AN EILENBERG–GANEA PHENOMENON FOR ACTIONS WITH VIRTUALLY CYCLIC STABILISERS

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ABSTRACT. In dimension 3 and above, Bredon cohomology gives an accurate purely algebraic description of the minimal dimension of the classifying space for actions of a group with stabilisers in any given family of subgroups. For some Coxeter groups and the family of virtually cyclic subgroups we show that the Bredon cohomological dimension is 2 while the Bredon geometric dimension is 3.

1. INTRODUCTION AND PRELIMINARIES

For a discrete group $G$, a family of subgroups $\mathfrak{F}$ is a non-empty collection of subgroups of $G$ that is closed under conjugation and taking subgroups. If $\mathfrak{F}$ is a family of subgroups of $G$ then a model for $E_{\mathfrak{F}}G$, the classifying space for $G$-actions with stabilisers in $\mathfrak{F}$, is a $G$-CW-complex $X$ such that for $H \leq G$, the fixed point set $X^H$ is empty if $H \notin \mathfrak{F}$ and is contractible if $H \in \mathfrak{F}$. For any $G$ and $\mathfrak{F}$ there is always a model for $E_{\mathfrak{F}}G$ and it is unique up to equivariant homotopy.

In the case when $\mathfrak{F}$ consists of just the trivial group, $E_{\mathfrak{F}}G$ is the same thing as $EG$, the universal cover of an Eilenberg–Mac Lane space for $G$. In the case when $\mathfrak{F}$ is the family $\mathfrak{F}_{\text{fin}}(G)$ (respectively the family $\mathfrak{F}_{\text{vc}}(G)$) of all virtually cyclic subgroups of $G$ we write $E_{\mathfrak{F}}G$ (respectively $E\mathfrak{F}G$) for $E_{\mathfrak{F}}G$. The minimal dimension of any model for $E_{\mathfrak{F}}G$ is denoted by $\text{gd}_{\mathfrak{F}}G$ and is called the Bredon geometric dimension of $G$.

Homological algebra over the group ring $\mathbb{Z}G$ can be used to study models for $EG$, and Bredon cohomology is the natural generalisation for studying models for $E_{\mathfrak{F}}G$. In Bredon cohomology the orbit category $\mathcal{O}_{\mathfrak{F}}G$ replaces the group $G$. The orbit category $\mathcal{O}_{\mathfrak{F}}G$ is the category with objects the $G$-sets $G/H$ with $H \in \mathfrak{F}$ and $G$ maps as morphisms. A (right) $\mathcal{O}_{\mathfrak{F}}G$-module is then a contravariant functor from the orbit category $\mathcal{O}_{\mathfrak{F}}G$ to the category of abelian groups. In the case when $\mathfrak{F}$ consists of just the trivial group, $\mathcal{O}_{\mathfrak{F}}G$ is a category with one object and morphism set $G$ and $\mathcal{O}_{\mathfrak{F}}G$-modules are the same as $\mathbb{Z}G$-modules.

The category of $\mathcal{O}_{\mathfrak{F}}G$-modules is an abelian category with enough projectives. The Bredon cohomological dimension $\text{cd}_{\mathfrak{F}}G$ is defined to be the projective dimension of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module $\mathbb{Z}$, which takes the value $\mathbb{Z}$ on any object of $\mathcal{O}_{\mathfrak{F}}G$ and which maps any morphism to the identity. The derived functors of the morphism functor in the category of Bredon modules over $\mathcal{O}_{\mathfrak{F}}G$ are denoted by $\text{Ext}^*_{\mathcal{O}_{\mathfrak{F}}G}(-,-)$. The Bredon cohomology groups of $G$ with coefficients the $\mathcal{O}_{\mathfrak{F}}G$-module $M$ are the abelian groups $H^*_G(\mathbb{Z}; M) = \text{Ext}^*_G(\mathbb{Z}, M)$. For details on Bredon cohomology we refer to [12] or [9].

