Asymptotically Normal Estimators of the Gerber-Shiu Function in Classical Insurance Risk Model

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Abstract: Nonparametric estimation of the Gerber-Shiu function is a popular topic in insurance risk theory. Zhang and Su (2018) proposed a novel method for estimating the Gerber-Shiu function in classical insurance risk model by Laguerre series expansion based on the claim number and claim sizes of sample. However, whether the estimators are asymptotically normal or not is unknown. In this paper, we give the details to verify the asymptotic normality of these estimators and present some simulation examples to support our result.

Keywords: Gerber-Shiu function; Laguerre series; classical risk model; asymptotic normality

1. Introduction

In this paper, we consider the classical risk model

\[ U_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \]

where \( u \geq 0 \) denotes the initial surplus level and \( c > 0 \) denotes the premium rate. The claim number \( \{N_t\}_{t \geq 0} \) follows a Poisson process with intensity \( \lambda > 0 \). \( X_1, X_2, \ldots \) denote the claim sizes which are independent and identically distributed random variables with density \( f \) and mean \( \mu \). Meanwhile, suppose that \( \{N_t\} \) and \( \{X_i\} \) are mutually independent. Let \( \tau = \inf\{t \geq 0 : U_t < 0\} \) denote ruin time and we set \( \tau = \infty \) when \( U_t \geq 0 \) for all \( t \geq 0 \).

Gerber and Shiu [1] considered the expected discounted penalty function about the ruin time \( \tau \), the surplus before ruin time \( U_{\tau-} \) and the deficit at ruin time \( |U_{\tau}| \) which is defined

\[ \phi(u) = \mathbb{E}[e^{-\delta \tau} w ((U_{\tau-}), |U_{\tau}|) 1_{\{\tau < \infty\}} | U_0 = u], \quad u \geq 0, \]

where \( w : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a measurable penalty function of \( U_{\tau-} \) and \( |U_{\tau}| \) is the indicator function, \( \delta \) is the interest force. In this paper, we verify asymptotically normal estimators of the Gerber-Shiu function when \( \delta = 0 \). Many researchers have made remarkable contributions to this model and its generalizations have been made in various risk model. Asmussen and Albercher [2] considered the classical risk model; Gerber and Landry [3], Tsai [4,5] considered the compound Poisson risk model perturbed by diffusion; Yu [6] and Yu et al. [7] investigated the absolute ruin risk model. Zhao and Yin [8] studied the Gerber-Shiu function in Lévy risk model; Xie and Zou [9] and Zhu et al. [10] studied the compound Poisson with delayed claims. The Gerber-Shiu function has also been studied by Yin and Wang [11], Shen et al. [12], Yin and Yuen [13], Cai et al. [14], Deng et al. [15], Dong et al. [16], Wang and Zhang [17], Yu et al. [18], Peng et al. [19] among others.

In all of these mentioned papers, they assumed that the parameter \( \lambda \) and the claim size density \( f \) are known. However, the probability characteristics of the surplus process are usually unknown.
Recently, statistical estimation of risk measures based on the observed data information of claim number and claim sizes has become a popular topic, see e.g., Shimizu [20] applied regularized Laplace transform to estimate the Gerber-Shiu function in Lévy risk model; Zhang and Yang [21,22] estimated the ruin probability based on high-frequency observation and low-frequency observation, respectively; Zhang [23] constructed an estimator of the Gerber-Shiu by Fourier-Sinc series expansion and Zhang [24] estimated the finite time ruin probability by double Fourier transform. Later, Zhang and Su [25] proposed a new method for estimating the Gerber-Shiu function by Laguerre basis and derived the convergence rate of the estimate. In this paper, we show that the above estimator for the Gerber-Shiu function is asymptotically normal. Asymptotic normality results allow us to establish confidence intervals. For more study on the statistical estimation of risk model, the interested readers are referred to the work of Shimizu and Zhang [26] Yu et al. [27], Peng and Wang [28], Yu et al. [29] and Zhang et al. [30].

The remainder of this paper is organized as follows. We present some preliminaries on Laguerre expansion of Gerber-Shiu function and the estimator of Gerber-Shiu function in Section 2. We give the details to verify the asymptotic normality of these estimators in Section 3. Finally, we present some numerical examples to verify the proposed estimator is asymptotically normal in Section 4.

