Non-Local Thirring model with backward and umklapp interactions

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Abstract

We extend a non local and non covariant version of the Thirring model in order to describe a many-body system with backward and umklapp scattering processes. We express the vacuum to vacuum functional in terms of a non trivial fermionic determinant. Using path-integral methods we find a bosonic representation for this determinant which allows us to obtain an effective action for the collective excitations of the system. By introducing a non local version of the self-consistent harmonic approximation, we get an expression for the gap of the charge-density excitations as functional of arbitrary electron-electron potentials. As an example we also consider the case of a non contact umklapp interaction.

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1 Introduction

In recent years there has been much interest in the study of low-dimensional field theories. In particular they are useful to describe the behavior of strongly anisotropic physical systems in condensed matter, such as organic conductors [1], charge transfer salts [2] and quantum wires [3]. Probably the two most widely studied 1d systems are the Hubbard model [4] and the so-called "g-ology" model [5]. They are known to display the Luttinger liquid [6] behavior characterized by spin-charge separation and by non-universal (interaction dependent) power-law correlation functions. In a recent series of papers [7], [8] an alternative, field-theoretical approach was developed to consider this problem. In these works a non-local and non-covariant version of the Thirring model [9] was introduced, in which the fermionic densities and currents are coupled through bilocal, distance-dependent potentials. This non-local Thirring model (NLT) contains the Tomonaga-Luttinger (TL) model [10] as a particular case. Although it constitutes an elegant framework to analyze the 1d many-body problem, one serious limitation appears if one tries to make contact with real systems. Indeed, one has to recall that the building blocks of the NLT are the forward-scattering (fs) processes which are supposed to dominate the scene only in the low transferred momentum limit. This means that in its present form it can only provide a very crude description of the Luttinger liquid equilibrium and transport properties. In ref. [11] we started to develop an improved version of the NLT in which larger momentum transfers are taken into account by including back-scattering (bs) [12] and umklapp (us) [13]. This formulation led us to a highly non-trivial action whose physical content was very hard to extract. In the present paper we reconsider this problem from a different point of view which allowed us to get insight on the spectrum of the collective modes. By using a non local version of the self-consistent harmonic approximation (SCHA) [14] we were able to obtain general expressions for the bosonic gaps as functionals of arbitrary forward, backward and umklapp scattering coupling functions. In Section 2 we present the model and review the steps that allow us to write a purely bosonic action describing the dynamics of collective excitations. In Section 3 we analyze the bosonic action by using a self-consistent harmonic approximation. We derive our main formal result, i.e. the general formula for the gap, valid for pure backscattering ($g_1 \neq 0, g_3 = 0$) and pure umklapp scattering ($g_3 \neq 0, g_1 = 0$). In Section 4, as an example, we study in some detail the case of a non local umklapp interaction. In Section 5 we present our conclusions. In the Appendix we review the main points of the non local SCHA.

2 The Model

In this section we begin the study of an extended version of the NLT which includes the contribution of forward, backward and umklapp scattering. Following the formulation
proposed in [1] we shall attempt to describe these interactions by means of a fermionic (1+1)-dimensional Quantum Field Theory with Euclidean action given by

\[ S = S_0 + S_{fs} + S_{bs} + S_{us} \]  

where

\[ S_0 = \int d^2x \bar{\Psi} i \partial \Psi \]  

is the unperturbed action associated to a linearized free dispersion relation. The contributions of the different scattering processes will be written as

\[ S_{fs} = -\frac{g^2}{2} \int d^2x d^2y (\bar{\Psi} \gamma_\mu \Psi)(x) V(\mu)(x, y) (\bar{\Psi} \gamma_\mu \Psi)(y) \]  

and

\[ S_{bs} + S_{us} = -\frac{g'^2}{2} \int d^2x d^2y (\bar{\Psi} \Gamma_\mu \Psi)(x) U(\mu)(x, y)(\bar{\Psi} \Gamma_\mu \Psi)(y) \]  

where the \( \gamma_\mu \)s are the usual two-dimensional Dirac matrices and \( \Gamma_0 = 1, \Gamma_1 = \gamma_5 \). Please keep in mind that no sum over repeated indices is implied when a subindex \( (\mu) \) is involved. Let us also mention that the coupling potentials \( V(\mu) \) and \( U(\mu) \) are assumed to depend on the distance \( |x - y| \) and can be expressed in terms of Solyom’s ”g-ology” [5] as

\[ V(0)(x, y) = \frac{1}{g^2}(g_2 + g_4)(x, y) \]  

\[ V(1)(x, y) = \frac{1}{g^2}(g_2 - g_4)(x, y) \]  

\[ U(0)(x, y) = \frac{1}{g'^2}(g_3 + g_1)(x, y) \]  

\[ U(1)(x, y) = \frac{1}{g'^2}(g_3 - g_1)(x, y) \]  

