From Quantum State Targeting to Bell Inequalities

H. Bechmann-Pasquinucci
UCCLIT, via Olmo 26, I-23888 Rovagnate, Italy

25 February, 2004

Dedicated to Asher Peres on his 70th birthday

Abstract

Quantum state targeting is a quantum game which results from combining traditional quantum state estimation with additional classical information. We consider a particular version of the game and show how it can be played with maximally entangled states. The optimal solution of the game is used to derive a Bell inequality for two entangled qutrits. We argue that the nice properties of the inequality are direct consequences of the method of construction.

1 Introduction

Many areas of quantum mechanics have received renewed interest in the framework of quantum information theory. Here we will consider two aspects: quantum state estimation [1, 2] and Bell inequalities [3, 4]. These areas have been given completely new meaning in the light of quantum information theory.

Traditionally quantum state estimation is considered in the following way: A system is prepared in one of a known set of states and is given to an estimator. The task of the estimator is to identify as best as possible the state of the system and make an announcement about the state. Already here there are choices: the estimator could decide that he wants to make the best guess on the identity of every system submitted to him, in which case he will optimize his procedure to give him the maximum
probability for guessing correctly the state — and sometimes making an error \[1,2\]. But it could also be that it is not acceptable to make any errors, in which case the estimator can optimize his procedure such that when he identifies the state it is with certainty, by paying the price of sometimes obtaining an inconclusive answer \[2,5\].

With the development of quantum information theory, quantum state estimation has in many cases been given a twist. It is no longer simply a question of identifying a particular state out of a set of states, but there might be additional classical information available after the interaction with the ‘unknown’ quantum state or even after a measurement has actually been performed. For example, in the BB84 protocol \[6\] for quantum cryptography \[7]-[11]\ the eavesdropper knows that the quantum system is prepared with equal probability in a state belonging to a set of states made by two mutually unbiased bases. But she also knows that after her eavesdropping, i.e. after the interaction with the ‘unknown’ quantum state, she will learn in which of the two bases the system was originally prepared. This means that the eavesdropper knows that she will later receive additional classical information, which she can use to gain more information about the initial state of the system.

This example with quantum state estimation in connection with quantum cryptography, also shows how quantum state estimation in the field of quantum information often becomes only a part of a bigger picture. And naturally this also means that different aspects are added to the subject, for example in eavesdropping in quantum cryptography it is no longer just a question of making the best identification of the ‘unknown’ quantum system — it should also be done causing minimum disturbance! This is the problem which lies at the center of quantum cryptography, namely that any interaction with the ‘unknown’ state which will lead to a higher probability of identifying it correctly will automatically lead to a disturbance of the state. This is what makes quantum cryptography safe for the legitimate users, since any eavesdropping attempt can be detected.

One can imagine different ways to combine the elements of quantum state estimation and additional classical information: recently it was presented in a new form called ‘quantum state targeting’ \[12\]. Briefly described, quantum state targeting works as follows: There exist a set of target states which is known to both players, called Alice and Bob. The task of Alice is to prepare a quantum system and submit it to Bob, after that Bob will reveal the target state he has chosen. After receiving this information Alice makes an announcement concerning the system she prepared and finally Bob performs a fail/pass test on the quantum
system in accordance with Alice's statement.

Clearly, Alice uses her complete knowledge of the set of target states to prepare the quantum system she sends to Bob. However, there are many strategies she could adopt to prepare the system and her choice depends on what she really wants to achieve. She may just optimize her control, which means the probability that her quantum system will pass the test for being the target state chosen by Bob. She could also be interested in optimizing the control-disturbance trade-off. And there is also the possibility that she is given the option to decline to make an announcement.

It is immediately clear that there can be many different setups and situations both involving pure and mixed states, several of which have already received attention [12]. Here we are only interested in the particular situation where the set of target states corresponds to two non-orthogonal pure states, Alice goes for maximum control but at the same time she has the option to decline to make an announcement.

It is possible to view quantum state targeting as a game by itself, but it can also be incorporated into other games like for example weak coin flipping [12] or it can be related with different parts of quantum mechanics. Recently there has been a big interest in investigating Bell inequalities in higher dimensions [13, 14]. Here we will show how playing the game of quantum state targeting using maximally entangled states can lead to one of the generalized Bell inequality for ququits which was recently presented [13, 14].

