Random Market Models with an $H$-Theorem

Ricardo Lópe-Ruiz  
Dept. of Computer Science & BIFI  
University of Zaragoza  
Email: rilopez@unizar.es

Elyas Shivanian  
Dept. of Mathematics  
Imam Khomeini Int. University  
Email: shivanian@ikiu.ac.ir

José-Luis López  
Dept. of Math. Engineering  
Public University of Navarre  
Email: jl.lopez@unavarra.es

Abstract—In this communication, some economic models given by functional mappings are addressed. These are models for random markets where agents trade by pairs and exchange their money in a random and conservative way. They display the exponential wealth distribution as asymptotic equilibrium, independently of the effectiveness of the transactions and of the limitation of the total wealth. Also, the entropy increases with time in these models and the existence of an H-theorem is computationally checked.

I. INTRODUCTION

In the last years, it has been reported [1], [2] that in western societies, around the 95% of the population, the middle and lower economic classes of society arrange their incomes in an exponential wealth distribution. The incomes of the rest of the population, around the 5% of individuals, fit a power law distribution, around the 5% of individuals, fit a power law distribution. The incomes of the rest of the population, around the 5% of individuals, fit a power law distribution. The incomes of the rest of the population, around the 5% of individuals, fit a power law distribution. The incomes of the rest of the population, around the 5% of individuals, fit a power law distribution.

A kind of models considering the randomness associated to markets are the gas-like models [4]. These random models interpret economic exchanges of money between agents similarly to collisions in a gas where particles share their energy [5].

In this communication, we consider a continuous version of an homogeneous gas-like model [6], [7] which we generalize to a situation where the agents present a control parameter to decide the degree of interaction with the rest of economic agents [8] and also to another new situation where there is an upper limit of the total richness.

The appearance of the exponential (Gibbs) distribution as a fixed point for all these three cases is mathematically explained [7]. Also, the increasing of the entropy when these systems evolve toward the asymptotic equilibrium is checked. This is associated with the existence of an H-theorem for all these economic models [9], [10]. Despite their apparent simplicity, these models based on functional mappings can help to enlighten the reasons of the ubiquity of the exponential distribution in many natural phenomena but in particular in the random markets.

II. THE CONTINUOUS GAS-LIKE MODEL

We consider an ensemble of economic agents trading with money by pairs in a random manner. The discrete version of this model is as follows [5]. For each interacting pair $(m_i, m_j)$ of the ensemble of $N$ economic agents the trading rules can be written as

$$m'_i = \epsilon (m_i + m_j),$$

$$m'_j = (1 - \epsilon)(m_i + m_j),$$

where $\epsilon$ is a random number in the interval $(0, 1)$. The agents $(i,j)$ are randomly chosen. Their initial money $(m_i, m_j)$, at time $t$, is transformed after the interaction in $(m'_i, m'_j)$ at time $t + 1$. The asymptotic distribution $p_f(m)$, obtained by numerical simulations, is the exponential (Boltzmann-Gibbs) distribution,

$$p_f(m) = \beta \exp(-\beta m), \quad \text{with} \quad \beta = 1/ <m>_\text{gas},$$

(2)

where $p_f(m) dm$ denotes the PDF (probability density function), i.e. the probability of finding an agent with money (or energy in a gas system) between $m$ and $m + dm$. Evidently, this PDF is normalized, $\int_0^\infty p_f(m) dm = 1$. The mean value of the wealth, $<m>_\text{gas}$, can be easily calculated directly from the gas by $<m>_\text{gas} = \sum m_i/N$.

The continuous version of this model [6] considers the evolution of an initial wealth distribution $p_0(m)$ at each time step $n$ under the action of an operator $T$. Thus, the system evolves from time $n$ to time $n + 1$ to asymptotically reach the equilibrium distribution $p_f(m)$, i.e.

$$\lim_{n \to \infty} T^n(p_0(m)) \to p_f(m).$$

(3)

In this particular case, $p_f(m)$ is the exponential distribution with the same average value, $<p_f>$, than the initial one, $<p_0>$, due to the local and total richness conservation.

