THE LINE BUNDLES ON MODULI STACKS OF PRINCIPAL
BUNDLES ON A CURVE

INDRANIL BISWAS AND NORBERT HOFFMANN

Abstract. Let $G$ be an affine reductive algebraic group over an algebraically
closed field $k$. We determine the Picard group of the moduli stacks of principal
$G$–bundles on any smooth projective curve over $k$.

1. Introduction

As long as moduli spaces of bundles on a smooth projective algebraic curve $C$
have been studied, their Picard groups have attracted some interest. The first case
was the coarse moduli scheme of semistable vector bundles with fixed determinant
over a curve $C$ of genus $g_C \geq 2$. Seshadri proved that its Picard group is infinite
cyclic in the coprime case [26]; Drézet and Narasimhan showed that this remains
valid in the non–coprime case also [8].

The case of principal $G$–bundles over $C$ for simply connected, almost simple
groups $G$ over the complex numbers has been studied intensively, motivated also
by the relation to conformal field theory and the Verlinde formula [1, 11, 18]. Here
Kumar and Narasimhan [17] showed that the Picard group of the coarse moduli
scheme of semistable $G$–principal bundles over a curve $C$ of genus $g_C \geq 2$
embeds as a subgroup of finite index into the Picard group of the affine Grassmannian,
which is canonically isomorphic to $\mathbb{Z}$; this finite index was determined recently in
[5]. Concerning the Picard group of the moduli stack $\mathcal{M}_G$ of principal $G$–bundles
over a curve $C$ of any genus $g_C \geq 0$, Laszlo and Sorger [21, 28] showed that its
canonical map to the Picard group $\mathbb{Z}$ of the affine Grassmannian is actually an
isomorphism. Faltings [12] has generalised this result to positive characteristic,
and in fact to arbitrary noetherian base scheme.

If $G$ is not simply connected, then the moduli stack $\mathcal{M}_G$ has several connected
components which are indexed by $\pi_1(G)$. For any $d \in \pi_1(G)$, let $\mathcal{M}_G^d$ be
the corresponding connected component of $\mathcal{M}_G$. For semisimple, almost simple groups
$G$ over $\mathbb{C}$, the Picard group $\text{Pic}(\mathcal{M}_G^d)$ has been determined case by case by Beauville,
Laszlo and Sorger [2, 20]. It is finitely generated, and its torsion part is a direct
sum of $2g_C$ copies of $\pi_1(G)$. Furthermore, its torsion–free part again embeds as
a subgroup of finite index into the Picard group $\mathbb{Z}$ of the affine Grassmannian.
Together with a general expression for this index, Teleman [29] also proved these
statements, using topological and analytic methods.

In this paper, we determine the Picard group $\text{Pic}(\mathcal{M}_G^d)$ for any reductive group
$G$, working over an algebraically closed ground field $k$ without any restriction on

2000 Mathematics Subject Classification. 14C22, 14D20, 14H10.
Key words and phrases. principal bundle, moduli stack, Picard group.

The second author gratefully acknowledges the support of the SFB/TR 45 "Perioden, Mod-
ulräume und Arithmetik algebraischer Varietäten".

1
the characteristic of $k$ (for all $g_C \geq 0$). Endowing this group with a natural scheme structure, we prove that the resulting group scheme $\text{Pic}(\mathcal{M}_G^d)$ over $k$ contains, as an open subgroup, the scheme of homomorphisms from $\pi_1(G)$ to the Jacobian $J_C$, with the quotient being a finitely generated free abelian group which we denote by $\text{NS}(\mathcal{M}_G^d)$ and call it the Néron–Severi group (see Theorem 5.3.1). We introduce this Néron–Severi group combinatorially in § 5.2 in particular, Proposition 5.2.11 describes it as follows: the group $\text{NS}(\mathcal{M}_G^d)$ contains a subgroup $\text{NS}(\mathcal{M}_G^{ab})$ which depends only on the torus $G^{ab} = G/[G, G]$; the quotient is a group of Weyl–invariant symmetric bilinear forms on the root system of the semisimple part $[G, G]$, subject to certain integrality conditions that generalise Teleman’s result in [29].

We also describe the maps of Picard groups induced by group homomorphisms $G \rightarrow H$. An interesting effect appears for the inclusion $\iota_G : T_G \hookrightarrow G$ of a maximal torus, say for semisimple $G$: Here the induced map $\text{NS}(\mathcal{M}_G^d) \rightarrow \text{NS}(\mathcal{M}_G^{d'})$ for a lift $\delta \in \pi_1(T_G)$ of $d$ involves contracting each bilinear form in $\text{NS}(\mathcal{M}_G^d)$ to a linear form by means of $\delta$ (cf. Definition 4.3.5). In general, the map of Picard groups induced by a group homomorphism $G \rightarrow H$ is a combination of this effect and of more straightforward induced maps (cf. Definition 5.2.7 and Theorem 5.3.1.iv). In particular, these induced maps are different on different components of $\mathcal{M}_G$, whereas the Picard groups $\text{Pic}(\mathcal{M}_G^d)$ themselves are essentially independent of $d$.

Our proof is based on Faltings’ result in the simply connected case. To deduce the general case, the strategy of [2] and [20] is followed, meaning we “cover” the moduli stack $\mathcal{M}_G^d$ by a moduli stack of “twisted” bundles as in [2] under the universal cover of $G$, more precisely under an appropriate torus times the universal cover of the semisimple part $[G, G]$. To this “covering”, we apply Laszlo’s [20] method of descent for torsors under a group stack. To understand the relevant descent data, it turns out that we may restrict to the maximal torus $T_G$ in $G$, roughly speaking because the pullback $\iota_G^*$ is injective on the Picard groups of the moduli stacks.

We briefly describe the structure of this paper. In Section 2 we recall the relevant moduli stacks and collect some basic facts. Section 3 deals with the case that $G = T$ is a torus. Section 4 treats the “twisted” simply connected case as indicated above. In the final Section 5 we put everything together to prove our main theorem, namely Theorem 5.3.1 Each section begins with a slightly more detailed description of its contents.

Our motivation for this work was to understand the existence of Poincaré families on the corresponding coarse moduli schemes, or in other words to decide whether these moduli stacks are neutral as gerbes over their coarse moduli schemes. The consequences for this question will be spelled out in a subsequent paper.

2. The stack of $G$–bundles and its Picard functor

Here we introduce the basic objects of this paper, namely the moduli stack of principal $G$–bundles on an algebraic curve and its Picard functor. The main purpose of this section is to fix some notation and terminology; along the way, we record a few basic facts for later use.

2.1. A Picard functor for algebraic stacks. Throughout this paper, we work over an algebraically closed field $k$. There is no restriction on the characteristic of $k$. We say that a stack $\mathcal{X}$ over $k$ is algebraic if it is an Artin stack and also locally of finite type over $k$. Every algebraic stack $\mathcal{X} \neq \emptyset$ admits a point $x_0 : \text{Spec}(k) \rightarrow \mathcal{X}$ according to Hilbert’s Nullstellensatz.
A 1–morphism \( \Phi : \mathcal{X} \to \mathcal{Y} \) of stacks is an equivalence if some 1–morphism \( \Psi : \mathcal{Y} \to \mathcal{X} \) admits 2–isomorphisms \( \Psi \circ \Phi \cong \text{id}_\mathcal{X} \) and \( \Phi \circ \Psi \cong \text{id}_\mathcal{Y} \). A diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{A} & \mathcal{X}' \\
\downarrow \Phi & & \downarrow \Phi' \\
\mathcal{Y} & \xrightarrow{B} & \mathcal{Y}'
\end{array}
\]
of stacks and 1–morphisms is 2–commutative if a 2–isomorphism \( \Phi' \circ A \cong B \circ \Phi \) is given. Such a 2–commutative diagram is 2–cartesian if the induced 1–morphism from \( \mathcal{X} \) to the fibre product of stacks \( \mathcal{X}' \times_{\mathcal{Y}} \mathcal{Y} \) is an equivalence.

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be algebraic stacks over \( k \). As usual, we denote by \( \text{Pic}(\mathcal{X}) \) the abelian group of isomorphism classes of line bundles \( L \) on \( \mathcal{X} \). If \( \mathcal{X} \neq \emptyset \), then \( \text{pr}^*_2 : \text{Pic}(\mathcal{Y}) \to \text{Pic}(\mathcal{X} \times \mathcal{Y}) \) is injective because \( x_0^* : \text{Pic}(\mathcal{X} \times \mathcal{Y}) \to \text{Pic}(\mathcal{Y}) \) is a left inverse of \( \text{pr}^*_2 \).

**Definition 2.1.1.** The Picard functor \( \text{Pic}(\mathcal{X}) \) is the functor from schemes \( S \) of finite type over \( k \) to abelian groups that sends \( S \) to \( \text{Pic}(\mathcal{X} \times S)/\text{pr}^*_2 \text{Pic}(S) \).

If \( \text{Pic}(\mathcal{X}) \) is representable, then we denote the representing scheme again by \( \text{Pic}(\mathcal{X}) \). If \( \text{Pic}(\mathcal{X}) \) is the constant sheaf given by an abelian group \( \Lambda \), then we say that \( \text{Pic}(\mathcal{X}) \) is discrete and simply write \( \text{Pic}(\mathcal{X}) \cong \Lambda \). (Since the constant Zariski sheaf \( \Lambda \) is already an fppf sheaf, it is not necessary to specify the topology here.)

**Lemma 2.1.2.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be algebraic stacks over \( k \) with \( \Gamma(\mathcal{X}, \mathcal{O}_X) = k \).

i) The canonical map
\[
\text{pr}^*_2 : \Gamma(\mathcal{Y}, \mathcal{O}_Y) \to \Gamma(\mathcal{X} \times \mathcal{Y}, \mathcal{O}_{\mathcal{X} \times \mathcal{Y}})
\]
is an isomorphism.

ii) Let \( \mathcal{L} \in \text{Pic}(\mathcal{X} \times \mathcal{Y}) \) be given. If there is an atlas \( u : U \to \mathcal{Y} \) for which \( u^* \mathcal{L} \in \text{Pic}(\mathcal{X} \times U) \) is trivial, then \( \mathcal{L} \in \text{pr}^*_2 \text{Pic}(\mathcal{Y}) \).

**Proof.** i) Since the question is local in \( \mathcal{Y} \), we may assume that \( \mathcal{Y} = \text{Spec}(A) \) is an affine scheme over \( k \). In this case, we have
\[
\Gamma(\mathcal{X} \times \mathcal{Y}, \mathcal{O}_{\mathcal{X} \times \mathcal{Y}}) = \Gamma(\mathcal{X}, (\text{pr}_1)_* \mathcal{O}_{\mathcal{X} \times \mathcal{Y}}) = \Gamma(\mathcal{X}, A \otimes_k \mathcal{O}_X) = A = \Gamma(\mathcal{Y}, \mathcal{O}_Y).
\]

ii) Choose a point \( x_0 : \text{Spec}(k) \to \mathcal{X} \). We claim that \( \mathcal{L} \) is isomorphic to \( \text{pr}^*_2 \mathcal{L}_{x_0} \) for \( \mathcal{L}_{x_0} := x_0^* \mathcal{L} \in \text{Pic}(\mathcal{Y}) \). More precisely there is a unique isomorphism \( \mathcal{L} \cong \text{pr}^*_2 \mathcal{L}_{x_0} \) whose restriction to \( \{x_0\} \times \mathcal{Y} \cong \mathcal{Y} \) is the identity. To prove this, due to the uniqueness involved, this claim is local in \( \mathcal{Y} \). Hence we may assume \( \mathcal{Y} = U \), which by assumption means that \( \mathcal{L} \) is trivial. In this case, statement (i) implies the claim. \( \square \)

**Corollary 2.1.3.** For \( \nu = 1, 2 \), let \( \mathcal{X}_\nu \) be an algebraic stack over \( k \) with \( \Gamma(\mathcal{X}_\nu, \mathcal{O}_{\mathcal{X}_\nu}) = k \). Let \( \Phi : \mathcal{X}_1 \to \mathcal{X}_2 \) be a 1–morphism such that the induced morphism of functors \( \Phi^* : \text{Pic}(\mathcal{X}_2) \to \text{Pic}(\mathcal{X}_1) \) is injective. Then
\[
\Phi^* : \text{Pic}(\mathcal{X}_2 \times \mathcal{Y}) \to \text{Pic}(\mathcal{X}_1 \times \mathcal{Y})
\]
is injective for every algebraic stack \( \mathcal{Y} \) over \( k \).
Proof. Since \( \mathcal{Y} \) is assumed to be locally of finite type over \( k \), we can choose an atlas \( u : U \to \mathcal{Y} \) such that \( U \) is a disjoint union of schemes of finite type over \( k \). Suppose that \( \mathcal{L} \in \text{Pic}(\mathcal{X}_2 \times \mathcal{Y}) \) has trivial pullback \( \Phi^* \mathcal{L} \in \text{Pic}(\mathcal{X}_1 \times \mathcal{Y}) \). Then \((\Phi \times u)^* \mathcal{L} \in \text{Pic}(\mathcal{X}_2 \times U)\) is also trivial. Using the assumption on \( \Phi^* \) it follows that \( u^* \mathcal{L} \in \text{Pic}(\mathcal{X}_2 \times U) \) is trivial. Now apply Lemma 2.1.2(ii). \( \square \)

We will also need the following stacky version of the standard see–say principle.

**Lemma 2.1.4.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two nonempty algebraic stacks over \( k \). If \( \text{Pic}(\mathcal{X}) \) is discrete, and \( \Gamma(\mathcal{Y}, O_{\mathcal{Y}}) = k \), then

\[
pr_1^* \oplus pr_2^* : \text{Pic}(\mathcal{X}) \oplus \text{Pic}(\mathcal{Y}) \to \text{Pic}(\mathcal{X} \times \mathcal{Y})
\]

is an isomorphism of functors.

**Proof.** Choose points \( x_0 : \text{Spec}(k) \to \mathcal{X} \) and \( y_0 : \text{Spec}(k) \to \mathcal{Y} \). The morphism of functors \( pr_1^* \oplus pr_2^* \) in question is injective, because

\[
y_0^* \oplus x_0^* : \text{Pic}(\mathcal{X} \times \mathcal{Y}) \to \text{Pic}(\mathcal{X}) \oplus \text{Pic}(\mathcal{Y})
\]

is a left inverse of it. Therefore, to prove the lemma it suffices to show that \( y_0^* \oplus x_0^* \) is also injective.

So let a scheme \( S \) of finite type over \( k \) be given, as well as a line bundle \( \mathcal{L} \) on \( \mathcal{X} \times \mathcal{Y} \times S \) such that \( y_0^* \mathcal{L} \) is trivial in \( \text{Pic}(\mathcal{X}) \). We claim that \( \mathcal{L} \) is isomorphic to the pullback of a line bundle on \( \mathcal{Y} \times S \).

To prove the claim, tensoring \( \mathcal{L} \) with an appropriate line bundle on \( S \) if necessary, we may assume that \( y_0^* \mathcal{L} \) is trivial in \( \text{Pic}(\mathcal{X} \times S) \). By assumption, \( \text{Pic}(\mathcal{X}) \cong \Lambda \) for some abelian group \( \Lambda \). Sending any \( (y, s) : \text{Spec}(k) \to \mathcal{Y} \times S \) to the isomorphism class of

\[
(y, s)^*(\mathcal{L}) \in \text{Pic}(\mathcal{X})
\]

we obtain a Zariski–locally constant map from the set of \( k \)-points in \( \mathcal{Y} \times S \) to \( \Lambda \). This map vanishes on \( \{y_0\} \times S \), and hence it vanishes identically on \( \mathcal{Y} \times S \) because \( \mathcal{Y} \) is connected. This means that \( u^* \mathcal{L} \in \text{Pic}(\mathcal{X} \times U) \) is trivial for any atlas \( u : U \to \mathcal{Y} \times S \). Now Lemma 2.1.2(ii) completes the proof of the claim.

If moreover \( x_0^* \mathcal{L} \) is trivial in \( \text{Pic}(\mathcal{Y}) \), then \( \mathcal{L} \) is even isomorphic to the pullback of a line bundle on \( S \), and hence trivial in \( \text{Pic}(\mathcal{X} \times \mathcal{Y}) \). This proves the injectivity of \( y_0^* \oplus x_0^* \), and hence the lemma follows. \( \square \)

2.2. **Principal \( G \)--bundles over a curve.** We fix an irreducible smooth projective curve \( C \) over the algebraically closed base field \( k \). The genus of \( C \) will be denoted by \( g \). Given a linear algebraic group \( G \to \text{GL}_n \), we denote by

\[
\mathcal{M}_G
\]

the moduli stack of principal \( G \)--bundles \( E \) on \( C \). More precisely, \( \mathcal{M}_G \) is given by the groupoid \( \mathcal{M}_G(S) \) of principal \( G \)--bundles on \( S \times C \) for every \( k \)--scheme \( S \). The stack \( \mathcal{M}_G \) is known to be algebraic over \( k \) (see for example [21], Proposition 3.4], or [22] Théorème 4.6.2.1] together with [27], Lemma 4.8.1]).

Given a morphism of linear algebraic groups \( \varphi : G \to H \), the extension of the structure group by \( \varphi \) defines a canonical 1–morphism

\[
\varphi_* : \mathcal{M}_G \to \mathcal{M}_H
\]

which more precisely sends a principal \( G \)--bundle \( E \) to the principal \( H \)--bundle

\[
\varphi_* E := E \times^G H := (E \times G)/H,
\]
following the convention that principal bundles carry a right group action. One has a canonical 2–isomorphism \((\psi \circ \varphi)_* \cong \psi_* \circ \varphi_*\) whenever \(\psi : H \to K\) is another morphism of linear algebraic groups.