2. Preliminaries

First, we introduce the following notation:

- \( \mathbb{R}_+ = [0, \infty), \mathbb{N} = \{0, 1, 2, \cdots \}, \mathbb{N}_0 = \{1, 2, \cdots \}. \)
- \( \mathbb{L}^2(\mathbb{R}_+) = \{ f : \mathbb{R}_+ \to \mathbb{R} \mid \int_{\mathbb{R}_+} |f(x)|^2 dx < \infty \}. \)
- \( \mathcal{L}f(s) \) denotes the Laplace transform of claim size density \( f, \)
  \[ \mathcal{L}f(s) = \int_{\mathbb{R}_+} e^{-sx}f(x)dx. \]
- \( \Phi_X = \mathbb{E}[\exp(it^T X)] \) denotes the characteristic function of random vector \( X \in \mathbb{R}^k, \) where \( t \) is a random vector in \( \mathbb{R}^k, k \in \mathbb{N}_0. \)
- For \( \forall f, g \in \mathbb{L}^2(\mathbb{R}_+), \) let \( \langle f, g \rangle = \int_{\mathbb{R}_+} f(x)g(x)dx \) be the scalar product and \( \| f \| = \int_{\mathbb{R}_+} f^2(x)dx \) be \( \mathbb{L}^2 \)-norm.
- For positive function \( f_1, f_2, \) let \( f_1 \preceq f_2 \) be \( f_1(x) \leq C f_2(x) \), where \( C \) is a positive constant.
- \( \top \) means the transpose of matrix.
- \( 0 \) is a zero vector.
- Let \( \overset{p}{\rightarrow} \) be convergence in probability and \( \overset{D}{\rightarrow} \) be convergence in distribution.
- \( f = O(g) \) means that \( |f| \leq C|g| \) for some constant \( C. \)

Meanwhile, we need the following conditions, which have also been considered in Zhang and Su [25]:

- **Condition 1** The premium rate \( c > \lambda \mu; \)
- **Condition 2** Suppose that \( \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (1 + x)w(x, y)f(x + y)dydx < \infty; \)
- **Condition 3** For some \( k_1, k_2 \in \mathbb{R}_+, \) suppose that the penalty function \( w(x_1, x_2) \preceq \left( 1 + x_1^{k_1} \right) \left( 1 + x_2^{k_2} \right). \)

2.1. Laguerre Expansion of Gerber-Shiu Function

The Laguerre functions \( \{\psi_k\}_{k \in \mathbb{N}} \) are defined by
\[
\psi_k(x) = \sqrt{2}L_k(2x)e^{-x}, \quad x \in \mathbb{R}_+, \quad k \in \mathbb{N}.
\]
where \( L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{x^j}{j!}, \) \( x \geq 0. \) It follows from Abramowitz and Stegun [31] that

- \( \{ \psi_k \}_{k \in \mathbb{N}} \) form a complete orthogonal basis over \( L^2(\mathbb{R}_+) \). Then, when \( f \in L^2(\mathbb{R}_+) \), it can be expanded by the Laguerre function: \( f(x) = \sum_{k \in \mathbb{N}} (f, \psi_k) \psi_k(x); \)
- \( \{ \psi_k \}_{k \in \mathbb{N}} \) are uniformly bounded, i.e. \( |\psi_k(x)| \leq \sqrt{2}, \) \( x \in \mathbb{R}_+, k \in \mathbb{N}; \)
- \( \forall k, j \in \mathbb{N}, \int_0^\infty \psi_k(x-y)\psi_j(y)dy = \frac{1}{\sqrt{2}} [\psi_{k+j}(x) - \psi_{k+j+1}(x)], \langle \psi_k, \psi_k \rangle = 1 \) and \( \langle \psi_j, \psi_k \rangle = 0 \) for \( k \neq j. \)

