ON SOME PROPERTIES OF THE FUNCTORS $\mathcal{F}_P^G$ FROM LIE ALGEBRA TO LOCALLY ANALYTIC REPRESENTATIONS

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ABSTRACT. For a split reductive group $G$ over a finite extension $L$ of $\mathbb{Q}_p$, and a parabolic subgroup $P \subset G$ we examine functorial properties of the functors $\mathcal{F}_P^G$ introduced in [22, 21]. We discuss the aspects of faithfulness, projective and injective objects, Ext-groups and some kind of adjunction formulas. Here we apply the (naive) Jacquet functor and a more detailed study of the category $\mathcal{O}^P$ introduced in [21].

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1. INTRODUCTION

This paper is a continuation of the work done in [20, 21, 22]. In loc.cit. we constructed locally analytic representations in $K$-vector spaces of a $p$-adic reductive Lie group $G$ by introducing certain bi-functors $\mathcal{F}_P^G : \mathcal{O}^P \times \text{Rep}^\infty_\text{ad}(L_P) \to \text{Rep}^{\text{loc.an.}}_K(G)$. Here $P$ denotes a parabolic subgroup and $\mathcal{O}^P$ is a sort of locally analytic lift of the BGG-category $\mathcal{O}^p$ where $p = \text{Lie} P$. Further $\text{Rep}^\infty_\text{ad}(L_P)$ is the category of smooth admissible representations of the Levi group $L_P$. We proved among others that it is exact in both arguments and gave an irreducibility criterion for the objects lying in the image of $\mathcal{F}_P^G$. From these properties one can derive a Jordan-Hölder series of any locally analytic representation $\mathcal{F}_P^G(M, V)$ from the corresponding series of $M$ and $V$. 1
In this paper we want to concentrate on functorial properties of these functors for a split group \( G \). We shall show that they behave fully faithful if the objects of the category \( \mathcal{O}^P \) are integral (i.e., they are contained in the subcategory \( \mathcal{O}_\text{alg}^P \) of modules such that all non-zero weight spaces belong to integral weights) or generalized Verma modules. This aspect has been considered by Morita in the case of \( G = \text{SL}_2 \), cf. [14, 15, 16, 17]. Concretely, we shall show:

**Theorem 1:** Let \( M_1, M_2 \in \mathcal{O}^P \). Suppose that we are in one of the following situations:

i) \( M_2 = U(g) \otimes_{U(p)} W \) is a generalized Verma module for some finite-dimensional locally analytic \( L_P \)-representation \( W \).

ii) \( M_1, M_2 \) are contained in the subcategory \( \mathcal{O}_\text{alg}^P \).

Then the map

\[
\text{Hom}_{\mathcal{O}^P}(M_1, M_2) \to \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1))
\]

\[
f \mapsto \mathcal{F}_P^G(f)
\]

is bijective (where \( \mathcal{F}_P^G(M) := \mathcal{F}_P^G(M, 1) \) for the trivial \( L_P \)-representation 1).

To prove this statement we make use of the (naive) topological Jacquet functor of locally analytic representations and more generally of an analogue of the Casselman-Jacquet functor \( \mathcal{G}_P^G : U \mapsto \lim_{\to k} H^0(u_k, U') \) which behaves almost like a section for \( \mathcal{F}_P^G \). This topic is a continuation of the theory started in [19, 2].

By the above theorem we can characterize projective and injective objects which lie in the essential image \( \mathcal{F}_\text{alg}^P \) of the functor \( \mathcal{F}_P^G : \mathcal{O}_\text{alg}^P \to \text{Rep}_{\text{loc.an.}}^G (G) \). More precisely, it follows that \( M \in \mathcal{O}_\text{alg}^P \) is projective (resp. injective) as an object in \( \mathcal{O}^P \) if and only if \( \mathcal{F}_P^G(M) \) is injective (resp. projective) in \( \mathcal{F}_\text{alg}^P \). Hence if we denote for a given integer \( i \geq 0 \), by \( \text{Ext}^i_{\mathcal{F}_\text{alg}^P} \) the corresponding Ext-group then the natural morphism

\[
\text{Ext}^i_{\mathcal{O}^P}(M_1, M_2) \to \text{Ext}^i_{\mathcal{F}_\text{alg}^P}(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1))
\]

is bijective. These Ext-groups are of course different from those considered more generally in the category of locally analytic \( G \)-representations, cf. [13]. These can be seen as an analogue of relating the groups \( \text{Ext}^i_\mathfrak{g}(M_1, M_2) \) and \( \text{Ext}^i_\mathcal{O}(M_1, M_2) \) for two objects \( M_1, M_2 \in \mathcal{O} \).

For considering also smooth contributions in this context, we extend \( \mathcal{F}_P^G \) to a bi-functor \( \mathcal{F}_P^G : \mathcal{O}^P \times \text{Rep}_{\text{loc.an.}}^\infty (L_P) \to \text{Rep}_{\text{loc.an.}}^G (G) \) where \( \text{Rep}_{\text{loc.an.}}^\infty (L_P) \) denotes the category of smooth \( L_P \)-representations. The latter object has as is well known enough injectives and projectives. We let \( \text{Inf}^{-1} \mathcal{F}_P^G \) be the smallest abelian subcategory of \( \text{Rep}_{\text{loc.an.}}^G (G) \) which contains
the essential images of all bi-functors $\mathcal{F}^G_Q$ with $Q \supset P$. It turns out that $\mathcal{F}^P\mathcal{P}$ has enough injective and projective objects. More precisely, we deduce this fact from the following statement.

**Theorem 2:** Let $M \in \mathcal{O}^G_P$ be a projective (resp. injective) object and let $V$ be an injective (resp. projective) smooth $L_P$-representation. Then $\mathcal{F}^G_P(M, V)$ is injective (resp. projective) in $\mathcal{F}^P\mathcal{P}$.

As an application we are able to determine extensions of generalized Steinberg representations in the category $\mathcal{F}^B\mathcal{B}$. For a parabolic subgroup $P \subset G$ the associated representation is given by the quotient $V^G_P = \text{Ind}^G_P(1)/\sum_{Q \supset P} \text{Ind}^G_Q(1)$ where $\text{Ind}^G_P(1)$ is the locally analytic induction with respect to the trivial $P$-representation. For a subset $I \subset \Delta$ of a fixed set of simple roots, let $P_I$ be the corresponding standard parabolic subgroup. The next result has the same structure as in the smooth setting [5, 18].

**Theorem 3:** Let $G$ be semi-simple. Let $I, J \subset \Delta$. Then

$$\text{Ext}^i_{\mathcal{F}^B}(V^G_{P_I}, V^G_{P_J}) = \begin{cases} K; & |I \cup J \setminus I \cap J| = i \\ (0); & \text{otherwise} \end{cases}.$$  

Finally we deduce from the naive Jacquet functor applied to different Borel subgroups lying in the same apartment an adjunction formula (in the sense of Bernstein). Let $U_B$ be the unipotent radical of a fixed Borel subgroup $B$. If we denote for a given $G$-representation $V$ by $V_{U_B}$ its (naive) topological Jacquet module then the map below is defined as follows: For an element $f$ of the LHS, the corresponding element on the RHS is given by the composition of the inclusion $((w_0\tau\chi)^{-1})_{U_B} \hookrightarrow I^G_B(\chi)^{-1})_{U_B}$ with the map $f_{U_B}$.

**Theorem 4:** Let $\chi$ be a dominant locally analytic character of $T$ and let $M \in \mathcal{O}^Bw$. Then

$$\text{Hom}_G(I^G_B(\chi), \mathcal{F}^G_{Bw}(M)) = \text{Hom}_T(((w_0\tau\chi)^{-1})_{U_B}, \mathcal{F}^G_{Bw}(M_w)_{U_B})$$  

Here $M_w$ denotes the largest Verma module quotient of $M$.

**Notation and conventions.** We denote by $p$ a prime number and consider fields $L \subset K$ which are both finite extensions of $\mathbb{Q}_p$. Let $O_L$ and $O_K$ be the rings of integers of $L$, resp. $K$, and let $| \cdot |_K$ be the absolute value on $K$ such that $|p|_K = p^{-1}$. The field $L$ is our ”base field”, whereas we consider $K$ as our ”coefficient field”. For a locally convex $K$-vector space $V$ we denote by $V'_b$ its strong dual, i.e., the $K$-vector space of continuous
linear forms equipped with the strong topology of bounded convergence. Sometimes, in particular when \( V \) is finite-dimensional, we simplify notation and write \( V' \) instead of \( V'_b \). All finite-dimensional \( K \)-vector spaces are equipped with the unique Hausdorff locally convex topology.

We let \( G_0 \) be a split reductive group scheme over \( O_L \) and \( T_0 \subset B_0 \subset G_0 \) a maximal split torus and a Borel subgroup scheme, respectively. We denote by \( G_0, B_0, T_0 \) the base change of \( G_0, B_0 \) and \( T_0 \) to \( L \). By \( G_0 = G_0(O_L), B_0 = B_0(O_L) \), etc., and \( G = G(L), B = B(L) \), etc., we denote the corresponding groups of \( O_L \)-valued points and \( L \)-valued points, respectively. Standard parabolic subgroups of \( G \) (resp. \( G \)) are those which contain \( B \) (resp. \( B \)). For each standard parabolic subgroup \( P \) (or \( P \)) we let \( L_P \) (or \( L_P \)) be the unique Levi subgroup which contains \( T \) (resp. \( T \)) and \( U_P \) (or \( U_P \)) its unipotent radical. Finally, Gothic letters \( g, p \), etc., will denote the Lie algebras of \( G, P \), etc.: \( g = \text{Lie}(G), t = \text{Lie}(T), b = \text{Lie}(B), p = \text{Lie}(P), l_P = \text{Lie}(L_P) \), etc. Base change to \( K \) is usually denoted by the subscript \( K \), for instance, \( g_K = g \otimes_L K \).

We make the general convention that we denote by \( U(g), U(p) \), etc., the corresponding enveloping algebras, after base change to \( K \), i.e., what would be usually denoted by \( U(g) \otimes_L K, U(p) \otimes_L K \), and so on. All distribution algebras appearing in this paper are tacitly assumed to be distribution algebras with coefficient field \( K \), and we write \( D(H) \) for the distribution algebra \( D(H, K) \).

Denote by \( \text{Rep}_K^{\text{loc.an.}}(G) \) the category of locally analytic representations of \( G \) on barrelled locally convex Hausdorff \( K \)-vector spaces.

2. A review of earlier results

We repeat the construction of the functors together with its main properties, cf. \[22,21\], in a nutshell.

For a parabolic subgroup \( P \subset G \), let \( \mathcal{O}^P \) be the corresponding BGG-category of \( U(g) \)-modules of type \( p \) in the usual sense. Let \( D(g, P) \) be the subring of \( D(G) \) generated by \( U(g) \) and \( D(P) \) inside \( D(G) \). Let \( \mathcal{O}^P \) the category whose objects are pairs \( M = (M, \tau) \) where \( M \in \mathcal{O}^p \) and \( \tau : P \to \text{End}_K(M)^* \) is a homomorphism such that there is an increasing union \( M = \bigcup_{i\in\mathbb{N}} M_i \) by finite-dimensional locally analytic \( P \)-stable subspaces such that the derived action of \( p \) coincides with the induced action and such that the actions of \( P \) and \( g \) are compatible in the obvious sense, i.e., any \( M \in \mathcal{O}^P \) is equipped with a \( D(g, P) \)-module structure. Morphisms are then just \( D(g, P) \)-module homomorphisms.
As in loc.cit. we denote by

\[ \omega : \mathcal{O}_P^P \rightarrow \mathcal{O}_P^P, \]

\[ M = (M, \rho) \mapsto M , \]

the forgetful functor.

Let \( \mathcal{O}^P_{\text{alg}} \) be the subcategory of \( \mathcal{O}_P^P \) consisting of objects with integral weights. It is shown in [22] that every object \( M \in \mathcal{O}_{\text{alg}}^P \) carries a canonical action of \( P \) so that one has a fully faithful functor \( \mathcal{O}^P_{\text{alg}} \rightarrow \mathcal{O}^P \) which gives rise to a section of \( \omega \) with respect to this subcategory. For this reason we also write sometimes just \( M \) instead of \( M \) if it comes from \( \mathcal{O}^P_{\text{alg}} := \mathcal{O}^P_{\text{alg}} \). The category \( \mathcal{O}_P^P \) is abelian, artinian and noetherian [21].

