A FOURTH ORDER IMPLICIT SYMMETRIC AND SYMPLECTIC EXponentially FITTED RUNGE-KUTTA-NYSTRÖM METHOD FOR SOLVING OSCILLATORY PROBLEMS

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Abstract. In this paper, we derive an implicit symmetric, symplectic and exponentially fitted Runge-Kutta-Nyström (ISSEFRKN) method. The new integrator ISSEFRKN2 is of fourth order and integrates exactly differential systems whose solutions can be expressed as linear combinations of functions from the set \( \{ \exp(\lambda t), \exp(-\lambda t) | \lambda \in \mathbb{C} \} \), or equivalently \( \{ \sin(\omega t), \cos(\omega t) | \lambda = i\omega, \omega \in \mathbb{R} \} \). We analyse the periodicity stability of the derived method ISSEFRKN2. Some the existing implicit RKN methods in the literature are used to compare with ISSEFRKN2 for several oscillatory problems. Numerical results show that the method ISSEFRKN2 possess a more accuracy among them.

1. Introduction. In this paper we focus on the initial value problems (IVP) related to a system of second-order ODEs of the form

\[ y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad x \in [0, x_{\text{end}}], \]  

(1)

whose solutions exhibit an oscillatory character. Problems of this type are of great interest in applied sciences such as molecular dynamics, orbital mechanics, and electronics. High accuracy of integration is often required in these areas. Until now there are broadly two categories of approaches to numerical integration of the IVP (1): indirect and direct. On one hand, if a new variable \( u \) is introduced to represent the first derivative \( y' \), then IVP (1) is turned into the partitioned system of first order equations

\[ y' = u, \quad u' = f(x, y), \quad y(x_0) = y_0, \quad u(x_0) = y'_0, \]  

(2)

and the problem can be solved by the general Runge-Kutta (RK) methods or partitioned Runge-Kutta (PRK) methods (see Refs. [16, 4, 17, 5, 22, 23, 6]). On the
other hand, Runge-Kutta-Nyström (RKN) methods, initiated by Nyström in 1925, are designed for dealing with the second order problem (1) directly. Some variations of the standard RKN methods are given in [24, 19, 18] and others.

As is known, the symplectic methods share the pronounced property of being zero-dissipative, which is an important requirement for solving oscillatory. Simos and Vigo-Aguiar (see Ref. [21]) considered symplectic methods of RKN type which are adapted to certain types of oscillators. Symplectic methods has been applied to many problems such as pendulum, harmonic oscillator, Morse oscillator, Lennard-Jones oscillator, Kepler’s orbit problem for two bodies and so on. As is pointed out in Chap. V and Chap. XI of [8], symmetric methods show a better long time behavior than non symmetric ones when applied to reversible differential systems, as it is the case for conservative mechanical systems. So, some symmetric and symplectic RKN methods are proposed such as [15].

However, they did not consider exponential fitting technique. Exponentially fitted methods which intend to integrate exactly differential systems whose solutions can be expressed as linear combinations of functions from \( \{ \exp(\lambda t), \exp(-\lambda t) | \lambda \in \mathbb{C} \} \), or equivalently \( \{ \sin(\omega t), \cos(\omega t) | \lambda = i\omega, \omega \in \mathbb{R} \} \), share very good behaviors when applied to oscillatory problems. The construction of exponentially fitted RK(N)(EFRK(N)) methods is originally due to Paternoster [14] and a detailed exposition of exponentially fitted methods with an extensive bibliography on this subject can be found in Ixaru and Vanden Berghe [9]. There are many exponentially fitted methods such as [20, 21, 6, 7, 25].

Many researchers focused on the explicit methods because they are not only easy to program but also efficient. However, the implicit methods are more suitable for solving stiff ODEs than the explicit methods. There are some researchers working on the implicit RKN methods, such as [11, 12, 13, 10]. Their works mainly focused on the diagonal implicit methods and did not combine symmetric, symplectic conditions and exponentially fitted conditions. Motived by this, in this paper, we investigate the full implicit method satisfying the symmetry and symplecticity. In order to be as accurate as possible for the problem whose solutions are in form of combination of trigonometric functions, we derive a method called ISSEFRKN2 which is suitable for stiff ODEs.

This paper is organized as follows: In Section 2 we present the notations and definitions to be used in the rest of the paper as well as some previous results on symmetric and symplectic RKN methods. In Section 3 we make a study of the local truncation error and obtain the order conditions (up to fifth order) for the considered methods. In Section 4 we derive a new two-stage RKN integrator based on symmetry, symplecticity, and exponentially fitted conditions. Section 5 is devoted to analyze the stability properties of the new method. The generalized periodicity region for the classical second-order linear test model is depicted. In Section 6 we carry out four numerical experiments to show the accuracy and efficiency of the new method. Compared with other implicit RKN integrators given in [18, 15, 3], the new method shows some advantages. Finally, Section 7 is devoted to some conclusions.