According to Gerber and Shiu [1], the Gerber-Shiu function satisfies

\[
\phi(u) = \int_0^u \phi(u-x)g(x)dx + h(u), \quad u \geq 0,
\]

where \( g(x) = \frac{1}{c} \int_x^{\infty} f(y)dy, \) \( h(u) = \frac{1}{c} \int_x^{\infty} \int_x^{\infty} w(x, y-x) f(y)dydx. \) Since \( \phi, \) \( g, \) \( h \in L^2(\mathbb{R}_+) \) by the condition 1, 2 of Zhang and Su [25], \( \phi, \) \( g, \) \( h \) can be expressed by Laguerre basis as

\[
\phi(x) = \sum_{k} P_k \psi_k(x), \quad g(x) = \sum_{k} Q_k \psi_k(x), \quad h(x) = \sum_{k} R_k \psi_k(x), \quad x \in \mathbb{R}_+, k \in \mathbb{N},
\]

where \( P_k = \langle \phi, \psi_k \rangle, \) \( Q_k = \langle g, \psi_k \rangle, \) \( R_k = \langle h, \psi_k \rangle. \) It follows from Zhang and Su [25], we obtain the approximation of the Gerber-Shiu function

\[
\phi(u) = \sum_{k=1}^{\infty} P_k \psi_k(x) \approx \sum_{k=0}^{K} P_k \psi_k(x) := \phi_K(u) = \bar{P}_K \bar{\psi}_K(u),
\]

where \( K \in \mathbb{N}_0 \) denotes truncation parameter, \( \bar{\psi}_K(u) = (\psi_0(u), \psi_1(u), \ldots, \psi_K(u)) \) and \( \bar{P}_K = (P_0, P_1, \ldots, P_K)\). \( \bar{P}_K \) can be expressed as \( \bar{P}_K = \bar{A}_K^{-1} \bar{r}_K, \) where \( \bar{r}_K = (R_0, R_1, \ldots, R_K)\) and \( \bar{A}_K = (a_{ij})_{1 \leq i, j \leq K+1} \) is a lower triangular invertible Toeplitz matrix, whose components are given by

\[
a_{ij} = \begin{cases} 
1 - \frac{1}{\sqrt{2}} Q_0 & i = j; \\
\frac{1}{\sqrt{2}} (Q_{i-1} - Q_{i}) & i > j; \\
0 & i < j.
\end{cases}
\]

### 2.2. Coefficient \( Q_k \) and \( R_k \)

By changing the order of integrals, we have

\[
Q_k = \int_0^\infty g(x)\psi_k(x)dx = \frac{\lambda}{c} \int_0^\infty \int_x^{\infty} f_X(y)dy\psi_k(x)dx
= \frac{\lambda}{c} \int_0^\infty \psi_k(x)dx f_X(y)dy = \frac{\lambda}{c} \mathbb{E}[H_k^Q(X)], \quad k \in \mathbb{N}
\]

and

\[
R_k = \int_0^\infty h(u)\psi_k(u)du = \frac{\lambda}{c} \int_0^\infty \int_u^{\infty} w(x, y-x) f_X(y)dydu
= \frac{\lambda}{c} \int_0^\infty \int_u^{\infty} w(x, y-x)du f_X(y)dy = \frac{\lambda}{c} \mathbb{E}[H_k^R(X)], \quad k \in \mathbb{N}
\]

where \( H_k^Q(z) = \int_0^z \psi_k(x)dx, \) \( H_k^R(z) = \int_0^z \int_u^{\infty} w(x, z-x)du \psi_k(u)du. \) Meanwhile, by Condition 3 and \( \{ \psi_k \}_{k \in \mathbb{N}} \) are uniformly bounded, we have

\[
\sup_k |H_k^Q(z)| \leq \sqrt{2} \int_0^\infty dx \lesssim z, \quad z \in \mathbb{R}_+.
\]
and
\[
\sup_k |H_k^R(z)| \leq \sqrt{2} \int_0^z \int_0^x w(x, z - x) dx du \lesssim z^{k_1 + k_2 + 2}, \quad z \in \mathbb{R}_+.
\]

