Confining Phase of Three Dimensional Supersymmetric Quantum Electrodynamics

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Abelian theories in three dimensions can have linearly confining phases as a result of monopole-instantons, as shown, for SU(2) Yang-Mills theory broken to its abelian subgroup, by Polyakov. In this article the generalization of this phase for $\mathcal{N} = 2$ supersymmetric abelian theories is identified, using a dual description. Topologically stable BPS-saturated and unsaturated particle and string solitons play essential roles. A plasma of chiral monopoles of charge 1 and -1 (along with their antichiral conjugates) are required for a stable confining vacuum. $\mathcal{N} = 2$ SU(2) Yang-Mills theory broken to U(1) lacks this phase because its chiral monopoles all have the same charge, leading to a runaway instability. The possibility of analogous confining phases of string theory, and a dual field theoretic model thereof, are briefly discussed.

1 Introduction

Yuri Golfand was one of the first to construct supersymmetric versions of abelian gauge theories. The three dimensional version of supersymmetric quantum electrodynamics (SQED) turns out to be a very rich dynamical system whose non-perturbative properties have recently received much attention and are still being explored. It seems fitting that the presentation of the linearly confining regime of this theory, generalizing the work of Polyakov in the non-supersymmetric context, be part of this volume.

Duality has long been a powerful tool in studying confining gauge theories. ’t Hooft, Mandelstam and others discovered weakly coupled field theories with solitonic flux tubes that behave somewhat similar to the tubes of chromoelectric flux which occur in QCD. In three dimensions, where charged particles are naively confined by a logarithmically growing potential, it is still interesting to find phases with the potential grows linearly. In pure SU(2) Yang-Mills theory broken to its abelian subgroup, Polyakov showed, using the duality of gauge fields and compact scalar fields in three dimensions, that ’t Hooft-Polyakov monopoles acting as dynamical instantons in three dimensions lead to linear confinement.
The supersymmetric generalization of Polyakov’s result does not give a confining phase. As Affleck, Harvey and Witten showed, the monopole instantons in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theories cause the theory to develop an unstable potential. The resulting instability drives the scale of the $SU(2)$-to-$U(1)$ breaking off to infinity, where the theory is Gaussian. No linearly confining phase has been found for these theories.

Three dimensional gauge theories also have conformal fixed points, where naive logarithmic confinement is lost and charged sources have a $1/r$ potential. With a large number of charged matter fields, it can be shown using large $N_f$ techniques that many gauge theories have this property. Such fixed points are found in many supersymmetric examples even when the number of charged matter fields is small. For $\mathcal{N} = 4$ supersymmetry, all abelian gauge theories are believed to have conformal fixed points at the origin of moduli space. This extends to a very wide class of $\mathcal{N} = 2$ theories as well, including the theory central to this paper, which contains a photon, an electron, a positron, and their $\mathcal{N} = 2$ superpartners. The discovery of a new dual description at these fixed points, one which is distinct from the gauge field/scalar duality used by Polyakov, permits new insights into these gauge theories.

In this letter it will be shown that $\mathcal{N} = 2$ SQED has a confining phase, in analogy to other abelian non-supersymmetric examples. The mechanism is again that of Polyakov. It will be explained below why this mechanism does not work in the $\mathcal{N} = 2$ $SU(2)$ Yang-Mills theories: a stable confining phase in a supersymmetric theory requires monopoles with charges 1 and -1, along with their antimonopoles, but in $SU(2)$ one obtains only monopoles with charge 1 and antimonopoles of charge -1, and a stable phase is not obtained. It is also noted that string theory may well have phases in which two-form flux does not propagate freely and strings are linearly confined by domain walls of two-form flux. String duals of such phases would be interesting to explore.

2 Preliminaries: The Pure $U(1)$ Gauge Theory

Let us begin with a trivial analysis of the theory without matter. Consider $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory consisting of a single $U(1)$ vector multiplet $V$, which contains a photon, a photino, and a scalar $\phi$. The field strength $F^{\mu\nu}$ is contained in the gauge invariant multiplet $\Sigma = \epsilon_{\alpha\beta} D^\alpha D^\beta V$. The Lagrangian of the theory is $\mathcal{L} = \int d^4 \theta \frac{1}{8\pi^2} \Sigma^2$.

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a The superfield language used here is that of $\mathcal{N} = 1$ supersymmetry in four dimensions, of which three-dimensional $\mathcal{N} = 2$ is the dimensional reduction.
Classically the theory has a moduli space of vacua given by $\langle \phi \rangle$. Quantum mechanically things are more interesting, because we may replace the gauge field $A^\mu$ by its electromagnetic dual, a scalar $\tau$ periodic under $\tau \to \tau + 2\pi$. We may define a chiral superfield $T$ whose lowest component is $\phi/g^2 + i\tau$; then the full moduli space is the cylinder defined by $\langle T \rangle$, with a trivial metric, or equivalently the plane defined by $\langle e^T \rangle$.

