Intuitionistic fixed point theories over Heyting arithmetic *

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Abstract
In this paper we show that an intuitionistic theory for fixed points is conservative over the Heyting arithmetic with respect to a certain class of formulas. This extends partly the result of mine. The proof is inspired by the quick cut-elimination due to G. Mints

1 Introduction
Fixed points occur frequently in mathematical reasonings. Let us consider in this paper the fixed point predicate $I^{\Phi}(x)$ for positive formula $\Phi(X, x)$:

$$(FP)^\Phi \; \forall x[I^\Phi(x) \leftrightarrow \Phi(I^\Phi, x)] \tag{1}$$

Over the classical logic, the existence of fixed points strengthens theories. In [2] we have shown that a first-order logic calculus with the axioms [10] has non-elementary speed-ups over the classical first-order predicate logic.

As an extension of the first-order arithmetic PA, the theory $\overline{ID}$ for fixed points is stronger than PA. In $\overline{ID}$ one can readily define the truth definition of arithmetic formulas. Moreover $\overline{ID}$ proves the transfinite induction up to each ordinal less than $\varepsilon_0$ for arithmetic formulas, [4] and [1].

However intuitionistic theories for fixed points may be proof-theoretically equivalent to the intuitionistic arithmetic HA.

W. Buchholz[3] showed that an intuitionistic fixed point theory $\overline{ID}(\mathcal{M})$ is conservative over the Heyting arithmetic HA with respect to almost negative formulas(, in which $\lor$ does not occur and $\exists$ occurs in front of atomic formulas only). The theory $\overline{ID}(\mathcal{M})$ has the axioms [10] $(FP)^\Phi$ for fixed points for monotone formula $\Phi(X, x)$, which is generated from arithmetic atomic formulas and $X(t)$ by means of (first order) monotonic connectives $\lor, \land, \exists, \forall$. Namely

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nor $\neg$ does occur in monotone formula. The proof is based on a recursive realizability interpretation.

After seeing the result of Buchholz, we showed that an intuitionistic fixed point (second order) theory is conservative over HA for any arithmetic formulas. In the theory the operator $\Phi$ for fixed points is generated from $X(t)$ and any second order formulas by means of first order monotonic connectives and second order existential quantifiers $\exists f(\in \omega \to \omega)$. The proof in [1] is to interpret the fixed points by $\Sigma^1_1$-formulas as in [4]. In interpreting the fixed points by $\Sigma^1_1$-formulas, we need an axiom of choice $\text{AC}_{01}$. Therefore the proof does not work for strictly positive operators, e.g., $\Phi(X, x) :\equiv \neg \exists y \forall z A(x) \to X(x)$ since $\neg \exists y \forall z A(x) \to \exists f R(f, x) \leftrightarrow \exists f [\neg \exists y \forall z A(x) \to R(x)]$ is nothing but the independence of premiss, IP, which is not valid intuitionistically.

The crux is the fact that the axiom of choice adds nothing to HA, i.e., N. Goodman’s theorem [5], while the theorem is proved by a combination of a realizability interpretation and a forcing. Also cf. [6] for a proof-theoretic proof of the Goodman’s theorem.

I met Grisha first time in Hiroshima, Japan, September, 1995. I explained to him the result in [1]. He soon realized that it be followed from the Goodman’s theorem before I told the proof. Then he asked me "Can you prove it by means of proof-transformations, e.g., cut-elimination?". This paper is a partial answer to his question.

Now let $\widehat{\text{ID}}^i(\mathcal{H}M)$ denote an intuitionistic fixed point theory in which the operator $\Phi(X, x)$ is in a class $\mathcal{H}M$ of formulas, cf. Definition 2.1 below. The class $\mathcal{H}M$ contains properly the monotone formulas and typically is of the form $H(x) \to M(X, x)$ for a (Rasiowa-)Harrop formula $H$, in which there is no strictly positive occurrence of disjunction nor existential subformulas) and a monotonic formula $M$.

We show that the theory $\widehat{\text{ID}}^i(\mathcal{H}M)$ is conservative over HA with respect to the class $\mathcal{H}M$. Thus the result of the paper extends partly one in [1].

On the other side, C. Rüede and T. Strahm [8] extends significantly the results in [3] and [1]. They showed that the intuitionistic fixed point theory for strictly positive operators is conservative over HA with respect to negative and $\Pi^0_2$-formulas. Moreover they determined the proof-theoretic strengths of intuitionistic theories for transfinitely iterations of fixed points by strictly positive operators. The class of strictly positive formulas is wider than our class $\mathcal{H}M$. In this respect the result in [8] supersedes ours. A merit here is that the class $\mathcal{H}M$ is wider than the class concerned in [8]. For example any formula in prenex normal form is (equivalent to a formula) in $\mathcal{H}M$, but a $\Pi^0_3$-formula is neither negative nor $\Pi^0_2$.

Rather, I think that the novelty lies in our proof technique, which shows that, cf. Theorem 3.1 eliminating cut inferences with $\mathcal{H}M$-cut formulas from derivations of $\mathcal{H}M$-end formulas blows up depths of derivations only by one exponential, e.g., towers of exponentials are dispensable. This is seen from the fact that there exists an embedding from the resulting tree of cut-free derivation to such a derivation with cut inferences such that the embedding maps the
deeper nodes in the tree ordering to larger nodes with respect to Kleene-Brouwer ordering. In other words eliminating monotone cut formulas is essentially to linearize the well founded tree as in Kleene-Brouwer ordering. This is an essence of quick cut-elimination in [7].