2. Conditions of symmetry, symplecticity and exponential fitting. An s-stage implicit RKN method \((s \geq 2)\) for the second order ODEs (1) is a scheme of the form
as we do for exponentially fitted type methods (see Refs. [22, 23, 25]). Then the map of the original method with reversed time step \( h \) of the adjoint method. We denote a one-step method for second-order ODEs (1) as \( \text{symmetric method are given by} \)

\[
Y_i = y_0 + c_i h y_0' + h^2 \sum_{j=1}^{s} a_{ij} f(t_0 + c_j h, Y_j), \quad i = 1, \ldots, s,
\]

\[
y_1 = y_0 + h y_0' + h^2 \sum_{i=1}^{s} b_i f(t_0 + c_i h, Y_i),
\]

\[
y_1' = y_0' + h \sum_{i=1}^{s} b_i f(t_0 + c_i h, Y_i),
\]

which can be expressed in the Butcher tableau as

\[
\begin{bmatrix}
  c_1 & 1 & 1 & a_{11} & \cdots & a_{1s} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_s & 1 & 1 & a_{s1} & \cdots & a_{ss} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 1 & \cdots & \cdots & \cdots & \cdots \\
  1 & b_1 & \cdots & b_s \\
\end{bmatrix}
\]

The objective of this section is to specify when the RKN method (3) is symmetric, symplectic and exponentially fitted. This is the cornerstone of our paper. In the following subsections, we will put forward these three important properties step by step.

2.1. Symmetry conditions. The key to understanding symmetry is the concept of the adjoint method. We denote a one-step method for second-order ODEs (1) as \( \Phi_h : (y_0, y_0') \mapsto (y_1, y_1') \). Here, from \( y_0 \) to \( y_1 \), the variable goes forward with a step \( h \). Then the symmetry of \( \Phi_h \) is defined as follows.

**Definition 2.1.** The adjoint method \( \Phi_h^* \) of a one-step method \( \Phi_h \) is the inverse map of the original method with reversed time step \(-h\), i.e., \( \Phi_h^* := \Phi_{-h}^{-1} \). In other words, \( y_1 = \Phi_h^*(y_0) \) is implicitly defined by \( \Phi_{-h}(y_1) = y_0 \). A method for which \( \Phi_h^* = \Phi_h \) is called symmetric.

In the case of the \( s \)-stage RKN method (3), a set of sufficient conditions for the symmetric method are given by

\[
c_i = 1 - c_{s+1-i}, \quad b_i = b_{s+1-i}, \quad \tilde{b}_i = b_{s+1-i} - b_{s+1-i},
\]

\[
a_{ij} = a_{s+1-i,s+1-j} + b_{s+1-j} - c_{s+1-i} b_{s+1-j} - b_{s+1-j},
\]

(4)

In this paper we consider the method (3) whose coefficients are \( z \)-dependent, as we do for exponentially fitted type methods (see Refs. [22, 23, 25]). Then the method (3) is symmetric if its coefficients satisfy the following conditions

\[
c_i(z) = 1 - c_{s+1-i}(-z), \quad b_i(z) = b_{s+1-i}(-z), \quad \tilde{b}_i(z) = b_{s+1-i}(-z) - b_{s+1-i}(-z),
\]

\[
a_{ij}(z) = a_{s+1-i,s+1-j}(-z) + b_{s+1-j}(-z) - c_{s+1-i}(-z) b_{s+1-j}(-z) - b_{s+1-j}(-z),
\]

where \( z = i\omega h, \omega \) is the principal frequency of the problem. We assume that the coefficients of the method (3) are even functions of \( h \), as we frequently encounter in the case of EFRKN methods, so that these conditions reduce to the conditions (4).

2.2. Symplectic conditions. Now, we turn to the symplectic conditions for the scheme (3). Symplecticity is defined for a Hamiltonian system. On many occasions, the problem under consideration takes the form of a Hamiltonian system

\[
p = -\frac{\partial}{\partial q} U(x, q), \quad \dot{q} = S^{-1} p
\]
with the Hamiltonian
\[ H(x, p, q) = \frac{1}{2} p^T S^{-1} p + U(x, q), \]
where \( S \) is a symmetric positive definite constant matrix. This system is equivalent to the second-order equation (1) with \( f(x, q) = -S^{-1} \frac{\partial}{\partial q} U(x, q) \).