In the above equations \( g \) and \( g' \) are just numerical constants that could be set equal to one. We keep them to facilitate comparison of our results with those corresponding to the usual Thirring model. Indeed, this case is obtained by choosing \( g' = 0 \) and \( V(0)(x, y) = V(1)(x, y) = \delta^2(x - y) \). On the other hand, the non-covariant limit \( g' = 0, V(0)(x, y) = 0 \) gives one version \(((g_2 = g_4)\) of the TL model [10].
The terms in the action containing $g_2$ and $g_4$ represent forward scattering events, in which the associated momentum transfer is small. In the $g_2$ processes the two branches (left and right-moving particles) are coupled, whereas in the $g_4$ processes all four participating electrons belong to the same branch. On the other hand, $g_1$ and $g_3$ are related to scattering diagrams with larger momentum transfers of the order of $2k_F$ (bs) and $4k_F$ (us) respectively (this last contribution is important only if the band is half-filled). For simplicity, throughout this paper we will consider spinless electrons. The extension of our results to the spin-$1/2$ case with spin-flipping interactions, though not trivial, could be done by following the lines of ref. [8].

Let us now turn to the treatment of the partition function. At this point we recall that in ref. [7] we wrote the fs piece of the action in a localized way:

$$S_{fs} = -\frac{g^2}{2} \int d^2x J_\mu K_\mu. \quad (2.9)$$

where $J_\mu$ is the usual fermionic current, and $K_\mu$ is a new current defined as

$$K_\mu(x) = \int d^2y V(\mu)(x,y)J_\mu(y). \quad (2.10)$$

Using a functional delta and introducing auxiliary bosonic fields in the path-integral representation of the partition function $Z$, we were able to write (see [7] for details):

$$Z = \mathcal{N} \int D\bar{\Psi} D\Psi D\tilde{A}_\mu D\tilde{B}_\mu \exp \left\{ -\int d^2x \left[ \bar{\Psi} i\gamma^\mu \Psi + \tilde{A}_\mu \tilde{B}_\mu + \frac{g}{\sqrt{2}} (\tilde{A}_\mu J_\mu + \tilde{B}_\mu K_\mu) \right] \right\} \quad (2.11)$$

If we define

$$\tilde{B}_\mu(x) = \int d^2y V(\mu)(y,x)\tilde{B}_\mu(y), \quad (2.12)$$

$$\tilde{B}_\mu(x) = \int d^2y V^{-1}(\mu)(y,x)\tilde{B}_\mu(y), \quad (2.13)$$

with $V^{-1}(\mu)(y,x)$ satisfying

$$\int d^2y V^{-1}(\mu)(y,x)V(\mu)(z,y) = \delta^2(x-z), \quad (2.14)$$

and change auxiliary variables in the form

$$3$$
\[ A_\mu = \frac{1}{\sqrt{2}}(\tilde{A}_\mu + \tilde{B}_\mu), \]  
\[ (2.15) \]
\[ B_\mu = \frac{1}{\sqrt{2}}(\tilde{A}_\mu - \tilde{B}_\mu), \]  
\[ (2.16) \]
we obtain
\[ Z = \mathcal{N} \int D\bar{\Psi} D\Psi D A_\mu D B_\mu \exp \left[ -\int d^2x (\bar{\Psi}(i\partial - gA)\Psi) \right] \]
\[ (2.17) \]
where
\[ S(A, B) = \frac{1}{2} \int d^2x d^2y V^{-1}_{(\mu)}(x, y) [A_\mu(x)A_\mu(y) - B_\mu(x)B_\mu(y)] \]
\[ (2.18) \]
The Jacobian associated with the change \((\tilde{A}, \tilde{B}) \rightarrow (A, B)\) is field-independent and can then be absorbed in the normalization constant \(\mathcal{N}\). Moreover, we see that the \(B\)-field is completely decoupled from both the \(A\)-field and the fermion field. Keeping this in mind, it is instructive to try to recover the partition function corresponding to the usual covariant Thirring model \((V^{-1}_{(0)}(y, x) = V^{-1}_{(1)}(x, y) = \delta^2(x - y), \text{ and } g' = 0)\), starting from (2.17). In doing so one readily discovers that \(B_\mu\) describes a negative-metric state whose contribution must be factorized and absorbed in \(\mathcal{N}\) in order to get a sensible answer for \(Z\). This procedure parallels, in the path-integral framework, the operator approach of Klaiber [4], which precludes the use of an indefinite-metric Hilbert space. Consequently, from now on we shall only consider the \(A\) contribution.

At this stage we see that, when \(bs\) and \(us\) processes are disregarded, the procedure we have just sketched allows us to express \(Z\) in terms of a fermionic determinant. Now we will show that this goal can also be achieved when the larger momentum transfers are taken into account. To this end we write:

\[ S_{bs} + S_{us} = -\frac{g^2}{2} \int d^2x L_\mu M_\mu \]
\[ (2.19) \]
where \(L_\mu\) and \(M_\mu\) are fermionic bilinears defined as

\[ L_\mu(x) = \bar{\Psi}(x)\Gamma_\mu \Psi(x), \]  
\[ (2.20) \]
\[ M_\mu(x) = \int d^2y U_{(\mu)}(x, y)L_\mu(y). \]  
\[ (2.21) \]
Thus it is evident that we can follow the same prescriptions as above, with $L_\mu$ and $M_\mu$ playing the same roles as $J_\mu$ and $K_\mu$, respectively. After the elimination of a new negative metric state whose decoupled partition function is absorbed in the normalization factor, as before, one obtains

$$Z = \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}C_\mu \det (i\partial - gA - g'\Gamma_\mu C_\mu) \exp (-S[A] - S[C])$$

