Noether conservation laws in infinite order Lagrangian formalism

G. Sardanashvily
Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia
E-mail: sard@grav.phys.msu.su
URL: http://webcenter.ru/~sardan/

Abstract. Conservation laws related to the gauge invariance of Lagrangians and Euler–Lagrange operators in finite and infinite order Lagrangian formalisms are analyzed.

1 Introduction

Let us start from familiar finite order Lagrangian formalism. Let \( Y \to X \) be a smooth fibre bundle over an \( n \)-dimensional base \( X \). An \( r \)-order Lagrangian is defined as a density

\[
L : J^r Y \to \bigwedge^n T^* X
\]

on the \( r \)-order jet manifold \( J^r Y \) of sections of \( Y \to X \). Let \( u \) be a vertical vector field on \( Y \to X \) and \( J^r u \) its prolongation onto \( J^r Y \to X \). Let \( L_{J^r u} L \) denote the Lie derivative of \( L \) along \( J^r u \). The first variational formula provides its canonical decomposition

\[
L_{J^r u} L = u \rvert \delta L + d_H(J^r u \rvert H_L),
\]

where \( \delta L \) is the Euler–Lagrange operator of \( L \), \( d_H \) is the horizontal (total) differential (see (12) below) and \( H_L \) is a Poincaré–Cartan form of \( L \) (see [13, 17] for its explicit expressions). Let \( L_{J^r u} L \) vanishes everywhere on \( J^r Y \), i.e., a Lagrangian \( L \) is invariant under a one-parameter group of vertical bundle automorphisms (gauge transformations) of \( Y \to X \) whose infinitesimal generator is \( u \). Then, on \( \text{Ker} \delta L \), one has the Noether conservation law

\[
0 \approx d_H(J^r u \rvert H_L)
\]

of the Noether current

\[
\mathfrak{J}_u = J^r u \rvert H_L.
\]

Note that, unless \( r \leq 2 \), a Poincaré–Cartan form \( H_L \) is not unique. Moreover, one can put \( \mathfrak{J}_u = h_0(J^{2r-1} u \rvert \rho_L) \) where \( h_0 \) is the horizontal projection (see (11) below) and \( \rho_L \) is an arbitrary Lepagean equivalent of a Lagrangian \( L \).
However, it may happen that, though the Lie derivative $L_{J^r u}L$ does not vanish, a conservation law takes place. Indeed, let this Lie derivative be a horizontal differential

$$L_{J^r u}L = d_H\sigma. \quad (4)$$

Then, the first variational formula (1) on Ker $\delta L$ leads to the equality

$$0 \approx d_H(\mathcal{J}_u - \sigma), \quad (5)$$

regarded as a conservation law of the modified Noether current $\mathcal{J}_u - \sigma$.

In order to understand the condition (4), let us refer to the master identity

$$\delta(L_{J^r u}L) = L_{J^{2r} u}(\delta L) \quad (6)$$

(see Appendix). It follows that the Euler–Lagrange operator is invariant under a one-parameter gauge group generated by $u$ iff the Lie derivative $L_{J^r u}L$ is a variationally trivial Lagrangian. The Lie derivative (4) is such a Lagrangian as follows.

**Theorem 1.** An $r$-order Lagrangian $L$ (1) is variationally trivial iff it takes the form

$$L = d_H\xi + h_0\varphi \quad (7)$$

where $\xi$ is an $n - 1$-form of jet order $r - 1$ and $\varphi$ is a closed $n$-form on $Y$.

This assertion has been proved by a computation of cohomology of finite order variational sequences [1, 5, 15, 16, 26]. It is also reproduced by a computation of cohomology of the infinite order variational complex, but without minimizing the jet order of the form $\xi$ [11, 12, 20, 21, 25].

**Corollary 2.** It follows from the master identity (6) and Theorem 1 that the Euler–Lagrange operator $\delta L$ of a Lagrangian $L$ is invariant under a one-parameter group of gauge transformations generated by a vector field $u$ iff the Lie derivative $L_{J^r u}L$ of this Lagrangian takes the form

$$L_{J^r u}L = d_H\sigma + h_0\phi \quad (8)$$

where $\phi$ is a closed $n$-form on $Y$.

