Channel-state duality and the separability problem.

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Separability of quantum states is analyzed with the use of the Choi-Jamiolkowski isomorphism.

I. Introduction

Entanglement as a resource [1,2] is a central notion in quantum information theory. The important question is to tell whether a given quantum composite system state is entangled or separable. One of the first remarkable results in this direction was the positive partial transposition (PPT) criterion [3] as necessary condition for separability of bipartite mixed states. This simple but extremely useful observation by Asher Peres has generated further considerable research. It was proved that PPT condition is necessary and sufficient for separability of $2 \otimes 2$ and $2 \otimes 3$ states [4]. Over time several other necessary or/and sufficient criteria were developed [4,5,6,7,8,9], among which entanglement witnesses [5,7] and the CCNR criterion [8,9] proved to be important tools in detecting entanglement. The Bloch representation was first introduced into the separability problem by de Vicente [10], the idea, extended in a recent paper [11] where a necessary and sufficient separability criterion was obtained in terms of inequalities for singular values of the correlation matrix.

In this paper we present an approach to the separability problem inspired by the well-known correspondence between completely positive (CP) maps and states, the Choi-Jamiolkowski isomorphism [12,13]. We make use of the fact that a state is separable if and only if the corresponding CP map can have operator-sum representation with unit rank operators. Different operator-sum representations of the same CP map are connected by linear transformations of the specific type. Thus, the state is separable if and only if the operators of the corresponding CP map can all be transformed to those of unit rank. We analyze the conditions under which such transformations exist and show that this approach can be a powerful tool in investigating the separability problem. The method is illustrated by examples. In addition, spectrum based separability criterion is derived.

II. Channel-state duality and connections with separability

We begin this section with recalling the properties of the Choi-Jamiolkowski isomorphism between states and CP maps. Let $\rho_{AB}$ be a density operator acting on Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $\dim(\mathcal{H}_A) = m$, $\dim(\mathcal{H}_B) = n$, $m \leq n$. Let $\sigma$ be a density operator on $\mathcal{H}_A$. A CP map $\Lambda_\rho: \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$, corresponding to $\rho_{AB}$, can be defined by the action on an arbitrary state $\sigma$ as follows:

$$\Lambda_\rho[\sigma] = \text{Tr}_A{\sigma^T \otimes I_A \rho_{AB}},$$

(1)

where $\sigma^T$ – transposed operator $\sigma$, $I_A$ – identity operator acting on $\mathcal{H}_A$.

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1If a CP map is trace-preserving, they are called Kraus operators.
Let $\mathcal{H}_R$ be a reference Hilbert space isomorphic to $\mathcal{H}_A$, then the initial state $\rho$ is recovered by the action of the Choi operator [14]:
\[(I_R \otimes \Lambda_{\rho}) |\Gamma\rangle \langle \Gamma|_{RA} = \rho_{RB},\]
where $|\Gamma_{RA}\rangle$ - unnormalized maximally entangled vector:
\[|\Gamma_{RA}\rangle = \sum_{i=0}^{m-1} |i\rangle_R \otimes |i\rangle_A.\]
Suppose that $\rho$ is realized by a specific ensemble of bipartite pure states $|\Psi_a\rangle$ with probabilities $p_a$:
\[\rho_{RB} = \sum_a p_a |\Psi_a\rangle \langle \Psi_a|_{RB},\]
then, from eq. 1 and eq. 4 it is straightforward to see that
\[\Lambda_{\rho} |\varphi\rangle \langle \varphi|_A = \sum_a p_a \rho_a (\varphi^* | \Psi_a\rangle \langle \Psi_a| \varphi)^R,\]
where we extract the action on $\varphi_A$ using the dual vector $\rho_a (\varphi^*)$ in accordance with the relative-state method [1]. Now, given a vector $|\Psi_{aR}|_{RB},$ an operator $M_a$ mapping $\mathcal{H}_A$ to $\mathcal{H}_B$ can be defined by
\[M_a |\varphi\rangle_A = \sqrt{mp_a} \rho (\varphi^* | \Psi_{aR}\rangle_{RB}.\]
Eq. 5 gives an operator-sum representation of the CP map $\Lambda_{\rho}$ acting on a pure state projector $|\varphi\rangle \langle \varphi|_A$ and hence, by linearity, on any density operator $\sigma$:
\[\Lambda_{\rho}[\sigma] = \sum_a M_a \sigma M_a^\dagger.\]
Note that the dimensional factor $\sqrt{n}$ is added in eq. 5 (and, correspondingly, the factor $1/m$ should be put before $|\Gamma\rangle \langle \Gamma|_{RA}$ in eq. 2) for consistency with the fact that $\rho_{AB}$ and its reductions $\rho_A, \rho_B$ all have unit traces.

