PRESERVATION OF SUPERAMALGAMATION BY EXPANSIONS

PAOLO LIPPARINI

Abstract. The superamalgamation property is a strong form of the amalgamation property which applies to ordered structures; it has found many applications in algebraic logic. We show that superamalgamation has some interest also from the pure model-theoretical point of view. Under a completion assumption, we prove that the superamalgamation property for some class of ordered structures implies strong amalgamation for classes with added operations, including isotone, idempotent, extensive, antitone and closure operations.

Thus, for example, partially ordered sets, semilattices, lattices, Boolean algebras and Heyting algebras with an isotone extensive operation (or an operation as above) have the strong amalgamation property. The theory of join semilattices with a closure operation has model completion. The set of universal consequences of the theory of Boolean algebras (or posets, semilattices, distributive lattices) with a closure or isotone, etc., operation is decidable.

1. Introduction

The amalgamation property (AP) is a classical tool in algebra [17] and has found many applications in logic, particularly, in model theory [11] and algebraic logic [1, 5, 6, 24].

B. Jónsson [14] asked whether there are general results that assert that if an elementary class is characterized by axioms of such and such a form, then this class has the amalgamation property. Jónsson’s problem, as stated, seems to have a positive solution only in an incredibly small number of cases. On the other hand, there are many situations in which theories which are already known to have the amalgamation...
property can be combined or modified in order to produce many other theories with AP. Classical examples include adding operators, e.g., , , , , .

Here we present a general result in the above wake. If some class of ordered structures has the superamalgamation property and satisfies an appropriate completion property, then we can add operations with a number of properties (isotone, antitone, extensive, idempotent, closure...) in such a way that amalgamation is preserved. The added operations possibly depend on many arguments, e.g., a 4-ary operation which is isotone on the first two components and antitone on the last two components. The assumptions apply to the classes of Boolean algebras, Heyting algebras, lattices, join semilattices, meet semilattices and partially ordered sets (henceforth, posets, for short). See Theorem and Corollary below. Recall that the superamalgamation property is a natural strengthening of AP for ordered structures, and has found significant applications in algebraic logic. Our results show that superamalgamation has some interest also from the pure model-theoretical point of view, even when dealing with AP alone.

As applications, we show that the theory of join semilattices with a closure operation has model completion (Corollary ) and we generally get the existence of Fraïssé limits for the subclass of finite members of the classes under consideration (Corollary ). As a consequence of our main Extension Lemma we prove that the set of universal consequences of the theory of Boolean algebras (or posets, semilattices, distributive lattices) with a finite number of operations of the kind taken into consideration is decidable. See Corollary .

The paper is divided as follows. Section is devoted to preliminaries and auxiliary results. In Section we present the Extension Lemma which broadly generalizes former results. Given a partial function on some complete lattice, we devise the exact conditions under which can be extended to a total function satisfying one of the properties taken into account. Classical embedding results for posets then imply extension results for various kinds of posets with an operation, for example, semilattices or lattices with a closure operation. See Corollary .

In Section we prove our more general result about amalgamation, Theorem , if is a class of partially ordered structures, has the superamalgamation property and every member of can be embedded into a complete member of , then superamalgamation is preserved by adding a closure operation, or an isotone (extensive, idempotent, etc.) operation. Complete is always meant in the lattice theoretical
sense. In particular, the result applies when $\mathcal{S}$ is one of the following classes: posets, semilattices, lattices, Boolean algebras, Heyting algebras (Corollary 4.3(1)). We can also add simultaneously many operations at a time, possibly with some comparability conditions (Corollary 4.3(2)). As a consequence, we get the existence of Fraïssé limits in the case of a finite language (Corollary 4.3(3)). Since the theory of join semilattices with a closure operation is locally finite, we get the existence of a model completion (Corollary 4.6).

Section 5 exploits the power of the extension lemma from Section 3 and does not rely on the amalgamation property. In Theorem 5.1 we consider a universal locally finite theory $T$ with an order relation such that every finite model of $T$ is lattice-ordered (or, more generally, can be extended to a finite lattice-ordered model). If $T^+$ extends $T$ in a finite language with further operations and with axioms asserting that the operations satisfy one of the conditions taken into account, then the set of universal consequences of $T^+$ is decidable (this would be obvious for $T$, but is far from being obvious for $T^+$, which is not necessarily locally finite). This applies to Boolean algebras (or posets, semilattices, distributive lattices) with a finite number of further operations. See Corollary 5.2. Section 6 presents a few further remarks; in particular, we show that, in some special cases, the completeness assumptions in Sections 3 - 4 can be relaxed to some extent; nevertheless, counterexamples are provided showing that some degree of completeness is necessary.

2. Preliminaries

We shall consider the following properties of a unary operation $K$ on a partially ordered set, poset, for short:

\begin{align*}
  a &\leq Ka & \text{extensive}, & (2.1) \\
  Ka &\leq a & \text{contractive}, & (2.2) \\
  a &\leq b \implies Ka \leq Kb & \text{isotone}, & (2.3) \\
  a &\leq b \implies Kb \leq Ka & \text{antitone}, & (2.4) \\
  KKa & = Ka & \text{idempotent}, & (2.5) \\
  KKa & = a & \text{involutive}. & (2.6)
\end{align*}

The conditions are supposed to hold for all the elements $a, b$ of the poset $P$ under consideration. We frequently consider a partial operation $G$ defined on some subset $D$ of $P$ and we say that $G$ is extensive or contractive if (2.1) or (2.2) holds for all $a \in D$. 
We shall also deal with $n$-ary operations. An $n$-ary operation is *isotone on the $i^{th}$ component* if it is isotone as a unary operation when the argument of the $i^{th}$ component varies and the other arguments are kept fixed. Operations *antitone on the $i^{th}$ component* are defined correspondingly.

A *closure* (interior) operation is an extensive (contractive), isotone and idempotent operation. In the presence of a semilattice operation, some authors include an additivity or multiplicativity requirement in the definitions of a closure and an interior operation. We shall adopt the more general convention [2, 10] according to which no additivity or multiplicativity assumption is made, unless explicitly mentioned otherwise.

We shall generally deal with *ordered structures*, namely, posets with possibly additional operations or relations. For definiteness, such structures will be considered as *models* in the sense of classical model theory [11], but in Section 4 there are more general possibilities, e. g., topological or infinitary structures. The precise setting will generally not be relevant for our purposes, insofar as the meaning of type (or signature, or language) and of embedding are clear. When some poset is, say, a lattice, we shall explicitly mention whether we are considering order-embeddings, or the stronger notion of lattice-embeddings, that is, embeddings preserving the lattice operations. Lattice operations shall be denoted by $\lor$ and $\land$. Their infinitary extensions are indicated by $\sum$ and $\prod$.

For the sake of simplicity, in the following definitions classes of structures shall be always supposed to be closed under isomorphism. If $A$ and $B$ are structures of the same type, with base sets, respectively, $A$ and $B$, then $A$ is said to be a *substructure* of $B$, in symbols, $A \subseteq B$, if $A \subseteq B$ as sets and the inclusion is an embedding of $A$ into $B$.

**Definition 2.1.** A class $S$ of structures of the same type and closed under isomorphism has the *strong amalgamation property* (SAP) if the following holds. Whenever $A, B, C \in S$, $C \subseteq A$, $C \subseteq B$ and $C = A \cap B$, then there is a structure $D \in S$ such that $A \subseteq D$ and $B \subseteq D$.

\[
\begin{array}{ccc}
  & & D \\
  & A & \\
  & \cup & B \\
 C & \subseteq & \subseteq \\
\end{array}
\]

(with $C = A \cap B$)

In the case of ordered structures, the *superamalgamation property* asserts that, under the above assumptions, there exists some $D$ as
above with the additional property that, for every \( a \in A \setminus B \) and \( b \in B \setminus A \),
(a) if \( a \leq_D b \), then there is \( c \in C \) such that \( a \leq_A c \), \( c \leq_B b \) and, symmetrically,
(b) if \( b \leq_D a \), then there is \( c \in C \) such that \( b \leq_B c \), \( c \leq_A a \).

In particular, every class with the superamalgamation property has the strong amalgamation property.

A theory \( T \) has the strong amalgamation property (the superamalgamation property) if the class of models of \( T \) has such a property.

In principle, our results apply to structures with many partial orders at a time. In such a situation, we always suppose that some specific partial order is selected; the superamalgamation property is always meant to refer to such partial order.

We refer to [2, 5, 10, 11, 17, 22, 24] for more information about the above notions.

The following result is proved by standard arguments in fixed point theory; see, e. g., Chapter 12 in [25], in particular, Theorem 12.9 therein. We present the explicit proof since the construction shall be used in the course of some proofs in the next section. If \( K \) and \( J \) are two operations defined on the same poset \( P \), we say that \( K \) is (pointwise) larger than \( J \), or that \( J \) is smaller than \( K \), if \( Kx \geq Jx \), for every \( x \in P \). Here “larger” is always intended in the broader sense of “larger than or equal to”.

**Lemma 2.2.** Suppose that \( P \) is a poset and \( H \) is an isotone operation on \( P \) such that \( HHx \leq Hx \), for every \( x \in P \). If every nonempty infinite chain in the range of \( H \) has a meet in \( P \) (in particular, if every nonempty infinite chain in \( P \) has a meet), then there is the largest operation \( K \) among those isotone and idempotent operations in \( P \) which are smaller than \( H \).

**Proof.** For \( \alpha \) a nonzero ordinal, let
\[
K^1x = Hx, \\
K^{\alpha+1}x = HK^\alpha x \quad \text{and} \\
K^\beta x = \prod_{0<\alpha<\beta} K^\alpha x \quad \text{for } \beta \text{ limit.} \tag{2.7}
\]

We need to justify the limit case in definition (2.7). For this it is enough to prove by transfinite induction on \( \beta > 0 \) that, for every \( x \in P \), the sequence \((K^\beta x)_{0<\beta\leq\beta}\) is defined and decreasing (not necessarily in the strict sense).
The base case $β = 1$ is immediate.

If $β$ is a limit ordinal, then the inductive assumption implies that the sequence $(K^δ x)_{0 < δ < β}$ is defined and decreasing, so that $\{ K^δ x \mid 0 < δ < β \}$ is a chain, hence $K^β x$ is defined, by the assumption on $P$ (notice that, if $δ$ is limit, then $K^δ x$ does not necessarily belong to the range of $H$; however, since the sequence is decreasing and $β$ is a limit ordinal, the meet of $\{ K^δ x \mid 0 < δ < β, δ$ not limit $\}$ exists and only if the meet of $\{ K^δ x \mid 0 < δ < β \}$ exists, and if they exist, they are equal). Moreover, $K^β ≤ K^δ x$, for $δ < β$, by construction.

It remains to prove the induction step, that is, $K^{β+1} x ≤ K^β x$, for every ordinal $β > 1$ and $x ∈ P$, assuming the inductive hypothesis, that is, $(K^δ x)_{0 < δ ≤ β}$ defined and decreasing. When $β > 1$ is a successor ordinal the induction step is immediate from the assumption that $HH x ≤ H x$, for every $x ∈ P$. Indeed, in this case $K^β x = HK^γ x$, where $γ$ is the predecessor of $β$, thus $K^{β+1} x = HK^β x = HHK^γ x ≤ HK^γ x = K^β x$, by the assumption on $H$ with $K^γ x$ in place of $x$. We now show that $K^{β+1} x ≤ K^β x$ also when $β$ is a limit ordinal. For every $γ$ with $0 < γ < β$ we have $K^{β+1} x = HK^β x ≤ HK^γ x = K^{γ+1} x ≤ K^γ x$, by the definition of $K^β$, since $H$ is assumed to be isotone and since $K^β x ≤ K^γ x$. The inequality $K^{γ+1} x ≤ K^γ x$ follows from the inductive assumption. We have showed that $K^{β+1} x ≤ K^γ x$, for every nonzero $γ < β$ and this means exactly $K^{β+1} x ≤ K^β x$.

We have proved that the sequence $K^α x$, $α$ a nonzero ordinal, is decreasing, so that it eventually stabilizes. This justifies the next definition. For every $x ∈ P$ we set

$$K x = K^α x$$

(2.8)

where $α$ is the smallest nonzero ordinal such that $K^{α+1} x = K^α x$.

From (2.8) it follows that

$$HK x = HK^α x = K^{α+1} x = K^α x = K x.$$  

(2.9)

We now show that $K$ is the desired operation. Indeed, $K$ is idempotent, since if $K^{α+1} x = K^α x$, then $K^2 K x = HHK^α x = HK^α x = HK^α x = K^{α+1} K x$, thus $KK x = K^1 K x = HK x = K x$, by (2.9). Notice that, on the other hand, we are not assuming $H x ≤ x$. Moreover, $K$ is smaller than $H$, since, for every $x ∈ P$, $(K^α x)_{0 < α}$ is a decreasing sequence, thus, for $α$ as in (2.8), $K x = K^α x ≤ K^1 x = H x$. We now check that $K$ is isotone. Notice that, for every ordinal $β > 0$, each $K^β$ is isotone, by induction on $β$ and using the assumption that $H$ is isotone. If $x ∈ P$ and $K^{α+1} x = K^α x$, then $K^β x = K^α x$, for every $β ≥ α$. Thus if $x ≤ y ∈ P$, then, for some suitably large ordinal $β$, $K x = K^{β} x ≤ K^β y = K y$. Hence $K$ is isotone. Finally, if $J$
is an isotone and idempotent operation smaller than $H$, we check by
induction on $\delta > 0$ that $Jx \leq K^\delta x$, for every $x \in P$ and every nonzero
ordinal $\delta$. It follows that $J$ is smaller than $K$. The base case $\delta = 1$ is
exactly the assumption that $J$ is smaller than $H$, while the limit case is
immediate from the inductive hypothesis. To prove the successor step,
notice that if $Jx \leq K^\delta x$, then $Jx = JJx \leq JK^\delta x \leq HK^\delta x = K^{\delta+1}x$,
by idempotency and isotony of $J$, and again by the assumption that $J$
is smaller than $H$.
Thus $K$ is the largest isotone idempotent operation among those
smaller than $H$.  \hfill $\Box$

The assumption $HHx \leq Hx$ is necessary in Lemma 2.2; see Remark
6.1(a) below. It is necessary to iterate $H$ in the proof of Lemma 2.2;
see Remark 6.1(c) below.

