VALUE ITERATION FOR APPROXIMATE DYNAMIC PROGRAMMING UNDER CONVEXITY

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Abstract. This paper studies value iteration for infinite horizon contracting Markov decision processes under convexity assumptions and when the state space is uncountable. The original value iteration is replaced with a more tractable form and the fixed points from the modified Bellman operators will be shown to converge uniformly on compacts sets to their original counterparts. This holds under various sampling approaches for the random disturbances. Moreover, this paper will present conditions in which these fixed points form monotone sequences of lower bounding or upper bounding functions for the original fixed point. This approach is then demonstrated numerically on a perpetual Bermudan put option.

Keywords. Convexity, Infinite horizon, Markov decision processes, Value iteration

1. Introduction

Infinite horizon discounted Markov decision processes occur frequently in practical applications and so the literature devoted to this class of problems is already well developed. This paper refers reader to the textbooks [8, 7, 10, 4, 9, 1] and the great many references cited within them. Typical numerical approaches to these problems include policy iteration or value iteration. This paper focuses on the value iteration procedure for uncountable state spaces and finite action spaces. Value iteration has been used extensively over the decades and early works include [12, 2, 3]. The following gives a brief intuition of this approach and the issues that arises in our setting. A more formal description will be given in the next section. Suppose the Bellman operator \( T \) is given by

\[
(Tv)(x) = \max_{a \in A} (r(x, a) + \beta K^a v(x))
\]

where \( r \) is the reward function, \( a \) is the action, \( x \) is the state, \( \beta \) is the discount factor, and \( K^a \) is the transition operator which gives the expected value functions. Value iteration relies on the Bellman operator being a contraction mapping so that successive applications of the above operator converges to a fixed point. However, there are two main issues in numerical work:

- If the random variable driving \( K^a \) takes an uncountable number of possible values, one often needs a discretized version of \( K^a \) for numerical tractability, and
- If the state \( x \) takes a uncountable number of values, one may need to represent the reward and expected value functions using numerical tractable objects.

It turns out that under certain convexity assumptions, the above problems can be addressed. This is the main contribution of this paper. In this paper, the original problem is replaced with a more tractable approximation and the new fixed points are shown to converge uniformly on compact sets to the original solution. Under additional assumptions, the fixed points from the
modified value iteration form a non-decreasing sequence of lower bounding functions or a non-increasing sequence of upper bounding functions. In this way, they can provide error bounds for the approximation scheme. This paper is best seen as an extension of [13] where the author used convex function approximations for finite horizon Markov decision processes. This paper is organized as follows. In the next section, the problem setting and the convexity assumptions are presented. Section 3 then presents the modified value iteration which exploits these convexity assumptions. The convergence properties of this approach is then studied in Section 4. The lower and upper bounding behaviour of the fixed points from the modified value iteration are derived in Section 5 and Section 6. These methods are then demonstrated numerically on a perpetual Bermudan put option in Section 7. Finally, Section 8 concludes this paper.

2. Problem setting

The following introduces the infinite horizon discounted Markov decision process. Denote time by \( t \in \mathbb{N} \) and the state by \( X_t := (P_t, Z_t) \) consisting of a discrete component \( P_t \) taking values in some finite set \( P \) and a continuous component \( Z_t \) taking values in an open convex set \( \mathbb{R}^d \). The set \( X = P \times Z \) is referred to as the state space. At each time \( t \in \mathbb{N} \), an action \( a \in A \) is chosen by the agent from a finite set \( A \). Suppose the starting state is given by \( X_0 = x_0 \) almost surely. Assume the discrete component evolves as a controlled finite state Markov chain with transition probabilities \( \alpha^a_{p,p'} \) for \( a \in A \), and \( p,p' \in P \). The value \( \alpha^a_{p,p'} \) gives the probability of transitioning from \( P_t = p \) to \( P_{t+1} = p' \) after action \( a \) is applied at time \( t \). Suppose action \( a \) governs the evolution of the continuous component via

\[
Z_{t+1} = f(W^a_{t+1}, Z_t)
\]

for random variable \( W^a_{t+1} : \Omega \to W \subseteq \mathbb{R}^d \) and measurable transition function \( f : W \times Z \to Z \). The variable \( W^a_{t+1} \) is referred to as the random disturbance driving the evolution of the state after action \( a \) is chosen. These random variables are assumed to be identically and independently distributed across time and actions. The random variables \( W^a_{t+1} \) and \( f(W^a_{t+1}, z) \) are assumed to be integrable for all \( z \in Z \), \( t \in \mathbb{N} \), and \( a \in A \).

Decision rule \( \pi_t \) gives a mapping \( \pi_t : X \to A \) which allocates an action \( \pi_t(x) \in A \) for given state \( x \in X \). A policy is defined as a sequence of decision rules i.e. \( \pi = (\pi_t)_{t \in \mathbb{N}} \). For each starting state \( x_0 \in X \), each policy \( \pi \) induces a probability measure such that \( \mathbb{P}^{x_0,\pi}(X_0 = x_0) = 1 \) and

\[
\mathbb{P}^{x_0,\pi}(X_{t+1} \in B | X_0, \ldots, X_t) = K^\pi_b(X_t)(X_{t+1}, B)
\]

for each measurable \( B \subseteq X \) where \( K^\pi_b \) denotes our Markov transition kernel after applying action \( a \). The reward at all times \( t \in \mathbb{N} \) is given by the time homogenous function \( r : X \times A \to \mathbb{R} \). The controller’s aim is to maximize the expectation

\[
v^\pi(x_0) = \mathbb{E}^{x_0,\pi}(\sum_{t=0}^{\infty} \beta^t r_t(X_t, \pi_t(X_t)))
\]

over all possible policies \( \pi \) where \( 0 < \beta < 1 \) is known as the discount rate. A policy \( \pi^* = (\pi^*_t)_{t \in \mathbb{N}} \) is said to be optimal if it satisfies \( v^{\pi^*}(x_0) \geq v^{\pi}(x_0) \) for any possible policy \( \pi \) and \( x_0 \in X \). The function \( v^{\pi^*} \) is often referred to as the value function.

**Assumption 1.** There exists function \( b : P \times Z \to \mathbb{R}_+ \) with constants \( c_r, c_b \in \mathbb{R}_+ \) such that

\[
|r(p, z, a)| \leq c_r(b(p, z)) \quad \text{and} \quad \int_X b(p', z') K^\pi_z((p, z), d(p', z')) \leq c_b(b(p, z))
\]

for \( p \in P \), \( z \in Z \), and \( a \in A \). Assume \( \beta c_b < 1 \) and that \( b(p, z) \) is continuous in \( z \) for all \( p \in P \).
With the above iteration, the decision problem now becomes a so-called contracting Markov decision process. To reveal why, the following is needed. Define the weighted supremum norm \( \| \cdot \|_b \) of a function \( v : P \times Z \rightarrow \mathbb{R} \) by
\[
\|v\|_b := \sup_{p \in P, z \in Z} b(p, z) |v(p, z)|
\]
and denote \( \mathcal{B}_b \) to be the family of all measurable functions \( v : P \times Z \rightarrow \mathbb{R} \) such that \( \|v\|_b < \infty \).

Let \( W^a : \Omega \rightarrow W \) be an identical distributed and independent copy of \((W^a_t)_{t \in \mathbb{N}}\) and define \( \mathcal{K}^a \) to represent the one step transition operator which acts on functions \( v \) by
\[
(\mathcal{K}^a v)(p, z) = \sum_{p' \in P} \alpha_{p,p'} v(p', f(W^a, z))
\]
whenever the expectations are well defined. Introduce the Bellman operator \( T : \mathcal{B}_b \rightarrow \mathcal{B}_b \) by
\[
(Tv)(p, z) = \max_{a \in A} (r(p, z, a) + \beta \mathcal{K}^a v(p, z))
\]
for \( p \in P \) and \( z \in Z \). The following theorem establishes the Bellman operator as a contraction mapping on \((\mathcal{B}_b, \| \cdot \|_b)\).

