We use the T-duality transformation which relates M-theory on $T^3$ to M-theory on a second $T^3$ with inverse volume to test the Banks-Fischler-Shenker-Susskind suggestion for the matrix model description of M-theory. We find evidence that T-duality is realized as S-duality for $U(\infty) \ N = 4$ Super-Yang-Mills in 3+1D. We argue that Kaluza Klein states of gravitons correspond to electric fluxes, wrapped membranes become magnetic fluxes and instantonic membranes are related to Yang-Mills instantons. The T-duality transformation of gravitons into wrapped membranes is interpreted as the duality between electric and magnetic fluxes. The identification of M-theory T-duality as SYM S-duality provides a natural framework for studying the M-theory 5-brane as the S-dual object to the unwrapped membrane. Using the equivalence between compactified M(atrix) theory and SYM, we find a natural candidate for a description of the light-cone 5-brane of M-theory directly in terms of matrix variables, analogous to the known description of the M(atrix) theory membrane.
1. Introduction

During the past two years, evidence has been accumulating which indicates that a consistent quantum theory (M-theory) underlies 11D supergravity [1,2]. Recently, an exciting conjecture for a microscopic description of M-theory has been put forward by Banks, Fischler, Shenker and Susskind [3]. The BFSS model incorporates in a natural way the non-commutative nature of microscopic space time [4] and the quantization of the membrane [5].

The authors of [3] have shown that their model contains many of the features of M-theory: the supermembrane, correct graviton scattering amplitudes, toroidal compactification and partial 11D Lorentz invariance. Further evidence for the BFSS conjecture was supplied in [6] where the behavior of a membrane in a 5-brane background was studied. Questions which remain open include a general description of compactification, an intrinsic description of a 5-brane and a complete proof of 11D Lorentz invariance (a suggestion in this direction has been made in [7]).

The purpose of the present paper is to pass M(atrix)-theory through one more test by considering its behavior under T-duality. T-duality relates compactified type IIA to compactified type IIB so in order to get an “automorphism” of M-theory we need to compactify the type IIA theory on $T^2$ and apply T-duality twice. This gives a connection between M-theory on $T^3$ with volume $V$ and M-theory on $T^3$ with volume $1/V$. Under this duality, wrapped membrane states are exchanged with Kaluza-Klein states of the graviton and unwrapped membranes become wrapped 5-branes [8,9].

Our discussion begins with a review of the M(atrix)-theory description of toroidal compactification given in [3,10]. The resulting model is a large $N$ limit of $U(N)$ Super-Yang-Mills theory. We recast the derivation in a slightly different language and explain how twisted sectors of the $U(N)$ bundle appear. We then argue that T-duality is realized as S-duality in the SYM theory and that graviton $\Leftrightarrow$ membrane duality is related to electric $\Leftrightarrow$ magnetic duality. Finally, we discuss some issues related to finding an explicit construction of the 5-brane in M(atrix) theory.

The paper is organized as follows: Section 2 is a review of toroidal compactification of the M(atrix)-model. In Section 3 we relate S-duality of $\mathcal{N} = 4$ SYM to T-duality. In Section 4, the transformations of gravitons into membranes in M-theory are discussed. We relate membranes to magnetic fluxes and gravitons with KK momentum to electric fluxes. In Section 5 we use the interpretation of the membranes in terms of magnetic flux to obtain the energy and counting of membrane states. We also relate the Yang-Mills instanton to a Euclidean membrane. In Section 6 we suggest a formulation of a 5-brane wrapped around the light-cone directions in terms of BFSS matrix variables satisfying a certain relation. We also discuss the implications of the T-duality/S-duality equivalence for the search for a 5-brane in M(atrix) theory which extends along 5 transverse directions.
2. Review of Compactification