One important property of the operators $M_a$, which we will use in the present paper, is the connection with the reduced density matrices $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$. Suppose that the pure states $|\Psi_a\rangle$ from the ensemble admit the following decomposition in the given orthonormal bases $|i\rangle_A, |j\rangle_B$ of $\mathcal{H}_A$ and $\mathcal{H}_B$:
\[|\Psi_a\rangle_{AB} = \sum_{i,j} c_{ij}(a) |i\rangle_A \otimes |j\rangle_B, 0 \leq i \leq m-1, 0 \leq j \leq n-1.\]
Consider matrix element $\langle i|M_a^\dagger M_a|j\rangle$: with the use of eqs. 6 [8] it can be evaluated as follows:
\[\langle i|M_a^\dagger M_a|j\rangle_A = mp_{a AB} \langle |i\rangle_A |j\rangle_A |\Psi_a\rangle_{AB} = \sum_{m,n,k,l} mp_a c_{mn}^{(a) \ast} \langle m|i\rangle_A \langle n|j\rangle_B c_{kl}^{(a) \ast} \langle k|l\rangle_B = \sum_l mp_a c_{jl}^{(a) \ast} c_{jl}^{(a)} = mp_a (\rho_a)^{jl},\]
where $\rho_A^{(a)} = c^{(a)}c^{(a)*} = \text{Tr}_B\{ |\Psi_a\rangle \langle \Psi_a|_{AB} \}$ — reduction of $|\Psi_a\rangle \langle \Psi_a|_{AB}$ on subsystem $A$ such that $\rho_A = \sum_a p_a \rho_A^{(a)}$. We see that

$$M_a^\dagger M_a = mp_a (\rho_A^{(a)})^T,$$

and the following property holds: a CP map is trace-preserving ($\sum_a M_a^\dagger M_a = I$) iff the reduced density operator $\rho_A$ of the corresponding state is maximally mixed: $\rho_A = \frac{1}{m} I_A$.

In a similar way, it can be obtained that

$$M_a M_a^\dagger = mp_a \rho_B^{(a)}.$$

In addition, substituting $\frac{1}{m} I_A$ for $|\varphi\rangle \langle \varphi|_A$ in eq. 5 gives:

$$\Lambda_\rho \left\{ \frac{1}{m} I_A \right\} = \text{Tr}_A \{ \rho_{AB} \} = \rho_B,$$

hence a CP map is unital iff the reduced density operator $\rho_B$ of the corresponding state is maximally mixed: $\rho_B = \frac{1}{n} I_B$.

The second important property is the transformations of $M_a$. According to the Hughston-Jozsa-Wootters theorem [15], two different ensemble realizations of the same density operator

$$\sum_a p_a |\Psi_a\rangle \langle \Psi_a|_{AB} = \sum_\mu q_\mu |\Phi_\mu\rangle \langle \Phi_\mu|_{AB}$$

are related by

$$\sqrt{q_\mu} |\Phi_\mu\rangle = \sum_a \sqrt{p_a} V_{\mu a} |\Psi_a\rangle,$$

where $V_{\mu a}$ — a matrix with orthonormal columns. Therefore, from eqs. 6, 14 it follows that operators $N_\mu$ and $M_a$, corresponding to $|\Phi_\mu\rangle$ and $|\Psi_a\rangle$, are related by

$$N_\mu = \sum_a V_{\mu a} M_a.$$

Eq. 15 plays the main role in the development of our paper. Now, to analyze the separability of quantum states, we need one more statement that can be deduced from Theorem 4 of Ref. [16].

**Lemma 1.** A bipartite mixed state $\rho$ is separable if and only if the operators $\{M_a\}$ from the operator-sum representation of the corresponding completely positive map can all be transformed by means of eq. 15 to rank one operators $\{N_\mu\}$.

**Proof.** If $\rho$ is separable, then, according to eq. 2 the corresponding map $\Lambda_\rho$ transforms the maximally entangled state to a separable state, and, by the proposition $(C) \Rightarrow (D)$ of Theorem 4 from Ref. [16], $\Lambda_\rho$ can be written in operator sum using only operators of rank one, i.e., eq. 15 holds with $\{N_\mu\}$ of rank one. The converse statement is also true due to the equivalence of the clauses $(C)$ and $(D)$ of the above mentioned Theorem.

Note that if $\rho$ is separable, the CP map $\Lambda_\rho$ is not an entanglement-breaking map unless it is trace-preserving, i.e., $\rho_A = \frac{1}{m} I_A$.

As an indirect application of Lemma 1 we consider a separability criterion based on spectral properties.

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1It will be a unitary matrix if the numbers of terms in both ensembles are equal.
III. Applications of the main result

A. Spectral separability criterion

For now, let eq. 4 express an eigenvector decomposition of the density operator \( \rho \). The matrices \( c^{(a)} \) of eq. 8, which correspond to the eigenvectors \( |\Psi_a\rangle \), are orthonormal with respect to the Hilbert-Schmidt inner product:
\[
\text{Tr}\{c^{(a)}|c^{(b)}\} = \delta_{ab}.
\] (16)

From eqs. 6, 8, 16 it follows, then, that operators \( \{M_a\} \) are mutually orthogonal:
\[
\text{Tr}\{M_a^\dagger M_b\} = m p_a \delta_{ab}.
\] (17)

Now, consider eq. 15. We can take advantage of the eigenvector decomposition and express the coefficients \( V_{\mu a} \) using eq. 17:
\[
V_{\mu a} = \frac{\text{Tr}\{M_a^\dagger N_\mu\}}{m p_a}.
\] (18)

