Optimal decay of full compressible Navier-Stokes equations with potential force

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Abstract

In this paper, we aim to investigate the optimal decay rate for the higher order spatial derivative of global solution to the full compressible Navier-Stokes (CNS) equations with potential force in \( \mathbb{R}^3 \). We establish the optimal decay rate of the solution itself and its spatial derivatives (including the highest order spatial derivative) for global small solution of the full CNS equations with potential force. With the presence of potential force in the considered full CNS equations, the difficulty in the analysis comes from the appearance of non-trivial stationary solutions. These decay rates are really optimal in the sense that it coincides with the rate of the solution of the linearized system. In addition, the proof is accomplished by virtue of time weighted energy estimate, spectral analysis, and high-low frequency decomposition.

Keywords: Full compressible Navier-Stokes equations; potential force; optimal decay rate.

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1 Introduction

It is well-known that full compressible Navier-Stokes (CNS) equations can be used to describe the motion of compressible viscous and heat-conductive fluids. In this paper, we are concerned with the optimal decay rate of global small solution to the Cauchy problem for the following full CNS equations with external force in \( \mathbb{R}^3 \):

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\rho(u_t + u \cdot \nabla u) + \nabla p(\rho, \theta) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \rho \nabla \phi &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
c_v \rho(\theta_t + u \cdot \nabla \theta) + \theta p_\theta(\rho, \theta) \text{div} u - \kappa \Delta \theta - \Psi(u) &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3,
\end{aligned}
\]  

(1.1)

where \( \rho, u, \theta \) and \( p \) represent the density, velocity, temperature and pressure, respectively. And \( -\nabla \phi \) is the time independent potential force. The constant viscosity coefficients \( \mu \) and \( \lambda \) satisfy the following physical conditions:

\( \mu > 0, \quad 2\mu + 3\lambda \geq 0. \)

And \( \kappa > 0 \) is the coefficient of heat conduction, \( c_v > 0 \) is the specific heat at constant volume. The initial data

\[
(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \to (\rho_\infty, 0, \theta_\infty), \quad \text{as} \quad |x| \to \infty,
\]

(1.2)

where \( \rho_\infty \) and \( \theta_\infty \) are two positive constants. The pressure \( p(\rho, \theta) \) here is assumed to be a smooth function in a neighborhood of \( (\rho_\infty, \theta_\infty) \) satisfying \( p_\rho(\rho, \theta) > 0 \) and \( p_\theta(\rho, \theta) > 0 \). And \( \Psi(u) \) is the classical dissipation function satisfying

\[
\Psi(u) = \frac{\mu}{2}(\nabla u + \nabla^T u)^2 + \lambda (\text{div} u)^2.
\]

In this paper, we will investigate the optimal convergence rates in time to the stationary solution of the Cauchy problem (1.1)–(1.2). It is noted that the large time behavior of the solution is an important topic in the research of...
the fluid dynamics for achieving the goal of the computation, one may refer to [11][18]. And the stationary solution \((\rho^*(x), u^*(x), \theta^*(x))\) for the full CNS equations \((1.1)\) is given by \((\rho^*(x), 0, \theta^*_\infty)\) satisfying
\[
\int_{\rho^*_\infty}^{\rho^*(x)} \frac{p_\rho(s, \theta^*_\infty)}{s} ds + \phi(x) = 0. \tag{1.3}
\]
The derivation for the stationary solution was given by Matsumura and Nishida in [37], so we omit here.

We will give an overview some known results on the mathematical analysis on existence, stability, large time behavior, and convergence rates of solutions to the CNS equations.

**Some Results without External Force.** There are huge literatures on the well-posedness and large time behavior of solutions to the CNS equations without external force. It is well-known that the local existence and uniqueness of classical solutions were obtained in [17][40] in the absence of vacuum. For the case that the initial density may vanish in open sets, the similar results may refer to [5][7][29][44]. The first global well-posedness result goes back to Matsumura and Nishida [36]. It is noted that this famed result requires that the initial data is closed to a non-vacuum equilibrium in some Sobolev space \(H^s\). In other words, the solution is a small oscillation around a uniform non-vacuum state, which guarantees that the density is strictly away from vacuum. In the framework of general data, this is a challenging problem, due to the difficulty in the analysis comes from the possible appearance of vacuum. It is indicated in [27][43][58][59] that the strong (or smooth) solution for the CNS equations will blow up in finite time. Then some blow-up criteria of strong solutions were given in [24][50][25][55] and the references therein. When the vacuum is taken into account, the global existence and uniqueness of strong solution for the full CNS equations in \(\mathbb{R}^3\) was achieved by Huang et al.[23] for small initial energy. Similar result was obtained for CNS equations in \([20][28][32][50]\). It is worth noting that all results on the global dynamics about the stability and large time behavior are restricted to the regime that the solutions are close to the equilibrium. For large data, it is well-known that Feireisl constructed the so-called variational solutions for specific pressure laws excluding the perfect gas in [12], when the temperature equation is satisfied only as an inequality. Then for a special form of the viscosity coefficients depending on the density, Bresch and Desjardins [3] obtained the existence of global weak solutions by making use of a new entropy inequality (BD-entropy), which was proposed in [1], and the construction scheme of approximate solutions in [2]. With the aid of BD-entropy, there are Some other related results with respect to the global well-posedness theory of weak solution, one may refer to [20][31][52]. Recently, He et al. [21] investigated global stability of large solution and established the decay rate for the global solution as it tends to the constant equilibrium state under the assumption that \(\sup_{t \in \mathbb{R}^+} \|\rho(t, \cdot)\|_{C^\alpha} \leq M\) for some \(0 < \alpha < 1\). Later, the optimal decay rate for this class of global large solution itself and its derivatives was investigated, one may refer to [15][16][17].

In the past and recent years, important progress has been made in the large time behavior of the solutions to the CNS system in the near equilibrium regime. The decay result was first achieved by Matsumura and Nishida in [35] for the optimal \(L^2\) decay rate, and later by Ponce in [42] for the optimal \(L^p(p \geq 2)\) decay rate. Hoff [22], Liu and Wang [33] obtained the optimal \(L^p(1 \leq p \leq \infty)\) decay rates in \(\mathbb{R}^n(n \geq 2)\) by virtue of the good properties of Green function with the small initial perturbation, which bounded in \(H^s \cap L^1\) with the integer \(s \geq [n/2] + 3\). Furthermore, developed by Schonbek [15], Gao et al. [14] established optimal decay rate for the higher-order spatial derivative of global small solution by using the Fourier splitting method. The approach to proving all these decay results mentioned above relies heavily on the analysis of the linearization of the system. From another point of view, under the assumption that the initial perturbation is bounded in \(\dot{H}^{-s}(s \in [0, \frac{4}{n}])\), Guo and Wang [19] applied pure energy method to build the optimal decay rate of the solution and its spatial derivatives of the CNS system under the \(H^N(N \geq 3)\)–framework. However, the decay rate for the highest order spatial derivative of global solution obtained in articles mentioned above is still not optimal. Recently, this tricky problem is addressed simultaneously in a series of articles [4][54][57] by using the spectrum analysis of the linearized system, and [13] by combining the energy estimates with the interpolation between negative and positive Sobolev spaces.

**Some Results with Potential Force.** When there is an external potential force, there are also some results on the convergence rate for solutions to the CNS equations. For potential force, one has to face a tricky problem of the appearance of non-trivial stationary solutions. However, some seminal results on the existence and large behavior theory were still achieved. The global solutions was first obtained by Matsumura and Nishida in [37] as initial data is closed to the steady state \((\rho^*(x), 0, \theta^*_\infty)\) in the Sobolev space \(H^3\). In addition, they also proved that the global
solution converges to the stationary state as time tends to infinity. The background profile is non-trivial due to the effect of the external force, thus, unlike the problems without potential force, the analysis on the convergence rates is more delicate and difficult. The first work concerning the explicit decay estimate for solution was done by Deckelnick in [8]. More precisely, he considered the isentropic case and showed that

\[
\sup_{x \in \mathbb{R}^3} |(\rho(t, x) - \rho^*(x), u(t, x))| \lesssim (1 + t)^{-\frac{3}{4}}.
\]

This result was then improved by Shibata and Tanaka in [18, 19] for more general external forces to \((1 + t)^{-\frac{3}{4} + \kappa}\) for any small positive constant \(\kappa\) when the initial perturbation belongs to \(H^3 \cap L^\infty\), and later by Duan et al. in [9] for \(L^p - L^q\) convergence rates when the initial perturbation is also bounded in \(L^p\) with \(1 \leq p < \frac{4}{3}\). However, in [9], the decay estimates of the higher order spatial derivatives of the solution were obtained the same as that of the first order one. Recently, Gao et al. [13] improved this result under \(H^N\)-framework (\(N \geq 3\)). Specifically, they established the optimal decay rate of \(k\)-th \((k = 0, 1, \ldots, N)\) order spatial derivative (including the highest order spatial derivative) of the solution.

Most of the above results are for isentropic fluids without taking the effect of heat-conductivity into account. In many physical situations, the heat-conductivity is an important driving force to motions of fluids. When the heat-conductivity is taken into account for compressible flows, Duan et al. [10] obtained the optimal decay rates in \(H^3\)-framework for system (1.1)-(1.2) when the initial perturbation is also bounded in \(L^1\). To be more specific, they established the following decay estimates:

\[
\|\rho - \rho^*, u, \theta - \theta^\infty(t)\|_{L^p} \lesssim (1 + t)^{-\frac{3}{4}}(1 - \frac{1}{p}), \quad 2 \leq p \leq 6, \quad \|\nabla(\rho - \rho^*, u, \theta - \theta^\infty(t))\|_{H^2} \lesssim (1 + t)^{-\frac{3}{4}}. \tag{1.4}
\]

For the case that the initial perturbation belongs to \(H^2 \cap L^q(1 \leq q \leq 2)\), Wang [53] established the following optimal time decay rates for all \(t \geq 0\),

\[
\|\rho - \rho^*, u, \theta - \theta^\infty(t)\|_{L^p} \lesssim (1 + t)^{-\frac{3}{4}}(1 - \frac{1}{p}), \quad 2 \leq p \leq 6, \quad \|\nabla(\rho - \rho^*, u, \theta - \theta^\infty(t))\|_{H^1} \lesssim (1 + t)^{-\frac{3}{4}(\frac{2}{3} - \frac{1}{q})}. \tag{1.5}
\]

There are some results concerning the decay estimate for the CNS equations with potential force, as observed in [11, 12, 30, 38, 39]. Obviously, the decay rates of the higher order spatial derivatives in either (1.4) or (1.5) are still not optimal. Thus it is of interest to investigate the optimal decay rate for the higher order derivative of global solutions to system (1.1)-(1.2) in three dimensions. Based on the decay result (1.4), our main purpose in this paper is to establish the optimal decay rate for the \(k\)-th \((k = 2, 3)\) order spatial derivative of the solution to the full CNS equations with potential force.

Now, the optimal convergence rates for solutions and its spatial derivatives of Cauchy problem (1.1)-(1.2) in \(L^2\)-norm can be obtained and stated as follows:

**Theorem 1.1.** Let \((\rho^*(x), 0, \theta^\infty)\) be the stationary solution of initial value problem (1.1)-(1.2), if \((\rho_0 - \rho^*, u_0, \theta_0 - \theta^\infty) \in H^3\), there exists a constant \(\delta\) such that the potential function \(\phi(x)\) satisfies

\[
\sum_{k=0}^{4} \|(1 + |x|)^k \nabla^k \phi\|_{L^2 \cap L^\infty} \leq \delta, \tag{1.6}
\]

and the initial perturbation satisfies

\[
\|\rho_0 - \rho^*, u_0, \theta_0 - \theta^\infty\|_{H^3} \leq \delta. \tag{1.7}
\]

Then there exists a unique global solution \((\rho, u, \theta)\) of initial value problem (1.1)-(1.2) satisfying for any \(t \geq 0\),

\[
\|\rho - \rho^*, u, \theta - \theta^\infty(t)\|_{H^3} + \int_0^t \left(\|\nabla(\rho - \rho^* )\|_{H^2}^2 + \|\nabla u\|_{H^3}^2 + \|\nabla \theta\|_{H^3}^2\right) ds \leq C\|\rho_0 - \rho^*, u_0, \theta_0 - \theta^\infty\|_{H^3}^2, \tag{1.8}
\]

where \(C\) is a positive constant independent of time \(t\). If further

\[
\|\rho_0 - \rho^*, u_0, \theta_0 - \theta^\infty\|_{L^1} < \infty,
\]

then there exist constants \(\delta_0 > 0\) and \(\bar{C}_0 > 0\) such that for any \(0 < \delta \leq \delta_0\), we have

\[
\|\nabla^k(\rho - \rho^*, u, \theta - \theta^\infty)\|_{L^2} \leq \bar{C}_0(1 + t)^{-\frac{3}{4} - \frac{k}{4}}, \quad \text{for} \quad k = 0, 1, 2, 3. \tag{1.9}
\]
Remark 1.2. The global well-posedness theory of the full CNS equations with potential force in three-dimensional whole space was studied in [10] under the $H^3-$ framework. Furthermore, they also established the decay estimate (1.4) if the initial data also belong to $L^1$. Thus, the advantage of the decay rate (1.9) in Theorem 1.3 is that the decay rate of the global solution $(\rho - \rho^*, u, \theta - \theta_\infty)$ itself and its any order spatial derivative is optimal in the sense that it coincides with the rate of the solution of the linearized system.

