Classical ladder operators, polynomial Poisson algebras and classification of superintegrable systems

Ian Marquette
School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia
i.marquette@uq.edu.au

We recall results concerning one-dimensional classical and quantum systems with ladder operators. We obtain the most general one-dimensional classical systems respectively with a third and a fourth order ladder operators satisfying polynomial Heisenberg algebras. These systems are written in terms of the solutions of quartic and quintic equations. They are the classical equivalent of quantum systems involving the fourth and the fifth Painlevé transcendent. We use these results to present two new families of superintegrable systems and examples of trajectories that are deformation of Lissajous’s figures.

1 Introduction

The most well known system with ladder operators is the 1D harmonic oscillator with first order raising and lowering operators. Such operators first order creation and annihilation operators were also used in context of the N-dimensional isotropic and the anisotropic harmonic oscillator. The integrals of motion of these multi-dimensional systems can be written in terms of these operators [1,2,3]. The second order ladder operators of two-dimensional superintegrable Hamiltonians were obtained in the 60ties by Winternitz and al [4] and a classification of two-dimensional systems with second order ladder operators was performed later by Miller and Boyer [5]. More recently, systems with third order ladder operators were studied [6-19]. It was shown that a 1D quantum system with third order ladder operators can be con-
structured from supersymmetric quantum mechanics (SUSYQM) with a first and second order supercharges [11,16,19]. The potential, the supercharges, the ladder operators and the zero modes were written in terms of the fourth Painlevé transcendent \( \mathcal{P}_4 \). The case of quantum 1D systems with fourth order ladder operators and the relation with the fifth Painlevé transcendent was first explored in Ref.20 and Ref.21 and also later with extensions to 2D superintegrable systems and the relations with integrals of motion [22]. These systems were written explicitly in terms of the fifth Painlevé transcendent \( \mathcal{P}_5 \). These systems with higher order ladder operators can allow many zero modes for the annihilation and creation operators. It was discussed how superintegrable systems generated from such 1D systems can have very interesting spectrum with more complicated degeneracies [16,22] due to the presence of singlet, doublet and triplet states.

In classical mechanics, a particular case of a system with third order integral of motion [13,23] was related to third order ladder operators [24]. Let us mention that classical systems with third order ladder operators were also explored in an earlier paper of Eleonsky and Korolev [18]. The purpose of this paper is to present a classification of classical systems with third and fourth order ladder operators satisfying polynomial Heisenberg algebras and show how these results can be used to generate superintegrable systems.

The polynomial (i.e. polynomial in the momenta) ladder operators described in Ref.11,16,18 and 19 satisfy polynomial Heisenberg algebras. However, for many quantum systems we need to introduce nonpolynomial ladder operators and generalized Heisenberg algebras. For example, 1D systems with quadratic energy spectrum and their classical equivalent allow such nonpolynomial ladder operators [18,25-28]. The infinite well and the Morse potentials are examples of such systems. The existence of raising and lowering operators is also not restricted to systems with separation of variables in Cartesian coordinates [3] and Hamiltonians with a scalar potential. Ladder operators in context of systems with a monopole interaction were discussed recently in the case of generalized MICZ-Kepler [29] and generalized Kaluza-Klein monopole [30].
These ladder operators are also related to other approaches in quantum and classical mechanics. The relations between integrals, supersymmetric quantum mechanics and ladder operators were discussed by various authors [1-5,14,15,22,31-36]. These connections provide new methods to study known superintegrable systems, or even obtain new one and their corresponding polynomial algebra [14,15,22,36].

Let us present the organization of the paper. In Section 2, we recall results on 1D systems with ladder operators in quantum and classical mechanics. In Section 3, we obtain classical systems with third order ladder operators that satisfy a polynomial Heisenberg algebra. In Section 4, we discuss the case of fourth order ladder operators. In Section 5, we recall a method discussed in earlier paper to generate superintegrable systems in context of classical mechanics from analog of lowering and raising operators [14]. We obtain from results of Section 3 and Section 4 two new families of superintegrable systems and moreover we present examples of trajectories.

2 Quantum and classical systems with ladder operators

Let us consider the following 1D Hamiltonian of the form:

$$H = \frac{P^2}{2} + V(x), \quad P = -i\hbar \partial_x,$$

(2.1)

with a polynomial ladder operators of order $n$ (i.e. polynomial in momenta),

$$A^\pm = \begin{cases} \pm iP^n + f_{n-1}(x)P^{n-1} + \ldots \pm if_2(x)P^2 + f_1(x)P \pm if_0 & \text{even}, \\ P^n \pm if_{n-1}(x)P^{n-1} + \ldots \pm if_2(x)P^2 + f_1(x)P \pm if_0 & \text{odd}, \end{cases}$$

(2.2)

that satisfy a polynomial Heisenberg algebra in quantum mechanics or the classical equivalent Poisson algebra in the context of classical mechanics.
In quantum mechanics, a polynomial Heisenberg algebra satisfy the following algebraic relations:

\[
[H, A^\dagger] = \omega A^\dagger, \quad [H, A] = -\omega A, \quad (2.3)
\]
\[
AA^\dagger = Q(H + \omega), \quad A^\dagger A = Q(H), \quad [A, A^\dagger] = P(H), \quad (2.4)
\]
with \(P(H) = Q(H + \omega) - Q(H)\). The polynomial Heisenberg algebra generated by \(\{H, A, A^\dagger\}\) provides informations on the spectrum of \(H\). We have for \(Q(H)\) a \(n\) th-order polynomial in \(H\). The annihilation operator can allow at most \(n\) zero modes (i.e. a state such \(A\psi = 0\)). In each axis we can have at most \(n\) infinite ladders by acting iteratively with the creation operator. The creation operator can also allow zero modes (i.e. a state such \(A^\dagger\psi = 0\)) and we can have finite ladder operators. Such operators and their corresponding polynomial Heisenberg algebras are important in context of coherent states [37].