2.3. Statistical Inference

For insurer, the parameter $\lambda$ and the claim size density $f$ are usually unknown. But, they can be obtained by the following data information,
\[
\{N_T, X_1, X_2, \ldots, X_{N_T}\},
\]
where $N_T$ is the claim number over $[0, T]$ and $\{X_i\}_{1 \leq i \leq N_T}$ are individual claim sizes. We can estimate $Q_k$ and $R_k$ by
\[
\hat{Q}_k = \frac{\hat{\lambda}}{c} \frac{1}{N_T} \sum_{i=1}^{N_t} H_k^Q(X_i), \quad \hat{R}_k = \frac{\hat{\lambda}}{c} \frac{1}{N_T} \sum_{i=1}^{N_t} H_k^R(X_i),
\]
where $\hat{\lambda} = \frac{N_T}{T}$ is the estimator of $\lambda$. Furthermore, we can estimate the Gerber-Shiu function by
\[
\hat{\phi}_K(u) := \bar{\tilde{p}}_K \bar{\tilde{p}}_K(u),
\]
where $\bar{\tilde{p}}_K = (\hat{\bar{r}}_0, \hat{\bar{r}}_1, \ldots, \hat{\bar{r}}_K)^\top$ are the estimators of $\bar{p}_K$. $\bar{\tilde{p}}_K$ holds that $\bar{\tilde{p}}_K = \bar{A}_K \bar{r}_K$, where $\bar{r}_K = (\hat{a}_{ij})_{1 \leq i, j \leq K+1}$ are given by
\[
\hat{a}_{ij} = \begin{cases} 1 - \frac{1}{\sqrt{2}} \hat{Q}_0 & i = j; \\ \frac{1}{\sqrt{2}} (\hat{Q}_{i-j-1} - \hat{Q}_{i-j}) & i > j; \\ 0 & i < j. \end{cases}
\]

3. Asymptotically Normality

In this part, we show that the estimator is asymptotically normal. For this purpose, we introduce some lemmas for the asymptotic normality of Laguerre coefficients.

Lemma 1. Suppose $\mathbb{E}[X^2(2 + k_1 + k_2)] < \infty$ and $\sup_k |H_k(x)| \lesssim x^{k_1 + k_2 + 2}$, let
\[
H = \left(\frac{1}{\hat{\lambda} c} \sum_{i=1}^{N_T} H_0(X_i), \ldots, \frac{1}{\hat{\lambda} c} \sum_{i=1}^{N_T} H_d(X_i)\right)^\top \quad \text{and} \quad u = \left(\frac{\hat{\lambda} c}{\hat{\lambda}} \mathbb{E}[H_0(X)], \ldots, \frac{\hat{\lambda}}{c^{d} \Sigma} \mathbb{E}[H_d(X)]\right)^\top.
\]

Then
\[
\sqrt{N_T} (H - u) \overset{D}{\to} N_{d+1} \left(0, \frac{\lambda^2}{c^2} \Sigma\right), \quad T \to \infty,
\]
where $\Sigma = (\sigma^2_{ij})_{0 \leq i, j \leq d}$, $\sigma^2_{ij} := \mathbb{E} [(H_i(X) - \mathbb{E}[H_i(X)])(H_j(X) - \mathbb{E}[H_j(X)])] < \infty$.

Proof of Lemma 1. Since $N_T$, which is independent of $X_i$, follows Poisson distribution with intensity $\lambda T$, we have $\frac{\lambda}{c} \mathbb{E}[H_i(X)] = \frac{\lambda}{c} \frac{1}{N_T} \sum_{j=1}^{N_T} \mathbb{E}[H_i(X_j)]$ and
\[
H_i - u_i = \left(\frac{\hat{\lambda}}{c} - \frac{\hat{\lambda}}{\hat{\lambda}}\right) \frac{1}{N_T} \sum_{j=1}^{N_T} H_i(X_j) + \frac{\lambda}{c} \frac{1}{N_T} \sum_{j=1}^{N_T} (H_i(X_j) - \mathbb{E}[H_i(X_j)]), \quad 0 \leq i \leq d.
\]
Furthermore, since \( \frac{\bar{\lambda}}{\epsilon} \overset{p}{\rightarrow} \frac{1}{\epsilon} \) when \( T \to \infty \) and \( \mathbb{E} [\sup_k |H_k(X)|] < \infty \), we have
\[
\left( \frac{\bar{\lambda}}{\epsilon} - \frac{\lambda}{\epsilon} \right) \frac{1}{N_T} \sum_{j=1}^{N_T} H_i(X_j) \overset{p}{\to} 0, \ 0 \leq i \leq d.
\]

