N = 2 Rigid Supersymmetry with Gauged Central Charge

N. Dragon†, E. Ivanov‡, S. Kuzenko†‡, E. Sokatchev¶ and U. Theis†

† Institut f"ur Theoretische Physik, Universit"at Hannover
   Appelstr"asse 2, 30167 Hannover, Germany
   dragon,kuzenko,utheis@itp.uni-hannover.de

‡ Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research
   141980 Dubna, Russia
   eivanov@thsun1.jinr.ru

¶ Laboratoire de Physique Théorique LAPTH
   LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France
   sokatche@lapp.in2p3.fr

Abstract

We develop a general setting for $N = 2$ rigid supersymmetric field theories with gauged central charge in harmonic superspace. We consider those $N = 2$ multiplets which have a finite number of off-shell components and exist off shell owing to a non-trivial central charge. This class includes, in particular, the hypermultiplet with central charge and various versions of the vector-tensor multiplet. For such theories we present a manifestly supersymmetric universal action. Chern-Simons couplings to an external $N = 2$ super Yang-Mills multiplet are given, in harmonic superspace, for both the linear and nonlinear vector-tensor multiplets with gauged central charge. We show how to deduce the linear version of the vector-tensor multiplet from six dimensions.
1 Introduction

Supersymmetric theories with gauged central charge were introduced for the first time in the context of $N = 2$ supergravity\[1, 2\]. Recently, there was a revival of interest in such theories, mainly because of the conjecture\[3\] that one of the important $N = 2$ multiplets with a non-trivial central charge, the vector-tensor (VT) multiplet\[4\], describes the dilaton-axion complex in heterotic $N = 2$ four-dimensional supersymmetric string vacua. In particular, it was found that, besides the original ‘linear’ version of this multiplet, there exists its new ‘nonlinear’ version\[5\]. Chern-Simons couplings of both versions to external $N = 2$ vector multiplets\[6\] and $N = 2$ supergravity\[7\] were constructed. One of the important observations made in\[6\] is that in the case of the linear VT multiplet with gauged central charge for ensuring the rigid scale and chiral invariances of the action one needs at least one extra background abelian vector multiplet in addition to that associated with the central charge. No such extension is required in the case of the non-linear VT multiplet. The scale and chiral invariances are of crucial importance for a self-consistent coupling of the VT multiplet to conformal $N = 2$ supergravity.

All these studies were carried out in the component field approach making use of the superconformal multiplet calculus. The supersymmetry transformation laws and the invariant actions look rather complicated in such an approach. In this connection, the authors of\[7\] noticed: “...the complexity of our results clearly demonstrates the need for a suitable superspace formulation.”

Superspace formulations for the linear version of the VT multiplet with rigid central charge were constructed in $N = 2$ global central charge superspace\[8, 9, 10\] and in $N = 2$ harmonic superspace\[11\]. The latter approach seems to be most advantageous, since the $N = 2$ harmonic superspace\[12\] provides a universal framework for general $N = 2$ matter and super Yang-Mills theories, as well as $N = 2$ supergravity. The nonlinear VT multiplet with rigid central charge was formulated in harmonic superspace in\[13, 14\]. In\[14\] a new version of the nonlinear VT multiplet was proposed in which the component vector field (the field strength of an antisymmetric tensor field in the old version) cannot be expressed in terms of a potential.

First steps toward superspace formulations of globally $N = 2$ supersymmetric theories with gauged central charge have been undertaken in\[15\]. In particular, the $N = 2$ superspace description of the linear VT multiplet with gauged central charge was given for the first time.

In the present paper we develop a general formalism for globally $N = 2$ supersymmet-
ric theories with gauged central charge in the framework of harmonic superspace. Our study should be considered as a preparatory step on the way to the full local case with couplings to \( N = 2 \) supergravity. We propose a supersymmetric action which reproduces the component action given in [3]. We describe the superfield formulations of the linear and the ‘old’ nonlinear VT multiplet with gauged central charge. We present the harmonic superspace form of the Chern-Simons couplings to an external \( N = 2 \) super Yang-Mills multiplet for both the linear and nonlinear VT multiplets with gauged central charge. Finally, we show how all the results concerning the linear VT multiplet can be obtained by dimensional reduction from six dimensions.

We would like to point out that one should distinguish two classes of \( N = 2 \) multiplets with central charge. One of them is described by constrained superfields with a finite number of off-shell components. The constraints do not put the superfields on shell only owing to the presence of a non-trivial central charge. This class includes the Fayet-Sohnius hypermultiplet [16, 17] in the off-shell version of ref. [17] and all versions of the VT multiplet. Our consideration here is limited just to this class. Another type of \( N = 2 \) multiplets is represented by unconstrained analytic harmonic superfields with infinitely many auxiliary components like the universal \( q^+ \) hypermultiplet [12]. One can introduce and gauge the central charge for such superfields too, but its presence or absence has no impact on the relevant off-shell content.

The paper is organized as follows. In Section 2 we give a general discussion of \( N = 2 \) supersymmetric theories with gauged central charge and review the harmonic superspace technique [12] in the form adapted to our study. We derive the manifestly supersymmetric action underlying the dynamics of \( N = 2 \) theories with gauged central charge. We also discuss the supersymmetry transformations of the component fields and the limit of rigid central charge. In Section 3 we start by reviewing the Fayet-Sohnius hypermultiplet and the linear VT multiplet with gauged central charge. Further, the superfield consistency conditions are derived which must be fulfilled for any consistent VT multiplet superfield formulation. We then construct the Chern-Simons coupling to an external \( N = 2 \) vector multiplet for the linear VT multiplet with gauged central charge. The model of linear VT multiplet with scale and chiral invariance is considered in detail. In Section 4 the nonlinear VT multiplet (its ‘old’ version) with gauged central charge is described in harmonic superspace. We also present its Chern-Simons coupling to an external \( N = 2 \) vector multiplet. In Section 5 we demonstrate that the linear VT multiplet constraints described in the preceding sections have a natural origin in six dimensions.
2 Fundamentals

2.1 Preliminaries

The theories exhibiting invariance under rigid $N = 2$ supersymmetry and local central charge transformations can be formulated in $N = 2$ superspace $\mathbb{R}^{4|8}$ with coordinates

$$z^M = (x^m, \theta_i^\alpha, \bar{\theta}_i^\dot{\alpha})$$

where

$$\theta_i^\alpha = \bar{\theta}_i^\dot{\alpha} = 1, 2.$$ (2.1)

The basic objects of these formulations are gauge covariant derivatives

$$D_M \equiv (D_m, D_i^\alpha, \bar{D}_i^{\dot{\alpha}}) = D_M + A_M \Delta,$$ (2.2)

where $D_M = (\partial_m, D_i^\alpha, \bar{D}_i^{\dot{\alpha}})$ are the flat covariant derivatives

$$\{D_i^\alpha, D_j^\beta\} = \{\bar{D}_i^{\dot{\alpha}}, \bar{D}_j^{\dot{\beta}}\} = 0 \quad \{D_i^\alpha, \bar{D}_j^{\dot{\beta}}\} = -2i \delta_i^j \partial_{\alpha \dot{\beta}},$$ (2.3)

$\Delta$ is the generator of central charge transformations (the mass dimension of $\Delta$ is chosen to be +1) and $A_M(z)$ is the corresponding superfield gauge connection. The generator $\Delta$ is often interpreted as the derivative with respect to an extra bosonic coordinate $x^5$. In Section 3 we shall explain its origin from a six-dimensional point of view. For the time being, we simply require $\Delta$ to obey the Leibniz rule along with the conditions

$$[\Delta, D_M] = [\Delta, D_M] = 0.$$ (2.4)

The covariant derivatives are required to satisfy the constraints

$$\{D_i^\alpha, D_j^\beta\} = -2 \varepsilon_{\alpha \beta} \varepsilon^{ij} \bar{Z} \Delta \quad \{\bar{D}_i^{\dot{\alpha}}, \bar{D}_j^{\dot{\beta}}\} = -2 \varepsilon_{\dot{\alpha} \dot{\beta}} \varepsilon^{ij} \bar{Z} \Delta$$

$$\{D_i^\alpha, \bar{D}_j^{\dot{\beta}}\} = -2i \delta_i^j D_{\alpha \dot{\beta}},$$ (2.5)

which mean that $A_M$ describe an abelian vector multiplet $18$. The superfield strengths $Z$, $\bar{Z}$ obey the Bianchi identities

$$\bar{D}_{\dot{\alpha}} Z = 0 \quad D^{\alpha (i} D^j_{\alpha)} Z = \bar{D}^{\dot{\alpha} (i} D^j_{\dot{\alpha} \dot{\beta})} \bar{Z}.$$ (2.6)

Local central charge transformations are realized on $D_M$ and matter superfields $U$ as

$$\delta D_M = [\tau \Delta, D_M] \quad \delta A_M = -D_M \tau \quad \delta U = \tau \Delta U \equiv \tau U^{(\Delta)}.$$ (2.7)
with $\tau = \tau(z)$ being an unconstrained real gauge parameter. The interpretation of $\Delta$ as the derivative in an extra bosonic coordinate is useful for recognizing the fact that $U^{(\Delta)}$ cannot be represented, in general, as a linear combination of original superfields $U$ and their covariant derivatives. Applying the central charge transformations to $U^{(\Delta)}$ leads to new superfields $U^{(\Delta \Delta)}$, and so on and so forth. Generally, in a supersymmetric theory we have some set of basic (‘primary’) superfields $U$ and an infinite tower of descendants $\{U^{(\Delta)}, U^{(\Delta \Delta)}, \ldots\}$. If we wish to have an off-shell multiplet with a finite number of component fields, the superfields should be subject to some gauge-covariant constraints. The rôle of such constraints is not only to express the descendants in terms of a few basic superfields, but also to completely specify the central charge transformations. We will consider the case of a single central charge, though $N = 2$ supersymmetry in general admits two such charges (that would amount to a complex $\Delta$). The main motivation for this restriction is just the desire to have a finite multiplet; in the presence of two central charges the sequence $\{U, U^{(\Delta)}, U^{(\Delta \Delta)}, \ldots\}$ in many cases does not terminate at any finite step (for an alternative explanation see Section 5, where the two central charges are interpreted as two extra coordinates $x^{5,6}$).

It is worth saying that the superalgebra \((2.5)\), as it stands, looks like the algebra of covariant derivatives of some abelian $N = 2$ gauge theory with an unspecified gauge generator $\Delta$. The interpretation of the latter as a central charge generator necessarily requires non-zero background values of the gauge connections corresponding to $\langle Z \rangle = \text{const} \neq 0$. Just with this choice of the flat limit the algebra \((2.5)\) goes into that of the covariant spinor derivatives corresponding to $N = 2$ supersymmetry with $\Delta$ as the rigid central charge. This issue will be discussed in more detail in subsection 2.3.

### 2.2 Harmonic superspace formulation

The main virtue of the harmonic superspace method \([12]\) consists in providing the unique possibility to describe the off-shell $N = 2$ hypermultiplets in terms of unconstrained superfields. Also, it results in off-shell formulations of $N = 2$ gauge theories and supergravity in terms of the connections and vielbeins covariantizing the derivatives with respect to harmonic variables. The formulation of the vector gauge multiplet in the standard $N = 2$ superspace $\mathbb{R}^{4|8}$ turns out to be a gauge-fixed version of that in $N = 2$ harmonic superspace \([12]\).

The harmonic superspace is also indispensable for formulating $N = 2$ theories with gauged central charge in a manifestly supersymmetric way. Since our consideration will
be essentially based upon the harmonic superspace techniques, we start by recalling some salient features of this method.

The \( N = 2 \) harmonic superspace is an extension of ordinary \( N = 2 \) superspace \( \mathbb{R}^{4|8} \) by the harmonic variables \( u_i^- \), \( u_i^+ \) which parametrize the two-sphere \( S^2 = SU(2)/U(1) \), \( SU(2) \) being the automorphism group of \( N = 2 \) supersymmetry,

\[
(u_i^-, u_i^+) \in SU(2)
\]

\[
u_i^+ = \varepsilon_{ij} u^{+j}, \quad \overline{u}^{+i} = u_i^- \quad u^{+i} u_i^- = 1.
\]  

(2.8)

Tensor fields over \( S^2 \) are in a one-to-one correspondence with functions on \( SU(2) \) possessing definite harmonic \( U(1) \)-charges. A function \( \Psi^{(p)}(u) \) is said to have the harmonic \( U(1) \)-charge \( p \) if

\[
\Psi^{(p)}(e^{i\alpha} u^+, e^{-i\alpha} u^-) = e^{i\alpha p} \Psi^{(p)}(u^+, u^-) \quad |e^{i\alpha}| = 1.
\]

Such functions extended to the whole harmonic superspace \( \mathbb{R}^{4|8} \times S^2 \), that is \( \Psi^{(p)}(z, u) \), are called harmonic \( N = 2 \) superfields.

The operators

\[
D^{\pm\pm} = u^{\pm i} \partial / \partial u^{\mp i} \quad D^0 = u^{+i} \partial / \partial u^{-i} - u^{-i} \partial / \partial u^{+i}
\]

\[
[D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm} \quad [D^{++}, D^{--}] = D^0
\]  

(2.9)

are left-invariant vector fields on \( SU(2) \). \( D^{\pm\pm} \) are two independent harmonic covariant derivatives on \( S^2 \), while \( D^0 \) is the \( U(1) \) charge operator, \( D^0 \Psi^{(p)} = p \Psi^{(p)} \).

Using the harmonics, one can convert the spinor covariant derivatives into \( SU(2) \)-invariant operators on \( \mathbb{R}^{4|8} \times S^2 \)

\[
D^\alpha_\pm = D^i_\alpha u^\pm_i \quad \bar{D}^\alpha_\pm = \bar{D}^i_\alpha u^\pm_i.
\]  

(2.10)

Then the superalgebra (2.8) implies the existence of the anticommuting subset

\[
\{D^\alpha_+, D^\beta_+\} = \{\bar{D}^\alpha_+, \bar{D}^\beta_+\} = \{D^\alpha_+, \bar{D}^\beta_+\} = 0,
\]  

(2.11)

whence

\[
D^\alpha_+ \mathcal{A}_\beta + D^\beta_+ \mathcal{A}_\alpha = D^\alpha_+ \bar{\mathcal{A}}_\beta + \bar{D}^\beta_+ \mathcal{A}_\alpha = 0 \quad \Rightarrow \quad (\mathcal{A}_\beta, \bar{\mathcal{A}}_\beta) = (D^\alpha_+ \mathcal{G}, \bar{D}^\alpha_+ \mathcal{G})
\]

\[
\mathcal{D}^\alpha_+ = D^\alpha_+ + (D^\alpha_+ \mathcal{G}) \Delta, \quad \overline{\mathcal{D}}^\alpha_+ = \overline{D}^\alpha_+ + (\overline{D}^\alpha_+ \mathcal{G}) \Delta.
\]  

(2.12)

Here the superfield \( \mathcal{G} = \mathcal{G}(z, u) \), called ‘bridge’, has vanishing harmonic \( U(1) \)-charge and is real, \( \mathcal{G} = \mathcal{G} \), with respect to the generalized conjugation \( \sim \equiv \overline{\sim} \) \[12\], where the operation \( \sim \) is defined by

\[
(u^i_+)^\sim = u^-_i, \quad (u^-_i)^\sim = -u^+_i, \quad \Rightarrow \quad (u^\pm_i)^{**} = -u^\pm_i.
\]
Eq. (2.12) solves the constraints (2.3) under some additional restriction on $G(z,u)$ (see below). An obvious immediate consequence of the relations (2.11) and (2.12) is the existence of an important subclass of harmonic superfields, the covariantly analytic ones. They are defined by the constraints

$$D^+_\alpha \Phi^{(p)} = \bar{D}^+_\alpha \Phi^{(p)} = 0 ,$$  \tag{2.13}

whence

$$\Phi^{(p)} = e^{-G\Delta} \Phi^{(p)} , \quad D^+_\alpha \hat{\Phi}^{(p)} = \bar{D}^+_\alpha \hat{\Phi}^{(p)} = 0 .$$  \tag{2.14}

The superfields $\hat{\Phi}^{(p)}$ are functions over the so-called analytic subspace of the harmonic superspace parameterized by

$$\{\zeta, u^\pm_i\} \equiv \{x^m_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_i\} , \quad \hat{\Phi}^{(p)} \equiv \hat{\Phi}^{(p)}(\zeta, u) ,$$  \tag{2.15}

where

$$x^m_A = x^m - 2i\theta^i(\sigma^m \bar{\theta}^j)u_i^+u_j^- \quad \theta^\pm = \theta^\pm_i u_i^\pm \quad \bar{\theta}^\pm = \bar{\theta}^\pm_i u_i^\pm .$$  \tag{2.16}

That is why such superfields are called analytic. Note that the analytic subspace (2.15) is closed under $N = 2$ supersymmetry transformations and the generalized conjugation “$\tilde{}$”, i.e. it is real with respect to this conjugation. The analytic superfields with even $U(1)$ charge can so be chosen real.