The equality (8) locally reduces to the equality (4) known as the Noether–Bessel–Hagen equation [17]. If the equality (8) globally takes the form (4), the conservation law (5) holds.

The Lie derivative of a global Chern–Simons Lagrangian illustrates the formula (8) [6].
A differential operator \( \mathcal{E} \) on \( Y \to X \) is said to be locally variational if each point of \( Y \) admits an open neighbourhood such that, on this neighbourhood, \( \mathcal{E} \) is the Euler–Lagrange operator of some local Lagrangian.

**Theorem 3.** A \( 2k \)-order differential operator \( \mathcal{E} \) is locally variational iff

\[
\mathcal{E} = \delta L + \tau(\varphi),
\]

where \( L \) is a \( k \)-order Lagrangian, \( \tau \) is a certain differential operator such that \( \delta = \tau \circ d \) (see (13) below) and \( \varphi \) is a closed \((n + 1)\)-form on \( Y \).

For instance, if \( Y \to X \) is an affine bundle, its de Rham cohomology equals that of \( X \) and, consequently, any variationally trivial operator on \( Y \) is the Euler–Lagrange operator of some global Lagrangian. Then, Corollary 2 can be applied to this operator. The above mentioned global Chern–Simons model illustrates this fact.

Theorem 3 gives a solution of the global inverse problem in finite order Lagrangian formalism \cite{1} (see also \cite{5, 15, 17, 26}). This Theorem as like as Theorem 1 issues from a computation of cohomology of the infinite variational complex, but without minimizing the order of a Lagrangian \( L \) \cite{11, 12, 20, 21}. Infinite order jet formalism and the infinite variational complex is a convenient tool of studying Lagrangian systems both of infinite and finite order (see, e.g., \cite{10, 21}). Note that infinite order jets are also utilized in some quantum field models \cite{3, 8, 9, 11, 19}. Our goal here is the extension of the first variational formula (1), the Noether conservation law (2) and the master identity (6) to infinite order Lagrangians.

### 2 The differential calculus in infinite order jets

Smooth manifolds throughout are assumed to be real, finite-dimensional, Hausdorff, second-countable (i.e., paracompact), and connected. We follow the terminology of \cite{7, 14}, where a sheaf \( S \) is a particular topological bundle, \( \overline{S} \) denotes the canonical presheaf of sections of the sheaf \( S \), and \( \Gamma(S) \) is the group of global sections of \( S \).

Recall that the infinite order jet space of a smooth fibre bundle \( Y \to X \) is defined as a projective limit \((J^\infty Y, \{\pi^\infty_r\})\) of the inverse system

\[
X \leftarrow^{\pi} Y \leftarrow^{\pi_0^1} \cdots \leftarrow^{\pi_{r-1}^r} J^{r-1}Y \leftarrow^{\pi_{-1}^1} J^{r}Y \leftarrow \cdots
\]

of finite order jet manifolds \( J^r Y \) of \( Y \to X \). Endowed with the projective limit topology, \( J^\infty Y \) is a paracompact Fréchet manifold \cite{23}. A bundle coordinate atlas \( \{U, (x^\lambda, y^j)\} \) of
$Y \to X$ yields the manifold coordinate atlas
\[ \{(\pi^\infty_0)^{-1}(U), (x^\lambda, y^i_\Lambda)\}, \quad 0 \leq |\Lambda|, \]
of $J^\infty Y$, together with the transition functions
\[ y^i_{\lambda + \Lambda} = \frac{\partial x^\mu}{\partial x^\lambda} d\mu y^i_\Lambda, \]
where $\Lambda = (\lambda_k \ldots \lambda_1)$, $\lambda + \Lambda = (\lambda \lambda_k \ldots \lambda_1)$ are multi-indices and
\[ d_\Lambda = \partial_\Lambda + \sum_{|\Lambda| \geq 0} y^i_{\lambda + \Lambda} \partial_i^\Lambda \]
is the total derivative. We will also use the notation $d_\Lambda = d_{\lambda_k} \ldots d_{\lambda_1}$, $\Lambda = (\lambda_k \ldots \lambda_1)$.

