Abstract

This paper studies absolute retracts in congruence modular varieties of universal algebras. It is shown that every absolute retract with finite dimensional congruence lattice is a product of subdirectly irreducible algebras. Further, every absolute retract in a residually small variety is the product of an abelian algebra and a centerless algebra.

1 Introduction

Recall that an algebra \( R \) belonging to a variety \( \mathcal{V} \) of algebras is said to be an absolute retract in \( \mathcal{V} \) if and only if it is a retract of each of its extensions in \( \mathcal{V} \), that is, if for any embedding \( e: R \rightarrow A \) in \( \mathcal{V} \) there is a surjective homomorphism \( p: A \rightarrow R \) such that \( p \circ e = \text{id}_R \). Absolute retracts have been studied in a universal-algebraic context in a number of papers, particularly in association with related notions of injectivity, congruence extension, amalgamation [7], [2], [4], [10], [14], [9], [13].

We briefly describe why the investigation of absolute retracts is worth pursuing: In his survey on equational logic, Walter Taylor [16] states the following problem: *Does every residually small variety of algebras with the amalgamation property also have the congruence extension property?* This problem is still open, though it has been settled in the affirmative for congruence modular varieties by the efforts of Bergman, Kearnes and McKenzie [1], [3], [11]. Absolute retracts seem central to this problem. Firstly, the notion of absolute retract joins two model-theoretic properties: An algebra is an absolute retract in a variety if and only if it is both equationally compact and algebraically closed in that variety (cf. [13]). Secondly, every absolute retract in a variety is a member of the amalgamation class of that variety (cf. [2]). Thirdly, a variety \( \mathcal{V} \) is residually small if and only if every algebra in \( \mathcal{V} \) is embeddable into an absolute retract of \( \mathcal{V} \) (cf. [15]). Finally, it is easy to show that if a residually small variety \( \mathcal{V} \) satisfies the amalgamation property, then \( \mathcal{V} \) satisfies the congruence extension property if and only if the class of absolute retracts of \( \mathcal{V} \) is closed under products.

The investigation of absolute retracts has been most fruitful when restricted to congruence distributive varieties, where the Fraser-Horn property and Jónsson’s Lemma provide powerful tools for managing congruences on products. Davey and Werner [5] observe that “there are a number of papers in which it is shown that the injectives, and more generally the weak injectives, of a particular variety are precisely finite complete Boolean powers of appropriate subdirectly irreducible algebras”, and proceed to prove a statement which encompasses many such results at a single stroke. Products of Boolean powers also play a role in [14] and [9], where it is proved that, in a finitely generated congruence distributive variety, every absolute
retract is a product of Boolean powers (indeed, reduced powers) of maximal subdirectly irreducible algebras.

In the congruence modular case, congruences on products are less manageable. Davey and Kovács [3] do study absolute retracts in modular varieties, but restrict their attention to directly indecomposable absolute retracts, where they show that each such is finitely subdirectly irreducible. The main aim of the current paper is to partly extend this result to cover products: We prove that if \( R \) is an absolute retract in a congruence modular variety, and if the congruence lattice of \( R \) is of finite dimension, then \( R \) is a product of subdirectly irreducible algebras.

## 2 Preliminaries

In this section we fix notation and state some basic results that we require in the sequel.

We denote the congruence lattice of an algebra \( A \) by \( \text{Con}(A) \). \( 0_A \) and \( 1_A \) denote the smallest and largest congruences on \( A \), but we may omit the subscripts if these are clear from context. If \( \varphi \leq \psi \in \text{Con}(A) \), then \( I[\varphi, \psi] \) denotes the interval \( \{ \theta \in \text{Con}(A) : \varphi \leq \theta \leq \psi \} \) in \( \text{Con}(A) \).

If \( a \in A \) and \( \theta \in \text{Con}(A) \) we shall use two notations for the congruence class of \( a \) modulo \( \theta \): We shall denote by \([a] \theta \) the congruence class of \( a \) as a set: \([a] \theta := \{ b \in A : a \theta b \} \).

On the other hand, the element of the quotient algebra \( A/\theta \) corresponding to \( a \) will be denoted by \( a/\theta \). If \( B \subseteq A \), then we define \([B] \theta := \bigcup_{b \in B} [b] \theta \). If \( A \) is a subalgebra of \( B \) and \( \theta \in \text{Con}(B) \), then we shall denote the restriction of \( \theta \) to \( A \) by \( \theta \upharpoonright A \).

If \( \theta_i \in \text{Con}(A_i) \) for \( i \in I \), then \( \prod_{i \in I} \theta_i \) is the congruence on \( \prod_i A_i \) given by: \((a_i)_{i \in I} \in (\prod_{i \in I} \theta_i)(b_i)_{i \in I}\) if and only if \( a_i \theta_i b_i \) for all \( i \in I \). For finite products, we may write \( \theta_1 \times \theta_2 \times \cdots \times \theta_n \) instead of \( \prod_{i=1}^n \theta_i \). A congruence of the form \( \prod_{i \in I} \theta_i \) is called a product congruence.

The following result is implicit in [4]:

**Lemma 2.1** Suppose that \( \text{Con}(A \times B) \) is a modular lattice, and that \( \varphi_1 \leq \varphi_2 \in \text{Con}(A) \) and \( \psi \in \text{Con}(B) \). Then each \( \theta \in \text{Con}(A \times B) \) satisfying \( \varphi_1 \times \psi \leq \theta \leq \varphi_2 \times \psi \) is a product congruence of the form \( \varphi \times \psi \) for some \( \varphi \leq \varphi_2 \).

**Proof:** By modularity, since \((\varphi_2 \times \psi) \wedge (\varphi_1 \times 1_B) = \varphi_1 \times \psi \) and \((\varphi_2 \times \psi) \vee (\varphi_1 \times 1_B) = \varphi_2 \times 1_B \), we have \( I[\varphi_1 \times \psi, \varphi_2 \times \psi] \cong I[\varphi_1 \times 1_B, \varphi_2 \times 1_B] \), via inverse isomorphisms

\[ \theta \mapsto \theta \vee (\varphi_1 \times 1_B) \quad \gamma \wedge (\varphi_2 \times \psi) \leftarrow \gamma \]

Obviously, \( I[\varphi_1 \times 1_B, \varphi_2 \times 1_B] \cong I[\varphi_1, \varphi_2] \). Thus whenever \( \theta \in I[\varphi_1 \times \psi, \varphi_2 \times \psi] \), then \( \theta \vee (\varphi_1 \times 1_B) = \varphi_1 \times 1_B \) for some \( \varphi \in I[\varphi_1, \varphi_2] \), and hence \( \theta = (\varphi \times 1_B) \wedge (\varphi_2 \times \psi) = \varphi \times \psi \). \( \square \)

An algebra \( A \) is said to be finitely subdirectly irreducible if and only if \( 0_A \) is meet irreducible.

