QCD Critical Points and Their Associated Soft Modes

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The mean-field level calculation shows that the QCD matter can have multiple critical points incorporating the color superconductivity under charge neutrality constraint due to the repulsive vector interaction; this actually implies that the QCD matter is very soft for a simultaneous formation of diquark and chiral condensates coupled with the baryonic density. Dynamical density fluctuations are analyzed as possible soft modes around the QCD critical point using dissipative relativistic fluid dynamics. It is found that the entropy fluctuation solely gets enhanced while the sound modes due to mechanical density fluctuations are strongly attenuated around the QCD CP, which may suggest a suppression or even total disappearance of Mach cone at the CP.

§1. Introduction

A unique feature of the phase diagram of quantum chromodynamics (QCD) is the existence of a critical point (CP) see, for a review. In the first part of this talk, however, we shall describe possible variants of the QCD phase diagram when the color superconductivity (CSC) is taken into account, focusing on a possible important role of the vector-vector interaction between quarks. The mean-field level calculation shows that there can exist multiple critical points in the QCD phase diagram when the repulsive vector interaction is taken into account which feature can be enhanced when the charge neutrality constraint is imposed see also for a similar result in a different context. The possible emergence of multiple critical points actually implies that the QCD matter around the phase boundary of the chiral-to-CSC transition is soft for a combined excitation of the baryon density, scalar condensate and the diquark excitations.

Around a critical point of a second-order transition, large fluctuations of physical quantities are expected. It has been established that the QCD CP belongs to the same universality class ($Z_2$) as the liquid-gas CP, and hydrodynamic modes coupled to conserved quantities such as sound modes are softening modes at the CP. We show on the basis of the dissipative relativistic fluid dynamics that the dynamical density fluctuations or the Brillouin modes are greatly suppressed around the CP, while the energy fluctuations or the Rayleigh modes are enhanced when approaching the CP.

We shall conclude the report by mentioning some other examples of phase transitions that the QCD matter may undergoes and the respective soft modes, which could affect the quark spectrum drastically.
§2. QCD critical points; alternatives of QCD phase diagram with vector interaction and charge neutrality

When charge neutrality is imposed to the quark matter, a mismatch of the Fermi surfaces of the respective flavors arises unless the chiral limit is taken.\textsuperscript{12} For such a system, it is known that the diquark gap has an abnormal temperature dependence that the gap increases as the temperature is raised, and has the maximum at a finite temperature.\textsuperscript{13} This is due to the smearing of the Fermi surface by temperature. On the other hand, such a system tends to become unstable due to the color magnetic instability.\textsuperscript{14}

It is known that the repulsive vector interaction\textsuperscript{1}, 15)–19) in the Nambu-Jona-Lasinio (NJL) model\textsuperscript{20} as a low-energy effective model of QCD\textsuperscript{21}–23) postpones the chiral restoration towards larger chemical potential.\textsuperscript{1} We note that the renormalization-group analysis \textsuperscript{24}, 25) the chiral instanton-anti-instanton molecule model\textsuperscript{26} and the truncated Dyson-Schwinger model of QCD\textsuperscript{27} all support the existence of the vector-vector four-quark interaction. When the possible transition to a CSC phase is considered in the NJL model, it was found that there can appear another critical point in the QCD phase diagram.\textsuperscript{4} The relevance of the repulsive vector-vector interaction to the chiral and CSC transitions can be intuitively understood as follows.\textsuperscript{28} According to thermodynamics, when two phases I and II are in an equilibrium state, their temperatures $T_I$, $T_{II}$, pressures $P_I$, $P_{II}$ and the chemical potentials $\mu_I$, $\mu_{II}$ are the same: $T_I = T_{II}$, $P_I = P_{II}$, and $\mu_I = \mu_{II}$. If the two phases are the chirally broken and restored phase with quark masses satisfying $M_I > M_{II}$, the last equality implies that the chirally restored phase has a higher density than the broken phase, because $\mu_I$ at vanishing temperature are expressed as $\mu_i = \sqrt{M_i^2 + p_{F_i}^2}$, ($i = I, \ II$), and hence $p_{F_I} < p_{F_{II}}$, where $p_{F_i}$ is the Fermi momentum of the $i$-th phase. This means that chiral restoration at finite density is necessarily accompanied by a density jump to a higher density state with a large Fermi surface, which in turn favors the formation of Cooper instability leading to CSC. However, since the vector coupling in the zero-th component couples to the quark (or baryon) density $\rho_B = \langle \bar{q}^0 q \rangle$, it gives rise to a repulsive energy proportional to the density squared, i.e. $G_V \rho_B^2 / 2$ with $G_V$ being the vector coupling. This means that the restored phase is disfavored energetically due to the vector coupling, and hence the vector coupling weakens and delays the phase transition of the chiral restoration at low temperatures. Thus one also expects that the vector interaction postpones the formation of CSC to higher chemical potentials.