More explicitly, given any effective action $S_{\text{eff}}$ for $\Sigma$, we may introduce $T$ as follows. The path integral over $V$ can be replaced with a path integral over $\Sigma$ only if a Lagrange multiplier, a chiral superfield $T$, is added to implement the Bianchi identity via the terms $\Delta L = \int d^2\theta T\bar{D}^2\Sigma + \text{c.c.}$. Integrating by parts, we have

$$\int D\Sigma \, D\tau \exp \left( iS_{\text{eff}}(\Sigma) + \frac{i}{4\pi} \int d^3x d^4\theta \, \Sigma(T + T^\dagger) \right).$$  \hspace{1cm} (1)

By integrating out $\Sigma$, we obtain an effective action for $T$. For the free $U(1)$ theory, this effective action is trivial: $L = \int d^4\theta \, (g^2/8\pi^2)T^\dagger T$. Generally, however, the field $T$, although gauge invariant, is non-local. However, $e^T$ may be a local gauge invariant operator. If one inserts $e^T(x)$ into the above path integral, the Bianchi identity is violated by a delta function, $dF = 2\pi\delta(x)$; thus $e^T$ represents a pointlike instanton in the form of a Dirac magnetic monopole. (The normalization is chosen so that the Dirac quantization condition is satisfied.) We therefore identify $e^T$ and $e^{-T}$ with chiral operators $M, \tilde{M}$ of monopole charge 1, $-1$, and their complex conjugates with the conjugate antichiral operators $M^\dagger, \tilde{M}^\dagger$ of charge $-1, 1$.

Since the field $T$ takes expectation values on a cylinder, the fundamental group $\pi_1$ of the moduli space is non-zero and there exists the possibility of particle-like vortex solitons, or more precisely, of solutions to the equations of motion which have winding number. The configuration $\langle T(r, \theta, t) \rangle \propto iq\theta$, $q$ an integer, realizes this possibility. Unfortunately, it is singular at the origin. Since $g^2\partial_\theta \tau = rF_{rr}$, this singular soliton is nothing but the electric field outside of a point electric charge of charge $q$. Unless electrically charged fields are actually added to the theory, this solution plays no dynamical role. In three dimensions, the energy in the isotropic electric field around a point charge is logarithmically divergent; strictly speaking we should consider only configurations with total charge zero. This is the logarithmic confinement of QED$_3$.

\footnote{This has not actually been proven, although mirror symmetry implies it must be true.}
2.2 The linearly confining phase

We may make things more interesting by modifying the theory through the terms $\Delta L = \int \! d^2 \theta \ W(T) + c.c.$ with $W = h(e^{pT/2} + e^{-pT/2})$, making it a sine-Gordon model. If the radius of the $U(1)$ gauge group is $2\pi$, then $p$ must be an integer. For a trivial Kähler potential, this generates a scalar potential $V(T) = |p\sinh \frac{pT}{2}|^2$, which has $p$ supersymmetric vacua, located at $\langle T \rangle = 2\pi in/p$, $n$ an integer. (Note that the positions of these vacua do not change for a non-trivial but non-singular Kähler potential; the potential $V(T)$ will remain periodic in $T \rightarrow T + 2\pi i/p$.) This implies the existence of domain walls separating regions in different vacua. Consider a configuration with vacuum $n_1$ at $x \ll 0$ and vacuum $n_2$ at $x \gg 0$. There is a non-singular solution $T(x, y, t) = T(x)$ which interpolates between these two vacua, giving a domain wall — a string in two spatial dimensions — in the $y$ plane. Since $\partial_y \tau = F_{ty}$, the string carries electric flux: it is a linearly confining electric flux tube. The total flux in the tube is $\int \! dx F_{ty} = \Delta \tau = \frac{2\pi}{p}$, and the width of the tube is proportional to $1/p\sqrt{h}$.

Equivalently, note that the existence of $W(T) \neq 0$ changes the equations so that a constant electric field $F_{xt}(x, y, t) = E$ is no longer a solution; the equations of motion reduce in this case to

$$\frac{\partial S(F_{xt})}{\partial F_{xt}} + \partial_y \tau = 0 ; \quad \frac{df}{d\tau} = 0 \tag{2}$$

the second of which implies $\tau$ is everywhere at a minimum of $f$, while the first implies $\tau$ must vary linearly with $y$, inconsistent with the second if $f$ depends on $\tau$. The electric field between two parallel line charges will therefore not be uniform, but will instead break up into strings of electric flux.