**Lemma 2.2.1.** Suppose that the diagram of linear algebraic groups

\[
\begin{array}{ccc}
H & \xrightarrow{\psi_2} & G_2 \\
\downarrow{\psi_1} & & \downarrow{\varphi_2} \\
G_1 & \xrightarrow{\varphi_1} & G
\end{array}
\]

is cartesian. Then the induced 2–commutative diagram of moduli stacks

\[
\begin{array}{ccc}
\mathcal{M}_H & \xrightarrow{(\psi_2)_*} & \mathcal{M}_{G_2} \\
(\psi_1)_* & & (\varphi_2)_* \\
\mathcal{M}_{G_1} & \xrightarrow{(\varphi_1)_*} & \mathcal{M}_G
\end{array}
\]

is 2–cartesian.

**Proof.** The above 2–commutative diagram defines a 1–morphism

\[\mathcal{M}_H \to \mathcal{M}_{G_1 \times_{\mathcal{M}_G} \mathcal{M}_{G_2}}.\]

To construct an inverse, let \(E\) be a principal \(G\)–bundle on some \(k\)–scheme \(X\). For \(\nu = 1, 2\), let \(E_\nu\) be a principal \(G_\nu\)–bundle on \(X\) together with an isomorphism \(E_\nu \times^{G_\nu} G \cong E\); note that the latter defines a map \(E_\nu \to E\) of schemes over \(X\).

Then \(G_1 \times G_2\) acts on \(E_1 \times_X E_2\), and the closed subgroup \(H \subseteq G_1 \times G_2\) preserves the closed subscheme

\[F := E_1 \times_E E_2 \subseteq E_1 \times_X E_2.\]

This action turns \(F\) into a principal \(H\)–bundle. Thus we obtain in particular a 1–morphism

\[\mathcal{M}_{G_1 \times_{\mathcal{M}_G} \mathcal{M}_{G_2}} \to \mathcal{M}_H.\]

It is easy to check that this is the required inverse. \(\square\)

Let \(Z\) be a closed subgroup in the center of \(G\). Then the multiplication \(Z \times G \to G\) is a group homomorphism; we denote the induced 1–morphism by

\[\otimes : \mathcal{M}_Z \times \mathcal{M}_G \to \mathcal{M}_G\]

and call it tensor product. In particular, tensoring with a principal \(Z\)–bundle \(\xi\) on \(C\) defines a 1–morphism which we denote by

\[(1) \quad t\xi : \mathcal{M}_G \to \mathcal{M}_G.\]

For commutative \(G\), this tensor product makes \(\mathcal{M}_G\) a group stack.

Suppose now that \(G\) is reductive. We follow the convention that all reductive groups are smooth and connected. In particular, \(\mathcal{M}_G\) is also smooth [B 4.5.1], so its connected components and its irreducible components coincide; we denote this set of components by \(\pi_0(\mathcal{M}_G)\). This set \(\pi_0(\mathcal{M}_G)\) can be described as follows:

Let \(\iota_G : T_G \hookrightarrow G\) be the inclusion of a maximal torus, with cocharacter group \(\Lambda_{T_G} := \text{Hom}(\mathbb{G}_m, T_G)\). Let \(\Lambda_{\text{coroots}} \subseteq \Lambda_{T_G}\) be the subgroup generated by the coroots of \(G\). The Weyl group of \((G, T_G)\) acts trivially on \(\Lambda_{T_G}/\Lambda_{\text{coroots}}\), so this quotient is, up to a canonical isomorphism, independent of the choice of \(T_G\). We
denote this quotient by \( \pi_1(G) \); if \( \pi_1(G) \) is trivial, then \( G \) is called simply connected. For \( k = \mathbb{C} \), these definitions coincide with the usual notions for the topological space \( G(\mathbb{C}) \).

Sending each line bundle on \( C \) to its degree we define an isomorphism \( \pi_0(\mathcal{M}_{G,m}) \rightarrow \mathbb{Z} \), which induces an isomorphism \( \pi_0(\mathcal{M}_G) \rightarrow \Lambda_{T_G} \). Its inverse, composed with the map

\[
(\iota_G)_* : \pi_0(\mathcal{M}_{T_G}) \rightarrow \pi_0(\mathcal{M}_G),
\]

is known to induce a canonical bijection

\[
\pi_1(G) = \Lambda_{T_G}/\Lambda_{\text{coroots}} \sim \pi_0(\mathcal{M}_G),
\]

cf. \[9\] and \[14\]. We denote by \( \mathcal{M}_G^d \) the component of \( \mathcal{M}_G \) given by \( d \in \pi_1(G) \).

**Lemma 2.2.2.** Let \( \varphi : G \rightarrow H \) be an epimorphism of reductive groups over \( k \) whose kernel is contained in the center of \( G \). For each \( d \in \pi_1(G) \), the 1–morphism

\[
\varphi_* : \mathcal{M}_G^d \rightarrow \mathcal{M}_H^e, \quad e := \varphi_*(d) \in \pi_1(H),
\]

is faithfully flat.

**Proof.** Let \( T_H \subseteq H \) be the image of the maximal torus \( T_G \subseteq G \). Let \( B_G \subseteq G \) be a Borel subgroup containing \( T_G \); then

\[
B_H := \varphi(B_G) \subset H
\]
is a Borel subgroup of \( H \) containing \( T_H \). For the moment, we denote

- \( \mathcal{M}_G^d \subseteq \mathcal{M}_T_G \) and \( \mathcal{M}_{B_G}^d \subseteq \mathcal{M}_{B_G} \) the inverse images of \( \mathcal{M}_G^d \subseteq \mathcal{M}_G \), and

- \( \mathcal{M}_{T_H}^d \subseteq \mathcal{M}_{T_H} \) and \( \mathcal{M}_{B_H}^d \subseteq \mathcal{M}_{B_H} \) the inverse images of \( \mathcal{M}_H^d \subseteq \mathcal{M}_H \).

Let \( \pi_G : B_G \rightarrow T_G \) and \( \pi_H : B_H \rightarrow T_H \) denote the canonical surjections. Then

\[
\mathcal{M}_{B_G}^d = (\pi_G)_*^{-1}(\mathcal{M}_G^d) \quad \text{and} \quad \mathcal{M}_{B_H}^d = (\pi_H)_*^{-1}(\mathcal{M}_H^d),
\]

because \( \pi_0(\mathcal{M}_{T_G}) = \pi_0(\mathcal{M}_{B_G}) \) and \( \pi_0(\mathcal{M}_{T_H}) = \pi_0(\mathcal{M}_{B_H}) \) according to the proof of \[9\] Proposition 5. Applying Lemma 2.2.1 to the two cartesian squares

\[
\begin{array}{ccc}
T_G & \xrightarrow{\pi_G} & B_G \\
\downarrow{\varphi_T} & & \downarrow{\varphi_B} \\
T_H & \xrightarrow{\pi_H} & B_H
\end{array}
\]

of groups, we get two 2–cartesian squares

\[
\begin{array}{ccc}
\mathcal{M}_{T_G}^d & \xrightarrow{(\varphi_T)_*} & \mathcal{M}_{B_G}^d \\
\downarrow{\varphi_*} & & \downarrow{\varphi_*} \\
\mathcal{M}_{T_H}^d & \xrightarrow{(\varphi_B)_*} & \mathcal{M}_{B_H}^d
\end{array}
\]

of moduli stacks. Since \((\varphi_T)_*\) is faithfully flat, its pullback \((\varphi_B)_*\) is so as well. This implies that \( \varphi_* \) is also faithfully flat, as some open substack of \( \mathcal{M}_{B_H}^d \) maps smoothly and surjectively onto \( \mathcal{M}_H^d \), according to \[9\] Propositions 1 and 2.

□
3. THE CASE OF TORUS

This section deals with the Picard functor of the moduli stack $\mathcal{M}_G^0$ in the special case where $G = T$ is a torus. We explain in the second subsection that its description involves the character group $\text{Hom}(T, G_m)$ and the Picard functor of its coarse moduli scheme, which is isomorphic to a product of copies of the Jacobian $J_C$. As a preparation, the first subsection deals with the Néron–Severi group of such products of principally polarised abelian varieties. A little care is required to keep everything functorial in $T$, since this functoriality will be needed later.

3.1. ON PRINCIPALLY POLARISED ABELIAN VARIETIES. Let $A$ be an abelian variety over $k$, with dual abelian variety $A^\vee$ and Néron–Severi group

$$\text{NS}(A) := \text{Pic}(A)/A^\vee(k).$$

For a line bundle $L$ on $A$, the standard morphism

$$\phi_L : A \to A^\vee$$

sends $a \in A(k)$ to $\tau_a(L) \otimes L^\text{dual}$ where $\tau_a : A \to A$ is the translation by $a$. $\phi_L$ is a homomorphism by the theorem of the cube [25 §6]. Let a principal polarisation

$$\phi : A \to A^\vee$$

be given. Let

$$c^\phi : \text{NS}(A) \to \text{End} A$$

be the injective homomorphism that sends the class of $L$ to $\phi^{-1} \circ \phi_L$. We denote by $\dagger : \text{End} A \to \text{End} A$ the Rosati involution associated to $\phi$; so by definition, it sends $\alpha : A \to A$ to $\alpha^\dagger := \phi^{-1} \circ \alpha^\vee \circ \phi$.

Lemma 3.1.1. An endomorphism $\alpha \in \text{End}(A)$ is in the image of $c^\phi$ if and only if $\alpha^\dagger = \alpha$.

Proof. If $k = \mathbb{C}$, this is contained in [19, Chapter 5, Proposition 2.1]. For polarisations of arbitrary degree, the analogous statement about $\text{End}(A) \otimes \mathbb{Q}$ is shown in [25, p. 190]; its proof carries over to the situation of this lemma as follows.

Let $l$ be a prime number, $l \neq \text{char}(k)$, and let

$$e_l : T_l(A) \times T_l(A^\vee) \to \mathbb{Z}_l(1)$$

be the standard pairing between the Tate modules of $A$ and $A^\vee$, cf. [25 §20]. According to [25 §20, Theorem 2 and §23, Theorem 3], a homomorphism $\psi : A \to A^\vee$ is of the form $\psi = \phi_L$ for some line bundle $L$ on $A$ if and only if

$$e_l(x, \psi_y) = -e_l(y, \psi_x) \quad \text{for all} \quad x, y \in T_l(A).$$

In particular, this holds for $\phi$. Hence the right hand side equals

$$-e_l(y, \psi_x) = -e_l(y, \phi_x \phi^{-1}_x \psi_x) = e_l(\phi^{-1}_x \psi_x, \phi_x y) = e_l(x, \psi^\vee_{\phi} \phi^\vee_{\phi} y),$$

where the last equality follows from [25 p. 186, equation (1)]. Since the pairing $e_l$ is nondegenerate, it follows that $\psi = \phi_L$ holds for some $L$ if and only if

$$\psi_y = \psi^\vee_{\phi} (\phi^{-1})^\vee_{\phi} y \quad \text{for all} \quad y \in T_l(A),$$

hence if and only if $\psi = \psi^\vee \circ (\phi^{-1})^\vee \circ \phi$. By definition of the Rosati involution $\dagger$, the latter is equivalent to $(\phi^{-1} \circ \psi)^\dagger = \phi^{-1} \circ \psi$. \qed

Let $\Lambda$ be a finitely generated free abelian group. Let $\Lambda \otimes A$ denote the abelian variety over $k$ with group of $S$–valued points $\Lambda \otimes A(S)$ for any $k$–scheme $S$. 


**Definition 3.1.2.** The subgroup

\[ \text{Hom}^s(\Lambda \otimes \Lambda, \text{End } A) \subseteq \text{Hom}(\Lambda \otimes \Lambda, \text{End } A) \]

consists of all \( b : \Lambda \otimes \Lambda \to \text{End } A \) with \( b(\lambda_1 \otimes \lambda_2) = b(\lambda_2 \otimes \lambda_1) \) for all \( \lambda_1, \lambda_2 \in \Lambda \).

**Corollary 3.1.3.** There is a unique isomorphism

\[ c^\phi_\Lambda : \text{NS}(\Lambda \otimes A) \xrightarrow{\sim} \text{Hom}^s(\Lambda \otimes \Lambda, \text{End } A) \]

which sends the class of each line bundle \( L \) on \( \Lambda \otimes A \) to the linear map

\[ c^\phi_\Lambda(L) : \Lambda \otimes \Lambda \to \text{End } A \]

defined by sending \( \lambda_1 \otimes \lambda_2 \) for \( \lambda_1, \lambda_2 \in \Lambda \) to the composition

\[ A \xrightarrow{\lambda_1 \otimes} \Lambda \otimes A \xrightarrow{\phi} (\Lambda \otimes A)^{\vee} \xrightarrow{(\lambda_2 \otimes)_{\vee}} A^{\vee} \xrightarrow{\phi^{-1}} A. \]

**Proof.** The uniqueness is clear. For the existence, we may then choose an isomorphism \( \Lambda \cong \mathbb{Z}^r \); it yields an isomorphism \( \Lambda \otimes A \cong A^r \). Let

\[ \phi^r = \phi \times \cdots \times \phi : A^r \xrightarrow{\sim} (A^r)^r = (A^r)^{\vee} \]

be the diagonal principal polarisation on \( A^r \). According to Lemma 3.1.1,

\[ c^\phi^r : \text{NS}(A^r) \to \text{End}(A^r) \]

is an isomorphism onto the Rosati–invariants. Under the standard isomorphisms

\[ \text{End}(A^r) = \text{Mat}_{r \times r}(\text{End } A) = \text{Hom}(\mathbb{Z}^r \otimes \mathbb{Z}^r, \text{End } A), \]

the Rosati involution on \( \text{End}(A^r) \) corresponds to the involution \( (\alpha_{ij}) \mapsto (\alpha_{ji}) \) on \( \text{Mat}_{r \times r}(\text{End } A) \), and hence the Rosati–invariant part of \( \text{End}(A^r) \) corresponds to \( \text{Hom}^s(\mathbb{Z}^r \otimes \mathbb{Z}^r, \text{End } A) \). Thus we obtain an isomorphism

\[ \text{NS}(\Lambda \otimes A) \cong \text{NS}(A^r) \xrightarrow{\phi^r} \text{Hom}^s(\mathbb{Z}^r \otimes \mathbb{Z}^r, \text{End } A) \cong \text{Hom}^s(\Lambda \otimes \Lambda, \text{End } A). \]

By construction, it maps the class of each line bundle \( L \) on \( \Lambda \otimes A \) to the map

\[ c^\phi_\Lambda(L) : \Lambda \otimes \Lambda \to \text{End } A \]

prescribed above. □

3.2. **Line bundles on \( \mathcal{M}_T^0 \).** Let \( T \cong \mathbb{G}_m \) be a torus over \( k \). We will always denote by

\[ \Lambda_T := \text{Hom}(\mathbb{G}_m, T) \]

the cocharacter lattice. We set in the previous subsection this finitely generated free abelian group and the Jacobian variety \( J_C \), endowed with the standard principal polarisation \( \phi_\Theta : J_C \xrightarrow{\sim} J_C^{\vee} \).

**Definition 3.2.1.** The finitely generated free abelian group

\[ \text{NS}(\mathcal{M}_T) := \text{Hom}(\Lambda_T, \mathbb{Z}) \oplus \text{Hom}^s(\Lambda_T \otimes \Lambda_T, \text{End } J_C) \]

is the **Néron–Severi group** of \( \mathcal{M}_T \).

For each finitely generated abelian group \( \Lambda \), we denote by \( \text{Hom}(\Lambda, J_C) \) the \( k \)-scheme of homomorphisms from \( \Lambda \) to \( J_C \). If \( \Lambda \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s \), then

\[ \text{Hom}(\Lambda, J_C) \cong J_C^r \times J_C[n_1] \times \cdots \times J_C[n_s] \]

where \( J_C[n] \) denotes the kernel of the map \( J_C \to J_C \) defined by multiplication with \( n \).
Proposition 3.2.2. i) The Picard functor $\text{Pic}(M^0_T)$ is representable by a scheme locally of finite type over $k$.

ii) There is a canonical exact sequence of commutative group schemes

$$0 \rightarrow \text{Hom}(\Lambda_T, J_C) \xrightarrow{j_T} \text{Pic}(M^0_T) \xrightarrow{c_T} \text{NS}(M_T) \rightarrow 0.$$ 

iii) Let $\xi$ be a principal $T$–bundle of degree $0 \in \Lambda_T$ on $C$. Then the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\quad} & \text{Hom}(\Lambda_T, J_C) \\
| & | & | \\
\text{Pic}(M^0_T) & \xrightarrow{c_T} & \text{NS}(M_T) \\
| & | & | \\
0 & \xrightarrow{t^*_\xi} & \text{Pic}(M^0_T) \\
\end{array}
\]

commutes.

Proof. Given a line bundle $L$ on $M^0_T$, the automorphism group $T$ of each point in $M^0_T$ acts on the corresponding fibre of $L$, so we obtain a character $w(L) : T \rightarrow \mathbb{G}_m$ which is independent of the choice of the point in $M^0_T$. This character is called the weight $w(L)$ of $L$. Let

$$q : M^0_T \rightarrow \mathfrak{M}^0_T$$

be the canonical morphism to the coarse moduli scheme $\mathfrak{M}^0_T$, which is an abelian variety canonically isomorphic to $\text{Hom}(\Lambda_T, J_C)$. Line bundles of weight $0$ on $M^0_T$ descend to $\mathfrak{M}^0_T$, so the sequence

$$0 \rightarrow \text{Pic}(M^0_T) \xrightarrow{q^*} \text{Pic}(\mathfrak{M}^0_T) \xrightarrow{w} \text{Hom}(\Lambda_T, \mathbb{Z})$$

is exact. This extends for families. Since $\text{Pic}(A)$ is representable for any abelian variety $A$, the proof of (i) is now complete.

