Compactness of Kähler-Ricci solitons on Fano manifolds

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Abstract

In this short paper, we improve the result of Phong-Song-Sturm on degeneration of Fano Kähler-Ricci solitons by removing the assumption on the uniform bound of the Futaki invariant. Let $KR(n)$ be the space of Kähler-Ricci solitons on $n$-dimensional Fano manifolds. We show that after passing to a subsequence, any sequence in $KR(n)$ converge in the Gromov-Hausdorff topology to a Kähler-Ricci soliton on an $n$-dimensional $Q$-Fano variety with log terminal singularities.

1 Introduction

The Ricci solitons on compact and complete Riemannian manifolds naturally arise as models of singularities for the Ricci flow [7]. The existence and uniqueness of Ricci solitons has been extensively studied. A gradient Ricci soliton is a Riemannian metric satisfying the following soliton equation

$$Ric(g) = \lambda g + \nabla^2 u$$

for some smooth function $f$ with $\lambda = -1, 0, 1$. Such a soliton is called a gradient shrinking Ricci soliton if $\lambda > 0$. If we let the vector field $\mathcal{V}$ be defined by $\mathcal{V} = \nabla u$, the soliton equation becomes

$$Ric(g) = \lambda g + L_{\mathcal{V}}g,$$

where $L_{\mathcal{V}}$ is the Lie derivative along $\mathcal{V}$.

A Kähler metric $g$ on a Kähler manifold $X$ is called a Kähler-Ricci soliton if it satisfies the soliton equation (1.1) or equation (1.2) for $\mathcal{V} = \nabla u$. Any shrinking Kähler-Ricci soliton on a compact Kähler manifold $X$ must be a gradient Ricci soliton and such a Kähler manifold must be a Fano manifold, i.e. $c_1(X) > 0$. The vector field $\mathcal{V}$ must be holomorphic and it can be expressed in terms of the Ricci potential $u$, with

$$R_{ij} = g_{ij} - u_{ij}, \quad u_{ij} = u_{ij} = 0, \quad \mathcal{V}^i = -g^{ij} u_j.$$

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The well-known Futaki invariant associated to the Kähler-Ricci soliton \((X, g, \mathcal{V})\) on a Fano manifold \(X\) is given by
\[
\mathcal{F}_X(\mathcal{V}) = \int_X |\nabla u|^2 dV_g = \int_X |\mathcal{V}|^2 dV_g \geq 0.
\]

Let \(\mathcal{KR}(n, \mathcal{F})\) be the set of compact Kähler-Ricci solitons \((X, g)\) of complex dimension \(n\) with
\[
\text{Ric}(g) = g + L_{\mathcal{V}} g, \quad \mathcal{F}_X(\mathcal{V}) \leq \mathcal{F}.
\]
It is proved by Tian-Zhang \([17]\) that \(\mathcal{KR}(n, \mathcal{F})\) is compact in the Gromov-Hausdorff topology with an additional uniform upper volume bound. In \([10]\), Phong-Song-Sturm established a partial \(C^0\)-estimate on \(\mathcal{KR}(n, \mathcal{F})\), generalizing the celebrated result of Donaldson-Sun \([6]\) for the space of uniformly non-collapsed Kähler manifolds with uniform Ricci curvature bounds. An immediate consequence of the partial \(C^0\)-estimate in \([10]\) is that the limiting metric space must be a \(\mathbb{Q}\)-Fano variety equipped with a Kähler-Ricci soliton metric.

The purpose of this paper is to remove the assumption in \([10]\) on the bound of the Futaki invariant.

**Definition 1.1** Let \(\mathcal{KR}(n)\) be the set of compact Kähler-Ricci solitons \((X, g, \mathcal{V})\) of complex dimension \(n\) with
\[
\text{Ric}(g) = g + L_{\mathcal{V}} g.
\]

The following is the main result of the paper.

**Theorem 1.1** Let \(\{(X_i, g_i, \mathcal{V}_i)\}_{i=1}^\infty\) be a sequence in \(\mathcal{KR}(n)\) with \(n \geq 2\). Then after possibly passing to subsequence, \((X_i, g_i)\) converges in the Gromov-Hausdorff topology to a compact metric length space \((X_\infty, d_\infty)\) satisfying the following.

