GENERALIZED HOMOLOGIES
FOR THE ZERO MODES
OF THE \( SU(2) \) WZNW MODEL

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Dedicated to the memory of Moshé Flato.
Abstract

We generalize the BRS method for the (finite-dimensional) quantum gauge theory involved in the zero modes of the monodromy extended $SU(2)$ WZNW model. The generalization consists of a nilpotent operator $Q$ such that $Q^h = 0$ ($h = k + 2 = 2, 3, \ldots$ being the height of the current algebra representation) acting on an extended state space. The physical subquotient is identified with the direct sum $\bigoplus_{n=1}^{h-1} \ker(Q^n)/\operatorname{Im}(Q^{h-n})$.

1 Introduction

The Wess-Zumino-Novikov-Witten (WZNW) model was originally formulated [25] in terms of a multivalued action for a field $g(x, t)$ (which maps the cylinder $S^1 \times \mathbb{R}$ into a Lie group $G$). Its solution [22] came however within the axiomatic (or “bootstrap”) approach making use of the representation theory of affine Kac-Moody algebras. This solution exhibits some puzzling features like the appearance of non-integer (“quantum”) statistical dimensions (which appear as positive real solutions of the fusion rules’ equations) while the corresponding 2-dimensional (2D) fields satisfy local “Bose type” commutation relations. The gradual understanding of these features only began with the development of the canonical approach to the model ([11], [16], [1], [12]-[15]) in which one splits the field $g$ into left and right movers’ (chiral) components:

$$g^A_B(x, t) = u^A_\alpha(x - t) \bar{u}^\alpha_B(x + t) = u^A_\alpha(x^-) \bar{u}^\alpha_B(x^+) \quad (1.1)$$

These chiral components reveal a hidden quantum group symmetry (under “gauge transformations” $u^A_\alpha \mapsto u^A_\alpha T^\alpha_\sigma$ with non-commuting entries). The phase space of the theory is extended (compared to the 2D gauge invariant construction) by the chiral zero modes, including the monodromy degrees of freedom that appear in the twisted periodicity condition (cor-
responding to \( g(x + 2\pi, t) = g(x, t) \)

\[
u(x + 2\pi) = \nu(x)M, \quad \bar{\nu}(x + 2\pi) = M^{-1}\bar{\nu}(x).
\]

(1.2)

The resulting extended WZNW model is understood on the classical level \([11],[16]\) while its quantization has only been attempted in a lattice approach \([12]\) and has not been brought to a form yielding a satisfactory continuum limit. The direct investigation of the quantum model \([14],[15]\) has singled out a nontrivial finite-dimensional gauge theory problem.

The present paper constructs the physical state-space of the zero modes for \( G = SU(2) \) in terms of generalized Becchi-Rouet-Stora \([2]\) (BRS) generalized homologies thus providing a complete solution to this problem.

We proceed to describing the problem in more detail and to outlining the content of the paper. Following \([14],[15]\), let us expand \( u^A_\alpha \) and \( \bar{u}^\alpha_B \) into chiral vertex operators \( u^A_i, \bar{u}^\alpha_j \) which diagonalize the monodromy:

\[
u^A_\alpha(x) = a^i_\alpha u^A_i(x, p), \quad \bar{\nu}^{\alpha}_B(y) = \bar{a}^\alpha_j \bar{u}^j_B(y, \bar{p})
\]

(1.3)

(the repeated indices \( i, j \) are summed from 1 to 2, \( a^{1,2} \) standing for \( a^{+,-} \) of \([10]\) - cf. \([19]\)). Here \( p \) is the shifted Lie algebra weight (\( p = \Lambda + \rho \)) which labels the monodromy eigenvalues\(^2\). The irreducible representations of the quantum universal enveloping algebra \( U_q(\mathfrak{sl}_2) \) are labelled by \( q^\pm p \) with \( q = e^{i\pi} \), where \( h = 2, 3, \ldots \) is the height of the associated current algebra representation (\( h = k + 2 \) where \( k \) is the Kac-Moody level). The nontrivial finite-dimensional problem singled out by the canonical (hamiltonian) approach involves a pair of quantum matrix algebras \( \mathfrak{A}_h \otimes \bar{\mathfrak{A}}_h \) generated by the zero mode vertex operators \( a^i_\alpha \) and \( \bar{a}^\alpha_j \)

\(^2\)Such a simple picture only works for integrable highest weights, \( 0 < p < h \). Going beyond this limit requires dealing with indecomposable representations of the \( su(2) \) current algebra involving singular vectors in the associated Verma modules. The monodromy matrix would then contain a Jordan cell for each pair \( (p, 2h - p; 0 < p < h) \) of highest weights.
which reflects the basic properties of composition and braiding of current algebra modules

The problem is to develop a \( "q\)-gauge theory" approach that would allow to extract an \((h-1)\)-dimensional (generalized) BRS cohomology from the \( h^4 \)-dimensional \( q\)-Fock space module \( \mathcal{H} = \mathcal{F} \otimes \mathcal{F} \) of \( \mathfrak{A}_h \otimes \mathfrak{A}_h \). A step towards its solution was made in [10]. After singling out a \((2h-1)\)-dimensional subspace \( \mathcal{H}_I \) of quantum group invariant vectors of \( \mathcal{H} \), we proved that the nilpotent operator

\[
A = a_\alpha^2 \bar{a}_\beta^2
\] (1.4)

satisfying \( A^h = 0 \) in \( \mathcal{H} \) and \( A(\mathcal{H}_I) \subset \mathcal{H}_I \) has one-dimensional generalized homologies \( H(n)(\mathcal{H}_I, A) = \text{Ker}(A^n : \mathcal{H}_I \to \mathcal{H}_I)/A^{h-1}(\mathcal{H}_I) \) on \( \mathcal{H}_I \), \( n \in \{1, \ldots, h-1\} \). The direct sum \( \bigoplus_{n=1}^{h-1} H(n)(\mathcal{H}_I, A) \) was then identified with the \((h-1)\)-dimensional physical subquotient.

The objective of this paper is to extend this construction in such a way that quantum group invariance has not to be imposed as an extra constraint (in other words, we solve the first problem stated in the concluding Section 3 of [10]). It turns out that to this end it is necessary (as demonstrated in Section 3) to extend (in a suitable way) the space \( \mathcal{H} = \mathcal{F} \otimes \bar{\mathcal{F}} \); this fact is not completely unexpected since it corresponds in the usual situation to the addition of ghost’s states. A minimal canonical construction achieving our goal is presented in Section 4 and is related in Section 5 to a (generalized) Hochshild complex (see [6]-[9]). It is worth noticing here that the construction relies on a generalization, in the context of generalized (co)-homology, of an elementary spectral sequence’s argument. Finally, in Section 6 we identify the generalized homology of \( (\mathcal{H}_I, A) \) as a part of a generalized homology of Hochschild cochains and we compare our constructions and results with the BRS-like ones.

