Diamagnetic/Paramagnetic Phase Transitions in Non-equilibrium Confined Single-Component Plasmas

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In this letter, a phase transition of a non-equilibrated single component plasma evolving within a uniform constant magnetic field is demonstrated for the first time. We present classical fluid models that capture this phenomenon, confirm with N-particle simulations, derive constraints on parameters necessary to induce this phase transition, and suggest an experiment to test the novel predictions of the theory.

Single-component plasmas (SCPs) at equilibrium are known to be diamagnetic.1,2 The origin of this understanding can be traced at least back to the work of Brillouin, who presented a theory to describe the steady-state dynamics of a continuous ensemble of cold electrons in a uniform magnetic field, for which the repulsive space-charge and angular-motion effects are canceled by the focusing action of a magnetic field.3 Extension of this theory beyond the ideal Brillouin case provides a description of a rigid-rotor equilibrium that can be achieved by carefully tuning the external magnetic field, rotation rate, and density of the SCP.4,5 Since Brillouin’s description, the plasma and accelerator field have described SCP non-equilibrium dynamics including space-charge shifts and surface modes.6,7,8

FIG. 1: Diamagnetic/paramagnetic phase transition of an electron plasma radially oscillating in a constant uniform magnetic field along the z-axis, neglecting the interaction among the electrons. Time is in units of the cyclotron magnetic field at the origin, \( \tau_c = \frac{2\pi}{\omega_c} \). The lines represent theory and the scatter plots represent N-particle simulation for the magnetic moment, \( m_z \), (red and blue data, left axis) and the self-magnetic field at the origin, \( B_{\text{self},z} \), at the origin (orange and green data, right axis), respectively. Negative values correspond to a diamagnetic state; positive values to a paramagnetic state. Initial conditions are described in Section VII of the supplement.

Analytic models in accelerator physics typically assume continuous relativistic beams and employ statistical envelope parameters that allow the inclusion of normalized root-mean-square (rms) emittance.22,23 Such envelope parameters are related to standard fluid parameters (see Section I of the supplement) used in the plasma literature. Furthermore, many analytic models in both fields utilize the co-rotating Larmor frame in order to decouple the evolution in the two directions transverse to the magnetic field’s axis. We recently developed a theoretical lab-frame model to describe the periodic lensing effect of an electron plasma inside an ultrafast electron microscope.24 The lab-frame approach is more similar to earlier analyses1 and using this model we uncovered a surprising and fundamental effect that appears to be missing in the literature. In Fig. 1, the fundamental physical effect is illustrated using N-particle simulations for a non-interacting uniform spherical bunch of electrons in a constant, uniform magnetic field, and with an initial angular rotation frequency \( \omega_{r,0} = 0.25\omega_c \); where \( \omega_c = eB_{\text{ext}}/m \) is the cyclotron frequency, \( B_{\text{ext}} \) is the magnitude of the external magnetic field, and \( e/m \) is the charge-to-mass ratio of the electron. The figure plots the magnetic moment and self-magnetic field, defined in Section II of the supplement, showing the well-known radial “breathing mode”25, but the figure also shows that the magnetic field at the origin of the bunch changes sign twice every period, signaling periodic transitions between paramagnetic and diamagnetic phases. We note this is consistent with the Bohr-Van Leeuwen theorem26 as we consider non-equilibrium conditions. This periodic phase transition is the novel effect that we explore here.

While it is known that in SCPs the radial breathing mode couples to a related rotational breathing mode,6,7 the consequences for periodic magnetic transitions has not been studied. In this letter, we present simple theories for both non-interacting and interacting SCP systems to describe this novel phase transition in terms of fluid parameters. We find that interactions only quantitatively change the results. We discuss regions in parameter space composed of rotation frequency and interaction strength.
where such behavior occurs for cylindrically symmetric SCPs. We also suggest an experiment that should be able to test a fundamental prediction of the theory.

We consider the evolution of electrons distributed cylindrically symmetrically about the axis of a constant, uniform external magnetic field. We employ cylindrical coordinates $r$, $\phi$, and $z$ as well as standard Cartesian coordinates, and we use the terms longitudinal and transverse to refer to the axial and radial directions, respectively. For the continuous fluid model, the fluid velocity at the radial position $r$ can be written as

$$\vec{v}(r, t) = \frac{1}{R} \frac{dR}{dt} \hat{r} + \omega_r \hat{\phi},$$  \hspace{1cm} (1)$$

where $R$ is the transverse radius of the SCP and $\omega_r = \omega_r(t)$ is the instantaneous, coherent rate of rotation of the SCP. Explicit definitions of these quantities are presented in Section I of the supplement. By considering electrostatic fields and conservation of total angular momentum (details provided in Section III of the supplement) we get

$$\frac{\omega_r}{\omega_c} = \frac{1}{2} + \left(\frac{\omega_{r,0}}{\omega_c} - \frac{1}{2}\right) \frac{R^2}{R_0^2},$$  \hspace{1cm} (2)$$

where the subscript 0 indicates the value of the parameters at time zero. As the radial motion is captured in the model through the evolution of the parameter $R$, which is the sole time-dependent parameter in Eq. (43), we see that $\omega_r$ couples to the radial dynamics so that $\omega_r$ and $R$ oscillate coherently. In Fig. 2 the evolution of the rate of rotation as a function of $R^2$ from this prediction is confirmed in the simulation of a non-interacting ensemble initially with a Gaussian distribution.

We note that Eq. (43) is in essence equivalent to expressions that can be found in the literature describing Brillouin hollow beams (for instance, Eq. 5.36 of [23]). Such descriptions differ from our presentation due to their assumption of equilibrium: meaning that both $\dot{R}$ and $\omega_r$ are conserved in time. To satisfy these two constraints, the hollow beam must both satisfy Eq. (43) and adopt a non-uniform profile when the generalized angular momentum, sometimes called the canonical angular momentum [23], is non-zero. However, when the radial constraint is removed, the distribution profile is no longer confined in this manner, and Eq. (43) instead describes the manner in which the radial oscillation determines the angular oscillation.

