Quantum fields interacting with colliding plane waves: the stress-energy tensor and backreaction

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Abstract

Following a previous work on the quantization of a massless scalar field in a space-time representing the head on collision of two plane waves which focus into a Killing-Cauchy horizon, we compute the renormalized expectation value of the stress-energy tensor of the quantum field near that horizon in the physical state which corresponds to the Minkowski vacuum before the collision of the waves. It is found that for minimally coupled and conformally coupled scalar fields the respective stress-energy tensors are unbounded in the horizon. The specific form of the divergences suggests that when the semiclassical Einstein equations describing the backreaction of the quantum fields on the spacetime geometry are taken into account, the horizon will acquire a curvature singularity. Thus the Killing-Cauchy horizon which is known to be unstable under “generic” classical perturbations is also unstable by vacuum polarization. The calculation is done following the point splitting regularization technique. The dynamical colliding wave spacetime has four quite distinct spacetime regions, namely, one flat region, two single plane wave regions, and one interaction region. Exact mode solutions of the quantum field equation cannot be found exactly, but the blueshift suffered by the initial modes in the plane wave and interaction regions makes the use of the WKB expansion a suitable method of solution. To ensure the correct regularization of the stress-energy tensor, the initial flat modes propagated into the interaction region must be given to a rather high adiabatic order of approximation.
1 Introduction

Exact gravitational plane waves are very simple time dependent plane symmetric solutions to Einstein’s equations \[1\]. Yet, as a consequence of the non linearity of General Relativity these solutions show some nontrivial global features, the most conspicuous of which is the presence of a non-singular null Cauchy horizon (a Killing-Cauchy horizon, in fact). This horizon may be understood as the caustic produced by the focusing of null rays \[2\]. The inverse of the focusing time is a measure of the strength of the wave, thus for an Einstein-Maxwell plane wave such inverse time equals the electromagnetic energy per unit surface of the wave. This makes exact plane waves very different from their linearized counterparts, which have no focusing points and admit a globally hyperbolic spacetime structure. One expects that exact plane waves may be relevant for the study of the strong time dependent gravitational fields that may be produced in the collision of black holes \[3, 4\] or to represent travelling waves on strongly gravitating cosmic strings \[5\]. In recent years these waves have been used in classical general relativity to test some conjectures on the stability of Cauchy horizons \[6, 7\], and in string theory to test classical and quantum string behaviour in strong gravitational fields \[8, 9, 10\]. Their interest also stems from the fact that plane waves are a subclass of exact classical solutions to string theory \[11, 12, 13\].

When plane waves are coupled to quantum fields the effects are rather trivial since they produce neither vacuum polarization nor the spontaneous creation of particles, in that sense these waves behave very much as electromagnetic or Yang-Mills plane waves in flat spacetime \[14, 15\]. Still the classical focusing of geodesics has a quantum counterpart: when quantum particles are present the quantum field stress-energy tensor between scattering states is unbounded at the Cauchy horizon, i.e. where classical test particles focus after colliding with the plane wave \[16\]. This suggests that the Cauchy horizon of plane waves may be unstable under the presence of quantum particles. The classical instability of the null Cauchy horizons of plane waves is manifest when non-linear plane symmetric gravitational radiation collides with the background wave, i.e. when two plane waves collide. In this case the focusing effect of each wave distorts the causal structure of the spacetime near the previous null horizons and either a spacelike curvature singularity or a new regular Killing-Cauchy horizon is formed. However, it is generally believed that the Killing-Cauchy horizons of the colliding plane wave spacetimes are unstable in the sense that “generic” perturbations will transform them into spacelike curvature singularities. In fact, this has been proved under general plane symmetric perturbations \[17\]. Also exact colliding plane wave solutions with classical fields are known that have spacelike curvature singularities and which reduce, in the vacuum limit, to colliding plane wave solutions with a regular Killing-Cauchy horizon \[18\].

Colliding plane wave spacetimes are some of the simplest dynamical spacetimes and, as such, they have been used as a test bed for some problems in classical general relativity such as the, just mentioned, stability of the Killing-Cauchy horizons, or the cosmic censorship hypothesis \[4\]. Note that in a colliding plane wave spacetime with a Killing-Cauchy horizon inequivalent extensions can be made through the horizon, and this implies a breakdown of predictability since the geometry beyond the horizon is not uniquely determined by the initial data posed by the incoming colliding plane waves. The singularities in colliding wave spacetimes are also different from the more familiar cosmological and black hole singularities which are originated by the collapse of matter since they result from the non linear effects of
pure gravity. The type of singularities also differs in the sense that these are all encompassing, i.e. all timelike and null geodesics will hit the singularity in the future.

In a previous paper [19] which we shall refer to as paper I, we made a first step in the study of the interaction of massless scalar quantum fields with a gravitational background which represents the head on collision of two linearly polarized shock waves followed by trailing gravitational radiation which focus into a Killing-Cauchy horizon [20, 21]. The spacetime is divided into four distinct regions: a flat space region (which represents the initial flat region before the waves collide), two single plane wave regions (the plane waves before the collision) and the interaction region which is bounded by the previous three regions and a regular Killing-Cauchy horizon. The interaction region is locally isometric to a region inside the event horizon of a Schwarzschild black hole with the Killing-Cauchy horizon corresponding to the event horizon. The presence of the Killing horizon made possible the definition of a natural preferred “out” vacuum state [22] and it was found that the initial flat vacuum state contains a spectrum of “out” particles. In the long-wave length limit the spectrum is consistent with a thermal spectrum at a temperature which is inversely proportional to the focusing time of the plane waves. Of course, the definition of such “out” vacuum is not possible when we have a curvature singularity (i.e. in the “generic” case) instead of an horizon, whereas a physically meaningful “in” vacuum may be defined in all colliding plane wave spacetimes.

In this paper we compute the expectation value of the stress-energy tensor of the quantum field near the horizon in the initial flat space vacuum. The expectation value of the stress-energy tensor is the relevant observable in the quantization of a field in a gravitational background since it is the source of gravity (to be written to the right hand side in the semiclassical modification of Einstein’s equations). These semiclassical equations are interpreted as dynamical equations for both the quantum field and the gravitational field and determine the backreaction of the quantum field on the spacetime geometry. We find, not surprisingly, that the stress-energy tensor is unbounded at the horizon. The specific form of this divergence suggests that when the backreaction is taken into account the horizon will become a spacetime singularity, i.e. the Killing-Cauchy horizon is unstable under vacuum polarization. Note that this is a non perturbative effect, it is the result of the nonlinearity of gravity, since gravitational waves in the linear approximation do not polarize the vacuum. In fact the vacuum stress-energy tensor of a quantum field in a weakly inhomogeneous background was computed by Horowitz [23], and it is easy to see that such tensor can be written in terms of the linearized Einstein tensor only [24], which vanishes for gravitational waves.

The non perturbative evaluation of the expectation value of the stress-energy tensor of a quantum field in a dynamically evolving spacetime is generally a difficult task. So far the most relevant calculation in this respect is, probably, that of the expectation value of the stress-energy tensor in the Unruh vacuum state of a black hole near its horizon [25, 26]. Note that although such calculation is done in the extended Schwarzschild spacetime rather than in the dynamical spacetime describing the gravitational collapse, the Unruh vacuum describes in the extended spacetime the vacuum state with respect to modes which are incoming from infinity in the collapsing spacetime.

Even when the exact modes of the quantum field equation are known it may not be possible to perform the mode sums in order to get the quantum field two point function or, more precisely, the Hadamard function, which is the key ingredient in the evaluation of the
stress-energy tensor. In our colliding wave spacetime we do not even know the exact solution of the modes in the interaction region (a similar situation is produced in the Schwarzschild case). Fortunately the geometry of the colliding spacetime is such that the initial modes which come from the flat region are strongly blueshifted in their frequency in the interaction region near the horizon. This makes the use of the geometrical optics approximation and, more generally, of a systematic WKB expansion of the modes in the interaction region a very useful tool. Here we should mention that the blueshift of the initial modes in the interaction region is not exclusive of the particular colliding wave spacetime that we take, it is a general feature in colliding wave spacetimes and it is due to the focusing produced by the initial plane waves. This fact has also been exploited, for instance, to compute the production of particles in the so called Bell-Szekeres colliding spacetime [27].

The plan of the paper is the following. In section 2 the geometry of the colliding plane wave spacetime is briefly reviewed. In section 3 the mode solutions of the scalar field equation are given for the four different regions of the spacetime, it is only in the interaction region that exact solutions for these modes cannot be found. In section 4 the geometrical optics approximation is used to relate the parameters of the modes in the initial flat region with the parameters of the modes propagated to the interaction region, in this way a physical meaning for these parameters is found. In section 5 the initial flat spacetime modes which are propagated in the interaction region are given in terms of a WKB expansion up to the adiabatic order four. This is the order which will be required for the regularization of the Hadamard function. Then in section 6 the point splitting technique is reviewed for the computational purposes of this paper and in the short section 7 the renormalized stress-energy tensor is computed in the single plane wave regions, this is not necessary since we know from the literature [15, 28] that it vanishes, but this calculation is a simple and illustrative application of the point splitting regularization technique. In section 8, which is the core of the paper, this technique is used to regularize the Hadamard function and calculate its value by a mode sum near the Killing-Cauchy horizon. Finally, in section 9 the expectation value of the stress-energy tensor near the horizon is calculated. A summary and some consequences of our results, such as the backreaction problem, the quantum instability of the Killing-Cauchy horizon and the generality of these results are discussed in section 10. In order to keep the main body of the paper reasonably clear, many of the technical details of the calculations, as well as a short review on the algebra of bitensors, have been left to the Appendices.

2 Geometry of the colliding plane wave spacetime

Here we recall the main geometric properties of a spacetime describing the head on collision of two linearly polarized gravitational plane waves propagating in the z-direction and forming a regular Killing-Cauchy horizon; further details can be found in paper I. This spacetime has four regions (see Fig. 1): a flat region (or region IV) at the past, before the arrival of the waves, two plane wave regions (regions II and III) and an interaction region (region I) where the waves collide and interact nonlinearly. The geometry of these regions is described by the following four metrics, which are solutions of the Einstein’s field equations in vacuum and are written in coordinates adapted to the two commuting Killing vector fields $\partial_x$ and $\partial_y$.
\[ ds^2_I = 4L_1L_2 \left[1 + \sin (u + v) \right]^2 dudv - \frac{1 - \sin (u + v)}{1 + \sin (u + v)} dx^2 - \left[1 + \sin (u + v) \right]^2 \cos^2 (u - v) dy^2, \tag{1} \]

\[ ds^2_{II} = 4L_1L_2 (1 + \sin u)^2 dudv - \frac{1 - \sin u}{1 + \sin u} dx^2 - (1 + \sin u)^2 \cos^2 u dy^2; \tag{2} \]

\[ ds^2_{III} = 4L_1L_2 (1 + \sin v)^2 dudv - \frac{1 - \sin v}{1 + \sin v} dx^2 - (1 + \sin v)^2 \cos^2 v dy^2; \tag{3} \]

\[ ds^2_{IV} = 4L_1L_2 dudv - dx^2 - dy^2, \tag{4} \]

where \( u \) and \( v \) are two dimensionless null coordinates (\( v + u \) is a time coordinate and \( v - u \) a space coordinate) and \( L_1, L_2 \) are two arbitrary positive length parameters, which represent the focusing time (i.e. the inverse of the strength) of the plane waves. The boundaries of these four regions are: \{\( u = 0, v \leq 0 \}\) between regions IV and II, \{\( v = 0, u \leq 0 \}\) between regions IV and III, \{\( v = 0, 0 \leq u < \pi/2 \}\) between regions II and I and \{\( u = 0, 0 \leq v < \pi/2 \}\) between regions III and I. At these boundaries the matching of the metrics is such that the Ricci tensor vanishes, \( R_{\mu\nu} = 0 \) (i.e. we have a vacuum solution in the entire spacetime).

At the boundaries \( u = \pi/2 \) and \( v = \pi/2 \) on regions II and III, respectively, the determinants of the respective metrics vanish, these are the points of focusing of the plane waves and are coordinate singularities. The causal structure of these regions is best described with the use of appropriate coordinates (harmonic coordinates). In these new coordinates the boundaries \( u = \pi/2 \) and \( v = \pi/2 \) are seen to be spacetime lines.

Region I is locally isometric to a region of the Schwarzschild metric bounded by the event horizon. This is easily seen with the coordinate transformation,

\[
\begin{align*}
  t &= x, \\
  r &= M \left[1 + \sin(u + v) \right], \\
  \varphi &= 1 + y/M, \\
  \theta &= \pi/2 - (u - v)
\end{align*}
\]

where we have defined \( M = \sqrt{L_1L_2} \). Then the metric (1) becomes

\[
\begin{align*}
  ds^2 = (2M/r - 1)^{-1} dt^2 - \left( 2M/r - 1 \right) dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\end{align*}
\]

which is the interior of the Schwarzschild black hole. The surface \( u + v = \pi/2 \) corresponds to the black hole event horizon. The boundary \( v = 0 \) corresponds to \( r = M(1 + \cos \theta) \) and \( u = 0 \) corresponds to \( r = M(1 - \cos \theta) \), these are the boundaries of the plane waves. These boundaries join at \( r = M \) (spacetime point of the collision) and also at the surface \( u + v = \pi/2 \) at \( \theta = 0 \) and \( \theta = \pi \). This region of the black hole interior is outside the singularity \( r = 0 \) and thus the interaction region has no curvature singularities. The above local isometry is not global however, the coordinates \( \theta \) and \( \phi \) are cyclic in the black hole case but in the plane wave case, \(-\infty < y < \infty \) and \(-\infty < v - u < \infty \).

As in the Schwarzschild case it is convenient to introduce a set of Kruskal-Szekeres like coordinates to describe the interaction region, because the \((u, v, x, y)\) coordinates become singular at the horizon. Since the Klein-Gordon equation can be separated in the
new coordinates, these will play an important role in the quantization. First we introduce dimensionless time and space coordinates \((\xi, \eta)\)

\[
\xi = u + v, \quad \eta = v - u,
\]

with the range \(0 \leq \xi < \pi/2, \ -\pi/2 \leq \eta < \pi/2\). Then we introduce a new time coordinate \(\xi^*\) related to the dimensionless time coordinate \(\xi\) by

\[
\xi^* = 2M \ln \left( \frac{1 + \sin \xi}{2 \cos^2 \xi} \right) - M (\sin \xi - 1),
\]

and a new set of null coordinates \(\tilde{U} = \xi^* - x, \ \tilde{V} = \xi^* + x\). Note that the transversal coordinate \(x\), which behaves badly at the horizon, appears in this coordinate transformation. Finally, we define,

\[
U' = -2M \exp \left( -\tilde{U}/4M \right) \leq 0, \quad V' = -2M \exp \left( -\tilde{V}/4M \right) \leq 0,
\]

and the metric in the interaction region \([I]\) can be written as,

\[
ds_I^2 = \frac{2e^{(1-\sin \xi)/2}}{(1 + \sin \xi)} dU'dV' - M^2 (1 + \sin \xi)^2 d\eta^2 - (1 + \sin \xi)^2 \cos^2 \eta dy^2,
\]

with

\[
U'V' = 8M^2 \frac{\cos^2 \xi}{1 + \sin \xi} \exp \left( \frac{\sin \xi - 1}{2} \right), \quad \frac{U'}{V'} = \exp \left( \frac{x}{2M} \right).
\]

The curves \(\xi = \text{const}\). and \(x = \text{const}\). are, respectively, hyperbolae and straight lines through the origin of coordinates \((U' = V' = 0)\). The Schwarzschild horizon (which is a Killing-Cauchy horizon for our spacetime) corresponds to the limit of the hyperbolae when \(\xi \to \pi/2\) i.e. \(V' = 0\) or \(U' = 0\). Notice that the problem with the transversal coordinate \(x\) at the horizon is that all the lines \(x = \text{const}\). go through the origin of the \((U', V')\) coordinates, so that all the range of \(x\) collapses into the point \(V' = U' = 0\), whereas the lines \(U' = 0\) and \(V' = 0\) represent \(x = -\infty\) and \(x = \infty\) respectively. One should recall that we have not represented the coordinate \(x\) in Fig. 1 where only the \((u, v)\) coordinates are shown, \(x\) is a transversal coordinate perpendicular to the propagation of the waves.

To understand the global geometry of the spacetime a tridimensional picture like Fig. 3 of paper I, is helpful. It represents the boundary surfaces between the different regions in terms of the appropriate nonsingular coordinates adapted to each region. The spacetime lines \(L (u = \pi/2)\) and \(L' (v = \pi/2)\) are identified with points \(P\) and \(P'\) respectively, these points are known as folding singularities, and are avoided by all null geodesics except for a set of null measure. There is no spacetime beyond the lines \(L\) and \(L'\) (see Fig. 1).

