Mean-field backward stochastic differential equations and nonlocal PDEs with quadratic growth

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Abstract: In this paper, we study the general mean-field backward stochastic differential equations (BSDEs, for short) with quadratic growth. First, the existence and uniqueness of local and global solutions are proved with some new ideas for one-dimensional mean-field BSDEs when the generator $g(t,Y,Z,P_Y,P_Z)$ grows in $Z$ quadratically and the terminal value is bounded. Second, a comparison theorem for the general mean-field BSDEs is obtained with the Girsanov transform. Third, we prove the convergence of the particle systems to the mean-field BSDEs with quadratic growth, and in addition, the rate of convergence is given. Finally, in this framework, we use the mean-field BSDE to give a probabilistic representation for the viscosity solution of a nonlocal partial differential equation (PDE, for short), as an extended nonlinear Feynman-Kac formula, which yields the existence and uniqueness of the solution to the PDE.

Key words: backward stochastic differential equation, mean-field, quadratic growth, partial differential equation, McKean-Vlasov equation.

AMS subject classifications. 60H10, 60H30

1 Introduction

Mean-field stochastic differential equations (SDEs, for short), also called McKean-Vlasov equations, can be traced back to the work of Kac [26] in the 1950s. Recently, inspired by particle systems, the
mean-field backward stochastic differential equations (BSDEs, for short) were introduced by Buckdahn, Djehiche, Li, Peng [9] and Buckdahn, Li, Peng [10]. Since then, mean-field BSDEs and the related nonlocal partial differential equations (PDEs, for short) have received intensive attention. However, up to now, most works of mean-field BSDEs are based on the linear growth condition, which largely hinders the theory’s development and application. In this paper, we shall comprehensively study mean-field BSDEs with quadratic growth, including the existence and uniqueness, the comparison theorem, the particle systems, and the applications to PDEs. To be more precise, we describe the problem in detail.

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space on which a \(d\)-dimensional standard Brownian motion \(\{W_t : 0 \leq t < \infty\}\) is defined, where \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration of \(W\) augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Let \(T > 0\) be a time horizon, and consider the following general mean-field BSDE:

\[
Y_t = \eta + \int_t^T g(s, Y_s, Z_s, \mathbb{P}_{Y_s}, \mathbb{P}_{Z_s}) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T],
\]

where the random variable \(\eta\) is called the terminal value and the coefficient \(g\) is called the generator. The unknown processes, called an adapted solution of (1.1), are the pair \((Y, Z)\) of \(\mathbb{F}\)-adapted processes, with \(\mathbb{P}_{Y_s}, \mathbb{P}_{Z_s}\) being the laws of \(Y_s\) and \(Z_s\), respectively. Here by the word “general”, we stress that the generator \(g\) depends on the distribution of the solutions rather than their expectations. In what follows, BSDE (1.1) is called a quadratic mean-field BSDE or a mean-field BSDE with quadratic growth if the generator \(g\) in BSDE (1.1) grows quadratically in the second last argument \(Z\), and the terminal value \(\eta\) is called bounded if it is bounded.

When the laws of \(Y\) and \(Z\) appear as the expectations of \(Y\) and \(Z\), respectively, the mean-field BSDE (1.1) was studied by Buckdahn, Djehiche, Li, Peng [9] and Buckdahn, Li, Peng [10], where the existence, uniqueness, a comparison theorem, and the relation with a nonlocal PDE are given for the case of uniformly Lipschitz continuous coefficients. Recently, the derivative of a functional \(\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) with respect to the measure argument is introduced by Lions [33] at Collège de France. Since then, this definition is adopted by many works. For instance, Chassagneux, Crisan, and Delarue [15] (see also Carmona and Delarue [13]) studied the general mean-field BSDE (1.1) coupled with a McKean-Vlasov forward equation, and proved that this class of equations admits unique adapted solution under globally Lipschitz continuous coefficients. Buckdahn, Li, Peng and Rainer [11] studied the general forward mean-field stochastic differential equations and the associated PDEs. Li [32] studied the general mean-field forward-backward SDEs with jumps and associated nonlocal quasi-linear integral-PDEs. Besides, for the applications of the mean-field framework in stochastic control problems, Yong [41] studied a linear-quadratic optimal control problem of mean-field SDEs, and Buckdahn, Chen and Li [8] studied the partial derivative with respect to the measure and its application to general controlled mean-field systems.

On the one hand, when the generator \(g\) is independent of \((\mathbb{P}_{Y}, \mathbb{P}_{Z})\), the general mean-field BSDE
(1.1) is reduced to the following BSDE:

\[
Y_t = \eta + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T, \tag{1.2}
\]

which were introduced by Pardoux and Peng \[34\], where the existence and uniqueness were obtained for the case of Lipschitz continuous coefficients. From then on, BSDEs have received numerous developments in various fields of partial differential equations (see Pardoux and Peng \[35\]), mathematical finance (see El Karoui, Peng, and Quenez \[18\]), and stochastic optimal control (see Yong and Zhou \[42\]), to mention a few. At the same time, due to various applications as well as the open problems proposed by Peng \[37\], many efforts have been made to relax the conditions on the generator \( g \) of BSDE (1.2) for the existence and/or uniqueness of adapted solutions. For instance, Lepeltier and San Martin \[30\] obtained the existence of adapted solutions for BSDEs when the generator \( g \) is continuous and of linear growth in \((Y, Z)\). In 2000, Kobylanski \[27\] proved the existence and uniqueness for one-dimensional BSDEs (1.2) when the generator \( g \) is of quadratic growth in \( Z \) and the terminal value \( \eta \) is bounded. Along this way, for the existence and uniqueness of BSDEs (1.2) with quadratic growth, the one-dimensional situation with unbounded terminal value was obtained by Briand and Hu \[6, 7\] and Bahlali, Eddahbi, and Ouaknine \[1\]; the multi-dimensional situation with bounded terminal value was studied by Hu and Tang \[25\] and Xing and Zitkovic \[40\]; and the multi-dimensional situation with unbounded terminal value was investigated by Fan, Hu and Tang \[19\], under different conditions and using different methods. Some other recent developments concerning the quadratic BSDEs can be found in Barrieu and El Karoui \[2\], Delbaen, Hu and Bao \[17\], Fan, Hu and Tang \[20\], Hu, Li, and Wen \[24\], and so on.

On the other hand, in the last two decades, stimulated by the broad applications and the open problem proposed by Peng \[37\], a lot of efforts have been made to relax the conditions on the generator \( g \) of the mean-field BSDEs (1.1). When the generator \( g \) depends on the expectation of \((Y, Z)\), Cheridito and Nam \[16\] discussed the existence of a class of mean-field BSDE with quadratic growth. Hibon, Hu, and Tang \[23\] studied the existence and uniqueness of one-dimensional mean-field BSDEs with quadratic growth, and Hao, Wen, and Xiong \[22\] studied a class of multi-dimensional mean-field BSDEs with quadratic growth and with small terminal value. However, to our best knowledge, there are few works concerning the quadratic mean-field BSDEs when the generator \( g \) depends on the laws of \((Y, Z)\), let alone the particle systems.

In this paper, we study the one-dimensional mean-field BSDE (1.1) with quadratic growth and with bounded terminal value by introducing some new ideas. First, we construct a local solution for the general mean-field BSDE (1.1) (see Theorem 3.3) by borrowing some ideas from Hu and Tang \[25\] and by using the fixed-point argument. It should be pointed out that the generator \( g \) is of a general growth with respect to \( Y \) (see Remark 3.1), and the method to choose the radius of the central ball is simpler than that of Hu and Tang \[25, Theorem 2.2\]. Second, with the additional boundedness condition on the generator \( g \) with respect to the distribution of \( Z \), we obtain the existence and uniqueness of global solutions (see Theorem 3.5) by using the Picard iteration to stitch the local solutions. Third, for a broader application of the theory, a comparison theorem
for such a class of BSDEs (see Theorem 4.1) is obtained with the Girsanov transform. Fourth, we consider the following system of $N$ particles:

$$Y_t^i = \eta^i + \int_t^T g\left(s, Y_s^i, Z_s^i, a \frac{1}{N} \sum_{i=1}^N \delta Y_s^i, a \frac{1}{N} \sum_{i=1}^N \delta Z_s^i\right)ds - \sum_{j=1}^N \int_t^T Z_s^{ij} dW_s^j, \quad t \in [0, T],$$

(1.3)

where $\delta$ is the Dirac measure, $\{\eta^i; 1 \leq i \leq N\}$ are $N$ independent copies of $\eta$, and $\{W^j, 1 \leq j \leq N\}$ are $N$ independent $d$-dimensional Brownian motions. Following Lions's idea and the law of large numbers, we prove that the mean-field limit of the $N$-particle system (1.3) converges to the mean-field BSDE (1.1) (see Theorem 5.2) when $N$ tends to infinity. Moreover, we obtain the rate of convergence (see Theorem 5.5) when the generator $g$ does not depend on the law of $Z$. Finally, we use the mean-field BSDE (1.1) with quadratic growth to prove the existence and uniqueness of the viscosity solution of a nonlocal PDE, and thus extend the nonlinear Feynman-Kac formula of Buckdahn, Li and Peng [10] with linear growth to the case of quadratic growth (see Theorem 6.6).

In our nonlinear Feynman-Kac formula, the generator $g$ depends on the distribution only via the expectation of the state process $(Y, Z)$. Since Wasserstein space has no local compactness, it remains to be a challenging topic in the theory of viscosity solutions to allow the generator to depend on the distribution in a general way (see Wu and Zhang [39]).

The rest of this paper is organized as follows. In Section 2, we present some preliminary notations and results. In Section 3, we prove the existence and uniqueness of the local and global solutions to the general mean-field BSDE (1.1). In Section 4, a comparison theorem is proved. We study the particle systems for mean-field BSDEs in Section 5, where the convergence and its rate are given for the particle systems. In Section 6, we give the relationship between the solution of the mean-field BSDE with quadratic growth and the viscosity solution of the related nonlocal PDE. Section 7 concludes the results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$ be a complete filtered probability space on which a $d$-dimensional standard Brownian motion $\{W_t; 0 \leq t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t; 0 \leq t < \infty\}$ is the natural filtration of $W$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. The notion $\mathbb{R}^{m \times d}$ is the space of $m \times d$-matrix $C$ with Euclidean norm $|C| = \sqrt{tr(CC^*)}$. For two real numbers $a$ and $b$, denote by $a \land b$ and $a \lor b$ the minimum and maximum of them, respectively. Set $a^+ = a \lor 0$ and $a^- = -(a \land 0)$. Denote by $1_A$ the indicator of set $A$, and $\text{sgn}(x) = 1_{x > 0} - 1_{x < 0}$ with $\text{sgn}(0) = 0$. For some positive real number $b$, by $[b]$ we denote the largest integer not exceeding $b$. Let $M$ be a continuous local martingale, denote $\mathcal{E}(M)_0 = \exp \left(M_t - \frac{1}{2} t < M > t \right), \quad 0 \leq t < \infty$. The notation $\delta_{\{a\}}$ denotes the Dirac measure at $a$.

In addition, for any $p \geq 1$, $t \in [0, T)$, and Euclidean space $\mathbb{H}$, we introduce the following spaces:

$$L^p_{\mathcal{F}_t}(\Omega; \mathbb{H}) = \left\{ \xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_t \text{-measurable, } ||\xi||_{L^p(\Omega)} = \left(\mathbb{E}[|\xi|^p]\right)^{\frac{1}{p}} < \infty \right\},$$

where $\mathbb{E}[\cdot]$ denotes the expectation.
for every \( \mu \in P_2(\mathbb{R}^d) \) there is a \( \nu \in P_p(\mathbb{R}^d) \),
\[
\mathcal{W}_p(\mu, \nu) \triangleq \inf \left\{ \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \rho(dx, dy) \right)^{1/p} \mid \rho \in P_2(\mathbb{R}^{2d}), \, \rho(\cdot \times \mathbb{R}^d) = \mu, \, \rho(\mathbb{R}^d \times \cdot) = \nu \right\}.
\]

Now we let \( p = 2 \) and suppose that there exists a sub-\( \sigma \)-algebra \( \mathcal{G} \) of \( \mathcal{F} \) which is independent of \( \mathcal{F}_\infty \) and will be assumed “rich enough”, as explained below: for every \( \mu \in P_2(\mathbb{R}^d) \) there is a random variable \( \vartheta \in L^2_{\mathcal{G}}(\Omega; \mathbb{R}^d) \) such that \( P_\vartheta = \mu \). It is well known that the probability space \(((0, 1], \mathcal{B}([0, 1]), dx)\) has this property. Then we call that a function \( h : P_2(\mathbb{R}^d) \to \mathbb{R} \) is differentiable in \( \mu_0 \in P_2(\mathbb{R}^d) \), if there exists a \( \xi_0 \in L^2_{\mathcal{G}}(\Omega; \mathbb{R}^d) \) with \( \mu_0 = P_{\xi_0} \), such that the lifted function

\[
L^p_{\mathcal{F}}(\Omega; \mathbb{H}) = \left\{ \xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_t\text{-measurable, } \|\xi\|_\infty \triangleq \operatorname{esssup}_{\omega \in \Omega} |\xi(\omega)| < \infty \right\},
\]

\[
L^p_{\mathcal{F}}(t, T; \mathbb{H}) = \left\{ \varphi : \Omega \times [t, T) \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}\text{-progressively measurable, } \|\varphi\|_{L^p_{\mathcal{F}}(t, T; \mathbb{H})} \triangleq \mathbb{E}\left[ \left( \int_t^T |\varphi_s|^p \, ds \right)^{1/p} \right] < \infty \right\},
\]

\[
S^p_{\mathcal{F}}(t, T; \mathbb{H}) = \left\{ \varphi : \Omega \times [t, T) \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}\text{-adapted, continuous, } \|\varphi\|_{S^p_{\mathcal{F}}(t, T; \mathbb{H})} \triangleq \left\{ \mathbb{E}\left( \sup_{s \in [t, T]} |\varphi_s|^p \right) \right\}^{1/p} < \infty \right\},
\]

\[
S^\infty_{\mathcal{F}}(t, T; \mathbb{H}) = \left\{ \varphi : \Omega \times [t, T) \to \mathbb{H} \mid \varphi \text{ is } \mathcal{F}\text{-adapted, continuous, } \|\varphi\|_{S^\infty_{\mathcal{F}}(t, T; \mathbb{H})} \triangleq \sup_{(s, \omega) \in [t, T] \times \Omega} |\varphi_s(\omega)| < \infty \right\},
\]

\[
Z^2_{\mathcal{F}}(t, T; \mathbb{H}) = \left\{ Z \in L^2_{\mathcal{F}}(t, T; \mathbb{H}) \mid \|Z\|_{Z^2_{\mathcal{F}}(t, T; \mathbb{H})} \triangleq \sup_{\tau \in \mathcal{F}[t, T]} \|\mathbb{E}[\int_\tau^T |Z_s|^2 \, ds]\|_\infty^{1/2} < \infty \right\},
\]

where \( \mathcal{F}[t, T] \) is the set of all \( \mathcal{F} \)-stopping times \( \tau \) valued in \([t, T]\). It is well known that the probability space \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) with finite \( p \)-th moment, i.e., \( \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty \). Here \( \mathcal{B}(\mathbb{R}^d) \) denotes the Borel \( \sigma \)-field over \( \mathbb{R}^d \). In addition, the class \( \{ M : \|M\|_{BMO_P(\mathbb{P})} < \infty \} \) is denoted by \( BMO_P(\mathbb{P}) \). Note that \( BMO_P(\mathbb{P}) \) is a Banach space under the norm \( \| \cdot \|_{BMO_P(\mathbb{P})} \). In the sequel, we write \( BMO(\mathbb{P}) \) for the space \( BMO_2(\mathbb{P}) \). Note that the process \( t \mapsto \int_0^t Z_s dW_t \) on \([0, T)\) (denoted by \( Z \cdot W \)) belongs to \( BMO(\mathbb{P}) \) if and only if \( Z \in Z^2_{\mathcal{F}}(0, T; \mathbb{H}) \), that is,

\[
\|Z \cdot W\|_{BMO(\mathbb{P})} = \|Z\|_{Z^2_{\mathcal{F}}(0, T)}. \tag{2.1}
\]
\( \tilde{h} : L^2_\mathbb{F}(\Omega; \mathbb{R}^d) \to \mathbb{R} \) defined by \( \tilde{h}(\xi) \triangleq h(\mathbb{P}_\xi) \) has Fréchet derivative at \( \xi_0 \). In other words, there exists a continuous linear functional \( D\tilde{h}(\xi_0) : L^2_\mathbb{F}(\Omega; \mathbb{R}^d) \to \mathbb{R} \), such that for any \( \eta \in L^2_\mathbb{F}(\Omega; \mathbb{R}^d) \),

\[
\tilde{h}(\xi_0 + \eta) - \tilde{h}(\xi_0) = D\tilde{h}(\xi_0)(\eta) + o(||\eta||_{L^2(\Omega)}) \quad \text{with} \quad ||\eta||_{L^2(\Omega)} \to 0.
\] (2.2)

Riesz representation theorem and the argument of Cardaliaguet [12] show that there exists a Borel measurable function \( \psi : \mathbb{R}^d \to \mathbb{R}^d \) depending only on the law of \( \xi_0 \), but not on the random variable \( \xi_0 \) itself, such that the forward term of (2.2) can be written as

\[
h(\mathbb{P}_{\xi_0 + \eta}) - h(\mathbb{P}_{\xi_0}) = E[\psi(\xi_0) \cdot \eta] + o(||\eta||_{L^2(\Omega)}).
\] (2.3)

Then, according to (2.3), Buckdahn, Li, Peng and Rainer [11] first define \( \partial_\mu h(\mathbb{P}_{\xi_0}; a) \triangleq \psi(a) \) for every \( a \in \mathbb{R}^d \), which is called the derivative of \( h \) at \( \mathbb{P}_{\xi_0} \). Remark that the function \( \partial_\mu h(\mathbb{P}_{\xi_0}; a) \) is only \( \mathbb{P}_{\xi_0}(da) \)-a.e. uniquely determined.

