A NEW FIXED POINT APPROACH TO HYPERSTABILITY OF RADICAL-TYPE FUNCTIONAL EQUATIONS IN QUASI-(2, β)-BANACH SPACES

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Abstract. The main focus of this paper is to define the notion of quasi-(2, β)-Banach space and show some properties in this new space, by help of it and under some natural assumptions, we prove that the fixed point theorem [16, Theorem 2.1] is still valid in the setting of quasi-(2, β)-Banach spaces, this is also an extension of the fixed point result of Brzdek et al. [12, Theorem 1] in 2-Banach spaces to quasi-(2, β)-Banach spaces. In the next part, we give a general solution of the radical-type functional equation (1.2). In addition, we study the hyperstability results for these functional equation by applying the aforementioned fixed point theorem, and at the end of this paper we will derive some consequences.

1. Introduction

Let $E$ and $F$ be linear spaces over a real or complex scalar field $K$. We recall that a function $g : E \rightarrow F$ satisfies the general quadratic equation if

$$g(ax + by) + g(ax - by) = cg(x) + dg(y), \quad x, y \in E,$$

where $a, b, c, d \in K \setminus \{0\}$ are fixed numbers. We see that for $c = d = a^2 + b^2$ in (1.1) we get the Euler-Lagrange functional equation investigated by J.M. Rassias [42, 41] (see also [38]), while the quadratic functional equation corresponds to $a = b = 1$ and $c = d = 2$.

In the sequel, $\mathbb{N}$, $\mathbb{R}$ and $K$ denote the sets of all positive integers, real numbers and the field of real or complex numbers, respectively. Also, $W^V$ denotes the set of all functions from a set $V \neq \emptyset$ to a set $W \neq \emptyset$. We put $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

First of all, let us recall the history in the stability theory for functional equations. The story of the stability of functional equations dates back to 1925 when a stability result appeared in the celebrated book by Pólya and Szeg [39]. In 1940, Ulam [43] posed the famous Ulam stability problem which was partially solved by Hyers [27] in the framework of Banach spaces. Later, Hyers’ result was extended by Aoki [5] and next by Rassias [40]. Since then numerous papers on this subject have been published and we refer to [2, 14, 15, 19, 28, 32, 29, 35] for more details. On the other hand, fixed point theorems have been already applied in the theory of Hyers-Ulam stability by several authors (see for instance

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According to our best knowledge, the first hyperstability result was published in \[7\], and concerned ring homomorphisms. However, it seems that the term hyperstability was used for the first time in \[33\] (quite often it is confused with superstability, which admits also bounded functions). There are many researchers investigating the hyperstability results for functional equations in many areas (see, e.g., \[8, 9, 16, 21, 26\]).

The radical functional equation is one of the popular topics for investigating in the theory of stability. Nowadays, a lot of papers concerning the stability and the hyperstability of the radical functional equation in various spaces was appeared (see in \[3, 4, 17, 19, 20, 32\] and references therein).

In \[32\], Khodaei et al. solved the following functional equations

\[
\begin{align*}
 f \left( \sqrt[3]{ax^3 + by^3} \right) &= af(x) + bf(y), \quad x, y \in \mathbb{R}, \\
 f \left( \sqrt[3]{ax^3 + by^3} \right) + f \left( \sqrt[3]{ax^3 - by^3} \right) &= a^2 f(x) + b^2 f(y), \quad x, y \in \mathbb{R},
\end{align*}
\]

where \(a, b\) are fixed positive reals and proved generalized Ulam stability of these functional equations in 2-normed spaces.

In 2018, Dung et al. \[16\] extended the fixed point theorem of Brzdęk et al. \[11, Theorem 1\] in metric spaces to \(b\)-metric spaces, in particular to quasi-Banach spaces, and studied the hyperstability for the general linear equation in the setting of quasi-Banach spaces.

A function \(f : \mathbb{R} \to X\) is a solution of the radical-type functional equation if and only if

\[
\begin{align*}
 f \left( \sqrt[3]{ax^3 + by^3} \right) + f \left( \sqrt[3]{ax^3 - by^3} \right) &= cf(x) + df(y), \quad x, y \in \mathbb{R}, \quad (1.2)
\end{align*}
\]

where \(a, b, c\) and \(d\) are nonzero real constants.

We say that a function \(f : \mathbb{R} \to X\) fulfills the radical-type functional equation (1.2) on \(\mathbb{R}_0\) if and only if

\[
\begin{align*}
 f \left( \sqrt[3]{ax^3 + by^3} \right) + f \left( \sqrt[3]{ax^3 - by^3} \right) &= cf(x) + df(y), \quad x, y \in \mathbb{R}_0, \quad (1.3)
\end{align*}
\]

where \(\sqrt[3]{ax} \neq \pm \sqrt[3]{by}\) and \(a, b, c, d \in \mathbb{R}_0\).

The contents of the paper are as follows:

- In Sect. 2, we introduce a new space called \(\text{quasi-}(2, \beta)\)-Banach space and we investigate also some results about this space.
- In Sect. 3, we prove that fixed point theorem \[16, Theorem 2.1\] remains valid in the setting of quasi-\((2, \beta)\)-Banach space and we derive from it many particular cases.
- In Sect. 4, we achieve the general solution of the functional equation (1.2).
- In the last Sect. 5, we will apply our fixed point theorem to show the hyperstability results for the radical-type functional equation (1.2), and we finish this paper with some consequences.

It is well-known that the theory of 2-normed spaces was initially introduced by Gähler \[24, 25\] in the mid 1960s, and has been developed extensively in different subjects by others,
for example [22, 23, 37, 44]. Now, we recall by the definition of a \((2, \beta)\)-normed space and some preliminary results.

Let \(0 < \beta \leq 1\) be a fixed real number and let \(X\) be a linear space over \(\mathbb{K}\) with \(\text{dim } X \geq 2\). A function \(\|\cdot\|_\beta : X \times X \to \mathbb{R}_+\) is called a \((2, \beta)\)-norm on \(X\) if and only if it satisfies:

1. \(\|x, y\|_\beta = 0\) if and only if \(x\) and \(y\) are linearly dependent;
2. \(\|x, y\|_\beta = \|y, x\|_\beta\) for all \(x, y \in X\);
3. \(\|\lambda x, y\|_\beta = |\lambda|^\beta \|x, y\|_\beta\) for all \(x, y \in X\) and \(\lambda \in \mathbb{K}\);
4. \(\|x, y + z\|_\beta \leq \|x, y\|_\beta + \|x, z\|_\beta\) for all \(x, y, z \in X\).

The pair \((X, \|\cdot\|_\beta)\) is called a \((2, \beta)\)-normed space.

- If \(x \in X\) and \(\|x, y\|_\beta = 0\) for all \(y \in X\), then \(x = 0\). Moreover, the functions \(x \to \|x, y\|_\beta\) are continuous functions of \(X\) into \(\mathbb{R}_+\) for each fixed \(y \in E\).
- If \(\beta = 1\), then \((X, \|\cdot\|_\beta)\) becomes a linear 2-normed space.

In 2006, Park [36] introduced the concept of quasi-2-normed spaces and quasi-(2, \(p\))-normed spaces and studied the properties of these spaces.

**Definition 1.1.** Let \(X\) be a linear space over \(\mathbb{K}\) with \(\text{dim } X \geq 2\) and \(\|\cdot\|_q : X \times X \to \mathbb{R}_+\) be a function such that

1. \(\|x, y\|_q = 0\) if and only if \(\{x, y\}\) is linearly dependent;
2. \(\|x, y\|_q = \|y, x\|_q\) for all \(x, y \in X\);
3. \(\|\lambda x, y\|_q = |\lambda| \|x, y\|_q\) for all \(x, y \in X\) and all \(\lambda \in \mathbb{K}\);
4. There is a constant \(\kappa > 1\) such that \(\|x, y + z\|_q \leq \kappa (\|x, y\|_q + \|x, z\|_q)\) for all \(x, y, z \in X\).

Then \(\|\cdot\|_q\) is called a quasi-2-norm on \(X\), the smallest possible \(\kappa\) is called the modulus of concavity and \((X, \|\cdot\|_q, \kappa)\) is called a quasi-2-normed space.

A quasi-2-norm \(\|\cdot\|_q\) is called a quasi-\(p\)-2-norm \((0 < p \leq 1)\) if

\[\|x + y, z\|_q^p \leq \|x, z\|_q^p + \|y, z\|_q^p\]

for all \(x, y, z \in X\). The first difference between a quasi-2-norm and a 2-norm is that the modulus of concavity of a quasi-norm is greater than or equal to 1, while that of a 2-norm is equal to 1. This causes the quasi-2-norm to be not continuous in general, while a norm is always continuous. Moreover, by Aoki-Rolewicz Theorem [34], each quasi-norm is equivalent to some \(p\)-norm. In [36], Park has shown the following theorem.