Of course, Polyakov explained long ago why the above modification of the action causes linear confinement. The action of this theory contains the terms $\frac{1}{2} \int \! d^3 x \left[ \frac{\partial}{\partial \tau} dF + f(\tau) \right]$. Let us work in Euclidean space for a moment. The Bianchi identity has become $dF = 2\pi f'(\tau)$. Thus, at any $\tau$ which is not a local
minimum or maximum of \( f(\tau) \), there is a local density of magnetic monopoles in the vacuum. This means that the vacuum is a magnetic plasma. An attempt to force a nontrivial electric field \( F_{xt} \) through the Minkowski vacuum by introducing an electric source corresponds in the Euclidean description to trying to force a magnetic field through a magnetic plasma by introducing a wire carrying electric current. Such a field will of course be screened by magnetic charge separation (corresponding to \( df/d\tau \) taking positive and negative values in various places) taking place in a region of order the Debye length \( \ell_D \) (inversely proportional to the monopole charge times the square root of the density, \( \ell_D \sim 1/(p\sqrt{n}) \) in our example.) This immediately leads to linear confinement of the electric currents; see Fig. 1. In broken three-dimensional non-supersymmetric \( SU(2) \) Yang-Mills theory the monopole plasma is generated dynamically, but the low-energy effective \( U(1) \) gauge theory is similar to the one considered here.

Polyakov showed\(^8\) that Wilson loops acquire an area law when \( f(\tau) \) is non-zero and periodic; this is done by considering the area-dependent effects far from the edge of the loop.\(^7\) By considering the edge effects near the boundary of the loop, the analysis can straightforwardly be extended to show that for \( f(\tau) = 0 \) Wilson loops show logarithmic behavior when the kinetic term for

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\(^{5}\)The presentation contains many typographical errors; note particularly that Eq. (5.23) is the correct solution but neither (5.22) nor (5.24) is correct as written.
\[ F^\mu\nu \] is \( F^2 / g^2 \). Similar computations can be used to show that the Wilson loop has the required behavior in the conformal phase of SQED with \( N_f \) matter fields, where the effective action for \( F \) is \( F \square^{-1/2} F \) plus corrections which are small in the large \( N_f \) limit.

In this modified theory, the presence of a static point charge cannot lead to a configuration with \( \langle T(r, \theta, t) \rangle \propto i\theta \), since \( \tau \) has \( p \) preferred values. Instead, the point charge, around which \( \tau \) must wind once, will distribute that winding as in Fig. 2, by hopping from one vacuum to another until \( \tau \) has shifted by \( 2\pi \). The electric field \( \partial_\theta \tau \) of the source, instead of being isotropic, is now confined in the flux-carrying domain walls. Amusing configurations such as that in Fig. 3 can be constructed; note that similar configurations are lacking in QCD only because of the presence of light quarks.

A natural question is whether these domain walls are BPS saturated. First, consider the question of whether non-trivial BPS bounds exist. In analogy to kink solitons in two dimensions and domain-wall solitons in four dimensions, BPS bounds for domain-wall (stringlike) solitons in three dimensions can occur when the expectation value of the superpotential is different in the two vacua which the domain wall separates. Interestingly, because the superpotential chosen above depends on the vacuum of choice only up to a sign \( W(T = i\pi) = (-)^n 2\hbar \), walls separating vacua with \( n_1 - n_2 \) odd have a BPS bound while those with \( n_1 - n_2 \) even have no BPS bound, as in the two-dimensional version of the same model. This observation suggests that multiple strings with \( n_1 - n_2 = 1 \) will bind together to make strings carrying larger amounts of flux (whose energy cannot be calculated.) The physics behind this behavior is not obvious; in particular it is not clear why the strings in...
this confining abelian theory are so different from the domain walls in supersymmetric QCD in four dimensions, which have a BPS bound between any two vacua. This also prevents the junction of multiple strings from being BPS saturated, in contrast to the case of QCD domain wall junctions.

 Bounds having been established for $n_1 - n_2$ odd, the next question is whether such solitons exist. The BPS-saturated kink solitons for the sine-Gordon model in two-dimensions have been constructed, and can be directly lifted to this three-dimensional model.

3 The Abelian Theory with Matter

3.1 $\mathcal{N} = 2$ Supersymmetric QED

Consider now $\mathcal{N} = 2$ supersymmetric QED, consisting of a single $U(1)$ vector multiplet $V$ and chiral multiplets $Q, \tilde{Q}$ of charge $1, -1$. The vector multiplet in three dimensions contains a real scalar $\phi$ in addition to the photon and photino. The Lagrangian of the theory is

$$L_{SQED} = \int d^4\theta \frac{1}{4g^2} \Sigma^2 - (Q^\dagger e^{2V}Q - \tilde{Q}^\dagger e^{-2V}\tilde{Q}) - \left[ \int d^2\theta W(Q, \tilde{Q}) + c.c. \right]$$

Initially we take the superpotential of the theory $W$ to be zero.

