FACTORY THREEFOLD HYPERSURFACES

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Abstract. Let $X$ be a hypersurface in $\mathbb{P}^4$ of degree $d$ that has at most isolated ordinary double points. We prove that $X$ is factorial in the case when $X$ has at most $(d - 1)^2 - 1$ singular points.

1. Introduction

The Cayley–Bacharach theorem (see [7], [10]), in its classical form, may be seen as a result about the number of independent linear conditions imposed on forms of a given degree by a certain finite subset of $\mathbb{P}^n$. The purpose of this paper is to prove the following result.

Theorem 1.1. Let $\Sigma$ be a finite subset in $\mathbb{P}^n$, and let $\mu$ be a natural number such that

- the inequalities $\mu \geq 2$ and $|\Sigma| \leq \mu^2 - 1$ hold,
- at most $\mu k$ points in the set $\Sigma$ lie on a curve in $\mathbb{P}^n$ of degree $k = 1, \ldots, \mu - 1$, where $n \geq 2$. Then $\Sigma$ imposes independent linear conditions on forms of degree $2\mu - 3$.

Let $X$ be a hypersurface in $\mathbb{P}^4$ of degree $d \geq 3$ such that the threefold $X$ has at most isolated ordinary double points. Then $X$ can be given by the equation

$$f(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \text{Proj} \left( \mathbb{C} [x, y, z, t, u] \right),$$

where $f(x, y, z, t, u)$ is a homogeneous polynomial of degree $d$.

Remark 1.2. It follows from [12] and [9] that the following conditions are equivalent:

- every Weil divisor on the threefold $X$ is a Cartier divisor;
- every surface $S \subset X$ is cut out on $X$ by a hypersurface in $\mathbb{P}^4$;
- the ring

$$\mathbb{C} [x, y, z, t, u] / \langle f(x, y, z, t, u) \rangle$$

is a unique factorization domain;
- the set $\text{Sing}(X)$ imposes independent linear conditions on forms of degree $2d - 5$.

We say that $X$ is factorial if every Weil divisor on $X$ is a Cartier divisor.

Example 1.3. Suppose that $X$ is given by

$$xg(x, y, z, t, u) + yh(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \text{Proj} \left( \mathbb{C} [x, y, z, t, u] \right),$$

where $g$ and $h$ are general homogeneous polynomials of degree $d - 1$. Then

- the threefold $X$ has at most isolated ordinary double points,
- the equality $|\text{Sing}(X)| = (d - 1)^2$ holds, but $X$ is not factorial.

The assertion of Theorem 1.1 implies the following result (cf. [6], [2], [4]).

Theorem 1.4. Suppose that $|\text{Sing}(X)| < (d - 1)^2$. Then $X$ is factorial.

Proof. The set $\text{Sing}(X)$ is a set-theoretic intersection of hypersurfaces of degree $d - 1$. Then

- the inequalities $d - 1 \geq 2$ and $|\text{Sing}(X)| \leq (d - 1)^2 - 1$ hold.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$. 

• at most \((n-1)k\) points in the set \(\operatorname{Sing}(X)\) lie on a curve in \(\mathbb{P}^4\) of degree \(k = 1, \ldots, n-2\), which immediately implies that the points of the set \(\operatorname{Sing}(X)\) imposes independent linear conditions on forms of degree \(2d - 5\) by Theorem 1.1. Thus, the threefold \(X\) is factorial.  

The assertion of Theorem 1.4 is proved in [3] and [5] in the case when \(d \leq 7\).

**Remark 1.5.** Suppose that \(d = 4\) and \(X\) is factorial. Then it follows from [13] that

- the threefold \(X\) is non-rational,
- the threefold \(X\) is not birational to a conic bundle,
- the threefold \(X\) is not birational to a fibration into rational surfaces,

but general determinantal quartic hypersurfaces in \(\mathbb{P}^4\) are rational.

