Totally Acyclic Approximations

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Abstract
Let \( Q \to R \) be a surjective homomorphism of Noetherian rings such that \( Q \) is Gorenstein and \( R \) as a \( Q \)-bimodule admits a finite resolution by modules which are projective on both sides. We define an adjoint pair of functors between the homotopy category of totally acyclic \( R \)-complexes and that of \( Q \)-complexes. This adjoint pair is analogous to the classical adjoint pair of functors between the module categories of \( R \) and \( Q \). As a consequence, we obtain a precise notion of approximations of totally acyclic \( R \)-complexes by totally acyclic \( Q \)-complexes.

Keywords Totally acyclic complex · Approximation · Adjoint functors

Mathematics Subject Classification 16E05, 18G80, 16E65

1 Introduction

This paper is the result of a desire to approximate in a meaningful way totally acyclic complexes over a Noetherian ring \( R \) by simpler totally acyclic complexes, possibly even periodic ones. By definition, (nontrivial) totally acyclic complexes are (necessarily unbounded) exact complexes of finitely generated projective modules, whose dual complex is also exact. Nontrivial ones exist abundantly over any Gorenstein ring, and their behavior can be quite varied and unexpected (see, for example, [18].) Approximating such complexes by simpler ones was the motivation for our main theorem below. Our interest in this project was also motivated by recent work in [19] and [23] on approximations and adjoints in homotopy categories.
The homotopy category $K_{tac}(R)$ of totally acyclic $R$-complexes is a thick triangulated subcategory of the homotopy category of $R$-complexes. The main result of the paper is the establishment of an adjoint pair of functors which are triangle versions of the classical adjoint pair of functors between the module categories through change of rings.

**Theorem A** Assume that $\varphi : Q \to R$ is a surjective homomorphism of Noetherian rings such that $Q$ is Gorenstein, and such that $R$ as a $Q$-bimodule admits a finite resolution by modules which are projective as both left and right $Q$-modules. Then there exists an adjoint pair of triangle functors

$$K_{tac}(Q) \xleftarrow{S_\varphi} K_{tac}(R) \xrightarrow{T_\varphi}$$

As is the case in the classical setting, the functor $S_\varphi$ is simply the base change functor $R \otimes_Q -$. However, as nontrivial totally acyclic $R$-complexes are never totally acyclic $Q$-complexes when $R \neq Q$, obtaining the right adjoint $T_\varphi$ requires a modification of the forgetful functor. It is well known that adjoints of triangle functors are themselves triangulated (see, for example, [22, Lemma 5.3.6]), hence since $S_\varphi$ is triangulated, it follows immediately that so is $T_\varphi$. However, we find it instructive to give the direct arguments for $T_\varphi$. We remark also that the conditions of $Q$ being Gorenstein and the finiteness of the projective dimension of $R$ over $Q$ are essential to the existence of $S_\varphi$ and $T_\varphi$. Moreover, the existence of a finite bimodule resolution of $R$ by two-sided projective $Q$-modules is key to the adjunction. When $Q$ is a commutative local ring, then this condition amounts to saying that $R$ has finite projective dimension as a $Q$-module. We prove theorem A in Sect. 4 through explicit computations of the unit and counit natural transformations.

In Sect. 5 we discuss the approximations we sought. Our main result here is the following. **Theorem B** The isomorphism closure in $K_{tac}(R)$ of the image of $S_\varphi$ is functorially finite in $K_{tac}(R)$. In other words, both left and right approximations exist in $K_{tac}(R)$ by objects in the isomorphism closure of the image of $S_\varphi$.

That right approximations exist in $K_{tac}(R)$ by objects in the isomorphism closure of the image of $S_\varphi$ is an immediate consequence of Theorem A. To show that left approximations exist, we use the existence of right approximations and the duality properties inherent in $K_{tac}(R)$. We also give several examples illustrating aspects of Theorem B.

After an earlier version of this paper appeared on the arXiv, the paper [24] was published, with some similar results. The paper studies ring homomorphisms of Noetherian algebras, and the first part of its Theorem I, where an adjoint pair on the level of singularity categories is established, is analogous to our Theorem A.

## 2 Preliminaries

Let $R$ be an associative ring with unity. Unless otherwise stated, we assume all modules to be left modules. By an $R$-complex $C$ we mean a sequence of left $R$-module homomorphisms

$$C : \cdots \to C_{n+1} \xrightarrow{\partial^C_{n+1}} C_n \xrightarrow{\partial^C_n} C_{n-1} \to \cdots$$

graded homologically, so that $n$ is the homological degree of $C_n$. If $C$ is such a complex, then $\text{Hom}_R(C, R)$ is a complex of right $R$-modules via the canonical right action on each
Hom\(_R(C, R)\). We often regard \(R\)-modules as complexes concentrated in homological degree 0.

Given a complex \(C\), we denote by \(\Sigma C\) the shift of \(C\), where \((\Sigma C)_n = C_{n-1}\); one has the natural map \(\sigma : C \to \Sigma C\), where for \(x \in C_n\), the element \(\sigma (x)\) is the same element \(x\), but now has degree \(n + 1\) in \(\Sigma C\). This map \(\sigma\) has the obvious inverse \(\sigma^{-1} : \Sigma C \to C\). The differential of \(\Sigma C\) is taken to be \(\partial \Sigma C = -\sigma_{-2}\partial C\sigma_{-1}^{-1}\). For a morphism of complexes \(f : C \to D\) we have the induced morphism \(\Sigma f : \Sigma C \to \Sigma D\). We also have the inverse shift \(\Sigma^{-1}\), where \((\Sigma^{-1}C)_n = C_{n+1}\) and \(\partial_{n}^{-1} = - (\Sigma^{-1}\sigma)^{-1}\partial C\Sigma^{-1}\sigma\).

From this point on, unless stated to the contrary, we assume that \(R\) is a Noetherian ring (on both sides).

**Definition** Recall from [5] that an \(R\)-complex \(C\) of finitely generated projective modules is called *totally acyclic* if

\[
H(C) = 0 = H(\text{Hom}_R(C, R)).
\]

Note that if \(P\) is a finitely generated projective \(R\)-module and \(C\) is a totally acyclic \(R\)-complex, then \(H(\text{Hom}_R(C, P)) = 0\).

Recall that the homotopy category of \(R\)-complexes is a triangulated category with shift functor \(\Sigma\). We denote by \(K_{\text{fac}}(R)\) the subcategory of the homotopy category of \(R\)-complexes consisting of the totally acyclic \(R\)-complexes; the objects in \(K_{\text{fac}}(R)\) are the totally acyclic \(R\)-complexes, and the morphisms are homotopy equivalence classes of morphisms of \(R\)-complexes. This is a thick triangulated subcategory of the homotopy category. For a morphism \(f : C \to C'\) of \(R\)-complexes we write \([f]\) for its homotopy equivalence class. Thus for two morphisms \(f, g : C \to C'\) of \(R\)-complexes, one has \(f \sim g\) if and only if \([f] = [g]\).

The following two facts we use often in the rest of the paper. The proofs are standard, and left to the reader. Given an \(R\)-complex \(C\), we write from now on \(C^*\) for the \(R\)-complex \(\text{Hom}_R(C, R)\).

