A survey of topological Zimmer’s program

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Abstract
In this article, we survey the status of topological Zimmer’s conjecture on matrix group actions on manifolds.

1 Introduction

In mathematics, there are two important basic objects: the integers \(\mathbb{Z}\) and the real line \(\mathbb{R}\). The former is algebraic while the latter is geometric. If we take products, we could get the finitely generated torsion-free abelian group \(\mathbb{Z}^n\) and the Euclidean space \(\mathbb{R}^k\). The automorphism group of \(\mathbb{Z}^n\) is the general linear group \(\text{GL}_n(\mathbb{Z})\) consisting of all invertible integral matrices. Usually, we consider only the subgroup \(\text{SL}_n(\mathbb{Z})\) of determinant-one matrices, which is called the special linear group. Let \(M\) be a manifold, which is a geometric object locally homeomorphic to \(\mathbb{R}^k\). We have a natural question:

Problem 1.1 How does the algebraic object \(\text{SL}_n(\mathbb{Z})\) act on the geometric object \(M\)?

There is an obvious action: the linear action of \(\text{SL}_n(\mathbb{Z})\) on \(\mathbb{R}^n\). On one hand, the linear action preserves the origin. There is an induced action of \(\text{SL}_n(\mathbb{Z})\) on the sphere \(S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}\), given by

\[
(A, x) \mapsto \frac{Ax}{\|Ax\|}
\]

for any \(A \in \text{SL}_n(\mathbb{Z})\) and \(x \in S^{n-1}\). On the other hand, the linear action of \(\text{SL}_n(\mathbb{Z})\) preserves the integral points \(\mathbb{Z}^n\) in \(\mathbb{R}^n\). Therefore, there is another induced action of \(\text{SL}_n(\mathbb{Z})\) on the torus \(T^n = \mathbb{R}^n/\mathbb{Z}^n\). Except for these actions, what are other actions? The following conjecture says that when the dimension of the manifold \(M\) is smaller, any action should be a finite-group action. Let \(\text{Homeo}(M)\) (resp. \(\text{Diff}(M)\)) be the group consisting of all self-homeomorphisms (resp. self-diffeomorphisms) of \(M\).

Conjecture 1.2 Any group action of \(\text{SL}_n(\mathbb{Z})\) \((n \geq 3)\) on a connected compact manifold \(M^r\) \((r < n - 1)\) by homeomorphisms factors through a finite group. In other words, any group homomorphism from \(\text{SL}_n(\mathbb{Z})\) to the homeomorphism group \(\text{Homeo}(M)\) has a finite image.
Conjecture 1.2 is a special case of a general conjecture in Zimmer’s program [37], in which the special linear group is replaced by an arbitrary irreducible lattice $\Gamma$ in a semisimple Lie group $G$ of $\mathbb{R}$-rank at least 2, and the integer $n$ is replaced by a suitable integer $h(G)$. The smooth version of Conjecture 1.2 was formulated by Farb and Shalen [11], which is related to the original Zimmer’s program. The topological version of Conjecture 1.2 has been discussed by Weinberger [28]. These conjectures are part of a program to generalize the Margulis Superrigidity Theorem to a non-linear context. In general, it is difficult to prove these conjectures. A recent breakthrough is that Brown-Fisher-Hurtado [7] confirms Conjecture 1.2 for $(C^2)$-smooth actions. In this survey, we focus on topological actions. Note that the topological actions could be very different from smooth actions. Furthermore, not every topological manifold is smoothable and not every topological manifold is triangulizable (see Freedman and Quinn [13], Chapter 8). For more details of Zimmer’s program and related topics, see survey articles of Zimmer and Morris [40], Fisher [12] and Labourie [21].

2 Understand the conjecture

First of all, the inequality in Conjecture 1.2 is sharp since $\text{SL}_n(\mathbb{Z})$ acts non-trivially on the sphere $S^{n-1}$ when $n \geq 2$ (i.e. the group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Homeo}(S^{n-1})$ induced by the linear transformations has infinite image).

2.1 Why is it for compact manifolds?

The following result is well-known (eg. see [19]).

Lemma 2.1 Let $G = \langle g_1, g_2, \ldots, g_n \mid r_1, r_2, \ldots, r_m \rangle$ be a finitely presented group. There exists a closed 4-dimensional manifold $M$ with the fundamental group $\pi_1(M) \cong G$.