The following definition can be found in [8]. Here we only list it without explanation.

**Definition 2.2.** A one-step method is symplectic if for every smooth Hamiltonian function \( H \) and for every step size \( h \), the corresponding flow preserves the differential 2-form
\[ dp \wedge dq = \sum_{i=1}^{d} dp_i \wedge dq_i. \]

Accordingly, the scheme (3) for the problem (1) is symplectic if and only if
\[ dy_1 \wedge dy'_1 = dy_0 \wedge dy'_0. \] (5)

For the left side of this equation, we have
\[ dy_1 \wedge dy'_1 = dy_0 \wedge dy'_0 + h \sum_{i=1}^{s} b_i dy_0 \wedge df(Y_i) \]
\[ + h^2 dy'_0 \wedge dy'_0 + h^2 \sum_{i=1}^{s} b_i dy'_0 \wedge df(Y_i) \]
\[ + h^2 \sum_{i=1}^{s} \bar{b}_i df(Y_i) \wedge dy'_0 + h^3 \sum_{i,j=1}^{s} \bar{b}_i \bar{b}_j df(Y_j) \wedge df(Y_i). \]

Eliminating \( dy_0 \) in the second term of this equation by inserting Eq.(3) we obtain
\[ dy_1 \wedge dy'_1 = dy_0 \wedge dy'_0 + h^2 \sum_{i=1}^{s} (b_i - b_i c_i - \bar{b}_i) dy'_0 \wedge df(Y_i) \]
\[ + \frac{1}{2} h^3 \sum_{i,j=1}^{s} (\bar{b}_i b_j - b_i a_{ij}) df(Y_j) \wedge df(Y_i) \]
\[ + \frac{1}{2} h^3 \sum_{i,j=1}^{s} (\bar{b}_i b_j - b_j a_{ij}) df(Y_i) \wedge df(Y_j). \]

Therefore, Eq. (5) holds if the following conditions are satisfied
\[ \bar{b}_i + (c_j - 1)b_i = 0, \]
\[ b_i(\bar{b}_j - a_{ij}) = b_j(\bar{b}_i - a_{ij}), \quad i, j = 1, \ldots, s. \] (6)

### 2.3. Exponential fitting conditions

Following Albrecht’s approach (see Refs. [1, 2]), each stage of the scheme (3) can be viewed as a linear multistep method on a non-equidistant grid. With each stage one can associate a linear functions as follows:

- for the internal stages,
\[ \varphi_i[y(x); h; a] = y(x + c_i h) - y(x) - c_i y'(x) - h^2 \sum_{j=1}^{s} a_{ij} y''(x + c_j h), \quad i = 1, 2, \ldots, s; \]

- for the final stages,
\[ \varphi[y(x); h; \bar{b}] = y(x + h) - y(x) - h y'(x) - h^2 \sum_{i=1}^{s} \bar{b}_i y''(x + c_i h), \]
\[ \varphi[y(x); h; b] = y'(x + h) - y'(x) - h \sum_{i=1}^{s} b_i y''(x + c_i h). \]

By requiring the internal and final stages vanish for the functions from the set \( \{\exp(\pm iw x)\} \) leads to the following equations

\[
\begin{align*}
& e^{c_i z} = 1 \pm c_i z + z^2 \sum_{j=1}^{s} a_{ij}(z) e^{\pm c_j z}, \\& e^{\pm z} = 1 \pm z + z^2 \sum_{i=1}^{s} b_i(z) e^{\pm c_i z}, \\& e^{\pm z} = 1 \pm z \sum_{i=1}^{s} b_i(z) e^{\pm c_i z}, \quad z = i \omega h.
\end{align*}
\] (7)

Note that \( \cosh(z) = (e^z + e^{-z})/2 \) and \( \sinh(z) = (e^z - e^{-z})/2 \), then the equations (7) imply that

\[
\begin{align*}
& \sum_{j=1}^{s} a_{ij}(z) \cosh(c_j z) = \frac{\cosh(c_i z) - 1}{z^2}, \\
& \sum_{j=1}^{s} a_{ij}(z) \sinh(c_j z) = \frac{\sinh(c_i z) - c_i z}{z^2}, \quad i = 1, 2, \ldots, s, \\
& \sum_{i=1}^{s} b_i(z) \cosh(c_i z) = \frac{\cosh(z) - 1}{z}, \\
& \sum_{i=1}^{s} b_i(z) \sinh(c_i z) = \frac{\sinh(z) - z}{z}.
\end{align*}
\] (8)

In this paper, we call the method (3) satisfied the exponentially fitted (EF) conditions (8) and (9) as exponentially fitted Runge-Kutta-Nystöm (EFRKN) method.

