**O(α_s^2) Contributions to the fragmentation function g_1(x, Q^2) in polarized e^+e^- -annihilation**

P.J. Rijken and W.L. van Neerven

stitute-Lorentz,  
University of Leiden,  
PO Box 9506, 2300 RA Leiden,  
The Netherlands.

November 1997

Abstract

We present the calculation of the order α_s^2 contributions to the coefficient functions (singlet and non-singlet) belonging to the longitudinal spin fragmentation function g_1(x, Q^2) measured in polarized electron-positron annihilation. From this calculation we also obtain the two-loop contribution to the timelike spin splitting functions P^S_{qq} and P_{qq}. These splitting functions are in agreement with recent results in the literature which were obtained by using a different method. We show that in second order the renormalization constant needed for the HVBM prescription of the γ_5-matrix is process dependent.
1 Introduction

The measurement of the fragmentation functions $F_i^H(x, Q^2)$ in unpolarized electron-positron annihilation provides us with a new possibility to test the predictions of perturbative quantum chromodynamics (QCD). Here the reaction under consideration is given by $e^+ + e^- \rightarrow H + "X"$ where H represents the detected hadron and ”X” stands for any inclusive hadronic state. Because of the similarity between the above process and deep inelastic lepton-hadron scattering $e^- + H \rightarrow e^- + "X"$ one can consider the fragmentation functions as the timelike analogues of the well known deep inelastic structure functions measured in the latter process. This similarity becomes clear if one looks at the experimental and the theoretical characteristics of both functions. As far as experiment is concerned one has a full control of the background and a large amount of data so that the systematical and statistical errors are rather small. The data are collected over a wide range of $Q^2$ values which enable us to test the scaling violations exhibited by both functions (for the fragmentation function see e.g. [1], [2], [3]. From the theoretical viewpoint structure functions and fragmentation functions have the common feature that the Born approximation is determined by the electroweak interactions only. Therefore it is easier to calculate their QCD corrections at higher order in the strong coupling constant $\alpha_s$ than for other quantities.

It turns out that the most complicated virtual corrections to these reactions, which are represented by n-point functions with $n \geq 4$, appear in much higher order than for other processes like e.g. jet production. Furthermore the semi inclusive nature of the above processes leads to a considerable simplification of the phase space integrals. Another important feature is that one does not have to deal with problems like jet definitions and hadronization effects in the case of inclusive quantities. Because of the these properties it was possible to compute the order $\alpha_s^2$ contributions to the coefficient functions corresponding to the structure functions [4], [5], [6] and the fragmentation functions [7], [8], [9], [10]. In spite of the similarities between the semi inclusive quantities in deep inelastic scattering and electron-positron annihilation there is also one striking difference. This can be attributed to the current current correlation function which shows up in the expressions for the structure functions as well as in the fragmentation functions. In the case of deep inelastic scattering the current current correlation function is dominated by the light cone in the Bjørken limit whereas in the case of electron-positron
annihilation this property does not exist. Therefore the light cone expansion techniques cannot be applied to the fragmentation functions. These techniques allow us to relate parton densities to the expectation values of hadronic operators between hadronic states. It also enables us to compute sum rules of structure functions to even higher orders in perturbation theory than is possible for the coefficient functions (see e.g. [11]).

Up to now we have only discussed unpolarized scattering. It would be interesting to see whether QCD also yields good predictions when the hadron H is polarized. This has been investigated theoretically as well as experimentally during the last ten years for polarized deep inelastic scattering. The most effort was devoted to the longitudinal spin structure function $g_1(x, Q^2)$ because of the disagreement between the data [12] and its first moment given by the Ellis-Jaffe sum rule [13]. This discrepancy lead to a thorough investigation from which one can conclude that it is more difficult to make QCD predictions for spin dependent quantities than for spin averaged ones. Therefore we expect that the same will happen for electron-positron annihilation when the detected hadron H in the final state is polarized. This process allows us to study the timelike analogue of the above longitudinal spin structure function which we will denote by the fragmentation function $g_1^H(x, Q^2)$. Contrary to polarized deep inelastic scattering there do not exist data for the longitudinal spin fragmentation function. At this moment there are some data on polarized $\Lambda$-baryon production (see e.g. [14]) so that in the future one maybe able to measure the fragmentation densities $\Delta D^\Lambda_a(z, \mu^2)$ for $a = q, \bar{q}, g$ which appear in $g^\Lambda_1(x, Q^2)$. On the theoretical side considerable activity has been devoted in the past few years to understand the fragmentation in polarized $\Lambda$-baryon production in spacelike as well as in timelike processes. For recent work we refer to [15], [16], [17], [18], [20] (for older work see also [19]). Apart from the nonperturbative aspects it is also important to calculate the QCD corrections to the spin fragmentation functions. Recent progress has been made in the computation of the next-to-leading order (NLO) corrections to $g_1^H(x, Q^2)$. The order $\alpha_s$ contributions to the coefficient functions have been calculated in [16]-[20] and the two-loop contributions to the timelike spin splitting functions are presented for the first time in [20]. It is our aim to extend the above calculations by including the order $\alpha_s^2$ contributions to the coefficient functions which is a first step to obtain the next-to-next-to-leading (NNLO) corrections to $g_1^H(x, Q^2)$.

Our paper will be organized as follows. In the next section we introduce
our notations and present the framework of the computations of the longitudinal spin timelike coefficient functions. In section 3 we give an outline of the calculation. From the latter we also obtain two out of the four splitting functions computed in [20]. Further we discuss the intricacies of the prescription of the $\gamma_5$-matrix in n dimensions and show the process dependence of the renormalization constant needed to restore the Ward-identities when this matrix is treated in the scheme of HVBM [21], [22]. The long expressions for the order $\alpha_s^2$ corrected coefficient functions are presented in appendix A.
2 Fragmentation functions in polarized and unpolarized electron-positron annihilation

Single particle inclusive production in electron-positron annihilation is given by the reaction

\[ e^-(\lambda_1, l_1) + e^+(\lambda_2, l_2) \rightarrow V(q) \rightarrow H(s, p) + "X", \]

(2.1)

where "X" represents any inclusive final hadronic state. Further \( \lambda_i, l_i \) (\( i = 1, 2 \)) denote the spin and momentum of the incoming leptons and \( s, p \) stand for the spin and momentum of the hadron \( H \) detected in the final state. If we denote the momentum of the virtual vector boson \( V \), with \( V = \gamma \) or \( V = Z \), by \( q = l_1 + l_2 \) the Bjørken scaling variable is defined by

\[ x = \frac{2p \cdot q}{Q^2}, \quad q^2 = Q^2 > 0, \quad 0 < x \leq 1. \]  

(2.2)

The spin averaged differential cross section can be written as

\[ \frac{d\sigma^H}{dx d\Omega} = \frac{N}{4} \sum_V \sum_{V'} L_{(V,V')}(l_1; l_2) D_{(V,V')}(Q^2) W_{\mu\nu}^{\text{average},H,(V,V')}(p, q), \]  

(2.3)

where \( V \) and \( V' \) run over \( \gamma \) and \( Z \) and \( N \) stands for the colour factor of the local gauge group \( SU(N) \). Further we have denoted the spin averaged leptonic tensor by \( L_{(V,V')}(l_1; l_2) \) and \( D_{(V,V')}(Q^2) \) stands for the electroweak vector boson propagators. If we average over the spin of the detected hadron \( H \) one obtains the hadronic tensor \( W_{\mu\nu}^{\text{average},H,(V,V')}(p, q) \) which can be expressed into the fragmentation functions \( F_{i}^{H,(V,V')}(x, Q^2) \) (see below). An explicit expression for Eq. (2.3) can be found in [8] and [23]. In the case the electron is polarized we can choose \( \lambda_1 = \downarrow \) and average over the spin \( \lambda_2 \) of the positron in Eq. (2.3). The difference between the cross sections where the detected hadron is polarized parallel \( s = \downarrow \) and antiparallel \( s = \uparrow \) with respect to the electron is given by

\[ \frac{d\sigma^{H(\downarrow)}(\downarrow)}{dx d\Omega} - \frac{d\sigma^{H(\uparrow)}(\downarrow)}{dx d\Omega} = \]  

\[ \frac{N}{2} \sum_V \sum_{V'} L_{(V,V')}(\downarrow, l_1; l_2) D_{(V,V')}(Q^2) W_{\mu\nu}^{\text{spin},H,(V,V')}(s, p, q). \]  

(2.4)
For a polarized electron the leptonic tensor is given by $L_{\mu\nu}^{(\downarrow, l_1; l_2)}$ and the spin dependent hadronic tensor $W^{\text{spin},H,(V,V')}(s,p;q)$ can be expressed into the spin fragmentation functions $g_i^{H,(V,V')}(x,Q^2)$ (see below). Explicit formulae for Eq. (2.4) can be found in [17]. In general the hadronic tensor can be written as

$$W^{H,(V,V')}(s,p,q) =$$

$$\frac{1}{4\pi} \sum_X \int d^4y e^{iqy} \langle 0 \mid J^V_\mu(y) \mid H(s,p), X \rangle \langle H(s,p), X \mid J^{V'}_\nu(0) \mid 0 \rangle$$

$$= W^{\text{average},H,(V,V')}(p,q) + W^{\text{spin},H,(V,V')}(s,p,q). \quad (2.5)$$