If the family $\mathfrak{F}$ consists of the trivial subgroup only, then $\text{gd}_{\mathfrak{F}}G$ is the minimal dimension $\text{gd}G$ an Eilenberg–Mac Lane space for $G$ can have. If $\mathfrak{F}$ is the family $\mathfrak{F}_{\text{fin}}(G)$ (respectively $\mathfrak{F}_{\text{vc}}(G)$) then we use the notation $\widehat{\text{gd}}G$ (respectively $\underline{\text{gd}}G$) for $\text{gd}_{\mathfrak{F}}G$. 
As in the classical case a model for $EFG$ gives rise to a resolution of the trivial $OFG$-module $Z$ by projective $OFG$-modules. Therefore $\text{cd}_{OFG} \leq \text{gd}_{OFG}$ in general. If $\text{cd}_{OFG} \geq 3$, then $\text{cd}_{OFG} = \text{gd}_{OFG}$. In the classical case, that is when $\mathfrak{F} = \{1\}$ consists only of the trivial subgroup, this is due to Eilenberg–Ganea [7]. For $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$ this was proved in [12] and this proof generalises to arbitrary families $\mathfrak{F}$, cf. Theorem 0.1 in [13, p. 294]. In the classical case, that is when $F = \{1\}$ consists only of the trivial subgroup, this is due to Eilenberg–Ganea [7]. For $F = F_{\text{fin}}(G)$ this was proved in [12] and this proof generalises to arbitrary families $F$, cf. Theorem 0.1 in [13, p. 294]. In the classical case, the statement that the cohomological and geometric dimension always agree is known as the Eilenberg–Ganea Conjecture. Since the work of Stallings [14] and Swan [15] implies that $\text{cd}_{G} = 1$ if and only if $\text{gd}_{G} = 1$, this conjecture can only be falsified by a group $G$ with $\text{cd}_{G} = 2$ but $\text{gd}_{G} = 3$.

In [1] right-angled Coxeter groups $W$ such that $\text{cd}_{W} = 2$ but $\text{gd}_{W} = 3$ were exhibited. We show here that some, but not all, of these examples have a similar property for actions with virtually cyclic stabilisers.

**Main Theorem.** Let $(W, S)$ be a right-angled Coxeter system for which the nerve $L = L(W, S)$ is an acyclic 2-complex that cannot be embedded in any contractible 2-complex.

- If $W$ is word hyperbolic, then $\text{cd}_{W} = 2$ and $\text{gd}_{W} = 3$.
- If $W$ is not word hyperbolic, then $\text{cd}_{W} = \text{gd}_{W} \geq 3$.

A right angled Coxeter group $W$ is word hyperbolic if and only if its nerve $L$ satisfies the so called “flag no squares condition”, cf. [4, p. 233]. By Proposition 2.1 of [5] the “flag no squares conditions” puts no restriction on the homeomorphism type of the 2-complex $L$ (or see [1, p. 498] an explicit example for a suitably triangulated $L$). Therefore it follows from our theorem, that the Bredon analogue of the Eilenberg–Ganea Conjecture is false for the family of virtually cyclic subgroups.

The proof of the non-word hyperbolic case of our Main Theorem is the easy part and is described in Section 3. The word hyperbolic case is Theorem 6 and 7 combined.

As mentioned before, in the classical case $\text{cd}_{F} = 1$ implies $\text{gd}_{F} = 1$ by the work of Stallings and Swan. It follows from Dunwoody’s Accessibility Theorem [6], that the same implication is true in the case that $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$. In the light of this one may ask, whether this implication also holds in the case that $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$. The first author obtained in his thesis a positive answer for countable, torsion-free, soluble groups [9, p. 127]. In this class, the groups $G$ with $\text{cd}_{G} = 1$ are precisely the subgroups of the rational numbers which are not finitely generated and for these group $\text{gd}_{G} = 1$ holds. However, a general answer to this question is still open.