**Proposition 2.3.** Suppose that $P$ is a bounded complete lattice.

If $\{ K_i \mid i \in I \}$ is a family of isotone and idempotent operations on
$P$, then there is the largest isotone idempotent operation $K$ which is
smaller than each $K_i$.

In other words, the set of all the isotone and idempotent operations
on $P$ is a complete bounded lattice, under the ordering $\leq_p$ given by
$J \leq_p K$ if $J$ is pointwise smaller than $K$.

**Proof.** For $x \in P$, define $Hx = \prod_{i \in I} K_ix$.

For every $i \in I$, we have $HHx \leq K_iHx \leq K_iK_ix = K_ix$, since
$Hx \leq K_ix$, for every $x \in P$ (in particular, we can take $Hx$ in place
of $x$) and then using isotony and idempotency of $K_i$. We have showed
that $HHx \leq K_ix$, for every $i \in I$, hence $HHx \leq \prod_{i \in I} K_ix = Hx$.

Since each $K_i$ is isotone, then $H$ is isotone, hence we can apply
Lemma 2.2 in order to get the largest operation $K$ among those iso-
tone and idempotent operations which are smaller than $H$. Now the
conclusion follows from the fact that, by the definition of $H$, some op-
eration $K$ is smaller than $H$ if and only if $K$ is smaller than all the
$K_i$s.  \hfill $\Box$

The next theorem is a collection of folklore results, but some of them
can be hardly found explicitly mentioned in the literature.

**Theorem 2.4.** The classes of partially ordered sets, meet semilattices,
join semilattices, lattices, Boolean algebras and Heyting algebras all
have the superamalgamation property. The same applies to the classes
of finite such structures.

**Proof.** Though not explicitly stated, the proof of Lemma 3.3 in [13]
provides the superamalgamation property for posets. Full details are
presented, for example, in [21, Proposition 2.1]. For short, if \( A, B \) and \( C \) are posets to be amalgamated, then \( D = (A \cup B, \leq_D) \) is a superamalgamating structure, where \( \leq_D \) is the smallest transitive relation containing \( \leq_A \cup \leq_B \). The key argument in [13, Lemma 3.3] amounts to show that such a relation equals \( \leq_A \cup \leq_B \cup (\leq_A \circ \leq_B) \cup (\leq_B \circ \leq_A) \), where, say, \((a, b) \in \leq_A \circ \leq_B \) means that there is \( c \in A \cap B = C \) such that \( a \leq_A c \) and \( c \leq_B b \).

An argument in the proof\(^1\) of [13, Theorems 3.5] on page 205 then goes on by showing that if \( C \) is a lattice, then all existing meets and joins in \( A \) are preserved in \( D \), as constructed above, and similarly for existing meets and joins in \( B \). In particular, if \( A, B \) and \( C \) are lattices, then meets and joins of \( A \) and \( B \) are preserved in the poset \( D \). This does not mean that \( D \) is a lattice, since, say, the meet of some \( a \in A \) with some \( b \in B \) might not exist in \( D \). However, any poset can be embedded into some lattice in such a way that existing meets and joins are preserved, thus \( D \) can be order-embedded into some lattice \( L \) in such a way that \( A \) and \( B \) are lattice-embedded into \( L \) (the argument can be reformulated in terms of partial lattices, see [7, p. 455]). This proves strong amalgamation for lattices and, as noticed in the second paragraph of [3], the same argument works for semilattices. Since the definition of superamalgamation deals only with the order structure, the fact that \( D \) superamalgamates \( A \) and \( B \) over \( C \) as posets implies that \( L \) superamalgamates \( A \) and \( B \) over \( C \) as lattices, respectively, semilattices.

Thus we know that the classes of posets, lattices and semilattices all have the superamalgamation property.

Boolean algebras and Heyting algebras are well-known to have the superamalgamation property; see, e.g., [5]. In detail, [5, Theorem 6.1], among other, proves that a variety \( V \) of Heyting algebras has the amalgamation property if and only if \( V \) has the superamalgamation property. Notice that Heyting algebras are generally called pseudoboolean algebras in [5]. Theorem 6.1 in [5] is stated in terms of superintuitionistic logics; however, there is a bijective correspondence between superintuitionistic logics and varieties of Heyting algebras; see [5, p. 172]. Then [5, Proposition 6.8] goes on by showing that the variety of

\(^1\)The arguments in the proof of [13, Theorem 3.5] essentially amount to the proof of the strong amalgamation property (condition IV in the terminology from [13]) for the class of lattices; all the rest is a consequence of the former Theorem 2.11 in [13]. In particular, [13, Theorem 3.5] is stated under the assumption of the Generalized Continuum Hypothesis, but the assumption is not used directly in most arguments in the proof, it is used only when relying to 2.11.
all Heyting algebras has the amalgamation property. In \cite{5} the amalgamation property for Boolean algebras is obtained again from Theorem 6.1 therein; see \cite{5, p. 177}.

Superamalgamation for Boolean algebras can also be obtained more directly as a consequence of \cite{18, Corollary 11.22}. There it is shown that if $D$ is the free product of two Boolean algebras $A$ and $B$ with amalgamated subalgebra $C$, then, for every $a \in A$ and $b \in B$ such that $ab = 0$ in $D$, there is $c \in C$ such that $a \leq c$ and $b \leq -c$. Taking $-b$ in place of $b$, $a(-b) = 0$ means $a \leq b$ and then the mentioned corollary gives $-b \leq -c$, that is, $c \leq b$.

The last statement follows from the fact that all the above constructions preserve finiteness (this needs to be checked: for example, let $T$ be the theory in the pure language of equality asserting that if there are more than two objects, then there are infinitely many objects. Then $T$ has the strong amalgamation property, but the class of the finite models of $T$ has the amalgamation property but not the strong amalgamation property).

\section*{3. An extension lemma}

The next lemma is a broad generalization of earlier results, e. g., \cite[Lemma 2.3]{23}, \cite[Theorems 1 - 4]{27}, \cite[Proposition 3]{19}, \cite[Theorem 3.19]{2}, \cite[Lemma 8.1]{26}. The present treatment has several advantages over earlier versions: first, no assumption is made on the subset $D$; second, extensiveness is not necessarily assumed, though its presence simplifies statements and proofs. Finally, we do not need to work with complete Boolean algebras. In the simpler case it is enough to have just an arbitrary poset. In the general case a complete bounded lattice is enough and no form of distributivity is necessary. On the other hand, distributivity seems to be necessary when dealing with additive operations, as we plan to show in a further work.

The general form of the next lemma, in particular, the fact that no special assumption is imposed on $D$, will prove particularly useful in Section \cite{5} below, where it will lead to some decidability results.

In the next lemma we shall consider a poset $P$, a subset $D$ of $P$ and a function $G : D \to P$. We shall deal with the following properties of $G$.

\begin{align*}
\text{for all } a \in D, \text{ if } Ga \in D, \text{ then } GGa &= Ga, \quad (3.1) \\
\text{for all } a \in D, \text{ if } Ga \in D, \text{ then } GGa &= a, \quad (3.2) \\
\text{for all } a, b \in D, \text{ if } Ga = Gb, \text{ then } a &= b, \quad (3.3) \\
\text{for all } a, b \in D, \text{ if } a \leq b \text{ then } Ga &\leq Gb, \quad (3.4)
\end{align*}
for all $a, b \in D$, if $a \leq Gb$ then $Ga \leq Gb$. \hfill (3.5)
for all $a, b \in D$, if $Ga \leq b$ then $Ga \leq Gb$. \hfill (3.6)
for all $a, b \in D$, if $a \leq b$ then $Gb \leq Ga$. \hfill (3.7)

**Lemma 3.1.** Suppose that $P$ is a partially ordered set, $D \subseteq P$ and $G : D \rightarrow P$ is a function. Then, for each line in the following table, $G$ can be extended to an operation $K$ on $P$ satisfying the properties listed in the second column if and only if $G$ satisfies the properties listed in the third column.

| $G$ can be extended to $K$ such that | if and only if |
|-------------------------------------|----------------|
| A1e $K$ is extensive | $G$ is extensive |
| A1c $K$ is contractive | $G$ is contractive |
| A2 $K$ is idempotent | $G$ satisfies (3.1) |
| A2e $K$ is idempotent and extensive | $G$ satisfies (3.1) and is extensive |
| A2c $K$ is idempotent and contractive | $G$ satisfies (3.1) and is contractive |
| A3 $K$ is an involution | $G$ satisfies (3.2) and (3.3) |

and, under the further assumption that $P$ is a bounded complete lattice,

| $G$ can be extended to $K$ such that | if and only if |
|-------------------------------------|----------------|
| B1 $K$ is isotone | $G$ satisfies (3.4) |
| B1e $K$ is isotone and extensive | $G$ satisfies (3.4) and is extensive |
| B1c $K$ is isotone and contractive | $G$ satisfies (3.4) and is contractive |
| B2 $K$ is isotone and idempotent | $G$ satisfies (3.4), (3.5) and (3.6) |
| B3 $K$ is a closure operation | $G$ satisfies (3.5) and is extensive |
| B4 $K$ is an interior operation | $G$ satisfies (3.6) and is contractive |
| B5 $K$ is antitone | $G$ satisfies (3.7) |

Suppose further that $n \in \mathbb{N} \setminus \{0\}$, $i, j \in \mathbb{N}$, $i + j \leq n$, $X \subseteq P^n$ and $V : X \rightarrow P$ is a function. Then $V$ can be extended to an $n$-ary operation $F$ on $P$ such that

(C1) $F$ is isotone on the first $i$ components and antitone on the last $j$ components if and only if, for all $n$-uples $\bar{a} = \{a_1, \ldots, a_n\}$, $\bar{b} =$
\begin{align*}
\{b_1, \ldots, b_n\} &\in X, \\
& a_h \leq b_h, \text{ for every } h \leq i, \ a_h = b_h, \text{ for every } h \text{ with } i < h \leq n - j, \\
& \text{and } a_h \geq b_h, \text{ for every } h > n - j \quad \text{imply} \quad V\bar{a} \leq V\bar{b}. \\
\end{align*}
(3.8)

(C2) If \( i \geq 1 \), in (C1) we can additionally obtain \( x_1 \leq F\bar{x} \) for all \( \bar{x} \), provided \((3.8)\) holds and \( a_1 \leq V\bar{a} \), for all \( \bar{a} \in X \).

Similarly for any subset of the first \( i \) variables.

(C3) More generally, given an \( i \)-ary lattice term \( t \), in (C1) we can additionally obtain \( t(x_1, \ldots, x_i) \leq F\bar{x} \), provided \((3.8)\) holds and \( t(a_1, \ldots, a_i) \leq V\bar{a} \), for every \( n \)-uple \( \bar{a} \in X \).

In cases (B1) - (C3) the assumption that \( P \) is a bounded complete lattice can be weakened to “every subset of the range of \( G \) (resp., of \( V \)) has a meet in \( P \)”, except for cases (B1c) and (B4), in which cases the assumption can be weakened to “every subset of the range of \( G \) has a join in \( P \)”, and for case (C3) in which, by definiteness, we must further assume that \( P \) is a lattice.

**Proof.** In each case, necessity is immediate, without any assumptions on the completeness of \( P \), assuming that \( P \) is a lattice in case (C3).

Notice also that the order structure on \( P \) is not relevant in case (A3).

We now prove sufficiency.

(A1e) - (A2c) In each case define \( K : P \rightarrow P \) by

\begin{align*}
Kx &= Gx \quad \text{if } x \in D, \quad (3.9) \\
Kx &= x \quad \text{if } x \notin D. \quad (3.10)
\end{align*}

Clause \((3.9)\) assures that \( K \) extends \( G \). Moreover, \( K \) is extensive or contractive if and only if so is \( G \). Assume \((3.1)\). If \( a \in D \) and \( Ka \in D \), then \( KKa = Ka \) because of \((3.9)\) and \((3.1)\). If \( a \in D \) and \( Ka \notin D \), then \( KKa = Ka \) because of \((3.10)\), which implies \( KKa = Ka \) also when \( a \notin D \). Hence \( K \) extends \( G \) and is idempotent.

(A3) Define \( K : P \rightarrow P \) by

\begin{align*}
Kx &= Gx \quad \text{if } x \in D, \quad (3.11) \\
Kx &= a \quad \text{if } Ga = x, \text{ for some } a \in D, \quad (3.12) \\
Kx &= x \quad \text{otherwise, that is, neither } x \in D, \text{ nor } x \in \text{Im } G. \quad (3.13)
\end{align*}

Conditions \((3.12)\) and \((3.3)\) imply that the outcomes of \( K \) agree in all the overlapping cases. Clause \((3.11)\) implies that \( K \) extends \( G \) and \( K \) is an involution by construction.

(B1) Define, for each \( x \in P \),

\[ Kx = \prod \{Gb \mid b \in D, x \leq b\}. \quad \text{(Case B1)} \]
By definition, \( K \) is isotone. It remains to show that \( K \) extends \( G \). So let \( x \in D \). We can take \( b = x \) in the set defining \( Kx \), hence \( Kx \leq Gx \). On the other hand, for every \( b \in D \) such that \( x \leq b \), we have \( Gx \leq Gb \), by (3.4), hence \( Gx \leq \prod \{ Gb \mid b \in D, x \leq b \} \). Thus \( Kx = Gx \).

If \( G \) is extensive, then \( K \) is extensive. If \( G \) is contractive, then \( K \), as defined, need not be contractive. However, we can get a contractive \( K \) by performing the dual construction, namely, defining \( Kx = \sum \{ Gb \mid b \in D, b \leq x \} \).