Bell inequalities is indeed another area which has received renewed attention in the last years. Bell inequalities used to belong to the discussion of the completeness of quantum theory, and entanglement was viewed as a puzzle or even a problem to try to get rid off and not as a fantastic resource waiting to be explored [19]. But that changed dramatically with the birth of quantum information theory. Suddenly entanglement had to be explored, characterized and application had to be discovered, and in this process Bell inequalities too, became useful tools for example as a security measure in quantum cryptography [20, 14, 21] or identification of useful correlations between quantum systems [22].

The paper is organized as follows; in section two quantum state targeting is defined, and a special situation is investigated. In sec. 3 we show how to play the game of quantum state targeting by using maximally entangled states. In sec. 4 we establish a measure of how well Alice is playing the game of quantum state targeting and show that what we have obtained is actually a Bell inequality. Section 5 is devoted to a discussion of the obtained inequality and we present some intuitive
arguments about it’s optimality. Section 6 is left for concluding remarks.

2 Quantum state targeting

The concept of quantum state targeting was recently introduced [12], it is a particular way of combining quantum state estimation with additional classical information. In this section we review the basic idea of quantum state targeting and the results which are needed later. The rules are quite simple and they are here given as they were originally defined: First Alice and Bob decide on a set of target states, which means that the set of target states is known to both of them. After this initial setup, the game goes in the following steps:

(1) Alice submits a system to Bob
(2) Alice learn the identity of the target state (Bob reveal which of the possible target states he has chosen)
(3) Alice announces a state to Bob (a state from the set of target states, but not necessarily the target state chosen by Bob)
(4) Bob performs a pass/fail test for the state announced by Alice

the possible outcomes of this game are the following:

(A) Alice announces the target state and passes Bob’s test
(B) Alice announces the target state and fails Bob’s test
(C) Alice announces a non-target state and passes Bob’s test
(D) Alice announces a non-target state and fails Bob’s test

Here we are interested in a very simple situation, namely where the set of target states corresponds to two pure non-orthogonal states. In this case the target state is chosen uniformly from a pair of two (non-orthogonal) pure states, denoted $| \psi_1 \rangle$ and $| \psi_2 \rangle$. Which means that the situation is the following; Alice prepares a quantum state and sends it to Bob, who will then tell Alice if he has chosen target state $| \psi_1 \rangle$ or $| \psi_2 \rangle$. Alice will then make an announcement about the system that she prepared and Bob will finally perform a pass/fail test according to Alice’s statement.

Alice, of course, knows that Bob is going to chose either $| \psi_1 \rangle$ or $| \psi_2 \rangle$ as target state, which means that she can take this information into account when she prepares her quantum state. There are several ways of doing this, depending on what Alice wants to achieve. The situation of
interest here, is where Alice goes for maximum control, which means that she wants to optimize the probability that Bob tests and actually finds the target state. This corresponds to situation (A)-(B): Alice always announces the target state.

In this situation what Alice has to do is to prepare her system in such a way, that the probability that the system will pass the test for the target state is maximal and independent of which target state Bob chose. In [12] this particular situation was considered. It turns out that in order for Alice to have maximum control she has to submit a state which surprisingly enough is a pure state. Moreover, this pure state corresponds to what in connection with quantum cryptography [6]-[11] and Bell inequalities [13, 14] usually is denoted the intermediate state $|\psi_{int}\rangle$. The intermediate state, $|\psi_{int}\rangle$, is precisely what the name indicates, namely the state which lies exactly between the two states $|\psi_1\rangle$ and $|\psi_2\rangle$, it can be defined as follows:

$$|\psi_{int}\rangle = \frac{1}{\sqrt{N}}(|\psi_1\rangle + e^{-i\phi}|\psi_2\rangle)$$ (1)

where $N = 2(1 + |\langle\psi_1|\psi_2\rangle|)$ is the normalization constant and the phase comes from the overlap $\langle\psi_1|\psi_2\rangle = e^{i\phi}|\langle\psi_1|\psi_2\rangle|$, and assures that when computing the overlap between the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ and the intermediate state both terms have the same phase.

By submitting the intermediate state Alice’s control is maximized and identical for the two target states and it is given by

$$C_{\text{max}} = P(\psi_i|\psi_{int}) = \frac{1}{2}(1 + |\langle\psi_1|\psi_2\rangle|) \quad \text{for} \quad i = 1, 2$$ (2)

which naturally corresponds to the probability that Bob will find the target state.

