Probability distribution of magnetization in the one-dimensional Ising model: Effects of boundary conditions

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Finite-size scaling functions are investigated both for the mean-square magnetization fluctuations and for the probability distribution of the magnetization in the one-dimensional Ising model. The scaling functions are evaluated in the limit of the temperature going to zero (T → 0), the size of the system going to infinity (N → ∞) while N[1 − tanh(J/k_B T)] is kept finite (J being the nearest neighbor coupling). Exact calculations using various boundary conditions (periodic, antiperiodic, free, block) demonstrate explicitly how the scaling functions depend on the boundary conditions. We also show that the block (small part of a large system) magnetization distribution results are identical to those obtained for free boundary conditions.

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I. INTRODUCTION

Finite-size scaling has been developed intensively during the last few decades 1, 2, 3, 4 and it has become a standard tool in the studies of critical systems. An interesting application of the method is using the finite-size scaling of the distribution function of the order-parameter fluctuations as hallmarks of universality classes. The idea goes back to Bruce 5, 6, 7 who used it, e.g., to verify that the gas-liquid transition of the two-dimensional Lennard-Jones fluid belongs to the Ising universality class 8. The list of applications to thermodynamic critical points is long 9 and the idea has reemerged in nonequilibrium critical (or effectively critical) systems 10, 11, 12, 13, as well.

The usefulness of scaling functions as hallmarks of universality classes depends on their availability for comparisons. Indeed, a significant portion of the applications is about the Ising universality class where the scaling functions for the d = 2, 3, 4 dimensional distribution functions are well known from simulations 14, 15 and field-theoretic results in d = 4 − ε are also available 16. The picture gallery of scaling functions is clearly far from complete since these functions have been systematically worked out only for surface growth models 10 and for Gaussian 1/ f α type noise processes 13.

An important issue concerning the critical distribution functions is their dependence on the boundary conditions (BC). While theoretical calculations usually address periodic systems, the building of a histogram of a physical quantity (e.g., the magnetization) in an experimental system involves measuring the magnetizations in patches of a given size within the bulk of the system (corresponding to block-spin magnetizations in an Ising model). As has been shown by Binder 14, the block-spin distribution function at criticality depends on the BC on the block, a finding that is not entirely unexpected since the infinite-range critical correlations feel the boundaries of the system.

The BC dependence of the scaling functions is an interesting problem and, indeed, there has been a series of works where the PDF of the magnetization for the d = 2 Ising model at the critical point has been investigated for various BC 17 including some exotic ones (Möbius strip, Klein bottle). Similar problems have also been studied for the roughness distribution of 1/ f α type noise processes 13. Analytical results about BC dependence of the scaling functions are scarce: they are restricted to Gaussian models 13, expansions around d = 4 17 and around the spherical limit 18. This is why we decided to revisit the the d = 1 Ising model where the effect of BC can be seen in analytical detail.

Although the critical temperature of the d = 1 Ising model is zero, it displays nontrivial features in its finite-size scaling as the critical point is approached. The particular case of the distribution function of the magnetization in bulk blocks has already been discussed by Bruce 3. The purpose of this paper is to calculate the scaling function for the case of periodic (PBC), antiperiodic (APBC), free (FBC) and block (BBC) boundary conditions, and thus gauge the importance of the role played by the BC.

For pedagogical purposes, we also compute the finite-size scaling of the magnetization fluctuations. The calculation is elementary in this case and one can easily observe that the periodic, antiperiodic, and free BC yield distinct scaling functions. Furthermore, one can also see explicitly how the scaling function associated with the block BC emerges when a small part of a large system is used for measuring the fluctuations.

The evaluation of the magnetization distribution is somewhat more involved but can be carried out relatively simply by using a combinatorial approach. For the peri-
odic and free BC, the calculations yield nontrivial functions which are combinations of two delta peaks and a continuum background, with the relative weight of the delta functions reduced for the case of FBC. The delta functions disappear entirely for antiperiodic BC. Finally, the combinatorial approach reproduces Bruce’s result for the BBC and it turns out that the block scaling function is identical to that of the FBC case.