**Theorem 1.1.** [36, Theorem 3] Let \((X, \|\cdot\|_q, \kappa)\) be a quasi-2-normed space. There is \(p \in (0, 1)\) and an equivalent quasi-2-norm \(\|\cdot\|_q\) on \(X\) satisfying

\[\|x + y, z\|_q^p \leq \|x, z\|_q^p + \|y, z\|_q^p\]

for all \(x, y, z \in X\), with

\[\|x, z\|_q := \inf \left\{ \left( \sum_{i=1}^n \|x_i, z\|_q^p \right)^{1/p} : x = \sum_{i=1}^n x_i, x_i \in X, z \in X, n \in \mathbb{N} \right\}\]

and \(p = \log_{2\kappa} 2\).
2. Some properties of quasi-\((2, \beta)\)-Banach spaces

In this section, we generalize the concept of quasi-2-normed spaces and we prove some properties of the quasi-\((2, \beta)\)-Banach spaces.

**Definition 2.1.** Let \(0 < \beta \leq 1\) be a fixed real number and \(X\) be a linear space over \(\mathbb{K}\) with \(\dim X \geq 2\) and let \(|\cdot|_{q, \beta} : X \times X \to \mathbb{R}_+\) be a function such that

(B1) \(|x, y|_{q, \beta} = 0\) if and only if \(\{x, y\}\) is linearly dependent;

(B2) \(|x, y|_{q, \beta} = |y, x|_{q, \beta}\) for all \(x, y \in X\);

(B3) \(|\lambda x, y|_{q, \beta} = |\lambda|^\beta |x, y|_{q, \beta}\) for all \(x, y \in X\) and all \(\lambda \in \mathbb{K}\);

(B4) There is a constant \(\kappa \geq 1\) such that \(|x, y + z|_{q, \beta} \leq \kappa(|x, y|_{q, \beta} + |x, z|_{q, \beta}|\) for all \(x, y, z \in X\).

Then \(|\cdot|_{q, \beta}\) is called a quasi-\((2, \beta)\)-norm on \(X\), the smallest possible \(\kappa\) is called the modulus of concavity and \((X, |\cdot|_{q, \beta})\) is called a quasi-\((2, \beta)\)-normed space.

A quasi-2-norm \(|\cdot, \cdot|_{q, \beta}\) is called a quasi-\((-p, 2)\)-norm \((0 < p \leq 1)\) if

\[
|x + y, z|_{p, \beta}^p \leq |x, z|_{p, \beta}^p + |y, z|_{p, \beta}^p, \quad x, y, z \in X,
\]

and we have

\[
|\lambda|_{x, y} \leq |\lambda|_{y, x}, \quad |\lambda|_{x, y} + |\lambda|_{y, z} \leq |\lambda|_{x, z}.
\]

**Example 1.** Let \(X\) be a linear space with \(\dim X \geq 2\), and let \(|\cdot|_{q, \beta}\) be a \((2, \beta)\)-norm on \(X\). Then

\[
|x, y|_{q, \beta} = C|x, y|_{\beta}, \quad x, y \in X
\]

(with \(C > 0\) is a constant), is a quasi-\((2, \beta)\)-norm on \(X\), and \((X, |\cdot|_{q, \beta})\) is a quasi-\((2, \beta)\)-normed space.

**Example 2.** If \(X\) is a quasi-2-norm space with the quasi-2-norm \(|\cdot|_{q, \beta}\) and the modulus of concavity \(\kappa \geq 1\), then it is a quasi-\((2, \beta)\)-normed space with the quasi-\((2, \beta)\)-norm \(|x, y|_{q, \beta} = |x, y|_{q, \beta}\) for all \(x, y \in X\) and \(0 < \beta \leq 1\) is a fixed real number.

**Proof.** Indeed, for every \(x, y, z \in X\) and \(\lambda \in \mathbb{K}\), we have

\[
|x, y|_{q, \beta} = 0 \iff |y, x|_{q, \beta} = 0 \iff \{x, y\}
\]

is linearly dependent, and

\[
|y, x|_{q, \beta} = |y, x|_{q, \beta}^2 = |x, y|_{q, \beta}^2
\]

and

\[
|\lambda x, y|_{q, \beta} = |\lambda y, x|_{q, \beta} = |\lambda|^\beta |x, y|_{q, \beta}
\]

and

\[
|x + y, z|_{q, \beta} = |x + y, z|_{q, \beta}^2 \leq \kappa^\beta(|x, z|_{q} + |y, z|_{q})^2
\]

and

\[
\leq \kappa^\beta(|x, z|_{q}^2 + |y, z|_{q}^2)
\]

As \(\kappa \geq 1\), then \((X, |\cdot|_{q, \beta})\) is a quasi-\((2, \beta)\)-normed space.

**Lemma 2.1.** Let \((X, |\cdot|_{q, \beta})\) be a quasi-\((2, \beta)\)-normed space with \(0 < \beta \leq 1\). If \(x \in X\) and \(|x, y|_{q, \beta} = 0\) for all \(y \in X\), then \(x = 0\).

**Proof.** Suppose that \(x \neq 0\). Since \(\dim X \geq 2\), choose \(y \in X\) such that \(|x, y|\) is linearly independent, and by Definition 2.1 (B1) we have \(|x, y|_{q, \beta} \neq 0\). This is a contradiction, thus \(x = 0\).
Definition 2.2. Let $X$ be a quasi-$\langle 2, \beta \rangle$-normed space with $0 < \beta \leq 1$.

1. A sequence $(x_n)$ in $X$ is called a Cauchy sequence if there are two points $y, z \in X$ such that $y$ and $z$ are linearly independent,

$$\lim_{m,n \to \infty} \|x_m - x_n, y\|_{q,\beta} = 0 = \lim_{m,n \to \infty} \|x_m - x_n, z\|_{q,\beta}. $$

2. A sequence $(x_n)$ in $X$ is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\|_{q,\beta} = 0 \text{ for all } y \in X.$$

In this case we write we also write $\lim_{n \to \infty} x_n = x$.

3. A quasi-$\langle 2, \beta \rangle$-normed space in which every Cauchy sequence is a convergent sequence is called a quasi-$\langle 2, \beta \rangle$-Banach space.

Remark 1. The functions $x \to \|x, y\|_{q,\beta}$ are not necessary continuous functions of $X$ into $\mathbb{R}$ for each fixed $y \in X$.

Theorem 2.1. Let $(X, \|\cdot, \cdot\|_{q,\beta})$ be a quasi-$\langle 2, \beta \rangle$-normed space, with $0 < \beta \leq 1$. There is $p \in (0, 1]$ and an equivalent quasi-$2$-norm $\|\cdot, \cdot\|_{q,\beta}$ on $X$ satisfying

$$\|x + y, z\|_{q,\beta} \leq \|x, z\|_{q,\beta} + \|y, z\|_{q,\beta}$$

for all $x, y, z \in X$, with

$$\|x\|_{q,\beta} := \inf \left\{ \left( \sum_{i=1}^{n} \|x_i, z\|_{q,\beta}^{p/\beta} \right)^{\beta/p} : x = \sum_{i=1}^{n} x_i, x_i \in X, z \in X, n \in \mathbb{N} \right\}$$

and $p = \beta \log_{2\kappa} 2$.

Proof. Let $(X, \|\cdot, \cdot\|_{q,\beta})$ be a quasi-$\langle 2, \beta \rangle$-normed space with $0 < \beta \leq 1$, and let $x, z \in X$ such that

$$\|x, z\|_{q} := \inf \left\{ \left( \sum_{i=1}^{n} \|x_i, z\|^p_{q,\beta} \right)^{1/p} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \in \mathbb{N} \right\},$$

$p = \beta \log_{2\kappa} 2$.

We will demonstrate that $\|\cdot, \cdot\|_{q} := \|\cdot, \cdot\|_{q,\beta}$ is a quasi-$2$-norm on $X$ with the modulus of concavity $\frac{(2\kappa)^\beta}{2}$.

For that, we assume that $\|\cdot, \cdot\|_{q,\beta}$ is a quasi-$\langle 2, \beta \rangle$-norm on $X$. Then for every $x, y, z \in X$ and for every $\lambda \in \mathbb{K}$, we have

$$\|x, y\|_{q,\beta} = 0 \iff \|x, y\|_{q,\beta} = 0 \iff \|x, y\|_{q,\beta} = 0 \iff \{x, y\} \text{ is linearly dependent;}$$

$$\|\lambda x, y\|_{q,\beta} = \|\lambda x, y\|_{q,\beta} = |\lambda| \|x, y\|_{q,\beta};$$

and

$$\|x + y, z\|_{q,\beta} \leq \kappa^{1/\beta} \left( \|x, z\|_{q,\beta} + \|y, z\|_{q,\beta} \right)^{1/\beta}. \tag{2.1}$$

Let $g(t) = t^r$ for all $t > 0$ with $r > 1$. It is easy to check that $g$ is a convex function, hence by the definition of convexity, we have

$$g \left( \frac{t + s}{2} \right) \leq \frac{1}{2} (g(t) + g(s)) \text{ for all } t, s > 0,$$
i.e., \((t + s)^r \leq 2^{r-1}(t^r + s^r)\). So, \((2.1)\) becomes

\[
\|x + y, z\|_q \leq \frac{(2\kappa)^{1/\beta}}{2} (\|x, z\|_{q, \beta}^{1/\beta} + \|y, z\|_{q, \beta}^{1/\beta}) = \frac{(2\kappa)^{1/\beta}}{2} (\|x, z\|_q + \|y, z\|_q).
\]

