REFLEXIVE POLYTOPES ARISING FROM BIPARTITE GRAPHS WITH
\(\gamma\)-POSITIVITY ASSOCIATED TO INTERIOR POLYNOMIALS

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ABSTRACT. In this paper, we introduce polytopes \(\mathcal{B}_G\) arising from root systems \(B_n\) and finite graphs \(G\), and study their combinatorial and algebraic properties. In particular, it is shown that \(\mathcal{B}_G\) is a reflexive polytope with a regular unimodular triangulation if and only if \(G\) is bipartite. This implies that the \(h^*\)-polynomial of \(\mathcal{B}_G\) is palindromic and unimodal when \(G\) is bipartite. Furthermore, we discuss stronger properties, the \(\gamma\)-positivity and the real-rootedness of the \(h^*\)-polynomials. In fact, if \(G\) is bipartite, then the \(h^*\)-polynomial of \(\mathcal{B}_G\) is \(\gamma\)-positive and its \(\gamma\)-polynomial is given by an interior polynomial (a version of Tutte polynomial of a hypergraph). The \(h^*\)-polynomial is real-rooted if and only if the corresponding interior polynomial is real-rooted. From a counterexample of Neggers–Stanley conjecture, we give a bipartite graph \(G\) whose \(h^*\)-polynomial is not real-rooted but \(\gamma\)-positive, and coincides with the \(h\)-polynomial of a flag triangulation of a sphere.

INTRODUCTION

Ardila et al. [1] constructed a unimodular triangulation of the convex hull of the roots of the classical root lattices of type \(A_n, B_n, C_n\) and \(D_n\), and gave an alternative proof for the known growth series of these root lattices by using the triangulation. On the other hand, polytopes of the root system of type \(A_n\) arising from finite graphs are called symmetric edge polytopes and their combinatorial properties are well-studied ([20, 21, 28]). In this paper, we introduce polytopes arising from the root system of type \(B_n\) and finite graphs, and study their algebraic and combinatorial properties.

A lattice polytope \(\mathcal{P} \subset \mathbb{R}^d\) is a convex polytope all of whose vertices have integer coordinates. Let \(G\) be a finite simple undirected graph on the vertex set \([d] = \{1, \ldots, d\}\) with the edge set \(E(G)\). Let \(\mathcal{B}_G \subset \mathbb{R}^d\) denote the convex hull of the set

\[
B(G) = \{\pm e_1, \ldots, \pm e_d\} \cup \{\pm e_i \pm e_j : \{i, j\} \in E(G)\},
\]

where \(e_i\) is \(i\)-th unit coordinate vector in \(\mathbb{R}^d\). Then \(\dim \mathcal{B}_G = d\) and \(\mathcal{B}_G\) is centrally symmetric, i.e., for any facet \(F\) of \(\mathcal{B}_G\), \(-F\) is also a facet of \(\mathcal{B}_G\), and the origin \(0\) of \(\mathbb{R}^d\) is the unique interior lattice point of \(\mathcal{B}_G\). Note that, if \(G\) is a complete graph, then \(\mathcal{B}_G\) coincides with the convex hull of the roots of the root lattices of type \(B_n\) studied in [1]. Several classes of lattice polytopes arising from graphs have been studied from viewpoints of combinatorics, graph theory, geometric and commutative algebra. In particular, edge polytopes give interesting examples in commutative algebra ([16, 31, 32, 33, 39]). Note that edge polytopes of bipartite graphs are called root polytopes and play important roles in the study of generalized permutohedra ([37]) and interior polynomials ([23]).

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There is a good relation between $B_G$ and edge polytopes. In fact, one of the key properties of $B_G$ is that $B_G$ is divided into $2^d$ edge polytopes of certain non-simple graphs $\tilde{G}$ (Proposition 1.1). This fact helps us to find and show interesting properties of $B_G$. In Section 1 by using this fact, we will classify graphs $G$ such that $B_G$ has a unimodular covering (Theorem 1.3).

On the other hand, the fact that $B_G$ has a unique interior lattice point $0$ leads us to consider when $B_G$ is reflexive. A lattice polytope $P \subset \mathbb{R}^d$ of dimension $d$ is called reflexive if the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $P$ and its dual polytope $P^\vee := \{ y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in P \}$ is also a lattice polytope, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [2, 8]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([26]) and all of them are known up to dimension 4 ([25]). In Section 2, we will classify graphs $G$ such that $B_G$ is a reflexive polytope. In fact, we will show the following.

**Theorem 0.1.** Let $G$ be a finite graph. Then the following conditions are equivalent:

(i) $B_G$ is reflexive and has a regular unimodular triangulation;
(ii) $B_G$ is reflexive;
(iii) $G$ is a bipartite graph.

Moreover, by characterizing when the toric ideal of $B_G$ has a Gröbner basis consisting of quadratic binomials for a bipartite graph $G$, we can classify graphs $G$ such that $B_G$ is a reflexive polytope with a flag regular unimodular triangulation. In fact, we will show the following.

**Theorem 0.2.** Let $G$ be a bipartite graph. Then the following conditions are equivalent:

(i) The reflexive polytope $B_G$ has a flag regular unimodular triangulation;
(ii) Any cycle of $G$ of length $\geq 6$ has a chord (“chordal bipartite graph”).

Now, we turn to the discussion of the $h^*$-polynomial $h^*(B_G, x)$ of $B_G$. Thanks to the key property (Proposition 1.1), we can compute the $h^*$-polynomial of $B_G$ in terms of that of edge polytopes of some graphs. On the other hand, since it is known that the $h^*$-polynomial of a reflexive polytope with a regular unimodular triangulation is palindromic and unimodal ([6]), Theorem 1.3 implies that the $h^*$-polynomial of $B_G$ is palindromic and unimodal. In Section 3, we will show a stronger result, which is for any bipartite graph $G$, the $h^*$-polynomial $h^*(B_G, x)$ is $\gamma$-positive. The theory of interior polynomials (a version of Tutte polynomials of hypergraphs) introduced by Kálmán [22] and the theory of generalized permutohedra [29, 37] play important roles.

**Theorem 0.3.** Let $G$ be a bipartite graph on $[d]$. Then $h^*$-polynomial of the reflexive polytope $B_G$ is

$$h^*(B_G, x) = (x + 1)^d I_{\tilde{G}}\left(\frac{4x}{(x+1)^2}\right),$$

where $\tilde{G}$ is a connected bipartite graph defined in (11) later and $I_{\tilde{G}}(x)$ is the interior polynomial of $\tilde{G}$. In particular, $h^*(B_G, x)$ is $\gamma$-positive. Moreover, $h^*(B_G, x)$ is real-rooted if and only if $I_{\tilde{G}}(x)$ is real-rooted.
In addition, we discuss the relations between interior polynomials and other important polynomials in combinatorics.

- If \( G \) is bipartite, then the interior polynomial of \( \hat{G} \) is described in terms of \( k \)-matching of \( G \) (Proposition 3.3);
- If \( G \) is a forest, then the interior polynomial of \( \hat{G} \) coincides with the matching generating polynomial of \( G \) (Proposition 3.4);
- If \( G \) is a bipartite permutation graph associated with a poset \( P \), then the interior polynomial of \( \hat{G} \) coincides with the \( P \)-Eulerian polynomial of \( P \) (Proposition 3.5).

By using these results and a poset appearing in [43] as a counterexample of Neggers–Stanley conjecture, we will give an example of a centrally symmetric reflexive polytope such that the \( h^* \)-polynomial is \( \gamma \)-positive and not real-rooted (Example 3.6). This \( h^* \)-polynomial coincides with the \( h \)-polynomial of a flag triangulation of a sphere (Proposition 3.7). Hence this example is a counterexample of “Real Root Conjecture” that has been already disproved by Gal [10]. Finally, inspired by the simple description for the \( h^* \)-polynomials of symmetric edge polytopes of complete bipartite graphs [21], we will compute the \( h^* \)-polynomial of \( B_G \) when \( G \) is a complete bipartite graph (Example 3.8).

