Fluctuations in the coarsening dynamics of the $O(N)$ model with $N \to \infty$: are they similar to those in glassy systems?

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Received 15 June 2005
Accepted 13 December 2005
Published 12 January 2006

Abstract. We study spatio-temporal fluctuations in the non-equilibrium dynamics of the $d$-dimensional $O(N)$ model in the large $N$ limit. We analyse the invariance of the dynamic equations for the global correlation and response in the slow ageing regime under transformations of time. We find that these equations are invariant under scale transformations. We extend this study to the action in the dynamic generating functional, finding similar results. This model therefore falls into a different category to glassy problems in which full time reparametrization invariance, a larger symmetry that encompasses timescale invariance, is expected to be realized asymptotically. Consequently, the spatio-temporal fluctuations of the large $N$ $O(N)$ model should follow a pattern different from that for glassy systems. We compute the fluctuations of local, as well as spatially separated, two-field composite operators and responses, and we confront our results with the ones found numerically for the 3D Edwards–Anderson model and kinetically constrained lattice gases. We analyse the dependence of the fluctuations of the composite operators on the growing domain length and we compare to what has been found for supercooled liquids and glasses. Finally, we show that the development of time reparametrization invariance in glassy systems

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is intimately related to a well-defined and finite effective temperature, specified from the modification of the fluctuation-dissipation theorem out of equilibrium. We then conjecture that the global asymptotic time reparametrization invariance is broken down to timescale invariance in all coarsening systems.

**Keywords:** coarsening processes (theory), kinetic growth processes (theory), dynamical heterogeneities (theory), slow dynamics and ageing (theory)

**ArXiv ePrint:** cond-mat/0506297
1. Introduction

Many extended systems which consist of interacting microscopic degrees of freedom exhibit non-trivial slow dynamics at low temperatures. Macroscopic observables such as density–density or other relevant correlations have extremely slow relaxations. Magnetic, dielectric or other susceptibilities slowly evolve in time. A large amount of experimental and numerical data allow for a qualitative, and sometimes also quantitative, description of these macroscopic observables in a number of well-studied materials. A satisfactory understanding of the mechanism leading to such dramatic slowing down is, however, still lacking. In order to get a better insight into the relaxation of glassy systems it is important to investigate the dynamics at scales of length/time that range from the microscopic to the macroscopic, through proper experimental [1]–[3], numerical [4]–[8] and theoretical tools [9]–[23].

Systems with a clear mechanism for slow relaxations are the ones that evolve through coarsening of domains. They may thus provide a useful guideline for understanding the dynamics of, in principle, more complicated systems. After a transient, systems undergoing phase-ordering kinetics enter a scaling regime in which the order parameter morphology and its correlation functions depend on time only through a time dependent length $L(t)$, that characterizes the mean size of the domains [24]. Interestingly, all microscopic details are absorbed in $L(t)$. It is tempting to speculate that such space–time scaling also exists asymptotically in glassy systems. This is the starting point, for example, in the dynamic droplet theory of spin glasses [9] (see [10] for a detailed numerical examination).

Independently, analytical studies of dynamical mean-field theories of glassy systems demonstrated that the relaxation of global two-time correlation functions follows a self-similar structure, with a long time scaling given by a ratio between evaluations of a
function of time at the two times involved, $C(t, t') \approx f_C[h(t')/h(t)]$ [25]. In these models, there is no interpretation of the function $h(t)$ as a length scale. Even more generally, one can argue that any monotonic two-time correlation, independently of the origin of the slow dynamics, should depend on times only through a ratio $h(t')/h(t)$ within a given correlation scale [26].

It has been noticed by several authors [27]–[32] that the dynamic equations for the slow decay of the global correlations and responses of mean-field disordered models with glassy features acquire time reparametrization invariance once the time derivatives (and other irrelevant terms) are dropped in the long times limit, in which the scaling in $h(t')/h(t)$ actually holds. This symmetry is not exactly realized since one function $h(t)$ is selected by the dynamic evolution; in other words, the time derivative and other irrelevant terms act as (asymptotically vanishing) pinning fields that select the timescaling $h(t)$. The development, at long times, of an approximate invariance under generic reparametrization of time has hindered the complete solution of the dynamic problem, for fixing the choice of reparametrization involves a proper matching of the short time and long time dynamics that should be done by taking into account the effect of the time derivative—and other terms.

More recently it has been suggested that a global time reparametrization invariance may also exist in finite dimensional glassy systems and that it may be responsible for the main spatio-temporal fluctuations [20]–[23]. In this way, the inconvenience generated by the time reparametrization invariance was transformed into a tool with predictive power. Some consequences of this proposal were listed in these articles together with their numerical checks in finite dimensional spin glasses [21, 22] and kinetically facilitated models [23]. Interestingly enough, a kind of ‘universality’ emerged in the sense that the time evolution and form of the distributions of local correlations and responses followed a similar pattern for these rather different systems.

The global time reparametrization $t \to h(t)$ we are referring to acts on all spatial positions in identical way and it does not involve transforming space simultaneously. It is then simpler in form than the usual space–time rescaling that holds in coarsening systems at long times and large scales. In several stages of this paper we compare the time reparametrization invariance to the usual space–time rescaling. We also stress that time reparametrization invariance is a different transformation from Henkel’s local scale invariance hypothesis [33] (see [34]–[36] for a discussion on the validity of the latter).

The aim of this paper is to investigate the similarities and differences between fluctuations in simple coarsening and glassy systems. Specifically, we study analytically the coarsening dynamics of the $d$-dimensional O($N$) model in the large $N$ limit. This model has been studied in a large number of papers; see e.g. [35]–[42] and references therein. In section 2 we review its static and dynamic behaviour. We explain in special detail the separation of the field into two components, as presented by Corberi et al [41], and how this helps with understanding the condensation phenomenon and thermal fluctuations. Next, we analyse the fluctuating dynamics. In section 3 we derive the dynamic generating functional and write it in terms of the slow and fast fields introduced in section 2. We also derive closed dynamic equations for the global correlation and linear response of O($N$) models with translational invariant interactions in the large $N$ limit or the spherical model with similar quadratic energy. Then, in section 4 we examine the symmetries of the dynamic equations for the global correlation and response, and the
dynamical generating functional, under global transformations of time. We compare with the time reparametrization invariance suggested for glassy systems. Section 5 is devoted to the study of the probability distributions of the fluctuations at various mesoscopic scales of length/time through several dynamical observables. We confront the latter with the results obtained for disordered spin [20]–[22] and kinetically constrained models [23] and with the usual space–time scaling invariance of pure ferromagnetic coarsening. In section 6 we compute a four-point correlation function similar to the one that is usually used in the context of supercooled liquids [4]–[6], [11]–[16], [19] to extract a dynamic growing length. We study its behaviour as a function of the two times involved and discuss its relation to a response. Finally, in section 7 we present our conclusions together with some speculations.

2. The O(N) model

The d-dimensional O(N) non-linear sigma model is a coarse-grained approximation to a lattice spin model with nearest-neighbour ferromagnetic interactions. Its Hamiltonian reads

\[ H = \int_V d^d x \left[ \frac{1}{2} (\nabla \tilde{\phi}(\vec{x}))^2 + \frac{g}{4N} (\phi^2(\vec{x}))^2 + \frac{r}{2} \phi^2(\vec{x}) - \vec{h}(\vec{x}, t) \tilde{\phi}(\vec{x}) \right]. \]

The spatial dependence is given by the continuous d-dimensional vector \( \vec{x} = (x_1, \ldots, x_d) \) and \( V \) is the volume of the system. The field \( \tilde{\phi} \) is an N-dimensional vector, \( \tilde{\phi} = (\phi_1, \ldots, \phi_N) \) with \(-\infty < \phi_\alpha < \infty\). A subindex \( \alpha \) labels its N components, \( \alpha = 1, \ldots, N \).

The interplay between the quadratic and quartic terms (with couplings \( r \) and \( g > 0 \), respectively) favours the \( \phi^2(\vec{x}, t) \equiv \sum_{\alpha=1}^{N} \phi^2_{\alpha}(\vec{x}, t) = -Nr/g \) configurations for \( r < 0 \). \( h_\alpha \) is a magnetic field coupled linearly to the field. In the infinitesimal limit \( \vec{h} \) serves to compute the linear response; see equation (13). Soft Ising, XY and Heisenberg models correspond to \( N = 1, 2 \) and 3, respectively. In principle, the large \( N \) limit is the starting point for a systematic 1/N expansion, although this may be difficult to control [38].

In the absence of the magnetic field \( \vec{h} \), the Hamiltonian \( H \) is invariant under uniform rotations of \( \tilde{\phi} \):

\[ \phi_\alpha(\vec{x}) \rightarrow \tilde{\phi}_\alpha(\vec{x}) = R_{\alpha\beta} \phi_\beta(\vec{x}), \quad \forall \vec{x}, \]

with \( R \) and orthogonal matrix. The summation convention over repeated indices is used here and in what follows.

Dynamics is attributed to the field via the Langevin equations of motion:

\[ \gamma \dot{\phi}_\alpha(\vec{x}, t) = \nabla^2 \phi_\alpha(\vec{x}, t) - \left( \frac{g}{\sqrt{N}} \phi^2(\vec{x}, t) + r \right) \phi_\alpha(\vec{x}, t) + h_\alpha(\vec{x}, t) + \eta_\alpha(\vec{x}, t). \]

Henceforth we measure time in units of the inverse of the friction coefficient \( \gamma \). \( \eta_\alpha(\vec{x}, t) \) is a spatially uncorrelated Gaussian white noise with zero mean, \( \langle \eta_\alpha(\vec{x}, t) \rangle = 0 \) for all \( \vec{x} \) and \( t \), and variance

\[ \langle \eta_\alpha(\vec{x}, t) \eta_\beta(\vec{x}', t') \rangle = 2k_B T \delta_{\alpha\beta} \delta^d(\vec{x} - \vec{x}') \delta(t - t'), \]

where \( T \) is the temperature of the bath and \( k_B \) is the Boltzmann constant. It is convenient to regularize the spatial correlations of the noise including a finite short distance cut-off

\[ \langle \eta_\alpha(\vec{x}, t) \eta_\beta(\vec{x}', t') \rangle = 2k_B T \delta_{\alpha\beta} \frac{e^{-1/4(\vec{x}-\vec{x'})^2\Lambda^2}}{(4\pi\Lambda^2)^{d/2}} \delta(t - t'). \]

doi:10.1088/1742-5468/2006/01/P01006
This introduces correlations over a typical length $1/\Lambda$ simulating the lattice spacing and cures some short distance divergences. $1/(2\Lambda^2)$ will define a microscopic timescale $t_0$ that regularizes divergent equal-time correlations. Hereafter the angular brackets indicate an average over the thermal noise and we set $k_B = 1$.

The stochastic evolution has to be supplemented with the initial condition $\vec{x}(\vec{x},0)$. Since we are interested in phase-ordering dynamics, we typically choose initial conditions that are uncorrelated in the $N$-dimensional space, $[\phi_\alpha(\vec{x},0)\phi_\beta(\vec{x},0)]_{ic} \propto \delta_{\alpha \beta}$, and in real space, and have a Gaussian distribution

$$P[\vec{x}(\vec{x},0)] = (2\pi \Delta^2)^{-NV/2} \exp \left(-\frac{1}{2\Delta^2} \sum_\alpha \int d^d x \, \phi_\alpha^2(\vec{x},0) \right). \tag{1}$$

Hereafter we use square brackets, $[\ldots]_{ic}$, to represent an average over initial conditions.

In the large $N$ limit one expects that the sum over components in $\phi^2(\vec{x},t)$ averages away the $\vec{x}$ dependence. One then looks for a solution such that

$$z(\vec{x},t) \equiv \frac{\theta}{N} \phi^2(\vec{x},t) + r \approx z(t) \equiv \frac{\theta}{N}[\phi^2(\vec{x},t)]_{ic} + r, \tag{2}$$

where the average in the last term is taken over thermal histories and initial conditions. The functional form of $z(t)$ has to be determined self-consistently. As we shall see below the time dependence of $z(t)$ determines the scaling in time of most of the interesting dynamic quantities. Note that we are implicitly assuming that $N \to \infty$ in that we are not letting $z$ fluctuate. All results in this paper have been derived in this limit. As discussed by Newman and Bray [38], fluctuations of $z(t)$ appear at order $1/N$.

Under the assumption (2), that has to be verified a posteriori, one can Fourier transform the Langevin equation and the noise–noise correlation. We use the following conventions:

$$f(\vec{k}) = \int d^d x \, e^{-i\vec{k}\vec{x}} f(\vec{x}), \quad f(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\vec{x}} f(\vec{k}),$$

and we obtain

$$\dot{\phi}_\alpha(\vec{k},t) = -k^2 \phi_\alpha(\vec{k},t) - z(t)\phi_\alpha(\vec{k},t) + \eta_\alpha(\vec{k},t), \quad \langle \eta_\alpha(\vec{k},t)\eta_\beta(\vec{k}',t') \rangle = 2T \delta_{\alpha \beta} \exp(-k^2/\Lambda^2) \delta^d(\vec{k} + \vec{k}') \delta(t-t'). \tag{3}$$

In terms of the Fourier components $\vec{\phi}(\vec{k},0)$, the initial conditions are distributed according to

$$P[\vec{\phi}(\vec{k},0)] = (2\pi \Delta^2)^{-NV/2} \exp \left(-\frac{1}{2\Delta^2} \int \frac{d^d k}{(2\pi)^d} \vec{\phi}(\vec{k},0)\vec{\phi}(\vec{-k},0) \right). \tag{4}$$

Thus, the coupled dynamics in $x$ space transforms into a set of $N$ independent first-order differential equations for the $k$ components of the field. The label $\alpha$ is now superfluous and we omit it unless otherwise stated.

The $O(N \to \infty)$ model is intimately related to the spherical ferromagnet on a lattice and the fully connected spherical spin glass with two-body interactions. Actually, the full family of models with rotational symmetry on the $N$-dimensional space and differing only in the form of the space translational invariant quadratic interaction can be treated in the same way by introducing the density of states of the quadratic interaction matrix and

\[\text{doi:10.1088/1742-5468/2006/01/P01006}\]
examining how it decays to zero at its edge. In the case of the $O(N)$ model with $N \to \infty$ the density of states is given by

$$g(\epsilon) \sim \epsilon^\nu, \quad \nu = d/2 - 1,$$

for $\epsilon \gtrsim 0$ with $\epsilon \equiv k^2$. Many papers have been devoted to the study of the relaxation dynamics and global properties of the $O(N)$ model [35]–[42], the spherical ferromagnet [43]–[45] and the fully connected spin glass with two-body interactions [40], [45]–[52]. In the rest of this section we recall the main features of the statics and dynamics of the $O(N)$ model with $N \to \infty$ while in the rest of the paper we focus on the study of fluctuations and of symmetries under time transformations.

2.1. Statics

If the volume $V$ is finite, the system equilibrates in finite time and the probability distribution function (PDF) of the order parameter approaches a Gibbs–Boltzmann form in which the Fourier components are independent Gaussian random variables.

For $2 < d$ there is a finite critical temperature $T_c$ defined by

$$r + gT_c \int \frac{d^dk}{(2\pi)^d} \frac{e^{-k^2/\Lambda^2}}{k^2} = 0,$$

where the correlation length changes from a volume independent value at $T > T_c$ to a volume dependent one at $T \leq T_c$. For $d = 2$ the integral over $k$ has a logarithmic divergence and the critical temperature is pushed down to zero. Below $T_c$, the order parameter $m_{eq}$,

$$V^2 m_{eq}^2 \equiv N^{-1} \langle \phi^2(\vec{k} = 0) \rangle_{eq},$$

becomes non-zero and one finds

$$m_{eq}^2 = -r \frac{T_c - T}{g T_c},$$

with the correlation length $\xi^2 \sim m_{eq}^2 V/T$. Below $T_c$ the equilibrium susceptibility $\chi_{eq}$ to a uniform field $h_{eq}(\vec{x}) = h$ per unit volume is

$$\chi_{eq} \equiv \left. \frac{1}{V} \frac{\delta \langle \phi_{eq}(\vec{k} = 0) \rangle}{\delta h} \right|_{h=0} = \frac{m_0^2 - m_{eq}^2}{T} = -\frac{r}{g T_c} T_c^{-1},$$

$$= (4\pi)^{-d/2} \frac{2}{d-2} \Lambda^{d-2}$$

where

$$m_0^2 = -r/g.$$

In conclusion, the $O(N \to \infty)$ model has a phase transition from a paramagnetic to a ferromagnetic phase. The low temperature phase is characterized by a condensation phenomenon with $\langle \phi^2(\vec{k} = 0) \rangle_{eq} \propto V^2$ signalling ordering. The upper critical dimension is $d = 4$ and the lower critical dimension is $d = 2$. See [35, 41] for more details on the statics of this model.
2.1.1. Separation of the field. The nature of the phase transition and low temperature phase can be well understood by splitting the real space order parameter into a constant contribution and a position dependent one [41]:

\[ \tilde{\phi}(\vec{x}) = \tilde{\sigma} + \tilde{\psi}(\vec{x}) \]

with

\[ \tilde{\sigma} \equiv V^{-1} \tilde{\phi}(\vec{k} = 0) \quad \text{and} \quad \tilde{\psi}(\vec{x}) \equiv V^{-1} \sum_{\vec{k} \neq 0} \tilde{\phi}(\vec{k}) e^{i\vec{k} \cdot \vec{x}}. \]

The Gibbs–Boltzmann measure for the independent fields \( \tilde{\sigma} \) and \( \tilde{\psi} \) factorizes, \( P(\tilde{\phi}(\vec{x})) = P(\tilde{\sigma}) P(\tilde{\psi}(\vec{x})) \) with

\[ P(\tilde{\sigma}) = (2\pi m_{\text{eq}}^2)^{-N/2} e^{-\sigma^2/(2m_{\text{eq}}^2)}, \quad P(\tilde{\psi}(\vec{x})) \propto e^{-\beta/2} \int_v d^d x \left| \nabla \tilde{\psi}(\vec{x}) \right|^2. \]

The first factor describes the condensate with macroscopic variance \( \langle \sigma^2 \rangle_{\text{eq}} = m_{\text{eq}}^2 \) and the second one describes thermal fluctuations about it. Consequently, the static correlation function reads

\[ C_{\text{eq}}(\vec{r}) \equiv N^{-1} \langle \tilde{\phi}(\vec{x}) \cdot \tilde{\phi}(\vec{x} + \vec{r}) \rangle_{\text{eq}} = m_{\text{eq}}^2 + N^{-1} \langle \tilde{\psi}(\vec{x}) \cdot \tilde{\psi}(\vec{x} + \vec{r}) \rangle_{\text{eq}}. \quad (11) \]

2.2. Dynamics

The set of linear differential equation (3) can be easily solved:

\[
\phi(\vec{k}, t) = \exp \left( -k^2 t - \int_0^t dt' z(t') \right) \phi(\vec{k}, 0) 
+ \int_0^t dt' \exp \left( -k^2 (t - t') - \int_{t'}^t dt'' z(t'') \right) \left[ \eta(\vec{k}, t') + h(\vec{k}, t') \right], \quad (12)
\]

where we dropped the component index \( \alpha \), since all components satisfy the same equations due to rotational symmetry.

