Conformal Transformation in Gravity

Abstract

The conformal transformation in the Einstein - Hilbert action leads to a new frame where an extra scalar degree of freedom is compensated by the local conformal-like symmetry. We write down a most general action resulting from such transformation and show that it covers both general relativity and conformally coupled to gravity scalar field as the particular cases. On quantum level the equivalence between the different frames is disturbed by the loop corrections. New conformal-like symmetry in anomalous and, as a result, the theory is not finite on shell at the one-loop order.

1 Introduction

The frame dependence of the gravity theories is an important object for study on both classical and quantum levels. On classical level the theory may manifest a different physical properties in a different frames that leads to the nontrivial problems related with the "correct choice" of the field variables (see, for example, [1] [2] [3] [4] [5] [6] [7] [8] and the last work for a more complete list of references on the subject). On the other hand the study of different frames enables one to explore the relation between the different physical theories and thus generate the new exact solutions [9] [10] [11]. In this paper we construct and study the general action of the theory which is conformally equivalent to General Relativity with (or without) cosmological constant. The theory under consideration depends on the metric and also on the scalar field, whereas an extra scalar degree of freedom is compensated by additional conformal-like symmetry. In fact we exchange the theory with the Einstein - Hilbert action to the much more complicated but equivalent theory with the action depending on some arbitrary function of the scalar field. The form of the symmetry transformation depends on the form of this function.
On quantum level the equivalence between two formulations is disturbed by anomaly which is essential and manifest itself at the level of the one-loop divergences already. In a framework of Einstein gravity the one loop counterterms vanish on classical mass shell and the theory is finite \[12\]. Indeed this property does not hold if the matter fields are incorporated \[13\] or if the two-loop effects are taken into account \[14\]. In our new conformal frame the one-loop S-matrix is not finite because of anomaly. This fact can be interpreted as the noninvariance of the measure of path integral with respect to (generalized) conformal transformations. Earlier the similar objection have been made in a quantum conformal (Weyl) gravity \[15, 16\] which is power

Now, following \[3, 21\] we consider the simple particular case of the general action (1).

\[S = \int d^4 x \sqrt{-g} \left\{ A(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + B(\phi) R + C(\phi) \right\} \] (1)

that covers all special cases including the string inspired action, coupled with gravity scalar field and others. Now, following \[3, 21\] we consider the simple particular case of the general action (1) with

\[S = \int d^4 x \sqrt{-g'} \left\{ R' \Phi + V(\Phi) \right\} \] (2)

Here the curvature $R'$ corresponds to the metric $g'_{\mu\nu}$ and $g' = \det(g'_{\mu\nu})$. Transform this action to the new variables $g_{\mu\nu}$ and $\phi$ according to

\[g'_{\mu\nu} = g_{\mu\nu} e^{2\sigma(\phi)}, \quad \Phi = \Phi(\phi) \] (3)

where $\sigma(\phi)$ and $\Phi(\phi)$ are arbitrary functions of $\phi$. In a new variables the action becomes:

\[S = \int d^4 x \sqrt{-g} \left\{ \Phi(\phi) Re^{2\sigma(\phi)} + 6(\nabla_\phi)^2 e^{2\sigma(\phi)} [\Phi \sigma' + \Phi'] \sigma' + V(\Phi(\phi)) e^{4\sigma(\phi)} \right\} \] (4)

Therefore we are able to transform the particular action (3) to the general form (1) with

\[A(\phi) = 6 e^{2\sigma(\phi)} [\Phi \sigma_1 + \Phi_1] \sigma_1, \quad B(\phi) = \Phi(\phi) e^{2\sigma} \] (5)

Here and below the lower numerical index shows the order of derivative with respect to $\phi$. For instance,

\[B_1 = \frac{dB}{d\phi}, \quad A_2 = \frac{d^2 B}{d\phi^2}, \quad \sigma_1 = \frac{d\sigma}{d\phi}, \quad etc. \] (6)

It is possible to find the form of $\sigma(\phi)$ and $\Phi(\phi)$ that corresponds to the given $A(\phi)$ and $B(\phi)$. In this case $\sigma(\phi)$ and $\Phi(\phi)$ obey the equations

\[A = 6 B_1 \sigma_1 - 6 B(\sigma_1)^2, \quad \Phi = Be^{-2\sigma} \] (7)

Substituting (7) into (4) we find that in a new variables the action has the form

\[S = \int d^4 x \sqrt{-g} \left\{ A(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + B(\phi) R + \left( \frac{B}{\Phi} \right)^2 V(\Phi(\phi)) \right\} \] (8)

\footnote{One can consider the theory (4) as the four-dimensional analog of the Jakiw-Teitelboim model in two dimensions.}
where the last term is nothing but $C(\phi)$ from (1).