One way of understanding toroidal compactification of M(atrix)-theory is by considering a sector of the $N \rightarrow \infty$ 0-brane theory in which the $X$ matrices satisfy certain symmetry conditions. This description for compactification on the $d$-dimensional torus $T^d$ can be given as follows \[3,10\]: Infinite unitary matrices $U_i$ are chosen for $i = 1\ldots d$ that commute with each other,

$$U_i U_j = U_j U_i,$$  \hspace{1cm} (2.1)

and generate a subgroup of $U(\infty)$ isomorphic to $\mathbb{Z}^d$. The compactified theory is given by restricting to the subspace of $X$’s which are invariant under the $\mathbb{Z}^d$ action:

$$U_i : X^\mu \rightarrow U_i^{-1} X^\mu U_i + e_i^\mu$$  \hspace{1cm} (2.2)

where the $e_i$’s form a basis of the lattice whose unit cell is $T^d$. The countably infinite dimensional vector space on which the $X$’s act can be written as a tensor product

$$V = V_N \otimes H^d$$  \hspace{1cm} (2.3)

where $V_N$ is an $N$-dimensional space and $H$ is countably infinite dimensional. One can then take

$$U_i = I \otimes I_1 \otimes \cdots \otimes I_{i-1} \otimes S_i \otimes I_{i+1} \otimes \cdots \otimes I_d$$  \hspace{1cm} (2.4)

where $I_j$ is the identity on the $j$th $H$ and $S_i$ is a shift operator

$$(S_i)_{k,l} = \delta_{k+1,l}$$  \hspace{1cm} (2.5)

(the indices $k,l$ run over all integers $\mathbb{Z}$). By restricting the Lagrangian to the subspace of $X$’s invariant under (2.2) one obtains the Lagrangian for $(d+1)$ dimensional SYM theory. The gauge fields are

$$A^\mu \left( \sum_{i=1}^d x_i \hat{e}_i \right) = \sum_{l_1,\ldots,l_d \in \mathbb{Z}} e^{2\pi i \sum_{l_1,\ldots,l_d} l_i x_i X^\mu_{(0,l_1)\cdots(0,l_d)}}$$  \hspace{1cm} (2.6)

where $\hat{e}_i$ form a basis of the dual torus $\hat{T}^d$. When $d = 3$, the inverse squared coupling constant is the volume of $T^3$ (in 11-dimensional Planck units) and the $(3+1)$-dimensional theory is defined on the dual torus $\hat{T}^3$.

For later use, we will recast the derivation in a different language. We note that the resulting matrix $X^\mu$ invariant under (2.2) is just the matrix of the operator

$$\nabla^\mu = i \partial^\mu + A^\mu$$  \hspace{1cm} (2.7)
acting on fields in the fundamental representation

\[ \phi_k \left( \sum_{i=1}^{d} x_i \hat{e}_i \right), \quad k = 1 \ldots N \]  

(2.8)

which are sections of the trivial bundle over \( \hat{T}^d \). The requisite form of the \( X \) matrices is obtained by writing \( \nabla^\mu \) in the Fourier basis of \( \hat{\phi}_k(n_1 \ldots n_d) \):

\[ \phi_k \left( \sum x_i \hat{e}_i \right) = \tilde{\phi}_k(n_1 \ldots n_d) e^{2\pi i \sum n_i x_i} \]  

(2.9)

The operators \( U_j \) can be taken to act on sections by

\[ U_j \phi_k \left( \sum x_i \hat{e}_i \right) = e^{2\pi i x_j} \phi_k \left( \sum x_i \hat{e}_i \right) \]  

(2.10)

From (2.7) it is clear that if we also take \( X^\mu \) for \( \mu = d+1, \ldots, 9 \) to be the matrices of the operators

\[ \phi \rightarrow \Phi^\mu \phi \]  

(2.11)

where \( \Phi^\mu \) are the scalar fields of SYM, then the BFSS Lagrangian will reduce to the SYM Lagrangian.

3. Checking T-duality

A non-trivial duality of M-theory is obtained by compactifying on \( T^3 \), regarding the theory as type IIA on \( T^2 \) and T-dualizing twice (once along each direction of the \( T^2 \)).

3.1. Review of T-duality for M-theory

We will be using eleven dimensional Planck units \( l_p = 1 \). T-duality on M-theory is obtained from the relations (here \( G_{\mu\nu} \) is the metric in coordinates 0, \ldots, 6,11. Note that our space-time coordinates are \( x_0 \ldots x_9, x_{11} \)):

\[ \begin{align*}
\text{M – theory} & : \quad G_{\mu\nu}; R_7, R_8, R_9 \\
\text{Type IIA} & : \quad R_9 G_{\mu\nu}; R_9^{1/2} R_7, R_9^{1/2} R_8; \lambda_{st} = R_9^{3/2}
\end{align*} \]  