If the radial oscillation is of sufficient size, which we will quantify shortly, and if $\omega_{r,0} < \omega_c$, where $\omega_c = \frac{1}{2} \omega_L$ is the Larmor frequency, then the radial breathing can reverse the direction of the SCP rotation. This reversal is seen in Fig. 2 where the line describing the evolution of $\omega_r$ crosses 0. Moreover, the magnetic moment in our notation is given by

$$\vec{m} = -\frac{e}{2} \omega_r R^2 \hat{z}. \hspace{1cm} (3)$$

As the sign of the magnetic moment of the SCP is determined by the direction of rotation, which is reversing during the simulation, the SCP transitions between a diamagnetic and a paramagnetic phase.

Eq. (43) is derived without specific reference to the form of the inter-particle interactions. The oscillation is seen even in the non-interacting model. Assuming the oscillation starts at its maximum $R$ is

$$\frac{R^2}{R_0^2} = 1 + 2 \frac{\omega_{r,0}}{\omega_c} \left(1 - \cos(\omega_c t)\right). \hspace{1cm} (4)$$

Details of the derivation of this expression can be found in Section V of the supplement. We note that this equation also applies to ensembles with initially non-uniform distributions with our definition of $R$ (see Section I of the supplement). The ratio in Eq. (50) can be used in Eq. (43), and these two equations can then be used in Eq. (3) to obtain the magnetic moment of the SCP induced by the rotational breathing. The comparison of the magnetic moment and the magnetic field at the SCP’s barycenter calculated using an $N$-particle simulation for an initially uniform distribution and the above theory is what we presented in Fig. 1. As can be clearly seen, the agreement between theory and simulation is excellent.

For the non-interacting model, SCP’s are expected to undergo diamagnetic/paramagnetic phase transitions for half of the possible oscillations described by Eq. (50). This can be understood by appreciating the zero crossings of the envelope equation involving $R$, the derivation of which we describe in Section IV of the supplement. We adopt the dimensionless convention $\tilde{R} = \frac{R}{R_0}$ and $\tau = \omega_c t$, giving the corresponding envelope equation

$$\frac{d^2 \tilde{R}}{d\tau^2} = -\frac{1}{4} \frac{\omega_{r,0}}{\omega_c} - \frac{1}{2} \frac{1}{R^2}. \hspace{1cm} (5)$$

This equation is the Ermakov-Pinney equation with time-independent frequency, and a solution of the more general time-dependent equation was presented by Pinney

**FIG. 2:** Evolution of the rotation rate as a function of the squared ratio of the initial radius, $R_0$, to the radius, $R$, at time $t$ for an initially Gaussian distribution. The blue circles represent $N$-particle simulation and the red line is theory.

The dotted line is at $\frac{\omega_r}{\omega_c} = 0$. This change in rotation direction signals the magnetic phase transition illustrated in Fig. 1.
roughly 70 years ago\cite{27}. Notice that the right-hand side of this equation has zeros at \( \frac{\omega_L}{\omega_c} = 0 \) and 1 and that this equation can be thought of as an effective force on the envelope. If a SCP is radially oscillating, then the effective force on \( R \) must pass through 0 at some time. By Eq. \cite{43} we know that the rotational rate never spontaneously crosses \( \omega_L \), so all initial states with \( \omega_r,0 \geq \omega_L \) do not undergo phase transitions. On the other hand, all solutions with \( \omega_r,0 < \omega_L \) (except \( \omega_r,0 = 0 \) that does not breath) exhibit diamagnetic/paramagnetic phase transitions.

We now consider space-charge effects by using continuous-beam models as the simplest analytic approach. The continuous-beam model (see Section IV of the supplementary documentation) is given by

\[
\frac{d^2\tilde{R}}{d\tau^2} = \frac{1}{2} \frac{\omega_r^{2,0}}{\omega_c^2} \frac{1}{\tilde{R}} - \frac{1}{4} \tilde{R} + \left( \frac{\omega_r,0}{\omega_c} - \frac{1}{2} \right)^2 \frac{1}{\tilde{R}^3} \tag{6}
\]

where \( \omega_r^{2,0} = \sqrt{\epsilon_0 \frac{n_e}{\pi R_0^2}} \) is the initial plasma frequency with the 2 in the \( p^2 \) indicating the two-dimensional cylindrical coordinate system, \( n_e \) is the number density along the length of the beam, and \( \epsilon_0 \) is the vacuum permittivity. We note that equivalent models to this can be found in the literature.\cite{27,29} The continuous-beam envelope equation differs from the non-interacting envelope equation only by the inclusion of the term \( \frac{1}{2} \frac{\omega_r^{2,0}}{\omega_c^2} \frac{1}{\tilde{R}} \), which captures the effect of the collective repulsion of the distribution on the envelope. While this model depends on three parameters, \( \omega_r^{2,0}, \omega_r,0, \omega_c \), and the initial radial velocity, the freedom to assign what time is zero allows us to set one of the velocity degrees of freedom to zero. By convention, we choose the initial radial velocity to be zero. Thus the properties of the oscillations predicted by this model can be fully understood within the remaining parameter space that is standard in descriptions of Brillouin flow\cite{27}. Of course, the zeros of the effective force in Eq. \cite{6} of this model describe the well-known rigid-rotor states, which can be visualized as a parabola in this parameter space (as seen in Fig. 3). In the accelerator literature where continuous beam behavior near rigid-rotor states is discussed, small radial oscillations about the rigid-rotor state are described as rms-mismatched induced ripples along the longitudinal coordinate\cite{23}, however, such descriptions focus on the effect of \( R \) and not \( \omega_r \), which we emphasize here.

