GRAY CATEGORIES WITH DUALS AND THEIR DIAGRAMS

JOHN W. BARRETT, CATHERINE MEUSBURGER, AND GREGOR SCHAUMANN

Abstract. The geometric and algebraic properties of Gray categories with duals are investigated by means of a diagrammatic calculus. The diagrams are three-dimensional stratifications of a cube, with regions, surfaces, lines and vertices labelled by Gray category data. These can be viewed as a generalisation of ribbon diagrams. The Gray categories present two types of duals, which are extended to functors of strict tricategories with natural isomorphisms, and correspond directly to symmetries of the diagrams. It is shown that these functors can be strictified so that the symmetries of a cube are realised exactly. A new condition on Gray categories with duals called the spatial condition is defined. We exhibit a class of diagrams for which the evaluation for spatial Gray categories is invariant under homeomorphisms. This relation between the geometry of the diagrams and structures in the Gray categories proves useful in computations and has potential applications in topological quantum field theory.

1. Introduction

The aim of this paper is to develop the theory of duals for Gray categories. The principal tool is a diagrammatic calculus introduced in this paper. This can be viewed as a higher-categorical, three-dimensional analogue of the diagrams used for computations in pivotal categories. Many of the algebraic results on Gray categories with duals can be understood in terms of the geometry of the corresponding diagrams.

Our main motivation are the applications of higher categories in (extended) topological quantum field theory, quantum geometry and conformal field theory [16, 23]. For instance, one would like to construct topological quantum field theories with ‘defects’. These are theories in which certain embedded submanifolds are labelled with geometric data. In three dimensions, it is natural that the data on these decorated submanifolds should arise from a tricategory. An example of this is the work on the relation between Reshetikhin-Turaev and Turaev-Viro invariants [21, 33]. In this case, the relevant higher category is the center of a spherical category, which is a tricategory with a single object and a single 1-morphism. Another example is the tricategory of bimodule
categories, which plays an important role in conformal field theory, see e.g. [15, 8]. The notion of duals is required to incorporate orientation. If data from a tricategory is used to label distinguished submanifolds, orientation reversal of these submanifolds must be reflected in a corresponding structure in the tricategory, namely the duals.

Another important motivation is the benefit of diagrams for computations in higher categories that arises from a direct relation between geometry and structures in the category. That the diagrams have a non-trivial geometric content is familiar from the example of ribbon categories and knots, or more generally, ribbon graphs embedded in three-dimensional space. The Reshetikhin-Turaev invariants [27] define a functor which takes ribbon graphs in three-dimensional space decorated with data from a ribbon category and evaluates them in the category itself. The relations in the category imply invariance of the evaluation under homeomorphisms of three-dimensional space. In this way the homeomorphism invariance is the geometric expression of the relations in the category.

This article does not consider general tricategories, but restricts attention to Gray categories. As every tricategory is triequivalent to a Gray category [9, 11] and there is no stricter version of a tricategory with this property, Gray categories can be viewed as maximally strict tricategories. The practical reason for using Gray categories is the wish to avoid a degree of complexity that makes algebraic manipulations nearly impossible. The deeper reason is that the coherence data for Gray categories is precisely that part of the coherence data for tricategories that can be given a diagrammatic meaning. An analogous situation arises for pivotal tensor categories. There is a weak and a strict notion of monoidal structure and of duality and the diagrams for pivotal tensor categories reflect precisely the coherence data for a strict pivotal category with strict monoidal structure, while the rest of its coherence data is not given a diagrammatic representation.

Section 2 of the paper introduces diagrams for Gray categories without duals. This is a generalisation of the diagrammatic calculus for braided monoidal categories introduced by Joyal and Street [12]. The diagrams for Gray categories are located in a cube with the three coordinate axes corresponding to the three compositions in a Gray category. These diagrams consist of 3-, 2-, 1-, and 0-dimensional strata which are labelled, respectively, with objects and 1-, 2-, 3-morphisms in a Gray category. The categorical axioms are introduced in an ‘unpacked’ manner. We see this concreteness as an advantage, both in view of possible applications in state sum models and because this yields a direct link between categories and geometry.

Section 3 introduces Gray categories with duals using the definition of Baez and Langford [1] but with some minor modifications. The diagrammatic representation of the data for these duals is explained
in this section. The Gray categories possess two types of duals, * and #, which correspond, in a sense explained in this paper, to 180 degree rotations around two different coordinate axes. The * duals are familiar from pivotal or ribbon categories but the # duals are a feature of Gray categories that does not appear in the pivotal or ribbon cases. The coherence data for # is such that it matches the appearance of folds and cusps in projections of surfaces. As in the case of Gray categories, the duals considered in this paper are not the most general ones, and their axioms could be weakened. Again, the strictness of the axioms ensures that all their coherence data has a diagrammatic representation.

Our first central result in Section 4 concerns the algebraic structure of these duality operations.

**Theorem 1.1.** The duals extend in a canonical way to (partially contravariant) functors of strict tricategories *, # : G → G with ** = 1 and define natural isomorphisms Γ : *# → 1, Ʌ : ## → 1.

The structure maps for these natural isomorphisms are interpreted geometrically in terms of diagrams. By investigating a closely-related natural isomorphism Δ : # → ** we obtain two diagrams that are homeomorphic, but whose evaluations are not necessarily equal. This motivates the definition of a spatial Gray category as a Gray category with duals in which such an equality holds. This condition is a generalisation of the ribbon condition for a ribbon category.

Section 5 contains the second important result: a strictification theorem for the duals. While the functor of strict tricategories * : G → G satisfies the identity ** = 1, which corresponds to its geometrical interpretation, the functor # : G → G satisfies such an identity only up to a natural isomorphism. Similarly, the functor ** # whose geometrical counterpart is the identity rotation of \( \mathbb{R}^3 \), is not equal to the identity functor but only isomorphic to it. However, a spatial Gray category can be strictified (in the sense of [11]) to one in which these duality functors do indeed satisfy the relations for 180 degree rotations around different coordinate axes:

**Theorem 1.2.** Every spatial Gray category with duals can be strictified to a Gray category whose duals * , # : G → G satisfy ** = 1, ## = 1 and ** # = 1.

Thus the geometrical interpretation of the action of # as a rotation is restored for higher morphisms, which justifies the original set of duality axioms. The proof for this result is conceptually clear and may be of independent interest.

Section 6 explores in more depth the relation between Gray categories with duals and their diagrams. The diagrams in this paper have no framing. This is adequate to express all of the axioms for the category and also the structure maps for the duality functors. However it
does restrict the generality of the invariance results. Diagrams are labelled with category data using a generalisation of ‘blackboard framing’ familiar from knot theory. These diagrams are called standard. The general invariance result in this section holds for a large class of Gray category diagrams, called surface diagrams, whose 0-, 1- and 2-strata form a surface with a boundary.

**Theorem 1.3.** Let $D$ and $D'$ be standard surface diagrams that are labelled with a spatial Gray category. Let $f: D \to D'$ be an oriented isomorphism of standard surface diagrams and the labels of $D'$ induced from $D$ by $f$. Then the evaluations of $D$ and $D'$ are equal.

There is a caveat with this result. It depends on the conjecture that the moves on folds and cusps in the PL setting are the analogues of the moves in the smooth case. While this appears to be a reasonable conjecture we do not know of any previous work on this problem.

For surface diagrams, the result indicates that there are no further conditions other than the spatial condition on a Gray category with duals that are needed to prove invariance under homeomorphisms. Essentially it arises because surface diagrams have a uniquely determined notion of framing. It seems that to extends this result to all Gray category diagrams would require a general definition of framing; whilst this is an interesting problem we leave it for future work.

This relation between the geometry of the diagrams and the algebraic structures in a Gray category with duals is particularly apparent in the Gray category diagrams for closed oriented surfaces. In Section 7, these diagrams are related to a notion of trace in a Gray category with duals, which generalises the traces considered in [24] and defines invariants of oriented surfaces labelled with data from a Gray category with duals. These invariants allow one to compute the Euler characteristic of a surface.

In Section 8 we investigate examples of Gray categories with duals that arise from the Gray category $2\text{Cat}$. We analyse the structure of subcategories of $2\text{Cat}$ that carry the structure of a Gray category with duals and construct two concrete examples based on such subcategories.
2. Category, 2-category and Gray category diagrams

The aim of this section is to develop a diagrammatic calculus for Gray categories and to show that the evaluation of diagrams labelled with Gray category data is invariant under certain mappings of diagrams. This can be viewed as a higher-dimensional analogue of the diagrams that are called spin network diagrams in the physics literature, string diagrams in the mathematics literature and tangle diagrams in knot theory. They are dual to the more common pasting diagrams of the category theory literature.

The diagrams for an $n$-category are located in a geometrical ‘cube’ $[0,1]^n$. It should be possible to define diagrams analogous to the ones considered here for arbitrary dimension $n$. However, it is most practical to be guided by known examples rather than abstract formalism. Hence this work considers the cases only up to $n = 3$. The evaluation of an $n$-dimensional diagram labelled with $n$-category data will be defined inductively, in terms of a projection to an $(n-1)$-dimensional diagram labelled with $(n-1)$-category data.

For this reason, the definition of Gray category diagrams and their mappings, which corresponds to $n = 3$ requires a careful discussion.
of their lower-dimensional counterparts. We start with a discussion of one-dimensional diagrams, then we treat the two-dimensional case before introducing diagrams for Gray categories, which correspond to $n = 3$. At each stage, we discuss the $n$-dimensional diagrams, their mappings as well as their labelling with data from an $n$-category and their evaluation. Throughout the article, we work in the piecewise-linear framework, although diagrams are often drawn smoothly for better legibility.

2.1. Categories and diagrams. We start by considering the one-dimensional case, which is required for the definitions in higher dimensions, but also can be viewed as a toy-model that exhibits the general features of the construction.

**Definition 2.1** (One-dimensional diagrams). A one-dimensional diagram is a finite set of points, called vertices, in the interior of the unit interval $[0, 1]$. The complement of the vertices is a disjoint union of its connected components, which are called regions of the diagram.

As in higher dimensions, one-dimensional diagrams are a purely topological construction. They become category diagrams once decorated with data from a category $C$. A category diagram with a single vertex is called an elementary diagram. A general category diagram can then be defined in terms of the elementary diagrams.

**Definition 2.2** (Category diagrams). Let $C$ be a category. 

1. An elementary category diagram for $C$ is a one-dimensional diagram with one vertex together with a morphism $f: A \to B$ in $C$. The object $A$ is associated with the region containing 0, $B$ with the region containing 1, and $f$ with the vertex.

2. A category diagram for $C$ is a one-dimensional diagram together with a labelling of each region with an object in $C$ and a labelling of each vertex with a morphism in $C$. Every vertex is required to have a neighbourhood that is isomorphic to an elementary diagram. This means that the morphism at a vertex $v \in [0, 1]$ is a morphism from the object labelling the region ‘above’ $v$ (i.e. at values less than $v$) to the region below $v$.

The benefit of category diagrams and their higher-dimensional counterparts is that they allow one to visualise a calculation in the category $C$. This calculation is called the evaluation of the diagram.

**Definition 2.3** (Category diagram evaluation). The evaluation of a category diagram is the product of the morphisms at the vertices in the order of increasing values of $v$. It maps the object for the region containing 0 to the object for the region containing 1 (see Figure 1). A diagram without vertices is called an identity diagram and its evaluation is the identity on the single object. The evaluation is unique by associativity and is invariant under isotopies of $[0, 1]$. 

6
A category diagram and its evaluation are shown in Figure 1. The usefulness of diagrams for visualising calculations in (higher) categories is due to the fact that their evaluation is invariant under certain manipulations of diagrams such as homeomorphisms of diagrams and subdivisions. A precise formulation of this idea requires the notion of a mapping of diagrams.

**Definition 2.4** (Mapping of one-dimensional diagrams).

1. A mapping of one-dimensional diagrams $D \rightarrow D'$ is a PL embedding $m: [0, 1] \rightarrow [0, 1]$ such that $m(v)$ is a vertex of $D'$ for each vertex $v$ of $D$.
2. If the mapping has the property that $v$ is a vertex if and only if $m(v)$ is, then it is called a subdiagram.
3. A mapping $m: D \rightarrow D'$ of one-dimensional diagrams is called a homomorphism of diagrams if $m(0) = 0$ and $m(1) = 1$ and an isomorphism if $m^{-1}$ is also a mapping.
4. If $m : [0, 1] \rightarrow [0, 1]$ is the identity map, then the mapping is called a subdivision of $D$.

Mappings, homomorphisms, isomorphisms, subdiagrams and subdivisions of one-dimensional diagrams are shown in Figure 2. Note that if $m : D \rightarrow D'$ is a mapping, a homomorphism, a subdiagram or a subdivision, the image $D'$ can have more vertices than $D$. In the case of an isomorphism, there is a bijection between the vertices of $D$ and of $D'$. As the map $m : [0, 1] \rightarrow [0, 1]$ is an embedding, every homomorphism of diagrams is a homeomorphism and can be decomposed into an isomorphism of diagrams and a subdivision.
The concept of a mapping can be extended to mappings of diagrams that are decorated with data from a category. This amounts to imposing certain relations between the label of a point in the diagram and the label of its image.

**Definition 2.5** (Mapping of category diagrams). A mapping of category diagrams $D, D'$ is a mapping of one-dimensional diagrams $m : D \rightarrow D'$ that is orientation-preserving ($m(0) < m(1)$) and preserves the labelling:

1. If $x \in D$ and $m(x) \in D'$ are both vertices or both points in a region, then their labels are equal.
2. If $x \in D$ is a point in a region of $D$ labelled by $A$ and $m(x)$ is a vertex of $D'$, then $m(x)$ is labelled with the morphism $1_A : A \rightarrow A$.

A mapping of category diagrams is called a homomorphism, isomorphism, subdivision or subdiagram if the underlying mapping $m : D \rightarrow D'$ of one-dimensional diagrams is.

It is clear that if $D$ and $D'$ are category diagrams and $m : D \rightarrow D'$ a homomorphism of category diagrams, then the evaluations of $D$ and $D'$ are equal. It is also easy to see that, given a category $C$, the set of all category diagrams labelled by $C$, considered up to isomorphism, is the free category generated by $C$. 
2.2. 2-category diagrams. This section extends the notions of labelled diagrams, their evaluation and their mappings to two dimensions. In this case, the diagrams are labelled with data from a bicategory or a 2-category. As bicategories will not be required for the discussion of Gray category diagrams, we restrict attention to diagrams labelled with 2-categories and related data.

**Definition 2.6 (2-Category, [30]).** A 2-category \( C \) has objects and for each pair of objects \( A, B \) a category \( C(A, B) \), whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. The composition of morphisms in \( C(A, B) \) is denoted \( \cdot \) and called vertical composition. For each triple of objects \( A, B, C \), there is a horizontal composition functor \( \circ : C(B, C) \times C(A, B) \to C(A, C) \), which is strictly associative and unital. This implies that the objects and 1-morphisms form a category \( C_1 \). The unit 1-morphism for an object \( A \) is denoted \( 1_A \) and the unit 2-morphism for a 1-morphism \( f \) is denoted \( 1_f \).

It is possible to define a 2-category without using the horizontal composition of two 2-morphisms; all that is required is the horizontal composition of a 2-morphism with a 1-morphism. The notion of a 2-category can then be generalised by regarding the horizontal composition of two 2-morphisms as undefined and dropping the interchange law. This will be called a pre-2-category (it is also called a sesqui-category in [30, 5]), and will be useful for the discussion of Gray categories below. The definition of a pre-2-category is given next, followed by the interchange law. Taken together, these give an explicit definition of a 2-category.

**Definition 2.7 (Pre-2-category).** A pre-category consists of a set of objects, and for each pair of objects \( A, B \) a category \( C(A, B) \). There is a horizontal composition of 1-morphisms \( \circ : C_1(B, C) \times C_1(A, B) \to C_1(A, C) \) with units that makes the objects and 1-morphisms into a category \( C_1 \) and extends to a left action of \( C_1(B, C) \) on \( C(A, B) \) and a right action of \( C_1(A, B) \) on \( C(B, C) \) by functors. These actions are required to be unital and associative and to commute with each other.

The notation is the same as before: if \( f : A \to B \) is a 1-morphism, \( \Psi \in C(B, C) \) a 2-morphism and \( n : C \to D \) a 1-morphism, then \( \Psi \circ f \in C(A, C) \) and \( n \circ \Phi \in C(B, D) \) denote the horizontal composites. In a pre-2-category there are two possible definitions for the horizontal composite of two 2-morphisms. For 1-morphisms \( f, g : A \to B, h, k : B \to C \) and 2-morphisms \( \Phi \in C(f, g) \), \( \Psi \in C(h, k) \) these are

\[
(k \circ \Phi) \cdot (\Psi \circ f) \quad \text{and} \quad (\Psi \circ g) \cdot (h \circ \Phi).
\]
The interchange law in a 2-category states that these 2-morphisms are equal and define $\Psi \circ \Phi$. Thus a 2-category can be viewed as a pre-2-category with an interchange law.

Two-dimensional diagrams are a direct generalisation of one-dimensional diagrams. The unit interval $[0, 1]$ is replaced by the unit square $[0, 1]^2$, and the only additional condition is that lines meet the boundary of the square only at its top and bottom edge.

**Definition 2.8 (Two-dimensional diagrams).**

1. A two-dimensional diagram is a set of closed subspaces $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset X^2 = [0, 1]^2$ such that each $k$-skeleton $X^k \setminus X^{k-1}$ is a PL-manifold of dimension $k$, for $k = 0, 1, 2$ and all intersection points of $X^1 \setminus X^0$ with the boundary of the square are contained in $[0, 1] \times \{0, 1\}$. The connected components of $X^2 \setminus X^1$ are called regions, the connected components of $X^1 \setminus X^0$ lines, and the elements of $X^0$ vertices.

2. A two-dimensional diagram is called progressive if the projection of the diagram $p_1: (x, y) \mapsto y$ is regular, i.e. the mapping of each line is an isomorphism to its image in $\mathbb{R}$.

3. A progressive two-dimensional diagram is called generic if the $y$-coordinates of any two different vertices are different.

Mappings between two-dimensional diagrams are defined in direct analogy to the one-dimensional case; they are PL-embeddings that map $k$-skeleta to $k$-skeleta.

**Definition 2.9 (Mappings of two-dimensional diagrams).**

1. A mapping of two-dimensional diagrams $D \to D'$ is a PL-embedding $m: [0, 1]^2 \to [0, 1]^2$ that preserves the $k$-skeleta, i.e. $m(X^k) \subset X'^k$ for $k = 0, 1$.

2. If $m: [0, 1]^2 \to [0, 1]^2$ is a PL-homeomorphism that is the identity map on $\partial[0, 1]^2$, then $m: D \to D'$ is called a homomorphism of diagrams.

3. Isomorphisms, subdivisions and subdiagrams are defined analogously to the one-dimensional case (Definition 2.4).

Let $m: [0, 1]^2 \to [0, 1]^2$ an arbitrary PL-homeomorphism that is the identity on the boundary. Then a diagram $D'$ can be defined by applying $m$ to any diagram $D$. This gives $m$ the structure of a homomorphism of diagrams. Further examples can be constructed by subdividing $D'$, i.e. adding additional lines or vertices.
Progressive diagrams are the diagrams that are appropriate for 2-categories without any further structure, so these are the diagrams considered in the rest of this section. They were first studied in the context of monoidal categories by Joyal and Street [12] and have an important local structure: any vertex \( v = (x, y) \) in a progressive diagram has a rectangular neighbourhood \( [x - \varepsilon_1, x + \varepsilon_1] \times [y - \varepsilon_2, y + \varepsilon_2] \) which is, after a rescaling of the coordinates, a subdiagram that is an ‘elementary’ two-dimensional diagram with one vertex. For this, one chooses a sufficiently small neighbourhood of \( v \) in which all line segments are linear and \( \varepsilon_2 \) sufficiently small so that all line segments exit the rectangle through its top or bottom edge. The concept of a generic diagram is analogous to the notion of a generic projection of a knot or link in knot theory.

In analogy to the one-dimensional case, two-dimensional diagrams can be labelled with data from a 2-category. The definition of the labelling does not use the interchange law of a 2-category and hence can be stated in the more general context of a pre-2-category. This will be used in the definition of Gray category diagrams. Where (pre)-2-category appears, both cases are considered at once, the 2-category case being the one with ‘pre-’ deleted everywhere. As in the one-dimensional case, the labelling of a two-dimensional diagram is defined in terms of elementary diagrams.

**Definition 2.10 (2-Category diagrams).**

1. An elementary (pre)-2-category diagram is a progressive two-dimensional diagram with exactly one vertex, which meets every line in the diagram, together with:
   - a labelling of each region with an object in \( \mathcal{C} \),
   - a labelling of each line with a 1-morphism,
   - a labelling of the vertex with a 2-morphism.
   The top edge \([0, 1] \times \{0\}\) and the bottom edge \([0, 1] \times \{1\}\) of the diagram are required to be category diagrams for \( \mathcal{C}_1 \) and evaluate to 1-morphisms which are, respectively, the source and the target for the 2-morphism at the vertex.

2. A (pre)-2-category diagram for a (pre)-2-category \( \mathcal{C} \) is a progressive two-dimensional diagram together with a labelling of each region with an object in \( \mathcal{C} \), a labelling of each line with a 1-morphism and a labelling of each vertex with a 2-morphism. The top and bottom edges are required to be category diagrams for \( \mathcal{C}_1 \), and each vertex \( v \) is required to have a neighbourhood that is isomorphic to an elementary (pre)-2-category diagram.

An example of a (pre-)2-category diagram is shown in Figure 3. The requirement that vertices are locally isomorphic to an elementary vertex enforces the condition that the source and targets of 1- and
Figure 3. 2-Category diagram together with its projection onto the $y$-axis.

2-morphisms match. Important examples are the identity diagrams which have a number of vertical lines and no vertices. More precisely, an identity diagram is a diagram of the form $1_D = D \times [0, 1]$, where $D$ is a category diagram for $C_1$. The regions and lines of $1_D$ correspond to the regions and vertices of $D$.

Next we define the evaluation of a (pre-)2-category diagram. The evaluation of a diagram is simplest to give for the special case of a generic diagram (see Definition 2.8), so this case is treated first. The evaluation of a generic (pre)-2-category diagram consists of two steps. The first is to project the (pre-)2-category diagram to a category diagram via the projection map $p_1 : (x, y) \rightarrow y$. The second step is the evaluation of the resulting category diagram.

The category diagram $p_1 D$ is obtained as follows. Consider a generic (pre-)2-category diagram $D$ as in Figure 3 whose left-hand edge $\{0\} \times [0, 1]$ is labelled with an object $A$, and whose right-hand edge $\{1\} \times [0, 1]$ is labelled with an object $B$ in a (pre-)2-category $C$. Then the projection $p_1 D$ is labelled with data from the category $C(A, B)$. The labelling of a point $y \in [0, 1]$ depends on whether $p^{-1}(y)$ contains a vertex:

(1) If $p^{-1}(y)$ does not contain a vertex of $D$, then $y$ lies in a region of the category diagram $p_1 D$ and this is labelled with the horizontal composite of the 1-morphisms in $p_1^{-1}(y)$, composed as shown in Figure 3.

(2) If $p^{-1}(y)$ contains a vertex, then $y$ is a vertex of the category diagram $p_1 D$ and is labelled with the horizontal composite of the 1-morphisms and the single 2-morphism in $p_1^{-1}(y)$ as shown in Figure 3.
The evaluation of the category diagram $p_1D$ according to Definition 2.3 is a morphism in $C(A, B)$ and hence a 2-morphism in $C$. This defines the evaluation of a generic (pre-)2-category diagram.

**Definition 2.11** (Generic (pre)-2-category diagram evaluation). The evaluation of a generic (pre)-2-category diagram $D$ labelled with data from a (pre-)2-category $C$ is the 2-morphism in $C$ defined by the evaluation of the category diagram $p_1D$.

As in the case of tangle diagrams, there are two products for diagrams labelled with data from a (pre-)2-category $C$. Vertical composition is defined if the bottom edge of a diagram $D$, with its labelling, matches the top edge of another diagram $D'$ and consists of drawing one diagram above the other (along the $y$-axis). The evaluation of the composite diagram is then given by the vertical composite of the evaluations of $D$ and $D'$.

Horizontal composition consists of juxtaposing two diagrams along the $x$-axis and is defined only if the object on the left-hand side of one diagram matches the object on the right-hand side of the other. In the case of a pre-2-category, this product is defined only in the cases where one of the two diagrams is an identity diagram.

By analogy to the category of tangles, it seems plausible to expect that by taking a suitable quotient by isotopies it would be possible to make the diagrams into a (pre)-2-category. However we do not develop this idea here.

For a 2-category $C$, the 2-category diagrams are dual to the usual pasting diagrams considered in category theory. However the former contain more information than the latter, namely the values of the $y$-coordinate. It is therefore important to note that the evaluation is in fact independent of these values, which is a consequence of the interchange law. Formulating this precisely requires an appropriate notion of mappings between generic progressive diagrams and a proof that the evaluation of the diagrams is invariant under these mappings.

The homomorphisms in Definition 2.9 are too general to give a meaningful notion of mappings between generic progressive diagrams and to preserve their evaluation. For instance, there are examples of isomorphisms that change the order of the lines incident at a vertex. The appropriate notion of mappings for generic progressive diagrams was determined by Joyal and Street [12, Theorem 1.2] for the case of monoidal categories, which are 2-categories with a single object. The relevant mappings are the ones that are determined by a PL-isotopy from the identity mapping of the diagram. As the action of an isotopy on a 2-category diagram preserves all labels, this result has a direct generalisation to the context of 2-category diagrams.

**Theorem 2.12.** The evaluation of a generic 2-category diagram is invariant under a piecewise-linear isotopy that starts at the identity.
mapping and is a one-parameter family of isomorphisms of 2-category diagrams.

Proof. The important point in the statement of the theorem is that at every stage in the isotopy the diagram is progressive. Therefore the category diagram obtained by projection with \( p_1 \) changes only by an isotopy of \([0,1]\) if the order of the \(y\)-coordinates of the vertices does not change. In this case it follows that the evaluation is invariant under the isotopy.

If the isotopy does change the order of the \(y\)-coordinates of the vertices, then the isotopy can be perturbed slightly so that they change one at a time. This can be done by composing the isotopy with suitable isotopies of square neighbourhoods of each vertex. The invariance of the evaluation under an isotopy that changes the order of two neighbouring vertices then follows from the interchange law (see Figure 4).

By means of Theorem 2.12 it is possible to extend the definition of 2-category diagrams to non-generic diagrams by dropping the requirement that distinct vertices have different \(y\)-coordinates. This cannot be done for pre-2-category diagrams.

Figure 4. The interchange law: associated 2-category diagrams with associated projections.
Definition 2.13 (2-category diagram evaluation). The evaluation of a 2-category diagram is defined by perturbing it by an isotopy to a generic 2-category diagram and evaluating the resulting diagram. The result is independent of the choice of isotopy by Theorem 2.12.

In fact it is easy to see that the definition could also be extended by allowing the product of more than one 2-morphism in the projection in Definition 2.11 and this would give the same result. For pre-2-category diagrams, there is a result similar to Theorem 2.12 but where the isotopy is required to preserve the ordering of the vertices by the \( y \)-coordinate. This ensures that the interchange law is not required.

2.3. Gray categories. The three-dimensional categories considered in this article are Gray categories, the principal example being \( \text{2Cat} \). Although tricategories present a more general notion of a three-dimensional category, Gray categories have the advantage that their coherence data is stricter than that of a general tricategory and, consequently, the constructions are less involved.

Note that the focus on Gray categories is only a minor restriction on the generality of the constructions, since every tricategory \( T \) is triequivalent to a Gray category \( G \) [9, 11].

The standard definition of a Gray category [9] is a category enriched over the monoidal category \( \text{Gray} \), which is constructed using the Gray tensor product. This one-sentence definition, which is not given precisely here, is the same as a ‘strict cubical tricategory’, as shown by Gordon, Power and Street [9], see also [11].

In this work, we use a different convention, namely that of a strict opcubical tricategory, which is summarised in Definition A.5. It is worth noting that in general there are some differences between the definition of tricategory in [9] and ‘algebraic tricategory’ in [11]. However, for strict (op)cubical tricategories, and also their functors and transformations of functors, these definitions coincide. The definitions of functors and transformations for strict tricategories are summarised in Appendix A.

These conceptual definitions take some work to unpack; this is done in [5] and is summarised here, with some change in notation.

Definition 2.14 (Gray category data). A Gray category \( G \) has a set of objects, and for any pair of objects \( C, D \), a 2-category \( G(C, D) \) of 1-, 2- and 3-morphisms. In this 2-category, the notation is as defined previously: \( \circ \) for the horizontal composition and \( \cdot \) for the vertical composition.

The additional data is the Gray product \( \Box \) and the ‘tensorator’. The Gray product defines a product \( G \Box F \) of 1-morphisms \( F : C \to D \) and \( G : D \to E \), which extends to a product \( \Phi \Box F \) of a 1-morphism with a 2- or 3-morphism \( \Phi \in G(D, E) \) and to a product \( G \Box \Psi \) of a
2- or 3-morphism $\Psi \in G(C, D)$ with a 1-morphism $G$. These products are required to determine strict 2-functors, $- \Box F$ and $G \Box -$, and the $\Box$ product must be strictly unital and associative. The former means that each object $C$ has a unit 1-morphism $1_C$ and the 2-functors $- \Box 1_C$ and $1_C \Box -$ are the identity 2-functors. The associativity condition requires that all $\Box$-composable morphisms $P, Q, R$, two of which are 1-morphisms and the third a 1- 2- or 3-morphism, satisfy

\[(P \Box Q) \Box R = P \Box (Q \Box R).\]

The ‘tensorator’ or braiding consists of invertible 3-morphisms $\sigma_{\mu, \nu} : (\mu \Box F_2) \circ (G_1 \Box \nu) \Rightarrow (G_2 \Box \nu) \circ (\mu \Box F_1)$, for all composable 2-morphisms $\nu : F_1 \Rightarrow F_2 \in G(C, D)$ and $\mu : G_1 \Rightarrow G_2 \in G(D, E)$. It must be an identity 3-morphism if either $\mu$ or $\nu$ is an identity 2-morphism

\[\sigma_{\mu, 1_{F_1}} = 1_{\mu \Box F_1}, \quad \sigma_{1_{G_1}, \nu} = 1_{G_1 \Box \nu},\]

and must be natural in both arguments:

\[(3) \quad \sigma_{\mu, \nu'} \cdot ((\mu \Box F_2) \circ (G_1 \Box \nu)) = ((G_2 \Box \Phi) \circ (\mu \Box F_1)) \cdot \sigma_{\mu, \nu},\]

\[(\Psi \Box F_2) \circ (G_1 \Box \nu)) = ((G_2 \Box \nu) \circ (\Psi \Box F_1)) \cdot \sigma_{\mu, \nu}\]

for all 3-morphisms $\Phi : \nu \Rightarrow \nu'$, $\Psi : \mu \Rightarrow \mu'$. It is also required to be compatible with the horizontal composition $\circ$ of 2-morphisms:

\[(4) \quad \sigma_{\mu, \nu \circ \nu'} = ((G_2 \Box \nu') \circ \sigma_{\mu, \nu'}) \cdot ((\mu \Box F_1) \circ (G_1 \Box \nu)),\]

\[(\bar{\mu} \Box F_2) \circ (\sigma_{\mu, \nu}) = \sigma_{\bar{\mu} \circ \mu, \nu} \cdot ((G_2 \Box \nu) \circ (\mu \Box F_1)) \cdot \sigma_{\mu, \nu}\]

for all 2-morphisms $\bar{\nu} : F_2 \Rightarrow F_3$, $\bar{\mu} : G_2 \Rightarrow G_3$. In addition, for all 2-morphisms $\mu$ and $\nu$, and 1-morphisms $F$, the following equations must hold whenever the $\Box$ compositions are defined

\[(5) \quad \sigma_{\mu \Box F, \nu} = \sigma_{\mu, F \Box \nu} \quad \sigma_{F \Box \mu, \nu} = F \Box \sigma_{\mu, \nu} \quad \sigma_{\mu, \nu \Box F} = \sigma_{\mu, \nu} \Box F.\]

Note that Definition 2.14 implies the relations

\[(6) \quad 1_{\nu \Box G} = 1_{\nu \Box} G \quad 1_{F \Box \nu} = F \Box 1_{\nu},\]

for all 1-morphisms $F, G$ and 2-morphisms $\nu$ for which these expressions are defined. Using this definition, it can be checked that the 0- 1- and 2-morphisms of $G$ form a pre-2-category, which is denoted $G_2$. The 0- and 1-morphisms form a category denoted $G_1$. Where it is not ambiguous, the symbol $\Box$ may be omitted, so that the product of $G$ and $F$ may be written as just $GF$.

We will now show that the Gray category data in Definition 2.14 is equivalent to the Definition A.5 of a strict cubical or opcubical tricategory.
**Lemma 2.15.** Every strict cubical or opcubical tricategory determines a set of Gray category data and every set of Gray category data determines a strict cubical and a strict opcubical tricategory.

**Proof.** Let $\mathcal{G}$ be a strict (op)cubical tricategory according to Definition A.5 with composition $\Box : \mathcal{G}(\mathcal{D}, \mathcal{E}) \times \mathcal{G}(\mathcal{C}, \mathcal{D}) \to \mathcal{G}(\mathcal{C}, \mathcal{E})$ and coherence 3-morphisms

$$\square_{\mu, \nu} : (\mu_1 \Box \mu_2) \circ (\nu_1 \Box \nu_2) \to (\mu_1 \circ \nu_1) \Box (\mu_2 \circ \nu_2)$$

for all $\Box$-composable pairs of 2-morphisms $\mu = (\mu_1, \mu_2) : (H_1, H_2) \to (K_1, K_2)$, $\nu = (\nu_1, \nu_2) : (G_1, G_2) \to (H_1, H_2)$. Then $\Box$ defines the Gray product of 1-morphisms with 1-, 2- and 3-morphisms, and the tensorator is given by

$$\sigma_{\nu_1, \nu_2} = \Box^{-1}_{(1_{H_1}, \mu_2), (\nu_1, 1_{H_2})}$$

in case $\mathcal{G}$ is opcubical and by

$$\sigma_{\mu_1, \mu_2} = \Box_{(\mu_1, 1_{H_2}), (1_{H_1}, \nu_2)}$$

in case $\mathcal{G}$ is cubical. A direct computation shows that the axioms of a strict (op)cubical tricategory in Definition A.5 imply that the conditions in Definition 2.14 are satisfied.

Conversely, if $\mathcal{G}$ is a Gray category given by a set of Gray category data, then one obtains a strict opcubical tricategory by promoting the left-hand-side of (1) to the product of 2- and 3-morphisms

(7) \[ \Psi \Box \Phi = (\Psi \Box F_2) \circ (G_1 \Box \Phi) \]

for all 3- or 2-morphisms $\Phi \in \mathcal{G}(F_1, F_2), \Psi \in \mathcal{G}(G_1, G_2)$ and 1-morphisms $F_1, F_2 : \mathcal{C} \to \mathcal{D}, G_1, G_2 : \mathcal{D} \to \mathcal{E}$. The coherence morphisms for $\Box$ are then given by the collection of natural isomorphisms

$$\Box_{\mu, \nu} = 1_{\mu_1 \Box K_2} \circ \sigma_{\nu_1, \mu_2}^{-1} \circ 1_{G_1 \Box \nu_2} : (\mu_1 \Box \mu_2) \circ (\nu_1 \Box \nu_2) \to (\mu_1 \circ \nu_1) \Box (\mu_2 \circ \nu_2)$$

for all $\Box$-composable pairs of 2-morphisms $\mu = (\mu_1, \mu_2) : (H_1, H_2) \to (K_1, K_2)$, $\nu = (\nu_1, \nu_2) : (G_1, G_2) \to (H_1, H_2)$. That this determines a collection of weak 2-functors $\Box : \mathcal{G}(\mathcal{D}, \mathcal{E}) \times \mathcal{G}(\mathcal{C}, \mathcal{D}) \to \mathcal{G}(\mathcal{C}, \mathcal{E})$ with strict units (see Definition A.1) is a direct consequence of the axioms of the Gray category data. Consistency condition (1) of Definition A.1 follows from Definition 2.14 (2), and consistency condition (2) of Definition A.1 from Definition 2.14 (4). That the functor $\Box$ is opcubical follows directly from the definition.

Analogously, one obtains a strict cubical tricategory by promoting the right-hand-side of (1) to the product of 2- and 3-morphisms

(8) \[ \Psi \Box \Phi = (G_2 \Box \Phi) \circ (\Psi \Box F_1) \]

for all 3- or 2-morphisms $\Phi \in \mathcal{G}(F_1, F_2), \Psi \in \mathcal{G}(G_1, G_2)$ and 1-morphisms $F_1, F_2 : \mathcal{C} \to \mathcal{D}, G_1, G_2 : \mathcal{D} \to \mathcal{E}$. The coherence morphisms for $\Box$ are then given by the collection of natural isomorphisms

$$\Box_{\mu, \nu} = 1_{K_1 \Box \mu_2} \circ \sigma_{\mu_1, \nu_2} \circ 1_{\nu_1 \Box G_2} : (\mu_1 \Box \mu_2) \circ (\nu_1 \Box \nu_2) \to (\mu_1 \circ \nu_1) \Box (\mu_2 \circ \nu_2)$$
for all □-composable pairs of 2-morphisms \( \mu = (\mu_1, \mu_2) : (H_1, H_2) \to (K_1, K_2) \), \( \nu = (\nu_1, \nu_2) : (G_1, G_2) \to (H_1, H_2) \). The proof that this defines a strict cubical tricategory is analogous to the opcubical case.

The passage between a set of Gray category data and the associated cubical and opcubical tricategories can be viewed as a special case of the operation called “nudging” in \([9]\), which allows one to pass between cubical and opcubical tricategories and functors. For the case of strict tricategories, this gives rise to the following statement.

**Corollary 2.16.** For every strict cubical (opcubical) tricategory \( G \), there exists a canonical strict opcubical (cubical) tricategory \( \hat{G} \) and functors of strict tricategories \( \Sigma : G \to \hat{G}, \Sigma^{-1} : \hat{G} \to G \), that are the identity mappings on all objects and morphisms and satisfy \( \Sigma \circ \Sigma^{-1} = 1, \Sigma^{-1} \circ \Sigma = 1 \).

**Proof.** Let \( G \) be a strict opcubical tricategory. Then by Lemma 2.15 this defines a set of Gray category data. Define \( \hat{G} \) as the strict cubical tricategory determined by this set of Gray category data according to Lemma 2.15 and define the functor \( \Sigma : G \to \hat{G} \) of strict tricategories by taking the identity mappings on the objects and the identity functors for each 2-functor \( \Sigma_{C,D} : \hat{G}(C, D) \to \hat{G}(C, D) \). The only nontrivial data of \( \Sigma \) are the natural isomorphisms \( \kappa_{\mu,\nu} : \mu \Box \nu \to \mu \Box \nu \) from Definition A.6, where \( \Box \) and \( \Box \) denote, respectively, the products in the tricategories \( G \) and \( \hat{G} \). These are given by the tensorator:

\[
\kappa_{\mu,\nu} = \sigma_{\mu,\nu}^{-1}
\]

for all \( \Box \)-composable 2-morphisms \( \mu, \nu \). It follows directly from the properties of the tensorator in Definition 2.14 that this defines a functor \( \Sigma : G \to \hat{G} \) of strict tricategories. By taking again the identity mappings on the objects and the identity 2-functors together with the coherence isomorphisms \( \kappa_{\mu,\nu}^{-1} = \sigma_{\mu,\nu} \), we obtain a functor that is strictly inverse to \( \Sigma \).

This corollary implies in particular that the tricategories obtained via (7) and (8) are equivalent and one of these constructions can be chosen arbitrarily. In the following we take as our definition of a Gray category the strict opcubical tricategory defined by (7).

**Definition 2.17.** A Gray category is the strict opcubical tricategory constructed from Gray category data using the Gray product

\[
\Psi \Box \Phi = (\Psi \Box F_2) \circ (G_1 \Box \Phi)
\]

for all composable 2- and 3-morphisms \( \Psi \in G(G_1, G_2), \Phi \in G(F_1, F_2) \).
2.4. Example: 2Cat and MonCat. An example of a Gray category investigated in this paper is 2Cat. The objects of 2Cat are 2-categories, the 1-morphisms are strict functors of 2-categories, the 2-morphisms pseudo-natural transformations and 3-morphisms modifications. In the following, we recall these definitions, the compositions and the tensorator. For a proof that this defines a Gray category, see [10, §I,4.5].

To simplify the notation, and because it is the only case used in the following, it is assumed throughout that the 2-categories have only one object, i.e., are monoidal categories. Thus the Gray category defined here is MonCat. The definitions extend easily to the general case of a 2-category.

The objects of MonCat are strict monoidal categories. The objects of the monoidal category correspond to the 1-morphisms of the corresponding 2-category with one object and its morphisms to 2-morphisms of the corresponding 2-category with one object. The horizontal composition is given by the tensor product, denoted ◦ in the following, and the unit 1-morphism in the associated 2-category corresponds to the tensor unit e. The vertical composition of morphisms is denoted ·, as before. The 1-morphisms in MonCat are strict tensor functors.

Definition 2.18 (Tensor functor). A strict tensor functor $F : C \to D$ between strict monoidal categories $C, D$ is a functor $F : C \to D$ with $F(e_C) = e_D$ and $F(x \circ y) = F(x) \circ F(y)$ for all objects $x$ and $y$ of $C$.

The 2-morphisms in MonCat are pseudo-natural transformations between strict tensor functors. They can be viewed as a generalisation of natural transformations and are obtained by restricting the general definition of natural 2-transformations in Definition A.3 to 2-categories with a single object.

