Abstract. For a closed immersed minimal submanifold $M^n$ in the unit sphere $S^N$ ($n < N$), we prove
\[
\text{Vol}(M^n) \geq \frac{m}{2}\text{Vol}(S^n) + \frac{\sqrt{n} + 1}{n}\text{Vol}(S^{n-1}),
\]
where $m$ denotes the maximal multiplicity of intersection points of $M^n$ in $S^N$ and $\text{Vol}$ denotes the Riemannian volume functional. As an application, if the volume of $M^n$ is less than or equal to the volume of any $n$-dimensional minimal Clifford torus, then $M^n$ must be embedded, verifying the non-embedded case of Yau’s conjecture. In addition, we also get volume gaps for hypersurfaces under some conditions.

1. Introduction

In 1984, Cheng-Li-Yau \cite{7} proved that if $M^n$ is a closed minimal immersed submanifold in the unit sphere $S^N$ ($N = n + l \geq n + 1$) and $M^n$ is of maximal dimension ($M^n$ does not lie on any hyperplane of $\mathbb{R}^{N+1}$), then the volume of $M^n$ satisfies
\[
\text{Vol}(M^n) > \left(1 + \frac{2l + 1}{B_n}\right)\text{Vol}(S^n),
\]
where $B_n < 2n + 3 + 2\exp(2nC_n)$ and $C_n \leq \frac{1}{2}n^{n/2}\Gamma(n/2, 1)$. This means that there is a volume gap for minimal submanifolds in spheres. About the second smallest volume, Yau put forward the following famous conjecture in his Problem Section \cite{31}:

**Conjecture 1.1 (Yau’s Conjecture \cite{31}).** The volume of one of the minimal Clifford torus ($M_{k,n-k} = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n - 1$) gives the lowest value of volume among all non-totally geodesic closed minimal hypersurfaces of $S^{n+1}$.

For $n = 2$, Conjecture \cite{31} (also called the Solomon-Yau Conjecture \cite{13, 18}) is true, due to the following two results. On the one hand, Li and Yau \cite{21} proved that let $f : M^2 \hookrightarrow S^N$ be a minimal immersion of a closed surface into the unit sphere $S^N$.
If there exists a point such that its preimage set consists of \( m \) distinct points in \( M^2 \), then \( \text{Vol}(M^2) \geq m\text{Vol}(S^2) = 4\pi m \). More generally, Li and Yau \[21\] proved both of Yau’s Conjecture and Willmore’s Conjecture for non-embedded case in dimension 2. On the other hand, Marques and Neves \[22\] completely proved Willmore’s Conjecture and showed that any non-totally geodesic closed minimal embedded surface in \( S^3 \) has volume greater than or equal to \( 2\pi^2 \), the volume of the Clifford torus \( M_{1,1} \). The equality holds only for the Clifford torus \( M_{1,1} \). This completes the proof for embedded case in dimension 2. Another related rigidity result is Lawson’s Conjecture (proved by Brendle \[2\]), i.e., the only embedded minimal torus in \( S^3 \) is the Clifford torus. For more details of minimal surfaces, please see \[1, 3, 23\], etc.

For \( n \geq 3 \), among minimal rotational hypersurfaces, Yau’s Conjecture was verified for \( 2 \leq n \leq 100 \) by Perdomo and Wei \[26\], and for all dimensions by Cheng-Wei-Zeng \[6\]. Remarkably, in the asymptotic sense, Imanen and White \[18\] proved Yau’s Conjecture in the class of topologically nontrivial (at least one of the components of \( S^{n+1} \setminus M^n \) is not contractible) closed minimal embedded hypersurfaces whose hypercones are area-minimizing in \( \mathbb{R}^{n+2} \). Namely, they showed \( \text{Vol}(M^n) > \sqrt{2}\text{Vol}(S^n) \), where
\[
\sqrt{2} = \lim_{k \to \infty} \frac{\text{Vol}(M_{k,k})}{\text{Vol}(S^n)} = \lim_{k \to \infty} \frac{\text{Vol}(M_{k,k+1})}{\text{Vol}(S^n)} \quad \text{for} \quad n = 2k \quad \text{or} \quad n = 2k + 1 \text{ respectively.}
\]

In this paper, we study the monotonicity formula of Choe and Gulliver \[9\] (see Proposition 2.2) for any minimal submanifold in \( S^N \) similar to Euclidean space. For the classical monotonicity formula in Euclidean space, please see the excellent and elegant survey by Brendle \[4\]. As applications, we prove the following results. In particular, Theorem 1.2 generalizes the result of Li and Yau \[21\] for \( n = 2 \) to any dimension \( n \).

**Theorem 1.2.** Let \( f : M^n \hookrightarrow S^N \) be a closed minimal immersed submanifold in \( S^N \). If there exists a point such that its preimage set consists of \( m \) distinct points in \( M^n \), then
\[
\text{Vol}(M^n) \geq \frac{m}{2} \text{Vol}(S^n) + m\frac{\sqrt{n+1}}{n} \text{Vol}(S^{n-1}).
\]
Moreover, if \( M^n \) is invariant under the antipodal map, then \( \text{Vol}(M^n) \geq m\text{Vol}(S^n) \).