Zhang, Fukushima and Kunihiro\textsuperscript{7} found that the positive electric chemical potential $\mu_e$ inherent in the charge neutrality constraint plays a similar role with the repulsive vector interaction on the chiral phase transition in a four-quark interaction model: In a simple two-flavor NJL model, the same two-critical-point structure as found in \textsuperscript{4} emerges. In addition, positive $\mu_e$ also represents the magnitude of difference between the Fermi spheres of u and d quarks when taking into account the local charge neutrality constraint. For an asymmetric homogeneous system with the mismatched Fermi spheres, the energy gap of the Cooper paring between these two
flavor quarks can increase with temperature as mentioned before. For two-flavor neutral CSC phase, this unconventional thermal behavior of the diquark condensate can lead to a competition between chiral condensate and diquark condensate, which can be enhanced with increasing temperature. For some model parameters region, this abnormal competition induced by \( \mu_c \) can result in a phase structure with even three critical points. Thus one would expect that a simultaneous incorporation of the repulsive vector interaction and the electric chemical potential under neutrality constraint will weaken more significantly the phase transition from the chiral-broken to CSC phase.

Recently, Zhang and Kunihiro explored the effect of the repulsive vector-vector interaction combined with electric-charge neutrality in \( \beta \)-equilibrium on the chiral phase and CSC phase transitions within both two-flavor and two-plus-one-flavor NJL models. For the two-flavor case with \( u \) and \( d \) quarks, a nonlocal NJL model is adopted in (6): The vector-vector term that is \( U(2) \times U(2) \) invariant is chosen as

\[
\mathcal{L}_V = -G_V \sum_{i=0}^{3} \left[ (\bar{q}(x)\gamma^\mu \tau_i q(x))^2 - (\bar{q}(x)\gamma^\mu \gamma_5 \tau_i q(x))^2 \right],
\]

where \( q(x) = \int dy f(x - y)\psi(y) \). For other notations, we refer to (6). We remark that our choice of the chiral-invariant vector part involves also the axial vector part, and the vector (and axial vector) terms \( \sim (\bar{u}\gamma^\mu u)^2 + (\bar{d}\gamma^\mu d)^2 \) have no flavor mixing term. We should mention that the other chiral invariant vector interaction \( (\bar{q}\gamma^\mu q)^2 \) as adopted in (6) has a flavor mixing and may cause different density dependence of the phase diagram from those given in the present work. The nonlocal interaction is adopted to deal with the one loop ultraviolet divergence for the calculation of the Meissner mass squared at finite temperature. We take a Lorentzian-type form factor \( f^2(p = |p|) = g(p) = \frac{1}{1 + (\frac{p}{\Lambda^2})^2} \), with \( \Lambda = 10 \), where \( f(p) \) is the Fourier transformation of the form factor \( f(x) \). Other three model parameters are determined by the vacuum physical quantities of the pion mass \( M_\pi = 135 \) MeV, pion decay constant \( f_\pi = 92.4 \) MeV and the quark condensate \(-\langle \bar{u}u \rangle^{1/3} \approx 250 \) MeV.

We notice that the net quark chemical potential becomes dynamical as given by \( \tilde{\mu}_\alpha(p) = \mu_\alpha - 4G_V \rho_\alpha(g(p)) \), where \( \rho_\alpha \) being the quark density of the flavor \( \alpha \). It is to be noted that the induced chemical potential, \(-4G_V \rho_\alpha(g(p)) \) and hence the dynamical quark chemical potential \( \tilde{\mu}_\alpha(p) \) for \( u \) and \( d \) quarks can be different from each other because the electric-charge neutrality makes \( \mu_d \) larger than \( \mu_u \) and hence \( \rho_d \simeq \rho_u \). Thus the mismatch between the effective chemical potentials of \( u \) quark and \( d \) quark becomes \( \Delta \tilde{\mu} = \frac{1}{2}(\mu_c - 4G_V(\rho_d - \rho_u)g(p)) \), which tells us that the difference in the \( u \) and \( d \) quark densities in turn makes smaller the mismatch of their effective chemical potentials, thanks to the vector interaction. This effect is found to play an important role for the stability against the color magnetic instability.