Unlike the non-interacting model, the zeros of the effective force for the continuous-beam envelope equation do not correspond to a zero of \( \omega_r \), so some oscillations with \( \omega_r,0 < \omega_L \) do not oscillate sufficiently to experience the diamagnetic/paramagnetic phase transition. A diagram denoting all of the regions of parameter space where the initial conditions of the continuous-beam model will experience the phase transition is displayed in Fig. 3 One interesting aspect of this diagram is the fact that the two shaded regions, one inside the rigid-rotor parabola and a second describing the half-plane with \( \omega_r,0 < 0 \), describe the same oscillations. That is, our choice of time zero is ambiguous as there exists two times per period with zero radial velocity within every period of the radial oscillations: (1) the time where the radius is maximum (corresponding to the entire region of parameter space inside the rigid-rotor equilibrium parabola) and (2) the time at which the radius is minimum (that region’s complement). Deducing the fact that all initial conditions within the half plane where \( \omega_r,0 < 0 \) produce the phase transition is simple; we know that the point in the oscillation that corresponds to the maximum radius is inside the parabola and has \( \omega_r > 0 \), so all such initial states experience the phase transition. The initial states within the rigid-rotor equilibrium parabola that experience the phase transition requires more analysis, and we present the mathematical description of this region’s boundary in Section VI of the supplement.

As we see in the description of the non-interacting and cold continuous-beam models, the diamagnetic/paramagnetic phase transition is not uncommon in non-equilibrium systems, but we point out that these models do not predict that it arises from a system initially at rest. According to Eq. \cite{43}, this can only happen if \( R \) becomes less than \( R_0 \) at some point during the radial evolution of the SCP. For the non-interacting model initially at rest case, \( R = R_0 \) for all time; for the cold continuous-beam model, \( R > R_0 \) by a conservation of energy. So neither model gives rise to the phase transition if the ensemble starts from rest. On the other hand, if we consider a SCP of finite size undergoing Coulomb explosion in a constant uniform magnetic field, \( R \) can fall below \( R_0 \) and the phase transition can occur.

The spheroidal envelope model describing such dynam-
ics is given by
\[
\frac{d^2 \hat{R}}{dt^2} = \frac{1}{2} \frac{\omega_{p3,0}^2}{\omega_c^2} \Gamma_1(\alpha) \frac{1}{RZ} - \frac{1}{4} \hat{R} + \left( \frac{\omega_{r,0}}{\omega_c} - \frac{1}{2} \right)^2 \frac{1}{R^2} \tag{7a}
\]
\[
\frac{d^2 \hat{Z}}{dt^2} = \frac{\omega_{p3,0}^2}{\omega_c^2} \Gamma_2(\alpha) \frac{1}{R^2} \tag{7b}
\]
where \(\omega_{p3,0}^2 = \frac{e^2}{m_e c} \frac{N}{4\pi R_0 Z_0} \) is the initial plasma frequency of the SCP (in three dimensions, hence the p3), \(R_0\) represents the initial transverse radius, \(Z_0\) the initial longitudinal radius, \(\hat{Z} = \frac{Z}{Z_0}\), \(\alpha = \frac{Z}{\hat{R}}\), \(\Gamma_1(\alpha) = \frac{\alpha - \frac{\omega_{r,0}}{\omega_c}}{\sqrt{\alpha^2 - 1}}\), and
\[
\Gamma_2(\alpha) = \frac{-1 + \alpha \sqrt{\alpha^2 - 1}}{\alpha^2 - 1}. \]
A more general model in the Larmor frame corresponding to Eqs. (7a) and Eq. (7b) can be found in the plasma literature. Details of our model's derivation can be found in Section IV of the supplement. We point out that the radial portion of the model differs from the continuous-beam envelope equation primarily in the inclusion of the geometric \(\Gamma_1\) term. On the other hand, the model's inclusion of the evolution of the longitudinal radius is not present in the previous models. As the longitudinal radius parameter is coupled to the transverse dynamics through \(\Gamma_1\) and a SCP unconfined in the longitudinal direction will continue to expand, conservation of energy suggests that the envelope should still have kinetic energy when it returns to its initial radius. This results in the envelope becoming smaller than its original radius that in turn drives the phase transition.

Comparison of this model’s prediction to \(N\)-particle simulations for the magnetic moment and self-magnetic field at the SCP’s barycenter, the radial, the longitudinal, and the angular rate evolution can be seen in Fig. 4. Notice that the amplitude of the self-magnetic field oscillation gradually becomes smaller mostly due to the decrease in density as \(Z\) increases. On the other hand, the magnetic moment does not experience this damping as it is independent of the longitudinal evolution. Nonetheless, some damping is seen due to thermalization that can be quantified by normalized rms emittance growth, but we note that this growth becomes very slow in the spheroidal cases we examined. This is due to the longitudinal explosion that results in non-interacting like behavior once the density becomes sufficiently low; however, artificial normalized rms emittance growth effect\[23,20\] make such long-scale simulations unreliable.

The fundamental theory can be tested experimentally by careful accounting of the angular momentum of a beam. We note that generalized Courant-Snyder theory\[30\] provides tools to track the rotation angle of the envelope matrix in the presence of transverse-coupled dynamics, and this rotation can alternatively be conceptualized as the angular momentum of the beam as we’ve done here. Therefore, that theory and the tools we present here may be employed to account for the angular momentum. Employing such techniques suggests an experiment to validate the radial and angular coupling. A long solenoid can be used to introduce angular motion to a prolate SCP in an accelerator setting, and as the SCP evolves in a region without a magnetic field, the Eq. (43) be used with \(\omega_L = 0\) and the \(\omega_{r,0}\) introduced from the solenoid. The SCP could then be passed through a second long solenoid. By tuning the length of the first solenoid, the magnetic field strengths, and the separation distance between the solenoids for an SCP of a known length and number of electrons, it should be possible to induce a near equilibrium SCP once it enters the second solenoid. Of course, small ripples in the beam radius about this equilibrium state should be observed as has been discussed in the literature\[23\].

