Infinitely many Leray–Hopf solutions for the fractional Navier–Stokes equations

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ABSTRACT
We prove the ill-posedness for the Leray–Hopf weak solutions of the incompressible and ipodissipative Navier–Stokes equations, when the power of the diffusive term \((-\Delta)^\gamma\) is \(\gamma < 1/3\). We construct infinitely many solutions, starting from the same initial datum, which belong to \(C^{1/3}_0\) and strictly dissipate their energy in small time intervals. The proof exploits the “convex integration scheme” introduced by C. De Lellis and L. Székelyhidi for the incompressible Euler equations, joining these ideas with new stability estimates for a class of non-local advection-diffusion equations and a local (in time) well-posedness result for the fractional Navier–Stokes system. Moreover, we show the existence of dissipative Hölder continuous solutions of Euler equations that can be obtained as a vanishing viscosity limit of Leray–Hopf weak solutions of suitable fractional Navier–Stokes equations.

1. Introduction
In this article, we consider the Cauchy problem for the incompressible fractional Navier–Stokes equations

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p + (-\Delta)^\gamma v &= 0 \\
\text{div } v &= 0 \\
v(\cdot, 0) &= \tilde{v},
\end{align*}
\]

(1.1)
in the spatial periodic setting \(\mathbb{T}^3 = \mathbb{R}^3 \setminus \mathbb{Z}^3\), where \(v\) is a vector field representing the velocity of the fluid, \(p\) is the hydrodynamic pressure, \(\tilde{v} : \mathbb{T}^3 \to \mathbb{R}^3\) is any given solenoidal initial data and \(\gamma \in (0, 1/3)\). The operator \((-\Delta)^\gamma\) is the (non-local) diffusive operator, whose Fourier series is given by

\[(-\Delta)^\gamma \hat{v}(k) := \sum_{k \in \mathbb{Z}^3} |k|^2\gamma \hat{v}_k e^{ik \cdot x}.\]

We are interested in Leray–Hopf weak solutions of (1.1), namely solutions \(v \in L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^3)) \cap L^2(\mathbb{R}^+, H^\gamma(\mathbb{T}^3))\) satisfying (1.1) in the distributional sense, namely such that
\[
\int_0^\infty \int_{\mathbb{T}^3} \left[ (v \cdot \partial_t \varphi - v \cdot (-\Delta)\varphi + v \otimes v : D\varphi) (x, s) \right] dx ds = -\int_{\mathbb{T}^3} \vartheta(x) \cdot \varphi(x, 0) dx,
\]

every smooth test vector field \( \varphi \in C_c^\infty(\mathbb{T}^3 \times \mathbb{R}, \mathbb{R}^3) \) with \( \text{div} \varphi = 0 \) (note that \( p \) can be recovered uniquely as a distribution if we impose that \( \int p dx = 0 \)), and obeying to the global energy inequality
\[
\frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x, t) dx \leq \frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x, \tau) dx d\tau \leq \frac{1}{2} \int_{\mathbb{T}^3} |v|^2(x, s) dx, \quad \forall 0 \leq s < t. \tag{1.2}
\]

As for the Navier–Stokes equations (i.e. the case \( \gamma = 1 \)), it is known that such solutions exist. Indeed, we have (for the proof, see Theorem 1.1 by Colombo et al. [1].

**Theorem 1.1.** For any \( \vartheta \in L^2(\mathbb{T}^3) \) with \( \text{div} \vartheta = 0 \) and every \( \gamma \in ]0, 1[ \) there exists a Leray–Hopf weak solution of (1.1).

It is also known that, if the power \( \gamma \) of the Laplacian is suitably small, then these solutions are not unique. Indeed Colombo et al. [1] proved the ill-posedness in the case \( \gamma < 1/5 \). The question about uniqueness is still open if \( \gamma \geq 1/5 \). In this work, we partially answer this question, proving the non-uniqueness of such solutions in the range \( 0 < \gamma < 1/3 \). More precisely, the main result of this paper is the following

**Theorem 1.2.** Let \( \gamma < 1/3 \). Then there are initial data \( \vartheta \in L^2(\mathbb{T}^3) \) with \( \text{div} \vartheta = 0 \) for which there exist infinitely many Leray solutions \( \vartheta \) of (1.1) in \([0, +\infty) \times \mathbb{T}^3 \). More precisely, if \( \gamma < \beta < 1/3 \), there are initial data \( \vartheta \in C^\beta(\mathbb{T}^3) \) with \( \text{div} \vartheta = 0 \) and a positive time \( T \) such that

(a) there are infinitely many Leray–Hopf solutions of (1.1) and moreover \( \vartheta \in C^\beta(\mathbb{T}^3 \times [0, T]) \);

(b) such solutions strictly dissipate the total energy in \([0, T] \), i.e. the function (of time only)
\[
e_{\text{tot}}(t) := \frac{1}{2} \int_{\mathbb{T}^3} |\vartheta|^2(x, t) dx + \int_0^t \int_{\mathbb{T}^3} |(-\Delta)^{\gamma/2} \vartheta|^2(x, \tau) dx d\tau \tag{1.3}
\]
is strictly decreasing in \([0, T] \).

The proof of Theorem 1.2 is achieved by using the “convex integration methods” introduced by De Lellis and Székelyhidi [2] for the incompressible Euler equations, in particular the construction used by Buckmaster et al. [3], where the authors, thanks to the new ideas introduced by Isett in [4], proved the existence of \( C^{1/3} \) solutions of Euler equations with prescribed kinetic energy. This method can be also used to prove the ill-posedness for the distributional solutions of the Navier–Stokes equations (i.e. \( \gamma = 1 \)). Indeed, recently, Buckmaster and Vicol [5] proved the existence of infinitely many weak solutions of the Navier–Stokes equations with bounded kinetic energy. The solutions constructed by Buckmaster and Vicol [5] do not even have finite energy dissipation in the sense of \( e_{\text{tot}} \) thus they are not of Leray–Hopf type. These iterative methods have already been used to prove ill-posedness results in contexts of fractional powers of the Laplacian. For instance, by Buckmaster et al. [6] they produce infinitely many solutions of the SQG equation.
In order to use the argument proposed by Buckmaster et al. [3], we have to construct exact solutions of (1.1) in small time intervals. The corresponding stability estimates of such solutions, with respect to the initial data, are also needed. To this aim, we prove new stability estimates for classical solutions of non-local advection-diffusion equations.

Following Colombo et al. [1], we will see that if the exponent \( \gamma \) is not too large (in particular \( \gamma < 1/3 \)), then the methods used by Buckmaster et al. [3] to produce Hölder continuous solutions to the Euler equations with prescribed kinetic energy can be adapted to (1.1). Then, we will be able to produce (different) solutions with different kinetic energy profile, let all of them start from the same initial data and keep under control the dissipative part in the definition of \( e_{\text{tot}} \) (see (1.3)).

As already did by Colombo et al. [1], also in this case, the methods would give us infinitely many weak solutions bounded in \( L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^3)) \) in the range \( 1/3 \leq \gamma < 1/2 \), but (a priori) without any control on \( e_{\text{tot}} \). Since there will not be a big improvement with respect to Colombo et al. [1], we are not exploiting the details of the construction in this range.

In order to avoid confusion, for fractional Navier–Stokes equations with some viscosity \( \nu > 0 \) we mean the system
\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p + \nu(-\Delta)^\gamma v &= 0 \\
\text{div } v &= 0.
\end{align*}
\]

When \( \nu = 0 \), they are known as Euler equations. Using the main iterative proposition (Proposition 4.1), we are able to show the existence of dissipative solutions of Euler which can be obtained as a vanishing viscosity limit of solutions of (1.4). The main idea is taken from Buckmaster and Vicol [5] where the authors proved that Hölder continuous solutions of Euler arise as a strong limit in \( C^0_t(L^2) \) (as \( \nu \to 0 \)) of weak solutions of the classical Navier–Stokes equations.

Again by the restriction \( \gamma < 1/3 \), we are able to produce a sequence Leray–Hopf weak solutions of (1.4) converging to a dissipative solution of Euler, as \( \nu \to 0 \). More precisely we prove the following.

**Theorem 1.3.** Let \( \beta' < 1/3 \). There exist dissipative solutions \( v \in C^{\beta'}([0, T] \times \mathbb{T}^3) \) of Euler such that, if \( 0 < \gamma < \beta' \), there exists a sequence \( \nu_n \to 0 \) and a sequence \( v^{(\nu_n)} \) of Leray–Hopf weak solutions of (1.4) such that \( v^{(\nu_n)} \to v \) strongly in \( C^0([0, T], C^{\beta'}(\mathbb{T}^3)) \) for every \( \beta'' < \beta' \).

Also in this case, if we only want to require that the sequence \( v^{(\nu_n)} \) is just a sequence of weak solution of (1.4), bounded in \( L^\infty([0, T], L^2(\mathbb{T}^3)) \), we could also prove that for any \( \gamma < 1/2 \) there exists a sequence of solutions of (1.4) converging to any Hölder solution of Euler, as \( \nu \to 0 \), but in order to be consistent with the arguments of this work, we will not enter in this details.

2. Proof of Theorem 1.2

In order to show Theorem 1.2, we will prove a slightly more general result about (1.1). Indeed, using the inductive scheme proposed by Buckmaster et al. [3], we are able to prove the following.
Theorem 2.1. Let \( e : [0, 1] \to \mathbb{R}^+ \) with the following properties

(i) \( 1/2 \leq e(t) \leq 1, \forall t \in [0, 1] \);
(ii) \( \sup |e'(t)| \leq K \), for some \( K > 1 \).

Then for all \( \beta < \gamma < 1/3 \) there exists a couple \((v, p)\), solving

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p + (-\Delta)^\gamma v &= 0 \\
\text{div} \; v &= 0
\end{align*}
\] (2.1)

in the sense of distributions, such that \( v \in C^\beta(T^3 \times [0, 1]) \) and \(^1\)

\[
e(t) = \int_{T^3} |v|^2(x, t)dx,
\]

\[
\|v\|_\beta \leq C_\beta K^{4/9},
\]

where \( C_\beta \) is a constant depending only on \( \beta \). Moreover, given any two energy profiles \( e_1 \) and \( e_2 \) such that \( e_1(0) = e_2(0) \), then the two corresponding solutions \( v^{(1)} \) and \( v^{(2)} \) start from the same initial data, i.e. \( v^{(1)}(\cdot, 0) \equiv v^{(2)}(\cdot, 0) \).

We end this section proving Theorem 1.2, then the rest of the article will be devoted to the proof of Theorem 2.1.

Proof of Theorem 1.2. Elementary arguments produce for every \( K > 1 \) an infinite set \( E_K \) of smooth functions \( e : [0, 1] \to \mathbb{R} \) with the following properties:

(i) \( 1/2 \leq e(t) \leq 1, \forall t \in [0, 1] \);
(ii) \( \|e\|_{C^1([0,1])} \leq 2K + 2 \);
(iii) \( e(0) = 1 \);
(iv) \( e'(t) \leq -2K + 2, \forall t \in [0, \frac{1}{4K}] \);
(v) For any pair of distinct elements of \( E_K \), there is a sequence of times converging to 0 where they take different values.

For each \( e \in E_K \), we now use Theorem 2.1 to produce infinitely many weak solutions satisfying

(a) \( e(t) = \frac{1}{2} \int_{T^3} |v|^2(x, t)dx \);
(b) \( v \in C^\beta(T^3 \times [0, 1]), \quad \forall \beta < 1/3 \);
(c) \( v(\cdot, 0) = \tilde{v}, \text{ for some } \tilde{v} \in C^\beta(T^3) \);
(d) \( \|v\|_\beta \leq C_\beta K^{4/9} \).

