Arithmetically Cohen-Macaulay Curves cut out by Quadrics
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The following question was raised by M. Stillman.

Main Question: Let $C \subset \mathbb{P}^r = \mathbb{P}\mathbb{P}^r$ be a smooth arithmetically Cohen-Macaulay curve which is cut out scheme theoretically by quadrics. Is the homogeneous ideal of $C$ necessarily cut out by quadrics?

In [4], it was shown that the question has an affirmative answer if $r \leq 5$. The purpose of this note is to show that the question has a negative answer (there is a counterexample with $r = 7$).

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1 Homogeneous and Scheme-Theoretic Generation by Quadrics

Let $X$ be a projective variety. It is often of interest to know whether or not the homogeneous ideal of $X$ can be generated by quadrics, e.g. if $X$ is a general canonical curve. In such a case, $X$ is cut out scheme-theoretically by quadrics as well. It is usually easier to verify the scheme-theoretic statement—this amounts to ignoring the vertex of the affine cone over $X$.

Problem: Let $C \subset \mathbb{P}^r = \mathbb{P}\mathbb{P}^r$ be a smooth curve which is cut out scheme theoretically by quadrics. Is the homogeneous ideal of $C$ necessarily cut out by quadrics?

In [4], this problem was investigated. The answer is a resounding no. A counterexample was found with $r = 5$. However, positive results were found. The problem has an affirmative answer for curves on scrolls, all curves with
$r \leq 4$, and arithmetically Cohen-Macaulay curves which lie on projectively
normal K3 surfaces cut out by quadrics (this includes all arithmetically C-M
curves with $r = 5$). This leads to a more precise question, which we could
not answer:

**Question:** Let $C \subset \mathbb{P}^r = \mathbb{CP}^r$ be a smooth arithmetically Cohen-Macaulay
curve which is cut out scheme theoretically by quadrics. Is the homogeneous
ideal of $C$ necessarily cut out by quadrics?

It turns out that this question also has a negative answer.

**Proposition 1** Let $C \subset \mathbb{P}^7$ be a general degree 19 embedding of a general
genus 12 curve over an algebraically closed field of characteristic 0. Then $C$
is smooth and arithmetically Cohen-Macaulay, $C$ is cut out scheme-theoretically
by quadrics, and the homogeneous ideal of $C$ is not cut out by quadrics.

## 2 Candidates for a Counterexample

Let $C \subset \mathbb{P}^r$ be an arithmetically Cohen-Macaulay curve of degree $d$ and
genus $g$. Assume in addition that $\mathcal{O}_C(1)$ is non-special, i.e. $H^1(\mathcal{O}_C(1)) = 0$.
Then $d = g + r$.

It turns out that for certain values of $g$ and $r$, the homogeneous ideal of
such a curve $C$ *cannot* be cut out by quadrics, for simple dimension reasons.
Let $I$ denote the ideal sheaf of $C$. Then if

\[(1) \quad (r + 1)h^0(I(2)) < h^0(I(3)),\]

the natural map

\[H^0(I(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(I(3))\]

cannot be surjective, so that the homogeneous ideal of $C$ cannot be generated
by quadrics. Using Riemann-Roch, $(1)$ becomes

\[(2) \quad g > \frac{r(r - 2)}{3}.\]

On the other hand, if we want $C$ to be scheme-theoretically cut out by
quadrics, then we must have enough quadrics, i.e.
\[
\binom{r}{2} - g \geq r - 1.
\]

Equality holds if and only if \(C\) is a complete intersection of \(r - 1\) quadrics; but in this case the homogeneous ideal is cut out by quadrics as well. This can be improved slightly: in \([4, \text{Cor. 2.5}]\) it was shown that if \(C\) is cut out scheme theoretically by \(r\) quadrics, then necessarily

\[
g = (r - 1)d/2 + 1 - 2^{r-1}.
\]

So if \((3)\) does not hold, then

\[
g \leq \frac{r^2 - 3r - 2}{2}.
\]

There are no counterexamples to the main question for \(r \leq 5\) \([4]\). Suppose that there is a non-special counterexample with \(r = 6\). Then \(g \geq 9\) by \((4)\). Since \((3)\) gives \(g \leq 8\), it follows that \((3)\) holds, and \(g = 9\). But then \(d = 15\), and a contradiction is reached.