II. MODEL AND NOTATION

We consider the one-dimensional Ising chain of $N$ spins $(\sigma_i = \pm 1, i = 1, \ldots, N)$ with ferromagnetic ($J > 0$) coupling and with various BC. The interaction energy of a given configuration of spins $\{\sigma_i\}$ is given by

$$E(\{\sigma_i\}) = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - J_b \sigma_0 \sigma_1,$$  

(1)

where $J_b = J, -J, 0$ for periodic, antiperiodic and free BC respectively. The model is exactly solvable using, e.g., the transfer matrix formalism \[1\], and the partition functions for periodic BC (upper signs) and for antiperiodic BC (lower signs) are

$$Z^{(p)}(a) = 2^N \cosh^N K \pm \sinh^N K,$$  

(2)

while the correlations are given by

$$\langle \sigma_i \sigma_{i+n} \rangle^{(p), (a)}(a) = \frac{v^n \pm v^{-n}}{1 \mp v^n},$$  

(3)

where $v = \tanh K$ with $K = J/k_B T$ and, furthermore, $1 \leq i, i+n \leq N$. The above quantities are particularly simple for free BC

$$Z^{(f)} = 2^N \cosh^{N-1} K, \quad \langle \sigma_i \sigma_{i+n} \rangle^{(f)} = v^n.$$  

(4)

Note that the correlations only depend on the distance of the two spins and not on their particular positions within the chain. This is true not only in the periodic case but also for APBC and FBC. Also note that the $(p), (a),$ and $(f)$ superscripts refer to periodic, antiperiodic and free BC respectively throughout the paper.

III. FINITE-SIZE SCALING OF FLUCTUATIONS

The mean square fluctuations of the total magnetization $M = \sum_{i=1}^{N} \sigma_i$ can be calculated via the spin correlations as

$$\langle M^2 \rangle = \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle.$$  

(5)

We begin with the discussion of $\langle M^2 \rangle$ for the periodic-, free-, and antiperiodic BC, leaving the case of block BC for a separate subsection.

A. Periodic-, free-, and antiperiodic BC

Substituting the expressions for the spin correlations \[6\] into Eq. \[5\], one easily finds

$$\frac{\langle M^2 \rangle}{N} = \begin{cases} 1 + v & 1 - v^N \\ 1 - v & 1 + v^N \end{cases} \quad \text{PBC}$$

$$\frac{\langle M^2 \rangle}{N} = \begin{cases} 1 + v & 1 - v^N - \frac{4v}{N(1-v)^2} \\ 1 - v & 2v(1-v^N) \end{cases} \quad \text{APBC}$$

(6)

$$\frac{\langle M^2 \rangle}{N} = \begin{cases} 1 + v & 1 - v^N \end{cases} \quad \text{FBC}.$$  

In the thermodynamic limit ($N \to \infty$) all the above expressions reduce to

$$\frac{\langle M^2 \rangle}{N} = 1 + v \quad (1 - v).$$  

(7)

One can see that the fluctuations diverge as $T \to 0$, and they have a power-law singularity provided $t = 1 - v = 1 - \tanh J/k_B T$ is used as the control parameter. One can also read off \[7\] the value ($\gamma = 1$) of the susceptibility exponent \[20\].

The correlation-length exponent, $\nu$, is another exponent needed in finite-size scaling. It is obtained from the spin-spin correlations which decay exponentially in the thermodynamic limit. The correlation length, $\xi$, defined by the exponential decay is independent of the BC

$$\langle \sigma_i \sigma_{i+n} \rangle_{N \to \infty} = v^n = e^{-n \ln (1/v)} = e^{-n/\xi}. $$

(8)

and diverges for $T \to 0$ as

$$\xi = -\frac{1}{\ln [1 - (1 - v)]} \sim (1 - v)^{-1} = t^{-1}$$

(9)

thus providing us with $\nu = 1$. As noted by Chen and Domb \[21\] the above definition of the correlation length (rather than that found from the second-moment of the correlation function) is the appropriate one to discuss finite size scaling for this model.

The scaling form one expects in finite-size scaling in dimension $d$ is as follows \[11\]

$$\frac{\langle M^2 \rangle}{N^{1+\gamma/d}} = \Phi(t N^{1/dv})$$

(10)

with $N$ being the number of spins. Putting in the above expression $d = \gamma = \nu = 1$, the finite-size scaling suggestion takes the form

$$\frac{\langle M^2 \rangle}{N^2} = \frac{\langle m^2 \rangle}{N^2} = \Phi(tN)$$

(11)

where $m = M/N$ is the magnetization density.

