On sample complexity for computational pattern recognition

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March 11, 2008

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Abstract

In this work we consider the task of pattern recognition in which the target (labelling) function is known to be computable on some Turing machine. It is easy to show that there exist a pattern recognition method for which the number of examples needed to approximate the target function with certain accuracy is linear in the length of the (unknown) program computing the target function. We investigate the question whether any bounds of this kind exist if we consider only computable pattern recognition methods. We find that the number of examples required for a computable method to approximate an unknown computable function not only is not linear, but grows faster (in the length of the target function) than any computable function. No time or space constraints are put on the predictors or target functions; the only resource we consider is the training examples.

The task of pattern recognition is considered in conjunction with another learning problem — data compression. An impossibility result for the task of data compression allows us to estimate the sample complexity for pattern recognition.
1 Introduction

The task of pattern recognition consists in predicting an unknown label of some observation (or object). For instance, the object can be an image of a hand-written letter, in which case the label is the actual letter represented by this image. Other examples include DNA sequence identification, recognition of an illness based on a set of symptoms, speech recognition, and many others.

More formally, the objects are drawn independently from an object space $X$ (usually $X = [0, 1]^d$ or $\mathbb{R}^d$) according to some unknown but fixed probability distribution $P$ on $X$, and labels are defined according to some function $\eta : X \rightarrow Y$, where $Y$ is a finite set (often $Y = \{0, 1\}$). The task is to construct a function $\varphi : \{0, 1\}^* \rightarrow Y$ which approximates $\eta$, i.e. for which $P\{x : \eta(x) \neq \varphi(x)\}$ is small, where $P$ and $\eta$ are unknown but examples $x_1, y_1, \ldots, x_n, y_n$ are given; $y_i := \eta(x_i)$.

This task, in such a general formulation, is known to be hard. Although (simple) rules for which probability of error tends to zero exist, the rates of convergence can be arbitrary low (see [1], [2] Theorem 7.2). That is, for any pattern recognition method and for any sequence of real numbers $a_n \rightarrow 0$, there exist a distribution $P$ and a labelling function $\eta$ such that the probability of error on each step $n$ is bounded from below by $a_n$ (from some $n$ on). Thus less general formulations of the learning problem are often searched for. One of the most popular approaches is statistical learning theory of Vapnik and Chervonenkis [7, 8]. If the target function $\eta$ is known to belong to some class of functions $\mathcal{C}$, then good error bounds can be obtained, provided the class $\mathcal{C}$ is small enough. More formally, the number of examples required to obtain certain level of accuracy (or the sample complexity of $\mathcal{C}$) is linear in the VC-dimension of $\mathcal{C}$. How to choose the class (or classes) of labelling functions $\mathcal{C}$ is left to be specified for each learning problem separately.

In this work we study the task of pattern recognition when it is known that the labelling functions $\eta$ is computable (on some Turing machine). At first glance, this problem seems much more simple than in the general formulation. Indeed, although the class of all computable functions has infinite VC-dimension, the class of computable functions of length (or Kolmogorov complexity) not greater than $k$ has VC-dimension not greater than $k$, and thus there exist a method for which the number of examples needed to approximate the unknown function of length $k$ is linear in $k$ (even if $k$ is not known beforehand).

But what happens if we consider only computable methods for approxi-
mating computable functions? Are there still any linear bounds on sample complexity for this task? It appears that in this case the sample complexity of class of labelling functions of length not greater than $k$ grows faster than any computable function in $k$. Thus not only linear bounds do not exist, but also exponential, doubly exponential, etc.

To obtain the negative result we consider the task of data compression: an impossibility result for the task of data compression allows us to estimate the sample complexity for pattern recognition.

We also analyse how tight is the negative result, and show that for some simple computable rule (based on the nearest neighbour estimate) the sample complexity is finite in $k$, under different definitions of computational patterning recognition task.

