A SUSY SU(5) Grand Unified Model of Tri-Bimaximal Mixing from $A_4$

Guido Altarelli

Dipartimento di Fisica ‘E. Amaldi’, Università di Roma Tre
INFN, Sezione di Roma Tre, I-00146 Rome, Italy

and

CERN, Department of Physics, Theory Division
CH-1211 Geneva 23, Switzerland

Ferruccio Feruglio

Dipartimento di Fisica ‘G. Galilei’, Università di Padova
INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padua, Italy

Claudia Hagedorn

Max-Planck-Institut für Kernphysik
Postfach 10 39 80, 69029 Heidelberg, Germany

Abstract

We discuss a grand unified model based on SUSY SU(5) in extra dimensions and on the flavour group $A_4 \times U(1)$ which, besides reproducing tri-bimaximal mixing for neutrinos with the accuracy required by the data, also leads to a natural description of the observed pattern of quark masses and mixings.
1 Introduction

It is an experimental fact \cite{1} that within measurement errors the observed neutrino mixing matrix is compatible with the so called tri-bimaximal (TB) form, introduced by Harrison, Perkins and Scott (HPS) \cite{2}. The best measured neutrino mixing angle $\theta_{12}$ is just about $1\sigma$ below the HPS value $\tan^2 \theta_{12} = 1/2$, while the other two angles are well inside the $1\sigma$ interval \cite{1}. In a series of papers \cite{3,4,5,6,7} it has been pointed out that a broken flavour symmetry based on the discrete group $A_4$ appears to be particularly suitable to reproduce this specific mixing pattern as a first approximation. Other solutions based on alternative discrete or continuous flavour groups have also been considered \cite{8,9}, but the $A_4$ models have a very economical and attractive structure, e.g. in terms of group representations and of field content. In most of the models $A_4$ is accompanied by additional symmetries, either continuous like U(1) or discrete like $Z_N$, which are necessary to eliminate unwanted couplings, to ensure the needed vacuum alignment and to reproduce the observed mass hierarchies. In this way one can construct natural models where the corrections to TB mixing can be evaluated in a well defined expansion.

Recently much attention has been devoted to the question whether a model for HPS mixing in the neutrino sector can be suitably extended to also successfully describe the observed pattern of quark mixings and masses and whether this more complete framework can be made compatible with (supersymmetric (SUSY)) SU(5) or SO(10) grand unification. Early attempts of extending models based on $A_4$ to quarks \cite{10,6} and to construct grand unified versions \cite{11} so far have not been completely satisfactory, e.g. do not offer natural mechanisms for mass hierarchies and for the vacuum alignment. A direct extension of the $A_4$ model to quarks leads to the identity matrix for $V_{CKM}$ in the lowest approximation, which at first looks promising. But the corrections to it turn out to be strongly constrained by the leptonic sector, because lepton mixings are nearly TB, and are proven to be too small to accommodate the observed quark mixing angles \cite{6}. Also, the quark classification adopted in these models is not compatible with $A_4$ commuting with SU(5) \cite{3}. Due to this, larger discrete groups are considered for the description of quarks and for grand unified versions with approximate TB mixing in the lepton sector. A particularly appealing set of models is based on the discrete group $T'$, the double covering group of $A_4$ \cite{13}. In ref. \cite{14} a viable description was obtained, i.e. in the leptonic sector the predictions of the $A_4$ model are reproduced, while the $T'$ symmetry plays an essential role for reproducing the pattern of quark mixing. But, again, the classification adopted in this model is not compatible with grand unification. Unified models based on the discrete groups $T'$ \cite{15}, $S_4$ \cite{16} and $\Delta(27)$ \cite{17} have been discussed. Several models using the smallest non-abelian symmetry $S_3$ (which is isomorphic to $D_3$) can also be found in the recent literature \cite{18}.

In conclusion, the group $A_4$ is considered by most authors to be too limited to also describe quarks and to lead to a grand unified description. In the present work we show that this negative attitude is not justified and that it is actually possible to construct a viable model based on $A_4$ which leads to a grand unified theory (GUT) of quarks and

\footnote{In ref. \cite{12} an $A_4$ model compatible with the Pati-Salam group SU(4) $\times$ SU(2)$_L \times$ SU(2)$_R$ has been presented.}
leptons with TB mixing for leptons. At the same time our model offers an example of 
an extra dimensional GUT in which a description of all fermion masses and mixings is 
attempted. The model is natural, since most of the small parameters in the observed 
pattern of masses and mixings as well as the necessary vacuum alignment are justified by 
the symmetries of the model. For this, it is sufficient to enlarge the $A_4$ flavour symmetry 
by adding a U(1) of the Froggatt-Nielsen type and to suitably modify and extend the 
classification under the flavour group so that finally all fermions transform in an SU(5) 
compatible way. In addition, a $Z_3$ symmetry must be assigned to the fields of the model 
which is, however, flavour-independent. The formulation of SU(5) in extra dimensions 
has the usual advantages of avoiding large Higgs representations to break SU(5) and of 
solving the doublet-triplet splitting problem. A further ingredient of the model is a U(1)$_R$ 
symmetry which contains the discrete $R$-parity as a subgroup. A see-saw realization in 
terms of an $A_4$ triplet of right-handed neutrinos $N$ ensures the correct ratio of light neutrino 
masses with respect to the GUT scale. In the present model extra dimensional effects 
directly contribute to determine the flavour pattern, in that the two lightest tenplets $T_1$ 
and $T_2$ are in the bulk (with a doubling $T_i$ and $T'_i$, $i = 1, 2$ to ensure the correct zero mode 
spectrum), whereas the pentaplets $F$ and $T_3$ are on the brane. The hierarchy of quark 
and charged lepton masses and of quark mixings is determined by a combination of extra 
dimensional suppression factors for the first two generations and of the U(1) charges, while 
the neutrino mixing angles derive from $A_4$. The choice of the transformation properties of 
the two Higgses $H_5$ and $\tilde{H}_5$ is also crucial. They are chosen to transform as two different 
$A_4$ singlets 1 and 1’. As a consequence, mass terms for the Higgs colour triplets are not 
directly allowed and their masses are introduced by orbifolding, à la Kawamura [19]. 
Finally, in this model, proton decay is dominated by gauge vector boson exchange giving 
rise to dimension six operators. Given the relatively large theoretical uncertainties, the 
decay rate is within the present experimental limits.

The resulting model is shown to be directly compatible with approximate TB mixing 
for leptons as well as with a realistic pattern of fermion masses and of quark mixings in a 
SUSY SU(5) framework.

2 The Model

We consider a SUSY GUT based on SU(5) in 4+1 dimensions. Leaving aside extra dimen-
sional effects for a moment, from the four-dimensional (4D) point of view matter fields are 
chiral supermultiplets transforming as 10, 5 and 1 under SU(5). Part of the flavour sym-
metry is related to the discrete group $A_4$, whose properties are summarized, for instance, 
in section 2 of ref. [6], whose conventions are adopted here. The three 5 and the three 
singlets (corresponding to the right-handed neutrinos) are grouped into $A_4$ triplets $F$ and 
$N$, while the tenplets $T_1$, $T_2$ and $T_3$ are assigned to $1''$, $1'$ and 1 singlets of $A_4$, respectively 
(see table 1). The Higgs chiral supermultiplets that break the electroweak symmetry are 
$H_5$ and $\tilde{H}_5$, transforming as $5, 1$ and $\tilde{5}, 1'$ under SU(5) $\times A_4$. We also consider a set 
of flavon supermultiplets, all invariant under SU(5), that break the $A_4$ symmetry: two

\footnote{Even after $A_4$ breaking they are forbidden at all orders by the U(1)$_R$ symmetry.}
triplets $\varphi_T$ and $\varphi_S$ and two singlets $\xi$ and $\tilde{\xi}$. The alignment of their vacuum expectation values (VEVs) along appropriate directions in flavour space will be the source of TB lepton mixing. It is well-known that, for this to work, each triplet should mainly contribute to the mass generation of a specific sector. At the leading order and after spontaneous $A_4$ breaking, $\varphi_S$, $\xi$ and $\tilde{\xi}$ should give mass to neutrinos only, while $\varphi_T$ gives mass to charged leptons and to down quarks. This separation can be realized with the help of an additional spontaneously broken $Z_3$ symmetry under which $N$, $F$, $T_i$, $H_{5,\bar{5}}$, $\varphi_S$, $\xi$ and $\tilde{\xi}$ are multiplied by $\omega = \exp(i2\pi/3)$, while $\varphi_T$ is left invariant. The generation of the up quark masses as well as the quark mixings will be discussed below.