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1. A KEY PROPERTY OF \( B_G \) AND UNIMODULAR COVERINGS

In this section, we see a relation between \( B_G \) and edge polytopes. First, we recall what edge polytopes are. Let \( G \) be a graph on \( [d] \) (only here we do not assume that \( G \) has no loops) with the edge (including loop) set \( E(G) \). Then the edge polytope \( P_G \) of \( G \) is the convex hull of \( \{ e_i + e_j : \{i, j\} \in E(G) \} \). Note that \( P_G \) is a \((0, 1)\)-polytope if and only if \( G \) has no loops. Given a graph \( G \) on \([d]\), let \( \hat{G} \) be a graph on \([d + 1]\) whose edge set is

\[
E(G) \cup \{(1, d + 1), (2, d + 1), \ldots, (d + 1, d + 1)\}.
\]

Here, \( \{1, d + 1\} \) is a loop (a cycle of length 1) at \( d + 1 \). If \( G \) is a bipartite graph with a bipartition \( V_1 \cup V_2 = [d] \), let \( \hat{G} \) be a connected bipartite graph on \([d + 2]\) whose edge set is

\[
E(\hat{G}) = E(G) \cup \{ (i, d + 1) : i \in V_1 \} \cup \{ (j, d + 2) : j \in V_2 \cup \{d + 1\} \}.
\]

Now, we show the key proposition of this paper. Given \( \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{-1, 1\}^d \), let \( \bar{\mathcal{G}}_{\epsilon} \) denote the closed orthant \( \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \epsilon_i \geq 0 \text{ for all } i \in [d]\} \).

**Proposition 1.1.** Work with the same notation as above. Then we have the following:

\( a) \) Each \( B_G \cap \bar{\mathcal{G}}_{\epsilon} \) is the convex hull of the set \( B(G) \cap \bar{\mathcal{G}}_{\epsilon} \) and unimodularly equivalent to the edge polytope \( P_G \) of \( \hat{G} \). Moreover, if \( G \) is bipartite, then \( B_G \cap \bar{\mathcal{G}}_{\epsilon} \) is unimodularly equivalent to the edge polytope \( P_G \) of \( \hat{G} \). In particular, one has \( \Vol(B_G) = 2^d \Vol(P_G) \).

\( b) \) The edge polytope of \( G \) is a face of \( B_G \).

**Proof.** (a) Let \( \mathcal{P} \) be the convex hull of the set \( B(G) \cap \bar{\mathcal{G}}_{\epsilon} \). The inclusion \( B_G \cap \bar{\mathcal{G}}_{\epsilon} \subset \mathcal{P} \) is trivial. Let \( x = (x_1, \ldots, x_d) \in B_G \cap \bar{\mathcal{G}}_{\epsilon} \). Then \( x = \sum_{i=1}^d \lambda_i a_i \), where \( \lambda_i > 0 \), \( \sum_{i=1}^d \lambda_i = 1 \),
and each $\mathbf{a}_i$ belongs to $B(G)$. Suppose that $k$-th component of $\mathbf{a}_i$ is positive and $k$-th component of $\mathbf{a}_j$ is negative. Then $\mathbf{a}_i$ and $\mathbf{a}_j$ satisfy one of the following conditions:

$$
\mathbf{a}_j = -\mathbf{a}_i, \quad \text{and} \quad \mathbf{a}_i + \mathbf{a}_j = \mathbf{0} + \mathbf{0},
$$

$$
\mathbf{a}_i = \mathbf{e}_k, \quad \mathbf{a}_j = -\mathbf{e}_k \pm \mathbf{e}_{k'}, \quad \text{and} \quad \mathbf{a}_i + \mathbf{a}_j = \pm \mathbf{e}_{k'} + \mathbf{0},
$$

$$
\mathbf{a}_i = \mathbf{e}_k \pm \mathbf{e}_{k'}, \quad \mathbf{a}_j = -\mathbf{e}_k, \quad \text{and} \quad \mathbf{a}_i + \mathbf{a}_j = \pm \mathbf{e}_{k'} + \mathbf{0},
$$

$$
\mathbf{a}_i = \mathbf{e}_k \pm \mathbf{e}_{k'}, \quad \mathbf{a}_j = -\mathbf{e}_k \pm \mathbf{e}_{k''} \neq -\mathbf{a}_i, \quad \text{and} \quad \mathbf{a}_i + \mathbf{a}_j = \pm \mathbf{e}_{k'} \pm \mathbf{e}_{k''} \neq \mathbf{0}.
$$

By using the above relations for $\mathbf{a}_i + \mathbf{a}_j$ finitely many times, we may assume that $k$-th component of each vector $\mathbf{a}_i$ is nonnegative (resp. nonpositive) if $x_k \geq 0$ (resp. $x_k \leq 0$). Then each $\mathbf{a}_i$ belongs to $B(G) \cap \mathcal{O}_e$ and hence $\mathbf{x} \in \mathcal{P}$.

Next, we show that each $\mathcal{B}_G \cap \mathcal{O}_e$ is unimodularly equivalent to the edge polytope $P_G$. Set $\mathcal{D} = \mathcal{B}_G \cap \mathcal{O}(1,...,1)$. It is easy to see that each $\mathcal{B}_G \cap \mathcal{O}_e$ is unimodularly equivalent to $\mathcal{D}$ for all $e$. Moreover, one has

$$
B(G) \cap \mathcal{O}(1,...,1) = \{ \mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d \} \cup \{ \mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G) \}.
$$

Hence $P_G$ is unimodularly equivalent to $\mathcal{D} \times \{1\}^2$. Similarly, if $G$ is bipartite, it follows that $P_G$ is unimodularly equivalent to $\mathcal{D} \times \{1\}^2$.

(b) The edge polytope $P_G$ of $G$ is a face of $\mathcal{B}_G$ with a supporting hyperplane $\mathcal{H} = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 + \cdots + x_d = 2 \}$. □

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension $d$. We now focus on the following properties.

(VA) We say that $\mathcal{P}$ is very ample if for all sufficiently large $k \in \mathbb{Z}_{\geq 1}$ and for all $\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{P} \cap \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$.

(IDP) We say that $\mathcal{P}$ possesses the integer decomposition property (or is IDP for short) if for all $k \in \mathbb{Z}_{\geq 1}$ and for all $\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{P} \cap \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$.

(UC) We say that $\mathcal{P}$ has a unimodular covering if there exist unimodular lattice simplices $\Delta_1, \ldots, \Delta_n$ such that $\mathcal{P} = \bigcup_{1 \leq i \leq n} \Delta_i$.

(UT) We say that $\mathcal{P}$ has a unimodular triangulation if $\mathcal{P}$ admits a lattice triangulation consisting of unimodular lattice simplices.

These properties satisfy the implications

$$(\text{UT}) \Rightarrow (\text{UC}) \Rightarrow (\text{IDP}) \Rightarrow (\text{VA}).$$

On the other hand, it is known that the opposite implications are false. However, for edge polytopes, the first three properties are equivalent. We say that a graph $G$ satisfies the odd cycle condition if, for any two cycles $C_1$ and $C_2$ that belong to the same connected component of $G$ and have no common vertices, there exists an edge $\{i, j\}$ of $G$ such that $i$ is a vertex of $C_1$ and $j$ is a vertex of $C_2$. The following fact is known ([7, 31, 39]).

**Proposition 1.2.** Let $G$ be a finite (not necessarily simple) graph. Suppose that there exists an edge $\{i, j\}$ of $G$ if $G$ has loops at $i$ and $j$ with $i \neq j$. Then the following conditions are equivalent:

(i) $P_G$ has a unimodular covering;

(ii) $P_G$ is IDP;
(iii) $P_G$ is very ample;
(iv) $G$ satisfies the odd cycle condition.

We now show that the same assertion holds for $B_G$. Namely, we prove the following.

**Theorem 1.3.** Let $G$ be a finite simple graph. Then the following conditions are equivalent:

(i) $B_G$ has a unimodular covering;
(ii) $B_G$ is IDP;
(iii) $B_G$ is very ample;
(iv) $G$ satisfies the odd cycle condition.

**Proof.** First, implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold in general.

(iii) $\Rightarrow$ (iv): Suppose that $G$ does not satisfy the odd cycle condition. By Proposition 1.2, the edge polytope $P_G$ of $G$ is not very ample. Since $P_G$ is a face of $B_G$ by Proposition 1.1 (b), $B_G$ is not very ample.

(iv) $\Rightarrow$ (i): Suppose that $G$ satisfies the odd cycle condition. Then so does $\tilde{G}$. Hence Proposition 1.2 guarantees that $P_{\tilde{G}}$ has a unimodular covering. By Proposition 1.1 (a), $B_{\tilde{G}}$ has a unimodular covering. $\square$

**Example 1.4.** Let $G$ be a graph in Figure 1. Since $G$ satisfies the odd cycle condition, $B_G$ has a unimodular covering. However, since the edge polytope $P_G$ has no regular unimodular triangulations ([32]), so does $B_G$ by Proposition 1.1 (b). We do not know whether $B_G$ has a (nonregular) unimodular triangulation or not.

![Figure 1. A graph in [32]](image)

2. **Reflexive Polytopes and Flag Triangulations of $B_G$**

In the present section, we classify graphs $G$ such that

- $B_G$ is a reflexive polytope.
- $B_G$ is a reflexive polytope with a flag regular unimodular triangulation.