**Corollary 1.3.** Let \( M^n \) be a closed minimal non-embedded submanifold in \( S^N \). Then
\[
\text{Vol}(M^n) \geq (1 + p(n)) \text{Vol}(S^n),
\]
where \( p(n) = \frac{2\sqrt{n+1}}{\pi} \frac{\text{Vol}(S^{n-1})}{\text{Vol}(S^n)} \).

**Remark 1.4.** From Stirling’s approximation, we have
\[
\lim_{n \to +\infty} (1 + p(n)) = 1 + \sqrt{\frac{2}{\pi}} > 1.797.
\]
Besides, \( \text{Vol}(M_{k,n-k}) < (1 + p(n))\text{Vol}(S^n) \) for all \( 1 \leq k \leq n-1 \).

By Corollary 1.3 and Remark 1.4, we have
Theorem 1.5. Let $M^n$ be a closed minimal immersed submanifold in $S^N$. If
\[ \text{Vol}(M^n) \leq \max_{1 \leq k \leq n-1} \text{Vol}(M_{k,n-k}), \]
then $M^n$ must be embedded.

Remark 1.6. Theorem 1.5 shows that Yau’s Conjecture is correct for non-embedded hypersurfaces in $S^{n+1}$.

The preceding volume gaps make sense only for non-embedded submanifolds. In the following we give volume gaps that fit for embedded hypersurfaces, under some curvature conditions as in the famous Chern Conjecture (cf. [5, 25, 27, 30], etc), which is still open for $n > 3$ and states that a closed minimal immersed hypersurface in $S^{n+1}$ with constant scalar curvature is isoparametric.

Theorem 1.7. Let $M^n$ be a non-totally-geodesic closed minimal immersed hypersurface in $S^{n+1}$ with constant scalar curvature. If the third mean curvature is constant (or $M^n$ is Integral-Einstein, see Definition 4.3), then
\[ \text{Vol}(M^n) \geq \frac{n+2}{n+1} \text{Vol}(S^n). \]

Theorem 1.8. Let $M^n$ be a non-totally-geodesic closed minimal immersed hypersurface in $S^{n+1}$ with constant scalar curvature. If $M^n$ is invariant under the antipodal map, then
\[ \text{Vol}(M^n) \geq \frac{2n}{2n-1} \text{Vol}(S^n). \]

In fact, Theorem 1.8 is a special case of Theorem 1.5 where minimal hypersurfaces with non-constant scalar curvature are also considered. As an application, a new pinching rigidity result is obtained in Corollary 4.7.

2. The monotonicity formula for minimal submanifolds in spheres

Let $f : M^n \imath \hookrightarrow S^N \subset \mathbb{R}^{N+1}$ be a closed minimal immersed submanifold in the unit sphere $S^N$. For any fixed unit vector $a \in S^N$, we consider the height function on $M^n$,
\[ \varphi_a(x) = \langle f(x), a \rangle. \]

Then we have the following basic properties.

Proposition 2.1. [14, 29] For all $a \in S^N$, we have $-1 \leq \varphi_a \leq 1$, and
\[ \nabla \varphi_a = a^T, \quad \Delta \varphi_a = -n \varphi_a, \quad \int_M \varphi_a = 0, \]
where $a^T \in \Gamma(TM)$ denotes the tangent component of $a$ along $M^n$. 

In the following monotonicity formula for minimal submanifolds, our notations are different from that of Choe and Gulliver [9], therefore we give here a simple proof. Denote the level sets by
\[ \{ \varphi_a = t \} = \{ x \in \mathbb{R}^n : \varphi_a(x) = t \}, \quad \{ s \leq \varphi_a \leq t \} = \{ x \in \mathbb{R}^n : s \leq \varphi_a(x) \leq t \}. \]

**Proposition 2.2.** [9] For any fixed unit vector \(a \in \mathbb{S}^N\), if \(M \nsubseteq \{ \varphi_a = 0 \}\), then the function
\[ r \mapsto -\int_{\{ \varphi_a \geq r \}} \frac{\varphi_a}{(1 - r^2)^\frac{N}{2}} \]
is monotone increasing for \(-1 < r \leq 0\) and monotone decreasing for \(0 < r < 1\).

