Clique Coverings and Claw-free Graphs

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Abstract

Let \( C \) be a clique covering for \( E(G) \) and let \( v \) be a vertex of \( G \). The valency of vertex \( v \) (with respect to \( C \)), denoted by \( \text{val}_C(v) \), is the number of cliques in \( C \) containing \( v \). The local clique cover number of \( G \), denoted by \( \text{lcc}(G) \), is defined as the smallest integer \( k \), for which there exists a clique covering for \( E(G) \) such that \( \text{val}_C(v) \) is at most \( k \), for every vertex \( v \in V(G) \). In this paper, among other results, we prove that if \( G \) is a claw-free graph, then \( \text{lcc}(G) + \chi(G) \leq n + 1 \).

Keywords: edge clique covering; local clique covering; chromatic number; claw-free graph; sigma clique partition; Nordhaus-Gaddum inequality.

MSC: 05C70

1 Introduction

Throughout the paper, all graphs are simple and undirected. By a clique of a graph \( G \), we mean a subset of mutually adjacent vertices of \( G \) as well as its corresponding complete subgraph. The size of a clique is the number of its vertices. A clique covering for \( E(G) \) is defined as a family of cliques of \( G \) such that every edge of \( G \) lies in at least one of the cliques comprising this family.

Let \( C \) be a clique covering for \( E(G) \) and let \( v \) be a vertex of \( G \). Valency of vertex \( v \) (with respect to \( C \)), denoted by \( \text{val}_C(v) \), is defined to be the number of cliques in \( C \) containing \( v \). A number of different variants of the clique cover number have been investigated in the literature. The local clique cover number of \( G \), denoted by \( \text{lcc}(G) \), is defined as the smallest

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integer $k$, for which there exists a clique covering for $G$ such that $val_c(v)$ is at most $k$, for every vertex $v \in V(G)$.

This parameter may be interpreted as a variety of different invariants of the graph and the problem relates to some well-known problems such as line graphs of hypergraphs, intersection representation and Kneser representation of graphs. For example, lcc($G$) is the minimum integer $k$ such that $G$ is the line graph of a $k$-uniform hypergraph. By this interpretation, lcc($G$) $\leq 2$ if and only if $G$ is the line graph of a multigraph.

There is a characterization by a list of seven forbidden induced subgraphs and a polynomial-time algorithm for the recognition that $G$ is the line graph of a multigraph [3, 15]. On the other hand, L. Lovász in [16] proved that there is no characterization by a finite list of forbidden induced subgraphs for the graphs which are line graphs of some 3-uniform hypergraphs. Moreover, it was proved that the decision problem whether $G$ is the line graph of a $k$-uniform hypergraph, for fixed $k \geq 4$, and the problem of recognizing line graphs of 3-uniform hypergraphs without multiple edges are NP-complete [18].

For a vertex $v \in V(G)$, its (open) neighborhood $N(v)$ is the set of all neighbors of $v$ in $G$, while its closed neighborhood $N[v]$ is defined as $N[v] := N(v) \cup \{v\}$. Moreover, let $\overline{G}$ stand for the complement of $G$, and let $\Delta(G)$ and $\delta(G)$ be the maximum degree and the minimum degree of $G$, respectively. The subgraph induced by a set $Y \subset V(G)$ will be denoted by $G[Y]$. By the notations of $\alpha(G)$, $\omega(G)$, and $\chi(G)$ we mean the independence number, clique number, and chromatic number of $G$, respectively.

In 1956 E. A. Nordhaus and J. W. Gaddum proved the following theorem for the chromatic number of a graph $G$ and its complement, $\overline{G}$.

**Theorem 1.** [17] Let $G$ be a graph on $n$ vertices. Then $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

Later on, similar results for other graph parameters have been found which are known as Nordhaus-Gaddum type theorems. In the literature there are several hundred papers considering inequalities of this type for many other graph invariants. For a survey of Nordhaus-Gaddum type estimates see [1].

In this paper, we consider the following two conjectures on local clique cover number.

**Conjecture 2.** For every graph $G$ on $n$ vertices,

$$lcc(G) + lcc(\overline{G}) \leq n.$$ \hspace{1cm} (1)

This conjecture proposed by R. Javadi, Z. Maleki and B. Omoomi in 2012. Note that Conjecture 2 is a Nordhaus-Gaddum type inequality concerning the local clique cover number of $G$.