The function \( z(t) \) is self-consistently determined. Indeed,

\[ Y^2(t) \equiv \exp \left( 2 \int_0^t dt' z(t') \right) \]

satisfies a first-order differential equation complemented by the initial condition \( Y(0) = 1 \). One finds that \( Y^2 \) grows exponentially at high temperatures, as a power law at criticality, and it decays as a power law, \( Y^2(t) \sim t^{-d/2} \), below the critical temperature (see [35] for a careful study of the preasymptotic behaviour of \( Y(t) \) at low temperature).

The solution (12) takes a specially appealing form when written in terms of

\[ \frac{\delta \phi_\alpha(\vec{k}, t)}{\delta h_\beta(\vec{k'}, t')} \bigg|_{h=0} = \delta_{\alpha\beta} \delta^d(\vec{k} + \vec{k'}) \frac{Y(t')}{Y(t)} e^{-k^2(t-t')} \theta(t-t'), \quad (13) \]

which can be Fourier transformed to give

\[ \frac{\delta \phi_\alpha(\vec{x}, t)}{\delta h_\beta(\vec{x'}, t')} \bigg|_{h=0} = \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} e^{i\vec{k'} \cdot \vec{x}'} (2\pi)^d \frac{\delta \phi_\alpha(\vec{k}, t)}{\delta h_\beta(\vec{k'}, t')} \bigg|_{h=0}. \]
Note that these quantities depend on the noise realization and the initial condition only through the value of $Y(t)$. They are also identical to the linear response function,

$$R_{\alpha\beta}(\vec{x}, \vec{x}'; t, t') \equiv \left\langle \frac{\delta \phi_\alpha(\vec{x}, t)}{\delta h_\beta(\vec{x}', t')} \right|_{h=0} \right\rangle.$$

This property is peculiar to (quasi-)quadratic models. Calling now

$$r(k; t, t') \equiv \frac{Y(t')}{Y(t)} e^{-k^2(t-t')}$$

the solution (12) can be rewritten as

$$\phi(\vec{k}, t) = r(k; t, 0) \phi(\vec{k}, 0) + \int_0^t dt' r(k; t, t') \left[ \eta(\vec{k}, t') + h(\vec{k}, t') \right].$$

From this expression one easily sees that the field remains Gaussian distributed if the initial condition is also Gaussian.

### 2.3. Correlations and responses

Let us discuss the relaxation of the correlations

$$C_{\alpha\beta}(\vec{x}, \vec{x}'; t, t') \equiv \left\langle \phi_\alpha(\vec{x}, t) \phi_\beta(\vec{x}', t') \right\rangle_{ic},$$

and the linear response defined in equation (14). Due to the decorrelation of the initial conditions and noise in the $N$-dimensional space, these quantities are proportional to $\delta_{\alpha\beta}$ and we henceforth omit the internal indices assuming that we take $\alpha = \beta$ in all quantities studied.

#### 2.3.1. Asymptotic behaviour

For $2 < d$ one finds a dynamic phase transition at the static critical temperature, $T_c$ given in equation (6), where the asymptotic behaviour of $Y(t)$ changes. At high temperature each mode and, hence, the spatially averaged correlation decays exponentially. The linear response is related to the correlation by the fluctuation-dissipation theorem. At the transition one finds interrupted ageing. Since we shall not discuss the critical dynamics, we do not give a detailed description of the scaling laws at criticality. Below the transition, after a transient (i.e., $t' \gg \tau \gg 1$),

$$R(\vec{k}, \vec{k}'; t, t') \sim \delta^d(\vec{k} + \vec{k}') \left( \frac{t}{t'} \right)^{d/4} e^{-k^2(t-t')} e^{-k^2 t_0} \theta(t - t'),$$

$$C(\vec{k}, \vec{k}'; t, t') \sim \delta^d(\vec{k} + \vec{k}') (tt')^{d/4} e^{-k^2(t-t')} e^{-(k^2+k'^2)t_0} \times \left[ \Delta^2 + 2T \int_0^{\text{min}(t,t')} dt'' Y^2(t'') e^{(k^2+k'^2)t''} \right].$$

Note that the asymptotic linear response does not depend on temperature. The first term in the correlation represents the decay of the initial conditions while the second one has its origin in the thermal noise. Each Fourier component with $k > 0$ decays exponentially in time (with power law corrections). The $k = 0$ component behaves differently since the exponential factor disappears. The slow decay of the low wavevector components generates the non-trivial dynamics of the global correlation and response.
From the expressions above one easily recovers the real space behaviour of the response and correlation. Using the large wavevector cut-off Λ introduced as $e^{-(k^2 + k'^2)/2\Lambda^2}$ in the $k$ integrals the local response and correlation on the same spatial point, $\vec{x} = \vec{x}'$, are independent of $\vec{x}$; thus they are also equal to the global values, $R(\vec{x}, \vec{x}; t, t') = R(t, t')$ and $C(\vec{x}, \vec{x}; t, t') = C(t, t')$ with
\[
R(t, t') \sim \left( \frac{t}{t'} \right)^{d/4} (t - t' + t_0)^{-d/2},
\]
\[
C(t, t') \sim \left( \frac{t't}{(t + t')^2} \right)^{d/4} \Delta^2 + 2T \int_0^{\min(t, t') \leq \tau} d\tau'' \frac{Y^2(\tau'')}{[1 - (2\tau''/(t + t'))]^{d/2}},
\]
where we assumed that $t + t'$ is larger than $t_0 = 1/(2\Lambda^2)$ and we neglected the dependence of the correlation on this timescale.

The case of $d = 2$, the lower critical dimension, may seem slightly different [42]. Interesting dynamics occurs only at zero temperature, i.e. at the critical point. However, the dynamics is not typically critical but it corresponds to the zero-temperature limit of the coarsening phenomena observed in higher dimensions. More precisely, there is still an additive separation of timescales, as opposed to what occurs in critical relaxations where the separation is multiplicative and the ageing contribution to the correlation progressively disappears as time elapses.

One can check that these results are valid for all initial conditions with short range correlations. Initial configurations with long range correlations (as for an ordered configuration) lead to different scaling forms [38, 46].

2.3.2. Separation of timescales. At low temperature, $T < T_c$, and for very long waiting time, $t' \gg \tau$, the global linear response and correlation have two distinct two-time regimes depending on the relation between the times $t$ and $t'$. These are defined by
\[
t - t' \ll t' \quad \text{stationary regime,}
\]
\[
\lambda \equiv \frac{t'}{t} \in [0, 1] \quad \text{ageing regime.}
\]
In the limit $t' \gg \tau$ the global (and local) correlation and linear response are well described by an additive separation
\[
C(t, t') = C_{st}(t - t') + C_{ag}(t, t'),
\]
\[
C_{st}(t - t') \equiv (m_0^2 - m_{eq}^2) \left[(t - t')/t_0 + 1\right]^{1-d/2},
\]
\[
C_{ag}(t, t') \equiv m_{eq}^2 \left[\frac{4\lambda}{(1 + \lambda)^2}\right]^{d/4},
\]
and
\[
R(t, t') = R_{st}(t - t') + R_{ag}(t, t'),
\]
\[
R_{st}(t - t') \sim (t - t' + t_0)^{-d/2} = T^{-1}\partial_t C_{st}(t - t'),
\]
\[
R_{ag}(t, t') \sim t^{-d/2} \left[\left(\frac{t'}{t}\right)^{-d/4} - 1\right] \left[1 - \frac{t_0}{t} + \frac{t_0}{t'}\right]^{-d/2}.
\]
\( m_{eq} \) is the equilibrium magnetization given in (8) and \( m_0 \) is its value at \( T = 0 \) given in (10). The relation (9) between \( m_{eq}^2 - m_0^2 \) and \( t_0 \) ensures the validity of the fluctuation-dissipation theorem in the stationary regime. For long time differences, such that the ratio between the two times \( t \) and \( t' \) is held fixed, \( t'/t = \lambda \in [0, 1) \) the correlation and response ‘age’, i.e. they depend on the waiting time \( t' \).

The detailed scaling of the correlation of the field evaluated at different times and spatial points for the (simpler) Gaussian scalar model was presented in [53] and for the O(N) model with \( N \to \infty \) in [24]. In particular, the ageing contribution to the space averaged global correlation (20) scales as

\[
C_{ag}(t, t') = f_C \left( \frac{L(t')}{L(t)} \right) \quad \text{with} \quad f_C(x) = m_{eq}^2 \left( \frac{2x}{1 + x^2} \right)^{d/2},
\]

(24)

\( f_C(x) \sim x^{d/2} \) when \( x \sim 0 \) and \( f_C(x) \sim m_{eq}^2 (1 - d^2/4) \) when \( x \sim 1 - \epsilon \) and \( \epsilon \ll 1 \). \( L(t) \) is the ‘domain length’ at time \( t \), which in the relaxation \( O(N \to \infty) \) model with non-conserved order parameter is given by \( L(t) \sim \sqrt{t} \). Note that the regime of very separated times, \( t \gg t' \) or \( x \sim 0 \), is characterized by a power law decay with an exponent \( \lambda = d/2 \) [39, 57].

The global linear response (23) also takes a scaling form [35]:

\[
R_{ag}(t, t') \sim t^{-d/2}f_R(x, y), \quad \text{with} \quad f_R(x, y) \sim (x^{-d/4} - 1)(1 - x + y)^{-d/2}, \quad \text{and}
\]

\[
\lambda = t'/t = L^2(t')/L^2(t), \quad y = t_0/t.
\]

In the coarsening regime the global correlation and response are not related by the fluctuation-dissipation theorem. One defines the ratio [54]

\[
X(t, t') = \frac{T R_{ag}(t, t')}{\partial t C_{ag}(t, t')} \sim t^{-d/2}f_X(\lambda).
\]

(26)

Note that this is a decreasing function of time that tends to zero for all \( d > 2 \) and to a function of the times ratio, \( f_X(\lambda) \), taking finite values when \( d \to 2^+ \).

The dc susceptibility or zero-field cooled magnetization, defined as the integral of the linear response over a time period:

\[
\chi(t, t') = \int_{t'}^t dt'' R(t, t'')
\]

(27)

can be expressed as a sum of two terms, a stationary and an ageing contribution,

\[
\chi(t, t') \approx \chi_{st}(t, t') + \chi_{ag}(t, t') = \int_{t'}^t dt'' R_{st}(t - t'') + \int_{t'}^t dt'' R_{ag}(t, t'')
\]

(28)
given by [35]

\[
\chi_{st}(t - t') = \chi_{eq} \left\{ 1 - \left[ (t - t')/t_0 + 1 \right]^{1-d/2} \right\},
\]

(29)

\[
\chi_{ag}(t, t') \sim \begin{cases} 
   t^{1-d/2} & d < 4, \\
   t^{1-d/2} \ln(t/t_0) & d = 4, \\
   t^{-1}t_0^{1-d/2} & d > 4,
\end{cases}
\]

(30)
where $\chi_{\text{eq}} = (4\pi)^{-d/2} t_0^{-d/2}/(d/2 - 1)$ is the equilibrium susceptibility given in equation (9). There are several features to be noted in these expressions. The first one is that the stationary integrated response approaches a value proportional to $t_0^{1-d/2}$ in the long $t-t'$ limit. If one takes the cut-off $\Lambda$ to infinity this value diverges as a power law, $t_0^{1-d/2}$ for all $d > 2$. For $d = 2$, on the other hand, $\chi_{\text{st}}$ diverges as a logarithm of the time difference, $\chi_{\text{st}}(t,t') \sim \ln[(t-t')/t_0]$, for $t-t' \gg t_0$. The approach to this asymptotic value is given by a power law, $[(t-t')/t_0]^{1-d/2}/(1-d/2)$, that will play an important role in the analysis of the invariances of the slow dynamics. Above $d = 4$ the decay of the ageing part of the integrated linear response does not depend on dimensionality any longer. As discussed by Corberi et al a similar upper dimension $d_\chi$ is expected to exist in other coarsening systems [35]. For all $d > 2$ the ageing contribution to the total susceptibility vanishes at long times, i.e. when $t \to \infty$.

$d = 2$ is the lower critical dimension. But the dynamic behaviour at zero temperature can be reached as the zero-temperature limit of the finite temperature coarsening dynamics just described [42]. The additive separation of the correlation and response also holds in this case. In particular, the Edwards–Anderson parameter, $q_{\text{ea}} = m_{\text{eq}}^2$, that separates the stationary from the ageing regime in the correlation, remains finite (and equal to $m_0^2$ at zero temperature) in the limit of long waiting time, $t' \to \infty$. Note that the stationary response in (22) does not depend on temperature and thus these soft ‘spins’ respond even at zero temperature. In particular, one has

$$R_{\text{st}}(t-t') = -\lim_{t' \to 0} T^{-1} \frac{d}{dt} C_{\text{st}}(t-t'), \quad \text{for } t-t' > 0.$$  \hfill (31)

2.3.3. Separation of the field. Interestingly enough, Corberi et al showed that for $d > 2$ the above results can also be found by using a splitting of the space and time dependent field $\vec{\phi}(\vec{x},t)$ into two components [41]:

$$\vec{\phi}(\vec{x},t) = \vec{\sigma}(\vec{x},t) + \vec{\psi}(\vec{x},t).$$

Indeed, the solution (15) can be rewritten as

$$\vec{\phi}(\vec{k},t) = \vec{\sigma}(\vec{k},t) + \vec{\psi}(\vec{k},t),$$

$$\vec{\sigma}(\vec{k},t) \equiv r(k;t,1) \vec{\phi}(\vec{k},1) + \int_{t_1}^{t} dt' r(k;t,t') \vec{h}(\vec{k},t'),$$

$$\vec{\psi}(\vec{k},t) \equiv \int_{t_1}^{t} dt' r(k;t,t') \eta(\vec{k},t').$$

t_1 is an arbitrary time satisfying $t_1 \ll t$ and sufficiently long so that the scaling limit has been established between the initial quench and $t_1$, i.e. $t_1 \gg \tau$. The second term in $\vec{\sigma}$ represents the effect of an external field applied from $t_w$ (another long time $t_w \gg t_1$) on. From now on we call $\sigma = \sigma_0$.

The field $\vec{\sigma}$ is associated with local condensation of the order parameter while $\vec{\psi}$ describes thermal fluctuations within the domains. These fields are statistically independent ($\langle \vec{\sigma}(\vec{x},t) \vec{\psi}(\vec{x}',t') \rangle = 0$) and have zero average. The explicit calculations in [41] demonstrate that, in the long times limit, $t \geq t' \gg t_1$ with $t_1$ itself diverging, the global correlation of $\vec{\sigma}$, $NC_{\sigma}(t,t',t_1) = \langle \vec{\sigma}(t) \vec{\sigma}(t') \rangle$, yields the ageing component of the global correlation of the field $\vec{\phi}$, while the global correlation of $\vec{\psi}$, $NC_{\psi}(t,t') = \langle \vec{\psi}(t) \vec{\psi}(t') \rangle$, yields

\[\text{doi:10.1088/1742-5468/2006/01/P01006}\]
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the stationary components of the global correlation of the field $\tilde{\sigma}$. More precisely, for $t, t' \gg t_1$ one finds

$$C_\psi(t, t'; t_1) = (m_0^2 - m_{eq}^2) \left[ \frac{4t_1}{t_0} \right]^{1-d/2} \left( \frac{(4tt')^{d/4}}{(t+t')^{d/2}} + \left( \frac{2(t-t')}{t_0} + 1 \right)^{1-d/2} \right]$$

(32)

$$C_\sigma(t, t'; t_1) = \left[ m_{eq}^2 - (m_0^2 - m_{eq}^2)(4t_1/t_0)^{1-d/2} \right] \left( \frac{4tt'}{(t+t')^{d/2}} \right)^{d/4}.$$ 

(33)

In the limit $t_1/t_0 \gg 1$ and for $d > 2$ the first term in (32) vanishes and $C_\psi$ describes the time difference variation of the correlation in the stationary approach to the plateau at $m_0^2 - m_{eq}^2$. Similarly, $C_\sigma(t, t', t_1)$ becomes $C_{eq}(t, t')$. Indeed, in the stationary regime, $t-t' \ll t$, $C_\psi$ varies from $m_0^2 - m_{eq}^2$ to zero while $C_\sigma$ takes the constant value $m_{eq}^2$. In the ageing regime $t'/t = \lambda$, $C_\psi$ has already decayed to zero while $C_\sigma$ varies from $m_{eq}^2$ to zero.

The linear response is simply obtained as

$$\left. \left\langle \frac{\delta \phi_a(\vec{k}, t)}{\delta h_{\beta}(-\vec{k}', t')} \right\rangle \right|_{h=0} = \left. \left\langle \frac{\delta \sigma_{ho}(\vec{k}, t)}{\delta h_{\beta}(-\vec{k}', t')} \right\rangle \right|_{h=0}.$$ 

A similar separation has been used by Franz and Virasoro in a more general context [58].

3. The action

Let us now write the dynamic generating functional in terms of a path integral over the dynamic fields $\tilde{\sigma}_h$ and $\tilde{\psi}$. This will be useful to identify the symmetries of the ‘slow’ action under transformations of time.

The dynamic generating functional is

$$Z = \int D\tilde{\phi}(\vec{k}, t) D\tilde{\psi}(\vec{k}, t) D\tilde{\eta}(\vec{k}, t) \exp \left( -\int \frac{d^d k}{(2\pi)^d} \int_0^\infty dt S_{k1}^{(1)} \right),$$

$$S_{k1}^{(1)} = i\tilde{\phi}(\vec{k}, t) \left[ (\partial_t + k^2 + z(t)) \tilde{\phi}(\vec{k}, t) - \tilde{\eta}(\vec{k}, t) \right.$$

$$\left. - \tilde{h}(\vec{k}, t) \theta (t-t_w) \right] - (4T)^{-1} \tilde{\eta}(\vec{k}, t) \tilde{\eta}(\vec{k}, t)$$

(34)

where, for simplicity, we took the cut-off $\Lambda$ in the noise–noise correlation to infinity and we ignored all initial condition contributions. The external field $\tilde{h}$ is applied from $t_w$ onwards. We call $R^{-1}(k; t, t')$ the differential operator $\delta (t-t')[\partial_{t'} + k^2 + z(t')]$ whose inverse is the retarded linear response function (see equation (14))

$$\int dt' \delta (t-t') (\partial_{t'} + k^2 + z(t')) R(k; t', t'') = \delta (t-t''),$$

$$R(k; t, t') = r(k; t, t') \theta (t-t'),$$

for each $k$. We define

$$\tilde{\sigma}_h(\vec{k}, t) \equiv \int_{t_1}^{t_1} dt'' R(k; t, t'') \tilde{\eta}(\vec{k}, t'') + \int_{t_w}^{\infty} dt'' R(k; t, t'') \tilde{h}(\vec{k}, t''),$$

$$\tilde{\eta}(\vec{k}, t) \equiv \int_{t_1}^{t_1} dt'' R(k; t, t'') \tilde{\eta}(\vec{k}, t'') = \int_{t_1}^{t_1} dt'' r(k; t, t'') \tilde{\eta}(\vec{k}, t''),$$

where $r(k; t, t')$ is the retarded linear response kernel, $\theta (t-t_w)$ is the Heaviside step function, and $\tilde{h}(\vec{k}, t)$ is the external field.

