Abstract

The theory of learning in games is prominent in the AI community, motivated by several rising applications such as multi-agent reinforcement learning and Generative Adversarial Networks. We propose Mutation-driven Multiplicative Weights Update (M2WU) for learning an equilibrium in two-player zero-sum normal-form games and prove that it exhibits the last-iterate convergence property in both full- and noisy-information feedback settings. In the full-information feedback setting, the players observe their exact gradient vectors of the utility functions. On the other hand, in the noisy-information feedback setting, they can only observe the noisy gradient vectors. Existing algorithms, including the well-known Multiplicative Weights Update (MWU) and Optimistic MWU (OMWU) algorithms, fail to converge to a Nash equilibrium with noisy-information feedback. In contrast, M2WU exhibits the last-iterate convergence to a stationary point near a Nash equilibrium in both of the feedback settings. We then prove that it converges to an exact Nash equilibrium by adapting the mutation term iteratively. We empirically confirm that M2WU outperforms MWU and OMWU in exploitability and convergence rates.

1 Introduction

This paper considers learning algorithms for finding an (approximate) equilibrium in two-player zero-sum normal-form games. Motivated by the training for Generative Adversarial Networks (GANs) [Goodfellow et al., 2014] and multi-agent reinforcement learning [Busoniu et al., 2008], many algorithms have been developed to find a near-optimal solution to minimax optimization problems [Blum and Monsour, 2007; Daskalakis et al., 2018] in the form of \( \min_x \max_y f(x, y) \).

In this context, no-regret learning, which minimizes regret in repeated decisions, has been extensively studied [Banerjee and Peng, 2005; Zinkevich et al., 2007; Daskalakis et al., 2011]. The well-known Multiplicative Weights Update (MWU) exhibits the average-iterate convergence, i.e., the minimax solution (the Nash equilibrium) is obtained in an average sense. Still, it has been shown that the actual trajectory of updated strategies diverges or cycles [Mertikopoulos et al., 2018; Bailey and Piliouras, 2018]. This feature is unsatisfactory because averaging strategies is often prohibited in tasks such as training GANs with large neural networks [Wei et al., 2021a].

This paper focuses on whether the actual sequence of updated strategies converges to an equilibrium, i.e., the last-iterate convergence, which is inevitably a stronger notion than the average-iterate convergence. A series of optimistic no-regret learning algorithms is proven to exhibit the last-iterate convergence [Daskalakis et al., 2018; Mertikopoulos et al., 2019]. In particular, the optimistic MWU (OMWU) algorithm is guaranteed to converge to a Nash equilibrium at an exponential rate [Daskalakis and Panageas, 2019; Wei et al., 2021a].
Figure 1: Learning dynamics of RD and RMD in biased Rock-Paper-Scissors (the game matrix is given by
\[
\begin{bmatrix}
0 & -3 & 1 \\
3 & 0 & -1 \\
-1 & 1 & 0 \\
\end{bmatrix}
\]. The red star represents the Nash equilibrium point of the game.

However, it is required that players observe the exact gradient vectors of their utility functions at each
iteration, which we call full-information feedback.

We generalize the full-information feedback setting to the noisy-information feedback setting, where
players can only observe noisy estimates of the gradient vectors at each iteration; this setting is also called
semi-bandit feedback. The celebrated OMWU does not guarantee the last-iterate convergence and may
diverge or enter a limit cycle, as shown in Figure 5d. It has been so far guaranteed only in some restricted
games, such as those with a strict Nash equilibrium, in the noisy-information feedback setting [Heliou et al.,
2017, Giannou et al., 2021a].

Motivated by the aforementioned lines of work, we propose Mutant MWU (M2WU) as the first learning
algorithm that enjoys the last-iterate convergence with noisy-information feedback. M2WU is inspired by
the fact that MWU is tantamount to replicator dynamics (RD), which is widely used in evolutionary game
theory [Borgers and Sarin, 1997, Bloembergen et al., 2015]. Our M2WU is designed so that it corresponds to
replicator-mutator dynamics (RMD) [Hofbauer and Sigmund, 1998, Hofbauer et al., 2009, Zagorsky et al.,
2013, Bauer et al., 2019], where each player may mutate his/her action. It is known that introducing
mutation stabilizes the dynamics and empirically makes numerical errors in computation small [Zagorsky
et al., 2013]. Indeed, Figure 1 demonstrates that RMD clearly converges to a near-equilibrium in a biased
Rock-Paper-Scissors game, while RD oscillates around an equilibrium.

Starting with the full-information case, our main result is that M2WU with a constant learning rate
converges to a stationary point of RMD, which is known to be an approximate Nash equilibrium. The
amount of approximation is specified by the mutation parameters. We also show that convergence occurs at
an exponentially fast rate. Although OMWU achieves a similar convergence rate, it requires the equilibrium
in underlying games must be unique to establish convergence at that rate [Daskalakis and Panageas, 2019,
Wei et al., 2021a]. These consequences approximately apply to the noisy-information feedback setting under
mild conditions for noise influencing the player’s observations (zero-mean martingale noise with tame second-
moment tails). Specifically, M2WU converges to the stationary point almost surely.

Surprisingly, in both feedback settings, we successfully establish convergence to an exact Nash equilibrium
via iteratively adapting the mutation term according to the recently maintained strategy by M2WU. To
the best of our knowledge, the proposed M2WU with an appropriate choice of the update interval is the
first to exhibit the last-iterate convergence to an exact Nash equilibrium with noisy-information feedback.
Furthermore, we empirically demonstrate that M2WU outperforms MWU and OMWU in several games in
exploitability and convergence rate. In particular, M2WU achieves faster convergence rates than MWU and
OMWU, regardless of which feedback is applied.

2 Related Literature

Last-iterate convergence with full-information feedback. Recently, various optimistic learning al-
gorithms [Rakhlin and Sridharan, 2013a,b] such as optimistic Follow the Regularized Leader (FTRL) and
optimistic Mirror Descent have been proposed, and their last-iterate convergence guarantees are proven with
denote the finite action set for each player $i$. In normal-form games, a strategy profile $\pi$ can improve his/her expected utility by deviating from his/her specified strategy. In two-player zero-sum Nash equilibrium, which satisfies the following inequality:

$$\exists \epsilon > 0 \text{ s.t. } v_1^{\pi_1, \pi_2} \leq v_1^{\pi_1^*, \pi_2^*} + \epsilon. \quad (3.1)$$

We denote the set of Nash equilibria by $\Pi$. An $\epsilon$-Nash equilibrium $(\pi_1, \pi_2)$ is an approximation of a Nash equilibrium, which satisfies the following inequality:

$$\max_{\pi_1 \in \Delta(A_1)} \min_{\pi_2 \in \Delta(A_2)} v_1^{\pi_1, \pi_2} \leq \epsilon. \quad (3.2)$$

Furthermore, we define exploitation of the strategy profile $\pi$. Exploitation is widely used to assess how close $\pi$ is to Nash equilibrium in two-player zero-sum games and always takes a non-negative value.

Last-iterate convergence with noisy-information feedback. A few studies have been done to prove last-iterate convergence with noisy-information feedback. Most existing studies discuss last-iterate convergence under the assumption that the game's equilibrium is a pure (or strict). Helion et al. [2017] prove the convergence of an MWU-based algorithm with noise for the potential game, in which the game always has a pure Nash equilibrium, with the help of the stochastic approximation technique. There are also analyses with FTRL-based algorithms with noise Giannou et al. [2021]. Such results have been obtained under other strong assumptions, such as strict (or strong) monotonicity Bravo et al. [2018], Hsieh et al. [2019], Kannan and Shanbhag [2019], Azizian et al. [2021] and strict variational stability Mertikopoulos et al. [2019], Mokhtari et al., 2020. Another approach is to use a two-time scaling, i.e., fixing the strategies of the two players to obtain sufficient samples for accurate estimates of the expected value of the utility, e.g., Wei et al. [2021b]. Problem settings other than two-player zero-sum normal-form games have only recently been considered Hsieh et al. [2022].
A strategy profile $\pi$ has exploitability of 0 if and only if $\pi$ is a Nash equilibrium.

### 3.3 Problem Setting

In this study, we consider a setting where the following process is repeated: 1) At each iteration $t \in \mathbb{N}$, each player $i \in \{1, 2\}$ determines the (mixed) strategy $\pi_i^t \in \Delta(A_i)$ based on the previously observed feedback; 2) Each player $i$ observes the new feedback $\hat{q}_i^{\pi^t}$ with respect to the gradient vector of the expected utility function $\nabla_{\pi_i^t} v_i^{\pi_i^t} = q_i^{\pi^t}$. This study considers two feedback settings: full-information feedback and noisy-information feedback. In the full-information feedback setting, each player $i$ observes the conditional expected utility vector $q_i^{\pi^t}$ as feedback, i.e., $\hat{q}_i^{\pi^t} = q_i^{\pi^t}$. In the noisy-information feedback setting, at each iteration $t$, each player observes the noisy conditional expected utility vector

$$q_i^{\pi_t}(a) = q_i^{\pi^t}(a) + \xi_i^t(a) \quad \text{for } a \in A_i,$$

where the sequence of the noise vectors $(\xi_i(t))_{t \in A_i}$ is independent over $a$ and $t$. This type of noise-additive setting is standard in recent research [Heliou et al. 2017, Bravo et al. 2018, Giannou et al. 2021a,b].

Multiplicative Weights Update (MWU) is a widely used algorithm for learning a Nash equilibrium. In MWU, each player $i$ updates her strategy $\pi_i^t$ at iteration $t$ as follows:

$$\pi_i^{t+1}(a) = \frac{\pi_i^t(a) \exp \left( \eta q_i^{\pi_i^t}(a) \right)}{\sum_{a' \in A_i} \pi_i^t(a') \exp \left( \eta q_i^{\pi_i^t}(a') \right)},$$

where $\eta > 0$ is a learning rate.

### 3.4 Other Notations

We denote the probability simplex on $A_i$ by $\Delta(A_i)$, and the interior of $\Delta(A_i)$ by $\Delta^0(A_i) = \{ p \in \Delta(A_i) \mid \forall a \in A_i, \; p(a) > 0 \}$. The Kullback-Leibler divergence is defined by $\text{KL}(x, y) = \sum_i x_i \ln \frac{x_i}{y_i}$. Besides, with a slight abuse of notation, we denote the sum of Kullback-Leibler divergences as $\text{KL}(\pi, \pi') = \sum_{i=1}^2 \text{KL}(\pi_i, \pi'_i)$.

### 4 Mutant Multiplicative Weights Update

In this section, we propose a mutant Multiplicative Weights Update (M2WU) algorithm. M2WU is a variant of MWU, which adds a mutation (perturbation) term to the gradient vector. Specifically, M2WU updates each player’s strategy by the following update rule:

$$\pi_i^{t+1}(a) = \frac{\pi_i^t(a) \exp \left( \eta q_i^{\mu+t}(a) \right)}{\sum_{a' \in A_i} \pi_i^t(a') \exp \left( \eta q_i^{\mu+t}(a') \right)},$$

where $\mu > 0$ is the mutation parameter, and $r_i \in \Delta^0(A_i)$ is the reference strategy. The pseudo-code of M2WU is Algorithm 1 with $N = \infty$.

The mutation term $\frac{\mu}{\pi_i^t(a)} (r_i(a) - \pi_i^t(a))$ is inspired by RMD, which is governed by the following ordinary differential equation:

$$\frac{d}{dt} \pi_i^t(a) = \pi_i^t(a) \left( q_i^{\pi_i^t}(a) - v_i^{\pi_i^t} \right) + \mu \left( r_i(a) - \pi_i^t(a) \right).$$  \hspace{1cm} (RMD)

RMD is the continuous-time version of M2WU and has been reported to stabilize the learning dynamics [Bonze and Burger 1995, Bauer et al. 2019]. Figure 1a illustrates the trajectories of RD and RMD with $\mu = \{0.01, 0.1, 1.0\}$ in a biased version of the Rock-Paper-Scissors. From Figure 1a, the trajectory of RD cycles and fails to converge to a Nash equilibrium since the equilibrium is a mixed strategy with full support.
Algorithm 1 M2WU for player $i$. The algorithm with $N = \infty$ corresponds to M2WU with a fixed reference strategy.

