Well-posedness of the Ostrovsky–Hunter Equation under the combined effects of dissipation and short-wave dispersion

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Abstract. The Ostrovsky–Hunter equation provides a model for small-amplitude long waves in a rotating fluid of finite depth. It is a nonlinear evolution equation. In this paper we study the well-posedness for the Cauchy problem associated with this equation in presence of some weak dissipation effects.

1. Introduction

Many physical problems (such as nonlinear shallow-water waves and wave motion in plasmas) are described by the following nonlinear evolution equation

$$\partial_t u + \partial_x f(u) - \alpha \partial_{xx}^2 u - \beta \partial_{xxx}^3 u = 0, \quad \alpha, \beta \in \mathbb{R}, \quad f(u) = \frac{u^2}{2},$$

(1.1)

which was derived by Korteweg–deVries (see [12]). (1.1) is also known as the Korteweg–de Vries–Burgers equation (see [2,9,26]), where $\alpha \partial_{xx}^2 u$ is a viscous dissipation term. If (1.1) describes the evolution of nonlinear shallow-water waves, then the function $u(t,x)$ is the amplitude of an appropriate linear long wave mode, with linear long wave speed $C_0$. However, when the effects of background rotation through the Coriolis parameter $\kappa$ need to be taken into account, an extra term is needed, and (1.1) is replaced by

$$\partial_x (\partial_t u + \partial_x f(u) - \alpha \partial_{xx}^2 u - \beta \partial_{xxx}^3 u) = \gamma u,$$

(1.2)

where $\gamma = \frac{\kappa^2}{2C_0}$ (see [7,11]). If $\alpha = \beta = 0$, then (1.2) reads

$$\partial_x (\partial_t u + \partial_x f(u)) = \gamma u.$$

(1.3)

(1.3) is known under different names such as the reduced Ostrovsky equation [6,23,25], the Ostrovsky–Hunter equation [1], the short-wave equation [10], and the

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Vakhnenko equation [20,24]. The well-posedness of (1.3) in class of discontinuous solutions has been proved in [3,4].

If \( \alpha = 0 \), (1.2) reads

\[
\partial_x (\partial_t u + \partial_x f(u) - \beta \partial^3_{xxx} u) = \gamma u,
\]

(1.4)

which is known as the Ostrovsky equation (see [22]). Mathematical properties of (1.4) were studied recently in many details, including the local and global well-posedness in energy space [8,15,18,28], stability of solitary waves [13,16,19], wave breaking [17], and convergence of solutions in the limit of the Korteweg–deVries equation [14,19].

Let us assume, in (1.2), that \( \alpha = 1, \beta = 0 \). Therefore, we have

\[
\partial_x (\partial_t u + \partial_x f(u) - \partial^2_{xx} u) = \gamma u.
\]

(1.5)

(1.5) describes the combined effects of dissipation and short-wave dispersion, and is analogous to the (1.1) for dissipative long waves. It can be deduced considering two asymptotic expansions of the shallow-water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves (see [7,11]).

We are interested in the initial value problem for (1.5), so we augment (1.5) with the initial condition

\[
u(0, x) = u_0(x), \quad x \in \mathbb{R}.
\]

(1.6)

Integrating (1.5) on \((-\infty, x)\) we gain the integro-differential formulation of problem (1.5), and (1.6) (see [18])

\[
\begin{cases}
\partial_t u + \partial_x f(u) = \gamma \int_{-\infty}^x u(t, y) \, dy + \partial^2_{xx} u, & t > 0, \quad x \in \mathbb{R}, \\
 u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

(1.7)

that is equivalent to

\[
\begin{cases}
\partial_t u + \partial_x f(u) = \gamma P + \partial^2_{xx} u, & t > 0, \quad x \in \mathbb{R}, \\
 \partial_x P = u, & t > 0, \quad x \in \mathbb{R}, \\
 P(t, -\infty) = 0, & t > 0, \\
 u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]

(1.8)

On the initial datum we assume that

\[
u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \int_\mathbb{R} u_0(x) \, dx = 0.
\]

(1.9)

On the function

\[
P_0(x) = \int_{-\infty}^x u_0(y) \, dy, \quad x \in \mathbb{R},
\]

(1.10)
we assume that
\[ \|P_0\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y)dy \right)^2 dx < \infty, \]
\[ \int_{\mathbb{R}} P_0(x)dx = \int_{\mathbb{R}} \left( \int_{-\infty}^{x} u_0(y)dy \right) dx = 0. \] (1.11)

The flux \( f \) is assumed to be smooth, genuinely nonlinear, and subquadratic, namely:
\[ f \in C^2(\mathbb{R}), \quad |\{ f'' = 0\}| = 0, \quad |f'(u)| \leq C_0|u|, \quad u \in \mathbb{R}, \] (1.12)
for some a positive constant \( C_0 \).

The main result of this paper is the following theorem.

**THEOREM 1.1.** Let \( T > 0 \). Assume (1.9), (1.10), (1.11) and (1.12). Then there exists a unique classical solution for the Cauchy problem of (1.7), or (1.8), such that
\[ u \in L^\infty((0, T) \times \mathbb{R}) \cap C((0, T); H^\ell(\mathbb{R})), \quad \forall \ell \in \mathbb{N}, \]
\[ P \in L^\infty((0, T) \times \mathbb{R}) \cap L^2((0, T) \times \mathbb{R}), \]
\[ \int_{\mathbb{R}} u(t, x)dx = 0, \quad t \geq 0. \] (1.13)

Moreover, if \( u \) and \( v \) are two solutions of (1.7), or (1.8), the following inequality holds
\[ \|u(t, \cdot) - v(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t} \|u_0 - v_0\|_{L^2(\mathbb{R})}, \] (1.14)
for some suitable \( C(T) > 0 \), and every \( 0 \leq t \leq T \).

The existence argument is based on passing to limit using a compensated compactness argument [27] in the parabolic-elliptic approximation of (1.8):
\[ \partial_t u_\delta + \partial_x f(u_\delta) = \gamma P_\delta + \partial_{xx}^2 u_\delta, \quad -\delta \partial_{xx}^2 P_\delta + \partial_x P_\delta = u_\delta. \] (1.15)

In (1.8) \( P \) is not a real unknown of the problem, indeed we can rewrite (1.3) as the integro-differential problem (1.7). The same applies to (1.15). Indeed \( P_\delta \) has the integral form
\[ P_\delta(t, x) = \frac{1}{2\sqrt{\delta}} \int_{\mathbb{R}} e^{-\frac{|x-y|}{2\sqrt{\delta}}} u_\delta(t, y)dy \]
and we can rewrite (1.15) in the integro-differential form
\[ \partial_t u_\delta + \partial_x f(u_\delta) = \frac{\gamma}{2\sqrt{\delta}} \int_{\mathbb{R}} e^{-\frac{|x-y|}{2\sqrt{\delta}}} u_\delta(t, y)dy + \partial_{xx}^2 u_\delta. \]

The paper is organized as follows. In Sect. 2 we prove several a priori estimates on the parabolic-elliptic. Those play a key role in the proof of our main result, that is given in Sect. 3.
2. Parabolic-elliptic approximation

Our existence argument is based on passing to the limit in a parabolic-elliptic approximation. Fix $0 < \delta < 1$, and let $u_\delta = u_\delta(t, x)$ be the unique classical solution of the following mixed problem [5]:

\[
\begin{aligned}
\partial_t u_\delta + \partial_x f(u_\delta) = \gamma P_\delta + \partial_{xx}^2 u_\delta, & \quad t > 0, \ x \in \mathbb{R}, \\
-\delta \partial_{xx} P_\delta + \partial_x P_\delta = u_\delta, & \quad t > 0, \ x \in \mathbb{R}, \\
u_\delta(0, x) = u_{\delta, 0}(x), & \quad x \in \mathbb{R},
\end{aligned}
\tag{2.1}
\]

where $u_{\delta, 0}$ is a $C^\infty$ approximation of $u_0$ such that

\[
\begin{aligned}
\|u_{\delta, 0}\|_{L^2(\mathbb{R})} & \leq \|u_0\|_{L^2(\mathbb{R})}, \\
\|\partial_x u_{\delta, 0}\|_{L^2(\mathbb{R})} & \leq C_0, \\
\|\partial_{xx}^2 u_{\delta, 0}\|_{L^2(\mathbb{R})} & \leq C_0 \\
\|P_{\delta, 0}\|_{L^2(\mathbb{R})} & \leq \|P_0\|_{L^2(\mathbb{R})}, \\
\|\partial_x P_{\delta, 0}\|_{L^2(\mathbb{R})} & \leq C_0,
\end{aligned}
\tag{2.2}
\]

and $C_0$ is a constant independent on $\delta$.