It follows that \(\left(X, \|\cdot\|_q, \frac{(2\kappa)^{1/\beta}}{2}\right)\) is a quasi-2-normed space. From Theorem 1.1, we get a quasi-2-norm \(\|\cdot\|_q\) on \(X\) satisfying

\[
\|x + y, z\|_q^p \leq \|x, z\|_q^p + \|y, z\|_q^p, \quad x, y, z \in X,
\]

and there exist \(\mu_1, \mu_2 > 0\) such that

\[
\mu_1\|x, z\|_{q, \beta}^{1/\beta} = \mu_1\|x\|_q \leq \|x, z\|_q \leq \mu_2\|x, z\|_q = \mu_2\|x\|_{q, \beta}^{1/\beta} \quad \forall x, y \in X.
\]

Since \(0 < \beta \leq 1\) and by Example 2, we get \(\|\cdot\|_{q, \beta} := \|\cdot\|_{q, \beta}^\beta\) is also a quasi-(2, \(\beta\))-norm on \(X\) and

\[
\|x + y, z\|_{q, \beta}^\beta = \|x + y, z\|_q^\beta \leq (\|x, z\|_q^\beta + \|y, z\|_q^\beta)^\beta \leq \|x, z\|_{q, \beta}^\beta + \|y, z\|_{q, \beta}^\beta = \|x, z\|_{q, \beta}^\beta + \|y, z\|_{q, \beta}^\beta
\]

for all \(x, y, z \in X\). Also we have

\[
\mu_1^\beta\|x, z\|_{q, \beta} \leq \|x, z\|_q^\beta \leq \mu_2^\beta\|x, z\|_{q, \beta} \quad \forall x, z \in X,
\]

i.e.,

\[
C_1\|x, z\|_{q, \beta} \leq \|x, z\|_q \leq C_2\|x, z\|_{q, \beta} \quad \forall x, z \in X,
\]

(2.2)

with \(C_i = \mu_i^\beta\) for \(i \in \{1, 2\}\). This completes the proof. \(\Box\)

**Remark 2.** It follows from Theorem 2.1 that

(i) \(\|x, z\|_{q, \beta}^\beta - \|y, z\|_{q, \beta}^\beta \leq \|x - y, z\|_{q, \beta}, \quad x, y, z \in X\).

(ii) The functions \(x \to \|x, y\|_{q, \beta}\) are continuous functions of \(X\) into \(\mathbb{R}\) for each fixed \(y \in X\).

(iii) If \(\beta = 1\), we get the Theorem 1.1.

3. A NEW FIXED POINT THEOREM

In this section, we prove that the fixed point theorem \([16, \text{Theorem 2.1}]\) remains valid in the setting of quasi-(2, \(\beta\))-Banach space. Let us introduce the following four hypotheses:

(A1) \(W\) is a nonempty set, \((X, \|\cdot\|_{q, \beta}, \kappa)\) is a quasi-(2, \(\beta\))-Banach space.

(A2) \(f_i : W \to W\) and \(L_i : W \times X \to \mathbb{R}_+\) are given maps for \(i = 1, \ldots, j\).

(A3) \(T : X^W \to X^W\) is an operator satisfying the inequality

\[
\|(T\xi)(x) - (T\mu)(x), y\|_{q, \beta} \leq \sum_{i=1}^j L_i(x, y) \|\xi(f_i(x)) - \mu(f_i(x)), y\|_{q, \beta}
\]

for all \(\xi, \mu \in X^W\) and \((x, y) \in W \times X\).
(A4) \( \Lambda : \mathbb{R}_+^{W \times X} \rightarrow \mathbb{R}_+^{W \times X} \) is a linear operator defined by
\[
(\Lambda \delta)(x, y) := \sum_{i=1}^{j} L_i(x, y)\delta(f_i(x), y), \quad \delta \in \mathbb{R}_+^{W \times X}, \ (x, y) \in W \times X.
\]

**Theorem 3.1.** Let hypotheses (A1)-(A4) be valid, and let \( \varepsilon : W \times X \rightarrow \mathbb{R}_+ \), \( \varphi : W \rightarrow X \) satisfy the conditions
\[
\|(T \varphi)(x) - \varphi(x), y\|_\beta \leq \varepsilon(x, y), \ (x, y) \in W \times X,
\]
\[
\varepsilon^*(x, y) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)^{\theta}(x, y) < \infty, \ (x, y) \in W \times X,
\]
where \( \theta = \beta \log_2 \kappa \). Then the limit
\[
\psi(x) = \lim_{n \rightarrow \infty} T^n \varphi(x), \ x \in W
\]
extists and the function \( \psi : W \rightarrow X \) defined by (3.3) is a fixed point of \( T \) satisfying
\[
\|(\varphi(x) - \psi(x), y)\|^{q, \beta}_{q, \beta} \leq K \varepsilon^*(x, y), \ (x, y) \in W \times X.
\]
for some constant \( K > 0 \). Moreover, if
\[
\varepsilon^*(x, y) \leq \left( M \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x, y) \right)^{\theta} < \infty
\]
for some positive real number \( M \), then \( \psi \) is the unique fixed point of \( T \) satisfying (3.4).

**Proof.** It is easy to show by induction that, for any \( n \in \mathbb{N}_0 \)
\[
\|(T^{n+1} \varphi)(x) - (T^n \varphi)(x), y\|_{q, \beta} \leq (\Lambda^n \varepsilon)(x, y), \ (x, y) \in W \times X.
\]

Indeed, by (3.1), we have
\[
\|(\varphi(x) - (T \varphi)(x), y)\|_{q, \beta} \leq \varepsilon(x, y) = (\Lambda^0 \varepsilon)(x, y), \ (x, y) \in W \times X.
\]
Then the case \( n = 0 \) is true. Now, fix an \( n \in \mathbb{N}_0 \) and suppose that (3.6) is true. Then, by (A3)-(A4), for any \( (x, y) \in W \times X \), we get
\[
\|(T^{n+1} \varphi)(x) - (T^{n+2} \varphi)(x), y\|_{q, \beta} = \|(T(T^{m+1} \varphi)(x) - T(T^m \varphi)(x), y\|_{q, \beta}
\leq \sum_{1 \leq i \leq j} L_i(x, y)\left|\begin{array}{c}
T^n \varphi(f_i(x)) - T^{n+1} \varphi(f_i(x)), y
\end{array}\right|_{q, \beta}
\leq \sum_{1 \leq i \leq j} L_i(x, y)(\Lambda^n \varepsilon)(f_i(x), y) = (\Lambda^{n+1} \varepsilon)(x, y).
\]
So, (3.6) holds for all \( n \in \mathbb{N}_0 \).
Next, from (3.6), (2.2) and Theorem 2.1, for every $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $(x, y) \in W \times X$, we have

$$
\|\|(T^n \varphi)(x) - (T^{n+k} \varphi)(x), y\|\|_{q, \beta}^\theta = \|\|\sum_{i=0}^{k-1} ((T^{n+i} \varphi)(x) - (T^{n+i+1} \varphi)(x)), y\|\|_{q, \beta}^\theta \leq \sum_{i=0}^{k-1} \|\|(T^{n+i} \varphi)(x) - (T^{n+i+1} \varphi)(x), y\|\|_{q, \beta}^\theta \leq C^\theta_2 \sum_{i=0}^{k-1} \|\|(T^{n+i} \varphi)(x) - (T^{n+i+1} \varphi)(x), y\|\|_{q, \beta}^\theta \leq C^\theta_2 \sum_{i=0}^{k-1} (\Lambda^{n+i} \varepsilon)(x, y) = C^\theta_2 \sum_{i=n}^{n+k-1} (\Lambda^i \varepsilon)(x, y) \leq C^\theta_2 \varepsilon^*(x, y) \quad \text{for some } C_2 > 0. \quad (3.7)
$$

By the convergence of the series $\sum_{n \geq 0} (\Lambda^n \varepsilon)^\theta(x, y)$, it follows from (3.7) that, for every $x \in W$, $\{(T^n \varphi)(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, \|\cdot\|_{q, \beta})$. By Theorem 2.1, $\{(T^n \varphi)(x)\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in $(X, \|\cdot\|_{q, \beta}, \kappa)$. As $(X, \|\cdot\|_{q, \beta}, \kappa)$ is a quasi-$\beta$-Banach space, then the limit $\psi(x) := \lim_{m \to \infty}(T^m \varphi)(x)$ exists for any $x \in W$, so (3.3) holds.

Since $\|\cdot\|_{q, \beta}$ is continuous and taking $n = 0$ and $k \to \infty$ in (3.7), we get

$$
\|\varphi(x) - \psi(x), y\|_{q, \beta}^\theta \leq C^\theta_2 \varepsilon^*(x, y) \quad (3.8)
$$

for all $(x, y) \in W \times X$ and for some $C_2 > 0$. From (2.2), we find

$$
\|\varphi(x) - \psi(x), y\|_{q, \beta}^\theta \leq C^{-\theta}_1 \|\varphi(x) - \psi(x), y\|_{q, \beta}^\theta \leq (C_2/C_1)^\theta \varepsilon^*(x, y) \quad (3.9)
$$

for all $(x, y) \in W \times X$ and for some $C_1, C_2 > 0$, so (3.4) holds with $K := (C_2/C_1)^\theta$.

