PERSISTENCE OF AUTOREGRESSIVE SEQUENCES WITH LOGARITHMIC TAILS

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Abstract. We consider autoregressive sequences
\[ X_n = aX_{n-1} + \xi_n \]
and
\[ M_n = \max\{aM_{n-1}, \xi_n\} \]
with a constant \( a \in (0, 1) \) and with positive, independent and identically distributed innovations \( \{\xi_k\} \). It is known that if \( P(\xi_1 > x) \sim d \log x \) with some \( d \in (0, -\log a) \) then the chains \( \{X_n\} \) and \( \{M_n\} \) are null recurrent. We investigate the tail behaviour of recurrence times in this case of logarithmically decaying tails. More precisely, we show that the tails of recurrence times are regularly varying of index \( -1 - \frac{d}{\log a} \). We also prove limit theorems for \( \{X_n\} \) and \( \{M_n\} \) conditioned to stay over a fixed level \( x_0 \).

Furthermore, we study tail asymptotics for recurrence times of \( \{X_n\} \) and \( \{M_n\} \) in the case when these chains are positive recurrent and the tail of \( \log \xi_1 \) is subexponential.

1. Introduction.

Let \( \{\xi_n\}_{n\ge1} \) be a sequence of independent and identically distributed random variables. Let \( a \in (0, 1) \) be a constant. The corresponding AR(1)-sequence \( X = \{X_n\}_{n\ge0} \) is defined by
\[ X_n := aX_{n-1} + \xi_n, \quad n \ge 1, \]
where the starting position \( X_0 \) can be either random or deterministic.

Besides the Markov chain \( X \) we shall consider the so-called maximal autoregressive sequence \( M = \{M_n\}_{n\ge0} \), where
\[ M_n = \max\{aM_{n-1}, \xi_n\}, \quad n \ge 1. \]

The Markov chains \( X \) and \( M \) have rather similar properties. If, for example, the innovations are non-negative then these two chains are recurrent, positive recurrent or transient at the same time. Moreover, according to Theorem 3.1 in Zerner [19], the chains \( \{X_n\} \) and \( \{M_n\} \) are recurrent if and only if
\[ \sum_{n=0}^{\infty} \prod_{m=0}^{n} P(|\xi_1| \leq ta^{-m}) = \infty \quad (1) \]
for every \( t \) satisfying \( P(|\xi_1| \leq t) > 0 \). Furthermore, \( X \) and \( M \) are positive recurrent if and only if \( E[\log(1 + |\xi_1|)] \) is finite.

If the innovations \( \{\xi_n\} \) take only positive values, then we may define
\[ \eta_n := \log a \xi_n \quad \text{and} \quad R_n := \log a M_n, \]

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where \( A = a^{-1} \). Then the sequence \( R = \{ R_n \}_{n \geq 0} \) satisfies the recursive relation

\[
R_n = \max \{ R_{n-1} - 1, \eta_n \}, \quad n \geq 1.
\]

This Markov chain is a special random exchange process, see Helland and Nilsen [12] for the definition of this class of processes.

In this paper we shall consider the case when the tail of innovations decreases logarithmically. More precisely, the main part of the paper will deal with situation when

\[
P(\xi_1 > x) \sim \frac{d}{\log x} \quad \text{as } x \to \infty
\]

with some constant \( d > 0 \). This is equivalent to

\[
P(\eta_1 > y) \sim \frac{c}{y}, \quad c := \frac{d}{\log A}.
\]

(We shall explicitly mention one of these two conditions every time we need it.)

Notice also that if \( d > 0 \) then \( E \log(1 + \xi_1) = \infty \) and, consequently, the chains \( \{ X_n \}_{n \geq 0} \) and \( \{ M_n \}_{n \geq 0} \) are not positive recurrent. If (2) holds then, using the criterion (1), we conclude that

\[
\begin{align*}
\bullet & \quad d > \log A \quad (c > 1) \Rightarrow \{ X_n \}_{n \geq 0} \text{ and } \{ M_n \}_{n \geq 0} \text{ are transient;} \\
\bullet & \quad d < \log A \quad (c < 1) \Rightarrow \{ X_n \}_{n \geq 0} \text{ and } \{ M_n \}_{n \geq 0} \text{ are null-recurrent.}
\end{align*}
\]

In the critical case \( d = \log A \quad (c = 1) \) one has to consider further terms in the asymptotic representation for tails \( P(\xi_1 > x) \), \( P(\eta_1 > y) \). Assume that, for some \( k \geq 0 \),

\[
P(\eta_1 > y) = \sum_{j=0}^k \prod_{l=1}^j \frac{1}{\log(l) y} + (r_k + o(1)) \frac{1}{y} \prod_{l=1}^{k+1} \frac{1}{\log(l) y}, \quad y \to \infty,
\]

where \( \log(l) x \) is the \( l \)-th iteration of the logarithm. Then, applying (1) once again, we obtain

\[
\begin{align*}
\bullet & \quad r_k > 1 \Rightarrow \{ X_n \} \text{ and } \{ M_n \} \text{ are transient;} \\
\bullet & \quad r_k < 1 \Rightarrow \{ X_n \} \text{ and } \{ M_n \} \text{ are null-recurrent.}
\end{align*}
\]

A further similarity between the chains \( \{ X_n \}_{n \geq 0} \) and \( \{ M_n \}_{n \geq 0} \) consists in the joint scaling behaviour of these chains. More precisely, Buraczewski and Iksanov [8] have shown that if (2) is valid then

\[
Z(t) = (Z(t))_{t \geq 0}
\]

in the Skorohod \( J_1 \)-topology on the space \( D \). The limiting process \( Z \) is a self-similar Markov process. In [8] it is described with the help of an appropriate Poisson point process. One can describe this limiting process also via the transition probabilities:

\[
P_x((x-t)^+ \leq Z_t \leq y) = \left( \frac{y}{y+t} \right)^c, \quad y \geq (x-t)^+, \quad x \geq 0.
\]

It is easy to see that if \( X_0 = M_0 \) then

\[
M_k \leq X_k \leq (k+1)M_k \quad \text{for all } k \geq 1.
\]
This implies that (4) is equivalent to
\[
\left( \frac{\log A}{n} \right)_{n \geq 1} \Rightarrow Z. \tag{6}
\]
In its turn, (6) is equivalent to
\[
\left( \frac{R_{nt}}{n} \right)_{t \geq 0} \Rightarrow Z. \tag{7}
\]

The main purpose of this paper is to study the asymptotic behaviour of recurrence times
\[
T^{(X)}_x := \inf \{ k \geq 1 : X_k \leq x \},
T^{(M)}_x := \inf \{ k \geq 1 : M_k \leq x \},
T^{(R)}_x := \inf \{ k \geq 1 : R_k \leq x \}.
\]

Persistence of autoregressive processes has attracted a significant attention of many researchers in the recent past, but almost all results known in the literature deal with the case when some power moments of the innovations \( \xi_k \) are finite. It is known that the tail of \( T^{(X)}_x \) decreases exponentially fast
\[
- \frac{1}{n} \log P(T^{(X)}_x > n) \to \lambda \in (0, \infty)
\]
see [4], [13] and references there. If all power moments of innovations are finite then \( P(T^{(X)}_x > n) \sim C e^{-\lambda n} \) and the conditional distribution \( P(X_n \in \cdot | T^{(X)}_x > n) \) converges towards the corresponding quasi-stationary distribution, see [13]. It is worth mentioning that one can compute the persistence exponent \( \lambda \) in some special cases only. Some examples of autoregressive processes, for which there exist closed form expressions for \( \lambda \), can be found in [1] and in [4]. The authors of [3] have found a series representation for \( \lambda \) in the case of normally distributed innovations.

In the present paper we concentrate on the case when all power moments of innovations are infinite. This corresponds, as we shall show, to a subexponential decay of the tail of \( T^{(X)}_x \).

We start with the null-recurrent case. More precisely we consider first the innovations which satisfy (2). As we have mentioned before, the chains \( \{X_n\}_{n \geq 0} \), \( \{M_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) have the same scaling limit \( Z \) in this case. For that reason we first collect some crucial for us properties of the process \( Z \).

**Theorem 1.** (a) If \( c \leq 1 \) then the process \( Z \) is recurrent. If \( c < 1 \) then the stopping time \( T^{(Z)}_0 := \inf \{ s > 0 : Z_s = 0 \text{ or } Z_{s-} = 0 \} \) is almost surely finite and, furthermore,
\[
P \left( T^{(Z)}_0 > t \big| Z_0 = z \right) = \begin{cases} 1, & t < z \\ \frac{1}{B(c,1-c)} \int_0^{z/t} (1-u)^{c-1} u^{-c} du, & t \geq z. \end{cases} \tag{8}
\]

(b) The function \( u(z) = z^{1-c} \) is harmonic for \( Z \) killed at \( T^{(Z)}_0 \):
\[
u(z) = E_z[u(Z_t); T^{(Z)}_0 > t], \quad t, z > 0.
\]

(c) The sequence of distributions \( P_z \left( Z \in \cdot | T^{(Z)}_0 > 1 \right) \) on \( D[0,1] \) converges weakly, as \( z \to 0 \), towards a non-degenerate distribution \( Q \).
We now turn to the recurrence times of the chains \( \{M_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \). Since

\[ R_n = \log_A M_n, \]

\[ T^{(R)}_{x_0} = \inf\{n \geq 1 : R_n \leq x_0\} = \inf\{n \geq 1 : M_n \leq A^{-n_0}\} = T^{(M)}_{A^{-n_0}}. \]

Thus, it suffices to formulate the results for one of these processes.

Set

\[ u_0(x) := \int_0^x \mathbb{P}(\eta_1 > y)dy, \quad x \geq 0 \]

and

\[ U_0(x) := \int_0^x e^{-u_0(y)}dy, \quad x \geq 0. \]  

(9)

If (3) holds then \( u_0(x) \sim c \log x \) as \( x \to \infty \) and \( e^{-u_0(x)} \) is regularly varying of index \(-c\). Consequently, the function \( U_0(x) \) is regularly varying of index \( 1 - c \).

**Theorem 2.** Assume that \( x_0 \) is such that \( \mathbb{P}(\eta_1 \leq x_0)\mathbb{P}(\eta_1 > x_0) > 0 \). Then the equation

\[ G(x) = E_x[G(R_1); T^{(R)}_{x_0} > 1], \quad x > x_0 \]

has a non-trivial solution if and only if \( E\eta_1^+ = \infty \). In the latter case

\[ G(x) = C \left( 1 + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} \mathbb{P}(\eta_1 \leq x_0 + j)1_{(x_0+j+1,\infty)}(x) \right) \]

for every \( C \in \mathbb{R} \).

If (3) holds with some \( c \in (0,1) \) then

(i) \( G(x) \sim \gamma U_0(x) \) for some \( \gamma \in (0,\infty) \);

(ii) there exists a constant \( C > 0 \) such that

\[ \frac{1}{C} \frac{G(x \wedge n)}{G(n)} \leq P_x(T^{(R)}_{x_0} > n) \leq C \frac{G(x)}{G(n)}, \quad n \geq 1, \ x > x_0; \]

(iii) there exists a positive constant \( \varkappa = \varkappa(c) \) such that

\[ P_x(T^{(R)}_{x_0} > n) \sim \frac{G(x)}{G(n)}, \quad n \to \infty. \]

and the sequence of conditional distributions \( P_x \left( \frac{R_{[n]}_n}{n} \in [T^{(R)}_{x_0} > n] \right) \) on \( D[0,1] \) converges weakly to \( Q \) defined in Theorem 1.

We now state our main result for the chain \( \{X_n\}_{n \geq 0} \).

**Theorem 3.** Assume that (3) holds with some \( c \in (0,1) \). (This is equivalent to (2) with \( 0 < d < \log A \).) For every \( x_0 \) satisfying \( \mathbb{P}(ax_0 + \xi_1 \leq x_0) > 0 \) we have:

(i) There exists a strictly positive on \((x_0,\infty)\) function \( V \) such that

\[ V(x) = E_x[V(X_1); T^{(X)}_{x_0} > 1], \quad x > x_0. \]

In other words, \( V \) is harmonic for the chain \( \{X_n\} \) killed at leaving \((x_0,\infty)\). Furthermore, \( V(A^x) \sim U_0(x) \), where \( U_0 \) is defined in (1). (This is equivalent to (2) with \( 0 < d < \log A \).)

(ii) There exists a constant \( C \) such that

\[ \frac{1}{C} \frac{V(x \wedge A^n)}{V(A^n)} \leq P_x(T^{(X)}_{x_0} > n) \leq C \frac{V(x)}{V(A^n)} \]

(10)

for all \( n \geq 1 \) and all \( x > x_0 \).
(iii) There exists a positive constant $\varepsilon = \varepsilon(c)$ such that, for every $x > x_0$,
\[
P_x(T_{x_0}^\varepsilon > n) \sim \frac{V(x)}{V(A^n)}, \quad n \to \infty.
\]  

Furthermore, the sequence of conditional distributions
\[
P_x \left( \frac{\log A |x_n|}{n} \in \big| T_{x_0}^\varepsilon > n \big) \right)
\]
on $D[0,1]$ converges weakly to $Q$ defined in Theorem 2.

We now turn to the positive recurrent case: $E[\xi_1] < \infty$. To determine the tail behaviour of recurrence times we shall assume that $\overline{F}(y) := P(\xi_1 > y)$ is subexponential. We make use of the following class introduced in [14].

**Definition 4.** A distribution function $F$ with finite $\mu_+ = \int_0^\infty \overline{F}(y)dy < \infty$ belongs to the class $S^*$ of strong subexponential distributions if $\overline{F}(x) > 0$ for all $x$ and
\[
\int_0^x \frac{\overline{F}(x-y)\overline{F}(y)dy}{\overline{F}(x)} \to 2\mu_+, \quad \text{as } x \to \infty.
\]

This class is a proper subclass of class $S$ of subexponential distributions. It is shown in [15] that the Pareto, lognormal and Weibull distributions belong to the class $S^*$. An example of a subexponential distribution with finite mean which does not belong to $S^*$ can be found in [10].

**Theorem 5.** Assume that $x_0$ is such that $P(\xi_1 \leq x_0)P(\xi_1 > x_0) > 0$. Assume also that $E[\xi_1] < \infty$ and that $F \in S^*$. Then, for any $x > x_0$
\[
P_x(T_{x_0}^{(R)} > n) \sim E_x[T_{x_0}^{(R)}]P(\xi_1 > n). \tag{12}
\]
The expectation $E_x[T_{x_0}^{(R)}]$ can be computed explicitly: for every $n \geq 0$ and every $x \in (x_0 + n, x_0 + n + 1)$ one has
\[
E_x[T_{x_0}^{(R)}] = \frac{1}{\prod_{k=0}^{\infty} P(\xi_1 \leq x_0 + k)} \left( 1 + \sum_{j=1}^{n-1} \prod_{k=0}^{j-1} P(\xi_1 \leq x_0 + k) \right).
\]

Our approach to the proof of this theorem is based on a recursive equation for the tail of $T_{x_0}^{(R)}$, see Proposition 18 below. In the case of the chain $\{X_n\}_{n \geq 0}$ we do not have such an equation and we have to work with upper and lower estimates. This leads to more restrictive assumptions on the tail of innovations $\xi_k$.

