Abstract—This paper considers a status update communication system consisting of a source-destination link with timeliness requirements. First, we study the properties of a sample path of the age of information (AoI) process at the destination. Under the assumption of ergodicity, we obtain a general formula of the stationary distribution of the AoI. We relate this result to a discrete time queueing system and provide a general expression of the generating function of AoI in relation with the system time and the peak age of information (PAoI). Furthermore, we consider the first-come-first-served (FCFS) Geo/Geo/1 queue and we obtain closed-form expressions of the generating functions and the stationary distributions of the AoI and the PAoI. We built upon these results to provide a methodology for analyzing general non-linear age functions for this type of systems.

I. INTRODUCTION

In communication systems, it is common to deal with time critical information that needs to be transmitted from the generation point to a remote destination in the network. To address timeliness requirements, the notion of age of information (AoI) has been introduced to quantify the freshness of the received information at the destination [1]. At any moment, the AoI at the destination is the time that elapsed since the last received status update was generated by the source.

Systems with different availabilities of resources have been modeled through different queueing models and the time and age AoI was derived. In [2], the M/M/1, the M/D/1, and the D/M/1 queues were studied under the first-come-first-served (FCFS) discipline. The last-come-first-served (LCFS) queue discipline with or without the ability to preempt the packet in service has been considered in [3]–[7]. The effect of the buffer size and the available number of servers has been studied in [8]–[11]. The AoI in various continuous time and discrete time queueing systems, such as the FCFS G/G/1, the LCFS G/G/1, and the G/G/∞, has been analyzed in [12], [13].

Furthermore, some works depart from the average AoI and consider a complete characterization of the AoI distribution. In [14] new tools such as stochastic hybrid systems (SHS) are developed to analyze AoI moments and the moment generating function (MGF) of an AoI process in networks. In [15] a general formula of the stationary distribution of AoI is obtained and applied to a wide class of continuous-time single server queues with different disciplines. The distribution of AoI for the GI/GI/1/1 and GI/GI/1/2* systems, under non-preemptive scheduling is considered in [16]. In [17] the authors characterize the AoI distribution in bufferless systems. Delay and peak age of information (PAoI) violation guarantees are studied in [18] for the reliable transmission of short packets over a wireless channel.

The AoI is determined by two factors (i) the processing/transmission delay and (ii) the pattern that the source uses to generate status updates. To capture both the information characteristics of the source and these factors, it is meaningful to modify the definition of AoI to a non-linear cost function. The aim is to penalize the absence of updates at the destination according to the source characteristics by a non-negative, monotonically increasing category of functions.

The works in [19], [20], aim to expand the concept of information ageing by introducing the cost of update delay (CoUD) metric to provide a flexible measure of having stale information at the destination depending on the autocorrelation properties of the source. In [21], [22], sampling for data freshness is considered, and so-called age penalty and utility functions are employed to describe the level of dissatisfaction for having aged status updates at the destination. In [23], the authors use the mutual information between the real-time source value and the delivered samples at the receiver to quantify the freshness of the information contained in the delivered samples. In [24] it was proven that for a Wiener process sampling that minimizes AoI is not optimal for prediction. Recently, a relationship between a non-linear function of the AoI and the estimation error of the Ornstein-Uhlenbeck (OU) process was investigated in [25].

In this work, we are interested in the time average performance of non-linear functions of AoI in a discrete time system. To provide a methodology for analyzing general non-linear age functions for this type of systems, we consider a sample path of the AoI stochastic process. Studying the properties of a sample path of the AoI we can have a better and deeper understanding of the underlying AoI properties under general assumptions. More specifically, we determine the relation among AoI, the system delay, and the PAoI. Then, we invoke ergodicity that has been extensively used in the AoI literature and we apply our results to a discrete time queueing system. To that extent, we obtain general formulas of the stationary distributions and the z-transforms of the AoI and
PAI metrics. To illustrate the applicability of the results we consider an FCFS Geo/Geo/1 queue and derive the probability distribution function (PDF) and the probability mass function (pmf) of AoI and P AoI, that can be utilized to provide violation guarantees. Finally, we illustrate how our results can be used to obtain closed-form expressions of non-linear functions of AoI, providing some examples. Our results are general enough to be utilized to a variety of settings in terms of the queuing discipline, the arrival and service process, and the non-linear cost function at the destination.