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along the lines of what has been reviewed in section 2.3.3. In the end we shall focus on the behaviour of the two-time action for long times; more explicitly we shall take $t$ and $t'$ to be longer than a long but otherwise arbitrary time $t_1$ but, for the moment, $t_1$ is just an arbitrary timescale. After introducing these definitions with delta functions in the generating functional and some standard manipulations, one arrives at

$$Z \propto \int D\bar{\psi} D\bar{\sigma} D\psi \exp \left( \int \frac{dt}{(2\pi)^d} \int_0^\infty dt' \int_0^\infty dt' S_{ktt'}^{(2)} \right) \times \delta \left[ \int_0^t dt' R^{-1}(k; t, t') \left( \bar{\phi}(k, t') - \bar{\sigma}(k, t) - \bar{\psi}(k, t') \right) \right],$$

(35)

$$S_{ktt'}^{(2)} = -(4T)^{-1} \bar{\sigma}(k, t) \left[ \int_0^{t_1} dt'' r(k; t, t'') r(k; t', t'') \right]^{-1} \bar{\sigma}(-k, t')$$

$$- (4T)^{-1} \bar{\psi}(k, t) \left[ \int_0^{t_1} dt'' R(k; t, t'') R(k; t', t'') \right]^{-1} \bar{\psi}(-k, t').$$

From this action one easily derives the correlations of the Fourier components of the $\bar{\sigma}$ and $\bar{\psi}$ fields:

$$N^{-1} \langle \bar{\sigma}(k, t) \bar{\sigma}(-k, t') \rangle = 2T \int_0^{t_1} dt'' r(k; t, t'') r(k; t', t'')$$

(36)

$$= r(k; t, t_1) r(k; t', t_1) \langle \bar{\sigma}(k, t_1) \bar{\sigma}(-k, t_1) \rangle,$$

(37)

$$N^{-1} \langle \bar{\psi}(k, t) \bar{\psi}(-k, t') \rangle = 2T \int_{t_1}^{\min(t, t')} dt'' r(k; t, t'') r(k; t', t'').$$

In section 2.3 we showed that the separation into fast and slow timescales is clear in the spatial domain. Going back to real space one has

$$-2S_{xxtt'}^{(2)} = \bar{\sigma}(-\bar{x}, t) K_\sigma(-\bar{x}, -\bar{y}, t; t', t') \bar{\sigma}(\bar{y}, t') + \bar{\psi}(-\bar{x}, t) K_\psi(-\bar{x}, \bar{y}, t; t', t') \bar{\psi}(\bar{y}, t'),$$

(38)

with $K_\sigma(-\bar{x}, -\bar{y}, t; t', t') = N^{-1} \langle \bar{\sigma}(\bar{x}, t) \bar{\sigma}(\bar{y}, t') \rangle$ and $K_\psi(-\bar{x}, -\bar{y}, t; t') = N^{-1} \langle \bar{\psi}(\bar{x}, t) \bar{\psi}(\bar{y}, t') \rangle$ the Fourier transforms of (36) and (37), respectively. This result is obvious from the fact that $\sigma$ and $\psi$ are uncorrelated Gaussian variables.

Focusing on equal space points, $\bar{x} = \bar{y}$, and taking the limit $t \geq t' \gg t_1 \gg \tau$ as in [41], equation (38) is the action in the generating functional for the slow and fast components of the global correlation given in equations (19) and (20), respectively.

4. Time transformations

Long ago it was realized that the dynamic equations of motion of mean-field disordered models acquire, in the long waiting time limit and for large separations of times, an invariance under generic reparametrization of time [27]–[32]. This symmetry initially appeared as a nuisance since it was related to the impossibility of determining the equivalent of the scaling function $L(t)$ analytically. More recently, we tried to use this symmetry as a guideline to predict the main fluctuations in finite dimensional systems undergoing glassy dynamics [20]–[23]. With this aim we first analysed the symmetry properties of the action of the $d$-dimensional Edwards–Anderson spin glass [20]. Let us here recall the definition of the time reparametrization, how it acts on the fields, and check whether this invariance exists in the $O(N)$ model with $N \to \infty$. 

doi:10.1088/1742-5468/2006/01/P01006
4.1. Global time reparametrization

Global monotonic time reparametrization is defined as [20]

\[ t \rightarrow \tilde{t} = h(t) \]  

(39)

with \( h(t) \) any monotonic function of time. A particular subset of transformations are rescalings of time

\[ t \rightarrow \zeta t \]  

(40)

that correspond to \( h(t) = \zeta t \).

The transformation (39) acts on the fields \( \tilde{\phi}(\vec{x}, t) \) and \( \tilde{\hat{\phi}}(\vec{x}, t) \) as

\[ \tilde{\phi}(\vec{x}, t) \rightarrow \tilde{\phi}(\vec{x}, t) \equiv \tilde{\phi}(\vec{x}, h(t)), \]  

(41)

\[ \tilde{\hat{\phi}}(\vec{x}, t) \rightarrow \tilde{\hat{\phi}}(\vec{x}, t) \equiv \frac{d\tilde{h}(t)}{dt} \tilde{\phi}(\vec{x}, h(t)). \]  

(42)

Consequently, the space and time dependent two-point functions transform as

\[ C(\vec{x}, \vec{x}'; t, t') \rightarrow \tilde{C}(\vec{x}, \vec{x}'; t, t') \equiv C(\vec{x}, \vec{x}'; h(t), h(t')). \]  

(43)

\[ R(\vec{x}, \vec{x}'; t, t') \rightarrow \tilde{R}(\vec{x}, \vec{x}'; t, t') \equiv \frac{d\tilde{h}(t')}{dt'} R(\vec{x}, \vec{x}'; h(t), h(t')). \]  

(44)

Spatial positions remain untransformed, as do the corresponding Fourier transforms, under the simultaneous transformation of the two times.

The choice of the transformation of the fields is such that the integrated linear response transforms as the correlation under these reparametrizations of time:

\[
\chi(\vec{x}, \vec{x}'; t, t') \rightarrow \tilde{\chi}(\vec{x}, \vec{x}'; t, t') = \int_{t'}^{t} dt'' \tilde{R}(\vec{x}, \vec{x}'; t, t') \\
= \int_{t'}^{t} dt'' \left( \frac{d\tilde{h}(t'')}{dt''} \right) R(\vec{x}, \vec{x}'; h(t), h(t'')) \\
= \int_{h'}^{h} dh'' R(\vec{x}, \vec{x}'; h, h'') = \chi(\vec{x}, \vec{x}'; h, h').
\]

It is interesting to note that the transformation in (41) and (42) does not leave all terms in the Martin–Siggia–Rose action invariant. If we write this action in its most general form

\[
S \equiv \int d^4 x \int dt \left[ T(i\tilde{\phi}(\vec{x}, t))^2 + i\tilde{\phi}(\vec{x}, t) \frac{\partial \phi(\vec{x}, t)}{\partial t} + i\tilde{\phi}(\vec{x}, t) \frac{\delta V[\phi(\vec{x}, t)]}{\delta \phi(\vec{x}, t)} \right]
\]

we note that the first and second terms are not invariant while the last one is. This is not surprising since a particular evolution, i.e. a particular \( h(t) \), has to be chosen by the dynamic action. It is only the slow dynamics, which is generated in some models, that may acquire full time reparametrization (or a reduced) invariance. We shall come back to this important point below.

doi:10.1088/1742-5468/2006/01/P01006

J. Stat. Mech. (2006) P01006
4.2. Symmetries in the dynamic equations

Following the same route as in the study of the dynamics of disordered spin models, let us first examine whether the dynamic equations for the global correlation and response of the large $N$ $O(N)$ model become invariant under generic reparametrization of time in the scaling regime of long waiting time ($t' \gg \tau$) and for very separated times ($t - t' \gg t'$).

With this aim, we first derive closed-form dynamic equations for the global correlation and response of the $O(N)$ model with $N \to \infty$. When show that these are not invariant under the most generic time reparametrization defined in equations (43) and (44) but only under the subgroup of time rescalings given in equation (40).

4.2.1. Dynamic equations for the global correlation and response.

In appendix A we show that the dynamic equation for the global linear response takes the form

$$\frac{\partial R(t, t')}{\partial t} = -z(t) R(t, t') + \sum_{n=0}^{\infty} A_n \int dt_n \int dt_{n-1} \ldots \int dt_1 R(t, t_1) R(t_1, t_2) \ldots R(t_n, t')$$

(45)

for all spherical models with arbitrary two-body interactions and all $O(N)$ models in the limit $N \to \infty$ with arbitrary two-body elastic energy. The only requirement for this result to hold is that the energy band must have a finite edge. The coefficients $A_n$ are determined by the density of states of the interaction matrix or elastic 'coefficients' and thus depend on dimensionality. In particular, for the $p = 2$ spherical spin glass with interactions chosen from a Gaussian distribution with zero mean and variance of order $1/N$, $A_0 = 0$, $A_1 \neq 0$, $A_{n\geq 2} = 0$ and the series truncates at $n_{\text{max}} = 1$. For a general density of states $n_{\text{max}} \to \infty$. Dilute models are excluded from this family since their densities of states have long tails [55, 56]. It is interesting to note that the dynamics of the response is decoupled from that of the correlation for all these models.

Putting this equation in the Schwinger–Dyson form

$$\frac{\partial R(t, t')}{\partial t} = -z(t) R(t, t') + \int dt_n \Sigma(t, t_n) R(t_n, t')$$

(46)

allows us to identify the self-energy:

$$\Sigma(t, t') = \sum_{n=0}^{\infty} A_n \int dt_{n-1} \int dt_{n-2} \ldots \int dt_1 R(t, t_1) \ldots R(t_{n-1}, t').$$

(47)

Since we are not considering the possibility of applying non-potential forces, the Schwinger–Dyson equation for the global correlation should read

$$\frac{\partial C(t, t')}{\partial t} = -z(t) C(t, t') + \int dt_n [\Sigma(t, t_n) C(t_n, t') + D(t, t_n) R(t', t_n)]$$

(48)

with $D(t, t')$ the vertex kernel. If the model has an equilibrium high temperature phase the vertex should be related to the self-energy in such a way that the solution verifies the fluctuation-dissipation theorem. This is achieved by

$$\Sigma(t - t') = \frac{1}{T} \frac{\partial D(t - t')}{\partial t'} \theta(t - t')$$

doi:10.1088/1742-5468/2006/01/P01006
in the high $T$ phase. One can then guess that

$$D(t, t_n) = \sum_{n=0}^{\infty} A_n \int dt_1 \ldots \int dt_{n-1} R(t, t_1) \ldots R(t_{n-2}, t_{n-1}) C(t_{n-1}, t_n).$$  \hspace{1cm} (49)

Note that

$$\Sigma(t, t_n) = \int dt_a \int dt_b \frac{\delta D(t, t_n)}{\delta C(t_a, t_b)} R(t_a, t_b).$$

The equal-time global correlation $C(t, t) \equiv \int d^dx \langle \phi^2(\vec{x}, t) \rangle_{ic}$ may not, in general, be fixed to a constant value. The dynamic equation determining its time evolution is obtained by writing $d\Sigma(t, t_n) = \lim_{t' \to t} \left[ \partial_t C(t, t') + \partial_{t'} C(t, t') \right]$ using equation (48). The Lagrange multiplier $z(t)$ is in general determined by the equation fixing $Y^2(t)$ while one should use the equation for $C(t, t)$ to compute the average $\langle \phi^2(\vec{x}, t) \rangle_{ic}$.

### 4.2.2. Solution in the ordered phase.

Let us assume that the global correlation and response on the one hand, and the self-energy and the vertex on the other, separate into a fast and a slow component as in equations (18)–(23). We then introduce this Ansatz in equations (46) and (48) to derive dynamic equations for the fast and slow parts [25]. The equations of motion for the slow parts have the form

$$\frac{\partial C_{ag}(t, t')}{\partial t} = -z(t)C_{ag}(t, t') + \text{int}_C$$

$$\frac{\partial R_{ag}(t, t')}{\partial t} = -z(t)R_{ag}(t, t') + \text{int}_R$$

with $\text{int}_C$ and $\text{int}_R$ being two series of rather complicated terms involving $n$-order convolutions of the response and the correlation over the times. Clearly, the time derivatives on the left-hand side are not invariant under a generic reparametrization of time. A necessary step in trying to prove time reparametrization invariance is to assume that asymptotically they are much smaller than each term on the right-hand side, drop them, and check the invariance of the remaining terms. This is an assumption that should be checked a posteriori once the solution for $C_{ag}$ and $R_{ag}$ is derived from the remaining equations. In the case of the $O(N \to \infty)$ model we already know the exact solution for all times, from which we can derive the approximate form that holds in the scaling limit of very long times and separations among them, and check whether this form allows for the time reparametrization invariance of the equations.

Let us first focus on the equation for the global response which is easier to deal with. The slow ageing part of the linear response behaves asymptotically as $t^{-d/2} f_R(\lambda)$; see equation (25). Its time derivative is

$$\frac{\partial R_{ag}(t, t')}{\partial t} \sim -t^{-d/2-1} \left[ \frac{d}{2} f_R(\lambda) + \lambda f_R'(\lambda) \right].$$

The ‘mass’ $z(t)$ decays as

$$z(t) = \frac{1}{2} \ln Y^2(t) \sim \frac{1}{2} \frac{d}{dt} \ln t^{-d/2} = -\frac{d}{4} t^{-1},$$ \hspace{1cm} (50)
consequently, the first term on the right-hand side of equation (50) goes as
\[ z(t)R_{ag}(t, t') \sim -\frac{d}{4} t^{-d/2-1} f_R(\lambda) \]
and it is of the same order as the time derivative on the left-hand side of the same equation.

The terms in the series can also be analysed by separating the stationary and ageing contributions to the integrals in equation (45) (see [54] for a detailed explanation). Such separation, as carried out in appendix B, leads to
\[ \frac{\partial R_{ag}(t, t')}{\partial t} = -z(t)R_{ag}(t, t') + \sum_{n=0}^{\infty} \tilde{A}_n \int dt_n \int dt_{n-1} \cdots \int dt_1 R_{ag}(t, t_1)R_{ag}(t_1, t_2) \cdots R_{ag}(t_n, t'), \]
where the coefficients \( \tilde{A}_n \) are given by (see equation (B.1))
\[ \tilde{A}_n = -\frac{1}{(n+1)!} \left( \frac{d}{d\chi_{st}} \right)^n \epsilon(\chi_{st}), \]
with \( \chi_{st} \) the integrated stationary response evaluated at timescales of order \( \Delta t = t - t' \). The function \( \epsilon(h) \) is the inverse of the function
\[ h(\epsilon) = P \int_0^\infty d\epsilon' \frac{g(\epsilon')}{\epsilon' - \epsilon} \]
which is obtained from the density of states \( g(\epsilon) \) of the model (see appendices A and B).

In the \( O(N \to \infty) \) model the coefficients \( \tilde{A}_n \) scale as a power of \( \Delta t \), and the precise power is controlled by the form of the density of states at low energies, \( g(\epsilon) \propto \epsilon^\nu \), with \( \nu = d/2 - 1 \) (see equation (5)). The scaling of \( \tilde{A}_n \) at long time differences can be obtained as follows. First, notice that \( h(0) - h(\epsilon) \sim \epsilon^\nu \) by taking into account the power law dependence of the density of states at low energies. Then, the inverse function \( \epsilon(h) \) is \( \epsilon = h^{-1}(h(\epsilon)) \sim (h(0) - h(\epsilon))^{1/\nu} \). On the other hand, the stationary susceptibility \( \chi_{st}(\Delta t) \) is intimately related to the density of states. Using equation (A.5) one finds
\[ \chi_{st}(\Delta t) = \int_{-\infty}^{\Delta t} d\Delta t' G(\Delta t') = \int_0^\infty d\epsilon g(\epsilon) \frac{1}{\epsilon} \frac{1 - e^{-\epsilon \Delta t}}{\epsilon} = \chi_{st}(\infty) - \int_0^{\infty} d\epsilon g(\epsilon) e^{-\epsilon \Delta t}. \]
Using now the power law decay of the density of states at low energies one finds
\[ \chi_{st}(\infty) - \chi_{st} \sim (\Delta t)^{-\nu} \quad \text{or} \quad 1/\Delta t \sim [\chi_{st}(\infty) - \chi_{st}]^{1/\nu}, \]
with \( \chi_{st} \equiv \chi_{st}(\Delta t) \). From equation (A.5) one notices that \( \chi_{st}(\infty) = h(0) \). Thus,
\[ \epsilon(\chi_{st}) \sim [\chi_{st}(\infty) - \chi_{st}]^{1/\nu}, \]
so that \( d^n \epsilon / d\chi_{st}^n \sim [\chi_{st}(\infty) - \chi_{st}]^{1/\nu - n} \sim \Delta t^{-1+n\nu} \) which yields
\[ \tilde{A}_n \sim \Delta t^{-1+n\nu}. \]

Notice that the coefficients \( \tilde{A}_n \) keep a power law dependence on time differences. This anomalous dependence is in remarkable contrast to glassy systems, such as the p-spin disordered models with \( p \geq 3 \), in which the coefficients \( \tilde{A}_n \) reach finite constants as \( \Delta t \to \infty \) and \( \chi_{st} \to \chi_{st}(\infty) \). We discuss the consequences of this difference below, when we consider the scaling dimensions of the global correlation and response under time rescalings.

doi:10.1088/1742-5468/2006/01/P01006
4.2.3. Scaling dimensions. Consider the situation in which under, say, a scale transformation, the global correlation and response in the ageing regime transform according to

\[ t \rightarrow \zeta t \]  
\[ C_{ag}(t,t') \rightarrow \tilde{C}_{ag}(t,t') = C_{ag}(\zeta t, \zeta t') \]  
\[ R_{ag}(t,t') \rightarrow \tilde{R}_{ag}(t,t') = \zeta^{\Delta_R} R_{ag}(\zeta t, \zeta t') \],

where \( \Delta_R \) is the retarded dimension [31] of the response (the advanced dimensions for both response and correlation, as well as the retarded dimension for the correlation, are zero in the above). In systems in which correlation and response are related by an off-equilibrium fluctuation-dissipation relation with a finite effective temperature [61], the retarded dimension takes the value \( \Delta_R = 1 \). In this case, one can show that if the equations of motion are invariant under these scale transformations, they are also invariant under time reparametrization \( t \rightarrow h(t) \). In other words, for the special case \( \Delta_R = 1 \), scale invariance implies reparametrization invariance [20]. The situation is similar in character to what happens with scale invariant field theories, where in the special case of two-dimensional systems local scale invariance implies conformal invariance, a much larger symmetry.

Let us discuss concisely why the symmetry is larger when \( \Delta_R = 1 \). Consider the ageing contribution to a generic term \( I_n \) with \( n \) integrals and \( n + 1 \) responses as an example:

\[
I_n(t,t') = \int dt_n \int dt_{n-1} \ldots \int dt_1 \tilde{R}_{ag}(t,t_1) \tilde{R}_{ag}(t_1,t_2) \ldots \tilde{R}_{ag}(t_n,t')
\]

\[
= \int dt_n \ldots \int dt_1 \zeta^{(n+1)\Delta_R} R_{ag}(\zeta t, \zeta t_1) \ldots R_{ag}(\zeta t_n, \zeta t')
\]

\[
= \zeta^{(n+1)\Delta_R-n} \int d(\zeta t_n) \ldots \int d(\zeta t_1) R_{ag}(\zeta t, \zeta t_1) \ldots R_{ag}(\zeta t_n, \zeta t')
\]

\[
= \zeta^n(\Delta_R-1) \zeta^{\Delta_R} I_n(\zeta t, \zeta t').
\]

Thus, under the rescaling transformation, \( I_n(t,t') \) has dimension

\[ \Delta_{I_n} = n(\Delta_R - 1) + \Delta_R. \]

There are two special features that arise when \( \Delta_R = 1 \). The first is that all the \( I_n(t,t') \) have the same dimension, \( \Delta_{I_n} = 1 \) for all \( n \). The second is that the change of variables inside the integrals can be carried out for a more general change of variables \( t \rightarrow h(t) \) with an arbitrary monotonic function \( h(t) \), because

\[
\int dt_i \left( \frac{dh(t_i)}{dt_i} \right)^{\Delta_R=1} \ldots = \int dt_i \frac{dh(t_i)}{dt_i} \ldots = \int dh_i \ldots
\]

holds for each of the times that are being integrated over. Therefore, for \( \Delta_R = 1 \) the rescaling of the correlations and responses can be absorbed in the Jacobian for the changes of integration variables, resulting in

\[ I_n(t,t') = \left( \frac{dh}{dh'} \right) I_n(h(t), h(t')) , \]

which is, for case of a time reparametrization, the counterpart of the rescaling result equation (59). Because terms in the equation of motion for the response, including all \( I_n \)

\[
\text{Fluctuations in the coarsening dynamics of the O(N) model}
\]

\[
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independently of $n$, pick up the same transformation factor $(\partial h/\partial t')$ (that can then be cancelled), the equations remain invariant under the $t \rightarrow h(t)$ transformation.

