Brane-Wor(l)ds within Brans-Dicke Gravity*

M. C. B. Abdalla$^1$, M. E. X. Guimarães$^2$ and J. M. Hoff da Silva$^3$

1. Instituto de Física Teórica, Universidade Estadual Paulista,
Rua Pamplona 145 01405-900 São Paulo, SP, Brazil and
2. Instituto de Física, Universidade Federal Fluminense, Niterói-RJ, Brazil

We review some recent results obtained from the application of the Gauss-Codazzi formalism to brane-worlds models in the Brans-Dicke gravity. The cases of 4-branes embedded in a six-dimensional with and without $Z_2$ symmetry are both analyzed.

PACS numbers: 04.50.+h; 98.80Cq

I. INTRODUCTION

Soon after the establishment of the modern concepts of brane-worlds models [1] it was realized their importance in the study of gravitational systems [2], as well as their application in the analysis of cosmological problems [3]. For codimension one brane-world models there is a very powerful tool — the so-called Gauss-Codazzi formalism — developed in order to project the gravitational field equations from the bulk to the brane [4]. This procedure allows the exploration of brane cosmological signatures in a deep way. Roughly speaking, the existence of extra dimensions, within gravitational context, added new source terms to the brane projected gravitational field equation. Obviously, this also happens in the Brans-Dicke [5] gravity. Besides, new “source terms” appear due to the dynamics of the additional gravitational scalar field. These terms appearing in the projected gravitational equations in the Brans-Dicke theory certainly lead to new cosmological signatures. The reason to analyze brane-world models in such scalar-tensorial theory rests on the fact that, at least sufficiently high energies, the General Relativity is not able to fully describe mostly of the puzzled gravitational phenomena [6]. Apart of that, there is a strong interplay between Brans-Dicke theory and the gravity theory recovered from string theory at low energies [7]. In this vein, it is possible to obtain information about some systems in string theory by the analysis in Brans-Dicke framework. The relationship between...
the two theories is given by a — model depended — relabel of the Brans-Dicke parameter.

From the topological point of view, the brane(s) in the standard Randall-Sundrum model is (are) performed by domain wall(s) and the extra dimension is a $S^1/Z_2$ orbifold. It is familiar for cosmologists, however, that domain walls (with symmetry breaking scales greater than 1 MeV) are problematic and should not appear in a complete scenario. The program of an exotic compactification of extra dimensions using topological defects continued with global cosmic strings in General Relativity and with local and global cosmic strings in the Brans-Dicke gravity. Because of the defect used to generate the bulk-brane structure (the cosmic string), these models hold in six-dimensions. Models studied in the Brans-Dicke framework present only one transverse extra dimension (codimension one models), while the brane has five-dimensions with topology given by $R^4 \times S^1$. The presence of a transverse and a curled dimension in the same model is called hybrid compactification.

The aim of this work is to review the application of the Gauss-Codazzi formalism for hybrid compactification models in the Brans-Dicke framework. In Section II we develop the main lines of this approach in the case where the spacetime is endowed with $Z_2$ orbifold symmetry. In the Section III, we work in the case without such symmetry and in Section IV, we conclude with some final remarks and possible applications. We stress that the main results delivered here, as well as the details, are somehow described in [12, 13].

II. PROJECTED GRAVITATIONAL FIELD EQUATIONS IN $Z_2$ SYMMETRIC BRANE-WORLDS

From now on we consider the brane as a five-dimensional submanifold embedded in a manifold of six-dimensions — the bulk. As remarked before, the motivations to work in such dimensionality can be found in [9, 10] (and references therein). Denoting the covariant derivative in the bulk by $\nabla_\mu$ and the one associated to the brane by $D_\mu$ the Gauss equation reads

$$\begin{align}(5)R^\alpha_{\beta\gamma\delta} &= (6)R^\mu_{\nu\rho\sigma}q^\alpha_\mu q^\beta_\nu q^\gamma_\rho q^\sigma_\sigma + K^\alpha_\gamma K^\beta_\delta - K^\alpha_\delta K^\beta_\gamma,
\end{align}(1)$$