Proof. The manifold could be constructed explicitly. Let $X = \vee_{i=1}^n S^1$ be the wedge of $n$ circles. For each relator $r_i$, attach a 2-cell to $X$. The resulting space is denoted by $Y$, with $\pi_1(Y) \cong G$. Let $Y \hookrightarrow \mathbb{R}^3$ be an embedding with a tubular neighborhood $N$. The boundary $\partial N$ is a smooth closed 4-dimensional manifold with fundamental group $G$. □

It follows that the group $G$ will act freely on the universal cover $\hat{M}$, which is still a 4-dimensional manifold. When $n \geq 3$, the special linear group $\text{SL}_n(\mathbb{Z})$ has a finite presentation (see for example Milnor [24], Corollary 10.3, p.81). Therefore, the group $\text{SL}_n(\mathbb{Z})$ can act freely on a non-compact 4-dimensional manifold for any $n \geq 3$. This means that we have to assume the manifold $M$ is compact in Conjecture 1.2. However, we can still prove some results for non-compact manifolds with vanishing Euler characteristics (see Theorem 3.14, Theorem 3.16).
2.2 Why is it finite, not trivial?

Belolipetsky and Lubotzky [1] show that for every $n \geq 2$, every finite group $G$ is realized as the full isometry group of some compact hyperbolic $n$-manifold. The cases $n = 2$ and $n = 3$ have been proven by Greenberg [18] and Kojima [20], respectively.

**Lemma 2.2** For any $k \geq 2$, there exists a closed manifold $M^k$ such that $\text{SL}_n(\mathbb{Z})$ (for $n \geq 3$) acts on $M$ non-trivially.

**Proof.** Let $a > 1$ be a positive integer. The quotient ring homomorphism $\mathbb{Z} \to \mathbb{Z}/a$ induces a surjection $\text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/a)$. Note that $\text{SL}_n(\mathbb{Z}/a)$ is finite. Let $M$ be a hyperbolic manifold whose isometry group is $\text{SL}_n(\mathbb{Z}/a)$. Then $\text{SL}_n(\mathbb{Z})$ acts on $M$ non-trivially through $\text{SL}_n(\mathbb{Z}/a)$. ■

This means that we can only expect the finiteness, rather than the triviality of the image $\text{SL}_n(\mathbb{Z}) \to \text{Homeo}(M)$ in Conjecture 1.2. However, it is surprising that most confirmed cases for Conjecture 1.2 indeed have trivial images (see the next section for the status).

3 The status

There are many results on the smooth Zimmer program (see the book [6]). Since this survey focuses on the topological case, we consider only the actions by homeomorphisms. The following is a rough list of the status of Conjecture 1.2:

- When $r = 1$, Conjecture 1.1 is already proved by Witte [29] (see also Ghys [14] and Burger-Monod [8] for more general lattices, and Navas [25] for the $C^{1+\alpha}$-action ($\alpha > \frac{1}{2}$) of Kazhdan groups).
- Weinberger [27] confirms the conjecture when $M = T^r$ is a torus.
- Bridson and Vogtmann [4] confirm the conjecture when $M = S^r$ is a sphere.
- We confirm the conjecture for product of two spheres ([34]), flat manifolds [33], nilpotent manifolds [21] and
- Orientable manifolds with nonzero Euler characteristics modulo 6 (see [32]).

The proofs of the above mentioned results (except the cases of $M = S^1, T^r$) use torsion elements in $\text{SL}_n(\mathbb{Z})$. Actually, in the above cases stronger results can be proved: the group actions in Conjecture 1.2 are trivial, not just finite. However, the following question is still open:

**Conjecture 3.1** Any group action of a torsion-free finite-index subgroup of $\text{SL}_n(\mathbb{Z})$ on a $\mathbb{R}^r$ ($2 \leq r < n$) by homeomorphisms factors through a finite group.
3.1 Some basic facts

The following result on the structure of $\text{SL}_n(\mathbb{Z})$ will be helpful in later discussions.

**Lemma 3.2** Let $N$ be a normal subgroup $\text{SL}_n(\mathbb{Z})$, $n \geq 3$. Then either $N$ lies in the center of $\text{SL}_n(\mathbb{Z})$ (and hence it is finite) or the quotient group $\text{SL}_n(\mathbb{Z})/N$ is finite.

**Remark 3.3** The result holds more generally for an irreducible lattice $\Gamma$ in a connected irreducible semisimple Lie group $G$ of real rank $\geq 2$, which is called the Margulis-Kazhdan theorem (see [36], Theorem 8.1.2).

The following is a fact about the splitting of exact sequences of groups (for a proof see Brown [5], Theorem 6.6, p.105).

