Fast Consensus of High-Order Multiagent Systems

Jiahao Dai, Jing-Wen Yi, Member, IEEE, and Li Chai, Member, IEEE

Abstract—In this article, the fast consensus problem of high-order multiagent systems (MASs) under undirected topologies is considered. The direct link between the consensus convergence rate and the control gains is established. A gradient descent-based algorithm is proposed to optimize the convergence rate. By applying the Routh–Hurwitz stability criterion, the lower bound on the convergence rate is derived, and explicit control gains are derived as conditions to achieve the optimal convergence rate. Moreover, a protocol with time-varying control gains is designed to achieve the finite-time consensus. Explicit formulas for the time-varying control gains and the final consensus state are derived. Numerical examples and simulation results are presented to illustrate the obtained theoretical results.

Index Terms—Convergence rate, fast consensus, finite-time consensus, high-order, multiagent systems (MASs).

I. INTRODUCTION

Consensus is a fundamental problem in distributed coordination, which has been extensively studied in [1], [2], [3], and [4]. The main purpose of consensus is to design a control protocol, which uses the local information between each agent and its neighbors, such that the states of all agents can reach a common value over time.

Convergence rate is an important indicator to evaluate the consensus performance. There are many methods to accelerate convergence, which can be roughly summarized as: optimizing the weight matrix [5], [6], [7]; using the time-varying control [8], [9], [10], [11]; and introducing the agent’s memory [12], [13], [14], [15]. Recently, Yi et al. [11] gave an explicit formula for the optimal convergence rate of first-order multiagent systems (MASs) from the perspective of graph signal frequency-domain filtering. And Dai et al. [15] proposed a general control protocol with memory to accelerate the consensus of first-order MASs.

Most of the above methods to optimize the convergence rate are considered in the first-order system. However, a broad class of systems has multiple degrees of freedom in practical applications, where the input–output relationship needs to be illustrated by high-order dynamics [16], [17], [18], [19], [20]. Then, some researchers explored the convergence rate of high-order MASs. Li et al. [21] studied a consensus algorithm for MASs with double-integrator dynamics, and proved that the finite-time consensus can be achieved by using the Lyapunov stability theory. Under assumptions that the system matrix is controllable and the product of the unstable eigenvalues of the open-loop system matrix has an upper bound, You et al. [22] provided a lower bound of the optimal convergence rate for high-order discrete-time MASs. Eichler et al. [23] proposed a protocol for the consensus of MASs with discrete-time double-integrator dynamics, and derived the optimal control gain by minimizing the largest eigenvalue modulus of the closed-loop system matrix. Parlangeli et al. [24] proposed a control protocol in high-order continuous-time leader–follower networks, and indicated that the convergence can be achieved arbitrarily fast by allocating all the eigenvalues of the closed-loop system matrix.

In this article, the fast consensus problem of high-order MASs is considered. The control protocols with constant control gains and time-varying control gains are used to achieve the accelerated asymptotic consensus and the finite-time consensus, respectively. The main contributions of this article are summarized as follows.

1) Necessary and sufficient conditions for high-order MASs to achieve asymptotic consensus are given. The direct link between the consensus convergence rate and the control gains is established.

2) The accelerated asymptotic consensus problem is transformed into an optimization problem of the convergence rate, and a gradient descent-based algorithm is proposed to solve this problem. By applying the Routh–Hurwitz stability criterion, the lower bound on the convergence rate is given, and explicit control gains are derived as conditions to achieve the optimal convergence rate.

3) Explicit formulas of time-varying control gains to achieve the finite-time consensus are derived by applying the Cayley–Hamilton theorem.

It is shown that the step to achieve consensus is determined by the system’s order and the number of distinct nonzero eigenvalues of the Laplacian.

The rest of this article is organized as follows. In Section II, the problem statement is presented. Section III proposes...
some consensus conditions, designs a gradient descent-based algorithm to accelerate consensus, and derives the lower bound on the convergence rate. Section IV introduces a time-varying control protocol to achieve the finite-time consensus, and gives explicit formulas for the time-varying control gains. In Section V, numerical examples are given to verify the theoretical analysis. Finally, Section VI concludes this article.

Notations: The notations used in this article are standard. The notation $\text{diag}\{\cdots\}$ denotes a block-diagonal matrix. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ are the sets of column vectors of dimension $n$ and matrices of dimension $m \times n$ with real elements, respectively. $\|\cdot\|_2$ denotes the Euclidean norm. The symbol $\otimes$ stands for the Kronecker product. The symbol $C_p = \frac{p!}{q!(p-q)!}$ denotes the number of $q$-combinations from a given set of $p$ elements. Without special explanation, $0$ and $I$ represent the zero matrix and identity matrix with appropriate dimensions, and $1$ denotes the vector of all ones.

II. PRELIMINARIES

A. Graph Theory

The interactions among agents are modeled as an undirected graph $G(V, E, \Lambda)$, where $V = \{v_1, v_2, \ldots, v_N\}$ represents a set of agents or nodes, $E \subseteq V \times V$ represents a set of edges, and $\Lambda = [a_{ij}] \in \mathbb{R}^{N \times N}$ represents the weighted adjacency matrix. The adjacency element $a_{ij} = a_{ji} > 0$ if the edge between node $i$ and $j$ satisfies $e_{ij} \in E$. Denote the set of neighbors of node $i$ as $N_i = \{v_j \in V : (v_i, v_j) \in E\}$. Define the Laplacian matrix of $G$ as $\Lambda = \mathcal{D} - \Lambda$, where $\mathcal{D} = \text{diag}(d_1, d_2, \ldots, d_N)$ and $d_i = \sum_{j=1}^{N} a_{ij}$. For a connected graph, all the eigenvalues of $\Lambda$ are real in an ascending order as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$.