The second author with R. Javadi and B. Omoomi suggested the following weakening of Conjecture 2.
Conjecture 3. For every graph $G$ on $n$ vertices,

$$\text{lcc}(G) + \chi(G) \leq n + 1. \tag{2}$$

Let $G_1$ and $G_2$ be graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. The disjoint union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2).

Proof. Let $G$ be a graph on $n$ vertices. Then $\chi(G) \leq n$. Hence, assuming that each member of $G$ satisfies Conjecture 2, we have $\chi(G) \leq n - 1 - k$. Thus, inequality (2) holds for $G$. Also, if $G$ is a triangle-free graph, then for a vertex $v$ which has the maximum degree in $G$, $\chi(G) \leq n + 1 - \Delta(G)$. Since $\chi(G) \leq n + 1 - \Delta(G)$, Conjecture 3 is true for triangle-free graphs. In what follows we prove that not only (2) but also (1) holds if $G$ is triangle-free.

**Theorem 5.** Let $G$ be a graph on $n$ vertices. If $\alpha(G) = 2$, then $\text{lcc}(G) + \text{lcc}(\overline{G}) \leq n$. 

Proof. Clearly, $\text{lcc}(\overline{G}) \leq \Delta(\overline{G}) = n - 1 - \delta(G)$. It is enough to show that $\text{lcc}(G) \leq \delta(G) + 1$. Let $v$ be a vertex of minimum degree in $G$, and let $K \subseteq V(G)$ be the set of vertices which are not adjacent to $v$. Since $\alpha(G) = 2$, the induced subgraph on $K$, $G[K]$, is a clique in $G$. Now, for every vertex $u_i \in N(v)$, let $C_i := (N(u_i) \cap K) \cup \{u_i\}$ and define $C_{\delta(G)+1} := G[K]$. These cliques, along with the collection of those edges which are not covered by the cliques $C_1, \ldots, C_{\delta(G)+1}$ comprise a clique covering for $G$, say $\mathcal{C}$. It can be checked easily that $\text{val}_{\mathcal{C}}(v) = \delta(G)$ and $\text{val}(x) \leq \delta(G) + 1$, for every vertex $x \in V(G) - v$.

It is well-known that $\frac{n}{\alpha(G)}$ and $\omega(G)$ are lower bounds for $\chi(G)$, the chromatic number of $G$. We show that, if we replace $\chi(G)$ with any of these two general lower bounds in Conjecture 3, then the inequality holds.

**Proposition 6.** Let $G$ be a graph with $n$ vertices. Then $\text{lcc}(G) + \omega(G) \leq n + 1$. 

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Proof. Assume that $K \subset V(G)$ is a clique of size $\omega$. For every vertex $v_i \in V(G) - K$, $1 \leq i \leq n - \omega$, define $C_i := (N(v_i) \cap K) \cup \{v_i\}$, and let $C_{n+1} := G[K]$. Now, let $F$ be the set of all the edges which are not covered by the cliques $C_1, \ldots, C_{n-\omega+1}$. Clearly, the cliques $C_i$ for $1 \leq i \leq n - \omega + 1$ together with $F$ form a clique covering $\mathcal{C}$ for $G$. If $x \in K$, then $val_{\mathcal{C}}(x) \leq 1 + n - \omega(G)$, and for vertex $v_i \in V(G) - K$, $val_{\mathcal{C}}(v_i) \leq n - \omega(G)$. \hfill \Box

Before proving the other inequality $lcc(G) + \frac{n}{\alpha(G)} \leq n + 1$, we verify a stronger statement involving local parameters. Let $\alpha_G(v) = \alpha(G[N(v)])$ be the maximum number of independent vertices in the neighborhood of vertex $v$, and let the local independence number of graph $G$ be defined as $\alpha_L(G) = \max_{v \in V(G)} \alpha_G(v)$. Clearly, $\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)$. Further, $\alpha_G(v) \geq 1$ holds if and only if $v$ has at least one neighbor, while $\alpha_G(v) \leq 1$ is equivalent to that the closed neighborhood $N_G[v] = N(v) \cup \{v\}$ induces a clique.

**Theorem 7.** For every graph $G$ of order $n$, there exists a clique covering $\mathcal{C}$ such that for each non-isolated vertex $v \in V(G)$ the inequality $val_{\mathcal{C}}(v) + \frac{n}{\alpha_G(v)} \leq n + 1$ holds.