Notice that this result is independent of the dimension of the quantum system.
Quantum State Targeting with maximally entangled states

Quantum state targeting as defined in the previous section can either be considered a game by itself or it can be incorporated into other games, for example weak coin flipping \[12\]. Here we are interested in making the connection between quantum state targeting and Bell inequalities, in order to show that a Bell inequality at least in some cases can be derived as the result of playing a quantum game. This means that the next step is to introduce the concept of entanglement in connection with quantum state targeting, since until now the game has been played on single systems. In this section we show how the results from the previous section can be adopted in the case where Alice and Bob share maximally entangled states.

Suppose that Alice and Bob share many maximally entangled pairs of qutrits,

\[
| \psi_t \rangle = \frac{1}{\sqrt{3}} (|a_0, a_0 \rangle + |a_1, a_1 \rangle + |a_2, a_2 \rangle)
\]

here the maximally entangled state is described in two different bases, the \(A\) basis which is the computational basis and the \(A'\) basis which is the Fourier transformed:

\[
| a'_l \rangle = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \exp \left( \frac{2\pi i kl}{3} \right) | a_k \rangle.
\]

Notice that the two bases are mutually unbiased, i.e. \( \langle a_k | a'_l \rangle = \frac{\exp(2\pi i kl)}{\sqrt{3}} \), for all \( k, l = 0, 1, 2 \).

It is now possible to setup a quantum state targeting game similar to the one described in the previous section, but with the twist that Alice and Bob share maximally entangled states. One round of the game corresponds to using one of the maximally entangled state, hence Alice and Bob can play many times since they share many maximally entangled states.

In each round of the game i.e. for each qutrit pair, Alice and Bob first decide on the set of the two target states (corresponding to \( | \psi_1 \rangle \) and \( | \psi_2 \rangle \)). The target states are drawn from two sets of states corresponding to the basis states of the two bases \( A \) and \( A' \); They always selects a state...
Table 1: The intermediate states formed by the basis states from $A$ and $A'$

|   | $a_0'$ | $a_1'$ | $a_2'$ |
|---|--------|--------|--------|
| $a_0$ | $m_{00}$ | $m_{01}$ | $m_{02}$ |
| $a_1$ | $m_{10}$ | $m_{11}$ | $m_{12}$ |
| $a_2$ | $m_{20}$ | $m_{21}$ | $m_{22}$ |

from each basis, so that the set of target states corresponds to two non-orthogonal states. Notice that the two states $|a_k\rangle$ and $|a'_l\rangle$ are chosen at random and with equal probability and that there in total can be formed nine different sets of target states.

### 3.1 Alice’s measurement

We are now only at the beginning for the game of quantum state targeting, namely where the set of target states are known and Alice has to prepare a quantum system to submit to Bob. Here the difference is that Alice has to prepare the system by performing a measurement on one part of a maximally entangled state. This corresponds to step number (1) in the definition of state targeting.

If Alice want to optimize her control (this means that she is going for outcome (A)) we already know that she has to perform a measurement such that Bob’s qutrit is left in the state which corresponds to the intermediate state between the two target states. In table 1 the possible intermediate states are shown. The state $m_{kl}$ corresponds to the intermediate state between $|a_k\rangle$ and $|a'_l\rangle$, and with the first index always referring to the $A$ and the second to the $A'$-basis, and the general expression for the intermediate state $|m_{kl}\rangle$ is:

$$|m_{kl}\rangle = \frac{1}{\sqrt{N}} \left[ |a_k\rangle + \exp\left(-\frac{2\pi i \cdot kl}{3}\right) |a'_l\rangle \right]$$

(5)

where $N = 2(1 + 1/\sqrt{3})$. It should be mentioned that the intermediate states in general not are orthogonal.

A measurement on one part of an entangled pair can be considered a sophisticated way of making state preparation.

The question is what kind of measurement Alice has to perform on her qutrit in order to try to prepare Bob’s qutrit in the desired state. It turns out that she has to perform a binary measurement corresponding to a measurement of the intermediate states, related to the set of target states.
For example if the set of target states is $|a_0\rangle$ and $|a'_0\rangle$, we know that Alice wants to prepare Bob’s qutrit in the intermediate state $|m_{00}\rangle$. In order to do this Alice performs the binary measurement corresponding to a measurement of $|m_{00}\rangle$ on her own qutrit. A binary measurement is a ‘yes’/’no’ test, and in this case the question asked by Alice is ‘is the state of my qutrit $|m_{00}\rangle$?’. The question is asked by measuring $P_{00} = |m_{00}\rangle\langle m_{00}|$ and $P_{00}^c = 1 - P_{00}$. Each of the two possible outcomes appears with a certain probability, but the important thing is that when the answer to her test is ‘yes’, i.e. she finds the outcome $P_{00}$, then she knows that she has managed to prepare Bob’s qutrit in the desired state. Assuming that Alice measures on the first qutrit, the state of the system after the measurement, is (omitting normalization, when not needed),

$$P_{00} |\psi_t\rangle = |m_{00}\rangle\langle m_{00}| \left( \frac{1}{\sqrt{3}} (|a_0, a_0\rangle + |a_1, a_1\rangle + |a_2, a_2\rangle) \right) \propto |m_{00}\rangle \langle m_{00}|$$

Similarly can be shown for the other choices of sets of target states.