The breaking of the grand unified symmetry is a potential source of serious problems, like those related to the doublet-triplet splitting and to proton decay. One of the most efficient mechanisms to break SU(5) and avoid these problems is the one based on compactification of extra spatial dimensions [19]. The simplest setting is an SU(5) gauge invariant five-dimensional (5D) theory where the fifth dimension is compactified on a circle $S^1$ of radius $R$. The gauge fields, living in the whole 5D space-time, are assumed to be periodic along the extra dimension only up to a discrete parity transformation $\Omega$ such that the gauge fields of the SU(3)×SU(2)×U(1) subgroup are periodic, while those of the coset SU(5)/SU(3)×SU(2)×U(1) are antiperiodic. Only the gauge vector bosons of SU(3)×SU(2)×U(1) possess a zero mode. Those of SU(5)/SU(3)×SU(2)×U(1) form a Kaluza-Klein tower starting at the mass level $1/R$. From the viewpoint of a 4D observer, these boundary conditions effectively break SU(5) down to the Standard Model (SM) gauge group, at a GUT scale of order $1/R$. The transformation $\Omega$ is an automorphism of the SU(5) algebra, so that the whole construction can be carried out within an SU(5) invariant formalism. An important advantage of this mechanism is that it provides a simple solution to the doublet-triplet splitting problem. The parity $\Omega$ is consistently extended to the Higgs multiplets $H_5$ and $H_{\bar{5}}$, also assumed to live in the whole 5D space, in such a way that the electroweak doublets are periodic, whereas the colour triplets are antiperiodic. In this way we have zero modes only for the doublets and the lightest colour triplets get masses of order $1/R$. Notice that, if the model is supersymmetric as in the case under discussion here, we have an effective 4D $N = 2$ SUSY, induced by the original $N = 1$ SUSY in five dimensions. To reduce $N = 2$ down to $N = 1$ it is convenient to compactify the fifth dimension on the orbifold $S^1/Z_2$ rather than on the circle $S^1$. The orbifold projection eliminates all the zero modes of the extra states belonging to $N = 2$ SUSY and also those of the fifth component of the gauge vector bosons. The zero modes we are left with are the 4D gauge bosons of the SM, two electroweak doublets and their $N = 1$ SUSY partners. To complete the solution of the doublet-triplet splitting problem, we should also forbid a large mass term $H_5H_{\bar{5}}$, which would otherwise lift the doublet masses. As will be explained below, this is automatically guaranteed by the U(1)$_R$ symmetry that we specify in table 1.

For the gauge vector bosons and the Higgses $H_5$ and $H_{\bar{5}}$ we will adopt this setup, which is described in detail in refs. [20]. For the remaining fields we have much more freedom [20, 21]. Indeed the orbifold $S^1/Z_2$ corresponds to a segment where the fifth coordinate $y$ runs from 0 to $\pi R$. The boundaries of the segment determine two 4D slices...
of the original 5D space-time. When boundary conditions are consistently defined for the local parameters of SU(5) gauge transformations, we find that such transformations are generically non-vanishing only in the bulk and at $y = 0$. At the opposite endpoint of the segment, $y = \pi R$, the only gauge transformations that are different from zero are those of the SM. Therefore we have three qualitatively different possible locations for the remaining fields: in the bulk, at the SU(5) preserving brane $y = 0$, or at the SU(5) breaking brane $y = \pi R$. We choose to put the two tenplets $T_1$ and $T_2$ of the first and second family in the bulk. As explained in ref. [20, 21] to obtain the correct zero mode spectrum with intrinsic parities compatible with symmetry and orbifolding, one must introduce two copies of each multiplet with opposite parity $\Omega$ in the bulk. Therefore $T_1$ and $T_2$ come in two replicas with the same quantum numbers, except for the intrinsic parity $\Omega$. For simplicity, we only show one of them in the table.

An interesting feature of the 5D setup is the automatic suppression of the Yukawa couplings for the fields living in the bulk. Indeed, a bulk field $B$ and its zero mode $B^0$ are related by:

$$B = \frac{1}{\sqrt{\pi R}} B^0 + ...$$  \hspace{1cm} (1)

where dots stand for the higher modes. This expansion produces a suppression factor

$$s \equiv \frac{1}{\sqrt{\pi R \Lambda}} < 1$$  \hspace{1cm} (2)

Thereby, $\Lambda$ denotes the ultraviolet cut-off. Such a suppression factor enters the Yukawa couplings depending on the field $B^0$. As a result, the hierarchies among the charged fermion masses are partly due to the geometrical dilution of the Yukawa couplings involving $T_1$ and $T_2$. However this dilution cannot account for all the observed hierarchies and, to achieve a realistic mass spectrum, we also exploit the Froggatt-Nielsen mechanism. The tenplets $T_1$ and $T_2$ are charged under a U(1) flavour group, spontaneously broken by the VEVs of two fields $\theta$ and $\theta''$ both carrying U(1) charges $-1$. The elements of the charged fermion mass matrices are provided by higher-dimensional operators with powers of $\theta$ and $\theta''$ balancing

| Field | $N$ | $F$ | $T_1$ | $T_2$ | $T_3$ | $H_5$ | $H_\bar{5}$ | $\varphi_T$ | $\varphi_S$ | $\xi$, $\xi$ | $\theta$, $\theta$ | $\theta''$, $\theta''$ | $\varphi_0^T$, $\varphi_0^S$ | $\xi_0$ |
|-------|----|-----|-------|-------|-------|-------|--------|-----------|-----------|-------------|-----------------|-----------------|---------------|-----|
| SU(5) | 1  | 5   | 10    | 10    | 10    | 5     | 5      | 1         | 1         | 1           | 1               | 1               | 1              | 1  |
| $A_4$ | 3  | 3   | 1''   | 1'    | 1'    | 1     | 1      | 3         | 3         | 1           | 1               | 1               | 1              | 1  |
| U(1)  | 0  | 0   | 3     | 1     | 0     | 0     | 0      | 0         | 0         | 0           | 0               | 0               | 0              | 0  |
| $Z_3$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $\omega$ | $1$ | $\omega$ | $1$ | $1$ | $1$ | $1$ | $\omega$ | $\omega$ |
| U(1)$_R$ | 1 | 1   | 1     | 1     | 1     | 0     | 0      | 0         | 0         | 0           | 0               | 0               | 2              | 2  |

Table 1: Fields and their transformation properties under SU(5), $A_4$, U(1), $Z_3$ and U(1)$_R$. $T_1$ and $T_2$ come in two replicas with the same quantum numbers, except for the intrinsic parity $\Omega$. For simplicity, we only show one of them in the table.
the U(1) charge of the relevant combination of matter fields. Indeed, we need two fields, $\theta$ and $\theta''$, in order to reproduce a realistic pattern of quark masses and mixing angles. Under $A_4$, $\theta$ is invariant, while $\theta''$ transforms as $1''$. All this is summarized in table 1.