(B2) For \( x \in P \) let \( Hx = \prod \{ Gb \mid b \in D \text{ and either } x \leq b \text{ or } x \leq Gb \} \) \hspace{1cm} (3.14)

By definition, \( H \) is isotone. We now check that \( Hx \leq Hx \), for every \( x \in P \). \hspace{1cm} (3.15)

Indeed, if \( Gb \) belongs to the defining set for \( Hx \) in (3.14), then \( Hx \leq Gb \), by the very definition of \( Hx \); thus if in (3.14) we consider \( Hx \) in place of \( x \), then \( HHx \leq Gb \), since \( Hx \leq Gb \). We have showed that \( HHx \) is \( \leq \) than every factor in the set whose meet gives \( Hx \), hence it follows that \( HHx \leq Hx \).

Let \( K \) be the operation defined in (2.8) in the proof of Lemma 2.2, with respect to \( H \) defined in (3.14) above: recall that \( K \) is obtained by iterating \( H \). By Lemma 2.2 \( K \) is isotone and idempotent, hence it remains to check that if \( x \in D \), then \( Kx = Gx \). We first prove that if \( x \in D \), then \( Hx = Gx \). Indeed, suppose that \( x \in D \). If \( x \leq b \in D \), then \( Gx \leq Gb \), by (3.4). If \( x \leq Gb \) and \( b \in D \), then \( Gx \leq Gb \), by (3.5). Thus \( Gx \leq Hx = \prod \{ Gb \mid b \in D \text{ and either } x \leq b \text{ or } x \leq Gb \} \). Also \( Gx \geq Hx \), since we can take \( b = x \) in the defining set in (3.14). Hence \( Hx = Gx \).

We now prove that if \( x \in D \), then \( HHx = Hx \). By (3.15), \( HHx \leq Hx \). On the other hand, using the already proved fact that, for \( x \in D \), \( Gx = Hx \), we get that if \( b \in D \) and either \( Hx \leq b \) or \( Hx \leq Gb \), then \( Hx = Gx \leq Gb \) in each case, applying (3.6) in the former eventuality. Hence \( Hx \leq \prod \{ Gb \mid b \in D \text{ and either } Hx \leq b \text{ or } Hx \leq Gb \} = HHx \).

We have proved that if \( x \in D \), then \( Gx = Hx = HHx \). According to the definition in (2.7) in the proof of Lemma 2.2 this means that \( Gx = K^1 x = K^2 x \), thus \( Kx = K^1 x = Gx \), by the definition of \( K \) in (2.8).

(B3) follows from the proof of (B2), since if \( G \) is extensive, then \( H \), as defined by (3.14), turns out to be extensive, as well, hence \( K \) given by (2.8) is extensive. Since condition (3.6) is satisfied by an extensive \( G \) and, moreover, (3.5) implies (3.4) when \( G \) is extensive, then we get (B3).
Let us mention, however, that a direct proof of (B3) is much simpler, just define
\[
Kx = \prod \{ Gb \mid b \in D, x \leq Gb \}
\]
(Case B3)

The operation \( K \) is isotone and extensive. If \( b \in D \) and \( x \leq Gb \), then \( x \leq Kx \leq Gb \). Thus, for every \( b \in D \), \( x \leq Gb \) if and only if \( Kx \leq Gb \), that is \( KKx = Kx \), namely, \( K \) is idempotent.

Because of (3.5), if \( x \in D \), then \( Gx \leq Kx \). Conversely, since \( G \) is assumed to be extensive, we can take \( b = x \) in the defining set for \( Kx \), getting \( Kx \leq Gx \). Thus \( Kx = Gx \), for \( x \in D \), that is, \( K \) extends \( G \).

(Compare [23, Lemma 2.3], [2, Theorem 3.19] and [26, Lemma 8.1].)

(B3) is dual to (B3). In detail, we can choose
\[
Kx = \sum \{ Gb \mid b \in D, Gb \leq x \}
\]
in order to get an interior operation extending \( G \).

(B5) Define
\[
Kx = \prod \{ Gb \mid b \in D, b \leq x \}.
\]
(Case B5)

The operation \( K \) is antitone. Arguing as in case (B1) we see that \( K \) extends \( G \). Indeed, if \( x \in D \), then, taking \( b = x \) in the set defining \( Kx \), we get \( Kx \leq Gx \). If \( b \in D \) and \( b \leq x \), we have \( Gx \leq Gb \), by (3.7), hence \( Gx \leq Kx \).

(C1) It is enough to define
\[
F \bar{x} = \prod \{ Vb \mid \bar{b} \in X, x_h \leq b_h, \text{ for } h \leq i, x_h = b_h, \text{ for } h \text{ with } i < h \leq n - j, \text{ and } x_h \geq b_h, \text{ for } h > n - j \}.
\]
(Case C1)

(C2) follows from the above definition in (Case C1).

More generally (C3) follows from the fact that lattice terms are isotone on each component. Indeed, fix \( \bar{x} \in P^n \) and \( X^* \subseteq X \). If \( x_1 \leq b_1, \ldots, x_i \leq b_i \) and \( t(b_1, \ldots, b_i) \leq V\bar{b} \), for every \( \bar{b} \in X^* \), then \( t(x_1, \ldots, x_i) \leq t(b_1, \ldots, b_i) \leq V\bar{b} \), for every \( \bar{b} \in X^* \). Hence \( t(x_1, \ldots, x_i) \leq \prod \{ V\bar{b} \mid \bar{b} \in X^* \} \).

The last statement follows from the proof, since we have always taken meets (or joins) of subsets of the range of \( G \) or \( V \).

Some completeness assumptions are necessary in Lemma 3.1 in cases (B1) - (C3); see Remark 6.3(b) - (e).
By a comparability condition between two, say, unary operations \( H \) and \( K \) on the same set \( X \), we mean a condition of the form \( Hx \leq Kx \) for every \( x \in X \). Namely, a comparability condition is always a \( \leq \)-condition, we shall not deal with \( < \)-conditions here.

Lemma 3.1 holds when applied simultaneously to two or more partial functions, and comparability conditions can be preserved. This is the content of Lemma 3.2 below. In most cases the result is immediate from the proof of 3.1, however, some details need to be worked out in cases (A3), (B2) - (B4).

Lemma 3.2. Assume the hypotheses in Lemma 3.1.

(1) For each item (A1e) - (B5), assume that \( G_\circ : D \to P \) and \( G_* : D \to P \) are functions satisfying the corresponding sufficient condition. In case (A3) assume further that \( G_\circ b \in D \) and \( G_* b \in D \), for every \( b \in D \).

Under the above assumptions, if \( G_\circ b \leq G_* b \), for every \( b \in D \), then there are operations \( K_\circ \) and \( K_* \) defined on the whole of \( P \), extending respectively \( G_\circ \) and \( G_* \), satisfying the corresponding condition in the middle column in the tables in Lemma 3.1 and such that \( K_\circ x \leq K_* x \), for every \( x \in P \).

(2) For some fixed \( n \) and some \( X \subseteq P^n \), assume that \( V_\circ : X \to P \) and \( V_* : X \to P \) are functions satisfying the condition in Lemma 3.1(C), and let \( F_\circ \) and \( F_* \) be defined as in the proof of 3.1.

If \( V_\circ \bar{b} \leq V_* \bar{b} \), for every \( \bar{b} \in X \), then \( F_\circ \bar{x} \leq F_* \bar{x} \), for every \( \bar{x} \in P^n \).

Proof. In cases (A1e) - (B1c), (B5) let \( K_\circ \) and \( K_* \) be correspondingly defined as in the proof of Lemma 3.1. The conclusion is elementary in cases (A1e) - (A2c) and follows from the definition of \( K \) in cases (B1) - (B1c) and (B5). Say, in case (B1), we have \( K_\circ x = \prod \{ G_\circ b \mid b \in D, x \leq b \} \) and \( K_* x = \prod \{ G_* b \mid b \in D, x \leq b \} \). Since \( G_\circ b \leq G_* b \), for every \( b \in D \), then every element in the set whose meet gives \( K_* \) is bounded below by some element in the set whose meet gives \( K_\circ \) and this implies \( K_\circ x \leq K_* x \). The argument is similar in case (B5), as well as in part (2) of the present lemma. For case (A3) just notice that, under the additional assumption, condition (3.12) becomes redundant, while comparability is clearly preserved by clauses (3.11) and (3.13).

We now deal with case (B3) and shall postpone the more involved case (B2). Let \( K_\circ \) and \( K_* \) be given by Case B3 in the proof of Lemma 3.1. It is not necessarily the case that \( K_\circ x \leq K_* x \), for all \( x \in P \); see Remark 6.4(b) below. Hence we need consider another operation in place of \( K_\circ \). Let \( K_*^* x = K_\circ x \land K_* x \), for \( x \in P \). It
is elementary to see that $K^*$ is a closure operation, since both $K_o$ and $K_\bullet$ are. The argument is well-known; the only nontrivial part is idempotence. Let $x \in P$ and $y = K^*_o x$. Since $x \leq y \leq K_o x$, then $K_o x \leq K_o y \leq K_o K_o x = K_o x$, hence $K_o y = K_o x$. Similarly, $K_\bullet y = K_\bullet x$, thus $K^*_o K^*_\bullet x = K^*_o y \wedge K_\bullet y = K_o x \wedge K_\bullet x = K^*_\bullet x$, proving idempotence of $K^*_\bullet$.

Since $K_o$ and $K_\bullet$ extend, respectively, $G_o$ and $G_\bullet$, we have $K^*_o b = G_o b \wedge G_\bullet b = G_\bullet b$, for every $b \in D$, since, by assumption, $G_o b \leq G_\bullet b$. Hence $K^*_o$ extends $G_o$. Thus the operations $K^*_o$ (in place of $K_o$) and $K_\bullet$ witness the conclusion of the present lemma in case (B3). Case (B4) is dual.

In order to prove case (B2) we shall use Proposition 2.3. Let $K_o$ and $K_\bullet$ be given by the proof of Lemma 3.1. As in case (B3), we need to use another operation $K^*_o$ in place of $K_o$. Let $I = \{ o, \bullet \}$ in Proposition 2.3 and let $H$ and $K$ be given by the proofs of 2.2 and 2.3. Recall from the proof of Proposition 2.3 that, in this special case, $H x = K_o x \wedge K_\bullet x$; then the proof of Lemma 2.2 is applied and $K$ is obtained by transfinitely iterating $H$ until it assumes a constant value. By Proposition 2.3, $K$ is isitone, idempotent and smaller than $K_\bullet$, so that we can conclude the proof if we show that $K$ extends $G_o$, considering $K^*_o = K$ in place of $K_o$.

From the proof of Proposition 2.3 we have $H x = K_o x \wedge K_\bullet x$, for $x \in P$. Let us fix $b \in D$. By assumption, $G_o b \leq G_\bullet b$, thus $K_o b \leq K_\bullet b$ and $H b = K_o b \wedge K_\bullet b = K_\bullet b = G_\bullet b$, since $K_o$ and $K_\bullet$ extend, respectively, $G_o$ and $G_\bullet$.

We want to show that $H H b = H b$, so that (2.7) and (2.8) give $K b = H b = G_\bullet b$, the conclusion we need. We first recall the definition of $K_\bullet$ from the proof of Lemma 3.1. In detail, $K_\bullet$ is obtained by iterating the operation introduced in (3.14), recalled below with suitable relabelings:

\[ H_\bullet x = \prod \{ G_\bullet d \mid d \in D \text{ and either } x \leq d \text{ or } x \leq G_\bullet d \} . \quad (3.16) \]

We first check that $H_\bullet G_o b \geq G_o b$. So let $x$ be $G_o b$ in (3.16). If $G_\bullet d$ belongs to the set in (3.16) because $x = G_o b \leq d$, then $G_o b \leq G_\bullet d$, since $G_o$ is assumed to satisfy (3.6). Thus $G_o b \leq G_\bullet d \leq G_\bullet b$, by the comparability assumption between $G_o$ and $G_\bullet$. Otherwise, $G_\bullet d$ belongs to the set in (3.16) because $x = G_o b \leq G_\bullet d$; in conclusion, $H_\bullet G_o b$ is obtained as the meet of a set of elements which are all $\geq G_o b$, hence $H_\bullet G_o b \geq G_o b$. Since, recalling (2.7), $K_\bullet$ is obtained by iterating transfinitely $H_\bullet$, we get $K^\alpha_o G_o b \geq G_o b$, for every nonzero ordinal $\alpha$, since $H_\bullet$ is isitone, hence $K^\alpha_o G_o b \geq G_o b$. 

We now compute $HHb = HK\circ b = K\circ K\circ b \wedge K\circ G\circ b = G\circ b \wedge K\circ G\circ b = G\circ b = Hb$, since we already know that $Hb = G\circ b = K\circ b$, then using the definition of $H$, idempotency of $K\circ$ and the just proved inequality $K\circ G\circ b \geq G\circ b$.

Since in the above argument $b$ was an arbitrary element of $D$, we get $HHb = Hb = G\circ b$, for every $b \in D$. As remarked above, this means that $K$ extends $G\circ$, thus $K^* = K$ (in place of $K\circ$) and $K^*$ satisfy the desired conclusion. □

The additional assumption is needed in Lemma 3.2 in case (A3). See Remark 6.4(a) below.

Remark 3.3. We have stated Lemma 3.2 for just two operations only for simplicity: the analogue of Lemma 3.2 holds for any family of operations and an arbitrary set of comparability conditions. In detail, assume the hypotheses of Lemma 3.1 and fix some item (W) chosen from (A1) - (C3) in the statements there. Let $(Z, \leq)$ be a partially ordered set of indices, and $(G_z)_{z \in Z}$ be a $Z$-indexed sequence of functions from $D$ to $P$ such that, say in the unary case, $G_z(b) \leq G_{z'}(b)$, for every $b \in D$ and $z \leq z' \in Z$. In case (A3) assume further that $G_z(b) \in D$, for every $b \in D$ and $z \in Z$.