### 2.1 Extending homotopies

Suppose that \(s\) is an integer, \(C \in K_{\text{fac}}(R)\), \(C'\) is a complex of finitely generated projective \(R\)-modules and \(f : C \to C'\) is a morphism of \(R\)-complexes. If there exist maps \(h_n : C_n \to C'_{n+1}\) satisfying

\[
f_{n+1} = h_n \partial C_{n+1} + \partial C'_{n+2} h_{n+1}
\]

for all \(n > s\), then the \(h_n\) can be extended to a homotopy showing that \(f \sim 0\).

### 2.2 Extending morphisms

Suppose that \(s\) is an integer, \(C \in K_{\text{fac}}(R)\) and \(C'\) is a complex of finitely generated projective \(R\)-modules. If there exist maps \(f_n : C_n \to C'_n\) for \(n \geq s\) satisfying \(f_{n+1} \partial C = \partial C' f_n\) for all \(n > s\), then the \(f_n\) can be extended to a morphism of \(R\)-complexes \(f : C \to C'\). Any two such extensions are homotopic.

We now recall an important definition from [5].

**Definition** A complete resolution of a finitely generated \(R\)-module \(M\) is a diagram

\[
U \xrightarrow{\rho} P \xrightarrow{\pi} M
\]
such that $U \in \mathbf{K}_{\text{fic}}(R)$, $P$ is a projective resolution of $M$, $\rho$ is a morphism of $R$-complexes, and $\rho_n$ is bijective for all $n \gg 0$. We will often abuse terminology and call $U$ a complete resolution of $M$.

The following is [5, 5.3], and is key to defining our functor $T$. We include a different proof, using 2.1 and 2.2.

### 2.3 Comparison theorem

If $U \xrightarrow{\rho} P \xrightarrow{\pi} M$ and $U' \xrightarrow{\rho'} P' \xrightarrow{\pi'} M'$ are complete resolutions of finitely generated $R$-modules $M$ and $M'$, and $\mu : M \to M'$ is an $R$-module homomorphism, then there exists a unique up to homotopy morphism of $R$-complexes $\mu$ making the right-hand square of the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\rho} & P \\
\downarrow \mu & & \downarrow \pi \\
U' & \xrightarrow{\rho'} & P'
\end{array}
\begin{array}{ccc}
M \\
\downarrow \mu & & \downarrow M'
\end{array}
\]

commute, and for each choice of $\mu$ there exists a unique up to homotopy morphism $\hat{\mu}$ making the left-hand square commute up to homotopy. If two such $\mu$ are homotopic, then so are the respective $\hat{\mu}$. If $\mu = \text{Id}_M$, then $\mu$ and $\hat{\mu}$ are homotopy equivalences.

**Proof** The statement regarding $\mu$ is classical, so we first concern ourselves with the first statement regarding $\hat{\mu}$.

Suppose that the morphism $\mu$ has been chosen. Since $\rho_n$ and $\rho'_n$ are bijective for all $n \gg 0$, there obviously exist maps $\hat{\mu}_n : U_n \to U'_n$ such that $\hat{\mu}_{n-1} \partial_n U = \partial_n U' \hat{\mu}_n$ for all $n \gg 0$. Thus 2.2 says we obtain a unique up to homotopy morphism of $R$-complexes $\hat{\mu} : U \to U'$. Now since $\mu_n \rho_n = \rho'_n \hat{\mu}_n$ for all $n \gg 0$, 2.1 says that $\mu \rho \sim \rho' \hat{\mu}$, as desired.

Suppose that $\overline{\mu}_1 : P \to P'$ and $\overline{\mu}_2 : P \to P'$ are two morphisms such that $\overline{\mu}_1 \sim \overline{\mu}_2$. Since $\rho_n$ and $\rho'_n$ are bijective for all $n \gg 0$, we see that $\overline{\mu}_1$ and $\overline{\mu}_2$ are eventually homotopic. Therefore 2.1 shows that $\overline{\mu}_1 \sim \overline{\mu}_2$.

If $\mu = \text{Id}_M$, then one can also define morphisms $\overline{\mu} : P' \to P$ and $\overline{\mu}' : U' \to U$. Uniqueness up to homotopy then shows that $\overline{\mu} \mu' \sim \text{Id}_F$, $\overline{\mu} \overline{\mu}' \sim \text{Id}_F'$, $\mu' \hat{\mu} \sim \text{Id}_U$, and $\hat{\mu} \mu' \sim \text{Id}_U'$.

The final statement of 2.3 says that complete resolutions are uniquely defined up to homotopy equivalence.

One needs to know that complete resolutions exist. The following result follows from [5, Constructions 3.6 and 3.7]. Recall that a Noetherian (on both sides) ring is called Gorenstein if $R$ has finite injective dimension as a module over itself on both sides.

### 2.4 Existence of complete resolutions

Suppose that $R$ is Gorenstein, and let $M$ be a finitely generated $R$-module. Then for each projective resolution $P$ of $M$ there exists a complete resolution $U \xrightarrow{\rho} P \xrightarrow{\pi} M$ such that $\rho_n = \text{Id}_{\rho_n}$ for all $n \gg 0$; one may in addition choose $U$ such that $\rho_n$ is surjective for all $n$. 

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2.5 Duality

Suppose that \( Q \) and \( R \) are associative rings with unity, and \( \varphi : Q \to R \) is a surjective ring homomorphism. Let \( D \) be a complex of finitely generated projective left \( Q \)-modules. Then

\[
\text{Hom}_Q(D, Q) \otimes_Q R \quad \text{and} \quad \text{Hom}_R(R \otimes_Q D, R)
\]

are isomorphic as complexes of right \( R \)-modules. Similarly, if \( D \) is a complex of finitely generated projective right \( Q \)-modules, then

\[
R \otimes_Q \text{Hom}_Q(D, Q) \quad \text{and} \quad \text{Hom}_R(D \otimes_Q R, R)
\]

are isomorphic as complexes of left \( R \)-modules.

**Proof** We only prove the first isomorphism, as the second is similar. By Hom-tensor adjunction for complexes, and after making the canonical identification of \( R \)-\( R \) bimodules \( \text{Hom}_R(R, R) \) with \( R \), we immediately get that \( \text{Hom}_R(R \otimes_Q D, R) \) and \( \text{Hom}_Q(D, R) \) are isomorphic. Therefore it suffices to prove that \( \text{Hom}_Q(D, Q) \otimes_Q R \) and \( \text{Hom}_Q(D, R) \) are isomorphic. For this we recall the canonical tensor evaluation map of complexes (see, for example, [10, (A.2.10)])

\[
\text{Hom}_Q(D, Q) \otimes_Q R \to \text{Hom}_Q(D, Q \otimes_Q R)
\]

Since \( D_n \) is finitely generated projective for each \( n \), this map is an isomorphism in each degree. Finally, the natural identification \( \text{Hom}_Q(D, Q \otimes_Q R) \cong \text{Hom}_Q(D, R) \) finishes the proof. \( \square \)

3 The ‘forgetful’ triangle functor

From now on we assume that \( Q \) is a Gorenstein ring. We further assume that \( \varphi : Q \to R \) a surjective ring homomorphism such that \( R \) has a finite projective resolution over \( Q \).

The main objective of this section is to define the ‘forgetful’ functor

\[
T = T_\varphi : \text{K}_{\text{tac}}(R) \to \text{K}_{\text{tac}}(Q)
\]

and verify that it is a triangle functor. The definition of \( T \) is as follows.

