ZARISKI TOPOLOGIES ON GROUPS

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Abstract. The $n$-th Zariski topology $3_{G^n[x]}$ on a group $G$ is generated by the sub-base consisting of the sets $\{x \in G : a_0x^{\varepsilon_1}a_1x^{\varepsilon_2} \cdots x^{\varepsilon_n}a_n \neq 1\}$ where $a_0, \ldots, a_n \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. We prove that for each group $G$ the 2-nd Zariski topology $3_{G^2[x]}$ is not discrete and present an example of a group $G$ of cardinality continuum whose 2-nd Zariski topology has countable pseudocharacter. On the other hand, the non-topologizable group $G$ constructed by Ol’shanskii has discrete 665-th Zariski topology $3_{665[x]}$.

In this paper we study topological properties of groups endowed with the Zariski topologies $3_{G^n[x]}$. These topologies are defined on each group $G$ as follows. Let $G[x] = G \ast \langle x \rangle$ be the free product of the group $G$ and the cyclic group $\langle x \rangle$ generated by an element $x \notin G$. The group $G[x]$ can be written as the countable union

$$G[x] = \bigcup_{n \in \omega} G^n[x].$$

Here for a subset $A \subset G$ we put

$$A^0[x] = A \text{ and } A^{n+1}[x] = A^n[x] \cup \{wx : w, x^{-1}a : w \in A^n[x], a \in A\} \text{ for } n \in \omega.$$ 

The elements of the subset $A^n[x]$ are called monomials of degree $\leq n$ with coefficients in the set $A$.

Each monomial $w \in G[x]$ can be thought as a map $w(\cdot) : G \rightarrow G$, $w : g \mapsto w(g)$, where $w(g)$ is the image of $w$ under the group homomorphism $G[x] \rightarrow G$ that is identical on $G$ and maps the element $x \in G[x]$ onto $g \in G$. The subset

$$3_w = \{x \in G : w(x) \neq 1\}$$

is called the co-zero set of the monomial $w$. Here 1 is the neutral element of $G$.

Any family of monomials $W \subset G[x]$ induces a topology $3_W$ on $G$, generated by the sub-base $\{3_w : w \in W\}$ consisting of co-zero sets of the monomials from $W$. The topology $3_{G[x]}$ is called the Zariski topology on $G$ by analogy with the Zariski topology well-known in algebraic geometry. This topology was studied in [6], [2]. For $n \in \omega$ the topology $3_{G^n[x]}$ will be referred to as the $n$-th Zariski topology on $G$. It is clear that the Zariski topologies form a chain

$$3_{G^0[x]} \subset 3_{G^1[x]} \subset 3_{G^2[x]} \subset \cdots \subset 3_{G[x]}.$$ 

The 0-th Zariski topology $3_{G^0[x]}$ is antidiscrete while the 1-st Zariski topology $3_{G^1[x]}$ coincides with the cofinite topology on $G$ and hence satisfies the separation axiom $T_1$. The same is true for Zariski topologies $3_{G^n[x]}$ with $n \geq 1$.

It is easy to see that for every $n \in \omega$ the group $G$ endowed with the $n$th Zariski topology $3_{G^n[x]}$ is a quasi-topological group. The latter means that the group operation is separately continuous and the inversion is continuous with respect to the topology $3_{G^n[x]}$ on $G$.

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As we already know, for an infinite group $G$, the topology $T_2[G[x]]$, being cofinite, is non-discrete. The same is true for the topology $T_2[G^2[x]]$.

**Theorem 1.** For every infinite group $G$ the 2-nd Zariski topology $T_2[G^2[x]]$ is not discrete.

**Proof.** Assuming the converse, we would find a finite subset $W \subset G^2[x]$ such that $\{1\} = \bigcap_{w \in W} T_2[w]$. Find a countable subgroup $H \subset G$ such that $W \subset H \ast \langle x \rangle = H[x]$. By [17], the group $H$ is a quasitopological group with respect to some non-discrete Hausdorff (even regular) topology $τ$.