Remark 1.3. By using the Sobolev interpolation inequality, we can establish the following estimate:

$$\|\nabla^k(\rho - \rho^*, u, \theta - \theta_\infty)(t)\|_{L^p} \leq \hat{C}_0(1+t)^{-\frac{k}{2} + \frac{k}{p} - \frac{1}{2}},$$

for all $2 \leq p < +\infty$ and $k = 0, 1, 2, 3$. If $p = +\infty$, then $k = 0, 1$. Therefore, the global solution $(\rho, u, \theta)$ of Cauchy problem (1.1)-(1.2) tends to the constant equilibrium state $(\rho^*, 0, \theta_\infty)$ in $L^\infty-$norm at the $(1 + t)^{-\frac{5}{4}}$-rate. At the same time, we point out that the time derivative of density, velocity and temperature $(\rho_0, u_0, \theta_0)$ tends to $(0,0,0)$ in $L^2-$norm at the $(1 + t)^{-\frac{5}{4}}$-rate.

Remark 1.4. If the initial data $(\rho_0 - \rho^*, u_0, \theta_0 - \theta_\infty)$ in $H^N \cap L^1(N \geq 3)$, we also can get the similar decay result that

$$\|\nabla^k(\rho - \rho^*, u, \theta - \theta_\infty)(t)\|_{L^2} \leq \hat{C}_0(1+t)^{-\frac{k}{2} + \frac{k}{2}}, \quad \text{for } k = 0, \ldots N. \quad (1.10)$$

These decay rates of the solution itself and its spatial derivatives are optimal in the sense that it coincides with the rate of the solution of the heat equation. The decay estimate (1.10) was proven in this paper for $N = 3$ (see Theorem 1.7), however, the case $N > 3$ can be handled in the same way and so we omit the proof.

To end this section, we would like to introduce our strategies for deriving the optimal time-decay rates for the full CNS equations with potential force. We can only obtain the lower dissipation estimate about the density, which is essentially caused by the degenerate dissipative structure of the system (1.1)-(1.2) satisfying hyperbolic-parabolic coupling system. Therefore, we will focus on how to obtain the energy estimates which include only the highest-order spatial derivative of the solution. We point out that the equilibrium state of global solution will depend on the spatial variable caused by potential force. This will also create some fundamental and additional difficulties in the process of the energy estimates, see Lemmas 3.2 and 3.3. We can derive in a similar way as [10] in [10], by combining the energy estimate and the decay rate of linearized system, one can obtain the following decay estimates:

$$\|\nabla^k(\rho - \rho^*, u, \theta - \theta_\infty)(t)\|_{H^{3-k}} \lesssim (1+t)^{-(\frac{5}{4} + \frac{k}{2})}, \quad k = 0, 1, \quad (1.11)$$

if the initial data $(\rho_0 - \rho^*, u_0, \theta_0 - \theta_\infty)$ belongs to $H^3 \cap L^1$. We then prove that the decay estimate (1.11) for $k = 2$ by using the basic decay estimate (1.11). Motivated by [13], one can apply the time weighted method and the basic decay estimate (1.11) to achieve this goal. Indeed, by virtue of the classical energy estimate, we can establish following estimate:

$$\|\nabla^2(\rho - \rho^*, u_0, \theta_0 - \theta_\infty)\|_{H^1} + \int_0^t (1 + \tau)^{\frac{5}{2} + \epsilon_0}(\|\nabla^3(\rho - \rho^*)(\tau)\|_{L^2}^2 + \|\nabla^3(u, \theta - \theta_\infty)\|_{H^1}^2) d\tau$$

$$\lesssim \|\nabla^3(\rho_0 - \rho^*, u_0, \theta_0 - \theta_\infty)\|_{H^1} + \int_0^t (1 + \tau)^{\frac{5}{2} + \epsilon_0}\|\nabla^2(\rho - \rho^*, u, \theta - \theta_\infty)\|_{H^2}^2 d\tau. \quad (1.12)$$

where $\mathcal{E}_2(t)$ is equivalent to $\|\nabla^2(\rho - \rho^*, u, \theta - \theta_\infty)\|_{H^1}$. Thus, we need to control the second term on the right handside of (1.12). With the help of the decay estimate for $k = 1$ in (1.11), we can obtain that

$$\|\nabla^2(\rho - \rho^*, u, \theta - \theta_\infty)\|_{H^2} d\tau \lesssim (1+t)^{\epsilon_0}, \quad (1.13)$$

where $\mathcal{E}_1(t)$ is equivalent to $\|\nabla(\rho - \rho^*, u, \theta - \theta_\infty)\|_{L^2}$. The combination of (1.12) and decay estimate (1.13) gives the decay estimate (1.11) for $k = 2$ directly.

Due to the presence of potential force term $\rho \nabla \phi$, we can not apply the time weighted method mentioned above to build the optimal decay rate for the third order spatial derivative of global solution directly. In order to overcome this difficult, motivated by [57], we establish some energy estimate for the quantity $\int_{|\xi| \geq \tau} \nabla^3 u \cdot \nabla^3 (\rho - \rho^*) d\xi$, namely
the higher frequency part, rather than \( \int \nabla^2 u \cdot \nabla^3 (\rho - \rho^*) dx \). Here \( \nabla^2 u \) and \( \nabla^3 (\rho - \rho^*) \) stand for the Fourier part of \( \nabla^2 u \) and \( \nabla^3 (\rho - \rho^*) \), respectively. We point out that the advantage is that the quantity \( \| \nabla^3 (\rho - \rho^*, u, \theta - \theta_{\infty}) \|^2_{L^2} - \eta_2 \int_{|\xi| \geq q} \nabla^2 u \cdot \nabla^3 (\rho - \rho^*) d\xi \) is equivalent to \( \| \nabla^3 (\rho - \rho^*, u, \theta - \theta_{\infty}) \|^2_{L^2} \) for some small constant \( \eta_2 \). Hence, by virtue of some energy estimate and decay estimate, one can obtain the following inequality:

\[
\frac{d}{dt} \left\{ \| \nabla^3 (\rho - \rho^*, u, \theta - \theta_{\infty}) \|^2_{L^2} - \eta_2 \int_{|\xi| \geq q} \nabla^2 u \cdot \nabla^3 (\rho - \rho^*) d\xi \right\} \\
+ \| \nabla^3 (u^h, (\theta - \theta_{\infty})^h) \|^2_{L^2} + \eta_2 \| \nabla^3 (\rho - \rho^*)^h \|^2_{L^2} \\
\leq \| \nabla^3 ((\rho - \rho^*)^l, u^l, (\theta - \theta_{\infty})^l) \|^2_{L^2} + (1 + t)^{-6}. \tag{1.14}
\]

Thus, one has to estimate the decay rate of the low-frequency term \( \| \nabla^3 ((\rho - \rho^*)^l, u^l, (\theta - \theta_{\infty})^l) \|^2_{L^2} \). Indeed, the combination of Duhamel’s principle and decay estimate of \( k - \text{th} \ (0 \leq k \leq 3) \) order spatial derivative of solution obtained above help us to build that

\[
\| \nabla^3 ((\rho - \rho^*)^l, u^l, (\theta - \theta_{\infty})^l) \|^2_{L^2} \leq \delta \sup_{0 \leq \tau \leq t} \| \nabla^3 (\rho - \rho^*, u, \theta - \theta_{\infty}) \|^2_{L^2} + (1 + t)^{-\frac{9}{4}}.
\]

This, together with (1.14), by using the smallness of \( \delta \), we can obtain the optimal decay rate for the third order spatial derivative of global solution to the full CNS equations with potential force.

**Notation:** Throughout this paper, for \( 1 \leq p \leq +\infty \) and \( s \in \mathbb{R} \), we simply denote \( L^p(\mathbb{R}^3) \) and \( H^s(\mathbb{R}^3) \) by \( L^p \) and \( H^s \), respectively. And the constant \( C \) denotes a general constant which may vary in different estimates. \( A \lessgtr (\gtrsim) B \) stands for \( A \lessgtr (\gtrsim) C B \) for some constant \( C > 0 \) independent of \( A \) and \( B \). \( A \sim B \) stands for \( B \lessgtr A \lessgtr B \). \( \hat{f}(\xi) = \mathcal{F}(f(x)) \) represents the usual Fourier transform of the function \( f(x) \) with respect to \( x \in \mathbb{R}^3 \). \( \mathcal{F}^{-1}(\hat{f}(\xi)) \) means the inverse Fourier transform of \( \hat{f}(\xi) \) with respect to \( \xi \in \mathbb{R}^3 \). For the sake of simplicity, we write \( \int f dx := \int_{\mathbb{R}^3} f dx \).

The rest of the paper is organized as follows. In Section 2, we introduce some important lemmas and basic facts, which will be useful in later analysis. Finally, the proof of Theorem 1.1 is given in Section 3.

## 2 Preliminary

In this section, we recall some elementary inequalities, which will be of frequency use in next section. First of all, in order to deal with the terms about \( \bar{\rho}(x) \) in the CNS equations with a potential force in energy estimate, we need the following Hardy inequality.

**Lemma 2.1** (Hardy inequality). For \( k \geq 1 \), suppose that \( \frac{\nabla \phi}{(1 + |x|)^k} \in L^2 \), then \( \frac{\phi}{(1 + |x|)^{k-1}} \in L^2 \), with the estimate

\[
\| \frac{\phi}{(1 + |x|)^{k}} \|^2_{L^2} \leq C \| \frac{\nabla \phi}{(1 + |x|)^{k-1}} \|^2_{L^2}.
\]

The proof of Lemma 2.1 is simply and we omit it here. We then introduce the following Sobolev interpolation of Gagliardo-Nirenberg inequality, which will be used extensively in energy estimates. The proof and more details may refer to [19].

**Lemma 2.2** (Sobolev interpolation inequality). Let \( 2 \leq p \leq +\infty \) and \( 0 \leq l, k \leq m \). If \( p = +\infty \), we require furthermore that \( l \leq k + 1 \) and \( m \geq k + 2 \). Then if \( \nabla^l \phi \in L^2 \) and \( \nabla^m \phi \in L^2 \), we have \( \nabla^k \phi \in L^p \). Moreover, there exists a positive constant \( C \) depending only on \( k, l, m, p \) such that

\[
\| \nabla^k \phi \|_{L^p} \leq C \| \nabla^l \phi \|_{L^2}^\theta \| \nabla^m \phi \|_{L^2}^{1-\theta}, \tag{2.1}
\]

where \( 0 \leq \theta \leq 1 \) satisfying

\[
\frac{k}{3} - \frac{1}{p} = \left( \frac{l}{3} - \frac{1}{2} \right) \theta + \left( \frac{m}{3} - \frac{1}{2} \right)(1 - \theta).
\]

Then we introduce the following commutator estimate, which will be useful in following energy estimates and can be found in [34] for more details.
Lemma 2.3. Let \( k \geq 1 \) be an integer and define the commutator
\[
[\nabla^k, f] g \overset{\text{def}}{=} \nabla^k (fg) - f \nabla^k g.
\]

Then we have
\[
\| [\nabla^k, f] g \|_{L^2} \leq C \| \nabla f \|_{L^\infty} \| \nabla^{k-1} g \|_{L^2} + C \| \nabla^k f \|_{L^2} \| g \|_{L^\infty},
\]
where \( C \) is a positive constant dependent only on \( k \).

Finally, we end up this section with the following lemma. The proof and more details may refer to [4].

Lemma 2.4. Let \( r_1, r_2 > 0 \) be two real numbers, for any \( 0 < \epsilon_0 < 1 \), we have
\[
\int_0^t (1 + t - \tau)^{-r_1} (1 + \tau)^{-r_2} d\tau \leq C \begin{cases} (1 + t)^{-r_2}, & \text{for } r_2 > 1, \\ (1 + t)^{-r_1 + \epsilon_0}, & \text{for } r_2 = 1, \\ (1 + t)^{-(r_1 + r_2 - 1)}, & \text{for } r_2 < 1, \end{cases}
\]
where \( C \) is a positive constant independent of time.

3 Optimal decay of the full CNS equations with potential force

In this section, we will give the proof for Theorem 1.1 that include the global well-posedness theory and time decay estimates. First of all, we noted that the global small solution of the full CNS equations can be proven directly by taking the strategy of standard energy method in [10, 37], when the initial data is small perturbation near the equilibrium state. Thus, one can assume that the global solution \((\rho, u, \theta)\) in Theorem 1.1 exists and satisfies the energy estimate (1.3), i.e.,
\[
\|(\rho - \rho^*, u, \theta - \theta_\infty)\|_{H^3}^2 + \int_0^t (\|\nabla (\rho - \rho^*)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2 + \|\nabla \theta(t)\|_{H^3}^2) ds \leq C \|(\rho_0 - \rho^*, u_0, \theta_0 - \theta_\infty)\|_{H^3}^2, (3.1)
\]
for all \( t \geq 0 \). Secondly, similar to the decay estimate (1.4), the combination of the energy estimate and the decay rate of linearized system can help us to establish the following decay estimates:
\[
\|\nabla^k (\rho - \rho^*)(t)\|_{H^3-k} + \|\nabla^k u(t)\|_{H^3-k} + \|\nabla^k (\theta - \theta_\infty)(t)\|_{H^3-k} \leq C (1 + t)^{-\left(\frac{4}{3} + \frac{k}{2}\right)}, \quad k = 0, 1, (3.2)
\]
if the initial data \((\rho_0 - \rho^*, u_0, \theta_0 - \theta_\infty)\) belongs to \( L^1 \) additionally. Now, we only focus on establishing the optimal decay rate for the higher order spatial derivative of solution. More precisely, we would like to prove that the decay estimate (3.2) for the case \( k = 2, 3 \). Thus, we define
\[
n(t, x) \overset{\text{def}}{=} \rho(t, x) - \rho^*(x), \quad \bar{p}(x) \overset{\text{def}}{=} \rho^*(x) - \rho_\infty, \quad v(t, x) \overset{\text{def}}{=} \frac{\rho_\infty}{p_1} u(t, x), \quad q(t, x) \overset{\text{def}}{=} \frac{p_2}{p_1 p_3} (\theta(t, x) - \theta_\infty),
\]
where
\[
p_1 = \frac{p_\rho(\rho_\infty, \theta_\infty)}{\rho_\infty}, \quad p_2 = \frac{p_\theta(\rho_\infty, \theta_\infty)}{\rho_\infty}, \quad p_3 = \frac{\theta_\infty}{c_v \rho_\infty},
\]
then (1.1)–(1.2) can be rewritten in the following perturbation form
\[
\begin{align*}
n_t + \gamma \text{div} v &= F_1, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
v_t + \gamma \nabla n + \lambda \nabla q - \mu_1 \Delta n - \mu_2 \nabla \text{div} v &= F_2, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
q_t - \kappa \Delta q + \lambda \nabla \text{div} v &= F_3, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
(n, v, q)|_{t=0} \overset{\text{def}}{=} (n_0, v_0, q_0) &= (\rho_0 - \rho^*, \sqrt{\frac{p_2}{p_1}} u_0, \sqrt{\frac{p_2 p_\rho}{p_1 p_3}} (\theta_0 - \theta_\infty)) \to (0, 0, 0), \quad \text{as } |x| \to \infty,
\end{align*}
\]
where $\mu_1 = \frac{\mu}{\rho_\infty}, \mu_2 = \frac{\mu + \lambda}{\rho_\infty}, \gamma = \sqrt{p_1 \rho_\infty}, \lambda = \sqrt{p_2 \rho_\infty}$, and