For convenience, we set
\[
Z_1 := \left( \frac{1}{N_T} \sum_{j=1}^{N_T} (H_0(X_j) - \mathbb{E}[H_0(X_j)]), \ldots, \frac{1}{N_T} \sum_{j=1}^{N_T} (H_d(X_j) - \mathbb{E}[H_d(X_j)]) \right)^\top = \frac{1}{N_T} \sum_{j=1}^{N_T} Z_2^j;
\]
\[
Z_2 := (H_0(X_j) - \mathbb{E}[H_0(X_j)], H_1(X) - \mathbb{E}[H_1(X)], \ldots, H_d(X_j) - \mathbb{E}[H_d(X_j)])^\top; \quad Z_3 := (H_0(X) - \mathbb{E}[H_0(X)], H_1(X) - \mathbb{E}[H_1(X)], \ldots, H_d(X) - \mathbb{E}[H_d(X)])^\top.
\]

We can obtain \( \mathbb{E}Z_3 = 0 \) and for every random vector \( t \in \mathbb{R}^{d+1} \),
\[
\mathbb{E} \left[ e^{\sqrt{N_T} t^\top Z_1} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( i \frac{1}{\sqrt{N_T}} \sum_{j=1}^{N_T} t^\top Z_2 \right) \bigg| N_T \right]\right] = \mathbb{E} \left[ \Phi_{Z_3} \left( \frac{t^\top}{\sqrt{N_T}} \right)^{N_T} \right].
\]

It follows from (VI 2.13) in Cinlar [32] that \( N_T \to \infty \) almost everywhere when \( T \to \infty \). Furthermore, by dominated convergence theorem, we obtain
\[
\lim_{T \to \infty} \mathbb{E} \left[ e^{\sqrt{N_T} t^\top Z_1} \right] = \mathbb{E} \left[ \lim_{T \to \infty} \Phi_{Z_3} \left( \frac{t^\top}{\sqrt{N_T}} \right)^{N_T} \right] = e^{-\frac{1}{4} t^\top \Sigma t}. \tag{6}
\]

According to Theorem 2.13 in van der Vaart [33], we derive that \( \sqrt{N_T} (\mathbf{H} - \mathbf{u}) \overset{D}{\to} N_{d+1} \left( 0, \frac{\lambda^2}{\epsilon^2} \Sigma \right), \ T \to \infty \). This completes the proof. \( \square \)

Next, we have a weak consistency for Laguerre coefficients \( \hat{Q}_k, \hat{R}_k \).

**Lemma 2.** If \( \mathbb{E}X^{2(k_1+k_2+2)} < \infty \), then \( \hat{Q}_k, \hat{R}_k \overset{p}{\to} (Q_k, R_k) \) for all \( k \in \mathbb{N} \) when \( T \to \infty \).

**Proof of Lemma 2.** For \( \forall \ \epsilon > 0 \),
\[
P(|\hat{Q}_k - Q_k| > \epsilon) \leq \frac{\mathbb{E}|\hat{Q}_k - Q_k|^2}{\epsilon^2} = \frac{1}{c^2 T^2} \mathbb{E} \left\{ \sum_{j=1}^{N_T} \int_0^{X_i} \psi_k(x) dx \right\}^2 \leq \frac{\lambda}{c^2 T^2} \mathbb{E}X^2 \leq \frac{1}{\epsilon^2 T} \to 0, \ T \to \infty,
\]
and
\[
P(|\hat{R}_k - R_k| > \epsilon) \leq \frac{\mathbb{E}|\hat{R}_k - R_k|^2}{\epsilon^2} = \frac{1}{c^2 T^2} \mathbb{E} \left\{ \sum_{j=1}^{N_T} \int_0^{X_i} \int_u^{X_i} w(x, X_i - x) dx \psi_k(u) du \right\}^2 \leq \frac{\lambda}{c^2 T^2} \mathbb{E}X^{2(k_1+k_2+2)} \leq \frac{1}{\epsilon^2 T} \to 0, \ T \to \infty.
\]

It implies that \( \hat{Q}_k \overset{p}{\to} Q_k \) and \( \hat{R}_k \overset{p}{\to} R_k \). By Theorem 2.7 (vi) in van der Vaart [33], we have that \( (\hat{Q}_k, \hat{R}_k) \overset{p}{\to} (Q_k, R_k), \ T \to \infty. \) \( \square \)
Then, we prove that the Laguerre coefficients are asymptotically normal.

