On the edge-Wiener index of the disjunctive product of simple graphs

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Communicated by D. Simson

ABSTRACT. The edge-Wiener index of a simple connected graph \(G\) is defined as the sum of distances between all pairs of edges of \(G\) where the distance between two edges in \(G\) is the distance between the corresponding vertices in the line graph of \(G\). In this paper, we study the edge-Wiener index under the disjunctive product of graphs and apply our results to compute the edge-Wiener index for the disjunctive product of paths and cycles.

Introduction

Throughout this paper, we consider connected finite graphs without any loops or multiple edges. Let \(G\) be such a graph with vertex set \(V(G)\) and edge set \(E(G)\). A topological index (also known as graph invariant) is any function on a graph that does not depend on a labeling of its vertices. Several hundreds of different invariants have been employed to date with various degrees of success in QSAR/QSPR studies. We refer the reader to [12] for review.

The oldest topological index is the one put forward in 1947 by Harold Wiener [23] nowadays referred to as the Wiener index. Wiener used his index for the calculation of the boiling points of alkanes. The Wiener

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2010 MSC: 05C76, 05C12, 05C38.

Key words and phrases: distance in graphs, edge-Wiener index, disjunctive product of graphs.
The edge-Wiener index of the disjunctive product

index $W(G)$ of a graph $G$ is defined as the sum of distances between all pairs of vertices of $G$,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v | G),$$

where $d(u, v | G)$ denotes the distance between the vertices $u$ and $v$ of $G$ which is defined as the length of any shortest path in $G$ connecting them. We denote $d(u, v | G)$ simply by $d(u, v)$ when no ambiguity is present. Details on the mathematical properties of the Wiener index and its applications in chemistry can be found in [8, 10, 11, 13–15].

Motivated by definition of the Wiener index, the edge-Wiener index was introduced based on distance between all pairs of edges in a graph in 2009 [9, 19, 21]. The edge-Wiener index of a graph $G$ is defined as

$$W_e(G) = \sum_{\{f,g\} \subseteq E(G)} d_e(f, g | G),$$

where $d_e(f, g | G)$ denotes the distance between the edges $f$ and $g$ of $G$ which is defined as the ordinary distance between the corresponding vertices in the line graph $L(G)$ of $G$. So, $W_e(G) = W(L(G))$. It has been proved that [19], for each pair of edges $f = uv$ and $g = zt$ of $G$,

$$d_e(f, g | G) = \begin{cases} 
\min\{d(u, z), d(u, t), d(v, z), d(v, t)\} + 1 & \text{if } f \neq g, \\
0 & \text{if } f = g.
\end{cases}$$

For details on the theory of the edge-Wiener index and its applications see [2, 22, 24] and specially the recent paper [18].

Many graphs are composed of simpler graphs via various graph operations also known as graph products. These composite graphs have more complicated structures than their components. So, in general, computing their topological invariants is more difficult than computing the topological invariants of their components. So, it is important to understand how certain invariants of such composite graphs are related to the corresponding invariants of their components. The edge-Wiener index of some graph operations have been computed before [1, 3–7]. In this paper, we study the behavior of the edge-Wiener index under the disjunctive product of graphs and apply our results to compute the edge-Wiener index for the disjunctive product of paths and cycles. We refer the reader to [17] for details on the properties and applications of graph operations.
1. Definitions and preliminaries

For a simple connected graph $G$, let $N_G(u)$ denote the open neighborhood of a vertex $u$ in $G$ which is the set of all vertices of $G$ adjacent with $u$. The cardinality of $N_G(u)$ is called the degree of $u$ in $G$ and denoted by $d_G(u)$. If there is no confusion, we simply use $N(u)$ and $d(u)$ instead of $N_G(u)$ and $d_G(u)$, respectively. Let $\Delta(G)$ denote the number of all triangles (3-cycles) in $G$ and $M_1(G)$ denote the first Zagreb index of $G$ which is one the oldest topological indices introduced by Gutman and Trinajstić [16] as follow.