$$F_1 = -\frac{\mu_1 \gamma}{\mu} \text{div}[(n + \bar{p})v],$$

$$F_2 = -\frac{\mu \gamma}{\mu} \cdot \nabla v - \frac{\mu_1}{\mu_1} f(n + \bar{p}) (\mu_1 \Delta v + \mu_2 \nabla \text{div} v) - g(n + \bar{p}, q) \nabla n - h(n, \bar{p}, q) \nabla \bar{p} - r(n + \bar{p}, q) \nabla q,$$

$$F_3 = -\frac{\mu_1 \gamma}{\mu} v \cdot \nabla q + \kappa f(n + \bar{p}) \Delta q - m(n + \bar{p}, q) \text{div} v + \frac{\kappa \mu_1 \gamma^2}{\kappa \mu} \sqrt{\frac{p_2}{p_3}} \Psi(v) + \frac{\kappa \mu_1 \gamma^2}{\kappa \mu} \sqrt{\frac{p_2}{p_3}} f(n + \bar{p}) \Psi(v).$$

Here the nonlinear functions $f, g, h$ and $m$ are defined by

$$f(n + \bar{p}) \overset{\text{def}}{=} \frac{1}{n + \bar{p} + \rho_\infty} - \frac{1}{\rho_\infty},$$

$$g(n + \bar{p}, q) \overset{\text{def}}{=} \frac{\mu}{\mu_1 \gamma} \left( \frac{p_\rho(n + \bar{p} + \rho_\infty, \sqrt{p_2 p_3} q + \theta_\infty)}{n + \bar{p} + \rho_\infty} - \frac{p_\rho(p_\rho, \theta_\infty)}{\rho_\infty} \right),$$

$$h(n, \bar{p}, q) \overset{\text{def}}{=} \frac{\mu}{\mu_1 \gamma} \left( \frac{p_\rho(n + \bar{p} + \rho_\infty, \sqrt{p_2 p_3} q + \theta_\infty)}{n + \bar{p} + \rho_\infty} - \frac{p_\rho(p_\rho, \theta_\infty)}{\rho_\infty} \right),$$

$$m(n + \bar{p}, q) \overset{\text{def}}{=} \frac{1}{\epsilon_\nu} \sqrt{\frac{p_2}{p_3}} \left( \sqrt{\frac{p_2 p_3}{p_1 p_3}} q + \theta_\infty \right) \frac{p_\rho(n + \bar{p} + \rho_\infty, \sqrt{p_2 p_3} q + \theta_\infty)}{n + \bar{p} + \rho_\infty} - \theta_\infty p_\rho(p_\rho, \theta_\infty).$$

### 3.1. Energy estimates

In this subsection, we would like to establish the following differential inequality, which is the key to obtain the optimal decay rate for the higher order spatial derivative of solution. First of all, let us define the energy $E_1^3(t)$ as

$$E_1^3(t) \overset{\text{def}}{=} \sum_{k=1}^{3} \|\nabla^k (n, v, q)\|_{L^2}^2 + \eta_1 \sum_{k=1}^{2} \int \nabla^k v \cdot \nabla^{k+1} n dx, \quad 0 \leq l \leq 3,$$

where $\eta_1$ is a small positive constant. The smallness of parameter $\eta_1$ lead us to obtain the following equivalent relation

$$c_1 \|\nabla^l (n, v, q)\|_{H^{3-l}}^2 \leq E_1^3(t) \leq c_2 \|\nabla^l (n, v, q)\|_{H^{3-l}}^2.$$  

(3.4)

where $c_1$ and $c_2$ are positive constants independent of time. And one can deduce from the relation (3.3) and the condition (1.6) given in Theorem 1.1 that

$$\sum_{k=0}^{4} \| (1 + |x|)^k \nabla^k (\rho^* - \rho_\infty) \|_{L^2 \cap L^{\infty}} \leq \delta.$$  

(3.5)

The combination of (3.6) and the Sobolev interpolation inequality help us to deduce that

$$\sum_{k=0}^{4} \| (1 + |x|)^k \nabla^k (\rho^* - \rho_\infty) \|_{L^p} \leq \delta, \quad 2 \leq p \leq +\infty.$$  

(3.6)

This inequality will be used frequently in energy estimate in this section. Now we state the main result in this subsection.

**Proposition 3.1.** Under the assumptions in Theorem 1.1, for any $l = 1, 2$, it holds that

$$\frac{d}{dt} E_1^3(t) + \eta_1 \|\nabla^{l+1} n\|_{H^{2-l}}^2 + \|\nabla^{l+1} (v, q)\|_{H^{3-l}}^2 \leq 0,$$

(3.7)

where $\eta_1$ is a small positive constant.
Recalling the energy estimate (3.1), then there exists a positive constant $C$ such that for any $k \geq 1$,
\[
|f(n + \bar{\rho})| \leq C|n + \bar{\rho}|, \quad |g(n + \bar{\rho}, q)| \leq C|n + \bar{\rho} + q|, \quad |h(n, \bar{\rho}, q)| \leq C|n + q|,
\]
\[
|r(n + \bar{\rho}, q)| \leq C|n + \bar{\rho} + q|, \quad |m(n + \bar{\rho}, q)| \leq C|n + \bar{\rho} + q|,
\]
\[
|f^{(k)}(n + \bar{\rho})| \leq C, \quad |g^{(k)}(n + \bar{\rho}, q)| \leq C, \quad |h^{(k)}(n, \bar{\rho}, q)| \leq C, \quad |r^{(k)}(n + \bar{\rho}, q)| \leq C, \quad |m^{(k)}(n + \bar{\rho}, q)| \leq C.
\]
Thus, it is easy to check that
\[
F_1 \sim \div((n + \bar{\rho})v),
\]
\[
F_2 \sim v \cdot \nabla v + (n + \bar{\rho})(\Delta v + \nabla \div v) + (n + q)\nabla(n + \bar{\rho}(n + q)),
\]
\[
F_3 \sim v \cdot \nabla q + (n + \bar{\rho})\Delta q + (n + \bar{\rho})\div v + q\div v + (n + \bar{\rho})\Psi(v) + \Psi(v).
\]

Next, we will give the following three lemmas, which will play vital role to prove Proposition 3.1. The first one is Lemma 3.2 concerning the basic energy estimate for $k - \theta$ ($k = 1, 2$) order spatial derivative of solution.

**Lemma 3.2.** Under the assumptions in Theorem 1.1 for $k = 1, 2$, it holds that
\[
\frac{d}{dt}\|\nabla^k(n, v, q)\|_{L^2}^2 + \|\nabla^{k+1}(v, q)\|_{L^2}^2 \leq C\delta\|\nabla^{k+1}(n, v, q)\|_{L^2}^2, \tag{3.9}
\]
where $C$ is a positive constant independent of time.

**Proof.** We apply differential operator $\nabla^k$ to (3.3), (3.3), and (3.3), multiply the resulting equations by $\nabla^k n$, $\nabla^k v$ and $\nabla^k q$, respectively, then integrate over $\mathbb{R}^3$, to find
\[
\frac{1}{2} \frac{d}{dt}\|\nabla^k(n, v, q)\|_{L^2}^2 + \mu_1\|\nabla^{k+1}v\|_{L^2}^2 + \mu_2\|\nabla^k \div v\|_{L^2}^2 + \bar{k}\|\nabla^{k+1}q\|_{L^2}^2 = \int \nabla^k F_1 \cdot \nabla^k n dx + \int \nabla^k F_2 \cdot \nabla^k v dx + \int \nabla^k F_3 \cdot \nabla^k q dx. \tag{3.10}
\]
For $k = 1$, integrating by part and using the definition of $F_1$ and the estimates (3.6) and (3.8), one can obtain from Sobolev and Hardy inequalities that
\[
\int \nabla F_1 \cdot \nabla n dx \leq C\|F_1\|_{L^2}\|\nabla^2 v\|_{L^2} \leq C \int (v \cdot \nabla n + n \div v) \cdot \nabla^2 dx + C \int (v \cdot \nabla \bar{\rho} + \bar{\rho} \div v) \cdot \nabla^2 dx \tag{3.11}
\]
\[
\leq C \|(v\|_{L^3}\|\nabla n\|_{L^6} + |n||n|\|\nabla v\|_{L^6} + |(1 + |x|)\nabla \bar{\rho}|\|_{L^3}\|v\|_{L^6} + ||\bar{\rho}||_{L^3}\|\nabla v\|_{L^6}\|\nabla^2 n\|_{L^2} \leq C\delta\|\nabla^2(n, v)\|_{L^2}^2.
\]
It then follows from integrating by parts, Sobolev and Hardy inequalities, the estimates (3.6) and (3.8) that
\[
\int \nabla F_2 \cdot \nabla v dx \leq C\|F_2\|_{L^2}\|\nabla^2 v\|_{L^2} \leq C \int (v \cdot \nabla v + n\nabla v + n\div v + n\nabla q + q\nabla q)\nabla^2 v dx + C \int (\bar{\rho} \nabla v + \bar{\rho} \div v + n\nabla \bar{\rho} + q\nabla q)\nabla^2 v dx \tag{3.12}
\]
\[
\leq C\|(v\|_{L^3}\|\nabla v\|_{L^6} + |n||n|\|\nabla \nabla\|_{L^6} + |n||n|\|\nabla v\|_{L^6} + |(1 + |x|)\nabla \bar{\rho}|\|_{L^3}\|v\|_{L^6} + ||\bar{\rho}||_{L^3}\|\nabla v\|_{L^6}\|\nabla^2 v\|_{L^2} \leq C\delta\|\nabla^2(n, v)\|_{L^2}^2.
\]
In view of the estimate (3.6), Sobolev and Hardy inequalities, we deduce that

\[ \int \nabla F_3 \cdot \nabla q dx \leq C \| F_3 \|_{L^2} \| \nabla^2 q \|_{L^2} \]

\[ \leq C \int (v \cdot \nabla q + n \Delta q + n \text{div} \, v + g \text{div} \, v + n \Psi(v) + \Psi(v)) \nabla^2 q dx + C \int \left( \tilde{\rho} \Delta q + \tilde{\rho} \text{div} \, v + \tilde{\rho} \Psi(v) \right) \nabla^2 q dx \]

\[ \leq C \left( \| v \|_{L^3} \| \nabla q \|_{L^6} + \| n \|_{L^{\infty}} \| \nabla^2 q \|_{L^2} + \| \n\|_{L^{\infty}} \| \nabla v \|_{L^6} + \| q \|_{L^3} \| \nabla v \|_{L^6} + \| n \|_{L^{\infty}} \| \nabla v \|_{L^6} \right) \| \nabla v \|_{L^6} \]

\[ + \| \nabla v \|_{L^3} \| \nabla v \|_{L^6} + \| \tilde{\rho} \|_{L^3} \| \nabla^2 q \|_{L^2} + \| \tilde{\rho} \|_{L^3} \| \nabla v \|_{L^6} + \| \tilde{\rho} \|_{L^{\infty}} \| \nabla v \|_{L^6} \| \nabla v \|_{L^6} \| \nabla^2 q \|_{L^2} \]

\[ \leq C(\| (n, v, q) \|_{H^2} + \delta) \| \nabla^2 (n, v, q) \|_{L^2}^2 \leq C \delta \| \nabla^2 (n, v, q) \|_{L^2}^2. \tag{3.13} \]

Substituting three estimates (3.11)-(3.13) into (3.10) for \( k = 1 \), it holds true

\[ \frac{d}{dt} \| \nabla (n, v, q) \|_{L^2}^2 + \| \nabla^2 (n, v, q) \|_{L^2}^2 \leq C \delta \| \nabla^2 (n, v, q) \|_{L^2}^2. \tag{3.14} \]

As for \( k = 2 \), the Sobolev and Hardy inequalities yield directly

\[ \int \nabla^2 F_1 \cdot \nabla^2 v dx = - \int \nabla F_1 \cdot \nabla^3 n dx \leq C \| \nabla F_1 \|_{L^2} \| \nabla^3 n \|_{L^2} \]

\[ \leq C \left( \| (n, v) \|_{L^3} \| \nabla^2 (n, v) \|_{L^6} + \| \nabla (n, v) \|_{L^3} \| \nabla (n, v) \|_{L^6} + \| \tilde{\rho} \|_{L^3} \| \nabla^2 v \|_{L^6} \right) \]

\[ + \| (1 + |x|) \nabla \tilde{\rho} \|_{L^3} \| \nabla v \|_{L^6} + \| (1 + |x|) \| \nabla^2 \tilde{\rho} \|_{L^3} \| \nabla^2 n \|_{L^6} \| \nabla^3 n \|_{L^2} \]

\[ \leq C(\| (n, v) \|_{H^2} + \delta) \| \nabla^2 (n, v) \|_{L^2}^2 \leq C \delta \| \nabla^2 (n, v) \|_{L^2}^2, \tag{3.15} \]

where we have used the estimate (3.6) and the fact that

\[ \| \nabla (n, v) \|_{L^3} \| \nabla (n, v) \|_{L^6} \leq C \| \nabla^2 (n, v) \|_{L^2} \| \nabla^3 (n, v) \|_{L^2} \| \nabla^3 (n, v) \|_{L^2} \leq C \delta \| \nabla^3 (n, v) \|_{L^2}. \tag{3.16} \]