In comparison to the vast literature on pattern recognition and related learning problems surprisingly little attention had been paid to the “computable” version of the task; at least this concerns the task of approximating any computable function. There is a track of research in which a pattern recognition method is a function computable in polynomial time, or under other resource constraints. This approach leads to many interesting results, but it usually considers more specified settings of a learning problem, such as learning DNFs, finite automata, etc. See [3] for an introduction to this theory and for references.

The related task of making computable predictions for the next element of a sequence generated according to a computable probability distribution, have received much more attention, principal results being obtained in [9, 6]; an overview can be found in [5].

2 Main results

A (binary) string is a member of the set $\{0, 1\}^* = \cup_{i=0}^{\infty} \{0, 1\}^i$. The length of a string $x$ will be denoted by $|x|$, while $x^i$ denotes the $i$th element of $x$, $1 \leq i \leq |x|$. For a set $A$ the symbol $|A|$ denotes the number of elements in $A$. We will assume the lexicographical order on the set of strings, and when necessary will identify $\{0, 1\}^*$ and $\mathbb{N}$ via this ordering. Denote by $\mathbb{N}$ the sets of natural numbers. The symbol log is used for $\log_2$. For a real number $\alpha$ the symbol $\lceil \alpha \rceil$ denotes the largest natural number not greater than $\alpha$.

In pattern recognition a labelling function is usually a function from the interval $[0, 1]$ or $[0, 1]^d$ (sometimes more general spaces are considered) to a
finite space \( Y := \{0, 1\} \). As we are interested in computable functions, we have to refine this definition. We call a partial recursive function (or program) \( \eta \) a \textit{labelling function} if there exists such \( t =: t(\eta) \in \mathbb{N} \) that \( \eta \) accepts all string from \( X_t := \{0, 1\}^t \) and only such strings \(^1\). For an introduction to the computability theory see for example [4].

The sets \( X_t, \ t \in \mathbb{N} \) can be thought of as approximations of the interval \([0, 1]\) (say, by binary expansions) or \([0, 1]^d \) for some \( d \in \mathbb{N} \).

It can be argued that this definition of a labelling function is too restrictive to approximate well the notion of a real function. However, as we are after negative results (for the class of all labelling functions), this is not a disadvantage. Other possible definitions are discussed in Section 3, where we are concerned with tightness of our negative results.

Fix some effective one-to-one mapping \( \nu \) from \( \mathbb{N} \) to the set of all partial recursive functions. For a partial function \( \eta \) define the \textit{length} of \( \eta \) as \( l(\eta) := |n| \) where \( n \) is such that \( \nu(n) = \eta \).

Define the task of computational pattern recognition as follows. An (unknown) labelling function \( \eta \) is fixed. The objects \( x_1, \ldots, x_n \in X \) are drawn according to some distribution \( P \) on \( X_t(\eta) \). The labels \( y_i \) are defined according to \( \eta \), that is \( y_i := \eta(x_i) \).

A \textit{predictor} is a family of functions (indexed by \( n \))

\[
\varphi_n(x_1, y_1, \ldots, x_n, y_n, x),
\]

taking values in \( Y \), such that for any \( n \) and any \( t \in \mathbb{N} \), if \( x_i \in X_t \) for each \( i \), \( 1 \leq i \leq n \), then the marginal \( \varphi(x) \) is a total recursive function on \( X_t \) (that is, \( \varphi_n(x) \) accepts any \( x \in X_t \)). We will often identify \( \varphi_n \) with its marginal \( \varphi_n(x) \) when the values of other variables are clear.

Thus, given a \textit{sample} \( x_1, y_1, \ldots, x_n, y_n \) of labelled objects of the same size \( t \), a predictor produces a computable function; this function is supposed to approximate the labelling function \( \eta \) on \( X_t \).