3. Algebraic order conditions. In this section, we will present algebraic order conditions for exponentially fitted Runge-Kutta-Nystöm (EFRKN) methods. For an EFRKN method, the local truncation errors in the approximations of its solution and its derivative can be expressed as

\[
\begin{align*}
e_1 &= y(x_0 + h) - y_1 = \sum_{j=1}^{n-1} h^{j+1} (\sum_{i=1}^{k_j} d_i^{(j+1)} F^{(j)}(y_0)) + O(h^{p+1}), \\
e_1' &= y'(x_0 + h) - y_1' = \sum_{j=1}^{p-1} h^{j} (\sum_{i=1}^{k_j} d_i^{(j)} F^{(j)}(y_0)) + O(h^{p+1}),
\end{align*}
\]

where \( F^{(j)}(y_0) \) denotes an elementary differential and the terms \( d_i^{(j+1)} \) and \( d_i^{(j)} \) depend on the coefficients of the EFRKN method. A method (3) is of order \( p \) if, for every sufficiently smooth IVP(1) and for every small step size \( h \), the local truncation errors of the numerical solutions satisfy

\[
\begin{align*}
e_1 &= y(x_0 + h) - y_1 = O(h^{p+1}), \\
e_1' &= y'(x_0 + h) - y_1' = O(h^{p+1}),
\end{align*}
\]

or equivalently,

\[
\begin{align*}
d_i^{(j+1)} &= 0, \quad i = 1, \ldots, k_j, j = 1, \ldots, p - 1, \\
d_i^{(j)} &= 0, \quad i = 1, \ldots, k_j, j = 1, \ldots, p.
\end{align*}
\]
Since our RKN method is a particular case of the RKN method considered in [7], following the approach in [7] we consider the following assumptions

\[ \bar{b}_i(z) = \bar{b}_i^{(0)} + \bar{b}_i^{(2)} z^2 + \bar{b}_i^{(4)} z^4 + \cdots, \quad b_i(z) = b_i^{(0)} + b_i^{(2)} z^2 + b_i^{(4)} z^4 + \cdots, \]

\[ a_{ij}(z) = a_{ij}^{(0)} + a_{ij}^{(2)} z^2 + a_{ij}^{(4)} z^4 + \cdots. \]

The order conditions up to fifth order for the RKN method (3) are the following ones:

Order 1 requires:

\[ d_1^{(1)} := \sum_i b_i^{(0)} - 1 = 0. \]

Order 2 requires in addition:

\[ d_1^{(2)} := \sum_i b_i^{(0)} c_i - \frac{1}{2} = 0, \quad d_2^{(2)} := \sum_i \bar{b}_i^{(0)} - \frac{1}{2} = 0. \]

Order 3 requires in addition:

\[ d_1^{(3)} := \sum_i b_i^{(0)} c_i - \frac{1}{3} = 0, \quad d_2^{(3)} := \sum_i b_i^{(0)} \sum_k a_{ik}^{(0)} - \frac{1}{6} = 0, \]

\[ d_3^{(3)} := \sum_i \bar{b}_i^{(2)} = 0, \quad d_1^{(3)} := \sum_i b_i^{(0)} c_i - \frac{1}{6} = 0. \]

Order 4 requires in addition:

\[ d_1^{(4)} := \sum_i b_i^{(0)} c_i - \frac{1}{4} = 0, \quad d_2^{(4)} := \sum_i b_i^{(0)} \sum_k c_i a_{ik}^{(0)} - \frac{1}{8} = 0, \]

\[ d_3^{(4)} := \sum_i b_i^{(0)} \sum_k a_{ik}^{(0)} c_k - \frac{1}{24} = 0, \quad d_4^{(4)} := \sum_i \bar{b}_i^{(2)} = 0, \]

\[ d_1^{(4)} := \sum_i b_i^{(0)} c_i - \frac{1}{12} = 0, \quad d_2^{(4)} := \sum_i b_i^{(0)} \sum_k a_{ik}^{(0)} c_k - \frac{1}{24} = 0, \quad d_3^{(4)} := \sum_i \bar{b}_i^{(2)} = 0. \]