It is easy to see that the above transformations lead to some restrictions on the functions $A(\phi)$ and $B(\phi)$. All the consideration has to be modified if $\Phi = \text{const}$, that is the case when (2) is the Einstein-Hilbert action with cosmological constant. One can rewrite this condition in terms of $A(\phi)$ and $B(\phi)$. Note that

$$2AB - 3(B_1)^2 = -3 \left( \frac{d\Phi}{d\phi} \right)^2 e^{4\sigma(\phi)}$$

(9)

Hence it is clear that the theories with $2AB - 3(B_1)^2 = 0$ qualitatively differs from the other ones. Below we deal with the theories of this class. The only exception is the Appendix.

One can easily see that the Hilbert-Einstein action and the conformally coupled with gravity scalar field satisfy the condition (4) with $B = \text{const}$, $A = 0$, $C = \text{const}$ and $A = \frac{1}{2}, B = \frac{1}{12} \phi^2, C = \lambda \phi^4$ correspondingly. Therefore these two theories belongs to the special class of (1) which we are dealing with. Thus we have found that the theory of conformal scalar field and General Relativity are related with each other by the conformal transformation of the metric. The conformal symmetry compensate an extra degree of freedom which exists in the conformal scalar theory.

It is possible to show that this situation is typical. In fact all the models (1) with $2AB - 3(B_1)^2 = 0$ are related with each other by the conformal transformation of the metric. Next, all of them besides the degenerate case of General Relativity have some extra conformal-like symmetry that is called to compensate an extra degree of freedom related with scalar field. It is useful to formulate this as two theorems.

**Theorem 1.** If two sets of smooth functions $A(\phi), B(\phi), C(\phi)$ and $\bar{A}(\phi), \bar{B}(\phi), \bar{C}(\phi)$ satisfy the conditions

$$2AB - 3(B_1)^2 = 0, \quad C = \lambda B^2$$

(10)

then the corresponding models (1) are linked by the conformal transformation of the metric $\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2\sigma(\phi)}$ where the parameter of transformation $\sigma$ depends on the ratio $\bar{B}(\phi)/B(\phi)$.

**Theorem 2.** The action of any theory from the previous Theorem is invariant under the transformation which consists in an arbitrary reparametrization $\phi = \tilde{\phi}(\phi)$ and the conformal transformation $\bar{g}_{\mu\nu} = g_{\mu\nu} e^{2\sigma(\phi)}$ with

$$\sigma(\phi) = -\frac{1}{2} \ln \left[ \frac{B(\tilde{\phi}(\phi))}{B(\phi)} \right]$$

(12)

**Proof.** To cover both theorems let us consider the action (1), (10) with and make the conformal transformation of the metric with arbitrary $\sigma(\phi)$ and, simultaneously, an arbitrary reparametrization of the scalar field $\tilde{\phi} = \tilde{\phi}(\phi)$. The straightforward calculation gives the new action with the functions $\tilde{A}(\phi), \tilde{B}(\phi), \tilde{C}(\phi)$ which satisfy the conditions

$$\tilde{B}(\phi) = \tilde{B}(\tilde{\phi}(\phi)) e^{2\sigma(\phi)}$$

$$\tilde{C}(\phi) = \tilde{C}(\tilde{\phi}(\phi)) e^{4\sigma(\phi)}$$

(13)

$$A(\phi) = \left\{ \bar{A}(\tilde{\phi}(\phi)) \left[ \tilde{\phi}(\phi) \right]^2 + 6 \bar{B}(\tilde{\phi}(\phi)) \left[ \sigma_1(\phi) \right]^2 + 6 \frac{d\bar{B}(\tilde{\phi}(\phi))}{d\phi(\phi)} \left[ \sigma_1(\phi) \right] \left[ \tilde{\phi}_1(\phi) \right] \right\} e^{2\sigma(\phi)}$$

The first theorem results from (13) with $\tilde{\phi}(\phi) = \phi$. It is easy to see that if the functions $\tilde{A}(\phi), \tilde{B}(\phi), \tilde{C}(\phi)$ satisfy the conditions (11) then the functions $A(\phi), B(\phi), C(\phi)$ satisfy (10). Therefore for any given $\tilde{B}(\phi)$ and $B(\phi)$ one can take $\sigma(\phi) = \frac{1}{2} \ln \left( \frac{\tilde{B}(\phi)}{B(\phi)} \right)$ that complete the proof.