\(^3\)A BRS treatment of the Wakimoto module corresponding to the factor \( u_i^4(x,p) \) in (1.3) is contained in [3].
Throughout the paper $h$ is an integer greater or equal to 2 and $q = \exp(i\frac{\pi}{h})$. By an $h$-differential vector space, we mean a vector space $E$ equipped with a nilpotent endomorphism $d$, its $h$-differential, satisfying $d^h = 0$. The generalized homology of $(E, d)$ is then the family of vector spaces $H_{(k)}(E, d) = \text{Ker}(d^k)/\text{Im}(d^{h-k})$, $k \in \{1, \ldots, h-1\}$. An $h$-complex will be here an $h$-differential vector space which is $\mathbb{Z}$-graded and such that its differential is of degree 1. If $(E, d)$ is an $h$-complex with $E = \oplus_n E^n$, then the $H_{(k)}(E, d)$ are graded $H_{(k)}(E, d) = \oplus_n H_n^{(k)}(E, d) = \text{Ker}(d^k : E^n \rightarrow E^{n+k})/d^{h-k}(E^{n+k-h})$. More generally we use the notations of [7] for generalized (co)homology. Concerning $q$-numbers, we use here the convention of [10], that is $[n] = q^n - q^{-n}$ which differs in several respects from the one of [7]. In the next section we give a summary of relevant parts of earlier work ([15], [10], [19]).

2 Background, preliminaries

The quantum matrix algebra $\mathfrak{A}$ is characterized by $R$-matrix exchange relations and a determinant condition [19]. The exchange relations are written conveniently in terms of a pair of quantum antisymmetrizers $A$ and $A(p)$:

$$a_1 a_2 A = A(p) a_1 a_2, \quad \text{i.e.} \quad a_{\alpha_1} a_{\beta_2} A_{\alpha_1 \beta_2} = A(p) a_{\beta_1} a_{\alpha_2},$$

(2.1)

here

$$A_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = \varepsilon_{\alpha_1 \alpha_2} \varepsilon_{\beta_1 \beta_2} = q^{\delta_{\alpha_1 \alpha_2}} \delta_{\beta_1 \beta_2} - \delta_{\alpha_1 \alpha_2} \left( \delta_{\beta_1 \beta_2} \varepsilon_{\alpha_1 \alpha_2} = \delta_{\beta_1 \beta_2} \varepsilon_{\alpha_1 \alpha_2} \right)$$

$$\varepsilon_{\alpha \beta} = \begin{cases} 1 & \text{for } \alpha > \beta \\ 0 & \text{for } \alpha = \beta \\ -1 & \text{for } \alpha < \beta \end{cases}, \quad \left( \varepsilon_{\alpha \beta} \right) = \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix} = \left( \varepsilon_{\alpha \beta} \right),$$

$$A(p)^{ij}_{j_1 j_2} = \frac{[p+i_1-i_2]}{[p]} \varepsilon_{ij} \varepsilon_{j_1 j_2} = \frac{[p+i_1-i_2]}{[p]} \left( \delta_{j_1 j_2} - \delta_{j_1 j_2} \right);$$
both $\mathcal{A}$ and $\mathcal{A}(p)$ satisfy the Hecke algebra condition

$$\mathcal{A}^2 = [2] \mathcal{A}, \quad \mathcal{A}(p)^2 = [2] \mathcal{A}(p) \quad ([2] = q + q^{-1})$$

and a braiding (Temperley-Lieb-Martin) property

$$\mathcal{A}_{12} \mathcal{A}_{23} \mathcal{A}_{12} - \mathcal{A}_{12} \mathcal{A}_{23} \mathcal{A}_{12} = 0 \left( (\mathcal{A}_{12})_{\alpha_1 \alpha_2 \alpha_3} = \mathcal{A}_{\beta_1 \beta_2 \beta_3}^{\alpha_1 \alpha_2 \alpha_3} \right)$$

which involves a change of $p$ for $\mathcal{A}_{23}(p)$ (see [19]); in particular,

$$a_i^2 a_i^1 = q a_i^1 a_i^2, \quad [a_i^1, a_i^2] = 0.$$

The algebra $\mathfrak{A}$ contains, by definition, the field $\mathcal{R}$ of rational functions of $q^p$ (which enter the exchange relations (2.1)), $a_i^j$ shifting $p$ according to the law

$$q^p a_i^1 = a_i^1 q^{p+1}, \quad q^p a_i^2 = a_i^2 q^{p-1}.$$

The determinant condition allows to express a quadratic combination of $a_\alpha^i$ as a function of $q^p$

$$\det a := \frac{1}{[2]} \varepsilon_{ij} a_i^\alpha a_j^\beta e^{\alpha \beta} = [p] \Rightarrow \varepsilon_{ij} a_i^\alpha a_j^\beta = [p] e^{\alpha \beta}.$$ 

Identical relations are satisfied by $\bar{a}_\alpha^i$; we have, in particular,

$$\bar{a}_j^\alpha a_j^\alpha = q \bar{a}_j^\alpha a_j^\alpha, \quad [\bar{a}_1^\alpha, \bar{a}_2^\alpha] = 0, \quad q^p \bar{a}_1^\alpha \bar{a}_2^\alpha = \bar{a}_1^\alpha q^{p+1} e^{\alpha \beta}, \quad \varepsilon_{ij} \bar{a}_i^\alpha \bar{a}_j^\beta = [p] e^{\alpha \beta}.$$ 

The algebra $\mathfrak{A}$ admits a (two-sided) ideal $\mathfrak{I}_h$ generated by $(a_i^j)_h$, $\alpha, i = 1, 2$ and $[hp]$. The factor algebra $\mathfrak{A}_h = \mathfrak{A} / \mathfrak{I}_h$ is finite dimensional. It admits an $h^2$-dimensional Fock space module $\mathcal{F}$ with basis

$$|p, m > = (a_1^1)^m (a_2^1)^{p-1-m} |1, 0 >, \quad \text{where} \quad a_\alpha^2 |1, 0 > = 0,$$

$$0 < p < 2h \quad \text{and} \quad \max (0, p - h) \leq m \leq \min (p - 1, h - 1).$$

Similar statements are valid for $\bar{\mathfrak{A}}$.

The $h^4$ dimensional tensor product space $\mathcal{H} = \mathcal{F} \otimes \bar{\mathcal{F}}$ carries a representation of a tensor product of quantum universal enveloping algebras
(QUEA) which we proceed to define.