For any \( M = (M, \tau) \in \mathcal{O}_P^P \), there is a finite-dimensional locally analytic \( P \)-representation \( W \subset M \) which generates \( M \) as a \( U(g) \)-module. Thus we get an exact sequence

\[ 0 \rightarrow \mathfrak{d} \rightarrow U(g) \otimes_{U(p)} W \rightarrow M \rightarrow 0 \]

Let \( \text{Ind}_{\mathcal{P}}^G(W') \) be the locally analytic induction of the dual space \( W' \). There is a pairing

\[ \langle \cdot, \cdot \rangle_{C^\text{an}(G,K)} : \langle (D(G) \otimes_{D(P)} W) \otimes_{K} \text{Ind}_{\mathcal{P}}^G(W') \rightarrow C^\text{an}(G, K) \]

\[ (\delta \otimes w) \otimes f \rightarrow [g \mapsto \delta(x \mapsto f(gx)(w))] \]

which extends for any smooth admissible \( L_P \) representation, to a pairing

\[ \langle \cdot, \cdot \rangle_{C^\text{an}(G,V)} : \langle (D(G) \otimes_{D(P)} W \otimes_{K} V') \otimes_{K} \text{Ind}_{\mathcal{P}}^G(W' \otimes V) \rightarrow C^\text{an}(G, K). \]

Here we recall that \( V \) is equipped with the locally convex topology as follows, cf. [•]. Write \( V = \bigcup_H V^H \) as a union over its finite-dimensional fixsubspaces \( V^H \) where \( H \) ranges over all compact open subgroups \( H \) of \( G \). Then each \( V^H \) has a canonical Banach space structure and \( V \) is supplied with the locally convex limit topology. We set

\[ F^G_P(\mathcal{M}, V) = \text{Ind}_{\mathcal{P}}^G(W' \otimes_{K} V)^0 \]

\[ (2.1) \]

\[ = \{ f \in \text{Ind}_{\mathcal{P}}^G(W' \otimes_{K} V) | \forall \mathfrak{d} : \langle \mathfrak{d}, f \rangle_{C^\text{an}(G,V)} = 0 \} . \]

This object is a well defined locally analytic \( G \)-representation and gives rise to a bi-functor functor

\[ F^G_P : \mathcal{O}_P^P \times \text{Rep}_{K}^{\infty, a}(L_P) \rightarrow \text{Rep}_{K}^{\text{loc, an}}(G). \]
If $V = 1$ denotes the trivial representation, then we simply write $F^G_P(M)$ for $F^G_P(M, V)$. Then there are canonical isomorphisms

$$D(G) \otimes_{D(g, P)} M \cong D(G_0) \otimes_{D(g, P_0)} M \cong F^G_P(M).$$

For all $M \in \mathcal{O}^P$, and for all smooth admissible $L_P$-representations $V$ the $G$-representation $F^G_P(M, V)$ is admissible. If $V$ is of finite length, then $F^G_P(M, V)$ is even strongly admissible.

**Proposition 2.1.** a) The bi-functor $F^G_P$ is exact in both arguments.

b) (PQ-formula) If $Q \supset P$ is a parabolic subgroup, $q = \text{Lie}(Q)$, and $M$ an object of $\mathcal{O}^Q$, then

$$F^G_P(M, V) = F^G_Q(M, i^{L_Q}_{L_P(L_Q \cap U_P)}(V)),$$

where $i^{L_Q}_{L_P(L_Q \cap U_P)}(V) = i^Q_P(V) = \text{ind}_P^Q(V)$ denotes the corresponding induced representation in the category of smooth representations.

**Theorem 2.2.** Let $M = (M, \tau) \in \mathcal{O}^P$ be such that $M$ is simple, and suppose that $p$ is maximal for $M$. Let $V$ be a smooth and irreducible $L_P$-representation. Then $F^G_P(M, V)$ is topologically irreducible as a $G$-representation.

### 3. The category $\mathcal{O}^P$ revisited

This section is about some further properties of the category $\mathcal{O}^P$. In particular we discuss the question of simple objects in it. Some treated aspects can be also found in [1].

We start with an observation which is true for the underlying categories $\mathcal{O}^Q, \mathcal{O}^P$ and which was already proved in [1, Cor. 3.8].

**Lemma 3.1.** Let $Q \supset P$ be parabolic subgroups of $G$. Then the restriction functor $\mathcal{O}^Q \to \mathcal{O}^P$ is fully faithful.

**Proof.** We need to show that any $P \rtimes g$-module homomorphism $f : M \to N$ of objects $M, N \in \mathcal{O}^Q$ is in fact $Q$-equivariant. But $Q$ is generated as an abstract group by its unipotent elements together with the subgroup $T \subset P$. The action of the unipotent elements is induced by that of the corresponding nilpotent elements in the Lie algebra.

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1Here we assume that if the root system $\Phi = \Phi(g, t)$ has irreducible components of type $B, C$ or $F_4$, then $p > 2$, and if $\Phi$ has irreducible components of type $G_2$, we assume that $p > 3$. 
Since $Q$ acts on $M$ and $N$, we see that $f$ as a $P$-equivariant morphism is automatically $Q$-equivariant.

The following statement is the analogue of the classical situation [10, Proposition 9.3] dealing with Lie algebra representations in the category $\mathcal{O}^p$.

**Proposition 3.2.** Let $Q \supset P$ be parabolic subgroups of $G$.

i) Let $M \in \mathcal{O}^Q$ and let $N \in \mathcal{O}^P$ be a subobject or subquotient, respectively. Then $N \in \mathcal{O}^Q$.

ii) The category $\mathcal{O}^Q$ is closed under extensions in $\mathcal{O}^P$.

iii) Let $M \in \mathcal{O}^P$. Then $M \in \mathcal{O}^Q \iff$ all simple subquotients $L$ of $M$ are in $\mathcal{O}^Q$.

**Proof.**

i) By [10, ] we deduce that $N \in \mathcal{O}^q$. We apply again the reasoning of the proof in Lemma 3.1 since we only have to show that $N$ is closed with respect to the $Q$-action.

ii) Let $0 \to M_1 \to M \to M_2 \to 0$ be an extension in $\mathcal{O}^P$ such that $M_i \in \mathcal{O}^Q$, $i = 1, 2$. We consider the induced extension $0 \to \omega(M_1) \to \omega(M) \to \omega(M_2) \to 0$. Since the category $\mathcal{O}^q$ is closed under extensions [9] we deduce that $M = \omega(M)$ is an object of $\mathcal{O}^q$. We choose for $i = 1, 2$, finite-dimensional locally analytic $Q$-representations $W_i$ which generate $M_i$ as a $U(\mathfrak{g})$-module. Since $M$ is an object of $\mathcal{O}^q$ we may choose a a $\mathfrak{q}$-subspace $Z \subset M$ which maps bijectively onto $W_2$ (since $\mathfrak{q}$ is reductive). Hence the locally analytic $L_Q$-action on $W$ lifts to one on $Z$. It follows that $Z \oplus W_1$ is a locally analytic $Q$-representation which generates $M$ as a Lie algebra representation. The claim is an immediate consequence of that fact.

iii) follows from i) and ii).

From Proposition 3.2 i) we immediately deduce:

**Corollary 3.3.** Let $P \subset Q$ be parabolic subgroups of $G$ and let $M \in \mathcal{O}^Q$ be a simple object. Then $M$, considered as an object in $\mathcal{O}^P$ is simple, as well. In particular, the JH-series of an arbitrary object $M \in \mathcal{O}^Q$ in the category $\mathcal{O}^Q$ is the same as in $\mathcal{O}^P$ for any parabolic subgroup $P \subset Q$.

Let $M$ be an object in $\mathcal{O}^B$. If $M$ is even contained in $\mathcal{O}^P$, then by definition $\omega(M) \in \mathcal{O}^q$. The converse does not need to hold. This leads to the following notion.

**Definition 3.4.** Let $M \in \mathcal{O}^B$. We say that $M$ is equimaximal if for any parabolic $P$ we have $M \in \mathcal{O}^P$ if and only if $\omega(M) \in \mathcal{O}^q$.

By [9, Proposition 9.3] it suffices to check this definition for a single parabolic subgroup.
In the remainder of this section we want to determine some simple objects in $\mathcal{O}^B$. Let $\lambda : T \to K^*$ be a locally analytic character with derivative $d\lambda \in \text{Hom}(t, K)$. Let

$$M = M(d\lambda) = U(g) \otimes_U K_{d\lambda} \in \mathcal{O}$$

be the ordinary Verma module with respect to $d\lambda$. By integrating the action of $u_B$ to an action of $U_B$ on $M$ one verifies that there is a unique object $\overline{M}(\lambda)$ in $\mathcal{O}^B$ with the properties that

$$\omega(\overline{M}(\lambda)) = M(d\lambda)$$

and such $B$ acts on the highest weight vector $1 \otimes 1 \in M(d\lambda)$ via the locally analytic character $\lambda$. Let $L(d\lambda) \in \mathcal{O}$ be the unique simple quotient of $M(d\lambda)$.

**Lemma 3.5.** The $g$-representation $L(d\lambda) \in \mathcal{O}$ lifts to an object $\overline{L}(\lambda)$ of $\mathcal{O}^B$ which is moreover simple.

**Proof.** The last statement follows as the Lie algebra representation $\omega(\overline{L}(\lambda)) = L(d\lambda)$ is simple. Since by definition $\lambda$ is a lift of $d\lambda$ all weights spaces of $L(d\lambda)$ lift to $T$-representations. Further the action of $\text{Lie}(U_B)$ lifts always to an action of $U_B$ as elements of $\text{Lie}(U_B)$ are nilpotent. Hence we see that $L(d\lambda)$ lifts to an object $\overline{L}(\lambda)$ of $\mathcal{O}^B$. $\square$

The representations $M(d\lambda)$ and $L(d\lambda)$ are not uniquely determined by their lifts $\overline{M}(\lambda)$, $\overline{L}(\lambda)$. Indeed let $\chi$ be a smooth character of $T$. Then we may consider it as an object of $\mathcal{O}^B$ via inflation and with the trivial $g$-structure. Then $\overline{M}(\lambda) \hat{\otimes} \chi$ has the structure of a $D(g, B)$-module which is a lift of $M(d\lambda)$ as well. This observation is part of the next result.

**Lemma 3.6.** Let $\lambda : T \to K^*$ be a locally analytic character and let $\chi : T \to K^*$ be a smooth character. Then $\overline{M}(\lambda \cdot \chi) \in \mathcal{O}^B$. Moreover we have identities $\overline{M}(\lambda \cdot \chi) = \overline{M}(\lambda) \hat{\otimes} \chi$, $\overline{L}(\lambda) \hat{\otimes} \chi = \overline{L}(\lambda \cdot \chi)$ and $\omega(\overline{M}(\lambda)) = \omega(\overline{M}(\lambda \cdot \chi))$, $\omega(\overline{L}(\lambda)) = \omega(\overline{L}(\lambda \cdot \chi))$. Any object $\overline{M} \in \mathcal{O}^B$ with $\omega(\overline{M}) = M(d\lambda)$ resp. $\omega(\overline{M}) = L(d\lambda)$ is of the previous shape.

**Proof.** The proof is left as an exercise. $\square$

On the other hand we have a converse statement concerning the objects $\overline{L}(\lambda)$. Here we consider the following subcategory.

**Definition 3.7.** We denote by $\mathcal{O}^B_d$ the full subcategory of $\mathcal{O}^B$ consisting of objects such that the torus $T$ acts diagonalizable.

**Proposition 3.8.** Let $\overline{M} = (M, \tau) \in \mathcal{O}^B_d$ be a simple object. Then there is some locally analytic character $\lambda$ of $T$ such that $\overline{M} \cong \overline{L}(\lambda)$. 
ON SOME PROPERTIES OF THE FUNCTORS $F^\|_d$ ...

Proof. We need amongst other things to show that $\omega(M) = M$ is simple as a $g$-module. For this let $N \subset M$ be a simple submodule. Hence $N = L(\mu)$ for some $\mu \in \text{Hom}(t, K)$. But since $M$ comes from an object in $O^B_d$ we may choose a locally analytic lift of $\mu$ so that we see that $N$ lifts to an object $N$ in $O^B$. Hence $N = M$ and consequently $N = M$. The character $\lambda$ is induced by the locally analytic action of $T$ on the one-dimensional vector space $M^{U_B}$.

More generally we consider for a standard parabolic subgroup $P \subset G$ the full subcategory $O^P_d$ of $O^P$ consisting of objects which lie in $O^B_d$, too. Proposition 3.2 has then for these subcategories the following analogy.

Proposition 3.9. Let $Q \supset P$ be standard parabolic subgroups of $G$.

i) Let $M \in O^Q_d$ and let $N \in O^P_d$ be a subobject or subquotient, respectively. Then $N \in O^Q_d$.

ii) The subcategory $O^Q_d$ is closed under extensions in $O^P_d$.

iii) Let $M \in O^P_d$. Then $M \in O^Q_d \iff$ all simple subquotients $L$ of $M$ are in $O^Q_d$.

Proof. The proof follows from Proposition 3.2 and the definition of $O^B_d$. □

Corollary 3.10. Let $M$ be an object of $O^B_d$. Then it has a finite $JH$-series such that each simple subquotient is isomorphic to some $L(\lambda)$ where $\lambda$ is a locally analytic character of $T$.

Proof. This follows from Proposition 3.9 and Proposition 3.8. □

As in [10, Section 9] we set for a subset $I \subset \Delta$, $\Lambda^+_I = \{\lambda \in t^* | \langle \lambda, \alpha \rangle \in \mathbb{Z}^+ \forall \alpha \in I\}$.

Corollary 3.11. Let $P = P_I$ with $I \subset \Delta$ be a standard parabolic subgroup. Then $L(\lambda) \in O^P \Rightarrow d\lambda \in \Lambda^+_I$.

Proof. If $L(\lambda) \in O^P$ then $L(d\lambda) \in O^P$. Then the statement follows from [10, 9.3 e)]. □

Definition 3.12. A locally analytic character $\chi$ is called dominant, if $d\lambda \in t^*_K$ is dominant, i.e. if $d\lambda \in \Lambda^+_A$.

Corollary 3.13. Let $M \in O^G_d$ be a simple object. Then up to a locally analytic character of $G$, the object $M$ is induced by an algebraic $G$-representation.