The spin average hadronic tensor can be expressed into the fragmentation functions $F_i(x,Q^2)$ as follows

$$W^{\text{average},H,(V,V')}(p,q) = \left( -g_\mu \nu + \frac{q_\mu q_\nu}{q^2} \right) F_1^{H,(V,V')}(x,Q^2)$$

$$+ \hat{p}_\mu \hat{p}_\nu \frac{p \cdot q}{q^2} F_2^{H,(V,V')}(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda p^\sigma}{2p \cdot q} F_3^{H,(V,V')}(x,Q^2). \quad (2.6)$$

Similarly one can write the spin dependent hadronic tensor as

$$W^{\text{spin},H,(V,V')}(s,p,q) =$$

$$i\varepsilon_{\mu\nu\lambda\sigma} \frac{q^\lambda s^\sigma}{2p \cdot q} g_1^{H,(V,V')}(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma} q_\lambda \left( s_\sigma - \frac{s \cdot q}{p \cdot q} p_\sigma \right) g_2^{H,(V,V')}(x,Q^2)$$

$$+ \left( \hat{p}_\mu \hat{s}_\nu - \hat{s}_\mu \hat{p}_\nu \right) \frac{q \cdot p}{q^2} g_3^{H,(V,V')}(x,Q^2) + s \cdot q \left( \frac{\hat{p}_\mu \hat{p}_\nu}{(p \cdot q)^2} \right) g_4^{H,(V,V')}(x,Q^2)$$

$$+ \left( -g_\mu \nu + \frac{q_\mu q_\nu}{q^2} \right) \frac{s \cdot q}{p \cdot q} g_5^{H,(V,V')}(x,Q^2), \quad (2.7)$$

where $g_i^{H}(x,Q^2)$ denote the spin fragmentation functions. Further we have introduced the abbreviations

$$\hat{p}_\mu = p_\mu - \frac{p \cdot q}{q^2} q_\mu \quad , \quad \hat{s}_\mu = s_\mu - \frac{s \cdot q}{q^2} q_\mu. \quad (2.8)$$

The above fragmentation functions can be written as a product of the electroweak couplings, describing the interaction of the vector boson $V$ with the quarks, and the fragmentation functions $F_i^H$, $g_i^H$ so that the superscripts
\((V, V')\) can be dropped in the latter quantities. The QCD analysis of the fragmentation functions proceeds in the same way as is done for the structure functions in deep inelastic lepton-hadron scattering. Hence it follows that the leading contributions to the fragmentation functions \(F_i^H\) are all of type twist two. The same holds for the spin fragmentation functions \(g_i^H\) \((i = 1, 4, 5)\). However \(g_2^H\) and \(g_3^H\) also receive leading contributions of type twist three. This means that the latter cannot be simply expressed into convolutions of coefficient functions with fragmentation densities as can be done for twist two contributions. Limiting ourselves to twist two one can simply put \(s = p\) in Eq. (2.7) so that the functions \(g_2^H\) and \(g_3^H\) drop out of the equations and we are left with the longitudinal spin quantities \(g_i^H\) with \((i = 1, 4, 5)\). Further one can derive

\[
F_3^H(x, Q^2) = -F_3^\bar{H}(x, Q^2) \quad , \quad g_i^H(x, Q^2) = -g_i^\bar{H}(x, Q^2) \quad , \quad i = 4, 5 , (2.9)
\]

from charge conjugation invariance of the strong interactions. Hence it follows that the above fragmentation functions only receive non-singlet (with respect to flavour) contributions. The other quantities, given by \(F_i^H\), \(F_2^H\) and \(g_1^H\), also have singlet parts. The QCD corrections to the spin averaged fragmentation functions \(F_i^H\) \((i = 1, 2, 3)\) have been calculated up to second order in the strong coupling constant \(\alpha_s\) in [7], [8], [9], [10] (for the longitudinal fragmentation function \(F_L^H = F_2^H - 2xF_1^H\) see also [24]). Up to order \(\alpha_s^2\) the QCD corrections to \(g_4^H\) and \(g_5^H\) are the same as for the non-singlet parts of \(F_2^H\) and \(F_1^H\) respectively. The same holds for the nonsinglet part of \(g_1^H\) which is equal to the correction to \(F_3^H\). These equalities will be explained below. What is left is the singlet contribution to \(g_1^H\) which will be calculated up to order \(\alpha_s^2\) in this paper. This fragmentation function is the only one which shows up in the spin dependent differential cross section (2.4) if \(V = \gamma\) so that process (2.1) is purely electromagnetic. In this case Eq. (2.4) simplifies enormously and one gets

\[
\frac{d\sigma^{H(\downarrow)}(\downarrow)}{dx \, d\cos \theta} - \frac{d\sigma^{H(\uparrow)}(\downarrow)}{dx \, d\cos \theta} = N\pi \frac{\alpha^2}{Q^2} \cos \theta \, g_1^H(x, Q^2)
\]

after integration over the azimuthal angle. Notice that \(g_1(x, Q^2)\) can be also measured in the reaction where both the electron and positron are unpolarized. In this case it appears in the cross section (2.3) because of the axial
vector coupling of the Z-boson to the incoming leptons. In the purely electromagnetic case the longitudinal spin fragmentation function can be written as

\[ g_{1}^{H}(x, Q^{2}) = \frac{1}{n_{f}} \sum_{k=1}^{n_{f}} e_{k}^{2} \int_{x}^{1} \frac{dz}{z} \left[ \Delta D_{q}^{H,S}\left(\frac{x}{z}, M^{2}\right) \Delta c_{1,q}^{S}\left(z, \frac{Q^{2}}{M^{2}}\right) \right. \]

\[ + \Delta D_{g}^{H,S}\left(\frac{x}{z}, M^{2}\right) \Delta c_{1,g}^{S}\left(z, \frac{Q^{2}}{M^{2}}\right) \]

\[ + n_{f} \Delta D_{k}^{H,NS}(z, M^{2}) \Delta c_{1,k}^{NS}(z, \frac{Q^{2}}{M^{2}}) \] \hspace{1cm} (2.11)

The spin parton fragmentation densities are defined by

\[ \Delta D_{a}^{H}(z, M^{2}) = D_{a\downarrow}^{H}(z, M^{2}) - D_{a\uparrow}^{H}(z, M^{2}), \] \hspace{1cm} (2.12)

with \(a = q, \bar{q}, g\). The quark fragmentation densities can be distinguished into singlet (S) and non-singlet (NS) parts. The former is given by

\[ \Delta D_{q}^{H,S}(z, M^{2}) = \sum_{k=1}^{n_{f}} \left( \Delta D_{k}^{H}(z, M^{2}) + \Delta D_{k}^{H}(z, M^{2}) \right). \] \hspace{1cm} (2.13)

Here \(k = 1, 2, \cdots n_{f}\) means \(k = u, d, \cdots n_{f}\) where \(n_{f}\) stands for the heaviest light flavour appearing in the sums in Eqs. (2.11), (2.13). The spin non-singlet quark fragmentation densities are denoted by

\[ \Delta D_{k}^{H,NS}(z, M^{2}) = \Delta D_{k}^{H}(z, M^{2}) + \Delta D_{k}^{H}(z, M^{2}) - \frac{1}{n_{f}} \Delta D_{q}^{H,S}(z, M^{2}). \] \hspace{1cm} (2.14)

The fragmentation densities and the coefficient functions \(\Delta C_{1,a}(z, Q^{2}/M^{2})\) \((a = q, g)\) depend on the partonic Bjorken scaling variable \(z\) (see below) and on the factorization scale \(M\) which we have put equal to the renormalization scale.