### 3 Mode solutions of the field equation

In order to quantize a field in our background spacetime and to compute the field two-point function which will be needed later, we need the mode solutions of the field equations. Here we consider a massless scalar field \(\phi\), the Klein-Gordon equation is
\[ \Box \phi = 0, \]  
(8)

where \( \Box \phi = (-g)^{-1/2}[(g)^{1/2}g^\mu\nu \partial_\mu \phi]_{,\nu} \), and \( g \) is the determinant of the metric \( g_{\mu\nu} \). Note that since the curvature scalar \( R \) is zero in our case, the above equation is valid whatever coupling to the curvature we may consider. The mode solutions of these equations are different in the four spacetime regions we have. In paper I details of the equations in the different regions and their solution by separation of variables were given. Here we summarize the main results and introduce some new ones which will be of use later.

Our vacuum state is the “in” vacuum, this is the physically unambiguous vacuum defined in the flat region IV before the plane waves arrive. In this region a complete set of positive frequency modes with respect to the timelike Killing vector \( \partial_{u'+v'} \) is given by:

\[
u^\text{in}_k(u, v, x, y) = \frac{1}{\sqrt{2k_-(2\pi)^3}} e^{-i2L_2k_- v - i2L_1k_+ u + ik_x x + ik_y y}, \]

(9)

where \( u \) and \( v \) are the two dimensionless null coordinates related to a physical null coordinates \( u' \) and \( v' \) by \( u' = 2L_1u, v' = 2L_2v \). The labels \( k_x, k_y \) and \( k_- \) are independent separation constants for equation (8) and the label \( k_+ \) is determined by the relation,

\[
4k_+k_- = k_x^2 + k_y^2. \tag{10}
\]

It was shown in paper I that these modes are well normalized on the hypersurface \( \{u = 0, v \leq 0\} \cup \{v = 0, u \leq 0\} \). The labels \( k_x \) and \( k_- \) are continuous but \( k_y \) is discrete if we take a cyclic spacetime in the \( y \)-direction (this is convenient but not necessary), then we identify \( k_y \) with \( m\cdot M^{-1} \) where \( m \) is an integer.

These “in” modes, \( \nu^\text{in}_k \), which define our vacuum state, have to be propagated through the spacetime. Once we know the “in” modes in a certain region we use their expression on the boundaries with the next region as boundary conditions for the “in” modes in this next region. These will be given in terms of a complete set of solutions of the field equation in such region. In the plane wave regions II and III the “in” modes are easy to find:

\[
u^\text{in}_k(x) = \frac{1}{\sqrt{2k_-(2\pi)^3}} e^{ik_x x + ik_y y} \begin{cases} \frac{1}{\cos u} e^{-i2L_2k_- v - iA(u)/(2L_2k_-)}, & \text{Region II} \\ \frac{1}{\cos v} e^{-i2L_1k_+ u - iA(v)/(2L_1k_+)}, & \text{Region III} \end{cases} \tag{11}
\]

where \( A(x) = k_1^2 f(x) + k_2^2 g(x) \) and where the two functions \( f(x) \) and \( g(x) \) are

\[
f(x) = \frac{(1 + \sin x)^2}{2 \cos x} (9 - \sin x) + \frac{15}{2} \cos x - \frac{15}{2} x - 12, \quad g(x) = \tan x. \tag{12}
\]

We use for simplicity the dimensionless labels \( k_1 = M k_x \) and \( k_2 = M k_y \) where \( k_x, k_y \) and \( k_- \) have the same meaning as in the flat region IV because these expressions for the solutions of the Klein-Gordon equation in region II and III match smoothly (i.e in a continuous and differentiable way) with the respective solutions (9) on the boundary between regions II and IV, i.e \( \{u = 0, v \leq 0\} \) and the boundary between regions III and IV, i.e \( \{v = 0, u \leq 0\} \).

In the interaction region (region I), however, this kind of mode propagation is not that simple. First, we note that equation (8) in this region can be separated as
\[
\phi(\xi, \eta, x, y) = e^{ik_x x + ik_y y} \psi_{\alpha \xi}(\xi) \varphi_{\alpha \eta}(\eta),
\]
where coordinates \(\xi, \eta\) are related to the usual null coordinates \(u, v\), by (3) and the differential equations for \(\psi_{\alpha \xi}(\xi)\) and \(\varphi_{\alpha \eta}(\eta)\) are (dropping the indices)
\[
\psi,_{\xi\xi} - (\tan(\xi)) \psi,_{\xi} + \left(\alpha + k_1^2 \frac{(1 + \sin(\xi))^4}{\cos^2(\xi)}\right) \psi = 0,
\]
\[
\varphi,_{\eta\eta} - (\tan(\eta)) \varphi,_{\eta} + \left(\alpha - k_2^2 \frac{1}{\cos^2(\eta)}\right) \varphi = 0,
\]
where \(\alpha\) is a dimensionless separation constant and \(k_x, k_y\) (or \(k_1 = Mk_x, k_2 = Mk_y\)) are the same labels as in regions IV, II or III.

The solutions of (15) can be expressed in terms of associated Legendre functions and in the cyclic case they can be written in terms of spherical harmonics, \(Y^m_l(y/M, \pi/2 - \eta)\), with discrete labels related to \(\alpha\) and \(k_2\) simply by \(\alpha = l(l+1)\) and \(k_2 = m\). The solutions of (14), however, cannot be written in terms of known functions. If we use the variable \(\xi^*\) instead of \(\xi\) defined in (6), i.e. \(d\xi = \cos(\xi)M^{-1} (1 + \sin(\xi))^{-2} d\xi^*\), and introduce a new function \(\gamma\) defined by \(2\gamma = (1 + \sin(\xi)) \psi\), equation (14) becomes
\[
\gamma,_{\xi^*\xi^*} + \omega_1^2 \gamma = 0,
\]
where
\[
\omega_1^2(\xi) \equiv M^{-2} \left[k_1^2 + \frac{\cos^2(\xi)}{(1 + \sin(\xi))^4} \left(l(l+1) + \frac{2}{1 + \sin(\xi)}\right)\right] \equiv M^{-2}[k_1^2 + V_1(\xi)].
\]
Note that a new function \(V_1(\xi)\) has been defined. Equation (19) can be solved by a WKB method and the solutions \(\gamma(\xi^*)\) can be expanded up to any adiabatic order as
\[
\gamma(\xi^*) = \frac{1}{\sqrt{W_1(\xi)}} e^{\pm i \int_0^{\xi} W_1(\xi') d\xi'},
\]
where \(W_1(\xi)\) is given, up to adiabatic order 4 (this is the order we need) [29], by
\[
W_1(\xi) = \omega_1 + \frac{A_2}{\omega_1^3} + \frac{B_2}{\omega_1^5} + \frac{A_4}{\omega_1^7} + \frac{B_4}{\omega_1^9} + \frac{C_4}{\omega_1^{11}} + \frac{D_4}{\omega_1^{19}},
\]
where,
\[
A_2 = -\frac{1}{8} \frac{\ddot{V}_1}{M^2}, \quad B_2 = \frac{5}{35} \frac{\dot{V}_1^2}{M^4},
\]
\[
A_4 = \frac{1}{32} \frac{\dddot{V}_1}{M^2}, \quad B_4 = -\frac{1}{128} \frac{28 \dot{V}_1 \ddot{V}_1 + 19 \dot{V}_1^2}{M^4}, \quad C_4 = \frac{221}{258} \frac{\dddot{V}_1}{M^6}, \quad D_4 = -\frac{1105}{2048} \frac{\dddot{V}_1^2}{M^8},
\]
an overdot means derivative with respect to \(\xi^*\), and \(A_n, B_n, ...,\) denote the \(n\) adiabatic terms in \(W_1(\xi)\). From the two exponential factor signs in the WKB solution (18), we choose the
negative sign for consistency with the boundary conditions imposed by (11), as we will see later. This gives positive frequency solutions with respect to the timelike vector \( d/\partial \xi^* \) on the horizon since these solutions are asymptotically proportional to \( \exp(-i|k_x|\xi^*) \). A complete set of solutions of the Klein-Gordon equation in the interaction region is thus given, in the cyclic case, by

\[
\phi_k(x) = \frac{2C}{1 + \sin \xi} \frac{1}{\sqrt{W_1(\xi)}} e^{ik_xx - i \int_0^\xi W_1(\xi') d\xi'} Y^m_l(y/M, \pi/2 - \eta),
\]

where \( C \) is a normalization constant which can be easily calculated imposing that these solutions are well normalized on the horizon (\( \xi = \pi/2 \)). We will find its value in a next section.

Finally, the “in” modes in the interaction region, with the boundary conditions imposed by (11), can be written as,

\[
u^\text{in}_k(\xi, \eta, x, y) = \sum_l C_l \phi_k(\xi, \eta, x, y),
\]

where the coefficients \( C_l \) depend on \( l \) and on the separation constants used to label the modes in region II, i.e. \( k_x, k_y, k_- \). We devote the next two sections to find an explicit expression for (22).

### 4 Geometrical optics

The solutions of the Klein Gordon equation in the flat and single plane wave regions, (9) and (11) respectively, can be understood in terms of geometrical optics because the geometrical optics approximation is exact in these regions, i.e. the orthogonal curves to the surfaces of constant phase (the rays) follow exactly null geodesics. This will allow to “localize” the Klein-Gordon solutions in the spacetime and to determine which solutions will be relevant in later calculations. The geometrical optics also gives information on the meaning of the parameters which label the “in” modes (11), i.e. \( k_\pm, k_x, k_y \), and the parameters \( k_x, m, l \), which label the modes (21) in the interaction region, and on their connections.

In the flat region the normalized “in” modes (9) are purely flat modes which have positive frequency with respect to the timelike Killing vector \( \partial_t = \partial_{u'+v'} \). The energy of these modes, \( E_{IV} \), may be defined as the eigenvalue of \( \partial_t \), \( \partial_t u^\text{in(IV)}_k = -iE_{IV} u^\text{in(IV)}_k \). In terms of the null momenta \( k_\pm \) it is,

\[
E_{IV} = k_+ + k_-.
\]

We can now proceed in the same way in the single plane wave regions. In region II, the normalized “in” modes take the form (11), but now, instead of a timelike vector, \( \partial_t \), we have a null Killing vector \( \partial_{u'} \), and (11) are its eigenfunctions with positive eigenvalue \( k_- \), i.e. \( \partial_{u'} u^\text{in(II)}_k = -ik_- u^\text{in(II)}_k \). Similarly, in region III, \( k_\pm \) are eigenvalues of \( \partial_{u'} \). In this case, however, these eigenvalues cannot be directly interpreted as energies. The mode solutions in the plane wave regions II and III have the form \( Ce^{iS} \), where \( S = \text{constant} \) define the surfaces of constant phase of the modes and it satisfies exactly the null condition...
\( g^{\mu\nu} S_{\mu} S_{\nu} = 0 \). Thus, we might define the energy of these modes as the variation of \( S \) in the direction of the vector \(-\partial_t\), i.e. \( E_{II/III} = -\partial_t S_{II/III} \). From (12), we identify

\[
S_{II} = -2L_2 k_- v - \frac{f(u)k_1^2 + g(u)k_2^2}{2L_2 k_-} + k_x x + k_y y,
\]

and \( S_{III} \), by a similar expression. The tangent vector fields to the congruencies of null geodesics, expressed in \((u, v, x, y)\) components are, in region II,

\[
V_{II}^\mu = \frac{-1}{2M^2(1 + \sin u)^2} \left( 2L_2 k_-, \frac{\dot{f}(u)k_1^2 + \dot{g}(u)k_2^2}{2L_2 k_-}, 2Mk_1 \ddot{f}(u), 2Mk_2 \ddot{g}(u) \right),
\]

and a similar expression for \( V_{III}^\mu \) in region III.

By integration we obtain a congruence of null geodesics parametrized by their momenta \( k_\pm, k_x, k_y \) and their initial positions \( v_0, x_0, y_0 \) at \( u = 0 \) in region II and \( u_0, x_0, y_0 \) at \( v = 0 \) in region III. Each congruence at fixed values of the momenta represents the set of rays orthogonal to the surfaces \( S_{II/III} = \text{constant} \), so that these rays “localize” in the spacetime the mode labeled by \( k_\pm, k_x, k_y \). A simple inspection of the expression for the tangent vector fields above shows that in the plane wave region II the rays for each mode are peaked near the line \( \mathcal{L} (u = \pi/2) \) and reach the interaction region at points close to the folding singularity \( \mathcal{P} (u = \pi/2, v = 0) \) if the momenta satisfy the conditions,

\[
2L_2 k_- \gg 1, \quad 16k_1^2 + k_2^2 \ll 2L_2 k_-,
\]

where the second condition comes from the property,

\[
\dot{f}(u)k_1^2 + \dot{g}(u)k_2^2 \leq (16k_1^2 + k_2^2) \cos^2 u,
\]

of the functions \( f \) and \( g \), defined in (12). Similar conditions hold for modes in the plane wave region III, by the simple substitution of \( 2L_2 k_- \) by \( 2L_1 k_+ \), i.e. the rays for each mode are peaked near the coordinate singularity line \( \mathcal{L}' (v = \pi/2) \) and reach the interaction region at points close to the folding singularity \( \mathcal{P}' (u = 0, v = \pi/2) \) if

\[
2L_1 k_+ \gg 1, \quad 16k_1^2 + k_2^2 \ll 2L_1 k_+.
\]

Due to the relation (10), the two conditions (24) and (25) are mutually exclusive. These geometrical properties of the “in” modes indicate that any calculation involving “in” modes in regions close to the folding singularity \( \mathcal{P} \), requires only modes in region II with momenta satisfying (24) and no modes in region III. Equivalently any calculations in regions close to the folding singularity \( \mathcal{P}' \), requires only modes in region III with momenta satisfying (25) and no modes in region II. Note that the larger is the energy \( E_{IV} \) of the “in” modes in the flat region, the closer their rays get to the lines \( \mathcal{L} \) or \( \mathcal{L}' \), and this means that the “in” modes are blueshifted towards these lines and consequently towards the horizon. In fact, this is a general property of plane waves (16) and it will become very important in the evaluation of the vacuum expectation value of the stress-energy tensor when summations over “in” modes are performed.

Going back to the energy of the “in” modes in the plane wave region II we have, according to the previous definition, \( E_{II} = -\partial_t S_{II} = k_- + (\dot{f}(u)k_1^2 + \dot{g}(u)k_2^2)(4k_-)^{-1} \), which is not constant, because \( \partial_t \) is not a Killing vector, and cannot be interpreted properly as the energy
of the mode. However, close to the origin \( u = v = 0 \), i.e. just where the plain waves collide, \( E_{II} \) coincides with \( E_{IV} \), since
\[
E_{II}|_{u,v \sim 0} = k_- + \frac{k_x^2 + k_y^2}{4k_-} + O(u^2) = k_- + k_+ + O(u^2) \sim E_{IV},
\]
and close to the line \( \mathcal{L} \) we obtain, in powers of \( \cos u \),
\[
E_{II} = k_- + \frac{16k_x^2 + k_y^2}{4k_-} \cos^2 u + O(\cos^0 u).
\]
However, if we recall that for the “in” modes close to the line \( \mathcal{L} \) in region II the inequalities (24) are satisfied, then there is a large interval of the variable \( u \) for which,
\[
\frac{16k_x^2 + k_y^2}{2L_2k_-} \cos^2 u \ll 1.
\]
In this interval \( E_{II} \simeq k_- \simeq E_{IV} \), which is the eigenvalue of the null Killing vector \( \partial_\nu \), thus it can be interpreted unambiguously as the energy of the “in” modes in region II. Similarly for “in” modes close to de line \( \mathcal{L}' \) in region III there is a large interval of the variable \( v \) for which,
\[
\frac{16k_1^2 + k_2^2}{2L_1k_+} \cos^2 v \ll 1.
\]
and their energy is unambiguously defined by \( E_{III} \simeq k_+ \simeq E_{IV} \). We will see in the next section that these intervals, (26) and (27), play an important role.