**Lemma 3.4** Let

$$1 \to N \to G \to Q \to 1$$

be a short exact sequence of groups. Denote by $Z(N)$ the center of $N$. If the second cohomology group $H^2(Q; Z(N)) = 0$ and $Q$ acts trivially on $N$, then the exact sequence is splitting.

3.2 Group actions on 1-manifolds

In this section, we will discuss the results proved by Witte [29].

Recall that a group $G$ acts effectively on a space $X$ if the induced group homomorphism $G \to \text{Homeo}(X)$ is injective. The following result is well-known (cf. [20], Theorem 2.2.19, p.38).

**Lemma 3.5** If a countable group $G$ acts effectively on the real line $\mathbb{R}$ by orientation-preserving homeomorphisms, then the group $G$ is left orderable. In other words, there is a total order $<$ on $G$ such that $gh < gh'$ whenever $h < h'$ for any $g \in G$.

**Proof.** Choose a dense countable subset $\{x_n\}_{n=1}^{+\infty}$ (eg. rational numbers $\mathbb{Q}$) of $\mathbb{R}$. For two elements $f, g \in G$, we define $f < g$ if $f(x_n) < g(x_n)$ for the first $n$ with $f(x_n) \neq g(x_n)$ but $f(x_i) = g(x_i), i = 1, 2, \cdots, n-1$. It’s not hard to check that this an order. Since any $h \in G$ is orientation-preserving, we have $hf(x_n) < hf(x_n)$. This shows that the order is a left order. \[\blacksquare\]

Conversely, a countable left-orderable group $G$ acts effectively on a real line. But when $G$ is uncountable (and left-orderable), the result may not be true (cf. Mann [22]).

**Lemma 3.6** Let $\Gamma$ be a finite-index subgroup of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$). Then $\Gamma$ is not left-orderable.
Proof. Without loss of generality, we assume $n = 3$. Let $e_{ij}(k)$ denote the matrix with ones along diagonal, entry $k$ in the $(i, j)$-th position and zeros elsewhere. Since $\Gamma$ is finite-index, for a sufficiently large $k$ we assume that $\Gamma$ contains all $e_{ij}(k)$.

Suppose that $\Gamma$ has an order $<$, which is preserved by left multiplications. We choose $a = e_{12}(\pm k), b = e_{23}(\pm k)$ and $c = e_{13}(\pm k^2)$ such that $a, b, c > 1$. Note that $[a, b] = aba^{-1}b^{-1} = c^{-1}$ or $c$.

We claim that either $a > c^{s}$ or $b > c^{s}$ for any integer $s > 0$ (denoted as $a >> c$ or $b >> c$ in the following). To show this, suppose that $a < c^{m_1}$ and $a < c^{m_2}$ for some $m_1, m_2 > 0$. Hence $a < c^k, b < c^k$ for $k = m_1m_2$ since $c$ is positive. Let $d_m = a^mb^m(a^{-1}c^k)^m(b^{-1}c^k)^m > 1$ for $m > 0$. Note that $d_m = [a^m, b^m]e^{2km} = e^{-m^2+2km} < 1$ for sufficiently large $m$ (assuming $[a, b] = c^{-1}$). If $[a, b] = c$, we may switch the positions of $a$ and $b$.

This is a contradiction. We denote $e_{ij}$ the positive elements in $\{e_{ij}(\pm k)\}$.

We see that $e_{12} >> e_{13}$ or $e_{23} >> e_{13}$. For the first case, note that $[e_{13}, e_{32}] = e_{12}$ and similarly we have $e_{32} >> e_{12}$. We thus get

$$e_{13} << e_{12} << e_{32} << ... << e_{13},$$

a contradiction. The other case could be proved similarly. $lacksquare$

The previous two lemmas show that any action of a finite-index subgroup on the real line $\mathbb{R}$ factors a finite group action.

Theorem 3.7 Any action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) by homeomorphisms on $S^1$ is trivial. In particular, Conjecture 1.2 holds for $r = 1$.