The coefficient functions are computed from the partonic subprocesses which can be generally denoted by the reaction

\[ V(q) \rightarrow "a(s, p)" + a_{1}(s_{1}, p_{1}) + a_{2}(s_{2}, p_{2}) + \cdots + a_{k}(s_{k}, p_{k}), \] \hspace{1cm} (2.15)

where \(a\) represents the detected parton \((a = q, \bar{q}, g)\) with momentum \(p\) which fragments into the hadron \(H\). The remaining partons \(a_{k}\) belong to the inclusive state and one has to integrate over all momenta \(p_{k}\) and to sum over
all spins $s_k$. The reaction in Eq. (2.13) can be described by a partonic structure tensor, indicated by a hat, which has the same form as Eq. (2.3) where $H(s, p)$ and $X$ are replaced by $a(s, p)$ and $\{a_k\}$ respectively. In the case the detected parton in Eq. (2.15) is given by a massless (anti-) quark the partonic structure tensor for unpolarized scattering can be written as

$$
\hat{W}_{\mu\nu}^{\text{average}, q, (V, V')} (p, q) = \text{Tr} \gamma^{\mu} G_{\mu\nu} (p, q). \quad (2.16)
$$

In the above equation $G_{\mu\nu}$ denotes the amplitude squared where the Dirac spinors of the detected quark $q(p)$ are removed. The indices $\mu$ and $\nu$ refer to the (axial-) vector couplings of the vector bosons $V$ and $V'$ respectively. Further we have integrated over all momenta and summed over the spins of the inclusive state in reaction (2.13). If the gluon is detected in the final state of reaction (2.15) we can write

$$
\hat{W}_{\mu\nu}^{\text{average}, g, (V, V')} (p, q) = P_{\alpha\beta} G_{\mu\nu}^{\alpha\beta} (p, q). \quad (2.17)
$$

Here $G_{\mu\nu}^{\alpha\beta}$ is defined in an analogous way as $G_{\mu\nu}$ above and $\alpha, \beta$ refer to the Lorentz indices of the polarization vectors of the detected gluon i.e. $e^\alpha(p)$, $e^\beta(p)$ which are removed from $G_{\mu\nu}^{\alpha\beta}$. In unpolarized scattering $P_{\alpha\beta}$ stands for the spin averaged sum over the polarizations. The same can be done for polarized scattering. Choosing $s = p$ the partonic spin structure tensor reads

$$
\hat{W}_{\mu\nu}^{\text{spin}, q, (V, V')} (s, p, q) |_{s = p} = \text{Tr} \gamma^5 G_{\mu\nu} (p, q). \quad (2.18)
$$

If the gluon is detected in the final state of reaction (2.15) we can write

$$
\hat{W}_{\mu\nu}^{\text{average}, g, (V, V')} (p, q) = \Delta P_{\alpha\beta} G_{\mu\nu}^{\alpha\beta} (p, q). \quad (2.19)
$$

In polarized scattering $\Delta P_{\alpha\beta}$ stands for the difference between the right- and the lefthanded polarizations. Here we will choose (see [2])

$$
\Delta P_{\alpha\beta} = \frac{1}{p \cdot q} \varepsilon_{\alpha\beta\sigma\lambda} p^\sigma q^\lambda. \quad (2.20)
$$

From Eqs. (2.16)-(2.19) one can infer the partonic fragmentation functions $\hat{F}_{i,a}(z, Q^2)$ ($i = 1, 2, 3$) and $\hat{g}_{i,a}(z, Q^2)$ ($i = 1, 4, 5$) with $a = q, g$ and $z = 2p \cdot q/Q^2$ (see Eq. (2.15)) is the partonic Bjørken scaling variable. They are defined in a similar way as the hadronic fragmentation functions in Eqs.
Notice that the partonic fragmentation functions are collinearly divergent due to the higher order QCD radiative corrections. These divergences have to be removed via mass factorization so that one obtains the coefficient functions as e.g. shown in Eq. (2.11). The calculation of the spin averaged quantities $\hat{F}_{1,a}$ has been carried out up to order $\alpha_s^2$ in [7], [8], [9], [10]. Here one has applied the method of n-dimensional regularization to compute the many body phase space integrals and the loop integrals appearing in $G_{\mu\nu}$ (Eq. (2.14)) and $G^{\alpha\beta}_{\mu\nu}$ (Eq. (2.17)). We will use the same method to compute the partonic spin fragmentation function $\hat{g}_{1,a}$ which will be presented in the next section. In this case one has to find a suitable prescription for the $\gamma_5$-matrix and the Levi-Civita tensor in Eq. (2.20). Here we have adopted the HVBM prescription given by ’t Hooft and Veltman [21] which has been worked out in more detail by Breitenlohner and Maison [22] (see also [23], [24]). The computation of $\hat{g}_{1,a}$ is completely analogous to its deep inelastic counterpart described in [6] and the spin averaged partonic fragmentation functions $\tilde{F}_{1,a}$ in [6] so that we do not have to repeat the details in this paper. Before presenting the results in the next section one can derive several relations between the coefficient functions in the polarized and unpolarized case. However they are only valid up to order $\alpha_s^2$. The non-singlet coefficient functions satisfy the following equalities

$$
\Delta C_{1,q}^{NS} = C_{1,q}^{NS}, \quad \Delta C_{4,q}^{NS} = C_{2,q}^{NS}, \quad \Delta C_{5,q}^{NS} = C_{1,q}^{NS},
$$

(2.21)

For the singlet parts we have

$$
\Delta C_{4,q}^{PS} = 0, \quad \Delta C_{5,q}^{PS} = 0
$$

$$
\Delta C_{4,q}^{S} = 0, \quad \Delta C_{5,g}^{S} = 0,
$$

(2.22)

with the definition

$$
\Delta C_{i,q} = \Delta C_{i,q}^{NS} + \Delta C_{i,q}^{PS}.
$$

(2.23)

The same relations also hold for the partonic fragmentation functions from which the coefficient functions are derived via mass factorization. The relations in Eq. (2.21) follow from the anticommutativity of the $\gamma_5$-matrix which is manifest for a regularization method in four dimensions. In the case of n-dimensional regularization it occurs after the introduction of additional
renormalization constants which restore the various Ward-identities violated by the prescription used for the $\gamma_5$-matrix and the Levi-Civita tensor. The partonic fragmentation function $\hat{g}_{1,q}^{NS}$ is derived from Eq. (2.18) where the indices $\mu, \nu$ either stand for $\gamma_\mu \gamma_\nu$ or for $\gamma_\mu \gamma_5, \gamma_\nu \gamma_5$. Further if $G_{\mu\nu}$ originates from so called rainbow graphs (see e.g. fig. 1a in [11]) then the $\gamma_5$-matrix in Eq. (2.18) can be anticommuted in the string of $\gamma$-matrices so that either the index $\mu$ or the index $\nu$ represents an axial-vector. In this way Eq. (2.18) becomes equal to Eq. (2.10) which leads to $\mathcal{F}_{3,q}^{NS}$. In the case that either $\mu$ or $\nu$ (but not both) represent $\gamma_\mu \gamma_5$ or $\gamma_\nu \gamma_5$ one can anticommute the $\gamma_5$-matrix in Eq. (2.18) till it annihilates the other one in $G_{\mu\nu}$ and one obtains Eq. (2.10). In this way one obtains $\hat{g}_{4,q}^{NS} = \mathcal{F}_{2,q}^{NS}$ and $\hat{g}_{5,q}^{NS} = \mathcal{F}_{1,q}^{NS}$. The relations in Eq. (2.22) follow from charge conjugation invariance of QCD so that one gets

$$\mathcal{F}_{3,a} = -\mathcal{F}_{3,\bar{a}} \quad , \quad \hat{g}_{i,a} = -\hat{g}_{i,\bar{a}} \quad , \quad i = 4, 5 \ . \quad (2.24)$$

Notice that the relations in Eq. (2.21) do not hold anymore beyond order $\alpha_s^2$. A nice example is given by the graphs in fig. 1b of [11] (light by light scattering graphs) appearing in deep inelastic lepton-hadron scattering (spacelike process). These graphs contribute in order $\alpha_s^3$ to $\mathcal{F}_{3,q}^{NS}$ but not to $\hat{g}_{1,q}^{NS}$. In first order of $\alpha_s$ the coefficient functions $\Delta C_{4,q}^{NS}$ and $\Delta C_{5,q}^{NS}$ were calculated in [17] and they satisfy the relations in Eq. (2.21). The order $\alpha_s^2$ contributions are equal to $C_{2,q}^{NS}$ and $C_{1,q}^{NS}$ respectively and can be found in [4].
3 Order $\alpha_s^2$ contributions to the coefficient functions $\Delta C_{1,a}$

In this section we present the results for the coefficient functions $\Delta C_{1,a}$ which emerge from the computation of the partonic fragmentation functions $\hat{g}_{1,a}$ with $a = q, g$. The latter are collinearly divergent and the singularities are regularized via n-dimensional regularization so that they manifest themselves as pole terms of the type $(1/\varepsilon)^k$, with $\varepsilon = n - 4$, in $\hat{g}_{1,a}(z, Q^2, \varepsilon)$. For the definition of the $\gamma_5$-matrix we use the HVBM prescription [21], [22] (see also [25], [26]). This prescription destroys the anticommutativity of the $\gamma_5$-matrix in Eq. (2.18) so that $\hat{g}^{NS}_{1,q} \neq \hat{F}^{NS}_{3,q}$. This will lead to a violation of the relations between the coefficients in Eq. (2.21) and the nonrenormalization properties of the non-singlet axial-vector current. Furthermore as is shown in [25] the Adler-Bardeen theorem [27] will be also violated. These properties can be restored by introducing an additional renormalization constant $Z^r_{qq}$ ($r = NS, S$) which has the following unrenormalized form

$$Z^r_{qq} = \delta(1 - z) + \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^{\varepsilon/2} \left[ z^{(1)}_{qq}(z) + \varepsilon b^{(1)}_{qq}(z) \right]$$

$$+ \left( \frac{\hat{\alpha}_s}{4\pi} \right)^2 S_\varepsilon^2 \left( \frac{M^2}{\mu^2} \right)^{\varepsilon} \left[ - \frac{1}{\varepsilon} \beta_0 z^{(1)}_{qq}(z) + z^{r,(2)}_{qq}(z) - 2\beta_0 b^{(1)}_{qq}(z) \right], \quad (3.1)$$

up to order $\alpha_s^2$. Further we have defined the spherical factor as

$$S_\varepsilon = \exp \left[ \frac{1}{2} \varepsilon (\gamma_E - \ln 4\pi) \right], \quad (3.2)$$

which is an artefact of n-dimensional regularization. The same holds for the scale $\mu$ which originates from the dimensionality of the gauge coupling constant in n dimensions ($g \rightarrow g(\mu)^{-\varepsilon/2}$). The mass parameter $M$ denotes the renormalization scale which will be put equal to the factorization scale. Further $\beta_0$ is the lowest-order coefficient in the series expansion of the $\beta$-function which, up to order $g^5$, is given by

