High-precision calculation of the 4-loop QED contribution to the slope of the Dirac form factor

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Abstract

We have evaluated with 1100 digits of precision the contribution of all the 891 mass-independent 4-loop Feynman diagrams contributing to the slope of the Dirac form factor in QED. The total 4-loop contribution is

\[ m^2 F_1^{(4)}(0) = 0.886545673946443145836821730610315359390424032660064745 \ldots \left( \frac{\alpha}{\pi} \right)^4. \]

We have fit a semi-analytical expression to the numerical value. The expression contains harmonic polylogarithms of argument \( e^{i\pi/3}, e^{2i\pi/3}, e^{i\pi} \), one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits. We show the correction on the shift of the energy levels of the hydrogen atom due to the slope.

Keywords: Quantum electrodynamics; Dirac form factor; Hydrogen atom; Feynman diagram; High-precision calculation; Analytical fit;
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Recently in Ref.[1] the 4-loop contribution to the electron \( g-2 \) in QED was calculated numerically with very-high precision, and a semi-analytical fit was obtained. In this companion paper we use the same techniques to calculate the 4-loop QED contribution to the first derivative of the Dirac form factor.

In QED the amplitude for a vertex function can be written

\[
(-ie)\bar{u}(p_1)\Gamma_\mu(p_1,p_2)u(p_2) = (-ie)\bar{u}(p_1) \left( \gamma_\mu F_1(t) + \frac{\sigma_{\mu\nu}}{2m} q_\nu F_2(t) \right) u(p_2),
\]

where \( m \) is the electron mass, \( p_1, p_2 \) and \( q \) are the momenta of the electrons and the photon, satisfying

\[ p_1^2 = p_2^2 = -m^2, \quad q = p_1 - p_2, \quad t = -q^2. \]

\( F_1(t) \) and \( F_2(t) \) are the Dirac and Pauli form factors. At \( t = 0 \), charge conservation implies that

\[ F_1(0) = 1, \]

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whereas the value of the Pauli form factor is the $g^2$

$$F_2(0) = \frac{g^2}{2}.$$  \hfill (4)

The quantity \( \left. \frac{d}{dt} F_1(t) \right|_{t=0} = F_1'(0) \) is the slope of the Dirac form factor. The slope can be expanded perturbatively in powers of \((\alpha \pi)\)

$$m^2 F_1'(0) = A_1 \left( \frac{\alpha}{\pi} \right) + A_2 \left( \frac{\alpha}{\pi} \right)^2 + A_3 \left( \frac{\alpha}{\pi} \right)^3 + A_4 \left( \frac{\alpha}{\pi} \right)^4 + \ldots .$$  \hfill (5)

The coefficient \( A_1 \) in Eq. (5) is I.R. divergent:

$$A_1 = -\frac{1}{8} - \frac{1}{6\epsilon} ;$$  \hfill (6)

due to the on-mass-shell condition of the external electron\(^1\) from two loops onward, the coefficients are finite. The two-loop and three-loop coefficients are known in analytical form \[2–4\]

$$A_2 = \frac{4819}{5184} - \frac{49}{72}(2) - \frac{3}{4} \zeta(3) + 3\zeta(2) \ln 2 = 0.469 941 487 459 992 \ldots ;$$  \hfill (7)

$$A_3 = \frac{77513}{186624} - \frac{454979}{6480}(2) - \frac{2929}{288} \zeta(3) + \frac{41671}{360} \zeta(2) \ln 2 + \frac{3899}{288} \zeta(4) - \frac{103}{180} \zeta(2) \ln^2 2 - \frac{217}{9} \left( a_4 + \frac{1}{24} \ln^4 2 \right) + \frac{25}{8} \zeta(5) - \frac{17}{4} \zeta(3) \zeta(2) = 0.171 720 018 909 775 \ldots ,$$  \hfill (8)

where \( \zeta(n) \equiv \sum_{i=1}^{\infty} i^{-n} , \ a_n = \sum_{i=1}^{\infty} 2^{-i} i^{-n} \).

In this paper we present the result of the calculation of \( A_4 \) with a precision of 1100 digits. The first digits of the result are

$$A_4 = 0.88654567394644314583682173061031535939042403266006475436805590932084031664656282974548364863241773368 \ldots$$  \hfill (9)

The full-precision result is shown in Table 1. We note that \( A_2 , A_3 \) and \( A_4 \) are all

\[1\] In the calculation of the shift to energy levels due to the slope at one loop, the off-mass-shell condition has to be taken into account, and this removes the divergence and gives rise to the Bethe logarithms\[3 \text{ and } 4\].