Let us explain an idea of our proof more closely. First the finitary derivations in $\hat{D}^i_\infty(HM)$ is embedded to infinitary derivations, and eliminate cuts partially. This results in an infinitary derivation of depth less than $\varepsilon_0$, and in which there occurs cut inferences with cut formulas $I^\Phi(t)$ for fixed points only. Now the constrains on operator $\Phi$ and the end formula admits us to invert cut-free derivations of sequents with a Harrop antecedent and a monotonic succedent formula. Therefore the quick cut-elimination (and pruning) technique in Grisha’s[7] could work to eliminate cut inferences with cut formulas $I^\Phi(t)$. In this way we will get an infinitary derivation of depth less than $\varepsilon_0$, and in which there occurs no fixed point formulas.

By formalizing the arguments we see that the end formula is true in HA.

2 An intuitionistic theory $\hat{D}^i_\infty(HM)$

$L_{HA}$ denote the language of the Heyting arithmetic. $L_{HA}$ consists of the equality sign $=$, individual constants 0,1 for zero and one, function symbols $+,-,$ for addition and multiplication, and logical connectives $\lor,\land,\rightarrow,\exists,\forall$.

Let $X$ be a fresh predicate symbol, which is assumed to be unary for simplicity. $L_{HA}(X)$ denotes the language $L_{HA} \cup \{X\}$.

Definition 2.1 Define inductively two classes of formulas $\mathcal{H}$ in $L_{HA}$, and $\mathcal{HM}$ in $L_{HA}(X)$ as follows.

1. Any atomic formula $s = t$ belongs to both of $\mathcal{H}$ and $\mathcal{HM}$.
2. Any atomic formula $X(t)$ belongs to the class $\mathcal{HM}$.
3. If $H,G \in \mathcal{H}$, then $H \land G,\forall x H \in \mathcal{H}$.
4. If $H \in \mathcal{H}$, then $A \rightarrow H \in \mathcal{H}$ for any formula $A \in L_{HA}$.
5. If $R,S \in \mathcal{HM}$, then $R \lor S,R \land S,\exists x R,\forall x R \in \mathcal{HM}$.
6. If $L \in \mathcal{H}$ and $R \in \mathcal{HM}$, then $L \rightarrow R \in \mathcal{HM}$.

$\mathcal{H}$ denotes the class of (Rasiowa-)Harrop formulas, in which there occurs no strictly positive existential nor disjunctive subformula.

$\mathcal{HM}$ contains properly the monotone formulas, i.e., the class POS in [3], e.g., $\neg A \rightarrow X(a) \in \mathcal{HM}$ is not intuitionistically equivalent to any monotone formula, but there exists a strongly positive formula with respect to $X$ not in $\mathcal{HM}$, e.g., $(\forall x \exists y A \rightarrow \exists z B) \land X(a)$. Any formula in $\mathcal{HM}$ is strictly positive with respect to $X$.

Let $\hat{D}^i_\infty(HM)$ denote the following extension of HA. Its language is obtained from $L_{HA}$ by adding a unary set constant $I^\Phi$ for each $\Phi \equiv \Phi(X,x) \in \mathcal{HM}$, in
which only a fixed variable \( x \) occurs freely. Its axioms are those of HA in the expanded language, i.e., the induction axioms are available for any formulas in the expanded language plus the axiom \((FP)^\Phi\), \(\Pi\) for fixed points.

Now our theorem runs as follows.

**Theorem 2.1** \(\hat{ID}^-_i(\mathcal{HM})\) is conservative over HA with respect to formulas in \(\mathcal{HM}\) (in which the extra predicate constant \(X\) does not occur).

### 3 Infinitary derivations

Given an \(\hat{ID}^-_i(\mathcal{HM})\)-derivation \(D_0\) of an \(\mathcal{HM}\)-sentence \(R_0\), let us first embed it to an infinitary derivation in an infinitary calculus \(\hat{ID}^-_i(\mathcal{HM})\).

\(\neg A \iff A \rightarrow \bot\).

Let \(N\) denote a number which is big enough so that any formula occurring in \(D_0\) has logical complexity (which is defined by the number of occurrences of logical connectives) smaller than \(N\). In what follows any formula occurring in infinitary derivations which we are concerned, has logical complexity less than \(N\).

The derived objects in the calculus \(\hat{ID}^-_i(\mathcal{HM})\) are sequents \(\Gamma \Rightarrow A\), where \(A\) is a sentence (in the language of \(\hat{ID}^-_i(\mathcal{HM})\)) and \(\Gamma\) denotes a finite set of sentences, where each closed term \(t\) is identified with its value \(\bar{n}\), the \(n\)th numeral.

\(\bot\) stands ambiguously for false equations \(t = s\) with closed terms \(t, s\) having different values. \(\top\) stands ambiguously for true equations \(t = s\) with closed terms \(t, s\) having same values.

The initial sequents are

\[
\Gamma, I(t) \Rightarrow I(t); \quad \Gamma, \bot \Rightarrow A; \quad \Gamma \Rightarrow \top
\]

These are regarded as inference rules with empty premiss (upper sequent).

The inference rules are \((L\lor), (R\lor), (L\land), (R\land), (L\rightarrow), (R\rightarrow), (L\exists), (R\exists), (L\forall), (R\forall), (LI), (RI), (cut),\) and the repetition rule (Rep). These are standard ones.