In classical mechanics we can also define analog of raising and lowering operators [2] \((A^- \text{ and } A^+)\) and the corresponding polynomial Heisenberg algebra. We replace commutation relations by Poisson brackets. These operators satisfy the algebraic relations:

\[
\{H, A^+\}_p = \omega A^+, \quad \{H, A^-\}_p = -\omega A^- \quad , \quad (2.5)
\]
\[
\{A^-, A^+\}_p = P(H), \quad A^- A^+ = A^+ A^- = Q(H), \quad (2.6)
\]
where \(P(H)\) and \(Q(H)\) are polynomials. They are related by \(P(H) = -i\omega \frac{\partial Q(H)}{\partial H}\). Generalizations of the algebraic structure in classical and quantum were obtained by considering \(\omega = \omega(H)\). Ladder operators can also be useful in context of 1D systems and they can be used to generate time-dependent integrals of motion:

\[
Q^\pm = A^\pm e^{\mp i\omega t}, \quad (2.7)
\]
\[
\frac{dQ^\pm}{dt} = \{H, Q^\pm\}_p + \frac{\partial Q^\pm}{\partial t} = 0. \quad (2.8)
\]
The constant value of this integrals given by Eq.(2.7) can be written in the following form:

\[
q^\pm = c(E)e^{\pm i\theta_0}. \quad (2.9)
\]
We have \( A^\pm \) as given by Eq.(2.2) and thus
\[
A^\pm = c(E)e^{\pm(i\theta_0+\omega t)},
\]  
(2.10)
generate equations that can be used to obtain the trajectories on the phase space \((x(t), P(t))\) algebraically [27]. Ladder operators are thus useful in context of quantum mechanics but also classical mechanics.

2.1 Classification of systems in classical and quantum mechanics with ladder operators

The most general system with first and second order ladder operators in classical and quantum mechanics coincide [5]:

First order :
\[
V_1 = \frac{\omega^2}{2}x^2,
\]  
(2.11)

Second order :
\[
V_2 = \frac{\omega^2}{2}x^2 + \frac{b}{x^2},
\]  
(2.12)

These potentials also appear in context of superintegrable systems [4]. Recently systems with third and fourth order ladder operators were obtained and studied [11,16,21,22].

Third order :
\[
V_3(x) = \frac{\omega^2}{2}x^2 + \frac{\hbar\omega}{2}P_4' + \frac{\omega\hbar}{2}P_4^2 + \omega\sqrt{\hbar\omega}xP_4 + \frac{\hbar\omega}{3}(-\alpha + \epsilon),
\]  
(2.13)

with \((P_4 = P_4(\sqrt{\hbar}x, \alpha, \beta)\) and \(\epsilon = \pm 1\).

Fourth order :
\[
V_4(x) = \frac{\omega^2}{8}(1+\frac{4(P_5 + \epsilon P_5')^2 - 5P_5^2}{(P_6 - 1)^2P_5})x^2 + \frac{\hbar^2}{x^2}(a-b-\frac{1}{8} + \frac{b - aP_5^2}{P_5^2})\hbar\omega(1 + \frac{(2 - \epsilon + 2(1 + c)P_5)}{2(P_5 - 1)})
\]  
(2.14)
with \((P_5 = P_5(\frac{\omega}{\hbar}x^2, a, b, c, -\frac{1}{8})\) and \(\epsilon = \pm 1\). The second order differential equations (i.e. Painlevé equations) satisfied by the fourth and fifth Painlevé transcendents are the following:

\[
P''_4(z) = \frac{P'_4(z)}{2P_4(z)} + \frac{2}{2}P_4^3(z) + 4zP'_4(z) + 2(z^2 - \alpha)P_4(z) + \frac{\beta}{P_4(z)}, \tag{2.15}
\]

\[
P''_5(z) = (\frac{1}{2P_5(z)} + \frac{1}{P_5(z) - 1})P'_5(z)^2 - \frac{1}{z}P'_5(z) + \frac{(P_5(z) - 1)^2}{z^2}\left(\frac{aP_5^2(z) + b}{P_5(z)}\right)
+ \frac{cP_5(z)}{z} + \frac{dP_5(z)(P_5(z) + 1)}{P_5(z) - 1}. \tag{2.16}
\]

The Eq. (2.15) and (2.16) are two of the six Painlevé equations that serve to define the six Painlevé transcendents [38]. The fourth and the fifth Painlevé transcendents allow solutions in terms of rational functions or special functions [39]. These systems given the Eq.(2.13) and (2.14) are related with higher order supersymmetric quantum mechanics (SUSYQM) and we refer the reader to the ref.11, 16, 21 and 22 for more details concerning the explicit form of the supercharges, ladder operators, zero modes of the raising and lowering operators and the corresponding polynomial Heisenberg algebras. They possess respectively a polynomial Heisenberg algebra of order three and four. The energy of the zero modes are written respectively in terms of the parameters of the fourth and fifth Painlevé transcendents. The corresponding superintegrable systems were discussed in Ref. 14, 16 and 22. The Painlevé transcendents appear in many context in physics and the relation with quantum systems with ladder operator and superintegrability is very interesting. The study of the classical equivalent of such system is thus important and is an unexplored subject.