Notice that, once we have introduced all the fields with the quantum numbers displayed in table 1, there will be no contribution coming from colour triplet exchange to the dangerous dimension five operator that induces proton decay in SUSY theories. Actually that operator is strictly forbidden as long as the U(1)$_R$ symmetry remains unbroken. Indeed, the superpotential of the effective $\mathcal{N}=1$ SUSY should have U(1)$_R$ charge +2, to compensate the $R$-charge $-2$ coming from the Grassmann integration measure $d^2\theta$.

With the $R$ assignment in table 1, all superpotential couplings bilinear in the matter fields $N$, $F$ and $T$ have $R$-charge +2 and are allowed. At the same time dangerous operators are forbidden. First of all these include the mass term $H_5H_\overline{5}$, that would spoil the solution to the doublet-triplet splitting problem. Moreover, since U(1)$_R$ contains the discrete $R$-parity, also all renormalizable baryon and lepton number violating operators, such as $FH_5$ and $FFT$, are not allowed. Finally, the dimension five operator $FTTT$, leading to proton decay, has $R$-charge +4 and therefore is absent. As discussed in detail in ref. [6] and briefly recalled in section 4, the U(1)$_R$ symmetry plays also an important role in the dynamics that selects the correct vacuum of the theory, which is a crucial feature to reproduce nearly TB mixing in the lepton sector. The U(1)$_R$ symmetry is a remnant of the SU(2)$_R$ symmetry of the $\mathcal{N}=2$ SUSY bulk action, before compactification. By reducing $\mathcal{N}=2$ down to $\mathcal{N}=1$ the orbifold projection breaks SU(2)$_R$ down to U(1)$_R$. Eventually, after the inclusion of $\mathcal{N}=1$ SUSY breaking effects, the U(1)$_R$ symmetry will be broken down to the discrete $R$-parity, at the low energy scale $m_{\text{SUSY}}$. The operator $FTTT$ might be generated, but with a highly suppressed coupling of the kind $(m_{\text{SUSY}}/\Lambda)^n/\Lambda$, $n > 0$.

Therefore, the leading contribution to proton decay comes from gauge vector boson exchange and the corresponding proton decay rate is typically small enough, though suffering from considerable uncertainties [22].

### 3 Fermion Masses

The $\mathcal{N}=2$ SUSY invariance is broken down to $\mathcal{N}=1$ by the orbifold projection, but it still forbids 5D superpotential couplings. These couplings should be strictly localized at one of the two branes. By choosing the brane at $y = 0$, the brane action reads:

$$\int d^4 x \int_0^{\pi R} dy \int d^2 \theta \ w(x) \delta(y) + h.c. = \int d^4 x \int d^2 \theta \ w(x) + h.c. \ .$$

The superpotential $w$, which can be expressed in terms of $\mathcal{N}=1$ superfields, can be decomposed into several parts:

$$w = w_{up} + w_{down} + w_{\nu} + w_d + \ldots \ .$$

The first three contributions in eq. (4) give rise to fermion masses after $A_4$, U(1) and electroweak symmetry breaking. They are of the form:

$$w_{up} = \frac{1}{\Lambda^{1/2}} H_5 T_3 T_3 + \frac{\theta''}{\Lambda^2} H_5 T_2 T_3 + \frac{\theta'^2}{\Lambda^{7/2}} H_5 T_2 T_2 + \frac{\theta^2}{\Lambda^4} H_5 T_1 T_3$$
vanishing VEV. Therefore, it does not contribute to the neutron masses.

Following scheme:

We will discuss these issues in the next section. For the time being we assume that the scalar components of the supermultiplets acquire VEVs according to the scalar potential. Where \( \langle \phi \rangle \) is defined as the combination of the two independent \( \xi \)-type fields which has a vanishing VEV. Therefore, it does not contribute to the neutrino masses.

The last term in eq. (6), \( w_{\text{down}} \), is responsible for the alignment of the flavon fields \( \varphi_T \), \( \varphi_S \), \( \xi \) and \( \tilde{\xi} \). The fields \( \theta \) and \( \theta'' \) get VEVs from the minimisation of the D-term of the scalar potential. We will discuss these issues in the next section. For the time being we assume that the scalar components of the supermultiplets acquire VEVs according to the following scheme:

\[
\frac{\langle \varphi_T \rangle}{\Lambda} = (v_T, 0, 0) , \quad \frac{\langle \varphi_S \rangle}{\Lambda} = (v_S, v_S, v_S) , \quad \frac{\langle \xi \rangle}{\Lambda} = u , \quad \frac{\langle \theta \rangle}{\Lambda} = t , \quad \frac{\langle \theta'' \rangle}{\Lambda} = t'' .
\]

where \( \xi \) and \( \tilde{\xi} \) are the combinations of matter fields, as for instance \( T_1 T_2 \) to \( u,d \) and \( T_1 T_2 '' \) which are suppressed by the same power of \( \Lambda \), but can differ by order-one relative weights. It is important to keep this point in mind, since it allows to escape the too rigid mass relations between the first two generations of charged leptons and down quarks predicted by the minimal SU(5) GUT.

Neutrinos have both Dirac and Majorana mass terms, induced by:

\[
w_\nu = \frac{y^D V}{\Lambda} H_5(N F) + (x_\alpha \xi + \hat{x}_\alpha \tilde{\xi})(N N) + x_b(\varphi_S N N) ,
\]

The Higgs multiplets live in the bulk and what matters for the Yukawa couplings are the values of the VEVs at \( y = 0 \):

\[
\langle H_5(0) \rangle = \frac{v_u}{\sqrt{\pi R}} , \quad \langle H_5(0) \rangle = \frac{v_d}{\sqrt{\pi R}} ,
\]

where \( v_{u,d} \) have mass dimension 1. The electroweak scale is determined by the relation:

\[
v_u^2 + v_d^2 \approx (174 \text{ GeV})^2 , \quad v_u^2 \equiv \int_0^\pi R dy |\langle H_5(y) \rangle|^2 , \quad v_d^2 \equiv \int_0^\pi R dy |\langle H_5(y) \rangle|^2 .
\]
Notice that the electroweak gauge boson masses depend on the 5D averages of $|\langle H_{5,5}(y)\rangle|^2$, rather than on the values at $y = 0$. If the VEVs of $H_{5,5}$ are constant along the fifth dimension, then $v^0_u = v_u$ and $v^0_d = v_d$. However, if the profile of $\langle H_{5,5}(y)\rangle$ is not flat in $y$, the parameters $v_{u,d}^0$ are less constrained. In order to obtain $v_{u,d}^0 \neq v_{u,d}$, we need some special dynamics on the $y = 0$ and $y = \pi R$ branes, that we cannot control without detailing additional features of the model, such as the breaking of the residual $N = 1$ SUSY and the generation of a non-trivial potential for the electroweak doublets. In this section we consider $v_{u,d}^0 \neq v_{u,d}$ as an open possibility and we will discuss a possible application of it. All the other fields have vanishing VEVs.

