On the Signed (Total) $k$-Domination Number of a Graph*

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Abstract

Let $k$ be a positive integer and $G = (V, E)$ be a graph of minimum degree at least $k - 1$. A function $f : V \to \{-1, 1\}$ is called a signed $k$-dominating function of $G$ if $\sum_{u \in N_G[v]} f(u) \geq k$ for all $v \in V$. The signed $k$-domination number of $G$ is the minimum value of $\sum_{v \in V} f(v)$ taken over all signed $k$-dominating functions of $G$. The signed total $k$-domination function and signed total $k$-domination number of $G$ can be similarly defined by changing the closed neighborhood $N_G[v]$ to the open neighborhood $N_G(v)$ in the definition. The upper signed $k$-domination number is the maximum value of $\sum_{v \in V} f(v)$ taken over all minimal signed $k$-dominating functions of $G$. In this paper, we study these graph parameters from both algorithmic complexity and graph-theoretic perspectives. We prove that for every fixed $k \geq 1$, the problems of computing these three parameters are all $NP$-hard. We also present sharp lower bounds on the signed $k$-domination number and signed total $k$-domination number for general graphs in terms of their minimum and maximum degrees, generalizing several known results about signed domination.

1 Introduction

All graphs considered in this paper are simple and undirected. We generally follow [4] for standard notation and terminology in graph theory. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of $G$ is $|V(G)|$. For each vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, which are called the open neighborhood and closed neighborhood of $v$ (in $G$), respectively. The degree of $v$ (in $G$) is $d_G(v) = |N_G(v)|$. The minimum degree of $G$ is $\delta(G) = \min_{v \in V(G)} \{d_G(v)\}$, and the maximum degree of $G$ is $\Delta(G) = \max_{v \in V(G)} \{d_G(v)\}$. For an integer $r$, $G$ is called $r$-regular if $\Delta(G) = \delta(G) = r$, and is called nearly $r$-regular if $\Delta(G) = r$ and $\delta(G) = r - 1$. For $S \subseteq V(G)$, $G[S]$ is the subgraph of $G$ induced by $S$; that is, $G[S]$ is a graph with vertex set $S$ and edge set $\{uv \in E(G) \mid \{u, v\} \subseteq S\}$. For an integer $n \geq 1$, let $K_n$ denote the complete graph of order $n$; i.e., $K_n$ is an $(n - 1)$-regular graph of order $n$. For any function $f : V(G) \to \mathbb{R}$, we write $f(S) = \sum_{v \in S} f(v)$ for all $S \subseteq V(G)$, and the weight of $f$ is $w(f) = f(V(G))$.

Domination is an important subject in graph theory, and has numerous applications in other fields; see [11, 12] for comprehensive treatment and detailed surveys on (earlier) results in domination theory from both theoretical and applied perspectives. A set $S \subseteq V(G)$ is called a dominating set (resp. total dominating set) of $G$ if $\bigcup_{v \in S} N_G[v] = V(G)$ (resp. $\bigcup_{v \in S} N_G(v) = V(G)$). The domination number (resp. total domination number) of $G$, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum size of a dominating set (resp. total dominating set) of $G$.

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Let $k \geq 1$ be a fixed integer and $G$ be a graph of minimum degree at least $k - 1$. A function $f : V(G) \rightarrow \{-1, 1\}$ is called a signed $k$-dominating function of $G$ if $f(N_G[v]) \geq k$ for all $v \in V(G)$. The signed $k$-domination number of $G$, denoted by $\gamma_{kS}(G)$, is the minimum weight of a signed $k$-dominating function of $G$. When $G$ is of minimum degree at least $k$, the signed total $k$-dominating function and signed total $k$-domination number of $G$ (denoted by $\gamma^t_{kS}(G)$) can be analogously defined by changing the closed neighborhood $N_G[v]$ to the open neighborhood $N_G(v)$ in the definition. The concepts of signed $k$-domination number and signed total $k$-domination number are introduced in [16], where sharp lower bounds of these numbers are established for general graphs, bipartite graphs and $r$-regular graphs in terms of the order of the graphs. A related graph parameter called the upper signed $k$-domination number of $G$, denoted by $\Gamma_{kS}(G)$, is defined in [17] as the maximum weight of a minimal signed $k$-dominating function of $G$. (A signed $k$-dominating function $f$ of $G$ is called minimal if there exists no signed $k$-dominating function $f'$ of $G$ such that $f' \neq f$ and $f'(v) \leq f(v)$ for every $v \in V(G)$.) This parameter has also been studied in [3].

In the special case where $k = 1$, the signed $k$-domination number and signed total $k$-domination number are exactly the signed domination number [5] and signed total domination number [18], respectively. These two parameters have been extensively studied in the literature; see e.g. [1, 2, 4, 6, 7, 9, 13, 14, 18, 19] and the references therein.

