ENERGY CONDITIONS AND TWISTED LOCALIZATIONS OF OPERATORS

ERIC T. SAWYER

Abstract. We show that the energy conditions are not necessary for boundedness of fractional Riesz transforms $R^{\alpha,n}$ for $0 \leq \alpha < n$ in dimension $n \geq 2$.

We also give a weak converse, namely that the energy conditions are necessary for boundedness of families of twisted localizations of fractional singular integrals $T^\alpha$ having the positive gradient property - however, the kernels of these localizations satisfy only one-sided Calderón-Zygmund smoothness estimates.

Contents

1. Introduction 2
1.1. Statements of theorems 3

Part 1. Necessity of energy conditions for twisted localizations

2. Standard fractional singular integrals 4
2.1. Defining the norm inequality and testing conditions 4
2.2. Strong ellipticity and the positive gradient property 5
3. Poisson integrals and Muckenhoupt conditions 6
4. Strong, deep and bounded overlap energy constants 6
5. Twisted localizations and necessity of the deep energy condition 7
5.1. Family of localizations of an operator 8
5.2. Family of twisted localizations 10
5.3. Reversal of energy 11

Part 2. Failure of necessity of the energy condition for Riesz transforms

6. Construction of the counterexample pair of weights for the Cauchy operator 14
6.1. An estimate for the second component 15
6.2. The plan of attack 19
7. The $A_2^\alpha$ Condition 22
8. The pivotal and energy conditions 24
8.1. Failure of the backward pivotal and backward energy conditions 24
8.2. The forward energy condition 24
9. Testing conditions for the first component $R^{1,2}_1$ 25
9.1. The forward testing condition 25
9.2. The backward testing condition 27
10. Testing conditions for the second component $R^{1,2}_2$ 28
10.1. The backward testing condition 28
11. The norm inequality 30
12. The general case $0 \leq \alpha < n$ and $n \geq 2$ 30

References 31

Date: January 11, 2017.
Research supported in part by NSERC.
1. Introduction

An important ‘two weight theorem’ for the Hilbert transform was obtained early on by Nazarov, Treil and Volberg \cite{Vol}, who proved that the Hilbert transform $H$, with convolution kernel $K(x) = \frac{1}{x}$, was bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.

$$\int |H \sigma f(x)|^2 \omega(x) = \int \left| \int \frac{f(y)}{y-x} d\sigma(y) \right|^2 \omega(x) \leq \mathcal{H}_1 \int |f(y)|^2 d\sigma(y),$$

for all $f \in L^2(\sigma)$ uniformly over suitable truncations of the kernel $K$, provided that the following three conditions held:

1. the Muckenhoupt condition,
   $$A_2 \equiv \sup_{\text{intervals } I} \int_{I} \frac{|I|}{|I| + x^2} d\omega(x) \cdot \int \frac{|I|}{|I| + y^2} d\sigma(y) < \infty,$$

2. the testing conditions,
   $$\mathcal{T}_H \equiv \sup_{\text{intervals } I} \sqrt{\frac{1}{|I|}} \int_{I} |H \sigma 1| \omega < \infty \text{ and } \mathcal{T}_H \equiv \sup_{\text{intervals } I} \sqrt{\frac{1}{|I|}} \int |H \sigma 1|^2 \omega < \infty,$$

3. and the pivotal conditions,
   \begin{equation}
   \mathcal{V}_2 \equiv \sup_{\text{intervals } I} \int_{I} \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \int \frac{|I_r|}{|I_r|^2 + y^2} d\sigma(y) \right) |I_r|_{\omega} < \infty,
   \end{equation}

\begin{equation}
\mathcal{V}_2' \equiv \sup_{\text{intervals } I} \int_{I} \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \int \frac{|I_r|}{|I_r|^2 + y^2} d\omega(y) \right) |I_r|_{\sigma} < \infty.
\end{equation}

The first two conditions are necessary for boundedness of $H$, but the third condition is not. This was established in Lacey, Sawyer and Uriarte-Tuero \cite{LaSaUr2}, where a substitute for the pair of pivotal conditions was introduced, namely the pair of energy conditions,

\begin{equation}
\mathcal{E}_2 \equiv \sup_{\text{intervals } I} \int_{I} \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \int \frac{|I_r|}{|I_r|^2 + y^2} d\sigma(y) \right) |I_r|_{\omega} \mathcal{E}(I_r, \omega)^2 < \infty,
\end{equation}

\begin{equation}
\mathcal{E}_2' \equiv \sup_{\text{intervals } I} \int_{I} \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \int \frac{|I_r|}{|I_r|^2 + y^2} d\omega(y) \right) |I_r|_{\sigma} \mathcal{E}(I_r, \sigma)^2 < \infty,
\end{equation}

and this pair was shown to be not only necessary for boundedness of the Hilbert transform to hold, but in fact necessary for the Muckenhoupt and testing conditions to hold. The quantity

\begin{equation}
\mathcal{E}(J, \mu)^2 \equiv 2 \int_{[a,b]} \int_{J} \frac{1}{|J|} \int_{J} \frac{|x-x'|^2}{|I|} d\mu(x) d\mu(x'),
\end{equation}

is a one-dimensional $L^2$ version of the familiar normalized self-energy of the charge distribution $1_{I\mu}$ in physics, and $\mathcal{E}([a,b], \mu)$ takes values near 0 for highly concentrated distributions such as $\delta_a$, and values near 1 for highly spread out distributions such as $\mu = \delta_a + \delta_b$ (just the opposite from self-energy in 3-space, since the exponent $n-2$ of Laplace's fundamental solution changes sign when $n$ goes from 3 to 1).

This necessity of the energy condition reinforced the $T1$ conjecture of NTV \cite{Vol} that boundedness of the Hilbert transform is equivalent to the Muckenhoupt and testing conditions. And this conjecture was subsequently proved in the two part paper \cite{LaSaUr3, Lac} by Lacey, Sawyer, Shen and Uriarte-Tuero; Lacey, with the inclusion of common point masses by Hytönen in \cite{Hy}. The energy conditions played a crucial role in both parts \cite{LaSaUr3} and \cite{Lac}, and their $n$-dimensional counterparts have continued to play equally crucial roles in higher dimensional theorems of $T1$ type, \cite{SaShUr7, SaShUr10, LaSaShUrWi, LaWi} and \cite{LaWi}. The known proofs of necessity of the energy conditions broke down in higher dimensions,
leaving higher dimensional T1-type theorems in a state of limbo, not knowing if the energy conditions were necessary, or if another approach was needed.

In this paper we construct families of counterexample weight pairs to show that the energy conditions can indeed fail for a pair of weights, despite boundedness of the fractional Riesz transform - the prototypical fractional singular integral in higher dimensions. These families of counterexamples are motivated by a weak converse result that we also develop - namely that the boundedness of a large ‘twisted’ family of operators (related to a single ‘nice’ singular integral) does indeed imply the energy conditions. While this converse result may be of some theoretical interest, it is diminished by the requirement that the testing conditions be taken over too large a family \( \{ \Theta_1T^\alpha_\beta \} \) of twisted localizations, and by the fact that the kernels of these twisted localizations satisfy only one-sided Calderón-Zygmund smoothness estimates.

The counterexamples constructed here in dimension \( n \geq 2 \) are actually simpler than the subtle and complicated counterexample constructed in dimension \( n = 1 \) by Sawyer, Shen and Uriarte-Tuero in [SaShUr11]. Indeed, the counterexample in [SaShUr11] (which showed the energy conditions are not necessary for boundedness of a certain elliptic operator on the line) was obtained by modifying the example weight pair \((\sigma, \omega)\) in [LaSaUr12] consisting of a Cantor measure \( \omega \) and a discrete measure \( \sigma \). In order to fail the energy condition, the measure \( \sigma \) was modified into a measure \( \tilde{\sigma} \) by smearing out along the line each point mass in \( \sigma \), so that the local energies of the ‘smeared out’ measure \( \tilde{\sigma} \) no longer vanished. But this ‘smearing out’ destroyed the backward testing condition, which then required a modification of the Hilbert transform to a ‘flattened’ version \( H_j \), whose convolution kernel was still elliptic \( K_\beta(x) \approx \frac{1}{x} \), but no longer had strictly negative derivative. This in turn forced a redistribution \( \tilde{\omega} \) of the Cantor measure \( \omega \) in order that \( H_j \tilde{\omega} \) vanish on the support of \( \tilde{\sigma} \), resulting in a delicate and difficult recursion.

On the other hand, the extra dimension in \( \mathbb{R}^n \) for \( n \geq 2 \) permits a ‘spreading out’ of each point mass in \( \sigma \) into a new dimension, which then requires a matching ‘spreading out’ of the Cantor measure \( \omega \), something much simpler to deal with than that just outlined in dimension \( n = 1 \).

1.1. Statements of theorems. We state here our main theorem and proposition, but defer the definitions of some of the terminology used in the statements, until the sections where they are developed. First we show that the deep energy conditions are not necessary for boundedness of fractional Riesz transforms in general.

**Theorem 1.** Let \( 0 \leq \alpha < n \). Then there is a sequence of pairs \( \{ (\tilde{\sigma}_N, \tilde{\omega}_N) \}_{N=1}^\infty \) of locally finite positive Borel measures on \( \mathbb{R}^n \) (actually finite sums of point masses) such that the backward deep energy constants \( \mathcal{E}_{2,\alpha}^{\text{deep}}(\tilde{\sigma}_N, \tilde{\omega}_N) \) are unbounded in \( N \geq 1 \), and yet such that the vector Riesz transform \( R^{\alpha,n} \) of order \( \alpha \) is bounded from \( L^2(\tilde{\sigma}_N) \) to \( L^2(\tilde{\omega}_N) \) uniformly in \( N \geq 1 \).

In the converse direction, we can derive the deep energy conditions, and also the bounded overlap energy conditions, from uniform boundedness of a large enough family of operators with uniform one-sided Calderón-Zygmund norms.

**Proposition 2.** Let \( (\sigma, \omega) \) be a pair of locally finite positive Borel measures on \( \mathbb{R}^n \). Let \( 0 \leq \alpha < n \) and suppose that \( T^\alpha \) is a standard \( \alpha \)-fractional singular integral on \( \mathbb{R}^n \) that is both strongly elliptic and satisfies the positive gradient property. In particular we can take \( T^\alpha = R^{\alpha,n} \). If the family \( \{ \Theta_1T^\alpha_\beta \} \) of twisted localizations of the operator \( T^\alpha \) satisfies the testing conditions unconditionally in \( J = 3 \) and \( 1 \leq i, j \leq M \), then the deep and bounded overlap energy conditions hold, and moreover there is a positive constant \( C \), depending only on \( n, \alpha \), \( \| T^\alpha \|_{CZ_n} \), and the constants in the definitions of strongly elliptic and positive gradient property, such that

\[
\mathcal{E}_{2,\alpha}^{\text{deep}} \leq \mathcal{E}_{2,\alpha}^{\text{overlap}} \leq C \sup_{J \in \mathcal{J}} \sup_{1 \leq i, j \leq M} |T_{\Theta_iT^\alpha_\beta} T_{\Theta_j}| + C \beta A_2^\alpha.
\]

We also show in Lemma 10 below, that the kernels \( \Theta_iK^\alpha_\beta \Theta_j^{-1} \) of the operators \( \Theta_iT^\alpha_\beta \Theta_j \) satisfy one-sided Calderón-Zygmund estimates

\[
|\Theta_iK^\alpha_\beta \Theta_j^{-1}(x, y)| \lesssim \| K^\alpha \|_{CZ_n} |x - y|^{\alpha-n},
\]

\[
|\nabla_y \Theta_iK^\alpha_\beta \Theta_j^{-1}(x, y)| \lesssim \| K^\alpha \|_{CZ_n} |x - y|^{\alpha-n-\ell}, \quad \ell = 1, 2,
\]

with smoothness in the \( y \)-variable only, and then we show in Lemma 11 below, that the negative of the Riesz transform \( R^{\alpha,n} \) has the positive gradient property.
Proposition 2 is proved in Part 1 of the paper, while Theorem 1 is proved in Part 2. Each of these parts can essentially be read independently of the other.

Part 1. Necessity of energy conditions for twisted localizations

In the first part of this paper, we prove Proposition 2 by deriving the deep and bounded overlap energy conditions, as defined below, from testing conditions for the family of twisted localizations of the α-fractional Riesz transform $R^\alpha n$ in dimension $n$, and more generally for strongly elliptic convolution vector operators $T^\alpha$ in place of $R^\alpha n$ that enjoy the positive gradient property.

2. Standard fractional singular integrals

Let $0 \leq \alpha < n$ and $0 < \delta \leq 1$. We define a δ-standard α-fractional Calderón-Zygmund kernel $K^\alpha(x, y)$ to be a vector-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ whose components uniformly satisfy the following fractional size and smoothness conditions: For $x \neq y$ in $\mathbb{R}^n$,

\begin{equation}
|K^\alpha (x, y)| \leq C_{CZ, n} |x - y|^\alpha - n \quad \text{and} \quad |\nabla K^\alpha (x, y)| \leq C_{CZ, n} |x - y|^\alpha - n - 1,
\end{equation}

and where the last inequality also holds for the adjoint kernel in which $x$ and $y$ are interchanged. We define the Calderón-Zygmund norm $\|K^\alpha\|_{CZ, n}$ of $K^\alpha$ to be the least constant $C_{CZ, n}$ for which the above display holds.

2.1. Defining the norm inequality and testing conditions. We now recall the precise definition of the weighted norm inequality

\begin{equation}
\|T^\alpha_\sigma f\|_{L^2(\omega)} \leq \mathcal{R} \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma),
\end{equation}
as in [SaShUr10] for example. Let $\{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty}$ be a family of nonnegative functions on $[0, \infty)$ so that the truncated kernels $K^\alpha_{\delta, R}(x, y) = \eta_{\delta, R}^\alpha (|x - y|) K^\alpha (x, y)$ of the operator $T^\alpha$ are bounded with compact support for fixed $x$ or $y$. Then the truncated operators

\[ T^\alpha_{\sigma, \delta, R} f(x) = \int_{\mathbb{R}} K^\alpha_{\delta, R}(x, y) f(y) \, d\sigma(y), \quad x \in \mathbb{R}^n, \]

are pointwise well-defined, and we will refer to the pair $\left( K^\alpha, \{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty} \right)$ as an α-fractional singular integral operator, which we typically denote by $T^\alpha$, suppressing the dependence on the truncations. When $K^\alpha (x, y)$ is a δ-standard α-fractional Calderón-Zygmund kernel, we say that $T^\alpha = \left( K^\alpha, \{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty} \right)$ is a δ-standard α-fractional singular integral.

Definition 3. We say that an α-fractional singular integral operator $T^\alpha = \left( K^\alpha, \{\eta_{\delta, R}^\alpha\}_{0 < \delta < R < \infty} \right)$ satisfies the norm inequality (2.2) provided

\[ \|T^\alpha_{\sigma, \delta, R} f\|_{L^2(\omega)} \leq \mathcal{R} \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma), 0 < \delta < R < \infty. \]

It turns out that, in the presence of the Muckenhoupt conditions (3.1) below, the norm inequality (2.2) is essentially independent of the choice of truncations used (see e.g. [LaSaShUr3] in dimension $n = 1$), and this is explained in some detail in [SaShUr10]. Thus, as in [SaShUr10], we are free to use the tangent line truncations described there throughout this paper, and in particular we interpret the testing conditions below using the tangent line truncations:

\begin{equation}
\mathcal{E}_{T^\alpha} \equiv \sup_{\text{cubes } I} \left( \frac{1}{|I|} \int_I |T^\alpha_\sigma 1_I|^2 \, d\omega \right) < \infty \quad \text{and} \quad \mathcal{E}_{T^\alpha} \equiv \sup_{\text{cubes } I} \left( \frac{1}{|I|} \int_I |T^\alpha_\sigma 1_I|^2 \, d\sigma \right) < \infty.
\end{equation}
2.2. Strong ellipticity and the positive gradient property. Recall from [SaShUr7] that a standard \(\alpha\)-fractional vector singular integral \(T^\alpha\) on \(\mathbb{R}^n\) with vector kernel \(K^\alpha = (K^\alpha_j)_j\) is strongly elliptic if for each \(m \in \{1, -1\}^n\), there is a sequence of coefficients \(\{\lambda_j^m\}_{j=1}^J\) such that

\[
\left| \sum_{j=1}^J \lambda_j^m K^\alpha_j (x, x + tu) \right| \geq ct^{\alpha-n}, \quad t \in \mathbb{R},
\]

holds for all unit vectors \(u\) in the \(n\)-ant \(V_m\) (i.e. an \(n\)-dimensional quadrant) where

\[
V_m = \{ x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n \}, \quad m \in \{1, -1\}^n.
\]

We now define the positive gradient property. We say that a strongly elliptic standard convolution singular integral \(T^\alpha\) has the \textit{positive gradient property} if in addition there is a finite sequence of closed sectors \(\{S_j\}_{j=1}^M\) such that:

1. We have \(\mathbb{R}^n = \bigcup_{j=1}^M S_j\) and there is a positive constant \(\theta = \theta_{T^\alpha} > 0\) so that each sector \(S_j\) is a rotation \(\Theta_j\) of the unit sector \(S\) of aperture \(\theta\), where

\[
S = \left\{ y \in \mathbb{R}^n : y' = \frac{y}{|y|} \in B_{2^{n-1}}(e_1, \theta) \right\}.
\]

2. For each sector \(S_j = \Theta_j S\) there is a sequence of coefficients \(\{\lambda_j^m\}_{j=1}^J\) such that the scalar convolution kernel \(K(\xi) = \sum_{j=1}^J \lambda_j^m K^\alpha_j(\Theta_j^{-1}\xi)\) satisfies

\[
\frac{K(\xi) - K(\eta)}{\xi_1 - \eta_1} \approx \frac{\xi^{\alpha-n-1}}{\xi_1 - \eta_1}, \quad \text{for } \xi = \left(\xi_1, \xi \right), \eta = (\eta_1, \eta) \in S \text{ with } \frac{\xi - \eta}{\xi_1 - \eta_1} \leq \tan \theta.
\]

It is obvious that the vector Riesz transform \(R^\alpha\) is a strongly elliptic convolution singular integral, and we prove in Lemma [1] below that its negative has the positive gradient property. The rotation invariance of \(R^\alpha\) makes each of these two properties easy to establish.

**Remark 4.** In dimension \(n = 1\) the positive gradient property reduces to \(\frac{d}{dx} K^\alpha(x) \approx \frac{1}{|x|^{\alpha-n}}\), and in the case of the kernel \(K(x) = \frac{1}{x}\) for the Hilbert transform, we actually have equality, \(\frac{d}{dx} K(x) = -\frac{1}{x^2}\), a property exploited extensively in the proof of the NTV conjecture in [LaSaShUr3, Lac].

