Anomalous U(1)’s in Type I and Type IIB
D=4, N=1 string vacua

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Abstract

We study the cancellation of U(1) anomalies in Type I and Type IIB D = 4, N = 1 string vacua. We first consider the case of compact toroidal \(Z_N\) Type IIB orientifolds and then proceed to the non-compact case of Type IIB D3 branes at orbifold and orientifold singularities. Unlike the case of the heterotic string we find that for each given vacuum one has generically more than one U(1) with non-vanishing triangle anomalies. There is a generalized Green-Schwarz mechanism by which these anomalies are cancelled. This involves only the Ramond-Ramond scalars coming from the twisted closed string spectrum but not those coming from the untwisted sector. Associated to the anomalous U(1)’s there are field-dependent Fayet-Illiopoulos terms whose mass scale is fixed by undetermined vev’s of the NS-NS partners of the relevant twisted RR fields. Thus, unlike what happens in heterotic vacua, the masses of the anomalous U(1)’s gauge bosons may be arbitrarily light. In the case of D3 branes at singularities, appropriate factorization of the U(1)’s constrains the Chan-Paton matrices beyond the restrictions from cancellation of non-abelian anomalies. These conditions can be translated to constraints on the T-dual Type IIB brane box configurations. We also construct a new large family of N = 1 chiral gauge field theories from D3 branes at orientifold singularities, and check its non-abelian and U(1) anomalies cancel.
1 Introduction

One of the most inspiring features of string theory is that it describes consistent quantum theories of gravity and gauge interactions. For some vacua of the theory, where gauge and/or gravitational anomalies are potentially present, the claim above may be very non-trivial already at the one-loop level. However, string theory always provides the appropriate field content and interactions to yield an anomaly-free theory. The paradigmatic example of such property is the cancellation of anomalies in ten-dimensional heterotic of type I string theory, via the Green-Schwarz mechanism [1]. The usual contributions from fermions and the metric to gauge and gravitational anomalies are cancelled by further counterterms generated by the exchange of the two-form field.

Different versions of this mechanism play a key role also in compactifications of string theory to lower dimensions. The study of its precise form and its consequences in vacua with different numbers of spacetime dimensions and supersymmetries is an interesting subject.

For phenomenological reasons, most of the interest has centered in the study of $D = 4$, $N = 1$ compactifications of the $SO(32)$ and $E_8 \times E_8$ heterotic superstrings. Perturbatively, here the pattern of anomaly cancellation is quite restricted [1]. At most one $U(1)$ gauge factor is allowed to have triangle anomalies. This presents mixed gravitational and gauge anomalies, but they are precisely on the ratios adequate to allow for their cancellation through the exchange of the model-independent pseudoscalar partner of the dilaton [2,3,4]. This is a four-dimensional version of the GS mechanism mentioned above. The pseudo-anomalous $U(1)$ finally gets a large mass (slightly lower than the string scale) due to the presence of a Fayet-Iliopoulos D-term which triggers Higgs breaking to a one-loop stable vacuum [2,3]. This beautiful mechanism is quite model independent, and has allowed to draw a number of phenomenological interesting consequences valid for generic compactifications of this type (see e.g. [3,4,5] and references therein).

For $D = 4$, $N = 1$ compactifications of type I string theory, on the other hand, there has not been an analogous study, even though the issue is of similar phenomenological interest [4]. The purpose of the present paper is to improve the understanding of anomalous $U(1)$’s in such compactifications. We will first center on type IIB toroidal orientifolds [11,12,13,14,15,16,17,18,19,20,21,22], since they are simple con-

1As we comment on in our final remarks, this is also the pattern for type I compactifications on smooth Calabi-Yau threefolds.

2Refs. [8,9,10] have recently appeared concerning anomalous $U(1)$’s in the M-theory/Type I settings but their contents have no overlap with the present work.
structions whose world-sheet formulation is well understood, and they are expected to illustrate generic properties of type I compactifications.

We will be interested in determining the new features present in anomaly cancellation in type I vacua, as compared with perturbative heterotic vacua. A main novelty in type I compactifications, as compared with heterotic ones, is the presence of D-branes. The compact models we study contain D9 branes and possibly one set of D5 branes. On general grounds, one then expects a more complicated pattern of anomaly cancellation. Indeed, as we will discuss, these models have generically several anomalous $U(1)$’s. Their triangle anomalies will be cancelled again by a four-dimensional generalized version $\text{of the Green-Schwarz mechanism}$, but it will involve the exchange of RR twisted fields. In particular, untwisted fields like the partner of the dilaton do not take part in the cancellation of anomalies. Another marked difference with respect to the heterotic case is that the Fayet-Illiopoulos parameters are controlled by twisted fields, the NS-NS partners of the RR fields mentioned above. This allows one to tune the FI to any desired scale by merely tuning the vevs of these blowup modes. The $U(1)$’s are spontaneously broken, but their masses are not tied up to the string scale, and can be very light.

Thus, type I compactifications allow to understand the physics of cancellation of anomalies in the presence of D branes. This is an interesting point, since type I models with D5 branes are dual to heterotic compactifications with NS fivebranes. These last vacua are highly non-perturbative and the analysis of the dual type I can provide some insight into their properties. However, not much is known about their explicit construction, especially in four dimensions, and so this discussion is beyond the scope of the present paper.

Another application of the understanding of type I vacua with D-branes is the study of anomaly cancellation in the world-volume of decoupled D-branes. The interest is that, in the decoupling limit where gravity and other bulk modes are switched off, the theory reduces to a supersymmetric field theory. Thus string theory constructions can be used to learn about quantum field theory. For instance, in the context of $D = 6$, $N = 1$ field theories, the study of type IIB and type I D5 branes at singularities have provided a construction of large families of interacting superconformal field theories. A non-trivial check of the consistency of this construction is that the underlying string theory ensures the cancellation of anomalies in the world-volume field theory. In this respect, a key role is played by the cancellation of $U(1)$ anomalies through a six-dimensional version of the GS mechanism first uncovered in the study of

\footnote{An analogous mechanism for $D = 6$ theories was first considered in ref.\textsuperscript{23}.}
compact $D = 6$ type I models \cite{23,27,28}.

Recently, some configurations of branes in string theory have been proposed for the study of four-dimensional gauge theories. In the most interesting case of $N = 1$ super-symmetry, the construction of large families of chiral theories has been accomplished by the so-called brane box models, consisting of grids of two kinds of NS fivebranes in type IIB string theory, on which D5 branes are suspended \cite{29}. By applying T-duality to some of these models \cite{30}, these field theories are realized in the world-volume of D3 branes on a threefold singularity \cite{31,32}. This construction has become very popular, since, by use of the AdS/CFT correspondence \cite{33}, it allows a very simple construction of $N = 1$ superconformal field theories (in the limit of large number of D3 branes) \cite{34,32,35}.

These configurations of D3 branes at singularities are four-dimensional analogs of the six-dimensional theories mentioned above. So they can also be analyzed by perturbative type I string theory, and in particular it is a natural question to address how their cancellation of $U(1)$ anomalies occurs. We will fist analyze the system of D3 branes on top of orbifold singularities, and show explicitly how the string consistency conditions ensure the cancellation of $U(1)$ anomalies via the GS mechanism uncovered in compact models. Moreover, the generation of FI terms gives masses to these $U(1)$’s and explains why they are not present in the low energy dynamics. This ‘freezing’ of the $U(1)$’s was expected from the brane box point of view, in analogy with the result in \cite{36} for $N = 2$ theories, but had not been shown explicitly. Here we show it in the T-dual version.

Finally, we consider a family of new $N = 1$ chiral gauge field theories obtained from D3 branes on top of threefold orientifold singularities. We show that for each singularity of a certain type one can build a theory that becomes superconformal in the large $N$ limit. The stringy construction of this large family of models ensures their consistency, and we illustrate this by checking explicitly the cancellation of non-abelian and $U(1)$ anomalies.

The paper is organized as follows. In Section 2 we introduce the general idea for the GS mechanism in type I and type IIB $D = 4$, $N = 1$ vacua. In Section 3 we center on the case of compact toroidal orientifolds, where we show in detail the cancellation of gravitational, mixed gauge and cubic $U(1)$ anomalies in several non-trivial examples. In Section 4 we describe the construction of the field theories on the worldvolume of D3 branes on top of orbifold and orientifold singularities. We give general expressions for the triangle anomalies, and show explicitly how the consistency conditions for the underlying string theory ensure their cancellation via the GS mechanism. In Section 5
we comment on the generation of FI terms, and compare the situation with that in (perturbative) heterotic vacua. In Section 6 we make some final remarks. Details on the construction of the theory of D3 branes at orientifold singularities are left for the appendix.

2 Anomalous $U(1)$’s in Type I and Type IIB $D = 4$, $N = 1$ vacua

In what follows we are going to discuss four-dimensional vacua with $N = 1$ based on orientifolds [37, 38, 39] and/or orbifolds [40] of Type IIB theory. A first class of theories will consist of compact models from toroidal IIB orientifolds. Due to compactness the structure, type and number of D-branes present in the vacuum will be strongly constrained by tadpole cancellation conditions. We will concentrate here in models which contain nine-branes and at most one set of 32 fivebranes. The second class of theories we will be interested on will be Type IIB brane-box models in which appropriate configurations of D-fivebranes and NS-fivebranes are situated in such a way that one has an effective $D = 4$, $N = 1$ chiral theory on the worldvolume of the D-branes. These are non-compact theories which can also be studied by going to their T-dual which consists on sets of D3-branes sitting on $Z_N$ type singularities. Being non-compact, tadpole cancellation conditions are much milder for these theories and, e.g., the overall number of D-branes is undetermined.

In any of the above class of theories one obtains a general gauge group of the form

$$\prod_{\alpha} \left( \prod_{i=1}^{n_{\alpha}} U(1)_i \times \prod_{j=1}^{m_{\alpha}} G_j \right)$$  \hspace{1cm} (2.1)

where $G_j$ are general non-Abelian groups. Here $\alpha$ runs over the different sets of D-branes present in each model and $n_{\alpha}(m_{\alpha})$ are the number of $U(1)$’s (non-Abelian groups) present in the D-brane sector $\alpha$. Thus, for example, $\alpha$ runs over one set of 9-branes and one set of 5-branes for the compact orientifold models discussed bellow. In general we will have one $U(1)$ factor for each $SU(n)$ factor in the theory.

We are interested in the cancellation of $U(1)$ anomalies in these classes of theories. There will be mixed $U(1)_i \times G_j^2$ anomalies as well as cubic $U(1)_i \times U(1)_j^2$ anomalies. In addition, in the case of compact orientifolds we will have to care also about mixed $U(1)$-gravitational anomalies. The relevant graphs contributing to these anomalies are depicted in Figure 1, for the particular case of mixed $U(1)$ non-abelian anomalies.

The first graph corresponds to the annulus contribution and the second to the Moebius strip (the latter is only present in an orientifold setting). These two contributions
correspond to the usual effective low energy field theory triangle diagram calculation and we will not discuss it here any further. It is the third graph in Fig.1 which will be in charge of the cancellation of the $U(1)$ anomalies left over by the naive triangle graph calculation. Notice that this third graph is only relevant for anomalies involving at least one $U(1)$. Fig.2 shows the same graph but in the closed string exchange channel. It shows the field theory meaning of the cancellation mechanism. A $U(1)$ couples to some (Ramond-Ramond) closed string state which propagates and finally couples to a pair of gauge bosons (or gravitons).

The coupling of the $U(1)$ to a RR field described by an antisymmetric field $B^\mu_\nu_k$ gives rise to an effective term in the low energy action proportional to

$$Tr(\gamma_k\lambda_i)B_k \wedge F_{U(1)i}$$

(2.2)

where $\lambda_i$ is the CP matrix associated to the $U(1)_i$ generator and $\gamma_k$ is the matrix associated to the $\theta^k$ twist. Notice that for RR fields belonging to the untwisted sector this term is proportional to $Tr \lambda_i$ which vanishes for traceless CP generators, which is the case of the compact orientifolds discussed below. The coupling to the right hand side of Fig. 2 gives rise to effective low energy couplings proportional to

$$Tr(\gamma_k^{-1}\lambda_G^2)(\partial^\mu B_{k}^{\nu\rho})W_{\mu\nu\rho}^{CS}$$

(2.3)

where $\lambda_G$ is the CP matrix associated to the rightmost gauge bosons. $W_{\mu\nu\rho}^{CS}$ is the
Chern-Simons (CS) tensor of those gauge fields. In the case of mixed gravitational anomalies one replaces the gauge CS tensor by the Lorentz one and $\lambda_G$ by the unit matrix. Notice that sectors $k$ with $Tr \gamma_k = 0$ will thus not contribute to the cancellation of mixed gravitational anomalies.

The full low energy amplitude contributing to the mixed $U(1)_i \times G_j^2$ anomaly coming from graph 2 will be proportional to

$$A_{ij}^{\alpha\beta} = \frac{1}{|P|} \sum_k A_{ij}^{\alpha\beta}(k) = \frac{i}{|P|} \sum_{k \in \text{sectors}} C_k^{\alpha\beta} Tr(\gamma^\alpha_{i\theta} \lambda^\alpha_i) Tr((\gamma^\beta_{i\theta})^{-1}(\lambda^\beta_j)^2)$$  \hspace{1cm} (2.4)

where $|P|$ is the order of the orientifold or orbifold group and the sum runs over the different twisted sectors. The indices $\alpha(\beta)$ indicate the brane sector from which the $U(1)_i(G_j)$ are coming from. The coefficients $C_k$ depend on the particular twist $k$. For $Z_N$ they are given by

$$C_k^{\alpha\beta} = \prod_{a=1}^{3} 2 \sin \pi k v_a$$  \hspace{1cm} (2.5)

where the product extends only over complex planes $a$ with $NN$ or $DD$ boundary conditions. Here $v = (v_1, v_2, v_3)$ is the compact space twist vector, i.e., the $Z_N$ twist is generated by rotations by $\exp(2i\pi v_a)$ in the three $a = 1, 2, 3$ compact complex dimensions. The extension to $Z_N \times Z_M$ twists is straightforward. The existence of these $C_k$ factors is crucial in obtaining anomaly cancellation. Their existence can be inferred from the cylinder tadpole amplitudes shown in the appendix of ref.\cite{18}. We will also show later how they appear naturally in the context of D-branes sitting at orbifold/orientifold $Z_N$ singularities coming from a discrete Fourier transform.

The $U(1)$'s which turn out to be anomalous may be written as linear combinations of the form:

$$Q_k(\beta, p) = \sum_i \sum_\alpha A_{i\beta}^\alpha(k) Q_i^\alpha$$  \hspace{1cm} (2.6)

For each $k$ one gets a number of linear combinations labeled by $\beta$ and $p$. Only a subset of them are in general linearly independent. The number of anomalous $U(1)$'s in a given model will depend on the number of linearly independent $Q_k(\beta, p)$ generators one finds. Examples will be given below.

3 Anomalous $U(1)$s in compact Type IIB $D = 4$, $N = 1$ orientifolds

As a first application of the above discussion we are going to study cancellation of $U(1)$ anomalies in a class of $D = 4$, $N = 1$ Type IIB toroidal orientifolds (for details
about these models and notation see \[18\]). We will consider models obtained by compactifying Type IIB theory on orbifolds \(T^6/P\), \(P\) being either \(Z_N\) or \(Z_N \times Z_M\). In this compact case these discrete symmetries are restricted to act crystallographically on \(T^6\) which substantially reduces the possibilities. Those were classified in \[10, 11\]. In addition we are going to twist the theory by the world-sheet parity operator \(\Omega\). As a result one gets \(D = 4\) theories with \(N = 1\) unbroken supersymmetry. Orientifolds with only odd order \(Z_N\) twists have only 9-branes. Those with an order-two twist (acting e.g. along the first two \(a = 1, 2\) complex coordinates) will have in addition one set of 5-branes with their worldvolume filling the four non-compact dimensions plus the third complex plane. For simplicity we will restrict ourselves to the case of orientifolds with only one twisted sector of order two. These will have only one set of 5-branes.

Unlike the non-compact case the total number of each type of D-brane in the vacuum is limited due to tadpole cancellation conditions. The cases with maximal symmetry will admit a maximum of 32 9-branes and 32 5-branes. In their worldvolumes will live gauge fields with associated CP matrices \(\lambda^\alpha, \alpha = 9, 5\). Those will be \(32 \times 32\) hermitian matrices. As explained in \[18\] in this class of orientifolds with an \(\Omega\) action it is useful to use a Weyl-Cartan realization of the CP algebra. The 9-brane and 5-brane CP matrices in this case are restricted to be \(SO(32)\) generators. They can be organized into charged generators \(\lambda_a = E_a, a = 1, \cdots, 480\), and Cartan generators \(\lambda_I = H_I, I = 1, \cdots, 16\). The twist matrices \(\gamma^\alpha_k\) and its powers represent the action of the e.g. \(Z_N\) group on Chan Paton factors, and they correspond to elements of a discrete subgroup of the Abelian group spanned by the Cartan generators. Hence, we can write

\[
\gamma^\alpha_k \equiv \gamma^\alpha_0 = e^{-2\pi k V_\alpha H}
\]

This equation defines the 16-dimensional ‘shift’ vector \(V_\alpha\). One can see that \(\gamma^\alpha_1\) can be chosen diagonal and furthermore \((\gamma^\alpha_1)^N = \pm 1\). Cartan generators \(H_I\), are represented by tensor products of \(2 \times 2\) \(\sigma_3\) submatrices. We chose the normalization of the \(SO(32)\) generators \(\lambda\) in such a way that \(Tr \lambda^2 = 1\). The (unnormalized) generator \(\lambda_i\) of a given \(U(1)_i\) will be given by a linear combination

\[
\lambda_i = \sum_{\alpha=9,5} Q^\alpha_i \cdot H^\alpha
\]
have typically the form \(Q^\alpha_i = (0, 0, \ldots, 1, \ldots, 0, 0, \ldots, 0)\) where the \(n\) one-entries sit at the positions where a corresponding \(SU(n-1)\) lives. With these conventions and normalizations the relevant traces for anomaly cancellation are:

\[
\text{Tr} (\gamma^\alpha_k \lambda^\alpha_i) = \text{Tr} (e^{-2i\pi V^\alpha \cdot H} Q^\alpha_i \cdot H^\alpha) = (-i) 2n_i \sin 2\pi V^\alpha_i
\]

(3.3)

where \(n_i\) is the rank of the \(U(n)\) group containing this \(U(1)\) and \(V^\alpha_i\) is the component of the \(V^\alpha\) vector along any of the entries overlapping with that \(U(n)\). Notice that the \(n_i\) factor appears because we have not normalized the \(U(1)\). In the same way one obtains:

\[
\text{Tr} ((\gamma^\beta_k)^{-1} (\lambda^\beta_j)^2) = \text{Tr} (e^{2i\pi V^\beta \cdot H} (\lambda^\beta_j)^2) = \cos 2\pi V^\beta_j
\]

(3.4)

where again \(V^\alpha_j\) is the component of the shift vector along any entry overlapping with the group \(G_j\). Thus for the present class of compact orientifolds with \(|P| = 2N\) one gets a total contribution to the mixed anomalies from the graph in fig. 2

\[
A_{ij}^{\alpha \beta} = \frac{1}{N} \sum_{k \in \text{sectors}} C_k^{\alpha \beta} (v) n_i \sin 2\pi V^\alpha_i n_j \cos 2\pi V^\beta_j
\]

(3.5)

where \(k\) runs over twisted \(Z_N\) sectors, \(\alpha, \beta\) run over 5,9 (meaning 5- or 9-brane origin of the gauge boson). The coefficients are given by

\[
C_k^{\alpha \beta} (V) = \prod_{a=1}^3 2 \sin \pi k v_a \quad ; \quad \alpha = \beta
\]

\[
= 2 \sin \pi k v_3 \quad ; \quad \alpha \neq \beta
\]

(3.6)

Notice that the case \(\alpha \neq \beta\) corresponds to mixed anomalies mixing gauge groups coming from different brane systems (9-branes and 5-branes).

In the case of cubic \(U(1)\) anomalies a similar formula is obtained. The only difference is that, since we have not normalized the \(U(1)_j\) generators there will be an extra factor \((2n_j)\) in this expression. Care must be taken also to recall that there is an extra symmetry factor \(1/3\) when computing the triangle graphs for \(U(1)^3\) compared to those for \(U(1)_i \times U(1)_j^2, i \neq j\). Altogether one has an expression for the mixed \(U(1)_i \times U(1)_j^2\) anomalies:

\[
A_{ij}^{\alpha \beta} = \frac{2}{N} \sum_{k \in \text{sectors}} C_k^{\alpha \beta} (v) n_i \sin 2\pi V^\alpha_i n_j \cos 2\pi V^\beta_j
\]

(3.7)

In the case of mixed \(U(1)\) gravitational anomalies one has

\[
A_i^\alpha = \frac{3}{4N} \sum_{\beta} \sum_{k \in \text{sectors}} C_k^{\alpha \beta} (v) n_i \sin 2\pi V^\alpha_i \text{Tr} ((\gamma_k^\beta)^{-1})
\]

(3.8)

where \(\frac{3}{4} = \frac{24}{32}\) is a normalization factor. From the above expressions an interesting sum rule relating the mixed \(U(1)\)-gauge anomalies and the \(U(1)\)-gravitational anomalies can be obtained:

\[
A_i^\alpha = \frac{3}{2} \sum_j \sum_{\beta} n_j A_{ij}^{\alpha \beta}
\]

(3.9)
where \( n_j \) is the rank of the \( j^{th} U(n) \) or \( SO(m) \) gauge group in the theory and the \( j \)-sum goes over all of them. This formula is interesting because it is model independent, i.e., one only needs to know the massless spectrum of the theory and the \( U(1) \) charges in order to compute it, without any reference to the specific form of Chan-Paton matrices nor twist structure.

Notice the following points in these expressions:

i) Untwisted RR fields, like the partners of the dilaton \( \Re S \) and the partners of the untwisted moduli \( \Re T_b \) do not participate in anomaly cancellation since in this case \( \gamma_k = 1 \) and \( \Tr \lambda_i = 0 \).

ii) Due to the \( C_k \) coefficients, RR twisted fields associated to twists leaving one torus fixed do not contribute to anomaly cancellation for \( \alpha = \beta \). On the other hand they do in general contribute to cancellation of anomalies mixing different types of branes \((\alpha \neq \beta)\).

iii) Twisted sectors with \( \Tr \gamma_k = 0 \) do not contribute to the cancellation of the mixed \( U(1) \)-gravitational anomalies.

Up to now we have considered the most symmetric situation in which Wilson lines are absent and all 5-branes sit at the fixed point at the origin. Something analogous happens in more general cases. Consider for example the case of an orientifold with a quantized Wilson line background. As explained in [18], now the different fixed points of the orbifold will not be all equivalent since different fixed points will have associated different \( \gamma_k^{(9)} \) matrices. In particular the orbifold action is generated by the space group which involves elements \((\theta, 1)\), with \( \theta \) representing \( Z_N \) rotations, and elements \((1, e_m)\), with \( e_m \in \Lambda, m = 1, \ldots, 6 \), where \( T^6 = R^6/\Lambda \). The element \((\theta^k, 1)\) is embedded in the open string sector through unitary matrices \( \gamma_k^{(9)} \). In addition there can be background Wilson lines which correspond to embeddings of the elements \((1, e_m)\) through matrices \( W_m \) into the 9-brane sector. To a fixed point \( f \) of \( \theta^k \) there corresponds an element \((\theta^k, c_me_m)\) such that \((1 - \theta^k)f = c_me_m\), for some integers \( c_m \). The 9-brane monodromy associated to this fixed point \( f \) will thus be \( \gamma^f_k = (\prod_m W^{c_m}_m)^{\gamma^{(9)}_k} \). Thus different fixed points \( f \) will have different twist matrices acting on the CP factors. Thus in this case we will have

\[
A^{(99)}_{ij} = \frac{1}{N} \frac{1}{N_f} \sum_{f} \sum_{k \in \text{sectors}} C^{(99)}_{k}(v) \; n_i \sin 2\pi k V^f_i \cos 2\pi k V^f_j \quad (3.10)
\]

where the additional sum goes over the different fixed points which feel different monodromy \( \gamma^f_k = \exp(-i2\pi k V^f \cdot H) \). An example is provided below.
3.1 Examples

Let us see how the $U(1)$ anomaly cancellation proceeds in some specific orientifold examples.

i) The $Z_3$ $D = 4$, $N = 1$ orientifold

This is perhaps the simplest compact orientifold in four dimensions \[12, 13\]. It is obtained by modding Type IIB theory on a torus $T^6$ by the standard $Z_3$ action with $v = \frac{1}{3}(1,1,-2)$. In this case there are only 9-branes and tadpole cancellation conditions require $\text{Tr} \gamma_\theta = \text{Tr} \gamma_1 = -4$. The unique solution (up to irrelevant phases) is $\gamma_1 = \exp(-2i\pi V \cdot H)$ with a shift \[18\]

$$V = \frac{1}{3}(1,1,1,1,1,1,1,1,1,1,0,0,0,0)$$  \hspace{1cm} (3.11)

The gauge group is $U(12) \times SO(8)$ and the charged chiral matter fields transform as $3(12,8,v)_1 + 3(\overline{66},1)_{-2}$, where the subindex shows the $U(1)$ charges. With the standard normalization for generators of non-Abelian groups one finds mixed $U(1)$ triangle anomalies with respect to $SU(12), SO(8)$ and the $U(1)$ equal to $-18, 36$ and $-432$ respectively. Let us now compute the contribution of eq. (3.5) to these anomalies. Now we have only one constant $C_1^{\alpha\beta} = -C_2^{\alpha\beta} = -3\sqrt{3}$. On the other hand $V_i = 1/3$, $V_{SU(12)} = 1/3$ and $V_{SO(8)} = 0$. Thus altogether we get

$$A_{U(1)}(SU(12), SO(8), U(1)) = \frac{1}{3}2(-3\sqrt{3}) (12 \frac{\sqrt{3}}{2}) \times (-\frac{1}{2}, 1, 24(-\frac{1}{2})) = (18, -36, 432)$$  \hspace{1cm} (3.12)

where the factor 2 comes from the sum over $k = 1, 2$. This contribution exactly cancels that from the triangle graphs. Concerning the $U(1)$-gravitational anomaly the triangle graph gives a contribution proportional to $(-108)$ whereas eq. (3.8) yields

$$A_{grav} = \frac{3}{4}2(-3\sqrt{3})(6\sqrt{3})(-4) = 108$$  \hspace{1cm} (3.13)

as should be.

ii) The $Z_3$ orientifold with Wilson lines

The above example was quite simple since it has only one anomalous $U(1)$. Things get a little bit more involved when there are several $U(1)$s. Let us consider an orientifold obtained from the previous one by the addition of a discrete Wilson line associated to a shift $W = 1/3(0,0,1,1,2,2,0,0,1,1,2,2,1,1,1,1)$ e.g., around the first complex dimension \[18, 42\]. Then the 27 fixed points of the orbifold are split into three sets of nine each which have associated CP twists respectively:

$$V = \frac{1}{3}(1,1,1,1,1,1,1,1,1,1,0,0,0,0)$$
Let us now compute the contribution coming from eq. (3.5) respectively with the indices $U$ and $V$.

A similar fashion. Concerning the mixed gravitational anomalies, triangle graphs yield

$$3(1, 1, 0, 0, 2, 2, 1, 1, 0, 0, 2, 2, 2, 2, 2)$$

(3.14)

The gauge symmetry is $U(4)^4$ and the charged particle spectrum consists of chiral fields $3(1, 4, 1, 0)_{(0, -1, -1, 0)} + 3(1, 4, 1, 4)_{(0, 1, 0, -1)} + 3(1, 1, 4, 4)_{(0, 0, 1, 1)} + 3(6, 1, 1, 1)_{(-2, 0, 0, 0)}$. Here the four subindices correspond to the $U(1)$ charges. We label the $U(1)$'s and the $SU(4)$'s respectively with the indices $i = 1, \ldots, 4$ and $j = 1, \ldots, 4$. Computation of the $U(1)_i \times SU(4)^2_j$ mixed anomalies yields the matrix

$$A^0_{ij} = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & -6 \\ 0 & -6 & 0 & 6 \\ 0 & 6 & 6 & 0 \end{pmatrix}$$

Let us now compute the contribution coming from eq. (3.5)

$$A_{ij} = \frac{1}{3} \frac{1}{3} \frac{2}{3} (-3\sqrt{3}) \left\{ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & -2 & 0 \end{pmatrix} + (-\sqrt{3}) \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -2 & 2 & 0 & -2 \\ 1 & -1 & 0 & 1 \end{pmatrix} + \right. $$

$$+ \left. (-\sqrt{3}) \begin{pmatrix} 1 & 0 & -1 & -1 \\ -2 & 0 & 2 & 2 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \right) = -A^0_{ij}$$

(3.15)

so that mixed anomalies are exactly cancelled. In the present case three out of the four $U(1)$s are anomalous. It is easy to check that cubic anomalies do also cancel in a similar fashion. Concerning the mixed gravitational anomalies, triangle graphs yield $(-36, 0, 0, 0)$. The sum rule (3.9) leads to $\frac{3}{2} 4(6, 0, 0, 0)$ which exactly cancels that contribution.

**iii) An example with both 9-branes and 5-branes: the $Z_6$ orientifold**

In this example [16, 18] the twist $\theta$ is generated by $v = \frac{1}{6}(1, -3, 2)$. There is an order-two twist corresponding to $3v$ and tadpole cancellation conditions require the presence of 32 9-branes and 32 5-branes. Here we will consider a configuration with all 5-branes sitting at the fixed point at the origin. The tadpole cancellation conditions may be found in [16, 18]. The corresponding twist on CP matrices is generated by $\gamma_i^\alpha = \exp(-i2\pi V^\alpha \cdot H)$ with [18]

$$V^9 = V^5 = \frac{1}{12}(1, 1, 1, 1, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3).$$

(3.16)
The gauge group is \((U(4) \times U(4) \times U(8))^2\) and the spectrum may be found in table 3 of \([18]\). This model has three \(U(1)^9\), \(i = 1, 2, 3\), from the 9-brane sector and three \(U(1)^5\), \(i = 1, 2, 3\), from the 5-brane sector. Their mixed anomalies with respect to the six non-Abelian groups are found to be:

\[
A^{\alpha\beta}_{ij} = \begin{pmatrix}
2 & 2 & 8 & -2 & 0 & -4 \\
-2 & -2 & -8 & 0 & 2 & 4 \\
0 & 0 & 0 & 2 & -2 & 0 \\
-2 & 0 & -4 & 2 & 2 & 8 \\
0 & 2 & 4 & -2 & -2 & -8 \\
2 & -2 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

were the two \(3 \times 3\) sub-matrices in the diagonal correspond to anomalies not mixing fields from 9-branes to those from 5-branes. The off-diagonal boxes correspond to the contribution from particles being charged under both 9-brane and 5-brane gauge groups. The columns label the \(U(1)\)'s whereas the rows label the \(SU(N)\)’s. Let us now see how these anomalies are cancelled by the exchange of RR-fields. The \(C^{\alpha\beta}\) factors play now an important role. For the present case one finds:

\[
C^{99}_1 = C^{55}_1 = -C^{99}_5 = -C^{55}_5 = -2\sqrt{3} \\
C^{99}_2 = C^{55}_2 = C^{99}_4 = C^{55}_4 = 0 \\
C^{95}_3 = C^{55}_3 = C^{95}_3 = 0 \\
C^{95}_1 = C^{95}_2 = -C^{95}_4 = -C^{95}_5 = \sqrt{3}.
\]

(3.17)

From this we conclude that twisted sector \(\theta^3\) does not contribute to the cancellation. Twisted fields from sector \(k\) and \(N-k\) yield the same contribution thus altogether one gets twice the sum of two contributions from \(k = 1, 2\) sectors:

\[
-\frac{2}{6}(\sqrt{3})^2 \begin{pmatrix}
2 & 2 & 8 & -1 & -1 & -4 \\
-2 & -2 & -8 & 1 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -4 & 2 & 2 & 8 \\
1 & 1 & 4 & -2 & -2 & -8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 2 & -2 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(3.18)

which exactly cancels the triangle anomalies. Notice that the \(\theta^2(\theta^4)\) sector only contributes to the cancelation of mixed anomalies of 9(5)-brane \(U(1)\)'s with 5(9)-brane gauge groups. One can check that out of the six \(U(1)\)'s one can form two linear combinations which are anomaly free. Thus this orientifold has four anomalous \(U(1)\)s.
Equation (2.6) allows us to obtain which four linear combinations are anomalous:

\[ 2Q^0_1 + 2Q^0_2 + 8Q^0_3 - Q^5_1 - Q^5_2 - 4Q^5_3 \]
\[ -Q^0_1 - Q^0_2 - 4Q^0_3 + 2Q^5_1 + 2Q^5_2 + 8Q^5_3 \]
\[ -Q^5_1 + Q^5_2 ; \quad -Q^9_1 + Q^9_2 \]

The first two linear combinations couple to twisted RR fields with \( k = 1, 5 \) whereas the other two couple to RR fields with \( k = 2, 4 \). Notice that these linear combinations may be directly obtained from the linearly independent rows of the matrices in eq.(3.18).

The mixed \( U(1)_i \) gravitational anomalies are proportional to \( (12, -12, 0, 12, -12, 0) \). Using the sum rule (3.9) it is easy to check that they are also cancelled. The same is true for the cubic anomalies.

\[ \text{iv) Further compact orientifold examples} \]

The \( U(1) \) anomalies of the compact orientifolds discussed in refs.[18] all cancel in an analogous way. The number of anomalous \( U(1) \)'s in each model may be found by computing how many linearly independent generators \( Q_k(\beta, p) \) in eq.(2.6) the given orientifold has. In the case of odd order \( Z_N \) orientifolds the number of anomalous \( U(1) \)'s is easy to guess. There is one anomalous \( U(1) \) per twisted sector \( k \) (and its conjugate) leaving no fixed tori, yielding a total of \( (N-1)/2 \). Thus for \( Z_3 \) and \( Z_7 \) we have one and three anomalous \( U(1) \)'s, respectively. For \( Z_3 \times Z_3 \) there is only one anomalous \( U(1) \) since there is only one twisted sector leaving no fixed tori. In the presence of Wilson lines there are additional anomalous \( U(1) \)'s. Thus adding one Wilson line in an odd \( Z_N \) orientifold yields \( N \) more \( U(1) \)'s, with \( (N-1) \) of them anomalous and the other decoupled from the matter fields.

In the case of even \( Z_N \) orientifolds finding out the number of anomalous \( U(1) \)'s depends on the detailed structure of each twisted sector. We find three for \( Z_6 \), four for \( Z_6' \) and five for \( Z_{12} \). The total number of \( U(1) \)'s for these three cases is six, six and twelve respectively. The other \( Z_N \) cases which act crystallographically were shown to have tadpoles (end hence to be inconsistent) in ref.[18].

\[ \text{4 Brane-box models and D3-branes at } Z_N \text{ orbifold and orientifold singularities} \]

An interesting application of the formalism developed above is the study of \( U(1)'s \) in the field theory of D3 branes sitting at orbifold and orientifold singularities \( \mathbb{C}^3/\Gamma \). There are several motivations to study such system. First, they provide a simple construction
of $D = 4$, $N = 1$ finite field theories \[42, 34, 32, 35, 43\]. This has acquired certain relevance by the recent developments of the AdS/CFT correspondence \[33\], as has been analyzed in a number of papers (following the approach of \[34\]). Second, the configuration of D3 branes at abelian orbifold singularities is T-dual of the brane box models introduced in \[29\], as shown in \[30\]. These brane configurations have been introduced as a tool to construct large families of chiral $D = 4$, $N = 1$ gauge theories (other related configurations have been proposed in \[44\]). It is expected that a better understanding of these brane constructions will provide insight into the quantum effects in four-dimensional chiral gauge theories. Finally, it is useful to analyze the case of non-compact orbifolds/orientifolds (as opposed to the compact toroidal orientifolds studied above) since they are much less restricted. For instance, the orbifold group is not required to act crystallographically. Also, some tadpole cancellation conditions need not be imposed, due to the non-compactness of the space. Thus, one can study large families of examples, and the general formulae for the cancellation of their $U(1)$ anomalies illustrate very clearly the appearance and importance of each term in the factorization expression (2.4).

We start studying the case of D3 branes at orbifold singularities, and then turn to a family of new $D = 4$ $N = 1$ gauge theories from D3 branes at orientifold singularities.

### 4.1 D3 branes at orbifold singularities

In the following we will center on the theory of D3 branes sitting at $\mathbb{C}^3/Z_N$ orbifold singularities (the case of $\mathbb{C}^3/(Z_N \times Z_M)$ can be analyzed in complete analogy). These theories have been studied in \[31, 32\]. Let the generator $\theta$ of $Z_N$ act on $\mathbb{C}^3$ through the twist vector $v = (\ell_1, \ell_2, \ell_3)/N$, with $\ell_1 + \ell_2 + \ell_3 = 0$. Let us also define the action of $\theta$ on the Chan-Paton factors of the D3 branes to be given by a diagonal matrix $\gamma_\theta$ which has $n_j$ eigenvalues equal to $\exp(2i\pi j/N)$, with $j = 1, \ldots, N$.

The gauge group of the resulting $D = 4$, $N = 1$ field theory on the D3 brane world-volume is

\[\prod_{j=1}^{N} U(n_j)\] (4.1)

and there are chiral multiplets transforming as

\[
\bigoplus_{a=1}^{3} \bigoplus_{j=1}^{N} \bigoplus_{\nu} (\mathbf{1}, \nu, j + \ell_a) \bigoplus (\mathbf{3} \bigoplus_{j=1}^{N} (\mathbf{1}, \nu, j + \ell_3)) \bigoplus (\mathbf{3} \bigoplus_{j=1}^{N} (\mathbf{1}, \nu, j + \ell_2)) \bigoplus (\mathbf{3} \bigoplus_{j=1}^{N} (\mathbf{1}, \nu, j + \ell_1)) \bigoplus (\mathbf{3} \bigoplus_{j=1}^{N} (\mathbf{1}, \nu, j + \ell_0))
\] (4.2)

---

\[\text{In the case of orientifold singularities, the field theories are finite only in the limit of large number of D3 branes.}\]
where the subindices denote the group under which the field transforms. The $U(1)$ charges are normalized such that the fundamental representation $\mathbf{j}$ has charge +1 under $U(1)_j$.

As discussed in [30], these orbifold models are T-dual to some Type IIB configurations of NS, NS' and D5 branes (brane box configurations) introduced in [29]. It is a simple matter to obtain a brane box model yielding the above spectrum $\mathbf{j}$. First one constructs an infinite array of boxes using the NS and NS' branes. Next, one puts labels (ranging from 1 to $N$, and defined modulo $N$) in the boxes, in such a way that when one moves horizontally one box to the left the label shifts by a quantity $\ell_1$, and when one moves vertically one box upwards the label shifts by $\ell_2$ (this already ensures that when one moves diagonally one box from upper right to lower left, the label shifts by $\ell_3$). The number of D5 branes in the box with the $i^{th}$ label is set to be $n_i$. This construction ensures that, when one applies the rules derived in [29], the gauge group and matter content of the field theory obtained are as in (4.1), (4.2).

Boxes with identical labels should be identified, so the array must have a certain periodic structure. This means the corresponding two dimensional plane is compactified to a two-torus. These are the elliptic models first introduced in [45].

One of the open questions in the construction of brane box models is to determine the restrictions on possible sets $\{n_{i,j}\}$ of numbers of D5 branes in each box. From the field theory point of view, there are constraints from cancelation of non-abelian anomalies. From the viewpoint of consistency of the string theory construction, they are expected to follow from conservation of RR charge (as e.g. in the brane construction of chiral six-dimensional theories [46]). However, the brane box configurations are rather complicated, and all attempts to derive these restrictions from string theory have failed.

The situation in the T-dual picture is better suited for a string theory analysis. Here all NS fivebranes have been transformed into an orbifold singularity, where a set of D3 branes sit. The configuration can be studied in string perturbation theory, which allows to see the one-loop effects responsible for the anomalies. This analysis has been performed in [47], where it was shown that cancellation of certain tadpoles implied the cancellation of non-abelian anomalies. Specifically, the twisted tadpoles required to vanish correspond to twists whose only fixed point is the origin. For twists with a

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5Other brane box models may also give the same spectrum. All box models with the same field theory are related by permutations in the correspondence between the three complex coordinates in the singularity picture and the three kinds of arrows in the brane box construction. For details, see [30].
two real dimensional set of fixed points, there are non-compact dimensions along which the corresponding RR flux can escape to infinity, and tadpoles are related to physical quantities in the field theory, namely the one-loop beta functions of the gauge factors.

The outcome of this analysis is that the string consistency conditions, in the form of cancellation of mentioned tadpoles, amounts to the rule

\[ n_i = 0 \text{ or } A_i \equiv \sum_{a=1}^{3} (n_{i+a} - n_{i-a}) = 0. \]  

(4.3)

Observe that these conditions imply the cancellation of non-abelian anomalies, but are in fact slightly stronger. In particular, when \( n_i = 1, 2 \), the non-abelian part of the \( i^{th} \) group factor does not exist or is \( SU(2) \) and thus automatically anomaly-free. In such case, since \( n_i \neq 0 \), the conditions impose \( A_i = 0 \), even though there is no field theory reason for it. Our analysis of \( U(1) \) anomalies below will shed some light about why the condition \( A_i = 0 \) is required even in those cases.

In the following we show that the factorization of \( U(1) \) anomalies proposed in Section 2 follows from the string consistency conditions (4.3). We start by studying the mixed anomaly with non-abelian factors. For simplicity, let us first assume that all factors contain a non-abelian part. The basic argument goes as follows. The mixed anomaly between the \( j^{th} U(1) \) and the \( SU(n_l) \) factor, as computed from the spectrum (4.1), (4.2), is proportional to

\[ A_{jl} = \frac{1}{2} n_j \sum_{a=1}^{3} (\delta_{i,j+a} - \delta_{i,j-a}) + \frac{1}{2} \delta_{j,l} \sum_{a=1}^{3} (n_{j+a} - n_{j-a}) \]  

(4.4)

Notice that the second contribution is proportional to the non-abelian anomaly coefficient \( A_j \) which must vanish for consistency of the theory. The remaining contribution can be recast as

\[ A_{jl} = \frac{1}{2N} n_j \sum_{a=1}^{3} \sum_{k=1}^{N} \{ \exp[2i\pi(j + \ell_a - l)k/N] - \exp[2i\pi(j - \ell_a - l)/N] \} \]  

(4.5)

where we have used the discrete Fourier transform of the Kronecker delta.

Factorizing the term \( \exp[2i\pi(j - l)k/N] \), and using the identity \( \sum_{a=1}^{3} \sin 2\pi kv_a = -4 \prod_{a=1}^{3} \sin \pi kv_a \), where \( v_a = \ell_a/N \), we have

\[ A_{jl} = \frac{(-i)}{2N} \sum_{k=1}^{N} [n_j \exp(2i\pi k/j/N) \exp(-2i\pi \ell_a k/N) \prod_{a=1}^{3} 2 \sin \pi kv_a] \]  

(4.6)

Comparing with eq. (2.4), we see it has the adequate structure to be cancelled by the exchange of twisted RR fields (notice that, due to the absence of orientifold projection, in the orbifold case the traces in (2.4) amount to exponentials). A nice
insight our general formula reveals is that the coefficients $C_k$ in (2.3), that arise from
the cylinder diagram, are directly related (through the Fourier transform) to the delta
functions which define the matter content of the gauge theory.

It is easy to extend the conclusion to the case where some abelian or non-abelian
factor is absent. Then, the corresponding mixed anomalies vanish, but so do the
contribution from exchange of RR fields.

The analysis of the cubic $U(1)_j \times U(1)_l^2$ anomaly works similarly. Let us first
consider the case $n_i \neq 0$. The anomaly is proportional to

$$B_{jl} = \sum_{a=1}^3 (n_j n_{j+a} \delta_{l,j+l_a} - n_j n_{j-a} \delta_{l,j-l_a}) + \sum_{a=1}^3 (n_j n_{j+a} \delta_{j,l} - n_j n_{j-a} \delta_{j,j})$$ (4.7)

The second contribution is proportional to $A_j$, which vanishes due to tadpole can-
cellation. The remaining part can be written as

$$B_{j,l} = n_j n_l \sum_{a=1}^3 (\delta_{l,j+l_a} - \delta_{l,j-l_a})$$ (4.8)

which factorizes as in the previous case. Notice that, since no $l_a = 0 \mod N$, we have
$B_{i,i} = 0$ and the subtlety about the symmetry factor $1/3$ discussed in Section 3 is not
manifest in this family of models. Again, appropriate factorization is also found in
models where some abelian factor is absent, i.e. some $n_i = 0$.

We have shown how the string consistency conditions imply the factorization of
$U(1)$ anomalies. A nice result is that this provides a field theory interpretation for the
requirement of vanishing of $A_i$ even when $n_i = 1$. It is not required to cancel non-
abelian anomalies, but to ensure factorization of $U(1)$’s. Another interesting outcome
of our analysis is that the mechanism of Section 5 generates masses for the $U(1)$’s. This
phenomenon is responsible for the disappearance of the $U(1)$ factors from the low-
ergy dynamics, and is to be interpreted as the ‘freezing’ described in the language
of brane construction in [36]. The phenomenon is analogous to that present in six-
dimensional theories from D5 branes on singularities [26].

It is interesting that the requirement of factorization of $U(1)$’s imposes constraints
on the spectrum of the model beyond those following from cancellation of non-abelian
anomalies. These restrictions can be translated to constraints in the T-dual brane box
configurations. In order to illustrate this feature, consider a $\mathbb{C}^3/Z_3$ singularity, at which
we place D3 branes with the following Chan-Paton matrix

$$\gamma_\theta = \text{diag}(I_4, \alpha I_2, \alpha^2 I_2)$$ (4.9)
where $\alpha = e^{2i\pi/3}$ and $I_k$ is the $k \times k$ identity matrix. Thus we have $n_0 = 4$, $n_1 = n_2 = 2$. The spectrum of the models is

$$
U(4) \times U(2) \times U(2) \\
3(4,2,1) + 3(1,2,2) + 3(\overline{1},1,2)
$$

(4.10)

This is a priori a phenomenologically interesting model since it consists of three standard Pati-Salam $SU(4) \times SU(2)_L \times SU(2)_R$ generations plus three sets of Higgs fields coupling to them. Only a set $(4,1,2) + (\overline{4},1,2)$ is missing in order to further do the symmetry breaking to the standard model.

This model is free of non-abelian anomalies, by virtue of the special properties of $SU(2)$. However, it is not consistent from the string theory point of view, since $A_1$, $A_2$ are non-vanishing. This shows up as a non-vanishing contribution to the first term in (4.4), which spoils the factorization of $U(1)$’s.

### 4.2 D3 branes at orientifold singularities

In this subsection we study factorization of $U(1)$’s in a family of field theories arising from D3 branes at orientifold singularities (i.e. orbifold singularities with some world-sheet orientation reversing projection).

Such four-dimensional $N = 1$ theories have only been constructed for a few orientifold singularities [13]. Below we construct an infinite family of field theories corresponding to an infinite family of orientifold singularities. These field theories are very interesting, since by arguments from the AdS/CFT correspondence, they will be superconformal in the large $N$ limit. Moreover, they constitute the first example of an infinite family of $N = 1$ theories from D3 branes at orientifold singularities, and show that D3 branes at orientifold singularities generate a rich variety of $D = 4$, $N = 1$ field theories. We leave a more systematic study of other possible families of models for future work, and in what follows center on a particular large class. The details of the tadpole computation showing its consistency are left for the appendix. Here we just state the main properties of the spectrum.

We will consider a discrete group $Z_N$, with odd $N = 2P + 1$, generated by a twist $\theta$ acting on $\mathbb{C}^3$ as defined by $v = (\ell_1, \ell_2, \ell_3)$. The orientifold group will be generated by $\theta$ and $\Omega' \equiv \Omega(-1)^F R_1 R_2 R_3$, where $R_a$ denotes the inversion of the $a^{th}$ complex plane. As usual, $\Omega$ includes the element $J$ which exchanges oppositely twisted sectors. Notice that this orientifold does not require the presence of D7 branes.
Before the $Z_2$ orientifold projection, the spectrum of the model is
\[ \prod_{j=-P}^{P} U(n_j) \rightleftharpoons \bigoplus_{a=1}^{3} \bigoplus_{j=-P}^{P} (\Box_j, \Box_{j+l_a}) \] (4.11)
just like in (4.1), (4.2). The effect of $\Omega'$ is to exchange the factors $U(n_j)$ and $U(n_{-j})$, in such a way that the fundamental representation $\Box_j$ goes over to the anti-fundamental representation $\Box_{-j}$, and vice-versa. Notice that, in order to be a symmetry of the theory (4.11), we must require the ranks of exchanged groups to be equal, $n_j = n_{-j}$. The operation is an automorphism of the quiver diagram of the theory.

The final spectrum is found by keeping the fields invariant under the orientifold projection. The absence of D7 branes in the construction allows for two different projections. In one of them, the group $U(n_0)$, which is projected onto itself, becomes $SO(n_0)$ in the quotient. Also, when the two entries of some bi-fundamental are charged with respect to the same group in the quotient, the antisymmetric combination is to be taken. The second possibility is to project onto $USp(n_0)$, and symmetric representations. In the following, we will center on the ‘$SO$’ projection, and we stress the other case works in complete analogy.

Just to give a flavour of the type of theories that arise from the above construction, we give a simple example of a non-crystallographic case which has not been considered in the literature. It is a $Z_5$ model, generated by a twist $v = (1, 1, -2)/5$. The spectrum (4.11) before the orientifold projection is
\[ U(n_{-2}) \times U(n_{-1}) \times U(n_0) \times U(n_1) \times U(n_2) \]
\[ 2 \left[ (\Box_1, \Box_2) + (\Box_2, \Box_{-2}) + (\Box_{-2}, \Box_{-1}) + (\Box_{-1}, \Box_0) + (\Box_0, \Box_1) \right] + \]
\[ + (\Box_{-1}, \Box_1) + (\Box_1, \Box_0) + (\Box_0, \Box_{-1}) + (\Box_{-1}, \Box_{-2}) + (\Box_{-2}, \Box_{-1}) \] (4.12)

After the orientifold projection, the rules above yield the spectrum
\[ SO(n_0) \times U(n_1) \times U(n_2) \]
\[ 2 \left[ (\Box_1, \Box_2) + (\Box_2, \Box_1) \right] + \]
\[ + (\Box_1, \Box_0) + (\Box_0, \Box_1) \] (4.13)

The particular cases of $Z_3$ and $Z_7$ have appeared in [43], and, in the T-dual version of D9 branes in a compact orientifold, in [12, 18, 13, 14].

Since this family of theories has not appeared in the literature, we make here a brief comment concerning their non-abelian anomalies. The non-abelian anomaly coefficient, in the generic case in which no $n_i$ vanishes, is given by
\[ A_i = \sum_{a=1}^{3} \left[ (n_{i+\ell_a} - n_{i-\ell_a}) - 4(\delta_{i+\ell_a,-i} - \delta_{i-\ell_a,-i}) \right] \] (4.14)
where the second contribution takes into account the cases where the bifundamental is actually an antisymmetric. In this and following expressions, the formulae are valid for the ‘Sp’ projection by simply changing the sign of such contributions.

In analogy with the orbifold case [47], the conditions $A_i = 0$ can be stated as constraints on the Chan-Paton matrices, simply by taking a discrete Fourier transform. After a short computation, the conditions $A_i = 0$ are equivalent to

$$\prod_{a=1}^{3} \sin 2\pi kv_a \, \text{Tr} \, \gamma_{2k} - 4 \prod_{a=1}^{3} \sin \pi kv_a = 0 \quad (4.15)$$

or

$$\prod_{a=1}^{3} \sin \pi kv_a \left[ 2 \prod_{a=1}^{3} \cos \pi kv_a \, \text{Tr} \, \gamma_{2k} - 1 \right] = 0 \quad (4.16)$$

The results of the appendix show that string consistency conditions actually ensure the vanishing of this quantity. The first factor is non-zero when the twist has the origin as the only fixed point. This is precisely when tadpoles, which are proportional to the second factor, are required to vanish. This shows how string consistency implies the consistency of the gauge field theory on the D3 branes. Notice that in [43] only the $Z_3$ and $Z_7$ models were considered, since the indirect construction technique employed there (the system of D3 branes was obtained by T dualizing a set of D9 branes) does not allow to obtain the whole infinite family.

Being a bit more careful, it is possible to show that if some $n_i$ vanishes the condition for $A_i$ is not required. Notice that, as in the orbifold case, when some $n_i = 1, 2$ the condition $A_i$ does not have the interpretation of cancellation of non-abelian anomalies. Our arguments below will show that it is however needed to have appropriate factorization of $U(1)$ anomalies.

So, let us center on the study of mixed non-abelian anomalies. We first consider the case where no non-abelian factor is absent. The mixed anomaly between the $j^{th}$ $U(1)$ and the $l^{th}$ non-abelian factor can be computed from the spectrum of the theory to be

$$A_{jl} = \frac{1}{2} \delta_{jl, \ell} \sum_{a=1}^{n_{l+x-a}} \left( n_{l-x-a} - n_{l-x-a} \right) + \frac{1}{2} n_j \sum_{a=1}^{3} \left[ \left( \delta_{l,j+\ell_a} - \delta_{l,j-\ell_a} \right) + \left( \delta_{l,-j-\ell_a} - \delta_{l,-j+\ell_a} \right) \right] -$$

$$-2 \sum_{a=1}^{3} \delta_{j,l} \left( \delta_{j+\ell_a,-j} + \delta_{j-\ell_a,-j} \right) \quad (4.17)$$

The main difference with the equivalent expression for the orbifold case is that, besides the contribution from matter charged under the $j^{th}$ and $l^{th}$ factors, there is an additional contribution from matter charged under the $j^{th}$ and $(-l)^{th}$ factors. This last contribution arises in string theory from the second diagram in Figure 1, the Moebius
strip. The last line contains a correction that takes into account the cases when some bi-
fundamental is actually an antisymmetric (recall that in our normalization, the charge
of an antisymmetric representation is +2).

The term proportional to $\delta_{j,l}$ in the above expression is proportional to $A_l$, eq.(4.14),
and must vanish for string consistency. The remaining terms can be Fourier trans-
fomed in a by now familiar fashion, to yield

$$A_{j,l} = 1/N \sum_{k=-P}^{P} n_j \sin(2\pi k j/N) \cos(2\pi k l/N) \prod_{a=1}^{3} 2 \sin \pi k v_a$$  \hspace{1cm} (4.18)

This reproduces the structure depicted in (2.4), or (3.5). Also, it nicely shows how
the orientifold projection implies the appearance of sine and cosine functions instead
of exponentials.

Again, when some abelian or non-abelian factor is absent, the corresponding $A_{j,l}$
vanishes automatically, but so does the exchange of RR fields.

Let us finally discuss the structure of cubic anomalies. Since factorization works
analogously, the discussion will be brief. We center on the generic case of $n_i \neq 0$. The
mixed $U(1)_j \times U(1)_l^2$ anomaly is given by

$$B_{j,l} = \delta_{j,l} \sum_{a=1}^{3} (n_j n_{l+\ell_a} - n_{l-\ell_a} n_j) + n_j n_l \sum_{a=1}^{3} [ (\delta_{i,j+\ell_a} - \delta_{j-i-\ell_a}) + (\delta_{i-j-\ell_a} - \delta_{l,-j+\ell_a}) ] -$$

$$-\delta_{j,l} \sum_{a=1}^{3} [ (2n_j^2 - 4n_i)(\delta_{j+\ell_a,-i} - \delta_{j-\ell_a,-j}) ]$$ \hspace{1cm} (4.19)

As usual, some terms can be grouped to yield a contribution proportional to $A_j$,
eq (4.14), which vanishes. The remaining terms are

$$B_{j,l} = n_j n_l \sum_{a=1}^{3} [ \delta_{i,j+\ell_a} + \delta_{i,-j+\ell_a} - \delta_{i,j-\ell_a} - \delta_{i,-j+\ell_a} +$$

$$+ 2 \delta_{j,l} (\delta_{j,-\ell_a} - \delta_{j,-j+\ell_a} + \delta_{j+\ell_a} - \delta_{j-j-\ell_a}) ]$$ \hspace{1cm} (4.20)

The last two terms in the second line, which are vanishing because $\ell_a \neq 0 \mod N$, and
so $j \neq j \pm \ell_a$, have been introduced for convenience.

When $i \neq j$, the second line contribution vanishes, and the expression factorizes
in the same fashion as the mixed non-abelian anomalies (with the additional normalization
factor $n_l$). When $i = j$, the contribution from the second line has the same form as
that from the first, and both together give the usual factorized form with an additional
factor of 3, which cancels agains the symmetry factor $1/3$ mentioned in Section 3.

We have shown how this large family of field theories constructed from D3 branes
at orientifold singularities satisfy in a very non-trivial way all the constraints of can-
cellation of non-abelian anomalies, and appropriate factorization of $U(1)$ anomalies.
These properties follow directly from the cancellation of non-physical tadpoles (7.7). This family illustrates the rich variety of field theories from D3 branes at orientifold singularities, and hopefully will motivate further research in the field.

5 Fayet-Iliopoulos terms

The anomaly cancellation mechanism described in previous sections relies on the presence of the couplings (2.2) which mix the $U(1)$ fields with the RR two-forms $B_k$. In addition, supersymmetry requires the presence of terms of the form

$$D_i \sum_{k=1}^{N} (\text{Tr} \gamma_k \lambda_i) \Phi_k$$

(5.1)

where $D_i$ is the auxiliary field associated to the $U(1)_i$, and $\Phi_k$ is the NS partner of the corresponding RR fields $\phi_k$. These are nothing but field-dependent Fayet-Iliopoulos terms. They are similar to the FI terms found for $D = 6, N = 1$ in [24]. Notice that in the case of orientifolds one has

$$D_i \sum_{k=1}^{N} (n_i \sin 2\pi kV_i) \Phi_k$$

(5.2)

and untwisted ($k = 0$) NS fields (like e.g., the dilaton or untwisted moduli) do not contribute to FI-terms. This is quite different to the case of $N = 1, D = 4$ heterotic vacua in which it is only the dilaton which appears in the FI term [3, 4].

Other differences with the case of $D = 4, N = 1$ heterotic vacua are worth remarking. In the heterotic case there is at most one anomalous $U(1)$ whereas in the Type I and Type II vacua studied in this paper any number may appear. Furthermore, the counterterm (2.2) appears here at the disk level whereas in the heterotic such term is induced at the one-loop level.

Moreover, in the heterotic case, since the FI term is proportional to the heterotic dilaton, one cannot put it to zero without going to a non-interacting ($g = 0$) theory. Thus the scale of the heterotic FI term is of order of the string scale or slightly below.

In the Type I or Type II cases here considered we can in principle put the size of the FI-terms as small as we wish since $\Phi_k \to 0$ does not correspond in general to a non-interacting theory, but to the orbifold limit. Consider for example the case of the compact orientifolds discussed in chapter two. The gauge kinetic function for a $U(1)_j$ will have the general form

$$f_j = S + \sum_k n_j \cos 2\pi kV_j \psi_k$$

(5.3)
where \( S \) is the Type I untwisted dilaton chiral field and \( \Psi_k = \Phi_k + i\phi_k \) are the complex twisted scalars. For \( \Phi_k \rightarrow 0 \) there is a finite coupling given by the untwisted dilaton. The same is true for the kinetic functions of non-Abelian group factors for which one can write a similar expression. Notice that, since the cosine in eq.(5.3) may have both positive and negative sign, for some particular points in the dilaton/twisted NS fields moduli space, some of the gauge coupling constants may explode. At those points perturbation theory will fail and one expects the appearance of non-perturbative phenomena.

The couplings in eq.(2.2) (after a duality transformation) imply the presence of a Higgs mechanism by which \( U(1) \)'s get masses. The masses of these \( U(1) \)'s, like the FI-terms, are given by the vev’s of the NS \( \Phi_k \) fields. Thus the masses of the anomalous \( U(1) \)'s in these theories may in principle be as small as we wish. This is again to be contrasted to the heterotic case in which the mass of the unique anomalous \( U(1) \) is tied up to the string scale.

6 Final comments

In this paper we have shown that four-dimensional type IIB orientifold vacua have generically several anomalous \( U(1) \)'s. We have discussed how their triangle anomalies are cancelled through the exchange of twisted RR fields, in a version of the Green-Schwarz mechanism. This pattern is very different from that found in heterotic models, as we have already remarked, and seems to be worth of further study.

Here would like to stress another point concerning this mechanism. Four-dimensional \( N = 1 \) vacua can also be obtained by compactifying type I superstrings on smooth Calabi-Yau manifolds with a certain gauge bundle. These models can be analyzed as a Kaluza-Klein reduction of the ten-dimensional theory, and yield models with a pattern of \( U(1) \) anomalies identical to that in heterotic models. Namely, there is at most one anomalous \( U(1) \), and the anomaly is cancelled by exchange of the partner of the dilaton [52]. This is somewhat surprising, since the orientifold models we have analyzed (at least, those having no D5 branes) can be naively regarded as singular limits of such smooth compactifications. However, both constructions differ sharply in their pattern of anomaly cancellation.

Nevertheless, this is not the first time that such differences between compactifications on smooth and singular manifolds are found. Already in six-dimensional \( N = 1 \) vacua, smooth compactifications of type I string theory yield models with at most one tensor multiplet, containing the dilaton. On the other hand, orientifold models gener-
ically contain additional tensor multiplets, arising from the singular points \[49, 50\]. Understanding the relation between these two kinds of vacua has been the key to some of the new physics uncovered in six-dimensional string and field theory.

The situation we encounter in \(D = 4, N = 1\) vacua is certainly analogous. Four-dimensional orientifolds seem to be exploring regions of the moduli space which differ in their generic properties from those of smooth compactifications. One is led to expect new interesting insights on \(D = 4, N = 1\) vacua will be obtained by studying the relation between both descriptions.

On the phenomenological side, the anomalous \(U(1)\)’s found in the class of orientifold/orbifold Type IIB vacua studied in the present paper show characteristics totally different to the familiar single \(U(1)\)’s of perturbative heterotic vacua. It would be interesting to study in more detail possible phenomenological applications of these new anomalous \(U(1)\)’s to problems like fermion textures, supersymmetry-breaking and cosmology in which the heterotic anomalous \(U(1)\)’s have been suggested to play an important role.

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7 Appendix

Construction of the theories on D3 branes at orientifolds

In this appendix we construct the non-compact orientifold theories of Section 4.2 and discuss their tadpole cancellation conditions. Several remarks were already made in the main text, but we repeat them here for convenience.

The orientifold group has the structure $G_{\text{orient}} = G + \Omega' G$. Here $G$ is a $Z_N$ group with $N = 2P + 1$, odd, generated by a twist $\theta$ with vector $v = (v_1, v_2, v_3) = \frac{1}{N}(\ell_1, \ell_2, \ell_3)$. The world-sheet orientation reversing operation is $\Omega' \equiv \Omega(-1)^F R_1 R_2 R_3$, where the operator $R_a$ is the inversion of the $a$th complex plane. As usual $\Omega$ exchanges oppositely twisted sectors (it is $\Omega J$ in the notation of [48]).

The orientifolds do not contain D7 branes, and thus there are two possible $\Omega'$ projections on the D3 branes, which we will denote as the ‘SO’ and ‘Sp’ projections.

Notice also that these models are four-dimensional cousins of some $D = 6$ orientifolds considered in [49, 50].

The Chan-Paton matrices we will consider are

$$\gamma_\theta = \text{diag}(I_{n_0}, \alpha I_{n_1}, \ldots, \alpha^p I_{n_P}, \alpha^{-p} I_{n_P}, \ldots, \alpha^{-1} I_{n_1}) \quad \text{with} \quad \alpha = e^{2\pi i/N}$$

$$\gamma_{\Omega'} = \begin{pmatrix} I_{n_0} & & \\ & I_{n_1} & \\ & & I_{n_P} \\ & & & \\ & & & \\ & & & & I_{n_1} \\ & & & & & I_{n_P} \\ & & & & & & I_{n_0} \\ & & & & & & & -I_{n_1} \\ & & & & & & & & -I_{n_P} \\ & & & & & & & & & & & -I_{n_1} \\ & & & & & & & & & & & & I_{n_0} \\ & & & & & & & & & & & & & I_{n_1} \\ & & & & & & & & & & & & & & I_{n_P} \\ & & & & & & & & & & & & & & & I_{n_0} \\ & & & & & & & & & & & & & & & & I_{n_1} \\ & & & & & & & & & & & & & & & & & I_{n_P} \end{pmatrix} ; \quad \gamma_{\Omega'} = \begin{pmatrix} \epsilon_{n_0} & & \\ & \epsilon_{n_1} & \\ & & \epsilon_{n_P} \\ & & & \\ & & & & \epsilon_{n_1} \\ & & & & & \epsilon_{n_P} \\ & & & & & & \epsilon_{n_0} \\ & & & & & & & \epsilon_{n_1} \\ & & & & & & & & \epsilon_{n_P} \\ & & & & & & & & & \epsilon_{n_0} \\ & & & & & & & & & & \epsilon_{n_1} \\ & & & & & & & & & & & \epsilon_{n_P} \end{pmatrix}$$

where $\epsilon_{n_0}$ is block-diagonal with $n_0/2$ blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The two possibilities for $\gamma_{\Omega'}$ correspond to the $SO$ and $Sp$ projections, respectively. For future convenience, notice that the matrices verify

$$\text{Tr} (\gamma_{\Omega'}^{-1} \gamma_{\Omega'}^T) = \pm \text{Tr} (\gamma_{2k})$$

where the upper (lower) sign corresponds to the $SO$ ($Sp$) projection.

The spectrum before the $\Omega'$ projection is given by the orbifold theory

$$\prod_{i=-P}^{P} U(n_i) ; \quad \bigoplus_{a=1}^{3} \bigoplus_{i=-P}^{P} (\square, \square_{a+i})$$

(7.2)

The operation $\Omega'$ identifies the groups $U(n_i)$ and $U(n_{-i})$, such that the representation $\square$, is identified with $\square_{-i}$. In the $SO$ (resp. $Sp$)projection, $U(n_0)$ becomes $SO(n_0)$ (resp. $USp(n_0)$), and bi-fundamentals charged with respect to the same group in the quotient become antisymmetric (resp. symmetric) representations.
Computation of tadpoles

Since we are quotienting $\mathbb{C}^3$ and the resulting space is non-compact, string consistency only requires the cancellation of tadpoles for twisted sectors which have the origin as the only fixed point $^6$ We mainly refer to the general expressions derived in the appendix of [18], and only mention what changes should be taken into account.

The 33 cylinder amplitudes is identical to that of 99 or 55 cylinders. From equations (7.9) and (7.14) in [18], we have the $t \to 0$ contribution

$$C_{33} \to (1 - 1) \frac{V_4}{8N} \sum_{k=1}^{N} \int_{0}^{\infty} \frac{dt}{t} (8\pi\alpha')^{-2} t \prod_{a=1}^{3} |2 \sin \pi kv_a| (\text{Tr} \gamma_{k,3})^2$$

(7.3)

For the Klein bottle, there is only the contribution $Z_K(1, \theta R_1 R_2 R_3)$, where we must also include the twists implicit in $\Omega'$. From equations (7.1) and (7.6) in [18], we have the $t \to 0$ contribution

$$K \to (1 - 1) \frac{V_4}{8N} \sum_{k=1}^{N} \int_{0}^{\infty} \frac{dt}{t} (4\pi\alpha')^{-2} (2t) \prod_{a=1}^{3} |2 \sin[2\pi (kv_a + 1/2)]| \prod_{a=1}^{3} 4 \sin^2[\pi (kv_a + 1/2)]$$

(7.4)

The shifts by 1/2 arise from the twist $R_1 R_2 R_3$, and the denominator is the zero mode integration mentioned in [19] (in the compact examples in [18] it was taken into account by an explicit counting of fixed point sets).

Finally, the contribution from the Moebius strip $Z_3(\theta R_1 R_2 R_3)$ is analogous to eq.(7.24) in [18], but for the fact that there are Dirichlet boundary conditions all three complex planes. Using also (7.20) in [18], the leading contribution as $t \to 0$ is

$$M_3 \to (1 - 1) \frac{V_4}{8N} \sum_{k=1}^{N} \int_{0}^{\infty} \frac{dt}{t} (8\pi\alpha')^{-2} t \prod_{a=1}^{3} s_a 2 \cos[\pi (kv_a + 1/2)] \text{Tr} (\gamma_{a,1}^{-1} \gamma_{k})$$

(7.5)

where $s = \text{sign}(\sin 2\pi kv_a)$. Using the relations $t = \frac{1}{2\ell}$, $t = \frac{1}{4\ell}$, $t = \frac{1}{8\ell}$ for the cylinder, Klein bottle and Moebius strip [51], and the property (7.1), the amplitude is proportional to

$$\sum_{k=1}^{N} \left[ \prod_{a=1}^{3} 2 \sin 2\pi kv_a |(\text{Tr} \gamma_{2k})^2 + \prod_{a=1}^{3} s_a 2 \sin \pi kv_a \text{Tr} \gamma_{2k} + 16 \prod_{a=1}^{3} \left| \frac{\sin \pi kv_a}{\cos \pi kv_a} \right| \right]$$

(7.6)

with the upper (lower) sign for the $SO$ ($Sp$) projection. This can be factorized as

$$\sum_{k=1}^{N} \left[ \frac{1}{\prod_{a=1}^{3} |2 \sin 2\pi kv_a|} \left[ \prod_{a=1}^{3} 2 \sin 2\pi kv_a \text{Tr} \gamma_{2k} + 32 \prod_{a=1}^{3} \sin \pi kv_a \right] \right]^2$$

(7.7)

Each of the terms in the square bracket must vanish independently. Notice that when the twist leaves some complex plane fixed (i.e. some $\sin \pi kv_a$ vanishes) the equation is

$^6$However, in order to have a conformal theory in the limit of large number of D3 branes, all twisted tadpoles should vanish (at leading order)
automatically satisfied and the tadpole corresponding to that twist does not constrain the Chan-Paton matrices. For twists which have the origin as the only fixed point, the constraint reads

$$\text{Tr} \gamma_{2k} = \pm \frac{1}{2} \frac{1}{\prod_{a=1}^{3} \cos \pi k v_a}$$  \hspace{1cm} (7.8)$$

In the main text it is shown that these consistency conditions ensure the consistency of the field theory on the D3 branes. Namely, it implies cancellation of non-abelian anomalies, and appropriate factorization of $U(1)$ anomalies.

Notice that, even though in some cases these orientifolds are T-dual to models with only D9 branes, the general formula for the tadpoles conditions is quite different. This is always the case when the T-duality is performed along twisted coordinates. However, the result is consistent and for $Z_3$ and $Z_7$ the usual twisted tadpole conditions are recovered.

A last important point is that it is always possible to find solutions to the above tadpole equations. To show the existence of at least one for each $Z_{2P+1}$ singularity, we rewrite the condition (7.8) as

$$\text{Tr} \gamma_{2k} = \pm 4 \prod_{a=1}^{3} \frac{1}{1 + e^{2\pi i k v_a}}$$  \hspace{1cm} (7.9)$$

Since $N$ is odd, the ‘1’ in the numerator can be expressed as polynomial in $\exp(2\pi i k v_a)$ with an even number of non-zero terms. The denominator $(1 + e^{2\pi i k v_a})$ divides such polynomial, and thus Tr $\gamma_{2k}$ is a polynomial in $\exp(2\pi k/N)$. It is then straightforward to give $\gamma$ matrices with the appropriate traces. This procedure gives a solution that cancels all twisted tadpoles.

As a simple example, consider the $Z_5$ model with $v = \frac{1}{5}(1, 1, -2)$, whose spectrum is depicted in eq.(4.13). From (7.9), we have

$$\text{Tr} \gamma_{2k} = 4 \left( \frac{1}{1 + \alpha^k} \right)^2 \frac{1}{1 + \alpha^{3k}}$$  \hspace{1cm} (7.10)$$

with $\alpha = e^{2\pi i/5}$. Performing the trick mentioned above, the fractions can be expressed as a polynomial in $\alpha$, yielding

$$\text{Tr} \gamma_{2k} = 4(-\alpha^k - \alpha^{3k})^2(-\alpha^{3k} - \alpha^{9k}) = -4(1 + \alpha^{2k} + \alpha^{2k})$$  \hspace{1cm} (7.11)$$

The conditions for the twisted tadpoles are solved by

$$\gamma_1 = \text{diag} \left( I_{N-4}, \alpha I_{N-4}, \alpha^2 I_N, \alpha^3 I_N, \alpha^4 I_{N-4} \right)$$ \hspace{1cm} (7.12)$$

Using the spectrum in (4.13), one can check directly that non-abelian anomalies cancel.
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