Global Existence of Maxwell Klein Gordon Theory with General Couplings

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Abstract. In this paper, we prove the global existence of Maxwell Klein Gordon equation (MKG) with the addition of general couplings in 4-dimensional Minkowski space. It is well known that the MKG system is globally well-posed in consideration of the Coulomb gauge condition for finite energy, while here we use the temporal gauge condition to prove its existence. The result shows that the solution does not blow up in finite time. For complete proof of global existence, we show that \((H_2 \times H_1)\) norm in the form of "modification of energy" is not also blow up in finite time.

1. Introduction

For the past 30 years, the global existence of Maxwell Klein Gordon system has been intensively studied. It is well known that the solution is globally existed for finite-energy initial data in consideration of Coulomb gauge condition \([1]\). As we know that the MKG system satisfied the gauge transformation so we have freedom to choose a gauge condition for an appropriate case. In \([2]\), the MKG system is globally well-posed in Lorentz gauge conditions. It has been shown that the Lorentz gauge conditions have advantages compared to the Coulombs gauge conditions because of its Lorentz invariance and have more symmetric form. Other gauge conditions that are worth to deal with is the temporal gauge condition. The temporal gauge condition is mostly considered to prove the local and global solution of Yang-Mills Higgs equation \([3, 4]\).

For the sake of generality, other factors that need to be determined also is the global existence in the presence of general couplings between electromagnetic field and complex scalar field. It is important if we want to consider in the case of supersymmetry. For example, in \([5]\) the local existence is obtained for Bosonic part of 4\(d\) \(N = 1\) supersymmetric gauge theory with general couplings. Whereas in the global existence case has not been determined.

Inspired by the result above, in this paper we prove the global existence of a MKG system with the addition of general couplings in temporal gauge condition using the method as in \([4]\). We start with the Lagrangian of electromagnetic and complex scalar field with the addition of general couplings. Then, we get the solution in the integral form of its curvature \((^{(4)}F, D\phi)\) at an arbitrary point \(p\) using a spherical means method for \((3 + 1)\) dimension nonlinear wave equation as deriving in \([7]\). This expression makes our works in the light cone integral from \(p\) to the initial surface. Next, we show that the terms in the light cone integral can be bound by energy system and \(L^\infty\) estimates of its curvature \((\|^{(4)}F\|_{L^\infty}, \|D\phi\|_{L^\infty})\).
clearly that to get the positive energy the couplings \( h \) must be definite positive, \( h (\phi, \bar{\phi}) \geq 0 \).

We assume that we have local existence as in [4, 6] so we take the solution of MKG system lying in the \((H_2 \times H_1)\) where \( H_k \) denotes the Sobolev space. Furthermore, to get the global existence we just show that the norm of \((H_2 \times H_1)\) does not blow up in finite time. For this purpose the energy system is clearly insufficient even we choose the suitable gauge conditions since the energy could at most bound the norm \((H_2 \times L^2)\) of solution. So we must define a “modification of energy” that suitable for this purpose and show that it does not blow up in finite time.

### 2. The Maxwell Klein Gordon Equation with General Couplings

We begin with the Lagrangian of the Maxwell Klein Gordon equation (MKG) in presence of general gauge couplings as follows

\[
\mathcal{L} = -\frac{1}{4} h (\phi, \bar{\phi}) F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{4} k (\phi, \bar{\phi}) F_{\alpha\beta} \tilde{F}^{\alpha\beta} - D_\alpha \phi D^{\alpha} \bar{\phi} - V (\phi, \bar{\phi})
\]  

where \( F_{\alpha\beta} \) is the electromagnetic field tensor and \( D_\alpha \phi \) is the complex scalar field. Both related to real potential \( A_\alpha \) by equation

\[
\begin{align*}
F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha \\
D_\alpha \phi &= \partial_\alpha \phi + i A_\alpha \phi
\end{align*}
\]

\( \tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma} \) is the Hodge dual of \( F \), \( h (\phi, \bar{\phi}) \) and \( k (\phi, \bar{\phi}) \) are the general couplings between electromagnetic fields and complex scalar fields. Here we also use the Minkowski metric diag \((- , +, +, +)\), with Greek indices \( \alpha, \beta, ... \) run over \( 0, 1, 2, 3 \) and Roman indices \( i, j, ... \) run over \( 1, 2, 3 \).