We claim that for each monomial $w = a_0x^{±1}a_1x^{±2}a_2 \in W \subset H^2[x]$ with $±1, ±2 \in \{-1, 1\}$ the cozero set $T_2[w]$ contains a neighborhood $U_w \in τ$ of 1. Since $1 \in T_2[w]$, we get $a_01^{±1}a_11^{±2}a_2 \neq 1$ and hence $a_01^{±1}a_1 \neq a_2^{±1}1^{±2}$. Since the topology $τ$ on $H$ is Hausdorff, the points $a_01^{±1}a_1$ and $a_2^{±1}1^{±2}$ have disjoint neighborhoods $O(a_01^{±1}a_1), O(a_2^{±1}1^{±2}) \in τ$. Since $(H, τ)$ is a quasitopological group, there is a neighborhood $U_w \in τ$ of 1 such that $a_0U_w^{±1}a_1 \subset O(a_01^{±1}a_1)$ and $a_2^{±1}U_w^{±2} \subset O(a_2^{±1}1^{±2})$. For this neighborhood $U_w$ we get $1 \notin a_0U_w^{±1}a_1U_w^{±2}a_2$ and $U_w \subset T_2[w]$. Now we see from $\bigcap_{w \in W} U_w \subset \bigcap_{w \in W} T_2[w] = \{1\}$ that 1 is an isolated point of $H$ in the topology $τ$, which is a contradiction. □

Theorem 1 implies that for each infinite group $G$ the quasitopological group $(G, T_2[G^2[x]])$ has infinite pseudocharacter. By the pseudocharacter $ψ(x)(X)$ of a topological $T_1$-space at a point $x \in X$ we understand the smallest cardinality $|U_x|$ of a family $U_x$ of neighborhoods of the point $x$ such that $\cap U_x = \{x\}$. The cardinal $ψ(x) = \sup x ∈ X ψ(x)(X)$ is called the pseudocharacter of the topological space $X$.

Writing Theorem 1 in terms of the pseudocharacter, we get

**Corollary 1.** For any infinite group $G$ we get

$$ψ(G, T_2[G^2[x]]) = |G| \text{ and } ψ(G, T_2[G^2[x]]) ≥ ℵ_0.$$ 

It is interesting to observe that the lower bound $ψ(G, T_2[G^2[x]]) ≥ ℵ_0$ in this corollary cannot be improved to $ψ(G, T_2[G^2[x]]) = |G|$.

**Theorem 2.** There is a group $G$ of cardinality continuum such that $ψ(G, T_2[G^2[x]]) = ℵ_0$. In fact, the group $G$ contains two disjoint countable subsets $A, B \subset G$ such that $\{1\} = \bigcap_{w \in W} T_2[w]$ where $W = \{ax^{−1}b^{−1} : a \in A, b \in B\}$.

**Proof.** Let $T = \bigcup_{n \in ℤ} \{0, 1\}^n$ be the binary tree (consisting of finite binary sequences) and $\text{Aut}(T)$ be its automorphism group (which has cardinality of continuum). The action of $\text{Aut}(T)$ on $T$ induces an action of $\text{Aut}(T)$ on the free group $F(T)$ over $T$. In the free group $F(T)$ consider the set $L$ of words of the form $al$ where $a \in T$ and $l = a^{−0}$ is the “left” successor of $a$ in the binary tree $T$. By symmetry, let $R \subset F(T)$ be the set of words $ar$ where $a \in T$ and $r = a^{−1}$ is the “right” successor of $a$ in $T$. It is easy to check that the sets $L^F(T) = \{wxw^{−1} : x \in L, w \in F(T)\}$ and $R^F(T) = \{wxw^{−1} : x \in R, w \in F(T)\}$ are disjoint.

Let $F(T) \setminus \{1\} = \{w_n : n \in ℵ_0\}$ be an enumeration of the set of non-unit elements of $F(T)$. By induction for every $n \in ℵ_0$ we can find points $a_n, b_n \in F(T)$ such that

- $b_n = w_na_nw_n^{−1}$;
- $a_n \notin \{b_i : i ≤ n\}$;
- $b_n \notin \{a_i : i ≤ n\}$;
- $a_n \notin R^F(T)$;
- $b_n \notin L^F(T)$.
Consider the countable disjoint subsets
\[ A = L^{F(T)} \cup \{a_n : n \in \omega\} \text{ and } B = R^{F(T)} \cup \{b_n : n \in \omega\} \]
of the free group \( F(T) \).

Let \( G \) be the semidirect product of the groups \( F(T) \) and \( \text{Aut}(T) \). The elements of the group \( G \) are pairs \((w, f), \) where \( w \in F(T) \) and \( f \in \text{Aut}(T) \). The group operation on \( G \) is given by the formula:
\[
(w, f) \cdot (u, g) = (w \cdot f(u), f g).
\]

The groups \( F(T) \) and \( \text{Aut}(T) \) are identified with the subgroups \( \{(w, 1) : w \in F(T)\} \) and \( \{(1, f) : f \in \text{Aut}(T)\} \) of \( G \).