In the left panel of Fig. 1 we show an example of the phase diagram with a finite vector coupling under the charge neutrality: We use the abbreviations NG, CSC, COE, and NOR refer to the hadronic (Nambu-Goldstone) phase with \( \langle \bar{q}_a q_a^b \rangle \equiv \sigma_\alpha \neq 0 \) and \( \langle (\bar{q}c)^2 \rangle \equiv \Delta = 0 \), the CSC phase with \( \Delta \neq 0 \) and \( \sigma_{u,d} = 0 \), the coexisting phase with \( \sigma_{u,d} \neq 0 \) and \( \Delta \neq 0 \), and the normal phase with \( \sigma_{u,d} = \Delta = 0 \), respectively.
respectively, although they have exact meanings only in the chiral limit. We note that there appear four critical points, which are denoted by E, F, G and H. With an increase of the vector coupling, the lower two critical points, G and H disappear, while the upper two critical points, F and G, remain in the phase diagram. One should mention that the critical point H is located on the border between the stable region and the unstable region, while other critical points are free from the chromomagnetic instability.

A detailed analysis\cite{3} shows that for the parameter set which gives $M(p = 0) = 367.5$ MeV and the standard diquark coupling, five different types of chiral critical point structures may exist, and the number of the critical points changes as $1 \rightarrow 2 \rightarrow 4 \rightarrow 2 \rightarrow 0$ with the vector coupling being increased.

We should notice here that the repulsive vector interaction also suppresses the magnitude of the diquark condensate due to the reduced effective quark chemical potential. However, the direct effect of the vector interaction on $\delta \mu$ is more significant than that on $\Delta$, in particular for finite temperature. Thus, as shown in the right panel of Fig.\ref{fig:phase_diagram}, we reach a significant findings that an increased vector coupling tends to shrink the unstable region toward the high-density and lower-temperature region. Needless to say, however, the vector interaction may not totally remove the unstable region from the phase diagram and hence other mechanism will be still necessary for a thorough cure of the magnetic instability.

The above results obtained for the two-flavor case is not essentially altered even when the strange quark is taken into account\cite{3}, as shown in Fig.\ref{fig:phase_diagram_strange} where the so-called Kobayashi-Maskawa-'tHooft(KMT) term\cite{22,32} is also incorporated

$$L_{KMT} = -K \left\{ \det_f \left[ \bar{\psi} \left( 1 + \gamma_5 \right) \psi \right] + \det_f \left[ \bar{\psi} \left( 1 - \gamma_5 \right) \psi \right] \right\}. \quad (2.2)$$

We should, however, remark that the number of the critical points is sensitive to
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the ratio $G_V/G_S$ and other parameters; other choice of the ratio and other parameters would lead to different phase structures with a varying number of the critical points. The message of such results we should have is that the QCD matter around the phase boundary is very soft for the density fluctuations combined with the formation of the chiral and diquark condensates along the critical line when the color superconductivity is incorporated.

§3. Density and energy fluctuations around QCD CP

Now apart from possible variants of the QCD phase diagram, let us examine what would be good signatures or observables that reflect the existence of the QCD CP. Motivated by the fact that QCD CP belongs to the same universality class as the liquid-gas transition, Minami and Kunihiro have analyzed the dynamical density fluctuations using various dissipative relativistic fluid dynamic equations irrespective of the first or second order ones. The dynamical structure factor (or spectral function) for the density fluctuation is calculated to be

$$S_{nn}(k, \omega) = \frac{\gamma - 1}{\gamma} \frac{2 \Gamma_R k^2}{\omega^2 + \Gamma_R^2 k^4} + \frac{1}{\gamma} \left\{ \frac{\Gamma_B k^2}{(\omega - c_s k)^2 + \Gamma_B^2 k^4} + (\omega \rightarrow -\omega) \right\} (3.1)$$

where $\Gamma_R = \kappa / (n_0 c_p)$ and $\Gamma_B = \frac{1}{2} [\Gamma_R (\gamma - 1) + \nu_l] + \frac{1}{2} \zeta^2 T_0 (\frac{\kappa}{w_0} - 2 \Gamma_R \alpha_P)$ with $\nu_l = (\zeta + 4 \eta / 3) / w_0$ being the relativistic longitudinal kinetic viscosity. The enthalpy density in the equilibrium is denoted by $w_0$, while $\gamma = c_p / c_v$ denotes the ratio of the specific heats at constant pressure and volume. We refer to (11) for other notations. The spectral function has three peaks at frequencies $\omega = 0$ and $\omega = \pm c_s k$, which corresponds to the entropy fluctuations (Rayleigh peak), and mechanically induced density fluctuation (Brillouin peaks), respectively. Notice that the pre-factor for the spectral function for the sound modes is proportional to $1/\gamma$, which tends to vanish when approaching the CP: $c_p$ behaves like $\xi^2 - \eta$ in terms of the correlation length $\xi$ and the critical exponent $\eta \sim 0.03$. Thus one sees that the mechanical density fluctuations are attenuated owing to the divergence of the correlation length $\xi$ around
the QCD CP. On the other hand, the entropy fluctuation in turn is enhanced and tends to makes a single peak around the QCD CP in the dynamical structure factor of the density fluctuations.