3. Poisson integrals and Muckenhoupt conditions

Let \(\mu\) be a locally finite positive Borel measure on \(\mathbb{R}^n\), and suppose \(Q\) is a cube in \(\mathbb{R}^n\). Recall that \(|Q| = \ell(Q)^n\) where \(\ell(Q)\) is the side length of a cube \(Q\). The two \(\alpha\)-fractional Poisson integrals of \(\mu\) on a cube \(Q\) are given by the following expressions:

\[
P^\alpha (Q, \mu) = \int_{\mathbb{R}^n} \frac{|Q|^{\frac{\alpha}{2}}}{\left(|Q|^{\frac{\alpha}{2}} + |x - cQ|\right)^{n+1-\alpha}} d\mu(x),
\]

\[
\mathcal{P}^\alpha (Q, \mu) = \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{\alpha}{2}}}{\left(|Q|^{\frac{\alpha}{2}} + |x - cQ|\right)^2}\right)^{n-\alpha} d\mu(x),
\]

where \(|x - cQ|\) denotes distance between \(x\) and the center \(cQ\) of \(Q\), and \(|Q|\) denotes the Lebesgue measure of \(Q\). We refer to \(P^\alpha\) as the \textit{standard} Poisson integral and to \(\mathcal{P}^\alpha\) as the \textit{reproducing} Poisson integral. Note that for \(n - 1 \leq \alpha < n\), these two kernels satisfy

\[
P^\alpha (Q, \mu) \leq \mathcal{P}^\alpha (Q, \mu), \quad \text{for all intervals } Q \text{ and positive measures } \mu,
\]

and that the inequality is reversed for \(0 < \alpha \leq n - 1\).

We now define the \textit{one-tailed} \(A_\alpha^p\) constant using \(P^\alpha\). The energy constants \(\mathcal{E}_\alpha\) introduced in the next section will use the standard Poisson integral \(P^\alpha\). We denote the collection of cubes in \(\mathbb{R}^n\) with edges parallel to the coordinate axes by \(\mathcal{P}^\alpha\) (not to be confused with the Poisson integral \(\mathcal{P}^\alpha\)).
Definition 5. The one-sided constants $A_2^\alpha$ and $A_2^{\alpha,*}$ for the weight pair $(\sigma, \omega)$ are given by

$$
(3.1) \quad A_2^\alpha &= \sup_{Q \in \mathcal{P}_n} \mathcal{P}^\alpha(Q, 1_{Q'}) \frac{|Q|_\omega}{|Q|_\sigma^{1/2}} < \infty, \\
A_2^{\alpha,*} &= \sup_{Q \in \mathcal{P}_n} \mathcal{P}^\alpha(Q, 1_{Q'}) \frac{|Q|_\sigma}{|Q|_\omega^{1/2}} < \infty.
$$

Note that these definitions are the analogues of the corresponding conditions with ‘holes’ introduced by Hytönen [Hyt2] in dimension $n = 1$ - the supports of the measures $1_{Q'}\sigma$ and $1_{Q'}\omega$ in the definition of $A_2^\alpha$ are disjoint, and so the common point masses of $\sigma$ and $\omega$ do not appear simultaneously in each factor.

4. STRONG, DEEP AND BOUNDED OVERLAP ENERGY CONSTANTS

We begin with the strong energy constants (see e.g. [LaSaUr2] and [SaShUr7]).

Definition 6. Let $0 \leq \alpha < n$. Suppose $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}^n$. Then the strong energy constant $E_2^\alpha$ is defined by

$$
(4.1) \quad (E_2^\alpha)^2 &\equiv \sup_{I = I_r} \frac{1}{|I|} \sum_{\ell = 1}^\infty \left( \frac{\mathcal{P}^\alpha(I, 1_{I\sigma})}{|I|} \right)^2 \|x - m_I\|_{\widetilde{L}^2(1_{I, \sigma})}^2,
$$

where the supremum is taken over arbitrary decompositions of a cube $I$ using a pairwise disjoint union of subcubes $I_r$. Similarly, we define the dual strong energy constant $E_2^{\alpha,*}$ by switching the roles of $\sigma$ and $\omega$:

$$
(4.2) \quad (E_2^{\alpha,*})^2 &\equiv \sup_{I = I_r} \frac{1}{|I|} \sum_{\ell = 1}^\infty \left( \frac{\mathcal{P}^\alpha(I, 1_{I\omega})}{|I|} \right)^2 \|x - m_I\|_{\widetilde{L}^2(1_{I, \omega})}^2.
$$

In order to define the weaker notions of deep and bounded overlap energy constants, we must introduce additional notation. We say that a dyadic cube $J$ is $(r, \varepsilon)$-deeply embedded in a (not necessarily dyadic) quasicube $K$, which we write as $J \Subset_{r, \varepsilon} K$, when $J \subset K$ and both

$$
(4.3) \quad \ell(J) &\leq 2^{-r}\ell(K), \\
\text{dist}(J, \partial K) &\geq \frac{1}{2}\ell(J)^\varepsilon \ell(K)^{1-\varepsilon},
$$

Recall the collection

$$
\mathcal{M}_{(r, \varepsilon)\text{-deep}}(K) \equiv \{ \text{maximal } J \Subset_{r, \varepsilon} K \}
$$

of maximal $(r, \varepsilon)$-deeply embedded dyadic subcubes of a cube $K$ (a subcube $J$ of $K$ is a dyadic subcube of $K$ if $J \in \mathcal{D}$ when $\mathcal{D}$ is a dyadic grid containing $K$). This collection of dyadic subcubes of $K$ is of course a pairwise disjoint decomposition of $K$. Recall also the refinement and extension of the collection $\mathcal{M}_{(r, \varepsilon)\text{-deep}}(K)$ given in [SaShUr7] for certain $K$ and each $\ell \geq 1$ (where $\pi^tK'$ denotes the $t^{th}$ ancestor of $K'$ in the grid):

$$
\mathcal{M}_{(r, \varepsilon)\text{-deep}}'(K) \equiv \{ J \in \mathcal{M}_{(r, \varepsilon)\text{-deep}}(\pi^tK') \text{ for some } K' \in \mathcal{C}_{\mathcal{D}}(K) : J \subset L \text{ for some } L \in \mathcal{M}_{(r, \varepsilon)\text{-deep}}(K) \},
$$

where $\mathcal{C}_{\mathcal{D}}(K)$ is the set of $\mathcal{D}$-dyadic children of $K$. Thus $\mathcal{M}_{(r, \varepsilon)\text{-deep}}'(K)$ is the union, over all children $K'$ of $K$, of those cubes in $\mathcal{M}_{(r, \varepsilon)\text{-deep}}(\pi^tK')$ that happen to be contained in some $L \in \mathcal{M}_{(r, \varepsilon)\text{-deep}}(K)$. These collections of cubes satisfy the bounded overlap property (see e.g. [SaShUr7]),

$$
\sum_{J \in \mathcal{M}_{(r, \varepsilon)\text{-deep}}'(K)} 1_J \leq \beta 1_K, \quad \text{for each } \ell \geq 1.
$$

Finally, let $P_M^\omega \equiv \sum_{J \in \mathcal{D}} 1_{\mathcal{M}_{(r, \varepsilon)\text{-deep}}'(K)}$ be Haar projection onto the subspace of $L^2(\omega)$ consisting of those functions $f \in L^2(\omega)$ supported in $M$ with $\int_M f \, d\omega = 0$ - see e.g. [SaShUr7] for more detail on Haar expansions in $L^2(\omega)$. 
Definition 7. Suppose $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}^n$ and fix $\gamma > 1$. Then the deep energy condition constant $\mathcal{E}_{n,\text{deep}}^\gamma$ is given by

$$
\left( \mathcal{E}_{n,\text{deep}}^\gamma \right)^2 \equiv \sup_{D} \sup_{I=\bigcup I_r} \frac{1}{|I|^\gamma} \sum_{r=1}^\infty \sum_{M \in \mathcal{M}(I_r) - \text{deep}} \left( \frac{P^\alpha (M, 1_{I_r \cap \gamma M \sigma})}{|M|} \right)^2 \left\| \mathcal{P} \omega M x \right\|_{L^2(\omega)}^2,
$$

where $\sup_D \sup_{I=\bigcup I_r}$ is taken over

1. all dyadic grids $D$,
2. all $D$-dyadic cubes $I$,
3. and all subpartitions $\{I_r\}_{r=1}^N$ or $\infty$ of the cube $I$ into $D$-dyadic subcubes $I_r$.

The exact value of $\gamma > 1$ above is not too important in general, but when we wish to emphasize the value of $\gamma$, we will refer to $\mathcal{E}_{n,\text{deep}}^\gamma$ as the $\gamma$-deep energy condition constant.

Note that we could also define a slightly less restrictive notion of energy condition as in [LaWi] by taking the supremum over $I = \bigcup I_r$ for which there is bounded overlap of the expansions $\gamma I_r$.

Later, in Part 2 of the paper, we will have reason to consider the corresponding forward (bounded overlap) pivotal constant, which is defined by replacing $\left\| \mathcal{P} \omega M x \right\|_{L^2(\omega)}^2$ with its upper bound $|I_r|^{\frac{\theta}{\theta + 1}} |I_r|_\omega$ in (4.4):

$$(4.4) \quad \left( \mathcal{E}_{n,\text{overlap}}^\gamma \right)^2 \equiv \sup_{D} \sup_{I=\bigcup I_r} \frac{1}{|I|^\gamma} \sum_{r=1}^\infty \sum_{M \in \mathcal{M}(I_r) - \text{overlap}} \left( \frac{P^\alpha (I_r, 1_{I_r \cap \gamma M \sigma})}{|I_r|^{\frac{\theta}{\theta + 1}}} \right)^2 \left\| \mathcal{P} \omega I_r x \right\|_{L^2(\omega)}^2,$$

and we refer to finiteness of $\mathcal{E}_{n,\text{overlap}}^\gamma$ as the bounded overlap energy condition, or more precisely as the $\gamma$-overlap energy condition when we want to emphasize the choice of $\gamma$.

Later, in Part 2 of the paper, we will have reason to consider the corresponding forward (bounded overlap) pivotal constant, which is defined by replacing $\left\| \mathcal{P} \omega M x \right\|_{L^2(\omega)}^2$ with its upper bound $|I_r|^{\frac{\theta}{\theta + 1}} |I_r|_\omega$ in (4.4):

$$(4.5) \quad \left( \mathcal{V}_{n,\text{overlap}}^\gamma \right)^2 \equiv \sup_{D} \sup_{I=\bigcup I_r} \frac{1}{|I|^\gamma} \sum_{r=1}^\infty \sum_{M \in \mathcal{M}(I_r) - \text{overlap}} \left( \frac{P^\alpha (I_r, 1_{I_r \cap \gamma M \sigma})}{|I_r|^{\frac{\theta}{\theta + 1}}} \right)^2 |I_r|_\omega.$$

The backward pivotal constant $\mathcal{V}_{n,\text{overlap}}^\gamma$ is defined by interchanging the roles of the measures $\sigma$ and $\omega$.

5. Twisted localizations and necessity of the deep energy condition

Let $\{\Theta_j \}_{j=1}^M$ be a finite set of rotations such that $\mathbb{R}^n = \bigcup_{j=1}^M \Theta_j \hat{Q}$ where $\hat{Q}$ is the sector centered on the positive $x_1$-axis with aperture angle $\theta > 0$ as in the positive gradient property for the strongly elliptic convolution singular integral $T^\alpha$ in $\mathbb{R}^n$. Our goal here is to prove the (forward) $\gamma$-overlap energy condition with constant

$$\mathcal{E}_{n,\text{overlap}}^\gamma \lesssim \sup_{J} \sup_{1 \leq i,j \leq N} \sigma_{\Theta_j \Theta_j} + \mathcal{A}_2^\alpha,$$

where $\{\Theta_j, T^\alpha_{j} \Theta_j\}$ is a family of standard fractional singular integrals associated with $T^\alpha$. More precisely we will show

$$\sum_{r=1}^\infty \left( \frac{P^\alpha (J, 1_{I_r \cap \gamma M \sigma})}{|I_r|^{\frac{\theta}{\theta + 1}}} \right)^2 \left\| \mathcal{P} \omega J x \right\|_{L^2(\omega)}^2 \leq \left( \sup_{J} \sup_{1 \leq i,j \leq N} \sigma_{\Theta_j \Theta_j} \right)^2 + \beta \mathcal{A}_2^\alpha |I_r|_\sigma,$$

for all partitions of a dyadic cube $I = \bigcup_{r \geq 1} I_r$ into subcubes $I_r$ with $\sum_{r=1}^\infty 1_{I_r} \lesssim \beta 1$. We now turn to defining the twisted localizations $\Theta_j T^\alpha_{j} \Theta_j$ appearing on the right hand side of the inequality displayed above.

Let $Q \equiv [-\frac{1}{2}, \frac{1}{2}]^n$ be the unit cube of side length 1 centered at the origin, and let

$$\hat{Q} \equiv \left\{ y = (y_1, y) \in (\mathbb{R} \times \mathbb{R}^{n-1}) \setminus \gamma Q : |y| \leq \lambda |y_1| \right\}$$

$$\equiv \left\{ y \in \mathbb{R}^n \setminus \gamma Q : y' = \frac{y}{|y|} \in B_{2^{\gamma - 1}}(e_1, \theta) \right\}.$$
be the unit truncated sector of separation $\gamma$ and aperture $\theta = \arctan \lambda$ for $\gamma > 1$ and $\lambda > 0$ chosen as needed below. Let $\varphi$ be a smooth bump function that equals 1 on $Q$ and vanishes off an appropriate $\rho$-expansion

$$\hat{Q}^* = \left\{ y \in \mathbb{R}^n \setminus \frac{\gamma}{\rho} Q : y' = \frac{y}{|y|} \in \rho B_{|S|-1} (e_1, \theta) \right\},$$

where $1 < \rho < \gamma$, and such that

$$|\nabla \varphi(y)| \leq 1 \text{ and } |\nabla \varphi(y)| \leq C_\varphi |y|^{-1}, \quad y \in \mathbb{R}^n.$$

In particular, we can choose the bump function $\varphi$ so that the localized kernel $1_J (x) K^\alpha (x, y) \varphi (y)$ satisfies a one-sided Calderón-Zygmund condition, in which there is smoothness only in the $y$-variable. See below.

We also define such a bump function for each ‘rotated’ sector

$$\hat{Q}^j \equiv \left\{ y \in \mathbb{R}^n \setminus \gamma Q : y' = \frac{y}{|y|} \in B_{|S|-1} (\Theta_j e_1, \theta) \right\},$$

which with a small abuse of notation we denote by $\Theta_j \hat{Q}$, despite the fact that $\hat{Q}^j$ is not exactly a rotation of $Q$. But since the cube $Q$ is not rotation invariant, we cannot simply take a rotation of $\varphi$. Thus for each $1 \leq j \leq M$, we choose a bump function $\varphi^j$ that is equals 1 on the sector $\Theta_j \hat{Q} = \hat{Q}^j$ and is supported in the $\rho$-expansion of the sector,

$$\left\{ y \in \mathbb{R}^n \setminus \frac{\gamma}{\rho} Q : y' = \frac{y}{|y|} \in \rho B_{|S|-1} (\Theta_j e_1, \theta) \right\},$$

and satisfies appropriate estimates. To avoid clutter of notation, we will typically suppress the superscript $j$ and simply write $\varphi$ for each of these bump functions $\varphi_1, \ldots, \varphi_M$.

**Lemma 8.** With notation as above,

$$1_Q (x) K^\alpha (x, y) \varphi (y) \leq \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n},$$

$$\| \nabla_1 1_Q (x) K^\alpha (x, y) \varphi (y) \| \leq \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n - 1},$$

$$\| \nabla_2 1_Q (x) K^\alpha (x, y) \varphi (y) \| \leq \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n - 2}.$$

**Proof.** We trivially have the first line in (5.1),

$$1_Q (x) K^\alpha (x, y) \varphi (y) \leq |K^\alpha (x, y)| \leq \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n}.$$

If in addition $1_Q (x) \varphi (y) \neq 0$, then

$$3\sqrt{n} |y| \leq |x - y| \leq 5\sqrt{n} |y|,$$

and so

$$|\nabla_1 1_Q (x) K^\alpha (x, y) \varphi (y)| \leq |\nabla_1 K^\alpha (x, y)| |\varphi (y)| + |K^\alpha (x, y)| |\nabla_1 \varphi (y)| \leq \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n - 1} + \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n} |\nabla \varphi (y)| \leq \| K^\alpha \|_{CZ_\alpha} |1 + (5\sqrt{n})^{\alpha - \alpha} C_\varphi| |x - y|^{\alpha - n - 1} \lesssim \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n - 2},$$

Similarly

$$|\nabla_2 1_Q (x) K^\alpha (x, y) \varphi (y)| \lesssim \| K^\alpha \|_{CZ_\alpha} |x - y|^{\alpha - n - 2}. \quad \Box$$

Thus the localized kernel $1_Q (x) K^\alpha (x, y) \varphi (y)$ satisfies Calderón-Zygmund smoothness in the $y$-variable, but it fails to satisfy Calderón-Zygmund smoothness in the $x$-variable. This unfortunate omission diminishes the significance of the derivation of energy from localized families, but does help somewhat to narrow the focus on difficulties in obtaining necessity of energy from boundedness of families of operators.

### 5.1. Family of localizations of an operator

For any $\alpha$-fractional singular integral operator $T^\alpha$ with kernel $K^\alpha (x, y)$, and any cube $J$ with center $c_J$ and side length $\ell (J)$, we consider the vector operator $T^\alpha_J$ with kernel

$$K^\alpha_J (x, y) \equiv 1_J (x) \ K^\alpha (x, y) \ \varphi_J (y);$$

$$\varphi_J (y) = \varphi \left( \frac{y - c_J}{\ell (J)} \right),$$
which we refer to as a localization of \( T^\alpha \) to the cube \( J \) and sector \( \mathring{J} \), where \( \mathring{J} = \delta_{\ell(J)} \hat{Q} + c_J \) is the dilate by \( \ell \) \((J)\) and translate by \( c_J \) of the unit sector \( \hat{Q} \) with aperture \( \theta \) defined above.

Now we define the operator \( T^\alpha j^\Theta_j^{-1} \) with kernel
\[
K^\alpha_j \Theta_j^{-1}(x, y) = 1_J(x) K^\alpha(x, y) \varphi_j (\Theta_j^{-1} y),
\]
but where we must of course use \( \varphi_j \) in place of \( \varphi_j \) for each \( 1 \leq j \leq M \), since cubes are not invariant under rotations. As mentioned earlier, we will typically suppress the superscript \( j \) here. This operator \( T^\alpha j^\Theta_j^{-1} \) is referred to as a localization of \( T^\alpha \) to the cube \( J \) and sector \( \Theta_j^\ell J \), where \( \Theta_j^\ell J = \delta_{\ell(J)} \Theta_j^\ell \hat{Q} + c_J \) is the dilate by \( \ell \) \((J)\) and translate by \( c_J \) of the ‘rotation’ \( \Theta_j^\ell \hat{Q} \) of the unit sector \( \hat{Q} \) with aperture \( \theta \) (we say ‘rotation’ despite the fact that this is only approximately true).

