String topology on Gorenstein spaces

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Abstract

The purpose of this paper is to describe a general and simple setting for defining \((g,p + q)\)-string operations on a Poincaré duality space and more generally on a Gorenstein space. Gorenstein spaces include Poincaré duality spaces as well as classifying spaces or homotopy quotients of connected Lie groups. Our presentation implies directly the homotopy invariance of each \((g,p + q)\)-string operation as well as it leads to explicit computations.

Shriek maps play a central role in string topology and its generalizations. Following Dold, various presentations have been given ([2],[4],[10]). Usually shriek maps are defined in (co)homology for maps \(f : N \to M\) from a closed oriented \(n\)-manifold to a closed oriented \(m\)-manifold. Here we will consider shriek maps at the cochain level and in a more general setting.

More precisely we work in the category of (left or right) differential graded modules over a differential graded \(\mathbb{F}\)-algebra \((R,d)\) (\(\mathbb{F}\) is a fixed field), that we call for sake of simplicity the category of \((R,d)\)-modules. Its associated derived category is obtained by formally inverting quasi-isomorphisms, i.e. the maps that induce isomorphisms in homology and hereafter denoted by \(\simeq\) in the diagrams. In the derived category the vector space of homotopy classes of maps of degree \(q\) from the \((R,d)\)-module \((P,d)\) to the \((R,d)\)-module \((Q,d)\) is then denoted by \(\text{Ext}^q_R(P,Q)\). In other words an element \(\varphi \in \text{Ext}^q_R(P,Q)\) is represented by a morphism of differential \(R\)-modules \(P' \to Q\) (also denoted \(\varphi\) by abuse) where \((P',d)\) is some cofibrant replacement of \((P,d)\). The induced map \(H(P,d) \cong H(P',d) \to H(Q,d)\) will be denoted \(H(\varphi)\).

For recall, an oriented Poincaré duality space of (formal) dimension \(m\) is a path connected space \(M\) together with an orientation class \([M] \in H_m(M)\) such that the cap product by the orientation class,

\[- \cap [M] : H^*(M) \to H_{*-m}(M),\]

is an isomorphism. The fundamental class of \(M\) is the element \(\omega_M \in H^m(M)\) such that \(\langle \omega_M, [M] \rangle = 1\) where \(\langle , \rangle : H^*(M) \otimes H_*(M) \to \mathbb{F}\) denotes the Kronecker product. The definition of an oriented homotopy type is clear from the above definition.

We consider a pullback diagram,

\[
\begin{array}{ccc}
E' & \xrightarrow{g} & E \\
\downarrow{(*)} & & \downarrow{p} \\
N & \xrightarrow{f} & M
\end{array}
\]
where \( (H) \quad \begin{cases} N \text{ is an oriented Poincaré duality space of dimension } n, \\ M \text{ is a } 1\text{-connected oriented Poincaré duality space of dimension } m, \\ p : E \to M \text{ is a fibration,} \\ H^*(E) \text{ is a graded vector space of finite type.} \end{cases} \)

**Theorem A.** With the notation above there exist unique elements

\[ f^! \in \text{Ext}^{n-m}_{C^*(M)}(C^*(N), C^*(M)) \text{ and } g^! \in \text{Ext}^{n-m}_{C^*(E)}(C^*(E'), C^*(E)) \]

satisfying:

1. \( H^*(f^!)(\omega_N) = \omega_M; \)

2. In the derived category of \( C^*(M) \)-modules the following diagram commutes,

\[
\begin{array}{ccc}
C^*(E') & \xrightarrow{g^!} & C^{*+n-m}(E) \\
\downarrow C^*(p') & & \downarrow C^*(p) \\
C^*(N) & \xrightarrow{f^!} & C^{*+n-m}(M).
\end{array}
\]

Moreover, if the homotopy fiber of the map \( f \) is a Poincaré duality space then \( H^*(f^!) \)
and \( H^*(g^!) \) coincide with the respective integrations along the fiber.

It is immediate from theorem A that if \( M \) and \( N \) are closed oriented manifolds
then \( H^*(f^!) \) is the usual cohomology shriek map in the sense of [2, Chap.VI-Def.11.2].

As a first example, consider a smooth embedding of compact oriented manifolds
\( f : N^n \to M^m \). Let \( V \) be a tubular neighborhood of \( f(N) \) in \( M \). Denote by
\( \Omega \in C^{m-n}(V, \partial V) \) a cocycle representing the Thom class of the relative bundle
\( (V, \partial V) \to N \). Then we have a sequence of maps of \( C^*(M) \)-modules, the second one being the Thom quasi-isomorphism,

\[
C^*(N) \xrightarrow{\cap \Omega} C^{*+m-n}(V, \partial V) \xrightarrow{\cap} C^{*+m-n}(M, M \smallsetminus f(N)) \xrightarrow{\cap} C^{*+m-n}(M).
\]

This sequence induces a \( H^*(M) \)-linear map \( H^*(N) \to H^{*+(m-n)}(M) \) which maps the fundamental class to the fundamental class. Therefore this sequence defines in the derived category a representative of the shriek map \( f^! \). Another representative of the same class can be constructed as follows. Let \( \psi : (P, d) \xrightarrow{\simeq} C^*(N) \) be a cofibrant replacement of \( C^*(N) \) as a \( C^*(M) \)-module. We denote by \( \text{cap}_M \) and \( \text{cap}_N \) the cap products with representatives of the orientation classes in \( C_*(M) \) and \( C_*(N) \). The lifting property of cofibrant models (see §1.2-(SF3)) furnishes the following homotopy commutative diagram of \( C_*(M) \)-modules,

\[
\begin{array}{ccc}
(P, d) & \xrightarrow{\varphi} & C^{*+m-n}(M) \\
\downarrow \psi & & \downarrow \text{cap}_M \\
C^*(N) & \xrightarrow{\text{cap}_N} & C_{n-*}(N) \xrightarrow{\text{cap}_M} C_{n-*}(M).
\end{array}
\]

By construction, \( \varphi \) is unique up to homotopy. Moreover, \( H^*(\varphi) : H^*(N) \to H^*(M) \)
maps the fundamental class to the fundamental class. Therefore \( \varphi \) is also a representative of \( f^! \).
As a second example, consider a fibration \( p : E \to M \) with base a 1-connected oriented Poincaré duality space of dimension \( m \) and fibre \( F := p^{-1}(\{b_0\}) \). Applying Theorem A to the pullback diagram

\[
\begin{array}{c}
F \\
\downarrow j \\
\{b_0\} \\
\end{array}
\begin{array}{c}
E \\
\downarrow p \\
M \\
\end{array}
\]

yields a well defined homomorphism \( H^*(j^!): H^*(F) \to H^{*+m}(E) \) called the intersection map with the fiber. This morphism generalizes the intersection map defined by Chas and Sullivan ([3]). See [20] for another description.

Theorem A can be used to give a homotopy invariant definition of the loop product on a 1-connected oriented Poincaré duality space \( M \) of dimension \( m \). Consider the pullback diagram,

\[
\begin{array}{c}
LM \times_M LM \\
\downarrow p' \\
M \\
\end{array}
\begin{array}{c}
q \\
\downarrow p \times p \\
M \times M \\
\end{array}
\]

where \( \Delta : M \to M \times M \) denotes the diagonal map, \( LM \) is the free loop space, \( LM = M^{S^1} \), and \( p : LM \to M \) is the usual fibration that associates to a loop its base point. Diagram (**) satisfies conditions (H) above. Therefore, by Theorem A, there exists a unique class \( q^! \in \text{Ext}^{m}_{C^*(LM \times LM)}(C^*(LM \times_M LM), C^*(LM \times LM)) \).

Denote by \( c : LM \times_M LM \to LM \) the composition of free loops. Then the linear map of degree \( m \)

\[
H^*(q^!) \circ H^*(c) : H^*(LM) \to H^*(LM \otimes H^*(LM))
\]

coincides with the dual of the loop product (coefficients in the field \( \mathbb{k} \))

\[
\bullet : H_*(LM) \otimes H_*(LM) \to H_{*-m}(LM)
\]

when \( M \) is closed oriented manifold ([3], [9], [15]).

More generally, let \( S \) be a connected surface of genus \( g \) with \( p + q \) boundary components considered as a cobordism between the union of \( p \geq 1 \) incoming circles and \( q \geq 1 \) outgoing circles. The embedding of the incoming circles leads to a cofibration \( j : \Pi_p S^1 \to S \) inducing for each space \( M \) a fibration \( q_S : \text{Map}(S, M) \to (LM)^p \).

For instance, for \( g = 2 \), \( p = 3 \) and \( q = 2 \),
Using Sullivan chord diagrams ([7]) the fibration $q_S$ can be viewed as a pullback fibration in a diagram of fibrations

$$\begin{array}{ccc}
\text{Map}(S, M) & \to & M^r \\
q_S \downarrow & & \downarrow \Delta \\
(LM)^p & \xrightarrow{\psi} & M^t
\end{array}$$

Recall that a Sullivan chord diagram of the surface $S$ is a graph, $C$, homotopy equivalent to $S$ and composed of $p$ circles and $r$ trees $T_i$ whose extremities are $t$ distinct points on the circles. For instance in the chord diagram

\[ p = 3, \ r = 4 \text{ and } t = 11. \]

Then we have a commutative diagram of fibrations

$$\begin{array}{ccc}
\text{Map}(S, M) & \xrightarrow{\sim} & \text{Map}(C, M) \\
q_S \downarrow & & q_C \downarrow \\
(LM)^p & \xrightarrow{\psi} & (LM)^p \\
& q_T \downarrow & \downarrow \\
& & M^t
\end{array}$$

where $q_T$ and $\psi$ denote the evaluation maps at the extremities of the trees and $q_C$ denotes the restriction to $\prod_p S^1$. Since each $T_i$ is contractible, the map $q_T$ is homotopy equivalent to a product of diagonal maps $\Delta : M^r \to M^t$. This shows that the fibration $q_S$ is homotopy equivalent to the pullback of $\Delta$ along some map $\psi$.

Assume that $M$ is a 1-connected oriented Poincaré duality space. Then by Theorem A, there exists in the derived category of $C^*((LM)^p)$-modules a well defined (up to homotopy) morphism

$$(q_S)^! : C^*(\text{Map}(S, M)) \to C^{*-m\chi}((LM)^p),$$

of degree $-m\chi$ where $\chi = (2 - 2g - p - q)$ denotes the Euler characteristic of $S$. The cohomology $(g, p + q)$-string operation induced by the surface $S$,

$$H^*((LM)^q) \to H^{*-m\chi}((LM)^p),$$

is defined as the composition of $H^*((q_S)^!)$ with the morphism $H^*(\text{Map}(k, M))$ when $k$ denotes the inclusion of the outgoing circles into $S$. For closed oriented manifold the dual of these cohomology $(g, p + q)$-string operations coincide with those defined by Cohen-Godin, [7].

In [17] Gruher and Salvatore prove that the loop product is an oriented homotopy invariant. We generalize this result to all the string operations defined on a Poincaré duality space.
Theorem B. If \( M \) is a 1-connected oriented Poincaré duality space then all the \((g,p+q)\)-string operations on \( H_*(LM) \) depend only on the oriented homotopy type of \( M \).