On the one hand, \( V_{\mu a} \) are the entries of a matrix with orthonormal columns, and so we can write:
\[
\sum_{\mu} |V_{\mu a}|^2 = 1.
\] (19)

On the other hand, we can give an upper bound on \( |\text{Tr}\{M_a^\dagger N_\mu\}| \) using the Cauchy-Schwarz inequality or an even sharper bound using the following property [17, 18]:
\[
|\text{Tr}\{A^\dagger B\}| \leq \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B),
\] (20)

where \( A, B \) – complex \( m \times n \) matrices, \( q = \min\{m, n\} \), \( \sigma_i(A), \sigma_i(B) \) – singular values of \( A \) and \( B \) arranged in non-increasing order: \( \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_q(A) \).

Let \( \lambda_1^{(a)}, \ldots, \lambda_m^{(a)} \) denote the eigenvalues of the density operator \( \rho_A^{(a)} \) on subsystem \( A \), taken in non-increasing order. From eq. 10 we can see the connection between the singular values \( \sigma_i(M_a) \) of the operators \( M_a \) (thought of as matrices) and the eigenvalues \( \lambda_i^{(a)} \):
\[
\sigma_i(M_a) = \sqrt{m p_a \lambda_i^{(a)}}.
\] (21)

Eqs. 20, 21 give an upper bound on \( |V_{\mu a}| \):
\[
|V_{\mu a}| \leq \sum_{i=1}^{m} \sqrt{q_\mu \lambda_i^{(\mu)} \lambda_i^{(a)}} / p_a,
\] (22)

where \( q_\mu \) – probabilities of the second ensemble realization of the density operator \( \rho \), as in eq. 13 \( \lambda_i^{(\mu)} \) – eigenvalues of \( \hat{\rho}_A^{(\mu)} = \text{Tr}_B\{ |\Phi_{\mu}\rangle \langle \Phi_{\mu}|_{AB}\} \).
If \( \rho \) is separable, then, according to Lemma 1, there exist coefficients \( V_{\mu a} \) such that \( \{ N_\mu \} \) are rank one operators. Correspondingly, \( \tilde{\rho}_\lambda^{(\mu)} \) are also of rank one, and so each of them has only one non-vanishing eigenvalue:

\[
\tilde{\lambda}_1^{(\mu)} = 1, \quad \tilde{\lambda}_2^{(\mu)} = \ldots = \tilde{\lambda}_m^{(\mu)} = 0.
\]

Combining this fact with eqs. (19), (22) we obtain:

\[
1 = \sum_\mu |V_{\mu a}|^2 \leq \sum_\mu q_\mu \frac{\lambda_1^{(a)}}{p_a} \sum_\mu q_\mu = \frac{\lambda_1^{(a)}}{p_a}.
\] (24)

As a result, we have the following

**Separability criterion 1.** If a density operator \( \rho_{AB} \) with an eigenvector decomposition \( \rho_{AB} = \sum_a p_a |\Psi_a\rangle \langle \Psi_a|_{AB} \) is separable, then the largest eigenvalue of each \( \rho_A^{(a)} = \text{Tr}_B\{ |\Psi_a\rangle \langle \Psi_a|_{AB} \} \) is greater than or equal to the corresponding ensemble probability: \( \lambda_1^{(a)} \geq p_a \).

We note that the criterion was derived independently of Ref. [19], where the same property was obtained by a different method within the framework of the theory of entanglement witnesses.

As an example of application of the criterion, consider isotropic states [20] in arbitrary dimension \( d \) (\( m = n = d \)):

\[
\rho_{\text{iso}}^{(\alpha)} = \alpha |\Phi^+\rangle \langle \Phi^+| + \frac{1 - \alpha}{d^2} I_d \otimes I_d,
\] (25)

where \( 0 \leq \alpha \leq 1 \), and \( |\Phi^+\rangle \) – maximally entangled state:

\[
|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B.
\] (26)

In order to obtain an eigenvector decomposition of \( \rho_{\text{iso}}^{(\alpha)} \), we introduce mutually orthogonal auxiliary states:

\[
|\Phi_k^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i 2 \pi j k / d} |j\rangle_A \otimes |j\rangle_B, \quad k = 0, \ldots, d - 1,
\]

\[
|\Psi_{ij}^+\rangle = \frac{1}{\sqrt{2}} (|i\rangle_A \otimes |j\rangle_B \pm |j\rangle_A \otimes |i\rangle_B), \quad i < j, \quad i, j = 0, \ldots, d - 1,
\] (27)

such that \( |\Phi^+\rangle \equiv |\Phi_0^+\rangle \): \( d \) states \( |\Phi_k^+\rangle \) and \( d(d-1)/2 \) states \( |\Psi_{ij}^+\rangle \) belong to the symmetric subspace of \( \mathcal{H}_{AB} \), and \( d(d-1)/2 \) states \( |\Psi_{ij}^-\rangle \) belong to the antisymmetric subspace. Therefore, the identity operator \( I_d \otimes I_d \) can be decomposed in terms of \( d^2 \) mutually orthogonal states from eq. (27):

\[
I_d \otimes I_d = \sum_{k=0}^{d-1} |\Phi_k^+\rangle \langle \Phi_k^+| + \sum_{i < j}^{d-1} \left[ |\Psi_{ij}^+\rangle \langle \Psi_{ij}^+| + |\Psi_{ij}^-\rangle \langle \Psi_{ij}^-| \right],
\] (28)