In this paper, we continue the investigation of the signed $k$-domination number and signed total $k$-domination number of graphs, from both algorithmic complexity and graph theoretic points of view. In Section 2 we show that, for every fixed $k \geq 1$, the problems of computing the signed $k$-domination number, the signed total $k$-domination number, and the upper signed $k$-domination number of a graph are all $\mathcal{NP}$-hard. We then present, in Section 3, sharp lower bounds on the signed $k$-domination number and signed total $k$-domination number for general graphs in terms of their minimum and maximum degrees, from which several interesting results follow immediately.

## 2 Complexity Issues of Signed (Total) $k$-Domination

In this section we first show the $\mathcal{NP}$-hardness of computing the signed $k$-domination number and signed total $k$-domination number of a graph for all $k \geq 1$. Since the proofs for the two parameters are very similar, we only detail the proof for the signed total $k$-domination number, and merely point out the changes that need to be made for establishing hardness for the signed $k$-domination number. We now formally define the two decision problems corresponding to the computation of these two graph parameters.

**Signed $k$-Domination Problem (SkDP)**

*Instance:* A graph $G = (V, E)$ and an integer $r$.

*Question:* Is $\gamma_{kS}(G) \leq r$?

**Signed Total $k$-Domination Problem (STkDP)**

*Instance:* A graph $G = (V, E)$ and an integer $r$.

*Question:* Is $\gamma^t_{kS}(G) \leq r$?

**Theorem 1.** For every integer $k \geq 1$, the STkDP problem is $\mathcal{NP}$-complete.

**Proof.** Let $k \geq 1$ be a fixed integer. The STkDP problem is clearly in $\mathcal{NP}$. We now present a polynomial-time reduction from MINIMUM TOTAL DOMINATING SET (MTDS), which is a classical
\(\mathcal{NP}\)-complete problem \([8]\), to \(\text{STkDP}\). The MTDS problem is defined as follows: Given a graph \(G\) and an integer \(r\), decide whether \(G\) has a total dominating set of size at most \(r\).

Let \((G, r)\) be an instance of the MTDS problem. Construct another graph \(H\) as follows. First let \(H\) contain a copy of \(G\), which is denoted by \(G'\). Also, for each vertex \(v \in V(G)\), let \(v'\) denote its counterpart in \(G'\). For each \(v \in V(G)\), we add \(t(v)\) disjoint copies of \(K_{k+2}\) to \(H\), where \(t(v) = d_G(v) + k - 2\); call these copies \(K_{k+2}^{v_1}, K_{k+2}^{v_2}, \ldots, K_{k+2}^{v_{t(v)}}\). Then, for each \(i \in \{1, 2, \ldots, t(v)\}\), add an edge between \(v'\) and an (arbitrary) vertex from \(K_{k+2}^{v_i}\). This finishes the construction of \(H\). It is easy to verify that \(d_H(v') = 2d_G(v) + k - 2\) for all \(v \in V(G)\).

Let \(T = (k + 2) \sum_{v \in V(G)} t(v) = (k + 2) \sum_{v \in V(G)} (k + d_G(v) - 2)\) be the number of vertices in \(V(H) \setminus G'\). We will prove that \(\gamma_t(G) \leq r\) if and only if \(\gamma_{kS}^t(H) \leq 2r - |V(G)| + T\).

First consider the “if” direction. Assume that \(\gamma_{kS}^t(H) \leq 2r - |V(G)| + T\), and \(f: V(H) \to \{-1, 1\}\) is a signed total \(k\)-dominating function of \(H\) of weight \(\gamma_{kS}^t(H)\). Let \(S' = \{v' \in V(G') \mid f(v') = 1\}\). It is easy to see that, for each \(v \in V(G)\) and \(1 \leq i \leq t(v)\), all vertices in \(K_{k+2}^{v_i}\) must have function value “1” under \(f\). It follows that \(\gamma_{kS}^t(H) = w(f) = T + |S'| - (|V(G')| - |S'|) = 2|S'| - |V(G)| + T\). Since \(\gamma_{kS}^t(H) \leq 2r - |V(G)| + T\), we have \(|S'| \leq r\). Now define \(S = \{v \in V(G) \mid v' \in S'\}\); i.e., \(S\) is the counterpart of \(S'\) in \(G\). We show that \(S\) is a total dominating set of \(G\). Assume to the contrary that \(S\) is not a total dominating set of \(G\), and let \(v \in V(G)\) be such that \(N_G(v) \cap S = \emptyset\). By our definitions of \(S\) and \(S'\), \(f(u') = -1\) for all \(u \in N_G(v)\). Thus, \(\sum_{x \in N_H(v)} f(x) = t(v) - d_G(v) = k - 2\), contradicting with the fact that \(f\) is a signed total \(k\)-dominating function of \(H\). Therefore, \(S'\) is indeed a total dominating set of \(G\), from which \(\gamma_t(G) \leq |S'| \leq r\) follows. This completes the proof for the “if” direction.

