Abstract
In this paper, we obtain sufficient and necessary conditions of some classical convex sets as positively invariant sets for a continuous dynamical system, namely positive invariance conditions. The approach is based on Nagumo Theorem by deriving the tangent cones of these sets. We also propose approaches using optimization theory and models to verify the existence of these sufficient and necessary conditions.

Keywords:
Dynamical System, Invariant Set, Polyhedral Set, Ellipsoid, Lorenz Cone.

1. Introduction
Dynamical system has a wide range of applications in the real world. Positively invariant set is a key concept in dynamical system. A positively invariant set of a dynamical system is described as when the system emits from the set, it will always stay in the set. Invariant set is intuitively considered as the attracting region, also namely safety area, of the dynamical system. The applications of positively invariant set refer to [3, 4, 5]. Given a set and a dynamical system, to verify if the set is an invariant set for the system is a key problem in this area. Such verification criteria are usually used to construct the maximal invariant set, i.e., the maximal safety, in a controlled dynamical system.

Recently, some excellent surveys on the theoretical results and applications of invariant sets are published, e.g., [1, 17]. For specific classical sets, the invariance condition, i.e., the sufficient and necessary condition such that a given set is an invariant set for a given dynamical system, are widely studied. For polyhedral sets, one may refer to [6, 7, 11, 22] for various invariance conditions for linear continuous and discrete dynamical system. For quadratic type of sets, e.g., ellipsoidal and second order conic sets, one may refer to [5, 13, 19, 23]. For general convex set and nonlinear system, one may refer to a novel unified approach to derive invariance conditions for polyhedra, ellipsoids, and cones is presented in [11]. The connection between discrete and continuous dynamical systems for preserving the invariance of a set is studied, e.g., [10, 12]. Construction of some invariant sets for a given system is also an interesting topic in this area, e.g., [6, 18].
In this paper, we derive the sufficient and necessary conditions of some classical convex sets as positively invariant sets for a continuous dynamical system. These conditions are referred to as positive invariance condition for simplicity. The candidates of the sets are polyhedra, ellipsoids and cones. The approach is primarily based on Nagumo Theorem \([4, 14]\), which yields the positive invariance conditions to be deriving the tangent cones of these sets. We also propose approaches using optimization theory and models to verify the existence of these positive invariance conditions. The novelty of this paper is that we applied the theoretical result Nagumo Theorem into specific sets and dynamical system, as well as deriving the new positive invariance conditions or same invariance condition by using this new method. Also, the technique using optimization theory and algorithm is novel and the link between invariant set and optimization is built up.

**Notation and Conventions.** In this paper, we use the following notation and conventions to avoid unnecessary repetitions, e.g., \([1, 8]\):

- The inertia of a matrix is denoted by inertia\(\{Q\} = \{a, b, c\}\) that indicates the number of positive, zero, and negative eigenvalues of the matrix \(Q\), respectively.
- The basis in \(\mathbb{R}^n\) is denoted by \(e^1 = (1, 0, \ldots, 0)^T, e^2 = (0, 1, \ldots, 0)^T, \ldots, e^n = (0, 0, \ldots, 1)^T\). And we let \(e = (1, 1, \ldots, 1)^T\).
- The nonnegative quadrant of \(\mathbb{R}^n\) is denoted by \(\mathbb{R}^n_+\), i.e., any coordinates of \(x \in \mathbb{R}^n_+\) is nonnegative.
- Let a vector \(v \in \mathbb{R}^n\), we use \(v(\uparrow_k \alpha)\) to denote the \(k\)-th entry of \(v\) is replaced by \(\alpha\), i.e., \(v(\uparrow_k \alpha) = (v_1, \ldots, v_{k-1}, \alpha, v_{k+1}, \ldots, v_n)^T\).

The paper is structured as follows: Section

2. **Fundamental Definitions**

2.1. **Invariant Set and Nagumo Theorem**

We consider the continuous dynamical system, which is also named *initial value problem (IVP)*, as follows:

\[
\dot{x}(t) = f(t, x), \quad t \geq 0
\]

where \(x(t) \in \mathbb{R}^n\) are the state variables, \(t\) is the time variable, and \(f(t, x)\) is a real valued continuous function. For simplicity, we denote \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n\).

We now introduce the definition of positively invariant set of a dynamical system.