In summary, we have demonstrated the occurrence of a periodic, transient diamagnetic/paramagnetic phase transition in non-equilibrium SCPs subject to a uniform magnetic field. This observation is surprising given the established diamagnetism of SCPs in ubiquitous Brillouin
flow models. The model presented here lays the foundation for future theoretical and experimental investigations of these intriguing phenomena, with possible profound implications in the plasma, accelerator, and high-brightness electron source fields.

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II. SUPPLEMENT

A. Fluid and statistical parameters

For the purpose of comparing fluid results with $N$-particle simulations, the relation of fluid parameters extracted to the statistics of $N$-particle simulations needs to be clarified. The fluid parameter $R$ is typically measured using the square root of the variance, i.e.

$$ R = \sqrt{s_{x,x} + s_{y,y}}, $$

(8)

where we have taken the average to enforce cylindrical symmetry. This parameter is the continuous uniform beam radius but is related to the uniform ellipsoid radius by $\sqrt{\frac{5}{2}}R$. Here, $s_{a,b} = \overline{ab} - \overline{a}\overline{b}$, $\overline{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$, $N$ is the number of particles, $a_i$ is the value of the parameter $a$, which typically is $x$, $y$, $v_x$, or $v_y$, and $s_{a,a}$ is commonly known as the variance of $a$. Note, we use the sum of the two variances in determining $R$ in order to average out symmetry breaking stochastic effects between the two transverse dimensions which we assume are the same.

The rate of rotation, $\omega$, can be analogously expressed in terms of the statistical envelope parameters. First, write the velocity of the $i^{th}$ particle as $\vec{v}_i = (v_{x,i}, v_{y,i}, v_{z,i})$. If we assume there is a linear relationship between velocity and position, we obtain

$$ v_{x,i} = a_x x_i + b_x y_i + \delta_{v_{x,i}}, $$

(9a)

$$ v_{y,i} = b_y x_i + a_y y_i + \delta_{v_{y,i}}, $$

(9b)

where $x_i$ and $y_i$ are from the $i^{th}$ particle’s position, $x_i = (x_i, y_i, z_i)$ and the other six parameters, $(a_x, b_x, a_y, b_y, \delta_{v_{x,i}}, \delta_{v_{y,i}})$ are determined by finding the line of best fit through the data with $a$ and $b$ representing slopes and $\delta$ representing the residuals. If $x$, $y$, and the residuals are assumed to be uncorrelated, then

$$ s_{x,v_x} = a_x s_x^2 $$

(10a)

$$ s_{y,v_y} = b_y s_y^2 $$

(10b)

$$ s_{x,v_x} = b_y s_x^2 $$

(10c)

$$ s_{y,v_y} = a_y s_y^2 $$

(10d)

giving us

$$ a_x = \frac{s_{x,v_x}}{s_x^2} $$

(11a)

$$ b_x = \frac{s_{y,v_y}}{s_y^2} $$

(11b)

$$ b_y = \frac{s_{x,v_x}}{s_x^2} $$

(11c)

$$ a_y = \frac{s_{y,v_y}}{s_y^2} $$

(11d)

Assuming cylindrical symmetry, and comparing this cartesian representation to the standard equation of velocity drift (see Eqs. (25) and (30) below, we see $\frac{\dot{R}}{R} = \frac{s_{x,v_x} + s_{y,v_y}}{s_x^2 + s_y^2}$ and

$$ \omega = \frac{s_{x,v_y} - s_{y,v_x}}{s_x^2 + s_y^2}, $$

(12)

where we have again used the summation over identical terms under symmetry to eliminate symmetry-breaking effects.

Simulations were conducted using the velocity-Verlet algorithm employing the Fast Multipole Moment (FMM) algorithm to solve for the electric field at every step. The FMM algorithm has a computational complexity of $O(N \ln(N))$ allowing for the simulation of particle-particle interactions of moderate sized ($10^4$-$10^5$) particles in reasonable times. Simulations were run on the High Performance Computational Cluster at Michigan State University.

III. MAGNETIC MOMENT AND MAGNETIC FIELD

A. Magnetic moment

The magnetic moment, $\vec{m}$, is defined by

$$ \vec{m} = \frac{1}{2} \int \vec{r} \times (\vec{v} N q \rho_{3D}) dV. $$

(13)

where $N$ is the number of particles and $\rho_{3D}$ is the 3D density. On the other hand, the mechanical angular momentum is defined by

$$ \vec{L}_{\text{mech}} = \int \vec{r} \times (\vec{v} N m \rho_{3D}) dV $$

(14)

where $m$ in this equation represents the particle mass. It follows

$$ \vec{m} = \frac{1}{2} \frac{e}{m} \frac{\vec{L}}{L}. $$

(15)
We find explicit expressions for \( L_{z, \text{tot}} \) for the cylindrical case below, but here we write the discrete magnetic moment in terms of discrete statistical parameters in general. Notice

\[
L_{z, \text{mech}} = \frac{N}{m} \sum_{i=1}^{N} \left( (x_i - \bar{x})(y_{i} - \bar{y}) - (y_{i} - \bar{y})(x_{i} - \bar{x}) \right)
\]

= \( Nm(s_{x,y} - s_{y,x}) \)

= \( Nm\omega_{r}R^{2} \),

so it follows

\[
m_{z} = \frac{N}{2} \omega_{r}R^{2}.
\]  

(16)

We note that this equation is general for any distribution.