The gauge invariant operators of the theory include $Q\tilde{Q}$, but the theory also has the gauge-invariant operators $M$ and $\tilde{M}$ which are sources of magnetic flux. The theory with matter has an abelian Higgs phase $\langle Q\tilde{Q} \rangle \neq 0$, in which the gauge field is massive and the flux which emanates from $M$ and $\tilde{M}$ is trapped in magnetic flux vortices, solitons which behave like particles in three dimensions.
dimensions. We therefore identify \( M \) and \( \tilde{M} \) as vortex-creation operators.

The moduli space of the theory consists of the Higgs branch, where \( \langle Q \tilde{Q} \rangle \neq 0 \) and the gauge group is broken, and the Coulomb branch, which classically consists of expectation values for the real field \( \phi \). Quantum mechanically, the Coulomb branch corresponds to expectation values for \( \phi \) and \( \tau \), or more precisely for \( T \). For zero superpotential, the Coulomb branch splits into two pieces, one on which \( M \) has an expectation value, the other on which \( \tilde{M} \) is nonzero. At the origin these three branches touch and there is a conformal field theory.

3.2 The XYZ model

Consider the theory of three chiral superfields \( X, Y, Z \) and superpotential \( W = yXYZ \). The superpotential has dimension 2, while the fields and \( y \) have engineering dimension \( \frac{1}{2} \). There is a \( U(1) \) R-symmetry under which \( W \) has charge 2 and the fields have charge \( \frac{2}{3} \).

The classical scalar potential for the scalar fields, \( V = |(dW/dX)|^2 + |(dW/dY)|^2 + |(dW/dZ)|^2 \), takes the form \( |YZ|^2 + |ZX|^2 + |XY|^2 \). It has solutions \( X \neq 0, Y = Z = 0 \) and permutations thereof. Classically, then, the theory has a space of vacua consisting of three complex planes, each with one of the three fields as its coordinate, which touch at the point \( X = Y = Z = 0 \). At this special point the R-symmetry and permutation symmetry are unbroken, and the only scale in the problem is \( y \). It is believed that the theory flows to a conformal field theory in the infrared. The R-charges and permutation symmetry fix the dimensions of \( X, Y, Z \) to be \( 2/3 \). One way to understand this intuitively is to note that, in the superpotential, if \( y \) were transmuted by quantum dynamics to become a dimensionless coupling, then \( XYZ \) would have dimension 2. The moduli space is shown in Fig. 5.

The XYZ model and SQED are surprisingly in the same universality class. For this to be true the theories must have the same spectrum of gauge invariant operators and the same moduli space, and indeed they do: the chiral operators \( X, Y, Z \) map to the chiral operators \( M, \tilde{M}, Q \tilde{Q} \). The permutation symmetry which is a classical feature of the XYZ model is a quantum effect of the SQED conformal field theory.

3.3 The exact dual of \( \mathcal{N} = 2 \) SQED

An exact dual for \( \mathcal{N} = 4 \) SQED, valid at all energies, has been constructed and this dual can easily be extended to \( \mathcal{N} = 2 \) SQED. In particular, it involves coupling a pair of photon multiplets \( V_1, V_2 \) with an off-diagonal Chern-Simons
term $\int d^4\theta \ V_1 \Sigma_2$ (also known as a BF term) and using $V_1$ to gauge the symmetry which rotates the phases of $X$ and $Y$; thus the kinetic terms become $\int d^4\theta \ (X^\dagger e^{V_1} X + Y^\dagger e^{-V_1} Y)$. This description is strongly coupled at high energies (as expected, since SQED is weakly coupled there) and becomes the XYZ model at very low energies, plus irrelevant operators. The leading irrelevant operator is $(X^\dagger X - Y^\dagger Y)^2$. The addition of this and other non-singular operators to the Kähler potential does not change the phase of the theory, not does it change the topological issues which determine the existence of soliton solutions. For this reason we may use the XYZ model, whose low-energy Kähler potential is unknown in any case, for the investigations below, rather than the more complicated exact dual description.

The one subtlety which should be noted is that $X^p$ is a gauge invariant operator in the XYZ model, while in the exact dual of SQED it is not. Strictly speaking the operator $X$ in the latter must be supplemented with a Wilson line $C(\gamma)$, where $\gamma$ is a curve which extends from the insertion of $X$ to a point at infinity. Let us replace $V_2$ with the constrained field $\Sigma_2$ and the Lagrange multiplier $T_2$ which couples to it by $(T_2 + T_2^\dagger) \Sigma_2$. In the presence of the $V_1 \Sigma_2$ interaction, the field $T_2$ now shifts under the gauge invariance of $V_1$, and the Wilson line $C(\gamma)$ for $V_1$ along $\gamma$ can be replaced by $e^{T_2}$. Consequently the operator $[e^{T_2} X]$, and any half-integer power of it, is local and gauge invariant and can be added to the superpotential. This operator reduces to $X$ when the gauge fields are integrated out.
4 The conformal phase