II. THE AOI SAMPLE PATH IN A GENERAL SETTING

We consider a stochastic system consisting of a point-to-point communication link with a single transmitter node (source) that sends status updates to a single receiver node (destination). Time is assumed to be slotted. The AoI of the transmitter node at the receiver node is defined as the random process

\[ \Delta_t = t - u(t), \quad t \in \mathbb{Z}^*, \quad (1) \]

where \( u(t) \) is the time-stamp of the most recently received status update.

Any sample path of the AoI process \( \Delta_t \) can be constructed as follows. Let \( \{ t_n', n \geq 0 \} \) be a deterministic point process, with \( t_0' = 0 \) and \( t_n' < t_{n+1}' \leq \infty \). We interpret \( t_n' \) as the times at which the status updates are received at the destination. At time \( t \), the number of events in \([0,t]\) is denoted by \( N(t) = \max\{n : t_n' \leq t\}, \quad t \geq 0 \). We assume that \( t_n' \rightarrow \infty \) as \( n \rightarrow \infty \), so that there is a finite number of receptions in any finite time interval, and we note that, since \( t_n' < \infty \) for all \( n \geq 0 \), we have that \( N(t) \rightarrow \infty \) as \( n \rightarrow \infty \). Associated with each point \( t_n' \) is the mark \( T_n = \{ \Delta_{t_n'}, n \geq 0 \} \) denoting the value of AoI immediately after receiving the \( n \)th status update. Then, \( \{(t_n', T_n), n \geq 0 \} \) denotes the marked point process of AoI on \([0, \infty) \times [0, \infty)\). The AoI process is non-negative, piece-wise non-decreasing, right-continuous, with discontinuous jumps at times \( t_n' \). A sample path of the AoI process is shown in Fig. 1.

Thus, the AoI process \( \Delta_t \) is determined completely by \( \{(t_n', T_n), n \geq 0 \} \) as follows

\[ \Delta_t = T_{n-1} + (t - t_{n-1}), \quad t \in [t_{n-1}, t_n'), \quad n \geq 1. \quad (2) \]

Moreover, we define the PAI as the value of AoI achieved immediately before receiving the \( n \)th update

\[ A_n = T_{n-1} + (t_n' - t_{n-1}), \quad n \geq 1. \quad (3) \]

A. Main Result

Consider a fixed sample path of \( \{(t_n', T_n), n \geq 0 \} \) where the quantities are deterministic. Let \( \Delta'(x), A'(x), \) and \( T'(x) \), denote the asymptotic frequency distributions \( \{ A_t, t \geq 0 \}, \{ A_n, n \geq 1 \}, \) and \( \{ T_n, n \geq 1 \} \), respectively. When the limits exist these are given by

\[ \Delta'(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{1}_{\{\Delta_t \leq x\}}, \quad x \geq 0, \quad (4) \]

\[ A'(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{A_n \leq x\}}, \quad x \geq 0, \quad (5) \]

\[ T'(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{T_n \leq x\}}, \quad x \geq 0, \quad (6) \]

where \( \mathbb{1}_{\{\cdot\}} \) is the indicator function.

The following lemma is a sample-path analogue of the elementary renewal theorem.

Lemma 1 (26). Lemma 1.1. Let \( 0 < \lambda^\dagger \leq \infty \). Then \( t^{-1}N(t) \rightarrow \lambda^\dagger \) as \( t \rightarrow \infty \) if and only if \( n^{-1}t_n' \rightarrow 1/\lambda^\dagger \) as \( n \rightarrow \infty \).

Next, we have the following theorem.

Theorem 1. If the limits \( (5) \) and \( (6) \) exist for each \( x \geq 0 \), then the limit in \( (4) \) also exists and it is given by

\[ \Delta'(x) = \lambda^\dagger \sum_{u=0}^{x} (T'(u) - A'(u)). \quad (7) \]

Proof. The proof is given in Appendix A.

III. THE CASE OF A DISCRETE TIME QUEUE

In this section, we relate the sample-path of the previous section to a stationary, ergodic queueing system. We consider that the transmitter node has a buffer of infinite capacity to store incoming status updates in the form of packets. These packets are then sent through an error-free channel to the destination. Packets have equal length and time is divided into slots such that the transmission time of a packet from the buffer to the destination is equal to one slot. Each such packet is said to provide a status update and these two terms are used interchangeably. The status updates arrivals are modeled by an independent and identically distributed (i.i.d.) process with average arrival rate \( \lambda \). Moreover, we consider a general service process where the service times are i.i.d. with service rate \( \mu \), and a single server. We specify the interarrival distribution and service distribution in the next section.
Consider that the \( n \)th status update is generated at time \( t_n \), delivered through the transmission system, and received by the destination at time \( t_n \). Then, we denote by \( T_n = t_n' - t_n \) the system time of update \( n \). This corresponds to the sum of the queueing time and the queue service time. The interarrival time of update \( n \) is defined as the random variable \( Y_n = t_n - t_{(n-1)} \). Then, the PAoI achieved immediately before receiving the \( n \)th update in (3) can be defined alternatively as

\[
A_n = Y_n + T_n.
\]  