where $q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric on the brane ($n_\mu$ being the orthogonal unitary vector along to the extra transverse dimension) and $K_{\mu\nu} = q^\alpha_\mu q^\beta_\nu \nabla_\alpha n_\beta$ is the extrinsic curvature, which gives information about the embedding of the brane. Starting from equation (1), it is easy to see that the Einstein tensor on the brane in given by
\[ G_{\beta\delta} = G_{\nu\sigma} q_{\beta}^{\nu} q_{\delta}^{\sigma} + R_{\nu\sigma} n^{\nu} n^{\sigma} q_{\beta\delta} + KK_{\beta\delta} - K_{\delta}^{\gamma} K_{\beta\gamma} - \frac{1}{2} q_{\beta\delta} (K^2 - K^{\alpha\gamma} K_{\alpha\gamma}) - \tilde{E}_{\beta\delta}, \] (2)

where \( \tilde{E}_{\beta\delta} = R_{\nu\rho} n_{\mu} n^{\rho} q_{\beta}^{\mu} q_{\delta}^{\nu} \). Now, taking into account that the relation between the Riemann, Ricci and Weyl tensors in an arbitrary dimension \( (n) \) is

\[ (n) R_{\alpha\beta\mu\nu} = (n) C_{\alpha\beta\mu\nu} + \frac{2}{n-2} ( (n) R_{\alpha[\mu} g_{\nu]\beta] - (n) R_{\beta[\mu} g_{\nu]\alpha] ) - \frac{2}{n-1} (n-2) R g_{\alpha[\mu} g_{\nu]\beta], \] (3)

we can rewrite the equation \( (2) \) in the form

\[ G_{\beta\delta} = \frac{1}{2} (6) G_{\nu\sigma} q_{\beta}^{\nu} q_{\delta}^{\sigma} - \frac{1}{2} (6) R q_{\beta\delta} - \frac{1}{2} (6) R_{\nu\sigma} q^{\nu\sigma} q_{\beta\delta} + KK_{\beta\delta} - K_{\delta}^{\gamma} K_{\beta\gamma} - \frac{1}{2} q_{\beta\delta} (K^2 - K^{\alpha\gamma} K_{\alpha\gamma}) - E_{\beta\delta}, \] (4)

where \( E_{\beta\delta} = C_{\nu\rho\sigma} n_{\mu} n^{\rho} q_{\beta}^{\nu} q_{\delta}^{\sigma} \).

The idea, hereafter, is to express the geometric quantities of the brane in terms of the stress-tensor and the scalar dynamics of the bulk, in order to apply the Gauss-Codazzi formalism to the case in question. With this purpose, we remember that the Einstein-Brans-Dicke equation is given by

\[ G_{\mu\nu} = \frac{8\pi}{\phi} T_{M\mu\nu} + \frac{w}{\phi^2} \left( \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \nabla_{\alpha} \phi \nabla^{\alpha} \phi \right) + \frac{1}{\phi} \left( \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \Box^{2} \phi \right), \] (5)

where \( \phi \) is the Brans-Dicke scalar field — the dilaton —, \( w \) a dimensionless parameter and \( T_{M\mu\nu} \) the matter energy-momentum tensor, everything except \( \phi \) and gravity, in the bulk. The scalar equation of Brans-Dicke theory is given by

\[ \Box^{2} \phi = \frac{8\pi}{3+2w} T_{M}. \] (6)

Inserting the equation \( (6) \) into \( (5) \) and founding the Ricci tensor and the scalar of curvature, it is possible to express the equation \( (4) \) as

\[ G_{\beta\delta} = \frac{1}{2} \left[ \frac{8\pi}{\phi} T_{M\mu\nu} + \frac{1}{\phi} \nabla_{\nu} \nabla_{\rho} \phi + \frac{w}{\phi^2} \nabla_{\nu} \phi \nabla_{\rho} \phi \right] \left( q_{\beta}^{\nu} q_{\delta}^{\rho} - q^{\nu\rho} q_{\beta\delta} \right) + \frac{2\pi}{5\phi} q_{\beta\delta} T_{M} \left( \frac{13 + 27w}{3 + 2w} \right) - \frac{7w}{20\phi^2} q_{\beta\delta} \nabla_{\alpha} \phi \nabla^{\alpha} \phi + KK_{\beta\delta} - K_{\delta}^{\gamma} K_{\beta\gamma} \]

\[ - \frac{1}{2} q_{\beta\delta} (K^2 - K^{\alpha\gamma} K_{\alpha\gamma}) - E_{\beta\delta}. \] (7)
In order to extract information about this system we have to compute the quantities on the brane. It can be implemented by taking the limit of the extra transverse dimension tending to the brane, but we have to specify the behavior of the extrinsic curvature under such limit. This is a central piece in the application of the Gauss-Codazzi formalism and strongly depends whether or not the spacetime is endowed with a $\mathbb{Z}_2$ symmetry. In the case where there is such symmetry, it is possible to show, by application of distributional calculus tools, that the extrinsic curvature in one side of the brane, $K^+_{\mu\nu}$, is related with its other side partner, $K^-_{\mu\nu}$, by (see reference [12] for all the details)