We say that a congruence \( \alpha \in \text{Con}(A) \) is dense if and only if whenever \( \theta \neq 0_A \) in \( \text{Con}(A) \), then \( \theta \wedge \alpha \neq 0_A \) also. Clearly, if \( A \) is finitely subdirectly irreducible, then every non-zero congruence on \( A \) is dense.

**Lemma 2.2** Let \( A, B \) be algebras such that \( \text{Con}(A \times B) \) is a modular lattice. If \( \alpha \) is dense in \( \text{Con}(A) \), and \( \beta \) is dense in \( \text{Con}(B) \), then \( \alpha \times \beta \) is dense in \( \text{Con}(A \times B) \).

**Proof:** We first show that \( \alpha \times 1 \) is dense in \( \text{Con}(A \times B) \): For if \( \theta \wedge (\alpha \times 1) = 0 \), then \( [(\alpha \times 1)] \wedge \theta \vee (0 \times 1) = (0 \times 1) \), and hence \( (\alpha \times 1) \wedge [\theta \vee (0 \times 1)] = 0 \times 1 \), by modularity. But as \( \theta \vee (0 \times 1) \in I[0 \times 1, 1 \times 1] \), there is by Lemma 2.1 an \( \bar{\alpha} \in \text{Con}(A) \) such that \( \theta \vee (0 \times 1) = \bar{\alpha} \times 1 \).
Thus $(\alpha \land \bar{\alpha}) \times 1 = 0 \times 1$, so that we conclude first that $\bar{\alpha} \land \alpha = 0$, and then that $\bar{\alpha} = 0$, because $\alpha$ is dense. Then $\theta \lor (0 \times 1) = 0 \times 1$, and hence $\theta \leq 0 \times 1 \leq \alpha \times 1$. Thus $0 = \theta \land (\alpha \times 1) = \theta$, as required.

In the same way it follows that $1 \times \beta$ is dense in $\text{Con}(A \times B)$. Now observe that $(\alpha \times \beta) = (\alpha \times 1) \land (1 \times \beta)$, and conclude that $\alpha \times \beta$ is dense in $\text{Con}(A \times B)$.

An extension $A \hookrightarrow B$ is said to be an essential extension if and only if whenever $\theta \in \text{Con}(B)$ has $\theta \upharpoonright A = 0_A$, then $\theta = 0_B$. The following facts are well-known (and easy to show) (cf. [2]):

(i) An algebra $R \in \mathcal{V}$ is an absolute retract in $\mathcal{V}$ if and only if it has no proper essential extensions in $\mathcal{V}$.

(ii) An essential extension of a (finitely) subdirectly irreducible algebra is also (finitely) subdirectly irreducible.

A subdirectly irreducible algebra is said to be a maximal subdirectly irreducible in $\mathcal{V}$ if and only if it has no proper essential extensions in $\mathcal{V}$. Thus each maximal subdirectly irreducible is an absolute retract.

Henceforth, we work in a congruence modular variety $\mathcal{V}$, where we shall make use of the commutator theory. We enumerate the following basic facts, which are proved in [6], [8]:

1. If $A \in \mathcal{V}$ and $\varphi, \psi, \psi_i (i \in I) \in \text{Con}(A)$, then
   
   (i) $[\varphi, \psi] \leq \varphi \land \psi$;
   
   (ii) $[\varphi, \psi] = [\psi, \varphi]$;
   
   (iii) $[\varphi, \bigvee_{i \in I} \psi_i] = \bigvee_{i \in I} [\varphi, \psi_i]$;

2. If $B$ is a subalgebra of $A$ and $\varphi, \psi \in \text{Con}(A)$, then $[\varphi \upharpoonright B, \psi \upharpoonright B] \leq [\varphi, \psi] \upharpoonright B$.

3. If $\pi \in \text{Con}(A)$, then, with $\varphi, \psi \geq \pi$, we have that $[\varphi/\pi, \psi/\pi] = [\varphi, \psi] \lor \pi$ in $\text{Con}(A/\pi)$.

4. A congruence $\theta \in \text{Con}(A)$ is said to be abelian if $[\theta, \theta] = 0_A$, and central if $[\theta, 1_A] = 0_A$. Clearly every central congruence is abelian. The center of $A$ is the largest central congruence of $A$, and denoted by $\zeta_A$. The algebra $A$ is said to be centerless if $\zeta_A = 0_A$. Furthermore:

   (i) $\zeta_{A \times B} = \zeta_A \times \zeta_B$;
   
   (ii) If $B$ is a subalgebra of $A$, then $\zeta_A \upharpoonright B \leq \zeta_B$. In particular, the restriction of a central congruence is itself central.
   
   (iii) Abelian congruences permute with every congruence: If $\theta, \varphi \in \text{Con}(A)$ and $\theta$ is abelian, then $\theta \circ \varphi = \varphi \circ \theta = \theta \lor \varphi$.

5. Each congruence modular variety $\mathcal{V}$ has a ternary term $d(x, y, z)$, called the Gumm difference term, with the following properties:

   (i) $\mathcal{V} \models d(x, y, y) = x$;
   
   (ii) If $\theta$ is abelian, and $x \theta y$, then $d(x, x, y) = y$. 
6. A residually small congruence modular variety satisfies the commutator equation (C1) (cf. [8], Theorems 8.1 and 10.4):

\[ \alpha \land [\beta, \beta] = [\alpha \land \beta, \beta] \]  

(C1)

We will also need the following result:

**Theorem 2.3** ([8], Thm. 9.1) Suppose that \( \mathcal{V} \) is a congruence modular variety with Gumm difference term \( d(x, y, z) \). Let \( A \in \mathcal{V} \), and \( \varphi \geq \psi \in \text{Con}(A) \). Then \([\varphi, \psi] = 0_A\) if and only if for every term operation \( f \) (of arity \( n \)) and every \( x, \varphi y, \psi z \) (for \( 1 \leq i \leq n \)) we have

(i) \( d(y_i, y_i, z_i) = z_i \), and

(ii) \( f \left( d \left( \frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n} \right) \right) = d \left( \frac{f(x_1, x_2, \ldots, x_n)}{y_1, y_2, \ldots, y_n} \right) \)