Dividing both sides of Eqs. \[10\] by $N$, we indeed find that $\langle M^2 \rangle/N^2 = \langle m^2 \rangle$ yields well defined scaling functions in the limit of $t = 1 - v \to 0$ while $tN = 2\xi$ is kept
finite. The scaling variable \( \zeta \) has a simple meaning since, in the scaling limit

\[
2\zeta = Nt = N/\xi ,
\]

i.e., \( 2\zeta \) is the average number of domains (or domain walls) in the system. The actual scaling functions are given below

\[
\frac{\langle M^2 \rangle}{N^2} = \langle m^2 \rangle = \begin{cases} 
\frac{1}{\zeta} \tanh \zeta & \text{PBC} \\
\frac{1}{\zeta} \left( \coth \zeta - \frac{1}{\zeta} \right) & \text{APBC} \\
\frac{1}{\zeta} \left( 1 - \frac{1 - e^{-2\zeta}}{2\zeta} \right) & \text{FBC},
\end{cases}
\]

and they are shown on Fig. 1. The functions belonging to different BC are clearly distinct thus demonstrating explicitly the BC dependence of the scaling functions.

We close this section with a note on the speed of convergence of the scaling function. Looking at Eq. (11), one can see that \( \langle M^2 \rangle / N \) converges exponentially to its thermodynamic limit for PBC, while the convergence is only power law \( (N^{-1}) \) for FBC and APBC. As to the scaling function, it can be easily shown that the convergence is power law \( (N^{-1}) \) even in PBC case. More precisely, the correction term is of the form \( N^{-1} g(\zeta) \) where \( g(\zeta) \leq 0 \) with \( g(\zeta \to 0) = 0 \) and \( g(\zeta \to \infty) = -1 \). Fig. 2 displays \( \langle m^2 \rangle \) calculated from Eq. (10) for the PBC case (note that for finite \( N \) and using the scaling variable \( \zeta \), the results are meaningful only for \( \zeta < N \)). It should be clear from Fig. 2 that the convergence is slow although it may appear to be quite good at small \( \zeta \) due to the particular form of \( g(\zeta) \).

The scaling functions coincide for \( \zeta \to \infty \). This is understandable since large \( \zeta \) means large number of domain walls which means that the disordered regime is approached where the effects of the BC diminish. The differences in \( \zeta \to 0 \) limit can be accounted for by the number of domain walls in the system. In particular, the ground state is completely aligned for PBC and FBC systems while it contains an arbitrarily positioned single domain wall in case of APBC. As a result

\[
\langle m^2 \rangle^{(p)(f)} = 1 , \quad \langle m^2 \rangle^{(a)} = 1/3
\]

explaining the values of the scaling functions at \( \zeta = 0 \).

The small \( \zeta = 0 \) behaviour of \( \langle m^2 \rangle \) can be understood in terms of the smallest energy excitations above the ground states. These excitations are obtained from the ground state by adding a pair domain walls in case of PBC and APBC, while they consist of a single domain wall for FBC. Their effect is a quadratic (linear) decrease of \( \langle m^2 \rangle \) near \( \zeta = 0 \) for PBC and APBC (FBC) systems.

B. Block (window) boundary conditions

When studying magnetization fluctuations in an experiment, one usually divides the system into blocks and measures the block magnetizations. The corresponding theoretical construction in the \( d = 1 \) Ising model is to consider a block (window) of length \( \ell \) at a position \( k \) in a system of total length \( N \), and study the fluctuations of the block magnetization defined as

\[
\langle M^2_k \rangle = \sum_{i,j=k}^{k+\ell-1} \langle \sigma_i \sigma_j \rangle .
\]

Provided the full length of the block is within the system (i.e., it does not contain the boundary with the coupling \( J_{bc} \), the correlations entering Eq. (15) and, consequently, \( \langle M^2_k \rangle \) does not depend on the location of the block, \( k \),
and we can write

\[ \langle M_1^2 \rangle = \ell + 2 \sum_{n=1}^{\ell-1} (\ell - n) \langle \sigma_1 \sigma_{1+n} \rangle . \]  