**Definition 2.1.** A pair of processes \( (Y, Z) \in S^2_\mathbb{F}(0, T; \mathbb{R}^m) \times L^2_\mathbb{F}(0, T; \mathbb{R}^{m \times d}) \) is called an adapted solution of mean-field BSDE (1.1) if it satisfies (1.1) \( \mathbb{P} \)-almost surely, and a bounded adapted solution if it further belongs to \( S^\infty_\mathbb{F}(0, T; \mathbb{R}^m) \times Z^2_\mathbb{F}(0, T; \mathbb{R}^{m \times d}) \).

Now, we recall the following propositions concerning BMO-martingales, which are slightly different from those of Kazamaki [28, Chapters 2 and 3].

**Proposition 2.2** (The Reverse Hölder Inequality). Let \( p \in (1, \infty) \) and \( M \) be a one-dimensional continuous BMO martingale. If \( ||M||_{BMO(\mathbb{P})} < \Phi(p) \), then \( \mathcal{E}(M) \) satisfies the reverse Hölder inequality:

\[
E_\tau[\mathcal{E}(M)^\infty_\tau] \leq c_p,
\]

for any stopping time \( \tau \), with a positive constant \( c_p \) depending only on \( p \).

**Proposition 2.3.** For \( \tilde{K} > 0 \), there are constants \( c_1 > 0 \) and \( c_2 > 0 \) depending on \( \tilde{K} \) such that for any BMO-martingale \( M \) and any one-dimensional BMO-martingale \( N \) such that \( ||N||_{BMO(\mathbb{P})} \leq \tilde{K} \), we have

\[
c_1 ||M||_{BMO(\mathbb{P})} \leq ||\tilde{M}||_{BMO(\mathbb{P})} \leq c_2 ||M||_{BMO(\mathbb{P})},
\]

where \( \tilde{M} \triangleq M - \langle M, N \rangle \) and \( d\tilde{\mathbb{P}} \triangleq \mathcal{E}(N)^\infty_0 d\mathbb{P} \).

### 3 Existence and Uniqueness

In this section, we study the existence and uniqueness of mean-field BSDE (1.1) with quadratic growth and bounded terminal value. In particular, we consider the one-dimensional situation, i.e., \( m = 1 \). We shall study the local and global solutions and divide this section into two parts: Subsection 3.1 for the local solution and Subsection 3.2 for the global solution.

In the following, suppose that \( \theta : \Omega \times [0, T] \to \mathbb{R}^+ \) is an \( \mathcal{F}_t \)-progressively measurable process, \( \phi, \phi_0 : [0, +\infty) \to [0, +\infty) \) are two nondecreasing continuous functions, and \( \beta, \beta_0, \gamma, \gamma_0, \tilde{\gamma} \), and \( \alpha \in [0, 1) \) are all positive constants.
3.1 Local solution

In this subsection, we prove the existence and uniqueness of local adapted solutions of mean-field BSDE (1.1) with quadratic growth. For simplicity, we rewrite it as follows:

\[ Y_t = \eta + \int_t^T g(s, Y_s, Z_s, \mathbb{P}_{Y_s}, \mathbb{P}_Z) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T]. \]  

\[ (3.1) \]

**Assumption 1.** The terminal value \( \eta : \Omega \to \mathbb{R} \) and the generator \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) satisfy the following conditions:

(i) \( d\mathbb{P} \times dt \text{-a.e. } (\omega, t) \in \Omega \times [0, T], \) for \( y \in \mathbb{R}, \ z \in \mathbb{R}^d \) and \( \mu_1 \in \mathcal{P}_2(\mathbb{R}), \mu_2 \in \mathcal{P}_2(\mathbb{R}^d), \)

\[ |g(y, \omega, t, z, \mu_1, \mu_2)| \leq \theta_t(\omega) + \phi(|y|) + \frac{\gamma}{2} |z|^2 + \phi_0(W_2(\mu_1, \delta_{[0]})) + \gamma_0 W_2(\mu_2, \delta_{[0]})^{\frac{1+\alpha}{\alpha}}. \]

(ii) \( d\mathbb{P} \times dt \text{-a.e. } (\omega, t) \in \Omega \times [0, T], \) for \( y, \tilde{y} \in \mathbb{R}, \ z, \tilde{z} \in \mathbb{R}^d \) and \( \mu_1, \tilde{\mu}_1 \in \mathcal{P}_2(\mathbb{R}), \mu_2, \tilde{\mu}_2 \in \mathcal{P}_2(\mathbb{R}^d), \)

\[ |g(\omega, t, y, z, \mu_1, \mu_2) - g(\omega, t, \tilde{y}, \tilde{z}, \tilde{\mu}_1, \tilde{\mu}_2)| \leq \phi(|y| \vee |\tilde{y}| \vee W_2(\mu_1, \delta_{[0]}) \vee W_2(\tilde{\mu}_1, \delta_{[0]})) \]

\[ \cdot \left[ (1 + |z| + |\tilde{z}| + W_2(\mu_2, \delta_{[0]}) + W_2(\tilde{\mu}_2, \delta_{[0]})) \right] \cdot \left( |y - \tilde{y}| + |z - \tilde{z}| + W_2(\mu_1, \tilde{\mu}_1) \right) \]

\[ + \left( 1 + W_2(\mu_2, \delta_{[0]}))^{\alpha} + W_2(\tilde{\mu}_2, \delta_{[0]})^{\alpha} \right) W_2(\mu_2, \tilde{\mu}_2). \]

(iii) There are two positive constants \( K_1 \) and \( K_2 \) such that

\[ \|\eta\|_{\infty} \leq K_1 \quad \text{and} \quad \left\| \int_0^T \theta_t(\omega) dt \right\|_{\infty} \leq K_2. \]

**Remark 3.1.** Assumption 1 is more general than the condition (\( \mathcal{A}1 \)) of Hibon, Hu, and Tang [23] in that here we relax the growth and continuity of the generator \( g \) with respect to the first variable \( y \). For example, the following generator \( g \) satisfies Assumption 1, while does not satisfies the condition (\( \mathcal{A}1 \)) of [23]: for every \( t \in [0, T], \ (y, z) \in \mathbb{R} \times \mathbb{R}^d, \mu_1 \in \mathcal{P}_2(\mathbb{R}), \mu_2 \in \mathcal{P}_2(\mathbb{R}^d), \)

\[ g(t, y, z, \mu_1, \mu_2) = |y|^2 |z| + |z|^2 + W_2(\mu_1, \delta_{[0]})^3 \cos(W_2(\mu_2, \delta_{[0]})) + W_2(\mu_2, \delta_{[0]})^2. \]

The following proposition is an essential extension of Hu and Tang [25, Lemma 2.1] to the mean-field situation.

**Proposition 3.2.** Assume that for any given processes \( (P, Q) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathcal{Z}^2_{\mathbb{F}}(0, T; \mathbb{R}^d), \) the terminal value \( \eta : \Omega \to \mathbb{R} \) and the generator \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) satisfy the conditions:

(i) \( d\mathbb{P} \times dt \text{-a.e. } t \in [0, T] \) for any \( z \in \mathbb{R}^d, \)

\[ |g(\omega, t, z)| \leq \theta_t(\omega) + \phi(|P_t|) + \frac{\gamma}{2} |z|^2 + \phi_0(\|P_t\|_{L^2(\Omega)}) + \gamma_0 \|Q_t\|_{L^2(\Omega)}^{\frac{1+\alpha}{\alpha}}. \]
(ii) \( d \mathbb{P} \times dt \text{-a.e. } t \in [0, T] \) for every \( z, \bar{z} \in \mathbb{R}^d \),

\[
|g(\omega, t, z) - g(\omega, t, \bar{z})| \leq \phi\left(|P_t| \vee \|P_t\|_{L^2(\Omega)}\right) \cdot (1 + |z| + |ar{z}|) + 2\|Q_t\|_{L^2(\Omega)}|z - \bar{z}|.
\]

(iii) For two positive constants \( K_1 \) and \( K_2 \),

\[
\|\eta\|_{\infty} \leq K_1 \quad \text{and} \quad \left\| \int_0^T \theta_t(\omega) dt \right\|_{\infty} \leq K_2.
\]

Then the following backward stochastic differential equation

\[
Y_t = \eta + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]
\]

admits a unique adapted solution \((Y, Z) \in S_{\infty}^2(0, T; \mathbb{R}) \times Z_{\infty}^2(0, T; \mathbb{R}^d)\). Moreover, for any \( t \in [0, T] \) and stopping time \( \tau \in \mathcal{F}[t, T] \), one has

\[
|Y_t| \leq \|\eta\|_{\infty} + \left\| \int_0^T \theta_s(\omega) ds \right\|_{\infty} + \phi\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) \]

\[
+ \phi_0\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) + \gamma_0\|Q\|_{Z_{\infty}^2(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}},
\]

and

\[
\mathbb{E}_\tau\left[ \int_\tau^T |Z_s|^2 ds \right] \leq \frac{1}{\gamma^2} \exp\left\{2\gamma\|\eta\|_{\infty}\right\} + \frac{2}{\gamma} \exp\left\{2\gamma\|\bar{Y}\|_{S_\infty^\infty(t, T)}\right\} \cdot \left( \left\| \int_0^T \theta_s(\omega) ds \right\|_{\infty} \right)
\]

\[
+ \phi\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) + \phi_0\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) + \gamma_0\|Q\|_{Z_{\infty}^2(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}}.
\]

Proof. Using Hölder’s inequality, we have for \( t \in [0, T] \),

\[
\int_t^T \left( \mathbb{E}[|Q_s|^2] \right)^{\frac{1+\alpha}{2}} ds \leq \left( \int_t^T \mathbb{E}[|Q_s|^2] ds \right)^{\frac{1+\alpha}{2}} \left( T - t \right)^{\frac{1-\alpha}{2}} \leq \|Q\|_{Z_{\infty}^2(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}}.
\]

Then, with a constant \( \ell \geq 1 \), we have

\[
\mathbb{E}_t\left[ \exp\left\{ \ell\gamma_0 \int_t^T \|Q_s\|_{L^2(\Omega)}^{1+\alpha} ds \right\} \right] = \exp\left\{ \ell\gamma_0 \int_t^T \left( \mathbb{E}[|Q_s|^2] \right)^{\frac{1+\alpha}{2}} ds \right\}
\]

\[
\leq \exp\left\{ \ell\gamma_0\|Q\|_{Z_{\infty}^2(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}} \right\}.
\]

Consequently,

\[
\mathbb{E}_t \exp\left\{ p\gamma|\eta| + \ell\gamma \int_t^T \left[(\theta_s(\omega) + \phi(|P_s|)) + \phi_0(\|P_s\|_{L^2(\Omega)}) + \gamma_0\|Q_s\|_{L^2(\Omega)}\right] ds \right\}
\]

\[
\leq \exp\left\{ \ell\gamma|\eta|_{\infty} + \ell\gamma \int_0^T \theta_s(\omega) ds \right\}_{\infty} + \ell\gamma\phi\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) \]

\[
+ \ell\gamma\phi_0\left(\|P\|_{S_\infty^\infty(t, T)}\right)(T - t) + \ell\gamma_0\|Q\|_{Z_{\infty}^2(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}} \right\} < \infty.
\]
In view of the conditions (i)-(iii), we see from Briand and Hu \cite[Proposition 3 and Corollary 4]{7} that BSDE (3.2) admits a solution \((Y, Z)\) such that
\[
\mathbb{E} \int_0^T |Z_s|^2 ds < \infty
\]
and
\[
|Y_t| \leq \|\eta\| + \left\| \int_0^T \theta_s(\omega) ds \right\|_{\infty} + \phi(\|P\|_{S^\infty_{\mathbb{F}}(t,T)}) (T - t)
\]
\[
+ \phi_0(\|P\|_{S^\infty_{\mathbb{F}}(t,T)}) (T - t) + \gamma_0 Q^{1+\alpha}_{Z^2_{\mathbb{F}}(t,T)} (T - t)^{\frac{1-\alpha}{2}}
\]
which implies the inequality (3.3) holds and thus \(Y \in S^\infty_{\mathbb{F}}(0, T; \mathbb{R})\).

We now prove that \(Z \cdot W\) is a BMO martingale. Applying Itô-Tanaka’s formula to the term \(\exp \{2\gamma|Y_t|\}\) with \(r_T \in \mathcal{T}[t, T]\), we have that
\[
\exp \{2\gamma|Y_t|\} + 2\gamma^2 \int_r^T \exp \{2\gamma|Y_s|\} |Z_s|^2 ds
\]
\[
= \exp \{2\gamma|\eta|\} + 2\gamma \int_r^T \exp \{2\gamma|Y_s|\} g(s, Z_s) ds + 2\gamma \int_r^T \exp \{2\gamma|Y_s|\} Z_s dW_s. \tag{3.6}
\]
In view of the condition (i), we have
\[
\exp \{2\gamma|Y_t|\} + 2\gamma^2 \mathbb{E}_r \left[ \int_r^T \exp \{2\gamma|Y_s|\} |Z_s|^2 ds \right] \leq \mathbb{E}_r \left[ \exp \{2\gamma|\eta|\} \right]
\]
\[
+ 2\gamma \mathbb{E}_r \int_r^T \exp \{2\gamma|Y_s|\} \left[ \theta_s(\omega) + \phi(|P_s|) + \frac{\gamma}{2} |Z_s|^2 + \phi_0(\|P_s\|_{L^2(\Omega)}) + \gamma_0 Q_s \right] \right] ds.
\]
Consequently, we have
\[
\gamma^2 \mathbb{E}_r \int_r^T |Z_s|^2 ds \leq \gamma^2 \mathbb{E}_r \int_r^T \exp \{2\gamma|Y_s|\} |Z_s|^2 ds
\]
\[
\leq \exp \{2\gamma|\eta|\} + 2\gamma \exp \{2\gamma|Y_s|_{S^\infty_{\mathbb{F}}(t,T)}\} \left[ \left\| \int_0^T \theta_s(\omega) ds \right\|_{\infty} + \phi(\|P\|_{S^\infty_{\mathbb{F}}(t,T)}) (T - t)
\]
\[
+ \phi_0(\|P\|_{S^\infty_{\mathbb{F}}(t,T)}) (T - t) + \gamma_0 Q^{1+\alpha}_{Z^2_{\mathbb{F}}(t,T)} (T - t)^{\frac{1-\alpha}{2}} \right].
\]
Hence, we have inequality (3.4) and thereby \(Z \in Z^2_{\mathbb{F}}([0, T]; \mathbb{R}^d)\). In other words, \(Z \cdot W\) is a BMO martingale.

Finally, we prove the uniqueness. In fact, in view of the condition (ii) with \((U, V) \in S^\infty_{\mathbb{F}}(0, T; \mathbb{R}) \times Z^2_{\mathbb{F}}(0, T; \mathbb{R}^d)\), similar to the proof of Hu and Tang \cite[Lemma 2.1]{25}, we can use the Girsanov transform to derive a comparison result on the solutions of BSDE (3.2), which yields the desired result. \(\square\)

We now prove the existence and uniqueness of mean-field BSDE (3.1) in a subset of \(S^\infty_{\mathbb{F}}(T - \varepsilon, T; \mathbb{R}) \times Z^2_{\mathbb{F}}(T - \varepsilon, T; \mathbb{R}^d)\), where \(\varepsilon \in (0, T)\) is some constant. In addition, for positive constants
$L_1$ and $L_2$, we define the following Banach space:

$$\mathcal{B}_\varepsilon(L_1, L_2) \triangleq \left\{ (Y, Z) \in S_{\bar{\varepsilon}}(T - \varepsilon, T; \mathbb{R}) \times Z^2_{\bar{\varepsilon}}(T - \varepsilon, T; \mathbb{R}^d), \right.$$

$$\left. \|Y\|_{S_{\bar{\varepsilon}}(T - \varepsilon, T)} \leq L_1 \quad \text{and} \quad \|Z\|_{Z^2_{\bar{\varepsilon}}(T - \varepsilon, T)} \leq L_2 \right\}$$

eualled with the norm

$$\|(Y, Z)\|_{\mathcal{B}_\varepsilon(L_1, L_2)} = \left\{ \|Y\|_{S_{\bar{\varepsilon}}(T - \varepsilon, T)}^2 + \|Z\|_{Z^2_{\bar{\varepsilon}}(T - \varepsilon, T)}^2 \right\}^{1/2}.$$ 

The main result of this subsection is stated as follows.

