Field Theory reformulated without Self-energy Parts.
The dressing Operator

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Abstract

The reformulation of field theory for avoiding self-energy parts in the dynamical evolution has been applied successfully in the framework of the Lee model,[1] enabling a kinetic extension of the description. The basic ingredient is the recognition of these self-energy parts.[2] The original reversible description is embedded in the new one and appears now as a restricted class of initial conditions.[3] This program is realized here in the reduced formalism for a scalar field, interacting with a two-level atom, beyond the usual rotating wave approximation. The kinetic evolution operator, previously surmised,[4] is here derived from first principles, justifying the usual practice in optics where the common use of the so-called pole approximation[5] should no longer be viewed as an approximation but as an alternative description in the appropriate formalism. That model illustrates how some dressing of the atomic levels (and vertices), through an appropriate operator, finds its place naturally into the new formalism since the bare and dressed ground states do no longer coincide. Moreover, finite velocity for field propagation is now possible in all cases, without the presence of precursors for multiple detections.

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1 Introduction

Quantum optics is not an *ab initio* theory and requires a quantum modelization of the interacting atoms and fields. The natural point of view starts with an atom described by its energy levels and a dipolar interaction with the field, in a time reversible formalism, using hermitian Hamiltonians. Moreover, a welcome simplifying approximation (the so called rotating wave approximation) provides often an excellent approximation of the dynamics involved. Nevertheless, some problems subsists: they involve the description of the instability of the excited levels and the acausal behaviour in energy transfer between atoms or precursors in the case of a double photodetection. The first problem is tackled by the use of irreversible elements inside a reversible quantum mechanical description. Indeed, for instance, the explicit attribution inside the Hamiltonian of a lifetime to an atomic excited level is the natural way to take into account its unstability. The introduction of elements of a phenomenological origin into a microscopic description has then led to tremendous success in that field. The problems linked with the non-hermiticity are avoided by skilled use of the formalism. That practice has so far found no fundamental justification.

In a previous paper, we have analysed in details a renowned paper and we have shown that in fact, Mollow’s approximations were equivalent to the use of kinetic equations to provide the description of the system. Implicitly, such kinetic equations are naturally at the level of reduced distribution functions for the field. Indeed, when an arbitrary number of photons can be emitted from any initial state (except of course the true ground state), the most adequate description is a reduced formalism, applied currently for describing atoms in a fluid where quantal reduced distribution functions obey the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy in place of the Liouville-von Neumann equation.

The use of those (kinetic-like) equations in ordinary quantum mechanics can be criticized on two bases. The first one is that they do not belong to the (reversible) framework of ordinary quantum mechanics and their theoretical justification is still missing (despite their extraordinary success). The second one is that they are always presented as the result of a largely justified approximation and not as an intrinsic property of the system. It seems unsatisfactory that a radical change of the formalism (a transition from a time reversal invariant to an irreversible one) results simply from approximations.

Physicists associate the concept of an unstable state to an object that, outside external influence, would decay in a purely exponential way. Such
an unstable state is beyond the reach of ordinary quantum mechanics and the
tentatives to define it make trouble for normalization properties for
instance.[10] The previous attempts for obtaining a purely exponential de-
cay[11] were unsatisfactory since they did not incorporate the possibility
of an excitation mechanism. The main aim of this paper is to show that
an association between intuitive concepts and quantal description is possi-
ble, through the single subdynamics approach, for the interaction of a field
with a two-level atom, including the consideration of the counter-rotating
terms, keeping in mind the generalization to multilevel systems. Acas-
ual behaviours can moreover be excluded by an appropriate choice of a dress-
ing operator. Under compatibility conditions to be satisfied at initial times,
it is possible to show that the time reversal invariant and kinetic descrip-
tions bring simply different aspects to light while keeping their equivalence
(A spectral representation shares also that property. The derivation of a
spectral-like representation for the Lee model through our single subdy-
namics approach can be found in Ref. [12]).

A theory of subdynamics has been introduced thirty years ago by the
Brussels group (see e.g. [13], [9]) for a dynamics provided by the Liouville-
von Neuman equation. In that quest for the introduction of irreversibility
inside the formalization of dynamics, the subdynamics concept has been
shown to be fruitful. Different realizations are possible according to a choice
of the vacuum, i. e. the choice of the degrees of freedom that are included in
the resulting dynamics, while the other elements of the description become
functional of the vacuum ones. A setback of that approach is thus a limita-
tion on the class of possible initial conditions since they have to belong to
the subdynamics. Therefore, the initial formulation should be in some sense
overcomplete and contain degrees on freedom on which no control is possible.
A way out has been the introduction of a transformation theory, intensively
studied.[14] In the original ambition, an association of a so-called physical
representation with real (energy-conserving) processes between renormalized
(dressed) quantities. It could not be carried out in a general and consist-
tent fashion because of difficulties that have been reviewed.[13] Moreover,
the problematics of ensuring positivity of the density matrix is lacking in
late papers [14] where the positive character of the density matrices is no
more listed among the requirements on the Λ transformation.

In order to keep the completeness of the description, a new vacuum
concept has been introduced[2] for models in field theory, irrespectively of
their classical[17] or quantum character. It is based on a dynamical analy-
ysis of all possible contributions to the formal solution of the Liouville-von
Neumann equations. To avoid the previous trap, the so-called single subdynamics approach [2] is based on the existence of self-energy contributions to the dynamics. We can indeed accept that no control is possible on these processes. By definition, all self-energy parts have to be excluded from the vacuum. Their recognition implies that the initial dynamics is first extended by discriminating among the degrees of freedom according to their status with respect to preparation and observation. A subdynamics of the (extended) dynamics is then introduced such that it encompasses the original dynamics but does no longer contain dressing processes. In that way, we obtain a reformulation of field theory that excludes self-energy contributions in the dynamics. They are now driven by the other degrees of freedom and provided by time independent functionals of the other degrees of freedom that are the motor of the evolution. Therefore, the same mathematical tool (subdynamics) as the Brussels-Austin group is used, but with a different realization, leading to different physical content, although a similar aim is pursued.

In a previous paper in collaboration with C. George[3] we have been dealing with the Friedrichs model, equivalent to a specific sector of a two-level atom interacting with a scalar field within the rotating wave approximation (RWA). The Friedrichs-Lee model has been treated in two different approaches. In the first one[3] the existence of sectors has been used to perform the explicit construction of all the elements of the subdynamics super-operator inside the first non trivial sector. We have shown that a kinetic description exists that provides an exact and complete alternative to the (time reversal invariant) Schrödinger description. It is obtained by a double operation: An enlargement of dynamics, that enables the recognition of self-energy parts, followed by the use of an appropriate subdynamics projector. The resolution of the model enables the explicit verification of all the claimed properties. That proof is welcome since the very existence of the concept of subdynamics has been questioned for instance by P. Coveney and O. Penrose[18]. Their argument is the incompatibility between branch points (generating “long time tails”) and the kinetic description, preventing the subdynamics to provide the asymptotic behaviour. A general analysis of the situation, leading to their refutation, can be found in a previous paper[15]. In the second approach[1] the reduced formalism has been used to go beyond the limitation induced by the equivalence with the original description. The focus has been on the subdynamics evolution super-operator and we have shown the possibility of extending quantum theory in a satisfactory way. Positivity and normalization are automatically ensured in the new
kinetic description, starting from the reduced formalism. Such a construction rests entirely on the existence of poles, independent of branch points, for diagonal matrix elements of the Green’s function associated with the Hamiltonian. For this model, a clear-cut separation of poles and cuts does exist, leading to an intuitive description and justifying, through a derivation, a phenomenological approach.\cite{4}

To show the robustness of that intuitive description, we treat here a more general system, consisting of a two-level atom in interaction with a scalar field, keeping the counter-rotating terms. The system under study is interesting from a physical point of view since the existence of long time tails (non exponential contributions) and deviations from exponential behaviours for very short times has long been recognized in it. On the other hand, physicists analysing experiments in optics are accustomed to use the so-called poles approximations. How can those empirical rules be justified from first principles? Are they valid only in some approximate way or do they fully reflect the physical reality.

The associated subdynamics has to be constructed for the new system, outside the RWA. Such an extension involves formal modifications in the treatment, but the philosophy is the same. First of all, an analysis of the property of the Liouvillian using sectors \cite{3} is no longer valid: A “reduced formalism” is therefore naturally required. All sectors are now coupled and have to be considered together. As a consequence, the possibility of a complete resolution of the model is lost with respect to the RWA case.

From the expression of vacuum to vacuum elements of the resolvent of the generalized Liouvillian,\cite{19} we have investigated whether a pole can be associated with all the matrix elements that do not involve neither an incoming nor an outgoing field particle, the generalization of the property for other kinds of matrix elements being straightforward. We have proven \cite{19} that the notion of poles associated with the stable and unstable states is still relevant for the model under consideration treated inside the reduced formalism.

The construction of the evolution and projection super-operator for a subdynamics rests on an analysis of the kind of behaviour for the contribution of each term in a perturbation expansion of the resolvent (associated with the generator of motion) and the selection of the relevant behaviour.

When compatibility (or equivalence) conditions are satisfied,\cite{3} the kinetic description provides somehow a change of representation, analog to a change of basis in standard quantum mechanics. Therefore, it will always be possible to transfer the information on the system from the usual
density matrix operator (or its analog for the reduced description) to the kinetic description and vice versa. Moreover, the intuitive way of looking at the system is recovered in the kinetic description in terms of incident field, outgoing field, dressed atomic levels (including their attribute of unstability represented by their lifetime, see Ref. [6] for instance). Therefore, the common practice in optics will be justified from first principle and will no longer be the result of approximations. Let us underline the analogy of the present approach with the theory of renormalization. Indeed, in both theories, the aim is to take properly into account the self-energy contributions. The renormalization theory tackles the problem of removing ultraviolet divergences at the level of the wave function and to derive finite corrections.

The description of an unstable state requires the introduction of an imaginary part in the renormalized energy: The original framework cannot be respected. In the present theory, in terms of a reduced density operator, no such problem arises.

In this effective realization of our approach on a less simple non trivial example, we will not bother about the formal properties of the subdynamics super-operator that have been well established [9] and do not depend on the particular realization of the choice of the vacuum. We will not worry either of the compatibility conditions: we know that they do exist [3] and reflect the possible equivalence between the original and the kinetic descriptions. We focus on the derivation and the properties of the kinetic evolution super-operator, as defined by the single subdynamics approach [2]. The form of the obtained evolution generator leads us naturally to the introduction of dressing technique introduced long ago into the Brussels approach [14, 11]. A dressing operator enables to shape the form of the evolution generator corresponding to an intuitive description, out of the reach of the original time reversal invariant description, that accepts only unitary transformations.

In Ref. [4], a reduced formalism has been proposed to treat the interaction of a two-level atom with the electromagnetic field. Various kinetic equations have been accordingly surmised and justified on physical grounds. However, their derivation from first principles was outside the scope of that paper and we intend here to fulfill that missing part: In this way, we pursue the progressive introduction of the characteristics of the method to the effective description of the interaction between the electromagnetic fields and

\footnote{It has been recently noticed [17] that the application of the single subdynamics approach to a well adapted formalism of classical electrodynamics [2] leads directly to a divergence-free description, without any need to a substraction procedure.}
atoms.

In §2, the reduced formalism for the system is briefly recalled and the extension of the dynamics (distinction between the various photons) is treated. The formal properties of the subdynamics super-operator are briefly cited.

§3 is devoted to the elements of the subdynamics super-operator that enable the obtention of the kinetic operator for the elements describing the population of the atomic levels. They do not involve physical photons. The elements for the atomic dipolar moments are considered in §4.

The main difference with respect to RWA is already apparent in the elements of the evolution generator $\tilde{\theta}$ that do not involve photons (the purely atomic part of the evolution operator). Inside the RWA, the bare and dressed ground state coincide and the dressed excited state is then derived directly by the construction of the kinetic operator. Outside the RWA, the structure of the kinetic operator is no longer the same: both states that appear in it are susceptible of evolution. Indeed, already in usual quantum mechanics, we know that a change of basis is required with respect to the bare states outside RWA. Therefore, the stable ground state does not coincide with the state described by the kinetic operator. On physical grounds, it is required the true ground state being time independent and the excited state decaying. Since we are dealing in a reduced formalism, we have to translate such properties into similar ones for the matrix elements of the reduced density operator, in order to define a physical representation.

Through a dressing operator, usual in the subdynamics approach, a procedure enables to fix the problem in §5 in a completely satisfactory way. After dressing, the structures of the kinetic operators inside and outside the RWA are common. Let us note that the atomic model under consideration is the first example treated for which a dressing operator is required. Our dressing, in the single subdynamics approach formulated in reduced formalism, is not equivalent of defining new states in an Hilbert space formulation, such as in [20].

For completeness, the (somehow trivial) effect of passive photons is explicitly treated in §6. The one physical photon absorption process is also considered in the same §6 and leads, as in the case of the Lee model, to the introduction of dressed interactions between the atom and the field. The one physical photon emission process does not present unexpected new features.

The vertex dressing is considered in §7. We discuss the change in the interaction that can be induced by the dressing operator: Strict causality in
the exchange of photons between atoms can be ensured in all cases, without the usual presence of precursors due to a finite lower bound in the energy spectrum. The use of a kinetic description, as opposed to a reversible one, is a main ingredient to allow that property, beyond the reach of an hermitian generator of motion.

Concluding remarks are presented in the last part of this paper.

2 The model - Reduction - Indiscernability - Extended Dynamics - Subdynamics

The model considered is a two-level atomic system interacting with a field, without considering the rotating wave approximation nor explicitly the momentum change due the recoil of the atom. The Hamiltonian of the system can be written as

\[
H = \omega_1 a_1^+ a_1 + \omega_0 a_0^+ a_0 + \sum_k \omega_k a_k^+ a_k
+ \sum_k \left( V_{1|0k} a_1^+ a_0 (a_k + a_k^+) + V_{0|1k} a_0^+ a_1 (a_k + a_k^+) \right). \tag{2.1}
\]

In another paper, we have established the analytical properties of some Green’s functions associated with the Hamiltonian (2.1). Since the Hamiltonian enables an arbitrary number of field particles being present in the future of any state, due to the counter-rotating terms, we are in a situation similar to that met in statistical mechanics and a reduced formalism is required for the description of the degrees of freedom associated with the field. In the computation of the mean value of observables, such a formalism replaces the trace operation by a vacuum expectation value for the field. It has been developed in extenso in Ref. 4 and recalled in Ref. 1 and we will be satisfied with a reminder of the main features without entering into details.

The many body system of interest is described by a density operator \( \rho \) that obeys the Liouville-von Neuman equation

\[
i\partial_t \rho = L\rho, \tag{2.2}
\]

where the Liouvillian \( L \) is the commutator with the Hamiltonian \( H \) given by (2.1). In terms of factorizable superoperators \((A \times B)\), defined as \((A \times B)\rho \equiv A\rho B\), it is given by

\[
L = H \times I - I \times H, \tag{2.3}
\]
where \( I \) is the identity operator. A (factorizable) superoperator \((A \times B)\) will be called a connecting superoperator if both \( A \) and \( B \) are different from the identity operator. A superoperator such as in \( L \) that acts with an operator on one side of the density operator and with the identity operator on the other side will be called a non-connecting superoperator. It does not prevent from writing the formal solution of equation (2.1) for \( \rho \) under a factorized form \((\exp -iHt)\rho \exp iHt\).

The associated reduced density operator \( \bar{\rho} \) obeys an equation \(^2\)

\[
i\partial_t \bar{\rho} = \bar{L} \bar{\rho}.
\] (2.4)

The unperturbed part \( \bar{L}_0 \) of \( \bar{L} \) is the same as the unperturbed part \( L_0 \) of \( L \). The potential dependent part \( \bar{L}_V \) of \( \bar{L} \) contains, in addition to the potential dependent part \( L_V \), new connecting contributions \( \bar{L}_V' \) given by

\[
\bar{L}_V' = \sum_k \left( V_{1\theta_k} a_1^+ a_0 + V_{0\theta_k} a_0^+ a_1 \right) \times a_k^+ - \sum_k a_k \times \left( V_{1\theta_k} a_1^+ a_0 + V_{0\theta_k} a_0^+ a_1 \right).
\] (2.5)

If we note by a Roman letter the states of the Hilbert space (that letter defines the state of the atom and the wave numbers associated with the photons present) and if we use the notation \( <a|\rho|b> = \rho_{ab} \), the evolution equations take the form:

\[
i\partial_t \rho_{ab}(t) = \sum_{c,d} L_{ab,cd} \rho_{cd}(t), \quad i\partial_t \bar{\rho}_{ab}(t) = \sum_{c,d} \bar{L}_{ab,cd} \bar{\rho}_{cd}(t)
\] (2.6)

with an obvious definition for the matrix elements of the evolution super-operators \( L \) and \( \bar{L} \).

A formal solution of these equations is provided with the help of inverse Laplace transform as

\[
\rho_{ab}(t) = \sum_{c,d} \frac{-1}{2\pi i} \int_\gamma dz \ e^{-izt} \left( \frac{1}{z - \bar{L}} \right)_{ab,cd} \rho_{cd}(0),
\] (2.7)

\[
\bar{\rho}_{ab}(t) = \sum_{c,d} \frac{-1}{2\pi i} \int_\gamma dz \ e^{-izt} \left( \frac{1}{z - \bar{L}} \right)_{ab,cd} \bar{\rho}_{cd}(0),
\] (2.8)

\(^2\)We keep the notations of Ref. [1] where the operators inside the reduction procedure bears a bar accent \([1]\) while the symbol tilde is introduced to refer to the extended dynamics. With respect to Ref. [1], the reduced density operator is noted \( \bar{\rho} \) in place of \( \sigma \).
where the path $\gamma$ lies above the real axis. These forms enable immediately perturbation expansions in terms of the potential $V$. In the first part of this study, we limit ourselves to the case of the couples $(ab)$ and $(cd)$ referring to diagonal discrete states without field particles. The resolvent of $L$ that plays a role in (2.7) can be written as a convolution product of appropriate resolvents of $H$.