**Theorem 6.** Assume that $x_0$ is such that $P(ax_0 + \xi_1 \leq x_0) > 0$. Assume also that $E[\xi_1] < \infty$, that $F \in S^*$ and that
\[
P(\xi_1 > x) \sim P(\xi_1 > x - \log x), \quad \text{as } x \to \infty. \tag{13}
\]

Then, for any $x > x_0$,
\[
P_x(T_{x_0}^\varepsilon > n) \sim E_x[T_{x_0}^\varepsilon]P(\xi_1 > n). \tag{14}
\]

The rest of the paper is organised as follows. In Section 2 we discuss properties of $Z_t$ and prove Theorem 1. In Section 3 we construct harmonic functions for processes under consideration proving corresponding parts of Theorem 2 and Theorem 3. In Section 4 we derive lower and upper bounds for recurrence times Theorem 2 and Theorem 3, proving part (ii) of Theorem 2 and Theorem 3. In Section 5 we obtain the asymptotics for tails of recurrence times given in part (iii) of Theorem 2 and Theorem 3. In Section 6 we prove Theorem 5 and in Section 7 we prove Theorem 6.
2. Properties of the limiting process $Z$: proof of Theorem 1

It follows from (5) that if $t < x$ then

$$P_x(Z_t = x-t) = \left(\frac{x-t}{x}\right)^c \text{ and } \frac{P_x(Z_t \in dy)}{dy} = \frac{cy^{c-1}}{(t+y)^{c+1}} \quad y > x-t. \quad (15)$$

If $t \geq x$ then

$$P_x(Z_t \in dy) = \frac{cty^{c-1}}{(t+y)^{c+1}} \quad y > 0. \quad (16)$$

It is immediate from (5) that if $c \leq 1$ then

$$\int_0^\infty P_x(Z_t \leq y)dt = \infty$$

for all $x, y > 0$. Therefore, the process $Z$ is recurrent: it spends infinite amount of time in every interval $[0, y]$.

We next show that the state 0 is recurrent in the case $c < 1$. More precisely, we show that $P_x[\tau_{Z_0} < \infty] = 1$ for every $z > 0$. For that reason we compute first the generator of $Z$. Fix some $x > 0$ and a continuously differentiable bounded function $f$. It follows then from (15) that

$$E_x[f(Z_t)] = f(x-t)\left(\frac{x-t}{x}\right)^c + ct \int_{x-t}^{\infty} \frac{y^{c-1}}{(y+t)^{c+1}} f(y)dy, \quad t < x.$$

Therefore,

$$\frac{E_x[f(Z_t)] - f(x)}{t} = \frac{f(x-t) - f(x)}{t} + f(x-t)\left(\frac{x-t}{x}\right)^c - 1 + c \int_{x-t}^{\infty} \frac{y^{c-1}}{(y+t)^{c+1}} f(y)dy.$$

Letting now $t \to 0$, we conclude that

$$\mathcal{L}f(x) = -f'(x) - c \frac{f(x)}{x} + c \int_0^\infty f(y) \frac{y^{c-1}}{y^2} dy$$

$$= -f'(x) + c \int_0^\infty \frac{f(y) - f(x)}{y^2} dy, \quad x > 0. \quad (17)$$

It is easy to see that this generator can be represented as follows

$$\mathcal{L}f(x)$$

$$= -\left(1 - c \int_1^\infty \frac{\log u}{1 + \log^2 u} \frac{du}{u^2}\right) f'(x)$$

$$+ \frac{c}{x} \int_1^\infty \left(f(ux) - f(x) - \log u \frac{uf'(x)}{1 + \log^2 u}\right) \frac{du}{u^2}$$

$$= -\left(1 - c \int_1^\infty \frac{\log u}{1 + \log^2 u} \frac{du}{u^2}\right) f'(x) + \frac{1}{x} \int_1^\infty h^*(x, u) \frac{c\log^2 u}{u^2(1 + \log^2 u)} du,$$

where

$$h^*(x, u) = \left(f(ux) - f(x) - \log u \frac{uf'(x)}{1 + \log^2 u}\right) \frac{1 + \log^2 u}{\log^2 u}.$$

Then, according to Theorem 6.1 in Lamperti [15], $\{Z_t, t < \tau_{0}^Z\}$ can be represented as the exponential functional of a time-changed Lévy process with the following
Lévy-Khintchine exponent:

\[ \Psi(\lambda) = -i\lambda \left( 1 - c \int_1^{\infty} \frac{\log u}{1 + \log^2 u} \frac{du}{u^2} \right) + \int_0^{\infty} \left( e^{i\lambda y} - 1 - \frac{i\lambda y}{1 + y^2} \right) e^{-y} dy. \]

Simplifying this expression, we get

\[ \Psi(\lambda) = -i\lambda + \int_0^{\infty} (e^{i\lambda y} - 1) e^{-y} dy. \]

This corresponds to the process \( \zeta_t - t \), where \( (\zeta_t)_{t \geq 0} \) is a compound Poisson process with intensity \( c \) and with exponentially distributed jumps. In particular, \( \zeta_t - t \to -\infty \) a.s. as \( t \to \infty \) in the case \( c < 1 \) and \( \zeta_t - t \) is oscillating in the case \( c = 1 \). Then \( T_{0(Z)} \) is finite almost surely if \( c < 1 \).

To prove (8) we define

\[ g(t, z) := P_{z}(T_{0(Z)} > t). \]

It is clear that \( g(t, z) = 1 \) for all \( t \leq z \). Using (16), we see that \( g \) solves the equation

\[ g(t, z) = g(t-s, z-s) \left( \frac{z-s}{z} \right)^c + cs \int_{z-s}^{\infty} \frac{y^{c-1}}{(y+s)^{c+1}} g(t-s, y) dy, \quad s < z. \]

Letting \( s \to 0 \) we obtain the following decomposition for the expression on the right hand side:

\[ g(t-s, z-s) \left( 1 - \frac{cs}{z} \right) + cs \int_z^{\infty} \frac{g(t, y)}{y^2} dy + o(s). \]

Therefore,

\[ \lim_{s \to 0} \frac{g(t, z) - g(t-s, z-s)}{s} = -\frac{c}{z} g(t, z) + c \int_z^{\infty} \frac{g(t, y)}{y^2} dy. \]

As a result we have the following differential equation

\[ \frac{\partial}{\partial t} g(t, z) + \frac{\partial}{\partial z} g(t, z) = -\frac{c}{z} g(t, z) + c \int_z^{\infty} \frac{g(t, y)}{y^2} dy, \quad t > z. \quad (18) \]

Since the process \( Z \) is self-similar with index 1,

\[ g(t, z) = P(T_{0(Z)} > t | Z_0 = z) = P(T_{0(Z)} > t/z | Z_0 = 1) = g \left( \frac{t}{z}, 1 \right) =: h \left( \frac{t}{z} \right). \]

It follows then from (18) that the function \( h \) satisfies

\[ \frac{1}{z} h' \left( \frac{t}{z} \right) - \frac{t}{z^2} h' \left( \frac{t}{z} \right) = -\frac{c}{z} h \left( \frac{t}{z} \right) + c \int_z^{\infty} \frac{h(t/y)}{y^2} dy, \quad t > z. \]

Noting that \( h(r) = 1 \) for all \( r \leq 1 \) and substituting \( t/y = x \), we get

\[ \int_z^{\infty} \frac{h(t/y)}{y^2} dy = \int_z^{1/z} \frac{h(t/y)}{y^2} dy + 1 \frac{c}{t} \]

\[ = \frac{1}{t} \int_1^{1/z} h(x) dx + 1 \frac{c}{t}. \]

Therefore,

\[ (1-y)h'(y) = -ch(y) + \frac{c}{y} \left( 1 + \int_1^y h(x) dx \right), \quad y > 1. \]
Differentiating this equation, we get

\[
(1 - y)h''(y) - h'(y) = -ch'(y) + \frac{c}{y}h(y) - \frac{c}{y} \left( 1 + \int_1^y h(x)dx \right)
\]

\[
= -ch'(y) + \frac{c}{y}h(y) - \frac{1}{y}((1 - y)h'(y) + ch(y)).
\]

Rearranging the terms, we arrive at the equation

\[
(1 - y)h''(y) = \left( 1 - c - \frac{1 - y}{y} \right) h'(y).
\]

This is equivalent to

\[
(log h'(y))' = \frac{h''(y)}{h'(y)} = \frac{c - 1}{y - 1} - \frac{1}{y}.
\]

Consequently,

\[
h'(y) = C(y - 1)^{c-1} y^{-1} \quad \text{and} \quad h(y) = C \int_y^\infty (y - 1)^{c-1} y^{-1} dy.
\]

The boundary condition \(h(1) = 1\) leads to the equality

\[
h(x) = \frac{\int_x^\infty (y - 1)^{c-1} y^{-1} dy}{\int_1^\infty (y - 1)^{c-1} y^{-1} dy}, \quad x \geq 1.
\]

Substituting in these integrals \(y = 1/u\), we finally get

\[
h(x) = \frac{1}{B(c, 1 - c)} \int_0^{1/x} (1 - z)^{c-1} z^{-c} dz, \quad x \geq 1.
\]

As a result we have \(8\). This formula can be also obtained via the Lamperti transformation mentioned above. If \(Z_0 = 1\) then \(T_0^{(Z)}\) has the same distribution as \(I := \int_0^\infty e^{\xi} dt\) and \(1/I\) has the beta distribution with parameters \(c\) and \(1 - c\), see Bertoin and Yor \(7\).

We now turn to the proof of part (b). We start by computing the expectation \(E_x[Z_t^{1-c}]\). If \(t \leq x\) then, in view of \(10\),

\[
E_x[Z_t^{1-c}] = \int_{x-t}^\infty y^{1-c} \frac{1}{t} P_x(Z_t \in dy)
\]

\[
= (x - t)^{1-c} \left( \frac{x - t}{x} \right)^{c-1} + \int_{x-t}^\infty \frac{ct}{(t + y)^{c+1}} dy
\]

\[
= \frac{x - t}{x^{c-1}} + ct \int_x^\infty \frac{dy}{y^{c+1}} = x^{1-c}.
\]

If \(t > x\) then, by \(10\),

\[
E_x[Z_t^{1-c}] = \int_0^\infty \frac{ct}{(t + y)^{c+1}} dy = t^{1-c}.
\]
Using these equalities, we obtain

$$E_x[Z_t^{1-c}; T_0^{(Z)} > t] = E_x[Z_t^{1-c}] - E_x[Z_t^{1-c}; T_0^{(Z)} \leq t]$$

$$= (\max\{t, x\})^{1-c} - \int_0^t P_x(T_0^{(Z)} \in ds)E_0[Z_t^{1-c}]$$

$$= (\max\{t, x\})^{1-c} - \int_0^t (t-s)^{1-c}P_x(T_0^{(Z)} \in ds). \quad (19)$$

It follows from (8) that the integral in (19) is zero for $t \leq x$, and that for $t > x$ one has

$$\int_0^t (t-s)^{1-c}P_x(T_0^{(Z)} \in ds)$$

$$= \int_x^t (t-s)^{1-c}P_x(T_0^{(Z)} \in ds)$$

$$= \frac{1}{B(c, 1-c)} \int_x^t (t-s)^{1-c} \left(1 - \frac{x}{s}\right)^{c-1} \left(\frac{x}{s}\right)^{-c} \frac{x}{s^2} ds$$

$$= \frac{1}{B(c, 1-c)} \int_x^t (t-s)^{1-c} \left(1 - \frac{x}{s}\right)^{c-1} \left(\frac{s}{x}\right)^{c-1} \frac{1}{s} ds$$

$$= \frac{x^{1-c}}{B(c, 1-c)} \int_x^t (t-s)^{1-c} (s-x)^{-c} \frac{1}{s} ds.$$

With the help of the substitution $v = \left(\frac{s}{t-s}\right)$ we get

$$\int_x^t (t-s)^{1-c} (s-x)^{-c} 1 \frac{1}{s} ds = \int_0^\infty v^{c-1} \frac{1}{x+tv} \left(\frac{t}{1+v} - \frac{x+tv}{(1+v)^2}\right) dv$$

$$= t \int_0^\infty \frac{v^{c-1}}{x+tv} dv - \int_0^\infty \frac{v^{c-1}}{1+v} dv$$

$$= \left(\frac{t}{x}\right)^{1-c} - 1 \int_0^\infty \frac{v^{c-1}}{1+v} dv.$$

Noting now that $\int_0^\infty \frac{v^{c-1}}{1+v} dv = B(c, 1-c)$, we conclude that

$$\int_0^t (t-s)^{1-c}P_x(T_0^{(Z)} \in ds) = \max\{t^{1-c} - x^{1-c}, 0\}.$$

Plugging this into (19), we conclude that

$$E_x[Z_t^{1-c}; T_0^{(Z)} > t] = x^{1-c}$$

for all $x, t > 0$. Thus, (b) is proven.

To prove (c) we first consider one-dimensional marginals. For $t \leq x$ one has

$$P_x(Z_t \leq y; T_0^{(Z)} > t) = P_x(Z_t \leq y), \quad y > 0.$$

If $t > x$ then

$$P_x(Z_t \leq y; T_0^{(Z)} > t) = P_x(Z_t \leq y) - P_x(Z_t \leq y; T_0^{(Z)} \leq t)$$

$$= P_x(Z_t \leq y) - \int_x^t P_x(T_0^{(Z)} \in ds)P_0(Z_{t-s} \leq y).$$
Using now (5) and (8), we get
\[
P_x(z_t \leq y; T_0^{(Z)} > t) = \left(\frac{y}{y + t}\right)^c - \frac{1}{B(c, 1 - c)} \int_x^t \left(\frac{y}{y + t - s}\right)^c \left(1 - \frac{x}{s}\right)^{-c} \frac{x}{s^2} ds
\]
\[
= \left(\frac{y}{y + t}\right)^c - \frac{1}{B(c, 1 - c)} \int_x^t \left(\frac{y}{y + t - s}\right)^c \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds.
\]
This representation can be used to obtain an exact formula for the transition kernel
\[P_x(z_t \leq y; T_0^{(Z)} > t)\] in terms of the hypergeometric function of two variables. Instead of doing that we shall determine the asymptotic, as \(x \to 0\), behaviour of the distribution function \(P_x(z_t \leq y; T_0^{(Z)} > t)\). We start by noting that
\[
P_x(z_t \leq y; T_0^{(Z)} > t) = \left(\frac{y}{y + t}\right)^c P_x(T_0^{(Z)} > t) - \frac{1}{B(c, 1 - c)} \int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds, \tag{20}
\]
where
\[
\Delta_{y,t}(s) = \left(\frac{y}{y + t - s}\right)^c - \left(\frac{y}{y + t}\right)^c.
\]
Fix some \(\varepsilon > 0\). It is easy to see that
\[
\Delta_{y,t}(s) = \frac{y^c}{(y + t)^c} \left[\left(1 + \frac{s}{t + y - s}\right)^c - 1\right] \leq \frac{cy^c}{(y + t)^c} \frac{s}{y + t - \varepsilon}
\]
for all \(s \leq \varepsilon\). Therefore, for all \(x < \varepsilon\),
\[
\int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds \leq \frac{cy^c}{(y + t - \varepsilon)(y + t)^c} \int_x^t \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds
\]
\[
\leq \frac{y^c}{(y + t - \varepsilon)(y + t)^c} x^{1-c} \varepsilon^c. \tag{21}
\]
Furthermore, as \(x \to 0\),
\[
\int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds = x^{1-c} \int_x^t \Delta_{y,t}(s) (s - x)^{-c} \frac{1}{s} ds
\]
\[
= x^{1-c}(1 + o(1)) \int_x^t \Delta_{y,t}(s) s^{c-2} ds.
\]
Combining this with (21) and letting \(\varepsilon \to 0\), we conclude that
\[
\lim_{x \to 0} x^{c-1} \int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{-c} \frac{1}{s} ds = \int_0^t \Delta_{y,t}(s) s^{c-2} ds. \tag{22}
\]
Using the equality
\[
\Delta_{y,t}(s) = \int_{y/(t+y)}^{y/(t+y-s)} cu^{c-1} du
\]
and the Fubini theorem, we have
\[
\int_0^t \Delta_{y,t}(s) s^c ds = \int_0^t \left( \int_{y/(t+y)}^{y/(t+y+s)} u^{c-1} du \right) s^c ds \\
= \int_{y/(y+t)}^1 u^{c-1} \left( \int_{y+t-y/u}^t s^c ds \right) du \\
= \frac{c}{1-c} \int_{y/(y+t)}^1 u^{c-1} \left( (y+t-y/u)^{c-1} - t^{c-1} \right) du \\
= \frac{c}{1-c} \int_{y/(y+t)}^1 (u+t) - \frac{1}{1-c} t^{c-1} \left( 1 - \left( \frac{u}{y+t} \right)^c \right).
\]