Standard theory of abelian varieties and Corollary 3.1.3 together yield another short exact sequence

$$0 \rightarrow \text{Hom}(\Lambda_T, J_C) \rightarrow \text{Pic}(M^0_T) \rightarrow \text{Hom}(\Lambda_T, \mathbb{Z})$$

is exact. This extends for families. Since $\text{Pic}(A)$ is representable for any abelian variety $A$, the proof of (i) is now complete.

Finally, it is standard that $t^*_\xi$ (see (1)) is the identity map on $\text{Pic}^0(\mathfrak{M}^0_T) = \text{Hom}(\Lambda, J_C)$ (see [24, Proposition 9.2]), and $t^*_\xi$ induces the identity map on the discrete quotient $\text{Pic}(M^0_T)/\text{Pic}^0(M^0_T)$ because $\xi$ can be connected to the trivial $T$–bundle in $M^0_T$. \qed
Remark 3.2.3. The exact sequence in Proposition 3.2.2 ii) is functorial in $T$. More precisely, each homomorphism of tori $\varphi : T \to T'$ induces a morphism of exact sequences

$$
0 \to \text{Hom}(\Lambda^2, J_C) \xrightarrow{j_{T'}} \text{Pic}(M^0_{T'}) \xrightarrow{c_{T'}} \text{NS}(M^0_{T'}) \to 0
$$

$$
0 \to \text{Hom}(\Lambda, J_C) \xrightarrow{j_T} \text{Pic}(M^0_{T}) \xrightarrow{c_T} \text{NS}(M_{T}) \to 0.
$$

**Corollary 3.2.4.** Let $T_1$ and $T_2$ be tori over $k$. Then

$$
\text{pr}_T^* : \text{Pic}(M_{T_1}) \oplus \text{Pic}(M_{T_2}) \to \text{Pic}(M_{T_1 \times T_2})
$$

is a closed immersion of commutative group schemes over $k$.

**Proof.** As before, let $\Lambda_{T_1}$, $\Lambda_{T_2}$ and $\Lambda_{T_1 \times T_2}$ denote the cocharacter lattices. Then

$$
\text{pr}_T^* : \text{Hom}(\Lambda_{T_1}, J_C) \oplus \text{Hom}(\Lambda_{T_2}, J_C) \to \text{Hom}(\Lambda_{T_1 \times T_2}, J_C)
$$

is an isomorphism, and the homomorphism of discrete abelian groups

$$
\text{pr}_{T_1}^* \oplus \text{pr}_{T_2}^* : \text{NS}(M_{T_1}) \oplus \text{NS}(M_{T_2}) \to \text{NS}(M_{T_1 \times T_2})
$$

is injective by Definition 3.2.2.\qed

## 4. The twisted simply connected case

Throughout most of this section, the reductive group $G$ over $k$ will be simply connected. Using the work of Faltings [12] on the Picard functor of $M_G$, we describe here the Picard functor of the twisted moduli stacks $M_{G,L}$ introduced in [2]. In the case $G = \text{SL}_n$, these are moduli stacks of vector bundles with fixed determinant; their construction in general is recalled in Subsection 4.2 below.

The result, proved in that subsection as Proposition 4.2.1, is essentially the same: for almost simple $G$, line bundles on $M_{G,L}$ are classified by an integer, their so-called central charge. The main tool for that are as usual algebraic loop groups; what we need about them is collected in Subsection 4.1.

For later use, we need to keep track of the functoriality in $G$, in particular of the pullback to a maximal torus $T_G$ in $G$. To state this more easily, we translate the central charge into a Weyl–invariant symmetric bilinear form on the cocharacter lattice of $T_G$, replacing each integer by the corresponding multiple of the basic inner product. This allows to describe the pullback to $T_G$ in Proposition 4.4.7 (iii). Along the way, we also consider the pullback along representations of $G$: these just correspond to the pullback of bilinear forms, which reformulates — and generalises to arbitrary characteristic — the usual multiplication by the Dynkin index [18]. Subsection 4.3 describes these pullback maps combinatorially in terms of the root system, and Subsection 4.4 proves that these combinatorial maps actually give the pullback of line bundles on these moduli stacks.

### 4.1. Loop groups

Let $G$ be a reductive group over $k$. We denote

- by $LG$ the algebraic loop group of $G$, meaning the group ind–scheme over $k$ whose group of $A$–valued points for any $k$–algebra $A$ is $G(A((t)))$,
- by $L^+G \subseteq LG$ the subgroup with $A$–valued points $G(A[[t]]) \subseteq G(A((t)))$,
- and for $n \geq 1$, by $L^{\leq n}G \subseteq L^+G$ the kernel of the reduction modulo $t^n$. 
Note that $L^+G$ and $L^{\geq n}G$ are affine group schemes over $k$. The $k$-algebra corresponding to $L^{\geq n}G$ is the inductive limit over all $N > n$ of the $k$–algebras corresponding to $L^{\geq n}G/L^{\geq N}$. A similar statement holds for $L^+G$.

If $X$ is anything defined over $k$, let $X_S$ denote its pullback to a $k$–scheme $S$.

**Lemma 4.1.1.** Let $S$ be a reduced scheme over $k$. For $n \geq 1$, every morphism $\varphi : (L^{\geq n}G)_S \to (\mathbb{G}_m)_S$ of group schemes over $S$ is trivial.

**Proof.** This follows from the fact that $L^{\geq n}G$ is pro–unipotent; more precisely:

As $S$ is reduced, the claim can be checked on geometric points $\text{Spec}(k') \to S$. Replacing $k$ by the larger algebraically closed field $k'$ if necessary, we may thus assume $S = \text{Spec}(k)$; then $\varphi$ is a morphism $L^{\geq n}G \to \mathbb{G}_m$.

Since the $k$–algebra corresponding to $\mathbb{G}_m$ is finitely generated, it follows that $\varphi$ factors through $L^{\geq n}G/L^{\geq n}$ for some $N > n$. Denoting by $\mathfrak{g}$ the Lie algebra of $G$, [H II, §4, Theorem 3.5] provides an exact sequence

$$1 \to L^{\geq n}G \to L^{\geq n-1}G \to \mathfrak{g} \to 1.$$ 

Here $\varphi$ restricts to a character of $\mathfrak{g}$, which has to vanish; thus $\varphi$ also factors through $L^{\geq n}G/L^{\geq n-1}$. Iterating this argument shows that $\varphi$ is trivial. \hfill $\square$

**Lemma 4.1.2.** Suppose that the reductive group $G$ is simply connected, in particular semisimple. If a central extension of group schemes over $k$

$$1 \to \mathbb{G}_m \to \mathcal{H} \xrightarrow{\pi} L^+G \to 1$$

splits over $L^{\geq n}G$ for some $n \geq 1$, then it splits over $L^+G$.

**Proof.** Let a splitting over $L^{\geq n}G$ be given, i.e. a homomorphism of group schemes $\sigma : L^{\geq n}G \to \mathcal{H}$ such that $\pi \circ \sigma = \text{id}$. Given points $h \in \mathcal{H}(S)$ and $g \in L^{\geq n}G(S)$ for some $k$-scheme $S$, the two elements

$$h \cdot \sigma(g) \cdot h^{-1} \quad \text{and} \quad \sigma(\pi(h) \cdot g \cdot \pi(h)^{-1})$$

in $\mathcal{H}(S)$ have the same image under $\pi$, so their difference is an element in $\mathbb{G}_m(S)$, which we denote by $\varphi_h(g)$. Sending $h$ and $g$ to $h$ and $\varphi_h(g)$ defines a morphism

$$\varphi : (L^{\geq n}G)_\mathcal{H} \to (\mathbb{G}_m)_\mathcal{H}$$

of group schemes over $\mathcal{H}$. Since $L^+G/L^{\geq 1}G \cong G$ and $L^{\geq n-1}G/L^{\geq N}G$ is also smooth. Thus the limit $L^+G$ is reduced, so $\mathcal{H}$ is reduced as well. Using the previous lemma, it follows that $\varphi$ is the constant map 1; in other words, $\sigma$ commutes with conjugation. $\sigma$ is a closed immersion because $\pi \circ \sigma$ is, so $\sigma$ is an isomorphism onto a closed normal subgroup, and the quotient is a central extension

$$1 \to \mathbb{G}_m \to \mathcal{H}/\sigma(L^{\geq n}G) \to L^+G/L^{\geq n}G \to 1.$$ 

If $n \geq 2$, then this restricts to a central extension of $L^{\geq n-1}G/L^{\geq n}G \cong \mathfrak{g}$ by $\mathbb{G}_m$. It can be shown that any such extension splits.

(Indeed, the unipotent radical of the extension projects isomorphically to the quotient $\mathfrak{g}$. Note that the unipotent radical does not intersect the subgroup $\mathbb{G}_m$, and the quotient by the subgroup generated by the unipotent radical and $\mathbb{G}_m$ is reductive, so this this reductive quotient being a quotient of $\mathfrak{g}$ is in fact trivial.)

Therefore, the image of a section $\mathfrak{g} \to \mathcal{H}/\sigma(L^{\geq n}G)$ has an inverse image in $\mathcal{H}$ which $\pi$ maps isomorphically onto $L^{\geq n-1}G \subseteq L^+G$. Hence the given central
extension \((2)\) splits over \(L^{\geq n-1}G\) as well. Repeating this argument, we get a splitting over \(L^{\geq 1}G\), and finally also over \(L^+G\), because every central extension of \(L^+G/L^{\geq 1}G \cong G\) by \(\mathbb{G}_m\) splits as well, \(G\) being simply connected.

(To prove the last assertion, for any extension \(\tilde{G}\) of \(G\) by \(\mathbb{G}_m\), consider the commutator subgroup \([\tilde{G}, \tilde{G}]\) of \(\tilde{G}\). It projects surjectively to the commutator subgroup of \(G\) which is \(\tilde{G}\) itself. Since \([\tilde{G}, \tilde{G}]\) is connected and reduced, and \(G\) is simply connected, this surjective morphism must be an isomorphism.) □

4.2. **Descent from the affine Grassmannian.** Let \(G\) be a reductive group over \(k\). We denote by \(\text{Gr}_G\) the affine Grassmannian of \(G\), i.e. the quotient \(LG/L^+G\) in the category of fppf–sheaves. Given a point \(p \in C(k)\) and a uniformising element \(z \in \hat{\mathcal{O}}_{C,p}\), there is a standard 1–morphism

\[
\text{glue}_{p,z} : \text{Gr}_G \longrightarrow \mathcal{M}_G
\]

that sends each coset \(f \cdot L^+G\) to the trivial \(G\)–bundles over \(C \setminus \{p\}\) and over \(\hat{\mathcal{O}}_{C,p}\), glued by the automorphism \(f(z)\) of the trivial \(G\)–bundle over the intersection; cf. for example [21, Section 3], [12, Corollary 16], or [13, Proposition 3].

For the rest of this subsection, we assume that \(G\) is simply connected, hence semisimple. In this case, \(\text{Gr}_G\) is known to be an ind–scheme over \(k\). More precisely, [12, Theorem 8] implies that \(\text{Gr}_G\) is an inductive limit of projective Schubert varieties over \(k\), which are reduced and irreducible. Thus the canonical map

\[
(3) \quad \text{pr}_2^* : \Gamma(S, \mathcal{O}_S) \longrightarrow \Gamma(\text{Gr}_G \times S, \mathcal{O}_{\text{Gr}_G \times S})
\]

is an isomorphism for every scheme \(S\) of finite type over \(k\).

Define the Picard functor \(\text{Pic}(\text{Gr}_G)\) from schemes of finite type over \(k\) to abelian groups as in definition 2.1.1. The following theorem about it is proved in full generality in [12]. Over \(k = \mathbb{C}\), the group \(\text{Pic}(\text{Gr}_G)\) is also determined in [23] as well as in [18], and \(\text{Pic}(\mathcal{M}_G)\) is determined in [21] together with [28].

**Theorem 4.2.1** (Faltings). Let \(G\) be simply connected and almost simple.

i) \(\text{Pic}(\text{Gr}_G) \cong \mathbb{Z}\).

ii) \(\text{glue}_{p,z}^* : \text{Pic}(\mathcal{M}_G) \longrightarrow \text{Pic}(\text{Gr}_G)\) is an isomorphism of functors.

The purpose of this subsection is to carry part (ii) over to twisted moduli stacks in the sense of [2]; cf. also the first remark on page 67 of [12]. More precisely, let an exact sequence of reductive groups

\[
1 \longrightarrow G \longrightarrow \hat{G} \xrightarrow{\text{det}} \mathbb{G}_m \longrightarrow 1
\]

be given, and a line bundle \(L\) on \(C\). We denote by \(\mathcal{M}_{\hat{G},L}\) the moduli stack of principal \(\hat{G}\)–bundles \(E\) on \(C\) together with an isomorphism \(\text{det}_* E \cong L\); cf. section 2 of [2]. If for example the given exact sequence is

\[
1 \longrightarrow \text{SL}_n \longrightarrow \text{GL}_n \xrightarrow{\text{det}} \mathbb{G}_m \longrightarrow 1,
\]

then \(\mathcal{M}_{\text{GL}_n,L}\) is the moduli stack of vector bundles with fixed determinant \(L\).

In general, the stack \(\mathcal{M}_{\hat{G},L}\) comes with a 2–cartesian diagram

\[
\begin{array}{ccc}
\mathcal{M}_{\hat{G},L} & \longrightarrow & \mathcal{M}_G \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{L} & \mathcal{M}_{\mathbb{G}_m}
\end{array}
\]
from which we see in particular that $\mathcal{M}_{\hat{G}, L}$ is algebraic. It satisfies the following variant of the Drinfeld–Simpson uniformisation theorem [9, Theorem 3].

**Lemma 4.2.2.** Let a point $p \in C(k)$ and a principal $\hat{G}$--bundle $\mathcal{E}$ on $C \times S$ for some $k$--scheme $S$ be given. Every trivialisation of the line bundle $dt_\ast \mathcal{E}$ over $(C \setminus \{p\}) \times S$ can étalement--locally in $S$ be lifted to a trivialisation of $\mathcal{E}$ over $(C \setminus \{p\}) \times S$.

**Proof.** The proof in [9] carries over to this situation as follows. Choose a maximal torus $T \subseteq \hat{G}$. Using [9, Theorem 1], we may assume that $\mathcal{E}$ comes from a principal $T_{\hat{G}}$--bundle; cf. the first paragraph in the proof of [9, Theorem 3]. Arguing as in the third paragraph of that proof, we may change this principal $T_{\hat{G}}$--bundle by the extension of $G_m$--bundles along coroots $G_m \rightarrow T_{\hat{G}}$. Since simple coroots freely generate the kernel $T_G$ of $T_{\hat{G}} \rightarrow G_m$, we can thus achieve that this $T_{\hat{G}}$--bundle is trivial over $(C \setminus \{p\}) \times S$. Because $G_m$ is a direct factor of $T_{\hat{G}}$, we can hence lift the given trivialisation to the $T_{\hat{G}}$--bundle, and hence also to $\mathcal{E}$. □

Let $d \in \mathbb{Z}$ be the degree of $L$. Since $dt$ in (4) maps the (reduced) identity component $Z^0 \cong G_m$ of the center in $\hat{G}$ surjectively onto $G_m$, there is a $Z^0$--bundle $\xi$ (of degree 0) on $C$ with $dt_\ast (\xi) \otimes \mathcal{O}_C(dp) \cong L$; tensoring with it defines an equivalence

$$t_\xi : \mathcal{M}_{\hat{G}, \mathcal{O}_C(dp)} \xrightarrow{\sim} \mathcal{M}_{\hat{G}, L}.$$ 

Choose a homomorphism $\delta : G_m \rightarrow \hat{G}$ with $dt \circ \delta = d \in \mathbb{Z} = \text{Hom}(G_m, G_m)$. We denote by $t^d \in LG(k)$ the image of the tautological loop $t \in LG_m(k)$ under $\delta_\ast : LG_m \rightarrow LG$. The map

$$t^d \cdot : \text{Gr}_{\hat{G}} \rightarrow \text{Gr}_{\hat{G}}$$

sends, for each point $f$ in $LG$, the coset $f \cdot L^+ G$ to the coset $t^d f \cdot L^+ \hat{G}$. Its composition $\text{Gr}_G \rightarrow \mathcal{M}_{\hat{G}}$ with glue$_{p,z}$ factors naturally through a 1–morphism

$$\text{glue}_{p,z, \delta} : \text{Gr}_G \rightarrow \mathcal{M}_{\hat{G}, \mathcal{O}_C(dp)},$$

because $dt_\ast \circ (t^d \cdot) : LG \rightarrow LG_m \rightarrow LG_m$ is the constant map $t^d$, which via gluing yields the line bundle $\mathcal{O}_C(dp)$. Lemma 4.2.2 provides local sections of glue$_{p,z, \delta}$. These show in particular that

$$\text{glue}_{p,z, \delta}^* : \Gamma(\mathcal{M}_{\hat{G}, \mathcal{O}_C(dp)}, \mathcal{O}_{\mathcal{M}_{\hat{G}, \mathcal{O}_C(dp)}}) \xrightarrow{\sim} \Gamma(\text{Gr}_G, \mathcal{O}_{\text{Gr}_G})$$

is injective. Hence both spaces of sections contain only the constants, since $\Gamma(\text{Gr}_G, \mathcal{O}_{\text{Gr}_G}) = k$ by equation (3). Using the above equivalence $t_\xi$, this implies

(5) $$\Gamma(\mathcal{M}_{\hat{G}, L}, \mathcal{O}_{\mathcal{M}_{\hat{G}, L}}) = k.$$ 

**Proposition 4.2.3.** Let $G$ be simply connected and almost simple. Then

$$\text{glue}_{p,z, \delta}^* : \text{Pic}(\mathcal{M}_{\hat{G}, \mathcal{O}_C(dp)}) \xrightarrow{\sim} \text{Pic}(\text{Gr}_G)$$

is an isomorphism of functors.