$$M_1(G) = \sum_{u\in V(G)} d(u)^2 = \sum_{uv \in E(G)} (d(u) + d(v)). \quad (1)$$

It is easy to check that,

$$\sum_{uv \in E(G)} |N(u) \cap N(v)| = 3\Delta(G) \quad (2)$$

and

$$\sum_{u,v \in V(G)} |N(u) \cap N(v)| = M_1(G). \quad (3)$$

Using the inclusion–exclusion principle and then (1), (2), and (3), one can easily get the following equations.

$$\sum_{uv \in E(G)} |N(u) \cup N(v)| = M_1(G) - 3\Delta(G) \quad (4)$$

and

$$\sum_{u,v \in V(G)} |N(u) \cup N(v)| = 4ne - M_1(G), \quad (5)$$

where $n$ and $e$ denote the order and size of the graph $G$, respectively.

Here, we introduce some useful notations which will be used throughout the paper.

$$\nu(G) = \sum_{uv \in E(G)} |N(u) \cup N(v)|,$$  \hspace{1cm} (6)

$$\nu^*(G) = \sum_{u,v \in V(G)} |N(u) \cup N(v)|^2,$$  \hspace{1cm} (7)
\[ \mu(G) = \sum_{uv \in E(G)} \sum_{z \in V(G) \setminus (N(u) \cup N(v))} |N(z) \setminus (N(u) \cup N(v))|, \quad (8) \]

and

\[ \mu^*(G) = \sum_{u,v \in V(G)} \sum_{z \in V(G) \setminus (N(u) \cup N(v))} |N(z) \setminus (N(u) \cup N(v))|. \quad (9) \]

2. Results and discussion

Let \( G_1 \) and \( G_2 \) be two simple connected graphs. We denote by \( V(G_i) \) and \( E(G_i) \), the vertex set and edge set of \( G_i \), respectively, where \( i \in \{1, 2\} \). The disjunctive product \( G_1 \lor G_2 \) of graphs \( G_1 \) and \( G_2 \) is a graph with the vertex set \( V(G_1) \times V(G_2) \) and two vertices \((u_1, u_2)\) and \((v_1, v_2)\) of \( G_1 \lor G_2 \) are adjacent if and only if \( u_1 \) and \( v_1 \) are adjacent in \( G_1 \) or \( u_2 \) and \( v_2 \) are adjacent in \( G_2 \). The disjunctive product of two graphs is also known as their co-normal product or OR product. The distance between the vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) of \( G_1 \lor G_2 \) is given by

\[ d(u, v | G_1 \lor G_2) = \begin{cases} 
0 & \text{if } u_1 = v_1, \ u_2 = v_2, \\
1 & \text{if } u_1 v_1 \in E(G_1) \text{ or } u_2 v_2 \in E(G_2), \\
2 & \text{otherwise}.
\end{cases} \]

In this section, we compute the edge-Wiener index of the disjunctive product of \( G_1 \) and \( G_2 \). Throughout the section, for notational convenience, we let \( G = G_1 \lor G_2 \) be the disjunctive product of a pair of graphs \( G_1 \) and \( G_2 \), and \( n_1, e_1, n_2, e_2 \) denote the order of \( G_1 \), size of \( G_1 \), order of \( G_2 \), size of \( G_2 \), respectively.

At first, we consider three subsets of \( E(G) \) as follows.

\[ E_1 = \{(u_1, u_2)(v_1, v_2) \mid u_1 v_1 \in E(G_1), u_2, v_2 \in V(G_2)\}, \]
\[ E_2 = \{(u_1, u_2)(v_1, v_2) \mid u_2 v_2 \in E(G_2), u_1, v_1 \in V(G_1)\}, \]
\[ E_3 = \{(u_1, u_2)(v_1, v_2) \mid u_1 v_1 \in E(G_1), u_2 v_2 \in E(G_2)\}. \]