However, notice the form of the maximally entangled state when written in the two bases $A$ and $A'$. In the $A$ basis there is complete agreement between the states obtained by Alice and Bob if they both measured this basis, for example if Alice finds the state $|a_1\rangle$, Bob too will find the state $|a_1\rangle$. Whereas if they both measured the $A'$ basis and Alice obtains $|a'_1\rangle$, then Bob will obtain $|a'_2\rangle$. It is important to understand that this does not give rise to any problems, since the important point is that Alice and Bob in both bases are perfectly correlated. It only means that Alice has to keep this in mind when she choose her measurement; for example when Bob chooses the set of target states to $|a_1\rangle$ and $|a'_2\rangle$, Alice knows that for her $|a'_2\rangle$ corresponds to $|a'_1\rangle$ due to the way they are correlated in the $A'$ basis.

### 3.2 Bob’s choice of target state, Alice’s announcement and the final test

At this point in the game Alice has performed a measurement on her qutrit in order to try to prepare Bob’s qutrit in the intermediates state of the two target states, in this way Alice is optimizing her control. This corresponds to step (1), but with Alice determined to go for outcome (A).
Now Bob announces to Alice which of the two states is the target state of this round of the game, which means that Bob announce either |a_k⟩ or |a'_l⟩, this corresponds to step (2) where Alice learns the identity of the target state. Step (3) is an announcement from Alice and depending on the outcome of her measurement Alice will do one of two things: If she succeeded in her preparation so that Bob holds the state |m_{kl}⟩, then she will announce the same state as Bob, i.e. the target state, whereas if she has failed in making the desired preparation, then she will decline in making an announcement. Notice that Alice’s decline option is completely independent of Bob’s choice of target state, actually as soon as Alice has performed her measurement she knows if she will make an announcement or if she will decline\(^1\).

The cases of interest are of course when Alice has succeeded in making the right preparation and makes an announcement. Since we are considering the case where Alice is going for optimal control and hence is going for outcome (A), it means that she is always announcing the target state. This means that if Bob has chosen |a_k⟩, Alice will announce that she prepared the state |a_k⟩, whereas if Bob has chosen the state |a'_l⟩, Alice will claim that she prepared the state |a'_l⟩.

The final step (step (4)) in the game of quantum state targeting is for Bob to make a test of Alice’s announcement regarding the state. One way that he can make the test is by measuring either A or A' depending on the chosen target state. Since Alice in this scenario always announces the target state, it means that if the target state was |a_k⟩ then Bob will measure A. Naturally since the state of Bob’s qutrit is not |a_k⟩, but |m_{kl}⟩, there is a certain probability that Bob will actually find state |a_k⟩ and hence confirm Alice’s announcement, but there is also a certain probability that Bob will obtain one of the other basis states for A, which means that Alice has failed Bob’s test. Similar for the any other target state.

The probability that Alice’s announcement will pass Bob’s test is \(p(s) = \frac{1}{2} + \frac{1}{2\sqrt{3}}\), whereas the total probability that her announcement fails his test is \(p(f) = \frac{1}{2} - \frac{1}{2\sqrt{3}}\). Just as in the version of quantum state targeting using single particles, the only difference is that Alice has the option to decline making a statement.

\(^1\)In [12] there is also described a situation where Alice has the option to decline, but in that case it is dependent on Bob’s choice of target state.
4 From quantum state targeting to a Bell inequality for qutrits

4.1 The quantum mechanical limit

In the previous section it was shown how quantum state targeting can be played with the use of maximally entangled states, when considering the particular situation where Alice is going for maximum control but at the same time is given the option to decline to make an announcement.

The measurements which would give the maximum control to Alice was found: she performs a binary measurement given by the intermediate state corresponding to the set of target states. Bob for his part is just performing a standard von Neumann measurement corresponding to one of the two mutually unbiased bases $A$ or $A'$.