**Proof.** $LG$ acts on $\text{Gr}_G$ by multiplication from the left. Embedding the $k$--algebra $\mathcal{O}_{C \setminus \{p\}} := \Gamma(C \setminus \{p\}, \mathcal{O}_C)$ into $k((t))$ via the Laurent development at $p$ in the variable $t = z$, we denote by $L_{C \setminus \{p\}} G \subseteq LG$ the subgroup with $A$--valued points $G(A \otimes_k \mathcal{O}_{C \setminus \{p\}}) \subseteq G(A((t)))$ for any $k$--algebra $A$. The map glue$_{p,z}$ is a torsor under $L_{C \setminus \{p\}} G$;
According to Theorem 4.2.1(ii), this line bundle admits a Picard functor, because $\text{Pic}^\ast$ is a homomorphism of Picard groups. Hence it is also surjective as a morphism of total space $\text{Mum}_{\mathbb{G}} \twoheadrightarrow (\mathbb{G}_m)_S$, since the map $[\mathfrak{g}]$ is bijective. But $L_{\mathbb{C} \setminus \{p\}}^\delta G$ is isomorphic to $L_{\mathbb{C} \setminus \{p\}} G$, and every character $(L_{\mathbb{C} \setminus \{p\}} G)_S \rightarrow (\mathbb{G}_m)_S$ is trivial according to [12] p. 66f. This already shows that the morphism of Picard functors $\text{glue}^\ast_{\mathbb{P}_\mathbb{C}}$ is injective.

The action of $LG$ on $\text{Gr}_G$ induces the trivial action on $\text{Pic}(\text{Gr}_G) \cong \mathbb{Z}$, for example because it preserves ampleness, or alternatively because $LG$ is connected. Let a line bundle $\mathcal{L}$ on $\text{Gr}_G$ be given. We denote by $\text{Mum}_{\mathbb{G}}(\mathcal{L})$ the Mumford group. So $\text{Mum}_{\mathbb{G}}(\mathcal{L})$ is the functor from schemes of finite type over $k$ to groups that sends $S$ to the group of pairs $(f, g)$ consisting of an element $f \in LG(S)$ and an isomorphism $g : f^* \mathcal{L}_S \cong \mathcal{L}_S$ of line bundles on $\text{Gr}_G \times S$.

If $f = 1$, then $g \in \mathbb{G}_m(S)$ due to the bijectivity of $[\mathfrak{g}]$, while for arbitrary $f \in LG(S)$, the line bundles $\mathcal{L}_S$ and $f^* \mathcal{L}_S$ have the same image in $\text{Pic}(\text{Gr}_G)(S)$, implying that $\mathcal{L}_S$ and $f^* \mathcal{L}_S$ are Zariski–locally in $S$ isomorphic. Consequently, we have a short exact sequence of sheaves in the Zariski topology

\begin{equation}
1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Mum}_{\mathbb{G}}(\mathcal{L}) \longrightarrow LG \longrightarrow 1.
\end{equation}

This central extension splits over $L^+ G \subseteq LG$, because the restricted action of $L^+ G$ on $\text{Gr}_G$ has a fixed point. We have to show that it also splits over $L_{\mathbb{C} \setminus \{p\}}^\delta G \subseteq LG$.

Note that $L_{\mathbb{C} \setminus \{p\}}^\delta G = \gamma (L_{\mathbb{C} \setminus \{p\}} G)$ for the automorphism $\gamma$ of $LG$ given by conjugation with $t^5$. Hence it is equivalent to show that the central extension

\begin{equation}
1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Mum}_{\mathbb{G}}(\mathcal{L}) \longrightarrow LG \longrightarrow 1
\end{equation}

splits over $L_{\mathbb{C} \setminus \{p\}} G$. We know already that it splits over $\gamma^{-1}(L^+ G)$, in particular over $L^{\geq n} G$ for some $n \geq 1$. Thus it also splits over $L^+ G$, due to Lemma 4.1.2. Hence it comes from a line bundle on $LG/L^+ G = \text{Gr}_G$ (which associated $\mathbb{G}_m$–bundle has total space $\text{Mum}_{\mathbb{G}}(\mathcal{L})/L^+ G$, where $L^+ G$ acts from the right via the splitting). According to Theorem 4.2.1, this line bundle admits a $L_{\mathbb{C} \setminus \{p\}} G$–linearisation, and hence the extension splits indeed over $L_{\mathbb{C} \setminus \{p\}} G$.

Thus the extension (6) splits over $L_{\mathbb{C} \setminus \{p\}}^\delta G$, so $\mathcal{L}$ admits an $L_{\mathbb{C} \setminus \{p\}}^\delta G$–linearisation and consequently descends to $\mathcal{M}_{\mathbb{G}}(\mathcal{L})$. This proves that $\text{glue}^\ast_{\mathbb{P}_\mathbb{C}}$ is surjective as a homomorphism of Picard groups. Hence it is also surjective as a morphism of Picard functors, because $\text{Pic}(\text{Gr}_G) \cong \mathbb{Z}$ is discrete by Theorem 4.2.1(i).

\begin{remark} Put $G^{\text{ad}} := G/Z$, where $Z \subseteq G$ denotes the center. Given a representation $\rho : G^{\text{ad}} \rightarrow \text{SL}(V)$, we denote its compositions with the canonical
epimorphisms $G \to G^\text{ad}$ and $\hat{G} \to G^\text{ad}$ also by $\rho$. Then the diagram

$$
\begin{array}{ccc}
\text{Pic}(\mathcal{M}_{\text{SL}(V)}) & \xrightarrow{\text{glue}_{p,z}} & \text{Pic}(\text{Gr}_{\text{SL}(V)}) \\
\rho^* & & \rho^* \\
\text{Pic}(\mathcal{M}_{\hat{G},L}) & \xrightarrow{(t_\xi \circ \text{glue}_{p,z,\delta})^*} & \text{Pic}(\text{Gr}_{\hat{G}})
\end{array}
$$

commutes.

**Proof.** Let $t^{\rho \circ \delta} \in L_{\text{SL}(V)}$ denote the image of the canonical loop $t \in L_{G_m}$ under the composition $\rho \circ \delta : G_m \to \text{SL}(V)$. Then the left part of the diagram

$$
\begin{array}{ccc}
\text{Gr}_{\text{SL}(V)} & \xrightarrow{\text{glue}_{p,z,\delta}} & \mathcal{M}_{G,\text{C}(dp)} \\
\rho^* & & t_\xi \\
\text{Gr}_{\hat{G}} & \xrightarrow{t_\xi \circ \text{glue}_{p,z}} & \mathcal{M}_{\hat{G}} \\
\rho^* & & \rho^* \\
\text{Gr}_{\text{SL}(V)} & \xrightarrow{\text{glue}_{p,z}} & \mathcal{M}_{\text{SL}(V)}
\end{array}
$$

commutes. The four remaining squares are 2–commutative by construction of the 1–morphisms glue$_{p,z,\delta}$, glue$_{p,z}$ and $t_\xi$. Applying Pic to the exterior pentagon yields the required commutative square, as $L_{\text{SL}(V)}$ acts trivially on Pic(Gr$_{\text{SL}(V)}$).

**□**

### 4.3. Néron–Severi groups \(\text{NS}(\mathcal{M}_G)\) for simply connected \(G\)

Let $G$ be a reductive group over $k$; later in this subsection, we will assume that $G$ is simply connected. Choose a maximal torus $T_G \subseteq G$, and let

\[
\text{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^W
\]

denote the abelian group of bilinear forms $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z}$ that are invariant under the Weyl group $W = W_G$ of $(G, T_G)$.

Up to a *canonical* isomorphism, the group \(\text{(8)}\) does not depend on the choice of $T_G$. More precisely, let $T'_G \subseteq G$ be another maximal torus; then the conjugation $\gamma_g : G \to G$ with some $g \in G(k)$ provides an isomorphism from $T_G$ to $T'_G$, and the induced isomorphism from $\text{Hom}(\Lambda_{T'_G} \otimes \Lambda_{T'_G}, \mathbb{Z})^W$ to $\text{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^W$ does not depend on the choice of $g$.

The group \(\text{(8)}\) is also functorial in $G$. More precisely, let $\varphi : G \to H$ be a homomorphism of reductive groups over $k$. Choose a maximal torus $T_H \subseteq H$ containing $\varphi(T_G)$.

**Lemma 4.3.1.** Let $T'_G \subseteq G$ be another maximal torus, and let $T'_H \subseteq H$ be a maximal torus containing $\varphi(T'_G)$. For every $g \in G(k)$ with $T'_G = \gamma_g(T_G)$, there is an $h \in H(k)$ with $T'_H = \gamma_h(T_H)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
T_G & \xrightarrow{\gamma_g} & T'_G \\
\varphi & & \varphi \\
T_H & \xrightarrow{\gamma_h} & T'_H
\end{array}
$$
Proof. The diagram

\[
\begin{array}{ccc}
T_G & \xrightarrow{\gamma_g} & T'_G \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
T_H & \xrightarrow{\gamma_{\varphi(g)}^{-1}} & T'_H \\
\end{array}
\]

allows us to assume \( T'_G = T_G \) and \( g = 1 \) without loss of generality. Then \( T_H \) and \( T'_H \) are maximal tori in the centraliser of \( \varphi(T_G) \), which is reductive according to \([13] \text{ 26.2. Corollary A}\). Thus \( T'_H = \gamma_h(T_H) \) for an appropriate \( k \)-point \( h \) of this centraliser, and \( \gamma_h \circ \varphi = \varphi \) on \( T_G \) by definition of the centraliser.

Applying the lemma with \( T'_G = T_G \) and \( T'_H = T_H \), we see that the pullback along \( \varphi : \Lambda_{T_G} \to \Lambda_{T_H} \) of a \( W_H \)-invariant form \( \Lambda_{T_H} \otimes \Lambda_{T_H} \to \mathbb{Z} \) is \( W_G \)-invariant, so we get an induced map

\[
\varphi^* : \text{Hom}(\Lambda_{T_H} \otimes \Lambda_{T_H}, \mathbb{Z})^{W_H} \to \text{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^{W_G}
\]

which does not depend on the choice of \( T_G \) and \( T_H \) by the above lemma again.

For the rest of this subsection, we assume that \( G \) and \( H \) are simply connected.

Definition 4.3.2. i) The \( \text{Néron–Severi group} \) \( \text{NS}(\mathcal{M}_G) \) is the subgroup

\[
\text{NS}(\mathcal{M}_G) \subseteq \text{Hom}(\Lambda_{\mathcal{M}_G} \otimes \Lambda_{T_G}, \mathbb{Z})^{W_G}
\]

of symmetric forms \( b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z} \) with \( b(\lambda \otimes \lambda) \in 2\mathbb{Z} \) for all \( \lambda \in \Lambda_{T_G} \).

ii) Given a homomorphism \( \varphi : G \to H \), we denote by

\[
\varphi^* : \text{NS}(\mathcal{M}_H) \to \text{NS}(\mathcal{M}_G)
\]

the restriction of the induced map \( \varphi^* \) in (9).

Remarks 4.3.3. i) If \( G = G_1 \times G_2 \) for simply connected groups \( G_1 \) and \( G_2 \), then

\[
\text{NS}(\mathcal{M}_G) = \text{NS}(\mathcal{M}_{G_1}) \oplus \text{NS}(\mathcal{M}_{G_2}),
\]

since each element of \( \text{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^{W_G} \) vanishes on \( \Lambda_{T_{G_1}} \otimes \Lambda_{T_{G_2}} + \Lambda_{T_{G_2}} \otimes \Lambda_{T_{G_1}} \).

ii) If on the other hand \( G \) is almost simple, then

\[
\text{NS}(\mathcal{M}_G) = \mathbb{Z} \cdot b_G
\]

where the \textit{basic inner product} \( b_G \) is the unique element of \( \text{NS}(\mathcal{M}_G) \) that satisfies \( b_G(\alpha^\vee, \alpha^\vee) = 2 \) for all short coroots \( \alpha^\vee \in \Lambda_{T_G} \) of \( G \).

iii) Let \( G \) and \( H \) be almost simple. The \textit{Dynkin index} \( d_{\varphi} \in \mathbb{Z} \) of a homomorphism \( \varphi : G \to H \) is defined by \( \varphi^*(b_H) = d_{\varphi} \cdot b_G \), cf. \([10] \text{ §2}\). If \( \varphi \) is nontrivial, then \( d_{\varphi} > 0 \), since \( b_G \) and \( b_H \) are positive definite.

Let \( Z \subseteq G \) be the center. Then \( G_{\text{ad}} := G/Z \) contains \( T_{G_{\text{ad}}} := T_G/Z \) as a maximal torus, with cocharacter lattice \( \Lambda_{T_{G_{\text{ad}}}} \subseteq \Lambda_{T_G} \otimes \mathbb{Q} \).

We say that a homomorphism \( l : \Lambda \to \Lambda' \) between finitely generated free abelian groups \( \Lambda \) and \( \Lambda' \) is \textit{integral} on a subgroup \( \bar{\Lambda} \subseteq \Lambda \otimes \mathbb{Q} \) if its restriction to \( \Lambda \cap \bar{\Lambda} \) admits a linear extension \( \bar{l} : \Lambda \to \Lambda' \). By abuse of language, we will not distinguish between \( l \) and its unique linear extension \( \bar{l} \).

Lemma 4.3.4. Every element \( b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z} \) of \( \text{NS}(\mathcal{M}_G) \) is integral on \( \Lambda_{T_{G_{\text{ad}}}} \otimes \Lambda_{T_G} \) and on \( \Lambda_{T_G} \otimes \Lambda_{T_{G_{\text{ad}}}} \).
Proof. Let \( \alpha : \Lambda_{T_G} \otimes \mathbb{Q} \to \mathbb{Q} \) be a root of \( G \), with corresponding coroot \( \alpha^\vee \in \Lambda_{T_G} \).

Lemme 2 in [4, Chapitre VI, §1] implies the formula

\[
b(\lambda \otimes \alpha^\vee) = \alpha(\lambda) \cdot b(\alpha^\vee \otimes \alpha^\vee)/2
\]

for all \( \lambda \in \Lambda_{T_G} \). Thus \( b(\ast \otimes \alpha^\vee) : \Lambda_{T_G} \to \mathbb{Z} \) is an integer multiple of \( \alpha \); hence it is integral on \( \Lambda_{T_G} \), the largest subgroup of \( \Lambda_{T_G} \otimes \mathbb{Q} \) on which all roots are integral. But the coroots \( \alpha^\vee \) generate \( \Lambda_{T_G} \), as \( G \) is simply connected.

Now let \( \iota_G : T_G \hookrightarrow G \) denote the inclusion of the chosen maximal torus.

**Definition 4.3.5.** Given \( \delta \in \Lambda_{T_G} \), the homomorphism

\[
(\iota_G)^{NS,\delta} : NS(M_G) \to NS(M_{T_G})
\]

sends \( b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z} \) to

\[
b(-\delta \otimes -\delta) : \Lambda_{T_G} \to \mathbb{Z} \quad \text{and} \quad \text{id}_{J_C} \cdot b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \text{End} \, J_C.
\]

This map \((\iota_G)^{NS,\delta}\) is injective if \( g_C \geq 1 \), because all multiples of \( \text{id}_{J_C} \) are then nonzero in \( \text{End} \, J_C \). If \( g_C = 0 \), then \( \text{End} \, J_C = 0 \), but we still have the following

**Lemma 4.3.6.** Every coset \( d \in \Lambda_{T_G} / \Lambda_{T_G} = \pi_1(G^{ad}) \) admits a lift \( \delta \in \Lambda_{T_G} \) such that the map \((\iota_G)^{NS,\delta} : NS(M_G) \to NS(M_{T_G})\) is injective.

Proof. Using Remark 4.3.3 we may assume that \( G \) is almost simple. In this case, \((\iota_G)^{NS,\delta}\) is injective whenever \( \delta \neq 0 \), because \( NS(M_G) \) is cyclic and its generator \( b_G : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z} \) is as a bilinear form nondegenerate. \( \square \)

**Remark 4.3.7.** Given \( \varphi : G \to H \), let \( \iota_H : T_H \hookrightarrow H \) be a maximal torus with \( \varphi(T_G) \subseteq T_H \). If \( \delta \in \Lambda_{T_G} \), or if more generally \( \delta \in \Lambda_{T_G}^{ad} \) is mapped to \( \Lambda_{H^{ad}} \) by \( \varphi_* : \Lambda_{T_G} \otimes \mathbb{Q} \to \Lambda_{H} \otimes \mathbb{Q} \), then the following diagram commutes:

\[
\begin{array}{ccc}
NS(M_H) & \xrightarrow{(\iota_H)^{NS,\varphi_*\delta}} & NS(M_{T_H}) \\
\varphi_* \downarrow & & \varphi_* \downarrow \\
NS(M_G) & \xrightarrow{(\iota_G)^{NS,\delta}} & NS(M_{T_G})
\end{array}
\]

4.4. The pullback to torus bundles. Let \( \mathcal{L}^{\det} = \mathcal{L}_{\mathbb{Q}_{\text{det}}} \) be determinant of cohomology line bundle \[10\] on \( \mathcal{M}_{G_{m}} \), whose fibre at a vector bundle \( E \) on \( C \) is \( \det \mathbb{H}^0(E) = \det \mathbb{H}^0(E) \otimes \det \mathbb{H}^1(E)^{\text{dual}} \).