In accordance with eq. (2.12), the bridge possesses a more general gauge freedom than the original harmonic-independent $\tau$-group (2.7):

$$\delta G = \lambda - \tau \quad \lambda = \lambda(\zeta, u) \quad D^+_\alpha \lambda = \bar{D}^+_\alpha \lambda = 0 .$$  \tag{2.17}

Here the unconstrained analytic gauge parameter $\lambda$ has vanishing $U(1)$-charge and is real, $\tilde{\lambda} = \lambda$, with respect to the analyticity preserving conjugation. The set of all $\lambda$-transformations is called the $\lambda$-group. The $\tau$-group acts on $\Phi^{(p)}$ and leaves $\hat{\Phi}^{(p)}$ unchanged; the $\lambda$-group acts on $\hat{\Phi}^{(p)}$ as

$$\delta \hat{\Phi}^{(p)} = \lambda \Delta \hat{\Phi}^{(p)}$$  \tag{2.18}

and leaves $\Phi^{(p)}$ unchanged. Thus one can equivalently formulate the theory in the two frames related by the similarity operator $e^{-G\Delta}$: in the $\tau$-frame where the $\tau$-group is manifest and in the $\lambda$-frame with the $\lambda$-group manifest.

As we observe from the relation (2.14), the $N = 2$ harmonic analyticity is covariant in the $\tau$-frame, but it becomes manifest in the $\lambda$-frame. This can also be seen by comparing the covariant derivatives in both frames.
In the $\tau$-frame, the complete set of gauge-covariant derivatives reads
\[ \mathcal{D}_M \equiv (D_M, D^{++}, D^{-}, D^0) \quad D^{\pm\pm} = D^{\pm\pm} \quad D^0 = D^0 \] (2.19)
and their transformation law is the same as that of $\mathcal{D}_M$ given by (2.7). The transformation law of matter superfields $U$ is also given by (2.7).

In the $\lambda$-frame, the covariant derivatives
\[ \hat{\mathcal{D}}_M = e^{\Delta \mathcal{D}}_M e^{-\Delta} = \mathcal{D}_M - (D_M \mathcal{G}) \Delta \] (2.20)
and the matter superfields
\[ \hat{U} = e^{\Delta \mathcal{D}} U \] (2.21)
transform by the rule
\[ \delta \hat{\mathcal{D}}_M = [\lambda \Delta, \hat{\mathcal{D}}_M] \quad \delta \hat{U} = \lambda \Delta \hat{U} . \] (2.22)

In this frame we have
\[ \hat{\mathcal{D}}^+_\alpha = D^+_\alpha \quad \hat{\mathcal{D}}^+_\dot{\alpha} = \bar{D}^+_\dot{\alpha} \quad \hat{D}^0 = D^0 \]
\[ \hat{\mathcal{D}}^{\pm\pm} = D^{\pm\pm} - (D^{\pm\pm} \mathcal{G}) \Delta \equiv D^{\pm\pm} + \mathcal{V}^{\pm\pm} \Delta . \] (2.23)

We observe that the $u^+$-projections of the spinor covariant derivatives contain no central charge gauge connections in the $\lambda$-frame, so the conditions (2.13) imply exact harmonic analyticity (after passing to the analytic basis in $\mathbb{R}^{4|8} \times S^2$ according to eq. (2.16), these projections become partial derivatives in $\bar{\theta}^-, \dot{\bar{\theta}}^-$, and conditions (2.13) simply mean independence of these coordinates).

The algebra of covariant derivatives, with all the isospinor indices converted into $U(1)$ ones, clearly does not depend on the choice of the frame and/or the basis in $\mathbb{R}^{4|8} \times S^2$. It reads
\[ \{ \hat{\mathcal{D}}^+_\alpha, \mathcal{D}^-_\beta \} = -\{ \mathcal{D}^+_\alpha, \hat{\mathcal{D}}^-_\beta \} = 2i \mathcal{D}_{\alpha\dot{\beta}} \]
\[ \{ \mathcal{D}^+_\alpha, \mathcal{D}^-_\beta \} = 2\varepsilon_{\alpha\beta} \bar{Z} \Delta \quad \{ \hat{\mathcal{D}}^+_\dot{\alpha}, \hat{\mathcal{D}}^-_\dot{\beta} \} = -2\varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Z} \Delta \]
\[ [\mathcal{D}^{\pm\pm}, \mathcal{D}^+_\alpha] = \mathcal{D}^+_\alpha \quad [\mathcal{D}^{\pm\pm}, \hat{\mathcal{D}}^+_\dot{\alpha}] = \hat{\mathcal{D}}^+_\dot{\alpha} \]
\[ [\mathcal{D}^0, \mathcal{D}^{\pm\pm}] = \pm 2\mathcal{D}^{\pm\pm} \quad [\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 . \] (2.24)

All other (anti-)commutators vanish except those involving the vector covariant derivatives. The latter can be readily derived from the relations given above.

Consideration in the $\lambda$-frame allows one to reveal the basic unconstrained object of the theory. This is the connection $\mathcal{V}^{++}$ covariantizing the harmonic derivative $\mathcal{D}^{++}$ with respect to the $\lambda$-group.
Since \([\mathbf{D}^{++}, \mathbf{D}^{\alpha}] = [\mathbf{D}^{++}, \mathbf{D}^{\alpha}] = 0\), the connection \(\mathcal{V}^{++}\) is an analytic real superfield, \(D^{\alpha} \mathcal{V}^{++} = \mathbf{D}^{\alpha} \mathcal{V}^{++} = 0\), \(\dot{\mathcal{V}}^{++} = \mathcal{V}^{++}\), with the transformation law

\[
\delta \mathcal{V}^{++} = -D^{++} \lambda .
\] (2.25)

No other constraints on \(\mathcal{V}^{++}\) emerge.

To demonstrate that all other central-charge connections are expressed through the single object \(\mathcal{V}^{++}\), one should firstly solve for \(\mathcal{V}^{--}\) the zero-curvature condition

\[
D^{++} \mathcal{V}^{--} - D^{--} \mathcal{V}^{++} = 0,
\] (2.26)

which follows from considering the last commutator of harmonic derivatives in (2.24) in the \(\lambda\)-frame. Its solution exists and can be given explicitly \[19\]. Then, using the remainder of the (anti-)commutation relations (2.24) as well as the explicit form (2.23) of \(\mathbf{D}^{+}\) and \(\overline{\mathbf{D}}^{+}\), one easily expresses all the remaining connections in \(\mathbf{D}_{M}\) in terms of \(\mathcal{V}^{--}\) and, hence, \(\mathcal{V}^{++}\). The superfield strengths are expressed through \(\mathcal{V}^{--}\) as

\[
\mathcal{Z} = -\frac{1}{4}(\mathbf{D}^{+})^{2} \mathcal{V}^{--}, \quad \overline{\mathcal{Z}} = -\frac{1}{4}(\overline{\mathbf{D}}^{+})^{2} \mathcal{V}^{--}.
\] (2.27)

The Bianchi identities (2.6) can be shown to be identically satisfied. Note that the second identity can be written as

\[
(D^{+})^{2} \mathcal{Z} = (\overline{D}^{+})^{2} \overline{\mathcal{Z}} .
\] (2.28)

Thus, \(\mathcal{V}^{++}\) is indeed the fundamental unconstrained analytic prepotential of the theory. Note that the defining relation \(\mathcal{V}^{++} = -D^{++} \mathcal{G}\) should be treated as a constraint on the bridge \(\mathcal{G}\) serving to express \(\mathcal{G}\) through \(\mathcal{V}^{++}\).

To be convinced that \(\mathcal{V}^{++}\) accommodates the standard component fields content of the \(N = 2\) abelian vector multiplet, one should make use of the gauge freedom (2.25) and pass to the Wess-Zumino (WZ) gauge

\[
\mathcal{V}^{++}(\zeta, u) = (\theta^{+})^{2} \overline{Z}(x_{A}) + (\overline{\theta}^{+})^{2} Z(x_{A}) - 2i\theta^{+}\sigma^{m}\overline{\theta}^{+} A_{m}(x_{A}) - 2(\theta^{+})^{2} \theta^{+}\overline{\Psi}_{i}^{\alpha}(x_{A})u_{i}^{-} + 2(\theta^{+})^{2} \overline{\theta}^{+}\overline{\Psi}_{i}^{\dot{\alpha}}(x_{A})u_{i}^{-} + 3(\theta^{+})^{2}(\overline{\theta}^{+})^{2} Y^{(ij)}(x_{A})u_{i}^{-}u_{j}^{-} .
\] (2.29)

Here, \(A_{m}\) is a real vector field, and \(\Psi_{i}^{\alpha}\) and \(Y^{(ij)}\) satisfy the reality conditions

\[
\overline{\Psi}_{i}^{\alpha} = \Psi_{i}^{\dot{\alpha}}, \quad \overline{Y^{(ij)}} = Y^{(ij)} .
\] (2.30)
The residual gauge freedom is given by \( \lambda = \xi(x_A) \) describing the ordinary central charge transformations, with \( A_m \) being the corresponding gauge field. In the WZ gauge, \( \mathcal{V}^{--} \) reads
\[
\mathcal{V}^{--}(\zeta, \theta^-, \bar{\theta}^-) = (\bar{\theta}^-)^2 \tilde{Z}(x_A) + (\bar{\theta}^-)^2 Z(x_A) - 2i\theta^- \sigma^m \bar{\theta}^- A_m(x_A) \\
+ \{(\bar{\theta}^-)^2 \theta^- u^+_i - (\bar{\theta}^-)^2 \theta^+ u^-_i - 2(\bar{\theta}^+ \bar{\theta}^-) \theta^+ u^-_i \} \Psi^i(x_A) \\
+ \{(\bar{\theta}^-)^2 \bar{\theta}^+ u^-_i - (\bar{\theta}^-)^2 \bar{\theta}^- u^+_i + 2(\theta^+ \theta^-) \bar{\theta}^- u^+_i \} \bar{\Psi}^i(x_A) \\
+ \{(\theta^-)^2 (\bar{\theta}^+)^2 u^-_i u^-_j + (\theta^-)^2 (\bar{\theta}^-)^2 u^-_i u^-_j + (\theta^-)^2 (\bar{\theta}^+)^2 u^+_i u^+_j \} Y^{ij}(x_A) \\
- 2(\theta^+ \theta^-)(\bar{\theta}^-)^2 u^+_i u^-_j + (\theta^-)^2 (\bar{\theta}^+ \bar{\theta}^-) u^+_i u^-_j \} Y^{ij}(x_A) \\
+ 4(\theta^+ \theta^-)(\bar{\theta}^+ \bar{\theta}^-) Y^{ij}(x_A) u^-_i u^-_j + \cdots \tag{2.31}
\]
where the dots mean the terms involving derivatives of the fields. It is worth keeping in mind that in the analytic basis (2.16) \( D^{++} \) and \( D^{--} \) read
\[
D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} - 2i\theta^+ \sigma^m \bar{\theta}^+ \frac{\partial}{\partial x^m_A} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-} \\
D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} - 2i\theta^- \sigma^m \bar{\theta}^- \frac{\partial}{\partial x^m_A} + \theta^- \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^+} \tag{2.32}
\]
Using the above relations, one easily computes the components of the superfield strength \( \mathcal{Z} \) given by eqs. (2.27)
\[
\mathcal{Z} = Z \quad D^i \mathcal{Z} = \Psi^i \quad -\frac{1}{4} D^i D^j \mathcal{Z} = Y^{ij} \quad -\frac{1}{8} D^i D^j D^{ \alpha \beta} \mathcal{Z} = F_{\alpha \beta} \\
F_{mn} = \partial_m A_n - \partial_n A_m \quad (\sigma^m)_{\alpha \dot{\alpha}} (\sigma^n)_{\beta \dot{\beta}} F_{mn} = 2\varepsilon_{\alpha \dot{\alpha}} F_{\alpha \beta} + 2\varepsilon_{\alpha \dot{\alpha}} \tilde{F}_{\dot{\alpha} \beta} \tag{2.33}
\]
Here \( U \) denotes the \( \theta \)-independent component of a superfield \( U \).

Note that the formulation described so far actually coincides with the standard harmonic superspace formulation of abelian \( N = 2 \) gauge theory [12]. As was already mentioned in the previous subsection, the crucial assumption allowing to interpret the theory as that of gauged central charge amounts to specifying the appropriate flat limit characterized by a non-zero background value of \( \mathcal{Z} \) and, hence, of \( \mathcal{V}^{++} \). We discuss this point in the next subsection.

### 2.3 Supersymmetry transformations

Let us turn to a more detailed study of the supersymmetry transformations.

In setting up the invariant supersymmetric actions we will deal with the set of the matter basic superfields \( \tilde{U} \), their covariant derivatives (\( \Delta \) is to be included into the set
of covariant derivatives) and the harmonic gauge connection $\mathcal{V}^{++}$. The construction presented in the previous subsection is covariant under the local gauge transformations generated by $\Delta$ and given by eqs. (2.22). Obviously, it is also covariant under the standard rigid $N = 2$ supersymmetry transformations

$$\delta_\epsilon \hat{U} = i \left( \epsilon^\alpha Q^+_{\alpha} + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}^+_\dot{\alpha} - \epsilon^{+\alpha} Q^-_{\alpha} - \bar{\epsilon}^{+\dot{\alpha}} \bar{Q}^-_{\dot{\alpha}} \right) \hat{U}$$

(2.34)

and similar ones for $\mathcal{V}^{++}$, where

$$Q^-_{\alpha} = -i \frac{\partial}{\partial \theta^+\alpha} \quad Q^+_{\alpha} = i \frac{\partial}{\partial \theta^-\alpha} + 2 (\sigma^m \bar{\theta}^+)^\alpha \frac{\partial}{\partial x^m_A}$$

$$\bar{Q}^-_{\dot{\alpha}} = -i \frac{\partial}{\partial \bar{\theta}^+\dot{\alpha}} \quad \bar{Q}^+_{\dot{\alpha}} = i \frac{\partial}{\partial \bar{\theta}^-\dot{\alpha}} - 2 (\bar{\theta}^+ \sigma^m)^\dot{\alpha} \frac{\partial}{\partial x^m_A}$$

(2.35)

The invariance under these two sets of transformations (local central charge and rigid $N = 2$ supersymmetry) will be the basic requirement to be fulfilled by any off-shell superfield action we will deal with.

It is important to realize that these two basic invariances already imply the invariance under another type of rigid $N = 2$ supersymmetry, that with the operator $\Delta$ as the central charge, provided one chooses an appropriate ‘flat’ background value for $\mathcal{V}^{++}$.

The choice

$$\langle \mathcal{V}^{++} \rangle_{(1)} = 0$$

(2.36)

is clearly consistent with just the standard $N = 2$ supersymmetry

$$Q^+_\alpha \langle \mathcal{V}^{++} \rangle_{(1)} = \bar{Q}^+_{\dot{\alpha}} \langle \mathcal{V}^{++} \rangle_{(1)} = 0$$

(2.37)

With this option, $\Delta$ never appears in the algebra of $N = 2$ supersymmetry and should be treated as some extra U(1) gauge generator.

On the other hand, one can choose a more general background \[20, 21, 22\]

$$\langle \mathcal{V}^{++} \rangle_{(2)} = i \left[ (\theta^+)^2 - (\bar{\theta}^+)^2 \right].$$

(2.38)

Obviously, it does not vanish under the action of the generators of the standard $N = 2$ supersymmetry. However, the result of this action can be cancelled by an appropriate compensating transformation from the $\lambda$ group. Namely, (2.38) is stable,

$$\hat{\delta}_\epsilon \langle \mathcal{V}^{++} \rangle_{(2)} = 0$$

(2.39)

\[1\] In fact, the most general Ansatz consistent with the Poincaré and SU(2)$_A$ covariance reads (modulo $\lambda$ gauge group transformations) $\langle \mathcal{V}^{++} \rangle_{(2)} = b (\theta^+)^2 + \bar{b} (\bar{\theta}^+)^2$, $b$ being a complex constant. This option can be reduced to (2.38) by a proper chiral phase rotation of $\theta^\pm_{\alpha}, \bar{\theta}^\pm_{\dot{\alpha}}$ and rescaling of the generator $\Delta$. 


against the following modified $N = 2$ supersymmetry transformations

$$\tilde{\delta}_{\epsilon} V^{++} = \delta_{\epsilon} V^{++} - D^{++}\lambda_0 \quad \tilde{\delta}_{\epsilon} \hat{U} = \delta_{\epsilon} \hat{U} + \lambda_0 \Delta \hat{U}$$

$$\lambda_0 = -2i(\epsilon^{-\theta^+} - \tilde{\epsilon}^{-\tilde{\theta}^+}) . \quad (2.40)$$

It is easy to read off from (2.40) the expressions for the modified $N = 2$ supersymmetry generators in the realization on $\hat{U}$

$$\tilde{Q}^-_\alpha = Q^-_\alpha , \quad \tilde{Q}^+_\alpha = i \frac{\partial}{\partial \theta - \alpha} + 2(\sigma^m \tilde{\theta}^+)\alpha \partial_m - 2\theta^+\Delta ,$$

$$\tilde{Q}^-_\dot{\alpha} = Q^-_\dot{\alpha} , \quad \tilde{Q}^+_\dot{\alpha} = i \frac{\partial}{\partial \tilde{\theta} - \dot{\alpha}} - 2(\theta^+ \sigma^m )\dot{\alpha} \partial_m - 2\tilde{\theta}^+\Delta . \quad (2.41)$$

They form the $N = 2$ superalgebra with $\Delta$ as the central charge. What concerns the realization of this modified $N = 2$ supersymmetry on $V^{++}$, it can be reduced to the standard one (2.34) by shifting the harmonic connection

$$V^{++} = \langle V^{++} \rangle_{(2)} + \hat{V}^{++} , \quad \tilde{\delta}_{\epsilon} \hat{V}^{++} = \delta_{\epsilon} \hat{V}^{++} . \quad (2.42)$$

This is of course due to the fact that $V^{++}$ and $\hat{V}^{++}$ possess zero central charge, $\Delta V^{++} = \Delta \hat{V}^{++} = 0$.