**Proof.** By Proposition 2.1
\[ \nabla \varphi_a = a^T, \quad \Delta \varphi_a = -n \varphi_a. \]
Due to the divergence theorem and (2.1)
\[ |a^T|^2 + \varphi_a^2 \leq 1, \]
one has for all \(-1 < t < 1\),
\[ \int_{\{ \varphi_a \geq t \}} \varphi_a \leq \int_{\{ \varphi_a \geq t \}} \frac{|a^T|}{n} \leq \int_{\{ \varphi_a \geq t \}} \frac{\sqrt{1 - \varphi_a^2}}{n} = \int_{\{ \varphi_a = t \}} \frac{\sqrt{1 - t^2}}{n}. \]
For all \(0 < s \leq r < 1\), by the Co-Area formula and (2.1, 2.2), we obtain
\[ \int_{\{ s \leq \varphi_a \leq r \}} \varphi_a = \int_s^r dt \int_{\{ \varphi_a \geq t \}} \frac{\varphi_a}{|a^T|} \geq \int_s^r dt \int_{\{ \varphi_a = t \}} \frac{\varphi_a}{\sqrt{1 - \varphi_a^2}} \]
\[ = \int_s^r dt \int_{\{ \varphi_a = t \}} \sqrt{1 - t^2} \]
\[ \geq \int_s^r dt \int_{\{ \varphi_a \geq t \}} \frac{n t}{1 - t^2} \varphi_a. \]
Thus, by (2.3)
\[ \frac{d}{dr} \int_{\{ \varphi_a \geq r \}} \varphi_a \leq -\int_{s \leq r} \frac{\varphi_a - \int_{\{ \varphi_a \geq r \}} \varphi_a}{r - s} \geq -\frac{nr}{1 - r^2} \int_{\{ \varphi_a \geq r \}} \varphi_a. \]
Since (2.1) and
\[ \int_{\{ \varphi_a \geq 0 \}} \varphi_a = \int_{\{ \varphi_a \geq r \}} \varphi_a + \int_{\{ \varphi_a \geq 0 \}} \varphi_a, \]
we have
\[ \frac{d}{dr} \int_{\{ \varphi_a \geq r \}} \varphi_a \leq -\frac{nr}{1 - r^2}. \]
Hence, for \(0 < s_1 \leq s_2 < 1\), one has
\[ \int_{s_1}^{s_2} \frac{d}{dr} \int_{\{ \varphi_a \geq r \}} \varphi_a \leq -\int_{s_1}^{s_2} \frac{nr}{1 - r^2} dr = \frac{n}{2} \ln \left( \frac{1 - s_2^2}{1 - s_1^2} \right), \]
and
\[ \int_{\{\varphi_a \geq s_1\}} \varphi_a \geq \int_{\{\varphi_a \geq s_2\}} \varphi_a. \]
This shows the monotonicity for \( 0 < r < 1 \).

For \(-1 < r \leq 0\), we just need to change \( \varphi_a \) to \(-\varphi_a\). Similarly by the Co-Area formula and \([2.1, 2.2]\), for all \(-1 < s \leq r < 0\), we obtain
\[ \int_{\{\varphi_a \leq s\}} -\varphi_a = \int_s^r dt \int_{\{\varphi_a = t\}} \frac{-\varphi_a}{|a|^2} \geq \int_s^r dt \int_{\{\varphi_a \geq t\}} \frac{-nt}{1-t^2} \varphi_a. \]
Hence
\[ \frac{d\ln \int_{\{\varphi_a \geq r\}} \varphi_a}{dr} \geq -\frac{nr}{1-r^2}, \]
where \( \int_{\{\varphi_a \geq r\}} \varphi_a = -\int_{\{\varphi_a \leq r\}} \varphi_a > 0 \) for \( r > \min \varphi_a \). Thus, by integrating the above inequality we have the monotonicity: for \(-1 < s_1 \leq s_2 < 0\),
\[ \frac{\int_{\{\varphi_a \geq s_1\}} \varphi_a}{(1-s_1^2)^{\frac{n}{2}}} \leq \frac{\int_{\{\varphi_a \geq s_2\}} \varphi_a}{(1-s_2^2)^{\frac{n}{2}}}. \]
\[ \square \]

3. THE VOLUME GAP FOR CLOSED MINIMAL SUBMANIFOLDS

Let \( f : M^n \hookrightarrow S^N \subset \mathbb{R}^{N+1} \) be a closed minimal immersed submanifold in the unit sphere \( S^N \) and let \( B^n \) denote the unit ball in \( \mathbb{R}^n \).

**Definition 3.1.** We define a function \( \xi \) on \( S^N \) and a constant \( \Xi \) on \( M^n \):
\[ \xi(a) = \liminf_{t \to 1^{-}} \frac{\text{Vol}\{\varphi_a \geq t\}}{(1-t^2)^{\frac{n}{2}}} \text{Vol}(B^n), \quad \Xi = \sup_{a \in f(M)} \xi(a). \]

By Proposition \([2.2]\) it is easy to show that \( \xi(a) \) and \( \Xi \) are well-defined. The following lemma gives a lower bound estimate for \( \xi(a) \) and \( \Xi \).

**Lemma 3.2.** For any point \( p \in f(M) \), if its preimage set consists of \( m(p) \) distinct points in \( M^n \), then \( \xi(p) \geq m(p) \). In particular, \( \Xi \geq 2 \) if \( f \) is not an embedding.