Table 1: First 1100 digits of \( A_4 \).
positive, in contrast with the alternating signs observed in the $g-2$ up to 5 loops.

Let us now consider the shift to the hydrogen energy levels due to $A_4$. We express the energy shift in terms of the frequency shift $\Delta f = \Delta E/h$. For the level $nS$ the frequency shift is [5, 6]

$$\Delta f_{\text{slope}}(nS, 4\text{-loop}) = 4(Z\alpha)^4 mc^2 h n^3 \left[ \left( \frac{m_r}{m} \right)^3 \left( \frac{\alpha}{\pi} \right)^4 A_4 \right],$$

where $m_r$ is the reduced mass $m_r = mM/(m + M)$ and $M$ is the proton mass. Inserting the values of $m$, $M$, $c$, $h$ and $Z = 1$, the correction due to $A_4$ is

$$\Delta f_{\text{slope}}(nS, 4\text{-loop}) = \frac{36.11}{n^3} \text{ Hz},$$

and is comparable with the experimental error of the extremely precise measurement of $1S - 2S$ transition [7]

$$f(1S - 2S) = 2466061413187018 \pm 11 \text{ Hz}.$$

Eq. (11) is the first calculated 4-loop correction to energy levels, of the kind $\left( \frac{\alpha}{\pi} \right)^4 (Z\alpha)^4$; we note that there are some two-loop and three-loop radiative corrections which still have theoretical errors larger than Eq. (11), of the order of $10 \left( \frac{\alpha}{\pi} \right)^2 (Z\alpha)^6$ and $10 \left( \frac{\alpha}{\pi} \right)^3 (Z\alpha)^5$, respectively (see [8, 9]).
Now we consider the shift due to all the QED 4-loop contributions: $A_4$ from $F_1'(0)$, $\alpha_{e}^{(4)}$ from $g-2$ (see Eq.(2) of Ref.[1]) and $\Pi_{le}^{(4)}$ from vacuum polarization (see Eq.(4) of Ref.[10]). Writing

$$\Delta f_{4\text{-loop QED}}(nS) = \frac{(Z\alpha)^4 m^2}{h n^3} \left( \frac{m_e}{m} \right)^3 \left( \frac{\alpha}{\pi} \right)^4 D_{40} ,$$

then

$$D_{40} = 4A_4 + \alpha_{e}^{(4)} - \Pi_{le}^{(4)} = 3.546182 - 1.912245 - 1.583612$$

$$= 0.0503246508259024550858429619942750274917\ldots$$

Note the deep numerical cancellation. Therefore

$$\Delta f_{4\text{-loop QED}}(nS) = \frac{0.513}{n^3} \text{ Hz}.\quad (15)$$

There are 891 vertex diagrams contributing to $A_4$. They can be obtained by inserting an external photon in each possible electron line of the 104 4-loop self-mass diagrams shown in Fig.1. Because of the Furry’s theorem, the vertex diagrams with closed electron loops with an odd number of vertices do not contribute, and are not considered. The vertex diagrams can be arranged in 25 gauge-invariant sets (Fig.2). The sets are classified according to the number of photon corrections on the same side of the main electron line and the insertions of electron loops (see Ref.[11]). The numerical contributions of each set, truncated to 40 digits, are listed in the table 2. Adding the contributions of diagrams with and without closed electron loops one finds

$$A_4(\text{no closed electron loops}) =$$

$$0.3514798015766637774090446716934794695266,$$  \quad (16)

$$A_4(\text{closed electron loops only}) =$$

$$0.5350658723697793684277770589168358898637 .$$

By building systems of integration-by-parts identities\cite{12,13} and solving them\cite{14}, the contributions of all the diagrams to $A_4$ are expressed as linear combinations of 334 master integrals, the same ones as appeared in the calculation of 4-loop $g-2$\cite{1}. 

Figure 2: The 25 gauge-invariant sets. We show one single vertex diagram for each set.
Table 2: Contribution to $A_4$ of the 25 gauge-invariant sets of Fig.2.

