ALGEBRAIC DEGREE OF SERIES OF RECIPROCAL ALGEBRAIC INTEGERS

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Abstract. In this paper, I give sufficient conditions for any linear combination in \( \mathbb{Q} \) of numbers \( \sum_{n=1}^\infty b_{1,n} \alpha_{1,n} + \cdots + \sum_{n=1}^\infty b_{K,n} \alpha_{K,n} \) to have algebraic degree greater than an arbitrary fixed integer \( D \) when the numbers \( \alpha_{i,n} \) are algebraic integers of sufficiently rapidly increasing modulus and the \( b_{i,n} \) are positive integers that are not too large.

1. INTRODUCTION

In 1975, Erdős [3] gave a sufficient condition for an increasing series of reciprocal integers to be irrational, as stated in the below theorem.

Theorem 1 (Erdős). Let \( \{a_n\}_{n \in \mathbb{N}} \) be an increasing sequence of natural numbers such that \( a_n > n^{1+\varepsilon} \) for some \( \varepsilon > 0 \) for all \( n \) sufficiently large. If \( \limsup_{n \to \infty} a_n^{-1/2} a_n = \infty \), then \( \sum_{n=1}^\infty 1/a_n \) is irrational.

Later, in [4], Hančl extended this theorem to also cover the case of

\[
1 \leq \limsup_{n \to \infty} a_n^{1/2} a_n < \limsup_{n \to \infty} a_n^{-1/2} a_n < \infty,
\]

while also providing a related condition for finitely many series of fractions \( \sum_{n=1}^\infty b_{i,n} a_{i,n} \) (\( i = 1, \ldots, K \)) with sufficiently small and positive \( b_{i,n} \) to be irrational and linear independent over \( \mathbb{Q} \).

Theorem 2 (Hančl). Let \( K \in \mathbb{N} \), and let \( A_1, A_2, a, \varepsilon > 0 \) be real numbers such that \( a < 1 \leq A_1 < A_2 \). For \( i = 1, \ldots, K \), let \( \{a_{i,n}\}_{n \in \mathbb{N}} \) and \( \{b_{i,n}\}_{n \in \mathbb{N}} \) each be sequences of natural numbers. Suppose that

\[
\forall n \in \mathbb{N} : \quad n^{1+\varepsilon} \leq a_{1,n} < a_{1,n+1},
\]

\[
\limsup_{n \to \infty} a_{1,n}^{1/(K+1)} a_n = A_2,
\]

\[
\liminf_{n \to \infty} a_{1,n}^{1/(K+1)} a_n = A_1,
\]

\[
\forall n \in \mathbb{N} \forall 1 \leq i \leq K : \quad b_{i,n} < 2^{(\log_2 a_{1,n})^a},
\]

\[
\forall 1 \leq i < j \leq K : \quad \lim_{n \to \infty} b_{i,n} a_{j,n} = 0,
\]

\[
\forall 1 < i \leq K : \quad a_{i,n} 2^{-(\log_2 a_{1,n})^a} < a_{1,n} < a_{i,n} 2^{(\log_2 a_{1,n})^a}.
\]

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Then $\sum_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}}; \ldots; \sum_{n=1}^{\infty} \frac{b_{K,n}}{a_{K,n}}$ are irrational and linearly independent over $\mathbb{Q}$.

In 2017, Hančl and Nair [5] showed that integer sequences of the form $a_{n+1} = a_n^2 - a_n + 1$ with $a_1 \geq 2$ will satisfy both $\sum_{n=1}^{\infty} 1/a_n \in \mathbb{Q}$ and 
\[
1 < \sqrt[n]{a_1^2 - a_1} \leq \liminf_{n \to \infty} a_n^{1/2^n} = \limsup_{n \to \infty} a_n^{1/2^n} < \infty,
\]
which exemplifies that the requirement $A_1 < A_2$ cannot in general be omitted from Theorem 2. The main result of [5] was a variant of Theorem 1 where the $a_n$ may be square roots of positive integers when $\lim_{n \to \infty} a_1^{1/2^n} = \infty$. Two years later, this result was generalised to the below theorem by Andersen and Kristensen [1], which gives a sufficient condition for $\sum_{n=1}^{\infty} 1/\alpha_n$ to have large algebraic degree when $\alpha_n$ are algebraic integers of bounded degree.

**Theorem 3** (Andersen and Kristensen). Let $d, D \in \mathbb{N}$, $\varepsilon > 0$, and let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a series of algebraic integers of maximal degree $d$ such that
\[
\forall n \in \mathbb{N} : \quad n^{1+\varepsilon} \leq |\alpha_n| < |\alpha_{n+1}|,
\]
\[
\limsup_{n \to \infty} |\alpha_n|^{d_n} \prod_{i=1}^{d_n} \frac{1}{(d+i)^3} = \infty,
\]
\[
\forall n \in \mathbb{N} : \quad [\alpha_n] = |\alpha_n|.
\]
Suppose that $\Re(\alpha_n) > 0$ holds for all $n \in \mathbb{N}$ or that $\Im(\alpha_n) > 0$ holds for all $n \in \mathbb{N}$. Then $\deg \sum_{n=1}^{\infty} \frac{1}{\alpha_n}$ is strictly greater than $D$.

Both in the above theorem and for the remainder of this paper, $[\alpha]$ denotes the house of an algebraic number $\alpha$, which is defined as the maximum modulus among the conjugates of $\alpha$.