**Proof.** Since \( \varphi_p(x) = 1 \) if and only if \( f(x) = p \), it follows that
\[ \{x \in M^n : \varphi_p(x) = 1\} = f^{-1}(p) \neq \emptyset. \]
The properties of compactness and local embedding of \( M \) show that \( f^{-1}(p) \) is a finite set, say, \( f^{-1}(p) = \{q_1, q_2, ..., q_{m(p)}\} \) and \( q_i \in M \). For each \( i \), there is a neighborhood \( U_i \) of \( q_i \) such that \( f \) is a diffeomorphism from \( U_i \) to \( f(U_i) \), and by shrinking the \( U_i \)'s
if necessary, we may assume that they are pairwise disjoint. For any sufficiently small 
\( \delta > 0 \) and for any \( 1 - \delta < t < 1 \), we have 
\[ \{ \varphi_p \geq t \} \subset \bigcup_{i=1}^{m(p)} U_i. \]

Note that the intrinsic distance between two points of the submanifold is always greater 
than or equal to the distance of the ambient manifold. It follows that each \( \{ \varphi_p \geq t \} \cap U_i \) 
contains a geodesic ball \( B_r(q_i) \subset M \) of radius \( r \), where \( r = \arccos t \) is the radius of the 
geodesic ball \( \{ y \in S^N : \langle y, p \rangle \geq t \} \subset S^N \). Hence 
\[ \bigcup_{i=1}^{m(p)} B_r(q_i) \subset \{ \varphi_p \geq t \}. \]

Recalling the expansion formula of volume for the geodesic ball \( B_r(q_i) \) (cf. [16]) 
\[ \Vol(B_r(q_i)) = \Vol(B^n)r^n \left[ 1 - \frac{R_M(q_i)}{6(n+2)}r^2 + O(r^4) \right], \]
where \( R_M \) is the scalar curvature of \( M \), we have 
\[ \xi(p) = \liminf_{t \to 1^-} \frac{\Vol \{ \varphi_p \geq t \}}{(1-t^2)^{\frac{n}{2}} \Vol(B^n)} \geq \liminf_{r \to 0^+} \sum_{i=1}^{m(p)} \frac{\Vol(B_r(q_i))}{\sum_{i=1}^{m(p)} \sin^n r \Vol(B^n)} \]
\[ = \lim_{r \to 0^+} \sum_{i=1}^{m(p)} \frac{r^n (1 + O(r^2))}{\sin^n r} = m(p). \]

If \( f \) is not an embedding, we have 
\[ \Xi = \sup_{p \in f(M)} \xi(p) \geq \sup_{p \in f(M)} m(p) \geq 2. \]

\[ \square \]

**Lemma 3.3.** For any fixed unit vector \( a \in S^N \), if \( M \not\subset \{ \varphi_a = 0 \} \), then 
\[ \Vol \{ s \leq \varphi_a \leq r \} \geq n \xi(a) \Vol(B^n) \int_s^r (1 - t^2)^{\frac{n-2}{2}} dt, \]
for all \( 0 \leq s \leq r \leq 1 \).

**Proof.** By Proposition 2.2 for \( 0 \leq s_1 < s_2 < 1 \), we have 
\[ \frac{\int_{\{\varphi_a \geq s_1\}} \varphi_a}{(1 - s_1^2)^{\frac{n}{2}}} \geq \frac{\int_{\{\varphi_a \geq s_2\}} \varphi_a}{(1 - s_2^2)^{\frac{n}{2}}}. \]
Let $s_1 = t \geq 0$ and $s_2 \to 1^-$, then we have
\begin{equation}
\int_{\{\phi_a \geq t\}} \phi_a \geq (1 - t^2)^n \liminf_{s_2 \to 1^-} \frac{\int_{\{\phi_a \geq s_2\}} \phi_a}{(1 - s_2^2)^{n/2}} \geq (1 - t^2)^n \liminf_{s_2 \to 1^-} \frac{\int_{\{\phi_a \geq s_2\}} s_2}{(1 - s_2^2)^{n/2}}
\end{equation}
(3.1)
\begin{align*}
&= (1 - t^2)^n \liminf_{s_2 \to 1^-} \frac{\text{Vol} \{\phi_a \geq s_2\}}{(1 - s_2^2)^{n/2}} \liminf_{s_2 \to 1^-} s_2 \\
&= \xi(a) \text{Vol}(B^n) (1 - t^2)^n.
\end{align*}

Thus, similar to (2.3), we derive
\begin{align*}
\int_{\{s \leq \phi_a \leq r\}} 1 &= \int_s^r \int_{\{\phi_a = t\}} \frac{1}{|a|^2} \geq \int_s^r \int_{\{\phi_a = t\}} \frac{1}{\sqrt{1 - \phi_a^2}} \\
&= \int_s^r \int_{\{\phi_a = t\}} \frac{1}{\sqrt{1 - t^2}} \geq \int_s^r \int_{\{\phi_a \geq t\}} \frac{n}{1 - t^2} \phi_a \\
&\geq n\xi(a) \text{Vol}(B^n) \int_s^r (1 - t^2)^{n/2} dt.
\end{align*}

\[\square\]

**Proof of Theorem 1.2 and Corollary 1.3.** Without loss of generality, suppose $M \not\subset \{\phi_a = 0\}$ for any $a \in S^N$. Setting $s = 0$ and $r = 1$ in Lemma 3.3, we derive from Lemma 3.2 that for any $p \in f(M)$,
\begin{equation}
\int_{\{\phi_a \geq 0\}} 1 \geq n\xi(p) \text{Vol}(B^n) \int_0^1 (1 - t^2)^{n/2} dt \geq \frac{m(p)}{2} \text{Vol}(S^n).
\end{equation}
(3.2)

