Description and comparison of \(\text{Fat7}\) and HYP fat links

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We study various methods of constructing fat links based upon the HYP (by Hasenfratz & Knechtli) and \(\text{Fat7}\) (by W. Lee) algorithms. We present the minimum plaquette distribution for these fat links. This enables us to determine which algorithm is most effective at reducing the spread of plaquette values - a strong indicator of improved statistics for spectrum and static potential measurements, among other quantities.

1. INTRODUCTION

Despite its prominent role as a useful tool in the non-perturbative study of QCD, lattice field theory has long struggled with the difficulties of accurately representing a continuum field theory in a discrete spacetime. The occurrence of discretization errors has lead to the development of a great many improvement schemes, some more useful than others. One extremely popular method of improvement has been the use of “fat links”, in which each ordinary or “thin” link of the lattice, \(U_\mu(x)\), is replaced by a linear combination of the thin link and the links adjacent to it. Two particular fattening algorithms will be of interest to us in this report, HYP or hypercubic blocking \([1]\), and \(\text{Fat7}\) blocking (that is, SU(3)-projected Fat7 blocking \([2]\)). These two algorithms have been shown to be perturbatively equivalent at one-loop level \([3,4]\). We will present the results of a preliminary numerical investigation to determine whether there are any significant non-perturbative differences between the two algorithms. These results will point the way to more detailed investigations.

2. HYPERCUBIC BLOCKING

The HYP algorithm was proposed as a modification of standard APE smearing \([1]\). APE smearing involves constructing a fattened link \(V_\mu(x)\) by adding the sum of staples, weighted by some factor \(\alpha\), to the thin link,

\[V_\mu(x) = (1 - \alpha) U_\mu(x) + \frac{\alpha}{6} \sum_{\nu \neq \mu} W_\nu(x)\]

followed by projection back to SU(3) (where \(W_\nu(x)\) refers to the sum of three-link staples in the positive and negative \(\nu\) directions, with central link parallel to \(U_\mu(x)\)). Clearly \(n\) repeated applications of such an algorithm will access the gauge fields at a distance \(\sqrt{\sqrt{n}}\) lattice spacings from the original thin link (although smoothing algorithms such as cooling and smearing can be viewed as diffusive “random-walk” processes which primarily affect physics only within a range \(a\sqrt{n}\), as investigated in \([5]\)). To localize APE smearing within the smallest possible volume, HYP blocking was defined as being equivalent to three iterations of APE smearing with the caveat that the staples at each stage could not be constructed from links which are fattened in the same plane as the staple itself. The original thin link is therefore fattened by pairs of positively-and-negatively-oriented staples in three directions. Each of these staples is constructed from links fattened by staples in only two (positive and negative) directions, and each of these staples is constructed from links fattened in a single plane (for further details please refer to \([1]\)). HYP blocking therefore uses only the staples defined by the boundaries of hypercubes attached to the original thin link (Fig.1). Projection back to SU(3) is performed after each level of fattening.
Figure 1. HYP fattening. The bold line is the thin link, and the dashed lines are the hypercubes attached to it. The double and zig-zig lines are the staples at the final and middle levels of fattening respectively. The outer level (fourth direction, which must necessarily be constructed first) is omitted. Only positively directed staples are shown here.

We shall henceforth refer to the links fattened by the outermost set of staples as the initial-stage fat links, the links fattened by staples formed from the initial-stage links as the middle-stage fat links, and the completely fattened links (i.e. those fattened by staples formed from middle-stage fat links) as the final-stage links.

Since its introduction, HYP blocking has shown itself to be extremely effective at reducing taste-symmetry breaking effects, and has gained widespread attention and use.

3. FAT7 AND FAT7 BLOCKING

Fat7 blocking [6] is a form of fattening which incorporates not only standard three-link staples but also five-link and seven-link staples (Fig. 2). Clearly, with staples up to length seven it is possible to traverse all three directions orthogonal to the original thin link, as is the case with HYP blocking. However Fat7 blocking as it was originally envisaged did not incorporate SU(3) projection. Fat7 is a modification of this algorithm which incorporates SU(3) projection of the final sum of link and staples [2].

If we construct Fat7 fat-links in an analogous manner to the iterative construction of HYP links we have greater freedom in our application of SU(3)-projection. An iterative approach allows us to include SU(3) projection at each level of the construction of the link, rather than just at the final level. The only difference between HYP and Fat7 is then that in HYP blocking each link in each staple is fattened at the subsequent level, whereas in Fat7 it is only the links parallel to the original thin link that are fattened (see Fig. 3).