Lemma 1 [25]: For any connected undirected graph $G$, its Laplacian matrix has the following properties.

(i) $\Lambda$ has the spectral decomposition $\Lambda = \mathcal{V} \Lambda \mathcal{V}^T$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\mathcal{V} = [v_1, v_2, \ldots, v_N] \in \mathbb{R}^{N \times N}$.

(ii) Zero is a single eigenvalue of $\Lambda$, and the corresponding eigenvector is $v_1 = \frac{1}{\sqrt{N}} 1_N$.

B. Problem Formulation

Agents might only be able to interact with their neighbors intermittently rather than continuously because digital signals commute in discrete time. A discrete-time high-order MAS containing $N$ agents with order $n \geq 1$ is considered as follows:

\[
x_i^{(1)}(k + 1) = x_i^{(1)}(k) + \tau x_i^{(2)}(k) \\
x_i^{(2)}(k + 1) = x_i^{(2)}(k) + \tau x_i^{(3)}(k) \\
\vdots \\
x_i^{(n)}(k + 1) = x_i^{(n)}(k) + \tau u_i(k), \quad i = 1, 2, \ldots, N
\]  

(1)

where $x_i^{(l)}(k) \in \mathbb{R}$, $l = 1, 2, \ldots, n$ represents the $l$-order state of the agent $i$, $u_i(k) \in \mathbb{R}$ is the control input, and $\tau \in \mathbb{R}^+$ denotes the sampling period. Let $x_i(k) = [x_i^{(1)}(k), \ldots, x_i^{(n)}(k)]^T$ and rewrite system (1) into a matrix form

\[
x_i(k + 1) = Ax_i(k) + Bu_i(k), \quad i = 1, 2, \ldots, N
\]  

(2)

where

\[
A = \begin{bmatrix} 1 & \tau \\ & 1 & \cdots \\ & & 1 & \tau \\ & & & \tau \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\]  

(3)

Remark 1: Ma et al. [26] studied the consensus problem of high-order MASs, indicating that the state of each agent converges to zero without taking any control when the open-loop system is stable. It means that studying the consensus of open-loop stable systems is of little significance. For an unstable open-loop system, it is usually necessary to make some assumptions to achieve consensus. However, these assumptions make the conclusions obtained conservative. For example, the authors in [22] and [27] limit the range of eigenratio $\frac{\lambda_2}{\lambda_N}$. In fact, for an unstable open-loop system, each agent can use a local controller $u_i(k) = K_i x_i(k)$ for pole-placement, and make the open-loop system marginally stable. Then the neighbor information can be utilized to achieve consensus. Therefore, the marginally stable open-loop system considered in [16], [18], [19], and this article is not loss of generality. Instead, we think it is more suitable for practical applications.

Definition 1: Consider the high-order MAS (1) with arbitrary initial value.

(i) Consensus is said to be reached asymptotically if

\[
\lim_{k \to \infty} \left[ x_i^{(l)}(k) - x_j^{(l)}(k) \right] = 0, \quad i, j = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, n.
\]

(ii) Consensus is said to be reached at step $T$ if

\[
x_i^{(l)}(k) - x_j^{(l)}(k) = 0, \quad i, j = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, n
\]

holds for any $k \geq T$.

This article aims to design control protocols and corresponding control gains to achieve the accelerated asymptotic consensus and the finite-time consensus.

III. ACCELERATED ASYMPTOTIC CONSENSUS BY A TIME-IN Variant CONTROL PROTOCOL

In this section, the accelerated asymptotic consensus problem of high-order MASs is studied.

Consider the following time-invariant control protocol:

\[
u_i(k) = K \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k))
\]  

(4)

where $K = [K_1, K_2, \ldots, K_n] \in \mathbb{R}^{1 \times n}$ denotes the control gain. Denote $x(k) = [x_1(k)^T, x_2(k)^T, \ldots, x_N(k)^T]^T$. The system (2) can be written as

\[
x(k + 1) = (I_N \otimes A - \Lambda \otimes BK)x(k).
\]  

(5)
Let $H(\lambda_i, K) = A - \lambda_iBK$. According to Lemma 1, $\mathcal{L}$ has the spectral decomposition $\mathcal{L} = V\Lambda V^T$. Then we have

$$x(k) = (V \otimes I_n)\text{diag}\{A, H(\lambda_2, K), \ldots, H(\lambda_N, K)\}(V^T \otimes I_n)x(k-1)$$

$$= \frac{1}{N}(1_N \otimes I_n)A^k(1_N^T \otimes I_n)x(0) + \sum_{i=2}^{N}(v_i \otimes I_n)H^k(\lambda_i, K)(v_i^T \otimes I_n)x(0).$$

(6)

Note that $\frac{1}{N}(1_N \otimes I_n)A^k(1_N^T \otimes I_n)x(0)$ in (6) is the part to achieve consensus. We need to design the control gain $K$ so that

$$\lim_{k \to \infty} \sum_{i=2}^{N}(v_i \otimes I_n)H^k(\lambda_i, K)(v_i^T \otimes I_n)x(0) = 0.$$ 

The convergence rate is determined by the eigenvalue of $H(\lambda_i, K)$ with the largest modulus. Thus, the consensus convergence rate can be defined as [22]

$$r = r(K) = \max_{i=2, \ldots, N} \rho(H(\lambda_i, K))$$

(7)

where $\rho(\cdot)$ denotes spectral radius.