The corresponding equation of motion to this Lagrangian are

\[
\partial^\alpha F_{\alpha\gamma} = h^{-1} (\phi, \bar{\phi}) \left\{ 2 i \phi \bar{D}_\gamma \phi - F_{\alpha\gamma} \partial^\alpha h (\phi, \bar{\phi}) + \tilde{F}_{\alpha\gamma} \partial^\alpha k (\phi, \bar{\phi}) \right\}
\]  

\[
D_\alpha D^\alpha \phi = -\frac{1}{4} F_{\alpha\beta} \partial_{\phi} G^{\alpha\beta} + \partial_{\phi} V
\]  

with

\[
G_{\alpha\beta} = -h (\phi, \bar{\phi}) F_{\alpha\beta} + k (\phi, \bar{\phi}) \tilde{F}_{\alpha\beta}
\]

The energy-momentum tensor of MKG system for this case is

\[
T^{\mu\nu} = -F_{\gamma}^{\mu} G^{\gamma\nu} + 2 \Re (D^\mu \phi D^\nu \bar{\phi}) + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} G^{\alpha\beta} - \eta^{\mu\nu} D_\alpha \phi D^{\alpha} \phi - \eta^{\mu\nu} V
\]

where \( \eta^{\mu\nu} \) is the Minkowski metric. The energy sistem become

\[
E_0 = \int_{R^3} T^{00} r^2 dr d\Omega = \int_{R^3} r^2 dr d\Omega \left\{ \frac{h}{2} \left( |E|^2 + |B|^2 \right) + |D_0 \phi|^2 + |D_i \phi|^2 + V \right\}
\]  

clearly that to get the positive energy the couplings \( h \) must be definite positive, \( h (\phi, \bar{\phi}) \geq 0 \).

We can also get the derivative of equation (4) and (5)

\[
\partial^\mu \partial_\mu F_{\alpha\gamma} = h^{-1} \left\{ - (\partial_\mu \partial^\mu h) F_{\alpha\gamma} + (\partial_\mu \partial^\mu k) \tilde{F}_{\alpha\gamma} + 2 i \partial_\alpha (\phi \bar{D}_\gamma \phi) - 2 (\partial_\alpha h) \partial^\mu F_{\mu\gamma} \right\}
\]

\[
\partial^\alpha \partial_\alpha D_\mu \phi = A_\alpha A^\alpha (D_\mu \phi) - i \partial_\alpha A^\alpha (D_\mu \phi) - 2 i A_\alpha \partial^\alpha (D_\mu \phi) - \frac{1}{4} \left( D^\alpha F_{\alpha\beta} \partial_{\phi} G^{\beta\mu} + F_{\mu\beta} D_\alpha \left[ \partial_{\phi} G^{\alpha\beta} \right] \right) + D_\alpha \left( \partial_{\phi} V \right)
\]
where $\partial^\mu \partial_\mu = -\partial^2 + \Delta$ is the d’Alembertian operator. The equations (9) and (10) above are nonlinear wave equations which further provide an important role in the prove of global existence of MKG systems. Then, by using spherical means method as in [7] we can write the equation in the integral light cone form over point $p$ to initial surface as follows

$$\int_{K_p} \tau_{\alpha\gamma} = F_{\alpha\gamma}^{lin} + \frac{1}{4\pi} \int r d\Omega \left[-\left(h^{-1}\partial_\mu \partial^\mu h\right) F_{\alpha\gamma} + \left(h^{-1}\partial_\mu \partial^\mu k\right) \tilde{F}_{\alpha\gamma} + 2i h^{-1} \partial_\alpha (\phi D_\gamma \phi) - 2h^{-1} (\partial_\alpha h \partial^\mu F_{\mu\gamma})\right]_{t=-r}$$