(3.1)

together with type IIA T-duality twice (here \( l_i \) are lengths in string units):

\[ \begin{align*}
\frac{\text{Type IIA}}{g_{\mu\nu}; l_8, l_9; \lambda} & = \frac{\text{Type IIA}}{g_{\mu\nu}; l_8^{-1}, l_9^{-1}; l_8^{-1} l_9^{-1} \lambda}
\end{align*} \]  

(3.2)
To obtain
\[
\frac{M - \text{theory}}{G_{\mu\nu}; R_7, R_8, R_9} = \frac{M - \text{theory}}{V^{2/3}G_{\mu\nu}; V^{-2/3}R_7, V^{-2/3}R_8, V^{-2/3}R_9}
\]
(3.3)

where \( V = R_7R_8R_9 \) (the RHS actually comes out with \( R_7 \) and \( R_8 \) switched but in (3.3) we have combined a reflection). This can be generalized to slanted \( T^3 \)'s and to include the 3-form \( C_{\mu\nu\rho} \):

\[
\tau = iV_{7,8,9} + C_{7,8,9} \rightarrow -\frac{1}{\tau}.
\]

T-duality also acts non-trivially on D-brane states. Let us denote the internal torus in directions 7, 8, 9 by \( T^3 \). A state containing a graviton with momentum

\[
\vec{p} = n_7\hat{e}_7 + n_8\hat{e}_8 + n_9\hat{e}_9
\]

(3.5)

(where \( \hat{e}_7, \hat{e}_8, \hat{e}_9 \) are basis vectors for \( \hat{T}^3 \) and we will assume \( (n_7, n_8, n_9) \) to be relatively prime) is transformed by T-duality to a state with a membrane. When the T-duality is combined with the (78) reflection, the membrane is wrapped on the plane orthogonal to \( \vec{p} \) in the original \( T^3 \). This is easily seen by regarding the theory as type IIA on \( T^2 \) and using the T-duality transformations of Dirichlet \( p \)-branes into Dirichlet \((p \pm 2)\)-branes together with the identification of a D2-brane of type IIA with a membrane of M-theory and a 0-brane of type IIA with a KK state of a graviton.

Next, let us take a membrane wrapped on, say, the 8-9 directions and with momentum along the 7th direction. Regarding it as an elementary string of type IIA wrapped on the 8th direction and with momentum along the 7th direction we see that after T-duality and the (78) reflection, it becomes again a string wrapped on the 8th direction and with momentum along the 7th direction – so in M-theory it is another membrane wrapped on the 8-9 directions and with momentum along the 7th direction.

In general, a membrane wrapped on a plane orthogonal to \( \vec{p} \) and with momentum

\[
\vec{q} = m_7\hat{e}_7 + m_8\hat{e}_8 + m_9\hat{e}_9
\]

(3.6)

\((m_7, m_8, m_9 \text{ are integers, not all zero})\) becomes a membrane wrapped on a plane orthogonal to \( \vec{q} \) (wrapped \( \text{gcd}(m_7, m_8, m_9) \) times) and with momentum \( \vec{p} \).

We note that all these formulae are symmetric with respect to the 3 directions of the \( T^3 \) – as they should be. This symmetry is manifested by the fact that both membrane winding numbers and Kaluza-Klein momenta are parameterized by a vector in the dual lattice (whose unit cell is \( T^3 \)). T-duality, composed with the reflection, exchanges winding with KK momentum.

Finally, we can take a membrane that is extended in two non-compact directions, say 5, 6. Regarding it as the D2-brane of type IIA, we see that it becomes the D4-brane after T-duality. Since the D4-brane is a 5-brane of M-theory wrapped on the 9th direction, we see that a non-wrapped membrane becomes a 5-brane that is wrapped on all of the directions of \( T^3 \).
3.2. T-duality in the Matrix model

We have seen that M-theory on $\mathbb{T}^3$ is equivalent to $\mathcal{N} = 4$ $U(N)$ SYM on the dual torus $\hat{T}^3$:

$$H = \frac{1}{2} \int d^3x \text{tr} \left\{ g^2 \tilde{E}^2 + \frac{1}{g^2} \tilde{B}^2 + \frac{\theta}{8\pi^2} \tilde{E} \cdot \tilde{B} + g^2 \sum_{A=1}^{6} |D_i \Phi^A|^2 ight.$$  
$$
+ \frac{1}{g^2} \sum_{A=1}^{6} |D_i \Phi^A|^2 + \frac{1}{g^2} \sum_{1 \leq A < B \leq 6} |[\Phi^A, \Phi^B]|^2 + \text{(fermions)} \right\}. \tag{3.7}
$$