**Remark 2:** Denote

$$e(k) = x(k) - \frac{1}{N}(1_N \otimes I_n)A^k(1_N^T \otimes I_n)x(0).$$

According to (6), we have

$$\lim_{k \to \infty} \|e(k)\|_2 \leq (N - 1)\|x(0)\|_2 \lim_{k \to \infty} r^k.$$ 

Note that time $t = k\tau$ at step $k$. A smaller $r$ or $\tau$ can get faster convergence of consensus.

In this section, our goal is to design the control gain $K$ to make the convergence rate $r$ as small as possible.

### A. Conditions for Consensus

In this subsection, necessary and sufficient conditions for higher order MASs to achieve asymptotic consensus are proposed.

**Lemma 2:** (Schur Complement [28]) Given a matrix

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$$

with nonsingular $M_1 \in \mathbb{R}^{n \times n}$, $M_3 \in \mathbb{R}^{n \times n}$, $M_4 \in \mathbb{R}^{N \times \mu}$, and $M_1 \in \mathbb{R}^{n \times \mu}$. Then $\det M = \det M_1 \cdot \det(M_4 - M_1M_3^{-1}M_2)$.

**Lemma 3:** Consider the high-order MAS (1) on a connected graph $G$ with the control protocol (4). Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of the graph Laplacian matrix. Then

(i) consensus can be achieved asymptotically if and only if $r < 1$; (ii) the final consensus state is

$$\lim_{k \to \infty} x(k) = 1_N \otimes \left[ \lim_{k \to \infty} s_1(k), \ldots, \lim_{k \to \infty} s_n(k) \right]^T$$

(8)

where

$$s_j(k) = \frac{1}{N} \sum_{m=1}^{\infty} \tau^{m-1}C_k^{m-1} \sum_{p=1}^{N} x_p^{(m+j-1)}(0), j = 1, \ldots, n.$$ 

**Proof:** (i) Note that $\lim_{k \to \infty} [H(\lambda_i, K)]^k = 0_{n \times n}$ if and only if $\rho(H(\lambda_i, K)) < 1$. Then $\lim_{k \to \infty} [H(\lambda_i, K)]^k = 0_{n \times n}$ holds for all $i$, if and only if $r < 1$. It follows from (6) that

$$\lim_{k \to \infty} x(k) = \frac{1}{N}(1_N \otimes I_n) \lim_{k \to \infty} A^k(T_N \otimes I_n)x(0)$$

$$= \frac{1}{N} \lim_{k \to \infty} 1_N T_N \otimes A^k x(0)$$

(9)

holds if and only if $r < 1$. Equation (9) is equivalent to

$$\lim_{k \to \infty} x_i(k) = \frac{1}{N} \lim_{k \to \infty} A^k \sum_{p=1}^{N} x_p(0), i = 1, \ldots, N$$

(10)

which implies $\lim_{k \to \infty}[x_i^{(l)}(k) - x_j^{(l)}(k)] = 0$. Thus, consensus can be achieved asymptotically if and only if $r < 1$.

(ii) By direct computation, we have

$$A^k = \begin{bmatrix} 1 & C_1^1 \tau & C_1^2 \tau^2 & \cdots & C_1^{n-1} \tau^{n-1} \\ 1 & C_1^1 \tau & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \cdots & C_1^{n-1} \tau^{n-1} & 1 \end{bmatrix}.$$ 

(11)

Substituting (11) into (10), we get the consensus state, as shown in (8). 

**Remark 3:** Note that the final state (8) is a kind of dynamic consensus. The average consensus of the $I$-order state can be achieved, only if we set $\sum_{i=1}^{N} x_i^{(m)}(0) = 0, m = l + 1, \ldots, n$.

Based on Lemma 3, the following conclusion is obtained.

**Theorem 1:** Consider the high-order MAS (1) on a connected graph $G$ with the control protocol (4). Denote

$$R_i(z, K) = \det(zI - H(\lambda_i, K))$$

$$= z^n + b_1(\lambda_i)z^{n-1} + \cdots + b_{n-1}(\lambda_i)z + b_n(\lambda_i)$$

(12)

where

$$b_j(\lambda_i) = \lambda_i \sum_{p=1}^{j} (-1)^{j-p}r_p K_{n+1-p} - (-1)^j C_{n-j}$$

$$i = 2, \ldots, N, j = 1, \ldots, n.$$ 

(13)

Then consensus is achieved asymptotically if and only if all the roots of $R_i(z, K) = 0, i = 2, \ldots, N$ are within the unit circle.

**Proof:** Applying the Schur complement, we have

$$\det(zI - H(\lambda_i, K)) = \det P \cdot \det(U - YP^{-1}Q)$$

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
where
\[
P = \begin{bmatrix}
  z - 1 & -\tau & & & \\
  & z - 1 & -\tau & & \\
  & & \ddots & & \\
  & & & z - 1 & -\tau \\
  & & & & z - 1
\end{bmatrix}
\]
\[
U = z - 1 + \lambda_i \tau K_n, \\
Y = [\lambda_i \tau K_1, \lambda_i \tau K_2, \ldots, \lambda_i \tau K_{n-1}] \\
Q = [0, \ldots, 0, -\tau]^T.
\]
After some determinant calculations, we get
\[
det(zI - H(\lambda_i), K) = (z - 1)^n + \lambda_i \sum_{p=1}^{n} \tau^p K_{n-p+1} (z - 1)^{n-p} \tag{14}
\]
which can be expanded into (12). The roots of \(R_i(z, K) = 0\) are within the unit circle if and only if \(\rho(H(\lambda_i), K) < 1\). Finally, it follows from Lemma 3 that consensus is achieved if and only if the roots of \(R_i(z) = 0\), \(i = 2, \ldots, N\) are within the unit circle.

