Exponential tracking of general references and rejection of general disturbances for nonlinear control systems with applications to the blood glucose regulation system

Weijiu Liu∗
Department of Mathematics
University of Central Arkansas
201 Donaghey Avenue, Conway, AR 72035, USA

Abstract

In solving the problem of exponential tracking and disturbance rejection, it has been long always assumed that the reference to be tracked and the disturbance to be rejected are generated by an exosystem such as a finite dimensional system with pure imaginary eigenvalues. The aim of this note is to show that this assumption can be removed. For any nonlinear control system subject to a general disturbance, it can be split into a linear exponentially-stable system and a dynamical regulator system. If the dynamical regulator system has a solution, then there exists a feedback and feedforward controller such that an output of the control system exponentially tracks a desired general reference. The result is applied to the blood glucose regulation system.

Keywords: Nonlinear control system, exponential tracking, disturbance rejection, general reference and disturbance, dynamical regulator system, feedback and feedforward control, blood glucose regulation.

In solving the problem of exponential tracking and disturbance rejection, it has been long always assumed that the reference to be tracked and the disturbance to be rejected are generated by an exosystem such as a finite dimensional system with pure imaginary eigenvalues (see, e.g., [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]). In fact, this assumption is sufficient, but not necessary for making the problem solvable.

The aim of this note is to show that this assumption can be removed. Consider the nonlinear control system:

\[
\frac{dx}{dt} = f(x, u, v_d), \quad y = f_o(x, u, v_d), \quad e = y - r,
\]

where \( x \) is a state vector in \( \mathbb{R}^n \), \( u \) is a control vector, \( y \) is an output vector, \( v_d(t) \) is a disturbance vector, \( r(t) \) is a reference vector, and \( f \) and \( f_o \) are given vector functions. All functions in this note are assumed to have the required continuity and differentiability.

The problem of asymptotic tracking and disturbance rejection is to design a controller \( u \) such that

\[
\lim_{t \to \infty} e(t) = \lim_{t \to \infty} [y(t) - r(t)] = 0.
\]

Furthermore, if there exist positive constants \( C \) and \( a \) such that

\[
\|e(t)\| \leq Ce^{-at}
\]

for all \( t \geq 0 \), it is called exponential tracking.

To solve this tracking problem, we introduce the variable transform

\[
x = \hat{x} + X.
\]

Substituting this transform into (1)-(3) gives

\[
\frac{d\hat{x}}{dt} + \frac{dX}{dt} = f(\hat{x} + X, u, v_d), \quad e = f_o(\hat{x} + X, u, v_d) - r.
\]

This system can be split into a linear exponentially-stable system

\[
\frac{d\hat{x}}{dt} = A\hat{x}, \quad e = CX,
\]

and a dynamical regulator system

\[
\frac{dX}{dt} = f(\hat{x} + X, u, v_d) - A\hat{x}, \quad 0 = f_o(\hat{x} + X, u, v_d) - C\hat{x} - r.
\]

∗Corresponding author. Email: weijiul@uca.edu, Phone: 1-501-450-5661, Fax: 1-501-450-5662.
where \( \mathbf{A} \) is a constant matrix whose eigenvalues have negative real parts, and \( \mathbf{C} \) is a constant matrix.

**Theorem 1.** Assume that \( \mathbf{A} \) is a constant matrix whose eigenvalues have negative real parts and \( \mathbf{C} \) is a constant matrix. If the dynamical regulator system (11) and (12) has a solution, then there exists a feedback and feedforward controller

\[
\mathbf{u} = \mathbf{u}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d),
\]

such that the tracking error \( \mathbf{e}(t) \) satisfies the exponential tracking estimate (5).

**Proof.** Let

\[
\mathbf{X} = \mathbf{X}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d), \quad \mathbf{u} = \mathbf{u}(\hat{\mathbf{x}}, \mathbf{r}, \mathbf{v}_d)
\]

be a solution of the dynamical regulator system (11) and (12). Then, under the controller (13), we have

\[
\begin{align*}
\mathbf{e}(t) &= \mathbf{f}_p(\mathbf{x}, \mathbf{u}, \mathbf{v}_d) - \mathbf{r} \\
&= \mathbf{C} \hat{\mathbf{x}} + \mathbf{f}_p(\hat{\mathbf{x}} + \mathbf{X}, \mathbf{u}, \mathbf{v}_d) - \mathbf{C} \hat{\mathbf{x}} - \mathbf{r} \\
&= \mathbf{C} \mathbf{x}.
\end{align*}
\]

Under the assumption on the matrix \( \mathbf{A} \), the system (9) is exponentially stable. So \( \hat{\mathbf{x}}(t) \) converges to 0 exponentially as \( t \to \infty \), and then the tracking error \( \mathbf{e}(t) \) satisfies the exponential tracking estimate (5). This completes the proof.