Now let \( J = \{ J_k \}_{k=1}^\infty \) be a sequence of pairwise disjoint subcubes of a cube \( I \) satisfying the bounded overlap condition,
\[
(5.2) \quad \sum_{r=1}^\infty 1_{\gamma_{lr}} \lesssim \beta 1_I,
\]
and define the vector operators
\[
T^\alpha_j = \sum_{k=1}^\infty T^\alpha_{J_k}, \quad T^\alpha_j \Theta_j^{-1} = \sum_{k=1}^\infty T^\alpha_{J_k} \Theta_j^{-1},
\]
to have kernels
\[
(5.3) \quad K^\alpha_j (x, y) = \sum_{k=1}^\infty 1_{J_k}(x) K^\alpha(x, y) \varphi_{J_k}(y),
\]
\[
K^\alpha_j \Theta_j^{-1}(x, y) = \sum_{k=1}^\infty 1_{J_k}(x) K^\alpha(x, y) \varphi_{J_k}(\Theta_j^{-1} y).
\]
respectively. Here, for any cube \( J \),
\[
\Theta_j^\ell(x) = \Theta_j(x - c_J) + c_J, \quad x \in \mathbb{R}^n,
\]
is the conjugation by translation by \( c_J \) of the rotation \( \Theta_j \), resulting in a rotation about the point \( c_J \). We refer to the operator \( T^\alpha j^\Theta_j^{-1} \) as the localization rotated by \( \Theta_j \) of \( T^\alpha \) to the collection \( J \). Denote by \( \mathring{J} \), the infinite family of such collections of cubes, namely those collections \( J \) of pairwise disjoint subcubes of a cube \( I \), whose expansions have bounded overlap \((5.2)\). The corresponding infinite family of operators \( \{ T^\alpha_j \Theta_j^{-1} \}_{J \in \mathring{J} \text{ and } 1 \leq j \leq M} \) taken over all cubes \( I \) and decompositions \( J \) satisfying \((5.2)\) and all \( 1 \leq j \leq M \), is called the family of localizations of the operator \( T^\alpha \). The kernels \( \{ K^\alpha_j \Theta_j^{-1} \}_{J \in \mathring{J} \text{ and } 1 \leq j \leq M} \) uniformly satisfy a one-sided Calderón-Zygmund condition (in the \( y \)-variable only).

**Lemma 9.** Let \( K^\alpha_j \Theta_j^{-1} \) be as in the second line of \((5.3)\). Then for all \( J \in \mathring{J} \) and \( 1 \leq j \leq M \) we have
\[
(5.4) \quad |K^\alpha_j \Theta_j^{-1}(x, y)| \lesssim \|K^\alpha\|_{CZ_n} |x - y|^{\alpha - n},
\]
\[
|\nabla^\ell \Theta_j^\ell K^\alpha_j \Theta_j^{-1}(x, y)| \lesssim \|K^\alpha\|_{CZ_n} |x - y|^{\alpha - n - \ell}, \quad \ell = 1, 2.
\]

**Proof.** The first line in \((5.4)\) is automatic since the cubes \( J_k \) are pairwise disjoint:
\[
|K^\alpha_j \Theta_j^{-1}(x, y)| = \sum_{k=1}^\infty 1_{J_k}(x) K^\alpha(x, y) \varphi_{J_k}(y) \leq |K^\alpha(x, y)| \leq \|K^\alpha\|_{CZ_n} |x - y|^{\alpha - n}.
\]
Now note that

\[ |\nabla_y \varphi_J(y)| = |\nabla_y \left[ \varphi \left( \frac{y - c_J}{\ell(y)} \right) \right]| = \left| \nabla \varphi \left( \frac{y - c_J}{\ell(y)} \right) \right| \frac{1}{\ell(y)} \leq C |\varphi' \left( \frac{y - c_J}{\ell(y)} \right)| \frac{1}{\ell(y)} = C |y - c_J|^{-1} \cdot \]

For the second line we may suppose without loss of generality that \( j = 1 \) so that \( (\Theta^J_k)^{-1} \) is the identity rotation about \( c_{J_0} \), i.e. the identity map, and thus \( K^\alpha_j \Theta_1^{-1} = K^\alpha_j \). If \( K^\alpha_j (x, y) \neq 0 \), then \( x \in J_k \) for a unique \( k \geq 1 \) and

\[ 3 \sqrt{n} |y - c_{J_k}| \leq |x - y| \leq 5 \sqrt{n} |y - c_{J_k}|, \]

and then we have

\[ |\nabla_y K^\alpha_j \Theta_1^{-1}(x, y)| = |1_{J_k}(x) \nabla_y \{ K^\alpha_j(x, y) \hat{\varphi}_{J_k}(y) \}| \leq |\nabla_y K^\alpha_j(x, y)| |\hat{\varphi}_{J_k}(y)| + |\nabla_y \varphi_{J_k}(y)| \leq \|K^\alpha_j\|_{CZ_n} |x - y|^{\alpha - n - 1} + \|K^\alpha_j\|_{CZ_n} |x - y|^{\alpha - n} C_\varphi |y - c_{J_k}|^{-1} \]

Similarly we have \( |\nabla_y K^\alpha_j \Theta_1^{-1}(x, y)| \leq \|K^\alpha_j\|_{CZ_n} |x - y|^{\alpha - n - 2} \). \( \square \)

\section*{5.2. Family of twisted localizations.}

In order to derive the deep energy condition, it is not enough to assume the uniform boundedness of the family \( \{T^\alpha_j \Theta_1^{-1}\}_{j \in \mathbb{N}} \) of localizations of \( T^\alpha \), see Remark \( \text{[4]} \) at the end of the paper, but rather we must assume uniform boundedness of the larger family \( \{\Theta_i T^\alpha_j \Theta_1^{-1}\}_{j \in \mathbb{N}} \) of twisted localizations of \( T^\alpha \) given by

\[ (5.5) \quad [\Theta_i T^\alpha_j \Theta_1^{-1}] f(x) = \int \Theta_i K^\alpha_j \Theta_1^{-1}(x, y) f(y) d\sigma(y), \]

where we have pre-rotated the kernel by a rotation \( \Theta_i^J \) centered at \( c_J \), and post-rotated the kernel by a rotation \( \Theta_i^J \) centered at \( c_J \). For a single cube \( J \), we refer to \( \Theta_i T^\alpha_j \Theta_1^{-1} \) as a twisted localization of \( T^\alpha \) to the cube \( J \) and sector \( \tilde{J} = c_J + \Theta_i \tilde{Q} \), which is twisted by the post-rotation \( \Theta_i \). For a collection of cubes \( J \in \mathcal{J} \), we refer to the infinite sum \( \Theta_i T^\alpha_j \Theta_1^{-1} = \sum_{J \in \mathcal{J}} \Theta_i T^\alpha_j \Theta_1^{-1} \) as a twisted localization of \( T^\alpha \) to the collection of cubes \( \mathcal{J} \). Finally, we then refer to the family of operators \( \{\Theta_i T^\alpha_j \Theta_1^{-1}\}_{J \in \mathcal{J} \text{ and } 1 \leq i, j \leq M} \) as the family of twisted localizations of the operator \( T^\alpha \). Again, using \( |\Theta_i^J(x - c_J) = |x - c_J| \) together with the argument for the localized kernels \( K^\alpha_j \Theta_1^{-1} \) in the proof of Lemma \( \text{[3]} \) above, it is easy to obtain a one-sided Calderón-Zygmund kernel estimate for the twisted localizations.

**Lemma 10.** Let \( \Theta_i T^\alpha_j \Theta_1^{-1} \) be as in the second line of (5.5). Then

\[ |\Theta_i T^\alpha_j \Theta_1^{-1}(x, y)| \leq \|K^\alpha_j\|_{CZ_n} |x - y|^{\alpha - n}, \]

\[ |\nabla_y (\Theta_i T^\alpha_j \Theta_1^{-1})(x, y)| \leq \|K^\alpha_j\|_{CZ_n} |x - y|^{\alpha - n - \ell}, \quad \ell = 1, 2. \]

In applications to the necessity of the strong energies \( \mathcal{E}^\alpha_2 \) and \( \mathcal{E}^\alpha_2^{\ast, \ast} \) in Definition \( \text{[6]} \) one would take \( \mathcal{J} = \{I_r\}_{r=1}^\infty \).

Even more generally, given a sequence \( \mathcal{J} = \{J_k\}_{k=1}^\infty \) of pairwise disjoint subcubes of a cube \( I \) satisfying (5.2), and a choice of pre- and post-rotations \( \Theta_i^J \equiv \{\Theta_i^J_k\}_{k=1}^\infty \) and \( \Theta_i^J \equiv \{\Theta_i^J_k\}_{k=1}^\infty \), we define the vector operator

\[ \tilde{\Theta}_i^J \Theta_i^J \equiv \sum_{j=1}^\infty \{\Theta_i^J \Theta_i^J_j \} f, \quad 1 \leq i, j \leq M, \]
which has kernel

\[ \tilde{\Theta}_{\text{pre}} K_{\sigma}^\alpha \tilde{\Theta}_{\text{post}} (x, y) = \sum_{k=1}^{\infty} \varphi_{J_k} (x) \; K^\alpha \left( \Theta_{J_k}^k x, y \right) \varphi_{J} \left( \left( \Theta_{J_k}^k \right)^{-1} y \right), \]

whose rotations now vary with the subcube \( J_k \). We will show that for appropriate operators \( T^\alpha \), including the Riesz transform vector \( R^\alpha \), we can actually use reversal of energy for the single operator \( \tilde{\Theta}_{\text{pre}} T_{\sigma}^J \tilde{\Theta}_{\text{post}} \) to deduce the single inequality

\[ \sum_{k=1}^{\infty} \frac{\beta^\alpha (J_k, I_\sigma)}{|J_k|^2} \left\| \mathcal{P}_{J_k} x \right\|_{L^2 (\omega)}^2 \leq \left( \left( \mathcal{S}_{\tilde{\Theta}_{\text{pre}} T_{\sigma}^J \tilde{\Theta}_{\text{post}}} \right)^2 + \mathcal{A}_2^\alpha \right) |I_\sigma|, \]

when \( J \) is taken to be \( \{ J_k \}_{k=1}^{\infty} \), and \( \tilde{\Theta}_{\text{pre}} \) and \( \tilde{\Theta}_{\text{post}} \) are chosen appropriately depending on \( \sigma \) and \( \omega \) respectively.

5.3. **Reversal of energy.** Fix a cube \( J_k \) and indices \( 1 \leq i, j \leq M \). Let \( \theta = \Theta J^\alpha \) be the angle in the positive gradient property for the operator \( T^\alpha \), and set \( \lambda = \tan \theta \). Then set \( B_{S_{n-1}} \equiv B_{S_{n-1}} (e_1, \theta) \) and take \( x, z \in J_k \) with \( x - z \in \Theta_{i}^{-1} \Theta J, B_{S_{n-1}} \) so that

\[ \Theta_{J_k}^k x - \Theta_{J_k}^k z = \Theta_i (x - z) \in \Theta J, B_{S_{n-1}}, \]

\[ \Theta_{J_k}^k x - y \quad \text{and} \quad \Theta_{J_k}^k z - y \in \Theta J, B_{S_{n-1}}. \]

Let \( p = \Theta_{J_k}^k x \) and \( q = \Theta_{J_k}^k z \). Without loss of generality we can take \( \Theta J = Id \) the identity for this argument. Then for \( x = (x^1, \bar{x}) \) and \( z = (z^1, \bar{z}) \) in \( J_k \) with \( |x - z| \leq \lambda |x_1 - z_1| \) (equivalently \( \bar{x} = \bar{z} \in B_{S_{n-1}} \)), we claim the following ‘strong reversal’ of energy. Since \( 1_{J_k} (x) = 1 = 1_{J_k} (z) \), we can compute

\[ \frac{\left[ (\Theta, T^\alpha_{J_k} \Theta)_{\sigma} \right]}{\Theta_i (x - z)_{1}} 1_{I} (x) - \left[ (\Theta, T^\alpha_{J_k} \Theta)_{\sigma} \right] 1_{I} (z) \]

\[ \int \left\{ \left( \Theta_{J_k}^k (p, q, y) - 1_{J_k} (z) K^\alpha (q, y) \right) \varphi \left( y - \frac{c_{J_k}}{\ell (J_k)} (y) \right) 1_{I} (y) \right\} \; \sigma (y), \]

and since \( K^\alpha_i \) is a convolution operator, the term in braces satisfies

\[ K^\alpha_i (p - y) - K^\alpha_i (q - y) = \frac{K^\alpha_i (p_1 - y_1, \bar{p} - \bar{y}) - K^\alpha_i (q_1 - y_1, \bar{q} - \bar{y})}{p_1 - q_1} \]

\[ \frac{K^\alpha_i (s, \bar{p} - \bar{y}) - K^\alpha_i (t, \bar{q} - \bar{y})}{s - t} \]

with \( y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \), \( s = p_1 - y_1 \) and \( t = q_1 - y_1 \). Here \( K^\alpha_i (\xi) = K^\alpha_i (\xi_1, \xi) \) is the first component of the convolution kernel \( K^\alpha \) for \( \xi = (\xi_1, \xi) \).

Now we invoke the positive gradient property of \( K^\alpha_i \):

\[ \frac{K^\alpha_i (\xi_1, \xi_1) - K^\alpha_i (\eta_1, \eta_1)}{\xi_1 - \eta_1} \approx |\xi|^{\alpha - n-1}, \quad \text{for } \xi, \eta \in S \text{ with } |\xi_1 - \eta_1| \leq \lambda. \]

In particular, we then have

\[ \frac{K^\alpha_i (s, \bar{p} - \bar{y}) - K^\alpha_i (t, \bar{q} - \bar{y})}{s - t} \approx |(s, \bar{p} - \bar{y})|^{\alpha - n-1} \approx |c_{J_k} - y|^{\alpha - n-1}, \]

since

\[ \bar{p} - \bar{y} = \Theta_{J_k}^k x - \bar{y} = \Theta_i (\bar{x} - c_{J_k}) + \bar{c}_{J_k} - \bar{y} \]
satisfies
\[ \| \bar{p} - \bar{y} \| \leq \| \Theta_i (\hat{c} - c_{J_k}) \| + \| \hat{c}_{J_k} - \bar{y} \| \]
\[ \leq \| x - c_{J_k} \| + \| \hat{c}_J - \bar{y} \| \lesssim \lambda |p_1 - y_1| , \]
and similarly \( \| \bar{q} - \bar{y} \| \lesssim \lambda |p_1 - y_1| . \)
Thus we have
\[ \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (x) - \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (z) \]
\[ = \frac{K^n (s, \bar{p} - \bar{y}) - K^n (t, \bar{q} - \bar{y})}{s - t} \approx |c_{J_k} - y|^{n-1} , \]
and so in general,
\[ \left| \frac{\left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (x) - \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (z)}{\Theta_i (x-z)} \right| \lesssim \frac{P^n \left( J_k, 1_{\omega_\gamma \sigma} \right)}{|J_k|^{\frac{n}{n-1}}} , \]
for \( x, z \in J_k \) with \( \frac{x-z}{|x-z|} \in \Theta_i B_{\mathbb{R}^{n-1}} \).
Thus with \( \Phi_i = \left\{ (x, z) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{x-z}{|x-z|} \in \Theta_i B_{\mathbb{R}^{n-1}} \right\} \) and
\[ F_i (J_k, \omega) \equiv \frac{1}{|J_k|} \int \int_{J_k \times J_k \cap \Phi_i} |x-z|^2 \, d\omega (x) \, d\omega (z) , \]
we have
\[ \frac{1}{|J_k|} \int \int_{J_k \times J_k \cap \Phi_i} |x-z|^2 \, d\omega (x) \, d\omega (z) = \sum_{i=1}^{N} F_i (J_k, \omega)^2 \]
and so
\[ \sum_{k=1}^{\infty} \left( \frac{P^n \left( J_k, 1_{c_{J_k} + \Theta_j \hat{c}} 1_f 1_{\gamma \sigma} \right)}{|J_k|^{\frac{n}{n-1}}} \right)^2 \left( \frac{1}{|J_k|} \int \int_{J_k \times J_k} |x-z|^2 \, d\omega (x) \, d\omega (z) \right) \]
\[ = \sum_{i=1}^{N} \sum_{k=1}^{\infty} \left( \frac{P^n \left( J_k, 1_{c_{J_k} + \Theta_j \hat{c}} 1_f 1_{\gamma \sigma} \right)}{|J_k|^{\frac{n}{n-1}}} \right)^2 F_i (J_k, \omega)^2 \]
\[ \lesssim \sum_{i=1}^{N} \sum_{k=1}^{\infty} \frac{1}{|J_k|} \int \int_{J_k \times J_k \cap \Phi_i} \left| \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (x) - \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (z) \right|^2 \]
\[ \lesssim \sum_{i=1}^{N} \sum_{k=1}^{\infty} \int \left| \left[ \Theta_i T^\gamma_{J_k} \Theta_j \right]_1 1_f (x) \right|^2 \lesssim \sup_{1 \leq \gamma \leq N} \left( x_{\Theta_i}, T^\gamma_{\Theta_j} \right)^2 |I_{\gamma \sigma}| . \]
Finally then using \( 1_{I \cap J_k} \leq \sum_{j=1}^{M} 1_{c_{J_k} + \Theta_j \hat{c}} 1_f \), we have
\[ \sum_{k=1}^{\infty} \left( \frac{P^n \left( J_k, 1_{I \cap J_k} \right)}{|J_k|} \right)^2 \| P^n_{1_{I \cap J_k}} \|_{L^2 (\omega)}^2 \]
\[ \lesssim \sup_{1 \leq \gamma \leq N} \sum_{k=1}^{\infty} \left( \frac{P^n \left( J_k, 1_{c_{J_k} + \Theta_j \hat{c}} 1_f \right)}{|J_k|} \right)^2 \left( \frac{1}{|J_k|} \int \int_{J_k \times J_k} |x-z|^2 \, d\omega (x) \, d\omega (z) \right) \]
\[ \lesssim \sup_{1 \leq i, j \leq N} \left( x_{\Theta_i}, T^\gamma_{\Theta_j} \right)^2 |I_{\gamma \sigma}| . \]
which proves the forward deep and bounded overlap energy conditions with
\[ \mathcal{E}_2^{\alpha, \text{deep}} \leq \mathcal{E}_2^{\alpha, \text{overlap}} \lesssim \sup_{J} \sup_{1 \leq i,j \leq N} \mathcal{S}_{\Theta_i \Theta_j}, \]
where the supremum in \( J \) is taken over all sequences \( \{J_k\}_{k=1}^{\infty} \) of subcubes of \( I \) such that \( \sum_{k=1}^{\infty} 1_{J_k} \leq \beta 1_I \).
Indeed, we have
\[ \sup_{I \supseteq 0 \subsetneq I'} \sup_{r \geq 0} \sum_{I \in M_{\text{deep}}(I')} \left( \frac{\mathcal{P}_n \langle J, 1_{I} \sigma \rangle}{|J|^\frac{1}{2}} \right)^2 \| P_{J}^{n} \mathbf{x} \|_{L^2(\omega)}^2 \leq \left\{ \sup_{J} \sup_{1 \leq i,j \leq N} \left( \mathcal{S}_{\Theta_i \Theta_j} \right)^2 + \beta A_2^\alpha \right\} |I|_{\sigma}, \]
after writing \( 1_I = 1_{I_{\gamma,j}} + 1_{\gamma,J} \), and similarly for the bounded overlap energy condition. Thus we see that the deep and bounded overlap energy constants \( \mathcal{E}_2^{\alpha, \text{deep}} \) and \( \mathcal{E}_2^{\alpha, \text{overlap}} \) are controlled by the testing constants \( \mathcal{S}_{\Theta_i \Theta_j} \), for the family \( \{\Theta_i \Theta_j \}_{i,j} \) of twisted localizations of an operator \( T^\alpha \) with the positive gradient property. Proposition 2 is now proved save for the assertion regarding the Riesz transform, to which we now turn.