Let us prove some a priori estimates on $u_\delta$ and $P_\delta$, denoting with $C_0$ the constants which depend on the initial data, and $C(T)$ the constants which depend also on $T$.

**LEMMA 2.1.** For each $t \in (0, \infty)$,

\[
P_\delta(t, \infty) = \partial_x P_\delta(t, -\infty) = \partial_x P_\delta(t, \infty) = 0.
\tag{2.3}
\]

Moreover,

\[
\delta^2 \|\partial_{xx}^2 P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \|u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\tag{2.4}
\]

**Proof.** We begin by proving that (2.3) holds.

Differentiating the first equation of (2.1) with respect to $x$, we have

\[
\partial_x (\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta.
\tag{2.5}
\]

From the smoothness of $u_\delta$, it follows from (2.1) and (2.5) that

\[
\lim_{x \to -\infty} (\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta(t, -\infty) = 0,
\]

\[
\lim_{x \to -\infty} \partial_x (\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta(t, -\infty) = 0,
\]

\[
\lim_{x \to \infty} \partial_x (\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta(t, \infty) = 0,
\]

which gives (2.3).

Let us show that (2.4) holds. Squaring the equation for $P_\delta$ in (2.1), we get

\[
\delta^2 (\partial_{xx}^2 P_\delta)^2 + (\partial_x P_\delta)^2 - \delta \partial_x ((\partial_x P_\delta)^2) = u_\delta^2.
\]

Therefore, (2.4) follows from (2.3) and an integration on $\mathbb{R}$. □
LEMMA 2.2. For each $t \in (0, \infty)$,

$$\sqrt{\delta} \| \partial_x P_\delta(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq \| u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}, \quad (2.6)$$

$$\int_{\mathbb{R}} u_\delta(t, x) P_\delta(t, x) \, dx \leq \| u_\delta(t, \cdot) \|^2_{L^2(\mathbb{R})}, \quad (2.7)$$

**Proof.** We begin by proving that (2.6) holds.

Observe that

$$0 \leq (-\delta \partial^2_{xx} P_\delta + \partial_x P_\delta)^2 = \delta^2 (\partial^2_{xx} P_\delta)^2 + (\partial_x P_\delta)^2 - \delta \partial_x ((\partial_x P_\delta)^2),$$

that is,

$$\delta \partial_x ((\partial_x P_\delta)^2) \leq \delta^2 (\partial^2_{xx} P_\delta)^2 + (\partial_x P_\delta)^2. \quad (2.8)$$

Integrating (2.8) on $(-\infty, x)$, we have

$$\delta (\partial_x P_\delta)^2 \leq \delta^2 \int_{-\infty}^{x} (\partial^2_{xx} P_\delta)^2 \, dx + \int_{-\infty}^{x} (\partial_x P_\delta)^2 \, dx \leq \delta^2 \int_{\mathbb{R}} (\partial^2_{xx} P_\delta)^2 \, dx + \int_{\mathbb{R}} (\partial_x P_\delta)^2 \, dx. \quad (2.9)$$

It follows from (2.4) and (2.9) that

$$\delta (\partial_x P_\delta)^2 \leq \delta^2 \int_{\mathbb{R}} (\partial^2_{xx} P_\delta)^2 \, dx + \int_{\mathbb{R}} (\partial_x P_\delta)^2 \, dx = \| u_\delta(t, \cdot) \|^2_{L^2(\mathbb{R})}.\quad (2.10)$$

Therefore,

$$\sqrt{\delta} | \partial_x P_\delta(t, x) | \leq \| u_\delta(t, \cdot) \|_{L^2(\mathbb{R})},$$

which gives (2.6).

Finally, we prove (2.7). Multiplying by $P_\delta$ the equation for $P_\delta$ in (2.1), we get

$$-\delta P_\delta \partial^2_{xx} P_\delta + P_\delta \partial_x P_\delta = u_\delta P_\delta.$$ 

An integration on $\mathbb{R}$ and (2.3) give

$$\int_{\mathbb{R}} u_\delta P_\delta \, dx = \frac{1}{2} \int_{\mathbb{R}} \partial_t (P_\delta)^2 \, dx - \delta \int_{\mathbb{R}} P_\delta \partial^2_{xx} P_\delta \, dx$$

$$= -\delta \int_{\mathbb{R}} P_\delta \partial^2_{xx} P_\delta \, dx = \delta \int_{\mathbb{R}} (\partial_x P_\delta)^2 \, dx,$$

that is

$$\int_{\mathbb{R}} u_\delta P_\delta \, dx = \delta \int_{\mathbb{R}} (\partial_x P_\delta)^2 \, dx.$$ 

Since $0 < \delta < 1$, from (2.4), we have (2.7).  \qed
LEMMA 2.3. For each $t \in (0, \infty)$, the following inequality holds
\[
\|u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq e^{2\gamma t} \|u_0\|_{L^2(\mathbb{R})}^2. \tag{2.10}
\]
In particular, we have
\[
\|\partial_x P_\delta(t, \cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{\delta} \|\partial_x^2 P_\delta(t, \cdot)\|_{L^2(\mathbb{R})},
\]
\[
\|\partial_x P_\delta(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq e^{\gamma t} \|u_0\|_{L^2(\mathbb{R})}. \tag{2.11}
\]

Proof. Due to (2.1) and (2.7),
\[
\frac{d}{dt} \int_\mathbb{R} u_\delta^2 \, dx = 2 \int_\mathbb{R} u_\delta \partial_t u_\delta \, dx
\]
\[
= 2 \int_\mathbb{R} u_\delta \partial_x^2 u_\delta \, dx - 2 \int_\mathbb{R} u_\delta f'(u_\delta) \partial_x u_\delta \, dx + 2\gamma \int_\mathbb{R} u_\delta P_\delta \, dx
\]
\[
\leq -2 \int_\mathbb{R} (\partial_x u_\delta)^2 \, dx + 2\gamma \|u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
The Gronwall Lemma and (2.2) give (2.10). Finally, (2.11) follows from (2.4), (2.6) and (2.10). \qed

LEMMA 2.4. For each $t \geq 0$, we have that
\[
\int_0^{-\infty} P_\delta(t, x) \, dx = a_\delta(t), \tag{2.12}
\]
\[
\int_0^{\infty} P_\delta(t, x) \, dx = a_\delta(t), \tag{2.13}
\]
where
\[
a_\delta(t) = \frac{\delta}{\gamma} \partial_x^2 P_\delta(t, 0) - \frac{1}{\gamma} \partial_t P_\delta(t, 0) + \frac{1}{\gamma} f(0) - \frac{1}{\gamma} f(u_\delta(t, 0)) + \frac{1}{\gamma} \partial_x u_\delta(t, 0).
\]

In particular,
\[
\int_\mathbb{R} P_\delta(t, x) \, dx = 0, \quad t \geq 0. \tag{2.15}
\]

Proof. We begin by observing that, integrating the second equation of (2.1) on $(0, x)$, we have that
\[
\int_0^x u_\delta(t, y) \, dy = P_\delta(t, x) - P_\delta(t, 0) - \delta \partial_x P_\delta(t, x) + \delta \partial_x P_\delta(t, 0). \tag{2.16}
\]
It follows from (2.3) that
\[
\lim_{x \to -\infty} \int_0^x u_\delta(t, y) \, dy = \int_0^{-\infty} u_\delta(t, x) \, dx = \delta \partial_x P_\delta(t, 0) - P_\delta(t, 0). \tag{2.17}
\]
Differentiating (2.17) with respect to \( t \), we get
\[
\frac{d}{dt} \int_{-\infty}^{0} u_\delta(t, x)dx = \int_{0}^{-\infty} \partial_t u_\delta(t, x)dx = \delta \partial_{t x}^2 P_\delta(t, 0) - \partial_t P_\delta(t, 0). \tag{2.18}
\]

Integrating the first equation of (2.1) on \((0, x)\), we obtain that
\[
\int_{0}^{x} \partial_t u_\delta(t, y)dy + f(u_\delta(t, x)) - f(u_\delta(t, 0)) - \partial_x u_\delta(t, x) + \partial_x u_\delta(t, 0) = \gamma \int_{0}^{x} P_\delta(t, y)dy. \tag{2.19}
\]

Being \( u_\delta \) a smooth solution of (2.1), we get
\[
\lim_{x \to -\infty} \left( f(u_\delta(t, x)) - \partial_x u_\delta(t, x) \right) = f(0). \tag{2.20}
\]

Sending \( x \to -\infty \) in (2.19), from (2.18) and (2.20), we have
\[
\gamma \int_{0}^{-\infty} P_\delta(t, x)dx = \delta \partial_{t x}^2 P_\delta(t, 0) - \partial_t P_\delta(t, 0) + f(0) - f(u_\delta(t, 0)) + \partial_x u_\delta(t, 0),
\]
which gives (2.12).