1. \[
\frac{\Gamma, \Phi(I,t) \Rightarrow C}{\Gamma, I(t) \Rightarrow C} (LI) \quad \frac{\Gamma \Rightarrow \Phi(I,t)}{\Gamma \Rightarrow I(t)} (RI)
\]

2. \[
\frac{\Gamma, A_0 \Rightarrow C \quad \Gamma, A_1 \Rightarrow C}{\Gamma, A_0 \lor A_1 \Rightarrow C} (L\lor) \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \lor A_1} (R\lor) (i = 0, 1)
\]

3. \[
\frac{\Gamma, A_0 \land A_1, A_i \Rightarrow C}{\Gamma, A_0 \land A_1 \Rightarrow C} (L\land) (i = 0, 1) \quad \frac{\Gamma \Rightarrow A_0 \quad \Gamma \Rightarrow A_1}{\Gamma \Rightarrow A_0 \land A_1} (R\land)
\]
4. \[ \Gamma, A \rightarrow B \Rightarrow A \quad \Gamma, B \Rightarrow C \] \[ \Gamma, A \rightarrow B \Rightarrow C \quad (L \rightarrow) ; \quad \Gamma \Rightarrow A \rightarrow B \quad (R \rightarrow) \]

5. \[ \cdots \Gamma, B(\bar{n}) \Rightarrow C \quad \cdots (n \in \omega) \] \[ \Gamma, \exists x B(x) \Rightarrow C \quad (L \exists) ; \quad \Gamma \Rightarrow \exists x B(x) \quad (R \exists) \]

6. \[ \Gamma, \forall x B(x), B(\bar{n}) \Rightarrow C \] \[ \Gamma, \forall x B(x) \Rightarrow C \quad (L \forall) ; \quad \Gamma \Rightarrow \forall x B(x) \quad (R \forall) \]

7. \[ \Gamma \Rightarrow A \quad \Delta, A \Rightarrow C \] \[ (\text{cut}) \]

8. \[ \Gamma \Rightarrow C \] \[ (\text{Rep}) \]

The depth of an infinitary derivation is defined to be the depth of the well founded tree.

As usual we see the following proposition. Recall that \( N \) is an upper bound of logical complexities of formulas occurring in the given finite derivation \( D_0 \) of \( \mathcal{H}M \)-sentence \( R_0 \).

**Proposition 3.1** 1. There exists an infinitary derivation \( D_1 \) of \( R_0 \) such that its depth is less than \( \omega^2 \) and the logical complexity of any sentence, in particular cut formulas occurring in \( D_1 \) is less than \( N \).

2. By a partial cut-elimination, there exist an infinitary derivation \( D_2 \) of \( R_0 \) and an ordinal \( \alpha_0 < \varepsilon_0 \) such that the depth of the derivation \( D_2 \) is less than \( \alpha_0 \) and any cut formula occurring in \( D_2 \) is an atomic formula \( I(t) \), and the logical complexity of any formula occurring in it is less than \( N \).

Let \( \mathcal{H}M(I) \) denote the class of sentences obtained from sentences in \( \mathcal{H}M \) by substituting the predicate \( I^k \) for the predicate \( X \).

**Definition 3.1** The rank \( rk(A) \) of a sentence \( A \) is defined by

\[
rk(A) := \begin{cases} 
0 & \text{if } A \in \mathcal{H} \\
1 & \text{if } A \in \mathcal{H}M(I) \setminus \mathcal{H} \\
2 & \text{otherwise}
\end{cases}
\]

For inference rules \( J \), the rank \( rk(J) \) of \( J \) is defined to be the rank of the cut formula if \( J \) is a cut inference. Otherwise \( rk(J) := 0 \).

For derivations \( D \), the rank \( rk(D) \) of \( D \) is defined to be the maximum rank of the cut inferences in it.
Let $\vdash_\alpha \Gamma \Rightarrow C$ mean that there exists an infinitary derivation of $\Gamma \Rightarrow C$ such that its depth is at most $\alpha$, and its rank is less than $r$, and the logical complexity of any formula occurring in it is less than $N$.

**Theorem 3.1** Let $C_0$ denote an $\mathcal{HM}$-sentence, and $\Gamma_0$ a finite set of $\mathcal{H}$-sentences. Suppose that $\vdash_2 \Gamma_0 \Rightarrow C_0$. Then $\vdash_1 \omega_\alpha \Gamma_0 \Rightarrow C_0$.

Assuming the Theorem 3.1, we can show the Theorem 2.1 as follows. Suppose an $\mathcal{HM}$-sentence $C_0$ is provable in $\hat{\mathcal{ID}}(\mathcal{HM})$. By Proposition 3.1 we have $\vdash_2 \alpha_0 \Gamma_0 \Rightarrow C_0$ for a big enough number $N$ and an $\alpha_0 < \varepsilon_0$. Then Theorem 3.1 yields $\vdash_1 \omega_\alpha \Gamma_0 \Rightarrow C_0$ for $\beta_0 = \omega_\alpha + 1 < \varepsilon_0$.

Let $\text{Tr}_N(x)$ denote a partial truth definition for formulas of logical complexity less than $N$, cf. [9], 1.5.4. By transfinite induction up to $\beta_0$, cf. Lemma 4.1, we see $\text{Tr}_N(C_0)$. Note that any sentence occurring in the witnessed derivation for $\vdash_1 \beta_0 \Rightarrow C_0$ has logical complexity less than $N$, and it is either an $\mathcal{H}$-sentence or an $\mathcal{HM}$-sentence. Specifically there occurs no fixed point formula $I(t)$ in it.

Now since everything up to this point is formalizable in HA, we have $\text{Tr}_N(C_0)$, and hence $C_0$ in HA. This shows the Theorem 2.1.

A proof of Theorem 3.1 is given in the next section.

### 4 Quick cut-elimination for monotone cuts with Harrop side formulas

Our plan of the proof of Theorem 3.1 is as follows. Pick the leftmost cut $J$ of rank 1:

$$
\Gamma \Rightarrow A \quad \Delta, A \Rightarrow C \quad (\text{cut}) J
$$

with $rk(A) = 1$.