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2. **The Proof**

Let \(\Sigma\) be a finite subset in \(\mathbb{P}^n\), and let \(\mu\) be a natural number such that

- the inequalities \(\mu \geq 2\) and \(|\Sigma| \leq \mu^2 - 1\) hold,
- at most \(\mu k\) points in the set \(\Sigma\) lie on a curve in \(\mathbb{P}^n\) of degree \(k = 1, \ldots, \mu - 1\),

where \(n \geq 2\). Suppose that \(\Sigma\) imposes dependent linear conditions on forms of degree \(2\mu - 3\).

**Remark 2.1.** The inequality \(\mu \geq 3\) holds.

The following result is proved in [1] and [8].

**Theorem 2.2.** Let \(P_1, \ldots, P_\delta \in \mathbb{P}^2\) be distinct points such that

- at most \(k(\xi + 3 - k) - 2\) points in \(\{P_1, \ldots, P_\delta\}\) lie on a curve of degree \(k \leq (\xi + 3)/2\),
- the inequality

\[
\delta \leq \max \left\{ \left\lfloor \frac{\xi + 3}{2} \right\rfloor \left( \xi + 3 - \left\lfloor \frac{\xi + 3}{2} \right\rfloor \right) - 1, \left\lfloor \frac{\xi + 3}{2} \right\rfloor^2 \right\}
\]

holds, where \(\xi\) is a natural number such that \(\xi \geq 3\),

and let \(\pi: Y \to \mathbb{P}^2\) be a blow up of the points \(P_1, \ldots, P_\delta\). Then the linear system

\[
\left| \pi^* \left( \mathcal{O}_{\mathbb{P}^2}(\xi) \right) - \sum_{i=1}^\delta E_i \right|
\]

does not have base points, where \(E_i\) is the \(\pi\)-exceptional divisor such that \(\pi(E_i) = P_i\).

There is a point \(P \in \Sigma\) such that every hypersurface\(^1\) in \(\mathbb{P}^n\) of degree \(2\mu - 3\) that contains the set \(\Sigma \setminus P\) must contain the point \(P \in \Sigma\). Let us derive a contradiction.

**Lemma 2.3.** The inequality \(n \neq 2\) holds.

**Proof.** Suppose that \(n = 2\). Let us prove that at most \(k(2\mu - k) - 2\) points in \(\Sigma \setminus P\) can lie on a curve of degree \(k \leq \mu\). It is enough to show that

\[
k(2\mu - k) - 2 \geq k\mu
\]

for every \(k \leq \mu\). We must prove this only for \(k \geq 1\) such that

\[
k(2\mu - k) - 2 < |\Sigma \setminus P| \leq \mu^2 - 2,
\]

because otherwise the condition that at most \(k(2\mu - k) - 2\) points in the set \(\Sigma \setminus P\) can lie on a curve of degree \(k\) is vacuous. Therefore, we may assume that \(k < \mu\).

\(^1\)For simplicity we consider homogeneous forms on \(\mathbb{P}^n\) as hypersurfaces.
We may assume that $k \neq 1$, because at most $\mu \leq 2\mu k - 3$ points of $\Sigma \setminus P$ lie on a line. Then

$$k(2\mu - k) - 2 \geq k\mu \iff \mu > k,$$

which implies that at most $k(2\mu - k) - 2$ points in $\Sigma \setminus P$ can lie on a curve in $\mathbb{P}^2$ of degree $k \leq \mu$.

Thus, it follows from Theorem 2.2 that there is a curve of degree $2\mu - 3$ that contains all points of the set $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$, which is a contradiction. □

Moreover, we may assume that $n = 3$ due to the following result.

**Lemma 2.4.** Let $\Lambda \subseteq \Sigma$ be a subset, let $\psi: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a general linear projection, and let

$$\mathcal{M} \subseteq \mathcal{O}_{\mathbb{P}^n}(k)$$

be a linear subsystem that contains all hypersurfaces that pass through $\Lambda$. Suppose that

- the inequality $|\Lambda| \geq \mu k + 1$ holds,
- the set $\psi(\Lambda)$ is contained in an irreducible reduced curve of degree $k$,

where $n > m \geq 2$. Then $\mathcal{M}$ has no base curves, and either $m = 2$, or $k > \mu$.