**Lemma 1.** It holds that \( \|Tv' - Tv''\|_b \leq \beta c_b \|v' - v''\|_b \) for all \( v', v'' \in \mathcal{B}_b \).

**Proof.** See [1, Lemma 7.3.3].

**Theorem 1.** Let \( v^* \in \mathcal{B}_b \) be the unique fixed point of \( T \). If decision rule \( \bar{\pi}_0 \) satisfies
\[
v^*(p, z) = r(p, z, \bar{\pi}_0(p, z)) + \beta \mathcal{K}^{\bar{\pi}_0}(p, z) v^*(p, z)
\]
for all \( p \in P \) and \( z \in Z \), then \( v^{\pi^*} = v^* \) and \( \pi^* = (\bar{\pi}_0, \bar{\pi}_0, \ldots) \).

**Proof.** See [1, Theorem 7.3.4].

The above shows that an optimal policy is stationary and that the value function can be found through the well known approach of value iteration. In the following, some convexity assumptions are imposed.

**Assumption 2.** Let \( r(p, z, a) \) be convex in \( z \) for all \( a \in A \) and \( p \in P \). If \( h(p, z) \) is convex in \( z \) for \( p \in P \), then \( \mathcal{K}^a h(p, z) \) is also convex in \( z \) for \( a \in A \) and \( p \in P \).

The above assumption ensures that the value function is convex in \( z \) for all \( p \in P \).

**Lemma 2.** Let \( v_n, \hat{v} \in \mathcal{B}_b \) for \( n \in \mathbb{N} \). It holds that \( \lim_{n \rightarrow \infty} \|v_n - \hat{v}\|_b = 0 \iff v_n \) converges to \( \hat{v} \) uniformly on compact sets.

**Proof.** Recall \( b(p, z) \) is continuous in \( z \) and positive for \( p \in P \). On compact set \( K \subset P \times Z \),
\[
\bar{b}_K := \sup_{p \in P, z \in Z} b(p, z) < \infty \quad \text{and} \quad \underline{b}_K := \inf_{p \in P, z \in Z} b(p, z) < \infty.
\]
It can be seen that
\[
\sup_{(p, z) \in K} |v_n(p, z) - \hat{v}(p, z)| \leq \bar{b}_K \sup_{(p, z) \in K} \frac{|v_n(p, z) - \hat{v}(p, z)|}{b(p, z)} \leq \frac{\bar{b}_K}{\underline{b}_K} \sup_{(p, z) \in K} |v_n(p, z) - \hat{v}(p, z)|.
\]

**Theorem 2.** The value function \( v^{\pi^*}(p, z) \) is convex in \( z \) for all \( p \in P \).
Proof. Let \( v_0(p, z) \) be a function convex in \( z \) for all \( p \in \mathcal{P} \) and \( v_0 \in \mathcal{B}_b \). Observe that \((\mathcal{T} v_0)(p, z) = \max_{a \in \mathcal{A}} (r(p, z, a) + \mathcal{K}^a v_0(p, z)) \) is convex in \( z \) for \( p \in \mathcal{P} \) since the sum and pointwise maximum of a finite number of convex functions is convex. Therefore, by induction, successive value iteration always results in a function convex in \( z \) for \( p \in \mathcal{P} \). Let \([\mathcal{T}]^i\) represent \( i \in \mathbb{N} \) successive applications of the operator \( \mathcal{T} \). Since \( v^\star \) is the unique fixed point, it can be shown that
\[
\lim_{i \to \infty} \|v^\star - [\mathcal{T}]^i v_0\|_b = 0
\]
and so by Lemma 2, \([\mathcal{T}]^i v_0\) converges to \( v^\star \) uniformly on compact sets. Since \([\mathcal{T}]^i v_0(p, z)\) is convex in \( z \) for \( i \in \mathbb{N} \) and \( p \in \mathcal{P} \), \( v^\star(p, z) \) is also convex in \( z \) for \( p \in \mathcal{P} \) since convexity is preserved under pointwise convergence. \(\square\)

The following continuity assumption is imposed on the transition function and is needed for the convergence results presented later on.

Assumption 3. Assume \( f(w, z) \) is continuous in \( w \) for all \( z \in \mathcal{Z} \).

3. Modified value iteration

This section approximates the value iteration in (3) with a more tractable form. Let us first approximate the transition operator (2). For each action \( a \in \mathcal{A} \), choose a suitable \( n \)-point disturbance sampling \((W^a,n(k))_{k=1}^n\) with weights \((\rho^a,n(k))_{k=1}^n\). Define the modified transition operator by
\[
\mathcal{K}^a,n v(p, z) = \sum_{p' \in \mathcal{P}} \alpha^a_{pp'} \sum_{k=1}^n \rho^a,n(k) v(p', f(W^a,n(k), z)).
\]
Since Assumption 2 ensures that \( \mathcal{K}^a v(p, z) \) is convex in \( z \) if \( v(p, z) \) is convex in \( z \), it is only natural to make the same assumption for the above modified transition operator.

Assumption 4. If \( v(p, z) \) is convex in \( z \) for \( p \in \mathcal{P} \), assume \( \mathcal{K}^a,n v(p, z) \) is convex in \( z \) for all \( n \in \mathbb{N} , a \in \mathcal{A} \) and \( p \in \mathcal{P} \).

Now denote \( G^{(m)} \subset \mathcal{Z} \) to be a \( m \)-point grid. Assume that \( G^{(m)} \subset G^{(m+1)} \) and \( \cup_{n=1}^{\infty} G^{(m)} \) is dense in \( \mathcal{Z} \). Suppose \( h : \mathcal{Z} \to \mathbb{R} \) is a convex function and introduce some approximation scheme \( \mathcal{S}_{G^{(m)}} \) that approximates \( h \) using another more tractable convex function. With this, define the modified Bellman operator by
\[
\mathcal{T}^{(m,n)} v(p, z) = \max_{a \in \mathcal{A}} \left( \mathcal{S}_{G^{(m)}} r(p, z, a) + \beta \mathcal{S}_{G^{(m)}} \mathcal{K}^a,n v(p, z) \right).
\]
where \( v(p, z) \) is convex in \( z \) for all \( p \in \mathcal{P} \). The function approximation scheme \( \mathcal{S}_{G^{(m)}} \) is applied to the functions for each \( p \in \mathcal{P} \) and \( a \in \mathcal{A} \) above. The following assumption is used to ensure that (5) preserves convexity for all \( p \in \mathcal{P} \) and to use for the other theorems presented later on.

Assumption 5. Let \( h, h' : \mathcal{Z} \to \mathbb{R} \) be convex functions and assume for \( m \in \mathbb{N} , a \in \mathcal{A} \), and \( t \in \mathbb{N} \) that:
- \( \mathcal{S}_{G^{(m)}} h(z) \) is convex in \( z \),
- \( \mathcal{S}_{G^{(m)}} (ch(z) + h'(z)) = ch(z) + \mathcal{S}_{G^{(m)}} h(z) + \mathcal{S}_{G^{(m)}} h'(z) \) for \( z \in \mathcal{Z} \) and \( c \in \mathbb{R} \), and
- \( \lim_{m \to \infty} \mathcal{S}_{G^{(m)}} h(z) = h(z) \) for all \( z \in \cup_{m=1}^{\infty} G^{(m)} \).
Theorem 3. If $v_0(p, z)$ is convex in $z$ for $p \in \mathbf{P}$, then any successive applications of the modified Bellman operator $T^{(m,n)} \ldots T^{(m,n)}v_0(p, z)$ results in a function that is convex in $z$ for all $p \in \mathbf{P}$ and $m, n \in \mathbb{N}$.