(16)

Substituting now the correlations for PBC, one readily obtains an expression which depends both on the block size \( \ell \) and the system size \( N \):

\[ \frac{\langle M_1^2 \rangle^{(p)}}{\ell^2} = \frac{1 - v^N}{1 + v^N} \cdot \frac{1}{\ell(1 - v)} - 2v (1 - v^{N-\ell} - v^\ell) \frac{1 - v^\ell}{1 + v^N} \frac{1 - v^\ell}{\ell^2(1 - v^2)^2} \]  

(17)

where the superscript \( (p) \) denotes the periodic BC. Introducing the "aspect ratio" \( b = \ell/N \), and using the scaling variable \( \zeta = \ell(1 - v)/2 \), the scaling limit \( N \to \infty, \ell \to \infty \) with \( b \) and \( \zeta \) finite yields the following scaling function

\[ \frac{\langle M_1^2 \rangle^{(p)}}{\ell^2} = \langle m_1^2 \rangle^{(p)} (\zeta, b) = \frac{1}{\zeta} \left( \tanh \zeta/b - \frac{1 - e^{-2\zeta}}{2\zeta} \frac{1 - e^{-2(1-b)\zeta/b}}{1 + e^{-2\zeta/b}} \right) . \]  

(18)

As one can see, the scaling function goes over into the FBC case if \( b \to 0 \) while it becomes the scaling function for the PBC case if \( b = 1 \). The function smoothly interpolates between the limiting cases in \( 0 < b \leq 1 \).

If the block is embedded in a FBC system then \( \{ M_1^2 \}^{(f)} \) can be deduced from the observation that \( \langle \sigma_1 \sigma_{1+n} \rangle \) is independent of the system size (see Eq. 13). Namely, \( \langle M_1^2 \rangle^{(f)} \) must coincide with \( \langle M_2 \rangle \) for the FBC with \( N \) replaced by \( \ell \). Thus we have a scaling function \( \{ M_2 \}^{(f)} \) which does not depend on the "aspect ratio" \( b \)

\[ \frac{\langle M_2^2 \rangle^{(f)}}{\ell^2} = \langle m_2^2 \rangle^{(f)} (\zeta, b) = \langle m_2^2 \rangle^{(f)} (\zeta) . \]  

(19)

Finally, the fluctuations in a block embedded in an aperiodic chain, can also be calculated and the somewhat more complicated result has a similar structure as in case of embedding in a periodic chain. Namely, changing the aspect ratio from \( b \) to \( b = 0 \), the results interpolates between the APBC and the FBC scaling functions in Eq. 18.

The common feature of all the above results is that the BC become irrelevant in the limit \( b \to 0 \) where the block is much smaller than the system. Furthermore, we find that this bulk behaviour coincides with the FBC result. Whether this coincidence with the FBC result is a general feature of bulk fluctuations is not quite clear and should be investigated in more complicated and higher dimensional systems.

IV. MAGNETIZATION DISTRIBUTION

If at a given temperature, \( i.e. \), at a given correlation length, the system size goes to infinity then the magnetization distribution goes to a Gaussian around zero due to the Central Limit Theorem, and goes eventually to a Dirac delta function

\[ \lim_{T \to 0} \lim_{N \to \infty} P(m) = \delta(0) . \]  

(20)

On the other hand, at any given system size as the temperature goes to zero, \( i.e. \), the correlation length goes to infinity, the magnetization goes to either plus or minus one, and the distribution becomes

\[ \lim_{N \to \infty} \lim_{T \to 0} P(m) = \frac{1}{2} \left[ \delta(m + 1) + \delta(m - 1) \right] . \]  

(21)

The importance of the \( \zeta = Nt/2 \) scaling variable is that if \( N \to \infty \) and \( T \to 0 \) in such a way that the correlation length is always proportional to the system size \( i.e. \), \( \zeta \) is a constant \( \zeta \) then a nontrivial distribution arises even for the \( d = 1 \) Ising model.