By applying (A3), (3.3) and (2.2), we get

$$
\|\|T^{n+1} \varphi(x) - (T \psi(x), y\|_{q, \beta} \leq \sum_{i=1}^{j} L_i(x, y)\|\|(T^n \varphi)(f_i(x)) - \psi(f_i(x)), y\|_{q, \beta} \leq C^{-1}_1 \sum_{i=1}^{j} L_i(x, y)\|\|(T^n \varphi)(f_i(x)) - \psi(f_i(x)), y\|_{q, \beta} \quad \text{(3.9)}
$$

for all $(x, y) \in W \times X$, $n \in \mathbb{N}$ and for some $C_1 > 0$. Letting $n \to \infty$ in (3.9), we find

$$
\lim_{n \to \infty} \|\|T^{n+1} \varphi(x) - (T \psi)(x), y\|_{q, \beta} = 0
$$

for all $(x, y) \in W \times X$, that is, $\lim_{n \to \infty}(T^{n+1} \varphi)(x) = (T \psi)(x)$ for all $x \in W$. This proves $T \psi = \psi$. So $\psi$ is a fixed point of $T$ that satisfies (3.4).

It remains to prove the uniqueness of $\psi$. Let $\gamma$ be also a fixed point of $T$ satisfying (3.4). For every $m \in \mathbb{N}_0$, we show that

$$
\|\psi(x) - \gamma(x), y\|_{q, \beta} = \|\|T^m \psi(x) - (T^m \gamma)(x), y\|_{q, \beta} \leq (2K)^{1/\theta} M \sum_{i=m}^{\infty} (\Lambda^i \varepsilon)(x, y) \quad (3.10)
$$
for all \((x, y) \in W \times X\) and for some \(K, M > 0\). Indeed, for \(m = 0\) and from (3.8), we have
\[
\|\psi(x) - \gamma(x), y\|_{q, \beta}^\theta \leq \|\psi(x) - \phi(x), y\|_{q, \beta}^\theta + \|\phi(x) - \gamma(x), y\|_{q, \beta}^\theta \leq 2C_2^\theta \varepsilon^*(x, y)
\]
for all \((x, y) \in W \times X\) and for some \(C_2 > 0\). From (2.2) and (3.5) we have
\[
\|\psi(x) - \gamma(x), y\|_{q, \beta}^\theta \leq C_1^\theta \|\psi(x) - \gamma(x), y\|_{q, \beta}^\theta \leq (C_2/C_1)^\theta \varepsilon^*(x, y)
\]
\[
\leq 2K \left( M \sum_{i=0}^\infty (\Lambda^i \varepsilon)(x, y) \right)^\theta, \quad (x, y) \in W \times X.
\]
Thus, (3.10) holds for \(m = 0\).

Now, assume that (3.10) is valid for some \(m \in \mathbb{N}_0\). By (A3) and (3.10), we obtain
\[
\|\left((T^{m+1})\psi\right)(x) - (T^{m+1})\gamma(x), y\|_{q, \beta}^\theta \leq \sum_{k=1}^j L_k(x, y) \|\left(T^m \psi\right)(f_k(x)) - (T^m \gamma)(f_k(x), y\|_{q, \beta}^\theta
\]
\[
\leq \sum_{k=1}^j L_k(x, y) (2K)^{1/\theta} M \sum_{i=m}^\infty (\Lambda^i \varepsilon)(x, y)
\]
\[
= (2K)^{1/\theta} M \sum_{i=m+1}^\infty (\Lambda^i \varepsilon)(x, y), \quad x, y \in X.
\]
So, (3.10) holds for all \(m \in \mathbb{N}_0\). It follows from (2.2) and (3.10) that
\[
\|\left((T^m \psi\right)(x) - (T^m \gamma)(x), y\|_{q, \beta}^\theta \leq C_2^\theta \|\left(T^m \psi\right)(x) - (T^m \gamma)(x), y\|_{q, \beta}^\theta
\]
\[
\leq 2KC_2^\theta \left( M \sum_{i=m}^\infty (\Lambda^i \varepsilon)(x, y) \right)^\theta \tag{3.11}
\]
for all \((x, y) \in W \times X\) and all \(m \in \mathbb{N}_0\). Letting \(m \to \infty\) in (3.11) and from (3.5), we get
\[
\|\psi(x) - \gamma(x), y\|_{q, \beta}^\theta = 0
\]
for all \((x, y) \in W \times X\). That is \(\psi \equiv \gamma\). This completes the proof. \(\Box\)

**Corollary 3.1.** Assume that hypotheses (A1)-(A4) are satisfied. Suppose that there exist two functions \(\varepsilon : W \times X \to \mathbb{R}_+\) and \(\phi : W \to X\) such that (3.1) holds and
\[
(\Lambda \varepsilon)(x, y) = q \varepsilon(x, y), \quad (x, y) \in W \times X, \quad q \in [0, 1).
\]
Then the limit (3.3) exists and the function \(\psi : W \to X\) so defined is the unique fixed point of \(T\) with
\[
\|\phi(x) - \psi(x), y\| \leq \frac{K \varepsilon^*(x, y)}{1 - q^\theta}, \quad (x, y) \in W \times X, \quad q \in [0, 1)
\]
for some \(K > 0\), where \(\theta = \beta \log_{2N} 2\).

**Proof.** It follows from (3.12) that
\[
(\Lambda^m \varepsilon)(x, y) = q^m \varepsilon(x, y), \quad (x, y) \in W \times X, \quad n \in \mathbb{N}_0.
\]
Therefore, for every \((x, y) \in W \times X\),
\[
\varepsilon^*(x, y) = \sum_{n=0}^\infty (\Lambda^m \varepsilon)(x, y) = \sum_{n=0}^\infty q^\theta \varepsilon(x, y))^\theta \leq \frac{\varepsilon^*(x, y)}{1 - q^\theta} < \infty
\]
where \( \theta = \beta \log_{2\alpha} 2 \). So, condition (3.2) holds. Moreover
\[
\left( \sum_{n=0}^{\infty} (\Lambda^n \epsilon)(x,y) \right)^\theta = \left( \sum_{n=0}^{\infty} q^n \epsilon(x,y) \right)^\theta \leq \frac{\epsilon^\theta(x,y)}{(1-q)^\theta} < \infty
\]
for all \((x,y) \in W \times X\). As
\[
\frac{\epsilon^\theta(x,y)}{1-q^\theta} \leq \frac{\epsilon^\theta(x,y)}{(1-q)^\theta}, \quad (x,y) \in W \times X,
\]
then condition (3.5) holds and our assertion follows from Theorem 3.1 and its proof. \(\Box\)

**Remark 3.** (i) If \((X, \| \cdot \|, \kappa)\) is a quasi-2-Banach in Theorem 3.1, then \(\theta = \log_{2\kappa} 2\) (i.e., \(\beta = 1\)) and Theorem 3.1 remains true.

(ii) If \((X, \| \cdot \|, \beta)\) is a \((2, \beta)\)-Banach in Theorem 3.1, then \(\kappa = 1\) and Theorem 3.1 remains true with \(\| \cdot \|, \| \cdot \| = \| \cdot \|, \| \cdot \|\).

(iii) If \((X, \| \cdot \|, \| \cdot \|)\) is a 2-Banach in Theorem 3.1, then \(\kappa = 1 = \beta\) we obtain [12, Theorem 1] with \(\| \cdot \|, \| \cdot \| = \| \cdot \|, \| \cdot \|\). Moreover, we have
\[
\| \varphi(x) - \psi(x) \| \leq \epsilon^\theta(x,y).
\]

4. General Solution of Eq. (1.2)

In this section, we give the general solution of the radical-type functional equation (1.2) by using some results that are reported in [10].

**Proposition 4.1.** Let \(a, b, c, d \in \mathbb{R} \setminus \{0\}\) be fixed numbers and \(\mathcal{V}\) a real vector space. A function \(f : \mathbb{R} \to \mathcal{V}\) satisfies (1.2) if and only if there exists a solution \(g : \mathbb{R} \to \mathcal{V}\) of the equation
\[
g(ax + by) + g(ax - by) = cg(x) + dg(y), \quad x, y \in \mathbb{R},
\]
such that \(f(x) = g(x^3)\) for all \(x \in \mathbb{R}\).

**Proof.** It is clear that if \(f : \mathbb{R} \to \mathcal{V}\) has form \(f(x) = g(x^3)\) for \(x \in \mathbb{R}\), with \(g\) satisfies (4.1), then it is a solution of (1.2).