Proof. By assumption, $v_0(p, z)$ is convex in $z$ for $p \in \mathbf{P}$. Now $\beta S_{G(m)}^{\alpha(n)}v(p, z)$ is convex in $z$ for all $m, n \in \mathbb{N}$, $p \in \mathbf{P}$, and $a \in \mathbf{A}$. This is a consequence of Assumption 4 and Assumption 5. By Assumption 2 and Assumption 5, $S_{G(m)}^a r(p, z, a)$ is convex in $z$ for all $m \in \mathbb{N}$, $p \in \mathbf{P}$, and $a \in \mathbf{A}$. Thus, $T^{(m,n)}v_0(p, z)$ is convex in $z$ since the sum and pointwise maximum of a finite number of convex functions is also convex. Proceeding inductively for each successive application of the modified Bellman operator proves the desired result.

The next assumption ensures that the modified Bellman operator represents a contraction mapping on the Banach space $\mathfrak{B}_b, \| \cdot \|_b$. With this, there is a unique fixed point in $\mathfrak{B}_b$ for the modified value iteration represented by $T^{(m,n)}$.

Assumption 6. Assume for all $m, n \in \mathbb{N}$ that

$$|S_{G(m)}r(p, z, a)| \leq c_b b(p, z), \quad K^{a(n)}b(p, z) \leq c_b b(p, z), \quad \text{and} \quad \beta c_b \|S_{G(m)}b\|_b \leq 1$$

for $p \in \mathbf{P}$, $z \in \mathbf{Z}$, and $a \in \mathbf{A}$.

Theorem 4. It holds that

$$\|T^{(m,n)}v' - T^{(m,n)}v''\|_b \leq \beta c_b \|S_{G(m)}b\|_b \|v' - v''\|_b$$

for all $v', v'' \in \mathfrak{B}_b$.

Proof.\n
$$T^{(m,n)}v'(p, z) - T^{(m,n)}v''(p, z) \leq \beta \max_{a \in \mathbf{A}} \left( S_{G(m)}^{\alpha(n)}v'(p, z) - S_{G(m)}^{\alpha(n)}v''(p, z) \right)$$

$$\leq \beta \max_{a \in \mathbf{A}} \left( S_{G(m)}^{\alpha(n)}(v'(p, z) - v''(p, z)) \right)$$

$$\leq \beta \max_{a \in \mathbf{A}} \left( S_{G(m)}^{\alpha(n)} \frac{v'(p, z) - v''(p, z) b(p, z)}{b(p, z)} \right)$$

$$\leq c_b \beta \|v' - v''\|_b \|S_{G(m)}b(p, z)\|.$$

The first inequality above follows from $\max_{a \in \mathbf{A}} l(a) - \max_{a \in \mathbf{A}} l'(a)$ and the rest follows from Assumption 5 and the definition of $\| \cdot \|_b$. Using the same argument above, it can be shown that

$$T^{(m,n)}v''(p, z) - T^{(m,n)}v'(p, z) \leq c_b \beta \|v'' - v'\|_b \|S_{G(m)}b(p, z)\|.$$

Combining the two inequalities above gives

$$\|T^{(m,n)}v' - T^{(m,n)}v''\|_b \leq \beta c_b \|S_{G(m)}b\|_b \|v' - v''\|_b.$$

Theorem 5. The unique fixed point $v^{(m,n), *}(p, z)$ of the modified Bellman operator $T^{(m,n)}$ is convex in $z$ for all $p \in \mathbf{P}$, $m, n \in \mathbb{N}$.

Proof. The existence and uniqueness of the fixed point follows from Theorem 4 and the Banach fixed point theorem. The convexity can be proved in a similar manner as in the proof of Theorem 2 using Theorem 3.
Before proceeding, note that under the assumptions presented so far, the modified Bellman operator is not necessarily monotone. Additional assumptions are need on $S_{G(m)}$ for this. It turns out that the convergence results in the next section do not require this property. However, it is needed for the construction of the lower or upper bounding functions and so will be imposed then.

4. Convergence

This section proves that the fixed point in the modified value iteration represented by (5) converges to the value function uniformly on compact sets under different sampling schemes for the random disturbance $W^a$.

4.1. Disturbance sampling. The below concept of CCC sequences is crucial in this section.

**Definition 1.** Let $(h^{(n)})_{n \in \mathbb{N}}$ be a sequence of real-valued convex functions on $Z$ i.e. $h^{(n)} : Z \to \mathbb{R}$ for $n \in \mathbb{N}$. The sequence $(h^{(n)})_{n \in \mathbb{N}}$ is called a CCC (convex compactly converging) sequence in $z$ if $(h^{(n)})_{n \in \mathbb{N}}$ converges uniformly on all compact subsets of $Z$.

In the following, $(v^{(n)}(p, z))_{n \in \mathbb{N}}$ is assumed to be a sequence of functions convex in $z$ for $p \in P$ and $n \in \mathbb{N}$. So by Assumption 3, $v^{(n)}(p, f(w, z))$ is also continuous in $w$ for all $p \in P$, $z \in Z$, and $n \in \mathbb{N}$ since we have a composition of continuous functions. Moreover, suppose this sequence converges uniformly to function $\hat{v} \in \mathfrak{V}_0$ on all compact subsets of $X$. Thus, $(v^{(n)}(p, z))_{n \in \mathbb{N}}$ is a CCC sequence in $z$ converging to $\hat{v}(p, z)$ for $p \in P$. Lemma 3 below considers the use of random Monte Carlo samples for the disturbance sampling in the approximate value iteration. Let the choice of $p \in P$ and $a \in A$ below be arbitrary.

**Lemma 3.** Let $(W^a,(n)(k))_{k=1}^n$ be a collection of independently and identically distributed copies of $W^a$ and $\rho^{a,(n)}(k) = \frac{1}{n}$ for $k = 1, \ldots, n$. Assume these random variables lie on the same probability space as $W^a$. If $W$ is compact, then

$$\lim_{n \to \infty} K^{a,(n)}(v^{(n)}(p, z), z) = K^a \hat{v}(p, z), \quad z \in Z.$$ 

If $K^{a,(n)}(v^{(n)}(p, z))$ is also convex in $z$ for all $n \in \mathbb{N}$, then $(K^{a,(n)}(v^{(n)}(p, z)))_{n \in \mathbb{N}}$ also forms a CCC sequence in $z$.

*Proof.* See [13, Theorem 3].

Note that while the above assumes $W$ to be compact, it does not place any restriction on the size of $W$. So for unbounded $W$ cases, one can find a compact subset $\overline{W} \subset W$ that contains the vast majority of the probability mass. In this sense, this compactness assumption on $W$ is not restrictive from a numerical point of view. The above convergence when $W$ is not compact will be examined in future research. Now while the use of Monte Carlo sampling may be easier to implement, the user may desire finer control of the disturbance sampling. The following lemmas may be useful in this regard. Introduce partition of the disturbance space $W$ by $\Pi^{(a)} = \{\Pi^{(a)}(k) \subset W : k = 1, \ldots, n\}$. The situation where $W$ is compact is examined first.

**Lemma 4.** Suppose $W$ is compact. Denote the diameter of the partition by

$$\delta^{(n)} := \max_{k=1,\ldots,n} \sup \{\|w' - w''\| : w', w'' \in \Pi^{(a)}(k)\}$$
Lemma 6. Suppose this case occurs, for example, if the bounding function is constant or when there exists a positive constant $E$ possible misunderstandings, $\rho$ for all of local averages minimizes the mean square error from the discretization of $W$ uses local averages on each component of the partition. Later on, it will be shown that the use of local averages in the event $\{W^a \in \Pi(n)(k)\}$.