As already mentioned there exist nine different sets of target states and to each set of target states corresponds one particular binary measurement for Alice, namely a measurement of the intermediate state between the two possible target states. Alice can either succeed or fail in achieving what she wants by her measurement. From hereon we only consider the cases where Alice succeeds and makes an announcement, in other words when she managed to perform the desired 'preparation' of Bob's state.

Bob, on the other hand, has two possible measurements $A$ and $A'$, which one he performs depends on his choice of target state. In either case there are three possible outcomes of the measurement; with a certain probability Bob will find the target state and confirm Alice’s announcement, but there is also a certain probability that he will find one of the other two basis states and hence fail to confirm Alice’s announcement. In total this gives $9 \times 2 \times 3 = 54$ different probabilities.

Suppose that Alice and Bob share many maximally entangled states and they play the game of quantum state targeting many times, then we can ask the question: 'How well is Alice playing this game?’ Here we will measure how well Alice is doing by summing all the probabilities when Bob confirms Alice’s announcement and from this sum subtract all the probabilities when Alice’s announcement fails Bob’s test, in other words

\[
B_3 = \sum p(Alice \ passes \ Bob's \ test) - \sum p(Alice \ fails \ Bob's \ test)
\]

\[
= \sum p(s) - \sum p(f)
\]

It is by ordering all the involved probabilities in this particular way that we can obtain a Bell inequality for two entangled qutrits. In order
Table 2: Values assigned to the involved states

| value | \( A \) | \( A' \) | \( M_0 \) | \( M_1 \) | \( M_2 \) |
|-------|-------|-------|-------|-------|-------|
| \( 0 \) | \( |a_0\rangle \) | \( |a'_0\rangle \) | \( |m_{00}\rangle \) | \( |m_{01}\rangle \) | \( |m_{02}\rangle \) |
| \( 1 \) | \( |a_1\rangle \) | \( |a'_1\rangle \) | \( |m_{11}\rangle \) | \( |m_{12}\rangle \) | \( |m_{10}\rangle \) |
| \( 2 \) | \( |a_2\rangle \) | \( |a'_2\rangle \) | \( |m_{22}\rangle \) | \( |m_{20}\rangle \) | \( |m_{21}\rangle \) |

to write down the Bell inequality it is, however, convenient to assign values to the various states and organize them in different sets, since this simplifies the notation.

In table 2, it is shown the value which is assign to the various states. Notice that the value for the \( A \) and \( A' \) basis states is given by their index. The intermediate states has been organized into three different sets so that the value of the states is always given by the first index (i.e. corresponding to the value of the state from the \( A \) basis). The intermediate states are not orthogonal and the sets \( M_0 \), \( M_1 \) and \( M_2 \) do therefore not correspond to orthogonal bases; the states are only organized in sets in order to simplify the notation.

In the following we only consider the cases where Alice obtains a useful result from her measurement and hence makes an announcement. Suppose that the set of target states was chosen such that Alice has measured one of the projectors in the first set, \( M_0 \), and Bob following makes a measurement in the \( A \) basis, for this combination of measurements, there are the following contributions to the sum \( B_3 \):

\[
P(A = M_0) = \sum_{i=0}^{2} p(a_i \cap m_{ii}) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tag{7}
\]

\[
P(A \neq M_0) = \sum_{i,j=0,j \neq i}^{2} p(a_j \cap m_{ii}) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \tag{8}
\]

where \( P(A = M_0) \) should be read as follows: after Alice’s announcement, Bob measures \( A \) and finds the correct state and hence he can confirm Alice’s statement. Contrary \( P(A \neq M_0) \) means that Bob finds the wrong state and fails to confirm Alice’s announcement. The probability \( p(a_i \cap m_{ii}) \) is the joint probability for obtaining both \( m_{ii} \) and \( a_i \).

Similar for any of the other measurements that Alice can perform as long as Bob always chooses to measure in the \( A \) basis after Alice’s announcement. The same is also the case if the choice of target states is such that Alice will measure one of the projectors from the \( M_0 \) set and Bob subsequently measures in the \( A' \) basis. In total this leads to
the following contributions: \( P(A = M_i) = P(A' = M_0) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \), and \( P(A \neq M_i) = P(A' \neq M_0) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \).