**Proof.** A clique covering will be called **good** if it satisfies the requirement given in the theorem. Since the statement is true for all graphs of order $n \leq 3$, we may proceed by induction on $n$. Let $x$ and $y$ be two adjacent vertices of $G$. By the induction hypothesis, there is a good clique covering, $\mathcal{C}'$, for $G' = G - \{x, y\}$. We introduce the notations $N_1 := N(x) - N[y], N_2 := N(y) - N[x], \text{ and } N_{1,2} := N(x) \cap N(y)$. To obtain a good clique covering $\mathcal{C}$ of $G$ from $\mathcal{C}'$, we perform the following steps.

1. To handle vertices whose neighbors are completely adjacent, observe that every vertex $u$ from $N_1 \cup N_2 \cup N_{1,2}$ with $\alpha_G(u) = 1$ and $\deg_{G'}(u) \geq 1$ has $\alpha_{G'}(u) = 1$ and hence it is covered by the clique $N_G'[u]$ in the good covering $\mathcal{C}'$. Now, for each such vertex $u$, $N_{G'}'[u]$ is extended by $x$, by $y$ or by both $x$ and $y$ respectively, if $u \in N_1$, $u \in N_2$ or $u \in N_{1,2}$.

2. If $\alpha_G(x) = 1 < \alpha_G(y)$, take the clique $N_G[x]$; if $\alpha_G(y) = 1 < \alpha_G(x)$, take the clique $N_G[y]$; and if $\alpha_G(x) = \alpha_G(y) = 1$, take the clique $N_G[x] = N_G[y]$ into the covering $\mathcal{C}$ (if they were not included in step (1)).

3. If there still exist some uncovered edges between $x$ and $N_1$, we consider the set $N_1' = \{v \in N_1 \mid xv \text{ is uncovered}\}$ and partition it into some number of adjacent vertex pairs (inducing independent edges) and at most $\alpha(G(N_1'))$ isolated vertices. Then, we extend each of them with $x$ to a $K_2$ or $K_3$, and insert these cliques into the covering $\mathcal{C}$. This way, we get at most $\frac{|N_1| - \alpha(G(N_1'))}{2} + \alpha(G(N_1'))$ new cliques. Then, we define $N_2'$ and $N_{1,2}'$ analogously, and do the corresponding partitioning procedure for $N_2$ and $N_{1,2}$, extending every part of those partitions with $y$ or with $\{x, y\}$, respectively.

4. If the edge $xy$ remained uncovered, we take it as a clique into the covering $\mathcal{C}$.
It is easy to check that $C$ is a clique covering in $G$. We prove that it is good.

First note that after performing Step 1, each vertex $v \in V(G) - \{x, y\}$ has the same valency as in $C'$. Moreover, if two adjacent vertices, say $u$ and $x$, have $\alpha_G(u) = \alpha_G(x) = 1$, then $N_G[u] = N_G[x]$ must hold. Hence, if $u \in V(G) - \{x, y\}$ and $\alpha_G(u) = 1$, then $u$ is incident with only one clique from $C$. Thus, $\text{val}_C(u) + \frac{n}{\alpha_G(u)} = 1 + n$. If $v$ is a vertex from $V(G) - \{x, y\}$ and $\alpha_G(v) \geq 2$, then the valency of $v$ might increase in Step 2 or 3, but not in both. Therefore, $\text{val}_C(v) \leq \text{val}_C(v) + 1$, and clearly $\alpha_G(v) \leq \alpha_G(v)$. Since $C'$ is assumed to be good, these facts together imply

$$\text{val}_C(v) + \frac{n}{\alpha_G(v)} \leq \text{val}_C(v) + 1 + \frac{n - 2}{\alpha_G(v)} + \frac{2}{\alpha_G(v)} \leq n + 1.$$ 

Now, consider the vertex $x$. If $\alpha_G(x) = 1$, it is covered by only one clique (induced by its closed neighborhood), which was added to $C$ in Step 1 or 2. In this case $\text{val}_C(x) + \frac{n}{\alpha_G(x)} = n + 1$. Also if $\alpha_G(x) \geq \frac{n}{2}$, the trivial bound $\text{val}_C(x) \leq \deg(x) \leq n - 1$ implies the desired inequality. Hence, we may suppose $2 \leq \alpha_G(x) < \frac{n}{2}$.