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = \frac{4\pi}{\phi} \left( -T_{\mu\nu} + \frac{q_{\mu\nu}(1 + w)T}{2(3 + 2w)} \right),$$

and

$$K^+ = K^- = \frac{2\pi}{\phi} \left( \frac{w - 1}{3 + 2w} \right) T. \tag{9}$$

The relation (8) is the generalization of the so-called Israel-Darmois matching conditions [15] to the Brans-Dicke gravity. It is possible to split the matter stress-tensor in

$$T_{M\mu\nu} = -\Lambda g_{\mu\nu} + \delta(y)T_{\mu\nu}, \tag{10}$$

and

$$T_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}, \tag{11}$$

where $\Lambda$ is the cosmological constant of the bulk and $\lambda$ the tension of the brane\(^1\).

Now, substituting the equations (8)-(11) into (7) we obtain

$$(5) G_{\beta\delta} = \frac{1}{2} \left( \frac{1}{\phi} \nabla_\nu \nabla_\sigma \phi + \frac{w}{\phi^2} \nabla_\nu \phi \nabla_\sigma \phi \right) \left( q^\nu_\beta q^\sigma_\delta - q^\nu_\delta q^\sigma_\beta \right) + 8\pi \Omega \tau_{\beta\delta} - \Lambda_5 q_{\beta\delta}
+ 8 \left( \frac{\pi}{\phi} \right)^2 \Sigma_{\beta\delta} - E_{\beta\delta}, \tag{12}$$

where

$$\Omega = \frac{3\pi(w - 1)\lambda}{\phi^2(3 + 2w)} \tag{13}$$

$$\Lambda_5 = \frac{-4\pi\Lambda(21 - 41w)}{5\phi(3 + 2w)} + \left( \frac{\pi}{\phi} \right)^2 \left[ \frac{7w}{20\pi^2} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{24(w - 1)\lambda}{(3 + 2w)^2} [(w - 1)\lambda + \tau] \right], \tag{14}$$

and

$$\Sigma_{\beta\delta} = q_{\beta\delta} \tau_{\alpha\gamma} \tau_{\alpha\gamma} - 2\tau^\gamma_\delta \tau_{\gamma\beta} + \left( \frac{3 + w}{3 + 2w} \right) \tau_{\beta\delta} - \frac{(w^2 + 3w + 3)}{(3 + 2w)^2} q_{\beta\delta} \tau^2. \tag{15}$$

The main property to be noted from equation \([12]\) is that we do not recover Brans-Dicke gravity on the brane if the dilaton depends only on the extra transverse

\(^1\) We remark that the delta term appearing in such decomposition can lead to complications in a complete cosmological scenario.
dimension. Instead, Einstein equation is recovered with subtle but important modifications coming from both extra dimensions and dilaton dynamics. Hence, equation \[12\] can be used to extract deviations from usual General Relativity. We refer again the reader to reference \[12\] for more analysis and comments on the implications of the result encoded in \[12\]-\[15\].

III. LIFTING THE Z\(_2\) SYMMETRY

The Z\(_2\) symmetry has a multiple role in brane-worlds scenarios \[13\]. In what concerns to the gravitational aspects it determines univocally the jump of the extrinsic curvature across the brane \[8\]. In this vein, it is not surprising that the absence of such symmetry makes the calculations a little more involved. In this Section we shall present the guidelines of how to project the Einstein-Brans-Dicke equation on the brane without the Z\(_2\) symmetry. More details can be found in reference \[13\] for the Brans-Dicke case and in \[16\] for the context of General Relativity\(^2\). Let us start defining two quite important tools which determine the mean value of any tensorial quantity, say \(X\),

\[
\langle X \rangle = \frac{1}{2} (X^+ + X^-),
\]

and the jump across the brane

\[
[X] = X^+ - X^-,
\]

where \(X^\pm\) are both limits of \(X\) approaching the brane from both \(\pm\) sides. It is not difficult to see that the quantities defined by equations \[16\] and \[17\] lead to the algebra

\[
[AB] = \langle A \rangle \langle B \rangle + [A] \langle B \rangle,
\]

\[
\langle AB \rangle = \langle A \rangle \langle B \rangle + \frac{1}{4} [A] [B].
\]