From these VEVs, the superpotential terms in eqs. (5,6,7) and the volume suppression $s$ of eq. (2), it is immediate to derive the fermion mass matrices. In the up and down quark sector we get, up to unknown coefficients of order one for each matrix element and by adopting the convention $v_{u,d}^0 \approx u,d_0$, in this way, the Yukawa coupling of the top quark is of order one and, by

$$m_u = \begin{pmatrix}
    s^2 t^5 v'' + s^2 t^2 t''^4 & s^2 t^4 + s^2 t t''^3 & st t''^2 & st v_d^0 & sv_u^0 \\
    s^2 t^4 + s^2 t t''^3 & s^2 t^2 t''^2 & st'' & 1 & 1 \\
    st t''^2 & st'' & 1 & 1 & 1
\end{pmatrix}, \quad (11)$$

$$m_d = \begin{pmatrix}
    st^3 + st''^3 & \cdots & \cdots & \cdots & st v_d^0 \\
    st^2 t'' & st & \cdots & \cdots & 1 \\
    st t''^2 & st'' & 1 & 1 & 1
\end{pmatrix}, \quad (12)$$

where the dots stand for subleading contributions, that will be fully discussed in section 5. Here we explicitly see the interplay between the volume dilution and the Froggatt-Nielsen mechanism, to achieve the hierarchical pattern of the quark mass matrices. Realistic values of quark mass ratios and mixing angles are obtained by assuming

$$t \approx t'' \approx s \approx O(\lambda) \quad \text{with} \quad \lambda \equiv 0.22 \quad . \quad (13)$$

Indeed, with this choice we obtain

$$m_u = \begin{pmatrix}
    \lambda^8 & \lambda^6 & \lambda^4 & \lambda^2 & 1 \\
    \lambda^6 & \lambda^4 & \lambda^2 & 1 & 1 \\
    \lambda^4 & \lambda^2 & 1 & 1 & 1
\end{pmatrix} \lambda v_u^0, \quad (14)$$

$$m_d = \begin{pmatrix}
    \lambda^4 & \cdots & \cdots & \cdots & \lambda v_d^0 \\
    \lambda^4 & \lambda^2 & \cdots & \cdots & 1 \\
    \lambda^4 & \lambda^2 & 1 & 1 & 1
\end{pmatrix} v_T \lambda v_d^0 \quad . \quad (15)$$

We anticipate that, in the absence of corrections to the vacuum alignment given in eq. (8), the dots receive contributions from highly suppressed operators. In this case the entries 12, 13 and 23 of $m_d/(v_T v_d^0)$ would be of order $\lambda^7$, $\lambda^5$ and $\lambda^3$, respectively. Since $v_T \approx O(\lambda^2)$ (see below), $m_b/m_t \approx v_T v_d^0/v_u^0 \approx \lambda^2$ is easily reproduced by taking $v_u^0 \approx v_d^0$. Notice that there is an overall factor $s \approx O(\lambda)$, coming from the normalization of the Higgs VEVs, eq. (9), suppressing both $m_u$ and $m_d$. In order to avoid large dimensionless coefficients, we make use of the freedom related to the boundary values $v_{u,d}^0$ and we will assume that $v_{u,d} \approx \lambda v_{u,d}^0$. In this way, the Yukawa coupling of the top quark is of order one and, by
the patterns given in eqs. (14,15), also all the other couplings are of the same order. Alternatively, if the Higgs VEVs are flat along the fifth dimension and $v_{u,d}^0 = v_{u,d}$, we must assume that all Yukawa operators in $w$ have similar couplings of order $1/\lambda$. To correctly reproduce the quark mixing angle between the first and the second generation, a moderate tuning is needed in order to enhance the individual contributions from the up and down sectors, which are both of order $\lambda^2$.

The mass matrix for the charged lepton sector is of the type:

$$m_e = \begin{pmatrix} st^3 + st''^3 & st^2t'' & stt'' \cr ... & st & st'' \cr ... & ... & 1 \end{pmatrix} v_T s^0_d = \begin{pmatrix} \lambda^4 & \lambda^4 & \lambda^4 \cr ... & \lambda^2 & \lambda^2 \cr ... & ... & 1 \end{pmatrix} v_T \lambda v^0_d . \quad (16)$$

We observe that the minimal SU(5) relation $m_e = m_T^d$ is relaxed. Indeed, while the third column of $m_d$ exactly coincides with the third row of $m_e$, thus implying $m_b \approx m_\tau$ at the GUT scale, the remaining entries are only equal (up to a transposition) at the level of the orders of magnitude, since $T_1, T_2$ are doubled. This allows to evade the too rigid relations $m_\mu = m_s$ and $m_e = m_d$ of minimal SU(5). In our 5D setup these relations hold only up to order one coefficients and acceptable values of the masses for $e, \mu, d$ and $s$ can be accommodated.

In the neutrino sector, after the fields $\phi_S$ and $\xi$ develop their VEVs, the gauge singlets $N$ become heavy and the see-saw mechanism takes place. The mass matrix for light neutrinos is given by:

$$m_\nu = \frac{1}{3a(a+b)} \begin{pmatrix} 3a + b & 2ab + b^2 & b \\
\ & b - a & 2b - ab - 3a^2 \\
\ & b^2 - ab - 3a^2 & b - a \end{pmatrix} \frac{s^2(v_u^0)^2}{\Lambda} , \quad (17)$$

where

$$a \equiv \frac{2x_a u}{(y^D)^2} , \quad b \equiv \frac{2x_b v_S}{(y^D)^2} . \quad (18)$$

The neutrino mass matrix is diagonalized by the transformation:

$$U^T m_\nu U = \text{diag}(m_1, m_2, m_3) , \quad (19)$$

where, in units of $s^2(v_u^0)^2/\Lambda$,

$$m_1 = \frac{1}{(a + b)} , \quad m_2 = \frac{1}{a} , \quad m_3 = \frac{1}{(b - a)} \quad (20)$$

and $U$ is given by

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix} . \quad (21)$$

Note that, in the leading approximation, the model predicts the relation:

$$\frac{2}{m_2} = \frac{1}{m_1} - \frac{1}{m_3} . \quad (22)$$
It is expected to hold up to corrections of $O(\lambda^2)$, as will be discussed in section 5. Notice, that in our conventions $m_i$ ($i = 1, 2, 3$) are in general complex numbers, so that the previous relation cannot be used to exactly predict one physical neutrino mass in terms of the other two ones. Nevertheless, it provides a non-trivial constraint that the neutrino masses should obey.

To get the right solar mixing angle, we should impose $|m_2| > |m_1|$ and this requires $\cos \phi > -|z|/2$, where $z = b/a$ and $\phi$ is the phase difference between the complex numbers $a$ and $b$. The neutrino spectrum can have either normal or inverted mass ordering. If $\max(-1, -|z|/2) \leq \cos \phi \leq 0$ the ordering is inverted, $|m_3| \leq |m_1| < |m_2|$, while $|z|/2 \leq \cos \phi \leq 1$ gives rise to a normal ordering, $|m_1| < |m_2| \leq |m_3|$. By defining

$$r \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}, \quad \Delta m^2_{\text{sol}} \equiv |m^2_2 - |m^2_1|, \quad \Delta m^2_{\text{atm}} \equiv \left| |m^2_3 - |m^2_1| \right|,$$

we find

$$r = \frac{|1 - z|^2 |z + \bar{z} + |z|^2|}{2 |z + \bar{z}|}, \quad z \equiv \frac{b}{a}.$$

We see that a sufficiently small $r$ requires $z$ not to far from either $+1$ ($\cos \phi = 1$, normal hierarchy) or $-2$ ($\cos \phi = -1$, inverted hierarchy). If we expand $z$ around $+1$, we obtain:

$$|m_1|^2 = \frac{1}{3} \Delta m^2_{\text{atm}} r + \ldots$$
$$|m_2|^2 = \frac{4}{3} \Delta m^2_{\text{atm}} r + \ldots$$
$$|m_3|^2 = \left(1 + \frac{r}{3}\right) \Delta m^2_{\text{atm}} + \ldots$$
$$|m_{ee}|^2 = \frac{16}{27} \Delta m^2_{\text{atm}} r + \ldots,$$

where we have expressed the parameters in terms of $\Delta m^2_{\text{atm}}$ and $r$. Dots denote terms of order $r^2$ and $|m_{ee}|$ is the effective mass combination controlling the violation of the total lepton number in neutrinoless double beta decay. It is useful to estimate the cut-off $\Lambda$. We have roughly

$$\sqrt{\Delta m^2_{\text{atm}}} \approx \frac{s^2(v_0^0)^2}{|a| \Lambda \sqrt{r}}.$$

