Abstract

Via an elementary, exactly solvable example it is demonstrated that the physical meaning of quantum systems which are characterized by a $\mathcal{PT}$–symmetric Hamiltonian $H$ with real spectrum may prove dramatically different before and after some of the energy levels happen to cross. An elementary explanation of the paradox is provided.

keywords

crypto-Hermitian quantum Hamiltonians; parity times time-reversal symmetry; solvable model; complete set of inner products; paradox of unavoided crossing;
1 Introduction

In the various practical and realistic applications of quantum theory (say, in quantum chemistry [1]) the emergence of an unavoidable crossing of energy levels is currently being attributed to the presence of a hidden symmetry of the Hamiltonian. In a way, the very similar “rule of thumb” remains also applicable during the analysis of the recently very popular $\mathcal{PT}$—symmetric Hamiltonians $H$. They are characterized by the reality of spectrum (treated as a “spontaneously unbroken symmetry” of wave functions) and by the less usual symmetry property $H\mathcal{PT} = \mathcal{PT}H$ in which $\mathcal{P}$ is parity while antilinear $\mathcal{T}$ represents the operation of time reversal [2].

One of the simplest illustrative examples of a $\mathcal{PT}$—symmetric Hamiltonian which admits the unavoidable crossing of energy eigenvalues is provided by the generalized harmonic oscillator Hamiltonian $H^{(HO)}(\alpha)$ of Ref. [3],

$$H^{(HO)}(\alpha) = -\frac{d^2}{dx^2} + x^2 - 2icx + \frac{\alpha^2 - 1/4}{(x - ic)^2}, \quad x \in (-\infty, \infty). \quad (1)$$

For this toy model the centrifugal-like singularity is regularized by a shift $c > 0$ so that the spectrum of bound states proves well defined, discrete, real and available in closed form,

$$E = E_{(n,q)} = 4n + 2 - 2q\alpha + c^2, \quad n = 0, 1, \ldots, \quad q = \pm 1. \quad (2)$$

The unavoided energy-level crossings have the form of accidental degeneracies $E_{(m,1)} = E_{(n,-1)}$ at all of the integer couplings $\alpha = m - n$.

Puzzling as such a level-crossing phenomenon might seem, there exist several reasons why one need not feel too embarrassed by their occurrence. Even in Ref. [3] it has already been noticed that operators (1) with integer $\alpha$s are not diagonalizable so that they cannot represent any quantum observable [4]. In the standard mathematical terminology as introduced by Kato [3], the set of integer $\alpha$s forms simply the so called exceptional points (EP) of the toy model $H^{(HO)}(\alpha)$. For this reason, the integer-valued couplings $\alpha^{(EP)} \in \mathbb{Z}$ are most often exempted as “unphysical” while operators (1) themselves merely remain defined inside the punctured domain of couplings, $\alpha \in D^{(HO)} \equiv \mathbb{R} \setminus \mathbb{Z}$.

Due to the smooth $\alpha$—dependence of the energies in the vicinity of every EP it is usually believed that their presence does not really represent a truly deep characteristic of the quantum system in question. Such a point of view has further been supported, say, by a deeper analysis of another, even more elementary square-well-type Hamiltonian $H^{(SW)}(\alpha)$ of Ref. [6]. It has been revealed [7] that one may get rid of the mind-boggling punctures in the underlying domain $D^{(SW)}$ of admissible couplings, very easily, by the mere small deformation of the interaction in the Hamiltonian $H^{(SW)}(\alpha)$.

A principal weakness of the latter heuristic argument is that via the change of the interaction one merely circumvents the puzzle, tacitly accepting one of many possible ad hoc deformations of the model as a “unique” regularization recipe. In our present letter we intend to return to the problem, therefore. After a concise outline of the theory in section [2] we shall pick up a particularly
elementary model (see section 3) and we shall come to a few rather unexpected conclusions (see section 4 and the brief summary of our message in section 5).

2 The formalism

2.1 Quantum observables in crypto-Hermitian representation

In many textbooks on quantum mechanics it is recommended to replace a preselected, “physical” Hilbert space $\mathcal{H}^{(P)}$ (as chosen, say, in the traditional manybody form of a direct sum $\bigoplus L^2(\mathbb{R}^d)$) by a unitarily equivalent “sophisticated” alternative $\mathcal{H}^{(S)}$. In a way inspired by the needs of quantum physics of heavy nuclei \[8\] (but occurring also elsewhere \[9\]) the strategy has been further amended by the introduction of a third, auxiliary Hilbert space $\mathcal{H}^{(F)}$ which is “false but friendly” (our use of the P-, S- and F-superscripting convention is taken from Ref. \[10\]). In the literature, the resulting “three-Hilbert-space” (THS) formalism is being called, practically equivalently, quasi-Hermitian quantum mechanics \[8\], PT−symmetric quantum mechanics \[2\], pseudo-Hermitian quantum mechanics \[4\]) or crypto-Hermitian quantum mechanics \[11\].

In the nuclear-physics applications as reviewed in \[8\] the key advantage of the resulting two-step change of representation $\mathcal{H}^{(P)} \rightarrow \mathcal{H}^{(S)} \rightarrow \mathcal{H}^{(F)}$ has been found in the observation that one becomes allowed to transfer practically all calculations (i.e., typically, the variational estimates of the energy spectra) from the “sophisticated” space $\mathcal{H}^{(S)}$ to its “friendly” simplification $\mathcal{H}^{(F)}$. In this way one may work with the same Hamiltonian $H$ and, at the same time, with the different linear functionals $\langle \psi |^{(F,S)}$. In practice one must, of course, very carefully distinguish between the current Dirac’s, “transposed plus complex conjugate” bra vectors $\langle \psi |^{(F)} \equiv \langle \psi |$ in $\mathcal{H}^{(F)}$ and the highly unusual, non-Dirac, Hilbert-space-determining brabra vectors $\langle \psi |^{(S)} \equiv \langle \langle \psi | := \langle \psi | \Theta$ in $\mathcal{H}^{(S)}$. The key feature of the latter definition should be seen in the presence of operator $\Theta \neq I$ of the Hilbert-space metric (cf. \[10\] for details).

2.2 The concept of Hilbert-space metric $\Theta$

Successful implementations of the THS recipe are based on several mathematical conditions. Firstly, the time-independent generator $H$ of the time evolution must be Hermitian in $\mathcal{H}^{(S)}$, i.e.,

$$\sum_{k=1}^{N} \left[ H^\dagger_{jk} \Theta_{kn} - \Theta_{jk} H_{kn} \right] = 0, \quad j, n = 1, 2, \ldots, N, \quad N = \text{dim} \mathcal{H}^{(F,S)} \leq \infty.$$  \hspace{1cm} (3)

Although $H$ may be (though not necessarily \[12\]!\) non-Hermitian in $\mathcal{H}^{(F)}$, the spectrum must be real in a suitable physical domain $\mathcal{D}$ of parameters $\bar{\lambda}$. Inside this domain, our preselected Hamiltonian $H = H(\bar{\lambda})$ must be also diagonalizable \[4\]. For the sake of non-triviality of our considerations, let us also assume the non-emptiness $\partial \mathcal{D} \neq \emptyset$ of the EP boundary.
In practice, the spectrum of $H$ is often postulated discrete and bounded from below. This is a technical condition which will easily be satisfied here since we shall work with Hilbert space $H^{(F)}$ of a finite dimension $N < \infty$. This will enable us to consider the (complete) set of $N$ eigenstates $|\Xi_j\rangle$ of the conjugate operator $H^{\dagger}(\lambda)$,

$$H^{\dagger} |\Xi_n\rangle = E_n |\Xi_n\rangle, \quad n = 0, 1, \ldots, N - 1.$$  \hspace{1cm} (4)

Following Ref. [13], we may define the general matrix of the metric as the following sum

$$\Theta = \Theta(H, \kappa) = \sum_{j=0}^{N-1} |\Xi_j\rangle \kappa_n \langle\Xi_j|.$$ \hspace{1cm} (5)

The practically unrestricted variability of the optional parameters $\kappa_j > 0$ represents just the well known immanent degree of freedom of the theory which was well understood and well explained, for the first time, in Ref. [8].

3 The illustrative $N = 4$ toy model

3.1 Parametric domain $D$

Out of the family of the discrete, $N$–dimensional matrix Schrödinger equations as introduced in Ref. [14] at all $N < \infty$ let us pick up the simplest nontrivial $N = 4$ model which is characterized by the following two-parametric crypto-Hermitian Hamiltonian with real spectrum,

$$H = H^{(4)} = \begin{bmatrix}
0 & -1 + \beta & 0 & 0 \\
-1 - \beta & 0 & -1 + \alpha & 0 \\
0 & -1 - \alpha & 0 & -1 + \beta \\
0 & 0 & -1 - \beta & 0
\end{bmatrix} \neq H^{\dagger}. \hspace{1cm} (6)$$

In the spirit of review [2], this matrix is non-Hermitian but $\mathcal{PT}$–symmetric in terms of the most elementary time-reversal-simulating antilinear operator $\mathcal{T}$ of matrix transposition and of the antidiagonal parity-simulating $N$ by $N$ matrix

$$\mathcal{P} = \mathcal{P}^{(N)} = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix}. \hspace{1cm} (7)$$

For model (6) the bound-state energies are easily defined as the four roots of the secular equation

$$E^4 + (\alpha^2 - 3 + 2 \beta^2) E^2 + 1 - 2 \beta^2 + \beta^4 = 0.$$
These energies occur in pairs $E_{\pm,\varepsilon} = \pm \sqrt{Z_\varepsilon}$ where the symbol $Z_\varepsilon$ numbered by $\varepsilon = \pm$ denotes the two easily written roots of quadratic equation. Naturally, the latter two roots must be both non-negative inside the closure of the physical parametric domain $\mathcal{D}$.

From the secular equation one immediately deduces the double degeneracy $E \to 0$ of one of the pairs of the eigenenergies in the limit of $\beta^2 \to 1$. Under this constraint the complete quadruple degeneracy $E_{\pm,\pm} \to 0$ takes place in the second limit of $\alpha^2 \to 1$. In the vicinities of these extremes the exhaustive analysis of the character of the spectrum may start from its exact knowledge,

$$E_{\pm,\pm} = \pm \frac{1}{2} \sqrt{6 - 2 \alpha^2 - 4 \beta^2 \pm 2 \sqrt{\alpha^4 - 6 \alpha^2 + 4 \alpha^2 \beta^2 + 5 - 4 \beta^2}}.$$

In terms of the new variables $A = 1 - \alpha^2$, $B = 1 - \beta^2$ and $C = A + 4B$ the formula for the energies becomes more transparent,

$$2 E_{\pm,\pm} = \pm \sqrt{A + C \pm 2 \sqrt{AC}} = \pm \sqrt{(\sqrt{A} \pm \sqrt{C})^2} = \pm \sqrt{A} \pm \sqrt{C}.$$ (8)

This clarifies the root-complexification nature of the two EP lines $A = 0$ and $C = 0$.

![Parametric domains for Hamiltonian (6) in $A - B$ plane. The right upper wedge is, after exemption of the dashed EP line, physical (see the text for explanation).](image)

Figure 1: Parametric domains for Hamiltonian (6) in $A - B$ plane. The right upper wedge is, after exemption of the dashed EP line, physical (see the text for explanation).

Whenever needed, we may return to the old parameters $\alpha = \sqrt{1 - A}$ and $\beta = \sqrt{1 - B}$. Still, it is more rewarding to work in the new, more interesting $(A, B)$—parametrization of the model in which the whole $A - B$ plane decays, naturally, into nine subdomains. This is shown in Fig. 1. In the picture the simply connected closure $\mathcal{D} = \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_5 \cup \mathcal{D}_6$ of the real-spectrum domain with $A \geq 0$ and $C \geq 0$ is specified, according to Eq. (8), by the two thick half-lines of its energy-complexification boundaries.

From the dynamical point of view our Hamiltonian (6) is composed of a discrete kinetic-energy Laplacean $T$ and a slightly non-local potential $V$. The latter interaction only vanishes at the single point $A = B = 1$ and it remains nontrivial otherwise. The key observation is that the physical domain $\mathcal{D}$ itself as well as its two subdomains $\mathcal{D}_5$ and $\mathcal{D}_6$ are in fact not the simply connected
domains. In Fig. 1 we see that besides the two thick exceptional-point (i.e., unphysical) boundary half-lines with \( A = 0 \) and \( C = 0 \) one must also exclude the third, dashed thick EP half-line where \( B = 0 \) and \( A \geq 0 \). Before a deeper analysis of this feature, let us first return to the role of the metric \( \Theta \) in this context.

### 3.2 All metrics compatible with the Hamiltonian

Once we insert Hamiltonian (6) in the implicit linear algebraic definition (3) of the real, symmetric and positive definite metric matrix \( \Theta \), we obtain 16 equations for the 16 unknown matrix elements. As long as formula (5) indicates that there are strictly four free real parameters in the family of solutions, let us pick up, say, the quadruplet of elements \( \Theta_{1j} = t_j \) with \( j = 1, 2, 3, 4 \) as playing the role of these free parameters. We may then solve the system by the standard elimination technique, yielding

\[
\Theta_{22} = -\frac{-t_1 + t_1 \beta - t_3 - t_3 \alpha}{1 + \beta},
\]

\[
\Theta_{23} = \frac{t_2 - t_2 \alpha + t_4 + t_4 \beta}{1 + \beta},
\]

\[
\Theta_{24} = -\frac{t_3 (-1 + \beta)}{1 + \beta}
\]

in the second line,

\[
\Theta_{33} = \frac{t_1 - t_1 \alpha - t_1 \beta + t_4 \beta \alpha + t_3 - t_3 \alpha^2}{1 + \beta + \alpha + \alpha \beta} = \frac{(-t_1 + t_1 \beta - t_3 - t_3 \alpha)(-1 + \alpha)}{(1 + \beta)(1 + \alpha)},
\]

\[
\Theta_{34} = \frac{t_2 (1 - \alpha - \beta + \alpha \beta)}{1 + \beta + \alpha + \alpha \beta} = \frac{t_2 (-1 + \alpha)(-1 + \beta)}{(1 + \beta)(1 + \alpha)}
\]

in the third line and

\[
\Theta_{44} = -\frac{t_1 (\alpha \beta^2 - \beta^2 + 2 \beta - 2 \alpha \beta + \alpha - 1)}{\beta^2 + \alpha \beta^2 + 2 \beta + 2 \alpha \beta + 1 + \alpha} = -\frac{t_1 (-1 + \beta)^2 (-1 + \alpha)}{(1 + \beta)^2 (1 + \alpha)}
\]

in the fourth line. We arrived at an exhaustive, complete solution of our problem. Naturally, because of its matrix nature, it is difficult to display the result in print. At the same time, one can easily generate and store the formulae in the computer (e.g., using MAPLE). Even without such an assistance one can choose and analyze a few of the most interesting special cases of the general metric \( \Theta \).
4 Discussion

4.1 Characteristics of subdomains in $A - B$ plane

Due to the extreme simplicity of our $N = 4$ model in the $A - B$ parametrization of Fig. 1 we may immediately conclude that

- our (possibly, complex) Hamiltonian is Hermitian (in the usual matrix sense, i.e., in the usual complex vector space $\mathcal{H}^{(F)} \equiv \mathbb{C}^N$ endowed with trivial metric $\Theta^{(F)} = I$) strictly in subdomain $\mathcal{D}_3$;
- our Hamiltonian matrix remains crypto-Hermitian (i.e., Hermitizable, in $\mathcal{H}^{(S)}$, via a suitable \textit{ad hoc} metric $\Theta$) and, at the same time, real strictly in the doubly connected subdomain of parameters $\mathcal{D}_5$;
- the physical Hilbert space $\mathcal{H}^{(S)}$ and metric $\Theta$ cannot exist at any inner-coupling parameter $A < 0$ (i.e., in none of the irreparably non-Hermitian subdomains $\mathcal{D}_1, \mathcal{D}_4$ and $\mathcal{D}_7$ – one even has $\text{Re} \, E_n = 0$ in subdomain $\mathcal{D}_7$) nor at any coupling-mixing parameter $C < 0$ (i.e., in none of the remaining two irreparably non-Hermitian subdomains $\mathcal{D}_8$ and $\mathcal{D}_9$).

Curiously enough, the negativity of outer-coupling $B < 0$ need not suffice for the suppression of the crypto-Hermiticity of the Hamiltonian. In other words, the unphysical subdomains $\mathcal{D}_8$ and $\mathcal{D}_9$ lying below the $C = 0$ line are smaller than expected. Thus, there seems to exist an important difference between the two EP half-lines with $A = 0, C = 0$ and the third EP half-line with $B = 0$. Let us now test this expectation.

4.2 Two alternative unfoldings of the EP degeneracies

Quantitatively, some of the above-listed properties of our $N = 4$ model are particularly easily proved. In particular, along the $B = 0$ EP line and at $A > 0$ we arrive at a degenerate form of our Hamiltonian,

$$
H = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-2 & 0 & -1 + \alpha & 0 \\
0 & -1 - \alpha & 0 & 0 \\
0 & 0 & -2 & 0
\end{bmatrix}.
$$

(9)

It possesses two vanishing eigenvalues $E = 0$ but just a single related eigenvector. This means that matrix (9) is non-diagonalizable. Its Jordan-block canonical structure cannot be Hermitized by any metric $\Theta$, of course.

The $B \neq 0$ “unavoided crossing” unfolding of this $B = 0$ dashed-line EP degeneracy in domain $\mathcal{D}_5$ is illustrated in Fig. 2. This picture may be complemented by the closed-form calculations
Figure 2: The $A = 0.02$ sample of the spectrum of Hamiltonian (6). Exhibiting, in domain $\mathcal{D}_5$, the unfolding of the $B = 0$ EP degeneracy (= the “unavoided crossing” of two levels) as well as the unfolding of the other, $C = 0$ EP degeneracy (leading to the complexification of all of the four real energy levels in domain $\mathcal{D}_8$, i.e., below $B = -0.005$).

starting from the perturbed Hamiltonian

$$H = \begin{bmatrix}
0 & -\gamma & 0 & 0 \\
-2 + \gamma & 0 & -1 + \alpha & 0 \\
0 & -1 - \alpha & 0 & -\gamma \\
0 & 0 & -2 + \gamma & 0
\end{bmatrix}$$

with a small shift $\gamma$ in $\beta = 1 - \gamma$ (yielding $B = 2\gamma + \mathcal{O}(\gamma^2)$) and with the closed-form almost-vanishing eigenvalues

$$\pm 2E_{\pm,-} = \sqrt{2 - 2\alpha^2 + 8\gamma - 4\gamma^2 - 2\sqrt{\alpha^4 - 8\alpha^2\gamma + 4\alpha^2\gamma^2 - 2\alpha^2 - 4}\gamma + 8\gamma + 1}. \quad (10)$$

One quickly obtains the proof of the crossing phenomenon using Taylor-series expansion,

$$E_{\pm,-} \approx \pm \left(2 + \alpha^2 + 3/4 \alpha^4 + \ldots\right)\gamma \mp \left(5 + 13/2 \alpha^2 + \ldots\right)\gamma^2 \pm \ldots.$$ 

The real perturbations $\gamma$ of both signs yield, as they should, the pair of the real energies inside both the simply connected components of the physical, S-space-Hermiticity domain $\mathcal{D}_5$.

In comparison, there is no surprise in the unfolding of the pair of the double $A = 0$ EP degeneracies (cf. their $B = 1/50$ sample in Fig.3) nor in the unfolding of the quadruple $A = B = 0$ EP degeneracy as sampled, along the line of $B = A$, in Fig.4.
4.3 Phase transition and the change of physics at the $B = 0$ avoided crossings

In the general case, one must verify the positivity of the metric matrix at a given quadruplet $t_j$ of the free parameters. In our specific model, the task may be simplified, say, for the specific choice of $t_2 = t_3 = t_4 = 0$ for which our metric matrix becomes strictly diagonal with, say,

$$
\Theta_{11} = \frac{(1 + \alpha)(1 + \beta)}{1 - \beta}, \quad \Theta_{22} = 1 + \alpha, \quad \Theta_{33} = 1 - \alpha, \quad \Theta_{44} = \frac{(1 - \alpha)(1 - \beta)}{1 + \beta}
$$

under an optimal, symmetrized-normalization choice of $t_1 \neq 0$.

From the latter formula we may see that our special, diagonal metric remains positive definite for all of the real parameters such that $|\alpha| < 1$ and $|\beta| < 1$. The latter condition has an extremely
unexpected consequence. Below the EP line $B = 0$ but still inside the physical domain $\mathcal{D}_5$ of Fig. 1, we may put $\beta = 1 + \gamma^2$ (where $\gamma$ is small but real) and check that our diagonal matrix $\Theta$ loses the status of metric. It becomes converted into the mere indefinite diagonal pseudometric $\mathcal{P}$ which possesses two negative elements and/or eigenvalues,

$$
\mathcal{P}_{11} = -\frac{(1 + \alpha)(2 + \gamma^2)}{\gamma^2}, \quad \mathcal{P}_{22} = 1 + \alpha, \quad \mathcal{P}_{33} = 1 - \alpha, \quad \mathcal{P}_{44} = -\gamma^2 \frac{1 - \alpha}{2 + \gamma^2}.
$$

As long as this observation is based on the mere above-given complete solution of the hidden Hermiticity constraint (3), we must conclude that below the EP line $B = 0$, the correct physical metric must necessarily be chosen as non-diagonal, i.e., different.

5 Summary

In the context of quantum theory the later conclusion is truly far-reaching. Indeed, let us recall the fact that the physical contents of any quantum model is always determined by our specification of the pair of the operators $H$ and $\Theta$. Naturally, one must refer to the existence and/or to the (implicit or explicit) specification of all of the other physical operators of observables. Only the knowledge of these operators makes our metric formally unique, in principle at least [8]. In opposite direction, only the knowledge and uniqueness of the metric gives us the opportunity to deduce the probabilistic predictions from our theory.

In summary, even inside the “formally physical” domain $\mathcal{D}_5$ of Fig. 1, the explicit physical contents and meaning of our quantum system become different below and above the $B = 0$ EP line. In another formulation, once we decide to select and endow the physical Hilbert space $\mathcal{H}^{(S)}$ with the simplest, diagonal, $t_1 \neq 0$ metric $\Theta$ above the EP line $B = 0$, we are forced to change the metric (i.e., the Hilbert space). In the language of physics, the smooth crossing of the energies will necessarily be accompanied, at $B = 0$ and at any $A \in (0,1)$, by a sudden loss of the predictability of at least one other measurement. We must accept that the theories before and after the energy crossing remain fully and completely independent. In spite of the smoothness of the change of the energies one must speak about phase transition at $B = 0$.

Acknowledgments

Research supported by the GAČR grant Nr. P203/11/1433.
References

[1] S. Fernandez-Alberti, A. E. Roitberg, T. Nelson and S. Tretiak, J. Chem. Phys. 137 (2012) 014512.

[2] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[3] M. Znojil, Phys. Lett. A 259 (1999) 220.

[4] H. Langer and Ch. Tretter, Czech. J. Phys. 70 (2004) 1113;
   P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Theor. 40 (2007) R205;
   A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191;
   D. Krejčiřík and P. Siegl, J. Phys. A: Math. Theor. 43 (2010) 485204;
   F. Bagarello and M. Znojil, J. Phys. A: Math. Theor. 44 (2011) 415305.

[5] T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin, 1966.

[6] D. Krejčiřík, H. Bíla and M. Znojil, J. Phys. A 39 (2006) 10143.

[7] P. Siegl, Non-Hermitian quantum models, indecomposable representations and coherent states quantization, Univ. Paris Diderot, Paris, 2011 (PhD thesis);
   D. Krejčiřík, P. Siegl and J. Železný, Complex Anal. Oper. Theory, in print (doi 10.1007/s11785-013-0301-y).

[8] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[9] R. F. Bishop, Theor. Chim. Acta 80 (1991) 95.

[10] M. Znojil, SIGMA 5 (2009) 001 (arXiv overlay: 0901.0700).

[11] A. V. Smilga, J. Phys. A: Math. Theor. 41 (2008) 244026.

[12] M. Znojil and H. B. Geyer, Fort. d. Physik 61 (2013) 111.

[13] M. Znojil, SIGMA 4 (2008) 001 (arXiv overlay: 0710.4432).

[14] M. Znojil and J. Wu, Int. J. Theor. Phys. 52 (2013) 2152.