Underlying gauge symmetries of second-class constraints systems

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Gauge-invariant systems in unconstrained configuration and phase spaces, equivalent to second-class constraints systems upon a gauge-fixing, are discussed. A mathematical pendulum on an $n - 1$-dimensional sphere $S^{n-1}$ as an example of a mechanical second-class constraints system and the $O(n)$ non-linear sigma model as an example of a field theory under second-class constraints are discussed in details and quantized using the existence of underlying dilatation gauge symmetry and by solving the constraint equations explicitly. The underlying gauge symmetries involve, in general, velocity dependent gauge transformations and new auxiliary variables in extended configuration space. Systems under second-class holonomic constraints have gauge-invariant counterparts within original configuration and phase spaces. The Dirac’s supplementary conditions for wave functions of first-class constraints systems are formulated in terms of the Wigner functions which admit, as we show, a broad set of physically equivalent supplementary conditions. Their concrete form depends on the manner the Wigner functions are extrapolated from the constraint submanifolds into the whole phase space.

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I. INTRODUCTION

The specific feature of gauge theories is the occurrence of constraints which restrict the phase space of the system to a submanifold. A systematic study of the Hamiltonian formulation of gauge theories was made by Dirac \cite{dirac1, dirac2} who classified the constraints and developed the operator quantization schemes of the constraint Hamiltonian systems.

The Dirac’s theory of constraint systems was combined further with the Feynman path integral method. Faddeev \cite{faddeev} found an explicit measure on the constraint submanifolds, entering the path integrals. The Feynman diagram technique for non-abelian gauge theories was developed by Faddeev and Popov \cite{faddeev-popov}. The effective QCD Lagrangian involving ghost fields obeys the Becchi-Rouet-Stora (BRS) symmetry \cite{becchi-rouet-stora}. A completely general approach to quantization of gauge theories, in which the BRS transformation acquires an intrinsic meaning, is developed by Fradkin and his collaborators \cite{fradkin1, fradkin2}. An alternative symplectic scheme has been proposed by Faddeev and Jackiw \cite{faddeev-jackiw1, faddeev-jackiw2}. An introduction to quantization of the gauge theories with the use of the path integral method can be found in Refs. \cite{faddeev-popov, faddeev-jackiw1}. The path integral representation for the evolution operator satisfies the unitarity condition and meets requirements of the relativistic covariance.

According to the Dirac’s classification, constraint equations like $\Omega_A = 0$, appearing in the gauge theories, are of the first class. The Poisson bracket of first-class constraint functions is a linear combination of constraint functions

\[ \{\Omega_A, \Omega_B\} = C_{AB}^D \Omega_D \]  \hspace{1cm} (I.1)

where $C_{AB}^D$ are some structure functions. The quantization of gauge theories involves a set of gauge-fixing condition $\chi_A = 0$. The functions $\chi_A$ must be admissible, i.e., the Poisson bracket of the gauge-fixing and constraint functions is non-degenerate:

\[ \det\{\chi_A, \Omega_B\} \neq 0. \]  \hspace{1cm} (I.2)

The gauge-fixing functions are independent

\[ \{\chi_A, \chi_B\} = 0. \]  \hspace{1cm} (I.3)

Physical observables are gauge invariant and do not depend on the choice of $\chi_A$.

The wave functions satisfy the operator Dirac’s supplementary condition

\[ \hat{\Omega}_B \Psi = 0. \]  \hspace{1cm} (I.4)

Constraint functions $G_a$ of second-class constraints systems obey the Poisson bracket relations which form a non-degenerate matrix

\[ \det\{G_a, G_b\} \neq 0. \]  \hspace{1cm} (I.5)

The number of the constraints is even ($a = 1, \ldots, 2m$).

Among physical systems which belong to the second-class constraints systems are anomalous gauge theories \cite{fradkin1, fradkin2, fradkin3, fradkin4, fradkin5}, the $O(4)$ non-linear sigma model, which constitutes the lowest order of the chiral perturbation theory \cite{cheng-wu}, many-body systems involving collective and independent-particle degrees of freedom \cite{fujii}, and others.

The Dirac’s quantization method of such systems consists in constructing operators reproducing the Dirac bracket for canonical variables and taking constraints to
be operator equations of the corresponding quantum theory.

Batalin and Fradkin\(^[10]\) developed a quantization procedure for the second-class constraints systems, which converts constraints to the first class by introducing new canonical variables. The problem reduces thereby to the quantization of a first-class constraints system in an enlarged phase space. This method (see also\(^[20]\)) was found to be particularly useful for construction of the effective covariant Lagrangians in an extended configuration space\(^[21, 22]\).

The Hamilton-Jacobi scheme is also used for the quantization of constraint systems\(^[22, 23, 24, 25, 26]\).

Gauge systems are quantized by imposing gauge-fixing conditions. The initial system is reduced to a second-class constraints system. The evolution operator in the path representation can be written as\(^[3, 13]\)

\[
Z = \int \prod_i \frac{d^a q_i d^a p_i}{(2\pi\hbar)^n} \prod_a d(G_a) \sqrt{\det\{G_a, G_b\}} \\
\times \exp \left\{ \frac{i}{\hbar} \int dt (p_i \dot{q}_i - \mathcal{H}) \right\}
\]  

(1.6)

where \(\mathcal{H}\) is the Hamiltonian. The path integral representation\(^[16]\) allows not to solve the constraint equations explicitly and work in the unconstrained phase space.

The usefulness of the reduction of second-class constraints systems to equivalent gauged systems consists in getting the supplementary condition\(^[14]\) which is not evident otherwise.

The second-class constraints systems to which gauge systems are reduced obey specific requirements:

(i) The constraint functions \(G_a\) split naturally into canonically conjugate pairs \(G_a = (\chi_A, \Omega_B)\),

(ii) the wave functions satisfy Eq.\(^[14]\),

(iii) the gauge-fixing functions \(\chi_A\) are identically in involution Eq.\(^[12]\), and

(iv) the constraint functions \(\Omega_B\) associated to the gauge invariance are first class Eq.\(^[11]\).

As a consequence of Eq.\(^[12]\), the matrix \(\{\chi_A, \Omega_B\}\) is non-degenerate. The constraint functions are not defined uniquely. In particular, a linear transformations of \(G_a\) by a non-degenerate matrix leads to constraint functions \(G'_a\) describing the same constraint submanifold.

A question arises if constraints of an arbitrary second-class constraint systems can be redefined to fulfill (i) - (iv)?

This is the case for holonomic systems\(^[27]\). Such systems can be treated as being obtained upon a gauge-fixing of the corresponding gauge invariant systems. Within the generalized Hamiltonian formalism, constraints of holonomic systems are of the second class. In the Lagrangian formalism, the corresponding constraints do not depend on generalized velocities.

In this work, we review the underlying gauge symmetries of the holonomic systems and report new results connected to the quantization of more general second-class constraints systems.

The paper is organized as follows: In the next Section, we start from discussing a simple, but instructive example of a mechanical system under second-class holonomic constraints. The one-dimensional reduction of the \(O(n)\) non-linear sigma model is discussed, which is equivalent to a mathematical pendulum on \(n - 1\)-dimensional sphere in an \(n\)-dimensional Euclidean space. Lagrangian \(\mathcal{L}\) of the system is transformed on the constraint submanifold of the configuration space into an equivalent Lagrangian \(\mathcal{L}'\) to make explicit the appearance of an underlying gauge symmetry. The corresponding Hamiltonian, its constraints structure, and transformation properties are described in Sect. 3.

This example is analyzed further in Sect. 6 to construct a link with the Dirac’s quantization method.

In Sects. 4, 5, and 8, the quantization of systems under the second-class constraints is discussed.

The Poisson bracket of the constraint functions \(G_a\) forms a symplectic structure in the space of constraint functions. The corresponding symplectic basis is suitable for splitting the constraints into canonically conjugate pairs \((\chi_A, \Omega_A)\). In Sect. 4, an algorithm is proposed for constructing the functions \(\chi_A\) and \(\Omega_A\), describing the constraint submanifold of second-class constraints systems, for which the involution relations \(\{\chi_A, \chi_B\} = 0\), \(\{\Omega_A, \Omega_B\} = 0\), and \(\{\chi_A, \Omega_B\} = \delta_{AB}\) hold in the strong sense in an entire neighborhood of any fixed point of the constraint submanifold.

In Sect. 5, properties of the gradient projections into constraint submanifolds described by equations \(\chi_A = 0\), \(\Omega_A = 0\), and onto their intersection are discussed. The gradient projection is useful for calculation of the Dirac bracket and for constructing symplectic basis for the constraint functions.

The Dirac’s quantization of the model of Sect. 2 is performed in Sect. 6 and the equivalence with the results of Sect. 3 is established. In Appendix A, the model of a mathematical pendulum on \(n - 1\)-dimensional sphere is embedded further into a space of higher dimension to study the constraints structure and the associated gauge invariance. The equivalence with the results of Sects. 3 and 6 is proved.

In quantum mechanics, systems are described by wave functions or, equivalently, by density matrices or the Wigner functions. The Wigner functions are the most closely related to the classical probability densities in the phase spaces. In Sect. 8, we analyze constraints to be imposed on the Wigner functions within the framework of the generalized Hamiltonian dynamics. Such constraints are not unique and depend decisively on the way the Wigner functions are extrapolated from the constraint submanifold into the unconstrained phase space. The main result of this Section consists in demonstrating the fact that constraints imposed on the Wigner functions can be taken in a symmetric fashion with regard to permutations \(\chi_A \leftrightarrow \Omega_A, \chi_A \leftrightarrow -\Omega_A\) and, more generally, canonical transformations preserving the Poisson brackets. Those constraints are not symmetric in the Dirac’s
quantization scheme. We show that transition amplitudes are not affected by canonical transformations mixing the constraint functions.

In Sect. 9, the $O(4)$ non-linear sigma model is discussed as a field theory counterpart of the mathematical pendulum. We perform quantization in the straightforward way by solving the constraints from the outset and demonstrate the equivalence of the results with the method based on construction of the gauge-invariant model of the $O(4)$ non-linear sigma model. The Lagrange measure for the $O(4)$ non-linear sigma model, entering the path integral in the configuration space, is constructed and a parameterization is proposed for the pion fields, in terms of which the perturbation theory consistent with the mean field (MF) approximation can be developed. The covariance and the unitarity of the $S$-matrix are demonstrated.

The results are summarized in Sect.10.

II. GAUGED LAGRANGIAN FOR A SPHERICAL PENDULUM

We start from discussing a simple example of a mechanical system under holonomic constraints: A spherical pendulum on an $n−1$-dimensional sphere $S^{n−1}$ in $n$-dimensional Euclidean space. The trajectories of the system are determined as conditional extremals of the action functional $\mathcal{A} = \int L\,dt$ on the constraint submanifold $\chi = 0$, with

$$\chi = \ln \phi \quad \text{(II.1)}$$

where $\phi^2 = \phi^\alpha \phi^\alpha$ and $\alpha = 1, \ldots, n$. Lagrangian

$$L = \frac{1}{2} \phi^\alpha \dot{\phi}^\alpha \quad \text{(II.2)}$$

together with the constraint $\chi = 0$ defines a pointlike massive particle on an $n−1$-dimensional sphere $S^{n−1}$ as a mechanical analogue of the $O(n)$ non-linear sigma model. The constraint $\chi = 0$ implies

$$\phi^\alpha \dot{\phi}^\alpha = 0, \quad \text{(II.3)}$$

$$\dot{\phi}^\alpha \dot{\phi}^\alpha = \phi^\alpha \ddot{\phi}^\alpha = 0, \quad \text{(II.4)}$$

The dots stand for higher time derivatives of the constraint function $\chi$. Eq. (II.3) shows that the radial component of the velocity vanishes. The second equation shows that the radial component of the acceleration equals to the centrifugal force.

The equations of motion can be found using the D’Alembert-Lagrange variational principle for conditional extremals of the action functionals, or equivalently, the Euler-Lagrange variational principle for unconditional extremals with the constraints implemented through the Lagrange’s multipliers method.

Let us substitute $L \rightarrow L_1 = L + \lambda \chi$ and perform unconditional variations over the Lagrange’s multiplier $\lambda$ and the coordinates $\phi^\alpha$. One gets $\chi = 0$ and $\ddot{\phi}^\alpha = \lambda \phi^\alpha / \phi^2$. Multiplying further the last equation by $\phi^\alpha$ and substituting the result into Eq. (II.3), one gets $\lambda = -\phi^\alpha \ddot{\phi}^\alpha$. Given that the $\lambda$ is fixed from Eq. (II.3), the higher time derivatives of $\chi$ vanish identically. The radial component of the acceleration is determined by the constraint, while the tangent component of the acceleration vanishes. The equations of motion can be presented in the form

$$\Delta^{\alpha\beta}(\phi) \frac{d^2}{dt^2} (\phi^\beta / \phi) = 0 \quad \text{(II.5)}$$

where

$$\Delta^{\alpha\beta}(\phi) = \delta^{\alpha\beta} - \phi^\alpha \phi^\beta / \phi^2. \quad \text{(II.6)}$$

The tensor defined above obeys

$$\phi^\alpha \Delta^{\alpha\beta}(\phi) = 0, \quad \text{(II.7)}$$

$$\Delta^{\alpha\beta}(\phi) \Delta^{\beta\alpha}(\phi) = \Delta^{\alpha\alpha}(\phi). \quad \text{(II.8)}$$

It is invariant also with respect to dilatation

$$\Delta^{\alpha\beta}(\phi') = \Delta^{\alpha\beta}(\phi), \quad \text{(II.9)}$$

for

$$\phi^\alpha \rightarrow \phi'^\alpha = \exp(\theta) \phi^\alpha \quad \text{(II.10)}$$

where $\theta$ is an arbitrary parameter.

Eqs. (II.5) tell that the particle moves without tangent acceleration. In general, the acceleration orthogonal to a constraint submanifold $\chi = 0$ is fixed to keep the particle on it. In our case, the radial component of the acceleration is determined by the constraint. Eqs. (II.5) can also be derived using the D’Alembert-Lagrange variational principle.

On the submanifold $\chi = 0$, the radial components of the velocities are equal to zero Eq. (II.3), so one can replace $\dot{\phi}^\alpha$ by the tangent velocities $\Delta^{\alpha\beta}(\phi) \dot{\phi}^\beta$ in $L$.

The conditional extremals of the action functional $\mathcal{A} = \int L\,dt$ do not change also if we divide $L$ by $\phi^2 (= 1)$. The conditional variational problem for Lagrangian

$$L_* = \frac{1}{2} \Delta^{\alpha\beta}(\phi) \dot{\phi}^\alpha \dot{\phi}^\beta / \phi^2 \quad \text{(II.11)}$$

is thus completely equivalent to the conditional variational problem we started with.

The equations of motion (II.5) determine unconditional extremals of the action functional $\mathcal{A}_* = \int L_*\,dt$ on the configuration space $\mathcal{M} = (\phi^\alpha)$.

The extremals of the action functionals $\mathcal{A}$ and $\mathcal{A}_*$ under the constraint $\chi = 0$ coincide.

$\mathcal{A}_*$ depends on the spherical coordinates $\phi^\alpha / \phi$ which lie on an $n−1$-dimensional sphere of a unit radius. Eqs. (II.5) for unconditional extremals of the $\mathcal{A}_*$ in the coordinate space $\phi^\alpha$ coincide, as it should, with the D’Alembert-Lagrange equations for extremals of $\mathcal{A}_*$ in the space formed by the spherical coordinates under the
condition that they belong to an \(n - 1\)-dimensional sphere of a unit radius.

It is seen that Lagrangian (II.11) is invariant with respect to dilatation (II.10) where \(\theta\) is an arbitrary function of time. In the context of the dynamics defined by (II.11) with no constraints imposed, \(\phi\) turns out to be an arbitrary function of time. It can always be selected to fulfill the constraint \(\chi = 0\) or some other admissible constraint.

The constraint \(\chi = 0\) can therefore be treated as a gauge-fixing condition, the function \(\phi\) as a gauge degree of freedom, the ratios \(\delta^\alpha/\phi\) as gauge invariant observables.

The equations of motion (II.15) are formulated in terms of the gauge invariant observables.