In region I, the mode solutions of the Klein-Gordon equation are given by (21). Although the geometrical optics approximation is not exact in this region we can still make good use of it, since this approximation is compatible with the WKB expansion of equation (10), which depends on the dimensionless timelike coordinate \( \xi = u + v \) (or \( \xi^* \)). The energy of these modes can be defined by the variation of the surfaces of constant phase, \( S = -\int_0^\xi W_1(\xi')d\xi'^* + ik_xx + ik_yy \), along the timelike vector \( M^{-1}\partial_\xi \), i.e., \( E_I = -M^{-1}\partial_\xi S_\pm \). If we consider the range of values \( \rho \equiv |k_1|L(\cos \xi)^{-1} \ll 1 \) in analogy with (26) and (27), \( E_1 \sim M^{-1}[1 + O(\rho^2)] \sim L^{-1} \). In other words, the parameter \( M^{-1}L^{-1} = M^{-1}[l(l+1)]^{1/2} \), that labels the mode solutions of the Klein-Gordon equation in the interaction region, takes the role of an energy. Recall that this energy is positive just because we have choosen the minus sign in the WKB solution (13). This suggests that we can take the value for the “in” modes in the interaction region close to the horizon, \( u_k^{in} \), as a single function \( \phi_k \) (21) instead of the linear superposition (22), provided that the energy of \( \phi_k \), i.e \( E_I \) equals to \( E_{II} \) and to \( E_{III} \), at the boundaries of regions II and III with region I, respectively. In other words we can take \( u_k^{in} \sim \phi_k \) where \( \phi_k \) is labeled by a parameter \( L \) satisfying,
\[
E_I \sim M^{-1}L^{-1} \propto k_- + k_+.
\]
In the next section we will see that this identification is really possible and we will find the proportionality constant. Note that the “in” modes which are relevant at the boundary between region II and region I satisfy \( 2L_2k_- \gg 1 \) and \( 2L_1k_+ \sim 0 \), so that \( E_1 \sim E_{II} \) and \( L \propto L_2k_- \), and the modes which are relevant at the boundary between region III and region I satisfy \( 2L_1k_+ \gg 1 \) and \( 2L_2k_- \sim 0 \), then \( E_1 \sim E_{III} \) and \( L \propto L_1k_+ \).
5 Adiabatic expansion of the “in” modes

We seek a solution of the Klein-Gordon equation near the horizon that satisfies the boundary conditions imposed by the “in” modes, which have been propagated from the flat region through regions II and III. The problem is formally solved using a Bogoliubov transformation provided \( \{ u^\text{in}_k \} \), i.e. the “in” modes, and the set of modes \( \{ \phi_{k'}(x) \} \), of the field equation in the interaction region are complete and orthonormal, so that we can write,

\[
u^\text{in}_k(x) = \sum_{k'} \alpha_{kk'} [\phi_{k'}(x) + \beta_{kk'} \phi_{k'}^*(x)].
\]

All the information on the boundary conditions is contained in the Bogoliubov coefficients,

\[
\alpha_{kk'} = \langle u^\text{in}_k(x), \phi_{k'}(x) \rangle_\Sigma, \quad \beta_{kk'} = -\langle u^\text{in}_k(x), \phi_{k'}^*(x) \rangle_\Sigma,
\]

where the inner product is evaluated on the boundaries of regions II and III with region I and it is given, see paper I, by,

\[
\langle \phi_1, \phi_2 \rangle = -i \int dx \, dy \left[ \int_0^{\pi/2} \cos^2 u \left( \phi_1 \partial_u \phi_2^* \right)_{v=0}^u du + \int_0^{\pi/2} \cos^2 v \left( \phi_1 \partial_v \phi_2^* \right)_{u=0}^v dv \right].
\]

The coefficients \( \alpha_{kk'}^{(A)} \) and \( \beta_{kk'}^{(A)} \) can be calculated using (29), (30) and the expressions (21), (11) for the functions \( \phi_{k'}(x) \) and \( u^\text{in}_k \), respectively. The superscript \( (A) \) indicates that the values of \( \phi_{k'}(x) \) are known up to a certain adiabatic order \( A \). We have,

\[
\alpha_{kk'}^{(A)} = i M \sqrt{\pi} (-1)^m \delta(k_x - k'_x) \delta_{mm'}, A_{kk'}, \quad \beta_{kk'}^{(A)} = -i M \sqrt{\pi} \delta(k_x + k'_x) \delta_{m(-m')} B_{kk'},
\]

where \( A_{kk'}^{(A)} \) and \( B_{kk'}^{(A)} \) are given by,

\[
A_{kk'}^{(A)} = \frac{2 C}{\sqrt{k_-}} \int_0^{\pi/2} du \, \cos^2 u \left[ \frac{e^{-iA(u)/(2L_2k_-)} \partial_u \varphi(\eta)}{\cos u} \frac{e^{i \int_0^\xi W_1(\xi')d\xi'}}{\sqrt{W_1(\xi)}} \right]_{v=0}^u + \frac{2 C}{\sqrt{k_-}} \int_0^{\pi/2} dv \, \cos^2 v \left[ \frac{e^{-iA(v)/(2L_1k_+)} \partial_v \varphi(\eta)}{\cos v} \frac{e^{i \int_0^\xi W_1(\xi')d\xi'}}{\sqrt{W_1(\xi)}} \right]_{u=0}^v,
\]

\[
B_{kk'}^{(A)} = \frac{2 C}{\sqrt{k_-}} \int_0^{\pi/2} du \, \cos^2 u \left[ \frac{e^{-iA(u)/(2L_2k_-)} \partial_u \varphi(\eta)}{\cos u} \frac{e^{-i \int_0^\xi W_1(\xi')d\xi'}}{\sqrt{W_1(\xi)}} \right]_{v=0}^u + \frac{2 C}{\sqrt{k_-}} \int_0^{\pi/2} dv \, \cos^2 v \left[ \frac{e^{-iA(v)/(2L_1k_+)} \partial_v \varphi(\eta)}{\cos v} \frac{e^{-i \int_0^\xi W_1(\xi')d\xi'}}{\sqrt{W_1(\xi)}} \right]_{u=0}^v,
\]

where \( \varphi(\eta) \) is a shortened notation for

\[
\varphi(\eta) = Y^m_l(0, \pi/2 - \eta) = \left( \frac{2l + 1 (l - m)!}{4\pi (l + m)!} \right)^{1/2} P^m_l(-\sin \eta),
\]
and where $W_1(\xi)$ is known up to a certain adiabatic order $A$.

The problem now is to evaluate these integrals. Since all of them are of the type, $f \, dx \, f(x) \exp \{i\theta(x)\}$, we search for the intervals of the integration variable where the modulus of the integration function, $f(x)$, is much larger than the variation of its phase $\theta(x)$, i.e $\dot{\theta}(x) \ll f(x)$. In particular, we search for the phase stationary points, i.e. the values of the integration variable for which $\dot{\theta}(x) = 0$.

Using that $\xi = u + v$, $\eta = v - u$ and defining star variables $x^*$ by $dx^* = dx \, M(1 + \sin x)^2/\cos x$, the phases of the integration functions in (31) and (32) can be written, respectively, as

$$\frac{A(u)}{2L_2k_-} \pm \int_0^u W_1(u')du'^*, \quad \frac{A(v)}{2L_1k_+} \pm \int_0^v W_1(v')dv'^*, \quad (34)$$

We can study the problem qualitatively using the value of $W_1(\xi)$ up to adiabatic order zero,

$$W_1^{(0)} = \frac{1}{M} \left[ k_1^2 + \frac{\cos^2 u}{(1 + \sin u)^2} \left( \frac{1}{L^2} + \frac{2}{1 + \sin u} \right) \right]^{1/2}, \quad (35)$$

where we have used the notation $l(l+1) = L^2$. We have already seen in the previous section, using geometrical optics, that the relevant modes involved in any calculation near the horizon ($\xi = \pi/2$) are those which satisfy (26) in region II (or (27) in region III) and this implies (in region II) that the labels $L^{-1}$ and $L_2k_-$, which parametrize the Klein-Gordon modes (21) in the interaction region, and the “in” modes (14) are related. Let us assume that they are proportional, then it is easy to see that expressions (34) have stationary points only when the minus sign is involved and these stationary points are such that $L^{-2}\cos^2 u$ is of the order of $16k_1^2 + k_2^2$. This means, in particular, that $L^{-2}\cos^2 u$ is of the order of $k_2^2 = m^2$; but for these values, the associated Legendre function $P_1^m(-\sin u)$ has an oscillating behaviour that we must take into account in order to evaluate the phase stationary points. Note that for the range of values $16k_1^2 + k_2^2 \gg L^{-2}\cos^2 u$, the associated Legendre polynomial $P_1^m(-\sin u)$, with $k_2 = m$, behaves as $\cos^m u$ and in that case we have integrals $\int dx \, f(x) \exp \{i\theta(x)\}$ without stationary points and with $f(u) \sim \cos^{m-1} u \ll \dot{\theta}(u) \sim \cos^{-2} u$, which lead to negligible contributions.

In the following we will find the expressions of the Klein-Gordon solutions $\phi_k$ in the interaction region for the range of values such that $|k_1|L(\cos \xi)^{-1} \ll 1$ and $|k_2|L(\cos \eta)^{-1} \ll 1$. We have seen that solutions $\phi_k$ can be separated in two functions $\psi(\xi)$ and $\varphi(\eta)$, i.e. $\phi_k(\xi, \eta) = \psi(\xi) \varphi(\eta)$, which satisfy the differential equations (14) and (13). The first of these equations can be adiabatically solved up to any order, see (18), and the second one, although exactly solvable in terms of associated Legendre functions, can also be WKB solved up to any adiabatic order in the range of values such that $|k_2|L(\cos \eta)^{-1} \ll 1$. For this we define a new coordinate $\eta^*$ by $d\eta^* = d\eta \, M(\cos \eta)^{-1}$ in (13) and obtain,

$$\varphi_{\eta^*} + \omega_2^2 \varphi = 0, \quad (36)$$

where

$$\omega_2^2(\eta) \equiv \frac{1}{M^2} \left[ -k_2^2 + \frac{\cos^2 \eta}{L^2} \right] \equiv \frac{1}{M^2} \left[ -k_2^2 + V_2(\eta) \right], \quad (37)$$

which is WKB solved for the values $|k_2|L(\cos \eta)^{-1} \ll 1$ as
\[
\varphi(\eta) = \frac{C_1}{\sqrt{W_2(\eta)}} e^{-i \int_0^\eta W_2(\eta')d\eta'} + \frac{C_2}{\sqrt{W_2(\eta)}} e^{i \int_0^\eta W_2(\eta')d\eta'}
\]

(38)

where \( W_2(\eta) \) can be written up to adiabatic order 4 in exactly the same way as \[(19),
\]
\[
W_2(\eta) = \omega_2 + \frac{\hat{A}_2}{\omega_2^2} + \frac{\hat{B}_2}{\omega_2^6} + \frac{\hat{A}_4}{\omega_2^2} + \frac{\hat{B}_4}{\omega_2^6} + \frac{\hat{C}_4}{\omega_2^9} + \frac{\hat{D}_4}{\omega_2^{11}},
\]

(39)

where the expression of \( \hat{A}_2, \hat{B}_2, \cdots \) are formally the same expressions \[(20) \]for \( A_2, B_2, \cdots, \) but with \( V_1(\xi) \) substituted by \( V_2(\eta) \) (defined in \[(37), \]and with the overdot meaning derivative with respect to \( \eta' \). The two arbitrary constants \( C_1 \) and \( C_2 \) will be fixed imposing that the solution \[(38) \]is asymptotically equivalent to the associated Legendre function \( P_l^m(-\sin \eta) \) with \( L^2 = \lceil l(l+1) \rceil^{-1} \) and \( m = k_2 \), for the values \( |k_2|L(\cos \eta)^{-1} \ll 1 \) (see Appendix A).

Before going through, we need to introduce some useful definitions to relate the adiabatic expansions of \( W_1(\xi) \) and \( W_2(\eta) \). Let us define,

\[
q^2 = (16k_1^2 + k_2^2)^{-1}, \quad V^2 = 16k_2^2q^2, \quad U^2 = k_2^2q^2,
\]

note that in practice, and without loss of generality, we can take \( q^2 < 1 \), because the case \( q^2 \gg 1 \), which corresponds to \( k_1 \sim k_2 \sim 0 \), is equivalent to consider \( L \ll 1 \) in the equations \[(16) \]and \[(37) \]. Note also that \( 0 \leq U, V \leq 1 \), and that by construction, \( V^2 + U^2 = 1 \). With these definitions we can write,

\[
M^2\omega_1^2 = \frac{V^2}{16q^2} + \frac{\cos^2 \xi}{(1 + \sin \xi)^2} \left( \frac{1}{L^2} + \frac{2}{1 + \sin \xi} \right), \quad M^2\omega_2^2 = \frac{U^2}{q^2} + \frac{\cos^2 \eta}{L^2}.
\]

For parameters that satisfy \( |k_1|L(\cos \xi)^{-1} \ll 1 \) and \( |k_2|L(\cos \eta)^{-1} \ll 1 \), we introduce new functions \( p(\xi) \equiv Lq^{-1}(\cos \xi)^{-1} \) and \( p_1(\eta) \equiv Lq^{-1}(\cos \eta)^{-1} \), so that \( p \ll 1, p_1 \ll 1 \).

We can now expand \[(21) \]and \[(11) \]in powers of \( p \) and \( p_1 \) and substitute into \[(31) \]and \[(32) \]. Then the dominant integrals connecting the “in” modes between regions II and I can be written as (see Appendix A for details).

\[
2M \int du \exp \left\{ \int du \left[ p^2 \left( \frac{V^2}{32} (1 + \sin u)^4 + \frac{U^2}{2} \right) \left( \frac{1}{L_2k_-} - L \right) + O(p^4) \right] \right\} \left[ \frac{1}{L_2k_-} + L \right] + O(p^4) \right\} \right\} \sim \delta \left( L_2k_- - L^{-1} \right).
\]

The dominant integral connecting the “in” modes between regions III and I is obtained from this by changing \( L_2k_- \) by \( L_1k_+ \). This means that only the \( \alpha_{kk'} \) connection coefficients \[(29) \]are relevant in the propagation of the “in” modes through the boundaries between regions II and III with region I. In other words, the set of modes \( \{ \phi_k \} \) of the Klein-Gordon equation for region I correspond to the same initial vacuum state for calculations involving points near the horizon \( \xi = \pi/2 \). This important result, as we have pointed out in the geometrical optics analysis, can be seen as a consequence of the blueshift that the “in” modes suffer by a single plane wave. Although \( k_- \) (or \( k_+ \)) is a continuous label and \( L \) is a discrete label, this should not be a problem since we can use the wave packet formalism to discretize \( k_- \).
6 “Point splitting” regularization technique

In this section we briefly review, for the computational purposes of the following sections, the “point-splitting” regularization technique to calculate the expectation value of the stress-energy tensor of a scalar quantum field in some physical state. The stress-energy tensor of the field may be derived by functional derivation of the action for the scalar field with respect to the metric. When the field is massless and the Ricci tensor is zero it is \([31]\)

\[
T_{\mu\nu} = (1 - 2\xi) \phi,_{\mu} \phi,_{\nu} + (2\xi - \frac{1}{2}) g_{\mu\nu} g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} - 2\xi \phi,_{\mu\nu} \phi + \frac{1}{2} \xi g_{\mu\nu} \phi \nabla^\alpha \nabla_\alpha \phi,
\]

where \(\xi\) is the coupling parameter of the field to the curvature.

To quantize, the field \(\phi\) is promoted into a field operator acting over a given Hilbert space \(\mathcal{H}_\phi\) \([31], [33]\), \(\phi(x) = \sum_k a_k u_k(x) + a_k^\dagger u_k^*(x)\), where \(a_k, a_k^\dagger\) are the standard creation and annihilation operators and \(\{u_k(x)\}\) is a complete and orthonormal set of solutions of the Klein-Gordon equation \([8]\). Mathematically the field operator \(\phi(x)\) is a point distribution, therefore, the quantum version of the stress-energy tensor \((41)\) is mathematically pathological because it is quadratic in the field and its derivatives. One possible way to give sense to that expression is to note that the formula \((41)\) can be formally recovered as,

\[
\langle T_{\mu\nu}\rangle(x) = \lim_{x \to x'} \mathcal{D}_{\mu\nu} G^{(1)}(x, x'),
\]

where \(G^{(1)}(x, x')\) is a Green function of the field equation defined as the vacuum expectation value of the anticommutator of the field, and called the Hadamard function,

\[
G^{(1)}(x, x') = \langle \{\phi(x), \phi(x')\} \rangle = \sum_k \{u_k(x) u_k^*(x') + u_k^*(x') u_k(x)\}.
\]

As a product of distributions at different points this is mathematically well defined. The differential operator \(\mathcal{D}_{\mu\nu}\) is given in our case by,

\[
\mathcal{D}_{\mu\nu} = (1 - 2\xi) \frac{1}{4} (\nabla_{\mu'} \nabla_{\nu} + \nabla_{\nu'} \nabla_{\mu}) + (2\xi - \frac{1}{2}) g_{\mu\nu} \frac{1}{4} \left(\nabla_\alpha \nabla^\alpha + \nabla_{\alpha'} \nabla^{\alpha'}\right) - \xi \frac{1}{2} \left(\nabla_\alpha \nabla^\alpha + \nabla_{\alpha'} \nabla^{\alpha'}\right).
\]

However, the above differential operation and its limit has no immediate covariant meaning because \(G^{(1)}(x, x')\) is not an ordinary function but a \(\text{biscalar}\) and the differential operator \(\mathcal{D}_{\mu\nu}\) is \(\text{nonlocal}\), thus we need to deal with the nonlocal formalism of bitensors. In Appendix B we give a summary of the main properties of bitensors, how to parallel transport from points \(x\) to \(x'\) and how to average after the coincidence limit, \(x \to x'\), is taken.