1. The singular set \(\Sigma_\infty\) of the metric space \((X_\infty, d_\infty)\) is a closed set of Hausdorff dimension no greater than \(2n - 4\).

2. \((X_i, g_i, \mathcal{V}_i)\) converges smoothly to a Kähler-Ricci soliton \((X_\infty \setminus \Sigma_\infty, g_\infty, \mathcal{V}_\infty)\) satisfying
\[
\text{Ric}(g_\infty) = g_\infty + L_{\mathcal{V}_\infty} g_\infty, \quad (1.4)
\]
where \(\mathcal{V}_\infty\) is a holomorphic vector field on \(X_\infty \setminus \Sigma_\infty\).

3. \((X_\infty, d_\infty)\) coincides with the metric completion of \((X_\infty \setminus \Sigma_\infty, g_\infty)\) and it is a projective \(\mathbb{Q}\)-Fano variety with log terminal singularities. The soliton Kähler metric \(g_\infty\) extends to a Kähler current on \(X_\infty\) with bounded local potential and \(\mathcal{V}_\infty\) extends to a global holomorphic vector field on \(X_\infty\).
The assumption on the bound of the Futaki invariant in [10] is used to obtain a uniform lower bound of Perelman’s $\mu$-functional. We use the recent deep result of Birken [1] in birational geometry and show that there exists $\epsilon(n) > 0$ such that for any $n$-dimensional Fano manifold $X$, there exists a Kähler metric $g$ with $\text{Ric}(g) \geq \epsilon g$. In particular, the $\mu$-functional for $(X, g)$ is bounded below by a uniform constant that only depends on $n$. Then for any Kähler-Ricci soliton $(X, g) \in \mathcal{KR}(n)$, the $\mu$-functional for $(X, g)$ is uniformly bounded below because the soliton metric is the limit of the Kähler-Ricci flow. The proof of Theorem 1.1 also implies a uniform bound for the scalar curvature and the Futaki invariant for all $(X, g) \in \mathcal{KR}(n)$.

**Corollary 1.1** There exist $F = F(n), D = D(n)$ and $K = K(n) > 0$ such that for any $(X, g, u) \in \mathcal{KR}(n)$, the Futaki invariant, the diameter and scalar curvature $R$ of $(X, g)$ satisfy

$$ F_X \leq F, \text{diam}(X, g) \leq D, \ 0 < R \leq K. $$

We also derive some general compactness for compact or complete gradient shrinking solitons assuming a uniform lower bound of Perelman’s $\mu$-functional (see Section 3). For any closed or complete gradient shrinking soliton $(M, g, u)$, one can always normalize $u$ such that $\int_M e^{-u}dV_g = 1$. We define $\mathcal{RS}(n, A)$ to be the space of closed or complete shrinking gradient soliton $(M, g, u)$ of real dimension $n \geq 4$ satisfying

$$ \mu(g) \geq -A. \tag{1.5} $$

Then for any $A \geq 0$ and any sequence $(M_j, g_j, u_j, p_j) \in \mathcal{RS}(n, A)$ with $p_j$ being the minimal point of $u_j$, after passing to a subsequence, it converges in the pointed Gromov-Hausdorff topology to a compact or complete metric space $(M_\infty, d_\infty)$ of dimension $n$ with smooth convergence to a shrinking gradient Ricci soliton outside the closed singular set of dimension no greater than $n - 4$.

**2 Proof of Theorem 1.1**

Let us first recall the $\alpha$-invariant introduced by Tian on a Fano manifold [14].

**Definition 2.1** On a Fano manifold $(X, \omega)$ with $\omega \in c_1(X)$, the $\alpha$-invariant is defined as

$$ \alpha(X) = \sup\{ \alpha > 0 \mid \exists C_\alpha < \infty \text{ such that } \int_X e^{-\alpha(\varphi - \sup_X \varphi)}\omega^n \leq C_\alpha, \forall \varphi \in PSH(X, \omega) \}. $$

It is obvious that the $\alpha(X)$ does not depend on the choice $\omega \in c_1(X)$.