**Lemma 4.4.1.** Let \( \xi \) be a line bundle of degree \( d \) on \( C \). Then the composition

\[
\text{Pic}(\mathcal{M}_{G_{m}}) \xrightarrow{\iota_G^*} \text{Pic}(\mathcal{M}_{G_{m}}^{0}) \xrightarrow{\text{can}} \text{NS}(\mathcal{M}_{G_{m}}) = \mathbb{Z} \oplus \text{End}_{J_C}
\]

maps \( \mathcal{L}^{\det} \) to \( 1 - g_C + d \in \mathbb{Z} \) and \( -\text{id}_{J_C} \in \text{End}_{J_C} \).

Proof. For any line bundle \( L \) on \( C \) and any point \( p \in C(k) \), we have a canonical exact sequence

\[
0 \to L(-p) \to L \to L_p \to 0
\]

of coherent sheaves on \( C \). Varying \( L \) and taking the determinant of cohomology, we see that the two line bundles \( \mathcal{L}^{\det} \) and \( t_{\mathcal{L}}^{\mathcal{L}} \mathcal{L}^{\det} \) on \( \mathcal{M}_{G_{m}}^{0} \) have the same image in the second summand \( \text{End} \, J_C \) of \( \text{NS}(\mathcal{M}_{G_{m}}) \). Thus the image of \( t_{\mathcal{L}}^{\mathcal{L}} \mathcal{L}^{\det} \) in \( \text{End} \, J_C \) does not depend on \( \xi \); this image is \( -\text{id}_{J_C} \) because the principal polarisation \( \phi_\Theta : J_C \to J_C^\vee \) is essentially given by the dual of the line bundle \( \mathcal{L}^{\det} \).
The weight of \( t_\xi^* \mathcal{L}_{\text{det}} \) at a line bundle \( L \) of degree 0 on \( C \) is the Euler characteristic of \( L \otimes \xi \), which is indeed \( 1 - g_C + d \) by Riemann–Roch theorem.

Let \( \iota : T_{\text{SL}_n} \to \text{SL}_n \) be the inclusion of the maximal torus \( T_{\text{SL}_n} := G^n_m \cap \text{SL}_n \), where \( G^n_m \subseteq \text{GL}_n \) as diagonal matrices. Then the cocharacter lattice \( \Lambda_{T_{\text{SL}_n}} \) is the group of all \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) with \( d_1 + \cdots + d_n = 0 \). The basic inner product \( b_{\text{SL}_n} : \Lambda_{T_{\text{SL}_n}} \otimes \Lambda_{T_{\text{SL}_n}} \to \mathbb{Z} \) is the restriction of the standard scalar product on \( \mathbb{Z}^n \).

**Corollary 4.4.2.** Let \( \xi \) be a principal \( T_{\text{SL}_n} \)-bundle of degree \( d \in \Lambda_{T_{\text{SL}_n}} \) on \( C \). Then the composition

\[
\text{Pic}(\mathcal{M}_{T_{\text{SL}_n}}) \xrightarrow{c} \text{Pic}(\mathcal{M}_{T_{\text{SL}_n}}^0) \xrightarrow{t_\xi^*} \text{Pic}(\mathcal{M}_{T_{\text{SL}_n}}^0)^{\mathcal{L}_{\text{det}}} \to \text{NS}(\mathcal{M}_{T_{\text{SL}_n}})
\]

maps \( \mathcal{L}_{\text{det}} \) to \( b_{\text{SL}_n}(d \otimes \cdot) : \Lambda_{T_{\text{SL}_n}} \to \mathbb{Z} \) and \( \text{id}_{J_C} : \Lambda_{T_{\text{SL}_n}} \otimes \Lambda_{T_{\text{SL}_n}} \to \text{End} J_C \).

**Proof.** Since the determinant of cohomology takes direct sums to tensor products, the pullback of \( \mathcal{L}^\text{det}_{n} \) to \( \mathcal{M}_{G^n_m} \) is isomorphic to \( \text{pr}_1^* \mathcal{L}_{\text{det}} \otimes \cdots \otimes \text{pr}_{n-1}^* \mathcal{L}_{\text{det}} \), where \( \text{pr}_\nu : G^n_m \to G_m \) is the projection onto the \( \nu \)th factor. Now use the previous lemma to compute the image of \( \mathcal{L}^\text{det}_n \) in \( \text{NS}(\mathcal{M}_{G^n_m}) \) and then restrict to \( \text{NS}(\mathcal{M}_{T_{\text{SL}_n}}) \). □

**Corollary 4.4.3.** If \( \rho : \text{SL}_2 \to \text{SL}(V) \) has Dynkin index \( d_\rho \), then the pullback \( \rho^* : \text{Pic}(\mathcal{M}_{\text{SL}(V)}) \to \text{Pic}(\mathcal{M}_{\text{SL}_2}) \) maps \( \mathcal{L}_{\text{det}} \) to \( (\mathcal{L}_{\text{det}}^\text{det})^{\otimes d_\rho} \).

**Proof.** Let \( \iota : T_{\text{SL}_2} \hookrightarrow \text{SL}_2 \) be the inclusion of the maximal torus that contains the image of the standard torus \( T_{\text{SL}_2} \hookrightarrow \text{SL}_2 \). The diagram

\[
\text{Pic}(\mathcal{M}_{\text{SL}(V)}) \xrightarrow{\iota^*} \text{Pic}(\mathcal{M}_{T_{\text{SL}_2}(V)}) \xrightarrow{\iota^* \rho^*} \text{Pic}(\mathcal{M}_{T_{\text{SL}_2}(V)}^{\mathcal{L}_{\text{det}}}) \xrightarrow{\text{Pic}(\mathcal{M}_{T_{\text{SL}_2}(V)}^{\mathcal{L}_{\text{det}}})^\rho} \text{NS}(\mathcal{M}_{T_{\text{SL}_2}(V)})
\]

commutes for each principal \( T_{\text{SL}_2} \)-bundle \( \xi \) on \( C \). We choose \( \xi \) in such a way that \( \deg(\xi) \in \Lambda_{T_{\text{SL}_2}} \cong \mathbb{Z} \) is nonzero if \( g_{T_{\text{SL}_2}} = 0 \). Then the composition

\[
c_{T_{\text{SL}_2}} \circ t_\xi^* \circ \iota^* : \text{Pic}(\mathcal{M}_{\text{SL}_2}) \longrightarrow \text{NS}(\mathcal{M}_{T_{\text{SL}_2}})
\]

of the lower row is injective according to Theorem 4.2.1 and Corollary 4.4.2. The latter moreover implies that the two elements \( \rho^*(\mathcal{L}_{\text{det}}) \) and \( (\mathcal{L}_{\text{det}})^{\otimes d_\rho} \) in \( \text{Pic}(\mathcal{M}_{\text{SL}_2}) \) have the same image in \( \text{NS}(\mathcal{M}_{T_{\text{SL}_2}}) \).

Now suppose that the reductive group \( G \) is simply connected and almost simple. We denote by \( \mathcal{O}_{\text{Gr}_G}(1) \) the unique generator of \( \text{Pic}(\text{Gr}_G) \) that is ample on every closed subscheme, and by \( \mathcal{O}_{\text{Gr}_G}(n) \) its \( n \)th tensor power for \( n \in \mathbb{Z} \).

Over \( k = \mathbb{C} \), the following is proved by a different method in section 5 of [15].

**Proposition 4.4.4 (Kumar-Narasimhan-Ramanathan).** If \( \rho : G \to \text{SL}(V) \) has Dynkin index \( d_\rho \), then \( \rho^* : \text{Pic}(\text{Gr}_{\text{SL}(V)}) \to \text{Pic}(\text{Gr}_G) \) maps \( \mathcal{O}(1) \) to \( \mathcal{O}_{\text{Gr}_G}(d_\rho) \).

**Proof.** Let \( \varphi : \text{SL}_2 \to G \) be given by a short coroot. Then \( d_\varphi = 1 \) by definition, and [12] implies that \( \varphi^* : \text{Pic}(\text{Gr}_G) \to \text{Pic}(\text{Gr}_{\text{SL}(V)}) \) maps \( \mathcal{O}(1) \) to \( \mathcal{O}(1) \), for example because \( \varphi^* : \text{Pic}(\mathcal{M}_G) \to \text{Pic}(\mathcal{M}_{\text{SL}_2}) \) preserves central charges according to their definition [12] p. 59]. Hence it suffices to prove the claim for \( \rho \circ \varphi \) instead of \( \rho \). This case follows from Corollary 4.4.3 since \( \text{glue}_{p,z}^*(\mathcal{L}_{\text{det}}^\text{det}) = \mathcal{O}_{\text{Gr}_{\text{SL}(V)}}(-1) \). □
As in Subsection 4.2, we assume given an exact sequence of reductive groups
\[ 1 \rightarrow G \rightarrow \hat{G} \xrightarrow{dt} \mathbb{G}_m \rightarrow 1 \]
with \( G \) simply connected, and a line bundle \( L \) on \( C \).

**Corollary 4.4.5.** Suppose that \( G \) is almost simple. Then the isomorphism
\[
(t_\xi \circ \text{glue}_{p,z,\delta})^* : \text{Pic}(\mathcal{M}_{\hat{G},L}) \rightarrow \text{Pic}(	ext{Gr}_G)
\]
constructed in Subsection 4.2 does not depend on the choice of \( p, z, \xi \) or \( \delta \).

We say that a line bundle on \( \mathcal{M}_{\hat{G},L} \) has **central charge** \( n \in \mathbb{Z} \) if this isomorphism maps it to \( O_{\text{Gr}_G}(n) \); this is consistent with the standard central charge of line bundles on \( \mathcal{M}_G \), as defined for example in \cite{12}.

**Proof.** If \( \rho : G \rightarrow \text{SL}(V) \) is a nontrivial representation, then \( d_\rho > 0 \), as explained in Remark 4.3.3(iii). Using Proposition 4.4.4, this implies that \( \rho^* : \text{Pic}(	ext{Gr}_{\text{SL}(V)}) \rightarrow \text{Pic}(	ext{Gr}_G) \) is injective. Due to Remark 4.2.4, it thus suffices to check that
\[
\text{glue}_{p,z}^* : \text{Pic}(\mathcal{M}_{\text{SL}(V)}) \rightarrow \text{Pic}(\text{Gr}_{\text{SL}(V)})
\]
does not depend on \( p \) or \( z \). This is clear, since it maps \( L_{\text{det}} \) to \( O_{\text{Gr}_{\text{SL}(V)}}(-1) \). \( \square \)

The chosen maximal torus \( \iota_G : T_G \hookrightarrow G \) induces maximal tori \( \iota_{\hat{G}} : T_{\hat{G}} \hookrightarrow \hat{G} \) and \( \iota_{G^{\text{ad}}} : T_{G^{\text{ad}}} \hookrightarrow G^{\text{ad}} \) compatible with the canonical maps \( G \hookrightarrow \hat{G} \hookrightarrow G^{\text{ad}} \). Given a principal \( T_{\hat{G}} \)-bundle \( \hat{\xi} \) on \( C \) and an isomorphism \( dt_* \hat{\xi} \cong L \), the composition
\[
\mathcal{M}_{T_G} \xrightarrow{\iota_{\hat{\xi}}} \mathcal{M}_{T_{\hat{G}}} \xrightarrow{(\iota_G)_*} \mathcal{M}_{\hat{G}}
\]
factors naturally through a 1–morphism
\[
(10) \quad \iota_{\hat{\xi}} : \mathcal{M}^0_{T_G} \rightarrow \mathcal{M}_{\hat{G},L}.
\]

**Remark 4.4.6.** Given a representation \( \rho : G^{\text{ad}} \rightarrow \text{SL}(V) \), let \( \iota : T_{\text{SL}(V)} \hookrightarrow \text{SL}(V) \) be a maximal torus containing \( \rho(T_{G^{\text{ad}}}) \). Then the diagram
\[
\begin{array}{ccc}
\mathcal{M}^0_{T_G} & \xrightarrow{\iota_{\hat{\xi}}} & \mathcal{M}_{\hat{G},L} \\
\parallel & & \parallel \\
\mathcal{M}^0_{T_{\text{SL}(V)}} & \xrightarrow{\iota_{\rho_*\hat{\xi}}} & \mathcal{M}_{T_{\text{SL}(V)}} \xrightarrow{\iota_*} \mathcal{M}_{\text{SL}(V)}
\end{array}
\]
is 2–commutative, by construction of \( \iota_{\hat{\xi}} \).

**Proposition 4.4.7.**

i) \( \Gamma(\mathcal{M}_{\hat{G},L}, \mathcal{O}_{\mathcal{M}_{\hat{G},L}}) = k \).

ii) There is a canonical isomorphism
\[
e_G : \text{Pic}(\mathcal{M}_{\hat{G},L}) \sim \text{NS}(\mathcal{M}_G).
\]
iii) For all choices of $\iota_G : T_G \rightarrow G$ and of $\hat{\xi}$, the diagram

$$\begin{array}{ccc}
\text{Pic}(\mathcal{M}_{\hat{G},L}) & \xrightarrow{\iota_G^*} & \text{Pic}(\mathcal{M}_G^0) \\
\downarrow{c_G} & & \downarrow{c_{T_G}} \\
\text{NS}(\mathcal{M}_G) & \xrightarrow{(\iota_G)^{\text{NS,} \delta}} & \text{NS}(\mathcal{M}_{T_G})
\end{array}$$

commutes; here $\delta \in \Lambda_{T_G^{\text{ad}}}$ denotes the image of $\hat{\delta} := \deg \hat{\xi} \in \Lambda_{T_G^{\text{ad}}}$.

Proof. We start with the special case that $G$ is almost simple. Here part (i) of the proposition is just equation (5) from Subsection 4.2.

We let $c_G$ send the line bundle of central charge 1 to the basic inner product $b_G \in \text{NS}(\mathcal{M}_G)$. Due to Theorem 4.2.1(i), Proposition 4.2.3, Corollary 4.4.5 and Remark 4.3.3(ii), this defines a canonical isomorphism, and hence proves (ii).

To see that the diagram in (iii) then commutes, we choose a nontrivial representation $\rho : G^{\text{ad}} \rightarrow \text{SL}(V)$. We note the functorialities, with respect to $\rho$, according to Remark 4.4.6, Remark 4.2.4, Proposition 4.4.4, Remark 4.3.7 and Remark 3.2.3. In view of these, comparing Corollary 4.4.2 and Definition 4.3.5 shows that the two images of $\rho^* \mathcal{L}^{\text{det}} \in \text{Pic}(\mathcal{M}_{\hat{G},L})$ in $\text{NS}(\mathcal{M}_{T_G})$ coincide. Since the former generates a subgroup of finite index and the latter is torsionfree, the diagram in (iii) commutes.

For the general case, we use the unique decomposition

$$G = G_1 \times \cdots \times G_r$$

into simply connected and almost simple factors $G_i$. As $\hat{G}$ is generated by its center and $G$, every normal subgroup in $G$ is still normal in $\hat{G}$. Let $\hat{G}_i$ denote the quotient of $\hat{G}$ modulo the closed normal subgroup $\prod_{j \neq i} G_j$; then

$$\begin{array}{cccc}
0 & \rightarrow & G & \rightarrow & \hat{G} & \rightarrow & G_m & \rightarrow & 0 \\
& & \downarrow{\text{pr}_i} & & \downarrow{\text{dt}_i} & & \downarrow{\text{diag}} & & \downarrow{0} \\
0 & \rightarrow & G_i & \rightarrow & \hat{G}_i & \rightarrow & G_m & \rightarrow & 0
\end{array}$$

is a morphism of short exact sequences. Since the resulting diagram

$$\begin{array}{ccc}
\hat{G} & \rightarrow & \prod_i \hat{G}_i \\
\text{dt} & \downarrow & \text{diag} \\
G_m & \rightarrow & G_m
\end{array}$$

is cartesian, it induces an equivalence of moduli stacks

$$\mathcal{M}_{\hat{G},L} \xrightarrow{\sim} \mathcal{M}_{\hat{G}_1,L} \times \cdots \times \mathcal{M}_{\hat{G}_r,L}$$

due to Lemma 2.2.1. We note that equation (5), Lemma 2.1.2(i), Lemma 2.1.3, Remark 4.3.3(i) and Corollary 3.2.4 ensure that various constructions are compatible with the products in (11). Therefore, the general case follows from the already treated almost simple case. $\square$
5. The reductive case

In this section, we finally describe the Picard functor $\text{Pic}(\mathcal{M}_G^d)$ for any reductive group $G$ over $k$ and any $d \in \pi_1(G)$. We denote

- by $\zeta : Z^0 \hookrightarrow G$ the (reduced) identity component of the center $Z \subseteq G$, and
- by $\pi : \tilde{G} \rightarrow G$ the universal cover of $G' := [G, G] \subseteq G$.

Our strategy is to descend along the central isogeny

$\zeta \cdot \pi : Z^0 \times \tilde{G} \rightarrow G,$

applying the previous two sections to $Z^0$ and to $\tilde{G}$, respectively. The 1–morphism of moduli stacks given by such a central isogeny is a torsor under a group stack; Subsection 5.1 explains descent of line bundles along such torsors, generalising the method introduced by Laszlo [20] for quotients of $\text{SL}_n$. In Subsection 5.2, we define combinatorially what will be the discrete torsionfree part of $\text{Pic}(\mathcal{M}_G^d)$; finally, these Picard functors and their functoriality in $G$ are described in Subsection 5.3.