The above procedure still leads to a divergent quantity since we know that even in flat spacetime \(G^{(1)}(x, x')\) has a short-distance singularity and that a “vacuum” subtraction has to be performed to \(G^{(1)}(x, x')\) in order to obtain a regularized value. To regularize we assume that \(G^{(1)}(x, x')\), has a short-distance singular structure given by

\[
S(x, x') = \frac{2}{(4\pi)^2} \Delta^{1/2}(x, x') \left[ -\frac{2}{\sigma(x, x')} + v(x, x') \ln \sigma(x, x') + w(x, x') \right],
\]

\[x \to x',\]
where $\sigma(x, x')$ is the \textit{geodetic biscalar} (see Appendix B), $\Delta$ is the \textit{Van Vleck-Morette} determinant \cite{VanVleck1932}, which is singularity free in the coincidence limit, and where $v(x, x')$ and $w(x, x')$ are biscalars with a well-defined coincidence limit for which we assume the following covariant expansions,

$$v(x, x') = \sum_{l=0}^{\infty} v_l(x, x') \sigma^l(x, x'), \quad w(x, x') = \sum_{l=0}^{\infty} w_l(x, x') \sigma^l(x, x').$$ (46)

A Green function expressed in this form is usually called an \textit{elementary Hadamard solution}, the name of which comes from the work of Hadamard on the singular structure for elliptic and hyperbolic second order differential equations. Note, however, that this Hadamard singular structure is not a general feature of any Green function of the Klein-Gordon equation. In other words, although for an extensive range of spacetime and vacuum states, the vacuum expectation value of the anticommutator of the field, $G^{(1)}(x, x')$, has this singular form, this is not a general property. However, a theorem states that if $G^{(1)}(x, x')$ has the singular structure of an elementary Hadamard solution in a neighbourhood of a Cauchy surface of an arbitrary hyperbolic spacetime, then it has this structure everywhere \cite{Haag1967, O'Raifeartaigh1971}. As a corollary of this theorem, $G^{(1)}(x, x')$ has this singular structure if the spacetime is flat to the past of a spacetime Cauchy surface, as is the case of our colliding plane wave spacetime. This and other considerations lead to a proposal by Wald \cite{Wald1982} that any physically reasonable quantum state must be a \textit{Hadamard state}, that is to say, a state for which $G^{(1)}(x, x')$ takes the short-distance singular structure of an elementary Hadamard solution.

The coefficients $v_l(x, x')$ and $w_l(x, x')$ can be directly obtained by substitution in the differential equation, $\Box x S(x, x') = 0$. Recursion relations for $v_l(x, x')$ and $w_l(x, x')$ are then obtained \cite{HEI1985}. These relations uniquely determine all the $v_l(x, x')$ coefficients but the coefficients $w_l(x, x')$ can be written in terms of an arbitrary term $w_0(x, x')$. Up to order $\sigma$, $v(x, x')$ is given by

$$v(x, x') = -a_1(x, x') - \frac{1}{2} a_2(x, x') \sigma + \cdots,$$ (47)

where $a_1(x, x')$ and $a_2(x, x')$ are the Schwinger-DeWitt coefficients \cite{Wald1982}, which in our case ($R_{\mu\nu} = 0$) reduce to

$$a_1(x, x') = -\frac{1}{180} R^\alpha\beta\gamma\mu R_{\alpha\beta\gamma\nu} \sigma^\mu \sigma^\nu + O(\sigma^3), \quad a_2(x, x') = \frac{1}{180} R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta} + O(\sigma).$$ (48)

Only the coefficients $v_l(x, x')$ are related to the singular structure of $G^{(1)}(x, x')$ in the coincidence limit, and they are uniquely determined by the spacetime geometry. This means that given any two Hadamard elementary solutions in a certain spacetime geometry, both have the same singularity structure in the coincidence limit; therefore given two vacuum Hadamard states, $|0\rangle$ and $|\overline{0}\rangle$, $G^{(1)}(x, x') = \langle 0|\{\phi(x), \phi(x')\}|0\rangle$ and $\overline{G}^{(1)}(x, x') = \langle \overline{0}|\{\phi(x), \phi(x')\}|\overline{0}\rangle$, they have the same singular structure. Their finite parts, however, may differ because the two vacuum states are related to different boundary conditions, which are global spacetime features. Mathematically this comes from the fact that the term $w_0(x, x')$ in the elementary Hadamard solution is totally arbitrary, fixing $w_0(x, x')$ we fix a particular boundary condition. This suggests a possible renormalization procedure \cite{HEI1985, HEI1986, HEI1987}; we can eliminate
the non-physical divergences of any \( G^{(1)}(x, x') \) without alterations in the particular physical boundary conditions by subtracting an elementary Hadamard solution with the particular value \( w_0(x, x') = 0 \), which is the value that corresponds to the flat space case. In other words, we define the following regularized biscalar,

\[
G_B^{(1)}(x, x') = G^{(1)}(x, x') - S(x, x')|_{w_0=0}.
\]

(49)

Then by means of \( G_B^{(1)}(x, x') \) we can construct a \( \langle T_{\mu\nu}^B \rangle \) by differentiation with the nonlocal operator \( (44) \).

This regularization procedure, however, fails to give a covariantly conserved stress-energy tensor since it can be seen \([36]\) that for a massless conformal scalar field (i.e. \( \xi = 1/6 \)),

\[
\nabla^\nu \langle T_{\mu\nu}^B \rangle = \frac{1}{4} \lim_{x \to x'} \nabla_\mu \left( \Box_{x'} + \frac{1}{6} R(x') \right) G_B(x, x') = \frac{1}{64\pi^2} \nabla_\mu a_2(x),
\]

(50)

where \( a_2(x) \) is the coincidence limit of the Schwinger-DeWitt coefficient \( a_2(x, x') \) in \((48)\).

In particular for a spacetime with null curvature \( R = 0 \), such as our colliding spacetime, this is true for any coupling. Thus to ensure covariant conservation, we must introduce an additional prescription:

\[
\langle T_{\mu\nu}(x) \rangle = \langle T_{\mu\nu}^B(x) \rangle - \frac{a_2(x)}{64\pi^2} g_{\mu\nu}.
\]

(51)

Note that this last term is responsible for the trace anomaly in the conformal coupling, because even though \( \langle T_{\mu\nu}^B(x) \rangle \) has null trace when \( \xi = 1/6 \), the trace of \( \langle T_{\mu\nu}(x) \rangle \) is given by \( \langle T_{\mu}^\mu \rangle = -a_2(x)(16\pi^2)^{-1} \).

The regularization prescription just given satisfies the well known four Wald’s axioms \([37, 38, 39, 33]\), a set of properties that any physically reasonable expectation value of the stress-energy tensor of a quantum field should satisfy. There is still an ambiguity in this prescription since two independent conserved local curvature terms, which are quadratic in the curvature, can be added to this stress-energy tensor. This two-parameter ambiguity, however, cannot be resolved within the limits of the semiclassical theory, it may be resolved in a complete quantum theory of gravity \([33]\).

Note, however, that in some sense this ambiguity does not affect the knowledge of the matter distribution because a tensor of this kind belongs properly to the left hand side of Einstein equations, i.e. to the geometry rather than to the matter distribution.

### 7 Vacuum expectation value of the stress-energy tensor for a single plane wave

A simple nontrivial example that nicely illustrates the point splitting technique for the regularization of the vacuum expectation value of the stress-energy tensor is the particular case of a single gravitational plane wave, i.e. the case of regions II and III, \( \langle T_{\mu\nu}^{II} \rangle \) and \( \langle T_{\mu\nu}^{III} \rangle \).

The result is not new, it is known that the expectation value of the stress-energy tensor in the “in” state, i.e. the vacuum state at the flat region before the passing of the wave, is exactly zero \([13, 28]\). Let us start with the construction of the Hadamard function \( G^{(1)}(x, x') \), \([43]\), if the vacuum state is the “in” vacuum state, we have
\[ G^{(1)}(x, x') = \sum_k u_k(x) u_k^*(x') + \text{c.c.} = \frac{\sec u \sec u'}{2(2\pi)^3} \int_0^\infty \frac{dk_-}{k_-} e^{-i2Lz_k-} \left( v-v' \right) \times \int_{-\infty}^\infty dk_x e^{ik_x (x-x')} [f(u)-f(u')] k_x^2/k_- \int_{-\infty}^\infty dk_y e^{ik_y (y-y')} [g(u)-g(u')] k_y^2/k_- + \text{c.c.}, \]

where \( u_k \) are the “in” modes (11) of region II. It is easy to compute analytically the above “mode sum” since only Gaussian integrals are involved, the result is

\[
G^{(1)}(x, x') = \frac{2}{4M^2(2\pi)^2} \sec u \sec u' \sqrt{|f(u) - f(u')| [g(u) - g(u')] \left\{ 4M^2(v-v') + \left( \frac{(x-x')^2}{f(u) - f(u')} + \frac{(y-y')^2}{g(u) - g(u')} \right) \right\}^{-1}. \tag{52}
\]

We can now proceed with the point splitting formalism by setting the points \( x \) and \( x' \) at the endpoints of a non-null geodesic parametrized by its proper distance. Let us denote by \( \bar{\tau} \) the midpoint on the geodesic, which lies at equal proper distance from \( x \) and \( x' \), i.e.,

\[ x^\mu = x^\mu (\bar{\tau} + \epsilon), \quad \bar{x}^\mu = x^\mu (\bar{\tau}), \quad x'^\mu = x^\mu (\bar{\tau} - \epsilon), \]

where one should be aware that the notation \( x, x' \) may be a little ambiguous because it makes reference to the points \((u, v, x, y), (u', v', x', y')\) and also to their third components.

Now we can easily solve the geodesic equations, since the momenta \( p_x, p_y \) and \( p_v \) are three constants of the motion. The geodesics are a family of curves parametrized by \( p_x, p_y \) and \( p_v \),

\[
x = x(u) = x(0) - 2M^2 \frac{p_x}{p_v} f(u), \quad y = y(u) = y(0) - 2M^2 \frac{p_y}{p_v} g(u), \tag{53}
\]

\[
v = v(u) = v(0) + M^2 \frac{p_x^2}{p_v^2} f(u) + M^2 \frac{p_y^2}{p_v^2} g(u) + \frac{M^2}{p_v^2} \int_0^u du' (1 + \sin u)^2, \tag{54}
\]

where \( f(u) \) and \( g(u) \), are given in (12). If we substitute these curves into (52) using that \( x' = x(u') \), \( y' = y(u') \), \( v' = v(u') \) and take \( u, \bar{u}, u' \) as \( u = u(\bar{\tau} + \epsilon), \bar{u} = u(\bar{\tau}), u' = u(\bar{\tau} - \epsilon) \), then we only need to expand (52) in powers of \( \epsilon \) up to order \( \epsilon^2 \). The calculation is simple but tedious and finally one obtains,

\[
G^{(1)}(x, x') = -\frac{1}{4\pi^2} \left( \frac{1}{\sigma} - \frac{p_v^4}{160 M^8 \pi^2} \frac{\sigma}{(1 + \sin u)^{10}} \right. \tag{55}
\]

where \( \sigma = \Sigma s^2/2 = 2\epsilon^2 \Sigma \) and \( s \) is the proper distance along the geodesic between \( x \) and \( x' \), see Appendix B. The regularization follows from the subtraction of the elementary Hadamard solution (45)

\[ S(x, x')_{\text{w0}} = -\frac{1}{4\pi^2} \frac{1}{\sigma} \]
since the biscalars $a_2(x, x')$ and $a_1(x, x')$ are zero up to order $\sigma$. Note that, in particular, there is no trace anomaly because $a_2(x) = 0$, thus after regularization there remains a term proportional to

$$p^4 \sigma = 4 \frac{\sigma^4}{\sigma} = 4 \sigma^4 + O(\epsilon),$$

where we have used that $\sigma^\mu = 2\epsilon p^\mu$ being $p^\mu$ a tangent vector to the geodesic connecting $x$ with $x'$ such that $p^\mu p_\mu = \Sigma$. Under application of the differential operator (44) (details are given in section 9) one easily gets an expression which involves factor terms of the kind $p_v p_v$, $p_v p_v p_v p_\mu$, and $p_v p_v p_v p_v p_\mu p_v$. These terms are path dependent in the limit of coincidence $x \to x'$ therefore we must introduce the elementary averaging procedure of Appendix B, which gives,

$$\langle p_v p_v \rangle \propto g_{vv} = 0, \quad \langle p_v p_v p_\mu \rangle \propto g_{vv} = 0, \quad \langle p_v p_v p_v p_v p_\mu p_v \rangle \propto g_{vv} = 0.$$

Thus we recover from (42) the known result that the vacuum expectation value of the stress-energy tensor of a quantum scalar field, in a single plane wave region is exactly zero, $\langle T_{\mu\nu} \rangle_{11} = 0$, i.e. exact gravitational plane waves do not polarize the vacuum.

8 Hadamard function in the interaction region

Here we calculate the Hadamard function $G^{(1)}(x, x')$ in the “in” vacuum state in the interaction region, near the horizon. As we have seen in section 5, due to the blueshift of the “in” modes towards the horizon, the $u^{in}_{k}$ with large energy values (which are the relevant modes in calculations near the horizon) determine the same vacuum state as the modes $\phi_k$, (21). This means that near the horizon the Hadamard function in this vacuum state can be written as,

$$G^{(1)}(x, x') = \sum_k \left\{ u^{in}_{k}(x) u^{in*}_{k}(x') \right\} + c.c. = \sum_k \left\{ \phi_k(x) \phi^*_k(x') \right\} + c.c. \quad (56)$$

Since we do not have an exact expression for the solutions $\phi_k$ we need to work up to a certain adiabatic approximation. This means that we have the inherent ambiguity of where to cut the adiabatic series. Fortunately the boundary conditions of the problem impose that the asymptotic value for the “in” modes on the horizon is given by, $\exp(-i|k_x|\xi^*)$, see paper I for details. This is just the form for the “in” modes that we used in paper I to calculate the production of particles, and is equivalent to cut the adiabatic series at order zero. Although for the particle production problem it was enough to cut at order zero, it is not sufficient for the calculation of the vacuum expectation value of the stress-energy tensor. This is because $G^{(1)}$ at order zero does not reproduce the short-distance singular structure of a Hadamard elementary solution (45) in the coincidence limit $x \to x'$. The smallest adiabatic order which we need, to recover the singular structure of $G^{(1)}$, is order four.

