Adjacency Matrices of Configuration Graphs

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Abstract

In 1960, Hoffman and Singleton [7] solved a celebrated equation for square matrices of order \( n \), which can be written as

\[
(\kappa - 1) I_n + J_n - AA^T = A
\]

where \( I_n, J_n, \) and \( A \) are the identity matrix, the all one matrix, and a \((0,1)\)-matrix with all row and column sums equal to \( \kappa \), respectively. If \( A \) is an incidence matrix of some configuration \( C \) of type \( n_\kappa \), then the left-hand side \( \Theta(A) := (\kappa - 1) I_n + J_n - AA^T \) is an adjacency matrix of the non-collinearity graph \( \Gamma \) of \( C \). In certain situations, \( \Theta(A) \) is also an incidence matrix of some \( n_\kappa \) configuration, namely the neighbourhood geometry of \( \Gamma \) introduced by Lefèvre-Percsy, Percsy, and Leemans [9].

The matrix operator \( \Theta \) can be reiterated and we pose the problem of solving the generalised Hoffman–Singleton equation \( \Theta^n(A) = A \). In particular, we classify all \((0,1)\)-matrices \( M \) with all row and column sums equal to \( \kappa \), for \( \kappa = 3, 4 \), which are solutions of this equation. As a by–product, we obtain characterisations for incidence matrices of the configuration \( 10_3F \) in Kantor’s list [8] and the \( 17_4 \) configuration \#1971 in Betten and Betten’s list [2].

1 Preliminaries: \((0,1)\)-Matrices

Denote by \( I_n \) and \( J_n \) the identity matrix and the all one matrix of order \( n \), respectively. A \((0,1)\)-matrix is said to be \( J_2 \)-free if it does not contain an all one submatrix of order 2. In order to clearly display \((0,1)\)-matrices, we will often omit the entries 0.

For integers \( n \) and \( \kappa \) ranging over and \( \mathbb{N}^+ \) and \( \mathbb{Z} \) respectively, define

\[
\delta(\kappa) := n - \kappa^2 + \kappa - 1.
\]

Denote by \( \mathcal{M}_{n,\kappa}(\mathbb{Z}) \) and \( \mathcal{M}_{n,\kappa}(0,1) \) the subclasses of all matrices in \( \mathcal{M}_n(\mathbb{Z}) \) and \( \mathcal{M}_n(0,1) \), respectively, which have all row and column sums equal to \( \kappa \). Clearly, \( \mathcal{D}_{n,\kappa} \) is empty for \( \kappa < 0 \). Put

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Lemma 1.1 The application \( \Theta \) is a matrix operator in the class \( Z_n \).

Proof. We show that \( A = (a_{ij}) \in Z_n \) implies \( \Theta(A) \in Z_n \). Then the \( i^{th} \) row sum and the \( j^{th} \) column sum of \( AA^T \) respectively read

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} = \sum_{k=1}^{n} a_{ik}(\sum_{j=1}^{n} a_{jk}) = \sum_{k=1}^{n} a_{ik} \kappa = \kappa^2 \quad \text{and} \quad \kappa = \kappa^2.
\]

Hence all row and column sums of \( AA^T \) have constant value \( \kappa^2 \). This, in turn, implies that the summands \((\kappa - 1)I_n, J_n, -AA^T\) contribute \( \kappa - 1, n \) and \( -\kappa^2, \) respectively, to each row and column sum of \( \Theta(A) \).

For positive integers \( m \) and \( \kappa \), we are interested in the subclass \( G_\kappa \subseteq D_{\kappa + 1, \kappa} \) of solutions for the generalised Hoffman–Singleton matrix equation

\[
(gHS) \quad \Theta^m(A) = A.
\]

The proof of Lemma 1.1 shows that \( A \in D_{n, \kappa} \) is a solution of \((gHS)\) only if \( \delta^m(\kappa) = \kappa \). This Diophantine equation in \( n, m, \kappa \) has solutions \((\kappa^2 + 1, m, \kappa)\) for \( m \in \mathbb{N}^+ \) and \( \kappa \in \mathbb{Z} \) and in this paper we will consider this case only. We conjecture that the only other solutions to this equation are \((\kappa - 1)^2 + 2, 2\mu, \kappa)\) for \( \mu \in \mathbb{N}^+ \) and \( \kappa \in \mathbb{Z} \).