The following notation is used throughout this section. The reductive group $G$ yields semisimple groups and central isogenies $\tilde{G} \rightarrow G' \rightarrow \bar{G} := G/Z_0 \rightarrow G_{\text{ad}} := G/Z$.

We denote by $\tilde{\bar{G}} \subseteq \pi_1(\bar{G})$ the image of $\tilde{d} \in \pi_1(G)$. The choice of a maximal torus $\iota : T_G \hookrightarrow G$ induces maximal tori and isogenies

$T_G 
\rightarrow T_G' 
\rightarrow T_{\bar{G}} 
\rightarrow T_{G_{\text{ad}}}.$

Their cocharacter lattices are hence subgroups of finite index

$\Lambda_{T_G} 
\hookrightarrow \Lambda_{T_G'} 
\hookrightarrow \Lambda_{T_{\bar{G}}} 
\hookrightarrow \Lambda_{T_{G_{\text{ad}}}}.$

The central isogeny $\zeta \cdot \pi$ makes $\Lambda_{Z^0} \oplus \Lambda_{T_{\bar{G}}}$ a subgroup of finite index in $\Lambda_{T_G}$.

5.1. Torsors under a group stack. All stacks in this subsection are stacks over $k$, and all morphisms are over $k$. Following [6, 20], we recall the notion of a torsor under a group stack.

Let $\mathcal{G}$ be a group stack. We denote by 1 the unit object in $\mathcal{G}$, and by $g_1, g_2$ the image of two objects $g_1$ and $g_2$ under the multiplication 1–morphism $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$.

Definition 5.1.1. An action of $\mathcal{G}$ on a 1–morphism of stacks $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ consists of a 1–morphism

$\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}, 
(g, x) \mapsto g \cdot x,$

and of three 2–morphisms, which assign to each $k$–scheme $S$ and each object

- $x$ in $\mathcal{X}(S)$ an isomorphism $1 \cdot x \sim x$ in $\mathcal{X}(S)$,
- $(g, x)$ in $(\mathcal{G} \times \mathcal{X})(S)$ an isomorphism $\Phi(g \cdot x) \sim \Phi(x)$ in $\mathcal{Y}(S)$,
- $(g_1, g_2, x)$ in $(\mathcal{G} \times \mathcal{G} \times \mathcal{X})(S)$ an isomorphism $(g_1 \cdot g_2) \cdot x \sim g_1 \cdot (g_2 \cdot x)$ in $\mathcal{X}(S)$. 

These morphisms are required to satisfy the following five compatibility conditions:

\begin{align*}
(g \cdot 1) \cdot x & \sim \rightarrow g \cdot x \text{ in } \mathcal{X}(S), \\
(1 \cdot g) \cdot x & \sim \rightarrow g \cdot x \text{ in } \mathcal{X}(S), \\
\Phi(1 \cdot x) & \sim \rightarrow \Phi(x) \text{ in } \mathcal{Y}(S), \\
\Phi((g_1 \cdot g_2) \cdot x) & \sim \rightarrow \Phi(x) \text{ in } \mathcal{Y}(S), \\
(g_1 \cdot g_2 \cdot g_3) \cdot x & \sim \rightarrow g_1 \cdot (g_2 \cdot (g_3 \cdot x)) \text{ in } \mathcal{X}(S),
\end{align*}

and coincide for all \( k \)-schemes \( S \) and all objects \( g, g_1, g_2, g_3 \) in \( G(S) \) and \( x \) in \( \mathcal{X}(S) \).

**Example 5.1.2.** Let \( \varphi : G \rightarrow H \) be a homomorphism of linear algebraic groups over \( k \), and let \( Z \) be a closed subgroup in the center of \( G \) with \( Z \subseteq \ker(\varphi) \).

Then the group stack \( M_Z \) acts on the 1–morphism \( \varphi^* : M_G \rightarrow M_H \) via the tensor product \( \otimes : M_Z \times M_G \rightarrow M_G \).

From now on, we assume that the group stack \( G \) is algebraic.

**Definition 5.1.3.** A \( G \)-torsor is a faithfully flat 1–morphism of algebraic stacks \( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) together with an action of \( G \) on \( \Phi \) such that the resulting 1–morphism

\[ (g \cdot x) \mapsto \Phi(g \cdot x) \text{ in } \mathcal{Y}(S), \]

is an isomorphism.

**Example 5.1.4.** Suppose that \( \varphi : G \twoheadrightarrow H \) is a central isogeny of reductive groups with kernel \( \mu \). For each \( d \in \pi_1(G) \), the 1–morphism

\[ \varphi_* : \mathcal{M}^\mu_G \rightarrow \mathcal{M}_H, \quad e := \varphi_*(d) \in \pi_1(H) \]

is a torsor under the group stack \( \mathcal{M}_\mu \), for the action described in example 5.1.2.

**Proof.** The 1–morphism \( \varphi_* \) is faithfully flat by Lemma 2.2.2. The 1–morphism

\[ \mathcal{M}_\mu \times \mathcal{M}_G \rightarrow \mathcal{M}_G \times_{\mathcal{M}_H} \mathcal{M}_G, \quad (L, E) \mapsto (L \otimes E, E) \]

is an isomorphism due to Lemma 2.2.1. Since \( \varphi_* : \pi_1(G) \rightarrow \pi_1(H) \) is injective, \( \mathcal{M}^\mu_G \subseteq \mathcal{M}_G \) is the inverse image of \( \mathcal{M}_H^\mu \subseteq \mathcal{M}_H \) under \( \varphi_* \); hence the restriction

\[ \mathcal{M}_\mu \times \mathcal{M}^\mu_G \rightarrow \mathcal{M}^\mu_G \times_{\mathcal{M}_H^\mu} \mathcal{M}_G^\mu \]

is an isomorphism as well. \( \square \)

**Definition 5.1.5.** Let \( \Phi_\nu : \mathcal{X}_\nu \rightarrow \mathcal{Y}_\nu \) be a \( G \)-torsor for \( \nu = 1, 2 \). A morphism of \( G \)-torsors from \( \Phi_1 \) to \( \Phi_2 \) consists of two 1–morphisms

\[ A : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \quad \text{and} \quad B : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2 \]

and of two 2–morphisms, which assign to each \( k \)-scheme \( S \) and each object

\[ x \text{ in } \mathcal{X}_1(S) \quad \text{an isomorphism } \Phi_2A(x) \sim \rightarrow B\Phi_1(x) \text{ in } \mathcal{Y}_2(S), \]
\[ (g, x) \text{ in } (G \times \mathcal{X}_1)(S) \quad \text{an isomorphism } A(g \cdot x) \sim \rightarrow g \cdot A(x) \text{ in } \mathcal{X}_2(S). \]
These morphisms are required to satisfy the following three compatibility conditions: the two resulting isomorphisms
\[ A((1 \cdot x)) \sim \Phi_2 A(g \cdot x) \sim B \Phi_1 (x) \text{ in } X_2(S) \]
and \( A((g_1 \cdot g_2) \cdot x) \sim g_1 \cdot (g_2 \cdot A(x)) \text{ in } X_2(S) \) coincide for all \( k \)-schemes \( S \) and all objects \( g, g_1, g_2 \) in \( G(S) \) and \( x \) in \( X(S) \).

**Example 5.1.6.** Let a cartesian square of reductive groups over \( k \)

\[
\begin{array}{c}\begin{array}{c}G_1 \ra{\alpha} G_2 \\
\downarrow{\varphi_1} & \downarrow{\varphi_2} \\
H_1 \ra{\beta} H_2
\end{array}\end{array}
\]

be given. Suppose that \( \varphi_1 \) and \( \varphi_2 \) are central isogenies, and denote their common kernel by \( \mu \). For each \( d_1 \in \pi_1(G_1) \), the diagram

\[
\begin{array}{c}\begin{array}{c}M_{G_1}^{d_1} \ra{\alpha_*} M_{G_2}^{d_2} \\
\downarrow{\varphi_1} \downarrow{\varphi_2} \\
M_{H_1}^{\mu} \ra{\beta_*} M_{H_2}^{\mu}
\end{array}\end{array}
\]

\( d_2 := \alpha_*(d_1) \in \pi_1(G_2) \)

is then a morphism of torsors under the group stack \( M_\mu \).

**Proposition 5.1.7.** Let a \( G \)-torsor \( \Phi_\nu : X_\nu \ra Y_\nu \) with \( \Gamma(X_\nu, O_{X_\nu}) = k \) be given for \( \nu = 1, 2 \), together with a morphism of \( G \)-torsors

\[
\begin{array}{c}\begin{array}{c}X_1 \ra{\Phi_1} X_2 \\
\downarrow{\phi_1} & \downarrow{\phi_2} \\
Y_1 \ra{\phi_2} Y_2
\end{array}\end{array}
\]

such that the induced morphism of Picard functors \( A^* : \Pic(X_2) \ra \Pic(X_1) \) is injective. Then the diagram of Picard functors

\[
\begin{array}{c}\begin{array}{c}\Pic(X_1) \ra{A^*} \Pic(X_2) \\
\downarrow{\Phi_1} & \downarrow{\Phi_2} \\
\Pic(Y_1) \ra{B^*} \Pic(Y_2)
\end{array}\end{array}
\]

is a pullback square.

**Proof.** The proof of [20, Theorem 5.7] generalises to this situation as follows.

Let \( S \) be a scheme of finite type over \( k \). For a line bundle \( L \) on \( S \times X_\nu \), we denote by \( \Lin^G(L) \) the set of its \( G \)-linearisations, cf. [20, Definition 2.8]. According to Lemma [2.1.2(i)], each automorphism of \( L \) comes from \( \Gamma(S, O_{X_\nu}^*) \) and hence respects each linearisation of \( L \). Thus [20, Theorem 4.1] provides a canonical bijection between the set \( \Lin^G(L) \) and the fibre of

\[ \Phi_\nu^* : \Pic(S \times Y_\nu) \ra \Pic(S \times X_\nu) \]

over the isomorphism class of \( L \).
Let $\mathcal{T}$ be an algebraic stack over $k$. We denote for the moment by $\text{Pic}(\mathcal{T})$ the groupoid of line bundles on $\mathcal{T}$ and their isomorphisms. Lemma 2.1.2(i) and Corollary 2.1.3 show that the functor $A^*: \text{Pic}(\mathcal{T} \times X_2) \to \text{Pic}(\mathcal{T} \times X_1)$ is fully faithful for every $\mathcal{T}$. We recall that an element in $\text{Lin}_G(L)$ is an isomorphism in $\text{Pic}(G \times S \times X_\nu)$ between two pullbacks of $L$ such that certain induced diagrams in $\text{Pic}(S \times X_\nu)$ and in $\text{Pic}(G \times G \times S \times X_\nu)$ commute. Thus it follows for all $L \in \text{Pic}(S \times X_2)$ that the canonical map $A^*: \text{Lin}_G(L) \to \text{Lin}_G(A^*L)$ is bijective. Hence the diagram of abelian groups

\[
\begin{array}{ccc}
\text{Pic}(S \times X_1) & \xrightarrow{A^*} & \text{Pic}(S \times X_2) \\
\uparrow \phi_1 & & \uparrow \phi_2 \\
\text{Pic}(S \times Y_1) & \xleftarrow{B^*} & \text{Pic}(S \times Y_2)
\end{array}
\]

is a pullback square, as required. □

5.2. Néron–Severi groups $\text{NS}(M^d_G)$ for reductive $G$.

**Definition 5.2.1.** The Néron–Severi group $\text{NS}(M^d_G)$ is the subgroup

\[\text{NS}(M^d_G) \subseteq \text{NS}(M^d) \oplus \text{NS}(M_G)\]

of all triples $l_Z : \Lambda_{Z_0} \to \mathbb{Z}$, $b_Z : \Lambda_{Z_0} \otimes \Lambda_{Z_0} \to \text{End} \ J_C$ and $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z}$ with the following properties:

1. For every lift $\vec{\delta} \in \Lambda_{T_G}$ of $\vec{d} \in \pi_1(G)$, the direct sum

\[l_Z \oplus b(-\vec{\delta} \otimes \_): \Lambda_{Z_0} \oplus \Lambda_{T_G} \to \mathbb{Z}\]

is integral on $\Lambda_{T_G}$.

2. The orthogonal direct sum

\[b_Z \perp (\text{id}_{J_C} \cdot b) : (\Lambda_{Z_0} \oplus \Lambda_{T_G}) \otimes (\Lambda_{Z_0} \oplus \Lambda_{T_G}) \to \text{End} \ J_C\]

is integral on $\Lambda_{T_G} \otimes \Lambda_{T_G}$.

**Lemma 5.2.2.** If condition 2 above holds for one lift $\vec{\delta} \in \Lambda_{T_G}$ of $\vec{d} \in \pi_1(G)$, then it holds for every lift $\vec{\delta} \in \Lambda_{T_G}$ of the same element $\vec{d} \in \pi_1(G)$.

**Proof.** Any two lifts $\vec{\delta}$ of $\vec{d}$ differ by some element $\lambda \in \Lambda_{T_G}$. Lemma 4.3.4 states in particular that

\[b(-\lambda \otimes \_): \Lambda_{T_G} \to \mathbb{Z}\]

is integral on $\Lambda_{T_G}$, and hence admits an extension $\Lambda_{T_G} \to \mathbb{Z}$ that vanishes on $\Lambda_{Z_0}$. □

**Remark 5.2.3.** If $G$ is simply connected, then $\text{NS}(M^d_G)$ coincides with the group $\text{NS}(M_G)$ of definition 4.3.2. If $G = T$ is a torus, then $\text{NS}(M^d_T)$ coincides for all $d \in \pi_1(T)$ with the group $\text{NS}(M_T)$ of definition 3.2.1.

**Remark 5.2.4.** The Weyl group $W$ of $(G, T_G)$ acts trivially on $\text{NS}(M^d_G)$. Hence the group $\text{NS}(M^d_G)$ does not depend on the choice of $T_G$; cf. Subsection 4.3.
Definition 5.2.5. Given a lift \( \delta \in \Lambda_{T_G} \) of \( d \in \pi_1(G) \), the homomorphism 
\[
(i_G)^{NS, \delta} : NS(M^d_G) \longrightarrow NS(M^{\delta}_{T_G})
\]
sends \((l_Z, b_Z) \in NS(M^d_{Z^0}) \) and \( b \in NS(M^d_G) \) to the pair
\[
l_Z \oplus b(-\delta \otimes \cdot) : \Lambda_G \longrightarrow \mathbb{Z} \quad \text{and} \quad b_Z \perp (id_{J_G} \cdot b) : \Lambda_{T_G} \otimes \Lambda_T \longrightarrow \text{End} J_C
\]
where \( \delta \in \Lambda_{T_G} \) denotes the image of \( \delta \).

Note that this definition agrees with the earlier definition \([1.3.5]\) in the cases covered by both, namely \( G \) simply connected and \( \delta \in \Lambda_{T_G} \).

Lemma 5.2.6. Given a lift \( \delta \in \Lambda_{T_G} \) of \( d \in \pi_1(G) \), the diagram
\[
\begin{array}{c}
\text{NS}(M^d_G) \xrightarrow{(i_G)^{NS, \delta}} \text{NS}(M^{\delta}_{T_G}) \\
\downarrow \quad \downarrow \\
\text{NS}(M^d_{Z^0}) \oplus \text{NS}(M^d_G) \xrightarrow{id \oplus (i_G)^{NS, \delta}} \text{NS}(M^d_{Z^0}) \oplus \text{NS}(M^{\delta}_{T_G}) \\
\end{array}
\]
is a pullback square; here \( \tilde{\delta} \in \Lambda_{T_G} \) again denotes the image of \( \delta \).

Proof. This follows directly from the definitions. \( \square \)

Let \( e \in \pi_1(H) \) be the image of \( d \in \pi_1(G) \) under a homomorphism of reductive groups \( \varphi : G \longrightarrow H \). \( \varphi \) induces a map \( \varphi : G \longrightarrow H \) between the universal covers of their commutator subgroups. If \( \varphi \) maps the identity component \( Z^0_G \) in the center \( Z_G \) of \( G \) to the center \( Z_H \) of \( H \), then it induces an obvious pullback map
\[
\varphi^* : NS(M^\delta_H) \longrightarrow NS(M^d_G)
\]
which sends \( l_Z, b_Z \) and \( b \) simply to \( \varphi^* l_Z, \varphi^* b_Z \) and \( \varphi^* b \). This is a special case of the following map, which induces even without the hypothesis on the centers, and which also generalises the previous definition \([5.2.5]\).

Definition 5.2.7. Choose a maximal torus \( \iota_H : T_H \hookrightarrow H \) containing \( \varphi(T_G) \), and a lift \( \delta \in \Lambda_{T_H} \) of \( d \in \pi_1(G) \); let \( \iota \in \Lambda_{T_H} \) be the image of \( \delta \). Then the map
\[
\varphi^{NS, \delta} : NS(M^\iota_H) \longrightarrow NS(M^d_G)
\]
sends \((l_Z, b_Z, b) \in NS(M^\iota_H) \) and \( b \in NS(M^d_G) \) to the pullback along \( \varphi : Z^0_G \longrightarrow T_H \) of \((\iota_H)^{NS, \delta}(l_Z, b_Z, b) \in NS(M_{T_H}) \), together with \( \varphi^* b \in NS(M_G) \).

Lemma 5.2.8. The map \( \varphi^{NS, \delta} \) does not depend on the choice of \( T_G, T_H \) or \( \delta \).