The coupling constant becomes

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} = iV_{T^3} + C \tag{3.8}$$

where $C$ is the 3-form VEV on $T^3$. The S-duality transformation

$$\tau \rightarrow -\frac{1}{\tau} \tag{3.9}$$

agrees with (3.4). According to (3.3), the sides of $\hat{T}^3$ are rescaled by $(\text{Im} \, \tau)^{2/3}$ without changing its shape. Thanks to the exact conformal invariance of $\mathcal{N} = 4$ Yang-Mills we can rescale $\hat{T}^3$ and define the matrix model on a $\hat{T}^3$ of volume 1. This has to be accompanied by rescaling of the six scalars $X^\mu$ by $V^{-1/3}$. The S-dual $X^\mu$ should thus be rescaled by $V^{1/3}$. Since S-duality of SYM multiplies the Higgs VEVs by $V = 1/g^2$ we find that altogether $X^\mu$ is rescaled by $V^{1/3}$. This is in accord with the Weyl rescaling of $G_{\mu\nu}$ in (3.3).

4. T-duality action on branes

In this section we examine how the action of T-duality on states is manifested in the matrix model. Since S-duality exchanges electric and magnetic fluxes, a natural guess is that electric (magnetic) states correspond to graviton (membrane) states.

As described in Section 3, both the winding number of a membrane and the KK momentum are parameterized by vectors in the dual lattice with unit cell $\hat{T}^3$. Moreover, T-duality exchanges the winding and the momentum. Electric and magnetic fluxes are also parameterized by vectors in the dual lattice. Their exchange under S-duality is in accord with their identification with branes. We shall now examine the correspondence in more detail.
4.1. Review of electric and magnetic fluxes

We recall that $U(1)$ is a normal subgroup of $U(N)$ and

$$U(N) = (U(1) \times SU(N))/\mathbb{Z}_N$$  \hspace{1cm} (4.1)

We will consider $U(N)$ Super-Yang-Mills on $\hat{T}^3$. On $\hat{T}^2 \subset \hat{T}^3$ we can have non-trivial $U(N)$-bundles. States with one unit of $U(1)$ magnetic flux satisfy:

$$\int_{T^2} \text{tr}\{B\} = 2\pi.$$  \hspace{1cm} (4.2)

To build the corresponding bundle, we pick $[11]$:

$$U^{-1}V^{-1}UV = e^{-\frac{2\pi i}{N}}, \quad U, V \in SU(N).$$  \hspace{1cm} (4.3)

Let the coordinates on $\hat{T}^2$ be $x, y$ with period 1. The bundle is defined by the boundary conditions

$$\phi(x, y) = e^{-\frac{2\pi i}{N}} U^{-1} \phi(x + 1, y) = V^{-1} \phi(x, y + 1).$$  \hspace{1cm} (4.4)

A state with one unit of electric flux in the direction of $S^1 \subset T^3$ (where $S^1$ is some cycle of the torus) satisfies:

$$\int_{S^1} \text{tr}\{E\} = 2\pi.$$  \hspace{1cm} (4.5)

4.2. Fluxes and compactified M(atrix)-theory

In this section we show how the membrane of $[3]$ naturally becomes a state with magnetic flux after compactification. To see this, we begin by showing how twisted $U(N)$ bundles are obtained from the construction (2.2). The identification of $X^\mu$ as $(i\partial^\mu + A^\mu)$ at the end of Section 2 suggests that we write the same operator as a matrix acting on sections of the twisted bundle. Given this identification, we have

$$[X^7, X^8] = [\nabla^7, \nabla^8] = iB^{78}$$  \hspace{1cm} (4.6)

where in the sector with one unit of flux, $B^{78}$ is the scalar matrix $\frac{2\pi}{N}I$. As $N \to \infty$ it is natural to identify this state with the wrapped BFSS membrane.