Definition 2.19 (Pseudonatural transformation). A pseudo-natural transformation $\nu : F \Rightarrow G$ between strict tensor functors $F, G : C \to D$ consists of an object $x$ of $D$, together with a collection of isomorphisms $\nu_y : x \circ F(y) \to G(y) \circ x$ for all objects $y$ of $C$ such that

1. $\nu_e = 1_x : x \to x$ is the identity morphism.
2. $\nu$ is natural in $y$: for all morphisms $\alpha : y \to z$ in $C$, the following diagram commutes

\[
\begin{array}{ccc}
  x \circ F(y) & \xrightarrow{\nu_y} & G(y) \circ x \\
  \downarrow{1_x \circ F(\alpha)} & & \downarrow{G(\alpha) \circ 1_x} \\
  x \circ F(z) & \xrightarrow{\nu_z} & G(z) \circ x.
\end{array}
\]

3. $\nu$ is compatible with the tensor product: for all objects $y, z$

\[
\nu_{y\circ z} = (1_{G(y)} \circ \nu_z) \cdot (\nu_y \circ 1_{F(z)}).
\]
The 3-morphisms in MonCat are modifications between pseudo-natural transformations. Their definition is obtained by restricting Definition A.4 to 2-categories with a single object.

**Definition 2.20 (Modification).** Let $\mu, \nu : F \Rightarrow G$ be pseudo-natural transformations with component morphisms $\mu_z : x \circ F(z) \to G(z) \circ x$, $\nu_z : y \circ F(z) \to G(z) \circ y$ for all objects $z$ of $C$. A modification $\Phi : \mu \Rightarrow \nu$ is a morphism $\Phi : x \to y$ such that the following diagram commutes for all objects $z$ of $C$

\[
\begin{array}{c}
x \circ F(z) \\
\downarrow \Phi \circ 1_{F(z)}
\end{array}
\begin{array}{c}
\mu_z
\end{array}
\begin{array}{c}
\downarrow 1_{G(z) \circ \Phi}
\end{array}
\begin{array}{c}
y \circ F(z) \\
\downarrow \nu_z
\end{array}
\begin{array}{c}
\mu_z
\end{array}
\begin{array}{c}
G(z) \circ x \\
G(z) \circ y.
\end{array}
\]
The product operations and the tensorator of MonCat are obtained by specialising the ones in 2Cat to the case of a single object and are summarised in the following definition.

**Definition 2.21.** The product operations and the tensorator of MonCat are as follows:

1. **Gray product:**
   \[ \square : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \]
   - The composition of two functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{C} \to \mathcal{D} \) is given by the composition of the associated morphisms \( \Phi : w \to x \) and \( \Psi : x \to y \).

2. **horizontal composition:**
   - The horizontal composite \( \Phi \circ \Psi : w \to y \) is defined by the morphism \( \Phi \circ \Psi = (\Phi) \circ (\Psi) : w \to y \).

3. **vertical composition:**
   - The vertical composition of two modifications \( \Phi : \mathcal{C} \to \mathcal{D} \) and \( \Psi : \mathcal{D} \to \mathcal{E} \) with component morphisms \( (\Phi) : x \to y \) and \( (\Psi) : y \to z \) is given by the morphism \( \Phi \circ \Psi = (\Phi) \circ (\Psi) : x \to z \).

The defining properties of pseudo-natural transformations and modifications are depicted in Figure 5.

The product operations and the tensorator of MonCat and the tensorator of MonCat are obtained by specialising the ones in 2Cat to the case of a single object.

(1) Gray product:

(2) horizontal composition:

(3) vertical composition:
(4) tensorator: The tensorator $\sigma_{\mu,\nu}$ of two pseudo-natural transformations $\nu : F_1 \Rightarrow F_2$, $\mu : G_1 \Rightarrow G_2$ with associated morphisms $\nu : x \circ F_1(z) \to F_2(z) \circ x$ and $\mu : y \circ G_1(z) \to G_2(z) \circ y$ is the modification $\sigma_{\mu,\nu} : (\mu \Box F_2) \circ (G_1 \Box \nu) \Rightarrow (G_2 \Box \nu) \circ (\mu \Box F_1)$ given by the morphism $\mu : y \circ G_1(x) \to G_2(x) \circ y$.

Particular examples of subcategories of MonCat that are of interest are the subcategory obtained by restricting attention to 2-functors on a single category $C$, which is a monoidal 2-category, and the subcategory of this that consists of pseudo-natural transformations and modifications on a trivial functor. This is a braided monoidal category, called the center of $C$. Further examples arise when the monoidal categories $C$ have more structure. The example of a subcategory, in which all objects are pivotal categories is discussed in Section 8.

Other important examples of Gray categories are Gray groupoids, which are obtained from 2-crossed modules [20] and, more generally, Gray categories obtained from the strictification of tricategories.

2.5. Gray category diagrams. The definition of a diagrammatic calculus for Gray categories follows the pattern for categories and 2-categories. The diagrams are a three-dimensional generalisation of the two-dimensional diagrams defined above, and were previously studied informally by Trimble [31].

Gray category diagrams are located in the unit cube $[0,1]^3$ and consist of a number of points, lines, surfaces, etc. in the cube. It seems that the clearest way to organise the definition is in terms of a PL stratification. A stratification of the $n$-dimensional cube $[0,1]^n$ is a set of closed subspaces $\emptyset = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset \ldots \subset X^n = [0,1]^n$ called the $k$-skeleta, such that $X^k \setminus X^{k-1}$ is a PL-manifold of dimension $k$, for $k = 0, 1, \ldots, n$. Each component of $X^k \setminus X^{k-1}$ is called a stratum. In addition a PL stratification has an additional condition to ensure that the cross-section through each stratum is locally constant [29]. In the following we will always consider PL stratifications.

**Definition 2.22** (Three-dimensional diagrams).

1. A three-dimensional diagram is a stratification of $[0,1]^3$ so that
   (a) each $k$-stratum $C^k$ satisfies $\partial [0,1]^3 \cap C^k = C^k \cap (\{0,1\}^3 \setminus \partial [0,1]^k)$,
   (b) the side faces $[0,1] \times \{0\} \times [0,1]$ and $[0,1] \times \{1\} \times [0,1]$ are progressive diagrams.
   The 0-, 1-, 2- and 3-strata are called vertices, lines, surfaces and regions, of the diagram, respectively. The face $[0,1]^2 \times \{0\}$ is called the source and the face $[0,1]^2 \times \{1\}$ the target of the diagram.

2. A three-dimensional diagram is called progressive, if the projection $p_2 : (w,x,y) \mapsto (x,y)$ is a regular mapping of each surface,
and the projection \( p_1 \circ p_2 : (w, x, y) \mapsto y \) a regular mapping of each line.

(3) A progressive three-dimensional diagram is called generic if the following conditions on the image of the diagram under the projection \( p_2 : (w, x, y) \mapsto (x, y) \) hold:

(a) any two different vertices project to different points in \([0, 1]^2\).

(b) the images of any two lines meet only at interior points of \([0, 1]^2\), and at every point where they meet, they cross transversally.

(c) Vertices and crossings in the image do not coincide with the projection of points on other lines.

Condition (1) (a) in this definition states that lines in the diagram can intersect the boundary of the unit cube only in its top face \([0, 1]^2 \times \{0\}\) and its bottom face \([0, 1]^2 \times \{1\}\) and that surfaces of the diagram cannot intersect its front face \(\{0\} \times [0, 1]^2\) or its back face \(\{0\} \times [0, 1]^2\).

This is the three-dimensional analogue of the condition that lines in two-dimensional diagrams intersect only the top and bottom edge of the diagram. Condition (1) (b) is a new feature that does not appear in lower dimensions; in dimension two the side edges are required to be empty. For a generic three-dimensional diagram, the source and target are generic two-dimensional diagrams.

Mappings of three-dimensional diagrams are defined in analogy to the one- and two-dimensional case.

**Definition 2.23** (Mappings of \(n\)-dimensional diagrams).

1. A mapping \( D \to D' \) of three-dimensional diagrams is a PL-embedding \( m : [0, 1]^3 \to [0, 1]^3 \) that preserves the \(k\)-skeleta, i.e., \( m(X^k) \subset X'^k \), for \( k = 0, 1, \ldots, 3 \).

2. A mapping of three-dimensional diagrams is called a homomorphism if it is a PL-homeomorphism and is the identity map on the boundary \( \partial[0, 1]^3 \). An isomorphism of three-dimensional diagrams is a homomorphism that has an inverse.

3. Isomorphisms, subdivisions and subdiagrams are defined analogously to the one- and two-dimensional case (Definition 2.9).

To define diagrams labelled with data from a Gray category, it is necessary to restrict attention to progressive diagrams. As in the two-dimensional case, each vertex in a progressive three-dimensional diagram has a neighbourhood that is isomorphic to a diagram with a single vertex. The simplest type of one-vertex diagram has no crossings in its two-dimensional projection and is called an elementary diagram.

**Definition 2.24** ((Elementary) Gray category diagram). Let \( \mathcal{G} \) be a Gray category.
(1) An elementary Gray category diagram for $\mathcal{G}$ is a progressive three-dimensional diagram with one vertex such that the images of its lines under the projection $p_2 : (w, x, y) \rightarrow (x, y)$ do not intersect together with

- a labelling of each region with an object in $\mathcal{G}$
- a labelling of each surface with a 1-morphism in $\mathcal{G}$
- a labelling of each line with a 2-morphism in $\mathcal{G}$
- a labelling of the vertex with a 3-morphism in $\mathcal{G}$

The source and target of the diagram are required to be pre-2-category diagrams for $\mathcal{G}_2$ and evaluate to 2-morphisms which are the source and target for the vertex 3-morphism.

(2) A Gray category diagram for $\mathcal{G}$ is a progressive three-dimensional diagram together with a labelling of each region with an object in $\mathcal{G}$, each surface with a 1-morphism, each line with a 2-morphism and each vertex with a 3-morphism. The source and target are required to be pre-2-category diagrams for $\mathcal{G}_2$ and each vertex $v$ is required to have a rectangular neighbourhood that is an elementary Gray category subdiagram.

An elementary Gray category diagram and its projection are depicted in Figure 6. Note that the requirement on the vertex neighbours is both a restriction on the topology at a vertex, so that the plane projections of lines are locally non-intersecting, and a restriction on the vertex label. Any two such rectangular neighbourhoods give isomorphic elementary vertex subdiagrams, so that the choice of rectangular neighbourhood does not matter.

Note also that a diagram with a plane projection that has a single crossing and no other vertices is a Gray category diagram but not an elementary Gray category diagram. The requirement that the source and target are pre-2-category diagrams implies that the $x$-coordinates of the intersection points of lines with the source and target of the diagram are all different.

As in two dimensions, the evaluation of a Gray category diagram is obtained by projecting it to a two-dimensional 2-category diagram and then evaluating the resulting 2-category diagram according to Definition 2.11. The construction is described first for the case of a generic diagram.

The image of the diagram under the projection $p_2 : (w, x, y) \rightarrow (x, y)$ defines a two-dimensional diagram. This two-dimensional diagram has vertices given by the image of vertices or crossing points, and lines given by segments of images of lines between either vertices or crossing points.

Let $D$ be a generic Gray category diagram with initial region (containing the face $\{1\} \times [0, 1]^2$) labelled by an object $C$ in $\mathcal{G}$ and final region (containing the face $\{0\} \times [0, 1]^2$) labelled with an object $D$ of $\mathcal{G}$. Then its projection $p_2 D$ is a a 2-category diagram for the 2-category
The label at a point \((x, y) \in [0, 1]^2\) of \(p_2D\) depends on whether the set \(p_2^{-1}(x, y)\) contains a vertex, an interior point of a line, interior points of two different lines, or none of these:

1. If \(p_2^{-1}(x, y)\) contains a vertex, then the points in \(p_2^{-1}(x, y)\) define a sequence \(F_1, F_2, \ldots, F_j, \Phi, G_1, G_2, \ldots G_k\), where \(F_n, G_m\) are 1-morphisms labelling surfaces and \(\Phi\) is 3-morphism labelling the vertex, in order of increasing \(w\) coordinate. The point \((x, y)\) is a vertex of \(p_2D\) and is labelled with the 3-morphism "G_k□...G_2□G_1□Φ□F_j...F_2□F_1".

2. If \(p_2^{-1}(x, y)\) contains an interior point of a line, the labelling is analogous, but the 3-morphism \(\Phi\) is replaced by the 2-morphism \(\nu\) labelling the line. The point \((x, y)\) lies on a line of \(p_2D\) and is labelled with a 2-morphism.

3. If \(p_2^{-1}(x, y)\) contains no vertex and no interior points of lines, the labelling is as in (1) but with the 3-morphism \(\Phi\) removed. The point \((x, y)\) lies in a region of \(p_2D\) and is labelled with the corresponding 1-morphism.

4. If \(p_2^{-1}(x, y)\) contains interior points of two different lines, then the point \((x, y)\) is a crossing in \(p_2D\). In this case, the sequence associated with \(p_2^{-1}(x, y)\) is of the form

\[F_1, F_2, \ldots, F_j, \nu, G_1, G_2, \ldots G_k, \mu, H_1, H_2, \ldots H_l,\]

where \(F_i, G_m, H_n\) are 1-morphisms in \(\mathcal{G}\) that label the surfaces and \(\nu: A \to B, \mu: C \to D\) are the 2-morphisms labelling the two lines in the preimage of the crossing. In this case, there are two possible diagrams, whose labellings are given in Figure 7 a) and b). The vertex of the 2-category diagram is labelled, respectively, with \(H□\sigma_{\mu□G_\nu□F}\) and with \(H□\sigma_{\mu□G_\nu□F}\) where \(\sigma_{\mu□G_\nu□F}\) stands for the tensorator (see equation (1)) and we abbreviate \(F = F_j□...F_2□F_1, G = G_k□...G_2□G_1\) and \(H = H_l□...H_2□H_1\).

The 2-category diagram \(p_2D\) obtained in these four cases defines the evaluation of a generic Gray category diagram. By definition, the evaluation of \(p_2D\) is a 2-morphism in the 2-category \(\mathcal{G}(\mathcal{C}, \mathcal{D})\) and hence a 3-morphism in \(\mathcal{G}\).

**Definition 2.25** (Evaluation of a generic Gray category diagram). Let \(D\) be a generic Gray category diagram whose initial region is labelled with an object \(\mathcal{C}\) and whose final region is labelled with an object \(\mathcal{D}\) in \(\mathcal{G}\). Then the diagram \(D\) projects to a 2-category diagram \(p_2D\) for \(\mathcal{G}(\mathcal{C}, \mathcal{D})\), and the evaluation of \(D\) is the evaluation of \(p_2D\).

As in the one- and two-dimensional case, there is a relation between the composition of Gray category diagrams and the three compositions
Figure 6. Elementary Gray category diagram and its projection to a 2-category diagram for \(G(C, D)\):

- The regions are labelled with objects \(C, D, E\).
- The surfaces with 1-morphisms \(F, G, H : C \to D, J : D \to E, K : E \to D\) and \(L : C \to E\).
- The lines with 2-morphisms \(\rho : H \Rightarrow KL, \kappa : L \Rightarrow JF, \eta : KJ \Rightarrow 1D, \nu : H \Rightarrow G, \mu : G \Rightarrow F\).
- The vertex with a 3-morphism \(\Psi : (\eta F) \circ (K\kappa) \circ \rho \Rightarrow \mu \circ \nu\).

in a Gray category \(G\). Gray category diagrams can be composed in the \(w-, x-\) and \(y\)-direction as depicted in Figure 8.

The composition in the direction of the \(w\)-axis is defined if the object in \(G\) labelling the initial region at the face \(\{0\} \times [0, 1]^2\) of one diagram agrees with the object labelling the final region at the face \(\{1\} \times [0, 1]^2\) of the other diagram, as shown in Figure 8(c). This composition corresponds to the Gray product of two 3-morphisms in the Gray category \(G\). If \(D, D'\) are generic progressive Gray category diagrams that can be composed in this way such that their composite diagram \(\tilde{D}\) is again a generic progressive diagram, then the evaluation of \(\tilde{D}\) is the Gray product of the evaluation of \(D\) and \(D'\).

The composition in the direction of the \(x\)-axis is defined if the labelled progressive two-dimensional diagram at the face \([0, 1] \times \{0\} \times [0, 1]\) of
Figure 7. Gray category diagrams and projections for a crossing: a) the tensorator $\sigma_{\mu,\nu} G, k$ and b) its inverse. The labelling of regions by objects is omitted. The shortened notation omitting $\square$ is used.

one of the diagrams matches the labelled diagram at the face $[0, 1] \times \{0\} \times [0, 1]$ of the other, as shown in Figure 8 a). It corresponds to the horizontal composition in $G$. If two generic progressive Gray category diagrams $D, D'$ are composable in this sense and the resulting diagram $\tilde{D}$ is again generic and progressive, then the evaluation of $\tilde{D}$ is the horizontal composite of the evaluation of $D$ and of $D'$.

The composition in the direction of the $y$-axis is defined if the labelled two-dimensional diagram at the face $[0, 1]^2 \times \{0\}$ of one of the two diagrams matches the labelled two-dimensional at the face $[0, 1]^2 \times \{1\}$ of the other, as shown in Figure 8 b). This composition of diagrams corresponds to the vertical composition in $G$. If $D, D'$ are generic progressive Gray category diagrams which can be composed such that the resulting composite diagram $\tilde{D}$ is again a progressive generic diagram, then the evaluation of $\tilde{D}$ is the vertical composite of the evaluations of $D$ and of $D'$.

It seems plausible that by considering Gray category diagrams up to suitable isotopies, one should obtain a Gray category of Gray category diagrams, which generalises the well-known example of the tangle category. In this framework, the evaluation should define a functor from the Gray category of diagrams labelled with $G$ to the Gray category $G$. 27
Figure 8. Composition of Gray category diagrams:
a) Horizontal composition \( \circ \),
b) Vertical composition \( \cdot \),
c) Gray product \( \Box \).
The above relations between the composition of diagrams and the composition of their evaluations would then correspond to the axioms of a functor of strict tricategories. However, this aspect is not developed further in the paper.

In analogy to the lower-dimensional cases, each generic Gray category diagram can be viewed as a calculation in a Gray category $G$, which is given by the evaluation of the diagram. The benefit of such a diagrammatic calculus is that calculations in $G$ can be easily visualised. This requires a statement about the invariance of the evaluation under certain mappings of diagrams.

**Theorem 2.26.** Let $D, D'$ be generic Gray category diagrams that are isotopic by a one-parameter family of isomorphisms of progressive diagrams. Then the evaluations of $D$ and $D'$ are equal.

**Proof.** Joyal and Street [12] prove this for the case of a braided monoidal category. The proof extends directly to the case of a Gray category. This can be seen as follows: The proof in [12] relies on a decomposition of a three-dimensional isotopy into three-dimensional isotopies that induce isotopies of the two-dimensional diagrams obtained via the projection and three-dimensional isotopies that project to certain moves of two-dimensional diagrams. Invariance of the evaluation under the former follows directly from Theorem 2.12. The invariance under the second type of isotopies in [12] is a consequence of the properties of the braiding in a braided monoidal category. The associated diagrams in [12] have direct generalisations to progressive Gray category diagrams and the resulting set of moves is depicted in Figure 9a) to c). The invariance of the evaluation under these moves then follows from the properties of the tensorator in Definition 2.14.

By means of this theorem, it is possible to extend the definition of the evaluation to progressive Gray category diagrams which are not generic. The idea is the same as in the two-dimensional case (see Theorem 2.12 and Definition 2.13), namely to perturb a a non-generic progressive diagram into a generic progressive diagram by means of an isotopy. The resulting diagrams are then related by the moves from the proof of Theorem 2.26 in Figure 9a), b), c), and the properties of the tensorator in a Gray category ensure that their evaluations are equal.

**Definition 2.27** (Evaluation of progressive Gray category diagram). The evaluation of a progressive Gray category diagram that has generic source and target pre-2-category diagrams is defined by perturbing it by an isotopy that fixes the boundary to a generic progressive Gray category diagram. The result is independent of the choice of isotopy by Theorem 2.26.
Figure 9. Properties of the tensorator $\sigma_{\mu,\nu}$:

a) naturality in the first argument,

b), c) naturality in the second argument,

d) compatibility with the horizontal composition.
3. Gray categories with duals

This section introduces 2-categories and Gray categories with duals. The main aim is to investigate the structure and the diagrammatic representation of Gray categories with duals. As the associated diagrams and their evaluation are defined in terms of 2-category diagrams, this requires a careful investigation of diagrams for 2-categories with duals. Most of the material on 2-categories and the associated diagrams is standard \[13\], but the discussion of mappings of 2-dimensional diagrams contains the detail that is required in Section 6.1.

3.1. 2-categories with duals. As the duality operations introduced in this section reverse certain products in the Gray categories, we require two notions of opposites for 2-categories. We start by introducing the relevant notation. For a category \( C \) we denote by \( C^{\text{op}} \) the opposite category and for a morphism \( f \in C(A,B) \) by \( f^{\text{op}} \in (C^{\text{op}})(B,A) \) the corresponding morphism with source and target reversed. We denote by \( \cdot \) the composition of morphisms in \( C^{\text{op}} \), e. g.,

\[
f^{\text{op}} \cdot g^{\text{op}} = (g \cdot f)^{\text{op}}.
\]

Similarly, we denote for a functor \( F: C \to D \) by \( F^{\text{op}}: C^{\text{op}} \to D^{\text{op}} \) the opposite functor with \( F^{\text{op}}(f^{\text{op}}) = (F(f))^{\text{op}} \) and for a natural transformation \( \nu: F \to G \) by \( \nu^{\text{op}}: G^{\text{op}} \to F^{\text{op}} \) the opposite natural transformation defined by \( \nu^{\text{op}}(A) = \nu(A)^{\text{op}} \). In particular, if \( \nu \) is a natural isomorphism, then \((\nu^{\text{op}})^{-1}\) is a natural isomorphism from \( F^{\text{op}} \) to \( G^{\text{op}} \).

**Definition 3.1 (Opposite 2-categories).** Let \( C \) be a 2-category. Then \( C^{\text{op}} \) denotes corresponding 2-category with both products reversed

- \( (C^{\text{op}})(A,B) = C(B,A)^{\text{op}} \) for objects \( A, B \)
- \( \alpha^{\text{op}} \circ \beta^{\text{op}} = (\beta \circ \alpha)^{\text{op}} \) for composable 1- or 2-morphisms \( \alpha, \beta \),

and \( C_{\text{op}} \) the 2-category with the same vertical but opposite horizontal product

- \( (C_{\text{op}})(A,B) = C(A,B) \) for objects \( A, B \)
- \( \alpha_{\text{op}} \circ \beta_{\text{op}} = (\beta \circ \alpha)_{\text{op}} \) for composable 1- or 2-morphisms \( \alpha, \beta \).

In the sequel we will abuse notation and simply denote a morphism \( f^{\text{op}} \) by \( f \) whenever it is clear from the context to which category \( f \) belongs.

The appropriate notion of a 2-category with duals that will be used later in the definition of a Gray category with duals is that of a planar 2-category. A planar 2-category is a direct generalisation of a strict pivotal category, which in turn can be regarded as a planar 2-category with one object.

The definition uses a 2-category notion of contravariant functor. A strict 2-functor \( G: B \to C^{\text{op}} \) has an associated contravariant functor \( G: B \to C \) given by \( G(\alpha)^{\text{op}} = G(\alpha) \). In the sequel we will abuse notation and define a contravariant functor \( G: B \to C \) to as functor \( G: B \to C^{\text{op}} \), that we call again \( G \). The product \( FG \) of contravariant \( G \)
followed by functor $F : C \to D$ is defined as the contravariant functor associated to $F^{op} G$ and denoted again by $FG$. Similarly, if $F$ is a contravariant functor, then the product $FG = F^{op} G$ is an ordinary (covariant) functor.

**Definition 3.2** (Planar 2-category). A planar 2-category is a 2-category $C$ together with a contravariant functor $\ast : C \to C$ associated to a strict 2-functor $C \to C^{op}$ that is the identity on objects. There is also a collection of 2-morphisms $\epsilon_a : 1_{A'} \to a \circ a^*$ for all 1-morphisms $a : A \to A'$ of $C$ such that:

1. $\ast \ast = 1_C$ is the the identity functor
2. for all 1-morphisms $a, b, c$ and 2-morphisms $\alpha : a \to b$ for which these expressions are defined:

\[
\begin{align*}
(\alpha \circ 1_{a^*}) \cdot \epsilon_a &= (1_b \circ \alpha^*) \cdot \epsilon_b \quad (1_a \circ \epsilon_{a^*}) \cdot (\epsilon_a \circ 1_a) = 1_a \\
(1_a \circ \epsilon_c \circ 1_{a^*}) \cdot \epsilon_a &= \epsilon_{aoc}.
\end{align*}
\]

Note that the 2-functor $\ast$ and the collection of morphisms $\epsilon_a$ in a planar 2-category are not independent. The following lemma shows that the 2-morphisms $\epsilon_a$ determine the action of the functor $\ast$ on the 2-morphisms uniquely.

**Lemma 3.3.** [18, 3] For any 2-morphism $\alpha : a \to b$ in a planar 2-category, the dual $\alpha^* : b^* \to a^*$ is given by

\[
\alpha^* = (\epsilon_{b^*} \circ 1_{a^*}) \cdot (1_{b^*} \circ \alpha \circ 1_{a^*}) \cdot (1_{b^*} \circ \epsilon_a) = (1_{a^*} \circ \epsilon_{b^*}) \cdot (1_{a^*} \circ \alpha \circ 1_{b^*}) \cdot (\epsilon_{a^*} \circ 1_{b^*}),
\]

and the 2-morphism $\alpha$ satisfies the pivotal condition

\[
\alpha = (1_b \circ \epsilon_{a^*}) \cdot (1_b \circ 1_{a^*} \circ \epsilon_{b^*} \circ 1_a) \cdot (1_b \circ 1_{a^*} \circ \alpha \circ 1_{b^*} \circ 1_a) \cdot (1_b \circ \epsilon_{a^*} \circ 1_{b^*} \circ 1_a) \cdot (\epsilon_b \circ 1_a)
\]

**Proof.** The proof is a direct generalisation of the corresponding proof for pivotal categories, see [18, 3]. The identities in (13) follow from the first and second identity in (12) together with the exchange law. The pivotal condition (14) is then obtained by applying (13) twice and using the identity $\ast \ast = 1_C$. \[ \square \]

3.2. **Diagrammatic representation of the 2-category duals.** The $\ast$-duals in a planar 2-category are the extra data required to define 2-category diagrams that are not progressive. In this setting, the condition that a 2-category diagram $D$ is progressive can be relaxed to the condition that it is ‘piecewise progressive’, i. e. that there is a subdivision $m : D \to D'$ such that $D'$ is progressive. As for any two-dimensional diagram $D$ there is a subdivision $m : D \to D'$ such that each line in $D'$ is a straight line, this condition is satisfied if and only if the $y$-coordinates of different vertices of of $D'$ do not coincide. This motivates the definition of a generic diagram.
Definition 3.4 (Generic two-dimensional diagram). A two-dimensional diagram is called generic if the only singularities of the projection $p_1$ on lines are maxima and minima, and all vertices, maxima and minima have different $y$-coordinates.

Note that if $D$ is a generic two-dimensional diagram that is also progressive, then it is a generic progressive diagram in the sense of Definition 2.8. In particular, by promoting the maxima and minima of a generic 2-dimensional diagram $D$ to vertices, one obtains a subdivision $m : D \to D'$ with a generic progressive diagram $D'$. Such a subdivision, in which the only additional vertices in $D'$ are located at the maxima and minima of $D$ will be called minimal subdivision in the following. In the rest of this section, we will assume that two-dimensional diagrams are generic where appropriate. Note, however, that an isotopy between generic diagrams may fail to be generic at isolated points.

Subdividing generic two-dimensional diagrams to obtain generic progressive diagrams introduces a complication with the labelling. A line in $D$ that is labelled with an object $x$ and zig-zags upwards and downwards with respect to the $y$-coordinate corresponds to a collection of progressive line segments in $D'$, whose labels vary between $x$ to $x^*$, depending on their orientation. Keeping track of these labels and of the analogous labelling problems caused by the rotation of vertices motivated the introduction of a new structure into a diagram, namely a framing. This was introduced for knots by Kauffman [17] and for monoidal category diagrams by Reshetikhin and Turaev [27]. Nevertheless, it is possible to reformulate this theory in terms of unframed diagrams. Our experience is that this is much simpler in the case of three-dimensional diagrams, and we will work with unframed diagrams throughout the paper. However the issues that motivated the introduction of framing in previous works then appear in the action of mappings on diagrams.

By considering progressive subdivisions, it is possible to define the evaluation of generic two-dimensional diagrams labelled with data from a planar 2-category.

**Definition 3.5.** Let $C$ be a planar 2-category. A planar 2-category diagram for $C$ is a generic two-dimensional diagram $D$ together with a labelling of the image of its minimal progressive subdivision $m : D \to S$ with elements of $C$ such that $S$ is a 2-category diagram. This labelling must be such that the additional vertices are labelled with the canonical 2-morphisms $\epsilon_a$ or $\epsilon^*_a$ of $C$ as show in Figure 10. The evaluation of $D$ is defined as the evaluation of $S$.

Examples of planar 2-category diagrams are given in Figure 11. The aim is now to prove that the evaluation of planar 2-category diagrams is invariant under a much more general class of mappings of diagrams.
than the mappings in Theorem 2.12 namely under any homomorphism of diagrams. As the discussion is quite intricate, the result will be derived in several steps.

First, note that by considering minimal progressive subdivisions, it is possible to decompose any homomorphism of diagrams for planar 2-categories into two subdivisions and an isomorphism of progressive diagrams: If \( g : D \rightarrow D' \) is a homomorphism of diagrams for a planar 2-category, then the minimal progressive subdivisions \( m : D \rightarrow S \), \( m' : D' \rightarrow S' \) define progressive diagrams \( S \), \( S' \) with unique labellings given by Definition 3.5, and there is an isomorphism \( f : S \rightarrow S' \) such that \( f \circ m = m' \circ g \). As the evaluations of \( S \) and \( D \) and of \( S' \) and \( D' \) are equal by definition, it is sufficient to consider the associated isomorphism \( f : S \rightarrow S' \) of progressive diagrams.

Isomorphisms of progressive diagrams can be classified according to their action on the lines and vertices in the diagram. As explained in Section 2.2 each vertex \( q \) of a progressive diagram \( S \) has a neighbourhood that is isomorphic to an elementary diagram diagram \( E \). The incident at \( v \) lines fall into two sets, the input lines, which intersect the top edge \([0,1] \times \{0\}\) in \( E \), and the output lines that intersect the bottom edge \([0,1] \times \{1\}\) in \( E \). The sets of input lines and the set of output lines are ordered by increasing \( x \)-coordinate. An isomorphism of progressive diagrams can either preserve these two sets of lines and their orderings, or exchange lines between the two sets by a cyclic permutation. The cyclic order is preserved because the homeomorphism is orientation-preserving. The situation for the lines is simpler. The projection \( p_1 : (x,y) \rightarrow y \) induces an orientation on each line of \( S \), and an isomorphism of progressive diagrams either preserves or reverses these orientation.
Figure 11. Planar 2-category diagrams.

a) 2-morphism $\alpha^* : b^* \to a^*$ for a 2-morphism $\alpha : a \to b$.

b) The 2-morphism $\epsilon_a : 1_B \to a \circ a^*$.

c) The identity $(\alpha \circ 1) \cdot \epsilon_a = (1 \circ \alpha^*) \circ \epsilon_b$ from (12).

d) The identity $(\epsilon_a^* \circ 1_a) \cdot (\epsilon_a \circ 1_a) = 1_a$ from (12).

e) The identity $(1_a \circ \epsilon_c \circ 1_{a^*}) \cdot \epsilon_a = \epsilon_{aoc}$ from (12).

f) The identity $\alpha^* = (1_{a^*} \circ \epsilon_b^*) \cdot (1_{a^*} \circ \alpha \circ 1_{b^*}) \cdot (\epsilon_a^* \circ 1_{b^*})$

and g) condition (14) from Lemma 3.3.
If one considers labelled diagrams, it is directly apparent that homomorphisms of diagrams that preserve the ordered set of input and output lines at each vertex exhibit a particularly simple relation between the vertex labels in a diagram and in its image. This motivates the following definition.

**Definition 3.6.**

1. A homomorphism of progressive diagrams \( f : S \to S' \) is called vertex-preserving if, at each vertex \( v \) of \( S \) it maps input lines of \( v \) to input lines of \( f(v) \), output lines of \( v \) to output lines of \( f(v) \) and preserves the ordering of each set of lines.
2. The homomorphism \( f \) is called line-preserving at a line of \( S \) if it preserves orientation of the line induced by \( p_1 \), and line-reversing otherwise.
3. A vertex-preserving isomorphism of progressive planar 2-category diagrams \( f : S \to S' \) is a vertex-preserving isomorphism of progressive diagrams such that
   - the labels on a region of \( S \) and its image in \( S' \) agree
   - the labels on an line of \( S \) and its image in \( S' \) agree if the line is preserved and are related by \(*\) if the line is reversed.
   - the labels on a vertex of \( S \) and its image in \( S' \) agree.
4. A vertex-preserving homomorphism of planar 2-category diagrams is a homomorphism of planar 2-category diagrams that induces a vertex-preserving isomorphism via the minimal subdivisions.

We are now ready to prove that the evaluation of a planar 2-category diagram is invariant under homomorphisms of planar 2-category diagrams. The first step is to prove this result for homomorphisms of planar 2-category diagrams that induce vertex-preserving isomorphisms via their minimal subdivisions. In a second step, we will then account for homomorphisms which exchange elements of the sets of input and output lines at vertices or permute their order.

**Theorem 3.7.** Let \( D_1 \) and \( D_2 \) be planar 2-category diagrams that are related by a vertex-preserving homomorphism \( f : D_1 \to D_2 \). Then the evaluations of \( D_1 \) and \( D_2 \) are equal.

**Proof.** The theorem is proved by applying a result of Yetter [35, proposition 1.8], which gives a list of Reidemeister-type moves for a mapping of planar diagrams. These are the moves reproduced in Figure 12 plus the isotopies of progressive diagrams from Theorem 2.12. The moves on the vertices do not preserve the vertex, so it is necessary to replace the set of moves by another set of moves that is vertex-preserving. This is done by ‘rotating’ the vertex by a homeomorphism that is the identity outside a small neighbourhood of the vertex, on both sides of every move. An example is shown in Figure 13 b). Then the moves on
vertices become either the identity move or the pivotal move (Figure 11 g) combined in some cases with instances of the snake move of Figure 11 d) or f).

An example of a move is given in Figure 13 a). Yetter’s move in Figure 13 a) does not preserve the vertex. The vertex on the left-hand side of a) is rotated (as a result of the previous moves in the sequence) and is replaced by b), whereas in this example the rotation for the right-hand side of a) happens to be the identity, i.e., no rotation is required. The Yetter move is then carried out in c) using a snake move on the rotated vertex. The right-hand side of c) is equivalent to the right-hand side of a) by pivotal move d).
After subdivision, the moves become vertex-preserving isomorphisms of progressive planar 2-category diagrams. The invariance of the evaluation under the snake moves follows from (12), invariance under the pivotal moves from Lemma 3.3, and invariance under the isotopies of progressive diagrams by Theorem 2.12.

We now consider the case of homomorphisms of planar 2-category diagrams, which do not preserve the vertices. The general situation is that $D_1$ and $D_2$ are planar 2-category diagrams and $f : D_1 \to D_2$ is a homomorphism of the underlying two-dimensional diagrams. If $v$ is a vertex of $D_1$ such that the set of input edges and of output edges of $v$ or their ordering is not preserved by $f$, then one needs to specify the labelling at its image $f(v) \in D_2$. This can be done by the following construction:

As $f$ is a homomorphism of diagrams, there is an elementary diagram $E$ with a vertex $p$ and an isomorphism of diagrams $g : E \to D_1$, constructed by rotating a small neighbourhood of the vertex, such that $g(v') = v$ and $fg : E \to D_2$ is vertex-preserving. The homomorphism $g$ is chosen in such a way that it is the identity outside a small neighbourhood of $v'$. If $v \in D_1$ is labelled by a 2-morphism $\alpha$, then the 2-morphism $\alpha'$ labelling $v'$ is chosen in such a way that the evaluation of $E$ is equal to $\alpha$. The definition of $\alpha_2$ is independent of the

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig13.png}
\caption{An example of a vertex-preserving move.
  a) A move that does not preserve the vertex.
  b) The replacement of the vertex with a rotated one.
  c), d) snake and pivotal moves.}
\end{figure}
choice of the rotation at the vertex by the proof of Theorem 3.7, which uses the invariance of the evaluation under the moves in Figure 11. Consequently, the 2-morphism $\alpha'$ labelling $v'$ is unique. As the map $fg : E \to D_2$ is vertex-preserving, the label at the vertex $f(v)$ is also given by $\alpha'$.

**Definition 3.8.** Let $f : D_1 \to D_2$ be a homomorphism of planar 2-category diagrams and $v$ a vertex of $D_1$, labelled with a 2-morphism $\alpha_1$, and denote by $\alpha_2$ the label of its image $f(v) \in D_2$. Then $\alpha_2$ is said to be induced from $\alpha_1$ by $f$ if it is obtained by the above construction.

By combining this definition with Theorem 3.7 on vertex-preserving homomorphisms of planar 2-category diagrams, one can now prove that the evaluations of planar 2-category diagrams that are related by homomorphism are equal, provided that the labels on the vertices of $D_2$ are induced from the ones on $D_1$.

**Theorem 3.9.** Let $D_1$ and $D_2$ planar 2-category diagrams and $f : D_1 \to D_2$ homomorphism of planar 2-category diagrams such all vertex labels of $D_2$ are induced from the ones of $D_1$ by $f$. Then the evaluations of the two diagrams are equal.

**Proof.** Denote by $E$ the diagram in which all vertices are rotated as described before Definition 3.8 and by $g : E \to F$ the associated homomorphism of diagrams such that $f \circ g : E \to D_2$ is vertex-preserving. If the labels of $D_2$ are induced from the ones of $D_1$ by $f$, then by Definition 3.8, the evaluation of $D_1$ is the same as the evaluation of diagram $E$, in which all vertices are rotated. As the homomorphism $f \circ g : E \to D_2$ is vertex-preserving, the evaluation of $E$ is equal to the evaluation of $D_2$ by Theorem 3.7. □

### 3.3. Gray categories with duals.

The following definition is derived from the axioms that were first given by Baez and Langford [1] and Mackaay [24]. The main difference is that here only two independent duals are considered, whereas the previous authors defined three. The axioms are adapted from [1]. By referring to a planar 2-category these axioms can be cast into a more concise form.

**Definition 3.10 (Gray category with duals).** A Gray category with duals is a Gray category $G$ with the following additional structure:

1. For all objects $C, D$ of $G$, the 2-category $G(C, D)$ is planar, and its dual $\ast$ is compatible with the Gray product:

   $$(K \Box \mu \Box H)^\ast = K \Box \mu^\ast \Box H, \quad \text{and} \quad K \Box \epsilon_{\mu} \Box H = \epsilon_{K \Box \mu \Box H},$$

   for all 1-morphisms $H, K$ and 2-morphisms $\mu$ for which these expressions are defined.

2. For every 1-morphism $F : C \to D$, there is a dual 1-morphism $F^\# : D \to C$, a 2-morphism $\eta_{F} : 1_D \Rightarrow F \Box F^\#$, called fold,
and an invertible 3-morphism $T_F : (\eta_F^* \Box F) \circ (F \Box \eta_F^*) \Rightarrow 1_F$, called the triangulator, such that the following conditions are satisfied:

(a) $F^{##} = F$ for all 1-morphisms $F : C \to D$,
(b) $1_C^{##} = 1_C$, $\eta_{1C} = 1_{1C}$, $T_{1C} = 1_{1C}$ for all objects $C$,
(c) $(F \Box G)^{##} = G^{##} \Box F^{##}$, $\eta_{F \Box G} = (F \Box \eta_G \Box F^{##}) \circ \eta_F$,
\[
T_{F \Box G} = (T_F \Box G \circ F \Box T_G) \cdot (1_{\eta_F^* \Box F \Box \eta_G^*} \circ \sigma_{F \Box \eta_G \Box F^{##}} \circ 1_{1F \Box \eta_G^*})
\]
for all composable 1-morphisms $F : D \to D$, $G : C \to D$,
(d) $(1_{\eta_F^*} \circ T_F \Box F^*) \cdot (\sigma_{\eta_F^*, \eta_F^*} \circ 1_{F \Box \eta_F^*} \Box F^*) \cdot (1_{\eta_F^*} \circ F \Box T_F^*) = 1_{\eta_F^*}$.

The relation with the notation of \cite{1} is as follows. Firstly, the work \cite{1} considers monoidal 2-categories, so what is termed here an $n$-morphism is the present paper is called an $n-1$-morphism there. The duality on 2-morphisms (which they call 1-morphisms) is denoted $*$ in both works, and is extended here to 3-morphisms by the operation which is called an adjoint in \cite{1}. The duality on 3-morphisms in \cite{1} (which they call 2-morphisms) does not appear in this paper, and hence their constraints that specify that the tensorator and triangulator are unitary are relaxed here to the conditions that these morphisms are invertible.

The duality on 1-morphisms is denoted $#$ here and corresponds to the $*$-dual on 1-morphisms in \cite{1} (which they call objects). There are two axioms in Definition 3.10 that have no analogue in \cite{1}, namely the conditions $1_C^{##} = 1_C$ and $\eta_{1C} = 1_{1C}$.

Gray categories with duals can be viewed as a generalisation of braided pivotal tensor categories. As a direct consequence of the axioms in Definition 3.10, one obtains the following lemma.

**Lemma 3.11.** If $\mathcal{G}$ is a Gray category with duals, then for every object $C$ the category $\mathcal{G}(1_C, 1_C)$ is a braided strict pivotal tensor category. Conversely, a braided strict pivotal tensor category is a Gray category with duals with a single object and a single 1-morphism.

**Proof.** By Definition, the $##$-dual of the 1-morphism $1_C$ is trivial $##1_C = 1_C$ for each object $C$, and so are the associated 2-morphism $\eta_{1C}$ and the triangulator $T_{1C}$. The condition that $\mathcal{G}(1_C, 1_C)$ is a planar 2-category then becomes equivalent to the statement that $\mathcal{G}(1_C, 1_C)$ is a strict pivotal tensor category with the tensor product given by the horizontal composition $\circ$, the tensor unit by $1_{1C} : 1_C \Rightarrow 1_C$ and the pivotal structure by the 3-morphism $\epsilon_\mu : 1_{1C} \Rightarrow \mu \circ \mu^*$. As the Gray product of 1-, 2- and 3-morphisms with the 1-morphism $1_C$ is trivial, the axioms on the tensorator in Definition 2.14 reduce to the axioms for a braiding $\sigma_{\mu, \nu} : \mu \circ \nu \Rightarrow \nu \circ \mu$.

Conversely, if $\mathcal{G}$ is a Gray category with a single object $C$ and a single 1-morphism, then the 1-morphism is given by $1_C$. The axioms of a Gray
category with duals then imply that the \#-dual, the fold 2-morphisms and the triangulator are trivial and \( G(1_C, 1_C) \) is a braided strict pivotal tensor category.

As indicated by this lemma, some identities which are familiar from braided tensor categories have a direct analogue in Gray categories with duals. These similarities are also apparent in the diagrammatic calculus for Gray categories with duals introduced in the next subsection. A specific example is the following lemma and the associated diagrams in Figure 20.

**Lemma 3.12.** Let \( G \) be a Gray category with duals. Then for all 2-morphisms \( \mu, \mu' : F \Rightarrow G, \nu : H \Rightarrow K \) and all 3-morphisms \( \Phi : \mu \Rightarrow \mu' \) for which these expressions are defined, one has

\[
(K \Box \Phi \Box H)^* = K \Box \Phi^* \Box H, \quad \sigma_{\mu, \nu}^* = \sigma_{\mu^*, \nu^*},
\]

\[
(1_{(\mu \Box K) \circ (F \Box \nu)} \circ \epsilon_{\mu, \nu}^* \Box H) \cdot (1_{\mu \Box K} \circ \sigma_{\mu^*, \nu^*} \circ 1_{\mu \Box H}) \cdot (\epsilon_{\mu \Box K} \circ 1_{(G \Box \nu) \circ (\mu \Box H)}) = \sigma_{\mu, \nu}^{-1},
\]

\[
(1_{(G \Box \nu) \circ (\mu \Box H)} \circ \epsilon_{F \Box \nu} \Box F) \cdot (1_{G \Box \nu} \circ \sigma_{\mu^*, \nu^*} \circ 1_{F \Box \nu}) \cdot (\epsilon_{G \Box \nu} \circ 1_{(\mu \Box K) \circ (F \Box \nu)}) = \sigma_{\mu, \nu}.
\]

**Proof.** The first identity follows directly from the definition of the dual 3-morphisms in terms of the 3-morphism \( \epsilon_\mu \) in equation (13) and from condition (1) in Definition 3.10. The second identity follows from the third and the fourth. These two identities are direct consequences of the properties of the tensorator together with condition (1) in Definition 3.10.

\[\Box\]

3.4. **Diagrammatic representation of the Gray category duals.**

The set of progressive Gray category diagrams is sufficient to express the axioms of a Gray category with duals in diagrammatic form. The diagrammatic representation of non-progressive diagrams will be discussed in Section 6. Each of the canonical 2- and 3-morphisms in Definition 3.10 for a Gray category with duals determines a canonical diagrammatic element as follows:

- As for planar 2-categories, the 3-morphisms \( \epsilon_\mu : 1_G \Rightarrow \mu \circ \mu^* \) are canonical vertices that correspond to maxima and minima of the lines. The Gray category diagram for \( \epsilon_\mu \) is obtained by drawing the corresponding diagram for a planar category in Figure 11(b) on a plane labelled with two 1-morphisms \( F, G \) as shown in Figure 14(c). As shown, the vertex is not labelled by any morphism. By convention this means that the morphism at this vertex is \( \epsilon_\mu \). One of the lines meeting this vertex is labelled with \( \mu \) and the other with \( \mu^* \). It is therefore only necessary to show the label of one of these lines as the other label is then uniquely determined. This convention will be used in the following.

41
The compatibility condition (1) in Definition 3.10 which involves the Gray product of 1-morphisms with the 3-morphisms \( \epsilon_\mu : 1_G \Rightarrow \mu \circ \mu^* \) is shown in Figure 14 e).

- The diagram for the fold 2-morphism \( \eta_F : 1_D \Rightarrow F \square F^\# \) is a line with two planes attached to the right, as shown in Figure 15 c). The convention is that the label for this line is not shown, and only one of the two surfaces incident to the line is labelled. This diagram is in fact an identity diagram \( 1_D \), where \( D \) is the elementary pre-2-category diagram for \( \eta_F \) in \( G_2 \). The *-dual \( \eta_F^* \) is represented by a diagram with two planes on the left, as shown in Figure 15 d). The condition \( \eta_{1_C} = 1_{1_C} \) states that the 2-morphism \( \eta_{1_C} \) corresponds to an empty diagram. For better legibility of the diagrams, the 2-morphisms \( \eta_F \) and their *-duals will also sometimes be drawn as rounded lines in the following. This does not affect any of the results.

- The invertible 3-morphism \( T_F : (\eta_F^* \square F) \circ (F \square \eta_F^*) \Rightarrow 1_F \) for each 1-morphism \( F : C \to D \) corresponds to the Gray category diagram in Figure 15 g), and its inverse is depicted in Figure 16 a). As in the case of the 2-morphisms \( \eta_F \), it is not necessary to label the vertex in this diagram or the lines incident at the vertex, and only one of the incident surfaces is labelled.

The compatibility of the 2-morphisms \( \eta_F \) and the 3-morphism \( T_F \) with the Gray product \( \square \) (condition c) in Definition 3.10), relates the 2-morphisms \( \eta_{F \square G} \) and the 3-morphisms \( T_{F \square G} \) to the corresponding 2- and 3-morphisms \( \eta_F, \eta_G \) and \( T_F, T_G \). It states that the two diagrams in Figure 16 c), e) have a well-defined evaluation. Condition (d) in Definition 3.10 and the invertibility of the 3-morphisms \( T_F \) are depicted, respectively, in Figure 17 a), b) and c) and their projections in Figure 18.

By composing these diagrammatic elements, one obtains diagrams for all structural data and relations of a Gray category with duals. For instance, the diagrams for the canonical 3-morphisms \( \epsilon_{\eta_F} : 1_{F \square F^\#} \Rightarrow \eta_F \circ \eta_F^* \) and \( \epsilon_{\eta_F}^* : 1_D \Rightarrow \eta_F^* \circ \eta_F \) for a 1-morphism \( F : C \to D \) are given in Figure 19 a), b). The identities

\[
(\epsilon_{\eta_F}^* \circ 1_{\eta_F^*}) \cdot (1_{\eta_F^*} \circ \epsilon_{\eta_F}) = 1_{\eta_F^*}, \quad (\epsilon_{\eta_F}^* \circ 1_{\eta_F}) \cdot (1_{\eta_F} \circ \epsilon_{\eta_F}) = 1_{\eta_F}
\]

from condition 12 in the definition of a planar 2-category correspond to the diagrams in Figure 19 c), d).

As in the case of diagrams for tensor categories, Gray category diagrams prove useful for computations in a Gray category with duals. An example is the proof of the last three identities in Lemma 3.12 which is performed diagrammatically in Figure 20. This diagrammatic proof clearly exhibits the structural similarities between Gray categories with duals and braided pivotal tensor categories.

42
It also becomes apparent that diagrammatic calculations are much simpler than their symbolic counterparts. For these reasons, they will be used extensively in the following sections. Note, however, that at this stage the diagrammatic calculus is to be understood as a calculation in a Gray category with duals that is based on the evaluation of progressive Gray category diagrams.

Although the diagrams for Gray categories with duals involve lines with maxima and minima, planes with folds (denoting the 2-morphisms $\eta_F$) and cusps (denoting the 3-morphisms $T_F$) these diagrams are progressive. The maxima and minima, folds and cusps simply indicate a canonical labelling of the lines and vertices in a progressive Gray category diagram.

The diagrammatic calculations use the fact that the evaluation of a generic progressive Gray category diagram is invariant under certain isomorphisms of progressive diagrams (Theorem 2.26) and the axioms of a Gray category with duals. In particular, the diagrams in Figure 17 represent relations between certain 3-morphisms in a Gray category.
Figure 15. Diagrams for 

a) 1-morphism $F : \mathcal{C} \to \mathcal{D}$, b) its dual $F^\# : \mathcal{D} \to \mathcal{C}$, 
c) Fold $\eta_F : 1_D \Rightarrow F \Box F^\#$, d) its dual $\eta_F^* : F \Box F^\# \Rightarrow 1_D$.

e) Projection of c), f) projection of d).
g) Triangulator $T_F : (\eta_F^* \Box F) \circ (F \Box \eta_F^*) \Rightarrow 1_F$, h) its projection.
Figure 16. Diagrams for \#:
a) Inverse of the triangulator and b) its projection.
c) Fold $\eta_{FG}$ and d) its projection.
e) Triangulator $T_{FG}$ and f) its projection.
Figure 17. Consistency conditions for triangulators.
a) Diagrams for the identity \((1_{\eta^*_F} \circ T_F \square F^\#) \cdot (\sigma_{\eta^*_F, \eta^*_F} \circ F \square \eta^*_F \square \eta^*_F) \cdot (1_{\eta^*_F} \circ F \square T^*_F \square F^\#) = 1_{\eta^*_F}\).

b) Diagrams for the identity \(T_F \cdot T_F^{-1} = 1_{1_{\eta^*_F}}\).

c) Diagrams for the identity \(T_F^{-1} \cdot T_F = 1_{\eta^*_F \circ F \circ \eta^*_F \#}\).
Figure 18. Consistency conditions for triangulators.
a) Projection of the diagrams in Figure 17 a).
b) Projection of the diagrams in Figure 17 b).
c) Projection of the diagrams in Figure 17 c)
Figure 19. Gray category diagrams:

a) 3-morphism $\epsilon_{\eta_F}: 1_{F \square F^#} \Rightarrow \eta_F \circ \eta^*_F$,
b) 3-morphism $\epsilon_{\eta^*_F}: 1_{1_D} \Rightarrow \eta^*_F \circ \eta_F$,
c) identity \((\epsilon^*_{\eta^*_F} \circ 1_{\eta^*_F}) \cdot (1_{\eta^*_F} \circ \epsilon_{\eta^*_F}) = 1_{\eta^*_F}\),
d) identity \((\epsilon^*_{\eta^*_F} \circ 1_{\eta_F}) \cdot (1_{\eta_F} \circ \epsilon_{\eta^*_F}) = 1_{\eta_F}\).
Figure 20. Gray category diagrams for Lemma 3.12

a) identity \((1_{(\mu \square K) o (F \square \nu)} \circ \epsilon_{\mu, \nu}) \cdot (1_{\mu \square K} \circ \sigma_{\mu, \nu} \circ 1_{\mu \square H}) \cdot (\epsilon_{\mu \square K} \circ 1_{(G \square \nu) o (\mu \square H)}) = \sigma_{\mu, \nu}^{-1}\),
b) identity \((1_{(G \square \nu) o (\mu \square H)} \circ \epsilon_{\mu, \nu}^{-1}) \cdot (1_{G \square \nu} \circ \sigma_{\mu, \nu} \circ 1_{F \square \nu}) \cdot (\epsilon_{G \square \nu} \circ 1_{(\mu \square K) o (F \square \nu)}) = \sigma_{\mu, \nu},
c) diagrammatic proof of a),
d) diagrammatic proof of the identity \(*\sigma_{\mu, \nu} = \sigma_{\mu, \nu}.*

\[49\]
with duals and are not to be interpreted as invariance of the evaluation under certain homomorphisms of diagrams at this stage.

Non-progressive Gray category diagrams and the associated homomorphisms of diagrams will be investigated in Section 6. In particular, it will be shown there that the diagrams in Figure 17 have an interpretation as Whitney moves relating surface projections.

4. Duals as functors of strict tricategories

In this section it is shown that the duals * and # in Definition 3.10 define functors of strict tricategories (see Definition A.6 in Appendix A). As these functors will reverse different products in the Gray categories, this requires a notion of different opposites for Gray categories and functors thereof.

4.1. Higher opposites. The opposite 2-categories $C^{op}$, $C^{op}$ and contravariant functors associated to strict 2-functors $C \to D^{op}$ were introduced in Definition 3.1. Here, the discussion is extended to weak 2-functors. These constructions lead to the definition of two notions of the opposite of a Gray category.

A weak 2-functor of 2-categories $F : C \to D$ (Definition A.1) has the following data

- A function $F_0 : \text{Ob}(C) \to \text{Ob}(D)$.
- For all objects $G, H$ of $C$, a functor $F_{G,H} : C_{G,H} \to D_{F_0(G),F_0(H)}$.
- For all 1-morphisms $\nu : G \to H, \mu : H \to K$, a 2-isomorphism $\Phi_{\mu,\nu} : F_{H,K}(\mu) \circ F_{G,H}(\nu) \to F_{G,K}(\mu \circ \nu)$.
- For all objects $G$, a 2-isomorphism $\Phi_G : 1_{F_0(G)} \to F_{G,G}(1_G)$.

The opposite $F^{op} : C^{op} \to D^{op}$ is determined by the following data

- $(F^{op})_0 = F_0$
- $(F^{op})_{H,G} = (F_{G,H})^{op}$
- $(\Phi^{op})_{\nu,\mu} = (\Phi_{\mu,\nu}^{-1})^{op}$
- $(\Phi^{op})_G = (\Phi_G^{-1})^{op}$,

where the right-hand involves the 1-categorical opposites defined in Section 3.1. The corresponding opposite weak 2-functor $F^{op} : C^{op} \to D^{op}$ is determined by

- $(F^{op})_0 = F_0$
- $(F^{op})_{H,G} = F_{G,H}$
- $(\Phi^{op})_{\nu,\mu} = \Phi_{\mu,\nu}$
- $(\Phi^{op})_G = \Phi_G$.

Note that the choice of the coherence data for $F^{op}$ and $F^{op}$ is determined unambiguously by the coherence data of $F$ and the source and target of the coherence isomorphism. Hence it is justified to abuse notation and denote the functors $F^{op}$ and $F^{op}$ again $F$. In the following, we will use the notation $F^{op}$ and $F^{op}$ only to emphasise their relations.
These notions allow the definition of two types of opposite Gray category. The key point is the opposite of a Gray category is a tricategory but it is not necessarily a Gray category in the sense of Definition 2.17 since, depending on which products are reversed, the resulting strict tricategory can be cubical instead of opcubical.

**Definition 4.1.** Let $\mathcal{G}$ be a Gray category.

1. The tricategory $\mathcal{G}^{\text{op}}$ has the same Gray product but opposite horizontal and vertical composition. Thus the 2-categories are $\mathcal{G}^{\text{op}}(\mathcal{C}, \mathcal{D}) = (\mathcal{G}(\mathcal{C}, \mathcal{D}))^{\text{op}}$ and the Gray product is the collection of opposite weak 2-functors $\Box^{\text{op}}$. Thus, in the abbreviated notation introduced above and in Definition 3.1 compositions of $\mathcal{G}^{\text{op}}$ are denoted $\Box, \tilde{\circ}$ and $\tilde{\cdot}$.

2. The tricategory $\mathcal{G}^{\text{op}}$ has the same vertical composition but opposite Gray product and horizontal composition: $\mathcal{G}^{\text{op}}(\mathcal{C}, \mathcal{D}) = (\mathcal{G}(\mathcal{D}, \mathcal{C}))^{\text{op}}$. The Gray product for $\mathcal{G}^{\text{op}}$ is the collection of weak 2-functors $\tilde{\Box}^{\text{op}}$ where $\Phi \tilde{\Box} \Psi = \Psi \tilde{\Box} \Phi$ for all composable 2- and 3-morphisms $\Phi, \Psi$. Thus, in abbreviated notation, the compositions of $\mathcal{G}^{\text{op}}$ are $\tilde{\Box}, \tilde{\circ}$ and $\cdot$.

There is the analogous definition of the opposites $\mathcal{T}^{\text{op}}$ and $\mathcal{T}^{\text{op}}_{\text{op}}$ of a strict cubical tricategory $\mathcal{T}$. It is clear that $(\mathcal{T}^{\text{op}})^{\text{op}} = \mathcal{T}$ and $(\mathcal{T}^{\text{op}}_{\text{op}})^{\text{op}} = \mathcal{T}$. In the following, we will not distinguish the objects and morphisms of the opposite categories in our notation and only denote the compositions in $\mathcal{T}^{\text{op}}$ and $\mathcal{T}^{\text{op}}_{\text{op}}$ with the appropriate $\text{op}$-label.

The expressions for the coherence isomorphisms and tensorators of the opposite strict tricategories are obtained by unpacking these definitions. Using the notation introduced above, in Lemma 2.15 and Corollary 2.16 one finds that if $\mathcal{T}$ is a Gray category with has structure isomorphisms $\Box_{\mu, \nu}$, then $\mathcal{T}^{\text{op}}$ has the structure isomorphism $(\Box^{\text{op}})_{\mu, \nu} = (\Box_{\nu, \mu}^{\text{op}})^{\text{op}}$, and $\mathcal{T}^{\text{op}}_{\text{op}}$ has $\Box^{\text{op}, \mu, \nu} = \Box^{\mu', \nu'}$, with $\mu' = (\mu_2, \mu_1), \nu' = (\nu_2, \nu_1)$. The following statement then follows directly from the definitions.

**Lemma 4.2.** Let $\mathcal{T}$ be a strict cubical (opcubical) tricategory. Then $\mathcal{T}^{\text{op}}_{\text{op}}$ is again a strict cubical (opcubical) tricategory, while $\mathcal{T}^{\text{op}}$ is a strict opcubical (cubical) tricategory.

Finally, there are notions of opposite functors for strict tricategories.

Let $F: \mathcal{T} \rightarrow \mathcal{S}$ be a functor of strict tricategories. This has data (Definition A.6)

- a function $F_0 : \text{Ob}(\mathcal{T}) \rightarrow \text{Ob}(\mathcal{S})$,
- weak 2-functors $F_{\mathcal{C}, \mathcal{D}} : \mathcal{T}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{S}(F_0(\mathcal{C}), F_0(\mathcal{D}))$ for all objects $\mathcal{C}, \mathcal{D}$ of $\mathcal{T}$
- For $\Box$-composable 2-morphisms $\mu$ and $\nu$, 3-isomorphisms $\kappa_{\mu, \nu} : F(\mu) \Box F(\nu) \rightarrow F(\mu \Box \nu)$

The first opposite is the functor of strict tricategories $F^{\text{op}} : \mathcal{T}^{\text{op}} \rightarrow \mathcal{S}^{\text{op}}$ with data.
• $(F^\circ) = F_0$
• $(F^\circ)_{C,D} = (F_{C,D})^{op}$, using the opposite of weak 2-functors
• $(\kappa_{\circ})_{\nu,\mu} = (\kappa_{\nu,\mu})^{op}$.

The second opposite is the functor of strict tricategories $F_{op}: T_{op} \to S_{op}$ with data

• $(F_{op})_0 = F_0$
• $(F_{op})_{C,D} = (F_{C,D})_{op}$, using the opposite of weak 2-functors
• $(\kappa_{op})_{\nu,\mu} = \kappa_{\nu,\mu}$.

Again, all coherence data is unambiguous and we denote the functors $F^{op}$ and $F_{op}$ by $F$.

### 4.2. Duals as functors of strict tricategories.

To define a functor of strict tricategories $^\ast: \mathcal{G} \to \mathcal{G}^{op}$, the dual $^\ast$ in the planar 2-categories $\mathcal{G}(C, D)$is extended trivially to the objects and 1-morphisms of $\mathcal{G}$. Similarly, the dual $#$ is extended trivially to the objects of $\mathcal{G}$. To extend it to 2- and 3-morphisms, we define for each 2-morphism $\nu: F \Rightarrow G$ and 3-morphism $\Phi: \mu \Rightarrow \nu$ the associated #-duals $\#\nu: G^{#} \Rightarrow F^{#}$, $\#\Phi: \#\mu \Rightarrow \#\nu$ by

\[
\#\nu = (F^{#} \Box \eta_{G}^{\circ}) \circ (F^{#} \Box \nu \Box G^{#}) \circ (\eta_{F}^{#} \Box G^{#})
\]

\[
\#\Phi = 1_{F^{#} \Box \eta_{G}^{\circ}} \circ (F^{#} \Box \Phi \Box G^{#}) \circ 1_{\eta_{F}^{#} \Box G^{#}}.
\]

The diagrams for $\#\nu$ and $\#\Phi$ are given in Figure 21 a). They correspond to folding the plane segment labelled by the 1-morphism $F$ to the front and the plane segment labelled by $G$ to the back of the cube.

The operations $^\ast$ and $#$ reverse some of the products and so extend to contravariant functors of different types. The passage from a duality operation to a functor $F$ representing the duality is as follows. All the mappings in the definition of the functor $F$, i.e. $F_0$ and $F_{C,D}$ in the notation of Definition A.6, are given directly by the duality operation, with the result regarded as an object or morphism in the appropriate opposite category. For example, for the functor $^\ast: \mathcal{G} \to \mathcal{G}^{op}$, $^\ast\nu$ is regarded as $(\nu^{\circ})^{op}$ in $\mathcal{G}^{op}$ for a 2-morphism $\nu$ in $\mathcal{G}$.

### Theorem 4.3 (Duals as functors of strict tricategories).

1. The duality operation $^\ast$ extends to a strict functor of strict tricategories $^\ast: \mathcal{G} \to \mathcal{G}^{op}$ in the sense of Definition A.8.
2. The duality operation $#$ extends to a weak functor of strict tricategories $#: \mathcal{G} \to \mathcal{G}_{op}$ in the sense of Definition A.6.

**Proof.**

1. The data for the functor $^\ast$ is

- The identity mapping on objects,
- The strict 2-functors $^\ast_{C,D}: \mathcal{G}(C, D) \to \mathcal{G}^{op}(C, D)$ defined by $\phi \mapsto (\phi^{*})^{op}$ using the planar structure of $\mathcal{G}(C, D)$,
- natural isomorphisms $\rho_{B,C,D}: \Box^{op} (^\ast_{C,D} \times ^\ast_{B,C}) \to ^\ast_{B,D} \Box$ of weak 2-functors defined by components $\rho_{\mu,\nu} = (\sigma_{\mu,\nu}^{*})^{op}$.
To check that this is a strict functor of strict tricategories, we verify the axioms from Definition [A.6] and Definition [A.8]. Since for composable 2-morphisms $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow K$ in $\mathcal{G}$,

$$(\ast(\mu \Box \nu))' = ((F \Box \nu') \circ (\mu' \Box K))' = ((\mu')' \Box (\nu')')',$$

with $\Box$ defined as in Corollary [2.16], this shows that $\ast$ defines a functor of strict tricategories $\ast : \mathcal{G} \rightarrow \mathcal{G}^{op}$. The coherence data of the functor $\ast$ is obtained by composing $\ast$ with the functor $\Sigma : \mathcal{G}^{op} \rightarrow \mathcal{G}^{op}$ from Corollary [2.16] $\ast = \Sigma \circ \ast$. This produces precisely the natural isomorphism $\rho_{B,C,D}$ given above and shows that $\ast$ is indeed a strict functor of strict tricategories in the sense of Definitions [A.6] and [A.8].

2. As the duality $\#$ is the identity on the objects of $\mathcal{G}$, it is sufficient to show that for each pair of objects $C, D$ the dual $\#$ defines weak 2-functors $\#_{C,D} : \mathcal{G}(C, D) \rightarrow \mathcal{G}_{op}(C, D)$ and there are natural isomorphisms $\kappa_{B,C,D} : \Box(\#_{C,D} \times \#_{B,C}) \rightarrow \#_{B,D} \Box$ of weak 2-functors satisfying the conditions specified in Definition [A.6].

To show that $\#$ defines a weak 2-functor $\#: \mathcal{G}(C, D) \rightarrow \mathcal{G}_{op}(C, D)$, we note that for all objects $F$ in $\mathcal{G}(C, D)$

$$\#_{1_F} = (F^\# \square \eta^*_F) \circ (\eta_{F^\#} \square F^\#).$$

This implies that the $\ast$-dual of the triangulator defines an invertible 3-morphism $\Phi_F = T^*_F : 1_{F^\#} \Rightarrow \#_{1_F}$ with $\Phi_{1_C} = 1_{1_C}$ in $\mathcal{G}_{op}$. For each pair of composable 1-morphisms $\nu : F \Rightarrow G$, $\rho : E \Rightarrow F$ in $\mathcal{G}(C, D)$, one obtains a 2-morphism $\Phi_{\rho,\nu} : \# \rho \circ \# \nu \Rightarrow \# (\nu \circ \rho)$ in $\mathcal{G}_{op}(C, D)$ by

$$\Phi_{\rho,\nu} = (1_{E^\#} \eta^*_G \circ \#_{\nu G^\#} O E^\# T_F G^\# \circ 1_{E^\#} \rho G^\# \circ \eta_{G^\#} G^\#)
\cdot (\sigma_{E^\# \eta^*_F \circ \#_{\nu G^\#} G^\#} O 1_{E^\#} F \eta_{F^\#} G^\# \circ 1_{E^\#} \rho G^\# \circ \eta_{G^\#} G^\#)
\cdot (1_{E^\#} \eta^*_F \circ \sigma_{E^\# \rho G^\# \circ \eta_{G^\#} \#_{\nu G^\#}}).$$

The corresponding Gray diagram and its projection are given in Figure [21]. It follows directly from the invertibility of the triangulator and the tensorator that $\Phi_{\rho,\mu}$ is invertible, and the naturality of the tensorator implies that it is natural in both arguments. It remains to prove the identities

$$(16) \quad \Phi_{1_F,\nu} \cdot (\Phi_F \circ \#), = 1_{\#}, = \Phi_{\nu,1_C} \cdot (\# \circ \Phi_G)$$

and the commutativity of the diagram

$$(17) \quad \Phi_{\rho,\nu} \circ \# = \Phi_{\rho \circ \nu,1_C} \cdot \# (\mu \circ \nu)$$

which correspond to the two consistency conditions in Definition [A.1]. For this, note that for a 1-morphism $F : C \rightarrow D$, the 3-morphism $\Phi_{1_F,\nu}$
is given by

\[ \Phi_{F,\nu} = (1_{F^\mu G^\nu} \circ F^\Phi T_{F^\Phi} \circ 1_{F^\mu G^\nu}) \cdot (1_{F^\mu G^\nu} \circ 1_{F^\mu F^\mu G^\nu} \circ 1_{F^\mu G^\nu}) \cdot (1_{F^\mu G^\nu} \circ 1_{F^\mu G^\nu}). \]

Composing this expression with \( T_{F^\Phi} \circ 1_{#\nu} \), one finds that the conditions in (16) follow from the naturality of the tensorator \( \sigma \), together with the invertibility of \( T_F \) and identity (d) in Definition 3.10 - see also the third diagram in Figure 17. A diagrammatic proof is given in Figure 22. The commutativity of the diagram (17) follows from the naturality of the tensorator \( \sigma \) together with the invertibility of \( T_F \) and the exchange law for 2-categories. A diagrammatic proof is given in Figure 23. This shows that for all objects \( C, D \) of \( G \), the duality \( # \) defines a weak 2-functor \( #_{C, D} : G(C, D) \to G(C, D)_{op} \).

3. To show that the four consistency conditions in Definition A.6 are satisfied for the functor \( # \), note that the operation \( # \) satisfies \( 1_# = 1_\cdot \).

The natural isomorphisms \( \kappa_{B,C,D} : (\#C \times #B) \to #B \square C \) from Definition A.6 are thus specified by their component 3-morphisms

\[ \kappa_{\mu,\nu} : (#\mu) \square (#\nu) = (#\nu) \square (#\mu) \Rightarrow (#(\mu \square \nu)). \]

These 3-morphisms define natural isomorphisms of weak 2-functors if and only if they are natural in both arguments, invertible and the following two diagrams commute

\[ (\#\nu \square #\mu) \circ (#\tau \square #\rho) \xrightarrow{\Phi_{\mu,\nu \square \tau}} #((#\mu \square \nu) \circ (#(\rho \square \tau))) \]

\[ 1_{#G \square #1_{F^\Phi}} = 1_{(F \square G)^\Phi}, \]

where the two vertical arrows labelled with tensorators arise from the definition of the Gray products in Lemma 2.15 and Corollary 2.16.
Figure 21. Diagrams for \#:

a),b) 3-morphism \#\Phi : \#\nu \Rightarrow \#\mu \text{ and its projection.}

c),d) 3-morphism \Phi_{\rho,\nu} : \#\rho \circ \#\nu \Rightarrow \#(\nu \circ \rho), \text{ its projection.}

e),f) 3-morphism \kappa_{\mu,\nu} : \#\nu \Box \#\mu \Rightarrow \#(\mu \Box \nu) \text{ and its projection. Some labels are omitted for legibility.}
Figure 22. Diagrammatic proof of the identity
\[ \Phi_{1_F, \nu} \cdot (\Phi_F \circ 1_{#\nu}) = 1_{#\nu} = \Phi_{\nu, 1_G} \cdot (1_{#\nu} \circ \Phi_G) \]
Figure 23. Diagrammatic proof of the identity
\( \Phi_{\nu \circ \rho, \mu} \cdot (\Phi_{\rho, \nu} \circ 1_{\#\mu}) = \Phi_{\rho, \mu} \circ (1_{\#\rho} \circ \Phi_{\nu, \mu}). \)
Figure 24. Diagrammatic proof of the identity (18):
\[ \#(1 \square \sigma^{-1}_\tau \square 1) \cdot \Phi_{\mu \nu \rho \tau} \cdot (\kappa_{\mu \nu} \circ \kappa_{\rho \tau}) = (\kappa_{\rho \mu, \tau \nu}) \cdot (\Phi_{\nu \tau} \square \Phi_{\mu \rho}) \cdot \sigma^{-1} \cdot \sigma_{\# \tau, \# \mu}. \]
Figure 25. Diagrammatic proof of the identity (19):
\[ \Phi^{-1}_{FGG} \cdot \kappa_{1G}^{-1}_F = \Phi^{-1}_F \boxtimes \Phi^{-1}_G. \]
Figure 26. Diagrammatic proof of the identity (20):
\[ \kappa_{\rho,\nu} \Box_{\mu} \cdot (\kappa_{\nu,\mu} \Box 1) = \kappa_{\rho,\nu} \cdot (1 \Box_{\mu} \kappa_{\rho,\nu}). \]
Condition (1) in Definition A.6 is equivalent to the commutativity of diagram

\[
\begin{array}{c}
\square\mu \rightarrow \square\nu \rightarrow \square\rho \rightarrow \square\kappa \\
\downarrow \kappa_{\nu,\mu} \downarrow \kappa_{\rho,\nu} \downarrow \kappa_{\rho,\nu}
\end{array}
\]

and conditions (2), (3) to the equations \(\kappa_{\mu,1_C} = \kappa_{1_D,\mu} = 1_\mu\) for all \(\mu \in G(C,D)\).

For composable 2-morhisms \(\mu : F \Rightarrow G, \nu : H \Rightarrow K\), we define a 3-morphism \(\kappa_{\mu,\nu} : \#(\nu \square \mu) \rightarrow \#(\mu \square \nu)\) by

\[
\kappa_{\mu,\nu} = (1_{\nu\square F} \circ 1_{H\square F} \# G_{\nu}\# \sigma_{\mu\square H} \# G \cdot 1_{\nu\square H}) \cdot (1_{\nu\square F} \circ H \# G_{\nu} \sigma_{\mu\square H} \# G \cdot 1_{\nu\square H}) \cdot \sigma_{\nu\square H} \cdot \kappa_{\mu,\nu}.
\]

This 3-morphism and its projection are shown in Figure 21 e), f). It follows directly from the definition of the tensorator that the 3-morphisms \(\kappa_{\mu,\nu}\) are invertible, satisfy the conditions \(\kappa_{\mu,1_C} = \kappa_{1_D,\mu} = 1_\mu\) and are natural in both arguments. It is therefore sufficient to establish the commutativity of the diagrams in (18), (19) and (20). A diagrammatic proof of these identities is given, respectively, in Figures 24, 25 and 26.

**Lemma 4.4.** The functor of strict tricategories \(* : G \rightarrow G_{\text{op}}^{\text{op}}\) satisfies \(* * = 1\).

**Proof.** Note that strictly speaking this identity should be written \(*_{\text{op}} *_{\text{op}} = 1_G\), where we identify \((G_{\text{op}})^{\text{op}} = G\). That the mappings of the functor \(* *\) are given by the identity follows directly from the fact that \(*\) is trivial on the objects and 1-morphisms and the 2-categories \(G(C,D)\) are planar. It remains to show that the coherence morphisms of \(* *\) are the identities. Recall that the coherence data of \(*\) is given by \(\sigma\), i.e. the components of the transformation \(\rho_{G,C,D} : \square^{op} (\ast_{\text{op}} \times \ast_{\text{op}} C) \rightarrow \ast_{\text{op}}\square\) are \(\rho_{\mu,\nu} = (\sigma_{\mu,\nu})^{op}\). According to the definition of \(F^{op}\) for a functor \(F\) of strict tricategories, the components of the corresponding transformation for \(*^{op}\) are given by \(\sigma_{\mu,\nu}^{-1}\) and hence the composition \(* *\) is the identity functor.

**Theorem 4.3** gives a more conceptual understanding of the duals in terms of functors of strict tricategories rather than the concrete axioms in Definition 3.10. These functors of strict tricategories are related to certain symmetries of the cube. The \(*\)-dual does not affect the 1-morphisms in \(G\) and corresponds to a 180 degree rotation around the
$w$-axis of the diagrams, and the dual #-corresponds to a 180 degree
degree rotation around the $y$-axis.

Lemma 4.4 states that the operation $*$ satisfies $** = 1$, as expected
for the composite of two 180 degree rotations around the same axis.
However $###$ and $**#*#$ are not equal to the identity, although the
associated compositions of rotations are. The reason for this is that
in a higher category, one can in general only expect such relations to
hold up to higher morphisms. In the case at hand, one obtains natural
isomorphisms of functors of strict tricategories (Definition A.9).

**Theorem 4.5.** There are natural isomorphisms of functors of strict
tricategories $\Gamma : **#*# \to 1, \Theta : ### \to 1$.

**Proof.**
1. As the functor of strict tricategories $**#*# : \mathcal{G} \to \mathcal{G}$ acts trivially on
   objects and 1-morphisms, a natural isomorphism of functors of strict
   tricategories $\Gamma : **#*# \to 1$ corresponds to a collection of invertible
   3-morphisms $\Gamma_\mu : **#*# \Rightarrow \mu$ for each 2-morphism $\mu$ that satisfy the
   following conditions:

   - **naturality:** for each 3-morphism $\Psi : \mu \Rightarrow \nu$ the following dia-
     gram commutes

     \[
     \begin{array}{c}
     **#*# \mu \\
     \downarrow \hspace{2cm} \downarrow \Psi \\
     **#*# \nu
     \end{array}
     \xrightarrow{} \hspace{1cm}
     \begin{array}{c}
     \mu \\
     \downarrow \Gamma_\mu \\
     \nu
     \end{array}
     \]

   - **compatibility with the unit 2-morphisms:** for all 1-morphisms $F$, the following diagram commutes

     \[
     \begin{array}{c}
     **#*# \\
     \downarrow \Gamma_1 \\
     \#1_F
     \end{array}
     \xleftarrow{\Phi_F} \hspace{1cm}
     \begin{array}{c}
     1_F \\
     \downarrow \Gamma_{1_F} \\
     *\Phi_F
     \end{array}
     \]

     \[
     (21)
     \]

   - **compatibility with the horizontal composition:** for all compos-
     able 2-morphisms $\mu, \nu$, the following diagram commutes

     \[
     \begin{array}{c}
     **#*#(\mu \circ \nu) \\
     \downarrow \Gamma_{\mu \circ \nu} \\
     \#(\mu \circ \nu)
     \end{array}
     \xrightarrow{\Phi_{\mu \circ \nu}} \hspace{1cm}
     \begin{array}{c}
     **#*#(\mu \circ \nu) \\
     \downarrow \Gamma_{**#*#(\mu \circ \nu)} \\
     \#(\mu \circ \nu)
     \end{array}
     \]

     \[
     (22)
     \]
• compatibility with the Gray product: for all composable 2-morphisms \( \mu : F \Rightarrow G, \nu : H \Rightarrow K \), the following diagram commutes

\[
\begin{array}{ccc}
\ast\ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu & \xrightarrow{\Gamma_{\mu\square\nu}} & \mu \square \nu \\
\sigma_{\ast\ast\ast\ast\mu.,\ast\ast\ast\ast\nu}^{-1} \uparrow & & \uparrow \Gamma_{\mu\square\nu} \\
*\ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu & \xrightarrow{\ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu} & *\ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu \\
*\ast\ast\ast\ast\nu \ast\ast\ast\ast\mu & \xrightarrow{\ast\ast\ast\ast\nu \ast\ast\ast\ast\mu} & \ast\ast\ast\ast\nu \ast\ast\ast\ast\mu \\
\end{array}
\]

Note that the four arrows labelled with the invertible 3-morphisms \( \ast\ast\ast\ast\kappa_{\mu,\nu}, \ast\ast\ast\ast\sigma_{\mu,\ast\ast\ast\ast\nu}, \ast\ast\ast\ast\ast\kappa_{\mu,\nu}, \ast\ast\ast\ast\ast\ast\sigma_{\mu,\ast\ast\ast\ast\nu} \) in this diagram compose to the coherence 3-morphism \( \kappa_{\mu,\nu}^{\ast\ast\ast\ast} : \ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu \Rightarrow \ast\ast\ast\ast\ast\mu \square \ast\ast\ast\ast\nu \) of the functor \( \ast\ast\ast\ast : \mathcal{G} \Rightarrow \mathcal{G} \).

The natural isomorphism \( \Gamma : \ast\ast\ast\ast \Rightarrow 1 \) is defined by its component 3-morphisms \( \Gamma_{\nu} : \ast\ast\ast\ast \Rightarrow \nu \) for each 2-morphism \( \nu : F \Rightarrow G \):

\[
\Gamma_{\nu} = (T_{G} \circ 1_{\nu}) \cdot (1_{\nu}^{\ast\ast\ast\ast}_{G} \circ \sigma_{\nu,\eta_{\nu}}) \cdot (1_{\nu}^{\ast\ast\ast\ast} \circ T_{F} G^{\#} G \circ 1_{F_{0}^{\#} G})
\]

The associated Gray category diagram and its projection are given in Figure 27. From the definition of \( \Gamma_{\nu} \), it is clear that there is an inverse 3-morphism \( \Gamma_{\nu}^{-1} : \nu \Rightarrow \ast\ast\ast\ast \). The naturality of \( \Gamma_{\nu} \) follows directly from the naturality of the tensorator.

To show the compatibility of \( \Gamma_{\nu} \) with the unit 2-morphisms, recall that for each 1-morphism \( F \), the tensorator \( \sigma_{1,F,\eta_{F}^{\#}} \) is trivial. The associated 3-morphism \( \Gamma_{1,F} \) therefore reduces to:

\[
\Gamma_{1,F} = T_{F} \cdot (1_{\eta_{F}^{\#} F} \circ T_{F} F^{\#} F \circ 1_{F_{0}^{\#} G}) \cdot (\sigma_{\eta_{F}^{\#},\eta_{F}^{\#},\eta_{F}^{\#},F} \circ 1_{F_{0}^{\#} F \circ F_{0}^{\#} F})
\]

Composing this 3-morphism with the 3-morphism \( \ast\ast\ast\Phi_{F} = \ast\ast\ast T_{F}^{\#} \), one obtains the Gray category diagram in Figure 28. The commutativity of the diagram (21) is then a direct consequence of identity (d) in Definition 3.10. A diagrammatic proof is given in Figure 28.

The compatibility condition (22) between \( \Gamma \) and the horizontal composition follows from the definitions together with the invertibility of the triangulator, identity (d) in Definition 3.10, the naturality of the tensorator and the exchange identity. As the calculations are lengthy and technical, we give a diagrammatic proof in Figure 29.

The compatibility of \( \Gamma \) with the Gray product in equation (23) again follows from the definitions together with the naturality of the tensorator, the properties of the triangulator and the exchange identity. A diagrammatic proof is given in Figure 30. This concludes the proof that the 3-morphisms \( \Gamma_{\nu} : \ast\ast\ast\ast \Rightarrow \nu \) define a natural isomorphism of functors of strict tricategories \( \Gamma : \ast\ast\ast\ast \Rightarrow 1 \).
Figure 27. 3-morphism $\Gamma_{\nu} : \ast \# \# \# \nu \Rightarrow \nu$ a) and b) its projection. The three Gray category diagrams have the same evaluation.
Figure 28.
Diagrammatic proof of identity (21):
\[ \Gamma_{1_F} \cdot (\ast \# \Phi_F) = \ast \Phi_F\# . \]

2. As the functor of strict tricategories \( \# \# : \mathcal{G} \to \mathcal{G} \) acts trivially on the objects and 1-morphisms of \( \mathcal{G} \), a natural isomorphism of functors of strict tricategories \( \Theta : \# \# \to 1 \) is determined by a collection of invertible 3-morphisms \( \Theta_\mu : \# \# \mu \Rightarrow \mu \) for each 2-morphism \( \mu : F \to G \) that satisfy the following conditions:

- **naturality:** for each 3-morphism \( \Psi : \mu \Rightarrow \nu \) the following diagram commutes:

\[
\begin{array}{ccc}
\# \# \mu & \xrightarrow{\Theta_\mu} & \mu \\
\downarrow\Psi & & \downarrow\Psi \\
\# \# \nu & \xrightarrow{\Theta_\nu} & \nu
\end{array}
\]

- **compatibility with the unit 2-morphisms:** for all 1-morphisms \( F \) the following diagram commutes:

\[
\begin{array}{ccc}
\# \# 1_F & \xrightarrow{\Phi_F} & \# 1_F\# \\
\downarrow\Theta_{1_F} & & \downarrow\Phi_{1_F}\# \\
1_F & & 1_F
\end{array}
\]

(24)
Figure 29. Diagrammatic proof of identity (22):
\[ \Gamma_{\mu\nu} \cdot (\# \# \Phi_{\mu,\nu})^{-1} \cdot (\Phi_{\#\nu,\#\mu})^{-1} = \Gamma_{\mu} \circ \Gamma_{\nu}. \]
Figure 30. Diagrammatic proof of identity (23):
\[
\Gamma_{\mu\nu} : (* \# * \#_{\mu,\nu}) = (\Gamma_{\mu} \otimes \Gamma_{\nu}) \cdot \sigma_{\# \#_{\mu,\# \#_{\nu}}}^{-1} \cdot \sigma_{\# \#_{\mu,\# \#_{\nu}}} \cdot \#_{\mu,\# \#_{\mu}} \cdot \#_{\mu,\# \#_{\mu}}.
\]
67
Figure 31. a) Gray category diagram for the 3-morphism $\Theta_\nu : \#\#\nu \Rightarrow \nu$ and b) its projection.

- **compatibility with the horizontal composition:** for all composable 2morphisms $\mu, \nu$, the following diagram commutes:

\[
\begin{array}{ccc}
\#\#\mu \circ \#\#\nu & \xrightarrow{\Phi_{\#\mu,\#\nu}} & \#(\#\nu \circ \#\mu) \\
\downarrow \theta_\mu \circ \theta_\nu & & \downarrow \#\Phi_{\nu,\mu} \\
\mu \circ \nu & \xleftarrow{\theta_{\mu \circ \nu}} & \#\#(\mu \circ \nu)
\end{array}
\]

- **compatibility with the Gray product:** for all composable 2morphisms $\mu : F \Rightarrow G$, $\nu : H \Rightarrow K$ the following diagram commutes:

(26)
\[
\begin{array}{ccc}
\#\#(\mu \square \nu) & \xrightarrow{\theta_{\mu \square \nu}} & \mu \square \nu \\
\downarrow \#\kappa_{\mu,\nu} & & \downarrow \theta_{\mu \square \nu} \\
\#(\#\nu \square \#\mu) & \xrightarrow{\kappa_{\#\nu,\#\mu}} & \#\#\mu \#\#\nu.
\end{array}
\]
The natural transformation $\Theta : \# \rightarrow 1$ is defined by its component 3-morphisms. For each 2-morphism $\nu : F \Rightarrow G$ we set

$$
\Theta_{\nu} = (1_{\nu} \circ \epsilon_{F}^{*} \circ F_{\nu} \circ G^{*} \circ T_{F} G^{*} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{\nu} \circ (T_{F}^{*})^{-1} \circ \sigma_{F} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{\nu} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{\nu} \circ \sigma_{F} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{\nu} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*}).
$$

The corresponding diagram and its projection are given in Figure 31. The naturality of the 3-morphisms $\Theta_{\nu}$ is a direct consequence of the naturality of the tensorator (3) together with the first condition in (12) in the definition of a planar 2-category. Also, it follows from the invertibility of the triangulator, the invertibility of the tensorator and the identities satisfied by the 3-morphisms $\epsilon_{\nu}$ that $\Theta_{\nu}$ is invertible with inverse

$$
\Theta_{\nu}^{-1} = (1_{G_{\nu} F_{\nu} G^{*}} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{G_{\nu} F_{\nu} G^{*}} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{G_{\nu} F_{\nu} G^{*}} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{G_{\nu} F_{\nu} G^{*}} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*})
\cdot (1_{G_{\nu} F_{\nu} G^{*}} \circ \epsilon_{G} \circ 1_{\nu} \circ G_{\nu} F_{\nu} G^{*}).
$$

Figure 32. Inverse $\Theta_{\nu}^{-1} : \nu \Rightarrow \# \# \nu$ of the 3-morphism $\Theta_{\nu} : \# \# \nu \Rightarrow \nu$. 69
Figure 33. Diagrammatic proof of the relation
\[ \Theta_\nu \cdot \Theta_\nu^{-1} = 1_\nu. \]
Figure 34. Diagrammatic proof of identity (24):
\[ \Theta_{1_{F}} = \Phi_{F}^{-1} \cdot (\#F)^{-1}. \]
Figure 35. Diagrammatic proof of identity (25):
\[ \Theta_{\mu \nu} \cdot \# \Phi_{\nu, \mu} \cdot \Phi_{\# \mu, \# \nu} \cdot (\Theta^{-1}_{\mu} \circ \Theta^{-1}_{\nu}) = 1_{\mu \nu} \] - continued in Figure 36.
Figure 36. Diagrammatic proof of identity (25):
\[ \Theta_{\mu\nu} : \#\Phi_{\nu,\mu} : \Phi_{\#\mu,\#\nu} : (\Theta_{\mu}^{-1} \circ \Theta_{\nu}^{-1}) = 1_{\mu\nu} \] - continuation from Figure 35.
Figure 37. Diagrammatic proof of identity (26):
\[ \Theta_{\mu\square\nu} \cdot \#_{\kappa_{\mu,\nu}} \cdot \kappa_{\#\nu,\#\mu} = \Theta_{\mu\square\Theta_{\nu}}. \]
The 3-morphism $\Theta_{\nu}^{-1} : \nu \Rightarrow \# \# \nu$ is depicted in Figure 32, for a diagrammatic proof of the identity $\Theta_{\nu} \cdot \Theta_{\nu}^{-1} = 1_{\nu}$ see Figure 33.

To verify condition (24) on the compatibility of $\Theta$ with the unit 2-morphisms $1_F : F \Rightarrow F$, we note that the 3-morphism $\Theta_{1_F} : \# \# 1_F \Rightarrow 1_F$ is given by

$$
\Theta_{1_F} = \epsilon_{1_F}^* \cdot (1_{\eta_F^*} \circ T_F \circ 1_{\eta_F^*}) \cdot (\eta_F^* \circ \sigma_{1_F^*} \circ 1_{\eta_F^*})
$$

By applying condition (d) in Definition 3.10 together with the invertibility of the triangulator and the properties of the morphisms $\epsilon_{\nu}$ in a planar 2-category, one finds that the right hand side is equal to the 3-morphism $\Theta_{1_F}^{-1} \cdot (\# \Theta_F)^{-1}$. A diagrammatic proof is given in Figure 34.

The condition (25) on the compatibility of the 3-morphisms $\Theta_{\nu}$ with the horizontal composition of 3-morphisms is more involved. It is a consequence of the properties of the triangulator, the naturality of the tensorator and the properties of the morphisms $\epsilon_{\nu}$ in a planar 2-category. A diagrammatic proof is given in Figure 35 and 36.

To prove identity (26) which states the compatibility of $\Theta$ with the Gray product, consider the Gray category diagram for $\Theta_{\mu \Join \nu} \cdot \# \kappa_{\mu, \nu} \cdot \# \rightarrow 1$ in the upper left of Figure 37. A diagrammatic proof that the associated 3-morphism agrees with $\Theta_{\mu \Join \nu}$ is given in Figure 37.

4.3. Coherence properties of the duals. This section investigates the interaction of the functors of strict tricategories $\ast : \mathcal{G} \to \mathcal{G}^{op}$, $\# : \mathcal{G} \to \mathcal{G}^{op}$ with the natural transformations $\Theta : \# \# \to 1$, $\Gamma : *\# * \# \to 1$. The results are needed in the strictification of these functors in Section 5 and for the investigation of spherical Gray categories in Section 7.

The first result can be regarded as a coherence result for the functors of strict tricategories $\# * \#$ and $\ast$. By composing the natural transformations $\Theta : \# \# \to 1$ and $\Gamma : *\# * \to 1$ on the left and right with, respectively, the functors $\# * \#$ and $*\#$, one obtains pairs of natural transformations $\# \Theta, \Theta \# \to \# * \#$ and $\# * \Gamma^{-1}, \Gamma * \# \to * \#$. The following lemma shows that these natural transformations are equal.

**Lemma 4.6.** The natural isomorphisms $\Gamma : *\# * \to 1$ and $\Theta : \# \# \to 1$ satisfy

$$
\# \Theta = \Theta \\
(*\# \Gamma) \cdot (\Gamma * \#) = 1,
$$

and there is a natural isomorphism $\Delta : \# \to *\# *$ of functors of strict tricategories such that the following diagram commutes

$$
\begin{array}{ccc}
\# \# & \xrightarrow{\Delta} & *\# * \\
\downarrow & & \downarrow \\
\# * \# & \xrightarrow{\Theta} & * \# \ast \\
\downarrow & & \downarrow \\
\ast \# * \# & \xrightarrow{\Gamma \ast} & 1.
\end{array}
$$

(27)
Figure 38. Diagrammatic proof of the identity \( \# \Gamma_\nu = (\Gamma_\nu^{* \#})^{-1} \).

**Proof.** In terms of the associated component 3-morphisms \( \Theta_\nu : \# \# \nu \Rightarrow \nu \) and \( \Gamma_\nu : \# \# \nu \Rightarrow \nu \), the first two relations in the lemma read

\[
\# \Theta_\nu = \Theta_{\# \# \nu}, \quad \# \# \Gamma_\nu = \Gamma_{\# \# \nu}^{-1}.
\]
A diagrammatic proof of the second relation is given in Figure 38.

To construct the natural transformation $\Delta : \# \to *\#*$, it is sufficient to specify its component 3-morphisms $\Delta_\nu : \#\nu \Rightarrow *\# * \nu$ for each 2-morphism $\nu$ and to show that the following diagram commutes

$$
\begin{array}{ccc}
\#\#\nu & \xrightarrow{\# \Delta_\nu} & \# * \# * \nu \\
\downarrow \Delta_\nu & & \downarrow \Theta_\nu \\
*\# * \# \nu & \xrightarrow{(\Gamma_\nu)^{-1}} & \nu.
\end{array}
$$

For a 2-morphism $\nu : F \Rightarrow G$, we define the 3-morphism $\Delta_\nu : \#\nu \Rightarrow *\# * \nu$ as the composite

$$
(28) \quad \Delta_\nu = (T_{G\#} \circ 1_{*\#\nu}) \cdot (1_{G\#} \circ \eta_{G\#}^* \circ G^* \circ \epsilon_{\nu} G^* \circ 1_{\eta_{G\#} G\#} \circ 1_{*\#\nu})
$$

$$
\cdot (1_{G\#} \circ \sigma_{G\#} \circ \eta_{G\#} \circ \epsilon_{\nu} G^* \circ 1_{\eta_{G\#} G\#}) \circ 1_{*\#\nu})
$$

$$
\cdot (1_{G\#} \circ \sigma_{G\#} \circ \eta_{G\#} \circ \epsilon_{\nu} G^* \circ 1_{\eta_{G\#} G\#}) \cdot (1_{*\#\nu} \circ \epsilon_{\nu} G^*).
$$

The Gray category diagram for the 3-morphism $\Delta_\nu$ is given by the left diagram in Figure 38. After some computations, which are performed diagrammatically in Figures 39 and 40, one finds that the 3-morphism $\Theta_\nu : ***\nu \Rightarrow \nu$ is given in terms of $\Delta_\nu$ by

$$
(29) \quad \Theta_\nu = (\Gamma_\nu)^{-1} \cdot \# \Delta_\nu = \Gamma_\nu \cdot \Delta_{\#\nu}.
$$

This implies the commutativity of the diagram in (27). It also follows directly that the 3-morphisms $\Delta_\mu$ define a natural isomorphism of functors of strict tricategories.

By combining diagram (27) with the relation $*\# \Gamma_\nu = (\Gamma_{*\#\nu})^{-1}$, one obtains for all 2-morphisms $\nu$:

$$
\Theta_{*\#\nu} = (\Gamma_{*\#\nu})^{-1} \cdot \# \Delta_{*\#\nu} = \# \Gamma_\nu \cdot \# \Delta_{*\#\nu} = \# (\Gamma_\nu \cdot \Delta_{*\#\nu}) = \# \Theta_\nu,
$$

which proves the first identity in the lemma.

Lemma 4.6 has direct implications for the categories $G(F, G)$ associated 1-morphisms $F, G : C \to D$. The categories $G(F, F)$ have a canonical structure as strict monoidal categories with the tensor product given by the horizontal composition and the tensor unit by the 2-morphism $1_F : F \Rightarrow F$. The functors of strict tricategories $*$ and $\#$ induce functors $* : G(F, G) \to G(F, G)^{op}$ and $\# : G(F, G) \to G(G^#, F^#)$.

**Corollary 4.7.** For all 1-morphisms $F, G : C \to D$, the functors $* : G(F, G) \to G(F, G)^{op}$, $\# : G(F, G) \to G(G^#, F^#)$ are equivalences of categories. When $G(F, F)$ is equipped with its canonical monoidal structure, then $*$ defines a strict pivotal structure on $G(F, F)$, $\# : G(F, F) \to G(F^#, F^#)^{op}$ is a strong tensor functor to the tensor category $G(F, F)^{op}$, with the opposite tensor product, and the 3-morphisms $\Delta_\mu, \Delta_{\#\mu} : \#\mu \Rightarrow *\# * \mu$ define natural isomorphisms $\# \to *\#*$. 

77
Figure 39. Diagrammatic proof of the identity (39):
\[ \Theta_\nu = (\Gamma_{\nu^*})^{-1} \cdot \#\Delta_\nu. \]
Figure 40. Diagrammatic proof of the identity (39):
\[ \Theta_\nu = \Gamma_\nu \cdot \Delta_\nu. \]
The functor $\ast : \mathcal{G}(F, G) \to \mathcal{G}(F, G)^{\text{op}}$ is an equivalence of categories since it is invertible: $\ast\ast = 1_{\mathcal{G}(F, G)}$. It follows directly from the axioms of a planar 2-category that $\ast$ equips each monoidal category $\mathcal{G}(F, F)$ with a strict pivotal structure. To see that the functor $\# : \mathcal{G}(F, G) \to \mathcal{G}(G^\#, F^\#)$ is essentially surjective, note that for each object $\mu$ of $\mathcal{G}(G^\#, F^\#)$, the 3-morphism $\Theta_1 : \mu \Rightarrow \#\#\mu$ defines an isomorphism in $\mathcal{G}(G^\#, F^\#)$ from $\mu$ to an object in the image of $\#$. That $\# : \mathcal{G}(F, G) \to \mathcal{G}(G^\#, F^\#)$ is fully faithful follows from the fact that $\Theta : \#\# \to 1$ defines a natural isomorphism $\#\# \cong 1_{\mathcal{G}(F, G)}$.

To prove that $\# : \mathcal{G}(F, F) \to \mathcal{G}(F^\#, F^\#)^{\text{op}}$ is a strong tensor functor, consider the isomorphisms $\Phi_F : 1_{F^\#} \to \#1_F$ and the isomorphisms $\Phi_{\nu, \mu} : \#\nu \circ \#\mu \to \#(\mu \circ \nu)$ from the proof of Theorem 4.3. Identities (16) and (17) in the proof of Theorem 4.3 coincide with the axioms for a strong tensor functor. □

The last structural property of a Gray category with duals that will be required in the following is a relation between the natural transformation $\Delta : \# \to **\#$ from Lemma 4.6 and its double *-dual $*\Delta : \# \to **\*$.

**Definition 4.8 (Spatial Gray category).** A Gray category $\mathcal{G}$ with duals is called spatial if the natural transformations $\Delta : \# \to **\#$ and $*\Delta : \# \to **\*$ are equal.

**Corollary 4.9.** If $\mathcal{G}$ is a spatial Gray category, then for each object $C$ of $\mathcal{G}$, the category $\mathcal{G}(1_C, 1_C)$ is a ribbon category. Conversely, a ribbon category is a spatial Gray category with one object and one 1-morphism.

**Proof.** If $\mathcal{G}$ is a Gray category with duals, then by Lemma 3.11 the category $\mathcal{G}(1_C, 1_C)$ is a braided pivotal tensor category. As all Gray products with 1-morphism $1_C$, the 2-morphisms $\eta_{1_C}$ and the 3-morphism $\Upsilon_{1_C}$ are trivial, it follows that the 2-functor $\#: \mathcal{G}(1_C, 1_C) \to \mathcal{G}(1_C, 1_C)^{\text{op}}$ is trivial, and that the 3-morphisms from Theorem 4.5 and Lemma 4.6 satisfy $\Gamma_\mu = 1_\mu$ and $\Theta_\mu = \Delta_\mu$ for all 2-morphisms $\mu$ in $\mathcal{G}(1_C, 1_C)$. For each object $\mu$, the morphism $\Theta_\mu : \mu \to \mu$ reduces to the twist
in a pivotal braided category. The condition that \( \mathcal{G} \) is spatial ensures that the twist satisfies the condition that makes \( \mathcal{G}(1_\mathcal{C}, 1_\mathcal{C}) \) into a ribbon category.

\[ \square \]

4.4. **Geometrical interpretation of the duals.** The functors of strict tricategories \( * : \mathcal{G} \to \mathcal{G}^{op}, \# : \mathcal{G} \to \mathcal{G}^{op} \) and the natural isomorphisms \( \Gamma : *\# * \# \to 1 \) and \( \Theta : ## \to 1 \) have a direct geometrical interpretation in terms of Gray category diagrams. To see this, consider for each 2-morphism \( \mu : F \Rightarrow G \) the 3-morphism \( \Omega_\mu : \eta^*_G \circ (\mu \Box G^\#) \Rightarrow \eta^*_F \circ (F \Box # \mu) \)

\[
\begin{align*}
\Omega_\mu &= (\sigma^{-1}_{\eta^*_F, \eta^*_G \circ \mu \Box G^\#} \circ 1_F \eta^*_F \circ G^\#) \cdot (1_{\eta^*_G \circ \mu \Box G^\#} \circ T_F^{-1} G^\#) \\
&= (30) \\
&= (30)
\end{align*}
\]

The Gray category diagram for \( \Omega_\mu \) and its projection are given in Figure 42 b). By neglecting the expression for \( \Omega_\mu \) in terms of the data of a Gray category with duals and considering only its source and target 2-morphisms, one obtains the Gray category diagram in Figure 42 a). The 3-morphism \( \Omega_\mu \) thus allows one to let the lines labelled by 2-morphisms cross folds. It follows directly from the properties of the tensorator that \( \Omega_\mu \) satisfies the naturality condition

\[
\Omega_\nu \cdot (1_{\eta^*_G} \circ (\Psi \Box G^\#)) = (1_{\eta^*_F} \circ (F \Box # \Psi)) \cdot \Omega_\mu
\]

for all 2-morphisms \( \mu, \nu : F \Rightarrow G \) and all 3-morphisms \( \Psi : \mu \Rightarrow \nu \). This corresponds to sliding the dots labelled by 3-morphisms over the folds.
as shown in Figure 42 a). Moreover, the 3-morphism $\Omega_\mu$ is invertible with inverse

$$\Omega_\mu^{-1} = (1_{\eta_G^* \circ v G^*} \circ T_F G^*) \cdot (\sigma_{\eta_F^* \circ \eta_G^*} \circ 1_{F \# F})$$

The Gray category diagram for $\Omega_\mu^{-1}$ is given in Figure 43 b), and changing the orientation of the line leads to the Gray category diagram in Figure 43 c). The Gray category diagrams involving a fold that opens in the other direction are determined by the $*$-dual of $\Omega$ and are given in Figure 43 d) to g).

It remains to investigate the interaction of the 3-morphisms $\Omega_\mu$ with the unit 2-morphisms $1_F$, the horizontal composition $\circ$ and the Gray product $\Box$. For this, note that for every 1-morphism $F : C \to D$, the associated 3-morphism $\Omega_{1_F} : \eta_F^* \Rightarrow \eta_F^*$ is given in terms of the 3-morphism $\Phi_F : 1_F \Rightarrow \# 1_F$ from Theorem 4.3

$$\Omega_{1_F} = \sigma_{\eta_F^*, \eta_F^*}^{-1} \circ T_F^{-1} F = 1_{\eta_F^*} \circ F \Phi_F^{-1}$$
Figure 43. Gray category diagrams for $\Omega_\mu$:

a) naturality condition,
b) inverse 3-morphism $\Omega_\mu^{-1} : \eta^*_F \circ (F \Box # \mu) \Rightarrow \eta^*_G \circ (\mu \Box G^\#)$
c) 3-morphism $\Omega^*_\mu : \eta^*_F \circ (\mu^* \Box F^\#) \Rightarrow \eta^*_G \circ (G \Box \# \ast \ast \mu)$,
d) dual 3-morphism $\Omega^*_\mu^{-1} : (F \Box \# \mu) \circ \eta_F \Rightarrow (\mu^* \Box G^\#) \circ \eta_G$

e) 3-morphism $\Omega^*_\mu^*: 3$-morphism $\Omega^*_\mu^{-1}$,
f) 3-morphism $\Omega^*_\mu^*$,
g) 3-morphism $\Omega^*_\mu^{-1}$.
Figure 44. Gray category diagrams for the identities a) (31), b) (32) and c) (33). The 3-morphisms $\Phi_F$, $\Phi_{\mu,\nu}$ and $\kappa_{\mu,\nu}$ arise when lines corresponding to identity 2-morphisms or composite 2-morphisms cross a fold.
Figure 45. Diagrammatic proof of the identities a) (32) and b) (33).
which follows directly from the definition of $\Omega$ and the last property of the triangulator in Definition 3.10. The corresponding Gray category diagram is given in Figure 44 a).

The interaction of the 3-morphisms $\Omega_\mu$ with the horizontal composition is determined by the 3-morphisms $\Phi_{\mu,\nu} : \# \mu \circ \# \mu \Rightarrow \# (\nu \circ \mu)$ from Theorem 4.3. A direct calculation shows that for all composable 2-morphisms $\mu : F \Rightarrow G, \nu : H \Rightarrow K$, the following diagram commutes

\[
\eta^*_H \circ \nu F^# \circ \mu H^# \xrightarrow{\Omega_{\mu,\nu}} \eta^*_F \circ F^# (\nu \circ \mu) \\
\Omega_{\nu \circ 1 \mu,\nu} \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
3-morphisms obtained by projecting these diagrams are precisely the component 3-morphisms of the natural isomorphisms $\Gamma : * * * * \rightarrow 1$ and $\Theta : * * * * \rightarrow 1$. When expressed in terms of the 3-morphisms in the Gray category with duals, this corresponds to the identities

$$
\Gamma_\mu = (T_G \circ 1_\mu) \cdot (1_{\eta^*_G}G \circ \sigma_{\mu,\eta^*_G}) \cdot (\Omega^{-1}_G \circ 1_{F\eta^*_G}) \cdot (1_{\eta^*_G}G \circ FO^*_\mu) \\
\cdot (\sigma^{-1}_{\eta^*_F,+++\mu} \circ 1_{F\eta^*_F}) \cdot (1_{+++\mu} \circ T^{-1}_F)
$$

Figure 46. Diagrammatic proof of identity (34).
\begin{align*}
\Theta_\mu &= (1_\mu \circ T_F^{-1}) \cdot (\sigma_\mu^{-1} \circ 1_{\eta_F F}) \cdot (1_{G\eta_F^*} \circ \eta_F F \circ F \epsilon_{\# \mu}^* \circ 1_{\eta_F F}) \\
&\quad \cdot (1_{G\eta_F^*} \circ \eta_F^* F \circ F \mu \circ \Omega^{-1}_\mu F) \cdot (1_{G\eta_F^*} \circ \sigma_{\# \mu}^{-1} \circ 1_{\mu \eta_G F}) \\
&\quad \cdot (1_{G\eta_F^*} \circ \sigma_{\# \mu} \circ \epsilon_{\# \mu} G \circ 1_{\eta_G F}) \cdot (G \Omega^{-1}_{\# \mu} \circ 1_{\eta_G F}) \\
&\quad \cdot (1_{G\eta_F^*} \circ \sigma_{\# \mu} \circ 1_{\# \mu}) \cdot (T_G^* \circ 1_{\# \mu}).
\end{align*}

The natural isomorphisms \( \Gamma : ** \# \# \to 1 \) and \( \Theta : \# \# \to 1 \) thus have a direct geometrical interpretation. Each of them relates the evaluation of two Gray category diagrams that are obtained from each other by
sliding a line over a cusp. The coherence data of the functors of strict tricategories and the natural transformations $\Gamma$ and $\Theta$ thus arises naturally when one considers Gray category diagrams which involve lines crossing folds.

5. Strictification for Gray categories with duals

In this section, it is shown that for every spatial Gray category $G$, the functors of strict tricategories $\ast : G \to G^{op}$ and $\# : G \to G_{op}$ can be strictified to strict functors of strict tricategories $\ast : G \to G^{op}$ and $\# : G \to G_{op}$ in the sense of Definitions A.6 and A.8 which satisfy $\#\# = 1$, $\ast\ast = 1$ and $\ast\#\#\# = 1$. The Gray category $G$ is a Gray category with duals and equivalent to $G$ as a strict tricategory (see Definition A.11). The difference between $G$ and $G$ is that the structures from Definition 3.10 extend to strict functors of strict tricategories. This motivates the following definition.

**Definition 5.1** (Gray category with strict duals). A Gray category with strict duals is a Gray category with duals $G$ such that the operations $\#$ and $\ast$ extend to strict functors of strict tricategories $\ast : G \to G^{op}$, $\# : G \to G_{op}$ satisfying $\#\# = 1$, $\ast\ast = 1$ and $\ast\#\#\# = 1$, and for all 2-morphisms $\mu$,

$$\#\epsilon\mu = \epsilon\ast\#\mu.$$

To prove that it is possible to strictify the Gray with duals $G$ to a Gray category $G$ with strict duals, we first construct a Gray category $G$ with strict functors of strict tricategories $\ast : G \to G^{op}$ and $\# : G \to G_{op}$ satisfying $\#\# = 1$, $\ast\ast = 1$ and $\ast\#\#\# = 1$ that is equivalent to $G$. In a second step, we then show that the Gray category $G$ is a Gray category with duals in the sense of Definition 3.10 and satisfies the conditions in Definition 5.1.

**Theorem 5.2** (Strictification). Let $G$ be a spatial Gray category with duals. Then there exists a Gray category $G$ with the following properties:

1. $G$ is equipped with strict functors of strict tricategories $\# : G \to G_{op}$, $\ast : G \to G^{op}$ that satisfy $\#\# = 1$, $\ast\ast = 1$, $\ast\#\#\# = 1$.

2. $G$ is equivalent to $G$ as a Gray category: there are lax functors of strict tricategories $e : G \to G$ and $f : G \to G$ with $ef = 1_G$ and a natural isomorphism of lax functors of strict tricategories $\eta : fe \to 1_G$. The natural isomorphism satisfies $e\eta = 1_e$, $\eta fe = 1_{fe}$, and there is an invertible modification $\Psi : \eta f \Rightarrow 1_f$ with $e\Psi = 1_{1_G}$.

3. The lax functor of strict tricategories $f : G \to G$ satisfies $\ast f = f\ast$, and there are natural isomorphisms of lax functors of strict tricategories $\xi : \ast e \to e\ast$, $\chi : \# e \to e\#$ and $\tilde{\chi} : \# f \to f\#$. 89
Proof.

1. We construct $G$ in analogy to the strictification proof for tricategories in [11]. Note also that our proof can be viewed as a generalisation of the strictification proof for pivotal categories in [26].

The objects of $G$ are the objects of $G$. A basic 1-morphism in $G$ from $A$ to $B$ is a tuple $F = (F, z)$ of a number $z \in \{1, -1\}$ and a 1-morphism $F : A \to B$ in $G$ if $z = 1$ or a 1-morphism $F : B \to A$ in $G$ if $z = -1$.

The 1-morphisms of $G$ are composable strings $F = (F_n, \ldots, F_1)$ of basic 1-morphisms, including the empty strings $\emptyset_A : A \to A$.

The evaluation of a 1-morphism $F : A \to B$ in $G$ is the 1-morphism $e(F) : A \to B$ in $G$ determined by

$$e(F, 1) = F, \quad e(F, -1) = \#F, \quad e(\emptyset_A) = 1_A,$$

$$e(F_n, \ldots, F_1) = e(F_n) \Box \ldots \Box e(F_1).$$

A basic 2-morphism in $G$ is a tuple $\alpha = (\alpha, z) : F \Rightarrow G$ consisting of a number $z \in \{1, -1\}$ and a 2-morphism $\alpha : e(F) \Rightarrow e(G)$ in $G$ if $z = 1$ or a 2-morphism $\alpha : \#e(G) \Rightarrow \#e(F)$ if $z = -1$. The 2-morphisms in $G$ are composable strings $\alpha = (\alpha_n, \ldots, \alpha_1)$ of basic 2-morphisms in $G$, including the empty strings $\emptyset_F : F \Rightarrow F$.

The evaluation of a 2-morphism $\alpha : F \Rightarrow G$ in $G$ is the 2-morphism $e(\alpha) : e(F) \Rightarrow e(G)$ in $G$ determined by

$$e(\alpha, 1) = \alpha, \quad e(\alpha, -1) = \#\alpha, \quad e(\emptyset_F) = 1_{e(F)},$$

$$e(\alpha_n, \ldots, \alpha_1) = e(\alpha_1) \circ \ldots \circ e(\alpha_n),$$

for all basic 2-morphisms $\alpha_1, \ldots, \alpha_n$. A 3-morphism $\Gamma : \alpha \Rightarrow \beta$ in $G$ is given by a 3-morphism $e(\Gamma) : e(\alpha) \Rightarrow e(\beta)$ in $G$.

The vertical composition of 3-morphisms in $G$ is the vertical composition in $G$. The horizontal composition of 2-morphisms in $G$ is the concatenation of strings. This implies that the horizontal and vertical composition are strictly associative and the unit 2- and 3-morphisms are strict. As the evaluation is also strictly compatible with the horizontal and vertical composition and the units, the horizontal composition of 3-morphisms in $G$ is given by the horizontal composition of 3-morphisms in $G$. This shows that for each pair of objects $A, B$, $G(A, B)$ is a strict 2-category.

The Gray product $\Box$ in $G$ is defined as the concatenation of strings on 1-morphisms. This implies that it is strictly associative and the unit 1-morphisms are strict:

$$(F \Box G) \Box H = F \Box (G \Box H), \quad F \Box \emptyset_C = \emptyset_D \Box F = F$$

for all 1-morphisms $F : C \to D, G : B \to C, H : A \to B$ in $G$. It also follows that the Gray product of 1-morphisms is compatible with the evaluation. All composable 1-morphisms $F, G$ satisfy the relation $e(F \Box G) = e(F) \Box e(G)$. 

90
For 1-morphisms \( F \in \mathcal{G}(C, D) \), \( G \in \mathcal{G}(A, B) \) and a 2-morphism \( \alpha \in \mathcal{G}(B, C) \) we define the Gray product by

\[
F \boxtimes (\alpha, 1) = e(F) \square e(\alpha, 1), \quad F \boxtimes (\alpha, -1) = (\alpha \# e(F), -1),
\]

\[
(\alpha, 1) \square G = (\alpha \# e(G), 1), \quad (\alpha, -1) \square G = (\# e(G) \square \alpha, -1)
\]

\[
F \boxtimes \emptyset_G = \emptyset_F \square G, \quad F \boxtimes (\alpha_2, \ldots, \alpha_1) = (F \boxtimes \alpha_2, \ldots, F \boxtimes \alpha_1)
\]

This determines the Gray product of composable 2-morphisms \( \alpha : F \Rightarrow G, \beta : K \Rightarrow H \) in \( \mathcal{G} \), which is given by

\[
\alpha \boxtimes \beta = (\alpha \square K) \circ (F \boxtimes \beta).
\]

As a direct consequence of these definitions, the Gray product of 1- and 2-morphisms is again strictly associative, strictly compatible with the unit 2-morphisms and strictly compatible with the horizontal composition of 2-morphisms. To define the Gray product of two 3-morphisms, we compute the evaluation

\[
e(F \boxtimes (\alpha, 1)) = e(F) \square e(\alpha, 1), \quad e((\alpha, 1) \square G) = e(\alpha, 1) \square e(G)
\]

\[
e(F \boxtimes \emptyset_H) = e(F) \square e(\emptyset_H), \quad e(\emptyset_H \square G) = e(\emptyset_H) \square e(G),
\]

\[
e(F \boxtimes (\alpha, -1)) = \#(\alpha \# e(F)), \quad e(F) \square e(\alpha, -1) = e(F) \square \# \alpha,
\]

\[
e((\alpha, -1) \square G) = \#(\# e(G) \square \alpha), \quad e(\alpha, -1) \square e(G) = \# \alpha \square e(G).
\]

The Gray product of two 3-morphisms \( \Gamma : \alpha \Rightarrow \alpha', \Psi : \beta \Rightarrow \beta' \) between 2-morphisms \( \alpha, \alpha' : F \Rightarrow G, \beta, \beta' : H \Rightarrow K \) is defined as

\[
e(\Phi \boxtimes \Psi) = (\tilde{\iota}_{\alpha', K} \circ \tilde{\iota}_{F, \beta'}) \cdot (e(\Phi) \square e(\Psi)) \cdot (\tilde{\iota}_{\alpha, K}^{-1} \circ \tilde{\iota}_{L, \beta}^{-1}),
\]

where \( \tilde{\iota}_{\alpha, K} : e(\alpha) \square e(K) \Rightarrow e(\alpha \square K) \) and \( \tilde{\iota}_{L, \alpha} : e(L) \square e(\alpha) \Rightarrow e(L \square \alpha) \) are the invertible 3-morphisms given by

\[
\tilde{\iota}_{\alpha, K} = 1_{1_{(L) \square (K)}}, \quad \tilde{\iota}_{L, \alpha} = 1_{1_{(L) \square e(L)}}, \quad \tilde{\iota}_{\alpha, 1,K} = 1_{\alpha \square 1_K}, \quad \tilde{\iota}_{L, \alpha, 1} = 1_{e(L) \square \alpha},
\]

\[
\tilde{\iota}_{(\alpha, -1), K} = \kappa_{1 \# (\# e(K))} \cdot (1_{\# e(\Phi) \# e(K)}), \quad \tilde{\iota}_{L, (\alpha, -1)} = \kappa_{1 \# e(L) \# e(L)} \cdot (\Phi \# e(L) \square 1_{\# e(L)}),
\]

\[
\tilde{\iota}_{(\alpha, \ldots, \alpha), K} = \tilde{\iota}_{\alpha, K} \circ \cdots \circ \tilde{\iota}_{\alpha, K}, \quad \tilde{\iota}_{L, (\alpha, \ldots, \alpha)} = \tilde{\iota}_{L, \alpha} \circ \cdots \circ \tilde{\iota}_{L, \alpha},
\]

with the 3-morphisms \( \kappa_{\mu, \nu} : \# \mu \# \# \Rightarrow \# (\mu \# \nu) \) and \( \Phi_F : 1_{\# F} \Rightarrow \# 1_F \) from the proof of Theorem 4.3. In this, we used the identity \( \# F = F \) in a Gray category with duals. It follows that the Gray product of 3-morphisms is strictly compatible with their vertical composition and with the unit 3-morphisms. The Gray product is compatible with the horizontal composition of 3-morphisms if and only if the following two commutative diagrams and their counterparts with a 1-morphism on
the left are equal

\[(37)\]
\[
\begin{align*}
  e((\alpha \circ \beta) \Box K) &\xrightarrow{\iota_{\alpha \circ \beta \circ K}} e((\alpha' \circ \beta') \Box K) \\
e(\alpha \circ \beta) \Box e(K) &\xrightarrow{\iota_{\alpha \circ \beta}} e((\alpha' \circ \beta') \Box e(K)) \\
e((\alpha \circ \beta) \Box K) &\xrightarrow{\iota_{\alpha \circ \beta} \circ K} e((\alpha' \circ \beta') \Box K) \\
\end{align*}
\]

As the horizontal composition of 2-morphisms is strictly compatible with the evaluation, the equality of these two diagrams follows directly from the identities

\[(38)\]
\[
\iota_{\alpha \circ \beta \circ K} = \iota_{\alpha \circ K} \circ \iota_{\beta \circ K}, \quad \iota_{L \alpha \circ \beta} = \iota_{L \alpha} \circ \iota_{L \beta}
\]

which are satisfied by definition. That the Gray product is associative amounts to the commutativity of the following diagrams

\[(39)\]

\[(40)\]

and the analogue of diagram \[(39)\] for the composition with 1-morphisms from the left. In the diagram \[(39)\], all squares commute by definition.
of the Gray product. It remains to prove the identities

\[ \tilde{i}_{\alpha, F \Box G} = \tilde{i}_{\alpha} \Box F \Box G \cdot (\tilde{i}_{\alpha} \Box 1_{e(G)}) \]

\[ \tilde{i}_{H \Box K, \alpha} = \tilde{i}_{H, K \Box 1} \cdot (1_{e(H)} \Box \tilde{i}_{K, \alpha}) \]

for all 2-morphisms \( \alpha \) and 1-morphisms \( F, G, H, K \) for which these expressions are defined. Due to the identities (38), it is sufficient to prove this for basic 2-morphisms and the empty string of 2-morphisms. For the latter and for basic 2-morphisms \( \alpha = (\alpha, 1) \), this follows directly from the definition. For basic 2-morphisms \( \alpha = (\alpha, -1) \), inserting the definition of \( \tilde{i}_{\alpha, F} \) into these equations shows that this is the case for the first equation if and only if the outer paths in the diagram

\[
\begin{array}{cccc}
\# \alpha \Box F \Box G & \# \alpha \Box \Phi \Box G \Box F \Box G & \# \alpha \Box \# F \Box G & \# F \Box \# G \Box F \Box \alpha \\
\# \alpha \Box \Phi \Box G \Box F \Box G & \# \alpha \Box \# F \Box \Phi \Box G \Box F & \# \alpha \Box \# F \Box \# G & \# F \Box \# G \Box \alpha \\
\# \alpha \Box \# F \Box G & \# (1_{\# F} \Box \alpha) \Box \Phi \Box G & \# (1_{\# F} \Box \alpha) \Box \# G & \# (1_{\# F} \Box \alpha) \Box \# F \Box G \\
\end{array}
\]

are equal. The rectangle on the lower left commutes. The subdiagram on the right commutes due to equation (20), and the upper left rectangle due to the compatibility condition (19) between the 3-morphisms \( \kappa \) and \( \Phi_F, \Phi_G \). This shows that the diagram (39) commutes. The proof for the commutativity of the corresponding diagram with the 1-morphisms on the left is analogous.

In diagram (40), the three rectangles in the middle commute by definition of of the Gray product. It is therefore sufficient to prove that the two subdiagrams with curved arrows on the left and right commute. Using again the identities (38), one finds that it is sufficient to show that this is the case for the basic 2-morphisms and the empty string of 2-morphisms. In the cases \( \alpha = \emptyset_F \) and \( \alpha = (\alpha, 1) \) the commutativity of the subdiagrams is obvious. For \( \alpha = (\alpha, -1) \), we insert the definition of \( \tilde{i}_{\alpha, F} \) and \( \tilde{i}_{H, \alpha} \) and obtain diagram (41), in which we abbreviate \( F = e(F) \) and \( H = e(H) \). The outer paths in diagram (41) correspond to the 3-morphisms \( i_{H, \alpha \Box F} \cdot (1_{e(H)} \Box \tilde{i}_{\alpha, F}) \) and \( i_{H, \alpha \Box F} \cdot (1_{e(H)} \Box \tilde{i}_{\alpha, F}) \). It is directly apparent that the lower parallelogram in the middle and the two subdiagrams on the right and the left commute. The upper parallelogram in the middle commutes due to the pentagon axiom for \( \kappa \) in equation (20) and hence the outer paths on the left and on the right are equal. This proves the commutativity of the diagram (40) and completes the proof that the Gray product is strictly associative.
To conclude that $\mathcal{G}$ is a Gray category, we define the tensorator $\sigma_{\alpha,\beta} : (\alpha \Box K) \Rightarrow (\mathcal{G} \Box \beta) \Rightarrow (\alpha \Box H)$ in $\mathcal{G}$ by

$$\sigma_{\alpha,\beta} = (\check{\iota}_{G,\beta} \circ \check{\iota}_{\alpha,H}) \cdot \sigma_{e(\alpha),e(\beta)} \cdot (\check{\iota}_{\alpha,K} \circ \check{\iota}_{F,\beta})^{-1}.$$  

It follows from the definition, the properties of the tensorator in $\mathcal{G}$ and the identities proved above that $\sigma_{\alpha,\beta}$ satisfies the axioms for the tensorator in Definition 2.14. This shows that $\mathcal{G}$ is a Gray category.

2. To construct the strict functor of strict tricategories $\# : \mathcal{G} \rightarrow \mathcal{G}_{op}$, we set $\#(A) = A$ for objects, $\#(F,z) = (F,-z)$ for basic 1-morphisms $F = (F,z)$ of $\mathcal{G}$ and extend $\#$ to general 1-morphisms by $\#(F_1,\ldots,F_n) = (\# F_1,\ldots,\# F_n)$, $\#(\emptyset) = \emptyset$. It follows that $\#$ is strictly compatible with the Gray product of 1-morphisms, preserves the unit-1-morphisms, satisfies $\# \# F = F$ and is compatible with the evaluation: $\#e(F) = \#e(F)$ for all 1-morphisms $F$ in $\mathcal{G}$.

For basic 2-morphisms $\alpha = (\alpha, z) : F \Rightarrow \mathcal{G}$ we set $\#(\alpha, z) = (\alpha, -z)$ and extend $\#$ to general 2-morphisms by $\#(\alpha_1,\ldots,\alpha_n) = (\# \alpha_1,\ldots,\# \alpha_n)$, $\#(\emptyset F) = \emptyset \# F$. This implies that $\#$ is strictly compatible with the horizontal composition of 2-morphisms, preserves the unit 2-morphisms and satisfies $\# \# \alpha = \alpha$ for all 2-morphisms $\alpha$ in $\mathcal{G}$. Due to the identity $\# \# F = 1$ for all 1-morphisms $F$ in $\mathcal{G}$, it also follows that $\#$ is strictly compatible with the Gray product of 1- and 2-morphisms.

To define the action of $\#$ on 3-morphisms $\Gamma : \alpha \Rightarrow \beta$, we note that if $\alpha = (\alpha, z)$ is a basic 2-morphism, the 2-morphisms $\#e(\alpha)$ and $e(\# \alpha)$
are related by $\#e(\alpha, 1) = e(\#(\alpha, 1))$ and $\#e(\alpha, -1) = \#\#e(\#(\alpha, -1))$.

For general 2-morphisms $(\alpha_n, ..., \alpha_1)$, we have

$$\#e(\alpha_n, ..., \alpha_1) = \#(e(\alpha_n) \circ ... \circ e(\alpha_1)),$$

$$e(\#(\alpha_n, ..., \alpha_1)) = e(\#(\alpha_1)) \circ ... \circ e(\#(\alpha_n)).$$

We obtain an invertible 3-morphism $\chi_\alpha : \#e(\alpha) \Rightarrow e(\#(\alpha))$ by setting

$$\chi_\alpha = \Phi^{-1}_{e(\varepsilon[\mathcal{E}])}, \quad \chi_{(\alpha, 1)} = \varepsilon_{\alpha}, \quad \chi_{(\alpha, -1)} = \Theta_{\alpha},$$

$$\chi_{(\alpha_n, ..., \alpha_1)} = (\chi_{(\alpha_n, ..., \alpha_2)} \circ ... \circ \chi_{(\alpha_1)} \circ ... \circ \chi_{(\alpha_n)} \cdot \Phi^{-1}_{e(\varepsilon[\mathcal{E}])} ... e(\varepsilon[\mathcal{E}])),$$

where $\Phi_{\mu_1, ..., \mu_n} : \#(\mu_1 \circ ... \circ \mu_n) \Rightarrow \#(\mu_n \circ ... \circ \mu_1)$ denotes the invertible 3-morphism determined by $\Phi_{\mu_1} = 1_{\#(\mu_1)}$ and

$$\Phi_{\mu_1, ..., \mu_n} = \Phi_{\mu_1, \mu_n \circ ... \circ \mu_2} \cdot (1_{\#(\mu_1)} \cdot \Phi_{\mu_2, \mu_n \circ ... \circ \mu_3}) \cdot ... \cdot (1_{\#(\mu_1)} \circ ... \circ 1_{\#(\mu_n-3)} \cdot \Phi_{\mu_n-2, \mu_n \circ ... \circ \mu_n-1}) \cdot (1_{\#(\mu_1)} \circ ... \circ 1_{\#(\mu_n-2)} \cdot \Phi_{\mu_n-1, \mu_n})$$

and the 3-morphisms $\Phi_F : 1_F \Rightarrow 1_F$ and $\Phi_{\mu, \nu} : \#(\mu \circ \nu) \Rightarrow (\#(\nu \circ \mu)$ are given in the proof of Theorem 4.3. Note that it follows from identity (17) that the bracketing in the definition of $\Phi_{\mu_1, ..., \mu_n}$ is irrelevant, and for all composable 2-morphisms $\mu_1, ..., \mu_n$ and all $1 \leq k \leq n-1$

$$\Phi_{\mu_1, ..., \mu_n} = \Phi_{\mu_k \circ ... \circ \mu_1, \mu_n \circ ... \circ \mu_k+1} \cdot (\Phi_{\mu_1, ..., \mu_k} \circ \Phi_{\mu_{k+1}, ..., \mu_n}).$$

From this, it follows that the 3-morphisms $\chi_\alpha : \#e(\alpha) \Rightarrow e(\#(\alpha))$ satisfy the relation

$$\chi_{\alpha \circ \beta} = (\chi_\beta \circ \chi_\alpha) \cdot \Phi^{-1}_{e(\varepsilon[\mathcal{E}])}$$

for all composable 2-morphisms $\alpha, \beta$.

For a 3-morphism $\Gamma : \alpha \Rightarrow \beta$, we define $\# \Gamma : \#(\alpha) \Rightarrow \#(\beta)$ by

$$e(\#(\alpha)) = \chi_\beta \cdot e(\#(\Gamma)) \chi_{\alpha}^{-1}.$$

To show that $\#$ defines a functor of strict tricategories, we prove the identities

$$\#(\Psi : \Phi) = \#(\Psi : \Phi), \quad \#(\Psi) = \#(\Psi), \quad \#(\Psi \circ \Phi) = \#(\Psi \circ \Phi), \quad \#(\Psi \circ \Phi) = \#(\Psi \circ \Phi)$$

for all 3-morphisms $\Phi, \Psi$ for which these expressions are defined. The first follows directly from the definition. The identity $\#(\Psi \circ \Phi) = \#(\Psi \circ \Phi)$ follows from the commutative diagram

$$\begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array} \begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array} \begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array} \begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array} \begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array} \begin{array}{c}
\chi_{\alpha \circ \beta}^{-1} \\
\Phi(e(\beta) \circ e(\alpha))
\end{array}$$

for all 3-morphisms $\Phi : \beta \Rightarrow \beta'$ and $\Psi : \alpha \Rightarrow \alpha'$. The parallelogram in the middle of this diagram commutes due to the naturality of the
3-morphism $\Phi_{\mu,\nu} : \# \mu \circ \# \nu \Rightarrow \#(\nu \circ \mu)$ and the triangles on the left and right by identity $[12]$.

The identity $\#(\Psi \square \Phi) = \# \Phi \square \# \Psi$ is equivalent to the commutativity of the diagram

\[(44)\]

\[
\begin{array}{ccc}
\#e(\alpha \square \beta) & \xrightarrow{\chi_{\alpha \square \beta}} & \#e(\alpha' \square \beta') \\
\downarrow & & \downarrow \\
\#(\#_{\beta} \circ \beta) & \xrightarrow{\#(\Psi \circ \Phi)} & \#(\#_{\alpha'} \circ \beta') \\
\downarrow & & \downarrow \\
\#(\#_{\beta} \circ \beta) & \xrightarrow{\#(\Psi \circ \Phi) \circ \#(\Phi)} & \#(\#_{\alpha'} \circ \beta') \\
\end{array}
\]

In this diagram, the four rectangles in the middle commute by definition of the 3-morphisms $\#(\Psi \square \Phi), \Psi \square \Phi$ and $\# \Psi, \# \Phi$ and due to the naturality of the 3-morphism $\Phi_{\mu,\nu} : \# \mu \circ \# \nu \Rightarrow \#(\nu \circ \mu)$. It remains to show that the two pentagons on the left and the right commute. As a first step, we reduce the proof of the commutativity of these diagrams to the cases $\alpha = \emptyset_E$ or $\beta = \emptyset_K$. For this, we consider the diagram

\[(45)\]

\[
\begin{array}{ccc}
\#e(\alpha \square \beta) & \xrightarrow{\chi_{\alpha \square \beta}^{-1} \circ \chi_{\alpha \square \beta}^{-1}} & \#e(\alpha \square K) \\
\downarrow & & \downarrow \\
\#(\beta) \square \#e(\alpha) & \xrightarrow{(\#e(\beta) \circ \#1_E)} & \#(\beta) \square \#e(\alpha) \\
\downarrow & & \downarrow \\
\#(\beta) \square \#e(\alpha) & \xrightarrow{(\Phi_{\alpha} \square (\#1_E \circ \Phi_{\beta}))} & \#(\beta) \square \#e(\alpha) \\
\downarrow & & \downarrow \\
\#(\beta) \square \#e(\alpha) & \xrightarrow{(\Phi_{\beta} \square (\#1_E \circ \Phi_{\beta}))} & \#(\beta) \square \#e(\alpha) \\
\downarrow & & \downarrow \\
\#(\beta) \square \#e(\alpha) & \xrightarrow{(\Phi_{\beta} \square (\#1_E \circ \Phi_{\beta}))} & \#(\beta) \square \#e(\alpha) \\
\end{array}
\]
The path on the outside in this diagram corresponds to the pentagon in diagram (44). The two quadrilaterals in the diagram commute by the naturality of the 3-morphism $\Phi_{\mu,\nu} : \#\mu \circ \#\nu \Rightarrow \#(\nu \circ \mu)$ and by identity (18). The hexagon commutes if and only if the pentagon in diagram (44) commutes for the case where $\alpha = \emptyset_{\mathcal{E}}$ or $\beta = \emptyset_{\mathcal{K}}$.

It is therefore sufficient to prove that the pentagon in the diagram (44) commutes for $\alpha = \emptyset_{\mathcal{E}}$ or $\beta = \emptyset_{\mathcal{K}}$. In the latter, it reduces to the diagram

$$(46)$$

$$\xymatrix{ e(\#(\alpha \square K)) \ar[r]^-{\chi_{\alpha \square K}^{-1}} & \#e(\alpha \square K) \ar[r]^-{\#\iota_{\alpha \square K}^{-1}} & \#(\alpha \square e(K)) \ar[d]^-{\kappa_{\alpha \square K, 1, e(K)}} \cr \#e(K) \ar[u]^-{\iota_{\#K, \#\alpha}} \ar[r]^-{\#\iota_{\alpha \square K}} & \#e(K) \square \#e(\alpha) \ar[r]^-{\Phi_{e(K), \#e(\alpha)}} & \#1 e(K) \square \#e(\alpha).}$$

We start by proving that this diagram commutes for basic 2-morphisms and the empty string of 2-morphisms. For $\alpha = \emptyset_{\mathcal{E}}$, the 3-morphism $\iota_{\alpha \square K}$ is trivial, and we have

$$\chi_{\alpha} = \Phi_{e(\mathcal{E}), 1}^{-1}, \quad \chi_{\alpha \square K} = \Phi_{e(\mathcal{E} \square K), 1}^{-1}, \quad \kappa_{\alpha \square K, 1, e(K)} = \Phi_{e(\mathcal{E} \square K), 1, e(K)} \cdot (\Phi_{e(K), \#e(\alpha)} \Phi_{e(K), \#e(\alpha)})^{-1},$$

where the last identity follows directly with (19). Inserting this into (46), one finds that the diagram commutes. For $\alpha = (\alpha, 1)$, the 3-morphisms $\chi_{\alpha}, \chi_{\alpha \square K}, \iota_{\alpha \square K}$ are trivial and

$$\iota_{\#K, \#\alpha} = \kappa_{\alpha \square K, 1, e(K)} \cdot (\Phi_{e(K), \#e(\alpha)}),$$

which shows that the diagram commutes. For $\alpha = (\alpha, -1)$ diagram (46) corresponds to the boundary of the following diagram

$$\xymatrix{ \#e(K) \square \alpha \ar[r]^-{\Theta_{\#e(K) \square \alpha}} & \#(\#(\alpha \square e(K))) \ar[r]^-{\kappa_{\alpha, \#e(\alpha)}} & \#(\alpha \square \#1 e(K)) \ar[d]^-{\#(\alpha \square \Phi_{e(\mathcal{K}), 1}^{-1})} \cr \Phi_{e(K), \#e(\alpha)}^{-1} \ar[u]^-{\Phi_{e(K), \#e(\alpha)}} \ar[r]^-{\kappa_{\Phi_{e(K), \#e(\alpha)}^{-1}, \#e(\alpha)}} & \# \Phi_{e(K), \#e(\alpha)}^{-1} \square \# \alpha \ar[r]^-{\kappa_{\#e(\alpha), \#1 e(K)}} & \#(\alpha \square \#1 e(K)) \ar[u]^-{\#(\alpha \square \Phi_{e(\mathcal{K}), 1}^{-1})}.}$$

where we used identities (24) and (26) to express $\Theta_{\#e(K) \square \alpha}$ in terms of $\Theta_{\alpha}$. The pentagon in this diagram commutes due to identities (24) and (26) and the quadrilateral due to the naturality of the 3-morphism $\kappa_{\mu, \nu} : \#\nu \square \#\mu \Rightarrow \#(\mu \square \nu)$. This proves that diagram (46) commutes for basic 2-morphisms and the empty string of 2-morphisms.

The proof that this identity holds for general 2-morphisms $\alpha = (\alpha_1, \ldots, \alpha_n)$ is by induction over $n$. For $n = 1$, $\alpha$ is a basic 1-morphism and this identity was shown above. Suppose that the commutativity
of diagram (46) is established for all strings $\alpha = (\alpha_k, \ldots, \alpha_1)$ of basic 2-morphisms of length $k \leq n-1$ and let $\gamma = (\gamma_n, \ldots, \gamma_1)$ be a 2-morphism of length $n$. Set $\alpha = \gamma_n, \beta = (\gamma_{n-1}, \ldots, \gamma_1)$ and consider the diagram (47)

\[
\begin{array}{cccc}
\text{of diagram (46) is established for all strings } \alpha = (\alpha_k, \ldots, \alpha_1) \text{ of basic 2-morphisms of length } k \leq n-1 \text{ and let } \gamma = (\gamma_n, \ldots, \gamma_1) \text{ be a 2-morphism of length } n. \text{ Set } \alpha = \gamma_n, \beta = (\gamma_{n-1}, \ldots, \gamma_1) \text{ and consider the diagram (47).}
\end{array}
\]

\[
\begin{array}{cccc}
\Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)}
\end{array}
\]

\[
\begin{array}{cccc}
\Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)} & \Phi_{\nu(\alpha K), e(\beta K)} \circ \Phi_{\nu(\alpha K), e(\beta K)}
\end{array}
\]

The outer path in this diagram corresponds to the diagram (46) for $\gamma$. The quadrilateral at the top of the diagram commutes due to the naturality of the 3-morphisms $\Phi_{\mu, \nu} : \# \mu \circ \# \nu \Rightarrow \# (\nu \circ \mu)$, $\kappa_{\mu, \nu} : \# \nu \circ \# \mu \Rightarrow \# (\mu \circ \nu)$ and $\Phi_F : 1 \# F \Rightarrow 1 \# F$. The hexagon on the left of this diagram commutes because identity (46) holds for $\alpha$ and $\beta$. To show that the hexagon on the right of the diagram commutes, we set $\nu = \tau = 1_K$ in (18) and use the naturality of the tensorator. This yields the diagram

\[
\begin{array}{cccc}
\kappa_{\mu, 1_K} \circ \kappa_{\nu, 1_K} & \kappa_{\mu, 1_K} \circ \kappa_{\nu, 1_K} & \kappa_{\mu, 1_K} \circ \kappa_{\nu, 1_K} & \kappa_{\mu, 1_K} \circ \kappa_{\nu, 1_K}
\end{array}
\]

The triangle in this diagram commutes due to the naturality of the tensorator and the outer pentagon due to identity (18). This implies that the inner hexagon commutes as well and hence the hexagon on the right in diagram (47). This shows that identity (46) holds for $\gamma$ and concludes the proof that $\# : \mathcal{C} \to \mathcal{C}_{op}$ is a strict functor of strict tricategories for which all coherence data is trivial.

To prove the identity $\# \# = 1$, note that it is obvious for 1- and 2-morphisms. For 3-morphisms $\Psi : \alpha \Rightarrow \alpha'$, it holds if the following
diagram commutes

\[
\begin{align*}
\text{(48)} & \quad \Theta_{e(\omega)} = \chi_{\#_{\alpha}} \cdot \# \chi_{\alpha}.
\end{align*}
\]

For $\alpha = \emptyset_F$ this follows directly from identity \[24\], which implies

\[
\Theta_{e(\emptyset_F)} = \Theta_{e(F)} = \Phi_{\#_{e(F)}}^{-1} \cdot \Phi_{\#_{e(F)}}^{-1} = \chi_{\#_{\emptyset_F}} \cdot \# \chi_{\emptyset_F}.
\]

Similarly, for basic 2-morphisms $\alpha = (\alpha, z)$, we have

\[
\begin{align*}
\chi_{\#(\alpha,1)} \cdot \# \chi_{(\alpha,1)} &= \Theta_{\alpha} \cdot \# 1_{\#_{\alpha}} = \Theta_{\alpha} = \Theta_{e(\alpha,1)} \text{ and } \\
\chi_{\#(\alpha,-1)} \cdot \# \chi_{(\alpha,-1)} &= 1_{\#_{\alpha}} \cdot \# \Theta_{\alpha} = \Theta_{\#_{\alpha}} = \Theta_{e(\alpha,-1)}.
\end{align*}
\]

where we used the identity $\# \Theta_{\alpha} = \Theta_{\#_{\alpha}}$ from Lemma \[4.6\] in the second line. The proof that identity \[49\] holds for strings $\alpha = (\alpha_n, \ldots, \alpha_1)$ of basic 2-morphisms is by induction over the length $n$ of the string. For $n = 0, 1$ this was shown above. Suppose that the identity \[49\] is established for all 2-morphisms $\alpha = (\alpha_K, \ldots, \alpha_1)$ of length $0 \leq k < n$ and let $\gamma = (\gamma_n, \ldots, \gamma_1)$ be a string of basic 2-morphisms of length $n$. Then identity \[49\] holds for $\gamma$ if and only if the following diagram commutes for $\alpha = \gamma_n$ and $\beta = (\gamma_{n-1}, \ldots, \gamma_1)$

\[
\begin{align*}
\Theta_{e(\alpha,\beta)} = e(\alpha) \circ e(\beta) & \rightarrow e(\alpha,\beta) \\
\Phi_{\#_{e(\alpha),\#_{e(\beta)}}^{-1}}^{-1} \circ e(\alpha,\beta) & \rightarrow \Phi_{\#_{e(\alpha),\#_{e(\beta)}}^{-1}}^{-1} \circ e(\alpha,\beta)
\end{align*}
\]

The triangle at the bottom of the diagram commutes by induction hypothesis. The curved subdiagram at the left commutes due to identity
and the subdiagram at the right due to the naturality of the 3-morphism $\Phi_{\mu,\nu} : \#\mu\circ\#\nu \Rightarrow \#(\nu\circ\mu)$. This shows that the diagram (48) commutes for all 3-morphisms $\psi : \alpha \Rightarrow \alpha'$ in $\mathcal{G}$ and that the functor $\# : \mathcal{G} \to \mathcal{G}_{\text{op}}$ satisfies $\#\# = 1$.

3. To define the functor of strict tricategories $\star : \mathcal{G} \to \mathcal{G}_{\text{op}}$, we set $\star$ to be trivial on the objects and 1-morphisms of $\mathcal{G}$. For a basic 2-morphism $(\alpha, z)$ we set $\star(\alpha, z) = (\alpha^*, z)$ and extend $\star$ to general 2-morphisms via

$$\star(\alpha_2, ..., \alpha_1) = (\star\alpha_1, ..., \star\alpha_n), \quad \star(\emptyset) = \emptyset.$$

It follows that $\star$ is strictly compatible with the Gray product of 1- and 2-morphisms as well as the horizontal composition, preserves the unit 1- and 2-morphisms and satisfies $\star \star(\alpha) = \alpha$ for all 2-morphisms $\alpha$. We also have the identities

$$\begin{align*}
\star e(\emptyset) &= e(\star \emptyset) = 1e(\emptyset), \\
\star e(\alpha, -1) &= \# \star \alpha, \\
\star e(\alpha, 1) &= e(\star(\alpha, 1)) = \alpha^*.
\end{align*}$$

To obtain a 3-morphism $\star \Psi : \star \alpha' \Rightarrow \star \alpha$ for each 3-morphism $\Psi : \alpha \Rightarrow \alpha'$ we consider the 3-morphism $\xi_\alpha : \star e(\alpha) \Rightarrow e(\star \alpha)$ given by

$$\begin{align*}
\xi_{\emptyset} &= 1_{e(\emptyset)}, \\
\xi_{\alpha, 1} &= 1_{\star \alpha}, \\
\xi_{\alpha, -1} &= \Delta_{\star \alpha}^{-1} \\
\xi_{\alpha_2, ..., \alpha_1} &= \xi_{\alpha_1} \circ \cdots \circ \xi_{\alpha_n}
\end{align*}$$

and set

$$e(\star \Psi) = \xi_\alpha : \star e(\Psi) \cdot \xi_\alpha^{-1}.$$

The strict compatibility of $\star$ with the vertical composition of 3-morphisms is a direct consequence of the definition. The strict compatibility of $\star$ with the horizontal composition is equivalent to the commutativity of the diagram

$$\begin{align*}
e(\star(\alpha \circ \beta)) &= e(\beta) \circ e(\alpha) \circ e(\Psi) \circ e(\Phi) \\
e(\star(\alpha \circ \beta)) &= e(\star(\alpha \circ \beta)) \circ e(\alpha' \circ \beta')
\end{align*}$$

By definition, the paths on the boundary correspond to the 3-morphisms $e(\star(\Psi \circ \Phi))$ and $e(\star \Phi \circ \Psi)$, and the parallelogram in the middle commutes due to the identity $e(\Psi \circ \Phi) = e(\Psi) \circ e(\Phi)$. As we have $\xi_\beta \circ \xi_\alpha = \xi_{\alpha \circ \beta}$ for all 2-morphisms $\alpha, \beta$ by definition, the diagram commutes and we obtain $\star(\Psi \circ \Phi) = e(\star \Phi \circ \Psi)$ for all composable 3-morphisms $\Psi, \Phi$. The identity $\star \star \Psi = 1$ then follows from the identity $\Delta_{\star \alpha} = \Delta_\alpha$ for all 2-morphisms $\alpha$ in a spatial Gray category $\mathcal{G}$ and the compatibility of $\star$ with the horizontal composition.
The strict compatibility of the functor \( \ast \) with the Gray product corresponds to the commutativity of the diagram

\[
e(\ast(\Psi \Box \Phi)) \\
\xrightarrow{\xi^{-1}(\ast \beta \Box \ast \alpha)} e(\ast(\alpha \Box \beta)) \xrightarrow{\ast \epsilon(\alpha \Box \beta)} e(\ast(\Psi \Box \Phi)) \xrightarrow{\xi^{-1}(\ast \beta \Box \ast \alpha)} e(\ast(\alpha' \Box \beta'))
\]

(50)

for 3-morphisms \( \Phi : \beta' \Rightarrow \beta, \Psi : \alpha' \Rightarrow \alpha \) and 2-morphisms \( \alpha, \alpha' : F \Rightarrow G, \beta, \beta' : H \Rightarrow K \). In this diagram the expression \( \square \) denotes the opposite Gray product of 2-morphisms \( \beta \square \alpha = (G \Box \beta) \circ (\alpha \Box H) \) from (8) and from Corollary 2.16. The two rectangles in the middle of the diagram and the curved quadrilaterals at the top and bottom of the diagram commute by definition of the 3-morphisms \( \ast(\Psi \Box \Phi), \Psi \Box \Phi, \ast \Phi \Box \ast \Psi \). To show that the two curved quadrilaterals at the left and right of the diagram commute, we note that it is sufficient to prove this for the case where either \( \alpha = \emptyset_F \) or \( \beta = \emptyset_K \). In the latter, the diagram reduces to

\[
e(\ast(\alpha \Box K)) \xrightarrow{\xi^{-1}(\ast \alpha \Box K)} e(\ast(\alpha \Box K)) \xrightarrow{\ast \epsilon(\alpha \Box K)} e(\ast(\alpha \Box K))
\]

(51)

which clearly commutes if \( \alpha = \emptyset_F \) or \( \alpha = (\alpha, 1) \). For \( \alpha = (\alpha, -1) \), we consider the following diagram whose boundary corresponds to the
diagram (51)

\[
\begin{align*}
\# \ast (\# K \square \alpha) & \xrightarrow{\Delta_{\gamma,(K \square \alpha)}} \# (\# \alpha \square \# K) & \xrightarrow{\ast \kappa_{1,\# K,\alpha}} & \# (\# \alpha \square \# K) \\
\# \ast \Theta_{# K \square \alpha} & \xrightarrow{\ast \Gamma_{# K \square \alpha}} & \# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \kappa_{1,\# K,\alpha}} & \# \ast \# (\# K \square \alpha)
\end{align*}
\]

\[
\begin{align*}
\# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \kappa_{1,\# K,\alpha}} & \# \ast (\# K \square \# K) & \xrightarrow{\# \ast \kappa_{1,\# K,\alpha}} & \# \ast (\# K \square \# K) \\
\# \ast \Theta_{# K \square \alpha} \Phi_{K}^{-1} & \xrightarrow{\# \ast \Theta_{# K \square \alpha} \Phi_{K}^{-1}} & \# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \kappa_{1,\# K,\alpha}} & \# \ast (\# K \square \alpha)
\end{align*}
\]

\[
\begin{align*}
\# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha)
\end{align*}
\]

The two triangles in this diagram commute due to identity (27). The parallelogram at the top of the diagram commutes due to the naturality of the 3-morphisms \( \kappa_{\mu,\nu} : \# \nu \square \# \mu \Rightarrow \# (\mu \square \nu) \) and the quadrilateral on the right of the diagram due to the naturality of the 3-morphism \( \Gamma_{\mu} : \ast \# \ast \# \mu \Rightarrow \mu \). The heptagon in this diagram can be subdivided as

\[
\begin{align*}
\# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha)
\end{align*}
\]

The upper quadrilateral in this diagram commutes due to relation (26) and the rectangle below it by naturality of the 3-morphism \( \kappa_{\mu,\nu} : \# \nu \square \# \mu \Rightarrow \# (\mu \square \nu) \). The quadrilateral at the bottom of the diagram commutes due to the naturality of the tensorator and the hexagon on the right due to identity (23). The heptagon on the left commutes by

\[
\begin{align*}
\# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha) & \xrightarrow{\# \ast \# (\# K \square \alpha)} & \# \ast \# (\# K \square \alpha)
\end{align*}
\]
naturality of the tensorator and of the 3-morphisms \( \kappa_{\mu,\nu} : \#(\mu \square \nu) \Rightarrow \#(\mu \square \nu) \), \( \Phi_F : 1_F \Rightarrow 1_F \), \( \Gamma_\mu : *\# * \# \mu \Rightarrow \mu \) and \( \Theta_\mu : \# \# \Rightarrow \mu \) together with identity (21). Hence, the diagram commutes, which implies that diagram (52) commutes. This in turn proves the commutativity of diagram (51) for 2-morphisms \( \alpha = (\alpha, -1) \).

For general 2-morphisms \( \alpha = (\alpha_n, ..., \alpha_1) \) the commutativity of diagram (51) follows directly from the identities (38) and \( \xi_{\alpha \beta} = \xi_{\beta} \circ \xi_{\alpha} \). This proves that \( * \) defines a strict functor of strict tricategories \( * : \mathcal{G} \to \mathcal{G}^{op} \) with trivial coherence data and \( ** = 1 \).

4. It remains to prove the identity \( *\#*\# = 1 \). It is obvious that this identity holds for 1- and 2-morphisms. To prove that it holds for 3-morphisms \( \Psi : \alpha \Rightarrow \beta \), we consider the diagram

The three rectangles and the curved quadrilateral at the top of this diagram commute by definition of \( \# \) \( \Psi \), \( * \Psi \), and the curved quadrilateral at the bottom commutes due to the naturality of \( \Gamma \). It is therefore sufficient to show that the curved subdiagrams at the left and the right commute, which amounts to the relation

\[
\Gamma_{e(\alpha)} \cdot *\# * \chi^{-1}_\alpha \cdot *\# \xi_{\# \#} \cdot *\# \chi_{\# \#} \cdot \xi^{-1}_{\# \#} = 1_{e(\alpha)}
\]

for all 2-morphisms \( \alpha \) in \( \mathcal{G} \). For \( \alpha = 0_F \), the 3-morphisms \( \xi_{\# \#} \), \( \xi_{\# \#} \) are trivial, and this relation reduces to

\[
\Gamma_{1_{e(F)}} \cdot *\# * \Phi_{e(F)} \cdot *\Phi^{-1}_{e(F)} = 1_{1_{e(F)}}
\]
which holds by (21). For \( \alpha = (\alpha, 1) \), \( \xi_{\# \alpha} \) and \( \chi_\alpha \) are trivial, and from equation (27) one obtains
\[
\Gamma_\alpha \cdot \# \Delta^{-1}_\alpha \cdot \# \Theta_\alpha = 1.
\]
For \( \alpha = (\alpha, -1) \), equation (27) together with the identity \( \# \Theta_\alpha = \Theta_\# \alpha \) in Lemma 4.6 and the naturality of \( \Delta_\alpha \) implies
\[
\Gamma_\#_\alpha \cdot \# \cdot \Theta^{-1}_\alpha \cdot \Delta_\alpha = 1 \#_\alpha.
\]

To prove the identity for general 2-morphisms \( \alpha = (\alpha_1, \ldots, \alpha_n) \), it is sufficient to show that the following diagram commutes
\[
\begin{array}{c}
\ast \circ \ldots \circ \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast \# \ast \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast \# \ast \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast \# \ast \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\Gamma_{\ast \# \# \alpha_n \cdots \ast \# \alpha_1} \\
\Gamma_{\ast \# \# \alpha_n \cdots \ast \# \alpha_1} \\
\Gamma_{\ast \# \# \alpha_n \cdots \ast \# \alpha_1}
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]
\[
\begin{array}{c}
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1) \\
\ast \# \ast (\# \ast \# \alpha_n \cdots \ast \# \alpha_1)
\end{array}
\]

The upper two rectangles commute due to the naturality of the 3-morphism \( \Phi_{\mu, \nu} : \# \mu \circ \# \nu \Rightarrow \#(\nu \circ \mu) \). The rectangle at the bottom commutes due to identity (22). This shows that the diagram commutes and the functors of strict tricategories \( \# : \mathcal{G} \rightarrow \mathcal{G}_{op}^{\#}, \ast : \mathcal{G} \rightarrow \mathcal{G}_{op}^{\#} \) satisfy \( \ast \# = \# \ast = 1 \).

5. To show that the Gray category \( \mathcal{G} \) is equivalent to \( \mathcal{G} \), we note that the evaluation defines a strict functor of strict tricategories \( e : \mathcal{G} \rightarrow \mathcal{G} \). As the evaluation is strictly compatible with the horizontal and the vertical composition and with all unit morphisms, the only coherence data of this functor is given by the 3-morphisms \( \hat{\iota}_{\alpha K} : e(\alpha) \square e(K) \Rightarrow e(\alpha \square K) \), \( \hat{\iota}_E \beta : e(F) \square e(\beta) \Rightarrow e(F \square \beta) \). As in the proof of Theorem 4.3, it is therefore sufficient to show that for all composable 3-morphisms \( \alpha : F \Rightarrow G, \beta : H \Rightarrow K \) the 3-morphisms \( \hat{\iota}_{\alpha \beta} = \hat{\iota}_{\alpha K} \circ \hat{\iota}_E \beta \) are natural in both arguments and satisfy conditions analogous to (18), (19) and (20) as well as \( \hat{\iota}_{1_{\alpha \#}} = \hat{\iota}_{1_{\alpha \#}} = 1_{\#(\alpha)} \). The naturality and the compatibility with the unit-morphisms are a direct consequence of the definitions. Condition (18) follows from the commutative diagram (37) and Condition (20) from the commutative diagrams (39) and (40). This shows that the evaluation defines a functor of strict tricategories \( e : \mathcal{G} \rightarrow \mathcal{G} \).
We construct an embedding functor \( f : \mathcal{G} \to \mathcal{G} \) that will be a lax functor of strict tricategories. For this, we set \( f(\mathcal{A}) = \mathcal{A} \) for all objects, \( f(F) = (F, 1) \) for all 1-morphisms, \( f(\alpha) = (\alpha, 1) \) for all 2-morphisms of \( \mathcal{G} \) and \( f(\Gamma) = \Gamma \) for all 3-morphisms of \( \mathcal{G} \). This defines for all objects \( \mathcal{A}, \mathcal{B} \) a weak 2-functor \( f_{AB} : \mathcal{G}(\mathcal{A}, \mathcal{B}) \to \mathcal{G}(\mathcal{A}, \mathcal{B}) \) with the coherence data given by the invertible 3-morphisms \( \Gamma_{\mu\nu} : f(\mu) \circ f(\nu) \Rightarrow f(\mu \circ \nu) \) and \( 1_{1_F} : \emptyset_{f(F)} \Rightarrow f(1_F) \) for all 1-morphisms \( F \) and composable 2-morphisms \( \mu, \nu \) in \( \mathcal{G} \).

The 2-morphisms \( \iota_C : f(1_C) \Rightarrow 1_{f(C)} \) from Definition \( \text{A.6} \) and their inverses are given by \( \iota_C = \iota_C^{-1} = (1_1, 1) \). The invertible pseudo-natural transformation \( \kappa_{ABC} : \Box(f_B \times f_{AB}) \to f_{ABC} \) is determined by the 2-morphisms \( (1_{FGC}, 1) : f(G \Box f(F)) \Rightarrow f(G \Box F) \) and the invertible 3-morphisms \( 1_\mu \Box \nu : (1_{G\Box K}, 1) \Box (f(\mu) \Box f(\nu)) \Rightarrow f(\mu \Box \nu) \Box f(1_{F \Box H}) \) for all pairs of composable 1-morphisms \( G, F \) and \( H, K \) and 2-morphisms \( \mu, \nu : F \Rightarrow G, \nu : H \Rightarrow K \) in \( \mathcal{G} \). It is easy to show that the coherence conditions in Definitions \( \text{A.6} \) and \( \text{A.1} \) are satisfied and, consequently, \( f \) defines a lax functor of strict tricategories \( f : \mathcal{G} \to \mathcal{G} \).

It follows directly that \( ef = 1_{\mathcal{G}} \). The lax functor of strict tricategories \( fe : \mathcal{G} \to \mathcal{G} \) is given by \( fe(\mathcal{A}) = \mathcal{A}, \ f(e(\mathcal{F})) = (e(\mathcal{F}), 1), \ f(e(\alpha)) = (e(\alpha), 1) \) and \( f(e(\Gamma)) = \Gamma \). A natural isomorphism of lax functors of strict tricategories \( \eta : fe \Rightarrow 1 \) is given by the trivial 1-morphism \( \emptyset_A : A \to A \) for each object \( A \) of \( \mathcal{G} \) together with the invertible pseudo-natural transformation of weak 2-category functors \( 1_{AB} \to (f e)_{AB} \) that is determined by the 2-morphisms \( \eta_{f(\mathcal{F})} = (1_{f(\mathcal{F})}, 1) : f e(\mathcal{F}) \Rightarrow \mathcal{F} \) for each 1-morphism \( F : \mathcal{A} \to \mathcal{B} \) and the invertible 3-morphism \( \eta_\alpha = \Gamma_{\alpha(\mathcal{A})} : (1_{f(\mathcal{F})}, 1) \Box f e(\alpha) \Rightarrow \alpha \Box (1_{f(\mathcal{F})}, 1) \) for each 2-morphism \( \alpha : F \Rightarrow G \). A direct calculation shows that the consistency conditions in Definitions \( \text{A.9} \) and \( \text{A.3} \) are satisfied and that this defines a natural isomorphism \( fe \Rightarrow 1 \) of lax functors of strict tricategories. It also follows directly that \( e\eta = 1_e : e \Rightarrow e \) and \( \eta fe = 1_{fe} : fe \Rightarrow fe \).

The invertible pseudo-natural transformation \( \eta f : f \Rightarrow f \) is determined by the 2-morphisms \( (1_F, 1) : f(F) \Rightarrow f(F) \) for each 1-morphism \( F \) and the 3-morphisms \( 1_\alpha : ((\alpha, 1), (1_F, 1)) \Rightarrow ((1_G, 1), (\alpha, 1)) \). A modification \( \Psi : \eta f \Rightarrow 1_f \) is therefore given by the trivial 2-morphism \( \emptyset_{1_A} \) for each object \( A \) of \( \mathcal{G} \) and the invertible 3-morphisms \( 1_{1_{1_F}} : (\eta f)_F \Rightarrow \emptyset_{1_f} \). This implies \( e\Psi = 1_{1_g} : e\eta f = 1_{ef} = 1_{1_g} \Rightarrow e1_f = 1_{ef} = 1_{1_g} \) and it concludes the proof that the Gray categories \( \mathcal{G} \) and \( \mathcal{G} \) are equivalent.

By definition, the lax functor of strict tricategories \( f : \mathcal{G} \to \mathcal{G} \) satisfies \( * f = f * \). As the functors are the identity on the objects, a natural isomorphism \( \check{\chi} : f# \Rightarrow f\# \) is determined by an invertible pseudo-natural transformation of weak 2-functors \( \check{\chi} : (f#)_{\mathcal{A}, \mathcal{B}} \to (f\#)_{\mathcal{A}, \mathcal{B}} \) for each pair of objects \( \mathcal{A}, \mathcal{B} \). This natural isomorphism is given by the 2-morphisms \( \check{\chi}_F = (1_{f#}, 1) : f#(F) \Rightarrow f\#(F) \) for each 1-morphism \( F \).
in \( G \) and the invertible 3-morphisms \( \tilde{\chi}_\mu = 1_{\#G} : (1_{\#G}, 1) \otimes \#f(\mu) \to f(\#(\mu)) \otimes 1_{\#G} \) for each 2-morphism \( \mu : F \to G \). It follows directly that all coherence conditions in Definitions A.9 and A.3 are satisfied.

The natural isomorphisms \( \chi : \#e \to e \# \) and \( \xi : e^* \to e \#^* \) are obtained from the coherence data of \( G \). As the functors of strict tricategories \( \#e : G \to G_{op} \) and \( e^* : G \to G_{op} \), as well as \( *e : G \to G_{op} \) and \( e^* : G \to G_{op} \), agree on the objects and 1-morphisms of \( G \), such natural isomorphisms are specified uniquely by natural isomorphisms between the functors \( (\#e)_{F,G} \), \( (e^*)_{F,G} \), \( (\#e)_F, G \to G_{op}(\#e(\#), \#e(\#)) \) and between the functors \( (*e)_{F,G} \), \( (e^*)_{F,G} \) : \( G(F,G) \to G_{op}(*e(F), *e(G)) \). They are determined by the invertible 3-morphisms \( \chi_\alpha : \#e(\alpha) \to e(\#(\alpha)) \) and \( \xi_\alpha : *e(\alpha) \to e(\#(\alpha)) \) for each 2-morphism \( \alpha : F \to G \). That they satisfy the consistency conditions in Definitions A.9 and A.3 was shown, respectively, in the second and third part of the proof. □

Theorem 5.2 explicitly constructs a Gray category \( G \) and strictifications \( * e : G \to G_{op} \), \( \# e : G \to G_{op} \) of the functors of strict tricategories \( * e : G \to G_{op} \), \( \# e : G \to G_{op} \). This construction has the benefit that it is conceptually clear and concrete and allows one to verify the properties of the strictified functors by direct computations. It remains to show that the Gray category \( G \) with the strict functors of strict tricategories \( * e : G \to G_{op} \), \( \# e : G \to G_{op} \) is again a Gray category with duals in the sense of Definition 3.10 and to clarify which additional relations hold in the strictified Gray category.

**Theorem 5.3.** For every spatial Gray category \( G \), the associated Gray category \( G \) from Theorem 5.2 is a Gray category with strict duals in the sense of Definition 3.10.

**Proof.**
1. For each pair of objects \( C, D \) of \( G \), the functor \( * e : G \to G_{op} \) defines a strict 2-functor \( * e : G(C,D) \to G_{op}(C,D) \) that is trivial on the objects of \( G(C,D) \) and satisfies \( * e * e = 1 \). To show that this gives \( G(C,D) \) the structure of a planar 2-category, it is sufficient to construct for each 2-morphism \( \mu \) a 3-morphism \( \xi_\mu : 0_C \Rightarrow \mu \) satisfying the conditions in Definition 3.10 and in (12). This 3-morphism is defined by

\[
e(\xi_\mu) = (e(\xi_\mu) \circ \mu) \cdot e(\mu),
\]

where \( e(\xi_\mu) \) denotes the corresponding 3-morphism in \( G \). The identity \( H \square e(\mu) \square K = e(\mu) \square e(\mu) \) from Definition 3.10 follows from the commutative diagram (53) and the analogous diagram with the 1-morphism on the left.
That the 3-morphism $\epsilon_\mu : \Theta \Rightarrow \mu \otimes \mu$ satisfies the conditions \((12)\) is a consequence of the following three commutative diagrams and the analogue of the second diagram for the composite \((\epsilon_\mu \otimes \mu) : (\mu \otimes \epsilon_\mu)\).
This shows that for all objects \( C, D \) the 2-category \( \mathcal{G}(C, D) \) is planar and that the first condition in Definition 3.10 is satisfied.

2. The functor \( \# : \mathcal{G} \rightarrow \mathcal{G}_{op} \) defines the dual of each 1-morphism \( F : C \rightarrow D \) and by definition satisfies \( \# \# F = F, \#(F \square G) = \#G \square \#F, \#\emptyset_C = \emptyset_C \). It remains to construct the fold 2-morphisms \( \eta_F : \emptyset_D \Rightarrow F \square \#F \) and the triangulator 3-morphisms \( T_F : \star(\eta_F \square F) \circ (F \square \eta_F) \Rightarrow \emptyset_F \) and to show that they satisfy the conditions in Definition 3.10. We define

\[
\eta_{\emptyset_C} = \emptyset_C, \quad \eta_G = (\eta_{(\mathcal{G}), 1}) : \emptyset_D \Rightarrow F \square \#F, \quad \epsilon(T_F) = T_\epsilon(F).
\]

for all 1-morphisms \( F : C \rightarrow D \) and all non-empty 1-morphisms \( G : C \rightarrow D \). Conditions (2) (b) in Definition 3.10 then hold by definition. As the 3-morphisms \( i_{\eta_F \square \eta_F} \), \( i_{K_{\eta_F \square \eta_F}} \), and \( \xi_{\#F} \) and the analogous 3-morphisms for \( \star \eta \) are trivial, the remaining identities in (2) (c), (d) then follow directly from the corresponding properties of the fold 2-morphisms and the triangulator in \( \mathcal{G} \). This shows that \( (\mathcal{G}, \star, \#) \) is a Gray category with duals in the sense of Definition 3.10.

3. To prove the identity \( \# \epsilon = \epsilon \# \), consider the diagram in (54). The triangle at the top and the pentagon at the bottom of the diagram commute by definition of \( \epsilon \# \) and \( \# \epsilon \). The two triangles on the left commute by (12). The upper polygon in the middle commutes since it can be decomposed into diagrams whose commutativity was established in the proof of Theorem 5.2. The lower polygon in the middle which involves the 3-morphisms \( \# \epsilon_\mu \) and \( \epsilon_\mu \# \) commutes by definition of the 3-morphism \( \Delta_\epsilon_\mu \). This shows that the diagram (54) commutes and proves the identity \( \# \epsilon_\mu = \epsilon_\mu \# \).
The conditions in Definition 5.1 on a Gray category with strict duals have a clear geometrical interpretation in terms of Gray category diagrams. In the diagrams, the functor of strict tricategories $\ast$ corresponds to a 180 degree rotation around the $w$-axis, and the condition $\ast\ast = 1$ ensures that the evaluation is invariant under 360 degree rotation of the diagram. Similarly, the functor $\#\ast$ in a Gray category with strict duals corresponds to a 180 degree rotation around the $y$-axis and the condition $\#\# = 1$ ensures that the evaluation is invariant under a 360 degree rotation. The condition $\ast\#\ast\# = 1$ corresponds to the fact that the 180 degree rotations around the $w$- and $y$-axis commute. Together with the strictness of the functors $\ast$ and $\#$, these conditions on $\ast$ and $\#$ ensure that the functors of strict tricategories $\ast$ and $\#$ correspond to the symmetries of a cube. In contrast to the original Gray category $\mathcal{G}$, where these symmetries were realised up to higher morphisms, in the strictified Gray category $\mathcal{G}$ these symmetries are realised exactly.

The condition in Definition 5.1 that the 3-morphisms $\#\epsilon_{\mu}, \epsilon_{\ast\#\mu} : \#1_{\mathcal{G}} \rightarrow \#\mu \circ \#\#\mu$ agree ensures that the labelling of the minima and maxima of lines in the associated Gray category diagrams does not become ambiguous.

Note also that the strictification theorem implies a coherence result for the 3-morphisms $\Phi_{\mathcal{F}}, \Phi_{\mu, \nu}$ and $\kappa_{\mu, \nu}$ from Theorem 4.3 and
the 3-morphisms $\Theta_\mu$, $\Gamma_\mu$ that characterise the natural isomorphisms $\Theta : \#\# \to 1$ and $\Gamma : *\# *\# \to 1$ in Theorem 4.5. As shown in the proof of Theorem 5.2, these are precisely the 3-morphisms associated with the evaluation functor $e : \mathcal{G} \to \mathcal{G}$. The strictification theorem implies that any two 3-morphisms $\Psi, \Omega : \mu \Rightarrow \nu$ which are constructed from these 3-morphisms, their inverses and their $*$- and $\#$-duals via the Gray product, the horizontal and vertical composition and the tensorator are equal.

With respect to the discussion in Section 4.4 this suggests that in a spatial Gray category, it should be possible to omit the labelling by 3-morphisms $\Omega_\mu$ at the points where lines in the diagrams cross the fold lines, as these labellings are canonical. Similarly, the evaluation of two diagrams that can be transformed into each other by sliding lines over folds and cusps as in Figure 46, 47 should be related by a unique 3-morphism that is constructed from the 3-morphisms $\Gamma_\mu$ and $\Theta_\mu$, their $*$- and $\#$-duals and their inverses via the Gray product, the horizontal and vertical composition and the tensorator. However, a detailed exploration of this idea is beyond the scope of this paper.

6. Diagrams for Gray categories with duals

In this section, it is shown that under suitable additional assumptions, the evaluation of a Gray category diagram is invariant under homomorphisms of Gray category diagrams. For the Gray category, this additional assumption is is that the Gray category is spatial in the sense of Definition 4.8. We derive concrete conditions that ensure that a Gray category with duals is spatial in Section 7.1. For the diagram it is sufficient to assume that the topology of the entire diagram is a surface.

6.1. Non-progressive Gray category diagrams. In this subsection more general diagrams than the progressive ones are defined. This is familiar from two-dimensional diagrams, where the maxima and minima can be interpreted as points on a single line that changes direction. Exactly the same thing happens for lines in three-dimensional diagrams.

The analogous situation for surfaces is that they can have singularities of the projection. So the diagrams in Figure 15 can be interpreted as containing a single surface with folds and cusps. A singular point of the projection is called a fold if it is locally isomorphic to a subdiagram of any of the diagrams in Figure 19. A singular point is called a cusp if it is locally isomorphic to Figure 15 g), 16 a), or their rotations by $\pi$ around the $w$-axis. Note that the two folds meeting a cusp are either both input lines (lines meeting the top face of a small cube around the vertex) or both output lines (lines meeting the bottom face of a small cube around the vertex).
A diagram with folds and cusps can be subdivided to give a progressive diagram by introducing additional lines along the folds and additional vertices at the cusps and at the intersection of folds and lines. This remark is not entirely trivial because it is necessary to check that the additional lines at folds meet the boundary correctly according to Definition 2.22 (1a). This follows from condition (1b) of the same definition, since a fold line meeting a side face would imply that the side face is not a progressive diagram.

According to Whitney’s classification [34], the generic singularities of the projection of a smooth surface are the smooth folds and cusps. This motivates the analogous definition of generic singularity in the PL case.

Definition 6.1 (Generic three-dimensional diagram). A three-dimensional diagram is called generic if the only singularities of the projection $p_2$ on surfaces are folds and cusps, and the only singularities of $p_1 \circ p_2$ on lines and folds are maxima and minima. In addition, the subdivision obtained by additional lines at the folds and additional vertices at the cusps and the intersection of folds and lines is required to be a generic progressive diagram according to Definition 2.22.

The definition and evaluation of a diagram for a Gray category with duals follows the same pattern as for two-dimensional diagrams in Section 3.2. Again it will be assumed without further mention that diagrams are generic where this is appropriate.

Definition 6.2. Let $G$ be a Gray category with duals. A diagram for $G$ is a three-dimensional diagram $D$ together with a labelling of its minimal progressive subdivision $S$ with elements of $G$ that makes $S$ into a Gray category diagram. This labelling is required to be such that the fold lines are labelled appropriately with either $\eta_F$ or $\eta^*_F$, the cusps with $T_F$, $T^{-1}_F$, $*T_F$, or $*T^{-1}_F$, the minima and maxima for lines by $\epsilon_\mu$ or $\epsilon^*_\mu$, and the minima and maxima for folds by $\epsilon_\eta F$, $\epsilon^*_\eta F$, $\epsilon^*_\eta^* F$ or $\epsilon^*\eta^*_F$, as shown in Figure 19. The evaluation of $D$ is defined as the evaluation of $S$.

According to this definition, it is necessary to label any new vertices introduced at the intersection of folds and lines. It follows from the discussion in Section 4.4 that these labels must be given by the 3-morphism $\Omega_\mu : \eta^*_G \circ (\mu \square G^\#) \Rightarrow \eta_F^* \circ (F \square # \mu)$ defined in (30). As explained there, interaction of $\Omega_\mu$ with the unit 2-morphisms, the horizontal composition and the Gray product is determined by the coherence data of the functor of strict tricategories $# (see Figure 44) and the evaluations of diagrams related by sliding lines over a cusp are related by the natural isomorphisms $\Theta : ## \rightarrow 1$, $\Gamma : *## *# \rightarrow 1$ (see Figures 46 and 47). Together with the strictification result from Section 5, which implies that this data is coherent, this suggests that...
the labels $\Omega_\mu$ at the folds are canonical and can be omitted, since any two diagrams constructed from such labellings should be related by a unique 3-morphism. However, this aspect is not analysed systematically in this paper, and in the following we restrict attention to diagrams where there are no such vertices.

**Definition 6.3.** A three-dimensional diagram is called standard if there are no folds meeting vertices or lines.

Standard diagrams are familiar from knot theory. A ribbon knot is given by an embedding $e : S^1 \times [-1, 1] \to \mathbb{R}^3$. A ribbon knot with no folds is called blackboard-framed in knot theory. Ribbon knots were generalised to ribbon graphs by Reshetikhin and Turaev [27]. A ribbon graph consists of a graph, called the core, and a compact surface with boundary which contains the core. The ribbon graph is considered to be a thickening of the core. The ribbon graphs can be realised as diagrams in the sense of this paper in the following way.

**Definition 6.4** (Ribbon diagram). A ribbon diagram is a three-dimensional diagram $D$ with an embedded graph $\gamma \subset X^1$, called the core, such that

1. The core $\gamma$ is the union of all of the vertices $X^0$ and a subset of the set of lines.

2. There exists a two-dimensional PL-manifold $\Sigma \subset [0, 1]^3$ such that $X^2$ is a regular neighbourhood of $\gamma$ in $\Sigma$.

A standard result on regular neighbourhoods is that $X^2$ is a compact 2-manifold with boundary [28]. The lines of $D$ that are not in the core form a subset $l \subset \partial X^2$. The surfaces of $D$ are the components of $X^2 \setminus \{l \cup \gamma\}$. Note that $\Sigma$ is not part of the structure of the ribbon diagram. It is just required that a suitable PL-manifold $\Sigma$ exists. An example of a ribbon diagram is given in Figure 48 a).

As shown in Corollary 4.9, if $G$ is spatial, then for all objects $C$ of $G$ the 2-category $G(1_C, 1_C)$ is a ribbon category. Conversely, a ribbon category can be viewed as a spatial Gray category, with only one object $C$ and one 1-morphism $1_C$. This is the appropriate category data for labelling a ribbon diagram with no folds.

It is easy to see that the evaluation of a ribbon diagram labelled with such data coincides with the Reshetikhin-Turaev evaluation of the associated ribbon. For this, one labels the regions of the diagram with the object $C$, its surfaces with the 1-morphism $1_C$ and assigns the trivial 2-morphism $1_{1_C}$ to the lines in $l$. The vertices and lines in $\gamma$ are labelled with data from the ribbon category $G(1_C, 1_C)$. The evaluation of such a ribbon diagram with no folds according to Definition 6.2

---

1Note that, to avoid confusion, the definition uses the terminology ‘two-dimensional PL-manifold’ because the word ‘surface’ is reserved for the 2-dimensional strata of a diagram.
then coincides with the Reshetikhin-Turaev evaluation of the associated ribbon labelled with data from the ribbon category $\mathcal{G}(1_C, 1_C)$.

The ribbon diagrams can be modified to provide another interesting class of examples of Gray category diagrams.

**Definition 6.5.** A surface diagram is a three-dimensional diagram such that $X^2$ is a two-dimensional PL-manifold whose boundary is contained in the boundary of the cube, $\partial X^2 \subset \partial [0, 1]^3$.

An example of a surface diagram is given in Figure 48 b). Surface diagrams and ribbon diagrams are closely related. For every standard surface diagram $D$, taking a sufficiently small regular neighbourhood of $X^1 \subset X^2$ yields a ribbon diagram without folds. The projection plane is assumed to have a canonical orientation, and so the orientation of the ribbon is that induced by the projection map. This ribbon graph is called a ribbon neighbourhood of $X^1$.

Given a ribbon diagram with ribbon $X^2$, one can construct a surface diagram by embedding $\partial (X^2 \times [0, 1])$ in such a way that $X^2 \times \{0\}$ coincides with the ribbon. This corresponds to doubling the ribbon, placing one copy of the ribbon in front of the other and gluing the two copies of the ribbon together at their boundaries. The resulting surface is the boundary of a tubular neighbourhood of the core. For standard
diagrams, labelling with data from a ribbon category will result in the same evaluation.

6.2. **Invariance.** With the definitions from the last subsection, it is possible to consider the behaviour of the evaluation under mappings of the diagrams. The mappings of interest are the following.

**Definition 6.6.** A homomorphism of standard surface diagrams \( f: D \to D' \) is called an oriented homomorphism if \( f \) is an orientation-preserving map of a ribbon neighbourhood of \( X^1 \subset D \) to a ribbon neighbourhood \( X'^1 \subset D' \).

For diagrams that are labelled with a Gray category with duals, a mapping of diagrams determines the relation between the labels. The discussion is parallel to the two-dimensional case in Section 3.2. By subdividing the diagrams, one can restrict attention to isomorphisms of progressive diagrams. Surfaces are oriented by the projection \( p_2 \), and lines are oriented by the projection \( p_1 \circ p_2 \), by comparing with standard orientations of the coordinates. An isomorphism of progressive diagrams is thus either line-preserving or line-reversing on lines, and surface-preserving or surface-reversing on surfaces.

For general diagrams, the mapping of vertices has a complicated structure. However, for standard surface diagrams the situation simplifies. Let \( q \) be the projection map \( p_2 \) restricted to \( X^2 \subset D \) and \( q' \) the corresponding map for \( X'^2 \subset D' \). The mapping of the projection plane \( q' \circ f \circ q^{-1} \) is a local isomorphism near each vertex, and for an oriented homomorphism it is also orientation-preserving.

**Definition 6.7.** Let \( f: D \to D' \) be an oriented isomorphism of progressive surface diagrams. The labels of \( D' \) are called induced from the labels of \( D \) by \( f \) if

1. the labels on a region of \( D \) and its image in \( D' \) are equal
2. the labels on a line and its image are equal if the line is preserved; labels on a line and its image are related by \( * \) if the line is reversed
3. the labels on a surface and its image are equal if the surface is preserved; labels on an surface and its image are related by \( # \) if the surface is reversed
4. the label on a vertex of \( D' \) is induced from the label on the corresponding vertex of \( D \) by the mapping of the projection plane in a neighbourhood around the vertex, using Definition 3.8.

The main invariance result follows. The proof relies on the conjecture that the moves on a projection of a PL surface are the analogues of the moves in the smooth case. To our knowledge, this problem has not been investigated in the literature.
Theorem 6.8. Let \( D \) and \( D' \) be standard surface diagrams that are labelled with a spatial Gray category. Let \( f : D \to D' \) be an oriented isomorphism of standard surface diagrams and the labels of \( D' \) induced from \( D \) by \( f \). Then the evaluations of \( D \) and \( D' \) are equal.

Proof. For each homomorphism \( f \), there is an isotopy of the diagram from the identity to \( f \). As is standard in knot theory, this isotopy can be chosen in such a way that its effect on the two-dimensional diagram obtained by projection with \( p_2 \) is an isotopy of the two-dimensional projection plane punctuated by a finite sequence of moves that generalise the Reidemeister moves.

First, the isotopy is factored into a product of moves for the ribbon neighbourhood of \( X^1 \subset X^2 \). These moves will be called ribbon moves. At this stage, the action of the isotopy on the surfaces is not considered, except for the requirement that for each move the projection of the surface singularities avoids the move in the projection plane. If this condition is not satisfied, then the isotopy can be adjusted by a small perturbation so that it is satisfied.

Reidemeister moves for blackboard-framed links projected to the plane were given by Kauffman [14] and, more explicitly, by Freyd and Yetter [7]. These moves are the Reidemeister II and III moves (Figure 49 a) and 49 c), plus Kauffman’s double-twist cancellation move (Figure 49 b). The additional moves for ribbon graphs were given by Reshetikhin and Turaev [27, 32] and consist of sliding a line over or under a vertex (Figure 49 d).

The invariance of the evaluation under these moves is as follows. Invariance under the Reidemeister II and III moves follows from Theorem 2.26. Invariance under the Kauffman double-twist cancellation is the equation

\[
(*\Gamma_{\nu'})^{-1} \cdot \# \left( \Delta_{\nu'} \cdot \Delta_{\nu}^{-1} \right) \cdot *\Gamma_{\nu'} = 1_\nu
\]

for all 2-morphisms \( \nu \), in which \( \# \) is the canonical functor \([15]\). This equation, whose left-hand side is shown in Figure 50, follows from the spatial condition. Invariance under Reshetikhin and Turaev’s sliding move is a consequence of the naturality of the tensorator. It can be seen directly that each of these moves maps standard diagrams to standard diagrams.

It remains to consider the effect of the isotopies between the ribbon moves. Between these moves, the projection of the ribbon neighbourhood changes by an isotopy of the projection plane, and this also preserves the fact that the diagram is standard. However, the singularities of the projection of the surfaces change during these isotopies, and this is described next.

In a standard diagram, the singular points of the surfaces under projection lie outside a neighbourhood of \( X^1 \) and are thus in a compact subset of the 2-strata \( X^2 \setminus X^1 \). The proof requires a conjecture that the
Figure 49. Moves relating different projections of ribbon diagrams:

a) Reidemeister II move.
b) Double twist cancellation move.
c) Reidemeister III move.
d) The two additional moves from [27].

moves for singularities of projection of a PL surface are the analogues of the moves for the singularities of projection of a smooth surface.

**Conjecture 6.9.** An isotopy of a surface projection with singularities in a compact subset of the surface can be adjusted so that the moves are the Reidemeister II and III moves for the folds, sliding of folds over or under cusps, or the moves in which cusps appear or disappear in pairs given in Figures 17 and 18.

The evaluation is invariant under the moves in which cusps appear or disappear in pairs according to the equations depicted in Figure 17. These moves are called the surface moves.

Between surface moves, the diagram can be subdivided to a diagram \(E\) by introducing an additional vertex for each cusp and an additional
line for each fold, and labelling with the canonical 2- and 3-morphisms for folds and cusps. Note that $E$ need not be progressive, but the projection of each surface in $E$ is regular. Between the surface moves, the subdivision to $E$ is preserved by the isotopy.

The topology of the two-dimensional diagram that is given by the projection of $E$ is determined by the 1-skeleton of $E$. The associated moves induced by the isotopy are the moves for a projection of a graph. These moves are the Reidemeister II and III moves and the Reshetikhin-Turaev sliding move. Note that the Kauffman double-twist cancellation move does not occur. This is because this move is always accompanied by pairs of cusps appearing or disappearing, and all such cusp cancellations have already been accounted in the surface moves or ribbon moves. The invariance under the Reidemeister II and III moves and the sliding moves is proved in the same way as above for the ribbon moves.

Finally, between these moves, the diagram changes by isotopy of the projection plane. The diagram $E$ projects to a two-dimensional diagram for $\mathcal{G}(C, D)$, which is a planar 2-category, and so the evaluation of
the diagram is invariant, providing the labels for this two-dimensional diagram are induced according to Definitions 3.6 for regions and lines, and 3.8 for vertices. For the regions, lines and vertices projected from regions, lines and vertices of \( D \) this follows from Definition 6.7. For folds and cusps, this follows from the fact that \( * \) is a rotation in the projection plane, and for crossings from Lemma 3.12.

\[ \square \]

7. **Spherical Gray categories and traces**

7.1. **Traces in Gray categories with duals.** In this section we introduce a notion of trace for Gray categories with duals, define spherical Gray categories with duals and derive concrete conditions that ensure that a Gray category with duals is spatial. First recall the definition of a spherical category in [3]. For any morphism \( \alpha : x \to x \) in a pivotal category \( \text{tr}_L(\alpha) = \epsilon_{\mu} \circ (\alpha \circ 1_{\mu}) \circ \epsilon_{\mu} : 1_G \Rightarrow 1_G \) and \( \text{tr}_R(\alpha) = \epsilon_{\mu} \circ (1_{\mu} \circ \alpha) \circ \epsilon_{\mu} : 1_F \Rightarrow 1_F \), and the left- and right-dimension of a 2-morphism \( \mu : F \Rightarrow G \) are \( \text{dim}_L(\mu) = \text{tr}_L(1_{\mu}) \), \( \text{dim}_R(\mu) = \text{tr}_R(1_{\mu}) \).

The left- and right-trace of a 3-morphism \( \Psi : \mu \Rightarrow \mu \) in \( G \) are depicted in Figure 51. They satisfy identities similar to the trace in a pivotal category. In particular, the axioms of a planar 2-category and the definition of a Gray category with duals imply the identities

\[
\text{tr}_L(1_1) = 1_1, \quad \text{tr}_R(1_1) = 1_1,
\]

and for all 3-morphisms \( \Psi, \Phi, \Xi, \Upsilon \) for which these expressions are defined.

\[ (56) \quad \text{tr}_{L,R}(\Psi) = \text{tr}_{L,R}(\Psi^*), \quad \text{tr}_{L,R}(\Xi \cdot \Phi) = \text{tr}_{L,R}(\Phi \cdot \Xi), \]

\[ \text{tr}_{L,R}(1_{1_F}) = 1_{1_F}, \quad \text{tr}_{L,R}(\Psi \square \Upsilon) = \text{tr}_{L,R}(\Psi) \square \text{tr}_{L,R}(\Upsilon) \]

The left- and right-trace of a 3-morphism \( \Psi : \mu \Rightarrow \mu \) are depicted in Figure 51. They satisfy identities similar to the trace in a pivotal category. In particular, the axioms of a planar 2-category and the definition of a Gray category with duals imply the identities

\[ (56) \quad \text{tr}_{L,R}(\Psi) = \text{tr}_{L,R}(\Psi^*), \quad \text{tr}_{L,R}(\Xi \cdot \Phi) = \text{tr}_{L,R}(\Phi \cdot \Xi), \]

\[ \text{tr}_{L,R}(1_{1_F}) = 1_{1_F}, \quad \text{tr}_{L,R}(\Psi \square \Upsilon) = \text{tr}_{L,R}(\Psi) \square \text{tr}_{L,R}(\Upsilon) \]

for all 3-morphisms \( \Psi, \Phi, \Xi, \Upsilon \) for which these expressions are defined.

In the following, we will also require a relation between left- and right-traces which generalises the notion of sphericity to the context of a Gray category with duals. Note, however, that it is not possible to simply impose that the left- and right-trace of a 3-morphism \( \Psi : \mu \Rightarrow \mu \) in \( G(F, G) \) are equal, since the left-trace is a 3-morphism \( \text{tr}_L(\Psi) : 1_G \Rightarrow 1_G \) and the right-trace a 3-morphism \( \text{tr}_R(\Psi) : 1_F \Rightarrow 1_F \). Instead, we impose the following condition.

**Definition 7.2** (Spherical Gray category). A Gray category \( G \) with duals is called spherical if for all 2-morphisms \( \mu : F \Rightarrow G \) and 3-morphisms \( \Psi : \mu \Rightarrow \mu \)

\[ 1_{\eta_F} \circ (\text{tr}_L(\Psi) \square G^\#) = 1_{\eta_G} \circ (G \square \text{tr}_R(\# \Psi)), \quad \text{and} \]

\[ 1_{\eta_G} \circ (\text{tr}_R(\Psi) \square F^\#) = 1_{\eta_F} \circ (F \square \text{tr}_L(\# \Psi)). \]
The name spherical is justified by the fact that in the case of a Gray category with one object and one 1-morphism, these conditions reduce to the usual notion of a spherical category.

**Lemma 7.3.** If $\mathcal{G}$ is a spherical Gray category with duals, then the braided strict pivotal categories $\mathcal{G}(1_C, 1_C)$ are spherical for all objects $C$ of $\mathcal{G}$. Conversely, a spherical braided strict pivotal category is a spherical Gray category with duals with a single object and a single 2-morphism.

**Proof.** This follows directly from Lemma 3.11 and Corollary 4.9. The traces from Definition 7.1 coincide with the usual notion of left- and right-trace in a pivotal category. As the 2-morphism $\eta_1C$ are trivial, $1_C^\# = 1_C$ and $#\Psi = \Psi$, the conditions from Definition 7.2 then take the form $tr_L(\Psi) = tr_R(\Psi)$ for all morphisms $\Psi$. □

The conditions in Definition 7.2 also have a direct geometrical interpretation in terms of Gray category diagrams. They state that the pairs of diagrams in Figure 52 a), b) are equal. This amounts to imposing that the evaluation of a trace is invariant under sliding the associated ‘circle’ over a fold located at the right of the trace. Applying the $\ast$-dual to the equations in Definition 7.2 yields analogous conditions, which hold when $\mathcal{G}$ is spherical and the fold is on the left as in Figure 52 c), d):

\[
(57) \quad (tr_L(\Psi) \square G^\#) \circ 1_{\eta_C} = (G \square tr_R(#\Psi)) \circ 1_{\eta_C},
\]

\[
(57) \quad (tr_R(\Psi) \square F^\#) \circ 1_{\eta_F} = (F \square tr_L(#\Psi)) \circ 1_{\eta_F}.
\]

Note also that this concept of sphericality contains the notion of sphericity introduced in [24] which is based on another notion of trace. By
Figure 52. Diagrams for spherical Gray categories:
a) condition $1_{\eta^G} \circ (\text{tr}_L(\Psi) \Box G^\#) = 1_{\eta^G} \circ (G \Box \text{tr}_R(\#\Psi))$.
b) condition $1_{\eta^F} \circ (\text{tr}_R(\Psi) \Box F^\#) = 1_{\eta^F} \circ (F \Box \text{tr}_L(\#\Psi))$.
c) identity $(\text{tr}_L(\Psi) \Box G^\#) \circ 1_{\eta^G} = (G \Box \text{tr}_R(\#\Psi)) \circ 1_{\eta^G}$.
d) identity $(\text{tr}_R(\Psi) \Box F^\#) \circ 1_{\eta^F} = (F \Box \text{tr}_L(\#\Psi)) \circ 1_{\eta^F}$.
taking the left-traces of the 3-morphisms $1_{G^{\#}} \circ (\Psi \square G^\#)$, $1_{G^{\#}} \circ (G^\# \square \Psi)$ or the right traces of the 3-morphisms $(\Psi \square F^\#) \circ 1_{F^{\#}}$, $(F^\# \square \Psi) \circ 1_{F^{\#}}$, one obtains two 3-morphisms $1_L \Rightarrow 1_L$ and two 3-morphisms $1_D \Rightarrow 1_D$ that are depicted in Figure 53. These are the traces considered in [24], and the requirement in [24] is that these pairs of 3-morphisms are equal. This follows from the conditions in Definition 7.2.

**Lemma 7.4.** If $\mathcal{G}$ is a spherical Gray category, then all 3-morphisms $\Psi : \mu \Rightarrow \mu$ with $\mu : F \Rightarrow G$ satisfy

$$
\text{tr}_L(1_{G^{\#}} \circ (\Psi \square G^\#)) = \text{tr}_R((\Psi \square F^\#) \circ 1_{F^{\#}}),
$$

$$
\text{tr}_L(1_{G^{\#}} \circ (G^\# \square \Psi)) = \text{tr}_R((F^\# \square \Psi) \circ 1_{F^{\#}}).
$$

**Proof.** In a Gray category with duals, the invertibility of the cusp and the naturality of the 3-morphism $\epsilon_{\mu}$ imply

$$
\text{tr}_L(1_{F^{\#}} \circ (F \square \Psi)) = \text{tr}_L(1_{G^{\#}} \circ (\Psi \square G^\#)),
$$

$$
\text{tr}_R((\Psi \square F^\#) \circ 1_{F^{\#}}) = \text{tr}_R((F^\# \square \Psi) \circ 1_{F^{\#}}).
$$

A diagrammatic proof of the first identity is given in Figure 54 a). The proof of the second identity is analogous. If $\mathcal{G}$ is spherical, then it follows from identities

$$
\text{tr}_L(1_{F^{\#}} \circ (F \square \Psi)) = \epsilon_{F^{\#}}^* \cdot (1_{F^{\#}} \circ (F \square \text{tr}_L(\Psi \square G^\#)) \circ 1_{F^{\#}}) \cdot \epsilon_{F^{\#}},
$$

$$
\text{tr}_R((\Psi \square F^\#) \circ 1_{F^{\#}}) = \epsilon_{F^{\#}}^* \cdot (1_{F^{\#}} \circ (\text{tr}_R(\Psi \square F^\#) \circ 1_{F^{\#}}) \cdot \epsilon_{F^{\#}},
$$

together with (57) (see Figure 52 d)) that

$$
\text{tr}_L(1_{F^{\#}} \circ (F \square \Psi)) = \text{tr}_R((\Psi \square F^\#) \circ 1_{F^{\#}}).
$$

Combining this equation with the first line in (58) yields the first equation in the lemma. The proof of the second equation is similar. Combining the second equation in (58) with (57) and (59), one obtains

$$
\text{tr}_R((\Psi \square F^\#) \circ 1_{F^{\#}}) = \epsilon_{G^{\#}}^* \cdot (1_{G^{\#}} \circ G^\# \square \text{tr}_L(\Psi \square G^\#) \circ 1_{F^{\#}}) \cdot \epsilon_{G^{\#}},
$$

The naturality of the 3-morphisms $\Theta_{\mu} : \# \Rightarrow \mu$ implies

$$
\text{tr}_{L,R}(\Psi) = \text{tr}_{L,R}(\Theta_{\mu}^{-1} \cdot \Theta_{\mu}) = \text{tr}_{L,R}(\Theta_{\mu}^{-1} \cdot \Psi \cdot \Theta_{\mu})
$$

$$
= \text{tr}_{L,R}(\Psi \cdot \Theta_{\mu} \cdot \Theta_{\mu}^{-1}) = \text{tr}_{L,R}(\Psi).
$$

Inserting this into (60) and using again (59) then gives the second equation in the lemma. □
7.2. Semisimplicity, spherical and spatial Gray categories. As discussed in the previous subsections, spherical Gray categories can be viewed as a generalisation of spherical categories and spatial Gray categories as a generalisation of ribbon categories. In this subsection, we investigate the relation between these two types of Gray categories. The first result is a generalisation of the statement that every ribbon category is spherical.

**Lemma 7.5.** Every spatial Gray category with duals is spherical.

*Proof.* Let \( \mathcal{G} \) be a Gray category with duals. Then for every 1-morphism \( F : \mathcal{C} \to \mathcal{D} \) and every 3-morphism \( \Omega : 1_F \Rightarrow 1_F \) one has

\[
T_F^* \cdot (1_{F\# \eta_F^*} \cdot \# \Omega \circ 1_{\eta_F^* \# F\#}) \cdot T_F^{-1} = T_F^{*-1} \cdot (1_{\eta_F^* \# F\#} \cdot \# \# \circ \# \circ 1_{\eta_F^* \# F\#}) \cdot T_F^*.
\]

A diagrammatic proof of this relation is given in Figure 53. It uses the naturality and invertibility of the 3-morphism \( \Delta_{1_F} : \# 1_F \Rightarrow \# * 1_F \) and the identity \( \Delta_{1_F} \cdot T_{F\#}^{-1} = T_{F\#}^{*} \), which follows from the properties of the triangulator. In the following, we denote the 3-morphism in this relation by

\[
\Omega^+ = T_{F\#} \cdot (1_{F\# \eta_F^*} \cdot \# \Omega \circ 1_{\eta_F^* \# F\#}) \cdot T_{F\#}^{-1} : 1_{F\#} \Rightarrow 1_{F\#}.
\]
Figure 54.

a) diagrammatic proof of the identity
\[ \text{tr}_L(1_{\eta_F^*} \circ (F \Box \# \Psi)) = \text{tr}_L(1_{\eta_G^*} \circ (\Psi \Box G^\#)). \]
b) diagrammatic proof of identity
\[ \text{tr}_L((\eta_F^* \Box F) \circ (F \Box \eta_{F^*})) = 1_F. \]
The properties of the triangulator and the tensorator then imply
\[ 1_{\eta_F} \circ (F \Box \Omega^+) = 1_{\eta_F} \circ (\Omega \Box F^\#). \]

A diagrammatic proof of this relation is given in Figure 56. In particular, this shows that every 3-morphism \( \Psi : \mu \Rightarrow \mu \) with \( \mu : F \Rightarrow G \) satisfies the identities
\[
1_{\eta_F} \circ (F \Box \text{tr}_R(\Psi^+)) = 1_{\eta_F} \circ (\text{tr}_R(\Psi) \Box F^#) \\
1_{\eta_G} \circ (G \Box \text{tr}_L(\Psi^+)) = 1_{\eta_G} \circ (\text{tr}_L(\Psi) \Box G^#)
\]

If, additionally, \( G \) is spatial, then one has
\[
\text{tr}_L(#\Psi) = \text{tr}_R(\Psi^+), \quad \text{tr}_R(#\Psi) = \text{tr}_L(\Psi^+).
\]

A diagrammatic proof of the first identity is given in Figure 57. It is a direct generalisation of the corresponding proof for spherical categories and makes use of the fact that \( G \) is spatial together with the properties of the tensorator and the triangulator. The proof of the second identity is analogous. Combining these identities with the previous equations yields the equations in Definition 7.2
\[
1_{\eta_G} \circ (\text{tr}_L(\Psi) \Box G^#) = 1_{\eta_G} \circ (G \Box \text{tr}_R(#\Psi)) \\
1_{\eta_F} \circ (\text{tr}_R(\Psi) \Box F^#) = 1_{\eta_F} \circ (F \Box \text{tr}_L(#\Psi)).
\]

□

To determine under which additional conditions a spherical Gray category is spatial, we require a concept of semisimplicity for Gray categories that generalises the notion of semisimplicity in a monoidal category.

**Definition 7.6.** A Gray category \( G \) with duals is called semisimple \(^2\) if it has the following additional properties:

1. For all 1-morphisms \( F, G : C \to D \), the categories \( G(F, G) \) are semisimple categories that are locally finite over \( C \). This means that the isomorphism classes of objects form a set, all Hom-spaces are finite-dimensional vector spaces over \( C \), and there is a set \( J \) of non-zero, non-isomorphic objects such that for all objects \( x, y \in G(F, G) \) the canonical map given by composition
\[ \bigoplus_{a \in J} G(x, a) \otimes G(a, y) \to G(x, y), \]

is an isomorphism.

2. The horizontal composition \( \circ : G(F, G) \times G(G, H) \to G(F, H) \), the Gray product \( \Box : G(F, G) \times G(H, K) \to G(FH, GK) \) and the functors \( * : G(F, G) \to G(G, F)^{op} \), \( # : G(F, G) \to G(G^#, F^#) \) are \( C \)-bi-linear.

\(^2\)This is the notion of semisimplicity as used in [4]. In particular we do not require the existence of direct sums of objects, see also [25].
Figure 55. Diagrammatic proof of the identity
\[ T_{F^\#} \cdot (1_{F^\#} \square \eta_{F^\#} \circ \# \Omega \circ 1_{F^\#} \square F^\#) \cdot T_{F^\#}^{-1} = T_{F^\#}^{\ast} \cdot (1_{\eta_{F^\#} \square F^\#} \circ \# \ast \Omega \circ 1_{F^\#} \square \eta_{F^\#} \cdot ) \cdot T_{F^\#}^{\ast} \]
for a 3-morphism \( \Omega : 1_F \Rightarrow 1_F \).
Figure 56. Diagrammatic proof of the identity

\[ 1_{\eta_F^*} \circ (F \square \Omega^+) = 1_{\eta_F^*} \circ (\Omega \square F^#) \]

for a 3-morphism \( \Omega : 1_F \Rightarrow 1_F \).
Figure 57. Diagrammatic proof of the identity $\text{tr}_L(\#\Psi) = \text{tr}_R(\Psi)^+$ for a 3-morphism $\Psi : \mu \Rightarrow \mu$ and a 2-morphism $\mu : F \Rightarrow G$. 127
(3) For all objects $C$, the 2-morphisms $1_{1C}$ are simple objects of $G(1C, 1C)$.

The set $J$ is called a representative set of simple objects and any object that is isomorphic to one of the objects in $J$ is called simple. It is a consequence that an object $x$ in a semisimple locally finite category $C$ over $\mathbb{C}$ is simple if and only if $\text{Hom}_C(x, x) \simeq \mathbb{C}$ as vector spaces.

If $G$ is a semisimple Gray category with duals and $\mu : F \Rightarrow G$ is a simple object of $G(F, G)$, each 3-morphism $\Psi : \mu \Rightarrow \mu$ is a multiple of the unit 3-morphism $1_\mu$, and we write $\Psi = \langle \Psi \rangle 1_\mu$ with $\langle \Psi \rangle \in \mathbb{C}$. In particular, for all 3-morphisms $\Psi : \mu \Rightarrow \mu$ the traces
\[
\text{tr}_R((\Psi \square F^\#) \circ 1_{\eta_F}), \quad \text{tr}_L(1_{\eta_G} \circ (\Psi \square G^\#)),
\]
\[
\text{tr}_R((F^\# \square \Psi) \circ 1_{\eta_{F^\#}}), \quad \text{tr}_L(1_{\eta_{G^\#}} \circ (G^\# \square \Psi))
\]
in Figure 53 correspond to complex numbers. Moreover, if $G$ is a 1-morphism such that $1_G$ is simple, then for all 2-morphisms $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow H$ and 3-morphisms $\Phi : \mu \Rightarrow \mu$, $\Psi : \nu \Rightarrow \nu$
\[
\text{tr}_L(\Psi \circ \Phi) = \langle \text{tr}_L(\Phi) \rangle \text{tr}_L(\Psi) \quad \text{tr}_R(\Psi \circ \Phi) = \langle \text{tr}_R(\Psi) \rangle \text{tr}_R(\Phi).
\]

Another important consequence of semisimplicity is that each category $G(F, G)$ in a semisimple Gray category $G$ is equipped with partitions.

**Definition 7.7.** Let $A$ be a semisimple category that is locally finite over $\mathbb{C}$. If $J$ is a representative set of simple objects in $A$, then a partition of an object $\rho$ in $A$ with respect to $J$ is a set of morphisms $p^\rho_A : \mu \rightarrow \rho$ and $p^\rho_{1A} : \rho \rightarrow \mu$ with $\mu \in J$, $A \in \Lambda_{\mu, \rho}$, such that
\[
1_\rho = \sum_{\mu \in J} \sum_{A \in \Lambda_{\mu, \rho}} p^\rho_A \cdot p^\rho_{1A} \quad p^\rho_B \cdot p^\rho_{1B} = \delta^B_A \delta^\rho_{1B} 1_\mu.
\]

In a semisimple category that is locally finite over $\mathbb{C}$, every object has a partition, see for instance [25]. If $G$ is a Gray category with duals that satisfies conditions (1) and (2) in Definition 7.6, $F, G : C \rightarrow D$ are 1-morphisms in $G$ and $J$ is a representative set of simple objects in $G(F, G)$, then every 2-morphism $\rho : F \Rightarrow G$ has a partition. In the Gray category diagrams we denote the 3-morphisms $p^\rho_A : \mu \Rightarrow \rho$ and $p^\rho_{1A} : \rho \Rightarrow \mu$ by boxes labelled with $A$ as shown in Figure 58. The two defining identities of the partition correspond to the two diagrams in Figure 58 a), b). With the first identity in (61), every 3-morphism $\Psi : \rho \Rightarrow \rho$ can be expressed in terms of a partition of $\rho$ as
\[
\Psi = \sum_{\rho, \mu \in J} \sum_{A \in \Lambda_{\rho, \mu}} \sum_{B \in \Lambda_{\rho, \nu}} p^B_{\rho, \nu} \cdot (p^\rho_{\nu, \rho} \cdot \Psi \cdot p^A_{\rho, \mu}) \cdot p^\rho_{A, \mu}.
\]

Since $J$ is a representative set of simple objects in $G(F, G)$, the 3-morphism in brackets is a multiple of $1_\mu$, and one obtains
\[
\Psi = \sum_{\rho, \mu \in J} \sum_{A \in \Lambda_{\rho, \mu}} \sum_{B \in \Lambda_{\rho, \nu}} \langle p^B_{\rho, \nu} \cdot \Psi \cdot p^A_{\rho, \mu} \rangle p^B_{\rho, \nu} p^\rho_{A, \mu}.
\]

128
Lemma 7.8. A semisimple Gray category with duals is non-degenerate: For all simple 2-morphisms \( \mu : F \Rightarrow G \), \( \dim_L(\mu) \), \( \dim_R(\mu) \) \( \neq 0 \).

Proof. The proof is analogous to the proof of the corresponding statement for tensor categories with duals, see [2, Lemma 2.4.1]. Let \( \mu \) be a simple 2-morphism in a semisimple Gray category with duals \( G \). Then the vector space \( G(\mu,\mu) \) is isomorphic to \( \mathbb{C} \). It is easy to see that the map \( \psi \mapsto \psi \circ \epsilon_\mu \) provides an isomorphism of vector spaces \( G(\mu,\mu) \simeq G(1,\mu \circ \mu^*) \simeq \mathbb{C} \). Analogously, one obtains isomorphisms \( G(\mu \circ \mu^*,1) \simeq G(\mu,\mu) \simeq \mathbb{C} \). A partition of \( \mu \circ \mu^* \) provides a morphism \( \kappa : \mu \circ \mu^* \rightarrow 1 \) and a morphism \( \rho : 1 \rightarrow \mu \circ \mu^* \), such that \( \kappa \cdot \rho = 1_1 \). There are non-zero complex numbers \( a,b \) with \( \epsilon_\mu = a\kappa \) and \( \epsilon_\mu^* = b\rho \). Hence it follows that \( \dim_L(\mu) = ab1_1 \neq 0 \).

By Lemma 7.8, a 3-morphism \( \Psi \) in a semisimple Gray category with duals is characterised uniquely by the condition

\[
\text{tr}_R(p_B^{\tau \mu} \cdot \Psi \cdot p_{p,\mu}^A) = (p_B^{\tau \mu} \cdot \Psi \cdot p_{p,\mu}^A) \dim_R(\mu)
\]

for all simple objects \( \mu \). Using the semisimplicity conditions, one obtains a result which generalises the statement that semisimple pivotal categories are spherical if and only if for all simple objects the left and the right dimensions agree.

Lemma 7.9. A semisimple Gray category with duals is spherical if and only if all simple 2-morphisms \( \mu : F \Rightarrow G \) satisfy

\[
1_{\eta_G} \circ (\dim_L(\mu) \Box G^*) = 1_{\eta_G} \circ (G \Box \dim_R(G(\mu)))
\]

\[
1_{\eta_F} \circ (\dim_R(\mu) \Box F^*) = 1_{\eta_F} \circ (F \Box \dim_L(G(\mu))).
\]
Proof. If \( \mathcal{G} \) is a semisimple spherical Gray category with duals, then the identities in the lemma are satisfied by definition. To prove the converse, we express a 3-morphism \( \Psi : \rho \Rightarrow \rho \) in terms of a partition for \( \rho \) as in (62). Using the cyclic invariance of the trace and the second identity in (61), one obtains

\[
1_{\eta F} \circ (\text{tr}_L(\Psi) \square G^#) = \sum_{\mu \in I} \sum_{A \in \Lambda_{\rho,\mu}} \langle p^B_{\rho,\mu} \cdot \Psi \cdot p^A_{\rho,\mu} \rangle 1_{\eta F} \circ (\text{tr}_L(p^B_{\rho,\mu} \cdot p^A_{\rho,\mu}) \square G^#)
\]

and similarly

\[
1_{\eta F} \circ (G \square \text{tr}_R(\# \Psi)) = \sum_{\mu \in I} \sum_{A \in \Lambda_{\rho,\mu}} \langle p^B_{\rho,\mu} \cdot \Psi \cdot p^A_{\rho,\mu} \rangle 1_{\eta F} \circ (G \square \dim \text{tr}_R(\#))
\]

If all simple objects \( \mu \) in \( \mathcal{G} \) satisfy the first condition in the lemma, the two expressions agree. The proof for the other condition in Definition 7.2 is analogous. \( \Box \)

A semisimple spherical Gray category with duals can be viewed as a higher-dimensional analogue of a semisimple spherical category. To clarify the relation between these two concepts, we require a definition of a semisimple pivotal category.

Definition 7.10. A pivotal category is called semisimple (see [4]) if, as category, it is semisimple and locally finite over \( \mathbb{C} \), the tensor product and the duality \( * \) are \( \mathbb{C} \)-linear and the unit object is simple.

Corollary 7.11. Let \( \mathcal{G} \) be a semisimple spherical Gray category with duals. Then for every 1-morphism \( F \) in \( \mathcal{G} \) with the property that \( 1_F \) is simple and \( \dim \text{tr}_R(\eta_F) \neq 0 \), the category \( \mathcal{G}(F,F) \) is a semisimple spherical category. In particular, all categories \( \mathcal{G}(1_C,1_C) \) are semisimple spherical categories.

Proof. Lemma 4.7 implies that \( \mathcal{G}(F,F) \) is a strict pivotal category, and it follows directly from the definitions that \( \mathcal{G}(F,F) \) is semisimple. It is therefore sufficient to show that the left- and right-trace of all morphisms in \( \mathcal{G}(F,F) \) agree. As \( 1_F \) is simple, for every 3-morphism \( \Psi : \rho \Rightarrow \rho \) in \( \mathcal{G}(F,F) \) we have \( \text{tr}_{L,R}(\Psi) = \langle \text{tr}_{L,R}(\Psi) \rangle 1_{\eta F} \) and hence

\[
\text{tr}_L(1_{\eta F} \circ (\Psi \square F)) = \langle \text{tr}_L(\Psi) \rangle \dim \text{tr}_R(\eta_F)
\]

and

\[
\text{tr}_R((\Psi \square F^#) \circ 1_{\eta F}) = \langle \text{tr}_R(\Psi) \rangle \dim \text{tr}_R(\eta_F).
\]

130
Lemma 7.4 states that the two expressions agree, and if \( \dim_R(\eta_F) \neq 0 \) this implies \( \langle \tr_L(\Psi) \rangle = \langle \tr_R(\Psi) \rangle \). \[\square\]

We will show that every semisimple spherical Gray category with duals is spatial. The proof is a generalisation of the corresponding proof for the center of a spherical category given in [33].

**Lemma 7.12.** Every semisimple spherical Gray category with duals is spatial.

**Proof.** In a semisimple spherical Gray category \( \mathcal{G} \) with duals, each category \( \mathcal{G}(F,G) \) has a representative set \( J \) of simple objects, and for each object \( \rho \) of \( \mathcal{G}(F,G) \) there is a partition \( \{ p^\rho_A : \mu \Rightarrow \rho, \rho^\rho_A : \rho \Rightarrow \mu \}_{\rho \in J, A \in \Lambda_{p,\rho}} \). As \# : \( \mathcal{G}(F,G) \to \mathcal{G}(G^\#, F^\#) \) is an equivalence of categories (Corollary 4.7), for any representative set \( J' = \{ \# \mu : \mu \in J \} \) is a representative set of simple objects in \( \mathcal{G}(G^\#, F^\#) \). Moreover, as \# is compatible with direct sums, for every partition of an object \( \rho \) in \( \mathcal{G}(F,G) \), the morphisms \( \# p^\rho_A : \# \rho \Rightarrow \# \mu \) and \( \# p^\rho_A : \# \mu \Rightarrow \# \rho \) form a partition for the object \( \# \rho \) in \( \mathcal{G}(G^\#, F^\#) \). Using this partition and identity (62), we can rewrite the 3-morphism \( \Delta^*-1 \cdot \Delta_\rho : \# \rho \Rightarrow \# \rho \) as

\[ \Delta^*-1 \cdot \Delta_\rho = \sum_{\mu \in J} \sum_{A,B \in \Lambda_{p,\rho}} \langle \# p^\rho_A \cdot \Delta^*-1 \cdot \Delta_\rho \cdot \# p^\rho_B \rangle \cdot \#(p^\rho_A \cdot p^\rho_B). \]

To show that this 3-morphism is equal to \( 1_\rho \), it is sufficient to prove the identity

\[ \tr_R(\# p^\rho_B \cdot \Delta^*-1 \cdot \Delta_\rho \cdot \# p^\rho_A) = \tr_R(\#(p^\rho_B \cdot p^\rho_A)) \]

for all simple objects \( \mu \) in \( J \) and all \( A,B \). Using the naturality of the 3-morphism \( \Delta_\rho \) and of the tensoror together with the properties of a planar 2-category and the triangulator, one obtains after some calculations

\[ \tr_R(\# p^\rho_B \cdot \Delta^*-1 \cdot \Delta_\rho \cdot \# p^\rho_A) \]

\[ = T_G^\# \cdot (1_{\eta_{G,G^\#}} \circ (G^\# \square \tr L(p^\rho_B \cdot p^\rho_A) \square G^\#) \circ 1_{G^\# \eta_{G^\#}}) \cdot T_G^{-1}. \]

A diagrammatic proof is given in Figure 59. As \( \mathcal{G} \) is spherical, we can rewrite this expression as

\[ \tr_R(\# p^\rho_B \cdot \Delta^*-1 \cdot \Delta_\rho \cdot \# p^\rho_A) \]

\[ = T_G^\# \cdot (1_{\eta_{G,G^\#}} \circ (\tr R(\#(p^\rho_B \cdot p^\rho_A) \square G^\#) \circ 1_{G^\# \eta_{G^\#}}) \cdot T_G^{-1} \]

\[ = (T_G^* \cdot T_G^{-1}) \circ \tr R(\#(p^\rho_B \cdot p^\rho_A) = \tr R(\#(p^\rho_B \cdot p^\rho_A)) \]

where we used the naturality of the tensoror to obtain the last line. Since \( \Delta_\rho \) is an invertible 3-morphism, this proves the claim. \[\square\]
Figure 59. Gray category diagrams for the proof of Lemma 4.6. Diagrammatic proof of the identity
\[ \text{tr}_R(\#p^\rho_B \cdot \Delta_{p^\rho}^{-1} \cdot \Delta_{p^A} \cdot \#p^A_B) = T_{G^\#} \cdot (1_{G^\# \triangleright G^\#} \circ (G^\# \Box (p^\rho_B \cdot p^A) \Box G^\#) \circ 1_{G^\# \triangleright G^\#}) \cdot T_{G^\#}^{-1}. \]
7.3. Surface invariants from quantum dimensions. To conclude the discussion of traces in Gray categories with duals, we investigate the quantum dimensions of the fold 2-morphisms \( \eta_F : 1_D \Rightarrow F \Box F^\# \) and their relation to invariants of oriented surfaces.

The first observation is that for each 1-morphism \( F : C \rightarrow D \) the fold 2-morphism \( \eta_F : 1_D \Rightarrow F \Box F^\# \) and its dual define Frobenius algebras in the associated pivotal tensor categories \( \mathcal{G}(1_D, 1_D) \) and \( \mathcal{G}(F \Box F^\#, F \Box F^\#) \). The presence of non-trivial fold 2-morphisms is thus related to additional structure in these categories.

**Lemma 7.13.** Let \( \mathcal{G} \) be a Gray category with duals. Then for every 1-morphism \( F : C \rightarrow D \) in \( \mathcal{G} \), the 2-morphism \( \eta_F^* \circ \eta_F : 1_D \Rightarrow 1_D \) is a Frobenius algebra in \( \mathcal{G}(1_D, 1_D) \) and the 2-morphism \( \eta_F \circ \eta_F^* : F \Box F^\# \Rightarrow F \Box F^\# \) a Frobenius algebra in \( \mathcal{G}(F \Box F^\#, F \Box F^\#) \).

**Proof.** For the object \( \eta_F^* \circ \eta_F \) the unit morphism is given by \( \epsilon_{\eta_F^*} \), the counit by \( \epsilon_{\eta_F} \), the multiplication by \( 1_{\eta_F^*} \circ \epsilon_{\eta_F} \circ 1_{\eta_F} \) and the comultiplication by \( 1_{\eta_F} \circ \epsilon_{\eta_F^*} \circ 1_{\eta_F^*} \). The corresponding Gray category diagrams are given in, respectively, Figure 19 b), Figure 60 a), d), c). For the object \( \eta_F \circ \eta_F^* \), the unit is given by \( \epsilon_{\eta_F} \), the counit by \( \epsilon_{\eta_F^*} \), the multiplication by \( 1_{\eta_F} \circ \epsilon_{\eta_F^*} \circ 1_{\eta_F^*} \) and the comultiplication by \( 1_{\eta_F^*} \circ \epsilon_{\eta_F} \circ 1_{\eta_F} \). The corresponding Gray category diagrams are given, respectively, in Figure 19 a), Figure 60 b), f), e). That the axioms of a Frobenius algebra are satisfied is a direct consequence of the properties of the 3-morphisms \( \epsilon_\mu \) in a planar 2-category.

In particular, it follows that the Frobenius algebras defined by the 2-morphisms \( \eta_F : 1_D \Rightarrow F \Box F^\# \) are non-trivial, i.e. do not coincide with the tensor unit in \( \mathcal{G}(1_D, 1_D) \) or \( \mathcal{G}(F \Box F^\#, F \Box F^\#) \), unless both \( \eta_F^* \circ \eta_F \) and \( \eta_F \circ \eta_F^* \) are equal to the unit 2-morphisms \( 1_{1_D} \) and \( 1_{F \Box F^\#} \). This condition is investigated for concrete examples in Section 8.

As suggested by the diagrams in Figure 60, the fold 2-morphisms \( \eta_F \) define invariants of oriented closed surfaces embedded into the cube which are obtained by labelling these surfaces with \( F \) and evaluating the resulting diagrams. The invariance of the evaluation under suitable isomorphisms of diagrams then ensures that the evaluation does not depend on the embedding and defines an invariant of the surface.

**Lemma 7.14.** Let \( F : C \rightarrow D \) be a 1-morphism in a semisimple spherical Gray category with duals. Consider the surface diagram \( D \) obtained by embedding an oriented genus \( g \) surface \( \Sigma_g \) in a cube \([0, 1]^3\) and labelling its front side with \( F \). Then the evaluation of \( D \) is given by

\[
\lambda_{F,g} = \langle \epsilon_{\eta_F}^* \cdot \dim_L(\eta_F) \cdot \dim_L(\eta_F) \cdot \epsilon_{\eta_F} \rangle \in \mathbb{C}.
\]

If, additionally, the 2-morphism \( 1_{F \Box F^\#} \) is simple, then

\[
\lambda_{F,g} = \langle \dim_L(\eta_F) \rangle^g \langle \dim_R(\eta_F) \rangle = \langle \dim_L(\eta_F) \rangle^g \lambda_{F,0}
\]

133
Figure 60.
(a) 3-morphism $\epsilon_{\eta_F}^*$, b) 3-morphism $\epsilon_{\eta_F}^*$, c) 3-morphism $1_{\eta_F} \circ \epsilon_{\eta_F} \circ 1_{\eta_F}$, d) 3-morphism $1_{\eta_F} \circ \epsilon_{\eta_F} \circ 1_{\eta_F}$, e) 3-morphism $1_{\eta_F} \circ \epsilon_{\eta_F} \circ 1_{\eta_F}$, f) 3-morphism $1_{\eta_F} \circ \epsilon_{\eta_F} \circ 1_{\eta_F}$. 

134
and \( \lambda_{F,g} \cdot \lambda_{F^#,g} = 1 \).

**Proof.** An embedding of the surface \( \Sigma_g \) as a two-dimensional PL surface in the cube and its labelling with \( F \) defines a surface diagram (see Definition 6.5), and different labelled embeddings are related by orientation preserving isomorphisms of surface diagrams. It follows from Theorem 6.8 that the evaluation of the diagram is independent of the embedding.

It can be computed from an embedding in which the ‘holes’ of the surface are aligned parallel to the \( y \)-axis as in Figure 61 a), c), d). This yields the 3-morphism

\[
\Phi_{F,g} = \epsilon^{\eta_F^*} \cdot \dim L(\eta_F) \cdots \dim L(\eta_F) \cdot \epsilon_{\eta_F} : 1_D \Rightarrow 1_D.
\]

As all 2-morphisms \( 1_{1_D} \) in a semisimple Gray category with duals are simple, it follows that \( \Phi_{F,g} \) is a multiple of the unit 3-morphism \( 1_{1_D} \) and hence given by a complex number \( \lambda_{F,g} = \langle \Phi_{F,g} \rangle \). In particular, one has \( \lambda_{F,0} = \dim R(\eta_F) = \langle \dim R(\eta_F) \rangle_{1_D} \). If the 2-morphism \( 1_{F \Box F^#} \) is simple, then \( \dim L(\eta_F) = \langle \dim L(\eta_F) \rangle_{1_{F \Box F^#}} \), which proves the second equation in the lemma. The identity \( \lambda_{F,g} = \lambda_{F^#,g} \) then follows from the relation

\[
\dim_{L,R}(\eta^*_F \Box F) \circ (F \Box \eta^*_F) = 1_{1_D},
\]

which is a consequence of the invertibility of the cusps and the naturality of the 3-morphisms \( \epsilon_{\mu} \) in a Gray category with duals. A diagrammatic proof is given in Figure 54. If the 2-morphism \( 1_{F \Box F^#} \) is simple, this identity implies \( \langle \dim_{L,R}(\eta_F) \rangle \langle \dim_{R,L}(\eta_{F^#}) \rangle = 1 \), and inserting this into the second equation of the lemma yields \( \lambda_{F,g} \cdot \lambda_{F^#,g} = 1 \). \( \square \)

An alternative way to obtain these surface invariants from the evaluation of Gray category diagrams is to consider diagrams for the braided pivotal category \( G(1_D, 1_D) \) as shown in Figure 61 e), f), g). This yields standard diagrams without surfaces, whose evaluation agrees with the evaluation of the diagrams in Figure 61 a), c), d). The former are obtained by shrinking the surface \( \Sigma \) to a graph, which is a deformation retract of a three-dimensional manifold with boundary \( \Sigma \). Note, however, that if two embeddings of the surface are related by orientation preserving homeomorphisms, then the associated diagrams with graphs are not necessarily related by homomorphisms of diagrams. That the evaluations of these diagrams nevertheless agree follows from the properties of a Frobenius algebra.

8. **Examples**

In this section we discuss two main examples of Gray categories with duals. It is likely that another set of examples for Gray categories with duals could arise from the strictification of tricategories.
Figure 61. Gray category diagrams for surfaces labelled by 1-morphisms in G:
a), e) dim_R(η_F), b) dim_L(η_F),
c), f) ε_{η_F}^* \cdot (1_{η_F} \circ \text{tr}_L(η_F) \circ 1_{η_F}) \cdot ε_{η_F}^*,
d), g) ε_{η_F}^* \cdot (1_{η_F} \circ \text{tr}_L(η_F) \circ 1_{η_F}) \cdot (1_{η_F} \circ \text{tr}_L(η_F) \circ 1_{η_F}) \cdot ε_{η_F}^*.

The white vertices denote the unit and counit for the Frobenius algebra η_F^* \circ η_F, the unlabelled trivalent vertices the multiplication and comultiplication.
As every tricategory can be strictified to a Gray category [9], see also [11], and this strictification includes the strictification of tricategory functors [11] to functors of strict tricategories, it is plausible that a tricategory with suitable duality operations would give rise to a Gray category with duals. An obvious candidate is the tricategory of bimodule categories [6], which has important applications in topological field theory and conformal field theories [15, 8]. However, a detailed exploration of this idea is beyond the scope of this paper. Other obvious examples are given by Gray groupoids [20]. However, in that case, the 2-morphisms $\eta_F: 1_D \Rightarrow F \square F^\#$ associated with the folds are trivial for all 1-morphisms $F: \mathcal{C} \rightarrow \mathcal{D}$, which limits their interest as examples.

8.1. Pseudo-equivalences and pivotal functors. As for each object $\mathcal{C}$ in a Gray category $\mathcal{G}$ with duals the category $\mathcal{G}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ is a strict pivotal tensor category, it seems plausible to construct a concrete example of a Gray category with duals by considering a Gray category whose objects are strict pivotal categories. More precisely, we consider appropriate subcategories of the Gray category MonCat introduced in Section 2.4, in which the objects are strict pivotal categories and suitable restrictions are imposed on the functors, pseudo-natural transformations and modifications.

We show in the following that the conditions in Definition 3.10 are quite restrictive but nevertheless allow one to obtain non-trivial examples, which are discussed in Section 8.2. The first result regards restrictions on the 1-morphisms in MonCat that arises from the presence of fold 2-morphisms and the triangulator.

Definition 8.1. A pseudo-equivalence of pivotal categories $\mathcal{C} \rightarrow \mathcal{D}$ consists of tensor functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ and pseudo-natural transformations $\rho: FG \Rightarrow 1_D$, $\eta: 1_C \Rightarrow GF$, for which there exist invertible modifications $T_F: (\rho F)\circ (F \eta) \Rightarrow 1_F$, $T_G: (G \rho)\circ (\eta G) \Rightarrow 1_G$.

If $\mathcal{G}$ is a sub-Gray category of MonCat that has the structure of a Gray category with duals and whose objects are pivotal categories, then a 1-morphism $F: \mathcal{C} \rightarrow \mathcal{D}$ defines a pseudo-equivalence $\mathcal{C} \rightarrow \mathcal{D}$ with $G = F^\#$, $\rho = \eta_F^*$, $\eta = \eta_{F^\#}$, the triangulator as $T_F$ and $T_G = (T_{F^\#})^{-1}$. A classification of the pseudo-equivalences between pivotal categories therefore allows one to characterise the possible 1-morphisms, folds and triangulators of Gray categories with duals.

This requires some preliminary facts on invertible objects in pivotal categories. An object $x$ in a pivotal category $\mathcal{C}$ is called invertible if the morphism $\epsilon_x: e \rightarrow x \otimes x^*$ is an isomorphism, and it is called strictly invertible if $x \otimes x^* = e$. Note that strict invertibility of $x$ does not imply that $\epsilon_x: e \rightarrow x \otimes x^*$ is the identity morphism, but only $\epsilon_x \in \text{End}(e)$. It follows directly that every invertible object in a semisimple pivotal category is simple, i.e. satisfies $\text{Hom}(x, x) \cong \mathbb{C}$. In a semisimple pivotal
Lemma 8.2. Let $C$ be a semisimple pivotal category. If $\psi : x \otimes y \to e$ is an isomorphism, then $x$ and $y$ are invertible.

Proof. Assume that $x$ is not simple. By considering partitions of $x$ and $y$ as in in equation (61) one can show that there are two simple objects $c$ and $d$ and two non-zero morphisms $\alpha : c \to x$ and $\beta : x \to d$ with $\beta \circ \alpha = 0$.

This implies that the morphism $(\beta \otimes 1_y) \circ (\alpha \otimes 1_y) : c \otimes y \to d \otimes y$ is also the zero morphism, and with the isomorphism $\psi : x \otimes y \simeq e$, one obtains the following commuting diagram

$$
\begin{array}{ccc}
c \otimes y & \xrightarrow{\alpha \otimes 1_y} & x \otimes y & \xrightarrow{\beta \otimes 1_y} & d \otimes y \\
\downarrow{\psi} & & \downarrow{\psi} & & \\
e & & e
\end{array}
$$

This diagram expresses the zero morphism from $c \otimes y$ to $d \otimes y$ as the composition of two non-zero morphisms through $e$. Each partition of $d \otimes y$ defines a morphism $\gamma : d \otimes y \to e$, such that $\gamma(\beta \otimes 1_y) \circ (\alpha \otimes 1_y) = 1_e$.

This implies

$$0 = \gamma(\beta \otimes 1_y) \circ (\alpha \otimes 1_y) = \gamma(\beta \otimes 1_y) \circ (\alpha \otimes 1_y) \circ (\alpha \otimes 1_y) = \psi(\alpha \otimes 1_y),$$

which is a contradiction. Hence $x$ is simple, and an analogous argument shows that $y$ is simple. This implies that $y^*$ is simple and $x \otimes y \simeq e$ yields a non-zero morphisms $x \to y^*$ which must be an isomorphism.

It follows that $x$ and $y$ are invertible. \hfill \Box

If $x$ is an invertible object in $C$, the functor $A_{dx} : C \to C$ with $A_{dx}(f) = x \otimes f \otimes x^*$ for objects and morphisms $f$ in $C$ is a tensor functor. If $x$ is strictly invertible, the tensor functor $A_{dx}$ is strict. One finds that this functor is directly related to pseudo-equivalences between pivotal categories.

Lemma 8.3. Let $C$ and $D$ be pivotal categories.

(1) Assume that $C$ and $D$ are in addition semisimple. Let $(F, G, \rho, \eta) : C \to D$ be a pseudo-equivalence such that the pseudo-natural transformations $\rho : FG \Rightarrow 1_D$ and $\eta : 1_C \Rightarrow GF$ are given by component morphisms $\rho_x : d \otimes FG(x) \to x \otimes d$ and $\eta_y : c \otimes y \to GF(y) \otimes c$. Then the objects $d \in \text{Ob}(D)$ and $c \in \text{Ob}(C)$ are invertible, $F(c) \simeq d^*$ and $F$ together with $A_{dc^*} \circ G$ form an equivalence of tensor categories.

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3Note that it is not necessary to require that the pivotal category is abelian to ensure the existence of partitions. $C$-linearity together with semisimplicity is sufficient to guarantee the existence of partitions, see e.g. [25].
(2) Let \( F: \mathcal{C} \to \mathcal{D} \) be a tensor functor that is part of an equivalence of categories and \( c \in \mathcal{C} \) be an invertible object. Then the adjoint functor \( \tilde{G} \) of \( F \) naturally is a tensor functor and the adjunction defines a pseudo-equivalence \( (F, \tilde{G} = \text{Ad}_{c^*} \circ \tilde{G}, \rho, \eta) : \mathcal{C} \to \mathcal{D} \) with pseudo-natural transformations \( \rho_c : F(c) \otimes FG(a) \to a \otimes F(c) \) and \( \eta_c : c \otimes b \to GF(b) \otimes c. \)

Proof. 1. Let \( (F, G, \rho, \eta) \) be a pseudo-equivalence between \( \mathcal{C} \) and \( \mathcal{D} \) with pseudo-natural transformations \( \rho_x : d \otimes FG(x) \to x \otimes d \) and \( \eta_y : c \otimes y \to GF(y) \otimes c \) and choose invertible modifications \( T_F : (\rho F) \circ (F \eta) \Rightarrow 1_F \) and \( T_G : (G \rho) \circ (\eta G) \Rightarrow 1_G. \) Using the expressions for the composition of functors and pseudo-natural transformations from Section 2.4, one finds that \( T_F \) and \( T_G \) are given by isomorphisms \( t_F : d \otimes F(c) \to e_F \) and \( t_G : G(d) \otimes c \to e_C, \) respectively. By Lemma 8.2, the objects \( d \) and \( F(c) \) are invertible and \( t_F \) induces an isomorphism \( F(c) \simeq d^* \). Similarly, \( G(d) \) and \( c \) are invertible objects in \( \mathcal{C} \) and \( t_G \) induces an isomorphism \( G(d) \simeq c^*. \) This implies in particular that the functor \( \text{Ad}_{c^*}G \) is a tensor functor, as it is the composite of two tensor functors.

For any tensor functor \( F: \mathcal{C} \to \mathcal{D} \) between tensor categories with duals, there is a canonical natural isomorphism \( \chi^F : \ast_D F \Rightarrow F \ast_C \) with component morphisms
\[
\chi^F_x = (1_{F(x^*)} \otimes \epsilon^*_F(x)) \cdot (F(\epsilon_{x^*}) \otimes 1_{F(x^*)}) : F(x^*) \to F(x^*),
\]
whose inverse is given by
\[
(\chi^F_x)^{-1} = (1_{F(x^*)} \ast F(\epsilon^*_x)) \cdot (\epsilon_{F(x^*)} \ast 1_{F(x^*)}).
\]
The naturality of \( \chi^F \) and its compatibility with the tensor product are a direct consequence of the corresponding properties of the morphisms \( \epsilon_x : e \to x \otimes x^* \) in a pivotal category with duals. Using these isomorphisms, the fact that \( F \) is a tensor functor and the component morphisms of \( \rho \), one obtains the following chain of isomorphisms for all objects \( x \) of \( \mathcal{D} \):
\[
F \text{Ad}_{c^*}G(x) = F(c^* \otimes G(x) \otimes c) \simeq F(c^*) \otimes FG(x) \otimes F(c)
\]
\[
\simeq F(c)^* \otimes FG(x) \otimes F(c) \simeq d \otimes FG(x) \otimes d^* \simeq 1\mathcal{D}.
\]
All of these isomorphisms are natural in \( x \) and they define a natural isomorphism \( F \text{Ad}_{c^*}G \Rightarrow 1\mathcal{D} \). Similarly, the natural isomorphisms
\[
y \simeq c^* \otimes c \otimes y \simeq 1\mathcal{C} \otimes \eta_y \simeq c^* \otimes GF(y) \otimes c = \text{Ad}_{c^*}GF(y)
\]
define a natural isomorphism \( 1\mathcal{C} \Rightarrow \text{Ad}_{c^*}GF \) of tensor functors. This shows that the tensor functors \( F: \mathcal{C} \to \mathcal{D} \) and \( \text{Ad}_{c^*}G : \mathcal{D} \to \mathcal{C} \) form an equivalence of tensor categories.

2. To prove the second statement, consider a tensor functor \( F: \mathcal{C} \to \mathcal{D} \) that is part of an equivalence of categories. This implies that for
the (right) adjoint functor $\tilde{G} : \mathcal{D} \to \mathcal{C}$ there are natural isomorphisms $\tilde{\rho} : F\tilde{G} \to 1_{\mathcal{D}}$ and $\tilde{\eta} : 1_{\mathcal{C}} \to \tilde{G}F$, that satisfy $(\tilde{\rho}F) \circ (F\tilde{\eta}) = 1_F$ and $(\tilde{G}\tilde{\rho}) \circ (\tilde{\eta}\tilde{G}) = 1_{\tilde{G}}$. It is straightforward to see that the tensor functor structure of $F$ together with the natural isomorphisms $\tilde{\rho}$ and $\tilde{\eta}$ induces a tensor functor structure on $\tilde{G}$ such that $\tilde{\rho}$ and $\tilde{\eta}$ are tensor natural transformations. As $c$ is an invertible object of $\mathcal{C}$, the functor $G = Ad_c \circ \tilde{G}$ is also a tensor functor.

Define for each object $y$ of $\mathcal{C}$ isomorphisms $\eta_y : c \otimes y \to GF(y) \otimes c$ as the composites

$$c \otimes y \overset{1_c \otimes \eta_y}{\approx} c \otimes \tilde{G}F(y) \simeq c \otimes \tilde{G}F(y) \otimes c^* \otimes c = GF(y) \otimes c.$$

If follows directly that $\eta_y$ is natural in $y$ as well as compatible with the tensor product and hence defines a pseudo-natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$. Similarly, for $d := F(c)^*$, the following chain of isomorphisms defines a pseudo-natural transformation $\rho : FG \Rightarrow 1_D$:

$$d \otimes FG(x) = F(c)^* \otimes F(c \otimes \tilde{G}(x) \otimes c^*) \simeq F(c^*) \otimes F(c) \otimes F\tilde{G}(x) \otimes F(c^*)$$

$$\simeq F(c^*) \otimes c \otimes F\tilde{G}(x) \otimes F(c^*) \simeq F\tilde{G}(x) \otimes d \overset{\tilde{\rho}(x) \otimes 1_d}{\simeq} x \otimes d,$$

It remains to define invertible modifications $T_F : (\rho F) \circ (F\eta) \Rightarrow 1_F$ and $T_G : (G\rho) \circ (\eta G) \Rightarrow 1_G$. A lengthy but straightforward computation using the identity $(\tilde{\rho}F) \circ (F\tilde{\eta}) = 1_F$ shows that the isomorphism

$$F(c^*) \otimes F(c) \simeq (F(c^* \otimes c)$$

defines a modification $T_F : (\rho F) \circ (F\eta) \Rightarrow 1_F$. Similarly, it follows from the identity $(\tilde{G}\tilde{\rho}) \circ (\tilde{\eta}\tilde{G}) = 1_{\tilde{G}}$ that the isomorphism

$$G(d) \otimes c = c \otimes \tilde{G}(F(c)^*) \otimes c^* \otimes c \simeq c \otimes (F\tilde{G}(c))^* \otimes c \otimes c^* \simeq e$$

defines a modification $T_G : (G\rho) \circ (\eta G) \Rightarrow 1_G$. □

This lemma strongly restricts the possible subcategories of MonCat which can give rise to a Gray category with duals. It implies that all 1-morphisms between semisimple pivotal categories in this subcategory must be equivalences of tensor categories, that their $\#\text{-duals}$ must be closely related to their adjoint and that the fold 2-morphisms are associated with invertible objects in the underlying pivotal categories.

Further restrictions on the 1-morphisms arise from the condition that for each pair of objects $\mathcal{C}$, $\mathcal{D}$ in a Gray category with duals, the 2-category $\mathcal{G} \mathcal{C}, \mathcal{D}$ is a planar. If $\mathcal{C}$ is a pivotal category and $\nu : F \Rightarrow G$ a pseudo-natural transformation between tensor functors $F, G : \mathcal{C} \to \mathcal{D}$ with component morphisms $\nu_a : x \otimes F(a) \to G(a) \otimes x$, then its $\ast$-dual must be a pseudo-natural transformation $\nu^\ast : G \Rightarrow F$ with component morphisms $(\nu^\ast)_a : y \otimes G(a) \to F(a) \otimes y$. The associated 3-morphism $\epsilon_\mu : 1_G \Rightarrow \mu \circ \mu^\ast$ then corresponds to a morphism $e : x \to x \otimes y$ and must satisfy the conditions in (12).

The only natural choice for such a morphism are the 3-morphisms $\epsilon_x : e \to x \otimes x^\ast$ in the pivotal category $\mathcal{D}$, which require the condition
$y = x^*$. The first condition in Definition 3.10 then imposes the relation $H \Box x \Box K = \epsilon_{H \Box x \Box K}$, which implies $F(\epsilon_x) = \epsilon^D_F$ for all objects $x$ of $\mathcal{C}$. This motivates the following definition.

**Definition 8.4.** Let $\mathcal{C}, \mathcal{D}$ be pivotal categories. A pivotal functor from $\mathcal{C}$ to $\mathcal{D}$ is a strict tensor functor $F: \mathcal{C} \to \mathcal{D}$ with $\ast^D F = F \ast_C$ and $F(\epsilon_a) = \epsilon^D_F(a)$ for all objects $a$ of $\mathcal{C}$. We denote by $\text{Piv}(\mathcal{C}, \mathcal{D})$ the sub 2-category of $\text{MonCat}(\mathcal{C}, \mathcal{D})$ formed by pivotal functors $F: \mathcal{C} \to \mathcal{D}$, pseudo-natural transformations and modifications.

Note that this condition on functors is quite restrictive. In particular, it implies that a pivotal functor $F: \mathcal{C} \to \mathcal{D}$ preserves traces. For each object $a$ of $\mathcal{C}$ and each morphism $\alpha: a \to a$, one has

\[
\text{tr}^D_L(F(\alpha)) = \epsilon^*_F(a) \cdot (F(\alpha) \otimes 1_{F(a)}) \cdot \epsilon_F(a) = F(\epsilon^*_a(\alpha \otimes 1_{a^*}) \cdot \epsilon_a) = F(\text{tr}^C_L(\alpha))
\]

and analogously for the right-trace. If $\text{End}(e) \cong \mathcal{C}$ and $F$ is a strict tensor functor, it follows that $\text{tr}^D_{L,R}(F(\alpha)) = \text{tr}^C_{L,R}(\alpha)$. Nevertheless, there are non-trivial examples.

**Example 8.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be the spherical categories of finite-dimensional representations of groups $G$ and $H$. Then a group homomorphism $H \to G$ determines a pivotal functor $\mathcal{C} \to \mathcal{D}$.

Similar examples are given by homomorphisms of semisimple Hopf algebras, or more generally, homomorphisms of spherical Hopf algebras where the homomorphism preserves the spherical element. Another example based on groups will be discussed in detail in Section 8.2. A somewhat different example is the following

**Example 8.6.** Let $\mathcal{C}$ be a spherical category and $\widetilde{\mathcal{C}}$ its non-degenerate quotient (see [3]). Then the canonical functor $\mathcal{C} \to \widetilde{\mathcal{C}}$ is a pivotal functor.

If one restricts attention to pivotal functors, it is directly apparent from the considerations before Definition 8.4 how the 2-categories $\text{Piv}(\mathcal{C}, \mathcal{D})$ can be equipped with the structure of a planar 2-category. This planar 2-category structure is canonical, as it is induced by the pivotal structures of $\mathcal{C}, \mathcal{D}$.

**Lemma 8.7.** Let $\mathcal{C}, \mathcal{D}$ be strict (not necessarily $\mathcal{C}$-linear) pivotal categories. Then the 2-category $\text{Piv}(\mathcal{C}, \mathcal{D})$ has a canonical planar 2-category structure.

**Proof.** The $\ast$-dual of a pseudo-natural transformation $\nu: F \Rightarrow G$ with component morphisms $\nu_a: x \otimes F(a) \to G(a) \otimes x$ is the pseudo-natural transformation $\nu^*: G \Rightarrow F$ with component morphisms $(\nu^*)_a = \nu^*_a$, where $\ast$ denotes the dual in the pivotal category $\mathcal{D}$. That $\nu^*: G \Rightarrow F$ does indeed satisfy the conditions in Definition 2.19 follows by a direct computation from the corresponding properties of the pseudo-natural
transformation $\nu : F \Rightarrow G$ and the properties of the dual in a pivotal category. The dual of a modification $\Phi : \nu_1 \Rightarrow \nu_2$ is defined by the dual $\Phi^* : x^*_2 \rightarrow x^*_1$ of the corresponding morphism $\Phi : x_1 \rightarrow x_2$ in $\mathcal{D}$. For all pivotal categories $\mathcal{C}, \mathcal{D}$ this yields a strict 2-functor $^* : \text{Piv}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Piv}(\mathcal{C}, \mathcal{D})^{op}$ which is trivial on the objects and satisfies $^{**} = 1$.

It remains to show that there are modifications $\epsilon_{\nu} : 1 \Rightarrow \nu \circ \nu^*$ satisfying the conditions (12). These are given by the 3-morphism $\epsilon_x : e \rightarrow x \otimes x^*$ in $\mathcal{D}$. That the morphism $\epsilon_x : e \rightarrow x \otimes x^*$ defines a modification $\epsilon_{\nu} : 1 \Rightarrow \nu \circ \nu^*$ is equivalent to the commutativity of the diagram

$$
\begin{array}{ccc}
G(a) & \xrightarrow{1_{G(a)} \otimes \epsilon_x} & G(a) \otimes x \otimes x^* \\
\downarrow{\epsilon_x \otimes 1_{G(a)}} & & \downarrow{(\nu \circ \nu^*)_a} \\
x \otimes x^* \otimes G(a) & \xrightarrow{1_x \otimes (\nu^*)_a} & x \otimes F(a) \otimes x^*
\end{array}
$$

or, equivalently, to the condition

$$
(\nu_a \otimes 1_x)^* \cdot (1_x \otimes (\nu^*)_a) \cdot (\epsilon_x \otimes 1_{G(a)}) = 1_{G(a)} \otimes \epsilon_x
$$

The left-hand side of this equation is expressed diagrammatically in Figure 62 a), the right-hand side in Figure 62 d). To prove this identity, we use the definition of the spherical transformation $\nu^* : G \Rightarrow F$ together with identity (13) for the dual of a 2-morphism in a planar category. This yields

$$(\nu^*)_a = (\epsilon^*_x \otimes 1_{F(a)} \otimes 1_{x^*}) \cdot (1_{x^*} \otimes \epsilon^*_G(a) \otimes 1_x \otimes 1_{F(a)} \otimes 1_{x^*}) \cdot (1_{x^*} \otimes 1_{G(a)} \otimes \nu_{a^*} \otimes 1_{F(a)} \otimes 1_{x^*}) \cdot (1_{x^*} \otimes 1_{G(a)} \otimes 1_x \otimes \epsilon_{F(a^*)} \otimes 1_{x^*}) \cdot (1_{x^*} \otimes 1_{G(a)} \otimes \epsilon_x)
$$

Inserting this identity into the left-hand side of condition (65) yields the diagram in Figure 62 a). From the naturality of the tensorator and the second identity in (12) one then obtains the diagram in Figure 62 b). By applying again the second condition in (12), one transforms this diagram into the one in Figure 62 c). The naturality property and the invertibility of the tensorator then yield the diagrams in Figure 62 d). This shows that the morphism $\epsilon_x : e \rightarrow x \otimes x^*$ defines a modification $\epsilon_{\nu} : 1 \Rightarrow \nu \circ \nu^*$. The identities in (12) then follow directly from the properties of the 3-morphisms $\epsilon_x : e \rightarrow x \otimes x^*$ in a pivotal category. □

8.2. Gray categories with duals from pivotal categories. In this subsection, we consider two examples of Gray categories with duals constructed from pivotal categories. The first example is rather trivial. It is the subcategory of MonCat obtained by restricting attention to strict pivotal categories as objects and invertible pivotal functors as 1-morphisms.
Definition 8.8 (PivCat). The Gray category PivCat is the subcategory of MonCat with pivotal categories $C$ as objects and invertible pivotal functors $F : C \to D$ as 1-morphisms. Its 2-morphisms are pseudo-natural transformations $\nu : F \Rightarrow G$ between invertible pivotal functors and its 3-morphisms modifications $\Phi : \nu \Rightarrow \mu$.

An obvious way of defining a Gray category structure on PivCat is to take the inverse functor as the $\#$-dual of each 1-morphism $F : C \to D$. The identity $F F^\# = 1_D$ then allows one to take the unit 2-morphisms $1_{1_D}$ as the fold 2-morphisms and the unit 3-morphisms $1_{1_C}$ as triangulators. The planar 2-category from Lemma 8.7 then equips each 2-category $\text{PivCat}(C, D)$ with the structure of a planar 2-category, and the compatibility condition between the 3-morphisms $\epsilon_\mu$ and the Gray product follows from the fact that all 1-morphisms are pivotal functors.

Lemma 8.9. When equipped with the planar 2-category structure from Lemma 8.7 and the following additional structures

1. $\#$-dual: $F^\# = F^{-1}$ for all 1-morphisms $F : C \to D$.
2. Fold 2-morphisms: $\eta_F = 1_{1_D}$
3. Triangulators: $T_F = 1_{1_C}$,

the Gray category PivCat becomes a Gray category with duals.

A natural way of obtaining less trivial examples is to consider 1-morphisms that are equipped with additional structure, which enters
the definition of their $\#$-duals and hence of the fold 2-morphisms. In view of Lemma 8.3, it is natural that these additional structures should be related to invertible objects in the underlying pivotal categories. This motivates the following definition

**Definition 8.10.** Let $\mathcal{C}, \mathcal{D}$ be pivotal categories. A decorated pivotal functor is a pair $(F, c)$ of an invertible pivotal functor $F: \mathcal{C} \to \mathcal{D}$ and a strictly invertible object $c$ of $\mathcal{C}$.

By considering such decorated pivotal functors together with pseudonatural transformations and modifications, one obtains a Gray category that is closely related to $\text{PivCat}$ and will be denoted $\text{PivCat}_{\text{dec}}$ in the following.

**Definition 8.11** ($\text{PivCat}_{\text{dec}}$). The Gray category $\text{PivCat}_{\text{dec}}$ has as objects strict pivotal categories $\mathcal{C}$ and as 1-morphisms decorated pivotal functors $(F, c): \mathcal{C} \to \mathcal{D}$. The 2-morphisms $(F, c) \Rightarrow (F', c')$ are pseudonatural transformations $\nu: F \Rightarrow F'$ and the 3-morphisms modifications $\Phi: \nu \Rightarrow \nu'$. The vertical and the horizontal composition, the Gray product and the tensorator are defined as in $\text{MonCat}$, and the composition of 1-morphisms with 1-, 2- and 3-morphisms is given by

$$(H, d) \boxtimes (F, c) = (HF, c \otimes F^{-1}(d)) \quad (H, d) \boxtimes (K, f) = H\nu K$$

for all decorated pivotal functors $(F, c)$, $(H, d)$, $(K, f)$ and 2- or 3-morphisms $\nu$ for which these expressions are defined.

It is directly apparent that $\text{PivCat}$ is indeed a Gray category. Due to the results summarised in Section 2.4, it is sufficient to show that product $\boxtimes$ is strictly associative on the 1-morphisms and that for each object $\mathcal{C}$ there is a unit 1-morphism. The latter is given by $(1_{\mathcal{C}}, e)$, where $e$ denotes the tensor unit of $\mathcal{C}$ and $1_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ the identity functor. The proof of the associativity for the Gray product on 1-morphisms is a straightforward computation.

As the 2- and 3-morphisms in $\text{PivCat}_{\text{dec}}$ do not involve any additional structure, it is also clear that each 2-category $\text{PivCat}_{\text{dec}}(\mathcal{C}, \mathcal{D})$ inherits the canonical planar 2-category structure from Lemma 8.7. However, the additional data in the 1-morphisms allows one to obtain non-trivial $\#$-duals and fold 2-morphisms.

**Lemma 8.12.** The category $\text{PivCat}$ becomes a Gray category with duals when equipped with the planar 2-category structure from Lemma 8.7 and the following additional structures:

1. **$\#$-dual:** $(F, c)^\# = (F^\#, F(c))$ with $F^\# = A_d c F^{-1}$.
2. **fold 2-morphisms:** $(\eta_{(F, c)})_a = 1_{F(c) \otimes a}: F(c) \otimes a \to F(c) \otimes a$
3. **triangulators:** given by the morphisms $1_{F(c)}: F(c) \to F(c)$.

**Proof.** If $\mu: F \Rightarrow G$ is a pseudo-natural transformation with component morphisms $\mu_a: x \otimes F(a) \to G(a) \otimes x$, the first condition in
Definition 3.10 reads $K(\epsilon_x) = \epsilon_{K(x)}$, which holds for all pivotal functors $K$. This establishes the compatibility between the Gray product and the planar 2-category structure in PivCat dec.

It remains to show that the fold 2-morphisms and the triangulator are well-defined. Inserting the definition of the #-dual and the Gray product in PivCat, one obtains

$$ (F, c) \Box (F, c)^\# = (F \text{Ad}_c F^{-1}, F(c) \otimes F \text{Ad}_c^*(c)) = (\text{Ad}_{F(c)}, F(c) \otimes F(c)), $$

and it follows that the morphisms $(\eta_{(F, c)})_a = 1_{F(c) \otimes a}$ in $\mathcal{D}$ define a pseudo-natural transformation $\eta_{(F, c)} : 1_{\mathcal{D}} \Rightarrow (F, d) \Box (F, d)^\#$. The identity morphism $1_c$ on the unit object of $\mathcal{D}$ thus defines a modification $T_{(F, c)} : (\eta_{(F, c)}^* \Box (F, c)) \circ ((F, c) \Box \eta_{(F, c)^*}) \Rightarrow 1_{(F, c)}$.

The double dual of a 1-morphism is given by

$$ (F, c)^{\#\#} = (\text{Ad}_c F^{-1}, F(c)) = (\text{Ad}_{F(c)} F \text{Ad}_{c^*}, \text{Ad}_c F^{-1}(c)) = (F, c), $$

the dual of a the unit 1-morphism $(1_C, e)$ satisfies the identity $(1_C, e)^\# = (\text{Ad}_c, e) = (1_C, e)$. Similarly, one finds that $(\eta_{1_C, e})$ is given by the identity pseudo-natural transformation and by definition $T_{1_C} = 1_{1_C}$. The remaining conditions on the triangulator and the fold 2-morphisms are satisfied trivially. The identity $((H, d) \Box (F, c))^\# = (F, c)^{\#\#} \Box (H, d)^\#$ follows by a direct computation. \hfill \Box

Although the assumptions on the functors and invertible objects in the theorem are very restrictive, there is a nontrivial concrete example that provides functors and objects of this type.

**Example 8.13.** Let $G$ be a group and denote by $\mathcal{C}_G$ the category whose objects are group elements $g \in G$ and whose Hom-spaces are of the form $\text{Hom}(g, h) = \{0\}$ for $g \neq h$, $\text{Hom}(g, g) = \mathbb{C}$, where morphisms $f = \alpha 1_g : g \to g$ are identified with the numbers $\alpha \in \mathbb{C}$. The group multiplication equips $\mathcal{C}_G$ with the structure of a strict tensor category with tensor product $g \otimes h = g \cdot h$ and tensor unit $e$. The tensor product of two morphisms $f = \alpha 1_g : g \to g$ and $k = \beta 1_h : h \to h$ is given by $f \otimes k = \alpha \beta 1_{g \cdot h}$.

The duals are defined by $g^* = g^{-1}$ for all $g \in G$ and $(\lambda 1_g)^* = \lambda 1_{g^{-1}}$. This implies that every object of $\mathcal{C}_G$ is invertible and the dual satisfies the relation $** = 1$. Pivotal structures on $\mathcal{C}_G$ are in bijection with group homomorphisms $\lambda : G \to \mathbb{C}^\times$. This can be seen as follows: each morphism $\epsilon_g : e \to g \otimes g^* = e$ is of the form $\epsilon_g = \lambda(g) 1_e$ with $\lambda(g) \in \mathbb{C}$ and agrees with its dual $\epsilon_g^* = \epsilon_g$. The conditions in (12) imply

$$ \epsilon_{g \cdot h} = \lambda(g \cdot h) = (1_g \otimes \epsilon_h \otimes 1_{g^{-1}}) \cdot \epsilon_g = \lambda(g) \lambda(h) \quad \forall g, h \in G. $$

Conversely, if $\lambda : G \to \mathbb{C}^\times$ is a group homomorphism, then $\epsilon_g = \lambda(g) 1_e$ satisfies the axioms in (12). The left- and right-trace of a morphism $f = \alpha 1_g$ are then given by, respectively, $\text{tr}_L(f) = \alpha \lambda(g)^2 1_e$ and $\text{tr}_R(f) = \alpha \lambda(g^{-1})^2 1_e$, which implies that $\mathcal{C}_G$ is spherical if and only if $\lambda(G) \subset \{1, -1, i, -i\}$. 145
Each group homomorphism $\lambda: G \to \mathbb{C}^\times$ thus determines a pivotal structure on $\mathcal{C}_G$. In particular, if $\rho_V: G \to \text{End}(V)$ is a representation of $G$ on a finite-dimensional vector space $V$ over $\mathbb{C}$, then $\det(\rho): G \to \mathbb{C}^\times$ is a group homomorphism of the required form.

Although this example is rather simple, it gives rise to a Gray category with duals that exhibit non-trivial fold 2-morphisms and allows one to distinguish oriented surfaces of different genus via the associated surface invariants from Lemma 7.14.

**Lemma 8.14.** Let $F: \mathcal{C}_G \to \mathcal{C}_H$ be an invertible pivotal functor and $c \in G$. Then the surface invariant from Lemma 7.14 for a genus $g$ surface labelled with the decorated pivotal functor $(F, c)$ is

$$\dim_L(\eta(F,c))^g \dim_R(\eta(F,c)) = \lambda_H(F(c))^{2g-2}$$

**Proof.** If $G, H$ are groups with associated pivotal structures $\lambda_G: G \to \mathbb{C}^\times$, $\lambda_H: H \to \mathbb{C}^\times$, then strict tensor functors $F: \mathcal{C}_G \to \mathcal{C}_H$ correspond to group homomorphisms $F: G \to H$, invertible tensor functors to group isomorphisms, and a strict tensor functor $F: \mathcal{C}_G \to \mathcal{C}_H$ is pivotal if and only if $\lambda_G = \lambda_H \circ F$.

A decorated pivotal functor $\mathcal{C}_G \to \mathcal{C}_H$ is a pair $(F, g_0)$ of a group isomorphism $F: G \to H$ with $\lambda_G = \lambda_H \circ F$ and a group element $g_0 \in G$. A pseudo-natural transformation $\nu: F \Rightarrow F'$ between pivotal functors $F, F': \mathcal{C}_G \to \mathcal{C}_H$ corresponds to a group element $h \in H$ such that $F'(g) = h \cdot F(g) \cdot h^{-1}$ for all $g \in G$, and a modification $\Phi: h \Rightarrow h$ to a morphism $\mu: 1_h: h \to h$ with $\mu \in \mathbb{C}$.

The $#$-dual of a 1-morphism $(F, c): \mathcal{C}_G \to \mathcal{C}_H$ is given by the group homomorphism $F^# = \text{Ad}_c F: H \to G$, $h \mapsto c \cdot F^{-1}(g) \cdot c^{-1}$, and the fold 2-morphisms $\eta_{(F,c)}: 1_D \Rightarrow FF^#$ by the group element $F(g_0)$. If $\lambda_H: H \to \mathbb{C}^\times$ is the group homomorphism that determines the pivotal structure of $\mathcal{C}_H$, then one obtains $\dim_L(\eta_{(F,c)}) = \lambda_H(F(c))^2$ and $\dim_R(\eta_{(F,c)}) = \lambda_H(F(c))^{-2}$.

This implies that although the category $\mathcal{C}_G$ is not necessarily spatial, we can apply Lemma 7.14 to determine the surface invariants associated with a genus $g$ surface labelled with a decorated pivotal functor $(F, c)$. This is possible since the fold 2-morphisms are the only 2-morphisms that occur in the diagram. It follows directly from the above expressions that their braidings are trivial, which ensures invariance of the associated diagram under the ribbon moves [19]. Inserting these expressions into the formula from Lemma 7.14 then proves the claim.

\[\square\]

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Appendix A. Functors of strict tricategories, natural transformations and modifications

In this appendix, we define functors of strict tricategories and their natural transformations and modifications by specialising the associated definitions for tricategories in [9, 11]. For completeness, we also record the standard definitions for functors of (strict) 2-categories [19, 22].

Definition A.1 (Lax 2-functor). A lax 2-functor $F : C \to D$ between 2-categories $C, D$ is given by the following data

- A function $F_0 : \text{Ob}(C) \to \text{Ob}(D)$.
- For all objects $G, H$ of $C$, a functor $F_{G,H} : C_{G,H} \to D_{F_0(G),F_0(H)}$.
- For all objects $G, H, K$ of $C$ a natural transformation $\Phi_{GHK} : (F_{H,K} \times F_{G,H}) \to F_{G,K} \circ$. These determine, for all 1-morphisms $\nu : G \to H$, $\mu : H \to K$, a 2-morphism $\Phi_{\mu,\nu} : F_{H,K}(\mu) \circ F_{G,H}(\nu) \to F_{G,K}(\mu \circ \nu)$.
- For all objects $G$, a 2-morphism $\Phi_G : 1_{F_0(G)} \to F_{G,G}(1_G)$.

The function $F_0$, the functors $F_{G,H}$ and the 2-morphisms $\Phi_{\mu,\nu}$ and $\Phi_G$ are required to satisfy the following consistency conditions

1. For all 1-morphisms $\nu : G \to H$:
   $$\Phi_{1H,\nu} \cdot (\Phi_H \circ 1_{F_{G,H}(\nu)}) = \Phi_{\nu,1G} \cdot (1_{F_{G,H}(\nu)} \circ \Phi_G) = 1_{F_{G,H}(\nu)}.$$

2. For all 1-morphisms $\nu : G \to H$, $\mu : H \to K$, $\rho : K \to L$, the following diagram commutes
   $$\begin{array}{ccc}
   F_{K,L}(\rho) \circ F_{H,K}(\mu) \circ F_{G,H}(\nu) & \xrightarrow{1 \circ \Phi_{\mu,\nu}} & F_{H,L}(\rho \circ \mu) \circ F_{G,H}(\nu) \\
   \downarrow \Phi_{\rho,\mu} & & \downarrow \Phi_{\rho \circ \mu,\nu} \\
   F_{K,L}(\rho) \circ F_{G,K}(\mu \circ \nu) & \xrightarrow{\Phi_{\rho,\mu \circ \nu}} & F_{G,L}(\rho \circ \mu \circ \nu).
   \end{array}$$

A weak 2-functor (also called a strong 2-functor, pseudo-functor, or homomorphism) is a lax 2-functor in which all 2-morphisms $\Phi_{\mu,\nu}$ and $\Phi_G$ are invertible. A weak 2-functor is said to have strict units if the 3-morphisms $\Phi_G$ are all identities, and it is called strict if the 2-morphisms $\Phi_{\mu,\nu}$ and $\Phi_G$ are all identities. In this case, one has

$$F_{G,K}(\mu \circ \nu) = F_{H,K}(\mu) \circ F_{G,H}(\nu) \quad 1_{F_0(G)} = F_{G,G}(1_G).$$

There is an analogous definition with the arrows labelled by $\Phi_{\mu,\nu}$ and $\Phi_G$ reversed. In this case, the functor is called an op-lax 2-functor.
In the following, we will also require the notion of cubical and opcubical functors between certain 2-categories. Our definition is a special case of the definition of cubical and opcubical functors from [9, 11].

**Definition A.2** (Op)cubical functors. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be 2-categories. A functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with coherence isomorphisms

$$\Phi_{\mu,\nu} : F((\mu_1, \mu_2)) \circ F((\nu_1, \nu_2)) \to F((\mu_1 \circ \nu_1), (\mu_2 \circ \nu_2)),$$

for $\circ$-composable 1-morphisms $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ in $\mathcal{C} \times \mathcal{D}$ is called (op)cubical, if the 2-morphism $\Phi_{\mu,\nu}$ is the identity in case $\mu_1$ or $\nu_2$ ($\mu_2$ or $\nu_1$) is an identity 1-morphism.

The following notion of natural transformation of 2-functors adopts the convention of [9, 11] and is sometimes also referred to as ‘oplax 2-transformation’.

**Definition A.3** (Natural transformation of lax 2-functors). A natural transformation $\rho : F \to G$ between lax 2-functors $F = (F_0, F_{A,B}, \Phi_{\mu,\nu}, \Phi_A) : \mathcal{C} \to \mathcal{D}, G = (G_0, G_{A,B}, \Psi_{\mu,\nu}, \Psi_A) : \mathcal{C} \to \mathcal{D}$ is given by the following data:

- For all objects $A$ of $\mathcal{C}$, a 1-morphism $\rho_A : F_0(A) \to G_0(A)$.
- For all objects $A, B$ of $\mathcal{C}$ a natural transformation

$$\rho_{A,B} : (\rho_B \circ -)F_{A,B} \to (\circ \rho_A)G_{A,B},$$

where $\circ \rho_A : \mathcal{D}F_0(A),G_0(B) \to \mathcal{D}F_0(A),G_0(B)$ and $\rho_B \circ - : \mathcal{D}F_0(A),F_0(B) \to \mathcal{D}_F(A),G_0(B)$ denote the functors given by pre- and post-composition with $\rho_A$ and $\rho_B$. These natural transformations determine for all 1-morphisms $\mu : A \to B$ a 2-morphism $\rho_\mu : \rho_B \circ F_{A,B}(\mu) \to G_{A,B}(\mu) \circ \rho_A$.

The 1-morphisms $\rho_A$ and 2-morphisms $\rho_\mu$ are required to satisfy the following consistency conditions:

1. For all 1-morphisms $\nu : A \to B$ and $\mu : B \to C$ the following diagram commutes

$$\begin{array}{ccc}
\rho_C \circ F_{B,C}(\mu) \circ F_{A,B}(\nu) & \stackrel{\rho_\mu \circ 1}{\longrightarrow} & G_{B,C}(\mu) \circ \rho_B \circ F_{A,B}(\nu) \\
\downarrow 1 \circ \Phi_{\mu,\nu} & & \downarrow 1 \circ \rho_\nu \\
\rho_C \circ F_{A,C}(\mu \circ \nu) & \stackrel{\rho_{\mu \circ \nu}}{\longrightarrow} & G_{B,C}(\mu) \circ G_{A,B}(\nu) \circ \rho_A \\
\downarrow \rho_{\mu \circ \nu} & & \downarrow \Psi_{\mu,\nu} \circ 1 \\
G_{A,C}(\mu \circ \nu) \circ \rho_A & & \\
\end{array}$$

2. For all objects $A$ of $\mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc}
1_{G_0(A)} \circ \rho_A = \rho_A = \rho_A \circ 1_{F_0(A)} & \stackrel{1 \circ \Phi_A}{\longrightarrow} & \Psi_{A} \circ 1 \\
\downarrow 1 \circ \Phi_A & & \downarrow \Psi_A \circ 1 \\
\rho_A \circ F_{A,A}(1_A) & \stackrel{\rho_1 A}{\longrightarrow} & G_{A,A}(1_A) \circ \rho_A.
\end{array}$$
A pseudo-natural transformation \( \rho : F \to G \) of lax 2-functors \( F, G : C \to D \) is a natural transformation of lax 2-functors in which all 2-morphisms \( \rho \mu : \rho A \circ F A,B(\mu) \to G A,B(\mu) \circ \rho A \) are isomorphisms. A pseudo-natural transformation is called invertible if all the 2-morphisms \( \rho A \) are invertible. A natural isomorphism is a pseudo-natural transformation in which for every object \( A \), \( F_0(A) = G_0(A) \) and the 1-morphism \( \rho A \) is the identity.

It is easy to see that an invertible pseudo-natural transformation has indeed a unique inverse pseudo-natural transformation.

**Definition A.4 (Modification).** Let \( \rho = (\rho A, \rho A,B) : F \to G \) and \( \tau = (\tau A, \tau A,B) : F \to G \) be natural transformations between lax 2-functors \( F = (F_0, F A,B, \Phi_{\mu,\nu}, \Phi_A), G = (G_0, G A,B, \Psi_{\mu,\nu}, \Psi_A) : C \to D \). A modification \( \Psi : \rho \Rightarrow \tau \) is a collection of 2-morphisms \( \Psi A : \rho A \Rightarrow \tau A \) for every object \( A \) of \( G \) such that for all 1-morphisms \( \mu : A \to B \)

\[
\tau \mu \cdot (\Psi_A \circ 1_{F A,B(\mu)}) = (1_{G A,B(\mu)} \circ \Psi_B) \cdot \rho \mu
\]

A modification is called invertible if all 2-morphisms \( \Psi A \) are invertible.

In terms of these definitions for 2-categories, the concepts of functors of strict tricategories, natural transformations and modifications can be formulated. We start with the definition of a strict tricategory following [9] and [11]. Note that in general there are some differences between the definition of tricategory in [9] and ‘algebraic tricategory’ in [11]. However, for the strict tricategories considered in this paper these definitions coincide.

**Definition A.5 (Strict tricategory).** A strict tricategory is a tricategory \( (G, \Box, \circ, \cdot) \) in which the composition \( \Box \) is strictly associative and unital. A strict tricategory is called (op)cubical if the following additional conditions are satisfied

1. For all objects \( C, D \) the bicategory \( G(C, D) \) is a strict 2-category.
2. \( 1_{1_C} \circ 1_{1_C} = 1_{1_C} \).
3. Each functor \( \Box : G(D, E) \times G(C, D) \to G(C, E) \) is (op)cubical, i. e. the invertible coherence 3-morphisms

\[
\Box_{\mu,\nu} : (\mu_1 \Box \mu_2) \circ (\nu_1 \Box \nu_2) \to (\mu_1 \circ \nu_1) \Box (\mu_2 \circ \nu_2)
\]

for \( \mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in G(D, E) \times G(C, D) \) are identity 3-morphisms if \( \mu_1 \) or \( \nu_2 \) (\( \mu_2 \) or \( \nu_1 \)) is an identity 2-morphism.

In the following we call a 1-morphism \( F : C \to D \) in a strict tricategory invertible, if there exists a 1-morphism \( G : D \to C \) with \( F \Box G = 1_D \) and \( G \Box F = 1_C \). Similarly, a 2-morphism \( \mu : F \to G \) in a strict tricategory is called invertible, if there exists 2-morphism \( \nu : G \to F \) with \( \mu \circ \nu = 1_F \) and \( \nu \circ \mu = 1_F \). Note that the inverse 1-morphism \( G \) and 2-morphism \( \nu \) are determined uniquely.
A functor of strict tricategories is a functor of tricategories that is compatible with the extra requirements that hold in case of a strict tricategory. As we consider only tricategory functors between strict (op)cubical tricategories, we will not give the most general definition of a tricategory functor, but refer the reader to [9, 11] for details. Note that the definitions in [9, 11] differ slightly and that the definition of [11] is stronger than that of [9], since the coherence data of a functor $F$ consists of adjoint equivalences instead of just equivalences in certain bicategories. In particular, it contains an adjoint equivalence $\kappa : \square \circ (F \times F) \to F \circ \square$. A functor $F : G \to \tilde{G}$ of strict tricategories according our definition is a functor of tricategories according to [11], where all coherence data consists of identities and $\kappa$ is a natural isomorphism, which is automatically an adjoint equivalence.

**Definition A.6** (Functor of strict tricategories). A lax functor $F : G \to \tilde{G}$ between strict tricategories $G, \tilde{G}$ consists of

- a function $F_0 : \text{Ob}(G) \to \text{Ob}(\tilde{G})$,
- weak 2-functors $F_{\mathcal{C}, \mathcal{D}} : G(\mathcal{C}, \mathcal{D}) \to \tilde{G}(F_0(\mathcal{C}), F_0(\mathcal{D}))$ for all objects $\mathcal{C}, \mathcal{D}$ of $G$,
- an invertible pseudo-natural transformation of weak 2-functors $\kappa_{\mathcal{C}, \mathcal{D}, \mathcal{E}} : \tilde{\square}(F_{\mathcal{D}, \mathcal{E}} \times F_{\mathcal{C}, \mathcal{D}}) \to F_{\mathcal{C}, \mathcal{D}} \square$ for all objects $\mathcal{C}, \mathcal{D}, \mathcal{E}$ of $G$,
- an invertible 2-morphism $\iota_\mathcal{C} : F_{\mathcal{C}, \mathcal{C}}(1_{\mathcal{C}}) \to 1_{F_0(\mathcal{C})}$ for all objects $\mathcal{C}$ of $G$,

such that the following consistency conditions are satisfied

1. For all objects $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ of $G$

   $$(\kappa_{\mathcal{B}, \mathcal{C}, \mathcal{D}}) \circ (\tilde{\square}(\kappa_{\mathcal{C}, \mathcal{D}, \mathcal{E}} \times I)) = (\kappa_{\mathcal{B}, \mathcal{D}}(I \times \square)) \circ (\tilde{\square}(I \times \kappa_{\mathcal{C}, \mathcal{D}})).$$

   In this formula $I$ is the identity functor, the unnamed product is the Gray product in $\text{2Cat}$, and $\circ$ is the horizontal composition of pseudo-natural transformations.

2. For all objects $\mathcal{C}, \mathcal{D}$ of $G$, $\kappa_{\mathcal{C}, \mathcal{D}, \mathcal{C}}(I \times F_{\mathcal{C}, \mathcal{D}}(I_{\mathcal{C}})) = \tilde{\square}(1_{F_{\mathcal{C}, \mathcal{D}}} \times \iota_{\mathcal{C}})$, where $I_\mathcal{C}$ is the strict functor from the trivial 2-category that has image the object $1_\mathcal{C}$ of $G(\mathcal{C}, \mathcal{C})$ and $\iota_\mathcal{C}$ is considered as a natural transformation of 2-functors $\iota_\mathcal{C} : F_{\mathcal{C}, \mathcal{C}}(I_{\mathcal{C}}) \to I_{F_0(\mathcal{C})}$.

3. For all objects $\mathcal{C}, \mathcal{D}$ of $G$, $\kappa_{\mathcal{C}, \mathcal{D}, \mathcal{D}, \mathcal{D}}(I_D \times I) = \tilde{\square}(\iota_D \times 1_{F_{\mathcal{C}, \mathcal{D}}}).$

The lax functor is called a weak functor (or simply functor) of strict tricategories, if additionally

5. $\kappa$ is a natural isomorphism,

6. for all objects $\mathcal{C}$ of $G$, $F_{\mathcal{C}, \mathcal{C}}(1_{\mathcal{C}}) = 1_{F_0(\mathcal{C})}$ and $\iota_{\mathcal{C}} = \iota_{\mathcal{C}}^{-1} = 1_{\mathcal{C}}$.

We call the map $F_0$ and the maps of $F_{\mathcal{C}, \mathcal{D}}$ the mappings of $F$, while all the other data is called coherence morphisms or coherence data of $F$. 
Unpacking the definition of a weak functor of strict tricategories leads to the following explicit description of the coherence data for $F$. The weak 2-functors $F_{C,D}$ have as coherence data a collection of invertible 3-morphisms

$$\Phi_{\mu,\nu} : F_{C,D}(\mu) \circ F_{C,D}(\nu) \rightarrow F_{C,D}(\mu \circ \nu)$$

for all $\circ$-composable 2-morphisms $\mu$, $\nu$ in $G(C,D)$ and for each 1-morphism $G$ in $G(C,D)$ an invertible 3-morphism

$$\Phi_G : 1_{F_{C,D}(G)} \rightarrow F_{C,D}(1_G),$$

which satisfy the axioms in Definition A.1. The natural isomorphisms of weak 2-functors $F$ are characterized by invertible 3-morphisms

$$\kappa_{\mu,\nu} : F_{D,E}(\mu) \circ F_{C,D}(\nu) \rightarrow F_{C,E}(\mu \circ \nu)$$

for all $\square$-composable morphisms $\mu \in G(D,E)$, $\nu \in G(C,D)$. The conditions in Definition A.3 take the following form:

- For all 1-morphisms $G_1 \in G(D,E)$, $G_2 \in G(C,D)$ one has

$$F_{D,E}(G_1) \circ F_{C,D}(G_2) = F_{C,E}(G_1 \square G_2). \tag{66}$$

- For all 2-morphisms $\mu, \rho \in G(D,E)$ and $\nu, \tau \in G(C,D)$ such that $\mu, \rho$ and $\nu, \tau$ are $\circ$-composable, the following diagram commutes

$$\begin{array}{ccc}
(F_{D,E}(\mu) \circ F_{C,D}(\nu)) \circ (F_{D,E}(\rho) \circ F_{C,D}(\tau)) & \xrightarrow{\kappa_{\mu,\nu,\rho,\tau}} & F_{C,E}(\mu \circ \nu) \circ F_{C,E}(\rho \circ \tau) \\
\square_{F_{D,E}(\mu), F_{C,D}(\nu), F_{D,E}(\rho), F_{C,D}(\tau)} & \downarrow & \Phi_{\mu \circ \nu, \rho \circ \tau} \\
(F_{D,E}(\mu) \circ F_{C,D}(\nu)) \circ (F_{C,D}(\nu) \circ F_{C,D}(\tau)) & F_{C,E}((\mu \circ \nu) \circ (\rho \circ \tau)) \\
\Phi_{\mu, \nu, \rho, \tau} & \downarrow & \Phi_{\mu \circ \nu, \rho \circ \tau} \\
F_{D,E}(\mu \circ \rho) \circ F_{C,D}(\nu \circ \tau) & \xrightarrow{\kappa_{\mu \circ \rho, \nu \circ \tau}} & F_{C,E}((\mu \circ \rho) \circ (\nu \circ \tau)).
\end{array} \tag{67}$$

- All 1-morphisms $G \in G(D,E)$, $H \in G(C,D)$ satisfy

$$1_{F_{C,E}(G \square H)} = 1_{F_{D,E}(G)} \circ 1_{F_{C,D}(H)} \tag{68}$$

$$\begin{array}{ccc}
1_{F_{C,E}(G \square H)} & \xrightarrow{\phi_G \circ \phi_H} & 1_{F_{D,E}(G)} \circ 1_{F_{C,D}(H)} \\
\Phi_{G \square H} & \downarrow & \Phi_{G \square H} \\
F_{D,E}(1_G) \circ F_{C,D}(1_H) & \xrightarrow{\kappa_{1_G,1_H}} & F_{C,E}(1_G \square 1_H) = F_{C,E}(1_G \square 1_H).
\end{array}$$
Condition (1) in Definition A.6 states that the diagram

\[
\begin{array}{ccc}
F_{D,E}(\nu)\square F_{C,D}(\mu)\square F_{B,E}(\rho) & \overset{\kappa_{\rho,\mu,1}}{\rightarrow} & F_{D,E}(\nu)\square F_{B,D}(\mu\square \rho) \\
F_{C,E}(\nu\square \mu)\square F_{B,C}(\rho) & \overset{\kappa_{\mu,\nu,1}}{\rightarrow} & F_{B,E}(\nu\square \mu\square \rho)
\end{array}
\]

commutes for all 2-morphisms \(\rho \in \mathcal{G}(B,C)\), \(\mu \in \mathcal{G}(C,D)\), \(\nu \in \mathcal{G}(D,E)\), and conditions (2), (3) in Definition A.6 read

\[
\kappa_{\mu,1_C} = 1_{F_{C,D}(\mu)} : F_{C,D}(\mu) = F_{C,D}(1_C) \rightarrow F_{C,D}(\mu\square 1_C) = F_{C,D}(\mu),
\]

\[
\kappa_{1_D,\mu} = 1_{F_{C,D}(\mu)} : F_{C,D}(\mu) = F_{D,D}(1_D) \square F_{C,D}(\mu) \rightarrow F_{C,D}(1_D\square \mu) = F_{C,D}(\mu).
\]

The notion of a weak functor of strict tricategories in Definition A.6 thus corresponds to a trihomomorphism in [11, Def 3.3.1] for which the adjoint equivalence \(\chi\) in [11, Def 3.3.1] is a natural isomorphism given by the invertible 3-morphisms \(\kappa_{\mu,\nu}\) and for which the adjoint equivalence \(\iota\) and the invertible modifications \(\omega, \gamma, \delta\) in [11, Def 3.3.1] are trivial.

There is an obvious composition of functors of strict tricategories, that is a special case of the general composition of functors between tricategories considered in [11]. Although in general the composition of functors between tricategories is not strictly associative, this is the case for functors between strict tricategories.

**Lemma A.7.** Let \(F : \mathcal{G} \rightarrow \mathcal{H}\), \(G : \mathcal{H} \rightarrow \mathcal{K}\), \(H : \mathcal{K} \rightarrow \mathcal{L}\) be weak functors of strict tricategories. Then \(H(GF) = (HG)F\).

**Proof.** In [11, Prop. 4.2.3], explicit expressions for a natural transformation \(\alpha : H(GF) \rightarrow (HG)F\) are given. It is easy to see that for strict tricategories, the data from which \(\alpha\) is constructed consists entirely of identity mappings and morphisms. \(\square\)

In the following, we also require the notion of a strict functor of strict tricategories. The standard definition of a strict functor of strict tricategories is that of a weak functor of strict tricategories for which all 2-functors \(F_{C,D}\) in Definition A.6 are strict and all natural isomorphisms \(\kappa_{1_D,\mu,\nu}\) in Definition A.6 are identities. However, as we only consider functors between strict cubical or opcubical tricategories, we change this definition slightly to adapt it to our setting.

For this, note that a weak functor \(F : \mathcal{G} \rightarrow \mathcal{G}\) between an opcubical strict tricategory \(\mathcal{G}\) and a cubical strict tricategory \(\mathcal{G}\) can never be strict in the usual sense unless the coherence morphisms \(\square_{\mu,\nu}\) from Definition A.6 are trivial. The strictness of \(F\) and the fact that \(\mathcal{G}\) is opcubical
imply
\[ F_{\mathcal{C},\mathcal{E}}(\mu \Box \nu) = F_{\mathcal{C},\mathcal{E}}((\mu \Box 1_{H_2}) \circ (1_{G_1} \Box \nu)) \]
\[ = (F_{\mathcal{D},\mathcal{E}}(\mu) \Box 1_{F_{\mathcal{C},\mathcal{D}}(H_2)}) \Box (1_{F_{\mathcal{D},\mathcal{E}}(G_1)} \Box F_{\mathcal{C},\mathcal{D}}(\nu)) \]
for all composable 2-morphisms \( \mu : G_1 \Rightarrow G_2 \in \mathcal{G}(\mathcal{D}, \mathcal{E}) \), \( \nu : H_1 \Rightarrow H_2 \in \mathcal{G}(\mathcal{C}, \mathcal{D}) \). As \( \tilde{\mathcal{G}} \) is cubical, one has
\[ F_{\mathcal{D},\mathcal{E}}(\mu) \Box F_{\mathcal{C},\mathcal{D}}(\nu) = (1_{F_{\mathcal{D},\mathcal{E}}(G_2)} \Box F_{\mathcal{C},\mathcal{D}}(\nu)) \Box (F_{\mathcal{D},\mathcal{E}}(\mu) \Box 1_{F_{\mathcal{C},\mathcal{D}}(H_1)}). \]

The two expressions cannot agree for all composable 2-morphisms unless the coherence morphisms \( \square_{\mu,\nu} \) from Definition A.6 are trivial. For this reason, we modify the notion of strictness for the case of weak functors between opcubical and cubical strict tricategories and call such a functor strict if and only if its composition with the functor \( \Sigma \) from Corollary 2.16 is strict in the usual sense. This amounts to to the requirement that the coherence morphism \( \kappa_{\mathcal{C},\mathcal{D},\mathcal{E}} : \Box (F_{\mathcal{D},\mathcal{E}} \times F_{\mathcal{C},\mathcal{D}}) \to F_{\mathcal{C},\mathcal{E}} \Box \) from Definition A.6 is given by the coherence morphisms \( \square_{\mu,\nu} \) from Definition A.5.

**Definition A.8** (Strict functors of strict (op)cubical tricategories). Let \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) be strict tricategories that are either cubical or opcubical and \( F : \mathcal{G} \to \tilde{\mathcal{G}} \) a weak functor of strict tricategories. Then the functor \( F \) is called strict if for all objects \( \mathcal{C}, \mathcal{D} \) of \( \mathcal{G} \) the 2-functors \( F_{\mathcal{C},\mathcal{D}} \) are strict and the natural isomorphisms \( \kappa_{\mathcal{C},\mathcal{D},\mathcal{E}} \) from Definition A.6 are

- the identity morphisms in case \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) are both cubical or both opcubical,
- given by the 3-morphisms
  \[ \kappa_{\mu,\nu} = \tilde{\square}^{-1}(F_{\mathcal{D},\mathcal{E}}(1_G), F_{\mathcal{C},\mathcal{D}}(\nu)), (F_{\mathcal{D},\mathcal{E}}(\mu), F_{\mathcal{C},\mathcal{D}}(1_H)) \]
  for 2-morphisms \( \mu : G_1 \Rightarrow G_2 \in \mathcal{G}(\mathcal{D}, \mathcal{E}) \), \( \nu : H_1 \Rightarrow H_2 \in \mathcal{G}(\mathcal{C}, \mathcal{D}) \) in case \( \mathcal{G} \) is cubical and \( \tilde{\mathcal{G}} \) is opcubical,
- given by the 3-morphisms
  \[ \kappa_{\mu,\nu} = \tilde{\square}^{-1}(F_{\mathcal{D},\mathcal{E}}(\mu), F_{\mathcal{C},\mathcal{D}}(1_H)), (F_{\mathcal{D},\mathcal{E}}(1_G), F_{\mathcal{C},\mathcal{D}}(\nu)) \]
  for 2-morphisms \( \mu : G_1 \Rightarrow G_2 \in \mathcal{G}(\mathcal{D}, \mathcal{E}) \), \( \nu : H_1 \Rightarrow H_2 \in \mathcal{G}(\mathcal{C}, \mathcal{D}) \) in case \( \mathcal{G} \) is opcubical and \( \tilde{\mathcal{G}} \) is cubical.

**Definition A.9** (Natural transformations). A natural transformation \( \omega : F \to G \) between lax functors \( F = (F_0, F_{\mathcal{C},\mathcal{D}}, \kappa_{\mathcal{B}_G,\mathcal{C},\mathcal{D}}, \iota_{\mathcal{E}}) \), \( G = (G_0, G_{\mathcal{C},\mathcal{D}}, \kappa_{\mathcal{B}_G,\mathcal{C},\mathcal{D}}, \iota_{\mathcal{E}}) : \mathcal{G} \to \tilde{\mathcal{G}} \) of strict tricategories consists of the following data:

- For all objects \( \mathcal{C} \) of \( \mathcal{G} \) a 1-morphism \( \omega_{\mathcal{C}} : F_0(\mathcal{C}) \to G_0(\mathcal{C}) \)
- For all pairs of objects \( \mathcal{C}, \mathcal{D} \) of \( \mathcal{G} \) a natural transformation of weak 2-functors \( \omega_{\mathcal{C},\mathcal{D}} : (\omega_{\mathcal{E}} \Box -)F_{\mathcal{C},\mathcal{D}} \to (- \Box \omega_{\mathcal{C}})G_{\mathcal{C},\mathcal{D}} \)
such that for all triples $B, C, D$ the following diagrams commute

\[
\begin{array}{c}
(\omega_B \square -) \square (F_{B,C} \times F_{C,D}) \xrightarrow{(\omega_B \square -) \square (F_{B,C} \times F_{C,D})} (\omega_B \square -) F_{B,D} \\
\square (1 \times (\omega_B \square -)) (F_{B,C} \times F_{C,D}) \xrightarrow{\square (1 \times (\omega_B \square -)) (F_{B,C} \times F_{C,D})} (\omega_B \square -) G_{B,D} \\
\square ((\omega_C \square -) \times 1) (F_{B,C} \times F_{C,D}) \xrightarrow{\square ((\omega_C \square -) \times 1) (F_{B,C} \times F_{C,D})} (\omega_B \square -) G_{B,D} \\
\square (\omega_B \square 1) (G_{B,C} \times G_{C,D}) \xrightarrow{\square (\omega_B \square 1) (G_{B,C} \times G_{C,D})} (\omega_B \square -) G_{B,D}
\end{array}
\]

Here, $- \square \omega_C : \hat{\mathcal{G}}(G_0(C), D) \to \hat{\mathcal{G}}(F_0(C), D)$ and $\omega_D \square - : \hat{\mathcal{G}}(C, G_0(D)) \to \hat{\mathcal{G}}(C, G_0(D))$ denote the weak 2-functors defined by pre- and postcomposition with $\omega_C$ with respect to the Gray product. The natural transformations $\omega_{C,D}$ determine for all 1-morphisms $H : C \to D$ in $\mathcal{G}$ a 2-morphism $\omega_H : \omega_D \square F_{C,D}(H) \Rightarrow G_{C,D}(H) \square \omega_C$.

A natural transformation $\omega$ is called a natural isomorphism, if the 1-morphisms $\omega_C$ are invertible and all natural transformations $\omega_{C,D}$ are invertible pseudo-natural transformations.

**Definition A.10** (Modifications). Let $F = (F_0, F_{C,D}, \kappa_{C,D}, \xi_C, \lambda_C) : \mathcal{G} \to \hat{\mathcal{G}}$ and $G = (G_0, G_{C,D}, \kappa_{C,D}, \lambda_C, \iota_C) : \mathcal{G} \to \hat{\mathcal{G}}$ be functors of strict tricategories and $\omega = (\omega_C, \omega_{C,D})$, $\eta = (\eta_C, \eta_{C,D}) : F \Rightarrow G$ natural transformations. A modification $\Psi : \omega \Rightarrow \eta$ consists of the following data:

- For every object $C$ of $\mathcal{G}$ a 2-morphism $\Psi_C : \omega_C \Rightarrow \eta_C$ in $\hat{\mathcal{G}}$.
- For every pair of objects $C, D$ of $\mathcal{G}$, an invertible modification $\Psi_{C,D} : (1 \square \Psi_C) \circ \omega_{C,D} \Rightarrow \eta_{C,D} \circ (\Psi_D \square 1)$.

These determine for all 1-morphisms $H : C \to D$ a 3-morphism

$$\Psi_H : (1_{F_{C,D}(H)} \square \Psi_C) \circ \omega_H \Rightarrow \eta_H \circ (\Psi_D \square 1_{G_{C,D}(H)})$$

A modification $\Psi$ is called invertible if all 2-morphisms $\Psi_C$ are invertible.

**Definition A.11.** Two strict tricategories $\mathcal{G}, \hat{\mathcal{G}}$ are called equivalent, if there exist lax functors of strict tricategories $F : \mathcal{G} \to \hat{\mathcal{G}}$ and $G : \hat{\mathcal{G}} \to \mathcal{G}$ together with invertible pseudo-natural transformations $\eta : FG \to 1_{\hat{\mathcal{G}}}$ and $\varphi : GF \to 1_{\mathcal{G}}$. 

154
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School of Mathematical Sciences, University Park, Nottingham NG7 2RD, UK
E-mail address: john.barrett@nottingham.ac.uk

Department Mathematik, Friedrich-Alexander Universität Erlangen-Nürnberg, Cauerstrasse 11, D-91058 Erlangen, Germany
E-mail address: catherine.meusburger@math.uni-erlangen.de

Department Mathematik, Friedrich-Alexander Universität Erlangen-Nürnberg, Cauerstrasse 11, D-91058 Erlangen, Germany
E-mail address: gregor.schaumann@math.uni-erlangen.de