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and the isotropic state density operator is expressed as follows:

\[
\rho^{iso}(F) = F |\Phi^+_0\rangle\langle\Phi^+_0| + \frac{1 - F}{d^2 - 1}\left(\sum_{k=1}^{d-1} |\Phi^+_k\rangle\langle\Phi^+_k| + \sum_{i<j}^{d-1} |\Psi^+_{ij}\rangle\langle\Psi^+_{ij}| + |\Psi^-_{ij}\rangle\langle\Psi^-_{ij}| \right),
\]

(29)

\[
F = \frac{\alpha(d^2 - 1) + 1}{d^2}.
\]

Applying separability criterion 1 to the first term of the decomposition in eq. 29, we obtain that if \(\rho^{iso}\) is separable, then

\[
\lambda_1(|\Phi^+_0\rangle\langle\Phi^+_0|) = \frac{1}{d} \geq F.
\]

(30)

Conversely, if \(F > 1/d\), i.e., \(\alpha > 1/(d + 1)\), then \(\rho^{iso}\) is entangled. This fact was established by application of other separability criteria [20].

Although separability criterion 1 is able to detect all entangled isotropic states, it is not as much efficient in many other cases where it can be applicable. As an example, consider a \(2 \otimes 2\) state

\[
\rho^\pm = p |\psi^\pm\rangle\langle\psi^\pm| + (1 - p) |00\rangle\langle00|,
\]

(31)

where \(|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)|. In this case the criterion detects entanglement only when \(p > 1/2\), whereas \(\rho^\pm\) is entangled at each \(p \neq 0\).

As a consequence of separability criterion 1, the following inequality holds for a separable state:

\[
\sum_a \lambda_1^{(a)} \geq 1.
\]

(32)

This inequality can also be obtained from the majorization criterion [21], which states that the density matrix of a separable state is majorized by both of its reductions:

\[
\rho_{AB} \prec \rho_A, \rho_{AB} \prec \rho_B,
\]

(33)

where majorization \(A \prec B\) for arbitrary Hermitian operators \(A\) and \(B\) is defined in terms of their eigenvalues \(\lambda_1(A) \geq \ldots \geq \lambda_d(A)\) and \(\lambda_1(B) \geq \ldots \geq \lambda_d(B)\):

\[
\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B),
\]

(34)

for \(k = 1, \ldots, d - 1\), and with the inequality holding with equality when \(k = d\). Applying eq. 33 and eq. 34 (with \(k = 1\)) to the eigenvector decomposition of \(\rho_{AB}\), in which \(\lambda_j(\rho_{AB}) = p_j\), we obtain:

\[
p_1 \leq \lambda_1(\rho_A) = \lambda_1 \left( \sum_a p_a \rho^{(a)}_A \right) \leq \sum_a p_a \lambda_1^{(a)}.
\]

(35)

Here the \(\leq\) is due to the Ky Fan’s maximum principle [17, 18] in application to a sum of operators. Combination of eq. 35 with the fact that \(p_a \leq p_1\) for all \(a\) yields inequality 32.

Separability criterion 1 is not strong enough to detect entanglement in many important cases. In the next section we proceed to the direct application of Lemma 1 a more powerful tool.
B. Application to concrete states

Lemma 1 states that a bipartite density operator $\rho_{AB}$ is separable if and only if the corresponding completely positive map can be represented by unit rank operators $\{N_\mu\}$. Therefore, given any ensemble decomposition of $\rho_{AB}$ (not necessarily an eigenvector one) and a corresponding set $\{M_a\}$ of initial operators representing the CP map, by virtue of eq. 15 one can search for linear combinations of $M_a$ with coefficients $V_{\mu a}$ that produce unit rank operators. To do this, one can set the determinants of all second order minors of $N_\mu$ to null and analyze the resulting system of polynomial equations in variables $V_{\mu a}$. $\rho_{AB}$ is separable if and only if there exist solutions $V_{\mu a}$ which form a matrix with orthonormal columns.