Now comes the “only if” part of the reduction. Suppose \(\gamma_t(G) \leq r\) and \(S \subseteq V(G)\) is a total dominating set of \(G\) of size at most \(r\). Define a function \(f: V(H) \to \{-1, 1\}\) as follows: \(f(x) = -1\) if \(x = v'\) for some \(v \in V(G) \setminus S\), and \(f(x) = 1\) otherwise. The weight of \(f\) is \(T + |S| - (|V(G)| - |S|) = 2|S| - |V(G)| + T \leq 2r - |V(G)| + T\). We now verify that \(f\) is a signed total \(k\)-dominating function of \(H\). For each \(x \in V(H) \setminus G'\), \(f(N_H(x)) \geq (k + 1) - 1 = k\). For each \(v' \in V(G')\) (with \(v \in V(G)\)), since \(S\) is a total dominating set of \(G\), \(f(N_H(v')) \geq t(v) + 1 - (d_G(v) - 1) = t(v) + 2 - d_G(v) = k\). Hence, \(f\) is a signed total \(k\)-dominating function of \(H\) of weight at most \(2r - |V(G)| + T\). This completes the “only if” part of the reduction.

Therefore, \(\gamma_t(G) \leq r\) if and only if \(\gamma_{kS}^t(H) \leq 2r - |V(G)| + T\). This finishes the whole reduction, and hence concludes the proof of Theorem 1.

**Theorem 2.** For every integer \(k \geq 1\), the \(\text{SkDP}\) problem is \(\mathcal{NP}\)-complete.

**Proof.** The proof is very similar to that of Theorem 1 with two differences in the reduction. We reduce from the \(\mathcal{NP}\)-complete problem \(\text{MINIMUM DOMINATING SET}\) which, given a graph \(G\) and an integer \(r\), needs to decide whether \(G\) has a dominating set of size at most \(r\) to \(\text{SkDP}\). Let \((G, r)\) be an instance of \(\text{MINIMUM DOMINATING SET}\). Construct another graph \(H\) as follows. First let \(H\) contain a copy of \(G\), which is denoted by \(G'\). For each vertex \(v \in V(G)\), add \(s(v)\) disjoint copies of \(K_{k+1}\) to \(H\), where \(s(v) = d_G(v) + k - 1\); call these copies \(K_{k+1}^{v_1}, K_{k+1}^{v_2}, \ldots, K_{k+1}^{v_{s(v)}}\). Then, for each \(i \in \{1, 2, \ldots, s(v)\}\), add an edge between \(v'\) (the counterpart of \(v\) in \(G'\)) and an (arbitrary) vertex from \(K_{k+1}^{v_i}\). This finishes the construction of \(H\). Using similar argument to that in Theorem 1 we can prove that \(\gamma(G) \leq r\) if and only if \(\gamma_{kS}(H) \leq 2r - |V(G)| + T\), where \(T = (k + 1) \sum_{v \in V(G)} s(v)\). The \(\mathcal{NP}\)-completeness of \(\text{SkDP}\) is thus established.

\(\square\)
We now define the problem corresponding to the computation of the upper signed \(k\)-domination number of graphs as follows.

**Upper Signed \(k\)-Domination Problem (US\(k\)DP)**

*Instance:* A graph \(G = (V, E)\) and an integer \(r\).

*Question:* Is \(\Gamma_{kS}(G) \geq r\)?

**Theorem 3.** For every integer \(k \geq 1\), the US\(k\)DP problem is \(\mathcal{NP}\)-complete.

**Proof.** The US\(k\)DP problem is in \(\mathcal{NP}\) because given a function \(f : V(G) \rightarrow \{-1, 1\}\), we can verify in polynomial time whether \(f\) is a minimal signed \(k\)-dominating function of \(G\) using Lemma 4 in [3]. We will describe a polynomial time reduction from the 1-in-3 SAT problem to it. The 1-in-3 SAT problem is defined as follows: Given a Boolean formula in conjunctive normal form, each clause of which contains exactly three positive literals (i.e., variables with no negations), decide whether the formula is 1-in-3 satisfiable, i.e., if there exists an assignment of the variables such that exactly one variable of each clause is assigned TRUE. This problem is known to be \(\mathcal{NP}\)-complete [15].

