LOW MACH NUMBER LIMIT AND FAR FIELD CONVERGENCE RATES OF POTENTIAL FLOWS IN MULTI-DIMENSIONAL NOZZLES WITH AN OBSTACLE INSIDE

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Abstract. This paper considers the low Mach number limit and far field convergence rates of steady Euler flows with external forces in three-dimensional infinitely long nozzles with an obstacle inside. First, the well-posedness theory for both incompressible and compressible subsonic flows with external forces in multidimensional nozzle with an obstacle inside are established by several uniform estimates. The uniformly subsonic compressible flows tend to the incompressible flows as quadratic order of Mach number as the compressibility parameter goes to zero. Furthermore, we also give the convergence rates of both incompressible flow and compressible flow at far fields as the boundary of nozzle goes to flat even when the forces do not admit convergence rate at far fields. The convergence rates obtained for the flows at far fields clearly describe the effects of the external force.

Keywords: Subsonic flows, Potential equation, Nozzles, Obstacle, Low Mach number limit, Convergence rates

1. Introduction and main results

Both incompressible and compressible Euler equations can be used to describe the motion of the fluid dynamics and they give rise to many significant problems in mathematical theory. One of the important topics is the low Mach number limit problem which considers the convergence of compressible flows to the incompressible ones as the compressibility parameter goes to zero. The isentropic compressible Euler equations with the external force are described as follows

\[
\begin{aligned}
\text{div}(\rho \mathbf{u}) &= 0, \\
\text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \rho \mathbf{F},
\end{aligned}
\]

where \(\rho\) represents density, \(\mathbf{u} = (u_1, u_2, u_3)\) is the flow velocity, \(p\) is the pressure which is a function of \(\rho\), and \(\mathbf{F}\) describe the external forces, respectively. In this paper, denote

\[
p = \frac{\bar{p}(\rho) - \bar{p}(1)}{\epsilon^2},
\]

\(\epsilon\) is a small positive parameter, \(\bar{p}\) is a reference pressure. The convergence rates obtained for the flows at far fields clearly describe the effects of the external force.
where $\epsilon$ is the compressibility parameter \cite{33}. We always assume
\begin{equation}
\tilde{p}'(\rho) > 0 \quad \text{and} \quad \rho \tilde{p}''(\rho) + 2 \tilde{p}'(\rho) \geq 0, \quad \text{for} \ \rho > 0.
\end{equation}
The typical example is the polytropic gas for which the pressure is given by $\tilde{p} = \rho^\gamma$ with $\gamma > 1$ called the adiabatic exponent.

Assume the flow is irrotational, \textit{i.e.,}
\begin{equation}
\text{curl} \ u = 0,
\end{equation}
then, in the simply connected domain, there exists a potential function $\varphi$ such that
\begin{equation}
\nabla \varphi = u.
\end{equation}
Due to the irrotational condition (4), one may assume that the external force is conservative, which means that there exists a function $\phi$ such that
\begin{equation}
F = \nabla \phi.
\end{equation}

Denote
\begin{equation}
c(\rho) = \sqrt{\rho'\rho} = \frac{\sqrt{\tilde{p}'(\rho)}}{\epsilon},
\end{equation}
which is called the sound speed. And the Mach number is defined as
\begin{equation}
M = \frac{|u|}{c} = \frac{\epsilon |u|}{\sqrt{\tilde{p}'(\rho)}}.
\end{equation}
The flow is said to be subsonic, sonic, supersonic when $M < 1$, $M = 1$, $M > 1$, respectively. Therefore, if $\frac{|u|}{\sqrt{\tilde{p}'(\rho)}}$ is bounded and has a lower positive bound, one has $M \sim O(\epsilon)$.

We consider the domain to be a nozzle $\tilde{\Omega}$ which contains an obstacle $\Omega'$ inside, which models the wind tunnel in reality. Moreover, assume that $\partial \tilde{\Omega}$ and $\partial \Omega'$ are $C^{2,\alpha}$. Let $B_1(0) \subset \mathbb{R}^2$ be the unit disk centered at origin. Let
\begin{equation}
\mathcal{C} = B_1(0) \times (-\infty, +\infty)
\end{equation}
be the perfect cylinder. Suppose that there exists an invertible $C^{2,\alpha}$ map $\mathcal{R} : \tilde{\Omega} \to \mathcal{C}$ satisfying
\begin{equation}
\begin{cases}
\mathcal{R}(\partial \tilde{\Omega}) = \partial \mathcal{C}, \\
\mathcal{R}(\tilde{\Omega} \cap \{x_3 = k\}) = B_1(0) \times \{y_3 = k\} \quad \text{for any} \ k \in \mathbb{R}, \\
\|\mathcal{R}\|_{C^{2,\alpha}} + \|\mathcal{R}^{-1}\|_{C^{2,\alpha}} \leq C,
\end{cases}
\end{equation}
where $C$ is a constant. Furthermore, assume that there exists a $C^{2,\alpha}$ map $\mathcal{Y}$ such that
\begin{equation}
\mathcal{Y}(\Omega') \to B,
\end{equation}
where $B$ is the three dimensional unit ball with center at the origin. Using the cylindrical coordinates, $\tilde{\Omega}$ and $\Omega'$ can be written as

\begin{equation}
\tilde{\Omega} = \left\{ (r, \tau, x_3) \mid x_1 = r \cos \tau, x_2 = r \sin \tau, r < f_1(\tau, x_3), \tau \in [0, 2\pi), x_3 \in \mathbb{R} \right\}
\end{equation}

and

\begin{equation}
\Omega' = \left\{ (r, \tau, x_3) \mid x_1 = r \cos \tau, x_2 = r \sin \tau, r < f_2(\tau, x_3), \tau \in [0, 2\pi), L_1 \leq x_3 \leq L_2 \right\},
\end{equation}

respectively, where $L_1$ and $L_2$ are constants. Assume

\begin{equation}
0 \leq f_2 \leq C \quad \text{and} \quad \frac{1}{C} \leq f_1 - f_2 \leq C \quad \text{for any} \; \tau \in [0, 2\pi), \; x_3 \in [L_1, L_2]
\end{equation}

and

\begin{equation}
\frac{1}{C} \leq f_1 \leq C \quad \text{for any} \; \tau \in [0, 2\pi), \; x_3 \in \mathbb{R},
\end{equation}

where $C$ is a positive constant. Without loss of generality, assume the origin $O \in \Omega'$. In the rest of the paper, denote

\begin{equation}
\Omega = \tilde{\Omega} \setminus \Omega' \quad \text{and} \quad \Sigma_t = \Omega \cap \{x_3 = t\}.
\end{equation}

The slip boundary condition is supplemented on $\partial \Omega$, i.e.,

\begin{equation}
\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \; \partial \Omega,
\end{equation}

where $\mathbf{n} = (n_1, n_2, n_3)$ is unit outer normal of $\partial \Omega$. On any cross section $S_0$ of $\Omega$, it follows from the continuity equation in (1) that

\begin{equation}
\int_{S_0} \rho \mathbf{u} \cdot \mathbf{l} ds = m_0,
\end{equation}

where $\mathbf{l}$ is the unit normal pointed to the right of $S_0$. $m_0$ is called the mass flux of the flow across the nozzle, which is conserved through each cross section.

The well-posedness problem on compressible subsonic Euler flows in infinitely long nozzles was posed in [3]. The first rigorous proof for the well-posedness of irrotational flows in two dimensional nozzles was established in [37] via the stream function formulation. The results were extended to the three dimensional axially symmetric case in [39]. The uniformly subsonic flows in general multidimensional nozzles were obtained via potential formulation and variational method in [15]. Afterwards, the results were extended to the subsonic flows with external force in [23]. Recently, the optimal convergence rates of velocity at far field was given in [28]. When the vorticity of subsonic flow is not zero, the existence of solutions was established in [38] as long as the mass flux is less than a critical value and the variation of the Bernoulli's function is small. Later on, the well-posedness of subsonic flows with large
vorticity is proved in [14] under the condition that the velocity at upstream is convex. It is worth pointing that the optimal convergence rates of the flows at far fields are also established in [14]. When the conditions of the smallness of variation of Bernoulli’s function or the convexity of velocity are removed, the existence of general subsonic flows with characteristic discontinuity was obtained in [9]. There are many literatures on the subsonic flows in infinitely long nozzles, see [6, 12, 13, 16] and reference therein.

For the exterior problems, which is to study the flow past around the given obstacles, the existence of two dimensional subsonic flows was first studied in [20]. Later on, the existence of subsonic solution around a smooth body was established in [34] when the circulation is prescribed and the free stream Mach number is less than a critical value. Bers proved the existence of subsonic flows even when the body has a sharp corner [2]. In [18], the uniqueness and the convergence rates of subsonic plane flows were given. The existence of subsonic flows past a three dimensional body was first investigated in [19] and later established in [10, 11] when the Mach number of the free stream flow is less than a critical value. When there is an external force, the existence of subsonic flows past a body was established in [22].

A physical interesting problem is to study the low Mach number limit of the compressible subsonic flows [29, 33]. More precisely, as the compressibility parameter tends to zero, whether the solutions of compressible Euler equations (1) converge to that of the homogeneous incompressible Euler equations

\[
\begin{aligned}
\text{div} u &= 0, \\
\text{div}(u \otimes u) + \nabla p &= F,
\end{aligned}
\]

where \( u \) and \( p \) represent the velocity and pressure, respectively.

The first mathematical theory of the low Mach number limit for steady irrotational flows was studied in [32, Sect. 47] and [36] where the solutions were written as power series of Mach number. Klainerman and Majda [24, 25] proceeded with the study of the convergence of unsteady compressible flows to incompressible case for the suitable data. Later on, Ukai [35] improved the low Mach number limits for the general data with help of the decay property of acoustic waves. An important progress is made by Métivier and Schochet [30] where they proved the low Mach number limit in the whole space for general initial data of full Euler systems. These results were extended to the exterior problems in [1]. The low Mach number limit for one dimensional problem was investigated in the \( BV \) space [7]. The first rigorous analysis of subsonic flows for the steady Euler equations past a body, in infinitely long nozzles, and largely open nozzle were obtained in [26], [27], and [40], respectively.

The aim of this paper is to investigate the well-posedness of subsonic flows with force through infinitely long nozzles with an obstacle inside and the low Mach number limit of the
associated flows. The next key issue is to study the convergence rates of the velocity at far fields.

In order to study the low Mach number limit, we first investigate the incompressible flows in $\Omega$. Find $\bar{u}$ and $\bar{p}$ satisfy

$$
\begin{aligned}
\text{div} \bar{u} &= 0, & \text{in } \Omega, \\
\text{div}(\bar{u} \otimes \bar{u}) + \nabla \bar{p} &= \nabla \phi, & \text{in } \Omega, \\
\bar{u} \cdot n &= 0, & \text{on } \partial \Omega, \\
\int_{\Sigma_t} \bar{u} \cdot l \, ds &= m_0,
\end{aligned}
$$

where $n$ is the unit outer normal of $\Omega$ and $l$ is the unit normal pointed to the right of $\Sigma_t$.

**Theorem 1.** For any $m_0 > 0$, suppose

$$
\phi \in L^\infty(\Omega) \text{ and } \nabla \phi \in L^q(\Omega) \text{ for } q > 3,
$$

there exists a solution $(\bar{u}, \bar{p}) \in (C^\alpha(\Omega))^4$ to problem (19). Moreover, let $K$ be a large positive number and $q_0$ be the constant satisfying $q_0|B_1(0)| = m_0$.

(i) If the nozzle is flat at the downstream, i.e.,

$$
\Omega \cap \{x_3 > K\} = B_1(0) \times (K, +\infty),
$$

there exists a positive constant $d_1$ such that

$$
|\bar{u} - (0, 0, q_0)| \leq Ce^{-d_1 x_3}, \text{ for } x_3 > K;
$$

(ii) If the boundary of the nozzle satisfies

$$
\sum_{k=0}^{2} |x_3^{k} \partial_{x_3} \phi_1 - 1| \leq C x_3^{-a_1}, \text{ for } x_3 > K,
$$

with $a_1 > 0$, then the velocity satisfy

$$
|\bar{u} - (0, 0, q_0)| \leq C x_3^{-a_1},
$$

where $C$ is a constant independent of $x_3$.

Because of the irrotational condition (4), one has

$$
\nabla \bar{\phi} = \bar{u}.
$$

The straightforward computations yield the Bernoulli’s law, i.e.,

$$
\bar{p} = \phi - \frac{|\nabla \bar{\phi}|^2}{2} + C,
$$
where $C$ is a constant. Thus, the problem (19) is converted to the following problem

\[
\begin{align*}
\Delta \bar{\phi} &= 0, & \text{in } \Omega, \\
\frac{\partial \bar{\phi}}{\partial n} &= 0, & \text{on } \partial \Omega, \\
\int_{\Sigma} \nabla \bar{\phi} \cdot ld & = m_0.
\end{align*}
\]

Now, we turn to the compressible case. Suppose that $(\rho, u, p)$ satisfies

\[
\begin{align*}
\text{div}(\rho \epsilon u) &= 0, & \text{in } \Omega, \\
\text{div}(\rho \epsilon u \otimes u) + \nabla p & = \rho \epsilon \nabla \phi, & \text{in } \Omega, \\
u \cdot n & = 0, & \text{on } \partial \Omega, \\
\int_{\Sigma} \rho \epsilon u \cdot ld & = m_0,
\end{align*}
\]

with $p$ satisfies

\[
p = \frac{\bar{p}(\rho) - \bar{p}(1)}{\epsilon^2}.
\]

**Theorem 2.** For compressible flows, suppose (20) holds, for any $m_0 > 0$, there exists a constant $\epsilon_c$ such that if $0 < \epsilon < \epsilon_c$ the problem (28) admits a unique solution $(\rho^\epsilon, u^\epsilon, p^\epsilon) \in (C^\alpha(\Omega))^5$ for some $\alpha < 1$ with $M < 1$. Furthermore, as $\epsilon \to 0$ one has

\[
\rho^\epsilon = 1 + O(\epsilon^2), \quad u^\epsilon = \bar{u} + O(\epsilon^2), \quad p^\epsilon = \bar{p} + O(\epsilon^2) \quad \text{and} \quad M = O(\epsilon),
\]

where $(\bar{u}, \bar{p})$ solves the problem (19) in Theorem 1.