**Corollary 2.4** If \( x \varphi A \gamma y \), and \( a \in A \), then \( d(x, a, d(a, x, y)) = y \).

**Proof:** Note that \( 1_A \geq \varphi A \gamma \), and that \([1, \varphi A \gamma] = 0\). As \( x \varphi A \gamma y \), we may apply Theorem 2.3 with \( f = d \) to conclude that \( d \) commutes with itself on \( \left( \frac{x}{a}, \frac{a}{x}, \frac{x}{a}, \frac{a}{x}, \frac{x}{a} \right) \), i.e. that \( d \left( \frac{d(x, x, x)}{d(a, x, x)} \right) = \frac{d(x, x, x)}{d(a, x, y)} \)

\[ d \left( d \left( \frac{x}{a} \right), d \left( \frac{x}{a} \right), d \left( \frac{x}{a} \right) \right) \], so that \( d(x, a, d(a, x, y)) = d(x, x, d(x, y)) \). But as \( \varphi A \gamma \) is abelian, and \( x \varphi A \gamma y \), we have \( d(x, x, y) = y \), and hence \( d(x, a, d(a, x, y)) = y \). \( \square \)

3. **Subdirect Product–Essential Extensions are Essential**

We work throughout in a congruence modular variety \( \mathcal{V} \) with Gumm difference term \( d \).

**Definition 3.1** Let \( n \in \mathbb{N} \). An embedding \( e: A \hookrightarrow \prod_{i=1}^n A_i \) is said to be a **product–essential extension** if and only if whenever \( \varphi_i \in \text{Con}(A_i) \) are such that \( (\prod_{i=1}^n \varphi_i) |_A = 0_A \), then \( \varphi_i = 0_{A_i} \) (for \( i = 1, 2, \ldots, n \)).

**Remarks 3.2** (a) Note that if \( e: A \hookrightarrow \prod_{i=1}^n A_i \) is a subdirect product-essential extension, and if \( \eta_i := \ker \pi_i |_A \) is the kernel of the natural projection \( \pi_i \circ e: A \to A_i \), then

(i) \( \bigwedge_{i=1}^n \eta_i = 0_A \);

(ii) If \( \eta_i \leq \phi_i \) for \( i = 1, \ldots, n \), and \( \bigwedge_{i=1}^n \phi_i = 0_A \), then \( \phi_i = \eta_i \) for \( i = 1, \ldots, n \).

(b) If \( A \hookrightarrow \prod_{i=1}^n A_i \) is a subdirect embedding, then there are \( \bar{\varphi}_i \in \text{Con}(A_i) \) such that \( A \hookrightarrow \prod_{i=1}^n A_i/\bar{\varphi}_i \) is a subdirect product–essential embedding. Indeed, if \( A_i \cong A/\eta_i \), then by Zorn’s lemma there is \( \varphi_i \in \text{Con}(A) \) maximal with respect to the properties that \( \varphi_i \geq \eta_i \) and \( \varphi_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n = 0 \). Then choose \( \varphi_2 \in \text{Con}(A) \) maximal such that \( \varphi_2 \geq \eta_2 \) and \( \varphi_1 \wedge \varphi_2 \wedge \eta_3 \wedge \cdots \wedge \eta_n = 0 \), etc. Then \( \bar{\varphi}_i := \varphi_i/\eta_i \) will do.

The main technical result of this section is the following:
Theorem 3.3 If \( e : A \rightarrow \prod_{i=1}^{n} A_i \) is a subdirect product–essential embedding in a congruence modular variety, then it is an essential embedding.

Before we can tackle the proof, we will require some intermediary results, many of which owe a large debt to Davey and Kovács [4].

Proposition 3.4 Let \( A \) be a subalgebra of an algebra \( B \), and suppose that \( \theta \in \text{Con}(B) \) is a central congruence in \( B \) with \( \theta \mid A = 0_A \) and \([A] \theta = A \). Then \( \theta = 0_B \).

Proof: Let \( a \in A \). If \( x \theta y \), then \( d(a, x, y) \theta d(a, x, x) = a \), and hence \( d(a, x, y) \in A \). But then \( (d(a, x, y), a) \in \theta \mid A = 0_A \), so \( d(a, x, y) = a \). Thus \( x = d(x, a, a) = d(x, a, d(x, y)) = y \), by Corollary 2.4.

Proposition 3.5 Suppose that \( f : B \hookrightarrow \prod_{i=1}^{n} B_i \) is a product–essential embedding. Further suppose that \( \bar{\theta} \in \text{Con}(\prod_{i=1}^{n} B_i) \) is such that \( \theta \mid B = 0_B \). Then \( \bar{\theta} \) is a central congruence.

Proof: Define \( \bar{\nu}_i = 0 \times \cdots \times 0 \times 1 \times 0 \times \cdots \times 0, \) where 1 occurs in the \( i \)th place only, and 0 everywhere else. We begin by showing that \( [\bar{\theta}, \bar{\nu}_i] = 0 \) for each \( i = 1, \ldots, n \). Without loss of generality, take \( i = 1 \), so that \( \bar{\nu}_1 = 1 \times 0 \times \cdots \times 0 \). Note that since \( 0 \times 0 \times \cdots \times 0 \leq \bar{\theta} \wedge \bar{\nu}_1 \leq \bar{\nu}_1 = 1 \times 0 \times \cdots \times 0 \), Lemma 2.1 guarantees that there is a \( \bar{\varphi} \in \text{Con}(A_1) \) such that \( \bar{\varphi} \times 0 \times \cdots \times 0 \). Now since \( \theta \mid A = 0 \), we have \( (\bar{\varphi} \times 0 \times \cdots \times 0) \mid B = 0 \) as well, and as the embedding \( f \) is product–essential, we must have \( \bar{\varphi} = 0 \). It follows that \( [\bar{\theta}, \bar{\nu}_1] \leq \bar{\theta} \wedge \bar{\nu}_1 = 0 \).

In the same way, we see that \( [\bar{\theta}, \bar{\nu}_i] = 0 \) for all \( i = 1, \ldots, n \). Now \( [\bar{\theta}, 1 \times 0 \times \cdots \times 1] = [\bar{\theta}, \bigvee_{i=1}^{n} \bar{\nu}_i] = \bigvee_{i=1}^{n} [\bar{\theta}, \bar{\nu}_i] = 0 \).