**Theorem 3.3.** Under Assumption 1, there exists a positive constant $\varepsilon$, depending only on functions $(\phi, \phi_0)$ and constants $(\gamma, \gamma_0, \alpha, K_1, K_2)$, such that on the interval $[T - \varepsilon, T]$, the mean-field BSDE (3.1) possesses a unique local adapted solution $(Y, Z) \in \mathcal{B}_\varepsilon(L_1, L_2)$ with

$$L_1 = \frac{1}{2}(K_1 + K_2) \quad \text{and} \quad L_2 = \frac{1}{\gamma_2} e^{2\gamma_1 K_1} + \frac{2K_2}{\gamma} e^{2\gamma L_1}. \quad (3.7)$$

**Proof.** We construct a contraction mapping to prove the existence and uniqueness of (3.1). For any given processes $(P, Q) \in S_{\bar{\varepsilon}}(0, T; \mathbb{R}) \times Z^2_{\bar{\varepsilon}}(0, T; \mathbb{R}^d)$, we consider the following BSDE:

$$Y_t = \eta + \int_t^T g(s, P_s, Z_s, \mathbb{P}_{P_s}, \mathbb{P}_{Q_s})ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \quad (3.8)$$

We define the generator $g^{P,Q}$:

$$g^{P,Q}(t, z) = g(t, P_t, z, \mathbb{P}_{P_t}, \mathbb{P}_{Q_t}), \quad (t, z) \in [0, T] \times \mathbb{R}^d.$$ 

Under Assumption 1, we easily check that the generator $g^{P,Q}$ satisfies $d\mathbb{P} \times dt$-a.e., for every $z, \bar{z} \in \mathbb{R}^d$,

$$|g^{P,Q}(\omega, t, z)| \leq \theta_t(\omega) + \phi(|P_t|) + \frac{\gamma}{2}|z|^2 + \phi_0\left(\|P_t\|_{L^2(\Omega)}\right) + \gamma_0\|Q_t\|_{L^2(\Omega)}^{1+\alpha},$$

and

$$|g^{P,Q}(\omega, t, z) - g^{P,Q}(\omega, t, \bar{z})| \leq \phi\left(|P_t| \vee \|P_t\|_{L^2(\Omega)}\right) \cdot \left(1 + |z| + |\bar{z}| + 2\|Q_t\|_{L^2(\Omega)}\right)|z - \bar{z}|.$$

Thus according to Proposition 3.2, BSDE (3.8) admits a unique solution $(Y, Z) \in S_{\bar{\varepsilon}}(0, T; \mathbb{R}) \times Z^2_{\bar{\varepsilon}}(0, T; \mathbb{R}^d)$. Moreover, for any $t \in [0, T]$ and stopping time $\tau \in \mathcal{T}[t, T]$, we have

$$|Y_t| \leq \|\eta\|_\infty + \left\|\int_0^T \theta_s(\omega)ds\right\|_\infty + \phi\left(\|P\|_{S_{\bar{\varepsilon}}(t, T)}\right)(T - t)$$

$$+ \phi_0\left(\|P\|_{S_{\bar{\varepsilon}}(t, T)}\right)(T - t) + \gamma_0\|Q\|_{Z^2_{\bar{\varepsilon}}(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}} \right\}.$$ \hspace{1cm} (3.9)$$

and

$$\mathbb{E}_\tau \int_\tau^T |Z_s|^2ds \leq \frac{1}{2\gamma_2} \exp\left\{2\gamma\|\eta\|_\infty\right\} + \frac{1}{\gamma} \exp\left\{2\gamma\|Y\|_{S_{\bar{\varepsilon}}(t, T)}\right\} \cdot \left[2\left\|\int_0^T \theta_s(\omega)ds\right\|_\infty$$

$$+ 2\phi\left(\|P\|_{S_{\bar{\varepsilon}}(t, T)}\right)(T - t) + 2\phi_0\left(\|P\|_{S_{\bar{\varepsilon}}(t, T)}\right)(T - t) + 2\gamma_0\|Q\|_{Z^2_{\bar{\varepsilon}}(t, T)}^{1+\alpha}(T - t)^{\frac{1-\alpha}{2}}\right]. \quad (3.10)$$
From this, we define the mapping $\Theta$ on $S_{P}^{\infty}(0, T; \mathbb{R}) \times Z_{P}^{2}(0, T; \mathbb{R}^d)$ as follows:

$$
\Theta(P, Q) \triangleq (Y, Z), \quad (P, Q) \in S_{P}^{\infty}(0, T; \mathbb{R}) \times Z_{P}^{2}(0, T; \mathbb{R}^d).
$$

Next, we find a suitable subset $\mathcal{B}_\varepsilon(L_1, L_2)$ of $S_{P}^{\infty}(0, T; \mathbb{R}) \times Z_{P}^{2}(0, T; \mathbb{R}^d)$ such that $\Theta$ is stable and contractive in it. For this, note (3.9) and (3.10), thanks to Assumption 1, we have

$$
\|Y\|_{S_{P}^{\infty}(t, T)} \leq K_1 + K_2 + \phi(\|P\|_{S_{P}^{\infty}(t, T)})(T - t) + \phi_0(\|P\|_{S_{P}^{\infty}(t, T)})(T - t) + \gamma_0\|Q\|^{{1+\alpha}\over 2}\|\varepsilon_{P}^2(t, T)\|_{Z_{P}^{2}(t, T)}\|\varepsilon_{P}^2(t, T)\|_{S_{P}^{\infty}(t, T)}^{1+\alpha}(T - t)^{1-\alpha}\|\varepsilon_{P}^2(t, T)\|_{S_{P}^{\infty}(t, T)}^{1+\alpha}(T - t)^{1-\alpha}
$$

(3.11)

and

$$
\|Z\|^2_{Z_{P}^{2}(t, T)} \leq \frac{1}{\gamma^2} \exp(2\gamma\|\eta\|_{\infty}) + \frac{1}{\gamma} \exp(2\gamma\|Y\|_{S_{P}^{\infty}(t, T)}) \cdot \left[2K_2 + 2\phi(\|P\|_{S_{P}^{\infty}(t, T)})(T - t) + 2\phi_0(\|P\|_{S_{P}^{\infty}(t, T)})(T - t) + 2\gamma_0\|Q\|_{Z_{P}^{2}(t, T)}(T - t)^{1-\alpha}\right].
$$

(3.12)

Now, we define

$$
L_1 = 2(K_1 + K_2), \quad L_2 = \frac{2}{\gamma^2}e^{2\gamma K_1} + \frac{4K_2}{\gamma}e^{2\gamma L_1}.
$$

(3.13)

By $\varepsilon_1$ and $\varepsilon_2$, we denote the unique solutions to the following equations:

$$
[\phi(L_1) + \phi_0(L_1)]x + \gamma_0L_2^2 \left(x^{1-\alpha}\right) \cdot \frac{1}{2} = \frac{L_1}{2}
$$

and

$$
[2\phi(L_1) + 2\phi_0(L_1)]y + 2\gamma_0L_2^2 \left(y^{1-\alpha}\right) \cdot \frac{1}{2} = \frac{\delta L_2}{\gamma} e^{-2\gamma L_1},
$$

(3.14)

respectively. Hence, combining (3.11)-(3.14), we see that if

$$
\|P\|_{S_{P}^{\infty}(T-\varepsilon, T)} \leq L_1 \quad \text{and} \quad \|Q\|^2_{Z_{P}^{2}(T-\varepsilon, T)} \leq L_2,
$$

then

$$
\|Y\|_{S_{P}^{\infty}(T-\varepsilon, T)} \leq L_1 \quad \text{and} \quad \|Z\|^2_{Z_{P}^{2}(T-\varepsilon, T)} \leq L_2, \quad \forall \varepsilon \in (0, \varepsilon^*],
$$

where $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\} > 0$. This implies that,

$$
\Theta(P, Q) \in \mathcal{B}_\varepsilon(L_1, L_2), \quad \forall (P, Q) \in \mathcal{B}_\varepsilon(L_1, L_2).
$$

Therefore, the mapping $\Theta$ is stable in $\mathcal{B}_\varepsilon(L_1, L_2)$.

In the following, we prove that the mapping $\Theta$ is contractive in $\mathcal{B}_\varepsilon(L_1, L_2)$. Indeed, for any fixed $\varepsilon \in (0, \varepsilon^*)$, and any pairs $(P, Q) \in \mathcal{B}_\varepsilon(L_1, L_2)$ and $(P, Q) \in \mathcal{B}_\varepsilon(L_1, L_2)$, we set

$$
(Y, Z) = \Theta(P, Q) \quad \text{and} \quad (\tilde{Y}, \tilde{Z}) = \Theta(\tilde{P}, \tilde{Q}).
$$

In addition, set for any $t \in [T - \varepsilon, T]$

$$
\Delta Y_t = Y_t - \tilde{Y}_t, \quad \Delta Z_t = Z_t - \tilde{Z}_t, \quad \Delta P_t = P_t - \tilde{P}_t, \quad \Delta Q_t = Q_t - \tilde{Q}_t.
$$

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Then, we have
\[ \Delta Y_t = \int_t^T (I_{1,s} + I_{2,s}) ds - \int_t^T \Delta Z_s dW_s, \quad t \in [T - \varepsilon, T], \]  
(3.15)
where
\[
I_{1,s} = g(s, P_s, Z_s, P_{P_s}, P_{Q_s}) - g(s, \bar{P}_s, \bar{Z}_s, P_{P_s}, P_{Q_s}),
\]
\[
I_{2,s} = g(s, P_s, \bar{Z}_s, P_{P_s}, P_{Q_s}) - g(s, P_s, Z_s, P_{P_s}, P_{Q_s}),
\]
and thus
\[
I_{1,s} + I_{2,s} = g(s, P_s, Z_s, P_{P_s}, P_{Q_s}) - g(s, \bar{P}_s, \bar{Z}_s, P_{P_s}, P_{Q_s}).
\]

Note that the term \( I_{1,s} \) can be written as \( I_{1,s} = \Lambda_s (Z_s - \bar{Z}_s) \), where
\[
\Lambda_s = \begin{cases} 
\frac{(g(s, P_s, Z_s, P_{P_s}, P_{Q_s}) - g(s, P_s, \bar{Z}_s, P_{P_s}, P_{Q_s}))(Z_s - \bar{Z}_s)}{|Z_s - \bar{Z}_s|^2}, & \text{if } Z_s \neq \bar{Z}_s; \\
0, & \text{if } Z_s = \bar{Z}_s.
\end{cases}
\]

Clearly, the item (ii) of Assumption 1 implies that
\[
|\Lambda_s| \leq \phi(|P_s| \vee |P_{\bar{P}}|) (1 + |Z_s| + |\bar{Z}_s| + 2|Q_s|), \quad \forall s \in [t, T].
\]

Then, for
\[
\bar{W}_s = W_s - \int_0^s \Lambda_r dr \quad \text{and} \quad dQ = \delta \cdot dW, \quad (3.16)
\]
the probability measure \( Q \) is equivalent to \( P \) and the process \( \bar{W} \) is a Brownian motion under \( Q \). Furthermore, we can rewrite BSDE (3.15) as
\[
\Delta Y_t + \int_t^T \Delta Z_s d\bar{W}_s = \int_t^T I_{2,s} ds, \quad \forall t \in [T - \varepsilon, T];
\]
(3.17)
then we easily have
\[
|\Delta Y_t|^2 + \mathbb{E}^Q_t \int_t^T |\Delta Z_s|^2 ds = \mathbb{E}^Q_t \left[ \left( \int_t^T I_{2,s} ds \right)^2 \right].
\]

Since the following estimate follows from the item (ii) of Assumption 1
\[
|I_{2,s}| \leq \phi \left( |P_s| \vee |P_{\bar{P}}| \vee \mathbb{W}_2(P_{P_s}, \delta_{(0)}) \vee \mathbb{W}_2(P_{P_{\bar{P}}}, \delta_{(0)}) \right)
\]
\[
\cdot \left[ (1 + 2|\bar{Z}_s| + \mathbb{W}_2(P_{Q_s}, \delta_{(0)}) + \mathbb{W}_2(P_{P_{Q_s}}, \delta_{(0)})) \cdot (|\Delta P_s| + \mathbb{W}_2(P_{P_s}, P_{P_{\bar{P}}})) \right.
\]
\[
\left. + \left( 1 + \mathbb{W}_2(P_{Q_s}, \delta_{(0)})^\alpha + \mathbb{W}_2(P_{P_{Q_s}}, \delta_{(0)})^\alpha \right) \cdot \mathbb{W}_2(P_{Q_s}, P_{Q_s}) \right]
\]
\[
\leq \phi \left( |P_s| \vee |P_{\bar{P}}| \vee \|P_s\|_{L^2(\Omega)} \vee \|P_{\bar{P}}\|_{L^2(\Omega)} \right)
\]
\[
\cdot \left[ (1 + 2|\bar{Z}_s| + \|Q_s\|_{L^2(\Omega)} + \|\bar{Q}_s\|_{L^2(\Omega)}) \cdot (|\Delta P_s| + \|\Delta P_s\|_{L^2(\Omega)}) \right.
\]
\[
\left. + \left( 1 + \|Q_s\|_{L^2(\Omega)}^\alpha + \|\bar{Q}_s\|_{L^2(\Omega)}^\alpha \right) \cdot \|\Delta Q_s\|_{L^2(\Omega)} \right],
\]
(3.17)
\[
\leq \phi \left( \| P \|_{S^p_{\infty}}(t,T) \lor \| \bar{P} \|_{S^p_{\infty}}(t,T) \right) \left[ (1 + 2|\bar{Z}_s| + \| Q_s \|_{L^2(\Omega)} + \| \bar{Q}_s \|_{L^2(\Omega)}) \cdot 2\| \Delta P \|_{S^p_{\infty}}(t,T) \right. \\
\left. + (1 + \| Q_s \|_{L^2(\Omega)}^\alpha + \| \bar{Q}_s \|_{L^2(\Omega)}^\alpha) \| \Delta Q_s \|_{L^2(\Omega)} \right]
\]

we have

\[
|\Delta Y_t|^2 + E^Q_t \left[ \int_t^T |\Delta Z_s|^2 ds \right] \\
\leq E^Q_t \left\{ \left[ \int_t^T 2\phi(L_1) \left( 1 + 2|\bar{Z}_s| + \| Q_s \|_{L^2(\Omega)} + \| \bar{Q}_s \|_{L^2(\Omega)} \right) \| \Delta P \|_{S^p_{\infty}}(t,T) \right. \\
\left. + \phi(L_1) \left( 1 + \| Q_s \|_{L^2(\Omega)}^\alpha + \| \bar{Q}_s \|_{L^2(\Omega)}^\alpha \right) \| \Delta Q_s \|_{L^2(\Omega)} ds \right]^2 \right\} \\
\leq 8\phi(L_1)^2 E^Q_t \left[ \left( \int_t^T (1 + 2|\bar{Z}_s| + \| Q_s \|_{L^2(\Omega)} + \| \bar{Q}_s \|_{L^2(\Omega)}) \| \Delta P \|_{S^p_{\infty}}(t,T) ds \right)^2 \right] \\
+ 2\phi(L_1)^2 \left( \int_t^T (1 + \| Q_s \|_{L^2(\Omega)} + \| \bar{Q}_s \|_{L^2(\Omega)}) \| \Delta Q_s \|_{L^2(\Omega)} ds \right)^2.
\]

We now estimate the first term in the right hand side of the last inequality, using Hölder’s inequality and Proposition 2.3, and noting (2.1) for \( t \in [T - \epsilon, T] \),

\[
8\phi(L_1)^2 E^Q_t \left[ \left( \int_t^T (1 + 2|\bar{Z}_s| + \| Q_s \|_{L^2(\Omega)} + \| \bar{Q}_s \|_{L^2(\Omega)}) \| \Delta P \|_{S^p_{\infty}}(t,T) ds \right)^2 \right] \\
\leq 32\epsilon \phi(L_1)^2 \| \Delta P \|_{S^p_{\infty}}^2(\bar{Z}_s, t,F) \left[ T + 4E^Q_t \left[ \int_t^T |\bar{Z}_s|^2 ds \right] + \int_t^T E^\mathbb{F}[Q_s]^2 ds + \int_t^T E^\mathbb{F}[\bar{Q}_s]^2 ds \right] \\
\leq 32\epsilon \phi(L_1)^2 \| \Delta P \|_{S^p_{\infty}}^2(\bar{Z}_s, t,F) \left[ T + 4\| \bar{Z} \cdot \bar{W} \|_{BMO(\mathbb{F})}^2 + \| Q \cdot W \|_{BMO(\mathbb{F})}^2 + \| \bar{Q} \cdot W \|_{BMO(\mathbb{F})}^2 \right] \\
\leq 32\epsilon \phi(L_1)^2 \left( T + 4c_2L_2 + 2L_2 \right) \| \Delta P \|_{S^p_{\infty}}^2(\bar{Z}_s, t,F).
\]

Since \( \alpha \in [0, 1) \), using Hölder’s inequality and (2.1), we see that for \( t \in [T - \epsilon, T] \),

\[
\int_t^T \| Q_s \|_{L^2(\Omega)}^{2\alpha} ds = \int_t^T \left\{ E^\mathbb{F}[Q_s]^2 \right\}^{\alpha} ds \leq \left( \int_t^T E^\mathbb{F}[Q_s]^2 ds \right)^{\alpha} \left( T - t \right)^{1-\alpha} \\
\leq \| Q \|_{L^2(\bar{Z}_s, t,F)}^{2\alpha} (T - t)^{1-\alpha} \leq L_2^{\alpha} (T - t)^{1-\alpha} \leq L_2 (T - t)^{1-\alpha}.
\]

In the above, \( L_2 \) can be chosen to be greater than 1, by a careful choice of \( \gamma \), \( K_1 \) or \( K_2 \) in (3.13). Consequently, using Hölder’s inequality, we estimate the second term in the right hand side of (3.18)
as follows:
\[
2\phi(L_1)^2 \left( \int_t^T (1 + \|Q_s\|_{L^2(\Omega)}^\alpha + \|\bar{Q}_s\|_{L^2(\Omega)}^\alpha) \|\Delta Q_s\|_{L^2(\Omega)}^2 ds \right)^2 \leq 2\phi(L_1)^2 \int_t^T \|\Delta Q_s\|_{L^2(\Omega)}^2 ds \cdot \int_t^T \left( (1 + \|Q_s\|_{L^2(\Omega)}^\alpha + \|\bar{Q}_s\|_{L^2(\Omega)}^\alpha)^2 ds \right) \\
\leq 6\phi(L_1)^2 \|\Delta Q\|_{Z^2_{\bar{S},t}(t,T)}^2 \left[ T - t + 2L_2(T - t)^{1-\alpha} \right] \\
\leq 6\varepsilon^{1-\alpha} \phi(L_1)^2 (T\alpha + 2L_2^2) \|\Delta Q\|_{Z^2_{\bar{S},t}(t,T)}^2,
\]
where in the last inequality we have used the inequality \( \varepsilon \leq \varepsilon^{1-\alpha} T^\alpha \). Now from (3.18)-(3.19), we have
\[
|\Delta Y_t|^2 + \mathbb{E}_t^Q \int_t^T |\Delta Z_s|^2 ds \leq 32\varepsilon \phi(L_1)^2 (T + 4c_2L_2 + 2L_2^2) \|\Delta P\|^2_{S^\infty_{\bar{S},t}(T - \varepsilon, t)} + 6\varepsilon^{1-\alpha} \phi(L_1)^2 (T\alpha + 2L_2^2) \|\Delta Q\|_{Z^2_{\bar{S},t}(t,T)}^2.
\]
Thanks to Proposition 2.3, and noting that \( \alpha \in [0, 1) \) and \( t \in [T - \varepsilon, T] \), we have
\[
\|\Delta Y\|^2_{S^\infty_{\bar{S},(T-\varepsilon,T)}} + c_1^2 \|\Delta Z\|^2_{Z^2_{\bar{S},(T-\varepsilon,T)}} \leq 32\varepsilon \phi(L_1)^2 (T + 4c_2L_2 + 2L_2^2) \|\Delta P\|^2_{S^\infty_{\bar{S},(T-\varepsilon,T)}} + 6\varepsilon^{1-\alpha} \phi(L_1)^2 (T\alpha + 2L_2^2) \|\Delta Q\|^2_{Z^2_{\bar{S},(T-\varepsilon,T)}}.
\]
From (2.1), we know that \( \Theta \) is a contraction in \( \mathcal{B}_{\bar{S}}(L_1, L_2) \) for sufficiently small \( \varepsilon > 0 \). The proof is complete.

\[\square\]

### 3.2 Global solution

Based on the preceding result concerning the local solution of mean-field BSDE (3.1), in this subsection, we are going to study the global adapted solution of mean-field BSDE (3.1). Consider the following assumption, where the generator \( g \) is bounded with respect to \( \mu_2 \), slightly stronger than Assumption 1.