Outline of the proof. Let $f : \text{SL}_n(\mathbb{Z}) \to \text{Homeo}(S^1)$ be a group homomorphism. Suppose that $\ker f = 1$. By lifting the group action to the universal cover, we have an exact sequence

$$1 \to \mathbb{Z} \to G^* \to \text{SL}_n(\mathbb{Z}) \to 1,$$

where $G^* < \text{Homeo}(\mathbb{R})$. Note that $\text{GL}_1(\mathbb{Z}) = \mathbb{Z}/2$ and $\text{SL}_n(\mathbb{Z})$ is the same as its commutator subgroup when $n \geq 3$. This means that $\text{SL}_n(\mathbb{Z})$ acts trivially on the first term $\mathbb{Z}$. Since the second cohomology group $H^2(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$, the exact sequence is split by Lemma 3.4. Therefore, the group $\text{SL}_n(\mathbb{Z})$ acts effectively on the universal cover $\mathbb{R}$. This is impossible, since a finite cyclic group acting effectively on $\mathbb{R}$ is of order 2. If $\ker f$ is not trivial, the image $\text{Im} f$ is finite by the congruence subgroup property (see Lemma 3.2). Now a similar argument proves that $\text{Im} f$ is trivial. $lacksquare$

3.3 Group actions on surfaces
The study of group actions on surfaces has a long history. The following is well-known (see Edmonds [9], pp340–341).
Lemma 3.8 Let $\Sigma_g$ be a closed orientable surface. A finite subgroup $G < \text{Homeo}^+(\Sigma_g)$ (orientation-preserving homeomorphisms) is conjugate to a subgroup $G'$ preserving a complex structure.

By Greenberg [18], every finite group $G$ is a subgroup of $\text{Homeo}(\Sigma_g)$ for some surface $\Sigma_g$.

Example 3.9 A finite subgroup $G < \text{Homeo}(S^2)$ is a subgroup of $O(3)$ (cf. [23][9]). The set of all finite subgroups of $\text{Homeo}(T^2)$ can also be classified. For higher-genus surface $\Sigma_g, g > 1$, a finite subgroup $G < \text{Homeo}(\Sigma_g)$ is a quotient group of a non-Euclidean crystallographic group by the surface group $\pi_1(\Sigma_g)$ (cf. [17]).

Lemma 3.10 Suppose that Conjecture 3.1 holds for $r = 2$. Any group action of a torsion-free finite-index subgroup $\Gamma$ of $\text{SL}_n(\mathbb{Z})$ on a surface $\Sigma_g$ factors through a finite group.

Proof. Let $f : \Gamma \to \text{Homeo}(\Sigma_g)$ be a group homomorphism. If $\ker f$ is trivial, the image $\text{Im} f$ is finite by the congruence subgroup property (see Lemma 3.2). Suppose that $\ker f = 1$. Let $\text{Homeo}_0(\Sigma_g)$ be the identity component of $\text{Homeo}(\Sigma_g)$ and $\text{Mod}(\Sigma_g) = \text{Homeo}(\Sigma_g)/\text{Homeo}_0(\Sigma_g)$ be the mapping class group. Since the composite $\Gamma \to \text{Homeo}(\Sigma_g) \to \text{Mod}(\Sigma_g)$ has finite image (by Farb-Masur [19], Theorem 1.1), there is a finite-index subgroup $\Gamma'$ of $\Gamma$ such that $f(\Gamma')$ lies in $\text{Homeo}_0(\Sigma_g)$ the identity component of $\text{Homeo}(\Sigma_g)$. We thus have a lifting

$$1 \to \pi_1(\Sigma_g) \to G^* \to \Gamma' \to 1,$$

for some group $G^* < \text{Homeo}(\mathbb{H}^2)$, where $\mathbb{H}^2$ is the universal cover of $\Sigma_g$. Note that $\Gamma'$ acts trivially on $\pi_1(\Sigma_g)$ and $\pi_1(\Sigma_g)$ is centerless. Thus the second cohomology group $H^2(\Gamma'; \mathbb{Z}(\pi_1(\Sigma_g))) = 0$ and the above exact sequence is split by Lemma 3.4. Therefore, the group $\Gamma'$ could be lifted to be a subgroup of $G^*$ and thus acts on $\mathbb{H}^2$. However, the confirmation of Conjecture 3.1 for $r = 2$ implies the group action of $\Gamma'$ on $\mathbb{H}^2$ (which is homeomorphic to the plane $\mathbb{R}^2$) is finite. This is a contradiction. ■