Proof. We may write $M = L(\lambda)$ for a locally analytic character $\lambda$ of $T$. As $\lambda$ is dominant by the proof of Proposition 3.8, we may write $d\lambda = \eta_1 + \eta_2$ with an integral dominant weight $\eta_1 \in t^*_K$ and a weight $\eta_2 \in t^*_K$ which is induced by Lie$(G/G_{der})$ where $G_{der}$ is the
derived group of $G$. Now we choose lifts $\lambda_1$ of $\eta_1$ giving rise to an irreducible algebraic representation $L(\lambda_1)$. Then the locally analytic character $\lambda_2 := \lambda_1 \cdot \lambda_1^{-1}$ is a lift of $\eta_2$ and gives rise to a character of $G$. Then $M = L(\lambda_1) \otimes \lambda_2$.

**Remark 3.14.** The simple objects in $O^G_d$ are also considered in [1, 3.3] where they are called $\mathfrak{g}$-simple.

Because of the above lemma one cannot expect that simple objects are equimaximal. The following statement shows that this phenomena is the only possible one.

**Proposition 3.15.** Let $M \cong L(\lambda) \in O^B_d$ be a simple object. Then there is some locally analytic character $\zeta : T \to K^*$ and a smooth character $\chi : T \to K^*$ of $T$ such that $M \cong L(\zeta) \otimes_{K^*} \chi$ and such that $L(\zeta)$ is equimaximal. Moreover if $P$ is maximal for $M$, then $\chi$ is a character of $L_P$. The decomposition is unique up to twist by a smooth character of $L_P$.

**Proof.** Let $I \subset \Delta$ maximal such that $d\lambda \in \Lambda^+_I$. Then by [9, §9] the parabolic Lie algebra $\mathfrak{p}_I$ is maximal for $L(d\lambda)$. If $I = \emptyset$ there is nothing to prove by assumption resp. by Proposition 3.8. In the other extreme case $I = \Delta$, we know that $L(d\lambda)$ comes up to a locally analytic character of $G$ from an algebraic irreducible $G$-representation $L(\zeta)$. But then $\lambda$ and $\zeta$ differ by a smooth character $\chi$ and we are done, as well. So let $I$ be a proper subset of $\Delta$. Then we may write $L(d\lambda)$ as a quotient of a generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_I(\zeta)$ where $L_I(\zeta)$ lifts as in the case before to a finite-dimensional irreducible locally analytic representation of the Levi subgroup of $P_I$. Hence $L(\zeta)$ lifts to an object $L(\zeta) \in O^{P_I}$. Again by the same reasoning as above there is some smooth character $\chi$ such that $\lambda$ and $\zeta$ differ by $\chi$. \qed

4. Jacquet functors

The first part of this section deals with a generalization of results formulated in [19, 2], where the Jacquet functor of simple objects $F^G_P(M, V)$ with $M \in O^B_{alg}$ was discussed. We extend the known results to the categories $O^P_d$ and we consider also more generally non-simple objects in $O^B_{alg}$.

Let $P$ be a parabolic subgroup of $G$ with Levi decomposition $P = L_P U_P$. For a locally analytic $P$-representation $V$, let $V(U_P)$ be the subspace generated by the expressions $uv - v$, with $u \in U_P, v \in V$ and let $\overline{V(U_P)}$ be its topological closure which is a $P$-stable
subspace of $V$. Denote by
\[ \overline{H}_0(U_P, V) := V_{U_P} := V/V(U_P) \]
the corresponding quotient (the naive topological Jacquet module). It is the largest Hausdorff quotient of $V$ on which $U_P$ acts trivially.

**Lemma 4.1.** The space $\overline{H}_0(U_P, V)$ has the canonical structure of a locally analytic $P$-representation.

**Proof.** Since $V(U_P)$ is a closed subspace of $V$, the quotient is Hausdorff and again of compact type. Moreover the orbit maps $P \to \overline{H}_0(U_P, V)$ are clearly locally analytic since these are induced by the locally analytic orbit maps $P \to V$. \hfill \Box

On the other hand, let $V'$ be its dual which is a $K$-Fréchet space equipped with a continuous action of $P$. We let $H^0(U_P, V')$ be the subspace of $V'$ consisting of vectors which are fixed by $U_P$. This is a closed subspace so that $H^0(U_P, V')$ inherits the structure of a $K$-Fréchet space equipped with an action of $P$, as well.

**Lemma 4.2.** Under the duality pairing $V \times V' \to K$ the subspace $H^0(U_P, V')$ is the topological dual of $\overline{H}_0(U_P, V)$ as $P$-representations.

Let $Q$ be another parabolic subgroup with $P \subset Q$ and let $Q = L_Q \cdot U_Q$ be its Levi decomposition. In this sequel we want to determine for certain objects $\underline{M} \in \mathcal{O}_Q$ and smooth admissible $L_Q$-representations $V$ the $L_P$-representations $H^0(U_P, F_Q^G(\underline{M}, V'))$.

**Proposition 4.3.** Let $\underline{M} = (M, \tau)$ be an object of $\mathcal{O}_P$ and let $M_r = D_r(P, g)M$. We have an inclusion preserving bijection
\[ \left\{ \text{closed } U(I_P)\text{-invariant subspaces of } M_r \right\} \sim \left\{ \text{closed } U(I_P)\text{-invariant subspaces of } M \right\}. \]
\[ S \mapsto S \cap M \]
The inverse map is induced by taking the closure.

**Proof.** By [19] we have such an inclusion preserving bijection
\[ \left\{ \text{closed } U(t)\text{-invariant subspaces of } M_r \right\} \sim \left\{ \text{U(t)-invariant subspaces of } M \right\}. \]
\[ S \mapsto S \cap M \]
for $U(t)$-submodules. But for a closed $U(t)$-submodule $N \subset M_r$ the intersection $N \cap M$ is $I_P$-stable if and only if $N$ is $U(I_P)$-stable. Indeed, whereas one direction is obvious the other one follows by density arguments. The claim follows. \hfill \Box
Recall that if $M$ is a Lie algebra representation of $\mathfrak{g}$, then $H^0(\mathfrak{u}_Q, M) = \{ m \in M \mid \mathbf{r} \cdot m = 0 \ \forall \mathbf{r} \in \mathfrak{u}_Q \}$ denotes the subspace of vectors killed by $\mathfrak{u}_Q$. On the other hand we consider the quotient $H_0(\mathfrak{u}_Q, M) = M/\mathfrak{u}_Q M$. These are both $U(\mathfrak{q})$-modules.

**Corollary 4.4.** Let $\underline{M} = (M, \tau)$ be an object of $\mathcal{O}^P$. Then $H^0(\mathfrak{u}_P, M_r) = H^0(\mathfrak{u}_P, M)$. In particular, $H^0(\mathfrak{u}_P, M_r)$ is finite-dimensional.

**Proof.** We clearly have $H^0(\mathfrak{u}_P, M_r) \cap M = H^0(\mathfrak{u}_P, M)$. As $H^0(\mathfrak{u}_P, M_r)$ is closed in $M_r$ by the continuity of the action of $\mathfrak{g}$ and as $H^0(\mathfrak{u}_P, M)$ is finite-dimensional (!!!!) and therefore complete the statement follows by Proposition 4.3. $\blacksquare$

**Lemma 4.5.** Let $\underline{M} = (M, \tau)$ be an object of $\mathcal{O}^Q$ where $P \subset Q$ and let $V$ be a smooth admissible $L_Q$-representation. Then the identity

$$H^0(\mathfrak{u}_P, \mathcal{F}_Q^G(\underline{M}, V)) = H^0(\mathfrak{u}_P, \mathcal{F}_Q^G(\underline{M}')) \hat{\otimes}_K V'$$

is satisfied considered as Fréchet spaces.

**Proof.** The proof is the same as in [19] by replacing $\mathfrak{u}_B$ by $\mathfrak{u}_P$. $\blacksquare$

For $\underline{M} = (M, \tau) \in \mathcal{O}^Q$, we let $W \subset M$ be a finite-dimensional locally analytic $Q$-subrepresentation such that the map (a morphism in $\mathcal{O}^Q$)

$$\underline{M}(W) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W \to \underline{M}$$

is surjective. If $M$ is simple so that we may assume that $W$ comes via inflation from an irreducible $L_Q$-representation then $H^0(\mathfrak{u}_Q, M) = W$.

Now we are able to prove one of the main results of this section which is an analogue of a statement dealing with representations of real Lie groups and Harish-Chandra modules [8, 7].

**Theorem 4.6.** Let $\underline{M}$ be a simple equimaximal object of $\mathcal{O}^Q$ with $Q$ maximal for $\underline{M}$ and a finite-dimensional irreducible $L_Q$-representation $W$ as above. Let $V$ be a smooth admissible $L_Q$-representation. Then for $P \subset Q$ there are $L_P$-equivariant isomorphisms

$$H^0(U_P, \mathcal{F}_Q^G(\underline{M}, V)) = H^0(\mathfrak{u}_P, W) \otimes_K J_{U_P \cap L_Q}(V'),$$

and

$$\overline{\Pi}_0(U_P, \mathcal{F}_Q^G(\underline{M}, V)) = H_0(\mathfrak{u}_P, W') \otimes_K J_{U_P \cap L_Q}(V),$$

where $J_{U_P \cap L_Q}$ is the usual Jacquet functor for the unipotent subgroup $U_P \cap L_Q \subset L_Q$.
Proof. By the duality treated in Lemma 4.2 it suffices to check the first identity. Here we assume first that $V = 1$ is the trivial representation and that $P = Q$. By Proposition 3.8 we have $M = \mathcal{L}(\lambda)$ for some locally analytic character $\lambda$ of $T$.

We follow the proof of [19, Thm. 3.5]. Let $\mathcal{I} \subset G$ be the standard Iwahori subgroup. For $w \in W$, let $M^w = M$ be the $U(\mathcal{g}, \mathcal{I} \cap wP_0w^{-1})$-module with the twisted action given by conjugation with $w$. Let $I \subset \Delta$ be a subset with $P = P_I$. The Bruhat decomposition $G_0 = \coprod_{w \in W^I} \mathcal{I}wP_0$ induces a decomposition

$$D(G_0) \otimes_{U(\mathcal{g}, P_0)} M \cong \bigoplus_{w \in W^I} D(\mathcal{I}) \otimes_{U(\mathcal{g}, \mathcal{I} \cap wP_0w^{-1})} M^w \cong \bigoplus_{w \in W^I} D(w^{-1}\mathcal{I}w) \otimes_{U(\mathcal{g}, w^{-1}\mathcal{I}w \cap P_0)} M.$$

For each $w \in W^I$, we have

$$H^0(u_P, D(\mathcal{I}) \otimes_{U(\mathcal{g}, w^{-1}\mathcal{I}w \cap P_0)} M^w) \cong H^0(Ad(w^{-1})u_P, D(w^{-1}\mathcal{I}w) \otimes_{U(\mathcal{g}, w^{-1}\mathcal{I}w \cap P_0)} M).$$

We can write each summand in the shape

$$M^w = D(w^{-1}\mathcal{I}w) \otimes_{U(\mathcal{g}, w^{-1}\mathcal{I}w \cap P_0)} M = \lim_{\searrow r} M^w_r$$

where $M^w_r = D_r(w^{-1}\mathcal{I}w) \otimes_{U(\mathcal{g}, w^{-1}\mathcal{I}w \cap P_0)} M$. If we denote by $M^w_r$ the topological closure of $M$ in $M^w_r$, we get by [12, 1.4.2] finitely many elements $u \in U^-_P$ such that

$$M^w_r \cong \bigoplus_u \delta_u \otimes M^w_r$$

and the action of $r \in Ad(w^{-1})(u_P)$ is given by

$$\mathcal{r} \cdot \sum \delta_u \otimes m_u = \sum \delta_u \otimes (Ad(u^{-1})\mathcal{r}) m_u.$$

In [19] Thm 3.5 it is explained that for $w \neq 1$, there is a non-trivial element $r \in u^-_P \cap Ad(w^{-1})u_P$. Since $M$ is equimaximal and simple we deduce by [21 Corollary 5.5], that elements of $u^-_P$ act injectively on $M$, and as explained in Step 1 of [21 Theorem 4.7] they act injectively on $M^w_r$, as well. We conclude that $H^0(Ad(w^{-1})(Ad(w^{-1})(u_P)), M^w_r) = 0$ for $w \neq 1$ since $Ad(u^{-1})r \in u^-_P$. So $H^0(Ad(w^{-1})(u_P), M^w_r) = 0$. Hence by passing to the limit we get $H^0(Ad(w^{-1})(u_P), M^w) = 0$ for $w \neq 1$.

