The n-Hosoya Polynomials of the Square of a Path and of a Cycle

Ahmed M. Ali
ahmedgraph@uomousl.edu.iq

Department of Mathematics
College of Computer Science and Mathematics
University of Mosul, Mosul, Iraq

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ABSTRACT

The n-Hosoya polynomial of a connected graph G of order t is defined by:

\[ H_n(G; x) = \sum_{k=0}^{\delta} C_n(G,k) x^k \]

Where, \( C_n(G,k) \) is the number of pairs (v,S), in which

\[ |S| = n-1, \quad 3 \leq n \leq t, \quad v \in V(G), \quad S \subseteq V(G), \]

such that \( d_n(v,S) = k \), for each 0 \leq k \leq \delta = diam_n(G).

In this paper, we find the n-Hosoya polynomial of the square of a path and of the square of a cycle. Also, the n-diameter and n-Wiener index of each of the two graphs are determined.

**Keyword:** n-diameter, n-Hosoya polynomial, n-Wiener index, path square and cycle square.

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**Keyword:** n-diameter, n-Hosoya polynomial, n-Wiener index, path square and cycle square.
1. Introduction:

The n-distance [1] in a connected graph \( G = (V, E) \) of order \( t \) is the minimum distance from a singleton, \( v \in V \) to an \((n-1)\)-subset \( S \), \( S \subseteq V \), \( 3 \leq n \leq t \), that is,
\[
d_n (v, S) = \min \{ d(v, u) : u \in S \}, \quad 3 \leq n \leq t.
\]

It is clear that
\[
d_n (v, S) = 0 \quad \text{when} \quad v \in S,
\]
\[
d_n (v, S) \geq 1 \quad \text{when} \quad v \notin S.
\]

The n-Wiener index of a connected graph \( G = (V, E) \) is the sum of the minimum distances of all pairs \((v, S)\) in the graph \( G \), that is:
\[
W_n (G) = \sum_{(v, S), |S| = n-1} d_n (v, S), \quad 3 \leq n \leq t.
\]

The n-diameter of \( G \) is defined by:
\[
\text{diam}_n G = \max \{ d_n (v, S) : v \in V(G), |S| = n-1, S \subseteq V(G) \}.
\]

Now, let \( C_n (G, k) \) be the number of pairs \((v, S)\), \(|S| = n-1\), \( 3 \leq n \leq t \), \( v \in V \), \( S \subseteq V \), such that \( d_n (v, S) = k \), for each \( 0 \leq k \leq \delta_n = \text{diam}_n (G) \), then the n-Hosoya polynomial of \( G \) is defined by:
\[
H_n (G; x) = \sum_{k=0}^{\delta_n} C_n (G, k) x^k.
\]

We can obtain the n-Wiener index of \( G \) from the n-Hosoya polynomial of \( G \) as follows:
\[
W_n (G) = \left. \frac{d}{dx} H_n (G; x) \right|_{x=1} = \sum_{k=1}^{\delta_n} k C_n (G, k).
\]

For a vertex \( v \) of a connected graph \( G \), let \( C_n (v, G, k) \) be the number of \((n-1)\)-subsets \( S \) of vertices of \( G \) such that \( d_n (v, S) = k \), for \( n \geq 3 \), \( 0 \leq k \leq \delta_n \). The n-Hosoya polynomial of the vertex \( v \), denoted by \( H_n (v, G; x) \), is defined as:
\[
H_n (v, G; x) = \sum_{k=0}^{\delta_n} C_n (v, G, k) x^k.
\]

It is clear that for all \( k \geq 0 \),
\[
\sum_{v \in V(G)} C_n (v, G, k) = C_n (G, k),
\]
and
\[
\sum_{v \in V(G)} H_n (v, G; x) = H_n (G; x).
\]

For more information about these concepts, see the References [1, 2, 5, 6].

The next lemma will be used in proving our results.

Lemma 1.1:[1] Let \( v \) be any vertex of a connected graph \( G \). If there are \( r \) vertices of distance \( k \geq 1 \) from \( v \), and there are \( s \) vertices of distance more than \( k \) from \( v \), then, for \( n \geq 3 \),
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\[
C_n(v, G, k) = \left( \frac{r + s}{n - 1} \right) - \left( \frac{s}{n - 1} \right).
\] ...(1.1)

**Definition 1.2:** Let \( G \) be a connected non-trivial graph. The square \( G^2 \) of the graph \( G \), introduced by Harary and Ross [7], has \( V(G^2) = V(G) \) with \( u, v \) adjacent in \( G^2 \), whenever \( 1 \leq d_G(u, v) \leq 2 \).