Now we suppose that $\tilde{\phi}(\phi)$ is arbitrary function and require the action to be invariant under the reparametrization plus conformal transformation, that evidently gives (12).

It is interesting to consider a very simple example of that how Theorem 2 works. If $\tilde{\phi}(\phi) = \phi e^{-\rho(x)}$ where $\rho(x)$ is some arbitrary function of the spacetime variables, and $\tilde{B}(\phi) = \frac{1}{12} \phi^2$ then we find $\tilde{B}(\phi) = \frac{1}{12} \rho^2 e^{-2\rho(x)}$ from what follows $\sigma(\phi) = \rho(x)$. Therefore for the particular case of the conformally coupled with gravity scalar field the symmetry established in the Theorem 2 is nothing but usual conformal symmetry.

And so we have found that the theories (1) are distinguished (one can say labeled) by the form of the functions $A, B, C$. One can imagine some three dimensional space where the functions $A, B, C$ play the roles of coordinates. One-dimensional line in this space is composed by the theories which satisfy (10). All of

\footnote{Indeed we suppose that all the functions are mathematically acceptable for our manipulations.}
them are related with each other by the conformal transformation described in Theorem 1. Next, all of them with one exception of General Relativity possess an extra conformal-like symmetry according to the Theorem 2. Since all the models of this class are conformally equivalent to General Relativity they all have the same physical content and can be regarded as different frames for description of gravity. In particular, it is possible to obtain exact solutions for any of such theories with the use of conformal transformation in any of the known solutions of Einstein Gravity with (or without) cosmological constant.

It is interesting to consider the possibility of the soft breaking of the new conformal-like symmetry. To do this it is useful to construct it’s Noether identity. Taking into account the transformation rules from the Theorem 2, one can easily derive such identity in the form

$$B_1(\phi) g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} - B(\phi) \frac{\delta S}{\delta \phi} = 0 \quad (14)$$

where the factor $-\frac{2B}{B_1}$ stands for the conformal weight of the scalar field $\phi$ which depends on the form of the function $B(\phi)$. It is an analog (one can say generalization) of the ordinary conformal weight “−1” of the field $\phi$. The eq. (14) is the operator form of the symmetry transformation established by Theorem 2. It shows that in the presence of the conformal-like symmetry the equations of motion are linearly dependent.

The soft breaking of the symmetry means that the functions $A$ and $B$ satisfy the symmetry condition (14) whereas the restrictions on the potential term $C$ are not imposed. It turns out that only the invariant form of $C(\phi)$ is consistent with the equations of motion. From (14) follows that the kinetic (that is $A$ and $B$ dependent) parts of the equations of motion are linearly dependent. Substituting an arbitrary function $C(\phi)$ into (14) we arrive at the differential equation for $C$ : $C_1(\phi)B(\phi) = 2C(\phi)B_1(\phi)$ that leads to $C(\phi) = B^2(\phi) \cdot \text{const}$. Thus in a pure theory without matter only the symmetric form of $C(\phi)$ is consistent with the equations of motion and any soft symmetry breaking is forbidden. For the standard conformal symmetry this was pointed out by Ng [22]. One can easily check that this statement is correct even if we add the action of matter, if this matter does not depend on the field $\phi$. Thus if we consider the theory with $2AB - (B_1)^2 = 0$ then only in the case $C(\phi) = \lambda B^2(\phi), \lambda = \text{const}$ there can exist any solutions of the dynamical equations.