Now consider the case $w = 1$. Again we may write $D(\mathcal{I})_r = \bigoplus \delta_u U(\mathcal{g}, P_0)_r$ for a finite number of $u \in U^-_P$, so that $D(\mathcal{I})_r \otimes_{U(\mathcal{g}, P_0)_r} M^1_r = \bigoplus \delta_u \otimes M^1_r$. We will show that if $u \notin U^-_P \cap U(\mathcal{g}, P_0)_r$, then $H^0(Ad(w^{-1})u_P, M^r_1) = 0$. Here we will use Step 2 in the proof of [21 Theorem 4.7] where we use the equimaximality condition. Let $\hat{M}$ be the formal completion of $M$, i.e. $\hat{M} = \prod_{\mu} M_\mu$ which is a $\mathcal{g}$-module. The action of $u^-_P$ can be extended
to an action of $U^-$ as explained in loc.cit. If $\xi \in \mathfrak{g}$ and $u \in U^-$, the action of $\text{ad}(u)\xi$ on $M_r$ is the restriction of the composite $u \circ \xi \circ u^{-1}$ on $\hat{M}$. Let $\hat{M}$ be the formal completion of $M$, i.e. $\hat{M} = \prod_\mu M_\mu$ which is a $\mathfrak{g}$-module. The action of $\text{ad}(u)\hat{M}$ can be extended to an action of $U^-$ as explained in loc.cit. If $\xi \in \mathfrak{g}$ and $u \in U^-$, the action of $\text{ad}(u)\xi$ on $M_r$ is the restriction of the composite $u \circ \xi \circ u^{-1}$ on $\hat{M}$. As a consequence, we get

$$H^0(\text{ad}(u^{-1})u, M_r) = M_r^1 \cap u^{-1} \cdot H^0(u, \hat{M})$$

$$= M_r^1 \cap u^{-1} W$$

since $H^0(u, \hat{M}) = H^0(u, M) = W$ (Here and in the sequel we copy the argumentation of Breuil [2]). Let $v^+$ be a highest weight vector of $M$. If the term $H^0(\text{ad}(u^{-1})u, M_r^1) \neq (0)$ does not vanish, then we have consequently $u^{-1} W \cap M_r^1 \neq (0)$, we deduce that $u^{-1} W \subset M_r^1$ since $W$ is irreducible. In particular $u^{-1} v^+ \in M_r$. By the proof of [21, Theorem 4.7], this gives a contradiction if $u \notin U^- \cap U_r(\mathfrak{g}, P_0)$. Hence by passing to the limit and using Corollary 4.4 we obtain finally an isomorphism of Fréchet spaces

$$H^0(u, D(\mathcal{I}) \otimes_{U(\mathfrak{g}, P_0)} M) \simeq H^0(u, M) = W.$$

Next we consider the general situation where also a smooth representation is involved and where $P \subset Q$. Since $H^0(U, \mathcal{F}_Q^G(M, V))$ is a subspace of $H^0(u, \mathcal{F}_Q^G(M, V))$ the latter one is stable by the action of $U$. Thus we deduce by Lemma 4.5 that

$$H^0(U, \mathcal{F}_Q^G(M, V)) = H^0(U, H^0(u, \mathcal{F}_Q^G(M, V)))$$

$$= H^0(U, H^0(u, \mathcal{F}_Q^G(M'))) \otimes_K V'$$

$$= H^0(U, H^0(u, \mathcal{F}_Q^G(M'))) \otimes_K J_{U \cap L_Q}(V').$$

As $M = L(\lambda)$ is contained in $\mathcal{O}_d^P$, the last identity follows from the fact that the action of $U$ is induced by the one of $u_P$. \hfill $\Box$

**Corollary 4.7.** Let $M = L(\lambda)$ be an arbitrary simple object in $\mathcal{O}_d^P$. Write $\mathcal{L}(\lambda) = \mathcal{L}(\zeta) \otimes \chi$ as in Proposition 3.15 with an equimaximal object $L(\zeta) \in \mathcal{O}_d^Q$ together with an epimorphism $M(W) \rightarrow \mathcal{L}(\zeta)$ as before. Let $V$ be a smooth admissible $L_P$-representation of finite length. Then for any parabolic subgroup $R \subset P$, we have

$$H^0(u, \mathcal{F}_P^G(M, V)) = H^0(u, W) \otimes \iota_P^Q(\chi \otimes V)'$$

and

$$H^0(U, \mathcal{F}_P^G(M, V)) = H^0(u, W) \otimes J_{U \cap L_Q}(\iota_P^Q(\chi \otimes V)')$$. 
ii) For any \( V \), the restriction of \( V \) to any compact open subgroup \( C \) is a sum of finite-dimensional irreducible \( C \)-representations.

iii) For any \( v \in V \), there is a compact open subgroup \( C_v \subset G \) and a finite-dimensional subspace \( U \subset V \) with \( v \in U \) such that \( C_v \) leaves \( U \) invariant and acts on it via restriction to \( C_v \) of a finite-dimensional algebraic \( G \)-representation.

The main theorem of \([23]\) says that an irreducible locally algebraic \( G \)-representation is isomorphic to a tensor product \( V_{\text{alg}} \otimes V^{\infty} \) of some finite dimensional irreducible algebraic \( G \)-representation \( V_{\text{alg}} \) and a smooth irreducible \( G \)-representation \( V^{\infty} \). In \([23]\), Remark 1, Prasad points out that condition ii) is redundant in the semi-simple case. Moreover he gives the definition of a locally finite-dimensional analytic representation.
**Definition 4.9.** Let $V$ be a locally analytic $G$-representation. Then $V$ is called locally finite-dimensional if it satisfies condition i).

**Example 4.10.** The $L_P$-representations $\overline{H}_0(u_P, F^G_u(M, V))$ appearing in Theorem 4.6 are locally finite-dimensional analytic. In what follows it will become clear that this is also true for arbitrary modules $M$.

With the same proof as in loc.cit. one verifies the following statement:

**Proposition 4.11.** Let $V$ be a locally finite-dimensional analytic $G$-representation. Then $V$ is irreducible iff $V \cong V_f \otimes V^\infty$ for some irreducible smooth $G$-representation $V^\infty$ and some finite-dimensional irreducible locally analytic $G$-representation $V_f$.

It turns out that the decomposition above into a tensor product is compatible with respect to morphisms.

**Proposition 4.12.** Let $f : V \to W$ be a morphism of irreducible locally finite-dimensional analytic representations. Let $V = V_f \otimes V^\infty$ be a decomposition as above. Then there is a decompositions $W = W_f \otimes W^\infty$ such that $f$ has the shape $f = f_f \otimes f^\infty$ where $f_f : V_f \to W_f$ and $f^\infty : V^\infty \to W^\infty$ are morphisms of the corresponding $G$-representations.

**Proof.** Let $v = v_f \otimes v_\infty \in V$. Let $K$ be a compact open subgroup of $G$ such that $v_\infty$ is fixed by $K$. Suppose that $v_f \in V_f$ is a highest weight vector of weight $\lambda$. Hence $t \cdot v_f = \lambda(t) v_f$ for all $t \in T$. If we set $T_K = T \cap K$, then $t \cdot v = \lambda(t) v_f \otimes v_\infty$ for all $t \in T_K$ and therefore $tf(v) = f(tv) = \lambda(t)f(v)$ in this situation. It follows that if $f \neq 0$ then $w = f(v)$ is a weight vector of the same weight $\lambda$. But this weight space is one-dimensional as any $W_f$ is irreducible. Set $W_f := U(g)w$. We conclude that $f$ induces a map $f_f : V_f \to W_f$.

As for the construction of $f_\infty$ we consider the smoothing construction of Prasad [23]. The representations $V^\infty$ and $W^\infty$ can be identified with the direct limits $\varinjlim_K \text{Hom}_K(V_f, V)$ and $\varinjlim_K \text{Hom}_K(W_f, W)$ respectively. But $V_f = W_f$. Then the map $f_\infty$ is given by the obvious composition $\phi \mapsto f \circ \phi$. \hfill $\square$

**Remark 4.13.** If $V = V_{\text{alg}} \otimes V^\infty$ and $W = W_{\text{alg}} \otimes W^\infty$ are locally algebraic, then any morphism $f : V \to W$ automatically has the shape $f = f_{\text{alg}} \otimes f^\infty$ where $f_{\text{alg}} : V_{\text{alg}} \to W_{\text{alg}}$ and $f^\infty : V^\infty \to W^\infty$ are morphisms of the corresponding $G$-representations.

**Corollary 4.14.** Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an extension of locally analytic representations. Suppose that $V_1$ and $V_3$ are locally finite-dimensional analytic. Then $V_2$ is locally finite-dimensional analytic, as well.
Proof. To show that the entry in the middle is locally finite-dimensional analytic we apply dimension theory of locally analytic representations in the sense of Schneider and Teitelbaum [25]. The dimensions of $V_1$ and $V_3$ are both zero. Hence the same is true for $V_2$ as an extension of such representations [25]. Hence $V_2$ must be locally finite-dimensional analytic.

Corollary 4.15. Let $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ be an extension of locally analytic representations. Suppose that $V_1$ and $V_3$ are locally algebraic. Then $V_2$ is locally algebraic, as well.

Proposition 4.16. Let $M \in \mathcal{O}_d^P$ and let $V$ be a smooth $L_P$-representation. Then

$$
\overline{H}_0(u_P, F_G^P(M, V)) = \bigoplus_{W \subset H^0(u_P, M)'} W \otimes S_W
$$

where $S_W \subset i^P_W(V)$ is a smooth representation for some standard parabolic subgroup $P_W \supset P$ with $V \subset (S_W)|_P$. (Here the sum is over all simple $L_P$-subrepresentations $W$ of $H^0(u_P, M)'$.)

Proof. For simple objects $M = L(\lambda)$ we apply Corollary 4.7. If here the considered parabolic subgroups $P$ and $Q$ are identical then the claim is trivial. Otherwise, we apply Corollary 3.13 and [11, II, Prop. 2.11]. The latter reference says that for an algebraic simple $G$-module $M$ the fix space $M^{U_P}$ is a simple $L_P$-module, as well. Hence the module $L_P$-module $H^0(u_P, W_\zeta)$ is simple and contributes to the index family of the direct sum.

In general we fix a JH-series of $M$ and apply induction to the number of irreducible subquotients, cf. Corollary 3.10. Since $H^0(u_P, M)'$ is always contained in $\overline{H}_0(u_P, F_G^P(M))$ by the proof of Theorem 4.6 we see that $V \subset (S_W)|_P$ for all $W$. 

We can generalize the previous result as follows. Let $u_P^k \subset U(g)$ be the subspace generated by all the products $x_1 \ldots x_k$ with $x_i \in u_P$. With the same proof one checks:

Proposition 4.17. Let $M \in \mathcal{O}_d^P$ and let $V$ be a smooth $L_P$-representation. Then

$$
\overline{H}_0(u_P^k, F_G^P(M, V)) = \bigoplus_{W \subset H^0(u_P^k, M)'} W \otimes S_W
$$

where $S_W \subset i^P_W(V)$ is a smooth representation for some standard parabolic subgroup $P_W \supset P$ with $V \subset (S_W)|_P$. (Here the sum is over all indecomposable $P$-subrepresentations $W$ of $H^0(u_P^k, M)'$.)
Proof. As already mentioned the proof coincides with that of Proposition 4.10. Only for the start of induction which is essentially Theorem 4.6 one has to pay attention. Here we follow the proof of loc.cit. where \( k = 1 \). If \( w \neq 1 \), then some elements of \( u_P^k \) act injectively on \( M_w \), too. As for \( w = 1 \) we observe that \( H^0(\text{ad}(u^{-1})u_P^k, M_1^1) \neq 0 \) implies that \( H^0(\text{ad}(u^{-1})u_P, M_1^1) \neq 0 \). Hence we obtain for a simple and equimaximal object \( M \) the identity

\[
H^0(u_P^k, F_G^G(M, V)) = H^0(u_P^k, M) \otimes V'.
\]

The object \( H^0(u_P^k, M) \) is an indecomposable \( P \)-module which gives the claim in the simple case. \( \square \)

Remark 4.18. It is possible to make a more precise statement concerning the representations \( S_W \) by reentering the proof of Theorem 4.6 with non-simple objects \( M \). Indeed, if a contribution \( H^0(w^{-1}u_P w, M^w) \) does not vanish, then one checks easily that the same is true for the whole “Bruhat cell” \( U^-wP \), i.e. \( H^0(\text{Ad}(u^{-1}(w^{-1}u_P w)), M^w) \neq 0 \) for \( u \in U^-P \). Since the action of \( g \) on \( M \) is continuous, we see that the non-vanishing is also true for elements in the Zariski-closure \( \overline{U^-wP} \). Hence as a \( P \)-representation we can write \( S_W = C^\infty(Y, E) \) where \( Y \) is a union of “Schubert varieties” \( \overline{PwP} \). One might conjecture that these smooth representations \( S_W \) are induced representations, i.e., \( S_W = i_P^\overline{Pw}(V) \).

For a locally analytic \( T \)-representation \( V \) and a locally analytic character \( \lambda : T \to K^* \) we denote by

\[
V_\lambda := \{ v \in V \mid tv = \lambda(t)v \ \forall t \in T \}
\]

the \( \lambda \)-eigenspace of \( V \). We set

\[
V_{\text{alg}} := \bigoplus_{\lambda \in X^+(T)} V_\lambda.
\]

Corollary 4.19. Let \( M \in O^P_{\text{alg}} \) and \( k \geq 1 \). Then \( \overline{\Pi}_0(U_P, F_G^G(M))_{\text{alg}} = H^0(u_P^k, M)' \) and \( \overline{\Pi}_0(u_P^k, F_G^G(M))_{\text{alg}} = H^0(u_P^k, M)' \).

Proof. Since the weight spaces of \( M \) are algebraic we see that \( (W \otimes S_W)_{\text{alg}} = W \) for all contributions \( W \) in \( H^0(u_P^k, M) \). Hence the claim follows. \( \square \)

In the case of generalized Verma modules we can give a more precise statement.