Hence for $\Delta_R = 1$ scale invariance implies reparametrization invariance.

### 4.2.4. Fixing the retarded dimension $\Delta_R$.

Let us discuss now how the asymptotic behaviour of $z(t)$ fixes the retarded dimension $\Delta_R$.

**The case of glassy dynamics.** In mean-field spherical models displaying glassy dynamics, such as for example the spherical $p$-spin model for $p \geq 3$, the right-hand side of the dynamic equation for the response is much simpler that equation (45) in that the series actually has only one term of the form $I_1$. A supplementary difficulty arises from the fact that $C$ enters the integral but this is not very difficult to deal with if we assume that the scaling dimensions of $C$ vanish.

The function $z(t) \rightarrow z_\infty \neq 0$ as $t \rightarrow \infty$. In this case, the term

$$z(t) R_{ag}(t, t') \rightarrow z_\infty R_{ag}(t, t')$$

has the same scaling dimension as the response itself, i.e., $\Delta_R$. In these glassy systems, the coefficients $\tilde{A}_n$ in front of the integral terms are finite constants in the limit of $\Delta t = t - t' \rightarrow \infty$. (In the specific case of the $p$-spin models, only $\tilde{A}_0$ and $\tilde{A}_1$ are non-vanishing.)

The time derivative term

$$\frac{\partial}{\partial t} R_{ag}(t, t')$$

has scaling dimension $\Delta_{deriv} = \Delta_R + 1$, because the $\partial/\partial t$ has dimension $1$. So in this case the time derivative term is irrelevant, and it can be dropped in the long time limit, as long as one finds a non-trivial solution to the remaining equations. Indeed, there can be a non-trivial solution of the long time dynamical equations if at least one of the integral terms, $I_n(t, t')$ for some $n \geq 0$, can balance the $z_\infty R_{ag}(t, t')$ term. As we discussed previously, the contribution to a generic integral $I_n$ with $n \geq 0$ has scaling dimension $\Delta_{I_n} = n(\Delta_R - 1) + \Delta_R$. The cancellation can be achieved if and only if

$$\Delta_{I_n} = n(\Delta_R - 1) + \Delta_R = \Delta_R$$

for some $n \geq 0$.

There are two ways of satisfying this requirement. The contribution with $n = 0$ trivially satisfies this identity for any value of $\Delta_R$. Besides, terms of the same order arise from $n \geq 1$ only if $\Delta_R = 1$, in which case the $n$ dependence disappears and the condition is actually satisfied for all $n \geq 1$. The second possibility is realized by the $p > 2$ spherical Gaussian spin glass, a model with

$$\Delta_R = 1$$

and for which reparametrization invariance develops. The $p = 2$ spherical spin glass with Gaussian interactions is discussed in detail in appendix C.

One can argue that the scaling dimensions zero for the correlation (both retarded and advanced dimensions) and retarded dimension $\Delta_R = 1$ for the response are consistent with a factor $X(t, t') = T/T_{eff}$ that remains finite for fixed $C_{ag}$ in the long time limit. Consider the out-of-equilibrium fluctuation-dissipation relation:

$$R_{ag}(t, t') = \frac{X(t, t')}{T} \frac{\partial}{\partial t'} C_{ag}(t, t') \theta(t - t'). \quad (61)$$

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If the factor $X(t, t') \rightarrow X(C_{ag})$ for fixed $C_{ag}$ in the large $t, t'$ limit, without vanishing with some anomalous extra powers of $t$, then it follows that the retarded dimension of the response is one more than that of the correlation, because of the $\partial/\partial t'$. So if the correlation has retarded dimension zero, the response will have retarded dimension $\Delta_R = 1$ as long as $X$ remains finite and has no anomalous power law dependence on $t$ for fixed $C_{ag}$. This is the situation in glassy systems, where finite factors $X = T/T_{\text{eff}}$ have been observed in experiments and simulations (see [60] for a review).

The case of the $O(N \to \infty)$ model. Consider a case in which the function $z(t) \sim t^{-\Delta_z} \rightarrow 0$ as $t \to \infty$. Specifically, $\Delta_z = 1$ for the $O(N \to \infty)$ model (see equation (50)). Now, the term

$$z(t) R_{ag}(t, t')$$

has scaling dimension $\Delta_R + \Delta_z$. The time derivative term, just as in the case of glassy systems above, has scaling dimension $\Delta_{\text{deriv}} = \Delta_R + 1$. Therefore, in contrast to the cases discussed above, one cannot naively neglect the time derivative term, because it has the same scaling dimension as the $z(t) R_{ag}(t, t')$ term.

In the $O(N \to \infty)$ model the series in the r.h.s. of equation (45) does not truncate. The prefactor of the integral term $I_n(t, t')$ depends on $\Delta t = t - t'$, $A_n \sim \Delta t^{-1+n\nu}$ with $\nu = 1 - d/2$ (see equation (55)), and there is an additional scaling dimension arising from the anomalous scaling of the prefactors $A_n$:

$$\Delta A_n = 1 - n\nu.$$

In order to determine the dimension $\Delta_R$, one must balance at least one of the integral terms, $A_n(\Delta t) I_n(t, t')$ for some $n \geq 0$, against the $z(t) R_{ag}(t, t')$ term and the time derivative $\partial R_{ag}(t, t')/\partial t$. This can be achieved if and only if

$$\Delta A_n + \Delta I_n = n(\Delta_R - 1 - \nu) + \Delta_R + 1 = \Delta_R + 1 \quad \text{for some } n \geq 0. \quad (62)$$

Notice that this condition is satisfied in particular by $n = 0$, but it can also be satisfied for any $n$ if

$$\Delta_R = \nu + 1 = d/2,$$

which is indeed consistent with the exact result given in equation (23).

Notice that all terms in the equation of motion of $R_{ag}(t, t')$ have the same scaling dimension $\Delta_R + 1$ as the time derivative term, which thus cannot be dropped for any dimension $d$, in contrast to the case in glassy systems. Notice also that $\Delta_R \neq 1$ for $d > 2$, so reparametrization invariance does not develop; only scale invariance is a symmetry of the long time dynamical equations of motion. A retarded scaling dimension $\Delta_R > 1$ implies, using equation (61), that the factor $X(t, t') \rightarrow 0$ for long times and fixed $C_{ag}$, if the correlation has retarded and advanced dimensions zero. This result is in agreement with the direct calculation of the factor $X$ in equation (26).

For $d = 2$, one obtains that $\Delta_R = 1$, but in contrast to the case of glassy dynamics where the prefactors $A_n$ were constant, $\Delta A_n = 1$. For reparametrization invariance to develop, it is necessary that $\Delta_R = 1$ and that $\Delta A_n = 0$. Hence, reparametrization invariance does not develop even in the $d = 2$ case. Notice, however, that $\Delta_R = 1$ for $d = 2$ implies a non-trivial $X(t, t')$ which is actually found in the exact solution [35]; there is still

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an additive separation of correlation and linear response in a stationary and an ageing part at $T = 0$ (as opposed to the multiplicative scaling found in critical relaxations [42]). Nevertheless, it is important to remark that this $X(t, t')$ depends continuously on the ratio $t/t'$ (see equation (26)), which implies a $C_{ag}$ dependent effective temperature instead of a constant effective temperature; the latter is expected for a problem with a single correlation scale.

4.3. Conjecture

We argued that $\Delta_R = 1$ is a necessary condition for having an asymptotic time reparametrization invariance—though this condition is not sufficient, as shown by the $d = 2$ O($N \to \infty$) case. In addition, $\Delta_R = 1$ implies a finite integrated linear response and a finite effective temperature, as can be derived from equation (61).

The O($N \to \infty$) model has a weaker response than that of glassy models, as for example the $p$-spin spherical disordered system. Indeed, in the O($N \to \infty$) model the ageing contribution to the integrated response vanishes asymptotically for all $d > d_L = 2$—and this can be related to the development of an infinite effective temperature [61] at long times; while it approaches a finite $C$ dependent value for $d = d_L = 2$—and this cannot be interpreted in terms of an effective temperature since one would have a $C$ dependent value within a single correlation scale [26]. Other solvable coarsening problems have a similar integrated response (see e.g. [42]).

On the basis of the discussion above, we conjecture that models with a finite and well-defined effective temperature, such as the $p$-spin spherical disordered system with $p \geq 3$ or the more complex Sherrington–Kirkpatrick spin glass, develop time reparametrization invariance asymptotically, while this does not occur in systems with a diverging or ill-defined effective temperature, such as the large $N$ O($N$) model.

4.3.1. Space–time rescaling. For the sake of comparison, in the following we consider the standard dynamical scaling [24] which consists of simultaneous rescaling of time and space.

So far we discussed time rescaling and time reparametrization invariance properties in the real space representation. This is because we have been interested in making contact with glassy systems for which composite fields, which are related to the two-time correlation and response functions at equal space points, might be the natural order parameters [20]. In the case of the O($N \to \infty$) model, however, one knows that the original field $\phi_\alpha(\vec{x}, t)$ is already the natural order parameter. Especially, one expects its Fourier space representation $\phi_\alpha(\vec{k}, t)$ to be easier to handle.

Let us take the response function $\delta \phi_\alpha(\vec{k}, t)/\partial h_\beta(-\vec{k'}, t')|_{h=0} = r(k; t, t')\delta_{\alpha\beta}\delta(\vec{k}+\vec{k'})$ and the two-time composite field $[\phi_\alpha(\vec{k}, t)\phi_\beta(-\vec{k'}, t')] = c(k; t, t')\delta_{\alpha\beta}\delta(\vec{k}+\vec{k'})$ where $k = |\vec{k}|$.

In the $T \to 0$ limit these quantities satisfy the same evolution equation,

$$\frac{\partial}{\partial t}r(k; t, t') = -[k^2 + z(t)] r(k; t, t') ,$$

$$\frac{\partial}{\partial t}c(k; t, t') = -[k^2 + z(t)] c(k; t, t') .$$

5 This is general since the domain growth scaling is controlled by a ‘zero-temperature fixed point’ [24].

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Now let us suppose that these equations admit a set of asymptotic solutions with the following scaling forms:

\[ r(k; t, t') = \zeta^{\Delta_R^r + \Delta_A^r - d\Delta_s} r(k\zeta^{-\Delta_s}, \zeta t, \zeta t'), \quad (65) \]

\[ c(k; t, t') = \zeta^{\Delta_R^c + \Delta_A^c - d\Delta_s} c(k\zeta^{-\Delta_s}, \zeta t, \zeta t'), \quad (66) \]

and

\[ z(t) = \zeta^{\Delta_s} z(\zeta t). \]

\( \Delta_R^r \) and \( \Delta_A^r \) are the retarded and advanced dimensions of the response \( r \); similarly, \( \Delta_R^c \) and \( \Delta_A^c \) are the retarded and advanced dimensions of the correlation \( c \). The scaling dimensions are then fixed by inserting this Ansatz into equations (63) and (64). First, focusing on the \( k = 0 \) component, one finds \( \Delta_z = 1 \). Then considering the \( k > 0 \) components, one finds two possibilities. The first one is that the \( k^2 \) term has the same scaling dimension as the other two terms. In this case one finds the exponent for spatial scaling \( \Delta_s = 1/2 \). The other possibility is \( \Delta_s > 1/2 \) which means that the \( k^2 \) term becomes irrelevant. Which of the two cases appear in the large times regime depends on the initial conditions. For usual random initial condition of the form given in equation (4) with the same statistical weight on all \( k \) components, the exact solution summarized in section 2.3 tells us that \( \Delta_s = 1/2 \) is actually selected. In the following we only consider this case.

In the asymptotic regime all the terms in equations (63) and (64) have the same scaling dimensions. Thus none of them can be dropped irrespective of the scaling dimensions of the response and correlation functions which will be determined below.

We still need to determine the retarded and advanced scaling dimensions of the response and correlation functions. Since the solution of the Langevin equation at \( T = 0 \) can be written as (see equation (15))

\[ \phi(\vec{k}, t) = r(k; t, t') \phi(\vec{k}, t'), \]

the overall scaling factor in \( r \) must be identical to one, and one has that retarded and advanced scaling dimensions of the response functions must satisfy \( \Delta_R^r + \Delta_A^r = d\Delta_s = d/2 \). This is achieved, in particular, by \( \Delta_R^r = d\Delta_s = d/2 \) and \( \Delta_A^r = 0 \) though other choices are possible. The analysis of the self-consistent equation for \( z(t) \) fixes the scaling dimensions \( \Delta_R^c \) and \( \Delta_A^c \). Indeed, equation (2) reads

\[ z(t) = g \int \frac{d^d k}{(2\pi)^d} c(k, t, t) + r. \]

In the large time limit \( z(t) \to 0 \) and the integral converges to \(-r/g \). Therefore \( \Delta_R^c + \Delta_A^c = 0 \) which suggests the natural choice \( \Delta_R^c = \Delta_A^c = 0 \).

It is instructive to consider the inverse Fourier transform of the scaling Ansatz in equations (65) and (66) which reads

\[ R(|\vec{x} - \vec{y}|; t, t') = \zeta^{\Delta_A^r + \Delta_R^r} R(|\vec{x} - \vec{y}|\zeta^{\Delta_s}, \zeta t, \zeta t'), \]

\[ C(|\vec{x} - \vec{y}|; t, t') = \zeta^{\Delta_A^c + \Delta_R^c} C(|\vec{x} - \vec{y}|\zeta^{\Delta_s}, \zeta t, \zeta t'). \]

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\[ \text{J. Stat. Mech. (2006) P01006} \]
Thus the solution can be written in the scaling form

\[ R(|\overline{x} - \overline{y}|; t, t') \approx \frac{1}{L^d(t)} f_{R_c} \left( \frac{|\overline{x} - \overline{y}|}{L(t)}, \frac{L(t)}{L(t')} \right), \]

\[ C(|\overline{x} - \overline{y}|; t, t') \approx f_{C_r} \left( \frac{|\overline{x} - \overline{y}|}{L(t)}, \frac{L(t)}{L(t')} \right), \]

with the domain growth law

\[ L(t) \propto t^{\Delta_s} = \sqrt{t}. \]

Thus, the analysis of the invariance of the dynamic equations for the global \( C \) and \( R \) under rescaling of time presented in the previous sections, that serves to fix the scaling dimension of the global response \( \Delta_{R_c} \), can be extended to study the scaling dimensions of the space dependent correlation and response \( C(r; t, t') \) and \( R(r; t, t') \) under simultaneously rescaling of space and time. The analysis suggests \( \Delta_{R_c} = d/2 \) and \( \Delta_{A_c} = 0 \), and \( \Delta_{C_r} = 0 \), for any \( d \).

4.4. Symmetries of the (long time) action

We have already mentioned that a generic Martin–Siggia–Rose action is not invariant under reparametrizations of the times and fields defined in equations (41) and (42). Let us now analyse the symmetries of the action in the long times limit in which there is a separation of timescales into the global correlation and response.

Let us first focus on the case \( d > 2 \). Using the simple transformations described in section 3 [41] the dynamic generating function of the \( O(N) \) model can be expressed as a path integral over the fields \( \overline{\sigma}(\overline{x}, t) \) and \( \overline{\psi}(\overline{x}, t) \) only. On the one hand, one can argue that the ‘fast’ \( \overline{\psi} \) part ‘renormalizes’ to zero under generic time reparametrization and becomes asymptotically irrelevant. On the other hand, while the \( \overline{\sigma} \) field transforms in such a way that it ensures the correct transformation of the integration measure, the kernel in its action is just the inverse of the global correlation itself that is time dependent and transforms in a non-trivial manner under generic reparametrizations of time. The quadratic action for \( \overline{\sigma}(\overline{x}, t) \) at equal space points is not invariant under generic reparametrizations of time.

The local action of the \( \overline{\sigma} \) field is, however, invariant under a reduced subset of transformations, namely time rescalings. Since the kernel is a function of \( t'/t \) one finds that the slow action written in terms of the fields only (see equation (35)) is invariant under

\[ t \rightarrow \zeta t, \]

\[ \overline{\sigma}(\overline{x}, t) \rightarrow \zeta \overline{\sigma}(\overline{x}, \zeta t). \]

But one can also go one step back and check whether the action for the field, \( \overline{\sigma}(\overline{x}, t) \), and response field, \( i\overline{\sigma}(\overline{x}, t) \), is invariant under time rescalings that change the response field as

\[ i\overline{\sigma}(\overline{x}, t) \rightarrow \zeta \overline{\sigma}(\overline{x}, \zeta t), \] (67)
i.e. the reduction of (42) to time rescalings. Transforming the ‘slow’ part of $S_{k\delta t'}^{(4)}$ into
spatial coordinates and writing explicitly all integrals one has

$$S_{x}^{(4)} = \int d^{d}x \int d^{d}y \int dt \int dt' \left[ i\tilde{\sigma}(\bar{x}, t)K_{\sigma}(\bar{x} - \bar{y}, t, t')i\tilde{\sigma}(\bar{y}, t') + i\tilde{\sigma}(\bar{x}, t)\delta(t - t')\delta^{d}(\bar{x} - \bar{y})\tilde{\sigma}(\bar{y}, t') \right]$$

with

$$K_{\sigma}(\bar{x} - \bar{y}; t, t') \equiv T \int \frac{d^{d}k}{(2\pi)^{d}} e^{ik(\bar{x} - \bar{y})} e^{-k^{2}/\Lambda^{2}} \int_{0}^{t_{1}} dt'' r(k, t, t'')r(k, t', t'').$$

One can then easily check that the action at equal space points, $\bar{x} = \bar{y}$, remains invariant under the time rescalings proposed above. Indeed, in terms of the transformed fields the local action reads

$$\tilde{S}_{x}^{(4)} = \int dt \int dt' \left[ i\tilde{\sigma}(\bar{x}, t)K_{\sigma}(\bar{0}; t, t')i\tilde{\sigma}(\bar{x}, t') + i\tilde{\sigma}(\bar{x}, t)\delta(t - t')\tilde{\sigma}(\bar{x}, t') \right]$$

$$= \int dt \int dt' \left[ \zeta i\tilde{\sigma}(\bar{x}, \zeta t)K_{\sigma}(\bar{0}; \zeta t, \zeta t')\zeta i\tilde{\sigma}(\bar{x}, \zeta t') + \zeta i\tilde{\sigma}(\bar{x}, \zeta t)\zeta \delta(\zeta t - \zeta t')\tilde{\sigma}(\bar{x}, \zeta t') \right]$$

$$= S_{x}^{(4)}$$

where we used the fact that $K_{\sigma}(\bar{0}; t, t')$ is identical to the global correlation function, $C(t, t')$. In the limit $t, t' \gg t_{0}$ this is a function of $t'/t$ and thus invariant under time rescaling. The last identity follows simply from changing the integration variables form $t$ to $\zeta t$.

