Zero Modes and the Atiyah-Singer Index in Noncommutative Instantons

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ABSTRACT

We study the bosonic and fermionic zero modes in noncommutative instanton backgrounds based on the ADHM construction. In k instanton background in $U(N)$ gauge theory, we show how to explicitly construct $4Nk$ ($2Nk$) bosonic (fermionic) zero modes in the adjoint representation and $2k$ ($k$) bosonic (fermionic) zero modes in the fundamental representation from the ADHM construction. The number of fermionic zero modes is also shown to be exactly equal to the Atiyah-Singer index of the Dirac operator in the noncommutative instanton background. We point out that (super)conformal zero modes in non-BPS instantons are affected by the noncommutativity. The role of Lorentz symmetry breaking by the noncommutativity is also briefly discussed to figure out the structure of $U(1)$ instantons.

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1 Introduction

Instantons were found by Belavin, Polyakov, Schwartz and Tyupkin (BPST) [1] almost thirty years ago, as topologically nontrivial solutions of the duality equations of the Euclidean Yang-Mills theory with finite action. Immediately instantons were realized to describe the tunnelling processes between different $\theta$-vacua in Minkowski space and lead to the strong CP problem in QCD [2, 3]. (For the earlier development of instanton physics, see the collection of papers [4].) The non-perturbative chiral anomaly in the instanton background led to baryon number violation and a solution to the $U(1)$ problem [5, 6]. These revealed that instantons can have their relevance to phenomenological models like QCD and the Standard model [7].

Instanton solutions also appear as BPS states in string theory. They are described by Dp-branes bound to D(p+4)-branes [8, 9]. Subsequently, in [10, 11], low-energy excitations of D-brane bound states were used to explain the microscopic degrees of freedom of black-hole entropy, for which the information on the instanton moduli space has a crucial role. In addition the multi-instanton calculus was used for a nonperturbative test of AdS/CFT correspondence [12, 13, 14, 15], where the relation between Yang–Mills instantons and D-instantons was beautifully confirmed by the explicit form of the classical D-instanton solution in $AdS_5 \times S^5$ background and its associated supermultiplet of zero modes.

Recently instanton solutions on noncommutative spaces have been turned out to have more richer spectrums. While commutative instantons are always BPS states, noncommutative instantons admit both BPS and non-BPS states. Especially, instanton solutions can be found in $U(1)$ gauge theory and the moduli space of non-BPS instantons is smooth, small instanton singularities being resolved by the noncommutativity [16, 17]. Remarkably, instanton solutions in noncommutative gauge theory can also be studied by Atiyah-Drinfeld-Hitchin-Manin (ADHM) equation [18] slightly modified by the noncommutativity [19]. ADHM construction uses some quadratic matrix equations, hence noncommutative objects in nature, to construct (anti-)self-dual configurations of the gauge field. Thus the noncommutativity of space doesn’t make any serious obstacle for the ADHM construction of noncommutative instantons and indeed it turns out that it is a really powerful tool even for noncommutative instantons. Recently much progress has been made in this direction [16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

This paper is aiming to explain how to construct bosonic and fermionic zero modes in noncommutative instanton backgrounds based on the ADHM construction and how to relate them to the Atiyah-Singer index. Recently several papers [26, 36, 37, 38, 39, 40] discussed the instanton moduli space and the instanton calculus in noncommutative spaces. This paper is organized as follows. In next section we briefly review the ADHM construction on noncommutative spaces and explain how to construct the zero modes in noncommutative instanton backgrounds from the ADHM construction. In Section 3, we discuss the moduli space $\mathcal{M}_{k,N}$ of $k$ instantons in noncommutative $U(N)$ gauge theory and how to explicitly construct fermion
zero modes in adjoint and fundamental representations. In $U(1)$ instanton background the fermionic zero modes in the fundamental representation were briefly discussed by Nekrasov \[41\] recently, which are precisely reproduced from our solution for the $U(1)$ case. We point out that (super)conformal zero modes in the non-BPS background are affected by the noncommutativity and their explicit construction may be nontrivial, calling for further study. We also speculate that the $U(1)$ instanton may be understood by the structure of Lorentz symmetry breaking by the noncommutativity. In Section 4, we show that the number of the bosonic and the fermionic zero modes constructed in Section 3 is related to the Atiyah-Singer index of the noncommutative instantons. In Section 5 we discuss the results obtained and address some issues.

2 Zero Modes in Instantons

In this section we review briefly the formalism of the ADHM construction for instantons and discuss how to find the zero modes around the instanton background from this formalism \[42, 43, 44, 45, 46, 47\]. Here the ADHM construction is universally valid both for commutative and for noncommutative spaces. We will specify the noncommutative case if necessary. Although we discuss both self-dual and anti-self-dual instantons, we will often consider the anti-self-dual instantons as a specific problem. The self-dual instantons could be treated similarly.

Our interest is how to construct all the finite action solutions of the (anti-)self-duality equation in $U(N)$ gauge theory

$$F_{\mu\nu} = \pm * F_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}. \quad (2.1)$$

where the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (2.2)$$

In noncommutative space, for the field strength $F_{\mu\nu}$ to be gauge covariant, we need the commutator term in (2.2) even for $U(1)$ gauge group. ADHM construction provides an algebraic way for constructing (anti-)self-dual configurations of the gauge field in terms of some quadratic matrix equations on four manifolds \[18\].

In order to discuss the ADHM construction, it is convenient to introduce quaternions defined by

$$\mathbf{x} = x_\mu \sigma^\mu, \quad \mathbf{\bar{x}} = x_\mu \mathbf{\bar{\sigma}}^\mu, \quad (2.3)$$

where $\sigma^\mu = (i \tau^a, 1)$ and $\mathbf{\bar{\sigma}}^\mu = (-i \tau^a, 1)$ which have the basic properties

$$\sigma^\mu \mathbf{\bar{\sigma}}^\nu = \delta^{\mu\nu} + i \sigma^{\mu\nu}, \quad \sigma^\mu = \eta^a_{\mu\nu} \tau^a = * \sigma^{\mu\nu}, \quad \mathbf{\bar{\sigma}}^\mu \sigma^\nu = \delta^{\mu\nu} + i \mathbf{\bar{\sigma}}^{\mu\nu}, \quad \mathbf{\bar{\sigma}}^\mu = \bar{\eta}^a_{\mu\nu} \tau^a = - * \mathbf{\bar{\sigma}}^{\mu\nu}. \quad (2.4)$$
The $\sigma^\mu$ and $\bar{\sigma}^\mu$ can be used to construct the Euclidean Dirac matrices as
\[
\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
\[
\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = 2i \begin{pmatrix} \bar{\sigma}^{\mu\nu} & 0 \\ 0 & \sigma^{\mu\nu} \end{pmatrix}.
\] (2.5)