The first factor is the diagonal $U_q(\mathfrak{sl}_2)$ related to the left and right monodromies $M$ and $\bar{M}$ (appearing in (1.2)) as follows. Denoting by $\Delta$ the $U_q(\mathfrak{sl}_2)$ coproduct realized in $\mathcal{F} \otimes \bar{\mathcal{F}}$ we set

$$\bar{M}_+^{-1}M_+ = \begin{pmatrix} \Delta(K^{-1/2}) (q^{-1} - q)\Delta(F)\Delta(K^{1/2}) & 0 \\ 0 & \Delta(K^{1/2}) \end{pmatrix} \tag{2.2}$$

$$\bar{M}_-^{-1}M_- = \begin{pmatrix} \Delta(K^{1/2}) & 0 \\ (q - q^{-1})\Delta(K^{-1/2})\Delta(E) & \Delta(K^{-1/2}) \end{pmatrix} \tag{2.3}$$

where $M_\pm(\bar{M}_\pm)$ are the Gauss components of $M(\bar{M})$ defined by

$$q^{3/2}M = M_+M_-^{-1}, \quad q^{3/2}\bar{M}_-^{-1} = \bar{M}_+^{-1}M_-,$$

with the same diagonal elements in $M_+$ and $M_-^{-1}$ (as well as in $\bar{M}_+$ and $\bar{M}_-^{-1}$). Noting that $M_\pm^{\pm1}$ and $\bar{M}_\pm^{\pm1}$ satisfy identical exchange relations ([14], [15]), we parametrize them in the same way in terms of generators $X$ and $\bar{X}$ of the corresponding QUEA as the products (2.2),(2.3) are expressed in terms of $\Delta(X)$. As a result we obtain

$$\Delta(K^{\pm1/2}) = q^{\pm1/2}(H + \bar{H}) , \quad \Delta(E) = E + q^H \bar{E}, \quad \Delta(F) = Fq^{-\bar{H}} + \bar{F}$$

(where the two copies of $U_q(\mathfrak{sl}_2)$ labelled by $X$ and $\bar{Y}$ commute : $[X, \bar{Y}] = 0$). The second and the third QUEA are generated by $A, A'$ and $B, B'$ ($[A^{(t)}, B^{(t)}] = 0$) where

$$A = a_2^2a_2^\alpha, \quad A' = a_2^1a_1^\alpha \Rightarrow [A, A'] = [p + \bar{p}], \quad q^{p+\bar{p}}A = Aq^{p+\bar{p} - 2}$$

$$B = a_2^1a_2^\alpha, \quad B' = -a_2^2a_1^\alpha \Rightarrow [B, B'] = [p - \bar{p}], \quad q^{p-\bar{p}}B = Bq^{p-\bar{p} + 2}$$

We shall denote the QUEA generated by $\Delta(X)$ and by $B$ and $B'$ by $U_q(\mathfrak{sl}_2)_\Delta$ and $U_q(\mathfrak{sl}_2)_B$, respectively. In order to prove the $U_q(\mathfrak{sl}_2)_\Delta$

\[\]
invariance of $a_i^\alpha \bar{a}_j^\alpha$, we use the exchange relations

$$q^{H} a_i^\alpha = a_i^\alpha q^{H+\delta_i^\alpha-\delta_j^\alpha}, \quad q^{H} \bar{a}_i^\alpha = \bar{a}_i^\alpha q^{H-\delta_i^\alpha+\delta_j^\alpha},$$

$$[E, a_i^\alpha] = \delta_1^\alpha a_i^\alpha q^{H}, \quad q^{3-2\alpha} \bar{E} a_i^\alpha \bar{E} = -\delta_1^\alpha \bar{a}_i^\alpha, \quad \alpha = 1, 2,$$

which imply $[\Delta(\bar{X}), a_i^\alpha \bar{a}_j^\alpha] = 0$ for $\bar{X} = E, F, K, i, j = 1, 2$. The results of [10] (propositions 1, 2, 4) can be summarized as follows.

**THEOREM 0** (a) The set of vectors in $\mathcal{H}$ invariant under the pair of mutually commuting QUEA $U_q(\mathfrak{sl}_2)_\Delta$ and $U_q(\mathfrak{sl}_2)_B$ spans a $2h - 1$ dimensional space $\mathcal{H}_I$ with basis $\{|n+1> = (A')^n|1, 0 > \otimes |1, 0 >, n = 0, 1, \ldots, 2h - 2\}(\subset \mathcal{H}_I)$ where (for $A'_\alpha = a_1^\alpha \bar{a}_2^\alpha, \alpha = 1, 2$ (no summation in $\alpha$))

$$(A')^n = \frac{1}{[n]!!} (A')^n = \sum_{\ell=m}^{n-m} q^{\ell(n-\ell)} (A'_1)^\ell (A'_2)^{n-\ell}, m = \max(0, n - h + 1).$$

(b) The operators $A_1 = a_1^2 \bar{a}_2^1$ and $A_2 = a_2^2 \bar{a}_2^2$, as well as $A'_\alpha$ satisfy $A_2^{(\ell)} A_1^{(\ell)} = q^2 A_1^{(\ell)} A_2^{(\ell)}$ and $(A'_\alpha)^h = 0$ in $\mathcal{H} = F \otimes \bar{F}$ implying $(A'^{\ell})_h = 0$ for $A'^{\ell}$ standing for $A$ or $A'$; furthermore, the basis $\{|n> = \rangle \}$ is characterized by

$$A|n > = [n]|n - 1 >, \quad ([p] - [n])|n > = 0.$$

(c) Each of the generalized homologies of the nilpotent operator $A$ in $\mathcal{H}_I$ is one-dimensional and given by

$$H_{(\alpha)}(\mathcal{H}_I, A) \cong \{C|n\}, \quad n = 1, 2, \ldots, h - 1.$$

When $q$ is a root of the unity, $U_q(\mathfrak{sl}_2)$ has a huge Hopf ideal for which the quotient $\tilde{U}_q(\mathfrak{sl}_2)$ is a finite dimensional Hopf algebra (the reduced
Here we have $q^h = -1$ and it is not hard to see that the actions of $U_q(sl_2)_\Delta$ and $U_q(sl_2)_B$ on $\mathcal{H}$ are in fact actions of the corresponding finite dimensional quotients. In the following, $\mathcal{U}_q$ will denote their tensor product. It is this finite-dimensional Hopf algebra $\mathcal{U}_q$ which acts on $\mathcal{H}$ and $\mathcal{H}_I$ is the subspace of $\mathcal{U}_q$-invariant vectors.

3 Necessity to enlarge $\mathcal{H}$

In short, one has a vector space $\mathcal{H}$ on which act a Hopf algebra $\mathcal{U}_q$ and a nilpotent endomorphism $A$ satisfying $A^h = 0$. The action of the algebra $\mathcal{U}_q$ commutes with $A$, i.e. one has on $\mathcal{H}$ : $[A, X] = 0$, $\forall X \in \mathcal{U}_q$. It follows that the subspace $\mathcal{H}_I$ of $\mathcal{U}_q$-invariant vectors in $\mathcal{H}$ is stable by $A$, i.e. $A(\mathcal{H}_I) \subset \mathcal{H}_I$. Thus $(\mathcal{H}_I, A)$ is an $h$-differential subspace of the $h$-differential vector space $(\mathcal{H}, A)$ and it turns out that the “interesting object” (the physical space) is the generalized homology of $(\mathcal{H}_I, A)$. We would like to avoid the restriction to the invariant subspace $\mathcal{H}_I$ that is, in complete analogy with the BRS methods, we would like to define an extended $h$-differential space in such a way that the $\mathcal{U}_q$-invariance is captured by its $h$-differential in the sense that it has the same generalized homology as $(\mathcal{H}_I, A)$.