5.3.1. Positive gradient property of the Riesz transform. Finally we establish the positive gradient property for the negative of the vector Riesz transform \( R^{\alpha,n} \) with kernel \( K^{\alpha,n} \).

**Lemma 11.** The operator \(-R^{\alpha,n}\) has the positive gradient property.

**Proof.** For this we compute the gradient of the first component \( K^{\alpha,n}_1(u, w) \) for \((u,w) \in \mathbb{R} \times \mathbb{R}^{n-1}\). First, the \( u \) partial derivative of \( K^{\alpha,n}_1(u, w) \) is
\[
\frac{\partial}{\partial u} K^{\alpha,n}_1(u, w) = \left( u^2 + |w|^2 \right)^{-\frac{n+1-n}{2}} - \frac{n+1-\alpha}{2} \left( u^2 + |w|^2 \right)^{-\frac{n+1-\alpha}{2} - 1} 2u^2
\]

\[
= \left( u^2 + |w|^2 \right)^{-\frac{n+1-n}{2}} \left\{ \left( u^2 + |\xi|^2 \right) - \left( n+1-\alpha \right) u^2 \right\}
\]

\[
= \left( u^2 + |w|^2 \right)^{-\frac{n+1-n}{2} - 1} \left\{ |w|^2 - \left( n - \alpha \right) u^2 \right\},
\]
which satisfies
\[
\frac{\partial}{\partial u} K^{\alpha,n}_1(u, w) \approx \frac{(\alpha - n) u^2}{\left( u^2 + |w|^2 \right)^{\frac{n+1-\alpha}{2} + 1}} \approx (\alpha - n) u^{\alpha-n-1}
\]
provided \( |w| \leq \lambda u \), where \( \lambda > 0 \) is chosen sufficiently small depending on \( \gamma \) and \( \rho \). We also have,
\[
\nabla_w K^{\alpha,n}_1(u, w) = \nabla_w K^{\alpha,n}_1 \left( u^2 + |w|^2 \right)^{-\frac{n+1-n}{2}}
\]

\[
= - \frac{n+1-\alpha}{2} u \left( u^2 + |\xi|^2 \right)^{-\frac{n+1-n}{2} - 1} 2\xi
\]

\[
= (\alpha - n - 1) uw \left( u^2 + |\xi|^2 \right)^{-\frac{n+1-n}{2} - 1},
\]
which satisfies
\[
|\nabla_w K^{\alpha,n}_1(u, w)| \lesssim \frac{|uw|}{\left( u^2 + |w|^2 \right)^{\frac{n+1-n}{2} + 1}} \lesssim \lambda \frac{u^2}{\left( u^2 + |w|^2 \right)^{\frac{n+1-n}{2} + 1}},
\]
since \( |w| \leq \lambda u \).
Altogether then
\[
K_{1}^{\alpha,n}(p_{1} - y_{1}, \bar{p} - \bar{y}) - K_{1}^{\alpha,n}(q_{1} - y_{1}, \bar{q} - \bar{y})
\]
\[
= K_{1}^{\alpha,n} [\theta p_{1} + (1 - \theta) q_{1} - y_{1}, \theta \bar{p} + (1 - \theta) \bar{q} - \bar{y}]^{1}_{0}
\]
\[
= \int_{0}^{1} \frac{d}{d\theta} K_{1}^{\alpha,n} [\theta p_{1} + (1 - \theta) q_{1} - y_{1}, \theta \bar{p} + (1 - \theta) \bar{q} - \bar{y}] d\theta
\]
\[
= \int_{0}^{1} (p_{1} - q_{1}) \left( \frac{\partial}{\partial u} K_{1}^{\alpha,n} \right) [\theta p_{1} + (1 - \theta) q_{1} - y_{1}, \theta \bar{p} + (1 - \theta) \bar{q} - \bar{y}] d\theta
\]
\[
+ \int_{0}^{1} (\bar{p} - \bar{q}) \cdot (\nabla_{w} K_{1}^{\alpha,n}) [\theta p_{1} + (1 - \theta) q_{1} - y_{1}, \theta \bar{p} + (1 - \theta) \bar{q} - \bar{y}] d\theta,
\]
and since
\[
|\theta \bar{p} + (1 - \theta) \bar{q} - \bar{y}| \lesssim \lambda |p_{1} - q_{1}|,
\]
\[
||\bar{p} - \bar{q}|| \lesssim \lambda |p_{1} - q_{1}|,
\]
\[
u = \theta p_{1} + (1 - \theta) q_{1} - y_{1} \approx |c_{J_{k}} - y|,
\]
the above estimates give
\[
K_{1}^{\alpha,n}(p_{1} - y_{1}, \bar{p} - \bar{y}) - K_{1}^{\alpha,n}(q_{1} - y_{1}, \bar{q} - \bar{y})
\]
\[
\approx (p_{1} - q_{1}) \int_{0}^{1} (\alpha - n) u^{\alpha - n - 1} d\theta + o \left( \frac{u^{2}|p_{1} - q_{1}|}{\lambda^{2} + |\xi|^{2}} + 1 \right) \approx (p_{1} - q_{1}) (\alpha - n) u^{\alpha - n - 1},
\]
provided \( \lambda > 0 \) is chosen sufficiently small. This completes the proof of Lemma [1]. \( \square \)

We have now completed the proof of Proposition [2].

**Part 2. Failure of necessity of the energy condition for Riesz transforms**

In the second part of this paper, we prove Theorem [1] by constructing the families of counterexample weight pairs that demonstrate the failure of necessity of the energy conditions in higher dimensions.

In [LaSaUr], the authors constructed a weight pair \((\sigma, \omega)\) on the real line which demonstrated that the backward pivotal condition of NTV was not necessary for boundedness of the Hilbert transform. This pair was then modified in [SaShUr1], to demonstrate failure of necessity of the backward energy condition for boundedness of an elliptic operator on the line, by ‘smearing out’ the point masses of \( \sigma \) in order that the backward energy condition became equivalent with the backward pivotal condition. But this change then destroyed the backward testing condition for the Hilbert transform, and this necessitated a flattening of the kernel of the Hilbert transform, along with a delicate redistribution of the Cantor measure.

In this paper, we instead modify the weight pair \((\sigma, \omega)\) on the real line to obtain a family of weight pairs \(((\tilde{\sigma}_{N}, \tilde{\omega}_{N}))_{N=1}^{\infty}\) in a two-dimensional subspace of \( \mathbb{R}^{n} \), which demonstrate that the energy conditions are not necessary for boundedness of the vector Riesz transform \( R^{\alpha,n} \). This modification is suggested by the above derivation of the energy conditions from the testing conditions for the family of twisted localizations of \( R^{\alpha,n} \), and is accomplished by replacing the point masses of \( \sigma \) on the line with a ‘spread out’ pair of point masses extending off the real line (this is the twist), again resulting in failure of the backward energy condition.

While this spreading out of the point masses in \( \sigma \) leaves intact the testing conditions for the first component \( R_{1}^{\alpha,n} \) of the Riesz transform, it destroys the backward testing condition for \( R_{1}^{\alpha,n} \) - consistent with the fact that the energy conditions are necessary for boundedness of \( R^{\alpha,n} \) when the measure \( \omega \) is supported on a line - see [SaShUr1] and [LaSaShUrWi]. In order to circumvent this difficulty, we must carefully reposition the Cantor measure off the line to occupy the upper and lower half spaces of \( \mathbb{R}^{2} \subset \mathbb{R}^{n} \) in such a way that point masses associated with the repositioned Cantor measure appear near the spreadout point masses of \( \sigma \). This is needed in order to force zeroes of the function \( R_{1}^{\alpha,n} \tilde{\omega}_{N} \) to occur where we want them locally. Since the second component \( R_{2}^{\alpha,n} \) of the Riesz transform is essentially controlled by the Poisson operator, and the remaining components \( R_{j}^{\alpha,n} \), \( 3 \leq j \leq n \), vanish on the supports of these measures, we also obtain the testing conditions for the remaining \( R_{j}^{\alpha,n} \) when \( j \geq 2 \).
6. Construction of the counterexample pair of weights for the Cauchy operator

We begin the proof of Theorem 7 with the special case $\alpha = 1$ in dimension $n = 2$, where the components of the fractional Riesz transform $R_1^{1,2} = \left( R_1^{1,2}, R_2^{1,2} \right)$ are the real and imaginary parts of the Cauchy transform $C$ with convolution kernel $\frac{1}{z}$, for $z \in \mathbb{C}$. Note also that the restriction of the first component $R_1^{1,2}$ to the $x$-axis in $\mathbb{C}$ is precisely the Hilbert transform $H$ with convolution kernel $\frac{1}{t}$ on the real line, which explains the relevance of the one-dimensional weight pair in [LaSaUr2]. However, it is the additional dimension available in the plane that allows us to retain boundedness of the operator $R_1^{1,2}$ while spreading out both measures off the line, and arranging for the resulting backward energy condition to fail. The general case $0 \leq \alpha < n$ and $n \geq 2$ is considered at the very end of the paper.

Recall the middle-third Cantor set $E$ and Cantor measure $\omega$ on the closed unit interval $I^1 = [0, 1]$. At the $k$th generation in the construction, there is a collection $\{I^k_j\}_{j=1}^{2^k}$ of $2^k$ pairwise disjoint closed intervals of length $|I^k_j| = \frac{1}{3^k}$. With $K_k = \bigcup_{j=1}^{2^k} I^k_j$, the Cantor set is defined by $E = \bigcap_{k=1}^{\infty} K_k = \bigcap_{k=1}^{\infty} \left( \bigcup_{j=1}^{2^k} I^k_j \right)$. The Cantor measure $\omega$ is the unique probability measure supported in $E$ with the property that it is equidistributed among the intervals $\{I^k_j\}_{j=1}^{2^k}$ at each scale $k$, i.e.

$$\omega(I^k_j) = 2^{-k}, \quad k \geq 0, 1 \leq j \leq 2^k.$$  

Let $G^k_j = (a^k_j, b^k_j)$ be the open middle third of $I^k_j$ and let $(I^k_j)^{\text{left}}$ denote the interval $I^{k+1}_{j_1}$ with $j_1 = 2j - 1$ that has right hand endpoint equal to $a^k_j$, and more generally let $(I^k_{j_1})_{\infty}^{\text{left}}$ be the tower of intervals with right hand endpoint $a^k_j$. Similarly, let $(I^k_j)^{\text{right}}$ denote the interval $I^{k+1}_{j_1+1} = I^{k+1}_{2j}$ that has left hand endpoint equal to $b^k_j$, and let $(I^k_{j_1+1})_{\infty}^{\text{right}}$ be the tower of intervals with left hand endpoint $b^k_j$. Let $c^k_i \in G^k_i$ be the center of the interval $G^k_i = (a^k_i, b^k_i)$, which is also the center of the interval $I^k_j$.

Now we recall from [LaSaUr2] an important property of the Hilbert transform $H$ with respect to the Cantor measure $\omega$. We use the pairwise disjoint decomposition $I^{k+1}_{j_1} = \bigcup_{j=1}^{\infty} (I^k_{j_1})^{\text{left}}$ to compute

$$H \left( \int_{(I^k_j)^{\text{left}}} \omega \right) (a^k_j) = \int_{I^{k+1}_{j_1}} \frac{1}{y - a^k_j} d\omega (y) = \sum_{\ell=1}^{\infty} \int_{(I^k_j)^{\text{left}}} \frac{1}{y - a^k_j} d\omega (y),$$

and hence the estimate

$$H \left( \int_{(I^k_j)^{\text{left}}} \omega \right) (a^k_j) \approx -\sum_{\ell=1}^{\infty} \left| (I^k_{j_1})^{\text{left}} \right| \omega (y) = -\sum_{\ell=1}^{\infty} \frac{2^{-k-\ell}}{3^{-k-\ell}} = -\sum_{\ell=1}^{\infty} \left( \frac{3}{2} \right) \frac{k+\ell}{2} = -\infty.$$  

Since $H \left( \int_{(I^k_j)^{\text{left}}} \omega \right) (a^k_j) \lesssim \frac{1}{3^k} \approx \omega \left( (I^k_j)^{\text{left}} \right)$, we conclude that $H \omega (a^k_j) = -\infty$, and similarly $H \omega (b^k_j) = \infty$. Thus $H \omega (x)$ increases from $-\infty$ to $\infty$ on the interval $G^k_j$. We will later arrange for a similar result to hold for the first component $R_1^{1,2}$ of the Riesz transform with respect to a modification of $\omega$ into the plane.

We now extend certain approximations $\omega_N$ of the Cantor measure $\omega$ to the plane in the following way. Fix $N \in \mathbb{N}$. Recall that $K_N = \bigcup_{j=1}^{N} I^N_j$ and that

$$I^N_j = I^{N+1}_{2j-1} \cup G^N_j \cup I^{N+1}_{2j} \equiv I^N_j^{\text{left}} \cup G^N_j \cup I^N_j^{\text{right}} \cup I^N_j^{\text{right}}.$$

The Cantor measure $\omega$ charges each interval $I^N_j^{\text{left}}$ and $I^N_j^{\text{right}}$ with the same mass, namely $\left| I^N_j^{\text{left}} \right| = \left| I^N_j^{\text{right}} \right| = 2^{-(N+1)}$, and we now define the discrete approximation $\omega_N$ by

$$\omega_N = \sum_{j=1}^{2^N} 2^{-N-1} \left( \delta_{c^N_j \text{left}} + \delta_{c^N_j \text{right}} \right),$$

where we have relabelled the intervals $I^N_j^{\text{left}} = I^{N+1}_{2j-1}$ and $I^N_j^{\text{right}} = I^{N+1}_{2j}$, and have denoted their centers by $c^N_j \text{left} = c^N_{2j-1}$ and $c^N_j \text{right} = c^N_{2j}$ respectively.
We now embed the point mass $\delta_c$ on $\mathbb{R}$ as $\delta_{(c,0)}$ in the plane $\mathbb{R}^2$, and split each of the point masses $\delta_{c_{j, \text{left}}}, \delta_{c_{j, \text{right}}}$ for $1 \leq j \leq 2^N$ into a sum of two point masses located at equal distances $d_{j, \text{left}}^N$ and $d_{j, \text{right}}^N$ above and below the points $c_{j, \text{left}}^N$ and $c_{j, \text{right}}^N$ respectively. For $\delta_{c_{j, \text{left}}} = \delta_{c_{j-1, \text{left}}}$ we define $d_{j, \text{left}}^N$ to be one half the length of $I^N_{j, \text{left}}$ plus one quarter the length of the neighbouring open middle third $G_k^N$ to the left of $I^N_{j, \text{left}}$, i.e.

$$d_{j, \text{left}}^N = \frac{1}{2} 3^{-N-1} + \frac{1}{4} 3^{-k-1}.$$  

Note that $0 \leq k \leq N - 1$, and that the neighbouring open middle third to the right of $I^N_{j, \text{left}}$ is simply $G_j^N$. Similarly, we define $d_{j, \text{right}}^N$ to be one half the length of $I^N_{j, \text{right}}$ plus one quarter the length of the neighbouring open middle third $G_k^N$ to the right of $I^N_{j, \text{right}}$, i.e.

$$d_{j, \text{right}}^N = \frac{1}{2} 3^{-N-1} + \frac{1}{4} 3^{-k'-1},$$

where again $0 \leq k' \leq N - 1$, and the neighbouring open middle third to the left of $I^N_{j, \text{right}}$ is again $G_j^N$. Note that we have defined the lengths $d_{j, \text{left}}^N$ and $d_{j, \text{right}}^N$ so that

$$c_{j, \text{left}}^N - d_{j, \text{left}}^N = \frac{1}{4} 3^{-k-1},$$

$$c_{j, \text{right}}^N + d_{j, \text{right}}^N = \frac{1}{4} 3^{-k'-1}.$$  

We now define

$$\tilde{\omega}_N \equiv \sum_{j=1}^{2^N} 2^{-N-1} \left( \frac{\delta (c_{j, \text{left}}^N - d_{j, \text{left}}^N, -\frac{1}{3} 3^{-k-1}) + \delta (c_{j, \text{left}}^N - d_{j, \text{left}}^N, -\frac{1}{4} 3^{-k-1})}{2} \right)$$

$$+ \sum_{j=1}^{2^N} 2^{-N-1} \left( \frac{\delta (c_{j, \text{right}}^N + d_{j, \text{right}}^N, \frac{1}{3} 3^{-k'-1}) + \delta (c_{j, \text{right}}^N + d_{j, \text{right}}^N, \frac{1}{4} 3^{-k'-1})}{2} \right).$$

Note in particular that the point mass $\delta_{(c_{j, \text{left}}, 0)}$ has been replaced with the average of two point masses whose locations in the plane, $(c_{j, \text{left}}^N - d_{j, \text{left}}^N, \frac{1}{3} 3^{-k-1})$ and $(c_{j, \text{left}}^N - d_{j, \text{left}}^N, -\frac{1}{4} 3^{-k-1})$, lie at least $45^\circ$ angles from $c_{j, \text{left}}^N$ extending to the left in the upper and lower half planes respectively. In similar fashion, the point mass $\delta_{(c_{j, \text{right}}, 0)}$ has been replaced with the average of two point masses whose locations in the plane, $(c_{j, \text{right}}^N + d_{j, \text{right}}^N, \frac{1}{3} 3^{-k'-1})$ and $(c_{j, \text{right}}^N + d_{j, \text{right}}^N, -\frac{1}{4} 3^{-k'-1})$, lie at least $45^\circ$ angles from $c_{j, \text{right}}^N$ extending to the right into the upper and lower half planes respectively.