We will now explain how string operations can be extended to Gorenstein spaces. For recall, a differential graded augmented \( \mathbb{k} \)-algebra is called a Gorenstein algebra of dimension \( d \) if \( \text{Ext}^k_d(\mathbb{k}, A) = \{0\} \) if \( k \neq d \) and \( \text{Ext}^d_0(\mathbb{k}, A) = \{0\} \) has dimension one. (Here the field \( \mathbb{k} \) is considered as a trivial module while \( A \) acts on itself by left multiplication.) A path connected space \( M \) is called a lk-Gorenstein space of dimension \( d \) if the nondegenerated singular cochain algebra \( C^*(M) \) is a Gorenstein algebra of dimension \( d \). Gorenstein spaces have been introduced in [11] in relation with the Spivak fibration. Examples of Gorenstein spaces are given by Poincaré duality spaces, classifying spaces \( BG \) of compact connected Lie groups \( G \) and rational spaces with finite Postnikov tower. More generally, if \( F \) acts on a path connected Poincaré duality space \( M \) then the homotopy quotient \( EG \times_G M \) is a Gorenstein space since it is the total space of the Borel fibration \( M \rightarrow EG \times_G M \rightarrow BG \). String operations of those spaces have been recently studied using stacks and bivariant homology theory ([II]).

For a Poincaré duality space \( M \), the construction and the uniqueness of the string operations on \( LM \) were depending only on the existence and the uniqueness (up to homotopy), in the derived category of \( C^*(M^n) \)-modules, of a map of degree \( d(n-r) \), \( \Delta^l : C^*(M^r) \rightarrow C^*(M^n) \), for \( r \leq n \). The main tool to define string operations on Gorenstein spaces is the following Theorem C.

Theorem C. Let \( X \) be a 1-connected lk-Gorenstein space of dimension \( d \), and for \( r \leq n \) let \( \Delta : X^r \rightarrow X^n \) be the product of diagonal maps \( X \rightarrow X^{n_i}, \sum_{i=1}^r n_i = n \). Then,

\[
\text{Ext}_{C^*(X^n)}(C^*(X^r), C^*(X^n)) = s^{(n-r)d} H^*(X^r; \mathbb{k}) ,
\]

where \( C^*(X^r) \) is viewed as a \( C^*(X^n) \)-module via \( \Delta \).

This implies that, up to homotopy and up to the multiplication by a scalar, there is in the derived category of \( C^*(X^n) \)-modules a unique non trivial class of morphism of degree \( (n-r)d \),

\[
\Delta^l : C^*(X^r) \rightarrow C^*(X^n) ,
\]

and for each surface \( S \) of genus \( g \) with \( p + q \) boundary components, \( \Delta^l \) induces a map

\[
(qs)^l : C^*(\text{Map}(S, X)) \rightarrow C^*((LX)^p) ,
\]

in the derived category of \( C^*((LX)^p) \)-modules. This map \((qs)^l \) is uniquely defined up to homotopy and up to the multiplication by a scalar. In the same way as in the Poincaré duality case, we can now defined string operations for Gorenstein spaces. We give in section 5 some computations. In particular, considering the \((0,1+2)\)-string operation, we prove:

Theorem D. Let \( \mathbb{k} = \mathbb{Q} \) and let \( BG \) be the classifying space of a connected compact Lie group then the loop coproduct \( H_*(L(BG)) \rightarrow H_*(L(BG)) \otimes H^*(L(BG)) \) is injective.
Section 1 contains brief background material on cofibrant objects in the category of differential graded algebras and differential modules. In section 2, we define shriek maps for Poincaré duality spaces and prove Theorem A and B. In section 3 we consider string operations on Poincaré duality spaces. Section 4 is devoted to the characteristic zero case. We make there use of Sullivan minimal models for making computations. In section 5 and 6 we extend the previous definitions to Gorenstein spaces and we prove in particular Theorem C.

1 Models for algebras and modules

1.1 Conventions

All vector spaces are defined on a fixed field $\mathbb{k}$ and the unadorned $\otimes$ and Hom mean with respect to $\mathbb{k}$. Graduations are written either as superscripts or as subscripts, with the convention $V^k = V_{-k}$. We say that the graded vector space $V$ has finite type if each $V^i$ is finite dimensional. The graded dual of $V$ is denoted by $V^#$ i.e. $V^k = (V^k)^#$.

Differential graded algebras $(R, d)$ are assume to be of the form $\{R^k\}_{k \geq 0}$ with differential $d$ of (upper) degree $+1$. If $V$ is a graded vector space, $T(V)$ denotes the tensor algebra on $V$.

Unless we say otherwise, we shall use the word $(R, d)$-module for (left or right) differential $\mathbb{Z}$-graded module over a differential graded algebra $(R, d)$. Let $(Q, d)$ and $(Q', d')$ be $R$-modules, we denote by $\text{Hom}_R(Q, Q')$ the graded vector spaces

$$\text{Hom}_R^k(Q, Q') = \prod_{i \in \mathbb{Z}} \text{Hom}_R(Q^i, (Q')^{i+k})$$

with the differential $D$ defined by $D \varphi = d \circ \varphi - (-1)^k \varphi \circ d$.

Recall also that the $R$-linear maps $\varphi, \psi \in \text{Hom}_R^k(Q, Q')$ are homotopic if $\varphi - \psi = D \theta$ for some $\theta \in \text{Hom}_R^{k-1}(Q, Q')$.

1.2 Semifree modules in the category of $R$-modules

Definition. Let $(R, d)$ a differential graded algebra over the fixed field $\mathbb{k}$.

(i) A $R$-semifree extension of an $R$-module $(Q, d)$ is a morphism of $R$-modules of the form $(Q, d) \xrightarrow{i} (Q \otimes V, d)$ in which

(a) $i$ is the obvious inclusion.

(b) $V = \bigcup_k V(k)$ with $V(0) \subset \cdots \subset V(k) \subset \cdots$ sub vector spaces of $V$

(c) $d(V(0)) \subset Q$ and $d(V(k)) \subset Q \otimes V(k - 1)$, $k \geq 1$.

(ii) If $Q = R$ we say that $(R \otimes V, d)$ is a $R$-semifree module.

Semifree modules are cofibrant objects in the category of $R$-modules. We recall here their main properties, see [14 Section 6] or [13 § 2] for more details.

Let $\varphi : (Q, d) \to (Q', d')$ be a morphism of $R$-modules.
(SF1) φ admits a relative R-semifree model, i.e. there exists a commutative diagram

\[
\begin{array}{c}
P \xrightarrow{\psi} Q'' \\
\downarrow \psi \\
Q \xrightarrow{\varphi} Q' \\
i \xrightarrow{\sim} \end{array}
\]

where \((Q, d) \xrightarrow{i} (P, d)\) is a R-semifree extension of \((Q, d)\). In particular \((Q = R\) and \(\varphi = 0)\), every R-modules admits a R-semifree model.

(SF2) If \(P\) is a R-semifree module and if \(\varphi\) is a quasi-isomorphism then \(\text{Hom}_R(P, \varphi) : \text{Hom}_R(P, Q) \to \text{Hom}_R(P, Q')\) is a quasi-isomorphism.

(SF3) Given a diagram of morphisms of R-modules of the form

\[
\begin{array}{c}
P \xrightarrow{\psi} Q'' \\
\downarrow \psi \\
Q \xrightarrow{\varphi} Q' \\
i \xrightarrow{\sim} \end{array}
\]

where \(i : Q \hookrightarrow P\) is a R-semifree extension and \(\eta\) is a quasi-isomorphism, then there exists a morphism of R-modules \(\psi' : Q \to Q''\) (unique up to homotopy) such that \(\psi'i = \psi\) and \(\eta\psi' \sim \varphi\).

By definition, if \(P\) is a semifree model of \(N\), then,

\[\text{Ext}_R^n(N, Q) := H^n(\text{Hom}_R(P, Q))\]

while if \((Q', d)\) is a right module,

\[\text{Tor}_R^n(Q', N) := H_n(Q' \otimes_R P)\]

### 1.3 Bar constructions and Hochschild chain complex

Let \(A = k \oplus \hat{A}\) be a supplemented graded differential graded algebra, \(N\) a right \(A\)-module and \(M\) a left \(A\)-module. The two-sided (graded) bar construction on \(A, M\) and \(N, \mathbb{B}(N, A, M)\), is the differential bigraded vector space \(\mathbb{B}(N, A, M) = (N \otimes T(sA) \otimes M, d)\) defined as follows:

For \(k \geq 1\), we write \(n[a_1|a_2|...|a_k]m = n \otimes sa_1 \otimes ... \otimes sa_k \otimes m \in \mathbb{B}_k(N, A, M)\). If \(k = 0\), we write \(n[m]m = n \otimes 1 \otimes m \in N \otimes T^0(sA) \otimes M\). The differential \(d = d_0 + d_1\) is defined by

\[
d_0(n[a_1|a_2|...|a_k]m) = d(n)[a_1|a_2|...|a_k]m - \sum_{i=1}^{k}(-1)^{\epsilon_i}n[a_1|a_2|...|d(a_i)|...|a_k]m
\]

\[+(-1)^{\epsilon_{k+1}}n[a_1|a_2|...|a_k]d(m)\]

\[
d_1(n[a_1|a_2|...|a_k]m) = (-1)^{\epsilon_1}n[a_1|a_2|...|a_k]m + \sum_{i=2}^{k}(-1)^{\epsilon_i}n[a_1|a_2|...|a_{i-1}a_i|...|a_k]m
\]

\[-(-1)^{\epsilon_k}n[a_1|a_2|...|a_{k-1}]a_km\]

Here \(\epsilon_i = \lvert n \rvert + \sum_{j<i} \lvert sa_j \rvert\).  

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The (graded) bar construction on \( A \) is the differential graded coalgebra \( \mathbb{B}A = \mathbb{B}(\mathbb{A}, A, \mathbb{A}) \), (see \[13\], (4.2)), whose comultiplication \( \phi \) is defined by \( \phi([a_1 \cdots a_r]) = \sum_{i=0}^r [a_1] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_r] \).

For each \( A \)-module \( M \), \( \mathbb{B}(l \mathbb{A}, A, M) \) is a \( \mathbb{B}(A,A,M) \)-comodule and the morphism \( \varphi : \mathbb{B}(A,A,M) \to M \) defined by \( \varphi(a[m]) = am \) is a \( A \)-semifree model for \( M \), \[13\], Lemma 4.3.

Denote by \( A^{op} \) the opposite algebra and by \( A^e = A \otimes A^{op} \). The bar construction \( \mathbb{B}(A,A,A) \) is a \( A^e \)-semifree model of \( A \). The Hochschild chain complex of \( A \) is the complex \( CH^*(A) = (A,d) \otimes_{A^e} (\mathbb{B}(A,A,A),d) \) where \( A \) is viewed as a \( A^e \)-module via the multiplication. Its homology \( HH^*(A) = \text{Tor}_{A^e}(A,A) \) is called the Hochschild homology of \( A \) (with coefficients in \( A \)).

1.4 Free models

Definition.

(i) A free extension of a differential graded algebra \( (A,d) \) is a morphism of differential graded algebras of the form \( (A,d) \hookrightarrow (A[\bigoplus V],d) \) in which

(a) \( i \) is the obvious inclusion.

(b) \( V = \bigcup_k V(k) \) with \( V(0) \subset \cdots \subset V(k) \subset \cdots \) subspaces of \( V \)

(c) \( d(V(0)) \subset A \) and \( d(V(k)) \subset A \bigoplus T(V(k-1)) \), \( k \geq 1 \).

(ii) If \( A = \mathbb{A} \) we say that \( (R,d) \) is a free extension.