**Assumption 2.** The terminal value \( \eta : \Omega \to \mathbb{R} \) and the generator \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) satisfy the following conditions: there exists a positive constant \( K \) such that

(i) \( d\mathbb{P} \times dt \)-a.e. \( (\omega, t) \in \Omega \times [0, T] \), for every \( y \in \mathbb{R} \), \( z \in \mathbb{R}^d \) and \( \mu_1 \in \mathcal{P}_2(\mathbb{R}) \), \( \mu_2 \in \mathcal{P}_2(\mathbb{R}^d) \),
\[
|g(w, t, y, z, \mu_1, \mu_2)| \leq \theta_t(\omega) + K|y| + \frac{\gamma}{2}|z|^2 + KW_2(\mu_1, \delta_{[0]}).
\]

(ii) \( d\mathbb{P} \times dt \)-a.e. \( (\omega, t) \in \Omega \times [0, T] \), for every \( y, \bar{y} \in \mathbb{R} \), \( z, \bar{z} \in \mathbb{R}^d \) and \( \mu_1, \bar{\mu}_1 \in \mathcal{P}_2(\mathbb{R}) \), \( \mu_2, \bar{\mu}_2 \in \mathcal{P}_2(\mathbb{R}^d) \),
\[
|g(w, t, y, z, \mu_1, \mu_2) - g(w, t, \bar{y}, \bar{z}, \bar{\mu}_1, \bar{\mu}_2)| \leq K\left[ |y - \bar{y}| + \mathcal{W}_2(\mu_1, \bar{\mu}_1) \right] + \phi(|y| \vee |\bar{y}|) \vee \mathcal{W}_2(\mu_1, \delta_{[0]}) \vee \mathcal{W}_2(\bar{\mu}_1, \delta_{[0]}) \cdot (1 + |z| + |\bar{z}|)|z - \bar{z}| + \mathcal{W}_2(\mu_2, \bar{\mu}_2),
\]
\[
|g(w, t, y, z, \mu_1, \mu_2) - g(w, t, y, z, \delta_{[0]}, \delta_{[0]})| \leq K\left[ 1 + \mathcal{W}_2(\mu_1, \delta_{[0]}) \right].
\]
(iii) For two positive constants $K_1$ and $K_3$,
\[ \|\eta\|_\infty \leq K_1 \quad \text{and} \quad \left\| \int_0^T |\theta_t(\omega)|^2 \, dt \right\|_\infty \leq K_3. \]

Remark 3.4. (i) Using Hölder’s inequality, we have
\[ \left\| \int_0^T |\theta_t(\omega)| \, dt \right\|_\infty \leq \sqrt{T K_3} = K_2. \] (3.22)

(ii) Under Assumption 2, the generator $g$ is bounded and Lipschitz continuous with respect to the variable $\mu_2$ such that, for any $(\omega, t, y, z, \mu_1, \mu_2) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d),$
\[ |g(\omega, t, y, z, \mu_1, \mu_2) - g(\omega, t, 0, z, \delta_{\{0\}}, \delta_{\{0\}})| \leq K \left[ |y| + \mathcal{W}_{2}(\mu_1, \delta_{\{0\}}) + 1 \right]. \] (3.23)

On the other hand, Assumption 2 is weaker than the conditions (A2) and (A3) of Hibon, Hu, and Tang [23]. For example, the following generator $g$:
\[ g(\omega, t, y, z, \mu_1, \mu_2) = |y| + |z| + |z|^2 + \mathcal{W}_{2}(\mu_1, \delta_{\{0\}}) \sin(\mathcal{W}_{2}(\mu_2, \delta_{\{0\}})) \]
for every $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, \mu_1 \in \mathcal{P}_2(\mathbb{R}), \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfies Assumption 2, but it does not satisfy the conditions of [23].

In this subsection, the main result is the following theorem on the global adapted solution of mean-field BSDE (3.1).

**Theorem 3.5.** Under Assumption 2, on the whole interval $[0, T]$, BSDE (3.1) possesses a unique global solution $(Y, Z) \in S^\infty_F(0, T; \mathbb{R}) \times Z^2_F(0, T; \mathbb{R}^d)$. Moreover, there exist two positive constants $M_1$ and $M_2$ depending only on $(K, K_1, K_3, T)$ such that
\[ \|Y\|_{S^\infty_F(0, T)} \leq M_1 \quad \text{and} \quad \|Z\|_{Z^2_F(0, T)}^2 \leq M_2. \] (3.24)

**Proof.** We first consider the solvability of BSDE (3.1) on some interval $[T - \kappa \lambda, T]$ for a positive constant $\kappa \lambda$ that will be determined later. For this purpose, we define
\[ \overline{C} = K_1^2 + K_3 + 3K^2 + 2. \] (3.25)
Moreover, we let the function $\Gamma$ be the unique solution of the following ordinary differential equation
\[ \Gamma(t) = \overline{C} + \int_t^T \overline{C} \, ds + \int_t^T 3 \overline{C} \Gamma(s) \, ds, \quad s \in [0, T], \]
and define
\[ \lambda = \sup_{t \in [0, T]} \Gamma(t) = \Gamma(0). \]

Then, we see that the function $\Gamma$ is continuous and decreasing in $t$. 
Now, due to that $\|\eta\|_\infty \leq C = \Gamma(T)$, Theorem 3.3 implies that there exists a positive constant $\kappa_\lambda$, depending only on $\lambda$, such that BSDE (3.1) possesses a unique local solution $(Y, Z)$ on the interval $[T - \kappa_\lambda, T]$. Next, we would like to use the Picard iteration to construct the global solution.

Let us consider the following Picard iteration: for $j = 0$,

$$Y^0_t = \eta + \int_t^T Z^0_s dW_s, \quad t \in [T - \kappa_\lambda, T],$$

which is a classical BSDE with the solution $(\eta, 0)$. On the other hand, for $j \geq 0$,

$$Y^{j+1}_t = \eta + \int_t^T \left[ g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) + \int_t^T \left[ g(s, Y^{j}_s, Z^{j}_s, \mathbb{P}_{Y^{j}_s}, \mathbb{P}_{Z^{j}_s}) - g(s, 0, Z^{j}_s, \delta_{\{0\}}, \delta_{\{0\}}) \right] ds 
+ \int_t^T \left[ g(s, 0, Z^{j+1}_s, \delta_{\{0\}}, \delta_{\{0\}}) - g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) \right] ds - \int_t^T Z^{j+1}_s dW_s. \quad (3.26)$$

By the item (ii) of Assumption 2, there exists a process $\Lambda^{j+1}$ such that

$$\begin{cases} 
|\Lambda^{j+1}(s)| \leq \phi(0)(1 + |Z^{j+1}_s|); \\
g(s, 0, Z^{j+1}_s, \delta_{\{0\}}, \delta_{\{0\}}) - g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) = Z^{j+1}_s \Lambda^{j+1}_s. 
\end{cases} \quad (3.27)$$

Then, the process

$$\tilde{W}^{j+1}_t \triangleq W_t - \int_0^t \Lambda^{j+1}_s ds, \quad t \in [0, T] \quad (3.28)$$

is a Brownian motion with respect to the equivalent probability measure $\mathbb{P}^{j+1}$ defined by

$$d\mathbb{P}^{j+1} \triangleq e^{\varepsilon(\Lambda^{j+1} \cdot W)} d\mathbb{P},$$

which is denoted by $\tilde{\mathbb{P}}$ hereafter for simplicity, and whose expectation is denoted by $\tilde{E}$. Combining (3.26)-(3.28), we have that for any $t \in [T - \kappa_\lambda, T]$,

$$Y^{j+1}_t = \eta + \int_t^T \left[ g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) + \int_t^T \left[ g(s, Y^{j}_s, Z^{j}_s, \mathbb{P}_{Y^{j}_s}, \mathbb{P}_{Z^{j}_s}) - g(s, 0, Z^{j}_s, \delta_{\{0\}}, \delta_{\{0\}}) \right] ds 
- \int_t^T Z^{j+1}_s d\tilde{W}^{j+1}_s. \quad (3.29)$$

Next we show by induction the following inequality: for $j \geq 0$,

$$|Y^{j}_t|^2 \leq \Gamma(t), \quad t \in [T - \kappa_\lambda, T]. \quad (3.29)$$

Actually, it is easy to see that firstly $|Y^{0}_t|^2 \leq \Gamma(t)$, and then we would like to suppose that (3.29) holds for $Y^{j}$. So we just need to prove that (3.29) holds too for $Y^{j+1}$. Applying Itô’s formula to
$|Y^{j+1}|^2$ and using Assumption 2, in view of (3.23), we have that for any $r \in [T - \kappa_\lambda, t]$,

$$
\tilde{E}_r |Y^{j+1}_t|^2 + \tilde{E}_r \int_t^T |Z^{j+1}_s|^2 ds = \tilde{E}_r |\eta|^2 + \tilde{E}_r \int_t^T 2Y^{j+1}_s \left[ g(s,0,0,\delta_{(0)}, \delta_{(0)}) + g(s,Y^0_s, Z^0_s, \mathbb{P}_{Y^0_s}, \mathbb{P}_{Z^0_s}) - g(s,0, Z^0_s, \delta_{(0)}, \delta_{(0)}) \right] ds
$$

$$
\leq \tilde{E}_r |\eta|^2 + \tilde{E}_r \int_t^T \left[ 2|Y^{j+1}_s|^2 + |\theta_s|^2 + 3K^2 \left( |Y^j_s|^2 + \|Y^j_s\|_{L^2(\Omega)}^2 + 1 \right) \right] ds
$$

$$
\leq K^2 + K_3 + \int_t^T 3K^2 ds + \tilde{E}_r \int_t^T 3K^2 \left[ |Y^j_s|^2 + \|Y^j_s\|_{L^2(\Omega)}^2 \right] ds + \tilde{E}_r \int_t^T 2|Y^{j+1}_s|^2 ds.
$$

From the definition of $\mathcal{C}$ (see (3.25)), we have

$$
\tilde{E}_r |Y^{j+1}_t|^2 \leq \mathcal{C} + \int_t^T \mathcal{C} ds + \mathcal{C} \int_t^T \tilde{E}_r |Y^j_s|^2 + \mathbb{E} |Y^j_s|^2 ds + \mathcal{C} \int_t^T \tilde{E}_r |Y^{j+1}_s|^2 ds,
$$

which together with the inequality (3.29) for $Y^j$ implies that

$$
\tilde{E}_r |Y^{j+1}_t|^2 \leq \mathcal{C} + \mathcal{C} \int_t^T [1 + 2\Gamma(s)] ds + \mathcal{C} \int_t^T \tilde{E}_r |Y^{j+1}_s|^2 ds.
$$

Recall that

$$
\Gamma(t) = \mathcal{C} + \mathcal{C} \int_t^T (1 + 2\Gamma(s)) ds + \int_t^T \mathcal{C} \Gamma(s) ds, \quad 0 \leq t \leq T.
$$

From the comparison theorem, we have

$$
\tilde{E}_r |Y^{j+1}_t|^2 \leq \Gamma(t), \quad t \in [T - \kappa_\lambda, T].
$$

In particular, we have

$$
|Y^{j+1}_t|^2 \leq \Gamma(t), \quad t \in [T - \kappa_\lambda, T].
$$

Therefore, the inequality (3.29) holds for $Y^{j+1}$ too. Now, due to that $Y_t = \lim_{j \to \infty} Y^{j+1}_t$, so the constructed local solution $(Y,Z)$ in $[T - \kappa_\lambda, T]$ satisfies the following estimate,

$$
|Y_t|^2 \leq \Gamma(t), \quad t \in [T - \kappa_\lambda, T].
$$

In particular, $|Y_{T-\kappa_\lambda}|^2 \leq \Gamma(T - \kappa_\lambda)$.

Taking $T - \kappa_\lambda$ as the terminal time and $Y_{T-\kappa_\lambda}$ as the terminal value and applying Theorem 3.3, we obtain that the mean-field BSDE (3.1) admits a local solution $(Y,Z)$ on $[T - 2\kappa_\lambda, T - \kappa_\lambda]$ through the Picard iteration. Moreover, again using the above discussion, we have

$$
|Y_t|^2 \leq \Gamma(t), \quad t \in [T - 2\kappa_\lambda, T - \kappa_\lambda].
$$

Repeating the preceding process, we extend the pair $(Y,Z)$ to the whole interval $[0, T]$ within a finite of steps such that $Y$ is uniformly bounded by $\lambda = \Gamma(0)$. In other words, there exists a positive constants $M_1$ depending only on $(K, K_1, K_3, T)$ such that

$$
\|Y\|_{S^\infty_{0,T}} \leq M_1,
$$

(3.30)
Next, we show that $Z \cdot W$ is a $BMO(\mathbb{P})$-martingale. For this, we define that for $x \in \mathbb{R}$,
\[
\Phi(x) = \frac{1}{\gamma} \left[ \exp(\gamma |x|) - \gamma |x| - 1 \right],
\]
where $\gamma$ is a positive constant that comes from (3.21). Then it is easy to compute that for $x \in \mathbb{R}$,
\[
\Phi'(x) = \frac{1}{\gamma} \left[ \exp(\gamma |x|) - 1 \right] \text{sgn}(x), \quad \Phi''(x) = \exp(\gamma |x|), \quad \Phi''(x) - \gamma |\Phi'(x)| = 1.
\]
Using Itô’s formula and Assumption 2, we have
\[
\Phi(Y_t) = \Phi(Y_T) + \int_t^T \Phi'(Y_s) g(s, Y_s, Z_s, \mathbb{P}_{Y_s}, \mathbb{P}_{Z_s}) ds - \int_t^T \Phi'(Y_s) Z_s dW_s - \frac{1}{2} \int_t^T \Phi''(Y_s) |Z_s|^2 ds
\]
\[
\leq \Phi(Y_T) + \int_t^T |\Phi'(Y_s)| \{ \theta_s + K [ |Y_s| + \| Y_s \|_{L^2(\Omega)} ] \} ds - \int_t^T \Phi'(Y_s) Z_s dW_s
\]
\[
+ \frac{1}{2} \int_t^T [ |\Phi'(Y_s)| - \Phi''(Y_s) ] |Z_s|^2 ds.
\]
Then, from (3.32), we have
\[
\Phi(Y_t) + \frac{1}{2} \mathbb{E}_t \int_t^T |Z_s|^2 ds \leq \Phi(\| \eta \|_\infty) + \mathbb{E}_t \int_t^T |\Phi'(Y_s)| \{ \theta_s + K [ |Y_s| + \| Y_s \|_{L^2(\Omega)} ] \} ds
\]
\[
\leq \Phi(K_1) + |\Phi'(M_1)| \mathbb{E}_t \int_t^T (\theta_s + 2K M_1) ds
\]
\[
\leq \Phi(K_1) + |\Phi'(M_1)| [\sqrt{K_3 T + 2K M_1 T}].
\]
In other words,
\[
\mathbb{E}_t \int_t^T |Z_s|^2 ds \leq 2\Phi(K_1) + 2|\Phi'(M_1)| [\sqrt{K_3 T + 2K M_1 T}].
\]
Therefore, we have
\[
\| Z \|^2_{\mathbb{Z}_2^2(0, T)} = \| Z \cdot W \|^2_{BMO(\mathbb{P})} \leq 2\Phi(K_1) + 2|\Phi'(M_1)| [\sqrt{K_3 T + 2K M_1 T}] \triangleq M_2,
\]
which together with (3.30) implies that (3.24) holds.