Then $\vdash_1 \Gamma \Rightarrow A$ and $\vdash_2 \beta \Delta, A \Rightarrow C$ for some $\alpha$ and $\beta$. Moreover since for the end sequent $\Gamma_0 \Rightarrow C_0$, $\Gamma_0 \subseteq \mathcal{H}$ and $C_0 \in \mathcal{HM}$, we see that any sentence in $\Gamma \cup \Delta$ is in $\mathcal{H}$, and $C \in \mathcal{HM}$ by $\vdash_2 \alpha_0 \Gamma_0 \Rightarrow C_0$.

Since $\Gamma$ consists solely in Harrop formulas, we can invert the derivation $D_\ell$, and climb up the derivation $D_r$ with inverted $D_\ell$. This results in a derivation of $\Gamma, \Delta \Rightarrow C$ of depth $dp(D_\ell) + dp(D_r)$. Iterating this eliminations, we could get a derivation of rank 0, and of depth at most exponential of the depth of the given derivation.

Though intuitively this would suffice to believe in Theorem 3.1 we have to prove two facts: first why the iteration eventually terminates? second give a succinct argument to the estimated increase of depth. These are not entirely trivial tasks. It turns out that we need to proceed along the Kleene-Brouwer ordering on well founded trees instead of the depths. Let us explore this.

Let us fix a witnessed derivation $D_2$ of $\vdash_2 \alpha_0 \Gamma_0 \Rightarrow C_0$. Let $(T_2, <_{T_2})$ denote the wellordering, where $T_2 \subseteq <\omega \omega$ is the naked tree of $D_2$ and $<_{T_2}$ the Kleene-Brouwer ordering on $T_2$. 
Let us consider infinitary derivations equipped with additional informations as in [6].

**Definition 4.1** An infinitary derivation is a sextuple \( D = (T, \text{Seq}, \text{Rule}, rk, ord, kb) \) which enjoys the following conditions. The naked tree of \( D \) is denoted \( T = T(D) \).

1. \( T \subseteq ^{<\omega}\omega \) is a tree in the sense that there exists a root \( r \in T \) with \( \forall a \in T (r \subseteq a) \) and \( \forall a, b (r \subseteq a \subseteq b \Rightarrow a \in T) \).

   It is *not assumed* that the empty node \( \emptyset \) is to be the root nor \( a \star \langle n \rangle \in T \& m < n \Rightarrow a \star \langle m \rangle \in T \).

2. \( \text{Seq}(a) \) for \( a \in T \) denotes the sequent situated at the node \( a \).

   If \( \text{Seq}(a) \) is a sequent \( \Gamma \Rightarrow C \), then it is denoted \( a : \Gamma \Rightarrow C \).

3. \( \text{Rule}(a) \) for \( a \in T \) denotes the name of the inference rule with its lower sequent \( \text{Seq}(a) \).

4. \( rk(a) \) for \( a \in T \) denotes the rank of the inference rule \( rk(a) \).

5. \( \text{ord}(a) \) for \( a \in T \) denotes the ordinal \( < \varepsilon_0 \) attached to \( a \).

6. The quintuple \( (T, \text{Seq}, \text{Rule}, rk, \text{ord}) \) has to be locally correct with respect to \( \widehat{ID}_{\infty}(HM) \) and for being well founded tree \( T \).

Besides these conditions an extra information is provided by a *labeling function* \( kb \).

\[ kb : T \to T_2 \] is a function such that for \( a, b \in T \)

1. \[ kb(a) \neq kb(b) \Rightarrow [a \prec_T b \iff kb(a) \prec_{T_2} kb(b)] \] (2)

   for the Kleene-Brouwer ordering \( \prec_T \) on \( T \).

2. \[ a \subset b \Rightarrow kb(b) <_{T_2} kb(a) \] (3)

   where \( a \subset b \) means that \( a \) is a proper initial segment of \( b \) for \( a, b \in ^{<\omega}\omega \).

3. Let \( c \in T \) be a node with \( rk(c) = 1 \), and \( a, b \in T \) nodes such that \( c \star \langle \ell \rangle \subseteq a \) and \( c \star \langle r \rangle \subseteq b \) for \( \ell < r \). (This means that \( \text{Seq}(a) [\text{Seq}(b)] \) is in the left [right] upper part of the cut inference \( \text{Rule}(c) \).) Suppose that the right cut formula \( A \) in the antecedent of \( \text{Seq}(c \star \langle r \rangle) \) has an *ancestor* in \( \text{Seq}(b) \).

Then

\[ kb(a) \neq kb(b) \] (4)
The condition (3) to \( k_b \) ensures us that the depth of \( T \) is at most the order type of \( \prec_{T_2} \).

It is easy to see that Kleene-Brouwer ordering \( \prec_{T_2} \) is a well ordering, and its order type is bounded by \( \omega^\alpha + 1 \) for the depth \( \alpha \) of the primitive recursive and wellfounded tree \( T_2 \).

**Lemma 4.1** The transfinite induction schema (for arithmetical formulas) along the Kleene-Brouwer ordering \( \prec_{T_2} \) is provable in HA.

**Proof.** Since the transfinite induction schema along a standard \( \varepsilon_0 \)-ordering is provable in HA up to each \( \alpha < \varepsilon_0 \), the same holds for the tree ordering \( \{ (b, a) : a \subset b, a, b \in T_2 \} \) (bar induction).