$$D_\mu \phi = D_\mu \phi^{lin} + \frac{1}{4\pi} \int r d\Omega \left[A_\alpha A^\alpha (D_\mu \phi) - i \partial_\alpha A^\alpha (D_\mu \phi) - 2i A_\alpha \partial^\alpha (D_\mu \phi)\right]_{t=-r}$$

where

$$F_{\alpha\gamma}^{lin} = \frac{1}{4\pi} \int \Omega \left[r_0 m^\mu \partial_\mu F_{\alpha\gamma} + F_{\alpha\gamma}\right]_{t=0, r=r_0}$$

$$D_\mu \phi^{lin} = \frac{1}{4\pi} \int \Omega \left[r_0 m^\alpha \partial_\alpha D_\mu \phi + D_\mu \phi\right]_{t=0, r=r_0}$$

This integral form of curvature define over the interior of the past light cone $K_p$ from a point $p$ to the initial data surface $t = t_0$.

3. Estimates

3.1. Estimates For Electromagnetic Field

In this section we estimates the equation (11) using the $L^\infty$ norm and the energy of MKG system as define above. We start with the linear term of the solution as state in equation (13). The estimates for this term is

$$\left|F_{\alpha\gamma}^{lin}\right| \leq C \left(\|r_0 m^\mu \partial_\mu F(t_0)\|_{L^\infty} + \|F(t_0)\|_{L^\infty}\right)$$

$$\leq K_1(t_0) + r_0 K_2(t_0)$$

(15)

where $K_1(t_0)$ and $K_2(t_0)$ is a finite constant that depend on initial data only.

For the second term of equation (11), we use the Holder inequality so the second term can be written as

$$|I_2| \leq C \left(\int_0^{r_0} dr \left(\|F(-r)\|_{L^\infty}\right)^2\right)^{1/2} \left\|h^{-1} \partial_\mu \partial^\mu h\right\|_{L^2}$$

(16)

where

$$\left\|F(t)\right\|_{L^\infty} = \left|F_{\alpha\beta} (t) F_{\alpha\beta} (t)\right|^{1/2}.$$

(17)

Then, we attain

$$\left\|h^{-1} \partial_\mu \partial^\mu h\right\|_{L^2} \leq C \left(\left\|\partial_\mu \partial^\mu \phi\right\|_{L^2} + \left\|\partial_\mu \phi \partial^\mu \phi\right\|_{L^2} + \left\|\partial_\mu \phi \partial^\mu \phi\right\|_{L^2}\right)$$

(18)
the first term of equation (18) can be bounded if we substitute the equation of motion for complex scalar field and after that use the Minkowski inequality. The result is

\[ \| \partial_\mu \partial^\mu \phi \|_{L^2} \leq C \left( \| \partial_\phi \|_{L^2} + \left\| \frac{1}{4} F_{\alpha\beta} \partial_\alpha \partial_\beta G^{\alpha\beta} \right\|_{L^2} + \| A_\mu (D^\mu \phi) \|_{L^2} + \| \partial_\mu (A^\mu \phi) \|_{L^2} \right) \] (19)

We set that the potential energy is a bounded function so the first term of (19) bound by a positive constant. The second term of (19) can be express as

\[ \left\| \frac{1}{4} F_{\alpha\beta} \partial_\alpha \partial_\beta G^{\alpha\beta} \right\|_{L^2} \leq C \left( \| (4) F(-r) \|_{L^\infty}^2 \left( \| \partial_\phi h \|_{L^2} + \| \partial_\phi k \|_{L^2} \right) \right) \] (20)

To bound this equation we define the general couplings as

\[ h(\phi, \bar{\phi}) = g(\phi, \bar{\phi}) \left( 1 + |\phi|^2 \right) \] (21)

\[ k(\phi, \bar{\phi}) = l(\phi, \bar{\phi}) \left( 1 + |\phi|^2 \right) \] (22)

here we assume that \( g(\phi, \bar{\phi}) \) and \( l(\phi, \bar{\phi}) \) is also a bounded function. In fact we can choose a positive constant as the general couplings but we go with this for the sake of generality. Thus, we obtain

\[ \| \partial_\phi h \|_{L^2} \leq C \left\{ 1 + \mathcal{E}_0^{1/2} t + \left( \mathcal{E}_0^{1/2} t + \mathcal{E}_0^{1/2} + 1 \right)^2 \right\} \] (23)