The derivation of the operator $T$ is as follows [6]. Suppose that $p_n$ is the wealth distribution in the ensemble at time $n$. The probability to have a quantity of money $x$ at time $n + 1$ will be the sum of the probabilities of all those pairs of agents $(u, v)$ able to produce the quantity $x$ after their interaction, that is, all the pairs verifying $u + v > x$. Thus, the probability that two of these agents with money $(u, v)$ interact between them is $p_n(u) \ast p_n(v)$. Their exchange is totally random and then they can give rise with equal probability to any value $x$ comprised in the interval $(0, u + v)$. Therefore, the probability to obtain a particular $x$ (with $x < u + v$) for the interacting pair $(u, v)$ will be $p_n(u) \ast p_n(v)/(u + v)$. Then, $T$ has the form of a nonlinear integral operator,

$$p_{n+1}(x) = Tp_n(x) = \int_{u+v>x} \frac{p_n(u)p_n(v)}{u+v} \, du \, dv.$$
If we suppose $T$ acting in the PDFs space, it has been proved [7] that $T$ conserves the mean wealth of the system, $< Tp > = < p >$. It also conserves the norm $(|| \cdot ||)$, i.e. $T$ maintains the total number of agents of the system, $|| Ty || = || y || = 1$, that by extension implies the conservation of the total richness of the system. We have also shown that the exponential distribution $p_f(x)$ with the right average value is the only steady state of $T$, i.e. $T y = y$. Computations also seem to suggest that other high period orbits do not exist. In consequence, it can be argued that the relation (3) is true. We sketch some of these properties.

First, in order to set up the adequate mathematical framework, we provide the following definitions.

**Definition 2.1:** We introduce the space $L^+_1$ of positive functions (wealth distributions) in the interval $[0, \infty)$,

$$L^+_1[0, \infty) = \{ y : [0, \infty) \to R^+ \cup \{0\}, ||y|| < \infty \},$$

with norm $||y|| = \int_0^\infty y(x)dx$.

**Definition 2.2:** We define the mean richness $< x >_y$ associated to a wealth distribution $y \in L^+_1[0, \infty)$ as the mean value of $x$ for the distribution $y$. Then,

$$< x >_y = ||xy(x)|| = \int_0^\infty xy(x)dx.$$

**Definition 2.3:** For $x \geq 0$ and $y \in L^+_1[0, \infty)$ the action of operator $T$ on $y$ is defined by

$$T(y(x)) = \int_0^\infty \int_{S(x)} dudv \frac{y(u)y(v)}{u+v} ,$$

with $S(x)$ the region of the plane representing the pairs of agents $(u, v)$ which generate a richness $x$ after their trading, i.e.

$$S(x) = \{ (u, v), u, v > 0, u + v > x \} .$$

Now, we remind the following results recently presented in Ref. [7], [8].

**Theorem 2.4:** For any $y \in L^+_1[0, \infty)$ we have that $||Ty|| = ||y||^2$. In particular, consider the subset of PDFs in $L^+_1[0, \infty)$, i.e. the unit sphere $B = \{ y \in L^+_1[0, \infty), ||y|| = 1 \}$.

Observe that if $y \in B$ then $Ty \in B$. (It means that the number of agents in the economic system is conserved in time).

**Theorem 2.5:** The mean value $< x >_y$ of a PDF $y$ is conserved, that is $< x >_{Ty} = < x >_y$ for any $y \in B$. (It means that the mean wealth, and by extension the total richness, of the economic system are preserved in time).

**Theorem 2.6:** Apart from $y = 0$, the one-parameter family of functions $y_{\alpha}(x) = e^{-\alpha x}$, $\alpha > 0$, are the unique fixed points of $T$ in the space $L^+_1[0, \infty)$.

**Proposition 2.7:** For some members $y, w \in B$, $||Ty - Tw|| \geq ||y - w||$, hence $T$ is not a contraction.

**Example 2.8:** Take $y(x) = \frac{1}{\Gamma(\alpha + 1)}$ and $w(x) = e^{-x}$ which belong to $B$. By using Mathematica, it is seen that $||y - w|| = 0.407264$ and $||Ty - Tw|| = 0.505696$.