Let \( T = 1/4K \). We have to show that all these solutions strictly dissipate the total energy, which is equivalent to

\[
\frac{1}{2} (e(s) - e(t)) > \int_s^t \int_{T^3} \left| (-\Delta)^{\gamma/2} v \right|^2 (x, \tau)dx d\tau, \quad \forall 0 \leq s < t \leq T.
\] (2.4)

\(^1\)Here, \( \| \cdot \|_\beta \) denotes the Hölder norm, see next section for precise definition.
By our assumptions on the functions \( e(t) \) and using Corollary B.2 we have
\[
\frac{1}{2} \left( e(s) - e(t) \right) \geq (K-1)(t-s), \quad \forall 0 \leq s < t \leq T;
\]
\[
\int_s^t \left| (-\Delta)^{\gamma/2} \nu \right|^2(x, \tau) d\tau \leq (t-s) C_{\varepsilon} \| \nu \|_{\gamma+\varepsilon}^2.
\]
Choosing \( \varepsilon \) so that \( \gamma + \varepsilon = \beta \), we see that (2.4) holds if the constant \( K \) satisfies
\[
K-1 > C_{\beta, \gamma} K^{8/9},
\]
where \( C_{\beta, \gamma} \) depends only on \( \gamma \) and \( \beta \), but not on \( K \). It is clear that there exists a \( K \) (big enough) such that (2.5) is satisfied. Thus, we have proved the existence of infinitely many Leray–Hopf solutions in the interval \( [0, T]\) satisfying (a) and (b) of Theorem 1.2. Finally, using Theorem 1.1, it is not difficult to show that all these solutions can be prolonged to Leray–Hopf solutions for every \( t \geq 0 \), thus the proof is concluded. \( \square \)

3. Stability estimates for classical solutions of non-local advection-diffusion equations and classical solutions of the fractional Navier–Stokes equations

In the following, \( m = 0, 1, 2, ..., \quad \alpha \in (0, 1) \) and \( \theta \) is a multi-index. We introduce the usual (spatial) Hölder norms as follows. First of all, the supremum norm is denoted by \( \| f \|_0 := \sup_{T^3 \times [0,T]} |f| \). We define the Hölder seminorms as
\[
[f]_m = \max_{|\theta|=m} \| D^\theta f \|_0,
\]
\[
[f]_{m+\alpha} = \max_{|\theta|=m} \sup_{x \neq y, t} \frac{|D^\theta (f(x, t) - D^\theta f(y, t))|}{|x - y|^\alpha},
\]
where \( D^\theta \) are space derivatives only. The Hölder norms are then given by
\[
\| f \|_m = \sum_{j=0}^m [f]_j
\]
\[
\| f \|_{m+\alpha} = \| f \|_m + [f]_{m+\alpha}.
\]
Moreover, we will write \( [f(t)]_\alpha \) and \( \| f(t) \|_\alpha \) when the time \( t \) is fixed and the norms are computed for the restriction of \( f \) to the \( t \)-time slice.

3.1. Maximum principle and stability estimates

We begin by stating a maximum principle result for a non-local operator. The proof is standard, since, as for the local case (i.e. using the Laplacian), we have that \( (-\Delta)^{\gamma} u(x_0) \geq 0 \) whenever \( x_0 \) is a global maximum point of \( u \) (see for instance the integral representation formula given by Roncal and Stinga [7, Theorem 1.5]).

**Theorem 3.1.** (Maximum principle). Define \( Q_T := T^3 \times (0, T] \). Let \( L \) be the pseudodifferential operator defined as \( Lu = (v \cdot \nabla) u + \nu(-\Delta)^{\gamma} u \), where \( u : T^3 \times [0, T] \to \mathbb{R}, \quad v : T^3 \times [0, T] \to \mathbb{R}^3 \) is a given vector field and \( \nu > 0, \quad 0 < \gamma \leq 1 \). The following holds:
if \( u_t + Lu \leq 0 \) in \( Q_T \), then \( \max_{Q_T} u = \max_{T^3 \times \{0\}} u \)

(ii) if \( u_t + Lu \geq 0 \) in \( Q_T \), then \( \min_{Q_T} u = \min_{T^3 \times \{0\}} u \).

In using Theorem 3.1, we can prove a stability estimate for a general class of non-local parabolic equations. Indeed we have

**Proposition 3.2.** Let \( u : T^3 \times [t_0, T] \to \mathbb{R}^3 \) be a solution of the Cauchy problem

\[
\begin{aligned}
&u_t + Lu = f & \text{in } T^3 \times (t_0, T) \\
&u(\cdot, t_0) = u_0 & \text{in } T^3.
\end{aligned}
\]

Then for any \( t \in [t_0, T] \), we have

\[
\|u(t)\|_0 \leq \|u_0\|_0 + \int_{t_0}^{t} \|f(s)\|_0 ds,
\]

and, more generally, for any \( N \geq 2 \) there exists a constant \( C = C_N \) so that

\[
[u(t)]_N \leq ([u_0]_N + C(t-t_0)[v]_N[u_0]_1) e^{C(t-t_0)[v]_1} + \int_{t_0}^{t} e^{(t-s)[v]_1} ([f(s)]_N + C(t-s)[v]_N[f(s)]_1) ds.
\]

**Proof.** We may assume that \( u \) and \( f \) are two scalar functions, indeed we can work on each component of (3.1). Note also that Theorem 3.1 is invariant under the time shifting \( t \mapsto t + t_0 \).

Defining

\[ w := u - \int_{t_0}^{t} \|f(s)\|_0 ds, \]

we have

\[
\begin{aligned}
&\begin{cases}
& \quad w_t + Lw = f - \|f(t)\|_0 \leq 0 \\
& \quad w(\cdot, t_0) = u_0.
\end{cases}
\end{aligned}
\]

Thus, by Theorem 3.1, we have

\[
\|u(x, t)\|_0 \leq \|u_0\|_0 + \int_{t_0}^{t} \|f(s)\|_0 ds.
\]

Applying the same argument to the function \( \tilde{w} := u + \int_{t_0}^{t} \|f(s)\|_0 ds \), we get the bound from below, showing (3.2).

Next, differentiate (3.1) in the \( x \) variable to obtain

\[(Du)_t + LDu = Df - DvDu.\]

Applying (3.2) to \( Du \) yields

\[ [u(t)]_1 \leq [u_0]_1 + \int_{t_0}^{t} ([f(s)]_1 + [v]_1[u(s)]_1) ds, \]

and by Gronwall’s inequality, we get (3.3). Now, differentiating (3.1) \( N \) times yields
\[(D^N u)_t + LD^N u = D^N f + \sum_{k=0}^{N-1} c_{k,N} D^{k+1} u D^{N-k} v. \]  \hspace{1cm} (3.6)

Using again (3.2) we can estimate
\[\|u(t)\|_x \leq \|u_0\|_x + \int_{t_0}^t \left( (f(s))_x + C([v]_N [u(s)]_1 + [v]_1 [u(s)])_x \right) ds,\]
and plugging the estimate (3.3), we get
\[\|u(t)\|_x \leq \|u_0\|_x + C(t-t_0) [v]_N [u_0]_1 e^{(t-t_0)[v]} + \int_{t_0}^t ((f(s))_x + [v]_1 [u(s)]) ds,
\]
and Gronwall’s inequality finally leads to (3.4). \hspace{1cm} \Box

Using Proposition 3.2, we also get the following.

**Proposition 3.3.** Assume \(0 \leq (t-t_0)[v]_1 \leq 1\). Then, any solution \(u\) of (3.1) satisfies
\[\|u(t)\|_x \leq \|u_0\|_x + \int_{t_0}^t \|f(\cdot, \tau)\|_2 d\tau, \hspace{1cm} (3.7)\]
for all \(0 \leq x \leq 1\), and, more generally, for any \(N \geq 1\) and \(0 \leq x < 1\)
\[\|u(t)\|_{N+x} \leq \|u_0\|_{N+x} + (t-t_0) [v]_{N+x} [u_0]_1 + \int_{t_0}^t \left( ([f(\cdot)]_{N+x} + (t-\tau)[v]_{N+x} [f(\cdot)]_1 \right) d\tau, \hspace{1cm} (3.8)\]
where the implicit constant depends only on \(N\) and \(x\).

**Proof.** For any \(x \in [0, 1]\), let
\[w(x, t; h) := \frac{\delta_h u(x, t)}{|h|^x} = \frac{u(x + h, t) - u(x, t)}{|h|^x}.\]

We have that this new function \(w\) satisfies (see Equation (4.13) in Ref. [8])
\[(\partial_t + \nu(-\Delta)^\gamma + \nu \cdot \nabla x + \delta_h v \cdot \nabla_h) w = \frac{\delta_h v}{|h|} \frac{h}{|h|} w + \frac{\delta_h f}{|h|},\]
Thus, by the maximum principle\(^2\) (3.2) and since \(\sup_{h,x} |w(x, t; h)| = |u(t)|_x\), we get
\[\|u(t)\|_x \leq \|u_0\|_x + \int_{t_0}^t (\nu[v(s)]_1 [u(s)]_x + [f(s)]_x) ds,\]
from which, by Gronwall’s inequality (3.7) follows.

To get the higher order bounds (3.8) just differentiate the equation \(N\) times as in (3.6) and apply the previous argument with

\(^2\)Here, the maximum principle is applied in both the variables \(x, h\).
\[ w(x, t; h) := \frac{\delta_t D^N u(x, t)}{|h|^2} = \frac{D^N u(x + h, t) - D^N u(x, t)}{|h|^2}, \]

then (3.8) is again a consequence of (3.2) and Gronwall’s inequality.

### 3.2. Local existence of smooth solutions

We want to consider exact (smooth) solutions to the fractional Navier–Stokes equations

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p + \nu (-\Delta)^\gamma v &= 0 \\
\text{div } v &= 0 \\
v(\cdot, 0) &= u_0,
\end{align*}
\tag{3.9}
\]

in the periodic setting \(\mathbb{T}^3 \times [0, T]\), where \(\gamma \in (0, 1)\) and \(\nu > 0\). We define the space

\[ V^m := \{ v \in H^m(\mathbb{T}^3) : \text{div } v = 0 \}. \]

We start with the following

**Theorem 3.4.** For any \(m \geq 3\) there exists a constant \(c_m = c(m)\) such that the following holds. Given any initial condition \(u_0 \in V^m\) and \(T_m := c_m \|u_0\|_{V^{-\frac{1}{2}}}^{-1}\) there exists a unique solution \(v \in C([0, T_m], V^m) \cap C^1([0, T_m], V^{m-2})\). Moreover, we have the estimate

\[ \|v(t)\|_{V^m} \leq \|u_0\|_{V^m} e^{c_m \int_0^t \|\nabla v(s)\|_{H^1} ds} \quad \forall t \in [0, T_m]. \tag{3.10} \]

For a proof of Theorem 3.4, we refer to Majda and Bertozzi [9] (Theorem 3.4 in Chapter 3). Notice that that theorem is stated for the classical Navier–Stokes equations.

We now want to prove that there exists a maximal time of existence (independent on \(m\)) of such solution. In particular, if the initial datum is smooth, we get the local existence of a smooth solution of (3.9). We also prove some stability estimates of such solution in Hölder spaces, since they will play a crucial role in the iterative construction.

**Proposition 3.5.** For any \(\nu > 0\) and any \(0 < \alpha < 1\) there exists a constant \(c = c(\alpha) > 0\) with the following property. Given any initial data \(u_0 \in C^\infty\), and \(T \leq c \|u_0\|_{H^{\frac{1}{1+\alpha}}}^{-1}\) there exists a unique solution \(v: \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3\) of (3.9). Moreover, \(v\) obeys the bounds

\[ \|v\|_{N+\alpha} \leq \|u_0\|_{N+\alpha} \]

for all \(N \geq 1\), where the implicit constant depends on \(N\) and \(\alpha > 0\).