Let us denote by $s$ the number of cliques covering $x$ which were added to $C$ in Step 1. Choose one vertex $u_i$ with $\alpha_G(u_i) = 1$ from each of these $s$ cliques. The closed neighborhoods $N[u_i]$ are pairwise different cliques. Thus, the obtained vertex set $S$ is independent. By the definitions of $N'_1$ and $N'_{1,2}$, there exist no edges between $S$ and $N'_1 \cup N'_{1,2}$. Thus, $\alpha(G(N'_1)) \leq \alpha_G(x) - s$ and $\alpha(G(N'_{1,2})) \leq \alpha_G(x) - s$. Also, $|N'_1| + |N'_{1,2}| \leq |N_1| + |N_{1,2}| - s = \deg(x) - 1 - s$ follows.

- If $N_{1,2} \neq \emptyset$ and $\alpha_G(y) > 1$, then

$$\text{val}_C(x) \leq \frac{|N'_1| - \alpha(G(N'_1))}{2} + \alpha(G(N'_1)) + \frac{|N'_{1,2}| - \alpha(G(N'_{1,2}))}{2} + \alpha(G(N'_{1,2})) + s$$

$$= \frac{|N'_1| + |N'_{1,2}|}{2} + \frac{\alpha(G(N'_1)) + \alpha(G(N'_{1,2}))}{2} + s$$

$$\leq \frac{\deg(x)}{2} - s + \frac{2\alpha(x) - 2s}{2} + s \leq \frac{n - 2}{2} + \alpha_G(x).$$

On the other hand, our assumption $2 \leq \alpha_G(x) < \frac{n}{2}$ implies that $\alpha_G(x) + \frac{n}{\alpha_G(x)} \leq 2 + \frac{n}{2}$. Thus,

$$\text{val}_C(x) + \frac{n}{\alpha_G(x)} \leq \frac{n - 2}{2} + \alpha_G(x) + \frac{n}{\alpha_G(x)} \leq \frac{n - 2}{2} + 2 + \frac{n}{2} = n + 1.$$ 

- If $N_{1,2} \neq \emptyset$ and $\alpha_G(y) = 1$, all edges between $N_{1,2}$ and $x$ are covered by the clique $N_G[y]$, which was added to $C$ in Step 2 (or maybe earlier, in Step 1). Hence, $N'_{1,2} = \emptyset$
and we have
\[
\text{val}_C(x) \leq \frac{|N'_1| - \alpha(G(N'_1))}{2} + \alpha(G(N'_1)) + 1 + s
\]
\[
= \frac{|N'_1|}{2} + \frac{\alpha(G(N'_1))}{2} + 1 + s
\]
\[
\leq \frac{\text{deg}(x) - 1 - s}{2} + \frac{\alpha_G(x) - s}{2} + 1 + s \leq \frac{n - 2}{2} + \alpha_G(x).
\]

Again, we may conclude \(\text{val}_C(x) + \frac{n}{\alpha_G(x)} \leq n + 1\).

- If \(N_{1,2} = \emptyset\), the clique \(xy\) was added to \(C\) in Step 4, and the same estimation holds as in the previous case.

One can show similarly that \(\text{val}_C(y) + \frac{n}{\alpha_G(y)} \leq n + 1\). This completes the proof. \(\square\)

Since for every \(v \in V(G)\), \(\alpha_G(v) \leq \alpha_L(G) \leq \alpha(G)\), we have the following immediate consequences.

**Corollary 8.** Let \(G\) be a graph of order \(n\). Then

(i) \(\text{lcc}(G) + \frac{n}{\alpha_L(G)} \leq n + 1\);

(ii) \(\text{lcc}(G) + \frac{n}{\alpha(G)} \leq n + 1\).

On the other hand, \(\text{val}_C(v) \geq \alpha_G(v)\), for every arbitrary clique covering \(C\). Hence, \(\text{lcc}(G) \geq \alpha_L(G)\). (But \(\text{lcc}(G) < \alpha(G)\) may be true.) Also, it is easy to see that \(\text{lcc}(G) \geq \frac{\Delta(G)}{\omega - 1}\). Next we observe that replacing \(\text{lcc}(G)\) with \(\alpha(G)\) or \(\frac{\Delta(G)}{\omega - 1}\) in Conjecture 3, valid inequalities are obtained.

**Proposition 9.** If \(G\) is a graph on \(n\) vertices, then

1. \(\frac{\Delta(G)}{\omega - 1} + \chi(G) \leq n + 1\), and equality holds if and only if \(G\) is the complete graph \(K_n\) or the star \(K_{1,n-1}\);

2. \(\alpha(G) + \chi(G) \leq n + 1\), and equality holds if and only if there exists a vertex \(v \in V(G)\) such that \(N(v)\) induces a complete graph and \(V(G) \setminus N(v)\) is an independent set.

