HOMOLOGY THEORIES FOR COMPLEXES OF FINITE GORENSTEIN DIMENSIONS

LI LIANG

Abstract. In this paper, we study unbounded homology and Tate homology of complexes of modules. We give some balance results for these homology theories. In the case of module arguments, we give some relations between unbounded homology and Gorenstein relative homology.

1. Introduction

Tate (co)homology was initially defined for representations of finite groups, and extended by Buchweitz [5] to finitely generated modules over Gorenstein rings, by Avramov and Martsinkovsky [3] to finitely generated modules of finite Gorenstein dimension over noetherian rings, and by Veliche [23] to complexes of finite Gorenstein projective dimension. Christensen and Jorgensen [9] further studied Tate homology of complexes of finite Gorenstein projective dimension, where they gave a balance result for Tate homology using so called pinched complexes. In this paper, we reprove the Christensen and Jorgensen’s balance result using some different methods. Also our methods yield another balance result for Tate homology based on flats. More precisely, we prove the following result; see Theorem 5.4.

Theorem A. Let $M$ be a bounded above $R^\bullet$-complex of finite Gorenstein projective dimension and let $N$ be a bounded above $R$-complex.

(a) (Christensen and Jorgensen; see [9, Theorem 3.7]) If $N$ has finite Gorenstein projective dimension with $T \rightarrow P \rightarrow N$ a complete projective resolution, then for each $i \in \mathbb{Z}$ there is an isomorphism

$$\widehat{\text{Tor}}^R_i(M, N) \cong H_i(M \otimes_R T).$$

(b) If $N$ has finite Gorenstein flat dimension with $(T', F')$ a Tate flat resolution, then for each $i \in \mathbb{Z}$ there is an isomorphism

$$\widehat{\text{Tor}}^R_i(M, N) \cong H_i(M \otimes_R T').$$

Here, the definitions of Gorenstein projective dimension and Gorenstein flat dimension for complexes were proposed by Veliche [23] and Christensen, Köksal and

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Liang [10], respectively. In this paper, we give an upper bound for Gorenstein projective (resp., Gorenstein flat) dimension of complexes. Our result shows that if $M$ is a bounded $R$-complex then there is an inequality

$$\text{Gpd}_R M \leq \max\{\text{Gpd}_R M_i \mid i \in \mathbb{Z}\} + \sup M.$$  

In particular, if $\text{Gpd}_R M_i < \infty$ for each $i \in \mathbb{Z}$ then $\text{Gpd}_R M < \infty$; see Theorem 3.12. The similar result for Gorenstein flat dimension holds if $R$ is a right coherent ring; see Theorem 3.9.

As a broad generalization of Tate homology to the realm of associative rings, stable homology $\widetilde{\text{Tor}}$ was introduced by Vogel and Goichot [13], and further studied by Celikbas, Christensen, Liang and Piepmeyer [6, 7], Emmanouil and Manousaki [11], and Liang [20]. There are tight connections between stable homology $\widetilde{\text{Tor}}$, absolute homology $\text{Tor}$ and so called unbounded homology $\widetilde{\text{Tor}}$; see [6, 2.5]. That is, there is an exact sequence of functors

$$\cdots \to \text{Tor}_i \to \text{Tor}_i \to \widetilde{\text{Tor}}_i \to \widetilde{\text{Tor}}_{i-1} \to \cdots.$$  

If one compare the above exact sequence with the one given by Iacob [16] Theorem 1 and Proposition 6, one may wonder if there are some relations between unbounded homology $\widetilde{\text{Tor}}$ and Gorenstein relative homology $\text{Tor}^{GP}$ (resp., $\text{Tor}^{GF}$) given by Holm [15] for modules of finite Gorenstein projective (resp., Gorenstein flat) dimension. In fact, in many ways it looks that the unbounded homology functor behaves like the Gorenstein relative homology functor; see, e.g., Corollaries 4.4 and 4.5 and Theorem 4.6. Our result Proposition 4.9 shows that if $R$ is a left coherent ring with $\text{splf}^R < \infty$ and $M$ is an $R^c$-module of finite Gorenstein projective dimension, then for each $R$-module $N$ and each $i \geq 2$ there is an isomorphism

$$\text{Tor}^{GP}_i(M, N) \cong \text{Tor}^{GP}_i(M, N),$$  

and an exact sequence

$$0 \to \text{Tor}^{GP}_1(M, N) \to \text{Tor}^R_1(M, N) \to \widetilde{\text{Tor}}^R_0(M, N).$$  

Moreover, if $R$ is noetherian and $M$ is a finitely generated Gorenstein projective $R^c$-module, then $\text{Tor}^{GP}_1(M, N) \cong \text{Hom}_R(\text{Hom}_{R^c}(M, R), N)$; see Corollary 4.13. The similar results for $\text{Tor}^{GF}_i(M, N)$ hold if $R$ is a left coherent ring and $M$ is a cotorsion $R^c$-module of finite Gorenstein flat dimension; see Proposition 4.10.

As shown in [6], unbounded homology is a useful tool to study stable homology. In this paper, we give some methods to compute unbounded homology; see Proposition 4.3. As an application, we give a balance result for unbounded homology of complexes of modules. That is, we prove that if $R$ is a coherent ring, and $M$ is an $R^c$-complex and $N$ is an $R$-complex, both of which are of finite Gorenstein flat dimension, then there is an isomorphism $\text{Tor}^{GP}_i(M, N) \cong \text{Tor}^{GP}_i(N, M)$ for each $i \in \mathbb{Z}$; see Theorem 4.6.

The paper is organized as follows: In Section 2 we set the notation and recall some background material. Sections 3 focuses on Gorenstein flat dimension of complexes of modules. Sections 4 deals with unbounded homology of complexes of modules. Finally, Section 5 is devoted to studying balancedness of Tate homology of complexes of modules, and Theorem A is proved there.
2. Preliminaries

2.1. Throughout this work, all rings are assumed to be associative algebras over a commutative ring $k$. Let $R$ be a ring; by an $R$-module we mean a left $R$-module, and we refer to right $R$-modules as modules over the opposite ring $R^{\text{op}}$. By an $R$-complex $M$ we mean a complex of $R$-modules as follows:

$$\cdots \to M_n \xrightarrow{\partial^{M}_n} M_{n-1} \to \cdots .$$

We frequently (and without warning) identify $R$-modules with $R$-complexes concentrated in degree 0. For an $R$-complex $M$, we set $\text{sup} M = \sup \{i \in \mathbb{Z} \mid M_i \neq 0\}$ and $\text{inf} M = \inf \{i \in \mathbb{Z} \mid M_i \neq 0\}$. An $R$-complex $M$ is bounded above if $\text{sup} M < \infty$, and it is bounded below if $\text{inf} M > -\infty$. An $R$-complex $M$ is bounded if it is both bounded above and bounded below. For $n \in \mathbb{Z}$, the symbol $\Sigma^n M$ denotes the complex with $(\Sigma^n M)_i = M_{i-n}$ and $\partial^{\Sigma^n M}_i = (-1)^n \partial^n_{i-n}$ for all $i \in \mathbb{Z}$. We set $\Sigma M = \Sigma^1 M$.

For $n \in \mathbb{Z}$, the symbol $C_n(M)$ denotes the cokernel of $\partial^n_{n+1}$, and $H_n(M)$ denotes the $n$th homology of $M$, i.e., $\text{Ker} \partial^n_{n}/\text{Im} \partial^n_{n+1}$. An $R$-complex $M$ with $\text{H}(M) = 0$ is called acyclic, and a morphism of $R$-complexes $M \to N$ that induces an isomorphism $\text{H}(M) \to \text{H}(N)$ is called a quasi-isomorphism. The symbol $\simeq$ is used to denote quasi-isomorphisms; it is also used for isomorphisms in derived categories.

The symbol $M_{\leq n}$ denotes the subcomplex of $M$ with $(M_{\leq n})_i = M_i$ for $i \leq n$ and $(M_{\leq n})_i = 0$ for $i > n$, and the symbol $M_{\geq n}$ denotes the quotient complex of $M$ with $(M_{\geq n})_i = M_i$ for $i \geq n$ and $(M_{\geq n})_i = 0$ for $i < n$. The symbol $M_{\lt n}$ denotes the quotient complex of $M$ with $(M_{\lt n})_i = M_i$ for $i \leq n-1$, $(M_{\lt n})_n = C_n(M)$ and $(M_{\lt n})_i = 0$ for $i > n$.