With the inverse system (10), one has the direct system
\[ O^*(X) \xrightarrow{\pi^*_0} O^*_0 \xrightarrow{\pi^*_1} O^*_1 \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_{r-1}} O^*_r \to \cdots \]
of graded differential $\mathbb{R}$-algebras $O^*_r$ of exterior forms on finite order jet manifolds $J^r Y$, where $\pi^*_{r-1}$ are the pull-back monomorphisms. The direct limit of this direct system is the graded differential algebra $O^*_\infty$ of exterior forms on finite order jet manifolds modulo the pull-back identification. However, $O^*_\infty$ does not exhaust all exterior forms on $J^\infty Y$.

Let $\mathfrak{G}^*_r$ be a sheaf of germs of exterior forms on the $r$-order jet manifold $J^r Y$ and $\mathfrak{G}^*_r$ its canonical presheaf. There is the direct system of canonical presheaves
\[ \overline{\mathfrak{G}}^*_X \xrightarrow{\pi^*_0} \overline{\mathfrak{G}}^*_0 \xrightarrow{\pi^*_1} \overline{\mathfrak{G}}^*_1 \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_{r-1}} \overline{\mathfrak{G}}^*_r \to \cdots, \]
where $\pi^*_{r-1}$ are the pull-back monomorphisms. Its direct limit $\overline{\mathfrak{G}}^*_\infty$ is a presheaf of graded differential $\mathbb{R}$-algebras on $J^\infty Y$. Let $\mathfrak{Q}^*_\infty$ be a sheaf constructed from $\overline{\mathfrak{G}}^*_\infty$, $\overline{\mathfrak{G}}^*_\infty$ its canonical presheaf, and $\mathfrak{Q}^*_\infty = \Gamma(\mathfrak{Q}^*_\infty)$ the structure algebra of sections of the sheaf $\overline{\mathfrak{G}}^*_\infty$. There are $\mathbb{R}$-algebra monomorphisms $\overline{\mathfrak{G}}^*_\infty \to \overline{\mathfrak{G}}^*_\infty$ and $O^*_\infty \to Q^*_\infty$.

The key point is that, since the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\mathfrak{Q}^*_\infty$ [23], the sheaves of $\mathfrak{Q}^*_\infty$-modules on $J^\infty Y$ are fine and, consequently, acyclic. Therefore, the abstract de Rham theorem on cohomology of a sheaf resolution [14] can be called into play in order to obtain cohomology of the graded differential algebra $Q^*_\infty$. In turn, $O^*_\infty$ is proved to possess the same cohomology as $Q^*_\infty$ (see Theorem 10 below) [11, 12, 20, 21].

For short, we agree to call elements of $Q^*_\infty$ the exterior forms on $J^\infty Y$. Restricted to a coordinate chart $(\pi^\infty_0)^{-1}(U_Y)$ of $J^\infty Y$, they can be written in a coordinate form,
where horizontal forms \( \{dx^\lambda\} \) and contact 1-forms \( \{\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+A}dx^\lambda\} \) provide local generators of the algebra \( Q^{*}_\infty \). There is the canonical decomposition
\[
Q^{*}_\infty = \bigoplus_{k,s} Q^{k,s}_\infty, \quad 0 \leq k, \quad 0 \leq s \leq n,
\]
of \( Q^{*}_\infty \) into \( Q^{0}_\infty \)-modules \( Q^{k,s}_\infty \) of \( k \)-contact and \( s \)-horizontal forms, together with the corresponding projections
\[
h_k : Q^*_\infty \to Q^{*,k}_\infty, \quad 0 \leq k, \quad h^s : Q^*_\infty \to Q^{*,s}_\infty, \quad 0 \leq s \leq n. \quad (11)
\]
Accordingly, the exterior differential on \( Q^*_\infty \) is split into the sum \( d = d_H + d_V \) of horizontal and vertical differentials such that
\[
d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H(\phi) = dx^\lambda \wedge d\lambda(\phi),
\]
\[
d_V \circ h^s = h^s \circ d \circ h^s, \quad d_V(\phi) = \theta^i_\Lambda \wedge \partial^A\Lambda_i \phi, \quad \phi \in Q^{*}_\infty. \quad (12)
\]