**Definition 2.2** Let $X$ be a normal projective variety and $\Delta$ an effective $\mathbb{Q}$-Cartier divisor, the pair $(X, \Delta)$ is said to be log canonical if the coefficients of components
of \( \Delta \) are no greater than 1 and there exists a log resolution \( \pi : Y \to X \) such that \( \pi^{-1}(\text{supp} \Delta) \cup \text{exc}(\pi) \) is a divisor with normal crossings satisfying

\[
K_Y = \pi^*(K_X + \Delta) + \sum_j a_j F_j, \quad Q \ni a_j \geq -1, \forall j.
\]

**Definition 2.3** Let \( X \) be a projective manifold and \( D \) be a \( \mathbb{Q} \)-Cartier divisor. The log canonical threshold of \( D \) is defined by

\[
lct(X, D) = \sup \{ t \in \mathbb{R} \mid (X, tD) \text{ is log canonical} \}.
\]

It is proved by Demailly that the \( \alpha \)-invariant is related to the log canonical thresholds of anti-canonical divisors through the following formula (see Theorem A.3. in the Appendix A of [5]).

**Theorem 2.1** For any Fano manifold \( X \),

\[
\alpha(X) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |-mK_X|} \inf \{ t \mid (X, tD) \text{ is log canonical} \}.
\]

Recently Birkar (Theorem 1.4 of [1]) obtains a uniform positive lower bound of the log canonical threshold and the following is an immediate corollary of Birkar’s result.

**Theorem 2.2** There exists \( \varepsilon_0 = \varepsilon_0(n) > 0 \) such that for any \( n \)-dimensional Fano manifold \( X \)

\[
\alpha(X) \geq \varepsilon_0(n).
\]

From the Harnack inequality in [14], for any fixed Kähler metric \( \omega \in c_1(X) \), the curvature equation for \( \omega_t \) along the continuity method

\[
\text{Ric}(\omega_t) = t(\omega_t) + (1 - t)\omega
\]

and the equation (2.1) can be solved for all \( t \in [0, (n+1)\alpha(X)/n] \). As a consequence, we have the following corollary.

**Corollary 2.1** There exists \( \varepsilon_1 = \varepsilon_1(n) > 0 \) such that for any \( n \)-dimensional Fano manifold \( X \), there exists a Kähler metric \( \hat{\omega} \in c_1(X) \) satisfying

\[
\text{Ric}(\hat{\omega}) \geq \varepsilon_1 \hat{\omega}.
\]

We can also assume that \( \hat{\omega} \) is invariant under the group action of the maximal compact subgroup \( G \) of \( \text{Aut}(X) \) by choosing a \( G \)-invariant Kähler metric \( \omega \) in the equation (2.1).

The greatest Ricci lower bound \( R(X) \) for a Fano manifold \( X \) is introduced in \([16, 12]\) and is defined by

\[
R(X) = \sup \{ t \in \mathbb{R} \mid \exists \omega \in c_1(X) \text{ such that } \text{Ric}(\omega) \geq t\omega \}.
\]

Immediately one has the following corollary.
Corollary 2.2  There exists an $r_0 = r_0(n) > 0$ such that for any $n$-dimensional Fano manifold $X$,

$$\mathcal{R}(X) \geq r_0.$$  \hspace{1cm} (2.3)

We are informed by Xiaowei Wang that Corollary 2.2 is already a consequence of results in [8]. In fact, Theorem 5.2 and Proposition 5.1 in [8] will imply that there exist $m = m(n) > 0$ and $\beta = \beta(n) \in (0,1]$ such that for any $n$-dimensional Fano manifold $X$, there exists a smooth divisor $D \in | - mK_X|$ and a conical Kähler-Einstein metric $\omega \in c_1(X)$ satisfying

$$\text{Ric} (\omega) = \beta \omega + (1 - \beta)m^{-1}[D].$$

Then Corollary 2.2 immediately follows by the relation between $\mathcal{R}(X)$ and the existence of conical Kähler-Einstein metric established in [11].