**Proof of Proposition 3.5.** We first show that all solutions given by Theorem 3.4 exist in the interval \([0, T]\), for any \(T \leq \|u_0\|_{H^{\frac{1}{1+\alpha}}}^{-1}\). Fix any \(\alpha \in (0, 1)\) and let \(T^*\) be the maximal time such that

\[ T^* \sup_{0 \leq t \leq T^*} \|v(t)\|_{H^{\frac{1}{1+\alpha}}} \leq 1. \]

Suppose \(T^* < c \|u_0\|_{H^{\frac{1}{1+\alpha}}}^{-1}\), for some constant \(c = c(\alpha)\) to be fixed later (we will see that this contradicts the assumption on the maximality of \(T^*\), in particular \(T^* \geq c \|u_0\|_{H^{\frac{1}{1+\alpha}}}^{-1}\)). Using Schauder estimate on \(-\Delta p = tr(\nabla v \nabla v)\), we have
\[ \|p(t)\|_{2+\theta} \leq \|v(t)\|_{1+\theta}^2, \]

thus, differentiating the equation in the \( x \) variable, we get
\[ \left\| (\partial_t + v \cdot \nabla + \nu(-\Delta)^\gamma) Dv \right\|_2 \leq \|v(t)\|_{1+\theta}^2. \]

By Proposition 3.3, for any \( 0 \leq t \leq T^* \), we have
\[ \|v(t)\|_{1+\theta} \leq \|u_0\|_{1+\theta} + \int_0^t \|v(s)\|_{1+\theta}^2 ds. \]

Finally, using Gronwall’s inequality we get the estimate
\[ \|v(t)\|_{1+\theta} \leq \|u_0\|_{1+\theta} < \frac{1}{T^*} \ \forall t \in [0, T^*], \]

where in the last inequality we have chosen the constant \( c = c(x) \) to get it “strict”. Obviously, this contradicts the hypothesis on the maximality of \( T^* \), and also gives a priori estimate (3.11) for \( N = 1 \), which together with (3.10), gives the existence of a smooth solution in the interval \([0, T] \), for any \( T \leq c\|u_0\|_{1+\theta}^{-1} \).

We are left with the higher-order bounds (3.11) for \( N \geq 2 \). For any multi-index \( \theta \) with \( |\theta| = N \), we have
\[ \partial_t \partial^\theta v + v \cdot \nabla \partial^\theta v + \nu(-\Delta)^\gamma \partial^\theta v + [\partial^\theta, v \cdot \nabla] v + \nabla \partial^\theta p = 0. \]

Using again Schauder estimates for the pressure we obtain
\[ \|\nabla \partial^\theta p\|_2 \leq \|\text{tr}(\nabla v \nabla v)\|_{N-1+\theta} \leq \|v\|_{1+\theta} \|v\|_{N+\theta}. \]

Therefore
\[ \left\| (\partial_t + v \cdot \nabla + \nu(-\Delta)^\gamma) \partial^\theta v \right\|_2 \leq \|v\|_{1+\theta} \|v\|_{N+\theta}, \]

and (3.11) follows by applying (3.7) and Grönwall’s inequality.

4. The main inductive proposition and proofs of Theorems 1.3 and 2.1

As already outlined, the main construction is taken from Buckmaster et al. [3], thus we are not going to prove all technical details about the mechanism of the convex integration scheme. However, all the proofs of the propositions involving the structure of the Navier–Stokes equations (different from the Euler ones), are completely self-contained.

4.1. Inductive proposition

First of all, we impose for the moment that
\[ \sup_{t \in [0,1]} |v'(t)| \leq 1 \quad (4.1) \]

(We will see later that this can be done provided that we impose some conditions on the parameters appearing in the iteration).

Let then \( q \geq 0 \) be a natural number. At a given step \( q \), we assume to have a triple \((v_q, p_q, R_q)\) to the fractional Navier–Stokes Reynolds system, namely such that
\[
\begin{aligned}
\partial_t v_q + \text{div}(v_q \otimes v_q) + \nabla p_q + \nu(-\Delta)^\gamma v_q &= \text{div} \, R_q, \\
\text{div } v_q &= 0,
\end{aligned}
\]  
(4.2)

to which we add the constraints
\[
\begin{aligned}
\text{tr } R_q &= 0, \\
\int_{T^3} p_q(x,t) dx &= 0.
\end{aligned}
\]  
(4.3) (4.4)

In (4.2), the viscosity \(\nu\) is just some small constant (in particular \(\nu < 1\)) depending on some parameters of the inductive construction. In what follows, we will see that this coefficient comes from a “technical rescaling” on (1.1).

The size of the approximate solution \(v_q\) and the error \(R_q\) will be measured by a frequency \(\lambda_q\) and an amplitude \(\delta_q\), which are given by
\[
\begin{aligned}
\lambda_q &= 2\pi \lceil a^{(b_n)} \rceil, \\
\delta_q &= \lambda_q^{-2\beta}.
\end{aligned}
\]  
(4.5) (4.6)

where \(\lceil x \rceil\) denotes the smallest integer \(n \geq x\), \(a > 1\) is a large parameter, \(b > 1\) is close to 1 and \(0 < \beta < 1/3\) is the exponent of Theorem 2.1. The parameters \(a\) and \(b\) are then related to \(\beta\).

We proceed by induction, assuming the estimates
\[
\begin{aligned}
\|\hat{R}_q\|_0 &\leq \delta_q^{1/2} \lambda_q^{1/2} \delta_q, \\
\|v_q\|_1 &\leq M \delta_q^{1/2} \lambda_q, \\
\|v_q\|_0 &\leq 1 - \delta_q^{1/2}, \\
\delta_{q+1}\lambda_q^{-\alpha} &\leq e(t) \int_{T^3} |v_q|^2 dx \leq \delta_{q+1}.
\end{aligned}
\]  
(4.7) (4.8) (4.9) (4.10)

where \(0 < \alpha < 1\) is a small parameter to be chosen suitably (which will depend upon \(\beta\)), and \(M\) is a universal constant.

**Proposition 4.1.** There exists a universal constant \(M\) with the following property. Let \(0 < \beta < 1/3\), \(\alpha < \gamma < 1/3\) and
\[
1 < b < \min \left\{ \frac{1 - \beta}{2\beta}, \frac{4}{3} \right\}.
\]  
(4.11)

Then, there exists an \(a_0\) depending only on \(\beta\) and \(b\), such that for any \(0 < \alpha < a_0\) there exists an \(a_0\) depending on \(\beta\), \(a\) and \(M\), such that for any \(a \geq a_0\) the following holds: given a strictly positive function \(e : [0, T] \to \mathbb{R}^+\) satisfying (4.1), and a triple \((v_q, p_q, R_q)\) solving (4.2)–(4.4) and satisfying the estimates (4.7)–(4.10), then there exists a solution \((v_{q+1}, p_{q+1}, R_{q+1})\) to (4.2)–(4.4) satisfying (4.7)–(4.10) with \(q\) replaced by \(q + 1\). Moreover, we have
\[
\|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}} \|v_{q+1} - v_q\|_1 \leq M \delta_{q+1}^{1/2}.
\]  
(4.12)

Furthermore, \(v_{q+1}(\cdot, 0)\) depends only on \(e(0)\) and \(v_q(\cdot, 0)\).
The proof of Proposition 4.1 is summarized in Sections 5.1–5.3, but its details will occupy most of the article. We show next that this proposition immediately implies Theorem 2.1.

4.2. Proof of Theorem 2.1

First of all, we fix any Hölder exponent $\beta < 1/3$ and also the parameters $b$ and $\alpha$, the first satisfying (4.11) and the second smaller than the threshold given in Proposition 4.1. Next, we show that, without loss of generality, we may further assume the energy profile satisfies

$$\inf_t e(t) \geq \delta_1 \hat{\lambda}_0^{-\alpha}, \quad \sup_t e(t) \leq \delta_1, \quad \text{and} \quad \sup_t e'(t) \leq 1,$$  \hspace{1cm} (4.13)

provided the parameter $a$ is chosen sufficiently large.

To see this, we first make the following transformations:

$$\tilde{v}(x,t) := \mu v(x,\mu t) \quad \tilde{p}(x,t) := \mu^2 p(x,\mu t).$$

Thus if we choose

$$\mu = \delta_1^{1/2},$$

the stated problem reduces to finding a solution $(\tilde{v}, \tilde{p})$ of

$$\begin{cases}
\partial_t \tilde{v} + \tilde{v} \cdot \nabla \tilde{v} + \nabla \tilde{p} + \mu (-\Delta)^\beta \tilde{v} = 0, \\
\text{div} \ \tilde{v} = 0
\end{cases}$$

with the energy profile given by

$$\tilde{e}(t) = \mu^2 e(\mu t),$$

for which we have (using our assumptions on the function $e(t)$)

$$\inf_t \tilde{e}(t) \geq \delta_1 \inf_t e(t) \geq \frac{\delta_1}{2}, \quad \sup_t \tilde{e}(t) \leq \delta_1, \quad \text{and} \quad \sup_t \tilde{e}'(t) \leq \delta_1^{3/2} K.$$

If $a$ is chosen sufficiently large, in particular $a \geq a_0 K^{1/3\beta}$, then we can ensure

$$\sup_t \tilde{e}'(t) \leq \delta_1^{3/2} K \leq 1, \quad \text{and} \quad \frac{1}{2} \geq \hat{\lambda}_0^{-\alpha}.$$  

Now, we apply Proposition 4.1 iteratively with $(v_0, R_0, p_0) = (0,0,0)$. Indeed the pair $(v_0, R_0)$ trivially satisfies (4.7)–(4.9), whereas the estimate (4.10) and (4.1) follows as a consequence of (4.13). Notice that by (4.12) $v_q$ converges uniformly to some continuous $\tilde{v}$. Moreover, we recall that the pressure is determined by

$$\Delta p_q = \text{divdiv} \left( -v_q \otimes v_q + \tilde{R}_q \right).$$

and (4.4) and thus $p_q$ is also converging to some pressure $\tilde{p}$ (for the moment only in $L^r$ for every $r < \infty$). Since $\tilde{R}_q \to 0$ uniformly, the pair $(\tilde{v}, \tilde{p})$ solves (4.15).
Observe that using (4.12) we also infer
\[
\sum_{q=0}^{\infty} \| v_{q+1} - v_q \|_{\beta'} \leq \sum_{q=0}^{\infty} \| v_{q+1} - v_q \|_{0}^{-\beta'} \| v_{q+1} - v_q \|_{1}^{\beta'} \leq \sum_{q=0}^{\infty} \delta_{q+1}^{-\beta'} (\delta_{q+1}^{1/2} \lambda_q)^{\beta'} \leq \sum_{q=0}^{\infty} \lambda_q^{\beta'-\beta}
\]
and hence that \( v_q \) is uniformly bounded in \( C^0_t C^\beta_x \) for all \( \beta' < \beta \). Using the last inequality and the definitions of the parameters \( \lambda_q \) we also have that if \( a \) is chosen sufficiently large, then
\[
\| \tilde{v} \|_{\beta'} \leq 1, \quad \forall \beta' < \beta.
\]
Since \( \delta_{q+1} \to 0 \) as \( q \to \infty \), from (4.10), we have
\[
\int_{\mathbb{T}^3} |\tilde{v}|^2 \, dx = \tilde{\varepsilon}(t),
\]
If now we use the transformation
\[
v(x, t) := \mu^{-1} \tilde{v}(x, \mu^{-1} t) \quad \text{and} \quad p(x, t) := \mu^{-2} \tilde{p}(x, \mu^{-1} t),
\]
then it is clear that the pair \((v, p)\) solves (2.1) and it satisfies (2.2) and (2.3). To recover the time regularity we fix a smooth standard mollifier \( \psi \) in space, let \( q \in \mathbb{N} \), and consider \( \tilde{v}_q := v \ast \psi_{2^{-q}} \), where \( \psi_{\ell}(x) = \ell^{-3} \psi(x \ell^{-1}) \). From standard mollification estimates, we have
\[
\| \tilde{v}_q - v \|_0 \leq \| v \|_{\beta'} 2^{-q \beta'}, \tag{4.17}
\]
and thus \( \tilde{v}_q - v \to 0 \) uniformly as \( q \to \infty \). Moreover, \( \tilde{v}_q \) obeys the following equation:
\[
\partial_t \tilde{v}_q + \text{div}(v \otimes v) \ast \psi_{2^{-q}} + \nabla p \ast \psi_{2^{-q}} + (-\Delta)^{\gamma} \tilde{v}_q = 0.
\]
Next, since
\[
-\Delta p \ast \psi_{2^{-q}} = \text{div} \, \text{div}(v \otimes v) \ast \psi_{2^{-q}},
\]
using Schauder’s estimates, for any fixed \( \varepsilon > 0 \), we get
\[
\| \nabla p \ast \psi_{2^{-q}} \|_0 \leq \| \nabla p \ast \psi_{2^{-q}} \|_\varepsilon \| v \|_{\beta'} 2^{q(1+\varepsilon-\beta)},
\]
(where the constant in the estimate depends on \( \varepsilon \) but not on \( q \)). Similarly,
\[
\| (v \otimes v) \ast \psi_{2^{-q}} \|_1 \leq \| v \otimes v \|_{\beta'} 2^{q(1-\beta)} \leq \| v \|_{\beta'}^2 2^{q(1-\beta)}
\]
\[
\| (-\Delta)^{\gamma} \tilde{v}_q \|_0 \leq \| \tilde{v}_q \|_1 \leq \| v \|_{\beta'} 2^{q(1-\beta')}.\]
Thus the above estimates yield
\[
\| \partial_t \tilde{v}_q \|_0 \leq \| v \|_{\beta'}^2 2^{q(1+\varepsilon-\beta')}. \tag{4.18}
\]
\[\text{Throughout the manuscript, we use the notation } x \lesssim y \text{ to denote } x \leq Cy, \text{ for a sufficiently large constant } C > 0, \text{ which is independent of } a, b, \text{ and } q, \text{ but may change from line to line.}\]
Next, for \( \beta'' < \beta' \), we conclude from (4.17) and (4.18) that
\[
\| \tilde{v}_q - \tilde{v}_{q+1}\|_{C^0_t C^{\beta''}_x} \leq \left( \| \tilde{v}_q - v\|_0 + \| \tilde{v}_{q+1} - v\|_0 \right) \lambda \| \partial_t \tilde{v}_q\|_0 + \| \partial_t \tilde{v}_{q+1}\|_0 \leq \lambda \left( \| v\|_1^{\beta'} - 2 \| v\|_0 \right) \lambda \left( \| v\|_1^{\beta'} - 2 \| v\|_0 \right)
\]
\[
\leq \| v\|_1^{\beta'} 2 - q(\beta' - (1 + \delta b't))
\]
\[
\leq \| v\|_1^{\beta'} 2 - q\epsilon
\]