Analytical properties of the resolvent of $L$ and $\bar{L}$ in (2.7) have been studied extensively in Ref. [19] (in particular the elements $((z - L)^{-1})_{11,11}$, $((z - L)^{-1})_{00,00}$, $((z - L)^{-1})_{11,11}$ and $((z - L)^{-1})_{00,00}$). The main technical point involved was the ability of distinguishing the effects of the branch points and of the poles required for the construction of the single subdynamics. In the reduced formalism, the analytic structure of the resolvent of the evolution super-operator can no longer be examined a priori via a convolution involving the Green’s functions associated with the Hamiltonian operator and this has far reaching consequences on the level of the analytic properties.

We recall some of the properties that have been established.

The Green’s functions associated with the Hamiltonian (2.1) are noted as in the RWA case

$$\frac{1}{\eta_1(z)} = \left( \frac{1}{z - H} \right)_{11}, \quad \frac{1}{\eta_0(z)} = \left( \frac{1}{z - H} \right)_{00}. \quad (2.9)$$

Inside the RWA, the $\eta_0(z)$ function reduces to $(z - \omega_0)$. “Bar” functions $\bar{\eta}$ represent Green’s functions associated with the operator $(-H)$

$$\frac{1}{\bar{\eta}_1(z)} = \left( \frac{1}{z + H} \right)_{11}, \quad \frac{1}{\bar{\eta}_0(z)} = \left( \frac{1}{z + H} \right)_{00}, \quad (2.10)$$

so that the relation (2.7), for $a = b = c = d = 1$ or $a = b = c = d = 0$ involves the resolvent $R_{11,11}(z)$ and $R_{00,00}(z)$ given by

$$R_{11,11}(z) = \frac{-1}{2\pi i} \int_\gamma du \frac{1}{\eta_1(u)} \frac{1}{\bar{\eta}_1(z - u)}, \quad R_{00,00}(z) = \frac{-1}{2\pi i} \int_\gamma du \frac{1}{\eta_0(u)} \frac{1}{\bar{\eta}_0(z - u)}. \quad (2.11)$$

The $\eta(z)$, $\bar{\eta}(z)$ functions are required in (2.11) for a positive imaginary part of their argument ($0 < \Im \gamma < \Im z$). Therefore, those functions will be analytically contined from above into the lower half plane $\Im z < 0$. When branch points are met, the cuts will be placed parallelly to the imaginary axis. All
functions of complex arguments that we shall introduce share that property and we shall dispense them from a upper index “+” that would indicate the way the analytical continuations are performed. As a consequence of that convention, for instance, $\bar{\eta}_1(z)$ for $\Im z < 0$ cannot be computed directly from (2.10) but requires the computation of $((z + H)^{-1})_{11}$ for $\Im z > 0$ and then an analytical continuation. It is usual to note that fact by writing $\bar{\eta}_1^+(z)$ but to avoid too cumbersome notations, we use an implicit convention to keep the notation $\bar{\eta}_1(z)$. The same convention holds for other functions to be introduced soon.

The $\eta^{-1}$ functions are analytic in the upper halfplane $\Im z > 0$ and we first analyse their singularities in the lower halfplane $\Im z < 0$. The properties of the $\bar{\eta}$ functions are a translation of those of the $\eta$ functions. From the convolution product (2.11), the singularities of the resolvent into the lower halfplane $\Im z < 0$ are then inferred.

From a perturbation expansion with respect to the interaction $V$, the $\eta_{1}^{-1}$ function (2.9) can be written in general as $\eta_{1}^{-1} = \sum_{n=0}^{\infty}((z - H_0)^{-1})_{11} (f_1(z)((z - H_0)^{-1}))_{11}^n$ where $f_1(z)$ represents the sum of all the contributions leading from state“1” to itself with a condition of irreducibility with respect to “1” (all intermediate states implied in the sum have to be $\neq$ of “1”) :

$$f_1(z) = \sum_{m=0}^{\infty} ((V((z - H_0)^{-1}))_{11}^{m_{irr1}} V_{11}).$$

The index $irr1$ recalls the restriction on the intermediate states. The diagonal matrix elements of $(z - H_0)^{-1}$ are called propagators.

A visual representation of the contributions can be obtained using a diagrammatic representation, such as the one developed in appendix B of Ref. [8] for the same system. It corresponds in the present case to the Feynman diagrams. We have then trivially

$$\frac{1}{\eta_1(z)} = \frac{1}{z - \omega_1 - f_1(z)}, \quad \frac{1}{\eta_0(z)} = \frac{1}{z - \omega_0 - f_0(z)}.$$ (2.13)

The summations present in the expression of the functions $f$ have to be understood in the infinite volume limit where, for instance, $\sum_k |V_{1|0k}|^2 \ldots \rightarrow \int_0^{\infty} d\omega v^2(\omega) \ldots$. We assume that the function $v^2(\omega)$ vanishes as $\omega$ for small values of its argument (This may be understood as it is composed of a factor $k^2$ arising from the jacobian to obtain spherical coordinates and the square of a usual factor $\frac{1}{\sqrt{\omega_k}}$ arising from the matrix element of the potential.).
The thesis established in Ref. [19] is the following: Under this behaviour of the potential matrix elements $V_{1|0k}$, $V_{0|1k}$, $V_{0|10}$, we have the consistency of the following statements for the singularities of the functions $\eta$ and $f$.

1) The function $(\eta_1(z))^{-1}$ presents a pole at some point $z = \omega_1 + \zeta$ and a well defined residue $A_1$ at that point.

2) The function $(\eta_0(z))^{-1}$ presents a pole at some point $z = \omega_0 + \delta$ and a well defined residue $A_0$ at that point.

3) The functions $f_1(z)$ and $f_0(z)$ present logarithmic singularities at the points $z = \omega_1 + \zeta$ and $z = \omega_0 + \delta$.

4) The function $f_1(z)$ behaves as $(z - \omega_1 - \zeta)^3 \ln(z - \omega_1 - \zeta)$ in the vicinity of its singular at point $z = \omega_1 + \zeta$ and as $(z - \omega_0 - \delta) \ln(z - \omega_0 - \delta)$ in the vicinity of its singular point $z = \omega_0 + \delta$.

5) The function $f_0(z)$ behaves as $(z - \omega_0 - \delta)^3 \ln(z - \omega_0 - \delta)$ in the vicinity of its singular point $z = \omega_0 + \delta$ and as $(z - \omega_1 - \zeta) \ln(z - \omega_1 - \zeta)$ in the vicinity of its singular point $z = \omega_1 + \zeta$.

$\zeta$ has an negative imaginary part while $\delta$ is real. Inside the RWA [3], the pole of $(\eta_1(z))^{-1}$ has been noted $\theta_1$. Here, we bring to the fore the displacement with respect to the unperturbed value $\omega_1$. For the “bar” functions, we have the similar properties: the poles of $(\bar{\eta}_1(z) - 1)$ and $(\bar{\eta}_0(z))^{-1}$ are resp. at $z = -\omega_1 + \bar{\zeta}$ and $z = -\omega_0 + \bar{\delta}$, with $\delta = -\delta$, $i\zeta = (i\zeta)^*$.

The explicit demonstration has required a self-consistent analysis of the various contributions. For the system we consider here (outside the RWA), the previous proof of the existence of the poles inside the RWA [3] has been adapted: the structure of the Hamiltonian Green’s functions is no longer as simple and the diagonal elements of the Green’s functions present simultaneously a pole and a branch point for a same value in the complex plane $z$, while poles and branchpoints do not coincide inside the RWA.

An analysis of that new intertwining has enabled to desantangle it in order to be able to make a statement about the existence of poles independently of a choice of a particular Riemann sheet. The problematics to which an answer has been given is thus the existence of a “pole approximation” [3] beyond the RWA.

From the convolution form, (2.11) it is then established that $R_{11,11}(z)$ and $R_{00,00}(z)$ have respectively a pole at $z = \zeta + \bar{\zeta}$ and $z = 0$. In a similar way, it can be easily established that $R_{10,10}(z)$ and $R_{01,01}(z)$ have simple poles respectively for $z = \omega_1 + \zeta - \omega_0 + \delta$ and $z = \omega_0 + \delta - \omega_1 + \bar{\zeta}$.

That analysis could not be reproduced mutatis mutandis. Indeed, the new generator of motion in the reduced formalism is no longer likewise sim-
ply connected with the Hamiltonian since the notion of reduction leads outside the Hamiltonian formalism. Therefore, the existence of “connecting vertices” prevents a direct analysis in terms of a convolution involving matrix elements of the Green’s function associated with the Hamiltonian. Nevertheless, a further analysis has shown that both matrix elements $\bar{R}_{11,11}(z)$ and $\bar{R}_{00,00}(z)$ of the Green’s function associated with the reduced Liouvillian $\bar{L}$ have poles $z = \zeta + \bar{\zeta}$ and $z = 0$, with well defined residues. That result is not unexpected on physical ground: the natural time behaviour of the atom should not be modified by the reduction procedure. The existence of those poles is a necessary requisite for the construction of the subdynamics.

The indiscernability of the field and its consequences on the computations has already been treated in Ref. and it has been noted that they are minimal. We will not repeat the same considerations. Indeed, in Ref., care has been taken on that question.

The evolution for field and atom reduced distribution functions is governed by an evolution operator $\bar{L}$. Following the approach initiated with the potential scattering and the atom in interaction with the field inside RWA, an extended dynamics is now introduced, that rests on a distinction between all degrees of freedom, once the self-energy parts have been recognized. We use the same notations as in Ref., a Roman letter $l$ to represent a photon involved in the self-energy contribution, a Greek letter $\mu$ for a photon leaving the atom (emitted photon) and a Greek letter $\lambda$ for an incident photon. The number of reduced distribution functions to be considered is multiplied accordingly and the evolution is then governed by an evolution operator $\bar{L}$:

\[
\begin{align*}
\dot{\bar{L}} &= \omega_1 a_1^+ a_1 \times I - I \times \omega_1 a_1^+ a_1 + \omega_0 a_0^+ a_0 \times I - I \times \omega_0 a_0^+ a_0 \\
&+ \sum_l \omega_l a_l^+ a_l \times I - I \times \sum_l \omega_l a_l^+ a_l \\
&+ \sum_\lambda \omega_\lambda a_\lambda^+ a_\lambda \times I - I \times \sum_\lambda \omega_\lambda a_\lambda^+ a_\lambda + \sum_\mu \omega_\mu a_\mu^+ a_\mu \times I - I \times \sum_\mu \omega_\mu a_\mu^+ a_\mu \\
&+ \sum_l \left( V_{1|0} a_1^+ a_0 (a_l + a_l^+) + V_{0|1} a_0^+ a_1 (a_l + a_l^+) \right) \times I \\
&- I \times \sum_l \left( V_{1|0} a_1^+ a_0 (a_l + a_l^+) + V_{0|1} a_0^+ a_1 (a_l + a_l^+) \right) \\
&+ \sum_\lambda \left( V_{1|0\lambda} a_1^+ a_0 a_\lambda + V_{0\lambda|1} a_0^+ a_1 a_\lambda \right) \times I
\end{align*}
\]
The system is now described by a set of new reduced distribution functions \( \tilde{\rho} \) related to the extended dynamics. The constitutive relations connect the original set \( \bar{\rho} \) to the new one \( \tilde{\rho} \). Indeed, the new description contains obviously more degrees of freedom than the original one. Since the observables are defined originally in terms of \( \bar{\rho} \), we have to specify the relation between the two descriptions. The constitutive relations do not involve the "l" photons (involved in a self-energy process) and enable the interference between emitted and incident photons.

The construction of the subdynamics requires a classification of the states into two classes, the vacuum states and the correlated states. They are obtained traditionally by the introduction of a superoperator \( P \) that projects on the so called vacuum states. In our case, correlated states contain at least one intermediate field line (of the "l" type). Vacuum states contain thus only incoming and outgoing field lines.

The construction rule for the subdynamics operator can be formulated on the formal solution of the evolution equation in the extended dynamics:

\[
\tilde{\rho}_{ab}(t) = \sum_{c,d} \frac{-1}{2\pi i} \int_c dz e^{-izt} \left( \frac{1}{z - L} \right)_{ab,cd} \tilde{\rho}_{cd}(0).
\]  

(2.15)

The path \( c \) is chosen above the real axis for \( t > 0 \). Analytic continuation of the integrand from \( \Im z > 0 \) to \( \Im z < 0 \) is therefore required upon integration over \( z \). The symmetry with respect to time inversion is thus broken by the procedure. All integrations on intermediate field lines are (at least) formally performed to permit an analytical continuation from above of the functions of \( z \) so defined, placing all cuts parallely to the imaginary axis. The rule is to pick up the contribution of the poles associated with vacuum states.
mentioned analytical continuation from above enables to avoid accidental coincidence of the poles associated with the vacuum and correlation states. The physical poles are located at a value defined by the poles associated with the atomic states, i.e. \( z = \zeta + \bar{\zeta}, \ z = 0, \ z = \omega_1 + \zeta - \omega_0 + \delta, \ z = \omega_0 + \delta - \omega_1 + \bar{\zeta}, \) in addition to frequencies associated with physical field lines (incoming or outgoing). For instance, the vacuum-vacuum element \((z - \tilde{L})^{-1}\) may have the following physical poles: \( z = \zeta + \bar{\zeta} - \omega_\lambda, \ z = -\omega_\lambda, \ z = \omega_1 + \zeta - \omega_0 + \delta - \omega_\lambda + \omega_\mu, \ z = \omega_0 + \delta - \omega_1 + \bar{\zeta} - \omega_\lambda + \omega_\mu. \)

In the perturbation approach, we have the following expansion for the resolvent \( R_{ab,cd}(z) \) of \( L \) (involving the resolvent \( R^0(z) \) of the unperturbed hamiltonian \( H_0 \) and the interaction part \( L_V \) of the liouvillian):

\[
R_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( R^0(z) \left[ L_V R^0(z) \right] \right)_{ab,cd}^n. \tag{2.16}
\]

For the resolvent \( \tilde{R}_{ab,cd}(z) \) of \( \tilde{L} \) we have similarly the expansion:

\[
\tilde{R}_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( \tilde{R}^0(z) \left[ \tilde{L}_V \tilde{R}^0(z) \right] \right)_{ab,cd}^n. \tag{2.17}
\]

The singularities of the resolvent \( R(z) \) of \( L \) are defined on resummed expressions. Therefore, useful expressions are obtained by considering irreducible operators with respect to the vacuum. For the study of \( R_{ab,cd}(z) \), where no extension of dynamics has been defined, we may define here the set of vacuum states by the states without any field particles.\(^3\) Such a vacuum is useful since it will provide a point of comparison for the elements of \( \tilde{R}_{ab,cd} \) that do involve neither incident nor emitted photons. The collision operator \( \psi(z) \), is defined as the sum of irreducible fragments leading from a vacuum state to another: all intermediate states have to imply at least one field particle:

\[
\psi_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( L_V R^0(z) \right)_ {ab,cd(\text{irr})}^n \left. L_V \right)_{ab,cd(\text{irr})}. \tag{2.18}
\]

where the index \( \text{irr} \) means that al intermediate states involve at least one field line.

\(^3\)In the more usual approach by the Brussels group, the vacuum has been defined by the set of diagonal elements or by an adequate extension in the approach by the patterns of correlation.
We express also the perturbation expansion of the resolvent $\tilde{\bar{R}}(z)$ in terms of irreducible operators: For the complete Liouvillian $\tilde{\bar{L}}$, we introduce by analogy the irreducible operators $W_{ab,cd}(z)$ (that could also be noted by $\tilde{\bar{\psi}}_{ab,cd}(z)$)

$$W_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( \left[ \tilde{\bar{L}} V \tilde{\bar{R}}_0(z) \right]_n \tilde{\bar{L}} \right)_{ab,cd(\text{irr})} (2.19)$$

When $a$, $b$, $c$, $d$ represents states without photons, the operators $\tilde{\bar{R}}_{ab,cd}(z)$ and $\bar{R}_{ab,cd}(z)$ coincide. In order to be able to make the connection between the singularities of $R(z)$ and $\bar{R}(z)$, let us define irreducible operators that involve at least one of the connecting vertices $\tilde{\bar{L}}'_{V}$ (2.5) as:

$$T_{ab,cd}(z) = \left( \left[ \tilde{\bar{L}} V \tilde{\bar{R}}^0(z) \right]_n \tilde{\bar{L}} \right)_{ab,cd(\text{irr,con})} (2.20)$$

where the subscript $\text{con}$ implies that at least one of the vertices $\tilde{\bar{L}}$ is a connecting contribution $\tilde{\bar{L}}'_{V}$ (2.3). The link with the above introduced operators $\psi$ and $W$ is then obvious:

$$W_{ab,cd}(z) = \psi(z)_{ab,cd} + T(z)_{ab,cd} (2.21)$$

In view of the characteristics of the vacuum states (an incoming field line cannot be reintroduced and an outgoing field line never disappears), the contributions can be easily classified according to the number of vacuum field lines involved and computed in a recurrent way. Therefore, we first consider the contribution of terms that do not involve any field line.

### 3 Elements of $\tilde{\Sigma}$ without field

Our aim in this section is the computation of the elements of subdynamics super-operator $\tilde{\Sigma}_{ab,cd}(t)$ when $a$, $b$, $c$, $d$ represent states without photons. As in the case of the RWA,[1] they determine the evolution super-operator $\tilde{\Theta}_{at}$ that does not involve absorption nor emission processes. These elements of $\Sigma_{ab,cd}(t)$ are computed from the corresponding elements of the resolvent.