Based on the existence of subsonic solution to the problem (28), if the boundary of nozzle tends to be flat at far fields, we can also obtain the convergence rates of velocity fields. Let $C$ be the perfect cylinder defined in (8). According to [23], for any $m_0 > 0$, there exists a $\hat{\epsilon}_c > 0$ such that for any $\epsilon < \hat{\epsilon}_c$, as long as the force $\phi$ satisfies (20), there exists a unique uniformly subsonic solution $u_*$ satisfying

\[
\begin{align*}
\text{div}(\rho_* u_*) &= 0, & \text{in } C, \\
\text{div}(\rho_* u_* \otimes u_*) + \nabla p_* & = \rho_* \nabla \phi, & \text{in } C, \\
u_* \cdot n & = 0, & \text{on } \partial C, \\
\int_{B_1(0)} \rho_* u_* \cdot ld & = m_0.
\end{align*}
\]

It is worth pointing that if $\phi$ is a function independent of $x_3$, i.e., $\phi = \bar{\phi}(x_1, x_2)$ and satisfies (20), the straightforward computations yield that $\bar{u}_* = (0, 0, \bar{q})$ is the corresponding solution to (31), where $\bar{q}$ is a constant satisfying

\[
\int_{B_1(0)} \rho^\epsilon(\bar{q}^2, \bar{\phi}) \bar{q} dx = m_0 \quad \text{and} \quad \bar{q} < \frac{\sqrt{\bar{p}'(\rho^\epsilon)}}{\epsilon}.
\]
Theorem 3. Let $K$ be a large positive number. For any fixed $0 < \epsilon < \min\{\epsilon_c, \hat{\epsilon}_c\}$, $u^{\epsilon}$ is the subsonic solution of (28) and $u_\ast$ is the solution of (31).

(i) Suppose that the nozzle is flat, i.e., $\Omega$ satisfies (21). If $\phi$ satisfies (20), there exists a positive constant $d$ such that

$$\sum_{k=0}^{2} x_3^k \partial_3^k (\phi - \bar{\phi}) \leq C x_3^{-b_1} \quad \text{for } x_3 > K,$$

where $C$ is a constant independent of $x_3$.

(ii) If the boundary satisfies (23) with $a_1 > 1$, then the velocity satisfies

$$|u^{\epsilon} - u_\ast| \leq C x_3^{-a_1 + 1},$$

where $C$ is a constant independent of $x_3$.

(iii) For given $a_1 > 0$ and $b_1 > 0$, suppose that the boundary of the nozzle satisfies (23). If, in addition, the conservative force $\phi$ satisfies

$$\sum_{k=0}^{2} x_3^k \partial_3^k (\phi - \bar{\phi}) \leq C x_3^{-b_1} \quad \text{for } x_3 > K,$$

then the velocity satisfies

$$|u^{\epsilon} - \tilde{u}_\ast| \leq C x_3^{-b},$$

where $b = \min\{a_1, b_1\}$ and $C$ is a constant independent of $x_3$.

As long as the flows are irrotational, there also exists a potential $\varphi^{\epsilon}$ such that

$$\nabla \varphi^{\epsilon} = u^{\epsilon}.$$

Similarly, one has the following Bernoulli’s law

$$\frac{|\nabla \varphi^{\epsilon}|^2}{2} + \int_{1}^{\rho^{\epsilon}} \frac{(p^{\epsilon}(s))'}{s} ds = \phi + C_1,$$

where $C_1$ is a constant. Let $\tilde{h}$ be the enthalpy function satisfying

$$\tilde{h}'(\rho) = \frac{\tilde{p}'(\rho)}{\rho}.$$

Without loss of generality, assuming $C_1 = 0$, then (38) becomes

$$\frac{|\nabla \varphi^{\epsilon}|^2}{2} + h^{\epsilon}(\rho^{\epsilon}) - h^{\epsilon}(1) = \phi,$$

where $h^{\epsilon}(\rho) = \epsilon^{-2} \tilde{h}(\rho)$. The straightforward computations yield that $\tilde{h}(\rho)$ is a strictly increasing function with respect to $\rho$, so is $h^{\epsilon}(\rho)$. 
Now, one may introduce the critical speed for the flows. For each fixed $0 < \theta \leq 1$, one can follow [27] to define $q_\theta^c$ such that $u^\epsilon < q_\theta^c$ if and only if $M < \theta$. Specially, for the polytrotic case, define

$$
\mu^2 = \frac{(\gamma - 1)\theta^2}{2 + (\gamma - 1)\theta^2} \quad \text{and} \quad q_\theta^c = \mu \sqrt{2(\phi + h(1))},
$$

then the Bernoulli’s function (40) can be written in the following form

$$
\mu^2 |\nabla \varphi^\epsilon|^2 - (1 - \mu^2)\theta^2 c^2 = (q_\theta^c)^2.
$$

This implies that

$$
|\nabla \varphi^\epsilon|^2 - (q_\theta^c)^2 = (1 - \mu^2)(|\nabla \varphi^\epsilon|^2 - \theta^2 c^2).
$$

In particular, when $\theta = 1$, $q_\theta^c := q_1^c$ is called the critical speed. Obviously, the critical speed $q_\theta^c \sim O(\epsilon^{-1})$. It easy to see that $|u^\epsilon| < q_\theta^c(\phi)$ holds if and only if the flow is subsonic, i.e., $M < 1$. Similarly, when pressure satisfies (3), for each fixed positive $\theta < 1$, $q_\theta^c(\phi)$ satisfies $M < \theta$ holds if and only if $|u^\epsilon| < q_\theta^c(\phi)$. And, $q_\theta^c(\phi)$ is increasing with respect to $\theta \in (0,1)$.

In addition, $\epsilon q_\theta^c(\phi)$ and $\epsilon q_\theta^c(\phi)$ are bounded with respect to $\epsilon$.

Because of (40), the densty $\rho^\epsilon$ can be represented as follows

$$
\rho^\epsilon(|\nabla \varphi^\epsilon|^2, \phi) = \tilde{h}^{-1}\left(\frac{\epsilon^2(2\phi - |\nabla \varphi^\epsilon|^2)}{2} + \tilde{h}(1)\right).
$$

Thus for each fixed $\epsilon$, (28) is equivalent to the following problem

$$
\begin{cases}
\text{div} \left( \rho^\epsilon(|\nabla \varphi^\epsilon|^2, \phi) \nabla \varphi^\epsilon \right) = 0, & \text{in } \Omega, \\
\frac{\partial \varphi^\epsilon}{\partial n} = 0, & \text{on } \partial \Omega, \\
\int_{\Sigma_1} \rho^\epsilon(|\nabla \varphi^\epsilon|^2, \phi) \nabla \varphi^\epsilon \cdot l ds = m_0.
\end{cases}
$$

Furthermore, there exists a potential $\varphi^\ast$ such that the problem (31) is equivalent to the following

$$
\begin{cases}
\text{div} \left( \rho^\ast(|\nabla \varphi^\ast|^2, \phi) \nabla \varphi^\ast \right) = 0, & \text{in } C, \\
\frac{\partial \varphi^\ast}{\partial n} = 0, & \text{on } \partial C, \\
\int_{B_1(0)} \rho^\ast(|\nabla \varphi^\ast|^2, \phi) \nabla \varphi^\ast \cdot l ds = m_0, & \text{for } x_3 \in \mathbb{R}.
\end{cases}
$$

Therefore, if $\phi = \tilde{\phi}(x_1, x_2)$ and satisfies (20), then the straightforward computations yield that $\varphi^\ast = \tilde{q} x_3$ satisfies

$$
\begin{cases}
\text{div} \left( \rho^\ast(|\nabla \varphi^\ast|^2, \tilde{\phi}) \nabla \varphi^\ast \right) = 0, & \text{in } C, \\
\frac{\partial \varphi^\ast}{\partial n} = 0, & \text{on } \partial C, \\
\int_{B_1(0)} \rho^\ast(|\nabla \varphi^\ast|^2, \tilde{\phi}) \nabla \varphi^\ast \cdot l ds = m_0, & \text{for } x_3 \in \mathbb{R}.
\end{cases}
$$
In the rest of this paper, we mainly consider the problems (45)–(47). There are few remarks in order.

**Remark 1.** It easy to see that the gravity force $\phi = g x_i$ ($i = 1, 2$) satisfies (20). Also, it is easy to check the gravitational potential generated by the solid domain $\Omega^c$ (which is the complement of the fluid domain $\Omega$), i.e.,

$$\phi(x) = \int_{\Omega^c} \frac{\rho_s(y)}{|x-y|} dy$$

satisfies the conditions (20), where $x \in \Omega$ and $\rho_s \in L^1(\Omega^c)$ means the density distribution in $\Omega^c$ is of finite mass. Similarly, $\phi$ can also be the electric field.

**Remark 2.** By (20) and the Gagliardo-Nirenberg interpolation inequality, one has $\phi \in W^{1,q}(\Omega)$ for $q > 6$. Hence $\phi$ is an $L^\infty$ function by the Morrey’s inequality without extra condition.

**Remark 3.** For incompressible flows with external forces, the convergence rates (22) and (24) are similar to the compressible Euler flows in [28] without the force. It follows from the convergence rates obtained in Theorem 1 that the forces do not affect the convergence rates of velocity in the incompressible flows. However, the external force plays an important role in the convergence rates of the velocity field of the compressible flows in Theorem 3.

**Remark 4.** For the flows past a body, the order of the convergence rate of the velocity field at infinity is independent of the force [22], while the convergence rates of velocity field for subsonic flows in infinitely long nozzles depend on the force $\phi$, which is described in Theorem 3. It is worth to point out that cases (i) and (ii) of Theorem 3 show the flows have the precise far field asymptotic behavior even when $\phi$ does not admit convergence rate at far fields. Furthermore, there is 1 order loss of convergence rate in the polynomial case (ii), which is different from the exponential case (i) in Theorem 3. Finally, case (iii) shows the convergence rates of the velocity field matches the slower one between the rate of boundary and force $\phi$.

**Remark 5.** The convergence rates of the flow velocity at far fields is independent of the obstacle in nozzle. This implies that Theorem 3 can be applied to the flows through multidimensional nozzle studied in [23].

Here we give the main ideas for the proof of the main results. Inspired by [27], the existence of incompressible and compressible subsonic flows in the domain $\Omega$ is first established via the variational method. Then some uniform estimates are obtained which also implies that the order of low Mach number limit is $\epsilon^2$. The regularity of the solutions is improved since
the corresponding subsonic potential equation is elliptic. As long as there is an external force, both the boundary effect and the behavior of the external force at far fields have the strong influence on the convergence rates for the flows at far fields. Inspired by the delicate choice of weight function in [28,31], \( L^2 \)-norm of gradients of potential is obtained when the boundary effect and the external force effect are combined together. Finally, \( L^\infty \)-norm of the gradients of the potential established via the Nash-Moser iteration.

The rest of this paper is organized as follows. In Section 2, the approximate problems are introduced and the variational method is applied to get the existence of weak solutions for both the incompressible and compressible flows. Moreover, some uniform estimates and the regularity of weak solution are obtained. The existence and uniqueness of incompressible-compressible different function are established in Sections 3. In Section 4, the proof of the low Mach number limit is given. In the last section, the convergence rates of velocity at far fields are established.

2. Approximate Problems And Variational Approach

Since the domain is unbounded, the truncation of the domain is used to to study problems (27) and (45).

For any sufficiently large positive number \( L \) and any set \( U \), denote

\[
\Omega_L = \Omega \cap \{|x_3| < L\} \quad \text{and} \quad \int_U f \, dx = \frac{1}{|U|} \int_U f \, dx,
\]

where \( f \in L^1(U) \). Later on, denote

\[
\overline{S} = \inf_{t \in \mathbb{R}} |\Sigma_t| \quad \text{and} \quad \underline{S} = \sup_{t \in \mathbb{R}} |\Sigma_t|.
\]

Define the space

\[
(48) \quad \mathcal{H}_L = \{ \varphi \in H^1(\Omega_L) : \varphi = 0 \text{ on } \Sigma_{-L} \}.
\]

One can directly check that \( \mathcal{H}_L \) is a Hilbert space under \( H^1 \) norm.

Now we study the following problem about the incompressible flows in the truncated domain \( \Omega_L \)

\[
(49) \quad \begin{cases}
\Delta \varphi_L = 0, & \text{in } \Omega_L, \\
\frac{\partial \varphi_L}{\partial n} = 0, & \text{on } \partial \Omega_L, \\
\frac{\partial \varphi_L}{\partial x_3} = \frac{m_0}{\Sigma_L}, & \text{on } \Sigma_L, \\
\varphi_L = 0, & \text{on } \Sigma_{-L}.
\end{cases}
\]
\( \varphi_L \) is called a weak solution of the problem (49) in \( \mathcal{H}_L \) if

\[
\int_{\Omega_L} \nabla \varphi_L \cdot \nabla \psi \, dx - \frac{m_0}{|\Sigma_L|} \int_{\Sigma_L} \psi \, dx' = 0, \quad \text{for any } \psi \in \mathcal{H}_L.
\]

Define

\[
I_L(\psi) = \frac{1}{2} \int_{\Omega_L} |\nabla \psi|^2 \, dx - \frac{m_0}{|\Sigma_L|} \int_{\Sigma_L} \psi \, dx',
\]

where \( x' = (x_1, x_2) \). The straightforward computations show that if \( \varphi_L \) is a minimizer of \( I_L \), i.e.,

\[
I_L(\varphi_L) = \min_{\psi \in \mathcal{H}_L} I_L(\psi),
\]

then \( \varphi_L \) must satisfy (50).

