Dynamical Likelihood Method for Reconstruction of Quantum Process

Kunitaka Kondo

Advanced Research Institute for Science and Engineering,
Waseda University, Tokyo 169-8555, Japan

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Abstract

The dynamical likelihood method for analysis of high energy collider events is reformulated. The method is to reconstruct the elementary parton state from observed quantities. The basic assumption is that each of final state partons occupies a unit phase space. The parton kinematics is statistically reconstructed using (a) virtual masses of resonant partons and (b) parton kinematic quantities inferred from observed quantities. Generation of (b) is made with the transfer function which is the probability function for parton kinematics from a given set of observables. Corresponding to the unit parton phase space, the transfer variable spaces are also quantized. The likelihood of the reconstructed state is defined by the Poisson probability for a single event with the expected number of event that is the cross section per unit phase space times a luminosity factor. Applications of the method to selection of process, parton-observable identification, determinations of parton kinematics and dynamical parameters are discussed.
1 Introduction

Dynamical Likelihood Method (DLM) was originally proposed in Ref. [1] and developed in Refs. [2] and [3] as a method to determine dynamical parameters, e.g. masses, decay widths or coupling constants from measured quantities (observables). The basic idea is to use the differential cross section (d.c.s.) as a theoretical input for the event reconstruction. A formulation of the method to use the d.c.s. as a posterior probability was given in Ref. [4], and was used in Ref. [5].

An alternative formulation of the posterior probability with the d.c.s. is presented in this paper. The motivation is as follows.

In the traditional use of the d.c.s., it is defined per certain kinematic quantities and integrated by other kinematic variables. This is to match the detector arrangements devised for individual experiments. The d.c.s. there is the prior probability and can be essentially applied to a large number of events. In the general purpose collider experiments, however, the detectors are not designed to measure particular quantities but to make it possible to get over-all picture of each event. Hence the event reconstruction on event by event basis is feasible. To deal with a single event, we postulate that partons in the final state of hard scattering occupy a unit phase space. Thus the d.c.s. is defined per unit phase space of the final partons.

Leptons, quarks and gauge bosons are called partons in this paper. Parton process described by the Feynman diagram, i.e. a hard scattering process, is called the elementary or parton level process. A process from the initial beam state to the observables is called a path. A path that a real event has taken is unknown. The event reconstruction is to infer paths which the event could have taken.

The probability density function (p.d.f.) of the first half a path, i.e. from the beam to the parton state, is defined by the d.c.s. per unit phase space (quantum state) of the final partons in the elementary process. This is consistent with the original definition of the d.c.s.

The p.d.f. for the second half a path, i.e. from the parton state to observables, is the transfer function (T.F.) that relates the observables with the corresponding variables at the parton level. Corresponding to the quantum condition on the final parton state, the transfer variable spaces are also quantized.

The third quantum condition is about the number of events. The d.c.s. multiplied with a luminosity factor can be interpreted as an expectation value of the number of events, and the likelihood (posterior probability) of the event is defined by the Poisson probability for 1 event with the given expectation value.

This paper is organized as follows:

In Sec. 2, the d.c.s. for the final parton state and its use for the likelihood definition are discussed. The parton level likelihood works as the likelihood of the reconstructed parton process as we see in the following sections.

The subject of Sec. 3 is the T.F. A way to obtain the T.F. with fully simulated Monte Carlo events is proposed. The detection efficiency associated with measurements and event selection criteria is automatically included in the T.F. According to the quantum parton state, the Jacobian scaled transfer variables are introduced. As a result, the likelihood is essentially Jacobian free. The T.F. in its posterior form is used for parton level reconstruction.

An efficient way of inferring paths is discussed in Sec. 4. Momenta of observed final partons are inferred with the posterior T.F. using observed quantities as inputs. Virtual masses squared of resonant primary partons are inferred with their propagator factors.

In Sec. 5, the likelihood for reconstructed multiple parton states is discussed. To regenerate the unknown true path of a given event, one makes multiple inference of paths in the event. For the multiple inferences, the maximum, the expectation and the multiplicative values of the likelihood are defined. They are used statistically to select the process, the parton-observable identification and the kinematic solution of the secondary partons. The determination of the dynamical parameters is made by the joint likelihood of all events in the sample.

Summary of the formulation is given in Sec. 6.

2 Differential Cross Section and Parton Level Likelihood

In DLM, the d.c.s. for an inferred parton process is used as a theoretical input to evaluate the likelihood of the reconstruction.
2.1 Differential cross section for a final parton state

We assume that a process is described at the parton level by

\[ a/A + b/B \rightarrow \cdots \rightarrow c_1 + \cdots + c_n \equiv C, \]

where \( a \) and \( b \) are the initial partons, each representing a quark or an anti-quark or a gluon, in beam particles \( A \) and \( B \) respectively, and \( c(c_1, c_2, \ldots, c_n) \) are the final state partons. States of partons are after the initial- and before the final-state radiations. Throughout this paper, particle symbol \( p \) also represents its 4-momentum, and \( p \) its 3-momentum. The final partons are assumed to be on mass-shells, i.e. 3-momenta are enough to define their states. Process (1) as a whole, i.e. a set of momenta of all partons, is called parton state in this paper.

Beam particles \( A \) and \( B \) are assumed to make a head-on collision along the \( z \)-axis. Then the hadronic cross-section for process (1) is given by

\[ d\sigma = dz_adz_b dP_T f_{a/A}(z_a, \alpha) f_{b/B}(z_b, \alpha) f_T(P_T, \alpha) d\sigma(a + b \rightarrow C; \alpha), \]

where \( d\sigma \) is the parton level cross section.

\[ d\sigma(a + b \rightarrow C; \alpha) = \frac{(2\pi)^4 \delta^4(a + b - C)}{4\sqrt{(a-b)^2 - m_a^2 m_b^2}} |M(a + b \rightarrow C; \alpha)|^2 d\Phi_n^{(f)}. \]

In Eq. (2), symbol \( \alpha \) stands for a set of dynamical constants, e.g. masses, decay widths or coupling constants. Hereafter, we use symbol \( \alpha \) to represent only unknown parameters to be measured. Variables \( z_a = a_z/|A| \) and \( z_b = b_z/|B| \) are momentum fractions of \( a \) and \( b \) in hadrons \( A \) and \( B \) respectively, and \( P_T \) is the total momentum of the initial/final system of process (1) in the plane perpendicular to the beam axis. The probability density functions (p.d.f.) for \( z_a, z_b \) and \( P_T \) are denoted by \( f_{a/A}, f_{b/B}, \) and \( f_T \), respectively. Functions \( f_{a/A}, f_{b/B} \) and \( f_T \) are effective parton distribution functions for process (1) with the radiation parts removed. In practice, these p.d.f.’s are to be obtained by running Monte Carlo event generators.