In context of classical mechanics, the study of systems with higher order ladder operators (i.e. order higher than 2) is a relatively unexplored subject. System with third order ladder operators in classical mechanics were explored by Eleonsky and Korolev [18], and also in context of superintegrable systems [24]. Let us present the system considered in Ref.24:

\[
V(x) = \frac{\omega^2}{18}(2b + 5x^2 + \epsilon 4x\sqrt{b + x^2}). \tag{2.17}
\]
A new family of superintegrable systems [21] were also constructed using the existence of ladder operators. We will present the general systems possess a third order ladder operators and classify systems with fourth order ladder operators. As for higher order superintegrability, the classical and quantum systems with higher order ladder operators differ.

3 Classical systems with ladder operators of order 3

From Eq.(2.2), a general third order raising and lowering operators has the form:

\[ A_3^\pm = P_3^3 \pm if_2(x)P_2^2 + f_1(x)P_x \pm if_0(x). \] (3.1)

From Eq.(2.5) we obtain the following differential equations:

\[-i\omega + if_2'(x) = 0,\] (3.2)

\[-\omega f_2(x) - f_1'(x) + 3V'(x) = 0,\] (3.3)

\[-i\omega f_1(x) + if_0'(x) - 2if_2(x)V'(x) = 0,\] (3.4)

\[-\omega f_0(x) + f_1(x)V'(x) = 0.\] (3.5)

The first two equations give us (the terms that contain the two integration constants can be removed by an appropriate translations of \(x\) and \(V(x)\)):

\[ f_2(x) = \omega x, \quad f_1(x) = -\frac{1}{2}\omega^2 x^2 + 3V(x). \] (3.6)

When we put this in the last two equations we obtain:

\[-\omega f_0' + V'(x)(-\omega^2 + 3V'(x)) + (-\frac{1}{2}\omega^2 x^2 + 3V(x))V'(x) = 0,\] (3.7)

\[ f_0' = \frac{1}{2}(-\omega^3 x^2 + 6\omega V(x) + 4\omega x V'(x)). \] (3.8)
We can obtain a second order nonlinear differential equation for $V(x)$:

$$\frac{\omega^4 x^2}{2} - 3\omega^2 V(x) - 3\omega^2 x V'(x) + 3(V'(x))^2 - \frac{1}{2}\omega^2 x^2 V''(x) + 3V(x)V''(x). \quad (3.9)$$

This equation coincide with the one obtained in Ref.18 (i.e. the Eq.5.3) and the one obtained in context of 2D superintegrability with second and third order integrals by Gravel [13] (more precisely the Eq. (19) in the limit $\hbar \to 0$). As shown in Ref.13 and also Ref.18, this equation can be integrated by using the fact that it possesses a first integral.

However instead of giving the solution directly, let us take an other approach and impose the constraint that the ladder operators satisfy a polynomial Heisenberg algebra. The problem is reduced to solve a system of algebraic equations instead of a system of differential equations. Thus we consider the second relation of the Eq.(2.6). The polynomial $Q(H)$ can be at most a third order polynomial of the Hamiltonian with the following form:

$$Q(H) = 8(H - \epsilon_1)(H - \epsilon_2)(H - \epsilon_3). \quad (3.10)$$

Instead of a set of differential equations as the one obtained from Eq.(2.5), the Eq.(3.10) generates a set of algebraic relations among $f_0(x)$, $f_1(x)$, $f_2(x)$ and $V(x)$ involving three parameters ($\epsilon_1$, $\epsilon_2$ and $\epsilon_3$):

$$2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2f_1(x) + f_2^2(x) - 6V(x) = 0, \quad (3.11)$$

$$-4\epsilon_1\epsilon_2 - 4\epsilon_1\epsilon_3 - 4\epsilon_2\epsilon_3 + f_1^2(x) + 2f_0(x)f_2(x) + 8\epsilon_1 V(x) + 8\epsilon_2 V(x) + 8\epsilon_3 V(x) - 12V(x)^2 = 0, \quad (3.12)$$

$$8\epsilon_1\epsilon_2\epsilon_3 + f_0^2(x) - 8\epsilon_1\epsilon_2 V(x) - 8\epsilon_1\epsilon_3 V(x) - 8\epsilon_2\epsilon_3 V(x), \quad (3.13)$$

$$+ 8\epsilon_1 V(x)^2 + 8\epsilon_2 V(x)^2 + 8\epsilon_3 V(x)^2 - 8V(x)^3 = 0.$$