In Ref.[1] these master integrals were calculated numerically with precision ranging from 1100 to 9600 digits; analytical expressions were fit to all these master integrals (single or in particular combinations) by using the PSLQ algorithm[15,16]. For the scope of this work these results suffice, with the exception of a new combination of elliptic master integrals, which has been successfully fit by using the same basis used for the other master integrals. Therefore, the analytical expression of $A_4$ contains the same transcendentals appeared in the $g$-2 result: values of harmonic polylogarithms[17,18] with argument $1$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, a family of one-dimensional integrals of products of elliptic integrals, and the finite terms of the $\epsilon$-expansions of six master integrals belonging to the topologies 81 and 83 of Fig.1. The result of the analytical fit is written as follows:

$$A_4 = T_0 + T_2 + T_3 + T_4 + T_5 + T_7 + \sqrt{3} (V_{4a} + V_{6a}) + V_{6b} + V_{7b}$$

$$+ W_{4a} + W_{6b} + W_{7a} + \sqrt{3} (E_{4a} + E_{5a} + E_{6a} + E_{7a}) + E_{6b} + E_{7b} + U.$$  \hspace{1cm} (18)

The terms have been arranged in blocks with equal transcendental weight. The index number is the weight. The terms containing the “usual” transcendental constants are:
\[ T_5 = \frac{9035973}{134400} \zeta(5) + \frac{1173056009}{9072000} \zeta(3) \zeta(2) - \frac{8548241}{30240} \zeta(4) \ln 2 - \frac{68168}{135} t_5, \]
\[ T_6 = -\frac{244603373713}{522547200} \zeta(6) - \frac{8082848863}{24192000} \zeta^2(3) + \frac{159693503}{72000} \zeta(3) \zeta(2) \ln 2 - \frac{328317209}{302400} \zeta(4) \ln^2 2 + \frac{10415209}{189000} t_4 \zeta(2) - \frac{72000}{18215} t_6_1 + \frac{26062}{27} t_6_2, \]
\[ T_7 = -\frac{7224951103}{1741824} \zeta(7) - \frac{1267114025}{387072} \zeta(4) \zeta(3) - \frac{2749470791}{387072} \zeta(5) \zeta(2) + \frac{971827}{128} \zeta(6) \ln 2 - \frac{6242389}{6048} \zeta(3) \zeta(2) \ln^2 2 - \frac{427145}{504} t_4 \zeta(3) + \frac{1420289}{180} t_5 \zeta(2) + \frac{256321}{756} t_7_1 - \frac{116987}{63} t_7_2 + \frac{104041}{20} t_7_3, \]

where
\[ t_4 = a_4 + \frac{1}{24} \ln^4 2, \quad t_5 = a_5 + \frac{1}{12} \zeta(2) \ln^3 2 - \frac{1}{120} \ln^5 2, \]
\[ t_{6_1} = b_6 - a_5 \ln 2 + \zeta(5) \ln 2 + \frac{1}{6} \zeta(3) \ln^3 2 - \frac{1}{12} \zeta(2) \ln^4 2 + \frac{1}{144} \ln^6 2, \]
\[ t_{6_2} = a_6 - \frac{1}{48} \zeta(2) \ln^4 2 + \frac{1}{720} \ln^6 2, \]
\[ t_{7_1} = d_7 - 2 b_6 \ln 2 + 4 a_6 \ln 2 + 2 a_5 \ln^2 2 - \frac{49}{32} \zeta^2(3) \ln 2 - \frac{95}{32} \zeta(5) \ln^2 2 + \frac{1}{8} \zeta(4) \ln^3 2 - \frac{1}{3} \zeta(3) \ln^4 2 + \frac{1}{12} \zeta(2) \ln^5 2 - \frac{1}{120} \ln^7 2, \]
\[ t_{7_2} = b_7 - 3 a_7 - a_6 \ln 2 - \frac{1}{2} \zeta(5) \ln^2 2 + \frac{1}{48} \zeta(4) \ln^3 2 - \frac{1}{24} \zeta(3) \ln^4 2 + \frac{1}{120} \zeta(2) \ln^5 2 - \frac{1}{1680} \ln^7 2, \]
\[ t_{7_3} = \left( a_4 - \frac{1}{4} \zeta(2) \ln^2 2 + \frac{7}{16} \zeta(3) \ln 2 + \frac{1}{24} \ln^4 2 \right) \zeta(2) \ln 2. \]