(2.22)

where

$$S[A_\mu] = \frac{1}{2} \int d^2x d^2y A_\mu(x)V^{-1}_{(\mu)}(y)A_\mu(y)$$

$$S[C_\mu] = \frac{1}{2} \int d^2x d^2y C_\mu(x)U^{-1}_{(\mu)}(y)C_\mu(y)$$

(2.23)

and

$$\int d^2y U^{-1}_{(\mu)}(y,x)U_{(\mu)}(z,y) = \delta^2(x-z),$$

(2.24)

Then we have been able to express $Z$ in terms of a fermionic determinant. Let us stress, however, that this determinant is a highly non trivial one. Indeed, the term in $g'$ is not only a massive-like term (in the sense that it is diagonal in the Dirac matrices space) but it also depends on the auxiliary field $C_\mu(x)$. As shown in [11] one can combine a chiral change in the fermionic path-integral measure with a formal expansion in $g'$ in order to get a bosonic representation for the fermionic determinant. Let us start by performing the following transformation:

$$\Psi(x) = e^{g[\gamma_5\Phi(x)-i\eta(x)]} \chi(x)$$

(2.25)

$$\bar{\Psi}(x) = \bar{\chi}(x) e^{g[\gamma_5\Phi(x)+i\eta(x)]}$$

(2.26)

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = J[\Phi, \eta]\mathcal{D}\bar{\chi}\mathcal{D}\chi,$$

(2.27)

where $\Phi$ and $\eta$ are scalar fields and $J[\Phi, \eta]$ is the Jacobian of the transformation. As it is well known, the above transformation permits to decouple the field $A_\mu$ from the fermionic fields if one writes

$$A_\mu(x) = \partial_\mu \eta(x) + \epsilon_{\mu\nu} \partial_\nu \Phi(x)$$

(2.28)
which can also be considered as a bosonic change of variables with trivial (field independent) Jacobian. As a result we find

$$\det(i\phi - gA - g'\Gamma_\mu C_\mu) = J[\Phi, \eta] \det(i\phi - g'e^{2\gamma\gamma_5 \Phi} \Gamma_\mu C_\mu) \quad (2.29)$$

After a suitable regularization the fermionic Jacobian reads

$$J[\Phi, \eta] = \exp \frac{g^2}{2\pi} \int \frac{d^2p}{(2\pi)^2} \left[ -p_1^2\Phi(p)\Phi(-p) + p_1^2\eta(p)\eta(-p) - 2p_0p_1\Phi(p)\eta(-p) \right]. \quad (2.30)$$

The vacuum to vacuum functional is then expressed as

$$Z = \mathcal{N} \int D\Phi D\eta DC_\mu e^{-(S[\Phi, \eta]+S[C_\mu])} J[\Phi, \eta] \det(i\phi - g'e^{2\gamma\gamma_5 \Phi} \Gamma_\mu C_\mu) \quad (2.31)$$

where $S[\Phi, \eta]$ arises when one inserts (2.28) in $S[A_\mu]$ (See equation (2.23)):

$$S[\Phi, \eta] = \int \frac{d^2p}{(2\pi)^2} \left[ \Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) + \Phi(p)\eta(-p)C(p) \right] \quad (2.32)$$

with

$$A(p) = \frac{1}{2} \left[ p_1^2\hat{V}^{-1}_{(1)}(p) + p_0^2\hat{V}^{-1}_{(0)}(p) + \frac{g^2}{\pi} \right] \quad (2.33)$$

$$B(p) = \frac{1}{2} \left[ p_0^2\hat{V}^{-1}_{(0)}(p) + p_1^2\hat{V}^{-1}_{(1)}(p) - \frac{g^2}{\pi} \right] \quad (2.34)$$

$$C(p) = p_0p_1 \left( \hat{V}^{-1}_{(0)}(p) - \hat{V}^{-1}_{(1)}(p) + \frac{g^2}{\pi} \right). \quad (2.35)$$

The fermionic determinant in the above expression can be analyzed in terms of a formal perturbative expansion. Indeed, taking $g'$ as perturbative parameter, and using the fermionic fields $\chi$ and $\bar{\chi}$ defined in (2.26) one can write

$$Z_F = \sum_{n=0}^{\infty} \frac{g'^n}{n!} \left\langle \prod_{j=1}^{n} \int d^2x_j \bar{\chi}(x_j) C(x_j) \chi(x_j) \right\rangle_0 \quad (2.36)$$

where, for later convenience we have defined

$$Z_F = \det(i\phi - g'e^{2\gamma\gamma_5 \Phi} \Gamma_\mu C_\mu) \quad (2.37)$$
\[ C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} \] 

with

\[
\begin{cases}
C_+ = (C_0 + C_1) e^{2g\Phi(x)} \\
C_- = (C_0 - C_1) e^{-2g\Phi(x)}
\end{cases}
\]