Combining this with (22) and noting that
\[
P_x(T_0^{(Z)} > t) \sim \frac{x^c}{(1-c)B(c,1-c)} t^{c-1}, \quad x \to 0,
\]
we conclude that
\[
\lim_{x \to 0} \int_{y/(y+t)}^1 \Delta_{y,t}(s) \left( \frac{x}{x+y} - 1 \right)^{c-1} \frac{1}{x+y} ds \\
P_x(T_0^{(Z)} > t) = B(c,1-c) \left[ \left( \frac{y}{y+t} \right)^c - \frac{y}{y+t} \right].
\]

Combining this with (23), we finally obtain
\[
\lim_{x \to 0} P_x(Z_t \leq y|T_0^{(Z)} > t) = \frac{y}{y+t}, \quad y > 0.
\]

Using the harmonic function \( u(x) = x^{1-c} \) we now define the Doob \( h \)-transform of \( L \):
\[
\hat{L}f(x) := \frac{1}{u(x)} L(uf)(x), \quad x > 0.
\]
The corresponding probability measure is given by
\[
\hat{E}_x[g(Z)] := \frac{1}{u(x)} E_x[g(Z) u(Z_t); \tau_0^{(Z)} > t]
\]
for every bounded measurable functional \( g \) on \( D[0,t] \).

From (17) we infer that
\[
\hat{L}f(x) = \frac{1}{u(x)} \left[ -u(x)f'(x) - u'(x)f(x) - \frac{u(x)f(x)}{x} + c \int_x^\infty \frac{u(y)f(y)}{y^2} dy \right] \\
= -f'(x) - \frac{f(x)}{x} + \frac{c}{x^{1-c}} \int_x^\infty \frac{f(y)}{y^{1+c}} dy \\
= -f'(x) + \frac{c}{x^{1-c}} \int_x^\infty \frac{f(y) - f(x)}{y^{1+c}} dy.
\]

As a result we have the following representation:
\[
\hat{L}f(x) = -f'(x) + \frac{c}{x} \int_1^\infty \frac{f(u)}{u^{1+c}} du \\
= - \left( 1 - c \int_1^\infty \log u \frac{du}{1 + \log^2 u u^2} \right) f'(x) + \frac{1}{x} \int_1^\infty h^*(x,u) \frac{c \log^2 u}{u(1 + \log^2 u)} du.
\]
This implies that, under \( \hat{P} \), \( Z \) is self-similar and can be expressed via a Lévy process with the characteristic exponent
\[
\hat{\Psi}(\lambda) = -i\lambda + \int_0^\infty (e^{i\lambda y} - 1)e^{-cy}dy.
\]
This corresponds to \( \hat{\xi} - t \), where \((\hat{\xi}_t)_{t \geq 0}\) is a compound Poisson process with intensity \( c \) and with positive jumps, which have exponential with parameter \( c \) distribution. This Lévy process is clearly oscillating. Consequently, \( \hat{P}_x(T_0^{(Z)} = \infty) = 1, \quad x > 0. \)

According to Theorem 2 in Caballero and Chaumont [9], the sequence of measures \( \hat{P}_x \) converges weakly on \( D[0, 1] \), as \( x \to 0 \), to a non-degenerate probabilistic measure \( \hat{P}_0 \). We now show that this implies that \( \hat{P}_x \left( Z \in \cdot | T_0^{(Z)} > 1 \right) \) also converges weakly on \( D[0, 1] \).

It follows from the definition of \( \hat{P}_x \) that
\[
\hat{P}_0(Z_1 \leq y) = \lim_{x \to 0} \hat{P}_x(Z_1 \leq y) = \lim_{x \to 0} \frac{\hat{P}_x(T_0^{(Z)} > 1)}{u(x)} \hat{E}_x[u(Z_1)1\{Z_1 \leq y]\{T_0^{(Z)} > 1\}].
\]

Applying now (23) and (24), we obtain
\[
\hat{P}_0(Z_1 \leq y) = \frac{1}{(1 - c)B(c, 1 - c)} \int_0^y \frac{z^{1-c}}{(1 + z)^2}dz.
\]

Consequently, the density of \( Z_1 \) under \( \hat{P}_0 \) is proportional to \( \frac{z^{1-c}}{(1 + z)^2} \). Let \( g \) be a bounded and continuous functional on \( D[0, 1] \) and let \( \varepsilon \) be a fixed positive number. Since \( \hat{P}_0(Z_1 = \varepsilon) = 0 \), the weak convergence \( \hat{P}_x \Rightarrow \hat{P}_0 \) implies that
\[
\lim_{x \to 0} \hat{E}_x \left[ \frac{g(Z)}{u(Z_1)}; Z_1 > \varepsilon \right] = \hat{E}_0 \left[ \frac{g(Z)}{u(Z_1)}; Z_1 > \varepsilon \right]. \quad (26)
\]

Since \( g \) is bounded,
\[
\left| \hat{E}_x \left[ \frac{g(Z)}{u(Z_1)}; Z_1 \leq \varepsilon \right] \right| \leq C_g \hat{E}_x \left[ \frac{1}{u(Z_1)}; Z_1 \leq \varepsilon \right] \leq C_g \frac{\hat{P}_x(T_0^{(Z)} > 1)}{u(x)} \hat{P}_x(Z_1 \leq \varepsilon | T_0^{(Z)} > 1). \]

Using (23) and (24), we conclude that
\[
\limsup_{x \to 0} \left| \hat{E}_x \left[ \frac{g(Z)}{u(Z_1)}; Z_1 \leq \varepsilon \right] \right| \leq \frac{C_g}{(1 - c)B(c, 1 - c)} \varepsilon. \quad (27)
\]

Finally, recalling that the density of \( Z_1 \) under \( \hat{P}_0 \) is proportional to \( \frac{z^{1-c}}{(1 + z)^2} \), we get
\[
\left| \hat{E}_0 \left[ \frac{g(Z)}{u(Z_1)}; Z_1 \leq \varepsilon \right] \right| \leq C_g \hat{E}_0 \left[ \frac{1}{u(Z_1)}; Z_1 \leq \varepsilon \right] = C_g \int_0^\varepsilon (1 + z)^{-2}dz \leq C_g \varepsilon. \quad (28)
\]
Combining (26) and (28) and letting $\varepsilon \to 0$, we conclude that
\[
\lim_{x \to 0} \hat{\mathbb{E}}_x \left[ \frac{g(Z)}{u(Z)} \right] = \hat{\mathbb{E}}_0 \left[ \frac{g(Z)}{u(Z)} \right].
\]
Noting now that
\[
\mathbb{E}_x \left[ g(Z) | T_0^{(Z)} > 1 \right] = \frac{u(x)}{P_x(T_0^{(Z)} > 1)} \hat{\mathbb{E}}_x \left[ \frac{g(Z)}{u(Z)} \right]
\]
and taking into account (23), we obtain
\[
\lim_{x \to \infty} \mathbb{E}_x \left[ g(Z) | T_0^{(Z)} > 1 \right] = (1 - c) B(c, 1 - c) \hat{\mathbb{E}}_0 \left[ \frac{g(Z)}{u(Z)} \right].
\]
This completes the proof of the theorem.

3. Construction of harmonic functions

3.1. Harmonic function for the random exchange process and for the maximal autoregressive process. In this paragraph we shall consider the equation
\[
G(x) := \mathbb{E}_x [G(R_1); T_x^{(R)} > 1] = \mathbb{E}_x [G(R_1); R_1 > x_0], \quad x > x_0.
\]
Assume first that $x \in (x_0, x_0 + 1]$. In this case one has
\[
\{R_1 > x_0\} = \{R_1 = \eta_1 > x_0\}.
\]
Therefore,
\[
G(x) = \mathbb{E}[G(\eta_1); \eta_1 > x_0] \quad \text{for all} \quad x \in (x_0, x_0 + 1].
\]
For all $x > x_0 + 1$ one has $P_x(T_x^{(R)} > 1) = 1$. This implies that (29) reduces to
\[
G(x) = \mathbb{E}_x [G(R_1)]
= G(x - 1) P(\eta_1 \leq x - 1) + \mathbb{E}[G(\eta_1); \eta_1 > x - 1], \quad x > x_0 + 1.
\]
If $x \in (x_0 + 1, x_0 + 2]$ then $x - 1 \in (x_0, x_0 + 1]$ and, consequently, $G(x - 1) = G(x_0 + 1)$ for all $x \in (x_0 + 1, x_0 + 2]$. From this observation and from (30) we have
\[
G(x)
= G(x_0 + 1) P(\eta_1 \leq x - 1) + \mathbb{E}[G(\eta_1); \eta_1 > x - 1]
= G(x_0 + 1) P(\eta_1 \leq x_0 + 1) + \mathbb{E}[G(\eta_1); \eta_1 > x_0 + 1] + \mathbb{E}[G(\eta_1); \eta_1 > x_0 + 1]
= G(x_0 + 1) P(\eta_1 \leq x_0 + 1) + \mathbb{E}[G(\eta_1); \eta_1 > x_0 + 1] + \mathbb{E}[G(\eta_1); \eta_1 > x_0 + 1].
\]
This equality implies that $G(x) = G(x_0 + 2)$ for all $x \in (x_0 + 1, x_0 + 2]$. Note also that
\[
G(x_0 + 1) = \mathbb{E}[G(\eta_1); \eta_1 > x_0]
= G(x_0 + 1) P(\eta_1 \in (x_0, x_0 + 1]) + \mathbb{E}[G(\eta_1); \eta_1 > x_0 + 1].
\]
Combining this with (31), we conclude that
\[
G(x_0 + 2) = G(x_0 + 1) (1 + P(\eta_1 \leq x_0)).
\]
Fix now an integer $n$ and consider the case $x \in (x_0 + n, x_0 + n + 1]$. Assume that we have already shown that $G(y) = G(x_0 + n)$ for all $y \in (x_0 + n - 1, x_0 + n]$. Then we have from (30)

$$G(x) = G(x + 1)P(\eta_1 \leq x - 1) + E[G(\eta_1); \eta_1 > x - 1]$$

$$= G(x_0 + n)P(\eta_1 \leq x_0 + n) + E[G(\eta_1); \eta_1 > x_0 + n].$$

Therefore, $G(x) = G(x_0 + n + 1)$ for all $x \in (x_0 + n, x_0 + n + 1]$. This means that this property is valid for all $n$.

One has also equalities

$$G(x_0 + n + 1) = G(x_0 + n)P(\eta_1 \leq x_0 + n) + E[G(\eta_1); \eta_1 > x_0 + n]$$

and

$$G(x_0 + n) = G(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1) + E[G(\eta_1); \eta_1 > x_0 + n - 1]$$

$$= G(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1)$$

$$+ G(x_0 + n)P(\eta_1 \in (x_0 + n - 1, x_0 + n + 1]) + E[G(\eta_1); \eta_1 > x_0 + n].$$

Taking the difference we obtain

$$G(x_0 + n + 1) - G(x_0 + n)$$

$$= G(x_0 + n)P(\eta_1 \leq x_0 + n) - G(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1)$$

$$- G(x_0 + n)P(\eta_1 \in (x_0 + n - 1, x_0 + n))$$

$$= P(\eta_1 \leq x_0 + n - 1) (G(x_0 + n) - G(x_0 + n - 1)).$$

Consequently,

$$G(x_0 + n + 1) - G(x_0 + n) = G(x_0 + n) \prod_{k=0}^{n-1} P(\eta_1 \leq x_0 + k), \quad n \geq 1.$$ 

As a result we have

$$G(x) = G(x_0 + 1) \left(1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)\right), \quad x \in (x_0 + n, x_0 + n + 1]. \quad (32)$$

Finally, in order to get a non-trivial solution we have to show that the equation

$$G(x_0 + 1) = E[G(\eta_1); \eta_1 > x_0]$$

is solvable. In view of (32), the previous equation is equivalent to

$$G(x_0 + 1) = G(x_0 + 1) \sum_{n=0}^{\infty} \left(1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)\right) P(\eta_1 \in (x_0 + n, x_0 + n + 1]).$$

Now we infer that (29) has a non-trivial solution if and only if

$$1 = \sum_{n=0}^{\infty} \left(1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)\right) P(\eta_1 \in (x_0 + n, x_0 + n + 1]).$$
Clearly,
\[
\sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0 + n, x_0 + n + 1])
\]
\[
= P(\eta_1 > x_0) + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \sum_{n=j}^{\infty} P(\eta_1 \in (x_0 + n, x_0 + n + 1])
\]
\[
= P(\eta_1 > x_0) + \sum_{j=1}^{\infty} (1 - P(\eta_1 \leq x_0 + j)) \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k).
\]
Furthermore, for every \( N \geq 1, \)
\[
\sum_{j=1}^{N} (1 - P(\eta_1 \leq x_0 + j)) \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)
\]
\[
= \sum_{j=1}^{N} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) - \sum_{j=1}^{N} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)
\]
\[
= P(\eta_1 \leq x_0) - \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k).
\]
This implies that
\[
\sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0 + n, x_0 + n + 1])
\]
\[
= 1 - \lim_{N \to \infty} \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k).
\]
Thus, there is a non trivial solution \( G(x) \) if and only if
\[
\lim_{N \to \infty} \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k) = 0.
\]
Noting that this is equivalent to \( \mathbb{E}\eta_1^+ = \infty, \) we finish the proof of the first part of Theorem 2. We notice also that \( \mathbb{E}\eta_1^+ = \infty \) implies that \( \{R_n\} \) is either null recurrent or transient.