Free extensions are cofibrant objects in the category of differential graded algebras. We recall here some of their main properties, see \[13\], \[19\] or \[12\] for more details. Recall only that two morphisms of differential graded algebras \( \varphi, \psi : T(V) \to R \) are homotopic if there exists a \( (\varphi, \psi) \)-derivation \( \theta \) such that \( \varphi - \psi = d \circ \theta - \theta \circ d \).

Let \( (A,d) \) and let \( (A',d') \) be differential graded algebras and let \( \varphi : (A,d) \to (A',d') \) be a homomorphism of differential graded algebras.

(\text{FE1}) \( \varphi \) admits a relative free model, i.e. there exists a commutative diagram

\[
\begin{array}{ccc}
(A[\bigoplus V],d) & \xrightarrow{\sim} & (A,d) \\
\downarrow \varphi & & \downarrow i \\
(A,d) & \xrightarrow{i} & (A',d)
\end{array}
\]

where \( (A,d) \hookrightarrow (A[\bigoplus V]) \) is a free extension of \( (A,d) \). In particular every differential graded algebra admits a free model.

(\text{FE2}) Given a homotopy commutative diagram in the category of differential graded algebras of the form

\[
\begin{array}{ccc}
(T(V),d) & \xrightarrow{\psi} & (A,d) \\
\downarrow i & & \downarrow \sim \\
(T(V \oplus W),d) & \xrightarrow{\varphi} & (A',d)
\end{array}
\]

where \( i \) is a trivial splitting, \( \varphi \) is a homotopy equivalence and \( \sim \) is a homotopy.
where $i$ is a free extension of $(T(V),d)$ and $\eta$ a quasi-isomorphism, then there exists a morphism of differential graded algebras (unique up to homotopy) $\psi' : (T(V \oplus W),d) \to (A,d)$ such that $\psi' i = \psi$ and $\eta \psi' \simeq \varphi$.

Moreover, if $\eta$ is surjective and the diagram commutative, we can choose $\psi'$ such that $\eta \psi' = \varphi$.

(FE3) If $\eta : (T(V),d) \to A$ and $\eta' : (T(W),d) \to A'$ are free models of $(A,d)$, then there is a quasi-isomorphism $\alpha : (T(W),d) \to (T(V),d)$ such that $\eta \circ \alpha$ is homotopic to $\eta'$.

## 2 Poincaré duality spaces

### 2.1 Shriek maps

Let $M$ be an oriented Poincaré duality space of dimension $m$, $X$ any path connected space with homology of finite type and $f : X \to M$ be a map, then $C^*(X)$ is a $C^*(M)$-module via $f$. Therefore, for each integer $q$ we have a canonical linear map

$$
\theta : \text{Ext}^q_{C^*(M)}(C^*(X), C^*(M)) \to \text{Hom}^q(H^{m-q}(X), H^m(M)), \varphi \mapsto [\varphi].
$$

**Lemma 1.** ([21]) With the above notations, the linear map $\theta$ is an isomorphism.

**Proof.** Let $\psi : R \to C^*(X)$ be a $C^*(M)$-semifree model of $C^*(X)$. Then by property (SF-2) of semifree models, the cap product with a cycle representing the orientation class of $M$, $\text{cap}_M : C^*(M) \to C_*(M)$, induces a quasi-isomorphism

$$
\text{Hom}_{C^*(M)}(R, C^*(M)) \to \text{Hom}_{C^*(M)}(R, C_*(M)).
$$

Now let us consider a linear basis, $(\eta_i)_i$, of $\text{Hom}(H^{m-q}(X), H^m(M))$ and the corresponding basis $([a_i])_i$ of $H_{m-q}(X)$ under the linear isomorphism

$$
\text{Hom}(H^{m-q}(X), H^m(M)) \cong \text{Hom}(H^{m-q}(X), k) \cong H_{m-q}(X).
$$

By definition $a_i$ is a cycle in $C_{m-q}(X)$ and we denote by $\text{cap}_{a_i} : C^*(X) \to C_*(X)$ the cap product by $a_i$. We deduce, for each $i$, a homotopy commutative diagram in the category of $C^*(M)$-modules of the form

$$
\begin{array}{ccc}
R & \xrightarrow{\psi_i} & C^*(N) \\
\downarrow & & \downarrow \cong \text{cap}_M \\
C^*(X) & \xrightarrow{\text{cap}_{a_i}} & C_*(X) & \xrightarrow{\text{cap}(f)} & C_*(M)
\end{array}
$$

Since $\theta(\psi_i) = \eta_i$, $\theta$ is surjective. In fact this is a bijection since the two vector spaces have the same dimension. This last fact comes from the Cartan-Eilenberg associativity formulae. Indeed,

$$
\text{Ext}^q_{C^*(M)}(C^*(X), C^*(M)) \cong \text{Hom} \left( \text{Tor}^C_{C^*(M)}(C^*(X), C_*(M)), k \right) \cong \text{Hom} \left( \text{Tor}^C_{C^*(M)}(C^*(X), C^*(M)), k \right) \cong \text{Hom}(H^{m-q}(X), k).
$$


Lemma 1 implies directly the next result.

**Theorem 1.** Let $M$ and $N$ be two Poincaré duality spaces and $f : N \to M$ be a map. Then there is, in the derived category of $C^*(M)$-modules, a unique map (up to homotopy)

$$f^! : C^*(N) \to C^*(M)$$

such that $H^*(f^!)$ maps the fundamental class of $N$ to the fundamental class of $M$.

Now let consider diagram (*) and hypothesis (H) from the Introduction. Then,

**Theorem 2.** In the derived category of $C^*(E)$-modules there exists a unique map (up to homotopy) $g^! : C^*(E') \to C^*(E)$ making commutative the diagram

$$
\begin{array}{ccc}
C^*(E') & \xrightarrow{g^!} & C^*(E) \\
\downarrow C^*(p') & & \downarrow C^*(p) \\
C^*(N) & \xrightarrow{f^!} & C^*(M)
\end{array}
$$

**Proof.** Denote by $(P,d) \to C^*(N)$ a $C^*(M)$-semifree model of $C^*(N)$ and consider $C^*(E)$ as a right $C^*(M)$-module via $C^*(p)$. Since $M$ is simply connected, by ([14], Theorem 7.5), there exists a quasi-isomorphism of $C^*(E)$-modules

$$C^*(E) \otimes_{C^*(M)} (P,d) \to C^*(E')$$

that makes $C^*(E) \otimes_{C^*(M)} (P,d)$ a $C^*(E)$-semifree model of $C^*(E')$.

Now let $f^! : (P,d) \to C^*(M)$ be a shriek map associated to $f$ by Theorem 1. The tensor product

$$g^! = 1 \otimes f^! : C^*(E) \otimes_{C^*(M)} (P,d) \to C^*(E) \otimes_{C^*(M)} C^*(M)$$

is a morphism of left $C^*(E)$-modules making commutative the diagram

$$
\begin{array}{ccc}
C^*(E) \otimes_{C^*(M)} (P,d) & \xrightarrow{g^!} & C^*(E) \otimes_{C^*(M)} C^*(M) & \cong & C^*(E) \\
\downarrow h & & \downarrow C^*(p) & & \downarrow C^*(p) \\
(P,d) & \xrightarrow{f^!} & C^*(M) & &
\end{array}
$$

with $h(a) = 1 \otimes a$, $a \in R$.

We write $(P,d) = (C^*(M) \otimes V, d)$ with $d(V(q)) \subset C^*(M) \otimes V(q-1)$. Then $C^*(E) \otimes_{C^*(M)} (P,d) \cong (C^*(E) \otimes V, d)$ with $d(V(q)) \subset C^*(E) \otimes V(q-1)$. Since $g^!$ is completely determined by its values on $V$, there is only one way to extend $f^!$ into a map $g^!$ of $C^*(E)$-modules.

Consider once again the diagram (*) with the hypothesis (H). Denote by $(R,d)$ a semifree model for $C^*(N)$. We can then associate to each morphism of $C^*(M)$-module $k : R \to C^*(M)$ the morphism of $C^*(E)$-module $1 \otimes k : C^*E \otimes_{C^*(M)} R \to C^*E$. This correspondence defines a linear map

$$\Phi : \text{Ext}_{C^*(M)}(C^*(N), C^*(M)) \to \text{Ext}_{C^*(E)}(C^*(E'), C^*(E)).$$
Clearly from the definition of $g'$, we have

$$g' = \Phi(f').$$

## 2.2 Homotopy invariance of the shriek maps

Consider now the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & E \\
\downarrow{k'} & & \downarrow{k} \\
X' & \xrightarrow{g'} & E' \\
\downarrow{q} & & \downarrow{p} \\
N' & \xrightarrow{h'} & M' \\
\end{array}
$$

where

\begin{align*}
& \begin{cases} 
& \text{• } M \text{ and } M' \text{ are 1-connected} \\
& \text{• } M, M', N \text{ and } N' \text{ are Poincaré duality complexes} \\
& \text{• } H^*(E) \text{ and } H^*(E') \text{ are finite type vector spaces} \\
& \text{• } \text{The vertical maps are fibrations} \\
& \text{• } q \text{ and } q' \text{ are respectively pullback of } p \text{ and } p' \text{ along } f \text{ and } f'
\end{cases} \\
\end{align*}

### Theorem 3.

With the above notations, if $h, h', k$ and $k'$ are homotopy equivalences, then,

$$H^*(g') \circ H^*(k') = H^*(k) \circ H^*((g')').$$

### Proof.

The naturality of the isomorphism $\theta$ described at the beginning of section 2.1 gives the following commutative diagram of isomorphisms

$$
\begin{array}{ccc}
\text{Ext}_{C^*M}(C^*N, C^*M) & \xrightarrow{\rho} & \text{Ext}_{C^*M'}(C^*N', C^*M') \\
\downarrow{\theta} & & \downarrow{\theta} \\
\text{Hom}(H^{m-n}N, H^mM) & \xrightarrow{\bar{\rho}} & \text{Hom}(H^{m-n}N', H^mM')
\end{array}
$$

Here $\rho$ and $\sigma$ are the evident isomorphisms, $\bar{\rho}(f') = H^*(f') \circ H^*(h')$ and $\sigma(\bar{\theta}(f'')) = H^*(h) \circ H^*(f'')$. Now $\sigma(\bar{\theta}(f'')) = \rho(f')$ because their images by $\theta$ coincide. Now by the naturality of $\Phi$ we have a commutative diagram of graded vector spaces where $\rho'$ and $\sigma'$ are the evident isomorphisms induced by the homotopy equivalences $k$ and $k'$.

$$
\begin{array}{ccc}
\text{Ext}_{C^*E}(C^*X, C^*E) & \xrightarrow{\rho'} & \text{Ext}_{C^*E'}(C^*X', C^*E') \\
\downarrow{\phi} & & \downarrow{\phi} \\
\text{Ext}_{C^*M}(C^*N, C^*M) & \xrightarrow{\rho} & \text{Ext}_{C^*M'}(C^*N', C^*M') \\
\end{array}
$$

In particular $\rho'(g') = \sigma'(g'')$. This implies the result. \qed
2.3 Naturality of the shriek maps

We consider a diagram of fibrations over $B$,

\[ \begin{array}{c}
X 
\xrightarrow{g} 
Y \\
\downarrow{p} \\
B 
\end{array} \]