First, the existence of minimizer of \( I_L \) and the basic estimate for the minimizer are obtained in the following lemma.

**Lemma 1.** For any sufficiently large \( L > 0 \), \( I_L(\psi) \) has a minimizer \( \varphi_L \in \mathcal{H}_L \). Moreover, the following estimate holds,

\[
\int_{\Omega_L} |\nabla \varphi_L|^2 \, dx \leq C m_0^2,
\]

where \( C \) is a constant independent of \( L \).

**Proof.** Choose a subset \( U_L \subset \Omega_L \) such that \( U_L \cap \Omega' = \emptyset \) and \( \partial U_L \setminus (\Sigma_L \cup \Sigma_-) \) is \( C^{2,\alpha} \) (see Figure 1). Denote \( \mathcal{C}_L = B_1(0) \times \{-L \leq x_3 \leq L\} \). Then there exists an invertible \( C^{2,\alpha} \) map \( T_L: U_L \rightarrow \mathcal{C}_L, x \rightarrow y \) satisfying

(i) \( T_L(\partial U_L) = \partial \mathcal{C}_L \).

(ii) For any \( -L \leq k \leq L \), \( T_L(U_L \cap \{x_3 = k\}) = B_1(0) \times \{y_3 = k\} \).

(iii) \( \|T_L\|_{C^{2,\alpha}}, \|T_L^{-1}\|_{C^{2,\alpha}} \leq C \).

---

**Figure 1. Domain of \( U_L \)**
The straightforward computations yield that
\[
\left| \int_{\Sigma_L} \psi \, dx' \right| \leq C \int_{B_1(0)} |\psi(y', L)| \, dy' \leq C \int_{B_1(0)} \left( \int_{-L}^{L} |\partial_{y_3} \psi(y, L)| \, dy_3 \right) \, dy' \\
\leq C \int_{\Sigma_L} |\nabla \psi| \, dy \leq C \int_{U_L} |\nabla \psi| \, dx \leq C \int_{\Omega_L} |\nabla \psi| \, dx.
\]

Applying Hölder inequality gives
\[
\left| \int_{\Sigma_L} \psi \, dx' \right| \leq C |\Omega_L|^{\frac{1}{2}} \|\nabla \psi\|_{L^2(\Omega_L)}.
\]

The constant $C$ here and subsequently in the rest of the paper may change from line to line as long as what these constants depend on is clear. Substituting the estimate (55) into (51) yields that
\[
I_L(\psi) = \int_{\Omega_L} |\nabla \psi|^2 \, dx - \frac{m_0}{|\Sigma_L|} \int_{\Sigma_L} \psi \, dx' \geq \int_{\Omega_L} |\nabla \psi|^2 \, dx - C' \|\nabla \psi\|_{L^2(\Omega_L)} \geq \frac{1}{2} \|\nabla \psi\|_{L^2(\Omega_L)}^2 - C',
\]

where $C'$ depends on $m_0$, $\overline{S}$, $\underline{S}$ and $|\Omega_L|$. This implies that the functional $I_L(\psi)$ is coercive. Hence, as the same as the proof in [15, Theorem 4], $I_L(\psi)$ has a minimizer $\varphi_L$ satisfying (53). Moreover, the straight computations yield that $\varphi_L$ is a weak solution of (49). Therefore, the proof is completed.

From now on, denote $\Omega(t_1, t_2) = \Omega \cap \{t_1 < x_3 < t_2\}$. Note that for any $f \in H^1(\Omega(t_1 - 1, t_2 + 1))$,
\[
\left| \int_{\Omega(t_1 - 1, t_1)} f \, dx - \int_{\Omega(t_2, t_2 + 1)} f \, dx \right| \leq C \int_{\Omega(t_1 - 1, t_2 + 1)} |\nabla f| \, dx,
\]

where $C$ is a constant which depends on $\Omega$ but is independent of $t_1$ and $t_2$. For the detailed proof of (57), one may refer to [15, Proposition 4]. Furthermore, the following Poincaré inequality
\[
\left\| f(x) - \int_{\Omega(t, t + 1)} f(x) \, dx \right\|_{L^p(\Omega(t, t + 1))} \leq C \|\nabla f(x)\|_{L^p(\Omega(t, t + 1))},
\]

holds [17, Section 5.8 Theorem 1], where $t \in \mathbb{R}$, $p \in [1, +\infty)$ and $C$ is independent of $t$.

In order to study the compressible flows, the subsonic coefficients truncations are also needed to avoid the degeneracy of the equation near the sonic points. For $0 < \epsilon_0 < 1$ and
0 < \theta < 1, there exists a \( q_0^\theta(\phi) \) such that \( M^*(\phi) < \theta \). Set \( q^\theta_0(\phi) = \inf_{0 < \epsilon < \epsilon_0} q_0^\theta(\phi) \). Denote

\[
(59) \quad \hat{q}(q^2, \phi) = \begin{cases} 
q^2 - 2\phi & \text{if } |q| \leq \hat{q}^{\theta_0}(\phi), \\
\text{monotone smooth function} & \text{if } \hat{q}^{\theta_0}_0 < |q| \leq \hat{q}^{\epsilon_0}_{\epsilon_0}, \\
\sup_{x \in \Omega} \left( (\hat{q}^{\epsilon_0}_{\epsilon_0})^2(\phi) - 2\phi \right) & \text{if } |q| > \hat{q}^{\epsilon_0}_{\epsilon_0}.
\end{cases}
\]

Define

\[
(60) \quad \hat{\rho}(\mathcal{G}, \phi) = \tilde{h}^{-1}\left( \tilde{h}(1) - \frac{\epsilon^2 q(\mathcal{G}, \phi)}{2} \right).
\]

We first study the following problem with the subsonic truncations,

\[
(61) \quad \begin{cases} 
\text{div}(\hat{\rho}^\epsilon(\nabla \varphi^\epsilon, \phi) \nabla \varphi^\epsilon) = 0, & \text{in } \Omega, \\
\frac{\partial \varphi^\epsilon}{\partial n} = 0, & \text{on } \partial \Omega, \\
\int_{\Sigma^L} \rho^\epsilon \nabla \varphi^\epsilon \cdot \mathbf{l} ds = m_0.
\end{cases}
\]

Denote

\[
\hat{\rho}_\varphi = \frac{\partial \hat{\rho}}{\partial \varphi} \quad \text{and} \quad \hat{\rho}_\phi = \frac{\partial \hat{\rho}}{\partial \phi}.
\]

By the straightforward calculations, the equation in (61) can be written as

\[
\begin{aligned}
\hat{a}_{ij}(\nabla \varphi^\epsilon, \phi) \partial_{ij} \varphi^\epsilon + \hat{b}_i(\nabla \varphi^\epsilon, \phi) \partial_i \varphi^\epsilon = 0,
\end{aligned}
\]

where

\[
\begin{aligned}
\hat{a}_{ij}(\nabla \varphi^\epsilon, \phi) &= \hat{\rho}^\epsilon(\nabla \varphi^\epsilon)^2, \\
\hat{b}_i(\nabla \varphi^\epsilon, \phi) &= \hat{\rho}_\varphi \partial_i \varphi.
\end{aligned}
\]

The directly calculations give that

\[
(62) \quad \lambda \xi^2 \leq \hat{a}_{ij} \xi_i \xi_j \leq \Lambda \xi^2 \quad \text{for } \xi \in \mathbb{R}^3 \quad \text{and} \quad |\hat{b}_i(\nabla \varphi^\epsilon, \phi)| \leq C|\partial_i \varphi|,
\]

where constants \( \lambda, \Lambda \) and \( C \) are independent of \( \varphi^\epsilon \).

Since the domain \( \Omega \) is unbounded, one can first study the following problem in truncated domains,

\[
(63) \quad \begin{cases} 
\text{div}(\hat{\rho}^\epsilon(\nabla \varphi^\epsilon_L, \phi) \nabla \varphi^\epsilon_L) = 0, & \text{in } \Omega_L, \\
\frac{\partial \varphi^\epsilon_L}{\partial n} = 0, & \text{on } \partial \Omega_L, \\
\frac{\partial \varphi^\epsilon_L}{\partial x_3} = \frac{m_0}{|\Sigma_L|}, & \text{on } \Sigma_L, \\
\varphi^\epsilon_L = 0, & \text{on } \Sigma_{-L}.
\end{cases}
\]

Denote

\[
(64) \quad G^\epsilon(\mathcal{G}, \phi) = \frac{1}{2} \int_0^{\mathcal{G}} \hat{\rho}^\epsilon(\alpha, \phi) d\alpha.
\]
For given solution \( \bar{\varphi}_L \) of (49) and the potential \( \phi \) of the force, define

\[
(65) \quad \mathcal{L}^\epsilon(\psi) = \epsilon^{-4} \int_{\Omega_L} G^\epsilon(|\nabla \psi|^2, \phi) - G^\epsilon(|\nabla \bar{\varphi}_L|^2, \phi) - \nabla \bar{\varphi}_L \cdot (\nabla \psi - \nabla \bar{\varphi}_L) \, dx \quad \text{for} \quad \psi \in \mathcal{H}_L,
\]

and \( J^\epsilon(\tilde{\psi}) = \mathcal{L}^\epsilon(\bar{\varphi}_L + \epsilon^2 \tilde{\psi}) \) for all \( \tilde{\psi} \in \mathcal{H}_L \).

**Lemma 2.** \( J^\epsilon(\tilde{\psi}) \) admits a unique minimizer \( \bar{\varphi}^\epsilon_L \in \mathcal{H}_L \) and \( \varphi^\epsilon_L \) is a weak solution of (63) where

\[
(66) \quad \varphi^\epsilon_L = \bar{\varphi}_L + \epsilon^2 \bar{\varphi}^\epsilon_L.
\]

Moreover, for any \( t \in \left( -\frac{L}{4}, \frac{L}{4} \right) \), one has

\[
(67) \quad \int_{\Omega(t,t+1)} \frac{1}{2} (|\nabla \bar{\varphi}_L|^2 + |\nabla \varphi^\epsilon_L|^2) \, dx \leq C m_0^2,
\]

where \( C \) is independent of \( L \) and \( \epsilon \).

The proof of Lemma 2 is the same to [27], so we omit the details here.

Since \( \bar{\varphi}_L \) and \( \varphi^\epsilon_L \) are weak solutions of quasilinear elliptic equations of divergence form, similar to [15, Lemmas 6 and 7] and [27, Lemma 4.5], using the Nash-Moser iteration yields that there exists a positive constant \( K' < \frac{L}{4} \) such that

\[
(68) \quad \|\nabla \bar{\varphi}_L\|_{C^{0,\alpha}(\Omega(-K', K'))} + \|\nabla \varphi^\epsilon_L\|_{C^{0,\alpha}(\Omega(-K', K'))} \leq C m_0,
\]

where \( C \) is a constant independent of \( K' \).

### 3. The Existence And Uniqueness Of The Solution In The Whole Domain

For any fixed \( \bar{x} \in \Omega \), choose \( L \) large enough such that \( \bar{x} \in \Omega(-\frac{L}{4}, \frac{L}{4}) \). With abuse notations, we still denote \( \varphi^\epsilon_L - \bar{\varphi}^\epsilon_L(\bar{x}) \) and \( \varphi_L - \bar{\varphi}_L(\bar{x}) \) by \( \varphi^\epsilon_L \) and \( \bar{\varphi}_L \), respectively. It follows from (68) that

\[
(69) \quad \|\nabla \bar{\varphi}_L\|_{C^{0,\alpha}(\Omega(-K', K'))} + \|\nabla \varphi^\epsilon_L\|_{C^{0,\alpha}(\Omega(-K', K'))} \leq C m_0.
\]

By the standard diagonal procedure, there exist functions \( \bar{\varphi} \in C^{1,\alpha}_\text{loc}(\Omega) \) and \( \varphi^\epsilon \in C^{1,\alpha}_\text{loc}(\Omega) \) such that for some \( \alpha' < \alpha \),

\[
\lim_{n \to \infty} \|\bar{\varphi}_{L_n} - \bar{\varphi}\|_{C^{1,\alpha'}(\Omega(-K', K'))} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\varphi^\epsilon_{L_n} - \varphi^\epsilon\|_{C^{1,\alpha'}(\Omega(-K', K'))} = 0.
\]

Therefore, \( \bar{\varphi} \) and \( \varphi^\epsilon = \bar{\varphi} + \epsilon^2 \bar{\varphi}^\epsilon \) are solutions of (27) and (61), respectively.

Now we are in position to prove the uniqueness of the solutions. We use a method different from that in [15] and some energy estimates obtained here are also useful for the proof of the convergence rates of velocity fields in Section 5.
Lemma 3. Assume $\Omega$ satisfies (9) and (10). Let $\varphi_1$ and $\varphi_2$ be the two solutions of (27). Let $\varphi^\epsilon_1$ and $\varphi^\epsilon_2$ be the two uniformly subsonic solutions of (61). Then $\nabla \varphi_1 = \nabla \varphi_2$ and $\nabla \varphi^\epsilon_1 = \nabla \varphi^\epsilon_2$ in $\Omega$.