**Definition** Let \( C \in \text{K}_{\text{tac}}(R) \). Then \( TC \in \text{K}_{\text{tac}}(Q) \) is a complete resolution of \( \text{Im} \delta_0^C \) over \( Q \) (which exists by 2.4 and is uniquely defined by 2.3.) Given a morphism \([f] : C \to C' \) in \( \text{K}_{\text{tac}}(R) \), we have the \( Q \)-module homomorphism \( \mu : \text{Im} \delta_0^C \to \text{Im} \delta_0^{C'} \) induced by the morphism \( f : C \to C' \) of \( R \)-complexes. Then \( T[f] : TC \to TC' \) is the homotopy equivalence class \([\hat{\mu}]\) of the comparison map \( \hat{\mu} : TC \to TC' \) between complete resolutions (which is uniquely determined by \( \mu \), by 2.3.)

The main result of this section is the following.

**Theorem 3.1** \( T : \text{K}_{\text{tac}}(R) \to \text{K}_{\text{tac}}(Q) \) is a triangle functor.

**Proof** We claim that \( T \) is the composition of three well-known or obvious triangle functors

\[
\begin{array}{cccccc}
\text{K}_{\text{tac}}(R) & \xrightarrow{\Omega_0} & TR(R) & \xrightarrow{\Phi} & D^b_{\text{sg}}(Q) & \xrightarrow{\beta^{-1}} & \text{K}_{\text{tac}}(Q) \\
\downarrow{T} & & & & & & \\
\end{array}
\]
which we now describe. The two intermediate categories are the stable category of totally reflexive $R$-modules $TR(R)$ and the singularity category of Buchweitz $D^b_{\mathrm{sg}}(Q) = D^b(Q)/\text{perf}(Q)$ of $Q$, consisting of the bounded derived category modulo the perfect complexes.

The triangle equivalence given by the triangle functor $\Omega_0$ is well-known. For example, it follows directly from [20, Proposition 7.2], applied to the abelian category $A = (\text{mod } R)^{\text{op}}$. It also already appears in [9] in the case where $R$ is Gorenstein (for then totally reflexive modules are exactly the maximal Cohen–Macaulay modules.) This functor is defined by

$$\Omega_0(C) = \text{Im} \delta^C_0 \in TR(R), \text{ for } C \in K_{\mathrm{tac}}(R).$$

The definition of $\Omega_0$ on morphisms is then obvious.

The triangle functor $\beta$ giving the triangle equivalence $K_{\mathrm{tac}}(Q) \xrightarrow{\beta} D^b_{\mathrm{sg}}(Q)$ is established in [9, Theorem 4.4.1], as $Q$ is Gorenstein. It assigns to $D \in K_{\mathrm{tac}}(Q)$ the hard truncation $D_{\geq 0}$ of $D$ to the right of homological degree 0. By [22, Lemma 5.3.6] we have that the quasi-inverse $\beta^{-1}$ of $\beta$ is automatically a triangle functor. Moreover, as specified in [9, Section 4.5], its action, for example, on a finitely generated projective $R$-module $M$ concentrated in degree 0 in $D^b_{\mathrm{sg}}(Q)$ gives a totally acyclic complex $U \in K_{\mathrm{tac}}(Q)$ which is part of a complete resolution $U \to P \to M$ of $M$ over $Q$ (which exists since $Q$ is Gorenstein.) The action of $\beta^{-1}$ on morphisms of such objects is then is clear.

Finally, one has an obvious forgetful functor $TR(R) \to D^b(Q)$ which takes a totally reflexive $R$-module to itself, now viewed as a $Q$-complex concentrated in degree 0. Now since $R$ has finite projective dimension, this functor maps finitely generated projective $R$-modules into $D^b_{\text{perf}}(Q)$. Therefore this forgetful functor induces the functor $\Phi : TR(R) \to D^b_{\mathrm{sg}}(Q)$ from the diagram above. The action of $\Phi$ on morphisms is obvious.

One now sees that $T = \beta^{-1} \circ \Phi \circ \Omega_0$, both on objects and morphisms.

\section{4 Adjunction}

We continue to assume that $\varphi : Q \to R$ is a surjective ring homomorphism with $Q$ Gorenstein. Moreover, we now assume that $R$ as a $Q$-bimodule admits a finite resolution by modules which are projective as both left and right $Q$-modules. The goal of this section is to compare $K_{\mathrm{tac}}(R)$ with $K_{\mathrm{tac}}(Q)$ by means of an adjoint pair of triangle functors.

The descension functor $S = S_{\varphi} : K_{\mathrm{tac}}(Q) \to K_{\mathrm{tac}}(R)$ is easy—and valid even without assuming $Q$ is Gorenstein; it is defined by

$$SC = R \otimes_Q C \text{ and } S[f] = [R \otimes_Q f]$$

for $C$ an object and $[f]$ a morphism in $K_{\mathrm{tac}}(Q)$. This is a triangle functor due in part to 2.5. Indeed, if $C$ is acyclic complex of projective $Q$-modules, then $R \otimes_Q C$ is an acyclic complex of projective $R$-modules (since $R$ has a finite resolution by projective right $Q$-modules), and $\text{Hom}_Q(C, Q)$ being acyclic implies $\text{Hom}_Q(C, Q) \otimes_Q R$ is acyclic. Hence $\text{Hom}_R(R \otimes_Q C, R)$ is acyclic by 2.5. It is easy to see that $S$ takes homotopic morphisms of complexes to homotopic morphisms of complexes, commutes with shifts and takes distinguished triangles to distinguished triangles.

The ascension functor $T = T_{\varphi} : K_{\mathrm{tac}}(R) \to K_{\mathrm{tac}}(Q)$ is the functor defined in Sect. 3. Our main result for this section is the following.
Theorem 4.1 The triangle functors $S$ and $T$ form an adjoint pair, that is, they satisfy the following property: for all $C \in \mathcal{K}_{\text{tac}}(R)$ and $D \in \mathcal{K}_{\text{tac}}(Q)$ there exist a bijection

$$\text{Hom}_{\mathcal{K}_{\text{tac}}(Q)}(D, TC) \rightarrow \text{Hom}_{\mathcal{K}_{\text{tac}}(R)}(SD, C)$$

which is natural in each variable.

In the discussion below, we fix a finite $Q$-bimodule resolution $K$ of $R$, of length $c$, by modules that are projective $Q$-modules on the left and on the right; we assume, without loss of generality, that $K_0 = Q$. Before engaging the proof, we observe that for $D \in \mathcal{K}_{\text{tac}}(Q)$ one has

$$TS D \simeq K \otimes_Q D$$

Indeed, since $K$ is a finite bimodule resolution of $R$ by one-sided projective $Q$-modules, one has that $K \otimes_Q D \geq 0$ is a complex of projective $Q$-modules. It is exact (except in degree 0) by the finiteness of $K$. Thus we see that $K \otimes Q D \geq 0$ is a projective resolution of $\text{Im} \partial^0_D \cong R \otimes_Q \text{Im} \partial^0 D$ over $Q$. The assertion is now clear.