The most natural thing to do is to try to construct a nilpotent endomorphism $Q$ of $\mathcal{H}$ with $Q^h = 0$ such that its generalized homology coincides with the one of $A$ on $\mathcal{H}_I$ i.e. such that one has

$$H_{(n)}(\mathcal{H}, Q) = H_{(n)}(\mathcal{H}_I, A), \quad \forall n \in \{1, \ldots, h - 1\} \quad (3.1)$$

where $H_{(n)}(\mathcal{H}, Q) = \ker(Q^n)/\mathrm{im}(Q^{h-n})$. Unfortunately this is not possible. Indeed let $Q$ be a nilpotent endomorphism of $\mathcal{H}$ as above and let us decompose $\mathcal{H}$ into irreducible factors\footnote{We are using here the terminology of [17], (Chapter XV, §3); (maximal) indecomposable factors would perhaps be better than irreducible factors.} for $Q$. One obtains an isomor-
homomorphism

\[ \mathcal{H} \simeq \bigoplus_{n=1}^{h} \mathbb{C}^n \otimes \mathbb{C}^{m_n}, \quad Q \simeq \bigoplus_{n=1}^{h} Q_n \otimes \text{Id}_{\mathbb{C}^{m_n}} \text{ with} \]

\[ Q_n = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \in M_n(\mathbb{C}), \quad (Q_1 = 0), \]

where \( m_n \) is the multiplicity of the irreducible representation in \( \mathbb{C}^n \).

The above decomposition is also the Jordan normal form of \( Q \). One has \( \dim(\mathcal{H}) = \sum_{n=1}^{h} n m_n \) and it is easy to compute \( \dim(H_{(n)}(\mathcal{H}, Q)) \) in terms of the multiplicities \( m_n \). The result is (Proposition 2 of \([7]\))

\[ \dim(H_{(n)}(\mathcal{H}, Q)) = \sum_{j=1}^{n} \sum_{i=j}^{h-j} m_i = \dim(H_{(h-n)}(\mathcal{H}, Q)) \quad (3.2) \]

for \( 1 \leq n \leq h/2 \). On the other hand we know from \([10]\) that one has \( \dim(H_{(n)}(\mathcal{H}_I, A)) = 1 \) for \( 1 \leq n \leq h-1 \). This implies, by using \((3.1)\) and \((3.2)\), that one must have either \( m_{h-1} = 1 \) and \( m_n = 0 \) for \( 1 \leq n \leq h-2 \) or \( m_{1} = 1 \) and \( m_n = 0 \) for \( 2 \leq n \leq h-1 \). It follows that one must have either \( \dim(\mathcal{H}) = h m_h + h - 1 \) or \( \dim(\mathcal{H}) = h m_h + 1 \). However \( \dim(\mathcal{H}) = \dim(\mathcal{F} \otimes \mathcal{F}) = h^4 \) as in Section 2 is not compatible with the above estimates. The same conclusion would hold for other natural choices for \( \mathcal{H} \). It is worth noticing here that \( \mathcal{H}_I \) is perfectly compatible with the first possibility since \( \dim(\mathcal{H}_I) = 2h - 1 = h + h - 1 \).

4 A minimal canonical construction

We recall that \( A \) is a nilpotent endomorphism of \( \mathcal{H} \) with \( A^h = 0 \) and \( A(\mathcal{H}_I) \subset \mathcal{H}_I \). Let us define the graded vector space \( \mathcal{H}^\bullet = \bigoplus_{n \geq 0} \mathcal{H}^n \) by \( \mathcal{H}^0 = \mathcal{H}, \mathcal{H}^n = \mathcal{H}/\mathcal{H}_I \) for \( 1 \leq n \leq h-1 \) and \( \mathcal{H}^n = 0 \) for \( n \geq h \). One then
defines an endomorphism $d$ of degree 1 of $\mathcal{H}^\bullet$ by setting $d = \pi : \mathcal{H}^0 \to \mathcal{H}^1$ where $\pi : \mathcal{H} \to \mathcal{H}/\mathcal{H}_I$ is the canonical projection, $d = \text{Id} : \mathcal{H}^n \to \mathcal{H}^{n+1}$ for $1 \leq n \leq h - 2$ where $\text{Id}$ is the identity mapping of $\mathcal{H}/\mathcal{H}_I$ onto itself and $d = 0$ on $\mathcal{H}^n$ for $n \geq h - 1$. One has $d^h = 0$ and therefore $(\mathcal{H}^\bullet, d)$ is an $h$-complex, so its generalized (co)homology is graded $H_{(k)}(\mathcal{H}^\bullet, d) = \bigoplus_{n \geq 0} H^n_{(k)}(\mathcal{H}^\bullet, d)$ with

$$H^n_{(k)}(\mathcal{H}^\bullet, d) = \text{Ker}(d^k : \mathcal{H}^n \to \mathcal{H}^{n+k}) / d^{h-k}(\mathcal{H}^{n+k-h})$$

It is given by the following proposition.

**PROPOSITION 1** One has $H^n_{(k)}(\mathcal{H}^\bullet, d) = 0$ for $n \geq 1$ and $H^0_{(k)}(\mathcal{H}^\bullet, d) = \mathcal{H}_I$, $\forall k \in \{1, \ldots, h - 1\}$.

**Proof.** Let $\varphi \in \mathcal{H}^n$ for $n \geq 1$ be such that $d^k \varphi = 0$. Then either $\varphi = 0$ or $k \geq h - n$. In the latter case one has $\varphi = d^n \psi = d^{h-k}(d^{n+k-h} \psi)$ which implies that the class of $\varphi$ vanishes in $H^n_{(k)}(\mathcal{H}^\bullet, d)$. This proves $H^n_{(k)}(\mathcal{H}^\bullet, d) = 0$ for $n \geq 1$. Let $\psi \in \mathcal{H}^0 = \mathcal{H}$ be such that $d^k \psi = 0$. Then, by definition this is equivalent to $\pi(\psi) = 0$ i.e. $\psi \in \mathcal{H}_I$ which achieves the proof of the proposition. $\boxtimes$

It is worth noticing here that given the vector space $\mathcal{H}$ together with the subspace $\mathcal{H}_I$, the $h$-complex $(\mathcal{H}^\bullet, d)$ is characterized (uniquely up to an isomorphism) by the following universal property (the proof of which is straightforward).