Proof. Denote $\Phi = \varphi^\epsilon_1 - \varphi^\epsilon_2$. Then $\Phi$ satisfies

$$\begin{cases} \partial_i (a_{ij} \partial_j \Phi) = 0, & \text{in } \Omega, \\ \frac{\partial \Phi}{\partial n} = 0, & \text{on } \partial \Omega, \end{cases}$$

where

$$a_{ij} = \int_0^1 \hat{\rho}'(q^2, \phi) \delta_{ij} + 2 \rho_0(q^2, \phi) (s \partial_j \varphi^\epsilon_1 + (1-s) \partial_j \varphi^\epsilon_2) (s \partial_i \varphi^\epsilon_1 + (1-s) \partial_i \varphi^\epsilon_2) \, ds$$

with

$$q^2 = |s \nabla \varphi^\epsilon_1 + (1-s) \nabla \varphi^\epsilon_2|^2.$$

The straightforward computations show that there exist two constants $\lambda$ and $\Lambda$ such that

$$\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^3.$$

Moreover, one can increase $\Lambda$ so that the following Poincaré inequality holds on each cross section,

$$\left\| Z - \int_{\Sigma_t} Z \, ds \right\|_{L^2(\Sigma_t)} \leq \Lambda \| \nabla Z \|_{L^2(\Sigma_t)} \quad \text{for any } Z \in H^2_{loc}(\Omega(t - \epsilon, t + \epsilon)).$$

For any $t_1 < t_2$, $h$ and $\beta$ are constants to be determined later. Denote

$$\zeta(x_3; t_1, t_2, \beta, h) = \begin{cases} 1 & x_3 \leq t_1 - h, \\ e^{\beta(x_3-t_1+h)} & t_1 - h < x_3 \leq t_1, \\ e^{\beta h} & t_1 < x_3 \leq t_2, \\ e^{\beta h} \cdot e^{-\beta(x_3-t_2)} & t_2 < x_3 \leq t_2 + h, \\ 1 & x_3 > t_2 + h. \end{cases}$$

Multiplying $\Phi(\zeta(x_3, t_1, t_2, \beta, h) - 1)$ on both sides of (70) and taking integral on $\Omega(t_1-h, t_2+h)$ yield

$$\int_{\Omega(t_1-h, t_2+h)} a_{ij} \partial_i \Phi \partial_j \Phi (\zeta - 1) \, dx + \int_{\Omega(t_1-h, t_2+h)} a_{ij} \partial_i \Phi \partial_j \zeta \, dx$$

$$= \int_{\partial \Omega \cap \Sigma(t_1-h, t_2+h)} a_{ij} \partial_i \Phi (\zeta - 1) n_i \, ds.$$

For the boundary term, one has

$$a_{ij} \partial_j \Phi n_i = (\hat{\rho}(|\nabla \varphi^\epsilon_1|^2, \phi) \partial_i \varphi^\epsilon_1 - \hat{\rho}(|\nabla \varphi^\epsilon_2|^2, \phi) \partial_i \varphi^\epsilon_2) \cdot n_i = 0.$$
Moreover, the conserved mass flux on each cross section implies
\[
\int_{\Sigma_t} a_{i3} \partial_t \Phi dx' = \int_{\Sigma_t} \hat{\rho}(|\nabla \varphi_1|^2, \varphi) \partial_3 \varphi_1 - \hat{\rho}(|\nabla \varphi_2|^2, \varphi) \partial_3 \varphi_2 dx' = 0.
\]
Set \( \tilde{\eta}(t) = \int_{\Sigma_t} \Phi dx' \). Combining (77) and (78) yields that
\[
\lambda \int_{\Omega(t_1 - h, t_2 + h)} |\nabla \Phi|^2 (\zeta - 1) dx \\
\leq - \int_{[t_1 - h, t_1] \cup [t_2, t_2 + h]} \left( \int_{\Sigma x_3} a_{i3} \partial_t \Phi dx' \right) dx_3 \\
- \int_{\Omega(t_1 - h, t_1) \cup \Omega(t_2, t_2 + h)} a_{i3} \partial_t \Phi \left( \Phi - \frac{\tilde{\eta}(x_3)}{|\Sigma x_3|} \right) \partial_3 \zeta dx \\
\leq \left( \int_{\Omega(t_1 - h, t_1) \cup \Omega(t_2, t_2 + h)} \left( \Phi - \frac{\tilde{\eta}(x_3)}{|\Sigma x_3|} \right)^2 (\partial_3 \zeta)^2 \zeta^{-1} dx \right)^{\frac{1}{2}} \left( \int_{\Omega(t_1 - h, t_1) \cup \Omega(t_2, t_2 + h)} (a_{i3} \partial_t \Phi)^2 \zeta dx \right)^{\frac{1}{2}}.
\]
It follows from (74) that one has
\[
\left( \int_{\Omega(t_1 - h, t_1) \cup \Omega(t_2, t_2 + h)} \left( \Phi - \frac{\tilde{\eta}(x_3)}{|\Sigma x_3|} \right)^2 (\partial_3 \zeta)^2 \zeta^{-1} dx \right)^{\frac{1}{2}} \\
= \left\{ \int_{t_1 - h}^{t_1} + \int_{t_2}^{t_2 + h} (\partial_3 \zeta)^2 \zeta^{-1} \left[ \int_{\Sigma x_3} \left( \Phi - \frac{\tilde{\eta}(x_3)}{|\Sigma x_3|} \right)^2 dx' \right] dx_3 \right\}^{\frac{1}{2}} \\
\leq \left\{ \int_{t_1 - h}^{t_1} + \int_{t_2}^{t_2 + h} \Lambda^2 (\partial_3 \zeta)^2 \zeta^{-1} \left( \int_{\Sigma x_3} |\nabla \Phi|^2 dx' \right) dx_3 \right\}^{\frac{1}{2}} \\
\leq \left( \int_{\Omega(t_1 - h, t_1) \cup \Omega(t_2, t_2 + h)} \Lambda^2 (\partial_3 \zeta)^2 \zeta^{-1} |\nabla \Phi|^2 dx \right)^{\frac{1}{2}}.
\]
Note that \( \partial_3 \zeta = \beta \zeta \) for \( x_3 \in [t_1 - h, t_1] \cup [t_2, t_2 + h] \), then
\[
\lambda \int_{\Omega(t_1 - h, t_2 + h)} |\nabla \Phi|^2 (\zeta - 1) dx \leq \Lambda^2 \beta \int_{\Omega(t_1, t_1) \cup \Omega(t_2, t_2 + h)} |\nabla \Phi|^2 \zeta dx.
\]
Set \( \beta = \frac{\lambda}{\Lambda^2} \). For any \( T > 0 \), let \( t_1 = -T, t_2 = T \) and \( h = T \). Hence,
\[
\int_{\Omega(-T,T)} |\nabla \Phi|^2 dx \leq C e^{-\beta T} \int_{\Omega(-2T,2T)} |\nabla \Phi|^2 dx \leq 2\Lambda^2 T e^{-\beta T} \max |\nabla \Phi|^2.
\]
Let \( T \to +\infty \) yields \( |\nabla \Phi| = 0 \ a.e. \ in \ \Omega \). This finishes the proof of uniqueness of compressible case. One can follow the above steps to obtain the uniqueness of the incompressible case where \( a_{ij} = \delta_{ij} \). \qed
4. Low Mach Number limit

This section devotes to the proof of Theorem 2.

In Lemma 2, when \( \epsilon_0 \) and \( \theta \) given, the solution \( \varphi^{\epsilon} \) satisfies

\[
|\nabla \varphi^{\epsilon}(x)| = |\epsilon^2 \nabla \tilde{\varphi}^{\epsilon}(x) + \nabla \bar{\varphi}(x)| \leq \max |\nabla \bar{\varphi}(x)| + C(\epsilon_0, \theta)\epsilon^2. 
\]

Hence, there exists a constant \( \epsilon_{0,\theta} \in (0, \epsilon_0) \) such that for any \( 0 < \epsilon < \epsilon_{0,\theta} \), \( |\nabla \varphi^{\epsilon}(x, \epsilon_0, \theta)| < q^{\epsilon}_{0,\theta}(\phi) \). Moreover, \( \{\epsilon_{0,\theta}\} \) is a non-decreasing sequence with respect to \( \theta \) which has an upper bound \( \epsilon_0 \). Set \( \epsilon_{0,cr} = \sup_{0<\theta<1} \epsilon_{0,\theta} \). Hence one has

\[
|\nabla \varphi^{\epsilon}(x)| = |\epsilon^2 \nabla \tilde{\varphi}^{\epsilon}(x) + \nabla \bar{\varphi}(x)| \leq \max |\nabla \bar{\varphi}(x)| + C(\epsilon_0, \theta)\epsilon^2 < q^{\epsilon}_{0,cr}(\phi). 
\]

This implies \( M^{\epsilon}(\phi) < 1 \). Therefore, \( \varphi^{\epsilon} \) is the solution of (45). Denote \( \epsilon_c = \sup_{0<\epsilon<1} \epsilon_{0,cr} \).

For convergence rates of density and pressure, the proof is the same to [40] and we omit the details. This completes the proof of Theorem 2.

5. Convergence rate of the velocity field

In this section, we investigate the convergence rates of the flows at far fields if the boundary of nozzle \( \Omega \) tends to a perfect cylinder. First, the \( L^2 \) convergence is obtained by energy estimate. \( L^\infty \) convergence is established via Nash-Moser iteration. The proof is divided into four steps.

5.1. Energy estimate for the case where the nozzle boundary satisfies (21). Let \( \varphi^{\epsilon} \) be the uniformly subsonic solution of (45), whose existence and uniqueness are established in Theorem 2. Let \( \varphi^{*} \) satisfy (46) and \( \bar{\varphi}^{*} = \bar{q} x_3 \) satisfy (47). Since the boundary is actually the cylinder, the desired convergence rates of the velocity field is exponential.

Denote \( \Psi = \varphi^{\epsilon} - \varphi^{*} \). The straightforward computations yield that \( \Psi \) satisfies

\[
\begin{aligned}
\partial_i (a_{ij} \partial_j \Psi) &= 0, \quad \text{in } \Omega \cap \{x_3 > K\}, \\
\frac{\partial \Psi}{\partial n} &= 0, \quad \text{on } \partial \Omega \cap \{x_3 > K\},
\end{aligned}
\]

where

\[
a_{ij} = \int_0^1 \rho^{\epsilon}(\hat{q}^2, \phi)\delta_{ij} + 2\rho^{\epsilon}(\hat{q}^2, \phi)(s\partial_i \varphi^{\epsilon} + (1-s)\partial_i \varphi^{*}) (s\partial_j \varphi^{\epsilon} + (1-s)\partial_j \varphi^{*}) ds
\]

with

\[
\hat{q}^2 = |s \nabla \varphi^{\epsilon} + (1-s) \nabla \varphi^{*}|^2.
\]
Obviously, $a_{ij}$ satisfies (73). Let $\zeta(x_3, t_1, t_2, \beta, h)$ be the function defined in (75). Multiplying $\Psi(\zeta - 1)$ on both sides of the equation on (86) as same as the estimates (81) one has

$$e^{\beta h} \int_{\Omega(t_1, t_2)} |\nabla \Psi|^2 dx \leq \int_{\Omega(t_1 - h, t_2 + h)} |\nabla \Phi|^2 dx.$$  

Choosing $t_1 = T$, $t_2 = T + 1$ and $h = \frac{T}{2}$ yields

$$e^{\beta T} \int_{\Omega(T, T+1)} |\nabla \Psi|^2 dx \leq \int_{\Omega(T, T+1)} |\nabla \Psi|^2 dx \leq C(T + 1).$$

If $T > 0$ is large enough, then there exists a constant $\bar{\alpha}$ such that

$$\int_{\Omega(T, T+1)} |\nabla \Psi|^2 dx \leq e^{-\bar{\alpha}T}.$$  

For the incompressible flows, if the boundary satisfies (22), then $\Psi = \bar{\varphi} - q x_3$ satisfies

$$\begin{align*}
\Delta \Psi &= 0, \quad \text{in } \Omega \cap \{x_3 > K\}, \\
\frac{\partial \Psi}{\partial n} &= 0, \quad \text{on } \partial \Omega \cap \{x_3 > K\}.
\end{align*}$$

Following the same arguments with $a_{ij} = \delta_{ij}$ in (86) to get the exponential decay, i.e.,

$$\int_{\Omega(T, T+1)} |\nabla \Psi|^2 dx \leq e^{-\bar{\alpha}_1 T},$$

where $\bar{\alpha}_1$ is a positive constant.

5.2. The energy estimate when the nozzle boundary satisfies (23) and the conservative force satisfies (35). In this case, since the decay rates of boundary and external force is algebraic, the one cannot expect the convergence rates are exponential. Actually, the $L^2$ decay can only be algebraically fast.