2.2. For an $R^{\text{op}}$-complex $M$ and an $R$-complex $N$, the tensor product $M \otimes_R N$ is a $k$-complex with degree-$n$ term $(M \otimes_R N)_n = \prod_{i \in \mathbb{Z}} (M_i \otimes_R N_{n-i})$ and differential given by $\partial(x \otimes y) = \partial^M_i(x) \otimes y + (-1)^i x \otimes \partial^N_{n-i}(y)$ for $x \in M_i$ and $y \in N_{n-i}$.

The next result is clear.

2.3 Lemma. Let $M$ be an $R^{\text{op}}$-complex and $N$ an $R$-complex. The following assertions hold for each $n \in \mathbb{Z}$.

(a) If $\text{sup} N \leq s$, then $H_n(M \otimes_R N) = H_n(M_{\geq n-s-1} \otimes_R N)$.

(b) If $\text{inf} N \geq t$, then $H_n(M \otimes_R N) = H_n(M_{\leq n-t+1} \otimes_R N)$.

2.4. A complex $P$ of projective $R$-modules is called semi-projective if the functor $\text{Hom}_R(P, -)$ preserves quasi-isomorphisms. A complex $I$ of injective $R$-modules is called semi-injective if the functor $\text{Hom}_R(-, I)$ preserves quasi-isomorphisms. A complex $F$ of flat $R$-modules is called semi-flat if the functor $- \otimes_R F$ preserves quasi-isomorphisms.

A semi-projective resolution of $M$ is a quasi-isomorphism $\pi: P \to M$, where $P$ is a semi-projective complex. Every $R$-complex $M$ has a semi-projective resolution. Moreover, $\pi$ can be chosen surjective, and if $\text{inf} \text{H}(M) > -\infty$ then $P$ can be
chosen such that inf $P = \inf H(M)$; see Avramov and Foxby [2]. Dually, A semi-injective resolution of $M$ is a quasi-isomorphism $\iota: M \to I$, where $I$ is a semi-injective complex. Every $R$-complex $M$ has a semi-injective resolution. Moreover, $\iota$ can be chosen injective, and if $\sup H(M) < \infty$ then $I$ can be chosen such that $\sup I = \sup H(M)$. A semi-flat replacement of $M$ is an isomorphism $F \simeq M$ in the derived category, where $F$ is a semi-flat complex. Every complex has a semi-projective resolution and hence a semi-flat replacement.

2.5. Let $M$ be an $R^e$-complex with $P \xrightarrow{\cong} M$ a semi-projective resolution, and let $N$ be an $R$-complex. For all $i \in \mathbb{Z}$ the modules
\[
\Tor_i^R(M, N) = H_i(P \otimes_R N)
\]
makes up the absolute homology of $M$ and $N$ over $R$. It is clear that the definition of $\Tor_i^R(M, N)$ is functorial and homological in either argument.

2.6. An acyclic complex $T$ of projective $R^e$-modules is called totally acyclic if $\Hom_{R^e}(T, P)$ is acyclic for every projective $R^e$-module $P$. An $R^e$-module $G$ is called Gorenstein projective if there exists a totally acyclic complex $T$ of projective $R^e$-modules such that $G \cong \Coker(T_1 \to T_0)$. A complete projective resolution of an $R^e$-complex $M$ is a diagram $T \xrightarrow{\tau} P \xrightarrow{\cong} M$, where $T$ is a totally acyclic complex of projective $R^e$-modules, $P \xrightarrow{\cong} M$ is a semi-projective resolution, and $\tau_i$ is an isomorphism for $i \gg 0$. The Gorenstein projective dimension of $M$ is defined by
\[
\Gpd_{R^e} M = \inf \left\{ g \in \mathbb{Z} \mid \begin{array}{c}
T \xrightarrow{\tau} P \xrightarrow{\cong} M \\
\tau_i : T_i \to P_i \text{ isomorphism for each } i \geq g
\end{array} \right\};
\]
see [23, Definition 3.1]. It is clear that $M$ has finite Gorenstein projective dimension if and only if $M$ admits a complete projective resolution.

Let $M$ be an $R^e$-complex with a complete projective resolution $T \to P \to M$. From [9 (2.4)], for an $R$-complex $N$ and $i \in \mathbb{Z}$, the $i$th Tate homology of $M$ and $N$ over $R$ is defined as
\[
\widehat{\Tor}_i^R(M, N) = H_i(T \otimes_R N).
\]

3. Gorenstein dimensions of complexes

We start by recalling the following definitions.

3.1. An acyclic complex $T$ of flat $R^e$-modules is called $F$-totally acyclic if $T \otimes R E$ is acyclic for every injective $R$-module $E$. An $R^e$-module $L$ is called Gorenstein flat if there exists a $F$-totally acyclic complex $T$ of flat $R^e$-modules such that $L \cong \Coker(T_1 \to T_0)$. A Tate flat resolution of an $R^e$-complex $M$ is a pair $(T, F)$ where $T$ is an $F$-totally acyclic complex of flat $R^e$-modules and $F \simeq M$ is a semi-flat replacement with $T_{\geq g} \cong F_{\geq g}$ for some $g \in \mathbb{Z}$; see Liang [19]. If furthermore, there exists a morphism $\tau : T \to F$ such that $\tau_i$ is an isomorphism for each $i \geq g$, then the Tate flat resolution $(T, F)$ is said to be a complete flat resolution of $M$.

The next definition can be found in [10].

3.2 Definition. Let $M$ be an $R^e$-complex. The Gorenstein flat dimension of $M$ is given by
\[
\Gfd_{R^e} M = \inf \{ g \in \mathbb{Z} \mid (T, F) \text{ is a Tate flat resolution of } M \text{ with } T_{\geq g} \cong F_{\geq g} \}.
\]
3.3 Remark. From a very recent result by Šaroch and Štovíček [22, Theorem 3.11], all rings are GF-closed in the terminology of Bennis [4], and so Definition 3.2 agrees with Iacob’s definition [17, Definition 3.2]; see [10, Remark 5.13]. Thus from [17, Remark 3], it extends the definitions in Holm [14, 3.9] and Christensen, Frankild and Holm [8, 1.9] of Gorenstein flat dimension for modules and complexes with bounded below homology.

3.4 Remark. Let \( M \) be an \( R^n \)-complex. It is clear that \( \text{Gfd}_{R^n} M < \infty \) if and only if \( M \) admits a Tate flat resolution, and \( \text{Gfd}_{R^n} M = -\infty \) if and only if \( M \) is acyclic.

The following is an analogue of [23, Theorem 3.8] for Gorenstein flat dimension.

3.5 Proposition. Let \( M \) be an \( R^n \)-complex of finite Gorenstein flat dimension. Then one has

\[
\text{Gfd}_{R^n} M = \sup \left\{ n \in \mathbb{Z} \mid \begin{array}{l}
\text{Tor}_{n+\sup H(N)}^R(M, N) \neq 0 \\
\text{for some } R \text{-complex } N \text{ with id}_R N < \infty \text{ and sup } H(N) < \infty
\end{array} \right\}
\]

\[
= \sup \left\{ n \in \mathbb{Z} \mid \begin{array}{l}
\text{Tor}_{n}^R(M, E) \neq 0 \\
\text{for some injective } R \text{-module } E
\end{array} \right\}
\]

Proof. Let \( \text{Gfd}_{R^n} M = g < \infty \). If \( g = -\infty \), then \( M \) is acyclic; see Remark 3.4. Thus \( \text{Tor}_n^R(M, -) = 0 \) for each \( n \in \mathbb{Z} \), and so one gets the equalities in the statement. We let \( g \in \mathbb{Z} \), and let \( s \) (resp., \( s' \)) denote the number on the right side of the first (resp., second) equality in the statement. Then there exists a Tate flat resolution \((T, F)\) such that \( T \) is an \( F \)-totally acyclic complex of flat \( R^n \)-modules, \( F \cong M \) is a semi-flat replacement and \( T_{\geq g} \cong F_{\geq g} \). For each \( R \)-complex \( N \) with \( \text{id}_R N < \infty \) and \( \sup H(N) < \infty \), choose a semi-injective resolution \( N \congto I \) such that \( \sup I = \sup H(N) \) and \( \inf I > -\infty \). Then for all \( i > g + \sup H(N) \) one has

\[
\text{Tor}_i^R(M, N) \cong H_i(F \otimes_R N)
\]

\[
\cong H_i(F \otimes_R I)
\]

\[
= H_i(F_{\geq i-\sup H(N)+1} \otimes_R I)
\]

\[
\cong H_i(T_{\geq i-\sup H(N)+1} \otimes_R I)
\]

\[
= H_i(T \otimes_R I)
\]

\[
= 0;
\]

where the first and second equalities hold by Lemma 3.3, and the last equality follows from [8, Lemma 2.13]. Thus one gets \( g \geq s \). Obviously, \( s \geq s' \), so it remains to prove \( s' \geq g \).