We form the pullback of this triangle along a map $f : B' \to B$:

\[ \begin{array}{c}
X' 
\xrightarrow{h} 
X \\
\downarrow{g'} \\
Y' 
\xrightarrow{k} 
Y \\
\downarrow{q'} \\
B' 
\xrightarrow{f} 
B 
\end{array} \]

**Theorem 4.** If $B$ and $B'$ are 1-connected oriented Poincaré duality spaces, and $H^*(X)$ and $H^*(Y)$ of finite type, then,

\[ H^*(g) \circ H^*(k^1) = H^*(h^1) \circ H^*(g') : H^*(Y') \to H^*(X) . \]

**Proof.** We put $(R, d) = C^*(B)$ and we denote by $\varphi : (R, d) \to (R \otimes V, D)$ an $R$-semifree model of $C^*(f)$. We denote also by $\psi : (R_1, d) \to (R_2, d)$ a model of $g$ in the category of differential graded $R$-algebras. Then a model of $g'$ in the category of $R_1$-modules is

\[ \psi \otimes 1 : (R_1, d) \otimes_R (R \otimes V, D) \to (R_2, d) \otimes_R (R \otimes V, D) . \]

Now denote by $\theta : (R \otimes V, D) \to (R, d)$ a representative for $f^!$. The maps $C^*(g) \circ k^1$ and $h^1 \circ C^*(g')$ are given respectively by

\[ (R_1, d) \otimes_R (R \otimes V, D) \xrightarrow{1 \otimes \theta} (R_1, d) \otimes_R (R, d) = (R_1, d) \xrightarrow{\psi} (R_2, d) . \]

and

\[ (R_1, d) \otimes_R (R \otimes V, D) \xrightarrow{\psi \otimes 1} (R_2, d) \otimes_R (R \otimes V, D) \xrightarrow{1 \otimes \theta} (R_2, d) . \]

Since $C^*(g) \circ k^1$ and $h^1 \circ C^*(g')$ are homotopic, they induce the same map in cohomology. \[ \square \]

2.4 Integration along the fiber

Let $F \to E \xrightarrow{p} B$ be a fibration in which $F$ is an oriented Poincaré duality space of cohomological dimension $d$ and $B$ a 1-connected space. The integration along the fiber is the linear map, $\int_F : H^*(E) \to H^{*-d}(B)$, defined as the composition of natural maps arising in the Serre spectral sequence of the fibration

\[ H^*(E) \to E_{*\infty}^{*,d} \subset E_*^{*,d} = H^*(B) \otimes H^d(F) \cong H^{*-d}(B) . \]
Theorem 5. With the notation above, if $E$ and $B$ are oriented Poincaré duality spaces, then $H^*(p^!): H^*(E) \to H^{*-d}(B)$ is the integration along the fiber.

Proof. The differential graded algebra $C^*(E)$ admits a semifree $C^*(B)$-module of the form $(C^*(B) \otimes H^*(F), d)$ with $d(H^*(F)) \subset C^*(B) \otimes H^{<d}(F)$ ([11], Lemma A.3). In this setting it is clear that the integration along the fiber is the map induced in homology by the composite

$$
\varphi \colon C^*(B) \otimes H^*(F) \longrightarrow C^*(B) \otimes (H^*(F)/H^{<d}(F)) \longrightarrow C^{*-d}(B).
$$

We remark that $\varphi$ is a morphism of $C^*(B)$-modules of degree $-d = b - e$ where $b = \dim B$ and $e = \dim E$. Moreover the image by $H^*(\varphi)$ of the fundamental class of $E$ is the fundamental class of $B$. The results follows now from the uniqueness property in Theorem 1.

Theorem 6. If the fibration $F \to E \overset{p}{\to} B$ is the pullback, along a map $f : B \to M$, of a fibration $F \to X \overset{p}{\to} M$ in which all the involved spaces are Poincaré duality spaces, then $H^*((p')^!)$ is the integration along the fiber.

Proof. Let $(C^*(M) \otimes H^*(F), d)$ be a semifree model for $C^*(X)$. Then the tensor product $(C^*(B) \otimes C^*(M), (C^*(M) \otimes H^*(F)), d)$ is a semifree model for $C^*(E)$ as a $C^*(B)$-module. The respective integrations along the fiber are then given by the morphisms $\varphi$ and $\varphi'$ in the following commutative diagram,

$$
\begin{array}{ccc}
C^*(M) \otimes H^*(F) & \longrightarrow & C^*(M) \otimes (H^*(F)/H^{<d}(F)) \longrightarrow C^{*-d}(M) \\
C^*(f) \otimes 1 & \downarrow & C^*(f) \\
C^*(B) \otimes H^*(F) & \longrightarrow & C^*(B) \otimes (H^*(F)/H^{<d}(F)) \longrightarrow C^{*-d}(B) \\
\varphi' & \downarrow & \varphi' \\
& &
\end{array}
$$

Since $\varphi$ and $\varphi'$ are respectively morphisms of $C^*(M)$ and $C^*(B)$-modules of degree $\dim M - \dim X$, the uniqueness property of Theorem 2 implies that $\varphi'$ is a representative of the shriek map $(p')^!$. □

2.5 The intersection map with the fiber

Consider the diagram (†) of the introduction.

$$
\begin{array}{ccc}
F & \overset{j}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
\{b_0\} & \overset{i}{\longrightarrow} & M
\end{array}
$$

By Theorem 2, in the derived category of $C^*(E)$-modules there exists a unique morphism (up to homotopy) $j^!$ making commutative the diagram,

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Let $j^!$ be a representative of $\omega$. Finally we denote by $H$ where $H^*({j^!})$ is the multiplication by the fundamental class of $M$. The morphism $H^*({j^!})$, is called the intersection map with the fiber, and the purpose of this section is the description of a model of $H^*({j^!})$ in terms of cochain complexes.

Let $(T(V), d)$ be a free model of $C^*(M)$ (cf section 1.4) and $i : (T(V), d) \to (W \otimes T(V), d)$ be a right $(T(V), d)$-semifree model for the fibration $p : E \to M$, i.e., there is a commutative diagram of right $T(V)$-modules, where the vertical lines are quasi-isomorphisms

\[
\begin{array}{ccc}
(T(V), d) & \xrightarrow{\rho} & (P \otimes T(V), d) \\
\downarrow \cong & & \downarrow \cong \\
C^*(M) & \xrightarrow{C^*(p)} & C^*(E).
\end{array}
\]

Since $M$ is 1-connected, we may suppose that $V = V^{\geq 1}$. We denote by $q : (T(V), d) \to (A, d)$ a surjective quasi-isomorphism of differential graded algebras, with $A^0 = \mathbb{k}$ and $A^m = 0$. Using $q$ we define the cochain complexes

\[(P, d) := (P \otimes T(V), d) \otimes_{T(V)} (\mathbb{k}, 0) \quad \text{and} \quad (P \otimes A, d) := (P \otimes T(V), d) \otimes_{T(V)} (A, d).\]

Finally we denote by $\omega$ a cocycle in $A$ representing the fundamental class of $H(A) = H^*(M)$.

**Theorem 7.** With the notation above, the map of right $A$-modules

\[(P, d) \to (P \otimes A, d) , \quad x \mapsto x \otimes \omega ,\]

is a representative of $j^!$.

**Proof.** Let $k : (T(V), d) \hookrightarrow (T(V \oplus W), d)$ be a relative free model for $C^*(p) : C^*(M) \to C^*(E)$, and $(T(V), d) \hookrightarrow (T(V \oplus E), d)$ be a relative free model for $C^*(i) : (C^*(M) \to C^*((\{b_0\}))$.

\[
\begin{array}{ccc}
(T(V), d) & \xrightarrow{k} & (T(V \oplus W), d) \\
\downarrow \cong & & \downarrow \cong \\
C^*(M) & \xrightarrow{C^*(p)} & C^*(E),
\end{array}
\]

\[
\begin{array}{ccc}
(T(V), d) & \xrightarrow{} & (T(V \oplus E), d) \\
\downarrow \cong & & \downarrow \cong \\
C^*(M) & \xrightarrow{C^*(i)} & C^*(\{x\}).
\end{array}
\]

Since $(T(V \oplus E), d)$ is a left $(T(V), d)$-semifree module, $i^! \in \mathrm{Ext}^d_{C^*(M)}(\mathbb{k}, C^*(M)) \cong \mathrm{Ext}^d_{T(V)}(\mathbb{k}, T(V))$ can be represented by a morphism of $T(V)$-modules of degree $q$,

\[
\nu : (T(V \oplus E), d) \to (T(V), d).
\]

Therefore by Theorem 2, $j^!$ is represented by the morphism of left $(T(V \oplus W), d)$-modules

\[1 \otimes \nu : (T(V \oplus W), d) \otimes_{T(V)} (T(V \oplus E), d) \to (T(V \oplus W), d) \otimes_{T(V)} (T(V), d) = T(V \oplus W, d).\]
Now observe that the projection \( q : (T(V), d) \to (A, d) \) makes \((A, d)\) a \((T(V), d)\)-module and we have a commutative diagram of \((T(V), d)\)-modules
\[
\begin{array}{ccc}
(T(V \oplus E), d) & \xrightarrow{\nu} & (T(V), d) \\
\varepsilon \downarrow & & \downarrow \varphi \\
(k, 0) & \xrightarrow{\approx q} & (A, d).
\end{array}
\]
where \( \varphi(1) = q\nu(1) \).

Since \((T(V \oplus W), d)\) is a \((T(V), d)\)-semifree module, taking the tensor product with \((T(V \oplus W), d)\) over \(T(V)\) yields a commutative diagram of \((T(V \oplus W), d)\)-modules where the vertical lines are quasi-isomorphisms
\[
\begin{array}{ccc}
(T(V \oplus W), d) \otimes_{T(V)} (T(V \oplus E), d) & \xrightarrow{1 \otimes \nu} & (T(V \oplus W), d) \\
1 \otimes \varepsilon \approx & & \approx 1 \otimes q \\
(T(V \oplus W), d) \otimes_{T(V)} k & \xrightarrow{1 \otimes \varphi} & (T(V \oplus W), d) \otimes_{T(V)} (A, d),
\end{array}
\]
and \(1 \otimes \varphi\) is a new representative for \(j^!\).

Since two semifree models of a differential module are always quasi-isomorphic, we have a quasi-isomorphism of \((T(V), d)\)-semifree modules
\[
\theta : (P \otimes T(V), d) \to T(V \oplus W), d).
\]
Thus tensoring by \((A, d)\) yields a quasi-isomorphism of \((A, d)\)-semifree modules
\[
\theta \otimes 1 : (P \otimes A, d) \to (T(V \oplus W), d) \otimes_{T(V)} (A, d).
\]

Now the commutativity of the diagram
\[
\begin{array}{ccc}
(T(V \oplus W), d) \otimes_{T(V)} k & \xrightarrow{1 \otimes \varphi} & (T(V \oplus W), d) \otimes_{T(V)} (A, d) \\
\theta \otimes 1 \downarrow & & \downarrow \theta \otimes 1 \\
(P, d) := (P \otimes A, d) \otimes_A k & \xrightarrow{1 \otimes \varphi} & (P \otimes A, d)
\end{array}
\]
shows that the map \(1 \otimes \varphi : (P, d) \to (P \otimes A, d)\) is a representative for \(j^!\).