**Definition 2.1.** Let us assume \(S\) be a set in \(\mathbb{R}^n\). The set \(S\) is called a **positively invariant set** of the dynamical system (7) if \(x(0) \in S\) implies \(x(t) \in S\) for all \(t \geq 0\).

Positively invariant set is also named as forward invariant set. For simplicity, we use invariant set to represent positively invariant set. In other words, an invariant set is a set that once the trajectory of the system enters the set, then it will never leave the set in the
future. One example of an invariant set of the linear system $\dot{x}(t) = Ax(t)$, where $A$ is a real matrix, is the span space of all eigenvectors of the matrix $A$.

A fundamental characterization of a close and convex set to be an invariant set for a continuous system is proposed by Nagumo [4, 14].

**Theorem 2.2. Nagumo [4, 14]:** Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and assume that the system $\dot{x}(t) = f(t, x)$, where $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is a continuous mapping, has a globally unique solution for every initial point $x(t_0) \in S$. Then $S$ is an invariant set for this system if and only if

$$f(t, x) \in \mathcal{T}_S(x), \text{ for all } x \in \partial S,$$

where $\mathcal{T}_S(x)$ is the tangent cone of $S$ defined as follows:

$$\mathcal{T}_S(x) = \left\{ y \in \mathbb{R}^n \mid \lim_{t \to 0} \inf \frac{\|x + ty, \partial S\|}{t} = 0 \right\}. \tag{3}$$

Note that the condition that the set is closed and convex is critical in this theorem. In fact, this theorem can be simply illustrated in a geometrical way: for any trajectory that emits from $S$, one only needs to consider the property of this trajectory hits the boundary $\partial S$. We can see that condition (2) ensures the trajectory points inside $S$ since $f(t, x)$ is the derivative of the trajectory at $x$, thus $x(t)$ will stay in $S$. Also, there is no requirement that the set $S$ needs a specific form such that the theorem holds, thus this theorem is a general result. In this paper, we will apply Nagumo theorem on specific types of sets to derive the sufficient and necessary conditions such that the set is an invariant set for the continuous system (1).

### 2.2. Convex Sets

In this subsection, we introduce the concepts of a family of convex sets which are considered as invariant sets for dynamical systems. In particular, these convex sets are polyhedra, polyhedral cones, ellipsoids, and Lorenz cones. These types of sets are common and used in many areas.

A polyhedron has two ways to define. The first way is given as the intersection of a finite number of half-spaces as follows:

$$\mathcal{P} = R[G, b] = \{ x \in \mathbb{R}^n \mid Gx \leq b \}, \tag{4}$$

where $G \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The second way is given as the convex combination of a finite number of points and a conic combination of some vectors as follows:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{\ell_1} \theta_i x_i + \sum_{j=1}^{\ell_2} \hat{\theta}_j \hat{x}_j, \quad \sum_{i=1}^{\ell_1} \theta_i = 1, \theta_i \geq 0, \hat{\theta}_j \geq 0 \right\}. \tag{5}$$

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1The solution of the system is $x(t) = x(0)e^{At}$. For any $x(0)$ in the span space, then $x(0)$ can be represented as $\sum_{i=1}^{k} \ell_i v_i$, where $\{v_i\}$ are the eigenvectors of $A$ and $\{\ell_i\}$ are some coefficients. Note that $e^{At}v_i = e^{\lambda_i t}v_i$, where $\lambda_i$ is the eigenvalue corresponding to $v_i$, it is easy to derive the proof.
where \( x^1, \ldots, x^\ell, \hat{x}^1, \ldots, \hat{x}^\ell \in \mathbb{R}^n \). A special type of polyhedra when it is bounded, i.e., \( \ell_2 = 0 \) in (5), is referred to as polytope.

A polyhedral cone with origin as its vertex can be considered as a special class of polyhedra, therefore we define polyhedral cone as follows

\[
C_P = \mathbb{R}[G, 0] = \{ x \in \mathbb{R}^n \mid Gx \leq 0 \},
\]

or equivalently

\[
C_P = \{ x \in \mathbb{R}^n \mid x = \sum_{j=1}^{\ell} \hat{\theta}_j \hat{x}^j, \ \hat{\theta}_j \geq 0 \},
\]

where \( G \in \mathbb{R}^{m \times n} \), and \( \hat{x}^1, \ldots, \hat{x}^\ell \in \mathbb{R}^n \). In particular, the positive quadrant in \( \mathbb{R}^n \), i.e., all coordinates are nonnegative, which therefore denoted by \( \mathbb{R}^n_+ \), is a special polyhedral cone and highly interesting as it has tremendous scientific and engineering applications.