Since \( \text{Gfd}_{R^n} M < \infty \), one has \( \text{Gid}_{R^n} \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) < \infty \); see [10, Proposition 5.8]. On the other hand, for each injective \( R \)-module \( E \) and all \( i > s' \), one has

\[
\text{Ext}_R^n(E, \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_Z(\text{Tor}_i^R(M, E), \mathbb{Q}/\mathbb{Z})
\]

\[
= 0,
\]

so \( \text{Gid}_{R^n} \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq s' \) by Asadollahi and Salarian [1, Theorem 2.4]. Thus by [11, Theorem 2.3], for all \( i > s' \) one has

\[
\text{Hom}_Z(H_i(M), \mathbb{Q}/\mathbb{Z}) \cong \text{H}_{-i}(\text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})) = 0,
\]
and hence $H_i(M) = 0$. That is, $\text{sup} H(M) \leq s'$. Since $\sum_{-s'}^{s'} F_{\geq s'} \xrightarrow{\sim} C_{s'}(F)$ is a flat resolution, for each injective $R$-module $E$ and all $i \geq 1$ we have

$$
\text{Tor}_i^R(C_{s'}(F), E) \cong H_i(\sum_{-s'}^{s'} F_{\geq s'} \otimes_R E)
\cong H_{i+s'}(F_{\geq s'} \otimes_R E)
= H_{i+s'}(F \otimes_R E)
\cong \text{Tor}_i^R(M, E)
= 0.
$$

We notice that $\text{Gfd}_{R^n} C_{s'}(F) < \infty$ and all rings are GF-closed; see [22, Theorem 3.11]. Then $C_{s'}(F)$ is Gorenstein flat by [4, Theorem 2.8], and so one has $s' \geq g$ by [10, Proposition 5.7].

3.6 Corollary. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R^n$-complexes. Then there is an inequality

$$
\text{Gfd}_{R^n} M \leq \max\{\text{Gfd}_{R^n} M', \text{Gfd}_{R^n} M''\}.
$$

Proof. We may assume that $\max\{\text{Gfd}_{R^n} M', \text{Gfd}_{R^n} M''\} < \infty$. Since all rings are GF-closed by [22, Theorem 3.11], one has $\text{Gfd}_{R^n} M < \infty$ by [17, Proposition 3.4]. For each injective $R$-module $E$ there is an exact sequence

$$
\cdots \to \text{Tor}_i^R(M', E) \to \text{Tor}_i^R(M, E) \to \text{Tor}_i^R(M'', E) \to \cdots.
$$

Thus one gets $\text{Gfd}_{R^n} M \leq \max\{\text{Gfd}_{R^n} M', \text{Gfd}_{R^n} M''\}$ by Proposition 5.5.

3.7 Theorem. Let $R$ be a left coherent ring and $g \in \mathbb{Z}$. For an $R^n$-complex $M$ the following conditions are equivalent.

(i) $\text{Gfd}_{R^n} M \leq g$.

(ii) $M$ admits a complete flat resolution $T \xrightarrow{\tau} F$ such that $\tau_i$ is split surjective for each $i \in \mathbb{Z}$ and $\tau_i = 1_{R_i}$ for all $i \geq g$.

Proof. The implication (ii) $\implies$ (i) is clear.

(i) $\implies$ (ii): Let $\text{Gfd}_{R^n} M \leq g$. By [10, Proposition 5.7] there exists a semi-flat replacement $F \simeq M$ such that $C_i(F)$ is Gorenstein flat for each $i \geq g$, and $H_i(M) = 0$ for all $i > g$. Since $R$ is left coherent, there is a flat preenvelope $f : C_g(F) \to G_{g-1}$; see Xu [24, Theorem 2.5.1]. Note that $C_g(F)$ is Gorenstein flat, then $f$ is a monomorphism, and so there is an exact sequence

$$
0 \longrightarrow C_g(F) \xrightarrow{f} G_{g-1} \longrightarrow C_{g-1} \longrightarrow 0
$$

of $R^n$-modules, which is $\text{Hom}_{R^n}(-, F)$ exact for each flat $R^n$-module $F$. For each injective $R$-module $I$, one has

$$
\text{Hom}_Z(\text{Tor}_i^R(C_{g-1}, I), \mathbb{Q}/\mathbb{Z}) \cong \text{Ext}_i^R(C_{g-1}, \text{Hom}_Z(I, \mathbb{Q}/\mathbb{Z})) = 0,
$$

where the equality holds since $\text{Hom}_Z(I, \mathbb{Q}/\mathbb{Z})$ is a flat and cotorsion $R^n$-module, and $f$ is a flat preenvelope. Hence $\text{Tor}_i^R(C_{g-1}, I) = 0$, and so $C_{g-1}$ is Gorenstein flat by [14, Proposition 3.8]. Thus the sequence (1) is $- \otimes_R I$ exact for each injective $R$-module $I$.

Continue the above process, one gets an exact sequence

$$
0 \longrightarrow C_g(F) \longrightarrow G_{g-1} \longrightarrow G_{g-2} \longrightarrow \cdots
$$
of \( R^\circ \)-modules with each \( G_i \) flat, such that the sequence is \(- \otimes_R I \) and \( \text{Hom}_{R^\circ}(-, F) \) exact for each injective \( R \)-module \( I \) and flat \( R^\circ \)-module \( F \). So we have the following commutative diagram:

\[
\begin{array}{c}
\cdots \to C_g(F) \to G_{g-1} \to G_{g-2} \to \cdots \\
\cdots \to C_g(F) \to F_{g-1} \to F_{g-2} \to \cdots
\end{array}
\]

Since \( C_i(F) \) is Gorenstein flat for each \( i \geq g \), and \( H_i(F) = 0 \) for all \( i > g \), the sequence

\[
\cdots \to F_{g+1} \to F_g \to C_g(F) \to 0
\]

is exact, and it is \(- \otimes_R I \) exact for each injective \( R \)-module \( I \).

Assembling the sequences (2) and (3), one gets an \( F \)-totally acyclic complex \( T' \) of flat \( R^\circ \)-modules and a morphism \( \alpha' : T' \to F \) such that \( \alpha'_i = \text{id}_{F_i} \) for each \( i \geq g \).

Set \( T'' = \sum_{i=0}^{-1} \text{Cone}(\text{id}_{F_{<g}}) \). Then \( T'' \) is a contractible complex and there is a degree-wise split surjective morphism \( \tau : T'' \to F_{<g} \). Let \( \alpha'' = \varepsilon \tau : T'' \to F \), where \( \varepsilon : F_{<g} \to F \) is the natural morphism. Then \( \alpha''_i \) is split surjective for each \( i < g \) and \( \alpha''_i = 0 \) for each \( i \geq g \). Let \( T = T' \oplus T'' \) and \( \tau = (\alpha', \alpha'') : T \to F \). Then \( T \) is an \( F \)-totally acyclic complex of flat \( R^\circ \)-modules, and \( \tau_i \) is split surjective for each \( i < g \) and \( \tau_i = \text{id}_{F_i} \) for each \( i \geq g \).

In the following we give an upper bound for Gorenstein flat (resp., Gorenstein projective) dimension of complexes.