In Eq. (3), \( M \) is the matrix element for process (1), and \( d\Phi_n^{(f)} \) is the differential factor,

\[ d\Phi_n^{(f)} = \prod_{i=1}^{n} \frac{d^3c_i}{(2\pi)^32E_i}. \]

of the Lorentz invariant \( n \)-body phase space element,

\[ d\Phi_n = \delta^4(a + b - C) d\Phi_n^{(f)}. \]

We call \( d\Phi_n^{(f)} \) the phase space element (P.S.E.) in this paper.

The d.c.s. for a final state defined by \( c(c_1, \cdots, c_n) \) is obtained by integrating Eq. (2) with the initial state variables \( z_a, z_b \) and \( p_T \) as

\[ d\sigma = I(a, b) |M(a + b \rightarrow C; \alpha)|^2 d\Phi_n^{(f)}, \]

namely,

\[ \frac{d\sigma}{d\Phi_n^{(f)}} = I(a, b) |M(a + b \rightarrow C; \alpha)|^2, \]

where

\[ I(a, b) = \frac{(2\pi)^4}{4|A||B|^2 \sqrt{(a-b)^2 - m_a^2 m_b^2}} f_{a/A}(z_a, \alpha) f_{b/B}(z_b, \alpha) f_T(P_T, \alpha). \]

The formulation in this paper is to define the likelihood to be proportional to \( d\sigma/d\Phi_n^{(f)} \).

2.2 Resonances (internal lines)

Assume a process where resonance \( r \), which corresponds to an internal line in the Feynman diagram, is produced as

\[ a + b \rightarrow r + c_{j+1} + c_{j+2} + \cdots + c_n, \]
and subsequently decays into channel $\rho$ as
\[ r \rightarrow \rho : c_1 + c_2 + \cdots + c_j. \]

**Propagator factor** If the matrix elements for processes (9) and (10) are given as $M_{\text{prod}}$ and $M_{\text{dec}}$ respectively, the matrix element squared for process (1) is factorized as
\[ |M(a + b \rightarrow r + c_{j+1} + \cdots + c_n \rightarrow C; \alpha)|^2 = |M_{\text{prod}}|^2 \Pi(s_r)|M_{\text{dec}}|^2, \tag{11} \]
where $s_r$ is the virtual mass squared of resonance $r$, given by
\[ s_r \equiv r^2 = (\sum_{i=1}^{j} c_i)^2. \tag{12} \]

The lowest order approximation for propagator factor $\Pi(s_r)$ is given by the Breit-Wigner form,
\[ \Pi(s_r) = \frac{1}{(s_r - M_r^2)^2 + M_r^2 \Gamma_r^2}, \tag{13} \]
where $\Gamma_r$ is the total decay width of $r$. Improved forms taking the higher order effects into account are discussed in Ref. [7].

In the event reconstruction, if all $c_i$’s are inferred from observables, $s_r$ is given by Eq. (12), while if $s_r$ is generated according to the propagator factor, Eq. (13), Eq. (12) gives a constraint to $(c_1, \cdots, c_j)$.

### 2.3 Parton level likelihood

Reconstruction of a parton level process by using $s_r$’s according to the propagator factors and $c_j$’s with the transfer function will be discussed in the following Sections. In this subsection, we define the parton level likelihood assuming that a path, i.e. a set of parton kinematics $P(P_1, \cdots, P_N)$, where $N$ is the total number of partons in the process, is given.

If parton kinematics is given, the differential cross section $\varsigma \equiv d\sigma/d\Phi_n(f)$ can be calculated. The expected number of events with cross section $\varsigma$ is
\[ \mu = l_1 \varsigma \equiv l_1 \frac{d\sigma}{d\Phi_n(f)}, \tag{14} \]
where $l_1$ stands for a luminosity factor independent of kinematics of the path. We assume that an event takes place according to the Poisson probability of a single event $P(1; \mu)$, i.e.
\[ dP(1; \mu) = P(1; \mu) d\mu = P(\varsigma) d\varsigma, \tag{15} \]
and define the likelihood of the path by
\[ L_1^{(0)} \equiv P(1; \mu) = \mu \exp(-\mu), \tag{16} \]
where suffix 1 and prefix (0) stand for a single path and the parton level, respectively. The p.d.f. for $\varsigma \equiv d\sigma/d\Phi_n^{(f)}$ is given by
\[ P(\varsigma) = l_1 L_1^{(0)} = l_1 \mu \exp(-\mu) = l_2^2 \varsigma \exp(-l_1 \varsigma), \tag{17} \]
from Eqs. (14),(15) and (16).

Events in a data set are mutually independent, hence the number of event distribution in a data set with the total number of events $N_{\text{tot}}$ is given as
\[ \frac{dN}{d\mu} = N_{\text{tot}} L_1^{(0)}. \tag{18} \]

**Peak value and the normalization of $L_1^{(0)}** Likelihood $L_1^{(0)}$ takes its maximum value $1/e$ for $\mu = 1$, since
\[ \frac{dL_1^{(0)}}{d\mu} = (1 - \mu) \exp(-\mu), \tag{19} \]
and it is normalized as
\[ \int_0^\infty L_1^{(0)} d\mu = \int_0^\infty \mu \exp(-\mu) d\mu = 1. \]  

**Luminosity factor** \( l_1 \) The expectation value of \( \varsigma = d\sigma/d\Phi_n^{(f)} \) is
\[ \left\langle \frac{d\sigma}{d\Phi_n^{(f)}} \right\rangle = \int_0^\infty \varsigma P(\varsigma) d\varsigma = \frac{1}{l_1} \int_0^\infty \mu^2 \exp(-\mu) d\mu = \frac{2}{l_1}. \]

or
\[ l_1 = 2 \left( \frac{d\sigma}{d\Phi_n^{(f)}} \right)^{-1}. \]  