We will now go through this construction explicitly. We begin by finding a complete basis of sections of the twisted bundle. Working on $T^2$ and taking the sections in the fundamental representation:

$$\phi(x + 1, y) = e^{\frac{2\pi i}{N}} U \phi(x, y),$$

$$\phi(x, y + 1) = V \phi(x, y),$$  \hspace{1cm} (4.7)
where
\[
U = \begin{pmatrix}
1 & e^{\frac{2\pi i}{N}} & \\
& \ddots & \\
e^{\frac{2\pi i(N-1)}{N}} & & 1
\end{pmatrix}, \quad V = \begin{pmatrix}
1 & & \\
& \ddots \ & \\
& & 1
\end{pmatrix}
\] (4.8)

\[
U^{-1}V^{-1}UV = e^{-\frac{2\pi i}{N}}
\] (4.9)

We define
\[
X^\mu = (\nabla^\mu)^{\text{twisted}} \equiv i\partial^\mu + A^\mu
\] (4.10)

The expansion of \(\phi_k(x, y)\) in terms of a complete basis is very different from the untwisted sector:
\[
\phi_k(x, y) = \sum_{p \in \mathbb{Z}} \hat{\phi}(y + k + Np)e^{\frac{2\pi i}{N}(y + k + Np)x}.
\] (4.11)

Here \(\hat{\phi}\) is some arbitrary continuous function defined on \((-\infty, \infty)\) with \(\int |\hat{\phi}|^2 < \infty\). So, the space on which \(X^\mu\) is defined is now continuous (though, since \(\hat{\phi} \in L^2(\mathbb{R})\) we can find a countable basis as before).

Writing \(X^\mu\) in the basis of \(\hat{\phi}(w)\) we find
\[
(X^7)_{kl}(w, w') = \frac{2\pi}{N} w \delta(w - w')\delta_{kl} - \sum_{p,q \in \mathbb{Z}, s \in \mathbb{Z}_N} \delta(w' + Np + k - l - w)e^{2\pi i(q + \frac{s}{N})w'}a^7_{p + \frac{k - l}{N}, q + \frac{s}{N}},
\] (4.12)
\[
(X^8)_{kl}(w, w') = i\delta'(w - w')\delta_{kl} - \sum_{p,q \in \mathbb{Z}, s \in \mathbb{Z}_N} \delta(w' + Np + k - l - w)e^{2\pi i(q + \frac{s}{N})w'}a^8_{p + \frac{k - l}{N}, q + \frac{s}{N}}.
\]

Here \(-\infty < w, w' < \infty\) are the arguments of \(\hat{\phi}\) and \(k, l\) are \(U(N)\) indices. \(a^7, a^8\) are the modes of the gauge fields:
\[
A^7_{kl}(x, y) = \sum_{p,q \in \mathbb{Z}, s \in \mathbb{Z}_N} a^7_{p + \frac{k - l}{N}, q + \frac{s}{N}}e^{2\pi i(p + \frac{k - l}{N})x}e^{2\pi i(q + \frac{s}{N})(y + l)},
\] (4.13)
\[
A^8_{kl}(x, y) = \frac{2\pi}{N} x \sum_{p,q \in \mathbb{Z}, s \in \mathbb{Z}_N} a^8_{p + \frac{k - l}{N}, q + \frac{s}{N}}e^{2\pi i(p + \frac{k - l}{N})x}e^{2\pi i(q + \frac{s}{N})(y + l)}.
\]

From (4.10) it is obvious that the BFSS action goes over to the SYM action. The \(U_1\) and \(U_2\) matrices which define the sector are still given by multiplication by \(e^{ix}\) and \(e^{iy}\). As \(N \times N\) matrices they are different from those for the untwisted sector because they act on sections \((4.11)\) rather than functions. Thus there are several ways to embed \(\mathbb{Z}^d\) subgroups which are not mutually conjugate.
We also note that there is another way to realize the twisted $X^\mu$’s. For compactifications on $T^2$, we take a vector space (on which $X^\mu$ will act) that is a product of a finite dimensional vector space $V_N$ of dimension $N$ and single Hilbert space $H$ with a countably infinite basis

$$V = V_N \otimes H. \quad (4.14)$$

Note the difference between (1.14) and (2.3) – here only one copy of $H$ (rather than two) is used for compactification on $T^2$. This difference may be related to the form of the expansion (4.11). We realize $U_1$ and $U_2$ as

$$U_1 = U \otimes e^{iP/\sqrt{N}}, \quad U_2 = V \otimes e^{iQ/\sqrt{N}}, \quad (4.15)$$

where $Q, P$ are canonical operators acting on $W$ ($[Q, P] = 2\pi i$) and $U, V$ are as in (4.8). $U_1$ and $U_2$ commute and one can check that the expansion of (2.2) agrees with (4.12).