![Graph showing spectral function](image)

**Fig. 3.** Left panel: The spectral function at \( t \equiv (T - T_c)/T_C = 0.5 \) and \( k = 0.1 \) [1/fm]. The solid line represents the results using the Landau equation (energy frame) and Israel-Stewart equation. The dashed line represents the result using a fluid dynamic equation in the particle frame. The strength of the Brillouin peaks becomes small due to the singularity of the ratio of specific heats. Right panel: The spectral function at \( t = 0.1 \) and \( k = 0.1 \) [1/fm]. The sound modes die out when approaching the CP, irrespective of the relativistic fluid dynamic equations used. Taken from 11).

Such an attenuation of the mechanical density fluctuations may lead to a suppression or even total disappearance of Mach cone at the QCD CP. If the Mach cone formation is confirmed at some incident energy in relativistic heavy-ion collisions, possible disappearance or strong suppression of a Mach cone along with the lowering of the incident energy can be a signal of the existence of the critical point, because it may mean that the created matter should have gone through the critical region of the CP.

Explicit calculations with equation of motion which admits the existence of the critical point is necessary for confirm the fate of Mach cone formation. To make a direct connection with experimental observations, we should analyze the density fluctuations with the expanding back ground.

**§4. Summary and concluding remarks**

In the first part, after emphasizing that the repulsive vector interaction should exist between the quarks, we have shown that there is a possibility that the QCD phase diagram may have multiple critical points when the color superconductivity and the vector interaction are incorporated in the mean-field level. The message of this findings is that the QCD matter in the vicinity of the phase boundary at low or moderate temperature is very soft for the formation of diquark and chiral condensates combined with the baryonic density, as described by \( a\bar{q}q + b\bar{q}q + c\bar{q}q \), along the critical line. An analysis of such a possibility may involve a dynamical Hartree(-Fock)-Bogoliubov theory in the relativistic kinematics, which should be an interesting theoretical challenge.

It was also shown that the vector interaction as given by Eq. (2.1) suppresses the
chromomagnetic instability related to asymmetric homogeneous 2CSC phase: With increasing vector interaction, the unstable region associated with chromomagnetic instability shrinks towards lower temperature and higher chemical potential. That means that the vector interaction can at least partially resolve the chromomagnetic instability problem.

The dynamical density fluctuations have been analyzed using relativistic fluid dynamics, in which the entropy fluctuations are automatically incorporated. The sound modes due to density fluctuations are attenuated, and the Rayleigh peak due to the entropy fluctuation in turn gets enhanced around the QCD critical point. The attenuation of the sound mode may lead to the suppression or even total disappearance of Mach cone at the CP. Once the Mach cone formation is confirmed in experiments of heavy-ion collisions with a high incident energy, possible disappearance or strong suppression along with the variation of the initial energy can be a signal of the existence of the critical point and the created matter went through the critical region of the CP. It is clear that there need further explicit calculations for a confirmation of this conjecture.

The present analysis is made in the fluid dynamical regime. In the very vicinity of the critical point, we need an analysis beyond the fluid dynamics to take into account the non-linear effects. For this purpose, the mode-mode coupling theory\textsuperscript{24} and/or dynamical renormalization group technique\textsuperscript{35} should be applied\textsuperscript{36, 37}.

There are other QCD phase transitions than those discussed in the present report, and if the phase transition is of a second order or close to that, there should exist specific soft modes, which may be easily thermally excited. Such a bosonic excitation with a small mass could in turn affect the quark spectral functions: In the high-temperature phase above the critical point $T_c$ of the color superconductivity, the quark spectral function naturally exhibits a pseudo gap\textsuperscript{38} owing to the coupling with the pre-formed diquark fluctuations above $T_c$.\textsuperscript{39} The chiral soft modes\textsuperscript{40} may also lead to a complicated quark spectral function with three-peak structure above but around the critical temperature of the chiral transition\textsuperscript{41}.

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