Lemma 1.2 Let \( d_{ii} \) be an entry on the main diagonal of \( \Theta(A) \) for some \( A \in Z_n \). Then \( d_{ii} \leq 0 \) and equality holds if and only if \( A \in D_{n, \kappa} \).

Proof. Each entry on the main diagonal of \( AA^T \), say \( b_{ii} \), is the product of the \( i^{th} \) row of \( A \) with itself. Hence \( b_{ii} = \sum_{j=1}^{n} a_{ij}^2 \) is a sum of squares. Collect the positive summands of \( \sum_{j=1}^{n} a_{ij} = \kappa \) and write it as

\[
\sum_{j=1}^{n} a_{ij} = \sum_{k \in K} p_k - \sum_{l \in L} n_l
\]

where \( p_k \geq 1 \) and \( n_l \geq 0 \) are non negative integers for suitable (possibly empty) index sets \( K \) and \( L \), respectively. Since \( p_k^2 \geq p_k \) and \( n_l^2 \geq n_l \), one has

\[
b_{ii} = \sum_{j=1}^{n} a_{ij}^2 = \sum_{k \in K} p_k^2 + \sum_{l \in L} n_l^2 \geq \sum_{k \in K} p_k - \sum_{l \in L} n_l = \sum_{j=1}^{n} a_{ij} = \kappa.
\]

Thus the summands \((\kappa - 1)I_n, J_n\), and \(-AA^T\) contribute \( \kappa - 1, 1 \) and \(-b_{ii}, \) respectively, to each entry \( d_{ii} \) on the main diagonal of \( \Theta(M) \). This implies \( d_{ii} = \kappa - 1 + 1 - b_{ii} = \kappa - b_{ii} \leq 0 \). Equality holds if and only if

\[
\sum_{k \in K} p_k^2 - \sum_{k \in K} p_k = - \sum_{l \in L} n_l^2 - \sum_{l \in L} n_l.
\]
Since the left-hand side \( \sum_{k \in K} (p_k^2 - p_k) \) is either zero or positive, whereas the right-hand side \( - \sum_{l \in L} (n_l^2 + n_l) \) is either zero or negative, both sides must be zero. Clearly, the right-hand side is zero if and only if \( n_l = 0 \) for all \( l \in L \), whereas the left-hand side is zero if and only if \( p_k^2 = p_k \) and thus \( p_k = 1 \) for all \( k \in K \). Hence \( b_{ij} \) assumes its minimum value, namely \( \kappa \), if and only if the \( i^{th} \) row of \( A \) is made up of \( \kappa \) entries 1 and \( n - \kappa \) entries 0. \( \square \)

**Remark 1.3** By Lemma 1.1, the matrix \( \Theta^m(A) \) has all row and column sums equal to \( \delta^m(\kappa) \). Hence \( \delta^m(\kappa) = \kappa \) for \( A \in \mathfrak{S}_\kappa \). For \( \kappa = 1 \) the two solutions of the Diophantine equation \( \delta^m(\kappa) = \kappa \) coincide. The second solutions \( ((\kappa - 1)^2 + 2, 2\mu, \kappa) \) do not contribute to our problem for \( \kappa \geq 2 \). In fact, for \( n = (\kappa - 1)^2 + 2 \), the value of \( \delta(\kappa) = 2 - \kappa \) is no longer positive. Applying Lemma 1.2 to \( \Theta(\Theta(A)) = \Theta^2(A) \), each entry on the main diagonal of \( \Theta^2(A) \) is negative, thus \( \Theta^2(A) \neq A \). Hence, by induction we get the contradiction \( \Theta^{2\mu}(A) \neq A \) for all \( \mu \in \mathbb{N}^+ \).