On the other hand, if \(f : \mathbb{R} \to \mathcal{V}\) is a solution to (1.2), and setting \(h(x) = f\left(\sqrt[3]{x}\right)\) for all \(x \in \mathbb{R}\), then from (1.2), we get
\[
h(ax^3 + by^3) + h(ax^3 - by^3) = f\left(\sqrt[3]{ax^3 + by^3}\right) + f\left(\sqrt[3]{ax^3 - by^3}\right)
\]
\[
= cf(x) + df(y) = ch(x^3) + dh(y^3), \quad x, y \in \mathbb{R},
\]
where \(a, b, c, d \in \mathbb{R} \setminus \{0\}\). So, \(h\) satisfies (4.1). Thus it suffices to take \(g \equiv h\). \(\Box\)

**Lemma 4.1.** Let \(\mathcal{V}\) be a real vector space and \(f : \mathbb{R} \to \mathcal{V}\) be a function satisfying (1.2) with \(a, b, c, d \in \mathbb{R} \setminus \{0\}\) and \(c + d \neq 2\). Then

(i) \(f\) satisfies the functional equation
\[
f\left(\sqrt[3]{x^3 + y^3}\right) + f\left(\sqrt[3]{x^3 - y^3}\right) = 2f(x) + 2f(y), \quad x, y \in \mathbb{R}.
\]

(ii) \(f\) is a sextic mapping and if \(f\) is continuous, then \(f(x) = x^6f(1)\) for all \(x \in \mathbb{R}_+\).  

Proof. (i) Suppose that $f$ satisfies (1.2), then by letting $x = y = 0$ in (1.2), we have $f(0) = 0$ because $c + d \neq 2$. Replacing $y$ by $-y$ in (1.2), we get

$$f\left(\sqrt[3]{ax^3 - by^3}\right) + f\left(\sqrt[3]{ax^3 + by^3}\right) = cf(x) + df(-y), \quad x, y \in \mathbb{R}.$$  

(4.3)

If we compare (1.2) with (4.3), we obtain that the function $f$ is even.

Taking first $y = 0$ and next $x = 0$ in (1.2), we get

$$f(\sqrt[3]{ax}) = \frac{c}{2}f(x), \quad f(\sqrt[3]{by}) = \frac{d}{2}f(y).$$

So,

$$f(\sqrt[3]{abx}) = \frac{cd}{4}f(x), \quad x \in \mathbb{R}.$$  

(4.4)

Replacing $(x, y)$ by $(\sqrt[3]{ax}, \sqrt[3]{by})$ in (1.2), we obtain

$$f\left(\sqrt[3]{abx^3 + aby^3}\right) + f\left(\sqrt[3]{abx^3 - aby^3}\right) = cf(\sqrt[3]{bx}) + df(\sqrt[3]{ay}), \quad x \in \mathbb{R}.$$  

(4.5)

It follows from (4.4) and (4.5) that $f$ is a solution of (4.2).

(ii) As $f$ satisfies (4.2) and by [17, Remark 1], we get the desired results. \qed

**Theorem 4.2.** Let $a, b, c, d \in \mathbb{R} \setminus \{0\}$ be fixed numbers and $V$ a real vector space. A function $f : \mathbb{R} \rightarrow V$ satisfies (1.2) if and only if:

(i) In the case $c + d \neq 2$, there exists a quadratic mapping $Q : \mathbb{R} \rightarrow V$ such that $f(x) = Q(x^3)$ for all $x \in \mathbb{R}$ and

$$Q(ax) = \frac{c}{2}Q(x), \quad Q(bx) = \frac{d}{2}Q(x), \quad x \in \mathbb{R}. $$  

(4.6)

(ii) In the case $c + d = 2$, there exist $w \in V$ and a quadratic mapping $Q : \mathbb{R} \rightarrow V$, such that (4.6) holds and $f(x) = Q(x^3) + w$ for all $x \in \mathbb{R}$.

**Proof.** (i) Assume that $f$ satisfies (1.2) and $c + d \neq 2$. Then by Lemma 4.1, we obtain that $f$ satisfies (4.2). Applying [17, Theorem 2.1], there exists a quadratic mapping $Q : \mathbb{R} \rightarrow V$ such that $f(x) = Q(x^3)$ for all $x \in \mathbb{R}$. Also, we have $f(0) = 0$ and

$$f(\sqrt[3]{ax}) = \frac{c}{2}f(x), \quad f(\sqrt[3]{bx}) = \frac{d}{2}f(x), \quad x \in \mathbb{R}.$$  

So,

$$Q(ax) = \frac{c}{2}Q(x) \quad \text{and} \quad Q(bx) = \frac{d}{2}Q(x), \quad x \in \mathbb{R}. $$  

(4.6)

(ii) Suppose that $f$ satisfies (1.2) and $c + d = 2$. Putting

$$f_0(x) := f(x) - f(0), \quad x \in \mathbb{R}.$$  

Then $f_0(0) = 0$ and $f_0$ satisfies (1.2). Similarly to the proof of Lemma 4.1, we get

$$f_0(\sqrt[3]{ax}) = \frac{c}{2}f_0(x), \quad f_0(\sqrt[3]{bx}) = \frac{d}{2}f_0(x), \quad x \in \mathbb{R},$$

and $f_0$ satisfies (4.2), so by [17, Theorem 2.1], there exists a quadratic mapping $Q : \mathbb{R} \rightarrow V$ such that $f_0(x) = Q(x^3)$ and $Q$ satisfies (4.6). Hence $f(x) = Q(x^3) + w$, with $w := f(0)$.

The converse is easy to check. This completes the proof. \qed
Remark 4. It is well known (see, e.g., [1]) that a mapping $Q : \mathbb{R} \to \mathcal{V}$ is quadratic if and only there exists $B : \mathbb{R}^2 \to \mathcal{V}$ that is symmetric (i.e., $B(x, y) = B(y, x)$ for all $x, y \in \mathbb{R}$) and biadditive (i.e., $B(x + y, z) = B(x, z) + B(y, z)$ for all $x, y, z \in \mathbb{R}$) such that $Q(x) = B(x, x)$ for all $x \in \mathbb{R}$.

We derive from Proposition 4.1, Theorem 4.2 and Remark 4 the following corollaries.

Corollary 4.1. Let $a, b, c, d \in \mathbb{R} \setminus \{0\}$ be fixed numbers and $\mathcal{V}$ a real vector space. A function $f : \mathbb{R} \to \mathcal{V}$ satisfies (1.2) if and only if:

(i) In the case $c + d \neq 2$, there is a symmetric biadditive mapping $B : \mathbb{R} \to \mathcal{V}$ such that $f(x) = B(x^3, x^3)$ for all $x \in \mathbb{R}$, and

$$B(ax, ay) = \frac{c}{2}B(x, y), \quad B(bx, by) = \frac{d}{2}B(x, y), \quad x, y \in \mathbb{R}. \quad (4.7)$$

(ii) In the case $c + d = 2$, there are $w \in \mathcal{V}$ and a symmetric biadditive mapping $B : \mathbb{R} \to \mathcal{V}$ such that (4.7) holds and $f(x) = B(x^3, x^3) + w$ for all $x \in \mathbb{R}$.

Corollary 4.2. Let $a, b, c, d \in \mathbb{R} \setminus \{0\}$ be fixed numbers and $\mathcal{V}$ a real vector space. A function $g : \mathbb{R} \to \mathcal{V}$ satisfies (1.1) if and only if:

(i) In the case $c + d \neq 2$, there is a symmetric biadditive mapping $B : \mathbb{R} \to \mathcal{V}$ such that $g(x) = B(x, x)$ and

$$B(ax, ay) = \frac{c}{2}B(x, y), \quad B(bx, by) = \frac{d}{2}B(x, y), \quad x, y \in \mathbb{R}. \quad (4.8)$$

(ii) In the case $c + d = 2$, there are $w \in \mathcal{V}$ and a symmetric biadditive mapping $B : \mathbb{R} \to \mathcal{V}$ such that (4.8) holds and $g(x) = B(x, x) + w$ for all $x \in \mathbb{R}$.

5. Hyperstability criterion of Eq. (1.3) and some consequences

By using Theorem 3.1, we study the hyperstability results of Eq. (1.3) in quasi-$(2, \beta)$-Banach space. In the following, we consider $(X, \|\cdot\|_X, \beta, \kappa)$ is a quasi-$(2, \beta)$-Banach space, $a, b, c, d \in \mathbb{R} \setminus \{0\}$ are fixed numbers, $\theta = \beta \log_2 2$ and $N_{m_0} := \{m \in \mathbb{N} : m \geq m_0\}$ with $m_0 \in \mathbb{N}$. A function $f : \mathbb{R} \to X$ fulfilling Eq. (1.3) $\gamma$-approximately if

$$\left\| f \left( \sqrt[3]{ax^3 + by^3} \right) + f \left( \sqrt[3]{ax^3 - by^3} \right) - cf(x) - df(y), z \right\|_{q, \beta} \leq \gamma(x, y, z) \quad (5.1)$$

for all $(x, y, z) \in \mathbb{R}^3 \times X$, where $\sqrt[3]{ax} \neq \pm \sqrt[3]{by}$ and $\gamma : \mathbb{R}^2 \times X \to \mathbb{R}_+$ is a given function.