Now note that \(\omega\) and \(\varphi(1)\) represent the fundamental class, the map \(\varphi\) is thus homotopic to the map \(\varphi' : (k, 0) \to (A, d)\) defined by \(\varphi'(1) = \omega\). The maps \(1 \otimes \varphi\) and \(1 \otimes \varphi'\) are then also homotopic and both represent \(j^!\).

\[\square\]

2.6 The integration map with the fiber in the free loop space fibration

Let \(M\) be a simply connected oriented Poincaré duality space, and \(\Omega M \to LM \to M\) be the free loop space fibration. We denote by \((A, d)\) a differential graded algebra quasi-isomorphic to \(C^*(M)\) and satisfying \(A^0 = k\) and \(A^{>m} = 0\), with \(m = \dim M\).

**Theorem 8.** If \(\omega \in A^m\) is a representative of the fundamental class of \(M\), then there is a commutative diagram
\[
\begin{array}{ccc}
H^*(\Omega M) & \xrightarrow{H(j^!)} & H^*(LM) \\
\cong \downarrow & & \cong \downarrow \\
H(\mathbb{E}(A)) & \longrightarrow & HH_*(A)
\end{array}
\]
where the lower map is induced by the multiplication by $\omega$,

$$\mathbb{H}(A) \to CH_*(A), \quad [a_1|\cdots|a_k] \mapsto \omega[a_1|\cdots|a_k].$$

**Proof.** Let $K$ be a 1-reduced finite simplicial set of dimension $m$ homotopy equivalent to $M$. We denote by $C_*(K)$ the normalized chain complex on $K$ and by $\omega \in C^m(K)$ a representative of the fundamental class. Clearly we have only to prove the result for $(A,d) = C^*(K)$. In [23] Szczarba gives an explicit quasi-isomorphism of chain algebras $\alpha : \Omega C_*(K) \to C_*(GK)$ where $\Omega$ denote the cobar construction and $GK$ the loop group associated to $K$. Since $C_*(K)$ is of finite type, by duality we get a quasi-isomorphism of coalgebras $\alpha^# : C_*(GK) \to B(C_*(K)).$ By dualizing the results of Hess, Parent and Scott ([18], Theorem 4.4 and Theorem 5.1), there is a multiplication (associative only up to homotopy) on the Hochschild complex $CH_*(C^*(K))$ and a commutative diagram,

$$
\begin{array}{ccc}
C^*(GK) & \xrightarrow{\alpha^#} & \mathbb{H}(C^*(K)) \\
\downarrow q & & \downarrow \tau \\
C^*(LK) & \xrightarrow{\tau} & CH_*(C^*(K)) \\
\downarrow C_*(\rho) & & \downarrow \tau \\
C^*K & \longrightarrow & C^*K \\
\end{array}
$$

where $\tau$ is a quasi-isomorphism preserving the products on $C^*(LK)$ and $CH_*(C^*(K))$, up to homotopy. Moreover $\tau(\omega \cup b) = \omega \otimes \alpha^#q(b)$ for $b \in C^+(LK)$. Denote now by $\varphi : (P \otimes C^*K,d) \to C^*(LK)$ a semifree resolution of $C^*(LK)$ as $C^*K$-module. Then the composition $\tau \circ \varphi : (P \otimes C^*K,d) \to CH_*(C^*K)$ makes commutative the diagram

$$
\begin{array}{ccc}
(P,\bar{d}) & \xrightarrow{\simeq} & \mathbb{H}(C^*(K)) \\
\downarrow \varphi & & \downarrow \psi' \\
(P \otimes C^*K,d) & \xrightarrow{\tau \circ \varphi} & CH_*(C^*K) \\
\end{array}
$$

where $\psi$ and $\psi'$ consist in the left multiplication by $\omega$. Since the horizontal arrows in this diagram are quasi-isomorphisms, we deduce that the multiplication by $\omega$, $\psi' : \mathbb{H}(C^*K) \to CH_*(C^*K)$ is a model for $j^!$ and induces in cohomology the intersection map with the fiber $H^*(j^!)$. \qed

3 String topology

3.1 Homotopy invariance of string operations

Since, as explained in the introduction the $a (g,p+q)$string operation in cohomology is defined by composition of $H^*(q^g_5)$ with $H^*(c)$ where $c$ is a natural map. It follows directly from Theorem 3 that a $(g,p+q)$string operation in cohomology is invariant with respect to orientation preserving homotopy equivalences between Poincaré duality spaces. Thus Theorem B of the Introduction is proved.
3.2 The loop product

The loop product, or $(0, 2 + 1)$-string operation, can be described as follows. The injection of $S^1 \times S^1$ into the "pair of pants" surface is homotopy equivalent to the injection of $S^1 \mid S^1$ into the subspace $X \subset \mathbb{R}^2$ that is the union of the interval $[0, 1] \times \{0\}$ with the two circles of radius 1/4 centered at the points $(0, 1/4)$ and $(1, 1/4)$. Since this injection is a cofibration, for any 1-connected oriented Poincaré duality space $M$ of dimension $d$, the induced map $q_X : \text{Map}(X, M) \to \text{Map}(S^1 \mid S^1, M)$ is a fibration that is the pullback fibration of the usual path fibration $M^{[0, 1]} \to M \times M$ along the projection $p \times p : LM \times LM \to M \times M$. We have recovered Diagram (**) of the introduction:

$$LM \times_M LM \simeq \text{Map}(X, M) \xrightarrow{q_X} M^{[0,1]} \xrightarrow{\cong} M \xrightarrow{\Delta} M \times M.$$  

Denote by $c : LM \times_M LM \to LM$ the composition of loops. The composite

$$H^*(LM) \to H^*(LM \times_M LM) \xrightarrow{H(q_X^1)} H^*(LM \times LM)$$

is, by definition, the dual of the loop product $\bullet$ on $H_*(LM)$.

Since $q_X^1$ is a morphism of $C^*(LM \times LM)$-modules, $H_*(q_X^1) : H_*(LM \times LM) \to H_*(LM \times_M LM)$ is a morphism of $H^*(LM \times LM)$-modules where the actions are given by the cap product. Remark now that the projection $p \times p : LM \times LM \to M \times M$ makes $H_*(LM \times LM)$ a $H^*(M \times M)$-module. We extend to the case of Poincaré duality spaces the following result of Tamanoi in (26, Theorem A),

**Theorem 9.** Let $M$ be a 1-connected oriented Poincaré duality space, then the loop product is a morphism of $H^*(M) \otimes H^*(M)$-modules:

$$(\alpha \cap b) \bullet (\alpha \cap c) = (-1)^{[d + |b|]}(\alpha \cap \alpha \cap (b \bullet c)), \quad b, c \in H_*(LM), \alpha_1, \alpha_2 \in H^*(M).$$

**Proof.** First remark that $H_*(q_X^1) : H_*(LM) \otimes H_*(LM) \to H_*(LM \times_M LM)$ is a morphism of $H^*(LM) \otimes H^*(LM)$-modules, and therefore a morphism of $H^*(M) \otimes H^*(M)$-module. This implies that

$$H_*(q_X^1)((\alpha \cap b) \otimes (\beta \cap c)) = (-1)^{d(|\alpha| + |\beta|) + |b| \cdot |\beta|}(\alpha \cup \beta) \cap H_*(q_X^1)((b \otimes c)).$$

Now, since the composition of loops $LM \times_M LM \to LM$ is a morphism over $M$, the induced map in homology is a morphism of $H^*(M)$-modules for the cap product. This implies the result. \qed

3.3 The loop coproduct

Let $M$ be a 1-connected oriented Poincaré duality space. Consider the fibration $q : Z \to LM$ where

$$Z = \{ (\omega, c) \mid \omega \in LM, c \in M^{[0,1]}, c(0) = \omega(0), c(1) = \omega(1/2) \}$$
and \( q(\omega, c) = \omega \). The space \( Z \) has the homotopy type of \( LM \times_M LM \). Denote by \( \pi : Z \to LM \times LM \) the canonical injection. The dual of the loop coproduct can be described as the composition of \( H^*(\pi) \) with the shriek map \( H^*(q') \),

\[
H^*(LM) \otimes H^*(LM) \xrightarrow{H^*(\pi)} H^*(Z) \xrightarrow{H^*(q')} H^*(LM).
\]

The fibration \( q \) is obtained as a pullback of the fibration \((p_0, p_1) : M^{[0,1]} \to M \times M\) along the map \( \ell : LM \to M \times M \) that evaluates a loop at the base point and at the middle point, \( \ell(\omega) = (\omega(0), \omega(1/2)) \).

\[
\begin{array}{ccc}
Z \ar@{^{(}->}[r] \ar[d]_q & M^{[0,1]} \ar[d]_{(p_0, p_1)} \\
LM \ar[r]^-\ell & M \times M
\end{array}
\]

Since \( \ell \) is clearly homotopic to the composite \( LM \xrightarrow{p} M \xrightarrow{\Delta} M \times M \), a model for \( q' \) is obtained by pulling back a model for \( (p_0, p_1)' = \Delta' \) along \( \Delta \circ p \).

### 3.4 The intersection map with the fiber in a monoidal fibration

A monoidal fibration is a fibration \( F \xrightarrow{j} E \xrightarrow{p} B \), with a multiplication \( \mu : E \times_B E \to E \) that extends a multiplication \( \mu_0 \) on \( F \). The basic example is the free loop space fibration. Denote by \( \Delta' : E \times_B E \to E \times E \) the natural injection. Then the composition

\[
H_*(E \times E) \xrightarrow{H_*(\Delta')} H_*(E \times_B E) \xrightarrow{H_*(\mu)} H_*(E)
\]

defines a \( \mu \)-intersection product on \( H_*(E) \). Generalizing the result of Chas and Sullivan for the usual loop product, we have,

**Theorem 10.** Suppose that \( F, E, B \) are oriented Poincaré duality spaces and that \( B \) is simply connected. Then the intersection map with the fiber

\[
H_*(j^!) : H_*(E) \to H_*(F)
\]

is a multiplicative map with respect to the \( \mu \)-intersection product and the product \( H_*(\mu_0) \).

**Proof.** Denote by \( k : \{b_0\} \to B \) the injection of the base point. Since the product \( \mu_0 \) is the pullback of the multiplication \( \mu \) along \( k \), we have the commutative diagram

\[
\begin{array}{ccc}
F \times F \ar[r]^-{j'} \ar[dl]_{\mu_0} & E \times_B E \ar[dl]_\mu \\
F \ar[r]^-j & E \ar[u]_\mu \\
\{b_0\} \ar[uu] & B \ar[uu]
\end{array}
\]
We deduce then from Theorem 4 that $H_*(\mu_0) \circ H_*(j'^!_*) = H_*(j^!_*) \circ H_*(\mu)$. On the other hand we consider the commutative diagram obtained by obvious pullbacks

![Diagram]

By uniqueness of the shriek map (Theorem 2) we obtain

$$H_*(j'^!_*) \circ H_*(\Delta'^!_*) = H_*(\Delta^!_*) \circ H_*(j^!_*) = H_*(j^!_*) \otimes H_*(j^!_*).$$

Then,

$$H_*(j^!_*) \circ (H_*(\mu) \circ H_*(\Delta'^!_*)) = H_*(\mu_0) \circ H_*(j'^!_*) \circ H_*(\Delta'^!_*) = H_*(\mu_0) \circ (H_*(j^!_*) \otimes H_*(j^!_*)).$$

□

4 Rational string topology

4.1 Rational homotopy theory

To make computations over a field $\mathbb{k}$ of characteristic zero, the good tool is the theory of minimal models introduced by Sullivan in [24] (see also [14]). For recall, Sullivan defines a functor $A(\cdot)$ from the category of topological spaces to the category of commutative differential graded $\mathbb{Q}$-algebras (for short cdga). One major property of the functor $A(\cdot)$ is the existence of a functor $E(\cdot)$ with values in differential graded algebras, and the existence of natural quasi-isomorphisms

$$A(X) \leftarrow E(X) \rightarrow C^*(X; \mathbb{Q}).$$

If $V$ is a graded vector space, $\wedge V$ denotes the free commutative graded algebra on $V$. A Sullivan cdga is a cdga of the form $(\wedge V, d)$ such that $V$ admits a basis $(v_i)$ indexed by a well ordered set such that $d(v_i) \in \wedge(v_j, j < i)$. A minimal cdga is a Sullivan cdga $(\wedge V, d)$ for which $d(V) \subset \wedge^{\geq 2}(V)$. Now if $(A, d)$ is a cdga such that $H^0(A, d) = \mathbb{Q}$, there exists a minimal cdga $(\wedge V, d)$ equipped with a quasi-isomorphism $\varphi : (\wedge V, d) \rightarrow (A, d)$. This property characterizes $(\wedge V, d)$ up to an isomorphism. The cdga $(\wedge V, d)$ is called the minimal model of $(A, d)$. In particular the minimal model of $A(X)$ is called the minimal model of $X$ and is denoted by $\mathcal{M}_X$. More generally we call model of $X$ any cdga which is quasi-isomorphic to $\mathcal{M}_X$. 