In the ADHM construction the gauge field with instanton number $k$ for $U(N)$ gauge group is given in the form
\[
A_\mu(x) = v(x)^\dagger \partial_\mu v(x)
\] (2.6)
where $v(x)$ is $(N+2k) \times N$ matrix defined by the equations
\[
v(x)^\dagger v(x) = 1, \tag{2.7}
\]
\[
v(x)^\dagger \Delta(x) = 0. \tag{2.8}
\]
In (2.8), $\Delta(x)$ is a $(N+2k) \times 2k$ matrix, linear in the position variable $x$, having the structure
\[
\Delta(x) = \begin{cases} 
  a - bx, & \text{self-dual instantons}, \\
  a - b\bar{x}, & \text{anti-self-dual instantons},
\end{cases}
\] (2.9)
where $a, b$ are $(N+2k) \times 2k$ matrices. $\Delta(x)$ can be thought of as a map from a $2k$-complex dimensional space $W$ to a $N+2k$-complex dimensional space $V$. The matrices $a, b$ are constrained to satisfy the conditions that $\Delta(x)^\dagger \Delta(x)$ be invertible and that it commutes with the quaternions. These conditions imply that $\Delta(x)^\dagger \Delta(x)$ as a $2k \times 2k$ matrix has to be factorized as follows
\[
\Delta(x)^\dagger \Delta(x) = f^{-1}(x) \otimes 1_2
\] (2.10)
where $f^{-1}(x)$ is a $k \times k$ matrix and $1_2$ is a unit matrix in quaternion space. Then the resulting field strength $F_{\mu\nu}$ obtained by (2.4) ensures the (anti-)self-duality equation (2.1) if and only if $v(x)$ and $\Delta(x)$ obey the completeness relation
\[
v(x)v(x)^\dagger + \Delta(x)f(x)\Delta(x)^\dagger = 1. \tag{2.11}
\]
Note that the matrix $v(x)$ in (2.4) and (2.8) is unique only up to a gauge transformation
\[
v(x) \to v(x)g(x), \quad \Delta(x) \to \Delta(x), \quad g(x) \in U(n) \tag{2.12}
\]
which generates usual $U(N)$ gauge transformations for the gauge field
\[
A_\mu(x) \to g(x)^\dagger A_\mu(x)g(x) + g(x)^\dagger \partial_\mu g(x).
\] (2.13)
\footnote{One can always normalize the matrix $v$ in usual way even for noncommutative $R^4_{NC}$ and $R^2_{NC} \times R^2_C$ as done in [30, 33], respectively.}
Given a pair of matrices \( a, b \), (2.7) and (2.8) define \( A_\mu \) up to gauge equivalence. Different pair of matrices \( a, b \) may yield gauge equivalent \( A_\mu \) since (2.7) and (2.8) are invariant under
\[
a \to QaK, \quad b \to QbK, \quad v \to Qv
\]
where \( Q \in U(N+2k) \) and \( K \in GL(k, \mathbb{C}) \). This freedom can be used to put \( a, b \) in the canonical forms
\[
a = \begin{pmatrix} \lambda \\ \xi \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1_{2k} \end{pmatrix},
\]
where \( \lambda \) is an \( N \times 2k \) matrix and \( \xi \) is a \( 2k \times 2k \) matrix. Here we decompose the matrix \( \xi \) in the quaternionic basis \( \bar{\sigma}^\mu \) as
\[
\xi = \xi_\mu \bar{\sigma}^\mu,
\]
where \( \xi_\mu \)'s are \( k \times k \) matrices. In the basis (2.15), the constraint (2.10) boils down to
\[
\text{tr}_2 \tau^a a^\dagger = \begin{cases} 
\theta_{\mu\nu} \bar{\eta}^a_{\mu\nu}, & \text{self-dual instantons,} \\
\theta_{\mu\nu} \gamma^a_{\mu\nu}, & \text{anti-self-dual instantons,}
\end{cases}
\]
where \( \tau^2 \) is the trace over the quaternionic indices. Here we work in general in flat noncommutative Euclidean space \( \mathbb{R}^4 \) represented by
\[
[x^\mu, x^{\nu}] = i\theta^{\mu\nu}
\]
where \( \theta^{\mu\nu} = -\theta^{\nu\mu} \). Note that there still exists a residual \( U(k) \) symmetry of the \( U(N+2k) \times GL(k, \mathbb{C}) \) symmetry group (2.14) to preserve the canonical form for \( b \) given in (2.15) which acts on \( \lambda \) and \( \xi_\mu \) as
\[
\lambda \to \lambda U, \quad \xi_\mu \to U^\dagger \xi_\mu U, \quad U \in U(k).
\]
Introduce two \( N \times k \) matrices \( I^\dagger, J \) to represent the \( N \times 2k \) matrix \( \lambda \) as
\[
\lambda = (I^\dagger, J).
\]
Then the ADHM constraints (2.17) for the anti-self-dual instantons, for example, have the following forms
\[
\begin{align*}
\mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \theta^{\mu\nu} \eta^3_{\mu\nu} \\
\mu_c &= IJ + [B_1, B_2] = \frac{1}{2} \theta^{\mu\nu} (\eta^1_{\mu\nu} + i\eta^2_{\mu\nu}),
\end{align*}
\]
where \( B_1 = \xi_2 + i\xi_1, B_2 = \xi_4 + i\xi_3 \). Now we can count the independent bosonic collective coordinates of multi-instanton solution. From the ADHM construction for \( k \) instantons in
\textit{U(N)} gauge theory, the matrices $\lambda$ and $\xi_{\mu}$ in (2.13) have $4Nk + 4k^2$ real degrees of freedom. However (2.17) imposes $3k^2$ constraints for $\lambda$ and $\xi_{\mu}$. In addition we have the $U(k)$ residual symmetry (2.20). These remove $4k^2$ degrees of freedom and thus totally $4Nk$ real degrees of freedom remain. They describe a $4Nk$-dimensional moduli space $\mathcal{M}_{k,N}$ which is still hyper-Kähler space. The ADHM constraints (2.22) and (2.23) are nothing but the (deformed) moment maps of the hyper-Kähler quotient construction. We will show that even for noncommutative spaces $\mathcal{M}_{k,N}$ is the moduli space of $k$ instantons in $U(N)$ gauge theory.

Let’s consider Yang-Mills theory with gauge group $U(N)$ with action
\begin{equation}
S = -\frac{1}{2} \int d^4x \text{tr} F_{\mu\nu} F^{\mu\nu} .
\end{equation}
Of course, for the noncommutative space (2.19), the integral in (2.24) may be replaced by an appropriate trace over the representation space $\mathcal{H}$ of the algebra (2.19):
\begin{equation}
\int d^4x \rightarrow \text{Tr}_{\mathcal{H}} .
\end{equation}
Consider small fluctuations about a classical instanton solution satisfying (2.1)
\begin{equation}
A_{\mu}(x) = A_{\mu}^{cl}(x) + \delta A_{\mu}(x) .
\end{equation}
Since we are only interested in physical excitations, the fluctuations $\delta A_{\mu}(x)$ are required to be square integrable \footnote{If the action is expanded to second order in $\delta A_{\mu}$, one can find the following result
\begin{equation}
S[A_{\mu}] \approx S[A_{\mu}^{cl}] - \frac{1}{4} \int d^4x \text{tr}(\delta F_{\mu\nu} \mp *\delta F_{\mu\nu})^2
\end{equation}
where $\delta F_{\mu\nu} = D_{\mu}\delta A_{\nu} - D_{\nu}\delta A_{\mu}$ with $D_{\mu}\delta A_{\nu} = \partial_{\mu}\delta A_{\nu} + [A_{\mu}^{cl}, \delta A_{\nu}]$. To derive the result (2.28) we have used the (anti-)self-duality condition (2.1) and the equations of motion for $A_{\mu}^{cl}$ and the global gauge rotations satisfying (2.28) which are non-normalizable on $\mathbb{R}^4$. To obtain purely physical zero modes, the global gauge rotations are removed, leaving the number of physical zero modes as $4Nk - N^2 + 1$ for $k \geq \frac{N}{2}$, and $4k^2 + 1$ for $k \leq \frac{N}{2}$ \footnote{For noncommutative spaces, this condition means that $\text{Tr}_{\mathcal{H}}|\delta A_{\mu}(x)|^2 < \infty$.} However the global gauge rotations appear in the $k$-instanton measure in the functional integral and in the calculation of the Atiyah-Singer index. So we will keep these modes.} and to be orthogonal to gauge transformations, i.e.
\begin{equation}
\int d^4x \text{tr} D_{\mu} \Lambda(x) \delta A_{\mu}(x) = - \int d^4x \text{tr} \Lambda(x) D_{\mu} \delta A_{\mu}(x) = 0 \tag{2.27}
\end{equation}
where $\Lambda(x)$ is an arbitrary gauge parameter in $U(N)$. Since the condition (2.27) should be satisfied for any $\Lambda(x)$, this orthogonality requirement is equivalent to the usual gauge condition
\begin{equation}
D_{\mu} \delta A_{\mu}(x) = 0 .
\end{equation}
If the action is expanded to second order in $\delta A_{\mu}$, one can find the following result
\begin{equation}
S[A_{\mu}] \approx S[A_{\mu}^{cl}] - \frac{1}{4} \int d^4x \text{tr}(\delta F_{\mu\nu} \mp *\delta F_{\mu\nu})^2 \tag{2.29}
\end{equation}
the fact that the boundary contribution at infinity is trivial since the fluctuations \( \delta A_\mu(x) \) are square integrable. The above result shows that as long as the fluctuations, satisfying the gauge condition (2.28), around a classical instanton solution \( A^{cl}_\mu(x) \) satisfy the (anti-)self-duality equation,