If $M^n$ is invariant under the antipodal map, then
\[\text{Vol}(M^n) = 2 \int_{\{\phi_a \geq 0\}} 1 \geq m(p) \text{Vol}(S^n).\]

For general case, we need to estimate the volume of $\{\phi_a \leq 0\}$. By Proposition 2.1, for any $a \in S^N$, $\int_M \phi_a = 0$, thus
\begin{equation}
\int_{\{\phi_a \geq 0\}} \phi_a = \int_{\{\phi_a \leq 0\}} -\phi_a.
\end{equation}
(3.3)

The divergence theorem shows that
\[\int_{\{\phi_a \leq 0\}} \Delta \phi_a^2 = 0,
\]
and by $\Delta \phi_a^2 = -2n\phi_a^2 + 2|a|^2$, one has
\begin{equation}
n \int_{\{\phi_a \leq 0\}} \phi_a^2 = \int_{\{\phi_a \leq 0\}} |a|^2.
\end{equation}
(3.4)

Then, due to (2.1) and (3.4), we have
\begin{equation}
(n + 1) \int_{\{\phi_a \leq 0\}} \phi_a^2 \leq \int_{\{\phi_a \leq 0\}} 1.
\end{equation}
(3.5)
By the Cauchy-Schwarz inequality and (3.5),

\[
\sqrt{\frac{1}{n+1}} \int_{\{\varphi_a \leq 0\}} 1 \geq \sqrt{\frac{1}{n+1}} \int_{\{\varphi_a \leq 0\}} \varphi_a^2 \geq \int_{\{\varphi_a \leq 0\}} -\varphi_a.
\]

Choose \(a = p \in f(M)\) and set \(t = 0\) in (3.1). Then

\[
\int_{\{\varphi_p \leq 0\}} \varphi_p \geq \xi(p) \Vol(B^n) \geq m(p) \Vol(B^n).
\]

Due to (3.3), (3.6) and (3.7), we get

\[
\sqrt{\frac{1}{n+1}} \int_{\{\varphi_p \leq 0\}} 1 \geq \int_{\{\varphi_p \leq 0\}} -\varphi_p = \int_{\{\varphi_p \geq 0\}} \varphi_p \geq m(p) \Vol(B^n).
\]

Hence, combining (3.2) and (3.8), we obtain

\[
\Vol(M^n) = \int_{\{\varphi_p \geq 0\}} 1 + \int_{\{\varphi_p \leq 0\}} 1 \geq \frac{m(p)}{2} \Vol(S^n) + m(p) \sqrt{n+1} \Vol(B^n).
\]

This completes the proof since \(\Vol(S^{n-1}) = n \Vol(B^n)\). If \(f\) is not an embedding, then \(m(p) \geq 2\) for some \(p \in f(M)\), which implies the corollary immediately. \(\Box\)

**Proof of Remark 1.4** For all \(k \geq 1\), by [18] one has

\[
\Vol(S^k) = (k+1) \Vol(B^{k+1}) = (k+1) \frac{\pi^{(k+1)/2}}{\Gamma\left(\frac{k+3}{2}\right)},
\]

where \(\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt\). Thus

\[
p(n) = \frac{2\sqrt{n+1}}{n} \Vol(S^{n-1}) = \frac{2}{\sqrt{n+1}} \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} < 1,
\]

\[
\Vol(M_{k,n-k}) = \left(\frac{k}{n}\right)^{\frac{1}{2}} \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \Vol(S^k) \Vol(S^{n-k})
\]

\[
= (k+1)(n-k+1) \left(\frac{k}{n-k}\right)^{\frac{1}{2}} \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \frac{\pi^{(n+2)/2}}{\Gamma\left(\frac{n+k+3}{2}\right)}.
\]

A direct calculation shows that

\[
\Vol(M_{k,n-k}) \leq \Vol(M_{1,n-1}), \quad 1 \leq k \leq n-1.
\]

Then for all \(n \geq 2\), one has

\[
\frac{(1 + p(n)) \Vol(S^n)}{\Vol(M_{k,n-k})} \geq \frac{(1 + p(n)) \Vol(S^n)}{\Vol(M_{1,n-1})}
\]

\[
= \frac{1}{\pi} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}} \left(1 + \frac{1}{p(n)}\right)
\]

\[
> \frac{2}{\pi} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}} > 1.
\]

\(\Box\)
4. The volume gap for closed minimal hypersurfaces

In this section, we give volume gaps for both immersed and embedded closed minimal hypersurfaces in $S^{n+1}$ under some conditions. Firstly, we prove Theorem 1.7 and we need the following lemmas.

**Lemma 4.1.** [12] Let $N^{n+1}$ be a complete, connected manifold with positive Ricci curvature. Let $V^n$ and $W^n$ be immersed minimal hypersurfaces of $N^{n+1}$, each immersed as a closed subset, and let $V^n$ be compact. Then $V^n$ and $W^n$ must intersect.