Proposition 4.20. Let \( M = U(g) \otimes_{U(p)} W \in O^P \) be a generalized Verma module for some parabolic subgroup \( P \) and let \( V \) be a smooth admissible \( L_P \)-representation. Then \( \overline{\Pi}_0(U_P, F_G^G(M, V)) = H^0(u_P, \omega(M))' \otimes V \) and \( \overline{\Pi}_0(u_P^k, F_G^G(M, V)) = H^0(u_P^k, \omega(M))' \otimes V \) for all \( k \geq 1 \).
Proof. We may suppose that $V$ is trivial. The start of the proof is the same as in Theorem 4.6. For $w \neq 1$ one checks that the contributions $H^0(u_P^k, D(I) \otimes U(g,I \cap wP_0w^{-1})M^w)$ vanishes well since a generalized Verma module is free over $U(u_P^w)$. Now consider the case $w = 1$. Here we shall show that if $u \in U_{P_0}^\circ \setminus \{1\}$, then we have $H^0(\text{Ad}(u^{-1})u_P, \omega(M)) = 0$. Indeed, let $u \neq 1$. Since the normaliser of $u_P$ under the adjoint action of $G$ is the parabolic subgroup $P$, there is some $v \in u_P$ such that $vuP \not\subseteq u_P$. Write $vuP = v_- + v_+$ where $v_- \in u_P^w$ and $v_+ \in p$. Let $m \in M_\chi, m \neq 0$ as we have already used above the action of $u_P^w$ is injective on $M$. Hence $v_- m \neq 0$. But the elements $v_-, v_+$ shift the weights of $M$ in opposite directions. Any identity $(v_- + v_+) \cdot m = 0$ would imply $0 \neq v_- m = -v_+ m$ which yields thus for weight reasons a contradiction. In general we decompose any element $m \in M$ into its weight components. For simplicity let $m = m_1 + m_2$ where $m_i \in M_{\chi_i}$. Again we consider the sequence $0 \neq v_- m = v_- m_1 + v_- m_2 = -v_+ m_1 - v_+ m_2$. Comparing weights and that the action of $u_P^w$ on $M$ is injective we see that this is not possible. Hence $H^0(\text{Ad}(u^{-1})u_P, M) = 0$.

By repeating the arguments above we obtain an isomorphism of Fréchet spaces

$$H^0(u_P, D(I) \otimes U(g,P_0) M) \simeq H^0(u_P, M).$$

The claim follows moreover for all $k \geq 1$ easily. \hfill \Box

Remark 4.21. The previous statement corroborates a conjecture of Kohlhaase made in [13, Remark 8.6].

Remark 4.22. The same statement holds true (with the same proof) for objects $M \in \mathcal{O}_P$ of the shape $M = U(g) \otimes U(p) W$ where $W$ is an arbitrary finite-dimensional locally analytic $P$-representation. In particular, it holds for objects $M$ such that $\omega(M)$ is projective in the category $\mathcal{O}$ since such an object it is free as a $U(u_P^w)$-module [10].

Next there is the following variant of the above proposition concerning the other parabolic subgroups of type $P$ lying in the same apartment. Let $P = P_I = L_P U_P$ and set for $w \in W^I, P^w = w^{-1} P w, L^w_P = w^{-1} L_P w, U^w_P = w^{-1} U_P w$. Here for a $L^w_P$-module $V$, we let $V^w$ be the $L_P$-module twisted by $w$, i.e. we consider the action induced by composing the given action with the homomorphism $L_P \rightarrow w^{-1} L_P w, g \mapsto w^{-1} g w$.

Proposition 4.23. With the above notation, let $M \in \mathcal{O}_P^{P^w}$ be a generalized Verma module with respect to $P^{w}$ or a simple module such that $P^w$ is maximal for $M$. Let $V$ be a smooth admissible $L^w_P$-representation. Then $\Pi_0(U_P, \mathcal{F}_P^{G}(M, V)) = (H_0(U^w_P, M))^w \otimes V^w$ and $\Pi_0(u_P^k, \mathcal{F}_P^{G}(M, V)) = (H_0((u^w_P)^k, M))^w \otimes V^w$ for all $k \geq 1$. 
Proof. The proof is the same as above. The difference is that this time all contributions $H^0(\text{Ad}(x^{-1})u_k^p, D(x^{-1}I_x) \otimes_{U(\mathfrak{g}, x^{-1}I_x \cap P^w)} \otimes M)$ with $x \neq w$ vanish. \hfill \Box

Next we consider an analogue of the Casselman-Jacquet functor [4], i.e., limits of the above functors $H^0(u_k^p, P, -)$ (resp. $\mathcal{F}_0^u(P, -)$ by duality) with varying $k$. For a locally analytic $G$-representation, the expression $\lim_{\rightarrow k} H^0(u_k^p, U')$ is a $g \ltimes P$-module as the same reasoning as in loc.cit. applies. We denote by $\mathcal{O}P: \text{Rep}_K(G)_{\text{alg}} \rightarrow \text{Mod}_{g \ltimes P}$ the induced functor. As before let $M$ be an object of $\mathcal{O}P$ and let $V$ be a smooth admissible $L_P$-representation. Then the object $\lim_{\rightarrow k} H^0(u_k^p, \mathcal{F}_P^G(M, V))$ is also a $D(g, P)$-module. Moreover, it defines a section of it for some objects in $\mathcal{O}P \times \text{Rep}_\infty(L_P)$, cf. Proposition 4.20 and Theorem 4.6.

**Proposition 4.24.** Let $U$ be some irreducible subquotient of some $\mathcal{F}_P^G(M, V)$ with $M \in \mathcal{O}P_{\text{alg}}$. Then $\mathcal{G}_P^G(U)$ is simple as $D(g, P)$-module.

**Proof.** Since $U$ is simple it must coincide by the JH-theorem applied to $\mathcal{F}_P^G(M, V)$ with some object of the shape $\mathcal{F}_Q^G(N, W)$ where $N$ is a simple subquotient of $M$, $Q$ is maximal for $N$ and $W$ is an irreducible subquotient of $i_P^Q(V)$. But for these objects we deduce by Theorem 4.6 that $\mathcal{G}_P^G(U) = \mathcal{G}_Q^G(U) = N \otimes W'$ which gives now easily the claim. \hfill \Box

As a by-product we get the following statement by applying the functor $\mathcal{G}_P^G$ and Proposition 4.17. One part of it was already given by Breuil [2, Cor. 2.5].

**Corollary 4.25.** Let $U$ be an irreducible subobject (quotient) of some $\mathcal{F}_P^G(M, V)$ with $M \in \mathcal{O}P_{\text{alg}}$. Then $U$ has the shape $\mathcal{F}_Q^G(N, W)$ where $P \subset Q$ and $N$ is a simple quotient of $M$ (submodule) and $W$ is a subrepresentation (quotient) of $i_P^Q(V)$.

**Proof.** If $U$ is a quotient then we get by the left exactness of the functor $\mathcal{G}_P^G$ an injection $\mathcal{G}_P^G(U) \hookrightarrow \mathcal{G}_P^G(\mathcal{F}_P^G(M, V))$. Since $\mathcal{G}_P^G(U) = N \otimes W'$ we obtain by Proposition 4.17 the claim. If $U$ is a subobject we get a morphism $\mathcal{G}_P^G(\mathcal{F}_P^G(M, V)) \rightarrow \mathcal{G}_P^G(U)$. As this morphism is non-trivial and the RHS is simple it is necessarily surjective and we argue as above. \hfill \Box
5. ARE THE FUNCTORS $\mathcal{F}_P^G$ FAITHFUL?

In this section we want to address the question whether the functors $\mathcal{F}_P^G$ are faithful resp. fully faithful. This aspect was discussed for $G = \text{SL}_2$ already in the series of papers by Morita [14, 15, 16].

**Theorem 5.1.** Let $M_1, M_2 \in \mathcal{O}^P$. Suppose that we are in one of the following situations:

i) $M_2 = M(W)$ is a generalized Verma module for some finite-dimensional locally analytic $L$-representation $W$.

ii) $M_1, M_2$ are contained in the subcategory $\mathcal{O}_\text{alg}^P$.

Then the map

$$\text{Hom}_{\mathcal{O}^P}(M_1, M_2) \to \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1))$$

$$f \mapsto \mathcal{F}_P^G(f)$$

is bijective.

**Proof.** i) The proof is divided into several steps.

1) Let $M_1 = M(Z)$ be another generalized Verma module for some finite-dimensional locally analytic $L$-representation $Z$. Then $\mathcal{F}_P^G(M_1) = \text{Ind}_P^G(Z')$ and $U$ acts trivially on $Z$. Now we have $\overline{H}_0(U, \mathcal{F}_P^G(M_2)) = H_0(u, M_2')$ by Lemma [14,20]. On the other hand we have $H_0(u, M_2')' = H^0(u, M_2)$ by duality. We consider the identities induced by Frobenius reciprocity and the previous observations

$$\text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) = \text{Hom}_P(\mathcal{F}_P^G(M_2), Z')$$

$$= \text{Hom}_L(\overline{H}_0(U, \mathcal{F}_P^G(M_2)), Z')$$

$$\cong \text{Hom}_L(H^0(u, M_2'), Z')$$

$$\cong \text{Hom}_L(Z, H^0(u, M_2))$$

$$= \text{Hom}_P(Z, M_2)$$

$$= \text{Hom}_{D(\mathfrak{g}, P)}(M_1, M_2).$$

2) Let $M_1$ be a quotient of some generalized Verma module, i.e., there is a surjective homomorphism $M(Z) \to M_1$ for some finite-dimensional locally analytic $L$-representation $Z$. Let $\mathfrak{d}$ be its kernel. Then by definition we have $\mathcal{F}_P^G(M_1) = \mathcal{F}_P^G(M(Z))^{\mathfrak{d}}$. We consider the commutative diagram
\[ \text{Hom}_{D(g,P)}(M_1, M_2) \hookrightarrow \text{Hom}_{D(g,P)}(M(Z), M_2) \]
\[ \downarrow \quad \downarrow \]
\[ \text{Hom}_G(F^G_P(M_2), F^G_P(M_1)) \hookrightarrow \text{Hom}_G(F^G_P(M_2), F^G_P(M(Z))). \]

By step 1) the right vertical map is an isomorphism. It follows that the left vertical map is injective. To show surjectivity we consider the dual objects, i.e. the commutative diagram

\[ \text{Hom}_{D(g,P)}(M_1, M_2) \hookrightarrow \text{Hom}_{D(g,P)}(M(Z), M_2) \]
\[ \downarrow \quad \downarrow \]
\[ \text{Hom}_D(M^D(Z), M^D_2) \hookrightarrow \text{Hom}_D(M^D(Z), M^D_2). \]

where we abbreviate \( M^D = D(G) \otimes_{D(g,P)} M \) for \( M \in \mathcal{O}^P \). Moreover the vertical maps are the obvious ones, i.e. induced by base change. For the surjectivity, let \( f \in \text{Hom}_D(M^D(Z), M^D_2) \) and consider it via the injection as an element in the set \( \text{Hom}_D(M^D(Z), M^D_2) \). Hence there is some morphism \( \tilde{f} : M(Z) \to M_2 \) with \( \tilde{f} \otimes \text{id} = f \). We need to show that \( \tilde{f}(\mathfrak{g}) = 0 \). By assumption we have that \( \tilde{f}(\mathfrak{g}) = 0 \). But we proved in [21 (3.7.6)] that if \( M \in \mathcal{O}^B, M \neq 0 \) then \( D(G) \otimes_{D(g,B)} M \neq 0 \). By applying this fact to \( M = \tilde{f}(\mathfrak{g}) \) the claim follows.

3) Let \( M_1 = U(g) \otimes_{U(p)} W \) for some finite dimensional locally analytic \( P \)-representation \( W \). We may view it as a successive extension of generalized Verma modules considered in Step 1. The proof of the statement is by dimension on \( \dim W \). Here step 1) serves as the start of induction. Write down an exact sequence

\[ 0 \to M(Z) \to M_1 \to M'_1 \to 0 \]

where \( M'_1 = U(g) \otimes_{U(p)} W' \) with \( \dim W' < \dim W \) and the induced exact sequence

\[ 0 \to F^G_P(M'_1) \to F^G_P(M_1) \to F^G_P(M(Z)) \to 0. \]

We consider the resulting diagram of long exact sequences.
ON SOME PROPERTIES OF THE FUNCTORS $\mathcal{F}_P^G$ ... 

\[ 0 \to \text{Hom}_{D(\mathfrak{g}, P)}(M'_1, M_2) \to \text{Hom}_{D(\mathfrak{g}, P)}(M_1, M_2) \to \text{Hom}_{D(\mathfrak{g}, P)}(M(Z), M_2) \]

\[ 0 \to \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M'_1)) \to \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) \to \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M(Z))) \]

\[ \delta \quad \text{Ext}^1(M'_1, M_2) \quad \to \quad \text{Ext}^1(M_1, M_2) \quad \to \quad \text{Ext}^1(M(M_1), M_2) \]

Here we consider the Ext groups as Yoneda-Ext groups. The maps $f'$ and $f_Z$ are by induction isomorphisms of finite-dimensional vector spaces. By diagram chase, it suffices to check that $\delta(g) \neq 0$ if and only if $\delta\mathcal{F}_P^G(g) \neq 0$. Concretely we have to show that if $\delta(g) \neq 0$ then $\delta\mathcal{F}_P^G(g) \neq 0$ since the other direction follows directly by diagram chase again. If $\delta\mathcal{F}_P^G(g) = 0$, then the extension

\[ 0 \to \mathcal{F}_P^G(M'_1) \to E_{\mathcal{F}_P^G(g)} \to \mathcal{F}_P^G(M_2) \to 0 \]

induced by $\mathcal{F}_P^G(g) \in \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M(Z)))$ splits. Then we apply Remark \[4.22\] to deduce that

\[ H^0(u, E_g) = H^0(u, E_{\mathcal{F}_P^G(g)}) = H^0(u, \mathcal{F}_P^G(M'_1)) \oplus H^0(u, \mathcal{F}_P^G(M_2)) \]

\[ = H^0(u, M'_1) \oplus H^0(u, M_2). \]

We conclude that the extension

\[ 0 \to M_2 \to E_g \to M'_1 \to 0 \]

splits as well. Indeed suppose for simplicity that $M'_1 = M(U)$ is a generalized Verma module. Then $U = H^0(u, M'_1)$ appears in $E_g$ so that we get a section of $E_g \to M'_1$.