3 Divergences and conformal anomaly

The next purpose of the present paper is to investigate the symmetric version of the theory (1), (10) on quantum level. According to the Theorems 1,2 all the models which possess an extra conformal-like symmetry are conformally equivalent to General Relativity with cosmological constant. The different versions of the symmetric conformally equivalent models can be labeled by the values of function $B(\phi)$ and constant $\lambda$, as $A(\phi) = \lambda B^2(\phi)$ and $C(\phi) = \lambda B^2(\phi)$. It is useful to denote the action of the symmetric theory as $S_{B(\phi),\lambda}$. General Relativity with cosmological constant corresponds to $S_{\gamma,\lambda}$ where “−$\gamma$” is an inverse Newtonian constant. Thus our calculation of the one-loop divergences in general $S_{B(\phi),\lambda}$ theory may be considered as the calculation for the special case of $S_{\gamma,\lambda}$ in a conformally transformed quantum variables. For the sake of brevity we shall denote as $S_{B(\phi),\lambda}$ only the action of the theory with a nonconstant $B(\phi)$ and preserve the notation $S_{B(\phi),\lambda}$ for General Relativity.

Before starting the calculations let us say some words about what result we can expect. The general theory $S_{B(\phi),\lambda}$ differs from $S_{\gamma,\lambda}$ in one respect. The first one has one more field variable that is compensated by an extra conformal-like symmetry. On classical level both theories are equivalent. However on quantum level the equivalence may be broken by anomaly which can violate the symmetry. The example of Weyl conformal gravity has learned us that in quantum gravity the conformal anomaly can affect the one-loop divergences already [13,16]. The Weyl (conformal) gravity is higher derivative theory where the renormalizability is disturbed only by the conformal anomaly. Our purpose here is to check whether the conformal anomaly exists for the second derivative theory under consideration. Since the source of anomaly is the noninvariance of the measure of the path integral [14] our study concerns the divergent anomalous part of the Jacobian of an arbitrary conformal transformation from $S_{\gamma,\lambda}$ to $S_{B(\phi),\lambda}$.

The simple consideration based on power counting shows that the theory $S_{B(\phi),\lambda}$ is nonrenormalizable just as General Relativity. The one-loop counterterms contain the terms of fourth order in derivatives. The

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5In fact this noninvariance is caused by the UV divergences because the regularization scheme doesn’t preserve both diffeomorphism and conformal invariance. It is quite possible that the IR effects can also be relevant, but this is not clear yet.
The most general action of this type has the form \(^{20, 21}\):

\[
\Gamma_{\text{div}}^{\text{1-loop}} = \frac{1}{16\pi^2(n-4)} \int d^4x \sqrt{-g} c_w C^2 + c_C R^2 + c_R R(\nabla \phi)^2 + c_3 R(\Box \phi) + c_6 R_{\mu \nu} \nabla_{\mu} \phi \nabla_{\nu} \phi + \\
c_7 R + c_8 (\nabla \phi)^4 + c_9 (\nabla \phi)^2 (\Box \phi) + c_{10} (\Box \phi)^2 + c_{11} (\nabla \phi)^2 + c_{12} + (s.t.) \tag{15}
\]

where \(n\) is the parameter of dimensional regularization, \(C^2 = C_{\mu \nu \alpha \beta} C^{\mu \nu \alpha \beta}\) is the square of Weyl tensor and \((\nabla \phi)^2 = g^{\mu \nu} \nabla_{\mu} \phi \nabla_{\nu} \phi\). "s.t." means "surface terms". The functions \(c_{w, r, 4, \ldots, 12}\) depend on \(B(\phi), \lambda\) and on the derivatives of \(B(\phi)\). One can easily check the surface form of the other possible structures (see also \(^{27, 21}\)).

We remark that the conformal-like invariance of the one-loop counterterms requires an enormous cancellation of divergences. Let us consider, for simplicity, the particular case of the conformal scalar theory \(B(\phi) = \frac{1}{12} \phi^2, A(\phi) = \frac{1}{2}, B(\phi) = \lambda \phi^4\). The conformal transformation has the form \(\phi \rightarrow \phi' = \phi e^{-\rho(x)}, g_{\mu \nu} \rightarrow g_{\mu \nu}' = g_{\mu \nu} e^{2\rho(x)}\). It is fairly easy to see that even for the global transformation \(\rho = \text{const}\) the expression \(^{14}\) is invariant only when \(c_4 = c_5 = \ldots = c_{11} = 0\). If one considers the local conformal transformation, then it is necessary to have \(c_4 = 0\) as well. And so the one-loop counterterms preserve the symmetries of the classical action if and only if they are given by the pure Weyl term \(\int d^4x \sqrt{-g} C_{\mu \nu \alpha \beta}^2\), and all other terms are cancelled for any form of \(B(\phi)\). As will be shown below it is not the case.

