Abstract

Let $G$ be a Lie group, let $M$ and $N$ be smooth connected $G$-manifolds, let $f: M \to N$ be a smooth $G$-map, and let $P_f$ denote the fiber of $f$. Given a closed and equivariantly closed relative 2-form for $f$ with integral periods, we construct the principal $G$-circle bundles with connection on $P_f$ having the given relative 2-form as curvature. Given a compact Lie group $K$, a biinvariant Riemannian metric on $K$, and a closed Riemann surface $\Sigma$ of genus $\ell$, when we apply the construction to the particular case where $f$ is the familiar relator map from $K^{2\ell}$ to $K$, which sends the $2\ell$-tuple $(a_1, b_1, \ldots, a_{\ell}, b_{\ell})$ of elements $a_j, b_j$ of $K$ to $\prod [a_j, b_j]$, we obtain the principal $K$-circle bundles on the associated extended moduli spaces which, via reduction, then yield the corresponding line bundles on possibly twisted moduli spaces of representations of $\pi_1(\Sigma)$ in $K$, in particular, on moduli spaces of semistable holomorphic vector bundles or, more precisely, on a smooth open stratum when the moduli space is not smooth. The construction also yields an alternative geometric object, distinct from the familiar gerbe, representing the fundamental class in the third integral cohomology group of $K$ or, equivalently, the first Pontrjagin class of the classifying space of $K$. 

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1 Introduction

Let $G$ be a Lie group with a bi-invariant Riemannian metric. Moduli spaces of homomorphisms or more generally twisted homomorphisms from the fundamental group of a surface to $G$ were connected with geometry through their identification with moduli spaces of holomorphic vector bundles [17]. Atiyah and Bott [2] initiated a new approach to the study of these moduli spaces by identifying them with moduli spaces of projectively flat constant central curvature connections on principal bundles over Riemann surfaces, which they analyzed by methods of gauge theory. A proof that the resulting symplectic form is closed, using group cohomology rather than gauge theory, was given by Karshon [14]; in [23], A. Weinstein reformulated Karshon’s proof in terms of the double complex of Bott [3] and Shulman [18]. A purely finite dimensional construction, including that of an ordinary finite-dimensional Hamiltonian $G$-space, referred to as an extended moduli space, from which the moduli spaces arise by ordinary finite-dimensional symplectic reduction, was announced in [12] and given in [8] and [13]. The construction has been extended in [7] to include surfaces with boundary and parabolic structures along the boundary, and in [10] the entire approach has been pushed further to handle an arbitrary gauge theory situation in terms of a similar construction. An application of the construction in [10] is a purely combinatorial construction of the Chern-Simons function over a 3-manifold.

The symplectic structure of the moduli space (more precisely: on the smooth stratum thereof) is known to be integral and, at the time, A. Weinstein raised the issue of constructing a corresponding line bundle or, equivalently, principal circle bundle. In this paper, we present a solution to this problem. More precisely, the line bundle or principal circle bundle not necessarily being defined on the moduli space itself, we shall construct the requisite $G$-equivariant circle bundle on the extended moduli space. We shall actually abstract from the particular case and explore the more general case of a $G$-equivariant smooth map $f : M \to N$, together with (i) a closed $G$-equivariant relative 2-form $(\zeta, \lambda)$ with integral periods where $\zeta$ is a $G$-invariant 2-form on $M$ and $\lambda$ a $G$-invariant 3-form on $N$ such that $d\zeta = f^*\lambda$ and with (ii) the requisite additional technical ingredient encapsulating the information to carry out the construction of the principal circle bundle $G$-equivariantly; this additional information is encoded in a $G$-equivariant linear map $\vartheta$ from the Lie algebra $\mathfrak{g}$ of $G$ to the space of 1-forms on $N$ and contains the information needed to construct a $G$-momentum mapping from $P_f$ to $\mathfrak{g}^*$, that momentum mapping being the additional constituent to arrive at an equivariantly closed 2-form.

Let $I$ denote the unit interval, let $I^2$ be the ordinary unit square, and let $j_1 : I \to I^2$ be the injection which sends the point $t$ of $I$ to $(t,0) \in I^2$. We shall construct the total space of the circle bundle on the fiber $P_f$ of the
map $f$ as a space of equivalence classes of strings of the kind

$$
\begin{array}{ccc}
\{0\} & \longrightarrow & I \\
\downarrow & \downarrow w & \downarrow \phi \\
\{\alpha\} & \longrightarrow & M \\
\end{array}
$$

In Section 2 below, we recall how a principal circle bundle can be recovered from the curvature; thereafter We refine the construction to an equivariant one. In Section 3 we generalize the construction to that of a principal circle bundle on the fiber $P_f$ of a map $f: M \to N$ and in Section 4 we give the equivariant extension of that construction. In Section 5 we explore the special case where the target $N$ of $f$ is a Lie group.

Given a closed Riemann surface $\Sigma$ of genus $\ell$ and a compact connected Lie group $K$, we take $M = K^{2\ell}$, $N = K$, and $f$ to be the familiar relator map $K^{2\ell} \to K$ which sends the $2\ell$-tuple $(a_1, b_1, \ldots, a_\ell, b_\ell)$ of elements $a_j, b_j$ of $K$ to $\prod[a_j, b_j]$; moreover we choose a biinvariant Riemannian metric on $K$ and take $\lambda$ to be the fundamental 3-form on $K$ and $\zeta$ and $\vartheta$ the corresponding forms explored in [7], [8], [12], [14], [23]. In that particular case the construction yields a $K$-equivariant principal circle bundle on the fiber of the relator map.

Via the holonomy, the fiber of the relator map is actually based homotopy equivalent to the space $\text{Map}^0(\Sigma, BK)$ of based maps from $\Sigma$ to the classifying space $BK$ of $K$. Each path component of $\text{Map}^0(\Sigma, BK)$ corresponds to a principal $K$-bundle on $\Sigma$ and in fact amounts to the classifying space of the associated group of based gauge transformations. Thus the $K$-equivariant principal circle bundle on $P_f$ induces a $K$-equivariant principal circle bundle on $\text{Map}^0(\Sigma, BK)$. This association can be made functorial in terms of geometric presentations of the surface variable $\Sigma$; the geometric object which thereby results represents the cohomology class given by the Cartan 3-form and may thus be viewed as an alternative to the familiar equivariant gerbe representing the first Pontrjagin class of the classifying space of $K$ [4]. We explain the details in Section 6 below.

The extended moduli spaces lie $K$-equivariantly in the fiber of the relator map, and the $K$-equivariant principal circle bundles on the extended moduli space we are looking for are then simply obtained by restriction. Details are given in Section 6 below. In a final section we illustrate our method in terms of equivariant circle bundles on coadjoint orbits of the loop group.

The approach in the present paper can be extended to a construction of principal circle bundles in the more general situation of equivariant plots for an arbitrary gauge theory situation of the kind developed in [10]. An extended moduli space is a special case of such an equivariant plot. We plan to come back to this situation elsewhere.
2 Reconstruction of a circle bundle from the curvature

Our main aim is the construction of principal circle bundles on the fiber of a map. In this section, we will explain the essence of the construction, but just over a space rather than over the fiber of a map. This will help understand the subsequent construction over the fiber of a map.

Let $I$ be the unit interval and let $N$ be a topological space having suitable local properties so that the constructions below make sense—$N$ being a CW-complex will certainly suffice. By a path in $N$ we mean a (continuous) map $u: I \to N$ as usual; then $u(0)$ is the starting point and $u(1)$ the end point. Occasionally we will refer to both the starting and end point as end points. Let $o$ be a point of $N$, taken henceforth as base point, let $P_o(N)$ be the space of paths in $N$ having starting point $o$, and let $p_o: P_o(N) \to N$ be the obvious projection which sends a path to its end point, well known to be a (Hurewicz) fibration onto the path component of $o$ having as fiber $p_o^{-1}(o)$ the space $\Omega_o(N)$ of closed based loops in $N$, based at $o$. The space $P_o(N)$, topologized as usual by the compact-open topology, is contractible, the standard contraction being given by the operation of contracting a path to its starting point. A familiar construction yields the universal cover $\tilde{N}$ of $N$: Identify two paths $w_1$ and $w_2$ having $o$ as starting point and having the same end point provided these paths are homotopic relative to the starting and end points. The space of equivalence classes, suitably topologized, yields the universal cover of $N$, the covering projection being the obvious map which sends a homotopy class of paths to the common end point. We will now recall how a variant of this construction yields the principal circle bundles on $N$.