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} \quad , \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_f, \quad g^2 = 4\pi \alpha_s, (3.3)$$

where $C_A = N$ and $T_f = 1/2$ are colour factors in $SU(N)$. Notice that in second order $z^{S,(2)}_{qq} = z^{NS,(2)}_{qq} + n_f z^{PS,(2)}_{qq}$ where $n_f z^{PS,(2)}_{qq}$ is coming from the
light by light scattering graphs in fig. 1 of [25]. In the non-singlet case $Z_{\text{NS}}^{qq}$ can be derived from

$$Z_{\text{NS}}^{qq}(z) = \frac{g_{1,q}^{\text{NS}}(z, Q^2, \mu^2)}{F_{3,q}^{\text{NS}}(z, Q^2, \mu^2)} |_{Q^2=M^2}. \quad (3.4)$$

The first order coefficients are given by

$$z_{qq}^{(1)} = C_F \left[ -8(1 - z) \right], \quad (3.5)$$

with $C_F = (N^2 - 1)/2N$ and

$$b_{qq}^{(1)} = C_F \left[ -8(1 - z) \ln z - 4(1 - z) \ln(1 - z) + 10 - 8z \right]. \quad (3.6)$$

The second order coefficient $z_{qq}^{\text{NS},(2)}$ is given in appendix A. We will now present the general form of the partonic fragmentation functions which follow from mass factorization and renormalization. The expressions are written in such a way that the non-logarithmic parts of the coefficient functions denoted by $\bar{c}_{1,k} (k = q, g)$ and the DGLAP splitting functions are presented in the $\overline{\text{MS}}$-scheme. This scheme holds for coupling constant renormalization as well as mass factorization.

In the Born approximation the partonic reaction in Eq. (2.15) is given by (see fig. 2 in [8])

$$V \rightarrow q + \bar{q}, \quad (3.7)$$

where either the quark or the anti-quark is detected in the final state. This leads to the contribution

$$\hat{g}_{1,q}^{(0)} = 1 \equiv \delta(1 - z). \quad (3.8)$$

In order $\alpha_s$ we have the following contributions. First we have to compute the one-loop virtual corrections to the Born reaction (3.7) (see fig. 3 in [8]) and add them to the gluon bremsstrahlung process (see fig. 4 in [8])

$$V \rightarrow q + \bar{q} + g. \quad (3.9)$$

In the case the quark or the anti-quark is detected in the final state the result becomes

$$\hat{g}_{1,q}^{(1)} = \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{Q^2}{\mu^2} \right)^{\varepsilon/2} \left[ \Delta F_{qq}^{(0)} \frac{1}{\varepsilon} + \hat{c}_{1,q}^{(1)} + z_{qq}^{(1)} + \varepsilon \left\{ \hat{a}_{1,q}^{(1)} + b_{qq}^{(1)} \right\} \right]. \quad (3.10)$$
The timelike spin splitting function is given by

$$\Delta P_{qq}^{(0)} = C_F \left[ 8 \left( \frac{1}{1-z} \right) + 4 - 4z + 6\delta(1-z) \right], \quad (3.11)$$

and the non-pole terms are given by

$$\bar{c}_{1,q}^{(1)} = C_F \left[ 4 \left( \frac{\ln(1-z)}{1-z} \right) + 3 \left( \frac{1}{1-z} \right) + 
-2(1+z)\ln(1-z) + 4 \frac{1+z^2}{1-z} \ln(z) + 1-z 
+ \delta(1-z) \left( -9 + 8\zeta(2) \right) \right], \quad (3.12)$$

$$\bar{a}_{1,q}^{(1)} = C_F \left[ \left( \frac{\ln^2(1-z)}{1-z} \right) + \frac{3}{2} \left( \frac{\ln(1-z)}{1-z} \right) + \left( \frac{7}{2} - 3\zeta(2) \right) \left( \frac{1}{1-z} \right) + 
+2 \frac{1+z^2}{1-z} \left( \ln(z) \ln(1-z) + \ln^2(z) \right) - 3 \frac{\ln(z)}{1-z} 
+(1+z) \left( \frac{3}{2}\zeta(2) - \frac{1}{2} \ln^2(1-z) \right) + (1-z) \left( \frac{1}{2} \ln(1-z) 
+ \ln(z) \right) - \frac{3}{2} + \frac{1}{2} z + \delta(1-z) \left( -9 + \frac{33}{4} \zeta(2) \right) \right]. \quad (3.13)$$

Notice that we also have to include in order $\alpha_s$ terms proportional to $\varepsilon$, which are represented by $b_{qq}^{(1)}$ Eq. (3.1) and $\bar{a}_{1,k}^{(1)} (k = q, g)$ since they contribute via mass factorization to the non-logarithmic parts of the coefficient functions. When the gluon is detected in reaction (3.9) we get

$$\hat{g}_{1,g}^{(1)} = \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{Q^2}{\mu^2} \right)^{\varepsilon/2} \left[ 2\Delta P_{gg}^{(0)} \frac{1}{\varepsilon} + \bar{c}_{1,g}^{(1)} + \varepsilon \bar{a}_{1,g}^{(1)} \right]. \quad (3.14)$$

Here the timelike spin splitting function is given by

$$\Delta P_{gg}^{(0)} = C_F \left[ 8 - 4z \right], \quad (3.15)$$

and the remaining coefficients are

$$\bar{c}_{1,g}^{(1)} = C_F \left[ (8 - 4z) \left( \ln(1-z) + 2\ln(z) \right) - 16 + 12z \right], \quad (3.16)$$
\[
\hat{a}_{1,\bar{q}}^{(1)} = C_F \left[ (2-z) \left\{ \ln^2(1-z) + 4 \ln(z) \ln(1-z) + 4 \ln^2(z) \right\} 
\right.
\]
\[
+ (-8 + 6z) \left\{ \ln(1-z) + 2 \ln(z) - 2 \right\} - \left. (6 - 3z) \zeta(2) \right] .
\]

In order \(\alpha_s^2\) we have the following subprocesses. The non-singlet partonic fragmentation function receives contributions from the two-loop corrections to reaction (3.7) (fig. 5 of [8]), the one-loop correction to reaction (3.9) (fig. 6 of [8]) and the double gluon bremsstrahlung process (fig. 7 of [8])

\[
V \rightarrow q + \bar{q} + g + g ,
\]

where either the quark or the anti-quark is detected in the final state. Further we have to add the contribution from the partonic subprocess

\[
V \rightarrow q + \bar{q} + q + \bar{q} ,
\]

with the condition that the two quarks or the two anti-quarks which belong to the inclusive state are identical (see figs. 8,10 in [8]). The result has the following form

\[
\hat{g}^{NS,(2)}_{1,q} = \left( \frac{\alpha_s}{4\pi} \right)^2 S_\varepsilon^2 \left( \frac{Q^2}{\mu^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon} \left\{ \frac{1}{2} \Delta P_{qq}^{(0)} \otimes \Delta P_{qq}^{(0)} - \beta_0 \Delta P_{qq}^{(0)} \right\} 
\right.
\]
\[
+ \frac{1}{\varepsilon} \left\{ \frac{1}{2} \left( \Delta P_{qq}^{NS,(1)} - \Delta P_{qq}^{NS,(1)} \right) \right\} - 2\beta_0 \left( \hat{c}_{1,q}^{(1)} + z_{qq}^{(1)} \right)
\]
\[
+ \Delta P_{qq}^{(0)} \otimes \left( \hat{c}_{1,q}^{(1)} + z_{qq}^{(1)} \right) \right\} + \hat{c}_{1,q}^{NS,(2),id} - \hat{c}_{1,q}^{NS,(2),id} + z_{qq}^{NS,(2)}
\]
\[
+ z_{qq}^{(1)} \otimes \hat{c}_{1,q}^{(1)} - 2\beta_0 \left( \hat{a}_{1,q}^{(1)} + b_{qq}^{(1)} \right) + \Delta P_{qq}^{(0)} \otimes \left( \hat{a}_{1,q}^{(1)} + b_{qq}^{(1)} \right) \right] .
\]