Notice that the square of complete graph, star graph, wheel graph, complete bipartite graph are complete graphs.

In [1,2,3,4] , the \( n \)-Hosoya polynomials for many special graphs and many compound graphs are obtained. In this paper, we continue such works by obtaining the \( n \)-Hosoya polynomials of the square of paths and cycles.

2. The \( n \)-Hosoya Polynomial of the Square of a Path:

In this section, we obtained the \( n \)-Hosoya polynomial of the square \( P_t^2 \) of a path \( P_t \) of order \( t \). We shall consider two main cases of \( P_t^2 \) according to the parity of \( t \).

**First Case:** Even \( t \), \( t = 2r, r \geq 2 \).

Let \( P_t; u_1, u_2, u_3, \ldots, u_t \), then \( P_t^2 \) is shown in Fig.2.1, and by relabeling its vertices, we have Fig. 2.2 for \( P_{2r}^2 \).

![Fig. (2.1). The Path Square \( P_t^2 \)](image)

**Second Case:** If \( t \) is odd, then there exists an integer \( r \) such that \( t = 2r + 1 \). The graph \( P_t^2 \) is shown in Fig.2.3.

![Fig. (2.2). The Path Square \( P_{2r}^2 \)](image)

**Fig. (2.3). The Path Square \( P_{2r+1}^2 \).**
Theorem 2.1: For $t \geq 5$ and $n \geq 2$, let $r = \left\lceil \frac{t}{2} \right\rceil$, then,

$$\text{diam}_n(P^2_t) = \begin{cases} r + 1 - \left\lceil \frac{n}{2} \right\rceil, & \text{for even } t, \\ r + 1 - \left\lfloor \frac{n}{2} \right\rfloor, & \text{for odd } t. \end{cases}$$

Proof:

(1). Let $t$ be even, then $t = 2r$.

From Fig. 2.2, we notice that $\text{diam}(P^2_{2r}) = d(v_1, v_{r+1}) = r$, then $\text{diam}_n(P^2_{2r}) = d_n(v_1, S)$, $n \geq 2$, where $S$ consists of the first $n-1$ vertices from the sequence $\{v_r, v_{r+1}, v_{r+2}, ... ; v_1, v_{2r}\}$. Thus, if $n$ is even, then

$$S = \{v_{r+1}, v_r, v_{r+2}, v_r, v_{r+3}, v_{r+2}, ... ; v_{r+2-n/2}, v_{r+1-n/2} \}, \quad n = 4, 6, 8, ... , 2r.$$ 

So, $d_n(v_1, S) = r + 1 - \frac{n}{2}$.

If $n$ is odd, then

$$S = \{v_{r+1}, v_r, v_{r+2}, v_{r+1}, v_{r+3}, v_{r+2}, ... ; v_{r+2-n/2}, v_{r+1-n/2} \}, \quad n = 3, 5, 7, ... , 2r - 1.$$ 

So, $d_n(v_1, S) = r + 1 - \frac{n+1}{2}$.

Therefore, $\text{diam}_n(P^2_{2r}) = r + 1 - \left\lfloor \frac{n}{2} \right\rfloor$, for all $n \geq 2$.

(2). Let $t$ be odd, then $t = 2r + 1$.

From Fig. 2.3, we notice that $\text{diam}(P^2_{2r+1}) = d(v_1, v_{r+1}) = r$. (or $d(v_1, v_{r+2})$, or $d(v_{2r+1}, v_{r+1})$), then $\text{diam}_n(P^2_{2r+1}) = d_n(v_1, S)$, $|S| = n-1, n \geq 2$, where $S$ consists of the first $n-1$ vertices from the sequence $\{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+4}, ... ; v_1, v_{2r+1}\}$. Thus, if $n$ is even, then

$$S = \{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+1}, v_{r+4}, ... ; v_{r+2-n/2}, v_{r+1-n/2} \}, \quad n = 4, 6, 8, ... , 2r.$$ 

So, $d_n(v_1, S) = d(v_1, v_{r+2-n/2}) = r + 1 - \frac{n-1}{2}$.