**Proof.** We may assume that there are linear subspaces $\Omega$ and $\Pi \subseteq \mathbb{P}^n$ such that

$$\psi: \mathbb{P}^n \dashrightarrow \Pi \cong \mathbb{P}^m$$

is a projection from $\Omega$, where $\dim(\Omega) = n - m - 1$ and $\dim(\Pi) = m$.

Suppose that there is an irreducible curve $Z \subset \mathbb{P}^n$ such that $Z$ is contained in the base locus of the linear system $\mathcal{M}$. Put $\Xi = Z \cap \Lambda$. We may assume that $\psi|_Z$ is a birational morphism, and

$$\psi(Z) \cap \psi(\Lambda \setminus \Xi) = \emptyset,$$

because the projection $\psi$ is general. Then $\deg(\psi(Z)) = \deg(Z)$.

Let $C \subset \Pi$ be an irreducible curve of degree $k$ that contains $\psi(\Lambda)$, and let $W \subset \mathbb{P}^n$ be the cone over the curve $C$ whose vertex is $\Omega$. Then $W \in \mathcal{M}$, which implies that $Z \subseteq W$. We have

$$\psi(Z) = C,$$

which immediately implies that $\Xi = \Lambda$ and $\deg(Z) = k$. But $|Z \cap \Sigma| \leq \mu k$, which is a contradiction. Therefore, the linear system $\mathcal{M}$ does not have base curves.

Now we suppose that $m \geq 3$ and $k \leq \mu$. Let us show that this assumption leads to a contradiction. Without loss of generality, we may assume that $m = 3$ and $n = 4$.

Let $\mathcal{Y}$ be the set of all irreducible reduced surfaces in $\mathbb{P}^4$ of degree $k$ that contains the set $\Lambda$, and let $\mathcal{Y}$ be a subset of $\mathbb{P}^4$ that consists of all points that are contained in every surface of the set $\mathcal{Y}$. Then $\Lambda \subseteq \mathcal{Y}$. Arguing as above, we see that $\mathcal{Y}$ is a finite set.

Let $\mathcal{S}$ be the set of all surfaces in $\mathbb{P}^3$ of degree $k$ such that

$$S \in \mathcal{S} \iff \exists Y \in \mathcal{Y} \text{ such that } \psi(Y) = S \text{ and } \psi|_Y \text{ is a birational morphism},$$

and let $\Psi \subset \mathbb{P}^3$ that consists of all points contained in every surface in $\mathcal{S}$. Then $\mathcal{S} \neq \emptyset$ and

$$\psi(\Lambda) \subseteq \psi(\mathcal{Y}) \subseteq \Psi.$$

For every point $O \in \Pi \setminus \Psi$ and for a general surface $Y \in \mathcal{Y}$, we may assume that the line passing through $O$ and $\Omega$ does not intersect $Y$. But $\psi|_Y$ is a birational morphism. Then

$$\psi(\mathcal{Y}) = \Psi,$$

and $\psi(\Lambda) \subseteq \Psi$ contains at least $\mu k + 1 \geq k^2 + 1$ points that are contained in a curve of degree $k$, which is impossible, because $\Psi$ is a set-theoretic intersection of surfaces of degree $k$. □

Fix a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let

$$\psi: \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.
Lemma 2.5. There is a curve $C \subset \Pi$ of degree $k \leq \mu - 1$ such that $|C \cap \Sigma'| \geq \mu k + 1$.

Proof. We suppose that at most $\mu k$ points of the set $\Sigma'$ are contained in a curve in $\Pi$ of degree $k$ for every $k \leq \mu - 1$. Then arguing as in the proof of Lemma 2.3 we obtain a curve $Z \subset \Pi \cong \mathbb{P}^2$ of degree $2\mu - 3$ that contains the set $\Sigma' \setminus P'$ and does not pass through the point $P'$.

Let $Y$ be the cone in $\mathbb{P}^3$ over the curve $Z$ whose vertex is the point $O$. Then $Y$ is a surface of degree $2\mu - 3$ that contains all points of the set $\Sigma \setminus P$ but does not contain the point $P \in \Sigma$. □

It immediately follows from Lemma 2.4 that $k \geq 2$.

Lemma 2.6. Suppose that $|C \cap \Sigma'| \geq 9$. Then $k \geq 3$.