By taking $\sqrt{\Delta m^2_{\text{atm}}} = 0.05$ eV, $s^2(v_0^0)^2 = (100 \text{ GeV})^2$ and $\sqrt{r} \approx 0.2$, we obtain $|a| \Lambda \approx 10^{15}$ GeV, not far from the unification scale. For $u \approx v_{S,T} \approx \lambda^2$ the cut-off $\Lambda$ is then above $10^{16}$ GeV. If we expand $z$ around $-2$, we get:

$$|m_1|^2 = \left(\frac{9}{8} + \frac{r}{12}\right) \Delta m^2_{\text{atm}} + \ldots$$
$$|m_2|^2 = \left(\frac{9}{8} + \frac{13}{12r}\right) \Delta m^2_{\text{atm}} + \ldots$$
$$|m_3|^2 = \left(\frac{1}{8} + \frac{r}{12}\right) \Delta m^2_{\text{atm}} + \ldots$$
$$|m_{ee}|^2 = \left(\frac{1}{8} - \frac{11}{108}r\right) \Delta m^2_{\text{atm}} + \ldots.$$

We now have

$$\sqrt{\Delta m^2_{\text{atm}}} \approx \frac{s^2(v_0^0)^2}{|a| \Lambda}.$$
By repeating the previous estimate, we find $|a| \Lambda \approx 10^{14}$ GeV and $\Lambda$ slightly below $10^{16}$ GeV.

Several remarks should be made:

Concerning the lepton mixing, this is dominated by $U$, eq. (21). The contribution from the charged lepton sector depends on the entries denoted by the dots in $m_e$. Putting all the dots to zero, the charged leptons affect the lepton mixing through rotations of order $\lambda^4$, $\lambda^8$ and $\lambda^4$ in the 12, 13 and 23 sectors, respectively. Operators of dimensions higher than the ones, considered so far, are strongly suppressed and provide contributions of order $\lambda^4$ to the mixing matrix. These are negligible, since the leading effect comes from the modification of the vacuum structure of eq. (8), due to higher order terms in the scalar potential. We shall discuss this in sections 4 and 5. Eventually, such terms modify only slightly the TB mixing pattern.

Apart from $w_\nu$ contributions to neutrino masses and mixing angles might come from higher dimensional operators, as for instance

$$\frac{\xi FF H_5 H_2}{\Lambda^4}.$$  \hspace{1cm} (29)

However, they are completely negligible compared to those discussed above. If we forced this type of operator to be the dominant one, by eliminating the singlets $N$ from our model, we would need a value of $\Lambda$ too small compared with the GUT scale.

Depending on the value of $z$, our model gives rise to two separate branches in the neutrino spectrum, both characterized by a nearly TB mixing. On the first branch, $z \approx +1$, we find a spectrum with normal hierarchy, while on the second branch, $z \approx -2$, we get an inverted hierarchy. A degenerate spectrum is actually disfavored in our construction, since it would require $z \ll 1$ (see eq. (20)) which leads to $r$ close to $1/2$, as can be read off from eq. (21). This can obviously not be reconciled with the data.

In our model the possibility of normal hierarchy is somewhat more natural than the one of inverted hierarchy. There is no reason a priori why $z$ should be close to $+1$ or to $-2$ and reproducing $r$ requires some amount of tuning. However, such a tuning is stronger for inverted hierarchy (ih) than for the normal one (nh), as can be seen by

$$\left. \frac{dr}{dz} \right|_{nh} \left. \frac{dz}{dr} \right|_{ih} \approx -\frac{4}{3\sqrt{3}} \sqrt{r} \approx -0.14.$$  \hspace{1cm} (30)

The derivatives are computed at the relevant value of $z$ in each case and $r$ is the experimental value. Moreover the solution with a normal hierarchy has a domain of validity in energy larger by a factor of $1/\sqrt{r} \approx 5.6$ and extends beyond $10^{16}$ GeV. In the normal hierarchy solution we find with the help of eq. (25)

$$\sum_i |m_i| \approx (0.06 - 0.07) \text{ eV} \quad \text{and} \quad |m_{ee}| \approx 0.007 \text{ eV}.$$  \hspace{1cm} (31)

It is interesting to see that $|m_{ee}|$ is close to the upper limit of the range expected in the normal hierarchy case, being not too far from the aimed for sensitivity of the next generation of neutrinoless double beta decay experiments, $0.01$ eV. This is partly attributed to the fact that $|m_1| \approx 0.005$ is different from zero and in part to the absence of a negative interference with the $m_3$ contribution, as $\theta_{13} = 0$. 
4 Vacuum Alignment

Here we discuss the minimisation of the scalar potential, in order to justify the VEVs assumed in the previous section. We work in the limit of exact SUSY. This will not allow us to analyse the electroweak symmetry breaking induced by $H_5$ and $\bar{H}_5$, whose VEVs are assumed to vanish in first approximation. Indeed all the VEVs we are interested in here, i.e. those of the flavon fields $\varphi_{S,T}$, $\xi$, $\bar{\xi}$, $\theta$ and $\theta''$, are relatively close in magnitude to the cut-off $\Lambda$ and therefore much larger than the electroweak scale, which will be consistently neglected. Moreover we work at leading order in the parameter $1/\Lambda$, that is we keep only the lowest dimensional operators in the superpotential shown in the previous section. Subleading effects will be discussed later on. All the multiplets but the flavon ones are assumed to have vanishing VEVs and set to zero for the present discussion. We regard the U(1) Froggatt-Nielsen flavour symmetry as local. Since the field content displayed in table 1 is anomalous under the U(1), we need additional chiral multiplets to cancel the anomaly. These multiplets can be chosen vector-like with respect to SU(5), so that they only contribute to the U(1) anomaly. Here we do not need to specify these fields, but we must presume that they do not acquire a VEV. Within these assumptions the relevant part of the scalar potential of the model is given by the sum of the F-terms and of a D-term:

$$V = V_F + V_D ,$$

$$V_F = \sum \left| \frac{\partial w}{\partial \varphi_i} \right|^2 ,$$

where $\varphi_i$ stands for the generic chiral multiplet. Only the last term in eq. (33), $w_d$, contributes to the VEVs we are looking for. It is given by:

$$w_d = M(\varphi_0^T \varphi_T) + g(\varphi_0^T \varphi_T \varphi_T) + g_1(\varphi_0^S \varphi_S \varphi_S) + g_2(\varphi_0^S \varphi_S \varphi_S) + g_3(\varphi_0 \xi_0 \varphi_S \varphi_S) + g_4(\varphi_0 \xi_0 \bar{\xi} + g_5(\varphi_0 \xi_0 \bar{\xi}) + g_6(\varphi_0 \xi_0 \bar{\xi})^2 .$$

Since also the terms in $w_d$ have to have R-charge +2, we introduce additional gauge singlets, so called driving fields, $\varphi_0^T$, $\varphi_0^S$ and $\xi_0$ with R-charge +2 (see table 1). Note that therefore all terms in $w_d$ are linear in these fields. Note further that due to U(1) invariance neither the multiplet $\theta$, nor the multiplet $\theta''$ is contained in $w_d$. Moreover the D-term $V_D$ does not depend on $\varphi_{S,T}$, $\xi$, $\bar{\xi}$, which are all singlets under the (gauged) U(1). The expression of $w_d$ and the minimisation procedure are exactly as described in ref. [6] and leads to the result anticipated in the previous section:

$$\langle \varphi_T \rangle = (v_T,0,0)\Lambda , \quad v_T \Lambda = -\frac{3M}{2g} ,$$

$$\langle \varphi_S \rangle = (v_S,v_S,v_S)\Lambda , \quad v_S = \frac{g_4}{3g_3} ,$$

$$\langle \xi \rangle = u\Lambda ,$$

$$\langle \bar{\xi} \rangle = 0$$

(34)

with $u$ undetermined and $g_3 \equiv 3g_0^2$, $g_4 \equiv -g_1^2$. In the following we take $v_T$, $v_S$ and $u$ to be of $O(\lambda^2)$. This order of magnitude is indicated by the observed ratio of up and down
or charged lepton masses, by the scale of the light neutrino masses and is also compatible with the bounds on the deviations from TB mixing for leptons.