Proof. Let \( W_G \) denote the Weyl group of \((G, T_G)\). It acts trivially on \( \Lambda_{Z^0_G} \), and without nontrivial coinvariants on \( \Lambda_{T_G} \); these two observations imply
\[
(13) \quad \text{Hom}(\Lambda_{T_G} \otimes \Lambda_{Z^0_G}, \mathbb{Z})^{W_G} = 0.
\]

Lemma \([1.3.4]\) states that \( b \) is integral on \( \Lambda_{T_H} \otimes \Lambda_{T_H} \); its composition with the canonical projection \( \Lambda_{T_H} \longrightarrow \Lambda_{T_H} \) is a Weyl–invariant map \( b_r : \Lambda_{T_H} \otimes \Lambda_{T_H} \longrightarrow \mathbb{Z} \).

As explained in Subsection \([2.3]\) Lemma \([1.3.1]\) implies that \( \varphi^* b_r : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z} \) is still Weyl–invariant; hence it vanishes on \( \Lambda_{T_G} \otimes \Lambda_{Z^0} \) by \((13)\).

Any two lifts \( \delta \) of \( d \) differ by some element \( \lambda \in \Lambda_{T_G} \); then the two images of \((l_Z, b_Z, b) \in NS(M^\iota_H) \) in \( NS(M_{T_H}) \) differ, according to the proof of Lemma \([5.2.2]\) only by \( b_r(-\lambda \otimes \cdot) : \Lambda_{T_H} \longrightarrow \mathbb{Z} \). Thus their compositions with \( \varphi : \Lambda_{Z^0_G} \longrightarrow \Lambda_{T_H} \)
coincide by the previous paragraph. This shows that the two images of \((l_Z, b_Z, b)\) have the same component in the direct summand \(\text{Hom}(\Lambda_{T_G}', \mathbb{Z})\) of \(\text{NS}(\mathcal{M}_{T_G}')\); since the other two components do not involve \(\delta\) at all, the independence on \(\delta\) follows.

The independence on \(T_G\) and \(T_H\) is then a consequence of Lemma 4.3.1, since the Weyl groups \(W_G\) and \(W_H\) act trivially on \(\text{NS}(\mathcal{M}_{T_G}')\) and on \(\text{NS}(\mathcal{M}_{T_H}')\).

**Lemma 5.2.9.** For all maximal tori \(\iota_G : T_G \hookrightarrow G\) and \(\iota_H : T_H \hookrightarrow H\) with \(\varphi(T_G) \subseteq T_H\), and all lifts \(\delta \in \Lambda_{T_G}\) of \(d \in \pi_1(G)\), the diagram

\[
\begin{array}{ccc}
\text{NS}(\mathcal{M}_{T_H}') & \xrightarrow{(\iota_H)^{\text{NS}, \eta}} & \text{NS}(\mathcal{M}_{T_H}) \\
\varphi^{\text{NS}, d} & & \varphi^* \\
\text{NS}(\mathcal{M}_{T_G}') & \xrightarrow{(\iota_G)^{\text{NS}, \delta}} & \text{NS}(\mathcal{M}_{T_G})
\end{array}
\]

commutes, with \(\eta := \varphi_*\delta \in \Lambda_{T_H}\) and \(e := \varphi_*d \in \pi_1(H)\) as in definition 5.2.7.

**Proof.** Given an element in \(\text{NS}(\mathcal{M}_{T_G}')\), we have to compare its two images in \(\text{NS}(\mathcal{M}_{T_H}')\). The definition 5.2.7 of \(\varphi^{\text{NS}, d}\) directly implies that both have the same pullback to \(\text{NS}(\mathcal{M}_{T_G}')\) and to \(\text{NS}(\mathcal{M}_{T_H}')\). Moreover, their components in the direct summand \(\text{Hom}^*(\Lambda_{T_G} \otimes \Lambda_{T_G} \otimes J_C)\) of \(\text{NS}(\mathcal{M}_{T_G}')\) are both Weyl–invariant due to Lemma 4.3.1 thus equation (13) above shows that these components vanish on \(\Lambda_{T_G} \otimes \Lambda_{T_G}\) and on \(\Lambda_{T_G} \otimes \Lambda_{T_G}\). Hence two images in question even have the same pullback to \(\text{NS}(\mathcal{M}_{T_G}' \times T_G)\). But \(\Lambda_{T_G} \otimes \Lambda_{T_G}\) has finite index in \(\Lambda_{T_G}\).

**Corollary 5.2.10.** Let \(\psi : H \twoheadrightarrow K\) be another homomorphism of reductive groups, and put \(f := \psi_*e \in \pi_1(K)\). Then

\[
\varphi^{\text{NS}, d} \circ \psi^{\text{NS}, e} = (\psi \circ \varphi)^{\text{NS}, d} : \text{NS}(\mathcal{M}_{K}') \longrightarrow \text{NS}(\mathcal{M}_{T_G}').
\]

**Proof.** According to the previous lemma, this equality holds after composition with \((\iota_G)^{\text{NS}, \delta} : \text{NS}(\mathcal{M}_{T_G}') \longrightarrow \text{NS}(\mathcal{M}_{T_G})\) for any lift \(\delta \in \Lambda_{T_G}\) of \(d\). Due to the Lemma 4.3.6 and Lemma 5.2.6 there is a lift \(\delta\) of \(d\) such that \((\iota_G)^{\text{NS}, \delta}\) is injective.

We conclude this subsection with a more explicit description of \(\text{NS}(\mathcal{M}_{C}')\). It turns out that genus \(g_C = 0\) is special. This generalises the description obtained for \(k = \mathbb{C}\) and \(G\) semisimple by different methods in [29, Section V].

**Proposition 5.2.11.** Let \(q : G \twoheadrightarrow G/\Gamma' := G^{ab}\) denote the maximal abelian quotient of \(G\). Then the sequence of abelian groups

\[
0 \longrightarrow \text{NS}(\mathcal{M}_{G')} \xrightarrow{q^*} \text{NS}(\mathcal{M}_{G}) \xrightarrow{\text{pr}_2} \text{NS}(\mathcal{M}_{C})
\]

is exact, and the image of the map \(\text{pr}_2\) in it consists of all forms \(b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}\) in \(\text{NS}(\mathcal{M}_{C})\) that are integral

- on \(\Lambda_{T_G} \otimes \Lambda_{T_G}\), if \(g_C \geq 1\);
- on \((\mathbb{Z}d) \otimes \Lambda_{T_G}\) for a lift \(\delta \in \Lambda_{T_G}\) of \(d \in \pi_1(G)\), if \(g_C = 0\).

The condition does not depend on the choice of this lift \(\delta\), due to Lemma 4.3.4.

**Proof.** Since \(q : Z^0 \longrightarrow G^{ab}\) is an isogeny, \(q^*\) is injective; it clearly maps into the kernel of \(\text{pr}_2\). Conversely, let \((l_Z, b_Z, b) \in \text{NS}(\mathcal{M}_{C})\) be in the kernel of \(\text{pr}_2\); this means \(b = 0\). Then condition (1) in the definition 5.2.1 of \(\text{NS}(\mathcal{M}_{C}')\) provides a map

\[
l_Z \oplus 0 : \Lambda_{T_G} \longrightarrow \mathbb{Z}
\]
which vanishes on Λ_{T_G}, and hence also on Λ_{T_{G'}}; thus it is induced from a map on Λ_{T_G}/Λ_{T_{G'}} = Λ_{G^a b}. Similarly, condition 2 in the same definition provides a map 
\[ b_Z \perp 0 \] on Λ_{T_G} ⊗ Λ_{T_G} which vanishes on Λ_{T_G} ⊗ Λ_{T_G} + Λ_{T_G} ⊗ Λ_{T_G}, and hence also on Λ_{T_{G'}} ⊗ Λ_{T_G} + Λ_{T_{G'}} ⊗ Λ_{T_G}; thus it is induced from a map on the quotient Λ_{G^a b} ⊗ Λ_{G^a b}.

This proves the exactness.

Now let \( b \in NS(M_G^d) \) be in the image of pr_2. Then \( b \) is integral on \((Z\hbar) ⊗ Λ_{G'}\) by condition 4 in definition 5.2.1 implies that

\[ 0 \oplus b : (Λ_{Z^0} ⊕ Λ_{G'}) ⊗ Λ_{T_G} \to Z \]

is injective with torsionfree cokernel; thus condition 2 in definition 5.2.1 implies that

\[ (\bar{-} \cdot id_{J_G} : Z \to \text{End} J_G \]

is injective with torsionfree cokernel; thus condition 2 in definition 5.2.1 implies that

\[ 0 \oplus b : (Λ_{Z^0} ⊕ Λ_{G'}) ⊗ Λ_{T_G} \to Z \]

is integral on Λ_{T_G} ⊗ Λ_{T_{G'}}, and hence, vanishing on Λ_{Z^0} ⊆ Λ_{T_G}, comes from a map on the quotient Λ_{T_{G'}} ⊗ Λ_{T_G}. This shows that \( b \) satisfies the stated condition.

Conversely, suppose that \( b \in NS(M_G^d) \) satisfies the stated condition. Then \( b \) is integral on \((Z\hbar) ⊗ Λ_{G'}\); since Λ_{T_{G'}} ⊆ Λ_{T_G} is a direct summand,

\[ b(-\hbar \otimes \cdot) : Λ_{T_{G'}} \to Z \]

can thus be extended to Λ_{T_G}. We restrict it to a map \( (l_Z : Λ_{Z^0} \to Z \in NS(M_G^d)) \) and hence an inverse image of \( b \).

It remains to consider \( g_C \geq 1 \). Then \( b \) is by assumption integral on \( Λ_{T_{G'}} \otimes Λ_{T_{G'}} \), so composing it with the canonical subjection \( Λ_{T_{G'}} \to Λ_{G'} \) defines a linear map Λ_{T_{G'}} ⊗ Λ_{T_{G'}} → Z. Since \( b \) is symmetric, this extends canonically to a symmetric linear map from

\[ Λ_{T_{G'}} \otimes Λ_{T_{G'}} + Λ_{T_{G'}} \otimes Λ_{T_{G}} \subseteq Λ_{T_{G'}} \otimes Λ_{T_{G}} \]

to \( Z \). It can be extended further to a symmetric linear map from Λ_{T_{G'}} ⊗ Λ_{T_{G}} to \( Z \), because Λ_{T_{G'}} ⊆ Λ_{T_G} is a direct summand. Multiplying it with \( id_{J_G} \) and restricting to \( Λ_{Z^0} \) defines an element \( b_Z \in \text{Hom}^* (Λ_{Z^0} ⊗ Λ_{Z^0}, \text{End} J_G) \). By construction, the triple \( (l_Z, b_Z, b) \) is in NS(M_G^d) and hence an inverse image of \( b \).

In particular, the free abelian group NS(M_G^d) has rank

\[ \text{rk} \text{NS}(M_G^d) = r \cdot r \cdot \text{rk} J_G + \frac{r(r-1)}{2} \cdot \text{rk} \text{End} J_G + s \]

if \( G^a b \cong G^r_m \) is a torus of rank \( r \), and \( G^{ad} \) contains \( s \) simple factors.

5.3. Proof of the main result.

**Theorem 5.3.1.**

i) \( \Gamma(M_G^d, O_{M_G^d}) = k \).

ii) The functor \( \text{Pic}(M_G^d) \) is representable by a \( k \)-scheme locally of finite type.

iii) There is a canonical exact sequence

\[ 0 \to \text{Hom}(\pi_1(G), J_G) \to \text{Pic}(M_G^d) \to \text{NS}(M_G^d) \to 0 \]

of commutative group schemes over \( k \).

iv) For every homomorphism of reductive groups \( \varphi : G \to H \), the diagram

\[
\begin{array}{ccc}
0 & \to & \text{Hom}(\pi_1(G), J_G) \\
\downarrow \varphi^* & & \downarrow \varphi^* \\
0 & \to & \text{Hom}(\pi_1(H), J_G)
\end{array}
\]

commutes.
commutes; here \( e := \varphi_*(d) \in \pi_1(H) \).

Proof. We record for later use the commutative square of abelian groups

\[
\begin{array}{ccc}
\pi_1(G) & \xrightarrow{pr} & \Lambda_{T_G} \\
\downarrow{\zeta_*} & & \downarrow{(\zeta \pi)_*} \\
\Lambda_{Z^0} & \xrightarrow{pr_1} & \Lambda_{Z^0 \times T_G}.
\end{array}
\]

The mapping cone of this commutative square

\[
(14) \quad 0 \to \Lambda_{Z^0} \oplus \Lambda_{T_G} \to \Lambda_{Z^0} \oplus \Lambda_{T_G} \to \pi_1(G) \to 0
\]

is exact, because its subsequence \( 0 \to \Lambda_{T_G} \to \Lambda_{Z^0} \to \pi_1(G) \to 0 \) is exact, and the resulting sequence of quotients \( 0 \to \Lambda_{Z^0} = \Lambda_{Z^0} \to 0 \to 0 \) is also exact.

Lemma 5.3.2. There is an exact sequence of reductive groups

\[
(15) \quad 1 \to \hat{G} \to \hat{G} \xrightarrow{dt} \mathbb{G}_m \to 1
\]

and an extension \( \hat{\pi} : \hat{G} \to G \) of \( \pi : \hat{G} \to G \) such that \( \hat{\pi} : \pi_1(\hat{G}) \to \pi_1(G) \) maps \( 1 \in \mathbb{Z} = \pi_1(\mathbb{G}_m) = \pi_1(\hat{G}) \) to the given element \( d \in \pi_1(G) \).

Proof. We view the given \( d \in \pi_1(G) \) as a coset \( d \subseteq \Lambda_{T_G} \) modulo \( \Lambda_{\text{coroots}} \). Let

\[
\Lambda_{T_G} \subseteq \Lambda_{T_G} \oplus \mathbb{Z}
\]

be generated by \( \Lambda_{\text{coroots}} \oplus 0 \) and \((d, 1)\), and let

\[
(\hat{\pi}, dt) : \hat{G} \to G \times \mathbb{G}_m
\]

be the reductive group with the same root system as \( G \), whose maximal torus \( T_{\hat{G}} = \hat{\pi}^{-1}(T_G) \) has cocharacter lattice \( \text{Hom}(\mathbb{G}_m, T_{\hat{G}}) = \Lambda_{T_G} \). As \( \pi_* \) maps \( \Lambda_{T_G} \) isomorphically onto \( \Lambda_{\text{coroots}} \), we obtain an exact sequence

\[
0 \to \Lambda_{T_G} \xrightarrow{\pi_*} \Lambda_{T_G} \xrightarrow{pr_2} \mathbb{Z} \to 0,
\]

which yields the required exact sequence \((15)\) of groups. By its construction, \( \hat{\pi}_* \) maps the canonical generator \( 1 \in \pi_1(\mathbb{G}_m) = \pi_1(\hat{G}) \) to \( d \in \pi_1(G) \).

Let \( \mu \) denote the kernel of the central isogeny \( \zeta : \pi : Z^0 \times \hat{G} \to G \). Then

\[
\psi : Z^0 \times \hat{G} \to G \times \mathbb{G}_m, \quad (z^0, \hat{g}) \mapsto (\zeta(z^0), \hat{\pi}(\hat{g}), dt(\hat{g}))
\]

is by construction a central isogeny with kernel \( \mu \). Hence the induced 1–morphism

\[
\psi_* : \mathcal{M}_{Z^0}^0 \times \mathcal{M}_G^1 \to \mathcal{M}_G^0 \times \mathcal{M}_{\mathbb{G}_m}^1
\]

is faithfully flat by Lemma 2.2.2. Restricting to the point \( \text{Spec}(k) \to \mathcal{M}_G^1 \) given by a line bundle \( L \) of degree 1 on \( C \), we get a faithfully flat 1–morphism

\[
(\psi_*)_L : \mathcal{M}_{Z^0}^0 \times \mathcal{M}_{\hat{G},L} \to \mathcal{M}_{\hat{G}}^0.
\]

Since \( \Gamma(\mathcal{M}_{Z^0}^0 \times \mathcal{M}_{\hat{G},L}, O) = k \) by Proposition 4.4.7(i) and Lemma 2.1.2(i), part (i) of the theorem follows. The group stack \( \mathcal{M}_\mu \) acts by tensor product on these two 1–morphisms \( \psi_* \) and \( (\psi_*)_L \), turning both into \( \mathcal{M}_\mu \)-torsors; cf. Example 5.1.3. The idea is to descend line bundles along the torsor \( (\psi_*)_L \).
We choose a principal $T_G$–bundle $\hat{\xi}$ on $C$ together with an isomorphism of line bundles $\delta^* \hat{\xi} \cong L$. Then $\xi := \hat{\pi}_+(\hat{\xi})$ is a principal $T_G$–bundle on $C$; their degrees $\hat{\delta} := \deg(\hat{\xi}) \in \Lambda T_G$ and $\delta := \deg(\xi) \in \Lambda T_G$ are lifts of $d \in \pi_1(G)$. The diagram

\begin{equation}
\begin{array}{ccc}
Z^0 \times T_G & \xrightarrow{\text{id} \times t_G} & Z^0 \times \hat{G} \\
\psi & & \psi \\
T_G \times \mathbb{G}_m & \xrightarrow{t_G \times \text{id}} & G \times \mathbb{G}_m
\end{array}
\end{equation}

of groups induces the right square in the 2–commutative diagram

\begin{equation}
\begin{array}{cccc}
\mathcal{M}^0_{Z^0} \times \mathcal{M}^0_{T_G} & \xrightarrow{\text{id} \times t_{\hat{\xi}}} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^\hat{\xi}_{T_G} & \xrightarrow{(\text{id} \times t_G)_*} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^1_{G} \\
\psi_* & & \psi_* & & \psi_* \\
\mathcal{M}^0_{T_G} \times \mathcal{M}^0_{\mathbb{G}_m} & \xrightarrow{t_G \times \text{id}} & \mathcal{M}^0_{T_G} \times \mathcal{M}^1_{\mathbb{G}_m} & \xrightarrow{(t_G)_* \circ t_\xi} & \mathcal{M}^0_{G} \times \mathcal{M}^1_{\mathbb{G}_m}
\end{array}
\end{equation}

of moduli stacks; note that $t_{\hat{\xi}}$ and $t_\xi$ are equivalences. Restricting the outer rectangle again to the point Spec($k$) $\rightarrow$ $\mathcal{M}^1_{\mathbb{G}_m}$ given by $L$, we get the diagram

\begin{equation}
\begin{array}{ccc}
\sim \mathcal{M}^0_{Z^0 \times T_G} & \xrightarrow{\text{id} \times t_{\hat{\xi}}} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^\hat{\xi}_{T_G} & \xrightarrow{(\text{id} \times t_G)_*} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^1_{G,L} \\
\sim \mathcal{M}^0_{T_G} \xrightarrow{(t_G)_* \circ t_\xi} & \mathcal{M}^0_{G,L} \\
\sim \mathcal{M}^0_{T_G} \xrightarrow{(\text{id} \times t_G)_*} & \mathcal{M}^0_{G,L}
\end{array}
\end{equation}

containing an instance $t_{\hat{\xi}}$ of the 1–morphism (10) defined in Subsection 4.4. According to the Proposition 4.4.7 and Proposition 4.4.7, $t_{\hat{\xi}}: \text{Pic}(\mathcal{M}^0_{G,L}) \rightarrow \text{Pic}(\mathcal{M}^0_{T_G})$ is a morphism of group schemes over $k$. This morphism is a closed immersion, according to Proposition 4.4.7(iii), if $g_{G} \geq 1$ or if $\hat{\xi}$ is chosen appropriately, as explained in Lemma 4.3.6 we assume this in the sequel. Using Lemma 2.1.4 and Corollary 3.2.4 it follows that

\[(\text{id} \times t_{\hat{\xi}})_*: \text{Pic}(\mathcal{M}^0_{Z^0}) \oplus \text{Pic}(\mathcal{M}^0_{G,L}) \cong \text{Pic}(\mathcal{M}^0_{Z^0} \times \mathcal{M}^0_{G,L}) \rightarrow \text{Pic}(\mathcal{M}^0_{Z^0 \times T_G})\]

is a closed immersion of group schemes over $k$ as well.