**Lemma 4.2.** Let $f : M^n \to S^{n+1}$ be a closed minimal immersed hypersurface. There exists a unit vector $a \in f(M)$ such that

$$\int_M \varphi_a^2 \geq \frac{1}{n+1} \text{Vol}(S^n).$$

In particular, if $M^n$ is invariant under the antipodal map, then (4.1) holds for any unit vector $a \in f(M)$.

**Proof.** By Lemma 4.1 (or Hsiang [17]), there exists a unit vector $a \in f(M)$ such that $-a \in f(M)$, since otherwise, we can find two disjoint minimal hypersurfaces by antipodal map. By the Co-Area formula, (2.2), Proposition 2.2 and Lemma 3.2, we have

$$\int_M \varphi_a^2 = \int_0^1 dt \int_{\{\varphi_a = t\}} \varphi_a^2 |a^T| \geq \int_0^1 dt \int_{\{\varphi_a = t\}} \varphi_a^2 |1 - \varphi_a^2|$$

$$= \int_0^1 dt \int_{\{\varphi_a = t\}} \frac{t^2}{\sqrt{1-t^2}} \geq \int_0^1 dt \int_{\{\varphi_a \geq t\}} \frac{n t^2}{1-t^2} |\varphi_a|$$

$$\geq n \lim_{u \to 1^-} \frac{\text{Vol}\{\{\varphi_a \geq u\}\}}{(1-u^2)^{\frac{n}{2}}} \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$= (\xi(a) + \xi(-a)) n \text{Vol}(B^n) \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$\geq 2n \text{Vol}(B^n) \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$= \frac{1}{n+1} \text{Vol}(S^n).$$

□

**Definition 4.3.** [14] Let $M^n$ ($n \geq 3$) be a compact submanifold in the Euclidean space $\mathbb{R}^N$. We call $M^n$ an Integral-Einstein (IE) submanifold if for any unit vector $a \in S^{N-1}$,

$$\int_M \left( \text{Ric} - \frac{R}{n} g \right) (a^T, a^T) = 0,$$

where $a^T \in \Gamma(TM)$ denotes the tangent component of the constant vector $a$ along $M^n$. 

**Proof.** By Lemma 4.1 (or Hsiang [17]), there exists a unit vector $a \in f(M)$ such that $-a \in f(M)$, since otherwise, we can find two disjoint minimal hypersurfaces by antipodal map. By the Co-Area formula, (2.2), Proposition 2.2 and Lemma 3.2, we have

$$\int_M \varphi_a^2 = \int_0^1 dt \int_{\{\varphi_a = t\}} \varphi_a^2 |a^T| \geq \int_0^1 dt \int_{\{\varphi_a = t\}} \varphi_a^2 |1 - \varphi_a^2|$$

$$= \int_0^1 dt \int_{\{\varphi_a = t\}} \frac{t^2}{\sqrt{1-t^2}} \geq \int_0^1 dt \int_{\{\varphi_a \geq t\}} \frac{n t^2}{1-t^2} |\varphi_a|$$

$$\geq n \lim_{u \to 1^-} \frac{\text{Vol}\{\{\varphi_a \geq u\}\}}{(1-u^2)^{\frac{n}{2}}} \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$= (\xi(a) + \xi(-a)) n \text{Vol}(B^n) \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$\geq 2n \text{Vol}(B^n) \int_0^1 t^2 (1-t^2) \frac{n-2}{2} dt$$

$$= \frac{1}{n+1} \text{Vol}(S^n).$$

□

**Definition 4.3.** [14] Let $M^n$ ($n \geq 3$) be a compact submanifold in the Euclidean space $\mathbb{R}^N$. We call $M^n$ an Integral-Einstein (IE) submanifold if for any unit vector $a \in S^{N-1}$,
Proof of Theorem 1.7. For any \( a \in S^{n+1} \), the height functions (cf. [14, 15], etc) are defined as
\[
\varphi_a(x) = \langle f(x), a \rangle, \quad \psi_a(x) = \langle \nu, a \rangle,
\]
where \( \nu \) is the unit normal vector field along \( x \in M_n \). In [14], we have shown that for a non-totally-geodesic closed minimal immersed hypersurface \( M_n \) in \( S^{n+1} \) with constant scalar curvature, \( M \) is IE if and only if one of the following equivalent conditions holds:

- \( \int_M \varphi_a^2 = \frac{1}{n+2} \text{Vol}(M^n) \) for all \( a \in S^{n+1} \);
- \( \int_M \psi_a^2 = \frac{1}{n+2} \text{Vol}(M^n) \) for all \( a \in S^{n+1} \);
- \( \int_M \varphi_a^2 = \int_M \psi_a^2 \) for all \( a \in S^{n+1} \);
- \( \int_M \varphi_a \psi_a f_3 = 0 \) for all \( a \in S^{n+1} \), where \( f_3 = \text{Tr}(A^3) = 3(n)H_3 \), \( A \) is the shape operator with respect to the unit normal vector field \( \nu \) and \( H_3 \) is the third mean curvature.