It is clear that, \( E(G) = \bigcup_{i=1}^{3} E_i \) and

\[ |E(G)| = |E_1| + |E_2| - 2|E_3| = e_1 n_2^2 + e_2 n_1^2 - 2e_1 e_2. \quad (10) \]

Since all distinct vertices of \( G \) are either at distance 1 or 2, so all distinct edges of \( G \) are either at distance 1, 2, or 3. Therefore, we can partition the set of all pairs of edges of \( G \) into three sets as follows.

\[ A = \{\{f, g\} \mid d_e(f, g \mid G) = 1\}, \]
\[ B = \{ \{ f, g \} \mid d_e(f, g | G) = 2 \}, \]
\[ C = \{ \{ f, g \} \mid d_e(f, g | G) = 3 \}. \]

In order to find the edge-Wiener index of \( G \), we should compute the cardinality of the above sets. It is clear that,
\[
|A| + |B| + |C| = \left( \frac{|E(G)|}{2} \right) = \left( \frac{e_1n_2^2 + e_2n_1^2 - 2e_1e_2}{2} \right). \tag{11}
\]

By (11), it is enough to find the cardinality of the sets \( A \) and \( C \).

In the following proposition, we compute the cardinality of the set \( A \).

**Proposition 1.** The cardinality of the set \( A \) is given by
\[
|A| = \frac{1}{2} \left[ n_2(n_2^2 - 4e_2)M_1(G_1) + n_1(n_1^2 - 4e_1)M_1(G_2) \right. \\
+ \left. M_1(G_1)M_1(G_2) + 8n_1n_2e_1e_2 - 2(e_1n_2^2 + e_2n_1^2 - 2e_1e_2) \right]. \tag{12}
\]

**Proof.** Clearly, \( A \) is the set of all pairs of adjacent edges of \( G \). So
\[
|A| = \sum_{u \in V(G)} \left( \frac{d(u)}{2} \right) = \frac{1}{2} \sum_{u \in V(G)} (d(u)^2 - d(u)) = \frac{1}{2} M_1(G) - |E(G)|.
\]

By Theorem 3 in [20], the first Zagreb index of the disjunctive product of \( G_1 \) and \( G_2 \) is given by
\[
M_1(G) = n_2(n_2^2 - 4e_2)M_1(G_1) + n_1(n_1^2 - 4e_1)M_1(G_2) \\
+ M_1(G_1)M_1(G_2) + 8n_1n_2e_1e_2.
\]

Now using (10), we can get (12). \( \square \)

Now we start to find the cardinality of the set \( C \). Suppose \( f = (u_1, u_2)(v_1, v_2) \) is an arbitrary edge of \( G \) and let \( C(f) \) be the set of all edges of \( G \) which are at distance 3 from \( f \),
\[
C(f) = \{ g \in E(G) \mid d_e(f, g \mid G) = 3 \}.
\]

In the following lemma, we compute the cardinality of the set \( C(f) \).

**Lemma 1.** For every arbitrary edge \( f = (u_1, u_2)(v_1, v_2) \) of \( G \), the cardinality of the set \( C(f) \) is given by
\[
|C(f)| = \frac{1}{2} \left[ (n_2 - |N(u_2) \cup N(v_2)|)^2 \right]. \tag{13}
\]
This implies that,

Hence,

Let \( f = (u_1, u_2)(v_1, v_2) \) be an arbitrary edge of \( G \) and let \( g = (z_1, z_2)(t_1, t_2) \) be an arbitrary element of \( C(f) \). By definition of the distance \( d_e \), we have

\[
1 + \min\{d((u_1, u_2), (z_1, z_2)), d((u_1, u_2), (t_1, t_2)), d((v_1, v_2), (z_1, z_2)), d((v_1, v_2), (t_1, t_2))\} = d_e(f, g|G) = 3.
\]

Hence,

\[
\min\{d((u_1, u_2), (z_1, z_2)), d((u_1, u_2), (t_1, t_2)), d((v_1, v_2), (z_1, z_2)), d((v_1, v_2), (t_1, t_2))\} = 2.
\]