Theorem 1 not only gives a common consensus condition, but also derives the explicit formulas of the characteristic polynomial. The explicit formulas of the characteristic polynomial will be used to derive the theoretical results of Theorem 2 and Theorem 3. In the next subsection, we will optimize the convergence rate under the consensus condition.

### B. Convergence Rate Optimization

In this subsection, an accelerated consensus algorithm based on gradient descent is designed to optimize the convergence rate. The lower bound of the convergence rate is given, and explicit control gains are derived as the necessary condition to achieve the optimal convergence rate.

The goal of the accelerated consensus algorithm is to design the control gain \(K\) so that the convergence rate \(r(K)\) is as small as possible under the consensus condition, that is,
\[
\min_{K} r(K) \quad \text{s.t.} \quad \rho(H(\lambda_i, K)) < 1, \quad i = 2, \ldots, N. \tag{15}
\]

Denote \(\nabla r = [\nabla r_1, \ldots, \nabla r_n]^T, \nabla r_m = \frac{r(K + \delta(m)) - r(K)}{\delta}\) where \(\delta(m) \in \mathbb{R}^{1 \times n}\) is a row vector whose elements are 0 except the \(m\)th term is a tiny positive scalar \(\delta\). Then the accelerated consensus algorithm described in Algorithm 1 can provide a numerical solution to the optimization problem (15).

The convergence rate in Algorithm 1 is bounded. Next, we give the lower bound on the convergence rate, and the necessary condition for reaching this convergence rate.

**Theorem 2:** Consider the high-order MAS (1) on a connected graph \(G\) with the control protocol (4). The following conclusions hold.

(i) The consensus convergence rate has a lower bound
\[
r \geq \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^{1/n} \tag{16}
\]

(ii) The convergence rate \(r^* = \left(\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}\right)^{1/n}\) can be achieved only if the control gains are
\[
K^*_i = \frac{1}{r^*(f_{n+1-j} + \sum_{i=1}^{n-j} K^*_i (-1)^{i+j+1} r^{-n-j-1-i} C_j^{i-1})} \tag{17}
\]
where
\[
f_q = \frac{(-1)^q}{2^{n+q}} \left[ r^n + 2q C_q^n (\lambda_N - \lambda_2) - C_n^{n-q} (\lambda_N + \lambda_2) \right] \tag{18}
\]

Algorithm 1: Accelerated Consensus Algorithm Based on Gradient Descent.

**Input:** Nonzero eigenvalues of graph Laplacian \(\lambda_i\); number of iterations \(T\); sampling period \(\tau\); a tiny positive scalar \(\delta\); the learning rate \(\eta\); initial parameters \(K^{(0)}\)

**Output:** \(r^* = r(K^{(T)})\), \(K^* = K^{(T)}\)

1: Calculate \(r(K^{(0)}) = \max_{i=2, \ldots, N} \rho(A - \lambda_i BK^{(0)})\)
2: Calculate \(\nabla r_m = \frac{r(K^{(0)} + \delta(m)) - r(K^{(0)})}{\delta}, m = 1, \ldots, n\)
3: for \(t = 1 \) to \(T\) do
4: \(K^{(t)} = K^{(t-1)} - \eta \nabla r^{(t-1)}\)
5: \(\nabla r_m = \frac{r(K^{(t)}) + \delta(m) - r(K^{(t)})}{\delta}, m = 1, \ldots, n\)
6: end for

**Proof:** (i) Let \(z = r^* + \frac{1}{s}\) in (12), and have
\[
\tilde{R}_i(s, K) = c_0(\lambda_i) s^n + c_1(\lambda_i) s^{n-1} + \cdots + c_{n-1}(\lambda_i) s + c_n(\lambda_i) \tag{19}
\]

where
\[
c_j(\lambda_i) = C_j^{n-j} + \sum_{q=1}^{n-j} w_{pq}(r) b_q(\lambda_i)
\]
\[
w_{pq}(r) = r^{n-q} \tilde{w}_{pq}
\]
\[
\tilde{w}_{pq} = \sum_{i=0}^{n-q-j} C_{n-q-j}^{p+1+j} [(\lambda_N - \lambda_2) - C_n^{n-q} (\lambda_N + \lambda_2)]
\]
and \(b_q(\lambda_i)\) is defined in (13). The roots of \(\tilde{R}_i(s, K) = 0\) are in the left plane, if and only if the roots of \(R_i(z, K) = 0\) are in the circle with radius \(r\). Let \(c(\lambda_i) = [c_0(\lambda_i), c_1(\lambda_i), \ldots, c_n(\lambda_i)]^T\). Substituting (13) into \(c_j(\lambda_i)\), the vector form of the linear relationship between \(c_j(\lambda_i)\) and \(K_j\) can be written as
\[
c(\lambda_i) = h(r) + \lambda_i W(r) MK^T \tag{19}
\]

where
\[
h(r) = [h_1(r), \ldots, h_{n+1}(r)]^T
\]
\[
h_p(r) = [r^n + (-1)^{n-p} C_n^{n-p+1} + \sum_{q=1}^{n-1} w_{pq}(r) (-1)^q C_n^{n-q}]
\]
\[
W(r) = [w_{pq}(r)] \in \mathbb{R}^{(n+1) \times n}
\]