For the next few lemmas we fix the following: We are given a subdirect product–essential embedding \( e : A \hookrightarrow \prod_{i=1}^{n} A_i \). For \( i = 1, \ldots, n \), let \( \eta_i \) be the kernel of the natural projection \( \pi_i \circ e : A \rightarrow A_i \), so that \( A/\eta_i \cong A_i \). Throughout, we will take \( A_i = A/\eta_i \), when convenient. We also assume that the embedding \( e : A \hookrightarrow \prod_{i=1}^{n} A_i \) is an inclusion, so that

\[
a \in A \quad \text{is identified with} \quad (a/\eta_1, \ldots, a/\eta_n) \in \prod_{i=1}^{n} A_i
\]

Further, define

\[
\alpha_i := \eta_i \lor (\bigwedge_{j \neq i} \eta_j) \in \text{Con}(A) \quad \text{and} \quad \bar{\alpha}_i := \alpha_i/\eta_i \in \text{Con}(A_i)
\]

Lemma 3.6 \( \bar{\alpha}_1 \times \cdots \times \bar{\alpha}_n \) is dense in \( \text{Con}(\prod_{i=1}^{n} A_i) \).

Proof: Let \( \bar{\mu}_i := 1 \times 1 \cdots \times 1 \times \bar{\alpha}_i \times 1 \times \cdots \times 1 \) in \( \text{Con}(\prod_{i=1}^{n} A_i) \), where \( \bar{\alpha}_i \) occurs in the \( i \)th place, and 1 everywhere else. Then \( \prod_{i=1}^{n} \bar{\alpha}_i = \bigwedge_{i=1}^{n} \bar{\mu}_i \), so by Lemma 2.2 it suffices to show that \( \bar{\mu}_i \) is dense for each \( i = 1, \ldots, n \).

Without loss of generality, we may assume that \( i = 1 \). Suppose therefore that \( \bar{\psi} \land \bar{\mu}_1 = 0 \). Then since \( 0 \times 1 \times \cdots \times 1 \leq (0 \times 1 \times \cdots \times 1) \lor \bar{\psi} \leq 1 \times 1 \times \cdots \times 1 \), there is by Lemma 2.1 a \( \bar{\varphi}_1 \in \text{Con}(A_1) \) such that \( (0 \times 1 \times \cdots \times 1) \lor \bar{\psi} = \bar{\varphi}_1 \times 1 \times \cdots \times 1 \). As \( A_1 = A/\eta_1 \), there is
\( \varphi_1 \geq \eta_1 \) in \( \text{Con}(A) \) such that \( \bar{\varphi}_1 = \varphi_1/\eta_1 \). Now
\[
\bar{\psi} \land \bar{\mu}_1 = 0
\]
\[
\implies (0 \times 1 \times \cdots \times 1) \lor (\bar{\psi} \land \bar{\mu}_1) = 0 \times 1 \times \cdots \times 1
\]
\[
\implies \left( (0 \times 1 \times \cdots \times 1) \lor \bar{\psi} \right) \land \bar{\mu}_1 = 0 \times 1 \times \cdots \times 1 \quad \text{by modularity}
\]
\[
\implies (\bar{\alpha}_1 \land \bar{\varphi}_1) \times 1 \times \cdots \times 1 = 0 \times 1 \times \cdots \times 1
\]
\[
\implies \bar{\alpha}_1 \land \bar{\varphi}_1 = 0
\]
\[
\implies \alpha_1 \land \varphi_1 \leq \eta_1
\]
\[
\implies (\eta_1 \lor (\eta_2 \land \cdots \land \eta_n)) \land \varphi_1 \leq \eta_1
\]
\[
\implies \varphi_1 \land \eta_2 \land \cdots \land \eta_n \leq \eta_1 \quad \text{by modularity}
\]
\[
\implies \varphi_1 \land \eta_2 \land \cdots \land \eta_n = 0
\]

Now since \( \eta_1 \leq \varphi_1 \), it follows by Remarks 3.2 that \( \varphi_1 = \eta_1 \). Hence \( \bar{\varphi}_1 = 0 \) in \( \text{Con}(A_1) \), i.e. \( (0 \times 1 \times \cdots \times 1) \lor \bar{\psi} = 0 \times 1 \times \cdots \times 1 \). Thus \( \bar{\psi} \leq 0 \times 1 \times \cdots \times 1 \leq \bar{\alpha}_1 \times 1 \times \cdots \times 1 \), i.e. \( \bar{\psi} \leq \bar{\mu}_1 \), and so \( 0 = \bar{\psi} \land \bar{\mu}_1 = \bar{\psi} \).

**Lemma 3.7** Suppose that for \( i = 1, \ldots, n \) we have \( \beta_i \in \text{Con}(A) \) such that

(i) \( \eta_i \leq \beta_i \leq \alpha_i \), and

(ii) \( \bar{\beta}_i := \beta_i/\eta_i \) is a central congruence in \( \text{Con}(A_i) \).

Then for each \( k = 1, \ldots, n - 1 \) we have
\[
(\beta_1 \land \cdots \land \beta_k) \circ \beta_{k+1} = (\eta_1 \land \cdots \land \eta_k) \circ \eta_{k+1}
\]

**Proof:** Since each \( \bar{\beta}_i \) is central in \( \text{Con}(A_i) \), the congruence \( 0 \times \cdots \times 0 \times \bar{\beta}_i \times 0 \times \cdots \times 0 \) is central in \( \text{Con}(\prod_{i=1}^n A_i) \). As restrictions of central congruences are central, we see that \( \xi_i := (0 \times \cdots \times 0 \times \bar{\beta}_i \times 0 \times \cdots \times 0) \downarrow A \) is central in \( \text{Con}(A) \). In particular, each \( \xi_i \) is abelian, and hence permutes with every congruence in \( \text{Con}(A) \).

Recall that \( A_i \) is identified with \( A/\eta_i \) and that elements \( a \in A \) are identified with tuples \( (a/\eta_1, \ldots, a/\eta_n) \in \prod_{i=1}^n A_i \). Now
\[
(a, b) \in \xi_i
\]
\[
\iff (a/\eta_1, \ldots, a/\eta_n) (0 \times \cdots \times \bar{\beta}_i \times \cdots \times 0) (b/\eta_1, \ldots, b/\eta_n)
\]
\[
\iff (a, b) \in \beta_i \land (\bigwedge_{j \neq i} \eta_j)
\]
and thus \( \xi_i = \beta_i \land (\bigwedge_{j \neq i} \eta_j) \), for all \( i = 1, \ldots, n \). From this, we obtain the following fact, which we will use several times in the sequel:

\[
\xi_i \leq \eta_j \quad \text{whenever} \quad i \neq j
\]

Next, observe that, since \( \eta_i \leq \beta_i \leq \alpha_i \), we have
\[
\beta_i = \beta_i \land \alpha_i = \beta_i \land (\bigwedge_{j \neq i} \eta_j) = (\beta_i \land \bigwedge_{j \neq i} \eta_j) \lor \eta_i = \xi_i \lor \eta_i = \xi_i \circ \eta_i
\]
We will now show by induction that for all $k = 1, \ldots, n$ we have

$$\beta_1 \wedge \cdots \wedge \beta_k = (\eta_1 \wedge \cdots \wedge \eta_k) \circ \xi_1 \circ \cdots \circ \xi_k$$