The D-term is given by:

\[ V_D = \frac{1}{2} (M_F^2 - g_{F_N} |\theta|^2 - g_{F_N} |\theta''|^2 + \ldots)^2 \]  

(35)

where \( g_{F_N} \) is the gauge coupling constant of U(1) and \( M_F^2 \) denotes the contribution of the Fayet-Iliopoulos term. We have omitted the SU(5) contribution to the D-term, whose VEV is zero. There are SUSY minima such that \( V_F = V_D = 0 \). The vanishing of \( V_D \) requires

\[ g_{F_N} |\theta|^2 + g_{F_N} |\theta''|^2 = M_F^2 . \]  

(36)

If the parameter \( M_F^2 \) is positive, the above condition determines a non-vanishing VEV for a combination of \( \theta \) and \( \theta'' \). Here we assume that the VEVs fulfil eqs. (3,13). The different order of \( t, t'' \) versus \( v_T, v_S \) and \( u \) can be attributed to the different couplings and mass parameters in \( V_D \) and \( V_F \).

Finally, we discuss the subleading corrections to the vacuum alignment. As already noticed above, the fields \( \theta \) and \( \theta'' \) cannot couple to the flavon fields, since the flavons \( \varphi_T, \varphi_S, \xi, \xi', \varphi_T^0, \varphi_S^0 \) and \( \xi_0 \) are not charged under the U(1) symmetry, responsible for the charged fermion mass hierarchy. Therefore, the subleading effects in the potential arise from terms made up of one driving field and three fields \( \varphi_T, \varphi_S, \xi \) and \( \xi' \). They induce shifts in the VEVs shown above and thereby influence the mass matrices, as discussed in the next section. Since the flavon field content of this model is essentially the same as the one in ref. [6], not only the renormalizable part of \( w_d \) coincides, but also the subleading terms are the same. Hence, we do not need to repeat this discussion and we only state the results found there. The shifted VEVs are

\[ \langle \varphi_T \rangle / \Lambda = (v_T + \delta v_{T1}, \delta v_{T2}, \delta v_{T3}) , \]
\[ \langle \varphi_S \rangle / \Lambda = (v_S + \delta v_{1}, v_S + \delta v_{2}, v_S + \delta v_{3}) , \]
\[ \langle \xi \rangle / \Lambda = u , \]
\[ \langle \xi' \rangle / \Lambda = \delta u' , \]  

(37)

where \( u \) remains undetermined and, once we have taken \( v_{T,S} \), \( u \sim O(\lambda^2) \), all shifts are suppressed by a factor of order \( \lambda^2 \): \( \delta v / v \sim O(\lambda^2) \). As found in ref. [6] the following relation holds:

\[ \delta v_{T2} = \delta v_{T3} . \]  

(38)

Higher order corrections to \( t \) and \( t'' \) simply amount to a rescaling that does not change their individual order of magnitude which remains of \( O(\lambda) \).

5 Subleading Corrections

In this section, we analyse the effects of the subleading corrections in terms of \( \lambda \) to the fermion masses and mixings. The corrections arise from additional insertions of the flavons \( \varphi_T, \varphi_S, \xi \) and \( \tilde{\xi} \) as well as from shifts of the VEVs shown above.

\(^3\)Note that \( |\theta''|^2 \) is a singlet under \( A_4 \), because \( \theta'' \sim 1'' \) and \( \theta''^* \sim 1' \) under \( A_4 \).
5.1 Corrections to $w_{up}$

In the up quark sector the leading order terms only involve the fields $\theta$ and $\theta''$, since they are the only fields which have a non-vanishing U(1) charge among the gauge singlets of the model. The subleading terms then additionally involve the fields $\varphi_T$, $\varphi_S$, $\xi$ and $\bar{\xi}$. As the tenplets transform as singlets under $A_4$ and the combinations $T_i T_j H_5 \theta^m \theta^{mm}$ are invariant under the $Z_3$ group, we cannot multiply the $w_{up}$ terms by a single flavon field. The most economic possibility is to insert two flavons, namely $\varphi_T \varphi_T$. Among the three contractions leading to a 1 or 1' or 1'' representation of $A_4$ only the 1 has a non-vanishing VEV, given that $\langle \varphi_T \rangle = (v_T, 0, 0) \Lambda$. Therefore the dominant subleading corrections to the up quark mass matrix have the same structure as the leading order results and are suppressed by an overall factor $v_T^2 \sim O(\lambda^4)$. The fields $\varphi_S$ and $\xi$, $\bar{\xi}$ can only couple at the level of three flavon insertions due to the requirement of $Z_3$ invariance. However, all contributions stemming from three flavon insertions are suppressed by $\lambda^6$ relative to the leading order term. Similarly, the corrections due to shifts in the VEVs contribute at most at relative order $\lambda^6$. For the up quark masses and the mixings all these corrections are negligible.

5.2 Corrections to $w_{down}$

In the down sector the main effect of the subleading corrections is to fill the zeros indicated by dots in the upper triangle of $m_d$. In order to maintain the $A_4$ invariance the leading order terms include one insertion of the flavon $\varphi_T$. The subleading corrections arise from two effects: a.) replacing $\varphi_T$ with products of flavon fields and b.) including the corrections to the VEVs of $\varphi_T$. The replacement of $\varphi_T$ with a product $\varphi_T \varphi_T$ is the simplest choice compatible with the $Z_3$ charges. Note that this is similar to the up quark sector. If the VEVs are unchanged this contribution to $m_d$ is of the same form as displayed in eq. (12) and suppressed by $v_T^2 \sim O(\lambda^2)$ compared to the leading result due to the additional flavon field. Therefore this type of correction does not fill the zeros in $m_d$. They are filled by the corrections coming from the VEV shifts inserted in the terms containing one flavon $\varphi_T$. Considering that we assumed all $\delta v / v \sim O(\lambda^2)$, the corrections to the matrix elements of $m_d$ are of the following order in $\lambda$:

$$
\delta m_d = \begin{pmatrix}
\lambda^6 & \lambda^4 & \lambda^2 \\
\lambda^6 & \lambda^4 & \lambda^2 \\
\lambda^6 & \lambda^4 & \lambda^2
\end{pmatrix} v_T \lambda v_d^0. 
$$

(39)

As said, the matrix elements which are already non-vanishing at the leading order, eq. (12), receive additional corrections from the two flavon insertion $\varphi_T \varphi_T$. These are of the same order as the corrections from the VEV shifts, e.g. for the element 11 also of order $\lambda^6$. In summary, the zeroes in the elements 12, 13 and 23 of $m_d$, appearing at leading order, are replaced by terms of order $\lambda^4$, $\lambda^2$ and $\lambda^2$, respectively, in units of $v_T \lambda v_d^0$.

In our model the relation $m_d = m_e^T$ is not valid for the first two families but it still holds at the level of orders of magnitude for each entry. So the powers of $\lambda$ are also the same for each matrix element of $m_d m_d^T$ and of $m_e m_e^T$. This is important as the matrix
is diagonalized by the unitary matrix $U_e$ that enters in determining the leptonic mixing matrix $U = U_e^\dagger U_\nu$. The results just described for the subleading corrections on $m_d$ and $m_e^\dagger$ imply that $U_e$ induces corrections of $O(\lambda^2)$ on all mixing angles in $U$, that is, in our case, corrections of $O(\lambda^2)$ to the TB values of each mixing angle.