**Require:** Learning rate sequence $\{\eta_t\}_{t \geq 0}$, mutation parameter $\mu$, update frequency $N$, initial strategy $\pi^0_i$, initial reference strategy $r^0_i$.

1: $k \leftarrow 0$, $\tau \leftarrow 0$
2: **for** $t = 0, 1, 2, \ldots$ **do**
3:  
4:  **if** $\tau = N$ **then**
5:  
6:   
7: Observe the (noisy) gradient vector $\hat{q}_{\pi^t_i}^\tau$.
8:  **for** $a \in A_i$ **do**
9:  
10:   
11:  **end for**
12:  
13: **end for**

On the other hand, Figure 1b-1d shows that RMD’s trajectory converges to a unique stationary point. Therefore, the mutation term is expected to make the stationary point to the asymptotically stable one.

We note that M2WU can be viewed as an instantiation of the discrete-time Mutant FTRL algorithm with entropy regularization Abe et al. [2022].

5 Convergence to an Approximate Nash Equilibrium

In this section, we mainly show that the updated strategy profile $\pi^t$ converges to a stationary point of (RMD). We denote the stationary point of (RMD) with fixed $\mu$ and $r = (r_i)_{i=1}^2$ by $\pi^{\mu,r}$.

5.1 Full-Information Feedback Setting

First, we establish the last-iterate convergence rate of M2WU with full-information feedback. Remind that in the full-information feedback setting, each player $i$ observes the conditional expected utility vector $\hat{q}_i^\tau = q_i^\tau$ as feedback. The following convergence result for M2WU with a constant learning rate $\eta = \eta_t$ is obtained in the full-information feedback setting:

**Theorem 5.1.** Let $\pi^{\mu,r} \in \prod_{i=1}^2 \Delta(A_i)$ be a stationary point of (RMD). If we use the constant learning rate sequence in M2WU, $\forall t \geq 0: \eta_t = \eta \in (0, C_1)$, the strategy $\pi^t$ updated by M2WU satisfies that:

$$\text{KL}(\pi^{\mu,r}, \pi^t) \leq \text{KL}(\pi^{\mu,r}, \pi^0)(1 - C_2)^t, \text{ for all } t \geq 0,$$

where $C_1 > 0$ and $C_2 \in (0, 1)$ are constants depend only on $\pi_0$, $\mu$, $\pi^{\mu,r}$, and $r$.

This result means that for a fixed $\mu$ and $r$, $\pi^t$ converges to $\pi^{\mu,r}$ exponentially fast. From this theorem, $\pi^t$ converges to a $2\mu$-Nash equilibrium because $\pi^{\mu,r}$ is a $2\mu$-Nash equilibrium [Bauer et al. 2019]:

5
Corollary 5.2. For any constant learning rate \( \eta_t = \eta \in (0, C_1) \), the exploitability for M2WU is bounded as:

\[
\text{exploit}(\pi^t) \leq \text{exploit}(\pi^{\mu,r}) + 2u_{\text{max}}\sqrt{\text{KL}(\pi^{\mu,r}, \pi^0)(1 - C_2)^2} \\
\leq 2\mu + 2u_{\text{max}}\sqrt{\text{KL}(\pi^{\mu,r}, \pi^0)(1 - C_2)^2},
\]

where \( C_1 \) and \( C_2 \) are the same constants in Theorem 5.1.

The proof of this corollary is shown in Appendix B.

5.1.1 Proof Sketch of Theorem 5.1

We sketch below the proof of Theorem 5.1. Complete proofs for the theorem and associated Lemmas are presented in Appendix A.

(1) Decomposing single-step variation of KL(\( \pi^{\mu,r}, \cdot \)). First, we derive the following difference equation for the Kullback-Leibler divergence between \( \pi^{\mu,r} \) and \( \pi^t \):

\[
\text{KL}(\pi^{\mu,r}, \pi^{t+1}) - \text{KL}(\pi^{\mu,r}, \pi^t) = \eta \sum_{i=1}^{2} \left( v_i^{\pi^{\mu,r}} + \mu - \mu \sum_{a \in A_i} r_i(a) \frac{\pi_i^{\mu,r}(a)}{\pi_i^t(a)} \right) + \text{KL}(\pi^t, \pi^{t+1}).
\]

Equation (2) stems from the fact that for any \( \pi \in \prod_{i=1}^{2} \Delta(A_i) \), \( \text{KL}(\pi, \pi^t) = \sum_{i=1}^{2} (\eta \sum_{s=1}^{t-1} q_i^{\mu,s} \pi_i^t - \pi_i) - \psi_i(\pi_i^t) + \psi_i(\pi_i)) \), where \( \psi_i(p) = \sum_{a \in A_i} p(a) \ln p(a) \). For the details of the proof, see Appendix A. Hereafter, we quantify the terms (A) and (B), respectively.

(2) Equivalence notation of (A) in quasi-metric form. First, we prove that the term (A) can be rewritten by the (pseudo) metric between \( \pi^{\mu,r} \) and \( \pi^t \).

Lemma 5.3. Let \( \pi^{\mu,r} \in \prod_{i=1}^{2} \Delta(A_i) \) be a stationary point of (RMD). Then, \( \pi^t \) updated by M2WU satisfies that:

\[
\sum_{i=1}^{2} \left( v_i^{\pi^{\mu,r}} + \mu - \mu \sum_{a \in A_i} r_i(a) \frac{\pi_i^{\mu,r}(a)}{\pi_i^t(a)} \right) = -\mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \sqrt{\frac{\pi_i^t(a)}{\pi_i^{\mu,r}(a)}} - \sqrt{\frac{\pi_i^{\mu,r}(a)}{\pi_i^t(a)}} \right)^2.
\]

This result can be shown by using Lemma 5.6 in Abe et al. 2022.

(3) Quasi-metric upper bound on the term (B). Next, we upper bound the Kullback-Leibler divergence between \( \pi^t \) and \( \pi^{t+1} \) by the (pseudo) metric between \( \pi^{\mu,r} \) and \( \pi^t \):

Lemma 5.4. For any fixed learning rate \( \eta_t = \eta \in (0, C_1) \), M2WU ensures for any \( t \geq 0 \):

\[
\text{KL}(\pi^t, \pi^{t+1}) \leq 8\eta^2 u_{\text{max}}^2 \sum_{i=1}^{2} ||\pi_i^t - \pi_i^{\mu,r}||_1^2 + 8\eta^2 \mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a)^2 \left( \frac{1}{\pi_i^{\mu,r}(a)} - \frac{1}{\pi_i^t(a)} \right)^2,
\]

where \( C_1 \) is the same constant in Theorem 5.1.

(4) Putting it all together. By combining (2), Lemma 5.3 and Lemma 5.4, we get:

\[
\text{KL}(\pi^{\mu,r}, \pi^{t+1}) - \text{KL}(\pi^{\mu,r}, \pi^t) \leq -\eta \mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \sqrt{\frac{\pi_i^t(a)}{\pi_i^{\mu,r}(a)}} - \sqrt{\frac{\pi_i^{\mu,r}(a)}{\pi_i^t(a)}} \right)^2
\]

\[
+ 8\eta^2 \mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a)^2 \left( \frac{1}{\pi_i^{\mu,r}(a)} - \frac{1}{\pi_i^t(a)} \right)^2
\]

\[
+ 8\eta^2 u_{\text{max}}^2 \sum_{i=1}^{2} ||\pi_i^t - \pi_i^{\mu,r}||_1^2.
\]
From Pinsker's inequality \cite{tsybakov2009}, we can upper bound $\sum_{i=1}^{2} \sum_{a \in A_i} r_i(a)^2 \left( \frac{1}{\pi_i^t(a)} - \frac{1}{\pi_i^*} \right)^2$ and $\sum_{i=1}^{2} \| \pi_i^t - \pi_i^* \|_2^2$ by $\text{KL}(\pi_i^t, \pi_i^*)$, respectively. Furthermore, from Jensen's inequality, we can lower bound $\sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \frac{\pi_i^t(a)}{\pi_i^*} - \frac{\pi_i^t(a)}{\pi_i^*} \right)^2$ by $\text{KL}(\pi_i^t, \pi_i^t)$. Therefore, we can find a constant learning rate $\eta$ such that for some constant $C_2 \in (0, 1)$, we have:

\[
\text{KL}(\pi_i^t, \pi_i^{t+1}) - \text{KL}(\pi_i^t, \pi_i^t) \leq -C_2 \text{KL}(\pi_i^t, \pi_i^t).
\]

By transposing the term of $\text{KL}(\pi_i^t, \pi_i^{t+1})$ from the left-hand side to right-hand side, we get $\text{KL}(\pi_i^t, \pi_i^{t+1}) \leq (1 - C_2) \text{KL}(\pi_i^t, \pi_i^t)$. Thus, by mathematical induction, the statement of the theorem is concluded. 

\section{Convergence to an Exact Nash Equilibrium}

Sections 4 and 5 presented the M2WU with a fixed reference strategy profile $r$ and its convergence results. As shown in Corollary 5.2, the strategy $\pi^t$ updated by M2WU converges to the stationary point $\pi_i^*$. Therefore, if the exploitability of the stationary point $\pi_i^*$ converges to zero, the exact Nash equilibrium of the original game can be obtained. To this end, we control the exploitability of $\pi_i^*$ by adapting the reference strategy $r$. 

\subsection{5.2.1 Proof Sketch of Theorem 5.6}

First, we show that the update rule of the strategy $\pi_i^t$ by M2WU is an approximate Robbins-Monro algorithm. Then, inspired by the proofs of \cite{Kleinberg2009} and \cite{Heliou2017}, we prove that the strategy $\pi_i^t$ updated by M2WU is an asymptotic pseudo trajectory of the (continuous-time) replicator mutator dynamics (RMD) using the condition on the noise and the utility function. From Theorem 5.2 in \cite{Abe2022}, we can find a strict Lyapunov function of the continuous dynamics, and the stationary point of the dynamics is unique. From Lyapunov arguments and the results of \cite{Benaïm1999}, we can conclude that $\pi_i^t$ converges to the stationary point almost surely. 

\section{Noisy-Information Feedback Setting}

In this section, we consider a noisy-information feedback setting, where each player's observation is affected by noise. We assume the following mild condition on the noise distribution. Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by the random sequence $(\pi^t_i(a))_{a \in A_i})_{t=1,\ldots,T}$.

\textbf{Assumption 5.5.} For all player $i \in \{1, 2\}$, the noise process $(\xi^t_i(a))_{a \in A_i}$ satisfies the following two conditions.

(i) Zero-mean: $\mathbb{E}[\xi^t_i(a) \mid \mathcal{F}_{t-1}] = 0$, $\forall a \in A_i$, $\forall t \geq 1$, almost surely.

(ii) Moderate tails: For any $x > 0$, $\mathbb{P}[|\xi^t_i(a)|^2 \geq x \mid \mathcal{F}_{t-1}] \leq C/x^\alpha$, $\forall a \in A_i$, $\forall t \geq 1$, almost surely, with some constants $C > 0$ and $\alpha > 2$.

Assumption 5.5(i) means that the observation is unbiased: $\mathbb{E}[q^t_i(a) \mid \mathcal{F}_{t-1}] = q^t_i(a)$, almost surely. Assumption 5.5(ii) is a relatively weak assumption on the noise and is satisfied by a wide range of distributions, including bounded, sub-Gaussian, and sub-exponential distributions \cite{Heliou2017}. Assumption 5.5(ii) also implies that the variance of the noise is upper bounded by some constant. The following convergence results are obtained in the noisy feedback setting.

\textbf{Theorem 5.6.} Suppose that Assumption 5.5 is satisfied. Under M2WU with the step size $\eta_t \propto t^{-\beta}$ for some constant $\beta \in (1/\alpha, 1]$, the strategy $\pi^t$ converges to the stationary point $\pi_i^*$ almost surely.