Now let \( X \) be a formula, and assume that \( X \) is progressive with respect to \( \prec_{T_2} \):

\[
\forall a \in T_2[\forall b \prec_{T_2} a X(b) \rightarrow X(a)].
\]

Let

\[
j[X](a) : \iff \forall y \in T_2[y \supseteq a \rightarrow \forall x \prec_{T_2} y X(x) \rightarrow \forall x \prec_{T_2} a X(x)].
\]

Then we see that \( j[X] \) is progressive with respect to the tree ordering. Therefore \( j[X](r) \) for the root \( r \in T_2 \), and by letting \( y \) to be the leftmost leaf in \( T_2 \) we have \( \forall x \prec_{T_2} r X(x) \). The progressiveness of \( X \) with respect to \( \prec_{T_2} \) yields \( X(r) \).

\[ \square \]

The following Lemmas are seen as usual.

**Lemma 4.2** Let \( D = (T, \text{Seq}, \text{Rule}, \text{rk}, \text{ord}, k_b) \) be a derivation of rank 1, and of a sequent \( \Gamma \Rightarrow A \) such that \( \Gamma \subseteq H \) and \( A \in HM \& \text{rk}(A) = 1 \). For any \( \Delta \), there exists a derivation \( D \Delta = (T, \text{Seq} \Delta, \text{Rule}, \text{rk}, \text{ord}, k_b) \) of the sequent \( \Gamma, \Delta \Rightarrow A \).

**Lemma 4.3** (Inversion Lemma)

Let \( D = (T, \text{Seq}, \text{Rule}, \text{rk}, \text{ord}, k_b) \) be a derivation of rank 1, and of a sequent \( \Gamma \Rightarrow A \) such that \( \Gamma \subseteq H \) and \( A \in HM \& \text{rk}(A) = 1 \).

1. If \( A \equiv B_0 \lor B_1 \), then there exists a derivation \( D_i = (T, \text{Seq}_i, \text{Rule}_i, \text{rk}, \text{ord}, k_b) \) of rank 1 and of a sequent \( \Gamma \Rightarrow B_i \) for an \( i = 0, 1 \).
2. If \( A \equiv B_0 \land B_1 \), then there exist derivations \( D_i = (T_i, \text{Seq}_i, \text{Rule}_i, \text{rk}, \text{ord}, k_b) \) of rank 1 and of sequents \( \Gamma \Rightarrow B_i \) for any \( i = 0, 1 \), where \( T_i \subseteq T \) by pruning.
3. If \( A \equiv \exists x B(x) \), then there exists a derivation \( D_n = (T, \text{Seq}_n, \text{Rule}_n, \text{rk}, \text{ord}, k_b) \) of rank 1 and of a sequent \( \Gamma \Rightarrow B(\bar{n}) \) for an \( n \in \omega \).
4. If \( A \equiv \forall x B(x) \), then there exist derivations \( D_n = (T_n, \text{Seq}_n, \text{Rule}_n, \text{rk}, \text{ord}, k_b) \) of rank 1 and of a sequent \( \Gamma \Rightarrow B(\bar{n}) \) for any \( n \in \omega \), where \( T_n \subseteq T \) by pruning.
5. If \( A \equiv B_0 \rightarrow B_1 \), then there exist a derivation \( D' = (T, \text{Seq}', \text{Rule}', \text{rk}, \text{ord}, k_b) \) of rank 1 and of sequents \( \Gamma, B_0 \Rightarrow B_1 \).

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6. If \( A \equiv I(t) \), then there exists a derivation \( D' = (T, Seq', Rule', rk, ord, kb) \) of rank 1 and of sequents \( \Gamma \Rightarrow \Phi(I, t) \).

**Definition 4.2** For each cut inference \( J \) in a derivation \( D \), \( KB(J) \) denotes \( kb(a) \in T_2 \) with the left upper node \( a \) of \( J \):

\[
\begin{array}{c}
a : \Gamma \Rightarrow A \\
\Gamma, \Delta \Rightarrow C \\
\end{array}
\]

Let us define a cut-eliminating operator \( ce_1(D) \) for derivations \( D = (T, Seq, Rule, rk, ord, kb) \) of rank 1 and of an end sequent \( \Gamma_0 \Rightarrow C_0 \) with \( \Gamma_0 \subseteq H \) and \( C_0 \in HM \).

If \( D \) is of rank 0, then \( ce_1(D) := D \).

Assume that \( D \) contains a cut inference of rank 1. Pick the leftmost cut of rank 1:

\[
D = \begin{array}{c}
\begin{array}{c}
\vdash D_\ell \\
\vdash D_r \\
\vdash \Gamma, \Delta, A \Rightarrow C \\
\vdash \Gamma, \Delta \Rightarrow C \\
\end{array}
\end{array}
\]

The leftmostness means that \( KB(J) \) is least in the Kleene-Brouwer ordering \( <_{T_2} \).

By recursion on the depth \( 1 \) of the derivation \( D_r \) we define a derivation \( ce_2(D_\ell, D_r) \) of \( \Gamma, \Delta \Rightarrow C \). Then \( ce_1(D) \) is obtained from \( D \) by pruning \( D_\ell \) and replacing \( D_r \) by \( ce_2(D_\ell, D_r) \), i.e., by grafting \( ce_2(D_\ell, D_r) \) onto the trunk of \( D \) up to \( \Gamma, \Delta \Rightarrow C \).

As in Lemma 3.2, \[7\] the construction of \( ce_2(D_\ell, D_r) \) is fairly standard, leaving the resulted cut inferences of rank 0, but has to performed parallely.

Let \( \Gamma \cup \Delta \subseteq H \) and \( A = A_1, \ldots, A_k \) be a finite sequence of \( HM \)-sentences. Let \( D_\ell = D_{\ell,1}, \ldots, D_{\ell,k} \) be rank 0 derivations of \( \Gamma \Rightarrow A_i \), and \( D_r \) a rank 1 derivation of \( \Delta, A \Rightarrow C \). We will eliminate the cuts with the cut formulas \( A_i \) in parallel. \( ce_2(D_\ell, D_r) \) is defined from the resulting derivation, denoted \( E \) by recursion.

\[
D_a = \begin{array}{c}
\begin{array}{c}
\vdash D_\ell \\
\vdash D_r \\
b : \Gamma \Rightarrow A \\
\end{array}
\end{array}
\]

\[
A : \Gamma, \Delta \Rightarrow C 
\]

\[
\begin{array}{c}
\vdash D_{\ell,1} \\
\vdash D_r \\
b_1 : \Gamma \Rightarrow A_1 \\
\end{array}
\]  

\[
\begin{array}{c}
\vdash D_{\ell,k} \\
\vdash \Gamma, \Delta, A_2, \ldots, A_k \Rightarrow C \\
b_k : \Gamma \Rightarrow A_k \\
\end{array}
\]

\[
A : \Gamma, \Delta \Rightarrow C 
\]

\[
\begin{array}{c}
\vdash D_{\ell,1} \\
\vdash D_r \\
b_1 : \Gamma \Rightarrow A_1 \\
\end{array}
\]  

\[
\begin{array}{c}
\vdash D_{\ell,k} \\
\vdash \Gamma, \Delta, A_2, \ldots, A_k \Rightarrow C \\
b_k : \Gamma \Rightarrow A_k \\
\end{array}
\]

\[
A : \Gamma, \Delta \Rightarrow C 
\]

\[1\] As in \[6\] we see that the operators \( ce_1, ce_2 \) are primitive recursive. We don’t need this fact.
1. If $\Delta, A \Rightarrow C$ is an initial sequent such that one of the cases $C \equiv \top, \bot \in \Delta$ or $C \in \Delta$ occurs, then $\Delta \Rightarrow C$, and hence $\Gamma, \Delta \Rightarrow C$ is still the same kind of initial sequent. For example

\[
\begin{align*}
c_1 : \Gamma, \Delta \Rightarrow & \top \\
c_2 : \Gamma, \Rightarrow & \top \quad \text{(Rep)} \\
\vdots & \\
c_k : \Gamma, \Delta \Rightarrow & \top \\
a : \Gamma, \Delta \Rightarrow & \top \quad \text{(Rep)}
\end{align*}
\]

The $T(E)$ is defined by

\[d \in T(E) \iff d \in T(D) & \forall i(b_i \not\subseteq d).\]

2. If $\Delta, A \Rightarrow C$ is an initial sequent with the principal formula $A \ni A_i \equiv C \equiv I(t)$, then $E$ is defined to be

\[
\begin{align*}
\vdots & \\
D_{t,i} \ast & \Delta \\
b_i : \Gamma, \Delta \Rightarrow & C \quad \text{(Rep)} \\
c_{i+1} : \Gamma, \Delta \Rightarrow & C \\
\vdots & \\
c_k : \Gamma, \Delta \Rightarrow & C \\
a : \Gamma, \Delta \Rightarrow & C \quad \text{(Rep)}
\end{align*}
\]

where $D_{t,i} \ast \Delta$ is obtained from $D_{t,i}$ by weakening, cf. Lemma 4.2

\[d \in T(E) \iff d \in T(D) & \forall j \neq i(b_j \not\subseteq d) & c_i \not\subseteq d.\]

3. If $A_i \in A$ is of rank 0, then do nothing for the cut inference of $A_i$.

In each of the above cases $T(E) \subseteq T(D_a)$. The labeling function $kb_E$ for $E$ is defined to be the restriction of $kb_{D_a}$ to $T(E)$.

In what follows assume that $\Delta, A \Rightarrow C$ is a lower sequent of an inference rule $J$.

4. If the principal formula of $J$ is not in $A$, then lift up $D_{t}$:

\[
\begin{align*}
b : \Gamma \Rightarrow & A \\
\cdots & \cdots \\
c_{1,i} : \Delta_i, A \Rightarrow & C_i \\
c_{1} : \Gamma, \Delta \Rightarrow & C \\
\cdots & \cdots \\
a : \Gamma, \Delta \Rightarrow & C \quad \text{(J)}
\end{align*}
\]

where $b = b_k, \ldots, b_1$ with $b_j = c_{j+1} \ast \langle \ell_j \rangle \{c_{k+1} := a\}$, and $c_j = c_{j+1} \ast \langle r_j \rangle$ for some $\ell_j < r_j$, and $c_{1,i} = c_i \ast \langle n_i \rangle$ for some $n_i$ with $i < j \Rightarrow n_i < n_j$.

$E$ is defined as follows.

\[
\begin{align*}
b_i : \Gamma \Rightarrow & A \\
\cdots & \\
c_{1,i} : \Delta_i, A \Rightarrow & C_i \\
a_1 : \Gamma, \Delta_i \Rightarrow & C_i \\
\cdots & \\
a : \Gamma, \Delta \Rightarrow & C \quad \text{(J)}
\end{align*}
\]
where \( a_i = a \cdot \langle n_i \rangle \), \( b_i = b_{k,i}, \ldots, b_{1,i} \) with \( b_{j,i} = c'_{j+1,i} \cdot \langle \ell_j \rangle \) (\( c'_{j+1,i} = a_i \)), and \( c'_j = c'_{j+1,i} \cdot (r_j) \).