The above gauge symmetry is defined ”on-shell”, i.e., for \(\dot{\phi}^\alpha\) treated as a time derivative of \(\phi^\alpha\). In the tangent bundle \(T^\ast M = (\phi^\alpha, \dot{\phi}^\alpha)\) where the coordinates and their derivatives are independent, \(L_*\) is invariant with respect to a two-parameter set of transformations:

\[
\phi^\alpha \to \phi'^\alpha = \exp(\theta)\phi^\alpha, \quad (\text{II.12}) \\
\dot{\phi}^\alpha \to \dot{\phi}'^\alpha = \exp(\theta)\dot{\phi}^\alpha \quad (\text{II.13})
\]

and

\[
\phi^\alpha \to \phi'^{\alpha} = \phi^\alpha, \quad (\text{II.14}) \\
\dot{\phi}^\alpha \to \dot{\phi}'^\alpha = \dot{\phi}^\alpha + \epsilon \phi^\alpha. \quad (\text{II.15})
\]

The last two equations describe the invariance under variation of the radial component of the particle velocities. If the \(\dot{\phi}^\alpha\) is treated on-shell, then \(\epsilon \sim \dot{\theta}\).

A point-like particle on an \(n - 1\)-dimensional sphere \(S^{n-1}\) has therefore underlying gauge symmetry connected with dilatation of the coordinates \(\phi^\alpha\). Its physical origin is simple: We allow virtual displacements from the constraint submanifold and treat such displacements as unphysical gauge degrees of freedom. The constraint equation \(\chi = 0\) turns thereby into a gauge-fixing condition. The physical variables are specified by projections of the coordinates \(\phi^\alpha\) onto an \(n - 1\)-dimensional sphere of a unit radius. Those projections are the spherical coordinates \(\phi^\alpha/\phi\).

Let us consider the system (II.11) within the generalized Hamiltonian framework.

III. \textbf{THE GAUGED SPHERICAL PENDULUM HAMILTONIAN}

Lagrangian (II.11) is defined outside of the constraint submanifold \(\chi = 0\) and invariant with respect to the dilatation (II.10). The value \(\phi\) is an arbitrary function and becomes a gauge degree of freedom. The equations of motion (II.15) are derived using the constraint \(\chi = 0\). Being formulated, the equations of motion do not depend, however, on the constraint anymore and allow an extension to the unconstrained configuration space \(M\). The invariance under the dilatation, Eq. (II.10), is a consequence of the two-parameter set of global symmetry transformations, Eqs. (II.12)-(II.15), of \(L_*\) as a function defined on the tangent bundle \(T^\ast M\).

One can consider therefore \(\mathcal{A}_*\) without imposing any constraints and treat the equation \(\chi = 0\) as a gauge-fixing condition.

A. Gauged spherical pendulum within generalized Hamiltonian framework

The canonical momenta corresponding to \(\dot{\phi}^\alpha\) are defined by

\[
\pi^\alpha = \frac{\partial L_*}{\partial \dot{\phi}^\alpha} = \Delta^{\alpha\beta}(\phi)\dot{\phi}^\beta / \phi^2. \quad (\text{III.1})
\]

They satisfy constraints

\[
\pi^\alpha - \Delta^{\alpha\beta}(\phi)\pi^\beta \approx 0 \quad (\text{III.2})
\]

which are equivalent to the one primary constraint:

\[
\Omega = \phi \pi \approx 0. \quad (\text{III.3})
\]

The primary Hamiltonian can be obtained with the use of the Legendre transform:

\[
\mathcal{H} = \frac{1}{2} \phi^2 \Delta^{\alpha\beta}(\phi)\pi^\alpha \pi^\beta. \quad (\text{III.4})
\]

For \(n = 3\), \(\mathcal{H}\) is proportional to the orbital momentum squared. The non-vanishing Poisson bracket relations for the canonical variables are defined by

\[
\{\phi^\alpha, \pi^\beta\} = \delta^{\alpha\beta}. \quad (\text{III.5})
\]

The constraint function \(\Omega\) is stable with respect to the time evolution:

\[
\{\Omega, \mathcal{H}\} = 0. \quad (\text{III.6})
\]

The relations

\[
\{\phi^\alpha, \Omega\} = \phi^\alpha, \quad (\text{III.7}) \\
\{\pi^\alpha, \Omega\} = -\pi^\alpha \quad (\text{III.8})
\]

show that \(\Omega\) generates dilatation of \(\phi^\alpha\) and \(\pi^\alpha\). The transformation law for the canonical coordinates Eq. (III.7) is in agreement with Eq. (II.12) in its infinitesimal form. The transformation law for the canonical momenta, which follows from Eqs. (II.12), (II.13), and (II.14),

\[
\pi^\alpha \to \pi'^\alpha = \exp(-\theta)\pi^\alpha, \quad (\text{III.9})
\]

considered in its infinitesimal form, is in agreement with Eq. (III.5) either. The Hamiltonian \(\mathcal{H}\) is gauge invariant under the dilatation.

The roles of the gauge-fixing function \(\chi\) and the gauge generator \(\Omega\) are similar. The function \(\chi\) is identically in involution with the Hamiltonian:

\[
\{\chi, \mathcal{H}\} = 0. \quad (\text{III.10})
\]
The Poisson bracket relations
\[
\{\phi^\alpha, \chi\} = 0, \quad (\text{III.11})
\]
\[
\{\pi^\alpha, \chi\} = -\phi^\alpha \quad (\text{III.12})
\]
define the one-parameter set of transformations with respect to which \(\mathcal{H}\) is invariant. The function \(\chi\) generates shifts of the longitudinal component of the canonical momenta. This symmetry is connected to the invariance of \(\mathcal{L}_x\) described by Eqs. (II.13) and (II.15).

The gauge-fixing condition \(\chi = 0\) is admissible:
\[
\{\chi, \Omega\} = 1. \quad (\text{III.13})
\]

The equations of motion generated by the primary Hamiltonian look like
\[
\dot{\phi}^\alpha = \{\phi^\alpha, \mathcal{H}\} = \phi^\beta \Delta^\alpha_\beta (\phi) \pi^\beta, \quad (\text{III.14})
\]
\[
\dot{\pi}^\alpha = \{\pi^\alpha, \mathcal{H}\} = -\phi^\beta \Delta^\beta_\gamma (\phi) \pi^\beta \pi^\gamma. \quad (\text{III.15})
\]

### B. Quantization of spherical pendulum

The quantization of a mathematical pendulum on an \(n-1\)-dimensional sphere is discussed in Ref. [22] where the standard Batalin-Fradkin-Tyutin (BFT) algorithm [19, 20] for second-class constraints systems is applied. The second-class constraints appear if one starts directly from \(\mathcal{L}\) and formulate the conditional variational problem for \(\chi = 0\). By constructing auxiliary fields, it is possible to pass over to an equivalent first-class constraint system.

In our approach, we start from \(\mathcal{A}\), which is gauge invariant explicitly, so the constraints appear to be of the first class from the start. One can therefore quantize the pendulum as a gauge-invariant system without introducing auxiliary fields.

From the point of view of the generalized Hamiltonian framework, the gauge-fixing conditions and the gauge generators play similar roles. The function \(\chi\) does, however, not generate transformations in \(T^*\mathcal{M}\), so it appears just as a candidate for gauge-fixing function. If we pass to the Lagrangian framework, we can verify that \(\chi = 0\) is the gauge-fixing condition indeed.

The system has the gauge invariance described by the generator \(\Omega\) and the admissible gauge-fixing condition \(\chi = 0\).

The standard procedure for gauge theories can therefore be applied for quantization of the spherical pendulum.

The system is quantized by the algebra mapping \((\phi^\alpha, \pi^\alpha) \to (\hat{\phi}^\alpha, \hat{\pi}^\alpha)\) and \(\{,\} \to -i/\hbar [,,]\). Consequently, to any symmetrized function in the phase space variables one may associate an operator function. The function is symmetrized in such a way that quantal image is a hermitian operator.

The quantum hermitian Hamiltonian has the form
\[
\hat{\mathcal{H}} = \frac{1}{2} \phi^\beta \Delta^\alpha_\beta \hat{\pi}^\beta \phi^\gamma \pi^\gamma. \quad (\text{III.16})
\]

The vector \(i\phi \Delta^\alpha_\beta \hat{\pi}^\beta\) gives the angular part of the gradient operator. Although it is not conspicious, \(\mathcal{H}\) does not depend on the radial coordinate \(\phi\). The constraint operator can be defined as
\[
\hat{\Omega} = (\phi^\alpha \hat{\pi}^\alpha + \hat{\pi}^\alpha \phi^\alpha)/2. \quad (\text{III.17})
\]

It acts only on the radial component of \(\phi^\alpha\), so the relation
\[
[\hat{\Omega}, \hat{\mathcal{H}}] = 0 \quad (\text{III.18})
\]
holds.

The physical subspace of the Hilbert space is singled out by imposing the Dirac’s supplementary condition
\[
\hat{\Omega} \Psi = 0. \quad (\text{III.19})
\]

This condition implies
\[
\Psi = \phi^{-n/2} \Psi_1 (\phi^\alpha / \phi). \quad (\text{III.20})
\]

The physical information is contained in \(\Psi_1 (\phi^\alpha / \phi)\).

The path integral for the evolution operator becomes
\[
Z = \int \prod_t d\phi^\alpha d\pi^\alpha (2\pi\hbar)^{n-1} \delta(\chi) \delta(\Omega) \exp \left\{ \frac{i}{\hbar} \int dt (\pi^\alpha \dot{\phi}^\alpha - \mathcal{H}) \right\}. \quad (\text{III.21})
\]

Eqs. (III.19) and (III.21) solve the quantization problem for a mathematical pendulum on an \(n-1\)-dimensional sphere \(S^{n-1}\).

It is desirable that the quantization procedure does not destroy the classical symmetries which results in having the supplementary condition (III.19) satisfied by the state \(\Psi(t)\) for any value of \(t\). This feature can, in general, be violated either due to complex terms entering the Hamiltonian or by approximations adopted for treating the operator eigenvalues.

Given the initial state wave function \(\Psi(0)\) satisfying (III.19), the final wave function \(\Psi(t)\) can be found applying the evolution operator (III.21) on \(\Psi(0)\). Since \(\hat{\Omega}\) commute with the Hamiltonian, the final state wave function obeys Eqs. (III.19).

In the standard canonical frame [33], the first canonical coordinates and momenta are identical with the constraint functions. If the Poisson bracket of the Hamiltonian with the constraint functions vanishes, the Hamiltonian is independent of the first canonical coordinates and momenta. It means that the quantized Hamiltonian is commutative with operators associated to the constraint functions.

By initiating the quantization procedure in the standard canonical frame, one gets the classical symmetries preserved and the Dirac’s supplementary condition fulfilled at any time.

The fact that the dilatation symmetry for the spherical pendulum is preserved on the quantum level is connected with the fact that the canonical frame we used is simply related to the standard canonical frame.
The integral over the canonical momenta Eq. (III.21) can be simplified to give
\[ Z = \int \prod_i \sqrt{(\partial \chi / \partial \phi^i)^2} \delta(\chi)d\phi \exp \left( \frac{i}{\hbar} \int dt L \right). \] (III.22)
Lagrange's measure \( \sqrt{(\partial \chi / \partial \phi^i)^2} \delta(\chi)d\phi \) coincides with the volume element of \( S^{n-1} \) sphere. It can be rewritten as an invariant volume of the configuration space, e.g., in terms of the angular variables \( (\varphi_1 \ldots \varphi_{n-1}) \) with the help of the induced metric tensor. It is invariant under \( O(n) \) rotations and remains the same for all functions \( \chi \) vanishing at \( \phi = 1 \). For \( n = 4 \), Lagrange's measure matches Haar's measure of the group \( SU(2) \).

In gauge theories, the evolution operator is independent of the gauge-fixing conditions \[ ]\). We can insert \( \det{\{\chi, \Omega\}} = 1 \) into the integrand of Eq. (III.22) to bring the measure into the form identical with gauge theories. In the path integral, \( \chi \) can then be replaced with an arbitrary function. The condition (III.19) does not comprise the constraint \( \chi = 0 \) also.

It is remarkable that physical observables depend on \[ \frac{1}{2} \] of the number of the second-class constraints.

The main question to be raised here is if the quantization method described above is general enough or specific only for the spherical pendulum? In Sect. 7 we show that such a method works for mechanical systems and field theories under holonomic constraints. The quantization of more general second-class constraints is discussed in Sect. 8.

### IV. LOCAL SYMPLECTIC BASIS FOR SECOND-CLASS CONSTRAINTS FUNCTIONS

Two sets of the constraint functions are equivalent if they describe the same constraint submanifold. The Hamiltonian function admits transformations which do not change its value and its first derivatives on the constraint submanifold. This allows to make transformations of the constraint and Hamiltonian functions without changing the physical content of theory.

The second-class constraints satisfy \( \det{\{G_a, G_b\}} \neq 0 \). The Poisson bracket defines therefore a non-degenerate symplectic linear structure in the vector space of the constraint functions \( G_a \). Indeed, any linear transformation \( G'_a = \mathcal{O}_{ab} G_b \) with matrix \( \mathcal{O}_{ab} \) depending on the canonical variables transforms accordingly the Poisson bracket: \( \{G'_a, G'_b\} \approx \mathcal{O}_{ac} \mathcal{O}_{bd} \{G_c, G_d\} \). The Poisson bracket plays thereby a role of a skew-scalar product in the symplectic vector space of the constraint functions. Every symplectic space has a symplectic basis (see, e.g., \[ ]\), so the constraint functions can be brought by linear transformations into the form
\[ \{G_a, G_b\} \approx \mathcal{I}_{ab} \] (IV.1)
where
\[ \|\mathcal{I}_{ab}\| = \left\| \begin{array}{cc} 0 & E_m \\ -E_m & 0 \end{array} \right\|, \] (IV.2)
with \( E_m \) being the \( m \times m \) identity matrix. \( \mathcal{I}_{ab} \mathcal{I}_{bc} = -\delta_{ac} \). Using representation \( G_a = (\chi_A, \Omega_A) \), one has
\[ \{\Omega_A, \Omega_B\} \approx 0, \] (IV.3)
\[ \{\chi_A, \Omega_B\} \approx 0, \] (IV.4)
\[ \{\chi_A, \Omega_B\} \approx \delta_{AB}. \] (IV.5)

This basis is not unique. Indeed, there remains a group of symplectic transformations \( Sp(2m) \), which keeps the Poisson bracket of the constraint functions in the symplectic form.

At any given point of a neighborhood of the constraint submanifold one can find symplectic basis for second-class constraints functions in the weak form.

This result can be strengthened as shown below:

(A) If \( \xi \) is close enough to the submanifold, one has \( \det{\{\chi_A(\xi), \Omega_B(\xi)\}} \neq 0 \). By continuity there exists a finite neighborhood \( \Delta_\xi \) of \( \xi \), such that \( \det{\{\chi_A(\xi'), \Omega_B(\xi')\}} = 0 \) for all \( \xi' \in \Delta_\xi \). Let us assume that the intersection of \( \Delta_\xi \) with the constraint submanifold is not empty, i.e., one can find \( \xi_1 \in \Delta_\xi \) such that \( G_a(\xi_1) = 0 \). If it is not fulfilled, we start from another \( \xi \) closer to the constraint submanifold.

Let us chose an equivalent set of the constraint functions \( \chi_A \to \chi'_A = \{\chi_A, \Omega_B\}^{-1} \chi_B \) to ensure \( \{\chi'_A, \Omega_B\} \approx \delta_{AB} \) in the region \( \Delta_\xi \).

(B) We replace further \( \Omega_A \to \Omega'_A = \Omega_A - \frac{1}{2} C_{AB} \chi_B' \) to get equations
\[ \{\Omega'_A, \Omega'_B\} \approx 0, \] (IV.6)
These equations can be solved for matrix \( C_{AB} \) in terms of a power series of the matrix \( \{\chi'_D, \chi'_F\} \);
\[ C_{AB} = \sum_{k=0}^{\infty} C^{[k]}_{AB} \] (IV.7)
where \( C^{[k]}_{AB} = -C^{[k]}_{BA} \) and
\[ C^{[0]}_{AB} = \{\Omega_A, \Omega_B\}, \]
\[ C^{[k+1]}_{AB} = \frac{1}{4} \sum_j C^{[k-j]}_{AD} C^{[j]}_{BF} \{\chi'_D, \chi'_F\}, \]
with \( k = 0, 1, \ldots \).