The situation changes when Alice has measured one of the projectors from \( M_1 \) or \( M_2 \) and Bob subsequently measures \( A' \). This is due to the fact that in the \( A' \) basis Alice and Bob are correlated in a different way, as can be seen in eq. (3). However this point has already been discussed (see Sec. 3.1) and does not give rise to any problems — it just has to be taken into account. Consider the case where Alice has measured one of the projectors from \( M_1 \) and Bob measured in \( A' \), then the state that Alice possesses will consistently have a value which is 2 (mod 3) higher than the value of Bob’s state. To see this, assume for example that the set of target states were \( |a_2⟩ \) and \( |a_0'⟩ \), this means that in the case that Alice has succeeded in her projection, she has the state \( |m_{20}⟩ \) which has been assigned the value 2. At the same time, if Bob has chosen target state \( |a_0'⟩ \) and his qutrit passes the test for being in that state, Bob will find a state which has been assigned the value 0. Similarly for the other combinations, which leads to \( P(M_1 = A' + 2) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \) and \( P(M_1 \neq A' + 2) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \). Whereas in the situation that Alice uses one of the projectors from \( M_2 \) and Bob subsequently measures in \( A' \), Alice will consequently find a value which is 1 higher than the value which correlates her with Bob, i.e. \( P(M_2 = A' + 1) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \) and \( P(M_2 \neq A' + 1) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \).

Now all the probabilities can be written down in a nice and compact way, and it is easy to evaluate the total sum, based on the the specific set of measurements used above:

\[
B_3 = 2 \sum_{i=0}^{2} P(M_i = A) - 2 \sum_{i=0}^{2} P(M_i \neq A) \\
+ 2 \sum_{i=0}^{2} P(M_i = A' + (3 - i)) - 2 \sum_{i=0}^{2} P(M_i \neq A' + (3 - i)) \\
= 2 \times 3 \left( \left( \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) + \left( \frac{1}{2} - \frac{1}{2\sqrt{3}} \right) \right) \\
= 2\sqrt{3} \tag{9}
\]

### 4.2 The local variable limit

In order to show that the sum of probabilities \( B_3 \), is actually a Bell inequality it is necessary to consider what happens when a local variable model tries to attribute definite values to the observables use in the game.
Alice performs binary measurements where each $m_{kl}$ is measured independently, and hence in a local variable model they could all be true simultaneously. Bob on the other hand performs standard von Neumann measurements where $a_0$, $a_1$ and $a_2$ are measured simultaneously in a single measurement $A$, which means that only one of them can come out true in a local variable model. The same is the case for $a'_0$, $a'_1$ and $a'_2$ which are measured simultaneously in a single measurement $A'$. This means that if $a_i$ is true, meaning a measurement of $A$ will result in the outcome $a_i$, then all probabilities involving $a_j$ with $j \neq i$ must be zero.

Suppose that in a specific local variable model $a_i$ and $a'_j$ are true, in principle all the $m_{kl}$ can be true at the same time. What we need to investigate is what are the contributions from the various $m$-states. First notice that the only $m$-state which gives a positive contribution to $B_3$ is the one which identifies both $a_i$ and $a'_j$ correctly, in other words $m_{ij}$. This will give a contribution of $+2$. Whereas $m_{kj}$ and $m_{il}$, where only one index is correct will identify only one state correctly and the other one wrong; which in total will give a zero contribution to $B_3$. Instead the case where both indexes are wrong, and hence both states will be wrongly identified will give a negative contribution of $-2$ to $B_3$.

This means that the total maximum of the sum $B_3$ according to a local variable model is

$$B_3 \leq 2$$

(10)

5 Quantum state targeting leads to maximal violation

In the previous section we saw that an attempt to try to assign definite values to the observables will lead to $B_3 \leq 2$. But we have also seen (Sec. 4.1) that with the measurement settings proposed above and using the maximally entangled state, it is possible quantum mechanically to obtain $2\sqrt{3}$, which is obviously higher than 2. This means that we have obtained a Bell inequality and moreover shown that it can be violated with a $\sqrt{3}$.

However some important issues still need to be addressed; What is the maximal violation? For which state is it maximally violated? and What are the optimal measurement settings for the above inequality? These questions have received numerical attention [13, 14]; first of all it has been checked that $2\sqrt{3}$ indeed is the quantum mechanical limit to the above sum of probabilities and that this maximum is reached for the
maximally entangled state. Moreover, it has also been checked using "polytope software" \cite{24} that the inequality \( B_3 \leq 2 \) is optimal for the measurement settings presented.

We will try to give some intuitive arguments why this inequality possesses these properties. The inequality \( B_3 \), was originally developed to mimic as closely as possible the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality \cite{25} for qubits. This means that there were some very deliberate choices regarding for example the structure. Very often the CHSH inequality is presented in terms of the correlation coefficients, which basically is the sum of the probabilities for the results of the measurements being correctly correlated while subtracting the probabilities that the results are not correctly correlated. Hence the choice of the structure of the inequality can be considered a generalization of the CHSH inequality. And indeed by playing the game of quantum state targeting with qubits as it has been played above with qutrits, will indeed lead to the CHSH inequality.