5.3 Corrections to $w_\nu$

Also the $w_\nu$ term of the superpotential, eq. (7), is modified by terms with more flavon factors and by subleading corrections to the VEVs. The Dirac mass term, proportional to $H_5(NF)$, is mainly modified by a single $\varphi_T$ insertion, that produces corrective terms suppressed by a $O(\lambda^2)$ factor. These corrections are of the same order as those arising for Majorana mass terms. In fact, $NN$ can be in a $1, 1', 1''$ or $3_s$ combination. Since $NN \sim \omega^2$ under $Z_3$, the singlet 1 can be multiplied by $\xi$ (the singlet leading term) or by $(\varphi_T\varphi_S)$ (which can be absorbed into a redefinition of the leading term), $1'$ by $(\varphi_T\varphi_S)'$, $1''$ by $(\varphi_T\varphi_S)''$ and $3_s$ by $\varphi_S$ (the triplet leading term) or by $(\varphi_T\xi)$ or $(\varphi_T\varphi_S)_{3_s}$ or $(\varphi_T\varphi_S)_{3_s}$. All two flavon insertions lead to corrections of relative order of $O(\lambda^2)$ to the matrix elements of the Majorana matrix. In addition, the shifts of the $\varphi_S$ VEVs applied to the triplet leading term also produce $O(\lambda^2)$ corrective terms. As it is easy to check, in general there are enough parameters so that all 6 independent entries of the (symmetric) Majorana mass matrix receive a different correction at $O(\lambda^2)$.

The described corrections affect the neutrino masses and, together with the corrections to $m_e$, also all lepton mixing angles. To be compatible with the data, given the accuracy of the TB approximation, the dominant corrections must be of $O(\lambda^2)$ at most, and this is precisely the magnitude of the terms that we have just mentioned.

6 Conclusion

We have constructed a SUSY SU(5) grand unified model which includes the $A_4$ description of TB mixing for leptons. For this it is not only necessary to adopt an $A_4$ classification of quarks and leptons compatible with SU(5), but also to introduce additional U(1) and $Z_N$ symmetries and to suitably formulate the grand unification model. We find that the most attractive solution to cope with the different requirements from fermion mass and mixing hierarchies, from the problem of doublet-triplet splitting in the Higgs sector, from proton decay bounds and from maintaining bottom tau unification only, is a formulation in 5 space-time dimensions with a particular location of the different fields, with some of them on the brane at $y = 0$ and some in the bulk. The latter include the gauge and Higgs fields as well as the tenplets of the first two, i.e. lightest, families. The resulting model naturally leads to TB mixing in first approximation with corrections of $O(\lambda^2)$ from higher dimensional effective operators, together with reproducing the observed mass hierarchies for quarks and charged leptons and the CKM mixing pattern. In the quark sector, however, as is typical of U(1) models, only orders of magnitude are determined in terms of powers of $\lambda$ with exponents fixed by the charges. A moderate fine tuning is only needed to enhance the CKM mixing angle between the first two generations, which would generically be of
\(O(\lambda^2)\), and to suppress the value of \(r\), given in eq. \((24)\), which would typically be of order 1. The latter feature is also true in all purely leptonic \(A_4\) models, in which \(A_4\) leads to the correct mixing, but not directly to the spectrum of the neutrino masses. Actually the model allows for both types of neutrino mass hierarchy, the normal and the inverted one. The normal hierarchy is, however, somewhat more natural, since it requires less tuning to reproduce \(r\). Furthermore, it is consistent with a larger value of the cut-off \(\Lambda\). If the normal hierarchy is the correct one, the model predicts the sum of neutrino masses to be around \((0.06 - 0.07)\) eV and \(|m_{ee}|\) to be close to 0.007 eV. Therefore, \(|m_{ee}|\) is not far from the future aimed for experimental sensitivity. Finally, all subleading corrections to the leading order result of fermion masses and mixings have been carefully analysed.

In conclusion, we have demonstrated that the simple \(A_4\) approach to TB mixing is compatible with a grand unified picture describing all quark and lepton masses and mixings.

**Acknowledgements**

We thank S. King for very interesting discussions. We recognize that this work has been partly supported by the European Commission under contracts MRTN-CT-2004-503369 and MRTN-CT-2006-035505, and by the Italian Ministero dell’Universita’ e della Ricerca Scientifica, under the COFIN program for 2007-08.
References

[1] T. Schwetz, Acta Phys. Polon. B 36 (2005) 3203 [arXiv:hep-ph/0510331]; A. Strumia and F. Vissani, Nucl. Phys. B 726 (2005) 294 [arXiv:hep-ph/0503246]; G. L. Fogli, E. Lisi, A. Marrone, A. Melchiorri, A. Palazzo, P. Serra and J. Silk, Phys. Rev. D 70 (2004) 113003 [arXiv:hep-ph/0408045]; J. N. Bahcall, M. C. Gonzalez-Garcia and C. Peña-Garay, JHEP 0408, 016 (2004) [arXiv:hep-ph/0406294]; M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 6 (2004) 122 [arXiv:hep-ph/0405172].

[2] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530 (2002) 167 [arXiv:hep-ph/0202074]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209]; Z. z. Xing, Phys. Lett. B 533 (2002) 85 [arXiv:hep-ph/0204049]; P. F. Harrison and W. G. Scott, Phys. Lett. B 547 (2002) 219 [arXiv:hep-ph/0210197]; P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 [arXiv:hep-ph/0302025]; P. F. Harrison and W. G. Scott, [arXiv:hep-ph/0402006]; P. F. Harrison and W. G. Scott, Phys. Lett. B 594 (2004) 324 [arXiv:hep-ph/0403278].

[3] E. Ma and G. Rajasekaran, Phys. Rev. D 64 (2001) 113012 [arXiv:hep-ph/0106291].

[4] K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552 (2003) 207 [arXiv:hep-ph/0206292]; M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, [arXiv:hep-ph/0312244]; M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, Phys. Rev. D 69 (2004) 093006 [arXiv:hep-ph/0312265]; E. Ma, Phys. Rev. D 70 (2004) 031901 [arXiv:hep-ph/0404199]; E. Ma [arXiv:hep-ph/0409075]; E. Ma, New J. Phys. 6 (2004) 104 [arXiv:hep-ph/0405152]; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724 (2005) 423 [arXiv:hep-ph/0504181]; E. Ma, Phys. Rev. D 72 (2005) 037301 [arXiv:hep-ph/0505209]; K. S. Babu and X. G. He, [arXiv:hep-ph/0507217]; A. Zee, Phys. Lett. B 630 (2005) 58 [arXiv:hep-ph/0508278]; E. Ma, Mod. Phys. Lett. A 20 (2005) 2601 [arXiv:hep-ph/0508099]; E. Ma, Phys. Rev. D 73 (2006) 057304 [arXiv:hep-ph/0511133]; K. Kang, Z. z. Xing and S. Zhou, Phys. Rev. D 73 (2006) 013001 [arXiv:hep-ph/0511157]; X. G. He, Y. Y. Keum and R. R. Volkas, JHEP 0604 (2006) 039 [arXiv:hep-ph/0601001].

[5] G. Altarelli and F. Feruglio, Nucl. Phys. B 720 (2005) 64 [arXiv:hep-ph/0504165].

[6] G. Altarelli and F. Feruglio, Nucl. Phys. B 741 (2006) 215 [arXiv:hep-ph/0512103].

[7] G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B 775 (2007) 31 [arXiv:hep-ph/0610165].