(C) The transform \( \chi_A \to \chi''_A = \{\chi''_A, \Omega_B\}^{-1} \chi_B'' \) brings back the Poisson bracket \( \{\chi''_A, \Omega_B\} \) to the diagonal form \( \delta_{AB} \).

(D) The last transform looks like \( \chi''_A \to \chi'''_A = \chi''_A - \frac{1}{2} \{\chi''_A, \chi''_B\} \Omega''_B \). It provides \( \{\chi'''_A, \chi'''_B\} \approx 0 \) and keeps the relation \( \{\chi''_A, \Omega_B\} \approx \delta_{AB} \) unchanged.

Now, we remove primes from the notations. As a result of the steps (A)-(D), we obtain weak equations
\[ \{G_a(\xi), G_b(\xi)\} \approx \mathcal{I}_{ab} \] (IV.8)
∀ξ ∈ Δξ. It is manifest that Eq. (IV.9) is valid in some neighborhood of any point ξ₁ of the constraint submanifold too.

The symplectic basis for second-class constraints functions in the weak form exists in an entire neighborhood of any given point of the constraint submanifold.

The existence of the local symplectic basis in the weak form is on the line with the Darboux’s theorem (see, e.g., [33]) which states that around every point ξ in a symplectic space there exists a coordinate system in Δξ such that ξ ∈ Δξ where the symplectic structure takes the standard canonical form. The symplectic space can be covered by such coordinate systems.

This is in sharp contrast to the situation in Riemannian geometry where the metric at any given point x can always be made Minkowskian, but in any neighborhood of x the variance of the Riemannian metric with the Minkowskian metric is, in general, ∼ Δx². In other words, by passing to an inertial coordinate frame one can remove gravitation fields at any given point, but not in an entire neighborhood of that point. The Darboux’s theorem states, reversely, that the symplectic structure can be made to take the standard canonical form in an entire neighborhood Δξ of any point ξ ∈ Δξ. In Riemannian spaces, locally means at some given point. In symplectic spaces, locally means at some given point and in an entire neighborhood of that point.

Locally, all symplectic spaces are indistinguishable. Any submanifold in a symplectic space, including any constraint submanifold, is a plane. The possibility of finding the standard canonical frame [34] illustrates this circumstance.

In the view of this marked dissimilarity, the validity of Eqs. (IV.9) in a finite domain looks indespensible.

The global symplectic basis exists apparently for m = 1 as, e.g., in the case of the spherical pendulum. The global existence of the basis [IV.9] has been proved for systems with one primary constraint [26] and also, assuming that det{χA, ΩB} 0 holds globally [30].

Admissible transformations on the second-class constraints functions allow to bring Eqs. (IV.9) into a strong form. The Hamiltonian can also be modified to convert the Poisson bracket relations with the second-class constraints functions into the strong form without changing the dynamics. The arguments given below follow closely our discussion of holonomic systems [27]:

According to Eqs. (IV.9), at any given point of the constraint submanifold where det{Gₐ, Gₙ} 0 one can select a symplectic basis in which the constraint functions satisfy

\[ \{Gₐ, Gₙ\} = Iₐₙ + Cₐₙ Gₖ \] (IV.9)

in an entire neighborhood of that point. The first-class Hamiltonian H has relations

\[ \{Gₐ, H\} = Rₐₙ Gₙ. \] (IV.10)

The Jacobi identities for Gₐ, Gₙ, and Gₖ and for Gₐ, Gₙ, and H imply

\[ Cₐₙ + Cₙₐ + Cₐₖ Gₖ = 0, \] (IV.11)

\[ Rₐₙ - Rₙₐ = 0. \] (IV.12)

where Cₐₙ = IₐₙCₖₙ and Rₐₙ = IₐₙRₖₙ.

Let us define

\[ G'ₐ = Gₐ + \frac{3}{4} Cₐ₉ g₉ Gₖ, \] (IV.13)

\[ H' = H - \frac{1}{2} Rₐₙ Gₙ Gₖ. \] (IV.14)

The constraint functions Gₐ and G'ₐ coincide on and in vicinity of the constraint submanifold up to the second order in Gₐ, and generate accordingly identical phase flows on the submanifold Gₐ = 0. The Hamiltonian functions H and H' coincide on and in vicinity of the constraint submanifold up to the second order in Gₐ and generate on the submanifold Gₐ = 0 identical Hamiltonian phase flows.

Using Eqs. (IV.11) and (IV.12) one gets

\[ \{G'ₐ, G'ₙ\} = Iₐₙ + Cₐₙ Gₖ Gₖ, \] (IV.15)

\[ \{G'ₐ, H'\} = Rₐₙ Gₖ Gₖ. \] (IV.16)

where Cₐₙ and Rₐₙ are new structure functions. The first-order terms in Gₐ do not appear in the right sides of these equations. The pair (G'ₐ, H') describes the same Hamiltonian dynamics as (Gₐ, H), being at the same time in a stronger involution around the constraint submanifold.

The above procedure can be repeated to remove the quadratic terms in Gₐ, cubic terms in Gₐ, etc. In general, assuming

\[ \{G'ₐ, G'ₙ\} = Iₐₙ + Cₐₙ Gₖ Gₖ Gₖ [Gₖ, Gₖ], \] (IV.17)

\[ \{G'ₐ, H'\} = Rₐₙ Gₖ Gₖ Gₖ [Gₖ, Gₖ]. \] (IV.18)

we get as a consequence of the Jacobi identities and of the symmetry of structure functions Cₐₖν...κₙ and Rₐₖν...κₙ with respect to the upper indices:

\[ Cₐₙκₜ...κₚ + Cₙₜκₖ...κₚ + Cₙₚκₜ...κₖ ≈ 0, \] (IV.19)

\[ Rₐₙκₜ...κₚ - Rₙₜκₖ...κₚ ≈ 0. \] (IV.20)

The next-order constraint functions and the Hamiltonian are given by

\[ G'ₐ[k+1] = G'ₐ[k] + \frac{1}{k + 2} Cₐₙκₜ...κₙ Gₖ Gₖ [Gₖ, Gₖ], \] (IV.21)

\[ H'[^{k+1}] = H'[k] - \frac{1}{k + 1} Rₐₙκₜ...κₙ Gₖ Gₖ Gₖ [Gₖ, Gₖ]. \] (IV.22)

Using Eqs. (IV.17) and (IV.18), one can calculate the next-order structure functions Cₐₙκₜ...κₚ and Rₐₙκₜ...κₚ again and repeat the procedure. If structure functions vanish, we shift k by one unit and check Eqs. (IV.11) and (IV.12) again. At each step, the Poisson bracket relations get
closer to the normal form. In the limit \( k \to +\infty \), we obtain

\[
\{ \tilde{G}_a, \tilde{G}_b \} = \mathcal{I}_{ab}, \quad (IV.23)
\]

\[
\{ \tilde{G}_a, \tilde{H} \} = 0 \quad (IV.24)
\]

where \( \tilde{G}_a \lim_{k \to -\infty} G_a^{[k]} \) and \( \tilde{H} = \lim_{k \to -\infty} \mathcal{H}^{[k]} \). The matrix \( \mathcal{I}_{ab} \) defines splitting of the constraints into two groups \((\tilde{\chi}_A, \tilde{\Omega}_A)\), such that

\[
\{ \tilde{\chi}_A, \tilde{\chi}_B \} = 0, \quad (IV.25)
\]

\[
\{ \tilde{\Omega}_A, \tilde{\Omega}_B \} = 0, \quad (IV.26)
\]

\[
\{ \tilde{\chi}_A, \tilde{\Omega}_B \} = \delta_{AB}. \quad (IV.27)
\]

The progress made consists in extending the validity of Eqs. (IV.1) from one point into its neighborhood Eqs. (IV.28), and further, in passing from the weak to the strong form of the Poisson bracket relations Eqs. (IV.29).

The symplectic basis for second-class constraints functions exists in the strong form in an entire neighborhood of any given point of the constraint submanifold. There exists a Hamiltonian function \( \tilde{\mathcal{H}} \), describing the same dynamics on the constraint submanifold as the initial Hamiltonian function \( \mathcal{H} \), which is identically in involution with the constraint functions.

An independent geometric-based construction of \( \tilde{\mathcal{G}}_a \) and \( \tilde{\mathcal{H}} \) is given in Sect. 5. A similar local basis exists for first-class constraints \[10, 32\]. The arguments of Refs. \[10, 32\] apply to second-class constraints under special restrictions which are discussed in Appendix B.

For systems of pointlike particles under holonomic constraints, Eqs. (IV.9) hold globally, so \( \tilde{\mathcal{G}}_a \) and \( \tilde{\mathcal{H}} \) exist globally also. Furthermore, \( \tilde{\Omega}_A \) is linear in the canonical momenta, \( \tilde{\chi}_B \) does not depend on the canonical momenta, and \( \tilde{\mathcal{H}} \) splits into a sum of a kinetic energy term quadratic in the canonical momenta and a potential energy term depending on the canonical coordinates \[27\].

\[ \xi \]

V. GRADIENT PROJECTION

The concept of the gradient projection is useful for applications. It defines functions \( \xi_s(\xi) \) which project an arbitrary point \( \xi \) of the phase space onto a submanifold \( \mathcal{G}_a = 0 \) \((a = 1, \ldots, 2m)\) of the phase space along phase flows generated by the constraint functions \( \mathcal{G}_a \).

A. Full gradient projection

Let \( \mathcal{G}_a = 0 \) be second-class constraints, \( \det \{ \mathcal{G}_a, \mathcal{G}_b \} \neq 0 \), \( \xi^i (i = 1, \ldots, 2n) \) are canonical variables. In vicinity of the submanifold \( \mathcal{G}_a = 0 \), the projections can be constructed explicitly. Near the constraint submanifold, one can write \( \xi_s(\xi) = \xi + \{ \xi, \mathcal{G}_a \}\lambda_a \). The small parameters \( \lambda_a \) are determined by requiring \( \mathcal{G}_a(\xi_s(\xi)) = 0 \) to the first order in \( \mathcal{G}_a \). We get

\[
\xi_s(\xi) = \xi - \{ \xi, \mathcal{G}_a \} \{ \mathcal{G}_a, \mathcal{G}_b \}^{-1} \mathcal{G}_b. \quad (V.1)
\]

It is seen that \( \{ \mathcal{G}_a, \xi_s(\xi) \} = 0 \), consequently \( \{ \mathcal{G}_a, f(\xi_s(\xi)) \} = 0 \) for any function \( f \). This is natural, since \( \mathcal{G}_a \) generate phase flows along which the projections \( \xi_s(\xi) \) have been constructed.

The reciprocal statement is also true: If \( \{ \mathcal{G}_a, f \} = 0 \) for all \( \mathcal{G}_a \), then \( f = f(\xi_s(\xi)) \). Indeed, the coordinates on the constraint submanifold can be parameterized by \( \xi_s \). The coordinates describing shifts from the constraint submanifold can be parameterized by \( \mathcal{G}_a \). The functions \( f \) can in general be written as \( f = f(\xi_s, \mathcal{G}_a) \). If the Poisson brackets of \( f \) with all \( \mathcal{G}_a \) vanish, \( f \) depends on \( \xi_s \) only.

This can be summarized by

\[
\{ \mathcal{G}_a, f \} = 0 \leftrightarrow f = f(\xi_s(\xi)). \quad (V.2)
\]

Beyond the lowest order in \( \mathcal{G}_a \), the operation is unique provided the phase flows commute. This is always the case for the constraint functions taken to accomplish \[10, 32\] in a finite neighborhood of the constraint submanifold.

B. Partial gradient projection

Let \( \mathcal{G}_a \) split into \( \chi_A \) and \( \Omega_A \). We wish to construct functions \( \xi_u(\xi) \) which project an arbitrary point \( \xi \) of the phase space onto the gauge-fixing surface \( \chi_A = 0 \) with the use of the constraint functions \( \Omega_A \) associated to gauge transformations. In vicinity of the submanifold, one can write \( \xi_u(\xi) = \xi + \{ \chi, \Omega_A \}\lambda_A \). To the first order in \( \chi_A \), the parameters \( \lambda_A \) can be found from equation \( \chi_A(\xi_u(\xi)) = 0 \):

\[
\xi_u(\xi) = \xi - \{ \xi, \Omega_A \} \{ \chi_A, \Omega_B \}^{-1} \chi_B. \quad (V.3)
\]

The projection is made along the phase flows of \( \Omega_A \), so \( \{ \Omega_A, f(\xi_u(\xi)) \} = 0 \) for any function \( f \).

The reverse statement is also true. We write \( f = f(\xi_u, \chi_A) \) and conclude from \{ \Omega_A, f \} = 0 that the dependence on \( \chi_A \) drops out.

It can be summarized as follows:

\[
\{ \Omega_A, f \} = 0 \leftrightarrow f = f(\xi_u(\xi)). \quad (V.4)
\]

The second partial gradient projection can be made onto the submanifold \( \Omega_A = 0 \) with the use of the constraint functions \( \chi_A \). The result is similar to Eq. (V.3)

\[
\xi_v(\xi) = \xi + \{ \chi, \chi_A \} \{ \chi_A, \Omega_B \}^{-1} \Omega_B. \quad (V.5)
\]

The relation \( \{ \chi_A, f(\xi_v(\xi)) \} = 0 \) is valid for any function \( f \). Furthermore,

\[
\{ \chi_A, f \} = 0 \leftrightarrow f = f(\xi_v(\xi)). \quad (V.6)
\]
Combining the partial projections, e.g., \( \xi_s(\xi) = \xi_s(\xi_u(\xi)) \), one gets the full gradient projection. To the first order in \( G_\alpha \), the order in which the partial projections are applied does not matter, so \( \xi_s(\xi_u(\xi)) = \xi_s(\xi_u(\xi)) \). The full gradient projection constructed in this way coincides with that given by Eq. \( \text{(V.1)} \).

C. Example: Gauged spherical pendulum

The constraint function \( \Omega \) can be used to bring the vector \( \phi^\alpha \) onto a sphere of a unit radius \( \phi = 1 \) (\( \chi = 0 \)). Such a transformation has a meaning of a gradient projection. The functions \( \phi^\alpha_u \) and \( \pi^\alpha_u \) can be constructed in terms of the variables \( \phi^\alpha \) and \( \pi^\alpha \):

\[
\phi^\alpha_u = \exp(\theta)\phi^\alpha, \quad \pi^\alpha_u = \exp(-\theta)\pi^\alpha. \tag{V.7, V.8}
\]

The condition \( \chi(\phi_u, \pi_u) = 0 \) gives \( \exp(\theta) = 1/\phi \).

The Poisson bracket relations for the projected variables \( \text{(V.7)} \) and \( \text{(V.8)} \) can be found to be

\[
\{\phi^\alpha_u, \phi^\beta_u\} = 0, \quad \{\phi^\alpha_u, \pi^\beta_u\} = \Delta^\alpha\beta(\phi_u), \quad \{\pi^\alpha_u, \pi^\beta_u\} = \phi^\alpha_u\pi^\beta_u - \phi^\beta_u\pi^\alpha_u. \tag{V.9, V.10, V.11}
\]

where \( \Delta^\alpha\beta(\phi_u) = \delta^\alpha\beta - \phi^\alpha\delta^\beta\).

The following properties are worthy of mention:

(i) The Poisson bracket of the projected canonical variables coincides with the Dirac bracket associated to the constraints \( \chi = 0 \) and \( \Omega = 0 \) on the submanifold \( \chi = 0 \).

The Dirac bracket for the canonical variables can be calculated using Eq. \( \text{(V.26)} \) to give

\[
\{\phi^\alpha, \phi^\beta\}_D = 0, \quad \{\phi^\alpha, \pi^\beta\}_D = \Delta^\alpha\beta(\phi), \quad \{\pi^\alpha, \pi^\beta\}_D = (\phi^\beta\pi^\alpha - \phi^\alpha\pi^\beta)/\phi^2. \tag{V.12, V.13, V.14}
\]

The right sides of Eqs. \( \text{(V.9)} - \text{(V.11)} \) are reproduced at \( \chi = 0 \).

(ii) The relations \( \text{(V.9)} - \text{(V.11)} \) define a Poisson algebra in the space of functions \( f(\phi^\alpha_u, \pi^\beta_u) \) depending on the projected canonical variables, so they can be used to generate consistently a Hamiltonian phase flow on the constraint submanifold \( \chi = 0 \).