If each $G_z$ satisfies the condition on the right in (W), then there is a way of extending simultaneously each $G_z$ to a total function $K_z$ on $P$ in such a way that each $K_z$ satisfies the corresponding condition and moreover $K_z(x) \leq K_{z'}(x)$, for every $x \in P$ and $z \leq z' \in Z$.

This is proved as in Lemma 3.2 in cases (A1e) - (B1c), (B5) - (C3), since in such cases we can always consider the functions $K_z$ or $F_z$ provided by the proof of Lemma 3.1. To prove case (B3), let the functions $K_z$ ($z \in Z$) be given by Lemma 3.1. For every $x \in P$ and $z \in Z$, define

$$K_z^* x = \prod \{ K_z x \mid z' \in Z, z \leq z' \}. \quad (3.17)$$

The same arguments as in the proof of Lemma 3.2 show that each $K_z^*$ is a closure operator which extends $G_z$. The desired comparability conditions follow directly from (3.17), since if $z \not\leq w \in Z$, then the set defining $K_z^* x$ is contained in the set defining $K_z^* x$. Case (B4) is dual.

In case (B2) the equation (3.17) should be used to define some (not necessarily idempotent) functions $H_z^*$ ($z \in Z$) which need to be finitely iterated as in the proofs of Lemmas 2.2, 3.2 and Proposition 2.3, in order to obtain isotone and idempotent operations $K_z^*$ ($z \in Z$). The above argument for case (B3) shows that $H_z^* x \leq H_w^* x$, for every $z \not\leq w \in Z$ and $x \in P$. Since each $H_z^*$ is isotone, then, applying the definitions in (2.7) and (2.8) from the proof of Lemma 2.2 to both $H_z^*$ and $H_w^*$, we get $K_z^* x \leq K_w^* x$. 

The comparability assumptions on the $G_z$s imply that $H_z^* b = G_z b$, for every $z \in Z$ and $b \in D$. The arguments in the proof of Lemma 3.2 in this same case (B2) show that $K_w G_z b \geq G_z b$, for every $z \preceq w \in Z$ and $b \in D$ (here the index $w$ is in place of • and the index $z$ is in place of $\circ$). From $H_z^* b = G_z b$ and by the definition of $H_z^*$ we get $H_z^* H_z^* b = H_z^* G_z b = \prod \{ K_w G_z b \mid w \in Z, z \preceq w \} \geq G_z b = H_z^* b$. From the proof of Proposition 2.3 we have $H_z^* H_z^* b \leq H_z^* b$, hence $H_z^* H_z^* b = H_z^* b = G_z b$ and (2.8) gives $K_z^* b = H_z^* b = G_z b$, thus $K_z^*$ extends $G_z$, concluding the proof of case (B2).

**Definition 3.4.** In view of Lemma 3.1 we shall consider the following properties of a unary operation in an ordered structure: (A1e) extensive, (A1c) contractive, (A2) idempotent, (A2e) idempotent and extensive, (A2c) idempotent and contractive, (A3) involutive, (B1) isotone, (B1e) isotone and extensive, (B1c) isotone and contractive, (B2) isotone and idempotent, (B3) a closure operation (that is, isotone, extensive and idempotent), (B4) an interior operation (that is, isotone, contractive and idempotent), (B5) an antitone operation, and, for $n$-ary operations, (C1)$_{i,j,n}$ isotone on the first $i$ components and antitone on the last $j$ components, and, possibly, (C2)$_{i,j,n,h}$ satisfying also $x_1 \leq F x, \ldots, x_h \leq F x$, for some $h \leq i$, more generally, for lattice-ordered structures, (C3)$_{i,j,n,t}$ satisfying $t(x_1, \ldots, x_i) \leq F x$, for some given lattice term $t$.

**Corollary 3.5.** Suppose that $(W)$ is any one of the properties (A1e) - (C2) listed in Definition 3.4.

1. If $Q$ is a poset with an operation satisfying $(W)$ and $\iota$ is an order-embedding of $Q$ into some bounded complete lattice $P$, then $P$ can be expanded by adding an operation satisfying $(W)$ in such a way that $\iota$ is an embedding with respect to the operation.
2. Every poset $Q$ with an operation satisfying $(W)$ can be order-embedded into some complete bounded lattice $P$ with an operation satisfying $(W)$ and in such a way that the embedding preserves the operation and all existing, possibly infinitary, meets and joins in $Q$.
3. Every poset with an operation satisfying $(W)$ can be order-embedded into some complete atomic Boolean algebra with an operation satisfying $(W)$ and in such a way that the embedding preserves the operation and all existing, possibly infinitary, meets (alternatively, joins).
4. Every distributive lattice with an operation satisfying $(W)$ can be lattice-embedded into some complete atomic Boolean lattice.
with an operation satisfying \((W)\) and in such a way that the embedding also preserves the operation.

(5) In all the above cases we can add simultaneously any number of operations, possibly of distinct arities, and possibly satisfying distinct properties chosen from \((A1e) - (C2)\). The construction can be performed in such a way that it preserves comparability conditions among operations satisfying the same property.

Proof. (1) In cases \((A1e) - (B5)\) define \(G\) on \(\iota(Q)\) by \(G(\iota(a)) = \iota(K_Qa)\), where \(K_Q\) is the given operation on \(Q\). This is a good definition, since \(\iota\) is injective. The respective conditions among \((3.1) - (3.7)\) in Lemma 3.1 are satisfied by \(G\) on \(D = \iota(Q)\), since, by assumption, they are satisfied by \(K_Q\) in \(Q\) and \(\iota\) is an order-embedding. By Lemma 3.1 \(G\) can be extended on the whole of \(P\) to an operation satisfying the desired property. With respect to this operation \(\iota\) turns out to be an embedding by the very definition of \(G\). Of course, in cases \((A1e) - (A3)\) we do not need the completeness assumption on \(P\).

In cases \((C1) - (C2)\) define \(V\) on \((\iota(Q))^n\) by \(V(\iota(a_1), \ldots, \iota(a_n)) = \iota(F_Q(a_1, \ldots, a_n))\) and argue similarly.

(2) follows from (1), since every poset can be embedded into some bounded complete lattice in such a way that existing meets and joins are preserved \([10\text{, Ch. 1, Theorems 10.6, 10.7}]\).

(3) - (4) follow similarly by known results about embeddings of ordered structures, e. g., \([10\text{, Ch. 1, Theorems 9.9, 9.10}]\) and \([7\text{, Theorem 153}]\).

(5) The constructions of the lattice \(P\) in (2) and of the Boolean lattices in (3) - (4) do not depend on the operation, hence we can add as many operations as we want at the same time. The last statement follows from Lemma 3.2 and Remark 3.3.

Item (2) (under the further assumption that \(Q\) is a lattice) and item (4) in Corollary 3.5 apply also in case \((C3)\). Clause (2) for a closure operation appears in \([2\text{, Corollary 3.20}]\). Clause (3) for meet-semilattices with a closure operation appears in \([12\text{, Proposition 3.2 and Lemma 3.4}]\).

Notice that the case of join-semilattices is not the dual case, since the dual of a meet-semilattice with a closure operation is a join-semilattice with an interior operation. In detail, if the semilattice operation is written multiplicatively, extensiveness is equivalent to \(x \cdot Kx = Kx\) in join semilattices; to \(x \cdot Kx = x\), instead, in meet semilattices.

A reduct of some structure is a structure in which some operations or relations are forgotten. A subreduct is a substructure of some reduct. It follows from Corollary 3.5(3) that, say, if \(S\) is the class of all Boolean
algebras with a closure operator, then the class of all subreducts of members of $\mathcal{S}$ to the language of posets with an operator is the class of posets with a closure operator. As another example, if we consider lattice operations as ternary relations $\lor(x, y, z)$, $\land(x, y, z)$ and $\mathcal{H}$ is the class of all lattices (in the above relational sense) with a closure operator, then the class of all substructures of members of $\mathcal{H}$ is the class of partial lattices with a closure operator: use (2). Similar consequences can be obtained in all the other cases.

4. **Superamalgamation implies amalgamation for expanded structures**

As mentioned in the section on preliminaries, we shall deal with models in the classical model-theoretical sense. Classes of models are always meant to be of the same type and closed under isomorphism. The proof of the next Theorem 4.1 can be applied without essential modifications to a somewhat broader setting, for example, dealing with topological or infinitary structures. An even more general version in a categorical setting is possible; however, details become quite cumbersome and we know no significant application; hence we shall provide details (elsewhere) if and when some applications are found.

We require embeddings to be at least order-embeddings. If we expand $\mathcal{S}$ to some class $\mathcal{S}_1$ by adding one or more operations, an embedding for $\mathcal{S}_1$ is meant to be an embedding for $\mathcal{S}$ which in addition respects the new operations, say, $\iota(Kx) = K\iota(x)$, for unary operations. When dealing with case (C3) we shall always assume that structures are lattice-ordered and that embeddings are at least lattice-embeddings.

The assumption that $\mathcal{S}$ has the superamalgamation property in the next theorem cannot be weakened, in general, to the strong amalgamation property. See Example 4.2 below.

**Theorem 4.1.** Suppose that $\mathcal{S}$ is a class of ordered structures such that

1. $\mathcal{S}$ has the superamalgamation property, and
2. every structure $F \in \mathcal{S}$ can be extended to some structure $E \in \mathcal{S}$ such that every subset of $F$ has both a meet and a join in $E$ (in particular, this applies if every structure $F$ in $\mathcal{S}$ can be embedded into some structure $E \in \mathcal{S}$ such that the order on $E$ is a complete bounded lattice).

If $\mathcal{S}_1$ is the class of expansions of structures of $\mathcal{S}$ obtained by adding a new operation satisfying some fixed property chosen from (A1e) - (C2) in Definition 3.4, then $\mathcal{S}_1$ has the superamalgamation property, in particular, the strong amalgamation property. In particular, this applies
to adding an isotone, or an extensive, idempotent, closure, interior, antitone... operation.

More generally, the same applies to expansions obtained by adding families of such operations, possibly with comparability conditions among operations satisfying the same property.

Proof. Suppose that $A$, $B$ and $C$ are structures in $S_1$ to be amalgamated and with, say, a closure operation $K$ not in the type of $S$. By the superamalgamation property of $S$, the reducts $A^-, B^-, C^-$ to the type of $S$ can be superamalgamated to some structure $F^-$. By the assumption (2) we can extend $F^-$ to some $E^-$ in $S$ such that every subset of $F$, in particular, every subset of $A \cup B$ has both a meet and a join in $E^-$. We want to expand $E^-$ to some structure $E$ in $S_1$ in such a way that $E$ strongly amalgamates $A$ and $B$ over $C$. If this is possible, $K_E$ should agree with the following function $G$

$$Gd = \begin{cases} 
K_A d & \text{if } d \in A, \\
K_B d & \text{if } d \in B 
\end{cases} \quad (4.1)$$

defined on $D = A \cup B$. Notice that $K_A$ and $K_B$ agree on $C = A \cap B$, by the assumptions in the hypothesis of the strong amalgamation property.

We shall use the superamalgamation property to check that $G$, as given by (4.1), satisfies the assumptions of Lemma 3.1 in this specific instance, extensiveness and the condition (3.5). $G$ is obviously extensive, since both $K_A$ and $K_B$ are. To prove (3.5), first assume that $a \in A \setminus B$ and $b \in B \setminus A$. If $a \leq Gb$ in $E^-$, then $a \leq Gb$ in $F^-$, as well, since $Gb = K_B b \in B \subseteq F^-$. Since $F^-$ superamalgamates $A^-$ and $B^-$ over $C^-$, there is $c \in C$ such that $a \leq_A c$ and $c \leq_B Gb$. Then $Ga = K_A a \leq_A K_A c = Gc$, since $a, c \in A$ and $K_A$ is a closure operation on $A$. Similarly, from $c \leq_B Gb$, that is, $c \leq_B K_B b$, we get $Gc = K_B c \leq_B K_B K_B b = K_B b = Gb$, since $K_B$ is a closure operation on $B$. Since we assume that the embeddings from $A^-$ and $B^-$ to $E^-$ are at least order-embeddings, from $Ga \leq_A Gc$ and $Gc \leq_B Gb$ we get $Ga \leq Gc \leq Gb$ in $E^-$, hence $Ga \leq Gb$ by transitivity of $\leq$.

The case $b \in A \setminus B$, $a \in B \setminus A$ is symmetrical, while the cases when $a, b \in A$ or $a, b \in B$ follow from the assumption that $K_A$, $K_B$ are closure operations on $A$, $B$. We have proved (3.5) for $G$.

We have showed that $D$ and $G$, as chosen, satisfy the assumptions in Lemma 3.1(B3), hence $G$ can be extended to a closure operation $K$ on the whole of $E$. By (4.1), the expansion of $E^-$ obtained by adding
$K$ superamalgamates $A$ and $B$ over $C$, since we already know that $E^-$ superamalgamates $A^-$ and $B^-$ over $C^-$. The other cases in (A1e) - (C2) are entirely similar. In cases (C1) - (C2) take $X = A^n \cup B^n$. The point is that, in the conditions (3.4) - (3.8), each inequality involves only one element on the right and one element on the left, so that we can apply the superamalgamation property. In cases (A1e) - (A3) it is enough to assume the strong amalgamation property (this is necessary for $G$ to be well-defined). The proof is similar to the above arguments. For case (A3) notice that if both $K_A$ and $K_B$ are involutions, $a \in A$, $b \in B$ and $Ga = Gb$, then $Ga = Gb \in A \cap B = C$, say, $Ga = Gb = c \in C$. Then $a = K_A K_A a = K_A Ga = K_A c = K_B c = K_B K_B b = b$. Thus (3.3) holds in $D$, for $G$ defined by (4.1).