Since all distinct vertices of \( G \) are either at distance 1 or 2, so

\[
d((u_1, u_2), (z_1, z_2)) = d((u_1, u_2), (t_1, t_2)) = d((v_1, v_2), (z_1, z_2)) = d((v_1, v_2), (t_1, t_2)) = 2.
\]

This implies that, \( z_i \) and \( t_i \) are adjacent neither to \( u_i \) nor to \( v_i \) in \( G_i \), where \( i \in \{1, 2\} \). Consequently,

\[
|C(f)| = \frac{1}{2} \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} \left[ |N(z_1) \setminus (N(u_1) \cup N(v_1))| (n_2 - |N(u_2) \cup N(v_2)|) + |N(z_2) \setminus (N(u_2) \cup N(v_2))| (n_1 - |N(u_1) \cup N(v_1)|) - |N(z_1) \setminus (N(u_1) \cup N(v_1))| |N(z_2) \setminus (N(u_2) \cup N(v_2))| \right].
\]

\[
= \frac{1}{2} \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))|.
\]
Proposition 2. The cardinality of the set $C$ is given by

$$|C| = \frac{1}{4} \left[ \left( n_2^2 (n_2^2 - 10e_2) + \nu^*(G_2) - 2\nu(G_2) \right) \mu(G_1) \right. $$

$$+ \left. 6n_2 (M_1(G_2) - 2\Delta(G_2)) \right] \mu(G_1)$$

$$+ \left( n_1^2 (n_1^2 - 10e_1) + \nu^*(G_1) - 2\nu(G_1) \right) \mu(G_2)$$

$$+ \left( n_2^2 e_2 + \nu(G_2) - 2n_2 (M_1(G_2) - 3\Delta(G_2)) \right) \mu^*(G_1)$$

$$+ \left( n_1^2 e_1 + \nu(G_1) - 2n_1 (M_1(G_1) - 3\Delta(G_1)) \right) \mu^*(G_2)$$

$$- \mu(G_1) \mu^*(G_2) - \mu(G_2) \mu^*(G_1) + 2\mu(G_1)\mu(G_2) \right].$$

Now, (13) is obtained after simplifying the above expression. \qed

Let $f = (u_1, u_2)(v_1, v_2)$ be an edge of $G$. Then, $(u_1, v_2)(v_1, u_2)$ is also an edge of $G$. We denote the edge $(u_1, v_2)(v_1, u_2)$ by $\tilde{f}$.

Lemma 2. For every arbitrary edge $f = (u_1, u_2)(v_1, v_2)$ of $G$, $|C(\tilde{f})| = |C(f)|$.

Proof. The cardinality of $C(\tilde{f})$ can easily be obtained by changing the role of the vertices $v_2$ and $v_1$ in (13). On the other hand, one can easily check that changing the role of these two vertices does not influence the result. So $|C(\tilde{f})| = |C(f)| \quad \Box$

In the following proposition, we obtain the cardinality of the set $C$.

Proposition 2. The cardinality of the set $C$ is given by

$$|C| = \frac{1}{4} \left[ \left( n_2^2 (n_2^2 - 10e_2) + \nu^*(G_2) - 2\nu(G_2) \right) \mu(G_1) \right. $$

$$+ \left. 6n_2 (M_1(G_2) - 2\Delta(G_2)) \right] \mu(G_1)$$

$$+ \left( n_1^2 (n_1^2 - 10e_1) + \nu^*(G_1) - 2\nu(G_1) \right) \mu(G_2)$$

$$+ \left( n_2^2 e_2 + \nu(G_2) - 2n_2 (M_1(G_2) - 3\Delta(G_2)) \right) \mu^*(G_1)$$

$$+ \left( n_1^2 e_1 + \nu(G_1) - 2n_1 (M_1(G_1) - 3\Delta(G_1)) \right) \mu^*(G_2)$$

$$- \mu(G_1) \mu^*(G_2) - \mu(G_2) \mu^*(G_1) + 2\mu(G_1)\mu(G_2) \right].$$
Proof. For every $f \in E(G)$, there exist $|C(f)|$ elements in $C$. Furthermore, for every pair of edges $f, g$ in $G$, $g \in C(f)$ if and only if $f \in C(g)$. Hence,

$$|C| = \frac{1}{2} \sum_{f \in E(G)} |C(f)| = \frac{1}{2} \left[ \sum_{f \in E_1} |C(f)| + \sum_{f \in E_2} |C(f)| - \sum_{f \in E_3} (|C(f)| + |C(\bar{f})|) \right].$$