(iii) The Hamiltonian function

\[
\mathcal{H} = \frac{1}{2}\Delta^\alpha\beta(\phi_u)\pi^\alpha_u\pi^\beta_u. \tag{V.15}
\]

coincides with Eq. \( \text{(III.4)} \): \( \mathcal{H} = \mathcal{H}(\phi^\alpha_u, \pi^\beta_u) = \mathcal{H}(\phi^\alpha, \pi^\alpha) \).

The Hamiltonian \( \mathcal{H} = \mathcal{H}(\phi^\alpha_u, \pi^\beta_u) \) is thus the function of the projected variables \( \xi_u(\xi) \). It can be defined first on the submanifold \( \chi = 0 \) and then extended to the unconstrained phase space using the gradient projection parallel to the phase flow associated to \( \Omega \). The relation \( \{\Omega, \mathcal{H}\} = 0 \), Eq. \( \text{III.12} \), is the necessary and sufficient condition (see Eq. \( \text{III.1} \)) for \( \mathcal{H} \) to be a function of a fewer number of variables \( \mathcal{H} = \mathcal{H}(\phi^\alpha_u, \pi^\beta_u) \).

Let us consider the gradient projection onto the submanifold \( \Omega = 0 \) using the constraint function \( \chi \). The constraint \( \chi = 0 \) is responsible for shifts of the longitudinal component of the canonical momenta Eqs. \( \text{III.11} \) and \( \text{III.13} \). The functions \( \phi^\alpha_u \) and \( \pi^\beta_u \) have the form

\[
\phi^\alpha_u = \phi^\alpha, \quad \pi^\alpha_u = \pi^\alpha - \phi^\alpha\phi\pi/\phi^2. \tag{V.16, V.17}
\]

Equation \( \Omega(\phi_u, \pi_u) = 0 \) is fulfilled identically.

The Poisson bracket relations for the projected variables \( \text{V.16} \) and \( \text{V.17} \) can be found to be

\[
\{\phi^\alpha_u, \phi^\beta_u\} = 0, \quad \{\phi^\alpha_u, \pi^\beta_u\} = \Delta^\alpha\beta(\phi_u), \quad \{\pi^\alpha_u, \pi^\beta_u\} = (\phi^\beta_u\pi^\alpha_u - \phi^\alpha_u\pi^\beta_u)/\phi_u^2. \tag{V.18, V.19, V.20}
\]

One can see again:

(i) The Poisson bracket of the projected variables coincides with the Dirac bracket Eqs. \( \text{V.12} - \text{V.15} \) associated to the constraints \( \chi = 0 \) and \( \Omega = 0 \) on the submanifold \( \Omega = 0 \).

(ii) The Poisson bracket relations \( \text{V.18} - \text{V.20} \) are closed and define thereby a Poisson algebra in the space of functions depending on the projected canonical variables \( (\phi^\alpha_u, \pi^\beta_u) \).

(iii) The Hamiltonian function

\[
\mathcal{H} = \frac{1}{2}\phi^2\Delta^\alpha\beta(\phi_u)\pi^\alpha_u\pi^\beta_u. \tag{V.21}
\]

coincides with Eq. \( \text{III.13} \): \( \mathcal{H} = \mathcal{H}(\phi^\alpha_u, \pi^\beta_u) = \mathcal{H}(\phi^\alpha, \pi^\alpha) \).

\( \mathcal{H} \) given by Eq. \( \text{III.13} \) is the function of the gradient variables \( \xi_u(\xi) \). The relation \( \{\chi, \mathcal{H}\} = 0 \), Eq. \( \text{III.10} \), is the necessary and sufficient condition to present the Hamiltonian function as a function the projected variables: \( \mathcal{H} = \mathcal{H}(\phi^\alpha_u, \pi^\beta_u) \).

It is clear that the \( \mathcal{H} \) is defined finally on the intersection of the submanifolds \( \chi = 0 \) and \( \Omega = 0 \), being thus a function of the \( \xi_u(\xi) \). Eq. \( \text{III.13} \) represents its extension to the unconstrained phase space using the full gradient projection.

The statements (i) - (iii) are of the general validity for gradient projections. The statement (iii) has been proved as such above, the other two ones are proved below.

D. Dirac bracket calculated by gradient projection

The phase flow associated to a function \( g = g(\xi) \) defined on the submanifold \( G_\alpha = 0 \) has an ambiguity since one can add to \( g = g(\xi) \) a linear combination of the constraints \( \lambda_\alpha G_\alpha \) where \( \lambda_\alpha \) are undetermined parameters. The phase flow \( L_g[f] \) applied to a function \( f = f(\xi) \) suffers from this ambiguity also:

\[
L_g[f] = \{f, g\} + \lambda_\alpha\{f, G_\alpha\}. \tag{V.22}
\]
The submanifold $\mathcal{G}_a = 0$ should, however, be invariant, i.e., $L_a[\mathcal{G}_a] = 0$. This equation allows to find $\lambda_a = \lambda_a(\xi)$. Substituting $\lambda_a$ into Eq. (V.22), one gets the Dirac bracket

$$L_a[f] = \{f, g\} - \{f, \mathcal{G}_a\} ||\mathcal{G}_a, \mathcal{G}_b||^{-1}\{\mathcal{G}_b, g\}$$

(V.23)

where $f$, $g$, and $\mathcal{G}_a$ are functions of $\xi$. The Dirac bracket defines a phase flow generated by a function $g = g(\xi)$ within the constraint submanifold $\mathcal{G}_a = 0$.

Using Eq. (V.21), one finds at $\mathcal{G}_a = 0$

$$\{f(\xi), g(\xi)\}_D = \{f(\xi), g(\xi)\}$$

(V.24)

This is the analogue of the statement (i) made in the previous subsection for the full gradient projection. The gradient projection can therefore be used to calculate the Dirac bracket.

There is an analogue of this statement for the partial gradient projections also. Let us suppose that the second-class constraints $\mathcal{G}_a = 0$ split into the canonical pairs: $\chi_A = 0$ and $\Omega_A = 0$ such that $\{\chi_A, \chi_B\} = 0$, $\{\Omega_A, \Omega_B\} = 0$, and $\det ||\chi_A, \Omega_B|| \neq 0$. The Dirac bracket becomes

$$\{f, g\}_D = \{f, g\} + \{f, \chi_A\} ||\chi_A, \Omega_B||^{-1}\{\Omega_B, g\} - \{f, \Omega_A\} ||\chi_A, \Omega_B||^{-1}\{\chi_B, g\}.$$

(V.25)

Let $\xi_u(\xi)$ and $\xi_v(\xi)$ be partial gradient projections such that $\chi_A(\xi_u(\xi)) = 0$ and $\Omega_A(\xi_v(\xi)) = 0$ identically (cf. Eqs. (V.23) and (V.25)). If we replace arguments of the functions $f$ and $g$ with $\xi_u(\xi)$ or $\xi_v(\xi)$, the last two terms vanish due to (V.23) or (V.24), respectively. The Poisson bracket for the projected variables then coincides with the Dirac bracket for the canonical variables $\xi$ constrained to the submanifold $\chi_A = 0$ or $\Omega_A = 0$:

$$\{f(\xi_u(\xi)), g(\xi_u(\xi))\}_D = \{f(\xi_u(\xi)), g(\xi_u(\xi))\},$$

(V.26)

$$\{f(\xi_v(\xi)), g(\xi_v(\xi))\}_D = \{f(\xi_v(\xi)), g(\xi_v(\xi))\}.$$

(V.27)

The arguments of the functions represent the partial gradient projections like in Eqs. (V.26) - (V.21) and (V.15) - (V.20).

Eqs. (V.26) and (V.27) are sufficient to calculate the Dirac bracket given that the partial gradient projections are constructed.

This completes the proof of the statement (i) from the previous subsection, extended to arbitrary Hamiltonian systems.

Turning to the point (ii), it is sufficient to notice that the Poisson bracket, e.g., $\{\xi^i, \xi^j\}$ determines a variation of the $\xi^i$ along the submanifold $\mathcal{G}_a = 0$. This submanifold is parameterized with the $\xi_a$'s. The Poisson bracket is thus a function of the $\xi_a$ again. The involution relations for $\{\xi^i, \xi^j\}$ define therefore an algebra. Similar arguments apply for the partial gradient projections. Eqs. (V.26) - (V.21) and (V.15) - (V.20) represent therefore an illustration of this statement.

E. Constraint functions $\tilde{\mathcal{G}}_a$ constructed by gradient projection

The statements of Sect. 4 can be proved using the gradient projection method.

The vector fields

$$I^{ij} \frac{\partial \mathcal{G}_a}{\partial \xi^j}$$

(V.28)

determine phase flows associated to the constraint functions $\mathcal{G}_a$. These fields are non-singular, i.e., do not vanish in a neighborhood of the constraint submanifold $\mathcal{F} = \{\xi : \mathcal{G}_a(\xi) = 0 \ \forall a\}$. The opposite would mean $\exists a$ such that $\mathcal{G}_a$ vanish at some point $\xi \in \mathcal{F}$. It follows then that $\{\mathcal{G}_a, \mathcal{G}_b\} = 0 \ \forall b$. This is in contradiction with

$$\det\{\mathcal{G}_a, \mathcal{G}_b\} \neq 0$$

(V.29)

which holds, by assumption, everywhere on $\mathcal{F}$ and, by continuity, in a neighborhood of $\mathcal{F}$. In Eq. (V.28), $I^{ij} = -I_{ij}$ where

$$\|I_{ij}\| = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix},$$

(V.30)

with $E_n$ being the $n \times n$ identity matrix (cf. Eq. (V.22)). In what follows, we denote phase flows $\mathcal{V}_{28}$ briefly as $I_d\mathcal{G}_a$.

Let $\mathcal{F}_a = \{\xi : \mathcal{G}_a(\xi) = 0\}$. The condition $I_d\mathcal{G}_a(\xi) \neq 0 \ \forall \xi \in \mathcal{F}_a$ is stronger than the one mentioned above. It looks evident, since any constraint function $\mathcal{G}_a(\xi)$ in a neighborhood of $\mathcal{F}_a$ can be redefined to assign the gradient a definite direction.

$\mathcal{F}_a$ is a subspace of the dimension $2n - 1$. The intersection of all $\mathcal{F}_a$ gives $\mathcal{F}$. The tangent space to $\mathcal{F}_a$ at a point $\xi$, denoted as $T_\xi\mathcal{F}_a$, is skew-orthogonal to $I_d\mathcal{G}_a(\xi)$. Indeed, if $d\mathcal{G}_a = 0$ then $d\xi \in T_\xi\mathcal{F}_a$, and we obtain

$$I^{ij} \frac{\partial \mathcal{G}_a}{\partial \xi^j} I_{ik} d\xi^k = -\frac{\partial \mathcal{G}_a}{\partial \xi^i} = 0.$$  

(V.31)

The space $T_\xi\mathcal{F}_a$ has the dimension $2n - 1$. Among the vectors of $T_\xi\mathcal{F}_a$ one can find $I_d\mathcal{G}_a(\xi)$, since $I_d\mathcal{G}_a(\xi)$ is skew-orthogonal to itself. $T_\xi\mathcal{F}_a$ is therefore a skew-orthogonal complement of $I_d\mathcal{G}_a(\xi)$.

One can find $b$ such that $\{\mathcal{G}_a(\xi), \mathcal{G}_b(\xi)\} \neq 0$. As discussed above,

$$I_d\mathcal{G}_a(\xi) \in T_\xi\mathcal{F}_a,$$

(V.32)

$$I_d\mathcal{G}_b(\xi) \in T_\xi\mathcal{F}_b,$$

(V.33)

and furthermore,

$$I_d\mathcal{G}_a(\xi) \notin T_\xi\mathcal{F}_b,$$

(V.34)

$$I_d\mathcal{G}_b(\xi) \notin T_\xi\mathcal{F}_a.$$  

(V.35)

From the other side, any vector $d\xi \in T_\xi\mathcal{F}_a \cap T_\xi\mathcal{F}_b = T_\xi(\mathcal{F}_a \cap \mathcal{F}_b)$ is skew-orthogonal to $I_d\mathcal{G}_a(\xi)$ and $I_d\mathcal{G}_b(\xi)$. As a consequence of Eqs. (V.34) and (V.35), one gets

$$I_d\mathcal{G}_a(\xi) \notin T_\xi(\mathcal{F}_a \cap \mathcal{F}_b),$$

(V.36)

$$I_d\mathcal{G}_b(\xi) \notin T_\xi(\mathcal{F}_a \cap \mathcal{F}_b).$$

(V.37)
The subspace $F_a \cap F_b$ and, respectively, $T_c(F_a \cap F_b)$ have the dimension $2n - 2$. The vectors $IdG_a(\xi)$ and $IdG_b(\xi)$ are linearly independent and form a two-dimensional space $D_{ab}(\xi)$ which is a skew-orthogonal complement of $T_c(F_a \cap F_b)$ such that $D_{ab}(\xi) \cap T_c(F_a \cap F_b) = \emptyset$.

Let us consider motion of a particle with a Hamiltonian function $G_b$. In virtue of Eq. (V.35), the phase flow $IdG_b$ does not lie in the submanifold $F_a$ entirely and therefore crosses it. Let $t_a(\xi)$ be the time needed for particle located at $\xi \notin F_a$ in a neighborhood of $\xi \in F_a$ to cross $F_a$ at some point $\eta \neq \xi$. The equations of motion look like

$$\frac{d\eta}{dt_a} = IdG_b(\xi).$$

The derivative of $t_a(\xi)$ along the phase flow $IdG_b(\xi)$ with respect to time is, by definition, equal to unity:

$$\{t_a(\xi), G_b(\xi)\} = 1. \quad (V.38)$$

One may interpret, equivalently, $-G_b(\xi)$ as a time needed to cross the submanifold $F_b$ by a particle located at $\xi \notin F_b$. The motion of such a particle is described by a Hamiltonian function $t_a(\xi)$.

The function $t_a(\xi)$ vanishes for $\xi = \eta \in F_a$. At any point of $F_a$, $dG_a$ and $dt_a$ vanish for $d\eta \not\in T_\eta F_a$. There exists only one $d\eta \not\in T_\eta F_a$ such that $dG_a \neq 0$ and $dt_a \neq 0$. It means that $\forall d\eta \neq 0$ is proportional to $dt_a$ and $IdG_a$ is in turn proportional to $Idt_a$.

The first canonical pair $\tilde{G}_a = t_a$ and $\tilde{G}_b = G_b$ is thus constructed.

Let us consider the full gradient projection $\xi_1(\xi)$ onto the submanifold $F_a \cap F_b$, using the constraint functions $G_a$ and $G_b$. One gets $\tilde{G}_a(\xi_1(\xi)) = 0$, $\tilde{G}_b(\xi_1(\xi)) = 0$, whereas the equations $G_c(\xi_1(\xi)) = 0$ for $c \neq a, b$ are significant to determine the location of the constraint submanifold $F$, owing to shifts along the phase flows $Id\tilde{G}_a$ and $Id\tilde{G}_b$. A complete set of equations for $F$ can be taken to be

$$\tilde{G}_a(\xi) = 0, \quad (V.39)$$
$$\tilde{G}_b(\xi) = 0, \quad (V.40)$$
$$G_c(\xi_1(\xi)) = 0. \quad (V.41)$$

for $c \neq a, b$. In virtue of Eq. (V.22),

$$\{\tilde{G}_a(\xi), G_c(\xi_1(\xi))\} = 0, \quad (V.42)$$
$$\{\tilde{G}_b(\xi), G_c(\xi_1(\xi))\} = 0. \quad (V.43)$$

Eqs. (V.40), (V.41) determine $F$ uniquely, so the determinant of the Poisson bracket relations between the $2m$ constraint functions is not zero. The functions $\tilde{G}_a$ and $\tilde{G}_b$ have the vanishing Poisson brackets with the rest ones, so

$$\det\{G_c(\xi_1(\xi)), G_d(\xi_1(\xi))\} \neq 0 \quad (V.44)$$

where $c, d$ take $2m - 2$ values $(c, d \neq a, b)$.