In the mode sum of (54) we use the shortened notation $\sum_k \equiv \int_{-\infty}^{\infty} dk_x \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l}$, and we note from (21) that the sum over $m$ is straightforward using the property of the spherical harmonics.
\[ \sum_{m=-l}^{m=l} Y_i^m(\pi/2 - \eta, y/M)Y_i^{m*}(\pi/2 - \eta', y'/M) = \frac{2l + 1}{4\pi} P_l(\cos \Theta), \]

where \( P_l(x) \) is a Legendre polynomial of order \( l \) and \( \Theta \) is the angle between the points \((\theta, \varphi)\) and \((\theta', \varphi')\) on the sphere, i.e., \( \cos \Theta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi') \). The limit \((\theta', \varphi') \to (\theta, \varphi)\), or equivalently \( \Theta \to 0 \), can be taken right away, it is not singular and it is just related to the spherical symmetry of the horizon, provided the coordinate \( y \) is cyclic. Then, since \( P_l(1) = 1 \),

\[
G^{(1)}(x, x') = \frac{C^2}{\pi} \frac{1}{(1 + \sin \xi)(1 + \sin \xi')} \int_{-\infty}^{\infty} dk_x e^{ik_x(x-x')} \times \sum_{l=0}^{\infty} (2l + 1) \frac{1}{\sqrt{W_1(\xi)W_1(\xi')}} e^{-i \int_{\xi}^{\xi'} W_1(\xi'')d\xi''}. \tag{57}
\]

Let us start now with the point splitting procedure. We assume that the points \( x \) and \( x' \) are connected by a non null geodesic in such a way that they are at the same proper distance \( \epsilon \) from a third midpoint \( \bar{x} \). We parametrize the geodesic by its proper distance \( \tau \) and denote the end points by \( x \) and \( x' \) (it should not be confused with the third component of \((\xi, \eta, x, y)\)). Then we expand,

\[
x^{\mu} = x^{\mu}(\tau) = x^{\mu}(\bar{\tau} + \epsilon) = x_0^{\mu} + \epsilon x_1^{\mu} + \frac{\epsilon^2}{2!} x_2^{\mu} + \cdots, \tag{58}
\]

\[
x'^{\mu} = x'^{\mu}(\tau - \epsilon) = x_0'^{\mu} - \epsilon x_1'^{\mu} + \frac{\epsilon^2}{2!} x_2'^{\mu} - \cdots, \tag{59}
\]

where we have defined,

\[
x_0^{\mu}\equiv x^{\mu}(\tau), \quad x_1^{\mu}\equiv \frac{dx^{\mu}}{d\tau}|_{\bar{\tau}}, \quad x_2^{\mu}\equiv \frac{d^2x^{\mu}}{d\tau^2}|_{\bar{\tau}}, \cdots, \tag{60}
\]

i.e. the subindex of a given coordinate indicates the number of derivatives with respect to \( \tau \) at the point \( \bar{\tau} \).

Using these expansions in \( \epsilon \) we can write \( \int_{\xi}^{\xi'} W_1(\xi'')d\xi'' \), \( [W_1(\xi)W_1(\xi')]^{-1/2} \) and the prefactor \([1 + \sin \xi)(1 + \sin \xi')]^{-1}\) in (57) up to \( O(\epsilon^4) \). Note also that \( x - x' = 2 \epsilon x_1 + (\epsilon^3/3) x_3 + (\epsilon^5/60) x_5 + O(\epsilon^7) \). With these expressions we can expand the exponential term in (57) in powers of \( \epsilon \) as

\[
e^{-i \int_{\xi}^{\xi'} W_1(\xi'')d\xi'' + ik_x(x-x')} = e^{-i\epsilon(\delta_1 \omega - \delta_2 k_x)} \times (1 + \cdots),
\]

where we preserve from the expansion the zero adiabatic terms because they will be useful as new integration variables. These terms have been included in the coefficients \( \delta_1 \) and \( \delta_2 \) as,

\[
\delta_1 = 2\xi_1^* + \frac{\epsilon^2}{3} \xi_3^* + \frac{\epsilon^4}{60} \xi_3^*, \quad \delta_2 = 2x_1 + \frac{\epsilon^2}{3} x_3 + \frac{\epsilon^4}{60} x_3. \tag{61}
\]
Since the number of derivatives \( d/d\xi^* \) determine the adiabatic order, we introduce a new parameter \( T \), which will indicate the adiabatic order, then at the end of calculations we will take \( T = 1 \). We will also denote,

\[
\omega \equiv \omega_1(\xi), \quad V_l \equiv V_1(\xi), \quad W \equiv W_1(\xi), \quad \dot{W} \equiv \frac{dW_1(\xi)}{d\xi^*} \bigg|_{\xi^*}.
\]

Recall that from section 3 we know the function \( W_1 \) up to adiabatic order four, thus from (19) we have now,

\[
W = \omega + T^2 \left( \frac{A_2}{\omega^3} + \frac{B_2}{\omega^5} \right) + T^4 \left( \frac{A_4}{\omega^3} + \frac{B_4}{\omega^5} + \frac{C_4}{\omega^9} + \frac{D_4}{\omega^{11}} \right),
\]

where the coefficients \( \omega_1, A_n, B_n, C_n \), are given in (17) and (20).

We substitute these expansions in (57) and separate the different adiabatic terms by expanding in powers of the parameter \( T \). The process is quite simple but tedious and the result is,

\[
G^{(1)}(x, x') = \frac{C^2}{\pi} \left[ \frac{1}{(1 + \sin \xi)^2} + e^2 \left( \xi^{*2} \left( \frac{3 - \sin \bar{\xi}}{M^2(1 + \sin \xi)^2} - \frac{\cos^2 \bar{\xi}}{M(1 + \sin \xi)^5} \right) \right) + O(e^4) \right]
\times \int_{-\infty}^{\infty} dk_x \sum_{l=0}^{\infty} (2l + 1) e^{-ix\delta_{1}\omega - l k_x} \left[ A_0 + i e A_1 + e^2 A_2 + i e^3 A_3 + e^4 A_4 + O(e^5) \right]. \tag{62}
\]

where the coefficients \( A_n \) are given in Appendix C.

Let us now proceed to the integration in (62), evaluating first the summation over the discrete parameter \( l \). It is convenient to transform the above summation into an integral by means of the Euler-Maclaurin summation formula (34),

\[
\sum_{l=0}^{n-1} F(l) = \int_0^n F(x) \, dx + \frac{1}{2} [F(0) - F(n)] + \frac{1}{12} [F'(0) - F'(n)] + \frac{1}{720} [F'''(0) - F'''(n)] + \cdots, \tag{63}
\]

where \( F' = dF(l)/dl \). In our case \( n \to \infty \) and \( F^{(i)}(l) \) is given from (52) as, \( F^{(i)}(l) \propto \exp (-i\epsilon \delta_{1} \omega) \). Note that in the limit \( n \to \infty \) with finite geodesic distance \( \epsilon \), the phase in \( F^{(i)}(n) \) is highly oscillating but the modulus is either approaching to zero or is finite. In other words, the contribution at \( n \to \infty \) is negligible, this can be mathematically reproduced by introducing a small negative imaginary part in the geodesic splitting as, \( \epsilon \to \epsilon - i0^+ \), which allows us to regularize from the very beginning: \( \lim_{n \to \infty} F^{(i)}(n) = 0 \). Therefore we write,

\[
\sum_{l=0}^{n-1} F(l) = \int_0^n F(x) \, dx + \frac{1}{2} F(0) - \frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + \cdots \equiv \int_0^n F(x) \, dx + \mathcal{L}F(x)|_{x=0}, \tag{64}
\]

where the definition of the differential operator \( \mathcal{L} \) should be clear from the last identity. Then we can substitute the summation in (62) for an integral over a formally continuous parameter \( l \). This integral can be further simplified by using a new integration variable \( u \),
where we define, see (17), $a(\xi) = \cos^2 \xi (1 + \sin \xi)^{-4}$, $c(\xi) = 2 \cos^2 \xi (1 + \sin \xi)^{-5}$. Then $2u du = a(\xi) (2l + 1) dl$, and,

\[ l(l + 1) = \frac{u^2 - \Omega^2}{a(\xi)}, \quad \Omega \equiv k_l^2 + c(\xi), \]

where for simplicity we introduce a new dimensionless variable $\Omega$. With this one can directly substitute the derivatives $d^k V_l/d\xi^k \equiv V_l^{(i)}$ which appear in the coefficients $A_n$ by,

\[ V_l^{(i)} = \frac{1}{M^2} \left[ \frac{a(\xi)}{a(\xi)} (u^2 - \Omega^2) + c^{(i)}(\xi) \right]. \]

With all this (62) takes the form,

\[
G^{(1)}(x, x') = \frac{C^2}{\pi} \left[ \frac{1}{(1 + \sin \xi)^2} + \epsilon^2 \left( \xi_l^2 \left( \frac{3 - \sin \xi}{M^2(1 + \sin \xi)} \cos^2 \xi - \xi_l^4 \frac{\cos^2 \xi}{M(1 + \sin \xi)} \right) + O(\epsilon^4) \right) \right] \times \int_\infty \int \int_\infty \int \frac{2u}{a(\xi)} du e^{-i\delta_1 u/M} [A_0 + i\epsilon A_1 + \epsilon^2 A_2 + i\epsilon^3 A_3 + \epsilon^4 A_4 + O(\epsilon^5)] + e^{i\delta_2 k_s} \mathcal{L} \left[ (2l + 1) (A_0 + i\epsilon A_1 + \epsilon^2 A_2 + i\epsilon^3 A_3 + \epsilon^4 A_4 + O(\epsilon^5)) e^{-i\delta_1 \Omega} \right], \]

where the functions $A_n$ are now written in powers of $u^{-1}$ and $\Omega$. The integrals that we have in (63) are of the type,

\[ T_n = \int_\infty \int_\infty e^{-i\delta_1 u/M - \delta_2 k_s} \frac{\Omega^m}{u^n} du, \]

which can be easily integrated over $u$ using the following identities,

\[
\int_\Omega e^{-i\delta_1 u/M} u du = - \left( \frac{M^2}{\epsilon^2 \delta_1^2} + i \frac{M \Omega}{\epsilon \delta_1} \right) e^{-i\delta_1 \Omega/M}, \quad \int_\Omega e^{-i\delta_1 u/M} du = - \frac{iM}{\epsilon \delta_1} e^{-i\delta_1 \Omega/M},
\]

\[
\int_\Omega \frac{e^{-i\delta_1 u/M} du}{u^n} = - \left( -i \frac{\delta_1}{M} \right)^{n-1} E_n(-i\epsilon \delta_1 \Omega/M) + e^{-i\epsilon \delta_1 \Omega/M} \frac{1}{(n-1)!} \frac{1}{\Omega^{n-1}} \sum_{k=0}^{n-2} \left( -i \frac{\epsilon \delta_1 \Omega}{M} \right)^k (n - 2 - k)!, \quad n > 0.
\]

where we have used a small negative imaginary part in $\epsilon$, i.e. $\epsilon = i0^+$, and where $E_n(ix)$ is the integral-exponential function with imaginary argument [40]. The remaining integrals are,

\[ T_0^m \equiv T_m = \int_\infty e^{-i(\delta_1 \Omega - \delta_2 k_1)/M} \frac{dk_1}{\Omega^m} = O(e^{m-1}) \ell, \quad m > 0, \]

21
\[ I_0^m \equiv I_m = \int_{-\infty}^{\infty} e^{-i(\delta_1 \Omega - \delta_2 k_1)/M} \Omega^m dk_1 = O(\epsilon^{-(m+1)}) + \begin{cases} O(\epsilon) \ell & m \geq 0 \text{ even,} \\ [1 + O(\epsilon^2)] \ell & m \geq 0 \text{ odd,} \end{cases} \]

\[ I_m = \int_{-\infty}^{\infty} e^{i \delta_2 k_1/M} E_i(-i \epsilon \delta_1 \Omega/M) \Omega^m dk_1 = O(\epsilon^{-(m+1)}) + O(\epsilon) \ell, \quad m \geq 0 \text{ even,} \]

where \( \ell \) is a logarithmic term defined as \( \ell = \gamma + \ln(\epsilon/M) \), where \( \gamma \) is Euler’s constant. These integrals can be calculated using the indications of Appendix D.

Many of the terms in (67), however, are not relevant for our calculation because they give results of order beyond \( \epsilon^2 \). In fact, note that the series \( A_0 + i \epsilon A_1 + \epsilon^2 A_2 + i \epsilon^3 A_3 + \epsilon^4 A_4 + O(\epsilon^5) \) has dimension of \( M^{-1} \), \( \Omega \) and \( u \) are dimensionless variables, \( T \) has dimensions of \( M^{-1} \), and the geodesic coefficients \( d^n x^m / d\tau^n \) have dimensions of \( M^{-(n-1)} \). Now a simple dimensional analysis shows that the singularity \( \sigma^{-2} \) in the Hadamard function is recovered with the single term \( A_0 \) up to adiabatic order zero only, the logarithmic singularity, i.e. \( \ln \sigma \), is recovered also with the single term \( A_0 \), but now up to adiabatic order two, and the singularity \( \sigma \ln \sigma \) is recovered with the terms \( A_0, A_1, A_2 \) and \( A_3 \) up to adiabatic order four.

We can now substitute in (67) the values of the integrals given in Appendix D, the values for the functions \( a \) and \( c \) given in (63), the expression (61), for the coefficients \( \delta_1 \), \( \delta_2 \), and the relation between the geodesic coefficients \( \xi_x \) and \( x \) in terms of \( \xi_x \) and \( x_1 \) (see Appendix E). Also we use the identities, see Appendix B, \( \sigma = 2 \epsilon^2 \Sigma, \sigma^\mu = 2 \epsilon p^\mu, p^\mu p_\mu = \Sigma \), where \( \sigma^\mu \) is the geodesic tangent vector in the midpoint \( \bar{x} \) with modulus the proper distance on the geodesic between \( x \) and \( x' \). In particular, \( \sigma^\bar{x} = 2 \epsilon x_1 \). Finally, after a rather tedious calculation we obtain the expression for the Hadamard function,

\[ G^{(1)}(x, x') = -\frac{4 M^2 C^2}{\pi \sigma} + \bar{A} + \sigma \bar{B} + C_{2x} \sigma^2 \sigma^x + D_{xxxx} \frac{\sigma^2 \sigma^x \sigma^x}{\sigma}, \]

where the normalization constant \( C \), by comparison with (63), is necessarily \( C^2 = (16M^2 \pi)^{-1} \) and the explicit expressions for \( \bar{A}, \bar{B}, C_{2x} \) and \( D_{xxxx} \) are given up to adiabatic order four in Appendix C. Note that, as it is expected on general grounds, there is no logarithmic singularity. In fact the coefficient for the logarithmic term is,

\[ -\frac{2}{3} + \frac{c}{a} + \frac{M^2 \dot{a}^2}{3 a^2} - \frac{M^2 \dot{a}}{3 a^2}, \]

which is identicaly zero as one can easily see by direct substitution of the value for the functions \( a \) and \( c \), in (63).

According to (63), the Hadamard function can be regularized using the elementary Hadamard solution (43), which in our case reduces simply to,

\[ S(x, x')_{w_0} = -\frac{1}{4 \pi^2 \sigma}, \]

since the coefficient of the logarithmic divergence, see (17), exactly cancels up to order \( \sigma \) for a Schwarzshild geometry (the interaction region is locally isometric to Schwarzshild). In fact, for our geometry and using (18)
where before expanding, a homogenization of the indices by the parallel transport bivector can expand in the neighbourhood of \( x \) where it is understood that \( \overline{\alpha} \), \( \overline{\beta} \), \( \overline{\gamma} \), \( \overline{\delta} \) operators, \( \overline{\alpha} \overline{\beta} \overline{\gamma} \overline{\delta} \) are functions that depend on the endpoints \( x, x' \) but are evaluated at the midpoint \( \overline{x} \). Then, following the formalism of Appendix B, we can expand in the neighbourhood of \( x \) as,

\[
G_B^{(1)}(x, x') = \overline{A} + \sigma \overline{B} + C_{\overline{\alpha} \overline{\beta}} \sigma_{\overline{\alpha}} \sigma_{\overline{\beta}} + D_{\overline{\alpha} \overline{\beta} \overline{\gamma} \overline{\delta}} \frac{\sigma_{\overline{\gamma}} \sigma_{\overline{\delta}}}{\sigma},
\]

where it is understood that \( \overline{A}, \overline{B}, C_{\overline{\alpha} \overline{\beta}}, D_{\overline{\alpha} \overline{\beta} \overline{\gamma} \overline{\delta}} \) are functions that depend on the endpoints \( x, x' \) but are evaluated at the midpoint \( \overline{x} \). Then, following the formalism of Appendix B, we can expand in the neighbourhood of \( x \) as,

\[
G_B^{(1)}(x, x') = \overline{A} + \sigma \overline{B} + \overline{g}_\alpha^\overline{\alpha} \overline{g}_\beta^\overline{\beta} C_{\overline{\alpha} \overline{\beta}} \sigma_{\overline{\alpha}} \sigma_{\overline{\beta}} + \overline{g}_\alpha^\overline{\alpha} \overline{g}_\beta^\overline{\beta} \overline{g}_\gamma^\overline{\gamma} \overline{g}_\delta^\overline{\delta} D_{\overline{\alpha} \overline{\beta} \overline{\gamma} \overline{\delta}} \frac{\sigma_{\overline{\gamma}} \sigma_{\overline{\delta}}}{\sigma},
\]

where before expanding, a homogeneization of the indices by the parallel transport bivector \( \overline{g}_\alpha^\overline{\alpha} \), i.e, \( \sigma_{\overline{\alpha}} = \overline{g}_\alpha^\overline{\alpha} \sigma_\alpha \), has been applied. The covariant expansions at \( x \) are given by,

\[
\overline{A} = A - \frac{1}{2} A_{\alpha \alpha} \sigma_{\alpha} + \frac{1}{8} A_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta} + O(\epsilon^3), \quad \overline{B} = B + O(\epsilon),
\]

\[
\overline{g}_\alpha^\overline{\alpha} \overline{g}_\beta^\overline{\beta} C_{\overline{\alpha} \overline{\beta}} = C_{\alpha \beta} + O(\epsilon), \quad \overline{g}_\alpha^\overline{\alpha} \overline{g}_\beta^\overline{\beta} \overline{g}_\gamma^\overline{\gamma} \overline{g}_\delta^\overline{\delta} D_{\overline{\alpha} \overline{\beta} \overline{\gamma} \overline{\delta}} = D_{\alpha \beta \gamma \delta} + O(\epsilon).
\]