\[
\delta F_{\mu\nu} = \pm \ast \delta F_{\mu\nu},
\]

they are zero modes, i.e. the action is not changed by these collective excitations. Thus one can study the space of (anti-)self-dual solutions by considering small fluctuations \( A^{cl}_\mu(x) + \delta A_\mu(x) \) about a particular solution \( A^{cl}_\mu(x) \) and asking that the resulting field strength continue to be (anti-)self-dual.

Now our problem is how to find the fluctuations, the zero modes, satisfying (2.30) and (2.28). One strategy to characterize the general family of instanton solutions by all relevant collective coordinates is following [48]. One starts from some known solution and transforms it by applying all symmetry transformations. The symmetry transformations which act non-trivially generate new solutions and require an introduction of the collective coordinates. For example, for commutative \( SU(2) \) instantons, these are positions \( x^\mu_0 \) by translation symmetry, size \( \rho \) by dilatation and three orientation parameters \( \omega^a \) corresponding to global \( SU(2) \) rotations. However this method is not easy to generalize to noncommutative spaces although it may be interesting.

Instead we will use the ADHM construction to find the zero modes. As discussed in this section, the matrices \( a, b \) carry full information for collective coordinates up to the gauge transformation (2.14). Consider so varying \( \Delta(x) \) by an amount \( \delta \Delta(x) = \delta a - \delta b x \) (or \( \delta a - \delta b \bar{x} \)) within the space of solutions. \( \delta \Delta \) is given by \( \delta a \) and \( \delta b \) which describe collective variables in the moduli space \( \mathcal{M}_{k,N} \). After the gauge fixing (2.15), all the collective coordinates are encoded in the matrix \( a \) up to the residual \( U(k) \) symmetry (2.20). Here we will use the gauge fixed basis (2.15) and thus we will put \( \delta b = 0 \). Then \( v(x) \) will have to be changed to \( (v + \delta v)(x) \) where

\[
\delta v^\dagger \Delta + v^\dagger \delta \Delta = 0 \tag{2.31}
\]

and

\[
\delta v^\dagger v + v^\dagger \delta v = 0. \tag{2.32}
\]

The general solution to (2.31) is of the form [13]

\[
\delta v^\dagger = -v^\dagger \delta \Delta f^\dagger + \delta u^\dagger v^\dagger \tag{2.33}
\]

where \( \delta u = v^\dagger \delta v \) and (2.32) implies

\[
\delta u + \delta u^\dagger = 0. \tag{2.34}
\]

Since we are still in the space of solutions, \( (\Delta + \delta \Delta)(x) \) has to satisfy the factorization condition (2.10) and the completeness relation (2.11). By using (2.33) and (2.34), one can easily check
that \((\Delta + \delta \Delta)(x)\) automatically satisfies the completeness relation. Since \(\delta \Delta = \delta a\), this variation is equivalent to \(a \rightarrow a + \delta a\). Thus, from (2.17) and (2.18), we see that \(\delta a\) should satisfy
\[
\delta \mu^a = \text{tr}_2 \tau^a (a \delta a + a \delta \tau^a) = 0 \tag{2.35}
\]
or explicitly
\[
\delta \mu_r = \left[ \delta B_1, B_1^\dagger \right] + \left[ B_1, \delta B_1^\dagger \right] + \left[ \delta B_2, B_2^\dagger \right] + \left[ B_2, \delta B_2^\dagger \right] \]
\[
+ \delta I I^\dagger + I \delta J^\dagger J - J^\dagger J \delta = 0, \tag{2.36}
\]
\[
\delta \mu_c = \delta I J + I \delta J + \left[ \delta B_1, B_2 \right] + \left[ B_1, \delta B_2 \right] = 0. \tag{2.37}
\]
One sees that the ADHM constraints for the variation \(\delta a\) are never deformed by the noncommutativity and are completely equivalent to their commutative counterparts. However, we will see later that some zero modes are still affected by the noncommutativity since they are represented by the original moduli \(a\) which should satisfy the constraints (2.17).

From the solution (2.33),
\[
\delta A_\mu = \quad v^\dagger \partial_\mu \delta v + v^\dagger \partial_\mu v
\]
\[
= \quad v^\dagger (\delta \Delta f \partial_\mu \Delta^\dagger - \partial_\mu \Delta f \delta \Delta^\dagger) v + D_\mu \delta u. \tag{2.38}
\]
The term \(D_\mu \delta u = \partial_\mu \delta u + [A^d_\mu, \delta u]\) corresponds to an infinitesimal gauge transformation. By a suitable choice of gauge for \(A^d_\mu\), we will put \(\delta u = 0\). Thus the fluctuations \(\delta A_\mu\) are essentially determined by \(\delta \Delta = \delta a\) up to gauge transformation. As discussed in (2.28), we have to require the gauge condition (2.28) for \(\delta A_\mu(x)\). Let’s calculate the gauge condition (2.28) for the solution (2.38) using (2.7), (2.8), (2.10) and (2.11),
\[
D_\mu \delta A_\mu = \quad v^\dagger \partial_\mu (v \delta A_\mu v^\dagger) v
\]
\[
= \quad -v^\dagger (\partial_\mu \Delta f \Delta^\dagger \delta f \partial_\mu \Delta^\dagger - 2 \delta \Delta f \partial_\mu \Delta^\dagger \Delta f \partial_\mu \Delta^\dagger - \delta \Delta f \Delta^\dagger \partial_\mu \Delta f \partial_\mu \Delta^\dagger) v - \text{h.c.}
\]
\[
= \quad -2v^\dagger b f \left( \text{tr}_2 (\Delta^\dagger \delta \Delta) - \text{tr}_2 (\delta \Delta^\dagger \Delta) \right) f b^\dagger v = 0, \tag{2.39}
\]
where the following relations are used:
\[
\bar{\sigma}^\mu f \sigma^\mu = 4 f, \quad \sigma^\mu b^\dagger \Delta \sigma^\mu = -2 \Delta^\dagger b, \quad \text{for} \quad \Delta(x) = a - b \bar{x}, \tag{2.40}
\]
\[
\sigma^\mu f \bar{\sigma}^\mu = 4 f, \quad \bar{\sigma}^\mu b^\dagger \Delta \bar{\sigma}^\mu = -2 \Delta^\dagger b, \quad \text{for} \quad \Delta(x) = a - b \bar{x} \tag{2.41}
\]
and
\[
\bar{\sigma}^\mu \Delta^\dagger \delta \Delta \sigma^\mu = 2 \text{tr}_2 (\Delta^\dagger \delta \Delta), \quad \sigma^\mu \delta \Delta^\dagger \Delta \bar{\sigma}^\mu = 2 \text{tr}_2 (\delta \Delta^\dagger \Delta). \tag{2.42}
\]

\(^5\)Using the gauge degrees of freedom (2.12), we can always choose this gauge.
Thus the gauge condition (2.39) imposes another constraint \( \mu_g = \text{tr}_2(\Delta^\dagger \Delta) - \text{tr}_2(\Delta^\dagger \Delta) = 0 \), or more explicitly,
\[
\mu_g = [\delta B_1, B_1^\dagger] - [B_1, \delta B_1^\dagger] + [\delta B_2, B_2^\dagger] - [B_2, \delta B_2^\dagger] + \delta I I^\dagger - I \delta I^\dagger + \delta J^\dagger J - J^\dagger J = 0.
\]
(2.43)

Note that the global \( U(k) \) symmetry (2.20) is also fixed by the above gauge condition (2.43) since it generates the “global” gauge transformation \( a \rightarrow a + \delta a \).