A \textit{computable predictor} is a predictor which for any \( t \in \mathbb{N} \) and any \( n \in \mathbb{N} \) is a total recursive function on \( X_t \times Y \times \cdots \times X_t \times Y \times X_t \).

We are interested in what size sample is required to approximate a labelling function \( \eta \).

\(^1\)It is not essential for this definition that \( \eta \) is not a total function. An equivalent (for our purposes) definition would be as follows. A labelling function is any total function which outputs the string \( 00 \) on all inputs except on the strings of some length \( t =: t(\eta) \), on each of which it outputs either 0 or 1.
Moreover, for a (computable) predictor $\varphi$, a labelling function $\eta$ and $0 < \varepsilon \in \mathbb{R}$ define
\[
\delta_n(\varphi, \eta, \varepsilon) := \sup_{P_t} \left\{ \mathbb{P}_t \{ x_1, \ldots, x_n \in X_t : \right. \\
\mathbb{P}_t \{ x \in X_t : \varphi_n(x_1, y_1, \ldots, x_n, y_n, x) \neq \eta(x) \} > \varepsilon \left. \} \right\},
\]
where $t = t(\eta)$ and $P_t$ ranges over all distributions on $X_t$. For $\delta \in \mathbb{R}$, $\delta > 0$ define the sample complexity of $\eta$ with respect to $\varphi$ as
\[
N(\varphi, \eta, \delta, \varepsilon) := \min \{ n \in \mathbb{N} : \delta_n(\eta, \varepsilon) \leq \delta \}.
\]
The number $N(\eta, \delta, \varepsilon)$ is the minimal sample size required for a predictor $\varphi$ to achieve $\varepsilon$-accuracy with probability $\delta$ when the (unknown) labelling function is $\eta$.

We can use statistical learning theory \footnote{8} to derive the following statement

**Proposition 1.** There exists a predictor $\varphi$ such that
\[
N(\varphi, \eta, \delta, \varepsilon) \leq \max \left( l(\eta) \frac{8}{\varepsilon} \log \frac{13}{\varepsilon}, \frac{4}{\varepsilon} \log \frac{2}{\delta} \right)
\]
for any labelling function $\eta$ and any $\varepsilon, \delta > 0$.

Observe that the bound is linear in the length of $\eta$.

In what follows the proof of this simple statement, we investigate the question of whether any such bounds exist if we restrict our attention to computable predictors.

**Proof.** The predictor $\varphi$ is defined as follows. For each sample $x_1, y_1, \ldots, x_n, y_n$ it finds a shortest program $\tilde{\eta}$ such that $\tilde{\eta}(x_i) = y_i$ for all $i \leq n$. Clearly, $l(\tilde{\eta}) \leq l(\eta)$. Observe that the VC-dimension of the class of all functions of length not greater than $l(\eta)$ is bounded from above by $l(\eta)$, as there are not more than $2^l(\eta)$ such functions. Moreover, $\varphi$ minimises empirical risk over this class of functions. It remains to use the following bound (see e.g. \footnote{2}, Corollary 12.4)
\[
\sup_{\eta \in \mathcal{C}} N(\varphi, \eta, \delta, \varepsilon) \leq \max \left( V(\mathcal{C}) \frac{8}{\varepsilon} \log \frac{13}{\varepsilon}, \frac{4}{\varepsilon} \log \frac{2}{\delta} \right)
\]
where $V(\mathcal{C})$ is the VC-dimension of the class $\mathcal{C}$. \qed
The main result of this work is that for any computable predictor \( \varphi \) there is no computable upper bound in terms of \( l(\eta) \) on the sample complexity of the function \( \eta \) with respect to \( \varphi \):

**Theorem 1.** For any computable predictor \( \varphi \) and any total function \( \beta : \mathbb{N} \to \mathbb{N} \) there exist a labelling function \( \eta \), and some \( n > \beta(l(\eta)) \) such that

\[
P\{x \in X_{t(\eta)} : \varphi(x_1, y_1, \ldots, x_n, y_n, x) \neq \eta(x)\} > 0.05,
\]

for any \( x_1, \ldots, x_n \in X_{t(\eta)} \), where \( y_k = \eta(x_k) \) and \( P \) is the uniform distribution on \( X_{t(\eta)} \).