**Proof** In order to prove the theorem we define natural transformations

$$\eta : \text{Id } \mathcal{K}_{\text{tac}}(Q) \rightarrow TS$$

and

$$\epsilon : ST \rightarrow \text{Id } \mathcal{K}_{\text{tac}}(R)$$

—the unit and counit, respectively, of the adjunction—as follows. For $D \in \mathcal{K}_{\text{tac}}(Q)$ define $\eta_D : D \rightarrow TS D$ to be the morphism of complexes embedding $D_n$ into the first summand of $TS D_n = \bigoplus_{i=0}^c K_i \otimes_Q D_{n-i}$ via $x \mapsto 1 \otimes x$ for all $n$. And for $C \in \mathcal{K}_{\text{tac}}(R)$ define $\epsilon_C : STC \rightarrow C$ to be the morphism of complexes induced by the comparison map $F \rightarrow C \geq 0$, where $F$ is a projective resolution of $\text{Im} \partial^0_C$ over $Q$. It follows from 2.2 and 2.1 that $\eta$ and $\epsilon$ are natural in their arguments.

We just need to show that

$$T \epsilon_C \circ \eta TC \sim \text{Id } TC \quad \text{and} \quad \epsilon SD \circ S \eta D \sim \text{Id } SD$$

First we discuss the map $T \epsilon_C$. By definition we have the morphism of complexes

$$\cdots \rightarrow TC_1 \xrightarrow{\partial^1_{TC}} TC_0 \xrightarrow{p} \text{Im} \partial^0_{TC} \rightarrow 0$$

$$\cdots \rightarrow F_1 \xrightarrow{\partial^1_F} F_0 \xrightarrow{p'} \text{Im} \partial^0_C \rightarrow 0$$

where $F$ is a projective resolution of $\text{Im} \partial^0_C$ over $Q$; by 2.4 we can assume that $\rho_n = \text{Id } TC_n$ for all $n \gg 0$. Consider the induced map $\hat{\rho} : R \otimes_Q \text{Im} \partial^0_{TC} \rightarrow \text{Im} \partial^0_C$ where $\hat{\rho}(r \otimes a) = r \bar{\partial}(a)$. By lifting $\hat{\rho}$ one summand of $(K \otimes TC)_n$ at a time, for $n = 0, 1, \ldots$, we may achieve a morphism of complexes

$$\cdots \rightarrow (K \otimes TC)_1 \xrightarrow{\partial^{K \otimes TC}_{\geq 0}} K_0 \otimes_Q TC_0 \xrightarrow{\tau \otimes p} R \otimes_Q \text{Im} \partial^0_{TC} \rightarrow 0$$

$$\cdots \rightarrow F_1 \xrightarrow{\partial^1_F} F_0 \xrightarrow{p'} \text{Im} \partial^0_C \rightarrow 0$$
written succinctly as \( u : K \otimes_Q TC_{\geq 0} \to F \), the former complex being a \( Q \)-free resolution of \( R \otimes_K \text{Im} \partial^n_C \), such that \( u_n(x \otimes a) = x \rho_n(a) \) for \( x \otimes a \in K_0 \otimes_Q TC_n \). It follows that we may achieve the morphism of complexes \( T \epsilon_C : TSTC \to TC \) satisfying \( (T \epsilon_C)_n(x \otimes a) = xa \) for \( x \otimes a \in K_0 \otimes_Q TC_n \), and all \( n \gg 0 \). We also have the natural embedding \( \eta_{TC} : TC \to TSTC \) with \( (\eta_{TC})_n(a) = 1 \otimes a \in K_0 \otimes_Q TC_n \) for all \( a \in TC_n \). Thus we have shown that \( (T \epsilon_C)_n \circ (\eta_{TC})_n = \text{Id}_{TC_n} \) for all \( n \gg 0 \). It follows from 2.2 that \( T \epsilon_C \circ \eta_{TC} \sim \text{Id}_{TC} \).

The morphism \( \tilde{\eta}_D : SD \to STSD \) embeds \( SD_n \) into the first component of \( STSD_n \simeq \bigoplus_{i=0}^n R \otimes_Q (K_i \otimes_Q D_{n-i}) \) for all \( n \). And the morphism \( \epsilon_D : STSD \to SD \) takes the first component of \( STSD \simeq \bigoplus_{i=0}^n R \otimes_Q (K_i \otimes_Q D_{n-i}) \) to \( SD_n \) for all \( n \in \mathbb{Z} \). Thus we have \( \epsilon_D \circ \tilde{\eta}_D \sim \text{Id}_{SD} \).

\[ \text{Definition 5.1} \]

\textbf{5 Approximations of totally acyclic complexes}

Our main application of Theorem 4.1 is a resulting notion of approximation in the homotopy category of totally acyclic complexes. We now recall the notion of approximation we use, due to Auslander and Smalø [4], and independently, Enochs [16]. Let \( \mathcal{X} \) be a full subcategory of a category \( C \). Then a \textit{right} \( \mathcal{X} \)-\textit{approximation} of \( C \in C \) is a morphism \( X \xrightarrow{\epsilon} C \), with \( X \in \mathcal{X} \), such that for all objects \( Y \in \mathcal{X} \), the sequence

\[ \text{Hom}_C(Y, X) \xrightarrow{\text{Hom}(Y, \epsilon)} \text{Hom}_C(Y, C) \to 0 \]

is exact. Dually, one has the concept of left \( \mathcal{X} \)-approximations. Specifically, a morphism \( C \xrightarrow{\mu} X \), with \( X \in \mathcal{X} \), is called a \textit{left} \( \mathcal{X} \)-\textit{approximation} of \( C \in C \) if for all objects \( Y \in \mathcal{X} \), the sequence

\[ \text{Hom}_C(X, Y) \xrightarrow{\text{Hom}(\mu, Y)} \text{Hom}_C(C, Y) \to 0 \]

is exact. The full subcategory \( \mathcal{X} \) is called \textit{functorially finite} in \( C \) if for every object \( C \in C \), there exists a right \( \mathcal{X} \)-approximation of \( C \) and a left \( \mathcal{X} \)-approximation of \( C \).

We let \( SK_{\text{tac}}(Q) \) denote the isomorphism closure in \( K_{\text{tac}}(R) \) of \( \{ R \otimes_Q D \mid D \in K_{\text{tac}}(Q) \} \). Our main application of Theorem 4.1 is the following.

\textbf{Theorem 5.1} \quad \text{\( SK_{\text{tac}}(Q) \) is functorially finite in \( K_{\text{tac}}(R) \).}

\textbf{Proof} \quad That right \( SK_{\text{tac}}(Q) \)-approximations exist in \( K_{\text{tac}}(R) \) follows immediately from Theorem 4.1: it is well known that every adjoin pair of functors gives rise to one-sided approximations (see for example [3, Proposition 1.1]). It is instructive to repeat the argument in the current context: the morphism \( [\epsilon_C] : STC \to C \) is a right approximation in \( K_{\text{tac}}(R) \). Indeed, if \( [f] : SD \to C \) is any morphism in \( K_{\text{tac}}(R) \) with \( D \in K_{\text{tac}}(Q) \), then from the natural transformation \( \epsilon : ST \to \text{Id}_{K_{\text{tac}}(R)} \) we have the equality \( [\epsilon_C] \circ ST[f] = [f] \circ [\epsilon_D] \).

Composing on the right with \( S[\eta_D] \) we obtain \( [\epsilon_C] \circ ST[f] \circ S[\eta_D] = [f] \), and thus \( ST[f] \circ S[\eta_D] : SD \to STC \) is the morphism we seek.