Eqs. (2.40) and (2.41) yield the off-shell realization of the central-charge extended rigid $N = 2$ supersymmetry in the general case of unfixed $\lambda$-group gauge freedom.

In practice, for the component considerations, it is convenient to know how rigid $N = 2$ supersymmetry is realized in the WZ gauge (2.29). Actually, it makes no difference from which rigid supersymmetry transformations one starts, (2.34), (2.35) or (2.40), (2.41), since they differ by a $\lambda$-gauge transformation. Let us choose, e.g., (2.34) and (2.35). Then, to preserve the WZ gauge, every supersymmetry transformation should be accompanied by a special $\epsilon$-dependent gauge transformation with the parameter

$$\lambda(\epsilon) = -2(\epsilon^{-\theta^+})\tilde{Z} - 2(\tilde{\epsilon}^{-\tilde{\theta}^+})Z + 2i(\epsilon^{-\sigma^m \tilde{\theta}^+} + \theta^+ \sigma^m \epsilon^-)A_m + O(\theta^2) . \quad (2.43)$$

This leads to the standard transformation law for the component fields of the vector multiplet as well as modifies eq. (2.34) by

$$\delta_{\text{SUSY}} \hat{U} = i \left( \epsilon^{-\alpha} Q^+_\alpha - \epsilon^+\alpha \tilde{Q}^-_\alpha - \epsilon^{-\dot{\alpha}} \tilde{Q}^+_\dot{\alpha} - \epsilon^+\dot{\alpha} \tilde{Q}^-_\dot{\alpha} \right) \hat{U} , \quad (2.44)$$

where

$$Q^-_\alpha = Q^-_\alpha , \quad Q^+_\alpha = i \frac{\partial}{\partial \theta - \alpha} + 2i\theta^+\tilde{Z}(x_A)\Delta + 2(\sigma^m \tilde{\theta}^+)\alpha \nabla_m + O(\theta^2)$$

$$\tilde{Q}^-_\dot{\alpha} = \tilde{Q}^-_\dot{\alpha} , \quad \tilde{Q}^+_\dot{\alpha} = i \frac{\partial}{\partial \tilde{\theta} - \dot{\alpha}} - 2i\tilde{\theta}^+Z(x_A)\Delta - 2(\theta^+ \sigma^m )\dot{\alpha} \nabla_m + O(\theta^2)$$

$$\nabla_m \equiv \frac{\partial}{\partial x^m_A} + A_m(x_A)\Delta . \quad (2.45)$$
To compute the supersymmetry transformations of the component fields, it is sufficient to note the identity

\[
\delta_{\text{SUSY}} \dot{D}_{M_1} \cdots \dot{D}_{M_p}(\Delta)^q \dot{U}
= -\left( \epsilon^{-a} \dot{D}^+_a + \epsilon^{-\bar{a}} \dot{D}^+_{\bar{a}} - \epsilon^{+a} \dot{D}^-_a - \epsilon^{+\bar{a}} \dot{D}^-_{\bar{a}} \right) \dot{D}_{M_1} \cdots \dot{D}_{M_p}(\Delta)^q \dot{U} \, .
\] (2.46)

Along with the relation

\[
\dot{D}_{M_1} \dot{D}_{M_2} \cdots \dot{D}_{M_k} \dot{U} = D_{M_1} D_{M_2} \cdots D_{M_k} U \, ,
\] (2.47)

which holds in the above WZ gauge (and with the appropriately fixed $\tau$-gauge group freedom), this gives us a convenient practical recipe to compute the supersymmetry transformations of component fields.

Now, let us recall eqs. (2.5) and (2.33). It follows from (2.44), (2.45) that the commutator of two supersymmetry transformations involves not only a space-time translation, but also central charge terms proportional to $Z$ and $\bar{Z}$. Therefore, (2.44), (2.45) describe both situations discussed earlier: when $Z$ is assumed to have a zero vacuum expectation value, we get the WZ gauge-fixed form of the standard $N = 2$ supersymmetry with $\Delta$ being an extra gauge group generator; on the contrary, if $Z$ possesses a non-zero vacuum expectation value, we are left with a gauge-fixed form of rigid supersymmetry with the central charge $\Delta$. The precise correspondence with eqs. (2.38) and (2.41) emerges upon the choice

\[\langle Z \rangle = -i \, .\] (2.48)

As the last remark, we mention that with our choice of the flat background the reduction to the case of rigid central charge (which corresponds to a constant strength $Z$) goes by putting

\[\mathcal{V}^{++} = \langle \mathcal{V}^{++} \rangle = i \left[ (\theta^+)^2 - (\bar{\theta}^+)^2 \right] \, ,\] (2.49)
or, in the WZ gauge,

\[Z = \langle Z \rangle = -i \quad (Z = -\bar{Z} = -i) \, ,\] (2.50)

all other components of the vector multiplet being equal to zero. This yields, in particular, the following rigid central charge form of the covariant derivatives $D^{i}_\alpha$ and $\bar{D}^{i}_{\bar{\alpha}}$

\[
D^{i}_\alpha = \frac{\partial}{\partial \theta^i_\alpha} + i(\sigma^m \bar{\theta}^i_\alpha) \frac{\partial}{\partial \theta^m} - i\theta^i_\alpha \Delta \quad \bar{D}^{i}_{\bar{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^i_{\bar{\alpha}}} - i(\sigma^m \theta^i_{\bar{\alpha}}) \frac{\partial}{\partial \theta^m} - i\bar{\theta}^i_{\bar{\alpha}} \Delta \, .
\] (2.51)

As follows from the above consideration, $N = 2$ supersymmetric theories with gauged central charge can be obtained from those with rigid central charge by covariantizing
the constraints which define the multiplets in the rigid case. However, as a rule such a
covariantization requires adding some non-minimal terms in order to give rise to consistent
constraints (see Sect. 3).

2.4 Supersymmetric action

Now, we are prepared to present the main result of this Section – the action functional
rule underlying the dynamics of \( N = 2 \) supersymmetric theories with gauged central
charge. To avoid a possible confusion, let us point out that here and in the rest of
the paper we limit our consideration to matter superfields \( U \) with a \textit{finite} number of
auxiliary components and non-trivial central charge. Such superfields are necessarily
constrained, so are the relevant Lagrangian densities. Our consideration does not directly
apply to the theories with unconstrained superfields like the \( q^+ \) hypermultiplet \([12]\) having
infinitely many auxiliary fields, though non-trivial rigid and gauged central charges can
be introduced in this case too, e.g. via the Scherk-Schwarz mechanism \([23]\).

Let \( L^{ij}(z) \) be an isovector superfield which is built out of the basic dynamical super-
fields \( (U, V^{++}, \ldots) \) and possesses the following basic properties

\[
D^{ij} L^{jk} = \bar{D}^{\dot{i} \dot{j}} L^{\dot{k} \dot{l}} = 0 .
\]  

(2.52)

The same constraints written for the harmonic superfield

\[
L^{++} = L^{ij} u^+_i u^+_j
\]  

(2.53)

amount to the following set

\[
D^+_\alpha L^{++} = \bar{D}^\dot{\alpha} L^{++} = 0
\]  

(2.54)

\[
D^{++} L^{++} = 0 .
\]  

(2.55)

Thus, \( L^{++} \) is covariantly analytic and bilinear in the harmonics. In the \( \lambda \)-frame, eqs.
(2.54) and (2.53) turn into

\[
\hat{L}^{++} = \hat{L}^{++}(\zeta, u)
\]  

(2.56)

\[
\hat{D}^{++} \hat{L}^{++} = \left( \hat{D}^{++} + \hat{V}^{++} \Delta \right) \hat{L}^{++} = 0 .
\]  

(2.57)

Let us consider the integral over the analytic subspace

\[
S = \int du d\zeta(-4) \hat{V}^{++} \hat{L}^{++}
\]  

(2.58)
where \( d\zeta^{(-4)} = d^4x_A d^2\theta^+ d^2\bar{\theta}^+ \) and the integration over \( SU(2) \) is defined by [12]

\[
\int du = 1 \quad \int du u_{(i_1}^+ \ldots u_{i_n}^+ u_{j_1}^- \ldots u_{j_m}^- = 0 \quad n + m > 0 .
\]

Being manifestly \( N = 2 \) supersymmetric, functional (2.58) is also invariant under arbitrary local central charge transformations, since its variation

\[
\delta S = \int du d\zeta^{(-4)} \left\{ -(D^{++} + \lambda)\hat{L}^{++} + \lambda V^{++} \Delta \hat{L}^{++} \right\}
\]

(2.59) can be transformed, by integrating the first term by parts, to

\[
\delta S = \int du d\zeta^{(-4)} \lambda \hat{D}^{++} \hat{L}^{++}
\]

(2.60) that vanishes, as a consequence of (2.57). In the WZ gauge (2.29), it is easy to reduce \( S \) to components using the integration rule

\[
\int du d\zeta^{(-4)} \frac{1}{16} \int d^4x du (D^-)^2(D^-)^2
\]

and the relation (2.47). This gives

\[
S = -\frac{1}{12} \int d^4x \left\{ Z \hat{D}^a \hat{D}^a \mathcal{L}_{ij} + \bar{Z} \hat{D}_a \hat{D}^{\dot{a}} \mathcal{L}_{ij} - i A^{a\dot{a}} \left[ \hat{D}^a_{;i}, \hat{D}^{\dot{a}}_{;j} \right] \mathcal{L}_{ij} \\
+ 4 \Psi \bar{\Psi} \hat{D}^a \mathcal{L}_{ij} + 4 \bar{\Psi} \Psi \hat{D}^{\dot{a}} \mathcal{L}_{ij} - 12 Y \mathcal{L}_{ij} \right\} .
\]

(2.61) This is exactly the general form of the component action for \( N = 2 \) supersymmetric theories with gauged central charge, which was suggested in [4] (of course, the invariant action for the gauge superfield \( V^{++} \) itself should be added). After performing the reduction (2.49) to the case of rigid central charge, (2.58) goes into the supersymmetric action functional proposed in [11].

At this step we observe one more reason why \( V^{++} \) should necessarily contain a non-vanishing background part (2.38): just this part produces correct kinetic terms for the matter superfields \( U \) present in \( \hat{L}^{++} \).

The action (2.58) provides us with a universal rule for constructing invariants in \( N = 2 \) theories with gauged central charge and constrained matter superfields \( U \). It is reminiscent of the action rule in general relativity, with \( V^{++} \) being the analog of \( \sqrt{-g} \) and \( \hat{L}^{++} \) of the Lagrangian density.

Note that \( \Delta \hat{L}^{++}, \Delta^2 \hat{L}^{++}, \ldots \) satisfy the basic requirements (2.54) and (2.55) and so, at first sight, their \( \lambda \)-frame images could be equally acceptable for the rôle of Lagrangian densities. However, the harmonic constraint (2.57) implies

\[
V^{++} \Delta^n \hat{L}^{++} = -D^{++} \Delta^{n-1} \hat{L}^{++} .
\]
As a consequence, all such densities, except for $\hat{L}^{++}$ itself, produce full harmonic derivatives upon substitution into (2.58) and so do not contribute, both in the cases of gauged and rigid central charges. Recalling the interpretation of $\Delta$ as $\partial/\partial x^5$ (see Section 5), this amounts to saying that the analytic superspace action (2.58) does not depend on $x^5$, though the Lagrangian densities could bear such a dependence.

In the next Sections we will illustrate the general formalism given here by several examples of $N = 2$ supersymmetric theories with gauged central charge.

### 3 Models with linear central charge transformations

Here we consider the simplest case of the linearly realized central charge.

#### 3.1 Hypermultiplet with central charge

Let us start with gauging the central charge of the Fayet-Iliopoulos hypermultiplet [16, 17] coupled to an external $N = 2$ Yang-Mills superfield. It is described by a superfield $q_i(z)$

and its conjugate $\tilde{q}^i(z)$ subject to the constraints

$$D^i_{\alpha}q^j = \bar{D}^i_{\dot{\alpha}}\tilde{q}^j = 0 \quad (3.1)$$

where

$$D_M = D_M + A_M \quad A_M = A_M^a(z)T^a \quad [\Delta, D_M] = [\Delta, T^a] = 0 \quad (3.2)$$

with $D_M$ being the rigid central charge covariant derivatives (2.51), $A_M$ the $N = 2$ super Yang-Mills connection [18], and $T^a$ the generators of the gauge group. Eq. (3.1) implies that only the superfields $q_i$, $q_i^{(\Delta)}$ and their conjugates contain independent component fields, and the higher descendants are expressed in terms of $q_i$, $q_i^{(\Delta)}$ and their conjugates. Actually, it is easy to show that the components of $q_i^{(\Delta)}$ are expressed through those of $q_i$ and $x$-derivatives of the latter, leaving us with $(8 + 8)$ component fields off-shell.

To gauge the central charge, it is sufficient to naively covariantize the above constraints with respect to the central charge

$$D^i_{\alpha}q^j = \bar{D}^i_{\dot{\alpha}}\tilde{q}^j = 0 \quad (3.3)$$

Here the covariant derivatives

$$D_M = D_M + A_M \Delta + \bar{A}_M \quad (3.4)$$
satisfy now the algebra
\[
\{D^i_{\alpha}, D^j_{\beta}\} = -2\varepsilon_{\alpha\beta}\varepsilon^{ij}(\bar{Z}\Delta + \bar{W})
\]
\[
\{\bar{D}^\dot{i}_{\dot{\alpha}}, \bar{D}^\dot{j}_{\dot{\beta}}\} = -2\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{ij}(\bar{Z}\Delta + \bar{W})
\]
\[
\{D^i_{\alpha}, \bar{D}^\dot{i}_{\dot{\alpha}}\} = -2i\delta^i_j\delta_{\alpha\dot{\alpha}}
\]
(3.5)

with the Yang-Mills superfield strength \( W = W^a(z)T^a \) obeying the Bianchi identities of the form (2.6). The central charge transformations of the component fields can be determined in the same fashion as it has been done in [15] for the case \( A_M = 0 \). For the harmonic superfield \( q^+ \equiv q^i u^+_i \) the constraints (3.3) amount to
\[
D^+ q^+ = \bar{D}^+ q^+ = 0 \quad D^{++} q^+ = 0 .
\]
(3.6)

Therefore, \( q^+ \) is a covariantly analytic superfield. Passing to the \( \lambda \)-frame with respect to both the central charge and Yang-Mills gauge groups yields \( \hat{q}^+ \) which is analytic, \( \hat{q}^+ = \hat{q}^+(\zeta, u) \), and obeys the constraint
\[
(D^{++} + V^{++} \Delta + V^{++}) \hat{q}^+ = 0 .
\]
(3.7)

Here \( V^{++} \) is the analytic prepotential associated with \( A_M \) [12]. As the Lagrangian density \( L^{++} \) we can choose the same expression as in the case of rigid central charge [11]
\[
L^{++} = \frac{1}{2}(\hat{q}^+ \Delta q^+ - \Delta \hat{q}^+ q^+) - i m \hat{q}^+ q^+ .
\]
(3.8)

It is straightforward to check that both structures in the right-hand side of (3.8) solve the basic constraints (2.54) and (2.55). Using (3.7), one can rewrite the action in the form
\[
S = - \int dud\zeta d(\zeta^4)\left\{\hat{q}^+(D^{++} + V^{++})\hat{q}^+ + i m V^{++}\hat{q}^+\hat{q}^+\right\} .
\]
(3.9)

This action looks very similar to the action of the \( q \)-hypermultiplet with infinitely many auxiliary components [12]. The crucial difference lies, however, in that the \( q \)-hypermultiplet is an unconstrained analytic superfield, while the above Fayet-Sohnius \( \hat{q} \)-hypermultiplet is still subject to the off-shell harmonic constraint (3.7) which reduces the infinite tower of components appearing in the harmonic expansion of \( \hat{q}^+ \) to the irreducible (8 + 8) content. The presence of non-trivial central charge in the harmonic derivative in (3.7) is crucial for keeping the theory off-shell: putting \( \Delta \) equal to zero immediately makes (3.7) an equation of motion. Note that for the \( q \)-hypermultiplet of ref. [12] one can also introduce the central charge. This can be done by the Scherk-Schwarz mechanism, identifying \( \Delta \) with the generator of some U(1) symmetry of the action (e.g., of U(1) subgroup of the YM group). One can gauge such a central charge by introducing the
appropriate $V^{++}$. However, such a central charge does not lead to any reduction of the infinite number of off-shell degrees of freedom in $q^+$; its only effect is to provide a mass for the $q^+$ hypermultiplet (and a physical bosons potential in the case of self-interacting $q^+$).