As Andersen and Kristensen note in their paper, their proof only really needs $[\alpha_n]$ to be bounded by $C|\alpha_n|$ for some uniform constant $C > 0$. Similarly, the restriction on the sign of the real (or imaginary) value of $\alpha_n$ is only to enforce that all $\alpha_n$ are contained in an open half plane not containing 0, which ensures that the partial sums $\sum_{n=1}^{N} \frac{1}{\alpha_n}$ are non-zero and non-conjugate to $\sum_{n=1}^{\infty} \frac{1}{\alpha_n}$ for sufficiently large $N$.

The main result of this paper is a generalisation of Theorem 2 in the spirit of Theorem 3, and the proof will combine the arguments used by the respective papers. For the sake of clarity, we will, however, be slightly more explicit with the open half-plane containing all $\alpha_n$, as compared to Andersen’s and Kristensen’s proof of Theorem 3. For this purpose, I introduce notation $\mathbb{R}_\zeta(z)$ to denote $\Re(\zeta z)$ for $\zeta \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$, as $\Re_\zeta(z) > 0$ is then equivalent to $z$ lying in the open half-plane with 0 on its border and moving in direction of $\zeta$. 


**Theorem 4.** Let $D, K \in \mathbb{N}$, let $A_1, A_2, a, \varepsilon > 0$ be real numbers such that $a < 1 \leq A_1 < A_2$, and let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let $\{d_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{b_{i,n}\}_{n \in \mathbb{N}}$ be a sequence of natural numbers, and let $\{\alpha_{i,n}\}_{n \in \mathbb{N}}$ be sequences of algebraic integers such that

\begin{align*}
(1) & \quad \forall n \in \mathbb{N} : |\alpha_{1,n}| < |\alpha_{1,n+1}| \\
(2) & \quad \forall n \in \mathbb{N} : |\alpha_{1,n}| \geq n^{1+\varepsilon} \\
(3) & \quad \forall n \in \mathbb{N} : [\mathbb{Q}(\alpha_{1,n}, \ldots, \alpha_{K,n}) : \mathbb{Q}] \leq d_n \\
(4) & \quad \liminf_{n \to \infty} |\alpha_{1,n}|^{\frac{1}{D_n \prod_{i=1}^{n} (K D_i + b_i)}} = A_1 \\
(5) & \quad \limsup_{n \to \infty} |\alpha_{1,n}|^{\frac{1}{D_n \prod_{i=1}^{n} (K D_i + b_i)}} = A_2 \\
(6) & \quad \forall n \in \mathbb{N} \forall 1 < i \leq K : 2^{-\left(\log_2 |\alpha_{1,n}|\right)^n} < \frac{|\alpha_{1,n}|}{|\alpha_{i,n}|} < 2^{\left(\log_2 |\alpha_{1,n}|\right)^n} \\
(7) & \quad \forall n \in \mathbb{N} \forall 1 \leq i \leq K : b_{i,n}^{\frac{1}{\alpha_{i,n}}} \leq 2^{\left(\log_2 |\alpha_{1,n}|\right)^n} |\alpha_{i,n}| \\
(8) & \quad \forall n \in \mathbb{N} \forall 1 \leq i \leq K : \Re_\zeta \left( \frac{b_{i,n}}{\alpha_{i,n}} \right) > 0, \\
(9) & \quad \forall 1 \leq i < j \leq K : \lim_{n \to \infty} \left( \Re_\zeta \left( \frac{b_{i,n}}{\alpha_{i,n}} \right) / \Re_\zeta \left( \frac{b_{j,n}}{\alpha_{j,n}} \right) \right) = 0.
\end{align*}

Then $\deg \gamma > D$ when $\gamma$ is any non-trivial linear combination over $\mathbb{Q}$ of the numbers $\sum_{n=1}^{\infty} \frac{b_{1,n}}{\alpha_{1,n}}, \ldots, \sum_{n=1}^{\infty} \frac{b_{K,n}}{\alpha_{K,n}}$.

2. **Auxiliary Results**

The proof of Theorem 4 will be split into two parts, the first of which will be based around the Weil height and the Mahler measure. We recall the definitions below.

For $K$ being some finite field extension of $\mathbb{Q}$ of degree $d$, we define for $\alpha \in K$ the Weil height of $\alpha$ as the number

$$H(\alpha) := \prod_{\nu \in M_K} \max \{ 1, |\alpha|_\nu \}^d,$$

where $M_K$ denotes the set of places of $K$, and $d_\nu = [K_\nu : \mathbb{Q}_\nu]$ denotes the degree of the completion of $K$ with respect to place $\nu$ as an extension of the completion of $\mathbb{Q}$ with respect to $\nu$. With the normalisation in the exponent $d_\nu/d$, the definition is independent of the field $K$ containing $\alpha$. We define the Mahler measure of $\alpha$ as

$$M(\alpha) := |a_d| \prod_{i=1}^{n} \max \{ 1, |\alpha_i| \},$$

where $a_d$ here denotes leading coefficient of the minimal polynomial in $\mathbb{Z}[X]$ of $\alpha$, and $\alpha_1, \ldots, \alpha_d$ denote the conjugates of $\alpha$. 
The proof will furthermore use the following lemmas, the first of which relates Weil height, Mahler measure, and house of algebraic integers. The main part of the statement, $H(\alpha) = M(\alpha)^{1/d}$, is a classical result, which is presented in [8]. The rest is essentially a trivial consideration, see [1].