**Lemma 3.** Suppose that \( \mathbb{E}X^{2(k_1+k_2+2)} < \infty \). Then, for all \( K \in \mathbb{N} \) we have

\[
\sqrt{N_T} \left( \frac{\tilde{q}_K - \hat{q}_K}{\tilde{r}_K - \hat{r}_K} \right) \xrightarrow{D} \mathcal{N}_{2(k+1)}(0, \Sigma_{Q,R}), \ T \to \infty,
\]

where \( \Sigma_{Q,R} = \begin{pmatrix} \sigma_{QQ}^R & \sigma_{QR}^R \\ \sigma_{QR}^R & \sigma_{RR}^R \end{pmatrix} \), \( \sigma_{XY}(\sigma_{ij}^Y)_0 \leq i,j \leq K \), \( \sigma_{ij}^Y = \int_0^\infty (H_i^Y(z) - Y_i)(H_j^Y(z) - Y_j)f(z) \, dz < \infty \), \( X = Q \) or \( R \), \( Y = Q \) or \( R \).

**Proof of Lemma 3.** According to Lemma 1, \( \sqrt{N_T} \left( \frac{\tilde{q}_K - \hat{q}_K}{\tilde{r}_K - \hat{r}_K} \right) \) converges in distribution to a normal variable with mean 0. Then, we study its covariance matrix. For convenience, we set \( \tilde{Y} = (Y_0, Y_1, \ldots, Y_{2K+1})^\top \) and \( \tilde{Y} = (\tilde{Y}_0, \tilde{Y}_1, \ldots, \tilde{Y}_{2K+1})^\top \) with

\[
Y_i = \begin{cases} Q_i & 0 \leq i \leq K; \\ R_{i-(K+1)} & K + 1 \leq i \leq 2K + 1; \end{cases} \quad \tilde{Y}_i = \begin{cases} \hat{Q}_i & 0 \leq i \leq K; \\ \hat{R}_{i-(K+1)} & K + 1 \leq i \leq 2K + 1; \end{cases}
\]

where

\[
\sqrt{N_T} \left( \tilde{Y} - \tilde{Y} \right) \xrightarrow{D} \mathcal{N}_{d+1}(0, \Sigma_{Q,R}), \ T \to \infty.
\]

The components of \( \Sigma_{Q,R} = (\sigma_{ij})_{0 \leq i,j \leq d} \) are

\[
\sigma_{ij} = \begin{cases} \mathbb{E} \left[ (\frac{1}{2} H_i^Q(X) - Q_i) (\frac{1}{2} H_j^Q(X) - Q_j) \right], & \text{if } 0 \leq i \leq K, \quad 0 \leq j \leq K; \\ \mathbb{E} \left[ (\frac{1}{2} H_i^Q(X) - Q_i) (\frac{1}{2} H_j^R(X - R_{j-(K+1)}) - R_{j-(K+1)}) \right], & \text{if } 0 \leq i \leq K, \quad K + 1 \leq j \leq 2K + 1; \\ \mathbb{E} \left[ \left( \frac{\lambda H_{i-(K+1)}^R(X)}{2} - R_{i-(K+1)} \right) (\frac{1}{2} H_j^Q(X) - Q_j) \right], & \text{if } K + 1 \leq i \leq 2K + 1, \quad 0 \leq j \leq K; \\ \mathbb{E} \left[ \left( \frac{\lambda H_{i-(K+1)}^R(X)}{2} - R_{i-(K+1)} \right) (\frac{\lambda H_{j-(K+1)}^R(X)}{2} - R_{j-(K+1)}) \right], & \text{if } K + 1 \leq i \leq 2K + 1, \quad K + 1 \leq j \leq 2K + 1. \end{cases}
\]

Following from Lemma 1, we have \( \sqrt{N_T} \left( \frac{\tilde{q}_K - \hat{q}_K}{\tilde{r}_K - \hat{r}_K} \right) \xrightarrow{D} \mathcal{N}_{2(k+1)}(0, \Sigma_{Q,R}), \ T \to \infty. \)

This completes the proof. \( \square \)

Finally, we derive the asymptotic normality of Laguerre estimators.