**Proof.** To prove (1), first note that it is showed in [10] that there are only two types of graphs \(G\) for which \(\chi(G) + \chi(G) = n + 1\),

(a) if \(V(G) = K \cup S\) where \(K\) is a clique and \(S\) is an independent set, sharing a vertex \(K \cap S = \{u\}\), or
(b) $G$ is obtained from (a) by substituting $C_5$ into $u$.

Now, we estimate $\frac{\Delta(G)}{\omega-1} + \chi(G)$ as follows. We write $\theta$ for the clique covering number (minimum number of complete subgraphs whose union is the entire vertex set, that is the chromatic number of the complement graph). Let $x$ be a vertex of degree $\Delta = \Delta(G)$. We have

$$\frac{\Delta}{\omega - 1} \leq \theta(G[N(x)]) \leq \theta(G) \leq n + 1 - \chi(G),$$

where the last inequality is the Nordhaus-Gaddum theorem (Theorem [I]). Thus, in order to have $\frac{\Delta}{\omega - 1} + \chi = n + 1$, it is necessary that $G$ is of type (a) or (b). We shall see that (b) is not good enough, and (a) yields $G = K_n$ or $G = K_{1,n-1}$.

Note that equality does not hold for $G = C_5$, therefore in (b) we have $k > 0$. Let $|K - u| = k$ and $|S - u| = s$ in (a). Then after substitution of $C_5$, we have $n = k + s + 5$, $\Delta \leq n - 1$, $\omega = k + 2$ (with $k > 0$), and $\chi = k + 3$. Therefore, the most favorable case is $s = 0$, because increasing $s$ by 1 makes $n + 1$ increase by 1, while the left-hand side of the inequality increases by at most 1/2. Hence, in the best case we have $n = k + 5 \geq 6$, and

$$\frac{\Delta}{\omega - 1} + \chi = \frac{n - 1}{n - 3} + n - 2 < n + 1$$

Now, we consider case (a). Here, again we have $k > 0$ and $\Delta \leq n - 1$, moreover now $n = k + s + 1$, $\omega = k + 1$, and $\chi = k + 1$. Thus

$$\frac{\Delta}{\omega - 1} + \chi \leq \frac{(k + s)}{k} + k + 1 \leq k + s + 2$$

with equality if and only if $s/k = s$, that is $k = 1$ or $s = 0$, where for the case $k = 1$ we also have to ensure $\Delta = s + 1$. This completes the proof of (1).

To see (2), consider an independent set $A$ of cardinality $\alpha = \alpha(G)$. A proper $(n - \alpha + 1)$-coloring always exists as we can assign color 1 to all vertices from $A$ and the further $n - \alpha$ vertices are assigned with pairwise different colors. Hence, $\chi(G) \leq n - \alpha + 1$ holds for every graph. Moreover, if the graph induced by $V(G) \setminus A$ is not complete, we can color it properly by using fewer than $n - \alpha$ colors that yields a proper coloring of $G$ with fewer than $n - \alpha + 1$ colors. Therefore, $\chi(G) = n - \alpha + 1$ may hold only if $V(G) \setminus A$ induces a complete graph. In this case, $G$ is a split graph. Since split graphs are chordal and chordal graphs are perfect [I], $\omega(G) = \chi(G) = n - \alpha + 1$. Consequently, if (2) holds with equality, there exists a vertex $v \in A$ which is adjacent to all vertices from $V(G) \setminus A$. This vertex fulfills our conditions as $N(v)$ is a clique and $V(G) \setminus N(v)$ is an independent set.

On the other hand, if a vertex $v'$ with such a property exists in $G$, then the graph cannot be colored with fewer than $|N(v')| + 1$ colors. This implies $\chi = n - \alpha + 1$ and completes the proof of the second statement. 

\[\square\]
3 Claw-free graphs

Several related problems (say, perfect graph conjecture, to mention just the most famous one) are easier for claw-free graphs, i.e. for graphs not containing $K_{1,3}$ as an induced subgraph, other problems (say, complexity of finding chromatic number) are not. (For a survey of results on claw-free graphs see e.g. [9].) Concerning local clique cover number, R. Javadi et al. showed in [12] that if $G$ is a claw-free graph then $lcc(G) \leq c \frac{\Delta(G)}{\log(\Delta(G))}$, for a constant $c$. In this section, we are going to prove that Conjecture 3 does hold for claw-free graphs.