Now we show that every \( C \in K_{\text{tac}}(R) \) has a left approximation. This can be done by simply dualizing a right approximation. For this we will use several times the isomorphisms from 2.5. For \( C \in K_{\text{tac}}(R) \) we denote by \( C^* \) the dual complex \( \text{Hom}_R(C, R) \) of right \( R \)-modules, which we consider as a complex of left \( R \)-modules over the opposite ring \( R^{\text{op}} \), so that \( C^* \in K_{\text{tac}}(R^{\text{op}}) \). We do the same over \( Q \), so that for \( D \in K_{\text{tac}}(Q) \) we have \( D^* = \text{Hom}_Q(D, Q) \in K_{\text{tac}}(Q^{\text{op}}) \). For a morphism of complexes \( f : C \to C', C, C' \in K_{\text{tac}}(R) \), we
denote by \( f^* \) the dual morphism \( \text{Hom}_R(f, R) \). We let \( S^\text{op} \) and \( T^\text{op} \) denote the corresponding adjoint pair of functors between \( K_{\text{tac}}(R^\text{op}) \) and \( K_{\text{tac}}(Q^\text{op}) \).

We have the right approximation \( [\epsilon_{C^*}] : S^\text{op}T^\text{op}C^* \to C^* \) of \( C^* \). The claim is that \( [\epsilon_{C^*}] : C \cong C^* \to (S^\text{op}T^\text{op}C^*)^* \) is a left approximation of \( C \). Note that the target of \( \epsilon_{C^*} \) is in \( SK_{\text{tac}}(Q) \) by 2.5, that is, by the second isomorphism of 2.5 we have \( (S^\text{op}T^\text{op}C^*)^* \cong S((T^\text{op}C^*)^*) \). Now let \( E \in K_{\text{tac}}(Q) \) and \( f : C \to SE \) be a morphism in \( K_{\text{tac}}(R) \). Then we have the morphism \( f^* : (SE)^* \to C^* \), with \( (SE)^* \cong S^\text{op}E^* \) in \( S^\text{op}K_{\text{tac}}(Q^\text{op}) \). Therefore from the right approximation we have that \( f^* \sim \epsilon_{C^*}g \) for some morphism \( g : (SE)^* \to S^\text{op}T^\text{op}C^* \). Dualizing back we have that \( f \sim g^*\epsilon_{C^*} \), which is what we needed to show. \( \square \)

Motivated by results along the lines of [23, Proposition 1.4], we ask the following:

**Question 5.2** When is \( SK_{\text{tac}}(Q) \) a thick subcategory of \( K_{\text{tac}}(R) \)?

The following example from S. Lindokken shows that this is not always the case.

**Example 5.3** ([21]). Let \( Q = k[[x, y, z]]/(x^2 + yz) \) and \( R = Q/(y, z) \cong k[[x]]/(x^2) \), where \( k \) is an algebraically closed field. In this case \( K_{\text{tac}}(Q) \) has precisely one non-zero indecomposable element, namely the complex

\[
D : \cdots \to Q^2 \xrightarrow{(x - y) \ y \ z \ x} Q^2 \xrightarrow{(x \ y) \ -z \ x} Q^2 \xrightarrow{(x - y) \ z \ x} Q^2 \to \cdots
\]

Thus the complex \( SD \) has the summand \( \cdots \to R \xrightarrow{x} R \xrightarrow{x} R \to \cdots \), which is not in \( SK_{\text{tac}}(Q) \).

We illustrate Theorem 5.1 with an example.

**Example 5.4** Let \( R = k[x, y]/(x^2, y^2) \), and \( C \) be the totally acyclic \( R \)-complex with \( \text{Im}\delta_0^C = Rx y \cong k \):

\[
C : \cdots \to R^3 \xrightarrow{(x \ 0 \ -y) \ 0 \ y \ x} R^2 \xrightarrow{(x \ y) \ y} R \xrightarrow{(y)} R^2 \xrightarrow{(x \ y) \ y} R^2 \xrightarrow{(x \ y) \ y} R^2 \to \cdots
\]

Then a free resolution of \( \text{Im}\delta_0^C \) over \( Q = k[x, y]/(x^2) \) is given by

\[
F : \cdots \to Q^2 \xrightarrow{(x \ y) \ 0 \ x} Q^2 \xrightarrow{(x \ y) \ 0 \ x} Q^2 \xrightarrow{(x \ y) \ 0 \ x} Q^2 \xrightarrow{(x \ y) \ 0 \ x} Q \to 0
\]

The right approximation \( \epsilon_C : STC \to C \) takes the form

\[
\cdots \xrightarrow{R^2} \xrightarrow{R^2} \xrightarrow{R^2} \xrightarrow{R^2} \xrightarrow{R^2} \cdots
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\]
Since $C$ is self-dual in this example, that is $C \cong \Sigma^{-1}(C^*)$, the left approximation $[\epsilon_C^*] : C \to (STC)^*$ takes the form

$$
\cdots \longrightarrow R^3 \xrightarrow{(x\ 0\ -y\ 0)\ y\ x} R^2 \xrightarrow{(x\ y)\ 0\ y\ x} R \xrightarrow{(xy)\ 0\ 0\ y\ x} R \xrightarrow{(y)\ 0\ 0\ 0\ y\ x} R^2 \longrightarrow \cdots
$$

$$
\cdots \longrightarrow R^2 \xrightarrow{(y\ 0\ 0\ 0\ 0)} R^2 \xrightarrow{(y)\ 0\ 0\ 0\ 0} R \xrightarrow{(1)\ 0\ 0\ 0\ 0} R^2 \longrightarrow \cdots
$$

$$
\cdots \longrightarrow R^2 \xrightarrow{(x\ 0\ 0\ y\ x)} R^2 \xrightarrow{(x\ 0\ 0\ y\ x)} R^2 \xrightarrow{(x\ 0\ 0\ y\ x)} R^2 \longrightarrow \cdots
$$

Approximations may be trivial, in particular, when the projective dimension of $\text{Im} \partial_0^C$ is finite over $Q$, as is the case in the next example.

**Example 5.5** Let $R = k[x, y]/(x^2, y^2)$ and $C$ the totally acyclic $R$-complex with $\text{Im} \partial_0^C = R$: $C : \cdots \longrightarrow R \xrightarrow{(y)} R \xrightarrow{(y)} R \xrightarrow{(y)} R \longrightarrow \cdots$

Then for $Q = k[x, y]/(x^2)$, $\text{pd}_Q \text{Im} \partial_0^C < \infty$ and the approximation is $[\epsilon_C] : 0 \to C$.

Recall (from [4], for example) that a morphism $X \xrightarrow{\epsilon} C$ is called right minimal if for every morphism $X \xrightarrow{f} X$ such that $\epsilon f = \epsilon$, we have that $f$ is an isomorphism. We show that the right approximation $[\epsilon_C] : STC \to C$ may or may not be right minimal.

**Proposition 5.6** Suppose that $D \in K_{\text{tac}}(Q)$. Then $[\epsilon_{SD}] : STSD \to SD$ is not a minimal approximation whenever $D \neq 0$.

**Proof** Let $K$ be a bimodule $Q$-free resolution of $R$. As described in the proof of 4.1, $\epsilon_{SD} : STSD \to SD$ takes the first component of $STSD_n \cong \bigoplus_{i=0}^n R \otimes_Q (K_i \otimes_Q D_{n-i})$ to $SD_n$ for all $n \in \mathbb{Z}$. Thus taking as $[f] : STSD \to STSD$ the morphism sending $R \otimes_Q (K_0 \otimes_Q D_n)$ to itself and all other summands to zero, for each $n$, we have $[\epsilon_{SD}] \circ [f] = [\epsilon_{SD}]$ and $[f]$ is not an isomorphism in $K_{\text{tac}}(R)$ when $D \neq 0$. 