Using Lemma 2, we obtain

$$|C| = \frac{1}{2} \left[ \sum_{f \in E_1} |C(f)| + \sum_{f \in E_2} |C(f)| - 2 \sum_{f \in E_3} |C(f)| \right]. \quad (15)$$

Now, we compute $\sum_{f \in E_i} |C(f)|$, for every $i \in \{1, 2, 3\}$.

By definition of the set $E_1$ and (13), we have

$$\sum_{f \in E_1} |C(f)| = \frac{1}{2} \sum_{u_1v_1 \in E(G_1)} \sum_{u_2, v_2 \in V(G_2)} \left[ (n_2 - |N(u_2) \cup N(v_2)|)^2 \right.$$

$$\left. \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))| \right.$$

$$\left. + (n_1 - |N(u_1) \cup N(v_1)|)^2 \right.$$

$$\sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))|$$

$$\left. - \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))| \right.$$

$$\left. \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))| \right].$$

By simplifying the above expression, we obtain

$$\sum_{f \in E_1} |C(f)| = \frac{1}{2} \left[ \sum_{u_2, v_2 \in V(G_2)} (n_2 - |N(u_2) \cup N(v_2)|)^2 \right.$$

$$\sum_{u_1v_1 \in E(G_1)} \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))|$$

$$+ \sum_{u_1v_1 \in E(G_1)} (n_1 - |N(u_1) \cup N(v_1)|)^2 \right.$$

$$\sum_{u_2, v_2 \in V(G_2)} \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))| \right].$$
By symmetry, we obtain

\[- \sum_{u_1 v_1 \in E(G_1)} \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))|\]
\[\sum_{u_2, v_2 \in E(G_2)} \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))| \].

Now using \((4) - (9)\), we obtain

\[
\sum_{f \in E_1} |C(f)| = \frac{1}{2} \left( n_2^4 + n_1^2 \nu(G_1) - 2n_2 \left( 4n_2 e_2 - M_1(G_2) \right) \right) \mu(G_1) + \left( n_1^2 e_1 + n_1 \nu(G_1) - 2n_1 \left( M_1(G_1) - 3 \Delta(G_1) \right) \right) \mu^*(G_2) \quad (16)
\]
\[- \mu(G_1) \mu^*(G_2) \].

By definition of the set \(E_2\) and \((13)\), we have

\[
\sum_{f \in E_2} |C(f)| = \frac{1}{2} \sum_{u_2 v_2 \in E(G_2)} \sum_{u_1, v_1 \in V(G_1)} \left[ (n_2 - |N(u_2) \cup N(v_2)|)^2 \right.
\[
\sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))|
\[+ \left( n_1 - |N(u_1) \cup N(v_1)| \right)^2 \]
\[
\sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))|
\[\left. - \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))| \right) \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))| \].

By symmetry, we obtain

\[
\sum_{f \in E_2} |C(f)| = \frac{1}{2} \left( n_1^4 + n_1^2 \nu^*(G_1) - 2n_1 \left( 4n_1 e_1 - M_1(G_1) \right) \right) \mu(G_2) + \left( n_2^2 e_2 + n_2 \nu(G_2) - 2n_2 \left( M_1(G_2) - 3 \Delta(G_2) \right) \right) \mu^*(G_1) \quad (17)
\]
\[- \mu(G_2) \mu^*(G_1) \].