Let \(F\) be a Boolean formula with variables \(\{x_1, x_2, \ldots, x_n\}\), which is an input of the 1-in-3 SAT problem. Assume \(F = \bigwedge_{i=1}^{m} c_i\) where \(c_i = (x_{i_1} \lor x_{i_2} \lor x_{i_3})\) for each \(i \in \{1, 2, \ldots, m\}\). We construct a graph \(G\) as follows. Take \(m\) disjoint copies of \(K_{k+2}\), each of which corresponds to a clause \(c_i\) with \(i \in \{1, 2, \ldots, m\}\), and \(n\) disjoint copies of \(K_{k+3}\) (also disjoint from the copies of \(K_{k+2}\)’s) each of which corresponds to a variable \(x_j\) with \(j \in \{1, 2, \ldots, n\}\). Delete one edge from each copy of \(K_{k+3}\) (with one edge missing) corresponding to \(x_j\) the \(j\)-th variable block. For each \(i \in \{1, 2, \ldots, m\}\), let \(c'_i\) be an (arbitrary) vertex in the \(i\)-th clause block. For every \(j \in \{1, 2, \ldots, n\}\), let \(x'_j\) and \(x''_j\) be the two vertices in the \(j\)-th variable block for which the edge \(x'_j x''_j\) is removed. For each clause \(c_i = (x_{i_1} \lor x_{i_2} \lor x_{i_3})\), add three cross-block edges \(c'_i x'_{i_1}, c'_i x'_{i_2}, c'_i x'_{i_3}\). This finishes the construction of \(G\). Note that \(|V(G)| = (k + 3)n + (k + 2)m\).

We claim that \(\Gamma_{kS}(G) \geq (k + 1)n + (k + 2)m\) if and only if \(F\) is 1-in-3 satisfiable. First consider the “if” direction, and let \(A : \{x_1, x_2, \ldots, x_n\} \rightarrow \{\text{TRUE, FALSE}\}\) be an assignment that witnesses the 1-in-3 satisfiability of \(F\). Define \(f : V(G) \rightarrow \{-1, 1\}\) as follows: For each \(j \in \{1, 2, \ldots, n\}\), let

\[
f(x'_j) = \begin{cases} 
1 & \text{if } A(x_j) = \text{TRUE}; \\
-1 & \text{if } A(x_j) = \text{FALSE}. 
\end{cases}
\]

and

\[
f(x''_j) = \begin{cases} 
1 & \text{if } A(x_j) = \text{TRUE}; \\
-1 & \text{if } A(x_j) = \text{FALSE}. 
\end{cases}
\]

Let \(f(v) = 1\) for all \(v \in V(G) \setminus \bigcup_{j=1}^{n} \{x'_j, x''_j\}\).

Clearly, \(w(f) = (k + 1)n + (k + 2)m\). Since exactly one of \(A(x_{i_1}), A(x_{i_2})\) and \(A(x_{i_3})\) is TRUE for each \(1 \leq i \leq m\), it is easy to verify that \(f\) is a signed \(k\)-dominating function of \(G\). We next prove that \(f\) is minimal, that is, for every vertex \(v \in V(G)\) with \(f(v) = 1\) there exists \(u \in N_G[v]\) for which \(f(N_G[u]) \in \{k, k+1\}\) (see [3]). For every \(j \in \{1, 2, \ldots, n\}\), there is (at least) one vertex \(u\) in the \(j\)-th variable block such that \(u \notin \{x'_j, x''_j\}\). This vertex \(u\) is adjacent to all other vertices in the \(j\)-th variable block, and clearly \(f(N_G[u]) = k+1\). For every \(i \in \{1, 2, \ldots, m\}\), \(c'_i\) is adjacent to all other vertices in the \(i\)-th clause block, and \(f(N_G[c'_i]) = (k+2) + (1-2) = k+1\) since exactly one of \(f(x'_{i_1}), f(x'_{i_2})\) and \(f(x'_{i_3})\) is 1. Therefore, \(f\) is indeed a minimal signed \(k\)-dominating function of \(G\) with weight \((k+1)n + (k+2)m\), and the correctness of the “if” direction follows.

We now turn to the “only if” part of the claim. Assume that \(f\) is a minimal signed \(k\)-dominating function of \(G\) of weight at least \((k+1)n + (k+2)m\). If for some \(j \in \{1, 2, \ldots, n\}\), the vertices in the \(j\)-th variable block all have value 1 under \(f\), then \(f(N_G[v]) \geq k+2\) for every \(v \neq x'_j\) in the \(j\)-th
variable block. Thus, there is no \( u \in N_G[x^o_j] \) such that \( f(N_G[u]) \in \{k, k+1\} \), which violates the minimality of \( f \). Hence, at least one vertex from each variable block must have value \(-1\) under \( f \), implying that \( w(f) \leq (k+1)n + (k+2)m \). We thus have \( w(f) = (k+1)n + (k+2)m \), and therefore (1) \( f(v) = 1 \) for every vertex \( v \) in the clause blocks, and (2) for each \( j \in \{1, 2, \ldots, n\} \), \( f(v) = -1 \) for exactly one vertex \( v \) in the \( j \)-th variable block. Now produce an assignment \( A \) as follows: For each \( j \in \{1, 2, \ldots, n\} \), let \( A(x_j) = \text{TRUE} \) if \( f(x'_j) = 1 \), and \( A(x_j) = \text{FALSE} \) otherwise.

