Mass Spectrum of $D = 11$ Supergravity

on $\text{AdS}_2 \times S^2 \times T^7$

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ABSTRACT

We compute the Kaluza-Klein mass spectrum of the $D = 11$ supergravity compactified on $\text{AdS}_2 \times S^2 \times T^7$ and arrange it into representations of the $SU(1, 1|2)$ superconformal algebra. This geometry arises in M theory as the near horizon limit of a $D = 4$ extremal black-hole constructed by wrapping four groups of M-branes along the $T^7$. Via AdS/CFT correspondence, our result gives a prediction for the spectrum of the chiral primary operators in the dual conformal quantum mechanics yet to be formulated.
1 Introduction

Among all known examples of the $AdS$/CFT correspondence [1, 2, 3, 4], the least understood is the $AdS_2$/CFT$_1$ case. The $D = 1$ conformal field theory (CFT), or conformal quantum mechanics (CQM), has not been formulated and therefore no quantitative comparison between the two sides of the duality has been made. See [5, 6] for proposals on the CQM and [7, 8, 9, 10] for progress made in the bulk theory.

One of the most elementary check of the correspondence is to compare the spectrum of the two theories. In particular, the Kaluza-Klein (KK) mass spectrum of the supergravity (SUGRA) on $AdS$ is identified with the spectrum of chiral primary operators in the dual CFT. One may hope that the KK spectrum of a SUGRA on $AdS_2$ may give a clue to formulate the dual CQM.

The goal of this paper is to compute the KK spectrum in the cases where the $AdS_2$ is part of a string/M theory vacuum. We specialize in the example of $D = 11$ SUGRA compactified on $AdS_2 \times S^2 \times T^7$. \[1\] We consider only the zero modes in $T^7$. From the string theory point of view, this theory is a valid approximation when $R >> r, \tilde{r}$, where $r, \tilde{r}$ are the radius and the dual radius of $T^7$ respectively, and $R$ is the radius of the sphere. In what follows, we will put $R$ to 1 for simplicity. To obtain this geometry from M theory, one begin with compactifying M theory on $T^7$ with the following brane configuration [11]. \[1\]

\[
\begin{array}{cccccccccc}
\text{Brane} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
M2 & x & x & x & x & x & x & x & x & x & x & x \\
M2 & x & x & x & x & x & x & x & x & x & x & x \\
M5 & x & x & x & x & x & x & x & x & x & x & x \\
M5 & x & x & x & x & x & x & x & x & x & x & x \\
\end{array}
\]

With suitable choice of the orientation of the branes, this configuration breaks $\mathcal{N} = 8$ supersymmetry (SUSY) of the $D = 4$ theory to $\mathcal{N} = 1$. When the number of branes in each group is all equal, the background metric describes a direct product of an extremal $D = 4$ Reissner-Nordström black-hole and a $T^7$. See section 3. of [11] for more details. The $AdS_2 \times S^2$ spacetime arising as the near horizon geometry of this black-hole is known as the Bertotti-Robinson metric [12]. Note that the brane configuration at hand approaches the Bertotti-Robinson metric in the near horizon limit even when the four charges are not equal.

The number of SUSY is doubled in the near-horizon limit as usual, so we have $D = 4, \mathcal{N} = 2$ SUSY. The super-isometry group of the theory is $SU(1, 1|2)$. The KK spectrum form representations of the $SU(1, 1|2)$ superalgebra.

1We thank Seungjoon Hyun for bringing our attention to this example.
2There are many other brane configurations that are related to this one by U-duality. Three M5 branes intersecting over a line with momentum flowing along the line is one such example.
The methods of the computation used in this paper are well known from higher dimensional examples. There are two approaches to the problem; one is direct SUGRA calculation [14]-[18], and the other uses representation theory of superconformal algebra together with duality symmetry of SUGRA [19]-[24]. We will adopt the first approach and explicitly calculate the spectrum, starting from the $D = 11$ SUGRA lagrangian. Although we will be mainly interested in the modes which have bulk degrees of freedom. However, as was noted in ref.[29], we cannot ignore the boundary modes completely because one of them forms a multiplet with bulk modes. We will make further comments on this point later.

This paper is organized as follows. In section 2, we review the $SU(1, 1|2)$ superalgebra and its representation theory following [19, 20]. In section 3, as a warm-up exercise we compute the spectrum of a toy model, namely the minimal $D = 4, \mathcal{N} = 2$ SUGRA. This model illustrates many important aspects of the compactification on $AdS_2 \times S^2$ in a simple setting. In section 4, we present a summary of our main result. In section 5 and 6, we sketch the computation of bosonic and fermionic mass spectrum of the “realistic” model obtained from the $D = 11$ supergravity.

As this work was being completed, we received [29] which has overlap with section 3 of this paper. While this paper was being submitted to hep-th e-print archive, we received [30] which considered the same model in a manifestly U-duality covariant way.

2 Review of the $SU(1, 1|2)$ Superconformal Algebra

The $SU(1, 1|2)$ superconformal algebra is defined by the following commutation relations

\[ [L_m, L_n] = (m - n)L_{m+n}, \quad [J^a, J^b] = i\epsilon^{abc}J^c, \quad [L_m, J^a] = 0, \tag{2.1a} \]

\[ [L_m, G^{a\tilde{a}}_r] = (\frac{1}{2}m - r)G^{a\tilde{a}}_{m+r}, \quad [J^a, G^{a\tilde{a}}_r] = -\frac{i}{2}(\sigma^a)^{\alpha\beta}G^{\alpha\tilde{a}}_r, \tag{2.1b} \]

\[ \{G^{a\tilde{a}}_r, G^{\beta\tilde{\beta}}_s\} = \epsilon^{\tilde{a}\tilde{\beta}}\{\epsilon^{\alpha\beta}L_{r+s} - (r - s)(\sigma^a)^{\alpha\beta}J^a\}. \tag{2.1c} \]

and the Hermiticity conditions

\[ L^\dagger_m = L_{-m}, \quad (J^a)^\dagger = J^a, \quad (G^{a\tilde{a}}_r)^\dagger = \epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{\beta}}G^{\alpha\tilde{\beta}}_{-r}. \tag{2.2} \]

The bosonic generators $L_{+1,0,-1}$ and $J^{0,1,2}$ generate the $SL(2, \mathbb{R})$ conformal group and the $SU(2)$ $R$-symmetry group, respectively. We have eight supercharges all together; $G^{a\tilde{a}}_{\pm 1/2}$ carry $L_0$ charge $\mp \frac{1}{2}$ and transform in $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)_R \times SU(2)_{Aut}$, where the second $SU(2)$ is a global automorphism.
| Lowest weight states | \( j \) | \( L_0 \) | Degeneracy |
|----------------------|------|--------|-----------|
| \(|0\rangle\)         | \( n/2 \) | \( n/2 \) | \( n + 1 \) |
| \(G^{++}_{1/2}|0\rangle, G^{+-}_{1/2}|0\rangle\) | \((n - 1)/2\) | \((n + 1)/2\) | \(2 \times n\) |
| \(G^{++}_{-1/2}G^{+-}_{1/2}|0\rangle\) | \((n - 2)/2\) | \((n + 2)/2\) | \(n - 1\) |

Table 1: The short multiplets of \(SU(1, 1|2)\) superconformal algebra labeled by an integer \(n\).

One explicit way to find the representations of a superalgebra is the oscillator construction [25, 26, 27]. The oscillator representation of the generators of \(SU(1, 1|2)\) is given by

\[
L_1 = \vec{a}_1 \cdot \vec{a}_2, \quad L_0 = \frac{1}{2}(\vec{a}_1 \cdot \vec{a}_1 + \vec{a}_2 \cdot \vec{a}_2), \quad L_{-1} = \vec{a}_2 \cdot \vec{a}_1 \\
J^+ = \vec{\psi}_1 \cdot \vec{\psi}_2, \quad J^0 = \frac{1}{2}(\vec{\psi}_1 \cdot \vec{\psi}_1 - \vec{\psi}_2 \cdot \vec{\psi}_2), \quad J^- = \vec{\psi}_2 \cdot \vec{\psi}_1
\]

(2.3a) (2.3b)

\[
G_{-1/2} = \begin{bmatrix} G_{-1/2}^- & G_{-1/2}^+ \\ G_{-1/2}^+ & G_{-1/2}^+ \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\psi}_1 & -\vec{a}_2 \cdot \vec{\psi}_2 \\ \vec{a}_2 \cdot \vec{\psi}_2 & \vec{a}_1 \cdot \vec{\psi}_1 \end{bmatrix}, \quad G_{+1/2} = \begin{bmatrix} \vec{a}_1 \cdot \vec{\psi}_1 & -\vec{a}_2 \cdot \vec{\psi}_2 \\ \vec{a}_2 \cdot \vec{\psi}_2 & \vec{a}_1 \cdot \vec{\psi}_1 \end{bmatrix}
\]

(2.4)

where \(\vec{a}_1, \vec{a}_2\) are \(n\)-component vectors of bosonic creation and annihilation operators, and \(\vec{\psi}_1, \vec{\psi}_2\) are the fermionic counterparts. It is straightforward to see that they satisfy all the commutation relations and Hermiticity conditions (2.1a)-(2.2) except (2.1c), which is modified as

\[
\{G^{\alpha\bar{\alpha}}, G^{\beta\bar{\beta}}\} = \epsilon^{\bar{\alpha}\bar{\beta}}\{\epsilon^{\alpha\beta}L_{r+s} - (r - s)(\sigma^a)(\epsilon^{\alpha\beta}J^a + \epsilon^{\alpha\beta}I), \quad I \equiv \frac{1}{2}(\vec{a}_1 \cdot \vec{a}_2 - \vec{a}_2 \cdot \vec{a}_1) - \frac{1}{2}(\vec{\psi}_1 \cdot \vec{\psi}_1 - \vec{\psi}_2 \cdot \vec{\psi}_2).
\]

(2.5a) (2.5b)

The extra \(U(1)\) generator \(I\) must be added in order for the algebra to be closed. However, since \(I\) commutes with all the other generators, we may work in the restricted Fock space on which \(I = 0\), where the algebra precisely reduces to that of \(SU(1, 1|2)\).\footnote{We thank Jan de Boer for a correspondence on this point.}

For a given integer \(n\), the oscillator vacuum is identified with the lowest \(J^0\)-weight state of a chiral primary operator. We act \(G^{++}_{-1/2}\) on the vacuum to obtain the lowest weight states of other primary operators in the supermultiplet. Higher weight states of a given operator are obtained by acting \(J^+\) on the lowest weight state.