For the third term of equation (19), we use the temporal gauge condition \( A_0 = 0 \), so we get

\[ \| A_\mu (D^\mu \phi) \|_{L^2} \leq C \| A_t \|_{L^2} \| D\phi \|_{L^\infty} \leq C \left( 1 + \mathcal{E}_0^{1/2} t \right) \| D\phi \|_{L^\infty} \] (24)

Then, with the same method of estimation, we obtain the estimates for the second term of (11) as follows

\[ |I_2| \leq C \left\{ 1 + \left[ 1 + \mathcal{E}_0^{1/2} t + \left( \mathcal{E}_0^{1/2} t + \mathcal{E}_0^{1/2} + 1 \right)^2 \right] \left\| (4) F(-r) \right\|_{L^\infty}^2 \right. \]

\[ + \left( 1 + \mathcal{E}_0^{1/2} + \mathcal{E}_0^{1/2} t \right) \| D\phi \|_{L^\infty} + \mathcal{E}_0^{1/2} \| \phi \|_{L^\infty} \right\} \left( \int_0^{r_0} dr \left\| (4) F(-r) \right\|_{L^\infty}^2 \right)^{1/2} \] (25)

The same method apply for the next term of (11). Hence, we get the estimates for the electromagnetic field solution

\[ |F_{\alpha\gamma}| \leq K_1(t_0) + r_0 K_2(t_0) + C_1 \left( \mathcal{E}_0^{1/2} t + \mathcal{E}_0^{1/2} + 1 \right) \left( \int_0^{r_0} dr \| D\phi \|_{L^\infty}^2 \right)^{1/2} \]

\[ + C_2 \left( \int_0^{r_0} dr \left\| (4) F(-r) \right\|_{L^\infty}^2 \right)^{1/2} \left\{ 1 + \left[ 1 + \mathcal{E}_0^{1/2} t + \left( \mathcal{E}_0^{1/2} t + \mathcal{E}_0^{1/2} + 1 \right)^2 \right] \right. \]

\[ + \left( 1 + \mathcal{E}_0^{1/2} + \mathcal{E}_0^{1/2} t \right) \| D\phi \|_{L^\infty} + \mathcal{E}_0^{1/2} \| \phi \|_{L^\infty} + \| D\phi \|_{L^\infty} \mathcal{E}_0^{1/2} \right\} \] (26)
Since the right hand side of the equation above is independent of the spatial coordinates of \( p \), we can reexpress the equation by substitute \( t = -r \) and use the initial data for \( t_0 = 0 \) so we have
\[
\left\| (4) F(t) \right\|_{L^\infty}^2 \leq C_0(K_1(0) + tK_2(0))^2 + C_1y^2 \int_0^t dt \left\| (4) F(t) \right\|_{L^\infty}^2 + C_2f^2 \int_0^t dt \left\| D\phi(t) \right\|_{L^\infty}^2
\]
with
\[
y(t) = \left\{ 1 + \mathcal{E}_0^{1/2}t + f^2(t) + f(t) \left\| D\phi \right\|_{L^\infty} + \mathcal{E}_0^{1/2} \left\| \phi \right\|_{L^\infty} + \left\| D\phi \right\|_{L^\infty} \mathcal{E}_0^{1/2} \right\}
\]
\[
f(t) = \left( 1 + \mathcal{E}_0^{1/2} + \mathcal{E}_0^{1/2}t \right)
\]