Let us consider some perturbations of these theories. The first few of these are well-studied and serve as tests of the conjectured infrared equivalence of these theories. The later ones have not previously been explored in the literature. We begin for completeness with the conformal phase, found at the origin of moduli space. This section lies somewhat outside the main flow of the paper and can be skipped. The following sections address the phases of central interest.

\[ W_{XYZ} = y_{XYZ} + t(X^3 + Y^3 + Z^3), \quad W_{SQED} = t(M^3 + \bar{M}^3 + Q^3\bar{Q}^3) \]

Using well known techniques, it is straightforward to show that this perturbation is exactly marginal at the origin of moduli space. Specifically, within the space of physical couplings \( h, t \) (which must be distinguished from the holomorphic couplings appearing in the superpotential above, which do not run) there is a one-complex-dimensional subspace where the theory remains conformally invariant. From the SQED point of view, the naively-irrelevant tenth-order scalar potential for the charged matter balances the appearance of a plasma of charge-3 magnetic instantons to maintain conformal invariance. Whether this behavior has a simple physical interpretation is not known.

\[ W_{XYZ} = y_{XYZ} + tZ^3, \quad W_{SQED} = tQ^3\bar{Q}^3 \]

This marginal perturbation is marginally irrelevant. To see this, consider the beta functions for the theory \( W = y_{XYZ} + tZ^3 \).

\[ \beta_y = y[-\frac{1}{2} + \gamma_X(h, t) + \frac{1}{2}\gamma_Z(y, t)]; \quad \beta_t = t[-\frac{1}{2} + \frac{3}{2}\gamma_Z(y, t)] \] (4)

I have used the symmetry between \( X, Y \) here to write \( \gamma_Y = \gamma_X \). Note the following facts. First, there exists some \( y \equiv y^* \) for which \( W = y_{XYZ} \) is conformal; in short, \( y = y^*, t = 0, \gamma_X = \gamma_Z = \gamma^* = 1/3 \). Second, \( \gamma_X(y, t), \gamma_Z(y, t) \) are functions of two variables, and the conditions in Eq. (4) represent two linearly independent conditions on two variables, whose solutions will be isolated. The fixed point \( y = y^*, t = 0 \) is therefore isolated, and thus the perturbation is not exactly marginal. To see that it is irrelevant, note that the solutions to the condition \( \gamma_X = \gamma_Z \) form one or more lines in coupling space. One of these lines, by symmetry, is at \( t = 0 \) for any value of \( y \). Any other lines will generally lie at some distance from this one. For \( y = 0, t \neq 0, \gamma_Z > 0 \) by unitarity while \( X \) is noninteracting, so \( \gamma_Z - \gamma_X \) is positive. It must therefore be that for all \( y \) and small \( t \), \( \gamma_Z - \gamma_X > 0 \). In conclusion, for \( y = y^*, t = \epsilon, \beta_t > 0 \), and so the perturbation is irrelevant.
Figure 6: The moduli space of SQED with massive matter. The field around a massive charged field involves the winding of $\tau$ around the small circle at the center of the space.

\[ W^{XYZ} = y_{XYZ} + mZ^2, \quad W^{SQED} = m\tilde{Q}\tilde{Q}\tilde{Q}\tilde{Q} \]

Integrating out $Z$ leaves a low-energy effective superpotential $W^{XYZ} = -\frac{y^2}{4m}XYYXY$. This interaction, which gives a sixth-order scalar potential, is marginally irrelevant in three dimensions. Consequently, the above perturbation drives the XYZ theory to a free theory of $X$ and $Y$. The same perturbation pushes the $\mathcal{N} = 2$ SQED theory to $\mathcal{N} = 4$ SQED\footnote{Henceforth we set $y = 1$ for simplicity, since it plays no role in the following discussion.}, which is known to flow to a CFT that can be written as a free theory of its vortex solitons.

5 The Logarithmically Confining Phase: non-BPS solitons

\[ W^{XYZ} = XYZ + mZ, \quad W^{SQED} = m\tilde{Q}\tilde{Q} \]

A mass for $Q$ and $\tilde{Q}$ must leave the theory with a moduli space topologically equivalent to that of a free $U(1)$ gauge theory, which has only a Coulomb branch with the topology of a cylinder\footnote{Henceforth we set $y = 1$ for simplicity, since it plays no role in the following discussion.}. Although the metric cannot be computed exactly, as would be the case in $\mathcal{N} = 4$ SQED, it can be computed using perturbation theory for large $|\phi|$, while for small $\phi$ it can be constrained using perturbation theory and our knowledge of the theory with $m = 0$. The radius of the cylinder asymptotically approaches the gauge coupling. The charged matter reduces the effective value of the gauge coupling at small $|\phi|$, and thus the cylinder has smaller radius, as shown in Fig. 7 note its consistency with Fig. 4.