Turning next to \(r = 7\), \((3)\) gives \(g \geq 12\), and \((4)\) gives \(g \leq 13\). In the following section, we show that in fact the general curve of degree 19 and genus 12 in \(\mathbb{P}^7\) is a counterexample.

### 3 The counterexample

Pick 22 general points \(p_1, p_2, p_3, q_1, \ldots, q_7, r_1, \ldots, r_{12}\) in \(\mathbb{P}^2\). Let \(C'\) be a general plane curve of degree 9 passing through the \(p_i\) with multiplicity 3, through the \(q_i\) with multiplicity 2, and simply through the \(r_i\). The linear system \(|L|\) of degree 7 curves passing doubly through the \(p_i\) and simply through the \(q_i\) and \(r_i\) maps \(C'\) birationally to a smooth curve \(C\) of degree 19 and arithmetic genus 12 in \(\mathbb{P}^7\).

It is a simple matter to use MACAULAY \([3]\) to construct such a curve. In describing the calculation, I will informally say that a general curve has a certain property, when I mean that the property is satisfied for an example curve constructed using MACAULAY’s pseudo-random number generator. In fact, I repeated the construction several times with different pseudo-random coefficients, and the properties mentioned below held in each instance. Thus, as expected, a “general” curve has been constructed.
Macaulay’s pseudo-random number generator is used to construct 22 “general” points in $\mathbb{P}^2_{\mathbb{F}_{31991}}$, and from this the curve $C'$ (actually, there is no harm in supposing that the $p_i$ are $(1,0,0), (0,1,0), (0,0,1)$, to shorten computations). By calculating the Jacobian of $C'$, it is checked that the singular scheme of $C'$ has degree 19 as expected (triple points count at least 4 times). Hence $C'$ has the expected geometric genus 12. The equations of the image curve $C$ can then be explicitly calculated. $C$ is cut out ideal theoretically by 9 independent quadrics and 2 independent cubics, and has Hilbert function $(1 + 6t + 12t^2)(1 - t)^{-2}$. In particular $C$ has arithmetic genus 12; being the image of the normalization of $C'$ by the base point free system $|L|$ on the blowup of $\mathbb{P}^2$, it follows that $C$ is smooth. Let $\tilde{C} \subset \mathbb{P}^2$ be the scheme cut out by the 9 quadrics alone. Via Macaulay, $\tilde{C}$ has degree 19 and arithmetic genus 12. It follows easily that $C = \tilde{C}$, i.e. $C$ is cut out scheme-theoretically by quadrics.

Next, to see that $C$ is arithmetically Cohen-Macaulay, note that $C$ is non-special since the projective dimension of the embedding system is 7, is linearly normal by construction, and is quadratically normal by Riemann-Roch and $h^0(I_C(2)) = 9$ found by Macaulay. This suffices to show that $C$ is arithmetically Cohen-Macaulay by [1] P. 222 or the argument in the proof of Theorem 1.2.7 in [8].

**Proof of Proposition 1:** The key point is to show that the conditions “arithmetically Cohen-Macaulay” and “scheme-theoretically cut out by quadrics” are dense.

A curve is arithmetically Cohen-Macaulay if and only if it is projectively normal. So $C$ is arithmetically Cohen-Macaulay if and only if $H^1(I_C(n)) = 0$ for all $n \geq 0$, where $I_C$ is the ideal sheaf of $C$. By [2], $H^1(I_C(n)) = 0$ for all $n \geq 13$, so there are only finitely many cohomology groups that are required to vanish in addition. By upper semicontinuity of $h^1(I_C(n)) = \dim H^1(I_C(n))$, this is a Zariski open condition in the Hilbert scheme.

As to the condition of being scheme-theoretically cut out by quadrics, we may restrict to considering curves which are arithmetically Cohen-Macaulay. Let $V$ be the 9 dimensional space of quadrics containing $C$. Consider the maps

\[ V \otimes H^0(\mathcal{O}_{\mathbb{P}^7}(k)) \to H^0(I_C(k + 2)) \]

V cuts out $C$ scheme-theoretically if and only if (5) is surjective for some
\( k \geq 12 \) (since \( C \) is 14-regular by \cite{[7]}; a smaller bound for effective \( k \) can be given if desired). This is again an open condition.