If the integrated luminosity of the data set is \( L_{int} \), then
\[ N_{tot} = L_{int} \left\langle \frac{d\sigma}{d\Phi_n^{(f)}} \right\rangle = \frac{2L_{int}}{l_1}. \]  

assuming the detection efficiency is 1. Then,
\[ l_1 = \frac{2L_{int}}{N_{tot}} = \frac{2}{\sigma_T}, \]  

where \( \sigma_T \) is the total cross section for the process. In event reconstruction it may be interpreted as a function of the mass of particle in search, but since \( L_{int} \) and \( N_{tot} \) are measured/measurable quantities, we interpret them as observables which are intrinsic to the given data sample just as the observed kinematic variables are. Formulation including the detection efficiency will be discussed in the later section.

### 3 Transfer function

The p.d.f. for the second half a path, i.e. a path from the parton state to the observables, is the transfer function (T.F.).

#### 3.1 Observables in collider experiments

Typical collider detectors have calorimeters and the tracking system. Calorimeters and trackers with give energies and momenta of particles respectively. A jet is generally identified with a quark (anti-quark) or a gluon. An electromagnetic shower associated with or without a track is assigned to an electron or a photon. A track passing through calorimeters with a minimum ionizing signal is identified with a muon. We call these particles observable partons and their measured quantities observables.

**Electrons, muons and photons** These particles are relatively well identified and their momenta are measured within the detector resolutions.

**Jets** Jets are assigned to quarks or glus. Measured quantities of jets have uncertainties due to statistical nature of parton shower, hadronization, resolution of detectors and jet reconstruction algorithm.

As for the assignment of jets to partons and the relation between their momenta, we make following comments:

1. There are color flows between these partons, and the fragmentation is not independent among them. The effect can be integrated in the transfer function to be discussed later.

2. It is possible that a quark/gluon is observed as two or more jets. If two or more nearby jets are merged into one and identified with a single parton successfully, the effect can be remedied. Otherwise, the effect results as an inefficiency of the reconstruction and/or a shift in the values of dynamical parameters to be determined.

3. In general 4-momentum \((E_J, p_J)\) is measured on a jet. For quarks in the final partons, however, we assume their pole masses. Hence 3 quantities of a jet are enough to infer the quark 3-momentum. The selection of these quantities is not unique but is to be made according to the process, the purpose of analysis and detector properties.
Missing partons. For partons which do not interact with detectors, e.g. neutrinos, the missing transverse energy (MET) $E_T$ defined by

$$ E_T = -E_T^{\text{obs}} = -(E_T^{\text{cal}} + \sum \mu_T) $$

is measured, where $E_T^{\text{obs}}$ is the measured total transverse energy flow, $E_T^{\text{cal}}$ is the sum of the transverse energy flow measured by calorimeters, and $\sum \mu_T$ is the sum of transverse momenta of muons measured by the tracking detector. All vectors in Eq. (25) are in the plane perpendicular to the beam-axis.

3.2 Transfer functions for observable quantities

For a real event in experiment, the final observables are known, while the parton state in process (1) is unknown. The event reconstruction in DLM is to infer such unknown parton state that leads to an observed variable set $y(y_1, \cdots, y_{N_v})$. The parton variable set corresponding to $y$ is denoted by $x(x_1, \cdots, x_{N_p})$. We call $x$ and $y$ transfer variables.

Prior transfer function. The prior transfer function (T.F.) is a p.d.f. for $y$ when $x$ is given and denoted by $w(y|x||i_p, \alpha)$, where $i_p$ is an integer to specify the process. The probability for $(x, y)$ to be in $(dx, dy)$ is

$$ dP(x, y||i_p) = w(y|x||i_p, \alpha) \, dx \, dy. $$

where

$$ dx = \prod_{m=1}^{N_p} dx_m, \quad dy = \prod_{m=1}^{N_v} dy_m. \quad (27) $$

If $w(y|x||\alpha) > 0$, a certain value of $y$ should exist. Hence we require the normalization condition,

$$ \int_{w>0} w(y|x||\alpha) dy = 1, \quad (28) $$

for any $x$ with $w > 0$.

A typical example of a component of $y$ is the energy of a jet. The T.F. for a jet depends on models of parton-shower and fragmentation, the detector response and the jet reconstruction algorithm. Thus it is appropriate to derive the T.F. by using Monte Carlo event generators with full simulation, where the momentum of each parton and measured quantities associated with it are provided. Events are to be selected with the same criteria as applied to real data.

T.F. from Monte Carlo events. The T.F. is a function of multi-dimensional variables $x$ and $y$. We assume that the T.F. is factorisable as

$$ w(y|x||i_p, \alpha) = \prod_{m=1}^{N_v} w(y_m|x||i_p, \alpha). \quad (29) $$

To illustrate how to get T.F., we take a simple case where a T.F. for $y_m$ depends on the corresponding variable $x_m$ only, and $\alpha$ has a single component $\alpha$. We denote the T.F. by $w(y|x||\alpha)$, abbreviating process number $i_p$ and variable number $m$.

Let $n_{xy}$ denote the density of generated number of events at $(x, y)$, and $n_x$ that at $x$. T.F. $w(y|x||\alpha)$ is defined such that the number of events in $(dx, dy)$ is given by

$$ dN(x, y) = n_{xy} dx dy = n_x dx \times w(y|x||\alpha) dy. \quad (30) $$

With the integrated luminosity $L_{int}$, the number densities are given by

$$ n_{xy} = L_{int} \frac{d\sigma}{dx} w(y|x||\alpha), \quad n_x = \int n_{xy} dy = L_{int} \frac{d\sigma}{dx}. \quad (31) $$

The $y$ dependence of the detection efficiency is included in $w(y|x||\alpha)$.