We can use the expressions given by Eq.(3.6) for $f_1$ and $f_2$ and insert them in Eq.(3.11), (3.12) and (3.13). We obtain $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and the following two equations with 2 parameters $c$ and $d$ ( with $d = 4(\epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2)$ and $c = 32\omega^2(\epsilon_1^2\epsilon_2 + \epsilon_1\epsilon_2^2)$)
\[ \frac{\omega^4 x^4}{4} + d + 2 \omega x f_0(x) - 3 \omega^2 x^2 V(x) - 3 V(x)^2 = 0, \quad (3.14) \]
\[ -\frac{1}{4 \omega^2} c + f_0^2(x) + 2 d V(x) - 8 V(x)^3 = 0. \]

The solution is thus (including the Eq.(3.6)):
\[ f_0(x) = \frac{3}{2 \omega x} V^2(x) + \frac{3}{2} \omega x V(x) - \frac{1}{8} \omega^3 x^3 - \frac{d}{2 \omega x}, \quad (3.15) \]
\[ -9 V^4(x) + 14 \omega^2 x^2 V^3(x) + (6d - \frac{15}{2} \omega^4 x^4) V^2(x) + \frac{3}{2} \omega^6 x^6 - 2d) V(x) - (\frac{1}{2} d \omega^4 x^4 + d^2) = 0. \quad (3.16) \]

This solution satisfy all the equations given by Eq.(3.2)-(3.5) and (3.11)-(3.13). This solution agree with results obtained in Ref.18 and 13. The Eq.(3.14) coincide with Eq(5.5) of Ref.18 (up to a translation of the coordinates x and the potential. The Eq.(3.16) is a polynomial equation of degree 4 and thus can be solved. However, let us do not present explicitly this solution and only present a particular case. By imposing two of the roots of the polynomial to be equal (i.e. \( \epsilon_1 = \epsilon_2 \)) or (\( c^2 = \frac{64d^2 \omega^4}{27} \)) we obtain the following solutions
\[ f_0 = -6 \epsilon_2 \omega x + \omega^3 x^3, \quad V(x) = -2 \epsilon_2 + \frac{1}{2} \omega^2 x^2, \quad (3.17) \]
and
\[ f_0 = 6 \epsilon_2 \omega x + \frac{13}{27} \omega^3 x^3 \pm \frac{4 \epsilon_2 \sqrt{18 \epsilon_2 \omega^2 x^2 + \omega^4 x^4}}{3 \omega x} \pm \frac{14}{27} \omega x \sqrt{18 \epsilon_2 \omega^2 x^2 + \omega^4 x^4}, \quad (3.18) \]
\[ V(x) = 2 \epsilon_2 + \frac{5 \omega^2 x^2}{18} \pm \frac{2}{9} \sqrt{18 \epsilon_2 x^2 \omega^2 + \omega^4 x^4}. \]

We have the following relation \( \epsilon_2 = \sqrt{\frac{d}{12}} \). We can also use the parameter \( d = \frac{\omega^4 \tilde{d}}{27} \) to recover the potential obtained in Ref.13.
4 systems with fourth order ladder operators

Let us consider the ladder operators of the following form (i.e. as given by Eq.(2.2)):

\[ A_x^\pm = \pm iP^4 + f_3(x)p^3 \pm if_2(x)p^2 + f_1(x)p \pm if_0(x). \]  

(4.1)

We obtain from Eq.(2.5) the following five differential equations:

\[- \omega - f_3'(x) = 0, \]  

(4.2)

\[-i\omega f_3(x) + if_2(x) - 4iV'(x) = 0, \]  

(4.3)

\[- \omega f_2(x) - f_1'(x) + 3f_3(x)V'(x) = 0, \]  

(4.4)

\[-i\omega f_1(x) + if_0'(x) - 2if_2V'(x) = 0, \]  

(4.5)

\[- \omega f_0(x) + f_1(x)V'(x) = 0. \]  

(4.6)

From the Eq.(4.2) and (4.3) we obtain (again by an appropriate translations of \(x\) and \(V(x)\) we can remove the terms involving the two integration constants):

\[ f_2 = -\frac{1}{2} \omega^2 x^2 + 4V(x), \quad f_3(x) = -\omega x. \]  

(4.7)

We insert the expressions given by Eq.(4.7) in the Eq.(4.4), (4.5) and (4.6). From the Eq.(4.6), we can isolate the functions \(f_0(x)\) and \(f_0'(x)\) in terms of \(f_1(x), f_1'(x), V(x)\) and \(V'(x)\). Using these expressions, we obtain from the Eq.(4.4) and (4.5):

\[ - 4\omega V(x) - f_1'(x) + \frac{1}{2} \omega x (\omega^2 x - 6V'(x)) = 0, \]  

(4.8)

\[(\omega^3 x^2 - 8\omega V(x) + f_1'(x))V'(x) + f_1(x)(-\omega^2 + V''(x)) = 0. \]  

(4.9)

From the Eq.(4.8) and the equation obtained by taking the derivative of the Eq.(4.8), we can isolate \(f_1'(x)\) and \(f_1''(x)\). We can thus insert these expressions for \(f_1'(x)\) and \(f_1''(x)\) in the equation obtained by isolating \(f_1(x)\) in Eq.(4.9)
and taking the derivative. We thus obtain for $V(x)$ the following third order nonlinear differential equation:

$$
\omega^4 x^2 + 2(-4\omega^2 V(x) - 6\omega^2 x V'(x) + (15V'(x))^2 + (16V(x) + x(-2\omega^2 x + 9V'(x)))V'(x))
+ \frac{3V'(x)(8V(x) + x(-\omega^2 x + 2V'(x)))V''(x)}{\omega^2 - V''(x)} = 0,
$$

(4.10)

with $\omega^2 - V''(x) \neq 0$. The solution $V''(x) = \omega^2$ is the well known harmonic oscillator.