Here, we have chosen \( \epsilon > 0 \) sufficiently small (in terms of \( \beta' \) and \( \beta'' \)) so that that \( \beta' - (1 + \delta b't) \geq \epsilon \). Thus, the series
\[
v = \tilde{v}_0 + \sum_{q \geq 0} (\tilde{v}_{q+1} - \tilde{v}_q)
\]
converges in \( C^0_t C^{\beta''}_x \). Since we already know \( v \in C^0_t C^{\beta'}_x \), we obtain that \( v \in C^{\beta''}[0, 1] \times T^3 \) as desired, with \( \beta'' < \beta' < \beta < 1/3 \) arbitrary.

This concludes the proof of the theorem.

### 4.3. Proof of Theorem 1.3

Let \( v \in C^0_t C^{\beta'}_x \) be a dissipative solution of Euler, with the kinetic energy profile satisfying the assumptions (i)–(v) in the proof of Theorem 1.2 (note that the proof of the existence of such solution is given by Buckmaster et al. [3]). Using the rescaling (4.14), with \( \mu := (2|v|_0)^{-1} \), we can assume that \( |v|_0 \leq 1/2 \).

We fix two positive kernels (Friedrichs mollifiers) \( \varphi \) and \( \psi \), respectively in space and time. Let \( \delta_n := a^{-b'n+2} \) and \( \nu_n := \delta_n^{1/\beta} \). Since \( v \) solves Euler, the smooth function \( v_n := (v * \varphi_{\delta_n}) * \psi_{\delta_n} \) solves the following Navier–Stokes Reynolds equations
\[
\partial_t v_n + \text{div}(v_n \otimes v_n) + \nabla p_n + \nu_n (-\Delta)^{\gamma} v_n = \text{div} \tilde{R}_n,
\]
with
\[
\tilde{R}_n = v_n \otimes v_n - (v \otimes v)_n + \nu_n \mathcal{R}(-\Delta)^{\gamma} v_n
\]
where \( f \otimes g \) is the traceless part of the matrix \( f \otimes g \) and \( \mathcal{R} \) is the operator defined in (5.38). We also define the energy as
\[
e_n(t) := \int_{T^3} |v_n|^2 dx + \delta_n^{1/\beta - 1/2}.
\]
(4.19)

Using standard mollification estimates and Proposition A.2, we have
\[
|v_n|_1 \leq \delta_n^{\beta' - 1},
\]
\[
\| \tilde{R}_n\|_0 \leq \delta_n^{2\beta'} + \nu_n |v_n|_1 \leq \delta_n^{2\beta'}.
\]

Thus, if we chose \( \gamma < \beta < \beta' \) and the parameter \( a \) large enough, we can guarantee that (4.7)–(4.10) hold for \( q = n \), provided that \( b \) is sufficiently near 1 and \( \alpha \) is small.

We can now apply Proposition 4.1 (inductively for \( q \geq n \)) in order to obtain a solution \( v^{(n)} \) of (1.4), and since \( \gamma < \beta \) (as already done in the proof of Theorem 1.2) we can guarantee that \( v^{(n)} \) is indeed a Leray–Hopf weak solution.
Moreover, by (4.12), we have
\[
\|v^{(\nu_n)} - v_{n}\|_{\rho''} \leq \sum_{q \geq n} \|v_{q+1} - v_{q}\|_{\rho''} \approx \sum_{q \geq n} a^{(\beta'' - \beta) q+1}.
\]

Thus, provided that the parameter \(a\) is chosen even larger, we can ensure that
\[
\|v^{(\nu_n)} - v\|_{\rho''} \leq \|v^{(\nu_n)} - v_{n}\|_{\rho''} + \|v_{n} - v\|_{\rho''} \leq \frac{1}{n}, \quad \forall \beta'' < \beta,
\]
and this concludes the proof of the theorem. We also remark that \(e_n(t) \to \int_{\mathbb{R}^d} |v|^2 \, dx\) as \(n \to +\infty\).

### 5. The convex integration scheme and proof of the iterative proposition

The rest of the paper is devoted to the proof of Proposition 4.1. To simplify several estimates we will assume that \(\alpha\) is small enough so to have
\[
\lambda^3 \alpha \leq \left(\frac{\delta_q}{\delta_{q+1}}\right)^{3/2} \leq \frac{\lambda_{q+1}}{\lambda_q},
\]
in which we also need that \(a\) is big enough to nullify any constant from the ratio \(\lambda q / a^{(b)}\), which can be easily bounded as
\[
2\pi \leq \frac{\lambda_q}{a^{b}} \leq 4\pi.
\]

Following the construction of Buckmaster et al. [3], we subdivide the proof in three stages, in each of which we modify \(v_q\): mollification, gluing and perturbation.

#### 5.1. Mollification step

The first stage is mollification: we mollify \(v_q\) (in space) at length scale
\[
\ell := \frac{\delta_{q+1}^{1/2}}{\delta_q^{1/2}} 2\lambda_q^{1+3\alpha/2}.
\]

Fix a standard mollification kernel \(\psi\), we define
\[
v_\ell := v_q * \psi_\ell
\]
\[
\hat{R}_\ell := \hat{R}_{\ell} * \psi_\ell - \left(v_q \otimes v_q\right) * \psi_\ell + v_\ell \otimes v_\ell.
\]

These functions obey the equation
\[
\begin{cases}
\partial_t v_\ell + \text{div}(v_\ell \otimes v_\ell) + \nabla p_\ell + \nu(-\Delta)^7 v_\ell = \text{div} \hat{R}_\ell \\
\text{div} v_\ell = 0,
\end{cases}
\]
in view of (4.2).
Observe, again choosing $\alpha$ sufficiently small and $a$ sufficiently large we can assume
\[ \lambda_q^{-3/2} \leq \ell \leq \lambda_q^{-1}, \tag{5.5} \]
which will be used in order to simplify several estimates.

From standard mollification estimates, we obtain the following bounds\(^4\) (we refer to Buckmaster et al. [3] for a detailed proof).

**Proposition 5.1.**
\begin{align*}
\|v_t - v_q\|_0 &\leq \delta_q^{1/2} \lambda_q^{-\alpha}, & (5.6) \\
\|v_t\|_{N+1} &\leq \delta_q^{1/2} \lambda_q \ell^{-N} \quad \forall N \geq 0, & (5.7) \\
\|\hat{R}_t\|_{N+2} &\leq \delta_q^{1/2} \lambda_q \ell^{-N+2} \quad \forall N \geq 0, & (5.8) \\
\left|\int_{\mathbb{T}^3} |v_q|^2 - |v_t|^2 \, dx\right| &\leq \delta_q^{1/2} \lambda_q \ell^\alpha. & (5.9)
\end{align*}

**5.2. Gluing step**

In the second stage, we glue together exact solutions to the fractional Navier–Stokes equations in order to produce a new $\tilde{v}_q$, close to $v_q$, whose associated Reynolds stress error has support in pairwise disjoint temporal regions of length $\tau_q$ in time, where
\[ \tau_q = \frac{\ell^{2\alpha}}{\delta_q^{1/2} \lambda_q}. \tag{5.10} \]

Note that we have the CFL-like condition
\[ 2\tau_q \|v_t\|_{1+\alpha} \overset{(5.7)}{\leq} \tau_q \delta_q^{1/2} \lambda_q \ell^{-\alpha} \lesssim \ell^\alpha \ll 1 \tag{5.11} \]
as long as $a$ is sufficiently large.

More precisely, we aim to construct a new triple $(\tilde{v}_q, \tilde{R}_q, \tilde{p}_q)$ solving the Navier–Stokes Reynolds (4.2) such that the temporal support of $\tilde{R}_q$ is contained in pairwise disjoint intervals $I_i$ of length $\sim \tau_q$ and such that the gaps between neighboring intervals is also of length $\sim \tau_q$.

For each $i$, let $t_i = i\tau_q$, and consider smooth solutions of the fractional Navier–Stokes equations
\[ \begin{aligned}
\partial_t v_i + \text{div}(v_i \otimes v_i) + \text{grad} \nabla p_i + \nu(-\Delta)^\alpha v_i &= 0 \\
\text{div } v_i &= 0 \\
v_i(\cdot, t_i) &= v_i(\cdot, t_i)
\end{aligned} \tag{5.12} \]
defined over their own maximal interval of existence. An immediate consequence of (5.7), (5.10) and Proposition 3.5 is the following.

---

\(^4\)In the following, when considering higher order norms $\| \cdot \|_N$ or $\| \cdot \|_{N+1}$, the symbol $\lesssim$ will imply that the constant in the inequality might also depend on $N$. 
Corollary 5.2. If $a$ is sufficiently large, for $0 \leq (t-t_i) \leq 2\tau_q$, we have
\[ \|\nu_i\|_{N+2} \leq \delta_q^{1/2} \lambda_q \ell^{1-N/2} \leq \tau_q^{-1} \ell^{1-N/2} \] for any $N \geq 1$. \hspace{1cm} (5.13)

We will now show that for $0 \leq (t-t_i) \leq 2\tau_q$, $\nu_i$ is close to $\nu_\ell$ and by the identity
\[ \nu_i - \nu_{i+1} = (\nu_i - \nu_\ell) - (\nu_{i+1} - \nu_\ell), \]
the vector field $\nu_i$ is also close to $\nu_{i+1}$.