If $n$ is odd, then

$$S = \{v_{r+1}, v_{r+2}, v_r, v_{r+3}, v_{r+4}, ... ; v_{r+2-n/2}, v_{r+1-n/2} \}, \quad n = 3, 5, 7, ... , 2r.$$ 

So, $d_n(v_1, S) = d(v_1, v_{r+2-n/2}) = r + 1 - \frac{n}{2}$.

Therefore,

$$\text{diam}_n(P^2_{2r+1}) = r + 1 - \left\lceil \frac{n}{2} \right\rceil$$

for all $n \geq 2$. #
**Remark:** Throughout this work, we assume that \( \binom{a}{b} = 0 \), if \( a < b \).

**Theorem 2.2:** For any \( n \geq 3 \), the \( n \)-Hosoya polynomial of \( P_i^2 \), \( i \geq 6 \), is given by:

\[
H_n(P_i^2:x) = \sum_{k=0}^{\delta_n} C_n(P_i^2,k)x^k,
\]

where, \( \delta_n = \text{diam}_n(P_i^2) \),

\[
C_n(P_i^2,0) = t \binom{t-1}{n-2},
\]

\[
C_n(P_i^2,1) = t \binom{t-1}{n-1} - 2 \left( \binom{t-3}{n-1} + \binom{t-4}{n-1} \right) - (t-4) \binom{t-5}{n-1},
\]

\[
C_n(P_i^2,k) = 2 \left[ \binom{t-2k}{n-1} + \binom{t-2k+1}{n-1} \right] + (t-4k+2) \binom{t-4k+3}{n-1}
- 2 \sum_{i=0}^{k-2} \binom{t-4k+i}{n-1} - (t-4k) \binom{t-4k-1}{n-1},
\]

\[
2 \leq k \leq \left\lfloor \frac{\delta_n}{2} \right\rfloor,
\]

\[
C_n(P_i^2,k) = 2 \left[ \binom{t-2k}{n-1} + \binom{t-2k+1}{n-1} \right] + \left\lfloor \frac{\delta_n}{2} \right\rfloor + 1 \leq k \leq \delta_n.
\]

**Proof:** It is clear that \( C_n(P_i^2,0) = t \binom{t-1}{n-2} \).

From Fig.2.2, we notice that in \( P_i^2 \), there are two vertices of degree 2, two vertices of degree 3, and \( t - 4 \) vertices of degree 4. Thus, using formula (1.4.5) in [1], we obtain (2.2.2).

For each vertex \( w \) and given \( k \), let \( S_1(w,k) = \{ v \in V: d(v,w) = k \} \), \( S_2(w,k) = \{ v \in V: d(v,w) > k \} \).

**First**, we shall prove (2.2.3) and (2.2.4) for even \( t \), assuming \( t = 2r \), \( r \geq 4 \). It is clear, from Fig. 2.2, that for \( n \geq 3 \),

\[
C_n(v_i,P_i^2,k) = C_n(v_{1r},P_i^2,k),
\]

for \( i = 1, 2, \ldots, r \). Therefore, for \( 2 \leq k \leq \delta_n \),

\[
C_n(P_{2r}^2,k) = 2 \sum_{i=1}^{r} C_n(v_i,P_{2r}^2,k).
\]

Now, let \( 2 \leq k \leq \left\lfloor \frac{\delta_n}{2} \right\rfloor \), in which \( \delta_n \) is determined by Theorem 2.1, that is

\[
\delta_n = r + 1 - \left\lfloor \frac{n}{2} \right\rfloor.
\]

Since, \( n \geq 3 \), then \( \delta_n \leq r - 1 \), for \( r \geq 4 \).
But, in proving (2.2.3), we assume that $\delta_n \geq 4$.

According to the given value of $k$, we partition $\{v_1, v_2, \ldots, v_r\}$ into the following **four cases**:

(1) For $i = 1, 2, \ldots, k$, we notice, from Fig. 2.2, that:

$S_1(v_i, k) = \{v_{i+k}, v_{2r+2-i-k}\}$,

$S_2(v_i, k) = V(P^2_i) - \{v_1, v_2, \ldots, v_{r+k}, v_{2r+2-i-k}, v_{2r+3-i-k}, \ldots, v_{2r}\}$.

Thus,

$|S_1(v_i, k)| = 2$, $|S_2(v_i, k)| = t + 1 - 2k - 2i$.