By virtue of integration by parts and Sobolev inequality, we find

\[ \int \nabla^2 F_2 \cdot \nabla^3 v dx = - \int \nabla F_2 \cdot \nabla^3 v dx \leq C \| \nabla F_2 \|_{L^2} \| \nabla^3 v \|_{L^2}. \tag{3.17} \]

With the help of the definition of \( F_2 \), we have

\[ \| \nabla F_2 \|_{L^2} \leq C(\| \nabla (v \cdot \nabla v) \|_{L^2} + \| \nabla (f(n + \tilde{\rho}) \|_{L^2} + \| \nabla (g(n + \tilde{\rho}, q) \nabla n) \|_{L^2} + \| \nabla (h(n, \tilde{\rho}, q) \nabla \tilde{\rho}) \|_{L^2} \]

\[ = I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.18} \]

Sobolev inequality and (3.10) yield that

\[ I_1 \leq C \| v \|_{L^3} \| \nabla^2 v \|_{L^6} + C \| \nabla v \|_{L^3} \| \nabla v \|_{L^6} \leq C \delta \| \nabla^3 v \|_{L^2}. \]

In view of the estimate (3.6), Sobolev and Hardy inequalities, we deduce that

\[ I_2 \leq C \| f(n + \tilde{\rho}) \|_{L^\infty} \| \nabla^3 v \|_{L^2} + C \| \nabla f(n + \tilde{\rho}) \|_{L^3} \| \nabla^2 v \|_{L^6} \]

\[ \leq C \| (n + \tilde{\rho}) \|_{L^\infty} \| \nabla^3 v \|_{L^2} + C \| \nabla (n + \tilde{\rho}) \|_{L^3} \| \nabla^3 v \|_{L^2} \]

\[ \leq C \delta \| \nabla^3 v \|_{L^2}. \]

The application of the estimates (3.6) and (3.16), Sobolev and Hardy inequalities yields directly

\[ I_3 \leq C \| g(n + \tilde{\rho}, q) \|_{L^3} \| \nabla^2 n \|_{L^6} + C \| \nabla g(n + \tilde{\rho}, q) \nabla n \|_{L^2} \]

\[ \leq C \| (n + \tilde{\rho} + q) \|_{L^3} \| \nabla^2 n \|_{L^6} + C \| \nabla (n + q) \|_{L^3} \| \nabla n \|_{L^6} + C \| (1 + |x|) \| \nabla \tilde{\rho} \|_{L^3} \| \nabla \tilde{n} \|_{L^6} \]

\[ \leq C \delta \| \nabla^3 (n, q) \|_{L^2}. \]
By the estimate (3.6), Hardy and Sobolev inequalities, it is easy to check that
\[
I_4 \leq C\left\| \frac{h(n, \bar{\rho}, q)}{(1 + |x|)^2} \right\|_{L^4} \left(1 + |x|\right)^2 \nabla^2 \bar{\rho} \right\|_{L^2} + C\left\| \frac{\nabla h(n, \bar{\rho}, q)}{1 + |x|} \right\|_{L^6} \left(1 + |x|\right) \nabla \bar{\rho} \right\|_{L^3} \\
\leq C\left\| \frac{n + q}{(1 + |x|)^2} \right\|_{L^4} \left(1 + |x|\right)^2 \nabla^2 \bar{\rho} \right\|_{L^2} + C\left\| \frac{\nabla (n + q)}{1 + |x|} \right\|_{L^6} \left(1 + |x|\right) \nabla \bar{\rho} \right\|_{L^3} \\
\leq C\delta \left\| \nabla^3 (n, q) \right\|_{L^2}.
\]

Similar to the estimate of \( I_3 \), we apply the estimates (3.6) and (3.16), Sobolev and Hardy inequalities to obtain
\[
I_5 \leq C\left\| (n + \bar{\rho} + q) \right\|_{L^3} \nabla^2 q \right\|_{L^6} + C\left\| \nabla (n + q) \right\|_{L^3} \nabla q \right\|_{L^6} + C\left\| (1 + |x|) \nabla \bar{\rho} \right\|_{L^3} \frac{\nabla q}{1 + |x|} \right\|_{L^6} \\
\leq C\delta \left\| \nabla^3 (n, q) \right\|_{L^2}.
\]

Inserting the estimates of terms \( I_1 \) to \( I_5 \) into (3.18), it follows immediately
\[
\left\| \nabla F_2 \right\|_{L^2} \leq C\delta \left\| \nabla^3 (n, v, q) \right\|_{L^2}.
\]

Substituting this estimate into (3.17), we have
\[
\int \nabla^2 F_2 \cdot \nabla^2 v dx \leq C\delta \left\| \nabla^3 (n, v, q) \right\|_{L^2}.
\]

Integration by parts and Sobolev inequality imply that
\[
\int \nabla^2 F_3 \cdot \nabla^2 q dx = - \int \nabla F_3 \cdot \nabla^3 q dx \leq C\left\| \nabla F_3 \right\|_{L^2} \left\| \nabla^3 q \right\|_{L^2}.
\]

Remebering the definition of \( F_2 \), we find
\[
\left\| \nabla F_3 \right\|_{L^2} \leq C \left( \left\| \nabla (v \cdot \nabla q) \right\|_{L^2} + \left\| \nabla (f(n + \bar{\rho}) \nabla^2 q) \right\|_{L^2} + \left\| \nabla (f(n + \bar{\rho}, q) \nabla (v)) \right\|_{L^2} + \left\| \nabla \Psi (v) \right\|_{L^2} + \left\| \nabla (m(n + \bar{\rho}, q) \nabla v) \right\|_{L^2} \right) \\
\leq C \left( \left\| \nabla (v) \right\|_{L^2} + \left\| \nabla^3 q \right\|_{L^2} + \left\| \nabla (f(n + \bar{\rho}) \nabla (v)) \right\|_{L^2} + \left\| \nabla (m(n + \bar{\rho}, q) \nabla v) \right\|_{L^2} \right)
\leq C\delta \left\| \nabla^3 q \right\|_{L^2}.
\]

According to the Sobolev inequality and the estimate (3.16), we deduce that
\[
J_1 \leq C \left\| v \right\|_{L^3} \left\| \nabla^3 q \right\|_{L^6} + C \left\| \nabla v \right\|_{L^3} \left\| \nabla^3 q \right\|_{L^6} \leq C\delta \left\| \nabla^3 (v, q) \right\|_{L^2}.
\]

By Sobolev inequality and the estimate (3.6), one can deduce directly that
\[
J_2 \leq C \left\| f(n + \bar{\rho}) \right\|_{L^3} \left\| \nabla^3 q \right\|_{L^6} + C \left\| \nabla f(n + \bar{\rho}) \right\|_{L^3} \left\| \nabla^2 q \right\|_{L^6} \leq C \left\| (n + \bar{\rho}) \right\|_{L^3} \left\| \nabla^3 q \right\|_{L^6} + C \left\| \nabla (n + \bar{\rho}) \right\|_{L^3} \left\| \nabla^3 q \right\|_{L^6} \leq C\delta \left\| \nabla^3 q \right\|_{L^2}.
\]

Using the Sobolev and Hardy inequalities, the estimate (3.6) and (3.16), it holds
\[
J_3 \leq C \left\| f(n + \bar{\rho}) \right\|_{L^3} \left\| \nabla v \right\|_{L^3} \left\| \nabla^2 v \right\|_{L^6} + C \left\| \nabla f(n + \bar{\rho}) \right\|_{L^3} \left\| \nabla v \right\|_{L^3} \left\| \nabla v \right\|_{L^6} \leq C \left\| (n + \bar{\rho}) \right\|_{L^3} \left\| \nabla v \right\|_{L^3} \left\| \nabla^3 v \right\|_{L^2} + C \left\| \nabla (n + \bar{\rho}) \right\|_{L^3} \left\| \nabla v \right\|_{L^3} \left\| \nabla v \right\|_{L^6} \leq C\delta \left\| \nabla^3 v \right\|_{L^2}.
\]

According to Sobolev inequality and the estimate (3.16), we obtain immediately
\[
J_4 \leq C \left\| \nabla v \right\|_{L^3} \left\| \nabla^2 v \right\|_{L^6} \leq C\delta \left\| \nabla^3 v \right\|_{L^2}.
\]

In view of the estimates (3.6) and (3.16), Sobolev and Hardy inequalities, one deduces that
\[
J_5 \leq C \left\| m(n + \bar{\rho}, q) \right\|_{L^3} \left\| \nabla^2 v \right\|_{L^6} + C \left\| \nabla m(n + \bar{\rho}, q) \right\|_{L^2} \left\| v \right\|_{L^3} \left\| v \right\|_{L^6} \leq C \left\| (n + \bar{\rho} + q) \right\|_{L^3} \left\| \nabla^3 v \right\|_{L^2} + C \left\| \nabla (n + q) \right\|_{L^3} \left\| \nabla v \right\|_{L^6} + C \left\| (1 + |x|) \nabla \bar{\rho} \right\|_{L^3} \frac{\nabla v}{1 + |x|} \right\|_{L^6} \leq C\delta \left\| \nabla^3 (n, v, q) \right\|_{L^2}.
\]
Then, the combination of the estimates of terms $J_1$ to $J_5$ and (3.21) implies directly
\[ \| \nabla F_3 \|_{L^2} \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}. \]

We substitute this estimate into (3.20), to find
\[ \int \nabla^2 F_3 \cdot \nabla^2 q dx \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2. \tag{3.22} \]

Substituting (3.10), (3.14) and (3.19) into (3.10) for Lemma 3.3.

Under the assumptions in Theorem 1.1, it holds that
\[ J \]

Then, the combination of the estimates of terms $J_1$ to $J_5$ and (3.21) implies directly
\[ \| \nabla F_3 \|_{L^2} \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}. \]

We substitute this estimate into (3.20), to find
\[ \int \nabla^2 F_3 \cdot \nabla^2 q dx \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2. \tag{3.22} \]

Substituting (3.10), (3.14) and (3.19) into (3.10) for $k = 2$, we find
\[ \frac{d}{dt} \| \nabla^3(n, v, q) \|_{L^2}^2 + \| \nabla^4(v, q) \|_{L^2}^2 \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2, \]

which, together with (3.14), gives (3.9) directly. Thus, we complete the proof of this lemma.

We then derive the energy estimate for third order spatial derivative of the solution.

**Lemma 3.3.** Under the assumptions in Theorem 1.1, it holds that
\[ \frac{d}{dt} \| \nabla^3(n, v, q) \|_{L^2}^2 + \| \nabla^4(v, q) \|_{L^2}^2 \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2, \tag{3.23} \]

where $C$ is a positive constant independent of time.

**Proof.** Applying differential operator $\nabla^3$ to (3.9), (3.12) and (3.13), multiplying the resulting equations by $\nabla^3 n$, $\nabla^3 v$ and $\nabla^3 q$, respectively, and integrating over $\mathbb{R}^3$, it holds
\[ \frac{1}{2} \frac{d}{dt} \| \nabla^3(n, v, q) \|_{L^2}^2 + \| \nabla^4(v, q) \|_{L^2}^2 \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2, \tag{3.24} \]

Now we estimate three terms on the right hand side of (3.24) separately. In view of the definition of $F_1$, Sobolev inequality Lemma 2.3 and integration by parts, we have
\[ \int \nabla^3 F_1 \cdot \nabla^3 ndx = C \int \nabla^3(v \cdot \nabla n + n \div v + v \cdot \nabla \bar{\rho} + \bar{\rho} \div v) \cdot \nabla^3 ndx \]
\[ \leq - C \int \div v \| \nabla^3 n \|_{L^2} dx + (\| \nabla^3 v \|_{L^2} + \| \nabla^4(n \div v) \|_{L^2}^2) \| \nabla^3 n \|_{L^2}. \tag{3.25} \]

It is easy to check that
\[ \| \nabla^3 v \|_{L^2} \leq C \| \nabla v \|_{L^\infty} \| \nabla^3 n \|_{L^2} + C \| \nabla^2 v \|_{L^3} \| \nabla^2 n \|_{L^6} + C \| \nabla^3 v \|_{L^6} \| \nabla n \|_{L^3} \leq C\delta \| \nabla^3 n, \nabla^4 v \|_{L^2}. \tag{3.26} \]

It then follows from Sobolev inequality that
\[ \| \nabla^3(n \div v) \|_{L^2} \leq C \| n \|_{L^\infty} \| \nabla^4 v \|_{L^2} + C \| \nabla^3 n \|_{L^2} \| \nabla v \|_{L^\infty} \leq C\delta \| \nabla^3 n, \nabla^4 v \|_{L^2}. \tag{3.27} \]

The application of Sobolev and Hardy inequalities yields directly
\[ \| \nabla^3(v \cdot \nabla \bar{\rho} + \bar{\rho} \div v) \|_{L^2} \leq C \sum_{l=0}^{3} \left( \| (1 + |x|)^{4-l} \nabla^l \bar{\rho} \|_{L^\infty} \| \nabla^l v \| \frac{1}{(1 + |x|)^{4-l}} \|_{L^2} + \| (1 + |x|)^l \nabla^l \bar{\rho} \|_{L^\infty} \| \nabla^4-l v \| \frac{1}{(1 + |x|)^{4-l}} \|_{L^2} \right) \]
\[ \leq C\delta \| \nabla^4 v \|_{L^2}. \tag{3.28} \]