**PROPOSITION 2** Any linear mapping $\alpha : \mathcal{H} \to \mathcal{C}^0$ of $\mathcal{H}$ into the subspace $\mathcal{C}^0$ of elements of degree 0 of an $h$-complex $(\mathcal{C}^\bullet, d)$ which satisfies $d \circ \alpha(\mathcal{H}_I) = 0$ extends uniquely as a homomorphism $\bar{\alpha} : (\mathcal{H}^\bullet, d) \to (\mathcal{C}^\bullet, d)$ of $h$-complexes.

We now use this universal property to extend $A$ to $\mathcal{H}^\bullet$. 
PROPOSITION 3 \(\text{The endomorphism } A \text{ of } H = H^0 \text{ has a unique}
\text{extension to } H^0, \text{ again denoted by } A, \text{ as a homogeneous endomorphism}
of degree 0 satisfying } Ad - q^2 dA = 0. \text{ On } H^0, \text{ one has } A^h = 0 \text{ and}
(d + A)^h = 0.\)

\textbf{Proof.} Since \(A(H_I) \subset H_I\), one can apply the universal property (Propo-
sition 2) for \(\alpha = A : H \to H^0\) and one obtains a unique homomorphism
\(\bar{A} : H^0 \to H^0\) of \(h\)-complexes extending \(A\). One has \(\bar{A}d = d\bar{A}\) which is
equivalent to \(Ad - q^2 dA = 0\) for \(A = q^{2D} \bar{A} = \bar{A}q^{2D}\) where \(D\) denotes
the degree in \(H^0\). Again by uniqueness in Proposition 2, one has \(\bar{A}^h = 0\)
which is equivalent to \(A^h = 0\) on \(H^0\). It follows from \(Ad - q^2 dA = 0\)
and from the fact that \(q^2\) is a primitive \(h\)-root of the unity that one has
\((d + A)^h = d^h + A^h\) which implies \((d + A)^h = 0.\) □

Thus \(Q = d + A\) is an \(h\)-differential. The main result of this section,
Theorem 1, states that the generalized homology \(H_{(k)}(H^0, Q)\) coincides
with \(H_{(k)}(H_I, A)\). In order to prove the result we shall need the following
construction and lemma, (Lemma 1). Let \(E\) be a vector space equipped
with a nilpotent endomorphism \(L\) satisfying \(L^h = 0\), (i.e. \((E, L)\) is an
\(h\)-differential space), and let \(E^* = \bigoplus_n E^n\) be the graded vector space
defined by setting \(E^n = E\) for \(0 \leq n \leq h - 1\) and \(E^n = 0\) otherwise. Let \(\delta\)
and \(L\) be the endomorphisms of the vector space \(E^*\) defined by setting
\(\delta(\psi)_n = \psi_{n-1}\) and \(L(\psi)_n = q^{2n}L(\psi)_n\) for \(0 \leq n \leq h - 1\) with \(\psi = \bigoplus_n \psi_n\)
(\(\psi_n \in E^n\)). One has \((\delta + L)^h = 0\) because \(L\delta - q^2 \delta L = 0,\)
\(\delta^h = \mathcal{L}^h = 0.\)

Thus \((E^*, \delta + L)\) is an \(h\)-differential vector space.

\textbf{LEMMA 1} One has \(H_{(k)}(E^*, \delta + L) = 0\) for \(1 \leq k \leq h - 1\).

\textbf{Proof.} In view of Lemma 3 in [7] it is sufficient to prove that one has
\(H_{(1)}(E^*, \delta + L) = 0.\) So let \(\psi\) be such that \((\delta + L)(\psi) = 0.\) By definition
this means \(L(\psi_0) = 0\) and \(\psi_{n-1} + q^{2n}L(\psi_n) = 0\) for \(1 \leq n \leq h - 1\)
which is equivalent to \( \psi_k = (-1)^{h-1-k} q^{h(h-1)-k(k+1)} L^{h-1-k}(\psi_{h-1}) \) for \( 0 \leq k \leq h-1 \) (since \( L(\psi_0) = 0 \) follows then from \( L^h = 0 \)). On the other hand let \( \varphi = \bigoplus \varphi_n \in \mathcal{E}^\bullet \) be defined by \( \varphi_0 = (-1)^{h-1} q^{h(h-1)} \psi_{h-1} = \psi_{h-1} \) and \( \varphi_n = 0 \) otherwise; then one has \((\delta + \mathcal{L})^{h-1}(\varphi) = \psi\). This proves that \( H(1)(\mathcal{E}^\bullet, \delta + \mathcal{L}) = 0 \) and implies the result. □

**Theorem 1** The generalized \( Q \)-homology of \( \mathcal{H}^\bullet \) coincides with the generalized \( A \)-homology of \( \mathcal{H}_I \), i.e. one has \( H(k)(\mathcal{H}^\bullet, Q) = H(k)(\mathcal{H}_I, A) \) for \( 1 \leq k \leq h-1 \).

**Proof.** Let us consider the previous \( h \)-differential vector space \((\mathcal{E}^\bullet, \delta + \mathcal{L})\) for the choices \( \mathcal{E} = \mathcal{H}/\mathcal{H}_I \) and \( L = A_\pi \). One defines a surjective linear mapping \( \beta \) of \( \mathcal{H}^\bullet \) onto \( \mathcal{E}^\bullet \) by setting \( \beta = \text{Id} : \mathcal{H}^n \to \mathcal{E}^n \) for \( 1 \leq n \leq h-1 \) and \( \beta = \pi : \mathcal{H}^0 \to \mathcal{E}^0 \) where \( \text{Id} \) is the identity mapping of \( \mathcal{H}/\mathcal{H}_I \) onto itself and where \( \pi \) is the canonical projection of \( \mathcal{H} \) onto \( \mathcal{H}/\mathcal{H}_I \). The kernel of \( \beta \) is obviously \( \mathcal{H}_I \) so one has a short exact sequence

\[
0 \to \mathcal{H}_I \xrightarrow{\alpha} \mathcal{H}^\bullet \xrightarrow{\beta} \mathcal{E}^\bullet \to 0
\]

where \( \alpha \) is the composition of inclusions \( \mathcal{H}_I \subset \mathcal{H} = \mathcal{H}^0 \subset \mathcal{H}^\bullet \). It is straightforward to verify that one has \( \alpha \circ A = (d+A) \circ \alpha \) and \( \beta \circ (d+A) = (\delta + \mathcal{L}) \circ \beta \) so one has in fact a short exact sequence of \( h \)-differential vector spaces

\[
0 \to (\mathcal{H}_I, A) \xrightarrow{\alpha} (\mathcal{H}^\bullet, Q) \xrightarrow{\beta} (\mathcal{E}^\bullet, \delta + \mathcal{L}) \to 0
\]

By using Lemma 1 above and Lemma 2 of \([7]\), one obtains the exact sequences

\[
0 \xrightarrow{\partial} H(k)(\mathcal{H}_I, A) \xrightarrow{\alpha_*} H(k)(\mathcal{H}^\bullet, Q) \xrightarrow{\beta_*} 0
\]

where \( \alpha_* \) and \( \beta_* \) are induced by \( \alpha \) and \( \beta \) and where \( \partial \) is the connecting homomorphism \([7], [21]\). Thus \( \alpha_* \) is an isomorphism which allows the
canonical identifications of Theorem 1. □

Remark 1. Notice that the content of this section does only depend on the data \((\mathcal{H}, A, \mathcal{H}_I)\) where \((\mathcal{H}, A)\) is an \(\hbar\)-differential vector space and \(\mathcal{H}_I\) is a subspace of \(\mathcal{H}\) invariant by \(A\). The same remark applies to Section 3, except that, of course, the specific dimensions are also involved there.