Theorem 5.1. Let $h_i : \mathbb{R} \times X \to \mathbb{R}_+$ be a given function for $i \in \{1, 2, 3, 4\}$, and let

$$M_0 := \left\{ n \in \mathbb{N}_2 \mid P_n := \max\{A_n, B_n, C_n\} < 1 \right\} \neq \emptyset, \quad (5.2)$$

where

$$A_n = \kappa |c| \beta s_1(w_n^1)s_2(w_n^1) + \kappa^2 |d| \beta s_1(v_n^1)s_2(v_n^1) + \kappa^2 s_1(w_n^1)s_2(w_n^1)$$

$$B_n = \kappa |c| \beta s_3(u_n^1) + \kappa^2 |d| \beta s_3(v_n^1) + \kappa^2 s_3(u_n^1)$$

$$C_n = \kappa |c| \beta s_4(u_n^1) + \kappa^2 |d| \beta s_4(v_n^1) + \kappa^2 s_4(u_n^1)$$

$$u_n = \sqrt[n]{a}, \quad v_n = \sqrt[n]{1 - \frac{n^3}{b}}, \quad w_n = \sqrt[n]{2n^3 - 1}$$
\[
\lim_{n \to \infty} \max\{s_1(u_n^3)s_2(v_n^3), s_3(u_n^3), s_4(v_n^3)\} = 0
\]

and \(s_i(\rho) := \inf\{t \in \mathbb{R}_+: h_i(\rho x^3, z) < h_i(x^3, z)\} \text{ for all } (x, z) \in \mathbb{R} \times X\) for \(\rho \in \mathbb{R}_0\) and \(i \in \{1, 2, 3, 4\}\), such that one of the conditions holds:

(i) \(\lim_{|\rho| \to +\infty} s_1(\rho)s_2(\rho) = 0\) if \(P_n = A_n\),

(ii) \(\lim_{|\rho| \to +\infty} s_3(\rho) = 0\) if \(P_n = B_n\),

(iii) \(\lim_{|\rho| \to +\infty} s_4(\rho) = 0\) if \(P_n = C_n\).

If \(f : \mathbb{R} \to X\) satisfies (5.1) with \(\gamma(x, y, z) = h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z)\), then (1.3) holds.

Proof. It is clear that for \(\rho \in \mathbb{R}_0\) and \(i \in \{1, 2, 3, 4\}\), we have

\[
h_i(\rho x^3, z) \leq s_i(\rho)h_i(x^3, z), \quad (x, z) \in \mathbb{R} \times X.
\]

Replacing \((x, y)\) by \((u_m x, v_m x)\) in (5.1), with \(u_m = \frac{m}{\sqrt[3]{a}}\) and \(v_m = \frac{3^{1-m^3}}{b}\), we get

\[
\|c f(u_m x) + df(v_m x) - f(w_m x) - f(x), z\|_{q, \beta} \leq h_1(u_m^3 x^3, z)h_2(v_m^3 x^3, z) + h_3(u_m^3 x^3, z) + h_4(v_m^3 x^3, z)
\]

for all \(x \in \mathbb{R}_0\) and all \(z \in X\) with \(w_m = \sqrt[3]{2m^3 - 1}\) and \(m \in \mathbb{N}_2\). For each \(m \in \mathbb{N}_2\), we will define an operator \(T_m : X_{\mathbb{R}_0} \to X_{\mathbb{R}_0}\) by

\[
(T_m \xi)(x) := c \xi(u_m x) + d \xi(v_m x) - \xi(w_m x), \quad \xi \in X_{\mathbb{R}_0}, \ x \in \mathbb{R}_0.
\]

Putting

\[
\varepsilon_m(x, z) := h_1(u_m^3 x^3, z)h_2(v_m^3 x^3, z) + h_3(u_m^3 x^3, z) + h_4(v_m^3 x^3, z)
\]

for all \(x \in \mathbb{R}_0\) and \(z \in X\) with \(m \in \mathbb{N}_2\). Then by (5.3) we have

\[
\varepsilon_m(x, z) \leq \sigma_m \left[ h_1(x^3, z)h_2(x^3, z) + h_3(x^3, z) + h_4(x^3, z) \right], \quad (x, z) \in \mathbb{R}_0 \times X,
\]

with \(\sigma_m := \max\{s_1(u_m^3), s_2(v_m^3), s_3(u_m^3), s_4(v_m^3)\}\). Then the inequality (5.4) takes the form

\[
\|(T_m f)(x) - f(x), z\|_{q, \beta} \leq \varepsilon_m(x, z), \quad (x, z) \in \mathbb{R}_0 \times X, \ m \in \mathbb{N}_2.
\]

For each \(m \in \mathbb{N}_2\), we find that the operator \(\Lambda_m : \mathbb{R}_{\mathbb{R}_0} \times X \to \mathbb{R}_{\mathbb{R}_0} \times X\) defined by

\[
(\Lambda_m \delta)(x, z) = \kappa|c|^{\beta}\delta(u_m x, z) + \kappa^2|d|^{\beta}\delta(v_m x, z) + \kappa^2\delta(w_m x, z), \quad \delta \in \mathbb{R}_{\mathbb{R}_0} \times X, \ x \in \mathbb{R}_0
\]

has the shape given in (A4) with \(j = 3\),

\[
f_1(x) \equiv u_m x, \quad f_2(x) \equiv v_m x, \quad f_3(x) \equiv w_m x, \quad L_1(x, z) \equiv \kappa|c|^{\beta},
\]

\[
L_2(x, z) \equiv \kappa^2|d|^{\beta}, \quad L_3(x, z) \equiv \kappa^2; \quad (x, z) \in \mathbb{R}_0 \times X, \ m \in \mathbb{N}_2.
\]
Moreover, for every \( \xi, \mu \in X_{\mathbb{R}^2} \), \( m \in \mathbb{N} \) and \((x, z) \in \mathbb{R}_0 \times X\), we obtain
\[
\| (T_m \xi)(x) - (T_m \mu)(x), z \|_{q, \beta}
= \| \epsilon \| (u_m x) + d \xi (v_m x) - \epsilon (w_m x - c \mu (u_m x) - d \mu (v_m x) + \mu (w_m x), z) \|_{q, \beta}
\leq \kappa \epsilon | \beta [ Z_1 (f_1(x)) - \mu (f_1(x)), z], \|_{q, \beta}
+ \kappa \epsilon \epsilon [ Z (f_1(x)) - \mu (f_1(x)), z] \|_{q, \beta}
+ \kappa \epsilon \epsilon [ Z (f_1(x)) - \mu (f_1(x)), z] \|_{q, \beta}
\leq \sum_{i=1}^3 L_i (x, z) \| [ Z (f_1(x)) - \mu (f_1(x)), z] \|_{q, \beta}.
\]
So, (A3) is valid for \( T_m \) with \( m \in \mathbb{N} \). It is not hard to show that
\[
\Lambda_m \epsilon_m (x, z) \leq P_m \sigma_m \left[ h_1 (x^3, z) h_2 (x^3, z) + h_3 (x^3, z) + h_4 (x^3, z) \right]
\]
for all \((x, z) \in \mathbb{R}_0 \times X\). By induction, we will show that for each \( n \in \mathbb{N}_0 \) and \((x, z) \in \mathbb{R}_0 \times X\),
\[
\Lambda_m \epsilon_m (x, z) \leq P_m \sigma_m \left[ h_1 (x^3, z) h_2 (x^3, z) + h_3 (x^3, z) + h_4 (x^3, z) \right]
\]
where \( m \in \mathcal{M}_0 \). From (5.5), we obtain that the inequality (5.7) holds for \( n = 0 \). Next, we will assume that (5.7) holds for \( n = r \), where \( r \in \mathbb{N}_0 \). Then,
\[
\Lambda_m \epsilon_m (x, z) = \Lambda_n (\Lambda_m \epsilon_m (x, z))
\]
for all \((x, z) \in \mathbb{R}_0 \times X\) and \( m \in \mathcal{M}_0 \). This shows that (5.7) holds for all \( n \in \mathbb{N}_0 \).

By the definition of \( \mathcal{M}_0 \), we find that for each \((x, z) \in \mathbb{R}_0 \times X\) and \( m \in \mathcal{M}_0 \),
\[
\epsilon_m (x, z) = \sum_{n=0}^{\infty} (\Lambda_m \epsilon_m)^n (x, z) = \sum_{n=0}^{\infty} P^n \sigma_m \left[ h_1 (x^3, z) h_2 (x^3, z) + h_3 (x^3, z) + h_4 (x^3, z) \right]^n
\]
for all \((x, z) \in \mathbb{R}_0 \times X\) and \( m \in \mathcal{M}_0 \). Thus, according to Theorem 3.1, there exists a fixed point \( Q_m : \mathbb{R}_0 \rightarrow X \) of the operator \( T_m \) satisfying
\[
\| f (x) - Q_m (x, z) \|_{q, \beta} \leq K \epsilon_m (x, z)
\leq \frac{K \sigma_m \left[ h_1 (x^3, z) h_2 (x^3, z) + h_3 (x^3, z) + h_4 (x^3, z) \right]}{1 - P_m}.
\]
for all \((x, z) \in \mathbb{R}_0 \times X, \ m \in \mathcal{M}_0\) and for some constant \(K > 0\). That is,

\[Q_m(x) = cQ_m(u_m x) + dQ_m(v_m x) - Q_m(w_m x), \quad x \in \mathbb{R}_0, \ m \in \mathcal{M}_0,
\]

and (5.8) holds for all \((x, z) \in \mathbb{R}_0 \times X\). Moreover,

\[Q_m(x) := \lim_{n \to \infty} T_m^n f(x). \quad (5.9)
\]

Now, we show that for every \(n \in \mathbb{N}_0\) and \((x, y, z) \in \mathbb{R}^3_0 \times X\) with \(\sqrt[3]{ax} \neq \pm \sqrt[3]{by}\),

\[
\left\| T_m^n f \left( \sqrt[3]{ax^3 + by^3} \right) + T_m^n f \left( \sqrt[3]{ax^3 - by^3} \right) - cT_m^n f(x) - dT_m^n f(y), \ z \right\|_{q, \beta} \leq P_n \left[ h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z) \right]
\]

where \(m \in \mathcal{M}_0\) and \(\gamma(x, y, z) := [h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z)]\).