**3.8 Lemma.** Let \( R \) be a left coherent ring, and let \( M \) be a bounded \( R^\circ \)-complex such that \( \text{Gfd}_{R^\circ} M_i < \infty \) for each \( i \in \mathbb{Z} \). Then there exists an exact sequence \( 0 \to K \to G \to M \to 0 \) of \( R^\circ \)-complex such that the following conditions hold:

(a) \( G \) is a bounded complex of Gorenstein flat \( R^\circ \)-modules with \( G_i \) flat for \( i \geq \text{inf} G \), \( \text{sup} G = \text{inf} M \) and \( \text{sup} G \leq \max \{ \text{Gfd}_{R^\circ} M_i \mid i \in \mathbb{Z} \} + \text{sup} M \).

(b) \( K \) is a bounded acyclic \( R^\circ \)-complex with each \( C_i(K) \) cotorsion \( R^\circ \)-module of finite flat dimension.

**Proof.** Set \( g = \max \{ \text{Gfd}_{R^\circ} M_i \mid i \in \mathbb{Z} \} \). One has \( g < \infty \) since \( M \) is a bounded \( R^\circ \)-complex. Without loss of generality, we may assume that \( \text{inf} M = 0 \) and \( \text{sup} M = s \). We argue by induction on \( s \). If \( s = 0 \), then one has \( M = M_0 \) with \( \text{Gfd}_{R^\circ} M_0 = g \). Thus by [13] Theorem 3.23 there is an exact sequence

\[
0 \to K_1 \to G_0 \xrightarrow{\delta_0} M_0 \to 0
\]

such that \( G_0 \) is a Gorenstein flat and \( K_1 \) is cotorsion with \( \text{fd}_{R^\circ} K_1 = g - 1 \). For \( K_1 \), there is an exact sequence

\[
0 \to K_2 \to F_1 \xrightarrow{\delta_1} K_1 \to 0
\]

such that \( F_1 \) is flat and \( K_2 \) is cotorsion with \( \text{fd}_{R^\circ} K_2 = g - 2 \). Continue this process, one gets an exact sequence

\[
0 \to F_g \to F_{g-1} \xrightarrow{\delta_{g-1}} \cdots \to F_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} M_0 \to 0
\]
such that \( G_0 \) is Gorenstein flat, each \( F_i \) is flat and each \( \text{Ker} \delta_i \) is cotorsion with \( \text{fd}_{R^0} \text{Ker} \delta_i < \infty \). Let

\[
G = \cdots \to F_g \to \cdots \to F_1 \to G_0 \to 0 \to \cdots
\]

Then \( G \) satisfies the condition (a) in the statement, and there is a surjective morphism \( \alpha : G \to M \). Let \( K = \text{Ker} \alpha \). It is clear that \( K \) satisfies the condition (b) in the statement.

We let \( s > 0 \). Then there exists a morphism \( f : \Sigma^{s-1} M_s \to M_{<s-1} \). By induction hypothesis there exists exact sequences \( 0 \to K' \to G' \to \Sigma^{s-1} M_s \to 0 \) and \( 0 \to K'' \to G'' \to M_{<s-1} \to 0 \) such that the following conditions hold:

(a) \( K' \) and \( K'' \) are bounded acyclic \( R^0 \)-complexes with each \( C_i(K') \) and \( C_i(K'') \) cotorsion \( R^0 \)-modules of finite flat dimension.

(b) \( G' \) and \( G'' \) are bounded complexes of Gorenstein flat \( R^0 \)-modules with \( \inf G' = s-1 \), \( \inf G'' = 0 \), \( \sup G' \leq g+s-1 \) and \( \sup G'' \leq g+s-1 \).

(c) \( G'_i \) and \( G''_i \) are flat for \( i > s-1 \).

Since \( R \) is left coherent, one gets that \( \text{Ext}^1_{\text{Ch}(R)}(G', K'') = 0 \) by Yang and Liu [25, Lemma 3.12]. Thus there is a commutative diagram of \( R^0 \)-complexes

\[
\begin{array}{ccc}
0 & \to K' & \to G' & \to \Sigma^{s-1} M_s & \to 0 \\
& g \downarrow & & f \downarrow & \\
0 & \to K'' & \to G'' & \to M_{<s-1} & \to 0
\end{array}
\]

This yields an exact sequence

\[
0 \to \text{Cone} h \to \text{Cone} g \to \text{Cone} f \to 0
\]

of \( R^0 \)-complexes. We notice that \( \text{Cone} f = M \). Let \( G = \text{Cone} g \) and \( K = \text{Cone} h \). Then \( G \) and \( K \) satisfy the conditions (a) and (b) in the statement, respectively.

**3.9 Theorem.** Let \( R \) be a left coherent ring, and let \( M \) be a bounded \( R^0 \)-complex. Then there is an inequality

\[
\text{Gfd}_{R^0} M \leq \max\{\text{Gfd}_{R^0} M_i \mid i \in \mathbb{Z}\} + \sup M.
\]

In particular, if \( \text{Gfd}_{R^0} M_i < \infty \) for each \( i \in \mathbb{Z} \) then \( \text{Gfd}_{R^0} M < \infty \).

**Proof.** One may assume that \( M \neq 0 \). Let \( \sup M = s \in \mathbb{Z} \). We assume that \( \max\{\text{Gfd}_{R^0} M_i \mid i \in \mathbb{Z}\} = g < \infty \). By Lemma 3.8 there exists an exact sequence

\[
0 \to K \to G \xrightarrow{\pi} M \to 0
\]

such that \( K \) is a bounded acyclic \( R^0 \)-complex with each \( C_i(K) \) cotorsion \( R^0 \)-module of finite flat dimension, and \( G \) is a bounded complex of Gorenstein flat \( R^0 \)-modules with \( \sup G \leq g+s \). Fix a semi-projective resolution \( \pi' : P \xrightarrow{\sim} M \) such that \( P \) is bounded below. Since \( P \) is a semi-flat \( R^0 \)-complex, one has \( \text{Ext}^1_{\text{Ch}(R)}(P, K) = 0 \) by Gillespie [12, Proposition 3.6], and so there is a morphism \( \alpha : P \to G \) such that \( \pi \alpha = \pi' \). We notice that \( \pi \) and \( \pi' \) are quasi-isomorphisms. Then \( \alpha \) is a quasi-isomorphism. Let \( X = \Sigma^{-1} \text{Cone} \alpha \). Then \( X \) is an bounded below acyclic complex of Gorenstein flat \( R^0 \)-modules with \( X_i = P_i \) for \( i \geq g+s \), and so \( H_i(P) = H_i(X) = 0 \) for \( i > g+s \), and \( C_{g+s}(P) = C_{g+s}(X) \) is Gorenstein flat by [14, Theorem 3.7]. Thus one has \( \text{Gfd}_R M \leq g+s \) by [10, Proposition 5.12].

With the next fact we can give an alternate proof of Theorem 3.9. \qed
3.10. For bounded complexes, the notion of Gorenstein flat dimension in this paper is compatible with the one given in [8, 1.9]; see Remark 3.3. That is, if $M$ is a bounded $R^e$-complex then one has

$$Gfd_{R^e} M = \inf \left\{ \sup G \mid G \simeq M \text{ is a bounded below complex of Gorenstein flat } R^e\text{-modules} \right\}.$$

Alternate proof of 3.9. One may assume that $M \neq 0$. Let $\sup M = s \in \mathbb{Z}$. We assume that $\max\{Gfd_{R^e} M_i \mid i \in \mathbb{Z}\} = g < \infty$. By Lemma 3.8, there exists an exact sequence $0 \to K \to G \to M \to 0$ such that $K$ is an acyclic $R^e$-complex and $G$ is a bounded complex of Gorenstein flat $R^e$-modules with $\sup G \leq g + s$. Thus $\pi : G \to M$ is a quasi-isomorphism, and so by 3.10 one has $Gfd_{R^e} M \leq \sup G \leq g + s$.

Using similar methods as proved in Lemma 3.8 and Theorem 3.9, one gets the following results.

3.11 Lemma. Let $M$ be a bounded $R^e$-complex such that $Gpd_{R^e} M_i < \infty$ for each $i \in \mathbb{Z}$. Then there exists an exact sequence $0 \to K \to G \to M \to 0$ of $R^e$-complex such that the following conditions hold:

(a) $G$ is a bounded complex of Gorenstein projective $R^e$-modules with $G_i$ projective for $i > \sup M$, $\sup G \leq \max\{Gfd_{R^e} M_i \mid i \in \mathbb{Z}\} + \sup M$ and $\inf G = \inf M$.