The quantum numbers of each state is easily computed using the explicit oscillator representation of the generators (2.3a), (2.3b). The quantum numbers of the lowest weight state of each primary operator are summarized in Table 1. The total angular momentum \(j\) is defined by \(\hat{j}^2 = j(j + 1)\). The number of states for each primary operator, \(2j + 1\), is also included in the table.
Figure 1: The complete KK spectrum of the toy model. Each circle in the figure represents a state which has a definite value of \( h \) and \( j \). The crossed circles correspond to the boundary states. The degeneracy \((2j + 1)\) of each state is included in the circle. The states belonging to the same \( SU(1,1|2) \) multiplet is connected by a dotted line. The two KK towers on the top row satisfy \( h = j \) and correspond to chiral primary states.

3 Toy Model

As a warm-up exercise, we compute the mass spectrum of the minimal \( D = 4, \mathcal{N} = 2 \) SUGRA. It is the simplest SUGRA that admits the \( AdS_2 \times S^2 \) solution with the \( SU(1,1|2) \) superalgebra. The theory contains a single \( \mathcal{N} = 2 \) gravity multiplet whose component fields are a graviton, a massless vector and a complex gravitino.

3.1 Result

The computation in the following subsections show that the KK spectrum of the toy model contains the short multiplets in Table 1 for all even \( n \). We have two copies of each multiplet for \( n \geq 4 \) and one copy for \( n = 2 \). The result is depicted in Figure 1.

From the point of view of the SUGRA computation, each physical degree of freedom of the fields in \( D = 4 \) give a KK tower. That explains why we have four bosonic and four fermionic series of states in the spectrum. Depending on the spin and the polarization of a given field, the low lying modes \((j = 0, 1/2, 1)\) modes may be absent. Some of other low lying modes become massless and can be gauged away from the bulk spectrum. The absence of such modes is necessary in order for the KK spectrum to arrange itself into representations of \( SU(1,1|2) \).

In addition to the bulk degrees of freedom, there may be modes that are pure gauge in the bulk but can live on the boundary. The authors of [29] showed that the boundary modes indeed exist and form one \( n = 2 \) and one \( n = 1 \) representations of \( SU(1,1|2) \) algebra. We included these boundary modes in Figure 1 for completeness. In particular, we cannot ignore them since one of them forms \( n = 2 \) multiplet with bulk modes, as can be seen from the figure.
3.2 Bosonic mass spectrum

3.2.1 Setup

We normalize the fields such that the action reads

\[ 2\kappa^2 S = \int d^4x \sqrt{-G} \left\{ R - \frac{1}{4} F^2 - \bar{\psi}_m \Gamma^{mnp} \nabla_n \psi_p - \frac{i}{2} \bar{\psi}_m (F^{mn} + \frac{1}{2} F_{rs} \Gamma^{rs} \psi_n) \right\}, \quad (3.1) \]

up to terms quartic in fermions that are irrelevant to our computation. The bosonic equations of motion consist of the Einstein and Maxwell equations in vacuum.

\[ R_{mn} = \frac{1}{2} F_{ml} F^l_n - \frac{1}{8} G_{mn} F^2, \quad \nabla^m F_{mn} = 0. \quad (3.2) \]

These equations admit dyonic Reissner-Nordström black-hole solutions. The near horizon geometry of an extremal black-hole gives the \( AdS_2 \times S^2 \) solution. The radius of the \( S^2 \) is equal to the Schwarzchild radius of the black-hole. For simplicity, we consider an extremal electric black-hole with unit radius only. Then the \( AdS_2 \times S^2 \) solution reads

\[ R_{\mu\nu\lambda\sigma} = - (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}), \quad \bar{F}_{\mu\nu} = 2 \epsilon_{\mu\nu}, \]

\[ R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}, \quad \bar{F}_{\alpha\beta} = 0. \quad (3.3) \]

where the Greek letters \( \alpha, \beta \cdots \) and \( \mu\nu \cdots \) label two dimensional indices for \( AdS_2 \) and \( S^2 \) respectively. The fermions are set to zero. We are interested in the mass spectrum of the fluctuations of the fields around this background. We use the following parametrizations of the fluctuations.

\[ G_{\alpha\beta} = g_{\alpha\beta} + h_{(\alpha\beta)} + \frac{1}{2} h_2 g_{\alpha\beta}, \quad G_{\mu\nu} = g_{\mu\nu} + h_{(\mu\nu)} + \frac{1}{2} h_1 g_{\mu\nu}, \quad G_{\mu\alpha} = h_{\mu\alpha}, \quad (3.4a) \]

\[ F_{\alpha\beta} = \nabla_\alpha a_\beta - \nabla_\beta a_\alpha, \quad F_{\mu\nu} = 2 \epsilon_{\mu\nu} + \nabla_\mu a_\nu - \nabla_\nu a_\mu, \quad (3.4b) \]

where the paranthesis denotes the traceless part of a symmetric tensor.

3.2.2 Spherical harmonics decomposition and gauge choice

Each field can be decomposed into spherical harmonics. Unlike higher dimensional spheres, \( S^2 \) does not have genuine vector or tensor spherical harmonics. Vector and tensor fields are spanned by derivatives of the scalar spherical harmonics.

\[ h_{(\alpha\beta)} = \phi_j^I \nabla_{(\alpha} \nabla_{\beta)} Y^I + \phi_j^I \epsilon_{(\alpha} \nabla_{\beta)} \nabla^I Y^I \quad (j \geq 2), \quad h_2 = h_2^I Y^I, \]

\[ h_{(\mu\nu)} = h_{(\mu\nu)}^I Y^I, \quad h_1 = h_1^I Y^I, \]

\[ h_{\mu\alpha} = w^I_\mu \nabla_\alpha Y^I + v^I_\mu \epsilon_{\alpha\beta} \nabla^\beta Y^I \quad (j \geq 1), \]

\[ a_\alpha = q^I \nabla_\alpha Y^I + b^I \epsilon_{\alpha\beta} \nabla^\beta Y^I \quad (j \geq 1), \quad a_\mu = a_\mu^I Y^I. \quad (3.5) \]
The composite index $I$ specifies both the total angular momentum $j$ and the $J^0$ eigenvalue $m$. The restrictions on the value of $j$ for some fields are due to the fact that

$$\nabla_\alpha Y^{(j=0)} = \nabla (\alpha \nabla_\beta) Y^{(j=1)} = \epsilon^{\gamma (\alpha \nabla_\beta)} \nabla \gamma Y^{(j=1)} = 0. \quad (3.6)$$

Not all the modes in the above expansion are physical since the $(D = 4)$ graviton and gauge fields are subject to the gauge transformations,

$$\delta h_{mn} = \nabla_m \Lambda_n + \nabla_n \Lambda_m, \quad (3.7a)$$

$$\delta a_m = -\bar{F}_{mn} \Lambda^n + \nabla_m (\Sigma + \Lambda^n \bar{A}_n). \quad (3.7b)$$

The functions $\Lambda_m$ and $\Sigma$ are also expanded in spherical harmonics.

We need to make a choice of gauge. First, consider the case $j \geq 2$. We can gauge away $h^{(\alpha \beta)}$ completely by a suitable choice of $\Lambda_\alpha$. We then use $\Lambda_\mu$ to eliminate the $w^I_\mu$ terms. Lastly, we use $\Sigma$ to eliminate the $q^I$ terms. With this choice of gauge, we note that

$$\nabla_\alpha h_{\mu \alpha} = 0, \quad \nabla^\alpha a_\alpha = 0. \quad (3.8)$$

For $j = 1$, $h^{(\alpha \beta)}$ modes are absent, so $\Lambda_\alpha$ can be used to reduce other degrees of freedom. We find it convenient to parametrize $\Lambda_m$ and $\Sigma$ as

$$\Lambda^{(1)}_\mu = (K_\mu + \nabla_\mu X) \cdot Y, \quad (3.9a)$$

$$\Lambda^{(1)}_\alpha = P \cdot \epsilon_{\alpha \beta} \nabla^\beta Y - X \cdot \nabla_\alpha Y, \quad (3.9b)$$

$$\Sigma^{(1)} = Q \cdot Y. \quad (3.9c)$$

where the dot product means the sum over the three components of $j = 1$ spherical harmonics. We can use $X$, $K_\mu$, and $Q$ to gauge away $h_1$, $w_\mu$ and $q$, respectively. Under the gauge transformation by $P$, $v_\mu$ is shifted by $\nabla_\mu P$. This indicates that $v_\mu$ is a massless gauge field in $AdS_2$. Indeed, the mass term for $v_\mu$ is absent as we will see below. Being a gauge field in $D = 2$, $v_\mu$ has no propagating degree of freedom in the bulk. Also $h_2$ can be locally gauged away by residual gauge symmetry.