(8)

Let the PDF of the random variables \( T \) and \( A \) be denoted by \( T(x) \) and \( A(x) \), respectively. The stationary distribution \( \Delta(x) \), of the AoI is defined as the long-run fraction of time in which the AoI is not greater than an arbitrary fixed value \( x \), i.e.,

\[
\Delta(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{I}(\Delta_t \leq x).
\]  

(9)

Furthermore, let \( \Delta^*(z) \), \( A^*(z) \), and \( T^*(z) \), denote the z-transforms (also known as the generating functions) of the AoI, PAoI, and system time, respectively. The z-transform for a function of discrete time \( f_n \) is defined as

\[
F^*(z) = \sum_{n=0}^{\infty} f_n z^n.
\]  

(10)

**Lemma 2.** For stationary and ergodic systems \( \Delta^I(x) = \Delta(x) \), \( A^I(x) = A(x) \), and \( T^I(x) = T(x) \), with probability 1.

**Proof.** We invoke the ergodic theorem in [26, Theorem A.4] to guarantee that the asymptotic frequency distributions exist and coincide with the corresponding stationary probabilities.

Next, we have the following theorem.

**Theorem 2.** The z-transform of the AoI of the source at destination is given by

\[
\Delta^*(z) = \frac{\lambda T^*(z) - A^*(z)}{1 - z}.
\]  

(11)

**Proof.** The proof is straightforward from Theorem 1 and properties of the z-transform.

We proceed by deriving a general formula of the z-transform of the PAoI \( A^*(z) \). We observe that the random variables \( Y_n \) and \( T_n \) in (8) are dependent and provide an alternative definition of PAoI that is

\[
A_n = \max(Y_n, T_{n-1}) + S_n,
\]  

(12)

where \( S_n \) denotes the service time of the \( n \)th update. Then, the random variables \( Y_n \) and \( T_{n-1} \) are independent and we define \( Z = \max(Y_n, T_{n-1}) \). The probability distribution of \( Z \) is given by

\[
Z(x) = \Pr(Z \leq x) = \Pr(\max(Y_n, T_{n-1}) \leq x) = \Pr(Y_n \leq x, T_{n-1} \leq x) = \Pr(Y_n \leq x) \Pr(T_{n-1} \leq x) = Y(x)T(x).
\]  

(13)

Moreover, the pmf of \( Z \) is given by

\[
P_Z(x) = \Pr(Z = x) = Z(x) - Z(x - 1) = Y(x)T(x) - Y(x - 1)T(x - 1).
\]  

(14)

As a result, the z-transform of the PAoI is obtained as

\[
A^*(z) = Z^*(z)S^*(z) = \sum_{n=1}^{\infty} (Y(n)T(n) - Y(n - 1)T(n - 1))z^n S^*(z).
\]  

(15)

IV. THE FCFS GEO/GE/OI QUEUE

Consider a discrete time Geo/Geo/1 queue, where the arrival process is modeled as Bernoulli with average probability \( \lambda \in (0, 1) \). The probability distribution of time until successful delivery is assumed to be geometric with mean \( \mathbb{E}[S] = 1/\mu \) slots, where \( \mu \) is referred as the service rate. In the following theorem we obtain the generating functions of PAoI and AoI.

**Theorem 3.** The z-transforms of the PAoI and the AoI for the Geo/Geo/1 queue with an FCFS queue discipline are given by

\[
A^*(z) = \frac{\lambda \mu (\mu - 1)z^2(1 - \mu z)}{(1 - (1 - \lambda)z)(1 - (1 - \mu)z^2(1 - \mu)(1 - \mu z))},
\]  

(16)

and

\[
\Delta^*(z) = \frac{\lambda (\lambda - 1)z^2(1 - \mu z)(2\lambda - 1 - \mu z)}{(1 - (1 - \lambda)z)(1 - (1 - \mu)z^2(1 - \lambda)(1 - \mu z))},
\]  

(17)

respectively.