\[
M = \begin{bmatrix}
  \tau^2 & -\tau C_n^{n-1} & \cdots & \cdots & \cdots \\
  \tau^3 & \tau^2 C_n^{n-3} & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  \tau & \cdots & \cdots & \cdots & \cdots \\
  \tau^n & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}_{n \times n}
\]
Before the next step of the proof, we make some notes on the coefficients of polynomial (18). \( \tilde{w}_{pq} \) represents the coefficient of \( s^{n-p+1} \) of the polynomial \( (s + 1)^{n-q}(s - 1)^q \), and 
\[
(1)^{p-1} \tilde{w}_{pq} = \tilde{w}_{p(q-n)}.
\]
By setting \( s = 1 \), we have \( \sum_{p=1}^{n+1} \tilde{w}_{pq} = 0 \), that is, the column sum of the matrix \( W(r) = [w_{pq}(r)] \in \mathbb{R}^{(n+1) \times n} \) is zero. Similarly, by setting \( s = 1 \) and \( s = -1 \), we can get that for the \( q \)-th \( (q \neq n) \) column of matrix \( W(r) \), the sum of odd elements is zero, and the sum of even elements is zero. These two properties will be used to obtain (23) and (24), respectively.

Applying the Routh–Hurwitz stability criterion [29], the polynomial \( (18) \) is stable or marginally stable only if 
\[
c(\lambda_i) \geq 0, \quad i = 2, \ldots, N.
\]  
(20)
Note that (20) are linear inequalities about \( \lambda_i \). We only need to require 
\[
c(\lambda_i) \geq 0, \quad i = 2, N
\]  
(21)
to satisfy (20). Assume that \( n \) is odd. According to the inequality properties, the following inequality (22) holds: 
\[
\frac{c_0(\lambda_2) + c_2(\lambda_2) + \cdots + c_{n-1}(\lambda_2)}{\lambda_2} + \frac{c_1(\lambda_N) + c_3(\lambda_N) + \cdots + c_n(\lambda_N)}{\lambda_N} \geq 0.
\]  
(22)
Since the column sum of \( W(r) \) is zero, the sum of the vector \( W(r)MK^T \) is zero. According to (19), the left side of the inequality (22) can be written as 
\[
\frac{h_1(r) + h_3(r) + \cdots + h_n(r)}{\lambda_2} + \frac{h_2(r) + h_4(r) + \cdots + h_{n+1}(r)}{\lambda_N}.
\]  
(23)
By some algebraic calculations, we get 
\[
h_1(r) + h_3(r) + \cdots + h_n(r) = \frac{2^{n-1}(r^n - 1)}{\lambda_2},
\]
\[
h_2(r) + h_4(r) + \cdots + h_{n+1}(r) = \frac{2^{n-1}(r^n + 1)}{\lambda_N}.
\]  
(24)
Then inequality (22) can be written as 
\[
\frac{2^{n-1}}{\lambda_2} \left( \frac{r^n - 1}{\lambda_2} + \frac{r^n + 1}{\lambda_N} \right) \geq 0.
\]  
(25)
It follows from (25) that \( r \geq \left( \frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2} \right)^{1/n} \). Similarly, for the case where \( n \) is even, we can obtain the same lower bound. The proof of part (i) is completed.

(ii) For the case where \( n \) is odd, \( r = r^* \) holds only if 
\[
c_0(\lambda_2) = 0, c_2(\lambda_2) = 0, \ldots, c_{n-1}(\lambda_2) = 0
\]
\[
c_1(\lambda_N) = 0, c_3(\lambda_N) = 0, \ldots, c_n(\lambda_N) = 0.
\]  
(26)
Similarly, for the case where \( n \) is even, \( r = r^* \) holds only if 
\[
c_1(\lambda_2) = 0, c_3(\lambda_2) = 0, \ldots, c_{n-1}(\lambda_2) = 0
\]
\[
c_0(\lambda_N) = 0, c_2(\lambda_N) = 0, \ldots, c_n(\lambda_N) = 0.
\]  
(27)
To combine (26) and (27), we denote 
\[ J = \text{diag} \{ J_1, J_2, \ldots, J_{n+1} \} \in \mathbb{R}^{(n+1) \times (n+1)} \]
where 
\[
J_i = \frac{2\lambda_2 \lambda_N}{(\lambda_N + \lambda_2) + (-1)^{i-n+1}(\lambda_N - \lambda_2)}, i = 1, \ldots, n + 1.
\]

Then \( r = r^* \) holds for any \( n \) only if (28) holds 
\[
h(r^*) + JW(r^*)M K^T = 0.
\]  
(28)
Next, we solve \( K \) in (28). Since \( J \) is invertible, (28) is equivalent to 
\[
W(r^*)MK^T = -J^{-1}h(r^*).
\]  
(29)
The \( \phi \)-th row of \( -J^{-1}h(r^*) \) can be written as 
\[
\frac{1}{\lambda_2 \lambda_N} \sum_{q=1}^{n-1} w_{pq}(r^*) \cdot (r^*)^q \cdot (\lambda_N - \lambda_2)^{-q} + (-1)^{n+p-2}(\lambda_N + \lambda_2)^{-q} \frac{2}{\lambda_2 \lambda_N} \cdot \left( 1 \right).
\]  
(30)
It follows from (30) that:
\[
\sum_{q=1}^{n} w_{pq}(r^*) f_q = -\frac{1}{\lambda_p} h_p(r^*), \quad p = 1, \ldots, n + 1.
\]  
(31)
Then we have 
\[
f = MK^T
\]  
(32)
where \( f = [f_1, \ldots, f_n]^T \). (32) can be expanded to 
\[
f_1 = K_n \tau,
\]
\[
f_2 = K_{n-1} \tau^2 - K_n \tau C_{n-2}^{n-2}
\]
\[
\cdots
\]
\[
f_i = K_{n-i} \tau^i - K_{n-i+2} \tau^{i-1} C_{n-i+1}^{n-i+1} + \cdots + K_n (-1)^{i-1} \tau C_{n-i-1}^{n-i}
\]
\[
\cdots
\]
\[
f_n = K_1 \tau^n - K_2 \tau^{n-1} + \cdots + K_n (-1)^{n-1} \tau.
\]  
(33)
It follows from (33) that the solution of (28) is (17). The proof of part (ii) is completed. \( \square \)