Again, let us take the same approach than for systems with third order ladder operators and impose for the raising and lowering operators to generate a polynomial Heisenberg algebra (i.e. impose the Eq.(2.6) to be satisfied). The product of these operators can be at most a quartic polynomial of the following form:

$$
A_x A_x^+ = 16(H - \epsilon_1)(H - \epsilon_2)(H - \epsilon_3)(H - \epsilon_4).
$$

(4.11)

Using the explicit form of the ladder operators given by Eq.(4.1), we obtain the following set of equations:

$$
2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) + 2f_2(x) + f_3^2(x) - 8V(x) = 0,
$$

(4.12)

$$
-4(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 + \epsilon_1\epsilon_4 + \epsilon_2\epsilon_4 + \epsilon_3\epsilon_4)
$$

(4.13)

$$
+ 2f_0(x) + f_2^2(x) + 2f_1(x)f_3(x) + 12(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)V(x) - 24V(x)^2 = 0,
$$

$$
8(\epsilon_1\epsilon_2\epsilon_3 + \epsilon_1\epsilon_2\epsilon_4 + \epsilon_1\epsilon_3\epsilon_4 + \epsilon_2\epsilon_3\epsilon_4) + f_1^2(x) + 2f_0(x)f_2(x) - 16(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3)
$$

(4.14)

$$
+ \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_1\epsilon_4 + \epsilon_2\epsilon_4 + \epsilon_3\epsilon_4) V(x) + 24(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)V(x)^2 - 32V(x)^3 = 0,
$$

$$
-16\epsilon_1\epsilon_2\epsilon_3\epsilon_4 + f_0(x)^2 + 16(\epsilon_1\epsilon_2\epsilon_3 + \epsilon_1\epsilon_2\epsilon_4 + \epsilon_1\epsilon_3\epsilon_4 + \epsilon_2\epsilon_3\epsilon_4)V(x)
$$

(4.15)

$$
-16(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 + \epsilon_1\epsilon_4 + \epsilon_2\epsilon_4 + \epsilon_3\epsilon_4)V(x)^2 + 16(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)V(x)^3 - 16V(x)^4 = 0.
$$

We will insert the expression given in Eq.(4.7) for $f_3(x)$ and $f_2(x)$ in Eq.(4.12)-(4.15). We obtain the following relation $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$
and using the following parameters:

\[ d = \epsilon_1^2 + \epsilon_1 \epsilon_2 + \epsilon_2^2 + \epsilon_1 \epsilon_3 + \epsilon_2 \epsilon_3 + \epsilon_3^2, \]  

(4.16)

\[ c = \epsilon_1^2 \epsilon_2 - \epsilon_1 \epsilon_2^2 - \epsilon_1^2 \epsilon_3 - 2 \epsilon_1 \epsilon_2 \epsilon_3 - \epsilon_2^2 \epsilon_3 - \epsilon_1 \epsilon_3^2 + \epsilon_2 \epsilon_3^2, \]  

(4.17)

\[ e = \epsilon_1^2 \epsilon_2 \epsilon_3 + \epsilon_1 \epsilon_2^2 \epsilon_3 + \epsilon_1 \epsilon_3^2 \epsilon_2, \]  

(4.18)

we obtain three equations for \( f_0(x) \), \( f_1(x) \) and \( V(x) \):

\[-d + \frac{\omega^4 x^4}{4} + 2f_0(x) - 2\omega x f_1(x) - 4\omega^2 x^2 V(x) - 8V^2(x) = 0, \]  

(4.19)

\[ 8c - \omega^2 x^2 f_0(x) + f_1^2(x) - 4dV(x) + 8f_0(x)V(x) - 32V(x)^3 = 0, \]  

(4.20)

\[-16e + f_0(x)^2 + 16cV(x) - 4dV(x)^2 - 16V^4(x) = 0. \]  

(4.21)

The solution is thus (including Eq.(4.7)):

\[ f_0(x) = \sqrt{-16e + 16cV(x) - 16dV(x)^2 + 16V(x)^4}, \]  

(4.22)

\[ f_1(x) = \frac{1}{8\omega x}(16d + \omega^4 x^4 + 8f_0(x) - 16\omega^2 x^2 V(x) - 32V^2(x)), \]  

(4.23)

\[ V(x) = -32\omega^2 x^2 V^5(x) + (128d + 17\omega^4 x^4)V^4(x) + (-96c - \frac{128d^2}{\omega^2 x^2} - \frac{512e}{\omega^2 x^2} - 32d^2 \omega^2 x^2 - \frac{7\omega^6 x^6}{2})V^3(x) \]  