Proposition 5.3. (Stability and estimates on $\nu_i - \nu_\ell$). For $0 \leq (t-t_i) \leq 2\tau_q$, $N \geq 0$ and $0 < \nu < 1$ we have
\[ \|\nu_i - \nu_\ell\|_{N+2} \leq \tau_q \delta_{q+1} \ell^{1-N/2}, \] (5.14)
\[ \|\nabla(p_\ell - p_i)\|_{N+2} \leq \delta_{q+1} \ell^{1-N/2}, \] (5.15)
\[ \|L_{t,\ell,;}(\nu_i - \nu_\ell)\|_{N+2} \leq \delta_{q+1} \ell^{1-N/2}, \] (5.16)
\[ \|D_{t,\ell}(\nu_i - \nu_\ell)\|_{N+2} \leq \delta_{q+1} \ell^{1-N/2}, \] (5.17)

where we write
\[ D_{t,\ell} = \partial_t + \nu_\ell \cdot \nabla \quad L_{t,\ell,;} = D_{t,\ell} + \nu (-\Delta)^\gamma. \] (5.18)

Proof. Let us first consider (5.14) with $N= 0$. From (5.4) and (5.12), we have
\[ L_{t,\ell,;}(\nu_i - \nu_\ell) = (\nu_i - \nu_\ell) \cdot \nabla \nu_i - \nabla (p_\ell - p_i) + \text{div} \, \hat{R}_\ell. \] (5.19)

In particular, using
\[ \Delta(p_\ell - p_i) = \text{div}(\nabla \nu_i (\nu_i - \nu_\ell)) + \text{div}(\nabla \nu_i (\nu_i - \nu_\ell)) + \text{div} \, \text{div} \hat{R}_\ell, \] (5.20)

estimates (5.8) and (5.13), and Proposition C.1 (recall that $\partial_t \partial_\ell (-\Delta)^{-1}$ is given by $1/3\delta_{ij} + \text{a Calderón–Zygmund operator}$), we conclude
\[ \|\nabla(p_\ell - p_i)(\cdot, t)\|_2 \leq \delta_q^{1/2} \lambda_q \ell^{-3} \|\nu_i - \nu_\ell\|_2 + \delta_{q+1} \ell^{-1+\gamma}. \]

Thus, using (5.8) and the definition of $\tau_q$, we have
\[ \|L_{t,\ell,;}(\nu_i - \nu_\ell)\|_2 \leq \delta_{q+1} \ell^{-1+\gamma} + \tau_q^{-1} \|\nu_i - \nu_\ell\|_2. \] (5.21)

By applying (3.7) we obtain
\[ \|\nu_i - \nu_\ell\|_2 \|\nu_i - \nu_\ell\|_2 \leq |t - t_i| \delta_{q+1} \ell^{-1+\gamma} + \int_{t_i}^t \tau_q^{-1} \|\nu_i - \nu_\ell\|_2 \, ds. \]

Applying Grönwall’s inequality and using the assumption $0 \leq (t-t_i) \leq 2\tau_q$, we obtain
\[ \|\nu_i - \nu_\ell\|_2 \leq \tau_q \delta_{q+1} \ell^{-1+\gamma}, \] (5.22)
i.e. (5.14) for the case $N=0$. Then as a consequence of (5.21), we obtain (5.16) for $N= 0$.

Next, consider the case $N \geq 1$ and let $\theta$ be a multi index with $|\theta| = N$. Commuting the derivative $\partial^\theta$ with the material derivative $\partial_t + \nu_\ell \cdot \nabla$, we have
where we have used \((5.22)\). Furthermore, from \((5.20)\) we also obtain, using Corollary \((5.17)\),
\[
\text{we deduce with \((5.14)\). From \((5.24)\) and \((5.23)\) we then also conclude \((5.15)\) and \((5.16)\). We are only left }
\[
\text{where}
\]
\[
\text{is the Biot–Savart operator, so that }
\]
\[
\text{–}
\]
\[
\text{is chosen sufficiently large we can ensure }
\]
\[
\text{is the assumption } 0 \leq (t-t_i) \leq 2\tau_q \text{ we obtain }
\]
\[
\text{From \((5.24)\) and \((5.23)\) we then also conclude \((5.15)\) and \((5.16)\). We are only left with }
\]
\[
\text{By Theorem B.1 and estimate \((5.14)\) we have }
\]
\[
\text{If } a \text{ is chosen sufficiently large we can ensure } \ell^{-1} \leq \lambda_{q+1} \text{ and, using \((4.11)\), we get }
\]
\[
\text{from which we deduce }
\]
\[
\text{Finally, combining \((5.16)\), \((5.25)\) and triangular inequality, we get \((5.17)\). Make the vector potentials to the solutions } v_i 
\]
\[
\text{Define the vector potentials to the solutions } v_i \text{ as }
\]
\[
\text{where } B \text{ is the Biot–Savart operator, so that }
\]
\[\text{Div } z_i = 0 \quad \text{and} \quad \text{Curl } z_i = v_i - \int_{\mathcal{T}_i} v_i. \quad (5.27)\]

Our aim is to obtain estimates for the differences \(z_i - z_{i+1}\).

**Proposition 5.4.** (Estimates on vector potentials). For \(0 \leq (t-t_i) \leq 2\tau_q\), we have that

\[
\|z_i - z_{i+1}\|_{N+2} \leq \tau_q \delta_{q+1}^{\epsilon-N+\alpha}, \quad (5.28)
\]

\[
\|D_t,\ell (z_i - z_{i+1})\|_{N+2} \leq \delta_{q+1}^{\epsilon-N+\alpha}. \quad (5.29)
\]

**Proof.** Set \(\tilde{z}_i := B(v_i - v_\ell)\) and observe that \(z_i - z_{i+1} = \tilde{z}_i - \tilde{z}_{i+1}\). Hence, it suffices to estimate \(\tilde{z}_i\) in place of \(z_i - z_{i+1}\).

The estimate on \(\|\nabla \tilde{z}_i\|_{N-1+2}\) for \(N \geq 1\) follows directly from (5.14) and the fact that \(\nabla B\) is a bounded operator on Hölder spaces:

\[
\|\nabla \tilde{z}_i\|_{N-1+2} \leq \|\nabla B(v_i - v_\ell)\|_{N-1+2}\|v_i - v_\ell\|_{N+2} \leq \tau_q \delta_{q+1}^{\epsilon-N+\alpha}. \quad (5.30)
\]

Next, observe that

\[
\partial_t(v_i - v_\ell) + v_\ell \cdot \nabla (v_i - v_\ell) + (v_i - v_\ell) \cdot \nabla v_i + \nabla (p_i - p_\ell) + v_\ell (-\Delta) (v_i - v_\ell) + \text{div } \hat{R}_\ell = 0. \quad (5.31)
\]

Since \(v_i - v_\ell = \text{curl} \tilde{z}_i\) with \(\text{div} \tilde{z}_i = 0\), we have\(^5\)

\[
\nabla (v_i - v_\ell) = \text{curl}((v_i \cdot \nabla) \tilde{z}_i) + \text{div}((\tilde{z}_i \times \nabla)v_i) \\
(v_i - v_\ell) \cdot \nabla \nabla = \text{div}((\tilde{z}_i \times \nabla)v_i^T),
\]

so that we can write (5.31) as

\[
\text{curl}(\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i + v_\ell (-\Delta) \tilde{z}_i) = -\text{div}((\tilde{z}_i \times \nabla)v_i + (\tilde{z}_i \times \nabla)v_i^T) - \nabla (p_i - p_\ell) - \text{div } \hat{R}_\ell. \quad (5.32)
\]

Taking the curl of (5.32) the pressure term drops out. Using in addition that \(\text{div} \tilde{z}_i = \text{div} (v_i - v_\ell) = 0\) and the identity \(\text{curl} \text{curl} = -\Delta + \nabla \text{div}\), we then arrive at

\[-\Delta (\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i + v_\ell (-\Delta) \tilde{z}_i) = F,\]

where \(F = -\nabla \text{div}((\tilde{z}_i \cdot \nabla)v_i) - \text{curl} \text{div}((\tilde{z}_i \times \nabla)v_i + (\tilde{z}_i \times \nabla)v_i^T) - \text{curl} \text{div } \hat{R}_\ell.\)^6

Consequently, \begin{equation}
\begin{aligned}
\|\partial_t \tilde{z}_i + (v_\ell \cdot \nabla) \tilde{z}_i + v_\ell (-\Delta) \tilde{z}_i\|_{N+2} &\leq \left(\|v_i\|_{N+1+2} + \|v_\ell\|_{N+1+2}\right)\|\tilde{z}_i\|_x \\
&\quad + \left(\|v_i\|_{1+2} + \|v_\ell\|_{1+2}\right)\|\tilde{z}_i\|_{N+2} + \|\hat{R}_\ell\|_{N+2} \\
&\leq \tau_q^{-1}\|\tilde{z}_i\|_{N+2} + \tau_q^{-1}\|\tilde{z}_i\|_x + \delta_{q+1}^{\epsilon-N+\alpha}. \quad (5.33)
\end{aligned}
\end{equation}

Setting \(N=0\) and using (3.7) and Grönwall’s inequality, we obtain

\[
\|\tilde{z}_i\|_x \leq \tau_q \delta_{q+1}^{\epsilon-N+\alpha},
\]

\(^5\)Here, we use the notation \([z \times \nabla]v^\alpha = \epsilon_{\alpha i j} \partial_j v^\alpha\) for vector fields \(z, v\).\n\(^6\)In deriving the latter equality, we have used the identity \(\nabla \text{div}((v_i \cdot \nabla) \tilde{z}_i) = \nabla \text{div}((\tilde{z}_i \cdot \nabla)v_i)\), which follows easily from the fact that both \(v_i\) and \(\tilde{z}_i\) are divergence free.
which together with (5.30) gives (5.28). Using (5.28) into (5.33), we get
\[ \| \partial_t \tilde{z}_i + (v_i \cdot \nabla) \tilde{z}_i + \nu(-\Delta) \tilde{z}_i \|_{N+\alpha} \leq \delta_{q+1} \ell^{-N+\alpha}. \]

Thus we conclude
\[ \| \partial_t \tilde{z}_i + (v_i \cdot \nabla) \tilde{z}_i \|_{N+\alpha} \leq \delta_{q+1} \ell^{-N+\alpha} + \| (-\Delta) \tilde{z}_i \|_{N+\alpha} \leq \delta_{q+1} \ell^{-N+\alpha} + \| \tilde{z}_i \|_{N+2\gamma+2\alpha} \]
\[ \leq \delta_{q+1} \ell^{-N+\alpha} \left( 1 + \tau_q \ell^{-2\gamma-2\alpha} \right) \leq \delta_{q+1} \ell^{-N+\alpha}. \]

Proceeding as by Buckmaster et al. [3], we now glue the solutions \( v_i \) together in order to construct \( \tilde{v}_q \). Let
\[ t_i = i\tau_q, \quad I_i = \left[ t_{i+1} + \frac{1}{3} \tau_q, t_{i+1} + \frac{2}{3} \tau_q \right] \cap [0, T], \]
\[ J_0 = \left[ 0, t_1 + \frac{1}{3} \tau_q \right], \quad J_i = \left( t_{i+1} - \frac{1}{3} \tau_q, t_{i+1} + \frac{1}{3} \tau_q \right) \cap [0, T] \quad i \geq 1. \]

Note that \( \{I_i, J_i\} \) is a decomposition of \([0, T]\) into pairwise disjoint intervals. Note also that this definition of \( J_0, I_i \) is slightly different from the one used by Buckmaster et al. [3]. The reason is that our stability estimates for smooth solutions of the fractional Navier–Stokes equations hold for \( 0 \leq t - t_i \leq \tau_q \) as opposed to \( |t - t_i| \leq \tau_q \) by Buckmaster et al. [3].