2.1 The topological construction

We will use the notation $B^I = \text{Map}(I, B)$. Let $p: E \to B$ be a map, let $p_0: B^I \to B$ be the map which sends a path $u: I \to B$ to its starting point $u(0)$, let $E \times_B B^I$ be the associated fiber product, and let $p^I: E^I \to E \times_B B^I$ be the obvious map which sends a path $w: I \to E$ to the pair $(u(0), p \circ w)$. Recall that $p$ is a Hurewicz fibration if and only if it admits a lifting function $\lambda: E \times_B B^I \to E^I$, that is to say, a function $\lambda$ that is required to be a right-inverse for $p^I$, so that $p^I \circ \lambda$ is the identity of $E \times_B B^I$; cf. e. g. [20] (Chap. 2, Theorem 8 p. 92). Pick a base point $o$ of $B$ and a base point $o$ of $E$ such that $p(o) = o$, where the notation $o$ is slightly abused. This choice of base points induces an injection $j_o: P_o(B) \to E \times_B B^I$ in an obvious manner. Given the lifting function $\lambda: E \times_B B^I \to E^I$ for $p$ so that, in particular, $p: E \to B$ is a fibration, the composite

$$
\gamma: P_o(B) \xrightarrow{j_o} E \times_B B^I \xrightarrow{\lambda} E^I \xrightarrow{p_o} E
$$

(2.1)
is a map over $B$ and hence a morphism of fibrations.

Let $S^1$ be the circle group and let $\tau: S \to N$ be a topological principal circle bundle. Choose a lifting function for $\tau$ and pick a pre-image $o$ in $S$ of $o$. The above construction yields a map $\gamma_o: \Omega_o(N) \to S^1$ which is a homomorphism relative to composition of loops, and we will refer to $\gamma_o$ as the topological holonomy of $\tau$ determined by the lifting function. The topological holonomy $\gamma_o$ represents an integral class in $H^1(\Omega_o(N))$. For dimensional reasons, the transgression from $H^1(\Omega_o(N))$ to $H^2(N)$, i.e. the inverse of the suspension, is an isomorphism and, under transgression, the class of $\gamma_o$ goes to the topological characteristic class of $\tau$ in $H^2(N)$.

A standard construction recovers the circle bundle $\tau$ from a topological holonomy of the kind $\gamma_o$. Identify the two paths $w_1$ and $w_2$ in $N$ having $o$ as starting point and having the same end point provided the composite $w_2^{-1}w_1$, which is a closed path in $\Omega_o(N)$, has value $1 \in S^1$ under $\gamma_o$. The above map $\gamma$ from $P_o(N)$ to $S$ passes to a homeomorphism from the space of equivalence classes in $P_o(N)$ onto $S$.

We mention in passing that this notion of topological holonomy led Stasheff to the development of parallel transport in fiber spaces [21].

2.2 The differential-geometric construction

We now suppose that $N$ is a smooth manifold. Then lifting functions are provided by the operation of horizontal lift relative to a connection. A variant of the construction of the universal covering, similar to the topological reconstruction of a principal circle bundle from its topological holonomy reproduced above, yields the principal $S^1$-bundles on $N$ with connection having prescribed curvature.

Let $w_1$ and $w_2$ be two piecewise smooth paths in $N$ having $o$ as starting point and having the same end point. We define a piecewise smooth homotopy from $w_1$ to $w_2$ relative to the endpoints to be a piecewise smooth map

$$h: I \times I \to N$$

where the term “piecewise smooth” is to be interpreted in terms of some paving of $I \times I$ consisting of polygons, such that

- $h(t, 0) = w_1(t), h(t, 1) = w_2(t)$, for every $0 \leq t \leq 1$,
- $h(0, s) = o$, $h(1, s)$ is independent of $s$, for every $0 \leq s \leq 1$.

With this preparation out of the way, let $c$ be a closed 2-form on $N$ with integral periods. Let $P_o(N)$ now denote the space of piecewise smooth paths in $N$ having starting point $o$, let $\Omega_o(N)$ denote the space of piecewise smooth closed based loops in $N$, based at $o$, and let $\Omega_o(N)_0$ be the subspace of piecewise smooth closed loops which are homotopic to zero relative $o$. Standard smoothing arguments show that the inclusions from the various
piecewise smooth path spaces into the corresponding merely continuous path spaces are homotopy equivalences \[22\]. Identify two piecewise smooth paths \(w_1\) and \(w_2\) that are homotopic under a piecewise smooth homotopy \(h\) from \(w_1\) to \(w_2\) relative to the endpoints such that \(\int_{I \times I} h^*c\) is an integer. Since \(c\) has integral periods, this condition does not depend on the choice of homotopy \(h\).

Let \(\overline{S}\) denote the space of equivalence classes, let \(\tau: \overline{S} \to \tilde{N}\) and \(\tilde{\tau}: \overline{S} \to N\) be the obvious projection maps, and let \(\Gamma\) be the space of equivalence classes of closed loops at \(o\).

**Proposition 2.1.** Composition of closed loops turns \(\Gamma\) into a group.

**Proof.** The argument consists in copying the construction of the fundamental group. \(\square\)

The assignment to a closed path \(u: I \to N\) with \(u(o) = o\) of its class in \(\Gamma\) yields a surjective map \(\Omega_o(N) \to \Gamma\), that to such a closed path \(u: I \to N\) with \(u(o) = o\) which is, furthermore, null-homotopic relative to \(o\) of an integral of the kind \(\int_{I \times I} h^*c\) modulo \(Z\) yields a surjective map \(\Omega_o(N)_0 \to S^1\), and the two maps fit together in the commutative diagram

\[
\begin{array}{ccc}
\Omega_o(N)_0 & \longrightarrow & \Omega_o(N) \\
\downarrow & & \downarrow \\
S^1 & \longrightarrow & \Gamma \\
\end{array}
\]

whose bottom row is a central extension

\[
1 \longrightarrow S^1 \longrightarrow \Gamma \longrightarrow \pi_1(N) \longrightarrow 1 \tag{2.2}
\]

of Lie groups. Moreover, the familiar composition of paths

\[
P_o(N) \times \Omega_o(N) \longrightarrow P_o(N)
\]

induces a principal \(\Gamma\)-action on \(\overline{S}\) turning

\[
\tilde{\tau}: \overline{S} \longrightarrow N \tag{2.3}
\]

into a principal \(\Gamma\)-bundle, and the restriction of the action to \(S^1\) turns \(\overline{\tau}\) into a principal \(S^1\)-bundle.

Let \(\tau: S \to N\) be a principal \(S^1\)-bundle with a connection 1-form \(\omega\) having curvature \(c\). The operation of horizontal lift relative to \(\omega\) furnishes a unique map from \(P_oN\) to \(S\) which passes to a map from \(\overline{S}\) to \(S\) which, in turn, fits into a morphism \(\tilde{\tau} \to \tau\) of principal bundles on \(N\), i. e. into a commutative diagram of the kind

\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \overline{S} \\
\downarrow & & \downarrow \\
S^1 & \longrightarrow & S
\end{array}
\]

\[
\begin{array}{ccc}
\tau & \longrightarrow & \overline{\tau} \\
\downarrow & & \downarrow \text{Id} \\
N & \longrightarrow & N. \tag{2.4}
\end{array}
\]
Here the left-hand unlabelled vertical homomorphism from $\Gamma$ to $S^1$ is induced from the holonomy $\Omega_\omega(N) \to S^1$ of $\omega$. This homomorphism splits the exact sequence (2.2).

We will denote the de Rham algebra of differential forms on $N$ by $\mathcal{A}(N)$. We use the (nowadays) slightly unusual notation $\mathcal{A}$ to avoid notational conflict with the notation $\Omega$ for a based loop space.

Henceforth we will exploit the theory of differentiable spaces [5], see also [19] where these spaces are referred to as “espaces diffeologiques”. Below, when we refer to a form on a space which is not a smooth finite-dimensional manifold in an obvious manner, the term “form” is understood relative to an obvious differentiable structure. In this vein, view $P_o(N)$ as a differentiable space in the obvious way, and let $\mathcal{A}^*(P_o(N))$ be the algebra of differential forms on $P_o(N)$, relative to the differentiable structure. Let

$$\eta: \mathcal{A}^*(P_o(N)) \to \mathcal{A}^{*-1}(P_o(N))$$

be the homotopy operator given by integration along the paths which constitute the points of $P_o(N)$, so that

$$d\eta + \eta d = \text{Id}.$$ (2.6)

Thus, integration of $c$ along the paths which constitute the points of $P_o(N)$ yields the 1-form $\vartheta_c = \eta(p^*_o c)$ on $P_o(N)$ such that $p^*_o c = d\vartheta_c$, and this 1-form descends to a $\Gamma$-connection form

$$\varpi_c: T\overline{S} \to \mathbb{R}$$
on whose curvature coincides with the 2-form $c$ on $N$. Under (2.4), the connection form $\varpi_c$ descends to $\omega$.