By carefully analyzing each term in the series we found a selection rule quite similar to the one obtained in the path-integral treatment of (1+1) massive fermions with local \[16\] \[17\] and non-local interactions \[18\]. Indeed, due to the fact that in (2.36) \(\langle \rangle_0\) means v.e.v. with respect to free massless fermions, the v.e.v.’s corresponding to \(j = 2k + 1\) are zero. Thus, we obtain

\[
Z_F = \sum_{k=0}^{\infty} \frac{(g'c\rho)^{2k}}{(k!)^2(2\pi)^{2k}} \int \prod_{i=1}^{k} d^2x_i d^2y_i \\
\times \prod_{i=1}^{k} [C_0(x_i) + C_1(x_i)] [C_0(y_i) - C_1(y_i)] \\
\times \exp 2g \sum_{i=1}^{k} [\Phi(x_i) - \Phi(y_i)] \\
\times \prod_{i>j}^{k} (c\rho)^4|x_i - x_j|^2|y_i - y_j|^2 \\
\prod_{i,j}^{k} (c\rho)^2|x_i - y_j|^2
\]

where \(\rho\) is a normal ordering parameter and \(c\) is related to Euler’s constant.

In order to obtain a bosonic description of the present problem we shall now propose the following bosonic Lagrangian density:

\[
\mathcal{L}_B = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{\alpha_0}{2\beta^2} (m_+ e^{i\beta \varphi} + m_- e^{-i\beta \varphi})
\]

with \(\beta, m_+(x)\) and \(m_-(x)\) to be determined. The quantity \(\alpha_0\) is just a constant that we include to facilitate comparison of our procedure with previous works on local bosonization \[16\] \[17\]. Please notice that for \(m_+ = m_- = 1\) this model coincides with the well known sine-Gordon model that can be used to describe a neutral Coulomb gas. In this context \(\frac{\alpha_0}{\beta^2}\) is nothing but the corresponding fugacity \[19\]. We shall now consider the partition function
\[ Z_B = \int \mathcal{D} \varphi \ e^{-\int d^2 x \ \mathcal{L}_B} \] (2.42)

and perform a formal expansion taking the fugacity as perturbative parameter. It is quite straightforward to extend the analysis of each term, already performed for \( m_+ = m_- = 1 \), to the present case in which these objects are neither equal nor necessarily constants. The result is

\[ Z_B = \sum_{l=1}^{\infty} \frac{1}{l!^2} \left( \frac{\alpha_0}{2\beta^2} \right)^{2l} \int \left( \prod_{i=1}^{l} d^2 x_i d^2 y_i \right) \left( \prod_{i=1}^{l} m_+(x_i) m_-(y_i) \right) \left( \frac{\rho}{\Lambda} \right)^{2l^2 \pi \alpha_0 / \beta^2} \times \frac{\prod_{i>j} \left[(c\rho)^2 |x_i - x_j| |y_i - y_j| \right]^{\beta^2 / 2\pi}}{\prod_{i,j} \left[(c\rho)|x_i - y_j| \right]^{\beta^2 / 2\pi}}. \] (2.43)

Comparing this result with equation (2.40), we see that both series coincide if the following identities hold:

\[ \beta = \pm 2\sqrt{\pi} \]
\[ \frac{\alpha_0}{\beta^2} = \frac{g'\Lambda c}{\pi} \] (2.44)

and

\[ m_+(x_i) = (C_0(x_i) + C_1(x_i)) \ e^{2g\Phi(x_i)} \]
\[ m_-(y_i) = (C_0(y_i) - C_1(y_i)) \ e^{-2g\Phi(y_i)} \] (2.45)

Thus we have found a bosonic representation for the fermionic determinant (2.40). This is given by (2.43) together with the identities (2.44) and (2.45). Let us emphasize that equations (2.44) are completely analogous to the bosonization formulae first obtained by Coleman [16] whereas equations (2.45) constitute a new result, specially connected to the present problem.

3 Derivation of gap equations through self consistent harmonic approximation

Inserting (2.43) in (2.31) we can express the partition function of this system in terms of five scalars: \( \Phi, \eta, C_0, C_1 \) and \( \varphi \). We are thus led to a completely bosonized action \( S_{bos} \):

\[ S_{\text{bos}} = \int \frac{d^2 p}{(2\pi)^2} [\Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) + \Phi(p)\eta(-p)C(p) + \varphi(p)\varphi(-p)\frac{p^2}{2}] \]

\[ + \frac{1}{2} \int d^2 x d^2 y C_\mu(x)U_{(\mu)}(x,y)C_\mu(y) + \frac{g'\Lambda c}{\pi} \int d^2 x [C_0(x)f_0(x) + iC_1(x)f_1(x)] \] (3.1)

where

\[ f_0(x) = \cos((\sqrt{4\pi}\varphi - 2ig\Phi)(x)) \] (3.2)

\[ f_1(x) = \sin((\sqrt{4\pi}\varphi - 2ig\Phi)(x)) \] (3.3)