We define a partition of unity \( \{ \chi_i \}_i \) in time with the following properties:

- The cut-offs form a partition of unity
\[ \sum_i \chi_i \equiv 1 \quad (5.34) \]
- \( \text{supp} \, \chi_i \cap \text{supp} \, \chi_{i+2} = \emptyset \) and moreover
\[ \text{supp} \, \chi_0 \subset [0, t_1 + 2/3 \tau_q] \quad (5.35) \]
\[ \text{supp} \, \chi_i \subset I_{i-1} \cup J_i \cup I_i, \]
\[ \chi_i(t) = 1 \quad \text{for} \ t \in J_i \quad (5.36) \]
- For any \( i \) and \( N \), we have
\[ \| \partial_t^N \chi_i \|_0 \leq \tau_q^{-N}. \quad (5.37) \]

We define
\[ \tilde{v}_q = \sum_i \chi_i v_i \]
\[ \tilde{P}_q^{(1)} = \sum_i \chi_i P_i \]

Observe that \( \text{div} \, \tilde{v}_q = 0 \). Furthermore, if \( t \in I_i \), then \( \chi_i + \chi_{i+1} = 1 \) and \( \chi_j = 0 \) for \( j \neq i, i+1 \), therefore on \( I_i \):
\[ \tilde{v}_q = \chi_i v_i + (1 - \chi_i) v_{i+1} \]
\[ \tilde{P}_q^{(1)} = \chi_i P_i + (1 - \chi_i) P_{i+1} \]
\[ \partial_t \bar{\nu}_q + \text{div}(\bar{\nu}_q \otimes \bar{\nu}_q) + \nabla \bar{p}_q^{(1)} + \nu(-\Delta) \bar{\nu}_q = \partial_i \chi_i (v_i - v_{i+1}) - \chi_i (1 - \chi_i) \text{div}((v_i - v_{i+1}) \otimes (v_i - v_{i+1})). \]

On the other hand, if \( t \in I_i \) then \( \chi_i = 1 \) and \( \chi_j(\bar{t}) = 0 \) for all \( j \neq i \) for all \( \bar{t} \) sufficiently close to \( t \) (since \( I_i \) is open). Then for all \( t \in I_i \) we have
\[ \bar{\nu}_q = v_i, \quad \bar{p}_q^{(1)} = p_i, \]
and, from (5.12),
\[ \partial_t \bar{\nu}_q + \text{div}(\bar{\nu}_q \otimes \bar{\nu}_q) + \nabla \bar{p}_q^{(1)} + (-\Delta) \bar{\nu}_q = 0. \]

In order to define the new Reynolds tensor, we recall the operator \( \mathcal{R} \) from De Lellis and Székelyhidi [10], which can be thought of as an “inverse divergence” operator for symmetric tracefree 2-tensors. The operator is defined as
\[
(Rf)^{ij} = R^{ijk} f^k = -\frac{1}{2} \Delta^{-2} \partial_i \partial_j \partial_k - \frac{1}{2} \Delta^{-1} \partial_i \partial_j \partial_k + \Delta^{-1} \partial_i \partial_j \partial_k.
\]
when acting on vectors \( f \) with zero mean on \( T^3 \) and has the property that \( Rf \) is symmetric and \( \text{div}(Rf) = f \).

Thus, we define
\[
\bar{\mathcal{R}}_q = \partial_i \chi_i \mathcal{R}(v_i - v_{i+1}) - \chi_i (1 - \chi_i) (v_i - v_{i+1}) \otimes (v_i - v_{i+1})
\]
\[
\bar{p}_q^{(2)} = -\chi_i (1 - \chi_i) \left( |v_i - v_{i+1}|^2 - \int_{T^3} |v_i - v_{i+1}|^2 \, dx \right),
\]
for \( t \in I_i \) and \( \bar{\mathcal{R}}_q = 0, \bar{p}_q^{(2)} = 0 \) for \( t \not\in \cup_i I_i \).

Furthermore, we set
\[ \bar{p}_q = \bar{p}_q^{(1)} + \bar{p}_q^{(2)} \]

It follows from the preceding discussion and the definition of the operator \( \mathcal{R} \) that

- \( \bar{\mathcal{R}}_q \) is a smooth symmetric and traceless 2-tensor;
- For all \((x, t) \in T^3 \times [0, T] \)
\[
\begin{cases}
\partial_t \bar{\nu}_q + \text{div}(\bar{\nu}_q \otimes \bar{\nu}_q) + \nabla \bar{p}_q + \nu(-\Delta) \bar{\nu}_q = \text{div} \bar{\mathcal{R}}_q, \\
\text{div} \bar{\nu}_q = 0;
\end{cases}
\]
- \( \text{supp} \bar{\mathcal{R}}_q \subset T^3 \times \cup_i I_i \).

Next, we estimate the various Hölder norms of \( \bar{\nu}_q \) and \( \bar{\mathcal{R}}_q \).

**Proposition 5.5.** (Estimates on \( \bar{\nu}_q \) and \( \bar{\mathcal{R}}_q \).) The velocity field \( \bar{\nu}_q \) and the new Reynolds stress tensor \( \bar{\mathcal{R}}_q \) satisfy the following estimates
\[
\| \bar{\nu}_q - v_t \|_2 \lesssim \delta_{q+1}^{1/2} \ell^2,
\]
(5.39)
\[ \| \bar{v}_q - v \|_{N+\varepsilon} \leq \tau_q \delta_{q+1} \ell^{-1-N+\varepsilon}, \]  
\[ \| \bar{v}_q \|_{1+N} \leq \delta_{q+1} \lambda_q \ell^{-N}, \]  
\[ \| \bar{R}_q \|_{N+\varepsilon} \leq \delta_{q+1} \lambda_q \ell^{-N+\varepsilon}, \]  
\[ \| (\partial_t + \bar{v}_q \cdot \nabla) \bar{R}_q \|_{N+\varepsilon} \leq \delta_{q+1} \ell^{1/2} \lambda_q \ell^{-N-\varepsilon}. \]  

for all \( N \geq 0 \). Moreover, the difference of the energies of \( \bar{v}_q \) and \( v \) satisfies
\[
\left| \int_{\mathbb{T}^3} |\bar{v}_q|^2 - |v|^2 \right| \, dx \leq \delta_{q+1} \ell^\alpha. \tag{5.44}
\]

**Proof.** The estimates (5.39)–(5.43) are consequence of Propositions 5.3 and 5.26 (the proof can be found by Buckmaster et al. [3]). However, we prove explicitly (5.44) since it involves the structure of the dissipative term.

Observe that for \( t \in I_i \)
\[ \bar{v}_q \otimes \bar{v}_q = (\chi_i v + (1-\chi_i) v_{i+1}) \otimes (\chi_i v + (1-\chi_i) v_{i+1}) \]
\[ = \chi_i v_i \otimes v_i + (1-\chi_i) v_{i+1} \otimes v_i + \chi_i (1-\chi_i) (v_i - v_{i+1}) \otimes (v_i - v_{i+1}), \]
so that, taking the trace:
\[ |\bar{v}_q|^2 - |v|^2 = \chi_i (|v_i|^2 - |v_i|^2) + (1-\chi_i) (|v_{i+1}|^2 - |v_{i+1}|^2) - \chi_i (1-\chi_i) |v_i - v_{i+1}|^2. \]

Next, recall that \( v_i \) and \( v \) are smooth solutions of (5.12) and (5.4) respectively, therefore
\[ \left| \frac{d}{dt} \int_{\mathbb{T}^3} |v_i|^2 - |v|^2 \, dx \right| = 2 \int_{\mathbb{T}^3} \nabla v : \hat{R}_i \, dx + 2 \nu \int_{\mathbb{T}^3} \left( \left| (-\Delta)^{\gamma/2} v_i \right|^2 - \left| (-\Delta)^{\gamma/2} v \right|^2 \right) \, dx. \]

Using (5.8) and (5.13), we estimate
\[ \left| \int_{\mathbb{T}^3} \nabla v : \hat{R}_i \, dx \right| \leq \| \nabla v \|_2 \| \hat{R}_i \|_0 \leq \delta_{q+1} \lambda_q \delta_{q+1} \ell^{-1-\varepsilon}. \]

Moreover, since \( \| v_q \|_\gamma \leq 1 \) for every \( \gamma < \beta \) (as already exploited in the proof of Proposition 4.1), by (5.14), Theorem B.1 and Cauchy–Schwarz inequality, we have
\[ \left| \int_{\mathbb{T}^3} \left( \left| (-\Delta)^{\gamma/2} v_i \right|^2 - \left| (-\Delta)^{\gamma/2} v \right|^2 \right) \, dx \right| \leq \| v_i - v \|_{\gamma+\varepsilon} \leq \tau_q \delta_{q+1} \ell^{-1-\gamma} \leq \tau_q \delta_{q+1} \ell^\alpha, \]
where in the last inequality (remember the restriction \( \gamma < 1/3 \)), we have used
\[ \ell^{-1-\gamma} \leq \ell^{-4/3} (5.3) \leq \left( \frac{\delta_{q+1} \lambda_q}{\delta_{q+1}^2} \right)^{4/3} \lambda_q^{2\alpha} (5.10) \leq \frac{\tau_q}{\delta_{q+1}^2} \ell^2 \delta_{q+1}^2 \lambda_q (\tau_q \delta_{q+1} \ell^{-1-\gamma})^{2/3} \leq \tau_q \delta_{q+1} \ell^\alpha. \]

Moreover, \( v_i = v \) for \( t = t_i \). Therefore, after integrating in time we deduce
\[ \left| \int_{\mathbb{T}^3} |v_i|^2 - |v_t|^2 \, dx \right| \leq \delta_{q+1} \ell^\alpha. \]
Furthermore, using (5.14) and
\[ d_{q+1} = 2q \alpha_1 \leq \frac{2}{\lambda_{q+1}^2} \leq 1, \]

\[ \int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx \leq \|v_i - v_{i+1}\|_{L^2}^2 \leq \lambda_{q+1}^2 e_{q+1}^2 + 2^x \leq \delta_{q+1} e_{q+1}^2, \]

Therefore,

\[ \left| \int |\overline{v}_q|^2 - |\overline{v}_\ell|^2 \, dx \right| \leq \delta_{q+1} e_{q+1}^2, \]

which concludes the proof.

\[ \square \]

### 5.3. Perturbation and Mikado flows

We will now outline the construction of the perturbation \( w_{q+1} \), where

\[ v_{q+1} := w_{q+1} + \overline{v}_q. \]

The perturbation \( w_{q+1} \) is highly oscillatory and will be based on the Mikado flows introduced by Daneri and Székelyhidi [11].

First of all note that as a corollary of (4.10), (5.9) and (5.44), by choosing \( a \) sufficiently large we can ensure that

\[ \frac{\delta_{q+1}^2}{2\lambda_{q}^2} \leq e(t) - \int_{\mathbb{T}^3} |\overline{v}_q|^2 \, dx \leq 2\delta_{q+1}. \]  

(5.45)

Starting with the solution \( (\overline{v}_q, \overline{P}_q, \overline{R}_q) \), we then produce a new solution \( (v_{q+1}, p_{q+1}, R_{q+1}) \) of the Navier–Stokes Reynolds system (4.2) with estimates

\[ \|v_{q+1} - v_q\|_0 + \lambda_{q+1}^{-1} \|v_{q+1} - v_q\|_1 \leq \frac{M}{2} \delta_{q+1}^{1/2}, \]

(5.46)

\[ \|R_{q+1}\|_2 \leq \frac{\delta_{q+1}^2 \lambda_{q+1}^{1/2}}{\lambda_{q+1}^2}. \]

(5.47)

\[ \left| e(t) - \int_{\mathbb{T}^3} |v_{q+1}|^2 \, dx - \frac{\delta_{q+1}^{3/2} \lambda_{q+1}^{1/2} e_{q+1}^2}{\lambda_{q+1}^2} \right| \leq \frac{\delta_{q+1}^{1/2} \lambda_{q+1}^{-1} e_{q+1}^{2x}}{\lambda_{q+1}^2}. \]

(5.48)

cf. Propositions 5.11–5.13.

Then Proposition 4.1 is just a consequence of estimates (5.46)–(5.48), Propositions 5.1 and 5.5 (again, a detailed proof can be found by Buckmaster et al. [3]).

We now recall the construction of Mikado flows given by Daneri and Székelyhidi [11].

**Lemma 5.6.** For any compact subset \( \mathcal{N} \subset \mathbb{S}^3 \) there exists a smooth vector field

\[ W : \mathcal{N} \times \mathbb{T}^3 \rightarrow \mathbb{R}^3, \]

such that, for every \( R \in \mathcal{N} \)

\[ \begin{cases} \div \overline{z}(W(R, \xi) \otimes W(R, \xi)) = 0 \\ \div \overline{z}W(R, \xi) = 0, \end{cases} \]

(5.49)
and

\[ \int_{\mathbb{T}^3} W(R, \xi) d\xi = 0, \quad (5.50) \]

\[ \int_{\mathbb{T}^3} W(R, \xi) \otimes W(R, \xi) d\xi = R. \quad (5.51) \]

Using the fact that \( W(R, \xi) \) is \( \mathbb{T}^3 \)-periodic and has zero mean in \( \xi \), we write

\[ W(R, \xi) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} a_k(R) e^{ik \cdot \xi}, \quad (5.52) \]

for some smooth functions \( R \rightarrow a_k(R) \in \mathbb{C}^3 \), satisfying \( a_k(R) \cdot k = 0 \). From the smoothness of \( W \), we further infer

\[ \sup_{R \in \mathcal{N}} |D_R^N a_k(R)| \leq \frac{C(N, N, m)}{|k|^m}, \quad (5.53) \]

for some constant \( C \), which depends, as highlighted in the statement, on \( N, N \) and \( m \).