As no physical field lines are involved, the extension of dynamics plays no role: only virtual photons, involved in self-energy contributions, play a role: We may use the Liouvillian $\tilde{\bar{L}}$ in place of $\tilde{\bar{L}}$. The elements of the operators $\tilde{\bar{R}}_{ab,cd}(z)$ and $\bar{R}_{ab,cd}(z)$ coincide and the analytical properties of $\tilde{\bar{R}}_{ab,cd}(z)$ have been established in Ref. [19] for some diagonal-diagonal elements and the extension of the proof to the other matrix elements is straightforward.
The elements of $\hat{\Sigma}$ are expressed in terms of the irreducible operators $W$, $\psi$, $T$. For the elements of these operators without photons, some useful relations may be considered. The last vertex (at the extreme left) in the perturbative expansion \((2.19)\) for $W_{ab,cd}(z)$ corresponds to the absorption of the last virtual photon. For that vertex, the replacement of the element of $\bar{L}_V'$ by the element of $L_V$ that absorbs also the last photon, or vice versa, transforms the contribution to $W_{ab,cd}(z)$ into a contribution to $W$ with other values of the first two atomic indices, leading to the relations:

\[
W_{11,cd} + W_{00,cd} = 0, \quad (3.1)
\]
\[
W_{10,cd} + W_{01,cd} = 0. \quad (3.2)
\]

Since $\psi_{11.00}(z)$, $\psi_{00,11}(z)$, $\psi_{10.01}(z)$ and $\psi_{01.10}(z)$ vanish, the links \((2.21)\), with the previously introduced operators $\psi$ and $T$, provides the following relations

\[
T_{11.00}(z) = -W_{00.00}(z) = -\psi_{00.00}(z) - T_{00.00}(z),
\]
\[
T_{00.11}(z) = -W_{11.11}(z) = -\psi_{11.11}(z) - T_{11.11}(z), \quad (3.3)
\]
\[
T_{10.01} = -W_{01.01}(z) = -\psi_{01.01} - T_{01.01},
\]
\[
T_{01.10} = -W_{10.10}(z) = -\psi_{10.10} - T_{10.10}. \quad (3.4)
\]

This section is devoted to the elements $\hat{\Sigma}_{11,11}$, $\hat{\Sigma}_{00,00}$, $\hat{\Sigma}_{11,00}$, $\hat{\Sigma}_{00,11}$ of the subdynamics super-operator $\hat{\Sigma}(t)$ in order to obtain the elements $\hat{\Theta}_{11,11}$, $\hat{\Theta}_{00,00}$, $\hat{\Theta}_{11,00}$, $\hat{\Theta}_{00,11}$ of the evolution operator $\hat{\Theta}$. The other elements without physical field lines, such as $\hat{\Sigma}_{10,10}$, are not dynamically connected to the previous and are treated in the next §4.

The relevant elements of $\hat{\Sigma}$ are the same as those of $\Sigma_{11,11}$, $\Sigma_{00,00}$, $\Sigma_{11,00}$, $\Sigma_{00,11}$ computed by choosing the diagonal atomic states as the only elements of the vacuum. In terms of the irreducible operators, the following relations hold:

\[
\bar{R}_{11,11}(z) = R_{11,11} + R_{11,11}T_{11,11}\bar{R}_{11,11} + R_{11,11}T_{11,00}\bar{R}_{00,11},
\]
\[
\bar{R}_{00,11}(z) = R_{00,00}T_{00,11}\bar{R}_{11,11} + R_{00,00}T_{00,00}\bar{R}_{00,11},
\]
\[
\bar{R}_{11,00}(z) = R_{11,11}T_{11,11}\bar{R}_{11,00} + R_{11,11}T_{11,00}\bar{R}_{00,00},
\]
\[
\bar{R}_{00,00}(z) = R_{00,00} + R_{00,00}T_{00,11}\bar{R}_{11,00} + R_{00,00}T_{00,00}\bar{R}_{00,00}. \quad (3.5)
\]

For instance, the first one expresses the possibility of transition from \((11)\) to itself either directly (without connecting vertices), or with at least one
connecting vertex that involves $T_{11,11}$ or $T_{11,00}$. The system (3.3) can be solved easily, leading to [cf. Ref. [19], §4]

\[
\tilde{R}_{11,11}(z) = \frac{z - W_{00,00}(z)}{z(z - W_{11,11}(z) - W_{00,00}(z))}, \tag{3.6}
\]

\[
\tilde{R}_{00,00}(z) = \frac{z - W_{11,11}(z)}{z(z - W_{11,11}(z) - W_{00,00}(z))}, \tag{3.7}
\]

\[
\tilde{R}_{11,00}(z) = \frac{T_{11,00}(z)}{z(z - W_{11,11}(z) - W_{00,00}(z))}, \tag{3.8}
\]

\[
\tilde{R}_{00,11}(z) = \frac{T_{00,11}(z)}{z(z - W_{11,11}(z) - W_{00,00}(z))}. \tag{3.9}
\]

The relevant analytical properties of those expressions have been studied in Ref. [19]. The coincidence of the pole $z = \bar{\theta}$ with the corresponding pole at $z = \theta$ for the Liouvillian $L$ has been established. The residues at those poles are well defined, although the functions $W_{11,11}(z)$ and $W_{00,00}(z)$ present also branch points for the values of $z = 0$ and $z = \theta = \theta = \zeta + \bar{\zeta}$.

$\tilde{\Sigma}_{aa,bb}(t)$ is then computed from

\[
\tilde{\Sigma}_{aa,bb}(t) = \frac{-1}{2\pi i} \oint dz e^{-itz} \tilde{R}_{aa,bb}(z) = \frac{-1}{2\pi i} \oint dz e^{-itz} \left( \frac{1}{z - L} \right)_{aa,bb}, \tag{3.10}
\]

where the path contains only the two relevant physical poles: The first one is the obvious $z = 0$, the other one is a pole at $z = \bar{\theta} (= \theta = \zeta + \bar{\zeta})$. The value $\bar{\theta}$ satisfies formally the equation

\[
\bar{\theta} = W_{11,11}(\bar{\theta}) + W_{00,00}(\bar{\theta}), \tag{3.11}
\]

the solution does not depend of the chosen Riemann sheet.

We have by direct integration

\[
\tilde{\Sigma}_{11,11}(t) = \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} + \exp -i\bar{\theta}t |\bar{\mathcal{A}}_1|^2 \frac{W_{11,11}(\bar{\theta})}{\bar{\theta}}. \tag{3.12}
\]

The residue at the pole $z = \theta$ has been noted as $|\bar{\mathcal{A}}_1|^2$. That manner of writing has been introduced by analogy with the similar computation inside the rotating wave approximation [1] and does not imply a possible existence of $\bar{\mathcal{A}}_1$ for instance. Indeed, we have shown in Ref. [19] the existence of $|\bar{\mathcal{A}}_1|^2$ as a sum of residues and not as a product of separate contributions.

The main difference with the treatment inside the RWA can be seen on that expression: two poles play a role in the expression of this (doubly)
diagonal-diagonal element of $\tilde{\Sigma}$. Therefore, we have no longer the association of a temporal behaviour with a state. Inside the RWA, the bare ground state coincides with the true ground state while we have now to establish the connection.

Let us make some comments on the relative values of the contributions in a perturbation expansion in powers of the potential $V$. Indeed, it is easily realized that $W_{00,00}(0)$ vanishes at the second order while $W_{11,11}(0)$ is finite at that order and coincides with the second order value of $\tilde{\theta}$. This difference of behaviour has its origin in the stability of the state $0$ at the lowest order. Therefore, the second term starts as $1$ in a perturbation expansion in the potential while the first one is of order $V^2$.

The other elements of $\tilde{\Sigma}$ are evaluated in appendix A. The vacuum-vacuum elements of the super-operator $\tilde{\Sigma}(t)$ for the value $t = 0$ define the operator usually called $\tilde{A}$ in the subdynamics theory. Its derivation and the obtention of its inverse is now straightforward. The corresponding elements of the evolution operator $\tilde{\Theta}$ in the vacuum subdynamics can be obtained from the relation ($P$ is the projector on the vacuum elements):

$$\tilde{P} \tilde{\Sigma}(t) \tilde{P} = \exp -i \tilde{\Theta} t \tilde{A}. \quad (3.13)$$

Therefore, by multiplying the time derivative of $\tilde{\Sigma}(t)$ at $t = 0$ by the inverse of the operator $\tilde{A}$, we get:

$$\tilde{\Theta}_{11,11} = -\tilde{\theta} \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))},$$
$$\tilde{\Theta}_{00,00} = \tilde{\theta} \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))},$$
$$\tilde{\Theta}_{00,11} = -\tilde{\theta} \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))},$$
$$\tilde{\Theta}_{11,00} = -\tilde{\theta} \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))}. \quad (3.14)$$

Some obvious comments are a direct consequence of the form (3.14) of the $\tilde{\Theta}$ operator. A first remarkable point is that this result does not obviously depend on a choice of a Riemann sheet: that expression requires only the existence of the pole $\tilde{\theta}$: the $z = 0$ values for the elements of the irreductible operator $W$ are obviously well defined. That operator $\tilde{\Theta}$ preserves the norm of $\tilde{\rho}$, thanks the obvious relations $\tilde{\Theta}_{11,11} = -\tilde{\Theta}_{00,11}$ and $\tilde{\Theta}_{11,00} = -\tilde{\Theta}_{00,00}$. One of the eigenvalue of $\tilde{\Theta}$ vanishes while the other one is the sum $\tilde{\theta}$ of its
diagonal elements. A unitary transformation can lead to a Jordan form, with the cancelation of the elements $\Phi_{11,00}$ and $\Phi_{00,00}$ of a new evolution operator $\Phi$. The structure obtained inside the rotating wave approximation would then be rederived in an exact way. This point will be considered (see §5) after the examination of the properties of the other elements of $\tilde{\Sigma}$ that do not involve field particles and describe atomic dipolar moments.

4 The atomic dipolar moment in the reduced formalism

We are interested in this section in the elements without field $\tilde{\Sigma}_{10,10}$, $\tilde{\Sigma}_{01,01}$, $\tilde{\Sigma}_{10,01}$, $\tilde{\Sigma}_{01,10}$ that are not dynamically connected with the previous ones. Those elements play a role in determining the properties of the dipolar moment of the atom. As in the previous section, the extension of dynamics plays no role, the index “tilde” can be dropped and express the elements in the reduced formalism in terms of the elements in the unreduced one’s. Our starting expressions are therefore the formal solutions [formulae (2.7–2.8)] for the evolution of $\rho$ and $\tilde{\rho}$, where the couples $(ab)$ and $(cd)$ refer to the off diagonal discrete states (10) and (01). This can be easily symbolized by using a notation $(\tilde{a}\bar{a})$: $\bar{a}$ is one when $a$ is 0 and vice versa. An element of comparison is provided by studying first $R(z)$. That resolvent of $L$ that plays a role in (2.7) can be written as a convolution of the corresponding resolvents of $H$ and $-H$. Since the interaction involves the change of occupation numbers of field particles by one unit for each absorption or emission process, the states $a$ and $c$ on one hand, $b$ and $d$ on the other hand have to be identical. Only diagonal elements of the resolvent of $H$ and $-H$ are thus to be considered and we have

$$R_{\tilde{a}\bar{a},\tilde{a}\bar{a}}(z) = \left(\frac{1}{z - L}\right)_{\tilde{a}\bar{a},\tilde{a}\bar{a}} = \frac{-1}{2\pi i} \int_{0 < \Im u < \Im z} du \left(\frac{1}{u - H}\right)_{\tilde{a}\bar{a}} \left(\frac{1}{z - u + H}\right)_{\tilde{a}\bar{a}},$$

(4.1)

involving the functions $\eta$ and $\bar{\eta}$ given in (2.9–2.10). The analysis of the convolution enables [24, 19] to ascertain the presence of an isolated pole to both diagonal elements of $R$. In terms of the corresponding poles of the Green’s functions associated with the Hamiltonians $H$ and $-H$ (see §2 or Ref. [19]), they are given respectively by $z = \omega_1 + \zeta - \omega_0 + \delta$ with a residue $\mathcal{A}_1\mathcal{A}_0$ for $(\tilde{a}\bar{a}) = (10)$ and by $z = \omega_0 + \delta - \omega_1 + \zeta$ with a residue $\mathcal{A}_0\mathcal{A}_1$ for the other case $(\tilde{a}\bar{a}) = (01)$. 

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The computation of the elements of $\tilde{\Sigma}$ is detailed in appendix C. The poles that play a role are still located at $z = \omega_1 + \zeta - \omega_0 + \delta \equiv \delta_{10}$ and $z = \omega_0 + \delta - \omega_1 + \tilde{\zeta} \equiv \delta_{01}$. They now appear together as poles of each element. The result can be symbolized as:

$$\tilde{\Sigma}_{a\bar{a},a\bar{a}}(t) = e^{-i\delta_{10}t}a_{a\bar{a},a\bar{a}} + e^{-i\delta_{01}t}\beta_{a\bar{a},a\bar{a}},$$
$$\tilde{\Sigma}_{\bar{a}a,a\bar{a}}(t) = e^{-i\delta_{10}t}a_{\bar{a}a,a\bar{a}} + e^{-i\delta_{01}t}\beta_{\bar{a}a,a\bar{a}},$$

(4.2)

where the values of $\alpha$ and $\beta$ can be read on the equations (C.8). In a perturbation expansion, $\alpha_{10,10}$ and $\beta_{01,01}$ start as 1 while the other elements behave at least as $V^2$, since they involve connecting vertices.

The corresponding elements of the evolution operator $\tilde{\Theta}$ in the vacuum subdynamics can also be obtained from the relation (3.13) through the previous procedure ($|\tilde{\Theta}D|\equiv|\tilde{\Theta}D|$ is defined in (C.12))

$$\tilde{\Theta}_{a\bar{a},a\bar{a}} = \frac{1}{|\tilde{A}_D|}(\delta_{10}a_{a\bar{a},a\bar{a}} + \delta_{01}a_{\bar{a}a,a\bar{a}})(\alpha_{a\bar{a},a\bar{a}} + \beta_{\bar{a}a,a\bar{a}}),$$
$$\tilde{\Theta}_{\bar{a}a,a\bar{a}} = -\frac{1}{|\tilde{A}_D|}(\delta_{10}a_{\bar{a}a,a\bar{a}} + \delta_{01}a_{a\bar{a},a\bar{a}})(\alpha_{\bar{a}a,a\bar{a}} + \beta_{a\bar{a},a\bar{a}}).$$

(4.3)

In opposition with the case met in the RWA, the operator $\tilde{\Theta}$ is not diagonal inside the sectors defined by the atomic dipolar moments.

The value of $\left(e^{-i\tilde{\Theta}t}\right)_{a\bar{a},b\bar{b}}$ can be obtained directly from the knowledge of the elements of $\tilde{\Sigma}$ and $\tilde{A}^{-1}$:

$$\left(e^{-i\tilde{\Theta}t}\right)_{a\bar{a},b\bar{b}} = \tilde{\Sigma}^{-1}_{a\bar{a},b\bar{b}}(t) = \left(\tilde{A}^{-1}\right)_{a\bar{a},b\bar{b}} + \left(\tilde{A}^{-1}\right)_{01,b\bar{b}} + \left(\tilde{A}^{-1}\right)_{10,b\bar{b}}.$$

(4.4)

We know from the expression of $\tilde{\Sigma}(t)$ that the two eigenvalues of that operator $\tilde{\Theta}$ are $\delta_{10}$ and $\delta_{01}$ for the elements under consideration. Therefore, an invertible transformation can provide a diagonal evolution operator that has these two values as the only non-vanishing elements.
5 Physical representation for the purely atomic part of $\tilde{\rho}$

Denoting $\tilde{\rho}_{at}(t)$ the elements of $\tilde{\rho}$ that do not involve photons, in the absence of an incident field, we have, by definition of $\tilde{\Theta}^{at}$, the set of equations:

$$\frac{\partial}{\partial t}\tilde{\rho}_{at}(t) = -i\tilde{\Theta}^{at}\tilde{\rho}_{at}(t),$$  \hspace{1cm} (5.1)

where the relevant elements of the operators $\tilde{\Theta}^{at}$ are given in (3.14) for the diagonal-diagonal elements and in (4.3) for the off-diagonal ones. Let us examine first the diagonal-diagonal elements. The elements $\tilde{\Theta}_{00,00}$ and $\tilde{\Theta}_{11,00}$ would vanish if $W_{00,00}(0) = 0$. We may check explicitly this property in a perturbation expansion. The second order value $W^{(2)}_{00,00}(0)$ involves only contributions that are present in the unreduced formalism and vanishes accordingly. We have therefore to consider the fourth order value $W^{(4)}_{00,00}(0)$. After some algebra, a non-vanishing contribution, that involves the connecting vertices $\tilde{L}^I_V$ (2.3) is obtained (see Appendix B for details of computation).

$$W^{(4)}_{00,00}(0) = -2\pi i \sum_k \sum_{k'} |V_1|_{0k}|^2 |V_1|_{0k'}|^2 \delta(\omega_1 - \omega_0 - \omega_k) \frac{1}{(\omega_1 - \omega_0 + \omega_{k'})^2}. \hspace{1cm} (5.2)$$

Since we have

$$W^{(2)}_{11,11}(0) = -2\pi i \sum_k |V_1|_{0k}|^2 \delta(\omega_1 - \omega_0 - \omega_k) = \tilde{\theta}^{(2)}, \hspace{1cm} (5.3)$$

the first non-vanishing contribution to $\tilde{\Theta}_{00,00}$ is of the fourth order

$$\tilde{\Theta}^{(4)}_{00,00} = \tilde{\theta}^{(2)} \sum_{k'} |V_1|_{0k'}|^2 \frac{1}{(\omega_1 - \omega_0 + \omega_{k'})^2}. \hspace{1cm} (5.4)$$