In Theorem 2, we give the necessary condition to achieve the lower bound on the convergence rate. When \( n = 1 \), this condition is sufficient and necessary [5]. When \( n = 2 \), we can prove that the condition (17) is also sufficient and necessary, as shown in Corollary 1. Moreover, if the high-order MAS is on a star graph, then the condition (17) is sufficient and necessary for any \( n \), as shown in Corollary 2.
**Corollary 1:** Consider the MAS (1) on a connected graph $G$ with the control protocol (4). The optimal convergence rate of $n = 2$ is

$$r^* = \sqrt{\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}}$$

(34)

with the following control gains:

$$K_1^* = \frac{2\lambda_2}{\tau^2(\lambda_2 + \lambda_N)}, \quad K_2^* = \frac{2}{\lambda_N \tau}.$$  

(35)

**Proof:** When $n = 2$, the polynomial (18) becomes

$$\tilde{R}_i(s, K) = [s^2 + (\lambda_i \tau K_2 - 2)r + 1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2]s^2 + [2r^2 - 2(1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2)]s + r^2 - (\lambda_i \tau K_2 - 2)r + 1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2.$$  

(36)

According to the Routh–Hurwitz stability criterion, (36) is stable or marginally stable if and only if

$$r^2 + (\lambda_i \tau K_2 - 2)r + 1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2 \geq 0,$$

$$2r^2 - 2(1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2) \geq 0,$$

$$r^2 - (\lambda_i \tau K_2 - 2)r + 1 + \lambda_i \tau^2 K_1 - \lambda_i \tau K_2 \geq 0.$$  

(37)

Since all constraints in (37) are all linear with respect to $\lambda_i$, the inequalities can be reduced to

$$\tau K_1 - (1-r)K_2 + \frac{(1-r)^2}{\lambda_N \tau} \geq 0,$$

$$-2\tau K_1 + 2K_2 - \frac{2r^2}{\lambda_2 \tau} \geq 0,$$

$$\tau K_1 - (1+r)K_2 + \frac{(1+r)^2}{\lambda_N \tau} \geq 0.$$  

(38)

Adding the three inequalities in (38), we have

$$\frac{1 + r^2}{\lambda_N \tau} - \frac{1 - r^2}{\lambda_2 \tau} \geq 0.$$  

(39)

It follows from (39) that $r \geq \sqrt{\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}}$. The optimal convergence rate $r^* = \sqrt{\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}}$ is achieved if and only if

$$\tau K_1 - (1-r)K_2 + \frac{(1-r)^2}{\lambda_N \tau} = 0,$$

$$-2\tau K_1 + 2K_2 - \frac{2r^2}{\lambda_2 \tau} = 0,$$

$$\tau K_1 - (1+r)K_2 + \frac{(1+r)^2}{\lambda_N \tau} = 0.$$  

(40)

The solution to (40) is given by (35). The proof is completed.  

**Remark 4:** In Corollary 1, we apply the Routh–Hurwitz stability criterion to derive the optimal convergence rate $r^* = \sqrt{\frac{\lambda_N - \lambda_2}{\lambda_N + \lambda_2}}$. In [22] and [23], authors obtained the same convergence rate, by analyzing the eigenvalues of the closed-loop matrix in the complex plane.

**Corollary 2:** Consider the MAS (1) on a star graph $G$ with the control protocol (4). The optimal convergence rate is $r^* = \frac{1}{(\lambda_2 + z)^{1/n}}$ with the optimal control gains (17).

**Proof:** For a star graph with $N$ nodes, the Laplacian matrix has only two distinct nonzero eigenvalues $\lambda_2 = 1, \lambda_N = N$. For the case where $n$ is odd, when (26) holds, the Hurwitz matrices of $\tilde{R}_2(s, K)$ and $\tilde{R}_N(s, K)$ are

$$\Delta_1(\lambda_2) = c_1(\lambda_2) \geq 0, \quad \Delta_2(\lambda_2) = \begin{bmatrix} c_1(\lambda_2) & c_2(\lambda_2) \\ c_0(\lambda_2) & c_0(\lambda_2) & c_2(\lambda_2) \end{bmatrix} = 0,$$

$$\ldots, \Delta_{n-1}(\lambda_2) = 0, \quad \Delta_n(\lambda_2) = 0$$

and

$$\Delta_1(\lambda_N) = c_1(\lambda_N) = 0, \quad \Delta_2(\lambda_N) = \begin{bmatrix} c_1(\lambda_N) & c_3(\lambda_N) \\ c_0(\lambda_N) & (\lambda_2)^{1/n} \end{bmatrix} = 0,$$

$$\ldots, \Delta_{n-1}(\lambda_N) = 0, \quad \Delta_n(\lambda_N) = 0$$

respectively. Then the roots of $R_i(z, K) = 0, i = 2, N$ are on the circle with radius $r^*$ if and only if (26) holds. Similarly, for the case where $n$ is even, the roots of $R_i(z, K) = 0, i = 2, N$ are on the circle with radius $r^*$ if and only if (27) holds. Thus, the optimal convergence rate $r = r^*$ is achieved if and only if (28) holds, and the optimal control gains are given by (17).  

**IV. FINITE-TIME CONSENSUS BY A TIME-VARYING CONTROL PROTOCOL**

In this section, a protocol with time-varying control gains is presented for high-order MASs to achieve consensus in finite time.