The group stack $\mathcal{M}_G$ still acts by tensor product on the vertical 1–morphisms in (17) and in (18). Since the diagram (16) of groups is cartesian, (17) and (18) are morphisms of $G$–torsors; cf. Example 5.1.6. Proposition 5.1.7 applies to the latter morphism of torsors, yielding a cartesian square

\begin{equation}
\begin{array}{ccc}
\text{Pic}(\mathcal{M}^0_{G}) & \xrightarrow{t_{\hat{\xi}} \circ t_G^*} & \text{Pic}(\mathcal{M}^0_{T_G}) \\
\psi_\xi & & \psi_\xi \\
\text{Pic}(\mathcal{M}^0_{Z^0}) \oplus \text{Pic}(\mathcal{M}^0_{G,L}) & \xrightarrow{(\text{id} \times t_{\hat{\xi}})_*} & \text{Pic}(\mathcal{M}^0_{Z^0 \times T_G})
\end{array}
\end{equation}

of Picard functors. Thus $\text{Pic}(\mathcal{M}^0_{G})$ is representable, and $t_{\hat{\xi}} \circ t_G^*$ is a closed immersion; this proves part (ii) of the theorem.
The image of the mapping cone (14) under the exact functor $\text{Hom}(\cdot, J_C)$, and the mapping cones of the two cartesian squares given by diagram (19) and Lemma 5.2.6 are the columns of the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}(\pi_1(G), J_C) \\
\downarrow & & \downarrow \\
\text{Pic}(M^d_G) & \rightarrow & \text{NS}(M^d_G) \\
\downarrow & & \downarrow \\
\text{Hom}(\Lambda_x^0, J_C) & \oplus & \text{Pic}(M^0_{Z^0}) \\
\downarrow & & \downarrow \\
\text{Hom}(\Lambda_{T_G}, J_C) & \oplus & \text{Pic}(M^0_{T_G}) \\
\downarrow & & \downarrow \\
\text{Hom}(\Lambda_{x^0 \times T_G}, J_C) & \rightarrow & \text{NS}(M^0_{x^0 \times T_G}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

whose two rows are exact due to Proposition 5.2.2(ii) and Proposition 4.3.7(ii). Applying the snake lemma to this diagram, we get an exact sequence

\[
0 \rightarrow \text{Hom}(\pi_1(G), J_C) \xrightarrow{j_G(\iota_G, \delta)} \text{Pic}(M^d_G) \xrightarrow{c_G(\iota_G, \delta)} \text{NS}(M^d_G) \rightarrow 0.
\]

The image of $j_G(\iota_G, \delta)$ and the kernel of $c_G(\iota_G, \delta)$ are a priori independent of the choices made, since both are the largest quasicompact open subgroup in $\text{Pic}(M^d_G)$. If $G$ is a torus and $d = 0$, then this is the exact sequence of Proposition 5.2.2 in general, the construction provides a morphism of exact sequences (20)

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}(\pi_1(G), J_C) \\
\downarrow & & \downarrow \\
\text{Pic}(M^d_G) & \rightarrow & \text{NS}(M^d_G) \\
\downarrow & & \downarrow \\
\text{Hom}(\Lambda_{T_G}, J_C) & \rightarrow & \text{NS}(M^0_{T_G}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

whose three vertical maps are all injective. Using Proposition 5.2.2(iii), this implies that $j_G(\iota_G, \delta)$ and $c_G(\iota_G, \delta)$ depend at most on the choice of $\iota_G : T_G \rightarrow G$ and of $\delta$, but not on the choice of $\tilde{G}$, $L$ or $\hat{\xi}$; thus the notation. Together with the following two lemmas, this proves the remaining parts (iii) and (iv) of the theorem.

**Lemma 5.3.3.** The above map $j_G(\iota_G, \delta) : \text{Hom}(\pi_1(G), J_C) \rightarrow \text{Pic}(M^d_G)$

i) does not depend on the lift $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$,

ii) does not depend on the maximal torus $\iota_G : T_G \rightarrow G$, and

iii) satisfies $\varphi^* \circ j_H = j_G \circ \varphi^* : \text{Hom}(\pi_1(H), J_C) \rightarrow \text{Pic}(M^d_G)$ for all $\varphi : G \rightarrow H$.

**Proof.** If $G$ is a torus, then $\delta$ and $\iota_G$ are unique, so (i) and (ii) hold trivially.
The claim is empty for \( g_C = 0 \), so we assume \( g_C \geq 1 \). Then the above construction works for all lifts \( \delta \neq d \), because \( \iota_{\xi}^* \) is a closed immersion for all \( \xi \).

Given \( \varphi : G \to H \) and a maximal torus \( \iota_T : T_H \hookrightarrow H \) with \( \varphi(T_G) \subseteq T_H \), we again put \( e := \varphi_* d \in \pi_1(H) \) and \( \eta := \varphi_* \delta \in \Lambda T_H \). Then the diagram

\[
\begin{array}{ccc}
\text{Hom}(\pi_1(H), J_G) & \xrightarrow{j^H_{\iota H, \eta}} & \text{Pic}(\mathcal{M}_H^e) \\
\downarrow \varphi^* & & \downarrow \varphi^* \\
\text{Hom}(\pi_1(G), J_G) & \xrightarrow{j^G_{\iota G, \delta}} & \text{Pic}(\mathcal{M}_G^d)
\end{array}
\]

commutes, because it commutes after composition with the closed immersion

\[
t^\iota_{\xi} \circ t_G : \text{Pic}(\mathcal{M}_G^d) \to \text{Pic}(\mathcal{M}_G^{d_{\theta}})
\]

from diagram (20), using Remark 3.2.3. In particular, (iii) follows from (i) and (ii).

i) For \( G = \text{GL}_2 \), it suffices to take \( \varphi = \det : \text{GL}_2 \to \mathbb{G}_m \) in the above diagram (21), since \( \det_* : \pi_1(\text{GL}_2) \to \pi_1(\mathbb{G}_m) \) is an isomorphism.

For \( G = \text{PGL}_2 \), it then suffices to take \( \varphi = \text{pr} : \text{GL}_2 \to \text{PGL}_2 \) in the same diagram (21), since \( \text{pr}_* : \pi_1(\text{GL}_2) \to \pi_1(\text{PGL}_2) \) is surjective.

As (i) holds trivially for \( G = \text{SL}_2 \), and clearly holds for \( G \times \mathbb{G}_m \) if it holds for \( G \), this proves (i) for all groups \( G \) of semisimple rank one.

In the general case, let \( \alpha^v \in \Lambda T_G \) be a coroot, and let \( \varphi : G_\alpha \hookrightarrow G \) be the corresponding subgroup of semisimple rank one. Then the diagram (21) shows \( j_G(\iota_G, \delta) = j_G(\iota_G, \delta + \alpha^v) \), since \( \varphi_* : \pi_1(G_\alpha) \to \pi_1(G) \) is surjective. This completes the proof of i, because any two lifts \( \delta \neq d \) differ by a sum of coroots.

ii) now follows from Weyl-invariance; cf. Subsection 4.3.

Lemma 5.3.4. The above map \( c_G(\iota_G, \delta) : \text{Pic}(\mathcal{M}_G^d) \to \text{NS}(\mathcal{M}_G^d) \)

i) does not depend on the lift \( \delta \in \Lambda T_G \) of \( d \in \pi_1(G) \),

ii) does not depend on the maximal torus \( \iota_G : T_G \hookrightarrow G \), and

iii) satisfies \( \varphi^{\text{NS, d}} \circ c_G = c_G \circ \varphi^* : \text{Pic}(\mathcal{M}_H^e) \to \text{NS}(\mathcal{M}_G^d) \) for all \( \varphi : G \to H \).

Proof. If \( G \) is a torus, then \( \delta \) and \( \iota_G \) are unique; if \( G \) is simply connected, then \( c_G(\iota_G, \delta) \) coincides by construction with the isomorphism \( c_G \) of Proposition 4.4.7(ii).

In both cases, (i) and (ii) follow, and we can use the notation \( c_G \) without ambiguity.

Given a representation \( \rho : G \to \text{SL}(V) \), the diagram

\[
\begin{array}{ccc}
\text{Pic}(\mathcal{M}_{\text{SL}(V)}) & \xrightarrow{c_{\text{SL}(V)}} & \text{NS}(\mathcal{M}_{\text{SL}(V)}) \\
\downarrow \rho^* & & \downarrow \rho^{\text{NS, d}} \\
\text{Pic}(\mathcal{M}_G^d) & \xrightarrow{c_G(\iota_G, \delta)} & \text{NS}(\mathcal{M}_G^d)
\end{array}
\]

commutes, because it commutes after composition with the injective map

\[
(\iota_G)^{\text{NS, d}} : \text{NS}(\mathcal{M}_G^d) \to \text{NS}(\mathcal{M}_{T_G})
\]
from diagram (20), using Lemma 5.2.9 Corollary 4.4.2 Remark 3.2.3 and the 2–commutative squares

\[
\begin{array}{ccc}
\mathcal{M}_{T_G}^0 & \xrightarrow{\iota_*} & \mathcal{M}_{T_G}^\delta \\
\downarrow{\rho^*} & & \downarrow{\rho^*} \\
\mathcal{M}_{T_{SL(V)}}^0 & \xrightarrow{\iota_{\rho^*}} & \mathcal{M}_{T_{SL(V)}}^{\rho^*,\delta} \\
\end{array}
\]

in which \(\iota : T_{SL(V)} \hookrightarrow SL(V)\) is a maximal torus containing \(\rho(T_G)\).

Similarly, given a homomorphism \(\chi : G \to T\) to a torus \(T\), the diagram

\[
\begin{array}{ccc}
\text{Pic}(\mathcal{M}_T^{\chi,\delta}) & \xrightarrow{c_G} & \text{NS}(\mathcal{M}_T) \\
\downarrow{\chi^*} & & \downarrow{\chi^*} \\
\text{Pic}(\mathcal{M}_G^{d}) & \xrightarrow{c_G(\iota_G,\delta)} & \text{NS}(\mathcal{M}_G^d) \\
\end{array}
\]

commutes, again because it commutes after composition with the same injective map \((\iota_G)^{NS,\delta}\) from diagram (20), using Lemma 5.2.9 Remark 3.2.3 and the 2–commutative squares

\[
\begin{array}{ccc}
\mathcal{M}_{T_G}^0 & \xrightarrow{\iota_*} & \mathcal{M}_{T_G}^\delta \\
\downarrow{\chi_*} & & \downarrow{\chi_*} \\
\mathcal{M}_T^0 & \xrightarrow{\iota_{\chi_*}} & \mathcal{M}_T^{\chi_*}\delta \\
\end{array}
\]

The two commutative diagrams (22) and (23) show that the restriction of \(c_G(\iota_G,\delta)\) to the images of all \(\rho^*\) and all \(\chi^*\) in \(\text{Pic}(\mathcal{M}_G^{d})\) modulo \(\text{Hom}(\pi_1(G),J_C)\) does not depend on the choice of \(\delta\) or \(\iota_G\). But these images generate a subgroup of finite index, according to Proposition 5.2.11 and Remark 4.3.3. Thus (i) and (ii) follow. The functoriality in (iii) is proved similarly; it suffices to apply these arguments to homomorphisms \(\rho : H \to SL(V)\), \(\chi : H \to T\) and their compositions with \(\varphi : G \to H\), using Corollary 5.2.10. 

References

[1] A. Beauville and Y. Laszlo, Conformal blocks and generalized theta functions, Commun. Math. Phys. 164 (1994), no. 2, 385–419.
[2] A. Beauville, Y. Laszlo, and C. Sorger, The Picard group of the moduli of \(G\)-bundles on a curve, Compositio Math. 112 (1998), no. 2, 183–216.
[3] K. Behrend, The Lefschetz trace formula for the moduli stack of principal bundles, Ph.D. thesis, University of California, Berkeley, 1991, available at http://www.math.ubc.ca/~behrend/thesis.html.
[4] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie. Chapitres IV, V et VI., Paris: Hermann & Cie., 1968.
[5] A. Boysal and S. Kumar, Explicit determination of the Picard group of moduli spaces of semistable \(G\)-bundles on curves., Math. Ann. 332 (2005), no. 4, 823–842.
[6] L. Breen, Bitorseurs et cohomologie non abélienne., The Grothendieck Festschrift, Vol. I. Progress in Mathematics, Vol. 86. Boston - Basel - Stuttgart: Birkhäuser, 1990.
[7] M. Demazure and P. Gabriel, Groupes algébriques. Tome I., Paris: Masson et Cie, Éditeur; Amsterdam: North-Holland Publishing Company, 1970.
[8] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1989), no. 1, 53–94.
[9] V. G. Drinfeld and C. Simpson, *B*-structures on *G*-bundles and local triviality*, Math. Res. Lett. 2 (1995), no. 6, 823–829.
[10] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sbornik N.S. 30(72) (1952), 349–462 (3 plates), English translation: AMS Transl. Ser. II, vol. 6 (1957), 111-244.
[11] G. Faltings, *A proof for the Verlinde formula*, J. Algebr. Geom. 3 (1994), no. 2, 347–374.
[12] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, J. Eur. Math. Soc. (JEMS) 5 (2003), no. 1, 41–68.
[13] J. Heinloth, *Uniformization of *G*-bundles*, preprint arXiv:0711.4450 available at http://www.arXiv.org.
[14] Y. I. Holla, *Parabolic reductions of principal bundles*, preprint math.AG/0204219 available at http://www.arXiv.org.
[15] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics. 21. New York - Heidelberg - Berlin: Springer-Verlag, 1975.
[16] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”*, Math. Scand. 39 (1976), no. 1, 19–55.
[17] S. Kumar and M. S. Narasimhan, *Picard group of the moduli spaces of *G*-bundles*, Math. Ann. 308 (1997), no. 1, 155–173.
[18] S. Kumar, M. S. Narasimhan, and A. Ramanathan, *Infinite Grassmannians and moduli spaces of *G*-bundles*, Math. Ann. 300 (1994), no. 1, 41–75.
[19] H. Lange and C. Birkenhake, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften. 302. Berlin: Springer-Verlag, 1992.
[20] Y. Laszlo, *Linearization of group stack actions and the Picard group of the moduli of *SL*_r/*µ*_s*-bundles on a curve*, Bull. Soc. Math. France 125 (1997), no. 4, 529–545.
[21] Y. Laszlo and C. Sorger, *The line bundles on the moduli of parabolic *G*-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 4, 499–525.
[22] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Astérisque, 159-160. Paris: Société Mathématique de France, 1988.
[23] J. S. Milne, *Abelian varieties*, Arithmetic Geometry. (G. Cornell and J.H. Silverman, ed.) , Berlin-Heidelberg-New York: Springer-Verlag, 1986.
[24] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics. 5. London: Oxford University Press, 1970.
[25] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. 200 (1973), 69–84.
[26] A. Ramanathan, *Moduli for principal bundles over algebraic curves. II*, Proc. Indian Acad. Sci. Math. Sci. 106 (1996), no. 4, 421–449.
[27] C. Sorger, *On moduli of *G*-bundles of a curve for exceptional *G*,* Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 1, 127–133.
[28] C. Teleman, *Borel-Weil-Bott theory on the moduli stack of *G*-bundles over a curve*, Invent. Math. 134 (1998), no. 1, 1–57.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
E-mail address: indranil@math.tifr.res.in

Mathematisches Institut der Freien Universität, Arnimallee 3, 14195 Berlin, Germany
E-mail address: norbert.hoffmann@fu-berlin.de