The denominator is positive defined and no regularisation (“i\epsilon”) appears. Obviously, $\tilde{\Theta}^{(4)}_{00,00}$ does not vanish. As a consequence, $\tilde{\rho}_{00}$ does not describe the atom in its ground state. For comparison, $\tilde{\Theta}_{11,11}^{(2)}$ starts at the second order and we have $\tilde{\Theta}_{11,11}^{(2)} = \tilde{\theta}^{(2)}$. We know from the expressions of $\tilde{\Sigma}$ that the operator $\tilde{\Theta}$, relevant for the diagonal-diagonal elements has two eigenvalues,
0 and \( \bar{\theta} \). If we take into account the conservation of the norm, the operator \( \tilde{\Theta} \) can be related to an operator \( \Phi \) that has the structure met inside the RWA:

\[
\Phi_{11,11} = \bar{\theta}, \quad \Phi_{00,11} = -\bar{\theta}, \quad \Phi_{00,00} = \Phi_{11,00} = 0. \quad (5.5)
\]

Let us note that the natural structure for the evolution operator does not require an hermitian operator (or star-hermitian), in opposition to the early attempts in the subdynamics approach.\{14\} We define a new reduced density operator \( \tilde{\rho}^P \) (in the so-called physical representation) connected to the original one \( \tilde{\rho} \) through an invertible dressing operator \( \chi \)

\[
\tilde{\rho}^P(t) = \chi^{-1}(t)\tilde{\rho}(t), \quad \tilde{\rho}(t) = \chi\tilde{\rho}^P(t), \quad (5.6)
\]

and such that the evolution of \( \tilde{\rho}^P \) is governed by the operator \( \Phi \). Therefore, the following relations have to hold:

\[
\Phi = \chi^{-1}\tilde{\Theta}\chi, \quad \tilde{\Theta} = \chi\Phi^{-1}. \quad (5.7)
\]

If we impose that the traces of \( \tilde{\rho} \) and \( \tilde{\rho}^P \) are the same, we have moreover

\[
(\chi^{-1})_{11,11} + (\chi^{-1})_{00,11} = 1, \quad (\chi^{-1})_{00,00} + (\chi^{-1})_{11,00} = 1,
\]

\[
\chi_{11,11} + \chi_{00,11} = 1, \quad \chi_{00,00} + \chi_{11,00} = 1. \quad (5.8)
\]

We have therefore to determine \( \chi \) by imposing conditions (5.8) and (5.7). The last conditions (5.7) can be made explicit, taking (5.6) into account:

\[
\tilde{\Theta}_{aa,bb} = \chi_{aa,11}\Phi_{11,11}(\chi^{-1})_{11,bb} + \chi_{aa,00}\Phi_{00,11}(\chi^{-1})_{11,bb}. \quad (5.9)
\]

The values of the elements of \( \chi \) are computed from these conditions in Appendix D and the unambiguous result is

\[
\chi_{11,11} = \chi_{00,00} = \frac{W_{11,11}(0)}{W_{11,11}(0) + W_{00,00}(0)}, \quad \chi_{11,00} = \chi_{00,11} = \frac{W_{00,00}(0)}{W_{11,11}(0) + W_{00,00}(0)},
\]

\[
(\chi^{-1})_{11,11} = (\chi^{-1})_{00,00} = \frac{W_{11,11}(0)}{W_{11,11}(0) - W_{00,00}(0)},
\]

\[
(\chi^{-1})_{00,11} = (\chi^{-1})_{11,00} = -\frac{W_{00,00}(0)}{W_{11,11}(0) - W_{00,00}(0)}. \quad (5.10)
\]
The form of the operator $\Phi$ and the condition on the norm determines thus entirely the dressing operator $\chi$ for these elements. The two diagonal elements of $\tilde{\rho}^P$ can thus be interpreted as the probability of finding the atom resp. in the ground and excited levels (cf. \[1\]). They are normalized and have the expected associated “free” motion (in the absence of an incident field).

We now turn to the elements describing the dipolar atomic momentum. We know from the expressions of $\tilde{\Sigma}$ that the operator $\tilde{\Theta}$ has two complex eigenvalues inside the considered subspace, namely $\delta_{10}$ and $\delta_{01}$. The operator $\tilde{\Theta}$ should be related to the operator $\Phi$, the elements of which are given by:

$$
\Phi_{10,10} = \delta_{10}, \quad \Phi_{01,01} = \delta_{01}, \quad \Phi_{01,10} = \Phi_{10,01} = 0.
$$

The natural structure for these elements evolution operator requires here an hermitian operator, as opposed to the case of the diagonal elements. The relevant elements of the reduced density operator $\tilde{\rho}^P$ are connected with the original one $\tilde{\rho}$ through the same relation (5.6) and its evolution is governed by the operator $\Phi$ (5.11). Therefore, the same relations (5.7) have to hold inside the sector, but no condition on the trace appears.

Denoting by $|\chi_D|$ the determinant of the $\chi$ matrix in the dipolar sectors, we have from the inversion of a standard $2 \times 2$ matrix

$$
(x^{-1})_{a\bar{a},a\bar{a}} = \chi_{a\bar{a},\bar{a}a} \frac{1}{|\chi_D|}, \quad (x^{-1})_{a\bar{a},\bar{a}a} = -\chi_{a\bar{a},\bar{a}a} \frac{1}{|\chi_D|}.
$$

We have therefore to determine $\chi$ by imposing conditions (5.7) that can be made explicit, taking (5.11) into account

$$
\tilde{\Theta}_{a\bar{a},c\bar{c}} = \sum_b \chi_{a\bar{a},b\bar{b}} \delta_{b\bar{b},c\bar{c}} (x^{-1})_{b\bar{b},c\bar{c}}.
$$

We know (4.3) the explicit value of the elements of $\Theta$, but we will not proceed by direct identification of (5.13) and (4.3). In order to obtain the values of the elements of $\chi$, it seems easier to compare the expressions of $\exp -i\tilde{\Theta}t$ deduced from $\chi(\exp -i\Phi t)\chi^{-1}$ and from its expression (4.4). We proceed in appendix D with the identification of the terms with the same exponential behaviour ($\exp -i\delta_{10}t$ or $\exp -i\delta_{01}t$). The elements of $\chi$ can be determined up to a free parameter. A remaining indetermination is not new inside the context of subdynamics.\[14\] It does not affect the evolution equations but concerns the relation between $\tilde{\rho}^P$ and $\tilde{\rho}$. That indetermination is linked to a choice of relative phase between the basic states used for the
description. The dressing operator is indeed apt to change the original choice. We can fix it by connecting, for instance, the value of $|\chi|$ with that of $|\tilde{A}_D|$ in a “usual way”: $|\chi|^2 = |\tilde{A}_D|$ or introduce a criterion that ensures the positivity of the density operator if required. We will not elaborate here further on this point.

For the elements considered, the reduced density operator $\tilde{\rho}^P$ satisfies the kinetic equations that we have postulated in Ref. [4]. That derivation is exact (no approximation has been required). The kinetic equations appear naturally when the physical representation is introduced. As already stressed in Ref. [4], physicists in optics use them naturally, without realizing their profound origin, while considering conceptually such equations as useful approximated equations arising from some pole approximation. The formulation of observables in the formalism $\tilde{\rho}^P$ is more natural than in the original representation. What we call the ground state and excited states of the atom are the objects that behave as the diagonal elements of $\tilde{\rho}^P$: in the absence of an incident field, the excited state decays in a purely exponential way while the evolution of the ground state is only due to the transfer arising from the decay of the excited state. Such a decay takes place with the lifetime as it is usually computed from $S$-matrix or Green’s function formalism. Our formalism, that eliminates the reabsorption of the field by its source, leads automatically to a kinetic description while preserving the equivalence with the original Liouville-von Neumann equation, providing the compatibility conditions are satisfied at the initial time. [3] The atomic observables, computable with $\tilde{\rho}_{at}$ (= $\tilde{\rho}_{at}$ by the constitutive relations), have also to be modified by the dressing operator ($\chi$ or more precisely $\chi^{-1}$) if they are defined in the original representation, so that the conservation of their mean value is ensured. If the observables are defined by the kinetic properties, such as the probability of finding the atom in the ground or excited state, they can be expressed directly in the physical representation. Our approach enables moreover a consistent departure from the original time-reversible description while preserving normalizability and positivity. [1] The element $\tilde{\rho}^P_{11}$ has a clear physical meaning as the probability of finding the atom in the excited state, irrespectively of the state of the field. Inside RWA, in [4], we have shown that an initial condition describing the atom in the excited state (only $\tilde{\rho}^P_{11}$ does not vanish) does not belong to an admissible initial condition if the equivalence conditions are satisfied. Such an initial condition is however fit to discuss normalizability and positivity in the irreversible departure from quantum mechanics but its relevance in physics has to be established.
on an experimental basis.

6 One incident field

6.1 One passive incident field

We look first for the description of the elements of the evolution operators \( \Theta \) and \( \Phi \) involving an incident field line that is not absorbed. They can be obtained by considering the elements of \( \tilde{\Sigma}(t) \) that involve one incident field line at the extreme right and the same field line at the left. The atomic states are explicit in our notations and are represented by the letters \( a, b, c, d \) (\( a, b, c, d \) represent the atomic state \( 0 \) or \( 1 \)), while the present photons are written at the right of the atomic state. We will consider therefore the following elements of \( \tilde{\Sigma} \): \( \tilde{\Sigma}^a_{\lambda b . c . \lambda d} \), \( \tilde{\Sigma}^{ab}_{\lambda cd} \). The couple of atomic states \((ab)\) and \((cd)\) involved at the right \((cd)\) and at the left \((ab)\) of the elements of \( \tilde{\Sigma} \) are of the same nature to get a non-vanishing contribution, namely a diagonal couple or an off diagonal couple on both sides.

Our starting point is the formal expression of the contributions to \( \tilde{\rho}_{a\lambda b}(t) \) arising from the part of initial conditions involving only one incident field line:

\[
\tilde{\rho}_{a\lambda b}(t) \leftarrow \sum_{c,d} \frac{-1}{2\pi i} \int_c dz e^{-izt} \left( \frac{1}{z - L} \right)^{a\lambda b . c . \lambda d} \tilde{\rho}_{c\lambda d}(0) \quad (6.1)
\]

The detailed calculation can be found in Appendix E and provides the expected result. If we define \( I_{ab,cd} \) as \( 1 \) when \( a = c \) and \( b = d \) \( (I_{ab,cd} \) vanishes for the other possibilities) we obtain

\[
\tilde{\Theta}^{a\lambda b . c . \lambda d} = \omega_{\lambda} I_{ab,cd} + \tilde{\Theta}^{ab}_{cd} \quad \tilde{\Theta}^{ab}_{\lambda cd} = -\omega_{\lambda} I_{ab,cd} + \tilde{\Theta}^{ab,cd}_{\lambda} \quad (6.2)
\]

The dressing does not depend on the incident photon and the natural choice for the dressing operator \( \chi \) is

\[
\chi^{a\lambda b . c . \lambda d} = \chi^{ab,cd} \quad \chi^{ab}_{\lambda cd} = \chi^{ab,cd}_{\lambda} \quad (6.3)
\]

and therefore

\[
\Phi^{a\lambda b . c . \lambda d} = \omega_{\lambda} I_{ab,cd} + \Phi^{ab,cd} \quad \Phi^{ab}_{\lambda cd} = -\omega_{\lambda} I_{ab,cd} + \Phi^{ab,cd}_{\lambda} \quad (6.4)
\]

The generalization of that property in the case of the presence of an arbitrary numbers of field lines (incident or emitted) is obvious as long as they do not interact with the atomic variables.
6.2 One absorbed incident field line

We look for the elements of the evolution operators $\Theta$ and $\Phi$ that describe the absorption of one incident field line. They can be obtained by considering the elements of $\tilde{\Sigma}(t)$ that involve one incident field at the extreme right and no field at the left. We consider therefore the following elements of $\tilde{\Sigma} (a, b, c, d$ represent the atomic states 0 or 1) $\tilde{\Sigma}_{ab.c\lambda d}, \tilde{\Sigma}_{ab.cd\lambda}$. The couple of atomic states $(ab)$ and $(cd)$ involved at the right $(cd)$ and at the left $(ab)$ of $\tilde{\Sigma}$ are of different natures: We have a diagonal couple on one side and an off-diagonal couple on the other side. The intrinsic evolution of that kind of couples have been studied in previous sections. If the diagonal couple is at the left, we have to consider: $\tilde{\Sigma}_{11.0\lambda 0}, \tilde{\Sigma}_{00.0\lambda 1}, \tilde{\Sigma}_{11.0\lambda 0}, \tilde{\Sigma}_{00.0\lambda 1}, \tilde{\Sigma}_{11.01\lambda}, \tilde{\Sigma}_{00.10\lambda}$. Similar elements have to be considered in the case where the diagonal elements are at the right: $\tilde{\Sigma}_{10.1\lambda 0}, \tilde{\Sigma}_{10.0\lambda 0}, \tilde{\Sigma}_{01.0\lambda}, \tilde{\Sigma}_{11.01\lambda}, \tilde{\Sigma}_{00.11\lambda}, \tilde{\Sigma}_{00.00\lambda}$.

Our starting point is the formal expression of the contributions to $\tilde{\rho}_{ab}(t)$ arising from the part of initial conditions involving only one incident field

$$\tilde{\rho}_{ab}(t) \leftarrow \sum_{c,d,\lambda} \frac{-1}{2\pi i} \int_c dz e^{-izt} \times \left[ \left( \frac{1}{z - \tilde{L}} \right)_{ab.c\lambda d} \tilde{\rho}_{c\lambda d}(0) + \left( \frac{1}{z - \tilde{L}} \right)_{ab.cd\lambda} \tilde{\rho}_{cd\lambda}(0) \right]. \quad (6.5)$$

The details of the computation can be found in Appendix E and $\tilde{\Theta}$ takes the form:

$$\tilde{\Theta}_{aa.b\lambda\bar{b}} = \sum_{c,d,e} (\tilde{\Theta} \tilde{A})_{aa.cc}(\tilde{A}^{-1})_{cc.dd}\tilde{A}_{dd.e\lambda e}(\tilde{A}^{-1})_{ee.\bar{b}\bar{b}} + \sum_c (\tilde{\Theta} \tilde{A})_{aa.c\lambda e}(\tilde{A}^{-1})_{e\bar{e}.\bar{b}\bar{b}}$$

$$= \sum_{d,e} \tilde{\Theta}_{aa.dd}\tilde{A}_{dd.e\lambda e}(\tilde{A}^{-1})_{ee.\bar{b}\bar{b}} + \sum_c (\tilde{\Theta} \tilde{A})_{aa.c\lambda e}(\tilde{A}^{-1})_{e\bar{e}.\bar{b}\bar{b}}. \quad (6.6)$$

Similar expressions hold for the other elements $\tilde{\Theta}_{aa.b\bar{b}\lambda}, \tilde{\Theta}_{aa.b\bar{b}b}, \tilde{\Theta}_{aa.b\bar{b}b}$. The derivation of the elements describing an emission process can be performed in a completely similar way. We will not dwell on these terms since they do not introduce new ideas or properties.

These expressions may serve to determine the corresponding elements of the dressing operator $\chi$ and of the evolution operator $\Phi$ (see next §7).
7 Vertex dressing

We are interested in the possibilities for the evolution operator $\Phi$ that describes the absorption of one incident field line. We use our previous knowledge of the elements of $\chi$, that do not involve the field. These elements have been determined in previous sections, using physical requirements. For the computation of the element of $\Phi$ describing the disappearance of the incident field line, we may use the link (5.7) between $\tilde{\Theta}$ and $\Phi$ and introduce a new element $\chi_{\text{gh,cd}}$ for the dressing operator to get

$$
\Phi_{ab,\text{cd}} = \left(\chi^{-1}\tilde{\Theta}\chi\right)_{ab,\text{cd}}
= \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}} + \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}}
+ \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}}.
$$

(7.1)

Let us call $\Phi^\text{at}$ the part of $\Phi$ that does not involve transitions in the field. The elements of $\Phi^\text{at}$ have been determined in §5 [(5.3) and (5.11)] and in §6 (6.4). Using this known value of $\Phi^\text{at}$, the property $\chi_{g\lambda h, cd} = \chi_{gh, cd}$ (see (6.3)) and the value of the element $(\chi^{-1})_{ab,\lambda f}$ of the inverse of $\chi$, that can be computed in the same manner as the corresponding element of the inverse of $\tilde{A}$ in appendix E, we get

$$
\Phi_{ab,\text{cd}} = \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}} + \sum_{e,f,g,h} \Phi^\text{at}_{ab,ef}(\chi^{-1})_{ef,gh}\chi_{gh,\text{cd}}
- \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\chi_{ef,\text{cd}}\Phi^\text{at}_{g\lambda h,\text{cd}}.
$$

(7.2)

Defining the $X$ operator such that its only non-vanishing elements are

$$
X_{ab,\text{cd}} = \sum_{e,f} \left(\chi^{-1}\right)_{ab,ef}\chi_{ef,\text{cd}},
X_{ab,\text{cd}} = \sum_{e,f} \left(\chi^{-1}\right)_{ab,ef}\chi_{ef,\text{cd}}.
$$

(7.3)

we can express the last two terms in (7.2) as a commutator of $X$ with the operator $\Phi^\text{at}$.

$$
\Phi_{ab,\text{cd}} = \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}} + \left(\Phi^\text{at}, X\right)_{ab,\text{cd}}
\Phi_{ab,\text{cd}} = \sum_{e,f,g,h} \left(\chi^{-1}\right)_{ab,ef}\tilde{\Theta}_{ef,gh}\chi_{gh,\text{cd}} + \left(\Phi^\text{at}, X\right)_{ab,\text{cd}}.
$$

(7.4)
The elements $\Phi_{ab,c\lambda b}$ and $\Phi_{ab,c\lambda b}$ are expressed as the corresponding elements of $\tilde{\Theta}$, with a dressing bearing on the atomic levels, plus the commutator of $\Phi^a_t$ with an undetermined $X$ operator. The role of that indetermination can be understood from the introduction of the dressed reduced density operator $\tilde{\rho}^P$ (5.6). Indeed, the determination of the elements of $X$ is equivalent to that of $\chi$ [from (7.3)]. The presence of a non-vanishing $X$ operator means that the elements of the physical reduced density operator $\tilde{\rho}^P$ are also dressed by the elements $\tilde{\rho}_{ab}$ and $\tilde{\rho}_{ab\lambda}$: The dressing of the atomic states involves the incident photons. This is by no means mandatory: we can choose to dress the states by the self-field only. Therefore, the future determination of the $X$ operator requires a personal choice of the basic states for describing the atom.\footnote{Such a choice cannot be implied by the formalism.}

Due to the structure of the commutator, the contribution to $\Phi$ depending on $X$ vanishes for the resonance processes when the width of the states is neglected: This contribution plays a role in the off resonance processes. This point can be illustrated by considering some first order non-vanishing contributions to $\Phi$, the element $\Phi^{(1)}_{11,0\lambda 1}$ for instance. We can replace in (7.4) the $\chi$ and $\chi^{-1}$ operator by unity. The first order contribution to $\Theta_{11,0\lambda 1}$ is nothing but $V_{1|0\lambda}$. $\Phi_{11,11}$ is at least from the second order ($\tilde{\theta}^{(2)}$) while $\Phi_{0\lambda 1,0\lambda 1}$ provides at the lowest order a contribution independent of the coupling ($\Phi^{(0)}_{0\lambda 1,0\lambda 1} = \omega_0 + \omega_\lambda - \omega_1$). Therefore,

$$\Phi^{(1)}_{11,0\lambda 1} = V_{1|0\lambda} - (\omega_0 + \omega_\lambda - \omega_1)X_{11,0\lambda 1}. \quad (7.5)$$

If $X_{11,0\lambda 1}$ is not singular for $\omega_\lambda = \omega_1 - \omega_0$, its value is irrelevant for determining $\Phi^{(1)}_{11,0\lambda 1}$ at the bare energy resonance ($\omega_\lambda = \omega_1 - \omega_0$).