**Theorem 1.** Suppose that \( \mathbb{E}X^{2(k_1+k_2+2)} < \infty, \) \( \phi \in W(\mathbb{R}^+, r, B) \) and \( K = O(T^{-\frac{1}{2}}) \) for \( r > 2 \), then

\[
\sqrt{N_T} \left( \hat{q}_K(x) - \phi(x) \right) \xrightarrow{D} \mathcal{N}(0, \Sigma(x)), \ T \to \infty,
\]

where \( \Sigma(x) = \Psi_K(x) A^{-1}_K \mathbf{P}_K \Sigma_{Q,R}(\Psi_K(x) A^{-1}_K \mathbf{P}_K)^\top \), \( \Psi_K(x) = (\psi_0(x), \psi_1(x), \ldots, \psi_K(x)) \), \( \mathbf{P}_K = (\mathbf{P}_K^*) - I_{K+1} \) and \( W(\mathbb{R}^+, r, B) \) denotes Sobolev-Laguerre space (see Bongioanni and Torrea [34]). \( \mathbf{P}_K^* \) is given by \( \mathbf{P}_K^* = (\mathbf{P}_K^*)_{ij} \), where

\[
(\mathbf{P}_K^*)_{ij} = \begin{cases} -\frac{1}{\sqrt{2}} p_0, & \text{if } i = j, \\ -\frac{1}{\sqrt{2}} (p_i - p_{i-1}), & \text{if } i > j, \\ 0, & \text{if } i < j. \end{cases}
\]
Proof of Theorem 1. Since
\[ \sqrt{NT} (\hat{\phi}_K(x) - \phi(x)) = \sqrt{NT} (\bar{\phi}_K(x) - \phi_K(x)) + \sqrt{NT} (\phi_K(x) - \phi(x)), \]
according to Shimizu and Zhang [26], we have
\[ \sqrt{NT}(\hat{p}_K - p_K) = -A_K^{-1}(\hat{P}_K^* - I_{K+1})\sqrt{NT} \left( \frac{\hat{q}_K - q_K}{\hat{r}_K - r_K} \right), \]
where
\[ \hat{P}_K^* = (\hat{P}_K^*)_{ij} \overset{D}{=} \hat{P}_{K^*}, \text{ where } (\hat{P}_K^*)_{ij} = \begin{cases} -\frac{1}{\sqrt{2}}\hat{p}_{i}, & \text{if } i = j, \\ -\frac{1}{\sqrt{2}}(\hat{p}_{i-j} - \hat{p}_{j-i}), & \text{if } i > j, \\ 0, & \text{if } i < j. \end{cases} \]
Then
\[ \sqrt{NT}(\hat{p}_K - p_K) \overset{D}{\to} N(0, A_K^{-1}P_K^*\Sigma_{Q,R}(A_K^{-1}P_K^*)^\top), \]
and
\[ \sqrt{NT}(\hat{\phi}_K(x) - \phi_K(x)) = \Psi_K(x)\sqrt{NT}(\hat{p}_K - p_K) \overset{D}{\to} N \left( 0, \Psi_K(x)A_K^{-1}P_K^*\Sigma_{Q,R}(\Psi_K(x))A_K^{-1}P_K^*)^\top \right). \]