By definition of the set \(E_3\) and \((13)\), we have

\[
\sum_{f \in E_3} |C(f)| = \frac{1}{2} \sum_{u_1 v_1 \in E(G_1)} \sum_{u_2 v_2 \in E(G_2)} \left[ (n_2 - |N(u_2) \cup N(v_2)|)^2 \right.
\]
\[
\begin{align*}
\sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} & \mid N(z_1) \setminus (N(u_1) \cup N(v_1)) \mid \\
+ (n_1 - |N(u_1) \cup N(v_1)|)^2 & \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))| \\
& - \sum_{z_1 \in V(G_1) \setminus (N(u_1) \cup N(v_1))} |N(z_1) \setminus (N(u_1) \cup N(v_1))| \\
& \sum_{z_2 \in V(G_2) \setminus (N(u_2) \cup N(v_2))} |N(z_2) \setminus (N(u_2) \cup N(v_2))|.
\end{align*}
\]

Using (4), (6), and (8), we obtain

\[
\begin{align*}
\sum_{f \in E_3} |C(f)| = & \frac{1}{2} \left[ \left( n_2^2 e_2 + \nu(G_2) - 2n_2 \left( M_1(G_2) - 3\Delta(G_2) \right) \right) \mu(G_1) \\
& + \left( n_1^2 e_1 + \nu(G_1) - 2n_1 \left( M_1(G_1) - 3\Delta(G_1) \right) \right) \mu(G_2) \right] \left( 19 \right).
\end{align*}
\]

Now by (15)–(18), we can get (14).

Now, we are ready to compute the edge-Wiener index of the disjunctive product of \( G_1 \) and \( G_2 \).

**Theorem 1.** Assume that \( G_1 \) and \( G_2 \) are simple connected graphs, \( G_1 \vee G_2 \) is the disjunctive product of \( G_1 \) and \( G_2 \), and \( n_1, e_1, n_2, e_2 \) denote the order of \( G_1 \), size of \( G_1 \), order of \( G_2 \), size of \( G_2 \), respectively. Under the notation introduced earlier, the edge-Wiener index \( W_e(G_1 \vee G_2) \) of the disjunctive product \( G_1 \vee G_2 \) of \( G_1 \) and \( G_2 \) is given by

\[
W_e(G_1 \vee G_2) = \frac{1}{4} \left[ \left( n_2^2(n_2^2 - 10e_2) + \nu^*(G_2) - 2\nu(G_2) \right) \right. \\
+ 6n_2 \left( M_1(G_2) - 2\Delta(G_2) \right) ] \mu(G_1) \\
+ \left( n_1^2(n_1^2 - 10e_1) + \nu^*(G_1) - 2\nu(G_1) \right) \left. + 6n_1 \left( M_1(G_1) - 2\Delta(G_1) \right) ] \mu(G_2) \\
+ \left( n_2^2 e_2 + \nu(G_2) - 2n_2 \left( M_1(G_2) - 3\Delta(G_2) \right) \right) \mu^*(G_1) \\
+ \left( n_1^2 e_1 + \nu(G_1) - 2n_1 \left( M_1(G_1) - 3\Delta(G_1) \right) \right) \mu^*(G_2).
\]
\[
- \mu(G_1)\mu^*(G_2) - \mu(G_2)\mu^*(G_1) + 2\mu(G_1)\mu(G_2)
- 2n_2(n_2^2 - 4e_2)M_1(G_1) - 2n_1(n_1^2 - 4e_1)M_1(G_2)
- 2M_1(G_1)M_1(G_2) + 4(e_1n_2^2 + e_2n_1^2 - 2e_1e_2)^2 - 16n_1n_2e_1e_2.
\]