The terms containing harmonic polylogarithms of $e^{4 \pi i}$, $e^{2 \pi i}$:
\[ V_{4a} = -\frac{14186171}{194400} \zeta(4) \left( \frac{\pi}{3} \right) - \frac{103023803}{583200} \zeta(2) \zeta(2) \left( \frac{\pi}{3} \right), \]
\[ V_{6a} = \frac{916508}{76545} v_{6_1} + \frac{844333}{28350} v_{6_2} + \frac{178619489}{3980340} v_{6_3} - \frac{263673944}{295245} v_{6_4}, \]
\[ V_{6_b} = \frac{212671}{2400} v_{6_5} - \frac{1031987}{14400} \zeta(2) \zeta(2) \left( \frac{\pi}{3} \right), \]
\[ V_{70} = -\frac{507}{4} v_{71} - \frac{295}{4} v_{72} , \]  
where

\[ v_{61} = \text{Im} H_{0,0,0,1,-1,-1} \left( e^{i \frac{\pi}{2}} \right) + \text{Im} H_{0,0,0,1,-1,1} \left( e^{i \frac{3\pi}{2}} \right) + \text{Im} H_{0,0,0,1,-1,1} \left( e^{i \frac{5\pi}{2}} \right) + \frac{27}{26} \text{Im} H_{0,0,1,0,1,1} \left( e^{i \frac{3\pi}{2}} \right) + \frac{207}{104} \text{Im} H_{0,0,0,0,1,1} \left( e^{i \frac{3\pi}{2}} \right) + \frac{10}{3} u_4 c_2 \left( \frac{\pi}{3} \right) + \frac{7}{4} \zeta(3) \text{Im} H_{0,1,-1} \left( e^{i \frac{3\pi}{2}} \right) + \frac{21}{8} \zeta(3) \text{Im} H_{0,1,1} \left( e^{i \frac{3\pi}{2}} \right) - \frac{5}{72} \zeta(3) \zeta(2) \pi \\
- \frac{5}{6} c_2 \left( \frac{\pi}{3} \right) \zeta(2) \ln^2 2 + \frac{5}{36} c_2 \left( \frac{\pi}{3} \right) \ln^4 2 - \frac{27413}{67392} \zeta(5) \pi + \frac{4975}{11583} \zeta(4) c_2 \left( \frac{\pi}{3} \right) , \]  

\[ v_{62} = \zeta(2) \left( \text{Im} H_{0,1,1,-1} \left( e^{i \frac{\pi}{2}} \right) + \frac{3}{2} \text{Im} H_{0,1,1,-1,1} \left( e^{i \frac{3\pi}{2}} \right) - \frac{1}{6} \zeta(3) \pi + \frac{1}{108} \zeta(2) \pi \ln 2 \\
- \frac{5}{2} \text{Im} H_{0,1,1,-1} \left( e^{i \frac{\pi}{2}} \right) \ln 2 - \frac{15}{4} \text{Im} H_{0,1,1,1} \left( e^{i \frac{3\pi}{2}} \right) \ln 2 + \frac{25}{12} c_2 \left( \frac{\pi}{3} \right) \ln^2 2 \\
- \frac{661}{1188} c_2 \left( \frac{\pi}{3} \right) \zeta(2) \right) , \]  

\[ v_{63} = c_6 \left( \frac{\pi}{3} \right) - \frac{3}{4} \zeta(4) c_2 \left( \frac{\pi}{3} \right) , \quad v_{64} = c_4 \left( \frac{\pi}{3} \right) \zeta(2) - \frac{91}{66} \zeta(4) c_2 \left( \frac{\pi}{3} \right) , \]  

\[ v_{65} = \text{Re} H_{0,0,0,1,0,1} \left( e^{i \frac{\pi}{2}} \right) + c_2 \left( \frac{\pi}{3} \right) c_4 \left( \frac{\pi}{3} \right) , \]  

\[ v_{71} = \text{Re} H_{0,0,0,1,0,1,-1} \left( e^{i \frac{\pi}{2}} \right) + 4 \text{Re} H_{0,0,0,0,0,1,1,-1} \left( e^{i \frac{\pi}{2}} \right) - \frac{27}{8} \text{Re} H_{0,0,1,0,0,0,1,1} \left( e^{i \frac{3\pi}{2}} \right) \\
- \frac{135}{16} \text{Re} H_{0,0,0,1,0,1,1} \left( e^{i \frac{3\pi}{2}} \right) - \frac{27}{2} \text{Re} H_{0,0,0,0,0,1,1,1} \left( e^{i \frac{3\pi}{2}} \right) \\
+ \text{Im} H_{0,1,-1} \left( e^{i \frac{\pi}{2}} \right) c_4 \left( \frac{\pi}{3} \right) + \frac{3}{2} \text{Im} H_{0,1,1} \left( e^{i \frac{3\pi}{2}} \right) c_4 \left( \frac{\pi}{3} \right) + \frac{145}{132} c_6 \left( \frac{\pi}{3} \right) \pi , \]  