In passing, we remark that in a further work we shall see that, when dealing with additive operations, we get conditions involving more than one element on the right-hand side of the inequalities. We can adapt the above arguments anyway, by assuming a notion stronger than superamalgamation. We shall present details elsewhere.

Finally, the construction of $E^-$ does not depend on $K$, or on the other additional operations, hence we can repeat the above argument for as many operations as we want. Comparability conditions between operations satisfying the same property are preserved by Definition (4.1), hence can be maintained in view of Lemma 3.2 and Remark 3.3. Notice that the additional condition in the second sentence of Lemma 3.2(1) is verified here, since $D = A \cup B$, hence $d \in D$ implies $Gd \in D$, where $G$ is defined as in (4.1).

As in the last paragraph of Lemma 3.1 we only need assume that every subset of $F$ has a meet in $E$ in clause (2) in Theorem 4.1 unless we deal with cases (B1c) or (B4). Theorem 4.1 holds also in case (C3), under the assumption that $S$ is a class of lattice-ordered structures and that embeddings preserve the lattice operations.

In the following examples we show that the assumption (1) is necessary in Theorem 4.1.

**Examples 4.2.** Recall that a class of structures closed under isomorphism has the amalgamation property (AP) if, under the assumptions in Definition 2.1, we only obtain the weaker conclusion that there are a model $D$ and embeddings $\iota : A \to D$ and $\kappa : B \to D$ which agree on $C$. The difference is that we do not necessarily assume that $\iota$ and $\kappa$ are inclusions, in other words, possibly, some elements of $A$ need to be identified with elements of $B$. 

(a) It is then elementary to see that we need the strong amalgamation property in the hypothesis (1) of Theorem 4.1; AP alone does not suffice. Indeed, if SAP fails, additional operations might behave differently on elements to be identified, hence it is not possible to embed both $A$ and $B$ into the same structure. For example, let $C$ be a distributive lattice with some operation $K$ and with some element $c \in C$ which has no complement in $C$. Suppose that $A$ and $B$ are extensions of $C$ in which $c$ has a complement, call such complements $a$ and $b$, respectively. In any amalgamating structure in the class of distributive lattices, $a$ and $b$ should be identified, since complements are unique in distributive lattices. But if, say, $Ka = a$ in $A$ and $Kb \neq b$ in $B$, then it is not possible to embed $A$ and $B$ into the same structure. The argument applies to most kinds of operations; exceptional cases occur only when there are very tight assumptions on $K$. For instance, if in the above example we assume that $K$ is an isotone involution, then necessarily both $Ka = a$ and $Kb = b$, hence amalgamation is possible.

The above argument also explains why we need to deal with embeddings. The argument shows that we need to deal with, at least, injective homomorphisms. However, the class of posets with injective order preserving functions does not have AP, in the categorical sense from [17]. Indeed, if $C$ has two incomparable elements $c$ and $d$, we set $c \leq d$ in $A$ (this is compatible with the assumption that we deal with injective order preserving functions) and $d \leq c$ in $B$, then (the images of) $c$ and $d$ should be equal in any amalgamating structure $D$, thus injectivity is lost.

(b) A more involved example shows that we do need the superamalgamation property in the hypothesis (1) of Theorem 4.1; SAP is not enough. The classes of linearly ordered sets (linearly ordered sets with one isotone operation) have the strong amalgamation property \[20\] Theorem 3.1(a)], but not the superamalgamation property. Every linearly ordered set can be embedded into a complete bounded linearly ordered set, hence every linearly ordered set with one isotone operation can be embedded into a complete bounded linearly ordered set with one isotone operation, by Corollary 3.5(1).

On the other hand, the class of linearly ordered sets with two isotone operations does not have the amalgamation property \[20\] Theorem 3.1(c)]. This example shows that assumption (1) is necessary in Theorem 4.1 and cannot be weakened to the strong amalgamation property: take $S$ to be the class of linearly ordered sets with one isotone operation and let $S_1$ be obtained by adding another isotone operation.

The proof of Theorem 3.1(c) in \[20\] actually gives counterexamples for all cases (B1) - (B4). Indeed, the example in (c)(i) there provides
a triple of linearly ordered sets with two closure operations and which has no amalgamating model in the class of linearly ordered sets with two isotone operations. By [20, Lemma 5.1], the classes of linearly ordered sets with an isotone and extensive (isotone and idempotent, closure) operation have the strong amalgamation property. Notice that in [20] we used different terminology: isotone operations are called *order preserving* there, and we used *increasing* in place of extensive. Arguing as above, the counterexample in [20, Theorem 3.1(c)] takes care simultaneously of (B1), (B1e), (B2), (B3), while (B1c), (B4) are dual.

(c) To deal with case (B5), in [20, Remark 4.4] we noticed that the class of linearly ordered sets with an antitone operation with a fixed point (called a *center* in [20]) has the strong amalgamation property. On the other hand, the class of linearly ordered sets with two antitone operations, even with a common fixed point, does not have AP ([20, Theorem 4.3(b)]). Then argue as above.

(d) Notice that cases (B1), resp., (B1e), are the special unary cases of (C1) with $i = 1$, resp., of (C2). Moreover, for lattice ordered structures, case (C2) is the special case $t(x_1, \ldots, x_i) = x_1$ of (C3). Since the counterexample in (b) above is a linearly ordered set, in particular, a lattice, we get that the superamalgamation property is necessary also in cases (C1) - (C3).

(e) On the other hand, as we mentioned in the proof of Theorem 4.1, the strong amalgamation property is sufficient for cases (A1e) - (A3).

If $\mathcal{H}$ is a class of finitely generated structures, a Fraïssé limit of $\mathcal{H}$ is a countable universal homogeneous structure of age $\mathcal{H}$. For example, the ordered set of the rationals is the Fraïssé limit of the class of finite linearly ordered sets, the random graph is the Fraïssé limit of the class of finite graphs. See [11, Section 7.1] for details.

The joint embedding property is a necessary condition for the existence of a Fraïssé limit. Recall that a class $\mathcal{H}$ has the joint embedding property (JEP) if, for every $A, B \in \mathcal{H}$, there are a structure $E \in \mathcal{H}$ and embeddings $\iota : A \to E$ and $\kappa : B \to E$. For classes of structures in which it makes sense to consider empty structures, for example, posets or semilattices with no constant in the language, the amalgamation property implies JEP. On the other hand, for example, nontrivial Boolean algebras with additional operations generally do not have the joint embedding property since it may happen that $K^0 = 0$ in some algebra of the class, while $K^0 \neq 0$ in some other algebra.

However, given a class $\mathcal{S}$ with the amalgamation property, setting $A \sim B$ if $A, B \in \mathcal{S}$ and $A, B$ can be embedded into a same member
of \( S \), we get an equivalence relation such that each equivalence class has AP and JEP. If furthermore \( S \) is closed under substructures in a language with at least one constant, then \( A \sim B \) if and only if the \( \emptyset \)-generated substructures of \( A \) and \( B \) are isomorphic. For example, the class of nontrivial Boolean algebras with a closure operation satisfying \( K0 = 0 \) has the joint embedding property. Notice that if \( K \) is a closure operation, then necessarily \( K1 = 1 \).

In the next corollary embeddings are meant to preserve all the operations of the structures.

Corollary 4.3. Let \( (W) \) be any one of the properties \((A1e) - (C2)\) from Definition 3.4 and let \( S \) be any one of the following classes: the class of partially ordered sets, of meet semilattices, of join semilattices, of lattices, of Boolean algebras, of Heyting algebras. In the last three cases we may also allow \((W)\) to be \((C3)\). Then the following hold.

(1) If \( S^{(W)} \) is the class of structures obtained from members of \( S \) by adding a new operation satisfying \((W)\), then \( S^{(W)} \) has the superamalgamation property, in particular, the strong amalgamation property.

(2) More generally, the superamalgamation property is maintained if we add families of operations satisfying possibly distinct properties from \((A1e) - (C3)\). Further, we may possibly add comparability conditions among operations satisfying the same property.

(3) For each of the above classes, the class of their finite members has a Fraïssé limit, under the provisions that in \((2)\) above only a finite number of operations are added and that, in the case of Boolean and Heyting algebras, we consider a subclass of \( S^{(W)} \) consisting of structures having a fixed (modulo isomorphism) \( \emptyset \)-generated substructure.

Proof. Classical and well-known results [10, Ch. 1, Theorems 10.6 and 10.7], [18, Theorem 2.1], [5, Theorem 2.39], [9, Theorem 2.3] assert that, for every choice of \( S \) as in the statement, each member of \( S \) can be embedded into a complete bounded lattice in the corresponding class. Hence the corollary follows from Theorems 2.4 and 4.1.

Item \((3)\) follows from Fraïssé Theorem [11, Theorem 7.1.2]. The finiteness assumption is needed in order to have only a countable number of nonisomorphic finite structures. Under the assumptions, the joint embedding property follows from the amalgamation property, since it turns out to be equivalent to the case when \( C \) is the fixed
∅-generated substructure, for Boolean and Heyting algebras with operations, and since we can consider amalgamation over an empty structure in the other cases.

Corollary 4.3 applies with the same proof to the classes of bounded partially ordered sets, bounded lattices, bounded meet semilattices, bounded join semilattices (if maxima and minima are required to be preserved by embeddings, for example, when they are interpreted as constants, then in item (3) we need consider a subclass of $S^{(W)}$ with a fixed ∅-generated substructure).

As a way of example, the following classes have the superamalgamation property, and the classes of their finite members have a Fraïssé limit.

1. The class of non-trivial Boolean algebras with three closure operations $K$, $K_1$, and $K_2$ such that $K0 = K_10 = K_20 = 0$ and $Kx \leq K_1x$, $Kx \leq K_2x$ hold for every $x$.
2. The class of lattices with a closure operation, an antitone unary operation and a 3-ary operation which is isotone on each component.
3. The class of posets with two 4-ary operations $F_1$ and $F_2$ which are isotone on the first two components, antitone on the last two components and are such that $F_1(x, y, z, w) \leq F_2(x, y, z, w)$ always holds.

In the next section we shall prove that the sets of universal consequences of the corresponding first-order theories are decidable in cases (1) and (3). This holds in case (2), as well, if in place of lattices we consider distributive lattices.

The list of those classes $S$ to which Theorem 4.1 applies is illustrative and not intended to be exhaustive.

Recall that some first-order theory $T$ has model companion if $T$ has the same universal consequences of some model complete theory $T^*$. If in addition $T$ has the amalgamation property, the theory $T^*$ is a model completion of $T$. See [11] for details.

**Definition 4.4.** A universal theory $T$ is locally finite if every finitely generated model of $T$ is finite. If $T$ is universal in a finite language and $T$ is locally finite then $T$ is actually uniformly locally finite, to the effect that there is a function $g_T : \mathbb{N} \to \mathbb{N}$ such that, for every $m \in \mathbb{N}$, every model of $T$ generated by $m$ elements has cardinality $\leq g_T(m)$ [28, Lemma 5]. In particular, such a $T$ has a finite number of models generated by $m$ elements, since the language of $T$ is finite.

The next proposition is folklore; compare [15, Fact 2.1].
**Proposition 4.5.** Suppose that $T$ is a consistent first-order locally finite universal theory in a finite language. If $T$ has AP and JEP, then the class of finite models of $T$ has a Fraïssé limit $M$. The first-order theory $\text{Th}(M)$ of $M$ is $\omega$-categorical, has quantifier elimination and is the model completion of $T$.

*Proof.* The proposition follows from [11, Theorem 7.1.2 and Theorem 7.4.1]. The argument showing that $\text{Th}(M)$ is the model completion of $T$ can be found in [15, Fact 2.1(3)]. There the result is stated under the stronger assumption of uniform local finiteness, but, under the hypotheses of the proposition, it is equivalent to local finiteness by the mentioned Lemma 5 in [28].

In more detail, since $T$ is universal and locally finite, AP and JEP for $T$ imply AP and JEP for the class of finite models of $T$. The hereditary property holds since $T$ is assumed to be universal. Since $T$ locally finite in a finite language, then $T$ has countably many finite models up to isomorphism. Thus Fraïssé Theorem [11, Theorem 7.1.2] provides the existence of a Fraïssé limit $M$.

As we mentioned in Definition 4.4, $T$ is uniformly locally finite, by [28, Lemma 5]. Then, by [11 Theorem 7.4.1], $\text{Th}(M)$ is $\omega$-categorical and has quantifier elimination, in particular, $\text{Th}(M)$ is model complete. The model $M$ is constructed as the union of a chain of models of $T$, hence $M$ is a model of $T$, since $T$ is universal. Hence the theory $\text{Th}(M)$ contains $T$. Conversely, if some universal sentence $\varphi$ fails in some model of $T$, then $\varphi$ fails in a finite model of $T$, since $T$ is locally finite. But every finite model of $T$ can be embedded in $M$, since $M$ is a universal model, thus $\varphi$ fails in $M$. Hence $T$ and $\text{Th}(M)$ have the same universal consequences, and this means that $\text{Th}(M)$ is the model completion of $T$, since $\text{Th}(M)$ is model complete and $T$ has AP. \hfill \square

**Corollary 4.6.** The first-order theory $T$ of join semilattices with a closure operation has model completion.

In more detail, if $M$ is the Fraïssé limit of the class of finite join semilattices with a closure operation, then the first-order theory $\text{Th}(M)$ is $\omega$-categorical, has quantifier elimination and is the model completion of $T$.

Dually, the above results apply to meet semilattices with an interior operation.

*Proof.* By Corollary 4.3 the theory of join semilattices with a closure operation has AP and, as mentioned in the proof, this implies JEP, since we can consider $C$ as an empty model. In the next lemma we show that the theory of join semilattices with a closure operation is locally finite. The result then follows from Proposition 4.5. \hfill \square
Lemma 4.7. The theory of join semilattices with a closure operation is locally finite.