**Lemma 1.** Let $\alpha$ be an algebraic number of degree $d$. Then

$$H(\alpha) = M(\alpha)^{1/d} \leq M(\alpha) = H(\alpha)^d$$

The second lemma is a list of further classical results regarding the Weil height, see [8].

**Lemma 2.** Let $\alpha, \beta$ be algebraic numbers. Then

$$H(1/\alpha) = H(\alpha) \text{ if } \alpha \neq 0, \quad H(\alpha + \beta) \leq 2H(\alpha)H(\beta),$$

$$H(\alpha\beta) \leq H(\alpha)H(\beta)$$

Similar results are likewise true for the degree function, as seen by the below lemma.

**Lemma 3.** Let $\alpha, \beta$ be algebraic numbers. Then

$$\deg(1/\alpha) = \deg(\alpha) \text{ if } \alpha \neq 0,$$

$$\deg(\alpha + \beta) \leq \deg(\alpha)\deg(\beta), \quad \deg(\alpha\beta) \leq \deg(\alpha)\deg(\beta)$$

This is essentially trivial: Following the spirit of [6], the inequalities come from noting that $\alpha + \beta$ and $\alpha\beta$ both lie in the field extension $\mathbb{Q}(\alpha, \beta)$, which clearly has degree at most $\deg(\alpha)\deg(\beta)$ over $\mathbb{Q}$. Noting $1/\alpha \in \mathbb{Q}(\alpha)$ and $\alpha \in \mathbb{Q}(1/\alpha)$ for $\alpha \neq 0$, it is likewise obvious that $\deg(1/\alpha) = \deg(\alpha)$.

The below lemma is central for the first part of the proof of Theorem 4, and seems to originally be from [7]. A proof may also be extracted from the proof of Theorem A.1 in Appendix A of [2].

**Lemma 4.** Let $\alpha, \beta$ be non-conjugate algebraic numbers. Then

$$|\alpha - \beta| \geq \frac{1}{2^{\deg(\alpha)\deg(\beta)}M(\alpha)^{\deg(\beta)}M(\beta)^{\deg(\alpha)}}$$

In the second part of the proof of Theorem 4, we will occasionally need the below simple estimate related to the exponent of the limes superior and limes inferior.

**Lemma 5.** Let $D, K, N \in \mathbb{N}$ be natural numbers, and let $\{d_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers. Writing $D_n = \prod_{i=1}^n d_i$, we have

$$D^{N+1} \prod_{i=1}^N (KD_i + d_i) \geq KDD_N \sum_{n=1}^N D^n \prod_{i=1}^{n-1} (KD_i + d_i).$$
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Proof. The first statement proven by induction in $N$. Note that the statement clearly holds for $N = 1$. Suppose it holds for $N - 1$, for some $N > 1$. Then

$$D^{N+1} \prod_{i=1}^{N} (KD_i + d_i) = D(KD_N + d_N) \left( D^N \prod_{i=1}^{N-1} (KD_i + d_i) \right)$$

$$\geq KDD_N \left( D^N \prod_{i=1}^{N-1} (KD_i + d_i) \right)$$

$$+ d_N \left( KDD_{N-1} \sum_{n=1}^{N} D^n \prod_{i=1}^{n-1} (KD_i + d_i) \right)$$

$$= KDD_N \sum_{n=1}^{N} D^n \prod_{i=1}^{n-1} (KD_i + d_i).$$

Near the end of the proof, we will use a generalised version of a lemma from [3], which Erdős used for proving Theorem 1. The current version is presented and proven in [1].

Lemma 6. Let $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of real numbers satisfying $a_n > n^{1+\varepsilon}$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=N}^{\infty} \frac{1}{a_n} < \frac{2 + 1/\varepsilon}{a_N^{\varepsilon/(1+\varepsilon)}.}$$

3. Proof of Main Result

The proof of Theorem 4 will be split into two lemmas:

Lemma 7. Let $D, K \in \mathbb{N}$, let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, and let $a, c, A_2 > 0$ such that $c < a < 1 < A_2$. Let $\{d_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{\alpha_{i,n}\}_{n \in \mathbb{N}}$ be a sequence of algebraic integers, and $\{b_{i,n}\}_{n \in \mathbb{N}}$ be a sequence of natural numbers. Suppose that equations (3), (5), (6), (7), (8), (9) are satisfied, let $\beta_1, \ldots, \beta_K \in \mathbb{Z}$ be integers that are not all 0, and write

$$\gamma = \sum_{j=1}^{K} \beta_j \sum_{n=1}^{\infty} \frac{b_{j,n}}{\alpha_{j,n}}, \quad \gamma(N) = \sum_{j=1}^{K} \beta_j \sum_{n=N+1}^{\infty} \frac{b_{j,n}}{\alpha_{j,n}}$$

If $\deg \gamma \leq D$ and $c \in (a, 1)$, then

$$|\gamma(N)| \left( 2^{D^{N+1}} \prod_{i=1}^{N-1} (KD_i + d_i)^{K} \prod_{n=1}^{N} |\alpha_{1,n}|^{K} \right)^{DD_N} \geq 1$$

holds for all sufficiently large $N$. 
Lemma 8. Let $D, K \in \mathbb{N}$, and let $A_1, A_2, a, \varepsilon > 0$ be real numbers such that $a < 1 \leq A_1 < A_2$. Let $\{d_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{\alpha_{i,n}\}_{n \in \mathbb{N}}$ and $\{b_{i,n}\}_{n \in \mathbb{N}}$ be sequences of complex numbers. Suppose that equations (1), (2), (4), (5), (6) hold and that

\begin{align*}
\forall n \in \mathbb{N} \forall 1 \leq i \leq K : \quad |b_{i,n}| \leq 2^{(\log_2 |\alpha_{1,n}|)^a}.
\end{align*}