\[ v_{72} = \zeta(2) \left( \text{Re} H_{0,1,0,1,-1} \left( e^{i \frac{\pi}{2}} \right) + 2 \text{Re} H_{0,0,1,1,-1} \left( e^{i \frac{\pi}{2}} \right) + \frac{9}{4} \text{Re} H_{0,1,0,1,1} \left( e^{i \frac{3\pi}{2}} \right) \\
+ \frac{9}{2} \text{Re} H_{0,0,1,1,1} \left( e^{i \frac{3\pi}{2}} \right) + \text{Im} H_{0,1,-1} \left( e^{i \frac{\pi}{2}} \right) c_2 \left( \frac{\pi}{3} \right) \\
+ \frac{3}{2} \text{Im} H_{0,1,1} \left( e^{i \frac{3\pi}{2}} \right) c_2 \left( \frac{\pi}{3} \right) \right) . \]
The terms containing harmonic polylogarithms of $e^{i\pi/2}$:

\[ W_{4a} = -\frac{1117}{36} \zeta(2) \text{Cl}_2 \left( \frac{\pi}{2} \right), \]

\[ W_{6b} = \frac{38424}{125} \zeta(2) \text{Cl}_2^2 \left( \frac{\pi}{2} \right), \]

\[ W_{7b} = -472 v_{73}, \]

where

\[ v_{73} = \left( \frac{\pi}{2} \right)^2 \left( \Re H_{0,1,0,1,1} \left( e^{i\pi/2} \right) + \text{Cl}_2 \left( \frac{\pi}{2} \right) \right) - \zeta(2) \text{Cl}_4 \left( \frac{\pi}{2} \right) \pi \]

\[ + \frac{1}{4} \text{Cl}_2^2 \left( \frac{\pi}{2} \right) \ln 2 \]

A term $\zeta(2) \text{Cl}_2 \left( \frac{\pi}{2} \right)$ appears in Eq. (40); it did not appear in the 4-loop $g^{-2}$ result [1] because of cancellations in the final sum of all 4-loop diagrams. The terms containing elliptic constants:

\[ E_{4a} = \pi \left( \frac{5581729229}{362880000} B_3 + \frac{1233637481}{1399680000} C_3 \right), \quad E_{5a} = -\frac{11495611}{3265920} \pi f_2(0,0,1), \]

\[ E_{6a} = -\frac{365478661}{24494400} e_{61} + \frac{119022487}{5443200} e_{62}, \quad E_{6b} = -\frac{751}{729} \zeta(2) f_1(0,0,1), \]

\[ E_{7a} = \frac{98285}{248832} e_{71} - \frac{157753}{497664} e_{72}, \quad E_{7b} = \frac{157753}{41472} e_{73} - \frac{99731}{1944} e_{74}, \]

where

\[ e_{61} = \pi \left( f_2(0,2,0) - \frac{9}{4} \ln 2 f_2(0,0,1) \right), \]

\[ e_{62} = \pi \left( f_2(0,1,1) - \frac{3}{8} f_2(0,0,2) - \frac{3}{2} \ln 2 f_2(0,0,1) \right), \]

\[ e_{71} = \pi \left( f_2(2,1,0) + \frac{7}{3} f_2(1,2,0) - 2 f_2(1,1,1) + \frac{40}{27} f_2(0,3,0) - \frac{7}{3} f_2(0,2,1) \right. \]

\[ + f_2(0,1,2) - 30 \ln 2 f_2(0,2,0) + 45 \ln 2 f_2(0,1,1) - \frac{135}{8} \ln 2 f_2(0,0,2) \]

\[ + \frac{135}{8} \ln 2 f_2(0,0,2) \right), \]

\[ e_{72} = \pi \left( f_2(2,0,1) + \frac{14}{3} f_2(1,2,0) - 2 f_2(1,1,1) - 2 f_2(1,0,2) - \frac{370}{27} f_2(0,3,0) \right. \]

\[ + \frac{85}{3} f_2(0,2,1) - 22 f_2(0,1,2) + 7 f_2(0,0,3) + 11 \zeta(2) f_2(0,0,1) \]

\[ - 20 \ln 2 f_2(0,2,0) + 30 \ln 2 f_2(0,1,1) - \frac{45}{4} \ln 2 f_2(0,0,2) \right), \]
Figure 3: Master integrals known only numerically. \((f, f', f'')\) and \((g, g', g'')\) have numerators equal to \((1, p.k, (p.k)^2)\), respectively.