Next we show the relation between electric flux and graviton states. The operator that measures the total electric flux inside $\tilde{T}^3$ is given by

$$\int_{\tilde{T}^3} \text{tr}\{E_i\}. \quad (4.16)$$

Using

$$X^\mu = i\partial^\mu + A^\mu, \quad (4.17)$$

we can write the electric flux as

$$\int_{\tilde{T}^3} \text{tr}\{E_i\} = \int_{\tilde{T}^3} \dot{A}_i = \text{Tr}\{\dot{X}_i\} = \text{Tr}\{\Pi_i\} \quad (4.18)$$

where $\text{Tr}$ is the trace in the infinite basis of the matrix model. Thus, electric flux in SYM naturally corresponds to momentum along the $i$-th direction of the $T^3$ in M-theory.

5. More on membranes

In this section we describe further evidence in favor of the identifications between magnetic flux and Lorentzian membranes. We also propose a connection between Yang-Mills instantons and Euclidean membranes.
5.1. Energy and counting of wrapped membrane states.

Consider M-theory on $T^2$. We would like to calculate the energy of the membrane using the fact that the membrane states correspond to states in the sector with non-zero magnetic flux in the $U(1)$ subgroup. In “mathematical” conventions where there is an overall $g_{YM}^{-2}$ in front of the action, the unit of flux is

$$ \int trB = 2\pi. \quad (5.1) $$

Taking $B$ of the form $B_0$ times the identity matrix, we find that

$$ B_0 = \frac{1}{2\pi N R_1 Y M R_2 Y M} \quad (5.2) $$

where T-duality relates the Yang-Mills and M-theory lengths through $R_i^{Y M} = \frac{2\pi \alpha'}{R_i}$. In this subsection we restore the 11D Planck length

$$ l_p = g_{1/3}^{1/3} \sqrt{\alpha'}. \quad (5.3) $$

The coupling constant of the YM is given by

$$ g_{YM}^2 = \frac{g\sqrt{\alpha'}}{R_1 R_2}. \quad (5.4) $$

Calculating the energy we get :

$$ \frac{1}{2} g_{YM}^{-2} \int trB^2 = \frac{(2\pi)^{-4}(R_1 R_2)^2}{2gN(\alpha')^{5/2}} \quad (5.5) $$

As in the discussion of the membrane tension in [3], the energy of the membrane in the light-cone frame is given by

$$ E = \sqrt{p^2 + M^2} = \sqrt{\left(\frac{N}{g\sqrt{\alpha'}}\right)^2 + (TR_1 R_2)^2} \sim \frac{N}{g\sqrt{\alpha'}} + \frac{g\sqrt{\alpha'T^2}(R_1 R_2)^2}{2N}. \quad (5.6) $$

Here $T$ is the membrane tension and $\frac{N}{g\sqrt{\alpha'}}$ is the momentum along $X_{11}$. The energy of the state with magnetic flux in the large $N U(N)$ Yang Mills is interpreted as the excess kinetic energy. The total energy is obtained by adding the term $N/g\sqrt{\alpha'}$ due to the momentum along $R_{11}$ to the kinetic energy. This explains why the square of the area appears in the numerator, and gives the tension of the membrane,

$$ T^2 = \frac{1}{(2\pi)^4 g^2 \alpha'^3} = \frac{1}{(2\pi)^4 l_p^6}. \quad (5.7) $$
in agreement with [3].

M-theory on $R^9 \times T^2$ should have, for fixed momentum on $R^9$, one normalizable BPS multiplet (annihilated by half the supersymmetries) with the quantum numbers of an $n$-wrapped membrane. This is known [12] by using the relation between M-theory and type II strings. In the context of the relation between M-theory and large $N$ $U(N)$ Yang Mills, and the above description of membrane winding number as the magnetic flux in the $U(1)$ we can argue that this is the right counting. So we need to know the number of states coming from large $N$ $U(N)$ Yang Mills which sit in a $2^8$ dimensional representation of supersymmetry and carry $n$ units of magnetic flux. Following arguments in [4] and in [13], and in agreement with the discussion at the end of Section 4, precisely this Yang Mills question arises if we want to count the number of normalizable bound states carrying $N$ units of 2-brane charge and $n$ units of 0-brane charge in type IIA theory. We know that the unique bound state [14] of $N$ type IIA 2-branes can be given $n$ units of momentum along the eleventh dimension. We conclude that there should be a unique 1/2 BPS saturated state coming from the sector with $n$ units of magnetic flux in $U(N)$ Yang Mills. This statement in the large $N$ limit shows that M-theory on $T^2$ has exactly one BPS multiplet carrying the charge of a membrane wrapped $n$ times on the $T^2$.