Let $\beta_1, \ldots, \beta_K \in \mathbb{Z}$ be integers that are not all 0, and write

\begin{align*}
\gamma(N) = \sum_{j=1}^{K} \beta_j \sum_{n=N+1}^{\infty} \frac{b_{j,n}}{\alpha_{j,n}}.
\end{align*}

Let $c \in (a, 1)$. Then

\begin{align*}
\liminf_{N \to \infty} |\gamma(N)| \left( 2^{D^N} \prod_{i=1}^{N_K} (KD_i + d_i) \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{DD_N} = 0.
\end{align*}

One minor result that will be briefly used for proving both lemmas is that equation (5) implies

\begin{align*}
|\alpha_{1,n}| \leq (2A_2)^{D^N} \prod_{i=1}^{n} (KD_i + d_i)
\end{align*}

for $n$ sufficiently large.

We now prove Lemma 7:

Proof (Lemma 7). We introduce further notation

\begin{align*}
\gamma_N := \sum_{i=1}^{K} \beta_i \sum_{n=1}^{N} \frac{b_{i,n}}{\alpha_{i,n}}, \quad \beta := \prod_{i=1}^{K} H(\beta_i).
\end{align*}

By Lemma 3 and equation (3), we quickly find

\begin{align*}
\deg \gamma_N \leq \prod_{n=1}^{N} \deg \left( \sum_{i=1}^{K} \beta_i \frac{b_{i,n}}{\alpha_{i,n}} \right) \leq D_N.
\end{align*}

Applying Lemma 1 and Lemma 2 followed by equation (7), we then get

\begin{align*}
M(\gamma_N) = H(\gamma_N)^{\deg \gamma_N} \leq \left( 2^{NK} \prod_{i=1}^{K} H(\beta_i) \prod_{n=1}^{N} H(b_{i,n}) H\left( \frac{1}{\alpha_{i,n}} \right) \right)^{D_N} \\
= \left( 2^{KN} \beta \prod_{i=1}^{K} \prod_{n=1}^{N} H(\alpha_{i,n}) H(b_{i,n}) \right)^{D_N} \\
\leq \left( 2^{KN} \beta \prod_{i=1}^{K} \prod_{n=1}^{N} [\alpha_{i,n}]^{[\beta_i]} \right)^{D_N} \\
\leq \left( 2^{KN+KN(\log_2 |\alpha_{1,n}|)^a} \prod_{i=1}^{K} \prod_{n=1}^{N} [\alpha_{i,n}]^{D_N} \right)^{D_N},
\end{align*}
using that \( \alpha_{1,n} \) is non-decreasing and that \( b_{i,n} = b_{i,n} \) as \( b_{i,n} \in \mathbb{N} \). From equations (6) and (11), we then have for \( N \) sufficiently large that

\[
M(\gamma_N) < \left( 2^{2KN(\log_2(2A_2))D^N \prod_{i=1}^{N-1}(KD_i+d_i)^\alpha} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{D^N} \\
\leq \left( 2^{D^\alpha N \prod_{i=1}^{N-1}(KD_i+d_i)^\alpha} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{D_N}. 
\]

We now wish to apply Lemma 6 to get an estimate on \( |\gamma(N)| \). To do so, we need \( \gamma \neq \gamma_N \), which is ensured for sufficiently large \( N \) if the \( \gamma_N \) are mutually distinct from a some point.

**Claim** \( (\gamma_M \neq \gamma_N \text{ for } M > N \text{ sufficiently large}) \). To see this, let \( R \) be the maximal value of \( i \) such that \( \beta_i \neq 0 \) and assume without loss of generality that \( \beta_R > 0 \) (otherwise replace each \( \beta_i \) by \( -\beta_i \) for all \( i \)).

Using that \( \Re \zeta \) is clearly linear in \( R \), we find for each \( n \in \mathbb{N} \) that

\[
\Re \zeta \left( \sum_{j=1}^{K} \frac{\beta_j b_{j,n}}{\alpha_{j,n}} \right) = \sum_{j=1}^{K} \beta_j \Re \zeta \left( \frac{b_{j,n}}{\alpha_{j,n}} \right) = \Re \zeta \left( \frac{b_{R,n}}{\alpha_{R,n}} \right) \left( \beta_R + \sum_{j=1}^{R-1} \beta_j \Re \zeta \left( \frac{b_{j,n}}{\alpha_{j,n}} \right) \right)
\]

Equation (9) then implies that for \( n \) sufficiently large, we have

\[
\left| \sum_{j=1}^{R-1} \beta_j \Re \zeta \left( \frac{b_{j,n}}{\alpha_{j,n}} \right) \right| < \beta_R,
\]

and thus \( \Re \zeta \left( \sum_{j=1}^{K} \frac{\beta_j b_{j,n}}{\alpha_{j,n}} \right) > 0 \), by equation (8). For \( N \) sufficiently large, it hence follows that

\[
\Re \zeta \gamma_N = \sum_{n=1}^{N} \Re \zeta \sum_{j=1}^{K} \frac{\beta_j b_{j,n}}{\alpha_{j,n}} < \sum_{n=1}^{M} \Re \zeta \sum_{j=1}^{K} \frac{\beta_j b_{j,n}}{\alpha_{j,n}} = \Re \zeta \gamma_M,
\]

which implies the claim.