Alternatively one can suppose that the transformation rule for the scalar field is changed and in the counterterms it becomes inert to conformal transformation. This can be achieved by the use of the special conformal regularization \(^{14, 23, 13}\) in a manner similar to the last reference. In this case the invariance of \(^{15}\) under the global conformal transformation is fulfilled. The invariance under local transformation requires \(c_r = 0\) and moreover other terms have to compose the conformal invariant expressions established in \(^{14, 27}\). Note that the conformal regularization is not safe if we deal with the quantum gravity theory. In fact it corresponds to some change of variables in the path integral, that just leads to anomaly. Moreover, this change of variables is nonlocal and therefore it can give contribution to divergences \(^{16}\). In any case we can trace the \(R^2\) counterterm which will certainly indicate to the violation of the conformal-like symmetry in the one-loop divergences, just as it happens in the Weyl conformal gravity.

Now we comment the relation between the different versions of our theory with the conformal-like symmetry and General Relativity on quantum level. In a general (nonsymmetric) metric-dilaton theory \(^{6}\) the use of equations of motion enables one to reduce the one-loop counterterms to the structures of \(c_w, c_r, c_7, c_{12}\) types only \(^{21}\). Then one can fine tune three functions \(A(\phi), B(\phi), C(\phi)\) and provide the on shell renormalizability at one loop. It is not clear ad hoc that it is possible to make the same in a theory \(S_{B(\phi), \lambda}\) with an extra conformal-like symmetry. The symmetry results to that the equations of motion are linearly dependent \(^{14}\). Therefore one can not use those equations to cancel as much counterterms as in general (nonsymmetric) theory \(^{6}\). Hence the equivalence of the theory \(S_{B(\phi), \lambda}\) with the General Relativity on quantum level also requires the strong cancellation of the divergences. Moreover the equivalence with the General Relativity contradicts to the conformal-like invariance of the counterterms. The one-loop calculation below is called to check whether any of those cancellations really takes place or the conformal-like symmetry established in the Theorem 2 is anomalous.

### 4 One-loop calculation

In this section we shall present in some details the calculation of the one-loop counterterms of the theory \(S_{B(\phi), \lambda}\) with an arbitrary \(B(\phi)\). For our purposes we shall apply the background field method and the Schwinger-De Witt technique \(^{23}\) (see also \(^{24}\) for the introduction). The features of the metric-dilaton theory leads to the necessity of some modifications of the calculational scheme, basically developed in the similar two-dimensional theory \(^{25}\) and recently applied to the general theory \(^{6}\). The starting point of the calculations is the usual splitting of the fields into background \(g_{\mu \nu}, \phi\) and quantum \(h_{\mu \nu}, \varphi\) ones

\[
\phi \rightarrow \phi' = \varphi + \phi, \quad g_{\mu \nu} \rightarrow g_{\mu \nu}' + h_{\mu \nu}, \quad h_{\mu \nu} = \bar{h}_{\mu \nu} + \frac{1}{4} g_{\mu \nu} h, \quad h = h^{\mu}_{\mu} \tag{16}
\]

where we separated the trace and traceless parts of the quantum metric for the sake of convenience. The one-loop effective action is given by the standard general expression

\[
\Gamma = \frac{i}{2} \text{Tr} \ln \hat{H} - i \text{Tr} \ln \hat{H}_{\text{ghost}} \tag{17}
\]

\(^{6}\)It is not clear whether it is possible to do it for arbitrary \(B(\phi)\)
where $\hat{H}$ is the bilinear form of the action $S_{B(\phi),\lambda}$ with added gauge fixing term and $\hat{H}_{\text{ghost}}$ is the bilinear form of the gauge ghosts action. Since the theory under consideration is invariant under two – diffeomorphism and conformal-like symmetries, an additional gauge condition is necessary to fix the last one. We shall follow Fradkin and Tseytlin \cite{13} who have derived the counterterms in Weyl gravity and introduce this condition in the form $h = 0$. Some note is in order. The condition $h = 0$ does not touch the conformal-like invariance in the sector of background fields. However this invariance is violated by the covariant gauge fixing term

