(\mathcal{M}) \to \mathcal{H}_c^s(\mathcal{M})$ is such that $\|\ell'\| \leq 2^{1/s}$. From [26], there is a bounded projection $\pi' : L_s(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \to h_c^s(\mathcal{M})$. Then $\mu \pi' : L_s(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \to L_s(\mathcal{M} \overline{\otimes} B(\ell_2))$ is bounded. By interpolation, $\mu \pi' : E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \to E(\mathcal{M} \overline{\otimes} B(\ell_2))$ is bounded. As in the previous case, if $a \in h_E^c(\mathcal{M})$, then

$$\mu \pi'((da_n)_{n \geq 1}) = \sum_{n \geq 1} da_n \otimes e_{n,1} \in E(\mathcal{M} \overline{\otimes} B(\ell_2))$$

which again allows us to conclude that

$$\|a\|_{\mathcal{H}_E^c} \leq c'_E \|a\|_{h_E^c}$$

where $c'_E = \|\mu \pi' : E(\mathcal{M} \overline{\otimes} B(\ell_2(\mathbb{N}^2))) \to E(\mathcal{M} \overline{\otimes} B(\ell_2))\|$. The row version can be deduced by considering adjoint operators. The proof is complete.

The following consequence of Theorem 2.8 now follows from Proposition 4.3. It provides simultaneous decompositions for martingale Hardy spaces norms that are related to the noncommutative Burkholder-Gundy inequalities from [39].

**Corollary 4.4.** There exists a family of constants $\{\kappa'_p : 1 < p < 2\} \subset \mathbb{R}_+$ satisfying the following: if $x \in L_1(\mathcal{M}) \cap L_2(\mathcal{M})$, then there exist $y, z \in \cap_{1 < p < 2} L_p(\mathcal{M})$ such that

(i) $x = y + z$,
(ii) for every $1 < p < 2$, the following inequality holds:

$$\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^c} \leq \kappa'_p \|x\|_p$$

A direct alternative approach to Corollary 4.4 is to use the weak-type inequality for square functions from [42] (see also [37]) and then follow the argument used in the proof of Theorem 2.8. The above decomposition is related to another type of simultaneous decomposition considered by Junge and Perrin in [28, Theorem 3.3].

We note that Part I of the proof of Theorem 3.1 can be deduced from Proposition 4.3 and the noncommutative Burkholder-Gundy for symmetric spaces from [22] but we elect to present the more direct approach.

As a final remark, we provide an example involving Orlicz functions. Let $\Phi$ be an Orlicz function on $[0, \infty)$ i.e, a continuous increasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$. Two standard indices associated to the Orlicz function $\Phi$ are defined as follows: let

$$M_\Phi(t) = \sup_{s > 0} \frac{\Phi(ts)}{\Phi(s)}, \ t \in [0, \infty)$$
and
\[ p_\Phi := \lim_{t \to 0^+} \frac{\log M_\Phi(t)}{\log t}, \quad q_\Phi := \lim_{t \to \infty} \frac{\log M_\Phi(t)}{\log t}. \]
Then \( 1 \leq p_\Phi \leq q_\Phi \leq \infty \). The Orlicz space \( L_\Phi \) is the set of all Lebesgue measurable function \( f \) defined on \((0, \infty)\) such that for some constant \( c > 0 \),
\[ \int_0^\infty \Phi \left( \frac{|f(t)|}{c} \right) \, dt < \infty. \]
If we equip \( L_\Phi \) with the Luxemburg norm:
\[ \|f\|_{L_\Phi} = \inf \left\{ c > 0 : \int_0^\infty \Phi \left( \frac{|f(t)|}{c} \right) \, dt \leq 1 \right\}, \]
then \( L_\Phi \) is a symmetric Banach function space with the Fatou property. Moreover, the Boyd indices of \( L_\Phi \) coincide with the indices \( p_\Phi \) and \( q_\Phi \) (see for instance [35]). Thus, Theorem 3.1 and Corollary 4.1 apply to martingales in the noncommutative space \( L_\Phi(M) \) whenever \( 1 < p_\Phi \leq q_\Phi < 2 \) or \( 2 < p_\Phi \leq q_\Phi < \infty \). This example also motivates the consideration of the so-called \( \Phi \)-moment inequalities involving conditioned square functions. This direction will be explored in a forthcoming article. We refer to [3, 13, 15] for recent progress on moment inequalities for noncommutative martingales.

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