Note that the finite-time consensus will be reached if and only if $R_i(z, K) = z^n = 0$ holds for all $i = 2, \ldots, N$ in (12). However, the control protocol (4) with constant gains cannot achieve it. Therefore, we consider the following time-varying control protocol:

$$u_i(k) = K(k) \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k))$$

(41)

where $K(k) = [K_1(k), K_2(k), \ldots, K_n(k)] \in \mathbb{R}^{1 \times n}$.  

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Consensus error on different graphs.

λ at −τ = 0 nτ, n determines the overall convergence time. Consensus will, and then consensus will be achieved at step j0 m0 BK = 0 n×n.

The time-varying control sequence designed in (44) holds if we set

\[ K_m(nl) = \frac{C_n^{m-1}}{\lambda_{pl}^{m-n+1}}, m = 1, \ldots, n. \]  

According to the Cayley–Hamilton theorem, we have \([A - \lambda_{pl} BK(nl)]^n = 0_{n\times n}\) when applying the control gain (47). Thus, we design \(K(nl + j) \equiv K(nl)\) at \(j = 0, 1, \ldots, n - 1\), and take the control gains (47) to satisfy (45). Then the state at step \(k \geq nl\) is

\[
x(k) = \frac{1}{N}(1_N \otimes I_n)A^k(1_N^T \otimes I_n)x(0) = \frac{1}{N}(1_N 1_N^T) \otimes A^k x(0).
\]

By substituting (11) into (48), the consensus state (43) is obtained.

**Remark 5:** The time-varying control sequence designed in (42) is similar to graph filtering [11]. Large eigenvalues correspond to high frequencies, and small eigenvalues correspond to low frequencies. In order to avoid too much oscillation during the convergence, we suggest to filter out the part with large eigenvalues first, although the selection of the filtering order does not affect the final result in principle.

**Remark 6:** Since the high-order MAS (1) can achieve consensus at step nl by applying the control gain (42), the sampling period \(\tau\) determines the overall convergence time. Consensus will be achieved with arbitrarily fast convergence speed if \(\tau \to 0\), which implies the infinite band-width communication. In [24], authors have studied the accelerated consensus of high-order continuous-time systems and obtained the similar conclusion by allocating all the eigenvalues of the closed-loop system matrix.

**Remark 7:** Yi et al. [11] proposed that the first-order MASs can achieve consensus at step \(\bar{t}\) by applying the control gain \(K(k) = \frac{1}{\lambda_{pl}}\). This is a special case of \(n = 1\) in (42). Specially, if we consider a second-order MAS, it will reach the consensus state
Fig. 3. Finite-time consensus of the second-order MAS. (a) Position. (b) Velocity.

Fig. 4. Finite-time consensus of the third-order MAS. (a) Position. (b) Velocity. (c) Acceleration.

\[ x(k) = 1_N \otimes \left[ \frac{1}{N} \sum_{i=1}^{N} x_i^{(1)}(0) + k \tau \frac{1}{N} \sum_{i=1}^{N} x_i^{(2)}(0), \frac{1}{N} \sum_{i=1}^{N} x_i^{(2)}(0) \right]^T \]

\( k \geq 2\bar{l} \), by applying the control gain sequence

\[ K_1 = \left\{ \frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_{N-1}}, \frac{1}{\lambda_N} \right\} \]

\[ K_2 = \left\{ \frac{2}{\lambda_0}, \frac{2}{\lambda_1}, \frac{2}{\lambda_2}, \ldots, \frac{2}{\lambda_{N-1}}, \frac{2}{\lambda_N} \right\} . \]

V. NUMERICAL EXAMPLES

This section uses two examples to verify the effectiveness of the proposed theoretical results.

Example 1: In this example, Algorithm 1 is used to optimize the convergence rate on different graphs.

Consider a third-order MAS with ten agents on four unweighted graphs: the graph \( G_1 \) randomly generated by a small-world network model shown in Fig. 1, the cycle graph \( G_2 \), the path graph \( G_3 \), the complete bipartite graph \( G_4 \) with 4 + 6 vertices. Set \( \tau = 0.1, T = 5000, \eta = 0.01, \) and \( \delta = 10^{-6} \). In principle, the initial parameters \( K_i^{(0)} \), \( i = 1, \ldots, n \) in Algorithm 1 need to satisfy the consensus condition, that is, \( r(K^{(0)}) < 1 \).

For faster convergence of Algorithm 1, we initialize \( K_i^{(0)} \) around the control gains (17). Next, we run Algorithm 1, and record the convergence rate.

The convergence rate \( r^* \) obtained by Algorithm 1 is listed in Table I, where \( r_{lb} \) represents the lower bound \( (\frac{\lambda_N}{\lambda_2})^{1/3} \). It can be found that on different graphs, the convergence rate \( r^* \) of the third-order MAS can reach the lower bound \( r_{lb} \), and the smaller \( \lambda_N / \lambda_2 \) is, the smaller \( r^* \) is.

Next, we will plot the convergence of the consensus error \( \|e(k)\|_2 \). Randomly generate the initial state of each agent in the interval \([-1, 1]\). The convergence of the consensus error on different graphs is shown in Fig. 2. It can be observed that the error converges fastest on \( G_4 \) (\( \lambda_N / \lambda_2 = 2.5 \)) and the slowest on \( G_1 \) (\( \lambda_N / \lambda_2 = 4.4790 \)). The better the connectivity of the network is, the faster the consensus error converges.

| Graph \( G_1 \) | Graph \( G_2 \) | Graph \( G_3 \) | Graph \( G_4 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( \lambda_N / \lambda_2 \) | 4.4790 | 10.4721 | 39.8635 | 2.5000 |
| \( r_{lb} \) | 0.8595 | 0.9381 | 0.9834 | 0.7539 |
| \( r^* \) | 0.8595 | 0.9381 | 0.9834 | 0.7539 |
Example 2: In this example, the effectiveness of the control strategy in Theorem 3 is verified.