If \( \{R_n\} \) is recurrent and \( P(\eta_1 \leq x_0) > 0 \) then, according to (\( \mathbb{I} \)), the function \( G(x) \) grows unboundedly. Furthermore, if (\( \mathbb{II} \)) holds with some positive \( c \) then it follows from the Karamata representation theorem that there exists a slowly varying function \( L \) such that
\[
\prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \sim \frac{L(j)}{j^c} \quad \text{as} \quad j \to \infty.
\]
(33)
If we assume that \( c \in (0, 1) \) then \( \{R_n\} \) is null recurrent and
\[
G(x) \sim \frac{1}{1 - c} x^{1-c} L(x), \quad x \to \infty.
\]
(34)
If \( c = 1 \) then one has to take into account the asymptotic behaviour of the difference \( P(\eta_1 > y) - 1/y \). Assume, for example, that
\[
P(\eta_1 > y) = \frac{1}{y} + \frac{\theta + o(1)}{y \log y}
\]
for some \( \theta \in (0, 1) \). Then \( \{R_n\} \) is null recurrent and there exists a slowly varying function \( L_1 \) such that \( L(x) \sim (\log x)^{-\theta} L_1(\log x) \). This implies that
\[
G(x) \sim \frac{1}{1 - \theta} (\log x)^{1-\theta} L_1(\log x) \quad \text{if } \theta < 1.
\]

We conclude this paragraph with the following remark on the transient case. If \( \{R_n\} \) is transient then the function \( x \mapsto P_x(T_{x_0}^{(R)} = \infty) \) is harmonic and its limit, as \( x \to \infty \), is equal to one. Then, according to (32),
\[
P_x(T_{x_0}^{(R)} = \infty) = \frac{1 + \sum_{j \in \{1, x-x_0\}} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)}{1 + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k)}, \quad x > x_0.
\]

If (3) holds with some positive \( c > 1 \) then the chain is transient and, using (33), we obtain
\[
P_x(T_{x_0}^{(R)} < \infty) \sim \frac{1}{(c - 1) x^{c-1}}, \quad x \to \infty.
\]

3.2. Harmonic function for the autoregressive process: proof of Theorem 3(i).

**Lemma 7.** Let \( W \) be an increasing, regularly varying of index \( r \in (0, 1) \) function. We assume also that \( W'(x) = O \left( \frac{W(x)}{x} \right) \). If (3) holds then, as \( z \to \infty \),
\[
E[W(\log_A(A^{z-1} + A^{\eta_1}))] = W(z-1)P(\eta_1 \leq z - 1) + E[W(\eta_1); \eta_1 > z - 1] + o \left( \frac{W(z)}{z^2} \right).
\]

**Proof.** We start by decomposing the expectation into two parts:
\[
E[W(\log_A(A^{z-1} + A^{\eta_1}))] = E[W(\log_A(A^{z-1} + A^{\eta_1}); \eta_1 \leq z - 1) + E[W(\log_A(A^{z-1} + A^{\eta_1}); \eta_1 > z - 1]
\]
\[
= E[W(z-1)P(\eta_1 \leq z - 1)] + E[W(\log_A(1 + A^{\eta_1-z+1}); \eta_1 \leq z - 1]
\]
\[
E[W(\eta_1 + \log_A(1 + A^{z-1-\eta_1}); \eta_1 > z - 1].
\]

By the mean value theorem,
\[
E[W(z-1 + \log_A(1 + A^{\eta_1-z+1}); \eta_1 \leq z - 1)
\]
\[
= W(z-1)P(\eta_1 \leq z - 1) + E[W'(z-1 + \theta_1) \log_A(1 + A^{\eta_1-z+1}); \eta_1 \leq z - 1],
\]
where \( \theta_1 = \theta_1(z, \eta_1) \in (0, \log_A 2) \). Using now the assumption \( W'(x) = O \left( \frac{W(x)}{x} \right) \), we obtain
\[
E[W(z-1 + \log_A(1 + A^{\eta_1-z+1}); \eta_1 \leq z - 1)
\]
\[
= W(z-1)P(\eta_1 \leq z - 1) + O \left( \frac{W(z)}{z} \right) E[\log_A(1 + A^{\eta_1-z+1}); \eta_1 \leq z - 1].
\]
It is easy to see that
\[ \log_A(1 + A^{n-z+1}) = O\left(\frac{1}{z^2}\right) \]
if \( \eta_1 \leq z - 1 - 2\log_A z \). Furthermore, (3) implies that
\[ P(z - 1 - 2\log_A z < \eta_1 \leq z - 1) = o\left(\frac{1}{z}\right). \]
Combining these relations, we infer that
\[ E[\log_A (1 + A^{n-z+1}); \eta_1 \leq z - 1] = o\left(\frac{1}{z}\right). \]
As a result we have
\[ E[W(z - 1 + \log_A (A^{z-1} + A^{n-1})); \eta_1 \leq z - 1] = W(z - 1)P(\eta_1 \leq z - 1) + o\left(\frac{W(z)}{z^2}\right). \tag{36} \]
Using the mean value theorem and the assumption \( W'(x) = O\left(\frac{W(x)}{x}\right) \) once again, we get
\[ E[W(\eta_1 + \log_A (1 + A^{z-1} - \eta_1)); \eta_1 > z - 1] = E[W(\eta_1); \eta_1 > z - 1] + O\left(\frac{W(z)}{z^2}\right)E[\log_A (1 + A^{z-1} - \eta_1); \eta_1 > z - 1]. \]
Similar to the first part of the proof, we obtain the desired equality. \qed

For every \( \varepsilon \geq 0 \) we define
\[ u_\varepsilon(x) = (1 + \varepsilon) \int_0^x P(\eta_1 > y)dy, \quad x \geq 0 \]
and
\[ U_\varepsilon(x) = \begin{cases} 0, & x \leq 0 \\ \int_0^x e^{-u_\varepsilon(y)}dy, & x > 0. \end{cases} \]

**Lemma 8.** For every \( \varepsilon \in [0, \frac{1}{\Delta - 1}] \) one has
\[ E[U_\varepsilon(\log_A (A^{z-1} + A^{n}))] = U_\varepsilon(z) - \frac{\varepsilon}{1 + \varepsilon} e^{-u_\varepsilon(z)} + O\left(\frac{U_\varepsilon(z)}{z^2}\right). \]

**Proof.** (3) yields
\[ u_\varepsilon(x) \sim (1 + \varepsilon)c \log x \quad \text{as} \quad x \to \infty. \]
Furthermore, \( U_\varepsilon(x) \) is regularly varying of index \( 1 - \alpha(1 + \varepsilon) \) and that
\[ U_\varepsilon'(x) = e^{-u_\varepsilon(x)} \sim (1 - c(1 + \varepsilon)) \frac{U_\varepsilon(x)}{x}. \]
Therefore, we may apply Lemma 7 to the function $U_z$:

$$
\mathbb{E}[U_{\varepsilon}(\log_A(A^{z-1} + A^n))]
= U_{\varepsilon}(z - 1)P(\eta_1 \leq z - 1) + \mathbb{E}[U_{\varepsilon}(\eta_1); \eta_1 > z - 1] + o \left( \frac{U_z(z)}{z^2} \right).
$$

Integrating by parts, we have

$$
\mathbb{E}[U_{\varepsilon}(\eta_1); \eta_1 > z - 1] = U_{\varepsilon}(z - 1)P(\eta_1 > z - 1) + \frac{1}{1 + \varepsilon} \int_{z-1}^{\infty} e^{-\varepsilon u(y)}u'(y) dy
$$

$$
= U_{\varepsilon}(z - 1)P(\eta_1 > z - 1) + \frac{1}{1 + \varepsilon} e^{-\varepsilon u(z-1)}.
$$

Consequently,

$$
\mathbb{E}[U_{\varepsilon}(\log_A(A^{z-1} + A^n))] = U_{\varepsilon}(z - 1) + \frac{1}{1 + \varepsilon} e^{-\varepsilon u(z-1)} + o \left( \frac{U_z(z)}{z^2} \right).
$$

It remains now to notice that, by the Taylor formula,

$$
U_{\varepsilon}(z) = U_{\varepsilon}(z - 1) + U_{\varepsilon}'(z - 1) + \frac{1}{2} U_{\varepsilon}''(z - 1 + \theta)
$$

$$
= U_{\varepsilon}(z - 1) + e^{-\varepsilon u(z-1)} + O \left( \frac{U_z(z)}{z^2} \right).
$$

Applying Lemma 8 and noting that $U_{\varepsilon}(x) = o(U_0(x))$, we get

$$
\mathbb{E}_x[U_0(\log_A X_1) + U_{\varepsilon}(\log_A X_1)]
$$

$$
= \mathbb{E}[U_0(\log_A(A^{x-1} + A^n)) + U_{\varepsilon}(\log_A(A^{x-1} + A^n))]
$$

$$
= U_0(\log_A x) + U_{\varepsilon}(\log_A x) - \frac{\varepsilon}{1 + \varepsilon} e^{-\varepsilon u(\log_A x)} + O \left( \frac{U_0(\log_A x)}{(\log_A x)^2} \right).
$$

We know that $e^{-\varepsilon u(\varepsilon)}$ is regularly varying of index $-(1 + \varepsilon)c$ and that $\frac{U_0(z)}{z^2}$ is regularly varying of index $-c - 1$. Thus, for every $\varepsilon < \frac{1}{1 + \varepsilon}$ there exists $x^*$ such that

$$
\mathbb{E}_x[U_0(\log_A X_1) + U_{\varepsilon}(\log_A X_1)] \leq U_0(\log_A x) + U_{\varepsilon}(\log_A x), \quad x \geq x^*.
$$

This inequality implies that if $x_0 \geq x^*$ then the sequence

$$
Z_n := U_0(\log_A X_{n \wedge T_{x_0}^{(X)}}) + U_{\varepsilon}(\log_A X_{n \wedge T_{x_0}^{(X)}})
$$

is a supermartingale. We next notice that

$$
Z_{n+1}1\{T_{x_0}^{(X)} > n + 1\} - Z_n1\{T_{x_0}^{(X)} > n\}
$$

$$
= (Z_{n+1} - Z_n)1\{T_{x_0}^{(X)} > n\} - Z_{n+1}1\{T_{x_0}^{(X)} = n + 1\}
$$

$$
\leq (Z_{n+1} - Z_n)1\{T_{x_0}^{(X)} > n\}.
$$

This implies that $Z_n1\{T_{x_0}^{(X)} > n\}$ is also a supermartingale. Consequently, the function

$$
V_{\varepsilon}(x) := \lim_{n \to \infty} \mathbb{E}_x[U_0(\log_A X_n) + U_{\varepsilon}(\log_A X_n); T_{x_0}^{(X)} > n]
$$
Letting here $n$, it follows from the supermartinale property of $U_\varepsilon$.

Using now the integration by parts, we get

$$V_\varepsilon(x) = \begin{cases} 
U_0(\log_A x) + U_\varepsilon(\log_A x) \leq C U_0(\log_A x), & x > x_0.
\end{cases}$$

We now recall that $U_\varepsilon(z) = o(U_0(z))$. Thus, for every $\delta > 0$ there exists $B$ such that $U_\varepsilon(z) \leq \delta U_0(z)$ for all $z \geq B$. Therefore,

$$E_x[U_\varepsilon(\log_A x_n); T_{x_0}^{(X)} > n] = E_x[U_\varepsilon(\log_A x_n); \log_A x_n \leq B, T_{x_0}^{(X)} > n] + E_x[U_\varepsilon(\log_A x_n); \log_A x_n > B, T_{x_0}^{(X)} > n] \leq U_\varepsilon(B) P_x(T_{x_0}^{(X)} > n) + \delta E_x[U_0(\log_A x_n); T_{x_0}^{(X)} > n].$$

Recalling that $P_x(T_{x_0}^{(X)} > n) \to 0$, we get

$$\limsup_{n \to \infty} E_x[U_\varepsilon(\log_A x_n); T_{x_0}^{(X)} > n] \leq \delta V_\varepsilon(x).$$

Letting now $\delta \to 0$ we conclude that

$$\lim_{n \to \infty} E_x[U_\varepsilon(\log_A x_n); T_{x_0}^{(X)} > n] = 0$$

This means that $V_\varepsilon$ does not depend on $\varepsilon$. Thus we may set

$$V(x) := \lim_{n \to \infty} E_x[U_0(\log_A x_n); T_{x_0}^{(X)} > n].$$

Since $U_0$ and the chain $\{X_n\}$ are increasing, we infer that the function $V(x)$ is increasing as well.

By the Markov property,

$$E_x[U_0(\log_A x_{n+1}); T_{x_0}^{(X)} > n + 1] = \int_{x_0}^\infty P_y(X_1 \in dy) E_y[U_0(\log_A x_n); T_{x_0}^{(X)} > n].$$

It follows from the supermartinale property of $U_0(\log_A x_n) + U_\varepsilon(\log_A x_n)$ that

$$E_y[U_0(\log_A x_n); T_{x_0}^{(X)} > n] \leq E_y[U_0(\log_A x_n) + U_\varepsilon(\log_A x_n); T_{x_0}^{(X)} > n] \leq U_0(\log_A y) + U_\varepsilon(\log_A y), \quad n \geq 1$$

This allows one to apply the dominated convergence theorem and to conclude that

$$V(x) = E_x[V(X_1); T_{x_0}^{(X)} > 1], \quad x > x_0.$$ 

In other words, $V(x)$ is harmonic for $X_n$ killed at $T_{x_0}^{(X)}$. It is also clear that

$$V(x) \leq U_0(\log_A x) + U_\varepsilon(\log_A x) \leq C U_0(\log_A x).$$

To show that this function is strictly positive we notice that

$$E[U_0(\log_A (A^{z-1} + A^{n_1}))] \geq U_0(z - 1) P(\eta_1 \leq z - 1) + E[U_0(\eta_1); \eta_1 > z - 1].$$

Using now the integration by parts, we get

$$E[U_0(\log_A (A^{z-1} + A^{n_1}))] \geq U_0(z - 1) + e^{\log\gamma(z-1)} \geq U_0(z).$$

In other words, the sequence $U_0(\log_A X_n)$ is a submartingale. Then, by the optional stopping theorem,

$$E_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n] \geq U_0(\log_A x) - E_x[U_0(\log_A X_{T_{x_0}^{(X)}}); T_{x_0}^{(X)} \leq n].$$

Letting here $n \to \infty$, we conclude that

$$V(x) \geq U_0(\log_A x) - E[U_0(\log_A X_{T_{x_0}^{(X)}})] \geq U_0(\log_A x) - U_0(\log_A x_0).$$
Thus, \( V(x) > 0 \) for every \( x > x_0 \). Furthermore, one has the relation
\[
V(x) \sim U_0(\log_A x) \quad \text{as } x \to \infty.
\]
(37)

Summarizing, for each \( x_0 \geq x_* \) we have constructed a strictly positive on \((0, \infty)\), increasing harmonic function \( V(x) \) such that \( V(A^x) \sim U_0(x) \).