Proof. Suppose that $k = 2$. Let $\Phi \subseteq \Sigma$ be a subset such that $|\Phi| \geq 9$, but $\psi(\Phi)$ is contained in the conic $C \subset \Pi$. Then the conic $C$ is irreducible by Lemma 2.4.

Let $D$ be a linear system of quadric hypersurfaces in $\mathbb{P}^3$ containing $\Phi$. Then $D$ does not have base curves by Lemma 2.4. Let $W$ be a cone in $\mathbb{P}^3$ over $C$ with the vertex $\Omega$. Then

$$8 = D_1 \cdot D_2 \cdot W \geq \sum_{\omega \in \Phi} \text{mult}_\omega(D_1)\text{mult}_\omega(D_2) \geq |\Phi| \geq 9,$$

where $D_1$ and $D_2$ are general divisors in the linear system $D$. □

We may assume that $k$ is the smallest natural number such that at least $\mu k + 1$ points in $\Sigma'$ lie on a curve of degree $k$. Then there is a non-empty disjoint union

$$\bigcup_{j=k}^l \bigcup_{i=1}^{c_j} \Lambda^i_j \subset \Sigma$$

such that $|\Lambda^i_j| \geq \mu j + 1$, all points of the the set $\psi(\Lambda^i_j)$ are contained in an irreducible reduced curve of degree $j$, and at most $\mu \zeta$ points of the subset

$$\psi \left( \Sigma \setminus \left( \bigcup_{j=k}^l \bigcup_{i=1}^{c_j} \Lambda^i_j \right) \right) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve in $\Pi$ of degree $\zeta$ for every natural number $\zeta$. Put

$$\Lambda = \bigcup_{j=k}^l \bigcup_{i=1}^{c_j} \Lambda^i_j.$$

Let $\Xi^i_j$ be the base locus of the linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(j)|$ that contains all surfaces passing through the set $\Lambda^i_j$. Then $\Xi^i_j$ is a finite set by Lemma 2.3, and

$$|\Sigma \setminus \Lambda| \leq \mu \left( \mu - \sum_{i=k}^l c_i \mu i \right) - 2. \quad (2.7)$$

Corollary 2.8. The inequality $\sum_{i=k}^l ic_i \leq \mu - 1$ holds.

Put $\Delta = \Sigma \cap \left( \bigcup_{j=k}^l \bigcup_{i=1}^{c_j} \Xi^i_j \right)$. Then $\Lambda \subseteq \Delta \subseteq \Sigma$.

Lemma 2.9. The set $\Delta$ impose independent linear conditions on forms of degree $2\mu - 3$.

Proof. Let us consider the subset $\Delta \subset \mathbb{P}^3$ as a closed subscheme of $\mathbb{P}^3$, and let $\mathcal{I}_\Delta$ be the ideal sheaf of the subscheme $\Delta$. Then there is an exact sequence

$$0 \to \mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2\mu - 3) \to \mathcal{O}_{\mathbb{P}^3}(2\mu - 3) \to \mathcal{O}_\Delta \to 0,$$
which implies that $\Delta$ imposes independent conditions on forms of degree $2\mu - 3$ if and only if

$$h^1(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2\mu - 3)) = 0.$$ 

Suppose $h^1(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(2\mu - 3)) \neq 1$. Let us show that this assumption leads to a contradiction.

Let $\mathcal{M}$ be a linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(\mu - 1)|$ that contains all surfaces that pass through all point of the set $\Delta$. Then the base locus of $\mathcal{M}$ is zero-dimensional, because $\sum_{i=k}^{l} ic_i \leq \mu - 1$ and

$$\Delta \subseteq \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j,$$

but $\Xi_j$ is a zero-dimensional base locus of a linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(j)|$. Put

$$\Gamma = M_1 \cdot M_2 \cdot M_3,$$

where $M_1, M_2, M_3$ are general surfaces in the linear system $\mathcal{M}$. Then $\Gamma$ is a zero-dimensional subscheme of $\mathbb{P}^3$, and $\Delta$ is a closed subscheme of the scheme $\Gamma$.