For every \( i \in \{1, 2, \ldots, m\} \), we have \( k \leq f(N_G[c'_i]) = (k+2) + f(x_{i_1}) + f(x_{i_2}) + f(x_{i_3}) \), and thus at least one of \( f(x_{i_1}), f(x_{i_2}) \) and \( f(x_{i_3}) \) must be \( 1 \). Assume that at least two of the three values are \( 1 \). Then \( f(N_G[c'_i]) \geq k + 3 \), and obviously \( f(N_G[v]) = k + 2 \) for every other vertex \( v \) in the \( i \)-th clause block. This indicates, however, that a vertex \( v \neq c'_i \) in the \( i \)-th clause block does not have any neighbor (including itself) whose closed-neighborhood-sum is \( k \) or \( k + 1 \), contradicting with the minimality of \( f \). Accordingly, exactly one of \( f(x_{i_1}), f(x_{i_2}) \) and \( f(x_{i_3}) \) is \( 1 \), and thus exactly one of \( A(x_{i_1}), A(x_{i_2}) \) and \( A(x_{i_3}) \) is \( \text{TRUE} \), for every \( i \in \{1, 2, \ldots, n\} \). Therefore, \( F \) is \( 1 \)-in-\( 3 \) satisfiable, finishing the proof of the “only if” part of the reduction.

The reduction is completed and the \( \mathcal{NP} \)-completeness of \( \text{US}k\text{DP} \) is thus established. \( \square \)

3 Sharp Lower Bounds on \( \gamma_{kS}(G) \) and \( \gamma^t_{kS}(G) \)

In this section we present sharp lower bounds on \( \gamma_{kS}(G) \) and \( \gamma^t_{kS}(G) \) in terms of the minimum and maximum degrees of \( G \). Let \( k \geq 1 \) be a fixed integer throughout this section. For each integer \( n \), define \( I_n = 1 \) if \( n \equiv k \pmod{2} \), and \( I_n = 0 \) otherwise; that is, \( I_n \) is the indicator variable of whether \( n \) and \( k \) have the same parity.

**Theorem 4.** For every graph \( G \) with \( \delta(G) \geq k - 1 \),

\[
\gamma_{kS}(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + I_{\delta(G)} + I_{\Delta(G)}}{\delta(G) + \Delta(G) + 2 + I_{\delta(G)} - I_{\Delta(G)}}.
\]

**Proof.** Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq k - 1 \). For notational simplicity, we write \( \delta \) and \( \Delta \) to denote \( \delta(G) \) and \( \Delta(G) \) respectively. When \( \delta = \Delta \), it is easy to verify that the theorem degenerates to Theorem 5 in [16]. Thus, we assume in what follows that \( \Delta \geq \delta + 1 \). Let \( f \) be a signed \( k \)-dominating function of weight \( \gamma_{kS}(G) \). We need to introduce some notations. Let \( P = \{ v \in V(G) \mid f(v) = 1 \} \) and \( Q = V(G) \setminus P = \{ v \in V(G) \mid f(v) = -1 \} \). Furthermore, denote \( P_\delta = \{ v \in P \mid d_G(v) = \delta \} \), \( P_\Delta = \{ v \in P \mid d_G(v) = \Delta \} \), and \( P_m = P \setminus (P_\delta \cup P_\Delta) \). Define \( Q_\delta \), \( Q_\Delta \), and \( Q_m \) analogously. For each \( c \in \{\delta, \Delta, m\} \), let \( V_c = P_c \cup Q_c \). Notice that \( V_\delta \cap V_\Delta = \emptyset \) since \( \Delta > \delta \). Let \( R = \{ v \in V(G) \mid d_G(v) \equiv k \pmod{2} \} \). Clearly \( \sum_{y \in N_G(x)} f(y) \geq k + 1 \) for each \( x \in R \).
Thus, we have
\[
kn + |R| \leq \sum_{x \in V(G)} \sum_{y \in N_G(x)} f(y) = \sum_{x \in V(G)} (d_G(x) + 1)f(x)
= (\delta + 1)|P_\delta| + (\Delta + 1)|P_\Delta| + \sum_{x \in P_m} (d_G(x) + 1) - (\delta + 1)|Q_\delta| - (\Delta + 1)|Q_\Delta| - \sum_{x \in Q_m} (d_G(x) + 1)
\leq (\delta + 1)|P_\delta| + (\Delta + 1)|P_\Delta| + \Delta|P_m| - (\delta + 1)|Q_\delta| - (\Delta + 1)|Q_\Delta| - (\delta + 2)|Q_m|
\]
(since \(\delta + 1 \leq d_G(x) \leq \Delta - 1\) for each \(x \in P_m \cup Q_m\))
\[
= (\delta + 1)|V_\delta| + (\Delta + 1)|V_\Delta| + \Delta|V_m| - 2(\delta + 1)|Q_\delta| - 2(\Delta + 1)|Q_\Delta| - (\Delta + \delta + 2)|Q_m|
\]
\[
= (\Delta + 1)n - (\Delta - \delta)|V_\delta| - |V_m| - (\Delta + \delta + 2)|Q| + (\Delta - \delta)|Q_\delta| - (\Delta - \delta)|Q_\Delta|
\]
(note that \(n = |V(G)| = |V_\delta| + |V_\Delta| + |V_m|\) and \(|Q| = |Q_\delta| + |Q_\Delta| + |Q_m|\)).