Indeed, if \(n = 0\), then (5.10) is simply (5.1). So, fix \(n \in \mathbb{N}_0\) and assume that (5.10) holds for \(n\) and every \((x, y, z) \in \mathbb{R}^3_0 \times X\) with \(\sqrt[3]{ax} \neq \pm \sqrt[3]{by}\). Then, by the definition of \(T_m\) and (5.3), we get that

\[
\left\| T_m^{n+1} f \left( \sqrt[3]{ax^3 + by^3} \right) + T_m^{n+1} f \left( \sqrt[3]{ax^3 - by^3} \right) - cT_m^{n+1} f(x) - dT_m^{n+1} f(y), \ z \right\|_{q, \beta} \leq \kappa |c| \left\| T_m^n f \left( u_m \sqrt[3]{ax^3 + by^3} \right) + T_m^n f \left( u_m \sqrt[3]{ax^3 - by^3} \right) - cT_m^n f(x) - dT_m^n f(y), \ z \right\|_{q, \beta} + \kappa^2 |d| \left\| T_m^n f \left( v_m \sqrt[3]{ax^3 + by^3} \right) + T_m^n f \left( v_m \sqrt[3]{ax^3 - by^3} \right) - cT_m^n f(x) - dT_m^n f(y), \ z \right\|_{q, \beta} \leq P_n \max\{A_n, B_n, C_n\} \left[ h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z) \right] = P_n \left( h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z) \right)
\]

for all \((x, y, z) \in \mathbb{R}^3_0 \times X\) with \(\sqrt[3]{ax} \neq \pm \sqrt[3]{by}\). Thus, by induction, we have shown that (5.10) holds for all \(n \in \mathbb{N}_0\) and for all \((x, y, z) \in \mathbb{R}^3_0 \times X\) with \(\sqrt[3]{ax} \neq \pm \sqrt[3]{by}\).
From (5.10) and (2.2) we find that
\[
\left\| T_m^n f \left( \sqrt[n]{a x^3 + b y^3} \right) + T_m^n f \left( \sqrt[n]{a x^3 - b y^3} \right) - c T_m^n f(x) - d T_m^n f(y), z \right\|_{q,\beta}^\theta
\]
\[
\leq C_2 \left\| T_m^n f \left( \sqrt[n]{a x^3 + b y^3} \right) + T_m^n f \left( \sqrt[n]{a x^3 - b y^3} \right) - c T_m^n f(x) - d T_m^n f(y), z \right\|_{q,\beta}^\theta
\]
\[
\leq C_2 F_m \left[ h_1(x^3, z) h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z) \right] \tag{5.11}
\]
for some constant $C_2 > 0$. We conclude from the continuity of $\|\cdot,\cdot\|_{q,\beta}$, (5.9) and (5.11) that, for every $(x, y, z) \in \mathbb{R}_0^3 \times X$ with $\sqrt[n]{a x} \neq \pm \sqrt[n]{b y}$,
\[
\left\| Q_m \left( \sqrt[n]{a x^3 + b y^3} \right) + Q_m \left( \sqrt[n]{a x^3 - b y^3} \right) - c Q_m(x) - d Q_m(y), z \right\|_{q,\beta}^\theta
\]
\[
= \lim_{n \to \infty} \left\| T_m^n f \left( \sqrt[n]{a x^3 + b y^3} \right) + T_m^n f \left( \sqrt[n]{a x^3 - b y^3} \right) - c T_m^n f(x) - d T_m^n f(y), z \right\|_{q,\beta}^\theta
\]
\[
= 0,
\]
this means
\[
Q_m \left( \sqrt[n]{a x^3 + b y^3} \right) + Q_m \left( \sqrt[n]{a x^3 - b y^3} \right) = c Q_m(x) + d Q_m(y) \tag{5.13}
\]
for all $x, y \in \mathbb{R}_0$, with $\sqrt[n]{a x} \neq \pm \sqrt[n]{b y}$.

We want now to prove that the mapping $Q_m : \mathbb{R}_0 \to X$ is unique. So, let $G_m : \mathbb{R}_0 \to X$ be a solution of (5.13) and
\[
\left\| f(x) - G_m(x), z \right\|_{q,\beta}^\theta \leq \frac{K \sigma_m^\theta \left[ h_1(x^3, z) h_2(x^3, z) + h_3(x^3, z) + h_4(x^3, z) \right]}{1 - F_m^\theta} \tag{5.14}
\]
for all $(x, z) \in \mathbb{R}_0 \times X$. Thus, replacing $(x, y)$ by $(u_m x, v_m y)$ in (5.13), we get $T_m G_m(x) = G_m(x)$ for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$.

Moreover, we have
\[
\left\| Q_m(x) - G_m(x), z \right\|_{q,\beta} \leq C_2 \left\| Q_m(x) - G_m(x), z \right\|_{q,\beta} \leq \kappa C_2 \left( \left\| Q_m(x) - f(x), z \right\|_{q,\beta} + \left\| G_m(x) - f(x), z \right\|_{q,\beta} \right)
\]
\[
\leq L \left[ h_1(x^3, z) h_2(x^3, z) + h_3(x^3, z) + h_4(x^3, z) \right], \tag{5.15}
\]
where $L := \frac{2 \kappa C_2 \sqrt[n]{\sum \sigma_m^\theta}}{1 - F_m^\theta}$ and $m \in \mathcal{M}_0$. It is easy to prove by induction on $n$ that
\[
\left\| Q_m(x) - G_m(x), z \right\|_{q,\beta} \leq \left\| Q_m(x) - G_m(x), z \right\|_{q,\beta} \leq LP_m^\theta \left[ h_1(x^3, z) h_2(x^3, z) + h_3(x^3, z) + h_4(x^3, z) \right] \tag{5.16}
\]
for all $x \in \mathbb{R}_0$ and $m \in \mathcal{M}_0$. Letting $n \to \infty$ in (5.16), we get $Q_m = G_m$. So, the fixed point satisfying (5.8) of $T_m$ is unique.

Letting $m \to \infty$ in (5.8), we obtain
\[
\lim_{m \to \infty} \left\| f(x) - Q_m(x), z \right\|_{q,\beta} = 0, \text{ i.e.,}
\]
\[
\lim_{m \to \infty} Q_m(x) = f(x). \tag{5.17}
\]
Also, letting \( m \to \infty \) in (5.12), using (5.17) and the continuity of \( ||| \cdot ||| \), we have

\[
\left\| \frac{\partial}{\partial x} \right\| f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) - cf(x) - df(y), z \right\|_{q, \beta} = 0
\]

for all \((x, y, z) \in \mathbb{R}_0^3 \times X\), with \( \sqrt[\beta]{a x} \neq \pm \sqrt[\beta]{by} \), that is,

\[
f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) = cf(x) + df(y)
\]

for all \(x, y \in \mathbb{R}_0\), with \( \sqrt[\beta]{a x} \neq \pm \sqrt[\beta]{by} \). This proves that \( f \) is a solution of the radical-type functional equation (1.3). The proof of the theorem is complete. \( \square \)

**Remark 5.** Theorem 5.1 also provide hyperstability outcomes in each of the following cases:

1. \( \gamma(x, y, z) = h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) \)
2. \( \gamma(x, y, z) = h_1(x^3, z)h_2(y^3, z) + h_4(y^3, z) \)
3. \( \gamma(x, y, z) = h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) \)
4. \( \gamma(x, y, z) = h_3(x^3, z) + h_4(y^3, z) \)
5. \( \gamma(x, y, z) = h_1(x^3, z) \)
6. \( \gamma(x, y, z) = h_2(y^3, z) \)

for all \((x, y, z) \in \mathbb{R}_0^3 \times X\).

Applying Theorem 5.1 and the same technique, we get the following corollary.

**Corollary 5.1.** Under the hypotheses of Theorem 5.1, we consider two functions \( f : \mathbb{R} \to X \) and \( F : \mathbb{R}^2 \to X \) such that

\[
\left\| f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) - cf(x) - df(y) - F(x, y), z \right\|_{q, \beta} \leq \gamma(x, y, z)
\]

for all \((x, y, z) \in \mathbb{R}_0^3 \times X\), with \( \sqrt[\beta]{a x} \neq \pm \sqrt[\beta]{by} \) and \( \gamma(x, y, z) = h_1(x^3, z)h_2(y^3, z) + h_3(x^3, z) + h_4(y^3, z) \). Assume that the functional equation

\[
f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) = cf(x) + df(y) + F(x, y), \quad (5.18)
\]

admits a solution \( f_0 : \mathbb{R} \to X \). Then \( f \) is a solution of (5.18).

