Comment on: Thermal model for Adaptive Competition in a Market: Cavagna et al. [1] introduced an interesting model — called TMG as in [1] — similar to the minority game (MG), of N agents interacting in a market. Strategies of agents are represented by D dimensional vectors $\vec{R}_i$ with $i = 1, \ldots, N$ running through agents and $a = 1, \ldots, s$ through ith agent’s available choices. The strategy $\vec{R}_i$ used by i is selected drawing $a = \star$ from a Boltzmann distribution given by Eq. (4) of ref. [1] or (4) [1] for short — with “temperature” $T$ and energies $-P(\vec{R}_i^2)$ (see Eq. (3) [1]). Cavagna et al. [1] report numerical data showing an interesting collective behavior as a function of $T$ (figs. 2-1 and 3-1) and arrive at Eqs. (5,6)-[1] which are claimed to be the “exact dynamical equations for” the TMG. We show here that i) Eqs. (5,6)-[1] are incorrect ii) the correct continuum time dynamics is the same as that of the MG [4]. As a consequence the analytic solution of the MG of ref. [4] holds also for the TMG. Finally iii) the features found in [1] for $T \gg 1$ (figs. 2-3-[1]) are due to small simulation times and disappears if the system is in a steady state.

Cavagna et al. fail to define properly the continuum time limit (CTL) prescription, which is essential for stochastic differential equations such as Eq. (5)-[1]. It is crucial, in a proper derivation of the CTL, to observe that characteristic times in the TMG are proportional to $D$, as shown numerically in Fig. 1. This is natural because the adaptation of each agent’s strategy requires an optimization of all its $D$ components. This need sampling $\sim D$ values of $\vec{η}$, i.e. a time of order $D$. In order to eliminate the dependence of times on system size $N = D/d$, one has to rescale time by a factor $D$. The dynamics in the rescaled time $\tau = t/D$ is obtained iterating Eq. (3)-[1] from $t = D\tau$ to $D\tau'$

$$P(\vec{R}, \tau') - P(\vec{R}, \tau) = \frac{-d}{D(\tau' - \tau)} \sum_{t = D\tau}^{D\tau' - 1} A(t) \vec{R} \cdot \vec{η}(t). \quad (1)$$

The law of large numbers implies that, when $D = dN \to \infty$, the r.h.s. converges to $d\langle A \vec{R} \cdot \vec{η} \rangle$ where the average $\langle \ldots \rangle$ is both on the distribution $\pi^0_{\alpha}$ of $\vec{R}_i^*$ and on that of $\vec{η}$. If we then let $\tau' \to \tau$ the l.h.s. converges to the derivative $P$ of $P$ w.r.t. $\tau$. Hence, using Eq. (2)-[1] for $A(t)$ and $\langle \eta_{\alpha} \eta_{\beta} \rangle = \delta_{\alpha,\beta}/D$, Eq. (1) becomes $P = -\frac{d}{N} \sum \langle \vec{R}_i^* \rangle \cdot \vec{R}$ with $\langle \vec{R}_i^* \rangle = \sum \pi^0_{\alpha} \vec{R}_i$. The combination of Eq. (1) and Eq. (4)-[1] yields a dynamical equation for $\pi^\alpha_i$, which reads

$$\dot{\pi}^\alpha_i = -\frac{1}{NT} \pi^\alpha_i \sum_{j=1}^{N} (\langle \vec{R}_j^* \rangle - \langle \vec{R}_i^* \rangle) \cdot \left( \vec{R}_i^* - \langle \vec{R}_i^* \rangle \right). \quad (2)$$

Eq. (2) coincides with the continuum time equation of ref. [1] which leads to the exact solution of the MG for $N \to \infty$. This depends only on the first two moments of the distribution of components of $\vec{R}_i$, which plays the role of quenched disorder. Since, in the TMG, $\langle (\vec{R}_i^*)^2 \rangle = D$, these are the same as in the MG. Hence the two models have exactly the same collective behavior, as confirmed by Fig. 1.

Eq. (2) suggests that the dependence on $T$ disappears by time rescaling. This is true in the $d \geq d_c$ phase: The $T$ dependence for $T \gg 1$ reported in Figs 2,3-[1] is an artifact due to short simulation times (see also ref. [1]). For $d < d_c$ the CTL only holds for $T$ larger than a crossover $T_c(d)$, as discussed elsewhere [2]. Indeed for $d = 0.1 < d_c$ and $T$ large enough, data nicely collapses onto a single curve (see inset) once plotted against $T/\tau$. For $T < T_c(d)$ the solution of Eq. (2) becomes dynamically unstable and the system enters into a turbulent regime where the CTL breaks down [1].

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