To prove the main result of this section, we use the following definition and theorem of Balogh et al. [2].

Definition 10. [2] A graph $G$ is $(s, t)$-splittable if $V(G)$ can be partitioned into two sets $S$ and $T$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$. For $2 \leq s \leq \chi(G) - 1$, we say that $G$ is $s$-splittable if $G$ is $(s, \chi(G) - s + 1)$-splittable.

Theorem 11. [2] Let $s \geq 2$ be an integer. Let $G$ be a graph with $\alpha(G) = 2$ and $\chi(G) > \max\{\omega, s\}$. Then $G$ is $s$-splittable.

Now we prove:

Theorem 12. Let $G$ be a claw-free graph with $n$ vertices. Then $lcc(G) + \chi(G) \leq n + 1$. Moreover, for every $n \geq 4$, there exist several claw-free graphs with $n$ vertices such that equality holds.

Proof. We prove the theorem by induction on $n$. For small values of $n$, it is easy to check that a claw-free graph with $n$ vertices satisfies the inequality. Also, the assertion is obvious for $\alpha(G) = 1$.

Let $G$ be a claw-free graph on $n$ vertices. First, we consider the case that $\alpha(G) \geq 3$. Let $T$ be an independent set of size three. By the induction hypothesis, $G - T$ has a clique covering $C'$ such that every vertex $x \in V(G - T)$ has

$$\text{val}_{C'}(x) \leq (n - 3) + 1 - \chi(G - T) \leq n - 2 - (\chi(G) - 1) = n - 1 - \chi(G). \quad (3)$$

Now, for every vertex $u \in T$, partition $N(u)$ into the $\chi(G[N(u)])$ vertex-disjoint cliques. Then, add vertex $u$ to each clique to cover all the edges incident to $u$. These cliques along with cliques in an optimum clique covering of $G - T$ form a clique covering, say $C$, for $G$. Let $u \in T$ and $x \in G - T$. Then we have

$$\text{val}_C(u) = \chi(G[N(u)]) \leq \chi(G) \leq n + 1 - \chi(G),$$

$$\text{val}_C(x) \leq \text{val}_{C'}(x) + |N_G(x) \cap T|.$$

Since $G$ is claw-free, $|N_G(x) \cap T| \leq 2$. Thus, by Inequality (3), $lcc(G) \leq n + 1 - \chi(G)$. 

Consider now the case $\alpha(G) = 2$. By Proposition 6 we may assume that $\chi(G) > \omega(G)$. Moreover, as the statement clearly holds when $\chi(G) \leq 2$, we may also suppose that $\chi(G) \geq 3$. Then Theorem 11 with $s = 2$ implies that $V(G)$ can be partitioned into two parts, say $A$ and $B$, such that $\chi(G[A]) \geq 2$ and $\chi(G[B]) \geq \chi(G) - 1$. We assume, without loss of generality, that $A = \{u_1, u_2\}$, where the vertices $u_1$ and $u_2$ are adjacent. Then $\chi(G - \{u_1, u_2\}) \geq \chi(G) - 1$.

We will use the notation $N_1 := N(u_1) - N(u_2)$, $N_2 := N(u_2) - N(u_1)$, and $N_{1,2} := N(u_1) \cap N(u_2)$. Since $G$ is claw-free, $N_i \cup \{u_i\}$ induces a clique for $i = 1, 2$. Starting with an optimal clique covering $C''$ for $G - \{u_1, u_2\}$, we will construct a clique covering $C$ for $G$ such that $val_C(v) \leq n + 1 - \chi(G)$ holds for every vertex $v$.

If $N_{1,2} = \emptyset$, then $C := C'' \cup \{N_1 \cup \{u_1\}, N_2 \cup \{u_2\}, \{u_1, u_2\}\}$ is a clique covering for $G$. We observe that $val_C(u_i) \leq 2$ holds for $i = 1, 2$ and

$$val_C(v) \leq val_{C''}(v) + 1 \leq n - 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G)$$

for each vertex $v$ from $V(G - \{u_1, u_2\})$. Hence, $lcc(G) \leq n + 1 - \chi(G)$.