It looks like there are no conditions on the matrix \( \mathbf{C} \). In fact, the conditions on \( \mathbf{C} \) is implied in the dynamical regulator system (11) and (12). The existence of a solution of the system depends on the choice of \( \mathbf{C} \). For example, if \( \mathbf{C} = 0 \), in general, the dynamical regulator system may have no solutions. Thus \( \mathbf{C} \) should be chosen such that the system has a solution.

Note that the disturbance \( \mathbf{v}_d \) is directly present in the static controller \( \mathbf{u} \). Since the disturbance is unknown in reality, this is not reasonable. However, it seems that this is standard or cannot be avoided for such a static controller. For example, the disturbance is directly present in the static state feedback for an output regulation problem in Section 3.2 of [8] and in feedback control law for PDEs regulation in Problem 1.1 and Theorem 1.1 of [2]. To design a robust controller without the direct presence of the disturbance, we have to introduce a dynamical compensator (see, e.g., [8, 13]). But this is not the topic of this short note.

Compared with the usual static regulator partial differential equations (see, e.g., [8]), the dynamical regulator ordinary differential system (11) - (12) looks simpler, but still difficult to be solved. However, it can be solved for some important mathematical models such as the model of the blood glucose regulation system proposed by Bergman et al. [3]:

\[
\begin{align*}
\frac{dg}{dt} &= -m_1 g - h_a g + J, \\
\frac{dh_a}{dt} &= -m_2 h_a + m_3 h, \\
\frac{dh}{dt} &= -m_4 h + u, \\
e(t) &= g - r.
\end{align*}
\]

In the above equations, \( g \) and \( h \) denote concentrations of blood glucose and plasma insulin, respectively, \( h_a \) is the effect of the remote insulin on glucose, \( J \) is a rate of the exogenous glucose input from the intestine, \( u \) is a rate of insulin secreted from the endocrine system or infused externally, \( r \) is a glucose reference, and \( m_1, m_2, m_3, m_4 \) are positive rate constants. Let the exponentially-stable system be given by the simple linear system

\[
\begin{align*}
\frac{d\hat{g}}{dt} &= -\hat{g}, \\
\frac{d\hat{h}_a}{dt} &= -\hat{h}_a, \\
\frac{d\hat{h}}{dt} &= -\hat{h}, \\
e(t) &= \hat{h}.
\end{align*}
\]

Then the dynamical regulator system for the blood glucose regulation system is

\[
\begin{align*}
\frac{dG}{dt} &= -m_1(\hat{g} + G) - (\hat{h}_a + H_a)(\hat{g} + G) + \hat{g} + J, \\
\frac{dH_a}{dt} &= -m_2(\hat{h}_a + H_a) + m_3(\hat{h} + H) + \hat{h}_a, \\
\frac{dH}{dt} &= -m_4(\hat{h} + H) + \hat{h} + u, \\
0 &= \hat{g} + G - r - \hat{h}.
\end{align*}
\]

The solution of the system is given by

\[
\begin{align*}
G &= r + \hat{h} - \hat{g}, \\
H_a &= \frac{J + \hat{g} - (m_1 + \hat{h}_a)(\hat{g} + G) - G'}{\hat{g} + G}, \\
H &= \frac{H_a' + m_2(\hat{h}_a + H_a) - m_3\hat{h} - \hat{h}_a}{m_3}, \\
u &= H' + m_4(\hat{h} + H) - \hat{h}.
\end{align*}
\]

Thus, we have designed a feedback and feedforward controller (29) under which the blood glucose concentration \( g \) exponentially tracks the reference \( r(t) \).
We conduct a numerical computation to test the controller (29). In the computation, we take $m_1 = 0.0014 /\text{min}$, $m_2 = 0.0059 /\text{min}$, $m_3 = 0.012 /\text{min}$, $m_4 = 0.00023 /\text{min}$, $J = \sin(t)$, $r = 90 + 10 \cos(t)$, and $g(0) = 180 \text{ mg/dl}$. The figure 1 shows that the blood glucose concentration $g$ exponentially tracks the reference $r(t) = 90 + 10 \cos(t)$ under the feedback and feedforward controller (29).