For $j = 0$, $h^{(\mu \nu)}$, $h_1$, $h_2$ and $a_\mu$ are the only modes that remain. The gauge parameter $\Lambda_\alpha$ is absent. We can use $\Lambda_\mu$ to gauge away $h^{(\mu \nu)}$. The vector $a_\mu$ becomes a gauge field in $AdS_2$ with $\Sigma$ being the gauge transformation parameter, and again has no bulk degree of freedom.

### 3.2.3 Linearized field equations

The linearized Einstein and Maxwell equations read

$$R^{(1)}_{mn}(h) = F^{(l)}_n (\nabla_m a_l - \nabla_l a_m) + F^{(l)}_m (\nabla_n a_l - \nabla_l a_n) - g_{mn} F^{kl} \nabla_k a_l$$

$$- F_{mk} F_{nl} h^{kl} + \frac{1}{2} g_{mn} F_{j}^{k} F_{i}^{j} h_{kl} - \frac{1}{4} h_{mn} F^2, \quad (3.10)$$
\[ \nabla^m (\nabla_m a_n - \nabla_n a_m) - \frac{1}{2} (2 \nabla^m h_{ml} - \nabla_l h) \tilde{F}_n^l - \nabla_m h_{ln} \tilde{F}^{ml} = 0, \]  
where the linearized Ricci tensor is defined by
\[ R^{(1)}_{mn}(h) \equiv -\nabla^2 h_{mn} - \nabla_m \nabla_n k_k + \nabla^k \nabla_m h_{nk} + \nabla^k \nabla_n h_{mk}. \]

Upon spherical harmonics decomposition, the \( \alpha \beta \) component of the Einstein equation yields the following three equations. They are the coefficients of \( g_{\alpha \beta} Y^I \), \( \nabla_{(\alpha} \nabla_{\beta)} Y^I \), and \( \epsilon_{\gamma(\alpha} \nabla_{\beta)} \nabla^\gamma Y^I \), respectively.
\[ \nabla_x^2 h^I_2 - j(j+1)(h^I_1 + h^I_2) = 4 \epsilon^{\mu \nu} \nabla_\mu a^I_\nu - 2 h^I_2 + 4 h^I_1, \]  
\[ h^I_1 = 0, \]  
\[ \nabla_\mu v^I_\mu = 0. \]

We separate the trace and traceless part of the \( \mu \nu \) component of the Einstein equation. We replace \( h_{\mu \nu} \) by \( h_{(\mu \nu)} \) in all the equations below using the constraint (3.13b).
\[ \nabla_x^2 h^I_2 - 2 \nabla^\mu \nabla_\nu h^I_{(\mu \nu)} = -4 \epsilon^{\mu \nu} \nabla_\mu a^I_\nu, \]  
\[ \nabla_x^2 h^I_{(\mu \nu)} - j(j+1)h^I_{(\mu \nu)} + 2h^I_{(\mu \nu)} = \nabla_\mu \nabla^\lambda h^I_{(\lambda \nu)} + \nabla_\nu \nabla^\lambda h^I_{(\lambda \mu)} - g_{\mu \nu} \nabla^\sigma h^I_{(\lambda \sigma)} - \nabla_{(\mu} \nabla_{\nu)} h^I_2 \]  

The \( \mu \alpha \) component of the Einstein equation splits into two pieces. They are the coefficients of \( \nabla_\alpha Y^I \) and \( \epsilon_{\alpha \beta} \nabla^\beta Y^I \), respectively.
\[ \nabla_\mu h^I_2 - 2 \nabla^\nu h^I_{(\mu \nu)} = -4 \epsilon_\mu^\nu a^I_\nu, \]  
\[ \nabla_x^2 v^I_\mu - (j^2 + j - 3) v^I_\mu = 2 \epsilon_{\mu \nu} \nabla^\nu b^I. \]

The \( \alpha \) component of the Maxwell equation splits in the same way,
\[ \nabla_\mu a^I_\mu = 0, \]  
\[ \nabla_x^2 b^I - j(j+1) b^I = 2 \epsilon_{\mu \nu} \nabla_\mu v^I_\nu. \]

The \( \mu \) component of the Maxwell equation yields a single equation,
\[ \nabla_x^2 a^I_\mu - (j^2 + j - 1) a^I_\mu = \epsilon_{\mu \nu} \nabla^\nu h^I_2. \]

### 3.2.4 Computation of the mass spectrum: \( j \geq 2 \)

Altogether, we have ten equations of motion (3.13a) - (3.17). We already used (3.13b) to eliminate \( h^I_1 \). We also note that (3.15a) implies (3.14a) for \( j \geq 1 \). So, the number of independent equations is eight. We put off the discussion of (3.14b) and (3.15a) to the end of this subsection.
Among the other six equations, (3.13c) and (3.16a) are constraints, and the other four are dynamical equations for each physical field. We first use the constraints to set on shell.

\( v_\mu = 2 \epsilon_{\mu \nu} \nabla^\nu v, \quad a_\mu = \epsilon_{\mu \nu} \nabla^\nu a \) \hspace{1cm} (3.18)

To simplify notations, we are suppressing the superscripts \( I \) in the equations from here to the end of this subsection. Inserting these in (3.13a) and (3.16b) immediately yields

\[ \nabla_x^2 x^2 h_2 - (j^2 + j - 2) h_2 - 4 \nabla_x^2 a = 0, \] \hspace{1cm} (3.19a)

\[ \nabla_x^2 b - j(j + 1)b - 4 \nabla_x^2 v = 0. \] \hspace{1cm} (3.19b)

After some manipulations, the other two equations (3.15b) and (3.17) give

\[ \nabla_x^2 v - (j^2 + j - 2)v - b = 0, \] \hspace{1cm} (3.20a)

\[ \nabla_x^2 a - j(j + 1)a - h_2 = 0. \] \hspace{1cm} (3.20b)

They are diagonalized by the following linear combinations of the fields.

\[ s_1 = b - 2(j + 2)v, \quad s_2 = h_2 - 2(j + 1)a, \] \hspace{1cm} (3.21a)

\[ t_1 = b + 2(j - 1)v, \quad t_2 = h_2 + 2ja. \] \hspace{1cm} (3.21b)

They satisfy

\[ \nabla^2 s_i - j(j - 1)s_i = 0, \] \hspace{1cm} (3.22a)

\[ \nabla^2 t_i - (j + 1)(j + 2)t_i = 0. \] \hspace{1cm} (3.22b)

In \( AdS_2 \), the scaling dimension of the operator corresponding to a scalar field is given by \[2, 3]\n
\[ h = \frac{1}{2}(1 + \sqrt{1 + 4m^2}). \] \hspace{1cm} (3.23)

This implies that the fields \( s_{1,2} \) have \( h = j \) and are chiral primaries, while \( t_{1,2} \) have \( h = j + 2 \).

It remains to analyze (3.14b) and (3.15a). Inserting (3.15a) into (3.14b) and using (3.18), we find

\[ \nabla_x^2 h^I_{(\mu \nu)} - j(j + 1)h^I_{(\mu \nu)} + 2h^I_{(\mu \nu)} = 4 \nabla_{(\mu} \nabla_{\nu)} a. \] \hspace{1cm} (3.24)

It is also possible to show that in two dimensions, (3.15a) implies

\[ (\nabla^2 + 2)h_{(\mu \nu)} = \nabla_{(\mu} \nabla_{\nu)}(h_2 + 4a). \] \hspace{1cm} (3.25)

It can be derived most easily in a light-cone coordinate and a conformal gauge. Combining these two equations, we find that \( h_{(\mu \nu)} \) is algebraically determined by \( h_2 \) and hence has no degree of freedom. This argument is valid for \( j = 1 \) also, but not for \( j = 0 \).

\[ \text{This type of transformations appear in other compactifications with electric background field strength. For example, see [16].} \]
3.2.5 \( j = 1 \)

The computation for \( j = 1 \) differs from that for \( j \geq 2 \) in two ways. First, \( h_1 \) is removed by a gauge choice rather than the constraint (3.13b) which is absent because \( \nabla_{(\alpha} \nabla_{\beta)} Y^{(j=1)} = 0 \). Second, \( v_\mu \) and \( h_2 \) has no bulk degree of freedom and can be eliminated. The equations can be diagonalized as before, and the three eigenstates are identified with the \( j = 1 \) points of the \( t_1, t_2 \) and \( s_2 \) series. The absence of the corresponding point on the \( s_1 \) series is a consequence of the fact that \( v_\mu \) is a gauge field. Note also that \( s_2 \) can be gauged away \textit{on shell} by a residual gauge degree of freedom. By \( X \) in (3.9b) with \( \nabla^2 X = 0 \), \( s_2 \) is shifted to \( s_2 + X \). Therefore it is also a boundary degrees of freedom.