Finally, we prove the uniqueness. Let $(Y, Z)$ and $(\bar{Y}, \bar{Z})$ be two adapted solutions of BSDE (3.1). We denote
\[
\Delta Y = Y - \bar{Y}, \quad \Delta Z = Z - \bar{Z}.
\]
Then for arbitrary $\varepsilon > 0$ and for $t \in [T - \varepsilon, T]$, we have
\[
\Delta Y_t + \int_t^T \Delta Z_s d\bar{W}_s = \int_t^T g(s, Y_s, \bar{Z}_s, \mathbb{P}_{Y_s}, \mathbb{P}_{Z_s}) - g(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_{Y_s}, \mathbb{P}_{Z_s}) ds,
\]
where $\bar{W}$ is given in (3.16). Similar to (3.17)-(3.20), we have that for $t \in [T - \varepsilon, T]$,
\[
\| \Delta Y \|^2_{\mathbb{S}_2^\infty(\varepsilon, t, T)} + c_1^2 \| \Delta Z \|^2_{\mathbb{Z}_2^\infty(\varepsilon, t, T)}
\]
\[
\leq 32\varepsilon \phi(M_1)^2 \left( T + 4\varepsilon_2 M_2 + 2M_2 \right) \| \Delta Y \|^2_{\mathbb{S}_2^\infty(\varepsilon, t, T)} + 6\varepsilon^{1-\alpha} \phi(M_1)^2 \| \Delta Z \|^2_{\mathbb{Z}_2^\infty(0, T)} (T^\alpha + 2M_2),
\]
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where \(c_1\) and \(c_2\) are given in Proposition 2.3. Clearly, if \(\varepsilon\) is small enough, we can obtain \(Y = Y\) and \(Z = Z\) on the interval \([T - \varepsilon, T]\). Repeating this argument within a finite steps, the uniqueness can be obtained. 

**Remark 3.6.** Hibon, Hu, and Tang [23] obtained the existence and uniqueness result on mean-field BSDEs with quadratic growth when the generator \(g\) depends on the expectation of \((Y, Z)\). In comparison with it, our condition is weaker, our results are more general, and our method for the existence and uniqueness is more powerful.

## 4 Comparison Theorem

In this section, we give a comparison theorem for the solutions of the mean-field BSDEs (1.1) with quadratic growth. For simplicity, we rewrite it as follows:

\[
Y_t = \eta + \int_t^T g(s, Y_s, Z_s, P_{Y_s}, P_{Z_s}) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{4.1}
\]

BSDE (4.1) will be referred to BSDE (4.1) with parameters \((\eta, g)\) to indicate the generator \(g\) and the terminal value \(\eta\).

We have

**Theorem 4.1** (Comparison principle). Let the parameters \((g, \eta)\) and \((\bar{g}, \bar{\eta})\) satisfy Assumption 2. Denote by \((Y, Z)\) and \((\bar{Y}, \bar{Z})\) the global adapted solutions of mean-field BSDE (4.1) with parameters \((\eta, g)\) and \((\bar{\eta}, \bar{g})\), respectively. In addition, suppose that

(i) One of both generators \(g\) and \(\bar{g}\) does not depend on \(\mu_2\).

(ii) The other one of both generators \(g\) and \(\bar{g}\) is nondecreasing with respect to \(\mu_1\) in the following sense: there exists a positive constant \(K\) such that for every \((\omega, t) \in \Omega \times [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \xi, \xi' \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R})\) and \(\zeta \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^d),\)

\[
|g(\omega, t, y, z, \xi, \zeta) - g(\omega, t, y, z, \bar{\xi}, \bar{\zeta})| \leq K\|\xi - \bar{\xi}\|^+\|L^2(\Omega)\).
\]

Then, if \(\eta \leq \bar{\eta}\) and \(g \leq \bar{g}, \mathbb{P}\)-a.s., we have that for every \(t \in [0, T],\)

\[
Y_t \leq \bar{Y}_t, \quad \mathbb{P}\text{-a.s.}
\]

**Remark 4.2.** If the mean-field term in the generator appears as an expectation, Buckdahn, Li, and Peng [10, Example 3.2 and Theorem 3.2] impose the following condition

Condition (L) the generator \(g\) (or \(\bar{g}\)) is nondecreasing in the mean-field term \(y'\)

for the comparison property of mean-field BSDEs. One variant states that the derivative of \(g\) (or \(\bar{g}\)) with respect to the mean-field term \(y'\) (if exists) is not less than zero. The assumption (ii) in Theorem 4.1 takes the role of Condition (L) in the general context of measure-dependence.
Remark 4.3. Here are sufficient conditions for (ii) in Theorem 4.1. Assume that the function $g(\omega, t, y, z, \cdot, \mu_2)$ is differentiable with respect to $\mu_1$ for every $(\omega, t, y, z, \mu_2) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. If there is a positive constant $K$ such that for every $(\omega, t, y, z, \mu_1, \mu_2, a) \in \Omega \times [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}$,

$$0 \leq \partial_{\mu_1} g(s, y, z, \mu_1, \mu_2; a) \leq K,$$

then the assumption (ii) in Theorem 4.1 holds true. For more details, see Li, Liang, and Zhang [31, Remark 2.2].

Proof of Theorem 4.1. Without loss of generality, we assume that $g$ satisfies the item (i) and $\bar{g}$ satisfies the item (ii). Set for every $s \in [t, T]$,

$$\delta Y_s = Y_s - \bar{Y}_s, \quad \delta Z_s = Z_s - \bar{Z}_s, \quad \delta \eta = \eta - \bar{\eta},$$

$$\delta g(s) = g(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s) - \bar{g}(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s).$$

Then,

$$\delta Y_t = \delta \eta + \int_t^T \left[ g(s, Y_s, Z_s, \mathbb{P}_s) - \bar{g}(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s) \right] ds - \int_t^T \delta Z_s dW_s, \quad 0 \leq t \leq T. \quad (4.2)$$

On the generator of BSDE (4.2), we have that for $s \in [t, T]$,

$$g(s, Y_s, Z_s, \mathbb{P}_s) - \bar{g}(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s)$$

$$= \delta_y g(s) \delta Y_s + \delta_Z g(s) \delta Z_s + \delta_{\mu} \bar{g}(s) \|(\delta Y_s)^+\|_{L^2(\Omega)} + \delta g(s),$$

where

$$\delta_y g(s) = \begin{cases} \frac{g(s, Y_s, Z_s, \mathbb{P}_s) - g(s, \bar{Y}_s, Z_s, \mathbb{P}_s)}{Y_s - \bar{Y}_s}, & \text{if } Y_s \neq \bar{Y}_s; \\ 0, & \text{if } Y_s = \bar{Y}_s, \end{cases}$$

and

$$\delta_Z g(s) = \begin{cases} \frac{(g(s, \bar{Y}_s, Z_s, \mathbb{P}_s) - g(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s))(Z_s - \bar{Z}_s)}{|Z_s - \bar{Z}_s|^2}, & \text{if } Z_s \neq \bar{Z}_s; \\ 0, & \text{if } Z_s = \bar{Z}_s, \end{cases}$$

and

$$\delta_{\mu} \bar{g}(s) = \begin{cases} \frac{\bar{g}(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s) - \bar{g}(s, \bar{Y}_s, \bar{Z}_s, \mathbb{P}_s, \mathbb{P}_s)}{\|(\delta Y_s)^+\|_{L^2(\Omega)}}, & \text{if } \|(\delta Y_s)^+\|_{L^2(\Omega)} \neq 0; \\ 0, & \text{if } \|(\delta Y_s)^+\|_{L^2(\Omega)} = 0. \end{cases}$$

From Assumption 2, the item (ii) of Theorem 4.1 and Theorem 3.5, we have

$$|\delta_y g(s)| \leq K, \quad |\delta_{\mu} \bar{g}(s)| \leq K, \quad 0 \leq t \leq s \leq T, \quad (4.3)$$

$$|\delta_Z g(s)| \leq \phi(M_1)(1 + |Z_s| + |\bar{Z}_s|), \quad 0 \leq t \leq s \leq T,$$
where $M_1$ is given in (3.24). Moreover, since both $Z$ and $\bar{Z}$ belong to the space $\mathcal{Z}^2[0, T]$, we have $\delta_z g \in \mathcal{Z}^2[0, T]$. Define

$$\tilde{W}_t = W_t - \int_0^t \delta_z g(r) dr, \quad t \in [0, T] \quad \text{and} \quad d\tilde{Q} = \delta(\delta_z g \cdot W)^T_0 d\mathbb{P}.$$  

Then, $Q$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{W}$ is a Brownian motion under $Q$. So BSDE (4.2) can be rewritten as

$$\begin{cases} 
\delta Y_t = -\left[ (\delta_y g(t) \delta Y_t + \delta_\mu \bar{g}(t) \| \delta Y_t \|^2_{L^2(\Omega)} + \delta g(t) \right] dt + \delta Z_t d\tilde{W}_t, \quad 0 \leq t < T; \\
\delta Y_T = \delta \eta. 
\end{cases}$$

Applying Itô’s formula to $e^{\int_t^T \delta_\mu \bar{g}(s) ds} \delta Y_s$, we have

$$\delta Y_t = \mathbb{E}_t^Q \left[ e^{\int_t^T \delta_\mu \bar{g}(s) ds} \delta \eta \right] + \mathbb{E}_t^Q \left[ \int_t^T e^{\int_s^T \delta_\mu \bar{g}(r) dr} \left( \delta_\mu \bar{g}(s) \| \delta Y_s \|^2_{L^2(\Omega)} + \delta g(s) \right) ds \right].$$

Now, since $\delta \eta \leq 0$ and $\delta g(s) \leq 0$, we have

$$\delta Y_t \leq \mathbb{E}_t^Q \left[ \int_t^T e^{\int_s^T \delta_\mu \bar{g}(r) dr} \delta_\mu \bar{g}(s) \| \delta Y_s \|^2_{L^2(\Omega)} ds \right].$$

According to the fact (4.3), using Hölder’s inequality, we have

$$\delta Y_t \leq e^{KT} K \sqrt{T} \left\{ \int_t^T \mathbb{E}^P \left[ \| \delta Y_s \|^2 \right] ds \right\}^{\frac{1}{2}}.$$  

Hence,

$$\left( \| \delta Y_t \|^2 \right) \leq e^{2KT} K^2 T \int_t^T \mathbb{E}^P \left[ \| \delta Y_s \|^2 \right] ds.$$

Taking the expectation $\mathbb{E}^P$ on both sides of the last inequality, we have the desired result from Gronwall’s lemma.

**Remark 4.4.** Theorem 4.1 generalizes the related one of Hao, Wen, and Xiong [22] where the generator depends on the expectations of $(Y, Z)$, and also generalizes the comparison theorem of Buckdahn, Li and Peng [10] where the generator $g$ grows linearly with respect to $Z$.

## 5 Particle systems

In this section, we study the particle systems for the mean-field BSDE (1.1) with quadratic growth. The convergence of the particle systems and the rate of convergence will be given.

Let $\{\eta^i; 1 \leq i \leq N\}$ be $N$ independent copies of $\eta$, and $\{W^j; 1 \leq j \leq N\}$ be $N$ independent $d$-dimensional Brownian motions. Denote by $(Y^i, Z^{i;j})$ and $(\bar{Y}^i, \bar{Z}^i)$ the adapted solutions to the following BSDEs

$$Y^i_t = \eta^i + \int_t^T g(s, Y^i_s, Z^{i,j}_s, \nu^i_s, \mu^i_s) ds - \int_t^T \sum_{j=1}^N Z^{i;j}_s dW^j_s, \quad t \in [0, T]$$

(5.1)
and
\[ Y^i_t = \eta^i + \int_t^T g(s, Y^i_s, \bar{Z}^i_s, \bar{\nu}_s, \bar{\mu}_s)ds - \int_t^T \bar{Z}^i_s dW^i_s, \quad t \in [0, T], \] (5.2)
respectively, where for each \( i, j = 1, ..., N \), \( Z^{i,j} \) is a \( 1 \times d \) matrix, and
\[ \nu^N_s = \frac{1}{N} \sum_{i=1}^N \delta_{Y^i_t}, \quad \mu^N_s = \frac{1}{N} \sum_{i=1}^N \delta_{Z^{i,j}_s}, \quad \bar{\nu}_s \triangleq \mathbb{P}_{Y^i_s}, \quad \bar{\mu}_s \triangleq \mathbb{P}_{Z^j_s}. \]

We shall show that the pair \( (Y^i, Z^{i,j}) \) is close to the pair \( (\bar{Y}^i, \bar{Z}^i) \). For simplicity, we set for \( i, j = 1, ..., N \),
\[ \Delta Y^i = Y^i - \bar{Y}^i, \quad \Delta Z^{i,j} = Z^{i,j} - \bar{Z}^{i,j} \quad \text{with} \quad \bar{Z}^{i,j} = \begin{cases} \bar{Z}^i, & i = j; \\ 0, & i \neq j. \end{cases} \] (5.3)

For the particle systems, we consider the following slightly stronger assumption than Assumption 2.

**Assumption 3.** For \( i = 1, 2, \cdots, N \), there exists a positive constant \( K \) such that the terminal value \( \eta^i : \Omega \to \mathbb{R} \) and the generator \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \), adapted the filtration of \( \mathbb{F}^i \), where \( \mathbb{F}^i = \{ \mathcal{F}^i_t \}_{t \geq 0} \) is the natural filtration of \( W^i \) augmented by all the \( \mathbb{P} \)-null sets in \( \mathcal{F}^i \), satisfy the following conditions:

(i) \( d\mathbb{P} \times dt \)-a.e. \( (\omega, t) \in \Omega \times [0, T] \), for every \( (y, z, \nu, \mu) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R}^d) \),
\[ |g(w, t, y, z, \nu, \mu)| \leq \theta_t(\omega) + K|y| + \frac{\gamma}{2}|z|^2 + KW_2(\nu, \delta_{\{0\}}). \]

(ii) \( d\mathbb{P} \times dt \)-a.e. \( (\omega, t) \in \Omega \times [0, T] \), for every \( y, \bar{y} \in \mathbb{R}, z, \bar{z} \in \mathbb{R}^d \) and \( \nu, \mu \in \mathcal{P}_2(\mathbb{R}), \bar{\nu}, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d) \),
\[ |g(\omega, t, y, z, \nu, \mu) - g(\omega, t, \bar{y}, \bar{z}, \bar{\nu}, \bar{\mu})| \leq K|y - \bar{y}| + W_2(\nu, \bar{\nu}) + W_2(\mu, \bar{\mu}) \]
\[ + \phi(|y| \vee |\bar{y}|) \vee W_2(\nu, \delta_{\{0\}}) \vee W_2(\bar{\nu}, \delta_{\{0\}}) \cdot (1 + |z| + |\bar{z}|)|z - \bar{z}|, \]
\[ |g(\omega, t, y, z, \nu, \mu) - g(\omega, t, y, z, \delta_{\{0\}}, \delta_{\{0\}})| \leq K[1 + W_2(\nu, \delta_{\{0\}})]. \]

(iii) There are two positive constants \( K_1 \) and \( K_3 \) such that
\[ \max_{1 \leq i \leq N} \|\eta^i\|_\infty \leq K_1 \quad \text{and} \quad \int_0^T |\theta_t(\omega)|^2 dt \|_\infty \leq K_3. \]

Similar to (3.24), the solution \( (Y^i, Z^{i,j}) \) of BSDE (5.1) has the following property.

**Proposition 5.1.** Under Assumption 3, there is a positive constant \( C \) depending only on \( (K, K_1, K_3, T) \), such that the solution \( (Y^i, Z^{i,j}) \) of BSDE (5.1) admits the following estimate: for each \( i, j = 1, ..., N \),
\[ \|Y^i\|_{S^\infty_p(0, T)} \leq C, \quad \|Z^{i,j}\|_{Z^2_p(0, T)} \leq C. \] (5.4)
Proof. We can rewrite BSDE (5.1) as follows:

\[ Y^i_t = \eta^i + \int_t^T \left[ g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) + g(s, Y^i_s, Z^{i,i}_s, \nu^N_s, \mu^N_s) - g(s, 0, Z^{i,i}_s, \delta_{\{0\}}, \delta_{\{0\}}) \right] ds - \int_t^T \sum_{j=1}^N Z^{i,j}_s dW^j_s \]

where \( \widetilde{W}^j_t = W^j_t - \int_0^t \Gamma(s) ds, \) is a Brownian motion under \( Q^j \) with

\[ \frac{dQ^j}{d\mathbb{P}} = \delta(T \cdot W^j)^T_0 \quad \text{and} \quad \Gamma(s) \leq \phi(0)(1 + |Z^{i,i}_s|), \]

\[ g(s, 0, Z^{i,i}_s, \delta_{\{0\}}, \delta_{\{0\}}) - g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) = \Gamma(s)Z^{i,i}_s. \]

Now, applying Itô’s formula to \(|Y^i_t|^2\), we have

\[ |Y^i_t|^2 + \mathbb{E}_t^Q \left[ \int_t^T \sum_{j=1}^N |Z^{i,j}_s|^2 ds \right] = \mathbb{E}_t^Q[|\eta^i|^2] \]

\[ + \mathbb{E}_t^Q \left[ \int_t^T 2Y^i_s \left( g(s, 0, 0, \delta_{\{0\}}, \delta_{\{0\}}) + g(s, Y^i_s, Z^{i,i}_s, \nu^N_s, \mu^N_s) - g(s, 0, Z^{i,i}_s, \delta_{\{0\}}, \delta_{\{0\}}) \right) ds \right]. \tag{5.5} \]

Thanks to Assumption 3 and (3.23), we have

\[ |Y^i_t|^2 \leq \mathbb{E}_t^Q[|\eta^i|^2] + \mathbb{E}_t^Q \left[ \int_t^T 2Y^i_s \left( \theta_s + K \left[ |Y^i_s| + W^N_2(\nu^N_s, \delta_{\{0\}}) + 1 \right] \right) ds \right] \]

\[ \leq \mathbb{E}_t^Q[|\eta^i|^2] + \mathbb{E}_t^Q \left[ \int_t^T \left( 2|Y^i_s|^2 + |\theta_s|^2 + 3K^2 \left[ |Y^i_s|^2 + W^N_2(\nu^N_s, \delta_{\{0\}}) + 1 \right] \right) ds \right] \]

\[ \leq K_1^2 + K_3 + 3K^2 T + \mathbb{E}_t^Q \left[ \int_t^T \left( 3K^2 + 2 \right) |Y^i_s|^2 + 3K^2 \frac{1}{N} \sum_{i=1}^N |Y^i_s|^2 \right] \]. \tag{5.6} \]

Summing over \( i \) on both sides of (5.6), using Gronwall’s inequality, we have

\[ ||Y^i||_{\mathbb{F}^Q_0(0,T)} \leq C, \quad i = 1, \ldots, N, \]

where \( C \) is a positive constant depending only on \((K, K_1, K_3, T)\).