For the noise process that can take an arbitrarily large value of $\alpha$, the value of $\beta$ can be arbitrarily close to 0. This suggests that learning is possible with a nearly-constant learning rate sequence for sub-Gaussian, sub-exponential, and bounded distributions. The proof of Theorem 5.6 is based on the method of stochastic approximation \cite{Benaïm1999, Borkar2009}. Here, we only present a sketch of the proof of Theorem 5.6. A complete proof is presented in Appendix C.
That is, we copy the updated strategy profile \( \pi^t \) to the reference profile \( r \) every \( N \) iterations. This technique is similar to the direct convergence method by Perolat et al. [2021]. The pseudo-code of M2WU with this technique corresponds to Algorithm 1 with finite \( N \).

Let us define \( r_k \) as the \( k \)-th reference strategy profile. From Theorem 6.1, \( \pi^t \) converges to \( \pi^{\mu,r} \) when \( N \) is set to a sufficiently large value. In this case, the following reference strategy \( r_{k+1} \) is set to the stationary point \( \pi^{\mu,r_k} \) of the (RMD) dynamics with reference strategy \( r_k \). In the remaining part of this section, we show that the sequence of stationary points \( \{ \pi^{\mu,r_{k-1}} \}_{k \geq 1} = \{ r_k \}_{k \geq 0} \) converges to \( \Pi^* \) of the original game.

**Theorem 6.1.** For any start point \( r^0 \in \prod_{i=1}^N \Delta^\circ(A_i) \), the sequence of stationary points \( \{ \pi^{\mu,r_{k-1}} \}_{k \geq 1} = \{ r_k \}_{k \geq 0} \) converges to the set of equilibria \( \Pi^* \) of the original game.

This result means that the exploitability of \( \pi^{\mu,r_k} \) converges to 0 because \( \exp(\pi^{\mu,r_k}) \leq \sum_{i \in \{1,2\}} \max_{\pi_i \in \Delta(A_i)} (v_{i}^{\pi_i,r_{k-1}} - v_{i}^{\pi_i,\pi^{\mu,r_k}}) \leq O\left(\sqrt{\text{KL}(\pi^*,\pi^{\mu,r_k})}\right) \), where \( \pi^* = \arg \min_{\pi \in \Pi^*} \text{KL}(\pi,\pi^{\mu,r_k}) \). We explain the proof sketch here; the complete proof of Theorem 6.1 is in Appendix D.

**Proof Sketch of Theorem 6.1.** Let \( F : \prod_{i=1}^N \Delta^\circ(A_i) \rightarrow \prod_{i=1}^N \Delta^\circ(A_i) \) be a function which maps the reference strategies \( r \) to the associated stationary point \( \pi^{\mu,r} \). We also define \( r_{k+1} = F(r_k) \), and \( r^0 \in \prod_{i=1}^N \Delta^\circ(A_i) \) as the starting reference strategy profile. Note that the function \( F \) is well-defined because the stationary point \( \pi^{\mu,r} \) of (RMD) is unique for each \( r \in \prod_{i=1}^N \Delta^\circ(A_i) \) [Abe et al. 2022]. First, we prove that the distance between \( \Pi^* \) and \( r_k \) decreases monotonically as \( k \) increases:

**Lemma 6.2.** For any \( k \geq 0 \), if \( r_k \in \prod_{i=1}^2 \Delta^\circ(A_i) \setminus \Pi^* \), then:

\[
\min_{\pi \in \Pi^*} \text{KL}(\pi^*, r_{k+1}) < \min_{\pi \in \Pi^*} \text{KL}(\pi^*, r_k).
\]

Otherwise, if \( r_k \in \Pi^* \), then \( r_{k+1} = r_k \in \Pi^* \).

We also show that \( F(\cdot) \) is a continuous function on \( \prod_{i=1}^N \Delta^\circ(A_i) \):

**Lemma 6.3.** Let \( F(r) : \prod_{i=1}^N \Delta^\circ(A_i) \rightarrow \prod_{i=1}^N \Delta^\circ(A_i) \) be a function which maps the reference strategies \( r \) to the associated stationary point \( \pi^{\mu,r} \) of (RMD). Then, \( F(\cdot) \) is a continuous function on \( \prod_{i=1}^N \Delta^\circ(A_i) \).

For these lemmas, we can use Lyapunov arguments to obtain a convergence result for \( \{ \pi^{\mu,r_{k-1}} \}_{k \geq 1} \).

### 7 Experiments

In this section, we conduct experiments to evaluate M2WU with a fixed reference strategy profile and M2WU with adaptive reference strategy profiles. For simplicity, we abbreviate M2WU with a fixed reference strategy profile (Algorithm 1 with \( N = \infty \)) as M2WU (fixed), and M2WU with adaptive reference strategy profiles as M2WU (adaptive). We compare their performance to those of MWU and OMWU.

We conduct our experiments in the following games: biased rock-paper-scissors (BRPS), a normal-form game with multiple Nash equilibria (M-Ne) [Wei et al. 2021a], and random utility games. BRPS and M-Ne have the following game matrices (Tables 1 and 2).

**Table 1: Biased RPS game matrix**

|   | R | P | S |
|---|---|---|---|
| R | 0 | -1 | 3 |
| P | 1 | 0 | -1 |
| S | -3 | 1 | 0 |

**Table 2: M-Ne game matrix**

|   | y1 | y2 | y3 | y4 | y5 |
|---|----|----|----|----|----|
| x1 | 0  | 1  | -1 | 0  | 0  |
| x2 | -1 | 0  | 1  | 0  | 0  |
| x3 | 1  | -1 | 0  | 0  | 0  |
| x4 | 1  | -1 | 0  | -2 | 1  |
| x5 | 1  | -1 | 0  | 1  | -2 |
Figure 2: Exploitability of $\pi^t$ for M2WU, MWU, and OMWU with full-information feedback.

Figure 3: Exploitability of $\pi^t$ for M2WU (fixed) with varying $\mu \in \{0.1, 0.01\}$ and $\eta \in \{0.1, 0.01, 0.001\}$ in BRPS with full-information feedback.

Note that the set of Nash equilibria in M-Ne is given as follows [Wei et al., 2021a]:

$$\Pi_1^* = \{(1/3, 1/3, 1/3, 0, 0)\},$$
$$\Pi_2^* = \{y \in \Delta^5 \mid y_1 = y_2 = y_3; 0.5y_5 \leq y_4 \leq 2y_5\}.$$

For random utility games, we consider the size of the game matrix to $25 \times 25$ and $100 \times 100$, then generate a random matrix with each component drawn from the standard gaussian distribution $\mathcal{N}(0, 1)$ in an i.i.d manner. For each game, the results are averaged over 100 instances with different random seeds. In each instance, we generate the initial strategy profile $\pi^0$ uniformly at random in $\prod_{i=1}^n \Delta^c(A_i)$.

7.1 Full-Information Feedback

In the first experiment, we compare the performance with full-information feedback. We set the learning rate $\eta = 0.1$ for all algorithms. For the M2WU algorithms, we set $\mu = 0.1$ and $r_{i}^0 = (1/|A_i|)_{a \in A_i}$. For M2WU (adaptive), we set $N = 100$ in Algorithm 4.

Figure 2 shows the average exploitability of the last-iterate strategy $\pi^t$ for each algorithm. We find that the exploitability of M2WU (fixed) quickly converges to a constant value smaller than $2\mu = 0.2$. This result highlights the upper bound on the exploitability of $\pi^t$ from Corollary 5.2. Figure 2 also shows that M2WU (adaptive) converges to a Nash equilibrium faster than MWU and OMWU. From this result, we can see that the sequence of stationary points of (RMD) converges to the Nash equilibrium (see Theorem 6.1 and Lemma 6.2).

Next, we examine the convergence rates of M2WU (fixed) with varying $\mu \in \{0.1, 0.01\}$ and $\eta \in \{0.1, 0.01, 0.001\}$. We conduct experiments on BRPS with full-information feedback. Figure 3 shows the average exploitability of $\pi^t$ for 100 instances. We find that the exploitability of $\pi^t$ converges to a smaller value for a smaller mutation parameter $\mu$. However, M2WU (fixed) with a smaller $\mu$ diverges when $\eta$ exceeds some value. On the other hand, a larger mutation parameter $\mu$ makes $\pi^t$ converges faster and enables to use of a larger learning rate.
In the second experiment, we analyze the performance of M2WU with noisy-information feedback. We set the initial strategy to \( \pi_t^0 = (1/|A_i|)_{i \in A}, \) for \( i \in \{1, 2\} \). The black point represents the equilibrium strategy. The blue/red points represent the initial/final points, respectively.

Figure 4 shows the average exploitability of \( \pi_t \) for M2WU, MWU, and OMWU with noisy-information feedback. We set the learning rate to \( \eta = 0.001 \). We set \( \mu = 0.1 \) for M2WU (fixed), and set \( \mu = 0.5 \) and \( N = 20,000 \) for M2WU (adaptive). In OMWU, we use the noisy gradient vector \( \tilde{q}_i^{t-1} \) at the previous step \( t-1 \) as the prediction vector. Figure 4 shows the exploitability of \( \pi_t \) for each algorithm. Observe that the exploitability for MWU/OMWU remains relatively high, whereas the exploitability of M2WU (fixed) almost converges to a relatively small value. Also, we find that M2WU (adaptive) achieves a strategy profile with smaller exploitability than other methods. We provide additional experimental results with decreasing learning rates in Appendix E.

Next, we compare the trajectories of the last-iterate strategies \( \pi_t \) of each algorithm. In the instance for this experiment, we set the initial strategy to \( \pi_t^0 = (1/|A_i|)_{i \in A}, \) for \( i \in \{1, 2\} \). Figure 5 shows the trajectories of \( \pi_t \) updated by each algorithm from an instance of BRPS. Surprisingly, the trajectory for M2WU (fixed) almost converges to a stationary point, while OMWU’s trajectory cycles and fails to converge to an equilibrium. Moreover, we find that M2WU (adaptive) converges to an exact Nash equilibrium.

8 Conclusion

In this paper, we proposed M2WU, an algorithm that utilizes a simple idea of stabilizing learning dynamics through mutation with a reference strategy. We proved that in both full- and noisy-information feedback settings, the last-iterate strategy converges to the stationary point of the replicator mutator dynamics. In particular, we showed that such convergence occurs exponentially fast with a constant learning rate in the full-information feedback setting. Furthermore, last-iterate convergence to an exact Nash equilibrium was also proven by iteratively reusing the converged stationary point as a subsequent reference strategy. The
numerical experiments showed that, even with the presence of noise, the last-iterate strategy by M2WU converges in a stable and faster manner. Future research could examine the convergence rate with noisy-information feedback and generalize the analyses to extensive-form games.