3.2.6 \( j = 0 \)

We have the field equations for \( h_1 \), \( h_2 \) and \( a_\mu \)

\[
\begin{align*}
\nabla_\nu (h_1 + h_2) &= -4 \epsilon^{\mu \nu} \nabla_\mu a_\nu - 2 h_1, \\
\nabla_\nu^2 h_2 &= 4 \epsilon^{\mu \nu} \nabla_\mu a_\nu - 2 h_2 + 4 h_1, \\
\nabla^\nu (\nabla_\nu a_\mu - \nabla_\mu a_\nu) &= \epsilon_{\mu \nu} \nabla^\nu (h_2 - h_1).
\end{align*}
\]

(3.26a) \hspace{1cm} (3.26b) \hspace{1cm} (3.26c)

Recall that we gauged away \( h_{(\mu \nu)} \). Its equation of motion then gives a “Gauss law” constraint,

\[
\nabla_{(\mu} \nabla_{\nu)} h_2 = 0.
\]

(3.27)

One can easily show that (in light cone coordinate, for example) the only normalizable solution to the constraint is \( h_2 = \text{constant} \). It is consistent to set \( h_2 \) to zero. We can eliminate the gauge field \( a_\mu \) from the \( h_1 \) equation and find that \( m^2 = 2 \). This is identified with the \( j = 0 \) point of the \( t_2 \) series. This completes the derivation of the bosonic spectrum in Figure 1.

3.3 \textbf{Fermionic mass spectrum}

The linearized field equation for the fermion reads

\[
\Gamma^{mn} \nabla_n \psi_p = -\frac{i}{2} \left( F_{mn} + \frac{1}{2} F_{rs} \Gamma^{mnrs} \right) \psi_n.
\]

(3.28)

The linearized SUSY transformation law plays the role of a gauge symmetry, that is, the following variation leaves the field equation invariant:

\[
\delta \psi_m = \nabla_m \epsilon - i F_{ml} \left( \frac{1}{4} \Gamma^l \delta^m_m - \frac{1}{8} \Gamma^m_{m l} \right) \epsilon.
\]

(3.29)

It is convenient to separate the “trace” and the “traceless” part of \( \psi_\mu \) and \( \psi_\alpha \).

\[
\psi_\mu = \psi_{(\mu)} + \frac{1}{2} \Gamma_\mu \lambda, \quad \psi_\alpha = \psi_{(\alpha)} + \frac{1}{2} \Gamma_\alpha \eta \quad (\Gamma_\mu \psi_{(\mu)} = \Gamma^\alpha \psi_{(\alpha)} = 0).
\]

(3.30)
We decompose the $D = 4$ gamma matrices in terms of the $D = 2$ gamma matrices as follows:

$$
\Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^\alpha = \bar{\gamma} \otimes \tau^\alpha \quad (\bar{\gamma} = \gamma^0 \gamma^1).
$$

(3.31)

Now we can split (3.28) into four components

$$
(\nabla_x + \bar{\gamma} \nabla_y) \eta + \bar{\gamma} \nabla_y \lambda - 2 \nabla^\alpha \psi(\alpha) = -i \bar{\gamma} \lambda,
$$

(3.32a)

$$
(\nabla_x + \gamma \nabla_y) \lambda + \nabla_x \eta - 2 \nabla^\mu \psi(\mu) = -i \gamma \eta,
$$

(3.32b)

$$
\nabla(\mu) \eta + \bar{\gamma} \nabla_y \psi(\mu) = -i \bar{\gamma} \psi(\mu),
$$

(3.32c)

$$
\nabla(\alpha) \lambda + \nabla_x \psi(\alpha) = +i \bar{\gamma} \psi(\alpha),
$$

(3.32d)

where the first two equations are the traceless parts of the $\mu$ and $\alpha$ components of (3.28), respectively. The other two are the trace parts. Here, $\nabla_x$, $\nabla_y$ are the two dimensional dirac operators. The gauge transformation law also divides into four pieces.

$$
\delta \psi(\mu) = \nabla(\mu) \epsilon, \quad \delta \lambda = \nabla_x \epsilon - i \bar{\gamma} \epsilon,
$$

(3.33a)

$$
\delta \psi(\alpha) = \nabla(\alpha) \epsilon, \quad \delta \eta = \bar{\gamma}(\nabla_y \epsilon + i \epsilon).
$$

(3.33b)

Consider the spherical harmonics decomposition.

$$
\lambda = \lambda_+^I \Sigma_+^I + \lambda_-^I \Sigma_-^I, \quad \psi(\mu) = \psi_+^{(\mu)_+} \Sigma_+^I + \psi_-^{(\mu)_-} \Sigma_-^I,
$$

$$
\eta = \eta_+^I \Sigma_+^I + \eta_-^I \Sigma_-^I, \quad \psi(\alpha) = \psi_+^{(\alpha)_+} \nabla(\alpha) \Sigma_+^I + \psi_-^{(\alpha)_-} \nabla(\alpha) \Sigma_-^I,
$$

(3.34)

$$
\epsilon = \epsilon_+^I \Sigma_+^I + \epsilon_-^I \Sigma_-^I.
$$

See Appendix B for the definition and properties of spinor spherical harmonics. For $j \geq 3/2$, it is clear that one can gauge away $\eta$ completely. Then (3.32c) sets $\psi_+^{\prime(\mu)_+} = 0$. In turn, we find in (3.32b) that $\lambda$ satisfies the eom for a free massless spinor in $d = 4$. Finally, (3.32a) determines $\psi_+^{(\alpha)_+}$ algebraically in terms of $\lambda$. As a consistency check, we substitute it into (3.32d) and find the same eom for $\lambda$. Thus all that remains is to find the mass spectrum of $\lambda$. After the spherical harmonics decomposition, the equation reduces to

$$
\nabla \lambda_+ + i(j + \frac{1}{2}) \bar{\gamma} \lambda_+ = 0, \quad \nabla \lambda_- - i(j + \frac{1}{2}) \bar{\gamma} \lambda_- = 0.
$$

(3.35)

The mass eigenstates are $\xi_1 = (1 + i \bar{\gamma})E$ and $\xi_2 = (1 - i \bar{\gamma})F$ both of which have $m = j + \frac{1}{2}$.  

The computation is slightly different for $j = 1/2$. To begin with, we note the following property of the $j = 1/2$ spherical harmonics.

$$
\nabla_\alpha \Sigma_\pm = \pm i \tau^\alpha \Sigma_\pm \quad \Rightarrow \quad \nabla_y \Sigma_\pm = \pm i \Sigma_\pm.
$$

(3.36)

\footnote{We may choose $(1 - i \bar{\gamma})E$ and $(1 + i \bar{\gamma})F$. The two choices are not independent, since one can multiply either of them by $\bar{\gamma}$ to get the other.}
It has three consequences. First, modes for $\psi(\alpha)$ are absent. Second, (3.32d) is trivially satisfied. Finally, the gauge variation of $\eta_-$ vanishes for arbitrary $\epsilon_-$. We choose to gauge away $\eta_+$ and $\psi(\mu)-$ using $\epsilon_+$ and $\epsilon_-$, respectively. With these in mind, we analyze the three field equations. From the coefficients of $\Sigma_+$ in (3.32a) and (3.32c), we find that
\[ \lambda_+ = 0, \quad \psi(\mu)+ = 0. \] (3.37)

The coefficients of $\Sigma_-$ of the same equations yield
\[ (\nabla_x - i\bar{\gamma})\eta_- = 0, \quad \nabla(\mu)\eta_- = 0. \] (3.38)

These two equations together imply that $\eta_-$ has no propagating degree of freedom and can be set to zero consistently. Finally (3.32b) gives
\[ (\nabla_x - i\bar{\gamma})\lambda_- = 0, \] (3.39)

which we recognize as the $j = 1/2$ point of the $\xi_2$ series.

The scaling dimension of the operator corresponding to a spinor field in $AdS_2$ is given by
\[ h = |m| + \frac{1}{2}, \] (3.40)

which implies that the fields $\xi_{1,2}$ have $h = j + 1$.

4 Summary of the Main Result

We now turn to the model which is the main interest of this paper, namely, the one obtained from the low energy M theory. We first dimensionally reduce $D = 11$ SUGRA to obtain $D = 4$. The resulting $\mathcal{N} = 8$ SUGRA contains 1 graviton, 8 real gravitini, 28 vectors, 56 real spinors and 70 scalars. Compactification on $AdS_2 \times S^2$ keeps $\mathcal{N} = 2$ SUSY unbroken. In the $\mathcal{N} = 2$ language, we have 1 gravity, 6 gravitino, 15 vector and 10 (complex) hyper multiplets. Each multiplet has 4 bosonic and 4 fermionic real degrees of freedom. The $4 + 4$ KK towers arrange themselves into representations of $SU(1,1 + 2)$ superalgebra. Figure 2 describes the KK spectrum of each multiplet. The gravity multiplet is identical to that of the toy model. The vector multiplet is similar to the gravity multiplet, but it has two copies of the $n = 2$ representation. Gravitino multiplet contains two copies of representations for all odd $n$ except for $n = 1$. Hyper multiplet includes the $n = 1$ representation.

The analysis for the boundary modes is more complicated since one has to keep track of modes which may be removed by fixing gauges. We will concentrate on obtaining bulk modes. As in the toy model, we included the boundary degrees of freedom for the gravity multiplet in the figure.
Figure 2: The KK spectrum of the $D = 11$ SUGRA on $AdS_2 \times S^2 \times T^7$. 
Boundary degrees of freedom can arise in the gravitino multiplet as well, but are not determined by the computation here.

