On the Algebraic Theory of Soliton and Antisoliton Sectors

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Abstract

We consider the properties of massive one particle states on a translation covariant Haag-Kastler net in Minkowski space. In two dimensional theories, these states can be interpreted as soliton states and we are interested in the existence of antisolitons. It is shown that for each soliton state there are three different possibilities for the construction of an antisoliton sector which are equivalent if the (statistical) dimension of the corresponding soliton sector is finite.

Introduction

The basic philosophy of algebraic quantum field theory is that the whole information about a physical system can be obtained from the observables, representing physical operations, and a special class of states, representing possible preparations of the physical system.

The physical operations are described by a Haag-Kastler net of C*-algebras in d-dimensional space-time, \( \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \) \([1, 11, 23]\). We denote the smallest C*-algebra containing all algebras \( \mathfrak{A}(\mathcal{O}) \) by \( \mathfrak{A} \). The elements in \( \mathfrak{A}(\mathcal{O}) \) are called local observables and the elements in \( \mathfrak{A} \) are called quasi-local observables. Furthermore, we select a class of states to consider aspects which are relevant in elementary particle physics. A state that describes a massive particle alone in the world (massive one-particle state) \( \omega \) is characterized by its spectral properties, i.e. the translations \( x \mapsto \alpha_x \) are implemented in the GNS-representation of \( \omega \) by unitary operators \( U_\omega(x) \), and the spectrum of \( U_\omega \) consists of
the mass shell \( H_m = \{ p : p^2 = m^2 \} \) and a subset of the continuum \( C_{m+\mu} = \{ p : p^2 > (m+\mu)^2 \} \). For a massive vacuum state, we have a different spectrum condition. Here, the spectrum consists of the value 0 and a subset of the continuum \( C_\mu \). The positive number \( \mu > 0 \) is called the mass gap.

It is shown by D. Buchholz and K. Fredenhagen that for quantum field theories in \( d > 1 + 1 \) space-time dimensions there is for any massive one particle states \( \omega \) a unique related massive vacuum state \( \omega_0 \). Moreover, one can find an endomorphism \( \rho \) acting on a suitable extension \( \mathcal{B} \) of the algebra of quasilocal observables \( \mathfrak{A} \), such that \( \rho|_\mathfrak{A} \) is unitarily equivalent to the GNS-representation of \( \omega \). These endomorphisms, let us call them BF-endomorphisms, are localized in space-like cones \( S \) in the sense that \( \rho(A) = A \) for any \( A \in \mathfrak{A}(\mathcal{O}) \) for which \( \mathcal{O} \) is contained in the space-like complement of \( S \).

We have to mention that there is an older description of particle states, due to S.Doplicher, R.Haag and J.E.Roberts. They select translation covariant representations \( \pi \) whose restrictions to an algebra that belongs to the space-like complement of a double cone \( \mathcal{O} \), are equivalent to the vacuum representation \( \pi_0 \), i.e.: \( \pi(A) = v^* \pi_0(A) v \), for each \( A \in \mathfrak{A}(\mathcal{O}') \). Here \( v \) is a unitary operator which depends on the double cone \( \mathcal{O} \). This leads to endomorphisms \( \rho : \mathfrak{A} \to \mathfrak{A} \), let us call them DHR-endomorphisms, which are localized in double cones.

In two dimensional quantum field theories the massive one particle states can have soliton properties, i.e. for each massive one particle state \( \omega \) there are two massive vacuum states, a left vacuum state \( \omega^- \) and a right vacuum state \( \omega^+ \), such that

\[
\lim_{|x| \to \pm \infty} \omega(\alpha_x A) = \omega_\pm(A).
\]

Here we have set for \( x \in \mathbb{R}^2 \), \( |x| := x^1 - |x^0| \) and for \( |x| \to \pm \infty \) converges \( x \) to \( \pm \)-space-like infinity. The sectors belonging to the states \( \omega_\pm \) are called the asymptotic vacua of \( \omega \).

It turns out that the most natural description of superselection sectors in two dimensional field theories is formulated in terms of algebra homomorphisms. For any vacuum sector \( a \) of \( \mathfrak{A} \) one obtains, in a natural way extensions \( \mathfrak{A}_a^\pm \) of the \( C^* \)-algebra \( \mathfrak{A} \) depending on the space-like directions \( |x| \to \pm \infty \). For each massive one particle state \( \omega \) with left vacuum \( a \) and right vacuum \( b \), one obtains *-homomorphisms \( \rho_+ : \mathfrak{A}_b^+ \to \mathfrak{A}_a^+ \) and \( \rho_- : \mathfrak{A}_a^- \to \mathfrak{A}_b^- \), such that \( \rho_\pm|_\mathfrak{A} \) is unitarily equivalent to the GNS-representation of \( \omega \). These *-homomorphisms,
let us call them one-soliton homomorphisms, are localized in space-like wedges of the type $W_\pm + x$, where $W_\pm$ are the regions given by $W_\pm := \{ x \in \mathbb{R}^2 : |x^0| < \pm x^1 \}$. Thus there are two types of soliton homomorphisms, namely homomorphisms with orientation $q = -$, i.e. $\rho_-$ is localized in a region $W_- + x$, and homomorphisms with orientation $q = +$, i.e. $\rho_+$ is localized in a region $W_+ + x$. Hence for each soliton superselection sector $\pi$ one can find a representative $\rho_+$ with orientation $q = +$ and a representative $\rho_-$ with orientation $q = -$.

In contrast to the DHR- and BF-case, it is not clear whether two soliton sectors can be always composed, but it is possible to compose soliton sectors $\theta_1$ and $\theta_2$, if the left vacuum corresponding to $\theta_1$ is equal to the right vacuum of $\theta_2$. To see this, we choose soliton homomorphisms $\rho_1 : \mathfrak{A}_a^+ \to \mathfrak{A}_b^+$, $\rho_1' : \mathfrak{A}_a^- \to \mathfrak{A}_b^-$, representing the sector $\theta_1$ and soliton homomorphisms $\rho_2 : \mathfrak{A}_c^+ \to \mathfrak{A}_d^+$, $\rho_2' : \mathfrak{A}_c^- \to \mathfrak{A}_d^-$, representing the sector $\theta_2$. The left vacuum of $\theta_1$ is $b$ and the right vacuum of $\theta_2$ is $c$. For $c = b$ we may compose $\rho_2$ and $\rho_1$ as well as $\rho_1'$ and $\rho_2'$. Thus there are two possibilities for the composition of soliton sectors, namely

$$\theta_1 \times \theta = [\rho_1] \times [\rho] := [\rho_1 \rho] \quad \text{and} \quad \theta \circ \theta_1 = [\rho'] \circ [\rho_1'] := [\rho' \rho_1]$$

It is shown in [14] that the homomorphisms $\rho_1 \rho$ and $\rho' \rho_1'$ represent the same superselection sector. Hence both ways to compose soliton sectors are equivalent, i.e.:

$$\theta_1 \times \theta = \theta \circ \theta_1$$

The structure of soliton superselection sectors is discussed in section 1.

We will see in section 2 that there are different ways to obtain an antisoliton sector. First, one can use the methods developed by D. Guido and R. Longo [22]. Let $J_a$ be the modular conjugation with respect to the von Neumann algebra $\pi_a(\mathfrak{A}(W_\pm))''$ and the cyclic and separating vector $\Omega_a$. We define $j_a := \text{Ad}(J_a)$ and interpret

\footnote{A soliton superselection sector is a class $\theta = [\rho]$ of unitarily equivalent soliton homomorphisms $\rho_1 \cong \rho_2$, i.e. the representations $\rho_1|_{\mathfrak{A}}$ and $\rho_2|_{\mathfrak{A}}$ are unitarily equivalent.}

\footnote{For each vacuum sector $a$ we choose a corresponding vacuum state $\omega_a$. $(\mathcal{H}_a, \pi_a, \Omega_a)$ is its GNS-representation. We consider the weak closure of $\pi_a(\mathfrak{A}(W_\pm + x))$ in the algebra of bounded operators $\mathcal{B}(\mathcal{H}_a)$ that is denoted by $\pi_a(\mathfrak{A}(W_\pm + x))''$. The vector $\Omega_a$ is cyclic for all algebras $\pi_a(\mathfrak{A}(W_\pm + x))''$ and hence cyclic and separating for $\pi(a(\mathfrak{A}(W_\pm))'')$. Thus one has modular data $\Delta_a, J_a$ with respect to the pair $(\pi(a(\mathfrak{A}(W_\pm))''\Omega_a))$.}
$j_a \circ \rho \circ j_b : \mathfrak{A}_a \to \mathfrak{A}_b^\dag$ as an antisoliton homomorphism. To show
that $j_a \circ \rho \circ j_b$ is localized in a wedge region, we assume that the
net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is equipped with a PCT-symmetry, i.e. there is an
anti-automorphism $j : \mathfrak{A} \to \mathfrak{A}$, that is implemented in each vacuum
sector $a$ by the modular conjugation $J_a$. If $\rho$ is localized in the wedge
region $W_\pm + x$, one finds that $j_a \circ \rho \circ j_b$ is localized in the PT-reflected
region $W_- - x$.

For our purpose, it is important to consider compositions of soliton
and antisoliton sectors and we wish to obtain an antisoliton homomor-
phism for $\rho$ that is localized in the same region as $\rho$. For this reason,
we choose a soliton homomorphism $\rho' : \mathfrak{A}_b^\dag \to \mathfrak{A}_a^\dag$ that is localized
in $W_- - x$ and represents the same sector as $\rho$. We shall show that
$j_a \circ \rho \circ j_b$ and $j_b \circ \rho' \circ j_a$ belong to the same superselection sector and
we can also choose $\rho_0 := j_b \circ \rho' \circ j_a : \mathfrak{A}_b^\dag \to \mathfrak{A}_a^\dag$ as an antisoliton
homomorphism. We call $\rho_0$ a $C_0$-conjugate for $\rho$ and shall prove that
$\rho_0$ has the following properties:

For a soliton homomorphism $\sigma$ which is unitarily equivalent to $\rho$,
the homomorphisms $\tilde{\sigma}_0$ and $\tilde{\rho}_0$ are unitarily equivalent. The state $\omega_{\rho_0} \circ
\rho_0$ is a massive one particle state possessing the same mass spec-
trum as $\rho$.