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2. **Coxeter Groups and the Davis Complex**

A *Coxeter matrix* is a symmetric matrix $M = (m_{st})$ indexed by a finite set $S$ and with entries integers or $\infty$ subject to the conditions that for all $s, t \in S$

1. $m_{st} = 1$ if $s = t$, and
Associated to a Coxeter matrix $M$ one has the *Coxeter group* $W$ given by the presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ with } m_{st} \neq \infty \rangle.$$  

The Coxeter group $W$ is *right-angled* if the finite off-diagonal entries of the Coxeter matrix are all equal to $2$. The elements of $S$ are called the fundamental Coxeter generators of the Coxeter group $W$ and the pair $(W, S)$ is called a *Coxeter system*. If $T \subset S$, then $W_T$ denotes the subgroup of $W$ generated by $T$ and these subgroups are called *special*.

The *nerve* $L = L(W, S)$ of a Coxeter system $(W, S)$ is the simplicial complex with vertex set $S$ and whose simplices are the non-empty subsets $T \subset S$ for which the special subgroup $W_T$ is finite.

Given a Coxeter system $(W, S)$ the *Davis Complex* $\Sigma = \Sigma(W, S)$ is a contractible simplicial complex on which $W$ acts with finite stabilisers; the action of the fundamental generators $S$ is by reflections. This complex has been introduced in [3] and it can be interpreted as the barycentric subdivision of a cell complex where the cells are in bijective correspondence with the cosets of finite special subgroups of $W$. This cell complex admits in a natural way a piecewise Euclidean metric and this metric can be shown to be CAT(0). The links of the 0-cells of this complex can be identified with the nerve $L$. The subcomplex generated by the cells corresponding to the maximal finite special subgroups is denoted by $K$. It is a fundamental domain of the action of $W$ and it can be realised as the cone of $L$, where $L$ is identified with the boundary $\partial K$ in $\Sigma$. For details see [4].

If $(W, S)$ is a right angled Coxeter system, then its nerve is a flag complex [4, p. 125]. Conversely, if we are given a finite flag complex $L$ and for $s \neq t$ set $m_{st} = 2$ if $s$ and $t$ are adjacent in $L$ and set $m_{st} = \infty$ if no edge connects $s$ and $t$ in $L$.

### 3. The Non-Hyperbolic Case

It suffices to show that $\text{cd} W \geq 3$. For this it is enough to show that $W$ contains a subgroup $H$ with $\text{cd} H \geq 3$. Since $W$ is not word hyperbolic it contains a subgroup isomorphic to $\mathbb{Z}^2$ [4, p. 241].

We show that $E\mathbb{Z}^2 = 3$ using an explicit 3-dimensional model $X$ for $E\mathbb{Z}^2$, which was first described by Farrell. See [8] for a general construction containing this as a special case, or see [11] for a description of $X$ and a computation of $H_r(X/\mathbb{Z}^2; \mathbb{Z})$ from which it follows that $H^3(X/\mathbb{Z}^2; \mathbb{Z})$ is a countable direct product of copies of $\mathbb{Z}$. Theorem 4.2 in [9, p. 83] states that $H^3(X/\mathbb{Z}^2) \cong H^3_{\delta_c}(\mathbb{Z}^2; \mathbb{Z})$. Hence it follows that $\text{cd} \mathbb{Z}^2 = 3$.

### 4. The Geometric Dimension in the Hyperbolic Case

Given a Coxeter system $(W, S)$ and a $W$-space $X$ we set

$$X^\# = \bigcup_{s \in S} X^s$$

and

$$X^{\text{sing}} = \{ x \in X \mid W_x \neq 1 \}.$$
Clearly $X^\# \subset X^\text{sing}$.

**Lemma 1.** Let $K \subset \Sigma$ be the fundamental chamber of $\Sigma$ and let $s \in S$. Then both $K$ and $K \cup sK$ are convex subsets of $\Sigma$.