Thus we conclude: The extension (2.2) splits and, given the splitting $\sigma: \Gamma \to S^1$, the induced principal $S^1$-bundle $\tau_\sigma = \sigma_*(\overline{\gamma}): S_\sigma \to N$ with connection $\omega_\sigma = \sigma_*(\omega_c)$ has curvature $c$. This recovers the following classical fact:

**Proposition 2.2.** The group $H^1(\pi_1(N), S^1) = \text{Hom}(\pi_1(N), S^1)$ acts simply transitively on the isomorphism classes of principal $S^1$-bundles with connection on $N$ having curvature $c$. Furthermore, two such principal $S^1$-bundles with connection are topologically equivalent if and only if their “difference” in $\text{Hom}(\pi_1(N), S^1)$ lifts to a homomorphism from $\pi_1(N)$ to $\mathbb{R}$. In particular, when $N$ is simply connected, up to gauge transformation, there is a unique principal $S^1$-bundle with connection on $N$ having curvature $c$.

### 2.3 The equivariant extension

Let $G$ be a Lie group, let $\mathfrak{g}$ denote its Lie algebra, and suppose that $N$ is a (left) $G$-manifold. Through the associated infinitesimal $\mathfrak{g}$-action $\mathfrak{g} \to \text{Vect}(N)$
induced by the $G$-action on $N$, the algebra $C^\infty(N)$ acquires a (right) $g$-module structure; let

$$d_g: \text{Alt}(g, C^\infty(N)) \to \text{Alt}(g, C^\infty(N))$$

be the resulting (Cartan-Chevalley-Eilenberg) Lie algebra cohomology operator.

Let $c$ be a $G$-invariant 2-form on $N$ and let $\tau: S \to N$ be a principal $S^1$-bundle on $N$ with connection $\nabla$ having curvature $c$. Let $G_\tau$ denote the group of pairs $(\phi, x)$ where $\phi: S \to S$ is a bundle automorphism which, on the base $N$, descends to the diffeomorphism $x_N$ induced from $x \in G$. Since $c$ is $G$-invariant, the obvious map from $G_\tau$ to $G$ is surjective and hence fits into the group extension

$$1 \to G(\tau) \to G_\tau \to G \to 1 \quad (2.7)$$

where $G(\tau) \cong \text{Map}(N, S^1)$ is the (abelian) group of gauge transformations of $\tau$. Here conjugation in $G_\tau$ induces the obvious action of $G$ on $G(\tau)$ coming from the $G$-action on $N$. Let $g(\tau) \cong \text{Map}(N, \mathbb{R}) = C^\infty(N)$ be the abelian Lie algebra of infinitesimal gauge transformations of $\tau$, made into a $G$- and hence $g$-module in the obvious manner. The Lie algebra extension associated with the group extension (2.7) takes the form

$$0 \to g(\tau) \to g_\tau \to g \to 0. \quad (2.8)$$

Through the infinitesimal $g$-action on $N$, the connection $\nabla$ induces a section $\nabla_g: g \to g_\tau$ for (2.8) in the category of vector spaces, and we denote by $c_g \in \text{Alt}^2(g, C^\infty(N))$ the $C^\infty(N)$-valued Lie algebra 2-cocycle on $g$ determined by $\nabla_g$ and the Lie algebra extension (2.8).

For $X \in g$, we will denote by $X_N$ the associated fundamental vector field on $N$. Recall that a momentum mapping for $c$ (whether or not $c$ is non-degenerate) is a $G$-equivariant map $\mu: N \to g^*$ such that the adjoint $\mu^g: g \to C^\infty(N)$ satisfies the identity

$$d(\mu^g(X)) = c(X_N, \cdot), \quad X \in g;$$

we will then refer to $\mu^g$ as a comomentum. The connection $\nabla$ and the 2-form $c$ being fixed, the comomenta are precisely the $G$-equivariant $C^\infty(N)$-valued 1-cochains $\delta$ on $g$ such that $d_g(\delta) = c_g \in \text{Alt}(g, C^\infty(N))$; in particular, each such comomentum

$$\delta: g \to g(\tau) \cong \text{Map}(N, \mathbb{R}) = C^\infty(N)$$

yields the Lie algebra section

$$\nabla_g + \delta: g \to g_\tau \quad (2.9)$$

for the Lie algebra extension (2.8). These observations entail the following well known fact:
Proposition 2.3. When $G$ is connected, a momentum mapping $\mu : N \to g^*$ induces a lift of the $G$-action to an action of a suitable covering group $\tilde{G}$ on $S$ compatible with the $S^1$-bundle structure and thus turning $\tau$ into a $\tilde{G}$-equivariant principal $S^1$-bundle, and every such lift induces a momentum mapping. Furthermore, the connection $\nabla$ on $\tau$ is then as well $\tilde{G}$-invariant.

3 Circle bundles on the fiber of a map

Let $M$ and $N$ be smooth connected manifolds, let $f : M \to N$ be a smooth map, and let $P_f$ denote the fiber of $f$, made precise below. Given a closed relative 2-form for $f$ (made precise below) with integral periods, using a variant of the method in the previous section, we will construct the principal $S^1$-bundles with connection on $P_f$ having the given relative 2-form as curvature.

Recall that, for a space $Y$ and a point $y$ of $Y$, the obvious projection map $\pi_y : P_y(Y) \to Y$ which sends a path to its end point is a Hurewicz fibration onto the path component of $y$ and the fiber over the point $y$ amounts to the space $\Omega_y(Y)$ of based loops in $Y$, based at $y$. We will denote by $i_y : \Omega_y(Y) \to P_y(Y)$ the corresponding inclusion.

Let momentarily $M$ and $N$ be merely pathwise connected topological spaces having suitable local properties, let $f : M \to N$ be a (continuous) map, let $o$ be a point of $M$, taken henceforth as base point, take $f(o)$ to be the base point of $N$, and let $P_f$ denote the fiber of the map $f$, that is,

$$P_f = M \times N P_{f(o)}(N) = \{(q, u); u(0) = f(o), u(1) = f(q)\} \subseteq M \times P_{f(o)}(N).$$

We will denote the obvious projection map from $P_f$ to $M$ which sends the pair $(u, q) \in P_f$ to $q \in M$ by $\pi_f : P_f \to M$ and we will denote by $j_f : P_f \to P_{f(o)}(N)$ the induced map. Since the resulting diagram

$$\begin{array}{ccc}
P_f & \xrightarrow{j_f} & P_{f(o)}(N) \\
\downarrow{\pi_f} & & \downarrow{\pi_{f(o)}} \\
M & \xrightarrow{f} & N
\end{array}$$

is a pull back square, $\pi_f$ a fibration; the fiber $\pi_f^{-1}(o)$ at the point $o$ of $M$ amounts to the based loop space $\Omega_{f(o)}(N)$, and we denote by

$$i_f : \Omega_{f(o)}(N) \to P_f$$

the corresponding injection. All these constructions and facts are well known and classical, cf. e.g. [20].
Let \( j_1 : I \rightarrow I^2 \) be the injection which sends \( t \in I \) to \((t, 0) \in I^2\), and let \( E_f \) denote the space of commutative diagrams

\[
\begin{array}{ccc}
\{0\} & \longrightarrow & I \\
\downarrow & & \downarrow j_1 \\
\{o\} & \longrightarrow & M
\end{array} \quad \begin{array}{ccc}
I & \longrightarrow & I^2 \\
\downarrow w & & \downarrow \phi \\
M & \longrightarrow & N
\end{array}
\]

having the property that, for every \( 0 \leq s \leq 1 \), (i) \( \phi(0, s) = f(o) \) and that
(ii) \( \phi(1, s) \) is independent of \( s \). Thus \( E_f \) is a space of strings in \( N \) which are subject to a boundary condition phrased in terms of the map \( f \).

Let \( \hat{\pi}_f : E_f \rightarrow P_f \) be the obvious map which sends \((w, \phi)\) to \((w(1), u_\phi)\).