**Proof.** The proof is given in Appendix B.

As a result, taking the inverse z-transform we can obtain the pmfs of the PAoI and the AoI as

\[
P_A(x) = \mu \left( \frac{(\mu - \lambda) (1 - \mu)}{\lambda (1 - \mu)} \right)^{1-x} + \mu (1 - x)(1 - \mu)^{x-2} + \frac{\lambda (1 - \mu)^{x-1}}{\mu - \lambda} + \frac{(\lambda^2 (\mu - 2) + 2\lambda \mu - \mu^2) (1 - \mu)^{x-2}}{\lambda (\mu - \lambda)}.
\]  

(18)

and

\[
P_{\Delta}(x) = \frac{(\mu - \lambda) (1 - \mu)^{1-x}}{1 - \mu} + \lambda \mu (1 - x)(1 - \mu)^{x-2} + \frac{\lambda (\mu - 1)^{x-1}}{\mu - \lambda} + \frac{((1 - \mu)^x (\lambda^2 (1 - \mu x) + \lambda \mu (x - 1) - 1) + \mu^2)}{\lambda (\mu - \lambda)}.
\]  

(19)

Furthermore, we derive the PDFs of the PAoI and the AoI as follows

\[
A(x) = \frac{1}{\lambda (1 - \mu)(\mu - \lambda)} \left( \frac{1 - \lambda}{1 - \mu} \right)^{-x} (\lambda - 1)\mu (\mu - \lambda) + \left( \frac{1 - \lambda}{1 - \mu} \right)^x \left( (1 - \mu)^x (\lambda^2 (1 - \mu x) + \lambda \mu (x - 1) + 1) + \mu^2 \right),
\]  

(20)
and
\[
\Delta(x) = \frac{1}{(1-\mu)(\mu-\lambda)} \left( \frac{1}{1-\mu} \right)^{-x} \left( -\mu + \lambda^2 \right) \\
(1-x)(1-\mu)^x \left( \frac{1}{1-\mu} \right)^x (1-x) - \mu (1-\mu)^x - 1 \\
(1-\lambda)^x + (\mu-1) \left( \frac{1}{1-\mu} \right)^2 x + \lambda \\
(\mu(x-2)(1-\mu)^x + \mu - 1) \left( \frac{\lambda-1}{\mu-1} \right)^x + \mu + 1 \right).
\]

(21)
These can be utilized for quantifying the AoI or PAoI violation probability that can in turn provide AoI and PAoI violation guarantees for the system.

V. NON-LINEAR AGE FUNCTIONS

Next, we consider a non-negative and monotonically increasing class of functions \( f_s(t) \), \( t \geq 0 \), with \( f_s(0) = 0 \). We define the cost of update delay (CoUD) to be

\[
C_t = f_s(\Delta t) = f_s(t-u(t)),
\]
and the peak cost of update delay (PCoUD) as

\[
C_{\text{peak,} n} = f_s(A_n) = f_s(t_n-t_{n-1}).
\]

In what follows, we provide a general formula that computes the time average CoUD and PCoUD and we derive closed-form expressions for two cases of the \( f_s(t) \) cost function.

**Theorem 4.** For any non-negative, non-decreasing function of age, \( f_s(t) \), \( t \geq 0 \), the time average PCoUD for the Geo/Geo/1 queue with an FCFS queue discipline is obtained as

\[
C_{\text{peak}} = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{N(T)} C_{\text{peak,} n} = \sum_{n=1}^{\infty} f_s(n) P_A(n),
\]
and the time average CoUD is obtained as

\[
C = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} C_t = \sum_{t=0}^{\infty} f_s(t) P_{\Delta}(t).
\]

**Proof.** The relations in (24) and (25) immediately follow from the definition of time average and the fundamental theorem of expectation [27, Page 379].

A. Two examples

For \( f_s(t) = \alpha t \), \( \alpha > 0 \), the average CoUD for the Geo/Geo/1 system with an FCFS queue discipline is given by

\[
C = \alpha \left( \frac{1}{\lambda} + \frac{1-\lambda}{\mu-\lambda} - \frac{\lambda}{\mu^2} + \frac{\lambda}{\mu} \right),
\]
and the average PCoUD is given by

\[
C_{\text{peak}} = \alpha \frac{\lambda^2 - \mu}{\lambda(\lambda-\mu)}.
\]

For \( \alpha = 1 \), the results in (26) and (27) are the AoI and PAoI of an Geo/Geo/1 queue that are found in [10], respectively.