**Theorem 1.4** Let \( A \in \mathfrak{S}_\kappa \) be a solution of \( \Theta^m(A) = A \), \( m \geq 1 \). Then:

(1) \( A \) is \( J_2 \)-free.

(2) \( A \) is symmetric;

(3) \( A \) has entries 0 on its main diagonal;

**Proof.** (1) By hypothesis \( \Theta^m(A) = A \), for some \( m \geq 1 \). If \( A \) contained entries 1 in positions \((i, j), (i, k), (l, j), \) and \((l, k)\) for \( i, j, k, l \in \{1, \ldots, n\} \) with \( i \neq l \) and \( j \neq k \), then the entry \((AA^T)_{il} = \sum_{r=1}^{n} a_{ir}a_{rl} \) would have at least two summands 1, namely for \( r = j \) and \( r = k \). This would imply \((AA^T)_{il} \geq 2 \) and \((\Theta(A))_{il} \leq -1 \), hence \( \Theta(A) \neq A \). The only remaining option would be \( \Theta^m(A) = A \) for some \( m \geq 2 \). On the other hand, \((\Theta(A))_{il} \leq -1 \) means \( \Theta(A) \notin \mathcal{D}_{\kappa^2+1,\kappa} \). Then Lemma 1.2 implies that each entry on the main diagonal of \( \Theta^2(A) \) is negative, hence \( \Theta^2(A) \notin \mathcal{D}_{\kappa^2+1,\kappa} \). Induction on \( m \) shows \( \Theta^m(A) \notin \mathcal{D}_{\kappa^2+1,\kappa} \) for all \( m \geq 2 \), a contradiction.

(2) Defined as a sum of symmetric matrices, \( \Theta(A) \) is symmetric for any \( A \in \mathfrak{M}_n(\mathbb{Z}) \).

Hence \( A \in \mathfrak{S}_\kappa \) implies that \( A = \Theta(\Theta^{m-1}(A)) \) is symmetric.

(3) follows immediately from Lemma 1.2. \( \square \)

Note that conditions (1), (2), and (3) do not characterise the class \( \mathfrak{S}_\kappa \). A counterexample will be presented in Remark 2.3(ii).

## 2 Connection to Configurations and Graphs

For notions from graph theory and incidence geometry, we respectively refer to [4] and [5]. We consider undirected graphs without loops or multiple edges.

A graph is said to be \( C_4 \)-free if it does not contain 4-cycles. With each permutation \( \pi \) in the symmetric group \( S_n \), we can associate its permutation matrix \( P_\pi = (p_{ij})_{1 \leq i, j \leq n} \) which is defined by \( p_{ij} = 1 \) if \( i^\pi = j \), and 0 otherwise.

We call an incidence structure (in the sense of [5]) linear if any two distinct points are incident with at most one line. A configuration \( C \) of type \( n_\kappa \) is a linear incidence structure consisting of \( n \) points and \( n \) lines such that each point and line is incident with \( \kappa \) lines and points, respectively. To individualise certain \( 10_3 \) and \( 17_4 \) configurations, we refer to the lists in [8] and [2], respectively.

Fix a labelling for the points and lines of a configuration \( C \) and consider the incidence matrix \( C \) of \( C \) (cf. e.g. [5] pp. 17–20)): there is an entry 1 and 0 in position \((i, j)\) of \( C \) if and only if the point \( p_i \) and the line \( l_j \) are incident and non-incident, respectively. The following result is well known.
Lemma 2.1 (i) A square $(0,1)$–matrix $C$ of order $n$ is an incidence matrix of some configuration $C$ of type $n_\kappa$ if and only if it is $J_2$–free and has all row and column sums equal to $\kappa$.
(ii) Any other incidence matrix of $C$ has the form $S_1^{-1}CS_2$ for permutation matrices $S_1$ and $S_2$ of order $n$, corresponding to re–labellings of points and lines.
(iii) $C$ is symmetric and has entries $0$ on its main diagonal if and only if $C$ admits a self–polarity $p_i \leftrightarrow l_i$ without absolute elements.