Example 1. A mixture of two $2 \otimes 2$ maximally entangled density operators

$$\rho = p |\psi^+\rangle \langle \psi^+| + (1-p) |\phi^-\rangle \langle \phi^-| ,$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

According to eq. 6, the CP map corresponding to $\rho$ can be represented by two operators $M_{\psi^+}$ and $M_{\phi^-}$ related to the states $|\psi^+\rangle$ and $|\phi^-\rangle$. Their matrix representations are obtained by substituting the elements of the computational basis for $|\varphi\rangle$ and the states $|\psi^+\rangle$ and $|\phi^-\rangle$ for $|\Psi_a\rangle$ in eq. 6:

$$M_{\psi^+} = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_{\phi^-} = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Eq. 15 gives the expression for $N_\mu$ in terms of $V_{\mu a}$:

$$N_\mu = \begin{pmatrix} \sqrt{1-p} V_{\mu 2} & -V_{\mu 1} \\ \sqrt{p} V_{\mu 1} & \sqrt{1-p} V_{\mu 2} \end{pmatrix} .$$

Operator $N_\mu$ is of unit rank when the determinant of its matrix vanishes:

$$(1-p)V_{\mu 2}^2 = -pV_{\mu 1}^2 .$$

Taking absolute value of both parts of eq. 39, one can see that it is consistent with the normalization condition 19 when $1 - p = p$. Therefore, the density operator in eq. 36 is entangled when $p \neq 1/2$. When $p = 1/2$, an example of a unitary matrix $V$ satisfying $V_{\mu 2}^2 = -V_{\mu 1}^2$ can easily be found:

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} .$$

Therefore, when $p = 1/2$, $\rho$ is separable. The proposed method also gives a separable decomposition in this case. Substituting the entries from the first row ($\mu = 1$) of the matrix of eq. 40 into eq. 38, we obtain $N_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, and, by eqs. 10 11 (with $N_1$ instead of $M_a$), the reductions

$$\rho_{A}^{(1)} = \rho_{B}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

along with the ensemble probability $q_1 = \frac{1}{m} \text{Tr}\{N_1^\dagger N_1\} = 1/2$. 7
In a similar way, it can be obtained that
\[ \rho^{(2)}_A = \rho^{(2)}_B = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \]
and so the separable decomposition (one of the many possible) of \( \rho \) is:
\[ \rho = \frac{1}{2} \rho^{(1)}_A \otimes \rho^{(1)}_B + \frac{1}{2} \rho^{(2)}_A \otimes \rho^{(2)}_B. \]

**Example 2.** A mixture of a maximally entangled density operator and a pure one
\[ \rho = p |\psi^+(\psi^+)^* + (1 - p) |00\rangle \langle 00|, \]
where \( |\psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \).

In this case the CP map is represented by two operators \( M_{\psi^+} \) and \( M_{00} \):
\[ M_{\psi^+} = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M_{00} = \sqrt{2(1 - p)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Operators \( N_\mu \), their linear combinations, are given by
\[ N_\mu = \begin{pmatrix} \sqrt{2(1 - p)} V_{\mu 2} & \sqrt{p} V_{\mu 1} \\ V_{\mu 1} & 0 \end{pmatrix}. \]

Again, \( N_\mu \) will be of unit rank when the determinant of its matrix vanishes:
\[ \det N_\mu = -p V_{\mu 1}^2 = 0. \]

The coefficients \( V_{\mu 1} \) cannot be equal to null simultaneously due to eq. 19, so the only possibility for \( \rho \) to be separable is \( p = 0 \). When \( p \neq 0 \), \( \rho \) is entangled.

As Examples 1 and 2 show, when the number of terms in the ensemble decomposition of a given density operator is small, the application of Lemma 1 is quite easy. The analysis of the next example will take much more effort.

**Example 3.** A qubit-qubit isotropic state \( \rho^{iso} \).

Ensemble decomposition of \( \rho^{iso} \), a \( 2 \otimes 2 \) density operator, is given by eq. 29 with \( d = 2 \) and \( 1/4 \leq F \leq 1 \). It consists of 4 terms determined by the states \( |\Phi_{0}^+\rangle, |\Phi_{1}^+\rangle, |\Psi_{01}^+\rangle, |\Psi_{01}^-\rangle \). Let \( M_1, M_2, M_3, M_4 \) denote the respective operators connected to these states by eq. 6. Their matrix representations are:
\[ M_1 = \sqrt{F} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \sqrt{\frac{1-F}{3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ M_3 = \sqrt{\frac{1-F}{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M_4 = \sqrt{\frac{1-F}{3}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
Operators \( \{ M_i \} \) give the representation of a CP map corresponding to \( \rho^{iso} \). Their transformations, \( \{ N_\mu \} \), are given by

\[
N_\mu = \begin{pmatrix}
\sqrt{F} V_{\mu 1} + \sqrt{\frac{1-F}{3}} V_{\mu 2} & \sqrt{\frac{1-F}{3}} (V_{\mu 3} - V_{\mu 4}) \\
\frac{1-F}{3} (V_{\mu 3} + V_{\mu 4}) & \sqrt{F} V_{\mu 1} - \sqrt{\frac{1-F}{3}} V_{\mu 2}
\end{pmatrix}.
\]

(49)

The rank of \( N_\mu \) is 1 when the determinant vanishes:

\[
F V_{\mu 1}^2 = 1 - \frac{F}{3} (V_{\mu 2}^2 + V_{\mu 3}^2 - V_{\mu 4}^2).
\]

(50)

Taking absolute value of both parts of this equation gives inequality

\[
F |V_{\mu 1}|^2 \leq \frac{1-F}{3} (|V_{\mu 2}|^2 + |V_{\mu 3}|^2 + |V_{\mu 4}|^2),
\]

(51)

which, after summing by \( \mu \) and using eq. 19, takes form:

\[
F \leq \frac{1-F}{3} 3 = 1 - F.
\]

(52)

Consequently, when \( F > 1/2 \), a two-qubit isotropic state is entangled.