Now let us recall Perelman’s entropy functional for a Fano manifold $(X, g)$ with the associated Kähler form $\omega(g) \in c_1(X)$. The $W$-functional is defined by

$$W(g, f) = \frac{1}{V} \int_X (R + |\nabla f|^2 + f - 2n)e^{-f}dV_g,$$

where $V = c_1^n(X)$, and the $\mu$-functional is defined by

$$\mu(g) = \inf_f \left\{ W(g, f) \left| \frac{1}{V} \int_X e^{-f}dV_g = 1 \right. \right\}.$$

Lemma 2.1  There exists $A = A(n) > 0$ such that for the Riemannian metric $\hat{g}$ associated to the form $\hat{\omega}$ in (2.2)

$$\mu(\hat{g}) \geq -A.$$

Proof  Since $\text{Ric}(\hat{g})$ is bounded from below by a uniform positive constant $\varepsilon_1(n)$, by Myers’ theorem and volume comparison,

$$\text{Vol}(X, \hat{g}) \leq C(n), \quad \text{diam}(X, \hat{g}) \leq C(n).$$

On the other hand, since $\hat{\omega} \in c_1(X)$ is in an integral cohomology class, in particular $\text{Vol}(X, \hat{g}) \geq c(n) > 0$. By Croke’s theorem, the Sobolev constant $C_S$ of $(X, \hat{g})$ is uniformly bounded. It is well-known that a Sobolev inequality implies the lower bound of $\mu$-functional. For completeness, we provide a proof below.

For any $f \in C^\infty$ with $\int_X e^{-f}d\hat{V}_\hat{g} = V$, we write $e^{-f/2} = \phi$. By Jensen’s inequality

$$\frac{1}{V} \int_X \phi^2 \log \phi \frac{\phi}{\phi^{n-1}} \leq \log \left( \frac{1}{V} \int_X \phi \frac{\phi}{\phi^{n-1}} \right) \leq \log \left( C_S \int_X (|\nabla \phi|^2 + \phi^2) \right) \leq \frac{4}{n-1} \int_X |\nabla \phi|^2 + C(n).$$
So
\[ W(\hat{g}, f) = \frac{1}{V} \int_X (R\phi^2 + 4|\nabla \phi|^2 - \phi^2 \log \phi^2) dV_{\hat{g}} - 2n \geq -C(n). \]

Let \( (X, g) \in \mathcal{KR}(n) \) be a gradient shrinking Kähler-Ricci soliton which satisfies the equation
\[ \text{Ric}(\omega_g) + \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} u = \omega_g, \quad \nabla \nabla u = 0. \] (2.4)

Let \( G \subset \text{Aut}(X) \) be the compact one-parameter subgroup generated by the holomorphic vector field \( \text{Im}(\nabla u) \). As we mentioned before, the metric \( \hat{\omega} \) in (2.2) can be taken to be \( G \)-invariant.

**Corollary 2.3** For any \( (X, g) \in \mathcal{KR}(n) \), we have
\[ \mu(g) \geq -A, \]
where \( A = A(n) \) is the constant in Lemma 2.1.

**Proof** We consider the normalized Kähler-Ricci flow with initial metric \( \hat{\omega} \) in (2.2) \( G \)-invariant.
\[ \frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) + \omega(t), \quad \omega(0) = \hat{\omega}. \]

By the convergence theorem for Kähler-Ricci flow ([17, 19]), \( \omega(t) \) converges smoothly to \( \omega_g \), modulo some diffeomorphisms. So \( \lim_{t \to \infty} \mu(g(t)) = \mu(g) \).

On the other hand, \( \mu(g(t)) \) is monotonically non-decreasing along the Kähler-Ricci flow ([9]). The lower bound of \( \mu(g) \) follows from this monotonicity and the lower bound of \( \mu(\hat{g}) \) established in Lemma 2.1.