Using the result (2.38), one can find \( \delta F_{\mu \nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu \) as
\[
\delta F_{\mu \nu} = v^\dagger \left\{ \partial_\mu (v \delta A_\nu v^\dagger) - \partial_\nu (v \delta A_\mu v^\dagger) \right\} v
\]
(2.44)

where \( \delta f = -f(\delta \Delta^\dagger \Delta + \Delta^\dagger \delta \Delta) f \). To get the result (2.44), we have used (2.7), (2.8), (2.10) and (2.11) as in the derivation of (2.39). One can easily check that the resulting \( \delta F_{\mu \nu} \) in (2.44) satisfies the (anti-)self-dual equation (2.30) if one notices that the function \( \delta f(x) \) as well as \( f(x) \) commutes with the quaternions since we required (2.36) and (2.37), i.e. the factorization condition for \( (\Delta + \delta \Delta)(x) \). So the fluctuations \( \delta A_\mu(x) \) in (2.38) describe the zero modes around a classical instanton solution (2.6) \[51\]. Explicitly, for example, for the anti-self-dual instanton with \( \Delta(x) = a - b \bar{x} \), the zero modes \( \delta A_\mu \) has the following form
\[
\delta A_\mu = -v^\dagger (\delta a f \sigma_\mu b^\dagger - b \bar{\sigma}_\mu f \delta a^\dagger) v
\]
(2.45)
and
\[
\delta F_{\mu \nu} = 2iv^\dagger \left\{ b \bar{\sigma}^{\mu \nu} \delta f b^\dagger - b \bar{\sigma}^{\mu \nu} f b^\dagger \Delta f \delta \Delta^\dagger - \delta \Delta f \Delta^\dagger b \bar{\sigma}^{\mu \nu} f b^\dagger \right\} v.
\]
(2.46)

Similarly, the expressions in the case of the self-dual background with \( \Delta(x) = a - b x \) can also be easily obtained.

The ADHM constraints (2.39) and (2.37) for zero modes can be synthesized with (2.43) into the constraints
\[
\delta \mu_k = [\delta B_1, B_1^\dagger] + [\delta B_2, B_2^\dagger] + \delta I I^\dagger - J^\dagger J = 0,
\]
(2.47)
\[
\delta \mu_c = \delta I J + I \delta J + [\delta B_1, B_2] + [B_1, \delta B_2] = 0.
\]
(2.48)

Since we now have the \( 4k^2 \) constraints, (2.47) and (2.48), \( \delta a \) has totally \( 4Nk \) real degrees of freedom, giving \( 4Nk \) zero modes as we already argued before. These are exactly given by (2.38).

As an example, in the anti-self-dual background, they take the form (2.45).

### 3 Noncommutative Instantons and Their Fermionic Zero Modes

Let’s start with a brief review of commutative \( SU(2) \) instantons. The standard ansatz for \( A_\mu(x) \), for example the BPST \[1\] or ’t Hooft ansatz \[6\], entangles the group and Lorentz indices.
This is basically based on the fact that the Euclidean Lorentz group $O(4)$ is isomorphic to $SU(2)_L \times SU(2)_R$, where two $SU(2)$ subgroups correspond to the left-handed and right-handed chiral rotations. More concretely, the Lorentz generators $L_{\mu\nu} = -L_{\nu\mu}$ can be decomposed into the self-dual ($L^+_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} L^+_{\rho\sigma}$) and the anti-self-dual ($L^-_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} L^-_{\rho\sigma}$) parts. According to the isomorphism $O(4) \cong SU(2)_L \times SU(2)_R$, the Lorentz generators $L^+_{\mu\nu}$ and $L^-_{\mu\nu}$ can be mapped to $SU(2)_L$ and $SU(2)_R$, respectively. The precise mapping can be achieved in terms of 't Hooft $\eta$-symbol:

$$L^+_{\mu\nu} = \eta^a_{\mu\nu} T^a, \quad L^-_{\mu\nu} = \bar{\eta}^a_{\mu\nu} T^a,$$

where $T^a$ are Lie algebra generators in $SU(2)_L$ or $SU(2)_R$. For spinors and vector fields, $T^a = \frac{1}{2} \tau^a$ and $\epsilon^{abc}$, respectively.

Under this decomposition the vector $A_\mu$ is indeed the representation $\left( \frac{1}{2}, \frac{1}{2} \right)$. Also the anti-symmetric tensor $F_{\mu\nu}$ in $O(4)$ having six components form a $\left( 1, 1 \right)$ representation of $SU(2)_L \times SU(2)_R$. Thus the (anti-)self-dual gauge field in $SU(2)_G$ gauge theory, i.e. $F^\pm_{\mu\nu} = \pm \ast F^\pm_{\mu\nu}$, may be obtained by reshuffling several $SU(2)$'s, i.e. $SU(2)_{L,R}$ and $SU(2)_G$:

$$F^+_{\mu\nu}(x) = \eta^a_{\mu\nu} f(x^2), \quad F^-_{\mu\nu}(x) = \bar{\eta}^a_{\mu\nu} f(x^2),$$

where $f(x^2)$ is a scalar function satisfying some harmonic equation. Thus group theory essentially determines the tensor structure of (anti-)self-dual gauge fields. (Actually this is also a reason why quaternions ($\cong SU(2)$) play a crucial role in the ADHM construction.)

When we try to generalize the above consideration to the noncommutative space (2.19), what happens? Since $\theta_{\mu\nu}$ are anti-symmetric tensor, let’s decompose them into self-dual and anti-self-dual parts:

$$\theta_{\mu\nu} = \eta^a_{\mu\nu} \xi^a + \bar{\eta}^a_{\mu\nu} \chi^a.$$  

Note that, as seen from (2.17), the ADHM construction depends only on the self-dual or the anti-self-dual part of $\theta_{\mu\nu}$, because $\eta^a_{\mu\nu} \bar{\eta}^b_{\mu\nu} = 0$. Since the self-duality condition is invariant under $SO(4)$ rotations (or more generally $SL(4, \mathbb{R})$ transformations), one can always make the matrix $\theta_{\mu\nu}$ to a standard symplectic form by performing the $SO(4)$ transformation $R$:

$$\theta = R \tilde{\theta} R^T,$$

where we choose $\tilde{\theta}$ as

$$\tilde{\theta}_{\mu\nu} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & -\theta_2 & 0 \end{pmatrix}.$$
There are four important cases to consider:

- \( \theta_1 = \theta_2 = 0 \) : commutative \( \mathbb{R}^4 \),
- \( \theta_1 = \theta_2 = \frac{\zeta}{4} \) : self-dual \( \mathbb{R}^4_{NC} \),
- \( \theta_1 = -\theta_2 = \frac{\zeta}{4} \) : anti-self-dual \( \mathbb{R}^4_{NC} \),
- \( \theta_1 \theta_2 = 0 \) but \( \theta_1 + \theta_2 \neq 0 \) : \( \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \).

By the noncommutativity (2.19), the original Lorentz symmetry is broken down to its subgroup. What are the remaining spacetime symmetries for each case?