Since an arbitrary ellipsoid is equivalent to an ellipsoid with origin as its center by a shifting transformation, we consider only an ellipsoid centered at origin defined as follows:

\[
E = \{ x \in \mathbb{R}^n \mid x^TQx \leq 1 \},
\]

where \( Q \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix.

A Lorenz cone is also refereed to as ice cream cone, or second order cone. Similar to the case of ellipsoids, an arbitrary Lorenz cone is equivalent to an Lorenz cone with vertex at origin by a shifting transformation, therefore we only consider a Lorenz cone with vertex at origin defined as follows:

\[
C_L = \{ x \in \mathbb{R}^n \mid x^TQx \leq 0, \ x^TQu_n \leq 0 \},
\]

where \( Q \in \mathbb{R}^{n \times n} \) is a symmetric nonsingular matrix with only one negative eigenvalue \( \lambda_n \). Thus, we have inertia \( \{ Q \} = \{ n - 1, 0, 1 \} \), which yields that there exists an orthonormal basis \( U = [u_1, u_2, \ldots, u_n] \) such that

\[
Q = U \Lambda \tilde{I} \Lambda^T U^T,
\]

where \( \Lambda = \text{diag} \{ \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{n-1}}, \sqrt{-\lambda_n} \} \) and \( \tilde{I} = \text{diag} \{ 1, ..., 1, -1 \} \). If we further operate an appropriate orthogonal transformation to \( C_L \), then a Lorenz cone with vertex at origin and axis at a coordinate axis, which refers to as standard Lorenz cone that denoted by \( C^*_L \), is generated and equivalent to \( C_L \). In particular, we have \( C^*_L = \{ x \in \mathbb{R}^n \mid x^T \tilde{I}x \leq 0, x^T \tilde{I}e_n \leq 0 \} \), where \( e_n = (0, \ldots, 0, 1)^T \).

\[\text{Recall that orthonormal basis means that } u_i^T u_j = \delta_{ij}, \text{ where } u_i \text{ is the eigenvector that corresponds to } \lambda_i \text{ and } \delta_{ij} \text{ is Kronecker delta function.}\]
3. Invariance Conditions

3.1. Tangent Cones

In this subsection, we will derive the formula of the tangent cones of polyhedra, polyhedral cone, ellipsoid, and Lorenz cone. According to Nagumo Theorem 2.2, the tangent cone is crucial for deriving the sufficient and necessary condition for an invariant set. For a given set $S$, it is easy to see that the tangent cone of a point $x$ in the interior of $S$ is $\mathbb{R}^n$, thus we only consider the case when $x$ is on the boundary of $S$.

**Theorem 3.1.** Let a polyhedron $P$ (or a polyhedral cone $C_P$) be in the form of (4) (or (6)). Assume $x$ is active at the $i_1$-th, $i_2$-th,..., $i_k$-th constraints, i.e., $g_{i_1}^T x = b_{i_1}, ..., g_{i_k}^T x = b_{i_k}$ (or $g_{i_1}^T x = 0, ..., g_{i_k}^T x = 0$), then the tangent cone at $x$ with respect to $P$ (or $C_P$) is

$$
T_P(x) \text{ (or } T_{C_P}(x)) = \{ y \in \mathbb{R}^n \mid g_{i_j}^T y \leq 0, j = 1, 2, ..., k \}. \tag{11}
$$

**Proof.** For an arbitrary point $\hat{x}$ in $P$, we have $g_{i_j}^T \hat{x} \leq b_{i_j}$, which yields $g_{i_j}^T (\hat{x} - x) \leq 0$. Then choosing $y = \hat{x} - x$, we have $x + ty \in P$ for sufficient small $t$, which deduces that $\|x + ty, P\| = 0$. Then we complete the proof. $\square$

Observing the formula of the tangent cone $T_P$ (or $T_{C_P}$) in (11), we can find that $T_P$ (or $T_{C_P}$) is a half space when $x$ is active at a single constraint. As the nonnegative quadrant is a special case of polyhedral cone, we have the following corollary.

