Blocks of Lie Superalgebras of Type $W(n)$

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0 Introduction

Let $g$ be a simple, finite-dimensional Lie superalgebra over $\mathbb{C}$. These have been classified by V. Kac. Unless $g$ is a Lie algebra or a Lie superalgebra of type $osp(1, 2n)$, the category of finite-dimensional representations of $g$ is not semisimple; q.v. [8]. This leads to a classification problem. For example, in [4], the representation theory of $sl(m, n)$ is worked out by showing it is wild when $m, n \geq 2$, and by giving an explicit description of the indecomposable finite-dimensional representations of $sl(1, n)$.

When $g$ is of type $W(0, n)$, the irreducible finite-dimensional $g$-modules are classified in [1]; in this paper, we investigate finite-dimensional indecomposable modules. We show that the category of finite-dimensional representations of $g$ is wild (i.e., as hard as classifying pairs of matrices; q.v. §2) when $g$ is of type $W(0, n)$ with $n \geq 3$. More precisely, the category of finite-dimensional representations decomposes into blocks parametrised by $(\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2$, and we show that each block is of wild type. This is done by explicitly exhibiting enough extensions between simple modules.

Secondly, we find the decomposition of the category of finite-dimensional representations into blocks. As an application, using an idea of Maria Gorelik, we prove that the centre of the universal enveloping algebra of $g$ is trivial.

When $n = 2$, there is a special isomorphism $W(0, 2) \cong sl(1, 2)$, in which case the representation theory is not wild, and the indecomposable representations are fully described in [4].

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The results in this paper are also related to results of Nakano [7] in the finite-characteristic case, for which he shows that the restricted universal enveloping algebra has a single block, and determines the structure of projective modules.

1 Conventions

By $\mathfrak{g}$ we shall denote a simple, finite-dimensional Lie superalgebra over the complex numbers. Our primary object of study will be the category of (graded) finite-dimensional $\mathfrak{g}$-modules, with even intertwiners.

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is an associative superalgebra; it satisfies the graded version of the usual universal property, thus a $\mathfrak{g}$-module is the same thing as a left $U(\mathfrak{g})$-module.

2 Quivers and representation type

A quiver is a directed graph, which consists of a set of vertices connected by various arrows (possibly including multiple arrows between two vertices, loops, etc.) Let $A$ be a unital $\mathbb{C}$-algebra, and denote by $\mathcal{M}$ some category of modules over $A$. Denote by $\text{Irr} \, \mathcal{M}$ the set of isomorphism classes of irreducible objects. The Ext-quiver of $\mathcal{M}$ is defined to be the quiver whose set of vertices is $\text{Irr} \, \mathcal{M}$ and where the number of arrows from $[S_1]$ to $[S_2]$ is equal to $\dim \text{Ext}^1(S_1, S_2)$. This is a combinatorial invariant of $\mathcal{M}$, whose structure gives information about the representation type. In particular, we have the following theorem, proven in [3]:

**Proposition 2.1.** Let $A$ be an algebra, and let $Q$ be a subquiver of the Ext-quiver of $A$. If $Q$ is a connected quiver containing no path of length 2, then there exists a fully faithful functor from the category of representations of $Q$ to the category of $A$-modules. In particular, the set of isomorphism classes of indecomposable representations of $Q$ embeds into the corresponding set for $A$.

The representation theory of quivers is well-established (see [2] for a comprehensive overview). In particular, if the underlying graph of a quiver is not of Dynkin or of affine type, then the representation theory of the quiver is wild. More precisely, a small $\mathbb{C}$-linear Abelian category $\mathcal{M}$ is defined to be wild if there exists a full exact embedding from the category of finite-dimensional representations of $\mathbb{C}\langle x, y \rangle$, the free associative
algebra on two generators, into \( \mathcal{M} \). This has the consequence that the objects of \( \mathcal{M} \) are unclassifiable in any finite sense. For example, if \( \mathcal{M} \) is wild, then it is possible to obtain any finite-dimensional algebra as the endomorphisms of some object.

3 Definition of \( W(n) \)

Assume that \( n \geq 2 \). The finite-dimensional Lie superalgebras \( W(n) := W(0, n) \) may be described as follows: let \( \bigwedge[\xi] = \bigwedge[\xi_1, \ldots, \xi_n] \) be the Grassmann algebra on \( n \) generators. It is a \( 2^n \)-dimensional associative \( \mathbb{Z} \)-graded algebra. We set \( W(n) = \text{Der} \bigwedge[\xi] \). It is a simple Lie superalgebra of dimension \( n2^n \). The \( \mathbb{Z} \)-grading on \( W(n) \) is given by \( W(n) = \bigoplus_{k=1}^{n} W_k \).