For a complete analysis, it is necessary to show that if \( 1/4 \leq F \leq 1/2 \), then \( \rho^{iso} \) is separable. Finding a matrix satisfying eq. 50 along with the orthonormal condition for columns is not an easy task if one tries to approach the problem by straightforward solving a system of polynomial equations. We will use a method for construction of unitary matrices described in [22]:

**Statement.** Let \( A_1, A_2, \ldots, A_n \) be \( m \times m \) unitary matrices and let

\[(a_{ij})_{i,j=1}^n, \text{ be a unitary matrix.}\]

Then the following matrix is a \( nm \times nm \) unitary matrix:

\[
B = \begin{bmatrix}
a_{11} A_1 & a_{12} A_2 & \cdots & a_{1n} A_n \\
a_{21} A_1 & a_{22} A_2 & \cdots & a_{2n} A_n \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} A_1 & a_{2n} A_2 & \cdots & a_{nn} A_n
\end{bmatrix}
\]

With the use of this statement we will show that for any \( F: 1/4 \leq F \leq 1/2 \), a unitary matrix with entries \( V_{\mu\alpha} \) satisfying eq. 50 can be constructed.

At first, we consider some particular case of eq. 50 for example, it could be

\[
2 V_{\mu 1}^2 = V_{\mu 2}^2 + V_{\mu 3}^2 - V_{\mu 4}^2.
\]

(53)

This choice corresponds to \( F = 2/5 \). Let \( U, W \) and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) — \( 2 \times 2 \) unitary matrices. We construct our solution, the \( 4 \times 4 \) unitary matrix \( V \), in accordance with the above statement:

\[
V = \begin{pmatrix}
a U & b W \\
c U & d W
\end{pmatrix}.
\]

(54)
The key is to keep all the coefficients as simple as possible. We can choose \(a, b, c, d\) to have the same absolute values: \((\frac{a}{b} c d) = \frac{1}{\sqrt{2}}(\frac{1}{1} - i)\). Substituting the ansatz for \(V\) into eq. 53, we obtain the following equations for the entries of \(U\) and \(W\):

\[
(2U_{11}^2 - U_{12}^2) - (W_{12}^2 - W_{11}^2) = 0,
\]
\[
(2U_{21}^2 - U_{22}^2) - (W_{22}^2 - W_{21}^2) = 0.
\] (55)

If we choose \(U\) in the simplest form: \(U = \frac{1}{\sqrt{2}}(\frac{1}{1} - i)\), then the matrix \(W\) satisfying eq. 55 can also be easily guessed: \(W = \frac{1}{2}(i\sqrt{3} - i)\). The solution for \(F = 2/5\) then will be:

\[
V = \frac{1}{2} \begin{pmatrix}
1 & -1 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{2}} \\
1 & 1 & \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}i} \\
1 & -1 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{2}} \\
1 & 1 & -i\sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}i}
\end{pmatrix}.
\] (56)

We can see a specific pattern here in \(V\):

\[
V = \frac{1}{2} \begin{pmatrix}
1 & -1 & -\alpha & \beta \\
1 & 1 & i\beta & i\alpha \\
1 & -1 & \alpha & -\beta \\
1 & 1 & -i\beta & -i\alpha
\end{pmatrix},
\] (57)

where \(\alpha, \beta\) - some positive numbers satisfying the orthonormal condition for columns (and rows) of \(V\): \(\alpha^2 + \beta^2 = 2\). This pattern works for the general solution, for any \(F, 1/4 \leq F \leq 1/2\): substituting \(V\) from eq. 57 into eq. 50 we can easily solve the resulting equations in variables \(\alpha\) and \(\beta\). The solution is as follows:

\[
V = \frac{1}{2} \begin{pmatrix}
1 & -1 & -\sqrt{\frac{2F+1}{2-2F}} & \frac{3-6F}{2-2F} \\
1 & 1 & \sqrt{\frac{3-6F}{2-2F}} & \frac{2F+1}{2-2F} \\
1 & -1 & \sqrt{\frac{3-6F}{2-2F}} & -\sqrt{\frac{2F+1}{2-2F}} \\
1 & 1 & -i\sqrt{\frac{3-6F}{2-2F}} & -i\sqrt{\frac{2F+1}{2-2F}}
\end{pmatrix}.
\] (58)

A unitary matrix satisfying eq. 50 is found, and we obtain that if \(1/4 \leq F \leq 1/2\), then \(\rho^{iso}\) is separable.