We have just shown this is true for $k = 1$. Assuming now that $(\beta_1 \wedge \cdots \wedge \beta_{k-1}) = (\eta_1 \wedge \cdots \wedge \eta_{k-1}) \circ \xi_1 \circ \cdots \circ \xi_{k-1}$, we see that

$$\beta_1 \wedge \cdots \wedge \beta_k \wedge \beta_{k+1} = (\eta_1 \wedge \cdots \wedge \eta_k) \circ \xi_1 \circ \cdots \circ \xi_k$$

We also have $\beta_{k+1} = (\eta_{k+1}) \circ \xi_{k+1}$. Thus

$$(\beta_1 \wedge \cdots \wedge \beta_k \wedge \beta_{k+1}) = (\eta_1 \wedge \cdots \wedge \eta_k) \circ \xi_1 \circ \cdots \circ \xi_k \circ (\eta_{k+1} \circ \xi_{k+1})$$

using the fact that each $\xi_i$ permutes with every congruence. This completes the induction step.

Now using the permutability of the $\xi_i$ we see that

$$(\beta_1 \wedge \cdots \wedge \beta_{k+1}) = (\eta_1 \wedge \cdots \wedge \eta_{k+1}) \circ \xi_1 \circ \cdots \circ \xi_{k+1}$$

since $\xi_k \leq \eta_1 \wedge \cdots \wedge \eta_{k-1}$. This completes the induction step.

Lemma 3.8 Suppose that we have $\beta_i$ as in as in Lemma 3.7. If $\bar{\theta} \in \text{Con}(\prod_{i=1}^n A_i)$ is such that $\bar{\theta} \leq \beta_1 \times \cdots \times \beta_n$, then $[A] \bar{\theta} = A$.

Proof: Since every $\bar{\theta}$-congruence class is contained in a $(\bar{\beta}_1 \times \cdots \times \bar{\beta}_n)$-class, it suffices to show that $[A](\bar{\beta}_1 \times \cdots \times \bar{\beta}_n) = A$.

Recall once more that in our subdirect product–essential embedding $e: A \hookrightarrow \prod_{i=1}^n A_i$, the algebras $A_i$ are identified with $A/\eta_i$ and elements $a \in A$ are identified with tuples $(a/\eta_1, \ldots, a/\eta_n) \in \prod_{i=1}^n A_i$. Now suppose that $(x_1/\eta_1, \ldots, x_n/\eta_n) \in [A](\bar{\beta}_1 \times \cdots \times \bar{\beta}_n)$. This means that there exists an $a \in A$ such that $(x_1/\eta_1, \ldots, x_n/\eta_n) = (\bar{\beta}_1 \times \cdots \times \bar{\beta}_n)(a/\eta_1, \ldots, a/\eta_n)$, i.e. that $\bigcap_{i=1}^n [x_i]_{\beta_i} \neq \emptyset$. We must show that $(x_1/\eta_1, \ldots, x_n/\eta_n) \in A_i$, i.e. that there exists an element $b \in A$ such that $(x_1/\eta_1, \ldots, x_n/\eta_n) = (b/\eta_1, \ldots, b/\eta_n)$, i.e. that $\bigcap_{i=1}^n [x_i]_{\eta_i} \neq \emptyset$.

It therefore suffices to prove that if $\bigcap_{i=1}^k [x_i]_{\beta_i} \neq \emptyset$, then $\bigcap_{i=1}^{k+1} [x_i]_{\eta_i} \neq \emptyset$, for $k = 1, \ldots, n$, and we do this by induction on $k$. There is nothing to prove for the case $k = 1$. Proceeding with the induction step, suppose that $\bigcap_{i=1}^{k+1} [x_i]_{\beta_i} \neq \emptyset$, and that $a \in \bigcap_{i=1}^{k+1} [x_i]_{\beta_i}$. Then also $\bigcap_{i=1}^k [x_i]_{\beta_i} \neq \emptyset$, so by induction hypothesis there exists an element $b \in \bigcap_{i=1}^k [x_i]_{\beta_i}$.

By Lemma 3.7 we see that there is $c \in A$ such that $b(\eta_1 \wedge \cdots \wedge \eta_k) c \eta_{k+1} x_{k+1}$. Then for $i = 1, \ldots, k$ we have $x_i/\eta_i = b/\eta_i = c/\eta_i$. We also have $x_{k+1}/\eta_{k+1} = c/\eta_{k+1}$. Thus $c \in \bigcap_{i=1}^{k+1} [x_i]_{\eta_i}$, so that the latter set is non–empty. This completes the induction step.

We can now prove the main result of this section. The proof adopts a strategy implicit in Davey and Kovács[4], to wit that to prove that an extension $A \hookrightarrow B$ is essential, one may proceed as follows: Given $a \in \text{Con}(B)$ such that $a \upharpoonright_A = 0_A$,
where $\chi, \psi > \phi$

It then follows that the natural subdirect embedding $\cong \phi$. Remarks 3.2(b), we may choose $A$ and hence essential. But then $A$ assume that $\phi$ and hence decomposable — contradiction. Hence $A$.

Proof of Theorem 3.3: Suppose that $\bar{\theta} \in \text{Con}(\prod_{i=1}^{n} A_i)$ is such that $\bar{\theta} \downarrow_{\bar{A}} = 0$. Define $\bar{\psi} := \bar{\theta} \land (\bar{\alpha}_1 \times \cdots \times \bar{\alpha}_n)$, so that also $\bar{\psi} \downarrow_{\bar{A}} = 0$. By Proposition 3.4 it follows that $\bar{\psi}$ is a central congruence in $\text{Con}(\prod_{i=1}^{n} A_i)$, and thus that $\bar{\psi} \leq \bar{\beta}_1 \times \cdots \times \bar{\beta}_n$, where $\bar{\beta}_i := \bar{\alpha}_i \land \bar{\zeta}_A$, and $\bar{\zeta}_A$ denotes the center of $A_i$. Hence by Lemma 3.8 we may conclude that $[A] \bar{\psi} = A$. It now follows from Proposition 3.4 that $\bar{\psi} = 0$, i.e. that $\bar{\theta} \land (\bar{\alpha}_1 \times \cdots \times \bar{\alpha}_n) = 0$. Finally, Lemma 3.6 shows that $\bar{\theta} = 0$, as required. □

4 Absolute Retracts

With Theorem 3.3 in hand, we can investigate absolute retracts in congruence modular varieties. To start, we obtain the following result:

Theorem 4.1 (Davey and Kovács[1]) If $A$ is a directly indecomposable absolute retract in a congruence modular variety, then it is finitely subdirectly irreducible.