Proof. Suppose that $S$ is a join semilattice with a closure operation $K$ and suppose that $S$ is generated by the elements $x_1, \ldots, x_n$. We claim that each element of $S$ can be written in the form

$$x_{j_1} \lor x_{j_2} \lor \cdots \lor x_{j_h} \lor K(x_{\ell_1,1} \lor x_{\ell_1,2} \lor \cdots \lor x_{\ell_1,h(1)}) \lor \cdots \lor K(x_{\ell_m,1} \lor x_{\ell_m,2} \lor \cdots \lor x_{\ell_m,k(m)}),$$

with $j_1, \ldots, j_h, \ell_{1,1}, \ldots, \ell_{m,k(m)} \leq n$, and where possibly $h = 0$, that is, we have a join of expressions with a closure, and possibly $m = 0$, that is, we have an expression without closures. The expression (4.2) is not ambiguous because of associativity of $\lor$. Because of commutativity and idempotence, we can assume that the $j_i$s are all distinct, and that, for each $p \leq m$, the indices $\ell_{p,1}, \ldots, \ell_{p,k(p)}$ are all distinct. Moreover, we can assume that, letting $p$ vary, the sets $\{\ell_{p,1}, \ldots, \ell_{p,k(p)}\}$ are all distinct. Hence, up to semilattice equivalence, we have at most $2n + 2^{2n}$ expressions of the form (4.2) (of course, this is an overestimated rough bound).

The join of two expressions of the form (4.2) has still the form (4.2) and can be reduced as above using associativity, commutativity and idempotence of $\lor$. It remains to show that if $\sigma$ is an expression of the form (4.2), then $K\sigma$ can be reduced to the form (4.2) by using the properties of a closure in a join semilattice. In fact we will show that

$$K(a_1 \lor \cdots \lor a_r \lor Kb_1 \lor \cdots \lor Kb_s) = K(a_1 \lor \cdots \lor a_r \lor b_1 \lor \cdots \lor b_s) \quad (4.3)$$

holds in every join semilattice with a closure operation, for all $a_1, \ldots, b_s$, thus if we apply $K$ to (4.2), we get $K(x_{j_1} \lor \cdots \lor x_{\ell_{m,k(m)}})$, a very special expression still of the form (4.2).

So let us prove (4.3). Since $Kb_1 \geq b_1, \ldots, \text{then } a_1 \lor \cdots \lor Kb_1 \lor \cdots \lor Kb_s \geq a_1 \lor \cdots \lor b_1 \lor \cdots \lor b_s$, thus, applying $K$ and by isotony, we get $K(a_1 \lor \cdots \lor Kb_s) \geq K(a_1 \lor \cdots \lor b_s)$. For the converse, by extensiveness and isotony, we have $a_1 \leq K(a_1 \lor \cdots \lor b_s), \ldots, Kb_s \leq K(a_1 \lor \cdots \lor b_s)$, hence $a_1 \lor \cdots \lor Kb_s \leq K(a_1 \lor \cdots \lor b_s)$. Applying $K$, we get $K(a_1 \lor \cdots \lor Kb_s) \leq KK(a_1 \lor \cdots \lor b_s) = K(a_1 \lor \cdots \lor b_s)$ by isotony and idempotence.

Notice that, in contrast with Lemma 4.7, the theory of meet semilattices with a closure operation is not locally finite. See [12, Section 2], in particular, p. 3 and Figure 1 on p. 13 therein.
5. Decidability of universal consequences

If $T$ is a universal locally finite theory, then a universal-existential sentence $\varphi$ is a consequence of $T$ if and only if $\varphi$ holds in every finite model of $T$. If we extend $T$ in a language with added operations, then the extended theory $T^+$ is not necessarily locally finite. However, we can retain the above characterization, limited to universal consequences, when we add operations of the kind considered in the present note and every finite model of $T$ can be extended to a finite lattice-ordered model. Compare [23, Appendix IV] for a special similar situation.

The present section relies only on Section 3 and does not deal with the amalgamation property.

**Theorem 5.1.** Suppose that $T$ is a locally finite universal theory in a language $\mathcal{L}$ with a specified order relation $\leq$ and suppose that every finite model of $T$ can be extended to a finite lattice-ordered model of $T$.

Suppose that $(W)$ is any one of the properties (A1e) - (C2) listed in Definition 3.4 and $\mathcal{L}' = \mathcal{L} \cup \{K\}$, where $K$ is a new operation symbol of corresponding arity. Let $T(W)$ in the language $\mathcal{L}'$ be the extension of $T$ obtained by adding axioms asserting that $K$ satisfies $(W)$. Then the following hold.

1. If $\varphi$ is a universal sentence in $\mathcal{L}'$ and $\varphi$ fails in some model of $T(W)$, then $\varphi$ fails in some finite model of $T(W)$.

2. Suppose that $\mathcal{L}'$ is finite and there is an effectively computable function $h_T$ such that, for every $n \in \mathbb{N}$, every model of $T$ generated by $n$ elements can be extended to a lattice-ordered model of $T$ of cardinality $\leq h_T(n)$. Then the set of all the universal consequences of $T(W)$ is decidable.

More generally, the above items (1) - (2) hold if we add any number of operations, possibly of distinct arities, possibly satisfying distinct properties chosen from (A1e) - (C2). If $T$ contains the axioms for (and in the language of) lattices, then $(W)$ might be chosen to be (C3), too.

**Proof.** (1) Suppose that $\varphi$ is $\forall \bar{x} \psi$, with $\psi$ quantifier-free, and $\varphi$ fails in some model of $T(W)$. If some term of the form $K(t_1, \ldots, t_n)$ occurs in $\varphi$ and $K$ does not occur in the terms $t_1, \ldots, t_n$, let $y$ be a new variable not occurring in $\varphi$. Then $\varphi$ is logically equivalent to $\forall \bar{x} y(K(t_1, \ldots, t_n) = y \Rightarrow \psi^*)$, where $\psi^*$ is obtained from $\psi$ by substituting all the occurrences of the term $K(t_1, \ldots, t_n)$ for $y$. Iterating the above procedure, it is no loss of generality to assume that $\varphi$ is of the
form $\forall x\bar{y}\psi$, where

$$\psi : \quad K(t_{1,1}, \ldots, t_{1,n}) = y_1 \land \ldots \land K(t_{m,1}, \ldots, t_{m,n}) = y_m \implies \psi^*$$

with $t_{1,1}, \ldots, t_{m,n}$, $\psi^*$ $K$-free and $\psi^*$ quantifier-free.

By assumption, there is some model $A$ of $T^{(W)}$ such that $\varphi$ fails, hence, for an appropriate assignment of elements of $A$ to the variables of $\psi$, the evaluation of $\psi$ fails in $A$. This means that, for the given assignment, $K(t_{1,1}, \ldots, t_{1,n}) = y_1, \ldots, K(t_{m,1}, \ldots, t_{m,n}) = y_m$ hold and $\psi^*$ fails in $A$. Let $t_{i,j}^A$ denote the evaluation of $t_{i,j}$ under the assignment and let $X$ be the set of the $n$-tuples of $A$ having the form $(t_{j,1}^A, \ldots, t_{j,n}^A)$, for $1 \leq j \leq m$. Let $V : X \to A$ be defined by $V(t_{j,1}^A, \ldots, t_{j,n}^A) = K_A(t_{j,1}^A, \ldots, t_{j,n}^A)$, which is also equal to $y_j^A$, since $K(t_{j,1}, \ldots, t_{j,n}) = y_j$ holds in $A$. Notice that in the unary case $X$ and $V$ are called $D$ and $G$ in Lemma 3.1.

Since $A$ is a model of $T^{(W)}$ and $K_A$ is an extension of $V$ satisfying (W), then the necessary condition in Lemma 3.1 for the satisfaction of property (W) holds (as we mentioned at the beginning of the proof of Lemma 3.1, no completeness assumption is needed to prove the necessary condition).

Let $B^-$ be the subreduct of $A$ generated in the language $\mathcal{L}$ by the elements assigned to the variables of $\psi$ under the given assignment. Thus $B^-$ is a finite model of $T$, since $T$ is universal and locally finite. By construction, $X \subseteq (B^-)^n$ and $V$ is actually a function from $X$ to $B^-$. By assumption, $B^-$ can be extended to a finite lattice-ordered model $C^-$ of $T$. Hence $C^-$ is complete, since $C^-$ is finite. We can apply Lemma 3.1 in order to extend $V$ on the whole of $(C^-)^n$ to an operation $K_C$ in such a way that $K_C$ extends $V$ and (W) holds in the expanded model $C$. Thus $C$ is a model of $T^{(W)}$, since $C^-$ is a model of $T$.

We have that $t_{i,j}^A = t_{i,j}^B = t_{i,j}^C$ hold, for all pairs of indices, since the terms $t_{i,j}$ are $K$-free, since the variables of $\psi$ are interpreted in $B^-$, because of the definition of $B^-$, and since $C \supseteq B$. Since $K_C$ extends $V$ and because of the definition of $V$, $K(t_{1,1}, \ldots, t_{1,n}) = y_1, \ldots, K(t_{m,1}, \ldots, t_{m,n}) = y_m$ hold in $C$ under the given assignment.

On the other hand, since $\psi^*$ is $K$-free, quantifier-free and false in $A$, then by the definitions of $B^-$ and $C$, $\psi^*$ is false in $C$. This shows that $\varphi$ is false in $C$, thus $\varphi$ fails in a finite model of $T^{(W)}$.

(2) Let $\varphi$ be a universal sentence in the language of $T^{(W)}$. The proof of (1) shows that $\varphi$ fails in some model of $T^{(W)}$ if and only if $\varphi$ fails in some lattice-ordered finite model of $T^{(W)}$ whose $\mathcal{L}$-reduct extends
a model of $T$ generated by $k$ elements, where $k$ can be effectively determined and depends only on the formula $\varphi$. In fact, if $\varphi$ contains $\ell$ variables and $K$ occurs $m$ times in $\varphi$, then $k \leq \ell + m$.

Thus $\varphi$ is a consequence of $T^{(W)}$ if and only if $\varphi$ holds in every model of $T^{(W)}$ of cardinality $\leq h_T(k)$. Since $h_T$ is effectively computable and the language of $T^{(W)}$ is finite, one can effectively check the validity of $\varphi$ in all these models. This provides a decision procedure for the validity of $\varphi$ in all models of $T^{(W)}$.

The last paragraph in the theorem is proved in the same way. In the general case, the premises in $\psi$ in (5.1) might involve distinct operations, but we can always manage to have all the terms $t_{i,j}$ to be $\mathcal{L}$-terms. In (2) the extended language is finite by assumption; as far as (1) is concerned, notice that a first-order formula involves only a finite set of symbols; then in $C$ all the remaining symbols can be interpreted in an arbitrary way, for example, as the projection onto the first component. Notice that in the case of many operations $T^{(W)}$ says nothing about the mutual relationships among the operations. To prove the last statement, observe that if $T$ is a theory of lattices in the language of lattices, then every lattice term is evaluated in the same way in $A$ and $B^{-}$, which in the present situation can be taken as $C^{-}$. Hence any condition of the form $t(x_1, \ldots, x_i) \leq V(\bar{x})$ is preserved. □

Corollary 5.2. Let $T$ be the extension of the theory of Boolean algebras in a language with a further finite set of operations, and with further axioms asserting that each operation satisfies some condition chosen among $(A1e)$ - $(C3)$ from Definition 3.4. Then the set of universal consequences of $T$ is decidable.

The same applies to distributive lattices in place of Boolean algebras, more generally, to any locally finite universal theory of lattices, and, excluding case $(C3)$, to partially ordered sets, join semilattices, meet semilattices.

In many cases Corollary 5.2, as well as the last paragraph in the statement of Theorem 5.1, can be generalized by adding comparability conditions among operations satisfying the same property. We leave details to the reader.

Generally, for a theory $T$ as in Corollary 5.2, the set of all the first order consequences of $T$ is not decidable. Indeed, the set of the first order consequences of the theory of Boolean algebras with an additive closure operation (called closure algebras in the literature) is not decidable [23, footnote 19], [8]. Were the consequences of a theory $T$ as in

\footnote{provided the function $g_T$ from Definition 4.4 is effectively computable.}
Corollary 5.2 decidable (except, possibly, for the cases of an involution and of an antitone operation), we could add as a premise a finite set of sentences characterizing closure algebras, which would produce a decision procedure for the consequences of the theory of closure algebras, a contradiction. Notice that the property that, say, a poset is (the order-reduct of) a Boolean algebra can be expressed by a first-order sentence in the language of posets.

As another observation, notice that the proof of Corollary 5.2 does not apply to the theory of lattices, which is not locally finite. On the other hand, the results in Section 4 do not apply to distributive lattices, which have the amalgamation property but not the strong amalgamation property [4]. This implies that the amalgamation property is generally destroyed by adding further operations, as exemplified in Example 4.2(a).

6. Further remarks

Remark 6.1. (a) The assumption that $HHx \leq Hx$, for every $x \in P$, is necessary in Lemma 2.2. Consider the 3-element chain $P = \{a, b, c\}$ with $a < b < c$. Let $Ha = b$, $Hb = Hc = c$, thus $H$ is isotone, but $c = HHa \not\leq Ha = b$. Let $K_1a = K_1b = b$, $K_1c = c$, $K_2a = a$, $K_2b = K_2c = c$, thus $K_1$ and $K_2$ are both isotone, idempotent and smaller than $H$. However, the only idempotent operation larger than both $K_1$ and $K_2$ is the constant function with value $c$, which is not smaller than $H$.

Notice that, in the above example, the operations $K_1$ and $K_2$ are also extensive. Thus in Lemma 2.2 the assumption $HHx \leq Hx$ is necessary also in the extensive case (in which case the assumption reads $HHx = Hx$, hence in this case the Lemma is trivially proved by taking $K = H$).

(b) The assumption that every nonempty infinite chain has a meet in $P$ is necessary in Lemma 2.2. For example, if $P$ is the ordered set $\mathbb{Z}$ of the integers and $H$ is the predecessor function, then in $P$ there is no idempotent operation smaller than $H$.