5 Extension to Hochschild cochains

Although the construction of last section is quite optimal, it is lacking a “geometrico-physical” interpretation. Our aim in the following is to cure that by casting the construction in a form which is closer to the BRS formulation in gauge theory or in constrained systems. To this end we recall that \(\mathcal{H}\) is a representation space for the Hopf algebra \(\mathcal{U}_q\) and that \(\mathcal{H}_I\) is the subspace of \(\mathcal{U}_q\)-invariant elements of \(\mathcal{H}\). An element \(\Psi \in \mathcal{H}\) is said to be \(\mathcal{U}_q\)-invariant, or simply invariant when no confusion arises, if it satisfies

\[ X \Psi = \Psi \varepsilon(X), \quad \forall X \in \mathcal{U}_q \]  

(5.1)

where \(\varepsilon\) denotes the counit of \(\mathcal{U}_q\). It turns out that (5.1) has a natural interpretation in terms of Hochschild cohomology. To see this, we equip \(\mathcal{H}\) with a structure of bimodule over \(\mathcal{U}_q\). One already has a structure of left \(\mathcal{U}_q\)-module on \(\mathcal{H}\) given by the representation of \(\mathcal{U}_q\) in \(\mathcal{H}\). We equip \(\mathcal{H}\) with a structure of right \(\mathcal{U}_q\)-module by using the scalar representation given by the counit \(\varepsilon\). Since one obviously has \((X \Psi) \varepsilon(Y) = X(\Psi \varepsilon(Y))\) for any \(\Psi \in \mathcal{H}\) and \(X, Y \in \mathcal{U}_q\), \(\mathcal{H}\) is a bimodule. One can introduce the graded space \(C(\mathcal{U}_q, \mathcal{H}) = \bigoplus_{n \geq 0} C^n(\mathcal{U}_q, \mathcal{H})\) of \(\mathcal{H}\)-valued Hochschild cochains of \(\mathcal{U}_q\), where \(C^n(\mathcal{U}_q, \mathcal{H})\) is the vector space of all linear mappings of \(\otimes^n \mathcal{U}_q\) into \(\mathcal{H}\), (i.e. \(n\)-linear mappings of \((\mathcal{U}_q)^n\) into \(\mathcal{H}\)). Equipped with the Hochschild differential \(d_H\), \(C(\mathcal{U}_q, \mathcal{H})\) is a complex and the \(\mathcal{H}\)-valued
Hochschild cohomology of $U_\theta$, $H(U_\theta, \mathcal{H}) = \bigoplus_{n \geq 0} H^n(U_\theta, \mathcal{H})$, is the homology of this complex. Now the condition \cite{5,1} for $\Psi$ to be in $\mathcal{H}_I$ also reads $d_H \Psi = 0$. Therefore $\mathcal{H}_I$ identifies with $H^0(U_\theta, \mathcal{H})$. However, except for $h = 2$, one cannot mix reasonably the Hochshild differential $d_H$ satisfying $d_H^2 = 0$ with (an extension of) the nilpotent $A$ satisfying $A^h = 0$. Fortunately, there is an $h$-differential $d$ on $C(U_\theta, \mathcal{H})$ which coincides with $d_H$ in degree 0. This $d$ was introduced in \cite{9} (with the notation $d = d_H$) and was analysed in details in \cite{7} (with the notation $d = d_1$; see Remark 3 below). It is given for $\omega \in C^n(U_\theta, \mathcal{H})$ by

$$d(\omega)(X_0, \ldots, X_n) = X_0 \omega(X_1, \ldots, X_n) + \sum_{k=1}^n q^{2k} \omega(X_0, \ldots, (X_{k-1}X_k), \ldots, X_n) - q^{2n} \omega(X_0, \ldots, X_{n-1}) \varepsilon(X_n).$$

(5.2)

**Lemma 2** Let $\Psi \in \mathcal{H} = C^0(U_\theta, \mathcal{H})$; the following conditions (i), (ii) and (iii) are equivalent

(i) $d^k(\Psi) = 0$ for some $k$ with $1 \leq k \leq h - 1$

(ii) $\Psi \in \mathcal{H}_I$

(iii) $d^n(\Psi) = 0$ for any $n \in \{1, \ldots, h - 1\}$.

**Proof.** We know that $\Psi \in \mathcal{H}_I$ is equivalent to $d_H \Psi = 0$ and, since $d = d_H$ on $C^0(U_\theta, \mathcal{H})$ this is equivalent to $d \Psi = 0$. The implication (ii)$\Rightarrow$(iii) follows (since $d \Psi = 0 \Rightarrow d^n \Psi = 0$ for $n \geq 1$). The implication (iii)$\Rightarrow$(i) is clear. It remains to show the implication (i)$\Rightarrow$(ii) to achieve the proof of the lemma. By induction on $n$ and by using definition (5.2) one has for $\Psi \in C^0(U_\theta, \mathcal{H})$

$$d^n \Psi(1, \ldots, 1, X) = (1 + q^2) \ldots (1 + q^2 + \cdots + q^{2(n-1)}) d \Psi(X)$$

(5.3)
for any \( n \geq 1, \, X \in \mathcal{U}_q \) where \( \mathbb{1} \) is the unit of \( \mathcal{U}_q \). Formula (5.3) shows that \( d^k \Psi = 0 \) for some \( k \in \{1, \ldots, h - 1\} \) implies \( d\Psi = 0 \), i.e. \( \Psi \in \mathcal{H}_I \) and thus the implication \((i) \Rightarrow (ii)\). □

As an easy consequence of this lemma one obtains the following result.

**PROPOSITION 4** The \( h \)-complex \( (\mathcal{H}^\bullet, d) \) can be canonically identified with the \( h \)-subcomplex of \( (C(\mathcal{U}_q, \mathcal{H}), d) \) generated by \( \mathcal{H} \) (that is with \( \mathcal{H} \oplus d\mathcal{H} \oplus \cdots \oplus d^{h-1} \mathcal{H} \subset C(\mathcal{U}_q, \mathcal{H}) \) for the \( h \)-differential \( d \)).