(b) $K$ is a bounded acyclic $R^e$-complex such that each $C_i(K)$ has finite projective dimension.

3.12 Theorem. Let $M$ be a bounded $R^e$-complex. Then there is an inequality

$$Gpd_{R^e} M \leq \max\{Gpd_{R^e} M_i \mid i \in \mathbb{Z}\} + \sup M.$$

In particular, if $Gpd_{R^e} M_i < \infty$ for each $i \in \mathbb{Z}$ then $Gpd_{R^e} M < \infty$.

3.13 Remark. The inequalities in Theorems 3.9 and 3.12 can be strict. For example, let $P$ be a projective $R^e$-module, and let $M$ be the acyclic complex with $P$ in degrees 0 and $-1$, and 0 elsewhere. Then one has $Gpd_{R^e} M = -\infty = Gfd_{R^e} M$. However, the numbers $\max\{Gpd_{R^e} M_i \mid i \in \mathbb{Z}\} + \sup M$ and $\max\{Gfd_{R^e} M_i \mid i \in \mathbb{Z}\} + \sup M$ are zero.

4. UNBOUNDED HOMOLOGY OF COMPLEXES

For an $R^e$-complex $M$ and an $R$-complex $N$, the unbounded tensor product $M \otimes_R N$ is a $k$-complex with degree-$n$ term

$$(M \otimes_R N)_n = \prod_{i \in \mathbb{Z}} (M_i \otimes_R N_{n-i})$$

and differential defined as in 2.2. It contains the tensor product $M \otimes_R N$ as a subcomplex. The quotient complex $(M \otimes_R N)/(M \otimes_R N)$ is called the stable tensor product, and it is denoted $M \otimes_R N$.

The following definitions can be found in [9, 13].
4.1 Definition. Let \( M \) be an \( R^\circ \)-complex and \( N \) an \( R \)-complex. Let \( P \xrightarrow{\sim} M \) be a semi-projective resolution and let \( N \xrightarrow{\sim} I \) be a semi-injective resolution. For each \( i \in \mathbb{Z} \), the \( k \)-module
\[
\widetilde{\text{Tor}}^{R}_i(M, N) = H_i(P \otimes_R I)
\]
is called the \( i \)th \emph{unbounded homology} of \( M \) and \( N \) over \( R \), and the \( i \)th \emph{stable homology} of \( M \) and \( N \) over \( R \) is
\[
\widetilde{\text{Tor}}^{R}_i(M, N) = H_{i+1}(P \otimes_R I).
\]

Using a similar method as proved in [6, Proposition 2.6], one gets the next result.

4.2 Proposition. Let \( M \) be an \( R^\circ \)-complex with \( F \xrightarrow{\sim} M \) a semi-flat replacement, and let \( N \) be a homologically bounded above \( R \)-complex with \( N \xrightarrow{\sim} I \) a semi-injective resolution such that \( \sup I < \infty \). Then for each \( i \in \mathbb{Z} \), there are isomorphisms
\[
\text{Tor}^{R}_i(M, N) \cong H_i(F \otimes_R I) \quad \text{and} \quad \widetilde{\text{Tor}}^{R}_i(M, N) \cong H_{i+1}(F \otimes_R I).
\]

Let \( R \) be a left coherent ring, and let \( M \) be an \( R^\circ \)-complex of finite Gorenstein flat dimension. We notice, from Theorem 3.7, that \( M \) admits a complete flat resolution \( \tau : T \to F \) such that \( \tau_i \) is split surjective. Set \( K = \text{Ker} \tau \). Then we have the following result.

4.3 Proposition. Let \( R \) be a left coherent ring, and let \( M \) be an \( R^\circ \)-complex of finite Gorenstein flat dimension and \( N \) a homologically bounded above \( R \)-complex with \( N \xrightarrow{\sim} I \) a semi-injective resolution such that \( \sup I < \infty \). Then for each \( i \in \mathbb{Z} \), there are isomorphisms
\[
\text{Tor}^{R}_i(M, N) \cong H_{i-1}(K \otimes_R I) \quad \text{and} \quad \widetilde{\text{Tor}}^{R}_i(M, N) \cong H_i(T \otimes_R I).
\]

If furthermore \( N \) is a bounded above complex, then there are isomorphisms
\[
\text{Tor}^{R}_i(M, N) \cong H_{i-1}(K \otimes_R N) \quad \text{and} \quad \widetilde{\text{Tor}}^{R}_i(M, N) \cong H_i(T \otimes_R N).
\]

Proof. Consider the degree-wise split exact sequence \( 0 \to K \to T \to F \to 0 \). By [6 1.5(c)] one has the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
0 & \longrightarrow & K \otimes_R I & \longrightarrow & T \otimes_R I & \longrightarrow & F \otimes_R I & \longrightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
0 & \longrightarrow & K \otimes_R I & \longrightarrow & T \otimes_R I & \longrightarrow & F \otimes_R I & \longrightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
0 & \longrightarrow & K \otimes_R I & \longrightarrow & T \otimes_R I & \longrightarrow & F \otimes_R I & \longrightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
0 & \longrightarrow & K \otimes_R I & \longrightarrow & T \otimes_R I & \longrightarrow & F \otimes_R I & \longrightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]
Since $\sup I < \infty$, $\overline{T \otimes_R} I$ is acyclic by [3] Proposition 1.7(a)]. We notice that $K = \ker \tau$ is bounded above. Thus $K \otimes_R I = 0$, and so for each $i \in \mathbb{Z}$, one has
\[
\overline{\text{Tor}}^i_R(M, N) \cong H_i(F \otimes_R I) \cong H_{i-1}(K \otimes_R I) = H_{i-1}(K \otimes_R I),
\]
and
\[
\overline{\text{Tor}}^i_R(M, N) \cong H_{i+1}(F \otimes_R I) = H_{i+1}(T \otimes_R I) \cong H_i(T \otimes_R I).
\]

If furthermore $N$ is a bounded above complex, then there are quasi-isomorphisms $K \otimes_R N \xrightarrow{\approx} K \otimes_R I$ and $T \otimes_R N \xrightarrow{\approx} T \otimes_R I$ by [5] Proposition 2.14 since both $K$ and $T$ are complexes of flat $R^e$-modules. Thus for each $i \in \mathbb{Z}$, one gets $\overline{\text{Tor}}^i_R(M, N) \cong H_{i-1}(K \otimes_R N)$ and $\overline{\text{Tor}}^i_R(M, N) \cong H_i(T \otimes_R N)$.

4.4 Corollary. Let $R$ be a left coherent ring, and let $M$ be an $R^e$-complex of finite Gorenstein flat dimension and $N$ a homologically bounded above $R$-complex. Then $\overline{\text{Tor}}^i_R(M, N) = 0$ for each $i > \text{Gfd}_{R^e} M + \sup H(N)$.

Proof. Let $\text{Gfd}_{R^e} M = q < \infty$. By Proposition 4.3, $M$ admits a complete flat resolution $\tau : T \to F$ such that $\tau_i$ is split surjective for each $i \in \mathbb{Z}$ and $\tau_i$ is an isomorphism for each $i \geq g$. Set $K = \ker \tau$. Then one gets $\sup K = g - 1$. Fix a semi-projective resolution $P \xrightarrow{\tau} M$ such that $\sup P = \sup H(N)$. Then $\sup(K \otimes_R I) \leq \sup K + \sup I = g - 1 + \sup H(N)$. Thus by Proposition 4.3, $\overline{\text{Tor}}^i_R(M, N) \cong H_{i-1}(K \otimes_R I) = 0$ for each $i > g + \sup H(N)$.