3.4 Group actions on spheres

In the study of group actions on high dimensional manifolds, there are several interesting phenomena. One of them is that the fixed point set of a finite group $G$ acting on a topological manifold $M$ is not necessary a topological manifold. Another one is that a cyclic group $\mathbb{Z}/pq$ ($p, q$ are two different primes) could act without fixed points on some Euclidean space $\mathbb{R}^n$ (cf. Bredon [3], Section I.8, p. 55). A good way to deal with these difficulties is to work in the category of (co)homology manifolds. Roughly speaking, a cohomology $n$-manifold mod $p$ is a locally compact Hausdorff space that has a local cohomology structure (with coefficient group $\mathbb{Z}/p$) resembling that of Euclidean $n$-space. Let $L = \mathbb{Z}$ or $\mathbb{Z}/p$. All homology groups in this section are Borel-Moore homology groups with
compact supports and coefficients in a sheaf $\mathcal{A}$ of modules over $L$. The homology
groups of $X$ are denoted by $H^*_c(X;\mathcal{A})$ and the Alexander-Spanier cohomology
groups (with coefficients in $L$ and compact supports) are denoted by $H^*_c(X;L)$.
We define the cohomology dimension $\dim_L X = \min\{n \mid H^{n+1}_c(U;L) = 0$ for
all open $U \subset X\}$. If $L = \mathbb{Z}/p$, we write $\dim_p X$ for $\dim_L X$. For integer $k \geq 0$,
let $\mathcal{O}_k$ denote the sheaf associated to the pre-sheaf $U \mapsto H^k_c(X,X\setminus U;L)$. An
$n$-dimensional homology manifold over $L$ (denoted $n$-hm$_L$) is a locally compact
Hausdorff space $X$ with $\dim_L X < +\infty$, and $\mathcal{O}_k(X;L) = 0$ for $p \neq n$ and
$\mathcal{O}_n(X;L)$ is locally constant with stalks isomorphic to $L$. The sheaf $\mathcal{O}_n$ is
called the orientation sheaf. There is a similar notion of cohomology manifold
over $L$, denoted $n$-cm$_L$ (cf. [2], p.373). Topological manifolds are (co)homology
manifolds over $L$.

The following is called the Smith theory, which is a combination of Corollary
19.8 and Corollary 19.9 (page 144) in [2] (see also Theorem 4.5 in [4]).

**Lemma 3.11** Let $p$ be a prime and $X$ be a locally compact Hausdorff space of
finite dimension over $\mathbb{Z}_p$. Suppose that $\mathbb{Z}_p$ acts on $X$ with fixed-point set $F$.

(i) If $H^*_c(X;\mathbb{Z}_p) \cong H^*_c(S^m;\mathbb{Z}_p)$, then $H^*_c(F;\mathbb{Z}_p) \cong H^*_c(S^r;\mathbb{Z}_p)$
for some $r$ with $-1 \leq r \leq m$. If $p$ is odd, then $r - m$ is even.

(ii) If $X$ is $\mathbb{Z}_p$-acyclic, then $F$ is $\mathbb{Z}_p$-acyclic (in particular non-empty and connected).

**Lemma 3.12** ([4], Theorem 4.7) If $m < d - 1$, the group $(\mathbb{Z}/2)^d$ cannot act ef-
fectively on a generalized $m$-sphere over $\mathbb{Z}/2$ or a $\mathbb{Z}/2$-acyclic $(m+1)$-dimensional
homology manifold over $\mathbb{Z}/2$.

If $m < 2d - 1$ and $p$ is odd, then $(\mathbb{Z}/p)^d$ cannot act effectively a generalized
$m$-sphere or a $\mathbb{Z}/p$-acyclic $(m + 1)$-dimensional homology manifold over $\mathbb{Z}/p$.

**Theorem 3.13** ([4], Theorem 1.1) Any action of $\text{SL}_n(\mathbb{Z})$ $(n \geq 3)$ on the sphere
$S^k$ $(k < n - 1)$ by homeomorphisms is trivial.

**Outline of the proof.** Note that the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ is of order 3.
When $n$ is even, $\text{SL}_n(\mathbb{Z})$ contains $\frac{n}{2}$ copies of $\mathbb{Z}/3$ generated by $A$ along the
diagonal. The previous lemma implies that this subgroup $(\mathbb{Z}/3)^\frac{n}{2}$ cannot act
effectively on $S^k$. The normal subgroup generated by this non-trivial element
acting trivially is the whole $\text{SL}_n(\mathbb{Z})$ (see Lemma 3.2). When $n$ is odd, an
additional argument proves that the bound for the rank of $(\mathbb{Z}/2)^{n-1}$ in the
previous lemma could be improved by one, considering the fact that the action
of $\text{SL}_n(\mathbb{Z})$ is orientation-preserving. Using the elementary 2-subgroup along the
diagonal of $\text{SL}_n(\mathbb{Z})$, a similar argument finishes the proof (for more details, see
[4], Proposition 4.13).