Proof: Suppose that $A$ is an absolute retract. If $A$ is not finitely subdirectly irreducible, then there exist $\eta_1, \eta_2 > 0$ in $\text{Con}(A)$ such that $\eta_1 \land \eta_2 = 0$. As in Remarks 3.2[b], we may assume that $\eta_1, \eta_2$ are maximal with respect to the property of having zero meet, so that the canonical embedding $A \hookrightarrow A/\eta_1 \times A/\eta_2$ is product essential, and hence essential. But an absolute retract has no proper essential extensions, so $A \cong A/\eta_1 \times A/\eta_2$, so that $A$ is directly decomposable — contradiction. Hence $A$ is finitely subdirectly irreducible. □

Theorem 4.2 If $A$ is an absolute retract in a congruence modular variety, and if $\text{Con}(A)$ is finite dimensional, then $A$ is a finite product of subdirectly irreducible algebras.

Proof: Suppose that $0_A = \Lambda_{i=1}^{n} \eta_i$ is a representation of $0_A$ as an irredundant meet (where the $\eta_i$ are not assumed to be meet irreducible). Then $\eta_1 > \eta_1 \land \eta_2 > \cdots > \Lambda_{i=1}^{n} \eta_i$, and hence $n \leq \text{height}(\text{Con}(A))$. Now let $m$ be the maximum integer for which there exists an irredundant meet representation $0_A = \Lambda_{i=1}^{m} \eta_i$ of length $m$. By the procedure outlined in Remarks 3.2[b], we may choose $\varphi_i \geq \eta_i$ (for $1 \leq i \leq m$) so that

(i) $\Lambda_{i=1}^{m} \varphi_i = 0_A$; this meet is then clearly irredundant.

(ii) If $\theta_i \geq \varphi_i$ for $1 \leq i \leq m$ are such that $\Lambda_{i=1}^{m} \theta_i = 0_A$ then each $\theta_i = \varphi_i$.

It then follows that the natural subdirect embedding $A \hookrightarrow \prod_{i=1}^{n} A/\varphi_i$ is product-essential, and hence essential. But then $A \cong \prod_{i=1}^{n} A/\varphi_i$, as $A$ does not have any proper essential extensions.

Moreover, each $\varphi_i$ is meet irreducible. To see this, suppose for example that $\varphi_1 = \chi \land \psi$, where $\chi, \psi > \varphi_1$. Then $0_A = \chi \land \psi \land \Lambda_{i=1}^{m} \varphi_i$. Since $\chi \land \Lambda_{i=2}^{m} \varphi_i > 0_A$ and $\psi \land \Lambda_{i=2}^{m} \varphi_i > 0_A$,
we see that the representation \( 0_A = \chi \land \psi \land \bigwedge_{i=2}^{m} \varphi_i \) is an irredundant meet of length \( m + 1 \), contradicting the maximality of \( m \). Hence \( \varphi_1 \) is meet irreducible, and as \( \text{Con}(A) \) has finite height, it is completely meet irreducible. By the same argument, it follows that each \( \varphi_i \) is completely meet irreducible, so that \( A \cong \prod_{i=1}^{m} A/\varphi_i \) decomposes \( A \) into a product of subdirectly irreducible algebras.

Davey and Kovács\(^{[1]} \) actually prove a stronger result than Theorem \( \ref{thm:finite-prod} \), namely that a directly indecomposable absolute retract in a congruence modular variety is a finitely subdirectly irreducible which is either centerless or abelian. We will now partly extend this result to include products, but with the added assumption that the base variety is residually small. Now any residually small congruence modular variety satisfies the commutator identity (C1): \( \alpha \land [\beta, \beta] = [\alpha \land \beta, \beta] \). Observe that, since \( \zeta_{A \times B} = \zeta_A \times \zeta_B \) and \( [1_A \times 1_B, 1_A \times 1_B] = [1_A, 1_A] \times [1_B, 1_B] \), the properties of being centerless and of being abelian are both preserved under finite products.

**Theorem 4.3** Let \( A \) be an absolute retract in a congruence modular variety which satisfies (C1). Then \( A \) is a product of a centerless algebra and an abelian algebra. In particular, if \( A \) is directly indecomposable, then \( A \) is either centerless or abelian.

**Proof:** Note that \( 0 = [\zeta, 1] = \zeta \land [1, 1] \). Now choose \( \theta \geq \zeta \) and \( \psi \geq [1, 1] \) in \( \text{Con}(A) \) maximal so that \( \theta \land \psi = 0 \), as discussed in Remarks \( \ref{rem:finite-prod} \). Then the subdirect embedding \( A \hookrightarrow A/\theta \times A/\psi \) is product–essential, hence essential. Since \( A \) is an absolute retract, we have \( A \cong A/\theta \times A/\psi \). But \( [\theta, 1] = \theta \land [1, 1] = 0 \), so \( \theta \) is central, i.e. \( \theta = \zeta \).

Furthermore, \( A/\zeta \) is centerless: For if \( \xi \geq \zeta \) in \( \text{Con}(A) \) is such that \( \xi/\zeta \) is central in \( \text{Con}(A/\zeta) \), then \( [\xi/\zeta, 1/\zeta] = 0 \), so \( [\xi, 1] \leq \zeta \). But then \( 0 = [[\xi, 1], 1] = [\xi, 1] \land [1, 1] = [\xi, 1] \), and hence \( \xi \) is central, so \( \xi = \zeta \). Thus \( \zeta/\zeta = 0 \) in \( \text{Con}(A/\zeta) \).

Finally, since \( \psi \geq [1, 1] \), \( A/\psi \) is abelian. Hence \( A \cong A/\zeta \times A/\psi \) is the product of a centerless and an abelian algebra. \( \square \)

**Theorem 4.4** Suppose that \( A \) is an absolute retract in a congruence modular variety satisfying (C1), such that \( \text{Con}(A) \) is finite dimensional. If \( A \) satisfies the unique factorization property, then \( A \) is a finite product of subdirectly irreducible algebras, each of which is either centerless or abelian.

In particular, this conclusion is valid when either (i) \( A \) is congruence–permutable with a one–element subalgebra, or when (ii) \( \text{Con}(A) \) is finite.