3 The infinite variational complex

Being nilpotent, the differentials \( d_V \) and \( d_H \) provide the natural bicomplex \( \{\Phi^{k,m}_\infty\} \) of the sheaf \( Q^{*}_\infty \) on \( J^\infty Y \). To complete it to the variational bicomplex, one defines the projection \( \mathbb{R} \)-module endomorphism
\[
\tau = \sum_{k>0} \frac{1}{k} \tau \circ h_k \circ h^n, \quad (13)
\]
of \( \Phi^{*}_\infty \) such that
\[
\tau \circ d_H = 0, \quad \tau \circ d = \tau \circ d - \tau \circ d = 0.
\]
Introduced on elements of the presheaf \( \Phi^{*}_\infty \) (see, e.g., [4, 10, 24]), this endomorphism is induced on the sheaf \( \Phi^{*}_\infty \) and its structure algebra \( Q^{*}_\infty \). Put
\[
\mathcal{E}_k = \tau(Q^{k,n}_\infty), \quad E_k = \tau(Q^{k,n}_\infty), \quad k > 0.
\]
Since \( \tau \) is a projection operator, we have isomorphisms
\[
\mathcal{E}_k = \tau(\Phi^{k,n}_\infty), \quad E_k = \Gamma(\mathcal{E}_k).
\]
The variational operator on $\Omega^*_{\infty}$ is defined as the morphism $\delta = \tau \circ d$. It is nilpotent, and obeys the relation

$$\delta \circ \tau - \tau \circ d = 0.$$  \hfill (14)

Let $R$ and $\mathcal{G}_X^*$ denote the constant sheaf on $J^\infty Y$ and the sheaf of exterior forms on $X$, respectively. The operators $d_V, d_H, \tau$ and $\delta$ give the following variational bicomplex of sheaves of differential forms on $J^\infty Y$:

\[
\begin{array}{ccccccccc}
0 & \to & \Omega^k_{\infty} & \to & \Omega^k_{\infty} & \to & \cdots & \Omega^k_{\infty} & \to & \mathcal{E}_k & \to & 0 \\
& & \delta & & & & & \tau & & & \\
0 & \to & \Omega^1_{\infty} & \to & \Omega^1_{\infty} & \to & \cdots & \Omega^1_{\infty} & \to & \mathcal{E}_1 & \to & 0 \\
& & \delta & & & & & \tau & & & \\
0 & \to & \Omega^0_{\infty} & \to & \Omega^0_{\infty} & \to & \cdots & \Omega^0_{\infty} & \to & \mathcal{E}_0 & \to & 0 \\
& & \delta & & & & & \tau & & & \\
0 & \to & \mathcal{G}_X^0 & \to & \mathcal{G}_X^1 & \to & \cdots & \mathcal{G}_X^m & \to & \mathcal{G}_X^n & \to & 0 \\
& & \delta & & & & & \tau & & & \\
0 & \to & 0 & \to & 0 & \to & \cdots & 0 & \to & 0 & \to & 0
\end{array}
\]  \hfill (15)

The second row and the last column of this bicomplex assemble into the infinite variational complex

$$0 \to R \to \Omega^0_{\infty} \to \Omega^1_{\infty} \to \cdots \to \Omega^m_{\infty} \to \Omega^n_{\infty} \delta \to \mathcal{E}_1 \delta \to \mathcal{E}_2 \to \cdots.$$  \hfill (16)

The corresponding variational bicomplex and variational complex of the graded differential algebra $Q^*_{\infty}$ (see (22) below) take place.

There are the well-known statements summarized usually as the algebraic Poincaré lemma (see, e.g., [18, 24]).

**Theorem 4.** If $Y$ is a contractible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, the variational bicomplex of the graded differential algebra $Q^*_{\infty}$ is exact.