Note that from the Gauss equation \[1\], we can write down the Ricci tensor on the brane in a more convenient way

\[
^{(5)}R_{\mu\nu} = Y_{\mu\nu} + KK_{\mu\nu} - K^\lambda_{\mu} K_{\lambda\nu},
\]

where

\[
Y_{\mu\nu} \equiv \frac{3}{4} \left(^{(6)}R_{\alpha\beta} q^\alpha_{\mu} q^\beta_{\nu} + \frac{1}{4} \left( ^{6}R_{\alpha\beta} q^{\alpha\beta} q_{\mu\nu} - \frac{1}{5} ^{6}R q_{\mu\nu} + E_{\mu\nu} \right) \right).
\]

\(^2\) Actually, the reference \[13\] is a first generalization of the work presented in \[16\] to the Brans-Dicke framework.
In order to obtain the projected equation on the brane, one needs to apply the limits defined in (16) and (17) into (21). Starting with \( [5] R_{\mu \nu} \), one has

\[
[5] R_{\mu \nu} = 0 = [Y_{\mu \nu}] + \langle K \rangle [K_{\mu \nu}] + [K] \langle K_{\mu \nu} \rangle - \langle K_{\mu}^{\alpha} \rangle [K_{\nu \alpha}].
\]  

(22)

Now, by using the same decomposition showed in equations (10) and (11), the equations (8) and (9) give, respectively

\[
[K_{\mu \nu}] = \frac{8 \pi}{\phi} \left( \tau_{\mu \nu} + \frac{q_{\mu \nu}}{2(3 + 2w)} ((w - 1) \lambda - (w + 1) \tau) \right),
\]  

(23)

and

\[
[K] = \frac{8 \pi (w - 1)}{2 \phi(3 + 2w)} (\tau - 5 \lambda).
\]  

(24)

Hence, in the light of (23) and (24), the equation (22) results in

\[
\left( \frac{8 \pi}{\phi} \right)^{-1} [Y_{\mu \nu}] = \frac{1}{4} \left( [K] [K_{\mu \nu}] - [K_{\mu}^{\alpha}] [K_{\nu \alpha}] \right) + \langle K \rangle [K_{\mu \nu}] - \langle K_{\mu}^{\alpha} \rangle [K_{\nu \alpha}].
\]  

(25)

This last equation will be useful helping to find the mean value of the extrinsic curvature. Firstly, however, let us derive the full projected equation. The mean operator acting on (21) gives

\[
\langle [5] R_{\mu \nu} \rangle = \langle [5] R_{\mu \nu} \rangle = \langle Y_{\mu \nu} \rangle + \frac{1}{4} \left( [K] [K_{\mu \nu}] - [K_{\mu}^{\alpha}] [K_{\nu \alpha}] \right) + \langle K \rangle [K_{\mu \nu}] - \langle K_{\mu}^{\alpha} \rangle [K_{\nu \alpha}].
\]  

(26)

Using the following decomposition of \( Y_{\mu \nu} \) and \( K_{\mu \nu} \) in the trace and traceless parts

\[
Y_{\mu \nu} = \frac{Y}{\phi} q_{\mu \nu} + \tilde{\omega}_{\mu \nu},
\]  

(27)

\[
K_{\mu \nu} = \frac{K}{\phi} q_{\mu \nu} + \zeta_{\mu \nu},
\]  

(28)

the projected equation on the brane reads

\[
[5] G_{\mu \nu} = -\bar{\Lambda}_5 q_{\mu \nu} + 8 \pi \Omega \tau_{\mu \nu} + 8 \left( \frac{\pi}{\phi} \right)^2 \Sigma_{\mu \nu} + \langle \tilde{\omega}_{\mu \nu} \rangle + \frac{3}{5} \langle K \rangle \langle \zeta_{\mu \nu} \rangle - \langle \zeta_{\mu}^{\alpha} \rangle \langle \zeta_{\nu \alpha} \rangle,
\]  