The space \( E_f \) is contractible. Indeed, the assignment to \((w, \phi, r) \in E_f \times I\) of \((w_r, \phi_r) \in E_f \) given by
\[
w_r = w \quad \text{and} \quad \phi_r(t, s) = \phi(t, (1-r)s)
\]
contracts \( E_f \) onto a subspace of \( E_f \) which amounts to the space \( P_o M \) of paths in \( M \) having starting point \( o \), and the space \( P_o M \), in turn, is contractible.

Suppose that \( P_f \) is pathwise connected. This can always be arranged for, in the following way: Since \( M \) and \( N \) are (supposed to be) pathwise connected, \( P_f \) being pathwise connected is equivalent to the induced map \( \pi_1(M) \rightarrow \pi_1(N) \) being surjective. When this map is not surjective, let \( N \) be the covering space of \( N \) having fundamental group the image of \( \pi_1(M) \) in \( \pi_1(N) \); a choice of pre-image \( \overline{\pi} \) of \( f(o) \) then determines a lift \( \overline{f} : M \rightarrow \overline{N} \) of \( f \) with \( \overline{f}(o) = \overline{o} \), the space \( P_f \overline{f} \), viewed as a subspace of \( P_f \) in the obvious way, is a path component of \( P_f \), and every path component of \( P_f \) arises in this manner. Thus we may always assume that \( P_f \) is pathwise connected.

With this preparation out of the way, \( P_f \) being pathwise connected, the map \( \hat{\pi}_f \) is surjective and hence a fibration. Since the total space \( E_f \) is contractible, the construction in the previous section, with \( E_f \) instead of \( P_o(P_f) \), will furnish the principal circle bundles on \( P_f \) with connection. We will now explain the details thereof.

Extending the notation \( I \) for the unit interval and \( I^2 \) for the unit square, let \( I^3 \) denote the unit cube and, for \( 1 \leq n \leq 3 \), let
\[
J^n = \partial I^{n-1} \times I \cup I^{n-1} \times \{1\} \subseteq I^n;
\]
here \( \partial I^n \) denotes the boundary of \( I^n \).

Let \( u_o \) be the constant path in \( N \) concentrated at the point \( f(o) \) of \( N \), and take \((o, u_o)\) as base point of \( P_f \). The fiber \( F_{(o, u_o)} \) over the point \((o, u_o)\) of \( P_f \) is the space of commutative diagrams or strings of the kind

\[
\begin{array}{ccc}
J^1 & \longrightarrow & \partial I^2 \\
\downarrow & & \downarrow w \\
\{o\} & \longrightarrow & M
\end{array} \quad \begin{array}{ccc}
\partial I^2 & \longrightarrow & I^2 \\
\downarrow \phi & & \downarrow \phi \\
M & \longrightarrow & N
\end{array}
\]

10
Here we do not distinguish in notation between the closed path \( w : I \to M \) and the map \( \partial I^2 \to M \) which coincides with \( w \) on \( I \times \{0\} \) and which is constant on \( J^1 \subseteq \partial I^2 \). Since \( E_f \) is contractible, the fiber \( F_{(o, u_o)} \) is homotopy equivalent to the based loop space \( \Omega_{(o, u_o)}P_f \) of \( P_f \). The loop multiplication corresponds to the familiar juxtaposition of strings of the kind (3.3).

Consequently the space of components of the fiber \( F_{(o, u_o)} \) underlies the fundamental group \( \pi_1(P_f) \), and the fundamental group of each path component of the fiber \( F_{(o, u_o)} \) is given by the second homotopy group \( \pi_2(P_f) \).

The obvious forgetful map from \( E_f \) to \( P_o(M) \) which sends \( (w, \phi) \in E_f \) to \( w \in P_o(M) \) is a fibration; the fiber \( F_{u_o} \) over the trivial path \( u_o \) concentrated at \( o \in M \) is the space of maps from \( I^2 \) to \( N \) having the property that \( \phi(0, s) = f(o) = \phi(1, s) = \phi(t, 0), \ 0 \leq t \leq 1, \ 0 \leq s \leq 1 \).

Given two commutative diagrams \((w_1, \phi_1)\) and \((w_2, \phi_2)\) of the kind (3.2), we define a homotopy \((h, H)\) from \((w_1, \phi_1)\) to \((w_2, \phi_2)\), written as

\[
(h, H) : (w_1, \phi_1) \simeq (w_2, \phi_2),
\]

to be a commutative diagram of the kind

\[
\begin{array}{ccc}
\{0\} \times I & \longrightarrow & I \\ \\
| & \downarrow h & | \\ \\
\{o\} & \longrightarrow & M \\
\end{array}
\xrightarrow{j_1 \times \text{Id}}
\begin{array}{ccc}
I^2 \times I & \longrightarrow & I \\
| & \downarrow H & | \\
M & \longrightarrow & N
\end{array}
\]

subject to the following requirements:

- \( h(t_1, 0) = w_1(t_1), \ 0 \leq t_1 \leq 1 \),
- \( h(t_1, 1) = w_2(t_1), \ 0 \leq t_1 \leq 1 \),
- \( H(t_1, t_2, 0) = \phi_1(t_1, t_2), \ 0 \leq t_1, t_2 \leq 1 \),
- \( H(t_1, t_2, 1) = \phi_2(t_1, t_2), \ 0 \leq t_1, t_2 \leq 1 \),
- \( h(1, s) \) is independent of \( s, \ 0 \leq s \leq 1 \),
- \( H(0, t_2, s) = f(o), \ 0 \leq t_2, s \leq 1 \),
- \( H(1, t_2, s) \) is independent of \( s, \ 0 \leq t_2, s \leq 1 \),
- \( H(t_1, 1, s) \) is independent of \( s, \ 0 \leq t_1, s \leq 1 \).

In the same vein, given two commutative diagrams \((w_1, \phi_1)\) and \((w_2, \phi_2)\) of the kind (3.3), a homotopy \((h, H)\) from \((w_1, \phi_1)\) to \((w_2, \phi_2)\), written as

\[
(h, H) : (w_1, \phi_1) \simeq (w_2, \phi_2),
\]
is a commutative diagram of the kind

$$
\begin{array}{ccc}
J^1 \times I & \longrightarrow & \partial I^2 \times I \\
\downarrow & & \downarrow h \\
\{o\} & \longrightarrow & M \\
\downarrow f & & \downarrow H \\
& & N.
\end{array}
$$

(3.5)

The homotopy classes relative to this homotopy relation are well known to underly the fundamental group $\pi_1(P_f)$ of $P_f$ or, equivalently, the second relative homotopy group $\pi_2(f)$ and hence constitute the space of path components of the fiber $F_{(o,u_o)}$ of $\hat{\pi}_f$. The group structure of $\pi_1(P_f)$ is induced by the familiar operation of juxtaposition relative to the first parameter of $I^2$ of two diagrams of the kind (3.3) and subsequent reparametrization.

We now suppose that $M$ and $N$ are ordinary smooth manifolds and that $f$ is a smooth map. With an abuse of notation, we will then denote by $P_{f(o)}(N)$ the space of piecewise smooth paths in $N$ having starting point $f(o)$, by $P_f$ the fiber of $f$ defined merely in terms of piecewise smooth paths in $N$, and by $E_f$ the space of piecewise smooth strings of the kind (3.2), where the term piecewise smooth is to be interpreted in terms of a paving of $I^2$.

In the obvious way, we view $P_{f(o)}(N)$ as a differentiable space in the sense of [3], and we denote by $\mathcal{A}^*(P_{f(o)}(N))$ the resulting algebra of differential forms on $P_{f(o)}(N)$, relative to the differentiable structure. Let

$$
\eta: \mathcal{A}^*(P_{f(o)}(N)) \rightarrow \mathcal{A}^{*-1}(P_{f(o)}(N))
$$

be the homotopy operator given by integration along the paths which constitute the points of $P_{f(o)}(N)$, so that

$$
d\eta + \eta d = \text{Id}.
$$

(3.7)

Let $\lambda$ be a closed 3-form on $N$. Then integration yields the 2-form

$$
\beta_\lambda = \eta(\pi_{f(o)}^*(\lambda))
$$

(3.8)
on $P_{f(o)}(N)$ such that $d\beta_\lambda = \pi_{f(o)}^*(\lambda)$, and $\iota_{f(o)}^*\beta_\lambda$ is a closed 2-form on $\Omega_{f(o)}(N)$.