**Proof.** Let \( G = G_1 \lor G_2 \). By applying Propositions 1-2, Lemmas 1-2, and definition of the edge-Wiener index \( W_e(G) \), we get

\[
W_e(G) = \sum_{\{f,g\} \subseteq E(G)} d_e(f, g | G) = \sum_{\{f,g\} \in A \cup B \cup C} d_e(f, g | G)
= \sum_{\{f,g\} \in A} d_e(f, g | G) + \sum_{\{f,g\} \in B} d_e(f, g | G) + \sum_{\{f,g\} \in C} d_e(f, g | G)
= |A| + 2|B| + 3|C| = 2(|A| + |B| + |C|) - |A| + |C|.
\]

Now, using (11), (12), and (14), we can get (19). \( \square \)

Let \( P_n \) and \( C_n \) denote the \( n \)-vertex path and cycle, respectively. It can be verified by a direct calculation that, for every \( n \geq 2 \),

\[
M_1(P_n) = 4n - 6; \quad \Delta(P_n) = 0; \quad \nu(P_n) = \begin{cases} 
4 & \text{if } n = 2, \\
16n - 30 & \text{if } n \geq 3;
\end{cases}
\]

\[
\nu^*(P_n) = \begin{cases} 
10 & \text{if } n = 2, \\
2(8n^2 - 27n + 31) & \text{if } n \geq 3;
\end{cases}
\]

\[
\mu(P_n) = \begin{cases} 
0 & \text{if } n = 2, \\
2(n - 3)(n - 4) & \text{if } n \geq 3;
\end{cases}
\]

\[
\mu^*(P_n) = \begin{cases} 
0 & \text{if } n = 2, \\
2(n - 3)(n^2 - 6n + 10) & \text{if } n \geq 3.
\end{cases}
\]

Also for every \( n \geq 3 \),

\[
M_1(C_n) = 4n;
\]

\[
\Delta(C_n) = \begin{cases} 
1 & \text{if } n = 3, \\
0 & \text{if } n \geq 4;
\end{cases}
\]

\[
\nu(C_n) = \begin{cases} 
27 & \text{if } n = 3, \\
16n & \text{if } n \geq 4;
\end{cases}
\]

\[
\nu^*(C_n) = \begin{cases} 
160 & \text{if } n = 4, \\
2n(8n - 13) & \text{if } n \neq 4;
\end{cases}
\]

\[
\nu^*(C_n) = \begin{cases} 
160 & \text{if } n = 4, \\
2n(8n - 13) & \text{if } n \neq 4;
\end{cases}
\]
\[
\mu(C_n) = \begin{cases} 
0 & \text{if } n \leq 4, \\
2n(n-5) & \text{if } n \geq 5;
\end{cases}
\]
\[
\mu^*(C_n) = \begin{cases} 
0 & \text{if } n \leq 4, \\
2n(n-4)^2 & \text{if } n \geq 5.
\end{cases}
\]

Now using (19), we easily arrive at:

**Corollary 1.** For every integers \( n \geq 2 \) and \( m \geq 3 \),

\[
W_e(P_n \vee C_m) = \begin{cases} 
150 & \text{if } n = 2, \ m = 3, \\
432 & \text{if } n = 2, \ m = 4, \\
m(m^3 + 3m^2 + m - 9) & \text{if } n = 2, \ m \geq 5, \\
\frac{1}{2}(18n^4 + 24n^3 - 39n^2 - 45n + 72) & \text{if } n \geq 3, \ m = 3, \\
8(2n^4 + 7n^3 - 2n^2 - 30n + 38) & \text{if } n \geq 3, \ m = 4, \\
\frac{1}{2}m\left[n^4(3m - 5) \\
+ 2n^3(3m^2 - 18m + 44) \\
+ n^2(3m^3 - 42m^2 + 280m - 722) \\
- n(11m^3 - 152m^2 + 1038m - 2524) \\
+ 2(7m^3 - 103m^2 + 683m - 1583)\right] & \text{if } n \geq 3, \ m \geq 5.
\end{cases}
\]

**Acknowledgments**

The authors would like to thank the referee for insightful comments and valuable suggestions. The partial support by the Center of Excellence of Algebraic Hyper-structures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledged by the second author.

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Received by the editors: 27.06.2016
and in final form 27.09.2017.