We claim that for every non-unit element \( x \in G \setminus \{1\} \) we get \( xA = 1 \cap B \neq 0 \). If \( x \in F(T) \), then \( x = w_n \) for some \( n \in \omega \) and then \( xA = 1 \cap B \ni b_n = w_n a_n w_n^{-1} \). If \( x \notin F(T) \), then \( x = (w, f) \) for some non-identity automorphism \( f \in \text{Aut}(T) \). It follows that \( f(t) \neq t \) for some node \( t \in T \).

We can assume that \( t \) has the smallest possible height. The node \( t \) is not the root of the tree \( T \) because the automorphism \( f \) of \( T \) does not move the root. So, \( t \) has an immediate predecessor \( a \) in the tree \( T \). The minimality of \( t \) guarantees that \( f(a) = a \) and hence \( \{t, f(t)\} = \{l, r\} \) where \( l = a^{-1} \) and \( r = a^{-1} \) are the “left” and “right” successors of \( a \) in the binary tree \( T \).

If \( al \in L \subset A \) and \( f(al) = f(a)f(l) = ar \in R \). Consequently,
\[
x \cdot al \cdot x^{-1} = (w, f) \cdot (al, 1) \cdot (f^{-1}(w^{-1}), f^{-1})
\]
\[
= (w \cdot f(al), f \cdot f^{-1}(w^{-1}), f^{-1}) = (w \cdot ar \cdot w^{-1}, 1) \in R^{F(T)} \subset B,
\]
witnessing that \( xA = 1 \cap B \neq 0 \) and hence \( 1 \in xA = 1 \cdot B^{-1} \).

\[ \square \]

**Theorem 3.** If \( G \) is an infinite abelian group or a free group, then \( \psi(G, 3_G) = |G| \). Moreover, for any family \( U \subset 3_G \) of neighborhoods of 1 with \( |U| < |G| \) the intersection \( \cap U \) contains a subgroup \( H \subset G \) of cardinality \( |H| = |G| \).

**Proof.** Let \( \kappa = |G| \). In order to prove that \( \psi(G, 3_G) = |G| \), take any family of monomials \( W \subset G[x] \) of size \( |W| < \kappa \) with \( 1 \in \bigcap_{w \in W} 3_w \). It follows that \( w(1) \neq 1 \) for every \( w \in W \) and hence the set \( W(1) = \{w(1) : w \in W\} \) is a subset of \( G \setminus \{1\} \) and has cardinality \( |W(1)| \leq |W| < \kappa \).

If the group \( G \) is abelian, then \( G \) contains a subgroup \( S \) of cardinality \( |S| = \kappa \) that is isomorphic to the direct sum \( \bigoplus_{n \in \mathbb{N}} S_n \) of cyclic groups, see [3], §1.6. Since the set \( S \) has cardinality \( < \kappa \), there is a subset \( A \subset \kappa \) of cardinality \( |A| < \kappa \) such that \( S \cap W(1) \subset \bigoplus_{n \in A} S_n \). Then the subgroup \( H = \bigoplus_{n \in A} S_n \) has cardinality \( |H| = \kappa \) and is disjoint with \( W(1) \).

Consequently, \( H \setminus W(1) \subset G \setminus \{1\} \). We claim that \( H \subset \bigcap_{w \in W} 3_w \). Indeed, for every \( w \in W \) and \( x \in H \), by the commutativity of \( G \) we get \( w(x) = w(1)x^n \) for some \( n \in \mathbb{Z} \) and hence \( w(x) \in w(1) \cdot H \subset W(1) \cdot H \subset G \setminus \{1\} \). Witnessing that \( H \subset 3_w \).

Next, assume that \( G = F(A) \) is a free group over an infinite alphabet \( A \subset G \). It follows that the set \( W(1) \) of cardinality \( < \kappa \) lies in the subgroup \( F(B) \) for some subset \( B \subset A \) of cardinality \( < \kappa \). Consider the subgroup \( H = F(A \setminus B) \) and observe that \( w(x) \neq 1 \) for every \( x \in H \). This means that \( H \subset \bigcap_{w \in W} 3_w \).

If \( G = F(A) \) is a non-commutative free group over a finite alphabet \( A \), then \( G \) contains a free subgroup \( F \subset G \) with infinitely many generators, see [7], II.1.2. By the preceding case the free group \( F \) contains a subgroup \( H \subset F \) of cardinality \( |H| = \aleph_0 = |G| \) such that \( H \subset F \cap \bigcap_{w \in W} 3_w \).