Let $\zeta$ be a 2-form on $M$ such that $d\zeta = f^*(\lambda) \in \mathcal{A}^2(M)$. Then

$$
\zeta_{(f,\lambda,\zeta)} = j_f^*(\beta_\lambda) - \pi_f^*(\zeta)
$$

(3.9)
is a closed 2-form on $P_f$ which restricts to $j_f^*(\beta_\lambda)$, that is,

$$
\iota_f^*\zeta_{(f,\lambda,\zeta)} = j_f^*(\beta_\lambda).
$$

Furthermore, $[\zeta_{(f,\lambda,\zeta)}] \in H^0(N, H^2(P_f))$ transgresses to $[\lambda] \in H^3(N, H^0(P_f))$. 

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Suppose that $\lambda$ has integral periods and that, furthermore, the pair $(\lambda, \zeta)$ has integral periods in the sense that, given a 3-manifold $C$ and a commutative diagram
\[
\begin{array}{ccc}
\partial C & \longrightarrow & C \\
\downarrow h & & \downarrow H \\
M & \longrightarrow & N
\end{array}
\] (3.10)
where $h$ and $H$ are piecewise smooth maps relative to a paving of $C$, the difference
\[
\int_C H^*(\lambda) - \int_{\partial C} h^*(\zeta)
\]
is an integer. Then the closed 2-form $\zeta_{f,\lambda,\zeta}$ on $P_f$ has integral periods. Hence there is a principal circle bundle on $P_f$ with a connection having curvature $\zeta_{f,\lambda,\zeta}$. Guided by the considerations in the previous section, we will now spell out an explicit construction for all such circle bundles, cf. Corollary 3.2 below.

Define the piecewise smooth strings $(w_1, \phi_1)$ and $(w_2, \phi_2)$ of the kind (3.2) to be equivalent whenever there is a piecewise smooth homotopy $(h, H)$ of the kind (3.4) such that
\[
\int_{I^3} H^*\lambda - \int_{I^2} h^*\zeta
\]
is an integer; here $(h, H)$ being piecewise smooth refers to a suitable paving of the cube $I^3$. Since the pair $(\lambda, \zeta)$ has integral periods, this equivalence relation is well defined. Let $\hat{P}_f$ denote the space of equivalence classes of such strings $(w, \phi)$ of the kind (3.2).

Likewise, let $\Gamma_f$ be the space of equivalence classes of piecewise smooth strings $(w, \phi)$ of the kind (3.3), the equivalence relation being defined in terms of piecewise smooth homotopies of the kind (3.3). The operation of juxtaposition relative to the first parameter of $I^2$ of two diagrams of the kind (3.3) and subsequent reparametrization, that is, the same operation of composition of diagrams as that which defines the group structure of $\pi_1(P_f)$, turns $\Gamma_f$ into a group. Moreover, denote by $F_{(o,U_o),0}$ the subspace of $F_{(o,U_o),0}$ of diagrams of the kind (3.3) that are null-homotopic; the assignment to such a diagram that is null-homotopic via a homotopy $(h, H)$ of the difference $\int_C H^*(\lambda) - \int_{\partial C} h^*(\zeta)$ modulo $\mathbb{Z}$ yields a map $F_{(o,U_o),0} \rightarrow S^1$, and this map and the canonical projection from $F_{(o,U_o)}$ to $\Gamma_f$ fit together in the commutative diagram
\[
\begin{array}{ccc}
F_{(o,U_o),0} & \longrightarrow & F_{(o,U_o)} \\
\downarrow & & \downarrow \text{Id} \\
S^1 & \longrightarrow & \Gamma_f \\
\downarrow & & \downarrow \text{Id} \\
& & 1
\end{array}
\]
\[
\pi_1(P_f)
\]
whose bottom row is a central extension

\[ 1 \longrightarrow S^1 \longrightarrow \Gamma_f \longrightarrow \pi_1(P_f) \longrightarrow 1 \]  

(3.11)
of Lie groups, necessarily split.

The operation

\[ E_f \times F_{(o,U_o)} \longrightarrow E_f \]  

(3.12)
of juxtaposition relative to the first parameter of \( I^2 \) of a diagram of the kind  

(3.2) and one of the kind  

(3.3) and subsequent reparametrization induces a \( \Gamma_f \)-action on \( \hat{P}_f \). The projection \( \hat{\tau}_f: \hat{E}_f \rightarrow P_f \) descends to a projection \( \hat{\tau}_f: \hat{P}_f \rightarrow P_f \).

Given the map \( f: M \rightarrow N \) with \( P_f \) not necessarily being pathwise connected, we can carry out the construction of \( \hat{\tau}_f \) separately for each path component of \( P_f \). The fundamental group \( \pi_1(P_f) \) does not depend on the choice of path component.

**Theorem 3.1.** The projection \( \hat{\tau}_f: \hat{P}_f \rightarrow P_f \) is a principal \( \Gamma_f \)-bundle, and the data determine a \( \Gamma_f \)-connection, with connection form \( \omega_{(f,\lambda,\zeta)} \) on \( \hat{P}_f \), having curvature \( \zeta_{(f,\lambda,\zeta)} \).

**Proof.** We noted above that (each path component of) the space \( E_f \) is contractible. The construction of the principal bundle \( \hat{\tau}_f \) is essentially the same as that of the principal bundle  

(2.3) for \( N = P_f \) in the previous section, but now carried out with the space \( E_f \) rather than with the space \( P_o(P_f) \). \( \square \)

Now, let \( \sigma: \Gamma_f \rightarrow S^1 \) be a splitting of  

(5.11). Then the induced principal \( S^1 \)-bundle \( \tau_\sigma = \sigma\ast(\hat{\tau}_f): S_\sigma \rightarrow P_f \) with connection \( \omega_{(f,\lambda,\zeta)} = \sigma\ast(\omega_{(f,\lambda,\zeta)}) \) has curvature \( \zeta_{(f,\lambda,\zeta)} \). Thus we obtain the following.

**Corollary 3.2.** The group \( \text{Hom}(\pi_1(P_f), S^1) = \text{Hom}(\pi_2(f), S^1) \) acts simply transitively on the isomorphism classes of principal \( S^1 \)-bundles with connection on \( P_f \) having curvature \( \zeta_{(f,\lambda,\zeta)} \).

The construction of the principal bundle \( \hat{\tau}_f \) applies to the particular case where \( M \) is a single point; we then denote by \( \Gamma_\Omega \) the resulting group written above as \( \Gamma_f \), and we write the resulting principal \( \Gamma_\Omega \)-bundle on \( \Omega_o(N) \) in the form

\[ \hat{\tau}_\Omega: \Omega_o(N) \longrightarrow \Omega_o(N) \]  

(3.13)

Since \( \pi_1(\Omega_o(N)) \cong \pi_2(N) \), the corresponding central extension  

(3.11) then takes the form

\[ 1 \longrightarrow S^1 \longrightarrow \Gamma_\Omega \longrightarrow \pi_2(N) \longrightarrow 1. \]  

(3.14)

This extension necessarily splits whence \( \Gamma_\Omega \) is an abelian group.

Return to a general smooth map \( f: N \rightarrow M \). Let \( \pi_M: \tilde{M} \rightarrow M \) and \( \pi_N: \tilde{N} \rightarrow N \) be the universal covering projections, pick a base point \( \tilde{o} \) of \( \tilde{M} \)
over \( o \), lift the map \( f \) to a map \( \tilde{f} : \tilde{M} \to \tilde{N} \), and take \( \tilde{f}(\tilde{o}) \) to be the base point of \( \tilde{N} \). We mention in passing that \( \tilde{f} \) is determined by the value \( \tilde{f}(\tilde{o}) \).

Any string of the kind (3.2) admits a unique lift to a string of the kind
\[
\{0\} \longrightarrow I \xrightarrow{j_1} I^2 \quad \{\tilde{o}\} \longrightarrow \tilde{M} \xrightarrow{\tilde{f}} \tilde{N}.
\]

Consequently the map \( \pi_f \) lifts to a unique based map \( \tilde{\pi}_f : \hat{P}_f \to \tilde{M} \), and the various fibrations fit into a commutative diagram of the kind
\[
\Gamma_{\Omega} \longrightarrow \Omega_o N \xrightarrow{\tilde{\pi}_{\Omega}} \Omega_o(N) \quad \Gamma_f \longrightarrow \hat{P}_f \xrightarrow{\tilde{\pi}_f} P_f \quad \pi_1(M) \longrightarrow \tilde{M} \xrightarrow{\pi_M} M.
\]

Here \( \tilde{\pi}_{\Omega} \), \( \tilde{\pi}_f \), and \( \pi_M \) are principal fiber bundle projections, \( \tilde{\pi}_f \) and \( \pi_f \) are fibrations, and the two left-hand unlabelled vertical arrows constitute an exact sequence of groups.