Adjacency matrices depend on the labelling of the vertices. If $G$ is a graph of order $n$ and $A$ an adjacency matrix for $G$, then any other adjacency matrix has the form $S^{-1}AS$ for a suitable permutation matrix $S$ of order $n$ (which represents a re–labelling of the vertices).

Any two matrices $M_1$ and $M_2$ of order $r$ are said to be permutationally equivalent or $p$–equivalent for short, denoted by $M_1 \sim M_2$, if there exists a permutation matrix $S$ of order $r$ such that $M_2 = S^{-1}M_1S$. The following result is again well known.

Lemma 2.2 (i) A square $(0,1)$–matrix $A$ of order $n$ is an adjacency matrix of some $\kappa$–regular graph $G$ of order $n$ if and only if it is symmetric, has entries $0$ on its main diagonal, and all row and column sums equal to $\kappa$.
(ii) Any other adjacency matrix of $G$ is $p$–equivalent to $A$.
(iii) A graph $G$ is $C_4$–free if and only if its adjacency matrix is $J_2$–free.

With each configuration $C$ of type $n_\kappa$, we can associate its configuration graph $\Gamma(C)$, known also as non–collinearity graph, as the result of the following operation $\Gamma$: the vertices of $\Gamma(C)$ are the points of $C$; any two vertices are joined by an edge if they are not incident with one and the same configuration–line (i). The number of points in $C$ not joined with an arbitrary point of $C$ is given by $\delta(\kappa) := n - \kappa^2 + \kappa - 1$, called the deficiency of $C$. Finite projective planes are characterised by deficiency $0$. Thus the configuration graph $\Gamma(C)$ is a $\delta(\kappa)$–regular graph on $n$ vertices. Since the following Lemma plays a key rôle, we will also quote its short proof:

Lemma 2.3 Let $C$ be a configuration of type $n_\kappa$ with incidence matrix $C$. Then the adjacency matrix $A$ of the configuration graph $\Gamma(C)$ is given by

$$A = (\kappa - 1) \, I_n + J_n - C \, C^T.$$  

Proof. Let $M := (m_{i,j}) := C \, C^T$. An arbitrary entry $m_{i,j}$ of $M$ is the result of the usual dot product (over $\mathbb{R}$) of the $i^{th}$ row and the $j^{th}$ row of $C$. Since the rows represent the $i^{th}$ and $j^{th}$ points of $C$, say $p_i$ and $p_j$, we have

$$m_{i,j} = \begin{cases} \kappa & \text{if } i = j; \\ 1 & \text{if } i \neq j \text{ and there is a line in } C \text{ joining } p_i, p_j; \\ 0 & \text{if } i \neq j \text{ and } p_i, p_j \text{ are not joined by any line of } C. \end{cases}$$

On the other hand, the adjacency matrix $A := (a_{i,j})$ of the configuration graph $\Gamma(C)$ has entries:

$$a_{i,j} = \begin{cases} 1 & \text{if the points } p_i, p_j \text{ are not joined by any line of } C; \\ 0 & \text{otherwise}. \end{cases}$$

This implies $A = (\kappa - 1) \, I_n + J_n - M$.  

Recently, Lefèvre-Percsy, Percsy, and Leemans introduced an operation $N$ which can be seen as a kind of “inverse” operation for $\Gamma$. It associates with each graph $G$ its
neighbourhood geometry $N(G) = (P, B, |)$: let $P$ and $B$ be two copies of $V(G)$, whose elements are called points and blocks, respectively; a point $x \in P$ is incident with a block $b \in B$ (in symbols $x|b$) if and only if $x$ and $b$, seen as vertices in $G$, are adjacent. On the other hand, $N$ has no effect in terms of $(0,1)$--matrices. In fact, it only reinterprets an adjacency matrix of $G$ as an incidence matrix of some configuration, namely $N(G)$.