We will refer only rotational symmetry since the noncommutative space (2.19) always respects a global translational symmetry. For the commutative \( \mathbb{R}^4 \), of course, we have full \( O(4) \) symmetry. However, for the self-dual and the anti-self-dual \( \mathbb{R}^4_{NC} \), the original Lorentz symmetry \( O(4) \cong SU(2)_L \times SU(2)_R \) is broken down to the subgroup:

\[
SU(2)_L \times SU(2)_R \rightarrow \begin{cases} 
SU(2)_R \times U(1)_L, & \text{self-dual } \mathbb{R}^4_{NC}, \\
SU(2)_L \times U(1)_R, & \text{anti-self-dual } \mathbb{R}^4_{NC}.
\end{cases}
\]

On the other hand, \( \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \) has more complicated Lorentz symmetry breaking:

\[
SU(2)_L \times SU(2)_R \rightarrow \begin{cases} 
SU(2)_D^\pm \times U(1)_D, & \text{for } \theta_1 \neq 0, \\
SU(2)_D^+ \times U(1)_D, & \text{for } \theta_2 \neq 0,
\end{cases}
\]

where \( SU(2)_D^\pm \) are the diagonal subgroups of \( O(4) \) generated by \( L^\pm_{\mu\nu} \pm L_{\mu\nu}^\pm \) and \( U(1)_D \) are their \( U(1) \) subgroups.

Let’s specify \( U(N) \) instantons in the noncommutative space (3.5). From (2.17), one sees that anti-self-dual (self-dual) instantons on self-dual (anti-self-dual) \( \mathbb{R}^4_{NC} \) are described by deformed ADHM equation and the singularities of instanton moduli space \( \mathcal{M}_{k,N} \) are resolved [16, 17]. While self-dual (anti-self-dual) instantons on self-dual (anti-self-dual) \( \mathbb{R}^4_{NC} \) are described by undeformed ADHM equation and the singularities of instanton moduli space \( \mathcal{M}_{k,N} \) still remain. While the former is not a BPS state, the latter is a BPS state. For the \( \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \) space discussed in [33, 30], both self-dual and anti-self-dual \( U(N) \) instantons are described by the deformed ADHM equation and their moduli space is always non-singular. This case is also a non-BPS state.

For \( U(1) \) gauge theory, according to a stability theorem by Nakajima [33], we can always put \( J = 0 \). So the ADHM contraints (2.22) and (2.23) in the \( U(1) \) case reduce to in the basis (3.3):

\[
\mu_r = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger = \theta^{\mu\nu} \eta^3_{\mu\nu}
\]

\[
\mu_c = [B_1, B_2] = 0.
\]

\footnote{Note that, by the canonical choice (3.5) of the noncommutativity, we already fixed our coordinate system. So we are considering symmetry properties with respect to this coordinate system.}
The most general solution for (3.12) and (3.13) may be obtained by simultaneously triangularizing the matrices $B_1$ and $B_2$ while they can be simultaneously diagonalized when all instanton separations become large, i.e.

$$B_1 = \text{diag}(z_1^1, \cdots, z_1^k), \quad B_2 = \text{diag}(z_2^1, \cdots, z_2^k).$$

(3.14)

A special solution is that one of them is identically zero, e.g. $B_2 = 0$. This solution describes $k$ instantons which sit along the complex $z_1$-plane, so-called elongated instantons \[20, 32\]. Note that the moduli space $\mathcal{M}_{k,1}$ for $k$ U(1) instantons is completely determined by translations of instantons, that is $4k$ position moduli.

Depending on the value of $\theta^{\mu\nu}$, U(1) instantons have different properties as discussed above. If $R^4_{NC}$ is self-dual, then $\mu_r = \zeta$. This case describes the Nekrasov-Schwarz instanton with fixed size $\sqrt{\zeta}$ \[10\]. On the other hand, in the case of anti-self-dual $R^4_{NC}$, $\mu_r = 0$ and $I = 0$, which describes localized instantons with zero size \[54, 55, 27, 56, 31\]. Since the structure of commutative instantons has been almost determined by symmetry property itself, the immediate question is whether one can understand noncommutative instantons, especially U(1) instantons, by the symmetry breaking (3.10) or (3.11). We think this may be the case although we don’t arrive at concrete answer yet. But, let’s just quote an observation that at least the anti-self-dual (or the self-dual) single U(1) instanton has this $SU(2)$ algebra structure in the self-dual (or in the anti-self-dual) $R^4_{NC}$, with the notation (22) in \[24\] (where $\zeta = 2$),

$$F_A = -\frac{2i}{x^2(x^2+1)(x^2+2)}\bar{\eta}^a T_a dx^\mu \wedge dx^\nu,$$

(3.15)

where $T^+ = -T^1 + iT^2 = z_1 \bar{z}_2$, $T^- = -T^1 - iT^2 = \bar{z}_1 z_2$ and $T^3 = \frac{1}{2}(z_1 \bar{z}_1 - z_2 \bar{z}_2)$ and thus $[T^a, T^b] = -i\varepsilon^{abc}T^c$. Since $T^a$’s commute with $x^2$, the solution (3.15) really looks like a commutative $SU(2)$ instanton where the gauge group is given by the unbroken $SU(2)_R$ spacetime symmetry. Besides, the (anti-)self-dual U(1) localized instantons are purely determined by the unbroken $U(1)_{L,R}$ symmetry.

As we discussed in Section 2, the whole family of the solutions of (2.47) and (2.48) gives $4Nk$ moduli. It is difficult to completely solve the constraints (2.47) and (2.48) to find the zero modes of U(1) instantons. However, when all distances between $k$ U(1) instantons (both BPS and non-BPS instantons) become large, $\delta B_{1,2} = \text{diag}(\delta z_{1,2}^1, \cdots, \delta z_{1,2}^k)$ and $\delta I = 0$ solve the constraints (2.47) and (2.48) since $B_1$ and $B_2$ can be diagonalized in the limit as in (3.14). The zero modes in the dilute $k$ anti-self-dual instantons are so given by (2.43) with $4k$ eigenvalues of $\xi_\mu$ and $\delta \lambda = \delta \bar{\lambda} = 0$. For the similar reason, it is difficult to solve the ADHM constraints

\footnote{We are grateful to Kimyeong Lee for pointing out a mistake in the earlier version of this paper.}

\footnote{This $SU(2)$ algebra may be viewed as a noncommutative Hopf fibration, as in \[58\], $\pi : S^3 \cong SU(2)_L \to S^2 \cong SU(2)_L/U(1)_L$ related to the symmetry breaking (3.10).}
\[ (2.47) \text{ and } (2.48) \text{ completely for } k \text{ } U(N) \text{ instantons for which } \delta B_{1,2} \text{ can be diagonalized only for dilute instanton gas limit.} \]

As observed in [21, 24], a \( U(N) \) instanton always contains a \( U(1) \) instanton, the Nekrasov-Schwarz instanton or the localized instanton, in the limit that the size of \( SU(N) \) instanton vanishes which corresponds to the limit \( J \to 0 \). 9 In this limit, the matrices \( B_1 \) and \( B_2 \) can be diagonalized like as \( (3.14) \) if all instantons are sufficiently distant. Then we see that \( \delta B_1 \) and \( \delta B_2 \) should also be diagonalized to satisfy \( (2.48) \) since they are independent of each other. This requires that \( \delta I = 0 \) because of \( (2.47) \). So, in the limit \( J \to 0 \) and in the dilute instanton gas limit, the moduli from \( \lambda \) have to be frozen and only 4\( k \) moduli from \( B_1 \) and \( B_2 \) remain. Eventually we thus arrive at the moduli space of \( U(1) \) instantons. At this stage, an intriguing question arises. Since \( M_{k,N} \) for \( k \) noncommutative instantons is \( 4Nk \)-dimensional, this implies that the size \( \rho \) of an \( SU(N) \) instanton should be one of the \( 4N \) moduli. According to [48], we can guess that the size moduli of noncommutative instantons should be generated by conformal transformation. This naturally implies that still there exist corresponding (super)conformal zero modes in the noncommutative instanton background. However we will show that the (super)conformal zero modes in non-BPS instantons are affected by the noncommutativity.

It is easy to find some special zero modes satisfying \( (2.47) \) and \( (2.48) \) which are related to supersymmetric fermionic zero modes [52, 59]. They are given by

\[ \delta a = b \bar{\sigma}_\mu c_\mu \] (3.16)

where \( c_\mu \) is a constant real four-vector. In order to check \( (2.47) \) and \( (2.48) \) for \( (3.10) \), it is more convenient to directly check using \( (2.4) \) that \( \delta \mu^a = \text{tr}_2 \tau^a (\delta a^1 + a^1 \delta a) = 0 \) and \( \mu_g = \text{tr}_2 (\Delta^1 \delta a^1 - \delta a^1 \Delta) = 0 \). The solution \( (3.10) \) gives \( \delta A_\mu = F_{\mu \nu}^{cl} c_\nu \) in \( (2.43) \) which have the anti-self-dual field strength \( \delta F_{\mu \nu} = -D_\lambda F_{\mu \nu}^{cl} c_\lambda \) proved using the Bianchi identity and \( D_\mu \delta A_\mu = D_\mu F_{\mu \nu}^{cl} c_\nu = 0 \) by the equations of motion.

In the case of commutative instantons, we can have additional zero modes which are related to “superconformal zero-modes” [52, 59]:

\[ \delta a = a \sigma_\mu d_\mu + \frac{b}{k} \text{tr}_k (b^1 a) \] (3.17)

where \( \text{tr}_k \) is the trace over \( k \times k \) matrices and \( d_\mu \) is a constant real four-vector. However, one can check that, in the case of non-BPS instantons, \( (3.14) \) doesn’t satisfy \( (2.47) \) and \( (2.48) \) and no longer gives zero modes. To see this, let’s calculate \( \delta \mu^a \) and \( \mu_g \) for the ansatz \( (3.17) \). It is sufficient to check the first term in \( (3.17) \) since the second term has the same structure with \( (3.10) \). A simple calculation gives

\[ \delta \mu^a = 2 \delta \mu^\mu (\eta^a_\mu d_4 + \varepsilon^{abc} b^a \eta^b_\mu d_c) \]

\[ \delta \mu^\mu = 2 \delta \mu^\mu \] (2.47) for single \( U(2) \) instanton, the ADHM constraints \( (2.22) \) and \( (2.23) \) can be solved by \( I = \sqrt{\mu^2 + \zeta (\cos \alpha e^{i\beta}, \sin \alpha e^{-i\beta}, J^1 = \rho (\sin \alpha e^{(\beta - \gamma)}, -\cos \alpha e^{-i(\beta + \gamma)} \) where \rho is the size of an \( SU(2) \) instanton and \( (\alpha, \beta, \gamma) \) are global \( SU(2) \) angles. Here we fixed the global \( U(1) \) symmetry in \( (2.20) \).
There is no solution satisfying $\delta \mu^a = 0$ and $\mu_g = 0$ if $\theta^\mu{}^\nu$ is self-dual or $\mathbb{R}^2_{NC} \times \mathbb{R}^2_L$. The same conclusion arises in the self-dual instanton. Thus, for the reason discussed in Section 2, the fluctuation (3.17) in the non-BPS background fails to give a field strength satisfying (2.30). Of course, for BPS instantons, (3.17) still gives zero modes because $\bar{\eta}^a{}^\mu \eta^b{}^\mu = 0$. It could be originated from the fact that the non-BPS instanton has a minimum size set by the noncommutative scale. Anyway it remains an open problem how to modify the ansatz (3.17) to satisfy $\delta \mu^a = 0$ and $\mu_g = 0$ for all cases.

Now let’s find the normalizable fermionic zero modes by solving a massless Dirac equation in the background of anti-self-dual $k$ instantons. Following the same strategy in [60], (2.28) and (2.30) can be united into

$$D_\mu \text{tr}_2(\sigma^\nu \Phi_1 \sigma^\nu) = 0$$

where the matrix $\Phi_1$ has the form

$$\Phi_1 = \bar{\sigma}^\mu \delta A_\mu = \begin{pmatrix} \delta A_4 - i \delta A_3 & -\delta A_2 - i \delta A_1 \\ \delta A_2 - i \delta A_1 & \delta A_4 + i \delta A_3 \end{pmatrix}$$

and thus $\Phi_1$ satisfies the reality condition

$$\sigma^2 \Phi_1^* \sigma^2 = -\Phi_1.$$ 

Since the $\sigma^\nu$ are a complete set of $2 \times 2$ matrices, (3.19) can be reduced to

$$\sigma^\mu D_\mu \Phi_1 = 0.$$ 

Since the field equation (3.22) independently acts on each column of $\Phi_1$ and the second column is uniquely determined by the reality condition (3.21) from the first column, one can solve the field equation (3.22) in terms of a two-component spinor

$$\chi = \begin{pmatrix} \delta A_4 - i \delta A_3 \\ \delta A_2 - i \delta A_1 \end{pmatrix}$$

that obeys

$$\sigma^\mu D_\mu \chi = 0.$$ 

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10 One may explicitly verify using $a^\dagger a = (f^{-1} - x^2 + 2 \xi \mu x_\mu) \otimes 1_2 + \frac{1}{2} \theta^\mu{}^\nu \sigma_{\mu\nu}$ that the first term in (2.44) breaks the self-duality property.

11 One may try to add a term to cancel the right-hand side of (3.18), for example, such as $-\frac{1}{2} a(a^\dagger a)^{-1} \theta^\mu{}^\nu \sigma_{\mu\nu} \bar{\sigma} \lambda d_\lambda$. Note that $a^\dagger a$ is invertible since $\Delta^\dagger \Delta$ is so in whole space, e.g. $x = 0$. One may find that now $\delta \mu^a = 0$ but $\mu_g \neq 0$. If one simply chooses $\delta a = a - \frac{1}{2} a(a^\dagger a)^{-1} \theta^\mu{}^\nu \sigma_{\mu\nu}$ instead, one will also get $\delta \mu^a = 0$, $\mu_g \neq 0$. 

13
Thus the spinor $-\sigma^2\chi^*$ which is the second column of $\Phi_1$ is also a solution of the field equation (3.24). From the solution $\chi$, one can construct a second, linearly independent matrix solution $\Phi_2$ obtained by replacing $\chi$ by $i\chi$:

$$\Phi_2 = \Phi_1\sigma^3 = \begin{pmatrix} \delta A_3 + i\delta A_4 & -\delta A_1 + i\delta A_2 \\ \delta A_1 + i\delta A_2 & \delta A_3 - i\delta A_4 \end{pmatrix}.$$  \hfill (3.25)

In sum, for each linearly independent spinor solution, there are precisely two linearly independent solutions about the fluctuation $\delta A_\mu$. So the problem of counting the number of bosonic zero modes is now translated to that of counting the fermionic zero modes in (3.24), or calculating the index of the Dirac operator.

As counted in Section 2, there are totally $4Nk$ bosonic zero modes in the $k$ instanton fields. Thus we should have $2Nk$ adjoint fermion zero modes satisfying (3.24). We have already known from (2.38) and (2.44) what $\delta A_\mu$ in (3.23) are and that they satisfy the self-duality condition (2.30). Thus if we plug the solution (2.38) in (3.23), then the resulting spinor $\chi$ automatically satisfies the Dirac equation (3.22). Thus we can easily construct $2Nk$ adjoint fermion zero modes in this way.