It has been long noticed\footnote{In Ref. [29], causality is claimed to be respected in the case of a double photodetection but that property is limited to two-level systems while multilevel systems were considered in Ref. [8].} that the kind of coupling (2.1) between atoms and field is not completely satisfactory with respect to the causal propagation of the field, i.e. its finite propagation. Indeed, in the problem of transfer of excitation from one atom to another, if the measurement concerns also the state of the initially excited atom (in a case of non local observables, as treated in case I of Ref. [7]), precursors appear and causality is not strictly respected. The same kind of acausal behaviour is met\footnote{In Ref. [8], in the description of a double photodetection of the light from an atom admitting a cascade decay that non causal behaviours are linked} in the description of a double photodetection of the light from an atom admitting a cascade decay. Those non causal behaviours are linked...
to the finite lower bound in the spectrum of the exchanged photon and disappear when the integration over the energy of the photons is extended to $-\infty$. This was common practice. Various authors [30]-[33] have claimed to have succeeded in proving causality of propagation of light: They all used at some stage an equivalent procedure. In the case of [33], for instance, the sign of a contribution is changed on the base of its smallness. In later works, that include the counter-rotating terms, that procedure has no longer been required for proving strict causality in either the framework of two-level systems for all observables,[34, 35, 29] or, for multilevel systems for local observables.[7] Nevertheless, the general case, beyond the two-level systems and local observables, still contains acausal behaviours, even when the counter-rotating contributions are considered.[8, 7]

In Ref. [8], a suggestion has been proposed to ensure a strict causality in all possible cases by proposing new terms of interaction between the atom and the field. The present formalism is more appropriate for realizing that possibility of including in a simple way causality into the equations of motion without modifying the energy spectrum of the field. With respect to the usual approach, we have indeed a supplementary degree of freedom since the symmetry between absorption and emission is lost.[4, 1] In the study of the coupling of the quantum field with a quasiclassical source,[4] it has been established that an appropriate association of super-operators for the creation of the field $\mathcal{E}_c$ in Ref. [4] and absorption process $\mathcal{E}$ in Ref. [4] leads to a strictly causal propagation, without precursor: The retarded solution for the field propagation appears automatically, irrespectively of the atomic state. We can use the possibility of dressing to meet the causality requirements. As we have seen, the dressing bears on non-resonant contributions. Therefore, we can require that the $\Phi$ elements, describing the interaction with an atom at some point $r$, have the appropriate structure for a local coupling ($k$ is the wave number associated with $\lambda$ or $\mu$)

$$
\Phi_{ab.cld} = \phi_{\text{abs}, ab, cd}^{\lambda} \frac{1}{\sqrt{\omega}} e^{i k \cdot r}, \quad \Phi_{ab.cd\lambda} = \phi_{\text{abs}, ab, cd}^{\lambda} \frac{1}{\sqrt{\omega}} e^{-i k \cdot r},
$$

$$
\Phi_{ab, cd} = \phi_{\text{em}, ab, cd} \frac{1}{\sqrt{\omega}} e^{-i k \cdot r}, \quad \Phi_{ab, cd\mu} = -\phi_{\text{em}, ab, cd}^{\mu} \frac{1}{\sqrt{\omega}} e^{i k \cdot r}. \quad (7.6)
$$

The dressing by $\chi$ cannot change the value of $\Theta$ for the resonant contribution at the lowest order in the coupling but can provide any chosen value for the other (off-resonant) elements. It is thus perfectly possible to choose $\chi$ such that these conditions are fulfilled. The change of relative sign between the two terms describing absorption and emission is capital. When an
emission, by a pointlike atom at some point \( r \), is combined with the absorption by another one, at some other point \( r_1 \), two processes lead to the same change in the atomic occupation numbers \((7.6)\). The field emitted by an atom through \( \Phi_{a b c d} \), and absorbed by the other atom through \( \Phi_{a' b' c' d'} \), provides a time dependence as \( \exp(-i\omega_k t) \) while the other process \((\Phi_{ab d} c d) \) involves \( \exp(i\omega_k t) \). Both contributions can be combined and their sum is equivalent of having a domain of integration over the photon spectrum from \(-\infty\) to \(+\infty\), but this procedure is now an integral part of the evolution equation. Therefore, the resulting contribution ignores the presence of a finite lower bound in the spectrum of the field and causality can be fully respected, without precursors or other oddities: \([8]\) The propagator corresponding to a retarded solution for an electromagnetic process appears naturally, corresponding to Ritz’s point of view on the origin of irreversibility, rather than Einstein’s conception. \([9]\) Of course, such a radical change in the form of the coupling has to take place while preserving positivity of the retrieved density operator.

The emphasis on the physical interpretation of the elements of the description is not the only difference with the Brussels-Austin group. \([16, 37]\) Indeed, as can be seen on the Friedrichs model, \([37]\) their approach is based on the consideration of Gamow vectors obtained through a generalized eigenvalues problem, outside Hilbert space, providing complex values. That obtention have been inspired by the early attempts of constructing a subdynamics. \([11]\) Besides problems linked to normalization (such states are of null norm and a set of left and right bicomplete and biorthogonal eigenvectors for the Hamiltonian has to be introduced), and positivity (that is not a requirement of their \( \Lambda \) transformation), such states can only decay (by construction) and the description of an excitation of the atom by an incident wave packet is outside the possibility of that approach. \([3]\) In contrast, the presence, in our kinetic operator \( \Phi \), of non-diagonal elements in the field (see \((7.6)\) for instance) enables a coupling between an incident wave packet and the atom. Therefore, the work of that Brussels-Austin group is more appropriate for an abstract discussion about irreversibility, \([38]\) through the introduction of “diagonal singularities”, than for the description of atom-field interactions required in quantum optics.

\(^5\) That criticism can also be addressed to our previous work \([11]\) and has been one of our motivations to depart from the early attempts of the Brussels group.
8 Concluding remarks

This paper enables to understand the link between reversible and irreversible formulations of interacting atoms and fields. This later formulation requires a description in a Schrödinger type description. The existence to an invertible transformation enabling a transition from one description to the other one may be perceived as a surprise: when compatibility conditions are satisfied, both formulations are equivalent. The physical representation enables physicists to provide in quantum optics a description conform to their intuition, without any loss of generality or the introduction of approximations. An atomic level is characterized by the values of its (dressed) energy and its lifetime. Dressing contributions are now excluded through in the structure of the evolution operator that involves the often used distinction between external and emitted photons. Quasiclassical state for the external field can be considered. The external field description naturally factorizes inside the reduced density operator. Acausal behaviours can now be excluded through the choice of the dressing operator. A modelization for an ideal photodetection device finds naturally his place in that framework. The relation with the original description involves the dressing operator, the constitutive relations and the compatibility conditions. A direct formulation of an initial condition for a problem inside the physical representation is nevertheless possible, with the possibility of dealing with an extension of quantum mechanics, as has been considered in Ref. The necessity of such an extension of quantum mechanics cannot be excluded a priori but should result from experiments.

This paper has focused on the role of dressing inside the single subdynamics approach. In the first papers within that approach, no need for a dressing has appeared. The main difference of the system inside and outside the RWA is indeed the following: In presence of counter-rotating terms, the obtention of the ground and excited states of the atom is no longer automatic and requires a dressing operator. Indeed, inside the RWA, the bare ground state can be identified with the true ground state but that property is lost outside the RWA: It will not convert itself into an invariant state and an adequate combination is required that enables the identification of the ground state and of the excited state through their temporal dependences. Renormalization through a dressing operator takes naturally place for dealing with physical ground and excited states, physical atomic dipolar
moments, pointlike interactions and causality. The recourse to these entities to make physical predictions depends of course of the observables considered. If they are defined in the original representation, no need to go into the physical representation is present, except for the simplicity and transparency of the evolution in that representation. If the observables are to be precized, they are more naturally defined inside the new representation. Indeed, the precise meanings of the ground state, the excited state, the atomic dipolar moments resort to $\hat{\rho}^P$ where temporal behaviours enable the identification. The recourse to the new description can also be justified according to the manner in which the initial conditions have to be formulated. If we intend to consider the scattering of a wave packet impinging on an atom in its ground state, we need to know how to describe correctly the atomic state and the present approach answers that question.

We have not considered in this paper the constitutive and compatibility conditions since they appear here in a fashion similar to our previous papers. We are aware that positivity should require a more detailed analysis to place restrictions on the undetermined elements of the dressing operator.

This paper shows moreover how the modelization in optics is justified from first principles without the need of approximations such as the pole approximation. In our kinetic equations, the explicit attribution of a lifetime to an atomic excited level is the consequence of the original Liouville-von Neumann equations when going into an “historical” representation.

The kinetic equations obtained in the present paper are at the level of reduced distribution functions for the field, and have been introduced in [4]. The kinetic equations that were surmised in that paper are now properly derived.

The dressing procedure introduced in the early approach of the subdynamics theory has proved to be still relevant and fruitful in the present work for more elaborate systems, closer to quantum field theory. We are nevertheless still facing some basic indetermination corresponding to the arbitrariness in a choice of basis vectors. That indetermination is not met for the purely atomic part, except for a (trivial) choice of the relative phase between the ground and excited atomic states: The atomic dipolar moment is not completely fixed. The indetermination can modify the form of the evolution operator when absorption and emission processes are considered and can be used to ensure a strict causality, in spite of a finite lower bound in the energy spectrum.

We have not dealt with simultaneous processes [1] that are to be present both inside and outside the RWA, the simultaneous absorption and emission
process appears at a lower order outside the RWA than inside. Since they involve necessarily nonresonant contributions, they could be of the same importance as higher order resonant contributions, not included into the model Hamiltonian.

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A Other elements of $\tilde{\Sigma}$ between diagonal atomic states

We now turn to the second element $\tilde{\Sigma}_{00.00}$. From the following relations obtained in Ref. [19],

$$R_{11.11}(z) + R_{00.00}(z) = \frac{1}{z} + \frac{1}{z - W_{11.11}(z) - W_{00.00}(z)}, \quad (A1)$$

we have

$$\tilde{\Sigma}_{11.11}(t) + \tilde{\Sigma}_{00.00}(t) = 1 + \exp(-i\theta t |\tilde{A}_1|^2), \quad (A2)$$

and therefore

$$\tilde{\Sigma}_{00.00}(t) = \frac{W_{11.11}(0)}{(W_{11.11}(0) + W_{00.00}(0))} + \exp(-i\theta t |\tilde{A}_1|^2) \frac{W_{00.00}(\bar{\theta})}{\theta}. \quad (A3)$$

The first term starts as 1 in a perturbation expansion in the potential while the second one is of order $V^2$.

We now turn to the next element $\tilde{\Sigma}_{11.00}$, computed from (3.5). The contributions to $\tilde{\Sigma}_{11.00}(t)$ emerge also from the poles at $z = 0$ and $z = \bar{\theta}$. The existence of those poles, established in Ref. [19] for $\tilde{\Sigma}_{11.11}(t)$ and $\tilde{\Sigma}_{00.00}(t)$ enables to prove their existence for $\tilde{\Sigma}_{11.00}(t)$. We can easily see that the residues of those poles are well defined, in particular that the numerator is uniquely defined for that value. Indeed, by comparing this perturbation expansion (the relevant part of the unperturbed propagator in reduced formalism $R^0(z)$ coincides with $R^0(z)$), we have

$$R_{00.00}(z) = \sum_{n=0}^{\infty} \left( R^0(z) \left[ \bar{L}_V R^0(z) \right]_0 \right)^n_{00.00},$$

33
\[ \hat{R}_{11,00}(z) = \sum_{n=0}^{\infty} \left( R^0(z) \left[ \bar{L}_V R^0(z) \right]^n \right)_{11,00}. \]  \hspace{1cm} (A4)

If in the second expression (of \( \hat{R}_{11,00}(z) \)), we modify the last vertex we get a contribution to \( \hat{R}_{00,00}(z) \) with a change of sign. The term \( n = 0 \), without vertex \( \bar{L}_V \), of \( \hat{R}_{00,00}(z) \) cannot be recovered in that way and we have to add its contribution. Similar considerations hold also when the last two indices are “11” in place of “00” and we have

\[ \hat{R}_{11,00}(z) = -\hat{R}_{00,00}(z) + \frac{1}{z}, \]  \hspace{1cm} (A5)

\[ \hat{R}_{00,11}(z) = -\hat{R}_{11,11}(z) + \frac{1}{z}. \]  \hspace{1cm} (A6)

We can check the compatibility of the two expressions (3.8) and (A5). Using the expression (3.7) for \( \bar{R}_{00,00}(z) \) we get:

\[ \bar{R}_{11,00}(z) = -\bar{R}_{00,00}(z) + 1 \frac{z}{W_{11,11}(0) + W_{00,00}(0))} + \exp -i\bar{\theta} t |\bar{A}_1|^2 T_{11,00}(\bar{\theta}) \]  \hspace{1cm} (A7)

If we take into account the relation (3.4) between \( W_{00,00}(z) \) and \( T_{11,00}(z) \), we get the equivalence between the two forms (3.8) and (A5) for \( \bar{R}_{11,00}(z) \). Its interest lies in the following remark: in Ref. [19], only the existence of the poles for the elements \( \bar{R}_{11,11}(z) \) and \( \bar{R}_{00,00}(z) \) has been considered. The formulae (A3), (A6) show directly that the poles of \( \bar{R}_{11,00}(z) \) and \( \bar{R}_{00,11}(z) \) are automatically well defined. We get therefore:

\[ \tilde{\Sigma}_{11,00}(t) = \frac{-T_{11,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} + \exp -i\tilde{\theta} t |\tilde{A}_1|^2 T_{11,00}(\tilde{\theta}) \]  \hspace{1cm} (A8)

and similarly:

\[ \tilde{\Sigma}_{00,11}(t) = \frac{-T_{00,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} + \exp -i\tilde{\theta} t |\tilde{A}_1|^2 T_{00,11}(\tilde{\theta}) \]  \hspace{1cm} (A9)

An alternative expression for \( \tilde{\Sigma}_{11,00}(t) \) and \( \tilde{\Sigma}_{00,11}(t) \) can be obtained from (A5), (A6) or from (3.3)

\[ \tilde{\Sigma}_{11,00}(t) = \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} - \exp -i\tilde{\theta} t |\tilde{A}_1|^2 W_{00,00}(\tilde{\theta}) \]  \hspace{1cm} (A10)

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\[
\tilde{\Sigma}_{00,11}(t) = \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} - \exp{-i\theta t} |\tilde{\mathcal{A}}_1|^2 \frac{W_{11,11}(\tilde{\theta})}{\theta}. \quad (A11)
\]

Let us note the obvious relations, that can be derived from (A3) and (A6)
\[
\tilde{\Sigma}_{00,11}(t) + \tilde{\Sigma}_{11,11}(t) = 1, \quad \tilde{\Sigma}_{00,00}(t) + \tilde{\Sigma}_{11,00}(t) = 1. \quad (A12)
\]

Those relations (3.12), (A3), (A10) and (A11) can be symbolized as:
\[
\tilde{\Sigma}_{aa,bb}(t) = \alpha_{aa,bb} + e^{-i\theta t} \beta_{aa,bb}, \quad (A13)
\]

where the values of \(\alpha\) and \(\beta\) can be read on the preceding equations. In a perturbation expansion, \(\alpha_{11,11}\) and \(\beta_{00,00}\) starts as 1 while the other elements behaves at least as \(V^2\).