A similar treatment for $d = 2$ is much more delicate. The explicit calculations in [41] show that the separation of the field is achieved by taking advantage of the fact that a factor proportional to $(\Lambda^{2}t_{1})^{1-d/2}$ vanishes (see equations (32) and (33)). This, however, is no longer true for $d = 2$. Besides, the interesting dynamics in this case arises only at $T = 0$, another non-trivial limit to be taken in the asymptotic expressions. For these reasons, we cannot simply carry through the arguments above to $d = 2$. Another way to attack the same problem would be to write an action in terms of $R(\bar{x}, \bar{x}; t, t')$ and $C(\bar{x}, \bar{x}; t, t')$ and use a similar reasoning to the one we used for the analysis of the equations of motion for $R(t, t')$ and $C(t, t')$. We shall not pursue this study here.

Let us note that the invariance under time rescaling discussed above can also be understood as a part of the usual space–time scaling invariance discussed in section 4.3.1. To this end, we perform a renormalization group (RG) analysis on the Fourier space representation of the slow part of the action,

$$S_{k<\Lambda}[i\tilde{\sigma}, \sigma] = \int_{0}^{\Lambda} d^{d}k \int_{0}^{\infty} dt \int_{0}^{\infty} dt' [i\tilde{\sigma}(k, t)K_{\sigma}(k, t, t')i\sigma(k, t') + \delta(t - t')i\tilde{\sigma}(k, t)\sigma(-k, t')]$$

$$+ \int_{-\Lambda}^{0} d^{d}k \int_{0}^{\infty} dt \int_{0}^{\infty} dt' [i\tilde{\sigma}(k, t)K_{\sigma}(k, t, t')i\sigma(k, t') + \delta(t - t')i\tilde{\sigma}(k, t)\sigma(-k, t')]$$

$$+ \int_{0}^{\Lambda} d^{d}k \int_{-\infty}^{0} dt \int_{0}^{\infty} dt' [i\tilde{\sigma}(k, t)K_{\sigma}(k, t, t')i\sigma(k, t') + \delta(t - t')i\tilde{\sigma}(k, t)\sigma(-k, t')]$$

$$+ \int_{-\Lambda}^{0} d^{d}k \int_{-\infty}^{0} dt \int_{0}^{\infty} dt' [i\tilde{\sigma}(k, t)K_{\sigma}(k, t, t')i\sigma(k, t') + \delta(t - t')i\tilde{\sigma}(k, t)\sigma(-k, t')]$$

$$= S_{k<\Lambda}^{(4)}[i\tilde{\sigma}, \sigma] + S_{\Lambda<\Lambda}^{(4)}[i\tilde{\sigma}, \sigma] + S_{\Lambda<\Lambda}^{(4)}[i\tilde{\sigma}, \sigma].$$

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$K_\sigma$ the Fourier transform of (69). First, by integrating out the ‘fast modes’ in $\Lambda/b < k < \Lambda$ we obtain $S^{(4)}_{k<\Lambda/b}$. Next, we choose a set of rescaled variables

$$
\tilde{k} = kb, \quad \tilde{t} = t/b^2, \\
i\tilde{\sigma}(\tilde{k}, \tilde{t}) = b^{-d/2+2}i\tilde{\sigma}(k, t), \quad \tilde{\sigma}(\tilde{k}, \tilde{t}) = b^{-d/2}\sigma(k, t), \\
\tilde{z}(\tilde{t}) = b^2z(t).
$$

In terms of the new variables the cut-off is put back to $\Lambda$ and the action of the original form is recovered. Converting the above results to the real space representation and equating the scaling parameter of space $b$ and time $\zeta$ as $b^2 = \zeta$. The space dependent slow action is invariant under simultaneous rescaling of space and time.

4.5. How the $O(N)$ model escapes reparametrization invariance but displays scale invariance

It was shown in [20] that under rather mild assumptions (namely, causality, a separation of timescales as the one discussed in section 2, the fact that the remaining free field action does not lead itself to slow dynamics and the use of the naive scaling dimensions of the fields) the slow part of the action of the 3D Edwards–Anderson spin glass, when written in terms of the two-time dependent dynamic order parameters, remains invariant under global time reparametrization. We have shown above that the action for the slow evolution of the $O(N \to \infty)$ model is not invariant under these transformations. One would like to identify which of the assumptions used in [20] is (are) violated in the $O(N \to \infty)$ model. Indeed, in [20] we used the naive dimensions for $Q^R$, that is to say $\Delta^A_{Q^R} = 0$ and $\Delta^R_{Q^R} = 1$. This assumption should be correct for systems that develop a finite and well-defined effective temperature in the ageing regime. The $O(N \to \infty)$ model falls out of this class and this assumption does not apply to it.

5. The distribution of local two-time observables

During the ageing relaxation of glassy systems one expects important temporal and spatial fluctuations. The distribution of local coarse-grained correlations and linear responses in spin glasses [21,22] and kinetically facilitated models [23] were computed numerically. The comparison of these probability distribution functions (PDFs) with the theoretical framework developed in [20]–[23] was also discussed. In short, the main features of these distributions are:

(i) The PDF of coarse-grained local two-time correlations is a function that depends on the two times and, when these are chosen to lie in the ageing regime, the PDF scales in time just as the global correlation itself.

(ii) The functional form of the PDF of coarse-grained local two-time correlations changes with the two times. It can be approximately described with a Gumbel-like function with a two-time dependent parameter, which in the ageing regime is simply a function of the global correlation [23]. The parameter $a$ characterizing the Gumbel-like form is positive for values of $C$ that are relatively large and close to the maximum given by $q_{\text{ea}}$: the distribution is negatively skewed. The parameter $a$ increases when decreasing $C$. 

\[ \text{doi:10.1088/1742-5468/2006/01/P01006} \]
and diverges at some value of $C$ signalling an approximately symmetric and Gaussian-like distribution. For still lower values of $C$ the PDF becomes positively skewed and this can be described with a negative value of the parameter $a$.

(iii) The joint PDF of local two-time correlations and linear responses follows the global $\chi(C)$ curve in the ageing regime. This means that the longitudinal fluctuations that take the points out of this ‘master’ curve become rare when the coarse-graining size increases while the transverse fluctuations along the master curve become more and more important when the waiting time increases.

In this section we compute these and similar distributions and we check whether the same features are observed in the $O(N \rightarrow \infty)$ model. For the sake of simplicity we work at $T = 0$ and we analyse the fluctuations induced by a Gaussian distribution of initial conditions keeping in mind that all calculations can be generalized to the finite temperature case. We take the $N \rightarrow \infty$ limit strictly and we do not let the ‘constraint’ $N^{-1} \sum_\alpha \phi_\alpha^2(\vec{x}, t)$ fluctuate (see [62, 63] for more details).

5.1. Coarse graining the field

The $O(N \rightarrow \infty)$ model yields provides a good description of ferromagnetic ordering. It is then worth starting by studying the distribution of the local magnetizations coarse grained within a region of volume $V_{x_0} \equiv \ell^d$ around $\vec{x}_0$. This quantity is defined as,

$$m_\ell(\vec{x}_0, t) \equiv \int \frac{d^d x}{(2\pi \ell^2)^{d/2}} \exp \left(-\frac{|\vec{x} - \vec{x}_0|^2}{2\ell^2}\right) \vec{\phi}(\vec{x}, t).$$

In terms of the Fourier transform $\vec{\phi}(\vec{k}, t)$ of the original field we find

$$m_\ell(\vec{x}_0, t) = \int \frac{d^d k}{2\pi d/2} \exp \left(i\vec{k}\cdot\vec{x}_0\right) \vec{\phi}(\vec{k}, t)e^{-k^2\ell^2/2}.$$

Using the solution of the equation of motion at $T = 0$ we find

$$m_\ell(\vec{x}_0, t) = m_\ell(t) \vec{\phi}(\vec{x}_0, t + \ell^2/2) \quad \text{with} \quad m_\ell(t) = \frac{Y(t + \ell^2/2)}{Y(t)}.$$

The coarse-grained local magnetization has the same statistical properties as the original field $\vec{\phi}(\vec{x}, t)$ but with time increased from $t$ to $t + \ell^2/2$ and amplitude reduced from 1 to $m_\ell(t)$. Namely, the fluctuations of each of its components obey a Gaussian distribution but the amplitude of the vector in $N$ space does not fluctuate at all due to the limit $N \rightarrow \infty$.

The dependence of the amplitude $m_\ell(t)$ on time $t$ and coarse-graining size $\ell$ is consistent with what one expects for a domain growth system. Firstly, if one fixes the coarse-graining size $\ell$, the amplitude approaches 1 as time $t$ is increased. So the system looks ‘more ordered’ at longer timescales. On the other hand, if the time $t$ is held fixed while the coarse-graining size $\ell$ is increased, the amplitude of the magnetization decreases meaning that the system looks ‘more disordered’ at larger length scales.

8 We do not find a non-trivial Gumbel-like distribution $P(m)$ of the amplitude of the magnetization as found for the finite volume two-dimensional $XY$ model in equilibrium in the KT phase [59]. This is again due to the $N \rightarrow \infty$ limit.
5.2. The distribution of coarse-grained two-time correlations

In the $O(N)$ model we can define local coarse-grained ‘correlations’ in the following way:

$$q\mathcal{V}_N(t, t') \equiv \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{V_x} \sum_{\vec{y}\in V_x} \phi_\alpha(\vec{y}, t) \phi_\alpha(\vec{y}, t').$$

Strictly speaking this is not a correlation function but rather a composite field. We shall use, however, both names in the following. The first sum is an average over components of the vector $\vec{\phi}$ in its internal space. Clearly, when $N = 1$ we test the single-component correlation while when $N = N$ we sum over all the components of the $\vec{\phi}$ vector. In the following we shall discuss these two limiting cases and the intermediate cases of finite $N$.

The second sum is a coarse graining in real space and it runs over a neighbouring region of the point $\vec{x}$. If $V_x = 1$ we have a strictly local quantity while for $V_x = V$ we recover the global correlation.

5.3. Local composite field

Let us start by studying the strictly local composite field

$$q_N \equiv q_{V_x=1,N}(t, t') = \frac{1}{N} \sum_{\alpha=1}^{N} \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t').$$

The PDF of $q_N$ is given by

$$p(q_N) = \frac{1}{Z_0} \int \mathcal{D}\phi \, \delta \left( q_N - \frac{1}{N} \sum_{\alpha=1}^{N} \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \right) \times \exp \left( -\frac{1}{2\Delta^2} \sum_{\alpha} \sum_{\vec{k}} \phi_\alpha^*(\vec{k}, 0) \phi_\alpha(\vec{k}, 0) \right)$$

$$= \int \frac{dn}{2\pi} e^{iqN} \frac{1}{Z_0} \int \mathcal{D}\phi \, \exp \left( -\frac{i\eta}{N} \sum_{\alpha} \phi_\alpha(x, t) \phi_\alpha(x, t') \right) \times \exp \left( -\frac{1}{2\Delta^2} \sum_{\alpha} \sum_{\vec{k}} \phi_\alpha^*(\vec{k}, 0) \phi_\alpha(\vec{k}, 0) \right)$$

$$= \int \frac{dn}{2\pi} e^{iqN} \frac{Z_\eta}{Z_0},$$

where

$$Z_\eta \equiv \int \mathcal{D}\phi \, \exp \left( -\frac{1}{2\Delta^2} \sum_{\alpha} \sum_{\vec{k}_1\vec{k}_2} \phi_\alpha^*(\vec{k}_1, 0) e^{-i\vec{k}_1\vec{x}} \mathcal{M}_\eta(\vec{k}_1, \vec{k}_2) e^{i\vec{k}_2\vec{x}} \phi_\alpha(\vec{k}_2, 0) \right),$$

with the symmetric matrix

$$\mathcal{M}_\eta(\vec{k}_1, \vec{k}_2) = \delta_{\vec{k}_1, \vec{k}_2} + \frac{i\eta\Delta^2}{N} [r(k_1, t, 0) r(k_2, t', 0) + r(k_1, t', 0) r(k_2, t, 0)].$$

(72)

Notice that the second term in $\mathcal{M}_\eta$ contains the $r(k, t_1, t_2)$ terms for the time evolution of the $k$ component. Also notice that $Z_0 = Z_{\eta=0}$.

doi:10.1088/1742-5468/2006/01/P01006

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The calculation of the function $Z$ can be done as follows. It is convenient to rescale the field $\tilde{\phi}$,

$$\tilde{\phi}(k) = \Delta^{-1} e^{i \vec{k} \vec{x}} \tilde{\phi}(\vec{k}, 0),$$

in such a way that

$$Z_\eta = \int D \tilde{\phi} \exp \left( -\frac{1}{2} \sum_\alpha \sum_{k_1, k_2} \tilde{\phi}_\alpha(\vec{k}_1, 0) M_\eta(\vec{k}_1, \vec{k}_2) \tilde{\phi}_\alpha(\vec{k}_2, 0) \right),$$

up to a trivial (independent of $\eta$) multiplicative constant coming from the change of measure. It follows that

$$\frac{Z_\eta}{Z_0} = \left( \frac{\det M_\eta}{\det M_0} \right)^{-N/2},$$

where we used that the $\alpha = 1, \ldots, N$ components are independent.

In appendix D.1 the eigenmodes of the matrix $M_\eta(\vec{k}_1, \vec{k}_2)$ defined in equation (72) are obtained; one finds two non-trivial eigenvalues $\lambda_\pm = 1 + i \eta \left[ C(t, t') \pm 1 \right]$ and $2L^d - 2$ trivial eigenvalues $\lambda = 1$. Using these results we have

$$\frac{Z_\eta}{Z_0} = \left\{ \left[ 1 + i \eta \frac{N}{N} (C(t, t') + 1) \right] \left[ 1 + i \eta \frac{N}{N} (C(t, t') - 1) \right] \right\}^{-N/2}$$

(73)

where $C(t, t')$ is the global correlation function. Thus, the PDF $p(q_N)$ is solely parametrized by the value of the global correlation function.

Lastly, let us note that it is straightforward to generalize the above result to the case of a composite field associated with two different points in space, $q_N(\vec{x}, \vec{y}) = N^{-1} \sum_{\alpha=1}^N \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{y}, t')$. One finds the same result as the one in equation (73) but with the global correlation function being replaced by the global two-point function $C(\vec{x}, \vec{y}; t, t')$.

### 5.3.1. $N$-component averaged local composite field.

When the average over all components of the $\vec{\phi}$ is considered, i.e. when $N = N$, and the $N \to \infty$ limit is also taken, equation (73) implies

$$\frac{Z_\eta}{Z_0} \to e^{-i \eta C(t, t')}$$

and

$$p(q_N) = \int \frac{d\eta}{2\pi} e^{i\eta q_N} Z_\eta Z_0 = \int \frac{d\eta}{2\pi} e^{i\eta q_N} e^{-i \eta C(t, t')} = \delta[q_N - C(t, t')] \cdot$$

As expected, the average over all internal components of the field erases all fluctuations and the local composite field is forced to take the value of the global correlation function on each site. Note that this is a special feature of the $N \to \infty$ limit. It is clear that a further coarse graining on real space will have no effect on the form of the distribution. Thus, the scaling in time is trivially dictated by the global correlation function in this case.

We emphasize that $p(q_N)$ computed above is valid in $N \to \infty$ limit. For some purposes $O(1/N)$ corrections [38] of $p(q_N)$ can be important. For example one may consider a spin glass susceptibility-like quantity $\chi_{SG} = N(\langle q_N^2 \rangle - \langle q_N \rangle^2)$ in an analogous way to the
equilibrium one. In equilibrium, the spin glass susceptibility may be defined as follows.

Consider two replicas, say A and B, coupled by an interaction term $N\epsilon q$ in the Hamiltonian where $q = (1/N) \sum_o \phi^A_o \phi^B_o$ is the overlap between the two replicas. The equilibrium spin glass susceptibility is then defined as $\chi_{SG}^{eq} = \partial \langle q \rangle_{eq} / \partial \epsilon = N(\langle q^2 \rangle_{eq} - \langle q \rangle_{eq}^2)$. Note that if $\langle q^2 \rangle_{eq} - \langle q \rangle_{eq}^2 = O(1/N)$, $\chi_{SG}^{eq}$ does not vanish in the $N \to \infty$ limit. We may expect a similar non-trivial result out of equilibrium.

5.3.2. One-component local composite field. Let us now consider the opposite limit in which we take $\mathcal{N} = 1$ and look at the distribution, $p(q_1)$, of the $x$ dependent composite field assembled from a single component of $\vec{\phi}$:

$$ q_1 \equiv \phi_1(\vec{x}, t) \phi_1(\vec{x}, t'). $$

By setting $\mathcal{N} = 1$ in equation (73)

$$ p(q_1) = \int \frac{d\eta}{2\pi} e^{im[1 + i\eta (C + 1)]^{1/2}} [1 + i\eta (C - 1)]^{-1/2}. $$

The distribution is non-Gaussian and it is a function of times only through the value of the global correlation function $C = C(t, t')$. In the above integral, the integrand has two branch points: one at $\eta = i/(1 + C)$ and the other at $\eta = -i/(1 - C)$. Performing the integral we obtain

$$ p(q_1) = \exp \left( -\frac{|q_1|}{1 + \text{sgn}(q_1)C} \right) \int_0^\infty \frac{dr}{\pi} \frac{e^{-|q_1|r}}{\sqrt{2r} + (1 - C^2)r^2} = e^{Cq_1/(1-C^2)} \frac{\pi^{1/2} K_0(1/1-C^2)}{\pi \sqrt{1-C^2}} K_0 \left( \frac{|q_1|}{1-C^2} \right), $$

with $K_0(x)$ the modified Bessel function which can be expressed as $K_0(z) = \int_1^\infty dx e^{-xz}(x^2 - 1)^{-1/2}$. This function does not depend on the dimension of space explicitly, it only depends on the form of $C$. It is sketched in figure 1 for four values of the global correlation that are given in the key.

In the special limit $C \to 1$ we find $p(q_1) = e^{-q_1/2}/\sqrt{2\pi q_1}$ for $q_1 > 0$ and $p(q_1) = 0$ for $q_1 < 0$. In the extreme limit of very separated times in which $C \to 0$, $p(q_1)$ becomes a symmetric function with respect to $q_1 = 0$ which is not, however, a delta function.

Note that this form is very similar to the result found by Fusco and Zannetti for the equilibrium overlap distribution, $P(q)$, of the mean-spherical model at zero temperature [44].