Therefore,
\[
(\Delta + 1 - k)n \geq |R| + |V_m| + (\Delta - \delta)(|V_\delta| - |Q_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|
= |R| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|.
\]

Since \(R = \{v \in V(G) \mid d(v) \equiv k \pmod{2}\}\), it holds that \(V_\delta \subseteq R\) if \(\delta \equiv k \pmod{2}\), and that \(V_\Delta \subseteq R\) if \(\Delta \equiv k \pmod{2}\). Recalling that \(V_\Delta \cap V_\delta = \emptyset\), we have \(|R| \geq I_\delta \cdot |V_\delta| + I_\Delta \cdot |V_\Delta|\). Thus,
\[
(\Delta + 1 - k)n \geq I_\delta \cdot |V_\delta| + I_\Delta \cdot |V_\Delta| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|
= I_\Delta(|V_m| + |V_\delta| + |V_\Delta|) + (1 - I_\Delta)|V_m| + (I_\delta - I_\Delta)|V_\delta|
+ (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|
= I_\Delta \cdot n + (1 - I_\Delta)|V_m| + (I_\delta - I_\Delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|.
\]

Observing that \(\Delta - \delta \geq 1 \geq \max\{I_\delta - I_\Delta, I_\Delta - I_\delta\}\) and \((1 - I_\Delta)|V_m| \geq (1 - I_\Delta)|Q_m| \geq (I_\delta - I_\Delta)|Q_m|\), we get
\[
(\Delta + 1 - k - I_\Delta)n \geq (I_\delta - I_\Delta)|Q_m| + (I_\delta - I_\Delta)|V_\delta| + (I_\Delta - I_\delta)|P_\delta| + (I_\delta - I_\Delta)|Q_\Delta| + (\Delta + \delta + 2)|Q|
= (I_\delta - I_\Delta)(|Q_m| + |V_\delta| + |P_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|
= (I_\delta - I_\Delta)(|Q_m| + |Q_\delta| + |Q_\Delta|) + (\Delta + \delta + 2)|Q|
= (\Delta + \delta + 2 + I_\delta - I_\Delta)|Q|.
\]

Hence, we deduce that
\[
|Q| \leq n \cdot \frac{\Delta - k + 1 - I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta},
\]
from which it follows that
\[
\gamma_{KS}(G) = n - 2|Q| \geq n \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta},
\]
which is exactly the desired inequality in Theorem 4. \(\square\)
A vertex of degree $k - 1$ or $k$ in a graph $G$ clearly has function value $1$ under all signed $k$-dominating functions of $G$. Thus, it is natural to consider graphs with minimum degree at least $k + 1$ (as is done in [3] for establishing sharp upper bounds for the upper signed $k$-domination number). We next show that Theorem 4 is sharp for all $\Delta \geq \delta \geq k + 1$. This level of sharpness is high as it applies not only to special values of minimum and maximum degrees.

**Theorem 5.** For any integers $\delta$ and $\Delta$ such that $\Delta \geq \delta \geq k + 1$, there exists an infinite family $\mathcal{F}$ of graphs with minimum degree $\delta$ and maximum degree $\Delta$, such that for every graph $G \in \mathcal{F}$,

$$\gamma_{kS}(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.$$  

**Proof.** Fix integers $\Delta$ and $\delta$ such that $\Delta \geq \delta \geq k + 1$. Let $H_1, H_2, \ldots, H_t$ be $t$ disjoint copies of the complete bipartite graph $K_{a,b}$ with vertex partition $(A, B)$, where $|A| = a = (\delta + k + 1 + I_\delta)/2$, $|B| = b = (\Delta - k + 1 - I_\Delta)/2$ (it is easy to verify that $a$ and $b$ are both integers), and $t$ is an arbitrary even integer larger than $\Delta$. It is also easy to check that $1 \leq a \leq \delta$ and $1 \leq b \leq \Delta$ (just note that $I_\delta = 0$ when $\delta = k + 1$). For each $1 \leq i \leq t$, let $A_i$ and $B_i$ denote the vertex partition of $H_i$ with size $a$ and $b$, respectively. Let $P = \bigcup_{i=1}^t A_i$ and $Q = \bigcup_{i=1}^t B_i$. Note that each vertex in $P$ is connected to exactly $b$ vertices in $Q$, and each vertex in $Q$ is adjacent to exactly $a$ vertices in $P$.