[8] S. F. King, JHEP 0508 (2005) 105 [arXiv:hep-ph/0506297]; I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B 733 (2006) 31 [arXiv:hep-ph/0507176]; S. F. King and M. Malinsky, JHEP 0611 (2006) 071 [arXiv:hep-ph/0608021]; Y. Koide, [arXiv:0707.0899 [hep-ph]].
[9] C. I. Low and R. R. Volkas, Phys. Rev. D 68 (2003) 033007 [arXiv:hep-ph/0305243]; J. Matias and C. P. Burgess, JHEP 0509 (2005) 052 [arXiv:hep-ph/0508156]; E. Ma, Phys. Lett. B 632 (2006) 352 [arXiv:hep-ph/0508231]; S. Luo and Z. z. Xing, Phys. Lett. B 632 (2006) 341 [arXiv:hep-ph/0509065]; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 644 (2007) 153 [arXiv:hep-ph/0512313]; N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. 97 (2006) 041601 [arXiv:hep-ph/0603116]; P. Kovtun and A. Zee, Phys. Lett. B 640 (2006) 37 [arXiv:hep-ph/0604169]; Z. z. Xing, H. Zhang and S. Zhou, Phys. Lett. B 641 (2006) 27 [arXiv:hep-ph/0706.2341 [hep-ph]]; C. S. Lam, Phys. Rev. D 74 (2006) 113004 [arXiv:hep-ph/0611017]; C. Luhn, S. Nasri and P. Ramond, Phys. Lett. B 652 (2007) 27 [arXiv:0708.3665 [hep-ph]]; E. Ma, arXiv:0709.0507 [hep-ph]; E. Ma, Europhys. Lett. 79 (2007) 61001 [arXiv:hep-ph/0701016]; C. S. Lam, arXiv:0711.3795 [hep-ph].

[10] E. Ma, Mod. Phys. Lett. A 17 (2002) 627 [arXiv:hep-ph/0203238].

[11] E. Ma, Mod. Phys. Lett. A 20 (2005) 2767 [arXiv:hep-ph/0506036]; E. Ma, Mod. Phys. Lett. A 21 (2006) 2931 [arXiv:hep-ph/0607190]; E. Ma, H. Sawanaka and M. Tanimoto, Phys. Lett. B 641 (2006) 301 [arXiv:hep-ph/0606103]; S. Morisi, M. Picariello and E. Torrente-Lujan, Phys. Rev. D 75 (2007) 075015 [arXiv:hep-ph/0702034]; W. Grimus and H. Kuhbock, [arXiv:0710.1585 [hep-ph]].

[12] S. F. King and M. Malinsky, Phys. Lett. B 645 (2007) 351 [arXiv:hep-ph/0610250].

[13] P. H. Frampton and T. W. Kephart, Int. J. Mod. Phys. A 10 (1995) 4689 [arXiv:hep-ph/9409330]; A. Aranda, C. D. Carone and R. F. Lebed, Phys. Lett. B 474 (2000) 170 [arXiv:hep-ph/9910392]; A. Aranda, C. D. Carone and R. F. Lebed, Phys. Rev. D 62 (2000) 016009 [arXiv:hep-ph/0002044]; P. D. Carr and P. H. Frampton, arXiv:hep-ph/0701034; P. H. Frampton and T. W. Kephart, JHEP 0709 (2007) 110 [arXiv:0706.1186 [hep-ph]]; A. Aranda, Phys. Rev. D 76 (2007) 111301 [arXiv:0707.3661 [hep-ph]].

[14] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775 (2007) 120 [arXiv:hep-ph/0702194].

[15] M. C. Chen and K. T. Mahanthappa, Phys. Lett. B 652 (2007) 34 [arXiv:0705.0714 [hep-ph]].

[16] D. G. Lee and R. N. Mohapatra, Phys. Lett. B 329 (1994) 463 [arXiv:hep-ph/9403201]; R. N. Mohapatra, M. K. Parida and G. Rajasekaran, Phys. Rev. D 69 (2004) 053007 [arXiv:hep-ph/0301234]; C. Hagedorn, M. Lindner and R. N. Mohapatra, JHEP 0606 (2006) 042 [arXiv:hep-ph/0602244]; Y. Cai and H. B. Yu, Phys. Rev. D 74 (2006) 115005 [arXiv:hep-ph/0608022].

[17] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 648 (2007) 201 [arXiv:hep-ph/0607045].
[18] R. Dermisek and S. Raby, Phys. Lett. B 622 (2005) 327 [arXiv:hep-ph/0507045]; S. Morisi and M. Picariello, Int. J. Theor. Phys. 45 (2006) 1267 [arXiv:hep-ph/0505113]; M. Picariello, arXiv:hep-ph/0611189; F. Caravaglios and S. Morisi, arXiv:hep-ph/0510321; S. Morisi, arXiv:hep-ph/0604106; F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234; N. Haba and K. Yoshioka, Nucl. Phys. B 739 (2006) 254 [arXiv:hep-ph/0511108]; M. Tanimoto and T. Yanagida, Phys. Lett. B 633 (2006) 567 [arXiv:hep-ph/0511336]; Y. Koide, Eur. Phys. J. C 48 (2006) 223 [arXiv:hep-ph/0508301]; R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 636 (2006) 114 [arXiv:hep-ph/0603020]; R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 639 (2006) 318 [arXiv:hep-ph/0605020]; J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, Prog. Theor. Phys. 109 (2003) 795 [Erratum-ibid. 114 (2005) 287] [arXiv:hep-ph/0302196]; J. Kubo, Phys. Lett. B 578 (2004) 156 [Erratum-ibid. B 619 (2005) 387] [arXiv:hep-ph/0309167]; W. Grimus and L. Lavoura, JHEP 0601 (2006) 018 [arXiv:hep-ph/0509239]; T. Teshima, Phys. Rev. D 73 (2006) 045019 [arXiv:hep-ph/0509094]; S. Kaneko, H. Sawanaka, T. Shingai, M. Tanimoto and K. Yoshioka, Prog. Theor. Phys. 117 (2007) 161 [arXiv:hep-ph/0609220]; Y. Koide, Phys. Rev. D 73 (2006) 057901 [arXiv:hep-ph/0509214]; Y. Koide, Eur. Phys. J. C 50 (2007) 809 [arXiv:hep-ph/0612058]; C. Y. Chen and L. Wolfenstein, [arXiv:0709.3767 [hep-ph]]; S. L. Chen, M. Frigerio and E. Ma, Phys. Rev. D 70 (2004) 073008 [Erratum-ibid. D 70 (2004) 079905] [arXiv:hep-ph/0404084]; L. Lavoura and E. Ma, Mod. Phys. Lett. A 20 (2005) 1217 [arXiv:hep-ph/0502181].

[19] E. Witten, Nucl. Phys. B 258 (1985) 75; Y. Kawamura, Prog. Theor. Phys. 105 (2001) 999 [arXiv:hep-ph/0012125]; A. E. Faraggi, Phys. Lett. B 520 (2001) 337 [arXiv:hep-ph/0107094] and references therein.

[20] L. J. Hall and Y. Nomura, Phys. Rev. D 64 (2001) 055003 [arXiv:hep-ph/0103125]; Y. Nomura, Phys. Rev. D 65 (2002) 085036 [arXiv:hep-ph/0108170]; L. J. Hall and Y. Nomura, Phys. Rev. D 66 (2002) 075004 [arXiv:hep-ph/0205067].

[21] G. Altarelli and F. Feruglio, Phys. Lett. B 511 (2001) 257 [arXiv:hep-ph/0102301]; A. Hebecker and J. March-Russell, Nucl. Phys. B 613 (2001) 3 [arXiv:hep-ph/0106166]; A. Hebecker and J. March-Russell, Phys. Lett. B 541 (2002) 338 [arXiv:hep-ph/0205143].

[22] R. Contino, L. Pilo, R. Rattazzi and E. Trincherini, Nucl. Phys. B 622 (2002) 227 [arXiv:hep-ph/0108102]; A. Hebecker and J. March-Russell, Phys. Lett. B 539 (2002) 119 [arXiv:hep-ph/0204037]; M. L. Alciati, F. Feruglio, Y. Lin and A. Varagnolo, JHEP 0503 (2005) 054 [arXiv:hep-ph/0501086].

[23] Y. Nomura, Phys. Rev. D 65 (2002) 085036 [arXiv:hep-ph/0108170].