5.3.3. Finite $\mathcal{N}$-component averaged local composite field. Let us now study the distribution of finite $\mathcal{N}$-component averaged local field. The integral in $Z_\eta$ can be transformed into multiple convolutions of the result for $\mathcal{N} = 1$. With the purpose of presenting the result graphically we prefer to perform the integral explicitly. For simplicity we consider only even $\mathcal{N}$, i.e. $\mathcal{N} = 2n$ with integer $n = 1, 2, 3, \ldots$. The integrand has two simple poles at $\eta = i\mathcal{N}/(1 + C)$ and $\eta = -i\mathcal{N}/(1 - C)$. We then obtain:

$$ p(q_n) = \frac{n}{4^{n-1}(1-C^2)^{n-1}} \exp \left( -\frac{2n|q_n|}{1 + \text{sgn}(q_n)C} \right) \sum_{l=0}^{n-1} \left( \frac{4n|q_n|}{1-C^2} \right)^{n-1-l} \frac{(n-1+l)!}{(n-1-l)!l!(n-1)}, $$

doi:10.1088/1742-5468/2006/01/P01006
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Figure 1. Probability distribution function of the one-component local two-field composite operator, $\phi_1(\vec{x}, t)\phi_1(\vec{x}, t')$, for several values of the pair of times $(t, t')$ such that the global correlation $C(t, t')$ takes the values given in the key.

with average $C$ and variance $\sigma^2$ given by

$$
\sigma^2 = q_+^2 + q_-^2 - C^2,
$$

$$
q_{\pm}^2 = \frac{n}{4^{n-1}} \left( \frac{1 \pm C}{2n} \right)^3 \sum_{l=0}^{n-1} \frac{(n - 1 + l)!}{(n - 1 - l)!!(n - 1)!} \left( \frac{2}{1 \mp C} \right)^{n-1-l} \Gamma(n - l + 2),
$$

where $\Gamma(x)$ is the gamma function. Again, we see that the distribution function is parametrized solely by the global correlation function $C = C(t, t')$.

Although the mean value of $p(q_N)$ is independent of $N$ and identical to the global correlation $C$, the functional form of this PDF depends strongly on $N$. In figure 2 we show the functional form for six values of the number of components $N$ given in the key and fixed global correlation $C$. It can be noted that for relatively small $N$, the position of the peak is different from $C$. As $N$ increases the position of the peak approaches $C$ and the width of the peak shrinks in such a way that the PDF becomes the delta function $\delta(q - C)$ obtained in section 5.3.1 in the $N = N \rightarrow \infty$ limit.

It is interesting to study the form of these PDFs in more detail. Figure 3 shows $p(q_N)$ against $x = (q_N - \langle q_N \rangle)/\sigma_{q_N}$. The number of components is $N = 4$ and different curves correspond to several values of the global correlation given in the key. The plot is on a double-logarithmic scale. The curves are positively skewed with the right tail of the distribution being approximately independent of the value of $C$ while the left one is not.

doi:10.1088/1742-5468/2006/01/P01006 31
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Figure 2. Probability distribution function of the $N$-component averaged local two-field composite operator, $q_N \equiv N^{-1} \sum_{\alpha=1}^{N} \phi_{\alpha}(\vec{x}, t)\phi_{\alpha}(\vec{x}, t')$, for several values of the number of components $N$ given in the key, at fixed global correlation $C(t, t') = 0.5$.

In figure 4 we compare the form of $p(q_N)$ to a Gaussian $e^{-x^2/2\pi}$ and a Gumbel curve with positive parameter $a$ in such a way to make it positively skewed. The normalized Gumbel distribution with mean zero and variance 1 is given by

$$
\Phi_a(x) = \frac{|\alpha|}{\Gamma(a)} e^{a \log a} e^{a(x-x_0) - e^{a(x-x_0)}},
$$

with

$$
\alpha = \sqrt{\Psi'(a)} \quad \text{and} \quad \alpha x_0 = \log a - \Psi(a),
$$

where $\Gamma(x)$ is the gamma function and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. For this intermediate value of $N$ the PDF is clearly not Gaussian. The right tail is well fitted with the Gumbel form while the left tail is not.

5.3.4. Effect of coarse graining the ‘correlation’. With a similar analysis one shows that coarse graining has no effect if $V_x \ll [L(t')^d, L(t)^d]$ while for $V_x \gg [L(t')^d, L(t)^d]$ the probability distribution function becomes a Gaussian as in the case in which we averaged over all components of the field. Consider again the case $N = 1$, but some general $V_x = \ell^d$:

$$
q_{V_x} \equiv q_{V_x, N=1}(\vec{x}, t, t') = \frac{1}{V_x} \sum_{\vec{y} \in V_x} \phi_{\alpha}(\vec{y}, t) \phi_{\alpha}(\vec{y}, t').
$$
Fluctuations in the coarsening dynamics of the $O(N)$ model

Figure 3. Probability distribution function of the coarse-grained $N$-component local two-field composite operator, $q_N = N^{-1} \sum_{a=1}^N \phi_a(x,t)\phi_a(x,t')$, with $N=4$ and several values of the pair of times $(t, t')$ such that the global correlation $C(t, t')$ takes the values given in the key. The plot is on a double-logarithmic scale and the $x$ axis has been put into the normal form to compare the forms of the PDF for different values of the global $C$.

The PDF can be computed similarly as before but now with

$$M_\eta(k_1, k_2) = \delta_{k_1, k_2} + \frac{i \eta \Delta^2}{V_x} \sum_{\vec{y} \in V_x} \left[ r(k_1, t, 0) e^{ik_1 \vec{y}} r(k_2, t', 0) e^{-ik_2 \vec{y}} + r(k_1, t', 0) e^{ik_1 \vec{y}} r(k_2, t, 0) e^{-ik_2 \vec{y}} \right].$$

The eigenmodes of this matrix are studied in appendix D.2. Diagonalizing is not easy for the general case but the following two limiting cases can be considered.

- $\ell \ll L(t), L(t')$. The eigenvalues are the same as in the $V_x = 1$ case: one finds two non-trivial eigenvalues $\lambda_\pm = 1 + i \eta \left[ C(t, t') \pm 1 \right]$ and $2V_x - 2$ trivial eigenvalues $\lambda = 1$. Thus, the PDF is the same as for $V_x = 1$ (see figure 1).

- $\ell \gg L(t) \sim L(t')$. For simplicity we consider $L = L(t) \sim L(t')$. Two non-trivial eigenvalues $\lambda_\pm \approx 1 + i \eta (L/l)^d \left( C(t, t') \pm 1 \right)$ and $2V_x - 2$ trivial eigenvalues $\lambda = 1$ are obtained. Note that this is equivalent to the case $V_x = 1$ if one substitutes

$N \sim (\ell/L)^d$. 

doi:10.1088/1742-5468/2006/01/P01006
Fluctuations in the coarsening dynamics of the $O(N)$ model: $C=0.5$

Gaussian Gumbel (maximum) $a=2.3$

$1e^{-06}$ $1e^{-05}$ $.1e^{-3}$ $.1e^{-2}$ $.1e^{-1}$ $1$ $10^{-8}$ $10^{-6}$ $10^{-4}$ $10^{-2}$ $10^{0}$

$P(q)$

$N=4$ $C=0.5$

Figure 4. Log–log probability distribution function of the coarse-grained $N$-component local ‘correlations’, $q_N$ with $N = 4$ and $C = 0.5$, compared to a Gaussian and to a Gumbel fit.

Thus, the PDF is the same as those corresponding to composite fields averaged over this number of field components (see figures 2–4).

5.4. The distribution of coarse-grained linear responses

The distribution of local linear responses is surprisingly trivial in quasi-quadratic systems such as the $O(N \to \infty)$ model. Indeed, the linear response of each thermal run in Fourier space is given by

$$ \frac{\delta \phi_{\alpha}(\vec{k}, t)}{\delta h_{\beta}(-\vec{k}'; t')} \bigg|_{h=0} = \delta_{\alpha\beta} \theta(\vec{k}; t, t') \delta(t-t'). $$

This implies

$$ \frac{\delta \phi_{\alpha}(\vec{x}, t)}{\delta h_{\beta}(\vec{x}', t')} = \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} r(k; t, t') = R_{\alpha\beta}(t, t'), $$

i.e. a uniform result in space that is just equal to the global value. Again, this is independent of the spatial dimension $d$.

5.5. The joint distribution of local correlations and responses

Using the results above one concludes that the projection of the joint PDF on the $(C_x, \chi_x)$ plane at fixed pair of times $(t, t')$ such that they fall in the ageing regime (this is the
extended FDT plot studied in [22]) is such that there are no fluctuations in the vertical direction while there are in the horizontal one.

### 5.6. Summary

In the strict $N \rightarrow \infty$ limit in which we have not taken into account fluctuations of the constraint $N^{-1} \sum_\alpha \phi^2_\alpha(\mathbf{x}, t)$, we found similarities and differences with the distributions of coarse-grained local correlations and responses found in glassy systems. Let us discuss the points enumerated in the introduction to this section in detail.

(i) In all cases the PDFs of local composite fields in the ageing regime depend on times only through the global correlation. This property appears to be independent of there being time reparametrization invariance.

(ii) The PDFs of one-component composite fields are definitely non-Gaussian but different from the ones observed in the 3D EA model [21, 22] and kinetically constrained particle systems on the lattice [23]. To understand the role of the large $N$ limit one should compare the above results to, for example, the same PDFs in the XY problem. Averaging over components or over real space washes out the weight on negative values ($q < 0$) just as found in the spin models. For finite $N$ or for coarse-graining boxes that do not go beyond the domain length, the PDFs remain, though, positively skewed for all values of $C$ even those corresponding to times that are close to each other (see figure 3). We have also checked whether the distributions of $q_M$ can be approximated by a Gumbel-like form with negative parameter. We find that while the tail on the right is quite well described with this functional form, the tail on the left is not (see figure 4). In the large coarse-graining volume, $\ell \gg [L(t), L(t')]$, or averaging over a diverging number of components, $N = N \rightarrow \infty$, the PDF becomes a delta function, $\delta(q - C)$.

(iii) There are no fluctuations of the linear responses. This is intimately related to the quasi-quadratic nature of the model in the limit $N \rightarrow \infty$. This result is clearly different from what is found in glassy models, in which the local response functions do fluctuate form site to site though constrained to follow the global $\chi(C)$ curve. In the O($N$) model the projection of the joint PDF of local correlations and responses also follows the global $\chi(C)$ curve but in a trivial manner, since the local responses take a single value.

It would be interesting to study whether these results are modified by $1/N$ corrections when the constraint is allowed to fluctuate and yields an additional contribution to the linear response [62].

### 6. Four-point correlation function

A coarsening system is one in which the growing length is easily identified as the typical domain length. A scaling theory then predicts that all correlations should depend on distance and on times only through the value of the typical domain length. This is explicitly realized by the O($N$) model.

In spin glasses and structural glasses the observation of such a growing length has been elusive. A growing correlation length in the supercooled liquid has been extracted...
from the analysis of the connected correlation of fluctuating local composite operators in a number of model systems [13, 15]. The analysis of numerical simulations of several models as well as some experiments indicate that this length takes very small values, of the order of a few nanometres in the supercooled liquid. A summary of these results appeared recently in [15].

In an out-of-equilibrium system, such as the problem at hand, this ‘four-point’ correlation function is naturally defined as [22, 16]

\[
C_4(\vec{x}, \vec{x}'; t, t') \equiv \frac{1}{V} \int d^d x \left[ \langle \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \rangle \right]_{ic}
- \frac{1}{V^2} \int d^d x \left[ \langle \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \rangle \right]_{ic} \int d^d x' \left[ \langle \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \rangle \right]_{ic}
\]

Note that this quantity is nothing but the connected spatial correlation function of the composite field \( q_\alpha \equiv \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \) (see section 5). Since noise and initial condition averaged quantities are expected to be invariant under translations of the space coordinates, this quantity should be equal to

\[
C_4(\vec{r}; t, t') \equiv \int d^d r \left[ \langle \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \rangle \right]_{ic}
- \frac{1}{V} \int d^d x \left[ \langle \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \rangle \right]_{ic} \int d^d x' \left[ \langle \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \rangle \right]_{ic}
\]

with \( \vec{r} \equiv \vec{x} - \vec{x}' \). \( C_4(\vec{x}, \vec{x}'; t, t') \) measures the probability that similar decorrelations taking place between \( t' \) and \( t \) occur at a spatial distance \( \vec{r} \) in the sample.

The volume integral of \( C_4 \) defines the quantity

\[
\chi_4(t, t') \equiv \int d^d r \ C_4(\vec{r}; t, t')
\]

that is loosely called a ‘susceptibility’ advocating the use of a fluctuation-dissipation theorem to relate the correlation of composite operators to a linear response. When the operators and the perturbations are composite ones depending on two (or more) times this is however highly non-trivial. Some examples have been exhibited in [56]. In particular, \( C_4 \) is not equal to the response of the composite observable \( \left[ \langle \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}, t') \rangle \right]_{ic} \) to an infinitesimal field that couples linearly to \( \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \) (see appendix E) as one would naively propose. In the low temperature phase, where equilibrium dynamics is lost, the relation between spontaneous and induced fluctuations is still more complicated due to the fact that these are not determined by the equilibrium measure.

With the aim of comparing to the results found in supercooled liquids we study the behaviour of \( \chi_4 \) during coarsening. The four-point correlation function equation (75) is easily obtained using the solution to the equation of motion, equation (12). Again, for simplicity we work at \( T = 0 \) and we find

\[
C_4(\vec{r}; t, t') = \left[ \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}', t) \right]_{ic} \left[ \phi_\alpha(\vec{x}, t') \phi_\alpha(\vec{x}', t') \right]_{ic}
+ \left[ \phi_\alpha(\vec{x}, t) \phi_\alpha(\vec{x}', t') \right]_{ic} \left[ \phi_\alpha(\vec{x}', t) \phi_\alpha(\vec{x}', t') \right]_{ic}
= e^{-(\rho/L(t))^2} e^{-(\rho/L(t'))^2} + C^2(t, t') e^{-2(\rho/L(t+t'))^2}
\]

(76)

where \( r = |\vec{x} - \vec{x}'| \), \( C(t, t') \) is the global correlation function and \( L(t) \propto \sqrt{t} \) is the usual domain size. The first term is a rather trivial contribution since it is just the product of
the (average) equal-time spatial correlation functions at \( t \) and \( t' \). The last term depends on the domain length evaluated at the sum of the two times involved, \( L(t + t') \). Note that if ‘reciprocity’ holds the last term becomes \([\phi_\alpha(\vec{x}, t)\phi_\alpha(\vec{x}', t')]^2\). In the ageing regime the length scales \( L(t) \) and \( L(t') \) are of the same order. Moreover, since \( L(t) \sim t^{1/2} \), \( L(t + t') \) is also of the same order. Using \( t' = \lambda t \) with \( \lambda \in [0, 1] \), \( L(t') = \lambda^{1/2}L(t) \) and \( L(t + t') \sim (1 + \lambda)^{1/2}L(t) \). Thus, for distances \( r \) of the order of \( L(t) \) all terms contribute. Note that \( C_4(\vec{r}; t, t) \) does not vanish.

Using equation (20) for the global correlation in the ageing regime we note that \( C_4(\vec{r}; t, t') \) can be put into the scaling form

\[
C_4(\vec{r}; t, t') = f_{C_4} \left( \frac{L(t)}{L(t')}, \frac{r}{L(t')} \right) = \tilde{f}_{C_4} \left( \frac{t}{t'}, \frac{r}{L(t')} \right)
\]

as expected from simple scaling arguments and found for the one-dimensional Ising chain [16].

From expression (76) we easily compute \( \chi_4(t, t') \):

\[
\chi_4(t, t') \propto L^d(t') f_{\chi_4} \left( \frac{t - t'}{t'} \right) \quad \text{with} \quad f_{\chi_4}(x) = \left( \frac{1 + x}{2 + x} \right)^{d/2}.
\]

This function has the form shown in figure 5. It does not have a maximum as a function of \( t - t' \) but it monotonically increases towards a finite \( t' \) dependent asymptote. In this respect the behaviour is rather different from what has been found in the supercooled liquid phase of a number of glassy systems [13, 15] and in the coarsening foam studied in [16].

It is interesting to analyse separately the behaviour of the second term in (76). If one assumes, based on scaling arguments, that

\[
C_{\text{ag}}(t, t') \sim \left( \frac{L(t')}{L(t)} \right)^{1/2},
\]

doi:10.1088/1742-5468/2006/01/P01006
i.e. that the very last decay \((t \gg t')\) is characterized by the \(\lambda\) exponent \([57,39]\), the behaviour of this term at very long time differences depends on whether \(\lambda\) is larger than or equal to \(d/2\), the lower bound conjectured by Fisher and Huse \([9]\). When \(\lambda = d/2\), as in the \(O(N)\) model, this term is also finite and contributes to the asymptotic value of \(\chi_4\). For other systems in which \(\lambda\) is larger than \(d/2\) this terms vanishes asymptotically (as implicitly assumed in \([16]\)).

Let us mention that the alternative definition of \(C_4(t, t')\) proposed in \([16]\) also has a finite asymptotic \((t \to \infty)\) value in the \(O(N)\) model (whereas it vanishes in the Ising chain). This is due to the fact that the last added term is equal to the second term discussed in the previous paragraph and does not vanish.

One could also define a connected spatio-temporal correlation function of the original field \(\phi_\alpha(\vec{x}, t)\). Due to the factorization rules for \(N \to \infty\), these quantities vanish. However, there are \(O(1/N)\) corrections which yield essential contributions to some related integral susceptibilities \([62]\).

We can now compare to what has been observed in numerical simulations of the 3D Edwards–Anderson model \([22]\) using a slightly different expression for the four-point correlation that differs from (75) just in a normalization. In \([22]\) \(C_4\) was normalized to be one at \(r = 0\) for all times. For the \(O(N)\) model this normalization factor is \(1 + C^2(t, t')\). Thus, the space integral of the \(O(N)\) model normalized four-point correlation also approaches a finite limit when \(t - t' \to \infty\) and \(t'\) is held fixed.

The normalized four-point correlation in the 3D Edwards–Anderson model was rather well described with the form \(e^{-\gamma/t\xi(t,t')}\). Even though the space and time dependence in the \(O(N)\) model is more complicated than a simple exponential decay, the qualitative behaviour of \(\xi(t, t')\) in \([22]\) is similar to that of \(\chi_4(t, t')\) for the \(O(N)\) model in that \(\xi(t, t')\) increases with both \(t'\) and \(t - t'\).

Finally, let us compare the four-point correlation function \(C_4\) to the integrated response of the composite field, \(\phi_\alpha(\vec{x}, t)\phi_\alpha(\vec{x}, t')\) to a composite perturbation \(h_\alpha(\vec{x}', t', t'')\) \([11]\). In appendix E we show

\[
\left[ \frac{\delta \phi(\vec{x}, t)\phi(\vec{x}, t')}{\delta h(\vec{x}', t', t'')} \right]_{ic} = R(\vec{r}; t, t'')C(\vec{r}; t', t'')\theta(t - t'') + R(\vec{r}; t, t'')C(\vec{r}; t', t'')\theta(t - t')
+ R(\vec{r}; t', t'')C(\vec{r}; t', t'')\theta(t' - t'') + R(\vec{r}; t', t'')C(\vec{r}; t, t'')\theta(t' - t').
\]

It is clear that this is not simply related to \(C_4\).

In summary, we found that a model undergoing coarsening has a \(\chi_4(t, t')\) that, as expected, depends on times only through \(L(t)\) and \(L(t')\), but does not decay to zero at long time differences. We conclude that the existence of a maximum in \(\chi_4(t, t')\) is not a necessary condition to have a growing correlation (though, of course, a maximum does provide evidence for a growing correlation length).

### 7. Conclusions

Neither the dynamic equations of the slow (coarsening) contributions to the global correlation and response nor their effective action in the large \(N\) \(O(N)\) model are invariant under generic time reparametrizations. This symmetry is reduced to uniform time rescalings \(t \to \zeta t\). The analysis in section 4.3.1 suggests the advanced and retarded

\[\text{doi}: 10.1088/1742-5468/2006/01/P01006\]
scaling dimensions of the global correlation and response, \( \Delta^A_C = \Delta_C^R = 0 \) and \( \Delta^A_R = 0 \), \( \Delta_R^R = d/2 \), respectively, and similarly for the corresponding fluctuating fields.