It follows that the variational bicomplex (15) and, consequently, the variational complex (16) are exact for any smooth bundle $Y \to X$. Moreover, the sheaves $\Omega^k_{\infty}$ and $\mathcal{E}_k$ are fine. Thus, the columns and rows of the bicomplex (15) as like as the variational complex (16) are sheaf resolutions, and the abstract de Rham theorem can be applied to them. The results are the following [1, 2, 12, 20, 21, 23].
Let us start from the following assertion.

**Proposition 5.** Since $Y$ is a strong deformation retract of $J^\infty Y$, there is an isomorphism

$$H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R}) = H^*(Y)$$

(17)

between cohomology $H^*(J^\infty Y, \mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$, that $H^*(Y, \mathbb{R})$ of $Y$, and the de Rham cohomology $H^*(Y)$ of $Y$.

Let us consider the de Rham complex of sheaves

$$0 \to \mathbb{R} \to \Omega_0^0 \xrightarrow{d} \Omega_1^0 \xrightarrow{d} \cdots$$

(18)
on $J^\infty Y$ and the corresponding de Rham complex of their structure algebras

$$0 \to \mathbb{R} \to \Omega^0_0 \xrightarrow{d} \Omega^1_0 \xrightarrow{d} \cdots .$$

(19)

The complex (18) is exact due to the Poincaré lemma, and is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$ since sheaves $\Omega^r_\infty$ are fine. Then, the abstract de Rham theorem and Lemma 5 lead to the following.

**Proposition 6.** The de Rham cohomology $H^*(\Omega^*_\infty)$ of the graded differential algebra $\Omega^*_\infty$ is isomorphic to that $H^*(Y)$ of the bundle $Y$.

It follows that every closed form $\phi \in \Omega^*_\infty$ is split into the sum

$$\phi = \varphi + d\xi, \quad \xi \in \Omega^*_\infty,$$

(20)

where $\varphi$ is a closed form on the fibre bundle $Y$.

Turn now to the rows of the variational bicomplex (15). We have the exact sequence of sheaves

$$0 \to \Omega^0_{\infty} \xrightarrow{d} \Omega^1_{\infty} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k,n}_{\infty} \xrightarrow{\tau} \mathcal{E}_k \to 0, \quad k > 0 .$$

Since the sheaves $\Omega^{k,m}_{\infty}$ and $\mathcal{E}_k$ are fine, this is a resolution of the fine sheaf $\Omega^{k,0}_{\infty}$. Then, the abstract de Rham theorem results in the following.

**Proposition 7.** The cohomology groups of the complex

$$0 \to \Omega^{0,0}_{\infty} \xrightarrow{d} \Omega^{1,0}_{\infty} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k,n}_{\infty} \xrightarrow{\tau} \mathcal{E}_k \to 0, \quad k > 0,$$

(21)

are trivial.
The variational complex (16) is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$. Then, from the abstract de Rham theorem and Proposition 5, we obtain the following.

**Proposition 8.** There is an isomorphism between $d_H$- and $\delta$-cohomology of the variational complex

$$0 \to \mathbb{R} \to \mathcal{Q}_\infty^0 \xrightarrow{d_H} \mathcal{Q}_\infty^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}_\infty^{0,n} \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_2 \to \cdots$$

and the de Rham cohomology of the fibre bundle $Y$, namely,

$$H^{k<n}(d_H; \mathcal{Q}_\infty^*) = H^{k<n}(Y), \quad H^{k-n}(\delta; \mathcal{Q}_\infty^*) = H^{k\geq n}(Y).$$

Moreover, the relation (14) for $\tau$ and the relation $h_0d = d_Hh_0$ for $h_0$ define a homomorphisms of the de Rham complex (19) of the algebra $\mathcal{Q}_\infty^*$ to its variational complex (22). The corresponding homomorphism of their cohomology groups is an isomorphism by virtue of Proposition 6 and Proposition 8. Then, the splitting (20) leads to the following decompositions.