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A Proofs for Theorem 5.1

A.1 Proof of Theorem 5.1

Proof of Theorem 5.1 Let us define the following notation:

\[ \Omega^{\mu,r} = \left\{ \pi \in \prod_{i=1}^{2} \Delta(A_i) \mid \text{KL}(\pi^{\mu,r}, \pi) \leq \text{KL}(\pi^{\mu,r}, \pi^0) \right\}, \]

\[ \rho = \min_{\pi \in \Omega^{\mu,r}} \min_{i \in \{1,2\}, a \in A_i} \pi_i(a_i) > 0, \]

\[ \zeta = \frac{1}{2u_{\text{max}} + \frac{\rho}{\rho} \max_{i \in \{1,2\}, a \in A_i} r_i(a)} > 0, \]

\[ \alpha = \mu \left( \min_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^{\mu,r}(a)} \right) > 0, \]

\[ \beta = 16 \left( \frac{\mu^2}{\rho^2} \left( \max_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^{\mu,r}(a)} \right)^2 + u_{\text{max}}^2 \right) > 0. \]

We consider the case of using a constant learning rate \( \eta_t = \eta \in (0, \min(\frac{\rho}{\rho}, \zeta)). \)

We prove the statement by mathematical induction. Clearly, for \( t = 0, \) we have \( \text{KL}(\pi^{\mu,r}, \pi^t) \leq \text{KL}(\pi^{\mu,r}, \pi^0) \) and \( \pi^0 \in \Omega^{\mu,r}. \) Let us assume that \( \pi^t \in \Omega^{\mu,r}, \) i.e., \( \text{KL}(\pi^{\mu,r}, \pi^t) \leq \text{KL}(\pi^{\mu,r}, \pi^0). \) Under the assumption that \( \text{KL}(\pi^{\mu,r}, \pi^t) \leq \text{KL}(\pi^{\mu,r}, \pi^0), \) we have \( \pi_i^t(a) \geq \rho \) for all \( i \in \{1,2\} \) and \( a \in A_i. \)

We first derive the difference equation for \( \text{KL}(\pi^{\mu,r}, \pi^t): \)

**Lemma A.1.** Let \( \pi^{\mu,r} \in \prod_{i=1}^{2} \Delta(A_i) \) be a stationary point of \( \text{RMD} \). Then, \( \pi^t \) updated by M2WU satisfies that:

\[ \text{KL}(\pi^{\mu,r}, \pi^{t+1}) - \text{KL}(\pi^{\mu,r}, \pi^t) = \eta_t \sum_{i=1}^{2} \left( v_i^{t^+, \mu,r} + \mu - \mu \sum_{a_i \in A_i} r_i(a_i) \pi_i^{\mu,r}(a_i) - \pi_i^t(a_i) \right) \text{KL}(\pi^t, \pi^{t+1}). \]

Moreover, under the assumption that \( \eta \in (0, \zeta), \) the statement of Lemma 5.4 holds. By combining Lemmas 5.3 and 5.4 and this lemma, we get:

\[ \text{KL}(\pi^{\mu,r}, \pi^{t+1}) - \text{KL}(\pi^{\mu,r}, \pi^t) \leq -\eta \mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \sqrt{\pi_i^t(a)} - \sqrt{\pi_i^{\mu,r}(a)} \right)^2 \]

\[ + 8 \eta^2 \mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a)^2 \left( \frac{1}{\pi_i^{t^+}(a)} - \frac{1}{\pi_i^t(a)} \right)^2 \]

\[ + 8 \eta^2 u_{\text{max}}^2 \sum_{i=1}^{2} \left\| \pi_i^t - \pi_i^{\mu,r} \right\|^2_{1}. \quad (3) \]

We prove the lower bounded on \( \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \sqrt{\pi_i^t(a)} - \sqrt{\pi_i^{\mu,r}(a)} \right)^2 \) as follows:

\[ \sum_{i=1}^{2} \sum_{a \in A_i} r_i(a) \left( \sqrt{\pi_i^t(a)} - \sqrt{\pi_i^{\mu,r}(a)} \right)^2 = \sum_{i=1}^{2} \sum_{a \in A_i} \frac{r_i(a) (\pi_i^t(a) - \pi_i^{\mu,r}(a))^2}{\pi_i^{\mu,r}(a)} \]

\[ \geq \left( \min_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^{\mu,r}(a)} \right) \sum_{i=1}^{2} \sum_{a \in A_i} (\pi_i^t(a) - \pi_i^{\mu,r}(a))^2 \]

\[ \geq \left( \min_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^{\mu,r}(a)} \right) \sum_{i=1}^{2} \ln \left( 1 + \sum_{a \in A_i} (\pi_i^t(a) - \pi_i^{\mu,r}(a))^2 \right) \]
where the second inequality follows from $x \geq \ln(1+x)$ for all $x > 0$, and the third inequality follows from the concavity of the $\ln(\cdot)$ function and Jensen’s inequality for concave functions. Next, $\sum_{i=1}^{2} \sum_{a \in A_i} \eta_i(a)^2 \left( \frac{1}{\pi_i^t(a)} - \frac{1}{\pi_i^r(a)} \right)^2$ is upper bounded as follows:

$$
\sum_{i=1}^{2} \sum_{a \in A_i} r_i(a)^2 \left( \frac{1}{\pi_i^r(a)} - \frac{1}{\pi_i^t(a)} \right)^2 = 2 \sum_{i=1}^{2} \sum_{a \in A_i} \left( \frac{r_i(a)}{\pi_i^r(a) \pi_i^t(a)} \right)^2 \left( \pi_i^t(a) \pi_i^r(a) \right) \leq \frac{1}{\beta^2} \left( \max_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^r(a)} \right)^2 \sum_{i=1}^{2} \left\| \pi_i^t - \pi_i^r \right\|_2^2
$$

where the last inequality follows from Pinsker’s inequality [Tsybakov 2009]. Similarly, $\sum_{i=1}^{2} \left\| \pi_i^t - \pi_i^r \right\|_1^2$ is upper bounded as

$$
\sum_{i=1}^{2} \left\| \pi_i^t - \pi_i^r \right\|_1^2 \leq 2 \text{KL}(\pi_i^t, \pi_i^t). \tag{6}
$$

By combining (3), (4), (5), and (6), we have:

$$
\text{KL}(\pi_i^t, \pi_i^{t+1}) - \text{KL}(\pi_i^r, \pi_i^t)
$$

$$
\leq -\eta \mu \left( \min_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^r(a)} \right) \text{KL}(\pi_i^t, \pi_i^t) + 16\eta^2 \left( \frac{\mu^2}{\beta^2} \left( \max_{i \in \{1,2\}, a \in A_i} \frac{r_i(a)}{\pi_i^r(a)} \right)^2 + u_{\text{max}}^2 \right) \text{KL}(\pi_i^t, \pi_i^t)
$$

$$
= (-\eta \alpha + \eta^2 \beta) \text{KL}(\pi_i^t, \pi_i^t).
$$

Thus, we have:

$$
\text{KL}(\pi_i^t, \pi_i^{t+1}) \leq (1 - (\eta \alpha - \eta^2 \beta)) \text{KL}(\pi_i^t, \pi_i^t),
$$

and then, for $\eta \in (0, \frac{\alpha}{\beta})$:

$$
\text{KL}(\pi_i^t, \pi_i^{t+1}) - \text{KL}(\pi_i^r, \pi_i^t) \leq 0.
$$

Thus, if $\eta \in (0, \frac{\alpha}{\beta})$, then $\text{KL}(\pi_i^t, \pi_i^{t+1}) \leq \text{KL}(\pi_i^t, \pi_i^0) \leq \text{KL}(\pi_i^t, \pi_i^0)$ and $\pi_i^{t+1} \in \Omega_i^r$ also hold. By mathematical induction, if $\eta \in (0, \frac{\alpha}{\beta})$, for all $t \geq 0$:

$$
\text{KL}(\pi_i^t, \pi_i^{t+1}) \leq (1 - (\eta \alpha - \eta^2 \beta)) \text{KL}(\pi_i^t, \pi_i^t) \leq \cdots \leq (1 - (\eta \alpha - \eta^2 \beta))^{t+1} \text{KL}(\pi_i^t, \pi_i^0).
$$

□
A.2 Proofs of Lemma 5.3

**Proof of Lemma 5.3.** First, we introduce the following lemma from [Abe et al. 2022]:

**Lemma A.2** (Lemma 5.6 of [Abe et al. 2022]). Let \( \pi_{i}^{\mu,r} \in \Delta(A) \) be a stationary point of \( \text{RMD} \) for \( i \in \{1,2\} \). Then, for any \( \pi_{i}^{t} \in \Delta(A_{i}) \):

\[
v_{i}^{t}(\pi_{i}^{t-1}) = v_{i}^{t}(\pi_{i}^{t}) + \mu \sum_{a_{i} \in A_{i}} r_{i}(a_{i}) \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})}.
\]

From this lemma, we have:

\[
\sum_{i=1}^{2} v_{i}^{t}(\pi_{i}^{t-1}) + 2\mu - \mu \sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} r_{i}(a_{i}) \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} = 2 \sum_{i=1}^{2} v_{i}^{t}(\pi_{i}^{t}) + 4\mu - \mu \sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} r_{i}(a_{i}) \left( \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} + \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} \right)
\]

\[
= 4\mu - \mu \sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} r_{i}(a_{i}) \left( \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} + \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} \right)
\]

\[
= -\mu \sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} r_{i}(a_{i}) \left( \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} - \frac{\pi_{i}^{t}(a_{i})}{\pi_{i}^{t-1}(a_{i})} \right)^{2},
\]

where the second equality follows from \( \sum_{i=1}^{2} v_{i}^{t}(\pi_{i}^{t-1}) = 0 \) by the definition of zero-sum games. \( \square \)

A.3 Proofs of Lemma 5.4

**Proof of Lemma 5.4.** Let us assume that \( \eta \in (0, \min(\frac{\alpha}{3}, \zeta)) \), where \( \alpha \), \( \beta \), and \( \zeta \) are defined in Appendix A.1.

First, we have:

\[
\text{KL}(\pi^{t}, \pi^{t+1}) = \sum_{i=1}^{2} \sum_{a \in A_{i}} \pi^{t}(a) \ln \frac{\pi^{t}(a)}{\pi^{t+1}(a)}
\]

\[
= 2 \sum_{i=1}^{2} \sum_{a \in A_{i}} \frac{1}{2} \pi^{t}(a) \ln \frac{\pi^{t}(a)}{\pi^{t+1}(a)} \leq 2 \ln \left( \frac{1}{2} \sum_{i=1}^{2} \sum_{a \in A_{i}} \pi^{t}(a) \frac{\pi^{t}(a)}{\pi^{t+1}(a)} \right),
\]

where the inequality follows from the concavity of the \( \ln(\cdot) \) function and Jensen’s inequality for concave functions. Here, from the update rule \([1]\):

\[
\frac{\pi^{t+1}(a)}{\pi^{t}(a)} = \sum_{a' \in A_{i}} \pi^{t}(a') \exp \left( \eta \left( q_{i}^{t}(a') + \mu r_{i}(a') \right) \right) \exp \left( -\eta \left( q_{i}^{t}(a) + \mu r_{i}(a) \right) \right),
\]

and then we get:

\[
\text{KL}(\pi^{t}, \pi^{t+1}) \leq 2 \ln \left( \frac{1}{2} \sum_{i=1}^{2} \sum_{a \in A_{i}} \pi^{t}(a) \sum_{a' \in A_{i}} \pi^{t}(a') \exp \left( \eta \left( q_{i}^{t}(a') + \mu r_{i}(a') \right) \right) \right) \exp \left( -\eta \left( q_{i}^{t}(a) + \mu r_{i}(a) \right) \right)
\]

\[
= 2 \ln \left( \frac{1}{2} \sum_{i=1}^{2} \sum_{a \in A_{i}} \sum_{a' \in A_{i}} \pi^{t}(a) \pi^{t}(a') \exp \left( \eta \left( q_{i}^{t}(a') + \mu r_{i}(a') - q_{i}^{t}(a) - \mu r_{i}(a) \right) \right) \right).
\]

Here, from the assumption for the learning rate \( \eta < \zeta = \frac{1}{2 \max + \frac{1}{2} \max_{i \in \{1,2\}, a \in A_{i}} r_{i}(a)} \), we have \( q_{i}^{t}(a') + \mu r_{i}(a') - q_{i}^{t}(a) - \mu r_{i}(a) \leq 0 \)
1. Thus, we can use the fact that \( \exp(x) \leq 1 + x + x^2 \) for \( x \leq 1 \), and then:

\[
\begin{align*}
\frac{1}{2} & \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \exp \left( \eta \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right) \right) \\
& \leq \frac{1}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( 1 + \eta \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right) \right) \\
& + \frac{1}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \eta^2 \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \\
& = 1 + \eta^2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \\
& \leq \exp \left( \eta^2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \right), \quad (8)
\end{align*}
\]

where the first equality follows from \( \sum_{a \in A_i} \sum_{a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( \eta \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right) \right) = 0 \), and the last inequality follows from \( 1 + x \leq \exp(x) \) for \( x \in \mathbb{R} \). By combining (7) and (8), we get:

\[
\begin{align*}
\text{KL}(\pi^t, \pi^{t+1}) & \leq 2 \ln \left( \exp \left( \eta^2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \right) \right) \\
& = \eta^2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2. \quad (9)
\end{align*}
\]

Here, by using the ordinary differential equation (RMD), we have for all \( i \in \{1, 2\} \) and \( a \in A \):

\[
q_i^\pi(a) = v_i^\mu(a) - \mu \frac{\pi_i^t(a)}{\pi_i^t(a)} \left( r_i(a) - \pi_i^t(a) \right).
\]

Thus,

\[
q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} = q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a') - \mu \frac{r_i(a')}{\pi_i^t(a')} + q_i^\pi(a') - q_i^\pi(a') + q_i^\pi(a') - q_i^\pi(a) + q_i^\pi(a).
\]

Then,

\[
\begin{align*}
\sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') + \mu \frac{r_i(a')}{\pi_i^t(a')} - q_i^\pi(a) - \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \\
& \leq 2 \mu^2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( \mu \frac{r_i(a)}{\pi_i^t(a)} - \mu \frac{r_i(a)}{\pi_i^t(a)} - \mu \frac{r_i(a)}{\pi_i^t(a)} + \mu \frac{r_i(a)}{\pi_i^t(a)} \right)^2 \\
& + 2 \frac{2}{2} \sum_{i=1}^{2} \sum_{a \in A_i, a' \in A_i} \pi_i^t(a) \pi_i^t(a') \left( q_i^\pi(a') - q_i^\pi(a') - q_i^\pi(a') + q_i^\pi(a') \right)^2
\end{align*}
\]
Lemma A.3.