Order 5 requires in addition:

\[ d_1^{(5)} := \sum_i b_i^{(0)} c_i^2 - \frac{1}{5} = 0, \quad d_2^{(5)} := \sum_i b_i^{(0)} \sum_k c_i^2 a_{ik}^{(0)} - \frac{1}{10} = 0, \]

\[ d_3^{(5)} := \sum_i b_i^{(0)} c_i \sum_k a_{ik}^{(0)} c_k - \frac{1}{30} = 0, \quad d_4^{(5)} := \sum_i b_i^{(0)} \sum_k a_{ik}^{(0)} c_k^2 - \frac{1}{60} = 0, \]

\[ d_5^{(5)} := \sum_i b_i^{(0)} c_i^2 = 0, \quad d_6^{(5)} := \sum_i b_i^{(0)} \sum_k a_{ik}^{(0)} c_k^2 = 0, \quad d_7^{(5)} := \sum_i \bar{b}_i^{(4)} = 0, \]

\[ d_1^{(5)} := \sum_i b_i^{(0)} c_i^3 - \frac{1}{20} = 0, \quad d_2^{(5)} := \sum_i b_i^{(0)} c_i \sum_k a_{ik}^{(0)} - \frac{1}{10} = 0, \]

\[ d_3^{(5)} := \sum_i \bar{b}_i^{(2)} c_i = 0, \quad d_4^{(5)} := \sum_i b_i^{(0)} \sum_j a_{ij}^{(0)} c_j = 0. \]

From Theorem 2.1 in [7], we know that the EFRKN method (3) has algebraic order at least 2.

4. Construction of implicit symmetric symplectic EFRKN method. In this section we construct implicit EFRKN method under the symmetry, symplecticity and exponential fitting conditions obtained in the previous section.
In order to derive a brief formula, we select \( s = 2 \), then the symmetry conditions (4) and the symplecticity conditions (6) reduce to

\[
c_1 + c_2 = 1, \quad b_1 = b_2, \quad b_1 + b_2 = b_1,
\]

\[
a_{12} = a_{21} + b_1(c_1 - c_2), \quad a_{21} = a_{12} + b_2(c_2 - c_1),
\]

(10)

\[
b_1 = b_1 c_2, \quad b_2 = b_2 c_1, \quad b_1 b_2 - b_1 a_{12} = b_2 b_1 - b_2 a_{21}.
\]

By the assumptions \( c_1 = \frac{1}{2} - \theta \), \( c_2 = \frac{1}{2} + \theta \), the equations (10) become

\[
c_1 = \frac{1}{2} - \theta, \quad c_2 = \frac{1}{2} + \theta, \quad b_2 = b_1, \quad b_1 = b_1 c_2, \quad b_2 = b_2 c_1, \quad a_{21} - a_{12} = 2\theta b_1. \quad (11)
\]

The EF conditions (9) and (8) with \( s = 2 \) give two conditions for the weights \( b_i \), two conditions for the weights \( a_i \), and six conditions for the coefficients \( a_{ij} \), but considering the condition (11), they reduce to two conditions for the weights \( b_i \) and three conditions for \( a_{ij} \) as follows

\[
b_i = \frac{\sinh(z/2)}{z \cosh(\theta z)}, \quad (12)
\]

\[
2b_i \theta \sinh(\theta z) = -\frac{2 \sinh(z/2) + z \cosh(z/2)}{z^2}, \quad (13)
\]

\[
(a_{11} + a_{12}) \cosh(\theta z) = \frac{\cosh(\theta z) - \cosh(z/2) + c_1 z \sinh(z/2)}{z^2}, \quad (14)
\]

\[
(a_{12} - a_{11}) \sinh(\theta z) = \frac{\sinh(z/2) - \sinh(\theta z) - c_1 z \cosh(z/2)}{z^2}, \quad (15)
\]

\[
(a_{21} + a_{22}) \cosh(\theta z) = \frac{\cosh(\theta z) - \cosh(z/2) + c_2 z \sinh(z/2)}{z^2}, \quad (16)
\]

\[
(a_{22} - a_{21}) \sinh(\theta z) = \frac{\sinh(z/2) + \sinh(\theta z) - c_2 z \cosh(z/2)}{z^2}. \quad (17)
\]

From (12) and (13), we can find two expressions of \( b_1 \) as follows

\[
b_1 = \frac{\sinh(z/2)}{z \cosh(\theta z)} \quad \quad b_1' = \frac{-2 \sinh(z/2) + z \cosh(z/2)}{2z^2 \theta \sinh(\theta z)}.
\]

The taylor expansions of the two expressions are

\[
b_1 = \frac{1}{2} + \frac{\theta^2}{4} + \frac{\theta^4}{3840} + \frac{5/4}{107520} + \cdots,
\]

\[
b_1' = \frac{1}{24\theta^2} + \frac{\theta^2}{960\theta^2} + \frac{1}{144} \theta^2 + \frac{1}{107520\theta^2} + \frac{7\theta^4}{8640} + \cdots.
\]