$$S_{gf} = \int d^4x \sqrt{-g} \chi_\mu \frac{\alpha}{2} \chi^\mu, \quad \chi_\mu = \nabla_\alpha \hat{h}^\alpha_\mu + \beta \nabla_\mu \varphi$$

where $\alpha, \beta$ are some functions of the background dilaton, which can be fine tuned to make the calculations more compact. For instance, if one choose these functions as follows

$$\alpha = -B, \quad \beta = -\frac{B_1}{B}$$

then the bilinear part of the action $S + S_{gf}$ and the operator $\hat{H}$ has especially simple (minimal) structure

$$(S + S_{gf})^{(2)} = \int d^4x \sqrt{-g} \omega \hat{H} \omega^T$$

$$\hat{H} = \hat{K} \square + \hat{L}_\rho \nabla^\rho + \hat{M}$$

Here $\omega = (\hat{h}_{\mu \nu}, \varphi)$, $T$ means transposition,

$$\hat{K} = \left( \begin{array}{ccc} \frac{B}{4} \left[ g^{\mu \nu} g_{\alpha \beta} - \frac{1}{4} g^{\mu \nu} g^{\alpha \beta} \right] & 0 & \frac{B_1^2}{B^2} \\
0 & \frac{B_2}{B} & 0 \\
\frac{B_1^2}{B^2} & 0 & \frac{B_2}{B} \end{array} \right)$$

and the components of $\hat{L}_\rho$ and $\hat{M}$ can be easily extracted from the similar expression \cite{21} for the general theory \cite{14} with the help of condition $h = 0$.

To separate the divergent part of $\text{Tr} \ln \hat{H}$ we rewrite this trace in a following way.

$$\text{Tr} \ln \hat{H} = \text{Tr} \ln \hat{K} + \text{Tr} \ln \left( \hat{1} \square + \hat{K}^{-1} \hat{L}_\mu \nabla_\mu + \hat{K}^{-1} \hat{M} \right)$$

The first term does not give contribution to the divergences whereas the second term has standard minimal form and can be easily estimated with the use of Schwinger-DeWitt method. The bilinear form of the ghost action also has the minimal form

$$\hat{H}_{\text{ghost}} = g^{\mu \alpha} \square + \gamma (\nabla^\alpha \phi) \nabla_\mu + \gamma (\nabla^\mu \nabla^\alpha \phi) + R^{\mu \alpha}$$

and it’s contribution to the divergences can be easily derived with the use of the standard technique.

Summing up the divergences of both terms of eq.\cite{13} we find that the one-loop divergences have the form \cite{13} that is in full accord with the power counting consideration. The coefficient functions $c$ have the form

$$c_w = -\frac{17}{120}, \quad c_r = \frac{5}{24} - \frac{t_1}{6t^2} + \frac{t_1^2}{8t^4}$$

$$c_4 = -\frac{708 t^6 + 368 t^4 t_1 - 247 t^2 t_1^2 + 45 t_1^3}{96 t^4} + \frac{(t^2 + 3t_1) t_2}{12 t^3}$$

$$c_5 = \frac{7t}{4} - \frac{11t_1}{12t} + \frac{t_1^2}{t^3}, \quad c_6 = \frac{5t^2}{2} - 3t_1, \quad c_7 = \frac{9C}{2B} + \frac{C_2}{B} \left( \frac{1}{6t^2} - \frac{t_1}{4t^4} \right)$$

$$c_8 = -\frac{1528 t^8 - 5584 t^6 t_1 + 3804 t^4 t_1^2 - 1804 t^2 t_1^3 + 97 t_1^4}{256 t^4} + \frac{(58 t_3^2 + 16 t_1^2 + 19 t_2^2 + 2 t_2^2) t_2}{32 t^3}$$

$$c_9 = \frac{27 t^2}{8} + \frac{31 t_1^2}{16 t^2}, \quad c_{10} = \frac{9 t^2}{2B} + \frac{C_2}{B} \left( 24 t_1^2 + 120 t_1^2 t_1 - 111 t_1^2 + 24 t_2^2 \right)$$

$$c_{11} = \frac{9C(t_1^2 - 32 t_1^2 + 4 t_1^2 t_2)}{16 B t^2} - \frac{9 t C_1}{2B} + \frac{C_2(24 t_1^4 + 120 t_1^2 t_1 - 111 t_1^2 + 24 t_2^2)}{32 B t^4} + \frac{C_3(2t_1 - t_2)^2}{2 B t^3}$$
where we have denoted $t = \frac{\phi_0}{B}$ for brevity. It is remarkable that the dimensionless divergences depend only on $t$. Moreover, $t$ enters in the denominators and hence we cannot put $B(\phi)$ equal to constant and so be back to General Relativity. The source of this is that the transformation (21) that we have used is singular at $t = 0$.