(29)

where \( \bar{\Lambda}_5 \) is given by

\[
\bar{\Lambda}_5 = \frac{3}{10} \langle Y \rangle + \frac{6}{25} \langle K \rangle^2 - \frac{1}{2} \langle \zeta_{\alpha \beta} \rangle \langle \zeta_{\alpha \beta} \rangle + \frac{3}{8} \left( \frac{\pi}{\phi} \right)^2 \lambda(w - 1) \left( \frac{3 \pi}{\phi} + 2 \right)(\tau + \lambda (w - 1)),
\]  

(30)

and

\[
\langle Y \rangle = -2 \left( \left\langle \frac{8 \pi}{\phi} T_{\mu \nu} n^\mu n^\nu \right\rangle + \left\langle \frac{w}{\phi^2} \phi_{,\mu} \phi_{,\nu} n^\mu n^\nu - \frac{1}{2} \phi_{,\alpha} \phi_{,\alpha} \right\rangle \right) + \left\langle \frac{1}{\phi} \left( \phi_{,\mu \nu} n^\mu n^\nu - \frac{8 \pi}{3 \phi} T_{\mu \nu} \right) \right\rangle,
\]  

(31)
with $\phi, \mu \equiv \nabla_\mu \phi$. Note the appearance of $\langle \zeta_{\mu\nu} \rangle$-like terms in (29). According to the decomposition (28) those terms arise only due to the absence of the $\mathbb{Z}_2$ symmetry and encodes information about the shear of the curled on-brane dimension. Therefore, it seems that the application of the Gauss-Codazzi formalism into non-$\mathbb{Z}_2$ symmetric brane-worlds describes hybrid compactification scenarios in a more natural way. Of course, by orbifolding the extra transverse dimension one reobtains the previous Section results.

It is necessary to go one step further in order to determine the mean value of the extrinsic curvature appearing in (29). To do so, let us define a convenient new brane matter stress-tensor by

$$\hat{\tau}_{\mu\nu} = \tau_{\mu\nu} + \frac{(3(1-w)\lambda-(w+3)\tau)}{4(3+2w)} q_{\mu\nu},$$

in terms of which the equation (22) reads

$$0 = \left[ Y_{\mu\nu} \right] + \langle K \rangle \left[ K_{\mu\nu} \right] + \frac{8\pi}{\phi} \langle K_{[\mu}^\alpha \rangle \hat{\tau}_{\mu\nu]}. \quad (32)$$

Now, after expressing $[K_{\mu\nu}]$ and $[K]$ in terms of that new stress-tensor $\hat{\tau}_{\mu\nu}$ and inserting it in equation (32) we find

$$\frac{8\pi}{\phi} \langle K \rangle = \frac{3(\hat{\tau}-1)^{\mu\nu}[Y_{\mu\nu}]}{9 - (\hat{\tau}-1)_{\mu}^\mu \hat{\tau}_{\nu}^\nu}, \quad (33)$$

and again from (32) we arrive at

$$-\frac{8\pi}{\phi} \langle K_{[\mu}^\alpha \rangle \hat{\tau}_{\mu\nu]\alpha} = \left[ Y_{\mu\nu} \right] + \frac{3(\hat{\tau}-1)^{\alpha\beta}[Y_{\alpha\beta}]}{9 - (\hat{\tau}-1)^{\sigma} \hat{\tau}_{\sigma}^\gamma} (-\hat{\tau}_{\mu\nu} + \frac{\hat{\tau}}{3} q_{\mu\nu}), \quad (34)$$

or, in a more compact way,

$$\frac{8\pi}{\phi} \langle K_{[\mu}^\alpha \rangle \hat{\tau}_{\mu\nu]\alpha} \equiv -\left[ Y_{\mu\nu} \right]. \quad (35)$$