Our desired graph $G$ has vertex set $P \cup Q$, and contains $\bigcup_{i=1}^t H_i$ as a subgraph. Furthermore, we add some edges between vertices in $P$ to make $G[P]$ become $(\Delta - b)$-regular (no edges need to be added if $\Delta = b$). This can be done in the following way: Imagine that there is a complete graph $K$ whose vertex set is $P$. Since $|P| = ta$ is even and every complete graph of even order is $1$-factorable (see e.g. Theorem 9.1 in [10]), the edges of $K$ can be partitioned into $|P| - 1 \geq \Delta$ perfect matchings of $K$. Taking $\Delta - b$ of these matchings and adding them to $G$ certainly makes $G[P]$ become $(\Delta - b)$-regular. Similarly, we add some edges between vertices in $Q$ to make $G[Q]$ $(\delta - a)$-regular. This finishes the construction of $G$. Note that all vertices in $P$ have degree $\Delta$ and those in $Q$ have degree $\delta$, and thus $G$ is of minimum degree $\delta$ and maximum degree $\Delta$. (Note also that by varying $t$, we get an infinite family of graphs with the desired properties.)

Define a function $f : P \cup Q \to \{-1, 1\}$ by letting $f(v) = 1$ for all $v \in P$ and $f(u) = -1$ for all $u \in Q$. Then, for each $v \in P$, $f(N_G[v]) = \Delta + 1 - 2b = k + I_\Delta \geq k$, and for each $u \in Q$, $f(N_G[u]) = 2a - (\delta + 1) = k + I_\delta \geq k$. Therefore, $f$ is a signed $k$-dominating function of $G$. Since $|V(G)| = |P| + |Q|$ and $|P|/|Q| = a/b = \frac{\delta + k + 1 + I_\delta}{\Delta - k + 1 - I_\Delta}$, we have

$$\gamma_{kS}(G) \leq w(f) = |P| - |Q| = (1 - \frac{2}{|P|/|Q| + 1})|V(G)| = |V(G)| \cdot \frac{\delta - \Delta + 2k + I_\delta + I_\Delta}{\delta + \Delta + 2 + I_\delta - I_\Delta}.$$  

By Theorem 3 we know that the equality holds in the above formula, which completes the proof of Theorem 5. 

We can also derive a sharp lower bound on the signed total $k$-domination number of a graph as follows.

**Theorem 6.** For every graph $G$ with $\delta(G) \geq k$,

$$\gamma_{kS}^t(G) \geq |V(G)| \cdot \frac{\delta(G) - \Delta(G) + 2k + 2 - I_\delta(G) - I_\Delta(G)}{\delta(G) + \Delta(G) + I_\Delta(G) - I_\delta(G)}.$$
Theorem 7. For any integers $\delta$ and $\Delta$ such that $\Delta \geq \delta \geq k + 2$, there exists an infinite family $F$ of graphs with minimum degree $\delta$ and maximum degree $\Delta$, such that for every graph $G \in F$,

$$\gamma_{kS}^t(G) = |V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta}.$$

The proofs of Theorems 6 and 7 are very similar to those of Theorems 4 and 5, and thus are put in the appendix.

Theorems 4 and 6 are generalizations of Theorem 5 in [16]. The following corollaries, which generalize some other known results regarding signed domination number and signed total domination number, are also immediate from the preceding theorems.

Corollary 1. For any nearly $r$-regular graph $G$ of order $n$ with $r \geq k$, $\gamma_{kS}(G) \geq kn/(r - 1)$ and $\gamma_{kS}^t(G) \geq kn/(r - 1)$.

Corollary 2. Let $c$ be a real number for which $-1 < c \leq 1$. Then $\gamma_{kS}(G) \geq cn$ for every graph $G$ of order $n$ with $\delta(G) \geq k - 1$ and $\Delta(G) \leq ((1 - c)\delta(G) + 2k - 2c)/(1 + c)$, and $\gamma_{kS}^t(G) \geq cn$ for every graph $G$ of order $n$ with $\delta(G) \geq k$ and $\Delta(G) \leq ((1 - c)\delta(G) + 2k)/(1 + c)$.

Corollary 3. Let $G$ be a graph with $\delta(G) \geq k$ and $\Delta(G) \leq \delta(G) + 2k$. Then $\gamma_{kS}(G) \geq 0$ and $\gamma_{kS}^t(G) \geq 0$.

