On some quadratic algebras

Anatol N. Kirillov

CRM, University of Montreal
C.P. 6128, Succursale A, Montreal (Quebec), H3C 3J7, Canada

and

Steklov Mathematical Institute,
Fontanka 27, St.Petersburg, 191011, Russia

Abstract

We study some quadratic algebras which are appeared in the low–dimensional topology and Schubert calculus. We introduce the Jucys–Murphy elements in the braid algebra and in the pure braid group, as well as the Dunkl elements in the extended affine braid group. Relationships between the Dunkl elements, Dunkl operators and Jucys–Murphy elements are described.

1 Introduction

In this paper we study some quadratic algebras which are naturally appeared in the low–dimensional topology [B-N], [Dr2], [Ko2] and Schubert calculus [FK], and investigate some of their properties. We define a quadratic algebra $G_n$, which is a further generalization of the quadratic algebras $E_n$ and $E^t_n$ introduced in [FK], Sections 2 and 15. Our main idea is to apply the results obtained for the braid algebra $B_n$, [Dr2], [Ko2], to the algebra $G_n$. The main observation is that some results which are well–known for the braid algebra $B_n$, can be (re)proven for the quadratic algebra $G_n$. For example, it happens that the quadratic algebra $G_n$ and the braid algebra $B_n$ have the same Hilbert series

$$H(B_n; t) = H(G_n; t) = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)},$$

and have the commutative subalgebras $K_n \subset B_n$ and $H_n \subset G_n$ which are both isomorphic to the ring of polynomials. The algebra $K_n$ is generated by the Jucys–Murphy elements, whereas the algebra $H_n$ is generated by the Dunkl ones. Both algebras $G_n$ and $B_n$ have the additive bases consisting of all monomials in normal
form (Theorem 2.4 and Corollary 4.3; cf. [L]). We expect also that the certain quotients $E_0^n$ and $B_0^n$ of the quadratic algebras $G_n$ and $B_n$ respectively, have the same Hilbert polynomials as well (see Section 9 for details).

One of the main goals of this paper is to describe the relationships between the Dunkl and Jucys–Murphy elements, and to compute the dimensions of homogeneous component of degree $\leq 6$ of the quadratic algebra $E_0^n$.

The structure of the paper is following:

In Section 2 we define the braid algebra $B_n$ as the infinitesimal deformation of the pure braid group $P_n$ (cf. T. Kohno [Ko2]). The algebra $B_n$ and its completion $\hat{B}_n$ have many important applications to the low–dimensional topology, see e.g. [B-N], [Dr2], [F], [Ko3], [L].

In Section 3 we define the Jucys–Murphy elements $d_k$ in the braid algebra $B_n$, and the multiplicative Jucys–Murphy elements $D_k$ in the pure braid group $P_n$. Follow to A. Ram, [Ra], we prove that the Jucys–Murphy element $d_k \in B_n$ is the quasi–classical limit of the element $D_k \in P_n$. We prove also, that the infinitesimal deformation of the multiplicative Jucys–Murphy element $D_k$ coincides with the element $d_k$.

In Section 4 we define the quadratic algebra $G_n$ and compute the Hilbert series for this algebra. In Section 5 we define the Dunkl elements $\theta_j$ in the quadratic algebra $G_n$ and describe a commutative subalgebra generated by $\theta_j$, $1 \leq j \leq n$.

In Section 6, which contains one of the main results of this paper, we study a relationship between the Jucys–Murphy elements and the Dunkl elements. We define the dual Dunkl elements $Y^*_1, \ldots, Y^*_n$ as certain elements in the extended affine braid group $\tilde{B}_n$, and prove that under a natural homomorphism $\pi : \tilde{B}_n \rightarrow B_n$, the dual Dunkl element $Y^*_k$ maps to the multiplicative Jucys–Murphy element $D_k$. We explain also a connection between Dunkl element $Y_k \in \tilde{B}_n$ and the Dunkl operator $Y_k \in D_{q,x}[W]$, where $D_{q,x}[W]$ is the algebra of $q$–difference operators with permutations.

In Section 7 we study the relations in the algebra $E^t_n$. In Section 8 we use the properties of ”normal basis” introduced in Section 4, to study the quotient algebra $E_0^n$, and compute the dimensions of homogeneous components $E_0^{n,k}$, $1 \leq k \leq 6$, and also the Hilbert polynomial $H(E_0^4; t)$. The last polynomial was also computed, using computer, by J.-E. Ross [Ro], who computed the Hilbert polynomial $H(E_0^5; t)$ as well.