If further \( M \) has the constant squared length of second fundamental form \( S := |A|^2 > n \) and has constant third mean curvature \( H_3 \), then by the fourth condition above, \( M \) is IE (cf. [14]). This is because \( \varphi_a \) and \( \psi_a \) are eigenfunctions of eigenvalues \( n \) and \( S \) respectively, and thus they are orthogonal. Hence, Lemma 4.2 and the first condition imply that for these IE hypersurfaces, there exists a unit vector \( a \in f(M) \) such that
\[
\text{Vol}(M^n) = (n+2) \int_M \varphi_a^2 \geq \frac{n+2}{n+1} \text{Vol}(S^n).
\]

On the other hand, Simons’ inequality [28] shows that if \( 0 \leq S \leq n \), then either \( S \equiv 0 \) or \( S \equiv n \) on \( M \). The case of \( S \equiv n \) was characterized by Chern-do Carmo-Kobayashi [8] and Lawson [19] independently: the Clifford torus \( M_{k,n-k} (1 \leq k \leq n-1) \) are the only closed minimal hypersurfaces in \( S^{n+1} \) with \( S \equiv n \). It is easy to verify that \( \text{Vol}(M_{k,n-k}) \geq \frac{2n^2}{n+1} \text{Vol}(S^n) \) by the volume formulas in the proof of Remark 1.4. □

Without assuming constant scalar curvature, i.e., \( S \neq \text{Constant} \), we are able to obtain Theorem 4.5 which implies Theorem 1.8.

Suppose
\[
S_{\text{max}} = \sup_{p \in M^n} S(p), \quad S_{\text{min}} = \inf_{p \in M^n} S(p), \quad C(n, S) = \max\{\theta_1, \theta_2\},
\]
where
\[
\theta_1 = \frac{\int_M S}{2nS_{\text{max}} \text{Vol}(M^n)}, \quad \theta_2 = \frac{n}{4n^2 - 3n + 1} \frac{(\int_M S)^2}{\text{Vol}(M^n) \int_M S^2}.
\]

Lemma 4.4. [14] Let \( M^n \) be a closed minimal hypersurface in \( S^{n+1} \).

(i) If \( S \neq 0 \), then
\[
\frac{\int_M S}{2nS_{\text{max}}} \leq \inf_{a \in S^{n+1}} \int_M \varphi_a^2.
\]
The equality holds if and only if $S \equiv n$ and $M$ is the minimal Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$.

(ii) 
\[
\frac{n}{4n^2 - 3n + 1} \left( \int_M S \right)^2 \leq \int_M S^2 \inf_{a \in S^{n+1}} \int_M \phi_a^2.
\]

The equality holds if and only if $M$ is an equator.

**Theorem 4.5.** Let $f : M^n \to S^{n+1}$ be a non-totally-geodesic closed minimal immersed hypersurface. If $M$ is invariant under the antipodal map, then
\[
\text{Vol}(M^n) \geq \frac{1}{1 - C(n,S)} \text{Vol}(S^n) \geq \frac{2n S_{\max}}{2n S_{\max} - S_{\min}} \text{Vol}(S^n).
\]

In particular, if $S$ is constant, then $C(n,S) = \frac{1}{2n}$.

**Proof.** By Proposition 2.1 similar to (3.4) one has
\[
n \int_M \phi_a^2 = \int_M |a^T|^2.
\]

Note that $|a^T|^2 + \phi_a^2 + \psi_a^2 = 1$ and (4.1) holds for all $a \in f(M)$ when $f(M)$ is symmetric about the origin. Then by (4.1) and (4.3), we have
\[
\int_M (1 - \psi_a^2) \geq \text{Vol}(S^n),
\]
for all $a \in f(M)$. Integrating $a = f(y)$ over $y \in M$ on both sides of (4.4), we have
\[
\text{Vol}(M^n) - \text{Vol}(S^n) \geq \frac{\int_{x \in M} \int_{y \in M} \langle \nu_x, f(y) \rangle^2}{\text{Vol}(M^n)} \geq \inf_{a \in S^{n+1}} \int_M \phi_a^2.
\]

By Lemma 4.4, we have
\[
\inf_{a \in S^{n+1}} \int_M \phi_a^2 \geq C(n,S) \text{Vol}(M^n) \geq \frac{\text{Vol}(M^n)}{2n S_{\max}} \geq \frac{S_{\min}}{2n S_{\max}} \text{Vol}(M^n).
\]

Combining (4.5) and (4.6) completes the proof. \qed

As applications, we obtain the following rigidity results.

**Corollary 4.6.** Let $M^n$ be a closed minimal immersed hypersurface in $S^{n+1}$ which is invariant under the antipodal map. For any $\delta \geq 0$, if $n \leq S \leq n + \delta$, then
\[
\text{Vol}(M^n) \geq \frac{2(n + \delta)}{2(n + \delta) - 1} \text{Vol}(S^n).
\]

**Proof.** Due to $n \leq S \leq n + \delta$, we have
\[
\frac{S_{\min}}{S_{\max}} \geq \frac{n}{n + \delta}.
\]
By Theorem 4.5, one has

\[ \text{Vol}(M^n) \geq \frac{2nS_{\text{max}}}{2nS_{\text{max}} - S_{\text{min}}} \text{Vol}(S^n) \geq \frac{2(n + \delta)}{2(n + \delta) - 1} \text{Vol}(S^n). \]

\[ \square \]

Corollary 4.7. Let \( M^n \) be a closed minimal immersed hypersurface in \( S^{n+1} \) which is invariant under the antipodal map. For any \( \delta \leq \frac{3}{8} n \), if the following conditions are satisfied:

(i) \( S \leq n + \delta \leq \frac{11}{8} n \),

(ii) \( \text{Vol}(M^n) \leq \frac{3(4n^2-3n+1)}{3(4n^2-4n+1)+8\delta} \text{Vol}(S^n) \),

then \( M \) is totally geodesic.