The labeling function \( kb_E \) is defined by

\[
kb_E(a \cdot \langle n_i \rangle) = kb_D(a \cdot \langle r_k \rangle) = kb_D(c_k),
\]
\[
kb_E(a \cdot \langle \langle r_k, \ldots, r_{j+1}, \ell_j \rangle \rangle \cdot \langle \rangle) = kb_D(a \cdot \langle \langle r_k, \ldots, r_{j+1}, \ell_j \rangle \rangle \cdot \langle \rangle) (1 \leq j \leq k),
\]
\[
kb_E(a \cdot \langle \langle r_k, \ldots, r_1 \rangle \rangle \cdot \langle \rangle) = kb_D(a \cdot \langle \langle r_k, \ldots, r_1 \rangle \rangle \cdot \langle \rangle)
\]

5. Finally suppose that the principal formula of \( J \) is a cut formula \( A_i \in A \) of \( rk(A_i) = 1 \). Use the Inversion Lemma 4.3.

(a) The case when \( A_i \equiv \exists x B(x) \in A \). For simplicity suppose \( i = 1 \).

\[
\begin{array}{c}
\vdots \\
c_{1,n} : \Delta, A_1, B(\bar{n}) \Rightarrow C \\
c_1 : \Delta, A \Rightarrow C
\end{array}
\]

(L\( \exists \))

where \( A_1 \notin A \).

By Inversion Lemma 4.33 pick an \( n \) such that \( \Gamma \Rightarrow B(\bar{n}) \) is provable without changing the naked tree.

\( E \) is defined as follows.

\[
\begin{array}{c}
\vdots \\
c_{1,n} : \Delta, A_1, B(\bar{n}) \Rightarrow C \\
b_1 : \Gamma \Rightarrow B(\bar{n}) \\
c_1 : \Delta, A_1, B(\bar{n}) \Rightarrow C
\end{array}
\]

Rep

\[
\begin{array}{c}
b_1 : \Gamma \Rightarrow A_1 \\
c_2 : \Gamma, \Delta, A_1 \Rightarrow C
\end{array}
\]

where \( A_1 \notin A \).

(b) The case when \( A_i \equiv H \Rightarrow A_0 \in A \) with an \( H \in H \) and an \( A_0 \in H \). For simplicity suppose \( i = 1 \).

\[
\begin{array}{c}
c_{1,\ell} : \Delta, A \Rightarrow H \\
c_{1,r} : \Delta, A_1, A_0 \Rightarrow C
\end{array}
\]

(L\( \rightarrow \))

where \( A_1 \notin A \), and for \( m = \ell, r \), \( c_{1,m} = c_1 \cdot (j_m) \) with \( j_\ell < j_r \).

\( E \) is defined as follows.

\[
\begin{array}{c}
b_0 : \Gamma \Rightarrow A \\
c_{1,0,l} : \Delta, A \Rightarrow H \\
\end{array}
\]

(Rep)

\[
\begin{array}{c}
b_{1,1} : \Gamma, H \Rightarrow A_0 \\
c_{1,1} : \Delta, A_1 \cup \{ A_0 \} \Rightarrow C
\end{array}
\]

(Rep)

\[
\begin{array}{c}
b_1 : \Gamma \Rightarrow A_1 \\
c_{2,1} : \Gamma, \Delta, H, A_1 \Rightarrow C
\end{array}
\]

\[
\begin{array}{c}
c_{k,0} : \Gamma, \Delta \Rightarrow H \\
c_{k,1} : \Gamma, \Delta, H \Rightarrow C
\end{array}
\]

where \( \Gamma, H \Rightarrow A_0 \) by inversion.

For \( m = 0, 1 \), \( c_{j,m} = c_{j+1,m} \cdot (2r_j + m) \) for \( 1 \leq j \leq k \) with \( c_{k+1,m} = a \), and \( b_m = b_{k,m}, \ldots, b_{1,m} \) with \( b_{j,m} = a \cdot (2r_k + m, \ldots, 2r_j + m, 2\ell_j + m) \) and \( c_{1,0} = c_{1,0} \cdot (\ell) \), \( c_{1,1,r} = c_{1,1} \cdot (r) \).
The labeling function is defined by

\[ kb_E(c_{j,m}) = kb_{D_n}(c_j) \quad (5) \]
\[ kb_E(b_{j,m} \ast d) = kb_{D_n}(b_j \ast d), \quad (m = 0, 1) \]

(c) The case when \( A_i \equiv \forall x B(x) \in A \). For simplicity suppose \( i = 1 \).

\[
\begin{align*}
& \vdash D_{r,n} \\
& c_{1,n} : \Delta, A, B(\tilde{n}) \Rightarrow C \\
& c_1 : \Delta, A \Rightarrow C \quad (L \forall)
\end{align*}
\]

with \( c_{1,n} = c_1 \ast \langle j_n \rangle \).

\( E \) is defined as follows.

\[
\begin{align*}
& b : \Gamma \Rightarrow A \\
& b_{1,1} : \Gamma \Rightarrow B(\tilde{n}) \\
& c_{1,n} : \Delta, A, B(\tilde{n}) \Rightarrow C \\
& c_1 : \Gamma, \Delta, A \Rightarrow C \\
& a : \Gamma, \Delta \Rightarrow C
\end{align*}
\]

where \( b_{1,1} = c_1 \ast \langle 2j_n \rangle \) and \( c_{1,n} = c_1 \ast \langle 2j_n + 1 \rangle \).

The labeling function is defined by

\[ kb_E(b_{1,1} \ast d) = kb_{D_n}(b_1 \ast d) \quad (6) \]

and

\[ kb_E(c_{1,n}') = kb_{D_n}(c_{1,n}). \]

(d) The case when \( A_i \equiv B_0 \lor B_1 \in A \). For simplicity suppose \( i = 1 \).

\[
\begin{align*}
& \vdash D_{r,0} \\
& c_{1,0} : \Delta, A_1, B_0 \Rightarrow C \\
& c_1 : \Delta, A \Rightarrow C \\
& \vdash D_{r,1} \\
& c_{1,1} : \Delta, A_1, B_1 \Rightarrow C
\end{align*}
\]

where \( A_1 \not\subseteq A_1 \).