1. Magnetic field at the center of a uniform spheroid due to rotation

In addition to the magnetic moment, another way to understand how this oscillation in the frequency causes a phase transition between a paramagnetic and diamagnetic state is to examine a characteristic magnetic field induced by the moving charges.

Consider a uniform, single-component, cylindrically-symmetric spheroid rotating with instantaneous rate \( \omega_{r} \) about the \( z \)-axis so that the particles have velocity \( \vec{v} = r\omega_{r}\hat{\phi} \), where \( \hat{\phi} \) is the azimuthal direction in cylindrical coordinates. The self-magnetic field for a spheroid is not uniform, but we know from symmetry considerations that it has maximum magnitude at the barycenter of the SCP; and we call the value of the magnetic field at this point \( B_{\text{self}}(\bar{0}) \). Denote \( R \) to be the radius transverse to \( \hat{z} \) and \( Z \) the radius along \( \hat{z} \); the resulting probability-like density can be written as \( \rho = \frac{1}{\sqrt{\pi R^{2}Z}} \) within the ellipsoid and 0 otherwise. The magnetic field at the barycenter of the ellipsoid can be written using the Biot-Savart law as and denoting the volume of the ellipse as \( V \) by

\[
\vec{B}_{\text{self}}(\bar{0}) = N \int_{V} \frac{\mu_{0}}{4\pi} \frac{q\vec{v} \times \vec{r}}{r^{3}} \rho dV
\]

= \( -\frac{3Nq\mu_{0}\omega_{r}}{16\pi^{2}R^{2}Z} \int_{V} \left( \frac{1}{\frac{1}{\sqrt{x^{2} + y^{2} + z^{2}}} + \frac{1}{\sqrt{x^{2} + y^{2} + z^{2}}} + \frac{1}{\sqrt{x^{2} + y^{2} + z^{2}}} - \frac{1}{\sqrt{x^{2} + y^{2} + z^{2}}}} \right) dxdydz \)

(18)

where \( N \) is the number of particles, \( q \) is the charge of the component, and \( \mu_{0} \) is the permeability of free space. Make the coordinate transformation \( x = Rx \sin \theta \cos \phi \), \( y = Rr \sin \theta \sin \phi \), and \( z = Zr \cos \theta \); we further denote \( \alpha = \frac{Z}{R} \) and solve the integrals separately below:

\[
I_{1} = \int_{V} \frac{1}{\sqrt{x^{2} + y^{2} + z^{2}}} dxdydz
\]

= \( R^{2}Z \int_{0}^{\pi} \int_{0}^{\pi} \frac{r^{2} \sin \theta}{\sqrt{R^{2}r^{2} \sin^{2} \theta + Z^{2}r^{2} \cos^{2} \theta}} d\phi d\theta dr
\]

= \( 2\pi RZ \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \frac{r \cos \phi \sin \theta}{\sqrt{1 + \alpha^{2} \cos^{2} \theta}} d\theta dr
\]

= \( 2\pi RZ \frac{\cos^{-1} \alpha}{\sqrt{\alpha^{2} - 1}} \)

(19)

and

\[
I_{2} = \int_{V} \frac{z^{2}}{(x^{2} + y^{2} + z^{2})^{\frac{3}{2}}} dxdydz
\]

= \( R^{2}Z^{3} \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \frac{r^{4} \sin \phi \cos \theta}{(R^{2}r^{2} \sin^{2} \theta + Z^{2}r^{2} \cos^{2} \theta)^{\frac{3}{2}}} d\phi d\theta dr
\]

= \( 2\pi RZ^{2} \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \frac{r \cos \phi \sin \theta}{(1 + (\alpha^{2} - 1) \cos^{2} \theta)^{\frac{3}{2}}} d\theta dr
\]

= \( 2\pi RZ^{2} \frac{\cos^{-1} \alpha}{(\alpha^{2} - 1) \frac{1}{\sqrt{\alpha^{2} - 1}}} \)

(20)

where we have used the fact that the integrand is even to halve the integral’s domain.

Introduce \( u = \cos \theta \) so \( du = -\frac{1}{\sin \theta} d\theta \), thus

\[
I_{2} = -\pi Z^{2} \alpha \int_{1}^{0} \frac{\sqrt{u}}{(1 + (\alpha^{2} - 1)u)^{\frac{3}{2}}} du
\]

= \( 2\pi Z^{2} \alpha \left( \frac{\cos^{-1} \alpha}{(\alpha^{2} - 1)^{\frac{3}{2}}} - \frac{1}{(\alpha^{2} - 1) \alpha} \right) \)

(21)

Putting this together gives

\[
\vec{B}_{\text{self}}(\bar{0}) = -\frac{3Nq\mu_{0}\omega_{r}}{8\pi R^{3}} \frac{1}{\alpha^{2} - 1} \left( \alpha - \frac{\cos^{-1} \alpha}{\sqrt{\alpha^{2} - 1}} \right) \hat{z}
\]

(22)

Notice that the dependence of this field on \( \omega_{r} \) and the reversal of this parameter due to the coupling to the breathing mode gives rise to the diamagnetic/paramagnetic phase transition.

B. Derivation of the instantaneous rotation oscillation expression

We derive Eq. (1) from the main text using the current density, the Lorentz force, and conservation of total angular momentum. We assume the distribution is uniform and stays uniform throughout its evolution. We note this assumption is exact for ideal cold uniform single-component plasmas ( SCPs) in the presence of electrostatics, but is only an approximation otherwise.