Hence, the combination of the estimates (3.24) gives
\[ \int \nabla^3 F_1 \cdot \nabla^3 ndx \leq C\delta \| \nabla^3(n, v, q) \|_{L^2}^2. \tag{3.29} \]
Integration by parts, by use of Sobolev inequality gives
\[ \int \nabla^3 F_2 \cdot \nabla^3 v \, dx = - \int \nabla^2 F_2 \cdot \nabla^4 v \, dx \leq C \| \nabla^2 F_2 \|_{L^2} \| \nabla^4 v \|_{L^2}. \]  
(3.30)

In view of the definition of \( F_2 \), we have
\[
\| \nabla^2 F_2 \|_{L^2} \leq C \left( \| \nabla^2 (v \cdot \nabla v) \|_{L^2} + \| \nabla^2 (f(n + \bar{\rho}) \nabla^2 v) \|_{L^2} + \| \nabla^2 (g(n + \bar{\rho}, q) \nabla n) \|_{L^2} 
+ \| \nabla^2 (h(n, \bar{\rho}, q) \nabla \bar{\rho}) \|_{L^2} + \| \nabla^2 (r(n + \bar{\rho}, q) \nabla q) \|_{L^2} \right)
\]
\[
d\equiv K_1 + K_2 + K_3 + K_4 + K_5.
\]

Thanks to the commutator estimate in Lemma \( \ref{2.3} \) and Sobolev inequality, one can deduce that
\[
K_1 \leq C \| v \|_{L^3} \| \nabla^3 v \|_{L^6} + C \| \nabla^2 v \|_{L^2} \leq C \| v \|_{L^2} \| \nabla^4 v \|_{L^2} + C \| v \|_{L^3} \| \nabla^2 v \|_{L^6} \leq C \delta \| \nabla^4 v \|_{L^2},
\]
where we have used the following estimate in the last inequality
\[
\| \nabla v \|_{L^3} \| \nabla^2 v \|_{L^6} \leq C \| \nabla^3 n \|_{L^6} \| \nabla^4 v \|_{L^2} \leq C \delta \| \nabla^4 v \|_{L^2}.
\]
(3.32)

According to the estimate \( \ref{3.30} \), Sobolev and Hardy inequalities, we deduce that
\[
K_2 \leq C \| (n + \bar{\rho}) \|_{L^\infty} \| \nabla^4 v \|_{L^2} + C \| \nabla^2 n \|_{L^6} \| \nabla^2 v \|_{L^3} + C \| \nabla n \|_{L^\infty} \| \nabla n \|_{L^6} \| \nabla^2 v \|_{L^6}
+ C \| (1 + |x|^2) \nabla^2 \bar{\rho} \|_{L^\infty} \| \nabla^2 v \|_{L^2} + C \| \nabla n \|_{L^6} \| (1 + |x|) \nabla \bar{\rho} \|_{L^\infty} \| \nabla^2 v \|_{L^6}
+ C \| (1 + |x|) \nabla \bar{\rho} \|_{L^2} \| \nabla^2 v \|_{L^2}
\leq C \delta \| (\nabla^2 n, \nabla^4 v) \|_{L^2},
\]
where we have used the following estimates in the last inequality
\[
\| \nabla n \|_{L^3} \| \nabla^2 v \|_{L^6} \leq \| \nabla n \|_{L^3} \| \nabla^3 n \|_{L^2} \| \nabla^4 v \|_{L^2} \leq C \delta \| (\nabla^3 n, \nabla^4 v) \|_{L^2}.
\]
(3.33)

Using the estimate \( \ref{3.30} \), Sobolev and Hardy inequalities, it holds
\[
K_3 \leq C \| (n + \bar{\rho} + q) \|_{L^\infty} \| \nabla^3 n \|_{L^6} + C \| \nabla^2 n \|_{L^6} \| \nabla n \|_{L^6} + C \| \nabla^2 q \|_{L^6} \| \nabla n \|_{L^6} + C \| \nabla (n, q) \|_{L^\infty} \| \nabla (n, q) \|_{L^2} \| \nabla n \|_{L^6}
+ C \| (1 + |x|^2) \nabla^2 \bar{\rho} \|_{L^\infty} \| \nabla n \|_{L^6} + C \| (1 + |x|) \nabla \bar{\rho} \|_{L^\infty} \| \nabla n \|_{L^6}
+ C \| (1 + |x|) \nabla \bar{\rho} \|_{L^2} \| \nabla n \|_{L^6}
\leq C \delta \| (\nabla^3 n, \nabla^4 v, \nabla^4 q) \|_{L^2},
\]
where we have used the estimate \( \ref{3.33} \) and the following estimates
\[
\| \nabla n \|_{L^3} \| \nabla n \|_{L^6} \leq \| \nabla^2 n \|_{L^2} \| \nabla^3 n \|_{L^2} \| \nabla^4 n \|_{L^2} \leq C \delta \| \nabla^3 n \|_{L^2},
\]
\[
\| \nabla q \|_{L^3} \| \nabla n \|_{L^6} \leq \| \nabla^2 q \|_{L^2} \| \nabla^4 q \|_{L^2} \| \nabla^3 q \|_{L^2} \| \nabla^3 n \|_{L^2} \leq C \delta \| (\nabla^3 n, \nabla^4 q) \|_{L^2}.
\]
(3.34)

We can use Sobolev and Hardy inequalities, the estimates \( \ref{3.32} \) and \( \ref{3.34} \) to find
\[
K_4 \leq C \| (1 + |x|^2) \nabla^2 \bar{\rho} \|_{L^\infty} \| \nabla n \|_{L^6} + C \| (1 + |x|^2) \nabla^2 \bar{\rho} \|_{L^2} \| \nabla n \|_{L^6} + C \| (1 + |x|) \nabla \bar{\rho} \|_{L^\infty} \| \nabla q \|_{L^6}
+ C \| (1 + |x|) \nabla \bar{\rho} \|_{L^6} \| \nabla \bar{\rho} \|_{L^\infty} \| \nabla (n, q) \|_{L^6} \| \nabla n \|_{L^6} + C \| (1 + |x|) \nabla \bar{\rho} \|_{L^\infty} \| \nabla q \|_{L^6} \| \nabla \bar{\rho} \|_{L^6}
\leq C \delta \| (\nabla^3 n, \nabla^4 q) \|_{L^2}.
\]
It then follows from the estimate (3.34), Sobolev and Hardy inequalities that

\[ K_5 \leq C\|(n + \rho + q)\|_{L^6}^2 \|\nabla^3 q\|_{L^6}^3 + C\|\nabla^2(n, q)\|_{L^6}^4 \|\nabla q\|_{L^6}^2 + C\|\nabla(n, q)\|_{L^6} \|\nabla q\|_{L^6}^3 \]

\[ + C\|(1 + |x|^2)\nabla q\|_{L^6} \|\nabla q\|_{L^6} \|\nabla q\|_{L^6}^3 \]

\[ + C\|(1 + |x|)\nabla \rho\|_{L^6} \|\nabla \rho\|_{L^6} \|\nabla \rho\|_{L^6} \|\nabla q\|_{L^6} \]

\[ \leq C\delta\|(\nabla^3 n, \nabla^4 q)\|_{L^2}, \]

where we have used the following estimates

\[ \|\nabla q\|_{L^6} \|\nabla q\|_{L^6}^2 \leq \|q\|_{L^6}^{\frac{1}{2}} \|\nabla^3 q\|_{L^6}^{\frac{1}{2}} \|\nabla^3 q\|_{L^6}^{\frac{1}{2}} \|\nabla^3 q\|_{L^6} \leq C\delta\|\nabla^4 q\|_{L^2}. \] (3.35)

Substituting the estimates of terms \( K_1 \) to \( K_5 \) into (3.31), we find

\[ \|\nabla^2 F_2\|_{L^2} \leq C\delta\|(\nabla^3 n, \nabla^4 v, \nabla^4 q)\|_{L^2}. \] (3.36)

Hence, the combination of the estimate above and (3.30) implies directly

\[ \int \nabla^3 F_2 \cdot \nabla^3 v dx \leq C\delta\|(\nabla^3 n, \nabla^4 v, \nabla^4 q)\|_{L^2}^2. \] (3.37)

It is easy to deduce by using integrating by parts and Sobolev inequality that

\[ \int \nabla^3 F_3 \cdot \nabla^3 q dx = - \int \nabla^2 F_3 \cdot \nabla^4 q dx \leq C\|\nabla^2 F_3\|_{L^2} \|\nabla^4 q\|_{L^2}. \] (3.38)

With the aid of the definition of \( F_3 \), we have

\[ \|\nabla^2 F_3\|_{L^2} \leq C\left(\|\nabla^2 v \cdot \nabla q\|_{L^2} + \|\nabla^2 (f(n + \rho)\nabla q)\|_{L^2} + \|\nabla^2 (f(n + \rho, q)\Phi(v))\|_{L^2} + \|\nabla^2 \Phi(v)\|_{L^2} \right) \]

\[ + \|\nabla^2 (m(n + \rho, q) \text{ div } v)\|_{L^2} \]

\[ \overset{\text{def}}{=} L_1 + L_2 + L_3 + L_4 + L_5. \] (3.39)

According to Sobolev inequality, the commutator estimate in Lemma 2.3 and the estimate (3.32), we obtain immediately

\[ L_1 \leq C\|v\|_{L^6} \|\nabla^3 q\|_{L^6} + C\|\nabla^2 v, v\| \cdot \nabla q\|_{L^2} \leq C\|v\|_{L^6} \|\nabla^4 q\|_{L^2} + C\|\nabla v\|_{L^6} \|\nabla^2 q\|_{L^6} \leq C\delta\|\nabla^4 q\|_{L^2}. \]

We employ Sobolev inequality and the estimate (3.33), to get

\[ L_2 \leq C\|(n + \rho)\|_{L^6} \|\nabla^3 q\|_{L^6} + C\|\nabla^2 n\|_{L^6} \|\nabla^2 q\|_{L^6} \|\nabla^3 q\|_{L^6} \|\nabla^3 q\|_{L^6} \|\nabla^2 q\|_{L^6} \]

\[ + C\|(1 + |x|^2)\nabla^2 \rho\|_{L^6} \|\nabla^2 q\|_{(1 + |x|^2)} \|\nabla \rho\|_{L^6} \|\nabla \rho\|_{L^6} \|\nabla^2 q\|_{1 + |x|^2} \|\nabla \rho\|_{L^6} \]

\[ \leq C\delta\|(\nabla^3 n, \nabla^4 q)\|_{L^2}. \]

Applying the estimate (3.35), Sobolev and Hardy inequalities, we obtain

\[ L_3 \leq C\|(n + \rho)\|_{L^6} \|\nabla^3 v\|_{L^6} \|\nabla^3 v\|_{L^6} + \|\nabla^2 v\|_{L^6} \|\nabla^2 v\|_{L^6} \|\nabla^2 v\|_{L^6} \]

\[ + C\|(1 + |x|^2)\nabla^2 \rho\|_{L^6} \|\nabla v\|_{L^6} \|\nabla v\|_{L^6} \|\nabla v\|_{L^6} \|\nabla (n + \rho)\|_{L^6} \|\nabla v\|_{L^6} \|\nabla v\|_{L^6} \]

\[ \leq C\delta\|(\nabla^3 n, \nabla^4 v)\|_{L^2}, \]

where we have used the following estimate in the last inequality

\[ \|\nabla^2 v\|_{L^6} \|\nabla^2 v\|_{L^6} \leq C\|\nabla^2 v\|_{L^6}^\frac{2}{3} \|\nabla^4 v\|_{L^6}^\frac{1}{3} \|\nabla^2 v\|_{L^6}^\frac{2}{3} \|\nabla^4 v\|_{L^2}^\frac{1}{2}, \] (3.40)
We then use the estimate (3.40), Sobolev and Hardy inequalities, to find
\[ L_4 \leq C\|\nabla v\|_{L^2}\|\nabla^3 v\|_{L^6} + C\|\nabla^2 v\|_{L^4}\|\nabla^2 v\|_{L^6} \leq C\|\nabla^4 v\|_{L^2}. \]

To deal with last term \( L_5 \), by virtue of the estimates (3.32)-(3.35), Sobolev and Hardy inequalities, we arrive at
\[ L_5 \leq C\|(n + \bar{\rho} + q)\|_{L^2}\|\nabla^3 v\|_{L^6} + C\|\nabla^2 (n + q)\|_{L^6}\|\nabla v\|_{L^3} + C\|\nabla (n + q)\|_{L^\infty}\|\nabla^3 v\|_{L^6} \]
\[ + C\|+(1+|x|)\nabla^2 \bar{\rho}\|_{L^6} + C\|n \bar{\rho}\|_{L^\infty}\|\nabla (n + q)\|_{L^6}\|\nabla v\|_{L^3} \]
\[ + C\|+(1+|x|)\nabla \bar{\rho}\|_{L^\infty}\|+(1+|x|)\nabla \bar{\rho}\|_{L^\infty}\|\nabla v\|_{L^3} \]
\[ \leq C\|\nabla^3 n, \nabla^4 v, \nabla^4 q\|_{L^2}. \]

Hence, the combination of estimates of terms \( L_1 \) to \( L_5 \) and (3.39) implies directly
\[ \|\nabla^2 F_3\|_{L^2} \leq C\|\nabla^3 n, \nabla^4 v, \nabla^4 q\|_{L^2}. \]

Inserting this estimate into (3.38), we thereby deduce that
\[ \int \nabla^3 F_3 \cdot \nabla^3 q dx \leq C\|\nabla^3 n, \nabla^4 v, \nabla^4 q\|_{L^2}^2. \]

Plugging the estimates (3.29), (3.37) and (3.41) into (3.24) gives (3.23) directly. Therefore, the proof of this lemma is completed. \( \square \)

Finally, we aim to recover the dissipation estimate for \( n \).