In the remaining set of the constraints, one can find $c, d$ such that $\{G_c(\xi_1(\xi)), G_d(\xi_1(\xi))\} \neq 0$ and repeat the arguments we used earlier. The analogue of Eq. (V.38) looks like

$$\{t_c(\xi), G_d(\xi_1(\xi))\} = 1. \quad (V.45)$$

The Poisson brackets of the left side of this equation with $\tilde{G}_a(\xi)$ and $\tilde{G}_b(\xi)$ vanish. The Jacoby identity yields

$$\{\{t_c(\xi), \tilde{G}_a(\xi)\}, G_d(\xi_1(\xi))\} = 0, \quad (V.46)$$
$$\{\{t_c(\xi), \tilde{G}_b(\xi)\}, G_d(\xi_1(\xi))\} = 0. \quad (V.47)$$

The Poisson brackets of $t_c(\xi)$ with $\tilde{G}_a(\xi)$ and $\tilde{G}_b(\xi)$ remain therefore constant along the phase flow $Id\tilde{G}_a(\xi_1(\xi))$.

At the submanifold $F'_c = \{\xi : G_c(\xi_1(\xi)) = 0\}$, $Idt_c(\xi)$ is proportional to $IdG_c(\xi_1(\xi))$. Those brackets vanish at $F'_c$, and furthermore, vanish for $\xi \notin F'_c$. Eq. (V.2) suggests then $t_c(\xi) = t_c(\xi_1(\xi))$.

The second canonical pair $\tilde{G}_c(\xi) = t_c(\xi_1(\xi))$ and $\tilde{G}_d(\xi) = G_d(\xi_1(\xi))$ is thus constructed.

The proof can be completed by induction. We consider the full gradient projection $\xi_2(\xi)$ onto the submanifold $F_a \cap F_b \cap F_c \cap F_d$ along the commutative phase flows generated by $\tilde{G}_a(\xi)$, $\tilde{G}_b(\xi)$, $\tilde{G}_c(\xi)$, and $\tilde{G}_d(\xi)$. These constraint functions have the vanishing Poisson brackets with the $2m - 4$ remaining ones $G_e(\xi_1(\xi))$ $(e \neq a, b, c, d)$. The latters constitute a complete non-degenerate set to determine the constraint submanifold $F$ uniquely, and so on.

At the end, one gets in a neighborhood of $\xi$ a symplectic basis (IV.23).

F. Hamiltonian $\tilde{H}$ constructed by gradient projection

The Hamiltonian $\tilde{H}$ of Sect. 4 can also be constructed with the help of Eq. (V.2):

$$\tilde{H}(\xi) = H(\xi_4(\xi)) \quad (V.48)$$

where $\xi_4(\xi)$ are gradient projections defined by $\tilde{G}_a$.

Eq. (V.48) and the algorithm described in Sect. 4 give, apparently, an equivalent Hamiltonian function, since the Hamiltonian flows on the constraint submanifold coincide. It can be demonstrated by the comparison of the Dirac brackets:

$$\{\xi^i, \tilde{H}(\xi)\}_D = \{\xi^i, H(\xi)\}_D \quad (V.49)$$

which holds due to Eq. (V.24).

Applications of the gradient projection method to constructing the quantum deformation of the Dirac bracket can be found in [10].

VI. DIRAC QUANTIZATION OF SPHERICAL PENDULUM

In Sect. 2, we modified the initial Lagrangian (II.2) on the constraint submanifold $\chi = \ln \phi = 0$ to make
more transparent the origin of the underlying dilatation gauge symmetry. Let us formulate the Hamiltonian dynamics of the spherical pendulum starting directly from Lagrangian $\mathcal{L} + \lambda \ln \phi$. The straightforward follow-up to the Dirac’s scheme (see also [27, 33]) leads to the Hamiltonian dynamics described in Sect. 3.

Using the Dirac’s scheme, we obtain the primary Hamiltonian $\mathcal{H}_p = \frac{1}{2} \pi^2 - \lambda \ln \phi$ and the primary constraint $\mathcal{G}_0 = \pi_\lambda \approx 0$, where $\pi^\alpha$ are canonical momenta associated to the canonical coordinates $\phi^\alpha$ and $\pi_\lambda$ is the canonical momentum associated to the Lagrange multiplier $\lambda$. The canonical Hamiltonian becomes $\mathcal{H}_c = \mathcal{H}_p + u \pi_\lambda$ where $u$ is an undetermined function of time. The secondary constraints $\mathcal{G}_{a+1} = \{ \mathcal{G}_a, \mathcal{H}_c \}$ can be found: $\mathcal{G}_1 = \ln \phi, \mathcal{G}_2 = \phi \pi / \phi^2, \mathcal{G}_3 = \pi^2 / \phi^2 - 2(\phi \pi)^2 / \phi^4 + \lambda / \phi^2$. The last constraint $\mathcal{G}_3 = 0$ allows to fix $\lambda$ in terms of $\phi^\alpha$ and $\pi^\alpha$, no new constraints then appear.

The dimension of the phase space can be reduced by eliminating the canonically conjugate pair $(\lambda, \pi_\lambda)$. We solve equations $\mathcal{G}_0 = 0$ with respect to $\pi_\lambda$ and $\mathcal{G}_3 = 0$ with respect to $\lambda$ and substitute solutions into $\mathcal{H}_c$. The result is

$$\mathcal{H}'_c = \frac{1}{2} \pi^2 + (\pi^2 - 2(\phi \pi)^2 / \phi^2) \ln \phi.$$  \hspace{1cm} (VI.1)

There remain two constraint functions $\mathcal{G}_1$ and $\mathcal{G}_2$, such that $\{ \mathcal{G}_1, \mathcal{G}_2 \} = 1 / \phi^2$. These are the second-class constraints. The Hamiltonian $\mathcal{H}_c'$ is first-class: $\{ \mathcal{G}_1, \mathcal{H}_c' \} = \mathcal{G}_2 - \mathcal{G}_1 \{ \mathcal{G}_1, \lambda \}$ and $\{ \mathcal{G}_2, \mathcal{H}_c' \} = - \mathcal{G}_1 \{ \mathcal{G}_2, \lambda \}$ where $\lambda$ is determined by $\mathcal{G}_3 = 0$.

In order to split the constraints into the gauge-fixing condition and the gauge generator, we should construct, according to the previous Section, $\mathcal{G}_a$ and $\mathcal{H}$ starting from $\mathcal{G}_a$ and $\mathcal{H}_c'$.

In Sect. 3, we already had the same constraint submanifold described by functions $\chi = \ln \phi$ and $\Omega = \phi \pi$ satisfying $\{ \chi, \Omega \} = 1$, so one can set $\mathcal{G}_1 = \ln \phi$ and $\mathcal{G}_2 = \phi \pi$.

The Hamiltonian $\mathcal{H}$ can be constructed with the help of Eq.(V.48). Let us combine Eqs.(V.7), (V.8) and (V.16), (V.17) to get the full gradient projection:

$$\phi^\alpha_s = \phi^\alpha / \phi, \hspace{1cm} (VI.2)$$

$$\pi^\alpha_s = \phi \pi^\alpha - \phi^\alpha \phi \pi / \phi. \hspace{1cm} (VI.3)$$

Replacing $\phi^\alpha$ and $\pi^\alpha$ in $\mathcal{H}_c'$ by variables $\phi^\alpha_s$ and $\pi^\alpha_s$, respectively, we get

$$\mathcal{H} = \frac{1}{2} \pi^2_s.$$ \hspace{1cm} (VI.4)

which coincides with the Hamiltonian (III.4).

The difference $\mathcal{H} - \mathcal{H}_c'$ is of the second order in the constraint functions. This is sufficient to have identical Hamiltonian flows on the constraint submanifold.

The equivalence is thus demonstrated, so further discussion can go in the parallel with Sect. 3. In the example considered, the constraint functions are of the second class, whereas in Sect. 3 they appear as the gauge-fixing condition, $\chi = 0$, and the gauge generator, $\Omega$. This suggests that the interpretation of second-class constraints of a holonomic system is a matter of convention.

VII. SECOND-CLASS CONSTRAINTS AS GENERATORS OF GAUGE SYMMETRIES AND GAUGE-FIXING CONDITIONS

The type of constraints appearing within the Hamiltonian framework depends on the form of the corresponding Lagrangian. In case of the spherical pendulum, starting from Eq.(II.12), one arrives to the constraints $\chi = 0$ and $\Omega = 0$ as to the second-class constraints of the Hamiltonian framework. Starting from Eq.(II.11), the constraint $\chi = 0$ appears as a gauge-fixing condition, whereas the constraint $\Omega = 0$ as a gauge generator of the symmetry group. Lagrangians (II.12) and (II.11) are equivalent at the classical level, so they lead to the same classical dynamics.

It is possible, therefore, at least in the case of spherical pendulum, to interpret second-class constraints as a gauge-fixing condition and a gauge generator, and vice versa.

We wish to discuss whether this is a common situation. Any set of admissible gauge-fixing conditions and gauge generators can be treated as second-class constraints. This statement is widely used for quantization of gauge theories (see, e.g., [11]).

The reverse statement represents apparent interest also. The second-class constraints systems are analyzed from this point of view by Mitra and Rajaraman [28]. In this Section, we discuss additional details about the underlying gauge symmetries of second-class constraints systems.

A. Converting a gauge system into a second-class constraints system

Let a primary Hamiltonian $\mathcal{H}$ of a system be gauge invariant. The generators of gauge transformations $\Omega_A = 0$ are such that

$$\{ \Omega_A, \mathcal{H} \} = R_A^B \Omega_B, \hspace{1cm} (VII.1)$$

$$\{ \Omega_A, \Omega_B \} = C_A^C \Omega_C. \hspace{1cm} (VII.2)$$

The gauge-fixing conditions $\chi_A = 0$ fulfill

$$\{ \chi_A, \chi_B \} = 0, \hspace{1cm} (VII.3)$$

$$\det \{ \chi_A, \Omega_B \} \neq 0. \hspace{1cm} (VII.4)$$

After imposing gauge-fixing conditions, the system becomes equivalent to a system described by a first-class Hamiltonian $\mathcal{H}'$ defined on the submanifold of second-class constraints $\mathcal{G}_a = (\chi_A, \Omega_A)$. The Hamiltonian $\mathcal{H}'$ can be constructed from the canonical Hamiltonian of the original system, $\mathcal{H}_c = \mathcal{H} + \lambda_B \Omega_B$, by requiring

$$\{ \chi_A, \mathcal{H}_c \} = \{ \chi_A, \mathcal{H} \} + \lambda_B \{ \chi_A, \Omega_B \} \approx 0. \hspace{1cm} (VII.5)$$
The gauge parameters $\lambda_B$ can be fixed from this equation in terms of the canonical variables. We thus get the first-class Hamiltonian $\mathcal{H}' = \mathcal{H}_c$ and second-class constraints $\xi_\alpha$ satisfying \{\$\xi_\alpha, \mathcal{H}'\} \approx 0$ and $\det(\xi_\alpha, \xi_\alpha) \neq 0$.

The gauge-fixing is equivalent to converting the original system into an equivalent one described by a first-class Hamiltonian and second-class constraints.

B. Converting a second-class constraints system into a gauge system

It was shown by Dirac that first-class constraints imply the presence of unphysical degrees of freedom the evolution of which is not fixed by the Hamilton’s equations. The dynamics is self-consistent provided the unphysical degrees of freedom do not belong to the set of physical observables. This is the case of gauge-invariant systems which have gauge degrees of freedom. According to the Dirac’s constraint dynamics, having first-class constraints is a necessary condition for the existence of a gauge symmetry. The reverse statement is less obvious:

Gauge transformations of the canonical coordinates have the form

$$\delta \phi^\alpha = \{\phi^\alpha, \Omega_A\} \theta_A$$  \hspace{1cm} (VII.6)

where $\theta_A$ are infinitesimal functions of time. Since $\Omega_A$ depend on the canonical coordinates and momenta, the variations $\delta \phi^\alpha$ depend on the coordinates and momenta also, unless $\Omega_A$ are first degree polynomials of the canonical momenta. Subject to a Legendre transform, $\delta \phi^\alpha$ become, in general, functions of the coordinates and velocities. The gauge theories such as QED and QCD have the constraint functions $\Omega_A$ as the first degree polynomials of the canonical momenta. In such cases $\delta \phi^\alpha = \xi_\alpha(\phi^\beta)$ do not depend on velocities and define thereby transformations on the configuration space $\mathcal{M} = (\phi^\alpha)$ which induce, in turn, transformations in the tangent bundle $T\mathcal{M} = (\phi^\alpha, \dot{\phi}^\alpha)$.

The first-class constraints systems correspond to more general class of gauge-invariant systems with $\delta \phi^\alpha = \delta \phi^\alpha(\phi^\beta, \dot{\phi}^\beta)$ and, probably, new auxiliary variables. In what follows, we distinguish thereby between gauge transformations of the form $\delta \phi^\alpha = \theta^\alpha(\phi^\beta)$ and generalized gauge transformations with velocity dependent parameters and/or new auxiliary variables.

Having a first-class constraint system is a necessary, but not a sufficient condition for an equivalent gauge-invariant system to exist in the original configuration space.

The possibility of constructing gauge-invariant systems in the unconstrained phase space, equivalent to second-class constraints systems upon a gauge-fixing, has been analyzed by Mitra and Rajaraman.

The applications discussed have constraint functions, associated to gauge transformations, as polynomials of the canonical momenta of the degree less or equal to unity. The gauge transformations are not velocity dependent, although involve new auxiliary variables. The gauged systems require generally an extended configuration space.

If global symplectic basis in the space of constraint functions can be found, equivalent generalized gauge systems for second-class constraints systems can be constructed in the unconstrained phase space.

By passing over to the standard canonical frame, one can always select $\Omega_A$ as first canonical momenta. This is, however, not sufficient to have gauge invariance on the physical configuration space. Before doing a Legendre transform, one has to pass first to a canonical coordinate system where the coordinates $\phi^\alpha$ constitute a physical configuration space. This will mix up the canonical coordinates and momenta, preventing from having $\Omega_A$ as the first degree polynomials.

It is worthwhile to notice that gauge invariant observables are not measurable if they involve auxiliary degrees of freedom. The sums of the vector potentials of the massive electrodynamics and the derivatives of the Stuckelberg scalar are gauge invariant. They do not belong, however, to the set of physical observables. The equivalence with the ordinary gauge systems, where sets of gauge invariant quantities and physical observables coincide, is therefore not complete.

C. Gauge-invariant systems as holonomic systems

The quantization of gauge theories which appear under first-class constraints is studied in many details (see, e.g., Refs. [10, 11]). The second-class constraints systems are usually quantized by employing the BFT formalism that converts second-class constraints into a first class by extending the original phase space to a higher dimension.

The BFT algorithm combined with the Fradkin-Vilkovisky quantization scheme if applied to a system in a 2n-dimensional phase space ends up with an extended phase space of a dimension larger or equal to $2n + 12m$, since the 2m constraints convert into first-class constraints, each of which requires an independent gauge-fixing condition. The remaining at least 8m degrees of freedom appear as ghost variables.

The problem of constructing gauge-invariant systems equivalent to second-class constraints systems in the original phase space is discussed in Ref. [21]. The existence of the global symplectic basis for the constraint functions is specified as sufficient conditions for the existence of gauged partners associated to the second-class constraints systems. The method being successful in removing auxiliary fields from the original phase space, generates auxiliary fields in the effective Lagrangians.

The systems of pointlike particles under holonomic constraints have the natural gauge counterparts both in the phase space and configuration space [27].

In holonomic systems, second-class constraints split...
into gauge-fixing conditions, \( \chi_A = 0 \), and gauge generators, \( \Omega_A \). Such systems can be quantized further as gauge theories. The BFT method, if applied, would result to an extended phase space of the dimension \( 2n + 6m \), containing \( 2m \) Lagrange multipliers and \( 4m \) ghost variables. The underlying gauge symmetry allows to reduce the number of auxiliary fields within the quantization scheme. 

D. Frobenius’ condition for holonomic constraints

The holonomic constraints are defined by a constraint submanifold \( \mathcal{N} \subset \mathcal{M} \) in the configuration space \( \mathcal{M} \). It comes then out automatically that particle velocities belong to the tangent space \( T\mathcal{N} \) of the constraint submanifold.

An \( n - m \)-dimensional tangent subspace \( T\mathcal{N} \subset T\mathcal{M} \) of an \( n \)-dimensional configuration space \( \mathcal{M} \) can be defined in general through \( m \) covector fields \( \omega_A \alpha \) with \( \alpha = 1, \ldots, n \) and \( A = 1, \ldots, m \) by imposing constraints

\[
\omega_A \alpha \dot{\phi}^\alpha = 0 \quad (VII.7)
\]

which are velocity dependent and therefore non-holonomic.