The vacuum-vacuum elements of the super-operator \(\tilde{\Sigma}(t)\) for the value \(t = 0\) define the operator usually called \(\tilde{\mathcal{A}}\) in the subdynamics theory. The corresponding elements are
\[
\tilde{\mathcal{A}}_{aa,bb}(t) = \alpha_{aa,bb} + \beta_{aa,bb}, \quad (A14)
\]

or more explicitly
\[
\begin{align*}
\tilde{\mathcal{A}}_{11,11} &= \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} + |\tilde{\mathcal{A}}_1|^2 \frac{W_{11,11}(\tilde{\theta})}{\theta}, \\
\tilde{\mathcal{A}}_{00,00} &= \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} + |\tilde{\mathcal{A}}_1|^2 \frac{W_{00,00}(\tilde{\theta})}{\theta}, \\
\tilde{\mathcal{A}}_{11,00} &= \frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} - |\tilde{\mathcal{A}}_1|^2 \frac{W_{00,00}(\tilde{\theta})}{\theta}, \\
\tilde{\mathcal{A}}_{00,11} &= \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} - |\tilde{\mathcal{A}}_1|^2 \frac{W_{11,11}(\tilde{\theta})}{\theta}. \quad (A15)
\end{align*}
\]

The corresponding elements of the inverse \(\tilde{\mathcal{A}}^{-1}\) of the \(\tilde{\mathcal{A}}\) operator can already be computed, independently of the elements of the subdynamics superoperator involving the field by the inversion of a two by two matrix. The determinant \(\mathcal{A}_D\) of that matrix is
\[
\mathcal{A}_D = \tilde{\mathcal{A}}_{11,11}\tilde{\mathcal{A}}_{00,00} - \tilde{\mathcal{A}}_{00,11}\tilde{\mathcal{A}}_{11,00} = |\tilde{\mathcal{A}}_1|^2, \quad (A16)
\]

as can be shown by direct computation using (3.11). We have therefore directly
\[
\left(\tilde{\mathcal{A}}^{-1}\right)_{11,11} = \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} |\tilde{\mathcal{A}}_1|^{-2} + \frac{W_{00,00}(\tilde{\theta})}{\theta},
\]

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\[
\begin{align*}
(\tilde{A}^{-1})_{00,00} &= \frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} |\bar{A}_1|^{-2} + \frac{W_{00,00}(\bar{\theta})}{\bar{\theta}}, \\
(\tilde{A}^{-1})_{11,00} &= -\frac{W_{00,00}(0)}{(W_{11,11}(0) + W_{00,00}(0))} |\bar{A}_1|^{-2} + \frac{W_{00,00}(\bar{\theta})}{\bar{\theta}}, \\
(\tilde{A}^{-1})_{00,11} &= -\frac{W_{11,11}(0)}{(W_{11,11}(0) + W_{00,00}(0))} |\bar{A}_1|^{-2} + \frac{W_{11,11}(\bar{\theta})}{\bar{\theta}}.
\end{align*}
\]

B The fourth order contribution to \( W_{00,00}(0) \)

We derive in this appendix the expression (5.2) for \( W^{(4)}_{00,00}(0) \). We have to consider all irreducible contributions leading from the diagonal state 00 to itself. We have 16 contributions to evaluate. Our convention to denote a matrix element <a|\(\bar{\rho}\)|b> = \(\bar{\rho}_{ab}\) of the (reduced) density operator \(\bar{\rho}\) is to write for a and b in the first place the state of the atom. This convention avoids the need of a separator when writing the indices \(ab\). Using that convention, the possible succession of correlated states to be considered to evaluate \(W_{00,00}\) from (C.10) can be described in the following way.

We first consider contributions that do not involve the connecting vertex \(\bar{L}_V\) and are included in \(\psi^{(4)}_{00,00}\). They correspond to the following successions that involve as last vertex (extreme left) a change of the first index:

\((00;1k0;0kk'0;0k0;00), (00;1k0;0kk'0;1k'0;00), (00;1k0;1k1k';1k0;00), (00;1k0;1k1k';01k';00)\). The corresponding contributions are called \( C_1(z) \), \( C_2(z) \), \( C_3(z) \), \( C_4(z) \). They are:

\[
C_1(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \frac{1}{z - \omega_k - \omega_k'},
\]

\[
C_2(z) = \sum_k |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \frac{1}{z - \omega_k - \omega_k'} \frac{1}{z - \omega_1 + \omega_0 - \omega_k'},
\]

\[
C_3(z) = \sum_k |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \frac{1}{z - \omega_1 + \omega_0 - \omega_k'} \frac{1}{z - \omega_k + \omega_k'},
\]

\[
C_4(z) = \sum_k |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \frac{1}{z - \omega_1 + \omega_0 - \omega_k'} \frac{1}{z - \omega_k + \omega_k'}.
\]
\[ C_4(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \frac{1}{z - \omega_1 + \omega_0 + \omega_{k'}} \times \frac{1}{z + \omega_1 - \omega_0 + \omega_{k'}}. \]  

(B.1)

The contributions involving as last vertex a change of the second index (such as (00;01k;00kk';01k;00)) can be obtained from these expressions by replacing in the propagators \( \omega \rightarrow -\omega \) for all values of the indices. The corresponding contributions are called \( C_1', C_2', C_3', C_4' \). The limit \( z \rightarrow 0 \) has to be considered to get the contributions to \( W_{00,00}^{(4)}(0) \). For the terms \( C_1 \) and \( C_2 \), the limit \( z \rightarrow 0 \) can be taken in a harmless way since no propagator can be resonant. Therefore, we have

\[ C_1(0) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \frac{1}{-\omega_1 + \omega_0 + \omega_k} \]

\[ C_2(0) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \frac{1}{-\omega_1 + \omega_0 + \omega_k}. \]

(B.2)

For obvious reasons, we have therefore:

\[ C_1(0) + C_1'(0) = 0 \quad C_2(0) + C_2'(0) = 0 \]  

(B.3)

We now turn to the contributions \( C_3 \) and \( C_4 \). The limit \( z \rightarrow 0 \) has to be taken carefully for the propagator \((z - \omega_k + \omega_{k'})^{-1}\) since it is resonant for \( \omega_k = \omega_{k'} \). Let us consider the sum \( C_{34} \) of \( C_3 \) and \( C_4 \) and take the limit \( z \rightarrow 0 \) whenever it is harmless:

\[ C_{34}(0) = \lim_{z \rightarrow 0} \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_k + \omega_{k'}} \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \times \left( \frac{1}{-\omega_1 + \omega_0 - \omega_k} + \frac{1}{\omega_1 - \omega_0 + \omega_{k'}} \right) \]

\[ = \lim_{z \rightarrow 0} \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_k + \omega_{k'}} \frac{1}{z - \omega_1 + \omega_0 - \omega_k} \times \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{\omega_1 - \omega_0 + \omega_{k'}} \]

\[ = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{\omega_1 - \omega_0 + \omega_{k'}} \]  

(B.4)
The limit $z \to 0$ has become harmless: the resonant factor plays no role since the fraction has a well defined limit. Therefore, we have anew the cancellation of $C_{34}(0)$ with $C_{3'4'}(0)$.

We now turn to the contributions that involve the connecting vertex $\bar{L'}_V$. They correspond to the following successions that involve as last vertex (extreme left) a change of the second index:

\[
(00;0k1;0k'0;1k0;00), (00;0k1;0kk'0;1k'0;00), (00;0k1;1k1k';1k0;00), (00;0k1;1k1k';01k';00).
\]

The corresponding contributions are called $C_5(z)$, $C_6(z)$, $C_7(z)$, $C_8(z)$. They are:

\[
C_5(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z + \omega_1 - \omega_0 - \omega_k - \omega_{k'}} \frac{1}{z + \omega_1 - \omega_0 - \omega_k - \omega_{k'}}
\]

\[
C_6(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z - \omega_1 + \omega_0 - \omega_k - \omega_{k'}} \frac{1}{z - \omega_1 + \omega_0 - \omega_k - \omega_{k'}}
\]

\[
C_7(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z + \omega_1 - \omega_0 - \omega_k + \omega_{k'}} \frac{1}{z + \omega_1 - \omega_0 - \omega_k + \omega_{k'}}
\]

\[
C_8(z) = \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z + \omega_1 - \omega_0 - \omega_k + \omega_{k'}} \frac{1}{z + \omega_1 - \omega_0 + \omega_k - \omega_{k'}}
\]

(B.5)

The contributions involving as last vertex a change of the second index can be obtained from those expression by replacing in the propagators $\omega \to -\omega$ for all values of the indices. The corresponding contributions are called $C_{5'}$, $C_{6'}$, $C_{7'}$, $C_{8'}$. With respect to $C_1(z)$, $C_2(z)$, $C_3(z)$, $C_4(z)$, we note that in $C_5(z)$, $C_6(z)$, $C_7(z)$, $C_8(z)$ the first propagator is resonant. Let us consider the sum $C_{56}(z)$ of $C_5(z)$ and $C_6(z)$ at the limit $z \to 0$. Anew, the limit is taken whenever it is harmless.

\[
C_{56}(0) = \lim_{z \to 0} \sum_k \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{z + \omega_1 - \omega_0 - \omega_k - \omega_{k'}} \frac{1}{z + \omega_1 - \omega_0 - \omega_k - \omega_{k'}}
\]
If we add the contribution $C_{5'6'}(0)$, we get:

\[
C_{56}(0) + C_{5'6'}(0) = \lim_{\zeta \to 0} \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \times \left( \frac{1}{\zeta + \omega_1 - \omega_0 - \omega_k} + \frac{1}{\zeta - \omega_1 + \omega_0 + \omega_k} \right) \frac{1}{-\omega_k - \omega_{k'}} \\
\times \left( \frac{1}{-\omega_1 + \omega_0 - \omega_k} + \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \right) = -2\pi i \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_1 - \omega_0 - \omega_k) \frac{1}{-\omega_k - \omega_{k'}} \\
\times \left( \frac{1}{-\omega_1 + \omega_0 - \omega_k} + \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \right) = -2\pi i \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_1 - \omega_0 - \omega_k) \frac{1}{-\omega_1 + \omega_0 - \omega_k} \\
\times \frac{1}{2(-\omega_1 + \omega_0)} = \lim_{\zeta \to 0} \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_1 - \omega_0 - \omega_k) \frac{1}{-\omega_1 + \omega_0 - \omega_k} \\
\times \frac{1}{2(-\omega_1 + \omega_0)}. \tag{B.7}
\]

We have used the Dirac delta function to simplify some propagators to get the last relation.

We now turn to the sum $C_{78}(z)$ of $C_7(z)$ and $C_8(z)$ at the limit $z \to 0$. Anew, the limit is taken whenever it is harmless.

\[
C_{78}(0) = \lim_{\zeta \to 0} \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{\zeta + \omega_1 - \omega_0 - \omega_k} \frac{1}{\zeta - \omega_1 + \omega_0 + \omega_k} \times \left( \frac{1}{-\omega_1 + \omega_0 - \omega_k} + \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \right) \\
= \lim_{\zeta \to 0} \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \frac{1}{\zeta + \omega_1 - \omega_0 - \omega_k} \frac{1}{\zeta + \omega_1 - \omega_0 - \omega_k} \times \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}} \\
\times \frac{1}{-\omega_1 + \omega_0 - \omega_k} \frac{1}{-\omega_1 + \omega_0 - \omega_{k'}}. \tag{B.8}
\]
If we add the contribution \( C_{78}(0) \), we get
\[
C_{78}(0) + C_{78'}(0) = \lim_{z \to 0} \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \\
\times \left( \frac{1}{z + \omega_{1} - \omega_{0} - \omega_{k}} + \frac{1}{z - \omega_{1} + \omega_{0} + \omega_{k}} \right) \\
\times \frac{1}{-\omega_{1} + \omega_{0} - \omega_{k}} \frac{1}{\omega_{1} - \omega_{0} + \omega_{k}} \\
= -2\pi i \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_{1} - \omega_{0} - \omega_{k}) \\
\times \frac{1}{-\omega_{1} + \omega_{0} - \omega_{k}} \frac{1}{\omega_{1} - \omega_{0} + \omega_{k}} \\
= -2\pi i \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_{1} - \omega_{0} - \omega_{k}) \\
\times \frac{1}{2(-\omega_{1} + \omega_{0}) \omega_{1} - \omega_{0} + \omega_{k}}.
\] (B.9)

If we combine the non-vanishing contributions, we are left with
\[
W_{00,00}^{(4)}(0) = -2\pi i \sum_{k} \sum_{k'} |V_{1|0k}|^2 |V_{1|0k'}|^2 \delta(\omega_{1} - \omega_{0} - \omega_{k}) \\
\times \frac{1}{(\omega_{1} - \omega_{0} + \omega_{k})^2}.
\] (B.10)

that is the looked-after expression. It is therefore manifest that \( W_{00,00}^{(4)}(0) \) does not vanish.

\section{C \ Element of \( \tilde{\Sigma} \) between off diagonal-off diagonal atomic states}

The matrix elements of the resolvent of \( R \) can also be expressed in terms of the irreducible collision operator \( \psi \): Only elements \( \psi_{a\bar{a},a\bar{a}} \) play a role while elements \( \psi_{a\bar{a},\bar{a}a} \) and \( \psi_{a\bar{a},\bar{a}a} \) vanish. We have the alternative forms with respect to (4.1):
\[
R_{a\bar{a},a\bar{a}}(z) = \frac{1}{z - \omega_{a} + \omega_{\bar{a}} - \psi_{a\bar{a},a\bar{a}}(z)}.
\] (C.1)
The analytical properties of $\psi_{a\bar{a},a\bar{a}}(z)$ are determined by comparing the two expressions of $R_{a\bar{a},a\bar{a}}(z)$ and are directly connected with the analytical properties of the resolvents of $H$ and $-H$.

For the reduced formalism, the role of $\psi$ is fulfilled by the operator $W$ (2.19). The elements of $R(z)$ are expressed in terms of these of $R(z)$ and $T(z)$

\[
R_{a\bar{a},a\bar{a}}(z) = R_{a\bar{a},a\bar{a}} + R_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}R_{a\bar{a},a\bar{a}} + R_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}R_{a\bar{a},a\bar{a}},
\]

\[
R_{a\bar{a},a\bar{a}}(z) = R_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}R_{a\bar{a},a\bar{a}} + R_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}R_{a\bar{a},a\bar{a}},
\]

(2.20)

A little algebra enables to solve this system of equations, using the relation (2.21) for $W_{ab,cd}(z)$:

\[
R_{a\bar{a},a\bar{a}}(z) = \frac{z - \omega_a + \omega_a - W_{a\bar{a},a\bar{a}}}{(z - \omega_a + \omega_a - W_{a\bar{a},a\bar{a}})(z - \omega_a + \omega_a - W_{a\bar{a},a\bar{a}}) - T_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}},
\]

\[
R_{a\bar{a},a\bar{a}}(z) = \frac{T_{a\bar{a},a\bar{a}}}{(z - \omega_a + \omega_a - W_{a\bar{a},a\bar{a}})(z - \omega_a + \omega_a - W_{a\bar{a},a\bar{a}}) - T_{a\bar{a},a\bar{a}}T_{a\bar{a},a\bar{a}}},
\]

(C.2)

This form shows that all elements share poles due to the common denominator (we are interested only to these poles that go either to $\omega_1 - \omega_0$ or $\omega_0 - \omega_1$ as $V^2 \to 0$, and not to the singular points arising from the numerators of (C.3)). The two off-diagonal elements ($R_{a\bar{a},a\bar{a}}$ for $a = 1$ and $a = 0$) are also linked with the diagonal ones by (cf. (A5))

\[(z - \omega_a + \omega_a - \psi_{a\bar{a},a\bar{a}})R_{a\bar{a},a\bar{a}} = -(z - \omega_a + \omega_a - \psi_{a\bar{a},a\bar{a}})R_{a\bar{a},a\bar{a}} + 1 \quad \text{(C.4)}\]

These relations can be interpreted by considering the last apparition, in $R_{a\bar{a},a\bar{a}}$, of an element of $\bar{L}'_V$, leading to a state without field particle (an odd number of $\bar{L}'_V$ vertex has to be present), and its replacement by an element of $L'_V$, leading to an even number of vertex $\bar{L}'_V$. The new expression can be placed in relation with contributions to $R_{a\bar{a},a\bar{a}}$. A supplementary contribution of $R_{a\bar{a},a\bar{a}}$ has to be treated separately: Those relations can be checked directly from (C.3).