4) Let $M_1$ be arbitrary. Then there is a surjective homomorphism $M(Z) \to M$ for some finite dimensional locally analytic $P$-representation $Z$. Then we proceed as in Step 2).

ii) Here we proceed as in the first case. In Step 1) and Step 3) we use Proposition \[4.19\] instead of the property that $M_2$ is a generalized Verma module as the smooth part does not matter. \qed

Proposition 5.2. Let $M_1, M_2 \in \mathcal{O}_{\text{alg}}^P$ and let $V_1, V_2$ be smooth $L_P$-representations. Assume that $Z \subset M_1$ is a finite-dimensional indecomposable $P$-representation which generates $M_1$ as a $U(\mathfrak{g})$-module. Then the natural map

\[ \text{Hom}_{\mathcal{O}_P}(M_1, M_2) \otimes \text{Hom}(V_2, V_1) \to \text{Hom}_G(\mathcal{F}_P^G(M_2, V_2), \mathcal{F}_P^G(M_1, V_1)) \]
induced by the functor \( F_p^G \) is injective and extends to a bijection

\[
\text{Hom}_{O^P}(\underline{M}_1, \underline{M}_2) \otimes \text{Hom}(S_Z(V_2), i^{P_p}_p(V_1)) \rightarrow \text{Hom}_G(F_p^G(\underline{M}_2; V_2), F_p^G(\underline{M}_1, V_1))
\]

**Proof.** Indeed we consider Step 3) in the modified situation. Then we argue as in Step 4) for the general case. So, let \( M_1 = U(g) \otimes_{U(p)} Z \) for some indecomposable finite-dimensional \( P \)-module \( Z \). Let \( k \geq 1 \) be an integer such that \( H^0(u^k_p, Z) = Z \). Then we apply Proposition 4.17 to deduce that

\[
\text{Hom}_G(F_p^G(\underline{M}_2, V_2), F_p^G(\underline{M}_1, V_1)) = \text{Hom}_{D(P)}(Z \otimes V'_1, F_p^G(\underline{M}_2, V_2))
\]

\[\cong \text{Hom}_{D(P)}(Z \otimes V'_1, H^0(u^k_p, F_p^G(\underline{M}_2, V_2))) \]

\[\cong \text{Hom}_{D(P)}(Z \otimes V'_1, \bigoplus_{W \subset H^0(u^k_p, \underline{M}_2)} W \otimes S_W(V_2)) \]

\[= \text{Hom}_{D(g, P)}(\underline{M}_1, \underline{M}_2) \otimes \text{Hom}_{P^p}(S_Z(V_2), i^{P_p}_p(V_1)). \]

\( \square \)

If Conjecture 4.18 is satisfied, then we may replace \( S_Z(V_2) \) by \( i^{P_p}_p(V_2) \) in the above formula.

**Remark 5.3.** The statement above is also true (with the same proof) if we consider additionally a parabolic subgroup \( Q \supset P \) such that \( \underline{M}_2 \in \mathcal{O}^Q_{\text{alg}}, V_2 \in \text{Rep}^\infty(L_Q) \) i.e. we have a bijection

\[
\text{Hom}_{O^P}(\underline{M}_1, \underline{M}_2) \otimes \text{Hom}(S_Z(V_2), i^{P_p}_p(V_1)) \rightarrow \text{Hom}_G(F_Q^G(\underline{M}_2; V_2), F_p^G(\underline{M}_1, V_1)).
\]

The following example shows that in the general case of objects in \( O^B \), the map in Theorem 5.1 need not to be surjective.

**Example 5.4.** i) Let \( G = \text{SL}_2, B \subset G \) the Borel subgroup of upper triangular matrices and let \( T = \{ \text{diag}(a, a^{-1}) \mid a \in L^* \} \) be the diagonal torus. We consider the smooth character \( \chi \) of \( T \) given by

\[
\chi(\text{diag}(a, a^{-1})) = |a|(-1)^{\text{val}_v(a)}
\]

where \( \pi \) is our fixed uniformizer of \( O_L \) and \( v \) is the normalized valuation, i.e. \( v(\pi) = 1 \). Let \( \underline{M} \) be the one-dimensional trivial Lie(G)-representation which we equip with a \( B \)-action induced by \( \chi^{-1} \) and inflation. Then the object \( F_B^G(\underline{M}) \) is just the smooth representation \( i_B^G(\chi) \). But the character \( \chi \) is chosen in such a way that it decomposes as a direct sum of two irreducible representations [3 Cor. 9.4.6 (b)]. Hence \( \text{Hom}_G(F_B^G(\underline{M}), F_B^G(\underline{M})) \) is two-dimensional whereas \( \text{Hom}_{O^B}(\underline{M}, \underline{M}) \) is one-dimensional.
ii) Let $G = \text{SL}_2$ and let $\delta$ be the non-trivial smooth character appearing in the Jacquet module of $i_B^G$. Put $M_1 = M(\delta)$ and $M_2 = L(0)$. Then $\mathcal{F}_B^G(M_1) = \text{Ind}_B^G(\delta^{-1})$ and $\mathcal{F}_B^G(M_2) = i_B^G$ so that $\text{Hom}_{O_B}(M_1, M_2) = 0$ whereas $\dim \text{Hom}_G(\mathcal{F}_B^G(M_2), \mathcal{F}_B^G(M_1)) = 1$.

Recall that for $w \in W$, we denote by $P_w$ the conjugated parabolic subgroup $w^{-1}Pw$. If $Z$ is a finite-dimensional locally analytic representation of $L$ we let $M_w(Z)$ be the corresponding generalized Verma module with respect to $P_w$, i.e. $M_w(Z) = U(g) \otimes U(p_w)Z$.

**Proposition 5.5.** Let $Z$ be a finite-dimensional locally analytic $L_P$-representation and let $w \in W$. Then for any finite-dimensional locally analytic $L^w_P$-representation $Y$ there is an identity

$$\text{Hom}_G(\text{Ind}_{P^w}(Y'), \text{Ind}_{P^w}(Z')) = \text{Hom}_{O^w}(M_w(Z'^{-1}), M_w(Y)).$$

**Proof.** We argue as in Step 1) in the proof of Theorem 5.1 and use additionally Proposition 4.23

$$\text{Hom}_G(\text{Ind}_{P^w}(Y'), \text{Ind}_{P^w}(Z')) = \text{Hom}_P(\text{Ind}_{P^w}(Y'), Z')$$
$$= \text{Hom}_L(H_0(U_P, \text{Ind}_{P^w}(Y'))(Z'))$$
$$= \text{Hom}_L(H_0(U_P, M^w(Y'))(Z'))$$
$$= \text{Hom}_L(Z^w, H^0(U_P, M^w(Y)))$$
$$= \text{Hom}_{P^w}(Z^w, M^w(Y))$$
$$= \text{Hom}_{D(g, P^w)}(M^w(Z^w^{-1}), M^w(Y)).$$

6. Applications

In the remaining paper we discuss some applications of the material collected in the previous sections. For this we recall a definition of [21]. Let $\lambda, \mu : T \to K^*$ be two locally analytic characters with derivatives $d\lambda$, $d\mu$, respectively. We write $\mu \uparrow_B \lambda$ if and only if $d\mu \uparrow_b d\lambda$ in the sense of [10] and $\mu - \lambda \in X^*(T)$ is an algebraic character. Then the natural homomorphism $M(d\mu) \to M(d\lambda)$ lifts to a morphism $M(\mu) \to M(\lambda)$. More
precisely, one has

\[
\dim_K \text{Hom}_{O^B}(\mathcal{M}(\mu), \mathcal{M}(\lambda)) = \begin{cases} 
1 & \mu \uparrow_B \lambda \\
0 & \text{otherwise}
\end{cases}.
\]

Analogously to the above definition we extend the “dot” action of $W$ on $X^*(T)$ to all locally analytic characters. Let $\lambda$ be a locally analytic character and let $w \in W$. The difference between $w \cdot_B (d\lambda)$ and $d\lambda$ is algebraic. Hence there is some algebraic character $\chi \in X^*(T)$ such that $w \cdot_B (d\lambda) = d\lambda + d\chi$. We set

\[w \cdot_B \lambda := \lambda \cdot \chi.\]

If $\lambda \in \Lambda^+$ is $B$-dominant, then $w \cdot_B \lambda \uparrow_B \lambda$ for all $w \in W$.

**Lemma 6.1.** The above construction induces an action of $W$ on the space of locally analytic characters.

**Proof.** The proof is left as an exercise. \qed

On the other hand, we let $\lambda^w := w(\lambda)$ be the character given by the ordinary action of $W$.

**Corollary 6.2.** Let $P = B$ and let $\lambda, \mu : T \to K^*$ be locally analytic characters. Then

\[
\dim_K \text{Hom}_G(\mathcal{F}^G_B(\mathcal{M}(\lambda)), \mathcal{F}^G_B(\mathcal{M}(\mu))) = \begin{cases} 
1 & \mu \uparrow_B \lambda \\
0 & \text{otherwise}
\end{cases}.
\]

**Proof.** This follows from Theorem 5.1 together with identity (6.1). \qed

For a standard parabolic subgroup $P \subset G$, we let $\mathcal{F}_{\text{alg}}^P$ be the full subcategory of $\text{Rep}_{K}^{\text{loc.an.}}(G)$ consisting of locally analytic representations which lie in the essential image of the functor $\mathcal{F}^G_P : \mathcal{O}^p_{\text{alg}} \to \text{Rep}_{K}^{\text{loc.an.}}(G)$.

**Corollary 6.3.** i) The category $\mathcal{F}_{\text{alg}}^P$ is abelian and has enough injective and projective objects. For a morphism $f : N \to M$ we have $\mathcal{F}^G_P(\text{coker}(f)) = \ker(\mathcal{F}^G_P(f))$ and $\mathcal{F}^G_P(\ker(f)) = \text{coker}(\mathcal{F}^G_P(f))$.

ii) Let $\mathcal{M}$ be a projective (resp. injective) object in $\mathcal{O}^p_{\text{alg}}$. Then $\mathcal{F}^G_P(\mathcal{M})$ is injective (resp. projective) in the category $\mathcal{F}_{\text{alg}}^P$.

**Proof.** The category $\mathcal{O}^p_{\text{alg}}$ is abelian and has enough projective and injective objects. This follows for $\mathcal{O}^p$ from [10]. But the proof shows that for an object $M \in \mathcal{O}^p_{\text{alg}}$ the construction of a projective cover $N$ of $M$, that $N$ is again in the subcategory $\mathcal{O}^p_{\text{alg}}$, hence the claim
is true for the category $\mathcal{F}_P^G$. Since the functor $\mathcal{F}_P^G$ induces an equivalence of categories between $\mathcal{O}_P^p_{\text{alg}}$ and $\mathcal{F}_P^G$ we get the first part of i) and ii). The remaining statements follow directly be the exactness of $\mathcal{F}_P^G$. □

We define a dual object for objects lying in the functor. In light of Theorem 5.1 it is well-defined.

**Definition 6.4.** Let $\underline{M} \in \mathcal{O}_P^p_{\text{alg}}$ and let $\underline{M}^\vee \in \mathcal{O}_P^p_{\text{alg}}$ be its dual object. Set

$$\mathcal{F}_P^G(\underline{M})^\vee := \mathcal{F}_P^G(\underline{M}^\vee).$$

It follows from the previous corollary that for an object $\underline{M} \in \mathcal{O}_P^p_{\text{alg}}$ the locally analytic $G$-representation $\mathcal{F}_P^G(\underline{M})$ is projective (resp. injective) object in $\mathcal{F}_P^G$ if and only if $\mathcal{F}_P^G(\underline{M})^\vee$ is injective (resp. projective) object in $\mathcal{F}_P^G$.

**Definition 6.5.** Let $V_1, V_2 \in \mathcal{F}_P^G$ be two locally analytic representations. We denote by $\text{Ext}_{\mathcal{F}_P^G}^i(V_1, V_2)$ the corresponding Ext-group in degree $i$.

These Ext-groups are of course different from those considered more generally in the category of locally analytic $G$-representations, cf. [13]. This can be seen as an analogue of relating the groups $\text{Ext}_O^i(M_1, M_2)$ and $\text{Ext}_O^i(M_1, M_2)$ for two objects $M_1, M_2 \in \mathcal{O}$ as the next statement confirms.