We now turn to the case \( x_0 \leq x_* \). Let \( V_* \) be the function corresponding to the stopping time \( T_{x_*}^{(X)} \), i.e.
\[
V_*(x) = E_x[V_*(X_1); X_1 > x_*], \quad x > x_*.
\]
Define
\[
V(x) = V_*(x)1\{x > x_*\} + \sum_{j=0}^{\infty} \int_{x_0}^{x_*} P_x(X_j \in dz, T_{x_0}^{(X)} > j)g(z), \quad (38)
\]
where
\[
g(z) := E_x[V_*(X_1); X_1 > x_*].
\]
Then one has
\[
E_x[V(X_1); X_1 > x_0]
= E_x[V_*(X_1); X_1 > x_*] + \int_{x_0}^{\infty} P_x(X_1 \in dy) \sum_{j=0}^{\infty} \int_{x_0}^{x_*} P_y(X_j \in dz, T_{x_0}^{(X)} > j)g(z)
= E_x[V_*(X_1); X_1 > x_*] + \sum_{j=1}^{\infty} \int_{x_0}^{x_*} P_x(X_j \in dz, T_{x_0}^{(X)} > j)g(z).
\]

If \( x > x_* \) then
\[
E_x[V_*(X_1); X_1 > x_*] = V_*(x) = V_*(x) + \int_{x_0}^{x_*} P_x(X_0 \in dz, T_{x_0}^{(X)} > 0)g(z).
\]
Moreover, for \( x \in (x_0, x_*] \) we have
\[
E_x[V_*(X_1); X_1 > x_*] = \int_{x_0}^{x_*} P_x(X_0 \in dz, T_{x_0}^{(X)} > 0)g(z).
\]
As a result,
\[
E_x[V(X_1); X_1 > x_0] = V(x), \quad x > x_0,
\]
i.e. \( V \) is harmonic.

Since \( V(x) \) is strictly positive on the half-line \((0, \infty)\), we may perform the corresponding Doob h-transform via the transition probabilities:
\[
\hat{P}^{(V)}_x(X_1 \in dy) = \frac{V(y)}{V(x)} P_x(X_1 \in dy), \quad x, y > x_0. \quad (39)
\]
The chain \( X_n \) becomes transient under this new measure. To see this we consider the sequence \((V(X_n))^{-1/2}/2\). It is immediate from the definition of \( \hat{P} \) that
\[
\frac{1}{\sqrt{V(X_1)}} \left[ \frac{1}{\sqrt{V(X_1)}} \right] = \frac{1}{V(x)} E_x \left[ \frac{V(X_1)}{\sqrt{V(X_1)}} 1\{T_{x_0}^{(X)} > 1\} \right]
= \frac{1}{V(x)} E_x \left[ \sqrt{V(X_1)} 1\{T_{x_0}^{(X)} > 1\} \right].
\]
Applying now the Jensen inequality, we obtain

\[
\hat{E}_x^{(V)} \left[ \frac{1}{\sqrt{V(X_1)}} \right] \leq \frac{1}{V(x)} \sqrt{\hat{E}_x \left[ V(X_1)1\{T_{x_0}^{(X)} > 1 \} \right]} = \frac{1}{\sqrt{V(x)}}.
\]

In other words, the sequence \((V(X_n))^{-1/2}\) is a positive supermartingale. Due to the Doob convergence theorem, this sequence converges almost surely. Noticing that \(\hat{P}_x(\lim \sup X_n = \infty) = 1\) implies that this limit of \((V(X_n))^{-1/2}\) is zero. This means that

\[X_n \to \infty \quad \hat{P}^{(V)} - \text{a.s.}\]

We now show that (37) holds also in the case when the harmonic function is defined by (38). Since the function \(g(z)\) is increasing,

\[
\sum_{j=0}^{\infty} \int_{x_0}^{x_*} P_x(\{X_j \in dz, T_{x_0}^{(X)} > j\}) g(z)
\]

\[
\leq g(x_*) \sum_{j=0}^{\infty} \int_{x_0}^{x_*} P_x(\{X_j \leq x_*, T_{x_0}^{(X)} > j\})
\]

\[
\leq g(x_*) \sup_{x \in (x_0, x_*)} V(x) \sum_{j=0}^{\infty} \int_{x_0}^{x_*} \hat{P}_x(\{X_j \leq x_*, T_{x_0}^{(X)} > j\})
\]

\[
= C(x_0, x_*) \sum_{j=0}^{\infty} \int_{x_0}^{x_*} \hat{P}_x(\{X_j \leq x_*, T_{x_0}^{(X)} > j\}).
\]

Due to the transience of \(\{X_n\}\) under \(\hat{P}\),

\[
\sum_{j=0}^{\infty} \int_{x_0}^{x_*} \hat{P}_x(\{X_j \leq x_*, T_{x_0}^{(X)} > j\}) < \infty.
\]

Therefore,

\[V(A^x) \sim V_*(A^x) \sim U_0(x)\]

Thus, the proof of Theorem 3(i) is complete.

4. LOWER AND UPPER BOUNDS FOR TAILS OF RECURRENCE TIMES.

We shall consider the chain \(X_n\) only and prove bounds in Theorem 3(ii). The proofs of corresponding estimates for chains \(M_n\) and \(R_n\) are simpler.

4.1. A lower bound for the tail of \(T_{x_0}^{(X)}\). We first consider the case \(x_0 \geq x_*\). As we have seen in the previous section, \(V(x)\) is increasing in this case.

Let \(\check{P}\) denote the Doob \(h\)-transform of \(P\), for its definition see (35). Define

\[\sigma_y := \inf\{n \geq 1 : X_n \geq A^y\}.\]
By the total probability formula, for \( x < A^{2n} \) and \( B > 2 \),
\[
\hat{P}_x(x_{2n} \leq A^{Bn}, \sigma_{2n} \leq n) = \sum_{k=1}^{n} \int_{z_0}^{A^{2n}} \hat{P}_x(\sigma_{2n} > k - 1, x_{k-1} \in dz) \hat{P}_x(x_1 \in (A^{2n}, A^{Bn}]) \\
\geq \inf_{z < A^{2n}} \hat{P}_z(x_1 \in (A^{2n}, A^{Bn}]) \sum_{k=1}^{n} \int_{z_0}^{A^{2n}} \hat{P}_x(\sigma_{2n} > k - 1, x_{k-1} \in dz) \hat{P}_z(x_1 > A^{2n}) \\
= \inf_{z < A^{2n}} \frac{\hat{P}_z(x_1 \in (A^{2n}, A^{Bn}])}{\hat{P}_z(x_1 > A^{2n})} \hat{P}_z(x_{2n} \leq n).
\]

For every \( r \geq 2 \), using the integration by parts, we get
\[
\hat{P}_z(x_1 > A^{rn}) = \frac{1}{V(z)} \int_{A^{rn}}^{\infty} V(y) \hat{P}_z(x_1 \in dy) = \frac{1 + o(1)}{V(z)} \int_{A^{rn}}^{\infty} U_0(\log A y) \hat{P}_z(x_1 \in dy) \\
= \frac{1 + o(1)}{V(z)} \left( U_0(rn) \hat{P}_z(x_1 > A^{rn}) + \int_{A^{rn}}^{\infty} U_0'(\log A y)(\log A y) \hat{P}_z(x_1 > y) dy \right).
\]

According to (3),
\[
\hat{P}_z(x_1 > y) = \mathbb{P}(\eta_1 > \log_A(y - az)) \sim \frac{c}{\log_A y}
\]
uniformly in \( z \leq A^{2n}, y \geq A^{2n} \). Therefore,
\[
\hat{P}_z(x_1 > A^{rn}) = \frac{1 + o(1)}{V(z)} \left( U_0(rn) \frac{c}{rn} + \int_{rn}^{\infty} e^{-u_0(t)} \mathbb{P}(\eta_1 > t) dt \right) \\
= \frac{1 + o(1)}{V(z)} \left( U_0(rn) \frac{c}{rn} + e^{-u_0(rn)} \right) = \frac{1 + o(1)}{V(z)} c e^{-u_0(rn)} = \frac{1 + o(1)}{V(z)} c e^{-u_0(rn)} \frac{n}{n}.
\]

This implies that
\[
\inf_{z < A^{2n}} \frac{\hat{P}_z(x_1 \in (A^{2n}, A^{Bn}])}{\hat{P}_z(x_1 > A^{2n})} = 1 - \left( \frac{2}{B} \right)^c + o(1).
\]

Taking \( B = 2^{1+2/c} \), we conclude that
\[
\hat{P}_x(x_{2n} \leq A^{2+2/c}, \sigma_{2n} \leq n) \geq \frac{1}{2} \hat{P}_x(\sigma_{2n} \leq n), \quad x \leq A^{2n}
\]
for all \( n \) large enough. Using this bound, we obtain
\[
\hat{P}_x(x_n \leq A^{Bn}) \\
\geq \hat{P}_x(\sigma_{2n} > n) + \hat{P}_x(x_n \leq A^{Bn}, x_{\sigma_{2n}} \leq A^{Bn}, \sigma_{2n} \leq n) \\
\geq \hat{P}_x(\sigma_{2n} > n) + \frac{1}{2} \hat{P}(\sigma_{2n} \leq n) \hat{P}_x(x_n \leq A^{Bn} | x_{\sigma_{2n}} \leq A^{Bn}, \sigma_{2n} \leq n).
\]

By the strong Markov property,
\[
\hat{P}_x(x_n \leq A^{Bn} | x_{\sigma_{2n}} \leq A^{Bn}, \sigma_{2n} \leq n) \geq \inf_{z \in (A^{2n}, A^{Bn})} \hat{P}_z(x_j \leq X_{j-1} \text{ for all } j \leq n).
\]
If $X_0 = z \geq A^{2n}$ then $X_j \geq A^{2n-j}$ for every $j \geq 1$. If $A^n \geq x_0$ then $\mathbb{P}_z(T_{x_0}^n) = 1$ and, consequently,

$$\mathbb{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \geq \frac{V(A^n)}{V(z)} \mathbb{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n)$$

For every $y$ we have

$$\mathbb{P}_y(X_1 \leq y) = \mathbb{P}(\eta_1 \leq \log_A y + \log_A(1 - a)).$$

Thus, by the Markov property,

$$\mathbb{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \geq \mathbb{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n-1) \mathbb{P}(\eta_1 \leq n + \log_A(1 - a))$$

$$\geq \prod_{j=0}^{n-1} \mathbb{P}(\eta_1 \leq 2n - j + \log_A(1 - a))$$

$$\geq (\mathbb{P}(\eta_1 \leq n + \log_A(1 - a)))^n \sim e^{-c}.$$

This implies that

$$\inf_{z \in (A^{2n}, A^n)} \mathbb{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \geq \frac{V(A^n)}{V(A^{2n})} \frac{e^{-c}}{2} \geq C_0.$$

Therefore,

$$\mathbb{P}_x(T_{x_0}^n) \geq \mathbb{P}_x(\sigma_{2n} > n) + \frac{C_0}{2} \mathbb{P}_x(\sigma_{2n} \leq n) \geq \frac{C_0}{2}$$

for all sufficiently large $n$. Recalling that $V$ is increasing, we obtain the bound

$$\mathbb{P}_x(T_{x_0}^n) \geq V(x) \mathbb{P}_x(T_{x_0}^n) \geq \frac{V(x)}{4} \frac{e^{-c}}{V(A^{2n})} \frac{V(A^n)}{V(A^{2n})}, \quad x \leq A^{2n}.$$
4.2. Upper bound for $P(T_{x_0}^{(X)} > n)$.

**Lemma 9.** If $V(x)$ is increasing on $(x_0, \infty)$ then

$$P_x(T_{x_0}^{(X)} > n) \leq C \frac{V(x)}{V(A^n)}, \quad x > x_0, \ n \geq 1.$$ 

**Proof.** Since $P_x(X_n > y)$ is monotonically increasing in $x$,

$$P_x(X_n > y | T_{x_0}^{(X)} > n) \geq P_x(X_n > y) \quad \text{for all } x, y > x_0.$$

Consequently,

$$E_x[W(X_n) | T_{x_0}^{(X)} > n] \geq E_x[W(X_n)] \geq E_x[W(X_n)1\{X_n > x_0\}]$$

for every nonnegative increasing function $W$. (To prove this, one approximates $W$ by functions of the form $\sum k \cdot 1_{(y_k, \infty)}$.) In particular, for $W = V$ one gets

$$V(x) = E_x[V(X_n) | T_{x_0}^{(X)} > n] \geq E_x[V(X_n)1\{X_n > x_0\}]$$

and can conclude

$$P_x(T_{x_0}^{(X)} > n) \leq \frac{V(x)}{E_x[V(X_n)1\{X_n > x_0\}]}. \quad (41)$$

As we already know, $\frac{\log_A X_n}{n}$ converges weakly to the distribution with density

$$cy^{c-1} / (y + 1)^{c+1} 1_{\mathbb{R}^+}(y).$$

The asymptotic behaviour in (41) is obtained most conveniently if one assumes that $\frac{\log_A X_n}{n}$ converges almost everywhere to some $Z$ with this distribution. (On a suitable probability space, the sequence can always be constructed in such a way.) Then, as $v(x) := V(\log_A x)$ varies regularly with index $1 - c$,

$$\frac{V(X_n)}{V(A^n)} = \frac{v(\log_A X_n)}{v(n)} = \frac{v(\frac{\log_A X_n}{n})}{v(n)} \sim \frac{v(Z_n)}{v(n)} \rightarrow Z^{1-c}.$$ 

(More precisely, due to the monotonicity of $V$, one first gets for every fixed $N \in \mathbb{N}$

$$\limsup_{n \to \infty} \frac{v(\frac{\log_A X_n}{n})}{v(n)} \leq \limsup_{n \to \infty} v\left(\sup_{k: k \geq N} \frac{\log_A X_k}{k}\right)^{-1} = \left(\sup_{k: k \geq N} \frac{\log_A X_k}{k}\right)^{-1}.$$ 

$N \to \infty$ shows $\limsup_{n \to \infty} \frac{v(\log_A X_n)}{v(n)} \leq Z^{1-c}$ and likewise one checks that the lower limit has at least this value.)

Now one can apply the Fatou lemma:

$$\liminf_{n \to \infty} E_0[V(X_n)1\{X_n > x_0\}] = \int_0^\infty y^{1-c} \frac{cy^{c-1}}{(y + 1)^{c+1}} = 1,$$

so (41) yields the desired bound. \hfill \Box

**Remark 10.** We know from the construction of $V$ that this function is increasing for $x_0 \geq x_2$. We now notice that, using Theorem (3(iii)), one can infer that $V$ is increasing for all $x_0$. Indeed, by the monotonicity of the chain $\{X_n\}$,

$$P_x(T_{x_0}^{(X)} > n) \leq P_y(T_{x_0}^{(X)} > n)$$
for all \( n \) and for all \( x \leq y \). Combining this with the asymptotic relation \( \mathbb{P}_x(T_{x_0}(X) > n) \sim \varphi(c)\frac{V(x)}{V(A^n)} \), we conclude that \( V(x) \leq V(y) \). Thus, the bound in Lemma 9 holds for each \( x_0 \) and, consequently, the upper bound in Theorem 8(ii) is valid.

We next prove an alternative upper bound, which is valid without monotonicity assumption.