3.2. Estimates For Complex Scalar Field
To estimate equation (12) we use the same method as in the electromagnetic case. The results is
\[
\left\| D\phi \right\|_{L^\infty}^2 \leq C_0'(l_1(0) + tl_2(0))^2 + C_1'p^2 \left( \int_0^t dt \left\| D\phi(t) \right\|_{L^\infty}^2 \right) + C_2'n^2 \left( \int_0^t dt \left\| (4) F(t) \right\|_{L^\infty}^2 \right)
\]
where
\[
n(t) = \left\{ (1 + \left\| \phi \right\|_{L^\infty}) \mathcal{E}_0^{1/2} \left\| (4) F(-r) \right\|_{L^\infty} + w(t) \left\| A_i \right\|_{L^\infty} \left\| (4) F(-r) \right\|_{L^\infty} \right\}
\]
\[
p(t) = \left( \mathcal{E}_0 + \mathcal{E}_0^{1/2} + \mathcal{E}_0t + r_0^2 \mathcal{E}_0^{1/2} \right)
\]
\[
w(t) = 1 + \mathcal{E}_0^{1/2}t + \left[ 1 + \mathcal{E}_0^{1/2}t + \mathcal{E}_0^{1/2} \right]^2
\]
The next step, we must show that the norm \( \left\| D\phi(t) \right\|_{L^\infty} \) and \( \left\| (4) F(t) \right\|_{L^\infty} \) are finite. Thus, we define a function as follows
\[
\Psi(t) = \left\| (4) F(t) \right\|_{L^\infty}^2 + \left\| D\phi(t) \right\|_{L^\infty}^2
\]
we see that form equation (27) and (30),
\[
\Psi(t) \leq p(t) + \sigma(t) \int_0^t dt \left( \left\| (4) F(t) \right\|_{L^\infty}^2 + \left\| D\phi(t) \right\|_{L^\infty}^2 \right)
\]
with
\[
p(t) = C_0(K_1(0) + tK_2(0))^2 + C_1'(l_1(0) + tl_2(0))^2
\]
\[
\sigma(t) = \left( C_1y^2 + C_2'n^2 \right) \left( C_2f^2 + C_1'p^2 \right)
\]
Then, to get bound on \( \Psi(t) \) we apply Gronwall Lemma so we just need to show that the norm \( \left\| D\phi(t) \right\|_{L^\infty} \) and \( \left\| (4) F(t) \right\|_{L^\infty} \) continuous. By Cauchy method we prove the continuity as follows
Let there is $\varepsilon > 0$ so for $\| (4) F(t + \varepsilon) \|_{L^\infty}$ and $\| (4) F(t) \|_{L^\infty}$ and using triangle inequality then the Sobolev inequality we have

$$
\| (4) F(t + \varepsilon) \|_{L^\infty} - \| (4) F(t) \|_{L^\infty} \leq \| (4) F(t + \varepsilon) - (4) F(t) \|_{L^\infty}
$$

(38)

As if $\varepsilon \to 0$, $\| (4) F(t + \varepsilon) - (4) F(t) \|_{H^2} \to 0$. So we get the continuity of $(4) F(t)$ as a curve in $H^2$. For the norm $\| D\phi(t) \|_{L^\infty}$ the same method obviously apply. Now we have proven that the norm of its curvature in the form of $L^\infty$ norm does not blow up in finite time. Thus, the bound on $\| D\phi(t) \|_{L^\infty}$ and $\| (4) F(t) \|_{L^\infty}$ implies that $\| \phi(t) \|_{L^\infty}$ and $\| A(t) \|_{L^\infty}$ are finite. We can show by using the temporal gauge condition as follows

$$
\| A_i(t, x) \|_{L^\infty} = \| A_i(0, x) \|_{L^\infty} + \int_0^t \| E_i(s) \|_{L^\infty} ds
$$

(39)

$$
\| \phi(t, x) \|_{L^\infty} = \| \phi(0, x) \|_{L^\infty} + \int_0^t \| \partial_0 \phi(s) \|_{L^\infty} ds
$$

(40)