Namely, we prove Theorems 0.1 and 0.2. First, we see some examples that $B_G$ is reflexive.

**Examples 2.1.** (a) If $G$ is an empty graph, then $B_G$ is a cross polytope.

(b) Let $G$ be a complete graph with 2 vertices. Then $B_G \cap \mathbb{Z}^2$ is the column vectors of the matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & -1 & 0 & -1 & -1 \\
0 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 1
\end{pmatrix},
$$
and $\mathcal{B}_G$ is reflexive and has a regular unimodular triangulation. Since the matrix

$$
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & -1
\end{pmatrix}
$$

is not unimodular, we cannot apply [34, Lemma 2.11] to show this fact.

In order to show that a lattice polytope is reflexive, we use an algebraic technique on Gröbner bases. We recall basic materials and notation on toric ideals. Let $K[t_{1}^{\pm 1}, s] = K[t_{1}^{\pm 1}, \ldots, t_{d}^{\pm 1}, s]$ be the Laurent polynomial ring in $d + 1$ variables over a field $K$. If $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, then $t^a s$ is the Laurent monomial $t_1^{a_1} \cdots t_d^{a_d} s \in K[t_{1}^{\pm 1}, s]$. Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope and $\mathcal{P} \cap \mathbb{Z}^d = \{a_1, \ldots, a_n\}$. Then, the toric ring of $\mathcal{P}$ is the subalgebra $K[\mathcal{P}]$ of $K[t_{1}^{\pm 1}, s]$ generated by $\{t^a s \mid a \in \mathbb{Z}^d\}$ over $K$. We regard $K[\mathcal{P}]$ as a homogeneous algebra by setting each deg $t^a s = 1$. Let $K[x] = K[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over $K$. The toric ideal $I_{\mathcal{P}}$ of $\mathcal{P}$ is the kernel of the surjective homomorphism $\pi : K[x] \rightarrow K[\mathcal{P}]$ defined by $\pi(x_i) = t^a s$ for $1 \leq i \leq n$. It is known that $I_{\mathcal{P}}$ is generated by homogeneous binomials. See, e.g., [44]. Let $<$ be a monomial order on $K[x]$ and in$<_<(I_{\mathcal{P}})$ the initial ideal of $I_{\mathcal{P}}$ with respect to $<$. The initial ideal in$<_<(I_{\mathcal{P}})$ is called squarefree (resp. quadratic) if in$<_<(I_{\mathcal{P}})$ is generated by squarefree (resp. quadratic) monomials. Now, we introduce an algebraic technique to show that a lattice polytope is reflexive.

**Lemma 2.2 ([44, Lemma 1.1]).** Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension $d$ such that the origin of $\mathbb{R}^d$ is contained in its interior and $\mathcal{P} \cap \mathbb{Z}^d = \{a_1, \ldots, a_n\}$. Suppose that any lattice point in $\mathbb{Z}^{d+1}$ is a linear integer combination of the lattice points in $\mathcal{P} \times \{1\}$ and there exists an ordering of the variables $x_i < \cdots < x_n$ for which $a_i = 0$ such that the initial ideal in$<_<(I_{\mathcal{P}})$ of $I_{\mathcal{P}}$ with respect to the reverse lexicographic order $<$ on $K[x]$ induced by the ordering is squarefree. Then $\mathcal{P}$ is reflexive and has a regular unimodular triangulation.

By using this technique, several families of reflexive polytopes with regular unimodular triangulations are given in [13, 14, 15, 17, 18, 19, 35]. In order to apply Lemma 2.2 to show Theorem 0.1 we see a relation between the toric ideal of $\mathcal{B}_G$ and that of $P_G$. Let $G$ be a simple graph on $[d]$ with edge set $E(G)$ and let $R_G$ denote the polynomial ring

$$
R_G = K[z, x_i+, y_{ij+}, y_{ij-}, y_{ij-}, y_{ij-}, y_{ij-}: 1 \leq i < j \leq d, \{i, j\} \in E(G)]
$$
in $2d + 1 + 4|E(G)|$ variables over a field $K$. Then the toric ideal $I_{\mathcal{B}_G}$ of $\mathcal{B}_G$ is the kernel of a ring homomorphism $\pi : R_G \rightarrow K[t_{1}^{\pm 1}, \ldots, t_{d}^{\pm 1}, s]$ defined by $\pi(z) = s$, $\pi(x_i+) = t_i s$, $\pi(x_i-) = t_i^{-1} s$, $\pi(y_{ij+}) = t_j s$, $\pi(y_{ij-}) = t_i^{-1} t_j^{-1} s$, $\pi(y_{ij-}) = t_i^{-1} t_j^{-1} s$, and $\pi(y_{ij-}) = t_i^{-1} t_j^{-1} s$. Let $S_G$ denote the polynomial ring

$$
S_G = K[z, x_i+, y_{ij+}: 1 \leq i < j \leq d, \{i, j\} \in E(G)]
$$
in $d + 1 + |E(G)|$ variables over $K$. Then the toric ideal $I_{P_G}$ of $P_G$ is the kernel of $\pi|_{S_G}$. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, 1\}^d$, we define a ring homomorphism $\varphi_{\varepsilon} : S_G \rightarrow R_G$ by $\varphi_{\varepsilon}(x_i+) = x_i \alpha$ and $\varphi_{\varepsilon}(y_{ij+}) = y_{ij+} \beta$ where $\alpha$ is the sign of $\varepsilon_i$, and $\beta$ is the sign of $\varepsilon_j$. In particular, $\varphi_{(1, \ldots, 1)} : S_G \rightarrow R_G$ is an inclusion map.
Lemma 2.3. Let $\mathcal{G}$ be a Gröbner basis of $I_{P_{G}}$ with respect to a reverse lexicographic order $<_{S}$ on $S_{G}$ such that $z < \{x_{i+}\} < \{y_{ij+}\}$. Let $<_{R}$ be a reverse lexicographic order such that $z < \{x_{i+}, x_{i-}\} < \{y_{ij+}, y_{ij-}, y_{i+}, y_{i-}\}$ and that (i) $\varphi_{\varepsilon}(x_{i+}) <_{R} \varphi_{\varepsilon}(x_{i-})$ if $x_{i+} <_{S} x_{i-}$ and (ii) $\varphi_{\varepsilon}(y_{ij+}) <_{R} \varphi_{\varepsilon}(y_{kl+})$ if $y_{ij+} <_{S} y_{kl+}$ for all $\varepsilon \in \{-1,1\}^{d}$. Then

$$G' = \left( \bigcup_{\varepsilon \in \{-1,1\}^{d}} \varphi_{\varepsilon}(\mathcal{G}) \right) \cup \left\{ x_{i\alpha}y_{ij\beta} - x_{j\gamma\zeta} : (i,j) \in E(G), \alpha \neq \beta \right\}$$

$$\cup \left\{ y_{ij\alpha}y_{ik\beta\delta} - x_{j\gamma\zeta}v_{k\delta} : (i,j), (i,k) \in E(G), \alpha \neq \beta \right\} \cup \left\{ x_{i+}x_{i-} - z^{2} : 1 \leq i \leq d \right\}$$

is a Gröbner basis of $I_{\mathcal{B}_{G}}$ with respect to $<_{R}$, where the underlined monomial is the initial monomial of each binomial. (Here we identify $y_{ij\alpha\beta}$ with $y_{ij\alpha\beta'}$.)

In particular, if in$_{<_{S}}(I_{P_{G}})$ is squarefree (resp. quadratic), then so is in$_{<_{R}}(I_{\mathcal{B}_{G}})$.

Proof. It is easy to see that $G'$ is a subset of $I_{\mathcal{B}_{G}}$. Assume that $G'$ is not a Gröbner basis of $I_{\mathcal{B}_{G}}$ with respect to $<_{R}$. Let $\mathcal{I}(G') = \langle \text{in}_{<_{R}}(g) : g \in \mathcal{G}' \rangle$. Then there exists a non-zero irreducible homogeneous binomial $f = u - v \in I_{\mathcal{B}_{G}}$ such that neither $u$ nor $v$ belongs to $\mathcal{I}(G')$. Since both $u$ and $v$ are divided by none of $x_{i+}x_{i-}$, $x_{i\alpha}y_{ij\beta}$, $y_{ij\alpha}y_{ik\beta\delta}$ ($\alpha \neq \beta$), they are of the form

$$u = \prod_{i \in I} (x_{i\alpha})^{u_{i}} \prod_{(j,k) \in E_{1}} (y_{jk\alpha\beta})^{u_{jk}}, \quad v = \prod_{i \in I'} (x_{i\alpha'})^{v_{i}} \prod_{(j,k) \in E_{2}} (y_{jk\alpha'\beta'})^{v_{jk}},$$

where $I, I' \subseteq [d]$, $E_{1}, E_{2} \subseteq E(G)$ and $0 < u_{i}, u_{jk}, v_{j}, v_{jk} \in \mathbb{Z}$. Since $f$ belongs to $I_{\mathcal{B}_{G}}$, the exponents of $t_{i}$ in $\pi(u)$ and $\pi(v)$ are the same. Hence, one of $x_{i\alpha}$ and $y_{i\alpha}y_{ik\beta\delta}$ appears in $u$ if and only if one of $x_{i\alpha'}$ and $y_{i\alpha'}y_{ik\beta\delta'}$ appears in $v$ with $\alpha' = \alpha_{l} \in \{-1,1\}^{d}$ such that the sign of the $l$th component of $\varepsilon$ is $\alpha_{l}$ if one of $x_{i+}$ and $y_{ij\alpha\beta}$ appears in $u$. Then $f$ belongs to the ideal $\varphi_{\varepsilon}(I_{P_{G}})$. Let $f' \in I_{P_{G}}$ be a binomial such that $\varphi_{\varepsilon}(f') = f$. Since $\mathcal{G}$ is a Gröbner basis of $I_{P_{G}}$, there exists a binomial $g \in \mathcal{G}$ whose initial monomial in $\text{in}_{<_{R}}(g)$ divides the one of the monomials in $f'$. By the definition of $<_{R}$, we have $\text{in}_{<_{R}}(\varphi_{\varepsilon}(g)) = \varphi_{\varepsilon}(\text{in}_{<_{R}}(g))$. Hence in$_{<_{R}}(\varphi_{\varepsilon}(g))$ divides one of the monomials in $f = \varphi_{\varepsilon}(f')$. This is a contradiction. \qed

Using this Gröbner basis with respect to a reverse lexicographic order, we verify which $\mathcal{B}_{G}$ is a reflexive polytope. Namely, we prove Theorem 0.1.

Proof of Theorem 0.1 The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Suppose that $G$ is not bipartite. Let $G_{1}, \ldots, G_{s}$ be connected components of $G$ and let $d_{i}$ be the number of vertices of $G_{i}$. In particular, we have $d = \sum_{i=1}^{s} d_{i}$. Since $G$ is not bipartite, we may assume that $G_{1}$ is not bipartite. Let $w = \sum_{i=1}^{s} w_{i}$, where $w_{i} = \sum_{k=1}^{\ell} e_{pk} \in \mathbb{R}^{d}$ if $G_{i}$ is a non-bipartite graph on the vertex set $\{p_{1}, \ldots, p_{\ell}\}$, and $w_{i} = \sum_{k=1}^{\ell} 2 e_{pk} \in \mathbb{R}^{d}$ if $G_{i}$ is a bipartite graph whose vertices are divided into two independent sets $\{p_{1}, \ldots, p_{\ell}\}$ and $\{q_{1}, \ldots, q_{m}\}$. It then follows that $\mathcal{H} = \{ x \in \mathbb{R}^{d} : w \cdot x = 2 \}$ is a supporting hyperplane of $\mathcal{B}_{G}$ and the corresponding face $\mathcal{F}_{w} = \mathcal{B}_{G} \cap \mathcal{H}$ is the convex hull of $H = \bigcup_{i=1}^{s} H_{i}$, where $H_{i} = \{ e_{u} + e_{v} : (u,v) \in E(G_{i}) \}$ if $G_{i}$ is not bipartite, and $H_{i} = \{ e_{u} + e_{v} : (u,v) \in E(G_{i}) \} \cup \{ e_{p_{1}} \ldots e_{p_{\ell}} \} \cup \{ e_{pk} - e_{v} : \{p_{k}, v\} \in E(G_{i}) \}$ if $G_{i}$ is a bipartite graph with $w_{i} = \sum_{k=1}^{\ell} 2 e_{pk} \in \mathbb{R}^{d}$. We will show that $\mathcal{F}_{w}$ is a facet of $\mathcal{B}_{G}$. The
convex hull of \( \{e_u + e_v : \{u, v\} \in E(G_i)\} \) is the edge polytope \( P_{G_i} \) of \( G_i \) and it is known \cite{33} Proposition 1.3] that
\[
\dim P_{G_i} = \begin{cases} 
  d_i - 1 & \text{if } G_i \text{ is not bipartite}, \\
  d_i - 2 & \text{otherwise}.
\end{cases}
\]

If \( G_i \) is not bipartite, then the dimension of \( \operatorname{conv}(H_i) = P_{G_i} \) is \( d_i - 1 \). If \( G_i \) is bipartite, then \( d_i - 2 = \dim P_{G_i} < \dim \operatorname{conv}(H_i) \) since \( P_{G_i} \subset \operatorname{conv}(H_i) \) and a hyperplane \( \mathcal{H} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 + \cdots + x_d = 2\} \) satisfies \( e_{P_i} \notin \mathcal{H} \supset P_{G_i} \). Hence the dimension of the face \( \mathcal{F}_w \) is at least \( s - 1 + \sum_{i=1}^s (d_i - 1) = d - 1 \), i.e., \( \mathcal{F}_w \) is a facet of \( \mathcal{B}_G \). Since \( G_1 \) is not bipartite, we have \( \frac{1}{2} \cdot w \notin \mathbb{Z}^d \). Thus \( \mathcal{B}_G \) is not reflexive.

(iii) \( \Rightarrow \) (i): Suppose that \( G \) is bipartite. Let \( <_S \) and \( <_R \) be any reverse lexicographic orders satisfying the condition in Lemma \ref{2.3}. It is well-known that any triangulation of the edge polytope of a bipartite graph is unimodular. By \cite{44} Corollary 8.9], the initial ideal of the edge polytope of \( P_G \) with respect to \( <_S \) is squarefree. Thanks to Lemmas \ref{2.2} and \ref{2.3} we have a desired conclusion.

We now give a theorem on quadratic Gröbner bases of \( I_{\mathcal{B}_G} \) when \( G \) is bipartite. This theorem implies that Theorem \ref{0.2}. The same result is known for edge polytopes \cite{33}.

**Theorem 2.4.** Let \( G \) be a bipartite graph. Then the following conditions are equivalent:

(i) The toric ideal \( I_{\mathcal{B}_G} \) of \( \mathcal{B}_G \) has a squarefree quadratic initial ideal (i.e., \( \mathcal{B}_G \) has a flag regular unimodular triangulation);

(ii) The toric ring \( K[\mathcal{B}_G] \) of \( \mathcal{B}_G \) is a Koszul algebra;

(iii) The toric ideal \( I_{\mathcal{B}_G} \) of \( \mathcal{B}_G \) is generated by quadratic binomials;

(iv) Any cycle of \( G \) of length \( \geq 6 \) has a chord ("chordal bipartite graph").

**Proof.** Implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) hold in general.

(iii) \( \Rightarrow \) (iv): Suppose that \( G \) has a cycle of length \( \geq 6 \) without chords. By the theorem in \cite{33}, the toric ideal of \( P_G \) is not generated by quadratic binomials. Since the edge polytope \( P_G \) is a face of \( \mathcal{B}_G \), the toric ring \( K[P_G] \) is a combinatorial pure subring \cite{30} of \( K[\mathcal{B}_G] \). Hence \( I_{\mathcal{B}_G} \) is not generated by quadratic binomials.

(iv) \( \Rightarrow \) (i): Suppose that any cycle of \( G \) of length \( \geq 6 \) has a chord. By Lemma \ref{2.3} it is enough to show that the initial ideal of \( I_{\mathcal{B}_G} \) is squarefree and quadratic with respect to a reverse lexicographic order \( <_S \) such that \( z < \{x_{i+}\} < \{y_{k+}\} \). Let \( A = (a_{ij}) \) be the incidence matrix of \( G \) whose rows are indexed by \( V_1 \) and whose columns are indexed by \( V_2 \). Then the incidence matrix of \( \hat{G} \) is
\[
A' = \begin{pmatrix}
1 & \\
A' & \\
1 \cdots 1 & 1
\end{pmatrix}
\]
By the same argument as in the proof of \cite{33}, we may assume that \( A' \) contains no submatrices \( \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 \end{pmatrix} \) if we permute the rows and columns of \( A \) in \( A' \). Each quadratic binomial in \( I_{\mathcal{B}_G} \) corresponds to a submatrix \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) of \( A' \). The proof of the theorem in \cite{33}
guarantees that the initial ideal is squarefree and quadratic if the initial monomial of each quadratic binomial corresponds to \( \binom{1}{1} \). It is easy to see that there exists a such reverse lexicographic order which satisfies \( z < \{x_i\} < \{y_k\} \).