**Lemma 11.** Assume that there exist \( x_1 \) and a subexponential distribution \( F \) such that

\[
\mathbb{P}_x(T_{x_1}^{(X)} > n) \leq C(x) F(n), \quad n \geq 0, \ x > x_1.
\]

If \( x_0 < x_1 \) is such that \( \mathbb{P}(ax_1 + \xi_1 < x_0) > 0 \) then there exists \( C(x_0, x) \) such that

\[
\mathbb{P}_x(T_{x_0}^{(X)} > n) \leq C(x, x_0) F(n), \quad n \geq 0, \ x > x_0.
\]

**Proof.** The assumption \( \mathbb{P}(ax_1 + \xi_1 < x_0) > 0 \) implies that

\[
p := \mathbb{P}_{x_1}(X_{T_{x_1}^{(X)}} \leq x_0) > 0.
\]

Then we can represent the law of \( T_{x_1}^{(X)} \) as a mixture of two distributions:

\[
\mathbb{P}_{x_1}(T_{x_1}^{(X)} \in B) = p\mathbb{P}_{x_1}(T_{x_1}^{(X)} \in B|X_{T_{x_1}^{(X)}} \leq x_0) + (1 - p)\mathbb{P}_{x_1}(T_{x_1}^{(X)} \in B|X_{T_{x_1}^{(X)}} > x_0)
\]

\[=: p\mathbb{P}(\theta \in B) + (1 - p)\mathbb{P}(\zeta \in B).\]

Noting that \( \{X_n\} \) may visit \((x_0, x_1]\) several times before \( T_{x_0}^{(X)} \) and using the monotonicity of the chain, we get

\[
\mathbb{P}_{x_1}(T_{x_0}^{(X)} > n) \leq p \sum_{k=0}^{\infty} (1 - p)^k \mathbb{P}(\zeta_1 + \zeta_2 + \ldots + \zeta_k + \theta > n),
\]

where \( \{\zeta_k\} \) are independent copies of \( \zeta \). Under the assumptions of the lemma we have

\[
\mathbb{P}(\zeta > n) \leq C_1 F(n) \quad \text{and} \quad \mathbb{P}(\theta > n) \leq C_2 F(n).
\]

Then, by Proposition 4 in [5],

\[
\mathbb{P}_{x_1}(T_{x_1}^{(X)} > n) \leq C F(n).
\]

If the starting point \( x \) is smaller than \( x_1 \) then

\[
\mathbb{P}_x(T_{x_0}^{(X)} > n) \leq \mathbb{P}_{x_1}(T_{x_0}^{(X)} > n) \leq C F(n).
\]

If the starting point \( x \) is bigger than \( x_1 \) then \( \mathbb{P}_x(T_{x_0}^{(X)} > n) \) is bounded by the tail of the convolution of \( \mathbb{P}_x(T_{x_1}^{(X)} \in \cdot) \) and \( \mathbb{P}_{x_1}(T_{x_0}^{(X)} \in \cdot) \). Since the tails of these two distributions are \( O(F(n)) \), the tail of their convolution is also \( O(F(n)) \). This completes the proof of the lemma.

**Corollary 12.** If [8] holds then

\[
\mathbb{P}_x(T_{x_0}^{(X)} > n) \leq C \frac{V(x)}{V(A^n)}
\]

for all \( x > x_0 \) and all \( n \geq 1 \).
Proof. It suffices to consider the case $x_0 < x_*$. Since $V_*(A^n)$ is regularly varying then, in view of Lemma 9, the conditions of Lemma 11 are valid for $x_1 = x_*$ and $F(n) \sim CU_0(n)$. Combining now Lemmata 9 and 11, we have, for $x > x_*$,

$$P_x(T_{x_0}^{(X)} > n) \leq P_x(T_{x_*}^{(X)} > n/2) + P_{x_*}(T_{x_0}^{(X)} > n/2) = C_1 V_* (x) + C_2 V_*(A^{n/2})$$

Recalling that $V_*(A^n)$ is regularly varying and that $V(x) \leq V_* (x) + C$ in the case $x_0 < x_*$, we have the desired estimate for $x > x_*$. In the case $x \leq x_*$ it suffices to apply Lemma 11. $\square$

5. Proof of asymptotic relations

In this section we shall prove asymptotic relations in Theorem 3(iii). Exact asymptotics in Theorem 2 can be derived by exactly the same arguments, and we omit their proof.

We are going to apply Theorem 3.10 from Durrett \[11\] to the sequence of Markov processes

$$v(n) := \log A X_{[nt]}, \quad t \geq 0.$$ Since this sequence converges weakly to the process $Z$, which is non-degenerate and $P_x(T_0^{(Z)} > t)$ is strictly positive for all $x, t > 0$, we conclude that the conditions (i)-(iii) from \[11\] are fulfilled. Moreover, we have already shown that $P_x \left( \cdot | T_{x_0}^{(X)} > n \right)$ converges, as $x \to 0$, to a non-degenerate limit. Thus, it remains to check that

- $P_{A^n} \left( T_{x_0}^{(X)} > nt_n \right) \to P_x \left( T_0^{(Z)} > t \right)$ if $x_n \to x > 0$ and $t_n \to t > 0$;
- $P_{A^n} \left( T_{x_0}^{(X)} > nt_n \right) \to 0$ whenever $x_n \to 0$ and $t_n \to t > 0$;
- the sequence $v(n)$ is tight; and
- $\lim_{h \to 0} \liminf_{n \to \infty} P_{A^n} (v(n) > h | T_{x_0}^{(X)} > n) = 1$ for every $t > 0$.

We start with the first condition.

**Lemma 13.** If $x_n \to x > 0$ and $t_n \to t > 0$ then

$$P_{A^n} \left( T_{x_0}^{(X)} > nt_n \right) \to P_x \left( T_0^{(Z)} > t \right).$$

**Proof.** Since $P_y(T_{x_0}^{(X)} > y)$ is increasing in $y$ and decreasing in $m$, it suffices to prove the lemma in the special case $x_n = x$ and $t_n = t$. We are going to apply Theorem 2.1 from \[11\]. We set

$$A_0 := \left\{ f \in D[0, t] : \inf_{s \leq t} f(s) > 0 \right\}$$
and

$$A_n := \left\{ f \in D[0, t] : \inf_{s \leq t} f(s) > \frac{x_0}{n} \right\}, \quad n \geq 1.$$ Furthermore, for every $\varepsilon > 0$ we define

$$G_\varepsilon := \left\{ f \in D[0, t] : \inf_{s \leq t} f(s) > \varepsilon \right\}.$$ Then we have

$$P_x (Z \in \partial G_\varepsilon) = 0 \quad \text{for all } \varepsilon, t > 0.$$ (42)
It is clear that $G_x \subset A_n$ for all $n > x_0/\varepsilon$ and $G_{1/n} \uparrow A_0$. Thus, in order to apply Theorem 2.1 from [11] we have only to show that

$$\limsup_{n \to \infty} P_{A^n}(T_{x_0}^{(X)} > nt) \leq P_x(T_0^{(Z)} > t). \quad (43)$$

Fix some $\varepsilon < x$ and $\delta < t$. Then, using the monotonicity of the chain $\{X_n\}$, we get

$$P_{A^n}(T_{x_0}^{(X)} > nt) \leq P_{A^n} \left( \inf_{s \leq t} X_{[sn]} > A^{\varepsilon n} \right) + P_{A^n}(T_{x_0}^{(X)} > \delta n).$$

According to the upper bound in [10],

$$P_{A^n}(T_{x_0}^{(X)} > \delta n) \leq C \frac{V(A^{\varepsilon n})}{V(A^{\delta n})}.$$ 

Recalling that $V(A^x)$ is regularly varying of index $1 - c$, we conclude

$$\limsup_{n \to \infty} P_{A^n}(T_{x_0}^{(X)} > \delta n) \leq C \left( \frac{\varepsilon}{\delta} \right)^{1-c}.$$ 

Furthermore, combining (41) and (42), we get

$$P_{A^n} \left( \inf_{s \leq t} X_{[sn]} > A^{\varepsilon n} \right) \to P_x \left( \inf_{s \leq t} Z_s > \varepsilon \right).$$

Consequently,

$$\limsup_{n \to \infty} P_{A^n}(T_{x_0}^{(X)} > nt) \leq P_x \left( \inf_{s \leq t} Z_s > \varepsilon \right) + C \left( \frac{\varepsilon}{\delta} \right)^{1-c} 
\leq P_x (T_0^{(Z)} > t - \delta) + C \left( \frac{\varepsilon}{\delta} \right)^{1-c}.$$ 

Letting here first $\varepsilon \to 0$ and then $\delta \to 0$, we arrive at (43). Thus, the proof is complete.

Lemma 14. If $t_n \to t > 0$ and $x_n \to 0$ then

$$P_{A^{x_{n}}} (T_{x_0}^{(X)} > nt) \to 0.$$ 

This is a simple consequence of the upper bound in [10] and we omit its proof.

Lemma 15. For all $x > x_0$ and all $t > 0$ one has

$$\lim_{h \to 0} \limsup_{n \to \infty} P_x (v_t^{(n)} > h | T_{x_0}^{(X)} > n) = 1.$$ 

Proof. By the definition of $v_t^{(n)}$, 

$$P_x (v_t^{(n)} \leq h | T_{x_0}^{(X)} > n) = \frac{P_x (X_{[nt]} \leq A^{hn}, T_{x_0}^{(X)} > n)}{P_x (T_{x_0}^{(X)} > n)}.$$ 

Set $s = \min \{1, t\}/2$. Then, by the monotonicity of $X_n$,

$$P_x (X_{[nt]} \leq A^{hn}, T_{x_0}^{(X)} > n) \leq P_x (T_{x_0}^{(X)} > ns) P_0 (X_{[nt-s]} \leq A^{hn}).$$ 

Therefore,

$$P_x (v_t^{(n)} > h | T_{x_0}^{(X)} > n) \leq \frac{P_x (T_{x_0}^{(X)} > ns) P_0 (X_{[nt-s]} \leq A^{hn})}{P_x (T_{x_0}^{(X)} > n)}.$$
Taking into account (4), (10) and (5), we get
\[
\limsup_{n \to \infty} P_x(\xi_1^{(n)} > h | T_{x_0}^{(X)} > n) \leq C s^{-1} P_0(Z_{1-s} \leq h) \leq C s^{-1} \left( \frac{h}{h + t - s} \right)^c.
\]
This yields the desired relation. \(\square\)

To show the tightness we shall use the following upper bound for the conditional distribution of \(X_n\).

**Lemma 16.** There exists a constant \(C\) such that
\[
P_x(X_n \geq A^y | T_{x_0}^{(X)} > n) \leq C \frac{n}{y} \quad y \geq 2 \log_A \left( x + \frac{1}{1 - a} \right).
\]

**Proof.** If \(\xi_k < A^{y/2} \) for all \(k \leq n\) then
\[
X_n = a^nx + a^{n-1}\xi_1 + a^{n-2}\xi_2 + \ldots + \xi_n \\
\leq x + A^{y/2} \sum_{j=0}^{n-1} a^j \leq x + A^{y/2} \frac{1}{1 - a} \leq A^y
\]
for all \(y \geq 2 \log_A \left( x + \frac{1}{1 - a} \right)\). Therefore,
\[
P_x(X_n \geq A^y, T_{x_0}^{(X)} > n) \leq \sum_{k=1}^{n} P_x(\xi_k \geq A^{y/2}, T_{x_0}^{(X)} > n) \\
\leq \sum_{k=1}^{n} P_x(\xi_k \geq A^{y/2}, T_{x_0}^{(X)} > k-1) \\
\leq P(\xi_1 \geq A^{y/2}) \sum_{k=1}^{n} P_x(T_{x_0}^{(X)} > k-1)
\]
Using the upper bound in (10) and recalling that \(V(A^x)\) is regularly varying with index \(1 - c\), we conclude that
\[
\sum_{k=1}^{n} P_x(T_{x_0}^{(X)} > k-1) \leq 1 + C \sum_{j=1}^{n-1} \frac{V(x)}{V(A^y)} \leq C \frac{nV(x)}{V(A^n)}.
\]
Consequently,
\[
P_x(X_n \geq A^y, T_{x_0}^{(X)} > n) \leq C \frac{nV(x)}{V(A^n)} P(\eta_1 \geq y/2).
\]
Combining this with (3) and with the lower bound in (10), we obtain the desired estimate. \(\square\)

**Lemma 17.** The sequence \(v^{(n)}\) is tight.

**Proof.** According to Theorem 3.6 in [11], it suffices show that
\[
\lim_{K \to \infty} \limsup_{n \to \infty} P_x(X_n > A^{nK} | T_{x_0}^{(X)} > n) = 0 \quad (44)
\]
and
\[
\lim_{t \to 0} \limsup_{n \to \infty} P_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > n) = 0, \quad h > 0. \quad (45)
\]
Consequently, the distribution introduced in Theorem 1. To show (15) we first notice that, for every $t < 1$,

$$
\mathbb{P}_x(X_{[nt]} > A^{nh}|T_{x_0}^{(X)} > n) \leq \mathbb{P}_x(X_{[nt]} > A^{nh}|T_{x_0}^{(X)} > nt) \cdot \frac{\mathbb{P}_x(T_{x_0}^{(X)} > nt)}{\mathbb{P}_x(T_{x_0}^{(X)} > n)}.
$$

Applying Lemma 16 to the first probability term on the right hand side, we get

$$
\mathbb{P}_x(X_{[nt]} > A^{nh}|T_{x_0}^{(X)} > n) \leq C \frac{t}{h} \frac{\mathbb{P}_x(T_{x_0}^{(X)} > nt)}{\mathbb{P}_x(T_{x_0}^{(X)} > n)}.
$$

Using again (10), we have

$$
\limsup_{n \to \infty} \frac{\mathbb{P}_x(T_{x_0}^{(X)} > nt)}{\mathbb{P}_x(T_{x_0}^{(X)} > n)} \leq Ct^{-1}.
$$

As a result we have the estimate

$$
\limsup_{n \to \infty} \mathbb{P}_x(X_{[nt]} > A^{nh}|T_{x_0}^{(X)} > n) \leq C \frac{t^c}{h},
$$

which implies (15). \qed

We have checked all the conditions in Theorem 3.10 in [11]. Therefore, the sequence of distributions $\mathbb{P}_x(u^{(n)} \in \cdot|T_{x_0}^{(X)} > n)$ on $D[0,1]$ converges weakly towards the distribution $Q$ introduced in Theorem 1.

Therefore, it remains to prove (11). Since $V$ is harmonic,

$$
V(x) = E_x[V(X_n); T_{x_0}^{(X)} > n]
= E_x[V(X_n); T_{x_0}^{(X)} > n, X_n \leq A^{Kn}] + E_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}]
$$

for every $K > 0$.