**Remark 2.4** (i) Given a configuration $C$ of type $(\kappa^2 + 1)_x$, its configuration graph $\Gamma(C)$ need not be $C_4$–free (i.e. it may contain a 4–cycle $p_1, l_1, p_2, l_2$). If this happens, the neighbourhood geometry of $\Gamma(C)$ contains a di-gon (\{p_1, p_2\}, \{l_1, l_2\}, \{(p_i, l_j) | i, j = 1, 2\}) which is forbidden for a linear incidence structure. Hence we say that $\Gamma(C)$ is not $N$–admissible.

(ii) A $C_4$–free $\kappa$–regular graph $G$ on $\kappa^2 + 1$ vertices is $(\Gamma \circ N)$--admissible, whereas a $(\kappa^2 + 1)_x$ configuration $C$ is said to be $(N \circ \Gamma)$–admissible if its configuration graph $\Gamma(C)$ is $C_4$–free (cf. (i)).

(iii) Recall that a Terwilliger graph is a non–complete graph $G$ such that, for any two vertices $v_1, v_2 \in V(G)$ at distance 2 from each other, the induced subgraph $G[N_G(v_1) \cap N_G(v_2)]$ is a clique of size $\mu$, for some fixed $\mu \geq 0$ (cf. e.g. [8 p. 34]). Thus the class of $(\Gamma \circ N)$–admissible $\kappa$–regular graphs coincides with the class of $\kappa$–regular Terwilliger graphs for $\mu = 1$.

(iv) The conditions $(L)$, $(S)$, and $(Z)$ of Theorem 1.4 do not characterise the class $\mathfrak{S}_\kappa$. A counterexample is given by the following adjacency matrix $A_1$ of the Terwilliger graph $T_1$ since $A_1 \neq \Theta(A_1)$, but $\Theta(A_1) = \Theta^m(A_1)$ for all $m \geq 1$.

\[A_1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
\begin{array}{cccccccc}
c & c_1 & c_2 & c_3 & c_{11} & c_{12} & c_{21} & c_{22} & c_{31} & c_{32}
\hline
\end{array}
\]

Note that $\Theta(A_1)$ is an adjacency matrix for the Petersen graph, see [11 Proposition 4.2].

### 3 Standard Forms

Motivated by the next result, this section is dedicated to finding some standard representatives within each class of $p$–equivalency in $\mathfrak{S}_\kappa$.

**Lemma 3.1** Let $A$ be a solution of $(gHS)$. Then any $p$–equivalent matrix $B$ is also a solution of $(gHS)$.

**Proof.** Suppose $B = S^{-1}AS$ for some permutation matrix $S$. Then, in general,

$$\Theta(S^{-1}AS) = (\kappa - 1)I_n + J_n - (S^{-1}AS)^2 = S^{-1}((\kappa - 1)I_n + J_n - A^2)S = S^{-1}\Theta(A)S$$

and hence

$$\Theta^m(S^{-1}AS) = \Theta(\Theta^{m-1}(S^{-1}AS)) = \Theta(S^{-1}\Theta^{m-1}(A)S) = \Theta(S^{-1}\Theta^{m-1}(A)S)$$
\[ S^{-1} \Theta(B) = S^{-1} \Theta(A) S \]
by induction on \( m \geq 1 \) and, in particular, \( \Theta^m(B) = B \) if \( \Theta^m(A) = A \) for some \( m \geq 1 \). \( \square \)

In the sequel we will use the following result which holds for all \( \kappa \geq 2 \):

**Proposition 3.2** [1] **Proposition 2.7** Let \( G \) be a \( \kappa \)-regular graph on \( \kappa^2 + 1 \) vertices whose adjacency matrix fulfils condition \((L)\). Then \( G \) has diameter \( \text{diam}(G) \leq 3 \). In particular, \( \text{diam}(G) = 2 \) if and only if \( G \) has girth 5.