For \( f_s(t) = \alpha t^2 \), \( \alpha > 0 \), the average CoUD for the Geo/Geo/1 system with an FCFS queue discipline is given by

\[
C = \frac{\alpha}{\lambda^2 \mu^4 (\lambda-\mu)^2} \left( \lambda^5 (\mu-1)(\mu+4) + \lambda^3 \mu((\mu-8)\mu+8) + \lambda^2 (\mu-2)^2 \mu^2 + \lambda^4 (\mu+2) + 2\mu^2 \right),
\]
and the average PCoUD is given by

\[
C_{\text{peak}} = \frac{\alpha}{\lambda^2 \mu^2 (\lambda-\mu)^2} \left( \lambda^4 (\mu+2) + \lambda^2 (\mu(6)\mu + 4) + \lambda^2 \mu^3 - \lambda^3 (\mu+2) + 2\mu^4 \right).
\]

In order to find the optimal value of \( \lambda \) that minimizes the average CoUD and PCoUD, for a given \( \mu \), we proceed as follows. We differentiate \( C \) and \( C_{\text{peak}} \) with respect to \( \lambda \) to obtain \( \frac{\partial C}{\partial \lambda} \) and \( \frac{\partial C_{\text{peak}}}{\partial \lambda} \), respectively.

Case 1: For \( f_s(t) = \alpha t \), by setting \( \frac{\partial C}{\partial \lambda} = 0 \) we can obtain the value of \( \lambda \) that minimizes the CoUD and satisfies the equation \( \alpha \lambda(\lambda-1) - 2\lambda^2 (\lambda-1) - \lambda^2 \mu^2 + 2\lambda \mu^2 - \mu^4 = 0 \). Moreover, by setting \( \frac{\partial C_{\text{peak}}}{\partial \lambda} = 0 \) we can obtain the value of \( \lambda \) that minimizes the PCoUD and satisfies the equation \( \alpha \mu(\lambda-2)\lambda + \mu) = 0 \). For both CoUD and PCoUD, taking the second derivative \( \frac{\partial^2 C}{\partial \lambda^2} \) and \( \frac{\partial^2 C_{\text{peak}}}{\partial \lambda^2} \), it can be easily seen that \( C \) and \( C_{\text{peak}} \) are convex functions of \( \lambda \) for a given service rate \( \mu \), if \( \lambda < \mu \) is not violated.

Case 2: For \( f_s(t) = \alpha t^2 \), by setting \( \frac{\partial C}{\partial \lambda} = 0 \) we can obtain the value of \( \lambda \) that minimizes the CoUD and satisfies the equation \( \alpha \lambda^6 (\mu-1)(\mu+4) - 3\lambda^5 (\mu-1)\mu(\mu+4) + \lambda^4 \mu^2 ((15-2\mu)\mu-14) + \lambda^3 \mu^3 (2-3\mu) + \lambda^2 \mu^2 (\mu+2) - \lambda \mu^5 (\mu+10) + 4\mu^6 = 0 \). Moreover, by setting \( \frac{\partial C_{\text{peak}}}{\partial \lambda} = 0 \) we can obtain the value of \( \lambda \) that minimizes the PCoUD and satisfies the equation \( \alpha(\lambda(2)-2)\lambda + \mu(\mu+2) - 4\mu^2 = 0 \). For both CoUD and PCoUD, taking the second derivative \( \frac{\partial^2 C}{\partial \lambda^2} \) and \( \frac{\partial^2 C_{\text{peak}}}{\partial \lambda^2} \), it can be shown that \( C \) and \( C_{\text{peak}} \) are also convex functions of \( \lambda \) for a given service rate \( \mu \), if \( \lambda < \mu \) is not violated.

**Remark.** An interesting observation is that in case of PCoUD both functions are minimized by the same value of \( \lambda^* \) and independently of the choice of the parameter \( \alpha \). Furthermore, for both CoUD and PCoUD we note that the optimal \( \lambda \) is independent of \( \alpha \), that is a scaling factor in the derivative equations.

VI. NUMERICAL RESULTS

In Fig. 2 we depict the PDF of the AoI in (21) as a function of the AoI, for different values of \( \lambda \), and \( \mu = 0.9 \). In addition, we develop a MATLAB-based behavioral simulator where each case runs for \( 10^6 \) time slots, to validate the analytical results. We observe that as the average probability of arrival \( \lambda \) increases, the probability of AoI being smaller that a given value increases as well. However, AoI is not a monotonically decreasing function of \( \lambda \). For \( \lambda = 0.8 \), when \( \lambda \) approaches the service rate \( \mu \), and the queue tends to become unstable, AoI gets larger values. These results can be utilized to provide AoI violation guarantees.
the AoI and PAoI metrics. To illustrate the applicability of the results we consider an FCFS Geo/Geo/1 queue and derive the PDF and the pmf of AoI and PAoI, that can provide violation guarantees. Finally, we illustrate how our results can be used to obtain closed-form expressions of non-linear functions of the average AoI by providing some examples.