**Proof.** In view of Proposition 2 (i.e. in view of the universal property of \( (\mathcal{H}^\bullet, d) \)), the identity mapping of \( \mathcal{H} \) onto itself extends uniquely as a homomorphism of \( h \)-complexes of \( (\mathcal{H}^\bullet, d) \) into \( (C(\mathcal{U}_q, \mathcal{H}), d) \). Lemma 2 then implies that this homomorphism is injective. □

Thus one has \( \mathcal{H}^\bullet \subset C(\mathcal{U}_q, \mathcal{H}) \) and the \( h \)-differential \( d \) of \( C(\mathcal{U}_q, \mathcal{H}) \) extends the one of \( \mathcal{H}^\bullet \); we now extend \( A \) to \( C(\mathcal{U}_q, \mathcal{H}) \).

**PROPOSITION 5** Let us extend \( A \) to \( C(\mathcal{U}_q, \mathcal{H}) \) as a homogeneous endomorphism \( \omega \mapsto (A\omega) \) of degree 0 by setting

\[
(A\omega)(X_1, \ldots, X_n) = q^{2n}A\omega(X_1, \ldots, X_n)
\]

for \( \omega \in C^n(\mathcal{U}_q, \mathcal{H}) \) and \( X_i \in \mathcal{U}_q \). On \( C(\mathcal{U}_q, \mathcal{H}) \) one has \( Ad - q^2dA = 0 \), \( A^h = 0 \) and \((d + A)^h = 0 \).

**Proof.** Consider first the extension \( \bar{A} \) defined by \((\bar{A}\omega)(X_1, \ldots, X_n) = A\omega(X_1, \ldots, X_n)\) for \( \omega \in C^n(\mathcal{U}_q, \mathcal{H}) \) and \( X_i \in \mathcal{U}_q \). Then, by using the fact that the action of \( \mathcal{U}_q \) on \( \mathcal{H} \) commutes with \( A \) (see Section 2), one obtains \( \bar{A}d = d\bar{A} \); more generally, if the cofaces of \( C(\mathcal{U}_q, \mathcal{H}) \)

\[
f_\alpha : C^n(\mathcal{U}_q, \mathcal{H}) \to C^{n+1}(\mathcal{U}_q, \mathcal{H}), \quad \alpha \in \{0, \ldots, n + 1\}
\]

are defined by

\[
(f_0\omega)(X_0, \ldots, X_n) = X_0\omega(X_1, \ldots, X_n),
\]

\[
(f_i\omega)(X_0, \ldots, X_n) = \omega(X_0, \ldots, (X_{i-1}X_i), \ldots, X_n)
\]
for $i \in \{1, \ldots, n\}$ and

$$(f_{n+1}\omega)(X_0, \ldots, X_n) = \omega(X_0, \ldots, X_{n-1})\varepsilon(X_n),$$

then one has $\bar{A}f_\alpha = f_\alpha\bar{A}, \forall \alpha \in \{0, \ldots, n+1\}$. This implies that $A = q^{2D}\bar{A} = \bar{A}q^{2D}$ satisfies in particular $Ad - q^2dA = 0$, $D$ being the cochain’s degree; more generally $A$ satisfies $Af_\alpha = q^2f_\alpha A$. Furthermore $\bar{A}^h = 0$, which is straightforward, implies $A^h = 0$. The last equality $(d+A)^h = 0$ follows by the same argument as in the proof of Proposition 3. $\square$

We have now extended to $C(\mathcal{U}_q, \mathcal{H})$ the whole structure defined on $\mathcal{H}^\bullet$ in the previous section. Indeed the uniqueness in Proposition 3 implies that $A$ defined on $C(\mathcal{U}_q, \mathcal{H})$ in Proposition 5 is an extension of $A$ defined on $\mathcal{H}^\bullet$ in Proposition 3. One then extends to $C(\mathcal{U}_q, \mathcal{H})$ the definition of $Q$ by setting again $Q = d + A$. Next section will be devoted to the formulation and the discussion of the appropriate extension to $C(\mathcal{U}_q, \mathcal{H})$ of Theorem 1.

Remark 2. Lemma 2 implies : $H^0_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d) = H^0(\mathcal{U}_q, \mathcal{H}), \forall k \in \{1, \ldots, h - 1\}$. This is a special case of Theorem 4 (1) of [7] which reads here : $H^{hr}_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d) = H^{2r}(\mathcal{U}_q, \mathcal{H}), H^{h(r+1)-k}_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d) = H^{2r+1}(\mathcal{U}_q, \mathcal{H}),$ and $H^n_{(h)}(C(\mathcal{U}_q, \mathcal{H}), d) = 0$ otherwise.

Remark 3. Given primitive $h$-th root of the unity $q^2$, one can construct on $C(\mathcal{U}_q, \mathcal{H})$ several $h$-differentials of degree 1 which coincide with the Hochschild differential when $q^2 = -1$. A whole sequence $(d_n)_{n \in \mathbb{N}}$ of such $h$-differentials has been introduced in [7] where their generalized cohomologies were computed in terms of the ordinary Hochschild cohomology. For the case of $d_0$, a $h$-differential which has be considered by several authors ([20], [21]), this computation has been done independently by Kassel and Wambst by using very interesting generalizations of concepts of homological algebra. However here only $d_1 = d$ should be used for $h > 2$ because $d_0$ does not coincide with the Hochschild
differential $d_H$ on $\mathcal{H}$ and, on the other hand, although $d_n$ coincides with $d_H$ on $\mathcal{H}$ for $n \geq 1$ it also coincides with $d_H$ on the 1-cochains for $n \geq 2$ and thus $d_n^2$ vanishes on $\mathcal{H}$ whenever $n \geq 2$.