The point of incorporating these less than $45^\circ$ angle translations of locations is to obtain the following crucial property for all pairs of points $y = (y_1, y_2)$ and $z = (z_1, z_2)$ in the support of $\tilde{\omega}_N$ with $y_1 \neq z_1$:  

$$|y_2 - z_2| \leq |y_1 - z_1| \text{ for all } y, z \text{ such that } y_1 \neq z_1, \tilde{\omega}_N (y) \neq 0 \text{ and } \tilde{\omega}_N (z) \neq 0.$$  

This property is evident from another useful description of these measures that derives from an extension of the observation that $I^N_{j, \text{left}}$ and $I^N_{j, \text{right}}$ are the left and right neighbours of $G_j^N$ at level $N$. More precisely, the support of $\tilde{\omega}_N$ is contained in the union $\bigcup_{k=0}^{N-1} \bigcup_{i=1}^{2^k} G_i^k$ of the squares $G_i^k = G_i^1 \times (\frac{1}{4} 3^{-k-1}, \frac{1}{4} 3^{-k-1})$ corresponding to the open middle thirds of the intervals $I^k_i$ up to level $N - 1$. Moreover, for each $G_i^k$ with $0 \leq k \leq N - 1$ and $1 \leq i \leq 2^k$, there are exactly four point masses from $\omega_N$ contained in $G_i^k$, and by (6.2), they are located at the points $(c_{i}^k, \pm \frac{1}{4} 3^{-k-1}, \pm \frac{1}{4} 3^{-k-1})$. Thus we can rewrite $\tilde{\omega}_N$ as

$$\tilde{\omega}_N = 2^{-N-1} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left( \frac{\delta (c_{i}^k + \frac{1}{4} 3^{-k-1}, \frac{1}{4} 3^{-k-1}) + \delta (c_{i}^k + \frac{1}{4} 3^{-k-1}, -\frac{1}{4} 3^{-k-1})}{2} \right)$$

$$+ 2^{-N-1} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left( \frac{\delta (c_{i}^k - \frac{1}{4} 3^{-k-1}, \frac{1}{4} 3^{-k-1}) + \delta (c_{i}^k - \frac{1}{4} 3^{-k-1}, -\frac{1}{4} 3^{-k-1})}{2} \right).$$
A simple picture in the plane of the support of $\hat{\omega}_N$ using this representation of $\hat{\omega}_N$ demonstrates the property (6.3). Indeed, the slopes of the lines joining pairs of the six points consisting of the four point supports of $\hat{\omega}_N$ in $G^k_i$, namely $(c^k_i \pm \frac{1}{4}3^{-k}, \pm \frac{1}{4}3^{-k})$, and the two ‘endpoints’ of $G^k_i \times \{0\}$, namely $(c^k_i \pm \frac{1}{3}3^{-k}, 0)$, are either infinite or at most 1 in modulus. In fact, the slopes of segments joining pairs of points in $\text{supp} \hat{\omega}_N$ are strictly less than 1 in modulus unless the pair of points lie in a common square $G^k_i$ on opposite sides of the $x_1$-axis.

We will now define three measures $\hat{\sigma}_N, \hat{\sigma}_N, \hat{\sigma}_N^+$ in the plane loosely motivated by the two measures $\hat{\sigma}, \sigma$ on the line constructed in [LaSaUr2]. Recall that $G^k_i$ is the removed open middle third of $I^k_i$. The measure $\hat{\sigma}$, restricted to a square $G^k_i$, will consist of a multiple of the single point mass $\delta_{(c^k_i, 0)}$ located on the real axis at the center $c^k_i$ of $G^k_i$, while the measure $\hat{\sigma}$, restricted to a square $G^k_i$, will consist of multiples of two point masses lying equidistant above and below the real axis. The measure $\hat{\sigma}_N^+$ will be the restriction of $\hat{\sigma}_N$ to the upper half plane. We now turn to describing these measures explicitly.

Let $H^k_i = \frac{1}{4}G^k_i$ be the middle half of the open middle third $G^k_i$ of $I^k_i$ (in other words the open middle sixth of $I^k_i$), and note that by construction $\hat{\omega}_N$ does not charge the open rectangle $H^k_i \times \mathbb{R}$. On the other hand there are four points $(c^k_i \pm \frac{1}{4}3^{-k-1}, \pm \frac{1}{4}3^{-k-1})$ in the support of $\hat{\omega}_N$ that lie on the boundary of the strip $H^k_i \times \mathbb{R}$, two on the left edge and two on the right edge. If we let $H^k_i = \{u^k_i, v^k_i\}$, then these four points are $P^k_i = (u^k_i, \pm \frac{1}{4}3^{-k-2})$ and $Q^k_i = (v^k_i, \pm \frac{1}{4}3^{-k-2})$.

It is convenient to also define the minimal closed intervals $L^k_j \supseteq I^k_j$ so that the closed square $\hat{L^k_j} = L^k_j \times [\frac{1}{2}3^{-k-1}, -\frac{1}{2}3^{-k-1}]$ contains all the point masses in $\hat{\omega}_N$ that were constructed from the point masses of $\omega_N$ lying in $I^k_j$ by the procedure of splitting into two point masses. In other words, if $I^k_j$ is adjacent to $G^k_i$ on the left and to $G^k_{i-1}$ on the right, then $L^k_j = [c^k_j + \frac{1}{4}3^{-k}, c^k_{j-1} - \frac{1}{4}3^{-k+1}]$; while if $I^k_j$ is adjacent to $G^k_{i+1}$ on the left and to $G^k_i$ on the right, then $L^k_j = [c^k_{j+1} - \frac{1}{4}3^{-k}, c^k_j + \frac{1}{4}3^{-k+1}]$. Thus $L^k_j$ sticks out beyond $I^k_j$ on each side a distance $\frac{1}{4} |G^k_i|$ determined by the length of the adjacent middle third $G^k_{i+1}$ on that side.

Recall the $A^1_2$ condition in the plane $\mathbb{R}^2$:

$$A^1_2(\hat{\sigma}, \hat{\omega}) \equiv \sup_{Q \in \mathbb{R}^2} \frac{|Q|_{\hat{\omega}_N}}{Q \left| \frac{1}{2} |Q| \right|_{\hat{\sigma}_N}}$$

**Notation 12.** For an interval $I$ denote by $\hat{I}$ the square $I \times [-\frac{1}{2} |I|, |I|]$.

Define

$$(6.5) \quad \hat{\sigma}_N = \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} s^k_i \delta_{(c^k_i, 0)},$$

where the sequence of positive numbers $s^k_i$ is chosen to satisfy the following precursor of the $A^1_2$ condition involving the squares $\hat{L}^k_i \equiv L^k_i \times [-\frac{1}{2} |L^k_i|, \frac{1}{2} |L^k_i|]$:

$$\frac{s^k_i \hat{\omega}_N(\hat{L}^k_i)}{|L^k_i|^2(\frac{1}{2} - \frac{1}{2})} = s^k_i \hat{\omega}_N(I^k_i) \approx \frac{s^k_i 2^{-k}}{3^{-2k}} = 1.$$ 

Note that we also have a similar estimate for the squares $\hat{I}^k_i$,

$$\frac{\hat{\omega}_N(\hat{I}^k_i)}{|I^k_i|^2(\frac{1}{2} - \frac{1}{2})} = s^k_i \hat{\omega}_N(I^k_i) \times [-\frac{1}{2} 3^{-k}, \frac{1}{2} 3^{-k}] \approx \frac{2^{-k}}{3^{-2k}} \approx \hat{\omega}_N(\hat{L}^k_i),$$ 

since $\hat{\omega}_N(I^k_i) \times [-\frac{1}{2} 3^{-k}, \frac{1}{2} 3^{-k}] \approx 2^{-k}$ for $0 \leq k \leq N - 1$ because only a fixed proportion of the mass of $\omega_N$ escapes $I^k_i$ when the point masses in $\omega_N$ at the extreme left and right inside $I^k_i$ were spread out at less than $45^\circ$ angles away from $I^k_i$ into the upper and lower half planes. Thus we define

$$s^k_i = \frac{2^k}{3^{2k}} = \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^k \quad k \geq 0, 1 \leq i \leq 2^k,$$
which agrees with the weights $s_k^i$ used in LaSaUr2 when $\alpha = 0$ and $n = 1$. The definition of the measures $\tilde{\sigma}_N$ and $\tilde{\sigma}_N^2$ will depend on the fractional Riesz transform $R^{1,2}$ with convolution kernel $K^{1,2}(\xi) = \frac{\xi_1}{|\xi|^2}$, and is closely related to the structure of the function $R^{1,2}_k\tilde{\omega}_N$, where $R^{1,2} = \left( R^{1,2}_1, R^{1,2}_2 \right)$.

We focus on the kernel

$$K^{1,2}_1(\xi) = \frac{\xi_1}{|\xi|^2} = \frac{\xi_1}{\xi_1^2 + \xi_2^2},$$

and the four points $P^{k}_{i, \pm} = (u^{k}_i, \pm \frac{1}{4 \cdot 3^k + 1})$ and $Q^{k}_{i, \pm} = (v^{k}_i, \pm \frac{1}{4 \cdot 3^k + 1})$ that are the vertices of the square $\tilde{H}^k_i$. Fix a horizontal segment $H^k_i \times \{x_2\}$ with $x_2 \in \{\pm \frac{1}{4 \cdot 3^k + 1}\}$, i.e. either the top or bottom edge of the square $\tilde{H}^k_i$. Then the function

$$F(x_1) \equiv R^{1,2}_k\tilde{\omega}_N(x_1, x_2) = \int K^{1,2}_1 d\tilde{\omega}_N = \int_{-1}^{1} \int_{-1}^{1} \frac{x_1 - y_1}{(x_1 - y_1)^2 + (x_2 - y_2)^2} d\tilde{\omega}_N(y_1, y_2)$$

is monotonically decreasing for $x_1$ in $[u^{k}_i, v^{k}_i]$ from the value

$$F(u^{k}_i) = \int_{-1}^{1} \int_{-1}^{1} \frac{u^{k}_i - y_1}{|u^{k}_i - y_1|^2 + |x_2 - y_2|^2} d\tilde{\omega}_N(y_1, y_2)$$

at the left hand endpoint of $G^{k}_i = [u^{k}_i, v^{k}_i]$, to the value

$$F(v^{k}_i) = \int_{-1}^{1} \int_{-1}^{1} \frac{v^{k}_i - y_1}{|v^{k}_i - y_1|^2 + |x_2 - y_2|^2} d\tilde{\omega}_N(y_1, y_2)$$

at the right hand endpoint.

Indeed, to see this, fix $y = (y_1, y_2) \in \text{supp}\tilde{\omega}_N \setminus \{P^{k}_{i, \pm}, Q^{k}_{i, \pm}\}$. Then using (6.3) it is easy to see that $|x_2 - y_2| < |x_1 - y_1|$ for all $x_1 \in H^k_i$, and so

$$\frac{d}{dx_1} \frac{x_1 - y_1}{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2 - 2(x_1 - y_1)^2}{[(x_1 - y_1)^2 + (x_2 - y_2)^2]^2}$$

$$= \frac{(x_2 - y_2)^2 - (x_1 - y_1)^2}{[(x_1 - y_1)^2 + (x_2 - y_2)^2]^2} < 0.$$

Now consider the integral corresponding to the sum of the two points $\{P^{k}_{i, \pm}\}$ in the support of $\tilde{\omega}_N$. This integral is a positive multiple of the following sum:

$$\frac{x_1 - u^{k}_i}{(x_1 - u^{k}_i)^2 + (\frac{1}{3 \cdot 3^k + 1} - \frac{1}{4 \cdot 3^k + 1})^2} + \frac{x_1 - u^{k}_i}{(x_1 - u^{k}_i)^2 + (\frac{1}{3 \cdot 3^k + 1} + \frac{1}{4 \cdot 3^k + 1})^2}$$

$$= \frac{1}{t} + \frac{t}{t^2 + A^2}, \quad t = x_1 - u^{k}_i \text{ and } A = \frac{1}{2 \cdot 3^{k+1}},$$

where

$$\frac{d}{dt} \left( \frac{1}{t} + \frac{t}{t^2 + A^2} \right) = -\frac{1}{t^2} + \frac{1}{t^2 + A^2} - \frac{2t^2}{(t^2 + A^2)^2} = -\frac{A^2 + t^4}{t^2 (t^2 + A^2)^2} < 0.$$

Similarly, the integral corresponding to the sum of the two points $\{Q^{k}_{i, \pm}\}$ is a positive multiple of the sum

$$\frac{x_1 - v^{k}_i}{(x_1 - v^{k}_i)^2 + (\frac{1}{3 \cdot 3^k + 1} - \frac{1}{4 \cdot 3^k + 1})^2} + \frac{x_1 - v^{k}_i}{(x_1 - v^{k}_i)^2 + (\frac{1}{3 \cdot 3^k + 1} + \frac{1}{4 \cdot 3^k + 1})^2}$$

$$= \frac{1}{t} + \frac{t}{t^2 + A^2}, \quad t = x_1 - v^{k}_i \text{ and } A = \frac{1}{2 \cdot 3^{k+1}},$$

whose $t$ derivative was shown above to be negative. This completes the proof that $F(x_1)$ is monotonically decreasing for $x_1$ in $[u^{k}_i, v^{k}_i]$.

Now we have $\lim_{x_1 \to u^{k}_i} F(x_1) = \infty$ since the integrand $\frac{x_1 - y_1}{|x_1 - y_1|^2 + |x_2 - y_2|^2} \to \infty$ as $(x_1, \pm \frac{1}{4 \cdot 3^k + 1}) \to P^{k}_{i, \pm} = (u^{k}_i, \pm \frac{1}{4 \cdot 3^k + 1}) \in \text{supp}\tilde{\omega}_N$. Similarly, $\lim_{x_1 \to v^{k}_i} F(x_1) = -\infty$, and we conclude that $F(x_1) \equiv R^{1,2}_k\tilde{\omega}_N(x_1, x_2)$
strictly decreases from $-\infty$ to $\infty$ along the horizontal segment $H_j^\ell \times \{x_2\}$ when $x_2 \in \{\pm \frac{1}{4^k 3^{k+1}}\}$. In particular, $R_1^{1,2} \omega_0$ has a unique zero $z_i^k$ on the horizontal segment $H_j^\ell \times \{x_2\}$, which is independent of the two choices $x_2 = \pm \frac{1}{4^k 3^{k+1}}$ by symmetry. With this choice of $z_i^k$, we define the measures $\tilde{\sigma}_N$ and $\tilde{\sigma}_N^+$ by

$$\tilde{\sigma}_N = \sum_{k,i} s_i^k \left( \delta(z_i^k, \frac{1}{4^k 3^{k+1}}) + \delta(z_i^k, -\frac{1}{4^k 3^{k+1}}) \right);$$

$$\tilde{\sigma}_N^+ = \sum_{k,i} s_i^k \delta(z_i^k, \frac{1}{4^k 3^{k+1}}).$$

### 6.1. An estimate for the second component

From the representation (6.3) of $\tilde{\omega}_N$, it is clear that there are $2N$ horizontal lines on which $\tilde{\omega}_N$ is supported, namely the lines $\{L_\beta\} = \{L_\beta \in \{x \mid x \in \mathbb{R}\} \}_{k=0}^{N-1}$ where $L_\beta \equiv \{(x, \beta) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ is the $x$-axis translated vertically by $\beta$. We now estimate the second component $R_2^{1,2} \tilde{\omega}_N$ of the Riesz transform of $\tilde{\omega}_N$ on the support of the measure $\tilde{\sigma}_N$. So fix a point $(z_j^\ell, \frac{1}{4^k 3^{k+1}})$ in the strip $H_j^\ell \times \mathbb{R}$ that lies in the support of $\tilde{\sigma}_N^+$ with $N \geq 3$. Then we have

$$R_2^{1,2} \tilde{\omega}_N (z_j^\ell, \frac{1}{4^k 3^{k+1}}) = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{|z_j^\ell - y_1|^2 + \frac{1}{4^k 3^{k+1}} - y_2^2} \, d\tilde{\omega}_N (y_1, y_2)$$

$$= 2^{-N-2} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left\{ \frac{1}{|z_j^\ell - (c_{left}^N (G_i^k) + \frac{1}{4^k 3^{k+1}})|^2 + \frac{1}{4^k 3^{k+1}} - \frac{1}{4^k 3^{k+1}} |^2} + \frac{1}{|z_j^\ell - (c_{right}^N (G_i^k) - \frac{1}{4^k 3^{k+1}})|^2 + \frac{1}{4^k 3^{k+1}} + \frac{1}{4^k 3^{k+1}} |^2} \right\}.$$ 

The negative terms are those with numerator $\frac{1}{4^k 3^{k+1}} - \frac{1}{4^k 3^{k+1}}$ for $0 \leq k \leq \ell$, and the sum of those terms corresponding to the left edge of $G_i^k$ is

$$2^{-N-2} \sum_{k=0}^{\ell} \sum_{i=1}^{2^k} \left\{ \frac{1}{|z_j^\ell - (c_{left}^N (G_i^k) + \frac{1}{4^k 3^{k+1}})|^2 + \frac{1}{4^k 3^{k+1}} - \frac{1}{4^k 3^{k+1}} |^2} \right\}.$$ 

The analogous sum of positive parts for $0 \leq k \leq \ell$ corresponding to the left edge of $G_i^k$ is given by

$$2^{-N-2} \sum_{k=0}^{\ell} \sum_{i=1}^{2^k} \left\{ \frac{1}{|z_j^\ell - (c_{right}^N (G_i^k) - \frac{1}{4^k 3^{k+1}})|^2 + \frac{1}{4^k 3^{k+1}} + \frac{1}{4^k 3^{k+1}} |^2} \right\}.$$
Adding the two fractions appearing in these sums gives, with $A^{(\ell, j), (k, i)}_{\text{left}} = z_j^\ell - \left( c^{N+1}_{\text{left}} \left( G^k_\ell \right) + \frac{1}{4 \cdot 3^{k+1}} \right)$,

$$\left| A^{(\ell, j), (k, i)}_{\text{left}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} < \frac{1}{4 \cdot 3^{k+1}} \right|^2 + \left| A^{(\ell, j), (k, i)}_{\text{left}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right|^2 \approx \frac{2}{4 \cdot 3^{k+1}}$$

which equals

$$\frac{1}{4 \cdot 3^{k+1}} \left[ \left| A^{(\ell, j), (k, i)}_{\text{left}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right]^2 \approx \frac{2}{4 \cdot 3^{k+1}}$$

Finally, if we sum this over $\sum_{k=0}^\ell \sum_{i=1}^{2^k}$ and multiply by $2^{N-2}$ we get

$$2^{N-2} \frac{1}{4 \cdot 3^{k+1}} \sum_{k=0}^\ell \sum_{i=1}^{2^k} \left| A^{(\ell, j), (k, i)}_{\text{left}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right|^2 \approx 2^{N-3}\ell,$$

since the main term here occurs when $k = \ell$ and $i$ and $j$ are such that $A^{(\ell, j), (k, i)}_{\text{left}} = z_j^\ell - \left( c^{N+1}_{\text{left}} \left( G^k_\ell \right) + \frac{1}{4 \cdot 3^{k+1}} \right) \approx \frac{1}{4 \cdot 3^{k+1}}$. A similar estimate, with $A^{(\ell, j), (k, i)}_{\text{right}} = z_j^\ell - \left( c^{N+1}_{\text{right}} \left( G^k_\ell \right) - \frac{1}{4 \cdot 3^{k+1}} \right)$, is obtained for the negative terms corresponding to the right edge of $G^k_\ell$, namely

$$2^{N-2} \frac{1}{4 \cdot 3^{k+1}} \sum_{k=0}^\ell \sum_{i=1}^{2^k} \left| A^{(\ell, j), (k, i)}_{\text{right}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right|^2 \approx 2^{N-3}\ell.$$

Now if we sum over the remaining terms for $\ell < k \leq N - 1$, and use the crude estimate $\left| \frac{1}{4 \cdot 3^{k+1}} \leq 2 \frac{1}{4 \cdot 3^{k+1}} \right| \leq 2 \frac{1}{4 \cdot 3^{k+1}}$ for $\ell < k$, we get approximately

$$2^{N-2} \sum_{k=\ell+1}^{N-1} \sum_{i=1}^{2^k} \left[ \left| A^{(\ell, j), (k, i)}_{\text{left}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right] + \frac{1}{4 \cdot 3^{k+1}} \right]^2 \right|^2 + \left| A^{(\ell, j), (k, i)}_{\text{right}} \right|^2 + \frac{1}{4 \cdot 3^{k+1}} \right|^2 \approx 2^{N-3}\ell,$$
since the main term here occurs when \( k = \ell + 1 \) and \( i \) and \( j \) are such that \( A_{\text{left}}^{(\ell,i), (\ell+1,i)} = z_j^f - (c_1^{N+1} G_i^{(\ell+1)} + \frac{1}{4 \cdot 3^{k+1}}) \approx \frac{1}{3} \). So altogether we have the estimate

\[
R_2^{1,2} \omega_N \left( z_j^f, \frac{1}{4 \cdot 3^{k+1}} \right) \approx 2^{-N} 3^{\ell},
\]

which will suffice to prove the backward testing condition for \( R_2^{1,2} \) below.

6.2. The plan of attack.

- From our choice of \( s_j^k \) we will obtain the Muckenhoupt conditions \( A_2^1 \) and \( A_2^1 \ast \) for the measure pairs \((\hat{\sigma}_N, \hat{\omega}_N)\) uniformly in \( N \geq 1 \).
- The points \((z_1^i(x_2), \pm \frac{1}{4 \cdot 3^{k+1}})\) lie in the zero set of \( R_1^{1,2} \hat{\omega}_N \), and the resulting cancellation in the backward testing condition for \( R_1^{1,2} \) with respect to the weight pair \((\hat{\sigma}_N, \hat{\omega}_N)\) is enough to obtain it uniformly in \( N \geq 1 \).
- The self-similarity of the measure \( \hat{\sigma}_N \) will aid in computing the forward testing condition for \( R_1^{1,2} \) with respect to the measure pairs \((\hat{\sigma}_N, \hat{\omega}_N)\), and then a perturbation argument will establish the forward testing condition for \( R_1^{1,2} \) with respect to the measure pairs \((\hat{\sigma}_N, \hat{\omega}_N)\) uniformly in \( N \geq 1 \).
- Then we will use the estimate (6.6) to show that the testing conditions for \( R_1^{1,2} \) hold uniformly in \( N \geq 1 \) for the measure pairs \((\hat{\sigma}_N, \hat{\omega}_N)\).
- We next establish the forward and backward energy conditions uniformly in \( N \) for the measure pairs \((\hat{\sigma}_N, \hat{\omega}_N)\).
- Then we establish the forward and backward testing conditions uniformly in \( N \) for the second component \( R_2^{1,2} \) of the Riesz transform.
- Using \( A_2^1 (\hat{\sigma}_N, \hat{\omega}_N) \leq A_2^1 (\hat{\sigma}_N, \hat{\omega}_N) \leq C < \infty \), we can then conclude from the T1 theorem in [SaShUr7] that \( \mathfrak{g}_{R^{1,2}} (\hat{\sigma}_N, \hat{\omega}_N) < \infty \) uniformly in \( N \).
- Finally, we show by a direct computation that \( \mathfrak{g}_{R^{1,2}} (\hat{\sigma}_N, \hat{\omega}_N) \leq 2 \mathfrak{g}_{R^{1,2}} (\hat{\sigma}_N, \hat{\omega}_N) \) for all \( N \), and that the backward energy condition fails with respect to the measure pair \((\hat{\sigma}_N, \hat{\omega}_N)\) for each \( N \).
- The result is that the two-dimensional Riesz transform \( R^{1,2} \) is bounded from \( L^2 (\hat{\sigma}_N) \) to \( L^2 (\hat{\omega}_N) \) uniformly in \( N \geq 1 \), yet the energy constants for the weight pairs \((\hat{\sigma}_N, \hat{\omega}_N)\) are unbounded for \( N \geq 1 \).

In order to execute this strategy in the next four subsections, although not necessarily in the order specified above, we will follow as closely as we can the line of argument in [LaSaUr2], adapting to the plane as necessary. We begin by calculating the rate at which \( R_1^{1,2} \hat{\omega}_N (\cdot, \frac{1}{4 \cdot 3^{k+1}}) \) blows up at the endpoints of the intervals \( H^k_j \).

**Lemma 13.** Let \( H^k_j = (u^k_j, v^k_j) \). We have

\[
R_1^{1,2} \hat{\omega}_N \left( u^k_j - c3^{-k}, \frac{1}{4 \cdot 3^{k+1}} \right) \approx \left( \frac{3}{2} \right)^k, \quad k \geq 0, \ 1 \leq j \leq 2^k,
\]

and a similar approximate equality, with signs reversed, holds for \( v^k_j \).

This in particular shows that the zeros \( z_j^k \) cannot move too far from the middle:

\[
\sup_{j,k} \frac{|z_j^k - c_k|}{|H^k_j|} < \zeta < 1.
\]

**Proof.** Fix \( k \), and consider the numbers \( R_1^{1,2} \hat{\omega}_N (u^k_j \pm c3^{-k}, \frac{1}{4 \cdot 3^{k+1}}) \) for \( 1 \leq j \leq 2^k \). These numbers are monotonically increasing as the point of evaluation \( u^k_j \pm c3^{-k} \) moves from left to right across the interval \([0, 1]\). So it suffices to verify that

\[
C_1 \left( \frac{3}{2} \right)^k \leq R_1^{1,2} \hat{\omega}_N \left( u_1^k - c3^{-k}, \frac{1}{4 \cdot 3^{k+1}} \right) \leq R_1^{1,2} \hat{\omega}_N \left( u_{2^k}^k + c3^{-k}, \frac{1}{4 \cdot 3^{k+1}} \right) \leq C_2 \left( \frac{3}{2} \right)^k
\]
We consider first the right hand inequality, and write

$$R_{1,2}^1 \omega_N \left( \frac{k}{2^k + c 3^{-k} \cdot \frac{1}{4 \cdot 3^{k+1}}} \right) = \int_{(H_{2k})^c} \left( \frac{a_{2k}^k + c 3^{-k} - y_1}{\frac{a_{2k}^k + c 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right) d\omega_N (y_1, y_2) \leq \int \left( \frac{a_{2k}^k + c 3^{-k} - y_1}{\frac{a_{2k}^k + c 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right)^2 d\omega_N (y_1, y_2).$$

Here we have discarded that part of the domain of the integral where the integrand is nonpositive. Now, on the square $[0, u_{2k}^k]$, the support of $\omega_N$ is contained in the set $\bigcup_{k=1}^{2^{-1}} \chi_{2k}^k$. Using this, we continue the estimate above as

$$R_{1,2}^1 \omega_N \left( \frac{k}{2^k + c 3^{-k} \cdot \frac{1}{4 \cdot 3^{k+1}}} \right) \leq \sum_{\ell=1}^{k} \omega_N \left( L_{2^{k+1}}^k \right) \sup_{y \in \chi_{2^{k+1}}^k} \frac{a_{2k}^k + c 3^{-k} - y_1}{\frac{a_{2k}^k + c 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right)^2 \leq c^{-1} \omega_N \left( L_{2^{k+1}}^k \right) \sup_{y \in \chi_{2^{k+1}}^k} \frac{a_{2k}^k + c 3^{-k} - y_1}{\frac{a_{2k}^k + c 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right)^2 \leq c^{-1} 2^{-k} 3^k = c^{-1} \left( \frac{3}{2} \right)^k.$$

It is useful to record for use below, that in this sum, the summand associated with $\ell = k$ is the dominant one.

Now we consider the left hand inequality in (6.9). We split the support of $\omega_N$ into the sets $I_{1,2}^k$, $I_{2,2}^k$, $I_{2,2}^{k-1}$, $\ldots$, $I_{2}^k$. By the argument above, we have

$$\sum_{\ell=1}^{k-1} R_{1,2}^1 \left( 1_{\chi_{2^{k+1}}^k} \omega_N \right) \left( \frac{k}{2^k + c 3^{-k} \cdot \frac{1}{4 \cdot 3^{k+1}}} \right) \leq A \left( \frac{3}{2} \right)^k,$$

where $A$ is an absolute constant, and we have yet to select the constant $c$. But we also have

$$R_{1,2}^1 (1_{\chi_{2}^k \cup \chi_{2}^k} \omega_N) \leq \int_{\chi_{2}^k} \left( \frac{a_{2k}^k + c 3^{-k} - y_1}{\frac{a_{2k}^k + c 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right)^2 \frac{a_{2k}^k + (1 + c) 3^{-k} - y_1}{\frac{a_{2k}^k + (1 + c) 3^{-k} - y_1}{4 \cdot 3^{k+1}}} - (y_1, y_2) \right)^2 d\omega_N (y) \geq c^{-1} 3^k \omega(L_{1}^k) = c^{-1} \left( \frac{3}{2} \right)^k.$$

The choice $0 < c \ll (2A)^{-1}$ then concludes the proof of Lemma [13].

7. The $A_2^1$ Condition

We recall from (3.3) the one-tailed constant with holes $A_2^1$ in the plane $\mathbb{R}^2$ using the reproducing Poisson kernel $P^1$. Suppose $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}$. Then the one-tailed constants $A_2^1$ and $A_2^{1,*}$ with holes for the weight pair $(\sigma, \omega)$ are given by

$$A_2^1 \equiv \sup_{Q \in P} P^1 (Q, 1_{Q^c} \sigma) \frac{|Q|_{\sigma}}{|Q|^{1 - \frac{1}{2}}} < \infty,$$

$$A_2^{1,*} \equiv \sup_{Q \in P} P^1 (Q, 1_{Q^c} \omega) \frac{|Q|_{\omega}}{|Q|^{1 - \frac{1}{2}}} < \infty.$$

For pairs of measures that share no common point masses, we will also use the classical Muckenhoupt condition

$$A_2^1 \equiv \sup_{Q \in P} \frac{|Q|_{\omega} |Q|_{\sigma}}{|Q|^{2(1 - \frac{1}{2})}} < \infty.$$
We will first verify that the $A_1^2$ condition holds for the weight pair $(\tilde{\omega}_N, \tilde{\sigma}_N)$. The same argument will apply to the weight pair $(\tilde{\omega}_N, \tilde{\sigma}_N)$. Recall that we have $|L^k_r|_{\tilde{\omega}} \approx 2^{-k}$. Now we use the definition

$$s^k_j = \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^k,$$

to compute the estimate

$$\hat{\sigma}_N \left( \hat{L}^k_r \right) = \sum_{(k,j): \; z^k_j \in L^k_r} s^k_j \approx \sum_{k=\ell} 2^{k-\ell} \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^k \approx \left( \frac{1}{3} \right)^\ell \left( \frac{2}{3} \right) = s^\ell_r,$$

by the ratio test since

$$\frac{2^{k+1} \left( \frac{4}{3} \right)^{(k+1)} \left( \frac{2}{3} \right)^{k+1}}{2^k \left( \frac{4}{3} \right)^k \left( \frac{2}{3} \right)^k} = 4 \left( \frac{1}{3} \right)^2 < 1.$$

From this, it follows that we have

$$\frac{\hat{\sigma}_N \left( \hat{L}^k_r \right) \hat{\omega}_N \left( \hat{L}^k_r \right)}{|L^k_r|} \approx \frac{s^k_j \hat{\omega}_N \left( \hat{L}^k_r \right)}{|L^k_r|} \approx 1.$$

The analogous condition $A_2^2$ with a tail also holds, namely

$$A_2^2 \left( \hat{\sigma}_N, \hat{\omega}_N \right) = \sup_{Q \in \mathcal{P}^2} \mathcal{P}^1 \left( Q, \hat{\sigma}_N \right) \mathcal{P}^1 \left( Q, \hat{\omega}_N \right) < \infty,$$

where

$$\mathcal{P}^1 \left( Q, \mu \right) = \int_{\mathbb{R}^2} \frac{\ell \left( Q \right)}{\ell \left( Q \right)^2 + |x-c_Q|^2} d\mu \left( x \right) = \int_{\mathbb{R}^2} \frac{|Q| \hat{\mathcal{P}}}{|Q| + |x-c_Q|^2} d\mu \left( x \right).$$

Indeed, using $|L^k_r|_{\tilde{\omega}_N} \approx 2^{-k}$, one can verify

$$\mathcal{P}^1 \left( \hat{L}^k_r, \hat{\omega}_N \right) = |L^k_r|_{\tilde{\omega}_N}^{-1} |L^k_r|_{\tilde{\omega}_N} + \sum_{k=1}^{\infty} 2^{-2k} |L^k_r|_{\tilde{\omega}_N}^{-1} 2^{k-1} L^k_r \lesssim \hat{\omega}_N \left( \hat{L}^k_r \right) \lesssim \left( \frac{4}{3} \right)^{\ell} \left( \frac{2}{3} \right)^k \lesssim \left( \frac{4}{3} \right)^{\ell} \left( \frac{2}{3} \right)^k \lesssim \left( \frac{1}{3} \right)^\ell \left( \frac{2}{3} \right)^k \lesssim s^\ell_r.$$

From this and (7.3), we see that

$$\mathcal{P}^1 \left( \hat{L}^k_r, \hat{\omega}_N \right) \mathcal{P}^1 \left( \hat{L}^k_r, \hat{\sigma}_N \right) \lesssim \hat{\omega}_N \left( \hat{L}^k_r \right) \hat{\sigma}_N \left( \hat{L}^k_r \right) \lesssim 1.$$
8. The pivotal and energy conditions

In this subsection, we show that the backward energy constants with respect to the weight pairs \((\hat{\sigma}_N, \hat{\omega}_N)\) are unbounded in \(N\). On the other hand, we show that the weight pairs \((\sigma_N, \omega_N)\) satisfy the forward energy condition uniformly in \(N \geq 1\). Recall that \(I \equiv I \times [-\frac{1}{2}, \frac{1}{2}]\) is the square centered on the \(x_1\)-axis whose intersection with the \(x_1\)-axis is the interval \(I \subset \mathbb{R}\). Recall also that

\[
P^1 (Q, \mu) \equiv \int_{\mathbb{R}^2} \frac{|Q|^{\frac{1}{2}}}{(|Q|^{\frac{1}{2}} + |x - c_Q|)} d\mu (x),
\]

and that the forward pivotal constant \(\nu_2^1, \text{overlap} (\hat{\sigma}_N, \hat{\omega}_N)\) in the plane \(\mathbb{R}^2\) for the weight pair \((\hat{\sigma}_N, \hat{\omega}_N)\) is given by

\[\nu_2^1 \text{overlap} (\hat{\sigma}_N, \hat{\omega}_N) = \sup \frac{1}{|I| \hat{\sigma}_N} \sum_{r=1}^{\infty} P^\alpha (I, 1_{I \setminus \gamma I \hat{\sigma}_N})^2 |I_r \hat{\omega}_N|.
\]

The backward pivotal constant \(\nu_2^1, \text{overlap} (\hat{\sigma}_N, \hat{\omega}_N)\) is obtained by interchanging the roles of \(\hat{\sigma}\) and \(\hat{\omega}_N\).

8.1. Failure of the backward pivotal and backward energy conditions. Failure of the backward pivotal condition uniformly in \(N \geq 1\) is straightforward. Indeed, \(I_1 \subset I_1 \subset \ldots \subset I_1\) and so

\[
P^1 (H_{I_1}', \hat{\omega}_N) \approx P^1 (I_{I_r}', \hat{\omega}_N) \approx \sum_{k=0}^{\ell} \frac{|I_{1_k}|}{|I_{1_r}|} \hat{\omega}_N (I_{1_k}) \approx \sum_{k=0}^{\ell} \frac{3^k}{3^{2k}} 2^{-k} \approx \left(\frac{3}{2}\right)^\ell,
\]

and similarly

\[
P^1 (H_{I_r}', \hat{\omega}_N) \approx P^1 (I_{I_r}', \hat{\omega}_N) \approx \left(\frac{3}{2}\right)^\ell, \quad \text{for all } r.
\]

We also have \(|H_{\gamma I_r}'|_{\hat{\sigma}_N} \approx \left(\frac{1}{3}\right)^\ell \left(\frac{2}{3}\right)^\ell\). Considering the decomposition \(\bigcup_{\ell, r} H_{\gamma I_r}' \subset [0, 1] = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}]\) where the squares \(\gamma H_{\gamma I_r}'\) are contained in \(G_{\ell, r}\) and the \(G_{\ell, r}\) are pairwise disjoint, we thus have

\[
\sum_{\ell, r} |H_{\gamma I_r}'|_{\hat{\sigma}_N} P^1 (H_{\gamma I_r}', \hat{\omega}_N) \approx \sum_{\ell=0}^{N-1} 2^\ell \left(\frac{1}{3}\right)^\ell \left(\frac{2}{3}\right)^\ell \left(\frac{3}{2}\right)^2 \approx \sum_{\ell=0}^{N-1} 1 \approx N,
\]

which shows that the backward pivotal constants \(\nu_1, \text{overlap}^* (\hat{\sigma}_N, \hat{\omega}_N)\) are unbounded in \(N \geq 1\).

Now we consider the backward energy condition. The sum corresponding to the above is

\[
\sum_{\ell, r} |H_{\gamma I_r}'|_{\hat{\sigma}_N} E (H_{\gamma I_r}', \hat{\sigma}_N)^2 P^1 (H_{\gamma I_r}', \hat{\omega}_N)^2
\]

where

\[
E (H_{\gamma I_r}', \hat{\sigma}_N)^2 = \frac{1}{|H_{\gamma I_r}'|_{\hat{\sigma}_N}} \frac{1}{|H_{\gamma I_r}'|_{\hat{\sigma}_N}} \int_{H_{\gamma I_r}'} \int_{H_{\gamma I_r}'} |x - z|^2 \hat{\sigma}_N (x) \hat{\sigma}_N d(z) \approx 1
\]

since \(1_{H_{\gamma I_r}'}\hat{\sigma}_N\) consists of four point masses separated by a distance of approximately \(|H_{\gamma I_r}'|^2 = |H_{\gamma I_r}'|\). Thus the backward energy constants fail to be bounded in \(N \geq 1\) as well.

8.2. The forward energy condition. It remains to verify that the pair of measures \((\hat{\sigma}_N, \hat{\omega}_N)\) satisfy the forward energy conditions. We will actually establish the stronger forward pivotal condition \((\text{8.1})\), which then implies that \(E_2^1 < \infty\). For this it suffices to show that the forward maximal inequality

\[
\int M (f \hat{\sigma}_N)^2 d\hat{\omega}_N \leq C \int |f|^2 d\hat{\sigma}_N
\]

holds for the pair \((\hat{\sigma}_N, \hat{\omega}_N)\), and \((\text{8.2})\) in turn follows from the testing condition

\[
\int M (1_Q \hat{\sigma}_N)^2 d\hat{\omega}_N \leq C \int d\hat{\sigma}_N,
\]
for all squares $Q$ (see \[Saw1\]). We will show (8.3) when $Q = \hat{L}_r^\ell$, the remaining cases being an easy consequence of this one. For this we use the fact that

\begin{equation}
\mathcal{M} \left( \mathbf{1}_{\hat{L}_r^\ell} \hat{\sigma}_N \right)(x) \leq C \left( \frac{2}{3} \right)^\ell, \quad x = (x_1, x_2) \in \text{supp} \hat{\omega}_N.
\end{equation}

To see (8.4), note that for each $x = (x_1, x_2) \in \text{supp} \hat{\omega}_N \cap \hat{L}_r^\ell$, we have

\begin{equation}
\mathcal{M} \left( \mathbf{1}_{\hat{L}_r^\ell} \hat{\sigma}_N \right)(x) \leq \sup_{(k,j): x \in \hat{L}_j^k} \frac{1}{\hat{L}_j^k} \int_{\hat{L}_j^k \cap \hat{L}_r^\ell} d\hat{\sigma}_N \approx \sup_{(k,j): x \in \hat{L}_j^k} \frac{\left( \frac{1}{3} \right)^{k\ell} \left( \frac{2}{3} \right)^{k\ell}}{\left( \frac{2}{3} \right)^k} \approx \left( \frac{2}{3} \right)^\ell.
\end{equation}

Thus we have

\begin{equation}
\int_{\hat{L}_r^\ell} \mathcal{M} \left( \mathbf{1}_{\hat{L}_r^\ell} \hat{\sigma}_N \right)^2 d\hat{\omega}_N \leq C \left( \frac{2}{3} \right)^{2\ell} \hat{\omega}_N \left( \hat{L}_r^\ell \right) \approx C \left( \frac{2}{3} \right)^{2\ell}. \quad C \approx C \int_{\hat{L}_r^\ell} d\hat{\sigma}_N.
\end{equation}

This yields the case $Q = \hat{L}_r^\ell$ of (8.3), and completes our proof of the pivotal condition, and hence also of the forward energy conditions uniformly in $N \geq 1$.

9. Testing Conditions for the First Component $R_1^{1,2}$

In this section we establish both testing conditions for $R_1^{1,2}$ with respect to the weight pairs $(\hat{\sigma}_N, \hat{\omega}_N)$ uniformly in $N \geq 1$. We consider first the forward testing condition.

9.1. The Forward Testing Condition. As an initial step in verifying the forward testing condition in (2.3) with respect to the weight pair $(\hat{\sigma}_N, \hat{\omega}_N)$, namely

\begin{equation}
\mathfrak{T}_{R_1^{1,2}} \left( \hat{\sigma}_N, \hat{\omega}_N \right) \equiv \sup_{\text{squares } I} \sqrt{\frac{1}{|I|^2}} \int_{I} |H_{\hat{\sigma}_N} I|^2 \bar{d}\hat{\omega}_N < \infty,
\end{equation}

we replace $\hat{\sigma}_N$ by the self-similar measure $\hat{\sigma}_N$, and exploit the self-similarity of both measures $\hat{\sigma}_N$ and $\hat{\omega}_N$ in the following repainting identities:

\begin{align}
\hat{\omega}_N &= \frac{1}{2} \text{Dil}_{\frac{1}{9}} \hat{\omega}_N + \frac{1}{2} \text{Trans}_{\left(\frac{1}{9}, 0\right)} \text{Dil}_{\frac{1}{9}} \hat{\omega}_N \equiv \hat{\omega}_{N,1} + \hat{\omega}_{N,2}, \\
\hat{\sigma}_N &= \frac{2}{9} \text{Dil}_{\frac{1}{9}} \hat{\sigma}_N + \frac{1}{2} \hat{\sigma}_N + \frac{2}{9} \text{Trans}_{\left(\frac{1}{9}, 0\right)} \text{Dil}_{\frac{1}{9}} \hat{\sigma}_N \equiv \hat{\sigma}_{N,1} + \hat{\sigma}_{N,2},
\end{align}

where $\text{Trans}_{\left(\frac{1}{9}, 0\right)}$ is translation in the plane by the vector $\left(\frac{1}{9}, 0\right)$, and $\text{Dil}_{\frac{1}{9}}$ is dilation in the plane by the factor $\frac{1}{9}$. But now we note that

\begin{equation}
\int \left| R_1^{1,2} \hat{\sigma}_{N,1} \right|^2 \hat{\omega}_{N,1} = \frac{1}{2} \int \left| R_1^{1,2} \hat{\sigma}_{N,1} (x) \right|^2 \text{Dil}_{\frac{1}{9}} \hat{\omega}_N (x) = \frac{1}{2} \int \left| R_1^{1,2} \hat{\sigma}_{N,1} \left( \frac{x}{3} \right) \right|^2 \hat{\omega}_N (x) = \frac{2}{9} \int \left| \frac{1}{3} \right|^2 \hat{\sigma}_N (z) \hat{\omega}_N (x) = \frac{2}{9} \int \left| R_1^{1,2} \hat{\sigma}_N \right|^2 \hat{\omega}_N,
\end{equation}

and similarly $\int \left| R_1^{1,2} \hat{\sigma}_{N,2} \right|^2 \hat{\omega}_{N,2} = \frac{2}{9} \int \left| R_1^{1,2} \hat{\sigma}_N \right|^2 \hat{\omega}_N$. We then claim as in \[LaSaUr2\] that

\begin{equation}
\int \left| R_1^{1,2} \hat{\sigma}_N \right|^2 \hat{\omega} = \int \left| R_1^{1,2} \left( \hat{\sigma}_{N,1} + \hat{\sigma}_{N,2} \right) \right|^2 \hat{\omega}_{N,1} + \int \left| R_1^{1,2} \left( \hat{\sigma}_{N,1} + \hat{\sigma}_{N,2} \right) \right|^2 \hat{\omega}_{N,2} = (1 + \epsilon) \left\{ \int \left| R_1^{1,2} \hat{\sigma}_{N,1} \right|^2 \hat{\omega}_{N,1} + \int \left| R_1^{1,2} \hat{\sigma}_{N,2} \right|^2 \hat{\omega}_{N,2} \right\} + \mathcal{R}_\epsilon,
\end{equation}

where \[\mathcal{R}_\epsilon\] is a remainder term.
for any $\varepsilon > 0$ where the remainder term $R_\varepsilon$ is easily seen to satisfy

$$R_\varepsilon \lesssim \varepsilon A_2^2 \left( \int \hat{\sigma}_N \right),$$

since the supports of $\delta_{\frac{1}{2}} + \hat{\sigma}_{N,2}$ and $\hat{\omega}_{N,1}$ are well separated, as are those of $\delta_{\frac{1}{2}} + \hat{\sigma}_{N,1}$ and $\hat{\omega}_{N,2}$. This is proved exactly as in [LaSaUr2] so we will be brief. We have

$$\int |R_1^{1,2} (\hat{\sigma}_{N,1} + \delta_{\frac{1}{2}} + \hat{\sigma}_{N,2})|^2 \hat{\omega}_{N,1} \lesssim \int \left\{ (1 + \varepsilon) |R_1^{1,2} \hat{\sigma}_{N,1}|^2 + (1 + \frac{1}{\varepsilon}) |R_1^{1,2} (\delta_{\frac{1}{2}} + \hat{\sigma}_{N,2})|^2 \right\} \hat{\omega}_{N,1},$$

and then,

$$\int |R_1^{1,2} \hat{\sigma}_{N,2}|^2 \hat{\omega}_{N,1} = \int_{[0,1]} \left| \int_{\frac{1}{2} - \varepsilon}^{\frac{1}{2} + \varepsilon} \frac{1}{y-x} \hat{\omega}_{N,2} (y) \hat{\omega}_N (x) \right|^2 A_2^2 \int \hat{\sigma}_N.$$

But now we note that

$$\int |R_1^{1,2} \hat{\sigma}_{N,1}|^2 \hat{\omega}_{N,1} = \frac{2}{9} \int |R_1^{1,2} \hat{\sigma}_N|^2 \hat{\omega}_N,$$

and similarly $\int |R_1^{1,2} \hat{\sigma}_{N,2}|^2 \hat{\omega}_{N,1} = = \frac{2}{9} \int |R_1^{1,2} \hat{\sigma}_N|^2 \hat{\omega}_N$. Thus we have

$$\int |R_1^{1,2} \hat{\sigma}_N|^2 \hat{\omega}_N = \frac{2}{9} \int \left( 1 + \varepsilon \right) \left| R_1^{1,2} \hat{\sigma}_N \right|^2 \hat{\omega}_N + \frac{2}{9} (1 + \varepsilon) \int \left| R_1^{1,2} \hat{\sigma}_N \right|^2 \hat{\omega}_N + R_\varepsilon,$$

and since $\int |R_1^{1,2} \hat{\sigma}_N|^2 \hat{\omega}_N$ is finite we conclude that

$$\int |R_1^{1,2} \hat{\sigma}_N|^2 \hat{\omega}_N = \frac{1}{1 - \frac{4}{9} (1 + \varepsilon)} R_\varepsilon \lesssim \varepsilon A_2^2 \left( \int \hat{\sigma}_N \right),$$

for $\varepsilon > 0$ so small that $1 - \frac{4}{9} (1 + \varepsilon) > 0$. This completes the proof of the forward testing condition in (2.3) for the cube $I = [0,1)$ with respect to the weight pair $\left( \hat{\sigma}_N, \hat{\omega}_N \right)$ uniformly in $N \geq 1$. The proof for the case $\hat{L} = \hat{L}_j^k$ is similar using $R_\varepsilon \left( \hat{L}_j^k \right) \leq C_\varepsilon A_2^2 \left( \int \hat{\sigma}_N \right)$, and the general case now follows without much extra work.

Having verified the forward testing condition for the weight pair $\left( \hat{\sigma}_N, \hat{\omega}_N \right)$, we now show that the forward testing condition in (2.3) holds for $\left( \hat{\sigma}_N, \hat{\omega}_N \right)$. For this, we estimate the difference as in [LaSaUr2] by

$$\int_{\hat{L}_j^k} \int_{\hat{L}_j^k} |R_1^{1,2} \hat{L}_j^k \hat{\sigma}_N|^2 \hat{\omega}_N \leq C \int_{\hat{L}_j^k} \left( \sum_{(k,j): z_j^k \in \ell_j^k} \sum_{s_j^k} \left| \frac{|I_k|}{|x - (z_j^k, \pm \frac{i}{3} + \eta_{j,s_j^k})|^2} \right|^2 \hat{\omega}_N (x) \right).$$

Now for any fixed $x$ in the support of $\hat{\omega}_N$ inside $\hat{L}_j^k$, we have just as in [LaSaU]2 that

$$\sum_{(k,j): z_j^k \in \ell_j^k} \sum_{s_j^k} \left| \frac{|I_k|}{|x - (z_j^k, \pm \frac{i}{3} + \eta_{j,s_j^k})|^2} \right|^2 \lesssim \left( \frac{2 \varepsilon}{3} \right)^{\ell}.$$

Thus we get

$$\int_{\hat{L}_j^k} \int_{\hat{L}_j^k} |R_1^{1,2} \hat{L}_j^k \hat{\sigma}_N|^2 \hat{\omega}_N \lesssim \left( \frac{2 \varepsilon}{3} \right)^{2\ell} \hat{\omega}_N (\hat{L}_j^k) = C^2 \left( \frac{2 \varepsilon}{3} \right)^{2\ell} \approx \hat{\sigma}_N (\hat{L}_j^k),$$

which yields

$$\left( \int_{\hat{L}_j^k} \int_{\hat{L}_j^k} |R_1^{1,2} \hat{L}_j^k \hat{\sigma}_N|^2 \hat{\omega}_N \right)^{\frac{1}{2}} \lesssim \left( \int_{\hat{L}_j^k} \int_{\hat{L}_j^k} |R_1^{1,2} \hat{L}_j^k \hat{\sigma}_N|^2 \hat{\omega}_N \right)^{\frac{1}{2}} + \left( \int_{\hat{L}_j^k} \int_{\hat{L}_j^k} \left| \hat{L}_j^k \hat{\sigma}_N \right|^2 \hat{\omega}_N \right)^{\frac{1}{2}} \lesssim C \sqrt{\sigma_N (\hat{L}_j^k)}.$$

This is the case $I = \ell_j^k$ of the forward testing condition in (2.3) for the weight pair $\left( \hat{\sigma}_N, \hat{\omega}_N \right)$ uniformly in $N \geq 1$, and the general case follows from this by an additional argument. This additional argument is
given explicitly in a related situation in [SaShUr11, Subsection 5.2.4 Completion of the proof for general intervals], to which we refer the reader for details.

9.2. The backward testing condition. Finally, we turn to the dual testing condition for \( R_{1}^{1,2} \) in (2.3) with respect to the weight pair \((\sigma_{N}, \bar{\omega}_{N})\), namely

\[
\mathcal{T}_{R_{1}^{1,2}}^{*} = \sup_{\text{squares } I} \left\{ \int_{I} \left| \frac{1}{|I|} \int_{I} |H_{\bar{\omega}_{N}}| \, d\sigma_{N} \right| \right\}.
\]

For an interval \( I_{r}' \) with \( z_{j}^{k} \in I_{r}' \), we claim that

\[
\left| R_{1}^{1,2} \left( 1_{\bar{\omega}_{N}} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right| \lesssim P^{1} \left( \bar{\omega}_{N} \right).
\]

This is a substantial improvement over the estimate \( \left| R_{1}^{1,2} \left( 1_{\bar{\omega}_{N}} \right) \left( c_{j}^{k}, 0 \right) \right| \lesssim \left( \frac{3}{2} \right)^{k} \), and is a consequence of the fact that the points \( \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \) are zeroes of the function \( R_{1}^{1,2} \bar{\omega}_{N} \). To see (9.4) let \( I_{r}^{s-1} \) denote the parent of \( I_{r}' \), and let \( I_{r+1}^{s-1} \) denote the other child of \( I_{r}^{s-1} \). Then we have using \( R_{1}^{1,2} \bar{\omega}_{N} \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) = 0 \),

\[
R_{1}^{1,2} \left( 1_{\bar{\omega}_{N}} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) = -R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) - R_{1}^{1,2} \left( 1_{\bar{\omega}_{N}} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right).
\]

Now we have using \( R_{1}^{1,2} \bar{\omega}_{N} \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) = 0 \) that

\[
R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) = R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) - \left\{ R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) - R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right\}.
\]

where

\[
A \equiv R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) - R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right).
\]

Combining equalities yields

\[
R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) = R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) + A - R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) + A - R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right).
\]

We then have for \((k, j)\) such that \( z_{j}^{k} \in I_{r}' \),

\[
\left| R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right| \lesssim \bar{\omega}_{N} \left( \frac{I_{r}'}{L_{s}^{1-1}} \right),
\]

\[
|A| \lesssim \int_{I_{r}'} \left| K \left( x - \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right) \right| \bar{\omega}_{N} (x),
\]

\[
\lesssim \int_{I_{r}'} \left| x - \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right| \bar{\omega}_{N} (x),
\]

\[
\left| R_{1}^{1,2} \left( 1 \right) \left( \bar{\omega}_{N} \right) \left( z_{j}^{k}, \pm \frac{1}{4 \cdot 3^{k+1}} \right) \right| \lesssim \bar{\omega}_{N} \left( \frac{I_{r}'}{L_{s}^{1-1}} \right).
\]
which proves (9.4).

Now we compute, using (9.4) and the estimate $P^1(I_r^\ell, \omega) \lesssim \widehat{\sigma}_N(I_r^\ell, \omega)$ proved above, that

$$
\int_{L_r^\ell} \left| R_1^{1.2} \left( 1_{L_r^\ell} \widehat{\omega}_N \right) \right|^2 \, d\sigma_N = \sum_{(k,j): z_k^j \in I_r^\ell} \left| R_1^{1.2} \left( 1_{L_r^\ell} \widehat{\omega}_N \right) \left( z_k^j, \pm \frac{1}{4 \cdot 3^k+1} \right) \right|^2 s_k^j \leq C \sum_{(k,j): z_k^j \in I_r^\ell} \left| P^1 \left( I_r^\ell, \widehat{\omega}_N \right) \right|^2 s_k^j
$$

$$
\lesssim \widehat{\sigma}_N(I_r^\ell) \left( \left| \frac{\widehat{\omega}_N(I_r^\ell)}{|I_r^\ell|} \right| \right)^2 \lesssim A_2 \widehat{\omega}_N(I_r^\ell).
$$

This is the case $I = I_r^\ell$ of the dual testing condition in (2.3) for the weight pair $(\widehat{\sigma}_N, \widehat{\omega}_N)$ uniformly in $N \geq 1$, and the general case follows from this, just as for the forward testing condition above, using the additional argument given in [SaShUr11, Subsubsection 5.2.4 Completion of the proof for general intervals], but adapted to the backward testing condition.

10. TESTING CONDITIONS FOR THE SECOND COMPONENT $R_2^{1.2}$

In this section we establish both forward and backward testing conditions for $W_r^{1.2}$ with respect to the weight pairs $(\widehat{\sigma}_N, \widehat{\omega}_N)$ uniformly in $N \geq 1$. Recall that $K^{1.2}(w_1, w_2) = (\frac{w_1}{w_1^2+w_2^2}, \frac{w_2}{w_1^2+w_2^2})$, and $s_k^j = (\frac{1}{3})^k (\frac{2}{3})^k$. We consider first the backward testing condition for $R_2^{1.2}$.