From Eq. (30), the T.F. is given by

$$ w(y|x||\alpha) = \frac{n_{xy}}{n_x} = \frac{n_{xy}}{n_x} = 1. \quad (32) $$
Thus \( w(y|x|\alpha) \) is obtained by filling the \((x, y)\) histogram with weight \(1/n_x\) for each event. Weighting by \(1/n_x\) is to avoid the double counting of the cross section factor which exists in the parton level likelihood. Integrating Eq. (32) by \(y\) and using Eq. (31), one obtains the normalization condition, Eq. (28). Note that the correction for the detection inefficiency associated with measurements and event selection conditions is automatically made by deriving the T.F. with the Monte Carlo events.

### 3.3 Quantization of the transfer variable space

We consider how the quantum condition \(\Delta \Phi^{(f)}_{n} = 1\) characterizes the transfer variable spaces. This condition applies to all final partons, while the transfer variables make sense only for observable partons. Hence we first discuss the case of observable final partons.

**Jacobian scaled variables for observable final parton** We denote transfer variables of the \(l\)-th observable final parton by \(x_l\) and the corresponding observables by \(y_l\) \((l = 1, \ldots, N_{\text{obs}})\). Variable \(x_l\) is a 3-component function of \(p_l\), and generally \(N_{\text{obs}}^* \leq n\), where \(n\) is the number of final state partons. For the \(l\)-th observable final parton, we introduce variables \((X_l, X_l)\) by

\[
dX_l \equiv \prod_{k=1}^{3} dX_{lk} = \frac{d\Phi^{(l)}_{1}}{dx_l}dx_l = J_{xl}dx_l, \quad (33)
\]

\[
dY_l \equiv \prod_{k=1}^{3} dY_{lk} = \frac{d\Phi^{(l)}_{1}}{dy_l}dy_l = J_{yl}dy_l, \quad (34)
\]

where \(J_{xl}\) and \(J_{yl}\) are the phase space Jacobian factors,

\[
J_{xl} \equiv \left(\frac{1}{(2\pi)^{3/2}E_l}\left|\frac{\partial(p_{lx}, p_{ly}, p_{lz})}{\partial(x_{l1}, y_{l2}, x_{l3})}\right|\right), \quad (35)
\]

\[
J_{yl} \equiv J_{xl}|_{x=y}. \quad (36)
\]

Obviously, the unit phase volume \(\Delta \Phi^{(f)}_{1} = 1\) corresponds to the unit variable spaces

\[
\Delta X_l = 1, \quad \Delta Y_l = 1. \quad (37)
\]

A time-ordered path may be described as follows. A single path specifies a unit phase volume (cell) of final parton \(l\), which one-to-one corresponds to a unit volume (cell) of \(X_l\), and picks up that of \(Y_l\) statistically according to the T.F. In other words, by condition \(\Delta \Phi^{(f)}_{1} = 1\), variable spaces \(x_l\) and \(y_l\) are quantized. The elements of these spaces become from real (continuous) to countable (discrete) almost-infinite numbers. The width of the quantized single path is \(\Delta X_l = \Delta Y_l = \Delta \Phi^{(f)}_{1} = 1\).

**Transfer functions for Jacobian scaled variables** We denote T.F. for Jacobian scaled variable \((X, Y)\) by \(W(Y|X||\alpha)\). To compare the two T.F.’s, \(w\) and \(W\), we again treat a case of a single variable set \((x, y)\) and \((X, Y)\). The number of generated event in \((dX, dY)\) is expressed in terms of \(W(Y|X||\alpha)\) as

\[
dN(X, Y) = L_{\text{init}} \frac{d\sigma}{dX} W(Y|X||\alpha) dX dY. \quad (38)
\]

But \(dN\) should be proportional to the outlet path width, i.e.

\[
dN(X, Y) = J_{y} dN(x, y). \quad (39)
\]

Comparing Eqs. (30),(31), (38) and (39), one gets a scale invariance of T.F.,

\[
W(Y|X||\alpha) = w(y|x||\alpha). \quad (40)
\]

**Posterior T.F.** Posterior T.F. \(w(x|y||\alpha)\) for a single component set \((x, y)\) is given by

\[
w(x|y||\alpha) = \frac{w(y|x||\alpha)}{\int_{w > 0} w(y|x||\alpha) dx} \quad (41)
\]
and is to be used to infer parton variable $x$ from observable $y$. The posterior T.F. for Jacobian scaled variables $W(X|Y||\alpha)$ is obtained by
\[ W(X|Y||\alpha) = \frac{W(Y|X||\alpha)}{\int_{W>0} W(Y|X||\alpha) \, dX}. \] (42)

For a given $Y$, the value of $X$ is to be inferred by the probability,
\[ dP(X; Y) = W(X|Y) \, dX. \] (43)

Using the scale invariance, Eq. (40), one gets
\[ dP(X; Y) = J_x \bar{J}_x w(x|y||\alpha) \, dx, \] (44)

where \( \bar{J}_x \) is the mean value of $J_x$ defined by
\[ \bar{J}_x = \int_{W>0} J_x w(y|x||\alpha) \, dx / \int_{W>0} w(y|x||\alpha) \, dx. \] (45)

The dominant part of $w(x|y||\alpha)$ is symmetric with respect to $x - \bar{x}$, where
\[ \bar{x} = \int x \, w(x|y||\alpha) \, dx. \] (46)

Hence the effect of $J_x - \bar{J}_x$ is cancelled out in the first order, and $J_x \approx \bar{J}_x$. In this approximation,
\[ dP(X; Y) \approx w(x|y||\alpha) \, dx. \] (47)

Thus the variable quantization is required only conceptually, and in practice one can use the posterior T.F. $w(x|y||\alpha)$ instead of $W(X|Y||\alpha)$.