4.5 Corollary. Let $R$ be a left coherent ring, and let $M$ be an $R^e$-complex of finite Gorenstein flat dimension. Then one has
\[
\text{Gfd}_{R^e} M = \sup \left\{ n \in \mathbb{Z} \mid \begin{array}{l}
\overline{\text{Tor}}^n_R(M, N) \neq 0 \\
\text{for some } R\text{-complex } N \\
\text{with } \sup H(N) < \infty
\end{array} \right\}
\]
\[
= \sup \left\{ n \in \mathbb{Z} \mid \overline{\text{Tor}}^n_R(M, N) \neq 0 \\
\text{for some } R\text{-module } N \right\}
\]

Proof. Let $\text{Gfd}_{R^e} M = g < \infty$. If $g = -\infty$, then $M$ is acyclic; see Remark 3.3. If fix a semi-projective resolution $P \xrightarrow{\tau} M$, then $P$ is contractible, and so $P \otimes_R I$ is acyclic for each $R$-complex $I$ by [5, 1.5]. Thus $\overline{\text{Tor}}^n_R(M, -) = 0$ for each $n \in \mathbb{Z}$, and so the equalities in the statement hold. We let $g \in \mathbb{Z}$, and let $s$ (resp., $s'$) denote the number on the right side of the first (resp., second) equality in the statement. Corollary 4.4 yields $g \geq s$. Obviously, $s \geq s'$, so it remains to prove $s' \geq g$. By Proposition 4.5, there is an injective $R$-module $E$ such that $\overline{\text{Tor}}^i_g(M, E) \neq 0$. Since $\overline{\text{Tor}}^i_R(M, E) = 0$ for each $i \in \mathbb{Z}$, one has $\overline{\text{Tor}}^i_g(M, E) \cong \overline{\text{Tor}}^i_R(M, E) \neq 0$; see [6] (2.5)]. This yields $s' \geq g$.

The following is a balance result for unbounded homology of complexes.

4.6 Theorem. Let $R$ be a coherent ring (that is, both left and right coherent), and let $M$ be an $R^e$-complex and $N$ an $R$-complex, both of which are of finite Gorenstein flat dimension. Then for each $i \in \mathbb{Z}$ there is an isomorphism $\overline{\text{Tor}}^i_R(M, N) \cong \overline{\text{Tor}}^i_R(N, M)$.

Proof. Let $\text{Gfd}_{R^e} M = s$ and $\text{Gfd}_R N = t$. Then by [10] Proposition 5.7 one has $\sup H(M) \leq s < \infty$ and $\sup H(N) \leq t < \infty$. It follows from Proposition 4.7 that $M$ admits a complete flat resolution $\tau : T \to F$ such that $\tau_i$ is split surjective for
each \( i \in \mathbb{Z} \) and \( \tau_i \) is an isomorphism for each \( i \geq s \), and \( \mathbf{N} \) admits a complete flat resolution \( \tau' : \mathbf{T}' \to \mathbf{F}' \) such that \( \tau'_i \) is split surjective for each \( i \in \mathbb{Z} \) and \( \tau'_i \) is an isomorphism for each \( i \geq t \). Fix semi-injective resolutions \( \mathbf{M} \xrightarrow{\sim} \mathbf{I} \) and \( \mathbf{N} \xrightarrow{\sim} \mathbf{I}' \) such that \( \sup \mathbf{I} = \sup \mathrm{H}(\mathbf{M}) \) and \( \sup \mathbf{I}' = \sup \mathrm{H}(\mathbf{N}) \). Set \( \mathbf{K} = \ker \tau \) and \( \mathbf{K}' = \ker \tau' \). Then for each \( i \in \mathbb{Z} \), one has

\[
\overline{\mathrm{Tor}}_i^R(\mathbf{M}, \mathbf{N}) \cong \mathrm{H}_{i-1}(\mathbf{K} \otimes_R \mathbf{I}')
\]

\[
= \mathrm{H}_{i-1}(\mathbf{K}_{s_{i-2-t}} \otimes_R \mathbf{I}')
\]

\[
\cong \mathrm{H}_{i-1}(\mathbf{K}_{s_{i-2-t}} \otimes_R \mathbf{F}')
\]

\[
\cong \mathrm{H}_{i-1}(\mathbf{K}'_{s_{i-2-s}} \otimes_R \mathbf{K}')
\]

\[
\cong \mathrm{H}_{i-1}(\mathbf{K}'_{s_{i-2-s}} \otimes_R \mathbf{K})
\]

\[
= \mathrm{H}_{i-1}(\mathbf{K}'_{s_{i-2-s}} \otimes_R \mathbf{I})
\]

\[
= \mathrm{H}_{i-1}(\mathbf{K}' \otimes_R \mathbf{I})
\]

\[
\cong \overline{\mathrm{Tor}}_i^R(\mathbf{N}, \mathbf{M})
\]

where the first and the last isomorphisms follow from Proposition \[\text{[23]}\] the four equations hold by Lemma \[\text{[23]}\] since \( \sup \mathbf{I}' \leq t \), \( \sup \mathbf{K}' = s - 1 \) and \( \sup \mathbf{I} \leq s \), the remaining isomorphisms hold since both \( \mathbf{K}_{s_{i-2-t}} \) and \( \mathbf{K}'_{s_{i-2-s}} \) are semi-flat complexes. \( \square \)

**4.7.** Let \( \mathbf{M} \) be an \( R^c \)-module of finite Gorenstein projective dimension. Then there exists a proper Gorenstein projective resolution of \( \mathbf{M} \), that is, a quasi-isomorphism \( \pi : \mathbf{G} \xrightarrow{\sim} \mathbf{M} \) with \( G_i \) Gorenstein projective for each \( i \geq 0 \) and \( G_i = 0 \) for each \( i < 0 \), such that \( \mathrm{Hom}_{R^c}(L, \mathrm{Cone} \pi) \) is acyclic for each Gorenstein projective \( R^c \)-module \( L \); see \[\text{[13]}\] Theorem 2.10. Following \[\text{[15]}\] Section 4], for each \( R \)-module \( \mathbf{N} \) and each \( i \geq 0 \), the *Gorenstein relative homology* is defined as

\[
\mathrm{Tor}_i^{GP}(\mathbf{M}, \mathbf{N}) = \mathrm{H}_i(\mathbf{G} \otimes_R \mathbf{N}).
\]

If \( R \) is a left coherent ring and \( \mathbf{M} \) is an \( R^c \)-module of finite Gorenstein flat dimension, then by \[\text{[13]}\] Theorem 3.23], \( \mathbf{M} \) admits a proper Gorenstein flat resolution \( \mathbf{G}^r \xrightarrow{\sim} \mathbf{M} \). Following \[\text{[15]}\] Section 4], for each \( R \)-module \( \mathbf{N} \) and each \( i \geq 0 \), the *Gorenstein relative homology based on flats* is defined as

\[
\mathrm{Tor}_i^{GF}(\mathbf{M}, \mathbf{N}) = \mathrm{H}_i(\mathbf{G}^r \otimes_R \mathbf{N}).
\]

**4.8.** We recall the invariant \( \text{splf} R = \sup \{\mathrm{pd}_R F \mid F \text{ is a flat } R\text{-module} \} \). Since an arbitrary direct sum of flat \( R \)-modules is flat, the invariant \( \text{splf} R \) is finite if and only if every flat \( R \)-module has finite projective dimension. If \( R \) is commutative noetherian of finite Krull dimension \( d \), then one has \( \text{splf} R \leq d \) by Jensen \[\text{[18]}\] Proposition 6]. Osofsky \[\text{[21]}\] 3.1] gives examples of rings for which the splf invariant is infinite.