**Proof.** For each $t \in S$ the fixedpoint set $\Sigma_t$ separates $\Sigma$ into two connected half spaces. Denote by $H_t^-$ the half space which does not intersect $K$ and denote by $H_t^+$ the complement of $H_t^-$. Then $H_t^+$ is a convex subset of $\Sigma$ containing $K$. Then $K = \bigcap_{t \in S} H_t^+$ is a convex subset of $\Sigma$. Finally, $K \cap sK$ is convex since $K \cup sK = K \cap sK_0$ where $K_0$ is the convex set $K_0 = \bigcap_{t \in S \setminus \{s\}} H_t^+$. □

**Lemma 2.** Let $X$ be a model for $\mathbb{E}W$. Then $X^\#$ is homotopy equivalent to $L$.

**Proof.** Since $X$ is $W$-homotopy equivalent to $\Sigma$ it follows that $X^\#$ is homotopy equivalent to $\Sigma^\#$. Thus it is enough that $\Sigma^\#$ is homotopy equivalent to $L$.

Let $K$ be the fundamental chamber of $\Sigma$. Then $K$ is complete and compact and due to Lemma 1 also convex. Therefore, since $\Sigma$ is a CAT(0) space, there exists a retraction of $\Sigma$ onto $K$ which sends every point $x \in \Sigma \setminus K$ to the unique point $\pi(x)$ of $K$ which is nearest to $x$, cf. [2, p.176f.].

Let $K^S$ the union of all mirrors of $K$, that is

$$K^S = \{x \in K \mid x \in K \cap sK \text{ for some } s \in S\},$$

cf. [4, p. 63, p. 127]. The set $K^S$ is homotopy equivalent to $L$ [4, p. 127].

Let $s \in S$ and $x \in X^S \setminus K$. Let $y = \pi(x)$. Then $sy \in sK$ and since $K \cup sK$ is convex it follows that the midpoint $m$ of the geodesic joining $y$ and $sy$ is contained in $K \cup sK$. Since $y$ and $sy$ have the same distance from $K \cap sK$ it follows that $m \in K \cap sK$. In particular $m \in K$. Since $x \in X^S$ it follows that $d(x, y) = d(sx, sy) = d(x, sy)$. Since the metric of $\Sigma$ is CAT(0) it follows that $d(x, m) \leq \max(d(x, y), d(x, sy)) = d(x, y)$. By the uniqueness of the point $\pi(x)$ it follows that $m = y$. Hence $y \in K^S$.

It follows that the homotopy equivalence $\Sigma \simeq K$ restricts to a homotopy equivalence $\Sigma^\# \simeq K^S$. Thus $X^\# \simeq L$. □

**Remark 3.** The above lemma could be used to give a slightly different proof of the main assertion of Proposition 4 of [1, p. 497].

**Lemma 4.** Let $X$ be a model for $\mathbb{E}W$. If $W$ is word hyperbolic, then $X^\#$ is homotopy equivalent to

$$L \vee \bigvee_{i \in I} S^1$$

where the index set $I$ consists of all maximal infinite virtually cyclic subgroups of $W$ which contain at least two non-commuting Coxeter generators.

**Proof.** Let $Y$ be the model for $\mathbb{E}W$ which is obtained from $\Sigma$ as described in [11]. This construction yields for every maximal infinite virtually cyclic subgroup $H$ of $W$ a 1-dimensional model $Z_H$ for $\mathbb{E}H$ together with an $H$-equivariant embedding $f_H$: $Z_H \to \Sigma$. We identify $Z_H$ with its image in $\Sigma$ under this embedding. Then $Y$ is obtained by coning of the sets $Z_H$ and extending the $W$-action suitably.

Since $X$ is $W$-homotopy equivalent to $Y$ it follows that $X^\#$ is homotopy equivalent to $Y^\#$. The set $Y^\#$ is obtained from $\Sigma^\#$ by coning of the intersection $\Sigma^\# \cap Z_H$ for every maximal infinite virtually cyclic subgroup $H$ of $W$.