It is important to notice that the above idea can be directly applied to partial differential equations such as the convection diffusion equation, the wave equation, and the Burgers’ equation.

References

[1] Brandon Ashley and Weijiu Liu, Asymptotic tracking and disturbance rejection of blood glucose regulation system, Mathematical Biosciences, 289, 2017, 78-88.

[2] E. Aulisa and D. Gilliam, A Practical Guide to Geometric Regulation for Distributed Parameter Systems, Chapman and Hall/CRC, Boca Raton, FL, 2015.

[3] R. N. Bergman, L. S. Phillips, and C. Cobelli, Measurement of insulin sensitivity and $\beta$-cell glucose sensitivity from the response to intravenous glucose, J. Clin. Invest., vol. 68 (1981), 1456-1467.

[4] C. I. Byrnes, I. G. Lauko, D. S. Gilliam, V. I. Shubov, Output regulation for linear distributed parameter systems, IEEE Trans. Autom. Control, 45, no. 12, pp. 2236-2252, 2000.

[5] J. Deutscher, A backstepping approach to the output regulation of boundary controlled parabolic PDEs, Automatica, 57 (2015), 56-64.

[6] J. Deutscher, Backstepping design of robust output feedback regulators for boundary controlled parabolic PDEs, IEEE Trans. Autom. Control, 61 (2016), 2288-2294.

[7] J. Deutscher, Finite-time output regulation for linear $2 \times 2$ hyperbolic systems using backstepping, Automatica, 75 (2017), 54-62.

[8] J. Huang, Nonlinear Output Regulation, Theory and Applications. Society for Industrial and Applied Mathematics, Philadelphia (2004)

[9] W. Liu, Elementary Feedback Stabilization of the Linear Reaction Diffusion Equation and the Wave Equation, Mathematiques et Applications, Vol. 66, Springer, 2010.

[10] W. Liu, Boundary feedforward and feedback control for the exponential tracking of the unstable high-dimensional wave equation, Journal of Mathematical Analysis and Applications, vol 499, issue 1, July, 2021, https://doi.org/10.1016/j.jmaa.2021.125010

[11] W. Liu, Independence of convergence rate of the wave tracking error on structures of feedforward controllers, Automatica, https://doi.org/10.1016/j.automatica.2020.109264

[12] W. Liu, Feedforward boundary control for the regulation of a passive and diffusive scalar in 2-D unsteady flows, IEEE Transactions on Automatic Control, vol. 65, no. 11, pp. 4882 - 4886, 2020.

[13] W. Liu, A mathematical model for the robust blood glucose tracking. Mathematical Biosciences and Engineering, 16 (2), 2019, 759 - 781.

[14] Florian Malchow and Oliver Sawodny, Feedforward Control of Inhomogeneous Linear First Order Distributed Parameter Systems. 2011 American Control Conference, San Francisco, CA, USA, 2011, 3597 - 3602.

[15] T. Meurer and M. Zeitz, Feedforward and Feedback Tracking Control of Nonlinear Diffusion-Convection-Reaction Systems Using Summability Methods. Ind. Eng. Chem. Res. 44, 2532 - 2548, 2005.

[16] Thomas Meurer and Andreas Kugi, Trajectory Planning and Feedforward Control Design for the Temperature Distribution in a Cuboid. Proc. Appl. Math. Mech. 6, 825 - 826, (2006)
[17] Thomas Meurer, Control of Higher–Dimensional PDEs: Flatness and Backstepping Designs. Springer, New York, 2013

[18] Thomas Meurer, Flatness-based motion planning and tracking. Lecture Notes for the Workshop ”New Trends in Control of Distributed Parameter Systems” at the 2016 IEEE CDC, Las Vegas (NV), USA.

[19] V. Natarajan, D. S. Gilliam, and G. Weiss, The State Feedback Regulator Problem for Regular Linear Systems, IEEE Trans. Autom. Control 59, pp. 2708-2722 (2014)

[20] Tilman Utz and Andreas Kugi, Flatness-based feedforward control design of a system of parabolic PDEs based on finite difference semi-discretization. Proc. Appl. Math. Mech. 12, 731 – 732 (2012).

[21] M. Wagner, T. Meurer, and A. Kugi, Feedforward control design for a semilinear wave equation, Proc. Appl. Math. Mech., 9, pp. 7-10, 2000.

[22] H.-C. Zhou and G. Weiss: Solving the regulator problem for a 1-D Schrödinger equation via backstepping, IFAC PapersOnLine, 50-1 (2017), 4516-4521.