The complete decoupling of $\langle K_{\mu\nu} \rangle$ can be obtained from the vielbein decomposition. Therefore, let us introduce a complete basis $h^i_\mu$ ($i = 0, 1, \ldots, 4$) of orthonormal vectors constructed by the contraction of an orthonormal matrix set which represents a local Lorentz transformation and turns $\hat{\tau}_{\mu\nu}$ (and consequently $\tau_{\mu\nu}$) diagonal. The orthonormality conditions are given by

$$h^i_\mu h^i_\nu = \eta(i)(j),$$

$$\sum_{i,j=0}^{4} \eta(i)(j) h^i_\mu h^j_\nu = \sum_{j=0}^{4} h^j_\mu h^j_\nu = q_{\mu\nu}, \quad (36)$$

where $\eta(i)(j)$ is the Minkowski metric$^3$. Expressing $\hat{\tau}_{\mu\nu}$ in terms of the vielbein, $\hat{\tau}_{\mu\nu} = \sum_i \hat{\tau}_{(i)} h^i_\mu h^i_\nu$, we have from (35)

$$\frac{8\pi}{\phi} \langle K_{\mu\nu} \rangle = -\sum_{i,j} \frac{h^i_\mu h^j_\nu}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} [\hat{Y}_{(i)(j)}], \quad (37)$$

$^3$ Note that we are not assuming Einstein’s summation convention over the tangent indices.
after a contraction with $h^\mu_{(i)}h^\nu_{(j)}$. In the equation (37), $[\hat{Y}_{(i)(j)}] \equiv h^\mu_{(i)}h^\nu_{(j)}[\hat{Y}_{\mu\nu}]$. So, since the diagonal term of (37) is given by $\sum_{i=j} h^\mu_{(i)}h^\nu_{(j)}[\hat{Y}_{(i)(j)}] = \frac{1}{2} (\hat{\tau}^{-1})^{\mu}_{\mu} [\hat{Y}_{\mu\mu}]$, the generalized matching condition to the mean value of the extrinsic curvature reads

$$\frac{8\pi}{\phi} \langle K_{\mu\nu} \rangle = \frac{1}{2} (\hat{\tau}^{-1})^{\mu}_{\mu} [\hat{Y}_{\mu\nu}] + \frac{3(\hat{\tau}^{-1})^{\mu\nu}}{2(9 - (\hat{\tau}^{-1})^{\mu\nu})} \left( q_{\mu\nu} - \frac{\hat{\tau}^{\mu\nu}(\hat{\tau}^{-1})^{\mu\nu}}{3} \right)$$

$$- \sum_{i \neq j} \frac{h^\mu_{(i)}h^\nu_{(j)}}{\hat{\tau}(i) + \hat{\tau}(j)} [\varpi_{(i)(j)}].$$

(38)

From equation (38) one can find the expression for $\langle K \rangle$, while from the decomposition (28) one finds $\langle \zeta_{\mu\nu} \rangle$. After all, the projected Einstein-Brans-Dicke equation in the orthonormal frame has the following diagonal terms

$$(5)G_{(i)(i)} = -\bar{\Lambda} + 8\pi\Omega \tau_{(i)} + \Sigma_{(i)} + \langle \varpi_{(i)(i)} \rangle + \frac{3}{5} \langle K \rangle \langle \zeta_{(i)(i)} \rangle + \sum_{k} \langle \zeta_{(i)(k)} \rangle \langle \zeta_{(i)(k)} \rangle,$$

(39)

where $\Sigma_{(i)} = \frac{1}{4} \left( \frac{8\pi}{\phi} \right)^2 \left( \frac{(w+3)}{2(3+2w)} \tau_{(i)} - \hat{\tau}_{(i)}^2 + \frac{1}{2} \left( \sum_{j} \hat{\tau}_{(j)}^2 \right) - \frac{(w^2+3w+3)}{2(3+2w)^2} \hat{\tau}^2 \right)$. Moreover, the absence of the $\mathbb{Z}_2$ symmetry allows the existence of off-diagonal terms of the Einstein brane tensor, given by

$$(5)G_{(i)(j)} = \langle \varpi_{(i)(j)} \rangle + \frac{3}{5} \langle K \rangle \langle \zeta_{(i)(j)} \rangle + \sum_{k} \langle \zeta_{(i)(k)} \rangle \langle \zeta_{(j)(k)} \rangle.$$

(40)

the equations (39) and (40) are the result of the generalization of the Gauss-Codazzi formalism to brane-worlds without the orbifold symmetry. A general characteristic of this procedure is the appearance of terms proportional to the shear of the extrinsic curvature, as well as the existence of off-diagonal elements in the Einstein projected tensor. These last two properties enables one to say that this formalism can extract more physical information when applied to hybrid compactification models.