Now we can apply the same argument as in [10] because the assumption of the uniform bound for the Futaki invariant in [10] is to obtain a uniform lower bound for the \( \mu \)-functional. This will complete the proof of Theorem 1.1. The argument in [10] also implies the uniform bound for the scalar curvature and diameter of \( (X, g, u) \in \mathcal{KR}(n) \) as well as \( |\nabla u|^2 \) and hence the Futaki invariant of \( (X, g) \). This implies Corollary 1.1.

**3 Generalizations**

We generalize our previous discussion to Riemannian complete gradient shrinking Ricci solitons \( (M^n, g, u) \) satisfying the equation
\[ \text{Ric}(g) + \nabla^2 u = \frac{1}{2} g. \]

By [3] we can always normalize \( u \) such that \( \int_M e^{-u} dV_g = 1 \).
Definition 3.1 We denote $\mathcal{RS}(n, A)$ to the set of $n$-dimensional closed or complete shrinking gradient Ricci solitons $(M, g, u)$ satisfying

$$\mu(g) \geq -A$$

with the normalization condition $\int_M e^{-u} dV_g = 1$.

The following proposition is the main result of this section and most results in the proposition are straightforward applications of the compactness results [20, 21] with Bakry-Emery Ricci curvature bounded below.

Proposition 3.1 Let $\{(M_i, g_i, u_i, p_i)\}_{i=1}^{\infty}$ be a sequence in $\mathcal{RS}(n, A)$ with $n \geq 4$, where $p_i$ be a minimal point of $u_i$. Then after possibly passing to subsequence, $(M_i, g_i, u_i, p_i)$ converges in the Gromov-Hausdorff topology to a metric length space $(M_\infty, d_\infty, u_\infty)$ satisfying the following.

1. The singular set $\Sigma_\infty$ of the metric space $(M_\infty, d_\infty)$ is a closed set of Hausdorff dimension no greater than $n - 4$.

2. $(M_i, g_i, u_i)$ converges smoothly to a gradient shrinking Ricci soliton $(M_\infty \setminus \Sigma_\infty, g_\infty, u_\infty)$ satisfying

$$\text{Ric}(g_\infty) = \frac{1}{2} g_\infty + \nabla^2 u_\infty.$$ 

3. $(M_\infty, d_\infty)$ coincides with the metric completion of $(M_\infty \setminus \Sigma_\infty, g_\infty)$.

Furthermore, if there exists $V > 0$ such that $\text{Vol}_{g_i}(M_i) \leq V$ for all $i = 1, 2, \ldots$, the limiting metric space $(M_\infty, d_\infty)$ is compact.

Proof For any $(M, g, u) \in \mathcal{RS}(n, A)$, $R = \frac{n}{2} - \Delta u \geq 0$ ([23]), the potential function $u$ satisfies

$$\Delta u - |\nabla u|^2 + u = a, \quad a = \int_M u e^{-u} dV_g.$$ 

We denote $\tilde{u} = u - a$. From $\Delta u \leq \frac{n}{2}$ and immediately we have $|\nabla \tilde{u}|^2 \leq \frac{n}{2} + \tilde{u}$. By [3], the minimum of $\tilde{u}$ is achieved at some finite point $p \in M$, so $\min \tilde{u} = \tilde{u}(p) \geq -\frac{n}{2}$. Applying maximum principle to $\tilde{u}$ which satisfies $\Delta \tilde{u} - |\nabla \tilde{u}|^2 + \tilde{u} = 0$ at a minimum point $p \in M$, we obtain that $\min_M \tilde{u} = \tilde{u}(p) \leq 0$.