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Every continuous map between connected spaces $f : X \to Y$ induces a unique map up to homotopy $M_f : \mathcal{M}_Y \to \mathcal{M}_X$ making commutative up to homotopy the diagram

\[
\begin{array}{ccc}
A(Y) & \xrightarrow{A(f)} & A(X) \\
\uparrow & & \uparrow \\
\mathcal{M}_Y & \xrightarrow{M_f} & \mathcal{M}_X
\end{array}
\]

The map $M_f$ is called the *minimal model of f*. More generally we call model of $f$ every map of cdga’s $h : (A, d) \to (B, d)$ such that there are quasi-isomorphisms $\varphi$ and $\psi$ making commutative, up to homotopy, the diagram

\[
\begin{array}{ccc}
\mathcal{M}_Y & \xrightarrow{M_f} & \mathcal{M}_X \\
\varphi \downarrow & & \psi \downarrow \\
(A, d) & \xrightarrow{h} & (B, d)
\end{array}
\]

Of course $\varphi$ and $\psi$ are parts of the structure of the model $h$.

A *relative Sullivan model* for a homomorphism of cdga’s $f : (A, d) \to (B, d)$ is an injection $(A, d) \hookrightarrow (A \otimes \wedge V, d)$ equipped with a quasi-isomorphism $\varphi : (A \otimes \wedge V, d) \to (B, d)$ satisfying $\varphi(a \otimes 1) = f(a)$, and where $V$ admits a basis indexed by a well ordered set such that $D(v_i) \subset A \otimes \wedge (v_j, j < i)$. The relative model is called *minimal* if $d(V) \subset \wedge^{\geq 2}(V) \oplus (A^{\geq 1} \otimes \wedge V)$.

Let now $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration, $(A, d)$ a model for $B$ and $\varphi : (A, d) \to (A \otimes \wedge V, d)$ be a relative model for $p$. The morphism $\varphi$ is called a relative model for the fibration $p$. The cdga $(\wedge V, d) := \mathbb{Q} \otimes_A (A \otimes \wedge V, d)$ is then a model for $F$.

For instance a model for the diagonal map $\Delta : X \to X \times X$ is given by the multiplication of $\mathcal{M}_X = (\wedge V, d)$, $\mu : \mathcal{M}_X \otimes \mathcal{M}_X \to \mathcal{M}_X$. A relative minimal model for the diagonal has the form

\[\rho : (\wedge V \otimes \wedge \wedge sV, d) \to (\wedge V, d)\]

where $(sV)^n = V^{n+1}$ and $\rho(sV) = 0$. The cdga

\[
(\wedge V \otimes \wedge sV, D) := (\wedge V, d) \otimes_{\wedge V \otimes \wedge V} (\wedge V \otimes \wedge V \otimes \wedge V, d)
\]

is then a minimal model for the free loop space on $X$. Then,

\[D(sv) = -Sd(v),\]

where $S$ is the derivation defined by $S(v) = sv$ and $S(sv) = 0$ ([27], [11, page 206]).

A *Poincaré duality cdga of dimension m* is a cdga $(A, d)$ satisfying:

(i) $A$ is finite dimensional, $A^{>m} = 0$, $A^m = \mathbb{Q}\omega$,

(ii) the map $\theta : A^r \to \text{Hom}(A^{m-r}, \mathbb{Q})$ defined by $ab = \theta(a)(b) \cdot \omega$ is an isomorphism for $0 \leq r \leq m$.

By a result of Lambrechts and Stanley [22] any simply connected space whose rational cohomology satisfies Poincaré duality admits a Poincaré duality model.
4.2 The intersection map with the fiber with rational coefficients

Here we will use Sullivan models to give a rational description of $H^*(j^!)$ for a fibration $F \overset{j}{\to} E \to M$ with base a 1-connected oriented Poincaré duality space.

Let $(A, d) \to (A \otimes \Lambda V, D) \to (\Lambda V, \overline{D})$ be a relative Sullivan model for the fibration $p : E \to M$. We suppose that $A^0 = \mathbb{Q}$ and $A^{>m} = 0$. If $\omega \in A^m$ is a cocycle representing the fundamental class of $M$, then a model of $i^!$ is given by

$$i^! : lk \to A, \quad i^!(1) = \omega.$$

Therefore, following the proof of Theorem 7 we obtain:

**Proposition 1.** The map $H^*(j^!) : H^*(F; \mathbb{Q}) \to H^*(E; \mathbb{Q})$ is induced by the morphism of $(A \otimes \Lambda V, D)$-modules

$$(\Lambda V, \overline{D}) \to (A \otimes \Lambda V, D), \quad \alpha \mapsto (-1)^{|\omega||\alpha|} \omega \otimes \alpha.$$

4.3 The rational loop product

Let $M$ be a Poincaré duality space of dimension $m$ and let $(A, d)$ be a Poincaré duality model for $M$. Denote by $(a_i)_i$ a homogeneous basis of $A$ and by $(a'_i)_i$ its Poincaré dual basis: $a_i \cdot a'_j = \delta_{ij} \omega$. The diagonal element

$$D = \sum_i (-1)^{|a_i|} a_i \otimes a'_i \in A \otimes A$$

is a cocycle such that, for each element $a \in A$, $(a \otimes 1)D = (1 \otimes a)D$. The multiplication by $D$ is thus a non-trivial morphism of $(A \otimes A)$-modules

$$\mu_D : A \to A \otimes A.$$

Since $\mu_D$ maps the fundamental class to the fundamental class, by Theorem 1, $\mu_D$ is a representative for $\Delta!$ where $\Delta : M \to M^2$ denotes the diagonal map.

Denote by $(A \otimes \Lambda sV, D)$ be a Sullivan model for the free loop space $LM$. As a corollary of Theorem 2, we obtain

**Proposition 2.** With the above notation, the morphism

$$\mu_D \otimes 1 : A \otimes_{A \otimes A} (A \otimes \Lambda V, D)^{\otimes 2} \to (A \otimes A) \otimes_{A \otimes A} (A \otimes \Lambda V)^{\otimes 2} = (A \otimes \Lambda V)^{\otimes 2}$$

induces in cohomology the map $H^*(q^!) : H^*(LM \times_M LM) \to H^*(LM \times LM)$.

This allow us to extend to Poincaré duality space the next result of [16]:

**Theorem 11.** There exists an isomorphism of graded algebras between the Hochschild cohomology of $A$, $HH^*(A)$, and the loop homology $H_*(LM)$. This isomorphism identifies in cohomology the dual of the loop product with the map induced by the composite

$$CH_*(A) \xrightarrow{\Delta} A \otimes_{A^{\otimes 2}} (CH_*(A))^{\otimes 2} \xrightarrow{\mu_D \otimes 1} A^{\otimes 2} \otimes_{A^{\otimes 2}} (CH_*(A))^{\otimes 2} = (CH_*(A))^{\otimes 2}.$$
where $\nabla : CH_*(A) \to A \otimes A \otimes (CH_*(A))^\otimes$ is the morphism of complexes defined by

$$\nabla (a \otimes [a_1|\cdots|a_n]) = \sum_{i=0}^{n} a \otimes [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

Proof. As proved in [15], the map $\nabla$ is a semifree model for the composition of loops $c : LM \times_M LM \to LM$. On the other hand, denoting by $(A \otimes \wedge V, D)$ the model for the free loop space described above, there is a quasi-isomorphism of semifree $A$-modules $\psi : (A \otimes \wedge V, D) \to CH_*(A)$.

Denote by $m$ the multiplication in $A$ and by $\theta : A \to A^\# = \text{Hom}(A, \mathbb{Q})$ a representative of the cap product with the orientation class. Then we obtain the commutative diagram whose vertical lines are quasi-isomorphisms

$$
\begin{array}{ccc}
A & \xrightarrow{\mu_D} & A \otimes A \\
\downarrow \psi & & \downarrow \theta \otimes \theta \\
A^\# & \xrightarrow{m^\vee} & A^\# \otimes A^\#.
\end{array}
$$

This is the main fact which allows to end the proof as in (15).

\[\Box\]

4.4 The rational loop coproduct

Suppose $M$ is a 1-connected oriented Poincaré duality space and $k = \mathbb{Q}$. We use the notations of section 3.3. To obtain a model for the loop coproduct we need first to obtain a model for $q^!$ in the derived category of $C^*(LM)$-modules. This model is obtained from a model of $(p_0, p_1)^!$ by tensorization with a model of the map $\ell$. We will recover in this way the model described in [5].

Let $(A, d)$ be a Poincaré duality model for $M$ with fundamental class $\Omega$, and $\theta^! : (A \otimes \wedge Z, d) \to (A, d)$,

$$\theta(a \otimes a' \otimes 1) = aa' \quad \text{and} \quad \theta(Z) = 0$$

be a relative Sullivan model for the product $A \otimes A \to A$. Let $(A \otimes A \otimes \wedge Z, D)$ be a copy of $(A \otimes A \otimes \wedge Z, D)$ and form the tensor product

$$(A \otimes A \otimes \wedge Z \otimes \wedge Z', D) := (A \otimes A \otimes \wedge Z, D) \otimes_{A \otimes A} (A \otimes A \otimes \wedge Z', D).$$

Then the injection

$$\rho : (A, d) \otimes (A, d) \to (A \otimes A \otimes \wedge Z \otimes \wedge Z', D)$$

is a model for $\ell : LM \to M \times M$. Recall now that the shriek map associated to $(p_0, p_1)$ is represented by the multiplication by the diagonal class $\mu_D : A \to A \otimes A$ (see section 4.1). Therefore a model for the shriek map associated to $q$ is given by $q^!$ in the diagram

$$\begin{array}{ccc}
(A \otimes \wedge Z \otimes \wedge Z', D) & \xrightarrow{\mu_D \otimes 1} & (A \otimes A) \otimes_{A \otimes A} (A \otimes A \otimes \wedge Z \otimes \wedge Z', D) \\
\downarrow q^! & & \downarrow (A \otimes A) \otimes_{A \otimes A} (A \otimes A \otimes \wedge Z \otimes \wedge Z', D)
\end{array}$$
Now observe that the projection \( \theta \otimes 1 : (A \otimes A \otimes Z \otimes Z', D) \to (A \otimes Z', D) \) is a quasi-isomorphism, and form the composition
\[
\psi = (\theta \otimes 1) \circ q^1 : (A \otimes Z \otimes Z', D) \to (A \otimes Z', D).
\]
We have
\[
\psi(a \otimes b \otimes c) = \begin{cases} 
0, & \text{if } a \otimes b \in (A \otimes Z)^+ \\
\chi(M) \cdot \Omega \otimes c, & \text{if } a \otimes b = 1 \otimes 1
\end{cases}
\]
Here \( \chi(M) \) denotes the Euler-Poincaré characteristic of \( M \). Therefore, if \( \chi(M) = 0 \) then \( H^*(q^1) = H^*(\psi) = 0 \), and thus the loop coproduct is trivial.