For example, we can take \( \beta(n) = 2^n \), or \( 2^{2^n} \).

**Corollary 1.** For any computable predictor \( \varphi \), any total function \( \beta : \mathbb{N} \to \mathbb{N} \) and any \( \delta < 1 \)

\[
\sup_{\eta, l(\eta) \leq k} N(\varphi, \eta, \delta, 0.05) > \beta(k)
\]

from some \( k \) on.

Observe that there is no \( \delta \) in the formulation of Theorem 1. Moreover, it is not important how the objects \( (x_1, \ldots, x_n) \) are generated — it can be any individual sample. In fact, we can assume that the sample is chosen in any manner, for example by some algorithm. This means that no computable upper bound on sample complexity exists even for active learning algorithms.

It appears that the task of pattern recognition is closely related to another learning task — data compression. Moreover, to prove Theorem 1 we need a similar negative result for this task. Thus before proceeding with the proof of the theorem, we introduce the task of data compression and derive some negative results for it.

We call a total function \( \psi : \{0,1\}^* \to \{0,1\}^* \) an archiver if it is an injection (i.e. \( x_1 \neq x_2 \) implies \( \psi(x_1) \neq \psi(x_2) \)). We say that an archiver compresses the string \( x \) if \( |\psi(x)| < |x| \). Clearly, for any natural \( n \) any archiver compresses not more than a half of strings of size not greater than \( n \).

We will now present a definition of Kolmogorov complexity; for fine details see [4, 9]. We call a machine any computable function that takes a partial recursive function (that is, its number in some effective enumeration) and
outputs its output on the empty input. The complexity of a string \( x \in \{0, 1\}^\ast \) with respect to a machine \( \zeta \) is defined as

\[
C_{\zeta}(x) = \min_{p} \{ l(p) : \zeta(p) = x \},
\]

where \( p \) ranges over all partial functions. There exists such a machine \( \zeta \) that

\[
C_{\zeta}(x) \leq C_{\zeta'}(x) + c_{\zeta'} \quad \text{for any } x \text{ and any machine } \zeta' \quad \text{(the constant } c_{\zeta'} \text{ depends on } \zeta' \text{ but not on } x \).
\]

Fix any such \( \zeta \) and define the Kolmogorov complexity of a string \( x \in \{0, 1\}^\ast \) as

\[
C(x) := C_{\zeta}(x).
\]

Clearly, \( C(x) \leq |x| + b \) for any \( x \) and for some \( b \) depending only on \( \zeta \). A string is called \( c \)-incompressible if \( C(x) \geq |x| - c \).

Obviously, any archiver can not compresses many \( c \)-incompressible strings, for any \( c \). However, highly compressible strings (that is, strings with Kolmogorov complexity low relatively to their length) might be expected to be compressed well by some sensible archiver. The following lemma shows that it can not be always the case, no matter what we mean by “relatively low”.

The proof of this lemma is followed by the proof of Theorem 1.

**Lemma 1.** For any archiver \( \psi \) and any total function \( \gamma : \mathbb{N} \to \mathbb{N} \) such that \( \gamma \) goes monotonically to infinity there exists a binary string \( x \) such that \( C(x) \leq \gamma(|x|) \) and \( \psi(x) \geq |x| \).

For example, we can take \( \gamma(n) = \log \log n \).