Now we can apply the differential operator (44) to (69). We will consider the two physically relevant cases of the minimal coupling (\( \xi = 0 \)) which should provide a good qualitative description for gravitons, and of the conformal coupling (\( \xi = 1/6 \)) which should provide a good qualitative description for photons (the use of \( \xi \) as the coupling parameter should not be confused with the coordinate \( \xi \) of the interaction region). If we introduce the operators,

\[
\mathcal{L}_{\mu \nu}^{(1)} = \nabla_\mu \nabla_\nu + \nabla_\nu \nabla_\mu, \quad \mathcal{L}_{\mu \nu}^{(2)} = \nabla_\mu \nabla_\nu + \nabla_\nu \nabla_\mu,
\]

the operator (44), for the minimal and conformal cases, is

\[
-a_1 - \frac{1}{2} a_2 \sigma = \frac{1}{180} R^\alpha_{\beta \gamma} R_{\alpha \beta \gamma} \sigma^\mu \sigma^\nu - \frac{1}{4} \frac{1}{180} R^\alpha_{\beta \gamma \delta} R_{\alpha \beta \gamma \delta} \sigma = 0,
\]
By the properties of the geodetic interval bivector $\sigma^\mu$ given in the Appendix B, which can be also written as $\sigma^\mu = 2\varepsilon\, p^\mu$ ($p^\mu p_\mu = \Sigma$) we can prove the following identities,

$$
\lim_{x \to x'} L_{\mu\nu}^{(1)} \sigma = - \lim_{x \to x'} L_{\mu\nu}^{(2)} \sigma = -2 g_{\mu\nu}, \quad \lim_{x \to x'} L_{\mu\nu}^{(1)} \sigma^\alpha \sigma^\beta = - \lim_{x \to x'} L_{\mu\nu}^{(2)} \sigma^\alpha \sigma^\beta = -4 \delta^\alpha_{(\mu} \delta^\beta_{\nu)} ,
$$

$$
\lim_{x \to x'} L_{\mu\nu}^{(1)} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta = - \lim_{x \to x'} L_{\mu\nu}^{(2)} \sigma^\alpha \sigma^\beta \sigma^\gamma \sigma^\delta = - \frac{48}{\Sigma} \delta^{(\alpha}_{(\mu} \delta^{\beta}_{\nu)} p^\gamma p^\delta + 64 \delta^{(\alpha}_{(\mu} \delta^{\beta}_{\nu)} p^\gamma p^\delta + 8 p^\alpha p^\beta p^\gamma p^\delta \left[ g_{\mu\nu} - 4 P_{\mu\nu} / \Sigma \right] ,
$$

where $(\cdots)$ is the usual symmetrization operator. Note that by a straightforward application of Synge’s theorem (see Appendix B), the coincidence limits of $L_{\mu\nu}^{(1)}$ and $L_{\mu\nu}^{(2)}$ differ in a sign when they are applied over bitensors for which the first covariant derivative has null coincidence limit. But when they are applied over the biscalar $\bar{L}$ of Synge’s theorem (see Appendix B), the coincidence limits of products of $\sigma^\mu$ (as is the case), we only need the following averages, provided we work with a diagonal metric (as is the case), we only need the following averages,

$$
\langle p^\mu p^\nu \rangle = \frac{\Sigma}{4} g^{xx} , \quad \langle p^\mu p^\nu p^\rho p^\sigma \rangle = \frac{1}{8} (g^{xx})^2 \delta^x_{\mu} ,
$$

$$
\langle p^\mu p^\nu p^\rho p^\sigma p^\rho p^\sigma \rangle = \frac{\Sigma}{16} (g^{xx})^3 \delta^x_{\mu} \delta^x_{\nu} + \frac{\Sigma}{64} g^{\mu\nu} (g^{xx})^2 .
$$

We now have all we need to evaluate the vacuum expectation value of the stress-energy tensor by application of the differential operator (14) to the expression (71) of $G^{(1)}_B (x, x')$. In the orthonormal basis, $\theta_0 = M (1 + \sin \xi) \, d\xi$, $\theta_1 = M (1 + \sin \xi) \, d\eta$, $\theta_2 = \cos \xi (1 + \sin \xi)^{-1} \, dx$, $\theta_3 = (1 + \sin \xi) \cos \eta \, dy$, we obtain the following expectation values $\langle T^B_{\mu\nu} \rangle$ for the minimal and conformal couplings,

$$
\langle T^B_{\mu\nu} \rangle^{\xi=0} = \lim_{x \to x'} D_{\mu\nu}^{\xi=0} G^{(1)}_B (x, x') = \text{diag} (\rho_1, -\rho_1, \rho_2, -\rho_1) ,
$$

$$
\langle T^B_{\mu\nu} \rangle^{\xi=1/6} = \lim_{x \to x'} D_{\mu\nu}^{\xi=1/6} G^{(1)}_B (x, x') = \text{diag} (\rho, -\rho, 3\rho, -\rho) ,
$$

where $\rho$, $\rho_1$ and $\rho_2$ are positive definite functions given by,

$$
\rho_1 = \frac{\cos^{-4} \xi}{256 \pi^2 M^4} \, \frac{10771}{2880} \left( 1 + \lambda_1 \cos^2 \xi + O(\cos^4 \xi) \right) ,
$$

$$
\rho_2 = \frac{\cos^{-4} \xi}{256 \pi^2 M^4} \, \frac{3341}{9216} \left( 1 + \lambda_2 \cos^2 \xi + O(\cos^4 \xi) \right) ,
$$

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\[ \rho = \frac{\cos^{-4} \xi}{256 \pi^2 M^4} \frac{6869}{2880} \left( 1 + \lambda \cos^2 \xi + O(\cos^4 \xi) \right), \]

which are unbounded at the horizon \((\xi = \pi/2)\), and the approximate values for \(\lambda_1, \lambda_2\) and \(\lambda\) are \(\lambda_1 = 1.474 \cdots, \lambda_2 = 1.466 \cdots\) and \(\lambda = 1.469 \cdots\). From (51) this is not the final result of the stress-energy tensor in region I, \(\langle T_{\mu \nu} \rangle_I\), since the trace anomaly term must be included. But this term depends on the spacetime curvature which is finite at the horizon,

\[ \frac{a_2}{64 \pi^2} g_{\mu \nu} = \left[ \frac{1}{240 M^4 \pi^2} \frac{1}{64} + O(\cos^2 \xi) \right] \text{diag} (1, -1, -1, -1), \]

and it does not modify the dominant contributions of (73) or (74), which increase as \(\cos^{-4} \xi\) near the horizon. In the conformally coupled case the trace \(\langle T^\mu_\mu \rangle_I\) is finite and can be obtained from the previous term. Inspection of (75)-(76) shows that for both couplings the weak energy condition is satisfied \([44]\), which means that the energy density is nonnegative for any observer, and that the strong energy condition is only satisfied for the conformal coupling.

## 10 Conclusions

We have evaluated the expectation value of the stress-energy tensor of a massless scalar field in the state, which corresponds to the physical vacuum state before the collision of the plane waves. This vacuum state has an analogue in the Schwarzschild black hole case as the empty state at large radius from the hole, the Boulware vacuum state, for which the vacuum expectation value of the stress-energy tensor also diverges at the horizon. In this case, however, it is argued that the Boulware vacuum state is not physical since it does not correspond to the vacuum of the gravitational collapse problem \([26, 25]\).

Our results are the following. Before the collision of the plane waves (in region IV) \(\langle T_{\mu \nu} \rangle_{IV} = 0\) by the definition of our vacuum state, in the plane wave regions (regions II and III) we have found in section 7 that \(\langle T_{\mu \nu} \rangle_{II} = \langle T_{\mu \nu} \rangle_{III} = 0\), since the plane waves do not polarize the vacuum, and finally, in the interaction region (region I) \(\langle T_{\mu \nu} \rangle_I\) becomes unbounded at the regular Killing-Cauchy horizon \((\xi = \pi/2)\) in the conformally and minimally coupled cases. In both cases the weak energy condition is satisfied, the rest energy density is positive and diverges as \(\cos^{-4} \xi\) and two of the principal pressures are negative and of the same order of magnitude of the energy density. The strong energy condition is satisfied for the conformal coupling, in this case \(\langle T^\mu_\mu \rangle_I\) is finite but \(\langle T_{\mu \nu} \rangle_I\) diverges at the horizon and we may use ref \([41]\) on the stability of Cauchy horizons to argue that the horizon will acquire by backreaction a curvature singularity too. Thus contrary to the simple plane waves which do not polarize the vacuum \([13, 14]\), the nonlinear collision of these waves polarize the vacuum and the focusing effect that the waves exert produce at the focusing points an unbounded positive energy density. Therefore when the colliding waves produce a Killing-Cauchy horizon that horizon is unstable by vacuum polarization.

In the more generic case when the wave collision produces a spacelike singularity it seems clear that the vacuum expectation value of the stress-energy tensor will also grow unbounded near the singularity. The reason is that such unboundness is, essentially, a consequence of the blueshift suffered by the mode solutions as they enter the plane wave regions, and it is easy to see \([16]\) that any plane wave produces a similar blueshift on mode solutions. In view
of our results it seems very unlikely that the negative pressures associated to the quantum fields could prevent the formation of the singularity.

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A Adiabatic expansion of “in” modes in region I

With the parameters of section 5, defining \( p \equiv Lq^{-1}(\cos \xi)^{-1} \) and \( p_1 \equiv Lq^{-1}(\cos \eta)^{-1} \) and assuming that \( p \ll 1, p_1 \ll 1 \), we can expand (21) up to adiabatic order 4 in powers of \( p \) and \( p_1 \). The process requires to expand the WKB functions (18) and (38). These expansions are rather easy to perform and, in particular, the expansion for (38) allow us to fix the two arbitrary coefficients \( C_1 \) and \( C_2 \) by requiring that (38) corresponds asymptotically to the function (33) which is, modulus a constant factor, an associated Legendre function. Using that the asymptotic expansion of an associated Legendre function

\[
P_m^l(\sin \eta),
\]

in the range of values, \( Lm(\cos \eta)^{-1} \ll 1, \) is \[40\]

\[
P_m^l(\sin \eta) = \frac{1}{4\pi \sqrt{M}} \sin^{l-m}(\eta) \frac{1}{W_2^{(4)}(\eta)} \left\{ e^{-i \int W_2^{(4)}(\eta) d\eta^*} + (-1)^{l+m} e^{i \int W_2^{(4)}(\eta) d\eta^*} \right\}.
\]

With these results, we can divide each term (31) or (32) in four integrals such that the variation of the phase of their integrand functions are given by

\[
\tilde{A}(u) \pm W^{(4)}(u), \quad \tilde{A}(u) \pm W^{(4)}(u),
\]

and the modulus of their integrand functions are given, respectively, by

\[
F_1(u) \left( \frac{\tilde{A}(u)}{2L_2k_-} \pm W^{(4)}(u) \right) + F_2(u), \quad F_1(u) \left( \frac{\tilde{A}(u)}{2L_2k_-} \pm W^{(4)}(u) \right) + F_2(u),
\]

\[
F_1(u) \left( \frac{\tilde{A}(u)}{2L_1k_-} \pm W^{(4)}(u) \right) + F_2(u), \quad F_1(u) \left( \frac{\tilde{A}(u)}{2L_1k_-} \pm W^{(4)}(u) \right) + F_2(u).
\]
Note that when a sign $\pm$ is taken in a phase term, then in the modulus term we have the opposite sign $\mp$. Up to adiabatic order 4 all these functions, i.e. $W_{\pm}^{(4)}(u)$, $F_1(u)$, $F_2(u)$ and $A(u)$, can be expanded in powers of $p$ on the boundaries between regions II and III with region I, where $\xi = \pm \eta$ and thus $p = p_1$, and where we use for simplicity $z \equiv \sin u$. We get,

$$W_{+}^{(4)}(u) = L^{-1} \left\{ 2 + p^2 \left( \frac{q^2}{4}(2 - z^2) + \frac{V^2}{32}(1 + z)^4 - \frac{U^2}{2} \right) + 
+ p^4 \left[ -\frac{q^4}{64}(12 + 12z^2 + z^4) + q^2 \left( -\frac{V^2}{256}(1 + z)^4(30 - 24z + 7z^2) + \frac{U^2}{16}(6 + 7z^2) \right) - \frac{V^4}{2048}(1 + z)^8 - \frac{U^4}{8} \right] \right\} + O(L^{-1}p^6),$$

$$W_{-}^{(4)}(u) = L^{-1} \left\{ p^2 \left( \frac{V^2}{32}(1 + z)^4 + \frac{U^2}{2} \right) + 
+ p^4 \left[ q^2 \left( -\frac{V^2}{256}(1 + z)^4(30 - 24z + 7z^2) - \frac{U^2}{16}(6 + 7z^2) \right) - \frac{V^4}{2048}(1 + z)^8 + \frac{U^4}{8} \right] \right\} + O(L^{-1}p^6),$$

$$F_1(u) = LM \left\{ 2 + p^2 \left( -\frac{q^2}{4}(2 - z^2) - \frac{V^2}{32}(1 + z)^4 + \frac{U^2}{2} \right) + 
+ p^4 \left[ \frac{q^4}{64}(20 + 4z^2 + 3z^4) + q^2 \left( \frac{V^2}{256}(1 + z)^4(34 - 24z + 5z^2) - \frac{5U^2}{16}(2 + z^2) + \frac{5V^4}{4096}(1 + z)^8 + \frac{5U^4}{16} - \frac{V^2U^4}{128}(1 + z)^4 \right) \right] \right\} + O(Lp^6),$$

$$F_2(u) = Lp^3 M \left\{ \frac{q^2}{2}z - \frac{V^2}{16}(1 + z)^3(2 + z - z^2) + U^2 z \right\} + O(Lp^5).$$

$$A(u) = L^{-2} \left\{ \frac{V^2}{16}(1 + z)^4 + U^2 \right\}. $$

Among the integrals in (31)-(32), there is only one integral which contains stationary points and it is always in the term $A_k^{(A)}$ in (31) when either $L^{-1} = L_2k_-$ or $L^{-1} = L_1k_+$, because the variation of the phase takes in this case its minimum value with respect to the modulus, and its contribution becomes dominant; consequently we can neglect the remaining integrals. The dominant integrals connecting the “in” modes between regions II and I can be written as,

$$\int du F_1(u) \left( \frac{A(u)}{2L_2k_-} + W_-^{(4)}(u) \right) e^{\int du \left( A(u)/(2L_2k_-) - W_-^{(4)}(u) \right)},$$

which can be written in powers of $p$ as (40). The dominant integrals connecting the “in” modes between regions III and I can be written in the same way but changing $L_2k_-$ for $L_1k_+$. 

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B Bitensor algebra

A bitensor with \( n + m \) indices, \( \alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_m; T_{\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_m} \), is an object which transforms, under coordinate changes, as the product of two ordinary tensors, \( A_{\alpha_1, \alpha_2, \ldots, \alpha_n} \) and \( B_{\beta_1, \beta_2, \ldots, \beta_m} \), each at a different spacetime point \( B, A \). The standard tensorial operations can be straightforwardly extended to bitensors with the simple precaution of contracting only indices referring to the same spacetime point, and noting that when a covariant derivative is performed over one of the points, the indices referring to the other point are irrelevant.

If we have an arbitrary bitensor evaluated at two close spacetime points \( x \) and \( x' \) it is possible to expand it in a covariant way at one of the two points. To achieve this we make use of the geodetic interval, \( s(x, x') \), a basic object in geodesic theory, which describes the proper distance between the spacetime points \( x \) and \( x' \) along a non null geodesic connecting them. Let us restrict ourselves to a neighbourhood of \( x \) (or \( x' \)) where the geodesics emanating from \( x \) do not cross (mathematically this means that the unique geodesic joining \( x \) and \( x' \) is an extremal curve \([44]\)), then \( s(x, x') \) is single-valued. The basic properties of \( s(x, x') \) are

\[
g^{\mu\nu} s_{\mu} s_{\nu} = g^{\mu'\nu'} s_{\mu'} s_{\nu'} = \pm 1, \tag{77}
\]

together with the symmetry \( s(x, x') = s(x', x) \) and \( \lim_{x \to x'} s(x, x') = 0 \).