Proof. See [13, Theorem 4].

For the case where $W$ is not compact, the following may be used instead. The next lemma points to a simple case where this holds and avoids any possible misunderstandings, $E[W^a \mid W^a \in \Pi(n)(k)]$ refers to the expectation of $W^a$ conditioned on the event $\{W^a \in \Pi(n)(k)\}$.

Lemma 5. Suppose generated sigma-algebras $\sigma_a^{(n)} = \sigma(\{W^a \in \Pi(n)(k)\}, k = 1, \ldots, n)$ satisfy $\sigma(W^a) = \sigma(\cup_{n \in \mathbb{N}} \sigma_a^{(n)})$. Choose sampling $(W^{a,(n)}(k))_{k=1}^n$ such that

$$W^{a,(n)}(k) = E[W^a \mid W^a \in \Pi(n)(k)]$$

with $\rho^{a,(n)}(k) = P(W^a \in \Pi(n)(k))$ for $k = 1, \ldots, n$. If $(v^{(n)}(p', f(W^{a,(n)}, z)))_{n \in \mathbb{N}}$ is uniformly integrable for $p' \in P$ and $z \in \mathbb{Z}$, then:

$$\lim_{n \to \infty} K^{a,(n)}(p, z) = K^a \hat{\nu}(p, z), \quad z \in \mathbb{Z}.$$ 

If $K^{a,(n)} v^{(n)}(p, z)$ is also convex in $z$ for all $n \in \mathbb{N}$, then $(K^{a,(n)} v^{(n)}(p, z))_{n \in \mathbb{N}}$ also forms a CCC sequence in $z$.

Proof. See [13, Theorem 5].

In the above lemma, the assumption that $(v^{(n)}(p', f(W^{a,(n)}, z)))_{n \in \mathbb{N}}$ is uniformly integrable for all $p' \in P$, and $z \in \mathbb{Z}$ is used. The next lemma points to a simple case where this holds and this case occurs, for example, if the bounding function is constant or when $W$ is compact.

Lemma 6. Suppose $\sup_{n \in \mathbb{N}} \|v^{(n)}\|_b < \infty$. If $(b(p', f(W^{a,(n)}, z')))_{n \in \mathbb{N}}$ is uniformly integrable, then $(v^{(n)}(p', f(W^{a,(n)}, z')))_{n \in \mathbb{N}}$ is uniformly integrable for $p' \in P$ and $z' \in \mathbb{Z}$.

Proof. There exists a positive constant $c$ such that

$$\sup_{n \in \mathbb{N}} \sup_{p' \in P, z' \in \mathbb{Z}} \frac{|v^{(n)}(p', z')|}{b(p, z)} < c \quad \Rightarrow \quad \sup_{n \in \mathbb{N}} \frac{|v^{(n)}(p', f(W^{a,(n)}, z'))|}{cb(p', f(W^{a,(n)}, z'))} < 1$$

holds with probability one. Thus, $v^{(n)}(p', f(W^{a,(n)}, z'))$ is dominated by a family of uniformly integrable random variables and so is also uniformly integrable.

4.2. Choice of sampling. There are two sources of approximation error induced by the modified Bellman operator. The first originates from the function approximation $S_{G^{(m)}}$ and the other stems from the discretization of the disturbance $W^a$ using sampling $(W^{a,(n)}(k))_{k=1}^n$ with weights $(\rho^{a,(n)}(k))_{k=1}^n$. This subsection will briefly examine the latter where Lemma 3, Lemma 4, and Lemma 5 offer three possible choices. We will see that the use of the local averages in
Lemma 5 represents the best choice. Suppose that on a component $\Pi^{(n)}(k)$ of the partition, we wish to represent the $W^a$ with a constant $\theta_k$. The resulting mean square error is given by

$$\mathbb{E}[(W^a - \theta_k)^2 | W^a \in \Pi^{(n)}(k)].$$

It is well known that the choice of the conditional expectation $\theta_k = \mathbb{E}[W^a | W^a \in \Pi^{(n)}(k)]$ minimizes the above mean square error. It is therefore clear that the disturbance sampling in Lemma 5 tends to represent the best choice. However, the calculation of $\mathbb{E}[W^a | W^a \in \Pi^{(n)}(k)]$ may become impractical in some settings e.g. high dimension state spaces and so the use of Lemma 3 or Lemma 4 may be the only practical alternatives.

4.3. Convergence of fixed points. This subsection proves that fixed points from the modified value iteration (5) converge to the fixed point of the original value iteration (3) under the different sampling schemes presented earlier. Let $(m_n)_{n \in \mathbb{N}}$ and $(n_m)_{m \in \mathbb{N}}$ be sequences of natural numbers increasing in $n$ and $m$, respectively.

**Lemma 7.** Let $(h^{(n)})_{n \in \mathbb{N}}$ be a sequence of real-valued convex functions on $\mathbb{Z}$. If the sequence converges pointwise to $h$ on a dense subset of $\mathbb{Z}$, then the sequence $(h^{(n)})_{n \in \mathbb{N}}$ converges uniformly to $h$ on all compact subsets of $\mathbb{Z}$.

**Proof.** See [11, Theorem 10.8].

**Lemma 8.** Let $(h^{(n)}_1)_{n \in \mathbb{N}}$ and $(h^{(n)}_2)_{n \in \mathbb{N}}$ be CCC sequences on $\mathbb{Z}$ converging to $h_1$ and $h_2$, respectively. Define $h^{(n)}_3(z) := \max(h^{(n)}_1(z), h^{(n)}_2(z))$ and $h_3(z) := \max(h_1(z), h_2(z))$. It holds that:

- $(h^{(n)}_1 + h^{(n)}_2)_{n \in \mathbb{N}}$ is a CCC sequences on $\mathbb{Z}$ converging to $h_1 + h_2$,
- $(\mathcal{S}_G h^{(n)}_1)_{n \in \mathbb{N}}$ is a CCC sequences on $\mathbb{Z}$ converging to $h_1$, and
- $(h^{(n)}_3)_{n \in \mathbb{N}}$ is a CCC sequences on $\mathbb{Z}$ converging to $h_3$.

**Proof.** This follows from the definition of uniform convergence on compact sets, Assumption 5, and Lemma 7.

**Theorem 6.** Let $v^{(m,n),*}$ be the unique fixed point of $T^{(m,n)}$. Using the disturbance samplings in either Lemma 3, Lemma 4 or Lemma 5, it holds that both $v^{(m,n),*}$ and $v^{(m,n+1),*}$ converge uniformly to $v^\pi$ on compact sets as $n \to \infty$ and $m \to \infty$, respectively.