**Proof:** By Theorem 4.2 \( A \) is a product of subdirectly irreducible algebras \( A = \prod_{i=1}^{k} A_i \). Furthermore, \( A = A/\zeta \times A/\psi \) is also a product of a centerless and an abelian algebra, by Theorem 4.3. Now subdirectly irreducible algebras are directly indecomposable, and hence, by unique factorization, \( A/\zeta \) must be the product of some of the \( A_i \), and \( A/\psi \) the product of the remaining \( A_i \). By reindexing, we may assume that \( A/\zeta = \prod_{i=n+1}^{k} A_i \), and that \( A/\psi = \prod_{i=n+1}^{k} A_i \). Now since \( A/\zeta \) is centerless and \( A/\psi \) is abelian, we see that \( A_i \) is centerless when \( i \leq n \), and abelian when \( n + 1 \leq i \leq k \).

The Birkhoff–Ore Theorem states that if \( A \) has a one–element subalgebra, and a finite–dimensional congruence lattice with permuting congruences, then \( A \) has the unique factorization property. A theorem of Jónsson reaches the same conclusion in the case that \( \text{Con}(A) \) is modular and finite. See Chapter 5 of \( \cite{12} \) for a proof of both assertions. \( \square \)
Remarks 4.5  (a) The above–mentioned results of Birkhoff–Ore and Jónsson admit a common generalization: Is every algebra with finite dimensional modular congruence lattice is uniquely factorable? This problem is still open, cf. [12].

(b) It is easy to show that if \( A = \prod_{i \in I} A_i \) is an absolute retract with a one–element subalgebra, then each \( A_i \) is an absolute retract also. In that case we can deduce from Theorems [4,3] and [4,2] that in a residually small congruence modular variety \( V \):

(i) If \( A \) is an absolute retract, then \( A/\zeta \) is an absolute retract. Moreover, if \( \zeta \neq 0 \), then some non–trivial abelian image of \( A \) is an absolute retract.

(ii) If \( A \) is an absolute retract with finite–dimensional congruence lattice in a congruence modular variety, then \( A \) is a finite product of maximal subdirectly irreducibles, each of which is either centerless or abelian.

(c) Note that every (weakly) injective algebra is an absolute retract. Furthermore, a finite algebra is algebraically closed in a variety if and only if it is an absolute retract (cf. [13]). Thus the above results also have implications for (weakly) injective and algebraically closed algebras.

A Auxiliary Results, Not For Publication

The following result is from Davey and Kovácś [4], who base their argument on a quite complicated isomorphism (obtained by Gumm [8]) between the lattice of central congruences of an algebra \( A \) and the lattice of subalgebras of a certain algebra defined on a congruence class of the center of \( A \). The direct proof given here incorporates arguments of from Gumm [8].

Proposition A.1 Suppose that \( \bar{A} \) is an algebra in a congruence modular variety, and that \( A \) is a subalgebra of \( \bar{A} \). Let \( a \in A \), and let \( \bar{a} \leq \bar{\zeta} \) in \( \text{Con}(\bar{A}) \), where \( \bar{\zeta} \) is the center of \( \bar{A} \). Suppose further that \( \beta \leq \bar{a} \rest A \) in \( \text{Con}(A) \). Now define

\[
\bar{\beta} := \{(x, y) \in \bar{\zeta} : d(a, x, y) \in [a] \beta\}
\]

Then \( \bar{\beta} \in \text{Con}(A) \), \( \beta \leq \bar{\alpha} \) and \( \bar{\beta} \rest A = \beta \).

Proof: Suppose that \( (x, y) \in \bar{\beta} \). Then \( d(a, x, y) \in [a] \beta \subseteq [a] \bar{\alpha} \) and hence

\[
x = d(x, a, a) \bar{\alpha} d(x, a, d(a, x, y)) = y. \text{ But } d(x, a, d(a, x, y)) = y \text{ by Corollary } 2.4 \text{ as } \bar{\beta} \subseteq \bar{\zeta}.
\]

Thus also \( (x, y) \in \bar{\alpha} \), which shows that \( \bar{\beta} \subseteq \bar{\alpha} \).

Let \( (x, y) \in \beta \). Then \( d(a, x, y) \beta, d(a, x, x) = a \), and hence \( (x, y) \in \bar{\beta} \), i.e. \( \beta \subseteq \bar{\beta} \rest A \).

Conversely, if \( (x, y) \in \bar{\beta} \rest A \), then \( d(a, x, y) \beta, a \) and hence \( x = d(x, a, a) \beta d(x, a, d(a, x, y)) = y \) by Corollary [2,4]. Thus also \( \bar{\beta} \rest A \subseteq \beta \).

It remains to show that \( \bar{\beta} \) is a congruence on \( \bar{A} \). As \( d(a, x, x) = a \), it is clear that \( \bar{\beta} \) is reflexive.

Now suppose that \( (x, y) \in \bar{\beta} \), i.e. that \( d(a, x, y) \in [a] \beta \subseteq A \). Since \( \bar{\beta} \leq \bar{\zeta} \), we see that \( d \) commutes with itself on \( \begin{pmatrix} a & a & a \\ a & x & y \\ a & x & x \end{pmatrix} \), and conclude that \( d(a, y, x) = d(a, d(a, x, y), a) \). Now as \( a, d(a, x, y) \in A \), it follows that \( d(a, y, x) \in A \) also. Clearly \( d(a, d(a, x, y), a) \beta d(a, a, a) \) yields \( d(a, y, x) \in [a] \beta \). Hence \( \bar{\beta} \) is symmetric.
Next, suppose that \((x, y), (y, z) \in \beta\). Since \(\beta \leq \zeta\), \(d\) commutes with itself on \(\begin{pmatrix} a & x & y \\ a & y & z \\ a & y & z \end{pmatrix}\) and hence \(d(a, x, z) = d(d(a, x, y), a, d(a, y, z))\). Since \(d(a, x, y), d(a, y, z) \in [a]\beta \subseteq A\), see that \(d(a, x, z) \in A\) also. Then \(d(d(a, x, y), a, d(a, y, z)) \beta d(a, a, a)\) yields that \(d(a, x, z) \in [a]\beta\). Hence \(\beta\) is transitive.

That \(\beta\) is compatible with all the operations follows once again from Theorem 2.3.

Proposition A.2 Suppose that \(f\) is an \(n\)-ary operation, and that \((x_i, y_i) \in \beta\) for \(i = 1, \ldots, n\). Let \(x := (x_1, x_2, \ldots, x_n)\), with \(y\) defined similarly, and let \(a = (a, a, \ldots, a)\) denote an \(n\)-tuple consisting entirely of \(a\)’s. Since \(\beta \leq \zeta\), we have \(f(x) \zeta f(y)\), and hence \(d\) commutes with itself on \(\begin{pmatrix} a & f(x) & f(x) \\ f(a) & f(x) & f(x) \\ f(a) & f(x) & f(y) \end{pmatrix}\), so that \(d(a, f(x), f(y)) = d(a, f(a), d(a, f(x), f(y)))\). But as \(f\) commutes with \(d\) on \(\begin{pmatrix} a & x_1 & y_1 \\ \vdots & \vdots & \vdots \\ a & x_n & y_n \end{pmatrix}\), we have \(d(a, f(x), f(y)) = f(d(a, x_1, y_1), \ldots, d(a, x_n, y_n))\).