By Inversion Lemma \( \text{II} \) pick an \( n = 0, 1 \) such that \( \Gamma \Rightarrow B_n \) is provable without changing the naked tree.

Suppose that \( n = 0 \). \( E \) is defined as follows.

\[
\begin{align*}
& b_1 : \Gamma \Rightarrow A_1 \\
& b_1 : \Gamma \Rightarrow B_0 \\
& c_{1,0} : \Delta, A_1, B_0 \Rightarrow C \\
& c_1 : \Delta, A_1, B_0 \Rightarrow C \\
& \vdash D_{r,0} \\
& c_2 : \Gamma, \Delta, A_1 \Rightarrow C
\end{align*}
\]

(\( \text{Rep} \))

Note that the new cut inference for \( B_0 \) may be of rank 0.
The case when \( A_i \equiv B_0 \land B_1 \in A \) is treated as in the case (5c) for universal quantifier.

**Claim 4.1** The resulting derivation \( ce_1(D) \) can be labeled enjoying the conditions (2), (3) and (4).

**Proof.** Let \( D_a \) be the trunk ending with the leftmost cut of rank 1 in \( D \). First observe that the labels \( \{ kb_E(b) : b \in T(E) \} \subseteq \{ kb_{D_a}(b) : b \in T(D_a) \} \). Therefore it suffices to see that \( E \), and hence \( ce_2(D_t, D_r) \) enjoys the three conditions if \( D_a \) does.

Note that (the naked tree of) \( E \) is constructed from \( D_r \) by appending trees \( D_t \) only where a right cut formula \( A_i \) has an ancestor which is either a formula of rank 0 or a principal formula of an initial sequent \( \Phi, I(t) \Rightarrow I(t) \). In the latter case the ancestor has to be the formula \( I(t) \) in the antecedent.

\( E \) enjoys the first (2) since \( D_a \) does the first (2). \( E \) enjoys the second (3) since \( D_a \) does the third (1). \( E \) enjoys the third (1) since \( D_a \) does the third (1), and the first (2).

Let us examine cases. Consider the case (4) when \( D_t \) is lifted up. We have to show

\[ kb_E(e) <_{T_2} kb_E(c'_{j,i}) \]

for \( e \) such that \( b_{j,i} \subseteq e \). Then \( kb_E(e) = kb_E(a \ast (u_i) \ast (r_k, \ldots, r_{j+1}, \ell_j) \ast d) = kb_{D_a}(a \ast (r_k, \ldots, r_{j+1}, \ell_j) \ast d) \) and \( kb_E(c_{j,i}) = kb_{D_a}(c_j) \).

Since the right cut formula \( A_i \) has an ancestor in \( c_{1,i} : \Delta, A \Rightarrow C_i, kb_E(e) <_{T_2} kb_E(c'_{j,i}) \) follows from (1) for \( D_a \).

Next consider the case (5b) Although \( b_{j,m}, c_{j,m} \) are duplicated for \( m = 0, 1 \), and \( kb_E(c_{j,0}) = kb_{D_a}(c_j) = kb_E(c_{j,1}), kb_E(b_{j,1} \ast d) = kb_{D_a}(b_j \ast d) = kb_E(b_{j,0} \ast d) \) by (5), (6) these are harmless for (1) since the juncture is a cut of rank 0, \( rk(H) = 0 \).

Finally consider the case (5c)

By (1) we have

\[ kb_E(b_{1,1} \ast d) = kb_{D_a}(b_1 \ast d) = kb_E(b_1 \ast d) \]

but the right cut formula \( A_i \) has no ancestor in \( b_{1,1} : \Gamma \Rightarrow B(\bar{u}) \). Thus (1) is enjoyed.

Next for (3) we have

\[ kb_{D_a}(c_j) <_{T_2} kb_{D_a}(b_1 \ast d) = kb_E(b_{1,1} \ast d) \]

by (3) and (1) in \( D_a \).

Finally for (2) assume \( j \neq 1 \) and

\[ kb_{D_a}(b_1 \ast d) = kb_E(b_{1,1} \ast d) \neq kb_E(b_j \ast e) = kb_{D_a}(b_j \ast e) \]

Then by (2) in \( D_a \) we have \( b_j \ast e <_{T(D_a)} b_1 \ast d \), and hence \( kb_{D_a}(b_j \ast e) <_{T_2} kb_{D_a}(b_1 \ast d) \).

This ends the construction of the cut-eliminating operator \( ce_1(D) \).
Finally we show Theorem 3.1. Given a derivation $D_2 = (T_2, \text{Seq, Rule, rk, ord, kb})$ of $\Gamma_0 \Rightarrow C_0$ of rank 1, and assume $\Gamma_0 \subseteq H$ and $C_0 \in H_M$. $(T_2, <_{T_2})$ denotes the Kleene-Brouwer ordering on the naked tree $T_2$.

Let $KB(D) := KB(J)$ for the leftmost cut inference $J$ of rank 1 if such a $J$ exists. Otherwise $KB(D)$ denote the largest element in $T_2$ with respect to $<_{T_2}$, i.e., the root of $T_2$. Then we see that $KB(D) <_{T_2} KB(ce_1(D))$ if $D$ contains a cut inference of rank 1.

Suppose as the induction hypothesis that any cut inferences $J$ of rank 1 has been eliminated for $KB(J) < a$, and let $D$ denote such a derivation. Also assume that $a$ is a node of a cut inference of rank 1. Then in $ce_1(D)$ the cut inference is eliminated. This proves the Theorem 3.1 by induction along the Kleene-Brouwer ordering $<_{T_2}$, cf. Lemma 4.1.

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