The convolution symbol denoted by \(\otimes\) is defined by

\[
(f \otimes g)(z) = \int_0^1 dz_1 \int_0^1 dz_2 \, \delta(z - z_1 z_2) f(z_1) g(z_2) .
\]

The terms \(\hat{c}_{1,q}^{(1)}\) and \(\hat{a}_{1,q}^{(1)}\) already appeared in \(\hat{g}^{(1)}_{1,q}\) (Eq. (3.10)). The order \(\alpha_s^2\) non-singlet timelike spin splitting functions \(\Delta P_{qq}^{NS,(1)}\) and \(\Delta P_{qq}^{NS,(1)}\) are calculated in [28] (see also [20]) and can be found in Eqs. (4.11) and (4.12)
of \[8\]. Notice that they are equal to the spin averaged ones contrary to the singlet case which we will present below. The splitting function \(\Delta P_{q\bar{q}}^{\text{NS}(1)}\) originates from process \((3.19)\) with two identical (anti-)quarks in the inclusive state. The same holds for the non-pole term \(\hat{c}_{1,q}^{\text{NS}(2),\text{id}}\) Notice that \(\hat{g}_{1,q}^{\text{NS}(2)}\) only appears if the amplitude \(G_{\mu \nu}\) (see Eq. \((2.18)\)) is projected on the non-singlet part. If we project on the singlet part we get

\[\hat{g}_{1,q}^{S,(2)} = \hat{g}_{1,q}^{\text{NS}(2)} + \hat{g}_{1,q}^{\text{PS}(2)},\]  

(3.22)

with

\[
\hat{g}_{1,q}^{\text{PS}(2)} = n_f \left(\frac{\hat{\alpha}_s}{4\pi}\right)^2 S^2 \left(\frac{Q^2}{\mu^2}\right) \frac{1}{\epsilon} \left\{ \frac{1}{2} \Delta P_{q\bar{q}}^{(0)} \otimes \Delta P_{q\bar{q}}^{(0)} \right\} + \frac{1}{\epsilon} \left\{ \frac{1}{2} \Delta P_{q\bar{q}}^{\text{PS}(1)} \right\}
\]

\[+ \frac{1}{2} \Delta P_{q\bar{q}}^{(0)} \otimes \hat{c}_{1,q}^{(1)} + \hat{c}_{1,q}^{\text{PS}(2)} + \frac{1}{2} \Delta P_{q\bar{q}}^{(0)} \otimes \hat{a}_{1,g}^{(1)} + z_{q\bar{q}}^{\text{PS}(2)},\]  

(3.23)

where \(\hat{c}_{1,q}^{(1)}\) and \(\hat{a}_{1,g}^{(1)}\) are given in Eq. \((3.14)\). The purely singlet (PS) part originates from process \((3.19)\) which proceeds by the exchange of two gluons in the t-channel (see figs. 8,9 in \[8\]). Further \(z_{q\bar{q}}^{S,(2)} = z_{q\bar{q}}^{\text{NS}(2)} + n_f z_{q\bar{q}}^{\text{PS}(2)}\) where \(n_f z_{q\bar{q}}^{\text{PS}(2)}\) originates from the light by light scattering graphs in fig. 1 in \[24\].

In the second order expression above the lowest order splitting function \(\Delta P_{q\bar{q}}^{(0)}\) appears for the first time and it reads

\[\Delta P_{q\bar{q}}^{(0)} = T_f (-8 + 16z).\]  

(3.24)

From our calculations and the expression given in Eq. \((3.23)\) one can also infer the splitting function

\[\Delta P_{q\bar{q}}^{\text{PS}(1)} = C_F T_f \left[ (16 + 16z) \ln^2 z - (48 + 112z) \ln z - 176 + 176z \right].\]  

(3.25)

The expression above agrees with the recent result obtained in \[20\]. The complete singlet spin timelike splitting function is then given by

\[\Delta P_{q\bar{q}}^{S,(1)} = \Delta P_{q\bar{q}}^{\text{NS}(1)} - \Delta P_{q\bar{q}}^{\text{NS}(1)} + \Delta P_{q\bar{q}}^{\text{PS}(1)}.\]  

(3.26)

\(^1\) For the explicit expressions of \(\hat{c}_{1,q}^{\text{NS}(2),\text{id}}\) and \(\hat{c}_{1,q}^{\text{NS(2),id}}\) see Eqs. (4.A.32) - (4.A.37) in \[10\].
If the gluon is detected in the final state the one-loop corrections to reaction (3.9) and the contribution due to the gluon bremsstrahlung process in Eq. (1.18) lead to the answer.

\[
\hat{g}^{(2)}_{1,g} = \left( \frac{\hat{\alpha}_s}{4\pi} \right)^2 S_z \left( \frac{Q^2}{\mu^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon^2} \left\{ \Delta P_{gg}^{(0)} \otimes (\Delta P_{gg}^{(0)} + \Delta P_{qq}^{(0)}) - 2\beta_0 \Delta P_{gg}^{(0)} \right\} 
\right.
\]

\[
\left. + \frac{1}{\varepsilon} \left\{ (\Delta P_{gg}^{(1)} - \Delta P_{gg}^{(0)} \otimes z_{qq}^{(1)}) - 2\beta_0 \hat{c}_{1,g}^{(1)} + \Delta P_{gg}^{(0)} \otimes \hat{c}_{1,g}^{(1)} \right\} + \hat{c}_{1,g}^{(2)} - 2\beta_0 \hat{a}_{1,g}^{(1)} + \Delta P_{gg}^{(0)} \otimes \hat{a}_{1,g}^{(1)} \right]
\]

\[
+ 2\Delta P_{gg}^{(0)} \otimes \left( \hat{c}_{1,g}^{(1)} + z_{qq}^{(1)} \right) \right] \right) + \left( \Delta P_{gg}^{(0)} \otimes \hat{a}_{1,g}^{(1)} \right)
\]

In the second order expression above one encounters for the first time the lowest order splitting function

\[
\Delta P_{gg}^{(0)} = C_A \left( 8 \left( \frac{1}{1-z} \right)_+ + 8 - 16z + \frac{22}{3} \delta(1-z) \right) - \frac{8}{3} n_f T_f \delta(1-z).
\]

From our calculation we also infer the second order spin timelike splitting function

\[
\Delta P_{qq}^{(1)} = C_F^2 \left[ (128 - 64z) \left( \text{Li}_2(1-z) + \frac{1}{2} \ln z \ln(1-z) 
\right.
\right.
\]

\[
\left. - \frac{1}{16} \ln^2 z + \frac{1}{8} \ln^2(1-z) - \zeta(2) \right) + (64 - 36z) \ln z
\]

\[
- 32(1-z) \ln(1-z) + 76 - 56z
\]

\[
+ C_A C_F \left[ (32 + 16z) \left( \text{Li}_2(-z) + \ln z \ln(1+z) \right) + (-128 + 64z)
\right.
\]

\[
\times \left( \text{Li}_2(1-z) + \frac{1}{4} \ln z \ln(1-z) + \frac{1}{8} \ln^2 (1-z) \right) - (48 + 8z) \ln^2 z
\]

\[
+ (128 - 48z) \zeta(2) + (-32 + 88z) \ln z + 32(1-z) \ln(1-z)
\]

\[
+ 40 - 32z \right],
\]
which agrees with the recent result in \[20\].

To obtain the finite coefficient functions in the $\overline{\text{MS}}$-scheme one has to perform mass factorization. The relations between the partonic fragmentation functions and the coefficient functions are then given by

\[ \hat{g}^\text{NS}_{1,q} = Z^\text{NS}_{qq} \otimes \hat{\Gamma}^\text{NS}_{qq} \otimes \Delta \mathcal{C}^\text{NS}_{1,q}, \]

\[ \hat{g}^S_{1,q} = Z^S_{qq} \otimes \hat{\Gamma}^S_{qq} \otimes \Delta \mathcal{C}^S_{1,q} + n_f \hat{\Gamma}^S_{gg} \otimes \Delta \mathcal{C}^S_{1,g}, \]

\[ \hat{g}_{1,g} = 2 Z^S_{qq} \otimes \hat{\Gamma}^S_{qq} \otimes \Delta \mathcal{C}^S_{1,q} + \hat{\Gamma}^S_{gg} \otimes \Delta \mathcal{C}^S_{1,g}. \]

Here the renormalization factor $Z_{qq}^r$ is needed to restore the anticommutativity of the $\gamma_5$-matrix on the level of the coefficient functions so that they are presented in the genuine $\overline{\text{MS}}$-scheme. If we expand the transition functions in the unrenormalised coupling constant $\hat{\alpha}_s$ they read as follows

\[ \hat{\Gamma}^\text{NS}_{qq} = 1 + \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^\varepsilon/2 \left[ \frac{1}{\varepsilon} \Delta P^{(0)}_{qq} \right] + \left( \frac{\hat{\alpha}_s}{4\pi} \right)^2 S^2_\varepsilon \left( \frac{M^2}{\mu^2} \right) \varepsilon \times \left[ \frac{1}{\varepsilon^2} \left\{ \frac{1}{2} \Delta P^{(0)}_{qq} \otimes \Delta P^{(0)}_{qq} - \beta_0 \Delta P^{(0)}_{qq} \right\} + \frac{1}{\varepsilon} \left\{ \frac{1}{2} \left( \Delta P^{\text{NS},(1)}_{qq} - \Delta P^{\text{NS},(1)}_{qq} \right) \right\} \right]. \]