The meaning of \( \zeta \) suggests the development of a small \( \zeta \) (small number \( g \) of domain walls) expansion. Thus we shall first calculate \( P_g(M) \), the probability of a given magnetization in the presence of \( g \) domain walls. Once \( P_g(M) \) is known, the probability of a given magnetization \( P(M) \) can be obtained by summing up \( P_g(M) \)

\[ P(M) = \sum_g P_g(M) . \]  

(22)

The states with \( g \) domain walls are degenerate as their energy \( E_g \) depends only on the number of walls

\[ E_g = \begin{cases} J(2g - N) & \text{PBC, APBC} \\ J(2g - N + 1) & \text{FBC} \end{cases} . \]  

(23)

Thus, the calculation \( P_g(M) \) reduces to counting all possible spin configurations \( \Omega_g(M) \) at given \( g \) and \( M \) values

\[ P_g(M) = \frac{e^{-E_g/k_B T}}{Z} \Omega_g(M) . \]  

(24)

Here \( Z \) is the partition function corresponding to a given BC which, in the scaling limit, becomes

\[ Z = \begin{cases} 2e^{NK} \cosh \zeta & \text{PBC} \\ 2e^{NK} \sinh \zeta & \text{APBC} \\ 2e^{(N-1)K} e^\zeta & \text{FBC} \end{cases} . \]  

(25)

For \( g = 0 \), \( i.e. \), if there is no domain wall in the configurations

\[ \Omega_0(M) = \delta_{M,N} + \delta_{M,-N} . \]  

(26)

For \( g > 0 \), let \( W_j \) be the position of the \( j \)th wall (say \( W_j = i \) when it is just before \( \sigma_i \) with \( j = 1, \ldots, g \), \( i.e. \), \( W_j < W_{j+1} \) for \( j = 1, \ldots, g - 1 \), and also note that by definition \( W_1 > 0 \) and \( W_g \leq N \). Let us denote the magnetization of the domain between \( W_1 \) and \( W_2 \) by \( M_1 \).
Now, it is sufficient to obtain the number of configurations with the restriction \( M_1 > 0 \) and denote it as \( \Omega^+_g(M) \) as

\[
\Omega_g(M) = \Omega^+_g(M) + \Omega^-_g(-M). \tag{27}
\]

Note that fixed magnetization also means fixed number of upspins \( N^+_\uparrow = (N + M)/2 \) and downspins \( N^-_\downarrow = (N - M)/2 \).

### A. Periodic boundary conditions

PBC enforce the number of domain walls to be always even \( g = 2s \), with \( s = 0, \ldots, [N/2] \), where \([ \cdot ]\) stands for the integer part. As there is at least one spin after each detached upspins among the \( N^-_\uparrow \) and independently distributing the \( N^+_\downarrow \) among the \( N^-_\downarrow \) downspins among the \( s + 1 \) down domains (between \( W_{2j} \) and \( W_{2j+1} \), also in front of \( W_1 \) and behind \( W_{2s} \))

\[
\Omega^+_{2s}(M) = \binom{N^+_\uparrow - 1}{s - 1} \binom{N^-_\downarrow}{s}. \tag{29}
\]

Note that the binomial coefficient \( \binom{a}{b} = 0 \) for \( a < b \), which reflects the fact that there are no configurations with more domain walls than either \( 2N^+_\uparrow \) or \( 2N^-_\downarrow \), i.e., \( s \leq \min(N^+_\uparrow, N^-_\downarrow) \). With the formula \( \binom{N}{s} \) one can easily arrive at \( \Omega_{2s} \) without restriction on the sign of \( M_1 \)

\[
\Omega_{2s}(M) = \binom{N^+_\uparrow - 1}{s - 1} \binom{N^-_\downarrow}{s} + \binom{N^-_\downarrow - 1}{s - 1} \binom{N^+_\uparrow}{s}. \tag{30}
\]

In the \( N \to \infty \) limit, for fix \( n^+_\uparrow = N/2 \) and \( n^-_\downarrow = N/2 \) the number of configurations becomes

\[
\Omega_{2s}(M) = N^{2s-1} \frac{\binom{n^+_\uparrow n^-_\downarrow}{s-1}}{s!(s-1)!}. \tag{31}
\]