5.2. Euclidean membranes

Since Lorentz invariance is not manifest in the BFSS formulation, it is interesting to check what an instantonic membrane looks like. Since the Minkowski metric is crucial to the IMF formulation, we will think of a Euclidean membrane as a transition between two different vacua which we will soon identify. Let’s take a Euclidean membrane that wraps all of $T^3$. Since it is localized in directions $1 \ldots 6$, the corresponding SYM solution must have all six adjoint scalars set to a scalar matrix, say:

$$\Phi^1 = \ldots = \Phi^6 = 0.$$  \hspace{1cm} (5.8)

This allows for an ordinary Yang-Mills instanton solution in the gauge fields. It is thus tempting to guess that a fully wrapped membrane corresponds to a transition between two vacua which differ by a large gauge transformation related to $\pi_3(U(N))$. In favor of this claim we recall from (3.4) that the $\theta$-angle is related to $C_{7,8,9}$. A membrane instanton that wraps $T^3$ is a transition in which (in an appropriate gauge):

$$\int dx_{1-6} dx_{11} \frac{d}{dt} C_{7,8,9} \sim \frac{d}{dt} \theta.$$  \hspace{1cm} (5.9)

changes by one unit. If we replace $\frac{d}{dt} \theta$ by the dual variable we find that in the SYM formulation the winding number of the vacuum increases by one unit. The action of the YM instanton, $\frac{4 \pi}{g^2} + \frac{i \theta}{2 \pi}$, agrees with the action $V_{7,8,9} + iC_{7,8,9}$ of the membrane instanton.
6. 5-branes in M(atrix) theory

We now discuss several issues related to the construction of a 5-brane in M(atrix) theory. A 5-brane which extends along the light-cone directions $x^\pm$ (and four more directions) has been discussed in [6]. However, the 5-brane described by these authors was essentially given as a background for the 0-brane theory. By using the relation between the 0-brane fields on a torus and the covariant derivative operator on the dual torus, we find a natural description of the light-cone 5-brane which is intrinsic to the 0-brane variables of M(atrix) theory. We also discuss the possibility of using T-duality to describe a 5-brane that occupies five transverse directions and is boosted along the BFSS preferred direction.

6.1. The light-cone 5-brane

It was shown by Witten [15], and in a more general form by Douglas [16], that an instanton on a $(p + 4)$-brane carries $p$-brane charge. This result is essentially due to the fact that the world volume theory on the $(p + 4)$-brane includes a Chern-Simons term

$$\int_{\Sigma_{p+5}} C \wedge e^F. \quad (6.1)$$

where $C$ is a sum over RR fields. The term

$$\int C^{(p+1)} \wedge F \wedge F \quad (6.2)$$

in particular couples an instanton to the field $C^{(p+1)}$, under which $p$-branes are electrically charged. As one application of this result, Yang Mills instantons embedded in a three-brane world-volume theory appear as D-instantons. After a T-duality which converts three branes to zero branes, these D-instantons are converted to membrane instantons. Relating this to the transformation from the 0-brane variables of M-theory on $T^3$ to the Yang Mills variables, this gives another piece of evidence in favour of the identification proposed in section 5.2.