Obviously, the unit outer normal of $\partial \Omega$ is

$$\mathbf{n} = (n_1, n_2, n_3) = \frac{1}{\sqrt{G}} \left( \cos \tau + \frac{\partial f_1}{\partial \tau} \frac{1}{r}, \sin \tau - \frac{\partial f_1}{\partial x_3}, -\frac{\partial f_1}{\partial x_3} \right),$$

where

$$G = 1 + \left( \frac{\partial f_1}{\partial \tau} \right)^2 \frac{1}{r^2} + \left( \frac{\partial f_1}{\partial x_3} \right)^2.$$

Denote $\mathcal{U} = \varphi - \bar{\varphi}_*$. Then $\mathcal{U}$ satisfies

$$\begin{align*}
\partial_t (a_{ij} \partial_i \mathcal{U} + b_i) &= 0, \quad \text{in } \Omega \cap \{x_3 > K\}, \\
\frac{\partial \mathcal{U}}{\partial n} &= -\bar{q} n_3, \quad \text{on } \partial \Omega \cap \{x_3 > K\},
\end{align*}$$

where

$$\begin{align*}
\frac{\partial \mathcal{U}}{\partial n} &= -\bar{q} n_3, \quad \text{on } \partial \Omega \cap \{x_3 > K\}.
\end{align*}$$
where
\[ a_{ij} = \int_0^1 \rho'(q^2, E)\delta_{ij} + 2\rho'_\phi(q^2, E)(s\partial_i\varphi' + (1-s)\partial_i\bar{\varphi}_*))(s\partial_j\varphi' + (1-s)\partial_j\bar{\varphi}_*)ds \]
and
\[ b_i = \int_0^1 \rho'_\phi(q^2, E)(\phi - \bar{\phi})(s\partial_i\varphi' + (1-s)\partial_i\bar{\varphi}_*)ds \]
with
\[ q^2 = |s\nabla\varphi' + (1-s)\nabla\bar{\varphi}_*|^2 \quad \text{and} \quad E = s\phi + (1-s)\bar{\phi}. \]
And \( a_{ij} \) satisfies (73). The straightforward computations yield that
\[ |b_i|_{C^1(\Sigma_{x_3})} \leq \frac{C}{x_3^{s_1}}. \]
In fact, it follows from (23) that one has
\[ (a_{ij}\partial_i\mathcal{U} + b_i)n_i = (\rho'(|\nabla\varphi'|^2, \phi)\partial_i\varphi' - \rho'(|\nabla\bar{\varphi}_*|^2, \bar{\phi})\partial_i\bar{\varphi}_*)n_i \]
\[ = -\rho'(|\bar{q}|^2, \bar{\phi})\bar{q}n_3 \leq \frac{C}{x_3^{s_1+1}}. \]
Moreover, on each cross section, one has
\[ \int_{\Sigma_{x_3}} a_{ij}\partial_j\mathcal{U} + b_3dx' = \int_{\Sigma_{x_3}} \rho'(|\nabla\varphi'|^2, \phi)\partial_i\varphi' - \rho'(|\nabla\bar{\varphi}_*|^2, \bar{\phi})\partial_i\bar{\varphi}_* dx' \]
\[ \leq \int_{\Sigma_{x_3}} \rho'(|\nabla\varphi'|^2, \phi)\partial_3\varphi' dx' - \int_{\Sigma_{x_3}} \rho'(|\nabla\bar{\varphi}_*|^2, \bar{\phi})\partial_3\bar{\varphi}_* dx' \]
\[ + \int_{B_1(0)} \rho'(|\nabla\bar{\varphi}_*|^2, \bar{\phi})\partial_3\bar{\varphi}_* dx' - \int_{B_1(0)} \rho'(|\nabla\varphi'|^2, \phi)\partial_3\varphi' dx' \]
\[ \leq C \left| \Sigma_{x_3} - B_1(0) \right| \leq Cx_3^{-s_1}. \]

Choosing \( t_1 = T \) and \( t_2 = t_1 + \hat{K} \) with \( \hat{K} \) a positive integer to be determined later. Let \( \zeta(x_3, t_1, t_2, \hat{\beta}, 1) \) be the function define in (75) with \( h = 1 \) and \( \hat{\beta} \) to be determined later. Denote
\[ s_1 = \int_{\Omega(t_1-1, t_1)} \mathcal{U}dx \quad \text{and} \quad s_2 = \int_{\Omega(t_2, t_2+1)} \mathcal{U}dx. \]
Let
\[ \hat{\mathcal{U}}(x; t_1, t_2, s_1, s_2) = \begin{cases} 
\mathcal{U}(x) - s_1, & x_3 < t_1, \\
\mathcal{U}(x) - s_1 - \frac{s_2 - s_1}{t_2 - t_1}(x_3 - t_1), & t_1 \leq x_3 \leq t_2, \\
\mathcal{U}(x) - s_2, & x_3 > t_2. 
\end{cases} \]
Multiplying \( \hat{U}(\zeta - 1) \) on both sides of (94) and integrating on \( \Omega(t_1 - 1, t_2 + 1) \) yield that

\[
\int_{\Omega(t_1-1,t_2+1) \cap \partial \Omega} (a_{ij} \partial U + b_i) \hat{U}(\zeta - 1) \, ds
\]

\[
= \int_{\Omega(t_1-1,t_2+1)} (a_{i3} \partial U + b_3) \hat{U} \partial_3 \zeta \, dx + \int_{\Omega(t_1-1,t_2+1)} a_{ij} \partial U \partial_j \zeta \, dx
\]

\[
+ \int_{\Omega(t_1-1,t_2+1)} b_i (\zeta - 1) \partial U \, dx - \int_{\Omega(t_1,t_2)} (a_{i3} \partial U + b_3)(\zeta - 1) \frac{s_2 - s_1}{t_2 - t_1} \, dx,
\]

where \( \zeta - 1 = 0 \) at \( x_3 = t_1 - 1 \) and \( x_3 = t_2 + 1 \) has been used. Then the direct calculations give

\[
\lambda \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 (\zeta - 1) \, dx
\]

\[
\leq - \int_{\Omega(t_1-1,t_1)} (a_{i3} \partial U + b_3) \hat{U} \partial_3 \zeta \, dx - \int_{\Omega(t_2,t_2+1)} (a_{i3} \partial U + b_3) \hat{U} \partial_3 \zeta \, dx
\]

\[
- \int_{\Omega(t_1-1,t_2+1)} b_i (\zeta - 1) \partial U \, dx + \int_{\Omega(t_1,t_2)} (a_{i3} \partial U + b_3)(\zeta - 1) \frac{s_2 - s_1}{t_2 - t_1} \, dx
\]

\[
+ \int_{\Omega(t_1-1,t_2+1) \cap \partial \Omega} (a_{ij} \partial U + b_i) n_i \hat{U}(\zeta - 1) \, ds = \sum_{k=1}^{5} I_k.
\]

Now \( I_k \) \((k = 1, 2, \ldots, 5)\) can be estimated one by one. Set

\[
\chi_1(x_3) = \int_{\Sigma_{x_3}} (U - s_1) \, dx' \quad \text{and} \quad \chi_2(x_3) = \int_{\Sigma_{x_3}} (U - s_2) \, dx'.
\]

The straightforward computations yield that

\[
|I_1| \leq \left| \int_{\Omega(t_1-1,t_1)} (a_{i3} \partial U + b_3) \partial_3 \zeta \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right) \, dx + \int_{\Omega(t_1-1,t_1)} (a_{i3} \partial U + b_3) \partial_3 \zeta \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \, dx \right|
\]

\[
\leq \left| \int_{\Omega(t_1-1,t_1)} a_{i3} \partial U \partial_3 \zeta \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right) \, dx \right| + \left| \int_{\Omega(t_1-1,t_1)} b_3 \partial_3 \zeta \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right) \, dx \right|
\]

\[
+ \left| \int_{\Omega(t_1-1,t_1)} (a_{i3} \partial U + b_3) \partial_3 \zeta \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \, dx \right| = \sum_{k=1}^{3} I_{1k}.
\]
Noting that \((\partial_3 \zeta)^2 = \beta^2 \zeta^2\) when \(x_3 \in [t_1 - h, t_1] \cup [t_2, t_2 + h]\). It follows from the Hölder inequality and Poincaré inequality (74) that one has

\[
|I_{11}| \leq \left[ \int_{\Omega(t_1-1,t_1)} (a_{i3} \partial_i U)^2 \zeta dx \right]^{\frac{1}{2}} \cdot \left[ \int_{\Omega(t_1-1,t_1)} \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right)^2 (\partial_3 \zeta)^2 \zeta^{-1} dx \right]^{\frac{1}{2}}
\]

\[
\leq \left[ \int_{\Omega(t_1-1,t_1)} \Lambda^2 \nabla U^2 \zeta dx \right]^{\frac{1}{2}} \cdot \left[ \int_{t_1-1}^{t_1} \int_{\Sigma_{x_3}} \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right)^2 dx' (\partial_3 \zeta)^2 \zeta^{-1} dx_3 \right]^{\frac{1}{2}}
\]

\[
\leq \Lambda^2 \beta \int_{\Omega(t_1-1,t_1)} |\nabla U|^2 \zeta dx.
\]

Using (35) gives

\[
|I_{12}| \leq \left( \int_{\Omega(t_1-1,t_1)} b_3^2 (\partial_3 \zeta)^2 dx \right)^{\frac{1}{2}} \cdot \left( \int_{t_1-1}^{t_1} \int_{\Sigma_{x_3}} \left( \hat{U} - \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right)^2 dx' dx_3 \right)^{\frac{1}{2}}
\]

\[
\leq \frac{CA}{(t_1-1)^{b_1}} \left( \frac{\beta}{2} (e^{2\beta} - 1) \right)^{\frac{1}{2}} \left( \int_{\Omega(t_1-1,t_1)} |\nabla U|^2 dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C \beta (e^{2\beta} - 1)}{\epsilon_1 (t_1-1)^{2b_1}} + \frac{\epsilon_1}{2} \int_{\Omega(t_1-1,t_1)} |\nabla U|^2 dx,
\]

where \(\epsilon_1\) is a small positive constant to be determined later. Applying the estimate (100) yields

\[
|I_{13}| \leq \left| \int_{t_1-1}^{t_1} \left( \int_{\Sigma_{x_3}} (a_{i3} \partial_i U + b_3) dx' \right) \partial_3 \zeta \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} dx_3 \right|
\]

\[
\leq \left[ \int_{t_1-1}^{t_1} \left( \int_{\Sigma_{x_3}} (a_{i3} \partial_i U + b_3) dx' \partial_3 \zeta \frac{\chi_1(x_3)}{|\Sigma_{x_3}|} \right)^2 dx_3 \right]^{\frac{1}{2}} \cdot \left[ \int_{t_1-1}^{t_1} \chi_1^2(x_3) dx_3 \right]^{\frac{1}{2}}
\]

\[
\leq \frac{C}{(t_1-1)^{a_1}} \left( \frac{\beta}{2} (e^{2\beta} - 1) \right)^{\frac{1}{2}} \left[ \int_{t_1-1}^{t_1} \left( \int_{\Sigma_{x_3}} (U - s_1)^2 dx' \right) dx_3 \right]^{\frac{1}{2}}
\]

\[
\leq \frac{C \beta (e^{2\beta} - 1)}{\epsilon_1 (t_1-1)^{2a_1}} + \frac{\epsilon_1}{2} \int_{\Omega(t_1-1,t_1)} |\nabla U|^2 dx.
\]
Therefore, combining (105), (106), and (107) together shows

\begin{equation}
|I_1| \leq \Lambda^2 \hat{\beta} \int_{\Omega(t_{i-1}, t_i)} |\nabla U|^2 \zeta dx + \frac{C \hat{\beta} (e^{2\hat{\beta}} - 1)}{\epsilon_1 (t_1 - 1)^{2a_1}} + \frac{C \hat{\beta} (e^{2\hat{\beta}} - 1)}{\epsilon_1 (t_1 - 1)^{2b_1}} + \epsilon_1 \int_{\Omega(t_{i-1}, t_i)} |\nabla U|^2 dx.
\end{equation}

Similarly, one has

\begin{equation}
|I_2| \leq \Lambda^2 \hat{\beta} \int_{\Omega(t_{2}, t_{2+1})} |\nabla U|^2 \zeta dx + \frac{C \hat{\beta} (e^{2\hat{\beta}} - 1)}{\epsilon_1 (t_2)^{2a_1}} + \frac{C \hat{\beta} (e^{2\hat{\beta}} - 1)}{\epsilon_1 (t_2)^{2b_1}} + \epsilon_1 \int_{\Omega(t_{2}, t_{2+1})} |\nabla U|^2 dx.
\end{equation}

For $I_3$, it follows from (35) that one has

\begin{equation}
|I_3| \leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^b_1} \int_{\Omega(t_{1-1}, t_{2+1})} |\nabla U| dx
\end{equation}

\begin{equation}
\leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^b_1} |\Omega(t_1 - 1, t_2 + 1)|^{\frac{1}{2}} \left( \frac{1}{|\Omega(t_{1-1}, t_{2+1})|} \right) \frac{1}{2} |\nabla U|^2 dx.
\end{equation}

It follows from (100) that

\begin{equation}
|I_4| = \left| \int_{t_{2} - t_1}^{t_2} \left( \int_{\Sigma_{s_3}} a_{33} \partial_s U + b_3 dx' \right) (\zeta - 1) dx_3 \right| \leq \frac{C e^{\hat{\beta}}}{t_1^{b_1}} |s_2 - s_1|.
\end{equation}

With the aid of estimate (57), one has

\begin{equation}
|I_4| \leq \frac{C e^{\hat{\beta}}}{t_1^{b_1}} \int_{\Omega(t_{1-1}, t_{2+1})} |\nabla U| dx \leq \frac{C e^{2\hat{\beta}} (t_2 - t_1 + 2)}{\epsilon_1 t_1^{2a_1}} + \frac{\epsilon_1}{2} \int_{\Omega(t_{1-1}, t_{2+1})} |\nabla U|^2 dx.
\end{equation}

For $I_5$, it follows from (99) that one has

\begin{equation}
|I_5| \leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^{a_1+1}} \int_{\Omega(t_{i}, t_{i+1})} \hat{U} ds
\end{equation}

\begin{equation}
\leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^{a_1+1}} \sum_{i=0}^{K+1} \int_{\Omega(t_{i-1}+i, t_{i+1}+i)} \hat{U} ds
\end{equation}

\begin{equation}
\leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^{a_1+1}} \sum_{i=0}^{K+1} \left( \int_{\Omega(t_{i-1}+i, t_{i+1}+i)} |\nabla \hat{U}| + |\hat{U} ds| \right)
\end{equation}

\begin{equation}
\leq \frac{C e^{\hat{\beta}}}{(t_1 - 1)^{a_1+1}} \left( \int_{\Omega(t_{i-1}+i, t_{i+1}+i)} |\nabla \hat{U}| dx + |s_2 - s_1| + \int_{\Omega(t_{i-1}+i, t_{i+1}+i)} |\hat{U} ds| \right).
\end{equation}