Since the integrals in \( C_0 \) and \( C_1 \) are quadratic these fields are easily integrated out and one gets

\[ Z = N \int D\Phi \, D\eta \, D\varphi \, e^{-S_{\text{eff}}[\Phi,\eta,\varphi]} \] (3.4)

with

\[ S_{\text{eff}}[\Phi,\eta,\varphi] = S_0 + S_{\text{int}} \] (3.5)

where

\[ S_0 = \int \frac{d^2 p}{(2\pi)^2} [\Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) + \Phi(p)\eta(-p)C(p) + \varphi(p)\varphi(-p)\frac{p^2}{2}] \] (3.6)

\[ S_{\text{int}} = -\frac{(\Lambda c)^2}{2\pi^2} \int d^2 x d^2 y g_1(x,y) \cos \left[ \sqrt{4\pi}(\varphi(x) - \varphi(y)) - 2ig(\Phi(x) - \Phi(y)) \right] \]

\[ - \frac{(\Lambda c)^2}{2\pi^2} \int d^2 x d^2 y g_3(x,y) \cos \left[ \sqrt{4\pi}(\varphi(x) + \varphi(y)) - 2ig(\Phi(x) + \Phi(y)) \right]. \] (3.7)

It is now convenient to diagonalize the quadratic part of the effective action by introducing the fields \( \zeta, \chi \) and \( \xi \):

\[ \Phi = \frac{i\zeta}{\bar{g}} + \frac{2i\bar{g}Bp^2}{\Delta + 2B\bar{g}^2p^2} \xi \] (3.8)

\[ \eta = \frac{-iC}{2B\bar{g}} \zeta - \frac{i\bar{g}Cp^2}{\Delta + 2B\bar{g}^2p^2} \xi + \frac{1}{\bar{g}} \chi \] (3.9)

\[ \varphi = -\zeta + \frac{\Delta}{\Delta + 2B\bar{g}^2p^2} \xi, \] (3.10)
where we have defined $\tilde{g}^2 = g^2/\pi$ and $\Delta(p) = C(p)^2 - 4A(p)B(p)$. We then obtain

$$S_0 = \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ \zeta(p) \left( p^2 + \frac{\Delta}{2B\tilde{g}^2} \right) \zeta(-p) + \chi(p) \frac{2B}{g^2} \chi(-p) + \xi(p) \frac{p^2 \Delta}{\Delta + 2B\tilde{g}^2 p^2} \xi(-p) \right],$$

(3.11)

$$S_{\text{int}} = -\frac{(\Lambda c)^2}{2\pi^2} \int d^2 x \, d^2 y \, g_1(x, y) \cos \sqrt{4\pi} [\xi(x) - \xi(y)]$$

$$- \frac{(\Lambda c)^2}{2\pi^2} \int d^2 x \, d^2 y \, g_3(x, y) \cos \sqrt{4\pi} [\xi(x) + \xi(y)].$$

(3.12)

One can see that the $\zeta$ and $\chi$ fields become completely decoupled from $\xi$. Moreover, it becomes apparent that the $\xi$-dependent piece of the action $S_{\text{int}}$ is the only one containing relevant contributions (i.e. gapped modes):

$$S[\xi] = \int \frac{d^2 p}{(2\pi)^2} \xi(p) F(p) \frac{\xi(-p)}{2} - \frac{(\Lambda c)^2}{2\pi^2} \int d^2 x \, d^2 y \, g_1(x, y) \cos \sqrt{4\pi} [\xi(x) - \xi(y)]$$

$$- \frac{(\Lambda c)^2}{2\pi^2} \int d^2 x \, d^2 y \, g_3(x, y) \cos \sqrt{4\pi} [\xi(x) + \xi(y)].$$

(3.13)

with

$$F(p) = \frac{1}{Kv} \left( p_0^2 + v^2 p_1^2 \right),$$

(3.14)

$$K = \sqrt{\frac{1 + g_4/\pi - g_2/\pi}{1 + g_4/\pi + g_2/\pi}},$$

(3.15)

$$v = \sqrt{(1 + \frac{g^2 \dot{V}_0(p)}{\pi})(1 - \frac{g^2 \dot{V}_1(p)}{\pi})}.$$  

(3.16)

Unfortunately, even if we succeeded in simplifying the original action, $S[\xi]$ is not yet soluble. We introduce a non-local version of the self-consistent harmonic approximation [14]. Basically, this amounts to replacing the so called true action (3.13) by a trial action in which the cosine terms are approximated as

$$- \frac{(\Lambda c)^2}{2\pi^2} g_{1,3}(x, y) \cos \sqrt{4\pi} [\xi(x) \pm \xi(y)] \rightarrow \frac{\Omega_{1,3}(x, y)}{2} \xi(x)\xi(y).$$