In the following two sections, we explain the dimensional reduction of the field equations from \( D = 11 \) to \( D = 4 \), how different fields fall into \( \mathcal{N} = 2 \) multiplets and how each multiplet produces the KK spectrum given in Figure 2.

## 5 Bosonic Mass Spectrum

### 5.1 Setup

We normalize the fields such that the action reads

\[
2\kappa^2 S = \int d^{11}x \sqrt{-G} \left\{ R - \frac{1}{24!} F^2 - \bar{\Psi}_I \Gamma^{IJK} \nabla_J \Psi_K \right\} + \frac{1}{3!} \int A \wedge F \wedge F + \int d^{11}x \sqrt{-G} \left\{ \frac{1}{4} \bar{\Psi}_I (\Gamma^{IJKLMN} \Psi_J F_{KLMN} + 12 \Gamma^{KL} G^{M}{}^{J} F^I{}_{KLM}) \Psi_J \right\}.
\]

(5.1)

The terms quartic in \( \Psi_M \) are not relevant to this paper and have been omitted. Bosonic equations of motion consist of the Einstein and Maxwell equations in vacuum.

\[
R_{MN} = \frac{1}{2 \cdot 3!} F^{M}{}_{IJK} F^{N}{}_{IJK} - \frac{1}{6 \cdot 4!} G_{MN} F^2,
\]

(5.2a)

\[
\nabla^M F_{MIJK} = \frac{1}{2 \cdot 4!} \epsilon_{IJKL}{}^{MN} F^{L}{}_{1234} F^{N}{}_{1234} F^{M}{}_{1234} F^{N}{}_{1234}.
\]

(5.2b)

The \( AdS_2 \times S^2 \times T^7 \) solution is given by

\[
d s^2_{11} = g_{\mu\nu} dx^\mu dx^\nu + g_{\alpha\beta} dx^\alpha dx^\beta + \delta_{ab} dz^a dz^b + \delta_{st} dw^s dw^t,
R_{\mu\nu\lambda\sigma} = -(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}), \quad R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma},
\]

(5.3)

\[
\bar{F}_{\mu\nu} = \bar{F}_{\nu\mu} = \epsilon_{\mu\nu}, \quad \bar{F}_{\alpha\beta\gamma\delta} = \bar{F}_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta}.
\]

All other fields are set to zero. This is the near horizon geometry of the brane configuration shown in the introduction. The notational conventions for the indices are summarized in Appendix A.

Note that the background fields are self-dual in the \( SO(4) \) for the four coordinates along which the branes lie. Equivalently, it transforms in the \( (0, 1) \) representation of \( SO(4) \simeq SU(2)_+ \times SU(2)_- \). This follows from the requirement of partially unbroken supersymmetry. If any one of the relative sign for the gauge field is flipped, the supersymmetry is completely broken, even though it is still a solution to the equations of motion. This background breaks the \( SO(7) \) isometry of the internal \( T^7 \) down to \( SU(2)_+ \times SU(2)_3 \), where \( SU(2)_3 \) is the rotation of \( w^{8,9,10} \). These internal symmetries will play a crucial role in grouping the fields, as will be shown in the next section.

In fact, supersymmetry requires that the product of the signs of the gauge field be +1. We set all the signs to be +1 using coordinate redefinition and parity transformation.
5.2 Linearized field equations and reduction to $D = 4$

We linearize the equations in $D = 11$ in fluctuations around the background,

$$G_{MN} = g_{MN} + h_{MN}, \quad F_{IJKL} = \tilde{F}_{IJKL} + 4\nabla_I a_{JKL},$$  \hspace{1cm} (5.4)

and then dimensionally reduce it to $D = 4$ by keeping only the zero modes of the fluctuations in internal $T^7 = T^4 \times T^3$. We then redefine some of the fluctuation fields,

$$h_{mn}^{(11)} = h_{mn}^{(4)} - \frac{1}{2}(B_a^a + B_s^s)g_{mn}, \quad h_{ab} = B_{ab}, \quad h_{ma} = V_{ma},$$

$$a_{abc} = C_{abc}^{}, \quad a_{mab} = A_{mab}, \quad a_{mna} = D_{mna},$$  \hspace{1cm} (5.5)

The definitions of $B, V, C, A, D$ remains valid when we replace the $a, b$ indices by $s, t$ indices. The shift in $h_{mn}$ is the linearized version of the Weyl rescaling which is necessary to absorb the volume factor of the internal dimensions and put the action into the standard Einstein-Hilbert form.

Also, one can do Hodge dual transformation to reduce the indices of the tensor fields. The tensor field with three index, $a_{mn}$ is the most trivial one, its dual field having rank $-1$ formally. This implies it has no dynamics. Indeed, one can show explicitly from its equations of motion that it has no degree of freedom and decouples from all the other fields. The next one we consider is the rank two tensor field $D_{mn}^a$ whose linearized equation of motion is

$$\nabla^m \{ \nabla_m D_{ij}^a + \tilde{F}_{ij}^{ab} V_{mn}^b \} = 0$$

which turns into an identity if we introduce the dual scalar

$$3\nabla^l D_{mn}^{a[l]} + 3\tilde{F}_{ij}^{ab} V_{mn}^b = \epsilon^{lmnk} \nabla_k D_a^a$$

Then the Bianchi identity for the original $D_{mn}^{a[l]}$ turns into the equation of motion for $D_a^a$,

$$\nabla^2 D_a^a = \frac{1}{4}\epsilon^{klmn} F_{kl} W_{mn}^b,$$  \hspace{1cm} (5.8)

where $W_{mn}^a$ is the field strength of $V_{mn}^a$. The equation remains valid when $a$ is replaced by $s$, except that in this case the right-hand side vanishes.

The quantum numbers of the various fluctuation fields with respect to the internal symmetries are summarized below, along with that of the background gauge field $\tilde{F}_{mn}^{ab}$. Using this table, one can divide the fields into small groups, where the fields belonging to the same group can couple to each other. The fields within a group must have the same quantum numbers except the broken $SU(2)_{-}$ charge, which can be shifted by 1 by the background field. We label these groups by capital roman letters. Note that we separated the self-dual and the anti-self-dual parts of the fields which are rank two tensors in $SO(4)$ by

$$A^{ab\pm} \equiv \frac{1}{2}(A^{ab} \pm \epsilon^{abcd} A_{cd}).$$  \hspace{1cm} (5.9)
| Field       | \( SU(2)_+ \) | \( SU(2)_- \) | \( SU(2)_3 \) | Group |
|------------|----------------|----------------|----------------|-------|
| \( h_{mnm} \) | 0              | 0              | 0              | F     |
| \( V^a_m \)  | 1/2            | 1/2            | 0              | D     |
| \( V^s_m \)  | 0              | 0              | 1              | B     |
| \( B_{(ab)} \) | 1              | 1              | 0              | E     |
| \( 3B^a_a + 2B^s_s \) | 0              | 0              | 0              | A     |
| \( B_{as} \) | 1/2            | 1/2            | 1              | C     |
| \( B_{(st)} \) | 0              | 0              | 2              | A     |
| \( B^s_S \)  | 0              | 0              | 0              | F     |
| \( A^m_{ab^+} \) | 0              | 1              | 0              | F     |
| \( A_{ab^-} \) | 1              | 0              | 0              | E     |
| \( A^m_{as} \) | 1/2            | 1/2            | 1              | C     |
| \( A_{st} \)  | 0              | 0              | 1              | B     |
| \( C_{abc} \) | 1/2            | 1/2            | 0              | D     |
| \( C^{sab^+} \) | 0              | 1              | 1              | B     |
| \( C^{sab^-} \) | 1              | 0              | 1              | A     |
| \( C_{ast} \) | 1/2            | 1/2            | 1              | C     |
| \( C_{stu} \) | 0              | 0              | 0              | F     |
| \( D^a \)  | 1/2            | 1/2            | 0              | D     |
| \( D^s \)  | 0              | 0              | 1              | A     |
| \( F_{mn} \) | 0              | 1              | 0              |       |

Table 2: Internal quantum numbers of the bosonic fields.
The linearized equations of motion in $D = 4$ are given by

**Group A**

\[ \nabla^2 B_{(st)} = \nabla^2 C_{sab} = \nabla^2 D^s = \nabla^2 (3B_a^s + 2B^s_s) = 0. \]  (5.10)

**Group B**

\[ \begin{align*}
\nabla^2 C_{sab}^+ &= \frac{1}{2} F_{mn}^{ab} W_{mn}^s - \frac{1}{4} \epsilon^{klmn} F_{kl}^{ab} F_{mn}^s, \\
\nabla^n F_{nm}^s &= -\frac{1}{4} \epsilon_m^{nk} F_{nk}^{ab} \nabla_l C_{sab}^+ \quad (F_{mn}^s \equiv \frac{1}{2} \epsilon^{stu} F_{mn}^t), \\
\nabla_n W_{nm}^s &= \frac{1}{2} F_{mn}^{ab} \nabla n C_{sab}^+. \quad (5.11) 
\end{align*} \]

**Group C**

\[ \begin{align*}
\nabla^2 C_{as} &= \frac{1}{4} \epsilon^{klmn} F_{kl}^{ab} F_{mn}^s \quad (C_{as} \equiv \frac{1}{2} \epsilon^{stu} C_{atu}), \\
\nabla^2 B_{as} &= \frac{1}{2} F_{mn}^{ab} F_{mn}^s, \\
\nabla^n F_{nm}^{as} &= -F_{mn}^{ab} \nabla n B_{bs} - \frac{1}{2} \epsilon_m^{nk} F_{nk}^{ab} \nabla l C_{bs}. \quad (5.12) 
\end{align*} \]