10.1. The backward testing condition. Recall the estimate (6.6),

$$
R_2^{1.2} \widehat{\omega}_N \left( z_k^j, \frac{1}{4 \cdot 3^k+1} \right) \approx 2^{-N} \beta^k,
$$

and the definitions

$$
\widehat{\sigma}_N^+ = \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} s_i^k \delta(z_k^i, \frac{1}{4 \cdot 3^k+1}),
$$

and

$$
\widehat{\omega}_N = 2^{-N-2} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left( \delta \left( \frac{\epsilon_{i,3}^{N+1}(G_i^+) + \frac{1}{4 \cdot 3^k+1}}{4 \cdot 3^k+1} \right) + \delta \left( \frac{\epsilon_{i,3}^{N+1}(G_i^+) - \frac{1}{4 \cdot 3^k+1}}{4 \cdot 3^k+1} \right) \right) + 2^{-N-2} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left( \delta \left( \frac{\epsilon_{i,3}^{N+1}(G_i^+) - \frac{1}{4 \cdot 3^k+1}}{4 \cdot 3^k+1} \right) - \delta \left( \frac{\epsilon_{i,3}^{N+1}(G_i^+) - \frac{1}{4 \cdot 3^k+1}}{4 \cdot 3^k+1} \right) \right) + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}.
$$

Thus we have the estimate

$$
\int_{[0,1]} \int_{[0,1]} \left| R_2^{1.2} \left( \frac{1}{[0,1]} \widehat{\omega}_N \right) \right|^2 \, d\sigma_N = \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left| R_2^{1.2} \widehat{\omega}_N \left( z_k^i, \frac{1}{4 \cdot 3^k+1} \right) \right|^2
$$

$$
\approx \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^k \left( 2^{-N} \beta^k \right)^2
$$

$$
= 2^{-2N} \sum_{k=0}^{N-1} \sum_{i=1}^{2^k} \approx 1 = \left| \widehat{\omega}_N \right|_{[0,1]}.
$$

More generally, for any square $L_j^\ell$, we have

$$
\int_{L_j^\ell} \left| R_2^{1.2} \left( 1_{L_j^\ell} \widehat{\omega}_N \right) \right|^2 \, d\sigma_N = \sum_{k=\ell}^{N-1} \sum_{i: L_i^\ell \subset L_j^\ell} \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right)^k \left( 2^{-N} \beta^k \right)^2 = \sum_{k=\ell}^{N-1} 2^{k-\ell} 2^{k-2N} \approx 2^{-\ell} \approx |L_j^\ell| \widehat{\omega}_N,
$$

and the general case follows easily.
10.2. The forward testing condition. Now we turn to the forward testing condition for $R^{1,2}_2$, namely the inequality
\[
\int \int_Q \left| R^{1,2}_2 (1_Q \delta_{\tilde{N}}) \right|^2 d\tilde{\omega}_N \lesssim |Q|_{\tilde{\sigma}_N}, \quad \text{for all squares } Q \subset [0,1],
\]
and just as for the first component $R^{1,2}_1$ above, we first prove the forward testing condition for the measure pair $\left( \delta_{\tilde{N}}, \tilde{\omega}_N \right)$,
\[
\int \int_Q \left| R^{1,2}_2 (1_Q \delta_{\tilde{N}}) \right|^2 d\tilde{\omega}_N \lesssim |Q|_{\tilde{\sigma}_N}, \quad \text{for all squares } Q \subset [0,1].
\]
We have already noted in (9.1) the replication identities satisfied by the measure pair $\left( \delta_{\tilde{N}}, \tilde{\omega}_N \right)$. Just as for $R^{1,2}_1$ we compute for the operator $R^{1,2}_2$ that
\[
\int \left| R^{1,2}_2 \sigma_{\tilde{N},2} \right|^2 \tilde{\omega}_{N,1} = \frac{2}{9} \int \left| R^{1,2}_2 \sigma_{\tilde{N}} \right|^2 \tilde{\omega}_N = \int \left| R^{1,2}_2 \sigma_{\tilde{N},2} \right|^2 \tilde{\omega}_{N,2},
\]
and then that
\[
\int \left| R^{1,2}_2 \sigma_{\tilde{N}} \right|^2 \omega = \int \left| R^{1,2}_2 \left( \sigma_{\tilde{N},1} + \delta_{\frac{1}{2}} + \tilde{\sigma}_{N,2} \right) \right|^2 \tilde{\omega}_{N,1} + \int \left| R^{1,2}_2 \left( \sigma_{\tilde{N},2} + \delta_{\frac{1}{2}} + \tilde{\sigma}_{N,2} \right) \right|^2 \tilde{\omega}_{N,2} \leq \left( 1 + \varepsilon \right) \left\{ \int \left| R^{1,2}_2 \sigma_{\tilde{N},1} \right|^2 \tilde{\omega}_{N,1} + \int \left| R^{1,2}_2 \sigma_{\tilde{N},2} \right|^2 \tilde{\omega}_{N,2} \right\} + R_\varepsilon,
\]
where $R_\varepsilon$ is easily seen to satisfy
\[
\left| R_\varepsilon \right| \lesssim \left| \delta_{\tilde{N}} \right|,
\]
for $\varepsilon > 0$ where the remainder term $R_\varepsilon$ is easily seen to satisfy
\[
\left| R_\varepsilon \right| \lesssim \left| \delta_{\tilde{N}} \right|,
\]
since the supports of $\delta_{\frac{1}{2}} + \tilde{\sigma}_{N,2}$ and $\tilde{\omega}_{N,1}$ are well separated, as are those of $\delta_{\frac{1}{2}} + \tilde{\sigma}_{N,1}$ and $\tilde{\omega}_{N,2}$. Now we simply proceed as before, and leave the details to the interested reader. Finally, just as we did for $R^{1,2}_1$ above, we use a perturbation argument to obtain the forward testing condition for the measure pair $\left( \delta_{\tilde{N}}, \tilde{\omega}_N \right)$, uniformly in $N \geq 1$.

11. The norm inequality

Here we show that the norm inequality for $R^{1,2}$ holds with respect to the weight pair $\left( \tilde{\sigma}_N, \tilde{\omega}_N \right)$ uniformly in $N$. We first observe that we have already established above the following facts for the weight pairs $\left( \tilde{\sigma}_N, \tilde{\omega}_N \right)$ uniformly in $N \geq 1$. Let $\tilde{\sigma}_N$ denote the reflection of $\tilde{\sigma}_N$ across the $x_1$-axis.

1. The Muckenhoupt/NTV condition $A_2$ holds:
\[
\sup_{Q \in P^2} \left\{ \frac{|Q|_{\tilde{\sigma}_N^+}}{\sqrt{|Q|}} : \mathcal{P}^1(Q, \tilde{\omega}_N) + \mathcal{P}^1(Q, \tilde{\sigma}_N^+ \mathcal{P}^1(Q, \tilde{\omega}_N)) \right\} = A_2 < \infty.
\]

2. The forward testing condition holds:
\[
\int |R_{1,2} (1_Q \delta_{\tilde{N}})|^2 d\tilde{\omega}_N \lesssim |Q|_{\tilde{\sigma}_N^+}.
\]
Indeed, if a square $Q$ is symmetric about the $x_1$-axis, then both $|Q|_{\tilde{\sigma}_N^+} = |Q|_{\tilde{\sigma}_N^-} = \frac{1}{2} |Q|_{\tilde{\sigma}_N}$ and
\[
\int |R_{1,2} \delta_{\tilde{N}}|^2 d\tilde{\omega}_N = \int |R_{1,2} \delta_{\tilde{N}}|^2 d\tilde{\omega}_N
\]
by symmetry. Since the testing condition holds for the weight pair $\left( \delta_{\tilde{N}}, \tilde{\omega}_N \right)$, we easily obtain (11.1) for such symmetric squares $Q$. The general case now follows easily from this.

3. The backward testing condition holds:
\[
\int |R_{1,2} \tilde{\sigma}_N|^2 d\tilde{\sigma}_N^+ \lesssim |Q|_{\tilde{\omega}_N}.
\]
since it holds for the larger measure $\tilde{\sigma}_N$ in place of $\tilde{\sigma}_N^+$. 

(4) The forward energy condition holds:
\[
\sum_{\cup_{r=1}^{\infty}, Q_r \subset R} \left( \frac{P^1_Q \left( \frac{1}{R} \hat{\sigma}_N^+ \right)}{\sqrt{|Q_r|}} \right)^2 \left\| P_Q x \right\|_{L^2(\hat{\sigma}_N)}^2 \lesssim |R| \hat{\sigma}_N^+.
\]

(5) The backward energy condition holds:
\[
\sum_{\cup_{r=1}^{\infty}, Q_r \subset R} \left( \frac{P^1_Q \left( \frac{1}{R} \hat{\sigma}_N^+ \right)}{\sqrt{|Q_r|}} \right)^2 \left\| P_Q x \right\|_{L^2(\hat{\sigma}_N)}^2 \lesssim |R| \hat{\sigma}_N^+.
\]

Now we can apply our T1 theorem with an energy side condition in [SaShUr7] (or see [SaShUr6] or [SaShUr4]) to obtain the dual norm inequality
\[
\int |R^{1,2} (g \hat{\sigma}_N)|^2 d\hat{\sigma}_N^+ \lesssim \int |g|^2 d\hat{\sigma}_N,
\]
\[\text{i.e. } \mathcal{M}_{R^{1,2}} (\hat{\sigma}_N^+, \hat{\sigma}_N^+) < \infty.\]

Consider now the weight pair \((\hat{\sigma}_N, \hat{\omega}_N)\). We have \(\hat{\sigma}_N = \hat{\sigma}_N^+ + \hat{\sigma}_N^-\) and \(\mathcal{M}_{R^{1,2}} ((\hat{\sigma}_N^+, \hat{\omega}_N)) = \mathcal{M}_{R^{1,2}} (\hat{\sigma}_N^+, \hat{\omega}_N)\) by symmetry, and so
\[
\mathcal{M}_{R^{1,2}} ((\hat{\sigma}_N^+, \hat{\omega}_N)) \leq \mathcal{M}_{R^{1,2}} (\hat{\sigma}_N^+, \hat{\omega}_N) + \mathcal{M}_{R^{1,2}} ((\hat{\sigma}_N^-, \hat{\omega}_N)) = 2\mathcal{M}_{R^{1,2}} (\hat{\sigma}_N^+, \hat{\omega}_N) < \infty.
\]

Thus we have shown that the two weight norm inequality for the Riesz transform \(R^{1,2}\) holds in the plane with respect to the weight pair \((\hat{\sigma}_N, \hat{\omega}_N)\) uniformly in \(N \geq 1\), and in Subsubsection 8.1 above, we showed that the backward energy constants with respect to the weight pairs \((\hat{\sigma}_N, \hat{\omega}_N)\) are unbounded in \(N \geq 1\). This completes the proof of Theorem [4] in the special case \(\alpha = 1\) and \(n = 2\).

12. THE GENERAL CASE \(0 \leq \alpha < n\) AND \(n \geq 2\)

The measure pair \((\hat{\sigma}_N, \hat{\omega}_N)\) just constructed above in the plane serves to show that the energy conditions are not implied by boundedness of the fractional Riesz transform \(R^{n-1,n}\) of order \(n-1\) in \(\mathbb{R}^n\) for \(n \geq 2\) - simply embed the measures in the two-dimensional subspace \(\mathbb{R}^2\) spanned by the unit coordinate vectors \(e_1\) and \(e_2\). The reason for this is that the restriction of the convolution kernel \(K^{n-1,n} (w) = \frac{(w_1, w_2)}{|(w_1, w_2)|^{n+1-(n-1)}}\) to \(\mathbb{R}^2\) is the kernel \(K^{1,2} (u) = \frac{(u_1, u_2)}{|(u_1, u_2)|^{n+1-(n-1)}}\). If we remain in dimension \(n = 2\), but permit \(0 \leq \alpha < 2\), then the argument above applies if we take
\[
s^1_k = \left( \frac{1}{3} \right)^k (3-2\alpha) \left( \frac{1}{3} \right)^k,
\]
along with similar arithmetic adjustments elsewhere.

In the general case \(0 \leq \alpha < n, n \geq 2\), we start with the computation that
\[
\frac{d}{dx_1} \left( \frac{1}{(x_1 - y_1)^2 + (x_2 - y_2)^2} \right)^{\frac{n+1-\alpha}{2}} \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{\frac{\alpha}{2}}
\]
\[
= \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{\frac{\alpha}{2}} \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 \right)^{\frac{n+1-\alpha}{2}}
\]
\[
= \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 - (n+1-\alpha)(x_1 - y_1)^2 \right)^{\frac{n+1-\alpha}{2}}
\]
\[
= \left( (x_1 - y_1)^2 + (x_2 - y_2)^2 - (n-\alpha)(x_1 - y_1)^2 \right)^{\frac{n+1-\alpha}{2}}
\]
\[
= \left( (x_2 - y_2)^2 - (n-\alpha)(x_1 - y_1)^2 \right)^{\frac{n+1-\alpha}{2}} < 0,
\]
provided
\[
|x_2 - y_2| < \sqrt{n-\alpha} |x_1 - y_1|.
\]
Thus in the subcase $0 \leq \alpha < n - 1$ and $|x_2 - y_2| < |x_1 - y_1|$, the $x_1$ derivative of the kernel $K^{\alpha,n}_1 (x - y)$ is negative, and the above construction of a family of weight pairs in the plane can be modified in a purely arithmetic way so as to show that the energy conditions are not necessary for boundedness of the fractional Riesz transform $R^{\alpha,n}$. The modified measure pair $(\hat{\sigma}_N, \hat{\omega}_N)$ lives in the two-dimensional subspace $\mathbb{R}^2$, and as a consequence, the components $R^{\alpha,n}_3, R^{\alpha,n}_4, \ldots, R^{\alpha,n}_n$ of $R^{\alpha,n}$ are all trivially bounded since both $R^{\alpha,n}_j \hat{\sigma}_N \equiv 0$ and $R^{\alpha,n}_j \hat{\omega}_N \equiv 0$ for $j \geq 3$.

However, in the subcase $n - 1 < \alpha < n$, we must alter the geometry as well, by translating the point masses of $\omega_N$ at an angle less than $\theta_{a,n}$ instead of less than $\frac{\pi}{4} = 45^\circ$, where

$$\tan \theta_{n,\alpha} = \gamma_{n,\alpha} = \sqrt{n - \alpha}.$$ 

The angle $\theta_{n,\alpha}$ is less than $\frac{\pi}{4} = 45^\circ$ precisely when $n - 1 < \alpha < n$, and with this geometric alteration, the above construction again goes through with only changes in arithmetic.

**Remark 14.** If $(\hat{\sigma}_N, \hat{\omega}_N)$ is the weight pair constructed above, then a very lengthy but straightforward computation shows that the family of localized operators $\{R^{\alpha,n}_j(\hat{\sigma}_N)\}_{j \in \mathbb{N}}$ is uniformly bounded from $L^2(\hat{\sigma}_N)$ to $L^2(\hat{\omega}_N)$. Indeed, the weight pair $(\hat{\sigma}_N, \hat{\omega}_N)$ satisfies the Muckenhoupt and energy conditions uniformly in $N \geq 1$ by Lemma 2 and the Calderón-Zygmund norms of the kernels of $R^{\alpha,n}_j(\hat{\sigma}_N)$ are uniformly bounded for $j \in \mathbb{N}$ and $1 \leq j \leq N$. Finally, the testing constants $\tilde{\Sigma}_{R^{\alpha,n}_j(\hat{\sigma}_N)}(\hat{\omega}_N)$ are uniformly bounded for $j \in \mathbb{N}$, $\hat{\sigma}_N \geq 1$ and $1 \leq j \leq M$. Thus from the T1 theorem in [SaShUr], with an energy side condition, we obtain the boundedness of the operators $R^{\alpha,n}_j(\hat{\sigma}_N)$ from $L^2(\hat{\sigma}_N)$ to $L^2(\hat{\omega}_N)$ uniform in $N \geq 1$. We leave details to the interested reader.

**References**

[DaJo] David, Guy, Journé, Jean-Lin, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. (2) 120 (1984), 371–397, MR763911 (85k:42041).

[DaJoSe] David, G., Journé, J.-L., and Semmes, S., Opérateurs de Calderón-Zygmund, fonctions para-acrétives et interpolation. Rev. Mat. Iberoamericana 1 (1985), 1–56.

[HuMuWh] R. Hunt, B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for the conjugate function and the Hilbert transform, Trans. Amer. Math. Soc. 176 (1973), 227-251.

[Hyt2] Hytönen, Tuomas, The two weight inequality for the Hilbert transform with general measures, [arXiv:1310.08531v1].

[Lac] Lacey, Michael T., Two weight inequality for the Hilbert transform: A real variable characterization, II, Duke Math. J. Volume 163, Number 15 (2014), 2821-2840.

[Lac2] Lacey, Michael T., The two weight inequality for the Hilbert transform: a primer, arXiv:1304.5004v1.

[LaMa] M. T. Lacey and H. Martikainen, Local Tb theorem with $L^2$ testing conditions and general measures: Calderón-Zygmund operators, arXiv:1310.0853v1.

[LaSaUr1] Lacey, Michael T., Sawyer, Eric T., Uriarte-Tuero, Ignacio, A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure, Analysis & PDE, Vol. 5 (2012), No. 1, 1-60.

[LaSaUr2] Lacey, Michael T., Sawyer, Eric T., Uriarte-Tuero, Ignacio, A Two Weight Inequality for the Hilbert transform assuming an energy hypothesis, Journal of Functional Analysis, Volume 263 (2012), Issue 2, 305-363.

[LaSaShUr1] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, The Two weight inequality for Hilbert transform, coronas, and energy conditions, arXiv: (2011).

[LaSaShUr2] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, [arXiv:1201.4319] (2012).

[LaSaShUr3] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, Two weight inequality for the Hilbert transform: A real variable characterization I, Duke Math. J, Volume 163, Number 15 (2014), 2795-2820.

[LaSaShUrWi] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, Wick, Brett D., Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}$, arXiv:1310.4820v4.

[LaWi] Lacey, Michael T., Wick, Brett D., Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}$, arXiv:1310.4820v4.

[LaWi1] Lacey, Michael T., Wick, Brett D., Two weight inequalities for Riesz transforms: uniformly full dimension weights, arXiv:1312.6163v1, arXiv:1312.6163v3.

[Naz] Nazarov, F., Treil, S., and Volberg, A., Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures, preprint (2004) arXiv:1003.1596.

[Saw1] E. Sawyer, A characterization of a two-weight norm inequality for maximal operators, Studia Math. 75 (1982), 1-11, MR(676801 (84i:2032)).

[Saw] E. Sawyer, A characterization of two weight norm inequalities for fractional and Poisson integrals, Trans. A.M.S. 308 (1988), 533-545, MR(930072 (89d:26009)).
Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight theorem for $\alpha$-fractional singular integrals with an energy side condition, arXiv:1302.5093v8.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms, arXiv:1310.4484v1.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A note on failure of energy reversal for classical fractional singular integrals, IMRN, Volume 2015, Issue 19, 9888-9920.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms, arXiv:1310.4484v1.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight theorem for $\alpha$-fractional singular integrals with an energy side condition and quasicube testing, arXiv:1302.5093v10.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight theorem for $\alpha$-fractional singular integrals with an energy side condition, quasicube testing and common point masses, arXiv:1505.07816v2.v3.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight theorem for $\alpha$-fractional singular integrals with an energy side condition, Revista Mat. Iberoam. 32 (2016), no. 1, 79-174.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, The two weight $T1$ theorem for fractional Riesz transforms when one measure is supported on a curve, arXiv:1505.07822v4.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight fractional singular integral theorem with side conditions, energy and $k$-energy dispersed, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory (Volume 2) (Celebrating Cora Sadosky’s life), Springer 2017 (see also arXiv:1603.04332v2).

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A good-$\lambda$ lemma, two weight $T1$ theorems without weak boundedness, and a two weight accretive global $Tb$ theorem, Harmonic Analysis, Partial Differential Equations and Applications (In Honor of Richard L. Wheeden), Birkhäuser 2017 (see also arXiv:1609.08125v2).

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A counterexample in the theory of Calderón-Zygmund operators, arXiv:1609.06071v3.

Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A two weight local $Tb$ theorem for the Hilbert transform, arXiv:1709.09595v6.

Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, Amer. J. Math. 114 (1992), 813-874.

E. M. Stein, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, N. J., 1993.

A. Volberg, Calderón-Zygmund capacities and operators on nonhomogeneous spaces, CBMS Regional Conference Series in Mathematics (2003), MR(2019058 (2005c:42015)).

Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1

E-mail address: sawyer@mcmaster.ca