Comparing the two taylor expansions, we can not find a constant \( \theta \) to make the two equations equivalent for any \( z \), but we can make them as close as possible. So we have

\[
\theta = \pm \sqrt[3]{\frac{6}{2}}.
\]

If \( a_{11} = a_{22} \), considering the condition \( a_{21} - a_{12} = 2\theta b_1 \), we can find that (14) – (16) is equivalent to (12) and (15) + (17) is equivalent to (13). So, the method (3) satisfying (12) – (15) is exponentially fitted. From (14) and (15), we have

\[
a_{11} = \frac{\sinh(2\theta z) - \sinh(c_2 z) + c_1 z \cosh(c_2 z)}{z^2 \sinh(2\theta z)}, \quad a_{12} = \frac{\sinh(c_1 z) - c_1 z \cosh(c_1 z)}{z^2 \sinh(2\theta z)}.
\]
Until now, we obtain an implicit symmetric and symplectic exponentially fitted Runge-Kutta-Nyström method which coefficients are given by
\[
\theta = \pm \frac{\sqrt{3}}{6}, \quad c_1 = \frac{1}{2} - \theta, \quad c_2 = \frac{1}{2} + \theta, \quad b_1 = \frac{\sinh(z/2)}{z \cosh(\theta z)}, \quad b_2 = b_1, \quad \bar{b}_1 = b_1 c_2, \quad \bar{b}_2 = b_2 c_1,
\]
\[
a_{11} = \frac{\sinh(2\theta z) - \sinh(c_2 z) + c_1 z \cosh(c_2 z)}{z^2 \sinh(2\theta z)}, \quad a_{12} = \frac{\sinh(c_1 z) - c_1 z \cosh(c_1 z)}{z^2 \sinh(2\theta z)},
\]
\[
a_{22} = a_{11}, \quad a_{21} = a_{12} + 2\theta b_1.
\]

We denote this method as ISSEFRKN2. In order to specify the algebra order of ISSEFRKN2, we give the Taylor expansions of the coefficients.
\[
b_1 = \frac{1}{2} + \frac{1-12\theta^2}{48} z^2 + \frac{1-40\theta^2+40\theta^4}{3840} z^4 + \cdots, \quad \bar{b}_1 = \frac{1+2\theta}{4} + \frac{1+2\theta-12\theta^2+24\theta^3}{96} z^2 + \cdots,
\]
\[
\bar{b}_2 = \frac{1-2\theta}{4} + \frac{1-2\theta-12\theta^2+24\theta^3}{96} z^2 + \cdots,
\]
\[
a_{11} = \frac{-1-12\theta^2+16\theta^3}{48\theta} + \frac{1+15\theta-80\theta^2-120\theta^3+720\theta^4-6560\theta^5}{3760\theta^3} z^2 + \cdots,
\]
\[
a_{12} = \frac{(2\theta-1)^3}{495} + \frac{3+30\theta-40\theta^2-240\theta^3+720\theta^4-5440\theta^5}{5760\theta^3} z^2 + \cdots,
\]
\[
a_{21} = \frac{-1+6\theta+36\theta^2+80\theta^3}{48\theta} + \frac{-3+30\theta+200\theta^2-240\theta^3-2160\theta^4-5440\theta^5}{3760\theta^3} z^2 + \cdots.
\]

From the Taylor expansions, we can verify that our method ISSEFRKN2 satisfies algebraic conditions up to fourth order, but doesn’t satisfy the fifth order condition $d^{(5)}_1 := \sum \ell_i^{(5)} c_i^4 - \frac{1}{5} = 0$. So, the method ISSEFRKN2 is of order 4.

The method ISSEFRKN2 is exponentially fitted. So when the solution of (1) can be expressed as linear combinations of functions from the set $\{\exp(\pm i\theta x)\}$, ISSEFRKN2 has higher efficiency and competence than other integrators which are not exponentially fitted. This will be shown in the numerical studies.