Parameterized the cylinder’s radius by \( R \), which is shorthand for \( \bar{R}(t) \). Using cylindrical coordinates, \( r \) and
φ, consider such a uniformly charged cylinder with density
\[
\rho(r, t) = \begin{cases} 
\frac{1}{\pi R^2}, & r \leq R \\
0, & r > R 
\end{cases} 
\] (23)
in a constant, uniform, axial, external magnetic \(\vec{B} = B_{ext} \hat{z}\). The solution of the continuity equation,
\[
\frac{\partial}{\partial t} \rho(r, t) = -\nabla \cdot \vec{J}(r, t)
\]
gives the isotropic current density
\[
\vec{J}(r, t) = \left( \frac{\dot{R}}{R} \hat{r} + g \hat{\phi} \right) n_z q \rho(r, t) 
\] (24)
where \(g = g(r, t)\) is a function that is constrained to be divergence free, \(n_z\) represents the line density, and \(q\) is the charge of the particles in the SCP. The corresponding drift velocity is
\[
\vec{v}(r, t) = \vec{J} = \frac{\vec{J}}{n_z q \rho} = \frac{\dot{R}}{R} \hat{r} + g \hat{\phi} 
\] (25)
and we see that \(g\) can be interpreted as the angular component of the drift velocity for the charged distribution.
The Lorentz force is
\[
\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) 
\] (26)
where \(\vec{E} = E_r \hat{r} = E_r(r, t) \hat{r} = E_r(R, t) \frac{\dot{R}}{R} \hat{r}\) for the symmetry we consider. We further discuss the properties of electrostatic fields within uniform distributions in Section III C.1 and in that section we define \(\omega_{pR}^2 = \frac{n_z \omega_r^2}{mR}\) and show how it is related to traditional plasma frequencies; from this equation, \(E_r = \frac{\omega_p^2}{2Q_c} \omega_r\). Also in the non-relativistic regime
\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \left( \frac{\dot{R}}{R} r - \frac{g^2}{r} \right) \hat{r} + \left( \frac{\partial g}{\partial t} + \frac{\dot{R}}{R} \frac{\partial g}{\partial r} + \frac{\dot{R}}{R} \frac{g^2}{r^2} \right) \hat{\phi} 
\] (27)
Equating the \(r\) component of the Lorentz force and this expression gives us the equations of motion
\[
\omega_{pR}^2 r - \omega_c g = \frac{\dot{R}}{R} - \frac{g^2}{r} 
\] (28)
where \(\omega_c = \frac{qB_{ext}}{mR}\) is the cyclotron frequency.
Notice Eq. (28) can be re-written as
\[
\frac{\dot{\dot{R}}}{R} - \omega_{pR}^2 - \omega_c g - \frac{g^2}{r^2} = 0. 
\] (29)
This condition must be satisfied for all \(r\)’s, so \(g\) must be linear in \(r\). We write
\[
g = g(r, t) = \omega_r r 
\] (30)
where \(\omega_r = \omega_r(t)\) is a function of time only and has units of frequency. Notice that plugging Eq. (29) back into Eq. (28) gives us a physical interpretation of \(\omega_{pR}\); it is the rate at which the ensemble is rotating at time \(t\). Plugging Eq. (29) back into Eq. (29) and solving for \(\dot{\dot{R}}\) gives
\[
\dot{R} = \omega_{pR}^2 R - \omega_r (\omega_r - \omega_c) R. 
\] (31)

C. Derivation of ODE’s

1. Uniform electric field properties and plasma frequencies

In this section, we present the linear relation of electric fields within uniform distributions, show how our notation relates to the three relevant models we discuss, and derive how the plasma frequency relates to the plasma frequency at a different time.

It is a well known consequence that uniform distributions result in linear fields. The electric field caused by particles in the non-interacting model is 0 by assumption, which is trivially linear with slope \(\omega_{pR}^2\). The electric field in the radial direction within a uniform cylinder is given
\[
E_r(r, t) = \frac{qn_z}{8\pi \epsilon_0 R^2} \Gamma_1(\alpha) R 
\]
\[
= \frac{3qN}{8\pi \epsilon_0 R^2} \Gamma_1(\alpha) R 
\]
\[
= \frac{m}{q} \omega_{pR}^2 \Gamma_1(\alpha) R 
\] (32)
where \(n_z\) represents the number density along the \(z\) coordinate and \(\omega_{pR} = \sqrt{\frac{q^2 n_z \omega_c}{m \pi \epsilon_0 R^2}}\). The electric field with a uniform ellipsoid is also well known
\[
E_r(r, t) = \frac{3qN}{8\pi \epsilon_0 R^2} \Gamma_1(\alpha) R 
\]
\[
= \frac{m}{q} \omega_{pR}^2 \Gamma_1(\alpha) R 
\] (33)
where \(Z\) is the length of the ellipsoidal axis along \(z\), \(\alpha = \frac{Z}{R}\), \(\omega_{p3} = \sqrt{\frac{q^2 n_z \omega_c}{m \pi \epsilon_0 N^3}}\), \(N\) is the number of particles, and \(\Gamma_1(\alpha) = \frac{\alpha - \sinh^{-1}\alpha}{\alpha^2 - 1}\). We point out a subtlety here — \(R = 2s_r\) where \(s_r\) is the transverse standard deviation, in the cylindrical model and \(R = \sqrt{3}s_c\) in the elliptical model; however, this is the standard notation, so we keep it in this work. Additionally, in the \(z\) direction, the field is given by
\[
E_z(z, t) = \frac{3qN}{4\pi \epsilon_0 R^2} \frac{\Gamma_2(\alpha) z}{\sqrt{N}} 
\]
\[
= \frac{m}{q} \omega_{p3}^{\delta} \Gamma_2(\alpha) z, 
\] (34)
where \(\Gamma_2(\alpha) = \frac{-1+\alpha \sinh^{-1}\alpha}{\alpha^2 - 1}\). We will use this to derive the evolution of \(Z\) in the cylindrical ellipsoidal model. In
In this work, we define
\[ \omega_{pR}^2 = \begin{cases} 0, & \text{non-int} \\ \frac{1}{2} \omega_{p2}^2, & \text{cylinder} \\ \frac{1}{2} \omega_{p3}^2 \Gamma_1(\alpha) & \text{ellipsoidal} \end{cases} \] (35)
for the purposes of a generic derivation of differential equations primarily used in the previous section.