**Lemma 3.4.** Under the assumptions in Theorem 1.1 for \( k = 1, 2 \), it holds that
\[ \frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} n dx + \|\nabla^{k+1} n\|_{L^2}^2 \leq C_1 \|\nabla^{k+1} v, \nabla^{k+2} v\|_{L^2}^2, \]
where \( C_1 \) is a positive constant independent of \( t \).

**Proof.** Applying differential operator \( \nabla^k \) to (3.32), multiplying the resulting equation by \( \nabla^{k+1} n \), and integrating over \( \mathbb{R}^3 \), one arrives at
\[ \int \nabla^k v_t \cdot \nabla^{k+1} n dx + \|\nabla^{k+1} n\|_{L^2}^2 \leq C\|\nabla^{k+2} v\|_{L^2}^2 + \int \nabla^k F_2 \cdot \nabla^{k+1} n dx. \]

The way we deal with \( \int \nabla^k v_t \cdot \nabla^{k+1} n dx \) is to turn the time derivative of the velocity to the density. Then, applying differential operator \( \nabla^k \) to the mass equation (3.31), we find
\[ \nabla^k n_t + \gamma \nabla^k \text{div} v = \nabla^k F_1. \]

Hence, we can transform time derivative to the spatial derivative, i.e.,
\[ \int \nabla^k v_t \cdot \nabla^{k+1} n dx = \frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} n dx - \int \nabla^k v \cdot \nabla^{k+1} n dx \]
\[ = \frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} n dx + \gamma \int \nabla^k v \cdot \nabla^{k+1} \text{div} v dx - \int \nabla^k v \cdot \nabla^{k+1} F_1 dx \]
\[ = \frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} n dx - \gamma \|\nabla^k \text{div} v\|_{L^2}^2 - \int \nabla^{k+1} \text{div} v \cdot \nabla^{k-1} F_1 dx \]

Substituting the identity above into (3.38) and integrating by parts yield
\[ \frac{d}{dt} \int \nabla^k v \cdot \nabla^{k+1} n dx + \|\nabla^{k+1} n\|_{L^2}^2 \]
\[ \leq C\|\nabla^{k+1} v, \nabla^{k+2} v\|_{L^2}^2 + C \int \nabla^{k+1} \text{div} v \cdot \nabla^{k-1} F_1 dx - C \int \nabla^{k+1} F_2 \cdot \nabla^{k+1} n dx. \]
As for the term of $F_1$, we have

$$\left| \int \nabla^{k+1} \text{div} \cdot \nabla^{k-1} F_1 dx \right| \leq C \| \nabla^{k+2} v \|_{L^2} \| \nabla^{k-1} F_1 \|_{L^2} \leq C \delta \| \nabla^{k+1} n \|_{L^2}^2 + C \|( \nabla^{k+1} v, \nabla^{k+2} v) \|_{L^2}^2,$$  \hspace{1cm} (3.45)

where $\| \nabla^{k-1} F_1 \|_{L^2} (k = 1, 2)$ can be controlled in a similar way to the estimates of terms from (3.11) and (3.15) in Lemma 3.2. To deal with the term of $F_2$, we then derive in a similar way in (3.30) in Lemma 3.3. Hence, we give the estimate as follow

$$\left| \int \nabla^k F_2 \cdot \nabla^{k+1} ndx \right| \leq C \| \nabla^k F_2 \|_{L^2} \| \nabla^{k+1} n \|_{L^2} \leq C \| (\nabla^{k+1} n, \nabla^{k+2} v, \nabla^{k+2} q) \|_{L^2}^2.$$  \hspace{1cm} (3.46)

We then utilize (3.45) and (3.46) in (3.44), to deduce (3.42) directly.

**The proof of Proposition 3.1.** With the help of Lemmas 3.2-3.4, it is easy to establish the estimate (3.7). Therefore, we complete the proof of Proposition 3.1.

### 3.2. Optimal decay of higher order derivative

In this subsection, we will establish the optimal decay rate for the second order spatial derivative of global solution. In order to achieve this goal, the optimal decay rate of higher order spatial derivative will be established by the lower one. In this aspect, developed by Schonbek (see [45]), the Fourier splitting method is applied frequently to establish the optimal decay rate for higher order derivative of global solution in [46, 14, 15]. However, we are going to use time weighted energy estimate to solve this problem.

**Lemma 3.5.** Under the assumption of Theorem 1.1, for $k = 0, 1, 2$, it holds that

$$\| \nabla^k (n, v, q) \|_{H^{3-k}} \leq C (1 + t)^{-\frac{1}{2} - \frac{k}{2}},$$  \hspace{1cm} (3.47)

where $C$ is a positive constant independent of time.

**Proof.** Actually, the decay rate (3.2) implies (3.47) holds true for the the case $k = 0, 1$. That is, the decay rate (3.47) holds on for the case $k = 1$, i.e.,

$$\| \nabla (n, v, q) \|_{H^2} \leq C (1 + t)^{-\frac{1}{2}}.$$  \hspace{1cm} (3.48)

It remains the case of $k = 2$ to be proven. We take the integer $l = 1$ in the estimate (3.7) and multiply it by $(1 + t)^{\frac{1}{2} + \epsilon_0} (0 < \epsilon_0 < 1)$, to discover

$$\frac{d}{dt} \left\{ (1 + t)^{\frac{1}{2} + \epsilon_0} E_1^3 (t) \right\} + (1 + t)^{\frac{1}{2} + \epsilon_0} \left( \| \nabla^2 n \|_{H^1}^2 + \| \nabla^2 (v, q) \|_{H^2}^2 \right) \leq C (1 + t)^{\frac{1}{2} + \epsilon_0} E_1^3 (t).$$

Integrating with respect to $t$, using the equivalent relation (3.3) and the decay estimate (3.18), one obtains

$$\begin{align*}
(1 + t)^{\frac{1}{2} + \epsilon_0} E_1^3 (t) + & \int_0^t (1 + \tau)^{\frac{1}{2} + \epsilon_0} \left( \| \nabla^2 n \|_{H^1}^2 + \| \nabla^2 (v, q) \|_{H^2}^2 \right) d\tau \\
\leq & E_1^3 (0) + C \int_0^t (1 + \tau)^{\frac{1}{2} + \epsilon_0} E_1^3 (\tau) d\tau \\
\leq & C \| \nabla (n_0, v_0, q_0) \|_{H^2}^2 + C \int_0^t (1 + \tau)^{\frac{1}{2} + \epsilon_0} \| \nabla (n, v, q) \|_{H^2}^2 d\tau \\
\leq & C \| \nabla (n_0, v_0, q_0) \|_{H^2}^2 + C \int_0^t (1 + \tau)^{-1 + \epsilon_0} d\tau \leq C (1 + t)^{\epsilon_0}.
\end{align*}$$  \hspace{1cm} (3.49)

On the other hand, taking $l = 2$ in the estimate (3.7), we have

$$\frac{d}{dt} E_2^3 (t) + \| \nabla^3 n \|_{L^2}^2 + \| \nabla^3 (v, q) \|_{L^2}^2 \leq 0.$$  \hspace{1cm} (3.50)
We then multiply (3.51) by \((1 + t)^\frac{3}{2} + \epsilon_0\), integrate over \([0, t]\) and use the estimate (3.49), to find
\[
\begin{align*}
(1 + t)^\frac{3}{2} + \epsilon_0 \mathcal{E}_2(t) + \int_0^t (1 + \tau)^{\frac{3}{2} + \epsilon_0} \left( \| \nabla^3 n \|_{L^2}^2 + \| \nabla^{k+2} (v, q) \|_{H^1}^2 \right) d\tau \\
\leq \mathcal{E}_2^2(0) + C \int_0^t (1 + \tau)^{\frac{3}{2} + \epsilon_0} \mathcal{E}_2^2(\tau) d\tau \\
\leq C \| \nabla^2 (n_0, v_0, q_0) \|_{H^1}^2 + C \int_0^t (1 + \tau)^{\frac{3}{2} + \epsilon_0} \| \nabla^2 (n, v, q) \|_{H^1}^2 d\tau \leq C(1 + t)^\epsilon_0.
\end{align*}
\]
This, together with the equivalent relation (3.4), yields immediately
\[
\| \nabla^2 (n, v, q) \|_{H^1} \leq C(1 + t)^{-\frac{3}{2}}.
\]
Then, the decay estimate (3.44) holds true for case of \(k = 2\). Therefore, we complete the proof of this lemma.  

3.3. Optimal decay of critical derivative

In this subsection, we aim to build the optimal decay rate for the third order spatial derivative of global solution \((n, v, q)\) as it tends to zero. The decay rate of the third order derivative of global solution \((n, v, q)\) obtained in Lemma 3.6 is not optimal since it is same as that of the second one. This is caused by the appearance of cross term to the equation (3.3), one obtains that
\[
\int_0^t (1 + \tau)^{\frac{3}{2} + \epsilon_0} \| \nabla^2 (n, v, q) \|_{H^1}^2 d\tau \leq C(1 + t)^{-\frac{3}{2}}.
\]

We then take the Fourier transform of (3.53) as follows:
\[
f^i(x) \overset{def}{=} F^{-1}(\varphi_0(\xi) \hat{f}(\xi)) \quad \text{and} \quad f^{h}(x) \overset{def}{=} f(x) - f^i(x).
\]

Lemma 3.6. Under the assumptions of Theorem 1.1, there exists a positive small constant \(\eta_2\), such that
\[
\begin{align*}
\frac{d}{dt} \left\{ \| \nabla^3 (n, v, q) \|_{L^2}^2 - \eta_2 \int_{|\xi| \geq \eta} \overline{\nabla^2 v} \cdot \overline{\nabla^3 n} d\xi \right\} &+ \| \nabla^3 (v^b, q^b) \|_{L^2}^2 + \eta_2 \| \nabla^3 n^b \|_{L^2}^2 \\
\leq C_2 \| \nabla^3 (n^i, v^i, q^i) \|_{L^2}^2 + C(1 + t)^{-6},
\end{align*}
\]
where \(C_2\) is a positive constant independent of time.