5. Periodicity region of the new method. Now we start to analyze the stability property of our new method. Stability means that the numerical solutions remain bounded as we move further away from the starting point. For classical RKN methods, the stability properties are checked using the second order linear test model
\[
y''(t) = -\mu y(t), \quad \text{with } \mu > 0.
\]
Recall that the new symmetric and symplectic exponentially fitted implicit RKN method derived in the previous section is dependent on the complex number $\lambda = i\omega$, where $\omega > 0$ is an estimate of the dominant frequency. Applying an $s$-stage ISSEFRKN method (3) to the test model (19) yields
\[
\begin{pmatrix}
y_1 \\
y_0
\end{pmatrix} = M(H^2, \nu^2) \begin{pmatrix}
y_0 \\
y_0
\end{pmatrix},
\]
where
\[
M(H^2, \nu^2) = \begin{pmatrix}
1 - H^2 b_1^T N^{-1} c & 1 - H^2 b_1^T N^{-1} c \\
-H^2 b_1^T N^{-1} c & 1 - H^2 b_1^T N^{-1} c
\end{pmatrix},
\]
\[
\nu = \omega h, \quad H = \mu h, \quad N = I + H^2 A, \quad b_1^T = (\bar{b}_1, \cdots, \bar{b}_s), \quad b_1^T = (b_1, \cdots, b_s),
\]
The stability behavior of the numerical solution depends on the eigenvalues or the spectrum of the stability matrix \( M = M(H^2, \nu^2) \). Eliminating \( y'_0 \) and \( y'_1 \) from (20) and the equation that is obtained from (20) by replacing the subscript 0 by 1 gives the difference equation

\[
y_2 - \text{tr}(M(H^2, \nu^2))y_1 + \text{det}(M(H^2, \nu^2))y_0 = 0.
\]

Accordingly, the characteristic equation is given by

\[
\xi^2 - \text{tr}(M)\xi + \text{det}(M) = 0,
\]

where \( \text{tr}(M) \) and \( \text{det}(M) \) are the trace and the determinant of \( M \), respectively.

The stability properties of an EFRKN method are characterized by the spectral radius \( \rho(M) \). Having in mind that the matrix \( M \) depends on the variables \( H \) and \( \nu \), the periodicity and stability intervals typical in the classical RKN methods now become two-dimensional regions in the \((H, \nu)\)-plane. So, for the new method we use an adaptation of the terminology introduced by Coleman and Ixaru (see Ref. [5]):

(i) \( R_s = \{ H > 0, \nu > 0 | \rho(M) < 1 \} \) is the region of stability;

(ii) \( R_p = \{ H > 0, \nu > 0 | \rho(M) = 1 \text{ and } \text{tr}(M)^2 < 4\text{det}(M) \} \) is the region of periodicity;

(iii) If \( R_s = (0, \infty) \times (0, \infty) \) except possibly for a discrete set of curves, the method is \( A \)-stable;

(iv) If \( R_p = (0, \infty) \times (0, \infty) \) except possibly for a discrete set of curves, the method is \( P \)-stable.

The periodicity region of the method ISSEFRKN2 is depicted in Figure 1.

![Figure 1. Periodicity regions for the method ISSEFRKN2.](image_url)

6. **Numerical experiments.** To test the numerical performance of the method ISSEFRKN2, we carry out experiments on four problems to illustrate the effectiveness and efficiency. The codes used in the comparisons are:
• DIRKNRaed: The embedded diagonally implicit RKN4(3) pair method proposed by Al-Khasawneh et al. in [3].
• DIRKNNora: The three-stage fourth-order diagonally implicit RKN method proposed by Senu et al. in [18].
• ISSRKN2: The symmetric and symplectic two-stage fourth-order implicit RKN method proposed by Qin et al. in [15] with \[ a_{11} = \frac{1}{48\theta} - \frac{\theta}{4} + \frac{\theta^2}{3} \] and \[ \theta = \pm \sqrt{\frac{3}{6}}. \] This method is not exponentially fitted.
• ISSEFRKN2: The symmetric and symplectic exponentially fitted two-stage fourth-order RKN method (18) proposed in this paper.

Compared with our method ISSEFRKN2, the method DIRKNRaed or DIRKNNora is neither symmetric, symplectic nor EF; ISSRKN2 is not EF.

In our numerical experiments we have solved the non-linear equations

\[
\begin{align*}
Y_1 &= y_0 + c_1 h y'_0 + h^2(a_{11} f(t_0 + c_1 h, Y_1) + a_{12} f(t_0 + c_2 h, Y_2)), \\
Y_2 &= y_0 + c_2 h y'_0 + h^2(a_{21} f(t_0 + c_1 h, Y_1) + a_{22} f(t_0 + c_2 h, Y_2)),
\end{align*}
\]

with the Newton iteration method and taking initial values \( Y_1^{(0)} = Y_2^{(0)} = y(0) \). The iteration is carried out until the difference between the Euclidean norm of two successive iterations attains \( 10^{-8} \). The maximum number of iterations is 1000.