The zero modes (3.16) can be translated into an adjoint fermion zero mode

$$\chi = \bar{\sigma}^\mu \delta A_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{i}{2} F^{cl}_{\mu\nu} \bar{\sigma}^{\mu\nu} \epsilon.$$  \hfill (3.26)

Here we expressed the translation parameters $c_\mu$ in terms of the two-component spinor $\epsilon$: $c_\mu := \sigma_\mu \epsilon$. Actually $\epsilon$ plays the role of supersymmetric partner of instanton position moduli [79]. Since the translational symmetry is still manifest in noncommutative space (2.19) and the supersymmetry is at most softly broken by the noncommutativity, the position moduli must have the supersymmetric partner. However, we observed that for the non-BPS instanton a supersymmetric partner corresponding to the size of instanton is affected by the noncommutativity. Since the non-BPS instanton has a minimum size set by the noncommutativity and this is just that of a $U(1)$ instanton, the superconformal symmetry as well as the conformal symmetry has to act nontrivially only on the space of $SU(N)$ instantons. But in noncommutative space it is not possible to separate $U(N)$ into $SU(N)$ and $U(1)$. So the (super)conformal transformation in the noncommutative instanton background may be a challenging problem. The issue on the instanton calculus in noncommutative super Yang-Mills theory, especially the zero modes related to the superconformal symmetry as well as the conformal symmetry, definitely deserves further study.

If $q$ denotes the fundamental representation of $U(N)$, the adjoint representation can be obtained by decomposing $q \otimes \bar{q}$, for which

$$D_\mu = \partial_\mu + A_\mu \otimes 1 + 1 \otimes \bar{A}_\mu.$$  \hfill (3.27)
Following the same strategy in [46] to get instantons in higher dimensional representation, one may show that the zero modes (3.24) for the adjoint fermion can be constructed by the zero modes in the fundamental representation according to the correspondence (3.27). So it will be interesting to consider the Dirac equation for a fermion \( \eta \) in the fundamental representation with positive chirality

\[
\sigma^\mu D_\mu \eta = \sigma^\mu (\partial_\mu + v^\dagger \partial_\mu v) \eta = 0. \tag{3.28}
\]

Transposing (3.28) with respect to spinor indices and then multiplying it by \( v \) on the left and \( \sigma^2 \) on the right, we can rewrite (3.28) as

\[
P \partial_\mu (\tilde{\eta} \sigma^\mu) = 0 \tag{3.29}
\]

where \( P = vv^\dagger \) and

\[
\tilde{\eta} = v\eta^T \sigma^2. \tag{3.30}
\]

Using (2.8) and (2.11), one can show that

\[
P \partial_\mu (Pb \sigma_\mu f) = 0. \tag{3.31}
\]

Hence uniqueness leads us to \( k \) independent solutions for \( \eta^T \) as an \( N \times 2 \) matrix

\[
\eta^T_{u,\alpha} = (v^\dagger b f \sigma^2)_{u,\alpha} \tag{3.32}
\]

where \( u = 1, \ldots, N, \ i = 1, \ldots, k \) and \( \alpha = 1, 2 \). Thus we have found \( k \) fermionic zero modes in the fundamental representation.

### 4 Atiyah-Singer Index in Noncommutative Instantons

If we introduce the Euclidean Dirac matrices (2.5), the spinor equation (3.24) or (3.28) is equivalent to the Dirac equation

\[
\gamma^\mu D_\mu \psi_+ = 0 \tag{4.1}
\]

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12It is quite plausible that the tensor product of instantons à la [46] is still applicable to noncommutative instantons since the ADHM construction is complete even for noncommutative instantons; the (anti-)self-dual solution for \( G_1 \times G_2 \) may be considered as a particular solution for a larger group \( G \) containing \( G_1 \times G_2 \). This speculation may be understood more clearly by considering the brane configuration of corresponding \( D4/D0 \) system in the 2-form \( B_{\mu\nu}^{NS} \) background. Even though explicit demonstration on this is an interesting problem, however, it goes beyond the scope of this paper.

13The proof of (3.31) is exactly the same as the commutative case [13, 44, 45, 51]. More generally, if we introduce \( k \) (complex) vectors in the fundamental representation of \( U(N) \), \( a_\mu = v^\dagger b f \sigma_\mu \), then \( D_\mu a_\nu = v^\dagger \partial_\mu (P b f \sigma_\nu) = v^\dagger b f \xi_\lambda \left( 2 \delta_{\lambda \mu} \sigma_\mu + 2 \delta_{\lambda \nu} \sigma_\nu - \sigma_\mu \sigma_\nu \sigma_\lambda \right) \), so that \( f_{\mu \nu} = - f_{\mu \nu} \), \( D_\mu a_\mu = 0 \) where \( f_{\mu \nu} = D_\mu a_\nu - D_\nu a_\mu \). Thus \( a_\mu \) gives \( 2k \) (real) vector zero modes in the fundamental representation [51].

14One can check that the zero modes (3.32) for \( U(1) \) case exactly reproduce the solution (4.5) in [41] if one notices that our \( v \) corresponds to \( \Psi \) in [41].
for a spinor field $\psi_+$ with definite chirality,

$$\gamma_5 \psi_+ = \psi_+$$  \hspace{1cm} (4.2)

and transforming in the adjoint or fundamental representation of $U(N)$, respectively. Here $D_\mu$ is the covariant derivative in the instanton background. In the previous section we showed that in noncommutative $k$ instanton background in $U(N)$ gauge theory the number of fermion zero modes in the adjoint representation and the fundamental representation are respectively $2Nk$ and $k$. In this section we will show that the number of the fermionic zero modes given by (4.1) is related to the topological charge of the instanton gauge field by the so-called Atiyah-Singer index theorem [61]. A reinterpretation of this theorem within the instanton calculus [60] shows that this theorem is perfectly equivalent to the anomaly in axial-vector current related to a regularization of the triangular graph. The interrelation between the Atiyah-Singer index, the chiral anomaly and the zero modes of Dirac operator in topologically nontrivial gauge fields form a Golden Triangle. Our calculation here is essentially the same as the chiral anomaly calculation in [62, 63].

We start with the Dirac equation (4.1) for a Dirac fermion instead in an arbitrary representation (fundamental, adjoint, etc) in the background of anti-self-dual instantons

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$  \hspace{1cm} (4.3)

In the basis (2.5)

$$\gamma_5 \psi_\pm = \pm \psi_\pm.$$  \hspace{1cm} (4.4)

Since

$$\begin{align*}
(\gamma D)^2 &= D^2 + \frac{i}{2} \begin{pmatrix} \sigma^{\mu\nu} F^{\text{cl}}_{\mu\nu} & 0 \\ 0 & \sigma^{\mu\nu} F^{\text{cl}}_{\mu\nu} \end{pmatrix},
\end{align*}$$

in the anti-self-dual instanton background, i.e. $\sigma^{\mu\nu} F^{\text{cl}}_{\mu\nu} = 0$, only the positive chirality spinor $\psi_+$ can be zero modes. \(15\) This means that $\text{ker} \bar{\mathcal{D}} = \text{ker} \bar{\mathcal{D}}$ where $\bar{\mathcal{D}} = \sigma^{\mu} D_\mu$, $\bar{\mathcal{D}} = \bar{\sigma}^{\mu} D_\mu$ since $\text{ker} \bar{\mathcal{D}} = 0$.