### 4 The equivariant extension

As before, let \( G \) be a Lie group, and denote its Lie algebra by \( \mathfrak{g} \). Suppose that \( M \) and \( N \) are \( G \)-spaces, that \( f \) is a \( G \)-map, and that the forms \( \lambda \) and \( \zeta \) are \( G \)-invariant. Then the spaces \( P_{f(o)}(N) \) and \( P_f \) are manifestly \( G \)-spaces, the commutative diagram (3.1) is one of \( G \)-spaces, the de Rham complexes \( A^*(M) \), \( A^*(N) \), \( A^*(P_{f(o)}(N)) \), \( A^*(P_f) \), are \( G \)-complexes, and the 2-form \( \zeta(f,\lambda,\zeta) \) is manifestly \( G \)-invariant. Thus, to carry out the construction of the principal \( \Gamma_f \)-bundle \( \tilde{\tau}_f \), cf. Theorem 3.1 above, in a \( G \)-equivariant manner by means of the method spelled out in subsection 2.3 above, we need a momentum mapping
\[
\mu : P_f \longrightarrow \mathfrak{g}^*.
\]

The resulting \( S^1 \)-bundles of the kind \( \tau_{\sigma} \), cf. Corollary 3.2 will then likewise be \( G \)-equivariant. We will now explain how such a momentum mapping arises.

For \( X \in \mathfrak{g} \), let \( X_N \) be the associated vector field on \( N \) and, given the vector field \( Y \) on \( N \), let \( i_Y \) denote the familiar operator of contraction with the
vector field $Y$. Further, let $\mathcal{A}^{2j,k}(N)$ denote the space of degree $j$ polynomial maps on $g$ with values in $\mathcal{A}^k(N)$, and let

$$\delta_G : \mathcal{A}^{2s,s}(N) \longrightarrow \mathcal{A}^{2s+2,s-1}(N)$$

be the familiar equivariant operator given by

$$(\delta_G(\alpha))(X) = -i_{X_N}(\alpha(X)).$$

As before, we denote by $\eta : \mathcal{A}^s(P_f(o)(N)) \rightarrow \mathcal{A}^{s-1}(P_f(o)(N))$ the homotopy operator given by integration along the paths which constitute the points of $P_f(o)(N)$. This homotopy operator is plainly $G$-equivariant.

**Theorem 4.1.** Given a $G$-equivariant linear map $\vartheta : g \rightarrow \mathcal{A}^1(N)$ satisfying the identities

$$\delta_G(\lambda) = -d\vartheta, \quad \delta_G(c) = f^*(\vartheta), \quad \delta_G(\vartheta) = 0,$$

$\vartheta$ being viewed as a $g^*$-valued 1-form on $N$, the map

$$\mu_{f,\vartheta} = -\eta(p_{f(o)}^*(\vartheta)) \circ j_f : P_f \rightarrow g^*$$

is a $G$-momentum mapping for $\zeta_{(f,\lambda,\zeta)}$.

**Proof.** The conditions on $\vartheta$ say that, somewhat more explicitly, given $X \in g$ and vector fields $Y$ and $Z$ on $N$,

$$\lambda(X_N, Y, Z) = -d(\vartheta(X))(Y, Z), \quad c(X_N, Y) = f^*(\vartheta(X))(Y).$$

Then

$$\delta_G(\zeta_{(f,\lambda,\zeta)}) = j_f^*\delta_G(\beta_{\lambda}) - \pi_{f(o)}^*(\delta_G(c))$$

$$= j_f^*\delta_G(\beta_{\lambda}) - \pi_{f(o)}^*(f^*\vartheta)$$

$$= j_f^*\delta_G(\beta_{\lambda}) - j_f^*(\pi_{f(o)}^*(\vartheta))$$

$$= j_f^*(\delta_G(\beta_{\lambda}) - \pi_{f(o)}^*(\vartheta))$$

View $\vartheta$ as a $g^*$-valued 1-form on $N$ and let $\mu_\vartheta = \eta(-\pi_{f(o)}^*(\vartheta)) : P_0(N) \rightarrow g^*$ be the $g^*$-valued function on $P_0(N)$, necessarily $G$-equivariant, which arises by integration of $-\pi_{f(o)}^*(\vartheta) \in \mathcal{A}^1(P_0(N), g^*)$. Then

$$\pi_{f(o)}^*(\vartheta) = d\eta(\pi_{f(o)}^*(\vartheta)) + \eta(d(\pi_{f(o)}^*(\vartheta)))$$

$$= -d\mu_\vartheta + \eta(\pi_{f(o)}^*(d\vartheta))$$

$$= -d\mu_\vartheta + \eta(\pi_{f(o)}^*(\delta_G(\lambda)))$$

$$= -d\mu_\vartheta + \delta_G(\eta(\pi_{f(o)}^*(\lambda)))$$

$$= -d\mu_\vartheta + \delta_G(\beta_{\lambda})$$

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whence
\[ d\mu_\vartheta = \delta_G(\beta_\lambda) - \pi^*_f(\vartheta). \]
By definition, \( \mu_{f,\vartheta} = \mu_\vartheta \circ j_f : P_f \to \mathfrak{g}^* \). This map is \( G \)-equivariant. Moreover, by construction,
\[ d\mu_{f,\vartheta} = j_f^*(\delta_G(\beta_\lambda) - \pi^*_f(\vartheta)) = \delta_G(\zeta_{(f,\lambda,\zeta)}), \]
that is, \( \mu_{f,\vartheta} \) is a \( G \)-momentum mapping for \( \zeta_{(f,\lambda,\zeta)} \).

Thus, when \( G \) is connected, as explained in the previous section, the momentum mapping \( \mu_{f,\vartheta} \) furnishes a lift of the \( G \)-action to \( \hat{P}_f \) turning \( \hat{\tau}_f : \hat{P}_f \to P_f, \omega_{f,\lambda,c} \) into a \( G \)-equivariant principal \( \Gamma_f \)-bundle with connection.

Now, let \( \sigma : \Gamma_f \to S^1 \) be a splitting of (3.11). Then the induced principal \( S^1 \)-bundle \( \tau_\sigma = \sigma_*(\hat{\tau}_f) : S_\sigma,f \to P_f \) with connection \( \omega_{\sigma,f,\lambda,\zeta} = \sigma_*(\omega_{(f,\lambda,\zeta)}) \) is a bundle in the category of \( G \)-spaces and has curvature \( \zeta_{(f,\lambda,\zeta)} \). In particular, the group Hom(\( \pi_1(\mathcal{P}_f), S^1 \)) acts simply transitively on the isomorphism classes of \( G \)-equivariant principal \( S^1 \)-bundles with connection on \( P_f \) having curvature \( \zeta_{(f,\lambda,\zeta)} \).

5 The case where the target is a Lie group

Let \( H \) be a Lie group and let \( \mathfrak{h} \) denote its Lie algebra. View \( H \) as an \( H \)-group via conjugation. Let \( \cdot \) be an invariant symmetric bilinear form on \( \mathfrak{h} \), for the moment neither necessarily non-degenerate nor positive. Let \( \omega_H \) denote the left-invariant \( \mathfrak{h} \)-valued Maurer-Cartan form on \( H \) and let \( \overline{\omega}_H \) be the right-invariant \( \mathfrak{h} \)-valued Maurer-Cartan form on \( H \). Let
\[ \lambda = \frac{1}{12} [\omega_H, \omega_H] \cdot \omega_H. \]
This is a closed bi-invariant 3-form on \( H \).

As before, we denote by \( P_\lambda(H) \) the space of piecewise smooth paths in \( H \) starting at the neutral element \( e \) of \( H \). Let
\[ \vartheta^\flat = \frac{1}{2}(\omega_H + \overline{\omega}_H) \in \mathcal{A}^1(H, \mathfrak{h}). \]
The resulting map
\[ \eta(\vartheta^\flat) : P_e(H) \to \mathfrak{h} \]
coming from integration sends the piecewise smooth path \( u : I \to H \) with \( u(0) = e \) to
\[ \eta(\vartheta^\flat)(u) = \frac{1}{2} \int_0^1 (u^{-1}\dot{u} + \dot{u}u^{-1})dt \in \mathfrak{h}. \]
Notice that when \( u \) is the 1-parameter subgroup generated by \( Y \in \mathfrak{h} \) the value \( \eta(\vartheta^\flat)(u) \) is just \( Y \), that is, the composite of \( \eta(\vartheta^\flat) \) with the canonical injection of \( \mathfrak{h} \) into \( P_e(H) \) is the identity of \( \mathfrak{h} \).