Define

\begin{equation}
D_i = \int_{\Omega(t_{i}, t_{i+1})} \hat{U} dx, \quad i = 0, 1, \cdots, K - 1.
\end{equation}
As same as the estimate (57), one has

\begin{equation}
|D_i - s_1| \leq \int_{\Omega(t_1-1,t_1+i+1)} |\nabla U| dx.
\end{equation}

Therefore, the direct computations give

\begin{align*}
\int_{\Omega(t_1,t_2)} |U - s_1| dx \\
\leq \sum_{j=0}^{K-1} \int_{\Omega(t_1+j,t_1+j+1)} |U - D_j| + |D_j - s_1| dx \\
\leq C \sum_{j=0}^{K-1} \int_{\Omega(t_1+j,t_1+j+1)} |\nabla U| dx + C \sum_{j=0}^{K-1} \int_{\Omega(t_1-1,t_1+j+1)} |\nabla U| dx \\
\leq C \sum_{j=0}^{K-1} \int_{\Omega(t_1+j,t_1+j+1)} |\nabla U| dx + C \sum_{j=0}^{K-1} \int_{\Omega(t_1-1,t_2+1)} |\nabla U| dx \\
\leq C(t_2 - t_1 + 2) \int_{\Omega(t_1-1,t_2+1)} |\nabla U| dx.
\end{align*}

This yields

\begin{align*}
\int_{\Omega(t_1-1,t_2+1)} |\hat{U}| dx &= \int_{\Omega(t_1-1,t_1)} |\hat{U}| dx + \int_{\Omega(t_1,t_2+1)} |\hat{U}| dx + \int_{\Omega(t_1,t_2+1)} |\hat{U}| dx + \int_{\Omega(t_1,t_2+1)} |\hat{U}| dx \\
&\leq C \left( \int_{\Omega(t_1-1,t_1) \cup \Omega(t_2,t_2+1)} |\nabla U| dx \right) + \int_{\Omega(t_1,t_2)} \left| U - s_1 - \frac{s_2 - s_1}{t_2 - t_1} (x_3 - t_1) \right| dx \\
&\leq C \left( \int_{\Omega(t_1-1,t_1) \cup \Omega(t_2,t_2+1)} |\nabla U| dx \right) + |s_2 - s_1| \int_{t_1}^{t_2} \int_{\Sigma_3} \frac{x_3 - t_1}{t_2 - t_1} dx' dx_3 + \int_{\Omega(t_1,t_2)} |U - s_1| dx \\
&\leq C \left( \int_{\Omega(t_1-1,t_1) \cup \Omega(t_2,t_2+1)} |\nabla U| dx \right) + C(t_2 - t_1) |s_2 - s_1| + C(t_2 - t_1 + 2) \int_{\Omega(t_1-1,t_2+1)} |\nabla U| dx \\
&\leq C(t_2 - t_1 + 2) \int_{\Omega(t_1-1,t_2+1)} |\nabla U| dx.
\end{align*}
Therefore, the estimate (113), together with (117), implies
\[
|I_5| \leq \frac{Ce\beta(t_2-t_1+2)}{(t_1-1)^{a_1+1}} \int_{\Omega(t_1-1,t_2+1)} |\nabla U| dx
\]
(118)
\[
\leq \frac{Ce\hat{\beta}(t_2-t_1+2)}{(t_1-1)^{a_1+1}} |\Omega(t_1-1,t_2+1)|^{\frac{1}{2}} \left( \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq \frac{Ce^{2\hat{\beta}}(t_2-t_1+2)^3}{\epsilon_1(t_1-1)^{2a_1+2}} + \frac{\epsilon_1}{2} \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx.
\]
Collecting (108), (109), (110), (112), and (118) together gives
\[
\lambda \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 (\zeta - 1) dx
\]
\[
\leq \Lambda^2 \hat{\beta} \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 \zeta dx + \epsilon_1 \int_{\Omega(t_1-1,t_1)\cup\Omega(t_2,t_2+1)} |\nabla U|^2 dx
\]
(119)
\[
+ \frac{3\epsilon_1}{2} \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx + \frac{C\hat{\beta}(e^{2\hat{\beta}}-1)}{\epsilon_1(t_1-1)^{2a_1}} + \frac{C\hat{\beta}(e^{2\hat{\beta}}-1)}{\epsilon_1(t_1-1)^{2b_1}} + \frac{C\hat{\beta}(e^{2\hat{\beta}}-1)}{\epsilon_1 t_2^{2a_1}}
\]
\[
+ \frac{C\hat{\beta}(e^{2\hat{\beta}}-1)}{\epsilon_1 t_2^{2b_1}} + \frac{Ce^{2\hat{\beta}}(t_2-t_1+2)}{\epsilon_1(t_1-1)^{2b_1}} + \frac{Ce^{2\hat{\beta}}(t_2-t_1+2)}{\epsilon_1(t_1-1)^{2a_1}} + \frac{Ce^{2\hat{\beta}}(t_2-t_1+2)^3}{\epsilon_1(t_1-1)^{2a_1+2}}.
\]
Choosing \( \hat{\beta} = \frac{\lambda}{\Lambda^2} \) and taking \( b = \min(a_1, b_1) \) yield
\[
\lambda e^{\hat{\beta}} \int_{\Omega(t_1,t_2)} |\nabla U|^2 dx
\]
(120)
\[
\leq (\lambda + 3\epsilon_1) \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx + \frac{C(t_2-t_1+2)}{\epsilon_1(t_1-1)^{2b_1}} + \frac{C(t_2-t_1+2)^3}{\epsilon_1(t_1-1)^{2b_1+2}},
\]
where \( C \) is a constant depending on \( \hat{\beta} \).

If one chooses \( \epsilon_1 \) small enough such that \( \frac{\lambda + 3\epsilon_1}{\Lambda^2\hat{\beta}} \leq \epsilon_0 < 1 \), then it holds that
\[
\int_{\Omega(t_1,t_2)} |\nabla U|^2 dx \leq \epsilon_0 \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx + \frac{C(t_2-t_1+2)}{(t_1-1)^{2b}} + \frac{C(t_2-t_1+2)^3}{(t_1-1)^{2b+2}}.
\]
Choosing \( K \) large enough such that \( \epsilon_0 \frac{t_2-t_1+2}{t_2-t_1} < c < 1 \) yields
\[
\frac{1}{t_2-t_1} \int_{\Omega(t_1,t_2)} |\nabla U|^2 dx
\]
(121)
\[
\leq c \frac{1}{t_2-t_1+2} \int_{\Omega(t_1-1,t_2+1)} |\nabla U|^2 dx + \frac{C(t_2-t_1+2)}{(t_2-t_1)(t_1-1)^{2b}} + \frac{C(t_2-t_1+2)^3}{(t_2-t_1)(t_1-1)^{2b+2}}.
\]
Let $J$ be an integer satisfying $\frac{T}{2} - 1 \leq J < \frac{T}{2}$. Denote $t_{1,i} = T - i$ and $t_{2,i} = T + \bar{K} + i$ ($i = 0, 1 \cdots J$). If $T > 0$ is large enough, one has

$$t_{2,i} - t_{1,i} + 2 \leq C \quad \text{for } i = 0, 1 \cdots J.$$  \hfill (123)

Substituting $t_{1,i}$ and $t_{2,i}$ into (122) yields

$$\frac{1}{t_{2,i} - t_{1,i}} \int_{\Omega(t_{1,i}, t_{2,i})} |\nabla U|^2 dx \leq C \frac{1}{t_{2,i} - t_{1,i}} + 2 \int_{\Omega(t_{1,i} - t_{1,i} + 2)} |\nabla U|^2 dx + C \frac{C}{(t_{1,i} - 1)^{2b}}.$$  \hfill (124)

Iterating (124) gives

$$\frac{1}{t_{2,0} - t_{1,0}} \int_{\Omega(t_{1,0}, t_{2,0})} |\nabla U|^2 dx \leq \frac{c^j}{t_{2,j} - t_{1,j}} \int_{\Omega(t_{1,j}, t_{2,j})} |\nabla U|^2 dx + \sum_{j=0}^{J} c^j C \frac{C}{(t_{1,j})^{2b}}.$$  \hfill (125)

Since $|\nabla U|$ is bounded and $T$ is large, one has

$$\frac{1}{K} \int_{\Omega(T, T + \bar{K})} |\nabla U|^2 dx \leq C c^j + C \frac{C}{T^{2b}} \leq \frac{C}{T^{2b}}.$$  \hfill (126)

Thus,

$$\int_{\Omega(T, T + 1)} |\nabla U|^2 dx \leq \int_{\Omega(T, T + \bar{K})} |\nabla U|^2 dx \leq \frac{C K}{T^{2b}} \leq \frac{C}{T^{2b}}.$$  \hfill (127)

For the incompressible flows, when the nozzle boundary satisfies (23), then $\bar{U} = \bar{\phi} - q x_3$ satisfies

$$\begin{cases}
    \Delta \bar{U} = 0, & \text{in } \Omega \cap \{x_3 > K\}, \\
    \frac{\partial \bar{U}}{\partial n} = -qn_3, & \text{on } \partial \Omega \cap \{x_3 > K\}.
\end{cases}$$  \hfill (128)

Let $a_{ij} = \delta_{ij}$ and $b_i = 0$ in (94). Then (99) and (100) can be written as

$$\int_{\Sigma x_3} \partial x_3 \bar{U} dx' \leq C x_3^{-a_1} \quad \text{and} \quad \left| \frac{\partial \bar{U}}{\partial n} \right| \leq C x_3^{-a_1 - 1} \quad \text{for } x_3 > K.$$  \hfill (129)

Similar to the proof for estimate (127), one can conclude that

$$\int_{\Omega(T, T + 1)} |\nabla \bar{U}|^2 dx \leq \frac{C}{T^{2a_1}}.$$  \hfill (130)
5.3. **Energy estimate where the boundary of the nozzle satisfies (23) with \(a_1 > 1\).**

In this case, the velocity at the downstream is not constant. Then convergence rates of the velocity at the boundary in normal direction is \(O(x_3^{-a_1})\). In order to get the convergence rates of velocity, we ask \(a_1 > 1\).

Let \(\varphi_*\) is the uniformly subsonic solution of (46). Obviously, \(\nabla \varphi_*\) is not a constant. Denote \(W = \varphi^* - \varphi_*\). Then \(W\) satisfies

\[
\begin{cases}
\partial_i (a_{ij} \partial_j W) = 0, & \text{in } \Omega \cap \{x_3 > K\}, \\
\frac{\partial W}{\partial n} = -\nabla \varphi_* \cdot n, & \text{on } \partial \Omega \cap \{x_3 > K\},
\end{cases}
\]  

where we abuse the notations

\[
a_{ij} = \int_0^1 \rho^\epsilon (\bar{q}^2, \phi) \delta_{ij} + 2 \rho^\epsilon (\bar{q}^2, \phi) (s \partial_i \varphi^* + (1-s) \partial_i \varphi_*) (s \partial_j \varphi^* + (1-s) \partial_j \varphi_*) ds
\]

with

\[
\bar{q}^2 = |s \nabla \varphi^* + (1-s) \nabla \varphi_*|^2.
\]

Obviously \(a_{ij}\) satisfies (73). On each cross section \(\Sigma_{x_3}\), one has

\[
\int_{\Sigma_{x_3}} a_{33} \partial_3 W \, dx = \int_{\Sigma_{x_3}} \rho^\epsilon (|\nabla \varphi^*|^2, \phi) \partial_3 \varphi^* - \rho^\epsilon (|\nabla \varphi_*|^2, \phi) \partial_3 \varphi_* \, dx'
\]

\[
= \int_{\Sigma_{x_3}} \rho^\epsilon (|\nabla \varphi^*|^2, \phi) \partial_3 \varphi^* \, dx' - \int_{\Sigma_{x_3}} \rho^\epsilon (|\nabla \varphi_*|^2, \phi) \partial_3 \varphi_* \, dx'
\]

\[
+ \int_{B_1(0)} \rho^\epsilon (|\nabla \varphi_*|^2, \phi) \partial_3 \varphi_* \, dx' - \int_{B_1(0)} \rho^\epsilon (|\nabla \varphi_*|^2, \phi) \partial_3 \varphi_* \, dx'
\]

\[
\leq C \left| \Sigma_{x_3} \right| - |B_1(0)| \leq C x_3^{-a_1}.
\]

Writing \(\hat{n} = (\cos \tau, \sin \tau, 0)\), then \(\nabla \varphi_* \cdot \hat{n} = 0\). In fact, on the boundary, one has

\[
(a_{ij} \partial_j W) n_i = \left( \rho^\epsilon (|\nabla \varphi^*|^2, \phi) \nabla \varphi^* - \rho^\epsilon (|\nabla \varphi_*|^2, \phi) \nabla \varphi_* \right) \cdot n
\]

\[
= -\rho^\epsilon (|\nabla \varphi_*|^2, \phi) \nabla \varphi_* \cdot n = -\rho^\epsilon (|\nabla \varphi_*|^2, \phi) \nabla \varphi_* \cdot (n - \hat{n}) \leq \frac{C}{x_3^{a_1}},
\]

where the assumption (23) is used. Note that the boundary convergence rates in (135) is \(-a_1\). This is main difference between the current case and the case where there is no external force decay as in (99). Therefore, with the help of the decay (134), the \(L^2\) convergence rates \(-(a_1 - 1)\) can be established.
It follows from (134), (135) and the same strategy to get the estimate (127) in Step 2 \((b_i = 0)\) that one has

\[
\int_{\Omega(T,T+1)} |\nabla W|^2 dx \leq \frac{C}{T^{2(a_1-1)}}.
\]