In this limit we also need to switch from the discrete probabilities \( P(M) \) to the probability density \( P(m) \), which brings in a factor \( N/2 \), and substituting Eq. \( \text{(31)} \) into Eq. \( \text{(29)} \) leads to

\[
P_{2s}(m) = \frac{e^{NK}}{2Z^{(p)}} e^{-4sK} N^{2s} \frac{\binom{n^+_\uparrow n^-_\downarrow}{s-1}}{s!(s-1)!}. \tag{32}
\]

Now the scaling limit can be finally taken using \( \text{(26)} \), and for \( s = 0 \) one realizes that the expression of Eq. \( \text{(26)} \) develops singularities

\[
P_0(m) = \frac{1}{2 \cosh \zeta} [\delta(m + 1) + \delta(m - 1)]. \tag{33}
\]

For \( s \neq 0 \), using \( \zeta = Ne^{-2K} \) leads to the final result for the magnetization distribution with a fixed \( 2s \) number of walls

\[
P_{2s}(m) = \frac{(\zeta/2)^{2s}(1 - m^2)^{s-1}}{s!(s-1)! \cosh \zeta}. \tag{34}
\]

It becomes clear at this point that we are doing a small \( \zeta \) expansion and that a fixed \( 2s \) number of domain walls belongs to the order \( 2s \) of the expansion.

![FIG. 3: Magnetization distribution](image)

The magnetization distribution without restrictions on the number of walls can be obtained from Eq. \( \text{(22)} \) which is valid also in the continuous limit

\[
P(m) = \sum_{s=0}^{\infty} P_{2s}(m). \tag{35}
\]

Using the series form of the modified Bessel functions

\[
I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\nu}}{k!(k+\nu)!}, \tag{36}
\]

leads to the final expression for periodic BC

\[
P^{(p)}(m) = \frac{1}{2 \cosh \zeta} \left[ \delta(m + 1) + \delta(m - 1) \right] \tag{37}
\]

\[
+ \frac{\zeta}{2 \sqrt{1 - m^2} \cosh \zeta} I_1 \left( \zeta \sqrt{1 - m^2} \right). \tag{38}
\]

One can easily check that \( P(m) \) is normalized, i.e., \( \int_{-1}^{1} dm \, P(m) = 1 \), by changing the integration variable \( m \) to \( \theta = \arcsin m \) and using the series form of Eq. \( \text{(36)} \). In the same way the expression for the second moment of \( P(m) \) in Eq. \( \text{(19)} \) can also be obtained.

One can investigate the speed of convergence in \( N \) of the magnetization distribution \( P(m) \) of Eq. \( \text{(37)} \). Instead
of making Monte Carlo simulations on the Ising model we calculated numerically the probabilities of possible magnetizations for finite chains, based on Eq. (24) and (35), and multiplied the results by \((N+1)/2\) for the sake of comparison with the probability density \(P(m)\) in the scaling limit

\[
P(m, N) = \frac{N+1}{2Z} \sum_{s=0}^{\lfloor N/2 \rfloor} e^{(N-4s)K} \Omega_{2s}(M), \quad (38)
\]

where \(\Omega_{2s}(M)\) is given by Eq. (26), (30), and \(K = \text{artanh}(1-2\zeta/N)\). Eq. (38) can be easily evaluated numerically, as the time computation increases linearly with \(N\) as opposed to the exponentially long time needed to encounter all possible configurations. One observes in Fig. 4 that the convergence is faster for smaller values of \(\zeta\), in agreement with Fig. 2.

![Magnetization distribution](image)

**FIG. 4:** Magnetization distribution \(P(m)\) for periodic BC in the \(N \to \infty\) limit [only the regular part of Eq. (38) is depicted] compared to its finite \(N = 20\) (square), 40 (circle), and 80 (triangle) forms of Eq. (38) with the scaling variable being \(\zeta = 3\) (closed symbols), and 6 (open symbols). Observe the evolving singularities at \(m = \pm 1\) for finite systems (closed triangles are out of range).