The criterion used in the numerical comparisons is the usual test based on computing the maximum global error in the solution over the whole integration interval. In Figures 2-6 we show the decimal logarithm of the maximum global error \( \log_{10}(\text{err}) \) versus the number of steps required by each code on a logarithmic scale \( \log_{10}(\text{nsteps}) \). All computations are carried out in double precision arithmetic (16 significant digits of accuracy).

**Problem 1.** We consider the linear problem with variable coefficients

\[
\begin{align*}
y'' + 4x^2 y &= (4x^2 - \omega^2) \sin(\omega x) - 2 \sin(x^2), \; x \in [0, x_{\text{end}}] \\
y(0) &= 1, \; y'(0) = \omega,
\end{align*}
\]

whose analytic solution is given by

\[ y(x) = \sin(\omega x) + \cos(x^2). \]

This solution represents a periodic motion that involves a constant frequency and a variable frequency. In our test we choose the parameter values \( \omega = 10, \lambda = 10i, x_{\text{end}} = 10, h = 1/2^m, m = 4, 5, 6, 7 \), and the numerical results are stated in Figure 2.

**Problem 2.** Consider the second order ODE

\[
\begin{align*}
y'' &= -30 \sin(30x), \; x \in [0, x_{\text{end}}] \\
y(0) &= 0, \; y'(0) = 1,
\end{align*}
\]

whose analytic solution is given by

\[ y(x) = \sin(30x)/30. \]

In our test we chose the parameter as \( \omega = 30, \lambda = 30i, x_{\text{end}} = 10 \), and the numerical results presented in the Figure 3 have been computed with the integration steps \( h = 1/2^m, m = 3, 4, 5, 6 \).
Problem 3. We consider the perturbed orbital problem

\[
\begin{align*}
    y_1'' &= -\frac{y_1}{(y_1^2 + y_2^2)^{3/2}} - \frac{(2\varepsilon + \varepsilon^2)y_1}{(y_1^2 + y_2^2)^{5/2}}, \quad x \in [0, x_{\text{end}}] \\
y_2'' &= -\frac{y_2}{(y_1^2 + y_2^2)^{3/2}} - \frac{(2\varepsilon + \varepsilon^2)y_2}{(y_1^2 + y_2^2)^{5/2}}, \\
y_1(0) &= 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1 + \varepsilon,
\end{align*}
\]

whose exact solution is

\[
y_1(x) = \cos(x + \varepsilon x), \quad y_2(x) = \sin(x + \varepsilon x).
\]

In this problem, we choose \( \omega = 1, \lambda = i, x_{\text{end}} = 10, h = 1/2^m, m = 1, 2, 3, 4, \) and the parameter values \( \varepsilon = 0(\text{Figure 4}) \) and \( \varepsilon = 10^{-3}(\text{Figure 5}) \).

Problem 4. We consider the second order ODEs

\[
\begin{align*}
y_1'' &= (\mu - 2)y_1 + (2\mu - 2)y_2, \quad x \in [0, x_{\text{end}}] \\
y_2'' &= (1 - \mu)y_1 + (1 - 2\mu)y_2, \\
y_1(0) &= 2, \quad y_2(0) = -1, \\
y_1'(0) &= 0, \quad y_2'(0) = 0,
\end{align*}
\]

where \( \mu \) is an arbitrary parameter. The exact solution is

\[
y_1(x) = 2\cos(x), \quad y_2(x) = -\cos(x).
\]

As we can see, its solution is independent on \( \mu \).

In this problem, the parameters are chosen as \( \mu = 0.25, \lambda = \sqrt{\mu}i = 0.5i, x_{\text{end}} = 10, \) and the numerical results presented in Figure 6 have been computed with the integration steps \( h = 1/2^m, m = 1, 2, 3, 4. \)

From Figures 2-6, we can find that symmetric and symplectic method ISSRKN2 is more efficient than the nonsymmetric or nonsymplectic methods. But, all of them are not better than the exponentially fitted RKN method ISSEFRKN2 when the exact solutions can be expressed in term of triangular functions.
7. Conclusions. In this paper a two-stage IEFRKN integrator which is symmetric and symplectic have been derived. Like the existing EFRKN integrators (see [25] for example), the coefficients of the new method depend on the product of the dominant frequency $\omega$ and the step size $h$. When the parameter $z(=\omega h)$ approaches to zero, the ISSEFRKN method reduces to the classical RKN method. The numerical experiments carried out show that the new method is more efficient than the two-stage classical symmetric and symplectic RKN integrator and other RKN methods used in the numerical studies.

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Figure 5. Maximum global error in the solution for problem 3 with \( \varepsilon = 10^{-3} \).

Figure 6. Maximum global error in the solution for problem 4.

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