With the help of eq. 58 a separable decomposition of \(\rho^{iso}\) can be obtained. First, substituting \(V_{iso}\) from eq. 58 into eq. 49 we obtain the expressions for 4 unit rank operators \(\{N_\mu\}\). Next, direct calculations with the use of eqs. 10 11 and the expression for the ensemble probabilities \(q_\mu = \frac{1}{m}\text{Tr}(N_\mu N_\mu)\) show that when \(1/4 \leq F \leq 1/2\), \(\rho^{iso}\) can be decomposed as:

\[
\rho^{iso} = \frac{1}{4}(\rho^{(1)}_{A} \otimes \rho^{(1)}_{B} + \rho^{(2)}_{A} \otimes \rho^{(2)}_{B} + \rho^{(3)}_{A} \otimes \rho^{(3)}_{B} + \rho^{(4)}_{A} \otimes \rho^{(4)}_{B}),
\] (59)
The density operator $\rho$ has a positive partial transpose, but in a 3 state having a positive partial transpose.

**Example 4. Detecting entanglement of a 3 ⊗ 3 PPT state.**

Let $\rho$ be a 3 ⊗ 3 density operator constructed with the use of the unextendible product bases (UPB) method [23]:

$$\rho = \frac{1}{4} \left( I_3 \otimes I_3 - \sum_{i=1}^{4} |\psi_i\rangle \langle \psi_i| - |S\rangle \langle S| \right),$$

where the states $|\psi_i\rangle$ and $|S\rangle$ are given by

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} (|2\rangle - |1\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle |2\rangle), \quad |\psi_4\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle |0\rangle),$$

$$|S\rangle = \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle).$$

The density operator $\rho$ has a positive partial transpose, but in a 3 ⊗ 3 case the Peres-Horodecki criterion is not sufficient for the separability of the state.

One possible ensemble decomposition of $\rho$ can be given as $\rho = 1/4 \sum_{i=1}^{4} |\phi_i\rangle \langle \phi_i|$, where

$$|\phi_1\rangle = \frac{1}{2} \left( |\psi_5\rangle + |\psi_6\rangle - |\psi_7\rangle - |\psi_8\rangle \right),$$

$$|\phi_2\rangle = \frac{1}{2} \left( |\psi_5\rangle - |\psi_6\rangle + |\psi_7\rangle - |\psi_8\rangle \right),$$

$$|\phi_3\rangle = \frac{1}{2} \left( |\psi_5\rangle - |\psi_6\rangle - |\psi_7\rangle + |\psi_8\rangle \right),$$

$$|\phi_4\rangle = \frac{1}{2} \left( |\psi_5\rangle + |\psi_6\rangle + |\psi_7\rangle + |\psi_8\rangle \right) - \frac{2\sqrt{2}}{3} |\psi_9\rangle.$$
and five product states

\[
|\psi_5\rangle = \frac{1}{\sqrt{2}} |0\rangle |0 + 1\rangle, \quad |\psi_6\rangle = \frac{1}{\sqrt{2}} |2\rangle |1 + 2\rangle,
\]

\[
|\psi_7\rangle = \frac{1}{\sqrt{2}} |0 + 1\rangle |2\rangle, \quad |\psi_8\rangle = \frac{1}{\sqrt{2}} |1 + 2\rangle |0\rangle,
\]

\[
|\psi_9\rangle = |1\rangle |1\rangle
\]

(65)

together with |\psi_1\rangle , |\psi_2\rangle , |\psi_3\rangle , |\psi_4\rangle from eq. 63 form a complete orthogonal product basis. Given the ensemble decomposition, we obtain the operator representation of a CP map corresponding to the density operator \( \rho \):

\[
M_{\phi_1} = \frac{\sqrt{3}}{4\sqrt{2}} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad M_{\phi_2} = \frac{\sqrt{3}}{4\sqrt{2}} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix},
\]

\[
M_{\phi_3} = \frac{\sqrt{3}}{4\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \quad M_{\phi_4} = \frac{1}{4\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

(66)

By eq. 65 operators \( N_\mu \) are expressed as follows:

\[
N_\mu = \frac{1}{4\sqrt{6}} \begin{pmatrix} 3(V_{\mu_1} + V_{\mu_2} + V_{\mu_3}) + V_{\mu_4} & V_{\mu_4} + 3(V_{\mu_3} - V_{\mu_2} - V_{\mu_1}) & V_{\mu_4} + 3(V_{\mu_3} - V_{\mu_1} - V_{\mu_2}) \\ V_{\mu_4} + 3(V_{\mu_1} + V_{\mu_2} + V_{\mu_3}) & -8V_{\mu_4} & V_{\mu_4} + 3(V_{\mu_1} - V_{\mu_2} - V_{\mu_3}) \\ V_{\mu_4} + 3(V_{\mu_2} - V_{\mu_1} - V_{\mu_3}) & V_{\mu_4} + 3(V_{\mu_1} - V_{\mu_2} - V_{\mu_3}) \end{pmatrix}.
\]

(67)

The matrix in eq. 67 is of rank 1 when the determinants of all its second order minors vanish. We obtain a system of 9 equations:

- \( 3V_{\mu_4}^2 - 10V_{\mu_3}V_{\mu_4} - 8V_{\mu_2}V_{\mu_4} - 8V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 + 3V_{\mu_2}^2 + 6V_{\mu_1}V_{\mu_2} + 3V_{\mu_1}^2 = 0 \),

\( \text{(68a)} \)