Let $s, t \in S$ such that $s, t \in H$ for some maximal infinite virtually cyclic subgroup $H$ of $W$. Then $x \in Z_H$ can be a common fixed point of $s$ and $t$ if and only if $s$ and $t$ commute. In particular $Z_H \cap X^\#$ can consist of at most 2 points as a virtually
cyclic subgroup of $W$ cannot contain more than 2 pairwise non-commuting Coxeter generators. Coning of a singleton set of path connected space does not change its homotopy type. And coning of a subset of a path connected space which has two points is homotopy equivalent to attaching a $S^1$ to it. Hence the claim of the of the lemma follows.

Lemma 5. Let $(X,A)$ be CW-pair complexes and let $B$ be a CW-complex which is homotopy equivalent to $A$. Then there exists a CW-pair $(Y,B)$ which is homotopy equivalent to $(X,A)$ such that the cells of $X \setminus A$ are dimension wise in a 1-to-1 correspondence to the cells of $Y \setminus B$.

Proof. This follows directly from Theorem 4.1.7 in [10, p. 104].

Theorem 6. Let $(W,S)$ be a Coxeter system with $W$ word hyperbolic and such that the nerve $L(W,S)$ of this Coxeter system is an acyclic complex, which is not homotopy equivalent to a subcomplex of a contractible 2-complex. Then $\text{gd} W = 3$.

Proof. Assume towards a contradiction that there exists a 2-dimensional model $X$ for $EW$. Then $X^d$ is homotopy equivalent to $L \cup S^1$ by Lemma 4. By Lemma 5 there exists a 2-dimensional CW-complex $Y$ which is homotopy equivalent to $X$ and which contains $L \cup S^1$. In particular $L$ is a subcomplex of $Y$ contradicting the assumption that $L$ does not embed into a contractible 2-complex. Thus $\text{gd} W \geq 3$.

On the other hand, the Davis complex $\Sigma$ is a model for $EW$ and $\dim \Sigma = \dim L + 1 = 3$. Since $W$ is word hyperbolic we can elevate $\Sigma$ to a model for $EW$ by attaching orbits of cells in dimension 2 and less, cf. [11]. Thus $\text{gd} W \leq 3$ and equality holds.

5. The Cohomological Dimension

Theorem 7. Let $(W,S)$ be a Coxeter system with $W$ word hyperbolic and such that the nerve $L(W,S)$ of this Coxeter system is an acyclic complex which is not homotopy equivalent to a subcomplex of a contractible 2-complex. Then $\text{cd} W = 2$.

Proof. Let $\mathfrak{F}$ be the family of virtually cyclic subgroups of $W$. Let $Z$ be the submodule of the trivial $\mathcal{O}_W$-module given by $Z(G/H) = \mathbb{Z}$ for any finite subgroup $H$ of $W$ and which is 0 otherwise. The complex $\Sigma^{\text{sing}}$ is acyclic and 2-dimensional by [1] and it follows that $\mathcal{C}_\mathfrak{F}(\Sigma^{\text{sing}})$ gives a projective resolution of $Z$ of length 2. Thus $\text{pd} Z \leq 2$.

On the other hand, if $X$ is a model for $EW$, then a model $Y$ for $EW$ can be obtained from $X$ by attaching orbits of cells in dimension 2 and less [11, Proposition 9]. It follows that $\mathcal{C}_\mathfrak{F}(Y,X)$ gives a free resolution of $Q = \mathbb{Z}/Z$ of length 2. Thus $\text{pd} Q \leq 2$.

Consider the short exact sequence

$$0 \rightarrow Z \rightarrow \mathbb{Z} \rightarrow Q \rightarrow 0$$

of $\mathcal{O}_W$-modules. Since $\text{pd} Z$ and $\text{pd} Q$ are bounded by 2 it follows by the Horseshoe Lemma that $\text{pd} Z \leq 2$, that is $\text{cd} W \leq 2$.

On the other hand, it follows from [11, Corollary 16] that the quotient space $EW/W$ has non-trivial cohomology in dimension 2, and thus $H^2_\mathfrak{F}(W;\mathbb{Z})$ must be non-trivial too, cf. Theorem 4.2 in [9, p. 83]. As a consequence we get $\text{cd} W \geq 2$ and therefore the claim follows.
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