The next two results give some relations between unbounded homology and Gorenstein relative homology (based on flats) of modules.
4.9 Proposition. Let $R$ be a left coherent ring with splf$R^c < \infty$, and let $M$ be an $R^c$-module of finite Gorenstein projective dimension and $N$ an $R$-module. Then there is an isomorphism

$$\text{Tor}^i_R(M, N) \cong \text{Tor}^i_{GP}(M, N)$$

for each $i \geq 2$, and an exact sequence

$$0 \to \text{Tor}^1_{GP}(M, N) \to \text{Tor}^R_1(M, N) \to \text{Tor}^R_0(M, N).$$

Proof. By Avramov and Martsinkovsky [3 Theorem 3.1], $M$ admits a complete projective resolution $\tau : T \to P$ such that $\tau_i$ is split surjective for each $i \in \mathbb{Z}$. This yields a degree-wise split exact sequence

$$0 \to \Sigma^{-1} G \to \tilde{T} \to P \to 0$$

such that $G \xrightarrow{\sim} M$ is a proper Gorenstein projective resolution and $(\Sigma^{-1} G)_{i=0} = (\text{Ker} \tau)_{i=0}$, and $\tilde{T}$ is acyclic with $\tilde{T}_{i=0} = T_{i=0}$ and $\tilde{T}_i = 0$ for each $i \leq -2$; see [3 (3.8)]. Since all Gorenstein projective $R^c$-modules are Gorenstein flat by the proof of [3 Proposition 3.4], $\tau : T \to P$ is a complete flat resolution of $M$, and so by Proposition [13] for each $i \geq 2$, one has

$$\text{Tor}^i_R(M, N) \cong H_{i-1}(\text{Ker} \tau \otimes_R N)$$

$$= H_{i-1}(\Sigma^{-1} G \otimes_R N)$$

$$\cong H_i(G \otimes_R N)$$

$$\cong \text{Tor}^i_{GP}(M, N).$$

Consider the exact sequence

$$0 \to \Sigma^{-1} G \otimes_R N \to \tilde{T} \otimes_R N \to P \otimes_R N \to 0.$$

Then one gets the exact sequence

$$H_1(\tilde{T} \otimes_R N) \to H_1(P \otimes_R N) \to H_0(\Sigma^{-1} G \otimes_R N) \to H_0(\tilde{T} \otimes_R N).$$

It is clear that $H_0(\tilde{T} \otimes_R N) = 0$ and $H_1(\tilde{T} \otimes_R N) = H_1(T \otimes_R N)$, and

$$H_0(\Sigma^{-1} G \otimes_R N) \cong \text{Tor}^1_{GP}(M, N).$$

Thus one has

(1) $\text{Tor}^1_{GP}(M, N) \cong \text{Coker} H_1(\tau \otimes_R N)$.

On the other hand, consider the exact sequence

$$0 \to (\text{Ker} \tau) \otimes_R N \to T \otimes_R N \to P \otimes_R N \to 0.$$

Then one gets the exact sequence

(2) $H_1(T \otimes_R N) \xrightarrow{H_1(\tau \otimes_R N)} H_1(P \otimes_R N) \to H_0((\text{Ker} \tau) \otimes_R N) \to H_0(T \otimes_R N)$.

Since $H_0((\text{Ker} \tau) \otimes_R N) \cong \text{Tor}^1_R(M, N)$ and $H_0(T \otimes_R N) \cong \text{Tor}^R_0(M, N)$ by Proposition [13] and [3 Theorem 3.10], (1) and (2) yield the exact sequence

$$0 \to \text{Tor}^1_{GP}(M, N) \to \text{Tor}^R_1(M, N) \to \text{Tor}^R_0(M, N).$$

$\square$
4.10 Proposition. Let $R$ be a left coherent ring, and let $M$ be a cotorsion $R^e$-module of finite Gorenstein flat dimension and $N$ an $R$-module. Then there is an isomorphism
\[ \overline{\text{Tor}}^R_i(M, N) \cong \text{Tor}^R_i(M, N) \]
for each $i \geq 2$, and an exact sequence
\[ 0 \to \text{Tor}^R_i(M, N) \to \overline{\text{Tor}}^R_i(M, N) \to \text{Tor}^R_0(M, N). \]

Proof. By [19, Lemma 4.2], $M$ admits a complete flat resolution $\tau : T \to F$ such that $\tau_i$ is split surjective for each $i \in \mathbb{Z}$. This yields a degree-wise split exact sequence
\[ 0 \to \Sigma^{-1} G \to \tilde{T} \to F \to 0 \]
such that $G \xrightarrow{\sim} M$ is a proper Gorenstein flat resolution and $(\Sigma^{-1} G)_{>0} = (\text{Ker} \tau)_{>0}$, and $\tilde{T}$ is acyclic with $\tilde{T}_{>0} = T_{>0}$ and $\tilde{T}_i = 0$ for each $i \leq -2$; see [19, Lemma 4.3]. Now using a similar method as proved in Proposition 4.9, one can complete the proof. \hfill \Box

In the following we give some developments relate to Auslander’s use of the transpose.

4.11 Lemma. Let $T$ be an $F$-totally acyclic complex of flat $R^e$-modules and $N$ a bounded above complex of $R$-modules. Then for all integers $i$ and $n$, there is an isomorphism
\[ \overline{\text{Tor}}^R_i(C_n(T), N) \cong H_{i+n-1}(T_{\leq n-1} \otimes_R N). \]
In particular, if $N$ is an $R$-module, then $\overline{\text{Tor}}^R_i(C_n(T), N) = 0$ for each $i \geq 1$.

Proof. Consider the degree-wise split exact sequence
\[ 0 \to T_{\leq n-1} \to T \to T_{\geq n} \to 0. \]

Let $N \xrightarrow{\sim} I$ be a semi-injective resolution such that $\sup I < \infty$. The complex $T \otimes_R I$ is acyclic by [9, Proposition 1.7], whence there is an isomorphism
\[ H(T_{\geq n} \otimes_R I) \cong \Sigma H(T_{\leq n-1} \otimes_R I). \]  
(1)

Since the canonical map $\Sigma^{-n} T_{>n} \to C_n(T)$ is a flat resolution, one has
\[ \overline{\text{Tor}}^R_i(C_n(T), N) \cong H_i((\Sigma^{-n} T_{>n}) \otimes_R I) \]
\[ \cong H_{i+n}(T_{>n} \otimes_R I) \]
\[ \cong H_{i+n-1}(T_{\leq n-1} \otimes_R I) \]
\[ \cong H_{i+n-1}(T_{<n-1} \otimes_R N), \]
where the first isomorphism holds by Proposition 4.12, the third one follows from the isomorphism [1], and the last one holds by [8, Proposition 2.14]. \hfill \Box

4.12 Corollary. Let $M$ be a Gorenstein flat $R^e$-module with $T$ an $F$-totally acyclic complex of flat $R^e$-modules such that $C_0(T) \cong M$, and let $N$ be a bounded above complex of $R$-modules. Then for each $i \in \mathbb{Z}$, there is an isomorphism
\[ \overline{\text{Tor}}^R_i(M, N) \cong H_{i-1}(T_{<i-1} \otimes_R N). \]

In particular, if $N$ is an $R$-module, then $\overline{\text{Tor}}^R_i(M, N) = 0$ for each $i \geq 1$. 

4.13 Corollary. Let $R$ be a noetherian ring, and let $M$ be a finitely generated Gorenstein projective $\text{R}^2$-module and $N$ a bounded above complex of $R$-modules. Then for each $i \in \mathbb{Z}$, there is an isomorphism

$$\widetilde{\text{Tor}}_i^R(M, N) \cong \text{Ext}_R^i(\text{Hom}_{\text{R}}(M, R), N).$$

In particular, if $N$ is an $R$-module, then $\widetilde{\text{Tor}}_0^R(M, N) \cong \text{Hom}_R(\text{Hom}_{\text{R}}(M, R), N)$ and $\widetilde{\text{Tor}}_i^R(M, N) = 0$ for each $i \geq 1$.

Proof. There exists an acyclic complex $T$ of finitely generated free $R^e$-modules, such that $\text{Hom}_{\text{R}}(T, R)$ is acyclic and $C_0(T) \cong M$. For each injective $R$-module $E$, one has

$$T \otimes_R E \cong \text{Hom}_R(\text{Hom}_{\text{R}}(T, R), R) \otimes_R E \cong \text{Hom}_R(\text{Hom}_{\text{R}}(T, R), E)$$

is acyclic, and so $T$ is an $F$-totally acyclic complex. Thus for every $i \in \mathbb{Z}$ one has

$$\widetilde{\text{Tor}}_i^R(M, N) \cong \text{H}_{i-1}(T_{\leq -1} \otimes_R N)$$

$$\cong \text{H}_{i-1}(\text{Hom}_{\text{R}}(T_{\leq -1}, R) \otimes_R N)$$

$$\cong \text{H}_{i-1}(\text{Hom}_{\text{R}}(T_{\leq -1}, N))$$

$$\cong \text{H}_i(\text{Hom}_{\text{R}}(\Sigma^{-1} \text{Hom}_{\text{R}}(T_{\leq -1}, R), N))$$

$$\cong \text{Ext}_R^i(\text{Hom}_{\text{R}}(M, R), N),$$

where the first isomorphism holds by Corollary 4.12 and the last one follows as

$$\Sigma^{-1} \text{Hom}_{\text{R}}(T_{\leq -1}, R) \rightarrow \text{Hom}_{\text{R}}(M, R)$$

is a projective resolution. \hfill \square

4.14 Theorem. Let $M$ be an $\text{R}^e$-complex of finite Gorenstein flat dimension with $(T, F)$ a Tate flat resolution of $M$, and let $N$ be a bounded above $R$-complex. For every $n \in \mathbb{Z}$ there is an exact sequence

$$\cdots \rightarrow \widetilde{\text{Tor}}_{n+1}^R(M, N) \rightarrow \text{Tor}_1^R(C_n(T), N) \rightarrow \widetilde{\text{Tor}}_1^R(C_n(T), N) \rightarrow \text{Tor}_n^R(M, N) \rightarrow \cdots .$$

Proof. Consider the degree-wise split exact sequence

(1) \hspace{1cm} 0 \rightarrow T_{\leq n-1} \rightarrow T \rightarrow T_{\geq n} \rightarrow 0 .