**Group D**

\[ \begin{align*}
\nabla^2 C_{abc} &= \frac{3}{2} W_{mn}^{[a} \tilde{F}_{mn}^{bc]}, \\
\nabla^2 D^a &= \frac{1}{4} \epsilon^{klmn} F_{kl}^{ab} W_{mn}^b, \\
\nabla_n W_{nm}^a &= -\frac{1}{2} \epsilon_m^{nk} F_{nk}^{ab} \nabla l D_b^a + \frac{1}{2} \tilde{F}_{mn}^{ab} \nabla n C_{abc}. \quad (5.13) 
\end{align*} \]

**Group E**

\[ \begin{align*}
\nabla^2 B^{(ab)} &= \frac{1}{2} F_{mn}^{ac} \tilde{F}_{mn}^{cb} + \frac{1}{2} F_{mn}^{bc} \tilde{F}_{mn}^{ca} + \frac{1}{2} \tilde{F}_{mn}^{ac} \tilde{F}_{mn}^{bd} B^{(cd)}, \\
\nabla^n F_{nm}^{ab} &= \tilde{F}_{mn}^{bc} \nabla n B^{(ca)} - \tilde{F}_{mn}^{ac} \nabla n B^{(cb)}. \quad (5.14) 
\end{align*} \]

**Group F**

\[ \begin{align*}
\nabla^2 C &= \frac{1}{8} \epsilon^{klmn} F_{kl}^{ab} F_{mn}^{ab+}, \quad (5.15a) \\
\nabla^2 B_s^a &= \frac{1}{2} F_{mn}^{ab} F_{mn}^{ab+} - \frac{1}{2} F_{mn}^{ab} F_{mn}^{ab+} \tilde{F}_{kl}^{kl}, \quad (5.15b) \\
\nabla^n F_{nm}^{ab+} &= \tilde{F}_{kl}^{ab} \nabla l h_{km} - \frac{1}{2} \tilde{F}_{mn}^{ab} (2\nabla_l h_{km} - \nabla m h_{lk} - \nabla n h_{kl} - \nabla n B_s^a) + \frac{1}{2} \epsilon_m^{klm} F_{kl}^{ab} \nabla n C, \quad (5.15c) \\
R_{mn}^{(1)}(h) &= \frac{1}{2} F_{mn}^{ab} F_{mn}^{ab+} + \frac{1}{2} \epsilon_m^{mnk} F_{mn}^{ab+} - \frac{1}{6} \epsilon_m^{nmk} F_{kl}^{ab} F_{mn}^{ab+} \\
&\quad - \frac{1}{2} F_{mn}^{ab} F_{mn}^{ab+} h_{kl} + \frac{1}{6} \epsilon_m^{nmk} F_{kj}^{ab} F_{mn}^{ab+} + \frac{1}{4} \epsilon_m^{nmk} F_{nm}^{ab} F_{mn}^{ab+} (5.15d) 
\end{align*} \]
5.3 Computation of mass spectrum in each multiplet

We have separated fields which decouple from one another using their internal quantum numbers. We should now disentangle the field equations further and find out which field belong to which $\mathcal{N} = 2$ multiplet. Obviously, the bosonic fields in the same multiplet satisfy the same field equations, and the same for the fermions.

In this section, we jump to solve the equations of motion of fields in each multiplet, except for the gravity multiplet which has been analyzed in detail in section 3. The reduction of the equations obtained in the previous subsection to the final form require somewhat lengthy algebra, and we put it off until the next subsection. The complication partly arises from the fact that we chose a specific $D = 11$ configuration from the beginning. Although the M theoretic origin of the geometry is manifest in this framework, the U-duality invariance of the $D = 4$ theory and its symmetry breaking pattern is obscured. A manifestly duality invariant approach sketched in [21] could simplify the process to a large extent.

5.3.1 Hyper multiplet

Minimally coupled scalars in $D = 4$ belong to this multiplet. Clearly, the KK modes have $m^2 = j(j + 1)$. It follows that $h = j + 1$. There is no gauge symmetry associated with the scalars.

5.3.2 Vector multiplet

A vector multiplet contains a vector $A_m$ and two real scalars $\phi_1, \phi_2$. In the simplest case, $\phi_1$ couples to $A_\alpha$ only and $\phi_2$ to $A_\mu$. The field equations for the first group are

\begin{align}
(\nabla_x^2 + \nabla_y^2 - 2) \phi_1 &= \frac{1}{2} \epsilon^{\alpha\beta} F_{\alpha\beta}, \\
\nabla^m F_{m\alpha} &= 4 \epsilon_{\alpha\beta} \nabla^\beta \phi_1.
\end{align}

In the same gauge as in the toy model, $A_\alpha$ is expanded in the spherical harmonics as

$$A_\alpha = b^I \epsilon_{\alpha\beta} \nabla^\beta Y^I.$$ (5.17)

We then get the equations

\begin{align}
\begin{pmatrix}
\nabla_x^2 - j(j + 1) - 2 & -j(j + 1) \\
-4 & \nabla_x^2 - j(j + 1)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
b
\end{pmatrix} = 0.
\end{align}

along with the constraint

$$\nabla_{\mu} A^\mu = 0.$$ (5.19)

For $j \geq 1$, one finds that the mass eigenvalues are

$$m^2 = j(j - 1), (j + 1)(j + 2) \implies h = j, j + 2.$$ (5.20)
For $j = 0$, $b$ is absent and $\phi_1$ has $m^2 = 2$, $h = 1$.

The field equations for $\phi_2$ and $A_\mu$ are

\[
(\nabla_x^2 + \nabla_y^2 + 2)\phi_2 = \frac{1}{2} \epsilon_{\mu\nu} F_{\mu\nu}, \tag{5.21a}
\]

\[
\nabla^n F_{n\mu} = 4 \epsilon_{\mu\nu} \nabla'' \phi_2. \tag{5.21b}
\]

As in the toy model, one can use the constraint $\nabla_{\mu} A_\mu = 0$ to set $A_\mu = \epsilon_{\mu\nu} \nabla_{\nu} a$. The equations then become,

\[
\begin{pmatrix}
\nabla_x^2 - j(j + 1) + 2 & -\nabla_x^2 \\
-4 & \nabla_x^2 - j(j + 1)
\end{pmatrix}
\begin{pmatrix}
\phi_2 \\
a
\end{pmatrix} = 0. \tag{5.22}
\]

For $j \geq 1$, one finds the same mass eigenvalues as for $\phi_1$ and $A_\alpha$. For $j = 0$, $A_\mu$ is a gauge field in $D = 2$ and can be eliminated, leaving $\phi_2$ with $m^2 = 2$, $h = 1$.

5.3.3 Gravitino multiplet

Minimally coupled vectors in $D = 4$ belong to this multiplet. One obtain two $D = 2$ scalars with $m^2 = j(j + 1)$ for all $j \geq 1$. For $j = 0$, the mode for $A_\alpha$ is absent and $A_\mu$ becomes a gauge field in $D = 2$, so there is no bulk degree of freedom.

5.3.4 Gravity multiplet

This multiplet was analyzed for the toy model case. Conformal weights of the bosonic states satisfy $h = j, j + 2$.

5.4 Grouping $D = 4$ fields into $\mathcal{N} = 2$ multiplets

Group A

All the fields in this group are minimally coupled scalars in $D = 4$ and belong to the hypermultiplet.

Group B

It is convenient to dualize $F_{mn}$ by defining

\[
\tilde{F}_{mn} \equiv -\frac{1}{2} \epsilon_{mn}^{kl} F_{kl} - F_{mn} C^{sab+}. \tag{5.23}
\]

Then the equation of motion of $F$ becomes the Bianchi identity for $\tilde{F}$, and the Bianchi identity for $F$ become

\[
\nabla^n \tilde{F}_{mn} = F_{mn} \nabla^n C^{sab+}. \tag{5.24}
\]
Note that the right-hand side is the same as that of (5.11c). In terms of \( \tilde{F}^s \), the field equation for \( C \) becomes
\[
\nabla^2 C_{sab}^{ab} = \frac{1}{2} \tilde{F}_{mn}^s (\tilde{F}_{mn}^s + W_{mn}^s + \tilde{F}_{mn}^{cd} C_{scd}^{cd}).
\] (5.25)

Clearly, \( \tilde{F}^s - W^s \) decouple from \( C \) and contribute to the 3/2 multiplet. Since \( C \) is coupled to \( \tilde{F}^s + W^s \) by \( \bar{F}^{ab} \), we find that \( C_{s47}^{47+} \) decouples and contribute to the hypermultiplet. The remaining fields belong to the vector multiplet. In particular, \( C_{s45}^{45+} \) couples to \( \tilde{A}_s + V^s \) and \( C_{s46}^{46+} \) couples to \( A_{\alpha}^s + V_{\alpha}^s \)

**Group C**

Writing down all the components of the field equations and collecting those which couple to one another, one finds twelve identical copies of the following set of coupled equations:
\[
\nabla^2 B = -\nabla^2 C = \frac{1}{2} (-\epsilon^{\mu \nu} F_{\mu \nu} + \epsilon^{\alpha \beta} F_{\alpha \beta}),
\] (5.26a)
\[
\nabla_{\mu} F_{n \mu} = \epsilon_{\mu \nu} \nabla^{\nu} (-B + C),
\] (5.26b)
\[
\nabla_{\alpha} F_{n \alpha} = \epsilon_{\alpha \beta} \nabla^{\beta} (B - C).
\] (5.26c)

As before, we set,
\[
A_{\alpha} = b_{\alpha} \epsilon_{\alpha \beta} \nabla^{\beta} Y^I, \quad A_{\mu} = A^I_{\mu} Y^I, \quad A_{\alpha} = \epsilon_{\mu \nu} \nabla^{\nu} a.
\] (5.27)

It is then easy to show that \( (B + C) \) belong to the hypermultiplet, \( (a + b) \) to the gravitino multiplet, and \( (B - C) \) and \( (a - b) \) together to the vector multiplet.