**Theorem 9.** Any $d_H$-closed form $\sigma \in \mathcal{Q}^{0,m}, m < n$, is represented by a sum

$$\sigma = h_0\varphi + d_H\xi, \quad \xi \in \mathcal{Q}^{m-1}_\infty,$$

where $\varphi$ is a closed $m$-form on $Y$. Any $\delta$-closed form $\psi \in \mathcal{Q}^{k,n}, k \geq 0$, is split into

$$\psi = h_0\varphi + d_H\xi, \quad k = 0, \quad \xi \in \mathcal{Q}^{n-1}_\infty,$$

$$\psi = \tau(\varphi) + \delta(\xi), \quad k = 1, \quad \xi \in \mathcal{Q}_\infty^1,$$

$$\psi = \tau(\varphi) + \delta(\xi), \quad k > 1, \quad \xi \in E_{k-1},$$

where $\varphi$ is a closed $(n + k)$-form on $Y$.

The variational complex (22) provides the algebraic approach to the calculus of variations in the class of exterior forms of locally finite jet order [4, 10, 24]. For instance, the variational operator $\delta$ acting on $\mathcal{Q}^{0,n}_\infty$ is the Euler–Lagrange map, while $\delta$ acting on $E_1$ is the Helmholtz–Sonin map. Accordingly, one can think of a horizontal density

$$L = L\omega, \quad \omega = dx^1 \wedge \cdots \wedge dx^n,$$

on $J^\infty Y$ as being a Lagrangian of locally finite order. Then, the expressions (24) – (25) in Theorem 9 give a solution of the global inverse problem of the calculus of variations.
on fibre bundles in the class of Lagrangians \( L \in Q^0_{\infty} \) of locally finite order. Namely, a Lagrangian \( L \in Q^0_{\infty} \) is variationally trivial iff it takes the form (24), while an Euler–Lagrange-type operator \( E \in E_1 \) satisfies the Helmholtz condition \( \delta(E) = 0 \) iff it takes the form (25).

In order to return to Theorems 1 and 3, let us consider the subalgebra \( O^*_{\infty} \subset Q^*_{\infty} \) of exterior forms of bounded jet order. It makes up a subcomplex of the variational complex (22). The key point is the following [11, 12, 20, 21].

**Theorem 10.** Graded differential algebra \( O^*_{\infty} \) has the same \( d^-, d_H^- \) and \( \delta \)-cohomology as \( Q^*_{\infty} \).

It follows that, if an exterior forms \( \psi \) in the formulas (23) – (26) are of finite jet order, then the exterior form \( \xi \) are so. In particular, we come to Theorems 1 and 3, but without minimizing the jet order of the exterior forms \( \xi \) and \( L \), respectively.

4 Conservation laws

The exactness of the complex (21) at the term \( Q^k_{\infty} \) implies that, if \( \tau(\phi) = 0, \phi \in Q^k_{\infty} \), then \( \phi = d_H\xi, \xi \in Q^{k-1}_{\infty} \). Since \( \tau \) is a projection operator, there is the \( \mathbb{R} \)-module decomposition

\[
Q^k_{\infty} = E_k \oplus d_H(Q^{k-1}_{\infty}).
\]

Given a Lagrangian \( L \in Q^0_{\infty} \), the decomposition (27) in the case of \( k = 1 \) reads

\[
dL = \tau(dL) + (\text{Id} - \tau)(dL) = \delta L + d_H(\phi),
\]

where \( \phi \in Q^1_{\infty} \) and

\[
\delta L = E_i \theta^i \land \omega = \sum_{|\Lambda| \geq 0} (-1)^{|\Lambda|} d_{\Lambda} (\partial_i^\Lambda L) \theta^i \land \omega
\]

is the Euler–Lagrange operator of an infinite order Lagrangian \( L \).

Let \( u = u^i \partial_i \) be a vertical vector field on a fibre bundle \( Y \to X \) seen as a generator of one-parameter gauge group. It defines the derivation

\[
J^\infty u = \sum_{|\Lambda| \geq 0} d_{\Lambda} u^i \partial_i^\Lambda
\]

of the ring \( Q^0_{\infty} \) regarded as an infinite order jet prolongation of \( u \) onto \( J^\infty Y \). We also have the contraction \( u \lfloor \phi \) and the Lie derivative

\[
L_{J^\infty u} \phi = J^\infty u \lfloor d\phi + d(J^\infty u \lfloor \phi)
\]
of elements of the differential algebra $\mathcal{Q}_\infty^\ast$. It is easily justified that

$$J^\infty u \vert d_H \phi = -d_H (J^\infty u \vert \phi), \quad \phi \in \mathcal{Q}_\infty^\ast.$$  