We can count the number of zero modes using index theorem. The index of the Dirac operator is defined as

$$\text{Ind} \mathcal{D} = \dim \{ \text{ker} \mathcal{D} \} - \dim \{ \text{ker} \bar{\mathcal{D}} \}.$$  \hspace{1cm} (4.6)

In order to calculate the index (4.6), let’s consider the quantity [80, 82]

$$T(m^2) = \text{Tr} \left\{ \frac{m^2}{-(\gamma D)^2 + m^2} \gamma_5 \right\} = \text{Tr} \left\{ \frac{m^2}{-\mathcal{D} \mathcal{D} + m^2} - \frac{m^2}{-\bar{\mathcal{D}} \bar{\mathcal{D}} + m^2} \right\}.$$  \hspace{1cm} (4.7)

\(15\) In fact, the zero modes $\psi_+$ will be given by $\chi$ in (3.23) in the adjoint representation, so $2Nk$ solutions, and by $\eta$ in (3.32) in the fundamental representation, so $k$ solutions. The space consisted of these modes is exactly $\text{ker} \mathcal{D}$ and thus they contribute to the index of the Dirac operator as we will discuss.
where the trace denotes a diagonal sum over all indices, e.g. group indices, spinor indices and Tr, an integration over spacetime. One can see that $T(m^2)$ is independent of $m^2$ and indeed exactly equal to the index defined in (4.6). The reason is following [15]. If $\psi$ is an eigenfunction of $\bar{\mathcal{D}}/\mathcal{D}$, then $\mathcal{D}/\psi$ is an eigenfunction of $\bar{\mathcal{D}}/\mathcal{D}$ with the same eigenvalue. Conversely, if $\psi$ is an eigenfunction of $\mathcal{D}/\bar{\mathcal{D}}$, then $\bar{\mathcal{D}}/\psi$ is an eigenfunction of $\mathcal{D}/\bar{\mathcal{D}}$ with the same eigenvalue. This means that there is a pairwise cancellation in (4.7) coming from the sum over eigenstates with non-zero eigenvalues. So the only contribution is coming from the zero modes, for which the first term simply gives one for each zero mode and the second term vanishes because $\ker \bar{\mathcal{D}}/\mathcal{D} = 0$. Thus the result is clearly independent of $m^2$ and moreover it is exactly equal to $\text{Ind}\mathcal{D}$ since $\ker \bar{\mathcal{D}}/\mathcal{D} = \ker \mathcal{D}$.

Since $T(m^2)$ is independent of $m^2$, we can evaluate it in two different limits, i.e. $m^2 \to 0$ and $m^2 \to \infty$,

$$\text{Ind}\mathcal{D} = \lim_{m^2 \to 0} T(m^2) = \text{Tr} P_0 \gamma_5 = \lim_{m^2 \to \infty} T(m^2) = \frac{1}{16\pi^2} \int d^4 x F^{cl a}_{\mu\nu} * F^{cl b}_{\mu\nu} \text{tr}(T^a T^b),$$

(4.8)

where $P_0$ is a projection operator into the subspace of zero modes. Here the calculation of $T(m^2)$ in the large $m^2$ limit is exactly the same as the commutative case. In this calculation we don’t meet any trouble due to the noncommutativity. So we will not repeat it here but, for example, see [60]. In the fundamental representation we normalize the generators $T^a$ of $U(N)$ so that $\text{tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}$. Other representations are constructed by decomposing tensor product of these and, in particular, for the adjoint representation of $U(N)$, $\text{tr}(T^a T^b) = -N \delta^{ab}$. So we get

$$\text{Ind}\mathcal{D} = n_+ - n_- = \begin{cases} -2Nk, & \text{adjoint} \\ -k, & \text{fundamental} \end{cases}$$

(4.9)

where $n_+(n_-)$ is the number of zero modes with positive (negative) chirality. In the anti-self-dual background, $n_- = 0$ while $n_+ = 0$ in the self-dual background as we discussed above.

5 Discussion

In this paper we discussed bosonic and fermionic zero modes in noncommutative instanton backgrounds based on the ADHM construction. We showed that the number of instanton zero modes is precisely equal to the Atiyah-Singer index of chiral Dirac operator. We pointed out that (super)conformal zero modes in the non-BPS background are affected by the noncommutativity and so their explicit construction in noncommutative space remains an open problem. We would like to discuss this problem a little more.

As shown in this paper, the moduli space $\mathcal{M}_{k,N}$ of $k$ instantons in $U(N)$ gauge theory is $4kN$-dimensional. If we specialize to single $U(2)$ instanton, $\mathcal{M}_{1,2}$ is 8-dimensional. By
the comparison to commutative \( SU(2) \) instanton, we know that the size parameter is still a modulus of the noncommutative instanton. Thus we can guess that it will be generated by the conformal transformation, more precisely, by dilatation, as in the commutative case [48]. However, as we discussed in Section 4, the size modulus should be frozen in the \( U(1) \) instanton limit which is defined by \( J \to 0 \) in this paper. This implies that the scale transformation has to act nontrivially only on the \( SU(2) \) instanton sector. In other words, the dilatation acting on the \( U(1) \) instanton probably generates only pure gauge transformation and thus it can be gauged away. However, because only \( U(N) \) algebra is closed in noncommutative space and so it is not possible to separate \( U(1) \) and \( SU(N) \), it is not obvious how to define the conformal transformation for the instanton moduli. The structure of Lorentz symmetry breaking in (3.10) and (3.11) may be helpful to shed light on this problem. Anyway we put this interesting problem for future work.

We speculated that \( U(1) \) instantons may be understood by the structure of Lorentz symmetry breaking in (3.10) and (3.11). Let’s discuss it a little more. For specification, let’s consider the self-dual \( R_{NC}^4 \) in which \( \theta_{\mu\nu} = \eta^a_{\mu\nu} \zeta_a \). In this case we have the following Lorentz symmetry breaking:

\[
SU(2)_L \times SU(2)_R \to SU(2)_R \times U(1)_L.
\]

(5.1)

We observed in (3.13) that the unbroken \( SU(2)_R \) plays a role of gauge group. Note that the size of non-BPS \( U(1) \) instanton is given by \( (\frac{1}{4} \theta^{\mu\nu} \theta_{\mu\nu})^{\frac{1}{2}} = (\zeta^a \zeta^a)^{\frac{1}{2}} := \sqrt{\zeta} \). From (5.1), we can imagine \( \zeta^a := \vec{\zeta} \) is a vector in the \( SU(2)_L \) space. Since we know that the magnitude of \( \vec{\zeta} \) is constant, the allowed values of \( \vec{\zeta} \) should lie on \( S^2 \) with radius \( \zeta \). Regardless of what direction \( \vec{\zeta} \) is pointing in \( SU(2)_L \) space, it is invariant with respect to \( U(1)_L \) rotations about the \( \vec{\zeta} \)-axis. Thus the allowed values of \( \vec{\zeta} \) are parameterized by \( SU(2)_L/U(1)_L \cong S^2 \) (compare with the footnote 8). This was the reason why the relative metric of two \( U(1) \) instantons is the Eguchi-Hanson [23]. In this spirit, the \( U(1) \) instanton here is very similar to the Abelian instanton obtained by a bundle reduction due to symmetry breaking for Einstein manifolds, as discussed by Soo in [63]. Also Braden and Nekrasov [20] argued that the noncommutative \( U(1) \) instanton corresponds to a non-singular \( U(1) \) gauge field on a commutative Kähler manifold obtained by a blowup of \( C^2 \) at a finite number of points. It will be interesting to study the noncommutative \( U(1) \) instantons from these perspectives.

In order to calculate instanton effects in quantum gauge theory, it is important to know the Green function in instanton backgrounds. In a pioneering work [65], Brown, Carlitz, Creamer, and Lee showed that the propagators for massless spinor and vector fields are determined by the massless scalar propagators. And the scalar propagator \( G(x,y) \) has a remarkably simple expression

\[
G(x,y) = v(x)^\dagger G^{(0)}(x,y)v(y)
\]

(5.2)

where \( G^{(0)}(x,y) = \frac{1}{4\pi(x-y)^2} \) is the Green function for the ordinary Laplacian, i.e. \(-\partial_\mu \partial_\mu G^{(0)}(x,y) = \delta(x-y)\). It is not difficult to prove [43, 14, 43] that \(-D_\mu D_\mu G(x,y) = \delta(x-y)\). An imme-
The immediate question is how to generalize the Green function (5.2) to noncommutative instanton backgrounds. The free scalar Green function was already defined in noncommutative space in [66, 67]. Using this Green function, the propagators in the noncommutative instanton background may be obtained as long as an annoying ordering problem is carefully treated. (Recently we solved this problem in [68].) We hope to address these problems mentioned here in near future.

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