From $|\nabla \tilde{u}|^2 \leq \tilde{u} + \frac{n}{2}$, we have $|\nabla \sqrt{\tilde{u} + \frac{n}{2}}| \leq \frac{1}{2}$. Thus for any $x \in M$

$$\tilde{u}(x) \leq \frac{1}{2} d(p, x)^2 + \tilde{u}(p) + C(n) \leq \frac{1}{2} d(p, x)^2 + C(n). \quad (3.1)$$

Immediately we have

$$|\nabla \tilde{u}|^2(x) \leq \frac{1}{2} d(p, x)^2 + C(n), \quad (3.2)$$

7
and
\[-n/2 \leq -\Delta \tilde{u}(x) \leq \frac{1}{2}d(p, x)^2 + C(n).\] (3.3)

When \((M, g)\) is closed and \(\text{Vol}(M, g) \leq V\). We note by Jensen’s inequality \(a \leq \log V\). The Ricci soliton \((M, g, u)\) gives rise to a Ricci flow \(g(t) = \varphi_t^* g\) with initial metric \(g(0) = g\), where \(\varphi_t\) is the diffeomorphism group generated by \(\nabla u\), 
\[
\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) + g(t).
\]
Combining with the fact that \(R(g) \geq 0\) and Perelman’s non-collapsing theorem, we see that \((M, g)\) is non-collapsed in the sense that if \(R \leq r^{-2}\) on \(B_r(x)\), then \(\text{Vol}(B_r(x)) \geq \kappa(n, A)r^n\), for all \(r \in (0, \bar{r}(n, A)]\). With this non-collapsing and equations (3.1), (3.2) and (3.3), we can apply the same argument of Perelman as in Section 3 of [13] to show that there exists a uniform constant \(C(n, A, V) > 0\) such that for any closed \((M, g, u) \in \mathcal{RS}(n, A)\) with the additional assumption \(\text{Vol}(M, g) \leq V\),
\[
\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty(M, g)} + \|R\|_{L^\infty} + \text{diam}(M, g) \leq C(n, A, V). \] (3.4)

The non-collapsing of \((M, g)\) also implies a uniform lower bound on \(\text{Vol}(M, g)\). Now we can apply the main theorem of [22].

In general, when \((M, g)\) is complete, applying [9] to the Ricci flow associated to \((M, g)\), there exists a \(\kappa = \kappa(A, n)\) such that \((M, g)\) is \(\kappa\)-noncollapsed. In particular, \(\text{Vol}(B(p, 1)) \geq c(A, n) > 0\). On any geodesic ball \(B(p, r)\) with \(p\) being the minimal point of \(u\), \(|\nabla u| \leq \frac{1}{2}r^2 + C(n, A)\). By the Cheeger-Colding theory for Bakry-Emery Ricci tensor \(\text{Ric}(g) + \nabla^2 u\) ([20, 21]), for any sequence of \((M, g, u, p_i) \in \mathcal{RS}(n, A)\) converges (up to a subsequence) in pointed Gromov-Hausdorff topology to a metric space \((M_\infty, d_\infty, p_\infty)\). Here we choose \(p_i\) to be a minimum point of \(u\). \(M_\infty\) has the regular-singular decomposition \(M_\infty = \mathcal{R} \cup \Sigma\). Recall a point \(y \in \mathcal{R}\) if all tangent cone of \((M_\infty, d_\infty)\) at \(y\) is isometric to \(\mathbb{R}^n\). From [21], we know the singular set \(\Sigma\) is closed and of Hausdorff dimension at most \(n - 4\) and \(d_\infty\) on \(\mathcal{R}\) is induced by a \(C^n\) metric \(g_\infty\). For any \(y \in \mathcal{R}\) and \(M_i \ni y_i \xrightarrow{GH} y\), when \(i\) is large enough there exists a uniform \(r_0 = r_0(y)\) such that \((B_{r_0}(y_i, r_0), g_i)\) has uniform \(C^n\) bound (Theorem 1.2 of [21]). By choosing \(r_0\) even smaller if possible, we may assume the isoperimetric constant of \((B_{r_0}(y_i, r_0), g_i)\) is very small so that we can apply Perelman’s pseudo-locality theorem ([9]) to the associated Ricci flow to derive uniform higher order estimates of \(g_i\) near \(y_i\), which in turn gives local estimates of \(u_i\). So locally near \(y_i\), the convergence is smooth and we conclude that the metric \(g_\infty\) in a small ball around \(y\) is a Ricci soliton.

\[\square\]

We remark that in the compact case, a compactness result is obtained earlier by Zhang [22] assuming a uniform upper bound for the diameter and a uniform lower bound for the volume.
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