**Corollary 3.2.** Let the nonnegative quadrant $\mathbb{R}^n_+$ be represented as $\{ x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, 2, ..., n \}$. Assume $x$ is on the boundary of $\mathbb{R}^n_+$, i.e., $x_i = 0$ for some $j = 1, 2, ..., k$, then the tangent cone at $x$ with respect to $\mathbb{R}^n_+$ is

$$
T_{\mathbb{R}^n_+}(x) = \{ y \in \mathbb{R}^n \mid y_j \geq 0, j = 1, 2, ..., k \}. \tag{12}
$$

We now turn to consider the second representation form of polyhedral sets given as in (5) and (7). As an arbitrary polyhedral set in the form of (5) is the union of a polytope and a polyhedral cone, we divide polyhedral sets into two basic classes, i.e., polytope and polyhedral cone. The difference between these two polyhedral sets is that a polytope is a bounded set, while a polyhedral cone is unbounded. There is one common characteristic between these two sets is that they are both represented by a finite number of vectors, which refers to as extreme points (or vertex) for polytope versus extreme ray for polyhedral cone. Therefore, it suffices to consider the tangent cones at these vectors instead of all points on the boundary. Similar to the proof in Theorem 3.1, the vector $x^j - x^i$ for any $j$ is in the tangent cone at $x^i$. Thus, the following theorem is immediate.

**Theorem 3.3.** Let a polytope $P$ be in the form of (7), i.e., all $\hat{\theta}_j = 0$. Then the tangent cone at vertex $x^i$ with respect to $P$ is

$$
T_P(x^i) = \{ y \in \mathbb{R}^n \mid y = \sum_{j=1, j \neq i}^{\ell_1} \alpha_j^{(i)} (x^j - x^i), \quad \alpha_j^{(i)} \geq 0, \text{ for } j = 1, 2, ..., \ell_1 \}. \tag{13}
$$
Theorem 3.4. Let a polyhedral cone \( C_P \) be in the form of (7). Then the tangent cone at extreme ray \( x^i \) with respect to \( C_P \) is

\[
\mathcal{T}_{C_P}(x^i) = \{ y \in \mathbb{R}^n | y = \hat{\alpha}_i^i x^i + \sum_{j=1, j \neq i}^k \alpha_j^i x^j, \hat{\alpha}_i^i \in \mathbb{R}, \alpha_j^i \geq 0, \text{ for } j = 1, 2, \ldots, \ell \}. \tag{14}
\]

Proof. One can prove that the tangent cone at extreme ray \( x^i \) with respect to \( \mathcal{T}_{C_P} \) is \( \mathcal{T}_P(x^i) = \{ y \in \mathbb{R}^n | y = \alpha_i^i x^i + \sum_{j=1, j \neq i}^k \alpha_j^i (x^j - x^i), \alpha_i^i \in \mathbb{R}, \alpha_j^i \geq 0, \text{ for } j = 1, 2, \ldots, \ell \} \), which is equivalent with (14) by letting \( \hat{\alpha}_i^i = \alpha_i^i - \sum_{j \neq i} \alpha_j^i \).

As a matter of fact, the region covered by the tangent cone at an extreme point with respect to a polytope contains no lines, while the region covered by the tangent cone at an extreme ray with respect to a polyhedral cone contains lines.

Corollary 3.5. Let the nonnegative quadrant \( \mathbb{R}_{+}^n \) be represented as \( \{ x \in \mathbb{R}^n | x = \sum_{i=1}^n \hat{\theta}_i e_i, \hat{\theta}_i \geq 0 \} \). Then the tangent cone at \( e_i \) with respect to \( \mathbb{R}_{+}^n \) is

\[
\mathcal{T}_{\mathbb{R}_{+}^n}(x) = \{ y \in \mathbb{R}^n | y_j \geq 0, j \neq i \}. \tag{15}
\]

We now analyze the tangent cones with respect to an ellipsoid and a Lorenz cone. As these two sets are both represented by a quadratic inequality, they can be considered simultaneously. Moreover, the two sets are both smooth at the boundaries except the vertex of Lorenz cone, thus there exists a tangent space at the boundaries except the vertex of Lorenz cone. Let us choose ellipsoids as an example, the outer norm of an arbitrary point \( x \in \partial E \) is \( Qx \), then the tangent space at \( x \) is represented as \( \{ y \in \mathbb{R}^n | y^T Q x = 0 \} \). Therefore, we have the following theorem. Note that a similar result for Lorenz cone can refer to [19].