Now the question is why the above inequality is actually optimal for the maximally entangled state and this odd choice of measurements, with two mutually unbiased basis on one side and nine binary measurements on the other! The answer at least intuitively lies in the interplay between this state and these measurement settings. The choice of two mutually unbiased bases as the possible measurement setting on one side means that there is no privileged state. Furthermore this symmetry is also preserved by the maximally entangled state. Indeed, there exists another Bell inequality for two entangled qutrits \cite{18}, which breaks the symmetry in the choice of bases and the maximal violation is reached for a pure but non-maximally entangled state.

Having chosen the measurement settings on one side, i.e. the two mutually unbiased bases \( A \) and \( A' \) and the maximally entangled state, we need to analyze what is the situation on the other side. It has already been discussed that we are trying to build a Bell inequality with a particular structure, which we can write as

\[
B_3 = \sum p(\text{results correlated}) - \sum p(\text{results not correlated})
\]

It is therefore clear that measurement setting on the other side has to correspond to the measurements which optimize the probability for Alice’s and Bob’s measurement results to be correlated — independently of whether the basis \( A \) or \( A' \) is being measured. Measuring the intermediate states indeed optimize the probability of being correlated, or correctly identifying the state as was discussed in section 2 about quantum states targeting.
Figure 2: The optimal measurement settings for the CHSH-inequality in two dimensions, drawn on the Block-sphere, but labeled according to the notation used for intermediate states. Remember that vectors pointing in opposite direction are orthogonal.

One very interesting point should be mentioned. In two dimensions, which means for qubits, the intermediate states actually turns out to be pairwise orthogonal and they therefore constitutes two bases. Even more the two intermediate bases are also mutually unbiased. This means that in two dimensions the Bell inequality which is obtained by performing the game of quantum state targeting, corresponds exactly to the CHSH-inequality with the optimal measurement settings.

In three and more dimensions, the situation becomes much more complicated, this is due to the fact that the intermediate states in general are not orthogonal. However, this problem is overcome by considering the intermediate states as binary measurements. But notice one interesting point, if the intermediate states would have formed orthogonal bases, then in three dimensions we would have had that when generalizing the CHSH inequality in this way, it would have resulted in two measurement settings on one side and three on the other!
6 Conclusions

In recent years, with the field of quantum information rapidly expanding, many aspects of quantum mechanics have received renewed interest. Here we have considered two things which a priory seem unrelated, namely the game of quantum state targeting and Bell inequalities.

Quantum state targeting is a particular way of combining quantum state estimation and additional classical information to obtain an interesting quantum game. Here we have taken the starting point that the players share many maximally entangled states of two qutrits, and that the two possible target states in each round is drawn at random from two mutually unbiased bases, $A$ and $A'$. Alice in this case 'submits' her quantum system by performing a measurement on her part of the maximally entangled state, since this can be viewed as state preparation for Bob’s part.

In the case that Alice goes for maximum control, her measurement corresponds to measuring the intermediate state between the target states and following Bob’s announcement of his choice of target state she announces the same state. This means that Bob will measure either $A$ or $A'$, depending on whether he has chosen the target state from $A$ or $A'$. Alice’s measurement of the intermediate state corresponds to a binary measurement, in the case where Alice obtains a positive result of her measurement she will continue to play the game, whereas in the case where she obtains a negative result she will decline to make an announcement of the state. It has been proven that for Alice to submit what corresponds to the intermediate state between the target states is what gives her the optimal control.

Following this we considered only the cases where Alice succeeds in the correct state preparation. Considering all the different measurement combinations, i.e. all possible sets of target states and all possible choices of target states, we formed a measure of how well Alice is in playing this game, by summing over all the probabilities that she will pass Bob’s test and from this subtract all the probabilities for failing his test. Assuming that Alice and Bob are playing by using the maximally entangled state and that Alice is measuring on of the nine possible intermediate states in a binary measurement and that Bob measures one of the two mutually unbiased bases $A$ or $A'$, we find that the sum is equal to $2\sqrt{3}$.

Interestingly it turns out the formed sum of probabilities actually is a Bell inequality, since a local variable model would for the involved sum of probabilities have a classical limit of 2. Moreover it turns out that the limit we have obtained as a result of using quantum state targeting
is actually the maximum violation, even more the maximum violation is reached for the maximally entangled state and for the specific measurement setting used in quantum state targeting.