(3.17)
where the functions $\Omega_i$ of the trial action can be variationally determined (See the Appendix for details). We will consider two cases: pure backscattering ($g_1 \neq 0, g_3 = 0$) and pure umklapp scattering ($g_3 \neq 0, g_1 = 0$). Once this is done, it is straightforward to obtain the charge spectrum. Indeed, going to momentum space, and back to real frequencies, $p_0 = \omega, p_1 = k$, the following equations are obtained:

$$Kv\Omega_i - \omega^2 + v^2k^2 = 0.$$  \hfill (3.18)

The gap equations satisfied by $\Omega_i$ are:

$$\Omega_{1,3}(p) = \frac{4\Lambda^2c^2}{\pi} \int d^2x g_{1,3}(x)e^{-4\pi[I_1(x)\mp I_1(0)]}(e^{ipx} \mp 1)$$  \hfill (3.19)

where

$$I_1(x) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{-ipx}}{F(p) + \Omega_i(p)}$$  \hfill (3.20)

Equation (3.19) is our main formal result. It gives, within the SCHA, a closed expression for the gap as functional of any arbitrary strength of the interactions (the $g_i's$). Let us study, as examples, the cases where umklapp and backscattering are local interactions (i.e. $g_3(x) = g_3\delta^2(x)$), but keeping arbitrary fs interactions. It is easy to see that the pure backscattering process is gapless ($\Omega_1 = 0$) and that the umklapp interaction opens a gap in the charge density spectrum ($\Omega_3 = \text{constant}$), as expected \cite{12} \cite{13}. For this last case we obtain

$$\Omega_3 = \frac{8\Lambda^2c^2}{\pi}g_3e^{-8\pi I_1(0)}$$  \hfill (3.21)

As $\Omega_3$ is a constant, i.e. it does not depend on $p$ we can calculate the propagator in terms of $\Omega_3$, and the equation then reduces to an algebraic equation for $\Omega_3$. If we consider also contact fs potentials (i.e. $g_{2,4}(x) = g_{2,4}\delta^2(x)$), the integral $I_1(0)$ can be explicitly performed. After a suitable regularization one obtains:

$$\Omega_3 = \frac{4\Lambda^2}{K} \left( \frac{2e^2K}{\pi}g_3 \right)^{1/\pi}.$$  \hfill (3.22)

An equation completely analogous to (3.21) was found in \cite{8} by working directly with a local coupling associated to spin-flipping interactions in that case. This provides a consistency check for our computation. Of course, having derived a general formula for the gap, like (3.19), it is interesting to analyze the effect of a non-contact interaction on the gap. We present this study in the next section.
4 Gap of charge-density modes for a non-local umklapp coupling

The purpose of this Section is to find a solution to equation (3.19) for a pure umklapp potential of the form

\[ g_3(p) = \lambda_0 + \frac{\epsilon}{\Lambda^2} p_1^2. \]  

being \( \epsilon \ll 1 \). It consists of the local potential plus a small non local correction. We start by writing the potential in coordinate space:

\[ g_3(x) = \lambda_0 \delta^2(x) - \frac{\epsilon}{\Lambda^2} \partial^2_1 \delta^2(x) \]  

where \( \partial^2_1 \delta^2(x) \) is the second derivative of the delta function with respect to \( x_1 \) only. By replacing this function in (3.19) and integrating we obtain

\[ \Omega(p) = \frac{4c^2 \Lambda^2}{\pi} \left[ 2\lambda_0 e^{-8\pi I_1(0)} - \frac{\epsilon}{\Lambda^2} \partial^2_1 \left( e^{-4\pi[I_1(0)+I_1(x)]}(1 + e^{ip_1}) \right)_{x=0} \right] \]

where we have dropped the subindex 3 in \( \Omega(p) \). Performing the derivation and rearranging terms we can write

\[ \Omega(p) = \frac{4\Lambda^2 c^2}{\pi} \left[ 2\lambda_0 + 8\pi \frac{\epsilon}{\Lambda^2} \partial^2_1 I_1(0) \right] e^{-8\pi I_1(0)} + \frac{4c^2 \epsilon}{\pi} e^{-8\pi I_1(0)} p_1^2 \]  

In this last equation we have disregarded terms proportional to \( \partial_1 I_1(0) \) which is zero by symmetry. We observe that \( \Omega \) retains the same structure as \( g_3(p) \) i.e. it is of the form

\[ \Omega(p) = \mu^2 + gp_1^2 \]  

where \( \mu^2 \) and \( g \) are two parameters to be determined in terms of \( \lambda_0 \) and \( \epsilon \). The fact that \( \Omega \) preserves its simple form allows to calculate the propagator, as occurs in the local case, and in consequence, to transform the equation into an algebraic one, easier to deal with. The objects needed are the propagator in the origin \( I_1(0) \) and its second derivative in the origin \( \partial^2_1 I_1(0) \). Theirs expression are

\[ I_1(0) = \frac{-Kv}{4\pi \sqrt{v^2 + Kg v}} \ln \left[ \frac{\mu^2 K v}{4\Lambda^2 (v^2 + Kg v)} \right], \]
and

$$\partial^2 I_1(0) = \frac{-\Lambda^2 K v}{8\pi \sqrt{v^2 + Kg v}} \left\{ \frac{\mu^2 K v}{\Lambda^2 (v^2 + Kg v)} \ln \left[ \frac{e\mu^2 K v}{4\Lambda^2 (v^2 + Kg v)} \right] - 2 \right\}. \quad (4.7)$$