Proof. Taking differential operator \(\nabla^2\) to the equation (3.53), one obtains that
\[
\begin{align*}
\nabla^2 n_1 + \gamma \nabla^2 \div v = \nabla^2 F_1, \\
\nabla^2 v_1 + \gamma \nabla^3 n + \lambda \nabla^3 q - \mu_1 \nabla^3 \div v - \mu_2 \nabla^3 \div v = \nabla^2 F_2, \\
\nabla^2 q_1 - \lambda \nabla^2 \div q + \lambda \nabla^2 \div v = \nabla^2 F_3.
\end{align*}
\]
We then take the Fourier transform of (3.53), multiply the resulting equation by \(\overline{\nabla^3 n}\) and integrate on \(|\xi| \geq \eta\), to discover
\[
\begin{align*}
&\int_{|\xi| \geq \eta} \overline{\nabla^2 v_1} \cdot \overline{\nabla^3 n} d\xi + \gamma \int_{|\xi| \geq \eta} |\overline{\nabla^3 n}|^2 d\xi \\
&= \int_{|\xi| \geq \eta} \left( \mu_1 \nabla^3 \div v + \mu_2 \nabla^3 \div v \right) \cdot \overline{\nabla^3 n} d\xi - \lambda \int_{|\xi| \geq \eta} \overline{\nabla^3 q} \cdot \overline{\nabla^3 n} d\xi + \int_{|\xi| \geq \eta} \overline{\nabla^2 F_2} \cdot \overline{\nabla^3 n} d\xi.
\end{align*}
\]
It follows from (3.53) that
\[
\nabla^2 v_t \cdot \nabla^3 n = -\xi \nabla^2 v_t \cdot \nabla^2 n = -\nabla^3 v_t \cdot \nabla^2 \nabla^2 v
\]
\[= -\partial_t (\nabla^3 v \cdot \nabla^2 n) + \nabla^3 v \cdot \nabla^2 n_t
\]
\[= -\partial_t (\nabla^3 v \cdot \nabla^2 n) - \gamma \nabla^3 v \cdot \nabla^2 \nabla^2 v + \nabla^3 v \cdot \nabla^2 F_1.
\]
Then, we substitute this identity into identity (3.54), to find
\[
-\frac{d}{dt} \int_{|\xi| \geq \eta} \nabla^3 v \cdot \nabla^2 n d\xi + \gamma \int_{|\xi| \geq \eta} |\nabla^3 n|^2 d\xi
\[= \int_{|\xi| \geq \eta} \left( \mu_1 \nabla^2 \triangle v + \mu_2 \nabla^3 \nabla \div v \right) \cdot \nabla^3 n d\xi - \bar{\lambda} \int_{|\xi| \geq \eta} \nabla^3 q \cdot \nabla^3 n d\xi + \gamma \int_{|\xi| \geq \eta} \nabla^3 v \cdot \nabla^2 \div v d\xi
\]
\[
= \int_{|\xi| \geq \eta} \nabla^3 v \cdot \nabla^2 F_1 d\xi + \int_{|\xi| \geq \eta} \nabla^2 F_2 \cdot \nabla^3 n d\xi
\]
\[
def \equiv M_1 + M_2 + M_3 + M_4 + M_5.
\]
The application of Cauchy inequality implies
\[
|M_1| \leq C \int_{|\xi| \geq \eta} |\xi|^7 |\hat{\xi}| |\hat{n}| d\xi \leq \epsilon \int_{|\xi| \geq \eta} |\xi|^6 |\hat{n}|^2 d\xi + C \epsilon \int_{|\xi| \geq \eta} |\xi|^6 |\hat{\xi}|^2 d\xi,
\]
for some small constant \( \epsilon \), which will be determined later. It then follows from a similar way that
\[
|M_2| \leq C \int_{|\xi| \geq \eta} |\xi|^6 |\hat{n}|^2 d\xi \leq \epsilon \int_{|\xi| \geq \eta} |\xi|^6 |\hat{n}|^2 d\xi + C \epsilon \int_{|\xi| \geq \eta} |\xi|^6 |\hat{\xi}|^2 d\xi.
\]
Obviously, it holds true
\[
|M_3| \leq C \int_{|\xi| \geq \eta} |\xi|^6 |\hat{\xi}|^2 d\xi.
\]
Using the Cauchy inequality and definition of \( F_1 \), one can show that
\[
|M_4| \leq C \int_{|\xi| \geq \eta} |\xi|^8 |\hat{\xi}|^2 d\xi + C \int_{|\xi| \geq \eta} |\xi|^2 |\hat{F}_1|^2 d\xi
\]
\[\leq C \int_{|\xi| \geq \eta} |\xi|^8 |\hat{\xi}|^2 d\xi + C \int_{|\xi| \geq \eta} |\xi|^2 |\nabla \div (n v) + n \nabla v||^2 d\xi + C \int_{|\xi| \geq \eta} |\xi|^2 |\nabla \div (\bar{\rho} v + \bar{\rho} \nabla v)|^2 d\xi
\]
\[
\equiv \int_{|\xi| \geq \eta} |\xi|^8 |\hat{\xi}|^2 d\xi + M_{41} + M_{42}.
\]
The Plancherel Theorem and Sobolev inequality yields directly
\[
M_{41} \leq C \left( |\nabla (\nabla \div (n v) + n \nabla v)|^2 \right)^{\frac{1}{2}}
\\leq C \left( |\nabla n|^2_{L^\infty} |\nabla v|^2_{L^2} + |\nabla^2 n|^2_{L^2} |v|^2_{L^\infty} + |n|^2_{L^\infty} |\nabla^2 v|^2_{L^2} \right)
\\leq C \left( |\nabla^2 n|^2_{L^1} |\nabla v|^2_{L^2} + \left( |\nabla (n, v)|^2_{L^1} + |\nabla v|^2_{L^2} \right)^{\frac{1}{2}} \right)
\]
\[\leq C (1 + t)^{-6},
\]
where we have used the decay (3.47) in the last inequality. We then apply Hardy inequality to obtain
\[
M_{42} \leq C \int_{|\xi| \geq \eta} |\xi|^4 |\nabla \div (\bar{\rho} v + \bar{\rho} \nabla v)|^2 d\xi \leq C |\nabla^2 (\nabla \div (\bar{\rho} v + \bar{\rho} \nabla v)|^2_{L^2}
\\leq C \sum_{0 \leq l \leq 2 \delta} \left( |(1 + |x|)^{l+1} \nabla^{l+1} \bar{\rho}||^2_{L^\infty} |\nabla^2 v|^2_{L^2} + |(1 + |x|)^{l} \nabla^{l} \bar{\rho}||^2_{L^\infty} |\nabla^3 v|^2_{L^2} \right)
\\leq C \delta |(\nabla^3 v, \nabla^4 v)|^2_{L^2},
\]
\[\text{where we have used the decay (3.47) in the last inequality. We then apply Hardy inequality to obtain}
\]
\[\text{M}_{42} \leq C \int_{|\xi| \geq \eta} |\xi|^4 |\nabla \div (\bar{\rho} v + \bar{\rho} \nabla v)|^2 d\xi \leq C |\nabla^2 (\nabla \div (\bar{\rho} v + \bar{\rho} \nabla v)|^2_{L^2}
\\leq C \sum_{0 \leq l \leq 2 \delta} \left( |(1 + |x|)^{l+1} \nabla^{l+1} \bar{\rho}||^2_{L^\infty} |\nabla^2 v|^2_{L^2} + |(1 + |x|)^{l} \nabla^{l} \bar{\rho}||^2_{L^\infty} |\nabla^3 v|^2_{L^2} \right)
\\leq C \delta |(\nabla^3 v, \nabla^4 v)|^2_{L^2},
\]
where we have used the fact that for any suitable function \( \phi \), there exists a positive constant \( C \) depending only on \( \eta \) such that
\[
\int_{|\xi| \geq \eta} |\xi|^2 |\hat{\phi}|^2 d\xi \leq C \int_{|\xi| \geq \eta} |\xi|^4 |\hat{\phi}|^2 d\xi.
\]

Substituting the estimates (3.60) and (3.61) into (3.59), it is easy to check that
\[
|M_4| \leq C\delta\|\nabla^3 v, \nabla^4 v\|_{L^2}^2 + C(1 + t)^{-6}.
\] (3.62)

Applying the definition of \( F_2 \) and Cauchy inequality, one can get that
\[
|M_5| \leq C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{F}_2| |\hat{n}| d\xi
\]
\[
\leq C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| |\hat{v} \cdot \nabla v| d\xi + C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| n(\Delta v + \nabla \div v) d\xi + C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| (n + q) \nabla (n + q) d\xi
\]
\[
+ C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| \rho(\Delta v + \nabla \div v) d\xi + C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| \rho\nabla (n + q) d\xi + C \int_{|\xi| \geq \eta} |\xi|^5 |\hat{n}| (n + q) \nabla \rho d\xi
\] (3.63)
\[
def = M_{51} + M_{52} + M_{53} + M_{54} + M_{55} + M_{56}.
\]

By virtue of Plancherel Theorem, Sobolev inequality, commutator estimate in Lemma 2.3, and the estimate (3.47), we obtain
\[
M_{51} \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla^2 (v \cdot \nabla v)\|_{L^2}^2
\]
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla v\|_{L^\infty}^2 \|\nabla^3 v\|_{L^2}^2 + C_\epsilon \|\nabla v\|_{L^2}^2 \|\nabla^2 v\|_{L^2}^2
\] (3.64)
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla v\|_{L^2}^2 \|\nabla^3 v\|_{L^2}^2 + C_\epsilon \|\nabla v\|_{L^2}^2 \|\nabla^2 v\|_{L^2}^2
\]
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon (1 + t)^{-6}.
\]

Similarly, it also holds that
\[
M_{52} \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla^2 (n(\Delta v + \nabla \div v))\|_{L^2}^2
\]
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla v\|_{L^\infty}^2 \|\nabla^4 v\|_{L^2}^2 + C_\epsilon \|\nabla^2 n\|_{L^2}^2 \|\nabla^2 v\|_{L^2}^2
\] (3.65)
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\nabla^4 v\|_{L^2}^2 + C_\epsilon (1 + t)^{-7}.
\]

One can deal with the term \( M_{53} \) in the manner of \( M_{52} \). It holds true
\[
M_{53} \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|n(q)\|_{L^\infty}^2 \|\nabla^3 (n, q)\|_{L^2}^2 + C_\epsilon \|\nabla (n, q)\|_{L^2}^2 \|\nabla^2 (n, q)\|_{L^2}^2
\]
\[
\leq \epsilon + C_\epsilon \delta \|\nabla^3 (n, q)\|_{L^2}^2.
\] (3.66)

As for \( M_{54} \), thanks to Hölder and Hardy inequalities, we find
\[
M_{54} \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\rho\|_{L^\infty} \|\nabla^4 v\|_{L^2}^2 + C_\epsilon \|\nabla^2 \rho\|_{L^2}^2 \|\nabla^2 v\|_{L^2}^2
\]
\[
\leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \delta \|\nabla^4 v, \nabla^4 v\|_{L^2}^2.
\] (3.67)

Finally, let us deal with the term \( M_{55} \) and \( M_{56} \) together. Indeed, the Hardy inequality yields directly
\[
M_{55} + M_{56} \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \|\rho\|_{L^\infty} \|\nabla^3 (n, q)\|_{L^2}^2 + C_\epsilon \|\nabla \rho\|_{L^2} \|\nabla^2 (n, q)\|_{L^2}^2
\]
\[
+ C_\epsilon \|(1 + |x|)^2 \nabla^2 \rho\|_{L^2} \|\nabla (n, q)\|_{L^2}^2 + C_\epsilon \|(1 + |x|)^2 \nabla \rho\|_{L^2} \|\nabla^3 (n, q)\|_{L^2}^2
\]
\[
\leq \epsilon + C_\epsilon \delta \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \delta \|\nabla^3 q\|_{L^2}^2.
\]

This bound, together with estimates (3.63)–(3.67), leads us to get
\[
|M_5| \leq \epsilon \|\nabla^3 n\|_{L^2}^2 + C_\epsilon \delta \|\nabla^3 (v, \nabla^4 v, \nabla^3 q)\|_{L^2}^2 + C_\epsilon (1 + t)^{-6}.
\] (3.68)
Substituting the estimates (3.56)-(3.58), (3.62) and (3.68) into (3.55), we find
\[- \frac{d}{dt} \int_{|\xi| \geq \eta} \nabla^3 v \cdot \nabla^2 nd\xi + \gamma \int_{|\xi| \geq \eta} |\nabla^3 n|^2 d\xi \leq (\epsilon + C\epsilon_\delta)\|\nabla^3 n\|_{L^2}^2 + C\epsilon_\delta\|\nabla^4 (v, \nabla^4 v, \nabla^3 q)\|_{L^2}^2 + C\epsilon(1 + t)^{-6}.
\]

Recalling the definition (3.51), there exists a positive constant $C$ such that
\[\|\nabla^3 n\|_{L^2}^2 \leq C\|\nabla^4 v\|_{L^2}^2, \quad \|\nabla^4 v\|_{L^2}^2 \leq C\|\nabla^3 v\|_{L^2}^2, \quad (3.69)\]
and choosing $\epsilon$ and $\delta$ suitably small, we deduce that
\[- \frac{d}{dt} \int_{|\xi| \geq \eta} \nabla^3 v \cdot \nabla^2 nd\xi + \gamma \int_{|\xi| \geq \eta} |\nabla^3 n|^2 d\xi \leq C\|\nabla^3 (n, v', q')\|_{L^2}^2 + C\|\nabla^4 v\|_{L^2}^2 + C(1 + t)^{-6}. \quad (3.70)\]

Recalling the estimate (3.74) in Lemma 3.6, the following estimate holds
\[\frac{d}{dt} \|\nabla^3 (n, v, q)\|_{L^2}^2 + \|\nabla^4 (v, q)\|_{L^2}^2 \leq C\delta \|\nabla^3 (n, v, q)\|_{L^2}. \quad (3.71)\]

We multiply (3.71) by $\eta_2$, then add to (3.74), and choose $\delta$ and $\eta_2$ suitably small, to discover
\[\frac{d}{dt} \left\{ \|\nabla^3 (n, v, q)\|_{L^2}^2 - \eta_2 \int_{|\xi| \geq \eta} \nabla^2 v \cdot \nabla^3 nd\xi \right\} + \|\nabla^4 (v, q)\|_{L^2}^2 + \eta_2\|\nabla^3 n\|_{L^2}^2 \leq C_2\|\nabla^3 (n, v', q')\|_{L^2}^2 + C(1 + t)^{-6}.
\]

Using (3.69) once again, we obtain that
\[\frac{d}{dt} \left\{ \|\nabla^3 (n, v, q)\|_{L^2}^2 - \eta_2 \int_{|\xi| \geq \eta} \nabla^2 v \cdot \nabla^3 nd\xi \right\} + \|\nabla^3 (v, q)\|_{L^2}^2 + \eta_2\|\nabla^3 n\|_{L^2}^2 \leq C_2\|\nabla^3 (n, v', q')\|_{L^2}^2 + C(1 + t)^{-6}.
\]

Thus, the proof of this lemma is completed. \(\square\)

It is noted that the low frequency of $\nabla^3 (n, v, q)$ in the right hand side of the estimate (3.72) in Lemma 3.6 need to be handled. For this purpose, we first analyze the initial value problem for the linearized system of (3.3):
\[
\begin{cases}
\bar{\eta}_t + \gamma \text{div} \bar{\nu} = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\bar{u}_t + \gamma \nabla \bar{n} + \lambda \nabla^3 \bar{q} - \mu_1 \Delta \bar{v} - \mu_2 \nabla \text{div} \bar{v} = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\bar{q}_t - \bar{\kappa} \Delta \bar{q} + \lambda \text{div} \bar{v} = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
(\bar{n}, \bar{v}, \bar{q})_{t=0} = (n_0, v_0, q_0), & x \in \mathbb{R}^3.
\end{cases}
\]

In terms of the semigroup theory for evolutionary equations, one can represent the solution $(\bar{n}, \bar{v}, \bar{q})$ of the linearized system (3.72) as follows:
\[
\begin{cases}
\bar{U}_t = A\bar{U}, & t \geq 0, \\
\bar{U}(0) = U_0,
\end{cases}
\]
where $\bar{U} \overset{\text{def}}{=} (\bar{n}, \bar{v}, \bar{q})^t$, $U_0 \overset{\text{def}}{=} (n_0, v_0, q_0)^t$ and the matrix-valued differential operator $A$ is given by
\[A = \begin{pmatrix} 0 & -\gamma \text{div} & 0 \\ -\gamma \nabla & \mu_1 \Delta + \mu_2 \nabla \text{div} & -\lambda \nabla \\ 0 & -\lambda \text{div} & \bar{\kappa} \Delta \end{pmatrix}.
\]

We then denote $S(t) \overset{\text{def}}{=} e^{tA}$, and recall the system (3.72), to find
\[\bar{U}(t) = S(t)U_0 = e^{tA}U_0, \quad t \geq 0. \quad (3.74)\]

Then, it is easy to deduce that the following estimate holds
\[\|\nabla^3 (S(t)U_0)\|_{L^2} \leq C(1 + t)^{-\frac{2}{3}}\|U_0\|_{L^1 \cap H^\infty}, \quad (3.75)\]
where $C$ is a positive constant independent of time. The proof of the estimate \[3.75\] can be found in \[4.9\], so we omit here. Finally, let us denote $F(t) = (F_1(t), F_2(t), F_3(t))^T$, then the system \[3.3\] can be rewritten as follows:

\[
\begin{cases}
U_t = AU + F, \\
U(0) = U_0.
\end{cases}
\tag{3.76}
\]

In term of the semigroup and Duhamel’s principle, the solution of system \[3.3\] can be expressed as

\[
U(t) = S(t)U_0 + \int_0^t S(t - \tau)F(\tau)d\tau.
\tag{3.77}
\]

Now, one can establish the following estimate for the low frequency of $\nabla^3(n, v, q)$ as follows:

**Lemma 3.7.** Under the assumption of Theorem \[1.1\] it holds that

\[
\|\nabla^3(n, v, q)\|_{L^2} \leq C\delta \sup_{0 \leq s \leq t} \|\nabla^3(n, v, q)(s)\|_{L^2} + C(1 + t)^{-\frac{4}{3}},
\tag{3.78}
\]

where $C$ is a positive constant independent of time.

