Bilateral Birth and death process in quantum calculus

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Abstract

In this paper I shall give the complete solution of the equations governing the bilateral birth and death process on path set \( \mathbb{R}_q = \{ q^n, \ n \in \mathbb{Z} \} \) in which the birth and death rates \( \lambda_n = q^{2\nu-2n} \) and \( \mu_n = q^{-2n} \) where \( 0 < q < 1 \) and \( \nu > -1 \). The mathematical methods employed here are based on \( q \)-Bessel Fourier analysis.

Keywords : Bilateral Birth and death process, \( q \)-Bessel function, \( q \)-Hankel transform.

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1 Introduction

Birth and death processes were introduced in [4] by W. Feller in (1939) and have since been used as models for population growth, queue formation, in epidemiology and in many other areas of both theoretical and applied interest. From the standpoint of the theory of stochastic processes they represent an important special case of Markov processes with countable state spaces and continuous parameters.

The birth–death process is a special case of continuous-time Markov process where the state transitions are of only two types: "births", which increase the state variable by one and "deaths", which decrease the state by one. The model’s name comes from a common application, the use of such models to represent the current size of a population where the transitions are literal births and deaths.

The purpose of this paper is to contribute to the knowledge of the connection between some classe of birth-death processes and the \( q \)-theory. We cite for example early result in this direction. The study of the time-dependent behavior of birth and death processes involves many intricate and interesting orthogonal polynomials, such as Charlier, Meixner, Laguerre, Krawtchouk, and other polynomials from the Askey scheme. For example the authors in [9, p.350] point out that the three-term recurrence relation of the \( q \)-Lommel polynomials can be viewed as a three-term recurrence relation as occurring in birth and death processes with value \( \lambda_m = w^{-2}q^{-m} \) and \( \mu_m = q^{-m} \).

In [1] the authors study the fundamental properties of classical and quantum Markov processes generated by \( q \)-Bessel operators and their extension to the algebra of all bounded operators on the Hilbert space \( L^2 \). They noticed the connection with a bilateral birth and death process considered in this paper but without an explicite solution to the minimal semigroup of the classical \( q \)-Bessel process. But they give an interesting result about the uniqueness of such semi groupe, which is an important tool in our study.

For birth and death processes with complicated birth and death rates, for example, when rates are state dependent or nonlinear, it is almost impossible to find closed form solutions of the transition functions. Due to the difficulties involved in analytical methods, it is pertinent to develop other techniques. In this paper, I shall give the complete solution of the equations governing the bilateral birth and death process on path set \( \mathbb{R}_q = \{ q^n, \ n \in \mathbb{Z} \} \) in which the birth and death rates \( \lambda_n = q^{2\nu-2n} \) and \( \mu_n = q^{-2n} \) where \( 0 < q < 1 \) and \( \nu > -1 \). The mathematical methods employed here are based on \( q \)-Bessel Fourier analysis. In particular the \( q \)-Bessel operator, \( q \)-Bessel fourier transform, \( q \)-translation operator and \( q \)-convolution product will appear in a natural way during our study.

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2 Preliminarily on $q$-Bessel Fourier analysis

Assume that $0 < q < 1$ and $\nu > -1$. Let $a \in \mathbb{C}$, the $q$-shifted factorial are defined by

\[(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),\]

and

\[\mathbb{R}_q = \{q^n, \quad n \in \mathbb{Z}\}.\]

The $q$-Bessel operator is defined as follows [2]

\[\Delta_{q, \nu} f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - (1 + q^{2\nu})f(x) + q^{2\nu}f(qx) \right].\]

The eigenfunction of $\Delta_{q, \nu}$ associated with the eigenvalue $-\lambda^2$ is the function $x \mapsto j_{\nu}(\lambda x, q^2)$, where $j_{\nu}(., q^2)$ is the normalized $q$-Bessel function defined by

\[j_{\nu}(x, q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}_2}{(q^{2\nu+2}; q^2)_n(q^2, q^2)_n} \cdot x^{2n}.\]

It satisfies the following estimate

\[|j_{\nu}(q^n, q^2)| \leq \left( \frac{-q^2; q^2_\infty}{(q^{2\nu+2}; q^2)_\infty} \right) \left\{ \begin{array}{ll} 1 \quad & \text{if } n \geq 0 \\ \frac{q^{n^2-(2\nu+1)n}}{n} \quad & \text{if } n < 0 \end{array} \right. \] (1)

which show the asymptotic decreasing at infinity on $\mathbb{R}_q$.

The $q$-Jackson integral of a function $f$ defined on $\mathbb{R}_q$ by [7]

\[\int_{0}^{\infty} f(t) dq t = (1 - q) \sum_{n \in \mathbb{Z}} q^n f(q^n).\]

We denote by $L_{q,p,\nu}$ the space of functions $f$ defined on $\mathbb{R}_q$ such that

\[\|f\|_{q,p,\nu} = \left[ \int_{0}^{\infty} |f(x)|^p x^{2\nu+1} dq x \right]^{1/p} < \infty.\]

The normalized $q$-Bessel function $j_{\nu}(., q^2)$ satisfies the orthogonality relation [2]

\[c_{q,\nu} \int_{0}^{\infty} j_{\nu}(xt, q^2) j_{\nu}(yt, q^2) t^{2\nu+1} dq t = \delta_q(x, y), \quad \forall x, y \in \mathbb{R}_q^+.\] (2)

where

\[\delta_q(x, y) = \left\{ \begin{array}{ll} 0 \quad & \text{if } x \neq y \\ \frac{1}{(1-q)x^{2\nu+1}} \quad & \text{if } x = y \end{array} \right. ,\]

and

\[c_{q,\nu} = \frac{1}{(1-q)} \left( \frac{q^{2\nu+2}, q^2_\infty}{(q^2, q^2)_\infty} \right).\]

Let $f$ be a function defined on $\mathbb{R}_q$ then

\[\int_{0}^{\infty} f(y) \delta_q(x, y) y^{2\nu+1} dq y = f(x).\]

The $q$-Bessel Fourier transform $F_{q,\nu}$ is defined by [2, 8]

\[F_{q,\nu} f(x) = c_{q,\nu} \int_{0}^{\infty} f(t) j_{\nu}(xt, q^2) t^{2\nu+1} dq t, \quad \forall x \in \mathbb{R}_q^+.\]
Let $f \in L_{q,1,\nu}$ then $F_{q,\nu}f \in C_{q,0}$ and we have
\[ \|F_{q,\nu}f\|_{q,\infty} \leq c_{q,\nu}\|f\|_{q,1,\nu}. \]

Let $f$ be a function belongs to $L_{q,p,\nu}$ where $p \geq 1$ then
\[ F_{q,\nu}^2f = f. \]  \hspace{1cm} (3)

If $f$ satisfies one of the following conditions :

i) $f \in L_{q,1,\nu}$ and $F_{q,\nu}f \in L_{q,1,\nu}$.

ii) $f \in L_{q,1,\nu} \cap L_{q,2,\nu}$ where $p > 2$.

iii) $f \in L_{q,2,\nu}$.

Then we have $\|F_{q,\nu}f\|_{q,2,\nu} = \|f\|_{q,2,\nu}$. Note that if we denote by
\[ \langle f, g \rangle_{q,\nu} = \int_0^\infty f(x)g(x)x^{2\nu+1}d_qx \]
the inner product in the space $L_{q,2,\nu}$ then we have
\[ \langle F_{q,\nu}(f), F_{q,\nu}(g) \rangle_{q,\nu} = \langle f, g \rangle_{q,\nu}, \ \forall f, g \in L_{q,2,\nu}. \]

The $q$-translation operator is defined by
\[ T_{q,\nu}^xf(y) = c_{q,\nu} \int_0^\infty F_{q,\nu}f(t)j_{\nu}(yt, q^2)j_{\nu}(xt, q^2)t^{2\nu+1}d_qt. \]

Let us now introduce
\[ Q_\nu = \{ q \in ]0,1[, \ T_{q,\nu}^x \text{ is positive for all } x \in \mathbb{R}_q \}, \]
the set of the positivity of $T_{q,x}$. We recall that $T_{q,x}^\nu$ is called positive if $T_{q,x}^\nu f \geq 0$ for $f \geq 0$.

In [5] it was proved that if $-1 < \nu < \nu'$ then $Q_\nu \subset Q_{\nu'}$. As a consequence :

- If $0 \leq \nu$ then $Q_\nu = ]0,1[.$

- If $-\frac{1}{2} \leq \nu < 0$ then $]0,q_0[ \subset Q_{-\frac{1}{2}} \subset Q_\nu \subset ]0,1[,$ $q_0 \simeq 0.43.$

- If $-1 < \nu \leq -\frac{1}{2}$ then $Q_\nu \subset Q_{-\frac{1}{2}}$.

Let $f \in L_{q,p,\nu}$ then $T_{q,x}^\nu f$ exist and we have
\[ \int_0^\infty T_{q,x}^\nu f(y)y^{2\nu+1}d_qy = \int_0^\infty f(y)y^{2\nu+1}d_qy. \]

If we assume that $T_{q,\nu}^x$ is positif then
\[ \|T_{q,x}^\nu f\|_{q,p,\nu} \leq \|f\|_{q,p,\nu}. \]

The $q$-convolution product is given as follows [2]
\[ f \ast_q g = F_{q,\nu}\left[ F_{q,\nu}f \times F_{q,\nu}g \right]. \]

Let $1 \leq p \leq 2$ and $1 \leq r, s$ such that
\[ \frac{1}{p} + \frac{1}{r} - 1 = \frac{1}{s}. \]
If \( f \in \mathcal{L}_{q,p,\nu} \) and \( g \in \mathcal{L}_{r,r,\nu} \) then \( f \ast_q g \) exist and we have
\[
 f \ast_q g(x) = c_{q,\nu} \int_0^\infty T_{q,x}^\nu f(y)g(y)y^{2\nu+1}\,dy.
\]
In addition, if \( 1 \leq r \leq 2 \) then
\[
 \mathcal{F}_{q,\nu}(f \ast_q g) = \mathcal{F}_{q,\nu}(f) \times \mathcal{F}_{q,\nu}(g).
\]
If \( s \geq 2 \) then \( f \ast_q g \in \mathcal{L}_{q,s,\nu} \) and
\[
 \|f \ast_q g\|_{q,s,\nu} \leq c_{q,\nu}\|f\|_{q,p,\nu} \times \|g\|_{q,r,\nu}
\]
If we assume that \( T_{q,x}^\nu \) is positive then (4) hold true for all \( s \geq 1 \).

In the sequel, we will always assume \( q \in Q_\nu \).

3 Bilateral birth and death processes on \( \mathbb{R}_q \)

We consider a bilateral birth and death processes \( X_t \) with parameter set \( \mathcal{T} = [0, \infty) \) on the path set
\[
 \mathbb{R}_q = \{q^n, \quad n \in \mathbb{Z}\}
\]
with stationary transition probabilities
\[
 p_{i,j}(h) = \Pr [X_{t+h} = q^j | X_t = q^i]
\]
which is not depending on \( t \). In addition we assume that \( p_{i,j}(h) \) satisfy
\[
 \begin{align*}
 \bullet & \quad p_{i,i+1}(h) = q^{2\nu-2i}h + o(h) \text{ as } h \downarrow 0, \quad \forall i \in \mathbb{Z}. \\
 \bullet & \quad p_{i,i-1}(h) = q^{-2i}h + o(h) \text{ as } h \downarrow 0, \quad \forall i \in \mathbb{Z}. \\
 \bullet & \quad p_{i,i} = 1 - (q^{2\nu-2i} + q^{-2i})h + o(h) \text{ as } h \downarrow 0, \quad \forall i \in \mathbb{Z}. \\
 \bullet & \quad p_{i,j}(h) = 0 \text{ if } |i - j| > 1.
\end{align*}
\]

The state space of this process is interpreted as jump rate from the point \( q^i \) to \( q^{i+1} \) or \( q^{i-1} \). The process will be parameterized by a continuous time \( t \), but its trajectories will not be continuous.

Fix an arbitrary state \( q^r \) and let
\[
 p_{nr}(t) = \Pr [X_t = q^n | X_0 = q^r].
\]
We use the following notation if there is no confusion
\[
 p_n(t) = p_{nr}(t).
\]
Then we obtain
\[
 p_n(t+h) = \Pr [X_{t+h} = q^n | X_0 = q^r] \\
 = \Pr [X_t = q^{n-1} | X_0 = q^i]p_{n-1,i}(h) + \Pr [X_t = q^n | X_0 = q^r]p_{n,n}(h) + \\
 \Pr [X_t = q^{n+1} | X_0 = q^r]p_{n+1,i}(h) \\
 = q^{2\nu-2(n-1)}h p_{n-1}(t) + [1 - (q^{2\nu-2n} + q^{-2n})h] p_n(t) + q^{-2(n+1)}h p_{n+1}(t) + o(h).
\]
Taking the limit \( h \to 0 \) we must have the following differential equation
\[
 \frac{d}{dt}p_n(t) = q^{2\nu-2(n-1)}p_{n-1}(t) - (q^{2\nu-2n} + q^{-2n})p_n(t) + q^{-2(n+1)}p_{n+1}(t).
\]
Let \( x_n = q^n, x_r = q^r \) and
\[
P_{x_r}(x_n, t) = \frac{1}{(1-q)^2} q^{2(\nu+1)n} p_n(t) = \frac{1}{(1-q)^2} x_{n+2(\nu+1)} p_n(t),
\]

We obtain the following equation:
\[
\frac{d}{dt} P_{x_r}(x_n, t) = \frac{P_{x_r}(x_n, t) - (1 + q^{2\nu}) P_{x_r}(x_n, t) + q^{2\nu} P_{x_r}(x_{n+1}, t)}{x_n^2}
\]
\[
= \frac{P_{x_r}(q^{-1} x_n, t) - (1 + q^{2\nu}) P_{x_r}(x_n, t) + q^{2\nu} P_{x_r}(q x_n, t)}{x_n^2}.
\]

Replacing \( x_n \) by an arbitrary \( x \in \mathbb{R}_q \) we deduce the \( q \)-Fokker-Planck equation
\[
\frac{\partial}{\partial t} P_{x_r}(x, t) = \Delta_{q,\nu} P_{x_r}(x, t).
\]
The solution is explicitly written as follows
\[
P_{x_r}(x, t) = c_{q,\nu}^2 \int_0^\infty e^{-ty^2} j_\nu(xy, q^2) j_\nu(x r, q^2) y^{2\nu+1} q^y dy
\]
\[
= c_{q,\nu} \mathcal{F}_{q,\nu} \left[ z \to e^{-tz^2} j_\nu(x r, q^2) \right] (x)
\]
\[
= c_{q,\nu} T_{q,x_r}^\nu \mathcal{F}_{q,\nu} \left[ z \to e^{-tz^2} \right] (x).
\]

where
\[
\rho_t(x) = \mathcal{F}_{q,\nu} \left[ z \to e^{-tz^2} \right] (x).
\]

In fact
\[
\frac{\partial}{\partial t} P_{x_r}(x, t) = c_{q,\nu}^2 \int_0^\infty -y e^{-ty^2} j_\nu(xy, q^2) j_\nu(x r, q^2) y^{2\nu+1} q^y dy
\]
\[
= c_{q,\nu}^2 \mathcal{F}_{q,\nu} \left[ \Delta_{q,\nu} j_\nu(xy, q^2) \right] j_\nu(x r, q^2) y^{2\nu+1} q^y dy
\]
\[
= \Delta_{q,\nu} \left[ c_{q,\nu}^2 \int_0^\infty -y e^{-ty^2} j_\nu(xy, q^2) j_\nu(x r, q^2) y^{2\nu+1} q^y dy \right]
\]
\[
= \Delta_{q,\nu} P_{x_r}(x, t).
\]
The derivative under integral sign follows from (1). Hence
\[
p_{nr}(t) = (1 - q) q^{2(\nu+1)n} P_{x_r}(x_n, t)
\]

(5)

**Proposition 1** The stationary transition probabilities \( p_{nr}(t) \) satisfies:

a. \( p_{nr}(t) \geq 0 \).

b. \( \sum_{n \in \mathbb{Z}} p_{nr}(t) = 1 \).

c. \( p_{nr}(0) = \delta(n, r) \).

d. \( p_{nr}(t + s) = \sum_{k \in \mathbb{Z}} p_{nk}(t) p_{kr}(s) \).

**Proof.** In fact
\[
p_{nr}(t) = (1 - q) q^{2(\nu+1)n} P_{x_r}(x_n, t) = c_{q,\nu} \left[ (1 - q) q^{2(\nu+1)n} T_{q,x_0}^\nu \mathcal{F}_{q,\nu} \left[ z \to e^{-tz^2} \right] (x_n) \right].
\]

In the proof of [5 Theorem 1] we have proved that \( \mathcal{F}_{q,\nu} \left[ e^{-tz^2} \right] \) is a positive function. The Positivity of the generalized translation operator \( T_{q,x_0} \) implies that \( p_{nr}(t) \geq 0 \).
It also satisfies the normalization of the total probability:

\[
\sum_{n \in \mathbb{Z}} p_{n r}(t) = (1 - q) \sum_{n \in \mathbb{Z}} x_n^{2(n+1)} P_{x_r}(x_n, t)
\]

\[
= \int_0^\infty P_{x_r}(x, t) x^{2\nu+1} d_q x
\]

\[
= c_{q, \nu} \int_0^\infty \mathcal{F}_{q, \nu} \left( z \rightarrow e^{-t z^2} \right) (x) x^{2\nu+1} d_q x
\]

\[
= c_{q, \nu} \int_0^\infty \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-t z^2} \right] (x) x^{2\nu+1} d_q x
\]

\[
= \mathcal{F}_{q, \nu}^2 \left[ z \rightarrow e^{-t z^2} \right] (0) = e^{-t y^2} \bigg|_{y=0} = 1.
\]

To show that it satisfies the initial conditions we use formula (2):

\[
p_{n r}(0) = (1 - q) x_n^{2(n+1)} P_{x_r}(x_n, 0)
\]

\[
= (1 - q) x_n^{2(n+1)} c_{q, \nu}^2 \int_0^\infty j_\nu(x_n y, q^2) j_\nu(x_r, q^2) y^{2\nu+1} d_q y
\]

\[
= (1 - q) x_n^{2(n+1)} \delta_q(x_r, x_n) = \delta(n, r).
\]

The processes satisfy the markovian property:

\[
\sum_{k \in \mathbb{Z}} p_{n k}(t)p_{k r}(s)
\]

\[
= (1 - q) x_n^{2(n+1)} \sum_{k \in \mathbb{Z}} x_k^{2(n+1)} \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-t z^2} j_\nu(x_n z, q^2) \right] (x_k) \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-s z^2} j_\nu(x_r z, q^2) \right] (x_k)
\]

\[
= (1 - q) c_{q, \nu}^2 \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-t z^2} j_\nu(x_n z, q^2) \right] (x) \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-s z^2} j_\nu(x_r z, q^2) \right] (x) x^{2\nu+1} d_q x
\]

\[
= (1 - q) c_{q, \nu}^2 \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-t z^2} j_\nu(x_n z, q^2) \right] (x) \mathcal{F}_{q, \nu} \left[ z \rightarrow e^{-s z^2} j_\nu(x_r z, q^2) \right] (x) x^{2\nu+1} d_q x
\]

\[
= (1 - q) c_{q, \nu}^2 \int_0^\infty e^{-(t+s) z^2} j_\nu(x_n z, q^2) j_\nu(x_r z, q^2) z^{2\nu+1} d_q z
\]

\[
= p_{n r}(t + s).
\]

\[\square\]

**Proposition 2** The solution given by (5) is unique if and only if \( \nu \geq 0 \).

**Proof.** In fact in [11, Theorem 3.5] it was given a necessary and sufficient condition that there is one and only one solution of a stationary transition probabilities \( p_{n r}(t) \) satisfies the conditions of Proposition 1. In our situation the result is an immediate consequence of the uniqueness result given by [11, Theorem 2]. \( \square \)

### 4 The transition semigroup

To introduce the transition semigroup we follows that given in [10]. But in our case this is a bilateral birth and death processes on the path set \( \mathbb{R}_q \) which must replace \( \mathbb{N} \).

We present here some connexion between the notations used in [10] and the standard notations of \( q \)-Bessel Fourier analysis. Let

\[
\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \ldots \lambda_{n-1}}{\mu_1 \ldots \mu_n}, \quad \forall n \in \mathbb{Z}^*
\]
In our case
\[ \lambda_n = q^{2\nu-2n} \quad \text{and} \quad \mu_n = q^{-2n} \Rightarrow \pi_n = q^{2(\nu+1)n} \]
The \( L^2(\pi) \) norm was introduced in \cite{10} as follows
\[ \|f\|^2 = \sum_{n \in \mathbb{Z}} |f(q^n)|^2 \pi_n = \sum_{n \in \mathbb{Z}} |f(q^n)|^2 q^{2(\nu+1)n} = \|f\|_{q,2,\nu}^2 \Rightarrow L^2(\pi) = L_{q,2,\nu}. \]
Also for a given functions \( f \) and \( g \) in \( L^2(\pi) \) we have
\[ \langle f, g \rangle = \sum_{n \in \mathbb{Z}} f(q^n) \overline{g(q^n)} \pi_n = \int_0^\infty f(x) g(x) x^{2\nu+1} d_q x = \langle f, g \rangle_{q,\nu}. \]
Now looking at the operator \( A \) defined by
\[ Af(q^n) = \sum_{k \in \mathbb{Z}} a_{n,k} f(q^k) \]
where
\[ a_{n,k} = \begin{cases} \mu_n & \text{if } k = n - 1 \\ -(\lambda_n + \mu_n) & \text{if } k = n \\ \lambda_n & \text{if } k = n + 1 \\ 0 & \text{otherwise} \end{cases} \]
A simple calculation
\[
Af(q^n) = a_{n,n-1} f(q^{n-1}) + a_{n,n} f(q^n) + a_{n,n+1} f(q^{n+1}) \\
= \mu_n f(q^{n-1}) - (\lambda_n + \mu_n) f(q^n) + \lambda_n f(q^{n+1}) \\
= q^{-2n} f(q^{n-1}) - (q^{2\nu-2n} + q^{-2n}) f(q^n) + q^{2\nu-2n} f(q^{n+1}) \\
= \frac{f(q^{n-1}) - (1+q^{2\nu}) f(q^n) + q^{2\nu} f(q^{n+1})}{q^{2n}}.
\]
lead to the fact that when \( x = q^n \) we obtain
\[ Af(x) = \Delta_{q,\nu} f(x) \Rightarrow A = \Delta_{q,\nu}. \]
Then
\[ \langle Af, g \rangle = \langle f, Ag \rangle \Rightarrow \langle \Delta_{q,\nu} f, g \rangle_{q,\nu} = \langle f, \Delta_{q,\nu} g \rangle_{q,\nu}. \]
The operator \( T_t \) introduced in \cite{10} p. 518 seem to be the appropriate choice for the transition semigroup. In our study we use \( P_t \) instead of \( T_t \). Let \( P_t f(x) \) given for \( t \geq 0 \) and \( x = q^r \) by
\[
P_t f(x) = \mathbb{E} [f(X_t)] = \sum_{n \in \mathbb{Z}} \text{Pr} [X_t = q^n | X_0 = q^r] f(q^n) \\
= \sum_{n \in \mathbb{Z}} p_{nr}(t) f(q^n) \\
= (1-q) \sum_{n \in \mathbb{Z}} q^{2(\nu+1)n} P_x(q^n, t) f(q^n) \\
= (1-q) c_{q,\nu} \sum_{n \in \mathbb{Z}} q^{2(\nu+1)n} T_{q,x}^\nu \rho_t(q^n) f(q^n) \\
= c_{q,\nu} \int_0^\infty T_{q,x}^\nu \rho_t(y) f(y) y^{2\nu+1} d_q y \\
= \rho_t *_q f(x),
\]
with initial condition
\[ P_0 f(x) = \sum_{n \in \mathbb{Z}} p_{nr}(0) f(q^n) = \sum_{n \in \mathbb{Z}} \delta(r, n) f(q^n) = f(q^r) = f(x). \]
From Proposition 1 we have the normalisation of total probability

\[ P_t 1 = \sum_{n \in \mathbb{Z}} p_{nr}(t) = 1, \]

and the positivity

\[ f \geq 0 \Rightarrow P_t f \geq 0. \]

From [10] we see that \( P_t \) define a bounded linear self-adjoint operator of \( L^2(\pi) \) into itself. The mapping

\[ t \mapsto P_t f \]

is continuous on \( 0 \leq t < \infty \) relative to the strong operator topology. Also by the use of Proposition 1 we have

\[ P_t P_s f(x) = \sum_{n \in \mathbb{Z}} p_{nr}(t) P_s f(q^n) \]

\[ = \sum_{n \in \mathbb{Z}} p_{nr}(t) \left[ \sum_{k \in \mathbb{Z}} p_{nk}(s) f(q^k) \right] = \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} p_{nr}(t) p_{kn}(s) \right) f(q^n) \]

\[ = \sum_{k \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} p_{nk}(s) p_{nr}(t) \right) f(q^k) = \sum_{k \in \mathbb{Z}} p_{kr}(t + s) f(q^k) \]

\[ = P_{t+s} f(x), \]

whenever \( f \) and \( g \) are with compact support. But functions with compact support are dense in \( L^2(\pi) \) and hence the semi group property established in \( L^2(\pi) \)

\[ P_t P_s f = P_{t+s} f, \quad \forall f \in L^2(\pi). \]

A direct consequence is the fact that \( P_t \) is positive definite

\[ \langle P_t f, f \rangle_{q,\nu} \geq 0, \quad \forall f \in L^2(\pi). \]

**Remark 1** The operator \( P_t \) was introduce in [5] and many of it’s properties was established.

**Theorem 1** A solution to the following \( q \)-heat equation:

\[ \frac{\partial}{\partial t} u(t, x) = \Delta_{q,\nu} u(t, x) \]

with initial condition

\[ u(0, x) = f(x), \quad f \in \mathcal{L}_{q,p,\nu}, \]

is given by \( u(t, x) = P_t f(x), \quad \forall x \in \mathbb{R}_q \). If \( f \in \mathcal{L}_{q,2,\nu} \) then there exists a unique solution if and only if \( \nu \geq 0 \).

**Proof.** In fact let

\[ u(t, x) = P_t f(x) = \rho_t *_q f(x) \]

Using the properties of the \( q \)-convolution product we obtain

\[ u(t, x) = \mathcal{F}_{q,\nu} \left[ \mathcal{F}_{q,\nu} \rho_t \right] f(x) = \int_0^\infty \mathcal{F}_{q,\nu} \rho_t(y) \mathcal{F}_{q,\nu} f(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_q y \]

\[ = \int_0^\infty e^{-ty^2} \mathcal{F}_{q,\nu} f(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_q y. \]

The inversion formula (3) lead to the initial condition

\[ u(0, x) = \frac{\partial}{\partial t} u(t, x) = \Delta_{q,\nu} u(t, x) \]

\[ u(0, x) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_q y = \int_0^\infty \mathcal{F}_{q,\nu} f(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_q y = \int_0^\infty \mathcal{F}_{q,\nu} f(y) j_{\nu}(xy, q^2) y^{2\nu+1} d_q y = \mathcal{F}_{q,\nu} f(x) = f(x). \]

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On the other hand
\[
\frac{\partial}{\partial t} u(t, x) = c_{q, \nu} \int_0^\infty (-y^2) e^{-ty^2} F_{q, \nu} f(y) j_\nu(xy, q^2) y^{2\nu+1} dy.
\]

\[
= c_{q, \nu} \int_0^\infty e^{-ty^2} F_{q, \nu} f(y) \left[ \Delta_{q, \nu} j_\nu(xy, q^2) \right] y^{2\nu+1} dy
\]

\[
= \Delta_{q, \nu} u(t, x).
\]

The uniqueness is a consequence of Proposition 2.

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