The singlet transition function is given by

\[ \hat{\Gamma}^S_{qq} = \hat{\Gamma}^\text{NS}_{qq} + 2n_f \hat{\Gamma}^\text{PS}_{qq}, \]

where the purely singlet (PS) part can be written as

\[ \hat{\Gamma}^\text{PS}_{qq} = \left( \frac{\hat{\alpha}_s}{4\pi} \right)^2 S^2_\varepsilon \left( \frac{M^2}{\mu^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon^2} \left\{ \frac{1}{4} \Delta P^{(0)}_{gg} \otimes \Delta P^{(0)}_{gg} \right\} + \frac{1}{\varepsilon} \left\{ \frac{1}{4} \Delta P^{\text{PS},(1)}_{qq} \right\} \right]. \]

Finally we have

\[ \hat{\Gamma}_{gg} = \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^\varepsilon/2 \left[ \frac{1}{\varepsilon} \Delta P^{(0)}_{gg} \right] + \left( \frac{\hat{\alpha}_s}{4\pi} \right)^2 S^2_\varepsilon \left( \frac{M^2}{\mu^2} \right) \varepsilon \]
\[
\times \left[ \frac{1}{\varepsilon^2} \left\{ \frac{1}{2} \Delta P_{gq}^{(0)} \otimes (\Delta P_{gg}^{(0)} + \Delta P_{qq}^{(0)}) - \beta_0 \Delta P_{gq}^{(0)} \right\} \\
+ \frac{1}{\varepsilon} \left\{ \frac{1}{2} \left( \Delta P_{gq}^{(1)} - \Delta P_{gg}^{(0)} \otimes \varepsilon_q^{(1)} \right) \right\} \right]. 
\]

(3.36)

We only need the following transition functions up to order \(\hat{\alpha}_s\)
\[
\Gamma_{gg} = \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^{\varepsilon/2} \left[ \frac{1}{2\varepsilon} \Delta P_{gg}^{(0)} \right], 
\]

(3.37)

\[
\Gamma_{qg} = \frac{\hat{\alpha}_s}{4\pi} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^{\varepsilon/2} \left[ \frac{1}{\varepsilon} \Delta P_{gq}^{(0)} \right]. 
\]

(3.38)

Further we have to perform coupling constant renormalization which is also carried out in the MS-scheme. Up to order \(\alpha_s^2\) it is sufficient to replace the bare coupling constant by
\[
\frac{\hat{\alpha}_s}{4\pi} = \frac{\alpha_s(M^2)}{4\pi} \left( 1 + \frac{\alpha_s(M^2)}{4\pi} \frac{2\beta_0}{\varepsilon} S_\varepsilon \left( \frac{M^2}{\mu^2} \right)^{\varepsilon/2} \right). 
\]

(3.39)

The coefficient functions (MS-scheme) have the following representations
\[
\Delta \bar{c}^{\text{NS}}_{1,q} = 1 + \frac{\alpha_s}{4\pi} \left[ \frac{1}{2} \Delta P_{gq}^{(0)} L_M + \bar{c}_{1,q}^{(1)} \right] + \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ \frac{1}{8} \Delta P_{gq}^{(0)} \otimes \Delta P_{gq}^{(0)} \right\} \\
- \frac{1}{4} \beta_0 \Delta P_{gq}^{(0)} \right\} L_M + \left\{ \frac{1}{2} \left( \Delta P_{gq}^{\text{NS},(1)} - \Delta P_{gq}^{\text{NS},(1)} \right) - \beta_0 \bar{c}_{1,q}^{(1)} \\
+ \frac{1}{2} \Delta P_{gq}^{(0)} \otimes \bar{c}_{1,q}^{(1)} \right\} L_M + \bar{c}_{1,q}^{\text{NS},(2),\text{id}} - \bar{c}_{1,q}^{\text{NS},(2),\text{id}} \right], 
\]

(3.40)

\[
\Delta \bar{c}^{\text{PS}}_{1,q} = n_f \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ \frac{1}{8} \Delta P_{gq}^{(0)} \otimes P_{gq}^{(0)} \right\} L_M^2 \\
+ \left\{ \frac{1}{2} \Delta P_{gq}^{\text{PS},(1)} + \frac{1}{4} \Delta P_{gq}^{(0)} \otimes \bar{c}_{1,q}^{(1)} \right\} L_M + \bar{c}_{1,q}^{\text{PS},(2)} \right], 
\]

(3.41)
\[\Delta \mathcal{C}_{1,g} = \frac{\alpha_s}{4\pi} \left[ \Delta P_{gq}^{(0)} L_M + \bar{c}_{1,g}^{(1)} \right] + \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ \left[ \frac{1}{4} \Delta P_{gq}^{(0)} \otimes (\Delta P_{gq}^{(0)} + \Delta P_{qq}^{(0)}) \right] - \frac{1}{2} \beta_0 \Delta P_{gq}^{(0)} \right\} L_M^2 + \left\{ \Delta P_{gq}^{(1)} - \beta_0 \bar{c}_{1,g}^{(1)} + \frac{1}{2} \Delta P_{gq}^{(0)} \otimes \bar{c}_{1,g}^{(1)} \right\} L_M + \bar{c}_{1,g}^{(2)}, \tag{3.42}\]

with the definitions

\[L_M = \ln \frac{Q^2}{M^2}, \quad \alpha_s \equiv \alpha_s(M^2), \tag{3.43}\]

In the above expressions \(M\) denotes the factorization scale which has been put equal to the renormalization scale defined by \(R\). In the case these scales are chosen to be different the corresponding coefficient functions can be derived from expressions (3.41)-(3.43) by the replacement

\[\alpha_s(M^2) = \alpha_s(R^2) \left[ 1 + \frac{\alpha_s(R^2)}{4\pi} \beta_0 \ln \frac{R^2}{M^2} \right]. \tag{3.44}\]

As has been indicated in Eq. (2.21) \(\Delta \mathcal{C}_{1,q}^{NS}\) (3.34) is equal to \(\bar{\mathcal{C}}_{3,q}^{NS}\) and the latter is calculated in [9] (see also [10]). Since \(z_{qq}^{PS,(2)}\) is not known yet, except for the first moment (see below), we can only present \(\bar{c}_{1,q}^{PS,(2)} + z_{qq}^{PS,(2)}\) in \(\mathcal{C}_{1,q}^{PS}\) (see Eqs. (3.41), (A.1)) but not \(\bar{c}_{1,q}^{PS,(2)}\) and \(z_{qq}^{PS,(2)}\) separately. Unfortunately we cannot infer the latter from a ratio of two partonic fragmentation functions like we could do for \(z_{qq}^{NS,(2)}\) in Eq. (3.4). In the purely singlet case one has to compute the three-loop operator matrix elements for general moments (spin) given by the light by light scattering graphs in fig. 1 of [25]. The renormalization of these graphs is ruled by the Adler-Bardeen theorem [27] which is hard to impose on the level of the second order partonic fragmentation functions. For that one has to calculate the latter quantities in third order which is a tantalizing enterprise. Fortunately this does not hamper the computation of \(\Delta \mathcal{C}_{1,g}\) (3.42) up to second order, in which \(z_{qq}^{PS,(2)}\) does not show up, and the result can be found in Eq. (A.2).

Finally we want to mention another problem which concerns the renormalization constant \(Z_{qq}^{r} (r = NS, S)\) in Eq. (3.1) which is needed to restore
the Ward-identities broken by the HVBM-prescription for the $\gamma_5$-matrix. After coupling constant renormalization it reads

$$Z_{qq}^r = \delta(1 - z) + \frac{\alpha_s}{4\pi} S_\varepsilon \left(\frac{M^2}{\mu^2}\right)^{\varepsilon/2} \left[ z_{qq}^{(1)}(z) + \varepsilon b_{qq}^{(1)}(z) \right]$$

$$+ \left(\frac{\alpha_s}{4\pi}\right)^2 S_\varepsilon \left(\frac{M^2}{\mu^2}\right)^{\varepsilon} \left[ -\frac{1}{\varepsilon} \beta_0 z_{qq}^{(1)}(z) + z_{qq}^{(2)}(z) \right],$$

(3.45)

where $z_{qq}^{(1)}$ and $b_{qq}^{(1)}$ are given in Eqs. (3.3) and (3.6) respectively. In the non-singlet case $z_{qq}^{NS, (2)}$ can be inferred from Eq. (3.4) and it is presented in Eq. (A.3). The first moment of Eq. (3.45) is equal to

$$Z_{qq}^{NS} = 1 + \frac{\alpha_s}{4\pi} S_\varepsilon C_F \left[ -4 + 13\varepsilon \right] + \left(\frac{\alpha_s}{4\pi}\right)^2 S_\varepsilon \left[ C_F^2 \{12 + 16\zeta(2)\} \right]$$

$$+ C_A C_F \left\{ -\frac{44}{3} \frac{1}{\varepsilon} - \frac{107}{9} \right\} + n_f C_F T_f \left\{ \frac{16}{3} \frac{1}{\varepsilon} + \frac{4}{9} \right\},$$