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A Proof of Theorem 6

Proof of Theorem 6. Let $G$ be a graph of order $n$ and $f$ be a signed total $k$-dominating function of $G$. Let $\delta$, $\Delta$, $P$, $Q$, $P_\delta$, $P_\Delta$, $P_m$, $Q_\delta$, $Q_\Delta$, $Q_m$, $V_\delta$, $V_\Delta$, $V_m$ be defined in the same way as in the proof of Theorem 4. Let $R = \{v \in V(G) \mid d(v) \not\equiv k \pmod{2}\}$ (which is different from the definition of $R$ in the proof of Theorem 4). Assume $\Delta > \delta$, otherwise the theorem just becomes Theorem 5 in [16]. Since $\sum_{y \in N_G(x)} f(y) \geq k + 1$ for all $x \in R$, we have:

$$kn + |R| \leq \sum_{x \in V(G)} \sum_{y \in N_G(x)} f(y)$$

$$= \sum_{x \in V(G)} d_G(x) f(x)$$

$$= \delta |P_\delta| + \Delta |P_\Delta| + \sum_{x \in P_m} d_G(x) - \delta |Q_\delta| - \Delta |Q_\Delta| - \sum_{x \in Q_m} d_G(x)$$

$$\leq \delta |P_\delta| + \Delta |P_\Delta| + (\Delta - 1)|P_m| - \delta |Q_\delta| - \Delta |Q_\Delta| - (\delta + 1)|Q_m|$$

$$= \delta |V_\delta| + \Delta |V_\Delta| + (\Delta - 1)|V_m| - 2\delta |Q_\delta| - 2\Delta |Q_\Delta| - (\Delta + \delta)|Q_m|$$

$$= \Delta n - (\Delta - \delta)|V_\delta| - |V_m| - (\Delta + \delta)|Q| + (\Delta - \delta)|Q_\delta| - (\Delta - \delta)|Q_\Delta|$$

(recall that $n = |V(G)| = |V_\delta| + |V_\Delta| + |V_m|$ and $|Q| = |Q_\delta| + |Q_\Delta| + |Q_m|$).
By our definition, it holds that \(|R| \geq (1 - I_\delta)|V_\delta| + (1 - I_\Delta)|V_\Delta|\). Therefore,

\[
(\Delta - k)n \geq |R| + |V_m| + (\Delta - \delta)(|V_\delta| - |Q_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|
\]

\[
= |R| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|
\]

\[
\geq (1 - I_\delta)|V_\delta| + (1 - I_\Delta)|V_\Delta| + |V_m| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|
\]

\[
= (1 - I_\Delta)(|V_m| + |V_\delta| + |V_\Delta|) + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta|
\]

\[
+ (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|
\]

\[
= (1 - I_\Delta)n + I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|.
\]

Noting that \(I_\Delta|V_m| \geq (I_\Delta - I_\delta)|Q_m|\) and \(\Delta - \delta \geq \max\{I_\Delta - I_\delta, I_\delta - I_\Delta\}\), we obtain

\[
(\Delta - k + I_\Delta - 1)n
\]

\[
\geq I_\Delta|V_m| + (I_\Delta - I_\delta)|V_\delta| + (\Delta - \delta)(|P_\delta| + |Q_\Delta|) + (\Delta + \delta)|Q|
\]

\[
\geq (I_\Delta - I_\delta)|Q_m| + (I_\Delta - I_\delta)|V_\delta| + (I_\delta - I_\Delta)|P_\delta| + (I_\Delta - I_\delta)|Q_\Delta| + (\Delta + \delta)|Q|
\]

\[
= (I_\Delta - I_\delta)|Q| + (\Delta + \delta)|Q|
\]

\[
= (\Delta + \delta + I_\Delta - I_\delta)|Q|.
\]

Hence, we have

\[
|Q| \leq n \cdot \frac{\Delta - k + I_\Delta - 1}{\delta + \Delta + I_\Delta - I_\delta},
\]

from which it follows that

\[
\gamma_{kS}(G) = n - 2|Q| \geq n \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta},
\]

completing the proof of Theorem 6.

\[
\square
\]

B Proof of Theorem 7

Proof of Theorem 7. Fix integers \(\Delta\) and \(\delta\) such that \(\Delta \geq \delta \geq k + 2\). We proceed with the same construction used in the proof of Theorem 5 except for setting \(a = (\delta + k - I_\delta + 1)/2\) and \(b = (\Delta - k + I_\Delta - 1)/2\) instead. (It is easy to check that \(a\) and \(b\) are integers satisfying that \(1 \leq a \leq \delta\) and \(1 \leq b \leq \Delta\).) The obtained graph \(G\) has vertex set \(P \cup Q\), where \(d_G(v) = \Delta\) for all \(v \in P\) and \(d_G(u) = \delta\) for all \(u \in Q\). Furthermore, each vertex \(v \in P\) is adjacent to exactly \(b\) vertices in \(Q\) and \(\Delta - b\) vertices in \(P\), while every vertex \(u \in Q\) is adjacent to precisely \(a\) vertices in \(P\) and \(\delta - a\) vertices in \(Q\). Now define a function \(f\) which assigns 1 to all vertices in \(P\) and \(-1\) to those in \(Q\). It is easy to verify that \(f\) is a signed total \(k\)-dominating function of \(G\) with weight \(|V(G)| \cdot \frac{\delta - \Delta + 2k + 2 - I_\delta - I_\Delta}{\delta + \Delta + I_\Delta - I_\delta}\), completing the proof of Theorem 7.

\[
\square
\]