To motivate another method for the construction of antisoli
ton sectors, let us return to the $d > 1 + 1$ dimensional case for which
K. Fredenhagen [15] has constructed antiparticle sectors. In order to
understand the physical idea behind his method, let us consider a
particle-antiparticle state which is a state in the vacuum sector. Now
suppose that $\omega$ is of the form

$$\omega(A) = \langle \psi, \pi_0(A) \psi \rangle ,$$

where $\psi$ is a vector in the Hilbert space $\mathcal{H}_0$ of the vacuum represent-
ation $\pi_0$. Since the vacuum can be approximated by shifting a one
particle state to space-like infinity [8], one obtains

$$\omega(A) = \lim_{|x| \to \infty} \langle \psi, \rho(\alpha_x A) \psi \rangle = \lim_{|x| \to \infty} \langle \psi, U_\rho(x) \rho(A) U_\rho(-x) \psi \rangle .$$

Now for observables $A$ that are localized in the space-like complement
of the localization region of $\rho$ one gets

$$\omega(A) = \lim_{|x| \to \infty} \langle \psi, U_\rho(x) A U_\rho(-x) \psi \rangle .$$

The particle which is described by $\rho$ is outside this region and what is
left is an antiparticle state. It is shown in [15] that this state can also
be described by an endomorphism $\bar{\rho}$, i.e. there is a vector $\psi' \in \mathcal{H}_a$ such that:

$$\langle \psi', \bar{\rho}(A) \psi' \rangle = \lim_{|x| \to \infty} \langle \psi, U_\rho(x) A U_\rho(-x) \psi \rangle$$

The endomorphism $\bar{\rho}$ can be chosen to be localized in the same region as $\rho$. Furthermore, by construction $\bar{\rho}\rho$ contains the vacuum representation. Using the statistics operator for $\rho$ and $\bar{\rho}$ we can conclude that $\rho\bar{\rho}$ too contains the vacuum representation and the statistical dimension of $\rho$ is finite. Before we return to the two dimensional situation, let us briefly summarize the results of [13]:

Let $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ be a quantum field theory in $d > 1 + 1$ dimensions. Then there exists for each massive one-particle endomorphisms $\rho$ a massive one-particle endomorphism $\bar{\rho}$, such that (1) $\text{sp}(U_\rho) = \text{sp}(U_{\bar{\rho}})$ and (2) both $\rho\bar{\rho}$ and $\bar{\rho}\rho$ contain the vacuum representation $\pi_0 = \pi$ precisely once.

Indeed, one can interpret $\bar{\rho}$ as an antiparticle for $\rho$. Property (1) shows that the antiparticle has the same mass as the particle and property (2) shows that the antiparticle carries the inverse charge of the particle with respect to the vacuum $\pi_0$.

The methods used by K. Fredenhagen can be also applied to the two dimensional situation, but here one has two possibilities to shift the isolated particle to space-like infinity. Firstly, for a soliton homomorphism $\rho : \mathfrak{A}_a^+ \to \mathfrak{A}_b^+$, one can consider the limit at positive space-like infinity

$$\langle \psi_+, \bar{\rho}_+(A) \psi_+ \rangle = \lim_{|x| \to +\infty} \langle \psi, U_\rho(x) A U_\rho(-x) \psi \rangle$$

Second, one can also consider the limit at negative space-like infinity

$$\langle \psi_-, \bar{\rho}_-(A) \psi_- \rangle = \lim_{|x| \to -\infty} \langle \psi, U_\rho(x) A U_\rho(-x) \psi \rangle$$

We shall show that in the two dimensional case one obtains two soliton homomorphisms $\bar{\rho}_+ : \mathfrak{A}_b^+ \to \mathfrak{A}_a^+$ and $\bar{\rho}_- : \mathfrak{A}_b^+ \to \mathfrak{A}_a^+$, let us call them $C_{\pm}$-conjugates for $\rho$, with the following properties:

The homomorphisms $\bar{\rho}_+$ and $\bar{\rho}_-$ are translation covariant and can be localized in a wedge region $W_+ + x$. Moreover, the homomorphism $\bar{\rho}_+ \rho$ contains the vacuum representation $\pi_a$ and the homomorphism $\rho \bar{\rho}_-$ contains the vacuum representation $\pi_b$.

In contrast to the DHR- and BF-case, we can not conclude that $\rho\bar{\rho}_+$ contains the vacuum representation $\pi_b$ and that $\bar{\rho}_- \rho$ contains the
vacuum representation \( \pi_a \), because for the soliton case there is no analogue of a statistics operator. Moreover, it is not clear whether \( \bar{\rho}_+ \) and \( \bar{\rho}_- \) are equivalent or not.

In section 3 we discuss the relations between the different constructions. We shall show that the \( C_0 \)-conjugate of a \( C_+ \)-conjugate for a soliton homomorphism \( \rho \) is equivalent to the \( C_- \)-conjugate of a \( C_0 \)-conjugate for \( \rho \), i.e.:

\[
 j_a \circ (\bar{\rho}_+) \circ j_b \equiv (j_b \circ \rho \circ j_a)_-
\]

It is shown by R. Longo that for a DHR- or BF-endomorphism \( \rho \) the index of the inclusion \( \rho(\mathcal{A}(S)) \subset \mathcal{A}(S) \), here \( S \) denotes region where \( \rho \) is localized, is precisely the square of the statistical dimension of \( \rho \) [26, 27]. Analogously, the square of the dimension of a soliton homomorphism \( \rho : \mathcal{A}_a^\pm \rightarrow \mathcal{A}_b^\pm \) is defined by the index of the inclusion \( \rho(\mathcal{A}(W_{\pm} + x)_a) \subset \mathcal{A}(W_{\pm} + x)_b \), i.e.:

\[
 d(\rho)^2 = \text{ind}(\rho(\mathcal{A}(W_{\pm} + x)_a), \mathcal{A}(W_{\pm} + x)_b)
\]

Here \( W_{\pm} + x \) is the localization region of \( \rho \).

We shall prove that the dimension of a soliton homomorphism \( \rho : \mathcal{A}_a^\pm \rightarrow \mathcal{A}_b^\pm \) is finite if and only if there exists a homomorphism \( \tilde{\rho} : \mathcal{A}_b^\pm \rightarrow \mathcal{A}_a^\pm \) such that \( \tilde{\rho} \rho \) contains the vacuum representation \( \pi_a \) and \( \rho \bar{\rho} \) contains the vacuum representation \( \pi_b \). We mention that this is a slight generalization of R. Longo’s result in [27] and we are using the same methods to prove it.

Suppose a massive one soliton homomorphism \( \rho : \mathcal{A}_a^\pm \rightarrow \mathcal{A}_b^\pm \) fulfills one of the following three conditions:

1: The dimension of \( \rho \) is finite.
2: The \( C_+ \)-conjugate and the \( C_- \)-conjugate of \( \rho \) are equivalent.
3: Either, the \( C_+ \)- or the \( C_- \)-conjugate and the \( C_0 \)-conjugate are equivalent.

Then we will show that there exists a soliton homomorphism \( \tilde{\rho} : \mathcal{A}_b^\pm \rightarrow \mathcal{A}_a^\pm \) with the following properties:

The homomorphism \( \tilde{\rho} \) is by one of the properties 1-3, up to unitary equivalence, uniquely determined. The state \( \omega_a \circ \tilde{\rho} \) is a massive one particle state possessing the same mass spectrum as \( \rho \). Furthermore, \( \tilde{\rho} \rho \) contains the vacuum representation \( \pi_a \) and \( \rho \bar{\rho} \) contains the vacuum representation \( \pi_b \).
1 Soliton Sectors

In this section, we discuss the mathematical structure of soliton sectors from the algebraic point of view. Based on the work of K. Fredenhagen [14], we describe the construction of soliton homomorphisms which are the analogue of the charged endomorphisms in the DHR- and BF-analysis [11, 8].

Before we start our discussion, we give some mathematical preliminaries. Let us denote by \( W_{\pm} \subset \mathbb{R}^2 \) the wedge region \( \{(t,x) : |t| < \pm x\} \).

A double cone \( O \subset \mathbb{R}^2 \) is a non-empty intersection of a left wedge region \( W_- + x \) and a right wedge region \( W_+ + y, O = W_- + x \cap W_+ + y \). The collection of all double cones is denoted by \( K \).

We list now the axioms which should be fulfilled by the theories under consideration.

**Axiom I:** There is a net of \( C^* \)-algebras, i.e. a prescription that assigns to each double cone \( O \in K \) a \( C^* \)-algebra \( \mathfrak{A}(O) \), called algebra of local observables. Furthermore, this prescription is isotonous, i.e.

\[
O \subset O_1 \implies \mathfrak{A}(O) \subset \mathfrak{A}(O_1)
\]

and local, i.e.

\[
O \subset O'_1 \implies [\mathfrak{A}(O), \mathfrak{A}(O_1)] = \{0\}
\]

where \( O'_1 \) is the space-like complement of \( O_1 \). We call the \( C^* \)-algebra

\[
\mathfrak{A} := \bigcup_{O \in K} \mathfrak{A}(O)
\]

the algebra of quasi local observables. Moreover, suppose there exists a group homomorphism \( \alpha \) from the two dimensional translation group \( T(\mathbb{R}^2) \cong \mathbb{R}^2 \) into the automorphism group \( \text{aut}(\mathfrak{A}) \) of \( \mathfrak{A} \), such that

\[
\alpha_x(\mathfrak{A}(O)) \subset \mathfrak{A}(O + x)
\]

for each \( x \in \mathbb{R}^2 \). Such a prescription \( O \in K \mapsto \mathfrak{A}(O) \) is called a two-dimensional Haag-Kastler-net with translation covariance.

**Axiom II:** There are translation covariant states on \( \mathfrak{A} \). These are states \( \omega \) such that there is a unitary strongly continuous representation \( U_\omega \) of the translation group on the GNS-Hilbert space.
\( \mathcal{H}_\omega \) of \( \omega \), with the property

\[
\pi_\omega(\alpha_x A) = U_\omega(x) \pi_\omega(A) U_\omega(x)^* \quad \forall A \in \mathfrak{A}; \quad \forall x \in \mathbb{R}^2.
\]

Here \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) is the GNS-triple, belonging to \( \omega \). Moreover, the set of massive one-particle and massive vacuum states is not empty. A massive one-particle state is a pure translation covariant state on \( \mathfrak{A} \), such that the spectrum \( \text{sp}(U_\omega) \) of the generators of the translation group \( U_\omega \) is consists of the mass shell \( H_m = \{ p : p^2 = m^2 \} \) and a subset of the continuum \( C_{m+\mu} = \{ p : p^2 > (m + \mu)^2 \} \). For a massive vacuum state, we have a different spectrum condition. Here, the spectrum consists of the value 0 and a subset of \( C_{\mu} \). The positive number \( \mu > 0 \) is called the mass gap.

We say that two states \( \omega, \omega_1 \) belong to the same sector if their GNS-representations \( \pi_\omega, \pi_{\omega_1} \) are unitarily equivalent, i.e., there is a unitary operator \( u : \mathcal{H}_\omega \to \mathcal{H}_{\omega_1} \), such that \( u \pi_\omega(A) = \pi_{\omega_1}(A) u \), for each \( A \in \mathfrak{A} \). A state \( \omega_1 \) belongs to the folium of a state \( \omega \) if \( \omega_1 \) is of the form \( \omega_1(A) = \text{tr}(T \pi_\omega(A)) \) for each \( A \in \mathfrak{A} \), where \( T \) is a trace class operator in the algebra \( \mathcal{B}(\mathcal{H}_\omega) \) of bounded operators on \( \mathcal{H}_\omega \).