It is interesting to compare Corollary [1] and Theorem [2] with
Theorem 4. For a finite subset $A$ of an infinite group $G$ and the family
\[ W = G^1[x] \cup \{bxax^{-1}c, bx^{-1}axc : a \in A, b, c \in G\}, \]
the group $G$ endowed with the topology $\mathfrak{Z}_W$ is a quasitopological group with pseudocharacter
\[ \psi(G, \mathfrak{Z}_W) = |G|. \]

Proof. It follows from the definition of the family $W$ that the inversion $(\cdot)^{-1} : G \to G$ and left shifts $l_b : G \to G, b \in G,$ are continuous with respect to the topology $\tau_W.$ So, $(G, \tau_W)$ is a left-topological group with continuous inversion. The continuity of right shifts follows from the continuity of the left shifts and the continuity of the inversion. Thus $(G, \tau_W)$ is a quasitopological group.

In order to show that $\psi(G, \mathfrak{Z}_W) = |G|$, fix a family $V \subset W$ of size $|V| < |G|$ with $1 \in \bigcap_{v \in V} \mathfrak{Z}_v.$ Each monomial $v \in V$ is of the form $b_v x a_v x^{-1} c_v$ or $b_v x^{-1} a_v x c_v$ for some $a_v \in A$ and $b_v, c_v \in G.$ Let $d_v = b_v^{-1} c_v^{-1}$ and observe that $b_v x a_v x^{-1} c_v \neq 1$ if and only if $x a_v x^{-1} \neq d_v.$ Since $1 \in \mathfrak{Z}_v,$ we get $a_v \neq d_v$ for every $v \in V.$

Let $A = \{a_1, \ldots, a_n\}$ be an enumeration of the finite set $A.$ For every $k \leq n$ let $D_k = \{d_v : v \in V, a_v = a_k\}.$ It follows that $a_k \notin D_k$ and $|D_k| \leq |V| < |G|.$

Let $G_0 = G$ and by induction for every $k < n$ define a subgroup $G_{k+1}$ of $G$ letting $G_{k+1} = G_k$ if the centralizer
\[ Z_{G_k}(a_{k+1}) = \{x \in G_k : xa_{k+1} = a_{k+1}x\} \]
has size $< |G|$ and $G_{k+1} = Z_{G_k}(a_{k+1})$ otherwise. So, $(G_k)_{k\leq n}$ is a decreasing sequence of subgroups of $G$ having size $|G|.$

Let $K$ be the set of positive numbers $k \leq n$ such that $|Z_{G_{k-1}}(a_k)| < |G|.$ For every $k \in K$ and $d \in D_k$ consider the set $X_{k,d} = \{x \in G_n : xa_k x^{-1} = d\}.$ We claim that $|X_{k,d}| < |Z_{G_{n}}(a_k)| \leq |Z_{G_{k-1}}(a_k)| < |G|.$ Indeed, for any $x, y \in X_{k,d}$ we get $xa_k x^{-1} = d = ya_k y^{-1}$ and thus $y^{-1}xa_k = a_k y^{-1}x,$ which implies $y^{-1}x \in Z_{G_n}(a_k)$ and thus $X_{k,d} = yZ_{G_n}(a_k).$ Then the set
\[ X_k = \bigcup_{d \in D_k} X_{k,d} \cup X_{k,-d} \]
has cardinality $|X_k| \leq 2 \cdot |D_k| \cdot |Z_{G_n}(a_k)| < |G|.$ Finally, consider the set $Y = G_n \setminus \bigcup_{k \leq K} X_k$ and observe that $Y = Y^{-1}$ and $|Y| = |G_n| = |G|.$

We claim that $Y \subset \mathfrak{Z}_v$ for every $v \in V.$ In the opposite case $v(y) = 1$ for some $v \in V$ and some $y \in Y.$ It follows that $b_v y a_v y^{-1} c_v = 1$ or $b_v y^{-1} a_v y c_v = 1.$ Since $Y = Y^{-1},$ we lose no generality assuming that $b_v y a_v y^{-1} c_v = 1$ and hence $ya_v y^{-1} = d_v.$ Find $k \leq n$ such that $a_v = k$ and then $d_v \in D_k.$ If $k \notin K,$ then $y \in G_n \subset G_k = Z_{G_{k-1}}(a_k)$ and then $D_k \ni d_v = ya_v y^{-1} = ya_k y^{-1} = a_k,$ which contradicts $a_k \notin D_k.$ If $k \notin K,$ then $ya_k y^{-1} = d_v \in D_k$ implies that $y \in X_k$ which contradicts the choice of $y \in Y = G_n \setminus \bigcup_{k \leq K} X_k.$ \hfill $\square$

Proposition 4 implies the following fact, proved in [1] by the technique of ultrafilters:

Corollary 2. For any disjoint finite subsets $A, B$ of an infinite group $G$ the set \[ \{x \in G : x A x^{-1} \cap B = \emptyset\} \]
is infinite.