**Proof.** Suppose the contrary, i.e. that there exist an archiver \( \psi \) and some function \( \gamma : \mathbb{N} \to \mathbb{N} \) monotonically increasing to infinity such that for any string \( x \) if \( C(x) \leq \gamma(|x|) \) then \( \psi(x) < |x| \). Denote by \( T \) the set of all strings which are not compressed by \( \psi \)

\[
T := \{ x : |\psi(x)| \geq |x| \}.
\]

Define the function \( \tau \) on the set \( T \) as follows: \( \tau(x) \) is the number of the element \( x \) in \( T \)

\[
\tau(x) := \#\{ x' \in T : x' \leq x \}
\]

for each \( x \in T \). Obviously, the set \( T \) is infinite. Moreover, \( \tau(x) \leq x \) for any \( x \in T \) (recall that we identify \( \{0, 1\}^\ast \) and \( \mathbb{N} \) via lexicographical ordering). Observe that \( \tau \) is a total function on \( T \) and onto \( \mathbb{N} \). Thus \( \tau^{-1} : \mathbb{N} \to \{0, 1\}^\ast \) is a total function on \( \mathbb{N} \). Hence, for any \( x \in T \),

\[
C(\tau(x)) \geq C(\tau^{-1}(\tau(x))) - c = C(x) - c > \gamma(|x|) - c,
\]

(1)
by definition of \( T \), where \( c \) is some constant depending only on \( \tau \).

It is a well-known result (see e.g. [4], Theorem 2.3.1) that for any partial function \( \delta \) that goes monotonically to infinity there is \( x \in \{0,1\}^* \) such that \( C(x) \leq \delta(|x|) \). In particular, allowing \( \delta(|x|) = \gamma(|x|) - 2c \), we conclude that there exist such \( x \in T \) that

\[
C(\tau(x)) \leq \gamma(|\tau(x)|) - 2c \leq \gamma(|x|) - 2c,
\]

which contradicts (1). \( \Box \)

**Proof of Theorem 1.** Suppose the contrary, that is that there exists such a computable predictor \( \varphi \) and a total function \( \beta : \mathbb{N} \to \mathbb{N} \) such that for any labelling function \( \eta \), and any \( n > \beta(l(\eta)) \) we have

\[
P\{x : \varphi(x_1, y_1, \ldots, x_n, y_n, x) \neq \eta(x)\} \leq 0.05,
\]

for some \( x_i \in X_{l(\eta)}, y_i = \eta(x_i), i \in \mathbb{N}, \) where \( P \) is the uniform distribution on \( X_{l(\eta)} \).

Not restricting generality we can assume that \( \beta \) is strictly increasing. Define the (total) function \( \beta^{-1}(n) := \max\{m \in \mathbb{N} : \beta(m) \leq n\} \). Denote \( \varepsilon := 0.05 \). Construct the archiver \( \psi \) as follows. For each \( \tilde{y} \in \{0,1\}^\ast \) denote \( t := \lceil \log |\tilde{y}| \rceil, m := 2^t, y := \tilde{y}^1 \ldots \tilde{y}^m \) and \( \hat{y} := \tilde{y}^{m+1} \ldots \tilde{y}^{|	ilde{y}|} \). Generate all strings of length \( t \) and denote them by \( x_i, 1 \leq i \leq m \), in the lexicographical order. Define the labelling function \( \eta_y \) as follows: \( t(\eta_y) = t \) and \( \eta_y(x_i) = y_i, 1 \leq i \leq m \). Clearly, \( C(\eta_y) \leq C(y) + c \), where \( c \) is some universal constant capturing the above description.

Let \( n := \frac{m}{\log m} \). Next we run the predictor \( \varphi \) on all possible tuples \( x = (x_1, \ldots, x_n) \in X_t^n \) and each time count errors that \( \varphi \) makes on all elements of \( X_t \):

\[
E(x) := \{x \in X_t : \varphi(x_1, y_1, \ldots, x_n, y_n, x) \neq \eta_y(x)\}\]  

If \( |E(x)| > \varepsilon m \) for each \( x \in X_t \) then \( \psi(\hat{y}) := 0\hat{y} \).