### 3.2 Linear vector-tensor multiplet

Now, let us turn to gauging the central charge of the linear free VT multiplet $[4, 3]$. It is described by a real superfield $L(z)$ subject to the constraints $[11]

$$D_\alpha^+ \bar{D}_\dot{\alpha}^+ L = D_\alpha^+ D_\dot{\alpha}^+ L = 0.$$  \quad (3.10)

Eq. (3.10) implies that only the superfields $L$ and $L^{(\Delta)}$ contain independent component fields, while the rest of the descendants is expressed in terms of $L$, $L^{(\Delta)}$. Simultaneously, (3.10) defines the action of $\Delta$ on $L(z)$ and thereby expresses the components of $L^{(\Delta)}(z)$ in terms of those of $L(z)$. In order to be able to treat the VT multiplet on an equal footing both in the $\tau$- and $\lambda$-frames, it is convenient to regard $L$ in (3.10) as a general harmonic superfield, $L = L(z, u)$, and to eliminate the dependence on the harmonics by the additional harmonic constraint

$$D^{++} L = 0.$$  \quad (3.11)

The VT multiplet requires more delicate treatment than the hypermultiplet, since in this case the naive minimal covariantization with respect to the gauged central charge yields inconsistent constraints. Namely, it turns out that such a covariantization is incompatible with an important consistency condition following from the harmonic constraint (3.11).

In order to demonstrate this, let us first note that (3.11), in the cases of both local and rigid central charges, implies $[14]

$$\mathcal{D}^{--} L = 0.$$  \quad (3.12)

The simplest way to see this is to realize that $\mathcal{D}^{++}$ and $\mathcal{D}^{--}$ are the raising and lowering operators of the right SU(2) group acting on the U(1) charges of the harmonic superfields (see (2.24)); $L$ is chargeless, so (3.11) as well as (3.12) mean that it is a singlet of this SU(2) group.
Equation (3.12) holds irrespectively of the precise form of the constraints (3.10), or their covariantization. It is important that the covariant constraints involve the gauged-central-charge-covariantized spinor derivatives $D^{\pm}_\alpha, \bar{D}^{\pm}_\dot{\alpha}$ satisfying the algebra (2.24) and commuting with $\Delta$. Successively acting on (3.12) by the derivatives $D^{\pm}_\alpha, \bar{D}^{\pm}_\dot{\alpha}$ and making use of this algebra, one gets a number of useful relations. Applying $(D^+D^+)$ and $(\bar{D}^+\bar{D}^+) \, L$ yields, respectively,

$$0 = D^+D^+D^-L = \left[ D^-D^+D^+ - 2D^-D^+ - 4\bar{Z}\Delta \right] L \tag{3.13}$$

$$0 = \bar{D}^+\bar{D}^+\bar{D}^+\bar{D}^-L = D^-\bar{D}^+\bar{D}^+\bar{D}^+\bar{D}^-L + 8i D^{\dot{\alpha}\alpha}_{\dot{\alpha}}D^{\dot{\alpha}}_\alpha D^{\dot{\alpha}}_\alpha L - 2D^-D^+\bar{D}^+\bar{D}^+\bar{D}^+L - 2D^-\bar{D}^+\bar{D}^+\bar{D}^+L - 4\Delta(LD^+D^+Z + 2D^+ZD^+L + 2\bar{D}^+\bar{D}^+L + Z\bar{D}^+\bar{D}^+L + \bar{Z}\bar{D}^+\bar{D}^+L) \tag{3.14}$$

Eq. (3.13) defines the action of the central charge $\Delta$ on $L$, and, because of the reality of $\Delta$, implies a kind of reality condition for $D^+_\alpha L$. Eq. (3.14) is the consistency condition mentioned earlier. For the case of rigid central charge, $Z = -i$, this consistency condition and eq. (3.13) are reduced to those found in [14].

Eq. (3.14) severely restricts the form of possible constraints on $L$. If, for instance, we naively covariantize the constraints of the free VT multiplet

$$D^+_\alpha \bar{D}^+_\dot{\alpha} L = 0 \quad \bar{D}^+_\dot{\alpha}D^+_\alpha L = 0$$

eq (3.14) would give

$$0 = \Delta(LD^+D^+Z + 2D^+ZD^+L + 2\bar{D}^+\bar{D}^+L)$$

which is fulfilled only if either $L$ is $\Delta$-invariant, thus putting the multiplet on-shell, or if $Z$ is a constant, which takes us back to rigid central charge.

To find the correct set of constraints, one should start from the general Ansatz

$$D^+_\alpha \bar{D}^+_\dot{\alpha} L = a_1D^+_\alpha Z\bar{D}^+_\dot{\alpha} L - a_1\bar{D}^+_\dot{\alpha}ZD^+_\alpha L + a_2D^+_\alpha Z\bar{D}^+_\dot{\alpha}\bar{Z} + a_3\bar{D}^+_\dot{\alpha}L\bar{D}^+_\alpha L$$

$$\bar{D}^+_\dot{\alpha}D^+_\alpha L = a_4D^+_\alpha ZD^+_\alpha L + a_5\bar{D}^+_\dot{\alpha}Z\bar{D}^+_\dot{\alpha}L + a_6D^+_\alpha Z + a_7D^+_\alpha ZD^+_\alpha Z$$

$$+a_8\bar{D}^+_\dot{\alpha}\bar{Z}\bar{D}^+_\dot{\alpha} + a_9\bar{D}^+_\dot{\alpha}L\bar{D}^+_\dot{\alpha}L + a_{10}\bar{D}^+_\dot{\alpha}L\bar{D}^+_\dot{\alpha}L \tag{3.15}$$

where all the coefficients are functions of $L, Z$ and $\bar{Z}$, and $a_2, a_3$ must be real. These constraints have to satisfy the obvious conditions

$$D^+_\alpha \bar{D}^+_\dot{\alpha}D^+_\alpha D^+_\alpha L = 0 \quad \bar{D}^+_\dot{\alpha}D^+_\alpha \bar{D}^+_\dot{\alpha}L = D^+_\alpha D^+_\alpha \bar{D}^+_\dot{\alpha}L \tag{3.16}$$
which produce a set of homogeneous differential equations for the coefficient functions. However, as was shown above, not all solutions turn out to be consistent. We have to check the condition (3.14) for each solution to single out the proper constraints.

Requiring the deformed constraints to reduce to the free ones (3.10) for $Z = \text{const}$, a particular solution reads \[15\]

$$D^+_\alpha \bar{D}^+_\alpha L = 0$$

$$D^+ D^+ L = \frac{2}{Z - \bar{Z}} \left( D^+ Z D^+ L + \bar{D}^+ \bar{Z} \bar{D}^+ L + \frac{1}{2} L D^+ D^+ Z \right).$$ \hspace{1cm} (3.17)

These constraints are linear in $L$ and its derivatives, and therefore the theory does not include any self-coupling. A superfield Lagrangian density which meets the requirements described in Section 2, is given by:

$$L^{++} = -\frac{i}{4} \left( D^+ L D^+ L - \bar{D}^+ \bar{L} \bar{D}^+ L + L D^+ D^+ L \right).$$ \hspace{1cm} (3.18)

This model has been investigated in detail in \[15\].

A more direct derivation of the constraints (3.17) (and their generalization) will be presented in Section 5. There the consistency condition (3.14) will be automatically solved in a six-dimensional framework.

### 3.3 Linear vector-tensor multiplet with Chern-Simons couplings

The linear VT multiplet coupled to an external $N = 2$ super Yang-Mills multiplet is described by a real superfield $L(z)$ constrained by \[9, 10, 11\]

$$D^+_\alpha \bar{D}^+_\alpha L = 0$$

$$D^+ D^+ L = \frac{g}{2} \text{tr} \left( (D^+)^2 \bar{W}^2 - (D^+)^2 W^2 \right)$$ \hspace{1cm} (3.19)

with $g$ a real coupling constant. The $N = 2$ super Yang-Mills field strength $W$ is defined as in eq. (3.5). The superfield redefinition

$$L \equiv \mathbb{L} - \frac{1}{2} g \text{tr} \left( W - \bar{W} \right)^2$$ \hspace{1cm} (3.20)

brings the above constraints to the form

$$D^+_\alpha \bar{D}^+_\alpha \mathbb{L} = -g \text{tr} \left( \mathbb{D}^\alpha \mathbb{W} \bar{D}^\alpha \mathbb{W} \right)$$

$$D^+ D^+ \mathbb{L} = g \text{tr} \left( \bar{D}^+ \bar{W} D^+ \bar{W} \right)$$ \hspace{1cm} (3.21)
given in \[11\].

A consistent deformation of the constraints \((3.19)\), which corresponds to the gauged central charge, reads

\[
D_\alpha \bar{D}_{\dot{\alpha}} L = 0 \\
D^+ D^+ L = \frac{2}{Z - \bar{Z}} \left( D^+ Z D^+ L + D^+ \bar{Z} \bar{D}^+ L + \frac{1}{2} L D^+ D^+ \bar{Z} \right) \\
\quad + \frac{1}{Z - \bar{Z}} \text{tr} \left( (\bar{D}^+)^2 \bar{W}^2 - (D^+)^2 W^2 \right) .
\] (3.22)

As the corresponding Lagrangian density we can again choose \(\mathcal{L}^{++}\) given by eq. \((3.18)\).

To solve the differential constraints on the field strengths of the vector and antisymmetric tensor contained in the VT multiplet, we first specify the component fields of the external super Yang-Mills strength

\[
\begin{align*}
\mathcal{W}| &= W \\
\mathcal{D}_\alpha^i \mathcal{W}| &= \lambda^i_\alpha \\
-\frac{1}{8} \mathcal{D}_\alpha^i \mathcal{D}_\beta^i \mathcal{W}| &= F_{\alpha \beta} \\
\mathcal{F}_{mn} &= \partial_m A_n - \partial_n A_m + [A_m, A_n] ,
\end{align*}
\] (3.23)

as well as those of \(L\)

\[
\begin{align*}
L &= L| \\
\lambda^i_\alpha &= \mathcal{D}_\alpha^i L| \\
\bar{\lambda}_{\dot{\alpha}i} &= \bar{D}_{\dot{\alpha}i} L| \\
U &= \Delta L| \\
G_{\alpha \beta} &= \frac{1}{4} [\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}] L| \\
\bar{G}_{\dot{\alpha} \dot{\beta}} &= \overline{G_{\alpha \beta}} \\
V_{\alpha \dot{\alpha}} &= -\frac{1}{4} [\mathcal{D}_\alpha^i, \bar{D}_{\dot{\alpha}i}] L| .
\end{align*}
\] (3.24)

The central charge transformations now read (hereafter, we keep only the purely bosonic contributions)

\[
\begin{align*}
\Delta (IV^m - L \partial_m R) &= \nabla^n G_{mn} \\
\Delta (I \tilde{G}_{mn} + RG_{mn} + LF_{mn}) &= -\varepsilon_{mnkl} \nabla^k V^l
\end{align*}
\] (3.25)

while the differential constraints on \(V_m\) and \(G_{mn}\) look as follows:

\[
\nabla^m (IV^m - L \partial_m R) = \frac{1}{2} F^{mn} G_{mn} - g \partial^m \text{tr} \left[ i(W - \bar{W}) D_m (W + \bar{W}) \\
\quad + 2 \varepsilon_{mnkl} (A^n \partial^k A^l + \frac{2}{3} A^n A^k A^l) \right] \\
\nabla^m (I \tilde{G}_{mn} + RG_{mn} + LF_{mn}) &= -\tilde{F}_m V^m + 2g \partial^m \text{tr} [i(W - \bar{W}) F_{mn} \\
\quad + (W + \bar{W}) \tilde{F}_{mn}] .
\] (3.26)
Here we denoted \( I = \text{Im} Z \) and \( R = \text{Re} Z \). The general solution of the constraints in terms of a 1-form \( T_m \) and a 2-form \( K_{mn} \) reads

\[
IV_m = \frac{1}{2} \varepsilon_{mnkl} (\partial^n K^{kl} - A^n \tilde{G}^{kl}) + L \partial_m R \\
- g \text{ tr } \left[ i(W - \bar{W})D_m(W + \bar{W}) + 2\varepsilon_{mnkl} (A^n \partial^k A^l + \frac{2}{3} A^n A^k A^l) \right]
\]

\[
I \tilde{G}_{mn} + RG_{mn} = \varepsilon_{mnkl} (\partial^k T^l - A^k V^l) - LF_{mn} \\
+ 2g \text{ tr } [i(W - \bar{W})F_{mn} + (W + \bar{W}) \tilde{F}_{mn}] .
\] (3.27)

In principle, all the consistency conditions are satisfied for more general constraints than those in (3.22):

\[
D^+ \bar{D}^+ L = 0 \quad \tilde{L} = L - ig (Z - \bar{Z})
\]

\[
D^+ D^+ L = \frac{2}{Z - \bar{Z}} \left( D^+ Z D^+ L + \bar{D}^+ \bar{Z} D^+ L + \frac{1}{2} LD^+ D^+ Z \right) \\
+ \frac{1}{Z - \bar{Z}} \left( (D^+)^2 F(W) + (\bar{D}^+)^2 F(\bar{W}) \right)
\] (3.28)

where \( F(W) \) is some holomorphic function of the chiral strength \( W \). But then we are unable, in general, to solve the constrained vector and antisymmetric tensor component field of \( L \) in terms of gauge two- and one-forms, respectively.

Let us choose \( W \) in eq. (3.22) just to be \( Z \). Then we obtain the following consistent constraints

\[
D^+ \bar{D}^+ L = 0 \quad \tilde{L} = L - ig (Z - \bar{Z})
\]

\[
D^+ D^+ L = \frac{2}{Z - \bar{Z}} \left( D^+ Z D^+ L + \bar{D}^+ \bar{Z} D^+ L + \frac{1}{2} LD^+ D^+ Z \right) \\
+ \frac{1}{Z - \bar{Z}} \text{ tr } \left( (D^+)^2 Z^2 - (D^+)^2 \bar{Z}^2 \right) .
\] (3.29)

which, however, are equivalent to the old ones (3.17). Indeed, the following redefinition

\[
\tilde{L} = L - ig (Z - \bar{Z})
\] (3.30)

brings the constraints (3.24) to the form (3.17).

\footnote{Likely, it is still possible to have a consistent dual formulation of such a more general theory in terms of a Lagrange multiplier vector multiplet in the spirit of ref. [14].}
3.4 Linear vector-tensor multiplet with scale and chiral invariance

If the background \( N = 2 \) vector multiplet in (3.13) is abelian (here we denote the corresponding superfield strength by \( \mathcal{Y} \)), then we are able to couple it to the VT multiplet in a different fashion\(^3\). Let us consider the constraints (supersymmetry with rigid central charge)

\[
\mathcal{D}_\alpha \bar{\mathcal{D}}^\alpha L = 0 \\
\mathcal{D}^+ \bar{\mathcal{D}}^+ L = -\frac{2}{\mathcal{Y} + \bar{\mathcal{Y}}} \left( \mathcal{D}^+ \mathcal{Y} \mathcal{D}^+ L + \bar{\mathcal{D}}^+ \bar{\mathcal{Y}} \bar{\mathcal{D}}^+ L + \frac{1}{2} L \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{Y} \right) 
\]

(3.31)

which satisfy all the superspace consistency conditions. In particular, it is easy to check the rigid central charge form of (3.14) using the fact that \( \mathcal{D}^+ \mathcal{D}^+ L = \bar{\mathcal{D}}^+ \bar{\mathcal{D}}^+ L \), as follows from (3.31). Substituting the components (3.24) of the VT multiplet and defining the components of \( \mathcal{Y} \) as in (3.23), we then arrive at the component constraints

\[
\partial^m [(W + \bar{W}) V_m - i L \partial_m (W - \bar{W})] = -\mathcal{F}^{mn} \tilde{G}_{mn} \\
\partial^m \tilde{G}_{mn} = 0 
\]

(3.32)

which can be easily solved.