### 6 Generalized homology of $Q$ on $C(\mathcal{U}_q, \mathcal{H})$

As explained in [7] (see Remark 2 above) the spaces $H^n_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d)$ can be computed in terms of the Hochschild cohomology $H(\mathcal{U}_q, \mathcal{H})$. In particular, one sees that $H^n_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d)$ does not generally vanish for $n \geq 1$. This implies that one cannot expect for the generalized homology of $Q$ on $C(\mathcal{U}_q, \mathcal{H})$ such a simple result as the one given by Theorem 1 for the generalized homology of $Q$ on $\mathcal{H}^\bullet$. Nevertheless, in view of Lemma 2, one has $H^0_{(k)}(C(\mathcal{U}_q, \mathcal{H}), d) = \mathcal{H}_I = H^0_{(k)}(\mathcal{H}^\bullet, d)$ and therefore one may expect $H^0_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q) = H_{(k)}(\mathcal{H}_I, A)(= H_{(k)}(\mathcal{H}^\bullet, Q))$. In fact, this is essentially true. However some care must be taken because $Q$ is not homogeneous so $H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)$ is not a graded vector space. Instead of a graduation, one has an increasing filtration $F^n H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)$, $(n \in \mathbb{Z})$, with $F^n H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q) = 0$ for $n < 0$ and where, for $n \geq 0$, $F^n H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)$ is the canonical image in $H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)$ of $\text{Ker}(Q^k) \cap \bigoplus_{r=0}^{\infty} C^r(\mathcal{U}_q, \mathcal{H})$. There is an associated graded vector space

$$
\text{gr} \, H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q) = \bigoplus_n F^n H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q) / F^{n-1} H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)
$$

which here is $\mathbb{N}$-graded. One has

$$
F^0 H_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q) = \text{gr} \, H^0_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)
$$

and it is this space which is the correct version of the $H^0_{(k)}(C(\mathcal{U}_q, \mathcal{H}), Q)$ above in order to identify $H_{(k)}(\mathcal{H}_I, A)$ in the generalized homology of $Q$ on $C(\mathcal{U}_q, \mathcal{H})$. 

18
THEOREM 2 The inclusion $\mathcal{H}^\bullet \subset C(U_q, \mathcal{H})$ induces the isomorphisms

$$H(k)(\mathcal{H}^\bullet, Q) \simeq F^0 H(k)(C(U_q, \mathcal{H}), Q)$$ for $1 \leq k \leq h - 1$.

In particular, with obvious identifications, one has

$$F^0 H(k)(C(U_q, \mathcal{H}), Q) = H(k)(\mathcal{H}_I, A), \quad \forall k \in \{1, \ldots, h - 1\}.$$ 

Proof. Let $\Psi \in \mathcal{H}$ be such that one has $Q^k\Psi = 0$ for some $k$ ($1 \leq k \leq h - 1$). By expanding $(d+A)^k\Psi = 0$, one obtains $d^k\Psi = 0$ for the highest degree and $A^k\Psi = 0$ for the lowest degree. In view of Lemma 2 this is equivalent to $\Psi \in \mathcal{H}_I$ and $A^k\Psi = 0$; this conversely implies $Q^k\Psi = 0$.

Thus $Q^k\Psi = 0$ for $\Psi \in \mathcal{H}$ is equivalent to $\Psi \in \mathcal{H}_I$ and $A^k\Psi = 0$. On the other hand $\Psi = Q^{h-k}\Phi$ for $\Psi \in \mathcal{H}$ implies $\Phi \in \mathcal{H}$ and $d^{h-k}\Phi = 0$ which by using again Lemma 2 implies $\Phi \in \mathcal{H}_I$ and $\Psi = A^{h-k}\Phi = Q^{h-k}\Phi$ ($\in \mathcal{H}_I$). This means that one has canonically:

$$F^0 H(k)(C(U_q, \mathcal{H}), Q) = H(k)(\mathcal{H}_I, A) = H(k)(\mathcal{H}^\bullet, Q)$$

which completes the proof of Theorem 2. □

If one compares this construction involving Hochschild cochains with the construction of Section 4, what has been gained here besides the explicit occurrence of the quantum gauge aspect is that the extended space $C(U_q, \mathcal{H})$ is a tensor product $\mathcal{H} \otimes \mathcal{H}'$ of the original space $\mathcal{H}$ with the tensor algebra $\mathcal{H}' = T(U_q^*)$ of the dual space of $U_q$. The factor $\mathcal{H}'$ can thus be interpreted as the state space for some generalized ghost. What has been lost is the minimality of the generalized homology, i.e. besides the “physical” $H(k)(\mathcal{H}_I, A)$, the generalized homology of $Q$ on $\mathcal{H} \otimes \mathcal{H}'$ contains some other non trivial subspace in contrast to what happens on $\mathcal{H}^\bullet$. In the usual homological (BRS) methods however such a “non minimality” also occurs. Indeed, for instance, in the homological approach to constrained classical systems, the relevant homology contains
besides the functions on the reduced phase space the whole cohomology of longitudinal forms \[5\]. The same is true for the BRS cohomology of gauge theory \[2\], \[4\].

In the usual situations where one applies the BRS construction (gauge theory, constrained systems) one has a Lie algebra \(g\) (the Lie algebra of infinitesimal gauge transformations) acting on some space \(\mathcal{H}\) and what is really relevant at this stage is the Lie algebra cohomology \(H(g, \mathcal{H})\) of \(g\) acting on \(\mathcal{H}\). The extended space is then the space of \(\mathcal{H}\)-valued Lie algebra cochains of \(g\), \(C(g, \mathcal{H})\). This extended space is thus also a tensor product \(\mathcal{H} \otimes \mathcal{H}'\) but now \(\mathcal{H}'\) is the exterior algebra \(\mathcal{H}' = \Lambda g^*\) of the dual space of \(g\). That is why this factor can be interpreted (due to antisymmetry) as a fermionic state space; indeed that is the reason why one gives a fermionic character to the ghost \[2\], \[4\], \[24\], \[5\]. There is however another way to proceed in these situations which is closer to what has been done in our case here. To understand it, we recall that any representation of \(g\) in \(\mathcal{H}\) is also a representation of the enveloping algebra \(U(g)\) in \(\mathcal{H}\). Thus \(\mathcal{H}\) is a left \(U(g)\)-module. Since \(U(g)\) is a Hopf algebra, one can convert \(\mathcal{H}\) into a bimodule for \(U(g)\) by taking as right action the trivial representation given by the counit. It turns out that the \(\mathcal{H}\)-valued Hochschild cohomology of \(U(g)\), \(H(U(g), \mathcal{H})\), coincides with the \(\mathcal{H}\)-valued Lie algebra cohomology of \(g\), \(H(g, \mathcal{H})\), i.e. one has \[18\], \[23\]: \(H(U(g), \mathcal{H}) = H(g, \mathcal{H})\). Since it is the latter space which is relevant one can as well take as extended space the space of \(\mathcal{H}\)-valued Hochschild cochains of \(U(g)\), \(C(U(g), \mathcal{H})\), and then compute its cohomology. Again this space is a tensor product \(\mathcal{H} \otimes \mathcal{H}'\) but now \(\mathcal{H}' = T(U(g)^*)\) is a tensor algebra as in our case.

In the above brief discussion of the usual BRS, we have oversimplified the situation. In general the extended space contains slightly more than
the Lie algebra cochains and the Chevalley-Eilenberg differential is only a part of the BRS operator. However the above picture is the essential point. The gist of our message is the realization that the construction of the present paper is in fact very close to the standard BRS procedure. The main difference (or extension) is the occurrence of a nilpotent $Q$ satisfying $Q^h = 0$ with $h > 2$ (instead of the usual $Q^2 = 0$) and, correspondingly, the occurrence of the generalized homology (instead of an ordinary homology).

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