Proof of Lemma A.1.

\[ \begin{align*}
\leq & 4\mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} \sum_{a' \in A_i} \pi_i'(a) \pi_i'(a') \left( \frac{r_i(a)}{\pi_i'(a)} - \frac{r_i(a)}{\pi_i'(a')} + \left( \frac{r_i(a')}{\pi_i'(a)} - \frac{r_i(a)}{\pi_i'(a')} \right)^2 \right) \\
+ & 4 \sum_{i=1}^{2} \sum_{a \in A_i} \sum_{a' \in A_i} \pi_i'(a) \pi_i'(a') \left( \frac{q_i^r(a') - q_i^r(a)}{\pi_i'(a')} + \frac{q_i^r(a) - q_i^r(a)}{\pi_i'(a)} \left( \sum_{b \in A_{i-1}} (\pi_{i-1}^t(b) - \pi_{i-1}^t(b)) u_i(a', b) \right) \right)^2 \\
= & 8\mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) r_i(a)^2 \left( \frac{1}{\pi_i'(a)} - \frac{1}{\pi_i'(a)} \right)^2 + 8 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) \left( \sum_{b \in A_{i-1}} (\pi_{i-1}^t(b) - \pi_{i-1}^t(b)) u_i(a', b) \right)^2 \\
\leq & 8\mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) r_i(a)^2 \left( \frac{1}{\pi_i'(a)} - \frac{1}{\pi_i'(a)} \right)^2 + 8 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) u_{\pi_{i-1}^t} \pi_i^{t-r} \pi_i^t \| \pi_i^t - \pi_{i-1}^{t-r} \|_1^2 \\
= & 8\mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) r_i(a)^2 \left( \frac{1}{\pi_i'(a)} - \frac{1}{\pi_i'(a)} \right)^2 + 8 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i'(a) u_{\pi_{i-1}^t} \pi_i^{t-r} \pi_i^t \| \pi_i^t - \pi_{i-1}^{t-r} \|_1^2, \\
\end{align*} \]

where the first and second inequalities follow from \((a + b)^2 \leq 2(a^2 + b^2)\) for \(a, b \in \mathbb{R}\), and the third inequality follows from Hölder’s inequality.

By combining (9) and (10), if \(\eta \in (0, \min(\frac{\eta}{2}, \zeta))\), for all \(t \geq 0\):

\[
\text{KL}(\pi^t, \pi^{t+1}) \leq 8\eta^2 \left( \mu^2 \sum_{i=1}^{2} \sum_{a \in A_i} \pi_i^t \pi_i^t \left( \frac{1}{\pi_i^t(a)} - \frac{1}{\pi_i^t(a)} \right)^2 + u_{\pi_{\pi_{i-1}}}^{t} \pi_{\pi_{i-1}}^{t-r} \pi_{\pi_{i-1}}^t \| \pi_{\pi_{i-1}}^t - \pi_{\pi_{i-1}}^{t-r} \|_1^2 \right).
\]

\[
\square
\]

A.4 Proof of Lemma A.3

Proof of Lemma A.3. We introduce the following lemma:

Lemma A.3. For any \(\pi \in \prod_{i=1}^{l} \Delta(A_i)\), \(\pi^t\) updated by M2WU satisfies that:

\[
\text{KL}(\pi^t, \pi^{t+1}) = \sum_{i=1}^{2} \left( \sum_{s=1}^{t-1} \eta q_i^{t-s, \pi_i^t} - \psi_i(\pi_i^t) - \sum_{s=1}^{t-1} \eta q_i^{t-s, \pi_i^t} + \psi_i(\pi_i^t) \right),
\]

where \(\psi_i(p) = \sum_{a \in A_i} p(a) \ln(p(a))\).

From Lemma A.3 we have:

\[
\begin{align*}
\text{KL}(\pi^{t-r}, \pi^{t+1}) - \text{KL}(\pi^{t-r}, \pi^t) \\
= & \sum_{i=1}^{2} \left( \sum_{s=1}^{t} \eta q_i^{t, \pi_i^s} - \psi_i(\pi_i^t) - \sum_{s=1}^{t} \eta q_i^{t, \pi_i^s} + \psi_i(\pi_i^{t-r}) \right) \\
- & \sum_{i=1}^{2} \left( \sum_{s=1}^{t-1} \eta q_i^{t-s, \pi_i^t} - \psi_i(\pi_i^t) - \sum_{s=1}^{t-1} \eta q_i^{t-s, \pi_i^t} + \psi_i(\pi_i^{t-r}) \right) \\
= & \sum_{i=1}^{2} \left( \sum_{s=1}^{t} \eta q_i^{t-s, \pi_i^{t+1}} - \psi_i(\pi_i^{t+1}) - \sum_{s=1}^{t} \eta q_i^{t-s, \pi_i^t} + \psi_i(\pi_i^t) \right) - \eta \sum_{i=1}^{2} (q_i^{t+1} \pi_i^{t-r} - q_i^t)
\end{align*}
\]

\[
= \text{KL}(\pi^t, \pi^{t+1}) - \eta \sum_{i=1}^{2} (q_i^{t+1} \pi_i^{t-r} - q_i^t)
\]

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\begin{align*}
\text{KL}(\pi^t, \pi^{t+1}) &+ \eta_t \sum_{i=1}^{2} \sum_{a \in A_i} \left( q_i^{\pi^t}(a) + \frac{\mu}{\pi_i^t(a)} (r_i(a) - \pi_i^t(a)) \right) (\pi_i^t(a) - \pi_i^{t+1}(a)) \\
= \text{KL}(\pi^t, \pi^{t+1}) &+ \eta_t \sum_{i=1}^{2} \sum_{a \in A_i} (\pi_i^t(a) - \pi_i^{t+1}(a)) \left( q_i^{\pi^t}(a) + \frac{\mu}{\pi_i^t(a)} r_i(a) \right) \\
= \text{KL}(\pi^t, \pi^{t+1}) &+ \eta_t \sum_{i=1}^{2} \sum_{a \in A_i} \left( v_i^{\pi^t} - v_i^{\pi^{t+1}} + \mu \sum_{a \in A_i} r_i(a) \frac{\pi_i^{t+1}(a)}{\pi_i^t(a)} - v_i^{\pi^t} \right) \\
= \text{KL}(\pi^t, \pi^{t+1}) &+ \eta_t \sum_{i=1}^{2} \left( -v_i^{\pi^t} - v_i^{\pi^{t+1}} + \mu \sum_{a \in A_i} r_i(a) \frac{\pi_i^{t+1}(a)}{\pi_i^t(a)} \right) \\
= \text{KL}(\pi^t, \pi^{t+1}) &+ \eta_t \sum_{i=1}^{2} \left( v_i^{\pi^t} + \mu \sum_{a \in A_i} r_i(a) \frac{\pi_i^{t+1}(a)}{\pi_i^t(a)} \right),
\end{align*}

where the seventh equality follows from \( \sum_{i=1}^{2} v_i^{\pi^t} = 0 \) and \( \mu \sum_{a \in A} \pi_i^t(a) r_i(a) = \mu \sum_{a \in A} c_i(a) = \mu \), and the last equality follows from \( -v_1^{\pi^t} - v_1^{\pi^{t+1}} = v_2^{\pi^t} - v_2^{\pi^{t+1}} \) by the definition of two-player zero-sum games.

\textbf{A.5 Proof of Lemma A.3}

\textit{Proof of Lemma A.3} From the definition of the Kullback-Leibler divergence, we have:

\begin{align}
\text{KL}(\pi, \pi_i^t) &= \sum_{i=1}^{2} \text{KL}(\pi_i, \pi_i^t) \\
&= \sum_{i=1}^{2} \left( \sum_{a \in A_i} (\pi_i^t(a) - \pi_i(a)) \ln \pi_i^t(a) - \sum_{a \in A_i} \pi_i(a) \ln \pi_i(a) + \sum_{a \in A_i} \pi_i(a) \ln \pi_i(a) \right) \\
&= \sum_{i=1}^{2} \left( \sum_{a \in A_i} (\pi_i^t(a) - \pi_i(a)) \ln \pi_i^t(a) - \psi_i(\pi_i^t) + \psi_i(\pi_i) \right). \tag{11}
\end{align}

Here, the update rule \[ \pi_i^t(a) = \frac{\exp \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}(a) \right) \pi_i(a)}{\sum_{a' \in A_i} \exp \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}(a') \right)} \]

and then we have:

\begin{align}
\sum_{a \in A_i} (\pi_i^t(a) - \pi_i(a)) \ln \pi_i^t(a) &= \sum_{a \in A_i} (\pi_i^t(a) - \pi_i(a)) \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}(a) - \ln \left( \sum_{a' \in A_i} \exp \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}(a') \right) \right) \right) \\
&= \sum_{a \in A_i} (\pi_i^t(a) - \pi_i(a)) \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}(a) \\
&= \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}, \pi_i^t \right) - \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}, \pi_i \right). \tag{12}
\end{align}

By combining (11) and (12), we get:

\begin{align}
\text{KL}(\pi, \pi_i^t) &= \sum_{i=1}^{2} \left( \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}, \pi_i^t \right) - \psi_i(\pi_i^t) - \left( \sum_{s=1}^{t-1} \eta_s q_i^{\mu,s}, \pi_i \right) + \psi_i(\pi_i) \right). \tag{13}
\end{align}
B Proof of Corollary 5.2

Proof of Corollary 5.2. From the definition of exploitability, we have:

\[
\text{exploit}(\pi^t) = \sum_{i=1}^{2} \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} \\
= \sum_{i=1}^{2} \left( \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} + \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} - \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} \right) \\
= \text{exploit}(\pi^\mu, \pi^r) + \sum_{i=1}^{2} \left( \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} - \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^t_{\pi}} \right) \\
\leq \text{exploit}(\pi^\mu, \pi^r) + \sum_{i=1}^{2} \left( \max_{\tilde{\pi} \in \Delta(A_i)} (\pi^t_{\pi} - \pi^t_{\tilde{\pi}}) \right) \\
\leq \text{exploit}(\pi^\mu, \pi^r) + \sum_{i=1}^{2} \left( \max_{\tilde{\pi} \in \Delta(A_i)} (\pi^t_{\pi} - \pi^t_{\tilde{\pi}}) \right) \\
\leq \text{exploit}(\pi^\mu, \pi^r) + \sum_{i=1}^{2} \left( u_{\pi^t} \sqrt{2 \text{KL}(\pi^\mu, \pi^r)} \right) \\
\leq \text{exploit}(\pi^\mu, \pi^r) + u_{\pi^t} \sqrt{2 \text{KL}(\pi^\mu, \pi^r)} \\
= \text{exploit}(\pi^\mu, \pi^r) + 2u_{\pi^t} \sqrt{\text{KL}(\pi^\mu, \pi^t)}.
\]

where the second inequality follows from Hölder’s inequality, the third inequality follows from Pinsker’s inequality [Tsybakov 2009], and the fourth inequality follows from $\sqrt{a + b} \leq \sqrt{2(a + b)}$ for $a, b > 0$. By combining (13) and Theorem 5.1, we have:

\[
\text{exploit}(\pi^t) \leq \text{exploit}(\pi^\mu, \pi^r) + 2u_{\pi^t} \sqrt{\text{KL}(\pi^\mu, \pi^t)}(1 - C_2)^{1/2}.
\]