If we gauge the central charge, the covariantized version of the constraints (3.31) compatible with all consistency conditions is as follows:

\[
\mathcal{D}_\alpha \bar{\mathcal{D}}^\alpha L = 0 \\
\mathcal{D}^+ \bar{\mathcal{D}}^+ L = \frac{2\bar{\mathcal{Y}}}{\mathcal{Z} \mathcal{Y} - \bar{\mathcal{Z}} \bar{\mathcal{Y}}} \left( \mathcal{D}^+ \mathcal{Z} \mathcal{D}^+ L + \bar{\mathcal{D}}^+ \bar{\mathcal{Z}} \bar{\mathcal{D}}^+ L + \frac{1}{2} L \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{Z} \right) \\
\quad - \frac{2\bar{\mathcal{Z}}}{\mathcal{Z} \mathcal{Y} - \bar{\mathcal{Z}} \bar{\mathcal{Y}}} \left( \mathcal{D}^+ \mathcal{Y} \mathcal{D}^+ L + \bar{\mathcal{D}}^+ \bar{\mathcal{Y}} \bar{\mathcal{D}}^+ L + \frac{1}{2} L \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{Y} \right). 
\]

(3.33)

The corresponding Lagrangian density for this model is given by

\[
\mathcal{L}^{++} = -\frac{i}{4} \left( \mathcal{Y} \mathcal{D}^+ L \mathcal{D}^+ L - \bar{\mathcal{Y}} \bar{\mathcal{D}}^+ L \bar{\mathcal{D}}^+ L \right) + \frac{i}{8} \frac{\mathcal{Y} \bar{\mathcal{Y}} + \mathcal{Z} \bar{\mathcal{Y}}}{\mathcal{Z} \mathcal{Y} - \bar{\mathcal{Z}} \bar{\mathcal{Y}}} \mathcal{L}^2 \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{Y} \\
\quad - \frac{i}{2} \frac{\mathcal{Y} \bar{\mathcal{Y}} L}{\mathcal{Z} \mathcal{Y} - \bar{\mathcal{Z}} \bar{\mathcal{Y}}} \left( \mathcal{D}^+ \mathcal{Z} \mathcal{D}^+ L + \bar{\mathcal{D}}^+ \bar{\mathcal{Z}} \bar{\mathcal{D}}^+ L + \frac{1}{2} L \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{Z} \right) \\
\quad + \frac{i}{2} \frac{L}{\mathcal{Z} \mathcal{Y} - \bar{\mathcal{Z}} \bar{\mathcal{Y}}} \left( \mathcal{Z} \mathcal{Y} \mathcal{D}^+ \mathcal{Y} \mathcal{D}^+ L + \bar{\mathcal{Z}} \bar{\mathcal{Y}} \bar{\mathcal{D}}^+ \bar{\mathcal{D}}^+ L \right). 
\]

(3.34)

\(^3\)The origin of this alternative coupling is most easily understood in the six-dimensional framework of Section 5.
As is seen from (3.31) and (3.33), the constraints are well defined only if \( Y \) has a non-vanishing vacuum expectation value. Freezing the external vector multiplet by setting \( Y = 1 \), constraints (3.33) reduce to (3.17) and Lagrangian density (3.34) to that given by eq. (3.18).

A remarkable feature of the system presented here is that it respects the invariance under global scale and chiral transformations of the superspace coordinates

\[
\theta' = e^{-\omega/2} \theta \quad \bar{\theta}' = e^{-\bar{\omega}/2} \bar{\theta} \quad x' = e^{-(\omega + \bar{\omega})/2} x
\]

\[
D'_{\alpha i} = e^{\omega/2} D_{\alpha i} \quad \bar{D}'_{\dot{\alpha} i} = e^{\bar{\omega}/2} \bar{D}_{\dot{\alpha} i} \quad D'_m = e^{(\omega + \bar{\omega})/2} D_m
\]

\[
Z' (z') = e^{\bar{\omega}} Z (z) \quad \bar{Z}' (z') = e^{\omega} \bar{Z} (z)
\] (3.35)

if we require \( Y \) and \( \bar{Y} \) to transform similarly to \( Z \) and \( \bar{Z} \)

\[
Y'(z') = e^{\bar{\omega}} Y(z) \quad \bar{Y}'(z') = e^{\omega} \bar{Y}(z)
\] (3.36)

and \( L \) to have a vanishing scale and chiral weight

\[
L'(z') = L(z).
\] (3.37)

This is in complete agreement with the results of [6].

Finally, we can couple the VT multiplet under consideration to an external \( N = 2 \) non-abelian vector multiplet \( \mathcal{W} \). The corresponding constraints read

\[
D^+_{\alpha} D^+_{\bar{\alpha}} L = 0
\]

\[
D^+ D^+ L = \frac{2Y}{Z Y - \bar{Z} Y} \left( D^+ Z D^+ L + \bar{D}^+ \bar{Z} D^+ L + \frac{1}{2} L D^+ D^+ Z \right)
\]

\[
\quad -\frac{2\bar{Z}}{Z Y - \bar{Z} Y} \left( D^+ Y D^+ L + \bar{D}^+ \bar{Y} D^+ L + \frac{1}{2} L D^+ D^+ Y \right)
\]

\[
\quad +i g \frac{\bar{Y}}{Z Y - \bar{Z} Y} \text{tr} \left( D^+ D^+ \frac{\bar{W}^2}{Y} - D^+ D^+ \frac{W^2}{Y} \right).
\] (3.38)

This system is also scale and chiral invariant if we require \( \mathcal{W} \) and \( \bar{\mathcal{W}} \) to have standard vector multiplet transformation laws

\[
\mathcal{W}' (z') = e^{\bar{\omega}} \mathcal{W} (z) \quad \bar{\mathcal{W}}' (z') = e^{\omega} \bar{\mathcal{W}} (z).
\] (3.39)

If we freeze the \( Y \)-multiplet, by specifying \( Y = 1 \), the constraints (3.38) reduce to (3.22).

We note that all the constraints in this subsection admit a simple derivation starting from six dimensions (see Section [3]).
4 Models with nonlinear central charge transformations

In the present section we consider two VT multiplet models whose main feature is the nonlinearity of the central charge transformations.

4.1 Nonlinear vector-tensor multiplet

The nonlinear VT multiplet [5] can be defined in harmonic superspace by the constraints

\[
D_\alpha^+ \bar{D}_\alpha^+ L = 0
\]

\[
D^+ D^+ L = -\frac{1}{L} \left( D^+ L D^+ L + \bar{D}^+ L \bar{D}^+ L \right). \tag{4.1}
\]

The superfield reparametrization

\[
L = \exp(-\kappa \tilde{L}) \tag{4.2}
\]

with \(\kappa\) a real coupling constant, brings these constraints into the form given in [13]

\[
D_\alpha^+ \bar{D}_\alpha^+ \tilde{L} = \kappa D_\alpha^+ \bar{L} \bar{D}_\alpha^+ \tilde{L}
\]

\[
D^+ D^+ \tilde{L} = \kappa \bar{D}^+ \bar{L} \bar{D}^+ \tilde{L} + 2\kappa D^+ \bar{L} \bar{D}^+ \tilde{L}. \tag{4.3}
\]

From eqs. (4.1), (4.3) it is obvious that in this case we are dealing with a self-interacting VT multiplet.

To gauge the central charge transformations, one can again start from the general Ansatz (3.15) and look for consistent solutions possessing the limit (4.1) for \(Z = -i\). A complete analysis will be given elsewhere. Here let us just present the constraints which underlie the component construction of [5] for the nonlinear VT multiplet with gauged central charge:

\[
D_\alpha^+ \bar{D}_\alpha^+ L = 0
\]

\[
D^+ D^+ L = -\frac{1}{Z} \left( 2D^+ Z D^+ L + \frac{1}{2} L D^+ D^+ Z + \frac{Z}{L} D^+ L D^+ L - \frac{Z}{L} \bar{D}^+ L \bar{D}^+ L \right). \tag{4.4}
\]

The consistency conditions (3.16) can be easily checked. The central charge transformation of \(D^+ L\) derived on the base of (4.4)

\[
Z \Delta D^+_\alpha L = iD_{\alpha\dot{\alpha}} \bar{D}^+ \dot{\alpha} L - D^+_\alpha Z \Delta L + \frac{1}{4Z} D_\alpha \left( 2\bar{D}^+ \bar{Z} \bar{D}^+ L \\
+ \frac{1}{2} L \bar{D}^+ D^+ Z - \frac{Z}{L} \bar{D}^+ L D^+ L + \frac{Z}{L} \bar{D}^+ L \bar{D}^+ L \right) \tag{4.5}
\]
passes the test \((3.14)\) too.

To discuss the differential constraints, we now turn to the component fields of \(L\). We choose them as in the linear case, with the exception of \(G_{mn}\):

\[
G_{\alpha\beta} = -i \left( \frac{1}{4} Z [\mathcal{D}_{\alpha i}, \mathcal{D}_j] + F_{\alpha\beta} \right) L | \quad \tilde{G}_{\dot{\alpha}\dot{\beta}} = \overline{G_{\alpha\beta}}. \tag{4.6}
\]

The constraints on \(G_{mn}\) and \(V_m\) (of which we again give only the bosonic parts, the complete expressions can be found in \([5]\)) then read

\[
\begin{align*}
\nabla^n \tilde{G}_{mn} &= -\tilde{F}_{mn} V^n \\
\nabla^m H_m &= -\frac{1}{4} G_{mn} \tilde{G}_{mn} - \frac{1}{4} (2L\tilde{G}_{mn} - L^2 F_{mn}) \tilde{F}^{mn} \tag{4.7}
\end{align*}
\]

and the central charge transformations of \(G_{mn}\) and \(V_m\) are

\[
\begin{align*}
\Delta G_{mn} &= -2 \nabla [m V_n] \\
\Delta H_m &= \tilde{G}_{mn} V^n - \frac{1}{4} \varepsilon_{mnkl} \nabla^n (2L\tilde{G}^{kl} - L^2 F^{kl}) \tag{4.8}
\end{align*}
\]

Here we have used the notation

\[
\nabla_m := \partial_m + A_m \Delta \quad H_m := |Z|^2 LV_m - \frac{i}{2} L^2 (\bar{Z} \partial_m Z - Z \partial_m \bar{Z}) \tag{4.9}
\]

The constraints are solved by introducing a 1-form \(T_m\) and a 2-form \(K_{mn}\) as in \([5]\)

\[
\begin{align*}
G_{mn} &= 2(\partial_m T_n - A_m V_n) \\
H_m &= \frac{1}{2} \varepsilon_{mnkl} [\partial^n K^{kl} - T^n \partial^k T^l - \frac{1}{2} A^n (2L\tilde{G}^{kl} - L^2 F^{kl})] \tag{4.10}
\end{align*}
\]

These equations are still coupled, however. They can be separated by analogy with \([24, 13]\), and we obtain

\[
\begin{align*}
V_m &= \frac{1}{E} \left[ T_m A^n + \frac{1}{2L} \varepsilon_{nktr} A^n (\partial^k K^{tr} - T^n \partial^k T^l + L^2 A^n \partial^k A^l) \\
&\quad - \frac{1}{2} |Z|^2 L A_m \varepsilon_{nktr} A^n (\partial^k K^{tr} - T^n \partial^k T^l) \\
&\quad + \frac{i}{2} L (\delta_m^n - |Z|^{-2} A_m A^n) (Z \partial_n Z - Z \partial_n \bar{Z}) \right] \\
G_{mn} &= T_{mn} - \frac{2}{E} A_m \left[ T_n A^k + \frac{i}{2} L (\bar{Z} \partial_n Z - Z \partial_n \bar{Z}) \\
&\quad + \frac{1}{2L} \varepsilon_{nktr} (\partial^k K^{tr} - T^n \partial^k T^l + L^2 A^k \partial^l A^r) \right] \tag{4.11}
\end{align*}
\]

where \(E = |Z|^2 - A^m A_m\), and \(T_{mn} = \partial_m T_n - \partial_n T_m\) is the field strength 2-form of \(T_m\).
The central charge transformations of the potentials finally read (up to gauge transformations)

\[ \Delta T_m = - V_m \]
\[ \Delta K_{mn} = T_m V_n - \frac{1}{2}(2L\tilde{G}_{mn} - L^2 F_{mn}) . \]  (4.12)

For the constraints (4.4) it is easy to find a supersymmetric Lagrangian density

\[ L^{++} = \frac{1}{4}L(ZD^+LD^+L + \bar{Z}D^+\bar{L}D^+L - \frac{1}{6}L^2D^+D^+Z) . \]  (4.13)

The action rule (2.58) then yields precisely the Lagrangian given in [5]:

\[ L^{++} = \left[ \frac{1}{2}|Z|^2(D^mLD_mL - V_mV_m - |Z|^2U^2) - \frac{1}{12}L^2Y_{ij}Y^{ij} \right. \]
\[ - \frac{1}{6}L^2(Z\square \bar{Z} + \bar{Z}\square Z) + \frac{1}{4}(LF_{mn} - \tilde{G}_{mn})^2 - |Z|^2U^A m D_m L \]
\[ - A_m D_n L(L\tilde{F}^{mn} - G^{mn}) - A_m V_n (L\tilde{F}^{mn} + G^{mn}) \]
\[ - \frac{1}{3}L^2 A_m \partial_n F^{mn} \]  \[ \equiv \partial_m M^m(Z, \bar{Z}, \cdots) \]
\[ L^{++} = \left. \frac{1}{4}(D^+D^+L + \bar{D}^+\bar{D}^+L) \right) . \]  (4.14)

The dynamical system given by constraints (4.4) and Lagrangian density (4.13) is invariant under global scale and chiral transformations (3.35) and (3.37).

There exist two more candidates for the rôle of Lagrangian density:

\[ L_1^{++} = \frac{1}{4}(D^+D^+L + \bar{D}^+\bar{D}^+L) \]
\[ L_2^{++} = \frac{i}{4}(D^+D^+L - \bar{D}^+\bar{D}^+L) . \]  (4.15)

However, the corresponding component Lagrangians are just total derivatives:

\[ L_1 = \partial_m \left[ \frac{\bar{Z} - Z}{|Z|^2} \frac{i}{2L} \varepsilon^{mnkl} (\partial_n K_{kl} - T_n \partial_k T_l - L^2 A_n \partial_k A_l) \right. \]
\[ - \frac{Z + \bar{Z}}{|Z|^2} \left( \frac{1}{2}L\partial^m |Z|^2 - LF^{mn} A_n + \tilde{T}^{mn} A_n \right) \]
\[ \equiv \partial_m M^m(Z, \bar{Z}, \cdots) \]
\[ L_2 = \partial_m M^m(-iZ, i\bar{Z}, \cdots) . \]  (4.16)
4.2 Nonlinear vector-tensor multiplet with Chern-Simons couplings

The nonlinear VT multiplet coupled to an external $N = 2$ super Yang-Mills multiplet is described by the constraints

$$D^+_\alpha \bar{D}^+_\alpha L = 0$$
$$D^+D^+L = -\frac{1}{L} \left( D^+L \bar{D}^+ L + \bar{D}^+ L \bar{D}^+ L \right)$$
$$-\frac{g}{2L} \text{tr} \left( (\bar{D}^+)^2 \bar{W}^2 + (D^+)^2 W^2 \right) \quad (4.17)$$

with $g$ a real coupling constant. If one implements the superfield redefinition (4.2), the constraints turn into

$$D^+_\alpha \bar{D}^+_\alpha \tilde{L} = \kappa D^+_\alpha \bar{L} \bar{D}^+_\alpha \tilde{L}$$
$$D^+D^+\tilde{L} = \kappa \bar{D}^+ \bar{L} \bar{D}^+ \tilde{L} + 2\kappa D^+ \bar{L} \bar{D}^+ \tilde{L}$$
$$+ \frac{g}{2\kappa} e^{2\kappa \tilde{L}} \text{tr} \left( (\bar{D}^+)^2 \bar{W}^2 + (D^+)^2 W^2 \right). \quad (4.18)$$

Rescaling here $g = \kappa g$ and considering the limit of vanishing self-coupling, $\kappa \to 0$, the latter constraints reduce to

$$D^+_\alpha \bar{D}^+_\alpha \tilde{L} = 0$$
$$D^+D^+\tilde{L} = \frac{g}{2} \text{tr} \left( (\bar{D}^+)^2 \bar{W}^2 + (D^+)^2 W^2 \right). \quad (4.19)$$

These constraints take the form given by eq. (3.21) if we redefine

$$\tilde{L} \equiv \mathbb{L} + \frac{1}{2} g \text{ tr } (\mathbb{W} + \bar{\mathbb{W}})^2. \quad (4.20)$$

A consistent deformation of the constraints (4.17), which corresponds to the gauged central charge, reads

$$D^+_\alpha \bar{D}^+_\alpha L = 0$$
$$D^+D^+L = -\frac{1}{Z} \left( 2D^+ Z D^+ L + \frac{1}{2} \bar{L} D^+ D^+ L + \frac{Z}{L} \bar{D}^+ L D^+ L - \frac{\bar{Z}}{L} D^+ L \bar{D}^+ L \right)$$
$$-\frac{1}{Z} \frac{g}{2L} \text{tr} \left( (\bar{D}^+)^2 \frac{W^2}{Z} - (D^+)^2 \frac{W^2}{Z} \right). \quad (4.21)$$