3. \( \gamma \)-POSITIVITY AND REAL-ROOTEDNESS OF THE \( h^* \)-POLYNOMIAL OF \( \mathcal{B}_G \)

In this section, we study the \( h^* \)-polynomial of \( \mathcal{B}_G \) for a graph \( G \). First, we recall what \( h^* \)-polynomials are. Let \( \mathcal{P} \subset \mathbb{R}^d \) be a lattice polytope of dimension \( d \). Given a positive integer \( n \), we define

\[
L_{\mathcal{P}}(n) = |n\mathcal{P} \cap \mathbb{Z}^d|.
\]

The study on \( L_{\mathcal{P}}(n) \) originated in Ehrhart [9] who proved that \( L_{\mathcal{P}}(n) \) is a polynomial in \( n \) of degree \( d \) with the constant term 1. We say that \( L_{\mathcal{P}}(n) \) is the Ehrhart polynomial of \( \mathcal{P} \). The generating function of the lattice point enumerator, i.e., the formal power series

\[
\text{Ehr}_{\mathcal{P}}(x) = 1 + \sum_{k=1}^{\infty} L_{\mathcal{P}}(k)x^k
\]

is called the Ehrhart series of \( \mathcal{P} \). It is well known that it can be expressed as a rational function of the form

\[
\text{Ehr}_{\mathcal{P}}(x) = \frac{h^*(\mathcal{P},x)}{(1-x)^{d+1}}.
\]

The polynomial \( h^*(\mathcal{P},x) \) is a polynomial in \( x \) of degree at most \( d \) with nonnegative integer coefficients [40] and it is called the \( h^* \)-polynomial (or the \( \delta \)-polynomial) of \( \mathcal{P} \). Moreover, one has \( \text{Vol}(\mathcal{P}) = h^*(\mathcal{P},1) \), where \( \text{Vol}(\mathcal{P}) \) is the normalized volume of \( \mathcal{P} \).

Thanks to Proposition 1.1 (a), we give a formula for \( h^* \)-polynomial of \( \mathcal{B}_G \) in terms of that of edge polytopes of some graphs. By the following formula, we can calculate the \( h^* \)-polynomial of \( \mathcal{B}_G \) if we can calculate each \( h^*(P_{\tilde{G}},x) \).

**Proposition 3.1.** Let \( G \) be a graph on \([d]\). Then the \( h^* \)-polynomial of \( \mathcal{B}_G \) satisfies

\[
h^*(\mathcal{B}_G,x) = \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in S_j(G)} h^*(P_{\tilde{H}},x),
\]

where \( S_j(G) \) denote the set of all induced subgraph of \( G \) with \( j \) vertices.

**Proof.** By Proposition 1.1 (a), \( \mathcal{B}_G \) is divided into \( 2^d \) lattice polytopes of the form \( \mathcal{B}_G \cap \mathcal{O}_e \). Each \( \mathcal{B}_G \cap \mathcal{O}_e \) is unimodularly equivalent to \( P_G \). In addition, the intersection of \( \mathcal{B}_G \cap \mathcal{O}_e \) and \( \mathcal{B}_G \cap \mathcal{O}_{e'} \) is of dimension \( d-1 \) if and only if \( e-e' \in \{ \pm 2e_1, \ldots, \pm 2e_d \} \). If \( e-e'=2e_k \), then \( (\mathcal{B}_G \cap \mathcal{O}_e) \cap (\mathcal{B}_G \cap \mathcal{O}_{e'}) = \mathcal{B}_G \cap \mathcal{O}_e \cap \mathcal{O}_{e'} \cong \mathcal{B}_G \cap \mathcal{O}_{e''} \), where \( G' \) is the induced subgraph of \( G \) obtained by deleting the vertex \( k \), and \( e'' \) is obtained by deleting \( k \)-th component of \( e \). Hence the Ehrhart polynomial \( L_{\mathcal{B}_G}(n) \) satisfies the following:

\[
L_{\mathcal{B}_G}(n) = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{H \in S_j(G)} L_{P_{\tilde{H}}}(n).
\]
Thus the Ehrhart series satisfies
\[
\frac{h^*(\mathcal{B}_G,x)}{(1-x)^{d+1}} = \sum_{j=0}^d 2^j (-1)^{d-j} \sum_{H \in \mathcal{S}_j(G)} \frac{h^*(P_{H^*},x)}{(1-x)^{j+1}},
\]
as desired. \qed

Let \( f = \sum_{i=0}^d a_ix^i \) be a polynomial with real coefficients and \( a_d \neq 0 \). We now focus on the following properties.

(RR) We say that \( f \) is real-rooted if all its roots are real.

(LC) We say that \( f \) is log-concave if \( a_i^2 \geq a_{i-1}a_{i+1} \) for all \( i \).

(UN) We say that \( f \) is unimodal if \( a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d \) for some \( k \).

If all its coefficients are nonnegative, then these properties satisfy the implications

(RR) \( \Rightarrow \) (LC) \( \Rightarrow \) (UN).

On the other hand, the polynomial \( f \) is said to be palindromic if \( f(x) = x^df(x^{-1}) \). It is \( \gamma \)-positive if there are \( \gamma_0, \gamma_1, \ldots, \gamma_{d/2} \geq 0 \) such that \( f(x) = \sum_{i=0}^{d/2} \gamma_i x^i (1+x)^{d-2i} \). The polynomial \( \sum_{i=0}^{d/2} \gamma_i x^i \) is called \( \gamma \)-polynomial of \( f \). We can see that a \( \gamma \)-positive polynomial is real-rooted if and only if its \( \gamma \)-polynomial had only real roots.

By the following proposition, we are interested in connected bipartite graphs.

**Proposition 3.2.** Let \( G \) be a bipartite graph and \( G_1, \ldots, G_s \) the connected components of \( G \). Then the \( h^* \)-polynomial of \( \mathcal{B}_G \) is palindromic, unimodal and
\[
h^*(\mathcal{B}_G,x) = h^*(\mathcal{B}_{G_1},x) \cdots h^*(\mathcal{B}_{G_s},x).
\]

**Proof.** It is known \([12]\) that the \( h^* \)-polynomial of a lattice polytope \( P \) with the interior lattice point \( 0 \) is palindromic if and only if \( P \) is reflexive. Moreover, if a reflexive polytope \( P \) has a unimodular triangulation, then the \( h^* \)-polynomial of \( P \) is unimodal (see \([6]\)). It is easy to see that, \( \mathcal{B}_G \) is the free sum of \( \mathcal{B}_{G_1}, \ldots, \mathcal{B}_{G_s} \). Thus we have a desired conclusion by Theorem \([0,1]\) and \([5\) Theorem 1]. \qed

In the rest of the present paper, we discuss the \( \gamma \)-positivity and the real-rootedness on the \( h^* \)-polynomial of \( \mathcal{B}_G \) when \( G \) is a bipartite graph. The edge polytope of a bipartite graph \( G \) is called a root polytope of \( G \) and it is shown \([22]\) that the \( h^* \)-vector \( P \) has a unimodular triangulation, then the \( h^* \)-polynomial of \( P \) is unimodal if it is reflexive. Moreover, if a reflexive polytope \( P \) has a unimodular triangulation, then the \( h^* \)-polynomial of \( P \) is unimodal (see \([6]\)). It is easy to see that, \( \mathcal{B}_G \) is the free sum of \( \mathcal{B}_{G_1}, \ldots, \mathcal{B}_{G_s} \). Thus we have a desired conclusion by Theorem \([0,1]\) and \([5\) Theorem 1].