Now we prove that \( Z^{i,j} \cdot W^j \) is a BMO martingale. For this, we let

\[ \Phi(x) = \frac{1}{2} \left[ \exp(\gamma|x|) - \gamma|x| - 1 \right]. \]
Again, for BSDE (5.1), applying Itô’s formula to $\Phi(Y^i_t)$, we have

$$
\Phi(Y^i_t) = \Phi(Y^i_T) + \int_t^T \Phi'(Y^i_s)g(s,Y^i_s,Z^i_s,\nu^N_s,\mu^N_s)\,ds
- \int_t^T \Phi'(Y^i_s)\sum_{j=1}^N Z^ij_s\,dW^j_s - \frac{1}{2} \int_t^T \Phi''(Y^i_s)\sum_{j=1}^N |Z^ij_s|^2\,ds
\leq \Phi(\eta^i) + \int_t^T |\Phi'(Y^i_s)|\left(\theta_s + K[|Y^i_s| + \mathcal{W}_2(\nu^N_s,\delta(0))]\right)\,ds
- \int_t^T \Phi'(Y^i_s)\sum_{j=1}^N Z^ij_s\,dW^j_s
+ \frac{1}{2} \int_t^T \left(\gamma|\Phi'(Y^i_s)| - \Phi''(Y^i_s)\right)\sum_{j=1}^N |Z^ij_s|^2\,ds.
$$

(5.7)

Taking the conditional expectation $\mathbb{E}_t$ on both sides and noting (3.32), we have

$$
\Phi(Y^i_t) + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T \sum_{j=1}^N |Z^ij_s|^2\,ds \right]
\leq \Phi(\eta^i) + \mathbb{E}_t \left[ \int_t^T |\Phi'(Y^i_s)|\left(\theta_s + K[|Y^i_s| + \mathcal{W}_2(\nu^N_s,\delta(0))]\right)\,ds \right]
\leq \Phi(K_1) + |\Phi'(C)|\mathbb{E}_t \left[ \int_t^T \left(\theta_s + K|Y^i_s| + K\left\{\frac{1}{N}\sum_{i=1}^N |Y^i_s|^2\right\}^\frac{1}{2}\right)\,ds \right]
\leq \Phi(K_1) + |\Phi'(C)|\left(\sqrt{K_3T + 2KCT}\right).
$$

Hence,

$$
\|Z^ij\|_{L^2(0,T)}^2 = \|Z^ij \cdot W^j\|_{\text{BMO}(\mathbb{P})}^2 \leq 2\Phi(K_1) + 2|\Phi'(C)|\left(\sqrt{K_3T + 2KCT}\right),
$$

which implies that $Z^ij \cdot W^j$ is a BMO martingale and thus (5.4) holds. \hfill \Box

The particle systems (5.1)-(5.2) has the following convergence.

**Theorem 5.2.** Under Assumption 3, for any $p \geq 2$, there exist two constants $q_0,q'_0 > 1$ and a constant $C$, depending only on $(K,K_1,K_3,T,p,q_0,q'_0)$, such that

(i) \( \mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^N \left\{ \sup_{t\in[0,T]} |\Delta Y^i_t|^p + \left( \int_0^T \sum_{j=1}^N |\Delta Z^i_t|^2\,dt \right)^\frac{p}{2} \right\} \right] \)

\[ \leq C \mathbb{E}\left[ \int_0^T \mathcal{W}_2^{p,q_0}(\nu^{N}_t,\bar{\nu}_t)\,dt + \int_0^T \mathcal{W}_2^{p,q'_0}(\mu^{N}_t,\bar{\mu}_t)\,dt \right] \frac{1}{q_0}, \]

(5.8)

(ii) \( \mathbb{E}\left\{ \sup_{t\in[0,T]} |\Delta Y^i_t|^p + \left( \int_0^T \sum_{j=1}^N |\Delta Z^i_t|^2\,dt \right)^\frac{p}{2} \right\} \)

\[ \leq C \mathbb{E}\left[ \int_0^T \mathcal{W}_2^{p,q_0}(\nu^{N}_t,\bar{\nu}_t)\,dt + \int_0^T \mathcal{W}_2^{p,q'_0}(\mu^{N}_t,\bar{\mu}_t)\,dt \right] \frac{1}{q'_0}. \]
Proof. From BSDE (5.1) and the notation (5.3), we see that the pair \((\Delta Y^i, \Delta Z^{i,j})\) satisfies the following equation

\[
\begin{cases}
-d\Delta Y^i_t = [g(t, Y^i_t, Z^{i,i}_t, \nu^N_t, \mu^N_t) - g(t, \bar{Y}^i_t, \bar{Z}^{i}_t, \bar{\nu}_t, \bar{\mu}_t)]dt - \sum_{j=1}^{N} \Delta Z^{i,j}_t dW^j_t, \quad t \in [0, T]; \\
\Delta Y^i_T = 0,
\end{cases}
\]  

(5.9)

where

\[
g(t, Y^i_t, Z^{i,i}_t, \nu^N_t, \mu^N_t) - g(t, \bar{Y}^i_t, \bar{Z}^{i}_t, \bar{\nu}_t, \bar{\mu}_t) = J_{1,t} + J_{2,t},
\]

with

\[
J_{1,t} \triangleq g(t, Y^i_t, Z^{i,i}_t, \nu^N_t, \mu^N_t) - g(t, \bar{Y}^i_t, \bar{Z}^{i}_t, \bar{\nu}_t, \bar{\mu}_t),
\]

and

\[
J_{2,t} \triangleq g(t, \bar{Y}^i_t, \bar{Z}^{i}_t, \bar{\nu}_t, \bar{\mu}_t) - g(t, \bar{Y}^i_t, \bar{Z}^{i}_t, \bar{\nu}_t, \bar{\mu}_t).
\]

On the term \(J_{2,t}\), by Assumption 3, there is a process \(\Gamma^i\) such that

\[
J_{2,t} = \Gamma^i_t \Delta Z^{i,i}_t,
\]

where

\[
|\Gamma^i_t| \leq \phi(||\bar{Y}^i_t|| \vee ||\bar{Y}^i_t||_{L^2(\Omega)}(1 + |Z^{i,i}_t| + |\bar{Z}^{i,i}_t|)).
\]  

(5.10)

By (3.24) and Proposition 5.1, for each \(i\), we have \(||\Gamma^i\||_{Z^i[0,T]} \leq C\) for a constant \(C\) depending only on \((K, K_1, K_3, T)\). Now, thanks to Girsanov theorem, we have

\[
-d\Delta Y^i_t = J_{1,t}dt - \sum_{j=1}^{N} \Delta Z^{i,j}_t d\bar{W}^{j,i}_t, \quad t \in [0, T].
\]

Here \(\bar{W}^{j,i}\) depends on \(i\), which means one only changes the \(i\)-th component of \((W^1, \cdots, W^N)\), i.e., \(\bar{W}^{j,i}_t = W^j_t - \int_{0}^{t} \Gamma^i_r dr, \quad j \neq i\); or else, \(\bar{W}^{i,i}_t = W^i_t\). Then \((\bar{W}^{j,i})_{1 \leq j \leq N}\) is a \(N\)-dimensional \(\mathbb{Q}^i\)-Brownian motion with \(\frac{d\bar{W}^{j,i}}{d\mathbb{P}} = \mathcal{E}^i(\Gamma \cdot W^j)_0^T\), where \(\Gamma = (0, \cdots, 0, \Gamma^i, 0 \cdots, 0)\). From Briand et al. [5], it follows for fixed \(i\),

\[
\mathbb{E}^{\mathbb{Q}^i} \left[ \sup_{t \in [0,T]} |\Delta Y^i_t|^p + \left( \int_{0}^{T} \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p}{2}} \right] 
\]

\[
\leq C \mathbb{E}^{\mathbb{Q}^i} \left[ \int_{0}^{T} \mathcal{W}^p_2(\nu^N_t, \bar{\nu}_t)dt + \int_{0}^{T} \mathcal{W}^p_2(\mu^N_t, \bar{\mu}_t)dt \right].
\]  

(5.11)

Notice that \(||\Gamma \cdot W^j\||_{BMO} \text{ and } ||\Gamma \cdot W^j||_{BMO_q}, q > 2\) are equivalent, from Proposition 2.2 there exists a constant \(p_0 > 1\) such that

\[
\mathbb{E} \left[ (\mathcal{E}^i(\Gamma \cdot W^j)_0^T)^{p_0} \right] \leq C_{p_0}.
\]
For this $p_0$, by summing over $i$ on both sides of (5.11) and then applying Hölder inequality, we have for $p \geq 2$,

$$
\frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{E}^{Q_i} \left[ \sup_{t \in [0,T]} |\Delta Y^i_t|^p + \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p}{2}} \right] \right\} 
\leq \frac{C}{N} \sum_{i=1}^{N} \mathbb{E}^{Q_i} \left[ \int_0^T \mathcal{W}^p_2(\nu_i^N, \bar{\nu}_i) dt + \int_0^T \mathcal{W}^p_2(\mu_i^N, \bar{\mu}_i) dt \right] 
\leq \frac{C}{N} \sum_{i=1}^{N} \mathbb{E} \left[ (\mathcal{E}^i(\Gamma \cdot W^j)_0)^{p_0} \right] \frac{1}{q_0} \mathbb{E} \left[ \left( \int_0^T \mathcal{W}^p_2(\nu_i^N, \bar{\nu}_i) dt + \int_0^T \mathcal{W}^p_2(\mu_i^N, \bar{\mu}_i) dt \right)^{q_0} \right]^{\frac{1}{q_0}} 
\leq C \mathbb{E} \left[ \int_0^T \mathcal{W}^{p_0}_{2p}(\nu_i^N, \bar{\nu}_i) dt + \int_0^T \mathcal{W}^{p_0}_{2p}(\mu_i^N, \bar{\mu}_i) dt \right]^{\frac{1}{q_0}},
$$

where $q_0 = \frac{p_0}{p_0 - 1} > 1$.

On the other hand, it is easy to see that $\frac{dp}{dq} = \mathcal{E}^i(-\Gamma \cdot \bar{W}^{j,i})_0^{T}$. Similar to the above argument, for fixed $i$, thanks to Proposition 2.2 there exists a constant $p'_0 > 1$ such that

$$
\mathbb{E}^{Q_i} \left[ \left( \mathcal{E}^i(-\Gamma \cdot \bar{W}^{j,i})_0^{T} \right)^{p'_0} \right] \leq C_{p'_0}.
$$

For the above $p'_0$, Cauchy-Schwarz inequality and Hölder inequality allow to show for any $p \geq 2$,

$$
\frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} |\Delta Y^i_t|^p + \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p}{2}} \right] \right\} 
\leq \frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{E}^{Q_i} \left[ \mathcal{E}^i(-\Gamma \cdot \bar{W}^{j,i})_0^{T} \cdot \left( \sup_{t \in [0,T]} |\Delta Y^i_t|^p + \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p}{2}} \right) \right] \right\} 
\leq \frac{C}{N} \sum_{i=1}^{N} \left\{ \mathbb{E}^{Q_i} \left[ \left( \mathcal{E}^i(-\Gamma \cdot \bar{W}^{j,i})_0^{T} \right)^{p'_0} \right] \right\} \frac{1}{q'_0} \left\{ \mathbb{E} \left[ \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{p'_0} \right] \right\}^{\frac{1}{q'_0}} 
\leq \frac{C}{N} \sum_{i=1}^{N} \left\{ \mathbb{E}^{Q_i} \left[ \left( \sup_{t \in [0,T]} |\Delta Y^i_t|^{p'_0} + \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p'_0}{2}} \right) \right] \right\} \frac{1}{q'_0} 
\leq C \left\{ \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{Q_i} \left[ \sup_{t \in [0,T]} |\Delta Y^i_t|^{p'_0} + \left( \int_0^T \sum_{j=1}^{T} |\Delta Z^{i,j}_t|^2 dt \right)^{\frac{p'_0}{2}} \right] \right\} \frac{1}{q'_0},
$$

where $q'_0 = \frac{p'_0}{p'_0 - 1} > 1$. 

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Finally, combining the above equalities we arrive at
\[
\frac{1}{N} \sum_{i=1}^{N} \left\{ E \left[ \sup_{t \in [0,T]} |\Delta Y_{t_i}^j|^p + \left( \int_0^T \sum_{j=1}^{N} |\Delta Z_{t_i}^{i,j}|^2 dt \right)^{\frac{p}{2}} \right] \right\} 
\leq CE \left[ \int_0^T \mathcal{W}_2^{\rho_0^q \nu_0^q} (\nu_t^N, \tilde{\nu}_t) dt + \int_0^T \mathcal{W}_2^{\rho_0^q \nu_0^q} (\mu_t^N, \tilde{\mu}_t) dt \right]^{\frac{1}{q_0^q}}.
\]

Thanks to the exchangeability of the \((Y_t, Z_t)\), one has the item (ii).

\[\square\]

**Remark 5.3.** In BSDE (1.1), if the generator \(g\) does not depend on the law of \(Z\), then the inequality (5.8) becomes that for any \(p \geq 2,\)

\[
(i) \quad E \left[ \frac{1}{N} \sum_{i=1}^{N} \left\{ \sup_{t \in [0,T]} |\Delta Y_{t_i}^j|^p + \left( \int_0^T \sum_{j=1}^{N} |\Delta Z_{t_i}^{i,j}|^2 dt \right)^{\frac{p}{2}} \right\} \right] \leq CE \left[ \int_0^T \mathcal{W}_2^{\rho_0^q \nu_0^q} (\nu_t^N, \tilde{\nu}_t) dt \right]^{\frac{1}{q_0^q}},
\]

\[
(ii) \quad E \left\{ \sup_{t \in [0,T]} |\Delta Y_{t_i}^j|^p + \left( \int_0^T \sum_{j=1}^{N} |\Delta Z_{t_i}^{i,j}|^2 dt \right)^{\frac{p}{2}} \right\} \leq CE \left[ \int_0^T \mathcal{W}_2^{\rho_0^q \nu_0^q} (\nu_t^N, \tilde{\nu}_t) dt \right]^{\frac{1}{q_0^q}}.
\]

**Lemma 5.4.** Let Assumption 3 hold true and assume that \(g\) does not depend on the law of \(Z\). Then, for \(p > 2\), there exist two constants \(q_1, q_1' > 1\) such that

\[
\sup_{t \in [0,T]} E \left[ \mathcal{W}_2^p (\nu_t^N, \tilde{\nu}_t) \right] \leq CN^{-\frac{1}{4q_1 q_1'}}.
\]

**Proof.** Let \((\tilde{Y}^i, \bar{Z}^i)\) be the solution of (5.2), and let \((\tilde{Y}^i, \bar{Z}^i), i = 1, 2, \cdots, N\) be i.i.d. copies of \((Y^i, Z^i)\) such that

\[
\tilde{Y}_t^i = \eta^i + \int_t^T g(s, \tilde{Y}_s^i, \bar{Z}_s^i, \tilde{\nu}_s^i) ds - \int_t^T \bar{Z}_s^i dW_s^i, \quad t \in [0, T]. \tag{5.12}
\]

Notice that

\[
\tilde{Y}_t^i - Y_t^i = \int_t^T g(s, \tilde{Y}_s^i, \bar{Z}_s^i, \tilde{\nu}_s) - g(s, Y_s^i, Z_s^i, \tilde{\nu}_s) + g(s, Y_s^i, \bar{Z}_s^i, \tilde{\nu}_s) - g(s, Y_s^i, Z_s^i, \tilde{\nu}_s) + g(s, Y_s^i, Z_s^{i,j}, \tilde{\nu}_s) - g(s, Y_s^i, Z_s^{i,j}, \tilde{\nu}_s) \nonumber \\
\quad + \int_t^T \sum_{j=1}^{N} \left( \delta_{ij} \bar{Z}_s^i - Z_s^{i,j} \right) dW_s^j, \tag{5.13}
\]

where \(\delta_{ij} = 1\), if \(i = j\); or else, it equals to 0.