Let us show that (2.13) holds. We begin by observing that, for (2.3) and (2.16),
\[
\int_{0}^{\infty} u_\delta(t, x)dx = \delta \partial_x P_\delta(t, 0) - P_\delta(t, 0).
\]

Therefore,
\[
\lim_{x \to \infty} \int_{0}^{x} \partial_t u_\delta(t, y)dy = \int_{0}^{\infty} \partial_t u_\delta(t, x)dx = \delta \partial_{t x}^2 P_\delta(t, 0) - \partial_t P_\delta(t, 0). \tag{2.21}
\]

Again by the regularity of \( u_\delta \),
\[
\lim_{x \to \infty} \left( f(u_\delta(t, x)) - \partial_x u_\delta(t, x) \right) = f(0). \tag{2.22}
\]

It follows from (2.19), (2.21) and (2.22) that
\[
\gamma \int_{0}^{\infty} P_\delta(t, x)dx = \delta \partial_{t x}^2 P_\delta(t, 0) - \partial_t P_\delta(t, 0) + f(0) - f(u_\delta(t, 0)) + \partial_x u_\delta(t, 0),
\]
which gives (2.13).

Finally, we prove (2.15). It follows from (2.12) that
\[
\int_{-\infty}^{0} P_\delta(t, x)dx = -a_\delta(t).
\]

Therefore, for (2.13),
\[
\int_{-\infty}^{0} P_\delta(t, x)dx + \int_{0}^{\infty} P_\delta(t, x) = \int_{\mathbb{R}} P_\delta(t, x)dx = -a_\delta(t) + a_\delta(t) = 0,
\]
that is (2.15). \qed
Lemma 2.4 says that $P_\delta(t,x)$ is integrable at $\pm \infty$. Therefore, for each $t \geq 0$, we can consider the following function

$$F_\delta(t,x) = \int_{-\infty}^{x} P_\delta(t,y)dy.$$  

(2.23)

**Lemma 2.5.** Let $T > 0$. There exists $C(T) > 0$, independent on $\delta$, such that

$$\|P_\delta\|_{L^\infty(I_{T,1})} \leq C(T),$$  

(2.24)

$$\|P_\delta(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$  

(2.25)

$$\delta \|\partial_x P_\delta(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$  

(2.26)

where

$$I_{T,1} = (0, T) \times \mathbb{R}.$$  

(2.27)

In particular, we have

$$\delta \left| \int_0^T \int_\mathbb{R} P_\delta \partial_x^2 P_\delta dsdx \right| \leq C(T), \quad 0 < t < T.$$  

(2.28)

**Proof.** Integrating the second equation of (2.1) on $(-\infty, x)$, for (2.3), we have that

$$\int_{-\infty}^{x} u_\delta(t,y)dy = P_\delta(t,x) - \delta \partial_x P_\delta(t,x).$$  

(2.29)

Differentiating (2.29) with respect to $t$, we get

$$\frac{d}{dt} \int_{-\infty}^{x} u_\delta(t,y)dy = \int_{-\infty}^{x} \partial_t u_\delta(t,y)dy = \partial_t P_\delta(t,x) - \delta \partial_x^2 P_\delta(t,x).$$  

(2.30)

It follows from an integration of the first equation of (2.1) on $(-\infty, x)$ and (2.23) that

$$\int_{-\infty}^{x} \partial_t u_\delta(t,y)dy + f(u_\delta(t,x)) - \partial_x u_\delta(t,x) = \gamma F_\delta(t,x).$$  

(2.31)

Due to (2.30) and (2.31), we have

$$\partial_t P_\delta(t,x) - \delta \partial_x^2 P_\delta(t,x) = \gamma F_\delta(t,x) - f(u_\delta(t,x)) + \partial_x u_\delta(t,x).$$  

(2.32)

Multiplying (2.32) by $P_\delta - \delta \partial_x P_\delta$, we have

$$(\partial_t P_\delta - \delta \partial_x^2 P_\delta)(P_\delta - \delta \partial_x P_\delta) = \gamma F_\delta(P_\delta - \delta \partial_x P_\delta)$$

$$- f(u_\delta)(P_\delta - \delta \partial_x P_\delta)$$

$$+ \partial_x u_\delta(P_\delta - \delta \partial_x P_\delta).$$  

(2.33)

Integrating (2.33) on $(0, x)$, we have

$$\int_0^x \partial_t P_\delta P_\delta dy = \delta \int_0^x \partial_t P_\delta \partial_x P_\delta dy$$.  

\[-\delta \int_0^x P_\delta \partial_{1x}^2 P_\delta \, dy + \delta^2 \int_0^x \partial_{1x}^2 P_\delta \partial_x P_\delta \, dy\]

\[= \gamma \int_0^x F_\delta P_\delta \, dy - \gamma \delta \int_0^x F_\delta \partial_x P_\delta \, dy\]

\[-\int_0^x f(u_\delta) P_\delta \, dy + \delta \int_0^x f(u_\delta) \partial_x P_\delta \, dy\]

\[+ \int_0^x \partial_x u_\delta P_\delta \, dy - \delta \int_0^x \partial_x u_\delta \partial_x P_\delta \, dy. \quad (2.34)\]

We observe that

\[-\delta \int_0^x \partial_x P_\delta \partial_t P_\delta \, dy = -\delta P_\delta \partial_t P_\delta + \delta P_\delta (t, 0) \partial_t P_\delta (t, 0) + \delta \int_0^x P_\delta \partial_{1x}^2 P_\delta \, dy. \quad (2.35)\]

Therefore, (2.34) and (2.35) give

\[\int_0^x \partial_t P_\delta \partial_t P_\delta \, dy + \delta^2 \int_0^x \partial_{1x}^2 P_\delta \partial_x P_\delta \, dy\]

\[= \delta P_\delta \partial_t P_\delta - \delta P_\delta (t, 0) \partial_t P_\delta (t, 0) + \gamma \int_0^x F_\delta P_\delta \, dy\]

\[-\gamma \delta \int_0^x F_\delta \partial_x P_\delta \, dy - \int_0^x f(u_\delta) P_\delta \, dy + \delta \int_0^x f(u_\delta) \partial_x P_\delta \, dy\]

\[+ \int_0^x \partial_x u_\delta P_\delta \, dy - \delta \int_0^x \partial_x u_\delta \partial_x P_\delta \, dy. \quad (2.36)\]