(c) If in the above example we add a minimum $-\infty$ to $\mathbb{Z}$ and set $H(-\infty) = -\infty$, the assumptions in Lemma 2.2 are met. The example of $\mathbb{Z} \cup \{-\infty\}$ shows that in the proof of Lemma 2.2 a finite iteration of the $K^\alpha$s is generally not sufficient.
(d) Some completeness assumption is necessary in Proposition 2.3.

Again on \( \mathbb{Z} \), define

\[
K_1 x = \begin{cases} 
  x - 1 & \text{if } x \text{ is even}, \\
  x & \text{if } x \text{ is odd},
\end{cases}
\]

\[
K_2 x = \begin{cases} 
  x & \text{if } x \text{ is even}, \\
  x - 1 & \text{if } x \text{ is odd}.
\end{cases}
\]

Both \( K_1 \) and \( K_2 \) are isotone and idempotent, but on \( \mathbb{Z} \) there is no idempotent operation smaller than both \( K_1 \) and \( K_2 \).

**Remark 6.2.**

(a) In the cases (A1c), (B1), (B1e), (B2), (B3) and (B5) in Lemma 3.1 there exists the largest operation \( K \) satisfying the conclusions. Recall that we say that some operation \( K \) is larger than \( J \) if \( K x \geq J x \), for every \( x \) in the domain. The largest operation is given by the corresponding formulae in the proof of Lemma 3.1. For posets with a maximum, the largest operation exists in cases (A1e) and (A2e), as well. In case (A1e) set \( Ka = Ga \) if \( a \in D \) and \( Ka \) to be the maximum of \( P \), otherwise. In case (A2e) set \( Ka = Ga \) if \( a \in D, Ka = a \) if \( a = Gb \), for some \( b \in D \), and \( Ka \) to be the maximum of \( P \) in the remaining cases.

Dually, in cases (A1e), (B1), (B1c), (B2), (B4), (B5) there is the smallest operation, given by the dual formulae.

(b) On the other hand, in case (A2) there does not necessarily exist the largest operation satisfying the conclusion in Lemma 3.1. Consider a five elements lattice with maximum 1, minimum 0 and three more elements \( a, b, c \) such that \( a \vee b = 1 \) and \( a \wedge b = c \) (a “diamond” with a new bottom element added). If \( D = \{0, 1, a, b\} \) and \( G1 = G0 = 0, Ga = a \) and \( Gb = b \), then we can extend \( G \) to an idempotent operation by taking \( Kc \in \{0, c, a, b\} \), but we cannot set \( Kc = 1 \), if \( K \) extends \( G \) and is idempotent. Hence there is no largest idempotent operation extending \( G \).

(c) In general, the largest operation does not exist in case (A3), either. Consider a “diamond” with maximum 1, minimum 0 and \( a, b \) such that \( a \vee b = 1 \) and \( a \wedge b = 0 \). Let \( D = \{1\} \) and \( G1 = 1 \). If \( K_o 1 = 1, K_o b = 0, K_o 0 = b \) and \( K_o a = a \), then \( K_o \) is an involution extending \( G \). Similarly, setting \( K_a 0 = 0, K_a 0 = a, K_a b = b \) and \( K_a 1 = 1 \), we get an involution extending \( G \). If \( K \) is larger than both \( K_o \) and \( K_a \), then \( K0 = 1 \), thus \( K \) is not an involution, if \( K \) extends \( G \).

**Remark 6.3.**

(a) We do not need the full assumption that \( P \) is a bounded and complete lattice in cases (B1e) and (B3) in Lemma 3.1. It is enough to assume that \( P \) is a poset such that, for every \( x \in P \), every subset of \( \{ y \in R \mid x \leq y \} \) has a meet, where \( R \) is the range of \( D \). This is some kind of a near-lattice completion.
If $R$ is cofinal in $P$, that is, for every $x \in P$, there is $y \in R$ such that $x \leq y$, then it is enough to assume that, for every $x \in P$, every nonempty subset of $\{ y \in R \mid x \leq y \}$ has a meet.

The dual assumptions are enough to deal with cases (B1c) and (B4).

The completeness assumption can thus be weakened as above in the corresponding cases in Corollary 3.5(1) and Theorem 4.1.

(b) On the other hand, some completeness assumption is necessary in Lemma 3.1, even in case (B3). Consider a poset $P$ with a descending chain $(c_n)_{n \in \mathbb{N}}$ and three elements $a, b, d$ smaller than all the $c_i$s and such that $a < b$, $a < d$ and $b, d$ incomparable.

Let $D = P \setminus \{ b \}$ and let $G : D \to P$ be defined by $Ga = d$ and $Gx = x$, for $x \in D \setminus \{ a \}$. The function $G$ satisfies (3.5); however, $G$ cannot be extended to a closure operation $K$ on the whole of $P$. Indeed, since $a < b$, we should have $Kb \geq Ka = Ga = d$. But we also want $Kb \geq b$, hence, since $b$ and $d$ are incomparable, then $Kb = c_i$, for some $i$. Then $c_i = Kb \leq Kc_{i+1} = c_{i+1}$, since $b \leq c_{i+1}$ and $K$ should be isotone. This is a contradiction, since we have assumed $c_{i+1} < c_i$.

The counterexample works also for case (B1e), since we have not used idempotence.

(c) In the above counterexample $P$ is not a lattice, but a more involved counterexample can be devised to treat the case when $P$ is a bounded (necessarily incomplete) lattice. The following example has also the advantage of working for all cases (B1) - (B4).

Let $F$ be the set of all the finite subsets of $\mathbb{N}$, $p$ the set of even natural numbers and $P = F \cup \{ p \cup f \mid f \in F \} \cup \{ c_i \mid i \in \mathbb{N} \}$, where the order among the subsets of $\mathbb{N}$ is inclusion and the $c_i$s are a descending chain of new elements taken to be greater than all the subsets of $\mathbb{N}$. Thus $P$ becomes a bounded distributive lattice with maximum $c_0$ and minimum the empty subset of $\mathbb{N}$.

Set $D = F \cup \{ c_i \mid i \in \mathbb{N} \}$ and $Gf = [0, \max f]$, for $f \in F$, and $Gc_i = c_i$, for every $i \in \mathbb{N}$. Then $G$ is extensive and satisfies (3.4) - (3.6). On the other hand, $G$ cannot be extended to an isotone operation $K$ on $P$, since $p \geq \{ 0, 2, \ldots, 2n \}$, for every $n \in \mathbb{N}$, hence we should have $Kp \geq K\{ 0, 2, \ldots, 2n \} = [0, 2n]$, for every $n \in \mathbb{N}$, hence $Kp = c_i$, for some $i$, but then we get a contradiction arguing as in (b).

In the present counterexample we have only used isotony of $K$, hence the counterexample (or its dual) applies to all cases (B1) - (B4). Moreover, $D$ and $G$ satisfy the stronger condition that if $x \in D$, then $Gx \in D$.

(d) The counterexample in (c) can be adapted in order to work for case (B5). Let $\mathbf{P}^+$ be the lattice described in (c); let $P^- = \{ x^- \mid x \in \mathbb{N} \}$.
\( P^+ \) be a disjoint copy of \( P^+ \) endowed with the reversed order and set \( P^* = P^+ \cup P^- \), letting every element of \( P^- \) be smaller than every element of \( P^+ \). Let \( D^+ = D \) as introduced in (c) and \( D^- \) correspond to the copy of \( D \) in \( P^- \); then set \( D^* = D^+ \cup D^- \). Given the function \( G \) introduced in (c), let \( G^*: D^* \to P^* \) be the function defined by \( G^*x = (Gx)^- \), for \( x \in P^+ \) and \( G^*(x^-) = Gx \), for \( x^- \in P^- \). Since \( G \) satisfies (3.4), then \( G^* \) satisfies (3.7). An argument similar to the one in (c) shows that \( G^* \) cannot be extended to an antitone operation on \( P^* \).

(e) By the comment in Example 4.2(d), and since the counterexample in (c) above is a lattice, some completeness assumption is necessary in cases (C1) - (C3).

Remark 6.4. (a) In Lemma 3.2 case (A3) the comparability condition is not necessarily preserved, unless the additional assumptions in Lemma 3.2 are satisfied. Consider the four element chain \( P \) with \( a < b < c < d \), \( D = \{a, b\} \), \( G_o a = b \), \( G_o b = a \), \( G_* a = c \) and \( G_* b = d \).

We have \( G_o x \leq G_* x \), for \( x \in D \). Moreover both \( G_o \) and \( G_* \), taken alone, satisfy the conditions (3.2) and (3.3), hence, by Lemma 3.1 both \( G_o \) and \( G_* \) can be extended to some involution. However, it is not possible to extend them in such a way that the comparability condition \( K_o x \leq K_* x \) is satisfied, since involutions are bijective, hence we must have \( K_o c \geq c \); on the other hand, since \( K_* \) must be an involution extending \( G_* \), then \( K_* c = a \), thus necessarily \( K_* c < K_o c \).

Notice that in the above example we have that \( x \in D \) implies \( G_o x \in D \) (not so for \( G_* \), of course, otherwise Lemma 3.2 would be contradicted).

(b) Without further assumptions, in cases (B2) and (B3) comparability conditions are not necessarily preserved by the operations defined in the proof of Lemma 3.1 (but, as shown in Lemma 3.2 we can maintain comparability by introducing different operations).

Let \( P = \{d, p, q\} \) with \( d < p < q \), \( D = \{d\} \) and let \( G_o d = a \), \( G_* d = p \). We have \( G_o d \leq G_* d \); however, if \( K_o \) and \( K_* \) are correspondingly defined by (Case B3), then \( K_o p = q \neq p = K_* p \). Similarly, if \( H_o \) and \( H_* \) are correspondingly defined by (3.14) in the proof of case (B2), then \( H_o d = d \), \( H_o p = H_o q = q \) and \( H_* d = H_* p = p \), \( H_* q = q \). In both cases \( HH x = H x \), for every \( x \), hence no iteration is needed, and \( K_o p = H_o p = q \neq p = H_* p = K_* p \), thus the example works also for case (B2).

(c) In Lemma 3.2 cases (B2) and (B3), the comparability conditions are satisfied by the operations defined in the proof of Lemma 3.1 under
the additional assumption that \( G \cdot b \in D \), for every \( b \in D \). In the dual case (B4) we need assume instead that \( G \cdot b \in D \), for every \( b \in D \).

We first prove the above claim in case (B3). Suppose that \( b \in D \). Since, by assumption, \( G \circ x \leq G \cdot x \), for every \( x \in D \), then \( G \circ G \cdot b \leq G \cdot G \cdot b \), by taking \( x = G \cdot b \) and since \( G \cdot b \in D \). Since \( G \circ \) is assumed to be extensive, we get \( G \cdot b \leq G \circ G \cdot b \leq G \cdot G \cdot b \leq G \cdot b \), where the last inequality is obtained by applying (3.5) to \( G \) with \( G \cdot b \) in place of \( a \), and using again the assumption that \( G \cdot b \in D \).

We have proved that \( G \cdot b = G \circ G \cdot b \), for all \( b \in D \). Now fix \( x \in P \) and let \( b \) vary in \( D \). Whenever \( b \) is such that \( x \leq G \cdot b \), then \( x \leq G \cdot b = G \circ G \cdot b \) and, since \( G \cdot b \in D \), we get \( \{ G \cdot b \mid b \in D, x \leq G \cdot b \} \subseteq \{ G \circ a \mid a \in D, x \leq G \circ a \} \), by considering \( a = G \cdot b \). This implies \( K \circ x \leq K \cdot x \).

We now consider case (B2). We first prove the above claim in case (B3). Suppose that \( b \) for certain \( \alpha \). Since \( \circ \) is isotone and idempotent, then \( (K \circ)_{\alpha} \circ \leq (K \circ)_{\alpha} \), and hence \( (K \circ)_{\alpha} \circ K \circ x \leq (K \circ)_{\alpha} \circ K \cdot x \), for every ordinal \( \alpha \), where \( (K \circ)_{\alpha} \) denotes the \( \alpha \)th stage of the construction of \( K \circ \). Since \( K \circ = (K \circ)_{\alpha} \), for some \( \alpha \), we get \( K \circ x \leq K \cdot x \).

Remark 6.5. In the proof of Lemma 3.1 case (B2) it is necessary to iterate \( H \). Suppose that \( b_1 \wedge b_2 = x \), \( b_1, b_2 \in D \), \( Gb_1 < b_1 \), \( Gb_2 < b_2 \) and \( Gb_1 \wedge Gb_2 < x \). According to (3.14), we have \( Hx \leq Gb_1 \wedge Gb_2 \), and we might assume to be in the situation in which \( Hx = Gb_1 \wedge Gb_2 \). It might happen that \( b_1 \wedge b_2 = Hx \), \( Gb_1 < b_1 \), \( Gb_2 < b_2 \) and \( Gb_1 \wedge Gb_2 < Hx \). for certain \( b_1, b_2 \in D \) incomparable with \( b_1 \) and \( b_2 \). The above relations entail \( HHx \leq Gb_1 \wedge Gb_2 < Hx \).
The above construction can be iterated transfinitely in order to get examples in which $K^{\alpha+1}x < K^\alpha x$, for an arbitrarily large ordinal $\alpha$.

Remark 6.6. In the situation described in Lemma 3.1 possible conditions for the existence of extensions of an isotone involution will necessarily be much more involved. In fact, if $a \leq b$ and $K$ is an isotone involution, then the order interval $[a, b]$ is isomorphic to $[Ka, Kb]$.

A similar remark applies to extensions of an antitone involution.