Our next proposition is again from Davey and Kovács [4], with only slight modifications. We include the proof to keep this exposition self-contained.

Proposition A.2 Suppose that \(A\) is non-abelian, and that its center \(\zeta\) is dense in \(\text{Con}(A)\). Then \(A\) has a proper essential extension.

Proof: Clearly \(B := \{(a, b, c) \in A^3 : a \zeta b \zeta c\}\) is a subalgebra of \(A\). By Theorem 2.3, \(d : B \rightarrow A\) is a surjective homomorphism. Let \(\theta := \ker d\). By Proposition A.1 there is a central congruence \(\Theta \leq \zeta^3\) such that \(\Theta \upharpoonright B = \theta\). We use the Davey and Kovács [4] strategy described just prior to the proof of Theorem 3.3 to prove that the induced embedding \(B/\theta \rightarrow A^3/\Theta\) is an essential extension.

(i) Suppose that \(\Psi \geq \Theta\) in \(\text{Con}(A^3)\) satisfies \(\Psi / \Theta \wedge \zeta^3 / \Theta = 0\), i.e. \(\Psi \wedge \zeta^3 = \Theta\). We begin by showing that \(\Psi\) is a central congruence. By Lemma 2.1 there is a \(\beta \in \text{Con}(A)\) such that \(\Psi \wedge (0 \times 1 \times 0) = 0 \times \beta \times 0\), so that \(0 \times (\beta \wedge \zeta) \times 0 \leq \Psi \wedge \zeta^3 = \Theta\). But if \((a, b) \in \beta \wedge \zeta\), then \((a, a, a) \times 0 \times (\beta \wedge \zeta) \times 0 (a, b, a)\), so that \((a, a, a) \Theta (a, b, a)\). But as \((a, a, a), (a, b, a) \in B\), it follows that \((a, a, a) \Theta (a, b, a)\), and hence \(b = d(b, a, a) = d(b, a, d(a, b, a)) = a\), by Corollary 2.4. It follows that \(\beta \wedge \zeta = 0\), and thus that \(\beta = 0\), as \(\zeta\) is dense in \(\text{Con}(A)\). Thus \(\Psi \wedge (0 \times 1 \times 0) = 0\), and hence \([\Psi, 0 \times 1 \times 0] = 0\). In an analogous — but slightly simpler — fashion does it follow that \([\Psi, 1 \times 0 \times 0] = 0\). Hence \([\Psi, 1 \times 1 \times 1] = 0\), so that \(\Psi\) is a central congruence on \(A^3\).

(ii) It follows that \(\Psi \leq \zeta^3\), and hence that \(\Theta = \Psi \wedge \zeta^3 = \Psi\). We thus see that if \(\Psi / \Theta \wedge \zeta^3 / \Theta = 0\) in \(\text{Con}(A^3/\Theta)\), then \(\Psi / \Theta = 0\). Hence \(\zeta^3 / \Theta\) is dense in \(\text{Con}(A^3/\Theta)\). Moreover, we clearly have \([B] \zeta^3 = B\), and thus also \([B/\theta] \zeta^3 / \Theta = B / \theta\).

With (i) and (ii) of the Davey and Kovács strategy satisfied, (iii) and (iv) follow. Hence \(B / \theta \rightarrow A^3 / \Theta\) is an essential extension. As \(\Theta \leq \zeta^3\), this embedding is surjective exactly when \(B = A^3\), i.e. when \(\zeta = 1\). But then \(A\) is abelian, contradiction. \(\square\)
It now follows easily that a directly indecomposable absolute retract $A$ is either centerless or abelian. For if $A$ is not centerless, then as $A$ is finitely subdirectly irreducible, we see that $\zeta_A$ is dense in $\text{Con}(A)$. But then as $A$ has no proper essential extension, it must be abelian.

References

[1] C. Bergman. On the relationship between AP, RS and CEP in congruence modular varieties. *Algebra Universalis*, 22:164–171, 1986.

[2] C. Bergman. Amalgamation classes of some distributive varieties. *Proc. Amer. math. Soc.*, 103:335–343, 1988.

[3] C. Bergman and R. McKenzie. On the relationship between AP, RS and CEP in congruence modular varieties II. *Proc. Amer. Math. Soc.*, 103:335–343, 1988.

[4] B.A. Davey and L.G. Kovács. Absolute subretracts and weak injectives in congruence modular varieties. *Trans. Amer. Math. Soc.*, 297:181–196, 1987.

[5] B.A. Davey and H. Werner. Injectivity and Boolean powers. *Math. Z.*, 166:205–223, 1979.

[6] R. Freese and R. McKenzie. *Commutator Theory for Congruence Modular Varieties*. Number 125 in London Mathematical Society Lecture Notes. Cambridge University Press, 1987.

[7] G. Grätzer and H. Lakser. The structure of pseudocomplemented distributive lattices III. *Trans. Amer. Math. Soc.*, 169:475–487, 1972.

[8] H.P. Gumm. Geometrical methods in congruence modular algebras. *Memoirs Amer. Math. Soc.*, 45(286), 1983.

[9] M. Jenner, P. Jipsen, P. Ouwehand, and H. Rose. Absolute retracts as reduced products. *Quaesiones Mathematicae*, 24:129–132, 2001.

[10] P. Jipsen and H. Rose. Absolute retracts and amalgamation in certain congruence distributive varieties. *Canadian Math. Bull.*, 32:309–313, 1989.

[11] K.A. Kearnes. On the relationship between AP, RS and CEP. *Proc. Amer. Math. Soc.*, 105:827–839, 1989.

[12] R. McKenzie, G. McNulty, and W. Taylor. *Algebras, Lattices, Varieties: Volume I*. Wadsworth and Brooks/Cole, 1987.

[13] P. Ouwehand. Algebraically closed algebras in certain small congruence distributive varieties. *Algebra Universalis*, 61:247–260, 2009.

[14] P. Ouwehand and H. Rose. Small congruence distributive varieties: retracts, injectives, equational compactness and amalgamation. *Periodica Math. Hung.*, 33:207–228, 1996.

[15] W. Taylor. Residually small varieties. *Algebra Universalis*, 2:33–53, 1972.

[16] W. Taylor. Equational logic. *Houston J. Math., Survey*, 1979.