Otherwise proceed as follows. Fix some tuple \( x = (x_1', \ldots, x_n') \) such that \( |E(x)| \leq \varepsilon m \), and denote by \( H := \{x_1', \ldots, x_n'\} \) the unordered tuple \( x \). Denote

\[
\kappa^i = \begin{cases} 
  e_0 & x_i \in E(x) \setminus H, y^i = 0 \\
  e_1 & x_i \in E(x) \setminus H, y^i = 1 \\
  c_0 & x_i \in H, y^i = 0 \\
  c_1 & x_i \in H, y^i = 1 \\
  * & \text{otherwise}
\end{cases}
\]

\[8\]
for \(1 \leq i \leq m\). Thus, each \(\kappa^i\) is a member of a five-letter alphabet (a five-element set) \(\{e_0, e_1, c_0, c_1, *\}\). Denote the string \(\kappa^1 \ldots \kappa^m\) by \(K\).

Observe that the string \(K\), the predictor \(\varphi\) and the order of \((x'_1, \ldots, x'_n)\) (which is not contained in \(K\)) are sufficient to restore the string \(y\). Furthermore, the \(n\)-tuple \((x'_1, \ldots, x'_n)\) can be obtained from \(H\) (the un-ordered tuple) by the appropriate permutation; let \(r\) be the number of this permutation in some fixed ordering of all \(n!\) such permutations. Using Stirling’s formula, we have \(|r| \leq 2n \log n\); moreover, to encode \(|r|\) with some self-delimiting code we need not more then \(4n \log n\) symbols (for \(n > 3\)). Denote such encoding of \(|r|\) by \(\rho\).

Next, as there are \((1 - \varepsilon - \frac{1}{\log m})m\) symbols * in the \(m\)-element string \(K\), it can be encoded by some simple binary code \(\sigma\) in such a way that

\[
|\sigma(K)| \leq \frac{1}{2}m + 7(\varepsilon m + n).
\]

Indeed, construct \(\sigma\) as follows. First replace all occurrences of the string ** with 0. Encode the rest of the symbols with any fixed 4-bit encoding such that the code of each letter starts with 1. Clearly, \(\sigma(K)\) is uniquely decodable. Moreover, it easy to check that (2) is satisfied, as there are not less than \(\frac{1}{2}(m - 2(\varepsilon m + n))\) occurrences of the string **.

Finally, \(\psi(\bar{y}) = 1\rho\sigma(K)\bar{y}\) and \(|\psi(\bar{y})| \leq |\bar{y}|\), for \(m > 2^{20}\).

Thus, \(\psi\) compresses any \(\bar{y}\) such that \(n > \beta(C(\eta_y))\); i.e. such that \(m/\log m > \beta(C(\eta_y)) \geq \beta(C(y) + c)\). This contradicts Lemma 1 with \(\gamma(k) := \beta^{-1}(\sqrt{k} - c)\).

3 On tightness of the negative results

In this section we discuss how tight are the conditions of the statements and to what extend they depend on the definitions.

Let us consider a question whether there exist any (not necessarily computable) sample-complexity function

\[
\mathcal{N}_\varphi(k, \delta, \varepsilon) := \sup_{\eta \in (\eta)} N(\varphi, \eta, \delta, \varepsilon),
\]

at least for some predictor \(\varphi\), or it is always infinity from some \(k\) on.

**Proposition 2.** There exist a predictor \(\varphi\) such that \(\mathcal{N}_\varphi(k, \delta, \varepsilon) < \infty\) for any \(\varepsilon, \delta > 0\) and any \(k \in \mathbb{N}\).
Proof. Clearly, $C(\eta) \geq C(t_\eta)$. Moreover, $\liminf_{t \to \infty} C(t) = \infty$ so that

$$\max\{t_\eta : l(\eta) \leq k\} < \infty$$

for any $k$. It follows that the “pointwise” predictor

$$\varphi(x_1, y_1, \ldots, x_n, y_n, x) = \begin{cases} y_i & \text{if } x = x_i, 1 \leq i \leq n \\ 0 & x \not\in \{x_1, \ldots, x_n\} \end{cases}$$

satisfies the conditions of the proposition.