Let us denote the collection of all massive vacuum sectors by \( \text{sec}_0 \). For each sector \( a \in \text{sec}_0 \), we fix one massive vacuum state \( \omega_a \) and write \( (\mathcal{H}_a, \pi_a, \Omega_a) \) for the GNS-triple of \( \omega_a \).

For later purpose, let us define the following algebras:

(a) The wedge C*-algebras

\[
\mathfrak{A}(W_{\pm} + x) := \bigcup_{\mathcal{O} \subset W_{\pm} + x} \mathfrak{A}(\mathcal{O})^{\| \cdot \|}. 
\]

(b) The von-Neumann-algebras with respect to the vacuum \( a \in \text{sec}_0 \)

\[
\mathfrak{A}(W_{\pm} + x)_a := \pi_a(\mathfrak{A}(W_{\pm} + x))'' \quad .
\]

Without loss of generality, we may assume that the representation \( \pi_a \) is faithful \([14]\). Thus we can interpret \( \mathfrak{A}(W_{\pm} + x) \) as a common weakly dense subalgebra of the von-Neumann-algebras \( \mathfrak{A}(W_{\pm} + x)_a \). Obviously, the algebras

\[
\mathfrak{A}_{\pm}^a := \bigcup_{x \in \mathbb{R}^2} \mathfrak{A}(W_{\pm} + x)_a^{\| \cdot \|}
\]

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are extensions of the C*-algebra of quasi-local observables $\mathfrak{A}$ and the vacuum representation $\pi_a$ has canonical extensions $\pi_a^\pm$ to the C*-algebras $\mathfrak{A}^\pm_a$, for which we write shortly $\pi_a$ instead of $\pi_a^\pm$.

**Definition 1.1** A state $\omega$ on $\mathfrak{A}$ is called a soliton state, if it satisfies the following conditions:

1. The state $\omega$ is translation covariant.
2. There are vacuum sectors $a, b \in \text{sec}_0$ and unitary operators $v : H_\omega \to H_b$ and $v' : H_\omega \to H_a$, such that:

$$\pi_\omega(A') = v^* \pi_a(A') v' \quad \text{and} \quad \pi_\omega(A) = v^* \pi_b(A) v$$

for all $A \in \mathfrak{A}(W_+)$ and for all $A' \in \mathfrak{A}(W_-)$.

The vacuum sectors $a, b$ are called asymptotic vacua associated with the state $\omega$.

One can deduce from the properties of soliton states the following statement [14]:

**Lemma 1.1** Let $\omega$ be a soliton state, $a, b \in \text{sec}_0$ the asymptotic vacua as in Definition 1.1 above and $\pi_\omega$ its GNS-representation. Then there are unique extensions of $\pi_\omega$,

$$\pi^+_\omega : \mathfrak{A}^+_b \to \mathcal{B}(H_\omega) \quad \text{and} \quad \pi^-_\omega : \mathfrak{A}^-_a \to \mathcal{B}(H_\omega)$$

For a soliton state $\omega$ with asymptotic vacua $a, b$ as above, we define $^*$-homomorphisms

$$\rho = \text{Ad}(v') \circ \pi^+_\omega : \mathfrak{A}^+_b \to \mathcal{B}(H_a) \quad \text{and} \quad \rho' = \text{Ad}(v) \circ \pi^-_\omega : \mathfrak{A}^-_a \to \mathcal{B}(H_b)$$

where $v, v'$ are given as in (1). By construction, the restrictions $\rho|_{\mathfrak{A}}$ and $\rho'|_{\mathfrak{A}}$ are unitarily equivalent. To obtain homomorphisms that map $\mathfrak{A}^+_a$ into $\mathfrak{A}^+_a$ or $\mathfrak{A}^-_a$ into $\mathfrak{A}^-_a$ we have to make an assumption.

**Assumption 1** The algebras $\mathfrak{A}(W_\pm + x)_a$ fulfill duality for wedges for all vacuum sectors $a \in \text{sec}_0$, namely

$$\mathfrak{A}(W_\pm + x)_a = \mathfrak{A}(W_\mp + x)'_a$$

It can then be shown by combining Assumption 1 with the methods in [14] that the image of $\rho$ is contained in $\mathfrak{A}^+_a$ and that the image of $\rho'$ is contained in $\mathfrak{A}^-_a$. 

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By construction, the \(*\)-homomorphism \(\rho\) is localized in \(W_+ + x\), i.e.:

\[
\rho|_{A(W_+ + x)} = \pi_{A(W_+ + x)}
\]

Moreover, \(\rho\) is implemented by a unitary operator \(u : \mathcal{H}_b \to \mathcal{H}_a\) on any von-Neumann-algebra \(\mathfrak{A}(W_+ + y)_b\). Using equation (1), one sees that for each \(A \in \mathfrak{A}(W_+ + x)_b\) one has

\[
\rho(A) = v'v^* Avv'^*.
\]

Since \(\rho\) is translation covariant, it is implemented in such a way on each algebra \(\mathfrak{A}(W_+ + y)_b\). Of course, the same is true if one exchanges \(\rho\) by \(\rho'\), \(a\) by \(b\) and the + by the − sign.

From the discussion above, we see that for each soliton state \(\omega\) there are two types of soliton homomorphisms, which represent the same superselection sector. The homomorphism \(\rho\) is localized in a right wedge region \(W_+ + x\), describing the creation of a soliton charge \([\rho]\) out of the vacuum \(a\) and connecting it to the vacuum \(b\). On the other hand, \(\rho'\) is localized in a left wedge region, describing the creation of the same charge \([\rho]\) out of the vacuum \(b\) and connecting it to the vacuum \(a\). In the following, we use the notation \(\rho : a \to b\), \(\rho' : b \to a\).

Now, consider a soliton state \(\omega\), with asymptotic vacua \(a, b\) and a soliton state \(\omega_1\) with asymptotic vacua \(c, d\). For each of the two states \(\omega, \omega_1\), we obtain a \(*\)-homomorphism localized in a left wedge and a \(*\)-homomorphism localized in a right wedge, i.e.:

For \(\omega\) we obtain \(\rho : \mathfrak{A}_b^+ \to \mathfrak{A}_a^+\) and \(\rho' : \mathfrak{A}_a^- \to \mathfrak{A}_b^-\).

For \(\omega_1\) we obtain \(\rho_1 : \mathfrak{A}_d^+ \to \mathfrak{A}_c^+\) and \(\rho'_1 : \mathfrak{A}_c^- \to \mathfrak{A}_d^-\).

For \(b = c\), the algebra \(\rho_1(\mathfrak{A}_d^+)\) is contained in \(\mathfrak{A}_b^+\) and \(\rho'(\mathfrak{A}_d^-)\) is contained in \(\mathfrak{A}_c^-\). Thus we may compose \(\rho\) with \(\rho_1\) and \(\rho'_1\) with \(\rho'\) and obtain further \(*\)-homomorphisms

\[
\rho \rho_1 : \mathfrak{A}_d^+ \to \mathfrak{A}_a^+ \quad \text{and} \quad \rho'_1 \rho' : \mathfrak{A}_a^- \to \mathfrak{A}_d^-.
\]

The proof of the following statement can be found in [14].

**Proposition 1.1** Both representations

\[
\rho \rho_1|_{\mathfrak{A}} \quad \text{and} \quad \rho'_1 \rho'|_{\mathfrak{A}}
\]

are unitarily equivalent.
From the results of [8, 14, 36] we may conclude that there is a special class of soliton states, namely states with particle character. This fact can be formulated by the following proposition:

**Proposition 1.2** If \( \omega \) is a massive one-particle state, then \( \omega \) is a soliton state.

We call massive one-particle states simply one soliton states and the corresponding homomorphisms one-soliton homomorphisms.

For \( q = \pm \), let us denote by \( \Delta(q, x) \), the set of all soliton homomorphisms which are localized in \( W_q + x \). In addition to that, the set of one-soliton homomorphisms is denoted by \( \Delta_1(q, x) \).

For each soliton homomorphism \( \rho : a \to b \) contained in \( \Delta(q, x) \), we define the source \( s(\rho) := a \) and the range \( r(\rho) := b \). The value \( q \) is called the orientation of a soliton homomorphism.

A soliton homomorphism \( \rho_1 \) is called a subobject of a soliton homomorphism \( \rho \), if there exists an isometric intertwiner \( v \) from \( \rho_1 \) to \( \rho \), i.e.:

\[
    v \rho_1(A) = \rho(A)v \quad ; \quad A \in \mathfrak{A}
\]

Furthermore, one has \( s(\rho) = s(\rho_1) \) and \( r(\rho) = r(\rho_1) \) if \( \rho_1 \) is a subobject of \( \rho \). Conversely, we are able to define the direct sum \( \rho_1 \oplus \rho_2 \), for each pair of soliton homomorphisms \( \rho_1, \rho_2 \) with \( s(\rho_1) = s(\rho_2) \) and \( r(\rho_1) = r(\rho_2) \). According to [3] we can find two isometries \( v_1, v_2 \in \mathfrak{A}(W_q + x) \) with complementary range and define \( \rho_1 \oplus \rho_2 \) as follows:

\[
    \rho(A) = (\rho_1 \oplus \rho_2)(A) := v_1 \rho_1(A) v_1^* + v_2 \rho_2(A) v_2^*
\]

If \( W_q + x \) contains the localization regions of \( \rho_1 \) and \( \rho_2 \), \( \rho \) is a soliton homomorphism which is also localized in \( W_q + x \).

**Lemma 1.2** 1: If \( \rho \) is a soliton homomorphism and \( \pi_1 \) a subrepresentation of \( \rho|_{\mathfrak{A}} \), i.e. there is an isometry \( v : \mathcal{H}_{\pi_1} \to \mathcal{H}_{s(\rho)} \) with

\[
    v \pi_1(A) = \rho(A)v \quad ; \quad A \in \mathfrak{A}
\]

then there exists a subobject \( \rho_1 \) of \( \rho \), such that \( \rho_1|_{\mathfrak{A}} \) is unitarily equivalent to \( \pi_1 \).