+(\begin{array}{c}
-40d^2 + 352e + \frac{256c^2}{\omega^4 x^4} + \frac{256cd}{\omega^2 x^2} + 32c\omega^2 x^2 + 3d\omega^4 x^4 + \frac{11\omega^8 x^8}{32} \\
+ (-16cd - \frac{128cd^2}{\omega^4 x^4} - \frac{512ce}{\omega^4 x^4} - \frac{256c^2}{\omega^2 x^2} + \frac{256de}{\omega^2 x^2} + \frac{256de}{\omega^2 x^2} - \frac{\omega^{10} x^{10}}{64})V^2(x) \\
+ 64c^2 + 4d^3 - 112de + \frac{16d^4}{\omega^4 x^4} + \frac{128d^2 e}{\omega^4 x^4} + \frac{256e^2}{\omega^4 x^4} - \frac{64cd^2}{\omega^2 x^2} + \frac{256ce}{\omega^2 x^2} - 8cd\omega^2 x^2 \\
+ \frac{3}{2} d^2 \omega^4 x^4 + \frac{17}{2} \epsilon \omega^4 x^4 - \frac{1}{4} \omega^6 x^6 + \frac{1}{64} d \omega^8 x^8 + \frac{\omega^{12} x^{12}}{4096}.
\end{array})V(x) \]  

These relations satisfy well the Eq.(4.2)-(4.6) and Eq.(4.12)-4.16). Let us again do not present explicitly the solution for the potential that can be obtained because this is a quintic equation. We present a particular solution.
by considering a triple roots (i.e. $\epsilon_1 = \epsilon_2$ and $\epsilon_3 = \epsilon_2$ or $c = \frac{2}{3}\sqrt{3}d^3$ and $e = \frac{d^2}{12}$).

We obtain the following solution:

$$f_1(x) = -\frac{(-8\epsilon_2 + \omega^2x^2)}{4\omega x}, f_0(x) = -\frac{(-8\epsilon_2 + \omega^2x^2)^3((8\epsilon_2 + \omega^2x^2)}{16\omega^4x^4},$$

$$V(x) = -\epsilon_2 + \frac{1}{8}\omega^2x^2 + \frac{8\epsilon_2^2}{\omega^2x^2},$$

and

the following 1D system:

$$f_0(x) = -3\epsilon_2^2 + \epsilon_2(-\frac{3}{2}\omega^2x^2\pm \frac{7}{16}\sqrt{64\epsilon_2^2\omega^2x^2 + \omega^4x^4}) + \frac{1}{512}(-15\omega^4x^4 + 17\omega^2x^2 \sqrt{64\epsilon_2^2\omega^2x^2 + \omega^4x^4}),$$

$$f_1(x) = -6\epsilon_2\omega x - \frac{3\epsilon_2^3}{32} \pm \frac{\epsilon_2\sqrt{64\epsilon_2^2\omega^2x^2 + \omega^4x^4}}{\omega x} \pm \frac{5}{32}\omega x \sqrt{64\epsilon_2^2\omega^2x^2 + \omega^4x^4},$$

$$V(x) = \frac{1}{64}(96\epsilon_2 + 5\omega^2x^2 \mp 3\sqrt{64\epsilon_2^2\omega^2x^2 + \omega^4x^4}).$$

5 New superintegrable systems in classical mechanics

Let us now consider a classical 2D Hamiltonian separable in Cartesian coordinates

$$H(x_1, x_2, P_1, P_2) = H_1(x_1, P_1) + H_2(x_2, P_2),$$

for which polynomial ladder operators ($A_{x_i}$ and $A_{x_i}^+$) of order $n_i$ exist and satisfy polynomial Heisenberg algebras. From the Eq.(2.2) the operators $f_1 = (A_{x_1}^+)^{m_1}(A_{x_2}^-)^{m_2}$ and $f_2 = (A_{x_1}^-)^{m_1}(A_{x_2}^+)^{m_2}$ (of order $n_1m_1 + n_2m_2$) Poisson commute with the Hamiltonian H given by Eq.(2.1) if $m_1\omega_{x_1} = \ldots$
\[ m_2 \omega x_2 = \omega \text{ with } m_1, m_2 \in \mathbb{Z}^+. \] The following products are also polynomial (i.e. polynomial in the momenta) integrals of the Hamiltonian \( H \):

\[ I_1 = (A_{x_1}^+)^{m_1} (A_{x_2}^-)^{m_2} - (A_{x_1}^-)^{m_1} (A_{x_2}^+)^{m_2}, \quad I_2 = (A_{x_1}^+)^{m_1} (A_{x_2}^-)^{m_2} + (A_{x_1}^-)^{m_1} (A_{x_2}^+)^{m_2}. \]  

(5.2)

By construction the Hamiltonian has the following second order integral related to separation of variables in Cartesian coordinates:

\[ K = H_1 - H_2. \]

(5.3)

The Hamiltonian \( H \) given by Eq.(5.1) is thus maximally superintegrable (i.e. there are 3 functionally independent integrals of motion including the Hamiltonian himself). We construct from integrals given by Eq.(5.2) and (5.3) the following polynomial Poisson algebra:

\[ \{ K, I_1 \}_p = 2 \omega I_2, \quad \{ K, I_2 \}_p = 2 \omega I_1, \quad \{ I_1, I_2 \}_p = 2 Q_1 \left( \frac{1}{2}(H + K) \right)^{m_1 - 1} Q_2 \left( \frac{1}{2}(H - K) \right)^{m_2 - 1} \left[ m_2^2 Q_1 \left( \frac{1}{2}(H + K) \right) P_2 \left( \frac{1}{2}(H - K) \right) - m_1^2 Q_2 \left( \frac{1}{2}(H - K) \right) P_1 \left( \frac{1}{2}(H + K) \right) \right]. \]

(5.4)

5.1 Two new families of superintegrable systems

We can consider a new following 2D superintegrable system \( H \) as given by Eq.(5.1) with \( H_1 \) and \( H_2 \) given by:

\[ H_1 = \frac{p_{x_1}^2}{2} + V_1(x_1; \omega_1), \quad H_2 = \frac{p_{x_2}^2}{2} + V_2(x_2; \omega_2), \]

(5.5)

with \( V_1(x_1; \omega_1) \) and \( V_2(x_2; \omega_2) \) that both satisfy Eq.(3.16). We impose the constraint \( m_1 \omega x_1 - m_2 \omega x_2 = 0 \) with \( m_1, m_2 \in \mathbb{Z}^+ \). From the construction at the beginning of the Section 5, this systems is thus maximally superintegrable.

Using the same form (i.e. the one given by Eq.(5.5)), we impose that \( V_1 \) and \( V_2 \) satisfy now the Eq.(4.24). We can thus generate a second family of 2D
superintegrable systems. These results show how new systems with ladder operators can generate new superintegrable Hamiltonian. These results can be extended in N dimensions.

Let us present trajectories for a particular case (i.e. satisfying Eq.(4.28)) respectively for $\epsilon = 1$ and $\epsilon = -1$. All bounded trajectories are closed and the motion is periodic. We choose for Fig.1 and Fig.2 the following parameters $\omega_1 = 1$, $\omega_2 = 2$, $\epsilon_{2x_1} = 4$, $\epsilon_{2x_2} = 16$, $x(0) = y(0) = 1$, $x'(0) = 1$, $y'(0) = -3$ and $t = [0, 20]$.

6 Conclusion

We presented the most general 1D classical system with a third order ladder operators and also the most general system with fourth order ladder operators. We obtain respectively potentials that satisfy a fourth and a fifth order algebraic equations. They are the classical equivalent of the quantum sys-
tems written in terms of the fourth and fifth Painlevé transcendents obtained and studied in earlier papers [11,16,19,21,22]. This point out how systems with higher order ladder operators differ in classical and quantum mechanics.

Using a method introduced in earlier articles [14,24] and extended by Miller, Kalnins and Kress [36] to systems with separation of variables in polar coordinates, we obtained two new families of superintegrable systems that are deformations of the anisotropic harmonic oscillator and Evans-Verrier systems [40]. This paper point out also how ladder operators can be used to generate new superintegrable systems. Let us mention other recent papers that discuss ladder operators in context of superintegrability [41,42].

Acknowledgments The research of I.M. was supported by a postdoctoral research fellowship from FQRNT of Quebec. referees for their comments and pointing out references.

References

[1] Jauch J.M. and Hill E.L., On the problem of degeneracy in quantum mechanics, Phys.Rev. 57 (1940) 641-645
[2] Dulock V.A. and McIntosh H.V., On the degeneracy of the two-dimensional harmonic oscillator, Am. J.Phys. 33 (1965) 109-118
[3] Cisneros A. and McIntosh H.V., Search for a Universal Symmetry Group in Two Dimensions, J.Math.Phys. 11 3 (1970) 870-895
[4] Winternitz P., Smorodinsky Ya.A., Uhlir M. and Fris I., Symmetry groups in classical and quantum mechanics, Sov. J.Nucl.Phys. 4 (1967) 444-450
[5] Boyer C.P. and Miller Jr. W., A classification of second order raising operators for Hamiltonians in two variables, J.Math.Phys. 15 9 (1974) 1484-1489
[6] Mielnik B., Factorization method and new potentials with the oscillator spectrum, J. Math. Phys. 25 12 (1984) 3387 3 pages
[7] Veselov A.P., Shabat A.B., Dressing chains and the spectral theory of the Schrödinger operator, Funct. Anal. Appl. 27 (2) (1993) 81-96
[8] Veselov A.P., On Stieltjes relations, Painlevé-IV hierarchy and complex monodromy, J.Phys. A 34 (2001) 3511-3519
[9] Spiridonov V., Universal superpositions of coherent states and self-similar potentials, Phys.Rev. A 52 3 (1995) 1909-1935
[10] Fushchych W.I. and Nikitin A.G., Higher symmetries of the Schrödinger equation, *J.Math.Phys.* 38 11 (1997) 5944 16 pages

[11] Andrianov A., Cannata F., Ioffe M. and Nishnianidze D, Systems with Higher-Order Shape Invariance: Spectral and Algebraic Properties, *Phys.Lett.A* 266 (2000) 341-349

[12] Mateo J. and Negro J., Third-order differential ladder operators and supersymmetric quantum mechanics, *J.Phys. A:Math. Theor.* 41 (2008) 045204 28 pages

[13] Gravel S., Hamiltonians separable in Cartesian coordinates and third-order integrals of motion, *J.Math.Phys.* 45 (3) (2004) 1003-1019

[14] Marquette I., Superintegrability and higher order polynomial algebras, *J.Phys. A: Math. Theor.* 43 (2010) 135203 15 pages