It can be argued that probably this statement is due to our definition of a labelling function. Next we will discuss some other variants of this definition.

First, observe that if we define a labelling function as any total function on $\{0, 1\}^*$ then some labelling functions will not approximate any real function; for example such is the function $\eta_+$ which counts bitwise sum of its input:

$$\eta_+(x) := \sum_{i=1}^{\lfloor |x| \rfloor} x_i \mod 2.$$ 

That is why we require a labelling function to be defined only on $X_t$ for some $t$.

Another way to define a labelling function (which perhaps makes labelling functions most close to real functions) is as a function which accepts any infinite binary string. Let us call an $i$-labelling function any total function $\eta : \{0, 1\}^\infty \to \{0, 1\}$. That is, $\eta$ is computable on a Turing machine with an input tape on which one way infinite input is written, an output tape and possibly some working tapes. The program $\eta$ is required to halt on any input. The next proposition shows that even if we consider such definition the situation does not change. The definition of a labelling function $\eta$ in which it accepts only finite strings is chosen in order to stay within conventional computability theory.

**Lemma 2.** For any $i$-labelling function $\eta$ there exist $n_\eta \in \mathbb{N}$ such that $\eta$ does not scan its input tape further position $n_\eta$. In particular, $\eta(x) = \eta(x')$ as soon as $x_i = x'_i$ for any $i \leq n_\eta$.

**Proof.** For any $x \in \{0, 1\}^*$ the program $\eta$ does not scan its tape further some position $n(x)$ (otherwise $\eta$ does not halt on $x$). For any $\chi \in \{0, 1\}^\infty$ denote
by \( n_\eta(\chi) \) the maximal \( n \in \mathbb{N} \) such that \( \eta \) scans the input tape up to the position \( n \) on the input \( \chi \).

Suppose that \( \sup_{\chi \in \{0,1\}^\infty} n_\eta(\chi) = \infty \), i.e. that the proposition is false. Define \( x^0 \) to be the empty string. Furthermore, let

\[
x^i = \begin{cases} 0 & \sup_{\chi \in \{0,1\}^\infty} n_\eta(x^1, \ldots, x^{i-1}, \chi) = \infty \\ 1 & \text{otherwise} \end{cases}
\]

By our assumption, \( x_i \) is defined for each \( i \in \mathbb{N} \). Moreover, it easy to check that \( \eta \) never stops on the input string \( x_1x_2 \ldots \).

Besides, it is easy to check that the number \( n_\eta \) is computable.

Finally, it can be easily verified that Proposition 2 holds true if we consider i-labelling functions instead of labelling functions.

**Proposition 3.** There exist a predictor \( \varphi \) such that \( iN_\varphi(k, \delta, \varepsilon) < \infty \) for any \( \varepsilon, \delta > 0 \) and any \( k \in \mathbb{N} \), where \( iN \) is defined as \( N \) with labelling functions replaced by i-labelling functions.

**Proof.** Indeed, if suffices to replace the “pointwise” predictor in the proof of Proposition 2 by the following predictor \( \varphi \), which assigns to the object \( x \) the label of that object among \( x_1, \ldots, x_n \) with whom \( x \) has longest mutual prefix:

\[
\varphi(x_1, y_1, \ldots, x_n, y_n, x) := y_k,
\]

where

\[
k := \argmax_{1 \leq m \leq n} \{ \max\{ i \in \mathbb{N} : x^i = x^i_{m_1} \ldots x^i_{m_m} \} \};
\]

to avoid infinite run in case of ties, \( \varphi \) considers only first (say) \( n \) digits of \( x_i \) and break ties in favour of the lowest index.

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