In Section 9 we consider a certain quotient $B_0^n$ of the braid algebra $B_n$, and make a conjecture that the quadratic algebra $E_0^n$ and $B_0^n$ have the same Hilbert polynomials. In particular, we expect that the algebra $B_0^n$ (as well as $E_0^n$) has a finite dimension.

In Section 10 we study the commutative quadratic algebra $A'_n$ (denote by $E_{C_n}$ in [FK], Section 4.3), which is a commutative quotient of $E^t_n$ (see Definition [0.1]), and explain some details which were omitted in [FK].

ACKNOWLEDGMENTS. This work was done during my stay at the CRM, University of Montreal. I would like to thank all my colleagues for discussions and support.
I would like also to thank M. Noumi and A.N. Varchenko for fruitful and stimulating discussions. My special thanks to Arun Ram, who explained me during my stay at the University of Wisconsin (1995) the importance of the Jucys–Murphy elements in the representation theory, and N.A. Liskova for an inestimable help on different stages of this paper and constructive criticism.

2 Quadratic algebra $\mathcal{B}_n$

We start with consideration of the quadratic algebra $\mathcal{B}_n$ which is the infinitesimal deformation of the pure braid group $P_n$, [Ko2]. This quadratic algebra is well–known as the algebra of (finite order) Vassiliev invariants of braids, [Ko3], [F], [B-N]. The completion $\hat{\mathcal{B}}_n$ of $\mathcal{B}_n$ with respect to the grading was considered in the papers of V. Drinfeld, [Dr1], [Dr2], in his study of quasi–Hopf algebras, and in the papers of T. Kohno, [Ko1]–[Ko3], in his study of monodromy representations of braid groups.

**Definition 2.1** ([Ko2], [Dr2]) Define the braid algebra $\mathcal{B}_n$ as the quadratic algebra (say over $\mathbb{Z}$) with generators $X_{ij}$, $1 \leq i < j \leq n$, which satisfy the following relations

1) $X_{ij} \cdot X_{jk} - X_{jk} \cdot X_{ij} = X_{ik} \cdot X_{ij} - X_{ij} \cdot X_{ik} = X_{ik} \cdot X_{jk} - X_{jk} \cdot X_{ik}$; 

(2.1)

2) $X_{ij} \cdot X_{kl} = X_{kl} \cdot X_{ij}$, if all $i, j, k, l$ are distinct. 

(2.2)

**Remark 2.2** The algebra $\mathcal{B}_n$ is denoted by $\mathcal{P}_n$ in [L], and as $A^n$ in [Ko1]–[Ko3].

The algebra $\mathcal{B}_n$ has a natural structure of a cocommutative Hopf algebra. Namely, a comultiplication $\Delta : \mathcal{B}_n \to \mathcal{B}_n \otimes \mathcal{B}_n$, antipode $S : \mathcal{B}_n \to \mathcal{B}_n$, and counit $\epsilon : \mathcal{B}_n \to \mathbb{Z}$, can be define as follows

\[
\Delta(X_{ij}) = 1 \otimes X_{ij} + X_{ij} \otimes 1,
\]

\[
S(X_{ij}) = -X_{ij},
\]

\[
\epsilon(X_{ij}) = 0, \quad \epsilon(1) = 1.
\]

Let us explain briefly an origin of the relations (2.1)–(2.2). For more details and proofs see [Dr1], [Dr2], [F], [Ko1]–[Ko3], [L]. Let us denote by

\[
\mathcal{M} = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{z_i = z_j\},
\]

the configuration space of $n$ distinct ordered points in $\mathbb{C}$. It is well–known that the fundamental group $\pi_1(\mathcal{M})$ of the space $\mathcal{M}$ coincides with the pure braid group $P_n$: 

\[
\pi_1(\mathcal{M}) = P_n.
\]
Let \[ w_{ij} = w_{ji} = \frac{1}{2\pi \sqrt{-1}} d\log(z_i - z_j) \]
be closed 1–form on \( \mathcal{M} \). Then, \( \{ w_{ij} \mid 1 \leq i < j \leq n \} \) represents a basis for \( H^1(\mathcal{M}) \), and the relations among \( w_{ij}, 1 \leq i < j \leq n \), are generated by the Arnold relations
\[ w_{ij} \land w_{jk} + w_{jk} \land w_{ik} + w_{ik} \land w_{ij} = 0 \]
for \( i < j < k \) (see, e.g. [A]).

Let \( \{ X_{ij} = X_{ji} \mid 1 \leq i < j \leq n \} \) be a set of non–commutative variables, and consider a formal connection
\[ \Omega = \sum_{1 \leq i < j \leq n} w_{ij} X_{ij}. \] (2.3)

**Lemma 2.3** (T. Kohno) The connection \( \Omega \) is integrable if and only if the following conditions are satisfied
\[
[X_{ij}, X_{ik} + X_{jk}] = [X_{ij} + X_{ik}, X_{jk}] = 0, \text{ for } i < j < k,
[X_{ij}, X_{kl}] = 0, \text{ for distinct } i, j, k, l.
\]

Let us remind that \([a, b] = ab - ba\) is the usual commutator.