Since

\[
\begin{align*}
& \left\{ g(s, Y_s^i, \bar{Z}_s^i, \tilde{\nu}_s) - g(s, Y_s^i, Z_s^i, \tilde{\nu}_s) = \Theta^i_s (\bar{Z}_s^i - Z_s^i), \\
& |\Theta^i_s| \leq \phi(|Y_s^i| \vee \|Y_s^i\|_{L^2(\Omega)})(1 + |\bar{Z}_s^i| + |Z_s^{i,j}|). \tag{5.14}
\end{align*}
\]

For given \(\tilde{\nu}_s\), the equation (5.12) is a standard BSDE with quadratic growth. Thanks to Hu and Tang [25, Theorem 2.3], \(|\bar{Z}^i|\) belongs to \(Z^2[0, T]\). Hence, according to Proposition 5.1, one can know that \(\Theta^i\) also belongs to \(Z^2[0, T]\). Define \(\tilde{W}^{i,j}(s) = W^j(s) - \int_0^s \Theta^j_r dr\). Then \((\tilde{W}^{i,j})_{1 \leq j \leq N}\) is an
$N$-dimensional Brownian motion under the probability $\hat{Q}^i$, which is defined by $d\hat{Q}^i = \varepsilon^i(\Theta \cdot W)^T d\mathbb{P}$ with $\Theta = (0, \ldots, 0, \Theta^i, 0, \ldots, 0)$. Consequently, (5.13) can be written as

$$
\bar{Y}_t^i - Y_t^i = \int_t^T g(s, \bar{Y}_s^i, \bar{Z}_s^i, \bar{\nu}_s) - g(s, Y_s^i, Z_s^i, \nu_s) + g(s, Y_s^i, Z_s^i, \nu_s) - g(s, Y_s^i, Z_s^i, \nu_s) ds 
- \int_t^T \sum_{j=1}^N \left( \delta_{ij} \bar{Z}_s^i - Z_s^i \right) d\tilde{W}_s^{j,i}.
$$

(5.14)

Thanks to Briand et al. [5, Proposition 3.2], we have for fixed $i$ and for $p > 2$,

$$
\mathbb{E}^{\hat{Q}^i} \left[ \sup_{t \in [0,T]} |\bar{Y}_t^i - Y_t^i|^p \right] \leq C \mathbb{E}^{\hat{Q}^i} \left[ \int_0^T \mathcal{W}_2^p (\nu_s^N, \bar{\nu}_s) ds \right]. \tag{5.15}
$$

Hence, Cauchy-Schwarz inequality allows to show, for $t \in [0, T]$,

$$
\mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\nu_t^N, \bar{\nu}_t^N) \right] \leq \mathbb{E}^{\hat{Q}^i} \left[ \left( \frac{1}{N} \sum_{i=1}^N |Y_t^i - \bar{Y}_t^i|^2 \right)^{\frac{p}{2}} \right] 
\leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ |Y_t^i - \bar{Y}_t^i|^p \right] \leq C \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \int_0^T \mathcal{W}_2^p (\nu_s^N, \bar{\nu}_s) ds \right]. \tag{5.16}
$$

By summing over $i$ on both sides of the past inequality, we have, for $t \in [0, T]$,

$$
\frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\nu_t^N, \bar{\nu}_t^N) \right] \leq C \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \int_0^T \mathcal{W}_2^p (\nu_s^N, \bar{\nu}_s) ds \right]. \tag{5.17}
$$

From this and the triangle inequality, we know, for $t \in [0, T]$,

$$
\frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\nu_t^N, \bar{\nu}_t) \right] \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \left( \mathcal{W}_2 (\nu_t^N, \bar{\nu}_t^N) + \mathcal{W}_2 (\nu_t^N, \bar{\nu}_t) \right)^p \right] 
\leq 2^p \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \left( \mathcal{W}_2 (\nu_t^N, \bar{\nu}_t^N) + \mathcal{W}_2 (\bar{\nu}_t^N, \bar{\nu}_t) \right) \right] 
\leq C \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \int_0^T \mathcal{W}_2^p (\nu_s^N, \bar{\nu}_s) ds \right] + 2^p \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\bar{\nu}_t^N, \bar{\nu}_t) \right]. \tag{5.18}
$$

Using Gronwall’s inequality, we deduce, for $t \in [0, T]$,

$$
\frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\nu_t^N, \bar{\nu}_t) \right] \leq C \frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\hat{Q}^i} \left[ \mathcal{W}_2^p (\bar{\nu}_t^N, \bar{\nu}_t) \right]. \tag{5.19}
$$

On the one hand, similar to Theorem 5.2, thanks to Proposition 2.2, there exists a constant $p_1 > 1$ such that

$$
\max_{1 \leq i \leq N} \mathbb{E} \left[ (\varepsilon^i (\Theta \cdot W)_0^T)^{p_1} \right] \leq C_{p_1}.
$$

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Define \( q_1 = \frac{p_1}{p_1 - 1} \). Clearly, \( q_1 > 1 \). From H"{o}lder inequality we have, for \( t \in [0, T] \),

\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{\hat{Q}} \left[ \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] \leq \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \mathcal{E}^i(\Theta \cdot W^j)_0^T \cdot \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] \leq \frac{1}{N} \sum_{i=1}^{N} \left( \left\{ \mathbb{E} \left[ \mathcal{E}^i(\Theta \cdot W^j)_0^T p_i \right] \right\}^{\frac{1}{p'_i}} \cdot \left\{ \mathbb{E} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}} \right) \leq C \left\{ \mathbb{E} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}}. \tag{5.20}
\]

On the other hand, clearly, \( d\mathbb{P} = \mathcal{E}^i((-\Theta) \cdot \hat{W}^{j,i})^T \tilde{d} \hat{Q}^i \). For fixed \( i \), from Proposition 2.2 there exists a constant \( p'_i > 1 \) such that

\[
\max_{1 \leq i \leq N} \mathbb{E}^{\hat{Q}} \left[ \mathcal{E}^i((-\Theta) \cdot \hat{W}^{j,i})^T p_i \right] \leq C_{p'_i}.
\]

Define \( q'_i = \frac{p'_i}{p'_i - 1} \). Obviously, \( q_1 > 1 \). Then one has, for \( t \in [0, T] \),

\[
\mathbb{E} \left[ \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{\hat{Q}} \left[ \mathcal{E}^i((-\Theta) \cdot \hat{W}^{j,i})^T \cdot \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] \leq \frac{1}{N} \sum_{i=1}^{N} \left( \left\{ \mathbb{E}^{\hat{Q}} \left[ \mathcal{E}^i((-\Theta) \cdot \hat{W}^{j,i})^T p'_i \right] \right\}^{\frac{1}{p'_i}} \cdot \left\{ \mathbb{E}^{\hat{Q}} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}} \right) \leq C \frac{1}{N} \sum_{i=1}^{N} \left\{ \mathbb{E}^{\hat{Q}} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}}. \tag{5.21}
\]

According to (5.19)-(5.21), we have from Cauchy-Schwarz inequality, for \( t \in [0, T] \),

\[
\mathbb{E} \left[ \mathcal{W}_2^p(\nu_t^N, \nu_t) \right] \leq C \left\{ \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{\hat{Q}} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}} \leq C \left\{ \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{\hat{Q}} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}} = C \left\{ \mathbb{E} \left[ \mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \right] \right\}^{\frac{1}{q_i}}. \tag{5.22}
\]

Next, we show for \( t \in [0, T] \) and \( p > 2 \), \( \mathbb{P} \)-a.s.,

\[
\mathcal{W}_2^{p'_i}(\nu_t^N, \nu_t) \leq \mathcal{W}_2^{p'_i}(\tilde{\nu}_t^N, \nu_t). \tag{5.23}
\]

In fact, from the definition of \( p \)-Wasserstein metric, for arbitrary \( \varepsilon > 0 \) there exists a \( \rho_0 \in \mathcal{H} \), where \( \mathcal{H} \) is the set of probability measures \( \rho \) on \( \mathbb{R}^n \times \mathbb{R}^n \) with the first and second marginal laws \( \tilde{\nu}_t^N \) and \( \nu_t \), such that

\[
\int_{\mathbb{R}^{2n}} |x - y|^{p'_i} \rho_0(dx, dy) \leq \mathcal{W}_2^{p'_i}(\tilde{\nu}_t^N, \nu_t) + \varepsilon.
\]
From Hölder inequality, we have for \( t \in [0, T] \) and \( p > 2 \), \( \mathbb{P} \)-a.s.,

\[
\mathcal{W}_2^{p_0q_0}(\tilde{\nu}_t^N, \tilde{\nu}_t) = \inf_{\rho \in \mathcal{K}} \left\{ \int_{\mathbb{R}^2} |x - y|^2 \rho(dx, dy) \right\}^{\frac{p_0q_0}{q_1q_1'}} \\
\leq \left\{ \int_{\mathbb{R}^2} |x - y|^2 \rho_0(dx, dy) \right\}^{\frac{p_0q_0}{q_1q_1'}} \\
\leq \int_{\mathbb{R}^2} |x - y|^{p_0q_0} \rho_0(dx, dy) \leq \mathcal{W}_2^{p_0q_0}(\tilde{\nu}_t^N, \tilde{\nu}_t) + \varepsilon.
\]

Letting \( \varepsilon \to 0 \) we have \( \mathbb{P} \)-a.s., \( \mathcal{W}_2^{p_0q_0}(\tilde{\nu}_t^N, \tilde{\nu}_t) \leq \mathcal{W}_2^{p_0q_0}(\tilde{\nu}_t^N, \tilde{\nu}_t) \).

Combining (5.22) and (5.23) we have from Fournier and Guillin [21, Theorem 1],

\[
\mathbb{E}\left[ \mathcal{W}_2^{p}(\nu_t^N, \tilde{\nu}_t) \right] \leq C \left\{ \mathbb{E}\left[ \mathcal{W}_2^{p_0q_0}(\tilde{\nu}_t^N, \tilde{\nu}_t) \right] \right\}^{\frac{q_1q_1'}{q_1'}} \leq C N^{-\frac{1}{2q_1q_1'}}, \quad t \in [0, T].
\] (5.24)

This completes the proof. \( \square \)

As an immediate consequence of Lemma 5.4 and Remark 5.3, we have the following result concerning the rate of convergence.

**Theorem 5.5.** Let Assumption 3 be in force and the generator \( g \) be independent of the law of \( Z \). Then, for \( p \geq 2 \), there exists four constants \( q_0, q_0', q_1, q_1' > 1 \) and a positive constant \( C \) depending only on \( (T, K, K_1, p, q_0, q_0', q_1, q_1') \) such that

\[
\mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^{N} \right\{ \sup_{t \in [0,T]} |\Delta Y_t^i|^p + \left( \int_0^T \sum_{j=1}^{N} |\Delta Z_t^{i,j}|^2 dt \right)^\frac{q_1}{2} \right\} \leq C N^{-\frac{1}{2q_0q_0'q_1q_1'}},
\] (5.25)

\[
\mathbb{E}\left\{ \sup_{t \in [0,T]} |\Delta Y_t^i|^p + \left( \int_0^T \sum_{j=1}^{N} |\Delta Z_t^{i,j}|^2 dt \right)^\frac{q_1}{2} \right\} \leq C N^{-\frac{1}{2q_0q_0'q_1q_1'}}.
\]

**Remark 5.6.** In Theorem 5.5, if the generator \( g \) depends on the law of \( Z \), the rate of convergence appeals to the estimate of \( \mathbb{E}[\mathcal{W}_2^{p_0q_0}(\mu_t^N, \mu_t)] \), which, however, is still an open problem now. The difficulty is that generally speaking, the process \( Z \) fails to have the following uniform Hölder continuity in time:

\[
\mathbb{E}[|Z_t - Z_s|^2] \leq C|t - s|, \quad 0 \leq s \leq t \leq T,
\]

which is true for the process \( Y \) as a crucial fact used in the proof of Lemma 5.4 (see also Lauriere and Tangpi [29, Lemma 3.2] and Briand et al. [4, Lemma 1]).

### 6 Applications to PDE

The master equation can be regarded as a PDE in the Wasserstein space, whose state variable refers to the distribution of some state process. However, the viscosity solution of the master
equation is always a challenging topic, due to Wasserstein space lacking the local compactness, which is a necessary element in the viscosity theory (see Wu and Zhang [39]). Moreover, the study of the general mean-field BSDEs and associated master equations with quadratic growth will further complicate the problem. For example, besides the quadratic growth, the nonlinearity of the coefficient $g$ of mean-field BSDEs can lead to time-inconsistency of the value function, and the regularity of the value function under weak formulation is more involved even for the Lipschitz case. Hence, instead of the distribution, we consider the situation where the coefficients depend on the expectation of the state process. In detail, for any initial pair $(t, x) \in [0, T] \times \mathbb{R}^n$ and frozen $x_0 \in \mathbb{R}^n$, consider the following mean-field forward-backward stochastic differential equation:

\[
\begin{align*}
    &dX_{t}^{t,x} = \mathbb{E}'[b(s, (X_{s}^{0,x_0})', X_{s}^{t,x})] ds + \mathbb{E}'[\sigma(s, (X_{s}^{0,x_0})', X_{s}^{t,x})] dW_s, \\
    &-dY_{t}^{t,x} = \mathbb{E}'[g(s, (X_{s}^{0,x_0})', X_{s}^{t,x}, (Y_{s}^{0,x_0})', Y_{s}^{t,x}, Z_{s}^{t,x})] ds - Z_{t}^{t,x} dW_s, \quad t \leq s \leq T, \\
    &X_{t}^{t,x} = x \in \mathbb{R}^n, \quad Y_{t}^{t,x} = \mathbb{E}'[\Phi((X_{t}^{0,x_0})', X_{t}^{t,x})],
\end{align*}
\]

where the coefficients $b: [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\sigma: [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times d}$ satisfy the following condition.

**Assumption 4.** There is a positive constant $C$ such that for any $t \in [0, T]$, $x', x, \bar{x} \in \mathbb{R}^n$,

\[
|b(t, x', x)| + |\sigma(t, x', x)| \leq C(1 + |x| + |x'|),
\]

and

\[
|b(t, x_1', x_1) - b(t, x_2', x_2)| + |\sigma(t, x_1', x_1) - \sigma(t, x_2', x_2)| \leq C(|x_1' - x_2'| + |x_1 - x_2|).
\]

Then under Assumption 4, it is well known that the forward equation of (6.1) admits a unique adapted solution, denoted by $\{X_{t}^{t,x}; t \leq s \leq T\}$, in the space $S_{\bar{F}}^2(t, T; \mathbb{R}^n)$. Besides, the following result holds (see Buckdahn, Li and Peng [10]).

**Lemma 6.1.** For every $p \geq 2$, there exists a positive constant $C_p$ such that for any $t \in [0, T]$, $\delta \in [0, T - t]$, and $x, x' \in \mathbb{R}^n$,

\[
\mathbb{E}_{t}\left[\sup_{s \in [t, T]} |X_{s}^{t,x} - X_{s}^{t,x'}|^p \right] \leq C_p|x - x'|^p;
\]

\[
\mathbb{E}_{t}\left[\sup_{s \in [t, T]} |X_{s}^{t,x}|^p \right] \leq C_p(1 + |x|^p);
\]

\[
\mathbb{E}_{t}\left[\sup_{s \in [t, t + \delta]} |X_{s}^{t,x} - x|^p \right] \leq C_p(1 + |x|^p)\delta^\frac{p}{2}.
\]

On the other hand, for the coefficients $g: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\Phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the backward equation of (6.1), we present the following assumptions.

**Assumption 5.** There exist positive constants $C$, $C_0$ and $C_1$ such that for any $t \in [0, T]$, $x', x, \bar{x} \in \mathbb{R}^n$, $y', y, \bar{y} \in \mathbb{R}$ and $\bar{z}, z \in \mathbb{R}^d$, the generator $g$ is differential with respect to $x, y, z$, and

\[
|g(t, x', x, y', y, z)| \leq C(1 + |z|^2) \quad \text{and} \quad |\Phi(x', x)| \leq C,
\]

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\[
|g(t, x', x, y', z) - g(t, \bar{x}, \bar{y}, \bar{z})| \\
\leq C(|x' - x| + |x - \bar{x}| + |y' - \bar{y}| + |y - \bar{y}|) + C(1 + |z| + |\bar{z}|)|z - \bar{z}|,
\]

\(g(t, x', x, y', y, z)\) is continuous with respect to \(t\),

\(g(t, x', x, y', y, z)\) is nondecreasing in \(y'\).

**Remark 6.2.** Under Assumption 5, it is easy to check that for any \(t \in [0, T]\), \(x', x \in \mathbb{R}^n\), \(y', y \in \mathbb{R}\), \(z', z \in \mathbb{R}^d\), and any \(\varepsilon > 0\), the following conditions, appeared in Kobylanski [27], hold:

\[
\left| \frac{\partial g}{\partial x}(t, x', x, y', y, z) \right| \leq C(1 + |z|) \leq C\left(\frac{3}{2} + |z|^2\right),
\]

\[
\left| \frac{\partial g}{\partial y}(t, x', x, y', y, z) \right| \leq C(1 + |z|) \leq C + \frac{C^2}{4\varepsilon} + \varepsilon|z|^2,
\]

\[
\left| \frac{\partial g}{\partial z}(t, x', x, y', y, z) \right| \leq C(1 + |z|).
\]

Due to that Assumption 5 is stronger then Assumption 2, then combining Theorem 3.5, it is not hard to verify that the backward equation of (6.1) admits a unique global solution in \(S^\infty_F(t, T; \mathbb{R}) \times Z^2_F(t, T; \mathbb{R}^d)\). In other words, we have the following result.

**Proposition 6.3.** Let Assumption 5 hold, then the backward equation of (6.1) possesses a unique global solution, denoted by \(\{(Y^{t,x}_s, Z^{t,x}_s); t \leq s \leq T\}\), in the space \(S^\infty_F(t, T; \mathbb{R}) \times Z^2_F(t, T; \mathbb{R}^d)\).