If we consider the restriction of $T$ for the subset $B_{x_0}$ of distributions with the same mean wealth $x_0$, i.e. $B_{x_0} = \{ y \in B | < x >_y = x_0 \}$, then by using the Laplace transform of the operator $T$, it has been proved [11] that $T$ is a contraction in $B_{x_0}$, hence the truth of the following relation:

$$\lim_{n \to \infty} T^n y(x) = \{ \begin{array}{ll} \delta e^{-\delta x} & \text{with } \delta = 1/x_0, \\
0^+ & \text{or } \text{when } < x >_y = +\infty . \end{array}$$

Let us observe that the above pointwise limit of $T^n y$ when $n \to \infty$ can be outside of $B$ in the case that $< x >_y = +\infty$.

See the next example.

**Example 2.9:** Take $y(x) = \frac{1}{\Gamma(\alpha + 1)}$, which belongs to $B$, with $< x >_y = +\infty$. Evidently, $T^n y \in B$ for all $n$. But it can be seen that $\lim_{n \to \infty} T^n y(x) = 0^+ \notin B$.

**Example 2.10:** Assume now the rectangular distribution:

$$y(x) = \frac{1}{2} \text{ if } 2 < x < 4, \text{ and } y(x) = 0 \text{ otherwise.}$$

So, $y \in B$ and $\delta = \frac{1}{2}$, then the steady state in this case is $\mu(x) = \frac{2}{3}e^{-\frac{x}{3}}$.

We find numerically that $||y - \mu|| > ||Ty - \mu|| > ||T^2 y - \mu|| > ||T^3 y - \mu||$, and so on. It is shown in Fig. 1. Then we can guess that $\lim_{n \to \infty} ||T^n y - \mu|| = 0$.

![Fig. 1. Plot of $y(x) = \frac{1}{2} \text{ if } 2 < x < 4, \text{ and } y(x) = 0 \text{ otherwise, } T$-iterates of $y$ and $\mu(x) = \frac{1}{2}e^{-\frac{x}{3}}$, (a) $||y - \mu||$, (b) $||Ty - \mu||$, (c) $||T^2 y - \mu||$.](image)

If we consider the entropy of $y(x)$ given by $H = -\int y(x) \log y(x)dx$, then it is found that $H$ increases in a monotonic way when $T$ is successively applied to an initial state $y_0(x)$. If we define $H_n = H(T^n y_0(x))$, then the following $H$-theorem [10] yields:

$$\lim_{n \to \infty} H_n = H(\delta e^{-\delta x}) \quad \text{with } \delta = 1/ < x >_{y_0} ,$$

$$H_n \leq H_{n+1} \quad \forall n .$$

Summarizing, the system has a fixed point, $\delta e^{-\delta x}$, which is asymptotically reached depending on the initial average value $< x >_{y_0}$ and following a trajectory of increasing entropy. This behavior is essentially maintained in the extension of this model for other similar random markets (see the next sections).
III. THE CONTINUOUS GAS-LIKE MODEL WITH HOMOGENEOUS EFFECTIVENESS

Let us think now that many of the economical transactions planned in markets are not successful and they are finally frustrated. It means that markets are not totally effective. We can reflect this fact in our model in a qualitative way by defining a parameter \( \lambda \in [0,1] \) which indicates the degree of effectiveness of the random market. When \( \lambda = 1 \) the market will have total effectiveness and all the operations will be performed under the action of the random rules (1). The evolution of the system in this case is given by the operator \( T \). When \( \lambda = 0 \), all the operations become frustrated, there is no exchange of money between the agents and then the market stays frozen in its original state. The operator representing this type of dynamics is just the identity operator. Therefore, we can establish a generalized continuous economic model whose evolution in the PDFs space is determined by the operator \( T_\lambda \), which depends on the parameter \( \lambda \) as follows:

Definition 3.1: \( T_\lambda y(x) = (1-\lambda) y(x) + \lambda T y(x) \), with \( \lambda \in [0,1] \).

Observe that the parameter \((1-\lambda)\) can also be interpreted as a kind of saving propensity of the agents, in such a way that for \( \lambda = 1 \) they do not save anything and they game all their resources, and for \( \lambda = 0 \) they save the totality of their money and then all the transactions are frustrated and the market stays in a frozen state.