Since \( \gamma \) can have at most \( D \) conjugates, it then follows that \( \gamma \) and \( \gamma_N \) must be non-conjugate for \( N \) sufficiently large, and we may apply Lemma 4, Lemma 1, and equation (12) (in that order) to find

\[
|\gamma(N)| = |\gamma - \gamma_N| \geq \frac{1}{2^{\deg(\gamma) \deg(\gamma_N) M(\gamma)^{\deg(\gamma_N)^{\deg(\gamma)}}}} \\
\geq \frac{1}{2^{DD_N \Re(\gamma) \Re(\gamma_N) \prod_{i=1}^{N-1}(KD_i+d_i)^\alpha} \prod_{n=1}^{N} |\alpha_{1,n}|^K} \left( 2^{D^\alpha N \prod_{i=1}^{N-1}(KD_i+d_i)^\alpha} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{D^N}. 
\]
This proves the lemma. □

Proof (Lemma 8). Applying equations (6) and (10) followed by (1), we find

\[
|\gamma(N)| = \left| \sum_{n=N+1}^{\infty} \sum_{j=1}^{K} \frac{b_{j,n}}{\alpha_{j,n}} \right| \leq \sum_{n=N+1}^{\infty} \sum_{j=1}^{K} \left| \frac{b_{j,n}}{\alpha_{j,n}} \right|
\]

\[
\leq \sum_{n=N+1}^{\infty} K \frac{2^{(\log_{2} |\alpha_{1,n}|)^c}}{|\alpha_{1,n}|} \leq \sum_{n=N+1}^{\infty} \frac{2^{(\log_{2} |\alpha_{1,n}|)^c}}{|\alpha_{1,n}|},
\]

for \( N \) sufficiently large.

We now split into two cases, both using the notation

\[
a_n := |\alpha_{1,n}|, \quad S_n := a_n \frac{d_{n} \prod_{i=n}^{n-1} (K D_i + d_i)}{a_{n+1}}.
\]

Case 1 \((a_n \geq 2^n \text{ for all } n \text{ sufficiently large})\). We continue on equation (13) and use that the function \( x^{(\log_{2} x)^c} / x \) is decreasing for \( x > 1 \) to find

\[
|\gamma(N)| \leq \sum_{N < n \leq \log a_{N+1}} \frac{2^{(\log_{2} a_n)^c}}{a_n} + \sum_{n > \log a_{N+1}} \frac{2^{(\log_{2} a_n)^c}}{a_n}
\]

\[
\leq \frac{2^{(\log_{2} a_{N+1})^c}}{a_{N+1}} + \sum_{n > \log a_{N+1}} \frac{2^{(\log_{2} 2^n)^c}}{2^n}
\]

\[
= \frac{2^{(\log_{2} a_{N+1})^c}}{a_{N+1}} + \sum_{n > \log a_{N+1}} \frac{1}{2^{n-cn}}
\]

\[
\leq \frac{2^{(\log_{2} a_{N+1})^c}}{a_{N+1}} + C \frac{1}{2^{(\log_{2} a_{N+1})^c}} \leq \frac{2^{(\log_{2} a_{N+1})^c}}{a_{N+1}},
\]

for sufficiently large \( N \), where \( C > 0 \) and \( \omega \in (c, 1) \) do not depend on \( N \). The above equation is (safe for notational differences) a direct transcription of equation (14) of [4], which is repeated here for clarity.

Next, we will make a choice of \( N \) that will later show the conclusion of Lemma 8. Let \( \delta > 0 \) be a “sufficiently” small number (we will later make uniform assumptions on its size). By equations (5) and (4), there exist \( s_0 \in N \) such that

\[
\max\{1, A_1 - \delta\} < S_n < A_2 + \delta
\]

holds for all \( n \geq s_0 \). For each such \( s_0 \), pick \( s_1 \in N \) minimal such that

\[
s_1 > D^{s_0} \prod_{i=1}^{s_0-1} (K D_i + d_i), \quad \max\{1, A_1 - \delta\} < S_{s_1} < A_1 + \delta,
\]
and pick then $s_2 \in \mathbb{N}$ minimal such that

\[(17)\quad s_2 > s_1, \quad A_2 - \delta < S_{s_2} < A_2 + \delta.\]

For sufficiently large $s_0$, pick $N = N(s_0) \in \mathbb{N}$ minimal such that

\[(18)\quad s_1 \leq N < s_2, \quad S_{N+1} > \left(1 + \frac{1}{(N+1)^2}\right) \max_{s_1 \leq j \leq N} \{S_j, A_2 - 2\delta\}.\]

This is doable as the contrary would imply

\[
A_2 - \delta < S_{s_2} \leq \left(1 + \frac{1}{s_2^2}\right) \max_{s_1 \leq j < s_2} \{S_j, A_2 - 2\delta\}
\]

\[
\leq \ldots \leq \max\{S_{s_1}, A_2 - 2\delta\} \prod_{j=s_1+1}^{s_2} \left(1 + \frac{1}{j^2}\right)
\]

\[
\leq (A_2 - 2\delta) \prod_{j=s_1+1}^{\infty} \left(1 + \frac{1}{j^2}\right),
\]

which would be a contradiction for large enough $s_0$ (and thus $s_1$), regardless of $\delta$. We then apply equation (18) along with Lemma 5 to find

\[
a_{N+1} = S_{N+1}^{D_{N+1} \prod_{i=1}^{N} (KD_i + d_i)}
\]