**Example 5.7** Let $R = k[x, y]/(x^2, y^2)$ and $C$ the totally acyclic complex

$C : \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$

Then $M = \text{Im} \partial_0^C = Rx$ and a free resolution of $M$ over $Q = k[x, y]/(x^2)$ is given by

$$
\cdots \longrightarrow Q^2 \xrightarrow{(x\ -y^2)\ 0\ x} Q^2 \xrightarrow{(x\ y^2)\ 0\ x} Q^2 \xrightarrow{(x\ -y^2)\ 0\ x} Q^2 \longrightarrow Q \longrightarrow 0
$$

Thus $STC$ takes the form

$$
\cdots \longrightarrow R^2 \xrightarrow{(x\ 0)\ 0\ x} R^2 \xrightarrow{(x\ 0)\ 0\ x} R^2 \xrightarrow{(x\ 0)\ 0\ x} R^2 \longrightarrow \cdots
$$
and $\epsilon_C : STC \to C$ is given by $(\epsilon_C)_n = (1\ 0)$ for all $n$. This is not a minimal right approximation. Indeed, consider the morphism $f : STC \to STC$ given by $f_n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then one has $\epsilon_C f = \epsilon_C$ and $f$ is not a homotopy equivalence.

We next state a few results for later reference. They have to do with compositions of approximations, in two different senses.

**Proposition 5.8** Consider a sequence of finite local ring homomorphisms $Q \xrightarrow{\phi} R' \xrightarrow{\psi} R$ such that $Q$ and $R'$ are Gorenstein, $pd_Q R' < \infty$, and $pd_{R'} R < \infty$. Then $S_{\psi \phi}$ and $T_{\psi \phi}$ are naturally isomorphic to $S_{\psi} S_{\phi}$ and $T_{\psi} T_{\phi}$, respectively.

**Proof** This follows from the fact that the assertion is clear for the $S$ functors, and from uniqueness of adjoints. \qed

4.9. Resolutions. Upon computing the right approximation $[\epsilon_C] : STC \to C$, one may iterate this process. Indeed, complete $[\epsilon_C]$ to a triangle in $\mathbb{K}_{\text{tac}}(R)$ and rotate it to obtain

$$\Sigma^{-1}\text{cone}([\epsilon_C]) \to STC \to C \to .$$

Now compute a right approximation of $\Sigma^{-1}\text{cone}([\epsilon_C])$, and repeat. One then obtains a sequence of maps in $\mathbb{K}_{\text{tac}}(R)$:

$$B : \cdots \to B_3 \to B_2 \to B_1 \to B_0 \to C$$

where $B_0$ is a right approximation of $C$, $B_1$ is a right approximation of $\Sigma^{-1}\text{cone}(B_0 \to C)$, etc. Note that since composing two consecutive maps in an exact triangle is the zero map, one has that the same holds for the maps in $B$.

6 Maximal Cohen–Macaulay modules

In this section we relate our functors $S$ and $T$ to other well-known triangle functors. We maintain the assumption that $Q$ is a Gorenstein ring and that $\phi : Q \to R$ is a surjective ring homomorphism such that $R$ admits a finite $Q$-bimodule resolution by one-sided projective $Q$-modules.

Recall that (in the current context) a finitely generated $Q$-module $M$ is called maximal Cohen–Macaulay (MCM for short) if $\text{Ext}^n_Q(M, Q) = 0$ for all $n > 0$. It is well-known that the stable module category of maximal Cohen–Macaulay $Q$-modules $\text{MCM}(Q)$ is a triangulated category, and is equivalent as such to $\mathbb{K}_{\text{tac}}(Q)$. Indeed, this was first shown by Buchweitz in [9], the functor being, in his notation, $\Omega_0 : \mathbb{K}_{\text{tac}}(Q) \to \text{MCM}(Q)$, where $\Omega_0(C) = \text{Im} \delta^C_0$ for $C \in \mathbb{K}_{\text{tac}}(Q)$ and $\Omega_0([f]) = \mu$ the induced map $\mu : \text{Im} \delta^C_0 \to \text{Im} \delta^{C'}_0$ for a morphism $[f] : C \to C'$.

Also shown in [9, Lemma 4.2.2] is that a $Q$-module $M$ is MCM if and only if its dual $M^* = \text{Hom}_Q(M, Q)$ is MCM as a $Q^{\text{op}}$-module. Furthermore, a MCM $Q$-module $M$ is reflexive, meaning that the natural biduality map $M \to M^{**}$ is an isomorphism. One recognizes that these properties are precisely those defining modules of $G$-dimension zero in [1, Proposition 3.8], also known as totally reflexive modules, which is the terminology introduced in [5, Section 2]. It turns out that the subcategory of the category of left $R$-modules consisting of the
totally reflexive \( R \)-modules is a Frobenius category (see, for example, [13, Proposition 2.2]). Just as in the case of MCM modules over a Gorenstein ring, the projective-injective objects in the subcategory are just the projective objects in the module category. Hence one may form the stable category of totally reflexive modules \( TR(R) \), which is a triangulated category. It is well-known that same functor \( \Omega_0 \) yielding the equivalence of triangulated categories \( K_{tac}(Q) \rightarrow MCM(Q) \) from [9] also yields an equivalence of triangulated categories \( K_{tac}(R) \rightarrow TR(R) \). Thus our functors \( S = S_\phi \) and \( T = T_\phi \) induce an adjoint pair of functors \( S \) and \( T \) which make the following diagram commute.

\[
\begin{array}{ccc}
K_{tac}(Q) & \xrightarrow{\Omega_0} & MCM(Q) \\
S & \uparrow T & S \uparrow L \\
K_{tac}(R) & \xrightarrow{\Omega_0} & TR(R)
\end{array}
\]

The functor \( S \) still simply the base-change functor, \( SN = R \otimes_Q N \) for \( N \in MCM(Q) \). The functor \( T \) is more interesting. For \( M \in TR(R) \), \( T(M) \) is the essential MCM approximation of \( M \), according to [2].

To put the functors \( S \) and \( T \) in further perspective, Let \( D_{sg}^b(Q) \) denote the singularity category of \( R \), which is the Verdier quotient of the bounded derived category \( D^b(Q) \) by the thick subcategory of perfect complexes \( \text{perf}(Q) \). Similarly, we let \( D_{sg}^b(R) \) denote the singularity category \( D^b(R)/\text{perf}(R) \) of \( R \). The top right horizontal arrow is the equivalence of categories proved in [9]; the functor \( \beta \) is defined by hard truncation of a totally acyclic complex to the right of zero. It is proved in [8] that the same functor is fully faithfully in general, that is, when the ring is not necessarily Gorenstein, which explains the middle right arrow.