Otherwise, if $N_{1,2} \neq \emptyset$, partition $N_{1,2}$ into at most $\chi(G - \{u_1, u_2\})$ cliques and extend each of them with the vertices $u_1$ and $u_2$. These cliques together with $N_1 \cup \{u_1\}, N_2 \cup \{u_2\}$, and with the cliques in $C''$ form a clique covering of $G$. We show that this clique covering $C$ satisfies $val_C(x) \leq n + 1 - \chi(G)$ for every vertex $x \in V(G)$. Note that $val_C(u_1) \leq \chi(G - \{u_1, u_2\}) + 1$, thus the Nordhaus-Gaddum inequality for chromatic number implies

$$val_C(u_1) \leq (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G - \{u_1, u_2\}) \leq n + 1 - \chi(G).$$

Similarly, we have $val_C(u_2) \leq n + 1 - \chi(G)$. For $v \in V(G - \{u_1, u_2\})$,

$$val_C(v) \leq val_{C''}(v) + 1 \leq (n - 2) + 1 - \chi(G - \{u_1, u_2\}) + 1 \leq n - \chi(G) + 1.$$

Finally, we note that $K_n$, $K_n - K_2$, and $K_n - K_{1,2}$ are examples of claw-free graphs with $n$ vertices such that $lcc(G) + \chi(G) = n + 1$. \hfill \Box

## 4 A Nordhaus-Gaddum type inequality

A clique partition of the edges of a graph $G$ is a family of cliques such that every edge of $G$ lies in exactly one member of the family. The sigma clique partition number of $G$, scp($G$), is the smallest integer $k$ for which there exists a clique partition of $E(G)$ where the sum of the sizes of its cliques is at most $k$.

It was conjectured by G. O. H. Katona and T. Tarján, and proved in the papers [4] [13] [11], that for every graph $G$ on $n$ vertices, scp($G$) \leq \lfloor n^2/2 \rfloor holds, with equality if and only if $G$ is the complete bipartite graph $K_{[n/2],[n/2]}$. 


Also, this parameter relates to a number of other well-known problems (see [6]). The second author and R. Javadi proved the following Nordhaus-Gaddum type theorem for scp.

**Theorem 13.** Let $G$ be a graph with $n$ vertices. Then

$$\frac{31}{50} n^2 + O(n) \leq \max \{ \text{scp}(G) + \text{scp}(\overline{G}) \} \leq \frac{9}{10} n^2 + O(n),$$

$$\frac{12}{125} n^4 + O(n^3) < \max \{ \text{scp}(G) \cdot \text{scp}(\overline{G}) \} < \frac{81}{400} n^4 + O(n^3).$$

In the following result we improve the upper bounds, from 0.9 to less than 0.77 and from 0.2025 to less than 0.15.

**Theorem 14.** For every graph $G$ with $n$ vertices,

$$\text{scp}(G) + \text{scp}(\overline{G}) \leq \frac{1203}{1568} n^2 + o(n^2) < 0.76722 n^2 + o(n^2)$$

and

$$\text{scp}(G) \cdot \text{scp}(\overline{G}) \leq \frac{1447209}{9834496} n^4 + o(n^4) < 0.1471564 n^4 + o(n^4).$$

**Proof.** Substantially improving on earlier estimates, P. Keevash and B. Sudakov [14] proved via a computer-aided calculation that every edge 2-coloring of $K_n$ contains at least $cn^2 - o(n^2)$ mutually edge-disjoint monochromatic triangles, where

$$c = \frac{13}{196} + \frac{1}{84} - \frac{1}{1568} = \frac{365}{4704}.$$ 

In our context this means that we can select approximately $cn^2$ triangles which together cover $3cn^2$ edges in $G$ and $\overline{G}$ at the cost of $3cn^2$. The remaining edges will be viewed as copies of $K_2$ in the clique partition to be constructed; they are counted with weight 2 in scp. In this way we obtain

$$\text{scp}(G) + \text{scp}(\overline{G}) \leq (1 - 3c) n^2 + o(n^2) = \frac{1203}{1568} n^2 + o(n^2).$$

This also implies the upper bound on $\text{scp}(G) \cdot \text{scp}(\overline{G})$. \hfill \Box