In a similar way, the following can be proved.

**Theorem 3.14** ([4], Theorem 1.2) Any action of $\text{SL}_n(\mathbb{Z})$ $(n \geq 3)$ on the acyclic
space $\mathbb{R}^k$ $(k < n)$ by homeomorphisms is trivial.
3.5 Manifolds with non-zero Euler characteristics

For a group $G$ and a prime $p$, let the $p$-rank be $\text{rk}_p(G) = \sup\{k \mid (\mathbb{Z}/p)^k \hookrightarrow G\}$. It is possible that $\text{rk}_p(G) = +\infty$. The following result relates the $p$-rank of the group acting effectively and the Euler characteristic of the acted manifold.

**Theorem 3.15** ([32]) Let $M^r$ be a first countable connected cohomology $r$-manifold over $\mathbb{Z}/p$ and $\text{Homeo}(M)$ the group of self-homeomorphisms. We adapt the convention that $p^n = 1$ when $n < 0$. Then the $p$-rank satisfies

$$p^{\text{rk}_p(\text{Homeo}(M)) - \lfloor \frac{r}{2} \rfloor} \mid \chi(M; \mathbb{Z}/p)$$

when $p$ is odd and

$$2^{\text{rk}_2(\text{Homeo}(M)) - r} \mid \chi(M; \mathbb{Z}/2)$$

when $p = 2$. If $M^r$ ($r \geq 1$) is an oriented connected cohomology $r$-manifold over $\mathbb{Z}$ and $\text{Homeo}_+(M)$ is the group of orientation-preserving self-homeomorphisms, we have

$$2^{\text{rk}_2(\text{Homeo}(M)) - r + 1} \mid \chi(M; \mathbb{Z}/2).$$

In particular, when $(\mathbb{Z}/p)^k$ acts effectively on a manifold $M$, we have that $p^{k-\lfloor \frac{r}{2} \rfloor} \mid \chi(M; \mathbb{Z}/p)$ when $p > 2$ and $p^{k-r} \mid \chi(M; \mathbb{Z}/2)$ when $p = 2$. Based on the previous result, we have the following criterion for non-trivial matrix group actions.

**Theorem 3.16** ([32]) Let $M^r$ be a connected (resp. orientable) manifold with the Euler characteristic $\chi(M) \not\equiv 0 \mod 3$ (resp. $\chi(M) \not\equiv 0 \mod 6$). Then any group action of $\text{SL}_n(\mathbb{Z})$ ($n > r + 1$) on $M^r$ by homeomorphisms is trivial.

**Outline of the proof.** Suppose that $\text{SL}_n(\mathbb{Z})$ acts effectively on $M$. Since the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ is of order 3, the group $\text{SL}_n(\mathbb{Z})$ contains $(\mathbb{Z}/3)^{[n/2]}$ as a subgroup. The previous theorem implies that $3 \mid \chi(M)$. Consider the elementary 2-group $(\mathbb{Z}/2)^{n-1} < \text{SL}_n(\mathbb{Z})$ given by the diagonal entries $\text{diag}(\pm 1, \cdots, \pm 1)$. When $n \geq 3$ and $M$ is orientable, the action of $\text{SL}_n(\mathbb{Z})$ is orientation-preserving. The previous theorem implies that $2 \mid \chi(M)$. Therefore, we have $6 \mid \chi(M)$. ■

Theorem 3.14 is a special case of Theorem 3.16 since the Euler characteristic of an acyclic space is one.

3.6 Group actions on flat manifolds

Let $\Gamma$ be a group acting freely, isometrically, properly discontinuously and co-compactly on the Euclidean space $\mathbb{R}^n$. The quotient space $M = \mathbb{R}^n/\Gamma$ is called a flat manifold. A classical result of Bieberbach implies that there is a short exact sequence of groups

$$1 \to \mathbb{Z}^n \to \Gamma \to \Phi \to 1,$$
where $\Phi < \text{GL}_n(\mathbb{Z})$ is called the holonomy group of $M$. Topologically, a (closed) flat manifold $M$ is covered by the torus $T^n$ with $\Phi$ as the deck transformation group.