**B. Antiperiodic boundary conditions**

The main difference from the periodic case is that the number of domain walls is odd \(g = 2s-1\), with \(s = 1, \ldots, ([N+1]/2)\). As there has to be at least one domain wall in each configuration there are no Dirac delta peaks at \(m = \pm 1\) in the distribution. A typical configuration looks like

\[
\downarrow (W_1 \uparrow) \uparrow \uparrow \downarrow (W_2 \downarrow) \downarrow \ldots \downarrow (W_{2s-1} \uparrow) \uparrow \uparrow . \quad (39)
\]

The number of configurations with the restriction of \(M_1 > 0\) is

\[
\Omega_{2s-1}^+(M) = \binom{N_\uparrow - 1}{s-1} \binom{N_\downarrow}{s-1}, \quad (40)
\]

as we have \(N_\uparrow - s\) free upspins to distribute into \(s\) up domains and \(N_\downarrow - s+1\) free downspins for \(s\) down domains. In the continuous limit the number of configurations becomes

\[
\Omega_{2s-1}(M) = 2N^{2s-2} \frac{(n_\uparrow n_\downarrow)^{s-1}}{(s-1)^2} , \quad (41)
\]

and the magnetization distribution with fixed number of walls \((24)\) reads

\[
P_{2s-1}(m) = \frac{(\zeta/2)^{2s-1}(1-m^2)^{s-1}}{(s-1)^2 \sinh \zeta} . \quad (42)
\]

Summing this expression up over the possible number of walls \((22)\) leads to the final result

\[
P^{(a)}(m) = \frac{\zeta}{2\sinh \zeta} I_0 \left( \zeta \sqrt{1-m^2} \right) . \quad (43)
\]

**C. Free boundary conditions**

In case of free BC the number of domain walls can be both even \(g = 2s\) and odd \(g = 2s-1\) with \(s = 1, \ldots, [N/2]\). The only difference in counting all possible configurations, with a given \(g, M\), and the condition \(M_1 > 0\), is the supplementary restriction that there always has to be a down spin before the first wall for obvious reasons, which can be visualized as \((\downarrow W_1 \uparrow)\) in the example \((28)\). Now the number of configurations can be easily obtained

\[
\Omega_{2s-1}^+(M) = \binom{N_\uparrow - 1}{s-1} \binom{N_\downarrow}{s-1} \quad (44)
\]

and one observes that in the scaling limit they are equal to the corresponding periodic \((21)\) or antiperiodic \((11)\) result, i.e., \(\Omega_{2s}^{(f)}(M) = \Omega_{2s}^{(p)}(M)\) and \(\Omega_{2s-1}^{(f)}(M) = \Omega_{2s-1}^{(a)}(M)\). The energies of both the even and odd states are greater than the energies of the corresponding periodic and antiperiodic states \((28)\) by \(J\) and thus we can write the scaling limit of the probability of a given \(M\) as

\[
P_g^{(f)}(M) = \frac{e^{-E_g^{(f)}/k_BT}}{Z^{(f)}} \Omega_g^{(f)}(M) = \begin{cases} 
\frac{e^{-K Z^{(a)}}}{Z^{(f)}} P_g^{(p)}(M) & g = 2s \\
\frac{e^{-K Z^{(a)}}}{Z^{(f)}} P_g^{(a)}(M) & g = 2s-1.
\end{cases} \quad (45)
\]

Using Eqs. (28), one finds that the prefactors of the distributions depend only on \(\zeta\), thus collecting the contributions with different number of domain walls \((22)\), we obtain an expression for \(P^{(f)}(m)\) through \(P^{(p) (a)}(m)\)

\[
P^{(f)}(m) = e^{-\zeta [\cosh \zeta P^{(p)}(m) + \sinh \zeta P^{(a)}(m)]} , \quad (46)
\]
which leads to

\[
P^{(f)}(m) = \frac{e^{-\zeta}}{2} \left[ \delta(m + 1) + \delta(m - 1) \right]
\]

\[
+ \frac{\zeta e^{-\zeta}}{2\sqrt{1 - m^2}} I_1 \left( \zeta \sqrt{1 - m^2} \right)
\]

\[
+ \frac{\zeta e^{-\zeta}}{2} I_0 \left( \zeta \sqrt{1 - m^2} \right)
\].

As we shall see in the next section, the above expression applies for bulk BC as well. The bulk result was obtained previously by Bruce (see (3.20)). Note also that the relationship established between the distributions for different BC is certainly valid for the fluctuations of the magnetization as well.

One should note that the coefficient of the singular part is larger for the periodic case, i.e., the fluctuations of the magnetization as well.