Sending \(x \to -\infty\), for (2.3), we get

\[\int_{-\infty}^0 \partial_t P_\delta \partial_t P_\delta \, dy + \delta^2 \int_{-\infty}^0 \partial_{1x}^2 P_\delta \partial_x P_\delta \, dy\]

\[= -\delta P_\delta (t, 0) \partial_t P_\delta (t, 0) + \gamma \int_{-\infty}^0 F_\delta P_\delta \, dy\]

\[-\gamma \delta \int_{-\infty}^0 F_\delta \partial_x P_\delta \, dy - \int_{-\infty}^0 f(u_\delta) P_\delta \, dy\]

\[+ \delta \int_{-\infty}^0 f(u_\delta) \partial_x P_\delta \, dy + \int_{-\infty}^0 \partial_x u_\delta P_\delta \, dy\]

\[-\delta \int_{-\infty}^0 \partial_x u_\delta \partial_x P_\delta \, dy, \quad (2.37)\]

while sending \(x \to \infty\),

\[\int_0^\infty \partial_t P_\delta \partial_t P_\delta \, dy + \delta^2 \int_0^\infty \partial_{1x}^2 P_\delta \partial_x P_\delta \, dy\]

\[= -\delta P_\delta (t, 0) \partial_t P_\delta (t, 0) + \gamma \int_0^\infty F_\delta P_\delta \, dy - \gamma \delta \int_0^\infty F_\delta \partial_x P_\delta \, dy\]
\[-\int_{0}^{\infty} f(u_{\delta}) P_{\delta} dy + \delta \int_{0}^{\infty} f(u_{\delta}) \partial_{x} P_{\delta} dy + \int_{0}^{\infty} \partial_{x} u_{\delta} P_{\delta} dy = -\int_{0}^{\infty} \partial_{x} u_{\delta} \partial_{x} P_{\delta} dy. \tag{2.38}\]

Since

\[
\int_{\mathbb{R}} P_{\delta} \partial_{t} P_{\delta} dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} P_{\delta}^2 dx,
\]

\[
\delta^2 \int_{\mathbb{R}} \partial_{tx}^2 P_{\delta} \partial_{x} P_{\delta} dx = \frac{\delta^2}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_{x} P_{\delta})^2 dx,
\]

it follows from (2.37) and (2.38) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} P_{\delta}^2 dx + \frac{\delta^2}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_{x} P_{\delta})^2 dx
= \gamma \int_{\mathbb{R}} F_{\delta} P_{\delta} dx - \gamma \delta \int_{\mathbb{R}} F_{\delta} \partial_{x} P_{\delta} dx
- \int_{\mathbb{R}} f(u_{\delta}) P_{\delta} dx + \delta \int_{\mathbb{R}} f(u_{\delta}) \partial_{x} P_{\delta} dx
+ \int_{\mathbb{R}} \partial_{x} u_{\delta} P_{\delta} dx - \delta \int_{\mathbb{R}} \partial_{x} u_{\delta} \partial_{x} P_{\delta} dx. \tag{2.39}\]

Due to (2.15) and (2.23),

\[
2\gamma \int_{\mathbb{R}} F_{\delta} P_{\delta} dx = 2\gamma \int_{\mathbb{R}} F_{\delta} \partial_{x} F_{\delta} dx = \gamma (F_{\delta}(t, \infty))^2
= \gamma \left( \int_{\mathbb{R}} P_{\delta}(t, x) dx \right)^2 = 0. \tag{2.40}\]

(2.39) and (2.40) give

\[
\frac{d}{dt} \left( \int_{\mathbb{R}} P_{\delta}^2 dx + \delta \int_{\mathbb{R}} (\partial_{x} P_{\delta})^2 dx \right)
= -2\gamma \delta \int_{\mathbb{R}} F_{\delta} \partial_{x} P_{\delta} dx - 2 \int_{\mathbb{R}} f(u_{\delta}) P_{\delta} dx
+ 2\delta \int_{\mathbb{R}} f(u_{\delta}) \partial_{x} P_{\delta} dx + 2 \int_{\mathbb{R}} \partial_{x} u_{\delta} P_{\delta} dx
- 2\delta \int_{\mathbb{R}} \partial_{x} u_{\delta} \partial_{x} P_{\delta} dx. \tag{2.41}\]

Thanks to (2.3), (2.15) and (2.23),

\[
-2\delta \gamma \int_{\mathbb{R}} \partial_{x} P_{\delta} F_{\delta} dx = 2\delta \gamma \int_{\mathbb{R}} P_{\delta} \partial_{x} F_{\delta} dx
= 2\delta \gamma \int_{\mathbb{R}} P_{\delta}^2 dx \leq 2\gamma \int_{\mathbb{R}} P_{\delta}^2 dx, \tag{2.42}\]

while for (2.3),

\[
2 \int_{\mathbb{R}} \partial_{x} u_{\delta} P_{\delta} dx = -2 \int_{\mathbb{R}} u_{\delta} \partial_{x} P_{\delta} dx. \tag{2.43}\]
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Thus, from (1.12), (2.42) and (2.43), we get

\[
\frac{d}{dt} \left( \int_{\mathbb{R}} P_\delta^2 \, dx + \delta^2 \int_{\mathbb{R}} (\partial_x P_\delta)^2 \, dx \right)
\leq 2\gamma \int_{\mathbb{R}} P_\delta^2 \, dx - 2 \int_{\mathbb{R}} f(u_\delta) P_\delta \, dx + 2\delta \int_{\mathbb{R}} f(u_\delta) \partial_x P_\delta \, dx
- 2 \int_{\mathbb{R}} u_\delta \partial_x P_\delta \, dx - 2\delta \int_{\mathbb{R}} \partial_x u_\delta \partial_t P_\delta \, dx
\leq 2\gamma \int_{\mathbb{R}} P_\delta^2 \, dx + 2 \int_{\mathbb{R}} f(u_\delta) P_\delta \, dx + 2\delta \int_{\mathbb{R}} f(u_\delta) \partial_x P_\delta \, dx
+ 2 \int_{\mathbb{R}} u_\delta \partial_x P_\delta \, dx + 2\delta \int_{\mathbb{R}} \partial_x u_\delta \partial_t P_\delta \, dx
\leq 2\gamma \int_{\mathbb{R}} P_\delta^2 \, dx + 2 \int_{\mathbb{R}} f(u_\delta) \| P_\delta \|_{L^2(\mathbb{R})} + 2\delta \int_{\mathbb{R}} \| f(u_\delta) \|_{L^\infty(\mathbb{R})} \| \partial_x P_\delta \|_{L^2(\mathbb{R})} \, dx
+ 2 \int_{\mathbb{R}} u_\delta \| \partial_x P_\delta \|_{L^2(\mathbb{R})} + 2\delta \int_{\mathbb{R}} \| \partial_x u_\delta \|_{L^2(\mathbb{R})} \| \partial_t P_\delta \|_{L^2(\mathbb{R})} \, dx.
\]

From the Young inequality,

\[
2 \int_{\mathbb{R}} |\partial_x P_\delta| |u_\delta| \leq \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \| u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2,
\]

\[
2\delta \int_{\mathbb{R}} |\partial_x u_\delta| |\partial_x P_\delta| \, dx = \int_{\mathbb{R}} \left| \frac{\partial_x u_\delta}{\sqrt{\gamma}} \right| \left| 2\sqrt{\gamma} \partial_x P_\delta \right| \, dx
\leq \frac{1}{2\gamma} \| \partial_x u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\delta^2 \gamma \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

Thus,

\[
\frac{d}{dt} G(t) - 2\gamma G(t) \leq \| u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2 C_0 \int_{\mathbb{R}} |P_\delta| u_\delta^2 \, dx
+ 2 C_0 \delta \int_{\mathbb{R}} |\partial_x P_\delta| u_\delta^2 \, dx + \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2
+ \frac{1}{2\gamma} \| \partial_x u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2,
\]

where

\[
G(t) = \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \delta^2 \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\]

We observe that, from (2.10),

\[
2 C_0 \int_{\mathbb{R}} |P_\delta| u_\delta^2 \, dx \leq C_0 e^{2\gamma t} \| P_\delta \|_{L^\infty(I_{T,1})},
\]

where \( I_{T,1} \) is defined in (2.27). Since \( 0 < \delta < 1 \), it follows from (2.10) and (2.11) that
\[ 2C_0 \delta \int_{\mathbb{R}} |\partial_x P_\delta| u_\delta^2 \, dx \leq 2C_0 \delta \| \partial_x P_\delta(t, \cdot) \|_{L^\infty(\mathbb{R})} \| u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ \leq 2\sqrt{3}C_0 e^{3\gamma t} \leq C_0 e^{3\gamma t}. \] (2.47)

Again by (2.11), we have that
\[ \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 e^{2\gamma t}. \] (2.48)