Because there are no longer any light charged fields in the theory to screen electric charge, the particles $Q$ and $\tilde{Q}$ should be visible as localized massive objects with long range fields. Actually this is not quite true; their electric charge means that they are still logarithmically confined, so they cannot be...
seen separately from one another. However, if we consider a pair of heavy particles of opposite charge, separated by a distance $L$, the energy associated with their electric fields may be much less than the energy associated with their cores; and so there are states in the theory with energy of order $2m + \log L \sim 2m$ which have their energy concentrated in two lumps of size $m^{-1}$ separated by distance $L$. Can such states be found in the XYZ theory?

The answer is yes, and they take the form of logarithmically confined states of BPS-unsaturated solitons. The moduli space of the theory is $XY = m$, a hyperbola, shown in Fig. 7. Since $\pi_1$ of the moduli space is non-zero, there are vortex solitons in which the fields $X, Y$ wind opposite to one another at spatial infinity. The kinetic terms for $X, Y$ ensure the energy of one of these solitons is logarithmically divergent, but a soliton-antisoliton pair would have finite energy of order twice a core energy plus a long-distance logarithm. More importantly, and in contrast to the pure $U(1)$ theory discussed earlier, the core energy is finite; the XYZ model has solitons because the point at the center of the soliton can have $X = 0, Y = 0$ while maintaining finite energy density. Note that the winding around the hyperbola will be energetically favored to occur for $|X| = |Y| = \sqrt{|m|}$, the region of maximum symmetry, since the spatial variation of the fields at infinity, and the associated stress energy, can be minimized there.

To confirm that the above vortex solitons are the fields $Q$ and $\tilde{Q}$, we should check that they are charged under the same fields. The winding of the phase of $X$ which defines the soliton corresponds in SQED to the winding of the phase of $M \sim e^\tau$, and thus the dual photon $\tau$ shifts by $2\pi$ while winding around the soliton. Since $\partial_\theta \tau$ is related by electric-magnetic duality to $F_{tr}$, the radial component of the electric field, the soliton emits the SQED electric field with

\*Of course we could avoid the problem of the logarithmic divergences by weakly breaking the gauge symmetry of the $U(1)$ theory; this method of regulating the theory will be discussed below.
total charge 1. This indeed corresponds to the properties of the field $Q$. The soliton winding the other way is either $\bar{Q}$ or $Q^\dagger$; to distinguish them one must examine the fermion zero modes of the solitons more carefully, which although interesting lies outside the scope of the present paper.

It is straightforward to verify that these solitons are meaningful, despite their logarithmic divergences, because the infrared logarithm may be easily regulated. There are several options. The easiest is to consider coupling the vector field $V$ of SQED to a second vector field $\hat{V}$, using an off-diagonal Chern-Simons (BF) term $\int d^4\theta \,(k/2\pi)\hat{V}\Sigma$ and adding a kinetic term $\int d^4\theta \,(1/4g^2)\hat{\Sigma}^2$. This gives a mass $kg^2$ to the vector fields, and means that the logarithmic potential confining $Q$ and $\bar{Q}$ now falls off exponentially. The state created by $Q$ alone now has finite energy. In the $k \to \infty$ limit the vector fields decouple, the theory is free and $Q$ has mass $m$. In fact, in this limit the theory has $N = 4$ supersymmetry and the particle $Q$ is BPS saturated. For small $k$ its mass diverges as $|\log k|$. The state created by $Q(x)Q(0)$ has mass of order $|\log k|$ or $\log g^2|x|$, whichever is smaller.

The mirror description of this same process requires the exact duality for SQED described in Sec. 3.3. The finite coupling $g$ in the original theory corresponds to coupling the XYZ model to topologically massive vector bosons $V_1, V_2$ with mass of order $g^2$. The field $\hat{V}$ couples to $V_2$ through a BF term with coupling $k$, and it therefore mixes with $V_1$, which couples to $V_2$ through a BF term with coupling 1. The mass matrix for the vectors $V_1, \hat{V}$ has a vanishing eigenvalue, so one linear combination is massive and the other, call it $V_0$, is massless. The fields $X, Y$ couple to the massless (massive) vector multiplet with a coupling proportional to $k$ (1), and the massless vector eliminates the logarithmic divergence in the soliton energy through the replacement of $\partial_\theta X$ with $\left(\partial_\theta + A_{0\theta}\right)X$. For $k = \infty$ this modified $\text{XYZ}$ theory is simply $N = 4$ SQED with a Fayet-Iliopolous term, which is well-known to be free with massive BPS vortices. For finite $k$ the solitons are not BPS saturated but still have finite mass; only for $k = 0$, when the massless vector multiplet $V_0$ completely decouples from $X, Y$, is the soliton mass divergent. Meanwhile, for small $k$ and large separation $|x|$, the mass of the soliton-antisoliton state is of order $\log g^2|x|$, where $g^2$ is the mass of the topologically massive vector multiplet; it is finite in the $k \to 0$ limit. This matches with the SQED description above.