Recall that a vertex \( v \) of a graph \( G \) is said to be a centre of \( G \) with radius 2 if the distance \( d_G(v,w) \leq 2 \) for each \( w \in V(G) \). In general, a graph \( G \) with \( \text{diam}(G) = 3 \) need not admit a centre with radius 2, but we can prove the following result, which also shows that the Conjecture posed in [1] p. 119] holds true.

**Proposition 3.3** Let \( G \) be a \( \kappa \)-regular graph on \( \kappa^2 + 1 \) vertices where \( \kappa = 2, 3, 4 \). If \( G \) admits an adjacency matrix \( A \) which is a solution of \((gHS)\), then \( G \) has a centre with radius 2.

**Proof.** First we verify the following claim: a vertex \( v \in V(G) \) does not lie in a 3-cycle of \( G \) if and only if \( v \) is a centre of \( G \) with radius 2. To see this, let \( v_1, \ldots, v_\kappa \) denote the \( \kappa \) vertices at distance 1 from \( v \). Then \( v \) does not lie in a 3-cycle of \( G \) if and only if we encounter further \( \kappa - 1 \) vertices \( v_{ij}, j = 1, \ldots, \kappa - 1 \), at distance 1 from each \( v_i \). (Since \( A \) is \( J_2 \)-free by Theorem [1] and hence \( G \) is \( C_4 \)-free, the vertices \( v_{ij} \) turn out to be distinct in pairs.) This, in turn, holds true if and only if there are no vertices at distance 3 from \( v \) since \( \{v, v_i, v_{ij} \mid i = 1, \ldots, \kappa, j = 1, \ldots, \kappa - 1 \} \) is all of \( V(G) \).

Secondly, a short calculation verifies that \( \kappa^2 + 1 \not\equiv 0 \pmod{3} \) for every integer \( \kappa \). Hence \( V(G) \) cannot be partitioned into vertex-disjoint 3-cycles.

If \( \kappa = 2 \), then \( G \) is a 5-cycle and every vertex is a centre with radius 2. If \( \kappa = 3 \) and hence \( |V(G)| = 10 \), then \( G \) contains at most three disjoint 3-cycles in \( G \) and the remaining vertex is a centre with radius 2.

Now let \( \kappa = 4 \) and suppose that \( G \) has no centre with radius 2. Then the above claim implies that \( G \) contains at least one vertex \( v_0 \) lying in two different 3-cycles of \( G \), say \( v_0v_1v_2v_0 \) and \( v_0v_3v_4v_0 \). With respect to a labelling of the vertices which starts with \( v_0, v_1, v_2, v_3, v_4, \ldots \), the first five rows and columns of the corresponding adjacency matrix \( B \) and \( \Theta(B) \) read

|   | \( v_0 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \ldots |
|---|---|---|---|---|---|---|
| \( v_0 \) | 0 | 1 | 1 | 1 | 1 | \ldots |
| \( v_1 \) | 1 | 0 | 1 | 0 | 0 | \ldots |
| \( v_2 \) | 1 | 1 | 0 | 0 | 0 | \ldots |
| \( v_3 \) | 1 | 0 | 0 | 0 | 1 | \ldots |
| \( v_4 \) | 1 | 0 | 0 | 1 | 0 | \ldots |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \ddots |

and

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

respectively. Denote by \( S \) the submatrix of order 5 \( \times 12 \) of \( \Theta(B) \) made up by the first five rows and the 6th through the 17th columns. Since \( B \) is \( p \)-equivalent to \( A \) and \( A \) is a solution of \((gHS)\), the image \( \Theta(B) \) is again an adjacency matrix of a \( 4 \)-regular graph on 17 vertices (cf. Theorem [1]). Thus \( S \) must contain four entries 1 in each row. A short argument shows that this is not compatible with Property \((L)\) of Theorem [1], namely that \( \Theta(B) \) and hence \( S \) is \( J_2 \)-free. In fact, in the best case we would need 13 columns to
fit four entries 1 into each row without producing a submatrix of $S$ isomorphic to $J_2$, e.g.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}.
$$

This contradiction shows that $G$ has a centre with radius 2.

\[ \square \]

**Corollary 3.4** Let $G$ be a $\kappa$–regular graph on $\kappa^2 + 1$ vertices where $\kappa = 2, 3, 4$ and let $v_0$ be a centre with radius 2. If $G$ admits an adjacency matrix $A$ which is a solution of $(gHS)$, then the vertex set of $G$ is

$$
\{v_0, v_1, \ldots, v_\kappa, v_{1,1}, \ldots, v_{1,\kappa-1}, v_{2,1}, \ldots, v_{2,\kappa-1}, \ldots, v_{\kappa,1}, \ldots, v_{\kappa,\kappa-1}\}
$$

where, $v_i$ and $v_{ij}$ denote the $\kappa$ neighbours of $v_0$ and the $\kappa - 1$ neighbours of each $v_i$ other than $v_0$, respectively, for $i = 1, \ldots, \kappa$ and $j = 1, \ldots, \kappa - 1$.

\[ \square \]

Denote by $0_\nu$ and $1_\nu$ row vectors of dimension $\nu$ all of whose entries are 0 and 1, respectively. Let $0_{\kappa,\kappa}$ be a copy of the zero matrix of order $\kappa$ and $K = I_\kappa \otimes 1_{\kappa-1}$ the Kronecker product of matrices. With respect to the labelling mentioned in Corollary 3.4, the adjacency matrix of $G$ gets the standard form

$$
S(P) := 
\begin{pmatrix}
0 & 1_\kappa & 0_{\kappa^2-\kappa} \\
1^T_\kappa & 0_{\kappa,\kappa} & K \\
0^T_{\kappa^2-\kappa} & K^T & P
\end{pmatrix}
$$

where $P$ is a symmetric $(0, 1)$–matrix of order $\kappa^2 - \kappa$ and all row and column sums equal to $\kappa - 1$. We regard $P$ as a block matrix $P = (P_{ij})_{1 \leq i,j \leq \kappa}$ of order $\kappa$ with square blocks $P_{ij}$ of order $\kappa - 1$.

**Theorem 3.5** Let $A$ be a solution of $(gHS)$ and suppose $\kappa \leq 4$.

(i) $A$ is $p$–equivalent to a standard form $S(P)$ where each block $P_{ij}$ is a $(0, 1)$–matrix which has at most one entry 1 in each row and column.

(ii) If there exists a zero block in some row and column of $P$, then all other blocks of $P$ in that row and column are permutation matrices.

(iii) If there exists a zero block on the diagonal of $P$, then we can write $S(P)$ in such a way that $P_{11} = 0_{\kappa-1,\kappa-1}$ and $P_{1,i} = P_{i,1} = I_{\kappa-1}$ for all $i = 2, \ldots, \kappa$.

**Proof.** (i) For $\kappa \leq 4$, Corollary 3.4 guarantees that $A$ has a $p$–equivalent standard form $S(P)$, which is a solution of $(gHS)$. Suppose that there are two entries 1 in one and the same row of some block $P_{ij}$, say in positions $(s, t)$ and $(s, u)$, for some $s, t, u \in \{1, \ldots, \kappa - 1\}$. Then in $S(P)$, we encounter entries 1 in the following four positions:

\begin{align*}
(1 + j, 1 + \kappa + (j - 1)(\kappa - 1) + t) \\
(1 + j, 1 + \kappa + (j - 1)(\kappa - 1) + u) \\
(1 + \kappa + (i - 1)(\kappa - 1) + s, 1 + \kappa + (j - 1)(\kappa - 1) + t) \\
(1 + \kappa + (i - 1)(\kappa - 1) + s, 1 + \kappa + (j - 1)(\kappa - 1) + u),
\end{align*}

thus obtaining a $J_2$ submatrix of $S(P)$, which contradicts Property $(L)$ of Theorem 1.4. By symmetry, an analogous reasoning works if there were two entries 1 in one and the same column of some block $P_{ij}$. 

\[ \square \]
(ii) follows for arithmetic reasons: $P$ is at once a block matrix of order $\kappa$ and a $(0,1)$–
matrix with all row and column sums equal to $\kappa - 1$.