**Missing final partons** The only observable about missing partons are the missing transverse energy, MET. The sum of transverse momenta of missing particles, $T(T_x, T_y)$, is inferred with T.F. for MET, $w(T|\not{E}_T||\alpha)$. The parton level cross section can be written as
\[ \frac{d\sigma}{d\Phi_{n}^{(f)}} = \frac{d\sigma}{d\Phi_{n}^{(f)}} \delta(T_x - \sum_{m=1}^{M} c_{mx}^{*}) \, dT_x \delta(T_y - \sum_{m=1}^{M} c_{my}^{*}) \, dT_y, \] (48)

where $c_{ix, iy}^{*}(i = 1, \ldots, M)$ are the $(x, y)$ components of missing partons. $\delta$-functions in Eq. (48) give constraints,
\[ \sum_{m=1}^{M} c_{mx}^{*} = T_x, \quad \sum_{m=1}^{M} c_{my}^{*} = T_y. \] (49)

Since the quantization requirement for each missing final parton, $\Delta \Phi_{1}^{(f)} = 1$, is for 3-dimensional variables, the requirement is compatible with the 2-dimensional constraint, Eq. (49). To summarize, the MET constraint Eq. (49) is free from the quantization condition, and the phase space of each reconstructed parton, whether observable or missing, is to be taken as 1. The value of $d\sigma/d\Phi_{n}^{(f)}$ is evaluated with the transverse momentum components determined with constraint (48).

## 4 Path Reconstruction

### 4.1 Primary and secondary partons in event reconstruction

DLM is a procedure to reconstruct the parton state, i.e. a set of momenta of all partons, $P(P_1, \ldots, P_N)$, including resonances and final partons. The parton kinematics is defined in general by giving momenta of $n$ out of $N$ partons. We call such $n$ partons the primary partons, denoting them by $p(p_1, \ldots, p_n)$. Momenta of residual partons are determined by the energy-momentum conservation at vertices of the Feynman diagram. We call these partons secondary partons. These names are only to specify roles of partons in the event reconstruction. The selection of the primary partons is optional, depending on the process and the reconstruction algorithm.
4.2 Specifications of process, topology and solution

Given an event with observable set \( y \), there are 3 integers to specify a path.

**Process**
First, one has to assign physics process \( i_p = 1, \cdots, N_p \) which \( y \) came from.

**Topology**
Some of observed partons in an event cannot be uniquely identified with final partons in the elementary process. Examples are the same sign electrons or muons, multiple photons or jets. In the event reconstruction, one has to assign some components of observable \( y \) to a set of parton species to define variable \( x \). We call each set of the parton assignment to \( y \) a *topology* in this paper, and denote the topology number by \( i_t = 1, \cdots, N_t \), where \( N_t \) depends on process \( i_p \). Variable \( x \) and hence the value of the T.F. depend on the assumed topology \( i_t \).

**Solution**
If a process includes resonance(s), whether daughters are missing or observed, one can infer their momenta and solve Eq. (12) for momentum component(s) of daughter parton(s). The solutions are sorted by the solution number \( i_s = 1, \cdots, N_s \), where \( N_s \) depends on \( i_p \) and \( i_t \).

4.3 Outline of path reconstruction

The procedure of a single path reconstruction is summarized below.

1. One specifies process \( i_p \) and infers \( \alpha \) uniformly,
2. One specifies topology \( i_t \), and infers parton kinematics as follows:
   a. One specifies an appropriate set of \( n \) primary partons. If all primary partons are observable, one infers their momenta \( p \) according to T.F. Totally missing partons are classified to primary partons, and a set of their momenta is to be inferred uniformly in their phase space.
   b. If a resonance is assigned to a primary parton, one infers its invariant mass squared \( s_r \) with the propagator factor \( \Pi(s_r) \), and determines a secondary parton momentum by Eq. (12).

   Such inferences of variables \( p \) and \( s_r \) are more efficient than scanning them uniformly. We call such inferences importance sampling (I.S.).

4.4 Inference of parton momentum from jet

Quarks and gluons in the final parton state are observed as jets \( (j_1, \cdots, j_{N_{jet}}) \). The parton momentum can be inferred from observables of corresponding jet by using T.F. \( w(y|x_i) \). In the following, we abbreviate parton/jet suffix \( i \). Variable \( x \) can be \((E, \theta, \phi), (E_T, \eta, \phi)\) of the parton, or any other set as long as it is observable and determines \( c \) uniquely.

An efficient way of inferring \( x \), a component of \( x \), is to make a variable transformation,

\[
u(x) = \frac{1}{x_{\max} - x_{\min}} \int_{x_{\min}}^{x} w(y|\xi) d\xi, \quad (50)\]

where \( u \) is a normalized uniform random number (n.u.r.n.: \( 0 < u < 1 \)), and range \( (x_{\min}, x_{\max}) \) is defined by \( w > 0 \). Generating \( u \), one can determine \( x \).

4.5 Inference of missing transverse energy

We denote the transverse energy flow of the \( i \)-th missing parton by \( t_i(t_i \cos \phi_i, t_i \sin \phi_i) \). The total transverse energy of \( m \) missing partons is

\[
T = \sum_{i=1}^{m} t_i, \quad (51)
\]

A simple example is a case where only one neutrino is involved in the process, where \( T = \nu_T \). \( T \) is a parton variable to be inferred with the transfer function.

The choice of the transfer variable set to infer \( T \) depends on whether process (1) includes partons going to jets or not. Let \( x_T \) and \( y_T = E_T \), if no jet is involved in the process,

(i) Take \( x_T = T \) and \( y_T = E_T \), if no jet is involved in the process,
(ii) If jets are involved in the process, the fluctuation of $E_T$ is strongly correlated with that of the jet energy. In this case, take

$$x_T = T + \sum_{j=1}^{N_{jet}} c_{Tj}, \quad y_T = E_T + \sum_{j=1}^{N_{jet}} c_{Tj}$$

(52)

where $j$ is the jet number, $E_{Tj}$ and $c_{Tj}$ are the jet and corresponding parton transverse energy, respectively. Parton transverse momenta $c_{Tj}$ are independently inferred from jets.