IV. OUTLOOKS

This work can be positioned in the middle road between the pure formalism and the application. A more formal approach must take into account the advises arising in the study of consistence conditions for Brans-Dicke brane-worlds (BDBW). Nevertheless, the use of Gauss-Codazzi formalism is necessary in order to construct a bridge between formalism and phenomenology.

Apart of the rather technical approach focused in this work, the result encoded in the equations (12), (39) and (40) is exhaustive: the presence of the dilatonic field, by all means, brings new signatures if applied to cosmological systems. In this vein, several possibilities are open to further investigation in the scope of BDBW. A
systematic study of the problems analyzed, for instance, in [3] in the context of BDBW models can certainly provide new insights about high energy physics, as well as about the physics of extra dimensions itself. Moreover, we hope that in the study of specific cosmological problems, as the galactic rotation curves for example, the ubiquitous presence of the Brans-Dicke parameter restricts the wide range of possible adjustments, coming from the projected gravitational field equations, and points out to a more phenomenologically viable scenario. Of course, such restriction is possible only if one is willing to accept that the current lower bound of the Brans-Dicke parameter, coming from Solar System experiments [18], is also valid in the brane-world framework.

To finalize we should remark that, since the Brans-Dicke theory can mimic gravity recovered from string theory at low energy, at least in some regimes, the study of the cosmological aspects of BDBW can also bring some information about problems in string cosmology.

Acknowledgments

The authors are very grateful for the invitation to make a contribution with this work. J. M. Hoff da Silva thanks to CAPES-Brazil for financial support. M. C. B. Abdalla and M. E. X. Guimarães acknowledge CNPq for support.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221]; L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].
[2] R. Maartens, Living Rev. Relativity, **7**, 7 (2004) [gr-qc/0312059].
[3] N. Chantillon, C. Macesanu, and M. Trodden, Phys. Rev. **D 74**, 124004 (2006) [gr-qc/0609093v2]; S. Kanno, D. Langlois, M. Sasaki, and J. Soda, Prog. Theor. Phys. **118**, 701 (2007) [0707.4510v2 [hep-th]]; M. K. Mak and T. Harko, Phys. Rev. **D 70**, 024010 (2004) [gr-qc/0404104]; F. Rahaman, M. Kalam, A. DeBenedicts, A. A. Usmani, and S. Ray, Mon. Not. R. Astron. Soc. **389**, 27 (2008).
[4] T. Shiromizu, K. Maeda, M. Sasaki, Phys. Rev. **D 62** 043523 (2000) [gr-qc/9910078v3].
[5] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[6] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, (Cambridge: Cambridge University Press), 1987.
[7] J. Scherk and J. Schwarz, Nucl. Phys. **B 81**, 223 (1974); C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Nucl. Phys. **262**, 593 (1985).
[8] Ya. B. Zel’dovich, I. Yu. Kobzarev, and L. N. Okun, Sov. Phys. JETP **40**, 1 (1975).
[9] R. Gregory, Phys. Rev. Lett. 84, 2564 (2000); A. G. Cohen and D. B. Kaplan, Phys. Lett. B 470, 52 (1999) [hep-th/9910132].

[10] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, Phys. Rev. D 75, 084028 (2007) [hep-th/0703234]; M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, JCAP 09, 021 (2008) [arXiv:0707.0233 [hep-th]].

[11] R. Koley and S. Kar, 2007 Class. Quantum Grav. 24, 79 (2007) [hep-th/0611074].

[12] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, Eur. Phys. J. C 55, 337 (2008) [arXiv:0711.1254 [hep-th]].

[13] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, to appear in the Eur. Phys. J. C (2008) [arXiv:0804.2834 [hep-th]].

[14] R. M. Wald, General Relativity, The University of Chicago Press, Chicago (1984).

[15] W. Israel, Nuovo Cim. B 44S10, 1 (1966).

[16] D. Yamauchi and M. Sasaki, Prog. Theor. Phys. 118 245 (2007) [gr-qc/0705.2443v3].

[17] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, [arXiv:0807.0580 [hep-th]].

[18] B. Bertotti, L. Iess, and P. Tortora, Nature 425, 374 (2003); M. W. Clifford, Living Rev. Relativity 9, 3 (2006) [gr-qc/0510072].