2: For each pair of soliton homomorphisms \( \rho_1, \rho_2 \in \Delta(q, x) \), the space of intertwiners \( I(\rho_1, \rho_2) \) contains nontrivial elements, only if the sources and ranges of \( \rho_1 \) and \( \rho_2 \) are equal.

\[
    I(\rho_1, \rho_2) \neq \{0\} \implies s(\rho_1) = s(\rho_2) \text{ and } r(\rho_1) = r(\rho_2)
\]
Proof. 1: We choose $\rho : a \rightarrow b$ to be a soliton homomorphism, localized in $W_+ + x$ and $\rho' : b \rightarrow a$ a soliton homomorphism localized in $W_- + x$, such that the representations $\rho|_A$ and $\rho'|_A$ are unitarily equivalent. As mentioned above, such a choice is always possible. For a subrepresentation $\pi_1$ of $\rho|_A$ there are isometries $v : H_{\pi_1} \rightarrow H_a$ and $v' : H_{\pi_1} \rightarrow H_b$ such that

$$v\pi_1(A) = \rho(A)v \quad \text{and} \quad v'\pi_1(A) = \rho'(A)v' .$$

The projection $E = vv^*$ is contained in $\mathfrak{A}(W_+ + x)_a$ and the projection $E' = v'v'^*$ is contained in $\mathfrak{A}(W_- + x)_b$, since $\rho$ is localized in $W_+ + x$ and $\rho'$ is localized in $W_- + x$. According to \[6\] there are isometries $w \in \mathfrak{A}(W_+ + x)_a$ and $w' \in \mathfrak{A}(W_- + x)_b$, such that $E = ww^*$ and $E' = w'w'^*$. Now we define the unitaries $v_1' := w^*v$ and $v_1 := w'^*v'$ and obtain for $A \in \mathfrak{A}(W_+ + x)$ and $A' \in \mathfrak{A}(W_- + x)$:

$$\pi_1(A) = v_1^*\pi_0(A)v_1 \quad \text{and} \quad \pi_1(A') = v_1'^*\pi_a(A')v_1'$$

Thus $\pi_1$ is a soliton representation which implies that there is a subobject for $\rho$ which is unitarily equivalent to $\pi_1$.

2: Suppose there are soliton homomorphisms $\rho_1 : a \rightarrow b$ and $\rho_2 : c \rightarrow d$, such that there is a nontrivial intertwiner $v : H_a \rightarrow H_c$ that intertwines the representations $\rho_1|_A$ and $\rho_2|_A$. If $\rho_1$ and $\rho_2$ have the same orientation, then we can conclude by 1 that there is a joint subobject $\rho$ for $\rho_1$ and $\rho_2$. Now we obtain

$$a = s(\rho) = c \quad \text{and} \quad b = r(\rho) = d$$

and the proof is complete. q.e.d.

Corollary 1.1 The set of soliton homomorphisms $\Delta(q, x)$ is closed under multiplication, taking direct sums and subobjects.

Definition 1.2 Two soliton homomorphisms $\rho \in \Delta(q, x)$ and $\rho \in \Delta(p, y)$ are called inner unitarily equivalent, if $p = q$ and there exists a unitary intertwiner $u \in I(\rho, \rho_1$). We call $\rho$ and $\rho_1$ unitarily equivalent, if the representations $\rho|_A$ and $\rho_1|_A$ are unitarily equivalent. We denote the set of inner unitary equivalence classes by $sec(q)$ and the set of unitary equivalence classes by $sec$. We call the elements of $sec(q)$
inner soliton sectors and the elements of sec soliton sectors. The projection which maps a soliton homomorphism onto its inner unitary equivalence class is denoted by $e : \rho \in \Delta(q,x) \mapsto e(\rho) \in \text{sec}(q)$, the projection which maps a soliton homomorphism onto its unitary equivalence class is denoted as usual by $[\cdot] : \rho \in \Delta(q,x) \mapsto [\rho] \in \text{sec}$.

According to Lemma 1.2, we may define the source and range maps and the composition for inner soliton sectors:

$$s(e(\rho)) := s(\rho) \quad r(e(\rho)) := r(\rho)$$

$$e(\rho_1)e(\rho_2) := e(\rho_1\rho_2) \quad \text{for} \quad s(\rho_2) = r(\rho_1)$$

The inner sector which belongs to the identity is denoted by $i_a := e(\pi_a)$, $a \in \text{sec}_0$.

Of course, the inner unitary equivalence is stronger than the unitary equivalence, because the inner unitary equivalence does not forget the information about the orientation of a soliton homomorphism whereas the unitary equivalence only sees the charge of a soliton homomorphism.

Both, the set of soliton homomorphisms $\Delta(q,x)$ and the set of inner soliton sectors $\text{sec}(q)$ are naturally equipped with a category structure. The objects are the vacuum sectors and the arrows are the soliton homomorphisms or inner soliton sectors. Moreover, there is a second natural category structure, namely the objects are soliton homomorphisms $\rho, \rho_1 \in \Delta(q,x)$ and the arrows are intertwiner $v \in I(\rho_1, \rho)$. Such a structure is also known as a 2-$C^*$-category. See [34] for this notion.

In the following lines we show that soliton homomorphisms which are unitarily equivalent and have the same orientation, are indeed inner unitarily equivalent.

**Lemma 1.3** Let $\rho \in \Delta(q,x)$ and $\rho' \in \Delta(p,y)$ be soliton homomorphisms with $[\rho] = [\rho']$, then for $p = q$ it follows: $e(\rho) = e(\rho') \in \text{sec}(p = q)$.

**Proof.** Since $\rho$ and $\rho'$ are localized in a $q$-wedge region, there is $z \in \mathbb{R}^2$, such that both $\rho$ and $\rho'$ are localized in $W_q + z$. The representations $\rho|_{\mathcal{A}}$ and $\rho'|_{\mathcal{A}}$ are unitarily equivalent, and there is a unitary intertwiner $u$ from $\rho$ to $\rho'$. But for $A \in \mathcal{A}(W_{-q} + z)$ we obtain $[A,u] = 0$. Thus, $u$ is contained in $\mathcal{A}(W_q + z)_{s(\rho)}$ and we conclude that $\rho$ and $\rho'$ are inner unitarily equivalent. q.e.d.
**Remark:** From the discussion above it follows that unitarily equivalent soliton homomorphisms with opposite orientation can not be inner equivalent.

Before we close this section, let us summarize the mathematical structure of the set of inner soliton sectors $\sec(q)$.

As mentioned above, one can interpret $\sec(q)$ as a category whose objects are the vacuum sectors $\sec_0$ and whose arrows are the inner soliton sectors $\sec(q)$. We write $\sec(q|a,b)$ for the set of arrows $\theta \in \sec(q)$ with $s(\theta) = a$ and $r(\theta) = b$.

With respect to the direct sum $\oplus$ of soliton homomorphisms, the sets of arrows $\sec(q|a,b)$ are commutative rings over the natural numbers.

For arrows $\theta_1, \theta_2 \in \sec(q|a,b)$, $\vartheta \in \sec(q|b,c)$ and $\hat{\vartheta} \in \sec(q|c,a)$ the following distributive laws are fulfilled:

$$(\theta_1 \oplus \theta_2) \vartheta = \theta_1 \vartheta \oplus \theta_2 \vartheta \quad \text{and} \quad \hat{\vartheta}(\theta_1 \oplus \theta_2) = \hat{\vartheta} \theta_1 \oplus \hat{\vartheta} \theta_2$$

The sets of arrows $\sec(q|a,b)$ are equipped with a partial order relation. We write $\theta_1 < \theta$ if $\theta_1$ is a subobject for $\theta$. This relation satisfies $\theta_1 < \theta_1 \oplus \theta$.

For later purpose, it is convenient to consider a special class of functorial operations from $\sec(q)$ to $\sec(p)$.

**Definition 1.3** A map $j : \sec(q) \rightarrow \sec(p)$ is called a conjugation or co-conjugation if $j$ fulfills the following conditions:

1 **Functor, Cofunctor Condition:** For $s(\theta_2) = r(\theta_1)$ we have:

   Functor : $j(\theta_1 \theta_2) = j(\theta_1) j(\theta_2)$ or Cofunctor : $j(\theta_1 \theta_2) = j(\theta_2) j(\theta_1)$

2 **Additivity:** For $s(\theta_1) = s(\theta_2)$ and $r(\theta_1) = r(\theta_2)$ we have:

   $$j(\theta_1 \oplus \theta_2) = j(\theta_1) \oplus j(\theta_2)$$

3 **Isotony:** Let $\theta_1$ be a subobject of $\theta$, then $j(\theta_1)$ is a subobject of $j(\theta)$.

4 **Involution:** The map $j$ is involutive, i.e.: $j \circ j = \pi$.

We will see later, that the charge conjugation is one example for a co-conjugation in the sense of Definition 1.3.
2 Candidates for Antisoliton Sectors

As described in the introduction, there are different constructions of antisoliton sectors available. The first construction uses the concepts of D.Guido and R.Longo [22].

For a soliton homomorphism \( \rho : a \rightarrow b \) which is localized in \( W_+ + x \), the PCT-conjugate homomorphism is given by the formula 
\[
j(\rho) = J_a \rho(J_b A J_b) J_a,
\]
where \( J_a \) is the PCT-antiunitary with respect to the vacuum \( a \). To obtain an antisoliton homomorphism which is also localized in \( W_+ + x \) one has to take \( \bar{\rho}_0 := J_b \rho'(J_a A J_a) J_b \) as an antisoliton candidate, where \( \rho' : b \rightarrow a \) is a soliton homomorphism localized in \( W_- - x \) and unitarily equivalent to \( \rho : a \rightarrow b \). We interpret the reflection of the localization region as a PT-conjugation and call such a \( \rho' : b \rightarrow a \) a PT-conjugate for \( \rho \).

We apply the methods which were used by K.Fredenhagen [15] to obtain an alternative construction. As described in the introduction, one has to control the following limit:
\[
\lim_{|x| \rightarrow +\infty} \langle \psi, U_\rho(x) A U_\rho(-x) \psi \rangle = \bar{\omega}_\rho(A) \quad ; \quad \psi \in \mathcal{H}_a \quad ; \quad ||\psi|| = 1
\]
Thus we get a state \( \bar{\omega}_\rho \) which belongs to a soliton sector \( \bar{\rho}_+ \), such that \( \bar{\rho}_+ \times [\rho] \) contains the vacuum \( b \). But there is another possibility left over. One can also take a PT-conjugate soliton homomorphism \( \rho' : b \rightarrow a \), and show that the limit
\[
\lim_{|x| \rightarrow -\infty} \langle \psi', U_{\rho'}(x) A U_{\rho'}(-x) \psi' \rangle = \bar{\omega}_{\rho'}(A) \quad ; \quad \psi' \in \mathcal{H}_b \quad ; \quad ||\psi'|| = 1
\]
exists. Now, \( \bar{\omega}_{\rho'} \) belongs to a sector \( \bar{\rho}_- \), such that \( [\rho] \times \bar{\rho}_- \) contains the vacuum \( a \). Thus this construction leads to two further candidates for antisoliton sectors which are represented by the states \( \bar{\omega}_\rho \) and \( \bar{\omega}_{\rho'} \).

Remark: That one has to distinguish the two constructions of \( \bar{\omega}_\rho \) and \( \bar{\omega}_{\rho'} \), is not an effect of a difference in the vacuum sectors \( a = s(\rho) \) and \( b = r(\rho) \). This problem also arises for improper soliton homomorphisms \( \rho \) with \( s(\rho) = r(\rho) \). The true reason for this distinction is the fact that the space-like complement of an arbitrarily small double-cone (in \( d = 1 + 1 \) dimensions) is a disconnected region.

In the sequel, we study the question which of the following properties are fulfilled by each of the candidates.