**Proof.** On the one hand, from Theorem 3.5, we see that for \((t, x) = (0, x_0)\), the backward equation of (6.1) possesses a global solution \(\{(Y^{0,x_0}_s, Z^{0,x_0}_s); 0 \leq s \leq T\}\) in the space \(S^\infty_F(0, T; \mathbb{R}) \times Z^2_F(0, T; \mathbb{R}^d)\). Moreover, \(\|Y^{0,x_0}\|_{S^\infty_F(0,T)}\) and \(\|Z^{0,x_0}\|_{Z^2_F(0,T)}\) are bounded. On the other hand, once knowing \((Y^{0,x_0}, Z^{0,x_0})\), we can define that for any \(s \in [t, T], y \in \mathbb{R} \) and \(z \in \mathbb{R}^d\),

\[
g^\#(s, X^{t,x}_s, y, z) = \mathbb{E}'[g(s, (X^{0,x_0}_s)', X^{t,x}_s, (Y^{0,x_0}_s)', y, z)], \quad \Phi^\#(x) = \mathbb{E}'[\Phi((X^{0,x_0}_T)', x)].
\]

Then, it is easy to check that \(g^\#\) and \(\Phi^\#\) also satisfy the condition (6) of Hibon, Hu, and Tang [23]. Hence, Lemma 2.1 of [23] (see also Theorem 2.3 of Hu and Tang [25]) implies that BSDE (1.2) with generator \(g^\#\) and terminal value \(\Phi^\#\) has a unique global solution \((Y, Z) \in S^\infty_F(t, T; \mathbb{R}) \times Z^2_F(t, T; \mathbb{R}^d)\). In other words, the backward equation of (6.1) admits a unique global solution \(\{(Y^{t,x}_s, Z^{t,x}_s); t \leq s \leq T\}\) in the space \(S^\infty_F(t, T; \mathbb{R}) \times Z^2_F(t, T; \mathbb{R}^d)\). \(\square\)

We define the value function as follows:

\[
u(t, x) \equiv Y^{t,x}_t, \quad t \in [0, T], \ x \in \mathbb{R}^n.
\] (6.2)

Note that \(u(t, x)\) is continuous with respect to \((t, x)\). In fact, on the one hand, Lemma 6.1 deduces that the flow \((t, x) \mapsto X^{t,x}_s\) is continuous. On the other hand, the stability result (see Kobylanski [27]) implies that \((t, x, s) \mapsto Y^{t,x}_s\) is continuous under Assumption 5. Thus, in particular, the deterministic function \(u(t, x) = Y^{t,x}_t\) is certainly continuous with respect to \((t, x)\).
Now, we would like to connect the forward-backward system (6.1) with quadratic growth to the following nonlocal partial differential equation:

\[
\begin{cases}
\frac{\partial v(t, x)}{\partial t} + \mathcal{L}v(t, x) + \mathbb{E}\left[ g\left(t, X^0_{t}, x, v(t, t, X^0_{t}), v(t, x), Dv(t, x)\mathbb{E}\left[ \sigma(t, X^0_{t}, x)\right]^T\right) \right] = 0, \\
(t, x) \in [0, T) \times \mathbb{R}^n,
\end{cases}
\]

(6.3)

where

\[
\mathcal{L}v(t, x) = \frac{1}{2} \text{tr}\left( \mathbb{E}\left[ \sigma(t, X^0_{t}, x)\right] \mathbb{E}\left[ \sigma(t, X^0_{t}, x)\right]^T D^2v(t, x) \right) + Dv(t, x)\mathbb{E}\left[ b(t, X^0_{t}, x)\right]^T.
\]

It should be pointed out that in (6.3), the terminology nonlocal means

\[
\mathbb{E}\left[ g\left(t, X^0_{t}, x, v(t, t, X^0_{t}), v(t, x), Dv(t, x)\mathbb{E}\left[ \sigma(t, X^0_{t}, x)\right]^T\right) \right] = \int_{\mathbb{R}^n} g\left(t, x' , x, v(t, x'), v(t, x), Dv(t, x)\mathbb{E}\left[ \sigma(t, X^0_{t}, x)\right]^T\right) \mathbb{P}_{_X^0}(dx'),
\]

where \(X^0_{t}\) is the solution of the forward equation of (6.1) with the initial pair \((0, x_0)\).

In the following, we prove that the value function \(u\) defined in (6.2) is the viscosity solution of PDE (6.3). For this, we extend the approach of Buckdahn, Li and Peng [10] developed in the framework of mean-field BSDEs with linear growth to that of quadratic growth. First, we recall the definition of a viscosity solution of PDE (6.3). For more details about the viscosity solutions, we refer the reader to Crandall, Ishii, and Lions [14].

Note that for Euclidean spaces \(\mathbb{H}\), denote by \(C^k_p(\mathbb{H})\) the set of continuous function on \(\mathbb{H}\), who grows at most like a polynomial function of the variable \(x\) at infinity; and denote by \(C^k_{l,b}(\mathbb{H})\) the set of functions of class \(C^k\) on \(\mathbb{H}\), whose partial derivatives of order less than or equal to \(k\) are bounded.

**Definition 6.4 (Viscosity solution).** A real-valued continuous function \(u \in C^p_p([0, T) \times \mathbb{R}^n)\) is called

(i) a viscosity subsolution to PDE (6.3), if \(v(T, x) \leq \mathbb{E}\left[ \Phi(X^0_{T}, x)\right]\) for all \(x \in \mathbb{R}^n\), in addition, if for arbitrary \(\psi \in C^3_{l,b}([0, T) \times \mathbb{R}^n)\) and \((t^*, x^*) \in [0, T) \times \mathbb{R}^n\) such that \(\psi(t^*, x^*) = v(t^*, x^*)\) and \(\psi(t, x) \geq v(t, x)\), we have

\[
\frac{\partial v(t^*, x^*)}{\partial t} + \mathcal{L}v(t^*, x^*)
\]

\[
+ \mathbb{E}\left[ g\left(t^*, X^{0,x^*}_{t^*}, x^*, v(t^*, X^{0,x^*}_{t^*}), v(t^*, x^*), \mathbb{E}\left[ \sigma(t^*, X^{0,x^*}_{t^*}, x^*)\right]^T Dv(t^*, x^*)\right) \right] \geq 0.
\]

(ii) a viscosity supersolution to PDE (6.3), if \(v(T, x) \geq \mathbb{E}\left[ \Phi(X^0_{T}, x)\right]\) for all \(x \in \mathbb{R}^n\), in addition, if for arbitrary \(\psi \in C^3_{l,b}([0, T) \times \mathbb{R}^n)\) and \((t^*, x^*) \in [0, T) \times \mathbb{R}^n\) such that \(\psi(t^*, x^*) = v(t^*, x^*)\) and \(\psi(t, x) \leq v(t, x)\), we have

\[
\frac{\partial v(t^*, x^*)}{\partial t} + \mathcal{L}v(t^*, x^*)
\]

\[
+ \mathbb{E}\left[ g\left(t^*, X^{0,x^*}_{t^*}, x^*, v(t^*, X^{0,x^*}_{t^*}), v(t^*, x^*), \mathbb{E}\left[ \sigma(t^*, X^{0,x^*}_{t^*}, x^*)\right]^T Dv(t^*, x^*)\right) \right] \leq 0.
\]
(iii) a viscosity solution to PDE (6.3), if it is both a viscosity subsolution and a viscosity supersolution.

**Lemma 6.5.** Under Assumption 4 and Assumption 5, for any \( t \in [0, T] \) and \( \xi \in L_{\mathcal{F}_t}^2(\Omega; \mathbb{R}^n) \), we have

\[
u(t, \xi) = Y_t^{t, \xi}.
\]

**Proof.** The proof is essentially an adaptation of Proposition 4.7 of Peng [36], so we sketch it. First, we assert that (6.4) holds true for a simple random variable \( \xi \in L_{\mathcal{F}_t}^2(\Omega; \mathbb{R}^n) \) with a form below

\[
\xi = \sum_{j=1}^M x_j 1_{B_j},
\]

where \( \{B_j\}_{j=1}^M \) is a finite partition of \((\Omega, \mathcal{F}_t)\) and \( x_j \in \mathbb{R}^n \) with \( j = 1, 2, \cdots, M \). For each \( x_j \), let \( X_s^{t, x_j} \) be the solution of the following mean-field SDE

\[
X_s^{t, x_j} = x_j + \int_t^s \mathbb{E}'[b(r, (X_r^{0,x_0})', X_r^{t, x_j})] dr + \int_t^s \mathbb{E}'[\sigma(r, (X_r^{0,x_0})', X_r^{t, x_j})] dW_r, \quad t \leq s \leq T,
\]

and let \( (Y_s^{t, x_j}, Z_s^{t, x_j}) \) be the solution of the mean-field BSDE

\[
Y_s^{t, x_j} = \mathbb{E}'[\Phi((X_T^{0,x_0})', X_T^{t, x_j})] + \int_t^s \mathbb{E}'[g(r, (X_r^{0,x_0})', X_r^{t, x_j}, (Y_r^{0,x_0})', Y_r^{t, x_j}, Z_r^{t, x_j})] dr
\]

\[
- \int_t^s Z_r^{t, x_j} dW_r, \quad t \leq s \leq T.
\]

Multiplying \( 1_{A_j} \) on both sides of the above two equations and then summing up with respect to \( j \), note that \( \sum_j \psi(x_j) 1_{A_j} = \psi(\sum_j x_j 1_{A_j}) \), one gets

\[
\sum_{j=1}^M 1_{A_j} X_s^{t, x_j} = \sum_{j=1}^M 1_{A_j} x_j + \int_t^s \mathbb{E}'[b(r, (X_r^{0,x_0})', \sum_{j=1}^M 1_{A_j} X_r^{t, x_j})] dr
\]

\[
+ \int_t^s \mathbb{E}'[\sigma(r, (X_r^{0,x_0})', \sum_{j=1}^M 1_{A_j} X_r^{t, x_j})] dW_r, \quad t \leq s \leq T,
\]

and for \( t \leq s \leq T \),

\[
\sum_{j=1}^M 1_{A_j} Y_s^{t, x_j} = \mathbb{E}'[\Phi((X_T^{0,x_0})', \sum_{j=1}^M 1_{A_j} X_T^{t, x_j})] - \int_t^s \sum_{j=1}^M 1_{A_j} Z_r^{t, x_j} dW_r
\]

\[
+ \int_t^s \mathbb{E}'[g(r, (X_r^{0,x_0})', \sum_{j=1}^M 1_{A_j} X_r^{t, x_j}, (Y_r^{0,x_0})', \sum_{j=1}^M 1_{A_j} Y_r^{t, x_j}, \sum_{j=1}^M 1_{A_j} Z_r^{t, x_j})] dr.
\]

Then from the existence and uniqueness of the mean-field BSDE with quadratic growth (see Theorem 3.5), we conclude that for \( t \leq s \leq T \),

\[
X_s^{t, \xi} = \sum_{j=1}^M 1_{A_j} X_s^{t, x_j}, \quad Y_s^{t, \xi} = \sum_{j=1}^M 1_{A_j} Y_s^{t, x_j}, \quad Z_s^{t, \xi} = \sum_{j=1}^M 1_{A_j} Z_s^{t, x_j}.
\]
Finally, from the definition of \( u(t, x) \), it yields
\[
Y_t^{t,ξ} = \sum_{j=1}^{M} 1_{A_j} Y_t^{t,x_j} = \sum_{j=1}^{M} 1_{A_j} u(t, x_j) = u(t, \sum_{j=1}^{M} x_j 1_{A_j}) = u(t, ξ).
\]

Now, for given \( ξ \in L_{2,F_t}^2(Ω; ℝ^n) \), there exists a sequence of simple variables \( ξ_j \) admitting the form (6.5) that converges to \( ξ \) in \( L_{2,F_t}^2(Ω; ℝ^n) \). Since the flow \((t, x) \mapsto X^{t,x}_s(6.6)\) and \((t, x, s) \mapsto Y^{t,x}_s\) are continuous, \( u(t, x) \) is continuous with respect to \((t, x)\) and note the fact \( Y^{t,ξ_j}_t = u(t, ξ_j) \), we have
\[
\mathbb{E}[|Y^{t,ξ}_t - u(t, ξ)|^2] = \mathbb{E}[|Y^{t,ξ}_t - Y^{t,ξ_j}_t + Y^{t,ξ_j}_t - u(t, ξ_j) + u(t, ξ_j) - u(t, ξ)|^2]
\leq 2\mathbb{E}[|Y^{t,ξ}_t - Y^{t,ξ_j}_t|^2] + \mathbb{E}[|u(t, ξ_j) - u(t, ξ)|^2] → 0 \quad \text{as} \quad j \to \infty.
\]
Hence, we have (6.4).

Based on the above preparation, now we state the main results of this section.

**Theorem 6.6** (Feynman-Kac formula). Under Assumption 4 and Assumption 5, the value function \( u \) defined in (6.2) is the unique viscosity solution of PDE (6.3).

**Proof.** From Lemma 6.5 and the uniqueness of the mean-field forward-backward SDE (6.1) with the initial pair \((t, x) = (0, x_0)\), we have that
\[
Y^{t,0,x_0}_t = Y^{t,X^{0,x_0}}_t = u(t, X^{0,x_0}_t), \quad t \in [0, T].
\]

Based on the value of \( X^{0,x_0} \) and \( Y^{0,x_0} \), one can define that
\[
\tilde{b}(t, x) = \mathbb{E}[b(t, X^{0,x_0}_t, x)], \quad \tilde{σ}(t, x) = \mathbb{E}[σ(t, X^{0,x_0}_t, x)],
\]
\[
\tilde{g}(t, x, y, z) = \mathbb{E}[g(t, X^{0,x_0}_t, x, Y^{0,x_0}_t, y, z)], \quad \tilde{Φ}(x) = \mathbb{E}[Φ(X^{0,x_0}_t, x)].
\]

By Remark 6.2, it is easy to check that the parameters \((\tilde{b}, \tilde{σ}, \tilde{g}, \tilde{Φ})\) satisfy the assumptions (H4) and (H5) of Kobylanski [27]. In view of Theorem 3.8 of [27], the function \( u \) is a viscosity solution to the following PDE
\[
\begin{aligned}
\frac{∂u(t, x)}{∂t} + \frac{1}{2} \text{tr}(\tilde{σ}(t, x)\tilde{σ}(t, x)^TD^2u(t, x)) + \tilde{b}(t, x)^TDu(t, x) \\
+ \tilde{g}(t, x, u(t, x), \tilde{σ}(t, x)^TDu(t, x)) = 0, \quad (t, x) \in [0, T] × ℝ^n,
\end{aligned}
\]
\[
u(T, x) = \tilde{Φ}(x), \quad x ∈ ℝ^n.
\]
Finally, from the definitions of the parameters \((\tilde{b}, \tilde{σ}, \tilde{g}, \tilde{Φ})\) and (6.6), we see that \( u \) is also a viscosity solution to PDE (6.3).

Next, we prove the uniqueness of PDE (6.3). Let both \( u^1 \) and \( u^2 \) be the viscosity solutions of PDE (6.3). For any \( t \in [0, T], x ∈ ℝ^n, y ∈ ℝ \) and \( z ∈ ℝ^d \), we set
\[
g^1(t, x, y, z) = \mathbb{E}[g(t, X^{0,x_0}_t, x, u^1(t, X^{0,x_0}_t), y, z)],
\]
\[ g^2(t, x, y, z) = E \left[ g \left( t, X_t^{0,x_0}, x, u^2(t, X_t^{0,x_0}), y, z \right) \right]. \]

Then, for \( i = 1, 2 \), \( u^i \) is a viscosity solution to the following PDE

\[
\begin{aligned}
\frac{\partial u^i(t, x)}{\partial t} + \frac{1}{2} \text{tr} \left( \hat{\sigma}(t, x) \hat{\sigma}(t, x)^T D^2 u^i(t, x) \right) + \hat{b}(t, x)^T Du^i(t, x) \\
+ g^i(t, x, u^i(t, x), \hat{\sigma}(t, x)^T Du^i(t, x)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n,
\end{aligned}
\]

where \( \hat{b}, \hat{\sigma}, \hat{\Phi} \) is given in (6.7). Again, thanks to Kobylanski [27], \( u^i \) admits the following probabilistic interpretation:

\[
u^i(t, x) = Y_{t,x,i}^t, \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\]

(6.8)

where \( (Y_{t,x,i}^t, Z_{t,x,i}^t) \) is the solution to the following quadratic BSDE

\[
\begin{aligned}
-dY_{s,x,i}^t &= g^i(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds - Z_s^{t,x,i} dW_s, \quad s \in [t, T], \\
Y_{T,x,i}^t &= \tilde{\Phi}(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

(6.9)

Now, we let \( (t, x) = (0, x_0) \) in (6.8). Similar to Lemma 6.5, from the continuity of \( u^i \) and the uniqueness of the solution to BSDE (6.9), one has

\[
u^i(t, X_0^{0,x_0}) = Y_{t,x_0,i}^t, \quad t \in [0, T].
\]

Finally, recall the definitions of \( (g^i, \hat{b}, \hat{\sigma}, \hat{\Phi}) \), one has that the pair \( (Y_{t,x,i}^t, Z_{t,x,i}^t) \) is the adapted solution to the following quadratic BSDE

\[
\begin{aligned}
-dY_{s,x,i}^t &= \mathbb{E} \left[ g^i(s, (X_s^{0,x_0})^T, X_{s,x}^t, (Y_{s,x_0,i})^T, Y_{s,x,i}^t, Z_{s,x,i}^t) \right] ds - Z_{s,x,i}^t dW_s, \quad s \in [t, T], \\
Y_{T,x,i}^t &= \mathbb{E} \left[ \Phi((X_T^{0,x_0})^T, X_{T,x}^t) \right], \quad x \in \mathbb{R}^n.
\end{aligned}
\]

Then Proposition 6.3 implies that

\[
Y_{s,x,1}^t = Y_{s,x,2}^t, \quad t \leq s \leq T.
\]

In particular, as \( s = t \), one gets that

\[
u^1(t, x) = Y_{t,x,1}^t = Y_{t,x,2}^t = u^2(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n.
\]

This completes the proof.

\[\square\]

Remark 6.7. Theorem 6.6 establishes the relation between the solution of mean-field BSDEs and the viscosity solution of the nonlocal PDEs with quadratic growth, which extends the related result of Kobylanski [27] to the mean-field framework and extends the nonlinear Feynman-Kac formula of Buckdahn, Li and Peng [10] with linear growth to that of quadratic growth.
7 Conclusion Remark

We initiate the study of the general mean-field BSDEs (1.1), and give with some new ideas the existence, uniqueness, and comparison results for one-dimensional mean-field BSDEs (1.1) with quadratic growth and with bounded terminal value. Besides, we obtain the convergence of particle systems for the mean-field BSDE (1.1) with quadratic growth, and give the rate of convergence when generator $g$ is independent of the law of $Z$. However, the rate of convergence is still an open problem if the generator $g$ depends on the law of $Z$. Finally, in this framework, when the generator $g$ depends on the expectation of the state process $(Y, Z)$, we used the mean-field BSDEs (1.1) to prove the existence and uniqueness of the viscosity solutions of the nonlocal PDEs (6.3), which extend the nonlinear Feynman-Kac formula of Buckdahn, Li and Peng [10] to that of quadratic growth. On the other hand, when the generator $g$ depends on the distribution of $Z$, the existence and uniqueness of viscosity solutions of related PDEs are interesting and challenging, which remains to be studied in the future.

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