**Proof.** The formula \[3.77\] yields directly

\[
\nabla^3(n, v, q) = \nabla^3[S(t)U_0] + \int_0^t \nabla^3[S(t - \tau)F(\tau)]d\tau.
\]

This implies that

\[
\|\nabla^3(n', v', q')\|_{L^2} \leq \|\nabla^3[S(t)U_0]\|_{L^2} + \int_0^t \|\nabla^3[S(t - \tau)F(\tau)]\|_{L^2}d\tau.
\tag{3.79}
\]

Since the initial data $U_0 = (n_0, v_0, q_0) \in L^1 \cap H^3$, it follows from the estimate \[3.60\] that

\[
\|\nabla^3[S(t)U_0]\|_{L^2} \leq C(1 + t)^{-\frac{4}{3}}\|U_0\|_{L^1 \cap H^3}.
\tag{3.80}
\]

We then apply Sobolev inequality to obtain that

\[
\int_0^t \|\nabla^3[S(t - \tau)F(\tau)]\|_{L^2}d\tau \leq \int_0^t \|\xi^3|\nabla|\tilde{S}(t - \tau)\|_{L^2(\|\xi\| \leq \eta)}d\tau
\]

\[
\leq \int_0^t \|\xi^3|\nabla|\tilde{S}(t - \tau)\|_{L^2(\|\xi\| \leq \eta)}\|\tilde{F}(\tau)\|_{L^\infty(\|\xi\| \leq \eta)}d\tau + \int_0^t \|\xi^3|\nabla|\tilde{S}(t - \tau)\|_{L^2(\|\xi\| \leq \eta)}\|\xi^2\tilde{F}(\tau)\|_{L^\infty(\|\xi\| \leq \eta)}d\tau
\]

\[
\leq \int_0^t (1 + t - \tau)^{-\frac{1}{3}}\|\tilde{F}(\tau)\|_{L^\infty(\|\xi\| \leq \eta)}d\tau + \int_0^t (1 + t - \tau)^{-\frac{1}{3}}\|\xi^2\tilde{F}(\tau)\|_{L^\infty(\|\xi\| \leq \eta)}d\tau
\]

\[
\overset{def}{=} N_1 + N_2.
\]

Now the first term on the right hand side of \[3.81\] can be estimated as follows:

\[
N_1 = \int_0^t (1 + t - \tau)^{-\frac{1}{3}}\|F\|_{L^1}d\tau \leq C\int_0^t (1 + t - \tau)^{-\frac{1}{3}}(\|F_1\|_{L^1} + \|F_2\|_{L^1} + \|F_3\|_{L^1})d\tau.
\tag{3.82}
\]

We compute by the definitions of $F_i (i = 1, 2, 3)$ and decay estimate \[3.47\] that

\[
\|F_1\|_{L^1} \leq C\|\nabla(n, v)\|_{L^2}\|n, v\|_{L^2} + C\|1 + |x|\nabla\tilde{\rho}\|_{L^2} + C\|\tilde{\rho}\|_{L^2}\|\nabla v\|_{L^2} \leq C\delta (1 + t)^{-\frac{4}{3}},
\]

\[
\|F_2\|_{L^1} \leq C\|v\|_{L^2}\|\nabla v\|_{L^2} + C\|n + \tilde{\rho}\|_{L^2}\|\nabla^2 v\|_{L^2} + C\|n + q\|_{L^2}\|\nabla(n + q)\|_{L^2}
\]

\[
\leq C\delta (1 + t)^{-\frac{4}{3}},
\]

\[
\|F_3\|_{L^1} \leq C\|v\|_{L^2}\|\nabla q\|_{L^2} + C\|n + \tilde{\rho}\|_{L^2}\|\nabla q\|_{L^2} + C\|n + q\|_{L^2}\|\nabla v\|_{L^2} + C\|n\|_{L^\infty} + 1\|\nabla v\|_{L^2}\|\nabla v\|_{L^2}
\]

\[
\leq C\delta (1 + t)^{-\frac{4}{3}}.
\]
Substituting three estimates above into (3.82), and using the estimate in Lemma 2.4, it holds that

$$N_1 \leq C \int_0^\tau (1 + t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{5}{2}} d\tau \leq C(1 + t)^{-\frac{7}{2}}.$$ (3.83)

Next, let us deal with the $N_2$ term. It follows directly

$$\|\xi^2 \hat{F}\|_{L^\infty(|\xi| \leq \eta)} \leq C\|\nabla \hat{F}_1\|_{L^1} + C\|\nabla \hat{F}_2\|_{L^\infty(|\xi| \leq \eta)} + C\|\xi \hat{F}_3\|_{L^\infty(|\xi| \leq \eta)}. $$ (3.84)

First of all, applying the decay estimate (3.47) and Hardy inequality, then the first term in the right handside of (3.84) can be estimated as follows

$$\|\nabla^2 \hat{F}_1\|_{L^1} \leq C\|\nabla^2((n + q)\nabla v + \hat{\rho} \nabla (n + q) + (n + q)\nabla \hat{\rho})\|_{L^1} \leq C\|\nabla ((n + \hat{\rho})(\triangle v + \nabla \text{div} v))\|_{L^1} \leq 2 \sum_{l=0}^2 (\|\nabla^l v\|_{L^2} \|\nabla^{3-l} v\|_{L^2} + \|\nabla^n (n + q)\|_{L^2} \|\nabla^{3-n} (n + q)\|_{L^2} + \| (1 + |x|)^l \nabla^l \hat{\rho}\|_{L^2} \|\nabla^{2-l} (n + q)\|_{L^2} \| (1 + |x|)^l \nabla^{l+1} \hat{\rho}\|_{L^2}) + \sum_{l=0,1} (\|\nabla^n n\|_{L^2} \|\nabla^{3-n} v\|_{L^2} + C\| (1 + |x|)^l \nabla^l \hat{\rho}\|_{L^2} \|\nabla^{3-l} v\|_{L^2} ) \leq C(1 + t)^{-\frac{5}{2}} + C\delta \|\nabla^3 (n, v, q)\|_{L^2}. $$ (3.86)

In view of the decay estimate (3.47) and Hardy inequality, we also have

$$\|\xi \hat{F}_3\|_{L^\infty(|\xi| \leq \eta)} \leq C\|\nabla((n + \hat{\rho})(\triangle v + \nabla \text{div} v + n \Psi(v) + \hat{\Psi}(v))\|_{L^1} \leq C\|\nabla ((n + \hat{\rho})(\triangle v + \nabla \text{div} v + \hat{\Psi}(v) + \nabla \text{div} v))\|_{L^1} \leq 2 \sum_{l=0}^2 (\|\nabla^l v\|_{L^2} \|\nabla^{3-l} q\|_{L^2} + \|\nabla^n (n + q)\|_{L^2} \|\nabla^{3-n} v\|_{L^2} + \|\nabla^{l+1} v\|_{L^2} \|\nabla^{3-l} v\|_{L^2} + \|\nabla^n \hat{\rho}\|_{L^2} \|\nabla^{3-n} v\|_{L^2} + \|\nabla^{l+1} \hat{\rho}\|_{L^2} \|\nabla^{3-l} v\|_{L^2} ) \leq C(1 + t)^{-\frac{7}{2}} + C\delta \|\nabla^3 (v, q)\|_{L^2}. $$ (3.87)
We then conclude from (3.84) - (3.87) that
\[ \| |\xi|^2 \tilde{F} \|_{L^\infty(\| \xi \| \leq \eta)} \leq C \delta \| \nabla^3(n, v, q) \|_{L^2} + C(1 + t)^{-\frac{3}{10}}, \]
which, together with the definition of term \( N_2 \) and the estimate in Lemma 2.4 yields directly
\[ N_2 \leq C \int_0^t (1 + t - \tau)^{\frac{1}{5}} \delta \| \nabla^3(n, v, q) \|_{L^2} + (1 + \tau)^{-\frac{3}{10}} d\tau \]
\[ \leq C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2} \int_0^t (1 + t - \tau)^{-\frac{3}{10}} d\tau + C(1 + t)^{-\frac{3}{10}} \]
\[ \leq C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2} + C(1 + t)^{-\frac{3}{10}}. \]
Substituting (3.83) and (3.88) into (3.81), one arrives at
\[ \int_0^t \| \nabla^3(S(t - \tau)F(U(\tau))) \|_{L^2} d\tau \leq C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2} + C(1 + t)^{-\frac{3}{10}}. \]
Inserting (3.80) and (3.81) into (3.79), one obtains immediately that
\[ \| \nabla^3(n', v', q') \|_{L^2} \leq C \delta \sup_{0 \leq s \leq t} \| \nabla^3(n, v, q) \|_{L^2} + C(1 + t)^{-\frac{3}{10}}. \]
Thus, we finish the proof of this lemma.

Finally, we aim to establish optimal decay rate for the third order spatial derivative of the solution.

**Lemma 3.8.** Under the assumption of Theorem 1.1, it holds that
\[ \| \nabla^3(n, v, q(t)) \|_{L^2} \leq C(1 + t)^{-\frac{3}{10}}, \]
where \( C \) is a positive constant independent of time.

**Proof.** We first rewrite the estimate (3.32) in Lemma 3.6 as
\[ \frac{d}{dt} \tilde{E}^3(t) + \| \nabla^3(n^h, q^h) \|_{L^2}^2 + \eta_2 \| \nabla^3 n^h \|_{L^2}^2 \leq C_2 \| \nabla^3(n, v, q) \|_{L^2}^2 + C(1 + t)^{-6}. \]
Here the energy \( \tilde{E}^3(t) \) is defined by
\[ \tilde{E}^3(t) \overset{df}{=} \| \nabla^3(n, v, q) \|_{L^2}^2 - \eta_2 \int_{|\xi| \geq \eta} \nabla^2 v \cdot \nabla \nabla^3 n d\xi. \]
Thanks to Young inequality, by choosing \( \eta_2 \) small enough, we obtain the following equivalent relation
\[ c_3 \| \nabla^3(n, v) \|_{L^2}^2 \leq \tilde{E}^3(t) \leq c_4 \| \nabla^3(n, v) \|_{L^2}^2, \]
where the constants \( c_3 \) and \( c_4 \) are independent of time. We then add on both sides of (3.91) by \( \| \nabla^3(n', v', q') \|_{L^2}^2 \)
and apply the estimate (3.78) in Lemma 3.7 to discover
\[ \frac{d}{dt} \tilde{E}^3(t) + \| \nabla^3(n, v, q) \|_{L^2}^2 \leq (C_2 + 1) \| \nabla^3(n', v', q') \|_{L^2}^2 + C(1 + t)^{-6} \leq C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2}^2 + C(1 + t)^{-\frac{3}{10}}. \]
In view of the equivalent relation (3.92), we have
\[ \frac{d}{dt} \tilde{E}^3(t) + \tilde{E}^3(t) \leq C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2}^2 + C(1 + t)^{-\frac{3}{10}}. \]
(3.93)
This, together with Gronwall inequality, gives immediately
\[ \tilde{E}^3(t) \leq e^{-t} \tilde{E}^3(0) + C \delta \sup_{0 \leq \tau \leq t} \| \nabla^3(n, v, q) \|_{L^2}^2 \int_0^t e^{-\tau} d\tau + C \int_0^t e^{-\tau}(1 + \tau)^{-\frac{3}{10}} d\tau. \]
By some direct calculations, we can deduce easily
\[
\int_0^t e^{-\tau} d\tau \leq C \quad \text{and} \quad \int_0^t e^{-\tau} (1 + \tau)^{-\frac{9}{2}} d\tau \leq C(1 + t)^{-\frac{9}{2}}.
\]
The equivalent relation (3.92) and (3.94) gives immediately
\[
\sup_{0 \leq \tau \leq t} \| \nabla^3 (n, v, q)(\tau) \|_{L^2}^2 \leq Ce^{-t} \| \nabla^3 (n_0, v_0, q_0) \|_{L^2}^2 + C\delta \sup_{0 \leq \tau \leq t} \| \nabla^3 (n, v, q) \|_{L^2}^2 + C(1 + t)^{-\frac{9}{2}}.
\]
By applying the smallness of \( \delta \), we have
\[
\sup_{0 \leq \tau \leq t} \| \nabla^3 (n, v, q)(\tau) \|_{L^2}^2 \leq C(1 + t)^{-\frac{9}{2}}.
\]
Consequently, this completes the proof of this lemma.

\[\Box\]

**The Proof of Theorem 1.1.** Combining the estimate (3.47) in Lemma 3.5 with estimate (3.90) in Lemma 3.8 we then can obtain the decay rate (1.9) in Theorem 1.1. Therefore, we complete the proof of Theorem 1.1.

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