We present some properties of the operator \( T_\lambda \), which shows a dynamical behaviour essentially similar to the behavior of \( T \). Concretely, the exponential distribution is also the asymptotic wealth distribution reached by the system governed by \( T_\lambda \), independently of the effectiveness \( \lambda \) of the random market.

Let us observe that \( T_\lambda \) is the identity operator when \( \lambda = 0 \) and \( T_\lambda = T \) for \( \lambda = 1 \), where \( I \) is the identity operator.

Proposition 3.2: \( T_\lambda \) conserves the norm, i.e. for each \( y \in B \), we have \( T_\lambda y \in B \).

Proposition 3.3: \( T_\lambda \) conserves the average value of \( y \in B \), i.e. \( \langle x \rangle_{T_\lambda} = \langle x \rangle \), where \( \langle x \rangle \) represents the mean value expressed in Definition 2.2.

Theorem 3.4: For any \( \lambda \in (0,1) \), the operators \( T \) and \( T_\lambda \) have the same fixed points.

Corollary 3.5: The function \( y(x) = 0 \) and the family of exponential distributions \( y(x) = \delta e^{-\delta x} \), \( \delta > 0 \), are the only fixed points of \( T \) in \( L^1(0,\infty) \), with \( \lambda \in (0,1) \).

Theorem 3.6: Suppose that for a given \( \lambda \in (0,1) \) we have \( \lim_{n \to \infty} ||T^n y(x) - \mu(x)|| = 0 \), with \( \mu(x) \) a continuous function, then \( \mu(x) \) should be the fixed point of the operator \( T_\lambda \) for the initial condition \( y(x) \in B \). In other words, \( \mu(x) = \delta e^{-\delta x} \) with \( \delta = \frac{1}{\lambda} \).

Example 3.7: Take the Gamma distribution \( y(x) = xe^{-x} \), so that \( y \in B \) and \( \delta = \frac{1}{\lambda} \), then in this case \( \mu(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \). For \( \lambda = 0.5 \), we find numerically that \( ||y - \mu|| = 0.368226 \), \( ||T y - \mu|| = 0.273011 \), \( ||T y - \mu|| = 0.206554 \), \( ||T y - \mu|| = 0.158701 \), and so on. It is shown in Fig. 2. Then we can guess that \( \lim_{n \to \infty} ||T^n y(x) - \mu|| = 0 \).

Also, the increasing of entropy in the system evolution can be checked. Then, similarly to the first model, this model has a fixed point, \( \delta e^{-\delta x} \), which is asymptotically reached depending on the initial average value \( <x> \geq y_0 \), and following a trajectory of increasing entropy. The difference with the first model remains in the transient towards equilibrium, that evidently is a longer time for a lower effectiveness \( \lambda \) of the random market.

IV. THE CONTINUOUS GAS-LIKE MODEL WITH LIMITATION OF THE RICHNESS

Here, we study the effect of a limitation in the maximum richness that an agent can have. We establish this upper limit to be \( \Lambda \) for \( x \in [0,\Lambda] \). Now, the mean wealth of the system is:

\[ <x> > y = \int_0^\Lambda x y(x) dx. \]

Evidently, \( <x> > y < \Lambda \).

The existence of the cut-off \( \Lambda \) in the economic system not only means that agents can not have more money than \( \Lambda \) if not the creation of some kind of control on the system that do not let the agents to perform trades that surpass the limit \( \Lambda \). In the case of interaction by pairs, it implies that only the pairs verifying \( u + v < \Lambda \) are allowed to trade and then they can exchange their money according to rules (1). In the rest of interactions surpassing the upper limit, that is \( u + v > \Lambda \), the agents are not allowed to trade and then they conserve their original money. Hence, the generalization of the operator \( T \) for this system is the following:

\[ [T_\Lambda y](x) = \int \int_{x \leq u + v \leq \Lambda} \frac{y(u)y(v)}{u + v} dv dy + y(x) \int_{\Lambda - x}^\Lambda y(v) dv, \]

where the first term integrates the allowed trades according to rules (1) and the second term gives account of the total probability of encounters with forbidden trades that an agent of richness \( x \) can have with other agents of the ensemble.
Observe that \( \lim_{\Lambda \rightarrow \infty} T_\Lambda = T \). Also:

\[
[T_\Lambda y](x) = \int_0^x du \int_{x-u}^{\Lambda-u} du' \frac{y(u)y(v)}{u + v} + \int_x^\Lambda du \int_0^{\Lambda-u} du' \frac{y(u)y(v)}{u + v} + y(x)\int_{\Lambda-x}^\Lambda y(v)dv.
\]