6 The confining phases: BPS and non-BPS strings

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The remainder of this section may be skipped; it is presented only for technical completeness.
$\tau = 0$

$\tau = \pi$

Figure 8: In the confining phase with $p = 2$, particles with charge 1 and -1 are confined by two strings carrying the minimal quantum of flux (which most likely bind into a single one with twice the flux.)

$$W^{XYZ} = XYZ + mZ + h(X + Y), \quad W^{SQED} = mQ\tilde{Q} + h(M + \tilde{M})$$

The presence of linear terms in $X$ and $Y$ implies that we will now have multiple isolated vacua, each with a mass gap. This means there will be regions in different vacua separated by domain walls, as we discussed earlier for pure $U(1)$ gauge theory. The vacuum equations are

$$XY + m = 0; \quad YZ + h = 0; \quad XZ + h = 0$$

which have two solutions

$$X = Y = \pm i\sqrt{m}; \quad Z = \pm i\frac{h}{\sqrt{m}}$$

These two isolated vacua are two points on the circle $|X| = |Y| = |\sqrt{m}|$. If $h$ were zero, then, as described above, a vortex soliton with logarithmically divergent energy could have been constructed with asymptotic behavior $X(\theta) = i\sqrt{m}e^{i\theta}$, $Y(\theta) = i\sqrt{m}e^{-i\theta}$. With $h \neq 0$, only two points of the moduli space remain, so it is no longer possible to have a soliton of this type. But we may ask, what if we had such a soliton for $h = 0$, and then adiabatically turned on a non-zero value for $h$? The winding of the soliton cannot be unwound without large energy cost. Instead, the system will minimize the energy by ensuring that the phases of $X, Y$ do all of their winding in two narrow domain walls, sitting in one or the other zero-energy vacuum in the regions between the walls, as in Fig. 2. These domain walls, in isolation (without solitons at the end) are BPS saturated.

In short, the vortex soliton will retain its core, but will find itself at the meeting point of two solitonic flux-carrying domain walls. A soliton-antisoliton pair will now be connected by two strings. This is shown in Fig. 8.
although we normally expect a pair of identical BPS-saturated solitons to have zero potential energy, a pair of domain wall solitons of this type is not BPS saturated, because in fact they are not the same: they connect different vacua. Consequently, there is no BPS bound for the pair of walls shown in Fig. 8 (even in the limit that their endpoints are taken to infinity), and it is likely the pair of strings in the figure attract one another and bind to form a single non-BPS string soliton.

\[
W^{\text{XYZ}} = XYZ + mZ + h(X^{p/2} + Y^{p/2}),
W^{\text{SQED}} = m\overline{Q} + h(M^{p/2} + \overline{M}^{p/2})
\]

Here we generalize the previous discussion. The vacuum equations are

\[
XY + m = 0 \ ; \ YZ + hX^{(p/2) - 1} = 0 \ ; \ XZ + hY^{(p/2) - 1} = 0
\]  

which has \( p \) solutions

\[
X = i\sqrt{me^{i\pi n/p}} \ ; \ Y = i\sqrt{me^{-i\pi n/p}} \ ; \ Z = (-1)^n(i\sqrt{m})^{(p/2) - 2}h
\]

where \( n = 1, \ldots, 2p \). There are thus \( p \) isolated vacua on the circle \( |X| = |Y| = |\sqrt{m}| \). In analogy to the case just discussed, this implies that a vortex of winding number 1 is the meeting place of \( p \) flux-carrying domain walls.

The physics of this case is straightforward. In \( \mathbb{C}^2 \), monopoles of charge \( p/2 \) and density \( h \) have been introduced. Since the Debye length in the plasma is inversely proportional to \( p \), dimensional analysis shows that the width of a solitonic tube and the amount of flux it can carry decrease linearly with \( p \). To confine an object of integer electric charge then requires \( p \) such strings.

\[
W^{\text{XYZ}} = XYZ + mZ + hX^{p/2} + \overline{h}Y^{p/2},
W^{\text{SQED}} = m\overline{Q} + hM^{p/2} + \overline{h}\overline{M}^{p/2}
\]

The explicit breaking of the symmetry between \( X \) and \( Y \) moves the \( 2p \) vacua. It is easiest to solve for the vacua by converting the above superpotential back to the previous one by rescaling \( X \rightarrow AX, Y \rightarrow Y/A \) where \( A = (\overline{h}/h)^{1/p} \). The resulting solutions are

\[
X = i\left(\frac{\overline{h}}{h}\right)^{1/p}\sqrt{me^{i\pi n/p}} \ ; \ Y = i\left(\frac{h}{\overline{h}}\right)^{1/p}\sqrt{me^{-i\pi n/p}} \ ; \ Z = (-1)^n(i\sqrt{m})^{(p/2) - 2}\sqrt{hh}
\]
where $n = 1, \ldots, 2p$. Notice that in the limit $\hat{h} \to 0$, $h \neq 0$ the vacua move to $|Y| = \infty$, $X = 0$.