**Proposition 3.2.** Let \( \mathcal{H} = (V,E) \), where \( E = \{e_1, \ldots, e_n\} \) is a finite multiset of non-empty subsets of \( V = \{v_1, \ldots, v_m\} \). Elements of \( V \) are called vertices and the elements of \( E \) are the hyperedges. Then we can associate \( \mathcal{H} \) to a bipartite graph \( \text{Bip.}\mathcal{H} \) with a bipartition \( V \cup E \) such that \( \{v_i, e_j\} \) is an edge of \( \text{Bip.}\mathcal{H} \) if \( v_i \in e_j \). Assume that \( \text{Bip.}\mathcal{H} \) is connected. A hypertree in \( \mathcal{H} \) is a function \( f : E \to \{0,1,\ldots\} \) such that there exists a spanning tree \( \Gamma \) of \( \text{Bip.}\mathcal{H} \) whose vertices have degree \( f(e) + 1 \) at each \( e \in E \). Then we say that \( \Gamma \) induce \( f \). Let \( B_\mathcal{H} \) denote the set of all hypertrees in \( \mathcal{H} \). A hyperedge \( e_j \in E \) is said to be internally active with respect to the hypertree \( f \) if it is not possible to decrease \( f(e_j) \) by 1 and increase \( f(e_{j'}) \) (\( j' < j \)) by 1 so that another hypertree results. We call a hyperedge internally inactive with respect to a hypertree if it is not internally active and denote the number of such hyperedges of \( f \) by \( \tau(f) \). Then the interior polynomial of \( \mathcal{H} \)
is the generating function $I_{\mathcal{M}}(x) = \sum_{x^{T(e)}}$. It is known [22, Proposition 6.1] that $\deg I_{\mathcal{M}}(x) \leq \min \{|V|, |E|\} - 1$. If $G = \text{Bip} \mathcal{M}$, then we set $I_G(x) = I_{\mathcal{M}}(x)$. Kálmán and Postnikov [23] proved that

$$I_G(x) = h^*(P_G, x)$$

for a connected bipartite graph $G$. Note that if $G$ is a bipartite graph, then the bipartite graph $\hat{G}$ arising from $G$ is connected. Hence we can use this formula to study equation (2) in Proposition 3.1. (Interior polynomials of disconnected bipartite graphs are defined in [24].) A $k$-matching of $G$ is a set of $k$ pairwise non-adjacent edges of $G$. Let

$$M(G, k) = \left\{ \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} : \text{there exists a } k\text{-matching of } G \text{ whose vertex set is } \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} \right\}.$$ 

For $k = 0$, we set $M(G, 0) = \{0\}$. Using the theory of generalized permutohedra [29, 37], we have the following important fact on interior polynomials:

**Proposition 3.3.** Let $G$ be a bipartite graph. Then we have

$$(3) \quad I_{\hat{G}}(x) = \sum_{k \geq 0} |M(G, k)| x^k.$$

**Proof.** Let $V \cup E$ denote a bipartition of $G$, where $V = \{v_2, \ldots, v_m\}$ and $E = \{e_2, \ldots, e_n\}$ with $d = m + n - 2$. Then $\hat{G}$ is a connected bipartite graph with a bipartition $V' \cup E'$ with $V' = \{v_1\} \cup V$ and $E' = \{e_1\} \cup E$. Recall that $\{v_i, e_j\}$ is an edge of $G$ if and only if either $(i-1)(j-1) = 0$ or $\{v_i, e_j\}$ is an edge of $G$. Let $\text{HT}(\hat{G})$ be the set of all hypertrees in the hypergraph associated with $\hat{G}$. Given a hypertree $f \in \text{HT}(\hat{G})$, let $G$ be a spanning tree that induces $f$. We now repeat the following procedure for $G$:

- For each $j = 1, 2, \ldots, n$, since $G$ is a spanning tree, there exists a unique path $e_j v_i \ldots v_j$ from $e_j$ to $v_j$. If $i > 1$, then remove $\{v_i, e_j\}$ from $G$ and add $\{v_1, e_j\}$ to $G$. Then it follows that the new $G$ is a spanning tree that induces $f$.

Hence we may assume that $\{v_1, e_j\}$ is an edge of $G$ for all $1 \leq j \leq n$. Note that the degree of each $v_j$ ($2 \leq i \leq m$) is 1.

By definition, $e_j$ is always internally active. We show that, $e_j$ ($j \geq 2$) is internally active if and only if $f(e_j) = 0$. By definition, if $f(e_j) = 0$, then $e_j$ is internally active. Suppose $f(e_j) > 0$. Then there exists $i \geq 2$ such that $\{v_i, e_j\}$ is an edge of $G$. Let $f' \in \text{HT}(\hat{G})$ be a hypertree induced by a spanning tree obtained by replacing $\{v_1, e_1\}$ with $\{v_i, e_1\}$ in $G$. Then we have $f'(e_j) = f(e_j) - 1$, $f'(e_1) = f(e_1) + 1$ and $f'(e_k) = f(e_k)$ for all $1 < k \neq j$. Hence $e_j$ is not internally active. Thus $\bar{t}(f)$ is the number of $e_j$ ($j \geq 2$) such that there exists an edge $\{v_i, e_j\}$ of $G$ for some $i \geq 2$.

In order to prove the equation (3), it is enough to show that, for fixed hyperedges $e_{j_1}, \ldots, e_{j_k}$ with $2 \leq j_1 < \cdots < j_k \leq n$, the cardinality of

$$\mathcal{S}_{j_1, \ldots, j_k} = \{f \in \text{HT}(\hat{G}) : e_{j_1}, \ldots, e_{j_k} \text{ are not internally active and } \bar{t}(f) = k\}$$

is equal to the cardinality of

$$\mathcal{M}_{j_1, \ldots, j_k} = \left\{ \{v_{i_1}, \ldots, v_{i_k}\} : \text{there exists a } k\text{-matching of } G \text{ whose vertex set is } \{v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_k}\} \right\}.$$
Let $G_{j_1,...,j_k}$ be the induced subgraph of $G$ on the vertex set $V \cup \{e_{j_1},...,e_{j_k}\}$. If $e_{j_\ell}$ is an isolated vertex in $G_{j_1,...,j_k}$, then both $\mathcal{J}_{j_1,...,j_k}$ and $\mathcal{M}_{j_1,...,j_k}$ are empty sets. If $v_i$ is an isolated vertex in $G_{j_1,...,j_k}$, then there is no relation between $v_i$ and two sets, and hence we can ignore $v_i$. Thus we may assume that $G_{j_1,...,j_k}$ has no isolated vertices.

It is known that $\mathcal{M}_{j_1,...,j_k}$ is the set of bases of a transversal matroid associated with $G_{j_1,...,j_k}$. See, e.g., [36]. For $i = 2, ..., m$, let

$$I_i = \{0\} \cup \{\ell : \{v_i, e_{j_\ell}\} \text{ is an edge of } G_{j_1,...,j_k} \subset \{0,1,\ldots,k\}\}.$$

Oh [29] define a lattice polytope $P_{\mathcal{M}_{j_1,...,j_k}}$ to be the generalized permutohedron [37] of the induced subgraph of $\tilde{G}$ on the vertex set $V \cup \{e_{j_1},...,e_{j_k}\}$, i.e., $P_{\mathcal{M}_{j_1,...,j_k}}$ is the Minkowski sum $\Delta_{\ell_2} + \ldots + \Delta_{\ell_m}$, where $\Delta_{\ell} = \text{Conv} (\{e_j : j \in I\}) \subset \mathbb{R}^{k+1}$ and $e_0, e_1,...,e_k$ are unit coordinate vectors in $\mathbb{R}^{k+1}$. By [29, Lemma 22 and Proposition 26], the cardinality of $\mathcal{M}_{j_1,...,j_k}$ is equal to the number of the lattice point $(x_0,x_1,\ldots,x_k) \in P_{\mathcal{M}_{j_1,...,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1,x_2,\ldots,x_k \geq 1$. In addition, by [37, Proposition 14.12], any lattice point $(x_0,x_1,\ldots,x_k) \in P_{\mathcal{M}_{j_1,...,j_k}} \cap \mathbb{Z}^{k+1}$ is of the form $e_2 + \ldots + e_{\ell_m}$, where $i_\ell \in I_\ell$ for $2 \leq \ell \leq m$. By a natural correspondence $(x_0,x_1,\ldots,x_k) \in P_{\mathcal{M}_{j_1,...,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1,x_2,\ldots,x_k \geq 1$ and $(f(e_{j_1}),f(e_{j_2}),...,f(e_{j_k}))$ with $f \in \mathcal{J}_{j_1,...,j_k}$, it follows that the number of the lattice point $(x_0,x_1,\ldots,x_k) \in P_{\mathcal{M}_{j_1,...,j_k}} \cap \mathbb{Z}^{k+1}$ with $x_1,x_2,\ldots,x_k \geq 1$ is equal to the cardinality of $\mathcal{J}_{j_1,...,j_k}$, as desired.

Now, we show that the $h^*$-polynomial of $\mathcal{B}_G$ is $\gamma$-positive if $G$ is a bipartite graph. In fact, we prove Theorem 0.3.