Let \( \vartheta \in A^1(H, \mathfrak{h}^*) \) be the \( \mathfrak{h}^* \)-valued 1-form on \( H \) which arises from combination of \( \vartheta^\flat \) with the adjoint \( \mathfrak{h} \to \mathfrak{h}^* \) of the given invariant symmetric bilinear form. Then

\[
\delta H \vartheta = 0, \quad \delta H \lambda = -d \vartheta,
\]

cf. [23] (4.1), (4.3). It is well known that, when \( H \) is compact, an invariant inner product on \( \mathfrak{h} \) exists that is even positive definite, and the resulting 3-form \( \lambda \), occasionally referred to in the literature as the Cartan 3-form, is well known to have integral periods. Thus Theorems 3.1 and 4.1 apply to maps of the kind \( f: M \to H \), with \( f(o) = e \) (which can always be arranged for), under suitable circumstances. In the remaining sections we spell out a number of examples.

In the present situation where the target of the map \( f \) is a Lie group, more structure is available: The spaces \( P_e(H) \) and \( \Omega_e(H) \) inherit group structures, and the projection \( \pi_e: P_e(H) \to H \) is a homomorphism whence this projection is, in particular, a principal fiber bundle with structure group \( \Omega_e(H) \). Consequently \( \pi_f: P_f \to M \) is necessarily a principal fiber bundle with structure group \( \Omega_e(H) \).

Suppose that \( H \) is connected. Since \( \pi_2(H) \) is zero, the extension (3.14) comes down to an isomorphism \( S^1 \to \Gamma \), and the diagram (3.16) takes the form

\[
\begin{array}{ccccccc}
S^1 & \to & \Omega_e(H) & \to & \Omega_e(H) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma_f & \to & \hat{P}_f & \to & P_f \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(M) & \to & \hat{M} & \to & M.
\end{array}
\]

When \( \Omega_e(H) \) acquires a group structure—this will always be so when \( H \) is simply connected—the resulting projection \( \hat{P}_f \to \hat{M} \) is likewise a principal fiber bundle, with structure group \( \Omega_e(H) \). The top row of the diagram (5.1) is then the universal central extension of the based loop group \( \Omega_e(H) \) of \( H \); cf. [15] and the literature there.

6 Application to moduli spaces

Let \( \Sigma \) be a closed surface of genus \( \ell \), let \( K \) be a compact connected Lie group, let \( \cdot \) be a positive definite invariant symmetric bilinear form on the Lie algebra \( \mathfrak{k} \) of \( K \), and let

\[
\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = \prod [x_j, y_j],
\]

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be the familiar presentation of the fundamental group $\pi_1(\Sigma)$ of $\Sigma$. The relator $r$ induces a relator map

$$r: K^{2\ell} \longrightarrow K.$$ 

Endow $K$ and $K^{2\ell}$ with the $K$-action given by conjugation in $K$. Then $r$ is plainly $K$-equivariant. The construction in Section 2 of [8] yields a $K$-invariant 2-form $\zeta$ on $M = K^{2\ell}$ (written there as $\omega_c$) such that

$$d\zeta = r^*\lambda,$$

cf. [8] (18). The pair $(\zeta, \lambda)$ arises from a certain form of total degree 4 on the simplicial model for the classifying space $BK$ of $K$ which, in turn, has been constructed by Bott [3], Dupont [6] and Shulman [18], and this form on $BK$ represents the universal Pontrjagin class and hence has integral periods. Consequently the pair $(\zeta, \lambda)$ has integral periods in our sense.

The map between the fundamental groups induced by the relator map is trivial. Hence a choice of central element $z$ of the universal covering group $\tilde{K}$ determines a lift $r_z: K^{2\ell} \longrightarrow \tilde{K}$.

The fiber $P_{r_z}$ is connected, even simply connected, since $\pi_2(K)$ is zero and, as $z$ ranges over the center of $\tilde{K}$ or, equivalently, over the fundamental group of $K$, the spaces $P_{r_z}$ range over the path components of the fiber $P_r$ of the original relator map $r$. Furthermore, $P_r$ is a $K$-space in an obvious fashion. We can thus take $f$ to be any of the maps $r_z$ as $z$ ranges over the center of $\tilde{K}$ and apply Theorems 3.1 and 4.1 with this choice of $f$.

Thus, let $\zeta_{(r,\lambda,\zeta)}$ be the closed $K$-invariant 2-form (3.8), constructed separately on each path component of $P_f$ of the kind $P_{r_z}$. This form has integral periods and is necessarily $K$-invariant. Theorem 3.1 exploited separately for each path component of $P_r$ yields the principal $S^1$-bundle $\tau_r: S \longrightarrow P_r$ together with the connection 1-form $\omega_{(r,\lambda,\zeta)}$ having curvature $\zeta_{(r,\lambda,\zeta)}$ and, since each path component of $P_r$ is simply connected, this $S^1$-bundle with connection is uniquely determined by the data up to gauge transformations. Moreover, let

$$\vartheta \in A^1(K, \mathfrak{p}^*) \cong A^{2,1}(K)$$

be the form introduced in the previous section, where now $K$ is substituted for $H$; it can be shown that

$$\delta_K(\zeta) = r^*(\vartheta) \in A^{2,1}(K^{2\ell}).$$

Theorem 4.1 exploited separately for each path component of $P_r$ of the kind $P_{r_z}$, yields a momentum mapping $\mu_{f,\vartheta}: P_r \longrightarrow \mathfrak{p}^*$ and hence a lift of the $K$-action to a $\tilde{K}$-action on the total space $S$ of $\tau_r$ turning $\tau_r$ into a $\tilde{K}$-equivariant principal $S^1$-bundle with connection. The construction is entirely rigid and natural in terms of the data.
The extended moduli space $\mathcal{H}(\mathcal{P}, K)$ constructed in [8] embeds $K$-equivariantly into $P_r$ in a canonical manner. The composite of the injection with the momentum mapping $\mu_{f,0}$ furnishes the momentum mapping $\mu_f^\#$ constructed in [8], and the 2-form $\zeta_{(r,\lambda,\zeta)}$ restricts to the 2-form denoted in [8] by $\omega_{c,P}$. Thus the present construction recovers the extended moduli space. New insight is provided by the functorial construction of the principal $S^1$-bundle $\tau_r$: This $S^1$-bundle restricts to a $\tilde{K}$-equivariant principal $S^1$-bundle on $\mathcal{H}(\mathcal{P}, K)$ having Chern class $[\omega_{c,P}]$ and, in particular, the construction furnishes, in a functorial manner, a connection having curvature $\omega_{c,P}$.

In a neighborhood of the zero locus of the momentum mapping, written in [8] as $M(\mathcal{P}, K)$, the 2-form $\omega_{c,P}$ is symplectic, and symplectic reduction yields the corresponding moduli spaces of (possibly) twisted representations of $\pi_1(\Sigma)$ in $K$ [8] or, equivalently, the corresponding moduli spaces of central Yang-Mills connections on $\Sigma$. Thus, reduction carries the principal $S^1$-bundle to an object which serves as a replacement for the (in general missing) principal $S^1$-bundle on the moduli space. When such a bundle exists, the corresponding line bundle is referred to in the literature as a Poincaré bundle.

On an open and dense stratum, the reduced object is an ordinary principal $S^1$-bundle, though.

7 A geometric object realizing the first Pontrjagin class

Let $o$ be a suitably chosen base point for $\Sigma$, see below. Under the circumstances of the previous section, we will construct a $K$-equivariant principal $S^1$-bundle on the space $\text{Map}^o(\Sigma, BK)$ of based maps from $\Sigma$ to $BK$. The group $K$ being supposed connected, each path component of this space corresponds to a principal $K$-bundle on $\Sigma$ and in fact amounts to the classifying space of the group of based gauge transformations of that principal $K$-bundle [2]. The space $\text{Map}^o(\Sigma, BK)$ is based homotopy equivalent to the space $P_r$ in the previous section.