The Frobenius’ integrability conditions (see, e.g., [28]),

\[
\frac{\partial \omega_A}{\partial \phi^\beta} \left( \dot{\phi}^\alpha_1 \dot{\phi}^\beta_2 - \dot{\phi}^\alpha_2 \dot{\phi}^\beta_1 \right) = 0, \quad (VII.8)
\]

for tangent vectors \( \dot{\phi}^\alpha_1 \) and \( \dot{\phi}^\beta_2 \) satisfying \( \text{(VII.7)} \), if fulfilled, implies the existence of a submanifold \( \mathcal{N} \) the tangent space of which coincides with the tangent subspace \( \text{(VII.7)} \). In such a case, the constraints \( \text{(VII.7)} \) can be replaced with holonomic constraints \( \chi_A = 0 \), identifying the constraint submanifold \( \mathcal{N} \subset \mathcal{M} \), without modifying the dynamics.

Eqs. \( \text{(VII.8)} \) specify non-holonomic systems which have gauged counterparts in the original configuration space.

For an \( n = 3 \) Euclidean space, Eqs. \( \text{(VII.7)} \) define, for \( m = 1 \), a submanifold orthogonal to the vector \( \omega_\alpha \). Eqs. \( \text{(VII.8)} \) tell us that the vector field \( \tilde{\omega} \) is a potential field provided that \( \text{rot} \tilde{\omega} = 0 \). One can find a potential function \( \chi \) such that \( \tilde{\omega} = \nabla \chi \). The sets of \( \chi = \text{constant} \) define, in our case, possible constraint submanifolds of an equivalent holonomic system.

The tangent subspace determined by Eq. \( \text{(VIII.6)} \) coincides with the tangent space of \( S^{n-1} \). The covector field \( \omega_\alpha = \phi^\alpha \) satisfies the Frobenius’ condition.

VIII. SUPPLEMENTARY CONDITIONS FOR WIGNER FUNCTIONS

The splitting of second-class constraints into constraints associated to gauge transformations and gauge-fixing conditions is not unique. Every pair \( (\chi_A, \Omega_A) \) can be transformed, e.g., as \( \chi_A \to \Omega_A, \Omega_A \to -\chi_A \). The symplectic group \( Sp(2m) \) of transformations mixes \( \chi_A \) and \( \Omega_A \) further, keeping the Poisson brackets invariant. Canonical transformations, furthermore, do not modify the Poisson bracket of the constraints also. There exists therefore an affluent spectrum of representations and we have to demonstrate that the physical content of theory is invariant with respect to the choice of representation.

In the general case of second-class constraints, we do not have any criterion how to discriminate between first-class constraints associated to gauge transformations and gauge-fixing conditions. We show, conversely, that the way the second-class constraints are split is not important for quantization.

A useful hint towards that conclusion is delivered by the path integral representation for the evolution operator \( \hat{\Omega}_A \), which is invariant with respect to linear transformations of \( G_\alpha \).

The problem is to demonstrate that the supplementary conditions \( \hat{\Omega}_A \Psi = 0 \) can be replaced by \( \hat{\chi}_A \Psi = 0 \). In case of the spherical pendulum, this is almost evident, since \( \hat{\Omega} \Psi = 0 \) means \( \Psi \sim \Psi(\phi^\alpha/\phi) \) whereas \( \hat{\chi} \Psi = 0 \) means \( \Psi \sim \delta(\chi)\Psi(\phi^\alpha/\phi) \). The essential dependence comes from the angular part of the wave function, while the radial part \( \delta(\chi) \) is factorized, being commutative, Eq. \( \text{(III.10)} \), with the Hamiltonian and the S-matrix. It can be absorbed thereby by an overall normalization factor.

These arguments can apparently be extended to an arbitrary case. According to the Dirac’s prescription, the physical wave functions are annihilated by the constraint operators associated to gauge transformations

\[
\hat{\Omega}_A \Psi = 0. \quad (VIII.1)
\]

To simplify the matter, we pass to the standard canonical coordinate system and choose the constraint functions \( \chi_A \) as first \( m \) canonical coordinates \( q_A \) and the constraint functions \( \Omega_A \) as first \( m \) canonical momenta \( p_A \). The remaining canonical variables are \( (q^*, p^*) \) constitute the physical phase space \( \Gamma_*^{2(n-m)} \). Eq. \( \text{(VIII.1)} \) gives then, in the coordinate representation, a wave function of the form \( \Psi = (\Psi(q^*)) \). Such a wave function has an infinite norm, since the integral \( \int dq^* |\Psi(q^*)|^2 \) diverges. In the momentum representation, we would get \( \Psi(p) = (2\pi \hbar)^m \prod_A \delta(p_A)\Psi(p^*) \). \( \Psi(p) \) has an infinite norm \( (2\pi \hbar)^m \delta^m(0) \) provided \( \Psi(p^*) \) is normalized to unity. An infinite but numerical factor can be included into the norm of \( \Psi(p) \).

Let us check whether wave functions satisfying

\[
\hat{\chi}_A \Psi = 0. \quad (VIII.2)
\]

have a physical sense. In the coordinate representation, one gets \( \Psi = \prod_A \delta(q_A)\Psi(q^*) \) and in the momentum space \( \Psi = \Psi(p^*) \). These states have infinite norms. There is an apparent symmetry \( p \to q, q \to -p \) between wave functions satisfying \( \text{(VIII.1)} \) and \( \text{(VIII.2)} \).
Owing to the factor $\prod A \delta(q_A)$, conditions (VIII.1) and (VIII.2) single out the same set of functions, the nontrivial dependence of which comes from the physical variables $q^*$ ($p^*$) only.

The Wigner functions provide complete information on quantum systems. The association rules of operators in the Hilbert space and functions in the phase space are discussed a long time ago [36, 37, 38, 39]. The quantum mechanics can be reformulated using the Groenewold star-product representing a deformation of the usual pointwise product of functions in the phase space. The Wigner functions and constraint functions are defined in the phase space, so it is natural to discuss supplementary conditions in terms of the Wigner functions. In this approach, the equivalence between Eqs. (VIII.1) and (VIII.2) becomes more transparent. In addition, more general group of supplementary conditions, equivalent to Eqs. (VIII.1) and (VIII.2), can be formulated in terms of the Wigner functions.

### A. Probability density localized on the constraint submanifold

Let us start from the standard canonical frame where the constraint functions $\chi_A$ are the first $m$ canonical coordinates $q_A$ and the constraint functions $\Omega_A$ are the first $m$ canonical momenta $p_A$. The probability density in the phase space $\Gamma^{2n}$ can be written as follows:

$$W(q, p) = (2\pi \hbar)^n \prod A \delta(q_A) \delta(p_A) W_\star(q^*, p^*).$$  

(VIII.3)

Identifying the $W(q, p)$ with the Wigner function in the unconstrained phase space and using the Wigner transform, one gets the density matrix

$$\rho(q, q') = \int W(q + q', p) e^{iq\pi(q - q')} \frac{d^n p}{(2\pi \hbar)^n} = \prod A \delta(q_A + q'_A) \rho_\star(q^*, q'^*).$$  

(VIII.4)

It satisfies the operator equations

$$\hat{q}_A \rho + \rho \hat{q}_A = 0,$$

(VIII.5)

and

$$\hat{p}_A \rho + \rho \hat{p}_A = 0.$$  

(VIII.6)

The first equation implies $\rho \sim \delta(q_A + q'_A)$. The second one implies that $\rho$ does not depend on $q_A - q'_A$.

The density matrix $\rho_\star(q^*, q'^*)$ is normalized to unity, so

$$\int \rho(q, q') d^n q = 1.$$  

(VIII.7)

One can make a unitary transformation to pass to an arbitrary set of operators associated to the canonical variables. In terms of

$$\hat{G}_a = \mathcal{U}(\hat{q}_A, \hat{p}_A) \mathcal{U}^+,$$

and $\rho$ replaced with $\mathcal{U} \rho \mathcal{U}^+$, Eqs. (VIII.5) and (VIII.6) become

$$\hat{G}_a \rho + \rho \hat{G}_a^\dagger = 0.$$  

(VIII.8)

Eqs. (VIII.8) are necessary and sufficient conditions to have the representation (VIII.4) in the standard canonical frame.

The supplementary conditions (VIII.8) cannot be formulated in terms of a wave function, since the density matrix in the unconstrained configuration space does not correspond to a pure state, even if system on the constraint submanifold is in a pure state.

In an arbitrary frame and in the classical limit, Eq. (VIII.3) looks like

$$W(\xi) = (2\pi \hbar)^m \prod_a \delta(G_a) \sqrt{\det \{G_a, G_b\}_\star W_\star(\xi_a(\xi)).}$$  

(VIII.9)

Note that $\prod_a \delta(G_a)$ acts as a projection operator, so that

$$\prod_a \delta(G_a) f(\xi) = \prod_a \delta(G_a) f(\xi_a(\xi)) \forall f(\xi).$$

In the classical limit, the Wigner function satisfies

$$G_a(\xi) W(\xi) = 0.$$  

(VIII.10)

The complete phase-space analogue of quantum Eqs. (VIII.8) can be formulated in terms of the symmetric part of the Groenewold star-product to give

$$G_a(\xi) \circ W(\xi) = 0.$$  

(VIII.11)

The star-product has the following decomposition:

$$f(\xi) \star g(\xi) = f(\xi) \circ g(\xi) + \frac{i}{\hbar} \{f(\xi), g(\xi)\}.$$  

(VIII.12)

where

$$f(\xi) \circ g(\xi) = f(\xi) \cos(\frac{\hbar}{2} P) \cos(\frac{\hbar}{2} P) g(\xi),$$

(VIII.13)

$$f(\xi) \wedge g(\xi) = f(\xi) \frac{\hbar}{2} \sin(\frac{\hbar}{2} P) g(\xi)$$

(VIII.14)

and

$$P = -\hbar \partial / \partial \xi^i \partial / \partial \xi^j$$

is the so-called Poisson operator. In the limit $\hbar \to 0$,

$$\lim_{\hbar \to 0} f(\xi) \wedge g(\xi) = \{f(\xi), g(\xi)\}.$$  

(VIII.15)

The Poisson bracket $\{f(\xi), g(\xi)\}$ coincides generally with a function associated to the commutator $-i/\hbar [\hat{f}, \hat{g}]$ to the lowest order in the Planck’s constant only. The skew-symmetric part of the Groenewold product provides a generalization of the Poisson bracket which is skew-symmetric with respect to two functions, real for real functions, coincides with the Poisson bracket to the lowest order in the Planck’s constant, satisfies the Jacobi identity, and keeps the association rule for commutators.
This part of the Groenewold product is known also as the "sine bracket" or Moyal bracket \[38\].

In the limit \( \hbar \to 0 \), the quantum condition \[8\] recovers the classical one \[10\]. These two conditions coincide in the standard canonical frame.

The normalization condition Eq.\[7\] holds in the quantum case. The Wigner function satisfies

\[
\int W(\xi) \frac{d^{2n}\xi}{(2\pi\hbar)^n} = 1. \tag{VIII.16}
\]

The classical limit of the Wigner function Eq.\[9\] provides the conventional normalization on the constraint submanifold for \( W_*(\xi_*(\xi)) \).

The Groenewold star-product is non-local. The usual pointwise product of two functions like in Eq.\[10\] has no quantum counterpart. It can be treated as a classical limit of a quantum operator equation only. By contrary, Eq.\[11\] is the phase space analogue of the quantum operator equation \[8\].

Eq.\[12\] accomplishes a trivial extrapolation of the Wigner function from the constraint submanifold: The density is set equal to zero when \( \xi \) does not belong to the constraint submanifold. The Wigner function is not a smooth function across the constraint submanifold, so it is hard to formulate using this approach an evolution equation similar to the Liouville equation in the unconstrained phase space.

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### B. Probability density localized on and outside of the constraint submanifold

One can require that at any given point \( \xi \) the density be the same as at \( \xi_* = \xi_*(\xi) \). The gradient projections \( \xi_* \) can be constructed, as discussed in Sect. 5, using phase flows generated by the constraint functions \( G_\alpha \) to solve equations \( G_\alpha(\xi_*(\xi)) = 0 \).

The analogue of Eq.\[13\] reads

\[
W(\xi) = W_*(\xi_*(\xi)). \tag{VIII.17}
\]

Such a density has infinite norm, since there are directions in the phase space \( \Gamma^{2n} \), crossing the submanifold \( \mathcal{G}_\alpha = 0 \), along which the density remains constant. These directions are determined by the constraint functions.

Eq.\[14\] tells

\[
\{\mathcal{G}_\alpha(\xi), W(\xi)\} = 0. \tag{VIII.18}
\]

In the standard canonical frame, where \( q_A = \chi_A \) and \( p_A = \Omega_A \), the gradient projections can be easily constructed:

\[
(0, q^*, 0, p^*) \tag{VIII.19}
\]

where \( \xi = (q_A, q^*, p_A, p^*) \). Eq.\[17\] then simplifies to

\[
W(q, p) = W_*(q^*, p^*). \tag{VIII.20}
\]

Note that

\[
\xi_*(\xi) = (0, q^*, p_A, p^*), \tag{VIII.21}
\]

\[
\xi_*(\xi) = (q_A, q^*, 0, p^*), \tag{VIII.22}
\]

so that \( \xi_*(\xi) = \xi_*(\xi_*(\xi)) = \xi_*(\xi_*(\xi)) \).

If we apply the Wigner transform, we get the density matrix

\[
\rho(q, q') = \int \rho(\frac{q + q'}{2}, p) e^{\frac{i}{\hbar} p(q - q')} \frac{d^n p}{(2\pi\hbar)^n} = \prod_A \delta(q_A - q'_A) \rho_*(q_A^*, q'_A^*). \tag{VIII.23}
\]

It satisfies

\[
\dot{\rho} - \dot{\rho} = 0, \tag{VIII.24}
\]

\[
\dot{\rho} - \dot{\rho} = 0. \tag{VIII.25}
\]

or, equivalently,

\[
\dot{G}_\alpha - \dot{G}_\alpha = 0. \tag{VIII.26}
\]

The phase space analogue of these operator equations looks like

\[
\mathcal{G}_\alpha(\xi) \wedge W(\xi) = 0. \tag{VIII.27}
\]

This condition is in agreement with its classical counterpart Eq.\[15\] by virtue of Eq.\[16\].

Eqs.\[26\] are distinct from Eqs.\[8\]. This is a consequence of different extrapolation schemes of the density to the unconstrained phase space. Eqs.\[26\] are the necessary and sufficient conditions for Eq.\[23\].

The dependence of the density matrix on \( (q_A, q'_A) \) does not factorize, although \( \rho^2 = \rho \) for \( \rho_*^2 = \rho_* \). The latter, even when fulfilled, is not sufficient to have a pure state. Eq.\[23\] describes, in particular, a noncoherent sum of pure states with different momenta \( p_A \). The system is thereby identified as a mixed state. It cannot be described by a wave function. The constraint equations \[26\], respectively, cannot be formulated in terms of a wave function \( \Psi(q) \).

If one works with density matrices in the representation \[13\] or \[20\], it is not important how the second-class constraints were split. Eqs.\[13\] and \[20\] are symmetric explicitly with respect to the interchange \( \chi_A \to \Omega_A, \Omega_A \to -\chi_A \), linear transformations of the constraint functions, and furthermore, with respect to unitary transformations in the Hilbert space. In the classical limit, they are invariant with respect to canonical transformations.

The normalization condition of the Wigner function involves a projection operator

\[
\Psi = \int \frac{d^{2m} \lambda}{(2\pi\hbar)^m} \prod_{a=1}^{2m} \exp\left(\frac{i}{\hbar} \mathcal{G}_a \lambda_a\right). \tag{VIII.28}
\]
in terms of which the norm is calculated as follows

\[ \int P(\xi) \ast W(\xi) \frac{d^{2n} \xi}{(2\pi \hbar)^n} = 1. \quad (\text{VIII.29}) \]

where \( P(\xi) \) is the Weyl’s symbol of the operator \( \chi \). Note that \( \Psi \) is commutative with \( \hat{\rho} \) in virtue of (VIII.26).