**Proof of Theorem 0.3** By Propositions 3.1 and 3.3, the $h^*$-polynomial of $\mathcal{B}_G$ is

$$h^*(\mathcal{B}_G,x) = \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in \mathcal{S}_j(G)} I_H(x)$$

$$= \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{H \in \mathcal{S}_j(G)} \sum_{k \geq 0} |M(H,k)| x^k.$$

Note that, for each $\{v_{i_1},\ldots,v_{i_k},e_{j_1},\ldots,e_{j_k}\} \in M(G,k)$ there exist $\binom{d-2k}{j-2k}$ induced subgraphs $H \in \mathcal{S}_j(G)$ such that $\{v_{i_1},\ldots,v_{i_k},e_{j_1},\ldots,e_{j_k}\} \in M(H,k)$ for $j = 2k, 2k+1, \ldots, d$. Thus we have

$$h^*(\mathcal{B}_G,x) = \sum_{k \geq 0} \sum_{j=2k}^{d} 2^j (x-1)^{d-j} |M(G,k)| \binom{d-2k}{j-2k} x^k$$

$$= \sum_{k \geq 0} |M(G,k)| 2^{2k} x^k (2 + (x-1))^{d-2k}$$

$$= (x+1)^d \sum_{k \geq 0} |M(G,k)| \left( \frac{4x}{(x+1)^2} \right)^k$$

$$= (x+1)^d I_G \left( \frac{4x}{(x+1)^2} \right).$$
as desired. □

By Proposition 3.3, it follows that, if $G$ is a forest, then $I_{\hat{G}}(x)$ coincides with the matching generating polynomial of $G$.

**Proposition 3.4.** Let $G$ be a forest. Then we have

$$I_{\hat{G}}(x) = \sum_{k \geq 0} m_k(G) x^k,$$

where $m_k(G)$ is the number of $k$-matching in $G$. In particular, $I_{\hat{G}}(x)$ is real-rooted.

**Proof.** Let $M_1$ and $M_2$ be $k$-matchings of $G$. Suppose that $M_1$ and $M_2$ have the same vertex set $\{v_1, \ldots, v_k, e_{j_1}, \ldots, e_{j_k}\}$. If $M = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$ is not empty, then $M$ corresponds to a subgraph of $G$ such that the degree of each vertex is 2. Hence $M$ has at least one cycle. This contradicts that $G$ is a forest. Hence we have $M_1 = M_2$. Thus $m_k(G)$ is the cardinality of $M(G, k)$. In general, it is known that $\sum_{k \geq 0} m_k(G) x^k$ is real-rooted for any graph $G$. See, e.g., [11, 27]. □

Next we will show that, if a bipartite graph $G$ is a “permutation graph” associated with a poset $P$, then the interior polynomial $I_{\hat{G}}(x)$ coincides with $P$-Eulerian polynomial $W(P)(x)$. A *permutation graph* is a graph on $[d]$ with edge set

$$\{\{i, j\} : L_i$ and $L_j$ intersect each other\},
$$

where there are $d$ points $1, 2, \ldots, d$ on two parallel lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in the plane, and the straight lines $L_i$ connect $i$ on $\mathcal{L}_1$ and $i$ on $\mathcal{L}_2$. If $G$ is a bipartite graph with a bipartition $V_1 \cup V_2$, the following conditions are equivalent:

(i) $G$ is permutation;

(ii) The complement of $G$ is a comparability graph of a poset;

(iii) There exist orderings $<_1$ on $V_1$ and $<_2$ on $V_2$ such that

$$i, i' \in V_1, i <_1 i', j, j' \in V_2, j <_2 j', \{i, j\}, \{i', j'\} \in E(G) \implies \{i, j'\}, \{i', j\} \in E(G);$$

(iv) For any three vertices, there exists a pair of them such that there exists no path containing the two vertices that avoids the neighborhood of the remaining vertex.

See [4] for details. On the other hand, let $P$ be a naturally labeled poset $P$ on $[d]$. Then the order polynomial $\Omega(P, m)$ of $P$ is defined for $0 < m \in \mathbb{Z}$ to be the number of order-preserving maps $\sigma : P \to [m]$. It is known that

$$\sum_{m \geq 0} \Omega(P, m + 1)x^m = \frac{\sum_{\pi \in \mathcal{L}(P)} x^{d(\pi)}}{(1 - x)^{d + 1}},$$

where $\mathcal{L}(P)$ is the set of linear extensions of $P$ and $d(\pi)$ is the number of descent of $\pi$. The $P$-Eulerian polynomial $W(P)(x)$ is defined by

$$W(P)(x) = \sum_{\pi \in \mathcal{L}(P)} x^{d(\pi)}.$$

See, e.g., [42] for details. We now give a relation between the interior polynomial and the $P$-Eulerian polynomial of a finite poset.
**Proposition 3.5.** Let $G$ be a bipartite permutation graph and let $P$ be a poset whose comparability graph is the complement of a bipartite graph $G$. Then we have

$$I_G(x) = W(P)(x).$$

**Proof.** In this case, $B_G \cap O_{(1,\ldots,1)}$ is the chain polytope $C_P$ of $P$. It is known that the $h^*$-polynomial of $C_P$ is the $P$-Eulerian polynomial $W(P)(x)$. See [41, 42] for details. Thus we have $I_G(x) = h^*(P_G, x) = W(P)(x)$, as desired. 

It was conjectured by Neggers–Stanley that $W(P)(x)$ is real-rooted. However this is false in general. The first counterexample was given in [41] (not naturally labeled posets). Counterexamples of naturally labeled posets were given in [43]. Counterexamples in these two papers are narrow posets, i.e., elements of posets are partitioned into two chains. It is easy to see that $P$ is narrow poset if and only if the comparability graph of $P$ is the complement of a bipartite graph. Since Stembridge found many counterexamples which are naturally labeled narrow posets, there are many bipartite permutation graphs $G$ such that $h^*(B_G, x)$ are not real-rooted. We give one of them as follows.

**Example 3.6.** Let $P$ be a naturally labeled poset in Figure 2 given in [43]. Then

$$W(P)(x) = 3x^8 + 86x^7 + 658x^6 + 1946x^5 + 2534x^4 + 1420x^3 + 336x^2 + 32x + 1$$

has a conjugate pair of zeros near $-1.85884 \pm 0.149768i$ as explained in [43]. Let $G$ be the complement of the comparability graph of $P$. Then $G$ is a bipartite graph with 17 vertices and 32 edges. The $h^*$-polynomial of $B_G$ is

$$h^*(B_G, x) = (x + 1)^{17} W(P) \left( \frac{4x}{(x + 1)^2} \right)$$

$$= x^{17} + 145x^{16} + 7432x^{15} + 174888x^{14} + 2128332x^{13} + 14547884x^{12} + 59233240x^{11} + 148792184x^{10} + 234916470x^9 + 234916470x^8 + 148792184x^7 + 59233240x^6 + 14547884x^5 + 2128332x^4 + 174888x^3 + 7432x^2 + 145x + 1$$

**Figure 2.** A counterexample of Neggers–Stanley conjecture [43]
and has conjugate pairs of zeros near $-3.88091 \pm 0.18448i$ and $-0.257091 \pm 0.0122209i$. (We used Mathematica to compute approximate values.) On the other hand, $h^*(B_G, x)$ is log-concave.

By the following proposition, it turns out that this example is a counterexample of “Real Root Conjecture” that has been already disproved by Gal [10].

**Proposition 3.7.** Let $G$ be a bipartite permutation graph. Then $h^*(B_G, x)$ coincides with the $h$-polynomial of a flag complex that is a triangulation of a sphere.

**Proof.** It is known [4, p.94] that any bipartite permutation graph satisfies the condition (iv) in Theorem 2.4. Hence there exists a squarefree quadratic initial ideal with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin. By the theory of regular unimodular triangulations [44, Chapter 8], this means that there exists a flag regular unimodular triangulation $\Delta$ such that the origin is a vertex of any maximal simplex in $\Delta$. Then $h^*(B_G, x)$ coincides with the $h$-polynomial of a flag unimodular triangulation of the boundary of a convex polytope $B_G$ arising from $\Delta$. \hfill \square

Let $G$ be a finite simple graph on $[d]$ with the edge set $E(G)$ and let $\mathcal{A}_G \subset \mathbb{R}^d$ denote the convex hull of the set

$$A(G) = \{e_i - e_j, e_j - e_i : \{i, j\} \in E(G)\}.$$

The lattice polytope $\mathcal{A}_G$ is called the symmetric edge polytope of $G$. In [21], for a $(p, q)$-complete bipartite graph $K_{p, q}$, the simple description for the $h^*$-polynomials of $\mathcal{A}_K_{p, q}$ was given and it is $\gamma$-positive and real-rooted. In fact, one has

$$h^*(\mathcal{A}_{K_{p+1, q+1}}, x) = \sum_{i=0}^{\min\{p, q\}} \binom{2i}{i} \binom{p}{i} \binom{q}{i} x^i(x+1)^{p+q+1-2i}.$$

Similarly, we can give the simple description for the $h^*$-polynomial of $B_{K_{p, q}}$ and show that it is $\gamma$-positive and real-rooted.