\[
> \left(1 + \frac{1}{(N+1)^2}\right)^{D_{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \max_{s_1 \leq j \leq N} \{S_j, A_2 - 2\delta\}^{D_{N+1} \prod_{i=1}^{N} (KD_i + d_i)}
\]

\[
\geq \left(1 + \frac{1}{(N+1)^2}\right)^{D_{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \left( \prod_{n=s_1+1}^{N} a_n \right)^{KDD_{N}}
\]

\[
= \left( \prod_{n=1}^{s_1} (A_2 - 2\delta)^{D_{n} \prod_{i=1}^{n-1} (KD_i + d_i)} \right)^{KDD_{N}}
\]

\[
\geq \left(1 + \frac{1}{(N+1)^2}\right)^{D_{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \left( \prod_{n=1}^{N} a_n \right)^{KDD_{N}}
\]

\[
\left( \prod_{n=s_0+1}^{s_1} a_{n} \right)^{KDD_{N}}
\]

\[(19)\]
for a small enough choice of \( \delta \). For sufficiently large \( s_0 \), equation (11) gives

\[
\frac{1}{\prod_{n=1}^{s_0-1} a_n} \geq \frac{1}{\prod_{n=1}^{s_0-1} (3A_2)^{D^n \prod_{i=1}^{a_n-1} (KD_i + d_i)}} \geq (3A_2)^{-N},
\]

by choice of \( s_1 \) and \( N \). Meanwhile, equations (15) and (16) followed by Lemma 5 give

\[
\prod_{n=s_0}^{s_1} \frac{(A_2 - 2\delta)^{D^n \prod_{i=1}^{a_n-1} (KD_i + d_i)}}{a_n} \geq \prod_{n=s_0}^{s_1-1} \frac{(A_2 - 2\delta)^{D^n \prod_{i=1}^{a_n-1} (KD_i + d_i)}}{(A_2 + \delta)(A_1 + \delta)} \geq 1,
\]

by choosing \( \delta > 0 \) small enough that \((A_2 - 2\delta)^2 > (A_2 + \delta)(A_1 + \delta)\). Notice that since \( d_i \) and \( K \) are all positive integers, we must have \( KD_i + d_i \geq 2 \), which ensures

\[
\prod_{i=1}^{N} (KD_i + d_i) \geq \log 2 \log \left( \frac{1}{1 + \frac{1}{(N+1)^2}} \right) N^{3} D_N \prod_{i=1}^{N-1} (KD_i + d_i)^{\omega},
\]

for large enough \( N \) (recall \( c < \omega < 1 \), using that \( 1/ \log \left( 1 + \frac{1}{(N+1)^2} \right) \) is dominated by the polynomial \((N+1)^2\). Thus

\[
\left( 1 + \frac{1}{(N+1)^2} \right)^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \geq 2^{N^{3} D^{N+1} D_N \prod_{i=1}^{N-1} (KD_i + d_i)^{\omega}}.
\]

Applying this as well as equations (20) and (21) to equation (19), we have

\[
a_{N+1} \geq \left( \prod_{n=1}^{N} a_n \right)^{KDD_N} 2^{N^{3} D^{N+1} D_N \prod_{i=1}^{N-1} (KD_i + d_i)^{\omega}} (3A_2)^{-KDD_N} \geq \left( \prod_{n=1}^{N} a_n \right)^{KDD_N} 2^{N^{2} D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)^{\omega}},
\]

for \( N(s_0) \) large enough, using that \((KD_N + d_N)/D_N \leq K + 1 \), and \( K \) is constant. Recalling equations (14) and (11), we now have

\[
|\gamma(N)| \leq \frac{2^{\log_2 a_{N+1}}^{\omega}}{a_{N+1}} \leq \left( \prod_{n=1}^{N} a_n \right)^{-KDD_N} 2^{\log_2(2A_2)D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)}^{\omega} \geq \left( \prod_{n=1}^{N} a_n \right)^{-KDD_N} 2^{-ND^{N+1} \prod_{i=1}^{N} (KD_i + d_i)^{\omega}},
\]
and so

$$|\gamma(N)| \left( \frac{2^{(D^N \prod_{i=1}^{N-1} (KD_i + d_i)) c}}{\prod_{n=1}^{N} |a_n|^K} \right)^{DD_N} \leq \frac{2^{D^N \prod_{i=1}^{N-1} (KD_i + d_i)^c}}{2^{N^D} \prod_{n=1}^{N} (KD_i + d_i)^{c_n}} \leq 2^{-(K+1)^\omega_N},$$

for all sufficiently large $N(s_0)$. As this becomes arbitrarily small as $s_0$ tends to infinity, the lemma follows.

**Case 2 ($a_n < 2^n$ infinitely often).** Put $A = (1 + A_2) / 2$. By equation (5), we may pick arbitrarily large $k \in \mathbb{N}$ such that

(23) $$S_k > A.$$  

For each such $k$, pick $k_0 \in \mathbb{N}$ maximal such that

(24) $$k_0 \leq k, \quad a_{k_0} < 2^{k_0}.$$  

Notice that the case assumption implies

(25) $$k_0 \xrightarrow{k \rightarrow \infty} \infty.$$  

As clearly $k_0 < k$ for just slightly large $k$, pick $N \in \mathbb{N}$ minimal such that

(26) $$k_0 \leq N < k, \quad S_{N+1} > \left(1 + \frac{1}{(N+1)^2}\right) \max_{k_0 \leq j \leq N} S_j.$$  