Therefore, (2.10), (2.47) and (2.48) give
\[ \frac{d}{dt} G(t) - 2\gamma G(t) \leq C_0 \left( \| P_\delta \|_{L^\infty(I_{T,1})} + 1 \right) e^{2\gamma t} + C_0 e^{3\gamma t} + \frac{1}{2\gamma} \| \partial_x u_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]

The Gronwall Lemma, (2.2), (2.10) and (2.45) give
\[ \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \delta^2 \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \| P_0 \|_{L^2(0,\infty)}^2 e^{2\gamma t} + \left( \| P_0 \|_{L^\infty(I_{T,1})} + 1 \right) te^{2\gamma t} + C_0 te^{3\gamma t} + \frac{e^{2\gamma t}}{2\gamma} \int_0^t e^{-2\gamma s} \| \partial_x u_\delta(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \]
\[ \leq \| P_0 \|_{L^2(0,\infty)}^2 e^{2\gamma t} + \left( \| P_0 \|_{L^\infty(I_{T,1})} + 1 \right) te^{2\gamma t} + C_0 te^{3\gamma t} + C_0 e^{2\gamma t}. \]

Hence,
\[ \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \delta^2 \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C(T) \left( \| P_\delta \|_{L^\infty(I_{T,1})} + 1 \right). \] (2.49)

Due to (2.11), (2.49) and the H"older inequality,
\[ P_\delta^2(t, x) \leq 2 \int_{\mathbb{R}} |P_\delta| |\partial_x P_\delta| \, dx \leq 2 \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_x P_\delta(t, \cdot) \|_{L^2(\mathbb{R})} \]
\[ \leq 2 \sqrt{C(T)} \left( \| P_\delta \|_{L^\infty(I_{T,1})} + 1 \right) \sqrt{C_0 e^{\gamma t}} \leq C(T) \left( \| P_\delta \|_{L^\infty(I_{T,1})} + 1 \right). \]

Therefore,
\[ \| P_\delta \|_{L^\infty(I_{T,1})} - C(T) \| P_\delta \|_{L^\infty(I_{T,1})} - C(T) \leq 0, \]
which gives (2.24), (2.25) and (2.26) follow from (2.24) and (2.49).

Let us show that (2.28) holds. Multiplying (2.32) by $P_\delta$, an integration on $\mathbb{R}$ and (2.40) give
\[ 2\delta \int_{\mathbb{R}} \partial_{xx}^2 P_\delta P_\delta \, dx = \frac{d}{dt} \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 - 2\gamma \int_{\mathbb{R}} f(x) P_\delta \, dx \]
\[ + 2 \int_{\mathbb{R}} f(u_\delta) P_\delta \, dx - 2 \int_{\mathbb{R}} \partial_x u_\delta P_\delta \, dx \]
\[ = \frac{d}{dt} \| P_\delta(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2 \int_{\mathbb{R}} f(u_\delta) P_\delta \, dx - 2 \int_{\mathbb{R}} \partial_x u_\delta P_\delta \, dx. \]
An integration on \((0, t)\) gives
\[
2\delta \int_0^t \partial_{tx}^2 P_\delta P_\delta \, dx = \|P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 - \|P_{\varepsilon, \delta, 0}\|_{L^2(\mathbb{R})}^2 \\
+ 2 \int_0^t \int_{\mathbb{R}} f(u_\delta) P_\delta \, dx - 2 \int_0^t \partial_x u_\delta P_\delta \, dx.
\]

It follows from (1.12), (2.10), (2.24) and (2.25) that
\[
2\delta \left| \int_0^t \partial_{tx}^2 P_\delta P_\delta \, ds \right| \leq \|P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|P_{\varepsilon, \delta, 0}\|_{L^2(\mathbb{R})}^2 \\
+ 2 \int_0^t \int_{\mathbb{R}} |f(u_\delta)| P_\delta \, ds \, dx \\
+ 2 \int_0^t \int_{\mathbb{R}} |\partial_x u_\delta| P_\delta \, ds \, dx \\
\leq \|P_{\delta, 0}\|_{L^2(\mathbb{R})}^2 + 2C(T) \int_0^t u_\delta^2 \, ds \, dx \\
+ 2 \int_0^t \int_{\mathbb{R}} |\partial_x u_\delta| P_\delta \, ds \, dx + C(T) \\
\leq \|P_{\delta, 0}\|_{L^2(\mathbb{R})}^2 + C(T) \\
+ 2 \int_0^t \int_{\mathbb{R}} |\partial_x u_\delta| P_\delta \, ds \, dx.
\]

Observe that, thanks to (2.10),
\[
\int_0^t \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \\
\leq e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T). \tag{2.50}
\]

Due to the Young inequality,
\[
2 \int_{\mathbb{R}} |\partial_x u_\delta| P_\delta \, ds \, dx \\
\leq \|P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + \|\partial_x u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2. \tag{2.51}
\]

Then, from (2.50) and (2.51), we have that
\[
2 \int_0^t |P_\delta| |\partial_x u_\delta| \, ds \, dx \\
\leq \int_0^t \|P_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds + \int_0^t \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T).
\]

Therefore,
\[
2\delta \left| \int_0^t \int_{\mathbb{R}} P_\delta \partial_{tx}^2 P_\delta \, ds \, dx \right| \leq \|P_{\varepsilon, 0}\|_{L^2(\mathbb{R})}^2 + C(T),
\]

which gives (2.28). \(\square\)
LEMMA 2.6. Let $T > 0$. Then,

$$
\|u_\delta\|_{L^\infty(I_{T,1})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T),
$$

(2.52)

where $I_{T,1}$ is defined in (2.27).

Proof. Due to (2.1) and (2.24),

$$
\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta \leq \gamma C(T).
$$

Since the map

$$
\mathcal{F}(t) := \|u_0\|_{L^\infty(\mathbb{R})} + \gamma C(T)t,
$$

solves the equation

$$
\frac{d\mathcal{F}}{dt} = \gamma C(T)
$$

and

$$
\max\{u_\delta(0, x), 0\} \leq \mathcal{F}(t), \quad (t, x) \in I_{T,1},
$$

the comparison principle for parabolic equations implies that

$$
u_\delta(t, x) \leq \mathcal{F}(t), \quad (t, x) \in I_{T,1}.
$$

In a similar way we can prove that

$$
u_\delta(t, x) \geq -\mathcal{F}(t), \quad (t, x) \in I_{T,1}.
$$

Therefore,

$$
|u_\delta(t, x)| \leq \|u_0\|_{L^\infty(\mathbb{R})} + \gamma C(T)t \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T),
$$

which gives (2.52). □

LEMMA 2.7. Let $T > 0$ and $0 < \delta < 1$. We have that

$$
\|\partial_x u_\delta(t, \cdot)\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_{xx}^2 u_\delta(s, \cdot)\|^2_{L^2(\mathbb{R})} ds \leq C(T).
$$

(2.53)

Proof. Let $0 < t < T$. Multiplying (2.1) by $-\partial_{xx}^2 u_\delta$, we have

$$
-\partial_{xx}^2 u_\delta \partial_t u_\delta + (\partial_{xx}^2 u_\delta)^2
= -\gamma P_\delta \partial_{xx}^2 u_\delta - f'(u_\delta) \partial_x u_\delta \partial_{xx}^2 u_\delta.
$$

(2.54)

Since

$$
-\int_\mathbb{R} \partial_{xx}^2 u_\delta \partial_t u_\delta dx = \frac{d}{dt} \left( \frac{1}{2} \int_\mathbb{R} (\partial_x u_\delta)^2 dx \right),
$$

integrating (2.54) on $\mathbb{R}$, we get

$$
\frac{d}{dt} \left( \int_\mathbb{R} (\partial_x u_\delta)^2 dx \right) + 2 \int_\mathbb{R} (\partial_{xx}^2 u_\delta)^2 dx
$$
\[ -2 \gamma \int_{\mathbb{R}} P_{\delta} \partial_{xx}^2 u_{\delta} \, dx \]
\[ -2 \int_{\mathbb{R}} f'(u_{\delta}) \partial_{x} u_{\delta} \partial_{xx}^2 u_{\delta} \, dx. \]

Due to (2.10), (2.25), (2.52) and the Young inequality,

\[ -2 \gamma \int_{\mathbb{R}} P_{\delta} \partial_{xx}^2 u_{\delta} \, dx \leq 2 \gamma \left| \int_{\mathbb{R}} P_{\delta} \partial_{xx}^2 u_{\delta} \, dx \right| \]
\[ \leq 2 \int_{\mathbb{R}} \sqrt{2} \gamma P_{\delta} \left| \frac{\partial_{xx}^2 u_{\delta}}{\sqrt{2}} \right| \, dx \]
\[ \leq 2 \gamma^2 \| P_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ \leq C(T) + \frac{1}{2} \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]
\[ -2 \int_{\mathbb{R}} f'(u_{\delta}) \partial_{x} u_{\delta} \partial_{xx}^2 u_{\delta} \, dx \]
\[ \leq 2 \int_{\mathbb{R}} \sqrt{2} f'(u_{\delta}) \partial_{x} u_{\delta} \left| \frac{\partial_{xx}^2 u_{\delta}}{\sqrt{2}} \right| \, dx \]
\[ \leq 2 \int_{\mathbb{R}} (f'(u_{\delta}))^2 (\partial_{x} u_{\delta})^2 + \frac{1}{2} \int_{\mathbb{R}} (\partial_{xx}^2 u_{\delta})^2 \, dx \]
\[ \leq 2 \| f' \|_{L^\infty(I_{T,2})}^2 \| \partial_{x} u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

where

\[ I_{T,2} = (-\| u_0 \|_{L^\infty(\mathbb{R})} - C(T), \| u_0 \|_{L^\infty(\mathbb{R})} + C(T)). \]

Therefore,

\[ \frac{d}{dt} \left( \| \partial_{x} u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + 2 \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ \leq \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \| f' \|_{L^\infty(I_{T,2})}^2 \| \partial_{x} u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C(T), \]

that is

\[ \frac{d}{dt} \left( \| \partial_{x} u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + \| \partial_{xx}^2 u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ \leq \| f' \|_{L^\infty(I_{T,2})}^2 \| \partial_{x} u_{\delta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C(T). \]
An integration on \((0, t)\) and (2.2) give
\[
\|\partial_x u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_x^2 u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq 2 \|f'\|_{L^\infty(IT, 2)}^2 \int_0^t \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds + C(T). \tag{2.56}
\]

(2.53) follows from (2.50) and (2.56).

**Lemma 2.8.** Let \(T > 0\) and \(0 < \delta < 1\). We have that
\[
\|\partial_x u_\delta\|_{L^\infty(IT, 1)} \leq C(T), \tag{2.57}
\]
where \(IT, 1\) is defined in (2.27). Moreover,
\[
\|\partial_x^2 u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_x^3 u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T). \tag{2.58}
\]

**Proof.** Let \(0 < t < T\). Multiplying (2.1) by \(\partial_{xxxx}^4 u_\delta\), we have
\[
\partial_{xxxx}^4 u_\delta \partial_t u_\delta - \partial_{xxxx}^4 u_\delta \partial_{xx}^2 u_\delta = \gamma P_\delta \partial_{xxxx}^4 u_\delta - f'(u_\delta) \partial_x u_\delta \partial_{xxxx}^4 u_\delta. \tag{2.59}
\]

Since
\[
\int_\mathbb{R} \partial_{xxxx}^4 u_\delta \partial_t u_\delta \, dx = \frac{d}{dt} \left( \frac{1}{2} \int_\mathbb{R} (\partial_{xx}^2 u_\delta)^2 \, dx \right),
\]
\[- \int_\mathbb{R} \partial_{xxxx}^4 u_\delta \partial_{xx}^2 u_\delta \, dx = \int_\mathbb{R} (\partial_{xx}^3 u_\delta)^2 \, dx,
\]
\[
\gamma \int_\mathbb{R} P_\delta \partial_{xxxx}^4 u_\delta \, dx = -\gamma \int_\mathbb{R} \partial_x P_\delta \partial_{xx}^3 u_\delta \, dx,
\]
\[- \int_\mathbb{R} f'(u_\delta) \partial_t u_\delta \partial_{xxxx}^4 u_\delta \, dx = \int_\mathbb{R} f''(u_\delta) (\partial_t u_\delta)^2 \partial_{xx}^3 u_\delta \, dx
\]
\[
+ \int_\mathbb{R} f'(u_\delta) \partial_{xx}^2 u_\delta \partial_{xxxx}^3 u_\delta \, dx,
\]
integrating (2.54) on \(\mathbb{R}\), we get
\[
\frac{d}{dt} \left( \int_\mathbb{R} (\partial_{xx}^2 u_\delta)^2 \, dx \right) + 2 \int_\mathbb{R} (\partial_{xx}^3 u_\delta)^2 \, dx
\]
\[- 2\gamma \int_\mathbb{R} \partial_x P_\delta \partial_{xx}^3 u_\delta \, dx
\]
\[
+ 2 \int_\mathbb{R} f''(u_\delta) (\partial_t u_\delta)^2 \partial_{xx}^3 u_\delta \, dx
\]
\[
+ 2 \int_\mathbb{R} f'(u_\delta) \partial_{xx}^2 u_\delta \partial_{xxxx}^3 u_\delta \, dx.
\]
Due to (2.11), (2.52), (2.53) and the Young inequality,

\[-2γ \int_\mathbb{R} \partial_x P_δ \partial_{xxx}^3 u_δ dx\]
\[\leq 2γ \left| \int_\mathbb{R} \partial_x P_δ \partial_{xxx}^3 u_δ dx \right|\]
\[\leq 2 \int_\mathbb{R} \sqrt{2γ} \partial_x P_δ \left| \frac{\partial_{xxx}^3 u_δ}{\sqrt{2}} \right| dx\]
\[\leq 3γ^2 \|∂_x P_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]
\[\leq C(T) + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 ,\]

\[2 \int_\mathbb{R} f''(u_δ) (\partial_x u_δ)^2 \partial_{xxx}^3 u_δ dx\]
\[\leq 2 \left| \int_\mathbb{R} f''(u_δ) (\partial_x u_δ)^2 \partial_{xxx}^3 u_δ dx \right|\]
\[\leq 2 \int_\mathbb{R} \sqrt{2} f''(u_δ) (\partial_x u_δ)^2 \left| \frac{\partial_{xxx}^3 u_δ}{\sqrt{2}} \right| dx\]
\[\leq 3 \int_\mathbb{R} (f''(u_δ))^2 (\partial_x u_δ)^4 dx + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]
\[\leq 3 \left\| f'' \right\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_δ\|_{L^\infty(I_{T,1})}^2 \|\partial_x u_δ(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]

\[2 \int_\mathbb{R} f'(u_δ) \partial_{xx}^2 u_δ \partial_{xxx}^3 u_δ dx\]
\[\leq 2 \left| \int_\mathbb{R} f'(u_δ) \partial_{xx}^2 u_δ \partial_{xxx}^3 u_δ dx \right|\]
\[\leq 2 \int_\mathbb{R} \sqrt{2} f'(u_δ) \partial_{xx}^2 u_δ \left| \frac{\partial_{xxx}^3 u_δ}{\sqrt{2}} \right| dx\]
\[\leq 3 \int_\mathbb{R} (f'(u_δ))^2 (\partial_{xx}^2 u_δ)^2 dx + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]
\[\leq 3 \left\| f' \right\|_{L^\infty(I_{T,2})}^2 \|\partial_{xx}^2 u_δ(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{3} \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \],

where \(I_{T,1}\) is defined in (2.27) and \(I_{T,2}\) is defined in (2.55). Therefore,

\[\frac{d}{dt} \left( \|\partial_{xx}^2 u_δ(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2 \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]
\[\leq \|\partial_{xxx}^3 u_δ (t, \cdot)\|_{L^2(\mathbb{R})}^2 \]
\[ + 3 \left\| f'' \right\|_{L^\infty(I_{T,2})}^2 C(T) \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 \]
\[ + 3 \left\| f' \right\|_{L^\infty(I_{T,2})}^2 \left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T), \]

that is
\[
\frac{d}{dt} \left( \left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + \left\| \partial_{xxx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T) \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 + C(T) \]
\[ + C(T) \left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \]

An integration on \((0, t), (2.2)\) and (2.53) give
\[
\left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_{xxx} u^\delta(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\leq \left( C(T) \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 + C(T) \right) \int_0^t ds
\]
\[ + C(T) \int_0^t \left\| \partial_{xx} u^\delta(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\leq C(T) \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 + C(T). \]

Thus,
\[
\left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_{xxx} u^\delta(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds
\leq C(T) \left( 1 + \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 \right). \tag{2.60} \]

Due to (2.53), (2.60) and the Hölder inequality,
\[
\left( \partial_x u^\delta(t, x) \right)^2 \leq 2 \int_\mathbb{R} |\partial_x u^\delta| |\partial_{xx} u^\delta| dx
\leq 2 \left\| \partial_x u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_{xx} u^\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}
\leq C(T) \sqrt{\left( 1 + \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 \right)}.
\]

Then,
\[
\left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^4 - C(T) \left\| \partial_x u^\delta \right\|_{L^\infty(I_{T,1})}^2 - C(T) \leq 0,
\]
which gives (2.57).