This nonsingular part also shows the expected sequence of a configuration of the spins for the segment of spins \(\ell\) and \(N\) is identical to the free end result for any \(\xi\). As we shall show the distribution \(P\) eventually evolves to a Dirac delta function which leads to the most disordered state.

In order to see this, consider the Boltzmann weight of a configuration of the spins for the segment of spins between \(\sigma_k\) and \(\sigma_{n=k+\ell}\)

\[
P(\{\sigma_k, ..., \sigma_n\}) = \frac{e^{K \sum_{j=k}^{n-1} \sigma_j \sigma_{j+1}}}{\sum_{\{\sigma_{1,N}\}} e^{K \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1}}}.
\]

This is the free end probability distribution for the spins in the segment \([k, k + \ell]\) thus the magnetization distribution \(P(m_\ell)\), with \(m_\ell = M_\ell/\ell\), is identical to the FBC case given by Eq.(17), i.e., \(P^{(f)}(m_\ell) = P^{(f)}(m)\).

The above derivation does not hold for chains with PBC and APBC and one expects that the \(P(M_\ell)\) depends on the aspect ratio \(b = \ell/N\). In the \(b \to 0\) limit, i.e., when the window size is relatively small, however, \(P(M_\ell)\) becomes independent from the BC imposed on the whole chain. More precisely, this happens in the scaling limit \(\ell, N \to \infty\), \(t \to 0\) with \(\zeta = \ell t/2\) and \(b = \ell / N\) kept constant, a limit where the correlation length \(\xi \sim \ell \ll N\) and the effects of the boundaries of the chain can be neglected.

E. Asymptotic regimes

The small \(\zeta\) expansion is already given by Eq.(22). For sufficiently small \(\zeta\) the distribution \(P(m)\) can be approximated by the first few terms, e.g., the Dirac delta functions (except for the antiperiodic case) plus a constant probability density for all BC.

For large values of \(\zeta\) a Gaussian approximation can be obtained using only the first term of the large \(x\) asymptotic of the Bessel functions

\[
I_1(x) \approx I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}.
\]

and using also the condition \(m \ll 1\), where \(P(m)\) significantly differs from zero, due to the factor \(\exp(\zeta \sqrt{1 - m^2})\)

\[
P(m) \approx \frac{1}{\sqrt{2\pi \zeta}} e^{-m^2 \zeta / 2}.
\]

\(P(m)\) eventually evolves to a Dirac delta function \(\delta(m)\) for \(\zeta \to \infty\).
V. FINAL REMARKS

The fluctuations of the magnetization in the one-dimensional Ising model have been investigated near its critical point in the limit of \( T \to 0 \) and \( N \to \infty \) with the average number of domain walls \( \langle 2\zeta = N \rangle \) kept constant. A simple combinatorial derivation has been presented for the magnetization distributions and it was showed that in this limit these distributions are nontrivial well defined functions of the magnetization and the control parameter \( \zeta \).

The \( \zeta \) dependence of the magnetization distribution should be emphasized, as it means that the order parameter distribution of a system can, in general, depend on the way the critical point is approached and the thermodynamic limit is taken.

We focused our attention on the effect of boundary conditions, namely we imposed periodic, antiperiodic, and free BC on the chain. The magnetization distributions are shown to be sensitive to the BC and well distinguishable for all values of \( \zeta \). For antiperiodic BC the distributions differ fundamentally from the periodic and free BC case (lack of Dirac delta peaks for APBC).

We also showed that the distribution of the magnetization of a segment (BBC) of the whole chain with FBC coincide with the distribution of the total magnetization, independently of the size of the segment. For PBC or APBC chains the above statement is true only if the relative segment size \( b \) goes to zero.

It is worth mentioning the analogies to a simple random walk. The corresponding quantity is the distribution of the width of the walk (mean square deviation of the position of the walker), and for that two distinct functions have been exactly derived for periodic and free BC [13]. The qualitative behaviour of BBC is also the same as that of the Ising model discussed above.

It would be very interesting to see similar calculations for the order parameter distribution in other equilibrium one-dimensional models, e.g., the classical and quantum XY and Heisenberg models, or nonequilibrium ones, e.g., absorbing state phase transitions.

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