A property of the plasma frequency we use is its relation to the initial plasma frequency. Specifically,
\[ \omega_{p2}^2 = \frac{q^2}{m \epsilon_0 \pi R^2} = \frac{q^2}{m \epsilon_0 \pi R_0^2} \frac{R_0^2}{R^2} = \omega_{p2,0}^2 \frac{R_0^2}{R^2} . \]
This may be trivial but will be used to simplify Eq. (31) in the next section; therefore it is worthy of note. Analogously,
\[ \omega_{p3}^2 = \omega_{p3,0}^2 \frac{R_0^2 Z_0}{R^2 Z}. \]

2. Conservation of generalized angular momentum

Motivated by relating the plasma frequency at \( t \) to the initial plasma frequency, we search for an analogous argument for replacing the time dependence within \( \omega_t \) by its initial value, \( \omega_{t,0} \). We do this argument for the 3D case in cylindrical coordinates; the argument for the cylindrical case is completely analogous ignoring the integral over \( Z \) and replacing \( N \) by \( n_z \). To get this relation, we consider the total angular momentum of this situation:
\[ \vec{L}_{\text{tot}} = \vec{L}_{\text{mech}} + \vec{L}_{\text{field}} \] (36)
where \( \vec{L}_{\text{mech}} \) is the mechanical angular momentum and \( \vec{L}_{\text{field}} \) is the field angular momentum. We need to determine both of these components.

The mechanical angular momentum is given by
\[ \vec{L}_{\text{mech}} = -\frac{m}{\epsilon} \int_{-Z}^{Z} \int_{0}^{R} \vec{r} \times \vec{J} 2\pi r drdz = mZN\omega_t R^2 \hat{z}. \] (37)

To determine the field angular momentum, we need to define a magnetic vector potential consistent with a uniform magnetic field:
\[ \vec{A}(r, t) = \frac{m}{e} (\omega_L r + h/r) \hat{\phi} \] (38)

where \( h \) depends on the initial conditions, is constant in time, and is as yet undetermined. Notice, we have neglected any magnetic field induced by the motion of the electrons — this is completely an electrostatic treatment. From this, the angular momentum of the field is given by
\[ \vec{L}_{\text{field}} = \int_{-Z}^{Z} \int_{0}^{R} \vec{r} \times \vec{A} 2\pi r drdz = \frac{-4mZ\omega_{p2}^2}{R^2} \int_{0}^{R} \left( \omega_L r + h/r \right) r^2 \hat{z} dr + mZN(\omega_L R^2 + h) \hat{z} \] (39)

This gives the total angular momentum
\[ \vec{L}_{\text{tot}} = mZN \left( (\omega_t - \omega_L) R^2 - h \right) \hat{z}. \] (41)

Requiring that this total angular momentum is conserved, we find that
\[ (\omega_t - \omega_L) R^2 = (\omega_{t,0} - \omega_L) R_0^2 \] (42)
or equivalently
\[ \omega_t = \omega_L + (\omega_{t,0} - \omega_L) R_0^2 R^2 R \] (43)

This is the equation for which we were searching. Fig. 5 shows how the oscillation of this parameter is caused by the cyclotron motion.

3. Derivation of the envelope models

Placing Eq. (42) into Eq. (31) gives the equation
\[ \vec{R} = \omega_{p2}^2 R - \omega_{t,0}^2 \frac{R_0^4}{R^3} \] non-int (44a)

Plugging in Eq. (35) and subbing the initial plasma frequencies described in Section III C 1 we get
\[ \vec{R} = \omega_{p2}^2 R - \omega_2^2 R \]
+ \( (\omega_{t,0} - \omega_L) R_0^2 R^2 R \) cylinder (45b)
\[ \vec{R} = \frac{1}{2} \omega_{p3,0}^2 \frac{R_0^2 Z_0}{RZ} \Gamma_1(\alpha) - \omega_L^2 R 
+ (\omega_{t,0} - \omega_L) R_0^2 R^2 R \) ellipsoidal (45c)

In the ellipsoidal case, the evolution of \( Z \) is also needed. This is easily obtained by
\[ \ddot{Z} = \frac{q}{m} \frac{E_z(z, t)}{Z} = \omega_{p3,0}^2 \frac{R_0^2 Z_0}{R^2} \frac{1}{R^2} \Gamma_2(\alpha). \] (45d)

D. Derivation of the solution to the non-interacting model ODE

We solve the non-interacting solution for the Ordinary Differential Equation (ODE) that we present later in Section III C 3. We present the solution first as this solution is referenced prior to the reference to the ODE’s in the main text.