Then for each $i \in \mathbb{Z}$, one has

(2) \hspace{1cm} \text{H}_i(T_{\leq n-1} \otimes_R N) \cong \text{Tor}_{i-n+1}^R(C_n(T), N)

by Lemma 4.11. It is clear that

(3) \hspace{1cm} \text{H}_i(T_{\geq n} \otimes_R N) \cong \text{Tor}_{i-n}^R(C_n(T), N).

From (1) one gets the exact sequence

$$0 \rightarrow T_{\leq n-1} \otimes_R N \rightarrow T \otimes_R N \rightarrow T_{\geq n} \otimes_R N \rightarrow 0 ,$$

and in view of [8] Theorem 3.10 and the isomorphisms (1) and (3) the associated sequence in homology is

$$\cdots \rightarrow \widetilde{\text{Tor}}_{n+1}^R(M, N) \rightarrow \text{Tor}_1^R(C_n(T), N) \rightarrow \widetilde{\text{Tor}}_1^R(C_n(T), N) \rightarrow \text{Tor}_n^R(M, N) \rightarrow \cdots ,$$

as desired. \hfill \square
4.15 Corollary. Let $M$ be an $R^e$-complex of finite Gorenstein flat dimension with $(T, F)$ a Tate flat resolution of $M$, and let $N$ be an $R$-module. For every $n \in \mathbb{Z}$ there is an exact sequence

$$0 \to \widetilde{\text{Tor}}^R_n(M, N) \to C_n(T) \otimes_R N \to \widetilde{\text{Tor}}^R_n(C_n(T), N) \to \widetilde{\text{Tor}}^R_{n-1}(M, N) \to 0.$$ 

In particular, if $M$ is a Gorenstein flat $R^e$-module, then there is an exact sequence

$$0 \to \widetilde{\text{Tor}}^R_0(M, N) \to M \otimes_R N \to \widetilde{\text{Tor}}^R_0(M, N) \to \widetilde{\text{Tor}}^R_{-1}(M, N) \to 0.$$ 

Proof. We notice that $\widetilde{\text{Tor}}^R_0(C_n(T), N) = 0$ by Lemma 4.11. The desired sequence now follows from Theorem 4.14. □

If $R$ is a noetherian ring and $M$ is a finitely generated Gorenstein projective $R^e$-module, then the stable homology $\text{Tor}_n^R(M, -)$ coincides with the Tate homology $\widetilde{\text{Tor}}^R_n(M, -)$ by [6, Theorem 6.4]. Thus the following result follows from Corollaries 4.13 and 4.15, which was proved by Christensen and Jorgensen over a commutative noetherian local ring; see [9, Proposition 6.3].

4.16 Corollary. Let $R$ be a noetherian ring, and let $M$ be a finitely generated Gorenstein projective $R^e$-module and $N$ an $R$-module. Then there is an exact sequence

$$0 \to \widetilde{\text{Tor}}^R_0(M, N) \to M \otimes_R N \to \text{Hom}_R(\text{Hom}_{R^e}(M, R), N) \to \widetilde{\text{Tor}}^R_{-1}(M, N) \to 0.$$ 

5. Balancedness of Tate homology of complexes

In this section, we give two balance results for Tate homology of $R$-complexes. We start with the following lemmas.

5.1 Lemma. Let $T$ be an acyclic complexes of flat $R^e$-modules, and let $N$ be a bounded above $R$-complex with $F' \cong N$ a semi-flat replacement. Then for each $n \geq \sup N$ there is an isomorphism $T \otimes_R N \cong \Sigma^n(T \otimes_R C_n(F'))$ in $D(k)$.

Proof. Since $n \geq \sup H(F)$, there is a quasi-isomorphism $F'_{\leq n} \cong N$. This yields a quasi-isomorphism

$$T \otimes_R F'_{\leq n} \cong T \otimes_R N$$

by [8, Proposition 2.14]. Consider the exact sequence

$$0 \to T \otimes R F'_{\leq n-1} \to T \otimes R F'_{\leq n} \to \Sigma^n(T \otimes_R C_n(F')) \to 0.$$ 

We notice that $T \otimes_R F'_{\leq n-1}$ is acyclic since $F'_{\leq n-1}$ is a semi-flat complex; one gets it by the exact sequence $0 \to F'_{\leq n-1} \to F' \to F'_{> n} \to 0$. Thus we have a quasi-isomorphism $T \otimes_R F'_{\leq n} \cong \Sigma^n(T \otimes_R C_n(F'))$. Now from the equation (1), one has $T \otimes_R N \cong \Sigma^n(T \otimes_R C_n(F'))$ in $D(k)$. □

The next result can be proved dually.

5.2 Lemma. Let $T'$ be an acyclic complexes of flat $R$-modules, and let $M$ be a bounded above $R^e$-complex with $F \cong M$ a semi-flat replacement. Then for each $m \geq \sup M$ there is an isomorphism $M \otimes_R T' \cong \Sigma^m(C_m(F) \otimes_R T')$ in $D(k)$. 

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5.3 Lemma. Let $M$ be a bounded above $R^\circ$-complex with $F \simeq M$ a semi-flat replacement and $T$ an acyclic complexes of flat $R^\circ$-modules such that for some $g \in \mathbb{Z}$ there is an isomorphism $T > g \cong F > g$. Let $N$ be a bounded above $R$-complex with $F' \simeq N$ a semi-flat replacement and $T'$ an acyclic complexes of flat $R$-modules such that for some $g' \in \mathbb{Z}$ there is an isomorphism $T' > g' \cong F' > g'$. Then there is an isomorphism $T \otimes_R N \cong M \otimes_R T'$ in $D(k)$.

Proof. Set $m = \max\{\sup M, g\}$ and $n = \max\{\sup N, g'\}$. One gets
\[
T \otimes_R N \cong \Sigma^n (T \otimes_R C_n(F')) \\
\cong \Sigma^n (T \otimes_R C_n(T')) \\
\cong \Sigma^m (C_m(T) \otimes_R T') \\
\cong \Sigma^m (C_m(F) \otimes_R T'),
\]
where the first and the last isomorphisms hold by Lemmas 5.1 and 5.2 respectively, and the middle one follows from [6, Lemma 4.1].

The following balance results for Tate homology are immediately by Lemma 5.3, where the first one was proved by Christensen and Jorgensen in [9, Theorem 3.7].

5.4 Theorem. Let $M$ be a bounded above $R^\circ$-complex of finite Gorenstein projective dimension and let $N$ be a bounded above $R$-complex.

(a) If $N$ has finite Gorenstein projective dimension with $T \to P \to N$ a complete projective resolution, then for each $i \in \mathbb{Z}$ there is an isomorphism
\[
\hat{\text{Tor}}_i^R (M, N) \cong H_i (M \otimes_R T).
\]

(b) If $N$ has finite Gorenstein flat dimension with $(T', F')$ a Tate flat resolution, then for each $i \in \mathbb{Z}$ there is an isomorphism
\[
\hat{\text{Tor}}_i^R (M, N) \cong H_i (M \otimes_R T').
\]

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Lanzhou Jiaotong University, Lanzhou 730070, China

E-mail address: liangnju@gmail.com