We ask, whether a particle and its antiparticle have the same mass, i.e.
For a soliton homomorphism $\rho \in \Delta(q,x)$, the antisoliton homomorphism $\bar{\rho} \in \Delta(q,x)$ satisfies:

$$\text{sp}(U_\rho) = \text{sp}(U_{\bar{\rho}})$$

We also study the question if the antiparticle carries the inverse charge of the corresponding particle, i.e.:

A2: For a soliton homomorphism $\rho \in \Delta(q,x)$ the antisoliton homomorphism $\bar{\rho}$ fulfills the following relations:

$$s(\bar{\rho}) = r(\rho) \quad \text{and} \quad r(\bar{\rho}) = s(\rho)$$

$$\rho \bar{\rho} > \pi_{s(\rho)} \quad \text{and} \quad \bar{\rho} \rho > \pi_{r(\rho)}$$

Furthermore, we try to find out whether the antiparticle of the antiparticle is the particle itself, i.e.:

A3: For a soliton homomorphism $\rho \in \Delta(q,x)$ the antisoliton fulfills following relation:

$$e(\bar{\rho}) = e(\rho)$$

### 2.1 The $C_0$-Conjugation for Soliton Sectors

**Definition 2.1** For a soliton homomorphism $\rho \in \Delta(q,x)$, we call a unitarily equivalent soliton homomorphism $\rho' \in \Delta(-q,-x)$ a PT-conjugate for $\rho$.

**Lemma 2.1** For each soliton homomorphism $\rho$ there exists a PT-conjugate $\rho'$, unique up to inner unitary equivalence, such that the map

$$j_{PT}: e(\rho) \in \sec(q) \mapsto e(\rho') \in \sec(-q)$$

is a well-defined co-conjugation.

**Proof.** According to Lemma 1.3, we conclude that $j_{PT}$ is a well-defined map. Since $j_{PT}$ preserves unitary equivalence, i.e. $[j_{PT}e(\rho)] = [\rho]$, it follows that $j_{PT}$ is additive and isotonous. It is also clear that $j_{PT}$ is involutive. For $\rho_j \in \Delta(q,x)$ we choose PT-conjugates $\rho'_j \in \Delta(-q,-x)$, $j = 1,2$. For $r(\rho_2) = s(\rho_1)$ we have $s(\rho'_2) = r(\rho'_1)$. As mentioned in the last section, the representations $\rho'_2\rho'_1|_A$ and $\rho_1\rho_2|_A$ are unitarily equivalent and $\rho'_2\rho'_1$ is indeed a PT-conjugate for $\rho_1\rho_2$. Thus $j_{PT}$ fulfills the cofunctor condition. q.e.d.
For each vacuum sector $a \in \text{sec}_0$, the algebra $\mathfrak{A}(W_q)_a$ is a von-Neumann-algebra with cyclic separating vector $\Omega_a$. The modular data of $(\mathfrak{A}(W_+)_a, \Omega_a)$ are denoted by $(J_a, \Delta_a)$. According to the theorem of Borchers [4], we obtain the relation

$$J_a U_a(x) J_a = U_a(-x)$$

for each $x \in \mathbb{R}^2$ and for each vacuum sector $a$. We define $j_a := \text{Ad}(J_a)$, which maps $\mathfrak{A}^+_{a}$ onto $\mathfrak{A}^-_{a}$ and vice versa. For a soliton homomorphism $\rho \in \Delta(q,x)$, we define the *-homomorphism

$$j(\rho) := j_s(\rho) \circ \rho \circ j_r(\rho)(A)$$

which maps $\mathfrak{A}^-_{r(\rho)}$ into $\mathfrak{A}^-_{s(\rho)}$.

**Assumption 2** Let us assume that there exists PCT-symmetry, i.e. an involutive antiautomorphism $j : \mathfrak{A} \mapsto \mathfrak{A}$ with $j(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(-\mathcal{O})$ and $j \circ \alpha_x = \alpha_{-x} \circ j$ which is implemented in each vacuum sector $a \in \text{sec}_0$ by the modular conjugation $J_a$, i.e.:

$$\pi_a(jA) = J_a \pi_a(A) J_a$$

**Proposition 2.1** Let $j(\rho) = j_s(\rho) \circ \rho \circ j_r(\rho)$ be the *-homomorphism defined as above. If the soliton homomorphism $\rho$ is a one soliton homomorphism, then the *-homomorphism $j(\rho)$ is also a one soliton homomorphism which has the same mass spectrum as $\rho$. If $\rho$ is a *-homomorphism localized in $W_q + x$, then $j(\rho)$ is localized in $W_{-q} - x$.

**Proof.** It is not hard to check the following equation:

$$j(\rho)(\alpha_x A) = J_s(\rho) U_\rho(-x) J_s(\rho) j(\rho)(A) J_s(\rho) U_\rho(x) J_s(\rho) ; \quad A \in \mathfrak{A}$$

Thus, $U_{j(\rho)}(x) := J_s(\rho) U_\rho(-x) J_s(\rho)$ implements the translation group in the representation $j(\rho)|_\mathfrak{A}$. Since $\text{sp}(U_\rho) = \text{sp}(U_{j(\rho)})$ and $\rho$ is a massive one-particle representation, $j(\rho)$ is also a massive one-particle representation which has the same mass spectrum as $\rho$. Now, for each $A \in \mathfrak{A}(\mathcal{O})$ with $\mathcal{O} \subset W_q + x$ we obtain

$$j_s(\rho) \circ \rho \circ j_r(\rho)(A) = j_s(\rho) \circ \rho \circ \pi_r(\rho)(jA)$$

$$= j_s(\rho) \circ \pi_s(\rho)(jA) = \pi_s(\rho)(jjA)$$

$$= \pi_s(\rho)(A)$$

Thus $j(\rho)$ is localized in $W_{-q} + x$. q.e.d.
Theorem 2.1 
i: The map $j_{PCT}: \text{sec}(q) \to \text{sec}(-q)$ which is defined by

$$j_{PCT}(\rho) := e(j(\rho))$$

is a conjugation (Definition 1.3), called PCT-conjugation.

ii: The PCT-conjugation commutes with the PT-conjugation, i.e.:

$$j_0 := j_{PCT} \circ j_{PT} = j_{PT} \circ j_{PCT} : \text{sec}(q) \to \text{sec}(q)$$

The co-conjugation $j_0$ is called $C_0$-conjugation and for a soliton homomorphism $\rho \in \Delta(q,x)$ we call $\tilde{\rho}_0 \in \Delta(q,x)$ with $e(\tilde{\rho}_0) = j_0 e(\rho)$ a $C_0$-conjugate for $\rho$.

Proof. i: By construction, the soliton homomorphism $j(\rho)$ is contained in the set $\Delta(-q,-x)$, if $\rho$ belongs to $\Delta(q,x)$. Thus the map $j$ is a well defined function. For $\rho_j \in \Delta(q,x_j)$, $j = 1, 2$, $s(\rho_2) = r(\rho_1)$, we have:

$$j(\rho_1 \rho_2) = j_{s(\rho_1)} \circ \rho_1 \rho_2 \circ j_{r(\rho_2)}$$

$$= j_{s(\rho_1)} \circ j_{s(\rho_1)} \circ j_{s(\rho_1)} \circ j_{s(\rho_1)} \circ j_{r(\rho_2)} = j_{r(\rho_1)} j(\rho_2)$$

Hence $j$ is a functor. Let $\rho, \hat{\rho}$ be soliton homomorphisms in $\Delta(q,x)$ and $w \in I(\rho, \hat{\rho})$ an intertwiner. It is easy to see that $J_{s(\rho)} w J_{s(\rho)}$ intertwines $j(\rho)$ and $j(\hat{\rho})$. From this we conclude that $j$ is additive and isotonous. Moreover, we have $j \circ j = \pi$ and hence $j$ is a conjugation. Since $j$ preserves inner unitary equivalence, we conclude that $j_{PCT}$ is also a well defined conjugation.

ii: To prove ii we have to show the following relation: For $\rho_1 \in \Delta(q,x)$ and $\rho_2 \in \Delta(-q,y)$:

$$[\rho_1] = [\rho_2] \Rightarrow [j(\rho_1)] = [j(\rho_2)] \quad (2)$$

From this we obtain for $\rho \in \Delta(q,x)$ that

$$e(j(\rho)) = j_{PT} e(j(\rho'))$$

where $\rho'$ is a PT-conjugate for $\rho$. Remember that $j(\rho)$ and $\rho'$ are both contained in $\Delta(-q,-x)$. Thus we have

$$j_{PCT} j_{PT} e(\rho) = j_{PT} j_{PCT} e(\rho)$$

and ii is proven.

It remains to prove the relation (2) above. For equivalent $\rho_1 \in \Delta(q,x)$ and $\rho_2 \in \Delta(-q,y)$ we have $a = s(\rho_1) = r(\rho_2)$ and $b = r(\rho_1) =$
Furthermore, there is a unitary operator \( u : \mathcal{H}_b \to \mathcal{H}_a \) which intertwines \( \rho_1|_{\mathfrak{A}} \) and \( \rho_2|_{\mathfrak{A}} \). In the following, we show that \( u^J := J_a u J_b \) intertwines the representations \( j(\rho_1)|_{\mathfrak{A}} \) and \( j(\rho_2)|_{\mathfrak{A}} \). Let \( \mathfrak{O} \) be an arbitrary double cone, then for any \( A \in \mathfrak{A}(\mathfrak{O}) \) we obtain:

\[
j(\rho_1)(A)u^J = J_a \rho_1(jA)uJ_b = J_a u \rho_1(jA)J_b = J_a u J_b J_b \rho_2(jA)J_b = u^J j(\rho_2)(A)
\]

Thus \( j(\rho_1) \) and \( j(\rho_2) \) are unitarily equivalent. q.e.d.

Using the statements of Theorem 2.1, we conclude this subsection and summarize the properties of a \( C_0 \)-conjugate in the following corollary:

**Corollary 2.1** Suppose \( \theta \) is a one-soliton sector, then the \( C_0 \)-conjugate \( j_0 \theta \) is also a one-soliton sector with the same mass spectrum as \( \theta \) (property A1). Moreover, the \( C_0 \)-conjugate of the \( C_0 \)-conjugate of an inner soliton sector \( \theta \) is \( \theta \) itself (property A3).

### 2.2 The \( C_{\pm} \)-Conjugation for Soliton Sectors

Beside the \( C_0 \)-conjugate for a one soliton homomorphism there are two further candidates for antisoliton sectors which can be represented by \(*\)-homomorphisms of soliton states. We start this subsection by presenting the main result to establish the existence of these \(*\)-homomorphisms. Afterwards, we apply the results in [15] to prepare the proof.

For convenience, it is sufficient to formulate the theorem for soliton homomorphisms with orientation \( q = + \). The formulation for the case \( q = - \) can be done in complete analogy.