So, by Lemma 1.1, we have, for $i = 1, 2, \ldots, k$,

$$C_n(v_i, P^2_i, k) = \binom{t + 3 - 2k - 2i}{n-1} - \binom{t + 1 - 2k - 2i}{n-1}.$$  ... (c1)

(2) For $i = 1, 2, \ldots, k-1$, we obtain, from Fig. 2.2,

$S_1(v_{r+1-i}, k) = \{v_{r-k+i+1}, v_{r+k+1}\}$,

$S_2(v_{r+1-i}, k) = V(P^2_i) - \{v_{r-k+i+1}, v_{r-k+i+2}, \ldots, v_r, v_{r+i}, \ldots, v_{r+k+i}\}$.

Thus,

$|S_1(v_{r+1-i}, k)| = 2$, $|S_2(v_{r+1-i}, k)| = t - 2k - 2i$.

So, using Lemma 1.1, we obtain, for $i = 1, 2, \ldots, k-1$,

$$C_n(v_{r+1-i}, P^2_i, k) = \binom{t + 2 - 2k - 2i}{n-1} - \binom{t - 2k - 2i}{n-1}.$$  ... (c2)

(3) For $v_{r-k+1}$, we have

$S_1(v_{r-k+1}, k) = \{v_{r+1}, v_{2r+k}, v_{r+1-2k}\}$,

$S_2(v_{r-k+1}, k) = V(P^2_i) - \{v_{r-2k+1}, v_{r-2k+2}, \ldots, v_r, v_{r+i}, \ldots, v_{r+2k}\}$.

Thus,

$|S_1(v_{r-k+1}, k)| = 3$, $|S_2(v_{r-k+1}, k)| = t - 4k$.

So, using Lemma 1.1, we get,

$$C_n(v_{r-k+1}, P^2_i, k) = \binom{t + 3 - 4k}{n-1} - \binom{t - 4k}{n-1}.$$  ... (c3)

(4) For $i = k+1, k+2, \ldots, r - k$,

$S_1(v_i, k) = \{v_{i-k}, v_{i+k}, v_{2r-k+i+1}, v_{2r-k+i+2}\}$,

$S_2(v_i, k) = V(P^2_i) - \{v_{i-k}, v_{i-k+1}, \ldots, v_{i+k}, v_{2r-k-i+2}, v_{2r-k-i+3}, \ldots, v_{2r-k+i+1}\}$.

Thus,

$|S_1(v_i, k)| = 4$, $|S_2(v_i, k)| = t - 4k - 1$.

Therefore, using Lemma 1.1, we get, for $i = k+1, k+2, \ldots, r - k$,

$$C_n(v_i, P^2_i, k) = \binom{t - 4k + 3}{n-1} - \binom{t - 4k - 1}{n-1}.$$  ... (c4)

Thus, from (2.2.6) and summing up the formulas (c1)-(c4) we get for $2 \leq k \leq \left\lfloor \frac{\delta_n}{2} \right\rfloor$, 

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\[
C_n(P^2_i, k) = 2 \sum_{i=1}^{k} \left[ \binom{t+3-2k-2i}{n-1} - \binom{t+1-2k-2i}{n-1} \right] \\
+ \sum_{i=1}^{k-1} \left[ \binom{t+2-2k-2i}{n-1} - \binom{t-2k-2i}{n-1} \right] \\
+ \left( \binom{t-4k+3}{n-1} - \binom{t-4k}{n-1} \right) + (r-2k) \left[ \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1} \right].
\]

\[
= 2 \left[ \binom{t-2k+1}{n-1} - \binom{t-4k+1}{n-1} \right] + \left[ \binom{t-2k}{n-1} - \binom{t-4k+2}{n-1} \right] \\
+ \left( \binom{t-4k+3}{n-1} - \binom{t-4k}{n-1} \right) + (r-2k) \left[ \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1} \right].
\]

\[
= 2 \left[ \binom{t-2k}{n-1} + \binom{t-2k+1}{n-1} \right] + (t-4k+2) \left( \binom{t-4k+3}{n-1} \right) \\
- 2 \sum_{j=0}^{n} \left( \binom{t-4k+j}{n-1} - (t-4k) \binom{t-4k-1}{n-1} \right).
\]

Now, we give the proof of (2.2.4) for \( \delta_n \) with \( 1 \leq k \leq \delta_n \). Here, we have two cases:

(a). For \( i = 1, 2, \ldots, r-k \),
\[
S_1(v_i, k) = \{ v_{i+k}, v_{2r+2-i-k} \},
\]
\[
S_2(v_i, k) = V(P^2_i) - \{ v_i, v_2, \ldots, v_{i+k}, v_{2r+2+i-k}, v_{2r+3+i-k}, \ldots, v_{2r} \}.
\]
Thus,
\[
|S_1(v_i, k)| = 2, \quad |S_2(v_i, k)| = t+1-2k-2i.
\]
So, by Lemma 1.1, we have, for \( i = 1, 2, \ldots, r-k \),
\[
C_n(v_i, P^2_i, k) = \left( \binom{t-2k-2i+3}{n-1} - \binom{t-2k-2i+1}{n-1} \right).
\] ... (d1)

(b). For \( v_{r+i-1} \), \( i = 1, 2, \ldots, r-k \), we have
\[
S_1(v_{r+i-1}, k) = \{ v_{r-k+i-1}, v_{r+k+i} \},
\]
\[
S_2(v_{r+i-1}, k) = V(P^2_i) - \{ v_{r-k+i-1}, v_{r-k+i+2}, \ldots, v_r, v_{r+i-1}, \ldots, v_{r+k+i} \}.
\]
Thus,
\[
|S_1(v_{r+i-1}, k)| = 2, \quad |S_2(v_{r+i-1}, k)| = t-2k-2i.
\]
So, by Lemma 1.1, we have, for \( i = 1, 2, \ldots, r-k \),
\[
C_n(v_{r+i-1}, P^2_i, k) = \left( \binom{t-2k-2i+2}{n-1} - \binom{t-2k-2i}{n-1} \right).
\] ... (d2)

Therefore, using (2.2.6) and summing up (d1) and (d2), we get for \( \frac{\delta_n}{2} \),
\[
1 \leq k \leq \delta_n,
\]

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The graph \( G \) is the square of a cycle of order \( t \), shown in Fig. 3.1. We shall find the \( n \)-distance regular graphs, and for the given value of \( n \), \( 2 \leq n \leq t \), \( H_n(G; x) = t H_n(v, G; x) \), where \( v \) is any vertex of \( G \) and \( t \) is the order of \( G \). The graph \( C^2_t \) is the square of a cycle of order \( t \), shown in Fig. 3.1. We shall find the \( n \)-diameter, \( n \)-Hosoya polynomial, and \( n \)-Wiener index of \( C^2_t \).
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Fig. (3.1) The Cycle Square $C_t^2$, $t \geq 6$.

**Lemma 3.1:** $\text{diam}_n (C_t^2) = \delta_n = 1 + \left\lfloor \frac{t-n}{4} \right\rfloor$, $n \geq 2$, $t \geq 6$.

**Proof:** Let $m = \left\lfloor \frac{t}{4} \right\rfloor$, then $t = 4m + r$, $r = 0, 1, 2, 3$.

For $r = 2$, $C_t^2$ is redrawn in Fig. 3.2.

Since, $C_t^2$ is vertex $n$-distance regular graph, then $\text{diam}_n (C_t^2) = e_n (v_i)$.

To find the $n$-eccentricity of $v_i$, we partition $V(C_t^2) - \{v_i\}$ into $S_1, S_2, \ldots, S_{m+1}$, where

$S_1 = \{v_2, v_3, v_1, v_{i-1}\}$,

$S_2 = \{v_4, v_5, v_{i-2}, v_{i-3}\}$,

$S_3 = \{v_6, v_7, v_{i-4}, v_{i-5}\}$,

\[\vdots\]

$S_j = \{v_{2j}, v_{2j+1}, v_{i-2(j-1)} , v_{i-2j+1}\}$,

\[\vdots\]

$S_m = \{v_{2m}, v_{2m+1}, v_{i-2m+2}, v_{i-2m+1}\}$,

$S_{m+1} = V(C_t^2) - \bigcup_{j=1}^{m} S_j \cup \{v_i\}$.
Fig. (3.2). The Cycle Square $C_t^2$, $t = 4m + 2$, $m \geq 1$.