This limit, for $p = 2$, resembles the result of Affleck, Harvey and Witten\textsuperscript{9} for $\mathcal{N} = 2$ SU(2) Yang-Mills theory. In turn this suggests why linear confinement does not occur in that case. In linearly confining SQED we have monopoles $M$ and $\tilde{M}$ of charge $1, -1$ and antimonopoles $M^\dagger$ and $\tilde{M}^\dagger$ of charge $-1, 1$. If they are both introduced with equal weight as in Eq. (7), then the theory has the symmetry $T \to -T$, implying vacua will be symmetric around $\phi = 0$. Reducing the effect of charge $-1$ monopoles $\tilde{M}$ relative to those with charge $1$, which can be done using the field rescaling given above, shifts the symmetry to $T \to -T + \log(h/h)^{2/p}$, essentially acting as a shift of $\phi$. In the limit of infinite rescaling, stable supersymmetric vacua have no natural location other than $\phi = \infty$. Since in $\mathcal{N} = 2$ SU(2) Yang-Mills we find only charge 1 't Hooft-Polyakov monopoles $M$ and charge $-1$ antimonopoles $M^\dagger$, the theory only has a vacuum for infinite $\langle \phi \rangle$.

7 A comment on string theory

I conclude with some limited remarks concerning the lifting of this physics to string theory.

Strings are sources for an abelian two-form gauge field $B^{\mu\nu}$, and it is an interesting question whether string theory permits its electric flux to be confined. In the context of the present paper, this is most easily discussed in four dimensions, where the two form is dual to a scalar $\tau$ (the dimensional reduction of the six-form which couples to Neveu-Schwarz five-branes.) In analogy to the discussion above, one needs to find physics which can generate a (stable) potential which depends non-trivially on $\tau$. The presence of such a potential will indicate that the vacuum has become a plasma of instantons with magnetic charge under $B^{\mu\nu}$. For example, in four dimensions such instantons would be given by Euclidean Neveu-Schwarz five-branes wrapped on the six compact dimensions.

It is easy to construct a four-dimensional field theory model with such behavior. The theory of a free string is given by considering the solitons of the abelian Higgs model, with gauge field $A^\mu$ and complex Higgs field $\Phi$ and a potential $V(\Phi) = (\Phi^*\Phi - v^2)^2$. The phase of $\Phi$ — let us call it $a$ — winds around the string.\textsuperscript{9} In fact the low-energy theory well below the scale $v$ can be

\textsuperscript{9}Free BPS strings can be defined in a suitable limit\textsuperscript{20} of the pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory studied by Seiberg and Witten.\textsuperscript{21} In the theory with dynamical scale $\Lambda$, the presence of an $\mathcal{N} = 2$ breaking parameter $\mu$ leads to confinement of electric flux into tubes. These tubes become $\mathcal{N} = 2$ BPS saturated strings in the limit $\mu \to 0$, $\Lambda \to \infty$ with $\mu\Lambda$ fixed; the parameter $\mu\Lambda$ becomes an $\mathcal{N} = 2$-preserving Fayet-Iliopoulos parameter.
written as the gauge field $A^\mu$ coupled to $a$ with the Lagrangian $(\partial_\mu a + A_\mu)^2$.

Shifting this phase as $a \to a + \Lambda(x)$ is a gauge symmetry.

We may then add a dynamical two-form field $B^{\mu\nu}$ with a dual scalar $\tau$, which couples to the magnetic flux in the string soliton via the interaction $B_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$. Since $\tau$ shifts under the gauge symmetry of $A^\mu$, in analogy to the discussion in Sec. 3.3, we can define a gauge invariant operator $\tau + a$, which has a global (not gauged) shift symmetry and is periodic under shifts by $2\pi$. The winding of this gauge-invariant operator around a string soliton corresponds to the gauge-invariant $B^{\mu\nu}$ electric flux $H^{0\mu\nu}$ (here $H = dB$), whose infrared-divergent energy causes logarithmic confinement of strings. Now, if a non-trivial periodic potential for $\tau + a$ is somehow added to the theory, then this global shift symmetry is broken. (For example, the global shift symmetry of $\tau + a$ can be broken through an anomaly by instantons in some other gauge group.) Consequently, as in Eq. (2), $H^{0\mu\nu}$ flux is confined, and the string solitons are linearly confined by axionic domain walls, as in Fig. 2.

Can a construction along these lines be carried out in a consistent string theory? Even if the shift symmetry of $\tau$ is broken, can the runaway behavior of three-dimensional supersymmetric Yang-Mills be avoided? It would be very interesting if a stable confining phase could be found, especially since string dualities would lead to a whole class of such phases in which confinement of various D branes by other (BPS-unsaturated) branes would occur.

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