**Example 3.8.** Let $K_{p, q}$ be a $(p, q)$-complete bipartite graph. Then the comparability graph of a poset $P$ consisting of two disjoint chains $1 < 2 < \cdots < p$ and $p + 1 < p + 2 < \cdots < p + q$ is the complement of $K_{p, q}$. It is easy to see that

$$W(P)(x) = \sum_{i=0}^{\min\{p, q\}} \binom{p}{i} \binom{q}{i} x^i.$$

Hence we have

$$h^*(B_{K_{p, q}}, x) = (x+1)^{p+q} W(P) \left( \frac{4x}{(x+1)^2} \right) = \sum_{i=0}^{\min\{p, q\}} 4^i \binom{p}{i} \binom{q}{i} x^i(x+1)^{p+q-2i}.$$

Simion [38] proved that $W(P)(x)$ is real-rooted if $P$ is a naturally labeled and disjoint union of chains. Thus the $h^*$-polynomial $h^*(B_{K_{p, q}}, x)$ is real-rooted.

In [21], for the proof of the real-rootedness of $h^*(\mathcal{A}_{K_{p, q}}, x)$, interlacing polynomials techniques were used. Let $f$ and $g$ be real-rooted polynomials with roots $a_1 \geq a_2 \geq \cdots$, respectively, $b_1 \geq b_2 \geq \cdots$. Then $g$ is said to interlace $f$ if

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots.$$
In this case, we write $f \preceq g$. In [21], it is shown that
\[ h^*(\mathcal{A}_{K_p}, x) \preceq h^*(\mathcal{A}_{K_{p+1}}, x). \]

By a similar way of [21], we can prove the following.

**Proposition 3.9.** For all $p, q \geq 1$, one has
\[ h^*(\mathcal{B}_{K_p}, x) \preceq h^*(\mathcal{B}_{K_{p+1}}, x). \]

**Proof.** Set $\gamma(\mathcal{B}_{K_p}, x) = \sum_{i \geq 0} 4^i \binom{n}{i} \left( \binom{q}{i} x^i \right)$. Since $\{ \binom{n}{i} \}_{i \geq 0}$ is a multiplier sequence (see [21]), and since $(4x+1)^q \preceq (4x+1)^{q+1}$, one has $\gamma(\mathcal{B}_{K_p}, x) \preceq \gamma(\mathcal{B}_{K_{p+1}}, x)$. By [21, Lemma 4.10], we obtain $h^*(\mathcal{B}_{K_p}, x) \preceq h^*(\mathcal{B}_{K_{p+1}}, x).$ □

**REFERENCES**

[1] F. Ardila, M. Back, S. Hoşten, J. Pfeifle and K. Seashore, Root Polytopes and Growth Series of Root Lattices, *SIAM J. Discrete Math.* 25 (2011), 360–378.

[2] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.*, 3 (1994), 493–535.

[3] P. Brändén, Counterexamples to the Neggers-Stanley conjecture, *Electron. Res. Announc. Amer. Math. Soc.* (2004), (electronic).

[4] A. Brandstädt, V.B. Le, and J.P. Spinrad, “Graph Classes: A Survey,” SIAM Monographs on Discrete Math. Appl., Vol. 3, SIAM, Philadelphia, 1999.

[5] B. Braun, An Ehrhart Series Formula For Reflexive Polytopes, *Electron. J. Combin.* 13 (2006), #N15.

[6] W. Bruns and T. Römer, $h$-Vectors of Gorenstein polytopes, *J. Combin. Theory Ser. A* 114 (2007), 65–76.

[7] D. A. Cox, C. Haase, T. Hibi and A. Higashitani, Integer decomposition property of dilated polytopes, *Electron. J. Combin.* 21 (4) (2014), #P4.28.

[8] D. Cox, J. Little and H. Schenck, “Toric varieties”, Amer. Math. Soc., 2011.

[9] E. Ehrhart, “Polynômes Arithmétiques et Méthode des Polyédres en Combinatorie”, Birkhäuser, Boston/Basel/Stuttgart, 1977.

[10] S. R. Gal, Real Root Conjecture fails for five and higher dimensional spheres, *Discrete Comput. Geom.*, 34 (2005), 269–284.

[11] O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Commun. Math. Physics* 25 (1972) 190–232.

[12] T. Hibi, Dual polytopes of rational convex polytopes, *Combinatorica* 12 (1992), 237–240.

[13] T. Hibi and K. Matsuda, Quadratic Gröbner bases of twinned order polytopes, *European J. Combin.* 54 (2016), 187–192.

[14] T. Hibi, K. Matsuda, H. Ohsugi, and K. Shibata, Centrally symmetric configurations of order polytopes, *J. Algebra* 443 (2015), 469–478.

[15] T. Hibi, K. Matsuda and A. Tsuchiya, Quadratic Gröbner bases arising from partially ordered sets, *Math. Scand.* 121 (2017), 19–25.

[16] T. Hibi, K. Matsuda and A. Tsuchiya, Edge rings with 3-linear resolutions, *Proc. Amer. Math. Soc.*, to appear.

[17] T. Hibi and A. Tsuchiya, Facets and volume of Gorenstein Fano polytopes, *Math. Nachr.* 290 (2017), 2619–2628.

[18] T. Hibi and A. Tsuchiya, Reflexive polytopes arising from perfect graphs, *J. Combin. Theory Ser. A* 157 (2018), 233–246.

[19] T. Hibi and A. Tsuchiya, Reflexive polytopes arising from partially ordered sets and perfect graphs, *J. Algebraic Combin.*, https://doi.org/10.1007/s10801-018-0817-3.

[20] A. Higashitani, M. Kummer and M. Michałek, Interlacing Ehrhart polynomials of reflexive polytopes, *Selecta Math.* 23 (2017), 2977–2998.
[21] A. Higashitani, K. Jochemko and M. Michałek, Arithmetic aspects of symmetric edge polytopes, arXiv:1807.07678.
[22] T. Kálmán, A version of Tutte’s polynomial for hypergraphs, Adv. Math. 244 (2013), 823–873.
[23] T. Kálmán and A. Postnikov, Root polytopes, Tutte polynomials, and a duality theorem for bipartite graphs, Proc. Lond. Math. Soc. 114 (2017), 561–588.
[24] K. Kato, Interior polynomial for signed bipartite graphs and the HOMFLY polynomial, arXiv: 1705.05063.
[25] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4 (2000), 1209–1230.
[26] J. C. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, Canad. J. Math. 43 (1991), 1022–1035.
[27] J. A. Makowsky, E. V. Ravve and Ni. K. Blanchard, On the location of roots of graph polynomials, European J. Combin. 41 (2014) 1–19.
[28] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, Roots of Ehrhart polynomials arising from graphs, J. Algebraic Combin. 34 (2011), 721–749.
[29] S. Oh, Generalized permutohedra, $h$-vectors of cotransversal matroids and pure $O$-sequences, Electron. J. Combin. 20 (3) (2013) P1.14.
[30] H. Ohsugi J. Herzog and T. Hibi, Combinatorial pure subrings, Osaka J. Math. 37 (2000), 745–757.
[31] H. Ohsugi and T. Hibi, Normal polytopes arising from finite graphs, J. Algebra 207 (1998), 409–426.
[32] H. Ohsugi and T. Hibi, A normal $(0,1)$-polytope none of whose regular triangulations is unimodular, Disc. Comput. Geom. 21 (1999), 201–204.
[33] H. Ohsugi and T. Hibi, Koszul bipartite graphs, Adv. Applied Math. 22 (1999), 25–28.
[34] H. Ohsugi and T. Hibi, Centrally symmetric configurations of integer matrices, Nagoya Math. J. 216 (2014), 153–170.
[35] H. Ohsugi and T. Hibi, Reverse lexicographic squarefree initial ideals and Gorenstein Fano polytopes, J. Commut. Alg. 10 (2018), 171–186.
[36] J. G. Oxley, “Matroid theory,” Oxford University Press, 2006.
[37] A. Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Notices 6 (2009), 1026–1106.
[38] R. Simion, A multi-indexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences, J. Combin. Theory Ser. A 36 (1) (1984), 15-22.
[39] A. Simis, W. V. Vasconcelos and R. H. Villarreal, The integral closure of subrings associated to graphs, J. Algebra 199 (1998), 281–289.
[40] R. P. Stanley, Decompositions of rational convex polytopes, Annals of Discrete Math. 6 (1980), 333–342.
[41] R. P. Stanley, Two poset polytopes, Disc. Comput. Geom. 1 (1986), 9–23.
[42] R. P. Stanley, “Enumerative Combinatorics, volume I.” Wadsworth & Brooks / Cole Advanced Books, Monterey, Calif., 1986.
[43] J. R. Stembridge, Counterexamples to the poset conjectures of Neggers, Stanley, and Stembridge, Trans. Amer. Math. Soc. 359 (2007), 1115–1128.
[44] B. Sturmfels, “Gröbner bases and convex polytopes,” Amer. Math. Soc., Providence, RI, 1996.