**Group D**

If we define
\[
C^a = \frac{1}{3!} \epsilon^{abcd} C_{bced},
\]
we find that the field equations have exactly the same structure as those in the previous group.

**Group E**

First, \( B_{45} + B_{67}, B_{46} - B_{75} \) and \( B_{44} - B_{55} - B_{66} + B_{77} \) decouple and contribute to the hypermultiplet. The other equations fall into six groups each of which contains one scalar and one vector. Each group gives the spectrum for half a vector multiplet. Explicitly, the six groups are
\[
(B^{44} + B^{55} - B^{66} - B^{77}, A_{\mu}^{45-}), \quad (B^{44} - B^{55} + B^{66} - B^{77}, A_{\alpha}^{46-}),
\]
\[
(B^{46} + B^{57}, A_{\mu}^{47-}), \quad (B^{45} - B^{67}, A_{\alpha}^{47-}),
\]
\[
(B^{47} - B^{56}, A_{\mu}^{46-}), \quad (B^{47} + B^{56}, A_{\alpha}^{45-}).
\] (5.28)

**Group F**
Table 3: This table summarizes the number of degrees of freedom a group of bosonic equations contribute to each of the four multiplets.

First, $h_2 \equiv h_\alpha^\alpha$, $B_\mu^\mu$, $A_\mu^{45+}$ and $A_\alpha^{46+}$ belong to the gravity multiplet. Second, $h_{\mu\alpha}$, $A_\mu^{46+}$ and $A_\alpha^{45+}, C$ belong to the vector multiplet. Finally, $A_\mu^{47+}$ and $A_\alpha^{47+}$ decouple and contribute to the gravitino multiplet.

## 6 Fermionic mass spectrum

### 6.1 Linearized field equations and reduction to $D = 4$

The linearized field equation for the gravitino in $D = 11$ reads

$$
\Gamma^{IJK} \nabla_J \Psi_K = \frac{1}{4!} \Gamma^{IJKLMN} \Psi_J F_{KLMN} + \frac{1}{8} \Gamma^{JK} \Psi^L F^{IJKL}.
$$

(6.1)

Throughout this section we suppress the bar on the background field strength. The linearized local SUSY transformation law plays the role of gauge symmetry for Fermions:

$$
\delta \Psi_M = \nabla_M \epsilon + \frac{1}{12 \cdot 4!} F_{IJKL} (8 \delta_M ^{IJKL} - \Gamma^{IJKL} M) \epsilon.
$$

(6.2)

In dimensional reduction to $D = 4$, it is convenient to define

$$
\lambda = \Gamma^a \Psi_a, \quad \Psi_a = \Psi_a - \frac{1}{4} \Gamma_a \lambda,
$$

$$
\chi = \Gamma^s \Psi_s, \quad \Psi_s = \Psi_s - \frac{3}{4} \Gamma_s \chi.
$$

(6.3)

The following shift in the $D = 4$ spin $3/2$ fields bring their kinetic term into the standard form.

$$
\Psi_m^{(11)} = \Psi_m^{(4)} - \frac{1}{2} \Gamma_m (\lambda + \chi).
$$

(6.4)

We then decompose the fermion into chiral and anti-chiral components with respect to $SO(4)$ of $T^4$,

$$
\Psi^\pm \equiv \frac{1}{2} (1 \pm \Gamma) \Psi
$$

(6.5)
| Field | $SU(2)_+$ | $SU(2)_-$ | $SU(2)_3$ | Group |
|-------|------------|------------|------------|-------|
| $\Psi^+_m$ | 1/2 | 0 | 1/2 | I |
| $\Psi^-_m$ | 0 | 1/2 | | J |
| $\Psi^+_{(a)}$ | 1 | 1/2 | 1/2 | H |
| $\Psi^-_{(a)}$ | 1/2 | 1 | 1/2 | I |
| $(3\lambda + 2\chi)^+$ | 1/2 | 0 | 1/2 | G |
| $(3\lambda + 2\chi)^-$ | 0 | 1/2 | | H |
| $\Psi^+_{(s)}$ | 1/2 | 0 | 3/2 | G |
| $\Psi^-_{(s)}$ | 0 | 1/2 | | H |
| $\chi^+$ | 1/2 | 0 | 1/2 | I |
| $\chi^-$ | 0 | 1/2 | | J |

Table 4: Internal quantum numbers of the fermionic fields.

where $\bar{\Gamma} \equiv \frac{1}{4!} \epsilon_{abcd} \Gamma^{abcd}$. As in the previous section, we can divide the field equations into a few groups using the internal symmetry. After some gamma matrix algebra, one finds that the field equations and gauge transformation laws in $D = 4$ are given by

**Group G**

$$\Gamma^n \nabla_n \Psi^+_{(s)} = \Gamma^n \nabla_n (3\lambda + 2\chi)^+ = 0,$$

$$\delta \Psi^+_{(s)} = \delta (3\lambda + 2\chi)^+ = 0.$$  \hspace{1cm} (6.6a)

**Group H**

$$\Gamma^n \nabla_n \Psi^+_{(a)} = \frac{1}{4} F^{ad}_{ij} \Gamma^{ij} \Psi^+_{(d)},$$

$$\Gamma^n \nabla_n (3\lambda + 2\chi)^- = \frac{1}{16} F^{cd}_{ij} \Gamma^{ij} \Gamma^{ad} (3\lambda + 2\chi)^-, $$

$$\Gamma^n \nabla_n \Psi^-_{(s)} = -\frac{1}{16} F^{cd}_{ij} \Gamma^{ij} \Gamma^{ad} \Psi^-_{(s)},$$

$$\delta \Psi^+_{(a)} = \delta (3\lambda + 2\chi)^- = \delta \Psi^-_{(s)} = 0.$$ \hspace{1cm} (6.6b)

**Group I**

$$\Gamma^n \nabla_n \Psi^-_{(a)} = -\frac{1}{8} F^{ab}_{ij} \Gamma^b \{ \Gamma^{ij} \chi^+ + (\Gamma^{ijk} - 2\Gamma^{ij}\delta^{jk}) \Psi^+_k \},$$

$$\Gamma^{mnk} \nabla_n \Psi^+_{k} = -\frac{1}{8} F^{ab}_{ij} (\Gamma^{mij} - 2\delta^{mi} \Gamma^j) \Gamma^a \Psi^-_{(b)},$$

$$\Gamma^n \nabla_n \chi^+ = \frac{1}{4} F^{ab}_{ij} \Gamma^{ij} \Gamma^a \Psi^-_{(b)},$$

$$\delta \Psi^-_{(a)} = \frac{1}{8} F^{ab}_{ij} \Gamma^b \Gamma^{ij} \epsilon^+, $$

$$\delta \Psi^+_{m} = \nabla_m \epsilon^+, \delta \chi^+ = 0.$$ \hspace{1cm} (6.6d)
6.2 Computation of the mass spectrum in each multiplet

6.2.1 Hyper multiplet

Spinors with nonzero mass generated by the background gauge field belong to this multiplet. After diagonalizing the mass matrix, they satisfy the equation of motion of the form

\[ \Gamma^m \nabla_m \psi + i \Gamma^{01} \psi = 0. \]  

(6.10)

One finds that \(|m| = j + \frac{1}{2} \pm 1\), which implies that \(h = j, j + 2\). There is no gauge symmetry acting on the spinors in this multiplet.

6.2.2 Vector multiplet

Minimally coupled massless spinors in \(D = 4\) belong to this multiplet. One easily finds that \(h = |m| + \frac{1}{2} = j + 1\) for all \(j \geq \frac{1}{2}\). There is no gauge symmetry acting on the spinors in this multiplet.

6.2.3 Gravitino multiplet

A gravitino multiplet contains a gravitino \(\psi\) and a spinor \(\chi\). In the same notation as in the toy model, their coupled equations of motion break up as follows

\[ (\nabla_x + \bar{\gamma} \nabla_y) \chi = -i \bar{\gamma} (\eta - \lambda), \]  

(6.11a)

\[ (\nabla_x + \bar{\gamma} \nabla_y) \eta + \bar{\gamma} \nabla_y \lambda - 2 \nabla^\alpha \psi^{(\alpha)} = -i \bar{\gamma} \chi, \]  

(6.11b)

\[ (\nabla_x + \bar{\gamma} \nabla_y) \lambda + \nabla_x \eta - 2 \nabla^\mu \psi^{(\mu)} = i \bar{\gamma} \chi, \]  

(6.11c)

\[ -\nabla^{(\mu)} \eta + \bar{\gamma} \nabla_y \psi^{(\mu)} = 0, \]  

(6.11d)

\[ -\nabla^{(\alpha)} \lambda + \nabla_x \psi^{(\alpha)} = 0. \]  

(6.11e)

where we expressed the four dimensional gamma matrices as tensor products of two dimensional ones as in the case of the toy model, and \(\nabla_x, \nabla_y\) are the two dimensional dirac operators. The gauge
transformation laws are given by

\[ \delta \psi_{(\mu)} = \nabla_{(\mu)} \epsilon, \quad \delta \lambda = \nabla_x \epsilon, \]
\[ \delta \psi_{(\alpha)} = \nabla_{(\alpha)} \epsilon, \quad \delta \eta = \bar{\gamma} \nabla_y \epsilon, \quad \delta \chi = -i \bar{\gamma} \epsilon. \] (6.12)

One can always gauge away \( \eta \). Then (6.11d) sets \( \psi_{(\mu)} \) to zero.