Proof. Without loss of generality, we suppose \( M^n \) is a non-totally geodesic closed minimal embedded hypersurface in \( S^{n+1} \) because of Corollary 1.3. Besides, if \( S \) is constant, then a contradiction follows directly from Theorem 4.5. Let \( h \) denote the second fundamental form of hypersurface with respect to the unit normal vector field \( \nu \). If \( \{\omega_1, \omega_2, \ldots, \omega_n\} \) is a local orthonormal coframe field, then \( h \) can be written as

\[ h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j. \]

The covariant derivative \( \nabla h \) with components \( h_{ijk} \) is given by

\[ \sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kij}\omega_k + \sum_k h_{ijk}\omega_k, \]

and \( \{\omega_{ij}\} \) are the connection forms of \( M \) with respect to \( \{\omega_1, \omega_2, \ldots, \omega_n\} \), which satisfy the following structure equations:

\[ d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \]

\[ d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \]

where \( \{R_{ijkl}\} \) are the coefficients of the Riemannian curvature tensor on \( M \). Hence

\[ S = \sum_{i,j} h_{ij}^2, \quad |\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2. \]

By the Cauchy-Schwarz inequality, one has

\[ |\nabla S|^2 = 4 \sum_k \left( \sum_{i,j} h_{ij} h_{ijk} \right)^2 \leq 4 \sum_k \left( \sum_{i,j} h_{ij}^2 \sum_{i,j} h_{ijk}^2 \right) = 4S|\nabla h|^2. \]
By Simons’ identity [28], we have
\begin{equation}
\frac{1}{2} \Delta S = |\nabla h|^2 + S(n - S). \tag{4.8}
\end{equation}
Due to (4.8) and
\begin{equation}
\frac{1}{2} \Delta S^2 = S \Delta S + |\nabla S|^2,
\end{equation}
we obtain
\begin{equation}
\int_M |\nabla S|^2 = 2 \int_M (S^2(S - n) - S|\nabla h|^2). \tag{4.9}
\end{equation}
By (4.7) and (4.9), one has
\begin{equation}
\int_M S^2(S - n) \leq \frac{3}{4} \int_M |\nabla S|^2 \leq \frac{4}{3} (S_{\text{max}} - n) \int_M S^2. \tag{4.10}
\end{equation}
Since $S$ is not constant, using Rayleigh’s formula, one has
\begin{equation}
\int_M |\nabla S|^2 \geq \lambda_1(M) \left( \int_M S^2 - \frac{(\int_M S)^2}{\text{Vol}(M^n)} \right), \tag{4.11}
\end{equation}
where $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian. In addition, Choi and Wang [10] proved that $\lambda_1(M) \geq n/2$. A careful argument (see [3, Theorem 5.1]) showed that the strict inequality holds, i.e., $\lambda_1(M) > n/2$. By (4.10) and (4.11), we obtain
\begin{equation}
\left( \frac{4}{3} (S_{\text{max}} - n) \int_M S^2 \right) > \frac{n}{2} \left( \int_M S^2 - \frac{(\int_M S)^2}{\text{Vol}(M^n)} \right),
\end{equation}
Hence
\begin{equation}
\frac{(\int_M S)^2}{\text{Vol}(M^n)} \int_M S^2 > 1 - \frac{8}{3n} (S_{\text{max}} - n) \geq \frac{3n - 8\delta}{3n} \geq 0,
\end{equation}
and
\begin{equation}
C(n, S) \geq \theta_2 = \frac{n}{4n^2 - 3n + 1} \frac{(\int_M S)^2}{\text{Vol}(M^n)} \int_M S^2 \geq \frac{3n - 8\delta}{3(4n^2 - 3n + 1)}.
\end{equation}
By Theorem 4.5, we get
\begin{equation}
\text{Vol}(M^n) > \frac{3}{3(4n^2 - 3n + 1)} \text{Vol}(S^n),
\end{equation}
a contradiction to the assumption of volume. \hfill \Box

**Remark 4.8.** The following rigidity result is well known (cf. [11, 20, 25], etc):
Let $M^n$ be a closed minimal immersed hypersurface in $S^{n+1}$ with $n \leq S \leq n + \delta$. If $\delta \leq \frac{n}{18}$, then $S \equiv n$ and $M^n$ is a Clifford torus.

In fact, due to some counterexamples of Otsuki [24], the condition $S \geq n$ is essential in the pinching result above. Comparing with Corollary 4.7, we have larger pinching constant $\frac{1}{4}n$ and do not need $S \geq n$, but we need to limit the symmetry and volume.
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