**Theorem 4.1:** For any \( y \in L_1^+[0, \infty) \) and \( \Lambda > 0 \) we have that \( ||T_\Lambda y|| = ||y||^2 \). In particular, if \( ||y|| = 1 \) then \( ||T_\Lambda y|| = 1 \).

**Theorem 4.2:** The mean richness is conserved by \( T_\Lambda \), that is \( \langle x \rangle_{T_\Lambda y} = \langle x \rangle_y \) for any \( y \in B \).

**Theorem 4.3:** The function

\[
y_{a,\Lambda}(x) = \frac{ae^{-ax}}{1 - e^{-a\Lambda}}
\]

has \( ||y_{a,\Lambda}|| = 1 \) and is a fixed point of the operator \( T_\Lambda \) for any \( a > 0 \). The mean richness for this function is

\[
\langle x \rangle_{y_{a,\Lambda}} = \frac{1}{a} + \frac{\Lambda}{1 - e^{-a\Lambda}}.
\]

**Proposition 4.4:** For the fixed point \( y_{a,\Lambda}(x) \), we have \( 2 < \langle x \rangle_{y_{a,\Lambda}} < \Lambda \).

Hence, if we define \( m = \frac{\langle x \rangle_{y_{a,\Lambda}}}{\Lambda} \), we can consider the **middle class, CM**, as all those agents having richness between \( m/2 \) and \( 2m \), that is

\[
CM(a,\Lambda) = \int_{m/2}^{2m} y_{a,\Lambda}(x)dx = \frac{e^{-am/2} - e^{-2am}}{1 - e^{-a\Lambda}}.
\]

The richness accumulated by the middle class is:

\[
\langle x CM \rangle (a,\Lambda) = \int_{m/2}^{2m} x y_{a,\Lambda}(x)dx = m\frac{(2 + am)e^{-am/2} - 2(1 + 2am)e^{-2am}}{2am[1 - e^{-a\Lambda}]},
\]

where

\[
am = 1 + \frac{a\Lambda}{1 - e^{a\Lambda}}, \quad x = a\Lambda.
\]

When we plot \( CM(x) \) or \( \langle x CM \rangle (x) \) for fixed \( m \), we see that it is always a decreasing function of \( x = a\Lambda \). Therefore, the smaller the richness limit \( \Lambda \) is, the larger the middle class is.

The same tendency can be observed if we calculate the mean wealth per individual of the middle class:

\[
\frac{\langle x CM \rangle (a,\Lambda)}{\langle CM \rangle (a,\Lambda)} = \frac{m}{am} + \frac{m[e^{-am/2} - 4e^{-2am}]}{2[e^{-am/2} - e^{-2am}]},
\]

which is also a decreasing function of \( x = a\Lambda \), for fixed \( m \).

The proportion of the total richness accumulated by the middle class is:

\[
\frac{\langle x CM \rangle (a,\Lambda)}{m} = \frac{(2 + am)e^{-am/2} - 2(1 + 2am)e^{-2am}}{2am[1 - e^{-a\Lambda}]},
\]

that is again a decreasing function of \( x = a\Lambda \).

Summarizing, an upper limit in the richness allowed in a random market provokes an enlargement of the middle class and also an enrichment of such a middle class.

**V. Conclusions**

Different versions of a continuous economic model [6] that takes into account idealistic characteristics of the markets have been considered. In these models, the agents interact by pairs and exchange their money in a random way. The asymptotic steady state of these models is the exponential wealth distribution. The system decays to this final distribution with a monotonic increasing of the entropy taking its maximum value just on the equilibrium. These are specific \( H \)-theorems that can be computationally checked, independently on the effectiveness of the markets or the limitation of the richness in the economic system.

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