The Wigner function appears as a smooth function, so it is an appropriate object to describe an evolution of the system on line with the Liouville equation in the unconstrained phase space. A quantum extension of the the Liouville equation for \( W(\xi) \) satisfying Eqs. (VIII.27) is given in Ref. [40].

\[ \Omega_A(\xi) \ast W(\xi) = 0, \quad (\text{VIII.36}) \]

\[ W(\xi) \ast \Omega_A(\xi) = 0. \quad (\text{VIII.37}) \]

As a consequence of Eqs. (VIII.34) and (VIII.35), we have \{\Omega_A, W\} = 0 in the classical limit. Eq. (V.4) tells then that \( W(\xi) = \rho(\xi, \xi) \). The second classical condition \( \Omega_A W = 0 \) results in the Wigner function \( W(\xi) = \prod \delta(\Omega_A(\xi)) \rho(\xi, \xi) \). Combining these two equations, we obtain

\[ W(\xi) = (2\pi \hbar)^m \prod \delta(\Omega_A(\xi)) W_*(\xi, \xi), \quad (\text{VIII.38}) \]

which is in agreement with Eq. (VIII.36). We used here \( \Omega_A(\xi, \xi) = \Omega_A(\xi) \) which is valid up to the second order in the constraint functions \( \chi_A \).

It is possible to establish a relationship with the results of the previous subsection. Indeed, one can check that

\[ \hat{\rho} = \Psi_\Omega \hat{\rho}_s, \quad (\text{VIII.39}) \]

with \( \hat{\rho}_s \) being the density matrix from the previous subsection, satisfies Eqs. (VIII.34) and (VIII.35). The operator \( \Psi_\Omega \) is defined by

\[ \Psi_\Omega = \int \frac{m! \lambda}{(2\pi \hbar)^m} \prod_{A=1}^m \exp\left( i \frac{\lambda}{\hbar} \Omega_A \lambda_A \right) \quad (\text{VIII.40}) \]

In order to get the normalization condition for \( W(\xi) \) one has to factorize \( \hat{\rho} \) according to (VIII.39), construct the Weyl’s symbol of \( \hat{\rho}_s \), and apply Eq. (VIII.29).

Let us consider the opposite localization:

\[ W(q, p) = \prod_A \delta(q_A) W_*(q^*, p^*). \quad (\text{VIII.41}) \]

The first \( m \) canonical coordinates are kept on the constraint submanifold, whereas the first \( m \) canonical momenta are projected.

The density matrix has the form

\[ \rho(q, q') = \prod_A \delta(q_A) \delta(q_A') \rho_s(q^*, q'^*). \quad (\text{VIII.42}) \]

It satisfies \( \hat{q}_A \hat{\rho} = \hat{p}_A \hat{\rho} = 0 \) or, equivalently,

\[ \hat{\chi}_A \hat{\rho} = 0, \quad (\text{VIII.43}) \]

\[ \hat{\rho} \hat{\chi}_A = 0. \quad (\text{VIII.44}) \]

These conditions are the necessary and sufficient conditions to have the density matrix of the form (VIII.42).

The system is allowed to appear in a pure state and can be described by a wave function \( \Psi(q) \) accordingly. Eqs. (VIII.34) and (VIII.35) can be reformulated in terms of wave functions to match the Dirac’s supplementary condition Eq. (VIII.2).

The mixed localization scheme breaks the symmetry \( \chi_A \rightarrow \Omega_A, \Omega_A \rightarrow -\chi_A \) from the outset. However, it allows to work with wave functions. The other methods we discussed lead, in the unconstrained configuration space, to mixed states.
As a consequence of Eqs. (VIII.43) and (VIII.44), one gets in the classical limit

\[ W(\xi) = \prod \delta(\chi_A(\xi)) W_\ast(\xi, \xi), \quad (VIII.45) \]

in agreement with Eq. (VIII.44). We used here the relation \( \chi_A(\xi, \xi) = \chi_A(\xi) \) which is valid up to the second order in \( \Omega_A \).

The quantum analogue of Eqs. (VIII.43) and (VIII.44) is given by

\[ \chi_A(\xi) \ast W(\xi) = 0, \quad (VIII.46) \]
\[ W(\xi) \ast \chi_A(\xi) = 0. \quad (VIII.47) \]

There exists a relationship with the Wigner function of the previous subsection. One can check that

\[ \hat{\rho} = \chi_\ast \hat{\rho} \]

satisfies Eqs. (VIII.43) and (VIII.44). The operator \( \chi_\ast \) is defined by

\[ \chi_\ast = \int d^m \lambda \prod_{A=1}^n \exp \left( \frac{i}{\hbar} \chi_A \lambda_A \right) \quad (VIII.49) \]

Note that \( \chi_\ast = \chi_\ast \chi_\ast = \Omega_A \chi_\ast = 0 \) in the operator sense.

The normalization of the Wigner function in arbitrary canonical coordinate system is quite involved:

*Given a wave function in the unconstrained configuration space, one should construct the density matrix, factorize it according to Eqs. (VIII.39) or (VIII.41) and integrate the Wigner function associated to \( \rho_s \) in order to extract the norm according to Eq. (VIII.22).*

Respectively, the Hermitian product of wave functions is calculated using the off-diagonal Wigner function and integrating it over the unconstrained phase space according to the same prescription.

### D. Discussion

In the classical constraint systems, physical quantities depend on the probability densities localized on the constraint submanifold only. In the standard canonical frame, the Wigner functions of quantum systems are localized on the constraint submanifolds also. This is mandatory, since any unconstrained system can be treated as a constrained system with constraints imposed to remove the added unphysical degrees of freedom. In doing so, the unphysical degrees of freedom should not modify dynamics of the initial system. This is achieved by attributing physical sense to the Wigner functions on the constraint submanifold. How to extrapolate Wigner functions from the constraint submanifold into the unconstrained phase space is a matter of convention.

We discussed the most evident extrapolations. Among them are those which allow to describe systems in the original phase space as pure states (mixed localization). One of them can be constructed using the Dirac’s prescription (VIII.2). The dual condition (VIII.2) was found to be possible also. If we work with density matrices or the Wigner functions, one arrives at conditions (VIII.8) or (VIII.15), both are symmetric under the permutations \( \chi_A \rightarrow \Omega_A, \Omega_A \rightarrow -\chi_A \). The constraints imposed on the physical states do not depend on the splitting \( G_a = (\chi_A, \Omega_A) \).

### IX. THE \( O(n) \) NON-LINEAR SIGMA MODEL

The \( O(n) \) non-linear sigma model represents the field theory analogue of the spherical \( n - 1 \)-dimensional pendulum. The \( n = 4 \) case corresponds to the chiral non-linear sigma model due to the isomorphism of algebras \( su(2)_L \oplus su(2)_R \sim so(4) \).

In Sect. 2, we started from the tangent bundle \( TM = (\phi^\alpha, \phi^\alpha) \) of the dimension \( 2n \), defined over the configuration space \( M = (\phi^\alpha) \). Lagrangian (II.2) depends on \( n \) velocities \( \dot{\phi}^\alpha \). On the constraint submanifold \( (II.1) \), it depends on \( n - 1 \) velocities \( \Delta^{\alpha\beta} \dot{\phi}^\beta \) tangent to the constraint submanifold. The dynamics of \( \phi \) turns out to be independent on other variables. The constraint \( (II.1) \) reduces the effective number of the degrees of freedom to \( n - 1 \). Thus, a \( 2n - 2 \) dimensional tangent bundle over the \( n - 1 \) configuration space can be constructed. It is described, e.g., by coordinates \( \vartheta^i \) analogous to the angular coordinates in three dimensional space. Lagrangian \( (II.11) \) is not degenerate with respect to \( \dot{\vartheta}^i \). The coordinates \( \vartheta^i \) constitute the physical configuration space.

The path integral for the evolution operator in terms of the angular coordinates \( \vartheta^i \) can be treated as a reference point for comparison to more involved quantization methods.

#### A. Path integral for the \( O(n) \) non-linear sigma model

Let us construct the path integral for the \( O(n) \) non-linear sigma model in terms of the angular variables \( \vartheta^i \), i.e., by solving the constraint equations from the outset and compare it with the expression of Sect. 2 derived using the underlying gauge symmetry of the \( O(n) \) non-linear sigma model.

The field theory extension of the spherical pendulum problem is rather straightforward. In what follows, the kinematic variables are functions of \( x_\mu = (t, \mathbf{x}) \).

Let us construct a basis

\[ e^\alpha_i = \frac{\partial}{\partial \vartheta^i} \phi^\alpha, \quad (IX.1) \]
\[ e_i^\alpha e_j^\alpha = g_{ij}, \quad (IX.2) \]
\[ g^{ij} e_i^\alpha e_j^\beta = \delta^{\alpha\beta} - \phi^\alpha \phi^\beta / \phi^2, \quad (IX.3) \]
where $g_{ij} = g_{ij}(\theta)$ is an induced metric tensor on the submanifold $\phi(x) = 1$, $\det \|g_{ij}\| \neq 0$. In terms of the coordinates $\phi^a$, the field theory extension of Lagrangian \[ (\text{IX.11}) \] takes the form

$$L_* = \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial_\mu \phi^j. \tag{IX.4}$$

The Legendre transformation of $L_*$ is well defined. In terms of the canonical momenta

$$\phi_1 = \frac{\partial L_*}{\partial \dot{\phi}^i} = g_{ij} \dot{\phi}^j \tag{IX.5}$$

the Hamiltonian density can be found to be

$$\mathcal{H}_* = \frac{1}{2} g^{ij} \phi_1 \phi_2 + \frac{1}{2} g^{ij} \partial_\alpha \partial_\alpha \phi^j \tag{IX.6}$$

where $a = 1, 2, 3$. The non-vanishing Poisson bracket for the canonical coordinates and momenta has the form

$$\{\phi^i(t, x), \phi_2(t, x')\} = \delta^i_2 (x - x'). \tag{IX.7}$$

The wavefunction relations for coordinates $\phi^a$ and momenta $\pi^a$ associated to the tangent velocities $\pi^a_1 = \partial_\alpha \phi^a_2$ agree with those discussed in Sect. 4.

The path integral in the phase space $(\phi^i, \phi_1)$ is given by

$$Z = \int \prod \frac{d\phi^i d\phi_1}{(2\pi\hbar)^{n-1}} \exp \left\{ \frac{i}{\hbar} \int d^4x (\phi^i \dot{\phi}^i - \mathcal{H}_*) \right\}. \tag{IX.8}$$

The Liouville measure $d\phi^i d\phi_1$ is consistent with Eq. (IX.7). The integrand over the canonical momenta in Eq. (IX.8) has a Gaussian form and can be calculated explicitly:

$$Z = \int \prod \sqrt{\det \|g_{ij}\|} d^{n-1}\theta \exp \left\{ \frac{i}{\hbar} \int d^4x L_* \right\}. \tag{IX.9}$$

The value $\sqrt{\det \|g_{ij}\|} d^{n-1}\theta$ gives volume of the configuration space, defined by the metric tensor $g_{ij}$. This measure is invariant under the $O(n)$ group.

The $S$-matrix (IX.3) can be written in an explicitly covariant form with respect to the $O(n)$ rotations and the Lorentz transformations. First, we rewrite Lagrangian density (IX.3) in terms of the coordinates $\phi^a$

$$L_* = \frac{1}{2} \Delta^{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b / \phi^2 \tag{IX.10}$$

and, second, rewrite the Lagrange measure

$$\sqrt{\det \|g_{ij}\|} d^{n-1}\theta = \sqrt{(\partial \chi / \partial \phi^a)^2 \delta(\chi) d^n \phi} \tag{IX.11}$$

where $\chi = \ln \phi$. The right side is the same for all functions vanishing at $\phi = 1$.

In this form we recover the result of Sect. 2.

### B. Pion field parameterization in chiral sigma model

The $n = 4$ case is especially interesting since it corresponds to the chiral non-linear sigma model. For $n = 4$, the angular coordinates are defined by

$$\phi^a = (\cos \psi, \sin \psi \times (\cos \theta, \sin \theta \times (\cos \varphi, \sin \varphi))) \tag{IX.12}$$

where $\vartheta^1 = \psi$, $\vartheta^2 = \theta$, and $\vartheta^3 = \varphi$.

The angular distance, $\Theta$, between two vectors $\phi^a$ and $\phi'^a$ is defined by scalar product $\cos \Theta = \phi \phi'$.

The distance element becomes

$$d\Theta^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{IX.13}$$

The components of the metric $g_{ij}$ can be found using the expansion $d\Theta^2 = g_{ij} d\vartheta^i d\vartheta^j$ or directly from Eqs. (IX.1) and (IX.2). The Lagrange measure of the path integral becomes

$$\sqrt{\det \|g_{ij}\|} d^3\theta = \sin^2 \psi \sin \theta d\psi d\theta d\varphi$$

$$= \sin^2 \psi dV, \tag{IX.14}$$

with $dV = \psi^2 d\psi \sin \theta d\theta d\varphi$ being an element of the Euclidean volume. The corresponding Lagrangian can be found from Eq. (IX.3) to give

$$L_* = \frac{1}{2} (\partial_\mu \psi)^2 + \frac{\sin^2 \psi}{2} ((\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \varphi)^2). \tag{IX.15}$$

The quantization of the chiral sigma model is made using an oscillator basis by expanding the $L_*$ around $\vartheta^i = 0$ breaking thereby the $O(4)$ symmetry down to its $O(3)$ subgroup. It can be successful provided that measure of the coordinate space is such that $\det \|g_{ij}\| = 1$. The path integrals convert then to the Gaussian integrals which can be calculated. The parameterization preserving the $O(3)$ symmetry and satisfying the above requirement is, apparently, unique. One should rescale the "radius" $\psi$ according to

$$\sin^2 \psi d\psi = \omega^2 d\omega. \tag{IX.16}$$

This elementary equation gives

$$\omega = \left( \frac{3}{2} (\psi - \sin \psi \cos \psi) \right)^{1/3}. \tag{IX.17}$$

Lagrangian $L_*$ then becomes

$$L_* = \frac{\omega^4}{2 \sin^4 \psi} (\partial_\mu \omega)^2 + \frac{\sin^2 \psi}{2} ((\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \varphi)^2) \tag{IX.18}$$

where $\psi$ is a function of $\omega$ Eq. (IX.17). The mass term breaking the $O(4)$ symmetry looks like

$$L_M = M^2 (\cos \psi - 1). \tag{IX.19}$$
The quadratic part of the Lagrangian used for the perturbation expansion can be selected as follows

$$\mathcal{L}_{\ast}^{[2]} = \frac{1}{2}(\partial_{\mu} \omega)^2 + \frac{\omega^2}{2}((\partial_{\mu} \theta)^2 + \sin^2 \theta (\partial_{\mu} \varphi)^2) - \frac{M^2}{2} \omega^2.$$ 

In terms of the pion fields

$$\pi^a = \omega \times (\cos \theta, \sin \theta \times (\cos \varphi, \sin \varphi)),$$

(IX.20) it takes the standard form

$$\mathcal{L}_{\ast}^{[2]} = \frac{1}{2}(\partial_{\mu} \pi^a)^2 - \frac{M^2}{2}(\pi^a)^2,$$

(IX.21) whereas the Lagrange measure is simply the Euclidean volume $d^3 \pi$. The difference $\delta \mathcal{L}_{\ast} = \mathcal{L}_{\ast} + \mathcal{L}_M - \mathcal{L}_{\ast}^{[2]}$ can be considered as a perturbation.

The pion fields can be parameterized in various ways. The problem of ambiguities of the transition amplitudes, connected to the arbitrariness of that choice, was discussed first in Refs. [11, 12]. It was shown that on-shell amplitudes do not depend on the choice of physical variables. This statement is known as the "equivalence theorem". The method proposed by Gasser and Leutwyler [17] associates the QCD Green functions to amplitudes of the effective chiral Lagrangian. Using this method, the QCD on- and off-shell amplitudes can be calculated in a way independent on the parameterization.

The $S$-matrix is invariant with respect to a symmetry group, if both the action functional and the Lagrange measure entering the path integral over canonical coordinates are invariant. The Liouville measure entering the path integral over canonical coordinates and momenta is always "flat", since we work in canonical basis. The quantization gives a non-trivial Lagrange measure, however (cf. Eq. (IX.9)). In case of the $O(n)$ non-linear sigma model, there is only one parameterization which makes the Lagrange measure flat, i.e., det $[\{\hat{g}_{ij}\}] = 1$. This requirement is useful for development of perturbation theory which uses an oscillator basis to convert path integrals into the Gaussian form.