(iii) is the result of a suitable relabelling of the rows and columns of $S(P)$. Consider
the vertices $v_0, v_i, v_{ij}$ introduced in Corollary 3.4 if $P_{ii} = 0_{\kappa-1, \kappa-1}$, then first exchange
the rôles of $v_1$ and $v_i$; secondly, for $i = 2, \ldots, \kappa$, relabel the vertices within each family
$\{v_{ij} \mid j = 1, \ldots, \kappa - 1\}$ such that $v_1,j$ is adjacent with $v_{ij}$ for all $j = 1, \ldots, \kappa$.

\begin{definition}
The standard form $S(P)$ is said to be a Hoffman–Singleton form, or an
HS–form for short, denoted by $S_{HS}(P)$, if all the diagonal blocks $P_{ii}$ are zero blocks and $P_{i,j} = P_{i,1} = I_{\kappa-1}$ for all $i = 2, \ldots, \kappa$.
\end{definition}

4 Classification

In 1960, Hoffman and Singleton [7] classified all $\kappa$–regular graphs on $\kappa^2 + 1$ vertices with
girth 5. By eigenvalue techniques, they actually proved the following

\begin{theorem}
Let $A \in D_{\kappa^2 + 1, \kappa}$ be a solution of the Hoffman–Singleton equation $\Theta(A) = A$. Then one of the following statements holds:

(i) $\kappa = 2$ and $A$ is an adjacency matrix for the 5–cycle;

(ii) $\kappa = 3$ and $A$ is an adjacency matrix for the Petersen graph as well
as an incidence matrix for the Desargues configuration;

(iii) $\kappa = 7$ and $A$ is an adjacency matrix for Hoffman–Singleton’s graph;

(iv) $\kappa = 57$ (no graph or configuration is known).
\end{theorem}

An HS–form for $\kappa = 7$, i.e. for an adjacency matrix of the Hoffman–Singleton graph, is
presented in [7 Figure 3]. The following two matrices are HS–forms for adjacency matrices
of the 5–cycle and the Petersen graph, respectively:

\begin{verbatim}
1 1 1
1 1 1
1 1 1
\end{verbatim}

\begin{verbatim}
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
\end{verbatim}

In [1], two further solutions for the generalised Hoffman–Singleton equation $\Theta^m(A) = A$, with $m \geq 2$, have been found; the first being an adjacency matrix of the following Terwilliger graph $T_2$:
Theorem 4.2 Let $A \in \mathcal{D}_{10,3}$. Then the following are equivalent:

(i) $\Theta^3(A) = A$, but $\Theta(A) \neq A$, i.e. $A$ is a proper solution of $(gHS)$ with $m = 3$;

(ii) $A$ is an incidence matrix of the $10_3$ configuration $10_3 F$;

(iii) $A$ is an adjacency matrix of the Terwilliger graph $T_2$;

(iv) $A$ is $p$–equivalent to the HS–form $A_2$.

Proof. This is an immediate consequence of [1, Proposition 4.2].

Theorem 4.3 Let $A \in \mathcal{D}_{17,4}$. Then the following are equivalent:

(i) $\Theta^2(A) = A$, i.e. $A$ is a solution of $(gHS)$ with $m = 2$;

(ii) $A$ is an incidence matrix of the $17_4$ configuration #1971 in Betten’s list [2];

(iii) $A$ is $p$–equivalent to the HS–form $S_{HS}(P)$, where

\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Proof. In Proposition 3.3 we showed that [1, Conjecture 4.3] is in fact a theorem. Then the statements follow by applying [1, Theorem 4.7].

Remark 4.4 The solutions of $(gHS)$ for $m = 1$ yield graphs which can be seen as association schemes. However, this is not the case in general. In fact, for $m = 3$, the graph $T_2$ from Theorem 4.2(iii), cannot be seen as an association scheme since its vertex $c$ is the only one not lying in a 3–cycle.
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