We have argued that the optimality of the obtained inequality arise from the interplay between the chosen structure of the inequality, the symmetry in using mutually unbiased bases together with the maximally entangled state and finally the intermediate states which optimize the probability of guessing correctly the state independently of the chosen basis. However, it should be stressed that these intuitive arguments are fully supported by numerical evidence.

Here we have played the game of quantum state targeting in three dimensions, and it leads to a Bell inequality for two entangled qutrits. However the same game could be played in any dimension $d$, the measurement settings in this case would be two mutually unbiased bases on one side and $d^2$ binary measurements on the other, and it would again lead to an optimal inequality where the classical limit is 2, but the quantum mechanical limit is $2\sqrt{d}$ \cite{13,14}. In other words we have described a way of obtaining a Bell inequality in arbitrary dimensions which has a violation which increases as $\sqrt{d}$. The obtained Bell inequality can be considered a generalization of the well-known Clauser-Horne-Shimony-Holt inequality, and indeed playing the game of quantum state targeting in two dimensions leads to the CHSH inequality with its optimal settings.

Here we have played a simple quantum game and used its optimal solution to arrive at a Bell inequality which in itself is optimal, at least in some respects. Notice that this is somehow backwards with respect to the usual presentation of Bell inequalities. Here we start from the quantum state, the measurement settings and a specific structure for the sum of probabilities and only afterward show that the classical limit is lower than the value which was obtained using quantum mechanics. We believe this leads to a whole new way of thinking about Bell inequalities and that in the future this will prove to be a powerful method to construct Bell inequalities.

**Acknowledgments**

On this special occasion I would like to take the opportunity to thank Asher Peres for all his suggestions and many interesting discussions over the past years.
References

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[2] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1995).

[3] J. S. Bell, *Physics* 1 (1964) 195

[4] A. Peres, *Found. Phys* 28 (1999) 589

[5] A. Peres, *Phys. Lett. A* (1988) 19

[6] C. H. Bennett and G. Brassard, in Proceedings on the IEEE International Conference on Computer, Systems, and Signal Processing, Bangalore, India (IEEE, New York, 1984), 175-179

[7] D. Bruss, *Phys. Rev. Lett* 81 (1998) 3018

[8] H. Bechmann-Pasquinucci and N. Gisin, *Phys. Rev. A* 59 (1999) 4238

[9] H. Bechmann-Pasquinucci and W. Tittel, *Phys. Rev. A* 61 (2000) 062308

[10] H. Bechmann-Pasquinucci and A. Peres, *Phys. Rev. Lett* 85 (2000) 3313

[11] N. Cerf, M. Bourennane, A. Karlsson and N. Gisin, *Phys. Rev. Lett.* 88 (2002) 127902

[12] T. Rudolph and R. W. Spekkens, *arXiv:quant-ph/0310060*

[13] H. Bechmann-Pasquinucci and N. Gisin, *QIC* 3 (2003) 157-164

[14] H. Bechmann-Pasquinucci and N. Gisin, *Phys. Rev. A* 67 (2003) 062310-062319

[15] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski and A. Zeilinger, *Phys. Rev. Lett.* 85 (2000) 4418

[16] T. Durt, D. Kaszlikowski and M. Zukowski, *arXiv:quant-ph/0101084*

[17] D. Kaszlikowski, L. C. Kwek, J.-L. Chen and M. Zukowski, *arXiv:quant-ph/0106016*

[18] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, *Phys. Rev. Lett.* 88 (2002) 040404

[19] For a good collection of the famous papers, see *Quantum Theory and Measurements* Edited by J. A. Wheeler and W. H Zurek, Princeton University Press (1983)
[20] C. A. Fuchs, N. Gisin, R. B. Griffiths, C.-S. Niu and A. Peres Phys. Rev. A 56 (1997) 1163

[21] A. Acin, N. Gisin, L. Masanes arXiv:quant-ph/0310166

[22] A. Acin, V. Scarani and M. M. Wolf Phys. Rev. A 66 (2002) 042323, idem J. Phys. A:Gen 36 (2003) L21

[23] See for example, C. Bennett, F. Bessette, G. Brassard, L. Salvain and J. Smolin, J. Cryptology (1992) 5:3-28

[24] R. M. Basoalto and I. Percival, arXiv:quant-ph/0012024

[25] J. F. Clauser, M. A. Horne, A Shimony and R. A. Holt, Phys. Rev. Lett 23 (1969) 880