The weight factor can always be exponentiated to generate an effective Lagrangian $\delta \mathcal{L}_H$, in which case $\chi^a = \phi^a$ provides desired parameterization also. The exponential parameterization of the pion matrix

$$U(\phi^a) = e^{i\pi^\alpha \phi^a}$$

(IX.22) gives, in particular,

$$\delta \mathcal{L}_H = -\frac{1}{a^2} \ln(\sin^2(\phi)\phi^{-2})$$

(IX.23)

where $a$ is a lattice size, $\delta \mathcal{L}_H$ diverges in the continuum limit. The non-linear sigma model is not a renormalizable theory, so divergences cannot be absorbed into redefinition of $F$ and $M_\pi$. Using the mean filed (MF) approximation, it is usually possible to keep renormalizations finite. The exponentiation of a variable weight factor breaks, in general, self-consistency of the MF approximation of the non-linear sigma model.

Divergences arising from $\delta \mathcal{L}_H$ could, however, be compensated by divergences coming from higher orders ChPT loops. From this point of view it looks natural to attribute $\delta \mathcal{L}_H$ to higher orders ChPT loop expansion starting from one loop. The MF approximation for ChPT implies then the tree level approximation for the non-linear sigma model with $\delta \mathcal{L}_H$ neglected. Such an approximation, however, neglects the Haar measure from the start. It is therefore hard to expect that such an approximation describes correctly the high temperature regime where the chiral invariance is supposed to be restored.

The self-consistency of the MF approximation of the non-linear sigma model survives with the one parameterization only.

The invariance under the chiral transformations admits, within the MF approximation, only the parameterization given by Eq. (IX.17). The parameterization based on the dilatation of $\phi^a$ gives $\delta \mathcal{L}_H = 0$, does not involve the higher orders ChPT loops, and allows to work in the continuum limit with finite quantities only.

The effective interaction terms in the Lagrangian which appear due to the presence of the Haar measure have been discussed earlier in QCD (see [13] and references therein).

The $SU(2)$ group has a finite group volume. The integration range of $\omega$ fields is therefore be restricted. According to the current paradigm, one can extend the integrals over $\omega$ from $-\infty$ to $+\infty$ within a perturbation theory framework. The modification of the result is connected to the integration over large field fluctuations, so the variance has, apparently, a non-perturbative nature and does thereby not affect the perturbation series. As a matter of fact, this justifies the standard loop expansion in ChPT.

X. SUMMARY

In this work, we discussed analogy between the second-class constraints systems and gauge theories with the equivalent structure of gauge generators and gauge-fixing conditions. Given the symplectic basis for the constraint functions exists globally, the second-class constraints systems can be interpreted as gauge invariant systems in the unconstrained phase space. Such systems can be quantized using the methods specific for gauge theories.

The second-class constraints $G_{\alpha}$ split in the symplectic basis into canonical pairs $(\chi_A, \Omega_B)$ satisfying $\{\chi_A, \chi_B\} \approx 0$, $\{\Omega_A, \Omega_B\} \approx 0$, and $\{\chi_A, \Omega_B\} \approx \delta_{AB}$. The constraint functions $(\chi_A, \Omega_B)$ can be transformed further, as discussed in Sects. 4 and 5, to fulfill the Poisson bracket relations in the strong sense in an entire neighborhood of any given point of the constraint submanifold. The Hamiltonian function can also be modified to be identically in
involution with the constraints. The new constraints define the same constraint submanifold, whereas the new Hamiltonian and its first derivatives coincide with the original ones on the constraint submanifold. The constrained dynamics is thus not modified. The new constraint functions $\chi_A$ and $\Omega_A$ are interpreted as gauge-fixing conditions and first-class constraints associated to gauge transformations.

We do not provide any criterion for what part of the constraints $G_A$ describes the gauge-fixing conditions and what part describes the first-class constraints associated to gauge transformations. By contray, we argue that transition amplitudes of the quantum theory do not depend on the interpretation of $G_A$.

The Dirac’s supplementary conditions $\Omega_A \Psi = 0$ depend on the way the constraints $G_A$ were split. These conditions are equivalent to $\chi_A \Psi' = 0$, since the corresponding Wigner functions coincide on the constraint submanifold. The supplementary conditions for the Wigner functions, furthermore, can be made to be explicitly invariant with respect to possible transformations of the constraint functions. The ambiguity reflects the freedom in extrapolation of the Wigner functions from the constraint submanifold into the unconstrained phase space.

We showed, finally, that the proposed quantization scheme applies to an $n-1$-dimensional spherical pendulum, which represents a mechanical version of the $O(n)$ non-linear sigma model. For this model, we demonstrated the existence of an underlying gauge symmetry and that the Wigner functions coincide on the constraint submanifold, which represents a mechanical version of the Wigner functions. The ambiguity reflects the freedom in extrapolation of the Wigner functions from the constraint submanifold into the unconstrained phase space.

APPENDIX A: TWO-CONSTRAINTS FORM OF GAUGED SPHERICAL PENDULUM

Let us consider another example. The system discussed in Sect. 3 is equivalent to a system described by a vector $\phi^i = (\phi^0, \phi^α)$ on the ”light-cone” submanifold $\phi^2 = \phi^0 \phi^0 - \phi^α \phi^α = 0$. The constraint $\phi^α \phi^α = 1$ is equivalent to the constraint $\phi^0 = 1$, while the transformations (1.12) are equivalent to Lorentz boosts of the null vector $\phi^0$ along the vector $\phi^α$,

$$\phi^i \rightarrow \phi'^i = \Lambda^i_j(\theta) \phi^j = \exp(\theta) \phi^j \quad (A.1)$$

where the ”boost velocity” $v = \tanh(\theta)$. The gauge invariance of the $L_*$ with respect to the dilatation reflects gauge invariance of the system with respect to the Lorentz boosts.

The ”light-cone” Lagrangian $L_2$ can be constructed by considering the requirement of the conditional maximum of the action

$$\max_{\phi^0, \phi^α} \{ \int L_2 dt \} |_{\phi^0 = 1} = \max_{\phi^0, \phi^i, \phi^j} \{ \int L_2 dt \} |_{\phi^2 = 0, \phi^0 = 1}. \quad (A.2)$$

If the $\phi^α \phi^α$ is treated as a gauge parameter, the problem simplifies. It is sufficient to require

$$\max_{\phi^0, \phi^α} \{ \int L_2 dt \} = \max_{\phi^0, \phi^i} \{ \int L_2 dt \} |_{\phi^2 = 0}. \quad (A.3)$$

The $L_2$ can be chosen as a straightforward extension of $L_*$:

$$L_2 = -\frac{1}{2} G_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j / ((\eta \phi)^2 - \phi^2) \quad (A.4)$$

where $\eta = (1, 0, ..., 0)$,

$$G_{ij}(\phi) = g_{ij} - \frac{\eta_i (\eta_i \phi_j + \eta_j \phi_i) - \phi^2 \eta_i \eta_j - \phi_i \phi_j}{(\eta \phi)^2 - \phi^2}. \quad (A.5)$$
and \( g_{ij} = \text{diag}(1, -1, \ldots, -1) \), \( g^{ij} = g_{ij} \).

The tensor \( G_{ij} \) obeys
\[
\eta^i G_{ij}(\phi) = \phi^i G_{ij}(\phi) = 0,
\]
(\ref{eq:G_shifts})
\[
G_{ij}(\phi)G_{jk}(\phi) = G_{ik}(\phi).
\]
(\ref{eq:G_gauge_inv}

It is invariant with respect to the dilatation (\ref{eq:G_shifts}) and the shifts
\[
\phi^i \rightarrow \phi^i + \epsilon \eta^i
\]
(\ref{eq:G_gauge_shifts})

where \( \epsilon \) is an arbitrary parameter.

Lagrangian (\ref{eq:G_lagrangian}) is well defined for \( \phi^2 \neq 0 \). It is invariant with respect to the dilatation (\ref{eq:G_shifts}) and the shifts (\ref{eq:G_gauge_shifts}).

The \( \phi^2 \) and the \( \eta \phi \) are thus gauge functions. They are not fixed by equations of motion and can be selected to satisfy admissible constraints. We consider therefore Lagrangian (\ref{eq:G_lagrangian}) without imposing any constraints. The initial and final conditions \( \phi^2 = 0 \) and \( \eta \phi = 1 \) are gauge-fixing conditions.

The canonical momenta corresponding to the \( \dot{\phi}^i \) are defined by
\[
\pi_i = \frac{\partial L}{\partial \dot{\phi}^i} = -G_{ij}(\phi) \dot{\phi}^j / ((\eta \phi)^2 - \phi^2).
\]
(\ref{eq:G_momenta})

They satisfy the primary constraints
\[
\pi_i - G_{ij} \pi^j = 0
\]
(\ref{eq:G_primary_constraints})

which are equivalent to two ones:
\[
\Omega_1 = \phi \pi \approx 0
\]
(\ref{eq:G Omega_1})
\[
\Omega_2 = \eta \pi \approx 0
\]
(\ref{eq:G Omega_2})

The primary Hamiltonian can be obtained with the use of Legendre transformation:
\[
H = -\frac{1}{2}((\eta \phi)^2 - \phi^2)G^{ij}(\phi)\pi_i \pi_j.
\]
(\ref{eq:G primary_Hamiltonian})

The Poisson bracket for canonical coordinates and momenta have the form
\[
\{ \phi_i, \pi_j \} = g_{ij}.
\]
(\ref{eq:G PB})

The primary constraints are stable with respect to the time evolution:
\[
\{ \Omega_1, H \} = 0
\]
(\ref{eq:G Omega_1_H})
\[
\{ \Omega_2, H \} = 0
\]
(\ref{eq:G Omega_2_H})

The Hamiltonian \( H \) is gauge invariant. The primary constraints are of the first class:
\[
\{ \Omega_1, \Omega_2 \} = 0.
\]
(\ref{eq:G primary_constraints})

The generators of gauge transformations constitute an algebra.

The relations
\[
\{ \phi^i, \Omega_1 \} = \phi^i, \quad \{ \phi^i, \Omega_2 \} = \eta^i
\]
(\ref{eq:G Omega_1_phi})
\[
\{ \pi_i, \Omega_1 \} = -\pi_i, \quad \{ \pi_i, \Omega_2 \} = 0
\]
(\ref{eq:G Omega_2_pi})

show that the \( \Omega_1 \) generates the dilatation of the \( \phi^i \) and the \( \pi_i \), while the \( \Omega_2 \) generates time-like shifts of \( \phi^i \).

The gauge-fixing conditions
\[
\chi_1 = \frac{1}{2} \ln((\eta \phi)^2 - \phi^2),
\]
(\ref{eq:G_gauge_fixing_1})
\[
\chi_2 = \eta \phi - 1
\]
(\ref{eq:G_gauge_fixing_2})

generate the following transformations:
\[
\{ \phi^i, \chi_1 \} = 0, \quad \{ \phi^i, \chi_2 \} = 0
\]
(\ref{eq:G_gauge_transforms_1})
\[
\{ \pi^i, \chi_1 \} = \frac{\phi^i - \eta^i \eta \phi}{(\eta \phi)^2 - \phi^2}, \quad \{ \pi^i, \chi_2 \} = -\eta^i.
\]
(\ref{eq:G_gauge_transforms_2})

They are identically in involution with the Hamiltonian:
\[
\{ \chi_1, H \} = 0, \quad \{ \chi_2, H \} = 0.
\]
(\ref{eq:G_gauge_fixing_Hamiltonian})

The equations of motion generated by the primary Hamiltonian look like
\[
\dot{\phi}^i = \{ \phi^i, H \} = -((\eta \phi)^2 - \phi^2)G^{ij}(\phi)\pi_j,
\]
(\ref{eq:G equations_of_motion_1})
\[
\dot{\pi}_i = \{ \pi_i, H \} \approx -((\phi_i - \eta_i \eta \phi))G^{jk}(\phi)\pi_j \pi_k.
\]
(\ref{eq:G equations_of_motion_2})

The main inference is that starting from the different Lagrangian, an equivalent first-class constraints system was constructed. There is no principal distinction between the system described here and the one discussed in Sect. 3. In particular, one can solve the constraints \( \chi_2 = 0 \) and \( \Omega_2 = 0 \) to remove the canonically conjugate pair \( (\phi^0, \pi_0) \) from the Hamiltonian. The system of Sect. 3 would be reproduced then explicitly.

**APPENDIX B: SYMPLECTIC BASIS FOR FIRST-CLASS CONSTRAINTS**

In Sect. 4, an equivalent system of second-class constraints satisfying involution relations Eqs. (\ref{eq:G_involution_1}) in the strong sense in an entire neighborhood of a given point of the constraint submanifold has been constructed. The existence of an equivalent Hamiltonian identically in involution with the new constraints has also been demonstrated Eq. (\ref{eq:G_involution_1}). The equivalence means that the constraint submanifolds and the phase space flows on the constraint submanifolds of the dynamical systems coincide.

Similar statements for first-class constraints systems are proved in Refs. (\ref{ref:10}, \ref{ref:32}). The arguments of Refs. (\ref{ref:10}, \ref{ref:32}) cannot be extended to second-class constraints without additional assumptions. Let us discuss the restrictions.
To replace the constraint functions $\Omega_A (A = 1, \ldots, m)$ by equivalent constraint functions $\tilde{\Omega}_A$ which are identically in involution $\{\tilde{\Omega}_A, \tilde{\Omega}_B\} = 0$, one can solve equations $\Omega_A = 0$ with respect to first $m$ canonical momenta $p_A = P_A$ where $P_A$ are functions of the $n$ canonical coordinates and the remaining $n - m$ canonical momenta. To have the constraints $\tilde{\Omega}_A = 0$ resolved, one has to require

$$\det || \frac{\partial \tilde{\Omega}_A}{\partial p_B} || \neq 0. \quad (B.1)$$

The phase space has a dimension $2n$, $n > m$. The functions $\tilde{\Omega}_A = P_A - P_A$ vanish on the submanifold $\tilde{\Omega}_A = 0$ only. The Poisson bracket $\{\tilde{\Omega}_A, \tilde{\Omega}_B\}$ vanishes weakly and does not depend on the first $m$ canonical momenta, therefore it vanishes identically.

The same prescription can be used to construct the functions $\chi_A$. The Poisson bracket $\{\chi_A, \tilde{\chi}_B\}$ vanishes identically. The new constraint functions satisfy $\det \{\chi_A, \tilde{\Omega}_B\} \neq 0$.

One can define an equivalent Hamiltonian $\tilde{\mathcal{H}}$. Let us substitute $p_A = P_A$ to the original Hamiltonian $\mathcal{H}'$ of the second-class constraints system. The resulting Hamiltonian $\tilde{\mathcal{H}}$ is first class with respect to the constraints $\tilde{\Omega}_\alpha = 0$, so $\{\tilde{\Omega}_A, \tilde{\mathcal{H}}\} \approx 0$. The difference $\tilde{\mathcal{H}} - \mathcal{H}'$ vanishes on the constraint submanifold $\tilde{\Omega}_A = 0$. The Hamiltonian $\tilde{\mathcal{H}}$ does not depend on the first $m$ canonical momenta, so the Poisson bracket does not depend on these canonical momenta either. The $\tilde{\mathcal{H}}$ is therefore identically in involution with the $\tilde{\Omega}_A$. The similar procedure can be applied for the $\tilde{\chi}_A$.

We should get finally constraint functions $\tilde{\chi}_A$ and $\tilde{\Omega}_A$ identically in involution with the $\tilde{\mathcal{H}}$.

The above arguments do not apply if some constraint functions do not depend on $p_A (\tilde{q}_A)$. The bracket $\{\tilde{\Omega}_A, \tilde{\Omega}_B\}$ can, e.g., be proportional to $\chi_C$ and $\chi_C$ can in turn be independent on $p_A$. In such a case, the weak equation $\{\tilde{\Omega}_A, \tilde{\Omega}_B\} \approx 0$ does not convert into the strong one, although the both sides do not depend on $p_A$. The similar restrictions appear in the construction of $\tilde{\chi}_A$ and $\tilde{\mathcal{H}}$. The systems under holonomic constraints have, in particular, constraints $\chi_A = 0$ which do not depend on the canonical momenta.

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