(2.58) follows from (2.57) and (2.60).

Arguing as in [5], we obtain the following result

**Lemma 2.9.** Let \( T > 0, \ell > 2 \) and \( 0 < \delta < 1 \). For each \( t \in (0, T) \),
\[
\partial_{x}^\ell u^\delta(t, \cdot) \in L^2(\mathbb{R}). \tag{2.61} \]

---

**[5]**
3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We begin by proving the following result

**Lemma 3.1.** Let $T > 0$. Assume (1.9), (1.10), (1.11) and (1.12). Then there exist

$$u \in L^\infty((0, T) \times \mathbb{R}) \cap C((0, T); H^\ell(\mathbb{R})), \quad \ell > 2, \quad (3.1)$$

$$P \in L^\infty((0, T) \times \mathbb{R}) \cap L^2((0, T) \times \mathbb{R}), \quad (3.2)$$

where $u$ is a classical solution of the Cauchy problem of (1.8).

**Proof.** Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex $C^2$ entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (2.1) with $\eta'(u_\delta)$ and using the chain rule, we get

$$\partial_t \eta(u_\delta) + \partial_x q(u_\delta) = \frac{\partial_x^2 \eta(u_\delta)}{\partial x^2} + \eta''(u_\delta)(\partial_x u_\delta)^2 + \gamma \eta'(u_\delta)P_\delta, \quad := \mathcal{L}_{1,\delta}$$

where $\mathcal{L}_{1,\delta}, \mathcal{L}_{2,\delta}, \mathcal{L}_{3,\delta}$ are distributions.

Let us show that

$$\{\mathcal{L}_{1,\delta}\}_\delta \text{ is compact in } H^{-1}((0, T) \times \mathbb{R}), \quad T > 0. \quad (3.3)$$

Since

$$\partial_x^2 \eta(u_\delta) = \partial_x (\eta'(u_\delta) \partial_x u_\delta),$$

we have to prove that

$$\{\eta'(u_\delta) \partial_x u_\delta\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}), \quad T > 0, \quad (3.4)$$

$$\{\eta''(u_\delta)(\partial_x u_\delta)^2 + \eta'(u_\delta) \partial_x^2 u_\delta\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}), \quad T > 0. \quad (3.5)$$

We begin by proving that (3.4) holds. Thanks to Lemmas 2.3 and 2.6,

$$\|\eta'(u_\delta) \partial_x u_\delta\|_{L^2((0, T) \times \mathbb{R})}^2 \leq \|\eta'\|_{L^\infty(I_{T,2})}^2 \int_0^T \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \|\eta'\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T} \int_0^T e^{-2\gamma s} \|\partial_x u_\delta(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \frac{1}{2} \|\eta'\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T} \|u_0\|_{L^2(\mathbb{R})}^2 \leq C(T),$$

where $I_{T,2}$ is defined in (2.55).

We claim that

$$\{\eta''(u_\delta)(\partial_x u_\delta)^2\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R)). \quad (3.6)$$
Due to Lemmas 2.3, 2.6, 2.8

\[ \left\| \eta''(u_\delta)(\partial_x u_\delta)^2 \right\|_{L^2((0,T) \times \mathbb{R})}^2 \leq \left\| \eta'' \right\|_{L^\infty(I_T,\mathbb{R})}^2 \int_0^T \left\| (\partial_x u_\delta(s,x))^4 \right\| ds \, dx \]

\[ \leq \left\| \eta'' \right\|_{L^\infty(I_T,\mathbb{R})}^2 \left\| \partial_x u_\delta \right\|_{L^\infty(I_T,\mathbb{R})}^2 \int_0^T \left\| \partial_x u_\delta(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \]

\[ \leq \frac{1}{2} \left\| \eta'' \right\|_{L^\infty(I_T,\mathbb{R})}^2 \left\| \partial_x u_\delta \right\|_{L^\infty(I_T,\mathbb{R})}^2 e^{2\gamma T} \left\| u_0 \right\|_{L^2(\mathbb{R})}^2 \leq C(T), \]

where \( I_{T,1} \) is defined in (2.27).

We claim that

\[ \{ \eta'(u_\delta)\partial_x^2 u_\delta \}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}). \]  

(3.7)

Thanks to Lemmas 2.6 and 2.7,

\[ \left\| \eta'(u_\delta)\partial_x^2 u_\delta \right\|_{L^2((0,T) \times \mathbb{R})}^2 \leq \left\| \eta' \right\|_{L^\infty(I_T,\mathbb{R})}^2 \int_0^T \left\| \partial_x^2 u_\delta(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \]

\[ \leq \left\| \eta' \right\|_{L^\infty(I_T,\mathbb{R})}^2 C(T) \leq C(T). \]

(3.6) and (3.7) give (3.5).

Therefore, (3.3) follows from (3.4) and (3.5).

We have that

\[ \{ \mathcal{L}_{2,\delta} \}_{\delta > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}). \]

Due to Lemmas 2.3, 2.6,

\[ \left\| \eta''(u_\delta)(\partial_x u_\delta)^2 \right\|_{L^1((0,T) \times \mathbb{R})} \leq \left\| \eta'' \right\|_{L^\infty(I_T,\mathbb{R})} \int_0^T \left\| \partial_x u_\delta(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \]

\[ \leq \left\| \eta' \right\|_{L^\infty(I_T,\mathbb{R})} \frac{e^{2\gamma T}}{2} \int_0^T e^{-2\gamma s} \left\| \partial_x u_\delta(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \]

\[ \leq \frac{1}{2} \left\| \eta' \right\|_{L^\infty(I_T,\mathbb{R})} \frac{e^{2\gamma T}}{2} \left\| u_0 \right\|_{L^2(\mathbb{R})}^2 \leq C(T). \]

We have that

\[ \{ \mathcal{L}_{3,\delta} \}_{\delta > 0} \text{ is bounded in } L^1_{loc}((0, T) \times \mathbb{R}). \]

Let \( K \) be a compact subset of \((0, T) \times \mathbb{R}\). By Lemmas 2.5 and 2.6,

\[ \left\| \gamma \eta'(u_\delta) P_\delta \right\|_{L^1(K)} = \gamma \int_K |\eta'(u_\delta)| |P_\delta| \, dt \, dx \]

\[ \leq \gamma \left\| \eta' \right\|_{L^\infty(I_T,\mathbb{R})} \left\| P_\delta \right\|_{L^\infty(I_{T,1})} |K|. \]

Therefore, Murat’s Lemma [21] implies that

\[ \{ \partial_t \eta(u_\delta) + \partial_x q(u_\delta) \}_{\delta > 0} \text{ lies in a compact subset of } H^1_{loc}((0, \infty) \times \mathbb{R}). \]  

(3.8)
The $L^\infty$ bound stated in Lemma 2.6, (3.8) and the Tartar’s compensated compactness method [27] give the existence of a subsequence $\{u_{\delta_k}\}_{k \in \mathbb{N}}$ and a limit function $u \in L^\infty((0, T) \times \mathbb{R})$ such that

$$u_{\delta_k} \to u \text{ a.e. and in } L^p_{loc}((0, T) \times \mathbb{R}), \quad 1 \leq p < \infty. \quad (3.9)$$

Hence,

$$u_{\delta_k} \to u \text{ in } L^\infty((0, T) \times \mathbb{R}). \quad (3.10)$$

Moreover, for convexity, we have

$$\|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),$$

$$\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_{xx} u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T), \quad (3.11)$$

$$\|\partial_{xxx} u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\partial_{xxxx} u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T).$$

We need only to observe that

$$2e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq 2e^{2\gamma t} \liminf_k \int_0^t e^{-2\gamma s} \|\partial_x u_{\delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),$$

$$\int_0^t \|\partial_{xx} u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \liminf_k \int_0^t \|\partial_{xx} u_{\delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),$$

$$\int_0^t \|\partial_{xxx} u(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq \liminf_k \int_0^t \|\partial_{xxx} u_{\delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T).$$

Moreover, it follows from convexity and Lemma 2.9 that

$$\partial_{\ell x}^\ell u(t, \cdot) \in L^2(\mathbb{R}), \quad \ell > 2, \quad t \in (0, T). \quad (3.12)$$

Therefore, (3.10), (3.11) and (3.12) give (3.1). (3.2) follows from Lemma 2.5.