\[
\begin{array}{ccc}
MCM(Q) & \xleftarrow{\Omega_0} & K_{tac}(Q) \\
\rotatebox[origin=c]{90}{S} & \uss \rotatebox[origin=c]{90}{T} & \rotatebox[origin=c]{90}{S} \uss \rotatebox[origin=c]{90}{L} \\
TR(R) & \xleftarrow{\Omega_0} & K_{tac}(R) \\
\rotatebox[origin=c]{90}{S} & \uss \rotatebox[origin=c]{90}{T} & \rotatebox[origin=c]{90}{S} \uss \rotatebox[origin=c]{90}{L} \\
\rotatebox[origin=c]{90}{D_{sg}^b(R)_{fGP}} & \xleftarrow{\Omega_0} & D_{sg}^b(R) \\
\rotatebox[origin=c]{90}{\sigma} & \uss \rotatebox[origin=c]{90}{\tau} & \rotatebox[origin=c]{90}{\sigma} \uss \rotatebox[origin=c]{90}{\tau}
\end{array}
\]

In fact, in [28, Theorem 3.4 (restricted to mod \( R \))] the essential image of \( \beta \) is determined; this is the triangulated full subcategory \( D_{sg}^b(R)_{fGP} = D_{sg}^b(R)/\text{perf}(R) \) of \( D_{sg}^b(R) \) consisting of complexes of finite Gorenstein projective dimension modulo the perfect complexes. Thus the diagonal arrow is an equivalence. Finally, the ascension functor \( \sigma \) is the derived base-change functor \( R \otimes_Q^L \), and the ascension functor \( \tau \) is the simply the forgetful functor; \( \sigma \) and \( \tau \) form an adjoint pair.

### 7 Relative codimension one approximations

In this section we assume that \( Q \) is a commutative local Gorenstein ring, and that \( \varphi : Q \rightarrow R \) is a map of local rings. We study the approximations \( [\epsilon_C] : STC \rightarrow \tilde{C} \) in the case of relative
codimension one, and give applications to Betti numbers. Later, we further assume that \( Q \) is a hypersurface ring, and study the approximations in this case.

We start with a concrete description of the approximation \( [\epsilon_C] : STC \to C \) in the case of relative codimension one. Recall the following construction from [14]

Let \( f \) be a non-zerodivisor contained in the maximal ideal \( n \) of \( Q, R = Q/(f) \), and \( C \in K_{\text{tac}}(R) \). Choose a sequence of free \( Q \)-modules \( \tilde{C}_n \) and maps \( \tilde{\partial}_n^C \) between them such that \( C \) and \( \tilde{C} \otimes Q R \) are isomorphic as \( R \)-complexes. One can then write for all \( n \)

\[
\tilde{\partial}_{n-1}^C \tilde{\partial}_n^C = f \cdot \tilde{t}_n
\]

for maps \( \tilde{t}_n : \tilde{C}_n \to \tilde{C}_{n-2} \).

Letting \( t_n = \tilde{t}_n \otimes_Q R \), Eisenbud [14] shows that \( t = \{ t_n \} \) defines a morphism of complexes \( t : C \to \Sigma^2 C \).

The following is from [26], see also [6].

**Theorem 6.2.** Let \( f \) be a non-zerodivisor contained in the maximal ideal of \( Q, R = Q/(f) \), and \( C \in K_{\text{tac}}(R) \). Set \( M = \text{Im} \partial_0^C \). Then the sequence

\[
\cdots \to \tilde{C}_3 \oplus \tilde{C}_2 \xrightarrow{\left( \begin{array}{cc} \tilde{\partial}_3^C & f \\ -\tilde{t}_3 & -\tilde{\partial}_2^C \end{array} \right)} \tilde{C}_2 \oplus \tilde{C}_1 \xrightarrow{\left( \begin{array}{cc} \tilde{\partial}_2^C & f \\ -\tilde{t}_2 & -\tilde{\partial}_1^C \end{array} \right)} \tilde{C}_1 \oplus \tilde{C}_0 \xrightarrow{\left( \begin{array}{c} \tilde{\partial}_1^C \\ f \end{array} \right)} \tilde{C}_0 \to \cdot \cdot \cdot
\]

is a \( Q \)-free resolution of \( M \).

**Corollary 6.3.** Let \( f \) be a non-zerodivisor contained in the maximal ideal of \( Q, R = Q/(f) \), and \( C \in K_{\text{tac}}(R) \). Then \( STC \) can be taken to be \( \Sigma^{-1} \text{cone}(t) \), and the approximation map

\[
[\epsilon_C] : \Sigma^{-1} \text{cone}(t) \to C
\]

the natural projection.

**Proof** We see from Thereom 6.2 that \( TC \) is given by

\[
\cdots \tilde{C}_{n+1} \oplus \tilde{C}_n \xrightarrow{\left( \begin{array}{cc} \tilde{\partial}_{n+1}^C & f \\ -\tilde{t}_{n+1} & -\tilde{\partial}_n^C \end{array} \right)} \tilde{C}_n \oplus \tilde{C}_{n-1} \xrightarrow{\left( \begin{array}{cc} \tilde{\partial}_n^C & f \\ -\tilde{t}_n & -\tilde{\partial}_{n-1}^C \end{array} \right)} \tilde{C}_{n-1} \oplus \tilde{C}_{n-2} \cdots
\]

and the approximation \( [\epsilon_C] : STC \to C \) takes the form

\[
\cdots C_{n+1} \oplus C_n \xrightarrow{\left( \begin{array}{cc} \partial_{n+1}^C & 0 \\ -\partial_n^C & -\partial_{n-1}^C \end{array} \right)} C_n \oplus C_{n-1} \xrightarrow{\left( \begin{array}{cc} \partial_n^C & 0 \\ -\partial_{n-1} \end{array} \right)} C_{n-1} \oplus C_{n-2} \cdots
\]

\[
\cdots C_{n+1} \xrightarrow{\partial_{n+1}^C} C_n \xrightarrow{\partial_n^C} C_{n-1} \cdots
\]

where the vertical maps are the natural projections. Finally, one recognizes that the top complex is \( \Sigma^{-1} \text{cone}(t) \).
Corollary 6.4. Let \( f \) be a non-zerodivisor contained in the maximal ideal of \( Q, R = Q/(f) \), and \( C \in \mathcal{K}_{\text{tac}}(R) \). If \( [\epsilon_C] : STC \to C \) is the right approximation of \( C \), then \( \text{cone}([\epsilon_C]) \) is isomorphic to \( \Sigma^2 C \) in \( \mathcal{K}_{\text{tac}}(R) \), and we have the distinguished triangle

\[
STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to \Sigma STC
\]

**Proof** Let \( t : C \to \Sigma^2 C \) be the morphism of complexes defined in 7.1, and consider the corresponding distinguished triangle

\[
C \xrightarrow{[t]} \Sigma^2 C \to \text{cone}([t]) \to \Sigma C
\]

Rotating this triangle we have

\[
\Sigma^{-1}\text{cone}([t]) \to C \xrightarrow{[t]} \Sigma^2 C \to \text{cone}([t])
\]

which by 6.3 is isomorphic to the triangle

\[
STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to \Sigma STC
\]

and this proves both claims of the corollary. \( \square \)

In the current case of relative codimension one, we have a concrete description of the form of resolutions defined in 4.9. The following proposition follows from the previous result, and may be regarded as an analogue in \( \mathcal{K}_{\text{tac}}(R) \) of the fact that modules over a hypersurface ring have free resolutions which are eventually periodic.