**Theorem 2.2** Let \( \rho : a \to b \in \Delta_1(+, x) \) be a one-soliton homomorphism. Then there are irreducible soliton homomorphisms \( \tilde{\rho}_{\pm} : b \to a \in \Delta(+, x) \), let us call them \( C_{\pm} \)-conjugates for \( \rho : a \to b \), with the following properties:

i: The energy momentum spectrum of \( U_{\tilde{\rho}_{\pm}} \) is contained in the closed forward light cone, i.e.: \( \text{sp}(U_{\tilde{\rho}_{\pm}}) \subset \mathbb{V}^+ \).

ii: The representation \( \tilde{\rho}_+ \rho \) contains \( \pi_b \) precisely once and the representation \( \rho \tilde{\rho}_- \) contains \( \pi_a \) precisely once.

Let \( \rho \) a massive one soliton homomorphism and let \( \tilde{\rho}_{\pm} \) \( C_{\pm} \)-conjugates for \( \rho \). Since \( \tilde{\rho}_{1,\pm} \) and \( \tilde{\rho}_{\pm} \) are inner unitarily equivalent, if \( \rho \) and
\( \rho_1 \) are inner unitarily equivalent, we may define the \( C_\pm \)-conjugation \( j_\pm \) by

\[ j_\pm : e(\rho) \in \sec_1(q) \mapsto e(\bar{\rho}_\pm) \in \sec(q) \, . \]

**Remark:** From Theorem 2.2, one cannot decide whether \( \bar{\rho}_\pm \) are massive one-particle representation. In addition, there is no reason why \( \bar{\rho}_+ \) and \( \bar{\rho}_- \) should be equivalent. Hence Theorem 2.2 does not imply that \( j_\pm \) are co-conjugations in the sense of Definition 3.1.

To prove Theorem 2.2, we apply the results and methods used in [13].

**Proposition 2.2** If \( \rho : a \to b \in \Delta_1(\,+,x) \) and \( \rho' : b \to a \in \Delta_1(-,\,-x) \) are PT-conjugate one soliton homomorphisms, then there are translation covariant states, \( \bar{\omega}_\rho : \mathfrak{A}_a^+ \to \mathbb{C} \) and \( \bar{\omega}_\rho' : \mathfrak{A}_b^- \to \mathbb{C} \) with the following properties:

1: Let \( e_\pm \in W_\pm \) be any space-like vector of unit length. Then, for all \( A \in \mathfrak{A}_a^+ \) one has

\[ \lim_{r \to \infty} \langle \psi, U_\rho(re_+)AU_\rho(re_+)^*\psi \rangle = \bar{\omega}_\rho(A) \quad \forall \psi \in \mathcal{H}_a ; \quad ||\psi|| = 1 \]

and for all \( A \in \mathfrak{A}_b^- \) one obtains

\[ \lim_{r \to \infty} \langle \psi', U_{\rho'}(re_-)AU_{\rho'}(re_-)^*\psi' \rangle = \bar{\omega}_{\rho'}(A) \quad \forall \psi' \in \mathcal{H}_b ; \quad ||\psi'|| = 1 \]

2: The state \( \bar{\omega}_\rho \) is normal on the wedge algebras \( \mathfrak{A}(W_+ + x)_a \), the similar statement holds also for \( \bar{\omega}_{\rho'} \).

3: For soliton homomorphisms \( \rho_1 \in \Delta(\,+,y) \) inner unitarily equivalent to \( \rho, \bar{\omega}_{\rho_1} \) and \( \bar{\omega}_\rho \) are equivalent states. An analogous statement holds for a pair of inner equivalent one soliton homomorphisms with orientation \( q = - \).

4: The GNS-representations \( (\bar{\mathcal{H}}_+, \bar{\pi}_+, \xi_+) \) of \( \omega_\rho \) and \( (\bar{\mathcal{H}}_-, \bar{\pi}_-, \xi_-) \) of \( \omega_{\rho'} \) can be localized in any wedge region, i.e. there are unitary operators \( v_{\pm,a} : \bar{\mathcal{H}}_\pm \to \mathcal{H}_a \) and \( v_{\pm,b} : \bar{\mathcal{H}}_\pm \to \mathcal{H}_b \) such that for \( A \in \mathfrak{A}(W_+) \) and \( A' \in \mathfrak{A}(W_-) \) the equations

\[ \bar{\pi}_\pm(A) = v_{\pm,a}^*\pi_a(A)v_{\pm,a} \quad \text{and} \quad \bar{\pi}_{\pm}(A') = v_{\pm,b}^*\pi_b(A')v_{\pm,b} \]

hold.

**Proof.** One can prove the proposition with the same methods used in the proof of Lemma 2.2 of ref. [15] (p. 144-146). Let us sketch only the main ideas.
Applying the results of [8], for a neighborhood $\Delta \subset \mathbb{R}^2$ of the mass shell with $\Delta \cap \text{sp}(U_\rho) \subset H_m$, there is a quasi local operator $B \in \mathfrak{A}_\rho^+$, such that $\Psi_\Delta := \rho(B)E_\rho(\Delta) \neq 0$. Here $E_\rho$ denotes the spectral measure with respect to the translation group $U_\rho$. Now one can show, that there are states $\varphi_{\pm}$ such that for each $\psi_\Delta = \Psi_\Delta \psi$, $\psi \in H_a$

$$\varphi_{\pm} : \cup_{x\rho}(\mathfrak{A}(W_{\pm} + x))' ||| \to \mathbb{C}$$

such that for each $\psi_\Delta = \Psi_\Delta \psi$, $\psi \in H_a$

$$r \mapsto \langle \psi_\Delta, (U_\rho(re_{\pm})AU_\rho(-e_{\pm}) - \varphi_{\pm}(A))\psi_\Delta \rangle$$

is a function of fast decrease. Moreover, $\varphi_{\pm}$ is a limit of states

$$\varphi_{\pm}(A) := \langle \psi_\Delta, (U_\rho(re_{\pm})AU_\rho(-e_{\pm})\psi_\Delta \rangle$$

which are normal on $\rho(\mathfrak{A}(W_{\pm} + x))'$ and hence $\varphi_{\pm}$ is normal on $\rho(\mathfrak{A}(W_{\pm} + x))'$. If $\rho$ is localized in $W_{+} + x$, then one gets

$$\cup_{x\rho}(\mathfrak{A}(W_{-} + x))' ||| = \mathfrak{A}_a^+$$

and we obtain the state $\tilde{\omega}_\rho := \varphi_{+} : \mathfrak{A}_a^+ \to \mathbb{C}$. On the other hand, if $\rho' := \rho$ is localized in $W_{-} + x$, then one gets

$$\cup_{x\rho'}(\mathfrak{A}(W_{-} + x))' ||| = \mathfrak{A}_b^-$$

and we obtain the state $\tilde{\omega}_{\rho'} := \varphi_{-} : \mathfrak{A}_b^- \to \mathbb{C}$.

By construction, the state $\tilde{\omega}_\rho$ is invariant under the charged translations, i.e.: $\tilde{\omega}_\rho(U_\rho(x)AU_\rho(-x)) = \tilde{\omega}_\rho(A)$. Thus one obtains via

$$T_{+}(x)(\pi_{+}(A)\xi_{+}) := \pi_{+}(U_\rho(x)A)\xi_{+}$$

a strongly continuous representation of the translation group. Applying the methods of [17], it can be shown that the spectrum of $T_{+}$ is contained in the closed forward light cone and that the vector $\xi_{+}$ is a ground-state of $T_{+}$, for the simple eigenvalue 0. Thus one concludes that $\pi_{+}$ is irreducible. We define now another unitary representation of the translation group by

$$x \mapsto U_{\pi_{+}}(x) := \pi_{+}(U_a(x)U_\rho(-x))T_{+}(x)$$

which implements $\alpha_x$ in the representation $\pi_{+}$. It can be shown that $U_{\pi_{+}}$ is strongly continuous. The analogous result is also true for $\tilde{\omega}_{\rho'}$ and we have established 1 and 2.
Let $\rho_1$ be a soliton homomorphism which is inner equivalent to $\rho$. Then one computes that $\bar{\omega}_\rho = \bar{\omega}_{\rho_1} \circ \text{Ad}(u)$ holds, where $u$ is a unitary intertwiner from $\rho_1$ to $\rho$. Thus 3 is proven.

By construction, one has $\bar{\omega}_\rho \circ \rho = \omega_b$. If $\rho$ is localized in $W_+ + x$, one obtains for $A' \in \mathfrak{A}(W_- + x)$:

$$\bar{\omega}_\rho(A') = \bar{\omega}_\rho \circ \rho(A') = \omega_b(A').$$

Hence there is a unitary operator $v_{+,b}$, such that for each $A' \in \mathfrak{A}(W_- + x)$ holds the equation:

$$v_{+,b}\pi_+(A') = \pi_b(A')v_{+,b}.$$

Since $\bar{\omega}_\rho$ is normal on $\mathfrak{A}(W_+ + x)_a$, there is a vector $\phi \in \mathcal{H}_a$, such that

$$\bar{\omega}_\rho(A) := \langle \phi, \pi_a(A)\phi \rangle$$

for each $A \in \mathfrak{A}(W_+ + x)$. Now we define an isometry $w_1$ by

$$w_1(\pi_+(A)x_+) := \pi_a(A)\phi \quad ; \quad A \in \mathfrak{A}(W_+ + x)_a$$

and obtain that $E_1 := w_1w_1^*$ is contained in $\mathfrak{A}(W_+ + x)_a$. According to [3] there is an isometry $w_a \in \mathfrak{A}(W_+ + x)_a$ such that $w_aw_a^* = E_1$ and for the unitary $v_{+,a} := w_a^*w_1$ holds the equation:

$$v_{+,a}\pi_+(A) = \pi_a(A)v_{+,a}.$$ 

The analogue can also be proven for $\bar{\omega}_{\rho'}$ and one gets 4. q.e.d.

To prove Theorem 2.2, it is sufficient to consider the case that $\rho$ has orientation $q = +$.

**Proof.** (Theorem 2.2:) i: By Proposition 2.2, we obtain that $\bar{\omega}_\rho$ and $\bar{\omega}_{\rho'}$ are translation covariant states, and if we use Proposition 2.2.4 it follows that $\bar{\omega}_\rho$ and $\bar{\omega}_{\rho'}$ are soliton states. Since the spectra of the translation groups $T_+, U_a$ and $U_\rho$ are contained in the closed forward light cone, one obtains by applying the additivity of the energy momentum spectrum (see [15, 10, 11]) that the spectrum of $U_{\pi_\pm}$ is also contained in the closed forward light cone.

ii: The statement ii follows immediately from the construction of the states $\bar{\omega}_\rho$ and $\bar{\omega}_{\rho'}$. q.e.d.