All consistency conditions are identically satisfied, as can be easily checked.
The constraint on $G_{mn}$ and its central charge transformation do not change, while the equations for $H_m$ now read

$$
\Delta H_m = \tilde{G}_{mn}V^n - \frac{1}{4}\varepsilon_{mnkl}\nabla^n(2L\tilde{G}^{kl} - L^2F^{kl}) \nonumber$$

$$-g\partial^n \text{tr} \left[ \frac{i}{2} \left( \frac{W^2}{Z^2} + \frac{\bar{W}^2}{\bar{Z}^2} \right) F_{mn} + \left( \frac{W^2}{Z^2} + \frac{\bar{W}^2}{\bar{Z}^2} \right) \tilde{F}_{mn} - 2i \left( \frac{W}{Z} - \frac{\bar{W}}{\bar{Z}} \right) F_{mn} \right] \nonumber$$

$$+2 \left( \frac{W}{Z} + \frac{\bar{W}}{\bar{Z}} \right) \tilde{F}_{mn} \nonumber$$

$$\nabla^m H_m = -\frac{1}{4}G^{mn}G_{mn} - \frac{1}{4}(2L\tilde{G}_{mn} - L^2F_{mn})\tilde{F}^{mn} \nonumber$$

$$-\frac{1}{2}gF_{mn}^{\text{tr}} \left[ \frac{i}{2} \left( \frac{W^2}{Z^2} - \frac{\bar{W}^2}{\bar{Z}^2} \right) F_{mn} + \left( \frac{W^2}{Z^2} + \frac{\bar{W}^2}{\bar{Z}^2} \right) \tilde{F}_{mn} - 2i \left( \frac{W}{Z} - \frac{\bar{W}}{\bar{Z}} \right) F_{mn} \right] \nonumber$$

$$-2 \left( \frac{W}{Z} + \frac{\bar{W}}{\bar{Z}} \right) \tilde{F}_{mn} - 2i \eta^m \text{tr} \left[ WD_m\bar{W} - \bar{W}D_mW - \frac{W^2}{Z}\partial_m\bar{Z} + \frac{\bar{W}^2}{\bar{Z}} \partial_mZ \right] \nonumber$$

$$+\frac{1}{2}D_m \left[ \frac{Z}{Z} \left( w^2 - \frac{\bar{Z}}{\bar{Z}} \bar{w}^2 \right) \right] - g \text{tr} (F^{mn} \tilde{F}_{mn}) , \quad (4.22)$$

the solution of which is

$$H_m = \frac{1}{2}\varepsilon_{mnkl} \left[ \partial^n K^{kl} - T^m \partial^k T^l - \frac{1}{2}A^n(2L\tilde{G}^{kl} - L^2F^{kl}) - 4g \text{tr} (A^n \partial^k A^l + \frac{2}{3}A^n A^k A^l) \right] \nonumber$$

$$-g \text{tr} \left[ \frac{i}{2} \left( \frac{W^2}{Z^2} - \frac{\bar{W}^2}{\bar{Z}^2} \right) F_{mn}A^n + \left( \frac{W^2}{Z^2} + \frac{\bar{W}^2}{\bar{Z}^2} \right) \tilde{F}_{mn}A^n - 2i \left( \frac{W}{Z} - \frac{\bar{W}}{\bar{Z}} \right) F_{mn}A^n \right] \nonumber$$

$$-2 \left( \frac{W}{Z} + \frac{\bar{W}}{\bar{Z}} \right) \tilde{F}_{mn}A^n + iW D_m W - \bar{W} D_m W - \frac{W^2}{Z} \partial_m \bar{Z} + i\frac{\bar{W}^2}{\bar{Z}} \partial_m Z \nonumber$$

$$+\frac{1}{2}D_m \left[ \frac{Z}{Z} \left( w^2 - \frac{\bar{Z}}{\bar{Z}} \bar{w}^2 \right) \right] . \quad (4.23)$$

The corresponding Lagrangian density is

$$\mathcal{L}^{++} = \frac{1}{4} \left( ZD^+L D^+L + \bar{Z}D^+L D^+L - \frac{1}{6}L^2D^+D^+Z \right) \quad (4.24)$$

$$+\frac{1}{8} g \text{tr} \left( L(D^+)^2 \frac{W^2}{Z} + L(D^+) \frac{\bar{W}^2}{\bar{Z}} - 2(D^+)^2 \left( L \frac{W^2}{Z} \right) - 2(D^+)^2 \left( L \frac{\bar{W}^2}{\bar{Z}} \right) \right) .$$

One can readily check that $\mathcal{L}^{++}$ satisfies the basic requirements (2.54) and (2.55). The dynamical system under consideration is invariant under global scale and chiral transformations (3.35), (3.37) and (3.39).

In principle, one could consider more general constraints than those given in eq. (1.21)

$$D^+_a \bar{D}^+_{\bar{a}} L = 0 \nonumber$$

$$D^+ D^+ L = -\frac{1}{Z} \left( 2D^+ Z D^+ L + \frac{1}{2} L D^+ D^+ Z + \frac{Z}{L} D^+ L D^+ L - \frac{\bar{Z}}{L} D^+ L D^+ L \right) \nonumber$$

$$+\frac{1}{2} \frac{1}{2L} \text{tr} \left( (D^+)^2 \mathcal{F}(W, Z) - (\bar{D}^+)^2 \bar{\mathcal{F}}(\bar{W}, \bar{Z}) \right) . \quad (4.25)$$
with $\mathcal{F}(\mathcal{W}, \mathcal{Z})$ a gauge invariant holomorphic function. It is obvious that all consistency conditions are identically satisfied. Like in the case of eq. (3.28), it is not possible, in general, to solve the constrained vector and antisymmetric tensor component fields of $L$ in terms of gauge two- and one-forms, respectively (see, however, footnote 2).

If we identify $\mathcal{W}$ in the constraints (4.21) with $\mathcal{Z}$, then we obtain exactly in the nonlinear VT multiplet constraints (4.4), as a consequence of the Bianchi identity (2.6).

5 The linear vector-tensor multiplet from six-dimensional harmonic superspace

5.1 Supersymmetry in six dimensions

5.1.1 Six-dimensional superspace

In this section we follow a number of standard references (see, e.g., [25, 26, 27]). There are two types of spinor indices $\hat{\alpha} = 1, 2, 3, 4$ of the group USp(4): ‘left-handed’ $\psi_{\hat{\alpha}}$ and ‘right-handed’ $\bar{\psi}_{\hat{\alpha}}$. Correspondingly, there are two types of gamma matrices, $\Gamma_{\hat{\alpha}\hat{\beta}}$ and $\Gamma_{\hat{m}\hat{\alpha}\hat{\beta}}$ ($\hat{m} = 0, 1, 2, 3, 5, 6$), which are related by raising and lowering the pair of antisymmetric spinor indices $\hat{\alpha}\hat{\beta}$:

$$\Gamma_{\hat{m}\hat{\alpha}\hat{\beta}} = \frac{1}{2} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \Gamma_{\hat{m}\hat{\gamma}\hat{\delta}} , \quad \Gamma_{\hat{m}\hat{\alpha}} = \frac{1}{2} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \Gamma_{\hat{m}\hat{\gamma}\hat{\delta}} .$$

(5.1)

The epsilon symbol is defined by $\varepsilon^{1234} = \varepsilon_{1234} = 1$ and it satisfies the identity

$$\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\rho}\hat{\sigma}} = 4 \delta_{\hat{\gamma}\hat{\delta}}^{\hat{\rho}\hat{\sigma}} .$$

(5.2)

The gamma matrices satisfy the usual algebra

$$\Gamma_{\hat{a}\hat{b}} \Gamma_{\hat{n}\hat{\beta}\hat{\gamma}} + \Gamma_{\hat{a}\hat{b}} \Gamma_{\hat{m}\hat{\alpha}\hat{\beta}} = 2 \eta_{\hat{m}\hat{n}} \delta_{\hat{\alpha}\hat{\beta}}$$

(5.3)

and have the property

$$\Gamma_{\hat{m}\hat{\alpha}} \Gamma_{\hat{m}\hat{\beta}} = -4 \delta_{\hat{\alpha}\hat{\beta}} .$$

(5.4)

Two useful corollaries of (5.3) and (5.4) are

$$\Gamma_{\hat{m}\hat{\alpha}\hat{\beta}} \Gamma_{\hat{m}\hat{\alpha}} = -4 \delta_{\hat{\alpha}\hat{\beta}} , \quad \Gamma_{\hat{m}\hat{\alpha}\hat{\beta}} \Gamma_{\hat{m}\hat{\beta}} = -2 \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} .$$

(5.5)

The Grassmann superspace coordinate $\theta^i_{\hat{\alpha}}$ has a left-handed spinor index of USp(4) and an SU(2) index $i$. The corresponding spinor derivative $\partial^i_{\hat{\alpha}} = \partial/\partial \theta^i_{\hat{\alpha}}$ has a right-handed
spinor index of USp(4). Besides the Weyl condition (which is taken care of by the above notation), these spinors satisfy a pseudo-Majorana reality condition (see [24]).

The algebra of the spinor covariant derivatives is

\[ \{D^i_\alpha, D^j_\beta\} = 2i \varepsilon^{ij} \Gamma^{\hat{m}}_{\alpha\beta} \partial_{\hat{m}} \equiv 2i \varepsilon^{ij} \partial_{\hat{\alpha}\hat{\beta}}. \] (5.6)

When the harmonic variables \( u^\pm \) are introduced, \( \theta^\pm_\alpha \) and \( D^i_\alpha \) split up into two U(1)-charged projections:

\[ \theta^\pm_\alpha = \theta^\mp_\alpha u^\pm_i, \quad D^\pm_\alpha = u^\pm_i D^i_\alpha, \] (5.7)

so that

\[ D^+_\alpha \theta^-_\beta = \delta^-_\beta. \] (5.8)

Thus, the algebra (5.6) becomes

\[ \{D^+_\alpha, D^+_\beta\} = \{D^-_\alpha, D^-_\beta\} = 0, \] (5.9)

\[ \{D^+_\alpha, D^-_\beta\} = -2i \partial_{\hat{\alpha}\hat{\beta}}. \] (5.10)

### 5.1.2 Analytic basis and sechsbein

The integrability condition (5.9) allows one to choose an analytic basis in which the spinor derivative \( D^+_\alpha \) becomes ‘short’,

\[ D^+_\alpha = \partial^+_\alpha. \] (5.11)

In this basis the harmonic derivatives acquire vielbeins:

\[ D^{\pm\pm} = \partial^{\pm\pm} + H^{\pm\pm\hat{m}} \partial_{\hat{m}} \] (5.12)

which satisfy the harmonic constraint

\[ D^{++} H^{--\hat{m}} = D^{--} H^{++\hat{m}} \] (5.13)

as a consequence of the commutation relation \([D^{++}, D^{--}] = D^0\). As long as the superspace remains flat, we have just

\[ H^{\pm\pm\hat{m}}|_0 = -i \theta^\pm \Gamma^{\hat{m}} \theta^\pm, \] (5.14)

but these vielbeins will become non-trivial later on, when we perform the reduction to four dimensions and gauge the two translations along the fifth and sixth directions. The minus projection of the spinor derivative is obtained by

\[ D^-_\alpha = [D^{--}, D^{\pm}_\alpha] = -\partial^-_\alpha - D^+_\alpha H^{--\hat{m}} \partial_{\hat{m}} \] (5.15)
and then (5.11) gives
\[ \{ D^+_{\dot{\alpha}}, D^-_\beta \} = -D^+_{\dot{\alpha}} D^-_\beta H^{-\hat{m}} \partial_{\hat{m}} \equiv -2i\epsilon_{\dot{\alpha}\dot{\beta}} \partial_{\hat{m}} \equiv -2iD^{\dot{\alpha}\dot{\beta}}. \] (5.16)

The sechsbein
\[ e^{\hat{m}}_{\alpha\beta} = -e^\dot{m}_{\dot{\alpha}\dot{\beta}} = -\frac{i}{2} D^+_{\dot{\alpha}} D^-_\beta H^{-\hat{m}} \] (5.17)
just coincides with the gamma matrices in the flat case,
\[ e^{\hat{m}}_{\alpha\beta} \big|_0 = \Gamma^{\hat{m}}_{\alpha\beta}, \] (5.18)
but it will play a key rôle in the context of gauged central charges. Note that the properties (5.4), (5.5) of the gamma matrices can serve as a formal definition of the inverse sechsbein:
\[ e^{\hat{m}}_{\alpha\beta} e^{\hat{m}}_{\gamma\delta} = -4\delta^\gamma_{\alpha}\delta^\delta_{\beta}, \quad e^{\hat{m}}_{\alpha\beta} e^{\hat{m}}_{\dot{\gamma}\dot{\delta}} = -4\delta^\dot{\gamma}_{\dot{\alpha}}\delta^\dot{\delta}_{\dot{\beta}}. \] (5.19)

5.2 Dimensional reduction to four dimensions and the origin of central charge

The dimensional reduction from six to four dimensions is obtained by replacing the four-component spinor index \( \dot{\alpha} \) of USp(4) by a pair of undotted and dotted spinor indices of SL(2, \( \mathbb{C} \)), \( \alpha = (\alpha, \dot{\alpha}) \). The gamma matrices \( \Gamma^m \) then split into:
\[ \Gamma^m_{\dot{\alpha}\dot{\beta}} = \sigma^m_{\alpha\beta}, \quad \Gamma^m_{\alpha\beta} = \Gamma^m_{\dot{\alpha}\dot{\beta}} = 0 \ (m = 0, 1, 2, 3) ; \]
\[ \Gamma^5_{\alpha\beta} = 0, \quad \Gamma^5_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\alpha\beta}, \quad \Gamma^5_{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}; \]
\[ \Gamma^6_{\alpha\beta} = 0, \quad \Gamma^6_{\dot{\alpha}\dot{\beta}} = i\varepsilon_{\alpha\beta}, \quad \Gamma^6_{\dot{\alpha}\dot{\beta}} = -i\varepsilon_{\dot{\alpha}\dot{\beta}}. \] (5.20)

The four-index epsilon symbol splits into a product of two two-index ones:
\[ \varepsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \rightarrow \varepsilon_{\alpha\beta}\varepsilon_{\dot{\gamma}\dot{\delta}} \] (5.21)
(the other components varnish or are obtained by permutations).

Accordingly, the spinor derivatives algebra (5.6) becomes
\[ \{ D_{\alpha i}, \bar{D}_{\dot{\alpha} j} \} = -2i\varepsilon_{ij}\partial_{\alpha\dot{\alpha}}, \] (5.22)
\[ \{ D_{\alpha i}, D_{\beta j} \} = 2\varepsilon_{ij}\varepsilon_{\alpha\beta}(i\partial_5 + \partial_6), \] (5.23)
\[ \{ \bar{D}_{\dot{\alpha} i}, \bar{D}_{\dot{\beta} j} \} = -2\varepsilon_{ij}\varepsilon_{\dot{\alpha}\dot{\beta}}(-i\partial_5 + \partial_6). \] (5.24)

The essential point is the appearance of the derivatives \( \partial_5, \partial_6 \) in the anticommutators (5.23) and (5.24). Comparing this with eqs. (2.5), one concludes that \( \partial_5, \partial_6 \) are the generators of
two real central charges (or one complex one). Thus, the dimensional reduction from six to four dimensions gives a natural explanation of the origin of the central charges in the $N = 2$ $D = 4$ supersymmetry algebra.

Let us now look at the harmonic vielbeins $H^{\pm \pm m}$. In the flat case they are simply

$$H^{\pm \pm m}|_0 = -2i\theta^\pm \sigma^m \bar{\theta}^\pm \ , \quad H^{\pm \pm 5}|_0 = i(\theta^\pm \theta^\pm - \bar{\theta}^\pm \bar{\theta}^\pm) \ , \quad H^{\pm \pm 6}|_0 = \theta^\pm \theta^\pm + \bar{\theta}^\pm \bar{\theta}^\pm \ . \quad (5.25)$$

However, if any of the central charges (or both) is gauged, the corresponding vielbein becomes a non-trivial harmonic superfield, $H^{\pm \pm 5,6}(x^\hat{m}, \theta, \bar{\theta}, u)$ and the harmonic derivatives $D^{\pm \pm}$ become covariant,

$$D^{\pm \pm} = \partial^{\pm \pm} + H^{\pm \pm 5,6} \partial_{5,6} \ . \quad (5.26)$$

It is crucial that the newly introduced gauge superfields $H^{\pm \pm 5,6}$ are not allowed to depend on the central charge coordinates $x^{5,6}$:

$$\partial_{5,6} H^{\pm \pm 5,6} = 0 \ . \quad (5.27)$$

In this way they give rise to two ordinary (i.e., central-charge independent) abelian gauge supermultiplets. In addition, the vielbeins $H^{++5,6}$ have to be analytic superfields:

$$[D_\alpha^+, D^{++}] = 0 \quad \Rightarrow \quad D_\alpha^+ H^{++5,6} = 0 \ . \quad (5.28)$$

This choice allows the spinor derivative $D_\alpha^+$ to remain flat,

$$D_\alpha^+ = D^{++}_\alpha \ . \quad (5.29)$$

These analytic superfields are the prepotentials of the two abelian gauge multiplets. In fact, one of them, $H^{++5}$, coincides with the central charge connection $V^{++}$ introduced in Section 3. The (non-analytic) vielbeins $H^{-5,6}$ are obtained, as usual, by solving the relation (5.13) which is the same as (2.20).