Such $N$ must exist as the contrary would imply

$$A < S_k \leq \left(1 + \frac{1}{k^2}\right) \max_{k_0 \leq j < k} S_j \leq \cdots \leq S_{k_0} \prod_{j=k_0}^{k} \left(1 + \frac{1}{j^2}\right)$$

$$< S_{k_0} \prod_{j=k_0}^{\infty} \left(1 + \frac{1}{j^2}\right)$$

for large enough $k$, as the number

$$C_k := S_{k_0} \prod_{j=k_0}^{\infty} \left(1 + \frac{1}{j^2}\right)$$

tends to 1 as $k$ (and thus $k_0$, by (25)) tends to infinity. Following the same argument, we may also conclude that $S_n < C_k$ for all $k_0 \leq n \leq N$ when $k$ is sufficiently large. That leads to

$$\prod_{n=1}^{N} a_n = \left(\prod_{n=1}^{k_0} a_n\right) \prod_{n=k_0+1}^{N} a_n < \left(\prod_{n=1}^{k_0} 2^{k_n}\right) \prod_{n=k_0+1}^{N} C_k^{D_1} \prod_{n=1}^{N-1} (KD_i + d_i)$$

$$\leq 2^{k_0} C_k^{D_1} \prod_{n=1}^{N-1} (KD_i + d_i) \prod_{n=k_0+1}^{N} C_k^{D_1} \prod_{n=1}^{N-1} (KD_i + d_i),$$
by using the choice of \( k_0 \) and equation (1). Applying Lemma 5 and the lower bound on \( N \), we reach

\[
(27) \quad \prod_{n=1}^{N} a_n < 2^{N^2} \left( C_k^{DN} \prod_{i=1}^{N-1} (KD_i + d_i) \right)^2 = 2^{N^2} \left( C_k^{2N} \right)^{DN} \prod_{i=1}^{N-1} (KD_i + d_i).
\]

Aiming for a lower bound on \( a_{N+1} \), we use equation (26) and Lemma 5 to find

\[
a_{N+1} = S_{N+1}^{DN+1} \prod_{i=1}^{N} (KD_i + d_i)
\geq \left( 1 + \frac{1}{(N + 1)^2} \right)^{DN+1} \prod_{i=1}^{N} (KD_i + d_i)
\geq \left( 1 + \frac{1}{(N + 1)^2} \right)^{KDDN} \sum_{n=1}^{N} D^n \prod_{i=1}^{N-1} (KD_i + d_i)
\geq \left( 1 + \frac{1}{(N + 1)^2} \right)^{DN+1} \prod_{i=1}^{N} (KD_i + d_i)
\left( \prod_{n=1}^{N} a_n \right)^{KDDN} \left( \prod_{n=1}^{N} \frac{1}{a_n} \right)^{KDDN}.
\]

By equation (1) and the choice of \( k_0 \), we have

\[
\left( \prod_{n=1}^{N} \frac{1}{a_n} \right)^{KDDN} \geq \left( \prod_{n=1}^{k_0-1} 2^{-k_0} \right)^{KDDN} \geq 2^{-KNN^2DDN}.
\]

Recalling equation (22) (which uses neither case assumption nor choice of \( N \)), we get for sufficiently large \( N \) that

\[
a_{N+1} > 2^{N^3DN+1DN} \prod_{i=1}^{N-1} (KD_i + d_i)^\omega \left( \prod_{n=1}^{N} a_n \right)^{KDDN} 2^{-KNN^2DDN}
\geq 2^{N^2DN+1\prod_{i=1}^{N} (KD_i + d_i)^\omega} \left( \prod_{n=1}^{N} a_n \right)^{KDDN}.
\]
Repeating equation (20) of [4], we use that the function $2^{c \log x}/x$ is decreasing combined with equation (2) to see that
\[
\sum_{n=k}^{\infty} 2^{c \log a_n} a_n = \sum_{k \leq n \leq a_k^c} 2^{c \log a_k} a_k + \sum_{n > a_k^c} 2^{c \log a_n} a_n \\
\leq a_k^{c a_k} a_k + \sum_{n > a_k^c} 2^{c \log a_n} a_n \\
\leq a_k^{(a-1)/2} + \sum_{n > a_k^c} \frac{1}{n^{1+\varepsilon/2}} \leq a_k^{(a-1)/2} + B_0 \frac{1}{(a_k^c)^{1+\varepsilon/2}} \\
\leq a_k^{(a-1)/2} + a_k^{-ae/3} \leq a_k^{-B},
\]
for $k$ sufficiently large, for some $0 < B < 1 < B_0$ not depending on $k$. By equations (13), (23) and (28), we then have
\[
|\gamma(N)| \leq \sum_{n=N+1}^{k-1} \frac{2^{c \log a_n}}{a_n} + \sum_{n=k}^{\infty} \frac{2^{c \log a_n}}{a_n} \leq \frac{2^{c \log a_{N+1}^c}}{a_{N+1}} + a_k^{-B}
\]
Thus
\[
|\gamma(N)| \left(2^{D \log N} \prod_{i=1}^{N-1} (K D_i + d_i)^{c} \prod_{n=1}^{N} a_n^k\right) \\
\leq \left(\frac{2^{c \log a_{N+1}^c}}{a_{N+1}} + a_k^{-B}\right) \left(2^{D \log N} \prod_{i=1}^{N-1} (K D_i + d_i)^{c} \prod_{n=1}^{N} a_n^k\right)
\]
It follows by $c < \omega$ and equations (13) and (28) that
\[
\frac{2^{c \log a_{N+1}^c}}{a_{N+1}} \left(2^{D \log N} \prod_{i=1}^{N-1} (K D_i + d_i)^{c} \prod_{n=1}^{N} a_n^k\right) \\
< \frac{2^{c \log (2A_2+1)D_0^{c(N+1)}} \prod_{i=1}^{N} (K D_i + d_i)^{c}}{2^{N^2D^{N+1} \prod_{i=1}^{N} (K D_i + d_i)^{c}}} < 2^{-(K+1)^{N}},
\]
for sufficiently large $N$. Meanwhile, equations (23) and (27) imply that
\[
a_k^{-B} \left(2^{((K+1)D_{N/2})^{c}N} \prod_{n=1}^{N} a_n^k\right) \\
< \left(2^{D \log N} \prod_{i=1}^{N-1} (K D_i + d_i)^{c}\right) \frac{2^{N^2 (C_k^D)^{D_{N}^{c} \prod_{i=1}^{N-1} (K D_i + d_i)^{c}}} \prod_{i=1}^{N} (K D_i + d_i)^{c}}{A^{BD_0^{c} \prod_{i=1}^{N-1} (K D_i + d_i)}} \\
= \left(2^{N^2+D \log N} \prod_{i=1}^{N-1} (K D_i + d_i)^{c}\right) \frac{(C_k^D)^{D_{N+1}^{c} \prod_{i=1}^{N} (K D_i + d_i)}}{A^{B}D_{N+1}^{c} \prod_{i=1}^{N} (K D_i + d_i)} \\
< 2^{D \log N} \prod_{i=1}^{N} (K D_i + d_i)^{c} / (A^{B/2}D_{N+1}^{c} \prod_{i=1}^{N} (K D_i + d_i)} \leq 2^{-(K+1)^{N}},
\]
using that \( C_k^2 < A^{B/2} \) for \( k \) (and thus \( N \)) sufficiently large. For \( k \) sufficiently large, we conclude