Remark 1. According to [11] or [13] each group $G$ with $\psi(G, \mathfrak{Z}_G) = |G|$ is topologizable, that is, admits a non-discrete Hausdorff group topology.

Remark 2. In [4] groups $G$ with $\psi(G, \mathfrak{Z}_G) \geq \kappa$ are called $\kappa$-unegebunden.

Groups $G$ with non-discrete $n$-th Zariski topology $\mathfrak{Z}_{G^n}$ have the following property:
Theorem 5. If the $n$-th Zariski topology $\mathcal{Z}_{G^n[x]}$ on a group $G$ is non-discrete, then any finite subsets $A_0, A_1, \ldots, A_n \subset G$ with $1 \notin A_0 A_1 \cdots A_n$ there is an infinite symmetric subset $X = X^{-1} \ni 1$ such that $1 \notin A_0 X A_1 \cdots X A_n$.

Proof. By induction on $k \in \omega$ we can construct a sequence $(x_k)_{k \in \omega}$ of points of the group $G$, a sequence $(C_k)_{k \in \omega}$ of finite subsets of $G$, and a sequence $(W_k)_{k \in \omega}$ of finite subsets of $G^n[x]$ such that for every $k \in \mathbb{N}$ the following conditions are satisfied:

1. $C_k = C_k^{-1} = (C_{k-1} \cup \{1, x_{k-1}, x_{k-1}^{-1}\})^n \subset G$;
2. $W_k = \{w \in C_k^n[x] : w(1) \neq 1\}$;
3. $w(x_k) \neq 1$ for any $w \in W_k$;
4. $x_k \notin \{x_i : i < k\}$.

We start the inductive construction letting $x_0 = 1$ and $C_0 = C_0^{-1}$ be any finite set containing $A_0 \cup \cdots \cup A_n$. For each $k$ the choice of the point $x_k$ is possible since the $n$th Zariski topology $\mathcal{Z}_{G^n[x]}$ is not discrete.

The conditions (1)--(3) of the inductive construction guarantee that the set $X = \{x_k, x_k^{-1} : k \in \omega\}$ has the desired property: $1 \notin A_0 X A_1 \cdots X A_n$. □

Looking at Theorem 1 one may ask what happens with the Zariski topologies $\mathcal{Z}_{G^n[x]}$ for higher $n$. For sufficiently large $n$ the topology $\mathcal{Z}_{G^n[x]}$ can be discrete.

The following criterion is due to A.Markov [6]:

Theorem 6 (Markov). For a countable group $G$ the following conditions are equivalent:

1. the topology $\mathcal{Z}_{G[x]}$ is discrete;
2. the topology $\mathcal{Z}_{G^n[x]}$ is discrete for some $n \in \omega$;
3. the group $G$ is non-topologizable.

We recall that a group $G$ is non-topologizable if $G$ admits no non-discrete Hausdorff group topology. For examples of non-topologizable groups, see [14], [9], [10], [8], [15], [16], [5]. In particular, the non-topologizable group from [9] yields the following example mentioned in [5]:

Example 1. There is a countable group $G$ whose 665-th Zariski topology $\mathcal{Z}_{G^{665}[x]}$ is discrete.

What happens for $n < 665$, in particular, for $n = 3$?

Problem 1. Is the 3-d Zariski topology $\mathcal{Z}_{G^3[x]}$ non-discrete on each infinite group $G$?

Also Corollary 1 and Example 2 suggest:

Problem 2. Is $\psi(G, \mathcal{Z}_{G^2[x]}) \geq \log |G|$ for every infinite group $G$?

Remark 3. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 4. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 5. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 6. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 7. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 8. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 9. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 10. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 11. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 12. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 13. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 14. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 15. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 16. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 17. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 18. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 19. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$. 

Remark 20. By [4] for every infinite cardinal $\kappa$ there is a topologizable group $G$ with $|G| = \kappa$ and $\psi(G, \mathcal{Z}_{G[x]}) = \aleph_0$.
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