The breakdown of time reparametrization invariance seems to be intimately related to the absence of a finite or well-defined effective temperature for \( d > 2 \) and \( d = 2 \), respectively. Indeed, the retarded scaling dimension \( \Delta^R_R = d/2 \) for \( d > 2 \) implies that the fluctuation-dissipation ratio vanishes asymptotically in the low temperature phase. Instead, \( \Delta^R_R = 1 \) for \( d = 2 \) implies that the fluctuation-dissipation ratio takes a non-trivial \( L(t')/L(t) \) dependent form but the evolution occurs in a single ageing scale in which the correlation itself varies as a function of this ratio. This is inconsistent with the natural requirement of having a single value of the effective temperature per correlation scale.

If we were to use the remaining time rescaling symmetry to characterize the fluctuations of local correlations of the \( O(N) \) model when \( N \to \infty \), as we did when we used time reparametrization invariance as a guideline to characterize fluctuations in glassy systems [20, 21], we should introduce the spatial dependence by rescaling time with a space dependent parameter: \( t \to \eta_x t \). However, a simple multiplicative rescaling of time disappears from the correlations:

\[
C_x(t, t') \sim f_C \left( \frac{\sqrt{\eta_x t'}}{\sqrt{\eta_x t}} \right) = f_C \left( \frac{\sqrt{t'}}{\sqrt{t}} \right) = f_C \left( \frac{L(t')}{L(t)} \right).
\]

This means that no such fluctuations are generated. Therefore spatio-temporal fluctuations in the \( O(N) \) model have a different origin, which is also expected from the fact that the dynamical exponent \( z = 2 \) for the \( O(N) \) model while it should be infinite in glassy systems.

We analysed several distribution functions with the aim of identifying similarities and differences with the ones generated by time reparametrization invariance. For simplicity we focused on the zero-temperature dynamics and we analysed the fluctuations induced by random initial conditions. We concentrated in times such that the dynamics is in the coarsening—ageing—regime. Let us now summarize and discuss our findings.

Each \( \phi_\alpha(\vec{x}, t) \), with \( \alpha \) any component of the \( N \)-dimensional vector \( \vec{\phi} \), obeys a Gaussian PDF. The local one-component composite field, \( \phi_\alpha(\vec{x}, t)\phi_\alpha(\vec{x}, t') \), has a non-Gaussian distribution. We derived the functional form of this PDF and we showed how it crosses over to a delta function under coarsening over a sufficient large volume, of linear size larger than the typical domain lengths, \( \ell \gg L(t'), L(t) \). We also found that the two-time observable made of a sum over a number, \( \mathcal{N} \), of components of the composite field has a similar behaviour to the distribution of the one-component quantity coarse grained over a volume of linear size \( \ell^d \sim L^d(t') \mathcal{N} \) when \( L(t) \) and \( L(t') \) are of the same order.

In all these cases, at zero temperature the PDF of local composite fields, \( q \), scales in time just as the global correlation itself; that is to say, it is a function of the ratio between the two characteristic scales \( L(t') \) and \( L(t) \):

\[
p(q; t, t') = p(q; C(t, t')) = p \left(q; f_C \left( \frac{L(t')}{L(t)} \right) \right).
\]

In [20]–[23] we argued that uniform time reparametrization invariance and the simplest choices of effective action for the local reparametrizations, \( h_x(t) \), imply this kind of scaling and, using numerical simulations, we found it in the 3D Edwards–Anderson model [21, 22] and a kinetically constrained lattice gas [23]. The solution of the \( O(N) \) model when

doi:10.1088/1742-5468/2006/01/P01006
$N \to \infty$ shows that this property is not unique to models with time reparametrization invariance.

The form of the PDF of these local two-time quantities is not the Gumbel-like form that we argued should describe the fluctuations of local correlations of spin-like variables that are associated with global time reparametrization. In particular, before any coarse graining—or even after averaging over a small number of components or a small coarse-graining box—the PDF has a peak at very small values of the argument, $q \sim 0$. Under further coarse graining the peak moves towards positive values of $q$ until reaching a Gaussian form centred on the average—global—value $C$, that eventually becomes a delta function. Note that the form of the PDFs does not depend on the dimension of space explicitly—it does only through $C$.

We may then conjecture that the reason for finding a strong weight at small values of $q$ is the continuous character of the order parameter and its large dimensionality. Indeed, a peak at small values of the two-time composite field should be present in the PDF of $q$ with $V_x \ll L^d$ and $N \ll N$ for all models with a continuous order parameter and a spherical constraint. But this peak should not be necessarily unique. Indeed, preliminary numerical simulations of the dynamics of the 2D XY model starting from a random initial condition and in the low temperature phase show that the PDF of, say, the horizontal component of the local composite field has a second peak at one, when the two times are not very far away and the global correlation takes a large value. The fate of the two peaks, and thus of the full PDF, under coarse graining needs to be analysed in more detail but it is not excluded that it may then take a form and evolution similar to the one observed in the 3D Edwards–Anderson and kinetically constrained lattice gas. Note, however, that the 2D XY model is critical in the full low temperature phase; its non-equilibrium dynamics is then typical of a critical point with a multiplicative separation of timescales and an ageing regime that eventually disappears in the long waiting time limit [64]. While in the ageing regime, this model has the very appealing feature of having a finite integrated response. It belongs to yet another class of models and it is then a very interesting case to study in the context of our discussion.

The PDF of local linear responses is deceptively trivial in the quasi-quadratic large $N$ O($N$) model: these quantities do not fluctuate at all and are just identical to the global value. We do not expect this result to survive in such a trivial manner when including $1/N$ corrections or for other coarsening problems that are not (almost) quadratic. In particular, if full time reparametrization invariance is broken there is no obvious reason why the joint probability distribution of local responses and correlations should follow the global $\chi(C)$ parametric curve between integrated response and correlation. This is a problem that deserves to be addressed analytically and/or numerically in other coarsening models.

We computed the four-point correlation, $C_4$, that is usually used to identify a growing correlation length in supercooled liquids, now during coarsening. Not surprisingly we found that it satisfies a scaling relation in which times enter only through the typical domain length, $L$. Contrary to what found in supercooled liquids and glasses, the integral over space of $C_4$ does not vanish at very long time differences, $t - t' \to \infty$ for any fixed $t'$. The reason for this is the fact that in coarsening systems the spatial correlation, $C(r, t)$, does not vanish. The same feature was highlighted in [16] in the context of the ferromagnetic Ising chain. We also stressed the fact that this quantity is not trivially related to a susceptibility (see [56,14] for a similar discussion).
It is interesting to compare the pure time transformations studied in this paper to the common space–time invariance of domain growth [24]. The exact solution of the O(N) model is invariant, in the long times and large scales limit, under simultaneously rescaling of time and space, and the slow part of the dynamic action is invariant under the related renormalization group transformation (see section 4.4). However, it is not this space–time invariance that is the relevant symmetry if one is interested in fluctuations within a given domain. One should consider separations r that are held fixed while the long time limit is taken. More precisely, one should consider fixed ratios t/t′ while L(t) → ∞, and thus r/L(t) → 0. It is in this limit that reparametrization invariance should be investigated, and there are a number of issues that one must consider specifically in the case of the O(N) model. First, reparametrization invariance cannot be an exact symmetry of the solution to the O(N)—or any other similar dynamic problem—since a particular function h(t) is bound to be chosen by the evolution. It may only arise as an approximate invariance in the asymptotic limit in which the non-invariant terms—that act as ‘pinning fields’ and fix the time scaling h(t)—become less and less important. This is indeed what happens in mean-field disordered models of the p-spin type and, we argued [20], in the 3D Edwards–Anderson spin glass. Second, we showed in this paper that reparametrization invariance does not develop in the O(N → ∞) model, only a smaller symmetry, simple timescale invariance, does. We arrived at these results by studying the equations of motion for the global C(t, t′) and R(t, t′) and the action for the slow fluctuating fields. Similar results would be obtained for the two-space-point correlation C(r; t, t′) and response R(r; t, t′), when r is held fixed while the long time limit is taken.

Let us finally stress the main issue arising from this study, i.e. the conjecture that an extreme violation of the fluctuation-dissipation theorem is intimately related to the breakdown of time reparametrization invariance at long times in general. If this is correct, systems with a finite or an asymptotically infinite ‘effective temperature’ belong to different ‘universality’ classes, as far as non-equilibrium fluctuations are concerned [65]. It would be interesting to put this conjecture to the test in other solvable models. In particular, by comparing to similar fluctuations in the XY model one should be able to identify the peculiar features due to the N → ∞ limit. The special d = 2 case should be particularly interesting. Another route is to analyse other coarsening systems with a discrete order parameter: one then should be able to disentangle the features that are due to X → 0 from those that are due to the continuous character of the field.

Acknowledgments

We thank G Biroli and M Picco for very useful discussions. LFC is a member of the Institut Universitaire de France. This research was supported in part by NSF grants DMR-0305482, DMR-0403997, and INT-0128922 (CC), an NSF-CNRS collaboration, the ACI-France ‘Algorithmes d’optimization et systèmes désordonnés quantiques’, the STIPCO European Community Network, and NSF Grant No. PHY99-07949 (LFC). H Yoshino acknowledges financial support from the Japanese Society of Promotion of Science and CNRS.

Appendix A. The equation of motion for the global linear response

In this appendix we show how to obtain the equations of motion for the correlation and response, expressed in terms of R(t, t′) and C(t, t′) themselves, starting from the exact
expressions for $R(t,t')$ and $C(t,t')$ obtained from the equations of motion for the field $\tilde{\phi}(t)$.

The exact solution for the correlation and response follows from the self-consistent solution of equations (12):

$$C(t,t') = Y^{-1}(t) Y^{-1}(t') \left[ \Delta^2 \langle \langle e^{-\epsilon_k(t+t')} \rangle \rangle + 2T \int_0^{\min(t,t')} dt'' \langle \langle e^{-\epsilon_k(t+t'-2t'')} \rangle \rangle Y^2(t'') \right]$$

(A.1)

$$R(t,t') = Y^{-1}(t) Y(t') \langle \langle e^{-\epsilon_k(t-t')} \rangle \rangle \theta(t-t'),$$

(A.2)

where $\langle \langle f(k) \rangle \rangle = \int (d^d k/(2\pi)^d) f(k)$. $\epsilon_k \geq 0$ is the dispersion; in the usual O($N$) model it is just $\epsilon_k = k^2$ but we keep here a more general notation in which $k$ is just a labelling for the states.

For $t > t'$ we can write

$$\frac{\partial C(t,t')}{\partial t} = -z(t) C(t,t') + Y^{-1}(t) Y^{-1}(t') \left[ \Delta^2 \langle \langle -\epsilon_k e^{-\epsilon_k(t+t')} \rangle \rangle + 2T \int_0^{\min(t,t')} dt'' \langle \langle -\epsilon_k e^{-\epsilon_k(t+t'-2t'')} \rangle \rangle Y^2(t'') \right]$$

(A.3)

$$\frac{\partial R(t,t')}{\partial t} = -z(t) R(t,t') + Y^{-1}(t) Y(t') \langle \langle -\epsilon_k e^{-\epsilon_k(t-t')} \rangle \rangle \theta(t-t'),$$

(A.4)

where $z(t) = \partial / \partial t \ln Y(t)$.

In order to express equations (A.3) and (A.4) in terms of Rs and Cs, all one needs to do is express the right-hand side of these equations in terms of convolutions of Rs and Cs. The first step is to express the function

$$D(t) = \langle \langle -\epsilon_k e^{-\epsilon_k t} \rangle \rangle \theta(t) = \int_0^\infty d\epsilon \, g(\epsilon) \, (-\epsilon) \, e^{-\epsilon t} \, \theta(t)$$

in terms of convolutions of the function

$$G(t) = \langle \langle e^{-\epsilon t} \rangle \rangle \theta(t) = \int_0^\infty d\epsilon \, g(\epsilon) \, e^{-\epsilon t} \, \theta(t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\omega),$$

where

$$\tilde{G}(\omega) \equiv \int_0^\infty d\epsilon \, \frac{g(\epsilon)}{\epsilon - i\omega}$$

(A.5)

and $g(\epsilon)$ is the density of states with $\epsilon = \epsilon_k$. In other words, we basically need to cast

$$D(t) = \sum_{n=1}^{\infty} A_{n-1} \underbrace{G * G * \ldots * G}_{n} (t).$$

(A.6)

We start by writing

$$G * G * \ldots * G (t) = \int_{-\infty}^\infty d\tau_1 \ldots \int_{-\infty}^\infty d\tau_n G(t - \tau_1) \ldots G(\tau_{n-1})$$

$$= \int_0^\infty d\epsilon_1 \ldots \int_0^\infty d\epsilon_n \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \prod_{a=1}^n \frac{g(\epsilon_a)}{\epsilon_a - i\omega}$$

$$= n \int_0^\infty d\epsilon g(\epsilon) e^{-\epsilon t} \left[ P \int_0^\infty d\epsilon' \frac{g(\epsilon')}{\epsilon' - \epsilon} \right]^{n-1} \theta(t).$$

$$\text{doi:10.1088/1742-5468/2006/01/P01006}$$
Thus, in short, we have
\[ G \ast G \ast \cdots \ast G_n(t) = n \int_0^\infty d\epsilon \ g(\epsilon) \ e^{-\epsilon t} \ [h(\epsilon)]^{n-1} \theta(t), \]
where the function \( h(\epsilon) \) is defined as
\[ h(\epsilon) = P \int_0^\infty d\epsilon' \frac{g(\epsilon')}{\epsilon' - \epsilon}, \]  
(A.7)
where \( P \) denotes the principal value. Next, let us expand \( \epsilon \) as a function of \( h(\epsilon) \):
\[ \epsilon = h^{-1} \circ h(\epsilon) = \sum_{n=0}^\infty a_n [h(\epsilon)]^n, \quad \text{with} \quad a_n = \frac{1}{n!} \left. \frac{d^n h^{-1}}{dz^n} \right|_{z=0}. \]
Therefore, we can write
\[ D(t) = -\theta(t) \int_0^\infty d\epsilon \ g(\epsilon) \ e^{-\epsilon t} \epsilon = -\theta(t) \int_0^\infty d\epsilon \ g(\epsilon) e^{-\epsilon t} \sum_{n=1}^\infty a_{n-1} [h(\epsilon)]^{n-1} \]
\[ = -\sum_{n=1}^\infty \frac{a_{n-1}}{n} \ G \ast G \ast \cdots \ast G_n(t), \]
which is exactly equation (A.6), with \( A_n = -a_n/(n + 1) \).

Let us show how one can write, for example, an integral–differential equation for \( R(t, t') \) (equation (A.4)) using this result. First, notice that from equation (A.2)
\[ G(t - t') = \frac{Y(t)}{Y(t')} \ R(t, t'). \]
Hence,
\[ G \ast G \ast \cdots \ast G_n(t - t') = \frac{Y(t)}{Y(t')} \ R \ast R \ast \cdots \ast R_n(t, t'), \]
which allows us to write the last term in equation (A.4) as
\[ \frac{Y(t')}{Y(t)} D(t - t') = \sum_{n=1}^\infty A_{n-1} \ R \ast R \ast \cdots \ast R_n(t, t'). \]
Thus finally we have
\[ \frac{\partial R(t, t')}{\partial t} = -z(t) \ R(t, t') + \sum_{n=0}^\infty A_n \ R \ast R \ast \cdots \ast R_n(t, t'). \]  
(A.8)

Finally, let us note that the above equations can be easily extended to the describe the evolution of the two-time two-point correlation function \( C(r; t, t') = C(\bar{x}, \bar{y}; t, t') \) and response function \( R(r; t, t') = R(\bar{x}, \bar{y}; t, t') \), with \( r = |\bar{x} - \bar{y}| > 0 \). One can easily verify that the generalization can be done by formally replacing the density of states \( g(\epsilon) \) by
\[ g(\epsilon; r) \equiv c g(\epsilon) \int_0^1 dy (1 - y^2)^{(d-3)/2} \cos(\sqrt{\epsilon r} y) \]  
(A.9)
for \( d \geq 3 \). Here \( c^{-1} = \int_0^1 dy (1 - y^2)^{(d-3)/2} = 2^{(d-2)} B((d - 1)/2, (d - 1)/2) \) is the normalization constant. For \( d = 1 \) and \( 2 \), one simply has to use \( g(\epsilon) \cos(\sqrt{\epsilon r}) \) and \( g(\epsilon) \int_0^\pi d\theta \cos(\sqrt{\epsilon r} \cos(\theta))/\pi \), respectively. The closed set of equations of motion for \( C(r; t, t') \) and \( R(r; t, t') \) are series expansions with coefficients \( A_n(r) \) which now depend on the distance \( r \) explicitly.
Appendix B. The ageing limit of the equations of motion

To obtain the equations of motion for the response in the ageing limit, one substitutes in equation (45) (or equation (A.8))

\[ R(t, t') = R_{st}(t - t') + R_{ag}(t, t') \]

and one uses that the stationary response decays to zero on timescales in which the ageing component remains roughly constant (see [54] for a detailed explanation of this separation). For example, in the term

\[ \int dt'' R(t, t'') R(t'', t') \sim \int_{t'}^{t} dt'' R_{ag}(t, t'') R_{ag}(t'', t') + 2 \chi_{st} R_{ag}(t, t') \]

with

\[ \chi_{st} = \int_{t'}^{t} dt'' R_{st}(t'' - t') = \int_{t}^{t'} dt'' R_{st}(t - t''). \]

\( t^+ \) represents the ‘end’ of the stationary regime close to \( t' \) while \( t^- \) denotes the ‘end’ of the stationary regime close to \( t \). Notice that if we start from a term with one time integral (the term with coefficient \( A_1 \)), then we collect in the ageing regime, in addition to the term with one time integral, a term with no time integrals. Similarly, starting from a term with \( n \) integrals (the term with coefficient \( A_n \)), we would generate in the ageing limit terms with \( n, n - 1, \ldots, 0 \) integrals. We can collect all these terms into a new series

\[ \sum_{n=0}^{\infty} \tilde{A}_n \int dt_n \int dt_{n-1} \cdots \int dt_1 R_{ag}(t_1)R_{ag}(t_2) \cdots R_{ag}(t_n, t'), \]

where the coefficients \( \tilde{A}_n \) are related to the original \( A_n \) by a simple combinatorial argument, that goes as follows. Terms with \( n \) time integrals and \( n + 1 \) \( R_{ag} \)s are obtained starting with terms with \( p \geq n \) integrals and \( p + 1 \) \( R_{st} \)s, where \( p - n \) of the \( R_{ag} \)s are replaced by \( R_{st} \)s and the remaining \( n + 1 \) \( R_{st} \)s are replaced by \( R_{ag} \)s. This allows us to write

\[ \tilde{A}_n = \sum_{p=n}^{\infty} A_p (p+1) (p-n) \chi_{st}^{p-n} = \frac{1}{(n+1)!} \left( \frac{d}{d\chi_{st}} \right)^n \sum_{p=0}^{\infty} A_p (p+1) \chi_{st}^p. \]

Now, from appendix A, \( a_p = -A_p (p+1) \) are the coefficients of the series expansion of the function \( \epsilon(h) \). Therefore we can simply write

\[ \tilde{A}_n = -\frac{1}{(n+1)!} \left( \frac{d}{d\chi_{st}} \right)^n \epsilon(\chi_{st}). \quad (B.1) \]

Appendix C. The spherical spin glass with Gaussian interactions

The spherical spin glass model with Gaussian distributed two-body interactions has been studied in a series of papers [40], [45]–[52] where it was there shown that the asymptotic solution in the ageing regime scales as

\[ R_{ag}(t, t') \sim t^{-3/2} f_R(\lambda), \quad C_{ag}(t, t') \sim f_C(\lambda), \quad (C.1) \]
and $0 \leq \lambda \equiv t'/t \leq 1$. Here, we look at this problem from a different angle, motivated by the generic discussion presented in section 4.2.2. Let us analyse each term in the equations for the global response and correlation by evaluating them in the ageing regime using the scaling forms in (C.1). The equation for the global response reads

$$\frac{\partial R(t, t')}{\partial t} = z(t)R(t, t') + \int_0^t dt'' R(t, t'')R(t', t'),$$

and the Lagrange multiplier $z(t)$ is fixed by the condition $C(t, t) = 1$ that yields:

$$z(t) = T + 2 \int_0^t dt' C(t, t')R(t, t').$$

(C.2)

In the ageing regime, the left-hand side scales as

$$-\left[\frac{2}{3} f_R(\lambda) + f'_R(\lambda)\right] t^{-5/2}.$$  

(C.3)

From the exact solution, one finds that the Lagrange multiplier scales asymptotically as

$$z(t) \sim 2 + ct^{-1}$$  

(C.4)

with $c$ a numerical coefficient. Let us derive this result from equation (C.2) using the forms in (C.1). If, proceeding as usual, we separate the integral in (C.2) into a stationary and an ageing part and we keep the leading contributions to each of these, we find

$$\lim_{t \to \infty} z(t) = z_\infty + \alpha t^{-1/2} \equiv T + \frac{1}{T} (1 - q^2_{ea}) + t^{-1/2} \int_0^1 d\lambda' f_R(\lambda') f_C(\lambda').$$

If one uses the relation between $q_{ea}$ and $T$, the time independent term is consistent with $z_\infty = 2$. However, the approach to the asymptotic value is incorrect. The mistake we have done is that we neglected the correction to the constant value of the stationary contribution that cancels the leading ageing one, and we neglected the correction to the leading ageing contribution that yields the correct $t^{-1}$ decay.