It is clear that these conditions are equivalent to the relations (2.1)–(2.2). A proof of Lemma 2.3 follows from the observations that \( d\Omega = 0 \), and the integrability condition for the connection \( \Omega \) is equivalent to the following one: \( \Omega \land \Omega = 0 \). Notice that the connection (2.3) is the formal version of the so–called Knizhnik–Zamolodchikov connection in conformal field theory.

The relations (2.1) and (2.2) can be also interpreted [Ch2], [Dr2], [Ko2], as the self–consistency conditions
\[ \frac{\partial A_j}{\partial z_i} - \frac{\partial A_i}{\partial z_j} = [A_i, A_j], \quad 1 \leq i, j \leq n, \]
of the Knizhnik–Zamolodchikov (system of) equation(s)
\[ \frac{\partial \Phi}{\partial z_i} = k A_i \Phi, \quad 1 \leq i \leq n, \]
where \( A_i = \sum_{j \neq i} \frac{X_{ij}}{z_i - z_j} \), and a function \( \Phi := \Phi(z), \quad z = (z_1, \ldots, z_n) \), takes the values in a certain algebra (say over \( \mathbb{C}((z)) \)) generated by the elements \( X_{ij} \).

Below we formulate the basic properties of the quadratic algebra \( \mathcal{B}_n \).
Theorem 2.4 ([Ko1], [Ko2]) The Hilbert series of the algebra $B_n$ is given by

$$H(B_n; t) = \frac{1}{(1-t)(1-2t) \cdots (1-(n-1)t)}. \quad (2.4)$$

Corollary 2.5 ([Dr1], [L]) A linear basis in the algebra $B_n$ is given by the following set of noncommutative monomials

$$Z_2 \cdot \ldots \cdot Z_n, \quad (2.5)$$

where

$$Z_2 = \{1, X_{12}\},$$

$$\ldots$$

$$Z_k = \{1\} \cup \{X_{i_1k} \ldots X_{i_k} | 1 \leq i_j < k, \ 1 \leq j \leq l\}, \ 2 \leq k \leq n.$$

A monomial in $X_{ij}$'s is called to be in normal form if it belongs to the set (2.5).

Remark 2.6 Corollary 2.5 was stated without a proof in [Dr1]. The formula (2.4) was obtained in [Ko1] by constructing a free resolution of $\mathbb{C}$ as a trivial $B_n$-module. Also, the formula (2.4) is a consequence of the fact that $B_n$ is a semi-direct product of free associative algebras as shown in [Dr2]; see also [L]. Note, that the relations (2.1) can be interpreted as the horizontal version of the 4-term relations in the theory of Vassiliev link invariants.

Let us consider a completion $\hat{B}_n$ of the algebra $B_n \otimes \mathbb{C}$ with respect to the powers of the ideal $I = (X_{ij})$ generated by $X_{ij}, 1 \leq i < j \leq n$. More precisely, let us denote by $\mathbb{C}\langle X_{ij} \rangle$ the ring of non–commutative formal power series with indeterminates $X_{ij}, 1 \leq i < j \leq n$, and let $J$ be an ideal generated by the relations (2.1) and (2.2). Then $\hat{B}_n = \mathbb{C}\langle X_{ij} \rangle/J$. It is well–known [Ko1], that there exists an isomorphism of complete Hopf algebras

$$\widehat{\mathbb{C}[P_n]} \cong \hat{B}_n,$$

where $\widehat{\mathbb{C}[P_n]}$ stands for the completion of the group ring of the pure braid group $P_n$ with respect to the powers of the augmentation ideal.

3 Jucys–Murphy elements

Definition 3.1 The Jucys–Murphy elements $d_j$, for $j = 2, \ldots, n$, in the quadratic algebra $B_n$ are defined by

$$d_j = \sum_{1 \leq i < j} X_{ij}. \quad (3.1)$$
It is clear that \(d_i\) is a primitive element, i.e. \(\Delta(d_i) = 1 \otimes d_i + d_i \otimes 1\).

**Lemma 3.2** Relations \([X_{ik} + X_{jk}, X_{ij}] = 0\) for \(i < j < k\), together with commutativity relations (2.2), imply that the Jucys–Murphy elements \(d_j\) commute pairwise.