The counterterms (23) are not invariant under the conformal-like transformation of Theorem 2. In particular, if we start with the conformal metric-scalar theory with $B(\phi) = \frac{1}{\lambda^2} \phi^2$, then (23) differs from conformal invariant dilaton action constructed in [16, 27] and this difference cannot be removed by the transformations from Theorems 1, 2. Let us now make some comments concerning the meaning of the $t$ dependence and the lack of conformal invariance. On classical level the model under consideration is conformally equivalent to General Relativity. On quantum level, within the background field method that we are using here, the difference consists in:

i) Change of quantum variables related with conformal transformation and with the consequent separation of fields into background and quantum parts.

ii) Conformal transformation and reparametrization of the background fields. Since we have only two arbitrary functions related with the last point, it is not possible to simplify (23) and thus provide the independence of $t$.

The crucial question is: whether the on shell one-loop finiteness of General Relativity is lost in a new frame? The positive answer indicates to the anomaly which break the conformal-like symmetry on quantum level and gives the nontrivial contribution to the one-loop counterterms. The detailed analysis of the on shell one-loop divergences in a general nonsymmetric theory [1] have been presented in [24]. Therefore we can discuss only the features of the conformal model. In conformal theory the amount of the independent equations of motion is less than in general one because of identity (14). That is why the equations of motion are insufficient to remove all the $\phi$ dependent counterterms in contrast to (21). One can remove, for instance, the $c_5, c_6, c_8, c_{10}$ type structures with the help of classical equations of motion. However the $c_\omega, c_7, c_4, c_8, c_{11}$ ones remain and violate the on shell one-loop finiteness [24].

5 Conclusion

We have considered the special class of the metric-dilaton theories [1] which satisfy the conditions (11). In the Theorem 1 we have proved that all the actions $S_{B(\phi), \lambda}$ of this class are converted into each other under the local conformal transformation of the metric field only. Two known actions namely the Einstein-Hilbert action of General Relativity and the action of conformal scalar field belong to this class and therefore all the $S_{B(\phi), \lambda}$ theories can be regarded as different frame for the description of Einstein gravity with (or without) cosmological constant. In the Theorem 2 we have shown that all the $S_{B(\phi), \lambda}$ possess some conformal-like symmetry including an arbitrary reparametrization of the scalar field supplemented by local conformal transformation of the metric. On quantum level this symmetry is disturbed by anomaly. The new anomalous degree of freedom starts to propagate because of quantum effects. Since the theory under consideration in not renormalizable by power counting, the anomalous contributions enlarge the amount of the divergent structures and finally the theories $S_{B(\phi), 0}$ with the nonconstant $B$ are not finite on shell whereas General Relativity is. Our result indicates to the conformal noninvariance of the $Diff$-invariant measure of the path integral in four dimensional space-time that was established earlier for the higher derivative Weyl gravity [15, 16]. Just as in the last case the conformal anomaly in the theory $S_{B(\phi), \lambda}$ gives contributions to the one-loop divergences. Since one can conclude that the effects of conformal anomaly in quantum gravity are essentially stronger as compared with the conformal theories of matter fields in an external gravitational field (one can see, for example, the review of Duff [24] for the references on the subject). In particular, the anomaly leads to the parametrization of gravity which we are using here does not satisfy the general theorem of Tyutin on the parametrization dependence of the effective action in quantum field theory [28]. The contribution of Jacobian of the conformal transformation breaks the symmetries of the theory. We remark that such things never happens with the gauge parameters dependence which one can keep under control at quantum level [24].

Not so far ago the conformal frame and conformal transformations has been investigated in 2 and $2 + \epsilon$ dimensional quantum gravity [21, 24, 25]. In that dimension the Einstein gravity and also the metric-dilaton model [1] are renormalizable by power counting. That is why the conformal anomaly does not lead to the

\[ c_{12} = \frac{9 C^2}{2 B^2} + \frac{C_2^2}{8 B^2 t^4} \]  (23)
nonrenormalizability in this $d = 2$. Thus one can regard our work as some investigation of the difference between quantum gravity theories in two and four dimensions.