Moreover, from Lemma 3.5 of Bauer et al. [2019], a stationary point $\pi^\mu, \pi^r$ of (RMD) satisfies that for all $i \in \{1, 2\}$ and $a_i \in A_i$, $q_i^{\pi^r}(a_i) - v_i^{\pi^r} \leq \mu$. Therefore, the term of $\text{exploit}(\pi^\mu, \pi^r)$ can be bounded as:

\[
\text{exploit}(\pi^\mu, \pi^r) = \sum_{i=1}^{2} \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^\mu, \pi^r} \\
= \sum_{i=1}^{2} \left( \max_{\tilde{\pi} \in \Delta(A_i)} v_i^{\tilde{\pi}, \pi^\mu, \pi^r} - v_i^{\pi^t} \right) \\
= \sum_{i=1}^{2} \left( \max_{a_i \in A_i} q_i^{\pi^r}(a_i) - v_i^{\pi^r} \right) \leq 2\mu.
\]

where the first equality follows from $\sum_{i=1}^{2} v_i^{\pi^r} = 0$ by the definition of zero-sum games. By combining (14) and (15), we have:

\[
\text{exploit}(\pi^t) \leq 2\mu + 2u_{\pi^t} \sqrt{\text{KL}(\pi^\mu, \pi^t)}.
\]

This concludes the statement.

\[\square\]

C Proofs for Theorem 5.6

C.1 Proof of Theorem 5.6

In preparation for the proof, we first define the notion of approximate Robbins-Monro algorithms.
Definition C.1. The stochastic approximation algorithm
\[ z(t + 1) = z(t) + \eta_t(F(z(t)) + U_t + \beta_t) \]
is refer to as an approximate Robbins-Monro algorithm if the following conditions are satisfied.
- \( F : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a continuous function
- \( (U_t)_{t \geq 1} \) s.t. \( U_n \in \mathbb{R}^m, \forall n \in \mathbb{N} \) is a martingale difference noise
- \( (\eta_t)_{t \geq 1} \) is a given sequence of numbers such that \( \sum_{t=1}^{\infty} \eta_t = \infty \) and \( \lim_{t \to \infty} \eta_t = 0 \)
- \( \lim_{t \to \infty} \beta_t = 0 \) almost surely

We provide the definition of the asymptotic pseudo-trajectory.

Definition C.2 (Benaim and Hirsch [1996]). A flow \( \phi \) on a metric space \((M, d)\) is a continuous mapping
\[ \phi : \mathbb{R} \times M \rightarrow M, \quad (t, x) \mapsto \phi_t(x) \]
such that \( \phi_0(x) = x \) and \( \phi_{t + \alpha} = \phi_t \circ \phi_\alpha \) for all \( t, \alpha \in \mathbb{R} \). For a metric space \((M, d)\), a continuous function \( X : \mathbb{R} \rightarrow M \) is an asymptotic pseudo trajectory for \( \phi \) if
\[ \lim_{t \to \infty} \sup_{s \in [0, T]} d(X(t + s), \phi_s(X(t))) = 0, \]
for every \( T > 0 \).

For each \( i \in \{1, 2\} \), let we define the logit function \( g_i : \mathbb{R}^{|A_i|} \rightarrow \Delta(A_i) \) as
\[ g_i(z_i) = \begin{pmatrix} \exp(z_i(a_i)) \\ \sum_{a_i' \in A_i} \exp(z_i(a_i')) \end{pmatrix}, \quad a_i \in A_i. \]
We write \( g_i(z_i)(a_i) \in \mathbb{R} \) be the \( a_i \)-th element of \( g_i(z_i) \) and \( \nabla g_i(z_i)(a_i) \in \mathbb{R}^{|A_i|} \) be the gradient vector of \( g_i(z_i)(a_i) \), respectively. As a first result, we prove that the dynamics of the strategy \( \{\pi^t_i\} \) updated by M2WU is an asymptotic pseudo trajectory of a continuous dynamics.

Lemma C.3. Suppose that the sequence \( \{\eta_t\}_{t \geq 1} \) satisfy \( \eta_t \propto t^{-\beta} \) for some \( \beta \in (1/\alpha, 1] \), where \( \alpha \) is a constant defined in Assumption 5.5 (ii). Then, for each \( i \in \{1, 2\} \), the sequence of strategies \( \{\pi^t_i\}_{t \geq 1} \) updated by M2WU is an asymptotic pseudo trajectory for the replicator mutator dynamics:
\[ \frac{d}{dt} \pi^t_i(a_i) = \pi^t_i(a_i) \cdot \left( \pi^t_i^{\mu_r}(a_i) - \pi^t_i^{\mu_i}(a_i) \right) + \mu \left( r_i(a_i) - \pi^t_i(a_i) \right). \]  \( \text{(RMD)} \)

We present the proof of Lemma C.3 in Appendix C.2. Furthermore, we have the following exponential convergence result to the stationary point \( \pi^{\mu_r}_i \) in the noiseless continuous time setting in [Abe et al. 2022].

Theorem C.4 (Theorem 5.2. of [Abe et al. 2022]). Let \( \pi^{\mu_r}_i \in \Delta(A_i) \) be a stationary point of \( \text{(RMD)} \) for all \( i \in \{1, 2\} \). Then, for all \( \mu > 0 \), the continuous-time dynamics \( \pi^t \) updated by \( \text{(RMD)} \) satisfies the following:
\[ \frac{d}{dt} \text{KL}(\pi^{\mu_r}, \pi^t) = -\mu \sum_{i=1}^{2} \sum_{a_i \in A_i} r_i(a_i) \left( \sqrt{\frac{\pi^t_i(a_i)}{\pi^{\mu_r}_i(a_i)}} - \sqrt{\frac{\pi^{\mu_r}_i(a_i)}{\pi^t_i(a_i)}} \right)^2. \]
Furthermore, \( \pi^t \) satisfies that:
\[ \frac{d}{dt} \text{KL}(\pi^{\mu_r}, \pi^t) \leq -\mu \xi \text{KL}(\pi^{\mu_r}, \pi^t), \]
where \( \xi = \min_{i \in \{1, 2\}, a_i \in A_i} \frac{c_i(a_i)}{\pi^{\mu_r}_i(a_i)}. \)

It can be observed that the function \( \text{KL}(\pi^{\mu_r} \cdot) \) is a strict Lyapunov function. Furthermore, the stationary point of \( \text{(RMD)} \) is unique. Therefore, \( \pi^t_i \) updated by M2WU converges to \( \pi^{\mu_r}_i \), almost surely.
C.2 Proof of Lemma C.3

For each $i \in \{1, 2\}$, for any $a_i', \tilde{a}_i \in A_i$,

$$\frac{\partial}{\partial z_i(a'_i)} (g_i(z_i)(a_i)) = g_i(z_i)(a_i) (\mathbb{I}_{\{a_i = a'_i\}} - g_i(z_i)(a_i'))$$

and

$$\frac{\partial^2}{\partial z_i(a_i) \partial z_i(a'_i)} (g_i(z_i)(a_i)) = g_i(z_i)(a_i) (\mathbb{I}_{\{a_i = a'_i = \tilde{a}_i\}} - g_i(z_i)(\tilde{a}_i)) (\mathbb{I}_{\{a_i = \tilde{a}_i\}} + \mathbb{I}_{\{a'_i = \tilde{a}_i\}} - 2g_i(z_i)(\tilde{a}_i)). \quad (20)$$

We write

$$\hat{q}_{i,t}^\mu = \left( q_{i,t}^{\pi}(a_i) + \frac{\mu}{\pi_i(a_i)} (r_i(a_i) - \pi_i(a_i)) \right)_{a_i \in A_i}$$

$$\hat{q}_{i,t}^\mu = \left( \eta \left( (\nabla g_i(z_i')(a_i)) \hat{q}_{i,t}^\mu + \frac{\eta}{2} (\hat{q}_{i,t}^\mu)^\top \text{Hess}(g_i(\zeta)(a_i)) \hat{q}_{i,t}^\mu \right) + \hat{\phi}_t \right),$$

where $\zeta$ is a point between $z_i'$ and $z_i'^1$.

$$(\nabla g_i(z_i')(a_i))^\top q_{i,t}^\mu = \sum_{a'_i \in A_i} g_i(z_i')(a_i) (\mathbb{I}_{\{a_i = a'_i\}} - g_i(z_i')(a_i')) \left( q_{i,t}^{\pi}(a_i') + \frac{\mu}{\pi_i(a_i')} (r_i(a_i') - \pi_i(a_i')) \right)$$

$$= g_i(z_i')(a_i) \left( q_{i,t}^\mu(a_i) - \sum_{a'_i \in A_i} g_i(z_i')(a_i') q_{i,t}^\mu(a_i') \right)$$

We can write the dynamics of $\pi_i(t)$ as follows

$$\pi_i^{t+1}(a_i) = \pi_i^t(a_i) + \eta_t(F(\pi_i^t) + U_t + \hat{\phi}_t),$$

where $F(\pi_i^t) = \pi_i^t(a_i) \left( q_{i,t}^\mu(a_i) - \sum_{a'_i \in A_i} \pi_i^t(a_i') q_{i,t}^\mu(a_i') \right)$ is a continuous function and $U_t = (\nabla g_i(z_i')(a_i))^\top (q_{i,t}^\mu - q_{i,t}^\mu)$. Let $\mathcal{E}_t$ be an event such that $\|q_{i,t}^\mu\|_2^2 \geq t^p$ with $p < 1/\alpha$, where $\alpha$ is Assumption 5.5 (ii) and $p$ is a value that satisfies $\eta_t = o(t^{-p})$ (note that $\eta_t \propto t^{-p}$ and $1/\alpha < \beta \leq 1$). Using Assumption 5.5 and the boundedness of the utility function,

$$\sum_{i=1}^\infty \mathbb{P}(\mathcal{E}_t) = \sum_{i=1}^\infty \mathbb{P}(\|q_{i,t}^\mu\|_2^2 \geq t^p \mid \mathcal{F}_{t-1}) = \sum_{i=1}^\infty \mathcal{O}(t^{-\alpha p}) < \infty.$$
From the Borel–Cantelli lemma, \( \mathbb{P}(\cap_{t=1}^{\infty} \cup_{s \geq t} \mathcal{E}_s) = 0 \). Therefore, the event \( \mathcal{E}_t \) occurs only for a finite number of \( t \), almost surely. Thus, as \( \eta t^p \propto t^{-\beta+p} = o(1) \),

\[
\phi_t = \mathcal{O}(\eta \| \phi_t^{\mu,t} \|^2) = \mathcal{O}(\eta t^p) = o(1),
\]

almost surely. Therefore, from Definition C.1 the update of \( \{\pi_t^i\}_{t \geq 1} \) is an approximate Robbins-Monro algorithm. From Proposition 4.2 of [Benaïm 1999], \( \{\pi_t^i\}_{t \geq 1} \) is an asymptotic pseudo-trajectory of the replicator mutator dynamics. \( \square \)

**D Proofs for Theorem 6.1**

**D.1 Proof of Theorem 6.1**

**Proof of Theorem 6.1** From Lemma 6.2 and the fact that \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^k) \) is bounded from below by zero, the sequence of \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^k) \) converges to \( b \geq 0 \). We show that \( b = 0 \) by a contradiction argument.

Suppose \( b > 0 \). For a given \( b' > 0 \), we define \( \Omega_{b'} = \{ r \in \prod_{i=1}^2 \Delta^0(A_i) \mid \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \leq b' \} \), and \( \bar{\Omega}_{b'} = \{ r \in \prod_{i=1}^2 \Delta^0(A_i) \mid \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) < b' \} \). Since \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \) is a continuous function on \( \prod_{i=1}^2 \Delta^0(A_i) \), the preimage \( \Omega_{b'} \) of the closed set \( [0, b'] \) is also closed. Similarly, the preimage \( \bar{\Omega}_{b'} \) of the open set \( (0, b') \) is open. Furthermore, since \( \prod_{i=1}^2 \Delta^0(A_i) \) is a bounded set, \( \Omega_{b'} \) and \( \bar{\Omega}_{b'} \) are bounded sets.