4. Global Existence

Until here we already show that the $\| D\phi(t) \|_{L^\infty}$, $\| (4) F(t) \|_{L^\infty}$, $\| \phi(t) \|_{L^\infty}$ and $\| A(t) \|_{L^\infty}$ does not blow up in finite time. Furthermore, let define the modification of energy by

$$
\mathcal{E}_0 = \frac{1}{2} \int_{\mathbb{R}^3} dx \left\{ E_i E_i + \partial_j A_i \partial_j A_i + mA_i A_i + \| \partial_0 \phi \|^2 + \| \partial_i \phi \|^2 + m|\phi|^2 \right\}
$$

(41)

$$
\mathcal{E}_1 = \frac{1}{2} \int_{\mathbb{R}^3} dx \left\{ (\partial_j E_i \partial_j E_i + \partial_j \partial_k A_i \partial_j \partial_k A_i) + \| \partial_i \partial_0 \phi \|^2 + \| \partial_j \partial_i \phi \|^2 \right\}
$$

(42)

where $m$ is a positive constant. We can see that $(\mathcal{E}_0 + \mathcal{E}_1)^{1/2}$ is equivalent to the $(H^2 \times H^1)^2$ norm. Then to prove the global existence of MKG system we need to show that $\mathcal{E}_0$ and $\mathcal{E}_1$ does not blow up in finite time. we derive the "modification of energy" equation with respect to time, then we have

$$
\left| \frac{d\mathcal{E}_0}{dt} \right| \leq Z(t) \mathcal{E}_0
$$

(43)

$$
\left| \frac{d\mathcal{E}_1}{dt} \right| \leq [\Sigma(t) + \Pi(t)] \mathcal{E}_1
$$

(44)

where

$$
Z(t) = \left\{ c_1 + c_2 \| D\phi(t) \|_{L^\infty} + c_3 \left( (4) F(t) \right)_{L^\infty} + c_4 (1 + \| \phi \|_{L^\infty}) \right\} \left( (4) F(t) \right)_{L^\infty}^2
$$

(45)

$$
\Sigma(t) = C_1 \left\{ \| D\phi \|_{L^\infty} \mathcal{E}_0^{1/2} + \| \phi \|_{L^\infty} \right\} \left( 1 + \mathcal{E}_0^{1/2} \right) + \left( (4) F(t) \right)_{L^\infty}
$$

(46)
\[ \Pi (t) = C_2 \left\{ \| D\phi \|_{L^\infty} a_0^{1/2} + \| \pi (t) \|_{L^\infty} \| (4) F(t) \|_{L^\infty} a_0^{1/2} + \| (4) F(t) \|_{L^\infty} \right\} \\
+ \| (4) F(t) \|_{L^\infty}^2 \left( E_0^{1/2} + E_0 + 1 \right) + \| \pi \|_{L^\infty} + E_0^{1/2} \right\} \]  

Finally, by Gronwall Lemma in the derivative form we obtain

\[ \mathcal{E}_0 (t) \leq \mathcal{E}_0 (0) \exp \left( \int_0^t Z (t) \, dt \right) \]  

\[ \mathcal{E}_1 (t) \leq \mathcal{E}_1 (0) \exp \left( \int_0^t [\Sigma (t) + \Pi (t)] \, dt \right) \]  

It is clear that the function \( Z(t), \Sigma(t), \) and \( \Pi(t) \) depend on \( \| D\phi (t) \|_{L^\infty}, \| (4) F(t) \|_{L^\infty}, \| \phi (t) \|_{L^\infty} \) and \( \| \pi (t) \|_{L^\infty} \). Here we have shown that the \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) does not blow up in finite time. Thus, we have prove that the solution of the Maxwell Klein Gordon system in presence of general gauge couplings and temporal gauge conditions is globally exist.

**Theorem 1** Let \( u_0 = (A_i, E_i, \phi, \partial_t \phi) \) is the initial data in \( (H_2 \times H_1)^2 \) then there is unique solution \( u(t) \in (H_2 \times H_1)^2 \) of Maxwell Klein Gordon (MKG) equation with general gauge couplings satisfy (21) and (22) define for \( t \in (-\infty, \infty) \) which the corresponding fields satisfy equation (4),(5),(9),and (10).

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