For \( \sqrt{NT}(\hat{\phi}_K(x) - \phi(x)) \), by Cauchy-Schwarz inequality,
\[ \sup_{x \in \mathbb{R}^+} \sqrt{NT} |\phi_K(x) - \phi(x)| = \sup_{x \in \mathbb{R}^+} \sqrt{NT} \left| \sum_{k \geq K+1} \langle \phi, \psi_k \rangle \psi_k(x) \right| \leq \sqrt{2NT} \sum_{k \geq K+1} k^\frac{r}{2} |\langle \phi, \psi_k \rangle | k^{-\frac{r}{2}} \]
\[ \leq \sqrt{2NT} \left( \sum_{k \geq K+1} k^r \langle \phi, \psi_k \rangle^2 \right)^\frac{1}{2} \left( \sum_{k \geq K+1} k^{-r} \right)^\frac{1}{2} \leq \left( 2NTB \int_K^\infty x^{-r} dx \right)^\frac{1}{2} \]
\[ = \left( \frac{2B}{r-1} \right)^\frac{1}{2} T^\frac{r}{2} K^{-\frac{r-1}{2}} \overset{p}{\to} 0, \quad T \to \infty, \quad r > 2. \]

It means that \( \sqrt{NT}(\hat{\phi}_K(x) - \phi(x)) \overset{p}{\to} 0 \). This completes the proof. \( \square \)

4. Simulation

In this part, we provide numerical examples to verify the asymptotic normality of Laguerre estimators. We set the premium rate \( c = 1.5 \), the Poisson density \( \lambda = 1 \) and we consider three claim size densities:
- Exponential: \( f(x) = e^{-x}, \quad x \in \mathbb{R}_+; \)
- Erlang(2): \( f(x) = 4xe^{-2x}, \quad x \in \mathbb{R}_+; \)
- Combination-of-exponentials: \( f(x) = 3e^{-1.5x} - 3e^{-3x}, \quad x \in \mathbb{R}_+. \)

Meanwhile, we estimate three special Gerber-Shiu functions:
- Ruin probability (RP);
- Expected claim size causing ruin (ECS);
- Expected deficit at ruin (ED).

By Asmussen and Albrecher [2] and Dickson [35], the explicit formulas for those function can be obtained by Laplace transform method. Note that under the above claim size densities and special Gerber-Shiu function assumptions, Conditions 1–3 hold true. We set \( T = 720, 1440 \) and cut-off parameter \( K = \lfloor 5T^{-\frac{1}{2}} \rfloor \), where \( \lfloor \cdot \rfloor \) means the integer part.
To check the asymptotic normality of the estimators, we present quantile-quantile plot (QQ-plot). QQ-plot displays each data point by ‘+’ marks, if the data points display a linear trend, then the distribution of the data is normal. To conclude this discussion, we show QQ-plot for different claim size densities and Gerber-Shiu functions with $T = 720, 1440$. In Figures 1 and 2, we plot QQ-plot of Ruin probability for exponential claim size density with different $\mu$ and $T$. The plot display an approximately straight line mean that the estimators are asymptotic normal. Then, we display the QQ-plot of ECS for Erlang(2) density in Figures 3 and 4. The data point produces a straight line means that the estimators are asymptotically normal. In the end, we present QQ-plot of ED for combination-of-exponentials density in Figures 5 and 6 with different $\mu$ and $T$. The results manifest the Laguerre estimators are asymptotically normal. In addition, it can be observed from Figures 1, 3 and 5 with $T = 720$ that the tails seem not converge to the normal distribution. Nevertheless, it is observed from Figures 2, 4 and 6 with $T = 1440$ that the results manifest asymptotic normality. In conclusion, we prove the asymptotic normality of our estimator as the value of $T$ becomes large.

**Figure 1.** QQ-plot of RP for Exponential claim size density with $T = 720$: (a) $\mu = 1$; (b) $\mu = 3$; (c) $\mu = 5$.

**Figure 2.** QQ-plot of RP for Exponential claim size density with $T = 1440$: (a) $\mu = 1$; (b) $\mu = 3$; (c) $\mu = 5$.

**Figure 3.** QQ-plot of ECS for Erlang(2) density with $T = 720$: (a) $\mu = 1$; (b) $\mu = 3$; (c) $\mu = 5$. 
5. Conclusions

Recently, Zhang and Su [25] proposed a novel method for estimating the Gerber-Shiu function by Laguerre series expansion. Based on the observed information, we obtain the estimator of the Gerber-Shiu function. In this paper, we analyze the asymptotic normality of the Laguerre estimator mathematically and present some simulation experiments to support our result.

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