We give a sufficient and necessary condition for a finite group to act effectively on a flat manifold:

**Theorem 3.17** ([33], Theorem 1.2) A finite group $G$ acts effectively on a closed flat manifold $M^n$ with the fundamental group $\pi$ and the holonomy group $\Phi$ by homeomorphisms if and only if there is an abelian extension

$$1 \to A \to G \to Q \to 1$$

such that

(i) $Q \cong \Phi^*/\Phi$ for a finite subgroup $\Phi^* < \text{GL}_n(\mathbb{Z})$;

(ii) there is a $(\Phi^*, Q)$-equivariant surjection $\alpha : \mathbb{Z}^n \to A$, and a commutative diagram

$$
\begin{array}{cccccc}
1 & \to & \mathbb{Z}^n & \to & G^* & \to & \Phi^* & \to & 1 \\
\alpha \downarrow & & f \downarrow & & \downarrow & & \\
1 & \to & A & \to & G & \to & Q & \to & 1
\end{array}
$$

with torsion-free kernel $\ker f = \pi$. Here $\alpha(gx) = \bar{g}\alpha(x)$ for any $x \in \mathbb{Z}^n$, $g \in \Phi^*$, where $\bar{g} \in Q$ acts on the abelian group $A$ through the exact sequence and $\ker(\Phi^* \to Q) = \Phi$.

The condition looks a bit complicated, partially because of the holonomy group $\Phi$. For torus $M = T^n$ (where $\Phi$ is trivial), we have the following simpler characterization. To the best of our knowledge, this characterization has so far not been stated explicitly in the literature, except possibly for low-dimensional cases (eg. $n = 1, 2$).

**Theorem 3.18** ([33], Theorem 1.3) A finite group $G$ acts effectively on a torus $T^n$ if and only if there is an abelian extension

$$1 \to A \to G \to Q \to 1$$

such that

(i) $Q < \text{GL}_n(\mathbb{Z})$;

(ii) there is a $Q$-equivariant surjection $\alpha : \mathbb{Z}^n \to A$ and the cohomology class representing of the extension lies in the image $\text{Im}(H^2(Q; \mathbb{Z}^n) \to H^2(Q; A))$.

**Example 3.19** When $n = 1$, $M = S^1$, we have that the group $Q < \text{GL}_1(\mathbb{Z}) = \mathbb{Z}/2$ and $A$ is a finite cyclic group. Therefore, any finite group $G$ acting effectively on the circle $S^1$ is a subgroup of a Dihedral group.
Example 3.20 Let $A_n$ be the alternating group. The group $A_4$ could act effectively on the 2-dimensional torus $T^2$.

Proof. Note that $A_4 \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$, where $(\mathbb{Z}/2)^2 = \langle (14)(23), (13)(24) \rangle$ and $\mathbb{Z}/3 = \langle (123) \rangle$, where $\mathbb{Z}/3$ acts on $(\mathbb{Z}/2)^2$ through matrix
\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\in \text{GL}_2(\mathbb{Z}).
\]
Take $Q = \mathbb{Z}/3$. Define $G^* = \mathbb{Z}^2 \rtimes \mathbb{Z}/3$, where $\mathbb{Z}^2$ acts on the free abelian group $\mathbb{Z}^2$ through matrix
\[
\begin{pmatrix}
0 & -1 \\
1 & -1
\end{pmatrix}
\in \text{GL}_2(\mathbb{Z}).
\]
Let $\alpha : \mathbb{Z}^2 \to (\mathbb{Z}/2)^2$ be the modulo 2 map. It is not hard to see that $\alpha$ is $Q$-equivariant. Moreover, the map $\alpha$ induces a map between the two split extensions. Theorem 3.18 implies that $A_4$ acts effectively on $T^2$. 

Based on this characterization of finite groups acting effectively on flat manifolds, we confirm Conjecture 1.2 for these manifolds. For a ring $R$, let $E_n(R)$ be the subgroup of the general linear group $\text{GL}_n(R)$ generated by elementary matrices. When $R = \mathbb{Z}$, we have $E_n(R) = \text{SL}_n(\mathbb{Z})$. Denote by $\text{EU}_n(R, \Lambda)$ the elementary quadratic group (for a definition, see [33]). Denote by $F_n$ the free group of $n$ letters and $\text{SAut}(F_n)$ (or $\text{SOut}(F_n)$) the unique index two subgroup of the automorphism group $\text{Aut}(F_n)$ (or outer automorphism group $\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$).

Theorem 3.21 ([33], Theorem 1.5) Let $G = E_n(R)$, $\text{EU}_n(R, \Lambda)$, $\text{SAut}(F_n)$ or $\text{SOut}(F_n)$, $n \geq 3$. Suppose that $M^r$ is a closed flat manifold. When $r < n$, any group action of the group $G$ on $M^r$ by homeomorphisms is trivial, i.e. is the identity homeomorphism.