The prepotentials are not gauge invariant objects. To find out which are the invariant ones, let us inspect the various components of the sechsbein. The vielbein $H^{--m}$ has remained flat, and so have the sechsbein components $e_{\alpha\beta}^m$:

$$e_{\alpha\beta}^m = \sigma_{\alpha\beta}^m \ . \quad (5.30)$$

However,

$$e_{\alpha\beta}^5 = -\frac{i}{2}D_\alpha^+ D_\beta^+ H^{--5} = -\frac{i}{4} \varepsilon_{\alpha\beta} D^+ D^+ H^{--5} \equiv i \varepsilon_{\alpha\beta} Z \ . \quad (5.31)$$
It is easy to see that the quantity
\[ \bar{Z} = -\frac{1}{4} D^+ D^+ H^{-5} \]  
(5.32)
is gauge invariant (it is the same as in eq. (2.27), after the identification \( H^{-5} = V^- \)).
Indeed, in the context of dimensional reduction the abelian gauge transformations are
generated by local shifts of the central-charge coordinates:
\[ \delta x^{5,6} = \lambda^{5,6}(x^m, \theta, \bar{\theta}, u) . \]  
(5.33)
The abelian character of these transformations is expressed in the fact that the parameters \( \lambda^{5,6} \) do not depend on \( x^{5,6} \):
\[ \partial_{5,6} \lambda^{5,6} = 0 . \]  
(5.34)
In addition, they must be analytic in order to preserve the ‘short’ form of the spinor
derivatives \( D^+_{\alpha,\dot{\alpha}} \):
\[ D^+_{\alpha,\dot{\alpha}} \lambda^{5,6} = 0 . \]  
(5.35)
Now, the vielbeins \( H^\pm_{5,6} \) have the typical transformation law:
\[ \delta H^\pm_{5,6} = D^\pm \lambda^{5,6} = D^\pm \lambda^{5,6} \]  
(5.36)
by virtue of (5.34). Then from (5.32), (5.35) and from the spinor derivatives algebra one
obtains
\[ \delta \bar{Z} = -\frac{1}{4} D^+ D^+ D^- \lambda^5 = \frac{1}{4} D^{+\alpha} D^{-\dot{\alpha}} \lambda^5 = 0 . \]  
(5.37)
Thus, \( \bar{Z} \) is the field strength of the abelian gauge multiplet. We must stress that it does
not depend on the central charge coordinates:
\[ \partial_{5,6} \bar{Z} = 0 , \]  
(5.38)
as expected from an ordinary abelian gauge supermultiplet. Further, it satisfies the harmonic constraint
\[ D^{++} \bar{Z} = D^{++} \bar{Z} = 0 \]  
(5.39)
(the proof is similar to (5.37) and makes use of the harmonic relation (5.13)). In addition,
\( \bar{Z} \) has all the properties (2.6) and (2.28) derived in Section 2. Note that the flat limit of
\( \bar{Z} \) (i.e., with \( H^- \) taken from (5.25)) is
\[ \bar{Z}|_0 = i . \]  
(5.40)
Similarly, the other non-trivial sechsbein component $e^6_{\alpha \beta}$ gives rise to the second abelian field strength:

$$e^6_{\alpha \beta} = i\varepsilon_{\alpha \beta} \tilde{Y}, \quad \tilde{Y} = -\frac{1}{4} D^+ D^+ H^{--6}$$

(5.41)

with the same properties as $\tilde{Z}$, except for the flat limit

$$\tilde{Y}|_0 = 1.$$  

(5.42)

As we shall see below, this field strength is the one which has been already introduced in subsection 3.4.

The conclusion of the above discussion is that gauging the central charges amounts to restricted diffeomorphisms in the six-dimensional superspace (only along the 5th and 6th dimensions; no dependence on $x^{5,6}$ is allowed). The corresponding components of the sechsbein are identified with the abelian field strengths.

### 5.3 Torsion in six dimensions

The procedure of obtaining the field strengths presented above is clearly not covariant in six dimensions. There the gauge (i.e. diffeomorphism invariant) object should be looked for one level higher in the covariant derivatives algebra. To keep six-dimensional covariance manifest, we shall treat all the prepotentials $H^{--\hat{m}}$ as non-trivial, i.e. no distinction will be made here between $H^{--m}$ and $H^{--5,6}$. Correspondingly, we shall consider analytic diffeomorphisms of all the six coordinates $x^{\hat{m}}$:

$$\delta x^{\hat{m}} = \lambda^{\hat{m}}(x^{\hat{\rho}}, \theta, u), \quad D^+_{\hat{a}} \lambda^{\hat{m}} = 0.$$  

(5.43)

This amounts to having the full $6 \times 6$ sechsbein matrix (5.17), as well as its inverse defined by (5.19). It should be stressed that even within this enlarged six-dimensional context the spinor derivatives $D^+_{\hat{a}}$ remain flat:

$$D^+_{\hat{a}} = D^+_{\hat{a}}.$$  

(5.44)

This fact expresses the fundamental postulate of preservation of Grassmann analyticity.

Now, consider the commutator

$$[D^+_{\hat{a}}, D^+_{\hat{b}}] = -\frac{i}{2} D^+_{\hat{a}} D^+_{\hat{b}} D^+_\gamma H^{--\hat{m}} \partial_{\hat{m}}$$

$$= -\frac{i}{2} \varepsilon_{\hat{a} \hat{b} \gamma \delta} (D^+)^{3\delta} H^{--\hat{m}} \partial_{\hat{m}}$$

$$= \frac{i}{8} \varepsilon_{\hat{a} \hat{b} \gamma \delta} (D^+)^{3\delta} H^{--\hat{m}} e_{\hat{m} \hat{\rho} \hat{\sigma}} D_{\hat{\rho} \hat{\sigma}}$$

$$= \frac{1}{2} \varepsilon_{\hat{a} \hat{b} \gamma \delta} T_{\hat{\rho} \hat{\sigma}}^{\hat{a} \hat{b}} D_{\hat{\rho} \hat{\sigma}}.$$  

(5.45)
Here \((D^+)^{\hat{\alpha}\hat{\beta}} \equiv -\frac{1}{6} \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} D^+_\hat{\alpha} D^+_\hat{\beta} D^+_\hat{\gamma}\) and the torsion tensor \(T^+\) is given by

\[
T^{+\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = \frac{i}{4} (D^+)^{\hat{\alpha\hat{\beta}}} \tilde{H}^{--m\hat{\delta}} e^{m\hat{\delta}\hat{\gamma}} .
\] (5.46)

By construction, this quantity is invariant under the diffeomorphisms (5.43). This can also be verified directly from the expression (5.46) using the algebra of the covariant derivatives. The presence of torsion reflects the fact that the superspace is not completely flat (recall the abelian curvature related to the gauged central charge in the four-dimensional context). Note, however, that it is not our aim here to develop a full-fledged supergravity formalism in six dimensions (that would require to consider the full superdiffeomorphism group and to introduce a complete set of supervielbeins; for details see [29]).

By raising the pair of indices \(\hat{\beta}\hat{\gamma}\), eq. (5.45) can be rewritten in the equivalent form

\[
[D^+_{\hat{\alpha}}, D^{\hat{\beta\hat{\gamma}}}] = \delta^{[\hat{\beta}}_{\hat{\alpha}} T^{+\hat{\gamma}]}_{\hat{\rho\hat{\sigma}}} D^{\hat{\rho\hat{\sigma}}} .
\] (5.47)

Then, anticommuting the left-hand side of eq. (5.47) with another spinor derivative \(D^+_{\hat{\kappa}}\), one derives the Bianchi identity

\[
D^+_{(\hat{\alpha}\hat{\beta}} T^{+\hat{\gamma})}_{\hat{\rho\hat{\sigma}}} - T^{+(\hat{\beta}}_{\hat{\alpha})\hat{\gamma}} T^{+\hat{\gamma}}_{\hat{\rho\hat{\sigma}}} = 0 ,
\] (5.48)

where \((\hat{\alpha}\hat{\beta})\) denotes the traceless part.

Another Bianchi identity is obtained by noting that

\[
[D^{++}, D_{\hat{\alpha}\hat{\beta}}] = \frac{i}{2} [D^{++}, \{D^+_{\hat{\alpha}}, D^+_{\hat{\beta}}\}] = \frac{i}{2} \{D^+_{\hat{\alpha}}, D^+_{\hat{\beta}}\} = 0 .
\] (5.49)

Applying this to eq. (5.47), one finds

\[
D^{++} T^{+\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = 0 .
\] (5.50)

5.4 The linear vector-tensor multiplet

5.4.1 Six-dimensional formulation

The six-dimensional analog of the linear VT multiplet is the so-called self-dual tensor multiplet [28]. In harmonic superspace it is described by a real superfield \(L\) satisfying the constraint [29]

\[
D^+_{\hat{\alpha}} D^+_{\hat{\beta}} L = 0 .
\] (5.51)
This constraint simply means that $L$ is a linear function of the Grassmann variables $\theta^-$:

$$L = l(x, \theta^+, u) + \theta^{-\dot{\alpha}} \psi^+_\alpha(x, \theta^+, u).$$

(5.52)

Note that the coefficient functions are analytic, but one of them ($l$) is not a superfield, since it transforms into $\psi^+$ under supersymmetry. One can also present the solution (5.52) in the superfield form

$$L = (D^+)^{3\dot{\alpha}} \Psi^{-3}_{\dot{\alpha}}$$

(5.53)

where $\Psi^{-3}_{\dot{\alpha}}$ is an arbitrary prepotential.

In addition to (5.51), $L$ satisfies the harmonic constraint

$$D^{++}L = 0$$

(5.54)

which reduces its harmonic dependence to a polynomial one and thus produces a finite supermultiplet. The two constraints (5.51) and (5.54) are clearly compatible. The harmonic constraint can also be written down in the equivalent form (see the discussion after eq. (3.12))

$$D^{--}L = 0.$$  

(5.55)

Acting on (5.53) with one, two, three or four spinor derivatives $D^+_{\dot{\alpha}}$ and using (5.51), one obtains the components of the self-dual tensor multiplet in six dimensions. Those are a scalar $\omega$, a self-dual antisymmetric tensor (in spinor notation $F^{\dot{\alpha}\dot{\beta}} = F^{\dot{\beta}\dot{\alpha}}$) and a spinor $\lambda_{\dot{\alpha}i}$ satisfying the on-shell equations

$$\Box \omega = 0, \quad \partial^{(\dot{\alpha}\dot{\beta})} F_{\dot{\beta}\dot{\gamma}} = 0, \quad \partial^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta}i} = 0.$$  

(5.56)

In addition, all these fields are harmonic-independent. Taking all this into account, one reduces the expansion of the superfield $L$ to

$$L = \omega(x) + \theta^{-\dot{\alpha}} \theta^{+\dot{\beta}} [F_{\dot{\alpha}\dot{\beta}}(x) + i \partial_{\dot{\alpha}\dot{\beta}} \omega(x)] + \theta^{+\dot{\alpha}} u^{-i}_i \lambda^i_{\dot{\alpha}}(x) - \theta^{-\dot{\alpha}} u^{+i}_i \lambda^i_{\dot{\alpha}}(x).$$  

(5.57)

Now we are going to covariantize the above picture with respect to the diffeomorphism group (5.43). While the spinor derivative constraint (5.51) needs no covariantization, the harmonic ones become covariant:

$$D^{++}L = 0$$  

(5.58)

and

$$D^{--}L = 0.$$  

(5.59)
This immediately raises the issue of compatibility between (5.51) and (5.59), just as in subsection 3.2, where the same problem arose in the four-dimensional context. Indeed, one should have

$$
\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} D^\hat{\alpha}_\beta D^\hat{\gamma}_\delta D^\hat{\delta}_\gamma D^{\hat{\gamma}} L = 0 .
$$

(5.60)

In the flat case this is a direct consequence of (5.51), but in the curved case the algebra of the covariant derivatives is complicated by the presence of torsion. Let us try to push all the spinor derivatives in (5.60) through $D^{\hat{\gamma}}$ until they reach $L$. This leads to the following compatibility condition, which is the counterpart of the four-dimensional constraint (3.14):

$$
\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} [D^{\hat{\gamma}} - D^{\hat{\beta}} D^\gamma_\delta D^\hat{\gamma} D^{\hat{\delta}} L - 4D^{\hat{\gamma}} D^\gamma_\delta D^\gamma_\alpha D^\delta_\gamma L + 12i D^{\hat{\alpha}} D^{\hat{\beta}} D^\gamma_\delta L - 4D^{\hat{\alpha}} D^{\hat{\beta}} D^\gamma_\delta L] 
$$

$$
24i T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} D^{\hat{\alpha}} L - 6i (D^{\hat{\gamma}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} - T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}}) D^{\hat{\gamma}} L = 0 .
$$

(5.61)

It is now obvious that keeping (5.51) in its flat form leads to an inconsistency. This means that we have to modify (5.51) in such a way that (5.61) becomes a corollary. The necessary modification is suggested by the comparison of the term $\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} D^{\hat{\alpha}} D^{\hat{\beta}} D^\gamma_\delta D^{\hat{\gamma}} L$ with the torsion terms in (5.61):

$$
D^{\hat{\alpha}} D^{\hat{\beta}} = -T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} D^{\hat{\gamma}} L + \frac{1}{4}(D^{\hat{\gamma}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} - T^{+\hat{\gamma}}\hat{\gamma}_{\hat{\alpha}_{\hat{\beta},\hat{\gamma}}}) L .
$$

(5.62)

Inserting this into the term $\varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} D^{\hat{\alpha}} D^{\hat{\beta}} D^{\hat{\gamma}} L$ leads to the cancellation of the torsion terms already present in (5.61), but produces two new ones:

$$
- 24i D^{\hat{\alpha}} D^{\hat{\beta}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} D^{\hat{\gamma}} L + 6i D^{\hat{\alpha}} (D^{\hat{\gamma}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} - T^{+\hat{\gamma}}\hat{\gamma}_{\hat{\alpha}_{\hat{\beta},\hat{\gamma}}}) L .
$$

(5.63)

The only way to get rid of these new terms is to impose the following constraint on the torsion:

$$
D^{\hat{\alpha}} T^{+\hat{\gamma}}\hat{\alpha}_{\hat{\beta},\hat{\gamma}} = 0 .
$$

(5.64)

Hitting (5.64) with $D^{\hat{\gamma}}$, we easily see that the second term in (5.63) vanishes as a corollary.

We have not yet finished checking the consistency conditions. The modified constraint (5.62) requires in its own turn a consistency test: hitting its right-hand side with $D^{\hat{\gamma}}$ should produce a result totally antisymmetric in $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, like the left-hand side. In fact, a simple calculation shows that the result is zero as a corollary of the Bianchi identity (5.48):

$$
D^{\hat{\gamma}} D^{\hat{\alpha}} D^{\hat{\beta}} L = 0 .
$$

(5.65)

As a direct consequence, the remaining three- and four spinor derivative terms in (5.61) vanish, so the consistency of the new constraint is fully established. Note that in the
process we have derived a condition on the torsion, eq. (5.64). Written out in terms of the prepotentials $H^{-\bar{m}}$, this condition reads

$$D^{\hat{\alpha} \hat{\beta}} T^{+ \bar{\gamma}} = -i (D^+)^{3\hat{\gamma}} \partial_{\bar{m}} H^{-\bar{m}} - \frac{1}{8} (D^+)^{3\bar{\gamma}} H^{-\bar{m}} \epsilon^{\hat{\alpha} \hat{\beta}} \epsilon^3 \epsilon^6 \partial_{\bar{m}} H^{-\bar{m}} = 0.$$  

(5.66)

One way to satisfy it is to put $\partial_{\bar{m}} H^{-\bar{m}} = 0$ and we shall see below that precisely this happens upon dimensional reduction to four dimensions.