**Remark 15.** The smallest number of cliques in a clique partition of $G$ is called the *clique partition number* of $G$. As a Nordhaus-Gaddum type inequality for parameter cp, D. de Caen et al. proved in [7] that

$$\text{cp}(G) + \text{cp}(\overline{G}) \leq \frac{13}{30} n^2 - O(n) \approx 0.43333 n^2 - O(n),$$

$$\text{cp}(G) \cdot \text{cp}(\overline{G}) \leq \frac{169}{3600} n^4 + O(n^3) \approx 0.0469444 n^4 + O(n^3).$$

\[1\] In the Abstract of [14] the authors announce the lower bound $n^2/13$, and in their Theorem 1.1 they state $n^2/12.89$ (the rounded form of $\frac{9}{130} n^2$, but actually on p. 212 they prove the even better lower bound displayed above.)
Note that if it is possible to select some $k$ edge-disjoint complete subgraphs in $G$ and $\overline{G}$ which together cover $m$ edges, then $\text{cp}(G) + \text{cp}(\overline{G}) \leq \left(\frac{365}{4704}n^2 - o(n^2)\right) + k - m$. As observed within the proof of Theorem 14, the choices $k = \frac{365}{4704}n^2 - o(n^2)$ and $m = 3k$ are feasible for every $G$ on $n$ vertices, thus

$$\text{cp}(G) + \text{cp}(\overline{G}) \leq \left(\frac{1}{2} - \frac{365}{2352}\right)n^2 + o(n^2) = \frac{811}{2352}n^2 + o(n^2) < 0.344813n^2 + o(n^2),$$

$$\text{cp}(G) \cdot \text{cp}(\overline{G}) \leq \frac{657721}{22127616}n^4 + o(n^4) < 0.029724n^4 + o(n^4).$$

These upper bounds improve the results of [7].

References

[1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Appl. Math.*, 161(4-5):466–546, 2013.

[2] J. Balogh, A. Kostochka, N. Prince, and M. Stiebitz, The Erdős-Lovász Tihany Conjecture for quasi-line graphs, *Discrete Math.*, 309:3985–3991, 2009.

[3] J. C. Bermond and J. C. Meyer, Graphe représentatif des arêtes d’un multigraphe, *J. Math. Pures Appl.*, 52(3):299–308, 1973.

[4] F. R. K. Chung, On the decomposition of graphs, *SIAM J. Alg. Disc. Meth.*, 2(1):1–12, 1981.

[5] A. Davoodi and R. Javadi, Nordhaus-Gaddum inequalities for sigma clique partition number of graphs, The 45th Annual Iranian Mathematics Conference, Semnan University, Aug. 26–29, 2014.

[6] A. Davoodi, R. Javadi, and B. Omoomi, Pairwise balanced designs and sigma clique partitions, *Discrete Math.*, 339:1450–1458, 2016.

[7] D. de Caen, P. Erdős, N. J. Pullmann, and N. C. Wormald, Extremal clique coverings of complementary graphs, *Combinatorica*, 6(4):309–314, 1986.

[8] G. A. Dirac, On rigid circuit graphs, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 25(1), 71–76, 1961.

[9] R. Faudree, E. Flandrin, and Z. Ryjáček, Claw-free graphs — A survey, *Discrete Math.*, 164(1-3):87–147, 1997.

[10] H. J. Finck, On the chromatic numbers of a graph and its complement, *Theory of Graphs*, 99–113, 1968.
[11] E. Győri and A. V. Kostochka, On a problem of G. O. H. Katona and T. Tarján, \textit{Acta Math. Acad. Sci. Hungar.}, 34(3-4):321–327, 1979.

[12] R. Javadi, Z. Maleki, and B. Omoomi, Local clique covering of claw-free graphs, \textit{J. Graph Theory}, 81(1):92–104, 2016.

[13] J. Kahn, Proof of a conjecture of Katona and Tarján, \textit{Period. Math. Hungar.}, 12(1):81–82, 1981.

[14] P. Keevash and B. Sudakov, Packing triangles in a graph and its complement, \textit{J Graph Theory}, 47:203–216, 2004.

[15] P. G. H. Lehot, An optimal algorithm to detect a line graph and output its root graph, \textit{J. Assoc. Comput. Mach.}, 21(4):569–575, 1974.

[16] L. Lovász, Problem 9: Beiträge zur Graphentheorie und deren Anwendungen, \textit{Technische Hochschule Ilmenau}, International Colloquium held in Oberhof, 1977.

[17] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, \textit{Amer. Math. Monthly}, 63:175–177, 1956.

[18] S. Poljak, V. Rödl, and D. Turzík, Complexity of representation of graphs by set systems, \textit{Discrete Appl. Math.}, 3(4):301–312, 1981.