Consider the cycle graph $G_2$ with ten nodes. The distinct nonzero eigenvalues of the Laplacian matrix $L_{G_2}$ are $\{0.3820, 1.3820, 2.6180, 3.6180, 4\}$, and $\tilde{l} = 5$. Let $\tau = 0.1$. According to Theorem 3, each agent will achieve consensus at step $5\tilde{n}$ by applying the control gains (42).

Next, we will plot the consensus convergence of second-order and third-order MASs, respectively. Randomly set the initial states of the agents in the interval $[-5, 5]$. For the case of second-order MASs ($n = 2$), the states of position and velocity achieve consensus at step $5\tilde{n} = 10$, as shown in Fig. 3, in Fig. 3, we can get that the final consensus state of agent $i$ is $x_i(k) = [1.3325 + 0.0963k, 0.9627]^T$, $k \geq 10$, which is consistent with (43). For the case of third-order MASs ($n = 3$), the states of position, velocity, and acceleration achieve consensus at step $5\tilde{n} = 15$, as shown in Fig. 4. In Fig. 4, we can get that the consensus state of agent $i$ is $x_i(k) = [1.3325 + 0.0850k + 0.0113k^2, 0.9627 + 0.2266k, 2.2662]^T$, $k \geq 15$, which is also consistent with (43).

VI. CONCLUSION

The problem of accelerated asymptotic and finite-time consensus of discrete-time high-order MASs has been studied in this article. First, a protocol with constant control gains has been introduced to achieve consensus asymptotically. The fast consensus problem has been transformed into an optimization problem of convergence rate, and a gradient descent-based algorithm has been designed to optimize the convergence rate. By using the Routh–Hurwitz stability criterion, the lower bound on the convergence rate has been derived, and the necessary conditions to achieve this convergence rate has been proposed. Due to the limitation of the constant control, consensus cannot be achieved in finite time. Hence, a protocol with time-varying control gains has been designed to achieve the finite-time consensus. Explicit formulas for the time-varying control gains and the final consensus state have been given. Numerical examples have demonstrated the validity and correctness of these results.

This article only considers the fast consensus on undirected graphs. For the case of directed graphs, the proposed method is valid in principle when the eigenvalues of the graph Laplacian matrix are all real. When the eigenvalues of the graph Laplacian matrix are complex, the optimization of the convergence rate is difficult. The problem of fast consensus on directed graphs will be studied in the future.

REFERENCES

[1] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” IEEE Trans. Ind. Inform., vol. 9, no. 1, pp. 427–438, Feb. 2013.
[2] J. Qin, Q. Ma, Y. Shi, and L. Wang, “Recent advances in consensus of multi-agent systems: A brief survey,” IEEE Trans. Ind. Electron., vol. 64, no. 6, pp. 4972–4983, Jun. 2017.
[3] A. Dorri, S. S. Kanhere, and R. Jurdak, “Multi-agent systems: A survey,” IEEE Access, vol. 6, pp. 28573–28593, 2018.
[4] F. Chen et al., “On the control of multi-agent systems: A survey,” Foundations Trends Syst. Control, vol. 6, no. 4, pp. 339–499, 2019.
Jiahao Dai received the B.E. degree in automation from the Wuhan Institute of Technology, Wuhan, China, in 2018. He is currently working toward the Ph.D. degree in control science and engineering with the School of Information Science and Engineering, Wuhan University of Science and Technology, Wuhan. His research interests include multiagent systems and distributed optimization.

Jing-Wen Yi (Member, IEEE) received the B.E. degree in automation from Yantai University, Yantai, China in 2010, and the Ph.D. degree in control science and engineering from Huazhong University of Science and Technology, Wuhan, China, in 2016. She was a Visiting Student with Nanyang Technological University, Singapore, from 2014 to 2015. In 2016, she joined the Wuhan University of Science and Technology, Wuhan, where she is currently an Associate Professor with the School of Information Science and Engineering. Her research interests include multiagent systems, distributed optimization, social networks, and graph signal processing.

Li Chai (Member, IEEE) received the B.S. degree in applied mathematics and the M.S. degree in control science and engineering from Zhejiang University, Hangzhou, China, in 1994 and 1997, respectively, and the Ph.D. degree in electrical engineering from the Hong Kong University of Science and Technology, Hong Kong, in 2002. From 2002 to 2007, he was with Hangzhou Dianzi University, Hangzhou, China. He was a Professor with the Wuhan University of Science and Technology, Wuhan, China, from 2008 to 2022. In 2022, he joined Zhejiang University, where he is currently a Professor with the College of Control Science and Engineering. He has been a Postdoctoral Researcher or Visiting Scholar with Monash University, Clayton, VIC, Australia; Newcastle University, Callaghan, NSW, Australia; and Harvard University, Cambridge, MA, USA. He has authored or coauthored over 100 fully refereed papers in prestigious journals and leading conferences. His research interests include distributed optimization, filter banks, graph signal processing, and networked control systems.

Dr. Chai was the recipient of the Distinguished Young Scholar of the National Science Foundation of China. He serves as the Associate Editor for IEEE TRANSACTIONS ON CIRCUIT AND SYSTEMS II: EXPRESS BRIEFS, Control and Decision, and Journal of Image and Graphs.