5.4.2 Four-dimensional formulation

The framework developed in the preceding subsection allows us to obtain the constraints of the linear VT multiplet in the presence of gauged central charges by straightforward dimensional reduction of the constraints for the self-dual tensor multiplet from six to four dimensions. Before doing this, we must stress the important difference between the VT superfield $L$ and all other superfields we are considering here (prepotentials $H$, sechsbein $e$, torsion $T$). While the latter ceases to depend on the central charge coordinates $x^{5,6}$ upon dimensional reduction (thus giving rise to ordinary supermultiplets), $L$ keeps its dependence on $x^5$, but not on $x^6$:

$$L = L(x^m, x^5, \theta, u).$$  

(5.67)

To see why we should only allow for one non-trivial central charge, we can look, for instance, at the six-dimensional equation of motion for the scalar $\omega(x)$ in (5.56). In four-dimensional notation it becomes

$$(\partial_5^2 + \partial_6^2) \omega = \Box \omega.$$  

(5.68)

This is not an equation of motion any more, but rather an equation allowing to relate the dependence on the extra central-charge coordinates to that on $x^m$. It is then clear that by putting, e.g., $\partial_6 \omega = 0$ we can completely fix the dependence on the remaining central-charge coordinate $x^5$, whereas keeping both $x^5$ and $x^6$ would leave a functional freedom in some combination of those coordinates. Thus, the VT multiplet carries only one central charge and is inert under the action of the second one.

All we need to do now is to compute the torsion $T^+$. The key ingredient in this is the sechsbein matrix $e_{\hat{\alpha} \hat{\beta}}$:

$$e^{\hat{m}}_{\hat{\alpha} \hat{\beta}} = \begin{pmatrix}
 e^m_{\alpha \beta} = \sigma^m_{\alpha \beta} & e^5_{\alpha \beta} & e^6_{\alpha \beta} \\
 e^m_{\bar{\alpha} \bar{\beta}} = 0 & e^5_{\bar{\alpha} \bar{\beta}} = i \varepsilon_{\bar{\alpha} \bar{\beta}} \bar{Z} & e^6_{\bar{\alpha} \bar{\beta}} = i \varepsilon_{\bar{\alpha} \bar{\beta}} \bar{Y} \\
 e^m_{\hat{\alpha} \bar{\beta}} = 0 & e^5_{\hat{\alpha} \bar{\beta}} = -i \varepsilon_{\hat{\alpha} \bar{\beta}} \bar{Z} & e^6_{\hat{\alpha} \bar{\beta}} = -i \varepsilon_{\hat{\alpha} \bar{\beta}} \bar{Y}
\end{pmatrix}.$$  

(5.69)

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Note that the elements $e^{5,6}_{\alpha\beta} = -\frac{1}{2} D^+_{\alpha} \bar{D}^+_{\beta} H^{--5,6}$ are not gauge invariant, but they will not show up in any of the expressions below. The elements of the inverse matrix $e_{\hat{m} \hat{n}}$ relevant to our calculation are

$$e^{5\hat{\alpha}\hat{\beta}} = 0, \quad e^{5\alpha\beta} = \frac{2i \bar{Y}}{\bar{Z}Y - Z\bar{Y}} \varepsilon_{\alpha\beta}, \quad e^{5\hat{\alpha}\hat{\beta}} = \frac{2i Y}{\bar{Y}Z - \bar{Z}Y} \varepsilon_{\hat{\alpha}\hat{\beta}};$$

$$e^{6\hat{\alpha}\hat{\beta}} = 0, \quad e^{6\alpha\beta} = \frac{-2i \bar{Z}}{\bar{Z}Y - Z\bar{Y}} \varepsilon_{\alpha\beta}, \quad e^{6\hat{\alpha}\hat{\beta}} = \frac{-2i Z}{\bar{Y}Z - \bar{Z}Y} \varepsilon_{\hat{\alpha}\hat{\beta}}. \quad (5.70)$$

Then the non-vanishing torsion components are

$$T^{+\alpha}_{\beta\gamma} = \frac{\bar{Z} D^{+\alpha} Y - \bar{Y} D^{+\alpha} Z}{\bar{Z} Y - Z\bar{Y}} \varepsilon_{\beta\gamma}, \quad T^{+\hat{\alpha}}_{\beta\gamma} = \frac{\bar{Y} D^{+\hat{\alpha}} \bar{Z} - \bar{Z} D^{+\hat{\alpha}} \bar{Y}}{\bar{Z} Y - Z\bar{Y}} \varepsilon_{\beta\gamma} \quad (5.71)$$

and complex conjugates.

Now we have to put all of this into the defining constraint (5.62). We should not forget that this constraint was only consistent under the condition (5.64) on the torsion. From (5.66) we see that the condition is satisfied if \( \partial_{\hat{m}} H^{--\hat{m}} = 0 \), but this is precisely what happens in our dimensional reduction scheme:

$$\partial_{\hat{m}} H^{--\hat{m}} = \partial_{\hat{m}} (-2i \bar{\theta} - \sigma^m \bar{\theta}) + \partial_{5} H^{--5} + \partial_{6} H^{--6} = 0. \quad (5.72)$$

Thus, we find the final form of the constraints of the linear VT multiplet in the presence of gauged central charges:

$$D^+_{\alpha} D^+_{\beta} L = 0, \quad (5.73)$$

$$D^+ D^+ L = \frac{2}{\bar{Z} Y - Z\bar{Y}} \left[ (\bar{Y} D^{+\alpha} Z - \bar{Z} D^{+\alpha} Y) D^+_{\alpha} L + (\bar{Y} \bar{D}^{\hat{\alpha}} \bar{Z} - \bar{Z} \bar{D}^{\hat{\alpha}} \bar{Y}) \bar{D}^{+\hat{\alpha}} L \right] + \frac{1}{2}(\bar{Y} D^+ D^+ Z - \bar{Z} D^+ D^+ Y) L. \quad (5.74)$$

They are the \( \lambda \)-frame form of those given in subsection 3.4, eq. (3.33).

### 5.5 Coupling to a super-Yang-Mills multiplet

#### 5.5.1 SYM in six dimensions

In six dimensions the SYM multiplet is described by an analytic prepotential \( V^{++} \) (the analog of \( H^{++5,6} \) in the case of the gauged (abelian) central charge) which serves as a gauge connection for the harmonic derivative:

$$D^{++} = D^{++} + i V^{++}. \quad (5.75)$$
It undergoes the gauge transformations
\[ \delta V^{++} = -D^{++}\Lambda = -D^{++}\Lambda - i[V^{++}, \Lambda] \] (5.76)
with an analytic gauge parameter \( \Lambda(x, \theta^+, u) \). Next, going through steps similar to those in the case of the prepotentials \( H^{++} \) and \( H^{--} \) above, we first define the connection \( V^{--} \) related to \( V^{++} \) by the constraint
\[ D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0 \] (5.77)
and from it we construct the field strength:
\[ W^{+\dot{\alpha}} = \frac{1}{2}(D^+)^{\dot{\alpha}\bar{\alpha}}V^{--} . \] (5.78)
Let us verify that this expression is gauge invariant if the vielbeins \( H \) are flat (5.25):
\[ \delta W^{+\dot{\alpha}} = \frac{1}{12}\varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}D^+_{\dot{\delta}}\Lambda = -\frac{1}{12}\varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}\Lambda = 0 \] (5.79)
since \([D^+_{\dot{\beta}}, D^+_{\dot{\gamma}}] = 0 \) and \( D^+_{\dot{\beta}}\Lambda = 0 \).

The field strength (5.78) satisfies the harmonic constraint
\[ D^{++}W^{+\dot{\alpha}} = 0 \] (5.80)
(the proof is similar to (5.79) and makes use of (5.77)), as well as the obvious spinor one
\[ D^{+\dot{\alpha}}W^{+\dot{\alpha}} = 0 . \] (5.81)

This very simple picture is distorted when we take into account the torsion introduced in subsection 5.3. First of all, the expression (5.78) is not gauge invariant any more:
\[ \delta(\varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}D^+_{\dot{\delta}}V^{--}) = \varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}\Lambda = -3T^{+\dot{\alpha}}_{\dot{\beta}\dot{\gamma}}D^+_{\dot{\beta}}\Lambda . \] (5.82)
This can be compensated for by adding the term
\[ \delta(T^{+\dot{\alpha}}_{\dot{\beta}\dot{\gamma}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}V^{--}) = -T^{+\dot{\alpha}}_{\dot{\beta}\dot{\gamma}}D^+_{\dot{\beta}}\Lambda . \] (5.83)
Thus, the combination
\[ W^{+\dot{\alpha}} = \frac{1}{2}(D^+)^{\dot{\alpha}\bar{\alpha}}V^{--} - \frac{1}{4}T^{+\dot{\alpha}}_{\dot{\beta}\dot{\gamma}}D^+_{\dot{\beta}}D^+_{\dot{\gamma}}V^{--} \] (5.84)
is gauge invariant and becomes the torsion-modified expression of the SYM field strength. This new field strength (5.84) satisfies the old harmonic constraint (5.80), but the spinor one (5.81) is modified by the torsion:

\[ D^{+\beta}(\hat{W}^{+\hat{\alpha}}) - T^{+(\hat{\alpha} \hat{\beta})\hat{\gamma}} \hat{W}^{+\hat{\gamma}} = 0. \] (5.85)

Although the construction outlined so far is sufficient to describe SYM in six dimensions, we point out that we may further modify the expression (5.84) (this will turn out useful when coupling the VT multiplet to SYM). The determinant of the sechsbein matrix \( e_{\hat{\alpha} \hat{\beta}} = \det(e_{\hat{\alpha} \hat{\beta}}) \) is a density, i.e. a quantity transforming homogeneously under the diffeomorphism group:

\[ \delta(\ln e) = \frac{1}{4} e_{\hat{\alpha} \hat{\beta}} e_{\hat{\gamma} \hat{\delta}} = \frac{1}{4} D^{+\alpha}_{\hat{\beta}} D^{-\hat{\alpha} \hat{\beta}} \lambda^{\hat{m}} e_{\hat{m} \hat{\gamma} \hat{\delta}} = \frac{1}{4} e_{\hat{\alpha} \hat{\beta}} \partial \lambda^{\hat{m}} e_{\hat{m} \hat{\gamma} \hat{\delta}} = \partial \lambda^{\hat{m}} e_{\hat{m} \hat{\gamma} \hat{\delta}} \] (5.86)

(note that \( \partial \lambda^{\hat{m}} e_{\hat{m} \hat{\gamma} \hat{\delta}} = 0 \) after dimensional reduction, so \( e \) is invariant in four dimensions). Then it is clear that its spinor derivative \( D^{+\alpha}_{\hat{\beta}} \ln e \) is invariant and as such must be related to the torsion tensor. Indeed,

\[ D^{+\alpha}_{\hat{\beta}} \ln e = \frac{1}{4} D^{+\alpha}_{\hat{\beta}} e_{\hat{m} \hat{\gamma} \hat{\delta}} = \frac{1}{8} D^{+\alpha}_{\hat{\beta}} D^{+\gamma}_{\hat{\delta}} H^{-\gamma \delta m} e_{\hat{m} \hat{\gamma} \hat{\delta}} = T^{+\hat{\beta}}_{\hat{\alpha} \hat{\gamma}}. \] (5.87)

All this allows us to redefine \( \hat{W}^{+\hat{\alpha}} \) by multiplying it by the density \( e \), for instance,

\[ \hat{W}^{+\hat{\alpha}} = e^{-1/2} W^{+\hat{\alpha}}. \] (5.88)

Then we find that the constraint (5.83) is further modified:

\[ D^{+\beta}(\hat{W}^{+\hat{\alpha}}) - T^{+(\hat{\alpha} \hat{\beta})\hat{\gamma}} \hat{W}^{+\hat{\gamma}} - \frac{1}{2} T^{+\hat{\gamma}}(\hat{\alpha} \hat{\beta}) \hat{W}^{+\hat{\gamma}} = 0. \] (5.89)

Note also that the harmonic constraint (5.80) acquires a new term:

\[ (D^{++} + \frac{1}{2} \partial \lambda^{++}) \hat{W}^{+\hat{\beta}} = 0, \] (5.90)

which serves as a connection for the density transformations (once again, this term vanishes upon dimensional reduction).

### 5.5.2 Chern-Simons coupling

The so-called Chern-Simons coupling of the VT multiplet to a SYM multiplet is realized in an obvious way in six-dimensional flat superspace. There its analog is the coupling of the self-dual tensor multiplet to SYM [30]:

\[ D^{\hat{\alpha}}_{\hat{\beta}} D^{\hat{\gamma}}_{\hat{\delta}} L = -\frac{1}{2} \varepsilon_{\hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\delta}} \text{tr} (W^{+\hat{\gamma}} W^{+\hat{\delta}}). \] (5.91)
Note that the superfield $L$ is a singlet with respect to the Yang-Mills group. The consistency condition for (5.91) is obtained by hitting it with $D^\gamma_\gamma$ and using the flat superspace SYM constraint (5.81):

$$D^\gamma_\gamma D^\gamma_\beta D^\gamma_\beta L = \frac{1}{4} \epsilon^{\gamma\delta\gamma\delta} \text{tr} \left( (D_{\hat{\rho}}^+ \hat{W}^{\hat{\rho}+}) \hat{W}^{\hat{\rho}+} \right).$$

(5.92)

Clearly, the right-hand side of eq. 5.92 is totally antisymmetric in $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, just like the left-hand side.

The problem now is to properly covariantize the coupling (5.91) in the presence of torsion. This is not too difficult. We just have to put together (5.62) with (5.91) and replace $W^{\hat{\alpha}}$ by $\hat{W}^{\hat{\alpha}}$ defined in (5.88):

$$D^\gamma_\gamma D^\gamma_\beta L = -T^\gamma_{\hat{\alpha}} D^\gamma_\beta L + \frac{1}{4} (D^\gamma_\gamma T^\gamma_{\hat{\alpha}} - T^\gamma_{\hat{\gamma}} T^\gamma_{\hat{\delta}}) L - \frac{1}{2} \epsilon^{\gamma\delta\gamma\delta} \text{tr} \left( \hat{W}^{\hat{\gamma}} \hat{W}^{\hat{\delta}} \right).$$

(5.93)

Repeating the consistency check (5.92), we find the following new terms:

$$T^\gamma_{\hat{\alpha}} \epsilon_{\hat{\beta}\hat{\gamma}\hat{\delta}} \text{tr} \left( \hat{W}^{\hat{\delta}} \hat{W}^{\hat{\beta}} \right) + 2 \epsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}} \text{tr} \left( (D^\gamma_\gamma \hat{W}^{\hat{\gamma}} \hat{W}^{\hat{\delta}}) \right)$$

(5.94)

which should be totally antisymmetric in $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$. It is a matter of a straightforward calculation to show that this is true as a consequence of the constraint (5.88) satisfied by the density-modified field strength $\hat{W}^{\hat{\alpha}}$. In addition, the constraint (5.93) passes the consistency check (5.61).

### 5.5.3 Four-dimensional formulation

Having achieved a consistent coupling in six dimensions, we just need to carry out the dimensional reduction. Here is a sketch. First, the determinant of the sechsbein is given in terms of the field strengths for the gauged central charges:

$$e = -4i (\bar{Z} \mathcal{Y} - Z \bar{\mathcal{Y}}).$$

(5.95)

Next, the expression (5.84) for the SYM field strength is shown to have two equivalent forms:

$$W^{\hat{\alpha}} = e D^{\hat{\alpha}} (\bar{Z} W - Z \bar{W}) = e D^{\hat{\alpha}} \left( \frac{\bar{Y} W - Y \bar{W}}{e \bar{Y}} \right).$$

(5.96)

According to (5.93), this leads to the following modification of the constraint (5.73):

$$D^\alpha_\alpha D^\beta_\beta L = D^\alpha_\alpha D^\beta_\beta \text{tr} \left( \frac{(\bar{Z} W - Z \bar{W})^2}{e \bar{Z} Z} \right).$$

(5.97)
which immediately suggests to redefine $L$:

$$L = L - \text{tr} \left( \bar{Z}W - Z\bar{W} \right)^2$$

(5.98)

in order to recover the simple form (5.73) of the constraint. Finally, the other constraint (5.74) turns into the second constraint in (3.38).

6 Conclusion

In the present paper we have developed the harmonic superspace setting for general $N = 2$ rigid supersymmetric theories with gauged central charge. We have constructed the superfield formulations for both the linear and nonlinear VT multiplets with gauged central charge and described their Chern-Simons couplings to $N = 2$ vector multiplets. The six-dimensional origin of the linear VT multiplet and its Chern-Simons couplings has been explained. Note that the constraints describing the nonlinear VT multiplet can also be written in a six-dimensional notation but using a constant tensor which breaks the six-dimensional Lorentz symmetry down to its four-dimensional part.

The VT multiplet superfield formulations developed in this paper allow one to couple the VT multiplet to $N = 2$ superfield supergravity according to the general rules given in [31, 32]. To do this one should find consistent curved superspace extensions of the VT multiplet constraints presented above.

The analysis of this paper was restricted to the case of a single VT multiplet but the whole consideration can be readily applied to theories with several VT multiplets. There remains, however, an interesting problem whether there exist consistent theories with several VT multiplets, linear or/and nonlinear ones, coupled to each other. It would also be interesting to study the gauged central charge version of the ‘new’ nonlinear VT multiplet proposed in [14].

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