Theorem 3.6. [19] Let an ellipsoid \( E \) (or a Lorenz cone \( C_L \)) be in the form of (8) (or (9)). Assume \( x \) is on the boundary of \( E \) (or \( C_L \)), then the tangent cone at \( x \) with respect to \( E \) (or \( C_L \)) is

\[
\mathcal{T}_E(x) \text{ (or } \mathcal{T}_{C_L} \text{)} = \{ y \in \mathbb{R}^n | y^T Q x \leq 0, \text{ for all } x \in E \text{ (or } C_L \)\}. \tag{16}
\]

3.2. Invariance Condition

In this subsection, we will investigate the sufficient and necessary conditions under which the involved convex sets in this paper are invariant sets with respect to a dynamical system as shown in [1].

Theorem 3.7. Let a polyhedron \( P \) (or a polyhedral cone \( C_P \)) be in the form of (4) (or (6)). Then \( P \) (or \( C_P \)) is an invariant set with respect to the dynamical system (1) if and only if any point \( x \) on the boundary of \( P \) (or \( C_P \)) holds the following condition

\[
g_j^T f(t_0, x) \leq 0, j = 1, 2, \ldots, k, \tag{17}
\]

where \( x \) is active at the \( i_1 \)-th, \( i_2 \)-th,..., \( i_k \)-th constraints.
Corollary 3.8. Let the nonnegative quadrant $\mathbb{R}_+^n$ be represented as \( \{ x \in \mathbb{R}^n | x_i \geq 0, i = 1, 2, ..., n \} \). Then $\mathbb{R}_+^n$ is an invariant set with respect to the dynamical system (1) if and only if any point $x$ on the boundary of $\mathbb{R}_+^n$ holds the following condition

\[ f_{ij}(t_0, x) \geq 0, j = 1, 2, ..., k, \]  

(18)

where $x_{ij} = 0$ for $j = 1, 2, ..., k$.

Theorem 3.9. Let a polytope $P$ be in the form of (5), i.e., all $\hat{\theta}_j = 0$. Then $P$ is an invariant set with respect to the dynamical system (1) if and only if for any extreme point $x_i$, there exists nonnegative scalars $\alpha_{ij}^{(i)} \geq 0$, for $j \neq i, j = 1, 2, ..., \ell_1$ such that the following condition holds

\[ f(t_0, x^i) = \sum_{j=1,j \neq i}^{\ell_1} \alpha_{ij}^{(i)} (x^j - x^i). \]  

(19)

We now investigate the way to verify the existence of the coefficients $\alpha_{ij}^{(i)}$ in (19), which might be not unique when $\ell_1 > n$, i.e., the number of vertices is greater than the dimension of points. Condition (19) is equivalently reformulated as

\[ f(t_0, x^i) = \sum_{j \neq i}^{\ell_1} \alpha_{ij}^{(i)} x^j - \left( \sum_{j \neq i}^{\ell_1} \alpha_{ij}^{(i)} \right) x^i = \sum_{j=1}^{\ell_1} \alpha_{ij}^{(i)} x^j, \text{ with } \sum_{j=1}^{\ell_1} \alpha_{ij}^{(i)} = 0, \]  

(20)

where $\alpha_{i}^{(i)} = -\sum_{j \neq i} \alpha_{j}^{(i)}$.

For the sake of simplicity, we denote $X = [x^1, x^2, ..., x^{\ell_1}], \hat{X} = [\hat{x}^1, \hat{x}^2, ..., \hat{x}^{\ell_1}] = [X^T, e]^T, \hat{f} = (f^T, 0)^T$, and $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, ..., \alpha_{\ell_1}^{(i)})^T$. Then two optimization models can be built to solve the coefficients in (19).

The first model that essentially is a linear feasibility problem is represented as follows:

\[
\begin{align*}
\min \quad & 0 \\
\text{s.t.} \quad & \hat{X} \alpha^{(i)} = \hat{f}, \quad \alpha_{ij}^{(i)} \geq 0, \quad j \neq i.
\end{align*}
\]  

(21)

As the objective function in (21) is a fixed number, the optimization model (21) has optimal solution, which is not necessary unique, if and only if the condition (19) has solution. Without loss of generality, we assume $X$ has the property that its column vectors are independent, and thereby $\hat{X}$ as well. Then the model (21) can be discussed into two cases: the number of rows of $\hat{X}$ is greater than or equal to the number of columns of $\hat{X}$, and the number of rows of $\hat{X}$ is less than the number of columns of $\hat{X}$. As a matter of fact, the first case is trivial, since the equation $\hat{X} \alpha^{(i)} = \hat{f}$, which assume it has solutions, has an unique solution, which is explicitly represented as

\[ \alpha^{(i)} = (\hat{X}^T \hat{X})^{-1} \hat{X}^T \hat{f}. \]

Then one only needs to check whether the solution satisfies the condition that $\alpha_{ij}^{(i)} \geq 0, j \neq i$.