Replacing (4.5), (4.6) and (4.7) in equation (4.4), and equating the coefficients of each power of \( p \), we obtain the pair of equations

$$x = \frac{gKv}{2\epsilon \sqrt{v^2 + Kg v}} \left[ \lambda_0 - \frac{\epsilon K v}{\sqrt{v^2 + Kg v}} (2x \ln ex + 1) \right] \quad (4.8)$$

$$g = \frac{4\epsilon c^2}{\pi} x^{2Kv/\sqrt{v^2 + Kg v}} \quad (4.9)$$

where we have introduced the adimensional quantity \( x = \mu^2 K v / 4\Lambda^2 (v^2 + Kg v) \). These coupled equations are to be solved for \( g \) and \( x \).

Of course, a complete solution of (4.8) and (4.9) is very difficult to obtain, but we can obtain an approximate solution in the limit \( \epsilon \ll 1 \). It corresponds to take a small non-local correction to the local potential. For \( \epsilon = 0 \) the solution is \( g = 0 \), and we expect no discontinuities as \( \epsilon \) grows from that value. So for small \( \epsilon \) the solution for \( g \) is also small.

This allows to approximate \( v^2 + Kg v \approx v^2 \), and one then gets

$$x = \frac{2c^2 K}{\pi} x^{2K} \left[ \lambda_0 - \epsilon K (2x \ln ex + 1) \right] \quad (4.10)$$

$$g = \frac{4\epsilon c^2}{\pi} x^{2K} \quad (4.11)$$

We notice that the first equation does not depend on \( g \), though it is still difficult to solve. We shall employ a perturbative approach. Suppose \( x_0 \) is the solution to the \( \epsilon = 0 \) equation:

$$x_0 = \left( \frac{2c^2 K \lambda_0}{\pi} \right)^{\frac{1}{2K}} \quad (4.12)$$

We can try a solution of the form \( x = x_0 + \delta x \) where \( \delta x \) goes to zero as \( \epsilon \) approaches zero. Replacing this form into the equation, and retaining terms up to first order in \( \delta x \) we obtain
\[ \delta x = \frac{-2\epsilon x_0(2x_0 \ln \epsilon x_0 + 1)}{\lambda_0(1 - 2K)} \quad (4.13) \]

\[ g = \frac{2\epsilon x_0}{K\lambda_0} \quad (4.14) \]

The first equation gives the correction to the gap of charge-density modes due to non-contact umklapp interactions. If one introduces a small non-local perturbation, it is enough to modify the value of the gap. Note that the numerator can be positive or negative, depending on the value of \( x_0 \), so the same happens with the modification in the gap. It can be lowered or increased by umklapp non-local interactions. The second equation gives the correction to the kinetic part of the spectrum.

5 Final result and next steps

In this paper we have presented a non-local and non-covariant extension of the Thirring model which can be used to describe a 1d many-body system when not only forward but also backward and umklapp scattering is considered. In this sense our results improve previous formulations reported in [7] and [11]. We were able to write the vacuum to vacuum functional of this model in terms of a non-trivial fermionic determinant. We obtained a bosonic representation for this determinant which led us to an effective action associated to the elementary collective excitations of the system. By employing the self-consistent harmonic approximation we found a closed expression for the gap of the charge-density spectrum as functional of arbitrary (not necessarily local) two-body potentials. This equation is valid for purely backward (\( g_1 \neq 0, g_3 = 0 \)) or pure umklapp scattering (\( g_3 \neq 0, g_1 = 0 \)). As an application of this formula we studied the effect of a non-local umklapp potential of the form \( g_3(p) = \lambda_0 + (\epsilon/\Lambda^2)p_1^2 \) for \( \epsilon \) small. Besides its illustrative purpose, the result obtained in this example is interesting in itself because most of the previous investigations involving umklapp scattering do not consider non-local effects [21].

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Appendix: details of the SCHA method

We shall give an sketch of the SCHA method. One usually starts from a partition function

\[ Z_{\text{true}} = \int \mathcal{D}\mu e^{-S_{\text{true}}} \]  

(6.1)

where \( \mathcal{D}\mu \) is a generic integration measure. An elementary manipulation leads to

\[ Z_{\text{true}} = \int \mathcal{D}\mu e^{-(S_{\text{true}} - S_{\text{trial}})} e^{-S_{\text{trial}}} \int \mathcal{D}\mu e^{-S_{\text{trial}}} = Z_{\text{trial}} \langle e^{-(S_{\text{true}} - S_{\text{trial}})} \rangle_{\text{trial}}. \]  

(6.2)

for any trial action \( S_{\text{trial}} \). Now, by means of the property

\[ \langle e^{-f} \rangle \geq e^{-\langle f \rangle}, \]  