We know that $V(x) \leq CU_0(\log_A x)$. Therefore,

$$
E_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}]
\leq CE_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}]
= CU_0(Kn)P_x(\log_A X_n \geq Kn; T_{x_0}^{(X)} > n)
+ C \int_{K_n}^{\infty} U_0'(y)P_x(\log_A X_n \geq y; T_{x_0}^{(X)} > n)dy.
$$

Combining Lemma 10 and 10, we have

$$
\mathbb{P}_x(\log_A X_n \geq y; T_{x_0}^{(X)} > n) \leq C \frac{nV(x)}{yV(A^n)}.
$$

Consequently,

$$
E_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}]
\leq C \frac{V(x)}{V(A^n)} \left( \frac{U_0(Kn)}{K} + n \int_{K_n}^{\infty} \frac{U_0'(y)}{y} dy \right)
\leq C \frac{V(x)}{V(A^n)} \left( \frac{U_0(Kn)}{K} + n \int_{K_n}^{\infty} \frac{U_0(y)}{y^2} dy \right) \leq C \frac{V(x)}{V(A^n)} \frac{U_0(Kn)}{K}.
$$
Recalling that $V(A^n) \sim U_0(n)$ and that $U_0$ is regularly varying, we finally get
\[
\limsup_{n \to \infty} \mathbb{E}_x[V(X_n); T^{(X)}_{x_0} > n, X_n > A^{K_n}] \leq \frac{C}{Kn} V(x).
\]
(47)

For the first summand on the right hand side of (56) we have
\[
\mathbb{E}_x[V(X_n); T^{(X)}_{x_0} > n, X_n \leq A^{K_n}]
\]
\[
= \mathbb{P}_x(T^{(X)}_{x_0} > n) \mathbb{E}_x[V(X_n) 1\{X_n \leq A^{K_n}\} \mid T^{(X)}_{x_0} > n]
\]
\[
= V(A^n) \mathbb{P}_x(T^{(X)}_{x_0} > n) \mathbb{E}_x \left[\frac{V(X_n)}{V(A^n)} 1\{X_n \leq A^{K_n}\} \mid T^{(X)}_{x_0} > n\right].
\]

It follows from the already proven conditional limit theorem and from (24) that
\[
\lim_{n \to \infty} \mathbb{P}_x \left(\frac{\log A X_n}{n} \leq y \mid T^{(X)}_{x_0} > n\right) = \frac{y}{y + 1}, \quad y > 0.
\]

Combining this with the regular variation property of $V$, we obtain
\[
\mathbb{E}_x \left[\frac{V(X_n)}{V(A^n)} 1\{X_n \leq A^{K_n}\} \mid T^{(X)}_{x_0} > n\right]
\]
\[
= (1 + o(1)) \mathbb{E}_x \left[\left(\frac{\log A X_n}{n}\right)^{1-c} 1\{X_n \leq A^{K_n}\} \mid T^{(X)}_{x_0} > n\right]
\]
\[
= (1 + o(1)) \int_0^K \frac{y^{c-1}}{(1+y)^2} dy.
\]

Consequently,
\[
\mathbb{E}_x[V(X_n); T^{(X)}_{x_0} > n, X_n \leq A^{K_n}]
\]
\[
= (1 + o(1)) V(A^n) \mathbb{P}_x(T^{(X)}_{x_0} > n) \int_0^K \frac{y^{c-1}}{(1+y)^2} dy.
\]
(48)

Plugging (47) and (48) into (46) and letting $K \to \infty$, we obtain
\[
\mathbb{P}_x(T^{(X)}_{x_0} > n) \sim \left(\int_0^\infty \frac{y^{c-1}}{(1+y)^2} dy\right)^{-1} \frac{V(x)}{V(A^n)}.
\]

Thus, (11) holds with
\[
\gamma(c) = \left(\int_0^\infty \frac{y^{c-1}}{(1+y)^2} dy\right)^{-1} = \frac{1}{(1 - c) B(c, 1 - c)}.
\]

6. Proof of Theorem 5

6.1. Expectation of hitting times for the maximal autoregressive process.

Put $u(x) = \mathbb{E}_x[T^{(R)}_{x_0}]$ and observe that the Markov property implies that it should satisfy
\[
u(x) = \mathbb{P}_x(T^{(R)}_{x_0} = 1) + \mathbb{E}_x[1 + u(R_1); T^{(R)}_{x_0} > 1]
\]
\[
= \mathbb{P}_x(R_1 \leq x_0) + \mathbb{E}_x[1 + u(R_1); R_1 > x_0], \quad x > x_0.
\]
(49)

Assume first that $x \in (x_0, x_0 + 1]$. In this case one has
\[
\{R_1 > x_0\} = \{R_1 = \eta_1 > x_0\}.
\]
Therefore,
\[ u(x) = P(\eta_1 \leq x_0) + E[1 + u(\eta_1); \eta_1 > x_0] \]
\[ = 1 + E[u(\eta_1); \eta_1 > x_0] \quad \text{for all } x \in (x_0, x_0 + 1). \]

For all \( x > x_0 + 1 \) one has \( P_x(T_{x_0}^1 R_1 = 1) = 0 \). This implies that \( (49) \) reduces to
\[ u(x) = E_x[1 + u(R_1)] \]
\[ = 1 + u(x - 1)P(\eta_1 \leq x - 1) + E[u(\eta_1); \eta_1 > x - 1], \quad x > x_0 + 1. \quad (50) \]
If \( x \in (x_0 + 1, x_0 + 2] \) then \( x - 1 \in (x_0, x_0 + 1] \) and, consequently, \( u(x - 1) = u(x_0 + 1) \) for all \( x \in (x_0 + 1, x_0 + 2] \). From this observation and from \( (50) \) we have
\[ u(x) = 1 + u(x_0 + 1)P(\eta_1 \leq x - 1) + E[u(\eta_1); \eta_1 > x - 1] \]
\[ + E[u(\eta_1); \eta_1 \in (x - 1, x_0 + 1)] + E[u(\eta_1); \eta_1 > x_0 + 1] \]
\[ = 1 + u(x_0 + 1)P(\eta_1 \leq x_0 + 1) + E[u(\eta_1); \eta_1 > x_0 + 1]. \quad (51) \]
This equality implies that \( u(x) = u(x_0 + 2) \) for all \( x \in (x_0 + 1, x_0 + 2] \). Note also that
\[ u(x_0 + 1) = 1 + E[u(\eta_1); \eta_1 > x_0] \]
\[ = 1 + u(x_0 + 1)P(\eta_1 \in (x_0, x_0 + 1]) + E[u(\eta_1); \eta_1 > x_0 + 1]. \]
Combining this with \( (51) \), we conclude that
\[ u(x_0 + 2) = u(x_0 + 1) (1 + P(\eta_1 \leq x_0)). \]

Fix now an integer \( n \) and consider the case \( x \in (x_0 + n, x_0 + n + 1] \). Assume that we have already shown that \( u(y) = u(x_0 + n) \) for all \( y \in (x_0 + n - 1, x_0 + n] \). Then we have from \( (50) \)
\[ u(x) = 1 + u(x_0 + n)P(\eta_1 \leq x - 1) + E[u(\eta_1); \eta_1 > x - 1] \]
\[ = 1 + u(x_0 + n)P(\eta_1 \leq x_0 + n) + E[u(\eta_1); \eta_1 > x_0 + n]. \]
Therefore, \( u(x) = u(x_0 + n + 1) \) for all \( x \in (x_0 + n, x_0 + n + 1] \). This means that this property is valid for all \( n \).

One has also equalities
\[ u(x_0 + n + 1) = 1 + u(x_0 + n)P(\eta_1 \leq x_0 + n) + E[u(\eta_1); \eta_1 > x_0 + n] \]
and
\[ u(x_0 + n) = 1 + u(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1) + E[u(\eta_1); \eta_1 > x_0 + n - 1] \]
\[ = 1 + u(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1) \]
\[ + u(x_0 + n)P(\eta_1 \in (x_0 + n - 1, x_0 + n]] + E[u(\eta_1); \eta_1 > x_0 + n]. \]
Taking the difference we obtain
\[ u(x_0 + n + 1) - u(x_0 + n) \]
\[ = u(x_0 + n)P(\eta_1 \leq x_0 + n) - u(x_0 + n - 1)P(\eta_1 \leq x_0 + n - 1) \]
\[ - u(x_0 + n)P(\eta_1 \in (x_0 + n - 1, x_0 + n]] \]
\[ = P(\eta_1 \leq x_0 + n - 1) (u(x_0 + n) - u(x_0 + n - 1)). \]
Consequently,

\[ u(x_0 + n + 1) - u(x_0 + n) = u(x_0 + 1) \prod_{k=0}^{n-1} P(\eta_1 \leq x_0 + k), \quad n \geq 1. \]

As a result, we have the following expression for the expectation of the hitting time

\[ u(x) = u(x_0 + 1) \left( 1 + \sum_{j=1}^{n-1} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right), \quad x \in (x_0 + n, x_0 + n + 1]. \quad (52) \]

Finally, in order to get a finite solution we have to show that the equation

\[ u(x_0 + 1) = 1 + E[u(\eta_1); \eta_1 > x_0] \]

is solvable. In view of (52), the previous equation is equivalent to

\[ u(x_0+1) = 1 + u(x_0+1) \sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0+n, x_0+n+1]). \]

Now we conclude that (49) has a finite solution if and only if

\[ \sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0+n, x_0+n+1]) < 1. \]

Clearly,

\[ \sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0+n, x_0+n+1]) = P(\eta_1 > x_0) + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \sum_{n=j}^{\infty} P(\eta_1 \in (x_0+n, x_0+n+1]) \]

\[ = P(\eta_1 > x_0) + \sum_{j=1}^{\infty} (1 - P(\eta_1 \leq x_0 + j)) \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k). \]

Furthermore, for every \( N \geq 1 \),

\[ \sum_{j=1}^{N} (1 - P(\eta_1 \leq x_0 + j)) \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \]

\[ = \sum_{j=1}^{N} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) - \sum_{j=1}^{N} \prod_{k=0}^{j} P(\eta_1 \leq x_0 + k) \]

\[ = P(\eta_1 \leq x_0) - \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k). \]
This implies that
\[
\sum_{n=0}^{\infty} \left( 1 + \sum_{j=1}^{n} \prod_{k=0}^{j-1} P(\eta_1 \leq x_0 + k) \right) P(\eta_1 \in (x_0 + n, x_0 + n + 1])
= 1 - \lim_{N \to \infty} \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k).
\]
Thus, there is a finite solution \( u(x) \) if and only if
\[
\lim_{N \to \infty} \prod_{k=0}^{N} P(\eta_1 \leq x_0 + k) > 0.
\]
Note that this is equivalent to \( E\eta_1^+ < \infty \). Then,
\[
u(x_0 + 1) = \prod_{k=0}^{\infty} P(\eta_1 \leq x_0 + k).
\]

6.2. Recursion for tails of exit times. We will consider now \( P_x(T_{x_0}^{(R)} > n) \).
Define
\[
v(n, k) = P_{x_0+k+1}(T_{x_0}^{(R)} > n), \quad n, k \geq 0
\]
and
\[
v_n = v(n, 0).
\]
Then the following result holds.

**Proposition 18.** Assume that \( 0 < P(\eta_1 < x_0) < 1 \). Then, for integer \( n, k \geq 0 \),
\[
P_x(T_{x_0}^{(R)} > n) = v(n, k), \quad x \in (x_0 + k, x_0 + k + 1].
\]
For \( n \leq k \) we have \( v(n, k) = 1 \) and for \( n > k \) the following recursive equality holds
\[
v(n, k) = v(n, 0) + \sum_{m=1}^{k} v(n-m, 0) \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j).
\]
Furthermore,
\[
v_n = P(\eta_1 > x_0 + n - 1) + v_{n-1} P(\eta_1 \in (x_0, x_0 + n - 1])
+ \sum_{m=1}^{n-2} v_{n-m-1} P(\eta_1 \in (x_0 + m, x_0 + n - 1]) \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j)
\]
and hence (53) and (55) allow us to find \( P_x(T_{x_0}^{(R)} > n) \) recursively.

**Proof.** It is clear that for \( n \leq k \) it holds \( P_x(T_{x_0}^{(R)} > n) = v(n, k) = 1 \). Hence, in the rest of the proof we will assume that \( n > k \).
Let \( x \in (x_0, x_0 + 1] \). Then, for \( n > 0 \) we have,
\[
P_x(T_{x_0}^{(R)} > n) = \int_{x_0}^{\infty} P_x(R_1 \in dy) P_y(T_{x_0}^{(R)} > n - 1)
= \int_{x_0}^{\infty} P(\eta_1 \in dy) P_y(T_{x_0}^{(R)} > n - 1).
\] Clearly this probability is the same for each \( x \in (x_0, x_0 + 1] \) and hence (53) holds for \( k = 0 \).
Next consider $x \in (x_0 + 1, x_0 + 2]$. For every $n > 1$ we have

$$P_x(T_{x_0}^{(R)} > n)$$

$$= P_{x-1}(T_{x_0}^{(R)} > n - 1)P(\eta_1 \leq x - 1) + \int_{x-1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= P_{x_0+1}(T_{x_0}^{(R)} > n - 1)P(\eta_1 \leq x - 1) + \int_{x_0+1}^{\infty} P(\eta_1 \in dy)P_{x_0+1}(T_{x_0}^{(R)} > n - 1)$$

$$+ \int_{x_0+1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= v(n-1,0)P(\eta_1 \leq x_0 + 1) + \int_{x_0+1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1). \quad (57)$$

This expression is constant for $x \in (x_0 + 1, x_0 + 2]$ and hence $[53]$ holds for $k = 1$. Note also that it follows from $[56]$ that

$$v(n,0) = v(n-1,0)P(\eta_1 \in (x_0, x_0 + 1]) + \int_{x_0+1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1).$$

Subtracting this expression from $[57]$ we obtain

$$v(n,1) - v(n,0) = v(n-1,0)P(\eta_1 \leq x_0). \quad (58)$$

We will now prove by induction that for $x \in (x_0 + k, x_0 + k]$ the tail $P_x(T_{x_0}^{(R)} > n)$ is constant and will simultaneously show that for $k \geq 2$ that

$$v(n,k) - v(n,k-1) = (v(n-1,k-1) - v(n-1,k-2))P(\eta_1 \leq x_0 + k - 1). \quad (59)$$

First consider the base of induction $k = 2$. In this case, for $n > 2$ and for $x \in (x_0 + 2, x_0 + 3]$, we have

$$P_x(T_{x_0}^{(R)} > n)$$

$$= P_{x-1}(T_{x_0}^{(R)} > n - 1)P(\eta_1 \leq x - 1) + \int_{x-1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= P_{x_0+2}(T_{x_0}^{(R)} > n - 1) + \int_{x_0+2}^{\infty} P(\eta_1 \in dy)P_{x_0+2}(T_{x_0}^{(R)} > n - 1)$$

$$+ \int_{x_0+2}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= v(n-1,1)P(\eta_1 \leq x_0 + 2) + \int_{x_0+2}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1). \quad (60)$$

This expression clearly does not depend on $x$. Thus,

$$v(n,2) = v(n-1,1)P(\eta_1 \leq x_0 + 2) + \int_{x_0+2}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1).$$

It also follows from $[57]$ that

$$v(n,1) = v(n-1,0)P(\eta_1 \leq x_0 + 1) + v(n-1,1)P(\eta_1 \in (x_0 + 1, x_0 + 2])$$

$$+ \int_{x_0+2}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1).$$

Subtracting this equation from $[60]$ we obtain

$$v(n,2) - v(n,1) = (v(n-1,1) - v(n-1,0))P(\eta_1 \leq x_0 + 1).$$
This is exactly (59) with $k = 2$. Thus, the base case is true.

We will now prove the induction step. Consider $x \in (x_0 + k, x_0 + k + 1]$. For $n > k$ we obtain, using the induction hypothesis,

$$P_x(T_{x_0}^{(R)} > n)$$

$$= P_{x-1}(T_{x_0}^{(R)} > n - 1)P(\eta_1 \leq x - 1) + \int_{x-1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= P_{x_0+k}(T_{x_0}^{(R)} > n - 1) + \int_{x-1}^{x_0+k} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$+ \int_{x_0+k}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1)$$

$$= v(n-1, k-1)P(\eta_1 \leq x_0 + k) + \int_{x_0+k}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1). \tag{61}$$

This expression clearly does not depend on $x$ and hence (53) holds. Thus,

$$v(n, k) = v(n-1, k-1)P(\eta_1 \leq x_0 + k) + \int_{x_0+k}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1). \tag{61}$$

The same expression is true for $k - 1$ by the induction hypothesis. Hence,

$$v(n, k - 1) = v(n-1, k-2)P(\eta_1 \leq x_0 + k - 1)$$

$$+ v(n-1, k-1)P(\eta_1 \in (x_0 + k - 1, x_0 + k])$$

$$+ \int_{x_0+k}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1).$$

Subtracting this expression from (61) we obtain (59).