- \( 3V_{\mu_4}^2 + 8V_{\mu_3}V_{\mu_4} - 10V_{\mu_2}V_{\mu_4} - 8V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 + 6V_{\mu_1}V_{\mu_3} + 3V_{\mu_2}^2 - 3V_{\mu_1}^2 = 0 \),

\( \text{(68b)} \)

- \( -V_{\mu_3}V_{\mu_4} + V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 - 3V_{\mu_2}V_{\mu_3} + 3V_{\mu_1}V_{\mu_2} + 3V_{\mu_1}^2 = 0 \),

\( \text{(68c)} \)

- \( 3V_{\mu_4}^2 - 8V_{\mu_3}V_{\mu_4} + 10V_{\mu_2}V_{\mu_4} - 8V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 - 6V_{\mu_1}V_{\mu_3} + 3V_{\mu_2}^2 - 3V_{\mu_1}^2 = 0 \),

\( \text{(68d)} \)

- \( -3V_{\mu_4}^2 + 10V_{\mu_3}V_{\mu_4} + 8V_{\mu_2}V_{\mu_4} - 8V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 + 3V_{\mu_2}^2 - 6V_{\mu_1}V_{\mu_2} + 3V_{\mu_1}^2 = 0 \),

\( \text{(68e)} \)

- \( V_{\mu_3}V_{\mu_4} + V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 - 3V_{\mu_2}V_{\mu_3} - 3V_{\mu_1}V_{\mu_2} + 3V_{\mu_1}^2 = 0 \),

\( \text{(68f)} \)

- \( V_{\mu_2}V_{\mu_4} + V_{\mu_1}V_{\mu_4} - 3V_{\mu_2}V_{\mu_3} - 3V_{\mu_1}V_{\mu_3} + 3V_{\mu_2}^2 - 3V_{\mu_1}^2 = 0 \),

\( \text{(68g)} \)

- \( -V_{\mu_2}V_{\mu_4} + V_{\mu_1}V_{\mu_4} - 3V_{\mu_2}V_{\mu_3} + 3V_{\mu_1}V_{\mu_3} + 3V_{\mu_2}^2 - 3V_{\mu_1}^2 = 0 \),

\( \text{(68h)} \)

- \( V_{\mu_1}V_{\mu_4} - 3V_{\mu_2}V_{\mu_3} = 0 \).

\( \text{(68i)} \)

As it turns out, the system can be easily analyzed. Adding eqs. 68a, 68h and also eqs. 68c, 68i with the use of eq. 68j we have:

\[
V_{\mu_2}^2 = V_{\mu_1}^2, \quad V_{\mu_3}^2 = V_{\mu_1}^2.
\]

(69)

Addition of eqs. 68b, 68d yields:

\[
3V_{\mu_4}^2 - 8V_{\mu_1}V_{\mu_4} - 3V_{\mu_3}^2 = 0,
\]

(70)

whereas addition of eqs. 68a, 68e gives:

\[
-3V_{\mu_4}^2 - 8V_{\mu_1}V_{\mu_4} + 3V_{\mu_4}^2 = 0.
\]

(71)
From the last three equations it follows that $V_{\mu 1}V_{\mu 4} = 0$, which in combination with eq. 68i and eq. 69 gives that $V_{\mu 1}^2 = 0$, i.e., the system has only trivial solution. This contradicts with the normalization condition, eq. 19. Consequently, by Lemma 1, $\rho$ is entangled.

IV. Conclusions

As it was shown in section III, the method developed in the present paper can be applied to a great variety of states. It seems to be quite operational when the number of terms in the ensemble decomposition of a density operator is small, but still in this case it can’t be better than the Peres-Horodecki criterion which is known to be necessary and sufficient for low rank density operators [17]. In the low rank case it can serve as a complementary method which allows to obtain separable decompositions of density matrices. In the high rank case, as examples with isotropic and PPT states showed, our approach is by no means operational, and it demands a detailed analysis of systems of polynomial equations. In many cases these polynomial systems can be reduced to simple ones; besides, numerical methods can be applied. The presented approach is also interesting from the theoretical point of view and can be used in derivation of separability criteria.

The open question is the connection of the presented approach with the Peres-Horodecki criterion in the qubit-qubit case. If each term in the ensemble decomposition of a given density operator is defined, as in eq. 8, by $c_{ij}^{(a)}$, a $2 \times 2$ matrix in this case, then the transformed operators $\{N_{\mu}\}$ have the following matrix representation:

$$ (N_{\mu})_{ij} = \sqrt{m} \sum_a V_{\mu a} \sqrt{p_a} c_{ji}^{(a)}, $$

(72)

and the unit rank condition yields only one equation for each $\mu$ in the qubit-qubit case:

$$ \sum_{a,b} \sqrt{p_a p_b} V_{\mu a} V_{\mu b} (c_{00}^{(a)} c_{11}^{(b)} - c_{01}^{(a)} c_{10}^{(b)}) = 0. $$

(73)

For a $2 \otimes 2$ state positivity of a partial transpose is sufficient for separability, but it is unclear how this fact implies existence of the solutions $V_{\mu a}$ of the above equation, such that $V_{\mu a}$ form a matrix with orthonormal columns.

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