The component \( W_0 \) is canonically isomorphic to \( \text{gl}(n) \); let us describe the structure of \( W(n) \) as a \( \text{gl}(n) \)-module. Let \( \text{std} \) be the standard representation of \( \text{gl}(n) \); then there is an isomorphism \( W(n) \cong \bigwedge(\text{std}) \otimes \text{std}^* \). If \( \text{std} \) is regarded as a \( \mathbb{Z} \)-graded vector space lying in degree 1, then this is an isomorphism of \( \mathbb{Z} \)-graded vector spaces, so that \( W_k \cong \bigwedge^{k+1}(\text{std}) \otimes \text{std}^* \); note that this has at most two irreducible components.

There is a bijection between simple finite-dimensional \( W(n) \)-modules and simple finite-dimensional \( \text{gl}(n) \)-modules, realized as follows: to an irreducible finite-dimensional \( W(n) \)-module \( V \), associate the \( \text{gl}(n) \)-module \( V^{W_{\geq 1}} \), and, conversely, given a simple finite-dimensional \( \text{gl}(n) \)-module \( M \), take the irreducible quotient of the induced module \( U(W(n)) \otimes_{U(W_{\geq 0})} M \) (q.v. [1]).

It is well-known that the irreducible finite-dimensional representations of \( \text{gl}(n) \), in the ungraded case, are parametrised by weights \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0} \) for \( i \leq j \). In the super case, this is true up to parity reversal. We fix, once and for all, the Cartan subalgebra \( \mathfrak{h} \) consisting of diagonal matrices in \( \text{gl}(n) \); it is also a Cartan subalgebra of \( W(n) \). Weights are written down with respect to the basis \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) of \( \mathfrak{h}^* \) dual to \( \{E_{11}, \ldots, E_{nn}\} \).

4 Kac modules

The Lie superalgebra \( W(n) \) is defined in Section 3. It contains subalgebras \( \text{gl}(n) \subset \text{sl}(1, n) \subset W(n) \), where we will fix the identifications \( \text{gl}(n) \cong W_0 \) and \( \text{sl}(1, n) \cong W_{-1} \oplus W_0 \oplus \text{span}\{\xi_i E \mid i = 1, \ldots, n\} \), where
\[ E = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} \] is the Euler vector field. Sometimes we will omit the parameters and write \( W \) for \( W(n) \), \( \text{sl} \) for \( \text{sl}(1, n) \), and \( \text{gl} \) for \( \text{gl}(n) \). Let \( \mathfrak{g} \) be either \( W(n) \) or \( \text{sl}(1, n) \). We fix the Cartan subalgebra \( \mathfrak{h} = \mathfrak{h}_0 \) consisting of diagonal matrices in \( \text{gl}(n) \), and consider all weights with respect to \( \mathfrak{h} \). Weights are written down with respect to the basis \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) of \( \mathfrak{h}^* \) dual to \( \{ E_{11}, \ldots, E_{nn} \} \); we then have \( \mathfrak{h}^* \cong \mathbb{C}^n \) and the root lattice \( Q \cong \mathbb{Z}^n \). We fix the Borel subalgebra \( \mathfrak{b}_0 \subset \text{gl}(n) \) of upper-triangular matrices, which has positive roots \( \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \} \). The corresponding set of highest weights of finite-dimensional irreducible representations (modulo parity if we consider graded representations) is

\[ \Lambda^+ = \{ (\lambda_1, \ldots, \lambda_n) \mid \forall 1 \leq i, j \leq n \lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0} \} . \]

Define the Borel subalgebra \( \mathfrak{b}_1 \) of \( \mathfrak{g} \) by \( \mathfrak{b}_1 = \mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1} \).

Let \( L_\lambda \) denote the simple, finite-dimensional \( \text{gl}(n) \)-module with highest weight \( \lambda \).

**Definition.** The Kac module \( K(\lambda) = K_\lambda \) is the induced representation

\[ K_\lambda = \text{ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} L_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} L_\lambda \]

of \( \mathfrak{g} \), where \( \mathfrak{g}_{\geq 1} \) acts trivially on \( L_\lambda \).

The module \( K(\lambda) \) is finite-dimensional, indecomposable, and has highest weight \( \lambda \) with respect to the Borel subalgebra \( \mathfrak{b}_1 \) of \( \mathfrak{g} \).

There is a bijection between simple finite-dimensional \( \mathfrak{g} \)-modules and simple finite-dimensional \( \text{gl}(n) \)-modules, realised as follows: to an irreducible finite-dimensional \( \mathfrak{g} \)-module \( V \), associate the \( \text{gl}(n) \)-module \( V^{\mathfrak{g}_{\geq 1}} \), and, conversely, given a simple finite-dimensional \( \text{gl}(n) \)-module \( M \), extend it to \( \mathfrak{g}_{\geq 0} \), then take the unique irreducible quotient of the induced module \( U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} M \). Denote the unique irreducible quotient of \( K(\lambda) \) by \( S(\lambda) \).

For \( \mathfrak{g} = W(n) \), the irreducible finite-dimensional representations are determined explicitly in [1]. For a generic weight \( \lambda \), the representation \( K_\lambda \) is irreducible.

For \( \text{sl}(1, n) \), the situation is similar, and we will need the following facts (see [5]). Consider the \( \text{sl}(1, n) \)-
There is a generic condition called *typicality* such that

\[ \lambda \text{ is typical } \iff K^\text{sl}(\lambda) \text{ is irreducible.} \]

Moreover, \( \lambda \) is typical iff \( K^\text{sl}(\lambda) \) is projective in the category of finite-dimensional \( \text{gl}(n) \)-semisimple \( \text{sl}(1, n) \)-modules, and the category of finite-dimensional \( \text{gl} \)-semisimple \( \text{sl}(1, n) \)-modules with typical subquotients is semisimple. For typical weights, we have \( K^\text{op}_\lambda = K^\text{sl}(\lambda + 1 \cdots 1) \). Note that if \( \lambda \notin \mathbb{Z}^n \), then the weight \( \lambda \) must be typical.

Now consider certain \( W \)-modules, the *big Kac modules*

\[ K'(\lambda) = \text{ind}_{W_{-1} \oplus W_0}^W L_{\lambda} = \text{ind}_{\text{sl}_1}^{\text{sl}_0} \text{ind}_{g_{-1} \oplus g_0}^g L_{\lambda}. \]

We have \( K'(\lambda) \cong \text{ind}_{\text{sl}_1}^{\text{sl}_0} K^\text{op}_\lambda = \text{ind}_{\text{sl}_1}^{\text{sl}_0} K^\text{sl}(\lambda + 1 \cdots 1) \). The \( K'(\lambda) \) are indecomposable modules with highest weight \( \lambda \) with respect to \( b_2 = W_{-1} \oplus b_0 \).

### 5 Extensions

In this section, we show the existence of certain non-split extensions between Kac modules. These will be realised as quotients of big Kac modules. More precisely, let \( \mathcal{M} \) be the category of \( W(n) \)-modules that are direct sums of finite-dimensional semisimple \( \text{gl}(n) \)-modules and are \( \text{sl}(1, n) \)-locally finite.

**Claim.** (1) \( \mathcal{M} \) has enough projectives, i.e., every module in \( \mathcal{M} \) is a quotient of a projective module in \( \mathcal{M} \);

(2) if \( P \) is a finite-dimensional projective \( \text{sl} \)-module, then \( \text{ind}_{\text{sl}_1}^{W} P \) is projective in \( \mathcal{M} \).

**Proof.** The main point is that induction takes projectives to projectives. All modules in this proof are assumed to be direct sums of simple finite-dimensional \( \text{gl}(n) \)-modules.
The category of locally finite sl-modules has enough projectives. This may be checked as follows. In the category of semisimple locally finite gl(n)-modules, every module is projective and injective. Therefore, if Z is any such gl(n)-module, then \( \text{ind}^{\text{sl}}_{\text{gl}}(Z) \) is projective in the category of locally finite sl-modules. In particular, if \( M_0 \) is a locally finite sl(1, n)-module, then \( M_0 \) will be a quotient of the standard projective \( \text{ind}^{\text{sl}}_{\text{gl}}(M_0) \), and therefore the category of locally finite sl-modules has enough projectives.

Finally, if \( M \in \text{Ob} \mathcal{M} \), then there will exist a projective cover \( P_0 \to \text{res}_{\text{sl}} M \), therefore we can get \( M \) as a quotient \( \text{ind}^{\text{W}}_{\text{sl}} (P_0) \to \text{ind}^{\text{W}}_{\text{sl}} (M) \to M \), where the last arrow is the canonical homomorphism \( \text{ind}^{\text{W}}_{\text{sl}} (M) \to M \), which is given by \( u \otimes m \mapsto um \). This proves (1).

For (2), we first note that, in this case, a locally finite sl-module will in fact be a direct sum of finite-dimensional sl-modules, and, therefore, if \( P \) is projective in the category of finite-dimensional sl-modules, then \( P \) is projective in this slightly larger category. Then induction makes \( \text{ind}^{\text{W}}_{\text{sl}} (P) \) a projective \( W \)-module. \( \square \)

Because of the claim, since there are enough projectives, any module in \( \mathcal{M} \) has a projective resolution. If

\[
\cdots \to P_{-1} \to P_0 \to M \to 0
\]

is a projective resolution of a module \( M \), we can calculate \( \text{Ext}_{\mathcal{M}}(M, N) \) as the cohomology of the complex

\[
\text{Hom}_{\mathcal{M}}(P_0, N) \to \text{Hom}_{\mathcal{M}}(P_{-1}, N) \to \text{Hom}_{\mathcal{M}}(P_{-2}, N) \to \cdots ,
\]

where the maps are obtained by composing a homomorphism with the appropriate map in the projective resolution. The following proposition is standard (e.g., see [4]):

**Proposition 5.1.** If \( \lambda \neq \mu \), then \( \text{Ext}^1_{C(W)} (S(\lambda), S(\mu)) = \text{Ext}^1_{\mathcal{M}} (S(\lambda), S(\mu)) \).

**Proof.** Let \( E \) be an extension of \( S(\lambda) \) by \( S(\mu) \), where \( \lambda \neq \mu \); then we need to show that the centre of \( \text{gl}(n) \) acts semisimply.

Let \( \pi : W \to \text{End}_C(E) \) be the action of \( W \). Let \( Z \) be the Euler vector field, and consider \( \pi(Z) \in \text{End}_C(E) \). The Jordan-Chevalley decomposition gives \( \pi(Z) = X + Y \), where \( X \) (resp. \( Y \)) is semisimple (resp. nilpotent). Furthermore, \( \text{ad}(X) \) and \( \text{ad}(Y) \) are respectively semisimple and nilpotent, they commute, the decomposition
is unique, and each of them maps $\pi(W)$ into $\pi(W)$. Note that $\text{ad}(Z)$ is semisimple, hence $\text{ad}(\pi(Z)) \in \text{End}(\pi(W))$ is semisimple. Then $\text{ad}(Y) = \text{ad}(\pi(Z)) - \text{ad}(X)$, the difference of two commuting semisimple endomorphisms, so $\text{ad}(Y)$ also acts semisimply on $\pi(W)$, therefore $\text{ad}(Y)$ restricted to $\pi(W)$ is zero.

Therefore we see that the nilpotent part $Y$ of the action of the Euler vector field is a nilpotent $W$-module endomorphism of $E$. Since $Y$ is nilpotent, we have $Y|_{S(\mu)} = 0$ and $Y : S(\lambda) \to S(\mu)$. If $\lambda \neq \mu$, the only morphism $S(\lambda) \to S(\mu)$ is zero.

Suppose that $\lambda \notin \mathbb{Z}^n$. Then $\lambda$ is typical for $\text{sl}(1,n)$, and we have $K^r(\lambda) \cong \text{ind}_{\text{sl}}^{W} K_{\lambda}^\text{op} = \text{ind}_{\text{sl}}^{W} K_{\lambda}(\lambda + 1 \cdots 1)$. Such big Kac modules are therefore projective, according to the above claim. Now let $M$ be a finite-dimensional $W(n)$-module all of whose simple subquotients are Kac modules with nonintegral highest weight. Then we construct a projective resolution of $M$ consisting of big Kac modules. Construct the standard Koszul resolution

$$
\cdots \to P_{-2} \to P_{-1} \to P_0 \to C \to 0,
$$

where $P_{-i} = U(W) \otimes_{U(\text{sl})} \wedge^i(W/\text{sl})$, which is exact. Tensoring by $M$, and using $\text{ind}_{\text{sl}}^{W}(X) \otimes_{C} Y = \text{ind}_{\text{sl}}^{W}(X \otimes_{C} \text{res} Y)$, we get a projective resolution

$$
\cdots \to \text{ind}_{\text{sl}}^{W} ((W/\text{sl}) \otimes_{C} M) \to \text{ind}_{\text{sl}}^{W} M \to M \to 0
$$

from which we would like to calculate extensions of $M$. It is projective, since each $\wedge^i(W/\text{sl}) \otimes M$ is a sl-module all of whose simple subquotients are typical, hence projective, Kac modules.

Note that the following results are vacuous unless $n \geq 3$.

**Theorem 5.2.** Let $\lambda \in \Lambda^+ \setminus \mathbb{Z}^n$, and let $\alpha$ be a root of $(W/\text{sl})_1$ (i.e., $\alpha$ is a root of the complement of $\text{sl}$ in $W$ and $\sum_{i=1}^n \alpha_i = 1$) such that $\lambda + \alpha \in \Lambda^+$. Then $\text{Ext}^1(K_{\lambda}, K_{\lambda+\alpha}) \neq 0$. Moreover, the dimension of the space of extensions is equal to the multiplicity of $K_{\lambda+\alpha}$ in $(W/\text{sl}) \otimes K_{\lambda}$.

**Corollary 5.3.** Let $\lambda \in \Lambda^+$ be any weight, and let $\alpha$ be a root of $(W/\text{sl})_1$. Then $\dim \text{Ext}^1(K_{\lambda}, K_{\lambda+\alpha}) \geq [(W/\text{sl}) \otimes K_{\lambda} : K_{\lambda+\alpha}]$.

**Proof of Theorem 5.2.** The condition on $\lambda$ ensures that all the Kac modules involved are sl-typical. By the
of $K_\lambda$, with $P_{-i} = \text{ind}_W^W (\bigwedge^i (W/\mathfrak{sl}) \otimes K_\lambda)$. The sl-module $\bigwedge^i (W/\mathfrak{sl}) \otimes K_\lambda$ is typical, therefore semisimple, so it is a direct sum of Kac modules.

We can calculate $\text{Ext}_{\mathcal{M}}(K_\lambda, N)$ as the cohomology of the complex

$$\text{Hom}_{\mathcal{M}}(P_0, N) \rightarrow \text{Hom}_{\mathcal{M}}(P_{-1}, N) \rightarrow \cdots$$

Now,

$$\text{Hom}_{\mathfrak{sl}}(\text{ind}_{\mathfrak{sl}}^W K_\mu, K_\xi) = \text{Hom}_{\mathfrak{sl}}(K_\mu, K_\xi) = \begin{cases} 0 & \text{if } \mu \neq \xi, \\ \mathbb{C} & \text{if } \mu = \xi. \end{cases}$$

Therefore, if we substitute $N = K_{\lambda+\alpha}$ into the above complex, we get

$$\text{Hom}_{\mathfrak{sl}}(K_\lambda, K_{\lambda+\alpha}) \rightarrow \text{Hom}_{\mathfrak{sl}}(W/\mathfrak{sl} \otimes K_\lambda, K_{\lambda+\alpha}) \overset{\delta}{\rightarrow} \text{Hom}_{\mathfrak{sl}}(\bigwedge^2 (W/\mathfrak{sl}) \otimes K_\lambda, K_{\lambda+\alpha}) \rightarrow \cdots,$$

so there are no coboundaries, and we claim that every $f \in \text{Hom}_{\mathfrak{sl}}(W/\mathfrak{sl} \otimes K_\lambda, K_{\lambda+\alpha})$ gives a cocycle. Indeed, $(\delta f)(x) = f(dx)$, where $d: \bigwedge^2 (W/\mathfrak{sl}) \otimes K_\lambda \rightarrow (W/\mathfrak{sl}) \otimes K_\lambda$, and it is easy to verify that there are no nonzero sl-module maps $\bigwedge^2 (W/\mathfrak{sl}) \otimes K_\lambda \rightarrow K_{\lambda+\alpha}$: define the height of a weight $(\lambda_1, \ldots, \lambda_n)$ to be $\sum_{i=1}^n \lambda_i$. Then the height of any $\nu$ such that $\bigwedge^2 (W/\mathfrak{sl}) \otimes K_\lambda = \bigoplus_{\nu} K_\nu$ is greater than or equal to $\text{ht}(\lambda) + 2$, while $\text{ht}(\lambda + \alpha) = \text{ht}(\lambda) + \text{ht}(\alpha) = \text{ht}(\lambda) + 1$. This weight calculation shows that $K_{\lambda+\alpha}$ simply does not occur among the $K_\nu$, and we are done.

Since all the Kac modules are typical, we can calculate the decomposition of $(W/\mathfrak{sl}) \otimes K_\lambda^\mathfrak{sl}$ into a direct sum of Kac modules as the decomposition of $(W/\mathfrak{sl}) \otimes L_\lambda$ into a direct sum of $\mathfrak{gl}(n)$-modules. In particular, for $\alpha$ as in Theorem 5.2, the multiplicity of $K_{\lambda+\alpha}$ is nonzero. □

Proof of Corollary 5.3. Consider the cohomology $H^1(W, \text{Hom}(K_{\lambda+(t\cdots t)}, K_{\lambda+\alpha+(t\cdots t)}))$ as $t \in \mathbb{C}$ varies. The
complex computing this cohomology is finite-dimensional, and shifting the weights by \( t \) does not change the dimension of the components. We can therefore view it as a complex with fixed terms with a differential that depends polynomially on \( t \). By Theorem 5.2, \( \dim H^1 = [(W/\mathfrak{sl}) \otimes K_{\lambda} : K_{\lambda+\alpha}] \) for generic values of \( t \).

By semicontinuity, \( \dim H^1 \) can only increase at special values.

\[ \text{6 Blocks and wildness} \]

A block of an Abelian category \( \mathcal{M} \) is defined to be an indecomposable full Abelian subcategory that is a direct summand. Given a subset \( \Gamma \subseteq \text{Irr} \mathcal{M} \), we denote by \( \mathcal{M}(\Gamma) \) the full subcategory of \( \mathcal{M} \) consisting of objects all of whose simple subquotients are in \( \Gamma \).

In this section, we take \( \mathcal{M} \) to be the category of all finite-dimensional representations of \( W(n) \). All objects of \( \mathcal{M} \) are generalised weight modules, therefore there is a decomposition according to weight and parity: define parameters \( t = (\bar{t}, p(t)) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2 \).

We need the following facts about representations of Lie superalgebras:

**Lemma 6.1.** Let \( \mathfrak{g} \) be a Lie superalgebra (with a fixed Cartan \( \mathfrak{h} \)), and \( M \) be a generalised weight module. Then \( M = \bigoplus_{t \in \mathfrak{h}_0^* / Q} M(t) \) as a \( \mathfrak{g} \)-module, where \( Q \) is the root lattice, and \( M(t) = \bigoplus \{ M(\lambda) \mid \lambda \in t \} \subseteq M \).

**Proof.** Consider a generalised weight module \( M \), so that \( M = \bigoplus_{\alpha} M(\alpha) \) as a vector space (recall that, for \( \alpha \in \mathfrak{h}_0^* \), the subspace \( M(\alpha) \subseteq M \) is simply \( \{ v \in M \mid \forall H \in \mathfrak{h}_0 \exists n \in \mathbb{Z}_+ (H - \alpha(H))^n v = 0 \} \) in this case).

We have the following simple fact: if \( \lambda, \mu \in \mathfrak{h}_0^* \), then \( U(\mathfrak{g})^{(\lambda)} M(\mu) \subseteq M(\lambda + \mu) \). An immediate consequence is that the \( M(t) \), defined above, are submodules.

**Lemma 6.2.** Let \( \mathfrak{g} \) be as in the previous lemma, and assume that, for every \( \alpha \in \Delta \), we have \( \dim \mathfrak{g}^{(\alpha)} = (0|k) \) or \( (k|0) \). Suppose \( M \) is a generalised weight module whose support is contained in a single \( Q \)-coset \( t = \lambda + Q \).

Then there exists a parity function \( \sigma : M \to M \), commuting with the action of \( \mathfrak{g} \), such that \( M = M' \oplus M'' \), defined by \( M' = \{ v \in M \mid \sigma(v) = v \} \), \( M'' = \{ v \in M \mid \sigma(v) = -v \} \).

**Proof.** First, define a parity \( p : \Delta \to \mathbb{Z}_2 \) by \( p(\alpha) = p(X_\alpha) \) for some \( X_\alpha \in \mathfrak{g}^{(\alpha)} \); this is well-defined, by our hypothesis. It extends linearly to a function \( p : Q \to \mathbb{Z}_2 \).
Now, suppose \( M \) is a generalised weight module whose support is contained in a single \( Q \)-coset \( t = \lambda + Q \). Shift the parity function to \( p: t \to \mathbb{Z}_2 \) by setting \( p(\lambda + \alpha) = p(\lambda) + p(\alpha) \), where \( p(\lambda) \in \mathbb{Z}_2 \) is fixed arbitrarily. Consider the linear map \( \sigma: M \to M \), uniquely defined by requiring that, if \( v \in M_{d}^{(\mu)} \), then \( \sigma(v) = (-1)^{p(\lambda) + d}v \). Finally, note that, if \( X_\alpha \in g(\alpha) \), then \( X_\alpha v \in M_{d+p(\alpha)}^{(\mu+\alpha)} \), so \( \sigma(X_\alpha v) = (-1)^{p(\lambda)-d}X_\alpha v = X_\alpha \sigma(v) \). Therefore \( \sigma \) commutes with the action of \( g \), and \( M \) breaks up into a direct sum of two submodules, determined by this new parity. 

**Proposition 6.3.** Let \( g = W(n) \), and let \( M \) be a generalised weight \( g \)-module. Then \( M \) decomposes, as a \( g \)-module, into a direct sum

\[
M = \bigoplus_{i \in \mathbb{C}/\mathbb{Z}_2} M(i) = \bigoplus_{i \in \mathbb{C}/\mathbb{Z}_2} M(i)' \oplus M(i)'',
\]

defined by Lemmas 6.1 and 6.2.

Since we are interested in finite-dimensional modules, we see that \( \Gamma_i \) consists of simple modules whose highest weight \( \lambda \) satisfies \( \lambda_1 \equiv \bar{t} \) \( (\mod \mathbb{Z}) \), where \( \bar{t} \in \mathbb{C}/\mathbb{Z} \). The categories \( M(\Gamma_i') \) and \( M(\Gamma_i'') \) are equivalent via parity-reversal, and for what follows we will not worry about whether a highest weight is “even” or “odd”, that being determined according to Lemma 6.2.

**Theorem 6.4.** The decomposition

\[
\mathcal{M} = \bigoplus_{t \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2} \mathcal{M}_t
\]

is the block decomposition of \( \mathcal{M} \), i.e., the categories \( \mathcal{M}_t = \mathcal{M}(\Gamma_t) \) are indecomposable.

**Proof.** We define a relation \( \rightsquigarrow \) on the set of highest weights, such that \( \lambda \rightsquigarrow \mu \) implies the existence of a finite-dimensional indecomposable having both \( S(\lambda) \) and \( S(\mu) \) as subquotients, and, consequently, that \( S(\lambda) \) and \( S(\mu) \) are in the same \( \mathcal{M} \)-block. Finally, we show that if \( S(\lambda), S(\mu) \in \Gamma_t \), i.e., if \( \lambda_1 \equiv \mu_1 \) \( (\mod \mathbb{Z}) \), then we can get from \( \lambda \) to \( \mu \) with a finite number of intermediate steps.

If \( \bar{t} \notin \mathbb{Z} \), then all Kac modules in \( \mathcal{M}_t \) are simple. In that case, by Theorem 5.2, there exists a non-split extension of \( S(\lambda) \) by \( S(\lambda + \varepsilon_i) \) as long as \( \lambda + \varepsilon_i \in \Lambda^+ \).

For the atypical case \( \bar{t} \in \mathbb{Z} \), Corollary 5.3 still ensures that there is an indecomposable module with
subquotients $K(\lambda)$ and $K(\lambda + \varepsilon_i)$, and hence both $S(\lambda)$ and $S(\lambda + \varepsilon_i)$ are subquotients, for $1 \leq i \leq n$ such that $\lambda + \varepsilon_i \in \Lambda^+$.

The relation $\lambda \rightsquigarrow \mu$ is now defined as follows: set $\lambda \rightsquigarrow \mu$ if $\mu = \lambda + \varepsilon_i$ for some $1 \leq i \leq n$. We have established that $\lambda \rightsquigarrow \mu$ implies that $\lambda$ and $\mu$ are in the same block. Finally, it is clear that the closure of $\rightsquigarrow$ to an equivalence relation on $\Lambda^+$ has equivalence classes which are exactly the cosets $(\lambda + Q) \cap \Lambda^+$. 

**Theorem 6.5.** Each block $M_t$ is wild.

**Proof.** Let $\lambda \in \Lambda^+$ be a weight such that $K(\lambda) \in \text{Ob}(M_t)$, $K(\lambda)$ is simple, $\lambda + \alpha \in \Lambda^+$ for every root $\alpha$ of $(W/\text{sl})_1$, and all the $K(\lambda + \alpha)$ are also simple. For example, any sufficiently dominant weight, i.e., $\lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_n$, with $\lambda_i \equiv \bar{t} \pmod{\mathbb{Z}}$, will do. Then, by Corollary 5.3, there exists a nontrivial extension of $K(\lambda) = S(\lambda)$ by $K(\lambda + \alpha) = S(\lambda + \alpha)$.

Therefore, the Ext-quiver of each block contains a subquiver consisting of a vertex $\lambda$ with arrows from it to $\lambda + \alpha$ for each root $\alpha$ of $(W/\text{sl})_1$. Not counting multiplicities, there are $3\binom{n}{3} + n$ such roots, namely, $\varepsilon_i + \varepsilon_j - \varepsilon_k$ with $1 \leq i, j, k \leq n$ and $i \neq j$. Since $n \geq 3$, we always have $3\binom{n}{3} + n > 5$, and the resulting quiver is already wild (q.v. Proposition 2.1):

![Quiver Diagram]

The original version of this paper used a much more cumbersome method to establish the block decomposition. The advantage of the present approach is that much less calculation is necessary. Also, it seems that the argument in Theorem 5.3 may be refined to calculate all extensions between two (generic) Kac modules, and that the projective resolution will give an easy proof of

**Conjecture 6.6.** All blocks not containing the trivial representation are equivalent.
7 The centre of $U(W(n))$

Consider the standard embedding $\mathfrak{sl}(1, n) \hookrightarrow W(n)$. Let $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{gl}(n) \subset \mathfrak{sl}(1, n)$ be the standard Cartan, and fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $W(n)$ such that $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ is also the positive Borel of a triangular decomposition of $\mathfrak{sl}(1, n)$.

Define the Verma modules

$$\mathfrak{sl}M(\lambda) = U(\mathfrak{sl}) \otimes_{U(\mathfrak{b}_+)} C_\lambda$$

$$M(\lambda) = U(W) \otimes_{U(\mathfrak{b}_+)} C_\lambda.$$

Induction by stages gives $M(\lambda) = U(W) \otimes_{U(\mathfrak{sl})} \mathfrak{sl}M(\lambda)$. Let $\mathfrak{sl}L(\lambda)$ be the unique irreducible quotient of $\mathfrak{sl}M(\lambda)$.

The following result is proved in [6], Section 3:

**Lemma 7.1.** Let $S \subseteq \mathfrak{h}_0^*$. Then

$$\bigcap_{\mu \in S} \text{Ann}_{U(\mathfrak{sl})} \mathfrak{sl}L(\mu) = 0$$

if and only if $S$ is Zariski dense in $\mathfrak{h}_0^*$.

**Corollary 7.2.** For any Zariski dense subset $S \subseteq \mathfrak{h}_0^*$ one has

$$\bigcap_{\mu \in S} \text{Ann}_{U(W)} M(\mu) = 0.$$

**Proof.** Suppose $S$ is Zariski dense. First of all, $\text{Ann}_{U(\mathfrak{sl})} \mathfrak{sl}M(\mu) \subseteq \text{Ann}_{U(\mathfrak{sl})} \mathfrak{sl}L(\mu)$ and therefore $\bigcap_{\mu} \text{Ann}_{U(\mathfrak{sl})} \mathfrak{sl}M(\mu) = 0$ by Lemma 7.1. Next, applying the Poincaré-Birkhoff-Witt Theorem, produce a basis $\{X_i\}$ of $U(W)$ over $U(\mathfrak{sl})$, and write $u \in \text{Ann}_{U(W)} M(\mu)$ as $u = \sum_i X_i Y_i$, with the $Y_i \in U(\mathfrak{sl})$. Applying $u$ to $v \in 1 \otimes \mathfrak{sl}M(\mu)$ gives

$$\sum_i X_i Y_i v = 0,$$

and, since the $X_i$ are linearly independent, each $Y_i v = 0$ and therefore $Y_i \in \text{Ann}_{U(\mathfrak{sl})} \mathfrak{sl}M(\mu)$. This shows
that \( \text{Ann}_{U(W)} M(\mu) \subseteq U(W) \text{Ann}_{U(sl)} M(\mu) \) and that

\[
\bigcap_{\mu} \text{Ann}_{U(W)} M(\mu) \subseteq U(W) \bigcap_{\mu} \text{Ann}_{U(sl)} M(\mu) = 0.
\]

\[ \square \]

**Proposition 7.3.** We have \( Z(U(W(0,n))) = \mathbb{C} \).

This is an immediate corollary of Corollary 7.2 and Lemma 7.4.

**Lemma 7.4.** If \( u \in Z \), then there exists a scalar \( c \in \mathbb{C} \) such that \( u - c \in \bigcap_{\mu \in \Gamma} \text{Ann} M(\mu) \), where \( \Gamma \subset \mathfrak{h}^* \) is Zariski dense.

**Proof.** For any \( \mu \in \mathfrak{h}^* \), we have \( \text{End}_{\mathfrak{h}} M(\mu) = \mathbb{C} \), therefore \( c \) acts by some scalar \( c_\mu \) on \( M(\mu) \). It therefore also acts by multiplication by \( c_\mu \) on the finite-dimensional quotient \( V_\mathfrak{h}(\mu) \) of \( M(\mu) \). Furthermore, if \( V_\mathfrak{h}(\mu) \) and \( V_\mathfrak{h}(\mu') \) are in the same block, then \( c_\mu = c_{\mu'} \). However, by Theorem 6.4, any block \( \Gamma \) is already a Zariski dense subset of \( \mathfrak{h}^* \).

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