We can use this same type of relation to show how a 5-brane of M-theory can be constructed directly from the $X$ matrix fields of BFSS. Wrapping the 5-brane around the light-cone directions, we should find a 4-brane in the resulting IIA theory. Let us compactify in dimensions 6-9. Since a 4-brane on a torus $T^4$ goes to a 0-brane on the dual torus $\hat{T}^4$, the connection [2.7] indicates that a configuration of the $X$ matrices satisfying

$$\text{Tr} \epsilon_{ijkl} X^i X^j X^k X^l = 8\pi \quad (6.3)$$
will have a unit of 4-brane charge. (The indices $i - l$ are summed over the compactified coordinates 6-9. The antisymmetric product of 4 $X$’s is just $\frac{1}{4}\epsilon_{ijkl}F^{ij}F^{kl}$ on the dual torus written in 0-brane language.) This condition for a 5-brane configuration is very similar to the condition
\[
\text{Tr} \ \epsilon_{ij}X^i X^j = 2\pi i \quad (6.4)
\]
for a membrane, which as in (6.4) is essentially the description given in [8]. Just as for the membrane, it is natural to expect that in the decompactified limit, (6.3) will be satisfied for a light-cone 5-brane. As for the 2-brane, this condition cannot be satisfied by finite dimensional matrices.

It is interesting to note that the trace in (6.3) is nonvanishing when there are membranes wrapped around for example the 67 and 89 directions. However, in this case the trace scales as $1/N$ and vanishes in the large $N$ limit. An explicit example of a configuration where (6.3) is satisfied can be constructed by simply taking the matrices corresponding to the action of the operators $i\partial^\mu + A^\mu$, where $A^\mu$ are gauge fields corresponding to an instanton on $\hat{T}^4$. Such a configuration presumably corresponds to a single 5-brane in M(atrix) theory.

6.2. Transverse 5-branes

Since T-duality transforms a 5-brane that wraps $T^3$ into a membrane that is unwarped we can relate the wave-function of a wrapped 5-brane to a wave-function for an unwrapped 2-brane through
\[
|5\text{-brane, } i, j, 7, 8, 9\rangle = S|\text{membrane, } i, j\rangle \quad (6.5)
\]
where $S$ is the S-duality operator and $|\text{membrane, } i, j\rangle$ is a state of $U(\infty)$ SYM in which the scalars $\Phi^i, \Phi^j$ (two of the six adjoint scalars) have condensed in the form
\[
\Phi^i = P + (\text{oscillators}), \\
\Phi^j = Q + (\text{oscillators}), \quad (6.6)
\]
where $P, Q$ are canonical $\infty \times \infty$ matrices.

The usefulness of formula (6.5), of course, depends upon progress in finding an explicit form for the S-duality operator $S$ and its generalization to $U(\infty)$. Once we have the wave function of the 5-brane on $T^3$ we can decompactify to 11D M-theory.

From (3.8) we see that this is done by taking the coupling constant of SYM to zero. At first sight one might worry that this gives a classical theory. However, we must first take $N \to \infty$ and only then take $g \to 0$. Thus, the effective coupling constant $g^2 N$ is never perturbative.
7. Conclusions

We have seen that T-duality is realized in a natural way in the M(atrix)-theory of BFSS as S-duality of large $N$ $U(N)$ $\mathcal{N} = 4$ Yang-Mills theory. The large/small volume duality of M-theory is mapped to the weak/strong coupling duality of SYM and graviton (0-brane)/membrane duality is mapped to electric/magnetic duality. We have also seen that there are different inequivalent embeddings of the $\mathbb{Z}^d$ symmetry group of translations in the BFSS model. The different embeddings give rise to different $U(N)$ bundles in the SYM theory. The wrapped membrane of M-theory is identified with states carrying magnetic flux.

The connections developed in this paper provide a natural framework in which to try to understand the 5-brane in M(atrix) theory. The unwrapped 5-brane is naturally related through S-duality of 3+1 dimensional Super-Yang-Mills theory to a 2-brane configuration which can be understood in matrix variables $X$. Furthermore, the 5-brane wrapped around the light-cone directions has a natural description as an “instanton” of 4+1 dimensional Super-Yang-Mills, which allows us to describe it in terms of a set of matrix variables satisfying the relation $\text{Tr} \, \epsilon_{ijkl} X^i X^j X^k X^l = 8\pi$. In the language of type IIA string theory, the fact that it is possible to describe the 4-brane in terms of fundamental 0-brane fields is essentially the T-dual of the result that instantons on a 4-brane carry 0-brane charge. Remarks along these lines were also made in [17]. It would seem that the ability of 0-branes to form the higher dimensional branes of M-theory is a strong argument in favor of the conjecture of BFSS that in fact 0-branes form a complete description of all the degrees of freedom in M-theory, at least in the IMF frame.

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Note added

As this work was completed, a paper by L. Susskind [18] appeared which discusses the same issue.
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