Let $L \in \mathcal{Q}_\infty^{0,n}$ be an infinite order Lagrangian. By virtue of the decomposition (28), we come to the first variational formula

$$L_{J^\infty u} L = J^\infty u \vert dL = u \vert \delta L - d_H (J^\infty u \vert \phi), \quad (31)$$

where

$$\mathcal{J}_u = -J^\infty u \vert \phi \in \mathcal{Q}_\infty^{0,n-1}$$

is the symmetry current along the vector field $u$. If $L$ is a finite order Lagrangian, this current is given by the expression (3) modulo a $d_H$-closed form. However, a glance at the explicit formulas for Lepagean equivalents [13, 17] shows that this expression can not be generalized to the case of infinite order Lagrangians. If the Lie derivative $L_{J^\infty u} L$ vanishes, the first variational formula (31) leads to the Noether conservation law

$$d_H J_u \approx 0 \quad (32)$$

on $\text{Ker} \delta L$, i.e., the global section $d_H J_u$ of the sheaf $\mathcal{Q}_\infty^{0,n}$ on $J^\infty Y$ takes zero values at points of the subspace $\text{Ker} \delta L \subset J^\infty Y$ given by the condition $\delta L = 0$.

There is the master identity

$$\delta (L_{J^\infty u} L) = L_{J^\infty u} (\delta L) \quad (33)$$

(see Appendix for its proof). It follows from this identity and Theorem 9 that the Euler–Lagrange operator $\delta L$ (29) of an infinite order Lagrangian $L$ is invariant under a one-parameter group of gauge transformations generated by a vector field $u$ iff the Lie derivative $L_{J^\infty u} L$ of this Lagrangian takes the form

$$L_{J^\infty u} L = d_H \sigma + h_0 \phi,$$

where $\phi$ is a closed $n$-form on $Y$.

In conclusion, let us say a few words on the cohomology of conservation laws in infinite (and finite) order jet formalism. If the conservation law (32) takes place, one can say that the horizontal differential $d_H J_u$ is a relative $d_H$-cocycle on the pair of topological spaces $(J^\infty Y, \text{Ker} \delta L)$. Of course, it is a $d_H$-coboundary, but need not be a relative $d_H$-coboundary
since $J_u \not\approx 0$. Therefore, the horizontal differential $d_H J_u$ of a conserved current $J_u$ can be characterized by elements of the relative $d_H$-cohomology group $H^\infty_{rel}(J^\infty Y, \text{Ker} \delta L)$ of the pair $(J^\infty Y, \text{Ker} \delta L)$.

For instance, any conserved Noether current in the Yang–Mills gauge theory on a principal bundle $P$ with a structure group $G$ is well known to reduce to a superpotential, i.e., $J_u = W + d_H U$ where $W \approx 0$. Its horizontal differential $d_H J_u$ belongs to the trivial element of the relative cohomology group $H^\infty_{rel}(J^2 Y, \text{Ker} \delta L_{YM})$, where $Y = J^1 P/G$.

Let now $N^n \subset X$ be an $n$-dimensional submanifold of $X$ with a compact boundary $\partial N^n$. Let $s$ be a section of the fibre bundle $Y \to X$ and $s = J^\infty s$ its infinite order jet prolongation, i.e., $y_\Lambda^i \circ s = d_\Lambda s^i$, $0 < |\Lambda|$. Let us assume that $s(\partial N^n) \subset \text{Ker} \delta L$. Then, the quantity

$$\int_{N^n} s^* d_H J_u = \int_{\partial N^n} s^* J_u$$

(34)

depends only on the relative cohomology class of the divergence $d_H J_u$. For instance, in the above mentioned case of gauge theory, the quantity (34) vanishes.