In the second case, since the equation $\hat{X} \alpha^{(i)} = \hat{f}$ always has solutions that might be unique
or not, which means the optimization model \((21)\) is always feasible, it is hard to obtain the solution of this equation directly.

There are normally two ways to solve the optimization model \((21)\): pivot methods or interior point methods (IPMs). The pivot methods, e.g., Simplex algorithm\([2]\), Criss-Cross algorithm\([20]\), have exponential complexity. The basic idea of pivot methods is enhancing, i.e., decreasing the objective function if it is a minimization problem, the optimization problem along the edges of the polyhedral set which is the feasible region of this optimization problem by pivoting from one vertex to another. The IPMs \([21]\) have polynomial complexity. The basic idea of IPMs is enhancing the optimization problem by along a central path interior the feasible region. Although IPMs have polynomial complexity, pivot methods shows great efficiency when the problem is not very large for linear optimization. The advantage of IPMs becomes significant when the size of the problem is large.

To solve the optimization model \((21)\), one can also consider its dual problem, which is also a linear method and represented as

\[
\begin{align*}
\max & \quad \tilde{f}^T y \\
\text{s.t.} & \quad (\tilde{\mathbf{x}}^i)^T y = 0, \\
& \quad (\tilde{\mathbf{x}}^j)^T y \leq 0, \quad j \neq i.
\end{align*}
\]  

\((22)\)

Then according to the knowledge in optimization, e.g., \([16]\), the optimality condition that is the sufficient and necessary condition of the existence of the optimal solution for an optimization problem, by introducing the artificial variables \(s^{(i)}_j\), is

\[
\tilde{\mathbf{x}}\alpha^{(i)} = \tilde{f}, \quad (\tilde{\mathbf{x}}^i)^T y = 0, \quad (\tilde{\mathbf{x}}^j)^T y + s^{(i)}_j = 0, \quad \alpha^{(i)}_j, s^{(i)}_j \geq 0, \quad j \neq i.
\]  

\((23)\)

The second model that is a quadratic optimization model is as follows:

\[
\begin{align*}
\min & \quad \frac{1}{2} \| \mathbf{x}\alpha^{(i)} - f \|^2_2 \\
\text{s.t.} & \quad e^T \alpha^{(i)} = 0, \quad \alpha^{(i)}_j \geq 0, \quad j \neq i.
\end{align*}
\]  

\((24)\)

One key difference between model \((21)\) and \((24)\) is that the former one might be infeasible, while the latter one is always feasible, which implies the latter model always has optimal solutions. The objective function in \((24)\) is the half of the square of the distance between \(f\) and \(\mathbf{x}\alpha^{(i)}\), therefore the optimal objective function value is exactly equal to 0 if model \((21)\) is feasible. Otherwise, the model \((24)\) will output the point that is closest to the polyhedral set defined by the vectors \(x^1, ..., x^\ell\), in which case, the optimal objective function value is strictly positive. To solve the quadratic model \((24)\), we consider its Lagrangian, which, by introducing the dual variables \(\eta^{(i)} = (\eta^{(i)}_1, \eta^{(i)}_2, ..., \eta^{(i)}_\ell)^T\) with \(\eta^{(i)}_j \geq 0\) for \(j \neq i\), is described as

\[
L = \frac{1}{2} \| \mathbf{x}\alpha^{(i)} - f \|^2_2 + \eta^{(i)} e^T \alpha^{(i)} - \sum_{j \neq i} \eta^{(i)}_j \alpha^{(i)}_j.
\]  

\((25)\)

The Karush-Kuhn-Tucker (KKT) condition \([15]\) is normally chosen as the first-order necessary optimality condition of an optimization problem. To verify the KKT condition of an
optimization problem, one needs to check whether this problem satisfies a so called linear independence constraint qualification (LICQ) holds. The LICQ is said to hold at a point $x$ if the gradients of the active constraint are linearly independent at $x$. It is easy to verify that optimization problem (24) holds LICQ. Then the KKT condition of (24) is shown as follows:

$$X^T(X\alpha^{(i)} - f) + \eta^{(i)}(\uparrow_i 0) = 0 \quad (26)$$
$$e^T\alpha^{(i)} = 0 \quad (27)$$
$$\alpha^{(i)}_j \geq 0, j \neq i \quad (28)$$
$$\eta^{(i)}_j \alpha^{(i)}_j = 0, j \neq i \quad (29)$$

where $\eta^{(i)}(\uparrow_i 0)$ denotes the $i$-th entry in $\eta^{(i)}$ is replaced by 0.