(6.3)

for \( f \) real, and taking natural logarithm in equation (6.2), we obtain Feynman’s inequality

\[ \ln Z_{\text{true}} \geq \ln Z_{\text{trial}} - \langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} \]  

(6.4)

We shall study a very general class of true bosonic actions of non local sine Gordon type, given by

\[ S_{\text{true}} = \int \frac{d^2p}{(2\pi)^2}\varphi(p)\frac{F(p)}{2}\varphi(p) - \int d^2x \, d^2y \frac{\alpha(x-y)}{b^2} \cos b\varphi(x) + \varphi(y), \]  

(6.5)

where \( F(p) \) and \( \alpha(x,y) \) are arbitrary functions. The usual local sine Gordon theory is obtained by taking \( F(p) = p^2 \) and \( \alpha(x,y) = \alpha_0 \delta^{(2)}(x-y) \) whereas (6.3) reduces to our model (3.13) without backscattering interactions (i.e. \( g_1 = 0 \)) by taking

\[ F(p) = \frac{1}{Kv}(p_0^2 + v^2p_1^2) \]  

(6.6)

\[ b^2 = 16\pi \]  

(6.7)

\[ \frac{\alpha(x,y)}{b^2} = \frac{(\Lambda c)^2}{2\pi^2}g_3(x,y) \]  

(6.8)
The trial action we will consider is a quadratic one:

$$S_{\text{trial}} = \int \frac{d^2 p}{(2\pi)^2} \varphi(p) \frac{F(p)}{2} \varphi(p) - \int d^2 x \, d^2 y \frac{\Omega(x - y)}{2} \varphi(x) \varphi(y)$$  \hspace{1cm} (6.9)

where the function $\Omega$ can be determined by maximizing the right hand side of equation (6.4). In order to achieve this goal we first write

$$\ln Z_{\text{trial}} = \ln \int \mathcal{D} \varphi \exp \left[ -\frac{1}{2} \int d^2 x \, \varphi(x)(\hat{A}\varphi)(x) \right]$$  \hspace{1cm} (6.10)

$$= \ln(\det \hat{A})^{-1/2} + \text{const}$$  \hspace{1cm} (6.11)

$$= -\frac{1}{2} \text{tr} \ln \hat{A} + \text{const}. \hspace{1cm} (6.12)$$

where the operator $\hat{A}$ is defined, in Fourier space, by

$$(\hat{A}\varphi)(p) = [F(p) + \Omega(p)]\varphi(p). \hspace{1cm} (6.13)$$

being $\Omega(p)$ is the Fourier transform of $\Omega(z)$. It is then easy to get

$$\text{tr} \ln \hat{A} = \mathcal{V} \int \frac{d^2 p}{(2\pi)^2} \ln[F(p) + \Omega(p)] \equiv \mathcal{V} I_0[\Omega] \hspace{1cm} (6.14)$$

where $\mathcal{V}$ is the volume (infinite) of the whole space $\int d^2 x$. On the other hand, it is straightforward to compute $\langle S_{\text{true}} - S_{\text{trial}} \rangle$, by following, for instance, the steps explained in ref. [18]. The result is

$$-\langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} = \mathcal{V} \int d^2 x \left[ \frac{\alpha(x)}{b^2} e^{-\frac{1}{2} b^2[I_1(x) + I_1(0)]} + \frac{\Omega(x)}{2} I_1(x) \right] \hspace{1cm} (6.15)$$

where $I_1(x)$ is a functional of $\Omega$ given by the propagator of the trial action:

$$I_1(x) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{-ipx}}{F(p) + \Omega(p)}. \hspace{1cm} (6.16)$$

and we have made use of the translational invariance of $\alpha$ and $\Omega$. Finally we can gather all the terms, and write them as
\[
\ln Z_{\text{trial}} - \langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} = \frac{\mathcal{V}}{2} \int d^2x \frac{\alpha(x)}{b^2} e^{-\frac{1}{2}b^2[f_1(x)+I_1(0)]} + \frac{\mathcal{V}}{2} \int \frac{d^2p}{(2\pi)^2} \frac{\Omega(-p)}{F(p)+\Omega(p)} - \frac{\mathcal{V}}{2} I_0[\Omega] + \text{const.} \quad (6.17)
\]

where we have turned the last term of equation (6.15) into Fourier space. Now extremizing expression (6.17) with respect to \( \Omega \) by functional derivation with respect to \( \Omega(k) \), we finally obtain the gap equation

\[
\Omega(p) = \frac{1}{2} \int d^2x \alpha(x) e^{-\frac{1}{4}b^2[f_1(x)+I_1(0)]}(e^{-ipx} + 1) \quad (6.18)
\]

If instead of considering only umklapp interactions we consider only backscattering interactions \( (g_3 = 0) \) then we have to change \( \varphi(x) + \varphi(y) \to \varphi(x) - \varphi(y) \) into the cosine of equation (6.5), and the resulting gap equation is

\[
\Omega(p) = \frac{1}{2} \int d^2x \alpha(x) e^{-\frac{1}{4}b^2[f_1(x)-I_1(0)]}(e^{-ipx} - 1) \quad (6.19)
\]

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