**Corollary 6.6.** Let $M_1, M_2 \in \mathcal{O}_P^p_{\text{alg}}$. The natural map

$$\text{Ext}_{\mathcal{O}_P}^i(M_1, M_2) \to \text{Ext}_{\mathcal{F}_P^G}^i(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1))$$

is bijective.

At this point one can derive many consequences on the above defined Ext-groups. Here we exemplary mention only the following:

**Corollary 6.7.** Let $\lambda \in \lambda_\emptyset^+ = \Lambda^+$ be dominant and let $w, w' \in W$.

a) Unless $w' \cdot \lambda \uparrow w \cdot \lambda$ we have for all $n > 0$,

$$\text{Ext}_{\mathcal{F}_P^G}^n(\mathcal{F}_B^G(M(w \cdot \lambda)), \mathcal{F}_B^G(M(w' \cdot \lambda))) = 0 = \text{Ext}_{\mathcal{F}_P^G}^n(\mathcal{F}_B^G(L(w \cdot \lambda)), \mathcal{F}_B^G(M(w' \cdot \lambda))).$$

b) If $w' \cdot \lambda \leq w \cdot \lambda$, then for all $n > \ell(w') - \ell(w)$

$$\text{Ext}_{\mathcal{F}_P^G}^n(\mathcal{F}_B^G(M(w \cdot \lambda)), \mathcal{F}_B^G(M(w' \cdot \lambda))) = 0 = \text{Ext}_{\mathcal{F}_P^G}^n(\mathcal{F}_B^G(L(w \cdot \lambda)), \mathcal{F}_B^G(M(w' \cdot \lambda))).$$

**Proof.** This is a consequence of [10] Proposition 6.11]. □
Next we consider additionally smooth representations as arguments in the functor $F^G_P$. Here we shall extend the parameter space in the second entry to the category of smooth $L_P$-representations $\text{Rep}_K^\infty(L_P)$ since it has enough injective and projective objects. So let $V$ be a smooth $G$-representation. Hence we may write $V = \bigcup_n V^{G_n}$ for a system of compact open subgroups $G_n \subset G$. We supply each $V^{G_n}$ with the finest locally convex topology and equip $V$ with the induced locally convex limit topology. This construction is compatible with the topology considered on admissible smooth representations since for a finite-dimensional Banach space any lattice is open \cite[Prop. 4.13]{24}. The resulting topology is Hausdorff \cite[Prop. 5.5 ii)]{24} and barrelled \cite[Cor. 6.16, Examples iii)]{24} (see also the construction in \cite[7.1]{6}). Moreover, for any $v \in V$ the orbit map $G \to V$ is locally constant and gives rise to an element of $C^{an}(G; V)$. Hence we may and will consider $V$ with the structure of a locally analytic $G$-representation. Then $F^G_P$ extends with the same definition as in (2.1) to a bi-functor

$$F^G_P : \mathcal{O}^P \times \text{Rep}_K^\infty(L_P) \longrightarrow \text{Rep}_K^{\text{loc.an.}}(G).$$

**Remark 6.8.** We stress that apart possible from the last two statements in §4 (since the proofs do not apply) all results of the previous sections are also valid for objects lying in the image of this enhanced functor.

We define $\infty F^P_{\text{alg}}$ to be the full subcategory of $\text{Rep}_K^{\text{loc.an.}}(G)$ consisting of locally analytic representations which lie in the essential image of this functor. The category $\infty F^P_{\text{alg}}$ is not abelian. For this reason we consider the smallest abelian subcategory $\overline{\infty F^P_{\text{alg}}}$ containing all categories $\infty F^Q_{\text{alg}}$ where $Q \supset P$ is a parabolic subgroup.

**Lemma 6.9.** Let $M_1, M_2, M \in \mathcal{O}^P_{\text{alg}}$ and $V_1, V_2, V \in \text{Rep}_K^\infty(L_P)$ such that $M_1, M_2$ are quotients of $M$ and $V_1, V_2$ are subrepresentations of $V$. Then

$$F^G_P(M_1, V_1) \cap F^G_P(M_2, V_2) = F^G_P(M_1 \oplus_M M_2, V_1 \cap V_2)$$

**Proof.** We have

$$F^G_P(M_1, V_1) \cap F^G_P(M_2, V_2) = F^G_P(M_1, V_1 \cap V_2) \cap F^G_P(M_2, V_1 \cap V_2)$$

$$= F^G_P(M_1 \oplus_M M_2, V_1 \cap V_2) \tag*{\Box}$$

**Lemma 6.10.** Let $M \in \mathcal{O}^P_{\text{alg}}$ be a simple object and let $V$ be a smooth $L_P$-representation. Then any subquotient of $F^G_P(M, V)$ has the shape $F^G_P(M, W)$ for some smooth subquotient $W$ of $V$. 


Proof. By the exactness of $\mathcal{F}_P^G$ it suffices to prove this statement for subobjects. Let $U \subset \mathcal{F}_P^G(M,V)$ be a subobject. We recall a construction of \[22,\] Thm. 5.8 which uses the simplicity of $M$. Set $U_{sm} = \lim_{\rightarrow H} \text{Hom}(\mathcal{F}_P^G(M)|_H, U|_H)$ where the limit is over all compact open subgroups $H$ of $G$. It is proved that $U_{sm}$ is a subrepresentation of $\mathcal{F}_P^G(M,V)_{sm}$ and that the latter object identifies with the smooth induction $\text{ind}_P^G(V)$ (for $V$ irreducible, but this holds also true in this general setting). Moreover, the natural map $\mathcal{F}_P^G(M) \otimes \text{ind}_P^G(V) \to \mathcal{F}_P^G(M,V)$ is surjective giving rise by the very definition of this map to a surjection $\phi : \mathcal{F}_P^G(M) \otimes U_{sm} \to U$. Set $W := \{ f(1) \mid f \in U_{sm} \}$. This is a smooth $L_P$-representation and the map $\phi$ factorizes over $\mathcal{F}_P^G(M,W)$. It follows that the image of the map $\phi$ coincides with $\mathcal{F}_P^G(M,W)$. Hence $U = \mathcal{F}_P^G(M,W)$.

Proposition 6.11. Every object $U$ in $\mathcal{F}_P^G_{\text{alg}}$ is a successive extension of objects of the shape $\mathcal{F}_Q^G(N,W)$ with $P \subset Q$.

Proof. As the direct sum of two objects of the kind $\mathcal{F}_Q^G(M_i,V_i)$, $i = 1, 2$, is contained in such an object we may suppose that $U$ is some subquotient of $\mathcal{F}_P^G(M,V)$. Indeed $\mathcal{F}_Q^G(M_i,V_i) \subset \mathcal{F}_P^G(M,(V_i)|_P)$ so that it suffices to treat the case $Q_1 = Q_2 = P$. But then $\mathcal{F}_P^G(M_1,V_1) \oplus \mathcal{F}_P^G(M_2,V_2) \subset \mathcal{F}_P^G(M_1 \oplus M_2, V_1 \oplus V_2)$.

The proof is by induction on the length on $M$. If $M$ is simple (where we may assume that $P$ is maximal for $M$ by the PQ-formula) then the statement follows from the above lemma. Otherwise, let $M_1 \subset M$ be some proper submodule and consider the exact sequence

$$0 \to \mathcal{F}_P^G(M/M_1,V) \to \mathcal{F}_P^G(M,V) \xrightarrow{p} \mathcal{F}_P^G(M_1,V) \to 0.$$ 

So let $U = U_1/U_2$ be some subquotient of $\mathcal{F}_P^G(M,V)$. We consider the induced exact sequence

$$0 \to \mathcal{F}_P^G(M/M_1,V) \cap U_1/\mathcal{F}_P^G(M/M_1,V) \cap U_2 \to U_1/U_2 \to p(U_1)/p(U_2) \to 0.$$ 

If $\mathcal{F}_P^G(M/M_1,V) \cap U_1/\mathcal{F}_P^G(M/M_1,V) \cap U_2 \in \{(0), U_1/U_2\}$ we may apply induction hypothesis to prove the claim. But also in the other case the inductive hypothesis applies.

Proposition 6.12. Let $M \in O_{\text{alg}}^P$ be projective (resp. injective) and let $V$ be a smooth injective (resp. projective) $L_P$-representation. Then $\mathcal{F}_P^G(M,V)$ is injective (resp. projective) in the category $\mathcal{F}_P^G_{\text{alg}}$.

Proof. We consider here the case of injective objects. The case of projective objects is treated in a dual sense. We consider thus an injection $Z_1 \hookrightarrow Z_2$ together with a morphism $Z_1 \to \mathcal{F}_P^G(M,V)$. Since any object in $\mathcal{F}_P^G_{\text{alg}}$ is subquotient of an object lying in the image
of our functor $\mathcal{F}_Q^G$ we may suppose by enlarging $Z_2$ that is has for simplicity the shape $\mathcal{F}_Q^G(N,W)$. Indeed if $Z_2$ is a submodule of $\mathcal{F}_Q^G(N,W)$ this is clear. If on the other hand, $Z_2$ is a quotient of $\mathcal{F}_Q^G(N,W)$ then we consider the preimages $\tilde{Z}_1 \to \tilde{Z}_2$ of $Z_1$ and $Z_2$ in $\mathcal{F}_Q^G(N,W)$. We get an induced map $f : \tilde{Z}_1 \to \mathcal{F}_Q^G(N,W)$ and if this extends to $\tilde{Z}_2$ then also to $Z_2$ since $\text{ker}(\tilde{Z}_1 \to Z_1) = \text{ker}(\tilde{Z}_2 \to Z_2)$ is mapped to zero under $f$.

By the PQ-formula we see that $\mathcal{F}_Q^G(N,W) \to \mathcal{F}_Q^G(N,i_Q^G(W|L_P)) = \mathcal{F}_Q^G(N,W|L_P)$. Hence we may suppose that $P = Q$. On the other hand, we may suppose that $Z_1$ has also the shape $\mathcal{F}_Q^G(N,W)$. Indeed using Lemma 6.9 we see that there are $N \in \mathcal{O}_{\text{alg}}^G$ and $W \in \text{Rep}_K^\infty(L_Q)$ such that $\mathcal{F}_Q^G(N,W)$ is a minimal object containing $Z_1$. By applying the functor $\mathcal{G}_Q^G$ we deduce that $N$ and $W$ appear in $\mathcal{G}_Q^G(U)$. Hence the morphism $Z_1 \to \mathcal{F}_Q^G(M,V)$ extends automatically to a morphism $\mathcal{F}_Q^G(N,W) \to \mathcal{F}_Q^G(M,V)$.

Hence we may think that our embedding $Z_1 \to Z_2$ is of the shape $\mathcal{F}_Q^G(M_1,V_1) \to \mathcal{F}_Q^G(M_2,V_2)$. It follows by the bi-exactness of $\mathcal{F}_Q^G$ and the exactness of the induction functor that it is induced by a surjection $M_2 \to M_1$ and a monomorphism $(V_1)_P \to V_2$. Indeed we consider first the morphism $\mathcal{F}_Q^G(M_1,(V_1)_P) \to \mathcal{F}_Q^G(M_2,V_2)$ and the embedding $\mathcal{F}_Q^G(M_1,V_1) \to \mathcal{F}_Q^G(M_1,(V_1)_P) = \mathcal{F}_Q^G(M_1,i_Q^G(V_1))$ given by the projection formula $i_Q^G(V_1) = V_1 \otimes i_P^G(1)$ and the obvious inclusion $V_1 = V_1 \otimes i_Q^G(1) \in V_1 \otimes i_P^G(1)$. Since any morphism $i_P^G((V_1)_P) \to i_P^G(V_2)$ which is injective for some parabolic subgroup $P_Z \supset P$ has to be induced by an injection $(V_1)_P \to V_2$ the claim stated above follows.

So for proving that $\mathcal{F}_Q^G(M,V)$ is injective let $\mathcal{F}_Q^G(M_1,V_1) \to \mathcal{F}_Q^G(M,V)$ be any morphism. By dividing out its kernel (from the very beginning) in the monomorphism above, we may assume that it is injective as well. Again it corresponds to a tuple of morphisms $M \to M_1$ and $(V_1)_P \to V$. Since $V$ is injective we see that there is an extension $V_2 \to V$. Further as $M$ is projective we have a lift $M \to M_2$. The claim follows.

Corollary 6.13. The category $\infty\mathcal{F}_Q^G_{\text{alg}}$ has enough injective and projective objects.

Proof. As above we consider here only the case of injectives.

Let $U \in \infty\mathcal{F}_Q^G_{\text{alg}}$. Suppose first that it has the shape $\mathcal{F}_Q^G(M,V)$. We choose a projective cover $N$ of $M$ and an embedding $V \hookrightarrow W$ into a smooth injective $L_P$-representation $W$. Then we have a topological embedding $\mathcal{F}_Q^G(M,V) \to \mathcal{F}_Q^G(N,W)$ and by the result above the object $\mathcal{F}_Q^G(N,W)$ is injective.

In general we know by Proposition 6.11 that it is a successive extension of such objects. As such it has an injective envelope, as well (Suppose that $0 \to A_1 \to U \to A_2 \to 0$ is exact and that $A_i \to I_i, i = 1, 2$ are monomorphism into injective objects. Then we get...
an exact sequence $0 \to I_1 \to I_1 \oplus A_1 U \to A_2 \to 0$ and the middle term is isomorphic to $I_1 \oplus A_2$ which embeds into the injective object $I_1 \oplus I_2$.)