Now it follows from (58) and (59) that

$$v(n, k) - v(n, k - 1) = v(n - k, 0) \prod_{j=0}^{k-1} P(\eta_1 \leq x_0 + j). \tag{62}$$

for $n > k$. Then the standard telescoping argument gives (54). Plugging (62) into (55) we obtain

$$v_n = \int_{x_0+1}^{\infty} P(\eta_1 \in dy)P_y(T_{x_0}^{(R)} > n - 1) + v_{n-1}P(\eta_1 \in (x_0, x_0 + 1])$$

$$= \sum_{l=1}^{n-2} P(\eta_1 \in (x_0 + l, x_0 + l + 1])v(n - 1, l) + P(\eta_1 > x_0 + n - 1)$$

$$+ v_{n-1}P(\eta_1 \in (x_0, x_0 + 1])$$

$$= \sum_{l=1}^{n-2} P(\eta_1 \in (x_0 + l, x_0 + l + 1]) \left( v_{n-1} + \sum_{m=1}^{l} v_{n-m} \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j) \right)$$

$$+ P(\eta_1 > x_0 + n - 1) + v_{n-1}P(\eta_1 \in (x_0, x_0 + 1]).$$

Swapping the order of summation we obtain (55).
6.3. **Heavy tails.** To analyse the heavy-tailed case we need first the following definition. We say that a non-negative sequence \((a_n)_{n \geq 0}\) is subexponential if

\[
\lim_{n \to \infty} \frac{a_n}{a_{n-1}} = 1, \quad a_\infty := \sum_{n=0}^{\infty} a_n < \infty
\]

\[
\sum_{k=0}^{n} a_k a_{n-k} \sim 2a_\infty a_n, \quad n \to \infty.
\]

We start by deriving an upper bound for \(v_n\).

**Lemma 19.** Assume that \(F \in S^\ast\), where \(F(x) = P(\eta_1 \leq x)\). Then there exists a constant \(C\) such that

\[
v_n \leq C P(\eta_1 > n), \quad n \geq 0.
\]

**Proof.** Note that it follows from (55) that \(v_n \leq w_n\), where the sequence \(\{w_n\}\) is given by

\[
w_1 = P(\eta_1 > x_0) \quad \text{and for } n \geq 2,
\]

\[
w_n = P(\eta_1 > x_0 + n - 1) + w_{n-1} P(\eta_1 > x_0)
\]

\[+ \sum_{m=1}^{n-2} w_{n-m-1} P(\eta_1 > x_0 + m) \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j).
\]

Set \(d_0 = P(\eta_1 > x_0)\) and

\[
d_m := P(\eta_1 > x_0 + m) \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j), \quad m \geq 1.
\]

Set also \(c_n = P(\eta_1 > x_0 + n - 1), n \geq 1\) Then we have \(w_1 = c_1\) and

\[
w_n = c_n + \sum_{m=0}^{n-2} w_{n-m-1} d_m, \quad n \geq 2. \tag{63}
\]

Using (63), we obtain the following equality for generating functions:

\[
\sum_{n=1}^{\infty} w_n s^n = \sum_{n=1}^{\infty} c_n s^n + \sum_{n=2}^{\infty} s^n \sum_{m=0}^{n-2} w_{n-m-1} d_m
\]

\[= \sum_{n=1}^{\infty} c_n s^n + s \sum_{m=0}^{\infty} d_m s^m \sum_{n=m+2}^{\infty} w_{n-m-1} s^{n-m-1}
\]

\[= \sum_{n=1}^{\infty} c_n s^n + s \sum_{m=0}^{\infty} d_m s^m \left( \sum_{n=1}^{\infty} w_n s^n \right).
\]

Set, for brevity,

\[
\hat{w}(s) = \sum_{n=1}^{\infty} w_n s^n, \quad \hat{c}(s) = \sum_{n=1}^{\infty} c_n s^n, \quad \text{and } \hat{d}(s) = \sum_{n=0}^{\infty} d_n s^n.
\]

Then we have

\[
\hat{w}(s) = \hat{c}(s) + s \hat{d}(s) \hat{w}(s).
\]

Solving this equality we obtain

\[
\hat{w}(s) = \frac{\hat{c}(s)}{1 - s \hat{d}(s)}.
\]
Noting that
\[
d_m = (1 - P(\eta_1 \leq x_0 + m)) \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j))
\]
\[
= \prod_{j=0}^{m-1} P(\eta_1 \leq x_0 + j)) - \prod_{j=0}^{m} P(\eta_1 \leq x_0 + j)),
\]
we get
\[
\sum_{m=0}^{\infty} d_m = 1 - \prod_{j=0}^{\infty} P(\eta_1 \leq x_0 + j)) < 1,
\]
where the last inequality follows from the assumption \(E\eta < \infty\). Also it is clear that
\[
\frac{d_{m+1}}{d_m} = P(\eta_1 \leq n + x_0) \to 1, \quad n \to \infty
\]
and
\[
d_n \sim P(\eta_1 > n) \prod_{j=0}^{\infty} P(\eta_1 \leq x_0 + j).
\]
Since \(F \in S^*\) we can see that \((d_n)_{n \geq 0}\) is a subexponential sequence.

Then, it follows from the results in the theory of locally subexponential distributions (see Corollary 2 and Proposition 4 in [5]) that \(\frac{1}{s_d(s)}\) is a generating function of subexponential sequence behaving like \(C_2P(\eta_1 > n)\). The same statement holds for \(\hat{c}(s)\). Hence \(w_n\) is obtained as a convolution of two subexponential sequences asymptotically equivalent to \(C_1P(\eta_1 > n)\) and \(C_2P(\eta_1 > n)\) and therefore behaves as \(C_3P(\eta_1 > n)\) for some \(C_3\). This implies the statement of the lemma. \(\Box\)

In the following lemma we complete the proof of Theorem 5.

**Lemma 20.** Assume that \(F \in S^*\), where \(F(x) = P(\eta_1 \leq x)\). Then, for any \(x > x_0\),
\[
P_x(T_{x_0}^{(R)} > n) \sim u(x)P(\eta_1 > n), \quad n \to \infty,
\]
where the function \(u(x) = E_x[T_{x_0}^{(R)}]\) has been computed in [52].

**Proof.** First we derive a lower bound. For every \(N \geq 1\) one has
\[
\{T_{x_0}^{(R)} > n\} \supset \bigcup_{k=1}^{N} \{T_{x_0}^{(R)} > n, \eta_k > x_0 + n\}
\]
\[
= \bigcup_{k=1}^{N} \{T_{x_0}^{(R)} > k - 1, \eta_k > x_0 + n\}.
\]
Therefore, by the inclusion-exclusion argument,
\[
P_x(T_{x_0}^{(R)} > n) \geq \sum_{k=1}^{N} P_x(T_{x_0}^{(R)} > k - 1) P(\eta_k > x_0 + n)
\]
\[
- \sum_{k=1}^{N-1} P_x(T_{x_0}^{(R)} > k - 1) P(\eta_k > x_0 + n) \sum_{j=k+1}^{N} P(\eta_j > x_0 + n)
\]
\[
\geq (1 - NP(\eta_1 > x_0 + n)) P(\eta_1 > x_0 + n) \sum_{k=1}^{N} P_x(T_{x_0}^{(R)} > k - 1).
\]

This implies that
\[
\lim \inf_{n \to \infty} \frac{P_x(T_{x_0}^{(R)} > n)}{P(\eta_1 > x_0 + n)} \geq \sum_{k=1}^{N} P_x(T_{x_0}^{(R)} > k - 1).
\]

Letting \(N\) to infinity we obtain
\[
\lim \inf_{n \to \infty} \frac{P_x(T_{x_0}^{(R)} > n)}{P(\eta_1 > x_0 + n)} \geq \sum_{k=1}^{\infty} P_x(T_{x_0}^{(R)} > k - 1) = E_x[T_{x_0}^{(R)}] = u(x). \quad (64)
\]

We next derive the corresponding asymptotic precise upper bound for \(v_n\). Fix \(\varepsilon > 0\). From Lemma 19 and from the subexponentiality of \(P(\eta_1 > x_0 + n)\) we conclude that there exists \(N\) such that
\[
\sum_{m=N+1}^{n-N} v_{n-m-1} P(\eta_1 \in (x_0 + m, x_0 + n - 1]) \leq C \sum_{m=N}^{n-N} F(n-m) F(m) \leq \frac{\varepsilon}{2} F(x_0 + n)
\]
for all \(n \geq 2N\). Also, since \(F \in S^*\) for any fixed \(i,\)
\[
P(\eta_1 \in (x_0 + n - i, x_0 + n]) = o(F(n))
\]
and therefore for all \(n \geq 2N\),
\[
\sum_{m=n-N}^{n-2} v_{n-m-1} P(\eta_1 \in (x_0 + m, x_0 + n - 1]) \leq \frac{\varepsilon}{2} F(x_0 + n).
\]

Combining these estimates with the representation (55), we get
\[
v_n \leq (1 + \varepsilon)c_n + \sum_{m=0}^{N} d_m v_{n-m-1}, \quad n \geq 2N,
\]
where the sequences \(\{c_n\}\) and \(d_n\) are defined in the proof of Lemma 19.

Set now \(w_n^{(N)} = v_n\) for \(n < 2N\) and
\[
w_n^{(N)} = (1 + \varepsilon)P(\eta_1 > x_0 + n - 1) + \sum_{m=0}^{N} d_m w_{n-m-1}^{(N)}.
\]
Clearly \(v_n \leq w_n^{(N)}\) for all \(n\).
This implies that we define the events \( \xi_k \leq \frac{A^{n-k}}{(n-k+1)^2 cy} \) for every \( y \geq x_0 \) we define the events

\[
\{ \xi_k \leq \frac{A^{n-k}}{(n-k+1)^2 cy} \}.
\]

7. Proof of Theorem 6.

The lower bound for the tail of \( T_{x_0}^{(N)} \) can be obtained by exactly the same arguments as the lower bound for \( T_{x_0}^{(R)} \) in Lemma 20.

We turn to the corresponding upper bound. Set \( c = \frac{1}{2 \sum_{j=1}^{n-1} r_j} \). For every \( y \geq x_0 \) we define the events

\[
\{ \xi_k \leq \frac{A^{n-k}}{(n-k+1)^2 cy} \}.
\]
On the intersection of these sets one has
\[ X_n = a^n X_0 + \sum_{k=1}^{n} a^{n-k} \xi_k \]
\[ \leq a^n X_0 + \sum_{k=1}^{n} \frac{cy}{(n-k+1)^2} \leq a^n X_0 + \frac{y}{2}. \]
If \( n \) is sufficiently large, say \( n \geq n_0 = n_0(X_0) \) then we infer that \( X_n \leq y \). Therefore,
\[ P_x(X_n > y, T^{(X)}_{x_0} > n) \leq \sum_{k=1}^{n} P_x(T^{(X)}_{x_0} > k - 1)c_{n-k}(y), \quad n \geq n_0, \tag{66} \]
where
\[ c_j(y) := P(\xi_1 > \frac{A_j}{(j+1)^2} y), \quad j \geq 0. \]
We first use this estimate with \( y = x_0 \). In this case we have
\[ P_x(T^{(X)}_{x_0} > n) \leq \sum_{k=1}^{n} P_x(T^{(X)}_{x_0} > k - 1)c_{n-k}(x_0), \quad n \geq n_0. \]
Consider the sequence \( \{w_n\} \) which is defined via the recursion
\[ w_n = \sum_{k=1}^{n} w_{k-1}c_{n-k}(x_0) \]
with initial condition \( w_0 = w_1 = \ldots = w_{n_0-1} = 1 \). Then clearly
\[ P_x(T^{(X)}_{x_0} > n) \leq w_n, \quad n \geq 0. \tag{67} \]
It is immediate from the definition of \( \{w_n\} \) that
\[ \sum_{n=n_0}^{\infty} w_n s^n = \sum_{n=n_0}^{\infty} s^n \sum_{k=1}^{n} w_{k-1}c_{n-k}(x_0) \]
\[ = \sum_{n=n_0}^{\infty} s^n \sum_{k=1}^{n_0-1} w_{k-1}c_{n-k}(x_0) + \sum_{n=n_0}^{\infty} s^n \sum_{k=n_0}^{n} w_{k-1}c_{n-k}(x_0) \]
Setting
\[ d_n(x_0) := \sum_{k=1}^{n_0-1} w_{k-1}c_{n-k}(x_0) \]
and interchanging the order of summation in the second series, we conclude that
\[ \sum_{n=n_0}^{\infty} w_n s^n = \frac{\sum_{n=n_0}^{\infty} d_n(x_0)s^n}{1 - s \sum_{j=0}^{\infty} c_j(x_0)s^j}. \]
Using once again the results from [5], we infer that
\[ w_n \sim C \mathbb{P}(\eta_1 > n) \]
provided that \( \sum_{j=0}^{\infty} c_j(x_0) < 1 \). Combining this with \( \text{[67]} \), we obtain
\[ P_x(T^{(X)}_{x_0} > n) \leq C \mathbb{P}(\eta_1 > n), \quad n \geq 0. \tag{68} \]
Using Lemma 11 we conclude that \( \text{[68]} \) is valid for all \( x_0 \) such that \( \mathbb{P}(ax_0 + \xi_1 < x_0) \) is strictly positive.
Combining now (63) and (68) and recalling that the sequences $P(\eta_1 > n)$ and $c_n(y)$ are subexponential, we conclude that

$$\limsup_{n \to \infty} \frac{P_x(X_n > y, T_{x_0}^{(X)} > n)}{P(\eta_1 > n)} \leq E_x[T_{x_0}^{(X)}] + C(y), \quad (69)$$

where

$$C(y) := \sum_{k=0}^{\infty} c_k(y).$$

This quantity is finite due to the assumption $E \eta_1 < \infty$. Furthermore, $C(y) \to 0$ as $y \to \infty$.

Fix now a integer-valued sequence $N_n \to \infty$ such that $P(\eta_1 > n) \sim P(\eta_1 > n - N_n)$. By the monotonicity of the chain $\{X_n\}$,

$$P_x(T_{x_0}^{(X)} > n) = P_x(X_n - N_n > y, T_{x_0}^{(X)} > n) + P_x(X_n - N_n \leq y, T_{x_0}^{(X)} > n) \leq P_x(X_n - N_n > y, T_{x_0}^{(X)} > n - N_n) + P_x(T_{x_0}^{(X)} > n - N_n)P_y(T_{x_0}^{(X)} > N_n).$$

Applying (68) and (69), we get

$$\limsup_{n \to \infty} \frac{P_x(T_{x_0}^{(X)} > n)}{P(\eta_1 > n)} \leq E_x[T_{x_0}^{(X)}] + C(y) + C \lim_{n \to \infty} P_y(T_{x_0}^{(X)} > N_n) \leq E_x[T_{x_0}^{(X)}] + C(y).$$

Letting now $y \to \infty$ and recalling that $\lim_{y \to \infty} C(y) = 0$, we finally obtain

$$\limsup_{n \to \infty} \frac{P_x(T_{x_0}^{(X)} > n)}{P(\eta_1 > n)} \leq E_x[T_{x_0}^{(X)}].$$

Thus, the proof is complete.

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