These ensure that the bitensor \( g^{\mu\nu}(x)s_{\mu}(x, x') \) is a tangent vector to the geodesic at the point \( x \) and a scalar at \( x' \). Note that \( g^{\mu\nu}(x)s_{\mu}(x, x') \) is an orthogonal vector to the surface \( s(x, x') = \) constant, and this surface is orthogonal to the geodesics which emanate from \( x' \) and therefore \( g^{\mu\nu}(x)s_{\mu}(x, x') \) is a tangent vector to the geodesic that crosses \( s(x, x') = \) constant at the point \( x \). Of course, if \( x \) is fixed and we draw the geometrical locus of the points \( x' \) at the same proper distance \( s(x', x) \), from \( x \) an equivalent description is found. If we take the signature \((+, -, -, -)\) then the positive sign in (77) refers to timelike separated points, and the negative to spacelike separated points. When \( s(x, x') = 0 \), then \( x \) and \( x' \) lie on a light cone.

Instead of \( s(x, x') \) it is convenient to define the geodetic biscalar, \( \sigma(x, x') \), which unlike \( s(x, x') \) has no branchpoints, as

\[
\sigma = \frac{s^2}{2} \Sigma, \tag{78}
\]

where we introduce the notation \( \Sigma = \pm \). For the signature \((+, -, -, -)\) the plus sign is for timelike separated points and the negative sign for spacelike separated points. Let us define \( \sigma_{\mu} \equiv \sigma_{,\mu} \), then the following relations are trivially satisfied,

\[
\frac{1}{2} g^{\mu\nu} \sigma_{\mu} \sigma_{\nu} = \frac{1}{2} g^{\mu'\nu'} \sigma_{\mu'} \sigma_{\nu'} = \sigma, \tag{79}
\]

together with the coincidence limits \( \lim_{x \to x'} \sigma = 0 \), and \( \lim_{x \to x'} \sigma_{\mu} = \lim_{x \to x'} \sigma_{,\mu} = 0 \).

Given an arbitrary bitensor \( A_{\mu\nu}(x, x') \), with both indices referring to the same point \( x \), it is possible to covariantly expand this tensor in the neighbourhood of \( x \), as

\[
A_{\alpha\beta}(x, x') = A_{\alpha\beta}(x) + A_{\gamma\alpha\beta}(x) \sigma_{\gamma} + \frac{1}{2} A_{\gamma\delta\alpha\beta}(x) \sigma_{\gamma} \sigma_{\delta} + \cdots, \tag{80}
\]
where the coefficients $A_{\alpha\beta}(x)$, $A^\gamma_{\alpha\beta}(x), \ldots$ are ordinary local tensors at the point $x$. These coefficients are easily calculated, provided that the successive covariant derivatives of the geodetic biscalar $\sigma$ are known. For this we derive (79) recursively:

\[
\sigma_{,\mu} = g^{\alpha\beta} \sigma_{\alpha\beta,\mu}, \quad \sigma_{,\mu\nu} = g^{\alpha\beta} (\sigma_{,\alpha\beta,\mu} + \sigma_{\alpha\beta,\mu,\nu}),
\]

\[
\sigma_{,\mu\nu\tau} = g^{\alpha\beta} (\sigma_{,\alpha\beta,\mu,\nu} + \sigma_{,\alpha\beta,\mu,\nu,\tau} + \sigma_{,\alpha\beta,\mu,\nu,\tau,\mu} + \sigma_{,\alpha\beta,\mu,\nu,\tau,\mu,\nu}).
\]

Then it is straightforward to find,

\[
A_{\alpha\beta} = [A_{\alpha\beta}], \quad A_{\alpha\beta\gamma} = [A_{\alpha\beta\gamma}] - A_{\alpha\beta,\gamma},
\]

\[
A_{\alpha\beta\gamma\delta} = [A_{\alpha\beta\gamma\delta}] - A_{\alpha\beta,\gamma\delta} - A_{\alpha\beta\gamma,\delta} - A_{\alpha\beta,\gamma\delta},
\]

where the notation $\dot{} = [\cdot]$ indicates the coincidence limit $x \to x'$ is used, i.e., $[T_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_m}] \equiv \lim_{x \to x'} T_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_m}(x, x')$. In particular, it is possible to covariantly expand the successive covariant derivatives of the geodetic biscalar $\sigma$ as,

\[
\sigma_{,\alpha\beta} = g_{\alpha\beta} + \frac{1}{3} R_{\alpha\beta}^{\quad \nu} \sigma_{,\nu} + O(s^3), \quad \sigma_{,\alpha\beta\gamma} = \frac{1}{3} (R_{\alpha\beta}^{\quad \gamma} + R_{\alpha\beta}^{\quad \gamma}) \sigma_{,\gamma} + O(s^2),
\]

\[
\sigma_{,\alpha\beta\gamma\delta} = \frac{1}{3} (R_{\alpha\beta}^{\quad \gamma} + R_{\alpha\beta}^{\quad \gamma}) \sigma_{,\gamma\delta} + O(s).
\]

Sometimes, for the sake of symmetry, it is convenient to covariantly expand the bitensor in terms of the midpoint $\bar{x}$ of the geodesic connecting $x$ and $x'$. For simplicity let us take a biscalar $G(x, x')$. Then a midpoint covariant expansion reads,

\[
G(x, x') = A(\bar{x}) + A_{\mu}(\bar{x}) \bar{\sigma}^\mu + A_{{(\mu\nu)}}(\bar{x}) \bar{\sigma}^\mu \bar{\sigma}^\nu + A_{{(\mu\nu\tau)}}(\bar{x}) \bar{\sigma}^\mu \bar{\sigma}^\nu \bar{\sigma}^\tau + \cdots,
\]

where $(\cdots)$ index symmetrization. The relation between the expansion coefficients $A(\bar{x})$ and the biscalar $G(x, x')$ is easily found to be,

\[
A = [G], \quad A_{\mu} = [G_{,\mu}] - \frac{1}{2} [G]_{,\mu}, \quad 2! A_{{(\mu\nu)}} \dot{=} [G_{{(\mu\nu)}}] - [G_{,\mu}]_{,\nu} + \frac{1}{4} [G]_{,\mu\nu},
\]

\[
3! A_{{(\mu\nu\tau)}} \dot{=} [G_{{(\mu\nu\tau)}}] - \frac{3}{2} [G_{,\mu\tau}]_{,\nu} + \frac{3}{4} [G_{,\mu\nu}]_{,\tau} - \frac{1}{8} [G]_{,\mu\nu\tau},
\]

where the notation $\dot{=} = [\cdot]$ means that the equality is modulus terms which vanish under symmetrizations.

The covariant expansion of a bitensor with indices referring to different points is slightly more sophisticated because it is necessary to homogenize the indices before the expansion. To do this we use the parallel transport bivector, $\bar{g}_{\mu}^\nu(x, x')$, which is defined by analogy to the parallel transport of ordinary vectors on a manifold. Given an ordinary vector $A^\mu(x)$ at the point $x$ it is possible to parallel transport it at the point $x'$ on a non-null geodesic, say $A^\mu(x')$, by a linear operator $\bar{g}_{\mu}^\nu(x, x')$ as $A^\mu(x') = \bar{g}_{\mu}^\nu(x, x') A^\nu(x)$, provided that $\bar{g}_{\mu}^\nu(x, x')$ is covariantly conserved along the geodesic, i.e., $\sigma^\nu(x, x') \bar{g}_{\mu}^\nu(x, x')_{,\mu} = 0$, and it satisfies the trivial boundary condition, $\left[ \bar{g}_{\mu}^\nu \right] = 0$. In exactly the same way, we can define a parallel transport for bitensors as $\bar{g}_{\mu}^\nu(x, x')$ if it verifies the previous differential condition not only at the point $x'$ but also at the point $x$, i.e,
\[
\sigma^\nu(x, x') \bar{g}^{\mu\nu}(x, x'),_{\nu'} = 0, \quad \sigma^\nu(x, x') \bar{g}^{\mu\nu}(x, x'),_{\mu'} = 0, \quad (83)
\]
and the same boundary condition \(\bar{g}^{\mu\nu}_x = \delta^{\mu\nu}_x\). The parallel transport bivector, defined in this a way, is unique because if we fix the point \(x\) one can integrate the second equation in (83) along each geodesic which emanates from \(x\) up to \(x'\) with the initial condition \(\bar{g}^{\mu\nu}_x = \delta^{\mu\nu}_x\).

Reciprocally, if we fix \(x'\), the first equation in (83) can be integrated from \(x'\) down to \(x\) with the same initial condition. This reciprocity can be expressed as, \(\bar{g}^{\mu\nu}_x(x, x') = \bar{g}^{\mu\nu}_x(x', x)\). From the geometric interpretation of \(\bar{g}^{\mu\nu}(x, x')\) it is straightforward to see that:

\[
\bar{g}^{\mu\nu}_x g_{\mu'\nu'} = g_{\mu\nu}, \quad \bar{g}^{\mu\nu}_x g_{\mu'\nu} = g_{\mu'\nu'}, \quad \bar{g}^{\mu\nu}_x \sigma_{\mu'} = -\sigma_\mu,
\]

\[
\bar{g}^{\mu\nu}_x \sigma_\mu = -\sigma_{\mu'}, \quad \bar{g}^{\mu\nu}_x \bar{g}^{\mu_2\nu_2}_x = \delta^{\mu_2\nu_2}_x, \quad \bar{g}^{\mu\nu}_x \bar{g}^{\mu_2\nu_2}_x = \delta^{\mu_2\nu_2}.\]

Thus given a non-homogeneous bitensor \(A_{\mu\nu'}(x, x')\), we first homogenize it as, \(\bar{A}_{\mu\nu'}(x, x') = \bar{g}^{\nu'}_\nu(x, x')A_{\mu\nu'}(x, x')\), and then we covariantly expand \(\bar{A}_{\mu\nu'}(x, x')\) in the same way as (80).

An interesting result in the algebra of bitensors, usually known as Synge’s theorem [43], is that given an arbitrary bitensor \(T_{\alpha_1\alpha_2...\alpha_n^\prime\beta_1^\prime...\beta_m^\prime}\), whose coincidence limit and the coincidence limit of its first covariant derivatives exist, then the following identity is satisfied [42]:

\[
\left[ T_{\alpha_1\alpha_2...\alpha_n^\prime\beta_1^\prime...\beta_m^\prime}\right] = -\left[ T_{\alpha_1\alpha_2...\alpha_n^\prime\beta_1^\prime...\beta_m^\prime}\right] + \left[ T_{\alpha_1\alpha_2...\alpha_n^\prime\beta_1^\prime...\beta_m^\prime}\right]_{\mu}. \quad (84)
\]

This is very useful when the calculations involve a lot of primed indices and we know the coincidence limits of the nonprimed quantities. A trivial application of this result implies that \([\sigma_{\mu\nu'}] = -g_{\mu\nu'}\).

When one is performing the covariant expansion of Green functions in preparation for the calculation of the regularized expectation value of the stress-energy tensor, ambiguities in the coincidence limits appear when one of the points, say \(x\), is held fixed and an arbitrary point \(x'\) is allowed to approach \(x\). These ambiguities are due to the different paths that \(x'\) may follow, therefore some type of averaging is required. The most elementary averaging is called \textit{four dimensional hyperspherical averaging} [13] and it consists in giving the same weight to all the geodesics which emanate from \(x\) as follows. First one analytically continues to an Euclidean metric the components of the tangent vectors to the geodesics which emanate from \(x\). Second, one averages over a 4-sphere, and third the results are continued back to the original metric. It is not very complicated to find the following averaging formulae which are useful in this paper,

\[
\langle \sigma_\mu \sigma_\nu \rangle = \frac{1}{2} \sigma g_{\mu\nu}, \quad \langle \sigma_\mu \sigma_\nu \sigma_\tau \sigma_\rho \rangle = \frac{1}{2} \sigma^2 g_{\mu(\nu}g_{\tau\rho)}, \quad \langle \sigma_\mu \sigma_\nu \sigma_\tau \sigma_\rho \sigma_\xi \sigma_\eta \rangle = \frac{5}{8} \sigma^3 g_{\mu(\nu}g_{\tau(\rho}g_{\eta)}.
\]

Note that from the symmetry of the averaging procedure, the average of an odd number of components \(\sigma^\mu\) vanishes identically.
C Useful adiabatic expansions

The adiabatic coefficients $A_n$ in formula (32) are given by,

\[ A_0 = \frac{1}{\omega} + T^2 \left( -\frac{5}{32} \frac{\dot{V}_l^2}{\omega^7} + \frac{1}{8} \frac{\ddot{V}_l}{\omega^5} \right) + T^4 \left( \frac{1155}{2048} \frac{\dot{V}_l^4}{\omega^{13}} - \frac{231}{256} \frac{\dot{V}_l^2 \ddot{V}_l}{\omega^{11}} + \frac{21}{128} \frac{\dddot{V}_l}{\omega^9} + \frac{7}{32} \frac{\dot{V}_l \dddot{V}_l}{\omega^9} - \frac{1}{32} \frac{\dddot{V}_l}{\omega^7} \right), \]

\[ A_1 = T^2 \left( -\frac{5}{16} \frac{\dot{V}_l^2}{\omega^6} + \frac{1}{4} \frac{\ddot{V}_l}{\omega^5} \right) \xi_1^* \]

\[ + T^4 \left( \frac{1155}{1024} \frac{\dot{V}_l^4}{\omega^{12}} - \frac{231}{128} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^{10}} + \frac{21}{64} \frac{\dddot{V}_l}{\omega^8} + \frac{7}{16} \frac{\dot{V}_l \dddot{V}_l}{\omega^8} - \frac{1}{16} \frac{\dddot{V}_l}{\omega^6} \right) \xi_1^*, \]

\[ A_2 = T \left( -\frac{1}{4} \frac{\dot{V}_l}{\omega^3} \right) \xi_2^* + T^2 \left( -\frac{1}{4} \frac{\dot{V}_l^2}{\omega^5} - \frac{1}{4} \frac{\ddot{V}_l}{\omega^3} \right) \xi_1^{*2} \]

\[ + T^3 \left( \frac{35}{128} \frac{V_l^3}{\omega^9} - \frac{5}{16} \frac{\dot{V}_l \ddot{V}_l}{\omega^7} + \frac{1}{16} \frac{\dddot{V}_l}{\omega^5} \right) \xi_2^* \]

\[ + T^4 \left[ \left( \frac{285}{2048} \frac{\dot{V}_l^4}{\omega^{12}} - \frac{77}{256} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^{10}} + \frac{7}{128} \frac{\dddot{V}_l}{\omega^8} + \frac{7}{96} \frac{\dot{V}_l \dddot{V}_l}{\omega^8} - \frac{1}{96} \frac{\dddot{V}_l}{\omega^6} \right) \xi_3^* \]

\[ + \left( -\frac{35}{64} \frac{\dot{V}_l^4}{\omega^{10}} + \frac{63}{64} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^8} - \frac{11}{48} \frac{\dddot{V}_l}{\omega^6} - \frac{11}{24} \frac{\dot{V}_l \dddot{V}_l}{\omega^6} + \frac{1}{24} \frac{\dddot{V}_l}{\omega^4} \right) \xi_1^{*3} \right], \]

\[ A_3 = T \left( -\frac{1}{2} \frac{\dot{V}_l}{\omega^2} \right) \xi_1^* \xi_2^* + T^2 \left[ \left( -\frac{5}{96} \frac{\dot{V}_l^2}{\omega^6} + \frac{1}{24} \frac{\dddot{V}_l}{\omega^2} \right) \xi_3^* + \left( \frac{1}{12} \frac{\dot{V}_l^2}{\omega^4} - \frac{1}{6} \frac{\dddot{V}_l}{\omega^2} \right) \xi_1^{*3} \right] \]

\[ + T^3 \left( \frac{35}{64} \frac{\dot{V}_l^3}{\omega^8} - \frac{5}{8} \frac{\dot{V}_l \dddot{V}_l}{\omega^6} + \frac{1}{8} \frac{\dddot{V}_l}{\omega^4} \right) \xi_1^* \xi_2^* \]

\[ + T^4 \left[ \left( \frac{285}{2048} \frac{\dot{V}_l^4}{\omega^{12}} - \frac{77}{256} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^{10}} + \frac{7}{128} \frac{\dddot{V}_l}{\omega^8} + \frac{7}{96} \frac{\dot{V}_l \dddot{V}_l}{\omega^8} - \frac{1}{96} \frac{\dddot{V}_l}{\omega^6} \right) \xi_3^* \]