Finally, we prove that

$$\int_{-\infty}^x u(t, y) \, dy = P(t, x), \quad \text{a.e. in } (t, x) \in I_{T, 1}. \quad (3.13)$$

Integrating the second equation of (2.1) on $(-\infty, x)$, for (2.3), we have that

$$\int_{-\infty}^x u_{\delta_k}(t, y) \, dy = P_{\delta_k}(t, x) - \delta_k \partial_x P_{\delta_k}(t, x). \quad (3.14)$$

We show that

$$\delta \partial_x P_{\delta}(t, x) \to 0 \text{ in } L^\infty((0, T) \times \mathbb{R}), \quad T > 0 \text{ as } \delta \to 0. \quad (3.15)$$
It follows from (2.11) that
\[ \delta \| \partial_x P_0 \|_{L^\infty((0,T) \times \mathbb{R})} \leq \sqrt{\delta} e^{\gamma T} \| u_{\epsilon,0} \|_{L^2(\mathbb{R})} = \sqrt{\delta} C(T) \to 0, \]
that is (3.15).
Therefore, (3.13) follows from (3.1), (3.2), (3.14) and (3.15). The proof is done. \(\square\)

**Lemma 3.2.** Let \(u(t, x)\) be a classical solution of (1.7), or (1.8). Then,
\[ \int_{\mathbb{R}} u(t, x) dx = 0, \quad t \geq 0, \]  \hspace{1cm} (3.16)

**Proof.** Differentiating (1.8) with respect to \(x\), we have
\[ \partial_x (\partial_t u + \partial_x f(u) - \partial^2_{xx} u) = \gamma u. \]  \hspace{1cm} (3.17)

Since \(u\) is a smooth solution of (1.8), an integration over \(\mathbb{R}\) gives (3.16). \(\square\)

We are ready for the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Lemma 3.1 gives the existence of a classical solution of (1.7), or (1.8), while Lemma 3.2 says that the solution has zero mean.

Let us show that \(u(t, x)\) is unique and (1.14) holds. Let \(u, v\) be two classical solutions of (1.7), or (1.8), that is
\[
\begin{align*}
\partial_t u + f'(u) \partial_x u &= \gamma P^u + \partial^2_{xx} u, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_x P^u &= u, \quad t > 0, \ x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
\partial_t v + f'(v) \partial_x v &= \gamma P^v + \partial^2_{xx} v, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_x P^v &= v, \quad t > 0, \ x \in \mathbb{R}, \\
v(0, x) &= v_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

Then, the function
\[ \omega(t, x) = u(t, x) - v(t, x) \]  \hspace{1cm} (3.18)
is solution of the following Cauchy problem
\[
\begin{align*}
\partial_t \omega + f'(u) \partial_x u - f'(v) \partial_x v &= \gamma \Omega + \partial^2_{xx} \omega, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_x \Omega &= \omega, \quad t > 0, \ x \in \mathbb{R}, \\
\omega(0, x) &= u_0(x) - v_0(x), \quad x \in \mathbb{R},
\end{align*}
\]  \hspace{1cm} (3.19)
where
\[
\begin{align*}
\Omega(t, x) &= P^u(t, x) - P^v(t, x) \\
&= \int_{-\infty}^{x} u(t, y) dy - \int_{-\infty}^{x} v(t, y) dy \\
&= \int_{-\infty}^{x} (u(t, y) - v(t, y)) dy = \int_{-\infty}^{x} \omega(t, y) dy.
\end{align*}
\]  \hspace{1cm} (3.20)
It follows from Lemma 3.2 and (3.20) that
\[
\Omega(t, \infty) = \int_{\mathbb{R}} u(t, y) dy - \int_{\mathbb{R}} v(t, y) dy = 0. 
\] (3.21)

Observe that, from (3.18),
\[
f'(u) \partial_x u - f'(v) \partial_x v = f'(u) \partial_x u - f'(u) \partial_x v + f'(u) \partial_x v - f'(v) \partial_x v \\
= f'(u) \partial_x (u - v) + (f'(u) - f'(v)) \partial_x v \\
= f'(u) \partial_x \omega + (f'(u) - f'(v)) \partial_x v.
\]

Therefore, the first equation of (3.19) is equivalent to the following one:
\[
\partial_t \omega + f'(u) \partial_x \omega + (f'(u) - f'(v)) \partial_x v = \gamma \Omega + \partial_{xx} \omega. \quad (3.22)
\]

Moreover, since \( u \) and \( v \) are in \( L^\infty((0, T) \times \mathbb{R}) \), we have that
\[
\left| f'(u(t, x)) - f'(v(t, x)) \right| \leq C(T) |u(t, x) - v(t, x)|, \quad (t, x) \in (0, T) \times \mathbb{R}, \quad (3.23)
\]
where
\[
C(T) = \sup_{(0, T) \times \mathbb{R}} \left\{ \left| f''(u) \right| + \left| f''(v) \right| \right\}. \quad (3.24)
\]

Therefore, (3.18) and (3.23) give
\[
\left| f'(u(t, x)) - f'(v(t, x)) \right| \leq C(T) |\omega(t, x)|, \quad (t, x) \in (0, T) \times \mathbb{R}. \quad (3.25)
\]

Multiplying (3.22) by \( \omega \), an integration on \( \mathbb{R} \) gives
\[
\frac{d}{dt} \int_{\mathbb{R}} \omega^2 dx = 2 \int_{\mathbb{R}} \omega \partial_t \omega dx \\
= 2 \int_{\mathbb{R}} \omega \partial_{xx} \omega dx - 2 \int_{\mathbb{R}} \omega f'(u) \partial_x \omega dx \\
- 2 \int_{\mathbb{R}} \omega (f'(u) - f'(v)) \partial_x v dx + 2 \gamma \int_{\mathbb{R}} \Omega \omega dx \\
= -2 \int_{\mathbb{R}} (\partial_x \omega)^2 dx + \int_{\mathbb{R}} \omega^2 f''(u) \partial_x u dx \\
- 2 \int_{\mathbb{R}} \omega (f'(u) - f'(v)) \partial_x v dx + 2 \gamma \int_{\mathbb{R}} \Omega \omega dx.
\]

It follows from the second equation of (3.19) and Lemma 3.2 that
\[
\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq \int_{\mathbb{R}} \omega^2 |f''(u)||\partial_x u| dx + 2 \int_{\mathbb{R}} |\omega||f'(u) - f'(v)||\partial_x v| dx. \quad (3.26)
\]
Since $u(t, \cdot), v(t, \cdot) \in H^\ell(\mathbb{R}), \ell > 2$, for each $t \in (0, T)$, then
\[
\partial_x u(t, \cdot), \partial_x v(t, \cdot) \in H^{\ell - 1}(\mathbb{R}) \subset L^\infty(\mathbb{R}), \quad t \in (0, T).
\] (3.27)

Therefore, thanks to (3.23), (3.24), (3.26) and (3.27),
\[
\frac{d}{dt} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \|\partial_x \omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\]
The Gronwall Lemma gives
\[
\|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2e^{C(T)t} \int_0^t e^{-C(T)s} \|\partial_x \omega(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq e^{C(T)t} \|\omega_0\|_{L^2(\mathbb{R})}^2.
\] (3.28)

Hence, (1.14) follows from (3.18), (3.19) and (3.28).

\[\Box\]

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