In both cases, we assume $(x_T, y_T)$ part of the transfer function can be factored out as

$$w(y|x) \propto w(y_T|x_T),$$

(53)

with a normalization condition,

$$\int_{y_{T_{min}}}^{y_{T_{max}}} w(y_T|x_T) d^2 y_T = 1.$$

(54)

Inference of $x_T$ is made by a 2-dim n.u.r.n. as

$$u = \int_{x_{T_{min}}}^{x_{T_{max}}} w(y_T|x) dx / \int_{x_{T_{min}}}^{x_{T_{max}}} w(y_T|x) dx.$$

(55)

4.6 Inference and use of $s_r$

Inference of $s_r$ We consider a case where resonance $r$ is selected as a primary parton and $s_r$ is inferred with the normalized propagator factor, $\Pi_N(s_r) = N\Pi(s_r)$, as a p.d.f. for $s_r$, i.e.

$$\int_0^\infty \Pi_N(s_r) ds_r = N \int_0^\infty \Pi(s_r) ds_r = 1,$$

(56)

$$N = 1 / \int_0^\infty \Pi(s_r) ds_r \approx \frac{M \Gamma}{\pi}.$$  

(57)

Multiplying $\delta(s_r - (\sum_{i=1}^j c_i)^2) ds_r (= 1)$ to Eq. (7), one gets

$$\frac{d\sigma}{d\Phi^{(f)}_n} = \left[ \frac{d\sigma}{d\Phi^{(f)}_n} \right]_c \delta(s_r - (\sum_{i=1}^j c_i)^2) ds_r,$$

(58)

$$= \left[ \frac{d\sigma}{d\Phi^{(f)}_n} \right]_c \delta(u - \bar{u}) du,$$

(59)

where

$$u(s_r) = \int_0^{s_r} \Pi_N(s) ds = \int_0^{s_r} \Pi(s) ds / \int_0^\infty \Pi(s) ds,$$

(60)

$$du = \Pi_N(s_r) ds_r,$$

(61)

$$\bar{u} = u((\sum_{i=1}^j c_i)^2).$$

(62)

Equation (61) indicates $\Pi_N(s)$ is a p.d.f. for $s_r$. Thus, in the reconstruction, scanning of $s_r$ can be made efficiently by generating a n.u.r.n. $u (0 < u < 1)$, and making a variable transformation from $u$ to $s_r$ by Eq. (60).

If there are a total of $n_r$ resonances, $|M(a + b \rightarrow C)|^2$ contains $n_r$ propagator factors, and one can choose $h$ ($h \leq n_r$) resonances as primary partons.

The d.c.s. in this case is written as

$$\frac{d\sigma}{d\Phi^{(f)}_n} = \frac{d\sigma}{d\Phi^{(f)}_n} \left[ \prod_{r=1}^h \delta(s_r - (\sum_{i=1}^{j_r} c_i^p)^2) ds_r \right].$$

(63)
The values of $s_r$ is inferred with $\Pi(s_r)$ ($r = 1, \cdots, h$), and $h$ components of daughters, one for each $r$, are determined by solving simultaneous equations,

$$s_r - \left( \sum_{i=1}^{j_r} c^{(i)}_r \right)^2 = 0 \quad (r = 1, \cdots, h).$$  \hfill (64)

The value of $d\sigma/d\Phi_n^{(l)}$ is to be evaluated using momentum components thus determined.

For multiple resonances, $s_r$'s can be scanned independently by Eq. (61).

$s_r$ for observable daughters When daughters of a resonance are all observable, one can evaluate $s_r$ by Eq. (12), using $c_i$'s inferred with T.F. and assumed masses of the final partons.

An alternative way of reconstruction is to infer $s_r$ according to Eq. (61). This is more efficient than scanning daughter momenta $c_i$'s independently, because independent scanning of $c_i$'s generally results in off-resonant value of $s_r$.

An example is process $W \rightarrow qq'$. We assume that directions of 2 partons are regenerated from those of 2 jets with their T.F., and ask energies of 2 jets. In this case, one regenerates $s_W$ by Eq. (61) and the energy of one parton by Eq. (50), then the energy of the other parton is given by solving equation $s_W = (q + q')^2$, and its T.F. is used as a factor of the likelihood.

$s_r$ for missing partons We consider a process, where there are $m$ missing partons, $q_1, \cdots, q_m$, and $n_r$ intermediate partons. The degree of freedom for missing partons is $3m$, while measurement of $E_T$ gives two constraints. Thus, if $n_r \geq 3m - 2$, one regenerates $s(s_1, \cdots, s_h)(h = 3m - 2)$ using Eq. (61) and solves Eq. (64) for $q$. Then all components of $q$ are determined.

If $n_r < 3m - 2$, the degree of freedom for $q$ is

$$d = 3m - 2 - n_r > 0,$$  \hfill (65)

and $d$ components of missing partons remain undetermined.

Examples of $d=0$ case Examples of 1 and 2 missing particles are given in the following.

Example 1: Single $W \rightarrow l\nu$ production associated with/without jets. In this case, $m = 1(\nu)$, $n_r = 1(W)$, hence if we regenerate $s_W$, then $h = 1, d = 0$, and Eqs. (12) and (51) lead to a quadratic equation for $\nu$. The parton kinematics is determined within two-fold ambiguity.

Example 2: Dilepton channel in $t\bar{t}$ production,

$$t\bar{t} \rightarrow l^+l^-b\bar{b}\nu\bar{\nu}. \hfill (66)$$

For this process, $m = 2 (\nu$ and $\bar{\nu})$, and $n_r = 4(t, \bar{t}, W^+, W^-)$, hence $d = 0$, if we regenerate $s_t, s_{\bar{t}}, s_{W^+}, s_{W^-}$ by propagator factors and $T$ by the transfer function. Six constraints by Eqs. (12) and (51) lead to a bi-quadratic equation for $\nu$. The parton kinematics is determined within 4-fold ambiguity\[3\].

Undetermined variables of missing partons ($d>0$) There are cases where some components of the parton momenta are left undetermined ($d>0$): e.g. in search for SUSY particles where many missing particles are involved in the process. If a parton momentum contains such component(s), the parton is to be assigned as primary, and the component(s) are to be scanned uniformly in the phase space.

Example 3: Charged Higgs production in $t\bar{t}$ channel

$$t\bar{t} \rightarrow (bW^+)(\bar{b}H^-) \rightarrow (bl^+\nu)(\bar{b}\tau^-\bar{\nu}_\tau) \hfill (67)$$

$$\rightarrow (bl^+\nu)(\bar{b}l^-\bar{\nu}_\tau\bar{\nu}_\tau),$$  \hfill (68)

where $l = e$ or $\mu$. Here, $m = 3(\bar{\nu}_l, \nu_l, \bar{\nu}_\tau)$, and $n_r = 5 (t, \bar{t}, W^+, H^-, \tau^-)$, hence with $h = 5, d = 2$. To determine the kinematics, one regenerates $s_t, s_{\bar{t}}, s_{W^+}, s_{H^-}, s_\tau$ with the propagator factors, $T_x, T_y$ with the transfer functions, and any 2 ($=d$) components of neutrino momenta uniformly in the phase space. Then neutrino equations are reduced to the case of Example 2.

5 Likelihood of Reconstructed Paths

5.1 Likelihood for a single path and multiple paths in an event

In this subsection, formulas are for each set of $(i_p, i_t, i_\nu), \nu$ which are abbreviated.
5.1.1 Luminosity factor and the likelihood for real events

To infer a set of single path kinematics, \( P(P_1, \ldots, P_n) \), for a given event, we use in general virtual mass squared of resonances \( s_r \) and parton kinematic variables \( x \). For a set of \( P \), we define the likelihood of the path similar to \( L_1^{(0)} \) of Eq. (16). The only modification required for real data is that for event detection efficiency (acceptance). Denoting the efficiency by \( \epsilon(\varsigma) \), the luminosity factor \( l_1 \) is to be replaced with

\[
\tilde{l}_1 = \frac{2 L_{\text{int}}}{N_{\text{tot}}} = \frac{2}{\epsilon(\varsigma)\sigma_T},
\]

and by replacing \( l_1 \) with \( \tilde{l}_1 \), the expected number of events has the same form,

\[
\tilde{\mu} = \tilde{l}_1 \frac{d\sigma}{d\Phi_a^{(f)}}.
\]

The likelihood for path \( k \) in event \( i \) is given by

\[
L_1(\alpha, P|y||i, k) = L_1^{(0)}(\tilde{\mu}_{ik}),
\]

if all components of \( x \) are used to define \( P \). If \( P \) is defined with unused components of \( x', x \), the T.F. for these components is to be multiplied to likelihood \( L_1(\mu) \), namely

\[
L_1(\alpha, P|y||i, k) = L_1^{(0)}(\mu_{ik}) w(y'|x'|\alpha, i, k)
\]

5.1.2 Likelihood for multiple paths in an event

To infer the unknown true path of an event, one makes multiple path reconstructions. Here we discuss three kinds of the likelihood for the true path. The advantage of one to the others depends on the process and the purpose of analysis.

**Maximum likelihood** The M.L.E. of \( x, P \) and \( \alpha \) in an event, which we denote by \( \hat{x}, \hat{P} \) and \( \hat{\alpha} \), are obtained by (a) using general purpose minimum search programs for \(-2\ln(L_1)\) or by (2) joint likelihood for multiple paths in an event, to be discussed in the following. By the use of \( \hat{P} \) and \( \hat{x} \), one can define a likelihood for the \( i \)-th event, as a function of \( \alpha \), \( L_1(\alpha|y, \hat{P}||i) \).

**Expectation value of likelihood** The expectation value of the likelihood for \( \alpha \) as obtained by a total of \( K \) paths for the \( i \)-th event is defined by

\[
\overline{L}_1(\alpha|y||i) = \frac{1}{K} \sum_{k=1}^{K} L_1(\alpha, P|y||i, k)
\]

The expectation values of \( x, s_r, P \) and \( \alpha \), which we denote by \( \overline{x}, \overline{s}_r, \overline{P} \) and \( \overline{\alpha} \), are obtained as their means weighted by \( L_1(P, \alpha|y||i, k) \).

**Joint likelihood** The value of true value of parton kinematics \( P, P_0 \), in an event is unknown but common to all reconstructed paths in an event, namely, \( P_0 \) is identified with a parameter set. Reconstructions of \( P \) can thus be interpreted as pseudo-experiments to determine \( P_0 \), where the single path likelihood plays a role of p.d.f. for \( P \). Formally, one inserts \( \delta(P - P_0)dP (=1) \) into the likelihood, interpreting \( P \) and \( P_0 \) as variable and parameter sets, respectively. The joint likelihood for \( K \) paths,

\[
L_1^{(K)}(\alpha, P|y||i) \equiv \left[ \prod_{k=1}^{K} L_1(\alpha, P|y||i, k) \right]^{1/K},
\]

can be used to get the M.L.E. \( \hat{P} \) by the method of maximum likelihood (m.m.l.)[9]. The likelihood as a function of \( \alpha \) with \( \hat{P} \) obtained from the joint likelihood is denoted by \( L_1^{(K)}(\alpha|y, \hat{P}||i) \).
5.2 Likelihood for process, topology and solution

In the preceding subsection, the likelihood is for a given set of \((i_p, i_t, i_s)\) in an event. We consider next the use of DLM for selection of these integers.

**Integer likelihood \(\Lambda(i_p, i_t, i_s)\)** We denote the likelihood for these integers by \(\Lambda(i_p, i_t, i_s)\). The integer likelihood can be normalized as

\[
\sum_{i_1}^{N_e} \sum_{i_s}^{N_t} \sum_{i_t}^{N_e} \Lambda(i, j, k) = 1. \tag{75}
\]

Individual likelihoods for \(i_p\), \(i_t\), and \(i_s\) are given by

\[
\Lambda_p = \sum_{i_1}^{N_e} \sum_{i_s}^{N_t} \Lambda(i_p, i_t, i_s), \tag{76}
\]

\[
\Lambda_t(i_p) = \sum_{i_s}^{N_t} \Lambda(i_p, i_t, i_s), \tag{77}
\]

\[
\Lambda_s(i_p, i_t) = \Lambda(i_p, i_t, i_s). \tag{78}
\]

Likelihood \(\Lambda_p\) is used to discriminate the background against the signal, \(\Lambda_t(i_p)\) to select topology in a signal-like event, and \(\Lambda_s(i_p, i_t)\) to choose solution of Eq. (12) for a likely topology in the signal-like event.

**Evaluation of \(\Lambda(i_p, i_t, i_s)\) by DLM** The values of \(\Lambda\)'s are often provided from other information, e.g. \(b\)-tagging with vertex measurement selects certain processes and topologies. We denote \(\Lambda\)'s from the other information by \(\Lambda^{(0)}\)'s, and define the integer likelihood as a function of \(\alpha\) for the \(i\)-th event by

\[
L_1^*(i_p, i_t, i_s, \alpha|y||i) = \Lambda^{(0)}(i_p, i_t, i_s)\lambda_i(\alpha)_{p,i_t,i_s}, \tag{79}
\]

where \(\lambda_i\) is the likelihood for the multiple inferences in an event as defined in the preceding subsection.

\[
\lambda_i(\alpha)_{p,i_t,i_s} = L_1^{(s)}(\alpha|y, \hat{P} ||i, i_t, i_s, i) = L_1^{(K)}(\alpha|y, \hat{P} ||i, i_t, i_s, i). \tag{80}
\]

The values of likelihood \(L_1^*\)'s defined by Eq. (79) and (80) are functions of \(\alpha\). Thus it is appropriate to take their mean value in the search range of \(\alpha\). Denoting their mean values by \(\overline{L}_1\), \(\Lambda\)'s are given by

\[
\Lambda(i_p, i_t, i_s) = \frac{\overline{L}_1}{\sum_{i_p} \sum_{i_t} \sum_{i_s} \overline{L}_1|\alpha|}. \tag{81}
\]

where suffix \([\alpha]\) stands for the search region.

If the search region is wide, the discrimination power for \((i_p, i_t, i_s)\) is weak. Thus evaluation of \(\Lambda\)'s and squeezing the search region of \(\alpha\) are to be alternately iterated. The M.L.E. of \(\alpha\) is obtained by using all events in data, as we discuss in the next subsection. If values of \(\Lambda\)'s converge after the iterations, statistical selection of \(i_p\), \(i_t\), and \(i_s\) can be made. The whole procedure studied with Monte Carlo events can be applied to real data.

5.3 Maximum likelihood estimate of \(\alpha\) from multiple events

The determination of \(\alpha\) is to be made by \(\hat{\alpha}_{Nev}\), i.e. M.L.E. from a total of \(N_{ev}\) events in the given sample. The simplest way is to fit the distribution of \(\hat{\alpha}\) for individual events, obtained from Eq. (77), with those of Monte Carlo events with known values of \(\alpha\). The minimum \(\chi^2\) of the fit gives \(\hat{\alpha}_{Nev}\) [10].

Since events are mutually independent, the \(\alpha_{Nev}\) search can also be made with the joint likelihood of \(N_{ev}\) events. Namely, \(\alpha_{Nev}\) is \(\alpha\) that maximizes the joint likelihood,

\[
L_{Nev}(\alpha) = \prod_{i=1}^{N_{ev}} \sum_{(i_p, i_t, i_s)} L_1^*(i_p, i_t, i_s, \alpha|y||i), \tag{82}
\]

with \(L_1^*\) given by Eqs. (79) and (80). If the selection of \((i_p, i_t, i_s)\) is not uniquely made, \(\alpha_{Nev}\) determined from Eq. (82) is generally shifted from true value \(\alpha_0\) because of remaining false sets of \((i_p, i_t, i_s)\) in the sum. This deviation is to be corrected by the Monte Carlo simulation.

As we discussed in the preceding subsection, alternate iterations of the \(\hat{\alpha}_{Nev}\) search and the selection of \((i_p, i_t, i_s)\) are to be made. If the value of \(\alpha_{Nev}\) converges, it can be used to redetermine \(\hat{P}\) and \(L_1^*\) in each event.
6 Summary and Comments

The dynamical likelihood method (DLM) is formulated as a procedure to reconstruct the quantum process. For a single event reconstruction, we require 3 quantum conditions:

1. The d.c.s. is per unit phase space, $d\sigma/d\Phi^{(f)}$, 2. the transfer variable spaces are quantized by Jacobian scaled variables, 3. the likelihood is defined by the Poisson probability for 1 event. In condition (1) the final state density which plays an important role in the traditional use of the d.c.s. is missing. By the Jacobian scaled variables in condition (2), the Jacobian factor, i.e. the final state density, is absorbed in the quantized path, and the use of T.F. with ordinary quantites are justified. The state density is resumed implicitly by condition (3), since the number of event distribution, which is the outcome of the traditional form of d.c.s., is given by the likelihood of our definition. In short, one can forget the Jacobian factor in the formulation given in this paper. Only exception is the totally missing particles, the reconstruction of which should be made per unit phase space. The integration by unknown variables is not to be made in this formulation.

The luminosity factor $l_1$ is a constant depending on the event detection efficiency. This factor can be obtained from the mean value of the d.c.s (for the reconstructed parton kinematics) of individual events, the integrated luminosity and the total number of candidate events. In this formulation, the absolute value of the likelihood, i.e. the coupling constant for the process, and the dynamical parameters are simultaneously determined.

The formulation is more suitable than the earlier ones, Refs. [1]~[5], to analyse events of the collider experiments with $4\pi$ detectors.

Procedure of path reconstruction

Given a set of observables of an event, one defines the primary partons and infers a path. A path is sorted by the physics process, the parton-observable identification (topology) and the solution for the momentum components of the secondary daughter partons.

Dynamical constants and parton kinematics in a path are inferred by random number generations: (a) dynamical parameters uniformly, (b) 3-momenta of observable primary partons according to transfer functions, and/or (c) virtual masses of intermediate partons with propagator factors. If there remain undetermined momentum components of missing partons, (d) they are to be inferred uniformly in the phase volume of the partons.

Applications

Selections of the process, the topology, and the solution for momentum components of the secondary partons, which are specified by integers, are made by the likelihood values for multiple inferences in an event. The parton kinematics for each event is given by the M.L.E. or the expectation value in the event. Dynamical parameters are given by the M.L.E. from the joint likelihood of all events. Iterations with alternate evaluation of the likelihood for the integers and for the continuous variables/parameters are important.

Finally, we comment on the use of DLM for new particle searches. Most theoretical models of new particles provide forms of the d.c.s. that can be used for DLM. In addition, the mass value does not strongly depend on details of the parton dynamics, but only on its essential part, i.e. the propagator factor of the particle in search. Thus the search for theoretically unpredicted new particles by DLM is also made possible.

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