**Proof.** Let us consider the case of $n \to \infty$ first. Let $v_0(p,z) \in \mathfrak{B}_b$ be convex in $z$ for all $p \in \mathcal{P}$ and denote $[T^{(m,n)}]^i$ to represent $i \in \mathbb{N}$ successive applications of the operator $T^{(m,n)}$. At $i = 1$, $(v_0(p,z))_{n \in \mathbb{N}}$ is a CCC sequence in $z$ and converges to $v_0(p,z)$ for $p \in \mathcal{P}$. Now, the use of Assumption 4 combined with either Lemma 3, Lemma 4 or Lemma 5 reveals that $(K^{a,(n)}v_0(p,z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and that it converges to $K^a v_0(p,z)$ for all $p \in \mathcal{P}$ and $a \in \mathcal{A}$. Since $\mathcal{A}$ is finite, it holds from Lemma 8 that $(T^{(m,n)}v_0(p,z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and converges to $T v_0(p,z)$ for all $p \in \mathcal{P}$. Now at $i = 2$, the same argument shows that $(K^{a,(n)}T^{(m,n)}v_0(p,z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and that it converges to $K^a T v_0(p,z)$ for all $p \in \mathcal{P}$ and $a \in \mathcal{A}$. Again, using the same argument above also reveals that $([T^{(m,n)}]^iv_0(p,z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and converges to $[T]^iv_0(p,z)$ for all $p \in \mathcal{P}$. Proceeding inductively for $i = 3, 4, \ldots$ proves that for all $i \in \mathbb{N}$ that $[T^{(m,n)}]^iv_0$ converges
uniformly on compact sets to $[\mathcal{T}]^i v_0$. From the triangle inequality, it can be seen that
\[
\lim_{n \to \infty} \|v^{(m,n),*} - v^\pi\|_b \\
\leq \lim_{n \to \infty} \lim_{i \to \infty} \|v^{(m,n),*} - [\mathcal{T}^{(m,n)}]^i v_0\|_b + \|[\mathcal{T}^{(m,n)}]^i v_0 - [\mathcal{T}]^i v_0\|_b + \|[\mathcal{T}]^i v_0 - v^\pi\|_b
\]
and so from Lemma 2, $v^{(m,n),*}$ converges to $v^\pi$ uniformly on compact sets. The proof for the $m \to \infty$ case follows the same lines as above. \hfill \Box

5. LOWER BOUNDING FIXED POINTS

This section examines the conditions in which the fixed points from (5) form a non-decreasing sequence of lower bounding fixed points for the true value function. To this end, the following assumption is required. Assumption 7 below induces monotonicity in the modified Bellman operator (5) i.e. for all $m, n \in \mathbb{N}$, $p \in \mathbb{P}$, and $z \in \mathbb{Z}$: $\mathcal{T}^{(m,n)}v'(p, z) \leq \mathcal{T}^{(m,n)}v''(p, z)$ if $v'(p', z) \leq v''(p', z)$ for all $p' \in \mathbb{P}$.

**Assumption 7.** Suppose for $z \in \mathbb{Z}$ and $m \in \mathbb{N}$ that $S_{G(m)} h'(z) \leq S_{G(m)} h''(z)$ for all convex functions $h', h'' : \mathbb{Z} \to \mathbb{R}$ such that $h'(z) \leq h''(z)$ for $z \in \mathbb{Z}$.

In the following, partition $\Pi^{(n+1)}$ is said to be a refinement of $\Pi^{(n)}$ if each component of $\Pi^{(n+1)}$ is a subset of a component in $\Pi^{(n)}$.

**Lemma 9.** Suppose $v(p, f(w, z))$ is convex in $w$ for all $p \in \mathbb{P}$ and that $\Pi^{(n+1)}$ refines $\Pi^{(n)}$. Using the disturbance sampling in Theorem 5 gives $\mathcal{K}^{(n)} v(p, z) \leq \mathcal{K}^{(n+1)} v(p, z)$.

**Proof.** See [13, Lemma 4]. \hfill \Box

**Theorem 7.** Using Theorem 5 gives for $p \in \mathbb{P}$, $z \in \mathbb{Z}$, and $m, n \in \mathbb{N}$:

- $v^{(m,n),*}(p, z) \leq v^\pi(p, z)$ if $v^\pi(p', f(w, z'))$ is convex in $w$ and if $S_{G(m')} h(z') \leq h(z')$ for $p' \in \mathbb{P}$, $z' \in \mathbb{Z}$, and all convex functions $h$.
- $v^{(m,n),*}(p, z) \leq v^{(m,n+1),*}(p, z)$ if $\Pi^{(n+1)}$ refines $\Pi^{(n)}$ and if $v^{(m',n'),*}(p', f(w, z'))$ is convex in $w$ for $p' \in \mathbb{P}$, $z' \in \mathbb{Z}$, and $m', n' \in \mathbb{N}$.
- $v^{(m,n),*}(p, z) \leq v^{(m+1,n),*}(p, z)$ if $S_{G(m')} h(z') \leq S_{G(m'+1)} h(z')$ for $z' \in \mathbb{Z}$, $m' \in \mathbb{N}$, and all convex functions $h$.

**Proof.** We prove the three inequalities separately.

1) Recall that $W^{a,n} = \mathbb{E}[W^a | a^{(n)}_z]$. The tower property and Jensen’s inequality gives for $a \in \mathbb{A}$, $p \in \mathbb{P}$, $z \in \mathbb{Z}$, and $n \in \mathbb{N}$:
\[
\mathbb{E}[v^\pi(p, f(W^a, z))] \geq \mathbb{E}[v^\pi(p, f(W^{a,n}(z), z))] = \sum_{k=1}^{n} \rho^{a,n}(k) v^\pi(p, f(W^{a,n}(k), z))
\]
\[
\implies k_{T-1} v^\pi(p, z) \leq k_{T-1} v^\pi(p, z) \implies \mathcal{T}^{(m,n)} v^\pi(p, z) \leq \mathcal{T} v^\pi(p, z) = v^\pi(p, z) \text{ for } p \in \mathbb{P}
\text{ and } z \in \mathbb{Z} \text{ by Assumption 7 and } S_{G(m)} \mathcal{T} r(p, z, a) \leq r(p, z, a). \text{ Now using the above and the monotonocity of the modified Bellman operator (5) gives}
\]
\[
[\mathcal{T}^{(m,n)}]^i v^\pi(p, z) \leq [\mathcal{T}^{(m,n)}]^i v^\pi(p, z) \leq [\mathcal{T}^{(m,n)}]^i v^\pi(p, z) = [\mathcal{T}]^i v^\pi(p, z) = v^\pi(p, z).
\]
In the same manner, it can be shown that for $i \in \mathbb{N}$:
\[
[\mathcal{T}^{(m,n)}]^i v^\pi(p, z) \leq [\mathcal{T}]^i v^\pi(p, z) = v^\pi(p, z)
\]
and so $v^{(m,n),*}(p, z) \leq v^\pi(p, z)$ for all $p \in \mathbb{P}$, $z \in \mathbb{Z}$, and $m, n \in \mathbb{N}$. 

2) Lemma 9 implies \( K^{a,(n)} v_{T}^{(m),*}(p, z) \leq K^{a,(n+1)} v_{T}^{(m),*}(p, z) \). From Assumption 7, it can be shown that \( T^{(m),*}(p, z) \leq T^{(m,n+1)} v_{T}^{(m),*}(p, z) \). Using this,

\[
v^{(m,n),*}(p, z) \leq T^{(m,n+1)} v^{(m,n),*}(p, z) \leq [T^{(m,n+1)}]^{2} v^{(m,n),*}(p, z)
\]

by the monotonicity of the modified Bellman operator. Therefore, it can be shown for \( i \in \mathbb{N} \):

\[
v^{(m,n),*}(p, z) \leq [T^{(m,n+1)}]^{i} v^{(m,n),*}(p, z)
\]

and so \( v^{(m,n),*}(p, z) \leq v^{(m,n+1),*}(p, z) \) for all \( p \in P \), \( z \in Z \), and \( m, n \in \mathbb{N} \).