If we take into account the vanishing of the off diagonal elements of $\psi$ to be able to replace in (C.3) the remaining $T$'s by $W$, the common denominator $D(z)$ in (C.3) can be written as:

\[
D(z) = z^2 - z(W_{10,10} + W_{01,01}) - (\omega_1 - \omega_0)^2 + (\omega_1 - \omega_0)(W_{01,01} - W_{10,10}) + W_{10,10}W_{01,01} - W_{10,01}W_{01,10} \quad \text{(C.5)}
\]
Using (3.1-3.2) the denominator $D(z)$ takes a simpler form:

$$D(z) = z^2 - z(W_{10.10} + W_{01.01}) - (\omega_1 - \omega_0)^2 + (\omega_1 - \omega_0)(W_{01.01} - W_{10.10}) \quad (C.6)$$

The analysis of the poles of the diagonal-diagonal elements in Ref. [19] can be repeated here for the elements relative to the dipole moments: $D^{-1}(z)$ has the two poles for the same values of $z$ for which the elements $R_{a\bar{a},\bar{a}a}(z)$ (4.1), for $a = 1$ and $a = 0$, are singular, i.e. for the values $z = \omega_1 + \zeta - \omega_0 + \delta \equiv \delta_{10}$ and $z = \omega_0 + \delta - \omega_1 + \zeta \equiv \delta_{01}$. We will note respectively by $A_{10}$ and $A_{01}$ the residue of $D^{-1}(z)$ at those poles, with the obvious property $A_{10} = A_{01}^*$. The difference with the unreduced formalism is that those residues are no longer given by simple products of $A_1, A_0$ defined from the Green’s functions associated with the Hamiltonian. The elements of $\tilde{R}(z)$ can now be written more explicitly as

$$\tilde{R}_{a\bar{a},\bar{a}a}(z) = \frac{z - \omega_\bar{a} + \omega_a - W_{\bar{a}a,\bar{a}a}(z)}{D(z)}$$
$$\tilde{R}_{a\bar{a},a\bar{a}}(z) = \frac{W_{\bar{a}a,a\bar{a}}(z)}{D(z)} = -\frac{W_{a\bar{a},a\bar{a}}(z)}{D(z)}, \quad (C.7)$$

from which the elements of $\tilde{\Sigma}(t)$ can now be computed. We have formally

$$\tilde{\Sigma}_{10.10}(t) = e^{-i\delta_{10}t} A_{10} (2\omega_1 + \zeta - 2\omega_0 - \delta - W_{01.01}(\omega_1 + \zeta - \omega_0 + \delta)) + e^{-i\delta_{01}t} A_{01} (-\zeta + \delta - W_{01.01}(-\omega_1 - \zeta + \omega_0 + \delta)),$$
$$\tilde{\Sigma}_{01.01}(t) = e^{-i\delta_{10}t} A_{01} (-2\omega_1 - \zeta + 2\omega_0 + \delta - W_{10.10}(\omega_1 - \zeta + \omega_0 + \delta)) + e^{-i\delta_{01}t} A_{10} (\zeta - \delta - W_{10.10}(\omega_1 + \zeta - \omega_0 + \delta)),$$
$$\tilde{\Sigma}_{01.10}(t) = -e^{-i\delta_{10}t} A_{10} W_{10.10}(\omega_1 + \zeta - \omega_0 + \delta) - e^{-i\delta_{01}t} A_{01} W_{10.10}(-\omega_1 - \zeta + \omega_0 + \delta),$$
$$\tilde{\Sigma}_{10.01}(t) = -e^{-i\delta_{01}t} A_{01} W_{01.01}(-\omega_1 - \zeta + \omega_0 + \delta) - e^{-i\delta_{10}t} A_{10} W_{01.01}(\omega_1 + \zeta - \omega_0 + \delta). \quad (C.8)$$

Those relations are symbolized in the main text as:

$$\tilde{\Sigma}_{a\bar{a},a\bar{a}}(t) = e^{-i\delta_{10}t} \alpha_{a\bar{a},a\bar{a}} + e^{-i\delta_{01}t} \beta_{a\bar{a},a\bar{a}},$$
$$\tilde{\Sigma}_{a\bar{a},a\bar{a}}(t) = e^{-i\delta_{10}t} \alpha_{a\bar{a},a\bar{a}} + e^{-i\delta_{01}t} \beta_{a\bar{a},a\bar{a}}. \quad (C.9)$$

where the values of $\alpha$ and $\beta$ can be read on the preceding equations (C.8).
\( \tilde{A} \) is evaluated from the vacuum-vacuum elements of the operator \( \tilde{\Sigma}(t) \) for the value \( t = 0 \). The corresponding elements are:

\[
\tilde{A}_{a\bar{a},a\bar{a}} = \alpha_{a\bar{a},a\bar{a}} + \beta_{a\bar{a},a\bar{a}}, \quad \tilde{A}_{\bar{a}a,a\bar{a}} = \alpha_{\bar{a}a,a\bar{a}} + \beta_{\bar{a}a,a\bar{a}}, \quad (\text{C.10})
\]

The elements of the inverse \( \tilde{A}^{-1} \) of the \( \tilde{A} \) operator can also be computed. This requires the inversion of a two by two matrix. The determinant \( |\tilde{A}_D| \) of that matrix is:

\[
|\tilde{A}_D| = (\alpha_{10.10} + \beta_{10.10})(\alpha_{01.01} + \beta_{01.01}) - (\alpha_{01.10} + \beta_{01.10})(\alpha_{10.01} + \beta_{10.01}). \quad (\text{C.11})
\]

We can compute partial contributions from \( (\text{C.8}) \) and obtain

\[
\alpha_{10.10}\alpha_{01.01} - \alpha_{10.01}\alpha_{01.10} = \mathcal{A}_{10}^2 \left[ 2(\omega_1 - \omega_0)(\zeta - \bar{\delta}) - 2(\omega_1 - \omega_0)(\omega_1 + \zeta - \omega_0 + \bar{\delta}) + (\zeta - \bar{\delta})^2 \right. \\
\left. - (\zeta - \bar{\delta})(W_{01.01}(\omega_1 + \zeta - \omega_0 + \bar{\delta}) + W_{10.10}(\omega_1 + \zeta - \omega_0 + \bar{\delta})) \right]. \quad (\text{C.12})
\]

We know that \( \delta_{10} \) satisfies the equation \( D(\delta_{10}) = 0 \), where \( D(z) \) is given by \( (\text{C.6}) \). Therefore, we have

\[
D(\delta_{10}) = \delta_{10}^2 - \delta_{10}(W_{10.10}(\delta_{10}) + W_{01.01}(\delta_{10})) - (\omega_1 - \omega_0)^2 \\
+ (\omega_1 - \omega_0)(W_{01.01}(\delta_{10}) - W_{10.10}(\delta_{10})) = 0. \quad (\text{C.13})
\]

Introducing the explicit value for \( \delta_{10} (\delta_{10} = \omega_1 + \zeta - \omega_0 + \bar{\delta}) \), this relation \( (\text{C.13}) \) becomes

\[
(\zeta - \bar{\delta})^2 + 2(\omega_1 - \omega_0)(\zeta - \bar{\delta}) - 2(\omega_1 - \omega_0)W_{10.10}(\delta_{10}) \\
+ (\zeta - \bar{\delta})(W_{10.10}(\delta_{10}) + W_{01.01}(\delta_{10})) = 0. \quad (\text{C.14})
\]

A direct comparison with \( (\text{C.12}) \) leads to the relation

\[
\alpha_{10.10}\alpha_{01.01} - \alpha_{10.01}\alpha_{01.10} = 0, \quad (\text{C.15})
\]

and a similar relation holds for the \( \beta \)'s.

\[
\beta_{10.10}\beta_{01.01} - \beta_{10.01}\beta_{01.10} = 0. \quad (\text{C.16})
\]

The determinant \( |\tilde{A}_D| \) takes the simplest form

\[
|\tilde{A}_D| = \alpha_{10.10}\beta_{01.01} + \alpha_{01.01}\beta_{10.10} - \alpha_{10.01}\beta_{10.01} - \alpha_{10.10}\beta_{10.10}. \quad (\text{C.17})
\]
A little algebra provides the explicit result

\[
|\tilde{A}_D| = A_{10,01} \left[ (2\omega_1 + \zeta - 2\omega_0 - \delta) (-2\omega_1 - \bar{\zeta} + 2\omega_0 + \delta) \\
+ (\zeta - \bar{\delta}) (-\bar{\zeta} + \delta) - 2(\omega_1 - \omega_0) (W_{10,10}(-\omega_1 - \bar{\zeta} + \omega_0 + \delta) \\
-W_{01,01}(\omega_1 + \zeta - \omega_0 + \delta)) \\
-(\zeta - \bar{\delta} - \bar{\zeta} - \delta) (W_{10,10}(-\omega_1 - \bar{\zeta} + \omega_0 + \delta) \\
+W_{01,01}(\omega_1 + \zeta - \omega_0 + \delta)) \right],
\]

(C.18)

In a perturbation expansion, \(|\tilde{A}_D|\) starts as \(-4(\omega_1 - \omega_0)^2\). The elements of the inverse \(\tilde{A}^{-1}\) are

\[
(\tilde{A}^{-1})_{aa.\bar{a}a} = \frac{1}{|\tilde{A}_D|}(\alpha_{\bar{a}a.aa} + \beta_{aa.\bar{a}a}),
\]

(\tilde{A}^{-1})_{\bar{a}a.aa} = -\frac{1}{|\tilde{A}_D|}(\alpha_{\bar{a}a.aa} + \beta_{aa.\bar{a}a}).

(C.19)

D Determination of \(\chi\)

Denoting by \(|\chi|\) the determinant of the \(\chi\) matrix, we have from the inversion of a standard 2 \(\times\) 2 matrix

\[
(\chi^{-1})_{11,11} = \chi_{00,00} \frac{1}{|\chi|}, \\
(\chi^{-1})_{00,00} = \chi_{11,11} \frac{1}{|\chi|}, \\
(\chi^{-1})_{00,11} = -\chi_{00,11} \frac{1}{|\chi|}, \\
(\chi^{-1})_{11,00} = -\chi_{11,00} \frac{1}{|\chi|}.
\]

(D.1)

Using the explicit value for the elements of \(\tilde{\Theta}\), we get for the diagonal and off diagonal elements of the relation (\(a = b\) and \(a \neq b\)):

\[
\chi_{11,11}(\chi^{-1})_{11,11} - \chi_{11,00}(\chi^{-1})_{11,11} = \frac{W_{11,11}(0)}{W_{11,11}(0) + W_{00,00}(0)}, \\
\chi_{00,11}(\chi^{-1})_{11,00} - \chi_{00,00}(\chi^{-1})_{11,00} = \frac{W_{00,00}(0)}{W_{11,11}(0) + W_{00,00}(0)},
\]

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\[ \chi_{11.11} (\chi^{-1})_{11.00} - \chi_{11.00} (\chi^{-1})_{11.00} = -\frac{W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}, \]
\[ \chi_{00.11} (\chi^{-1})_{11.11} - \chi_{00.00} (\chi^{-1})_{11.11} = -\frac{W_{11.11}(0)}{W_{11.11}(0) + W_{00.00}(0)}. \]

(\text{D.2})

Introducing the form (\text{D.1}) for the inverse of \( \chi \) into these relations (\text{D.2}), we get

\[ \chi_{11.11} \chi_{00.00} - \chi_{11.00} \chi_{00.00} = |\chi| \frac{W_{11.11}(0)}{W_{11.11}(0) + W_{00.00}(0)}. \]
\[ -\chi_{00.11} \chi_{11.00} + \chi_{00.00} \chi_{11.00} = |\chi| \frac{W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}, \]
\[ -\chi_{11.11} \chi_{11.00} + \chi_{11.00} \chi_{11.00} = -|\chi| \frac{W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}, \]
\[ \chi_{00.11} \chi_{00.00} - \chi_{00.00} \chi_{00.00} = -|\chi| \frac{W_{11.11}(0)}{W_{11.11}(0) + W_{00.00}(0)}. \]

(\text{D.3})

Introducing the form (\text{D.1}) for the inverse of \( \chi \) in (5.8), we get

\[ \chi_{00.00} - \chi_{00.11} = |\chi|, \quad \chi_{11.11} - \chi_{11.00} = |\chi|, \]
\[ \chi_{11.11} + \chi_{00.11} = 1, \quad \chi_{00.00} + \chi_{11.00} = 1. \]

(\text{D.4})

These relations enable to express all elements of \( \chi \) in terms of \( |\chi| \):

\[ \chi_{11.11} = \chi_{00.00} = \frac{1}{2}(1 + |\chi|) \]
\[ \chi_{11.00} = \chi_{00.11} = \frac{1}{2}(1 - |\chi|). \]

(\text{D.5})

We have indeed:

\[ \chi_{11.11} \chi_{00.00} - \chi_{11.00} \chi_{00.00} = \frac{1}{4}(1 + |\chi|)^2 - \frac{1}{4}(1 - |\chi|)^2 = |\chi|. \]

(\text{D.6})

Introducing those values (\text{D.5}) into (\text{D.3}), we get two relations (the third and fourth relations in (\text{D.3}) provide trivially the same relation as the first two ones)

\[ (1 + |\chi|)^2 - (1 + |\chi|)(1 - |\chi|) = 4|\chi| \frac{W_{11.11}(0)}{W_{11.11}(0) + W_{00.00}(0)}, \]
\[ -(1 - |\chi|)^2 + (1 + |\chi|)(1 - |\chi|) = 4|\chi| \frac{W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}. \]

(\text{D.7})
Let us add and subtract the two relations (D.7). We then get:

\[(1 + |\chi|)^2 - (1 - |\chi|)^2 = 4|\chi|,\]
\[(1 + |\chi|)^2 + (1 - |\chi|)^2 - 2(1 + |\chi|)(1 - |\chi|) = 4|\chi| \frac{W_{11.11}(0) - W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}.\]  

(D.8)

The first relation is trivially satisfied and we are left with the condition

\[4|\chi|^2 = 4|\chi| \frac{W_{11.11}(0) - W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)},\]  

(D.9)

therefore

\[|\chi| = \frac{W_{11.11}(0) - W_{00.00}(0)}{W_{11.11}(0) + W_{00.00}(0)}.\]  

(D.10)

From that expression, we get the explicit values for the first serie of elements of \(\chi\) reported in §5.

We now turn to the other elements \(\chi_{a\bar{a},b\bar{b}}\). In the expression \(\chi\exp(-i\Phi t)\chi^{-1}\) is diagonal and we get

\[
\left(e^{-i\hat{\Phi}t}\right)_{a\bar{a},c\bar{c}} = \sum_b \chi_{a\bar{a},b\bar{b}} e^{-i\delta_{b\bar{b}}t} (\chi^{-1})_{b\bar{b},c\bar{c}}.
\]  

(D.11)

Direct comparison between (D.11) and (4.4) provides us the conditions:

\[
\chi_{a\bar{a}.10} (\chi^{-1})_{10.c\bar{c}} = \sum_b \alpha_{a\bar{a}.b\bar{b}} \left(\chi^{-1}\right)_{b\bar{b},c\bar{c}},
\]
\[
\chi_{a\bar{a}.01} (\chi^{-1})_{01.c\bar{c}} = \sum_b \beta_{a\bar{a}.b\bar{b}} \left(\chi^{-1}\right)_{b\bar{b},c\bar{c}}.
\]  

(D.12)

We introduce the value of the inverse of the operators \(\chi\) and \(\hat{A}\). The form of the inverse depends on their diagonal or off-diagonal character and the two possible cases have to be distinguished

\[
\chi_{a\bar{a}.10} \chi_{01.01} = \frac{|\chi|}{|A_D|} \left(\alpha_{a\bar{a}.10}(\alpha_{01.01} + \beta_{01.01}) - \alpha_{a\bar{a}.01}(\alpha_{01.10} + \beta_{01.10})\right),
\]
\[
\chi_{a\bar{a}.10} \chi_{10.01} = -\frac{|\chi|}{|A_D|} \left(-\alpha_{a\bar{a}.10}(\alpha_{10.01} + \beta_{10.01}) + \alpha_{a\bar{a}.01}(\alpha_{10.10} + \beta_{10.10})\right),
\]
\[
\chi_{a\bar{a}.01} \chi_{01.10} = -\frac{|\chi|}{|A_D|} \left(\beta_{a\bar{a}.10}(\alpha_{01.01} + \beta_{01.01}) - \beta_{a\bar{a}.01}(\alpha_{01.10} + \beta_{01.10})\right),
\]
\[
\chi_{a\bar{a}.01} \chi_{10.10} = -\frac{|\chi|}{|A_D|} \left(-\beta_{a\bar{a}.10}(\alpha_{10.01} + \beta_{10.01}) + \beta_{a\bar{a}.01}(\alpha_{10.10} + \beta_{10.10})\right).
\]
\[ \chi_{a.01} \chi_{10.10} = \frac{|\chi|}{|\tilde{A}_D|} (-\beta_{a.10}(\alpha_{10.01} + \beta_{10.01}) + \beta_{a.01}(\alpha_{10.10} + \beta_{10.10})). \]

(D.13)

The last two equations provide us the same condition as the first two ones, if we take into account the relations (C.15-C.16). The first two conditions (D.13) provide, for \( a = 1 \) and \( a = 0 \), four relations that are simplified using the relations (C.15-C.16)

\[
\begin{align*}
\chi_{10.10} \chi_{01.01} &= \frac{|\chi|}{|\tilde{A}_D|} (\alpha_{10.10} \beta_{01.01} - \alpha_{10.01} \beta_{01.10}), \\
\chi_{01.10} \chi_{01.01} &= \frac{|\chi|}{|\tilde{A}_D|} (\alpha_{01.10} \beta_{01.01} - \alpha_{01.01} \beta_{01.10}), \\
\chi_{10.10} \chi_{10.10} &= -\frac{|\chi|}{|\tilde{A}_D|} (-\alpha_{10.10} \beta_{10.01} + \alpha_{10.01} \beta_{10.10}), \\
\chi_{01.10} \chi_{10.10} &= -\frac{|\chi|}{|\tilde{A}_D|} (-\alpha_{01.10} \beta_{10.01} + \alpha_{01.01} \beta_{10.10}).
\end{align*}
\]

(D.14)

Hermiticity of \( \tilde{\rho}_{a}(t) \) implies the obvious property \( \tilde{\rho}_{10}(t) = \tilde{\rho}_{01}^{*}(t) \) that we impose also on the matrix \( \tilde{\rho}_{a}^{P} \) and the following conditions have to hold:

\[
\begin{align*}
\chi_{10.10} &= \chi_{01.01}^{*}, & \chi_{01.10} &= \chi_{10.01}^{*}. 
\end{align*}
\]

(D.15)

Therefore, we are looking for a solution to (D.14) under a form that incorporates these conditions:

\[
\begin{align*}
\chi_{10.10} &= xe^{i\varphi}, & \chi_{01.01} &= xe^{-i\varphi}, & \chi_{10.10} &= ye^{i\psi}, & \chi_{01.10} &= ye^{-i\psi},
\end{align*}
\]

(D.16)

and we have \( |\chi| = x^2 - y^2 \). Introducing those values into (D.14), we get:

\[
\begin{align*}
x^2 &= \frac{x^2 - y^2}{|\tilde{A}_D|} (\alpha_{10.10} \beta_{01.01} - \alpha_{10.01} \beta_{01.10}), \\
xye^{i(\varphi-\psi)} &= \frac{x^2 - y^2}{|\tilde{A}_D|} (\alpha_{01.10} \beta_{01.01} - \alpha_{01.01} \beta_{01.10}), \\
xye^{-i(\varphi-\psi)} &= -\frac{x^2 - y^2}{|\tilde{A}_D|} (-\alpha_{10.10} \beta_{10.01} + \alpha_{10.01} \beta_{10.10}), \\
y^2 &= -\frac{x^2 - y^2}{|\tilde{A}_D|} (-\alpha_{01.10} \beta_{10.01} + \alpha_{01.01} \beta_{10.10}).
\end{align*}
\]

(D.17)
The first and fourth relations (D.17) are obviously identical when the expression of $|\tilde{A}_D|$ from (C.17) is taken into account: their subtraction provides an identity. The second and the third ones are complex conjugate of each other, and provide the value of the phase difference ($\varphi - \psi$). The first relation leads to

$$y^2 (\alpha_{10.10} \beta_{01.01} - \alpha_{01.01} \beta_{10.10}) = -x^2 (\alpha_{01.01} \beta_{10.10} - \alpha_{10.10} \beta_{01.01}) \quad (D.18)$$

and determines $y$ in function of $x$. A further relation is needed to fix the value of $x$. A remaining indetermination is not new inside the context of subdynamics. It does not affect the evolution equations but concerns the relation between $\tilde{\rho}^P$ and $\tilde{\rho}$. We can connect, for instance, the value of $|\chi|$ with that of $|\tilde{A}_D|$ in a “usual way”: $|\chi|^2 = |\tilde{A}_D|$ or introduce a criterion that ensures the positivity of the density operator. We will not elaborate here further on this point.