Let $\Upsilon$ be a closed subscheme of the scheme $\Gamma$ such that

$$\mathcal{I}_\Upsilon = \text{Ann}(\mathcal{I}_\Delta / \mathcal{I}_\Gamma),$$

where $\mathcal{I}_\Upsilon$ and $\mathcal{I}_\Gamma$ are the ideal sheaves of the subschemes $\Upsilon$ and $\Gamma$, respectively. Then

$$0 \neq h^1(\mathcal{O}_{\mathbb{P}^3}(2\mu - 3) \otimes \mathcal{I}_\Delta) = h^0(\mathcal{O}_{\mathbb{P}^3}(\mu - 4) \otimes \mathcal{I}_\Upsilon) - h^0(\mathcal{O}_{\mathbb{P}^3}(\mu - 4) \otimes \mathcal{I}_\Gamma)$$

by Theorem 3 in [7] (see also [10]). Thus, there is a surface $F \in |\mathcal{O}_{\mathbb{P}^3}(\mu - 4) \otimes \mathcal{I}_\Upsilon|$. Then

$$(\mu - 4)(\mu - 1)^2 = F \cdot M_1 \cdot M_2 \geq h^0(\mathcal{O}_\Upsilon) = h^0(\mathcal{O}_\Gamma) - h^0(\mathcal{O}_\Delta) = (\mu - 1)^3 - |\Delta|,$$

which implies that $|\Delta| \geq 3(\mu - 1)^2$. But $|\Delta| < |\Sigma| < \mu^2$, which is impossible, because $\mu \geq 3$. \hfill $\Box$

We see that $\Delta \subsetneq \Sigma$. Put $\Gamma = \Sigma \setminus \Delta$ and $d = 2\mu - 3 - \sum_{i=k}^{l} ic_i$.

**Lemma 2.10.** The set $\Delta$ imposes dependent linear conditions on forms of degree $d$.

**Proof.** Suppose that the points of the set $\Delta$ impose independent linear conditions on homogeneous polynomials of degree $d$. Let us show that this assumption leads to a contradiction.

The construction of $\Delta$ implies the existence of a homogeneous form $H$ of degree $\sum_{i=k}^{l} ic_i$ that vanishes at all points of the set $\Delta$ and does not vanish at any point of the set $\Gamma$.

Suppose that $P \in \Delta$. Then there is a homogenous form $F$ of degree $2\mu - 3$ that vanishes at every point of the set $\Delta \setminus P$ and does not vanish at the point $P$ by Lemma 2.9. Put

$$\Gamma = \{Q_1, \ldots, Q_\gamma\},$$

where $Q_i$ is a point in $\Gamma$. Then there is a homogeneous form $G_i$ of degree $d$ that vanishes at every point in $\Gamma \setminus Q_i$ and does not vanish at the point $Q_i$. Then

$$F(Q_i) + \mu_i H G_i(Q_i) = 0$$

for some $\mu_i \in \mathbb{C}$, because $G_i(Q_i) \neq 0$. Then the homogenous form

$$F + \sum_{i=1}^{\gamma} \mu_i H G_i$$

vanishes on the set $\Sigma \setminus P$ and does not vanish at the point $P$, which is a contradiction.

We see that $P \in \Gamma$. Then there is a homogeneous form $G$ of degree $d$ that vanishes at every point in $\Gamma \setminus P$ and does not vanish at $P$. Then $HG$ vanishes at every point of the set $\Sigma \setminus P$ and does not vanish at the point $P$, which is a contradiction. \hfill $\Box$

Put $\Gamma' = \psi(\Gamma)$. Let us check that $\Gamma'$ and $d$ satisfy the hypotheses of Theorem 2.2
Lemma 2.11. The inequality \( d \geq 3 \) holds.

Proof. Suppose that \( d \leq 2 \). It follows from Corollary 2.8 that

\[
2 \geq d = 2\mu - 3 - \sum_{i=k}^{l} ic_i \geq \mu - 2 \geq 1,
\]

but \( \mu \neq 3 \) by Lemma 2.10, because \( |\Gamma| \leq |\Sigma \setminus \Lambda| \leq \mu(\mu - \sum_{i=k}^{l} c_i) - 2 \).