It is clear that each vertex of $S_j$, $1 \leq j \leq m$, is of (standard) distance $j$ from $v_1$; and each of the other vertices (if exists) of $C_t^2$ (here in Fig. 3.2, we have \{v_{t+1}\} = \{v_{2m+2}\} = S_{m+1}$) is of the distance $m + 1$ from $v_1$. Notice that if $t = 4m + 1$, then $S_{m+1}$ is empty, and if $t = 4m$ then, $S_{m+1}$ is empty and $S_m$ consists of three elements; if $t = 4m + 2$, $t = 4m + 3$, then $S_{m+1}$ consists of one, respectively two, elements.

Let $k$ be the greatest positive integer such that the set $\bigcup_{i=k}^{m+1} S_i$ consists of at least $(n-1)$ vertices. Therefore, since $|S_1| \leq 4$.

$4(k - 1) + 1 + (n - 1) \leq t$,

$4k \leq t - n + 4$,

$k \leq \frac{t - n}{4} + 1$.

Therefore, $\text{diam}_n(C_t^2) = k = 1 + \left\lfloor \frac{t - n}{4} \right\rfloor$, \(\because k \) is positive integer. #

**Theorem 3.2:** For any $n \geq 3$, the $n$-Hosoya polynomial of $C_t^2$, $t \geq 6$ is given by:

$$H_n(C_t^2; x) = \left( \frac{t - 1}{n - 2} \right) + \sum_{k=1}^{\delta_n - 1} x^k + C_n(C_t^2, \delta_n)x^{\delta_n},$$

Where, $C_n(C_t^2, \delta_n)$ is determined in Remark 3.3, and $\delta_n$ is determined by Lemma 3.1.

**Proof:** Let $S$ be a set of (n-1) vertices of $V(C_t^2)$ such that $v_1 \not\in S$, $v_i \in V(C_t^2)$ and $d_n(v_1, S) = k$, $2 \leq k \leq \delta_n - 1$. Hence, $S$ does not contain any vertex from \{v_{t-2k+3}, \ldots, v_{t-1}, v_1, v_2, v_3, \ldots, v_{2k-1}\}, (see Fig. 3.1), but $S$ must contain, at least,
one vertex of \( \{ v_{2k}, v_{2k+1}, v_{t-2k+2}, v_{t-2k+1} \} \). Then, the number of vertices in \( C_t^2 \) of distance more than \( k \) from \( v_i \) is \((t-4k-1)\) and there are four vertices in \( C_t^2 \) of distance \( k \) from \( v_i \). Hence, by Lemma 1.1,

\[
C_n(v_i, C_t^2, k) = \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1}, \quad \text{for } 2 \leq k \leq \delta_n - 1.
\]

Moreover, it is clear that

\[
C_n(v_i, C_t^2, 1) = \binom{t-1}{n-1} - \binom{t-5}{n-1}.
\]

Since \( C_n(v_i, C_t^2, k) = C_n(v_i, C_t^2, k), \ 2 \leq i \leq t, \) then

\[
C_n(C_t^2, k) = \left[ \binom{t-4k+3}{n-1} - \binom{t-4k-1}{n-1} \right], \quad \text{for } 1 \leq k \leq \delta_n - 1. \quad \#
\]

**Remark 3.3:** From Fig. 3-2, we can easily obtain \( C_n(C_t^2, \delta_n) \), for \( n \geq 3 \).

1. If \( t = 4m + 3 \), then,

\[
C_n(C_t^2, \delta_n) = \binom{t}{n-1} \quad ; \quad n = 3
\]

\[
C_n(C_t^2, \delta_n) = \left[ \binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1} \right] \quad ; \quad n \geq 4.
\]

2. If \( t = 4m + 2, \ 4m+1 \), then,

\[
C_n(C_t^2, \delta_n) = \left[ \binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1} \right] \quad ; \quad n \geq 3.
\]

3. If \( t = 4m \), then,

\[
C_n(C_t^2, \delta_n) = \begin{cases} \binom{3}{n-1} & ; \quad n = 3, 4 \\ \binom{t-4\delta_n+3}{n-1} - \binom{t-4\delta_n-1}{n-1} & ; \quad n \geq 5. \end{cases}
\]

**Corollary 3.4:** The \( n \)-Wiener index of \( C_t^2 \) is given by:

\[
W_n(C_t^2) = \sum_{k=1}^{\delta_n} k C_n(C_t^2, k), \quad \text{where } C_n(C_t^2, k), \ 1 \leq k \leq \delta_n \text{ is given in Theorem 3.2 and Remark 3.3}. \quad \#
\]
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