Introduce the coordinate trasformation, \( u = \frac{R^2}{R_0^2} \). In terms of the envelope parameters, this transformation is
Moreover, as \( \tilde{R} \) is at its maximum radius and hence the velocity at time \( \omega \) is 0. This gives \( \ddot{u} = 2 \frac{R}{\tilde{R}_0^2} \dot{R} \) and

\[
\ddot{u} = 2 \frac{1}{\tilde{R}_0^2} (\dot{R})^2 + 2 \frac{R}{\tilde{R}_0^2} \ddot{R}
\]  
(46)

Moreover, as \( \tilde{R} = \frac{1}{2} \pi \tilde{R}_0 (\dot{R})^2 \), the second order ODE in Eq. (45a) can be integrated to give

\[
(\tilde{R})^2 = -\omega_c^2 \tilde{R}_0^2 \left( \frac{R^2}{\tilde{R}_0^2} - 1 \right) - (\omega_{r,0} - \omega_L)^2 \tilde{R}_0^2 \left( \frac{R^2}{\tilde{R}_0^2} - 1 \right)
\]  
(47)

Putting Eqs. (45a) and (47) into Eq. (46) gives

\[
\ddot{u} = -\omega_c^2 \left( u - 2 \frac{\omega_{r,0}}{\omega_c} \left( \frac{\omega_{r,0}}{\omega_c} - 1 \right) \right) - 1
\]

(48)

Introducing the second substitution \( u' = u - 2 \frac{\omega_{r,0}}{\omega_c} \left( \frac{\omega_{r,0}}{\omega_c} - 1 \right) \), we get the equation for simple harmonic motion

\[
\ddot{u}' = -\omega_c^2 u'.
\]

(49)

As we can choose our 0 time, we choose it so that the SCP is at its maximum radius and hence the velocity at time 0 is 0. This gives \( u' = A \cos(\omega_c t) \) where \( A \) represents a constant determined by the initial conditions. Using the initial condition, \( R = R_0 \) with \( R_0 \) as the maximum of \( R \) and writing this solution in terms of \( R \) we obtain

\[
\frac{R^2}{\tilde{R}_0^2} = 1 + 2 \frac{\omega_{r,0}}{\omega_c} \left( \frac{\omega_{r,0}}{\omega_c} - 1 \right) \left( 1 - \cos(\omega_c t) \right)
\]  
(50)

### E. Dia-/para- magnetic transition line in parameter space

In this section, we obtain the condition for the region within the Brillouin parabola where the cold continuous beam model predicts dia-/para- magnetic phase transition. From Eq. (13), the dia-/para- magnetic phase transition occurs at \( \frac{R_{pd}}{\tilde{R}_0} = \sqrt{1 - \frac{\omega_{r,0}}{\omega_L}} \), which is real only if \( \frac{\omega_{r,0}}{\omega_c} < \frac{1}{2} \). We are interested in the curve in parameter space that describes initial conditions that start within the Brillouin parabola and that turn around at \( R_{pd} \), i.e. they have zero velocity at \( R_{pd} \). We call this curve the PD curve for clarity. Initial conditions corresponding to the region of parameter space between the PD curve and the \( y \)-axis will have larger oscillations and will therefore experience dia-/para- magnetic phase transition.

We can obtain the dimensionless radial drift velocity, \( v_R = \frac{d\tilde{R}}{dt} \) by integrating Eq. (45b):

\[
v_R^2 = v_{R,0}^2 + \frac{\omega_{r,0}^2}{\omega_c^2} \ln \left( \frac{\tilde{R}}{R} \right) - \frac{1}{4} \left( \frac{R^2}{\tilde{R}_0^2} - 1 \right)
\]

(51)

As the initial conditions are from rest by assumption and the final velocity is also 0 at \( R_{pd} \), we obtain the constraint

\[
0 = \frac{\omega_{r,0}^2}{\omega_c^2} \ln \left( \frac{1}{\sqrt{1 - \frac{\omega_{r,0}}{\omega_L}}} + \frac{1}{4} \frac{\omega_{r,0}}{\omega_L} \right)
\]

(52)

Solving this for \( \frac{\omega_{r,0}^2}{\omega_c^2} \) we get

\[
\frac{\omega_{r,0}^2}{\omega_c^2} = \frac{1}{\ln \left( \frac{1}{1 - \frac{\omega_{r,0}}{\omega_L}} \right)} = \ln \left( \frac{1}{1 - \frac{\omega_{r,0}}{\omega_c}} \right)
\]

(53)
F. Simulation parameters

Fig. 1: We simulated (an analyzed) 10k electrons initially uniformly, spherically-symmetrically distributed within the radius $R_0 = 10 \mu$m with rotation rate $\omega_{r,0} = 0.25\omega_c$ about the axis of an external magnetic field of strength 1 T.

Fig. 4: We simulated (an analyzed) an initially spherically electron SCP with 10k electrons uniformly distributed in the radius $R_0 = 2 \mu$m starting from rest and undergoing Coulomb explosion within an external magnetic field of strength 4 T, tuned so that the plasma and cyclotron frequencies are comparable.

G. Validity domains of the theory and simulation

We must emphasize that the self-magnetic field does not appear in our derivation of Eq. (43), nor in standard models in both the accelerator field and the description of plasma traps[23]. Moreover, simulations employing electrostatics do not capture it without appropriate modification. Such treatment is valid if $B_{self} << B_{ext}$. Using this inequality, we obtain the relation

$$\frac{3}{2} \frac{N e^2 Q_1(\alpha)}{m c^2} \frac{\alpha_c}{4\pi\epsilon_0 R} w_c \ll mc^2.$$  (54)

for the entire evolution of the SCP. In this letter, we only examine situations where this condition is met. Thus, changes in the order parameter of the diamagnetic/paramagnetic phase transition are by assumption very small. A typical ratio is $\frac{B_{self}}{B_{ext}} = 3 \times 10^{-7} \frac{\omega_c}{c}$, based off the parameters published by Ahmadi et al[25]. This renders the magnetic phase transition difficult to detect.

Further, if the self-magnetic field is changing in time, as is the case during rotational breathing, it will also introduce a second-order electric field through the Maxwell-Faraday equation. If the field is not negligible, these effects should result in equilibrium distribution changes, such as hollowing of the beam, as the resulting non-linear force or the additional angular momentum would be compensated by the re-distribution of electrons.

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