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3 On the Existence of Antisoliton Sectors

The results of the last section show that a $C_0$-conjugate $\bar{\rho}_0$ for a one soliton homomorphism $\rho$ has the property $A1$, i.e. $\bar{\rho}_0$ is a one soliton homomorphism having the same mass spectrum as $\rho$, and property $A3$, i.e. a $C_0$-conjugate of $\bar{\rho}_0$ is inner unitarily equivalent to $\rho$. For the $C_-$-conjugate $\rho_-$ one can show the first part of $A2$, i.e. $\rho\bar{\rho}_-\rho$ contains the identity $\pi_{s(\rho)}$ precisely once. The $C_+$-conjugate $\bar{\rho}_+$ satisfies the second part of $A2$, i.e. $\bar{\rho}_+\rho$ contains the identity $\pi_{r(\rho)}$ precisely once.

As described in the introduction, the square of the dimension of a soliton homomorphism $\rho$ is defined by the index of the inclusion $\rho(\mathfrak{A}(W_q + x)_{r(\rho)}) \subset \mathfrak{A}(W_q + x)_{s(\rho)}$, i.e.:

$$d(\rho)^2 = \text{ind}(\rho(\mathfrak{A}(W_q + x)_{r(\rho)}), \mathfrak{A}(W_q + x)_{s(\rho)})$$

Here $q$ is the orientation of $\rho$ and $W_q + x$ its localization region.

We show that the index of an inner soliton sector $\theta$ is finite, if and only if there exist an inner soliton sector $\bar{\theta}$, such that $\theta\bar{\theta}$ contains the identity $\pi_{r(\theta)}$ and $\theta\bar{\theta}$ contains the identity $\pi_{s(\theta)}$. As mentioned in the introduction, this is a slight generalization of Longo’s result [27] (Theorem 4.1) and can be proven with similar methods. To make arguments clear, we shall give here an explicit proof.

Then one can use this statement to prove the main result of this section which states that for a soliton sector $\theta$ there is an antisoliton sector $\bar{\theta}$ with the properties $A1$ - $A3$, if one of the following three equivalent conditions is fulfilled:

1: The index or dimension of $\theta$ is finite.
2: The $C_+$-conjugate and the $C_-$-conjugate of $\theta$ are equal.
3: Either the $C_+$- or the $C_-$-conjugate equals the $C_0$-conjugate.

We have to mention that there might be a one soliton sector with infinite dimension. In such a case there is no unique choice for an antisoliton homomorphism.

3.1 Relation between $C_{\pm}$-Conjugate and $C_0$-Conjugate Homomorphisms

One obtains the following nice and useful relation between the $C_{\pm}$- and the $C_0$-conjugation.
Theorem 3.1 The equation
\[ j_0 \circ j_+ = j_- \circ j_0 : \sec_1(q) \to \sec(q) \]
holds on the set of all inner one soliton sectors.

Proof. Let \( \rho, q = \pm \), be a pair of PT-conjugate massive one soliton homomorphisms. Let us choose representatives \( \bar{\rho}_{qp} \) of \( j_p(e(\rho_q)) \); \( p, q = \pm \), where \( \bar{\rho}_{qp} \) is localized in the same region as \( \rho_q \), i.e. \( W_q + qx \).

We choose now a vector \( \psi \) with \( \|\psi\| = 1 \) which is contained in \( \mathcal{P}_{s(\rho_+)} := \{ A_{j_s(\rho_+)}(A)\Omega_{s(\rho_+)} \} \). For \( \psi \) we have \( J_{s(\rho_+)}(\psi) = \psi \) and thus we obtain for each observable \( A \in \mathcal{A} \):
\[
\lim_{|y| \to -\infty} \langle \psi, U_{j(\rho_+)}(y)AU_{j(\rho_+)}(-y)\psi \rangle = \bar{\omega}_{j(\rho_+)}(A)
\]
Since \( j(\rho_+) \) is a massive one soliton homomorphism, localized in \( W_- - x \), the limit above exists. Hence \( j(\rho_-) \) is a representative of \( j_- e(j(\rho_-)) \) and we obtain:
\[
[\bar{\omega}_{j(\rho_+)}] = [\omega_{r(\rho_+)} \circ j(\rho_-)]
\]
On the other hand we have:
\[
\lim_{|y| \to -\infty} \langle \psi, U_{j(\rho_+)}(y)AU_{j(\rho_+)}(-y)\psi \rangle = \lim_{|y| \to -\infty} \langle \psi, J_{s(\rho_+)}U_{j(\rho_+)}(y)AU_{j(\rho_+)}(-y)J_{s(\rho_+)}\psi \rangle = \lim_{|x| \to +\infty} \langle \psi, U_{\rho_+}(x)J_{s(\rho_+)}AJ_{s(\rho_+)}U_{\rho_+}(-x)\psi \rangle = \bar{\omega}_{\rho_+}(J_{s(\rho_+)}AJ_{s(\rho_+)})
\]
The state \( \bar{\omega}_{j(\rho_+)} \) is equivalent to \( \omega_{r(\rho_+)} \circ j(\bar{\rho}_+) \). Hence we obtain the equation:
\[
[j(\bar{\rho}_+)] = [\bar{j}(\rho_-)]
\]
Since \( j(\bar{\rho}_+) \) is localized in \( W_- - x \) and \( \bar{j}(\rho_-) \) is localized in \( W_+ + x \) and \( j(\bar{\rho}_+) \) and \( \bar{j}(\rho_-) \) are unitarily equivalent, we obtain
\[
j_0 j_+ e(\rho_+) = j_{PT} j_{PCT} e(\bar{\rho}_+) = j_{PT} e(j(\bar{\rho}_+)) = e(j(\rho_-)) = j_- e(j(\rho_-)) = j_0 e(\rho_+)
\]
which completes the proof. q.e.d.
3.2 The Dimension for Soliton Homomorphisms

Let \( \rho \in \Delta(q,x) \) be a *-homomorphism. Then, we define the square of the dimension of \( \rho \), by

\[
d(\rho)^2 := \text{ind}(\rho) := \text{ind}(\rho(\mathfrak{A}(W_q + x)_{r(\rho)})\mathfrak{A}(W_q + x)_{s(\rho)})
\]

Since the dimension of an inner soliton sector depends only on the inner equivalence class of a *-homomorphism, one can define the dimension of an inner soliton sector \( \theta = e(\rho) \in \text{sec}(q) \) by \( d(\theta) := d(\rho) \).

A consequence of the results proven in \([26, 27, 29, 30, 25]\) is the following proposition:

**Proposition 3.1** All *-homomorphisms \( \rho_1, \rho_2 \in \Delta(q,x) \) with finite dimension have the following properties:

i Multiplicativity: For \( s(\rho_2) = r(\rho_1) \):

\[
d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)
\]

ii Additivity: For \( s(\rho_1) = s(\rho_2) \) and \( r(\rho_1) = r(\rho_2) \) and one has:

\[
d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)
\]

iii Reducibility: If \( s(\rho_2) = r(\rho_1) \), then there are finitely many soliton homomorphisms \( \sigma_j \) with \( s(\sigma_j) = s(\rho_2) \) and \( r(\sigma_j) = r(\rho_1) \), such that

\[
\rho_1\rho_2 = \bigoplus_j N_{\rho_1,\rho_2}^j \sigma_j
\]

where \( N_{\rho_1,\rho_2}^j \) are natural numbers.

3.3 Conditions for the Existence of Antisolitons

We come now to the main result of this section:

**Theorem 3.2** For every massive inner one soliton sector \( \theta \in \text{sec}_1(q) \) the following statements are equivalent:

i: The dimension of \( \theta \) is finite:

\[
d(\theta) < \infty
\]
ii: The $C_+$-conjugate is equal to the $C_-$-conjugate of $\theta$:
\[ j_+ \theta = j_- \theta \]

iii: The $C_0$-conjugate is equal to the $C_+$- or the $C_-$-conjugate:
\[ j_0(\theta) = j_q(\theta) ; \quad q = \pm \]

We shall need some abstract index theoretical results to prove Theorem 3.2, and before we present them, we sketch the basic ideas of the proof.

We prove, if statement $i$ holds for $\theta$ then there exists an inner soliton sector $\bar{\theta}$ such that $\theta \bar{\theta}$ contains the identity $i_{s(\theta)}$ and $\bar{\theta} \theta$ contains the identity $i_{r(\theta)}$. Such an inner soliton sector exists if and only if the index of $\theta$ is finite. Since $j_\pm \theta$ are fixed uniquely by their properties, i.e. $j_+ \theta$ contains $i_{r(\theta)}$ and $\theta j_- \theta$ contains $i_{s(\theta)}$, we obtain $j_+ \theta = j_- \theta = \bar{\theta}$ which implies $ii$ and $iii$. Using the relation $j_+ j_0 \theta = j_0 j_- \theta$, we obtain from $iii$ $j_\pm = j_0$ and we conclude that the index of $\theta$ is finite.

The basic ingredient for the proof of Theorem 3.2 is the following Lemma:

**Lemma 3.1** The following two statements are equivalent:

1: The dimension of an inner sector $\theta \in \text{sec}_1(q)$ is finite, i.e. $d(\theta) < \infty$.

2: There exists an inner sector $\theta' \in \text{sec}(q)$ with $s(\theta) = r(\theta')$ and $r(\theta) = s(\theta')$, such that $\theta \theta'$ contains the identity $i_{s(\theta)}$ precisely once and $\theta' \theta$ contains the identity $i_{r(\theta)}$ precisely once, i.e.:
\[ \theta \theta' > i_{r(\theta)} \quad \text{and} \quad \theta' \theta > i_{s(\theta)} \]

Furthermore, if $\theta'$ exists, then $\theta' = j_0(\theta)$.

To prove the lemma, we need two results proven in [27]:

**Proposition 3.2** Let $\mathfrak{N} \subset \mathfrak{M}$ be a standard inclusion of infinite factors and denote by $C(\mathfrak{M}, \mathfrak{N})$ the collection of all normal conditional expectations $\epsilon : \mathfrak{M} \to \mathfrak{N}$. Furthermore, let $H(\gamma, \mathfrak{M})$ be the Hilbert space space of all intertwiners $\gamma(A)v = vA$, $A \in \mathfrak{M}$, where $\gamma$ is a normal endomorphism of $\mathfrak{M}$. Under this conditions, the maps
\[ \epsilon : H(\gamma, \mathfrak{M}) \to C(\mathfrak{M}, \mathfrak{N}) ; \quad v \mapsto \epsilon_v = v^* \gamma(\cdot)v \]
\[ \epsilon^* : H(\gamma, \mathfrak{M}) \to C(\mathfrak{M}_1, \mathfrak{N}) ; \quad w \mapsto \epsilon^*_w = w^* \gamma(\cdot)w \]
are bijective. Here $H(\gamma, \mathfrak{M})_1$ denotes the unit sphere in $H(\gamma, \mathfrak{M})$ and $\mathfrak{M}_1$ denotes the Jones basic construction, i.e. $\mathfrak{M}_1 = J_{\mathfrak{M}} \mathfrak{N} J_{\mathfrak{M}}$.
Proposition 3.3 Let $\mathcal{N} \subset \mathcal{M}$ an inclusion of infinite von-Neumann-algebras, then the following statements are equivalent:

1: The index of the inclusion $\mathcal{N}, \mathcal{M}$ is finite:

$$\text{ind}(\mathcal{N}, \mathcal{M}) < \infty$$

2: The relative commutant $\mathcal{N}' \cap \mathcal{M}$ is a finite dimensional algebra and there exists a faithful conditional expectation $\epsilon : \mathcal{M} \to \mathcal{N}$ and there exists a faithful conditional expectation $\epsilon' : \mathcal{N}' \to \mathcal{M}'$.