## 5 Gorenstein spaces

Let \( X \) be simply connected \( \mathbb{k} \)-Gorenstein space of dimension \( d \) whose cohomology \( H^*(X; \mathbb{k}) \) is of finite type. The diagonal map \( \Delta : X \to X^n \) makes \( C^*(X) \) into a \( C^*(X^n) \)-module, and we have

**Theorem 12.**

\[
\text{Ext}_{C^*(X^n)}(C^*(X), C^*(X^n)) \cong s^{(n-1)d} H^*(X).
\]

**Proof.** The proof will proceed in two main steps, the first one consisting to replace the statement concerning the cochains by a statement concerning their free models (§1.4). The second one will concern the computation of the corresponding Ext.

**Step 1 of the proof.** (Construction of models). Let
\[
\varphi_1 : (T(V), d) \cong C^*(X; \mathbb{k}) \quad \text{and} \quad \varphi_{n-1} : (T(W), d) \cong C^*(X^{n-1}; \mathbb{k})
\]
be free models of \( X \) and \( X^{n-1} \).

The tensor product \((T(V), d) \otimes (T(W), d)\) admits a free model of the form
\[
\varphi : (T(V \oplus W \oplus s(V \otimes W)), D) \to (T(V), d) \otimes (T(W), d)
\]
where the differential \( D \) extends the differentials \( d \) already defined on \( T(V) \) and \( T(W) \), and \( \varphi(v) = v, \varphi(w) = w \) and \( \varphi(s(V \otimes W)) = 0 \). For sake of simplicity we write \( V \otimes W := V \oplus W \oplus s(V \otimes W) \).

Consider the commutative diagram
\[
\begin{array}{ccc}
C^*(X) & \xrightarrow{C^*(q_1)} & C^*(X^n) \\
\downarrow{\cong 1 \otimes \text{id}} & & \downarrow{(EZ)^\#} \\
C^*(X^{n-1}) \otimes C^*(X) & \xrightarrow{\cong} & (C^*(X^{n-1}) \otimes C^*(X))^\#
\end{array}
\]
in which \( EZ \) denotes the Eilenberg-Zilber chain equivalence and \( q_1 \) denotes the projection on the last factor. Since \( (EZ)^\# \) is surjective, by (FE2) and the diagram (***) above the map
\[
\varphi_n := C^*(q_1) \circ \varphi_1 : (T(V), d) \to C^*(X^n)
\]
extends into a morphism of differential graded algebras, also denoted \( \varphi_n \), that makes commutative the next diagram.
Choose $e \in X$, and denote by $i_{n-1} : X^{n-1} \to X^n$ and $j_1 : X \to X^n$ the injections defined by $i_{n-1}(a) = (a, e)$, and $j_1(a) = (e, \ldots, e, a)$. Denote also by $\varepsilon : C^*(X) \to C^*(\{e\})$ the augmentation map. Since we work with normalized cochain complexes, $C^*(\{e\}) = k$.

The commutativity of the diagram
\[
\begin{array}{ccc}
C^*(X^n) & \xrightarrow{EZ^\#} & C^*(i_{n-1}) \\
\downarrow \cong & & \downarrow \cong \\
(C_*(X^{n-1}) \otimes C_*(X))^\# & \xrightarrow{C_*(i_{n-1})^\#} & C^*(X^{n-1}) \\
\downarrow \cong & & \downarrow \cong \\
C^*(X^{n-1}) \otimes C^*(X) & \xrightarrow{1 \otimes \varepsilon} & C^*(X^{n-1})
\end{array}
\]
implies the commutativity of the diagram of differential graded algebra,
\[
\begin{array}{ccc}
(T(V \odot W), D) & \xrightarrow{\varphi} & C^*(X^n) \\
\downarrow \cong & & \downarrow \cong \\
(T(W), d) \otimes (T(V), d) & \xrightarrow{\varphi_n} & C^*(X^n) \\
\downarrow \cong & & \downarrow \cong \\
(T(V), d) & \xrightarrow{\varphi_{n-1}} & C^*(X^{n-1})
\end{array}
\]
Therefore
\[
C^*(i_{n-1}) \circ \varphi_n(V \oplus s(V \otimes W)) = 0.
\]

We now apply to $C^*(\Delta)$ the lifting homotopy property for free models. Since $q_1 \circ \Delta = id_X$ and $\varphi_n(v) = C^*(q_1) \varphi_1(v)$ the identity on $T(V)$ extends into a morphism $\psi$ making commutative the following diagram of differential graded algebras
\[
\begin{array}{ccc}
(T(V \odot W), D) & \xrightarrow{\varphi} & C^*(X^n) \\
\downarrow \varphi & & \downarrow C^*(\Delta) \\
(T(V), d) & \xrightarrow{\varphi_1} & C^*(X)
\end{array}
\]

The morphism $\psi$ makes $T(V)$ a $T(V \odot W)$-module. By construction the induced structure of $T(V)$-module on the submodule $T(V)$ of $T(V \odot W)$ coincide with the usual multiplication on $T(V)$. We have the isomorphisms
\[
\text{Ext}_{C^*(X^n)}(C^*(X), C^*(X^n)) \cong \text{Ext}_{T(V \odot W)}(T(V), T(V \odot W))
\]
\[
\cong \text{Ext}_{T(V \odot W)}(T(V), T(W) \otimes T(V))
\]
\[
\cong H^*(\text{Hom}_{T(V \odot W)}((T(V \odot W) \otimes Z, D), T(W) \otimes T(V)))
\]
Here $(T(V \odot W) \otimes Z, D)$ denotes a $T(V \odot W)$-semifree model of $T(V)$.
Step 2 of the proof. (Computation of $\text{Ext}_{\mathcal{C}^*(X^n)}(\mathcal{C}^*(X), \mathcal{C}^*(X^n)))$.

We fix a non-negative integer $N$ and we filter the complex

$$\text{Hom}_{T(V \otimes W)} \left( T(V \otimes W) \otimes Z, T(W) \otimes \frac{T(V)}{T(V)^{>N}} \right)$$

by the sub vector spaces, $F^p \subset F^{p-1}$ defined by:

$$F^p = \{ \varphi | \varphi(Z) \subset T(W) \otimes \frac{T(V)^{>p}}{T(V)^{>N}} \}.$$

This filtration induces a converging spectral sequence with

$$E_0^{p,*} = \{ \varphi | \varphi(Z) \subset T(W) \otimes T(V)^p \},$$

Lemma 2. This spectral sequence collapses at the $E_2$-level and

$$E_2^{p,*} \cong s^{(n-1)d} H^p(T(V)/T(V)^{>N}).$$

Proof. Recall that $V \otimes W = V \oplus W \oplus s(V \otimes W)$ and denote by $I$ the ideal in $T(V \otimes W)$ generated by $V$ and $s(V \otimes W)$. Since $I$ acts trivially on $T(W) \otimes T(V)^p$, we have an isomorphism of complexes

$$\text{Hom}_{T(V \otimes W)}(T(V \otimes W) \otimes Z, T(W) \otimes T(V)^p) \simeq \text{Hom}_{T(W)}(T(W) \otimes_{T(V \otimes W)} (T(V \otimes W) \otimes Z), T(W) \otimes T(V)^p).$$

Here $T(W)$ is considered as a $T(V \otimes W)$-module via the morphism $(\varepsilon \otimes 1) \circ \varphi$. We have also quasi-isomorphisms between bar constructions:

$$\mathbb{B}(T(W), T(V \otimes W), \mathcal{I}_k) \simeq \mathbb{B}(T(W), T(W) \otimes T(V), \mathcal{I}_k) \cong \mathbb{B}(T(W), T(W), \mathcal{I}_k) \otimes \mathbb{B}(T(V)) \simeq \mathbb{B}(T(V)).$$

It follows that the natural injection

$$\mathbb{B}(T(V)) := \mathbb{B}(\mathcal{I}_k, T(V), \mathcal{I}_k) \hookrightarrow \mathbb{B}(T(W), T(V \otimes W), \mathcal{I}_k)$$

is a quasi-isomorphism of complexes. We consider then the injection

$$j : \mathbb{B}(\mathcal{I}_k, T(V), T(V \otimes W)) \hookrightarrow \mathbb{B}(T(W), T(V \otimes W), T(V \otimes W)),$$

and we filter the complexes $\mathbb{B}(\mathcal{I}_k, T(V), T(V \otimes W))$ and $\mathbb{B}(T(W), T(V \otimes W), T(V \otimes W))$ respectively by $\mathbb{B}(T(V))^p \otimes T(V \otimes W)$ and $(T(W) \otimes \mathbb{B}(T(V \otimes W)))^p \otimes T(V \otimes W)$. The morphism $j$ preserves the filtrations and induces an isomorphism at the $E_2$-level of the associated spectral sequences. Therefore $j$ is a quasi-isomorphism so that we deduce the isomorphisms

$$H_*(T(W) \otimes_{T(V \otimes W)} (T(V \otimes W) \otimes Z)) \cong \text{Tor}_*^{T(V \otimes W)}(T(W), T(V))$$

$$\cong H_*(\mathbb{B}(T(W), T(V \otimes W), T(V \otimes W)) \otimes_{T(V \otimes W)} T(V))$$

$$\cong H_*(\mathbb{B}(\mathcal{I}_k, T(V), T(V \otimes W)) \otimes_{T(V \otimes W)} T(V)) \cong H_*(\mathbb{B}(\mathcal{I}_k, T(V), T(V))) \cong \mathcal{I}_k.$$
This shows that $T(W) \otimes_{T(V \otimes W)} T(V \otimes W) \otimes Z$ is a semifree resolution of $\mathbb{k}$ as $T(W)$-module. Therefore,

$$E^p_1 \cong \text{Ext}_{T(W)}(\mathbb{k}, T(W)) \otimes T(V)^p \cong s^{(n-1)d}T(V)^p / T(V)^{>N}$$

and

$$E_2 \cong s^{(n-1)d}H^*(T(V)/T(V)^{>N}).$$

Now since $T(V)$ is of finite type, $T(W) \otimes T(V) = \lim_{\leftarrow N} T(W) \otimes \frac{T(V)}{T(V)^{>N}}$, and

$$\text{Hom}_{T(V \otimes W)}(T(V \otimes W) \otimes Z, T(W) \otimes T(V)) = \lim_{\leftarrow N} \text{Hom}_{T(V \otimes W)}\left((T(V \otimes W) \otimes Z, T(W) \otimes \frac{T(V)}{T(V)^{>N}}\right).$$

We get therefore the short exact sequence

$$0 \rightarrow \lim_{\leftarrow N} s^{(n-1)d}H^{p+1-(n-1)d}(T(V)/T(V)^{>N}) \rightarrow \text{Ext}^p_{T(V \otimes W)}(T(V), T(W) \otimes T(V)) \rightarrow \lim_{\leftarrow N} s^{(n-1)d}H^{p-(n-1)d}(T(V)/T(V)^{>N}) \rightarrow 0$$

Since the tower $(s^{(n-1)d}H^p(T(V)/T(V)^{>N}))_N$ satisfies the Mittag-Leffler condition, we have in fact isomorphisms

$$\text{Ext}^p_{T(V \otimes W)}(T(V), T(W) \otimes T(V)) \cong \lim_{\leftarrow N} s^{(n-1)d}H^p(T(V)/T(V)^{>N}) \cong H^{p-(n-1)d}(X).$$

\textbf{Corollary.} Let $\Delta : X^r \rightarrow X^n$ be the product of diagonal maps $X \rightarrow X^{n_i}, i = 1, \ldots, r$. Then

$$\text{Ext}_{C^*(X^r)}(C^*(X^r), C^*(X^n)) \cong s^{(n-r)d}H^*(X^r).$$

Here $C^*(X^r)$ is viewed as a $C^*(X^n)$-module via $C^*(\Delta)$.