5.4. \(L^\infty\)-estimate. Based on the \(L^2\) estimates obtained in the last sections, \(L^\infty\)-norm of velocity fields can be established via Nash-Moser iteration. Since the general case is that \(b_i \neq 0\) and the boundary estimate, we only consider the equation (94) in Section 5.2 and first prove the estimate near the boundary. For any point \(\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \partial \Omega\) and \(\tilde{x}_3 > 0\) sufficiently large, suppose that \((x_1(y_1, y_2), x_2(y_1, y_2), x_3(y_1, y_2)) \in C^{2,\alpha}\) is the standard parametrization of \(\Omega\) in a small neighborhood of \(\tilde{x}\). Then unit outer normal vector \(n\) satisfying

\[
\cos(n, x_i) = n_i(y_1, y_2) \in C^{1,\alpha} \quad \text{for} \quad i = 1, 2, 3.
\]

Define the map \(M_y : y \to x\) as follows

\[
x_i = x_i(y_1, y_2) + y_3^{-1} \int_{y_1}^{y_1+y_3} \int_{y_2}^{y_2+y_3} n_i(\alpha_1, \alpha_2) d\alpha_1 d\alpha_2, \quad \text{for} \quad i = 1, 2, 3.
\]

Then the map \(T_{\tilde{x}} = M_y^{-1} : x \to y\) makes the boundary flat and satisfies

\[
T_{\tilde{x}}(U_\delta \cap \Omega) \to B_R^+ \quad \text{and} \quad T_{\tilde{x}}(\partial U_\delta \cap \Omega) \to \partial B_R^+ \cap \{y_3 = 0\},
\]

where \(U_\delta\) is a neighborhood of \(\tilde{x}\), and \(B_R^+ = \{y_1^2 + y_2^2 + y_3^2 < R, \quad y_3 > 0\}\) with \(\delta\) and \(R\) uniform constants along the boundary of \(\partial \Omega\). Denote the jacobian \(\left(\frac{\partial y_i}{\partial x_j}\right) = D(x)\), then for any \(\xi \in \mathbb{R}^3\), there exists a constant \(C\) such that

\[
C^{-1} |\xi| \leq |D(x)\xi| \leq C |\xi| \quad \text{and} \quad C^{-1} |\xi| \leq |D^{-1}(x)\xi| \leq C |\xi|.
\]

Moreover, the map also satisfies for \(x \in \partial \Omega \ (i.e. \ y_3 = 0)\),

\[
\sum_{i=1}^{3} \frac{\partial y_j}{\partial x_i} \frac{\partial y_3}{\partial x_i} = 0, \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad \left(\frac{\partial y_3}{\partial x_1}, \frac{\partial y_3}{\partial x_2}, \frac{\partial y_3}{\partial x_3}\right) \times n = 0.
\]

On the boundary \(\partial \Omega\), denote

\[
g_1 = \sum_{i,j=1}^{3} a_{ij} \partial_j \mathcal{U} n_i, \quad g_2 = \sum_{i=1}^{3} b_i n_i \quad \text{and} \quad g_3 = -q n_3.
\]

It follows from (98) and (99) that

\[
|g_1| \leq \frac{C}{x_3^b}, \quad |g_2| \leq \frac{C}{x_3^{b_1}}, \quad \text{and} \quad |g_3| \leq \frac{C}{x_3^{a_1+1}} \quad \text{on} \ \partial \Omega,
\]

where \(b = \min(a_1, b_1)\).
In the rest of the paper, \( \tilde{f} \) denotes the function \( f \) in \( y \)-coordinates. Because of (140), one has

\[
\sum_{i,j,l=1}^{3} \hat{a}_{ij} \frac{\partial U}{\partial y_l} \frac{\partial y_3}{\partial x_j} \frac{1}{\mathcal{W}} = \bar{g}_1 \quad \text{and} \quad \sum_{i,l=1}^{3} \frac{\partial U}{\partial y_l} \frac{\partial y_3}{\partial x_i} \frac{1}{\mathcal{W}} = \frac{\partial U}{\partial y_3} = \bar{g}_3,
\]

where \( \mathcal{W} = \left( \sum_{i=1}^{3} \left| \frac{\partial y_3}{\partial x_i} \right|^2 \right)^{\frac{1}{2}}. \) It follows from (141) that

\[
|\bar{g}_1| \leq C \frac{x_3^{b_1}}{x_3^{a_1+1}}, \quad |\bar{g}_2| \leq C \frac{x_3^{b_1}}{x_3^{b_2}} \quad \text{and} \quad |\bar{g}_3| \leq C \frac{x_3^{b_1}}{x_3^{a_1+1}} \quad \text{on} \ \partial B^+_R \cap \{y_3 = 0\}.
\]

Denote \( g_4 = \sum_{i=1}^{3} \partial_i b_i, \) the straightforward computations give

\[
g_4 \leq C \bar{x}_3^{-b_1} \quad \text{and} \quad \tilde{g}_4 \leq C \bar{x}_3^{-b_1}.
\]

After changing variables, the problem (94) can be written as follows

\[
\begin{aligned}
&\sum_{i,j,l,s=1}^{3} \frac{\partial}{\partial y_s} \left( \hat{a}_{ij}(y) \frac{\partial y_3}{\partial y_l} \frac{\partial y_3}{\partial x_j} \right) \frac{\partial y_s}{\partial x_i} + \bar{g}_4 = 0, \quad \text{in} B^+_R, \\
&\sum_{i,s=1}^{3} \frac{\partial U}{\partial y_s} \frac{\partial y_3}{\partial x_i} \frac{\partial y_3}{\partial x_i} = \bar{g}_3 \mathcal{W}, \quad \text{on} \ B^+_R \cap \{y_3 = 0\}.
\end{aligned}
\]

For any \( \psi \in C^3_0(B^+_R), \) multiplying \( \psi \) on both sides of (145) and integrating by parts yield

\[
\sum_{i,j,s,l=1}^{3} \int_{B^+_R} \hat{a}_{ij} \frac{\partial U}{\partial y_l} \frac{\partial y_3}{\partial x_j} \frac{\partial \psi}{\partial x_i} dy - \int_{B^+_R} \bar{g}_4 \psi dy = \sum_{i,j,l=1}^{3} \int_{B^+_R \cap \{y_3 = 0\}} \hat{a}_{ij} \frac{\partial U}{\partial y_l} \frac{\partial y_3}{\partial x_j} \frac{\partial \psi}{\partial x_i} dy.
\]

Denote \( A_{sl} = \sum_{i,j=1}^{3} \hat{a}_{ij} \frac{\partial y_3}{\partial x_j} \frac{\partial y_3}{\partial x_i}. \) In virtue of (139), we still using \( \lambda \) and \( \Lambda \) such that

\[
\lambda |\xi|^2 \leq \sum_{s,l=1}^{3} A_{sl} \xi_s \xi_l \leq \Lambda |\xi|^2 \quad \text{and} \quad \left| \frac{\partial A_{ls}}{\partial y_r} \right| \leq C.
\]

It follows from (142) that

\[
\sum_{s,l=1}^{3} \int_{B^+_R} A_{ls} \frac{\partial U}{\partial y_l} \frac{\partial \psi}{\partial y_s} dy - \int_{B^+_R} \bar{g}_4 \psi dy = \int_{B^+_R \cap \{y_3 = 0\}} \psi \bar{g}_1 \mathcal{W} dy_1 dy_2.
\]

Denote \( g = \frac{\bar{g}_1 \mathcal{W}}{A_{33}}. \) By the definition of \( A_{33}, \) one has

\[
\lambda \mathcal{W}^2 \leq \hat{a}_{ij} \frac{\partial y_3}{\partial x_i} \frac{\partial y_3}{\partial x_j} = A_{33} \leq \Lambda \mathcal{W}^2.
\]
Given \( \zeta(z') \in C_0^2(\mathbb{R}^2) \) satisfying \( \int_{\mathbb{R}^2} \zeta(z')dz' = 1 \), define

\[
\vartheta(y) = y_3 \int_{\mathbb{R}^2} g(y' - y_3 z') \zeta(z')dz'.
\]

Then

\[
\vartheta(y', 0) = \frac{\partial \vartheta}{\partial y_1}(y', 0) = \frac{\partial \vartheta}{\partial y_2}(y', 0) = 0 \quad \text{and} \quad \frac{\partial \vartheta}{\partial y_3}(y', 0) = g(y').
\]

The straightforward computations yield

\[
\|\vartheta\|_{C^2(B^+_R)} \leq \frac{C}{x^3}.
\]

Define \( u = \partial_s (A_{sl} \partial_l \vartheta) \) and \( \kappa = u + \tilde{g}_3 \). It follows from (152) and the definition of \( A_{sl} \) that one has

\[
\|u\|_{L^\infty(B^+_R)} \leq \frac{C}{x^3}, \quad \|\kappa\|_{L^\infty(B^+_R)} \leq \frac{C}{x^3}
\]

and for any \( \psi \in C_0^3(B^+_R) \)

\[
-\sum_{s,l=1}^3 \int_{B^+_R} A_{ls} \partial_l \partial_s \psi + \sum_{l=1}^3 \int_{B^+_R \cap \{y_3=0\}} A_{3l} \partial_l \psi dx = \int_{B^+_R} (u + \tilde{g}_3) \psi dx.
\]

Combining (148) and (154) yields

\[
\sum_{s,l=1}^3 \int_{B^+_R} A_{ls} \frac{\partial (\mathcal{U} - \vartheta)}{\partial y_i} \frac{\partial \psi}{\partial y_s} dy = \int_{B^+_R} \kappa \psi dy,
\]

where the boundary conditions (151) have been used. Denote \( v = \mathcal{U} - \vartheta \). Replacing \( \psi \) by each \( \frac{\partial \psi}{\partial y_i} \) \((i = 1, 2, 3)\) in (155) and integrating by parts yield

\[
\int_{B^+_R} \kappa \frac{\partial \psi}{\partial y_i} dy = -\sum_{l,s=1}^3 \int_{B^+_R} A_{ls} \frac{\partial v}{\partial y_l} \frac{\partial \psi}{\partial y_s} dy + \sum_{l,s=1}^3 \int_{B^+_R \cap \{y_3=0\}} A_{ls} \frac{\partial v}{\partial y_l} \frac{\partial \psi}{\partial y_s} ds, \quad \text{for} \quad i = 1, 2, 3.
\]

Define

\[
\Theta = \max_{B^+_R \cap \{y_3=0\}} \left| \frac{\partial v}{\partial y_3} \right| , \quad w_1 = \frac{\partial v}{\partial y_1}, \quad w_2 = \frac{\partial v}{\partial y_2}, \quad \text{and} \quad w_3 = \frac{\partial v}{\partial y_3} - \Theta.
\]

It follows from (142) and (152) that

\[
\Theta \leq \max_{B^+_R \cap \{y_3=0\}} \left| \frac{\partial \mathcal{U}}{\partial y_3} \right| + \max_{B^+_R \cap \{y_3=0\}} \left| \frac{\partial \vartheta}{\partial y_3} \right| \leq \frac{C}{x^3}.
\]
Moreover, the expression (156) can be written as, for $i = (1, 2, 3)$

$$\sum_{l,s=1}^{3} \int_{B^+_R} A_{ls} \frac{\partial w_i}{\partial y_l} \frac{\partial \psi}{\partial y_s} dy + \sum_{l,s=1}^{3} \int_{B^+_R} w_i \frac{\partial A_{ls}}{\partial y_l} \frac{\partial \psi}{\partial y_s} dy$$

(158)

$$= \delta_3 \sum_{l,s=1}^{3} \int_{B^+_R \cap \{y_3 = 0\}} A_{ls} \frac{\partial v}{\partial y_l} \frac{\partial \psi}{\partial y_s} ds - \int_{B^+_R} \kappa \frac{\partial \psi}{\partial y_i} dy - \delta_3 \Theta \sum_{l,s=1}^{3} \int_{B^+_R} \frac{\partial A_{ls}}{\partial y_l} \frac{\partial \psi}{\partial y_s} dy.$$

Now we use Nash-Moser iteration to get the $L^\infty$-norm of $w_i$. We consider only the case $w_i \geq 0$. If $w_i \geq 0$ does not hold, one can repeat the proof for $w_i^+$ and $w_i^-$, respectively. It is easy to see that

(159) $w_3 = 0$ on $B^+_R \cap \{y_3 = 0\}$.