**Theorem 3.10.** Let a polyhedral cone $C_P$ be in the form of (7). Then $C_P$ is an invariant set with respect to the dynamical system (1) if and only if for any extreme ray $x^i$, there exists nonnegative scalars $\alpha^{(i)}_j \geq 0$, for $j \neq i, j = 1, 2, ..., \ell$, and $\alpha^{(i)}_i \in \mathbb{R}$, such that the following condition holds

$$f(t_0, x^i) = \alpha^{(i)}_i x^i + \sum_{j=1, j \neq i}^k \alpha^{(i)}_j x^j. \quad (30)$$

**Corollary 3.11.** Let the nonnegative quadrant $\mathbb{R}^n_+$ be represented as $\{x \in \mathbb{R}^n | x = \sum_{i=1}^n \hat{\theta}_i e^i, \hat{\theta}_i \geq 0\}$. Then $\mathbb{R}^n_+$ is an invariant set with respect to the dynamical system (1) if and only if for any extreme ray $e^i$, the following condition holds

$$f_j(t_0, e^i) \geq 0, j \neq i. \quad (31)$$

**Theorem 3.12.** Let an ellipsoid $E$ (or a Lorenz cone $C_L$) be in the form of (8) (or (9)). Then $E$ (or $C_L$) is an invariant set with respect to the dynamical system (1) if and only if any point $x$ on the boundary of $E$ (or $C_L$) holds the following condition

$$x^TQf(t_0, x) \leq 0. \quad (32)$$

According to Theorem 3.12 one has to check whether all points on the boundary of an ellipsoid or a Lorenz cone satisfy condition (32). But it is complicated if we directly examine condition (32) along the boundary of an ellipsoid or a Lorenz cone. We present an optimization method to solve this problem. For an ellipsoid $E$, we consider the following optimization model.

$$\max x^TQf(t_0, x) \quad (33)$$
$$\text{s.t. } x^TQx = 1,$$

where $Q$ is a symmetric positive definite matrix. This is not a convex problem, as the constraint is nonconvex.

$$\max x^TQf(t_0, x) \quad (34)$$
$$\text{s.t. } x^TQx = 1,$$
We consider the linear dynamical system, i.e., \( f(t_0, x) = Ax \). Then the optimization problem can be formulated as

\[
\begin{aligned}
\text{min} & \quad -\frac{1}{2} x^T (A^T Q + QA)x \\
\text{s.t.} & \quad x^T Q x = 1,
\end{aligned}
\]

The Lagrangian of optimization problem (35) is as follows:

\[
L = -\frac{1}{2} x^T (QA + A^T Q)x + \frac{\eta}{2} x^T Q x - 1.
\]

It is easy to check the optimization problem (35) satisfies LICQ condition, thus the first order optimality condition (KKT condition) is

\[
\begin{aligned}
(A^T Q + QA - \eta Q)x = 0 \\
x^T Q x = 1
\end{aligned}
\]

Now we consider the second order optimality condition, which can be written as

\[
\begin{aligned}
d^T (A^T Q + QA - \eta Q)d \leq 0 \\
d^T Q x = 0 \\
(A^T Q + QA - \eta Q)x = 0 \\
x^T Q x = 1
\end{aligned}
\]

We now show that (38)-(41) yield \( A^T Q + QA - \eta Q \preceq 0 \). Since \( A^T Q + QA - \eta Q \) is singular, the condition in (41) can be replaced by \( x \neq 0 \). The we consider the following two cases: if \( d = Q x \) also satisfies condition (38), then we can say that condition (38) satisfies for any \( d \in \mathbb{R}^n \), which is equivalent to \( A^T Q + QA - \eta Q \preceq 0 \). Otherwise, if \( d = Q x \) does not satisfy condition (38), we have

\[
(Qx)^T (A^T Q + QA - \eta Q)(Qx) > 0.
\]