3) By assumption \( S_{\bar{G}}(p, z, a) \leq S_{\bar{G}}(p, z, a) \) for \( p \in P \), \( a \in A \) and \( m \in \mathbb{N} \). Therefore, \( T^{(m),*}(p, z) \leq T^{(m+1),*}(p, z) \). Using a similar argument as before, it can be shown that for \( i \in \mathbb{N} \):

\[
v^{(m,n),*}(p, z) \leq [T^{(m+1,n)}]^{i} v^{(m,n),*}(p, z)
\]

and so \( v^{(m,n),*}(p, z) \leq v^{(m+1,n),*}(p, z) \) for all \( p \in P \), \( z \in Z \), and \( m, n \in \mathbb{N} \).

\[\square\]

6. Upper Bounding Fixed Points

This section examines conditions in which the fixed points from the modified value iteration (5) form a non-increasing sequence of upper bounding functions for the true value function. Suppose \( W \) is compact. Assume that the closures of each of the components in partition \( \Pi^{(n)} \) are convex and contain a finite number of extreme points. Denote the set of extreme points of the closure \( \Pi^{(n)}(k) \) of each component \( \Pi^{(n)}(k) \) in partition \( \Pi^{(n)} \) by

\[\mathcal{E}(\Pi^{(n)}(k)) = \{ e_{i}^{(n)} : i = 1, \ldots, L^{(n)}(k) \}\]

where \( L^{(n)}(k) \) is the number of extreme points in \( \Pi^{(n)}(k) \). Suppose there exist weighting functions \( q_{k,i}^{a,(n)} : W \rightarrow [0, 1] \) satisfying

\[
\sum_{i=1}^{q_{k,i}^{a,(n)}}(w) = 1 \quad \text{and} \quad \sum_{i=1}^{q_{k,i}^{a,(n)}}(w)e_{k,i}^{(n)} = w
\]

for all \( w \in \Pi^{(n)}(k) \) and \( k = 1, \ldots, n \). Suppose \( \rho^{a,(n)}(k) = \mathbb{P}(W^{a} \in \Pi^{(n)}(k)) > 0 \) for \( k = 1, \ldots, n \) and define random variables \( W_{k}^{a,(n)} \) satisfying

\[
\mathbb{P}
\left( W_{k}^{a,(n)} = e_{k,i}^{(n)} \right) = \frac{q_{k,i}^{a,(n)}}{\rho_{k,i}^{a,(n)}(k)} \quad \text{where} \quad q_{k,i}^{a,(n)} = \int_{\Pi^{(n)}(k)} q_{k,i}^{a,(n)}(w)\mu^{a}(dw)
\]

and \( \mu^{a}(B) = \mathbb{P}(W^{a} \in B) \). For the next theorem, define random variable

\[
W_{k}^{a,(n)} = \sum_{k=1}^{n} W_{k}^{a,(n)} 1 \left( W^{a} \in \Pi^{(n)}(k) \right)
\]

and introduce the following alternative modified transition operator:

\[
K^{a,(n)} v(p, z) = \sum_{p' \in P} \alpha_{p,p'}^{a} \mathbb{E}[v(p', f(W_{k}^{a,(n)}, z))].
\]

The following theorem proves convergence of the fixed points when the above is used in the modified value iteration.
Lemma 10. If \( \lim_{n \to \infty} \delta(n) = 0 \), then
\[
\lim_{n \to \infty} \mathbb{E}[v(n)(p, f(W^{(n+1)}_n, z))] = \mathbb{E}[\pi(p, f(W^n, z))], \quad z \in \mathbb{Z}.
\]
If \( \mathbb{E}[v(n)(p, f(W^{(n+1)}_n, z))] \) is also convex in \( z \) for \( n \in \mathbb{N} \), then \( \mathbb{E}[v(n)(p, f(W^{(n+1)}_n, z))]_{n \in \mathbb{N}} \) form a CCC sequence in \( z \).

**Proof.** See [13, Lemma 6].

**Theorem 8.** Suppose \( \lim_{n \to \infty} \delta(n) = 0 \). Let \( v^{(m,n),*} \) be the unique fixed point of \( T^{(m,n)} \) where the modified transition operator (6) is used. It holds that both \( v^{(m,n),*} \) and \( v^{(m,n+1),*} \) converge uniformly to \( v^{*} \) on compact sets as \( n \to \infty \) and \( m \to \infty \), respectively.

**Proof.** The proof mirrors the proof in Theorem 6 but using Lemma 10 instead.

To show upper bounding behaviour of the modified fixed points, the following lemma is handy.

**Lemma 11.** Suppose \( \Pi^{(n+1)} \) refines \( \Pi^{(n)} \) and that \( v(n)(p, f(w, z)) \) is convex in \( w \) for \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( n \in \mathbb{N} \). Assume \( v(n)(p, z) \geq v(n+1)(p, z) \geq \pi(p, z) \) for \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( n \in \mathbb{N} \). Then
\[
\mathbb{E}[v(n)(p, f(W^{(n+1)}_n, z))] \geq \mathbb{E}[v(n+1)(p, f(W^{(n+1)}_n, z))] \geq \mathbb{E}[\pi(p, f(W^n, z))]
\]
for \( a \in \mathcal{A} \), \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( n \in \mathbb{N} \).

**Proof.** See [6, Theorem 7.8] or [5, Theorem 1.3].

**Theorem 9.** Using the modified transition operator (6) gives for \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( m, n \in \mathbb{N} \):

- \( v^{(m,n),*}(p, z) \geq v^{*}(p', f(w, z')) \) is convex in \( w \) and if \( S\mathcal{G}(m', h(z')) \geq h(z') \) for \( p' \in \mathcal{P} \), \( z' \in \mathbb{Z} \), and all convex functions \( h \).
- \( v^{(m,n),*}(x) \geq v^{(m,n+1),*}(x) \) if \( \Pi^{(n+1)} \) is a refinement of \( \Pi^{(n')} \) and if \( v^{(m',n'),*}(p', f(w, z')) \) is convex in \( w \) for \( p' \in \mathcal{P} \), \( z' \in \mathbb{Z} \), and \( m', n' \in \mathbb{N} \).
- \( v^{(m,n),*}(x) \leq v^{(m+1,n),*}(x) \) if \( S\mathcal{G}(m,h(z')) \geq S\mathcal{G}(m+1,h(z')) \) for \( z' \in \mathbb{Z} \), \( m' \in \mathbb{N} \), and all convex functions \( h \).

**Proof.** The inequalities are basically reversed in the proof of Theorem 7:

1) Lemma 11 \( \implies \mathcal{K}(m,n)^* v^{*}(p, z) \geq \mathcal{K}(m,n)^* v^{*}(p, z) \implies T^{(m,n)} v^{*}(p, z) \geq T v^{*}(p, z) = v^{*}(p, z) \) for \( p \in \mathcal{P} \) and \( z \in \mathbb{Z} \) by Assumption 7 and the fact that \( S\mathcal{G}(m') v(p, z, a) \geq v(p, z, a) \).

Now using the above and the monotonicity of the modified Bellman operator (5) gives
\[
[T^{(m,n)}]_{1} v^{*}(p, z) \geq T v^{*}(p, z) \geq T^{(m,n)} v^{*}(p, z) \geq T v^{*}(p, z) = v^{*}(p, z).
\]

In the same manner, it can be shown that for \( i \in \mathbb{N} \):
\[
[T^{(m,n)}]_{i} v^{*}(p, z) \geq [T]_{i} v^{*}(p, z) = v^{*}(p, z)
\]
and so \( v^{(m,n),*}(p, z) \leq v^{*}(p, z) \) for all \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( m, n \in \mathbb{N} \).

2) Lemma 11 \( \implies \mathcal{K}(m,n)^* v^{(m,n),*}(p, z) \geq \mathcal{K}(m,n)^* v^{(m,n),*}(p, z) \). From Assumption 7, it can be shown that \( T^{(m,n)} v^{(m,n),*}(p, z) \geq T^{(m,n+1)} v^{(m,n),*}(p, z) \). Using this,
\[
v^{(m,n),*}(p, z) \geq T v^{(m,n+1),*}(p, z) \geq [T^{(m,n+1)}]_{1} v^{(m,n),*}(p, z)
\]
by the monotonicity of the modified Bellman operator. Therefore, it can be shown for \( i \in \mathbb{N} \):
\[
v^{(m,n),*}(p, z) \geq [T^{(m,n+1)}]_{i} v^{(m,n),*}(p, z)
\]
and so \( v^{(m,n),*}(p, z) \geq v^{(m,n+1),*}(p, z) \) for all \( p \in \mathcal{P} \), \( z \in \mathbb{Z} \), and \( m, n \in \mathbb{N} \).
3) By assumption \( S_{G(m)} r(p, z, a) \geq S_{G(m+1)} r(p, z, a) \) for \( p \in P \), \( a \in A \) and \( m \in \mathbb{N} \). Therefore, \( T^{(m,n)}_\pi (m,n)^* (p, z) \geq T^{(m+1,n)}_\pi (m,n)^* (p, z) \). Using a similar argument as for the first two cases, it can be shown that for all \( i \in \mathbb{N} \):
\[
v^{(m,n)^*} (p, z) \geq [T^{(m+1,n)}] v^{(m,n)^*} (p, z)
\]
and so \( v^{(m,n)^*} (p, z) \geq v^{(m+1,n)^*} (p, z) \) for all \( p \in P \), \( z \in \mathbb{Z} \), and \( m, n \in \mathbb{N} \).

7. Numerical Study

As way of demonstration, a perpetual Bermudan put option in the Black-Scholes world is considered. A Bermudan put option represents the right but not the obligation to sell the underlying asset for a predetermined strike price \( K \) at discrete time points. This problem is given by \( P = \{ \text{exercised, unexercised} \} \) and \( A = \{ \text{exercise, don't exercise} \} \). At \( P_1 = \text{“unexercised”} \), applying \( a = \text{“exercise”} \) and \( a = \text{“don’t exercise”} \) leads to \( P_{t+1} = \text{“exercised”} \) and \( P_{t+1} = \text{“unexercised”} \), respectively with probability one. If \( P_t = \text{“exercised”} \), then \( P_{t+1} = \text{“exercised”} \) almost surely regardless of any action. Let \( \kappa > 0 \) represent the interest rate per annum and let \( z \) represent the underlying stock price. Defining \( (z)^+ = \max(z, 0) \), the reward/payoff function is given by
\[
r(\text{unexercised}, z, \text{exercise}) = (K - z)^+
\]
for \( z \in \mathbb{R}_+ \) and zero for other \( p \in P \) and \( a \in A \). The fair price of the option is
\[
v^{\pi^*} (\text{unexercised}, z_0) = E^{(\text{unexercised}, z_0), \pi^*} \left( \sum_{t=0}^{\infty} \beta^t r_t(I_t, \pi_t^*(X_t)) \right)
\]
where \( \beta = e^{-\kappa \Delta} \) is the discount factor and \( \Delta \) is the time step. Assume
\[
Z_{t+1} = W_{t+1} Z_t = e^{(\kappa - \frac{\sigma^2}{2}) \Delta + \sqrt{\Delta N_{t+1}}} Z_t
\]
where \((N_t)_{t \in \mathbb{N}}\) are independent and identically distributed standard normal random variables and \( \rho \) is the volatility of stock price returns.

Note that the disturbance is not controlled by action \( a \) and so the superscript is removed from \( W_{t+1} \) for notational simplicity in the following subsections. It is clear that
\[
|r(p, z, a)| \leq b(p, z) = K, \quad E[K] \leq K, \quad \text{and} \quad \beta < 1
\]
for \( p \in P \), \( z \in Z \), and \( a \in A \). So by Assumption 1, the perpetual Bermudan put option is a contracting Markov decision process with a unique fixed point \( v^{\pi^*} \) that can be found via value iteration. In the next two subsections, different modified value iteration schemes are presented. The first results in the modified fixed points forming a lower bound and the second forms an upper bound. The same method were employed in a finite horizon with great success (see [13, Section 7]) and this also holds true in our infinite horizon setting.

7.1. Lower bound via tangents. The following scheme approximates the convex functions using the maximum of their tangents. From [11, Theorem 25.5], it is known that a convex real valued function is differentiable almost everywhere. Suppose convex function \( h : \mathbb{Z} \rightarrow \mathbb{R} \) holds tangents on each point in \( G^{(m)} \) given by \( \{h_1'(z), \ldots, h_m'(z)\} \). Set the approximation scheme \( S_{G^{(m)}} \) to take the maximising tangent to form a convex piecewise linear approximation of \( h \) i.e.
\[
S_{G^{(m)}} h(z) = \max\{h_1'(z), \ldots, h_m'(z)\}.
\]
It is not hard to see that the resulting approximation \( S_{G^{(m)}}h \) is convex, piecewise linear, and converges to \( h \) uniformly on compact sets as \( m \to \infty \). Thus, Assumption 5 is satisfied. Now observe that Assumption 3 and 4 both hold since \((Z_t)_{t \in \mathbb{N}}\) evolves in a linear manner. By definition of \( S_{G^{(m)}} \), Assumption 7 is obeyed. Let us now verify Assumption 6. Note that \( S_{G^{(m)}}h(z) \leq S_{G^{(m+1)}}h(z) \leq h(z) \) for all \( z \in \mathbb{Z} \). Observe that the reward and true value functions are bounded above by \( K \) and below by 0. Therefore, if the approximation scheme \( S_{G^{(m)}} \) includes a zero tangent i.e \( h'_{G^{(m)}}(z) = 0 \) for all \( z \in \mathbb{Z} \), then all the conditions in Assumption 6 are true. Therefore, the modified Bellman operator constructed using the above function approximation gives a contracting Markov decision process.

For the disturbance sampling, the space \( W = \mathbb{R}_+ \) is partitioned into sets of equal probability measure and then the local averages on each component are used. It is not difficult to verify that all the conditions in Theorem 7 hold and so the fixed point of this modified value iteration forms a non-decreasing sequence of lower bounding functions for the true value function. Before proceeding to the next scheme, let us address the issue of convergence in the above modified value iteration. Observe in (5) that if the expected value function converges to its fixed point, then the whole value iteration does so as well. Since \( \mathbb{Z} \) is uncountable, it is impossible to verify this convergence directly for each \( z \in \mathbb{Z} \). However, note that if

\[
S_{G^{(m)}}h_1(z) = S_{G^{(m)}}h_2(z) \quad \text{and} \quad \nabla_z S_{G^{(m)}}h_1 = \nabla_z S_{G^{(m)}}h_2
\]

for all \( z \in G^{(m)} \) where the \( \nabla_z \) operator gives the gradient/slope at \( z \), then \( S_{G^{(m)}}h_1(z) = S_{G^{(m)}}h_2(z) \) for all \( z \in \mathbb{Z} \). Therefore, there is convergence in the value iteration when there is convergence of the tangents on each of the finite number of grid points. In this manner, this approach is numerically tractable.

7.2. Upper bound via linear interpolation. The following constructs the upper bounds using Section 6. Let us approximate unbounded \( W = \mathbb{R}_+ \) with a compact set \( \overline{W} \) containing 99.9999999% of the probability mass i.e. \( \mathbb{P}(W_t \in \overline{W}) = 0.999999999 \) for all \( t \in \mathbb{N} \). Introduce the truncated distribution \( \overline{\mathbb{P}} \) defined by \( \overline{\mathbb{P}}(W_t \in B) = \alpha \mathbb{P}(W_t \in B) \) for all \( B \subseteq \overline{W} \) where \( \alpha = 1/\mathbb{P}(W_t \in \overline{W}) \) is the normalizing constant. Suppose the partition \( \Pi^{(n)} \) comprises \( n \) convex components of equal probability measure and that the extreme points are ordered \( e_{k,1}^{(n)} < e_{k,2}^{(n)} \) for all \( k = 1, \ldots, n \). For \( k = 1, \ldots, n \), define points \( e_k^{(n)} = e_{(k+1)/2}^{(n)} \) and \( e_{n+1}^{(n)} = e_{n,2}^{(n)} \) where \( [ ] \) denotes the integer part. Define \( \Lambda(a, b) := \mathbb{E}[W_1 \mathbb{I}(W_1 \in [a, b]) \] and set

\[
q_{k,1}^{(n)}(w) = \frac{e_{j,2} - w}{e_{j,2} - e_{j,1}} \quad \text{and} \quad q_{k,2}^{(n)}(w) = \frac{w - e_{j,1}}{e_{j,2} - e_{j,1}}.
\]