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APPENDIX A

PROOF OF THEOREM 1

The summation in (14) can be rewritten as a sum of disjoint parts. Starting from \( t = 0 \), the summation is decomposed into \( Q_n(x) \) for \( n = 1, 2, \cdots, N(T) \), defined as

\[
Q_n(x) = \sum_{t=t'_n}^{t'_{n+1}} \mathbb{I}_{\{\Delta_t \leq x\}} ,
\]

and the areas of width \( t'_n \) and \( T - t'_N(T) \) that we denote \( \tilde{Q} \), defined as

\[
\tilde{Q}(x) = \sum_{t=0}^{t'_1} \mathbb{I}_{\{\Delta_t \leq x\}} + \sum_{t=t'_N(T)}^{T} \mathbb{I}_{\{\Delta_t \leq x\}} .
\]

Using the definition of \( \Delta_t \) in (2) we have

\[
Q_n(x) = \sum_{t=t'_n}^{t'_{n+1}} \mathbb{I}_{\{\Delta_t \leq x\}} = \sum_{t=t'_n}^{t'_{n+1}} \mathbb{I}_{\{\Delta_{t-1}+(t-t'_{n-1}) \leq x\}} =
\]

\[
= \sum_{u=T_n}^{A_{n+1}} \mathbb{I}_{\{u \leq x\}} = \sum_{u=0}^{A_{n+1}} \mathbb{I}_{\{u \leq x\}} - \sum_{u=0}^{T_n} \mathbb{I}_{\{u \leq x\}} =
\]

\[
= \sum_{u=0}^{x} \mathbb{I}_{\{T_n \leq u\}} - \sum_{u=0}^{x} \mathbb{I}_{\{A_{n+1} \leq u\}},
\]

where the last equality follows from the following relation. For \( r \geq 0 \) and \( x \geq 0 \),

\[
\sum_{u=0}^{r} \mathbb{I}_{\{u \leq x\}} = \min(r, x) = \sum_{u=0}^{x} (1 - \mathbb{I}_{\{r \leq u\}}).
\]

Then, the decomposition yields

\[
\Delta^+(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{I}_{\{\Delta_t \leq x\}} =
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \left( \sum_{n=1}^{N(T)} Q_n(x) + \tilde{Q}(x) \right) =
\]

\[
= \lim_{T \to \infty} \left( \frac{N(T)}{T} \frac{1}{N(T)} \sum_{n=1}^{N(T)} Q_n(x) + \frac{\tilde{Q}(x)}{T} \right) =
\]

\[
= \lambda^+ \sum_{u=0}^{x} (T^+(u) - A^+(u)), \tag{34}
\]
Note that the term \( \frac{Q(x)}{T} \) goes to zero as \( T \) grows. In addition, from Lemma 1 we have
\[
\lim_{T \to \infty} \frac{N(T)}{\lambda} = 1.
\]

**APPENDIX B**

**Proof of Theorem 3**

Let \( \rho = \lambda(1-\mu)/\mu(1-\lambda) \). Since \( Y \) is geometrically distributed with parameter \( \lambda \) we know that
\[
Y(x) = \Pr(Y \leq x) = 1 - (1-\lambda)^x.
\]

To derive the pdf and cdf of the system time \( T \), we use the fact that the sum of \( N \) geometric random variables where \( N \) is geometrically distributed is also geometrically distributed, according to the convolution property of their generating functions [28]. Let \( S_j, j = 1, 2, \ldots \) be independent and identically distributed geometric random variables with parameter \( \mu \). If an arriving packet sees \( N \) packets in the system, then, the system time of that packet, using the memoryless property, is
\[
\mu
\]

To obtain the system time pmf we utilize Theorem 2, together with (16) and (37).

Finally, we know that
\[
S^*(z) = \frac{\mu z}{1 - (1-\mu) z}.
\]

Substituting all the relevant expressions to (15) we obtain the result in (16).

To obtain the z-transform of the AoI we utilize Theorem 2, together with (16) and (37).

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