\[
e_{73} = \zeta(2) \left( f_1(1,0,1) - f_1(0,1,1) + \frac{1}{4} f_1(0,0,2) \right), \tag{51}
\]

\[
e_{74} = \zeta(2) \left( f_1(0,2,0) - \frac{3}{2} f_1(0,1,1) + \frac{9}{16} f_1(0,0,2) \right). \tag{52}
\]

The term containing the \(\epsilon^0\) coefficients of the \(\epsilon\)-expansion of six master integrals (see \(f, f', f'', g, g', g''\) of Fig.3):

\[
U = \frac{174623}{28800} C_{\beta 1a} + \frac{29479}{7200} C_{\beta 1b} - \frac{43}{6} C_{\beta 1c} + \frac{10871}{14400} C_{\beta 3a} - \frac{157}{1620} C_{\beta 3b} - \frac{95}{24} C_{\beta 3c}. \tag{53}
\]

In the above expressions \(b_6 = H_{0,0,0,0,1,1}(\frac{1}{2}), b_7 = H_{0,0,0,0,0,1,1}(\frac{1}{2}), d_7 = H_{0,0,0,0,1,1,1}(1), Cl\_n(\theta) = \text{ImLi}_n(e^{i\theta}). H_{i_1,i_2,...}(x)\) are the harmonic polylogarithms. The integrals \(f_j\) are defined as follows:

\[
f_m(i,j,k) = \int_1^9 ds D_1(s) \text{Re} \left( \sqrt{3^{m-1} D_m(s)} \right) \left( s - \frac{9}{5} \right) \ln^i (9-s) \ln^j (s-1) \ln^k (s), \tag{54}
\]

\[
D_m(s) = \frac{2}{\sqrt{(s+3)(s-1)^3}} K \left( m - 1 - (2m-3) \frac{(\sqrt{s}-3)(\sqrt{s}+3)}{(s+3)(s-1)^3} \right); \tag{55}
\]

\(K(x)\) is the complete elliptic integral of the first kind. The constants \(B_3\) and \(C_3\) have the following hypergeometric representations \([21, 22]\):

\[
B_3 = \int_0^1 dx \frac{K^2_3(x)}{\sqrt{1-x}} = \frac{\pi}{27} \sqrt{3} \left( 4 \tilde{F}_3 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \right) - 4 \tilde{F}_3 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \right) \right), \tag{56}
\]

\[
C_3 = \int_0^1 dx \frac{E^2_3(x)}{\sqrt{1-x}} = \frac{\pi}{27} \sqrt{3} \left( 4 \tilde{F}_3 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \right) - 4 \tilde{F}_3 \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1 \right) \right). \tag{57}
\]
The numerical values of the constants appearing in Eq.(18) are listed in Table 3. The right-hand sides of Eqs.(20)-(23), Eqs.(31)-(33), Eq.(42) and Eqs.(45)-(46) have been written by using some suitable combinations of constants, $t_i$, $v_i$ and $e_i$, found by comparing the fits of several contributions of diagrams to $A_4$ and $F_2(0)$. In this way, we obtain a decomposition of $A_4$ as linear combinations of the elements of a basis of only 57 objects (the terms in the right-hand sides of Eqs.(19)-(23), Eqs.(30)-(33), Eqs.(40)-(42), Eqs.(44)-(46) and Eq.(53)). We have found that each one of the 891 contributions of the 4-loop vertex diagrams to $F_1'(0)$ and to $F_2(0)$ can be written as linear combination of the elements of this basis.

We briefly describe the method used to obtain $A_4$. It is the same used in Ref.[1]. The 104 self-mass diagrams are generated with a C program. The contribution to $A_4$ from the amplitude $M_\mu(p + q/2, p - q/2, q)$ of a vertex diagram is extracted by using

\[
4F_3 \left( \begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4
\end{array} ; x \right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)} F_3 \left( \begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4
\end{array} ; x \right),
\]

(58)

\[
K_c(x) = \frac{2\pi}{\sqrt{27}} F_1 \left( \begin{array}{c}
\frac{1}{2} \frac{1}{2} \frac{1}{2} ; x
\end{array} \right), \quad E_c(x) = 2\pi \frac{1}{\sqrt{27}} F_1 \left( \begin{array}{c}
\frac{1}{2} \frac{1}{2} \frac{1}{2} ; x
\end{array} \right).
\]

(59)

The numerical values of the constants appearing in Eq.(18) are listed in Table 3. The right-hand sides of Eqs.(20)-(23), Eqs.(31)-(33), Eq.(12) and Eqs.(43)-(46) have been written by using some suitable combinations of constants, $t_i$, $v_i$ and $e_i$, found by comparing the fits of several contributions of diagrams to $A_4$ and $F_2(0)$. In this way, we obtain a decomposition of $A_4$ as linear combinations of the elements of a basis of only 57 objects (the terms in the right-hand sides of Eqs.(19)-(23), Eqs.(30)-(33), Eqs.(40)-(42), Eqs.(44)-(46) and Eq.(53)). We have found that each one of the 891 contributions of the 4-loop vertex diagrams to $F_1'(0)$ and to $F_2(0)$ can be written as linear combination of the elements of this basis.

We briefly describe the method used to obtain $A_4$. It is the same used in Ref.[1]. The 104 self-mass diagrams are generated with a C program. The contribution to $A_4$ from the amplitude $M_\mu(p + q/2, p - q/2, q)$ of a vertex diagram is extracted by using...
projectors \[ \text{22} \quad \text{26} \]

\[
F_1'(0) = \text{Tr} \left( P^{(2)}_{\mu\nu}(p) \frac{\partial M_\mu(p + q/2, p - q/2, q)}{\partial q_\nu} \bigg|_{q=0} ight) + \text{Tr} \left( P^{(3)}_{\mu\nu\rho}(p) \frac{\partial^2 M_\mu(p + q/2, p - q/2, q)}{\partial q_\nu \partial q_\rho} \bigg|_{q=0} \right),
\]

(60)

analogously to the corresponding formula for \( g-2 \)

\[
F_2(0) = \text{Tr} \left( P^{(0)}_\mu(p) M_\mu(p, p, 0) + P^{(1)}_{\mu\nu}(p) \frac{\partial M_\mu(p + q/2, p - q/2, q)}{\partial q_\nu} \bigg|_{q=0} \right),
\]

(61)

we use a FORM program to perform this operation. For each self-mass diagram a large system of integration-by-parts identities \[ \text{12} \quad \text{13} \] is generated and solved by using the program SYS \[ \text{14} \]. Using this system of identities the contribution of each diagram is reduced to master integrals, which are the same of Ref.\[ \text{1} \].

The contribution of a diagram to the slope must be independent of the internal routing chosen for the external momentum of the photon \( q \). We compute the contributions with two different routings, one minimizing and the other maximizing the number of momenta containing \( q \). We check that both expressions are reduced to same combination of master integrals.

Let us compare the contributions to the slope and to \( g-2 \) of the same diagrams. Due to the second derivative appearing in Eq.\[ \text{60} \], the contribution to the slope contains Feynman integrals with sum of exponents increased by 2 in the numerators and increased by 1 in the denominators. The total number of Feynman integrals of a contribution increases typically of a factor \( \sim 10 \ldots 20 \).

For the same reason, the number of identities of the system necessary to reduce the contributions to the slope increases of a factor \( 10 \ldots 30 \) (up to \( 5 \times 10^8 \)), and the size increases of a factor 10 (up to \( 1.5 \text{TB} \)).

For example, let us consider the contributions from the vertex diagrams derived from the self-mass diagram 22 of Fig.\[ \text{1} \] in the sector with all the 11 denominators the Feynman integrals have maximum sum of the exponents of the scalar products equal to 7, and maximum sum of the exponents of the denominators minus the number of denominators equal to 3. The integrals which have maximum sum of exponents are generated by the derivative with respect to the external photon momentum; we have verified that it is not necessary to generate integration-by-parts identities which contain Feynman integrals with total sum of exponents greater than these maxima.

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