From the point of view of a perturbation expansion, the elements $x$, $y$, $\alpha_{10.10}$ and $\beta_{01.01}$ take the value 1 for $V \to 0$, while the other elements behave as $V^2$.

The relation (D.18) can be made more explicit by replacing the $\alpha$’s and the $\beta$’s by their formal value.

$$x^2 \left[(\zeta - \delta)(\zeta - \bar{\delta}) - (\zeta - \bar{\delta})W_{01.01}(\delta_{01}) - (\zeta - \delta)W_{10.10}(\delta_{10}) \right] = -y^2 \left[(2\omega_1 + \zeta - 2\omega_0 - \bar{\delta})(-2\omega_1 + \bar{\zeta} + 2\omega_0 - \delta)\right]$$

$$(2\omega_1 + \zeta - 2\omega_0 - \bar{\delta})W_{10.10}(\delta_{01}) - (-2\omega_1 + \bar{\zeta} + 2\omega_0 - \delta)W_{01.01}(\delta_{10}) \right]. \quad (D.19)$$

**E The presence of an incident field**

**E.1 A passive incident field**

To evaluate (6.1), the matrix elements of the resolvent $\tilde{R}(z)$ are required in terms of irreductible operators, starting from the perturbation expansion

$$\tilde{R}_{\alpha\lambda \beta \gamma \lambda \delta}(z) = \sum_{n=0}^{\infty} \left( \tilde{R}(z) \left[ \tilde{L} \tilde{R}(z) \right]^{n} \right)_{\alpha\lambda \beta \gamma \lambda \delta} \quad (E.1)$$

and a similar expression for $\tilde{R}_{ab \lambda \cdots \lambda}(z)$. 48
The irreductible operators \( W_{a\lambda b.c\lambda d} \) and \( W_{ab\lambda.cd\lambda} \) for the complete liouvil- 
villian \( \tilde{L} \) are defined through (2.19):

\[
W_{a\lambda b.c\lambda d}(z) = \sum_{n=0}^{\infty} \left( \left[ \tilde{L}_V \tilde{\tilde{R}}^0(z) \right]^n \tilde{L}_V \right)_{a\lambda b.c\lambda d(\text{irr})} \quad (E.2)
\]

\[
W_{ab\lambda.cd\lambda}(z) = \sum_{n=0}^{\infty} \left( \left[ \tilde{L}_V \tilde{\tilde{R}}^0(z) \right]^n \tilde{L}_V \right)_{ab\lambda.cd\lambda(\text{irr})} \quad (E.3)
\]

The irreductibility condition holds with respect to the vacuum chosen, namely the states involving the atom and the incident or emitted photons.

Since the field line \( \lambda \) is not involved in any vertex and plays no role in determining the irreductibility condition, we have the obvious property:

\[
W_{a\lambda b.c\lambda d}(z) = W_{ab.cd}(z - \omega_\lambda), \quad W_{ab\lambda.cd\lambda}(z) = W_{ab.cd}(z + \omega_\lambda). \quad (E.4)
\]

Those elements enable to write a compact form for the relevant elements of the resolvent \( \tilde{R} \):

\[
\tilde{R}_{a\lambda b.c\lambda d}(z) = \tilde{R}_{ab.cd}(z - \omega_\lambda), \quad \tilde{R}_{ab\lambda.cd\lambda}(z) = \tilde{R}_{ab.cd}(z + \omega_\lambda) \quad (E.5)
\]

The poles that have been considered for the computation of the atomic part of \( \tilde{\Sigma}(t) \) are the poles at \( z = 0, z = \theta, z = \delta_{10}, z = \delta_{01} \) that are present in \( \tilde{R}_{ab.cd}(z) \). These poles are now shifted in (6.6) by \( \pm \omega_\lambda \) and their residue is the same. We use the notations previously introduced:

\[
\tilde{\Sigma}_{a\lambda a.b\lambda b}(t) = e^{-i\omega_\lambda t} \alpha_{aa.bb} + e^{-i(\bar{\theta} + \omega_\lambda)t} \beta_{aa.bb} \quad (E.6)
\]

The values of the \( \alpha \)'s and the \( \beta \)'s can be obtained by identification with formulae (3.12), (A3), (A8), (A9) or they can be read in the expression (A15) for \( A \) in §3.

For the off-diagonal elements, we have

\[
\tilde{\Sigma}_{a\lambda a.b\lambda b}(t) = e^{-i(\delta_{10} + \omega_\lambda)t} \alpha_{a\bar{a}.b\bar{b}} + e^{-i(\delta_{01} + \omega_\lambda)t} \beta_{a\bar{a}.b\bar{b}} \quad (E.7)
\]

In a similar way, we have

\[
\tilde{\Sigma}_{aa\lambda.bb\lambda}(t) = e^{i\omega_\lambda t} \alpha_{aa.bb} + e^{-i(\bar{\theta} - \omega_\lambda)t} \beta_{aa.bb} \quad (E.8)
\]

\[
\tilde{\Sigma}_{a\bar{a}\lambda.\bar{b}\bar{b}\lambda}(t) = e^{-i(\delta_{10} - \omega_\lambda)t} \alpha_{a\bar{a}.b\bar{b}} + e^{-i(\delta_{01} - \omega_\lambda)t} \beta_{a\bar{a}.b\bar{b}} \quad (E.9)
\]
A direct identification is possible:

\[ \tilde{A}_{a\lambda,b,c\lambda} = \tilde{A}_{ab,cd} \quad \tilde{A}_{a\lambda,b,c\lambda} = \tilde{A}_{ab,cd} \]  
\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b\lambda} = \omega_\lambda \alpha_{aa,bb} + (\theta + \omega_\lambda) \beta_{aa,bb} \]
\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b\lambda} = (\delta_{10} + \omega_\lambda) \alpha_{aa,bb} + (\delta_{01} + \omega_\lambda) \beta_{aa,bb} \]
\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b\lambda} = -\omega_\lambda \alpha_{aa,bb} + (\theta - \omega_\lambda) \beta_{aa,bb} \]
\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b\lambda} = (\delta_{10} - \omega_\lambda) \alpha_{aa,bb} + (\delta_{01} - \omega_\lambda) \beta_{aa,bb} \]  
\[ (E.10) \]

The inverse \( \tilde{A}^{-1} \) of the operator \( \tilde{A} \) is known from the previous section thanks to the identification \((E.10)\). We have the obvious property (\( I_{ab,cd} \) is 1 when \( a = c \) and \( b = d \) and vanishes for the other possibilities)

\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b,c\lambda} = \omega_\lambda \tilde{A}_{ab,cd} + (\tilde{\Theta}\tilde{A})_{ab,cd} \]
\[ (\tilde{\Theta}\tilde{A})_{a\lambda,b,c\lambda} = -\omega_\lambda \tilde{A}_{ab,cd} + (\tilde{\Theta}\tilde{A})_{ab,cd} \]  
\[ (E.11) \]

from which the relations \((6.2)\) can be deduced.

**E.2 One absorbed incident field line**

The perturbation expansion of the resolvent \( \tilde{R}(z) \) can be written as

\[ \tilde{R}_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( \tilde{R}^0_{ab,cd} \left[ \tilde{L}_V \tilde{R}^0_{ab,cd} \right]^n \right) \]  
\[ (E.13) \]

A similar expression holds also for \( \tilde{R}_{ab,cd}(z) \) but only one kind of elements will be displayed in details. \( \tilde{R}_{ab,cd}(z) \) has to be expressed in terms of the irreducible operators \( W_{ab,cd}(z) \) defined in \((2.19)\) for the complete liouvillian \( \tilde{L} \):

\[ W_{ab,cd}(z) = \sum_{n=0}^{\infty} \left( \tilde{L}_V \tilde{R}^0_{ab,cd} \left[ \tilde{L}_V \tilde{R}^0_{ab,cd} \right]^n \right) \]  
\[ (E.14) \]

Those elements enable to write a compact form for the relevant elements of the resolvent \( \tilde{R} \):

\[ \tilde{R}_{ab,cd}(z) = \sum_{ef} \sum_{gh} \tilde{R}_{ab,ef}(z) W_{ef,gh}(z) \tilde{R}_{gh,cd}(z) \]  
\[ (E.15) \]
Since we have (the line λ is passive through the resolvent and the property (E.5) is used to get
\[ \tilde{R}_{ab,c\lambda d}(z) = \sum_{ef} \sum_{gh} \tilde{R}_{ab,ef}(z) W_{ef,g\lambda h}(z) \tilde{R}_{gh,cd}(z - \omega) \] (E.16)

As in §3, in absence of photons, \( \tilde{R}_{ab,ef}(z) = \tilde{R}_{ab,ef}(z) \) The elements of the resolvent in (E.16) are therefore the same one’s that have been studied previously.

The poles to be considered for the computation of \( \tilde{\Sigma}(t) \) are the poles at \( z = 0, z = \theta, z = \delta_{10}, z = \delta_{01} \) that are present in \( \tilde{R}_{ab,ef}(z) \) and \( \tilde{R}_{gh,cd}(z) \). These poles are well defined, as we have seen in the preceding sections. Moreover, we have seen in Ref. [19] that it is possible to write the elements of \( \tilde{R} \) as a sum of expressions such that each of them contains only one relevant pole. The computation of the residue does not therefore present conceptual problems. The residues can also be evaluated directly from the expression (E.16).

\[ \tilde{\Sigma}_{aa,d\lambda d}(t) = \sum_{b,c} \alpha_{aa,bb} W_{bb,c\lambda d}(0) \tilde{R}_{ce,dd}(-\omega) \]
\[ + e^{-i\delta_{10}t} \sum_{b,c} \beta_{aa,bb} W_{bb,c\lambda d}(\theta) \tilde{R}_{ce,dd}(\theta - \omega) \]
\[ + e^{-i(\delta_{01} + \omega_{\lambda})t} \sum_{b,c} \tilde{R}_{aa,bb}(\delta_{01} + \omega_{\lambda}) W_{bb,c\lambda d}(\delta_{01} + \omega_{\lambda}) \alpha_{ce,dd} \]
\[ + e^{-i(\delta_{01} + \omega_{\lambda})t} \sum_{b,c} \tilde{R}_{aa,bb}(\delta_{01} + \omega_{\lambda}) W_{bb,c\lambda d}(\delta_{01} + \omega_{\lambda}) \beta_{ce,dd} \] (E.17)

Let us make the following comments for the correct computation of \( \tilde{R}_{ce,dd}(-\omega) \), \( \tilde{R}_{ce,dd}(\theta - \omega) \), \( \tilde{R}_{aa,bb}(\delta_{10} + \omega_{\lambda}) \) and the similar expressions. Displaying the two relevant poles and their residue, we can indeed write

\[ \tilde{R}_{ce,dd}(z - \omega_{\lambda}) = \frac{1}{z - \omega_{\lambda} - \delta_{10}} + \frac{1}{z - \omega_{\lambda} - \delta_{01}} + r_{ce,dd}(z - \omega_{\lambda}) \] (E.18)

where the remaining function \( r_{ce,dd}(z - \omega_{\lambda}) \) is regular. When we take the residue at the point \( z = 0 \) and \( z = \theta \), we merely replace \( z \) in that expression
by its corresponding value.

\[
\tilde{R}_{cc,dd}(-\omega \lambda) = \alpha_{cc,dd} \frac{1}{-\omega \lambda - \delta_{10}} + \beta_{cc,dd} \frac{1}{-\omega \lambda - \delta_{01}} + r_{cc,dd}(-\omega \lambda)
\]

\[
\tilde{R}_{cc,dd}(\bar{\theta} - \omega \lambda) = \alpha_{cc,dd} \frac{1}{\bar{\theta} - \omega \lambda - \delta_{10}} + \beta_{cc,dd} \frac{1}{\bar{\theta} - \omega \lambda - \delta_{01}} + r_{cc,dd}(\bar{\theta} - \omega \lambda)
\]  

(E.19)

Therefore, we do not have to consider a possible deferred analytically continuation with respect to the integration variable \(\omega \lambda\). For consistency, when taking the residue at \(z = \delta_{10} + \omega \lambda\), we first write \(\tilde{R}_{aa,bb}(z)\) as:

\[
\tilde{R}_{aa,bb}(z) = \alpha_{aa,bb} \frac{1}{z} + \beta_{aa,bb} \frac{1}{z - \bar{\theta}} + r_{aa,bb}(z)
\]  

(E.20)

and we have:

\[
\tilde{R}_{aa,bb}(\delta_{10} + \omega \lambda) = \alpha_{aa,bb} \frac{1}{\delta_{10} + \omega \lambda}
\]  

(E.21)

We have therefore by direct identification from (E.17):

\[
\tilde{A}_{aa,d\lambda} = \sum_{b,c} \alpha_{aa,bb} W_{bb,cc}(0) \tilde{R}_{cc,dd}(-\omega \lambda)
\]  

(E.22)

\[
(\tilde{\Theta A})_{aa,d\lambda} = \bar{\theta} \sum_{b,c} \beta_{aa,bb} W_{bb,cc}(\bar{\theta}) \tilde{R}_{cc,dd}(\bar{\theta} - \omega \lambda)
\]  

\[
+ (\delta_{10} + \omega \lambda) \sum_{b,c} \tilde{R}_{aa,bb}(\delta_{10} + \omega \lambda) W_{bb,cc}(\delta_{10} + \omega \lambda) \alpha_{cc,dd}
\]  

\[
+ (\delta_{01} + \omega \lambda) \sum_{b,c} \tilde{R}_{aa,bb}(\delta_{01} + \omega \lambda) W_{bb,cc}(\delta_{01} + \omega \lambda) \beta_{cc,dd}
\]  

(E.23)
The inverse \( \tilde{A}^{-1} \) of the operator \( \tilde{A} \) can be computed easily. From \( \tilde{A} \tilde{A}^{-1} = I \)
we get
\[
0 = (\tilde{A} \tilde{A}^{-1})_{aa, \lambda \lambda} = \sum_{b} \tilde{A}_{aa, bb}(\tilde{A}^{-1})_{bb, \lambda \lambda} + \sum_{c} \tilde{A}_{aa, c \lambda \lambda}(\tilde{A}^{-1})_{c \lambda \lambda, \lambda \lambda} \tag{E.24}
\]
Since we have obviously (the field line \( \lambda \) is purely passive and that case has
been treated in the previous subsection):
\[
(\tilde{A}^{-1})_{c \lambda \lambda, \lambda \lambda} = (\tilde{A}^{-1})_{c \lambda \lambda, \lambda \lambda} \tag{E.25}
\]
and since \( \tilde{A}_{c \lambda \lambda, \lambda \lambda} \) are known from §4 while \( \tilde{A}_{aa, bb} \), \( (\tilde{A}^{-1})_{aa, bb} \) are
known from §3, we have
\[
(\tilde{A}^{-1})_{aa, \lambda \lambda} = -\sum_{b, c} (\tilde{A}^{-1})_{aa, bb}\tilde{A}_{bb, c \lambda \lambda}(\tilde{A}^{-1})_{c \lambda \lambda, \lambda \lambda} \tag{E.26}
\]
Those expressions enable the computation of the following elements of \( \Theta \):
\( \tilde{\Theta}_{aa, b \lambda \lambda} \), \( \tilde{\Theta}_{b \lambda, a \lambda \lambda} \), \( \tilde{\Theta}_{aa, b b \lambda} \), \( \tilde{\Theta}_{b b, a a \lambda} \). We have indeed:
\[
\tilde{\Theta}_{aa, b \lambda \lambda} = \left((\tilde{\Theta} \tilde{A})^{-1}\right)_{aa, b \lambda \lambda} = \sum_{c} (\tilde{\Theta} \tilde{A})_{aa, cc}(\tilde{A}^{-1})_{cc, b \lambda \lambda} + \sum_{c} (\tilde{\Theta} \tilde{A})_{aa, c \lambda \lambda}(\tilde{A}^{-1})_{c \lambda \lambda, b \lambda \lambda} \tag{E.27}
\]
Using the expressions \( \text{(E.24)} \) and \( \text{(E.26)} \) for \( (\tilde{A}^{-1})_{c \lambda \lambda, b \lambda \lambda} \) and \( (\tilde{A}^{-1})_{cc, b \lambda \lambda} \),
we get
\[
\tilde{\Theta}_{aa, b \lambda \lambda} = \sum_{c, d, e} (\tilde{\Theta} \tilde{A})_{aa, cc}(\tilde{A}^{-1})_{cc, dd}\tilde{A}_{dd, e \lambda \lambda}(\tilde{A}^{-1})_{e \lambda \lambda, b \lambda \lambda} + \sum_{c} (\tilde{\Theta} \tilde{A})_{aa, c \lambda \lambda}(\tilde{A}^{-1})_{c \lambda \lambda, e \lambda \lambda} \tag{E.28}
\]
The last result is reproduced in \( \text{(6.6)} \).

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