The easiest and most general way of deriving the result above is to go back to the general representation of the solution for $C$ and $R$ and plug these into the integral term in (C.2). After some algebra, and working at $T = 0$ for simplicity, one finds

$$\int_0^t dt' C(t, t')R(t, t') = Y^2(t) \int d\epsilon e^{-2\epsilon t} g(\epsilon) h(\epsilon)$$

(C.5)

where $g(\epsilon)$ is a generic density of states and $h(\epsilon)$ is the function defined in equation (A.7). Now, a density of states with a finite support in $[0, 1]$ and power law decays on the two ends can be mimicked by the form

$$g(\epsilon) \propto \epsilon^\nu (1 - \epsilon)^{1-\nu}$$

(C.6)

that allows us to do the calculations explicitly. In particular, the semicircle case is mimicked by $\nu = 1/2$. Close to $\epsilon \sim 0$ the function $h(\epsilon)$ then reads

$$h(\epsilon) \sim \frac{\pi}{\sin \pi \nu} [(1 - \nu) - \epsilon - \cos(\pi \nu)\epsilon^\nu + \cdots].$$

(C.7)

Replacing in (C.5) and using the asymptotic form of $Y(t)$ one has

$$z(t) \sim a(1 - \nu) - \alpha \cos \pi \nu t^{-\nu} - ct^{-1} + \cdots.$$  

(C.8)
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Thus, for the special case $\nu = 1/2$ the prefactor of the $t^{-\nu}$ term vanishes and one recovers the correct behaviour in $t^{-1}$. A similar phenomenon occurs in the integral over the two responses. The stationary contributions yields a term that is $O(t^{-3/2})$ and its cancellation with the constant asymptotic value of $z_{\infty}$ fixes the Edwards–Anderson order parameter as a function of temperature:

$$T + \frac{1}{T}(1 - q_{ea}^2) = \frac{2(1 - q_{ea})}{T}$$

\[\text{(C.9)}\]

that is equivalent to $T^2 = (1 - q_{ea})^2 \Rightarrow q_{ea} = 1 - T$ ($T \leq T_c = 1$). The next to leading order terms are $O(t^{-2})$ but their prefactor vanishes. Finally, one is left with a term that is $O(t^{-5/2})$, just as the time derivative and another term left from $z(t)R_{ag}(t, t')$. This non-trivial equation fixes the functions $f_C$ and $f_R$.

The analysis of the equation for $C$ is similar. The leading terms are $O(1)$; their cancellation leads to an equation identical to (C.9). The next to leading order terms are $O(t^{-1/2})$ but their overall prefactor vanishes. The time derivative term is $O(t^{-1})$ and it combines with the remaining terms to yield a non-trivial equation.

Note that in the analysis above we used the correct asymptotic behaviour of $R$ and $C$ in the ageing regime, that we know from the direct solution to the (linear set of) Langevin equations. If $p \geq 3$ one cannot solve the dynamics exactly and one is forced to do an asymptotic analysis of the equations for $R$ and $C$ assuming a decay of the linear response and searching for a consistent solution. When $p \geq 3$ one proposes [25] $R_{ag}(t, t') \sim t^{-1}f_R(\lambda)$ and $C_{ag}(t, t') \sim f_C(\lambda)$. In this case, the stationary and ageing contributions to the Lagrange multiplier are both finite. Moreover, all terms on the right-hand side of the equations for $R$ and $C$ are of the same order, $O(t^{-1})$ and $O(1)$, respectively, while the time derivatives are much smaller, $O(t^{-2})$ and $O(t^{-1})$, respectively. Dropping the time derivatives one finds a solution that is consistent with the scaling assumption. In the $p = 2$ model one could have proposed a similar (wrong) scaling and look for its consequences. It is interesting to note that naively pursuing this calculation one finds $X = 0$ as the unique possible asymptotic solution (see equation (26) for the definition of $X$) which is consistent with the exact result, $X \sim t^{-1/2}$, in the $t \to \infty$ limit.

Appendix D. Diagonalizing the matrix $M_\eta(\vec{k}_1, \vec{k}_2)$

In this appendix we study the eigenmodes the matrix $M_\eta(\vec{k}_1, \vec{k}_2)$ defined in equation (72) for the case of $V_x = 1$ (without coarse graining) and equation (75) with finite coarse-graining volume $V_x$.

Appendix D.1. Case $V_x = 1$ (without coarse graining)

First we study the matrix $M_\eta(\vec{k}_1, \vec{k}_2)$ defined in equation (72) for the case of $V_x = 1$. We show that two and only two eigenvalues of $M_\eta$ depend on $\eta$ and all the others are fixed to one. For convenience, let $\mathcal{R}_k(t) \equiv r(k, t, 0)$ and $\mathcal{R}_k(t') = r(k, t', 0)$ label the $k$ indexed row of column vectors $\mathbf{R}(t)$ and $\mathbf{R}(t')$ (these vectors live in $\mathcal{D} = L^d$ dimensions). Note that the length of this vector is constant in time $|\mathbf{R}(t)|^2 = R^2$ (which ensures conservation of the length of the $N$-component vector field in the $N \to \infty$ limit). Let $\mathbf{v}^\lambda$ be, within the
same notation, an eigenvector of $\mathcal{M}_\eta$ with eigenvalue $\lambda$. Then

$$\lambda v^\lambda_{k_1} = \sum_{k_2} \mathcal{M}_\eta(\vec{k}_1, \vec{k}_2)v^\lambda_{k_2}$$

$$= \sum_{k_2} \delta_{\vec{k}_1, \vec{k}_2} v^\lambda_{k_2} + i\eta \frac{\Delta_2}{N} \left[ \mathcal{R}_{\vec{k}_1}(t) \sum_{k_2} \mathcal{R}_{\vec{k}_2}(t') v^\lambda_{k_2} + \mathcal{R}_{\vec{k}_1}(t') \sum_{k_2} \mathcal{R}_{\vec{k}_2}(t) v^\lambda_{k_2} \right]$$

$$= v^\lambda_{k_1} + i\eta \frac{\Delta_2}{N} \left[ (\mathbf{R}(t')v^\lambda) \mathcal{R}_{\vec{k}_1}(t) + (\mathbf{R}(t)v^\lambda) \mathcal{R}_{\vec{k}_1}(t') \right] .$$

This equation is equivalent to

$$\lambda - 1 \quad v^\lambda = i\eta \frac{\Delta_2}{N} \left[ (\mathbf{R}(t')v^\lambda) \mathbf{R}(t) + (\mathbf{R}(t)v^\lambda) \mathbf{R}(t') \right] \quad \text{(D.1)}$$

and has $\mathcal{D}$ solutions.

Only two eigenvalues of $\mathcal{M}_\eta$ are changed by the presence of the $\eta$ term. One of them is

$$v^\lambda \parallel \mathbf{R}(t) + \mathbf{R}(t') \quad \text{with} \quad \lambda_+ = 1 + \frac{i\eta}{N} (C(t,t') + 1) .$$

the other is

$$v^\lambda \parallel \mathbf{R}(t) - \mathbf{R}(t') \quad \text{with} \quad \lambda_- = 1 + \frac{i\eta}{N} (C(t,t') - 1) .$$

Here we used $C(t,t') = \Delta^2 \mathbf{R}(t')\mathbf{R}(t) = \Delta^2 \sum_r r(k,t',0) \ r(k,t,0) \ \text{for the zero-}$

$$\text{temperature correlation, and } \Delta^2 \mathbf{R}^2 = 1. \ \text{The other } \mathcal{D} - 2 \text{ solutions are such that}$$

$$v^\lambda \perp 2d \ \text{plane spanned by the above two eigenvectors with } \lambda = 1.$$ 

**Appendix D.2. Case of finite $V_x$**

Next we study the matrix $\mathcal{M}_\eta(\vec{k}_1, \vec{k}_2)$ defined in equation (75) for the case of finite coarse-graining volume $V_x$. Let $\mathcal{R}_k(t, \vec{y}) \equiv r(k,t,0) e^{i\vec{k}\vec{y}}$ label the $k$ indexed row of the column vector $\mathbf{R}(t, \vec{y})$. This allows us to write an eigenvalue equation for $\mathcal{M}_\eta$, similarly to what we have done above for the case $l = 1$,

$$(\lambda - 1) \quad v^\lambda = i\eta \frac{\Delta_2}{V_x} \sum_{\vec{y}\in V_x} \left[ (\mathbf{R}(t', \vec{y})v^\lambda) \mathbf{R}(t, \vec{y}) + (\mathbf{R}(t, \vec{y})v^\lambda) \mathbf{R}(t', \vec{y}) \right] \quad \text{(D.2)}$$

where the inner (dot) product is here defined as $\mathbf{a}\mathbf{b} = \sum_k a_k^* b_k$.

This eigenvalue equation has $\mathcal{D} - 2V_x$ trivial solutions with $\lambda = 1$. The eigenvectors for such solutions satisfy $\mathbf{R}(t', \vec{y})v^\lambda = 0$ and $\mathbf{R}(t, \vec{y})v^\lambda = 0$, for $\vec{y} \in V_x$, and hence span the orthogonal subspace to that spanned by the $2V_x$ vectors $\mathbf{R}(t, \vec{y})$ and $\mathbf{R}(t', \vec{y})$ ($\vec{y} \in V_x$).

The remaining (non-trivial) eigenvectors can be written as

$$v^\lambda = \sum_{\vec{y}\in V_x} \alpha^\lambda(\vec{y}) \mathbf{R}(t, \vec{y}) + \beta^\lambda(\vec{y}) \mathbf{R}(t', \vec{y})$$

doi:10.1088/1742-5468/2006/01/P01006

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for some $2V_x$ expansion coefficients $\alpha^\lambda(\vec{y})$ and $\beta^\lambda(\vec{y})$ for $\vec{y} \in V_x$. Plugging this into equation (D.2) leads to

$$\begin{align*}
(\lambda - 1) \alpha^\lambda(\vec{y}) &= \imath \eta \frac{\Delta^2}{V_x} \sum_{\vec{y}' \in V_x} \left[ \alpha^\lambda(\vec{y}') \left( R(t', \vec{y}) R(t, \vec{y}') \right) + \beta^\lambda(\vec{y}') \left( R(t', \vec{y}) R(t, \vec{y}') \right) \right], \quad (D.3) \\
(\lambda - 1) \beta^\lambda(\vec{y}) &= \imath \eta \frac{\Delta^2}{V_x} \sum_{\vec{y}' \in V_x} \left[ \alpha^\lambda(\vec{y}') \left( R(t, \vec{y}) R(t, \vec{y}') \right) + \beta^\lambda(\vec{y}') \left( R(t, \vec{y}) R(t', \vec{y}') \right) \right]. \quad (D.4)
\end{align*}$$

Using that

$$\Delta^2 R(t, \vec{y}) R(t', \vec{y}') = C(\vec{y}, \vec{y}'; t, t') = C(t, t') \exp \left[ -\frac{|\vec{y} - \vec{y}'|^2}{L^2(t) + L^2(t')} \right],$$

with the length scales $L(t) = 2\sqrt{t}$ and $L(t') = 2\sqrt{t'}$, and substituting in equations (D.3) and (D.4), one obtains

$$\begin{align*}
(\lambda - 1) \alpha^\lambda(\vec{y}) &= \imath \eta \frac{1}{V_x} \sum_{\vec{y}' \in V_x} \left\{ \alpha^\lambda(\vec{y}') \left( C(t, t') \exp \left[ -\frac{|\vec{y} - \vec{y}'|^2}{L^2(t) + L^2(t')} \right] \right) + \beta^\lambda(\vec{y}') \exp \left[ -\frac{|\vec{y} - \vec{y}'|^2}{2L^2(t')} \right] \right\}, \quad (D.5) \\
(\lambda - 1) \beta^\lambda(\vec{y}) &= \imath \eta \frac{1}{V_x} \sum_{\vec{y}' \in V_x} \left\{ \alpha^\lambda(\vec{y}') \exp \left[ -\frac{|\vec{y} - \vec{y}'|^2}{2L^2(t)} \right] \right\} C(t, t') \exp \left[ -\frac{|\vec{y} - \vec{y}'|^2}{L^2(t) + L^2(t')} \right]. \quad (D.6)
\end{align*}$$

These equations are difficult to solve for generic ratios of the length scales $L(t)$ and $L(t')$ to the coarse-graining box size $\ell$. In the following we consider some limiting cases.

**Appendix D.2.1. Case $\ell \ll L(t), L(t')$.** A simple situation is given by $\ell \ll L(t), L(t')$, in which case $|\vec{y} - \vec{y}'| \ll L(t), L(t')$, and the eigenvalues can be found by adding and subtracting equations (D.5) and (D.6) and summing both sides over $\vec{y}$; one finds two non-trivial solutions

$$\lambda_{\pm} = 1 + \imath \eta \left[ C(t, t') \pm 1 \right],$$

and $2V_x - 2$ trivial solutions such that $\lambda = 1$ and $\sum_{\vec{y} \in V_x} \alpha^\lambda(\vec{y}) = \sum_{\vec{y} \in V_x} \beta^\lambda(\vec{y}) = 0$. Thus, in the case $\ell \ll L(t), L(t')$ we recover the same eigenvalues, and hence the same distribution as in the case $V_x = 1$. This result was to be expected since coarse graining of completely correlated regions should not affect the distribution obtained for a single site.

**Appendix D.2.2. Case $\ell \gg L(t), L(t')$.** One can seek approximate solutions of equations (D.5) and (D.6) in this limit if one assumes that the $\alpha^\lambda(\vec{y})$ and $\beta^\lambda(\vec{y})$ are...
slowly varying functions of $\vec{y}$, in which case one must solve the approximate equations

$$
(\lambda - 1) \alpha^\lambda(\vec{y}) \approx i\eta \frac{L^d}{V_x} \left[ C(t, t') \alpha^\lambda(\vec{y}) + \beta^\lambda(\vec{y}) \right],
$$

(D.7)

$$
(\lambda - 1) \beta^\lambda(\vec{y}) \approx i\eta \frac{L^d}{V_x} \left[ \alpha^\lambda(\vec{y}) + C(t, t') \beta^\lambda(\vec{y}) \right],
$$

(D.8)

where for simplicity we considered $L = L(t) \sim L(t')$. These equations admit non-trivial solutions

$$
\lambda_+ \approx 1 + i\eta \left( \frac{L}{\ell} \right)^d (C(t, t') \pm 1).
$$

Naively, there are as many of these solutions as the number of $\vec{y}$ points in $V_x$, for each of $\lambda_\pm$. However, the assumption that the $\alpha^\lambda(\vec{y})$ and $\beta^\lambda(\vec{y})$ are slowly varying correlates them, and thus one cannot expect that the non-trivial solutions span the whole of the $2V_x \times d$-dimensional space. The number of independent non-trivial solutions should be only order $V_x/L^d = (\ell/L)^d$ for each of $\lambda_\pm$.

**Appendix E. The response of composite operators**

In this appendix we compute the response of the composite operator $\phi(\vec{x}, t) \phi(\vec{x}, t')$ to a perturbation that couples to the same composite operator evaluated at a different spatial point and the same times [11]. In the Langevin equation such a perturbation is represented by an additional deterministic time dependent force:

$$
F(\vec{x}, t) = \int_0^t dt'' \left[ h(\vec{x}; t''; t; t') + h(\vec{x}; t, t'') \right] \phi(\vec{x}, t'').
$$

In the following we work at zero temperature. The perturbed field is

$$
\phi(\vec{k}, t) = r(k; t, 0) \phi(\vec{k}, 0) + \int_0^t dt'' r(k; t, t'') F(\vec{k}, t''),
$$

and the response we are interested in defined as

$$
\left[ \frac{\delta \phi(\vec{x}, t) \phi(\vec{x}, t')}{\delta h(\vec{x}, t'', t'')} \right]_{ic} = \left[ \phi(\vec{x}, t) \frac{\delta \phi(\vec{x}, t')}{\delta h(\vec{x}, t'', t'')} \right]_{ic} + \left[ \frac{\delta \phi(\vec{x}, t)}{\delta h(\vec{x}, t'', t'')} \phi(\vec{x}, t') \right]_{ic}.
$$

After some rather straightforward calculations one finds

$$
\left[ \frac{\delta \phi(\vec{x}, t) \phi(\vec{x}, t')}{\delta h(\vec{x}, t'', t'')} \right]_{ic} = R(\vec{x} - \vec{x}'; t, t'') C(\vec{x} - \vec{x}'; t, t'') \theta(t - t'')
$$

$$
+ R(\vec{x} - \vec{x}'; t, t'') C(\vec{x} - \vec{x}'; t, t'') \theta(t - t'')
$$

$$
+ R(\vec{x} - \vec{x}'; t', t'') C(\vec{x} - \vec{x}'; t, t'') \theta(t' - t'')
$$

$$
+ R(\vec{x} - \vec{x}'; t', t'') C(\vec{x} - \vec{x}'; t, t'') \theta(t' - t'')
$$

where $R$ and $C$ are the usual two-point, two-time linear response and correlation. One can readily verify that this expression is not simple related to time variations of the four-point correlation $C_4$ contrary to what one might have naively expected. Note that this expression has the expected $t = t'$ and $t'' = t'''$ limit.
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