**Remark 5.7.** Later in the proof, the estimates (5.53) will be used with a specific choice of the compact set \( \mathcal{N} \) and of the integers \( N \) and \( m \): this specific choice will then determine the universal constant \( M \) appearing in Proposition 4.1.

Using the Fourier representation we see that from (5.51)

\[ W(R, \xi) \otimes W(R, \xi) = R + \sum_{k \neq 0} C_k(R) e^{ik \cdot \xi}, \quad (5.54) \]

where

\[ C_k k = 0 \quad \text{and} \quad \sup_{R \in \mathcal{N}} |D_R^N C_k(R)| \leq \frac{C(N, N, m)}{|k|^m}, \quad (5.55) \]

for any \( m, N \in \mathbb{N} \).

It will also be useful to write the Mikado flows in terms of a potential. We note

\[ \text{curl}_z \left( \frac{ik \times a_k}{|k|^2} e^{ik \cdot \xi} \right) = -i \left( \frac{ik \times a_k}{|k|^2} \right) \times k e^{ik \cdot \xi} = - \frac{k \times (k \times a_k)}{|k|^2} e^{ik \cdot \xi} = a_k e^{ik \cdot \xi}. \quad (5.56) \]

Recall that \( \tilde{\mathcal{R}}_q \) is supported in the set \( \mathbb{T}^3 \times \bigcup_i I_i \), whereas, from (5.35) it follows that \( [0, T] \setminus \bigcup_i I_i = \bigcup_i J_i \), where the open intervals \( J_i \) have length \( |J_i| = \frac{2}{3} \tau_q \) each, except for the first \( J_0 \) and last one, which might be shortened by the intersection with \( [0, T] \), more precisely

\[ J_i = \left( t_{i+1} - \frac{1}{3} \tau_q, t_{i+1} + \frac{1}{3} \tau_q \right) \cap [0, T]. \]

We start by defining smooth non-negative cut-off functions \( \eta_i = \eta_i(x, t) \) with the following properties

(i) \( \eta_i \in C^\infty(\mathbb{T}^3 \times [0, T]) \) with \( 0 \leq \eta_i(x, t) \leq 1 \) for all \( (x, t) \);

(ii) \( \text{supp} \eta_i \cap \text{supp} \eta_j = \emptyset \) for \( i \neq j \);

(iii) \( \mathbb{T}^3 \times I_i \subset \{ (x, t) : \eta_i(x, t) = 1 \} \);

(iv) \( \text{supp} \eta_i \subset \mathbb{T}^3 \times I_i \cup J_i \cup J_{i+1} \).
(v) There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0, T]$

$$\sum_i \int_{\mathbb{T}^3} \eta_i^2(x, t) dx \geq c_0.$$ 

There exists a positive geometric constant $c_0 > 0$ such that for any $t \in [0, T]$

$$\sum_i \int_{\mathbb{T}^3} \eta_i^2(x, t) dx \geq c_0.$$ 

The next lemma is taken from Buckmaster et al. [3].

**Lemma 5.8.** There exist cut-off functions $\{\eta_i\}_i$ with the properties (i)--(v) above and such that for any $i$ and $n, m \geq 0$

$$\|\partial^\nu \eta_i\|_m \leq C(n, m) \tau^{-n}_q$$

where $C(n, m)$ are geometric constants depending only upon $m$ and $n$.

Define

$$\rho_q(t) := \frac{1}{3} \left( e(t) - \frac{\delta_q + 2}{2} - \int_{T^3} |\tilde{v}_q|^2 dx \right)$$

and

$$\rho_{q,i}(x, t) := \frac{\eta_i^2(x, t)}{\sum_j \int_{T^3} \eta_j^2(y, t) dy} \rho_q(t)$$

Define the backward flows $\Phi_i$ for the velocity field $\tilde{v}_q$ as the solution of the transport equation

$$\begin{cases} 
(\partial_t + \tilde{v}_q \cdot \nabla) \Phi_i = 0 \\
\Phi_i(x, t_i) = x. 
\end{cases}$$

Define

$$R_{q,i} := \rho_{q,i} \text{Id} - \eta_i^2 \tilde{R}_q$$

and

$$\tilde{R}_{q,i} = \frac{\nabla \Phi_i R_{q,i}(\nabla \Phi_i)^T}{\rho_{q,i}}. \quad (5.57)$$

We note that, because of properties (ii)--(iv) of $\eta_i$,

- $\text{supp } R_{q,i} \subseteq \text{supp } \eta_i$;
- on $\text{supp } \tilde{R}_q$ we have $\sum_i \eta_i^2 = 1$;
- $\text{supp } R_{q,i} \subset \mathbb{T}^3 \times I_i \cup I_i \cup J_{i+1}$;
- $\text{supp } R_{q,i} \cap \text{supp } R_{q,j} = \emptyset$ for all $i \neq j$. 

These results provide a foundation for the subsequent analysis of the dynamics of the velocity field within the framework of the transport equation.
Lemma 5.9. For $a \gg 1$ sufficiently large, we have
\[ |\nabla \Phi_i - \text{Id}|_0 \leq \frac{1}{2} \text{ for } t \in \text{supp}(\eta_i) \]  \\ (5.58)

Furthermore, for any $N \geq 0$
\[ \frac{\delta_q^{q+1}}{8\lambda_q^2} \leq |\rho_q(t)| \leq \delta_q^{q+1} \text{ for all } t, \]  \\ (5.59)

\[ \|\rho_{q,i}\|_0 \leq \frac{\delta_q^{q+1}}{c_0}, \]  \\ (5.60)

\[ \|\rho_{q,i}\|_N \leq \delta_q^{q+1}, \]  \\ (5.61)

\[ \|\partial_t \rho_q\|_0 \leq \delta_{q+1}^{1/2}\lambda_q, \]  \\ (5.62)

\[ \|\partial_t \rho_{q,i}\|_N \leq \delta_{q+1}\tau_q^{-1}. \]  \\ (5.63)

Moreover, for all $(x, t)$
\[ \tilde{R}_{q,i}(x, t) \in B_{1/2}(\text{Id}) \subset \mathcal{S}_+^{3 \times 3}, \]
where $B_{1/2}(\text{Id})$ denotes the metric ball of radius $1/2$ around the identity $\text{Id}$ in the space $\mathcal{S}_+^{3 \times 3}$.

Proof of Lemma 5.9. For the estimates (5.59)–(5.61) we refer to Buckmaster et al. [3]. Note that by the definition of the cut-off functions $\eta_i$
\[ c_0 \leq \sum_i \int_{\mathbb{T}^3} \eta_i^2(y, t)dy \leq 2. \]  \\ (5.64)

To prove (5.62) and (5.63) we first note that
\[ \left| \frac{d}{dt} \int |\nu_q(x, t)|^2 dx \right| \leq 2 \left| \int \nabla \nu_q \cdot \tilde{R}_q dx \right| + 2\nu \int |(-\Delta)^{1/2} \nu_q|^2 dx \leq \delta_{q+1}^{1/2}\lambda_q \nu. \]

Thus
\[ \|\partial_t \rho_q\|_0 \leq \delta_{q+1}^{1/2}\lambda_q. \]

Then, since $\|\partial_t \eta_j\|_N \leq \tau_q^{-1}$ and $\delta_{q+1}^{1/2}\lambda_q \leq \tau_q^{-1}$, using (5.64), the estimate (5.63) follows.

5.4. The constant $M$

The principal term of the perturbation can be written as
\[ w_0 := \sum_i \left( \rho_{q,i}(x, t) \right)^{1/2} (\nabla \Phi_i)^{-1} W \left( \tilde{R}_{q,i}, \lambda_{q+1} \Phi_i \right) = \sum_i w_{o,i}, \]  \\ (5.65)

where Lemma 5.6 is applied with $\mathcal{N} = B_{1/2}(\text{Id})$, namely the closed ball (in the space of symmetric $3 \times 3$ matrices) of radius $1/2$ centered at the identity matrix.

Note that $\|\partial_t e\|_0 \leq 1 \leq \delta_{q+1}^{1/2}\lambda_q$ since $\delta_{q+1}^{1/2}\lambda_q = \lambda_{q+1}^{1/2} \geq a^{1/2(1-\beta - 2\beta b)} \geq 1$ (recall that $b < \frac{1-\beta}{2\beta}$).
From Lemma 5.9, it follows that $W(\tilde{R}_{q,i}, \tilde{\lambda}_{q+1}\Phi_i)$ is well defined. Using the Fourier series representation of the Mikado flows (5.52), we can write

$$w_{0,i} = \sum_{k \neq 0} (\nabla \Phi_i)^{-1} b_{i,k} e^{i\lambda_{q+1}k \Phi_i},$$

where

$$b_{i,k}(x, t) := \left(\rho_{q,i}(x, t)\right)^{1/2} a_k \left(\tilde{R}_{q,i}(x, t)\right).$$

The following is a crucial point of our construction, which ensures that the constant $M$ of Proposition 4.1 is geometric and in particular independent of all the parameters of the construction.

**Lemma 5.10.** There is a geometric constant $\bar{M}$ such that

$$||b_{i,k}||_0 \leq \bar{M} \frac{1}{|k|^4} \delta_{q+1}^{1/2}.$$  \hspace{1cm} (5.66)

We are finally ready to define the constant $M$ of Proposition 4.1: from Lemma 5.10, it follows trivially that the constant is indeed geometric and hence independent of all the parameters entering in the statement of Proposition 4.1.

We can now define the geometric constant $M$ as follows:

$$M = 64\bar{M} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|^4},$$ \hspace{1cm} (5.67)

where $\bar{M}$ is the constant of Lemma 5.10.

We also define

$$w_c := -\frac{i}{\lambda_{q+1}} \sum_{i,k \neq 0} \left[ \text{curl} \left( (\rho_{q,i})^{1/2} \frac{\nabla \Phi_i^T \left( k \times a_k \left( \tilde{R}_{q,i} \right) \right)}{|k|^2} \right) \right] e^{i\lambda_{q+1}k \Phi_i} =: \sum_{i,k \neq 0} c_{i,k} e^{i\lambda_{q+1}k \Phi_i}.$$

Then by direct computations, one can check that

$$w_{q+1} = w_0 + w_c = -\frac{1}{\lambda_{q+1}} \text{curl} \left( \sum_{i,k \neq 0} (\nabla \Phi_i)^T \left( \frac{i k \times b_{k,i}}{|k|^2} \right) e^{i\lambda_{q+1}k \Phi_i} \right),$$ \hspace{1cm} (5.68)

thus the perturbation $w_{q+1}$ is divergence free. Note that the dependence of $w_{q+1}(\cdot, 0)$ on the function $e(t)$ is only through the value $e(0)$.

**5.5. The final Reynolds stress and conclusions**

Upon letting

$$\tilde{R}_q = \sum_i R_{q,i},$$

the new Reynolds stress will be split into two main component: the Euler error $\tilde{R}^E_{q+1}$ and the dissipative error $\tilde{R}^D_{q+1}$, i.e.
\[ \dot{R}_{q+1} = \dot{R}^E_{q+1} + \dot{R}^D_{q+1}, \]  

where

\[ \dot{R}^E_{q+1} := \mathcal{R}\left( w_{q+1} \cdot \nabla v_q + \partial_t w_{q+1} + \bar{v}_q \cdot \nabla w_{q+1} + \text{div}\left(-\bar{R}_q + (w_{q+1} \otimes w_{q+1})\right) \right), \]  

\[ \dot{R}^D_{q+1} := \nu \ \mathcal{R}\left(-\Delta\right)' w_{q+1}. \]  

Notice that all three terms in (5.69) are of the form \( \mathcal{R}f \), where \( f \) has always zero mean. Notice also that the definition of \( \dot{R}^E_{q+1} \) is the same as by Buckmaster et al. [3] and that due to the dissipative term \( \nu (-\Delta)' \) we have to put also \( \dot{R}^D_{q+1} \) in the definition of the new Reynolds stress in order to ensure that the system (4.2) is satisfied at the step \( q+1 \). Indeed, with this definition one may verify that

\[ \left\{ \partial_t v_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1} + \nu (-\Delta)' v_{q+1} = \text{div}\left(\dot{R}_{q+1}\right), \text{div} \ v_{q+1} = 0, \right\} \]

where the new pressure is defined by

\[ p_{q+1}(x, t) = \bar{p}_q(x, t) - \sum_i \rho_{q,i}(x, t) + \rho_q(t). \]  

We now state a proposition taken from Buckmaster et al. [3].