**Proposition 6.5.** Let \( f \) be a non-zerodivisor of \( Q, R = Q/(f) \), and \( C \in \mathcal{K}_{\text{tac}}(R) \). Then the triangle resolution, as in 4.9, of \( C \) in \( \mathcal{K}_{\text{tac}}(R) \) with respect to \( \mathcal{K}_{\text{tac}}(Q) \) has the form

\[
\cdots \to \Sigma^2 STC \to \Sigma STC \to STC \to C.
\]

A complex \( C \) of \( R \)-modules is minimal if \( \partial_i^C(C_i) \subseteq m C_{i-1} \) for all \( i \). A complex is called contractible if it is homotopically equivalent to the zero complex. We note that every totally acyclic complex \( C \) may be decomposed as \( C' \oplus Z \) where \( C' \) is minimal and \( Z \) is contractible. In this event we define the \( i \)th Betti number of \( C \) to be the rank of the free \( R \)-module \( C'_i \).

Note that this is equal to rank \( k(H_i(C \otimes_R k)) \).

For the maps \( t \) from 7.1, define \( r_i = \text{rank}_k(\text{Im}(t_{i+2} \otimes_R k)) \). The following is a consequence of 6.3. It compares the Betti numbers of a minimal totally acyclic \( R \)-complex \( C \) with those of its \( Q \)-approximation \( STC \), and can be seen as an analogue in \( \mathcal{K}_{\text{tac}}(R) \) of comparisons of Betti numbers of modules.

**Theorem 6.6.** Let \( C \) be a minimal totally acyclic \( R \)-complex. Set \( c_i = \text{rank} C_i \), and \( b_i = \text{rank}_k H_i(STC \otimes_Q k) \).

1. For all \( n \in \mathbb{Z} \) we have
   \[
b_n = c_n - r_{n-2} + c_{n-1} - r_{n-1}
   \]
2. Moreover, if for some \( n \), both \( t_n \) and \( t_{n+1} \) are surjective, then
   \[
b_n = c_n - c_{n-2}
   \]
3. and if for some \( n \), both \( t_n \) and \( t_{n+1} \) are injective, then
   \[
b_n = c_{n-1} - c_{n+1}
   \]
Proof By 6.3, the complex \( STC \otimes_Q k \) is isomorphic to
\[
\cdots \longrightarrow k^{c_{n+1}} \oplus k^{c_n} \xrightarrow{\begin{pmatrix} 0 & 0 \\ -u_{n+1} & 0 \end{pmatrix}} k^{c_n} \oplus k^{c_{n-1}} \xrightarrow{\begin{pmatrix} 0 & 0 \\ -u_n & 0 \end{pmatrix}} k^{c_{n-1}} \oplus k^{c_{n-2}} \longrightarrow \cdots
\]
where \( u_n \) represents \( t_n \otimes_R k \). The results follow. \( \square \)

The following is a triangulated analogue of the change of rings long exact sequence of cohomology for modules (see, for example, [12, Chap. XVI, Sect. 5]).

Theorem 6.7. Let \( f \) be a non-zerodivisor of \( Q, R = Q/(f) \), and \( C \in \text{K}_{\text{tac}}(R) \). If \([\epsilon_C] : STC \rightarrow C \) is the right approximation of \( C \), then for \( C' \in \text{K}_{\text{tac}}(R) \) we have the long exact sequences of cohomology
\[
\cdots \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(Q)}(\Sigma^{n+1} TC, TC') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^{n+2} C, C') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^n C, C') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(Q)}(\Sigma^n TC, T C') \longrightarrow \cdots
\]

Proof From the distinguished triangle \( STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[1]} \Sigma^2 C \rightarrow \cdots \) we have the standard long exact sequence of cohomology
\[
\cdots \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^{n+1} STC, C') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^{n+2} C, C') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^n C, C') \longrightarrow \text{Hom}_{\text{K}_{\text{tac}}(Q)}(\Sigma^n STC, T C') \longrightarrow \cdots
\]

Now using Theorem 4.1, we have for all \( n \) the isomorphisms
\[
\text{Hom}_{\text{K}_{\text{tac}}(R)}(\Sigma^{n+1} STC, C') \cong \text{Hom}_{\text{K}_{\text{tac}}(Q)}(\Sigma^{n+1} TC, T C')
\]
and the long exact sequence becomes what was advertised. \( \square \)

8 Approximations by periodic complexes and matrix factorizations

Recall that a commutative local ring \( Q \) is a hypersurface ring if \( Q \) is the quotient of a regular local ring by a principal ideal; hypersurface rings are Gorenstein. In this case, it follows from Eisenbud [14] that minimal totally acyclic complexes are always periodic of period at most two, and with constant Betti numbers. Thus when \( Q \) is a hypersurface (and, as always, that \( \text{pd}_Q R < \infty \) via \( \varphi : Q \rightarrow R \)), our approximations compare what are often aperiodic totally acyclic complexes with unbounded Betti numbers, to those of period two with constant Betti numbers. In fact, we will show that Theorem 5.1 allows us to approximate totally acyclic \( R \)-complexes by pairs of matrices.

Matrix Factorizations

We first recall the homotopy category of matrix factorizations. Matrix factorizations were first defined by Eisenbud in [14] in his investigation of maximal Cohen–Macaulay modules. Let \( P \) be a commutative local ring and \( x \) an element in the maximal ideal of \( P \). A matrix factorization \( (F, G, \phi, \psi) \) of \( x \) is a diagram
\[
F \xrightarrow{\phi} G \xrightarrow{\psi} F
\]
where \( F \) and \( G \) are finitely generated free \( P \)-modules, and \( \phi \) and \( \psi \) are homomorphisms satisfying \( \psi \circ \phi = x 1_F \) and \( \phi \circ \psi = x 1_G \). Since \( \phi \) and \( \psi \) are maps between free modules, one...
is welcome to think of them as matrices (with respect to fixed bases of $F$ and $G$) with entries in $P$. We refer the reader to [7, Section 2] for further properties of matrix factorizations. In particular, the homotopy category of matrix factorizations $\text{HMF}(P, x)$ is a triangulated category.

**Complete Intersections**

Now assume that $(P, n)$ is a regular local ring and $x_1, \ldots, x_c$ a regular sequence contained in $n^2$. Set $I = (x_1, \ldots, x_c)$ and define $R = P/I$; this is a complete intersection of codimension $c$. Let $x \in I - nI$, and $Q = P/(x)$. Then $Q$ is a hypersurface ring, and we have the natural projections $P \to Q \to R$.

**Theorem 8.1** We have the diagram of triangulated categories

\[ \begin{array}{ccc}
K_{\text{tac}}(Q) & \xrightarrow{\beta} & D^b_{\text{w}}(Q) & \xrightarrow{F} & \text{HMF}(P, x) \\
S & \downarrow & T & \downarrow & \\
K_{\text{tac}}(R) & & & \end{array} \]

where $S'$ and $T'$ are an adjoint pair of functors induced from $S$ and $T$.

**Proof** The equivalence $\beta$ is that of Buchweitz [9], and the equivalence $F$ was noted by Buchweitz and proved by Orlov [25].

Thus the functors $S$ and $T$ give approximations of totally acyclic $R$-complexes by matrix factorizations. We end with some questions for further study.

**Questions.** (1) How do the matrix factorization approximations in Theorem 8.1 compare with the higher matrix factorizations of Eisenbud and Peeva [15], as one varies the generator $x$ defining $Q$?

(2) Can one classify the objects of $K_{\text{tac}}(R)$ with finite data, in terms of the objects of $\text{HMF}(P, x)$ when $Q$ has finite Cohen–Macaulay type?

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