Let us define \( B = \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^0) \), \( b > 0 \) means that for all \( k \geq 0 \), \( r^k \) is in \( \Omega_{b} \setminus \bar{\Omega}_{b} \). Since \( \Omega_{b} \) is a closed and bounded set and \( \bar{\Omega}_{b} \) is a closed and bounded set. Thus, the set \( \Omega_{b} \setminus \bar{\Omega}_{b} \) is a compact set.

From Lemma 6.3 \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, F(r)) - \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \) is also a continuous function. Since a continuous function has a maximum over a compact set, the maximum \( \Delta = \max_{r \in \Omega_{b} \setminus \bar{\Omega}_{b}} \{ \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, F(r)) - \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \} \) exists. From Lemma 6.2 and the assumption that \( b > 0 \), we have \( \Delta < 0 \). It follows that:

\[
\min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^k) = \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^0) + \sum_{i=0}^{k-1} \left( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^{i+1}) - \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^i) \right) \\
\leq B + \sum_{i=0}^{k-1} \Delta = B + k\Delta.
\]

This implies that \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^k) < 0 \) for \( k > \frac{B}{\Delta} \), which is a contradiction because \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \geq 0 \). Therefore, the sequence of \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r^k) \) converges to 0, and \( r^k \) converges to some strategy profile in \( \Pi^* \). \( \square \)

**D.2 Proof of Lemma 6.2**

**Proof of Lemma 6.2**. First, we prove the first statement of the lemma using following two lemmas:

**Lemma D.1.** Let \( \pi_i^{\mu,r} \) be a stationary point of (RMD) with the reference strategy \( r_i \) for \( i \in \{1, 2\} \). Assuming that \( r \neq \pi_i^{\mu,r} \), for any Nash equilibrium \( \pi^* \) of the original game, we have:

\[
\text{KL}(\pi^*, \pi_i^{\mu,r}) - \text{KL}(\pi^*, r) < 0.
\]

**Lemma D.2.** Let \( \pi_i^{\mu,r} \) be a stationary point of (RMD) with the reference strategy \( r_i \) for \( i \in \{1, 2\} \). If \( r = \pi_i^{\mu,r} \), then \( r \) is a Nash equilibrium of the original game.

From Lemma D.2, when \( r \in \prod_{i=1}^2 \Delta^0(A_i) \setminus \Pi^* \), \( r \neq \pi_i^{\mu,r} \) always holds. Let us define \( \tilde{\pi} = \arg \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \).

From Lemma D.1 if \( r \neq \pi_i^{\mu,r} \) we have:

\[
\min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) = \text{KL}(\tilde{\pi}, r) > \text{KL}(\tilde{\pi}, \pi_i^{\mu,r}) \geq \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, \pi_i^{\mu,r}).
\]

Therefore, if \( r \in \prod_{i=1}^2 \Delta^0(A_i) \setminus \Pi^* \) then \( \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, \pi_i^{\mu,r}) < \min_{\pi^* \in \Pi^*} \text{KL}(\pi^*, r) \).
Next, we prove the second statement of the lemma. Assume that $r \in \Pi^*$ implies that $\pi^\mu,r \neq r$. In this case, we can apply Lemma 5.1, so we have for all $\pi^* \in \Pi^*$, $\KL(\pi^*, \pi^r) < \KL(\pi^*, r)$. On the other hand, since $r \in \Pi^*$, there exists a Nash equilibrium $\tilde{\pi}^*$ such that $\KL(\tilde{\pi}^*, r) = 0$. Thus, we have $\KL(\tilde{\pi}^*, r^r) < \KL(\tilde{\pi}^*, r) = 0$, which contradicts that $\KL(\tilde{\pi}^*, r^r) \geq 0$. Therefore, if $r \in \Pi^*$ then $\pi^r = r$. □

D.3 Proof of Lemma 6.3

Proof of Lemma 6.3: For a given $r \in \prod_{i=1}^{2} \Delta^\circ(A_i)$, let us consider that $\pi^t$ follows the following (RMD) dynamics with a reference strategy profile $r \in \prod_{i=1}^{2} \Delta^\circ(A_i)$:

$$\frac{d}{dt} \pi^t_i(a_i) = \pi^t_i(a_i) \left( q^t_i(a_i) - v^t_i \right) + \mu \left( r_i(a_i) - \pi^t_i(a_i) \right).$$

Therefore, we have for a given $r^r \in \prod_{i=1}^{2} \Delta^\circ(A_i)$ and the associated stationary point $\pi^\mu,r^r$:

$$\frac{d}{dt} \KL(\pi^\mu,r^r, \pi^t)$$

$$= \sum_{i=1}^{2} \sum_{a \in A_i} \pi^\mu,r^r_i(a) \frac{d}{dt} \ln \left( \frac{\pi^\mu,r^r_i(a)}{\pi^t_i(a)} \right)$$

$$= - \sum_{i=1}^{2} \sum_{a \in A_i} \pi^\mu,r^r_i(a) \frac{d}{dt} \ln \pi^t_i(a)$$

$$= - \sum_{i=1}^{2} \sum_{a \in A_i} \pi^\mu,r^r_i(a) \frac{d}{dt} \pi^t_i(a)$$

$$= - \sum_{i=1}^{2} \sum_{a \in A_i} \pi^\mu,r^r_i(a) \left( \pi^t_i(a_i) \left( q^t_i(a_i) - v^t_i \right) + \mu \left( r_i(a_i) - \pi^t_i(a_i) \right) \right)$$

$$= \sum_{i=1}^{2} \sum_{a \in A_i} \left( \pi^t_i(a) - \pi^\mu,r^r_i(a) \right) q^t_i(a) - \frac{\mu}{\pi^t_i(a)} \left( r_i(a_i) - \pi^t_i(a_i) \right) \pi^\mu,r^r_i(a) + \frac{\mu}{\pi^t_i(a)} \left( r_i(a_i) - \pi^t_i(a_i) \right) \pi^t_i(a)$$

$$= \sum_{i=1}^{2} \sum_{a \in A_i} \left( q^t_i(a) + \frac{\mu}{\pi^t_i(a)} \left( r_i(a_i) - \pi^t_i(a_i) \right) \right) \left( \pi^t_i(a) - \pi^\mu,r^r_i(a) \right)$$

$$+ \sum_{i=1}^{2} \sum_{a \in A_i} \left( \frac{\mu}{\pi^t_i(a)} \left( r_i(a_i) - \pi^t_i(a_i) \right) - \frac{\mu}{\pi^t_i(a)} \left( r_i'(a) - \pi^t_i(a) \right) \right) \left( \pi^t_i(a) - \pi^\mu,r^r_i(a) \right)$$

where the sixth equality follows from $v^t_i = \sum_{a \in A_i} \pi^t_i(a) q^t_i(a)$ and $\sum_{a \in A_i} \frac{\mu}{\pi^t_i(a)} \left( r_i(a_i) - \pi^t_i(a_i) \right) \pi^t_i(a) = \mu \sum_{a \in A_i} \left( r_i(a_i) - \pi^t_i(a_i) \right) = 0$. The first term of (21) can be written as:

$$\sum_{i=1}^{2} \sum_{a \in A_i} \left( q^t_i(a) + \frac{\mu}{\pi^t_i(a)} \left( r_i'(a) - \pi^t_i(a) \right) \right) \left( \pi^t_i(a) - \pi^\mu,r^r_i(a) \right)$$

$$= \sum_{i=1}^{2} \sum_{a \in A_i} \left( \pi^t_i(a) - \pi^\mu,r^r_i(a) \right) \left( q^t_i(a) + \mu \left( \frac{r_i'(a)}{\pi^t_i(a)} - 1 \right) \right)$$

24
\[
\sum_{i=1}^{2} \sum_{a \in A_i} \left( \pi_i^t(a) - \pi_i^{\mu,r'}(a) \right) \left( q_i^t(a) + \mu \pi_i^t(a) \frac{r_i'(a)}{\pi_i^t(a)} \right) \\
\sum_{i=1}^{2} \left( v_i^t - v_i^{\mu,r'} \pi_i^t + \mu - \mu \sum_{a \in A_i} r_i'(a) \frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)} \right) \\
\sum_{i=1}^{2} \left( -v_i^{\mu,r'} \pi_i^t + \mu - \mu \sum_{a \in A_i} r_i'(a) \frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)} \right) \\
\sum_{i=1}^{2} \left( v_i^t - v_i^{\mu,r'} \pi_i^t + \mu - \mu \sum_{a \in A_i} r_i'(a) \frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)} \right)
\]

where the fourth equality follows from \( \sum_{i=1}^{2} v_i^t = 0 \) and \( \mu \sum_{a \in A} \pi_i^t(a) r_i'(a) = \mu \sum_{a \in A} r_i'(a) = \mu \), and the last equality follows from \( -v_1^{\mu,r'}, v_2^t = v_2^{\mu,r'}, v_2 \) and \( -v_1^{\mu,r'}, v_1^t = v_1^{\mu,r'} \) by the definition of two-player zero-sum games. Here, from Lemma [A.2] for all \( i \in \{1, 2\} \):

\[
v_i^{\mu,r'} \pi_i^t = v_i^{\mu,r'} + \mu - \mu \sum_{a \in A_i} r_i'(a) \frac{\pi_i^t(a)}{\pi_i^{\mu,r'}(a)},
\]

and then we have:

\[
\sum_{i=1}^{2} \sum_{a \in A_i} \left( \pi_i^t(a) - \pi_i^{\mu,r'}(a) \right) \left( r_i'(a) - \pi_i^t(a) \right) \left( \pi_i^t(a) - \pi_i^{\mu,r'}(a) \right) \\
= 2 \sum_{i=1}^{2} v_i^{\mu,r'} + 4 \mu - \mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i'(a) \left( \frac{\pi_i^t(a)}{\pi_i^{\mu,r'}(a)} + \frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)} \right) \\
= 4 \mu - \mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i'(a) \left( \frac{\pi_i^t(a)}{\pi_i^{\mu,r'}(a)} - \frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)} \right) \\
= -\mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i'(a) \left( \sqrt{\frac{\pi_i^t(a)}{\pi_i^{\mu,r'}(a)}} - \sqrt{\frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)}} \right)^2,
\]

where the second equality follows from \( \sum_{i=1}^{2} v_i^{\mu,r'} = 0 \) by the definition of zero-sum games.