For a parabolic subgroup $P \subset G$, we abbreviate $I_P^G := \text{Ind}_P^G(1)$ and denote by $i_P^G$ the subspace of smooth vectors. The attached Steinberg representation is given by the quotient $V_P^G = \text{Ind}_P^G(1)/\sum_{Q \supset P} \text{Ind}_Q^G(1)$. We shall determine the Ext-groups of these objects in our compactified categories.

We recall a result from [18]. Here we denote by $\infty \text{Ext}^*$ the corresponding Ext-groups in the category of smooth representations.

**Proposition 6.14.** Let $I \subset \Delta$. Then we have

$$\infty \text{Ext}^*_L(I, I) = \Lambda^*(X^*(L_I)).$$

The next statement is contained in [9, Thm. 9.8].

**Lemma 6.15.** For a parabolic subgroup $Q$ of $G$, let $M = M_Q(0) = U(g) \otimes U(q) K$ be the generalized Verma module with respect to the trivial $Q$-module. Then $M$ is projective in $\mathcal{O}_Q^{\text{alg}}$.

**Proposition 6.16.** Let $G$ be semi-simple and let $I, J \subset \Delta$. Then we have

$$\text{Ext}^*_\infty_{\mathcal{F}_\text{alg}}(I^G_P, I^G_P) = \left\{ \begin{array}{ll}
\Lambda^*(X^*(L_J)) & \text{if } J \subset I \\
0 & \text{otherwise}
\end{array} \right.$$

**Proof.** We set $P = P_I$ and $Q = P_J$.

1. **Case.** Suppose that $J \not\subset I$. Let $I^\bullet$ be an injective resolution of the trivial $L_Q$-representation in the category of smooth $L_Q$-representations. Then by Lemma 6.15 and Proposition 6.12 $\mathcal{F}_Q^G(M_Q(0), I^\bullet)$ is an injective resolution of $I^G_Q$. Let $J^\bullet$ be an injective resolution of the trivial $T$-representation in the category of smooth representations. Then $i_B^Q(J^\bullet)$ is an injective resolution of $i_B^Q$ (in the category of smooth representations) since the induction functor is exact and has with the Jacquet functor an exact left adjoint. Hence the embedding $1_Q \to i_B^Q$ extends to a morphism of complexes $I^\bullet \to i_B^Q(J^\bullet)$. Here we may suppose by standard arguments that the maps in each degree are injective. We consider the induced (injective) maps $\mathcal{F}_Q^G(M_Q(0), I^\bullet) \to \mathcal{F}_Q^G(M_Q(0), i_B^Q(J^\bullet)) = \mathcal{F}_B^Q(M_Q(0), J^\bullet)$. We shall see that any map $I^G_P \to \mathcal{F}_Q^G(M_Q(0), J^\bullet)$ vanishes which is enough for our claim.

Indeed by Remark 5.3 it is induced on the Lie algebra part by a map $M_Q(0) \to M_P(0)$. Any such map vanishes if $Q \not\subset P$.

2. **Case.** Suppose that $J \subset I$. Then by applying Frobenius reciprocity any map $I^G_P \to \mathcal{F}_Q^G(M_Q(0), I^\bullet) = I^G_Q(I^\bullet)$ is given by a map $(I^G_P)_{U_Q} \to I^\bullet$. The left hand side coincides
by Proposition 4.20 with $H^0(u_Q, M_P(0))'$ which is a sum of algebraic representations and which contains the trivial representation. Since any map between an algebraic representation different from the trivial one and a smooth representation vanishes we see that any map $(I^G_P)_{u_Q} \to I^i$ corresponding to a map $1 \to I^i$. Hence the series of maps determines $\text{Ext}^*_{L,J}(1,1)$ which coincides with $\Lambda^*(X^*(L_J))$ by Proposition 6.14.

\textbf{Theorem 6.17.} Let $G$ be semi-simple. Let $I, J \subset \Delta$. Then

$$\text{Ext}^i_{\infty, F_P}(V^G_P, V^G_P) = \begin{cases} K^{(I)} & |I \cup J \setminus I \cap J| = i \\ (0) & \text{otherwise} \end{cases}.$$  

\textit{Proof.} In [19] we proved that the following complex is an acyclic resolution of $V^G_P$ by locally analytic $G$-representations,

$$0 \to I^G_G \xrightarrow{\bigoplus_{I \subset K \subset \Delta, |\Delta \setminus K| = 1} I^G_P} \bigoplus_{I \subset K \subset \Delta, |\Delta \setminus K| = 2} I^G_P \to \cdots \to \bigoplus_{I \subset K \subset \Delta, |K \setminus I| = 1} I^G_P \to I^G_P \to V^G_P \to 0.$$  

The smooth version of this complex was used in [18] together with the smooth version of Proposition 6.16 to get by formal arguments the smooth version of our theorem. Hence the rest of the proof is the same as in loc.cit. \qed

If $G$ is not necessarily semi-simple, then we have as in the smooth case a contribution of the center $Z(G)$. By using a Hochschild-Serre argument (cf. loc.cit.) we conclude:

\textbf{Corollary 6.18.} Let $G$ be reductive with center $Z(G)$ of rank $d$. Let $I, J \subset \Delta$. Then we have

$$\text{Ext}^i_{\infty, F_P}(V^G_P, V^G_P) = \begin{cases} K^{(I)} & : i = |I \cup J| - |I \cap J| + j, j = 0, \ldots, d \\ 0 & : \text{otherwise} \end{cases}.$$  

Next we want to discuss some adjunction formulas. For this we need some preparations.

\textbf{Lemma 6.19.} Let $x, w \in W$ and let $\chi : T \to K^*$ be a locally analytic character. Then

$$(x \cdot_B \chi)^w = \text{Ad}(w)(x) \cdot_B w^{-1} \chi^w.$$  

\textit{Proof.} First let $\chi \in X^*(T)$ be an algebraic character. Then we compute

$$(x \cdot_B \chi)^w = w(x(\chi + \rho_B) - \rho_B) = \text{Ad}(w)(x)(w(\chi + \rho_B) - w\rho_B) = \text{Ad}(w)(x)((\chi^w + \rho_{B^{-1}}) - \rho_{B^{-1}}) = \text{Ad}(w)(x) \cdot_B w^{-1} \chi^w.$$
If $\chi$ is arbitrary, then at least the above computations holds also true for its derivative $d\chi$. So let $x_B \cdot d\chi = d\chi + d\mu$ for some algebraic character $\mu$, so that $x \cdot_B \chi = \chi \cdot \mu$. Then $(x \cdot_B d\chi)^w = d\chi^w + d\mu^w$. But $(x \cdot_B d\chi)^w = \text{Ad}(w)(x) \cdot_{B^{w^{-1}}} d\chi^w$ by the first case. Hence $\text{Ad}(w)(x) \cdot_{B^{w^{-1}}} \chi^w = \chi^w \cdot \mu^w = (\chi \cdot \mu)^w = (x \cdot_B \chi)^w$. \hfill \square

**Definition 6.20.** Let $M$ be a indecomposable object in $\mathcal{O}$. We denote by $M_v \in \mathcal{O}$ the maximal Verma module quotient of $M$.

**Example 6.21.** i) Let $M = M(\lambda)$ be itself a Verma module. Then $M = M_v$. More generally, let $M$ be a successive extension of Verma modules $M(\lambda_i)_{i \in I}$. Then $M_v = M(\lambda)$ where $\lambda$ is minimal among the family $(\lambda_i)_{i}$ with respect to the partial order $\uparrow_B$.

ii) Let $M$ be a proper quotient of $M(\lambda)$. Then $M_v = (0)$.

**Lemma 6.22.** Let $M$ be an indecomposable object of $\mathcal{O}$ and let $\chi$ be a dominant weight. Then there is the identity $\text{Hom}_\mathcal{O}(M, M(\chi)) = \text{Hom}_\mathcal{O}(M_v, M(\chi))$.

**Proof.** Suppose first that $M$ is extension of Verma modules. The proof is by induction on the number of Verma modules appearing in $M$. We start with the case of a non-trivial extension $0 \to M(\lambda_1) \to M \to M(\lambda_2) \to 0$. Then $\lambda_2 \uparrow \lambda_1$. Let $f \in \text{Hom}(M, M(\chi))$ be a morphism. If $f(M(\lambda_1)) = (0)$ the map factorizes through $M(\lambda_2)$ and the statement is clear. Hence we suppose that $f(M(\lambda_1)) \neq (0)$. Since $\lambda_2 \uparrow \lambda_1$ we see that the image of $f$ is contained in $M(\lambda_1) \subset M(\chi)$. Hence we get a splitting of the embedding $M(\lambda_1) \hookrightarrow M$ which gives a contradiction. The case of more than two Verma modules is treated similarly.

In general $M$ is a quotient of an object $N$ considered before. Me may suppose that $N$ is a Verma module. If there is a non-trivial homomorphism $M \to M(\chi)$ we have a non-trivial homomorphism $N \to M(\chi)$. But the restriction to any subspace of $N$ has to be non-trivial again by formula \eqref{eq:6.1}. Hence we see that if $M_v = (0)$ then $\text{Hom}(M, M(\chi)) = (0)$ as well. The other case of $M_v = N \neq (0)$ is trivial. \hfill \square

Let $M = M_B(\chi)$ be a Verma module with respect to the opposite Borel subgroup $\overline{B}$. By Lemma \ref{lem:4.23} there is a natural homomorphism $(\chi^{-1})^{w_0} \to \overline{H}_0(U_B, \mathcal{F}_B^G(M))$. If further $\chi$ is $\overline{B}$-dominant, then we have moreover a natural homomorphism

$$(w_0 \cdot_{\overline{B}} \chi)^{-1}^{w_0} \to \overline{H}_0(U_B, \mathcal{F}_B^G(M)).$$

These maps lead by composing with the functor $V \mapsto V^\cdot_B = \overline{H}_0(U_B, V)$ to the following statements.
Theorem 6.23. Let $\chi$ be a $\overline{B}$-dominant locally analytic character. Then for $w \in W$ and any $M \in \mathcal{O}^B_w$ one has the identity

$$\text{Hom}_G(I^G_w(\chi^{-1}), F^G_w(M)) = \text{Hom}_T(((w_0 \cdot \overline{B} \chi)^{-1})_{w_0}, F^G_w(M_{v_B}))$$

Proof. We consider as be before different situations.

i) Let $M = M^B_w(\lambda)$ be a Verma module for some locally analytic character $\lambda$ of $T$. We start with the observation that both sides are at most one-dimensional. Indeed as for the LHS this follows from Proposition 5.5. As for the RHS we can identify it (see below) with some eigenspace of $H^0(u, M)$ for a Verma module. This eigenspace is one-dimensional, as well. The idea is in principal to apply Theorem 5.1 i), and Lemma 6.22. Since $\chi$ is $\overline{B}$-dominant we see that $\chi^{-1}_{w_0}$ is $B^w$-dominant. The left hand side does not vanish by Proposition 5.5 if and only if $\chi^{-1}_{w_0} \uparrow_{\overline{B}} \chi$. This is equivalent to $\lambda \uparrow_{B^w} \chi^{-1}_{w_0}$ by Lemma 6.19. Thus the RHS does not vanish iff the LHS does not vanish.

On the other hand, the Jacquet module $I^G_w(\lambda^{-1})_{\mathcal{U}}$ coincides by Proposition 4.23 with the direct sum $\bigoplus_{\mu \uparrow_{B^w} \lambda} K \mu^{-1} w$. Moreover, $(w_0 \cdot \overline{B} \chi)^{w_0} = w w^{-1} w_0 (w_0 \cdot \overline{B} \chi) = w (w^{-1} w_0 w \cdot B^w \chi^{-1}_{w_0})$ by Lemma 6.19. The LHS does not vanish iff the RHS does not vanish.

ii) Now let $M$ be a quotient of $M^B_w(\lambda)$. Then $F^G_w(M) \subset I^G_w(\lambda^{-1})$ so that both vector spaces in the above stated formula are at most one-dimensional. Moreover, we have a commutative diagram

$$\text{Hom}_G(I^G_w(\chi^{-1}), I^G_w(\lambda^{-1})) = \text{Hom}_T(((w_0 \cdot \overline{B} \chi)^{-1})_{w_0}, I^G_w(\lambda^{-1})_{U_B})$$

$$\uparrow \quad \uparrow$$

$$\text{Hom}_G(I^G_w(\chi^{-1}), F^G_w(M)) \rightarrow \text{Hom}_T(((w_0 \cdot \overline{B} \chi)^{-1})_{w_0}, F^G_w(M)_{U_B})$$

The upper line is an isomorphism by the first case. The LHS is an injection. In particular the lower line is an injection, as well. Since the spaces in question are at most one-dimensional the statement follows easily in this case. Note that if $M$ is a proper quotient of $M^B_w(\lambda)$, then the objects in the lower line vanishes and the claim is trivial.

iii) Let $W$ be a finite-dimensional $B$-representation and $M = M^B_w(W)$. We may suppose that $M$ is indecomposable. Here we proceed as in Step i). The left hand side coincides by Lemma 6.22 with $\text{Hom}_G(I^G_w(\chi^{-1}), F^G_w(M_v))$. Then we proceed as in Step 1).

iv) Let $M$ be a quotient of some $M^B_w(W)$. Then we proceed as in Step ii). \hfill \Box
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Remark 6.24. In [2] and [1] are presented adjunction formulas which use on the RHS Emerton Jacquet functor and which have a different style.

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