Finally, let \( \text{Hilb}_{19}^{0} \) be the subset of the Hilbert scheme parametrizing smooth, irreducible curves in \( \mathbb{P}^{7} \) of degree 19 and genus 12. It is open in the Hilbert scheme by \cite[p. 99]{[8]}. \( \text{Hilb}_{19}^{0} \) is defined over \( \text{Spec} \mathbb{Z} \) and is irreducible (its geometric fibers are equidimensional and irreducible; this follows from the irreducibility of \( M_{12} \) in arbitrary characteristic \cite{[8]}, and the non-speciality of \( |L| \)).

Hence the set of smooth arithmetically Cohen-Macaulay curves scheme-theoretically cut out by quadrics is non-empty and open, hence dense, in \( \text{Hilb}_{19}^{0} \). This completes the proof of Proposition \cite{[8]}. QED

It seems appropriate to conclude with some related questions.

In \cite{[1]}, \cite[§3]{[9]}, it was proven that a general linear system of degree \( d \geq \left\lceil (3g + 4)/2 \right\rceil \) on a curve \( C \) of genus \( g \) embeds \( C \) as an arithmetically Cohen-Macaulay curve. Rather than looking for a bound for all curves, instead one can ask:

Problem: Find the smallest possible \( d(g) \) such that for all \( d \geq d(g) \), a general curve of genus \( g \) admits a degree \( d \) complete embedding which is arithmetically Cohen-Macaulay.

Remark. Suppose that \( d \geq (2g + 1 + \sqrt{8g + 1})/2 \). Then the general degree \( d \) embedding of a general curve of genus \( g \) is arithmetically Cohen-Macaulay \cite{[9]}. This bound is in fact sharp for non-special embeddings. The inequality is just the solution of the inequality \( h^{0}(\mathcal{O}_{\mathbb{P}^{r}}(2)) \geq h^{0}(\mathcal{O}_{C}(2)) \) for a general non-special embedding.

Similarly, one can ask

Problem: Find the smallest possible \( d'(g) \) such that a general degree \( d \) embedding of a general curve of genus \( g \) is scheme theoretically cut out by quadrics if \( d \geq d'(g) \).

By work of Green and Lazarsfeld \cite[Prop. 2.4.2]{[8]}, \( d'(g) \leq \left\lceil (3g + 6)/2 \right\rceil \), and Proposition \cite{[8]} shows that this is not sharp.
Question: Is Proposition 4 true without restriction on the characteristic? Is there a counterexample to the main question with $r = 6$?

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris. Geometry of Algebraic Curves, Vol I, Springer 1985.

[2] E. Ballico and Ph. Ellia. On postulation of curves: Embeddings by complete linear series. Arch. Math. 43 (1984) 244–249.

[3] D. Bayer and M. Stillman. Macaulay, a computer algebra system. Available free from the authors for the Macintosh, VAX, SUN, and many other computers (1989) (with scripts by D. Eisenbud and M. Stillman).

[4] L. Ein, D. Eisenbud, and S. Katz. Varieties cut out by quadrics: Scheme-theoretic versus homogeneous generation of ideals. In: Algebraic Geometry, Sundance 1986, SLN 1311.

[5] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. Publ. Math. IHES 36 (1969), 75–110.

[6] M. Green and R. Lazarsfeld. On the projective normality of complete linear series on an algebraic curve. Inv. Math. 83 (1986) 73–90.

[7] L. Gruson, R. Lazarsfeld, and C. Peskine. On a theorem of Castelnuovo, and the equations defining space curves. Inv. math. 72 (1983) 491–506.

[8] R. Lazarsfeld. A sampling of vector bundle techniques in the study of linear series. Informal note for lectures at ICTP College on Riemann Surfaces, Trieste, December 1987.

[9] D. Mumford. Geometric Invariant Theory. Springer-Verlag, Heidelberg-Berlin-New York, 1965.