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Appendix

In this Appendix we consider an interesting symmetry property which takes place in the theory with the action

$$ S[g_{\mu\nu}; B(\phi), B(\phi); \lambda, \tau] = S_{B(\phi), \lambda} + S_{B(\phi), \tau} \quad (A1) $$

where we use our notation

$$ S_{B(\phi), \lambda} = \int d^4 x \sqrt{-\bar{g}} \left\{ A(\phi) g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + B(\phi) R + \lambda B^2 \right\} $$. \quad (A2)

for $A = A[B] = \frac{2B^2}{\gamma B^2 - 1} = 0$ and the same for $S_{B(\phi), \tau}$. Indeed the theory (A1) belongs to general nonsymmetric class of the models [9] rather than to the models with conformal-like symmetry. However this theory manifests a very interesting property under the transformation of Theorem 1.

At first we shall see how the general action (A1) behaves under the conformal transformation of the metric

$$ g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{2\sigma(\phi)} \quad (A3) $$

In four dimensions the geometrical quantities transform as

$$ \sqrt{-\bar{g}} = \sqrt{-\bar{g}} e^{4\sigma(\phi)}, \quad g^{\mu\nu} = \bar{g}^{\mu\nu} e^{-2\sigma(\phi)} $$

$$ R = e^{-2\sigma(\phi)} [\bar{\bar{R}} - 6\bar{\sigma} - 6(\nabla_{\mu}\bar{\sigma})(\nabla^{\mu}\bar{\sigma})] $$. \quad (A4)

Substituting (A4) into (A2) and then to (A1) we find the following transformation rule for the last one.

$$ S[g_{\mu\nu}; B(\phi), B(\phi); \lambda, \tau] = S\left[ \bar{g}_{\mu\nu}; B(\phi)e^{-2\sigma(\phi)}, B(\phi)e^{-2\sigma(\phi)}; \lambda, \tau \right] $$. \quad (A5)

Now we can explore an interesting particular case of the theory (A1) and transformation (A3). Let both things are chose in such a manner that $B(\phi) = \gamma = const$ and also $B(\phi)e^{2\sigma(\phi)} = \bar{\gamma} = const$. The last condition immediately gives $e^{2\sigma(\phi)} = \frac{\bar{\gamma}}{B(\phi)}$. Then (A5) becomes

$$ S[g_{\mu\nu}; \gamma, B(\phi); \lambda, \tau] = S\left[ \bar{g}_{\mu\nu}; \frac{\bar{\gamma}}{B(\phi)}; \bar{\gamma}; \lambda, \tau \right] $$. \quad (A6)

If one put $\bar{\gamma} = \gamma^{-1}$ then the last equation takes especially simple form

$$ S[g_{\mu\nu}; \gamma, B(\phi); \lambda, \tau] = S\left[ \bar{g}_{\mu\nu}; \frac{1}{\gamma}; \frac{1}{B(\phi)}; \lambda, \tau \right] $$. \quad (A7)

that deserve to be written in an explicit form

$$ \int d^4 x \sqrt{-\bar{g}} \left\{ \gamma R + \lambda \gamma^2 + A[B] g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + B(\phi) R + \tau B^2 \right\} $$

$$ = \int d^4 x \sqrt{-\bar{g}} \left\{ \frac{1}{\gamma} \bar{\bar{R}} + \frac{\tau}{\bar{\gamma}^2} + \frac{1}{B(\phi)} \bar{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{B(\phi)} \bar{\bar{R}} + \lambda \frac{B^2}{B(\phi)} \right\} $$

(A8)

The above transformation has a dual form. Since it describes the inverion of the coupling constant $\gamma$ and function $B(\phi)$ this transformation shows that there is some link between strong and week coupling regimes in some of the models [10].

The particular case of (A8) with $B(\phi) = \frac{1}{2} \phi^2$ and $\lambda = \tau = 0$ corresponds to the second theorem of Bekenstein [9] whereas the first theorem of [8] results from the conformal equivalence of the different versions of [1] which do not satisfy (10) (see also [10] for the case of the nonzero $\lambda, \tau$).
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