Now assume \( A^T Q + QA - \eta Q \not\preceq 0 \), then there exists an nonzero vector \( \tilde{x} \in \mathbb{R}^n \), such that \( (A^T Q + QA - \eta Q)\tilde{x} = \lambda \tilde{x} \), where \( \lambda > 0 \). By multiplying appropriate scalar for \( \tilde{x} \), we can have the following orthogonal decomposition of \( \tilde{x} \),

\[
\tilde{x} = \tilde{d} + Q x, \quad \text{where} \quad \tilde{d}^T Q x = 0.
\]

Then \( \tilde{d} \) satisfies condition (38). Substituting \( \tilde{d} = \tilde{x} - Q x \) into the left formula of condition (38), we have

\[
\lambda ||\tilde{x}||^2 - 2\lambda (Qx)^T \tilde{x} + (Qx)^T (A^T Q + QA - \eta Q)(Qx).
\]
Note that $\|\tilde{x}\|^2 = \|d\|^2 + \|Qx\|^2 + 2(Qx)^T \hat{x}$, we have $\|\hat{x}\|^2 > 2(Qx)^T \hat{x}$. Also, applying (12) to (44), we have that the formula in (44) is positive. This is a contradiction. Therefore, in this case, we also have $A^T Q + QA - \eta Q \preceq 0$.

Since $A^T Q + QA - \eta Q$ is singular, the last condition in (44) can be replaced by $x \neq 0$. By left multiplying $d^T$ to the third condition in (44), we have $d^T (A^T Q + QA) = 0$. In one can prove that (44) is equivalent with that

$$
\begin{align*}
  d^T (A^T Q + QA - \eta Q) & \leq 0 \\
  x^T (A^T Q + QA - \eta Q) & = 0 \\
  (A^T Q + QA - \eta Q) x & = 0
\end{align*}
$$

where $x \neq 0$.

For a Lorenz cone $C_L$, we consider the following optimization model.

$$
\begin{align*}
  \max & \quad x^T Q f(t_0, x) \\
  \text{s.t.} & \quad x^T Q x = 1, \\
  & \quad x^T Q u_n \leq 0
\end{align*}
$$

We consider the linear dynamical system, i.e., $f(t_0, x) = Ax$. Then the optimization problem can be formulated as

$$
\begin{align*}
  \min & \quad -\frac{1}{2} x^T (A^T Q + QA) x \\
  \text{s.t.} & \quad x^T Q x = 1 \\
  & \quad x^T Q u_n \leq 0
\end{align*}
$$

The Lagrangian of (47) is

$$
L = -\frac{1}{2} x^T (QA + A^T Q) x + \frac{\eta}{2} (x^T Q x - 1) + \alpha x^T Q u_n.
$$

The KKT condition is

$$
\begin{align*}
  (A^T Q + QA - \eta Q)x - \alpha x^T Q u_n &= 0 \\
  x^T Q x &= 1 \\
  x^T Q u_n &\leq 0 \\
  \alpha &\geq 0 \\
  \alpha x^T Q u_n &= 0
\end{align*}
$$

We can also prove that $\eta \leq 0$ as the discussion of ellipsoid.
Now we consider the second order optimality condition, which can be written as

\[ d^T (A^T Q + QA - \eta Q) d \leq 0 \]
\[ d^T Q x = 0 \]
\[ (A^T Q + QA - \eta Q)x - \alpha Qu_n = 0 \]
\[ x^T Q x = 1 \]
\[ x^T Qu_n \leq 0 \]
\[ \alpha \geq 0 \]
\[ \alpha x^T Qu_n = 0 \]

If \( x^T Qu_n < 0 \), which implies \( \alpha = 0 \), then this yields a similar condition as ellipsoid. Thus, \( A^T Q + QA - \eta Q \preceq 0 \). If \( x^T Qu_n = 0 \), a similar argument will be applied to derive the conclusion.

4. Conclusion

Positively invariant set is an important concept in dynamical system and has widely used in application in control. In this paper, we investigate Nagumo Theorem and apply it into specific convex sets, e.g., polyhedra, ellipsoids and cones. Then we derive the sufficient and necessary conditions of some classical convex sets as positively invariant sets for a continuous dynamical system. The method is to derive the tangent cones of these sets. We derive some new positive invariance conditions or similar invariance conditions obtained by other researchers. To verify the invariance conditions, we propose methods using optimization theory and models. The introduction of using optimization techniques brings a novel insight on studying invariant set for a continuous dynamical system.

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