We can see that the second covariant derivative operator [7] may be used to recursively define a fattened link

\[ L_\nu(\alpha) \cdot U_\mu(x) = (1 - 2\alpha) \cdot U_\mu(x) + \alpha W_\nu(x) \] (2)

(\(W_\nu(x)\) defined as in Eq. 1)). With the parameter \(\alpha\) taking the value one-quarter this operator can be interpreted as suppressing flavour-changing gluon interactions by vanishing in the limit as gluon momentum approaches the lattice cut-off. The Fat7 link can easily be constructed by repeated application of this operator as

\[ V_\mu = \frac{1}{6} \sum_{\text{perm}(\nu,\rho,\lambda)} L_\nu(\alpha) \cdot (L_\rho(\alpha) \cdot (L_\lambda(\alpha) \cdot U_\mu)) \] (3)

where we note explicitly that the sum over permutations of the directions is taken. This repeated use of the same operator accounts for the use of a single weighting parameter \(\alpha\) at each stage, whereas HYP blocking uses three parameters \(\alpha_1, \alpha_2, \alpha_3\).

As noted above, to construct the Fat7 links from the Fat7 links, SU(3) projection must be performed after the final summation is taken, but it can also optionally be applied after each or
either of the first two fattening stages, leaving us with four possible combinations of projection schemes to define our Fat\(_7\) link (i.e. projection at the final, initial+final, middle+final, and initial+middle+final stages).

4. COMPARISON OF FATTENING APPROACHES

Fat links have shown themselves to be extremely useful at overcoming problems associated with the lattice formulation. Specifically they have shown a reduction in the severity of taste-symmetry breaking associated with staggered fermions, and a reduction in the severity of the exceptional configuration problem associated with the Wilson fermion formulation. Both of these problems can be directly related to short-scale fluctuations of the link values. In [1], the values of the minimum plaquettes calculated on an ensemble of configurations were used as a probe of the most severe link fluctuations. The maximum plaquette values are bounded above, and hence we expect an increase in the minimum plaquette value of any configuration to indicate a smaller range of link fluctuations across that configuration, and hence a reduction in the scale of the largest short-range link fluctuations.

The process of fattening tends to increase the minimum plaquette values of different configurations to differing degrees, spreading the minimum values over quite a wide range (as can be seen by comparing Figs. 4 and Figs. 5 and 6). We expect that certain measurements will achieve a lower level of statistical uncertainty if performed on an ensemble of configurations having a narrow spread of minimum plaquette values. One of the aims of this preliminary study will be to identify whether the Fat\(_7\) blocking algorithm is capable of producing a narrower spread of minimum plaquette values than HYP, and is hence worthy of a more detailed investigation.

5. RESULTS

We present the minimum plaquette values for an ensemble of 194 quenched configurations generated on an \(8^3 \times 32\) lattice at \(\beta = 5.7\). We also present the minimum plaquette values for the ensembles obtained after blocking each configuration with the HYP algorithm (Figs. 5 and 6), and with the Fat\(_7\) algorithm (Figs. 7 to 10). As noted above the natural choice for the Fat\(_7\) weighting parameter is \(\alpha = 0.25\). For HYP we are interested in two choices of parameters, those found non-perturbatively in [1], namely \(\alpha_1 = 0.75, \alpha_2 = 0.6, \alpha_3 = 0.3\) (HYP-I), and those found perturbatively in [3], namely \(\alpha_1 = 7/8, \alpha_2 = 4/7, \alpha_3 = 1/4\) (HYP-II), although we note the caveat that choices of \(\alpha_1\) larger than 0.75 may tend to destabilize the smearing algorithm, making the blocked configuration rougher rather than smoother [8].
For the $\text{Fat}_7$-blocked configurations we have analysed the four different SU(3) projection options described in section 3. The histograms of minimum plaquette values are presented in Figs. 4 to 10. All figures are to the same scale.

We can see clearly that the initial unblocked data produce a very narrow peak. After HYP blocking the distribution of minimum plaquette values has been shifted towards (but not into) the positive region, and taken on a Poisson distribution-like outline. The data obtained from $\text{Fat}_7$ with SU(3) projection at the final stage, and at all three stages are especially interesting from an analytical basis [2]. It is clear that all four $\text{Fat}_7$ fattening schemes lead to minimum plaquette distributions that have a substantially narrower spread of values than the HYP-I and HYP-II data. Although the number of configurations used in this study is small, the results seem significant enough to warrant further investigations with $\text{Fat}_7$.

6. CONCLUSIONS

Although this work is only preliminary, we find encouraging signs that $\text{Fat}_7$ may be capable of producing results which are as good as HYP, and possibly better for some calculations. Both HYP and $\text{Fat}_7$ algorithms can be made substantially faster and more efficient by pre-calculating and storing the staples across the entire lattice, and using these to construct the links at the next level of fattening. This approach avoids calculating the same staple more than once, as one would be forced to do if one fattened in a naive site-by-site manner. Using this approach $\text{Fat}_7$ can be made less memory-intensive than HYP, as it only requires the staples in a limited number of directions to be pre-calculated at each stage of fattening. An efficiently-designed algorithm for $\text{Fat}_7$ does not sacrifice speed, and enables us to use only one-quarter of the memory that is needed to store the pre-calculated staples for HYP. In future work we hope to determine whether this cheaper form of fattening merely equals, or in fact can produce results which are superior to HYP for certain calculations.
7. ACKNOWLEDGEMENTS

We wish to thank Anna Hasenfratz for discussions concerning the relation of this work to her own, and Derek Leinweber for bringing reference [8] to our attention.

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