\[ + \left( -\frac{35}{64} \frac{\dot{V}_l^4}{\omega^{10}} + \frac{63}{64} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^8} - \frac{11}{48} \frac{\dddot{V}_l}{\omega^6} - \frac{11}{24} \frac{\dot{V}_l \dddot{V}_l}{\omega^6} + \frac{1}{24} \frac{\dddot{V}_l}{\omega^4} \right) \xi_1^{*3} \right], \]

\[ A_4 = T \left( -\frac{1}{48} \frac{\dot{V}_l}{\omega^3} \right) \xi_4^* + T^2 \left[ \left( \frac{3}{32} \frac{\dot{V}_l^2}{\omega^5} - \frac{1}{16} \frac{\dddot{V}_l}{\omega^3} \right) \xi_2^* + \left( \frac{1}{12} \frac{\dot{V}_l^2}{\omega^4} - \frac{1}{12} \frac{\dddot{V}_l}{\omega^3} \right) \xi_1^* \xi_3^* \right] \]

\[ + T^3 \left[ \left( \frac{35}{1536} \frac{\dot{V}_l^3}{\omega^9} - \frac{5}{128} \frac{\dot{V}_l \dddot{V}_l}{\omega^7} + \frac{1}{128} \frac{\dddot{V}_l}{\omega^5} \right) \xi_4^* + \left( -\frac{15}{32} \frac{\dot{V}_l^3}{\omega^7} + \frac{9}{16} \frac{\dot{V}_l \dddot{V}_l}{\omega^5} - \frac{1}{8} \frac{\dddot{V}_l}{\omega^3} \right) \xi_1^* \xi_2^* \right] \]

\[ + T^4 \left[ \left( -\frac{315}{1024} \frac{\dot{V}_l^4}{\omega^{11}} + \frac{245}{512} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^9} - \frac{5}{64} \frac{\dddot{V}_l}{\omega^7} - \frac{15}{128} \frac{\dot{V}_l \dddot{V}_l}{\omega^7} + \frac{1}{64} \frac{\dddot{V}_l}{\omega^5} \right) \xi_2^* \right. \]

\[ + \left( \frac{175}{512} \frac{\dot{V}_l^4}{\omega^{11}} + \frac{217}{384} \frac{\dot{V}_l^2 \dddot{V}_l}{\omega^9} - \frac{11}{96} \frac{\dddot{V}_l}{\omega^7} - \frac{13}{96} \frac{\dot{V}_l \dddot{V}_l}{\omega^7} + \frac{1}{48} \frac{\dddot{V}_l}{\omega^5} \right) \xi_1^* \xi_3^* \right] \]
\[ A_5 = T \left( -\frac{1}{12} \frac{\dot{V}}{\omega^2} \xi_3 \xi_3^* - \frac{1}{24} \frac{\dot{V}}{\omega} \xi_1 \xi_1^* \right) + O(T^2), \]

The coefficients for the Hadamard function in \([33]\), using \( z = \sin \xi \) are:

\[ 16M^2 \pi^2 \bar{A} = (19136 + 1523z + 99z^2 - 950z^3 - 422z^4 + 3z^5 + 3z^6) \frac{(1 + z)^{-3}}{14400}, \]

\[ 16M^4 \pi^2 \bar{B} = (-1093271 + 1317560z - 275260z^2 - 187012z^3 + 37110z^4 - 4908z^5 - 2396z^6 + 772z^7 + 57z^8 - 12z^9)(1 + z)^{-6} \frac{(1 - z^2)^{-2}}{115200} + 2(4z - 3)(1 + z)^{-6}(1 - z)^{-1} \ln \left[ (1 + z) 2^{-1/3} \right], \]

\[ 16M^4 \pi^2 C_{xx} = (-1474763 + 1403169z - 366148z^2 - 39568z^3 + 18686z^4 - 774z^5 - 196z^6 + 56z^7 + 21z^8 - 3z^9)(1 + z)^{-6} \frac{(1 - z^2)^{-1}}{57600} + 2(1 + z)^{-6} \ln \left[ (1 + z) 2^{-1/3} \right], \]

\[ 16M^4 \pi^2 D_{xxxx} = (-3 - 4z + 2z^2)^2 (1 + z)^{-8} \frac{(1 - z^2)^{-1}}{24}. \]

D. Some useful integrals.

Let us define

\[ I_n = \int_{-\infty}^{\infty} dk \Omega^n e^{-i(\delta_1 \Omega - \delta_2 k)}, \quad \Omega = \sqrt{k^2 + \lambda}, \quad \lambda > 0. \]  

(85)

For the particular value \( n = -1 \) this integral can be easily solved. Performing the following change of variable,

\[ t = t(k) = \delta_1 \Omega - \delta_2 k = \delta_1 \sqrt{k^2 + \lambda} - \delta_2 k, \]

(86)

we must consider separately the two possibilities \( |\delta_1| > |\delta_2| \) or \( |\delta_1| < |\delta_2| \). For \( |\delta_1| > |\delta_2| \) and without loss of generality we may take \( \delta_1, \delta_2 > 0 \), and invert the change (86) as,

\[ k = k(t) = \gamma \left[ \delta_2 t + \delta_1 \sqrt{t^2 - \beta^2} \right], \]

(87)

where we use two new variables \( \gamma \) and \( \beta \), defined as,

\[ \gamma^{-1} \equiv \delta_1^2 - \delta_2^2, \quad \beta^2 \equiv \lambda \gamma^{-1}. \]

(88)

Note that \( k(t) \), in (87), is a bivalued function and we have to be careful in changing the integration limits. Since \( |\delta_1| > |\delta_2| \), then \( \lim_{k \to \pm \infty} t(k) = +\infty \), and this means that (87) has an absolute minimum at the point \((+\gamma \beta \delta_2, \beta)\). With this it is easy to see that we have
to take as inverse function of (86) the function (87) with the plus sign to the right of the
minimum \( t = \beta \) and with the minus sign to the left of the minimum. Therefore we can split
the integral (85), for \( n = -1 \), in two parts at each side of \( t = \beta \), i.e,

\[
\mathcal{I}_{-1} = \int_{-\infty}^{\gamma \beta} \frac{dk}{\Omega} e^{-i\epsilon (\delta_1 \Omega - \delta_2 k)} + \int_{\gamma \beta}^{\infty} \frac{dk}{\Omega} e^{-i\epsilon (\delta_1 \Omega - \delta_2 k)}.
\]  

(89)

The change of variables (86) gives \( \Omega^{-1} dk = \pm \left(t^2 - \beta^2\right)^{-1/2} dt \), with the minus sign for the
first integral in (89) and the plus sign for the second integral. Then the integration can be
easily performed in terms of a zero order Bessel function [40], as

\[
\mathcal{I}_{-1} = 2 \int_{\beta}^{\infty} \frac{e^{-i\epsilon t}}{\sqrt{t^2 - \beta^2}} dt = 2 K_0 (i\epsilon \beta).
\]  

(90)

For the case \(|\delta_1| < |\delta_2|\), we define the parameter \( \gamma \), as, \( \gamma^{-1} \equiv \delta_2^2 - \delta_1^2 \), and the inverse
function of (86) is,

\[
k(t) = k(t) = -\gamma \left[ \delta_2 t \pm \delta_1 \sqrt{t^2 + \beta^2} \right],
\]  

(91)

which is again a bivalued function. Now, however, the function (86) has no extrema, and it
is easy to see that we can take a single inverse everywhere as,

\[
k(t) = -\gamma \left[ \delta_2 t - \delta_1 \sqrt{t^2 + \beta^2} \right], \quad \frac{dk}{\Omega} = -\frac{dt}{\sqrt{t^2 + \beta^2}},
\]

so that (85), for \( n = -1 \), can also be integrated in terms of a zero order Bessel function [40] as,

\[
\mathcal{I}_{-1} = \int_{-\infty}^{\infty} \frac{dk}{\Omega} e^{-i\epsilon (\delta_1 \Omega - \delta_2 k)} = \int_{-\infty}^{\infty} \frac{e^{-i\epsilon t}}{\sqrt{t^2 + \beta^2}} dt = 2 \int_{0}^{\infty} \frac{\cos (\epsilon t)}{\sqrt{t^2 + \beta^2}} dt 2 K_0 (\epsilon \beta).
\]  

(92)

Finally putting together the results (90) and (92), we can write

\[
\mathcal{I}_{-1} = 2 K_0 (i\epsilon \beta),
\]  

(93)

with the parameter \( \beta \) given by,

\[
\beta^2 = \lambda \gamma^{-1} = \lambda \left( \delta_2^2 - \delta_1^2 \right).
\]  

(94)

The integrals \( \mathcal{I}_n \) with \( n \geq 0 \) are divergent unless we adopt the standard prescription of
taking \( \epsilon \to \epsilon - i0^+ \) for \(|\delta_1| > |\delta_2|\) and \( \epsilon \to \epsilon + i0^+ \) for \(|\delta_1| < |\delta_2|\). In that case all these
integrals can be easily related to \( \mathcal{I}_{-1} \) by means of the following recursion formula,

\[
\mathcal{I}_n = \frac{i}{\epsilon} \frac{\partial \mathcal{I}_{n-1}}{\partial \delta_1}, \quad \text{then} \quad \mathcal{I}_n = \left( \frac{i}{\epsilon} \right)^{n+1} \frac{\partial^{n+1} \mathcal{I}_{-1}}{\partial \delta_1^{n+1}}.
\]  

(95)

Since \( \mathcal{I}_{-1} \) is essentially the Bessel function \( K_0 \) and the derivatives of the Bessel functions can
be written also in terms of Bessel functions [40], then all \( \mathcal{I}_n \) with \( n \geq 0 \) can be expressed as
a linear combination of Bessel functions. For example, using (94) for \( \beta \) and recalling that
\( K'_0(z) = dK_0(z)/dz = -K_1(z) \),
\[ I_0 = \frac{i}{\epsilon} \frac{\partial I_{-1}}{\partial \epsilon} = -2 K_1'(i\epsilon) \frac{\partial \beta}{\partial \epsilon} = 2 \gamma \beta \delta_1 K_1(i\epsilon \beta). \]

The integrals \( I_n \) with \( n \leq -2 \) are finite in the limit \( \epsilon \to 0 \) and they can also be recursively calculated by means of \( I_{-1} \) and the following recursion relation for \( n \leq -1 \):

\[ \frac{\partial I_n}{\partial \lambda} = \frac{n}{2} I_{n-2} - \frac{i}{\epsilon} \frac{\partial \epsilon}{\partial \epsilon} I_{n-1}. \]  

This allows us to calculate all the \( I_n \) with \( n \leq -1 \) only if we know at least another integral besides \( I_{-1} \). Fortunately it is not very difficult to evaluate \( I_{-2} \) from the recursion relation (95) and by a simple integration,

\[ I_{-2}(\delta_1) = -i e \int_{\delta_1}^{\delta'} d\delta' I_{-1}(\delta'_1) + I_{-2}(0), \]

where the integration constant \( I_{-2}(0) \), which corresponds to the value of \( I_{-2} \) for the special value \( \delta_1 = 0 \), can be calculated directly from \( (85) \) [40], then

\[ I_{-2}(\delta_1) = -i 2 e \int_{\delta_1}^{\delta'} d\delta' K_0(i\epsilon \beta(\delta'_1)) + \frac{\pi \lambda}{\lambda^{1/2}} e^{i\lambda^{1/2} \delta_2}. \]  

Let us denote,

\[ I_{E_{2n}} = \int_{-\infty}^{\infty} dk \Omega^{2n} \text{Ei}(-i\epsilon \delta_1 \Omega) e^{i\epsilon \delta_2 k}, \quad \Omega = \sqrt{k^2 + \lambda}; \quad \lambda > 0, \]  

where \( \text{Ei}(ix) \), with \( x \) real, is the integral-exponential function with complex argument [40]. We can easily calculate \( I_{E_0} \) since by the properties of \( \text{Ei}(ix) \), it is not difficult to relate \( I_{E_0} \) with \( I_{-2} \),

\[ \frac{\partial I_{E_0}}{\partial \lambda} = \frac{1}{2} I_{-2} \]  

This can be easily integrated as

\[ I_{E_0}(\lambda) = \frac{1}{2} \int_{0}^{\lambda} d\lambda' I_{-2}(\lambda') + I_{E_0}(0), \]  

where, as in the case for \( I_{-2} \), the arbitrary integration constant \( I_{E_0}(0) \) can be directly calculated from \( (88) \) [40], for the particular value \( \lambda = 0 \). The final result is,

\[ I_{E_0}(\lambda) = \frac{1}{2} \int_{0}^{\lambda} d\lambda' I_{-2}(\lambda') + \begin{cases} \frac{i}{\epsilon} \frac{\partial \epsilon}{\partial \epsilon} \ln[(\delta_1 + \delta_2)^2 \gamma], & \text{for } \delta_2 \neq 0. \\ -\frac{2i}{\epsilon \delta_1}, & \text{for } \delta_2 = 0. \end{cases} \]  

The integrals \( I_{E_{2n}} \) with \( n > 0 \) diverge unless we adopt the standard prescription \( \epsilon \to \epsilon - i0^+ \) for \( |\delta_1| > |\delta_2| \) and \( \epsilon \to \epsilon + i0^+ \) for \( |\delta_1| < |\delta_2| \) then they can be calculated with the known values of \( I_n \) and \( I_{E_0} \) by means of the following expression,

\[ \frac{\partial I_n}{\partial \delta_2} = i \epsilon \int_{-\infty}^{\infty} dk \left( \frac{e^{-i\epsilon \delta_1 \Omega}}{\Omega} \frac{k}{\Omega} \right) \Omega^{n+2} e^{i\epsilon \delta_2 k}, \]  

\[ \text{for } \delta_2 \neq 0. \]  

\[ \text{for } \delta_2 = 0. \]
which can be integrated by parts, using the properties of the Ei$(ix)$, to obtain the following recursion relation

$$\mathcal{I}_n^{+2} = \frac{1}{\epsilon^2 \delta_2} \frac{\partial}{\partial \delta_2} \left[ I_n + (n + 2) \mathcal{I}_0 \right].$$

(102)

\section{Geodesic coefficients $\xi^*_i$ and $x_i$}

We start with either a timelike or spacelike geodesic connecting the points $x'$, $\bar{x}$ and $x$, which can be written in terms of the proper geodesic distance $\tau$ as,

$$x'^\mu = x'^\mu (\bar{\tau} + \epsilon), \quad \bar{x}^\mu = x^\mu (\bar{\tau}), \quad x'^\mu = x^\mu (\bar{\tau} - \epsilon),$$

and define at the midpoint $\bar{x}$ the parameters $\xi^*_n = d^n x^*/d\tau^n|_{\bar{\tau}}$ and $x_n = d^n x/d\tau^n|_{\bar{\tau}}$, which determine the geodesic. From the metric in the interaction region keeping $\eta$ and $y$ fixed, by the symmetry of the problem, we get

$$ds^2 = \left( 1 - \sin \xi_1 \right) \left( d\xi^*_{\alpha} - dx^2 \right).$$

(103)

If we parametrize the geodesic with its proper distance, then (103) gives the following relation between the coefficients $\xi^*_1$ and $x_1$,

$$\xi^*_{\alpha} - x^2 = \pm \left( 1 + \frac{\sin \xi_1}{1 - \sin \xi_1} \right).$$

(104)

with the plus sign for timelike geodesics and the minus sign for spacelike geodesics. Since the coordinate $x$ does not appear in (103), it is easy to see that,

$$\frac{dx}{d\tau} = -p_x \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right), \quad \left( \frac{d\xi^*}{d\tau} \right)^2 = p_x^2 \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right)^2 \pm \frac{1 + \sin \xi}{1 - \sin \xi}.$$ 

(105)

Equations (105) and the relation between $\xi$ and $\xi^*$, $d\xi^* = d\xi (1 + \sin \xi)^2 / \cos \xi$, allow us to obtain the derivatives $d^n x^*/d\tau^n$ and $d^n x/d\tau^n$, which when evaluated at $\tau = \bar{\tau}$ produce the geodesic coefficients we need.

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Fig. 1: The four spacetime regions and their boundaries are represented in the coordinate system \((u, v)\), which does not preserve the causal structure. Region IV is the flat region before the arrival of the waves, regions II and III represent the plane waves and region I is the interaction region. The boundaries \(\mathcal{L}\) and \(\mathcal{L}'\), which are spacetime lines, must be identified with the folding singularities \(\mathcal{P}\) and \(\mathcal{P}'\) of the horizon \(u + v = \pi/2\) (there is no spacetime beyond these lines).