**Proposition 5.11.** For \( t \in I_i \) and any \( N \geq 0 \)

\[ \| (\nabla \Phi_i)^{-1} \|_N + \| \nabla \Phi_i \|_N \leq \ell^{-N}, \]  

\[ \| \bar{R}_{q,i} \|_N \leq \ell^{-N}, \]  

\[ \| b_{i,k} \|_N \leq \delta^{1/2}_q |k|^{-6} \ell^{-N}, \]  

\[ \| c_{i,k} \|_N \leq \delta^{1/2}_q \lambda^{-1}_q |k|^{-6} \ell^{-N-1}. \]  

Moreover assuming \( a \) is sufficiently large, the perturbations \( w_o, w_c \) and \( w_q \) satisfy the following estimates

\[ \| w_o \|_0 + \frac{1}{\lambda_{q+1}} \| w_o \|_1 \leq \frac{M}{4} \delta^{1/2}_{q+1} \]  

\[ \| w_c \|_0 + \frac{1}{\lambda_{q+1}} \| w_c \|_1 \leq \delta^{1/2}_{q+1} \lambda^{-1}_q \ell^{-1} \]  

\[ \| w_{q+1} \|_0 + \frac{1}{\lambda_{q+1}} \| w_{q+1} \|_1 \leq \frac{M}{2} \delta^{1/2}_{q+1} \]  

where the constant \( M \) depends solely on the constant \( c_0 \) in (5.64). In particular, we obtain (5.46).

We are now ready to complete the proof of Proposition 4.1 by proving the remaining estimates (5.47) and (5.48). The estimate (5.48) is a consequence of Proposition 5.11 and Lemma 5.9 and does not involve the different structure of the Navier–Stokes equations with respect to the Euler ones, thus for the proof of the next proposition we refer to Buckmaster et al. [3].
Proposition 5.12. The energy of $v_{q+1}$ satisfies the following estimate:

$$|e(t) - \int_{\mathbb{R}^3} |v_{q+1}|^2 dx - \frac{\delta_{q+2}}{2}| \leq \frac{\delta_{q+1/2}^{1/2} \delta_{q+1}^{1/2} \lambda_q^{1+2\gamma}}{\lambda_{q+1}^{1-4\gamma}}.$$ 

For the inductive estimate on $\hat{R}_{q+1}$ we have the following

Proposition 5.13. The Reynolds stress error $\hat{R}_{q+1}$ defined in (5.69) satisfies the estimate

$$||\hat{R}_{q+1}||_0 \leq \frac{\delta_{q+1/2}^{1/2} \delta_{q+1}^{1/2} \lambda_q^{1-4\gamma}}{\lambda_{q+1}^{1-4\gamma}}.$$ 

In particular (5.47) holds.

Proof. For the first term in the definition of the new Reynolds stress tensor, we have

$$||\hat{R}^E_{q+1}||_0 \leq \frac{\delta_{q+1/2}^{1/2} \delta_{q+1}^{1/2} \lambda_q^{1-4\gamma}}{\lambda_{q+1}^{1-4\gamma}}.$$ 

We are not going to give the proof of the last estimate because, as already explained, it can be found in Buckmaster et al. [3, Proposition 6.1]. To estimate $\hat{R}^D_{q+1}$ we first note that $\nu < 1$ and the two operators $R$ and $(-\Delta)^{1/2}$ commute, therefore we can first estimate $||Rw_{q+1}||_0$ and $||Rw_{q+1}||_1$ from which, using Theorem B.1 and interpolation in Hölder spaces, we conclude

$$||\hat{R}^D_{q+1}||_0 \leq ||Rw_{q+1}||_{\gamma+2} \leq ||Rw_{q+1}||_{1-\gamma+2} ||Rw_{q+1}||_{1}^{\gamma+2}.$$ 

By the definition of the new perturbations we have

$$w_c = \sum_{i,k \neq 0} c_{i,k} e^{i\lambda_{q+1} k \Phi_i},$$

$$w_o = \sum_{i,k \neq 0} L_{i,k} e^{i\lambda_{q+1} k \Phi_i},$$

where $L_{i,k} := (\nabla \Phi_i)^{-1} b_{i,k}$. Using Proposition 5.11 we have

$$||L_{i,k}||_N \leq ||(\nabla \Phi_i)^{-1}||_N ||b_{i,k}||_0 + ||(\nabla \Phi_i)^{-1}||_0 ||b_{i,k}||_N \leq \delta_{q+1}^{1/2} |k|^{-\gamma} \epsilon^{-N}.$$ 

Using Proposition C.2 and (5.82) we estimate

$$||\nabla w_o||_0 \leq ||\nabla w_o||_x \leq \sum_{i,k \neq 0} \frac{||L_{i,k}||_0}{\lambda_{q+1}^{1-\gamma} |k|^{1-x}} + \frac{||L_{i,k}||_N + ||L_{i,k}||_0 \Phi_i||_N + 2}{\lambda_{q+1}^{N-\gamma} |k|^{N-x}} \leq \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-x}} + \frac{\epsilon^{-N} \delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-x}},$$

where in the last inequality we have chosen $N$ big enough. It is not difficult to see that we also have

$$||\nabla w_o||_1 \leq \frac{\delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-x}},$$

since each term where we take a first derivative (in space) will gain a factor $\lambda_{q+1}$, thus by interpolation we conclude
\[\|(-\Delta)^{\gamma} R_{w_0}\|_0 \leq \frac{\delta^{1/2}}{\lambda^{1-\alpha}_{q+1}} \lambda^{\gamma}_{q+1}. \quad (5.83)\]

Now we observe that the estimate on the coefficients \(c_{i,k}\) are better than those for the \(L_{i,k}\)'s, so that we also bound
\[\|(-\Delta)^{\gamma} R_{w_1}\|_0 \leq \frac{\delta^{1/2}}{\lambda^{1-\alpha}_{q+1}} \lambda^{\gamma}_{q+1}. \quad (5.84)\]

Finally, combining (5.83), (5.84) and the restriction \(\gamma < 1/3\) we get
\[\|R_{q+1}^D\|_0 \leq \frac{\delta^{1/2}}{\lambda^{1-\alpha}_{q+1}} \lambda^{\gamma}_{q+1} \leq \frac{\delta^{1/2} \lambda^{1/2}}{\lambda^{1-\alpha}_{q+1}} \lambda^{\gamma}_{q+1}. \]

We remark that in the last step of the previous proof it was enough to require \((\gamma-3\alpha)b < 1 - \beta\) (instead of \(\gamma < 1/3\)) which would also allow bigger values of \(\gamma\).

**Disclosure statement**

No potential conflict of interest was reported by the author.

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Appendix A. Some estimates in Hölder spaces

Recall the following elementary inequalities

**Proposition A.1.** Let $f, g$ be two smooth functions. For any $r > s \geq 0$ we have

$$\left[fg\right]_r \leq C([f]_r\|g\|_0 + \|f\|_0[g]_r), \quad (A1)$$

$$\left[f\right]_r \leq C([f]_0^{s/r}[f]_r^s). \quad (A2)$$

We also recall the quadratic commutator estimate of Constantin et al. [12] (cf. also [13, Lemma 1]):

**Proposition A.2.** Let $f, g \in C^\infty(T^3 \times \mathbb{T})$ and $\psi$ a standard radial smooth and compactly supported kernel. For any $r \geq 0$ we have the estimate

$$\| (f \ast \psi_\epsilon)(g \ast \psi_\epsilon) - (fg) \ast \psi_\epsilon \| \leq C \epsilon^{2-\gamma}\|f\|_1\|g\|_1,$$

where the constant $C$ depends only on $r$.

Appendix B. Estimates on the fractional Laplacian

**Theorem B.1.** (Interaction with Holder spaces) Let $\gamma, \epsilon > 0$ and $\beta \geq 0$ such that $2\gamma + \beta + \epsilon \leq 1$, and let $f : T^3 \rightarrow \mathbb{R}^3$. If $f \in C^{0,2\gamma+\beta+\epsilon}$, then $(-\Delta)^\gamma f \in C^\beta$, moreover there exists a constant $C = C(\epsilon) > 0$ such that

$$\|(-\Delta)^\gamma f\|_\beta \leq C(\epsilon)\|f\|_{2\gamma+\beta+\epsilon}. \quad (B1)$$

**Proof.** The proof of (B.1) for $\beta = 0$ can be found by Roncal and Stinga [7], Theorem 1.4. Fix now $\beta > 0$. For any $h \in \mathbb{T}^3$ we have

$$\|(-\Delta)^\gamma (f(\cdot + h) - f(\cdot))\|_0 \leq C(\epsilon)\|f(\cdot + h) - f(\cdot)\|_{2\gamma+\epsilon} \leq C(\epsilon)\|h\|^\beta\|f\|_{2\gamma+\beta+\epsilon},$$

from which (B.1) follows.

**Corollary B.2.** Let $\gamma \in (0, 1)$, $\epsilon > 0$ be such that $0 < \gamma + \epsilon \leq 1$, and let $f : T^3 \rightarrow \mathbb{R}^3$. There exist a constant $C = C(\epsilon) > 0$ such that

$$\int_{T^3} \|(-\Delta)^{\gamma/2}f\|^2(x)dx \leq C(\epsilon)\|f\|^2_{T^3} \quad \forall f \in C^{\gamma+\epsilon}(T^3). \quad (B2)$$
Appendix C. Potential theory estimates

We recall the definition of the standard class of periodic Calderón–Zygmund operators. Let $K$ be an $\mathbb{R}^3$ kernel which obeys the properties

- $K(z) = \Omega(z)|z|^{-3}$, for all $z \in \mathbb{R}^3 \setminus \{0\}$
- $\Omega \in C^\infty(S^2)$
- $\int_{|z|=1} \Omega(\tilde{z})d\tilde{z} = 0$.

From the $\mathbb{R}^3$ kernel $K$, use Poisson summation to define the periodic kernel

$$K_T^\alpha(z) = K(z) + \sum_{\ell \in \mathbb{Z}^3 \setminus \{0\}} (K(z + \ell) - K(\ell)).$$

Then the operator

$$T_k f(x) = p.v. \int_{\mathbb{T}^3} K_T^\alpha(x-y) f(y)dy$$

is a $\mathbb{T}^3$-periodic Calderón–Zygmund operator, acting on $\mathbb{T}^3$-periodic functions $f$ with zero mean on $\mathbb{T}^3$. The following proposition, proving the boundedness of periodic Calderón–Zygmund operators on periodic Hölder spaces is classical (see [14]).

**Proposition C.1.** Fix $\alpha \in (0, 1)$. Periodic Calderón–Zygmund operators are bounded on the space of zero mean $\mathbb{T}^3$-periodic $C^\alpha$ functions.

The following is a simple consequence of classical stationary phase techniques. For a detailed proof, the reader might consult Daneri and Székelyhidi [11, Lemma 2.2].

**Proposition C.2.** Let $\alpha \in (0, 1)$ and $N \geq 1$. Let $a \in C^\infty(\mathbb{T}^3)$, $\Phi \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ be smooth functions and assume that

$$\hat{C}^{-1} \leq |\nabla \Phi|, |\nabla \Phi^{-1}| \leq \hat{C}$$

holds on $\mathbb{T}^3$. Then

$$\left| \int_{\mathbb{T}^3} a(x)e^{ik\Phi}dx \right| \leq \frac{\|a\|_N + \|a\|_0 \|\Phi\|_N}{|k|^N},$$

and for the operator $\mathcal{R}$ defined in (5.38), we have

$$\left\| \mathcal{R}\left( a(x)e^{ik\Phi} \right) \right\|_x \leq \frac{\|a\|_0}{|k|^{N-2}} + \frac{\|a\|_{N+2} + \|a\|_0 \|\Phi\|_{N+2}}{|k|^{N-2}},$$

where the implicit constant depends on $\hat{C}$, $\alpha$ and $N$, but not on $k$. 