For \( j \geq 3/2 \), (6.11b) determine \( \psi_{(\alpha)} \) algebraically. The only independent equations that remain are (6.11d) and (6.11e) with \( \eta \) and \( \psi_{(\mu)} \) removed. The mass eigenvalues are the same as those of hyper multiplet: \(|m| = j + \frac{1}{2} \pm 1, \ h = j, j + 2 \). For \( j = 1/2 \), the modes for \( \psi_{(\alpha)} \) are absent and (6.11e) is trivially satisfied. Eq. (6.11b) gives an algebraic relation between \( \lambda \) and \( \chi \). So the number of degrees of freedom is reduced by half. One finds \( h = j + 2 \) for all modes.

6.2.4 Gravity multiplet

The equations satisfied by this multiplet was analyzed for the toy model case. One finds \( h = j + 1 \), with number of degrees reduced by half for \( j = 1/2 \).

6.3 Grouping \( D = 4 \) fields into \( N = 2 \) multiplets

**Group G**

All the spinors in this group are massless and minimally coupled in \( D = 4 \) and belong to the vector multiplet.

**Group H**

We choose the following basis for \( SO(4) \) gamma matrices,

\[ \Gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^{5,6,7} = i \begin{pmatrix} 0 & -\sigma^{1,2,3} \\ \sigma^{1,2,3} & 0 \end{pmatrix}, \] (6.13)

where \( \sigma^i \) are the Pauli matrices. In this basis, an \( SO(4) \) spinor splits into two chiral spinors as

\[ \eta = \begin{pmatrix} \eta^- \\ \eta^+ \end{pmatrix}. \] (6.14)

Consider \((3\lambda + 2\chi)^-\) first. To simplify the equations, we use the letter \( \Psi \) to denote \((3\lambda + 2\chi)^-\) in the equations to follow. In the basis we chose, the field equation reduces to

\[ \Gamma^m \nabla_n \Psi = \frac{i}{2} (\sigma^1 \otimes \Gamma^{01} + \sigma^2 \otimes \Gamma^{23}) \Psi. \] (6.15)
We can further decompose the equation by setting

\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \]  

(6.16)

After splitting each of \( \Psi_{1,2} \) into two pieces according to their chirality in the non-compact \( D = 4 \) spacetime, and recombining those pieces which couple to each other, one finds that one linear combination belongs to the vector multiplet and the other one to the hyper multiplet.

The spectrum is exactly the same for \( \Psi^+ \) except that it has twice as many degrees of freedom as \( 3\lambda + 2\chi \). We find the same result even for \( \Psi^+ \) again except for the degeneracy. In counting the degeneracy, one should remember the constraint \( \Gamma^a \Psi_{(a)} = 0 \). In the basis we chose above, it reduces to

\[ \Psi_{(4)} - i\sigma^1\Psi_{(5)} - i\sigma^2\Psi_{(6)} - i\sigma^3\Psi_{(7)} = 0. \]  

(6.17)

**Group I**

Doing the same sort of recombination of spinors as above, one finds the following results.

1. A third of \( \Psi_{(a)}^− \) decouple from all the other fields. They belong to the vector multiplet.
2. Another third of \( \Psi_{(a)}^− \) couple to \( \chi^+ \). They contribute to the hypermultiplet.
3. The last third of \( \Psi_{(a)}^− \) couple to \( \Psi^+_m \). They belong to the spin \( 3/2 \) multiplet.

**Group J**

1. A half of \( \chi^− \) decouple and belong to the vector multiplet.
2. A half of \( \Psi^−_m \) decouple and satisfy the same equation as the gravitino in the toy model. Therefore they belong to the gravity multiplet.
3. The other components of \( \chi^− \) and \( \Psi^−_m \) couple to each other. They contribute to the spin \( 3/2 \) multiplet.

The following table summarizes the result of this section.

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Table 5: Summary of the fermionic spectrum.

A Notations and Conventions

We consider $D = 11$ SUGRA on $AdS_2 \times S^2 \times T^4 \times T^3$. Each manifold in the product is parametrized by $x^\mu (\mu = 0, 1), y^\alpha (\alpha = 2, 3), z^a (a = 4, 5, 6, 7)$ and $w^s (s = 8, 9, 10)$, respectively. We use the indices $(M, N, \cdots)$ to label all eleven coordinates together and $(m, n, \cdots)$ to label the coordinates of $AdS_2 \times S^2$. The signature of the metric is $(- + \cdots +)$. The field strength of a $p$-form potential in any dimension is defined by

$$F_{M_0 \cdots M_p} = p \nabla_{[M_0} A_{M_1 \cdots M_p]}.$$  (A.1)

B Spherical Harmonics

The spherical harmonics form a basis for the fields living on a sphere. In this appendix we consider only the case of $S^2$. We can construct them by considering the eigenstates of maximal commuting subalgebra of $SU(2)$ group, which are the total angular momentum $\vec{J}^2 = j(j+1)$, its $z$ component $J_z = m$, the orbital angular momentum $\vec{L}^2 = l(l+1)$, and the spin $\vec{S}^2 = s(s+1)$. The case for the scalar is easiest since $s = 0$ and $\vec{J}^2 = \vec{L}^2$, which we identify with the Laplacian on the sphere, $\nabla_y^2$, where $y$ indicates two dimensional coordinates parametrizing the two-sphere. This expression for $\vec{L}^2$ can be obtained by embedding $S^2$ into three dimensional space, writing down $\vec{L}^2$ in terms of the Cartesian coordinates, which is quite well known, and reexpressing these in terms of polar coordinates. Therefore, by construction, we have

$$\nabla_y^2 Y^{(j,m)}(y) = -j(j+1)Y^{(j,m)}(y),$$  (B.1)

where $Y^{(j,m)}(y)$ denotes the eigenstates with eigenvalues $(j, m)$, which were defined above.

Next consider the spinor spherical harmonics where $s = 1/2$. The easiest way to consider it is to embed the sphere in the three dimensional space and use cartesian coordinates. We construct them by taking the tensor product of scalar spherical harmonics with 2-component spinor, and
taking appropriate linear combinations. We then get the expression

\[\Sigma_{l}^{j=\pm l/2,m} = \frac{1}{\sqrt{2l+1}} \left( \pm \sqrt{\lambda \pm m + \frac{1}{2}} Y_{m-1/2}(\theta, \phi) \right), \quad (B.2)\]

where the labels indicates the eigenvalues as usual, and all the harmonics are normalized to unity. For given \( j \), the only possible values of \( l \) are \( j \pm \frac{1}{2} \), so the degeneracy is \( 2(2j+1) \). However, it is convenient for our purpose to group the spherical harmonics of given \( j \) according to the eigenvalue of \( \tilde{J} \equiv \nabla_{y} \) rather than \( m, l \), where \( \nabla_{y} \equiv \tau^{\alpha} \nabla_{\alpha} \) is the two dimensional dirac operator on the sphere with \( \tau^{\alpha} \) given by the usual Pauli matrices. One can show that

\[\tilde{J}^2 = \nabla_{y}^2 - \frac{1}{4}, \quad (B.3)\]

by comparing the two dimensional operator with the expression in the embedding three dimensional cartesian coordinates, so it is obvious that \( \tilde{J} \) commutes with \( \tilde{J}^2 \), and its eigenvalues are \( \pm (j + \frac{1}{2}) \). However, it turns out that neither of \( \tilde{L}^2 \) nor \( J_{z} \) commutes with \( \tilde{J} \). Therefore we have to find two other operators which commute with \( \tilde{J}^2 \), \( \tilde{J} \) in order to distinguish linearly independent spherical harmonics. We will not identify them since they are not needed for the present discussion. Note that the chirality operator \( \bar{\tau} \equiv \frac{1}{2} \epsilon_{\alpha\beta} \tau^{\alpha} \tau^{\beta} \) anticommutes with \( \tilde{J} \). Therefore given an eigenstate with \( \tilde{J} > 0 \), which we denote by \( \Sigma_{+}^{j} \), we have a counterpart \( \Sigma_{-}^{j} \equiv \bar{\tau} \Sigma_{+}^{j} \) and vice versa. \footnote{Our notations closely mimic those in [17], but the convention for the \( \pm \) sign is flipped.}

This immediately implies that there are same number of \( \Sigma_{+}^{j} \) states and \( \Sigma_{-}^{j} \) states for given \( j \). Since the degeneracy of total states is \( 2(2j+1) \), we have \( 2j + 1 \) \( \Sigma_{+}^{j} \) (or \( \Sigma_{-}^{j} \)) states.

We also state without proof that the lowest spinor spherical harmonics are killing spinors, satisfying the relation

\[ (\nabla_{\alpha} \mp i \frac{\tau_{\alpha}}{2}) \Sigma_{\pm}^{1/2} = 0. \quad (B.4)\]
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