\textbf{Proof.} Write $\Delta$ as the product of diagonal maps

$$\Delta = \Delta_1 \times \cdots \times \Delta_n : X \times \cdots \times X \rightarrow X^{n_1} \times \cdots \times X^{n_r}.$$

Then denoting by $T_{X^r}$ a free model for $C^*(X^r)$, we have

$$\text{Ext}_{C^*(X^r)}(C^*(X^r), C^*(X^n)) \cong \text{Ext}_{T_{X^r}}(T_{X^r}, T_{X^n})$$

$$\cong \lim_{\leftarrow r_1} T_{X^{n_1}} \otimes T_{X^{n_2}} \otimes \cdots \otimes T_{X^{n_r}}, (T_{X^{n_1}} \otimes T_{X^{n_2}} \otimes \cdots \otimes T_{X^{n_r}}) \cong \lim_{\leftarrow r_1} \text{Ext}_{T_{X^{n_1}}(T_{X^{n_2}} \otimes \cdots \otimes T_{X^{n_r}})}(T_{X^{n_1}}, T_{X^{n_2}} \otimes \cdots \otimes T_{X^{n_r}})$$

$$\cong s^{(n-r)d}H^*(X^r).$$

We denote by $\Delta^t$ the map defined in the derived category of $C^*(X^n)$-module from $C^*(X^r)$ to $C^*(X^n)$ corresponding to a generator of $\text{Ext}_{C^*(X^n)}^{(n-r)d}(C^*(X^r), C^*(X^n))$. This element is well defined up to homotopy and up to the multiplication by a scalar.
Let \( p : E \to X^n \) be a fibration and consider the homotopy pullback

\[
\begin{array}{ccc}
E' & \xrightarrow{g} & E \\
\downarrow{p'} & & \downarrow{p} \\
X^r & \xrightarrow{\Delta} & X^n
\end{array}
\]

In the same way that Theorem 2 and Theorem 3 are proved, we get

**Theorem 13.** If the spaces arising in the above diagram are Gorenstein spaces then in the derived category of \( C^*(E) \)-modules, there exists a unique morphism (up to homotopy and up to multiplication by a scalar) \( g^! : C^*(E') \to C^*(E) \) making commutative the diagram

\[
\begin{array}{ccc}
C^*(E') & \xrightarrow{g^!} & C^*(E) \\
C^*(p') & & C^*(p) \\
C^*(X^r) & \xrightarrow{\Delta^!} & C^*(X^n).
\end{array}
\]

When the fiber of \( \Delta \) is a Poincaré duality space, then \( H^*(\Delta^!) \) and \( H^*(g^!) \) coincide, up to multiplication by a scalar, with the integration along the fiber.

### 6 String operations on a Gorenstein space

Let \( S \) be a connected surface of genus \( g \) with \( p \geq 1 \) incoming boundary components and \( q \geq 1 \) outgoing boundary components. Then the injection of the incoming components defines for each Gorenstein space \( X \) a map \( q_S : \text{Map}(S, X) \to (LX)^p \) that is the homotopy pullback in a diagram

\[
\begin{array}{ccc}
\text{Map}(S, X) & \xrightarrow{q_S} & (LX)^p \\
\downarrow & & \downarrow \psi \\
X^r & \xrightarrow{\Delta} & X^t.
\end{array}
\]

By Theorem 12, in the derived category of \( C^*((LX)^p) \)-modules there exists a well defined map (up to homotopy and up to multiplication by a scalar)

\[
(q_S)^! : C^*(\text{Map}(S, X)) \to C^*((LX)^p).
\]

Thus, as described in the introduction in case of Poincaré duality spaces, \( (g, p + q) \)-string operations are well defined up to multiplication by a scalar.

#### 6.1 The loop product for a classifying space

First remark that if \( X \) is the classifying space \( BG \) of a compact connected Lie group \( G \) then the homotopy fiber of the diagonal map \( \Delta : X^r \to X^t \) is the space \( (\Omega BG)^{t-r} \cong G^{t-r} \) that is a finite dimensional Poincaré space. Therefore, by Theorem 3, the integration along the fiber gives a shriek map \( \Delta^! \) as required by Chataur and Menichi, [5].

**Theorem 14.** If \( k = \mathbb{Q} \) and \( G \) is a compact connected Lie group then loop product \( H_*(LBG) \otimes H^*(LBG) \to H_*(LBG) \) is trivial.
\textbf{Proof.} A Sullivan model for $BG$ is given by $A_G = (\wedge(x_1, \ldots, x_n), 0)$ where $n = \text{rank} G$ and the $x_i$ have even degree, $|x_i| = 2n_i$. A $A_G \otimes A_G$-semi-free model of $A_G$ is:

$$\varphi : (\wedge(x_1, \ldots, x_n, x'_1, \ldots, x'_n, \overline{x}_1, \ldots, \overline{x}_n), D) \to (\wedge(x_1, \ldots, x_n), 0),$$

where $|x'_i| = 2n_i$, $|\overline{x}_i| = 2n_i - 1$, $D(x_i) = D(x'_i) = 0$, the differential on the lefthand cdga is determined by: $D(\overline{x}_i) = x_i - x'_i$ and the morphism of cdga’s $\varphi$ is defined by: $\varphi(x_i) = \varphi(x'_i) = x_i$ and $\varphi(\overline{x}_i) = 0$. Then, a representative of $\Delta^1$ is given by the morphism of $(\wedge(x_i, x'_i), 0)$-modules

$$\Delta^1 : (\wedge(x_i, x'_i, \overline{x}_i), D) \to (\wedge(x_i, x'_i), 0)$$

defined by $\Delta^1(\overline{x}_i \overline{x}_j \cdots \overline{x}_r) = 0$ if $\overline{x}_i \overline{x}_j \cdots \overline{x}_r \neq \overline{x}_1 \cdots \overline{x}_n$ and is equal to 1 otherwise. Since the differential in the minimal model of $BG$ is zero, the Sullivan minimal model for the free loop space on $BG$ is $(\wedge(x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n), 0)$ with $|\overline{x}_i| = 2n_i - 1$. A model for the projection $p \times p : LM \times LM \to M \times M$ is given by the inclusion $\rho : (\wedge(x_i, x'_i), 0) \to (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), 0)$. A representative for $(q_X)^1$ is thus given by the tensor product

$$\psi = (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), 0) \otimes_{\wedge(x_i, x'_i)} \Delta^1 : (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), D) \to (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), 0).$$

Now remark that the injection of $(\wedge(x_i, \overline{x}_i, \overline{x}_i), 0)$ into the first factor is a quasi-isomorphism and the composition with $\psi$ is zero. This shows that the loop product is zero for $BG$. \hfill \Box

\textbf{Remark.} Denote by $M_1 \subset M_2 \cdots \subset M_n \subset BG$ be a sequence of compact manifolds in $BG$ such that $BG = \bigcup M_n$. Then by restriction to $M_n$ we get a fibration $\Omega BG \to L_nBG \to M_n$ and a loop product on $H_\ast L_nBG$. The injection $M_{n-1} \to M_n$ induces a multiplicative shriek map $H_\ast(L_nBG) \to H_\ast(L_{n-1}BG)$. In [17] Gruher and Salvatore define the loop product on $BG$ to be the graded algebra $\lim H_\ast(L_nBG)$. This so-called loop product is no more a product on $H_\ast(BG)$ and is different from our loop product.

### 6.2 The loop coproduct of a classifying space

\textbf{Theorem 15.} If $k = \mathbb{Q}$ and if $G$ is a compact Lie group then the loop coproduct $H_\ast(LBG; \mathbb{Q}) \to H_\ast(LBG) \otimes H^\ast(LBG)$ is an injective map.

\textbf{Proof.} Denote by $(\wedge(x_i), 0)$ a Sullivan minimal model for $BG$ as in the previous section. We consider anew the first diagram in § 3.4, with $X$ in place of $M$. We use the model of $\Delta^1$ described above,

$$\Delta^1 : (\wedge(x_i, x'_i, \overline{x}_i, D) \to (\wedge(x_i, x'_i), 0),$$

where $D(\overline{x}_i) = x_i - x'_i$. A model for $\ell$ is given by the inclusion

$$\varphi : (\wedge(x_i, x'_i), 0) \to (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), D)$$

where $|\overline{x}_i| = |\overline{x}_i| = 2n_i - 1$, $D(\overline{x}_i) = x_i - x'_i$ and $D(\overline{x}_i) = 0$. Therefore a model for $q^1$ can be chosen as

$$q^1 = \Delta^1 \otimes 1 : (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), D) \to (\wedge(x_i, x'_i, \overline{x}_i, \overline{x}_i), D).$$

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We consider now the following diagram where the vertical maps are quasi-isomorphisms of cdga’s

\[
\begin{array}{ccc}
(\wedge(x_i, x_i', \tilde{x}_i, \tilde{x}_i), D) & \xrightarrow{q'} & (\wedge(x_i, x_i', \tilde{x}_i), D) \\
\uparrow \tau \cong & & \downarrow \psi \cong \\
(\wedge(x_i, \tilde{x}_i), 0) & \rightarrow & (\wedge(x_i, \tilde{x}_i), D).
\end{array}
\]

where the quasi-isomorphisms \(\tau\) and \(\psi\) are defined by: \(\tau(x_i) = x_i, \quad \tau(x_i') = x_i - \tilde{x}_i, \quad \tau(\tilde{x}_i) = \tilde{x}_i, \quad \psi(x_i) = \psi(x_i') = x_i, \quad \psi(\tilde{x}_i) = 0\) and \(\psi(\tilde{x}_i) = \tilde{x}_i\). The composition \(\psi \circ q' \circ \tau\) is a surjective map and so is \(H^*(q')\).

A model for the injection \(LBG \times_{BG} LBG \rightarrow LBG \times LBG\) is given by the surjection

\[
(\wedge(x_i, \tilde{x}_i), 0) \otimes (\wedge(x_i', \tilde{x}_i), 0) \rightarrow (\wedge(x_i, \tilde{x}_i, x_i'), 0).
\]

The dual of the loop coproduct is the composition

\[
H^*(LBG \times LBG) \xrightarrow{H^*(\pi)} H^*(Z) \xrightarrow{H^*(q')} H^*(LBG).
\]

Since both maps are surjective, this is also true for the composition. \(\square\)

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