Proof. (Lemma 3.1:) First, let us remark that the relative commutant $\rho(\mathcal{M}_{r(\rho)})' \cap \mathcal{M}_{s(\rho)} = \mathbb{C} \cdot 1_{s(\rho)}$ is trivial, because $\rho$ is irreducible. $1 \Rightarrow 2$: Suppose the index of $\theta$ is finite and choose a representative $\rho$ of $\theta$. Then there exists a faithful conditional expectation $\epsilon \in C(\mathcal{M}_{s(\rho)}, \rho(\mathcal{M}_{r(\rho)}))$ and a faithful conditional expectation $\epsilon' \in C(\rho(\mathcal{M}_{r(\rho)})', \mathcal{M}_{s(\rho)}')$. There is also a faithful conditional expectation $\epsilon^1 \in C(\mathcal{M}_{s(\rho)}, 1, \mathcal{M}_{s(\rho)})$ and $\epsilon^1$ can be written in the form:

$$\epsilon^1 = \bar{\nu}^* \rho \bar{\rho}(\cdot) \bar{\nu}$$

with an isometry $\bar{\nu} \in \mathcal{H}(\rho \bar{\rho}, \mathcal{M}_{s(\rho)})$. Here $\bar{\rho}$ is any representative of $j_0(\theta)$. Now, $\bar{\nu}$ is an intertwiner from $\rho \bar{\rho}$ to $\pi_{s(\rho)}$, and since $\rho$ is irreducible we obtain that $\theta j_0(\theta)$ contains the identity precisely once. On the other hand, $\epsilon$ can be written in the form:

$$\epsilon = \rho(v^*) \rho \bar{\rho}(\cdot) \rho(v)$$

with an isometry $\rho(v) \in \mathcal{H}(\rho \bar{\rho}, \rho(\mathcal{M}_{r(\rho)}))$. Now, $v$ intertwines $\rho \bar{\rho}$ and $\pi_{r(\rho)}$, and since $\rho$ is irreducible we conclude, that $j_0(\theta) \theta$ contains the identity precisely once.

$2 \Rightarrow 1$: Suppose, there is an inner soliton sector $\theta'$, such that $\theta \theta' > i_{s(\theta)}$ and $\theta' \theta > i_{r(\theta)}$. If we choose representatives $\rho$ of $\theta$ and $\rho'$ of $\theta'$, then there are isometries $v \in \mathcal{M}_{r(\rho)}, \bar{\nu} \in \mathcal{M}_{s(\rho)}$, such that:

$$\rho \rho'(A) \bar{\nu} = \bar{\nu} A \quad A \in \mathcal{M}_{s(\rho)}$$

$$\rho' \rho(A) v = v A \quad A \in \mathcal{M}_{r(\rho)}$$

From this we obtain two faithful conditional expectations, namely

$$\epsilon_\rho = \rho(v^* \rho'(\cdot) v) \in C(\mathcal{M}_{s(\rho)}, \rho(\mathcal{M}_{r(\rho)}))$$

$$\epsilon_{\rho'} = \rho'(\bar{\nu}^* \rho(\cdot) \bar{\nu}) \in C(\mathcal{M}_{r(\rho)}, \rho'(\mathcal{M}_{s(\rho)}))$$
We get also a conditional expectation by
\[ \epsilon^1_\rho = \bar{v}^* \rho \rho' (\cdot) \bar{v} \in C(\mathcal{M}_{s(\rho)}^1, \mathcal{M}_{s(\rho)}) \]
and obtain the equation:
\[ \epsilon'_{\rho'} \circ \rho' = \rho' \circ \epsilon^1_\rho \]
Since both \( \rho' \) and \( \epsilon'_{\rho} \) are faithful, we conclude that \( \epsilon^1_\rho \) is faithful. Thus we have faithful conditional expectations
\[ \epsilon_\rho \in C(\mathcal{M}_{s(\rho)}, \rho(\mathcal{M}_{r(\rho)})) \]
\[ \epsilon'_\rho = j_{s(\rho)} \circ \epsilon^1_\rho \circ j_{s(\rho)} \in C(\rho(\mathcal{M}_{r(\rho)})', \mathcal{M}_{s(\rho)})' \]
and hence the index of \( \rho \) is finite. It remains to be proven that \( \theta' = j_0 \theta \).

One can prove this statement with the same methods used in [27] (Theorem 4.1). We sketch here only the main ideas. If we consider any representative \( \bar{\rho} \) of \( j_0 \theta \), then by construction of the \( C_0 \)-conjugate, \( \gamma_\rho = \rho \bar{\rho} \) is the canonical endomorphism mapping \( \mathcal{M}_{s(\rho)} \) into \( \rho(\mathcal{M}_{r(\rho)}) \). Now let \( u \) be a unitary operator implementing \( \rho \rho' = \rho' \) any representative of \( \theta' \) on \( \mathcal{M}_{s(\rho)} \). Furthermore, let \( v \) be the isometry which intertwines \( \rho \rho' \) and \( \pi_{s(\rho)} \). If we follow the argumentation of [27], we can conclude that \( v_0 := u^* \rho(v) u \) is contained in \( \rho(\mathcal{M}_{r(\rho)})' \) and we can find a unitary \( z \in \rho(\mathcal{M}_{r(\rho)})' \) such that
\[ \Gamma := z^* v_0 J_{s(\rho)} v_0^* z J_{s(\rho)} \]
implements the canonical endomorphism \( \gamma_\rho = \rho \bar{\rho} \). Thus we have \( \theta' = j_0 \theta \). q.e.d.

Proof. (Theorem 3.2) \( iii \Rightarrow ii \): If we suppose that statement \( iii \) is true, then we get:
\[ j_0 \theta = j_+ \theta \]
Since \( j_0 \) is involutive, we obtain together with Theorem 3.1:
\[ j_0 j_+ j_0 \theta = j_- \theta = j_+ \theta \]
\( ii \Rightarrow i \): Suppose now that statement \( ii \) is true. For each \( \theta \in \text{sec}_1(q) \), we obtain for \( \theta := j_+ \theta = j_- \theta \), by applying Theorem 2.2,
\[ \theta \bar{\theta} > i_{s(\theta)} \quad \text{and} \quad \bar{\theta} \theta > i_{r(\theta)} \]
and Lemma 3.1 implies that $d(\theta) < \infty$.

*i* $\Rightarrow$ *iii*: Suppose the dimension of $\theta$ is finite, then there exists an inner massive one-soliton sector $\bar{\theta}$, such that

$$\bar{\theta} \theta > i_{s(\theta)} \quad \text{and} \quad \bar{\theta} \theta > i_{r(\theta)}. $$

By Theorem 2.2.(ii) we have $\theta \sim \theta > i_{s(\theta)}$. Let $\bar{\rho}$ a representative of $\bar{\theta}$, $\bar{\rho}_-$ a representative of $j_- \theta$ and $\rho$ a representative of $\theta$. We choose $\rho, \bar{\rho}, \bar{\rho}_-$ to be localized in $W_+$. Since the index of the inclusion $\rho(\mathcal{A}(W_+)_{r(\theta)}) \subset \mathcal{A}(W_+)$ is finite, both $\gamma := \rho \bar{\rho}$ and $\gamma_- := \rho \bar{\rho}_-$ are inner unitarily equivalent to the canonical endomorphism $\gamma_0 = \rho \rho_0$ \textsuperscript{[26, 27]}. Thus we conclude $\bar{\theta} = j_- \theta$ and *iii* follows. q.e.d.

Furthermore applying the methods used in \textsuperscript{[27]} to the soliton case, one obtains that the dimension function is invariant under the charge conjugation.

**Lemma 3.2** Let $\theta \in \sec(q)$ be an inner soliton sector. If $d(\theta) < \infty$ then $d(j_0 \theta) < \infty$ and one has:

$$d(\theta) = d(j_0 \theta)$$

**Definition 3.1** The set of all inner soliton sectors generated \textsuperscript{[3]} by all massive inner one soliton sectors which fulfill one of the properties of Theorem 3.2, is denoted by $\sec_f(q)$ and called the set of inner soliton sectors with finite dimension.

**Remark:** By Proposition 3.1, the dimension map respects the multiplication and the direct sum of inner soliton sectors.

**Corollary 3.1** Let $\rho \in \Delta(q, x)$ be a soliton homomorphism such that $e(\rho)$ is contained in $\sec_f(q)$. Then there exists a co-conjugation, called charge-conjugation,

$$j^* : \sec_f(q) \rightarrow \sec_f(q)$$

such that each antisoliton homomorphism $\bar{\rho}$ satisfies the properties A1 - A3. In addition to that, the dimension map is invariant under charge-conjugation $j^*$, i.e. $d = d \circ j^*$.

\textsuperscript{3} A subset $\mathfrak{s}$ which generates $\sec(q)$ consists of a collection of inner sectors such that each element of $\sec(q)$ is a subobject of (or equal to) a finite direct sum of finite products of elements in $\mathfrak{s}$. 

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Proof. We define $j^* := j_0$ and on the set of all massive inner one
soliton sectors with finite dimension $\text{sec}_1 f(q)$ holds the equation

$$j^*|_{\text{sec}_1 f(q)} = j_0|_{\text{sec}_1 f(q)} = j_\pm$$

Since $\text{sec}_1 f(q)$ generates $\text{sec}_f(q)$ and $j_0$ is a co-conjugation, we con-
clude by applying Theorem 3.2 and Proposition 3.1 that the statement
of the corollary is true. q.e.d.

4 Concluding Remarks

We have seen that an antisoliton sector can be constructed by applying
the methods of D. Guido and R. Longo [22] on the one hand, or applying
the methods used by K. Fredenhagen [15] on the other hand.

We have learned that for soliton sectors with finite dimension,
each of the described constructions lead to the same antisoliton sector.
However, one cannot exclude the possibility that there are massive
one-soliton sectors with infinite dimension or not.

Indeed, there are examples for theories in which sectors of infinite
dimension (statistics) arise [17], but we have to mention that these
examples are related to models which describe massless particles.

The construction of $C_0$-conjugates is also possible for massless the-
ories, whereas for the construction of the $C_\pm$-conjugates one uses that
there is a mass gap in the energy-momentum spectrum of the theory.

To study the peculiar case where the dimension of a soliton sector
is infinite, one has either to construct models which describe mas-
sive one-particle states belonging to a sector of infinite dimension, or
one has to find out whether the construction of $C_\pm$-conjugates is also
possible for the massless examples described in [17].

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