On the other hand, the second term of (21) is written as:

\[
\sum_{i=1}^{2} \sum_{a \in A_i} \left( \frac{\mu}{\pi_i^t(a)} \left( r_i(a) - \pi_i^t(a) \right) - \frac{\mu}{\pi_i^{\mu,r'}(a)} \left( r_i'(a) - \pi_i^t(a) \right) \right) \left( \pi_i^t(a) - \pi_i^{\mu,r'}(a) \right) \\
= \mu \sum_{i=1}^{2} \sum_{a \in A_i} \frac{1}{\pi_i^t(a)} \left( r_i(a) - r_i'(a) \right) \left( \pi_i^t(a) - \pi_i^{\mu,r'}(a) \right) \leq \mu \sum_{i=1}^{2} \sum_{a \in A_i} \frac{1}{\pi_i^t(a)} \left| r_i(a) - r_i'(a) \right|.
\]

Combining (21), (22), and (23), we can obtain:

\[
\begin{align*}
\frac{d}{dt} \text{KL}(\pi^{\mu,r'}, \pi^t) & \leq -\mu \sum_{i=1}^{2} \sum_{a \in A_i} r_i'(a) \left( \sqrt{\frac{\pi_i^t(a)}{\pi_i^{\mu,r'}(a)}} - \sqrt{\frac{\pi_i^{\mu,r'}(a)}{\pi_i^t(a)}} \right)^2 + \mu \sum_{i=1}^{2} \sum_{a \in A_i} \frac{1}{\pi_i^t(a)} \left| r_i(a) - r_i'(a) \right| \\
\end{align*}
\]

Setting the start point as \( \pi^0 = \pi^r \), we have for all \( t \geq 0 \), \( \pi^t = \pi^r \). In this case, for all \( t \geq 0 \), we have \( \frac{d}{dt} \text{KL}(\pi^{\mu,r'}, \pi^t) = 0 \). Thus,

\[
\sum_{i=1}^{2} \sum_{a \in A_i} \frac{1}{\pi_i^t(a)} \left| r_i(a) - r_i'(a) \right|.
\]
Here, since \( r_i \) is in interior of \( \Delta(A_i) \), there exists \( \nu_1 > 0 \) such that \( \forall i, \forall a \in A_i, r_i(a) > \nu_1 \). Furthermore, from Lemma C.2 in Abe et al. 2022, \( \pi_i^r \) is also in interior of \( \Delta(A_i) \). Thus, there exists \( \nu_2 > 0 \) such that \( \forall i, \forall a \in A_i, \pi_i^r(a) > \nu_2 \). For a given \( \varepsilon > 0 \), let us define \( \delta = \frac{\varepsilon \nu_1 \nu_2}{4 + \varepsilon^2 \nu_2} \sqrt{\frac{1}{\sum_{i=1}^2 |A_i| - 1}} \). If \( \|r' - r\|_2 < \delta \), then \( \|r' - r\|_1 \leq \|r' - r\|_2 \sqrt{\sum_{i=1}^2 |A_i|} < \frac{\varepsilon \nu_1 \nu_2}{4 + \varepsilon^2 \nu_2} \). Thus, \( \forall i, \forall a \in A_i, r_i'(a) > \left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 > 0 \). So, if \( \|r' - r\|_2 < \delta \), we have then:

\[
\sum_{i=1}^2 \sum_{a \in A_i} r'_i(a) \left( \sqrt{\frac{\pi_i^{\mu, r'}(a)}{\pi_i^r(a)}} - \sqrt{\frac{\pi_i^r(a)}{\pi_i^{\mu, r'}(a)}} \right)^2 \geq \sum_{i=1}^2 \sum_{a \in A_i} r'_i(a) \left( \frac{\pi_i'^{\mu, r'}(a) - \pi_i'^r(a)}{\pi_i^r(a)} \right)^2 \geq \left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \sum_{i=1}^2 \sum_{a \in A_i} \left( \frac{\pi_i'^r(a) - \pi_i'^{\mu, r'}(a)}{\pi_i^r(a)} \right)^2 \geq \left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \sum_{i=1}^2 \left( \sum_{a \in A_i} \pi_i'^{\mu, r'}(a) \ln \left( \frac{\pi_i'^{\mu, r'}(a)}{\pi_i^r(a)} \right) \right) \geq \left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \sum_{i=1}^2 \text{KL}(\pi_i^{\mu, r'}, \pi_i^r),
\]

where the second inequality follow from \( x \geq \ln(1 + x) \) for all \( x > 0 \), and the third inequality follows from the concavity of the \( \ln(\cdot) \) function and Jensen’s inequality for concave functions. Moreover,

\[
\sum_{i=1}^2 \text{KL}(\pi_i^{\mu, r'}, \pi_i^r) \geq \frac{1}{2} \sum_{i=1}^2 \left\| \pi_i^{\mu, r'} - \pi_i^r \right\|_2^2 \geq \frac{1}{4} \left\| \pi_i^{\mu, r'} - \pi^r \right\|_2^2,
\]

where we use Pinsker’s inequality Tsybakov 2009, and the fact that \( \sum_{i=1}^2 x_i^2 \geq \frac{1}{2} \left( \sum_{i=1}^2 x_i \right)^2 \) for \( x_i \in \mathbb{R} \). Thus, we get:

\[
\left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \sum_{i=1}^2 \left\| \pi_i^{\mu, r'} - \pi^r \right\|_2^2 \leq \sum_{i=1}^2 \sum_{a \in A_i} \frac{1}{\pi_i^r(a)} |r_i(a) - r_i'(a)| < \frac{1}{\nu_2} \|r' - r\|_1.
\]

Therefore, if \( \|r' - r\| < \delta \), we have then:

\[
\left\| \pi^{\mu, r'} - \pi^r \right\|_2 \leq \left\| \pi^{\mu, r'} - \pi^r \right\|_1 < \left( \frac{4}{\left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \nu_2} \right) \|r' - r\|_1 \leq \left( \frac{4}{\left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \nu_2} \right) \|r' - r\|_2 \sum_{i=1}^2 |A_i| \leq \left( \frac{4}{\left( 1 - \frac{\varepsilon^2 \nu_2}{4 + \varepsilon^2 \nu_2} \right) \nu_1 \nu_2} \right) \delta \sum_{i=1}^2 |A_i| = 26.
\]
Since \( \sum \)\n
Combining (24) and (25), we have:\n
Thus, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( r' \in \prod_{i=1}^{2} \Delta^{q}(A_{i}) \), if \( \| r' - r \|_{2} < \delta \) then \( \| \pi^{r'} - \pi^{r} \|_{2} < \varepsilon \). Therefore, \( F(\cdot) \) is a continuous function on \( \prod_{i=1}^{2} \Delta^{q}(A_{i}) \). \( \square \)

D.4 Proof of Lemma [D.1]

Proof of Lemma [D.1] First, we have:

\[
\text{KL}(\pi^{*}, \pi'^{*}) - \text{KL}(\pi^{*}, r) = \sum_{i=1}^{2} \left( \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) \ln \frac{\pi_{i}^{*}(a_{i})}{\pi_{i}^{*}(a_{i})} - \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) \ln \frac{\pi_{i}^{*}(a_{i})}{r_{i}(a_{i})} \right) \\
= \sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) \ln \frac{r_{i}(a_{i})}{\pi_{i}^{*}(a_{i})} \\
\leq 2 \ln \left( \frac{1}{\sqrt{\sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) r_{i}(a_{i})}} \right),
\]

where the inequality follows from the concavity of the \( \ln(\cdot) \) function and Jensen’s inequality for concave functions. Since \( \ln(\cdot) \) is strictly concave, the equality holds if and only if \( \frac{r_{i}(a_{i})}{\pi_{i}^{*}(a_{i})} = \frac{r_{i}(a_{2})}{\pi_{i}^{*}(a_{2})} = \cdots = \frac{r_{i}(a_{A_{i}})}{\pi_{i}^{*}(a_{A_{i}})} \) for all \( i \in \{1, 2\} \). From the assumption that \( r \neq \pi^{r} \), there exists \( i \in \{1, 2\} \) and \( a_{i}, a'_{i} \in A_{i} \) such that \( \frac{r_{i}(a_{i})}{\pi_{i}^{*}(a_{i})} \neq \frac{r_{i}(a'_{i})}{\pi_{i}^{*}(a'_{i})} \). Therefore, we have:

\[
\text{KL}(\pi^{*}, \pi'^{*}) - \text{KL}(\pi^{*}, r) < 2 \ln \left( \frac{1}{\sqrt{\sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) r_{i}(a_{i})}} \right). \tag{24}
\]

Here, by using the ordinary differential equation (RMD), we have for all \( i \in \{1, 2\} \) and \( a_{i} \in A_{i} \):

\[
\pi_{i}^{*}(a_{i}) \left( q_{i}^{\pi^{r}}(a_{i}) - v_{i}^{\pi^{r}} \right) + \mu \left( r_{i}(a_{i}) - \pi_{i}^{*}(a_{i}) \right) = 0,
\]

and then:

\[
\frac{r_{i}(a_{i})}{\pi_{i}^{*}(a_{i})} = 1 - \frac{1}{\mu} \left( q_{i}^{\pi^{r}}(a_{i}) - v_{i}^{\pi^{r}} \right). \tag{25}
\]

Combining (24) and (25), we have:

\[
\text{KL}(\pi^{*}, \pi'^{*}) - \text{KL}(\pi^{*}, c) < 2 \ln \left( \frac{1}{\sqrt{\sum_{i=1}^{2} \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) \left( 1 - \frac{1}{\mu} \left( q_{i}^{\pi^{r}}(a_{i}) - v_{i}^{\pi^{r}} \right) \right)}} \right) \\
= 2 \ln \left( \frac{1}{\sqrt{\sum_{i=1}^{2} \left( 1 - \frac{1}{\mu} \left( v_{i}^{\pi_{i}^{*}}, v_{i}^{\pi^{r}_{i-1}} - v_{i}^{\pi^{r}_{i}} \right) \right)}} \right). 
\]

Since \( \sum_{a_{i} \in A_{i}} \pi_{i}^{*}(a_{i}) > 0 \), we have \( 1 - \frac{1}{\mu} \left( v_{i}^{\pi_{i}^{*}}, v_{i}^{\pi^{r}_{i-1}} - v_{i}^{\pi^{r}_{i}} \right) > 0 \). Also, since \( \pi^{*} \) is the Nash equilibrium,
we get:
\[
\sum_{i=1}^{2} \left( 1 - \frac{1}{\mu} \left( v_{i}^{\pi_{i}^{r}, \pi_{-i}^{r}} - v_{i}^{r} \right) \right) = 2 + \frac{1}{\mu} \sum_{i=1}^{2} \left( v_{i}^{r} - v_{i}^{\pi_{i}^{r}, \pi_{-i}^{r}} \right) \\
= 2 + \frac{1}{\mu} \sum_{i=1}^{2} \left( -v_{i}^{\pi_{i}^{r}, \pi_{-i}^{r}} - v_{i}^{r} \right) \\
= 2 + \frac{1}{\mu} \sum_{i=1}^{2} \left( v_{i}^{\pi_{i}^{r}, \pi_{-i}^{r}} - v_{i}^{r} \right) \leq 2,
\]
where the second equality follows from \( \sum_{i=1}^{2} v_{i}^{\pi_{i}^{r}, \pi_{-i}^{r}} = 0 \) and \( \sum_{i=1}^{2} v_{i}^{r} = 0 \), and the last equality follows from \( -v_{1}^{\pi_{1}^{r}, \pi_{2}^{r}} = v_{2}^{\pi_{1}^{r}, \pi_{2}^{r}} \) and \( -v_{2}^{\pi_{1}^{r}, \pi_{2}^{r}} = v_{1}^{\pi_{1}^{r}, \pi_{2}^{r}} \) by the definition of two-player zero-sum games. Thus, we get
\[
\text{KL}(\pi^{*}, \pi^{r}) - \text{KL}(\pi^{*}, r) < 2(\ln 1) \leq 0.
\]

D.5 Proof of Lemma D.2

Proof of Lemma D.2 By using the ordinary differential equation (RMD), we have for all \( i \in \{1, 2\} \) and \( a_{i} \in A_{i} \):
\[
\pi_{i}^{r}(a_{i}) \left( q_{i}^{r}(a_{i}) - v_{i}^{r} \right) + \mu (r_{i}(a_{i}) - \pi_{i}^{r}(a_{i})) = 0.
\]
Since \( r = \pi^{r} \), we have for all \( i \in \{1, 2\} \) and \( a_{i} \in A_{i} \):
\[
r_{i}(a_{i}) \left( q_{i}^{r}(a_{i}) - v_{i}^{r} \right) = 0.
\]
From the definition of the reference strategy, we have \( r_{i}(a_{i}) > 0 \), and then \( v_{i}^{r} = \max_{a_{i} \in A_{i}} q_{i}^{r}(a_{i}) \) for all \( i \in \{1, 2\} \). Therefore, each player \( i \)'s strategy \( r_{i} \) is a best response to the other player \(-i\)'s strategy \( r_{-i} \). Thus, \( r \) is a Nash equilibrium of the original game.

E Additional Experimental Results

In this section, we investigate the performance of M2WU with decreasing learning rates under the noisy-information feedback setting. We set the learning rates to \( \eta_{t} = t^{-\frac{3}{4}} \) for all algorithms. Other settings are equivalent to the noisy-information feedback experiments in Section 7.2. Figure 6 shows the average exploitability of \( \pi^{t} \) on 100 instances. Even with the decreasing learning rates, M2WU demonstrates better performance than MWU and OMWU.
Figure 6: Exploitability of $\pi_t$ for M2WU, MWU, and OMWU with decreasing learning rates under the noisy-information feedback setting.