**Proof.** Let \( g(x) := f(x) - f_0(x) \) for \( x \in \mathbb{R}_0 \). Then

\[
\left\| g \left( \sqrt[\beta]{a x^3 + by^3} \right) + g \left( \sqrt[\beta]{a x^3 - by^3} \right) - cg(x) - dg(y), z \right\|_{q, \beta} \leq \kappa \left\| f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) - cf(x) - df(y) - F(x, y), z \right\|_{q, \beta} + \kappa \left\| f_0 \left( \sqrt[\beta]{a x^3 + by^3} \right) + f_0 \left( \sqrt[\beta]{a x^3 - by^3} \right) - cf_0(x) - df_0(y) - F(x, y), z \right\|_{q, \beta} = \kappa \left\| f \left( \sqrt[\beta]{a x^3 + by^3} \right) + f \left( \sqrt[\beta]{a x^3 - by^3} \right) - cf(x) - df(y) - F(x, y), z \right\|_{q, \beta} \leq \kappa \gamma(x, y, z), \quad (x, y, z) \in \mathbb{R}_0^3 \times X, \quad \sqrt[\beta]{a x} \neq \pm \sqrt[\beta]{by}.\]
It follows from Theorem 5.1 that \( g \) satisfies the equation (1.3). Therefore,
\[
\begin{align*}
f \left( \sqrt[n]{ax^3 + by^3} \right) + f \left( \sqrt[n]{ax^3 - by^3} \right) - cf(x) - df(y) - F(x, y) \\
= g \left( \sqrt[n]{ax^3 + by^3} \right) + g \left( \sqrt[n]{ax^3 - by^3} \right) - cg(x) - dg(y) \\
+ f_0 \left( \sqrt[n]{ax^3 + by^3} \right) + f_0 \left( \sqrt[n]{ax^3 - by^3} \right) - cf_0(x) - df_0(y) - F(x, y) = 0
\end{align*}
\]
for all \( x, y \in \mathbb{R} \) with \( \sqrt[n]{ax} \neq \mp \sqrt[n]{by} \). \( \square \)

To end this section, we derive some particular cases from Theorem 5.1 and Corollary 5.1. Let \((X, \| \cdot \|_{q, \beta, \kappa})\) be a quasi-\((2, \beta)\)-Banach space, \((Y, \| \cdot \|_{q, \alpha, \kappa'})\) a quasi-\((2, \alpha)\)-normed space. Also, let \( \lambda_i : \mathbb{R} \to Y \setminus \{0\} \) be additive functions continuous at a point for \( i = 1, 2, 3, 4 \), and \( g : X \to Y \setminus \{0\} \) a surjective mapping. According to Theorem 5.1 and Corollary 5.1, we give some consequences with
\[
h_i(x, z) := c_i \| \lambda_i(x), g(z) \|_{q, \alpha}^{p_i}, \quad (x, z) \in \mathbb{R}_0 \times X,
\]
where \( c_1, p_1 \in \mathbb{R} \) for \( i \in \{ 1, 2, 3, 4 \} \).

**Corollary 5.2.** Let \( a, b, c, d \in \mathbb{R}_0 \) be fixed numbers, \( D_1, D_2, D_3 \geq 0 \) be three constants and let \( p_i \in \mathbb{R} \) such that \( p_1 + p_2, p_3, p_4 < 0 \) for \( i \in \{ 1, 2, 3, 4 \} \). If \( f : \mathbb{R} \to X \) satisfies the inequality
\[
\left\| f \left( \sqrt[n]{ax^3 + by^3} \right) + f \left( \sqrt[n]{ax^3 - by^3} \right) - cf(x) - df(y) \right\|_{q, \beta} \leq D_1 \| \lambda_1(x^3), g(z) \|_{q, \alpha}^{p_1} \| \lambda_2(y^3), g(z) \|_{q, \alpha}^{p_2} + D_2 \| \lambda_3(x^3), g(z) \|_{q, \alpha}^{p_3} + D_3 \| \lambda_4(y^3), g(z) \|_{q, \alpha}^{p_4}
\]
for all \( (x, y, z) \in \mathbb{R}_0^3 \times X \) with \( \sqrt[n]{ax} \neq \mp \sqrt[n]{by} \), then (1.3) holds.

**Proof.** Define \( h_i : \mathbb{R}_0 \times X \to \mathbb{R}_+ \) by \( h_i(x^3, z) = c_i \| \lambda_i(x^3), g(z) \|_{q, \alpha}^{p_i} \) for some \( c_i \in \mathbb{R}_+ \) and \( p_3, p_4, p_1 + p_2 < 0 \) with \( i \in \{ 1, 2, 3, 4 \} \) and \((D_1, D_2, D_3) = (c_1 c_2, c_3, c_4)\).

For each \( \rho \in \mathbb{R}_0 \), we have
\[
s_i(\rho) = \inf \left\{ t \in \mathbb{R}_+ : h_i(\rho x^3, z) \leq t h_i(x^3, z), \quad \forall (x, z) \in \mathbb{R} \times X \right\}
\]
\[
= \inf \left\{ t \in \mathbb{R}_+ : c_i \| \lambda_i(\rho x^3), g(z) \|_{q, \alpha}^{p_i} \leq t c_i \| \lambda_i(x^3), g(z) \|_{q, \alpha}^{p_i}, \quad \forall (x, z) \in \mathbb{R} \times X \right\}
\]
\[
= |\rho|^{\alpha p_i}
\]
for \( i \in \{ 1, 2, 3, 4 \} \). So,
\[
\lim_{n \to -\infty} \max \{ s_1(u_n^3), s_2(v_n^3), s_3(u_n^3), s_4(v_n^3) \}
\]
\[
= \lim_{n \to -\infty} \max \left\{ \frac{n^3}{a}, \frac{1-n^3}{b}, \frac{n^3}{a}, \frac{1-n^3}{b} \right\}
\]
\[
= \lim_{n \to -\infty} \max \left\{ \frac{n^3 - \alpha p_1}{a}, \frac{1-n^3 - \alpha p_2}{b}, \frac{n^3 - \alpha p_3}{a}, \frac{1-n^3 - \alpha p_4}{b} \right\} = 0
\]
where \( u_n = \frac{n}{\sqrt[n]{a}} \), \( v_n = \sqrt[n]{\frac{1-n^3}{b}} \) and \( n \in \mathbb{N}_2 \). Similarly, we have
\[
\lim_{n \to -\infty} P_n = \lim_{n \to -\infty} \max \{ A_n, B_n, C_n \} = 0
\]
where $A_n$, $B_n$ and $C_n$ are defined as in Theorem 5.1. Clearly, there is $n_0 \in \mathbb{N}_2$ such that $P_n < 1, \quad n \geq n_0$.

Thus, all the conditions in Theorem 5.1 are fulfilled. \hfill \□

**Corollary 5.3.** Let $a, b, c, d \in \mathbb{R}_0$ be fixed numbers, $D_1, D_2, D_3 \geq 0$ be three constants and let $p_i \in \mathbb{R}$ such that $p_1 + p_2, p_3, p_4 < 0$ for $i \in \{1, 2, 3, 4\}$. Let $f : \mathbb{R} \to X$ and $F : \mathbb{R}^2 \to X$ be two functions such that

$$
\|f(\sqrt[n]{ax^3 + by^3}) + f(\sqrt[n]{ax^3 - by^3}) - cf(x) - df(y) - F(x,y), z\|_{q, \beta}
\leq D_1 \|\lambda_1(x^3), g(z)\|_{p_1, q, \alpha} + D_2 \|\lambda_2(y^3), g(z)\|_{p_2, q, \alpha} + D_3 \|\lambda_3(y^3), g(z)\|_{p_3, q, \alpha}
$$

for all $(x, y, z) \in \mathbb{R}^2_0 \times X$ with $\sqrt[n]{ax} \neq \pm \sqrt[n]{by}$. Assume that the functional equation

$$
f \left( \sqrt[n]{ax^3 + by^3} \right) + f \left( \sqrt[n]{ax^3 - by^3} \right) = cf(x) + df(y) + F(x,y),
$$

(5.19)

$x, y \in \mathbb{R}_0, \quad \sqrt[n]{ax} \neq \pm \sqrt[n]{by}$

admits a solution $f_0 : \mathbb{R} \to X$. Then $f$ is a solution of (5.19).

**Note 1.** In the same way, we can find the general solution and the hyperstability results of the following functional equation:

$$
f \left( \sqrt[n]{ax^n + by^n} \right) + f \left( \sqrt[n]{ax^n - by^n} \right) = cf(x) + df(y), \quad x, y \in \mathbb{R},
$$

where $a, b, c$ and $d$ are nonzero real constants and $n$ is an odd positive integer.

**Question 1.** Considered as a future work concerning the solution and Ulam’s stability for the following functional equation:

$$
f \left( \sqrt[n]{ax^n + by^n} \right) + f \left( \sqrt[n]{ax^n - by^n} \right) = cf(x) + df(y), \quad x, y \in \mathbb{R},
$$

where $a, b, c$ and $d$ are nonzero real constants and $n$ is a positive integer.

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