Remark 3.22 The previous theorem does not hold for $n = 2$. The group $\text{SL}_2(\mathbb{Z})$ acts non-trivially on $S^1$ as shown in the introduction.

3.7 Group actions on aspherical manifolds

A manifold $M$ is aspherical if the universal cover $\tilde{M}$ is contractible. Note that the sphere $S^n$, $n > 1$, is not aspherical.

Conjecture 3.23 Any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on a closed aspherical $r$-manifold $M^r$ by homeomorphisms factors a finite group if $r < n$.

The natural action of $\text{SL}_n(\mathbb{Z})$ on the torus $T^n$ shows that the bound of $r$ in Conjecture 3.23 cannot be improved. Note that the upper bound of the dimension $r$ in Conjecture 3.23 is $n - 1$, while that of Conjecture 1.2 is $n - 2$. 

10
Lemma 3.24 (31, Theorem 1.2) Let $M^r$ be an aspherical manifold. A group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $M^r$ ($r \leq n - 1$) by homeomorphisms is trivial if and only if the induced group homomorphism $\text{SL}_n(\mathbb{Z}) \to \text{Out}(\pi_1(M))$ is trivial. In particular, Conjecture 3.23 holds if the set of group homomorphisms $	ext{Hom}(\text{SL}_n(\mathbb{Z}), \text{Out}(\pi_1(M))) = 1$.

Outline of the proof. It is clear that when the group action is trivial, the induced group homomorphism is trivial. Conversely, suppose that $\text{SL}_n(\mathbb{Z})$ acts on $M$ effectively. We have an exact sequence

$$1 \to \pi_1(M) \to G \to \text{SL}_n(\mathbb{Z}) \to 1,$$

where $G < \text{Homeo}(\tilde{M})$, the homeomorphism group of the universal cover. Since the dimension of $M$ is smaller than $n$, the rank of the center $Z(\pi_1(M))$ is smaller than $n$ as well by considering the cohomological dimension. This implies that the action of $\text{SL}_n(\mathbb{Z})$ on $Z(\pi_1(M))$ is trivial. It is known that $H^2(\text{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$. By Lemma 3.14 the exact sequence is splitting. Then we have a group action of $\text{SL}_n(\mathbb{Z})$ on the universal cover $\tilde{M}$, which is contractible. A result of Bridson and Vogtmann [4] (see also Theorem 3.14) implies that the action of $\text{SL}_n(\mathbb{Z})$ on the acyclic space $\tilde{M}$ (and thus on $M$) is trivial.

The previous lemma reduces Conjecture 3.23 to an algebraic question: whether any group homomorphism from $\text{SL}_n(\mathbb{Z})$ to $\text{Out}(\pi_1(M))$ is trivial. This is the case when $\pi_1(M)$ is nilpotent.

Theorem 3.25 (31, Theorem 1.4) Let $M^r$ be an aspherical manifold. If the fundamental group $\pi_1(M)$ is finitely generated nilpotent, any group action of $\text{SL}_n(\mathbb{Z})$ ($n \geq 3$) on $M^r$ ($r \leq n - 1$) by homeomorphisms is trivial.

Similarly, when $M$ is an almost flat manifold (i.e. a manifold finitely covered by a nil-manifold) with dihedral, symmetric or alternating holonomy group, it satisfies Conjecture 3.23 (see [31], Lemma 6.5).

Corollary 3.26 Let $M$ be a closed aspherical manifold. Any action of the real special linear group $\text{SL}_n(\mathbb{R}), n \geq 3$, on $M$ by homeomorphism is trivial.

Proof. When $M$ is compact, the fundamental group $\pi_1(M)$ is finitely presented and $\text{Out}(\pi_1(M))$ is countable. Therefore, any group homomorphism $\text{SL}_n(\mathbb{R}) \to \text{Out}(\pi_1(M))$ is trivial (note that $\text{PSL}_n(\mathbb{R})$ is a simple group). Lemma 3.24 implies that the action of $\text{SL}_n(\mathbb{Z})$ (and thus $\text{SL}_n(\mathbb{R})$) is trivial, when $\dim(M) < n$. Actually, a compact Lie group (including finite group) acting effectively and homotopically trivially on $M$ must be abelian (see [15], Theorem 2.5). The group $\text{SL}_n(\mathbb{R})$ contains a non-abelian finite subgroup (for example, the alternating group $A_{n+1}$). This implies that the action of $\text{SL}_n(\mathbb{R})$ is trivial for any $M$.

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