\[
|\gamma(N)| \left( 2^{(K+1)DDN/2} \prod_{n=1}^{N} a_n^{K_n} \right)^{DDN} < 2^{-(K+1)^N} + 2^{-(K+1)^N},
\]

which clearly tends to 0 as \( k \) (and thus \( N \)) grows large, and the lemma follows.

\[\square\]

**Proof (Theorem 4).** It is clear that the entire hypothesis of Lemma 8 is implied by the hypothesis of Theorem 4, as equation (7) implies equation (10). It is likewise clear that the only part of the hypothesis of Lemma 7 that is not also used in the hypothesis of Theorem 4 is the assumption that \( \deg \gamma \leq D \). As the conclusions of the two lemmas are mutually exclusive, we conclude \( \deg \gamma > D \). \[\square\]

**Concluding Remarks**

As in the case of Theorem 3, the requirements using \( \Re \zeta \) (i.e. equations (8) and (9)) are used solely to ensure that \( \gamma_N \) is non-zero and non-conjugate to \( \gamma \) for all sufficiently large \( N \). Consequently, these requirements may be replaced by any other set of conditions ensuring that property. Note, however, that the property is required as one might otherwise construct a sequence converging to a rational number while satisfying all other parts of the hypothesis.

In the case of \( K = 1 \), Theorem 4 implies

**Theorem 5.** Let \( D \in \mathbb{N} \) be a natural number, let \( \zeta \in \mathbb{C} \) with \( |\zeta| = 1 \), and let \( \alpha, \varepsilon > 0 \) be real numbers. Let \( \{\alpha_n\}_{n \in \mathbb{N}} \) be a sequence of algebraic integers, and let \( \{b_n\}_{n \in \mathbb{N}} \) be a sequences of rational integers. For \( n \in \mathbb{N} \), write \( d_n = \deg \alpha_n \) and \( D_n = \prod_{i=1}^{n} d_i \). Suppose that

\[
\forall n \in \mathbb{N} : \quad n^{1+\varepsilon} \leq |\alpha_n| < |\alpha_{n+1}|,
\]

\[
1 \leq \liminf_{n \to \infty} |\alpha_n|^{\frac{d_n-1}{\prod_{i=1}^{n-1}(d_i+d_i)}} < \limsup_{n \to \infty} |\alpha_n|^{\frac{d_n-1}{\prod_{i=1}^{n-1}(d_i+d_i)}} < \infty,
\]

\[
\forall n \in \mathbb{N} : \quad b_n|\alpha_n| \leq 2^{|\log_2|\alpha_n||} |\alpha_n|,
\]

\[
\forall n \in \mathbb{N} : \quad \Re \zeta(\alpha_n) > 0.
\]

Then \( \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \) has algebraic degree strictly greater than \( D \).

By doing the right modifications to the proof of Theorem 3, it may be improved so that the sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) only needs to satisfy the hypothesis of Theorem 5 where the requirement

\[
1 \leq \liminf_{n \to \infty} |\alpha_n|^{\frac{d_n-1}{\prod_{i=1}^{n-1}(d_i+d_i)}} < \limsup_{n \to \infty} |\alpha_n|^{\frac{d_n-1}{\prod_{i=1}^{n-1}(d_i+d_i)}} < \infty
\]
is replaced by \( \limsup_{n \to \infty} |\alpha_n|^{\frac{1}{\prod_{i=1}^{n-1}(D_i + d_i)}} = \infty \). This will in particular remove the restriction that the \( \alpha_n \) must be of bounded algebraic degree while also slacking the upper bound on \( \mathbb{F}_n \).

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