A choice of lifting function for the universal $K$-bundle induces a topological holonomy $\Omega BK \to K$. Alternatively, when $BK$ is taken to be the realization of the nerve $NK$ of $K$, in each simplicial degree $p$, a suitable connection on the corresponding principal $K$-bundle on $N_pK$ induces a $K$-valued holonomy map $\Omega N_pK \to K$, and these combine to a holonomy map $\Omega BK \to K$.

Let $B^2$ be the unit 2-disk, identify the circle $S^1$ with the boundary of $B^2$ in the standard fashion, take $1 \in S^1$ as base point of $S^1$ and $B^2$, let $\vee 2\ell S^1$ be the ordinary one-point union of $2\ell$ copies of the circle $S^1$, let $F: S^1 \to \vee 2\ell S^1$ be a map whose cofiber furnishes the surface $\Sigma$ and, for convenience, as base point $o$ for $\Sigma$ we take the obvious base point of $\vee 2\ell S^1$, viewed as a subspace
of $\Sigma$. Thus these spaces fit into the pushout diagram

\[
\begin{array}{ccc}
S^1 & \longrightarrow & B^2 \\
\downarrow F & & \downarrow \\
\vee_{2k} S^1 & \longrightarrow & \Sigma
\end{array}
\]

which we will refer to as a geometric presentation of $\Sigma$. Take

\[M = (\Omega BK)^{2k} = \text{Map}^\circ(\vee_{2k} S^1, BK), \quad N = \Omega BK = \text{Map}^\circ(S^1, BK),\]

and let

\[f = F^* : \text{Map}^\circ(\vee_{2k} S^1, BK) \longrightarrow \text{Map}^\circ(S^1, BK)\]

be the induced map. Then the fiber $P_f$ of the map $f$ coincides with the space $\text{Map}^\circ(\Sigma, BK)$. In particular, when the map $F$ is suitably chosen, the holonomy induces a map from $\text{Map}^\circ(\Sigma, BK)$ to the space $P_\ell$ in the previous section, and this map furnishes the asserted homotopy equivalence. Abusing notation, let $\lambda \in A^3(N)$ be the 3-form induced from the Cartan 3-form in $A^3(K)$ via the holonomy, and let $\zeta \in A^2(M)$ be the 2-form induced, via the holonomy, from the 2-form in $A^2(K^{2k})$ denoted by the same symbol. Moreover, let

\[\vartheta \in A^{2,1}(N)\]

be the form induced, via the holonomy, from the corresponding form in $A^1(K, \ast^*) \cong A^{2,1}(K)$ introduced in Section 5 and exploited already in the previous section and denoted by the same symbol. The construction given in Theorem 4.1 applied separately to each path component of $\text{Map}^\circ(\Sigma, BK)$, yields a $\tilde{K}$-equivariant principal $S^1$-bundle $\text{Map}^\circ(\Sigma, BK)$ with connection having curvature $\zeta_{(f, \zeta, \lambda)}$. The construction is natural in terms of the data.

This approach relies on the integrality of the fundamental degree four class, the Pontrjagin class, of the classifying space $BK$. Our procedure assigns a $\tilde{K}$-equivariant principal $S^1$-bundle over $\text{Map}^\circ(\Sigma, BK)$ to every closed surface $\Sigma$; this assignment is functorial in terms of geometric presentations of the kind (7.1) and can thus be viewed as an alternative to the equivariant gerbe representing the Pontrjagin class; see [4] for the latter.

8 Coadjoint orbits of the loop group

Let $M$ be a conjugacy class in $K$, let $N = K$, and let $f : M \rightarrow N$ be the inclusion. There is a 2-form $\zeta \in A^2(M)$ such that the pair $(\lambda, \zeta)$ has integral periods [5]. The fiber $P_f$ may be viewed as a coadjoint orbit of the loop group $L K = \text{Map}(S^1, K)$. All the requisite requirements are met: Theorem 3.1 furnishes a principal $S^1$-bundle with connection on this coadjoint orbit having curvature $\zeta_{(f, \zeta, \lambda)}$ and, when the additional ingredient $\vartheta \in A^{2,1}(K)$ introduced in Section 5 above is added, Theorem 4.1 applies equally and yields the structure of a $\tilde{K}$-equivariant principal $S^1$-bundle on $P_f$. 

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References

[1] M. F. Atiyah: The geometry and physics of knots. Cambridge University Press, Cambridge, U. K. (1990)

[2] M. F. Atiyah and R. Bott: The Yang-Mills equations over Riemann surfaces. *Phil. Trans. R. Soc. London A* 308 (1982), 523–615

[3] R. Bott: On the Chern-Weil homomorphism and the continuous cohomology of Lie groups. *Advances* 11 (1973), 289–303

[4] J. L. Brylinski and D. A. McLaughlin: Loop spaces, characteristic classes, and geometric quantization. Progress in Mathematics, 107, Birkhäuser-Verlag, Boston · Basel · Berlin, 1993

[5] K. T. Chen: Iterated path integrals. *Bull. Amer. Math. Soc.* 83 (1977), 831–879

[6] J. L. Dupont: Simplicial de Rham cohomology and characteristic classes of flat bundles. *Topology* 15 (1976), 233–245

[7] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein: Group systems, groupoids, and moduli spaces of parabolic bundles. *Duke Math. J.* 89 (1997), 377–412, [dg-ga/9510006](http://arxiv.org/abs/dg-ga/9510006)

[8] J. Huebschmann: Symplectic and Poisson structures of certain moduli spaces. *Duke Math. J.* 80 (1995), 737–756, [hep-th/9312112](http://arxiv.org/abs/hep-th/9312112)

[9] J. Huebschmann: Symplectic and Poisson structures of certain moduli spaces. II. Projective representations of cocompact planar discrete groups. *Duke Math. J.* 80 (1995), 757–770, [dg-ga/9412003](http://arxiv.org/abs/dg-ga/9412003)

[10] J. Huebschmann: Extended moduli spaces, the Kan construction, and lattice gauge theory. *Topology* 38 (1999), 555–596, [dg-ga/9505005](http://arxiv.org/abs/dg-ga/9505005), [dg-ga/9506006](http://arxiv.org/abs/dg-ga/9506006)

[11] J. Huebschmann: On the variation of the Poisson structures of certain moduli spaces. *Math. Ann.* 319 (2001), 267–310, [dg-ga/9710033](http://arxiv.org/abs/dg-ga/9710033)

[12] J. Huebschmann and L. Jeffrey: Group Cohomology Construction of Symplectic Forms on Certain Moduli Spaces. *Int. Math. Research Notices*, 6 (1994), 245–249

[13] L. Jeffrey: Symplectic forms on moduli spaces of flat connections on 2-manifolds. In Proceedings of the Georgia International Topology Conference, Athens, Ga. 1993, ed. by W. Kazez. *AMS/IP Studies in Advanced Mathematics* 2 (1997), 268–281
[14] Y. Karshon: An algebraic proof for the symplectic structure of moduli space. *Proc. Amer. Math. Soc.* **116** (1992), 591–605

[15] A. Losev, G. Moore, N. Nekrasov, S. Shatashvili Central extensions of gauge groups revisited. *Sel. math. New Series* **4** (1998), 117–123, hep-th/9511185

[16] J. Milnor: Construction of universal bundles. I. *Ann. of Math.* **63** (1956), 272–284

[17] M. S. Narasimhan and C. S. Seshadri: Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math.* **82** (1965), 540–567

[18] H. B. Shulman: On characteristic classes. Ph. D. Thesis, University of California, 1972

[19] J. M. Souriau: Groupes différentiels. In: Diff. geom. methods in math. Physics, Proc. of a conf., Aix en Provence and Salamanca, 1979, *Lecture Notes in Mathematics* **836**, 91–128, Springer-Verlag, Berlin · Heidelberg · New York · Tokyo, 1980

[20] E. Spanier: Algebraic Topology. *McGraw-Hill Series in Higher Mathematics*, McGraw-Hill Book Company, New York, 1966

[21] J. D. Stasheff: “Parallel” transport in fiber spaces, *Bol. Soc. Mat. Mexicana* (2) **11** (1966), 68–84

[22] F. W. Warner: Foundations of differentiable manifolds and Lie groups. Scott, Foresman and Company, Glenview, Illinois, London, 1971

[23] A. Weinstein: The symplectic structure on moduli space. In: The Andreas Floer Memorial Volume, H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds., *Progress in Mathematics* **133**, 627–635, Birkhäuser-Verlag, Boston · Basel · Berlin, 1995