**Proof.** For \(j < l\),
\[
[d_j, d_l] = \sum_{1 \leq i < j, 1 \leq k < l} [X_{ij}, X_{kl}] = \sum_{1 \leq i < j < l} \{[X_{ij}, X_{il}] + [X_{ij}, X_{jl}]\} = 0.
\]

Let us assume that additionally to the relations (2.1) and (2.2) the following relations are satisfied
\[
X_{ij}^2 = 1, \quad 1 \leq i < j \leq n.
\]
Then a map \((i < j)\)
\[
X_{ij} \rightarrow (i, j),
\]
where \((i, j) \in S_n\) is the transposition that interchanges \(i\) and \(j\) and fixes each \(k \neq i, j\), defines a representation of the relations (2.4), (2.2) and (3.2). In other words, there exists a homomorphism \(p\) from the braid algebra \(\mathcal{B}_n\) to the group ring of the symmetric group, \(p : \mathcal{B}_n \rightarrow \mathbb{Z}[S_n]\), such that \(p(X_{ij}) = (i, j)\). Under this homomorphism \(p\) the element \(d_j\) maps to the Jucys–Murphy element in the group ring of the symmetric group \(S_n\) (see, e.g. [J], [Mu], [Ra]). A proof follows from the following relations between the transpositions in the symmetric group \(S_n\): if \(1 \leq i < j < k \leq n\), then
\[
(ij)(ik) = (jk)(ij) = (ik)(jk),
\]
\[
(ij)(jk) = (ik)(ij) = (jk)(ik).
\]

**Theorem 3.3** Let \(K_n\) be a commutative subalgebra of \(\mathcal{B}_n\) generated by the Jucys–Murphy elements \(d_j, j = 2, \ldots, n\). Then a map \(d_j \mapsto x_{j-1}\) defines an isomorphism
\[
K_n \cong \mathbb{Z}[x_1, \ldots, x_{n-1}]\]

There exists a multiplicative analog, denoted by \(D_k\), of the Jucys–Murphy element \(d_k\). Our construction of the elements \(D_k\) follows to [Ra].

Let \(g_i, 1 \leq i \leq n-1\), be the standard generators of the braid group \(B_n\). Thus, the generators \(g_i\) satisfy the following relations
\[
g_i g_j = g_j g_i, \quad \text{if} \quad |i - j| \geq 2, \quad (3.3)
\]
\[
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \leq i \leq n - 2. \quad (3.4)
\]

Now let us define the multiplicative Jucys–Murphy elements \(D_k\):

6
**Definition 3.4** (cf. [Ra], (3.16)) The multiplicative Jucys–Murphy elements $D_k$ are defined by the following formulae

$$D_2 = g_1^2, \quad D_k = g_{k-1}g_{k-2}\cdots g_2g_1^2g_2\cdots g_{k-2}g_{k-1}, \quad 3 \leq k \leq n. \quad (3.5)$$

**Lemma 3.5** The elements $D_k$ commute pairwise.

**Proof.** First of all,

$$D_2D_3 = g_1^2g_2^3g_2 = g_1g_2g_1^2g_2g_2 = g_2g_1^2g_2g_1 = D_3D_2.$$

Now, using induction, we have ($3 \leq k \leq n - 1$)

$$D_kD_{k+1} = g_{k-1}D_{k-1}g_{k-1}D_{k-1}g_{k-1}D_k = g_{k-1}D_{k-1}g_{k-1}D_kg_{k-1} = g_{k-1}D_{k-1}D_kg_{k-1} = D_{k+1}D_k.$$  

Finally, if $l - k \geq 2$, then

$$D_kD_l = Dkg_{l-1}\cdots g_{k+1}D_{k+1}g_{k+1}\cdots g_{l-1} = g_{l-1}\cdots g_{l-1}D_{k+1}D_kg_{k+1}\cdots g_{l-1} = D_lD_k.$$  

The elements $D_k$, $2 \leq k \leq n$, generate a commutative subgroup $K_n$ in the braid group $B_n$. Now we are going to consider the following two problems: what is the infinitesimal deformation of the commutative subgroup $K_n$, and what is the quasi–classical limit

$$Z[B_n] \to H_n(q) \to Z[S_n] \quad (3.6)$$

of the multiplicative Jucys–Murphy elements $D_k$? In the quasi–classical limit (3.6) the group ring $Z[B_n]$ of the braid group $B_n$ is degenerated at first to the Iwahori–Hecke algebra $H_n(q)$, and then to the group ring $Z[S_n]$ of the symmetric group $S_n$.

**Proposition 3.6** The quasi–classical limit (3.6) of the multiplicative Jucys–Murphy element $D_k \in B_n$ is equal to the Jucys–Murphy element $d_k \in Z[S_n]$.

**Proof.** