Generalized quasi-statistical structures

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Abstract

Given a non-degenerate $(0,2)$-tensor field $h$ on a smooth manifold $M$, we consider a natural generalized complex and a generalized product structure on the generalized tangent bundle $TM \oplus T^*M$ of $M$ and we show that they are $\nabla$-integrable, for $\nabla$ an affine connection on $M$, if and only if $(M, h, \nabla)$ is a quasi-statistical manifold.

We introduce the notion of generalized quasi-statistical structure and we prove that any quasi-statistical structure on $M$ induces generalized quasi-statistical structures on $TM \oplus T^*M$. In this context, dual connections are considered and some of their properties are established. The results are described in terms of Patterson-Walker and Sasaki metrics on $T^*M$, horizontal lift and Sasaki metrics on $TM$ and, when the connection $\nabla$ is flat, we define prolongation of quasi-statistical structures on manifolds to their cotangent and tangent bundles via generalized geometry. Moreover, Norden and Para-Norden structures are defined on $T^*M$ and $TM$.

1 Introduction

Statistical manifolds were introduced in [1], [7]. They are manifolds of probability distributions, used in Information Geometry and related to Codazzi tensors and Affine Geometry. Let $h$ be a pseudo-Riemannian metric and let $\nabla$ be a torsion-free affine connection on a smooth manifold $M$. Then $(M, h, \nabla)$ is called a statistical manifold if

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(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), for all \(X, Y, Z \in C^\infty(TM)\). The definition can be extended to (0, 2)-tensor fields and affine connections \(\nabla\) with torsion, \(T^\nabla\). In this case, \((h, \nabla)\) is called a quasi-statistical structure on \(M\) if \(d^\nabla h = 0\), where \((d^\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h(T^\nabla(X, Y), Z)\), for all \(X, Y, Z \in C^\infty(TM)\), and the triple \((M, h, \nabla)\) is called a quasi-statistical manifold.

In this paper, given a non-degenerate (0, 2)-tensor field \(h\) and an affine connection \(\nabla\) on a smooth manifold \(M\), we consider a natural generalized complex and a generalized product structure on the generalized tangent bundle \(TM \oplus T^*M\) of \(M\) and we show that they are \(\nabla\)-integrable if and only if \((M, h, \nabla)\) is a quasi-statistical manifold. We introduce the notion of generalized quasi-statistical structure and we prove that any quasi-statistical structure on \(M\), defined by a symmetric or skew-symmetric tensor, induces two natural generalized quasi-statistical structures on \(TM \oplus T^*M\). We compute the dual connections and study some of their properties. The results are described in terms of Patterson-Walker and Sasaki metrics on \(T^*M\), horizontal lift and Sasaki metrics on \(TM\). In the case, the connection \(\nabla\) is flat we can define prolongation of quasi-statistical structures on manifolds to their cotangent and tangent bundles via generalized geometry. Moreover, in the last section, we construct Norden and Para-Norden structures on \(T^*M\) and \(TM\).

2 Quasi-statistical structures and generalized structures induced

Let \(M\) be a smooth manifold and \(h\) a non-degenerate (0, 2)-tensor field on \(M\). On the generalized tangent bundle \(TM \oplus T^*M\) of \(M\), we shall consider the generalized complex structure

\[(1) \quad J_c := \begin{pmatrix} 0 & -h^{-1} \\ h & 0 \end{pmatrix}\]

and the generalized product structure

\[(2) \quad J_p := \begin{pmatrix} 0 & h^{-1} \\ h & 0 \end{pmatrix},\]

where we denoted by \(h\) the musical isomorphism, \(\flat_h : TM \rightarrow T^*M\), \(\flat_h(X) := i_X h\), and by \(h^{-1}\) its inverse, \(\sharp_h : T^*M \rightarrow TM\).
Let
\[ < X + \alpha, Y + \beta > := -\frac{1}{2}(\alpha(Y) + \beta(X)) \]
be the natural indefinite metric on \( TM \oplus T^*M \) and
\[ (X + \alpha, Y + \beta) := -\frac{1}{2}(\alpha(Y) - \beta(X)) \]
be the natural symplectic structure on \( TM \oplus T^*M \).

**Remark 2.1.**
\begin{enumerate}
  \item If \( h \) is symmetric, then:
    \[ < \hat{J}_c \sigma, \hat{J}_c \tau > = -\sigma, \tau > \quad \text{and} \quad (\hat{J}_c \sigma, \hat{J}_c \tau) = (\sigma, \tau), \]
    or, equivalently:
    \[ < \hat{J}_c \sigma, \tau > = < \sigma, \hat{J}_c \tau > \quad \text{and} \quad (\hat{J}_c \sigma, \tau) = (\sigma, \hat{J}_c \tau), \]
    for any \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \).
  \item If \( h \) is skew-symmetric, then:
    \[ < \hat{J}_p \sigma, \hat{J}_p \tau > = -\sigma, \tau > \quad \text{and} \quad (\hat{J}_p \sigma, \hat{J}_p \tau) = (\sigma, \tau), \]
    or, equivalently:
    \[ < \hat{J}_p \sigma, \tau > = -\sigma, \hat{J}_p \tau > \quad \text{and} \quad (\hat{J}_p \sigma, \tau) = (\sigma, \hat{J}_p \tau), \]
    for any \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \).
\end{enumerate}

On \( TM \oplus T^*M \) we consider the bilinear form:
\[ \hat{h}(X + \alpha, Y + \beta) := h(X, Y) + h(h^{-1}(\alpha), h^{-1}(\beta)), \]
for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).

A direct computation gives the following:
Lemma 2.2. The structures $\hat{J}_c$ and $\hat{J}_p$ satisfy respectively:

$$\hat{h}(\hat{J}_c \sigma, \tau) = 2(\sigma, \tau),$$
$$\hat{h}(\sigma, \hat{J}_p \tau) = 2 < \sigma, \tau >,$$

for any $\sigma, \tau \in \mathcal{C}^\infty(TM \oplus T^*M)$.

For $\nabla$ an affine connection on $M$, we consider the bracket $[\cdot, \cdot]_\nabla$ on $\mathcal{C}^\infty(TM \oplus T^*M)$\,[6]:

$$[X + \alpha, Y + \beta]_\nabla := [X, Y] + \nabla_X \beta - \nabla_Y \alpha,$$

for all $X, Y \in \mathcal{C}^\infty(TM)$ and $\alpha, \beta \in \mathcal{C}^\infty(T^*M)$.

A generalized complex or product structure $\hat{J}$ is called $\nabla$-integrable if its Nijenhuis tensor field $N^\nabla_\hat{J}$ with respect to $\nabla$:

$$N^\nabla_\hat{J}(\sigma, \tau) := \hat{J}(\hat{J}(\sigma, \tau))_{\nabla} - \hat{J}(\hat{J}(\sigma, \tau))_{\nabla} - \hat{J}(\hat{J}(\sigma, \tau))_{\nabla} + \hat{J}^2(\sigma, \tau)_{\nabla}$$

vanishes for all $\sigma, \tau \in \mathcal{C}^\infty(TM \oplus T^*M)$.

Let $M$ be a smooth manifold with a non-degenerate $(0,2)$-tensor field $h$ and an affine connection $\nabla$.

Definition 2.3. [4] We call $(h, \nabla)$ a quasi-statistical structure (respectively, $(M, h, \nabla)$ a quasi-statistical manifold) if $d^\nabla h = 0$, where

$$(d^\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h(T^\nabla(X, Y), Z),$$

for any $X, Y, Z \in \mathcal{C}^\infty(TM)$ and $T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$.

We can state:

Proposition 2.4. The structures $\hat{J}_c$ and $\hat{J}_p$ are integrable if and only if $(M, h, \nabla)$ is a quasi-statistical manifold.

Proof. In this proof we will shortly denote $\hat{J}_\pm$ for $\hat{J}_= := \hat{J}_-$ and $\hat{J}_p := \hat{J}_+$.

Let us compute:

$$N^\nabla_{\hat{J}_\pm}(X, Y) = [\hat{J}_\pm X, \hat{J}_\pm Y]_{\nabla} - \hat{J}_\pm[\hat{J}_\pm X, Y]_{\nabla} - \hat{J}_\pm[X, \hat{J}_\pm Y]_{\nabla} + \hat{J}_\pm^2[X, Y]_{\nabla} =$$

$$= \pm h^{-1}((\nabla_X h)Y - (\nabla_Y h)X + h(\nabla_X Y - \nabla_Y X - [X, Y])) =$$
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\[ N_{j^{\pm}}^\nabla (X, h(Y)) = [\hat{J}_\pm X, \hat{J}_\pm h(Y)]_{\nabla} - \hat{J}_\pm [\hat{J}_\pm X, h(Y)]_{\nabla} - \hat{J}_\pm [X, \hat{J}_\pm h(Y)]_{\nabla} + \hat{J}_\pm^2 [X, h(Y)]_{\nabla} = \]
\[ = \mp((\nabla_X h)Y - (\nabla_Y h)X + h(\nabla_X Y - \nabla_Y X - [X, Y])) = \]
\[ = \mp(\nabla(Y h)(X, Y) = \mp(\nabla h)(X, Y)) = \]
\[ N_{j^{\pm}}^\nabla (h(X), h(Y)) = [\hat{J}_\pm h(X), \hat{J}_\pm h(Y)]_{\nabla} - \hat{J}_\pm [\hat{J}_\pm h(X), h(Y)]_{\nabla} - \hat{J}_\pm [h(X), \hat{J}_\pm h(Y)]_{\nabla} + \hat{J}_\pm^2 [h(X), h(Y)]_{\nabla} = \]
\[ = -h^{-1}((\nabla_X h)Y - (\nabla_Y h)X + h(\nabla_X Y - \nabla_Y X - [X, Y])) = \]
\[ = -h^{-1}(\nabla h)(X, Y)) , \]
for any \( X, Y \in C^\infty(TM) \). Therefore the proof is complete.

\[ \square \]

3 Generalized quasi-statistical structures

**Definition 3.1.** We call \( D : C^\infty(TM \oplus T^*M) \times C^\infty(TM \oplus T^*M) \to C^\infty(TM \oplus T^*M) \) an affine connection on \( TM \oplus T^*M \) if it is \( \mathbb{R} \)-bilinear and for any \( f \in C^\infty(M) \) and \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \), we have:

1. \( D_{f \sigma \tau} = f D_{\sigma \tau} \),
2. \( D_{\sigma}(f \tau) = \sigma(f) \tau + f D_{\sigma \tau} \),

where \((X + \alpha)(f) := X(f)\), for \( X + \alpha \in C^\infty(TM \oplus T^*M)\).

Let \( \hat{h} \) be a non-degenerate \((0, 2)\)-tensor field and \( D \) an affine connection on the generalized tangent bundle \( TM \oplus T^*M \) of the smooth manifold \( M \).

**Definition 3.2.** We call \((\hat{h}, D)\) a generalized quasi-statistical structure if \( d^D\hat{h} = 0 \), where

\[(d^D\hat{h})(\sigma, \tau, \nu) := (D_{\sigma \tau})(\nu) - (D_{\sigma \nu})(\tau) + \hat{h}(T^D(\sigma, \tau), \nu) , \]
for any \( \sigma, \tau, \nu \in C^\infty(TM \oplus T^*M) \) and \( T^D(\sigma, \tau) := D_{\sigma \tau} - D_{\tau \sigma} - [\sigma, \tau]_{\nabla} \), with \( \nabla \) a given connection on \( M \).
3.1 Generalized quasi-statistical structures induced by quasi-statistical structures

Let $h$ be a non-degenerate $(0, 2)$-tensor field and let $\nabla$ be an affine connection on $M$. We define the affine connection $\hat{\nabla}$ on $TM \oplus T^*M$ by:

$$\hat{\nabla}_{X+\alpha}Y + \beta := \nabla_XY + h(\nabla_Xh^{-1}(\beta)),$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$.

**Theorem 3.3.** $(TM \oplus T^*M, \hat{h}, \hat{\nabla})$ is a generalized quasi-statistical manifold if and only if $(M, h, \nabla)$ is a quasi-statistical manifold, where $\hat{h}$ is precisely $\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle$ given by (3) and (4) respectively, according as $h$ is symmetric or skew-symmetric, and $\hat{\nabla}$ is given by (5).

**Proof.** First notice that the torsion of $\hat{\nabla}$ equals to

$$T^{\hat{\nabla}}(X + \alpha, Y + \beta) := \hat{\nabla}_{X+\alpha}Y + \beta - \hat{\nabla}_{Y+\beta}X + \alpha - [X + \alpha, Y + \beta]_\nabla =$$

$$= T^\nabla(X, Y) + h(\nabla_Xh^{-1}(\beta) - \nabla_Yh^{-1}(\alpha)) - \nabla_X\beta + \nabla_Y\alpha.$$

We have:

$$(d^\nabla \hat{h})(X + \alpha, Y + \beta, Z + \gamma) := (\hat{\nabla}_{X+\alpha}\hat{h})(Y + \beta, Z + \gamma) - (\hat{\nabla}_{Y+\beta}\hat{h})(X + \alpha, Z + \gamma) +$$

$$+ \hat{h}(T^{\hat{\nabla}}(X + \alpha, Y + \beta), Z + \gamma) :=$$

$$:= X(h(Y + \beta, Z + \gamma)) - \hat{h}(\nabla_{X+\alpha}Y + \beta, Z + \gamma) - \hat{h}(Y + \beta, \nabla_{X+\alpha}Z + \gamma) -$$

$$- Y(h(X + \alpha, Z + \gamma)) + \hat{h}(\nabla_{Y+\beta}X + \alpha, Z + \gamma) + \hat{h}(X + \alpha, \nabla_{Y+\beta}Z + \gamma) +$$

$$+ \hat{h}(T^\nabla(X + \alpha, Y + \beta), Z + \gamma) :=$$

$$:= -\frac{1}{2}[X(\beta(Z) \pm \gamma(Y)) - h(\nabla_Xh^{-1}(\beta), Z) \mp \gamma(\nabla_XY) - \beta(\nabla_XZ) \mp h(\nabla_Xh^{-1}(\gamma), Y)] +$$

$$+ \frac{1}{2}[Y(\alpha(Z) \pm \gamma(X)) - h(\nabla_Yh^{-1}(\alpha), Z) \mp \gamma(\nabla_YX) - \alpha(\nabla_YZ) \mp h(\nabla_Yh^{-1}(\gamma), X)] -$$

$$- \frac{1}{2}[h(\nabla_Xh^{-1}(\beta), Z) - h(\nabla_Yh^{-1}(\alpha), Z) - (\nabla_X\beta)Z + (\nabla_Y\alpha)Z] \mp \frac{1}{2}\gamma(T^\nabla(X, Y)) :=$$

$$:= -\frac{1}{2}[\pm X(\gamma(Y)) \mp \gamma(\nabla_XY) \mp h(\nabla_Xh^{-1}(\gamma), Y) \mp$$

$$\mp Y(\gamma(X)) \pm \gamma(\nabla_YX) \pm h(\nabla_Yh^{-1}(\gamma), X) \pm \gamma(T^\nabla(X, Y)) :=$$

$$:= -\frac{1}{2}(d^\nabla \hat{h})(X, Y, h^{-1}(\gamma)).$$

Therefore the proof is complete. ∎
The couple \((\hat{h}, \hat{\nabla})\) with \(\hat{h}\) given by (3) or (4) respectively (according as \(h\) is symmetric or skew-symmetric) and \(\hat{\nabla}\) given by (6) will be called the \textit{generalized quasi-statistical structure induced} by \((h, \nabla)\).

A direct computation gives the expression of the \textit{generalized dual quasi-statistical connection} of \(\hat{\nabla}\), precisely:

**Proposition 3.4.** Let \((M, h, \nabla)\) be a quasi-statistical manifold and let \((\hat{h}, \hat{\nabla})\) be the generalized quasi-statistical structure induced on \(TM \oplus T^*M\). Then the generalized dual quasi-statistical connection, \(\hat{\nabla}^*\), defined by:

\[
\hat{h}(Y + \beta, \hat{\nabla}_{X+\alpha}^*Z + \gamma) = X(\hat{h}(Y + \beta, Z + \gamma)) - \hat{h}(\hat{\nabla}_{X+\alpha}Y + \beta, Z + \gamma),
\]

for all \(X, Y, Z \in C^\infty(TM)\) and \(\alpha, \beta, \gamma \in C^\infty(T^*M)\), is given by:

\[
\hat{\nabla}^*_{X+\alpha}Z + \gamma = h^{-1}(\nabla_X h(Z)) + \nabla_X \gamma.
\]

**Proof.** From the definition of the generalized dual quasi-statistical connection and using the definition of \(\hat{\nabla}\), we get:

\[
\hat{h}(Y + \beta, \hat{\nabla}_{X+\alpha}^*Z + \gamma) = X(\hat{h}(Y + \beta, Z + \gamma)) - h(\nabla_X h^{-1}(\beta), Z) \pm \gamma(\nabla_X Y),
\]

for any \(X, Y, Z \in C^\infty(TM)\) and \(\alpha, \beta, \gamma \in C^\infty(T^*M)\).

Let us denote \(\hat{\nabla}_{X+\alpha}^*Z + \gamma =: V + \eta\). Then we have:

\[
\beta(V) \pm \eta(Y) = X(\hat{h}(Z)) \pm X(\gamma(Y)) - h(\nabla_X h^{-1}(\beta), Z) \pm \gamma(\nabla_X Y),
\]

for any \(X, Y, Z \in C^\infty(TM)\) and \(\beta, \gamma \in C^\infty(T^*M)\).

Taking \(\beta := 0\), we obtain:

\[
\eta(Y) = X(\gamma(Y)) - \gamma(\nabla_X Y) := (\nabla_X \gamma)Y
\]

and taking \(Y := 0\), we obtain:

\[
\beta(V) = X(\hat{h}(Z)) - h(\nabla_X h^{-1}(\beta), Z)
\]

which is equivalent to:

\[
h(V, h^{-1}(\beta)) = X(h(Z, h^{-1}(\beta))) - h(Z, \nabla_X h^{-1}(\beta)) := (\nabla_X h)(Z, h^{-1}(\beta)) + h(\nabla_X Z, h^{-1}(\beta))
\]

and to:

\[
h(V) = (\nabla_X h)(Z, \cdot) + h(\nabla_X Z) = \nabla_X h(Z)
\]
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and to:

\[ V = h^{-1}(\nabla_X h(Z)). \]

Therefore the proof is complete. \(\square\)

**Proposition 3.5.** Let \((M, h, \nabla)\) be a quasi-statistical manifold. Then \(\hat{\nabla}^*\) is torsion-free.

**Proof.** For all \(X + \alpha, Y + \beta \in C^\infty(TM \oplus T^*M)\), we have:

\[
\begin{align*}
T^{\hat{\nabla}^*}(X + \alpha, Y + \beta) &= \hat{\nabla}^*_{X+\alpha}Y + \beta - \hat{\nabla}^*_{Y+\beta}X + \alpha - [X + \alpha, Y + \beta]_{\nabla} = \\
&= h^{-1}(\nabla_X h(Y)) - h^{-1}(\nabla_Y h(X)) - [X, Y] = \\
&= h^{-1}((\nabla_X h - (\nabla_Y h)X + h(T^\nabla(X,Y))) = \\
&= h^{-1}(d^\nabla(X,Y)) = 0.
\end{align*}
\]

\(\square\)

Let \(h\) be a non-degenerate, symmetric or skew-symmetric \((0, 2)\)-tensor field on \(M\) and let \(\nabla\) be an affine connection on \(M\). We have the following:

**Theorem 3.6.** \((TM \oplus T^*M, \hat{h}, \hat{\nabla})\) is a generalized quasi-statistical manifold if and only if \((M, h, \nabla)\) is a quasi-statistical manifold, where \(\hat{h}\) is given by (5) and \(\hat{\nabla}\) is given by (6).

**Proof.** We have:

\[
\begin{align*}
(d^{\hat{\nabla}}\hat{h})(X + \alpha, Y + \beta, Z + \gamma) := (\hat{\nabla}^*_{X+\alpha}\hat{h})(Y + \beta, Z + \gamma) - (\hat{\nabla}^*_{Y+\beta}\hat{h})(X + \alpha, Z + \gamma) + \\
&\quad + \hat{h}(T^{\hat{\nabla}}(X + \alpha, Y + \beta), Z + \gamma) := \\
&\quad := X(\hat{h}(Y + \beta, Z + \gamma)) - \hat{h}(\hat{\nabla}^*_{X+\alpha}Y + \beta, Z + \gamma) - \hat{h}(Y + \beta, \hat{\nabla}^*_{X+\alpha}, Z + \gamma) - \\
&\quad - Y(\hat{h}(X + \alpha, Z + \gamma)) + \hat{h}(\hat{\nabla}^*_{Y+\beta}X + \alpha, Z + \gamma) + \hat{h}(X + \alpha, \hat{\nabla}^*_{Y+\beta}, Z + \gamma) + \\
&\quad \quad + \hat{h}(T^{\hat{\nabla}}(X + \alpha, Y + \beta), Z + \gamma) := \\
&\quad := X(h(Y, Z)) + X(\beta(h^{-1}(\gamma))) - \hat{h}(\nabla_X Y + h(\nabla_X h^{-1}(\beta)), Z + \gamma) - \\
&\quad \quad - \hat{h}(Y + \beta, \nabla_X Z + h(\nabla_X h^{-1}(\gamma))) - \\
&\quad \quad - Y(h(X, Z)) - Y(\alpha(h^{-1}(\gamma))) + \hat{h}(\nabla_Y X + h(\nabla_Y h^{-1}(\alpha)), Z + \gamma) +
\end{align*}
\]
where the sign + is for $h$ symmetric, $-$ is for $h$ skew-symmetric. Therefore the proof is complete.

**Proposition 3.7.** Let $(M, h, \nabla)$ be a quasi-statistical manifold and let $(\tilde{h}, \hat{\nabla})$ be the generalized quasi-statistical structure induced on $TM \oplus T^*M$. Then the generalized dual quasi-statistical connection, $\hat{\nabla}^*_h$, defined by:

$$\tilde{h}(Y + \beta, (\hat{\nabla}^*_h)_{X + \alpha}Z + \gamma) = X(\tilde{h}(Y + \beta, Z + \gamma)) - \tilde{h}(\hat{\nabla}_{X + \alpha}Y + \beta, Z + \gamma),$$

for all $X, Y, Z \in \mathcal{C}^\infty(TM)$ and $\alpha, \beta, \gamma \in \mathcal{C}^\infty(T^*M)$, is given by:

$$(\hat{\nabla}^*_h)_{X + \alpha}Z + \gamma = h^{-1}(\nabla_X(h(Z))) + \nabla_X\gamma.$$

Therefore:

$$\hat{\nabla}^*_h = \hat{\nabla}^*.$$

**Proof.** We get:

$$\tilde{h}(Y + \beta, (\hat{\nabla}^*_h)_{X + \alpha}Z + \gamma) = X(\tilde{h}(Y, Z)) + X(\beta(h^{-1}(\gamma))) - h(\nabla_XY, Z) \mp \gamma(\nabla_Xh^{-1}(\beta)) =$$

$$= X(h(Y, Z)) - h(\nabla_XY, Z) \mp \gamma(\nabla_Xh^{-1}(\beta)) \pm X(\gamma(h^{-1}(\beta))),$$

for any $X, Y, Z \in \mathcal{C}^\infty(TM)$ and $\alpha, \beta, \gamma \in \mathcal{C}^\infty(T^*M)$.

Let us denote $(\hat{\nabla}^*_h)_{X + \alpha}Z + \gamma =: V + \eta$. Then we have:

$$h(Y, V) \pm \eta(h^{-1}(\beta)) = X(h(Y, Z)) - h(\nabla_XY, Z) \pm (\nabla_X\gamma)h^{-1}(\beta),$$

for any $X, Y, Z \in \mathcal{C}^\infty(TM)$ and $\beta \in \mathcal{C}^\infty(T^*M)$.

Taking $Y := 0$, we obtain:

$$\eta(h^{-1}(\beta)) = \nabla_Xh^{-1}(\beta)$$

and taking $\beta := 0$, we obtain:

$$h(Y, V) = (\nabla_Xh)(Y, Z) + h(Y, \nabla_XZ)$$
which is equivalent to:

\[ h(V) = (\nabla_X h)(Z, \cdot) + h(\nabla_X Z) = \nabla_X h(Z) \]

and to:

\[ V = h^{-1}(\nabla_X h(Z)). \]

Therefore the proof is complete. \( \square \)

Given an affine connection \( D \) on \( TM \oplus T^*M \), we define the curvature operator of \( D \), \( R^D : C^\infty(TM \oplus T^*M) \times C^\infty(TM \oplus T^*M) \times C^\infty(TM \oplus T^*M) \to C^\infty(TM \oplus T^*M) \), on \( \sigma, \tau, \nu \in C^\infty(TM \oplus T^*M) \), as in the following:

\[ R^D(\sigma, \tau)\nu = (D_\sigma D_\tau - D_\tau D_\sigma - D_{[\sigma,\tau]}\nu)\nu, \]

where \( \nabla \) is a given connection on \( M \).

**Proposition 3.8.** Let \((M, h, \nabla)\) be a quasi-statistical manifold and let \((\hat{h}, \hat{\nabla})\) be the generalized quasi-statistical structure induced on \( TM \oplus T^*M \). Then the curvature operators of \( \hat{\nabla} \) and \( \hat{\nabla}^* \) are given respectively by:

\[ \hat{R}^\nabla(X + \alpha, Y + \beta)Z + \gamma = \hat{R}^\nabla(X, Y)Z + h(\hat{R}^\nabla(X, Y)h^{-1}(\gamma)) \]

\[ \hat{R}^{\nabla^*}(X + \alpha, Y + \beta)Z + \gamma = h^{-1}(\hat{R}^\nabla(X, Y)h(Z)) + \hat{R}^\nabla(X, Y)\gamma, \]

where \( X, Y, Z \in C^\infty(TM) \), \( \alpha, \beta, \gamma \in C^\infty(T^*M) \) and \( \hat{R}^\nabla \) is the curvature operator of \( \nabla \). In particular, \( \hat{\nabla} \) and its dual \( \hat{\nabla}^* \) are flat if and only if \( \nabla \) is flat.

**Proof.** Let us compute:

\[ \hat{\nabla}_{X+\alpha}\hat{\nabla}_{Y+\beta}Z + \gamma - \hat{\nabla}_{Y+\beta}\hat{\nabla}_{X+\alpha}Z + \gamma - \hat{\nabla}_{[X+\alpha,Y+\beta]}\nu Z + \gamma := \]

\[ := \hat{\nabla}_{X+\alpha}(\nabla_Y Z + h(\nabla_Y h^{-1}(\gamma))) - \hat{\nabla}_{Y+\beta}(\nabla_X Z + h(\nabla_X h^{-1}(\gamma))) -
\]

\[ - \nabla_{[X,Y]}Z - h(\nabla_{[X,Y]}h^{-1}(\gamma)) := \]

\[ := \nabla_X \nabla_Y Z + h(\nabla_X \nabla_Y h^{-1}(\gamma)) - \nabla_Y \nabla_X Z - h(\nabla_Y h^{-1}(\gamma)) -
\]

\[ - \nabla_{[X,Y]}Z - h(\nabla_{[X,Y]}h^{-1}(\gamma)) := \]

\[ := R^\nabla(X, Y)Z + h(R^\nabla(X, Y)h^{-1}(\gamma)) \]

and:

\[ \hat{\nabla}^*_{X+\alpha}\hat{\nabla}^*_{Y+\beta}Z + \gamma - \hat{\nabla}^*_{Y+\beta}\hat{\nabla}^*_{X+\alpha}Z + \gamma - \hat{\nabla}^*_{[X+\alpha,Y+\beta]}\nu Z + \gamma := \]
\[ : \hat{\nabla}_{X+\alpha}^* \left( h^{-1}(\nabla_Y h(Z)) + \nabla_Y \gamma \right) - \hat{\nabla}_{Y+\beta}^* \left( h^{-1}(\nabla_X h(Z)) + \nabla_X \gamma \right) - \]
\[ - h^{-1}(\nabla_{[X,Y]} h(Z)) - \nabla_{[X,Y]} \gamma : = \]
\[ : = h^{-1}(\nabla_X \nabla_Y h(Z)) + \nabla_X \nabla_Y \gamma - h^{-1}(\nabla_Y \nabla_X h(Z)) - \nabla_Y \nabla_X \gamma - \]
\[ - h^{-1}(\nabla_{[X,Y]} h(Z)) - \nabla_{[X,Y]} \gamma : = \]
\[ : = h^{-1}(R^\nabla(X,Y,h(Z)) + R^\nabla(X,Y)) \gamma. \]

Therefore the proof is complete. \( \square \)

**Proposition 3.9.** The structures \( \hat{J}_c \) and \( \hat{J}_p \) are \( \hat{\nabla} \)-parallel and \( \hat{\nabla}^* \)-parallel.

**Proof.** In this proof we will shortly denote \( \hat{J}_\pm \) for \( \hat{J}_c =: \hat{J}_- \) and \( \hat{J}_p =: \hat{J}_+ \). Let us compute:

\[ (\hat{\nabla}_{X+\alpha} \hat{J}_\pm) Y + \beta : = \hat{\nabla}_{X+\alpha} (\mp h^{-1}(\beta) + h(Y)) - \hat{J}_\pm (\hat{\nabla}_{X+\alpha} Y + \beta) := \]
\[ := \mp \nabla_X h^{-1}(\beta) + h(\nabla_X h^{-1}(h(Y))) - \hat{J}_\pm (\nabla_X Y + h(\nabla_X h^{-1}(\beta))) := \]
\[ := \mp \nabla_X h^{-1}(\beta) + h(\nabla_X Y) \pm h^{-1}(h(\nabla_X h^{-1}(\beta))) - h(\nabla_X Y) = 0; \]

moreover:

\[ (\hat{\nabla}_{X+\alpha}^* \hat{J}_\pm) Y + \beta : = \hat{\nabla}_{X+\alpha}^* (\mp h^{-1}(\beta) + h(Y)) - \hat{J}_\pm (\hat{\nabla}_{X+\alpha}^* Y + \beta) := \]
\[ := \mp \nabla_X h^{-1}(\beta) + \nabla_X h(Y) - \hat{J}_\pm (h^{-1}(\nabla_X h(Y)) + \nabla_X \beta) := \]
\[ := \mp \nabla_X h^{-1}(\beta) + \nabla_X h(Y) - \nabla_X h(Y) \pm h^{-1}(\nabla_X \beta) = 0, \]

for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \). Therefore the proof is complete. \( \square \)

### 3.2 Generalized quasi-statistical structures induced by torsion-free connections

Another affine connection on the generalized tangent bundle \( TM \oplus T^*M \) is naturally defined by an affine connection \( \nabla \) on \( M \) by:

\[ (7) \quad \hat{\nabla}_{X+\alpha} Y + \beta : = \nabla_X Y + \nabla_X \beta, \]

for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \).
Remark 3.10. One can check that if \( h \) is a non-degenerate \((0,2)\)-tensor field on \( M \) which is \( \nabla \)-parallel, then the connections \( \hat{\nabla} \) and \( \tilde{\nabla} \) coincide (since we have \( \hat{\nabla}_{X+\alpha}Y + \beta - \tilde{\nabla}_{X+\alpha}Y + \beta = - (\nabla_X h)(h^{-1}(\beta), \cdot) \), for any \( X, Y \in C^\infty(TM) \) and \( \alpha, \beta \in C^\infty(T^*M) \)). In particular, \( \hat{\nabla}^* = \tilde{\nabla} = \hat{\nabla} \).

We have the following:

**Proposition 3.11.** \((TM \oplus T^*M, \hat{h}, \hat{\nabla})\) is a generalized quasi-statistical manifold if and only if \( \nabla \) is torsion-free, where \( \hat{h} \) is precisely \( < \cdot, \cdot > \) or \((\cdot, \cdot)\) given by (3) and (4) respectively and \( \tilde{\nabla} \) is given by (7).

**Proof.** First notice that the torsion of \( \tilde{\nabla} \) equals to

\[
T_{\tilde{\nabla}}(X + \alpha, Y + \beta) := \tilde{\nabla}_{X+\alpha}Y + \beta - \tilde{\nabla}_{Y+\beta}X + \alpha - [X + \alpha, Y + \beta] = T_{\nabla}(X, Y).
\]

We have:

\[
(d^* \hat{h})(X + \alpha, Y + \beta, Z + \gamma) := (\tilde{\nabla}_{X+\alpha}h)(Y + \beta, Z + \gamma) - (\tilde{\nabla}_{Y+\beta}h)(X + \alpha, Z + \gamma) + \hat{h}(T_{\tilde{\nabla}}(X + \alpha, Y + \beta), Z + \gamma) :=
\]

\[
:= X(h(Y + \beta, Z + \gamma)) - \hat{h}(\tilde{\nabla}_{X+\alpha}Y + \beta, Z + \gamma) - \hat{h}(Y + \beta, \tilde{\nabla}_{X+\alpha}, Z + \gamma) - \]

\[
-Y(\hat{h}(X + \alpha, Z + \gamma)) + \hat{h}(\tilde{\nabla}_{Y+\beta}X + \alpha, Z + \gamma) + \hat{h}(X + \alpha, \tilde{\nabla}_{Y+\beta}, Z + \gamma) +
\]

\[
+ \hat{h}(T_{\tilde{\nabla}}(X + \alpha, Y + \beta), Z + \gamma) :=
\]

\[
:= -\frac{1}{2}[X(\beta(Z) + \gamma(Y)) - (\nabla_X\beta)Z + \gamma(\nabla_XY) - \beta(\nabla_XZ) + (\nabla_X\gamma)Y] +
\]

\[
+ \frac{1}{2}[Y(\alpha(Z) + \gamma(X)) - (\nabla_Y\alpha)Z + \gamma(\nabla_YX) - \alpha(\nabla_YZ) + (\nabla_Y\gamma)X] + \frac{1}{2} \gamma(T_{\nabla}(X, Y)) :=
\]

\[
:= -\frac{1}{2}[X(\beta(Z))\pm X(\gamma(Y)) - X(\beta(Z))\pm \beta(\nabla_XZ)\pm \gamma(\nabla_XY) - \beta(\nabla_XZ)\pm X(\gamma(Y))] +
\]

\[
+ \frac{1}{2}[Y(\alpha(Z))\pm Y(\gamma(X)) - Y(\alpha(Z))\pm \alpha(\nabla_YZ)\pm \gamma(\nabla_YX) - \alpha(\nabla_YZ)\pm Y(\gamma(X))]\pm \gamma(\nabla_YX)] +
\]

\[
\mp \frac{1}{2} \gamma(T_{\nabla}(X, Y)) = \mp \frac{1}{2} \gamma(T_{\nabla}(X, Y)).
\]

Therefore the proof is complete.

**Proposition 3.12.** Let \( \nabla \) be a torsion-free affine connection on \( M \) and let \((< \cdot, \cdot >, \tilde{\nabla})\) and \((\cdot, \cdot), \hat{\nabla}\) be the canonical generalized quasi-statistical structures defined in Proposition 3.11. Then \( \nabla \) and its generalized dual quasi-statistical connection, \( \hat{\nabla}^* \), coincide.
Let us denote \( \hat{\nabla}_{X+\alpha}^* Z + \gamma =: V + \eta \). From the definition of the generalized dual quasi-statistical connection and using the definition of \( \hat{\nabla} \), we get:

\[
\beta(V) \pm \eta(Y) = X(\beta(Z)) \pm X(\gamma(Y)) - (\nabla_X \beta)Z \mp \gamma(\nabla_X Y),
\]

for any \( X, Y, Z \in C^\infty(TM) \) and \( \beta, \gamma \in C^\infty(T^*M) \).

Taking \( \beta := 0 \), we obtain:

\[
\pm \eta(Y) = \pm X(\gamma(Y)) \mp \gamma(\nabla_X Y) := \pm (\nabla_X \gamma)Y
\]

and taking \( Y := 0 \), we obtain:

\[
\beta(V) = X(\beta(Z)) - (\nabla_X \beta)Z := \beta(\nabla_X Z).
\]

Therefore the proof is complete. \( \square \)

**Proposition 3.13.** If \( \nabla \) is a torsion-free affine connection and \( h \) is a \( \nabla \)-parallel \((0, 2)\)-tensor field on \( M \), then \((\hat{h}, \hat{\nabla})\) is a generalized quasi-statistical structure, where \( \hat{h} \) is given by [3] and \( \hat{\nabla} \) is given by [7].

**Proof.** We have:

\[
(d\hat{h})(X + \alpha, Y + \beta, Z + \gamma) := (\hat{\nabla}_{X+\alpha} \hat{h})(Y + \beta, Z + \gamma) - (\hat{\nabla}_{Y+\beta} \hat{h})(X + \alpha, Z + \gamma) +
\]

\[
+ \hat{h}(T\hat{\nabla}(X + \alpha, Y + \beta), Z + \gamma) :=
\]

\[
:= X(\hat{h}(Y + \beta, Z + \gamma)) - \hat{h}(\hat{\nabla}_{X+\alpha} Y + \beta, Z + \gamma) - \hat{h}(Y + \beta, \hat{\nabla}_{X+\alpha} Z + \gamma) -
\]

\[
- Y(\hat{h}(X + \alpha, Z + \gamma)) + \hat{h}(\hat{\nabla}_{Y+\beta} X + \alpha, Z + \gamma) + \hat{h}(X + \alpha, \hat{\nabla}_{Y+\beta} Z + \gamma) +
\]

\[
+ \hat{h}(T\hat{\nabla}(X + \alpha, Y + \beta), Z + \gamma) :=
\]

\[
:= X(h(Y, Z) + h(h^{-1}(\beta), h^{-1}(\gamma)))-
\]

\[
- h(\nabla_X Y, Z) - h(h^{-1}(\nabla_X \alpha), h^{-1}(\gamma)) - h(Y, \nabla_X Z) - h(h^{-1}(\beta), h^{-1}(\nabla_X \gamma)) -
\]

\[
- Y(h(X, Z) + h(h^{-1}(\alpha), h^{-1}(\gamma)))+
\]

\[
+ h(\nabla_Y X, Z) + h(h^{-1}(\nabla_Y \alpha), h^{-1}(\gamma)) + h(X, \nabla_Y Z) + h(h^{-1}(\alpha), h^{-1}(\nabla_Y \gamma)) + h(T\hat{\nabla}(X, Y), Z) :=
\]

\[
:= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h(T\nabla(X, Y), Z) +
\]

\[
+ X(h^{-1}(\gamma)) - (\nabla_X \beta)h^{-1}(\gamma) - \beta(h^{-1}(\nabla_X \gamma)) -
\]

\[
- Y(h^{-1}(\alpha)) + (\nabla_Y \alpha)h^{-1}(\gamma) + \alpha(h^{-1}(\nabla_Y \gamma)) =
\]
\[= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h(T^\nabla)(X, Y), Z) + \]
\[+ \beta(\nabla_X h^{-1}(\gamma)) - \beta(h^{-1}(\nabla_X \gamma)) - \alpha(\nabla_Y h^{-1}(\gamma)) + \alpha(h^{-1}(\nabla_Y \gamma)). \]

Also, for any \( V \in C^\infty(TM) \), we have:
\[h(h^{-1}(\nabla_X \gamma) - \nabla_X h^{-1}(\gamma), V) = h(h^{-1}(\nabla_X \gamma), V) - h(\nabla_X h^{-1}(\gamma), V) = \]
\[= (\nabla_X \gamma)V - h(\nabla_X h^{-1}(\gamma), V) := X(\gamma(V)) - \gamma(\nabla_X V) - h(\nabla_X h^{-1}(\gamma), V) = \]
\[= X(h(h^{-1}(\gamma), V)) - h(h^{-1}(\gamma), \nabla_X V) - h(\nabla_X h^{-1}(\gamma), V) := (\nabla_X h)(h^{-1}(\gamma), V) = 0, \]
hence \( h^{-1}(\nabla_X \gamma) - \nabla_X h^{-1}(\gamma) = 0 \), for any \( X \in C^\infty(TM) \) and \( \gamma \in C^\infty(T^*M) \). Therefore, \( d\check{\nabla} h = 0 \) and the proof is complete.

**Proposition 3.14.** Let \((M, h, \nabla)\) be a quasi-statistical manifold with \(\nabla\) a torsion-free affine connection, \(h\) a \(\nabla\)-parallel \((0, 2)\)-tensor field on \(M\) and let \((\check{h}, \check{\nabla})\) be the generalized quasi-statistical structure on \(TM \oplus T^*M\), with \(\check{h}\) given by (7) and \(\check{\nabla}\) given by (7). Then \(\check{\nabla}\) and its generalized dual quasi-statistical connection, \(\check{\nabla}_h^*\), coincide.

**Proof.** We get:
\[\check{h}(Y + \beta, (\nabla_h^*)X + \alpha Z + \gamma) = X(h(Y, Z)) + X(h(h^{-1}(\beta), h^{-1}(\gamma))) - \]
\[h(\nabla_X Y, Z) - h(h^{-1}(\nabla_X \beta), h^{-1}(\gamma)) = \]
\[= h(Y, \nabla_X Z) + \beta(\nabla_X h^{-1}(\gamma)), \]
for any \( X, Y, Z \in C^\infty(TM) \) and \( \alpha, \beta, \gamma \in C^\infty(T^*M) \).

Let us denote \((\nabla_h^*)X + \alpha Z + \gamma =: V + \eta\). Then we have:
\[h(Y, V) + h(h^{-1}(\beta), h^{-1}(\eta)) = h(Y, \nabla_X Z) + \beta(\nabla_X h^{-1}(\gamma)), \]
for any \( X, Y, Z \in C^\infty(TM) \) and \( \beta, \gamma \in C^\infty(T^*M) \).

Taking \( Y := 0 \), we obtain:
\[\beta(h^{-1}(\eta)) = \beta(\nabla_X h^{-1}(\gamma)) \]
which is equivalent to:
\[\eta = h(\nabla_X h^{-1}(\gamma)) \]
and taking \( \beta := 0 \), we obtain:
\[h(Y, V) = h(Y, \nabla_X Z) \]
which is equivalent to:
\[V = \nabla_X Z. \]
From Remark 3.10 we get \(\check{\nabla}_h^* = \check{\nabla} = \check{\nabla} \). Therefore the proof is complete.
\[\square\]
4 The pull-back tensors on $TM \oplus T^*M$ of horizontal lifts, Sasaki and Patterson-Walker metrics

4.1 Patterson-Walker and Sasaki metrics on $T^*M$

Let $M$ be a smooth manifold and let $\nabla$ be an affine connection on $M$.

Let $\pi : T^*M \to M$ be the canonical projection and $\pi_* : T(T^*M) \to TM$ be the tangent map of $\pi$. If $a \in T^*M$ and $A \in T_a(T^*M)$, then $\pi_*(A) \in T_{\pi(a)}M$ and we denote by $\chi_a$ the standard identification between $T^*_{\pi(a)}M$ and its tangent space $T_a(T^*_{\pi(a)}M)$.

Let $\Phi^{\nabla} : TM \oplus T^*M \to T(T^*M)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$(8) \quad \Phi^{\nabla}(X + \alpha) := X^H + \chi_a(\alpha),$$

where $a \in T^*M$ and $X^H$ is the horizontal lift of $X \in T_{\pi(a)}M$.

Let $\{x^1, ..., x^n\}$ be local coordinates on $M$, let $\{\tilde{x}^1, ..., \tilde{x}^n, y_1, ..., y_n\}$ be respectively the corresponding local coordinates on $T^*M$ and let $\{X_1, ..., X_n, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\}$ be a local frame on $T(T^*M)$, where $X_i = \frac{\partial}{\partial \tilde{x}^i}$. The horizontal lift of $\frac{\partial}{\partial x^i}$ is defined by:

$$\left(\frac{\partial}{\partial x^i}\right)^H := X_i + y_k \Gamma^k_{il} \frac{\partial}{\partial y^l}$$

and we will denote $X_i^H := \left(\frac{\partial}{\partial x^i}\right)^H$. Moreover, the vertical lift of $\frac{\partial}{\partial x^i}$ is defined by:

$$\left(\frac{\partial}{\partial x^i}\right)^V := \frac{\partial}{\partial y^i},$$

where $i, k, l$ run from 1 to $n$ and $\Gamma^k_{il}$ are the Christoffel’s symbols of $\nabla$.

Let $\Phi^{\nabla} : TM \oplus T^*M \to T(T^*M)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$\Phi^{\nabla} \left(\frac{\partial}{\partial x^i}\right) = X_i^H$$

$$\Phi^{\nabla} (dx^j) = \frac{\partial}{\partial y_j}.$$
\[ \tilde{h}(X^V, Y^V) = 0 \]

\[ \tilde{h}(Y^V, X^H) = \tilde{h}(X^H, Y^V) = ((\Phi^\nabla)^{-1}(Y^V))(X), \]

where \( X, Y \in C^\infty(T^*M) \), \( X^H, Y^H \) are the horizontal lifts and \( X^V, Y^V \) are the vertical lifts of \( X, Y \) respectively.

The definition can also be given if \( \nabla \) has torsion and we define \( \tilde{h}_\pm \) on \( T^*M \) as in the following:

\[ \tilde{h}_\pm(X^H, Y^H) = 0 \]

\[ \tilde{h}_\pm(X^V, Y^V) = 0 \]

\[ \tilde{h}_\pm(Y^V, X^H) = ((\Phi^\nabla)^{-1}(Y^V))(X) \]

\[ \tilde{h}_\pm(X^H, Y^V) = \pm((\Phi^\nabla)^{-1}(Y^V))(X), \]

where \( X, Y \in C^\infty(T^*M) \), \( X^H, Y^H \) are the horizontal lifts and \( X^V, Y^V \) are the vertical lifts of \( X, Y \) respectively.

We denote by \( \tilde{\tilde{h}}_\pm \) the pull-back tensors of \( \tilde{h}_\pm \) on \( TM \oplus T^*M \):

\[ \tilde{\tilde{h}}_\pm(\sigma, \tau) := (\Phi^\nabla)^*(\tilde{h}_\pm)(\sigma, \tau) := \tilde{\tilde{h}}_\pm(\Phi^\nabla(\sigma), \Phi^\nabla(\tau)), \]

for any \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \). Remark that \( \tilde{\tilde{h}}_\pm \) are related to the indefinite metric or to the symplectic structure of \( TM \oplus T^*M \) as follows.

**Proposition 4.1.**

\[ \tilde{\tilde{h}}_+ = -2 < \cdot, \cdot > \]

\[ \tilde{\tilde{h}}_- = -2(\cdot, \cdot). \]

**Proof.** Let \( \sigma = X + \alpha, \tau = Y + \beta, X, Y \in C^\infty(TM) \), \( \alpha, \beta \in C^\infty(T^*M) \). Then:

\[ \tilde{\tilde{h}}_\pm(\sigma, \tau) = \tilde{h}_\pm(X^H + \Phi^\nabla(\alpha), Y^H + \Phi^\nabla(\beta)) = \]

\[ = \tilde{h}_\pm(\Phi^\nabla(\alpha), Y^H) + \tilde{h}_\pm(X^H, \Phi^\nabla(\beta)) = \]

\[ = \alpha(Y) \pm \beta(X). \]

Then we get the statement. \( \square \)
Let $h$ be a non-degenerate $(0,2)$-tensor field on $M$. The Sasaki $(0,2)$-tensor field $h^S$ on $T^*M$, with respect to $\nabla$, is naturally defined by:

$$h^S(X^H, Y^H) = h(X, Y)$$

$$h^S(\alpha^V, \beta^V) = h(h^{-1}(\alpha), h^{-1}(\beta))$$

$$h^S(\alpha^V, Y^H) = 0,$$

where $X, Y \in C^\infty(TM)$, $\alpha, \beta \in C^\infty(T^*M)$, $X^H, Y^H$ are the horizontal lifts of $X, Y$ and $\alpha^V, \beta^V$ are the vertical lifts of $\alpha, \beta$ respectively.

We denote by $\tilde{h}^S$ the pull-back tensor of $h^S$ on $TM \oplus T^*M$:

$$\tilde{h}^S(\sigma, \tau) := (\Phi^\nabla)^*(h^S)(\sigma, \tau) := h^S(\Phi^\nabla(\sigma), \Phi^\nabla(\tau)),$$

for any $\sigma, \tau \in C^\infty(TM \oplus T^*M)$.

**Proposition 4.2.** If $h$ is a non-degenerate $(0,2)$-tensor field on $M$, then:

$$\tilde{h}^S(X + \alpha, Y + \beta) = \hat{h}(X + \alpha, Y + \beta) = h(X, Y) + h(h^{-1}(\alpha), h^{-1}(\beta)),$$

for any $X, Y \in C^\infty(TM)$ and $\alpha, \beta \in C^\infty(T^*M)$.

**Proof.** Let $\sigma = X + \alpha$, $\tau = Y + \beta$, $X, Y \in C^\infty(TM)$, $\alpha, \beta \in C^\infty(T^*M)$. Then:

$$\tilde{h}^S(\sigma, \tau) = h^S(X^H + \Phi^\nabla(\alpha), Y^H + \Phi^\nabla(\beta)) =$$

$$= h^S(X^H, Y^H) + h^S(\Phi^\nabla(\alpha), \Phi^\nabla(\beta)) =$$

$$= h(X, Y) + h^S(\Phi^\nabla(\alpha), \Phi^\nabla(\beta)).$$

In local coordinates, let $\alpha = \alpha_k dx^k$, $\beta = \beta_l dx^l$ and we get:

$$h^S(\Phi^\nabla(\alpha), \Phi^\nabla(\beta)) = h^S(\alpha_k \frac{\partial}{\partial y_k}, \beta_l \frac{\partial}{\partial y_l}) =$$

$$= \alpha_k \beta_l h_{kl} = h(h^{-1}(\alpha), h^{-1}(\beta)).$$

Then we get the statement.\hfill\Box
4.2 Horizontal lift and Sasaki metrics on $TM$

Let $M$ be a smooth manifold, let $h$ be a non-degenerate $(0,2)$-tensor field on $M$, and let $\nabla$ be an affine connection on $M$. The horizontal lift $h^H$ of $h$ on $TM$ with respect to $\nabla$ is defined by:

$$h^H(X^H, Y^H) = 0$$
$$h^H(X^V, Y^V) = 0$$
$$h^H(X^H, Y^V) = h(X, Y),$$

where $X, Y \in C^\infty(TM)$, $X^H, Y^H$ are the horizontal lifts and $X^V, Y^V$ are the vertical lifts of $X, Y$ respectively.

Let $\pi: TM \to M$ be the canonical projection and $\pi_*: T(TM) \to TM$ be the tangent map of $\pi$. If $a \in TM$ and $A \in T_a(TM)$, then $\pi_*(A) \in T_{\pi(a)}M$ and we denote by $\chi_a$ the standard identification between $T_{\pi(a)}M$ and its tangent space $T_a(T_{\pi(a)}M)$.

Let $\Psi^\nabla: TM \oplus T^*M \to T(TM)$ be the bundle morphism defined by:

$$(\partial_{x^i})^H := X^H_i + \chi_a(h^{-1}(\alpha)),$$

where $a \in TM$ and $X^H_i$ is the horizontal lift of $X \in T_{\pi(a)}M$.

Let $\{x^1, ..., x^n\}$ be local coordinates on $M$, let $\{\tilde{x}^1, ..., \tilde{x}^n, y^1, ..., y^n\}$ be respectively the corresponding local coordinates on $TM$ and let $\{X_1, ..., X_n; \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$ be a local frame on $T(TM)$, where $X_i = \frac{\partial}{\partial \tilde{x}^i}$. The horizontal lift of $\frac{\partial}{\partial x^i}$ is defined by:

$$\left(\frac{\partial}{\partial x^i}\right)^H := X_i - y^k \Gamma^l_{ik} \frac{\partial}{\partial y^l}$$

and we will denote $X^H_i = \left(\frac{\partial}{\partial x^i}\right)^H$. Moreover, the vertical lift of $\frac{\partial}{\partial x^i}$ is defined by:

$$\left(\frac{\partial}{\partial x^i}\right)^V := \frac{\partial}{\partial y^i},$$

where $i, k, l$ run from 1 to $n$ and $\Gamma^l_{ik}$ are the Christoffel’s symbols of $\nabla$.

Let $\Psi^\nabla: TM \oplus T^*M \to T(TM)$ be the bundle morphism defined before (which is an isomorphism on the fibres). In local coordinates, we have the following expressions:

$$\Psi^\nabla \left(\frac{\partial}{\partial x^i}\right) = X^H_i$$
\[ \Psi^\nabla (dx^j) = h^{jk} \frac{\partial}{\partial y^k}. \]

We denote by \( \tilde{h} \) the pull-back tensor of \( h^H \) on \( TM \oplus T^*M \):

\[ \tilde{h}(\sigma, \tau) := (\Psi^\nabla)^*(h^H)(\sigma, \tau) := h^H(\Psi^\nabla(\sigma), \Psi^\nabla(\tau)), \]

for any \( \sigma, \tau \in C^\infty(TM \oplus T^*M) \). Remark that \( \tilde{h} \) is related to the indefinite metric or to the symplectic structure of \( TM \oplus T^*M \) as follows.

**Proposition 4.3.** If \( h \) is a symmetric tensor, then:

\[ \tilde{h} = -2 < \cdot, \cdot >. \]

If \( h \) is a skew-symmetric tensor, then:

\[ \tilde{h} = -2(\cdot, \cdot). \]

**Proof.** Let \( \sigma = X + \alpha, \tau = Y + \beta, X, Y \in C^\infty(TM), \alpha, \beta \in C^\infty(T^*M) \). Then:

\[ \tilde{h}(\sigma, \tau) = h^H(X^H + \Psi^\nabla(\alpha), Y^H + \Psi^\nabla(\beta)) = h^H(\Psi^\nabla(\alpha), Y^H) + h^H(X^H, \Psi^\nabla(\beta)). \]

In local coordinates, let \( X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}, \alpha = \alpha_k dx^k, \beta = \beta_l dx^l \) and we get:

\[ \tilde{h}(\sigma, \tau) = h^H(\alpha_k h^{kr} \frac{\partial}{\partial y^r}, Y^j X^H_j) + h^H(X^i X^H_i, \beta_l h^{ls} \frac{\partial}{\partial y^s}) = \alpha_k Y^j h^{kr} h_{rj} + X^i \beta_l h^{ls} h_{is} = \alpha_k Y^j \delta_j^k \pm X^i \beta_l \delta_i^l = \alpha(Y) \pm \beta(X), \]

where we denoted by \( \delta \) the Kronecker’s symbol and the sign + is for \( h \) symmetric, − is for \( h \) skew-symmetric. Then we get the statement. \( \square \)

The Sasaki \((0, 2)\)-tensor field \( h^S \) on \( TM \), with respect to \( \nabla \), is naturally defined by:

\[ h^S(X^H, Y^H) = h(X, Y) \]
\[ h^S(X^V, Y^V) = h(X, Y) \]
\[ h^S(X^H, Y^V) = 0, \]

\[ h^S(X^V, Y^H) = 0, \]

\[ h^S(X^V, Y^V) = 0, \]

\[ h^S(X^H, Y^H) = h(X, Y) \]

\[ h^S(X^V, Y^V) = h(X, Y) \]
\[ h^S(X^H, Y^V) = 0, \]

\[ h^S(X^V, Y^H) = 0, \]

\[ h^S(X^V, Y^V) = 0, \]
where \(X, Y \in C^\infty(TM)\), \(X^H, Y^H\) are the horizontal lifts and \(X^V, Y^V\) are the vertical lifts of \(X, Y\) respectively.

We denote by \(\bar{h}^S\) the pull-back tensor of \(h^S\) on \(TM \oplus T^*M\):

\[
\bar{h}^S(\sigma, \tau) := (\Psi^\nabla)^*(h^S)(\sigma, \tau) := h^S(\Psi^\nabla(\sigma), \Psi^\nabla(\tau)),
\]

for any \(\sigma, \tau \in C^\infty(TM \oplus T^*M)\).

**Proposition 4.4.** If \(h\) is a non-degenerate \((0,2)\)-tensor field on \(M\), then:

\[
\bar{h}^S(X + \alpha, Y + \beta) = \tilde{h}(X + \alpha, Y + \beta) = h(X, Y) + h(h^{-1}(\alpha), h^{-1}(\beta)),
\]

for any \(X, Y \in C^\infty(TM)\) and \(\alpha, \beta \in C^\infty(T^*M)\).

**Proof.** Let \(\sigma = X + \alpha, \tau = Y + \beta, X, Y \in C^\infty(TM), \alpha, \beta \in C^\infty(T^*M)\). Then:

\[
\bar{h}^S(\sigma, \tau) = h^S(X^H + \Psi^\nabla(\alpha), Y^H + \Psi^\nabla(\beta)) =
\]

\[
= h^S(X^H, Y^H) + h^S(\Psi^\nabla(\alpha), \Psi^\nabla(\beta)) =
\]

\[
= h(X, Y) + h^S(\Psi^\nabla(\alpha), \Psi^\nabla(\beta)).
\]

In local coordinates, let \(\alpha = \alpha_kdx^k, \beta = \beta_ldx^l\) and we get:

\[
h^S(\Psi^\nabla(\alpha), \Psi^\nabla(\beta)) = h^S(\alpha_kh^{kr}\frac{\partial}{\partial y^r}, \beta_lh^{ls}\frac{\partial}{\partial y^s}) =
\]

\[
= \alpha_k\beta_lh^{kr}h^{ls}h_{rs} = h(h^{-1}(\alpha), h^{-1}(\beta)).
\]

Then we get the statement. \(\square\)

### 4.3 Quasi-statistical structures on cotangent bundles

Given an affine connection on \(M\), the splitting in horizontal and vertical subbundles identifies \(T(T^*M)\) with the pull-back bundle \(\pi^*(TM \oplus T^*M)\), where \(\pi : T^*M \to M\) is the canonical projection map. In particular, given a connection on \(TM \oplus T^*M\), we can define the pull-back connection on \(\pi^*(TM \oplus T^*M)\).

A direct computation gives the following:
Proposition 4.5. The pull-back connection $\nabla$ of $\tilde{\nabla}$ on $T^*M$ is defined, in local coordinates, by:

$$
\tilde{\nabla} \left( \frac{\partial}{\partial x^i} \right) H \left( \frac{\partial}{\partial x^j} \right) H = \Gamma^k_{ij} \left( \frac{\partial}{\partial x^k} \right) H
$$

$$
\tilde{\nabla} \left( \frac{\partial}{\partial y^i} \right) \frac{\partial}{\partial y^j} = \left( \frac{\partial h_{jk}}{\partial x^i} + h_{jl} \Gamma^k_{ji} \right) \frac{\partial}{\partial y^r} \frac{\partial}{\partial y^r}
$$

$$
\tilde{\nabla} \frac{\partial}{\partial y^i} H = 0
$$

$$
\tilde{\nabla} \frac{\partial}{\partial y^i} = 0.
$$

In local coordinates, the torsion $T\tilde{\nabla}$ of $\tilde{\nabla}$ is:

$$
T\tilde{\nabla} \left( \frac{\partial}{\partial x^i} H, \frac{\partial}{\partial x^j} H \right) = \left( \Gamma^k_{ij} - \Gamma^k_{ji} \right) \left( \frac{\partial}{\partial x^k} \right) H - y_l R^l_{ijk} \frac{\partial}{\partial y^k}
$$

$$
T\tilde{\nabla} \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^j} H \right) = -\left( \frac{\partial h_{ik}}{\partial x^j} + h_{jl} \Gamma^k_{ji} \right) \frac{\partial}{\partial y^r} \frac{\partial}{\partial y^r} + \Gamma^i_{jk} \frac{\partial}{\partial y^r}
$$

$$
T\tilde{\nabla} \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0
$$

and the curvature $R\tilde{\nabla}$ of $\tilde{\nabla}$, which is the pull-back of $R\nabla$, is:

$$
R\nabla \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0
$$

$$
R\nabla \left( \frac{\partial}{\partial x^i} H, \frac{\partial}{\partial y^j} \right) = 0
$$

$$
R\nabla \left( \frac{\partial}{\partial x^i} H, \frac{\partial}{\partial x^j} H \right) \frac{\partial}{\partial y^k} = h_{ijr} R^l_{ijk} h_{ls} \frac{\partial}{\partial y^s}
$$

$$
R\nabla \left( \frac{\partial}{\partial x^i} H, \frac{\partial}{\partial x^j} H \right) \left( \frac{\partial}{\partial x^k} \right) H = \left( R\nabla \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right) \left( \frac{\partial}{\partial x^k} \right) H.
$$

Therefore we get:

Proposition 4.6. $\nabla$ is flat if and only if $\tilde{\nabla}$ is flat.

Theorem 4.7. Let $(M, h, \nabla)$ be a quasi-statistical manifold such that $\nabla$ is flat. Then $(T^*M, h^{S*}, \tilde{\nabla})$ is a flat quasi-statistical manifold.
**Proof.** Let us compute \( d\tilde{\nabla} h^{S^*} \). From the definition of \( h^{S^*} \) and \( \tilde{\nabla} \) we get immediately:

\[
(d\tilde{\nabla} h^{S^*})(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = 0
\]

\[
(d\tilde{\nabla} h^{S^*})(\frac{\partial}{\partial x^i})^H, \frac{\partial}{\partial y_j}) = 0
\]

\[
(d\tilde{\nabla} h^{S^*})(\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H, (\frac{\partial}{\partial x^k})^H = -y_l R_{ijk}^l h_{kr}
\]

Then we get the statement. \(\square\)

Moreover, considering the Patterson-Walker metric, \( \tilde{h}_\pm \), we get the following:

**Theorem 4.8.** Let \((M, h, \nabla)\) be a quasi-statistical manifold such that \( \nabla \) is flat. Then \((T^*M, \tilde{h}_\pm, \tilde{\nabla})\) is a quasi-statistical manifold.

**Proof.** Let us compute \( d\tilde{\nabla} \tilde{h}_\pm \). From the definition of \( \tilde{h}_\pm \) and \( \tilde{\nabla} \) we get immediately:

\[
(d\tilde{\nabla} \tilde{h}_\pm)(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = 0
\]

\[
(d\tilde{\nabla} \tilde{h}_\pm)(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = 0
\]

\[
(d\tilde{\nabla} \tilde{h}_\pm)(\frac{\partial}{\partial x^i})^H, \frac{\partial}{\partial y_j}) = 0
\]

\[
(d\tilde{\nabla} \tilde{h}_\pm)(\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H, (\frac{\partial}{\partial x^k})^H = -y_l R_{ijk}^l
\]

\[
(d\tilde{\nabla} \tilde{h}_\pm)(\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H, (\frac{\partial}{\partial y_k}) = \pm h^{kl}(d\nabla h)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(\frac{\partial}{\partial x^l}),
\]

where the sign + is for \( h \) symmetric, – is for \( h \) skew-symmetric. Then we get the statement. \(\square\)

**Definition 4.9.** A quasi-statistical manifold \((M, h, \nabla)\) such that \( \nabla \) is flat is called a Hessian manifold.

Therefore we get:

**Corollary 4.10.** If \((M, h, \nabla)\) is a Hessian manifold, then \((T^*M, h^{S^*}, \tilde{\nabla})\) and \((T^*M, \tilde{h}_\pm, \tilde{\nabla})\) are Hessian manifolds.
4.4 Quasi-statistical structures on tangent bundles

Given a non-degenerate \((0,2)-\)tensor field \(h\) on \(M\), we have an isomorphism between \(T(T^*M)\) and \(T(TM)\). The connection \(\tilde{\nabla}\) on \(TM\) corresponding to \(\tilde{\nabla}\) on \(T^*M\), is the following:

\[
\tilde{\nabla} \left( \frac{\partial}{\partial x^i} \right)^H = \Gamma^k_{ij} \left( \frac{\partial}{\partial x^k} \right)^H
\]

\[
\tilde{\nabla} \left( \frac{\partial}{\partial y^i} \right)^H = \Gamma^k_{ij} \frac{\partial}{\partial y^k}
\]

\[
\tilde{\nabla} \left( \frac{\partial}{\partial x^i} \right)^H = 0
\]

\[
\tilde{\nabla} \left( \frac{\partial}{\partial y^i} \right)^H = 0
\]

In local coordinates, the torsion \(T_{\tilde{\nabla}}\) of \(\tilde{\nabla}\) is:

\[
T_{\tilde{\nabla}}((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial y^j})^H) = (\Gamma^k_{ij} - \Gamma^k_{ji})((\frac{\partial}{\partial x^k})^H - y^l R^k_{lij} \frac{\partial}{\partial y^l})
\]

\[
T_{\tilde{\nabla}}((\frac{\partial}{\partial y^i})^H, (\frac{\partial}{\partial x^j})^H) = 0
\]

\[
T_{\tilde{\nabla}}((\frac{\partial}{\partial y^i})^H, (\frac{\partial}{\partial y^j})^H) = 0
\]

and the curvature \(R_{\tilde{\nabla}}\) of \(\tilde{\nabla}\) is:

\[
R_{\tilde{\nabla}}((\frac{\partial}{\partial y^i})^H, (\frac{\partial}{\partial y^j})^H) = 0
\]

\[
R_{\tilde{\nabla}}((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial y^j})^H) = 0
\]

\[
R_{\tilde{\nabla}}((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H) \frac{\partial}{\partial y^k} = R^l_{ijk} \frac{\partial}{\partial y^l}
\]

\[
R_{\tilde{\nabla}}((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H) (\frac{\partial}{\partial x^k})^H = R(\frac{\partial}{\partial x^i}, (\frac{\partial}{\partial x^j})^H)(\frac{\partial}{\partial x^k})
\]

Therefore we get:

**Proposition 4.11.** \(\nabla\) is flat if and only if \(\tilde{\nabla}\) is flat.
Theorem 4.12. Let \((M, h, \nabla)\) be a quasi-statistical manifold such that \(\nabla\) is flat. Then \((TM, h^S, \tilde{\nabla})\) is a flat quasi-statistical manifold.

**Proof.** Let us compute \(d\tilde{\nabla}h^S\). From the definition of \(h^S\) and \(\tilde{\nabla}\) we get immediately:

\[
(d\tilde{\nabla}h^S)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0
\]

\[
(d\tilde{\nabla}h^S)((\frac{\partial}{\partial x^i})^H, \frac{\partial}{\partial y^j}) = 0
\]

\[
(d\tilde{\nabla}h^S)((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H)(\frac{\partial}{\partial y^k}) = -y^l R_{ijl}^r h_{rk}
\]

Then we get the statement.

Moreover, considering the horizontal lift metric, \(h^H\), we get the following:

Theorem 4.13. Let \((M, h, \nabla)\) be a quasi-statistical manifold such that \(\nabla\) is flat. Then \((TM, h^H, \tilde{\nabla})\) is a quasi-statistical manifold if and only if \(\nabla h = 0\).

**Proof.** Let us compute \(d\tilde{\nabla}h^H\). From the definition of \(h^H\) and \(\tilde{\nabla}\) we get immediately:

\[
(d\tilde{\nabla}h^H)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = 0
\]

\[
(d\tilde{\nabla}h^H)((\frac{\partial}{\partial x^i})^H, \frac{\partial}{\partial y^j}) = \pm (\nabla \frac{\partial}{\partial x^i} h)\frac{\partial}{\partial x^j}
\]

\[
(d\tilde{\nabla}h^H)((\frac{\partial}{\partial x^i})^H, (\frac{\partial}{\partial x^j})^H)(\frac{\partial}{\partial x^k})^H = -y^l R_{ijl}^r h_{sk}
\]

where the sign + is for \(h\) symmetric, – is for \(h\) skew-symmetric. Then we get the statement.

Therefore we get:

Corollary 4.14. If \((M, h, \nabla)\) is a Hessian manifold, then \((TM, h^S, \tilde{\nabla})\) is a Hessian manifold. Moreover, if \(\nabla h = 0\), then \((TM, h^H, \tilde{\nabla})\) is a Hessian manifold.
5 Norden and Para-Norden structures on cotangent and tangent bundles

Norden manifolds, also called almost complex manifolds with B-metric, were introduced in [8]. They have applications in mathematics and in theoretical physics.

Definition 5.1. A Norden manifold, \((M, J, h)\), is an almost complex manifold \((M, J)\) with a pseudo-Riemannian metric, \(h\) (called Norden metric), such that \(J\) is \(h\)-symmetric.

Moreover, if \(J\) is integrable, then \((M, J, h)\) is called complex Norden manifold.

Definition 5.2. An almost Para-complex Norden manifold (or simply, almost Para-Norden manifold), \((M, J, h)\), is a real even dimensional smooth manifold \(M\) with a pseudo-Riemannian metric, \(h\), and a \((1,1)\)-tensor field, \(J\), such that \(J^2 = I\), the two eigenbundles \(T^+M, T^-M\), associated to the two eigenvalues \(+1, -1\), of \(J\) respectively have the same rank and \(J\) is \(h\)-symmetric.

Moreover, if \(J\) is integrable, then \((M, J, h)\) is called Para-Norden manifold.

5.1 Norden and Para-Norden structures on cotangent bundles

Let \((M, h)\) be a pseudo-Riemannian manifold and let \(\tilde{J}_c, \tilde{J}_p\) be the generalized complex structure and the generalized product structure defined by \(h\) in (1) and (2) respectively. Again we will denote \(\tilde{J}_\pm\) for \(\tilde{J}_c =: \tilde{J}_-\) and \(\tilde{J}_p =: \tilde{J}_+\).

Let \(\nabla\) be an affine connection on \(M\) and let \(\Phi^\nabla : TM \oplus T^*M \to T(T^*M)\) be the bundle morphism defined by (8). We define:

\[
\tilde{J}_\nabla^\pm =: \Phi^\nabla \circ \tilde{J}_\pm \circ (\Phi^\nabla)^{-1}.
\]

We have immediately that \((\tilde{J}_\nabla^\pm)^2 = \mp I\).

Proposition 5.3. Let \(\tilde{h}\) be the Patterson-Walker metric on \(T^*M\). Then \((T^*M, \tilde{J}_\nabla, \tilde{h})\) is a Norden manifold and \((T^*M, \tilde{J}_\nabla^+, \tilde{h})\) is an almost Para-Norden manifold. Moreover, if \((M, h, \nabla)\) is a flat quasi-statistical manifold, then \((T^*M, \tilde{J}_\nabla^-, \tilde{h})\) is a complex Norden manifold and \((T^*M, \tilde{J}_\nabla^+, \tilde{h})\) is a Para-Norden manifold.

Proof. In local coordinates, we get the following:

\[
\tilde{J}_\nabla^+(X_i^h) =: h_{ik} \frac{\partial}{\partial y_k}.
\]
\[ \tilde{J}^\nabla \left( \frac{\partial}{\partial y_j} \right) =: \mp h^{jk} X^H_k. \]

In particular, we have:
\[ \tilde{h}(\tilde{J}^\nabla (X_i^H), X_j^H) = h_{ij}, \]
\[ \tilde{h}(\tilde{J}^\nabla \left( \frac{\partial}{\partial y_i} \right), \frac{\partial}{\partial y_j}) = \mp h^{ij}, \]
\[ \tilde{h}(\tilde{J}^\nabla (X_i^H), \frac{\partial}{\partial y_j}) = 0, \]
\[ \tilde{h}(\tilde{J}^\nabla \left( \frac{\partial}{\partial y_i} \right), X_j^H) = 0, \]
therefore, from the symmetry of \( h \), we get the first statement.

Moreover, if we compute the Nijenhuis tensor field of \( \tilde{J}^\nabla \), we have:
\[ N_{\tilde{J}^\nabla} (X_i^H, X_j^H) = \pm (h^{kl} (d^\nabla h) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) - y_k R_{ijkl} \frac{\partial}{\partial y_l}), \]
\[ N_{\tilde{J}^\nabla} \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) = h^{ij} h^{kl} (d^\nabla h) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) X^H_s + y_s R_{klir} \frac{\partial}{\partial y_r}. \]

Then the proof is complete. \( \square \)

**Remark 5.4.** If \( h \) is a non-degenerate skew-symmetric \((0,2)\)-tensor field on \( M \), then the same construction gives rise to a Hermitian, respectively Para-Hermitian, structure on \( T^* M \).

### 5.2 Norden and Para-Norden structures on tangent bundles

Let \((M, h)\) be a pseudo-Riemannian manifold and let \( \hat{J}_c, \hat{J}_p \) be the generalized complex structure and the generalized product structure defined by \( h \) in (1) and (2) respectively. Again we will denote \( \hat{J}_\mp \) for \( \hat{J}_c =: \hat{J}_- \) and \( \hat{J}_p =: \hat{J}_+ \).

Let \( \nabla \) be an affine connection on \( M \) and let \( \Psi^\nabla : TM \oplus T^* M \to T(TM) \) be the bundle morphism defined by (9). We define:
\[ \tilde{J}^\nabla =: \Psi^\nabla \circ \hat{J}_\mp \circ (\Psi^\nabla)^{-1}. \]
Let $X \in C^\infty(TM)$ and let $X^H$, $X^V$ be respectively the horizontal and vertical lift of $X$. We have immediately that

\[ J_\mp(X^H) = X^V \\
J_\mp(X^V) = \mp X^H. \]

A direct computation gives the following:

**Proposition 5.5.** Let $h^H$ be the horizontal lift metric of $h$ on $TM$. Then $(TM, J_\mp, h^H)$ is a Norden manifold and $(TM, J_\mp, h^H)$ is an almost Para-Norden manifold.

**Remark 5.6.** The almost complex structure $J_\mp$ is the canonical almost complex structure of $TM$ defined in [3]. In particular, it is integrable if and only if $\nabla$ is flat and torsion-free.

**References**

[1] S. Amari, *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics, 28 Springer (1985).

[2] A. M. Blaga, A. Nannicini, *Generalized metallic structures*, arXiv:1807.08308, 2018.

[3] P. Dombrowski, *On the geometry of tangent bundles*, J. reine angew. Math. 210 (1962), 73-88.

[4] H. Matsuzoe, *Quasi-statistical manifolds and geometry of affine distributions*, Pure and Applied Differential Geometry 2012: In Memory of Franki Dillen, Berichte aus der Mathematik, ed. Joeri Van der Veken, Ignace Van de Woestyne, Leopold Verstraelen, Luc Vrancken, Shaker Verlag, 2013.

[5] A. Nannicini, *Almost complex structures on cotangent bundles and generalized geometry*, J. Geom. Phys. 60 (2010), 1781-1791.

[6] A. Nannicini, *Calibrated complex structures on the generalized tangent bundle of a Riemannian manifold*, J. Geom. Phys. 56 (2006), 903-916.

[7] M. Nogushi, *Geometry of statistical manifolds*, Differ. Geom. Appl. 2 (1992), 197-222.

[8] A. P. Norden, *On a class of four-dimensional A-spaces*, Russian Math. (Izv VUZ) 17 (4) (1960), 145-157.
[9] E. M. Patterson, A. G. Walker, *Riemann extension*, Quart. J. Math. Oxford 3 (1952), 19-28.

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