Thus, we see that \( \mu = 4 \). Then \( k = 3 \) by Lemma 2.6, which implies that

\[
|\Gamma| \leq |\Sigma \setminus \Lambda| \leq 14 - 4 \sum_{i=k}^{l} c_i \leq 2,
\]

which is impossible by Lemma 2.10, because \( d \geq 1 \).

It follows from the inequality 2.7 that \( |\Gamma'| = |\Gamma| \leq |\Sigma \setminus \Lambda| \leq \mu(\mu - \sum_{i=k}^{l} c_i) - 2 \). Then

\[
|\Gamma'| \leq \mu \left( \mu - \sum_{i=k}^{l} c_i \right) - 2 \leq \max \left\{ \left\lfloor \frac{d+3}{2} \right\rfloor \left( d + 3 - \left\lfloor \frac{d+3}{2} \right\rfloor \right) - 1, \left\lfloor \frac{d+3}{2} \right\rfloor^2 \right\},
\]

because \( d = 2\mu - 3 - \sum_{i=k}^{l} c_i \) and \( \mu \geq 3 \).

Lemma 2.12. At most \( d \) points of the set \( \Gamma \) is contained in a line.

Proof. Suppose that at least \( d + 1 \) points of the set \( \Gamma \) is contained in some line. Then

\[
\mu \geq d + 1 = 2\mu - 2 - \sum_{i=k}^{l} c_i,
\]

because at most \( \mu \) points of \( \Gamma \) is contained in a line. It follows from Corollary 2.8 that

\[
\mu - 1 \geq \sum_{i=k}^{l} c_i \geq \mu - 2.
\]

Suppose that \( \sum_{i=k}^{l} c_i = \mu - 2 \). Then \( |\Gamma| \leq 2\mu - 2 \). So, the set \( \Gamma \) imposes independent linear conditions on forms of degree \( d = \mu - 1 \) by Theorem 2 in [11], which is impossible by Lemma 2.10.

We see that \( \sum_{i=k}^{l} c_i = \mu - 1 \). Then \( |\Gamma| \leq \mu - 2 = d \), which is impossible by Lemma 2.10. Therefore, at most \( d \) points of the set \( \Gamma' \) lies on a line by Lemmas 2.12 and 2.4.

Lemma 2.13. For every \( t \leq (d + 3)/2 \), at most

\[
t(d + 3 - t) - 2
\]

points of the set \( \Gamma' \) lie on a curve of degree \( t \) in \( \Pi \cong \mathbb{P}^2 \).

Proof. At most \( \mu t \) points of the set \( \Gamma' \) lie on a curve of degree \( t \). It is enough to show that

\[
t(d + 3 - t) - 2 \geq \mu t
\]

for every \( t \leq (d + 3)/2 \) such that \( t > 1 \) and \( t(d + 3 - t) - 2 < |\Gamma'| \). But

\[
t(d + 3 - t) - 2 \geq t\mu \iff \mu - \sum_{i=k}^{l} c_i > t,
\]

because \( t > 1 \). Thus, we may assume that \( t(d + 3 - t) - 2 < |\Gamma'| \) and

\[
\mu - \sum_{i=k}^{l} c_i \leq t \leq \frac{d + 3}{2}.
\]
Let \( g(x) = x(d+3-x) - 2 \). Then

\[
g(t) \geq g\left( \mu - \sum_{i=k}^{l} c_i \right),
\]

because \( g(x) \) is increasing for \( x < (d+3)/2 \). Therefore, we have

\[
\mu \left( \mu - \sum_{i=k}^{l} ic_i \right) - 2 \geq |\Gamma' | > g(t) \geq g\left( \mu - \sum_{i=k}^{l} c_i \right) = \mu \left( \mu - \sum_{i=k}^{l} c_i \right) - 2,
\]

which is a contradiction. □

Thus, the set \( \Gamma' \) imposes independent linear conditions on forms of degree \( d \) by Theorem 2.2, which implies that the set \( \Gamma \) also imposes independent linear conditions on forms of degree \( d \), which is impossible by Lemma 2.10. The assertion of Theorem 1.1 is proved.

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