[15] Marquette I., Supersymmetry as a method of obtaining new superintegrable systems with higher order integrals of motion, *J.Math.Phys.* 50 (2009) 122102 10 pages

[16] Marquette I., Superintegrability with third order integrals of motion, cubic algebras and supersymmetric quantum mechanics II: Painlevé transcendent potentials, *J. Math. Phys.* 50 (2009) 095202 18 pages

[17] Marquette I., Superintegrability with third order integrals of motion, cubic algebras and supersymmetric quantum mechanics I: Rational function potentials, *J. Math. Phys.* 50 (2009) 012101 23 pages

[18] Eleonsky V.M. and Korolev V.G., On the nonlinear generalization of the Fock method, *J. Phys. A: Math. Gen.* 28 (1995) 4973-4985.

[19] Fernández D. J., Negro J. and Nieto L. M., Elementary systems with partial finite ladder spectra, *Phys. Lett. A* 234 (2004) 139–144

[20] Adler V. E., Nonlinear chains and Painlevé equations, *Physica D* 73 (1994) 335–351

[21] Carbello J. M., Fernández D. J., Negro J. and Nieto L. M., Polynomial Heisenberg algebras, *J. Phys. A: Math. Gen.* 37 (2004) 10349–10362.

[22] Marquette I., An infinite family of superintegrable systems from higher order ladder operators and supersymmetry, Proceeding of the GROUP28: The XXVIII International Colloquium on Group-Theoretical Methods in Physics, *J.Phys.:Conf. Series* 284 (2011) 012047 8 pages

[23] Marquette I. and Winternitz P., Polynomial Poisson algebras for superintegrable systems with a third order integral of motion, *J.Math. Phys.* 48 (2007) 012902 16 pages

[24] Marquette I., Construction of new two and three dimensional classical superintegrable systems, *J.Math.Phys.* 51 (2010) 092903 9 pages

[25] Hussin V. and Marquette I., Generalized Heisenberg algebras, SUSYQM and degeneracies: Infinite well and Morse potential, *SIGMA* 7 024 (2011) 16 pages

[26] Eleonsky V.M. and Korolev V.G., On the nonlinear Fock description of quantum systems with quadratic spectra, *J. Phys. A: Math. Gen.* 29 (1996) L241-L248.
[27] Kuru S. and Negro J., Factorizations of one dimensional classical systems, *Ann. Phys.* **323** (2008) 413-431.

[28] Cruz y Cruz S., Kuru S. and Negro J., Classical motion and Coherent States for Pöschl-Teller potentials, *Phys. Lett.* **372** (2008) 1391-1405.

[29] Salazar-Ramirez M., Martinez D., Granados V.D. and Mota R.D., The su(1,1) Dynamical Algebra for the Generalized MICZ-Kepler Problem from the Schrödinger Factorization, *Int. J. Theor. Phys.* **49** (2010) 967-973

[30] Marquette I., Generalized Kaluza-Klein monopole, quadratic algebras and ladder operators, *J. Phys. A: Math. Theor.* **44** (2011) 235203 12 pages

[31] Bonatsos D., Daskaloyannis C., Quantum groups and their applications in nuclear physics, *Prog. Part. Nucl. Phys.* **43** (1999) 537-618

[32] Rodriguez M.A., Tempesta P., and Winternitz P., Reduction of superintegrable systems: The anisotropic harmonic oscillator, *Phys. Rev. E* **78** (4) (2008) 046608 (6 pages)

[33] Rodriguez M.A., Tempesta P., and Winternitz P., Symmetry reduction and superintegrable Hamiltonian systems. In R. Campoamor-Stursberg et al., editor, *J. Phys. Conf. Series* **175** (2009) 012013 (8pp)

[34] Ranada M.F., Rodriguez M.A. and Santander M., A new proof of the higher-order superintegrability of a noncentral oscillator with inversely quadratic nonlinearities, *J. Math. Phys.* **51** 4 (2010) 042901 11 pages

[35] Mota R.D., Granados V.D., Queijeiro A. and Garcia J., *J. Math A: Math. Gen.* **35** (2002) 2979-2984

[36] Kalnins E.G., Kress J.M. and Miller Jr W., A Recurrence Relation Approach to Higher Order Quantum Superintegrability, *SIGMA* **7** 031 (2011) 24 pages

[37] Fernandez D. J. and Hussin V., Higher-order SUSY, linearized nonlinear Heisenberg algebras and coherent states, *J. Phys. A: Math. Gen.* **32** (1999) 3603–3619.

[38] Ince E L, *Ordinary Differential Equations* (Dover, New York, 1944).

[39] Gromak V, Laine I and Shimomura S, *Painlevé Differential Equations in the Complex Plane*, Walter de Gruyter (2002).

[40] Verrier P.E. and Evans N.W., *J.Math.Phys.* **49** (2008) 092902.

[41] Maciejewska A.J., Przybylska M. and Tsiganov A.V., On algebraic construction of certain integrable and super-integrable systems, *nlIn.SI* (In Press), [arXiv:1011.3249v1]

[42] Kalnins E.G. and Miller Jr W., Structure results for higher order symmetry algebras of 2D classical superintegrable systems, [arXiv:1101.5292]