(3.46)

where we have put $M = \mu$. Comparing the result above with the one obtained in [11] given by

$$Z_{qq}^{NS} = 1 + \frac{\alpha_s}{4\pi} S_\varepsilon C_F \left[ -4 + 5\varepsilon \right] + \left(\frac{\alpha_s}{4\pi}\right)^2 S_\varepsilon \left[ C_F^2 \{22\} \right]$$

$$+ C_A C_F \left\{ -\frac{44}{3} \frac{1}{\varepsilon} - \frac{107}{9} \right\} + n_f C_F T_f \left\{ \frac{16}{3} \frac{1}{\varepsilon} + \frac{4}{9} \right\},$$

(3.47)

we observe a discrepancy which shows up in the order $\alpha_s^2$ coefficient of the colour factor $C_F^2$. Notice that the result in Eqs. (8), (11) of [11] is obtained by computing the ratio between the vector and the axial-vector form factor where it was needed to restore the Ward-identity for the non-singlet axial vector current broken by the HVBM-prescription. A discrepancy is also observed when Eq. (3.3) is computed for deep inelastic scattering where $q^2 = -Q^2$ (see [1]). Here one gets an answer which differs from the result in Eq. (3.46) but agrees with Eq. (3.47). From the above we conclude that the renormalization constant $Z_{qq}^{NS}$ is process dependent except for leading order. The difference of this constant between deep inelastic structure functions (spacelike process) and the fragmentation functions (timelike process) is of the same origin as the one discovered for spacelike and timelike splitting functions in [28]. The process dependence of the renormalization constant above
is in contrast to what is known about the usual renormalization constants computed in cases where the regularization procedure does not violate the Ward-identities. In [25] also the first moment of $Z_{qq}^S$ has been computed. The difference between $Z_{qq}^{NS}$ and $Z_{qq}^S$ only shows up in the $n_f C_F T_f$-part of Eq. (3.47). In the singlet case one has to add to the latter expression the first moment of $n_f z_{qq}^{PS,(2)}$ which equals $3n_f C_F T_f$. Since the leading terms in $Z_{qq}^r (r = NS, S)$ are process independent we can assume that this also holds for $z_{qq}^{PS,(2)}$ so that at least we know the first moment for $\Delta C_{1,q}^{PS}$ (Eq. (3.41)).

Summarizing the above we have computed the order $\alpha_s^2$ contributions to the coefficient functions corresponding to the spin fragmentation functions $g_i^H(x, Q^2) (i = 1, 4, 5)$ measured in electron-positron annihilation in which the hadron H is polarized. In addition we obtained the second order splitting functions $\Delta P_{qq}^{PS,(1)}$, $\Delta P_{qq}^{(1)}$ which agree with the recent results published in [20]. Unfortunately we could not get the remaining ones given by $P_{qq}^{(1)}$ and $P_{gg}^{(1)}$ which could be calculated by the cut vertex method in [20]. This is because the splitting functions $P_{qq}^{(i)}$, $P_{gg}^{(i)}$ always appear one order higher in the coefficient functions than $P_{qq}^{(i)}$, $P_{gg}^{(i)}$. This property can be traced back to the fact that the electroweak vector bosons never directly couple to the gluon. Finally we want to mention that for a complete next-to-next-to-leading order analysis one also needs the three-loop splitting functions which have to be combined with the coefficient functions in Eqs. (3.40)-(3.42). The computation of these splitting functions for spacelike and timelike processes is one of the outstanding enterprises which have to be still carried out.

Acknowledgements

We would like to thank J. Smith for the critical reading of the manuscript and for giving us some useful comments.
Appendix A

In this section we only present the coefficient functions $\Delta C_{1,q}^{PS}$ and $\Delta C_{1,g}$ since the non-singlet ones, given by $\Delta C_{i,q}^{NS}$ ($i = 1, 4, 5$), are already known in the literature via the relations in Eq. (2.21). The purely singlet coefficient equals

$$\Delta C_{1,q}^{PS} = n_f (\frac{\alpha_s}{4\pi})^2 C_F T_f \left\{ 8(1+z) \ln z + 20(1-z) \right\} L_M^2$$

$$+ \left\{ 16(1+z) \left( \text{Li}_2(1-z) + \ln z \ln(1-z) \right) + 24(1+z) \ln^2 z ight\} L_M$$

$$+ (8-88z) \ln z + 40(1-z) \ln(1-z) - 136(1-z) \right\} L_M$$

$$+ 16(1+z) \left( 3S_{1,2}(1-z) - \text{Li}_3(1-z) + 3 \ln z \text{Li}_2(1-z) ight)$$

$$+ \ln(1-z) \text{Li}_2(1-z) + \frac{1}{2} \ln z \ln^2(1-z) + \frac{3}{2} \ln^2 z \ln(1-z)$$

$$+ \frac{11}{12} \ln^3 z - \zeta(2) \ln z \right\} + (8-88z) \left( \text{Li}_2(1-z) + \ln z \ln(1-z) \right)$$

$$- \left( \frac{32}{3z} + 32 + 3z^2 \right) \left( \text{Li}_2(-z) + \ln z \ln(1+z) \right) + \left( -26 \right.$$

$$- 58z + \frac{16}{3} z^2 \right) \ln^2 z + 20(1-z) \ln^2(1-z) + \left( -72 + 40z ight.$$

$$- \frac{32}{3} z^2 \zeta(2) + \left( -\frac{364}{3} + \frac{356}{3} \right) \ln z - 136(1-z) \ln(1-z)$$

$$+ \frac{472}{3} (1-z) \right\} - n_f (\frac{\alpha_s}{4\pi})^2 z_{qq}^{PS,(2)}(z), \quad (A.1)$$

with

$$z_{qq}^{PS,(2)}(z) = C_F T_f \left[-4(1+2z) \ln^2 z + 8(-1+3z) \ln z + 16(1-z) \right]. \quad (A.2)$$

The gluonic coefficient function becomes

$$\Delta C_{1,g}^{(2)} = (\frac{\alpha_s}{4\pi})^2 \left[C_F^2 \left\{ (-8+4z) \ln z + (16-8z) \ln(1-z) \right\} \right. \quad (A.3)$$

23
\begin{align}
+6z)L_M^2 + \left((48 - 24z)\text{Li}_2(1 - z) + (-24 + 12z) \ln^2 z + (48 - 24z) \ln z \ln(1 - z) + (32 - 16z) \ln^2(1 - z) + (-64 + 32z)\zeta(2)
\right.

+ (128 + 4z) \ln z + (-80 + 68z) \ln(1 - z) + 132 - 148z) \right) L_M

+ \left(- \frac{96}{z} + 64 - 96z\right) \text{Li}_3(-z) + \left(\frac{64}{z} + 128 + 64z\right) S_{1,2}(-z)

+ \left(\frac{32}{z} - 240 + 120z\right) S_{1,2}(1 - z) + \left(\frac{64}{z} + 128 + 64z\right)

\times \ln(1 + z) \text{Li}_2(-z) + (-48 + 24z) \ln z \text{Li}_2(1 - z) + (32 - 16z)

\times \ln(1 - z) \text{Li}_2(1 - z) + \left(\frac{32}{z} - 64 + 32z\right) \ln z \text{Li}_2(-z) + (8 - 4z)

\times \ln^2 z \ln(1 - z) - \left(\frac{16}{z} + 32 + 16z\right) \ln^2 z \ln(1 + z) + (40 - 20z)

\times \ln z \ln^2(1 - z) + \left(\frac{32}{z} + 64 + 32z\right) \ln z \ln^2(1 + z) + \left(- \frac{44}{3}\right)

+ \frac{22}{3} \ln^3 z + \left(\frac{40}{3} - \frac{20}{3} z\right) \ln^3(1 - z) + (16 - 8z)\zeta(2) \ln z

+ \left(- \frac{32}{z} + 48 - 24z\right)\zeta(2) \ln(1 - z) + \left(\frac{32}{z} + 64 + 32z\right)\zeta(2)

\times \ln(1 + z) + \left(- \frac{32}{z} + 80 - 104z\right)\zeta(3) + (-32 + 20z) \text{Li}_2(1 - z)

+ (-64 + 52z) \ln z \ln(1 - z) - \left(\frac{208}{3z} + 64 + \frac{64}{3} z^2\right) \text{Li}_2(-z)

+ \ln z \ln(1 + z) + \left(128 - 43z + \frac{32}{3} z^2\right) \ln^2 z + (-72 + 54z)

\times \ln^2(1 - z) + \left(32 - 60z - \frac{64}{3} z^2\right)\zeta(2) + \left(\frac{94}{3} + \frac{196}{3} z\right) \ln z
\end{align}
\[+ \left( \frac{16}{z} + 88 - 108z \right) \ln(1 - z) + \frac{98}{3} - \frac{110}{3} z \}
\]
\[+ C_A C_F \left\{ \left( - (32 + 8z) \ln z + (16 - 8z) \ln(1 - z) - 48(1 - z) \right) \right\} \]
\[\times L_M^2 + \left( - 96 \text{Li}_2(1 - z) - 32(1 + z) \ln z \ln(1 - z) + (32 + 16z) \right) \]
\[\times \left( \text{Li}_2(-z) + \ln z \ln(1 + z) \right) - (112 + 24z) \ln^2 z + (16 - 8z) \]
\[\times \ln^2(1 - z) + (96 - 32z)\zeta(2) + (-32 + 136z) \ln z + (-128 + 112) \ln(1 - z) + 232 - 224z \right) L_M + (32 + 16z) \left( \text{Li}_3 \left( \frac{1 - z}{1 + z} \right) \right) \]
\[- \text{Li}_3 \left( \frac{1 - z}{1 + z} \right) \right) + \left( 80 + 40z \right) \text{Li}_3(1 - z) - \left( \frac{16}{z} + 320 \right) \]
\[\times S_{1,2}(1 - z) + \left( \frac{48}{z} + 32 + 16z \right) \text{Li}_3(-z) - \left( \frac{32}{z} + 64 + 32z \right) \]
\[\times S_{1,2}(-z) - (288 + 16z) \ln z \text{Li}_2(1 - z) - (80 + 8z) \ln(1 - z) \]
\[\times \text{Li}_2(1 - z) + (32 + 16z) \ln(1 - z) \text{Li}_2(-z) + \left( \frac{16}{z} + 32 + 16z \right) \]
\[\times \ln z \text{Li}_2(-z) - \left( \frac{32}{z} + 64 + 32z \right) \ln(1 + z) \text{Li}_2(-z) - \left( \frac{16}{z} + 32 + 16z \right) \ln z \ln^2(1 + z) - (96 + 32z) \ln^2 z \ln(1 - z) - \left( \frac{248}{3} \right) \]
\[+ \frac{44}{3} z \right) \ln^3 z + (32 + 16z) \ln z \ln(1 - z) \ln(1 + z) - (24 + 12z) \]
\[\times \ln z \ln^2(1 - z) + \left( \frac{8}{z} + 48 + 24z \right) \ln^2 z \ln(1 + z) + \left( \frac{8}{3} - \frac{4}{3} z \right) \]
\[\times \ln^3(1 - z) + (256 - 48z)\zeta(2) \ln z + \left( \frac{16}{z} + 16z \right) \zeta(2) \ln(1 - z) \]
\[\]
\[-\left(\frac{16}{z} + 32 + 16z\right)\zeta(2) \ln(1 + z) + \left(\frac{16}{z} + 24 + 20z\right)\zeta(3)\]

\[+(64 + 56z)\text{Li}_2(1 - z) + \left(72 + 38z - \frac{16}{3}z^2\right)\ln^2 z +\]

\[(-56 + 48z) \ln^2(1 - z) + \left(\frac{176}{3z} + 64 + 32z + \frac{32}{3}z^2\right)\left(\text{Li}_2(-z)\right)\]

\[+ \ln z \ln(1 + z)\] \[+ (-48 + 136z) \ln z \ln(1 - z) + \left(48 - 16z\right)\]

\[+ \frac{32}{3}z^2\zeta(2) + \left(\frac{844}{3} - \frac{128}{3}z\right)\ln z + \left(- \frac{8}{z} + 276 - 240z\right)\]

\[\times \ln(1 - z) - \frac{124}{3} + \frac{76}{3}z\}\right]\] \hfill (A.3)

The renormalization constant $Z_{\text{NS}}^{qq}$ in Eq. (3.46) needed to restore the Ward-identities, which are broken by the HVBM prescription of the $\gamma_5$-matrix, is equal to

\[Z_{\text{NS}}^{qq}(z) = \delta(1 - z) + \frac{\alpha_s}{4\pi} C_F \left[-8(1 - z) + \varepsilon\{ -8(1 - z) \ln z\right.\]

\[-4(1 - z) \ln(1 - z) + 10 - 8z)\}\right]\]

\[+ \left(\frac{\alpha_s}{4\pi}\right)^2 \left[C_F^2 \left\{ -16(1 - z) - (8 + 16z) \ln z + 16(1 - z) \ln^2 z\right.\right.\]

\[-16(1 - z) \ln z \ln(1 - z)\]

\[+ C_A C_F \left\{ - \frac{188}{3}(1 - z) - \frac{592}{9}(1 - z) + 8(1 - z) \zeta(2)\right.\]

\[+ \left(-\frac{80}{3} + \frac{8}{3}z\right) \ln z - 4(1 - z) \ln^2 z\\}

\[+ n_f C_F T_f \left\{ \frac{132}{3}(1 - z) + \frac{16}{3}(1 - z) \ln z + \frac{80}{9}(1 - z)\right\}\]

\[-(C_F^2 - \frac{1}{2} C_A C_F) \left\{ 8(1 + z) \right.\left(4 \text{Li}_2(-z) + 4 \ln z \ln(1 + z) + 2\zeta(2)\right)\]
- \ln^2 z - 3 \ln z \right) - 56(1 - z) \right) \right]. \quad (A.4)

Where the last part originates from the process in Eq. (3.19) with two identical particles in the inclusive final state. In the expressions above the definitions of the Riemann zeta-functions \( \zeta(n) \) and the polylogarithms \( \text{Li}_n(z) \), \( S_{n,m}(z) \) can be found in [29].
References

[1] P. Abreu et al. (DELPHI), Phys. Lett. B311 (1993) 408.
[2] D. Buskulic et al. (ALEPH), Phys. Lett. B357 (1995) 487.
[3] R. Akers et al. (OPAL), Z. Phys. C68 (1995) 203.
[4] E.B. Zijlstra and W.L. van Neerven, Nucl. Phys. B383 (1992) 525.
[5] E.B. Zijlstra and W.L. van Neerven, Phys. Lett. B297 (1994) 377.
[6] E.B. Zijlstra and W.L. van Neerven, Nucl. Phys. B417 (1994) 61; Erratum: Nucl. Phys. B426 (1994) 245.
[7] P.J. Rijken and W.L. van Neerven, Phys. Lett. B386 (1996) 422.
[8] P.J. Rijken and W.L. van Neerven, Nucl. Phys. B487 (1997) 233.
[9] P.J. Rijken and W.L. van Neerven, Phys. Lett. B392 (1997) 207.
[10] P.J. Rijken, PhD thesis, Leiden University 1997.
[11] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345.
[12] J. Ashman et al. (EMC), Phys. Lett. B206 (1988) 364; Nucl. Phys. B328 (1989) 1.
    V. W. Hughes et al., Phys. Lett. B212 (1988) 511.
[13] J. Ellis and R. Jaffe, Phys. Rev. D9 (1974) 1444; Erratum: Phys. Rev. B10 (1974) 1669.
[14] D. Buskulic et al. (ALEPH), Phys. Lett. B374 (1996) 319;
    K. Ackerstaff et al. (OPAL), CERN-PPE/97-104, hep-ex/9708027.
[15] M. Burkardt and R.L. Jaffe, Phys. Rev. Lett. 70 (1993) 2537.
[16] V. Ravindran, Phys. Lett. B398 (1997) 169.
[17] V. Ravindran, Nucl. Phys. B490 (1997) 272.
[18] D. de Florian and R. Sassot, Nucl. Phys. B488 (1997) 367.
[19] R. Ramachandran and R.P. Bajpai, Phys. Lett. B115 (1982) 313;
    H.S. Mani and R. Ramachandran, Phys. Rev. D28 (1983) 1774.

[20] M. Stratmann and W. Vogelsang, Nucl. Phys. B496 (1997) 41.

[21] G. ’t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.

[22] F. Breitenlohner and D. Maison, Commun. Math. Phys. 52 (1977) 11,
    39, 55.

[23] P. Nason and B.R. Webber, Nucl. Phys. B421 (1994) 473.

[24] J. Binnewies, DESY-preprint 97-128 (hep-ph/9707269).

[25] S.A. Larin, Phys. Lett. B303 (1993) 113.

[26] D. Akyeampong and R. Delbourgo, Nuovo Cimento 17A (1973) 578;
    18A (1973) 94; 19A (1974) 219.

[27] S.L. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517.

[28] G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B175 (1980) 27;
    W. Furmanski and R. Petronzio, Phys. Lett. B97 (1980) 437; Z. Phys.
    C11 (1982) 293.

[29] L. Lewin, ”Polylogarithms and Associated Functions”, North Holland,
    Amsterdam, 1983;
    R. Barbieri, J.A. Mignaco and E. Remiddi, Nuovo Cimento 11A (1972)
    824;
    A. Devoto and D.W. Duke, Riv. Nuovo. Cimento Vol. 7, N. 6 (1984) 1.