Therefore,

\[
\mathbb{P}\left( W_1^{(n)} = e_1^{(n)} \right) = \frac{e_2^{(n)} - \Lambda(e_1^{(n)}, e_2^{(n)})}{e_2^{(n)} - e_1^{(n)}},
\]

\[
\mathbb{P}\left( W_1^{(n)} = e_{n+1}^{(n)} \right) = \frac{\Lambda(e_{n,1}^{(n)}, e_{n+1}^{(n)}) - e_n^{(n)}}{e_{n+1}^{(n)} - e_n^{(n)}},
\]

\[
\mathbb{P}\left( W_1^{(n)} = e_{j}^{(n)} \right) = \frac{e_{j+1}^{(n)} - \Lambda(e_j^{(n)}, e_{j+1}^{(n)})}{e_{j+1}^{(n)} - e_j^{(n)}} + \frac{\Lambda(e_{j-1}^{(n)}, e_j^{(n)}) - e_{j-1}^{(n)}}{e_j^{(n)} - e_{j-1}^{(n)}},
\]

for \( j = 2, \ldots, n \).
It is well known that \( v^{x^*}(\text{unexercised}, z) = r(\text{unexercised}, z, \text{exercise}) \) when \( z < z' \) for some \( z' \in \mathbb{Z} \) and that the value function is decreasing in \( z \). These features will be exploited. Suppose \( h : \mathbb{Z} \to \mathbb{R} \) is a decreasing convex function and \( h(z) = h'(z) \) when \( z < z' \). For \( G^m = \{ g^{(1)}, \ldots, g^{(m)} \} \) where \( g^{(1)} < g^{(2)} < \cdots < g^{(m)}, \) set

\[
S_{G^{(m)}} h(z) = \begin{cases} 
    h'(z) & \text{if } z \leq g^{(1)}; \\
    d_i(z - g^{(i)}) + h(g^{(i)}) & \text{if } g^{(i)} < z \leq g^{(i+1)}; \\
    h(g^{(m)}) & \text{if } z > g^{(m)},
\end{cases}
\]

where \( d_i = \frac{h(g^{(i+1)}) - h(g^{(i)})}{g^{(i+1)} - g^{(i)}} \) for \( i = 2, \ldots, m - 1 \). It is not difficult to verify that under this function approximation scheme, all the relevant assumptions are satisfied and so the above gives a contracting Markov decision process. Further, all the conditions in Theorem 9 is also satisfied and so the fixed points from this modified value iteration form a non-increasing sequence of upper bounding functions for the true value function. Finally, let us mention about the stopping criterion for the above modified criterion. It can be shown that if

\[
S_{G^{(m)}} h_1(z) = S_{G^{(m)}} h_2(z)
\]

for all \( z \in G^{(m)} \), then \( S_{G^{(m)}} h_1(z) = S_{G^{(m)}} h_2(z) \) for all \( z \in \mathbb{Z} \). Thus, there is convergence in the value iteration when the function values on each grid point converges.

### 7.3. Numerical results.

The following results were generated on a Bermudan put option with strike price 40. The put option is assumed to be exercisable every 3 months for perpetuity until exercised. The interest rate is set at 0.15 per annum. The reward function was used as the seeding function in the value iteration. If the values or gradients at each grid point is within 0.001 of the last iteration, the value iteration is stopped and we assume convergence. The computational times listed are for Linux Ubuntu 16.04 machine with Intel i5-5300U CPU @2.30GHz and 16GB of RAM. The following results can be reproduced using the R script listed in the appendix. For the disturbance sampling, a partition of \( n = 1000 \) components of equal probability measure is used.

| \( Z_0 \) | vol = 0.1 p.a. | vol = 0.2 p.a. | vol = 0.3 p.a. |
|-------|---------------|---------------|---------------|
|       | Lower Upper Gap | Lower Upper Gap | Lower Upper Gap |
| 32    | 8.00000 8.00000 0.00000 | 8.00000 8.00000 0.00000 | 8.00000 8.00000 0.00000 |
| 34    | 6.00000 6.00000 0.00000 | 6.00000 6.00000 0.00000 | 6.25550 6.30199 0.01649 |
| 36    | 4.00000 4.00000 0.00000 | 4.00000 4.00000 0.00000 | 5.23546 5.25366 0.01820 |
| 38    | 2.00000 2.00000 0.00000 | 2.45520 2.47724 0.02204 | 4.38277 4.40150 0.01874 |
| 40    | 0.34539 0.37316 0.02776 | 1.69317 1.71520 0.02203 | 3.69464 3.71292 0.01828 |
| 42    | 0.08485 0.09846 0.01361 | 1.17535 1.19501 0.01966 | 3.13829 3.15556 0.01727 |
| 44    | 0.02030 0.02556 0.00526 | 0.82723 0.84366 0.01643 | 2.68569 2.70162 0.01593 |
| 46    | 0.00508 0.00745 0.00237 | 0.59119 0.60451 0.01322 | 2.31435 2.32890 0.01455 |

Recall the the approximation schemes give lower and upper bounding functions for the fair price of the perpetual option. Table 1 list points on these curves at different starting asset prices. Columns 2 to 4 give an option with vol = 0.1 per annum. For this case, a grid of 51 equally spaced points from \( z = 20 \) to \( z = 70 \) was used. The computational times for each bounding function is around 0.025 cpu seconds (0.005 real world seconds) and it took roughly
10 iteration for both schemes to converge. Columns 5 to 7 gives an option with \( \text{vol} = 0.2 \). Here, a larger grid of 101 equally spaced points from \( z = 20 \) to \( z = 120 \) was used. The computational times for each bounding function is around 0.05 cpu seconds (0.01 to 0.03 real world seconds). Convergence took 45 and 28 iterations for the lower and upper bounding schemes, respectively. Columns 8 to 10 gives the case \( \text{vol} = 0.3 \). A grid of 401 equally spaced points from \( z = 20 \) to \( z = 420 \) was used. The computational times for each bounding function is around 0.2 cpu seconds (0.05 to 0.1 real world seconds). Convergence took 69 and 52 iterations for the lower and upper bounding schemes, respectively. The lower bounding and upper bounding curves are plotted in Figure 1 below.

![Figure 1](image)

**Figure 1.** Lower and upper bounding functions for option price with \( \text{vol} = 0.2 \) (left plot) and \( \text{vol} = 0.3 \) (right plot). The dashed lines indicate the upper bound while the unbroken curves give the lower bounds.

8. **Final thoughts**

In this paper, the original value iteration is replaced with a more tractable approximation (5). Under certain convexity assumptions, the fixed points from this modified value iteration converges uniformly on compact sets to the true value functions under different sampling schemes for the driving random disturbance. Moreover, the fixed points from the approximate value iteration (5) form a monotone sequence of functions that bound the true value function. The results in this paper can be modified for problems involving concave functions. For example, problems of the form

\[
\mathcal{T}v(p, z) = \min_{a \in A} (r(p, z, a) + K^a v(p, z))
\]

where the reward function is concave in \( z \) and the transition operator preserves concavity. Extensions to partially observable Markov decision processes will be considered in future research.

**Appendix A. R Script for Table 1**

The script (along with the R package) used to generate columns 2, 3, and 4 in Table 1 can be found at https://github.com/YeeJeremy/ConvexPaper. To generate the others, simply modify the values on Line 5, Line 11, and Line 12.
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