For $i = 1, 2, 3$, denote $\psi_i = \eta^2 w_i^{\mu+1}$ with some $\mu \geq 0$ and some nonnegative function $\eta \in C^2_0(B^+_R)$. Direct calculations give

$$\frac{\partial \psi_i}{\partial y_k} = 2\eta \frac{\partial \eta}{\partial y_k} w_i^{\mu+1} + \eta^2 (\mu + 1) \frac{\partial w_i}{\partial y_k} w_i^\mu, \quad \text{for } k = 1, 2, 3.$$

If one replaces $\psi$ by $\psi_i$ ($i = 1, 2, 3$) in (158), then it holds

$$\sum_{l,s=1}^{3} \int_{B^+_R} A_{ls} \frac{\partial w_i}{\partial y_l} \frac{\partial \psi_i}{\partial y_s} dy + \sum_{l,s=1}^{3} \int_{B^+_R} w_i \frac{\partial A_{ls}}{\partial y_l} \frac{\partial \psi_i}{\partial y_s} dy$$

(160)

$$= -\int_{B^+_R} \kappa \frac{\partial \psi_i}{\partial y_i} dy - \Theta \sum_{s=1}^{3} \int_{B^+_R} \frac{\partial A_{3s}}{\partial y_i} \frac{\partial \psi_i}{\partial y_s} dy$$

$$= -\sum_{s=1}^{3} \int_{B^+_R} (\delta_{is}\kappa + \Theta) \frac{\partial A_{3s}}{\partial y_i} \frac{\partial \psi_i}{\partial y_s} dy, \quad \text{for } i = 1, 2, 3.$$

where the boundary term vanishes due to (159). For $i = 1, 2, 3$, the straightforward computations give

$$\sum_{l,s=1}^{3} \int_{B^+_R} A_{ls} \frac{\partial w_i}{\partial y_l} \left(2\eta \frac{\partial \eta}{\partial y_s} w_i^{\mu+1} + \eta^2 (\mu + 1) \frac{\partial w_i}{\partial y_s} w_i^\mu\right) dy$$

(161)

$$\geq \lambda (\mu + 1) \int_{B^+_R} \eta^2 w_i^{\mu}|Dw_i|^2 dy - 2\Lambda \int_{B^+_R} \eta w_i^{\mu+1}|D\eta||Dw_i|dy$$

$$\geq \lambda (\mu + 1) \int_{B^+_R} \eta^2 w_i^{\mu}|Dw_i|^2 dy - \epsilon \int_{B^+_R} \eta^2 w_i^{\mu}|Dw_i|^2 - \frac{1}{\epsilon} \int_{B^+_R} |D\eta|^2 w_i^{\mu+2} dy$$
and
\[ \sum_{l,s=1}^{3} \int_{B_{R}^{+}} w_{l} \frac{\partial A_{ls}}{\partial y_{i}} \frac{\partial \psi_{i}}{\partial y_{s}} dy \leq C \sum_{l,s=1}^{3} \int_{B_{R}^{+}} \eta |D\eta| w_{l}^{\mu+1} \left| \frac{\partial A_{ls}}{\partial y_{i}} w_{l} \right| + \left| \frac{\partial A_{ls}}{\partial y_{i}} \right| \eta^{2} (\mu + 1) w_{l}^{\mu} |Dw_{i}| dy \]
\[ \leq C \sum_{l,s=1}^{3} \int_{B_{R}^{+}} |D\eta|^{2} w_{l}^{\mu+2} + \left( \eta^{2} + \frac{1}{\epsilon} \right) \left| \frac{\partial A_{ls}}{\partial y_{i}} \right|^{2} w_{l}^{\mu} + \epsilon \eta^{2} (\mu + 1)^{2} w_{l}^{\mu} |Dw_{i}|^{2} dy \]
\[ \leq C \sum_{l,s=1}^{3} \int_{B_{R}^{+}} |D\eta|^{2} w_{l}^{\mu+2} + \eta^{2} |\bar{w}|^{2} w_{l}^{\mu} + \epsilon \eta^{2} (\mu + 1)^{2} w_{l}^{\mu} |Dw_{i}|^{2} + \frac{1}{\epsilon} |\bar{w}|^{2} w_{l}^{\mu} dy, \]
where \(|\bar{w}|^{2} = w_{1}^{2} + w_{2}^{2} + w_{3}^{2} \).

Denote
\[ F_{ls} = \delta_{is} \kappa + \Theta \frac{\partial A_{3s}}{\partial y_{i}} \quad \text{and} \quad K = \max |F_{ls}|. \]

It follows from (153) and (157) that
\[ |K| \leq \frac{C}{x_{3}^{b}}. \]

If \(|w_{i}| \leq K\), the estimate (36) holds. Hence we assume
\[ |w_{i}| \geq K, \quad (i = 1, 2, 3). \]

Therefore, one has
\[ \sum_{s=1}^{3} \int_{B_{R}^{+}} F_{is} \frac{\partial \psi_{i}}{\partial y_{s}} dy \leq \int_{B_{R}^{+}} K\eta |D\eta| w_{i}^{\mu+1} + K\eta^{2} (\mu + 1) w_{l}^{\mu} |Dw_{i}| dy \]
\[ \leq \int_{B_{R}^{+}} \eta |D\eta| w_{i}^{\mu+2} + \epsilon \eta^{2} (\mu + 1)^{2} w_{l}^{\mu} |Dw_{i}|^{2} + \frac{1}{\epsilon} \eta^{2} K^{2} w_{l}^{\mu} dy \]
\[ \leq \int_{B_{R}^{+}} \eta |D\eta| w_{i}^{\mu+2} + \epsilon \eta^{2} (\mu + 1)^{2} w_{l}^{\mu} |Dw_{i}|^{2} + \frac{1}{\epsilon} \eta^{2} w_{l}^{\mu+2} dy. \]

Combining (161), (162), and (166) yields
\[ \lambda (\mu + 1) \int_{B_{R}^{+}} \eta^{2} w_{l}^{\mu} |Dw_{i}|^{2} dy - (\epsilon + 2\epsilon (\mu + 1)^{2}) \int_{B_{R}^{+}} \eta^{2} w_{l}^{\mu} |Dw_{i}|^{2} dy \]
\[ \leq (C + \frac{C}{\epsilon}) \int_{B_{R}^{+}} \eta^{2} |\bar{w}|^{2} w_{l}^{\mu} dy + (C + \frac{C}{\epsilon}) \int_{B_{R}^{+}} (\eta^{2} + |D\eta|^{2}) w_{l}^{\mu+2} dy. \]
If one chooses \( \epsilon = \frac{\lambda}{8(\mu+1)} \), then it holds that

\[
\int_{B_R^+} \eta^2 w_i^2 |Dw_i|^2 \, dy \leq C \int_{B_R^+} |D\eta|^2 |w_i|^\mu + \eta^2 |w_i|^\mu + \eta^2 \bar{w}^2 |w_i|^\mu \, dy.
\]

Therefore, one has

\[
\int_{B_R^+} \left| D(\eta w_i^\mu) \right|^2 \, dy \leq C(\mu + 2)^2 \int_{B_R^+} |D\eta|^2 |w_i|^\mu + \eta^2 |w_i|^\mu + \eta^2 \bar{w}^2 |w_i|^\mu \, dy.
\]

Applying Sobolev inequality yields

\[
\left( \int_{B_R^+} (\eta w_i^\mu)^6 \, dy \right)^{\frac{1}{6}} \leq C(\mu + 2)^2 \int_{B_R^+} |D\eta|^2 |w_i|^\mu + \eta^2 |w_i|^\mu + \eta^2 \bar{w}^2 |w_i|^\mu \, dy.
\]

Set

\[ R_j = \left( \frac{1}{2} + \frac{1}{2^{j+1}} \right) R \quad \text{and} \quad \gamma_j = 2 \cdot 3^j. \]

Let \( \eta_j \in C_0^\infty(B_{R_j}^+) \) satisfy

\[ \eta_j = 1 \text{ in } B_{R_j+1}^+ \quad \text{and} \quad |D\eta_j| \leq \frac{4}{R_j - R_{j+1}}. \]

Choose \( \mu = \gamma_j - 2 \), it follows from (170) that

\[
\left( \int_{B_{R_j+1}^+} w_i^{\gamma_j+1} \, dy \right)^{\frac{1}{\gamma_j+1}} \leq C\gamma_j^2 \int_{B_{R_j}^+} (2^{j+1} R)^2 w_i^{\gamma_j} + w_i^{\gamma_j} + |\bar{w}|^2 w_i^{\gamma_j-2} \, dy.
\]

Thus,

\[
\left( \int_{B_{R_j+1}^+} w_i^{\gamma_j+1} \, dy \right)^{\frac{1}{\gamma_j+1}} \leq \left( \int_{B_{R_j}^+} A_j w_i^{\gamma_j} + B_j w_i^{\gamma_j} + B_j |\bar{w}|^2 w_i^{\gamma_j-2} \, dy \right)^{\frac{1}{\gamma_j}},
\]

where \( A_j = C\gamma_j^2 (2^{j+1}/R)^2 \) and \( B_j = C\gamma_j^2 \). Note that

\[
\int_{B_{R_j}^+} |\bar{w}|^2 w_i^{\gamma_j-2} \, dy \leq \left( \int_{B_{R_j}^+} w_i^{\gamma_j} \right)^{\frac{\gamma_j-2}{\gamma_j}} \left( \int_{B_{R_j}^+} |\bar{w}|^{\gamma_j} \right)^{\frac{2}{\gamma_j}}.
\]
Therefore, one has

\[
\left( \int_{B_{R_j+1}}^+ w_i^{\gamma_j+1} \, dy \right)^{\frac{1}{\gamma_j+1}} 
\leq \left[ A_j \int_{B_{R_j}^+} w_i^{\gamma_j} + B_j \int_{B_{R_j}^+} w_i^{\gamma_j} + B_j \left( \int_{B_{R_j}^+} \bar{w}^{\gamma_j} \right)^{\frac{\gamma_j}{\gamma_j}} \left( \int_{B_{R_j}^+} \bar{w}^{\gamma_j} \right)^{\frac{\gamma_j}{\gamma_j}} \right] \frac{1}{\gamma_j}
\]

(173)

\[
\leq \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{\gamma_j-2}{\gamma_j}} \left[ (A_j + B_j) \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}^{2} + B_j \| \bar{w} \|_{L^{\gamma_j}(B_{R_j}^+)}^{2} \right] \frac{1}{\gamma_j}
\]

\[
\leq \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{\gamma_j-2}{\gamma_j}} (A_j + 2B_j) \frac{1}{\gamma_j} \| \bar{w} \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{2}{\gamma_j}}.
\]

This implies that

\[
\sum_{i=1}^3 \| w_i \|_{L^{\gamma_j+1}(B_{R_j+1}^+)}^{\gamma_j + 1} (B_{R_j+1}^+)
\leq \sum_{i=1}^3 \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{\gamma_j-2}{\gamma_j}} (A_j + 2B_j) \frac{1}{\gamma_j} \| \bar{w} \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{2}{\gamma_j}}
\]

(174)

\[
\leq \left[ \sum_{i=1}^3 \left( \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{\gamma_j-2}{\gamma_j}} (A_j + 2B_j) \frac{1}{\gamma_j} \| \bar{w} \|_{L^{\gamma_j}(B_{R_j}^+)}^{\frac{2}{\gamma_j}} \right) \right] \frac{\gamma_j-2}{\gamma_j} \frac{2}{\gamma_j}
\]

\[
\leq (9A_j + 18B_j) \frac{1}{\gamma_j} \left( \sum_{i=1}^3 \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)} \right)^{\frac{2}{\gamma_j}} \left( \sum_{i=1}^3 \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)} \right)^{\frac{\gamma_j-2}{\gamma_j}}
\]

\[
\leq (9A_j + 18B_j) \frac{1}{\gamma_j} \sum_{i=1}^3 \| w_i \|_{L^{\gamma_j}(B_{R_j}^+)}.
\]

Set

\[
Q_{j+1} = \sum_{i=1}^3 \| w_i \|_{L^{\gamma_j+1}(B_{R_j+1}^+)} \quad \text{and} \quad S_j = (9A_j + 18B_j) \frac{1}{\gamma_j}.
\]

Then the estimate (174) can be written as

(175) \quad Q_{j+1} \leq S_j Q_j.

Obviously,

(176) \quad S_j = (9A_j + 18B_j) \frac{1}{\gamma_j} \leq \left( C \gamma_j^2 (2^{j+1}/R)^2 \right)^{\frac{1}{\gamma_j}} \leq C \gamma_j^2 \frac{1}{\gamma_j} 16 \frac{1}{\gamma_j}.
Hence,
\[ Q_{j+1} \leq C \sum_{i=1}^{j} \frac{1}{\gamma_i} \sum_{i=1}^{j} \frac{i}{\gamma_i} Q_0. \]

Note that
\[ \sum_{i=1}^{j} \frac{1}{\gamma_i} \leq C \quad \text{and} \quad \sum_{i=1}^{j} \frac{i}{\gamma_i} \leq C. \]

Taking \( j \to \infty \) yields
\[ \sum_{i=1}^{3} \| w_i \|_{L^\infty(B_R^+)} \leq C \sum_{i=1}^{3} \| w_i \|_{L^2(B_R^+)}, \]
provided that (165) holds. Therefore, one has
\[ \sum_{i=1}^{3} \| w_i \|_{L^\infty(B_R^+)} \leq C \sum_{i=1}^{3} \| w_i \|_{L^2(B_R^+)} + \mathcal{K}. \]

It follows from the definition of \( w_i \), (152), (153) and (157) that
\[ \| \nabla U \|_{L^\infty(U_{\delta})} \leq C (\| \nabla U \|_{L^2(U_{\delta})} + \mathcal{K} + \| \vartheta \|_{C^2(B_R^+)}) . \]

As same as the estimate for (179) with \( \vartheta = 0 \), \( \kappa = u \) and \( \Theta = 0 \), for any \( B_R \in \Omega \), one has
\[ \| \nabla U \|_{L^\infty(B_R^+)} \leq C \| \nabla U \|_{L^2(B_R)}. \]

In a word, when the boundary satisfies the convergence rate (23) and the external force \( \phi \) satisfies (35), we have
\[ \| \nabla \varphi^\epsilon - (0,0,\bar{q}) \|_{L^\infty(\Omega(T,T+1))} \leq C x_3^{-b}. \]

For the case where the nozzle is a perfect cylinder if \( x_3 \) is sufficiently large, the same as (181) with \( g_1 = g_2 = g_3 = g_4 = 0 \), we can prove that there exists a positive constant \( \tilde{d} \) such that
\[ \| \nabla \varphi^\epsilon - \nabla \varphi_* \| \leq C e^{-\tilde{d} x_3}. \]

For the case where the nozzle boundary satisfies (23) with \( a_1 > 1 \), one can follow the proof of estimate (181) with \( g_2 = g_4 = 0 \), \( |g_1| \leq CT^{-a_1} \), and \( |g_3| \leq CT^{-a_1} \) to show
\[ \| \nabla \varphi^\epsilon - \nabla \varphi_* \|_{L^\infty(\Omega(T,T+1))} \leq C \| \nabla W \|_{L^2(\Omega(T,T+1))} + CT^{-a_1} \leq CT^{-a_1+1}. \]

Hence the proof of Theorem 3 is completed. Furthermore, one can also use the Nash-Moser iteration to get the desired estimates (22) and (24) in Theorem 1.

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