Let $N^{n-1}$ be a compact $(n-1)$-dimensional submanifold of $X$ without boundary, and $s$ a section of $Y \to X$ such that $s(N^{n-1}) \subset \text{Ker} \delta L$. Let $J_u$ and $J'_u$ be two currents in the first variational formula (31). They differ from each other in a $d_H$-closed form $\varphi$. Then, the difference

$$\int_{N^{n-1}} s^*(J_u - J'_u)$$

(35)

depends only on the homology class of $N^{n-1}$ and the de Rham cohomology class of $s^* \varphi$. The latter is an image of the $d_H$-cohomology class of $\varphi$ under the morphisms

$$H^{n-1}(d_H) \xrightarrow{h_0} H^{n-1}(Y) \xrightarrow{s^*} H^{n-1}(X).$$

In particular, if $N^{n-1} = \partial N^n$ is a boundary, the quantity (35) always vanishes.

5 Appendix

In order to prove the master identity (33), let us act on the first variational formula (31) by the variational operator $\delta$. Since $\delta \circ d_H = 0$, we obtain the equality

$$\delta(L_{J^\infty u} L) = \delta(u \mid \delta L).$$

Therefore, we aim to prove that

$$\delta(u \mid \delta L) = L_{J^\infty u} \delta L.$$

(36)
It suffices to show that, given an arbitrary point \( y \in Y \), there exists its open neighbourhood \( U \) such that the equality (36) holds on \((\pi_0^{-1}(U))\). Using the coordinate expressions (29) – (30), let us write

\[
\delta(u)[\delta L] = \delta(u^i \mathcal{E}_i \omega) = [\partial_k(u^i \mathcal{E}_i)]dy^k \wedge \omega,
\]

\[
\mathbf{L}_{J^\infty u}\delta L = d(u^i \mathcal{E}_i) \wedge \omega + J^\infty u]d(\delta L) = \\
\partial_k(u^i \mathcal{E}_i)dy^k \wedge \omega + u](\partial_k \mathcal{E}_i dy^i \wedge dy^j \wedge \omega) + \sum_{|\Lambda|>0} (d_{\Lambda}u^i)\partial_{\Lambda}^i \mathcal{E}_k dy^k \wedge \omega.
\]

Then, the equality (36) takes the form

\[
\sum_{|\Lambda|>0} (-1)^{|\Lambda|}d_{\Lambda} \partial_{\Lambda}^k (u^i \mathcal{E}_i)dy^k \wedge \omega = u](\partial_k \mathcal{E}_i dy^k \wedge dy^i \wedge \omega) + \sum_{|\Lambda|>0} (d_{\Lambda}u^i)\partial_{\Lambda}^i \mathcal{E}_k dy^k \wedge \omega. \quad (37)
\]

Let us further assume that \( u(y) \neq 0 \). In this case, there exists an open neighbourhood \( U \) of \( y \) provided with bundle coordinates \((x^\lambda, y'^i)\) such that \( u = \partial_1 \). With respect to these coordinates, the equality (37) reads

\[
\sum_{|\Lambda|>0} (-1)^{|\Lambda|}d_{\Lambda} \partial_{\Lambda}^k (\sum_{|\Sigma|\geq 0} (-1)^{|\Sigma|}d_{\Sigma} \partial_{\Sigma}^1 \mathcal{L})dy^k \wedge \omega = (\partial_1 \mathcal{E}_k - \partial_k \mathcal{E}_1)dy^k \wedge \omega.
\]

It is brought into the form

\[
\delta((\mathcal{E}_1 - \partial_1 \mathcal{L})\omega) = 0.
\]

This equality really holds since \((\mathcal{E}_1 - \partial_1 \mathcal{L})\omega\) is a variationally trivial Lagrangian due to the first variational formula (31) where \( u = \partial_1 \), i.e.,

\[
\partial_1 \mathcal{L} \omega = \mathcal{E}_1 \omega - d_H(\phi_1).
\]

If \( u(y) = 0 \). There exists a vertical vector field \( u' \) such that \( u'(y) \neq 0 \). The equality (36) holds both for \( u' \) and \( u + u' \) and, consequently, does so for \( u \).

If \( L \) is a finite order Lagrangian, we obtain the master identity (6).

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