Remark 6.7. The analogue of Corollary 3.5 generally fails for case (C3). Let $Q$ be the diamond with four elements 0, 1, $a$, $b$, with $a$ and $b$ not comparable. Let $F$ be the binary function $F(x, y) = x \land y$, which is isotone on both components. If $Q$ is obtained by adding a new element $c$ with $0 < c < a, b$, then the inclusion is an order-embedding (not a lattice-embedding), $x \land y \leq F(x, y)$ holds by construction in $Q$, but $a \land b = c > 0 = F(a, b)$ in $P$.

On the other hand, Corollary 3.5(3-case of meets)(4) apply also in case (C3).

Example 6.8. (a) Let $T$ be the theory of posets asserting that, for every $n \in \mathbb{N}$, if there are less than $n$ elements, then the poset is linearly ordered. $T$ has the superamalgamation property, the class of finite models of $T$ has the strong amalgamation property, but not the superamalgamation property.

If $T^+$ extends $T$ in a language with two further unary operations and asserts that the operations are isotone, then $T^+$ has the superamalgamation property, but the class of finite models of $T^+$ has not the amalgamation property.

(b) The above theories are not universal. Let $T'$ be a theory in a language with two unary relation symbols $U$, $V$, and a binary function $f$. The theory $T'$ asserts that if $U(x)$ and $V(y)$ hold, then all the elements $x, y, f(x, y), f(x, f(x, y)), f(x, f(x, f(x, y)))\ldots$ are distinct. $T'$ has the strong amalgamation property, but the class of finite models of $T'$ has not the amalgamation property.

(c) Let $T''$ be as above, with a further binary relation satisfying the axioms of posets. Since there is no axiom connecting $\leq$ with $U$, $V$ and $f$, then $T''$ has the superamalgamation property. $T''$ is universal and has JEP. On the other hand, the class of finite models of $T''$ has neither JEP, nor AP; in particular, it has not a Fraïssé limit.

It is an open problem whether the results in the present paper generalize to unary operations satisfying $K^n(x) = Kx$, or $K^n(x) = x$, or, more generally, $K^n(x) = K^m(x)$, for some $m, n \in \mathbb{N}$. Is it possible to consider more relationships connecting distinct operations, other than
comparability? For example, do the results in the present paper generalize when adding two commuting unary operations, that is, satisfying $K_\circ K_\bullet x = K_\bullet K_\circ x$?

Acknowledgements. We thank the referee for many useful comments and for detecting some inaccuracies.

References

[1] Czelakowski, J., Pigozzi, D., *Amalgamation and interpolation in abstract algebraic logic*, in Caicedo, X., Montenegro, C. H. (eds.), *Models, algebras, and proofs* (Bogotá, 1995), Lecture Notes in Pure and Appl. Math., 203, 187–265 (1999).
[2] Erné, M., *Closure*, in Mynard, F., Pearl E. (eds), *Beyond topology*, Contemp. Math. 486, Amer. Math. Soc., Providence, RI, 163–238 (2009).
[3] Fleischer, I., *Amalgamation for semilattices*, Algebra Universalis 6, 411–412 (1976).
[4] Fried, E., Grätzer, G., *Strong Amalgamation of Distributive Lattices*, J. Algebra 128, 446–455 (1990).
[5] Gabbay, D. M., Maksimova, L., *Interpolation and definability. Modal and intuitionistic logics*, Oxford Logic Guides 46, The Clarendon Press, Oxford University Press, Oxford (2005).
[6] Ghilardi, S., Gianola, A., *Modularity results for interpolation, amalgamation and superamalgamation*, Ann. Pure Appl. Logic 169, 731–754 (2018).
[7] Grätzer, G., *Lattice theory: foundation*, Birkhäuser/Springer Basel AG, Basel, 2011.
[8] Grzegorczyk, A., *Undecidability of some topological theories*, Fund. Math. 38, 137–152 (1951).
[9] Harding, J., Bezhanishvili, G., *MacNeille completions of Heyting algebras*, Houston J. Math. 30, 937–952 (2004).
[10] Harzheim, E., *Ordered sets*, Advances in Mathematics 7, New York, 2005.
[11] Hodges, W., *Model theory*, Encyclopedia of Mathematics and its Applications 42, Cambridge University Press, Cambridge, 1993.
[12] Jackson, M., *Semilattices with closure*, Algebra Universalis 52, 1–37 (2004).
[13] Jónsson, B., *Universal relational systems*, Math. Scand. 4, 193–208 (1956).
[14] Jónsson, B., *Extensions of relational structures*, in *Theory of Models* (Proc. Internat. AP Sympos. Berkeley, 1963), North-Holland, Amsterdam, 146–157 (1965).
[15] Kaplan, I., Simon, P., *Automorphism groups of finite topological rank*, Trans. Amer. Math. Soc. 372, 2011–2043 (2019).
[16] Kihara, H., Ono, H., *Interpolation properties, Beth definability properties and amalgamation properties for substructural logics*, J. Logic Comput. 20, 823–875 (2010).
[17] Kiss, E. W., Márki, L., Pröhle, P., Tholen, W., *Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residua,l smallness, and injectivity*, Studia Sci. Math. Hungar. 18, 79–140 (1982).
[18] Koppelberg, S., *Handbook of Boolean algebras. Vol. 1*, North-Holland Publishing Co., Amsterdam (1989).
7. Appendix. Superamalgamation into union

In this appendix we deal with the case when the superamalgamating structure can be taken over the set-theoretical union of the base sets of the models to be amalgamated. In this case no completion hypothesis is necessary and the framework is slightly more general, to the effect that we can work with a transitive binary relation, not necessarily an order.

There are situations in which the completion hypothesis (2) in Theorem 4.1 is not needed. First, cases (A1e) - (A3) are really elementary and do not need the assumption (2), since no completeness assumption is necessary in the proof of Lemma 3.1 in such cases.

More interestingly, we do not need completions when the superamalgamating structure can be taken over the set $D = A \cup B$. In this situation we can work with arbitrary transitive relations in place of orders, and we can also get a few additional results. Notice that the definitions in Section 2 in particular, the definition of superamalgamation, apply to an arbitrary binary relation in place of $\leq$.

The extension Lemma 3.1 is not needed in the rest of the present section.
**Definition 7.1.** Suppose that $\mathcal{S}$ is a class of structures for the same language and with a specified binary relation $R$. We say that $\mathcal{S}$ has the superamalgamation property into union if, under the assumptions in Definition 2.1 (with $R$ in place of $\leq$), a superamalgamating structure exists over the set $D = A \cup B$.

Many examples of classes with the superamalgamation property into union are presented in [14], for example, the classes of models with a binary relation, possibly satisfying some properties chosen among transitivity, reflexivity, symmetry, antireflexivity, antisymmetry. Other examples are classes with two binary transitive relations, one coarser than the other, each satisfying some set of the above properties.

**Proposition 7.2.** Suppose that $\mathcal{S}$ is a class of structures (or $T$ is a theory) with a specified binary relation $R$.

(a) If $\mathcal{S}$ is in a language without operations of arity $\geq 2$ and $\mathcal{S}$ is closed under taking substructures, then $\mathcal{S}$ satisfies the superamalgamation property into union if and only if $\mathcal{S}$ satisfies the superamalgamation property.

(b) If $T$ is a theory in some language $\mathcal{L}$, $\mathcal{L}' \supseteq \mathcal{L}$ and $T$ has the superamalgamation property into union, then the class of models of $T$ in the language $\mathcal{L}'$ has the superamalgamation property into union.

(c) Suppose that $\Sigma$ is a set of universal-existential sentences in which at most one variable is bounded by the universal quantifier. If $\mathcal{S}$ is a class of structures with the superamalgamation property into union, then the class of all structures in $\mathcal{S}$ which satisfy $\Sigma$ has the superamalgamation property into union.

(d) Suppose that $(T_i)_{i \in I}$ is a sequence of theories in languages $\mathcal{L}_i$, and suppose that $\mathcal{L}_i \cap \mathcal{L}_j = \{R\}$, for $i \neq j \in I$, where $R$ is a binary relation symbol.

If each $T_i$ has the superamalgamation property into union and asserts that $R$ is transitive, then $T = \bigcup_{i \in I} T_i$ has the superamalgamation property into union.

The proof of Proposition 7.2 is elementary, but the proposition is useful. Cases (a) - (c) hold for the strong amalgamation property, as well. See [14] for full details. As far as (d) is concerned, notice that, since $R$ is assumed to be transitive, the superamalgamation property determines the interpretation of $R$ on $A \cup B$.

If $A$ is a structure with a binary relation $R$, we shall write $a R b$ in place of $R(a, b)$ or $(a, b) \in R$. A unary operation $K : A \to A$ is $R$-isotone (or $R$-preserving) if $a R b$ implies $K a R K b$, for every
The operation $K$ is $R$-antitone (or $R$-reversing) if $a \ R \ b$ implies $Kb \ R \ Ka$, for every $a, b \in A$.

**Theorem 7.3.** Suppose that $S$ is a class of structures with a transitive binary relation $R$, and $S$ has the superamalgamation property into union.

If $S_1$ is the class of expansions of structures of $S$ obtained by adding an $R$-isotone (an $R$-antitone) unary operation, then $S_1$ has the superamalgamation property into union, in particular the strong amalgamation property.

More generally, the same applies when $S_1$ is obtained by adding families of such operations, possibly adding a set of comparability conditions. Moreover, for every set $\Sigma$ of universal-existential sentences in which at most one variable is bounded by the universal quantifier, the class of all structures in $S_1$ which satisfy $\Sigma$ has the superamalgamation property into union.

**Proof.** Given $A, B, C$ as in Definition 2.1, their $S$-reducts can be amalgamated into a structure $D^-$ over $D = A \cup B$, since the superamalgamation property is into union. Define $K$ on $D$ as in the proof of Theorem 4.1, namely

$$Kd = \begin{cases} K_A d & \text{if } d \in A, \\ K_B d & \text{if } d \in B. \end{cases} \quad (7.1)$$

We only need to check that, when $D^-$ is expanded by adding such a $K$, $R$-isotony is maintained. Indeed, if $a, b \in A$ and $a \ R \ b$, then $Ka \ R \ Kb$, by $R$-isotony on $A$. The case when $a, b \in B$ is similar. Otherwise if, say, $a \in A \setminus B$ and $b \in B \setminus A$, then, by superamalgamation, there exists $c \in C = A \cap B$ such that $a \ R_A c$ and $c \ R_B b$. By $R$-isotony of $K_A$ and $K_B$ on $A$ and $B$, we get $K_A a \ R_A K_A c$ and $K_B c \ R_B K_B b$, that is, $Ka \ R_D Kc$ and $Kc \ R_D Kb$, according to the definition (7.1) of $K$, and since $D$ extends $D^-$ which amalgamates the $S$-reducts of $A, B$ and $C$. Since $D^-$ belongs to $S$, then $R_D$ is transitive, thus we get $Ka \ R_D Kb$.

The case of $R$-antitony is similar.

As for the last paragraph, and as in the proof of Theorem 4.1 we can add many new operations at a time, since they do not influence each other. Clause (7.1) clearly preserves comparability conditions which already hold in $A$ and $B$. The last statement follows from Proposition 7.2(c).

Though simple (and simply proved), Proposition 7.2(c) is quite powerful. For example, we can add conditions asserting that, $x \ R \ Kx$, for
every $x$, or, possibly, $Kx = KKx$, or $x = KKx$. If we define recursively $K^n$ by $K^0x = x$ and $K^{n+1}x = KK^n x$, we can add conditions of the form $K^n x \, R \, K^m x$, or $K^n x \, = \, K^m x$, for some fixed $m$ and $n$. When dealing with more operations, all the universal closures of the following formulae can be taken in $\Sigma$ in (7.2): $KHx = HKx$, $K^m H^n x = H^n K^m x$, $K^1 K^2 x \, R \, K_3 x$, $K^1 K^2 K_3 x \, R \, K^2 K_4 x$, etc.

The assumption that $R$ is transitive is necessary in Theorem 7.3. The class of structures with a transitive relation $S$ and a coarser binary relation $R$ has the superamalgamation property into union, with respect to $R$, but if a unary $R$-isotone operation is added, then the amalgamation property is lost. See [L1, Theorem 4.1(A)(C)] in a slightly different terminology and with the role of $R$ and $S$ exchanged.

**Corollary 7.4.** Suppose that $T$ is a universal theory in a finite relational language $\mathcal{L}$ containing a binary relation symbol $R$, $T$ asserts that $R$ is transitive and $T$ has the superamalgamation property.

Let $T_1$ be the extension of $T$ in the language $\mathcal{L} \cup \{K\}$ obtained by adding an axiom saying that the unary operation $K$ is idempotent (idempotent and isotone, idempotent and extensive, a closure operation). We can also add a finite number of such operations, under the further assumption that they pairwise commute.

Then $T_1$ has model completion.

**Proof.** Since $T$ is universal in a relational language and $T$ has the superamalgamation property, then $T$ has the superamalgamation property into union, by Proposition (7.2)(a). Then, by Theorem 7.3, $T_1$ has the amalgamation property. Since $\mathcal{L}$ is relational and, in each case, $K$ is idempotent (and, if there are more operations, they pairwise commute), then $T_1$ is locally finite. The conclusion follows from the well-known fact that every universal locally finite theory with the amalgamation property in a finite language has model completion, e. g., [Wh].

□

If $T_1$ in Corollary 7.4 has the joint embedding property, we also get a Fraïssé model, arguing as in Corollary 4.3 (3).

The assumption that $\mathcal{L}$ is in a relational language in Corollary 7.4 can be somewhat relaxed; it is enough to assume that $T$ is locally finite with the superamalgamation property into union and that $T_1$ remains locally finite.

**Additional References**

[L1] Lipparini, P., *The strong amalgamation property into union*, arXiv:2103.00563v2, 1–38 (2021).
[Wh] Wheeler, W.H., *Model-companions and definability in existentially complete structures*, Israel J. Math. 25, 305–330 (1976).

Dipartimento di Supermatematica, Viale della Ricerca Scientifica, Università di Roma “Tor Vergata”, I-00133 ROME ITALY

Email address: lipparin@axp.mat.uniroma2.it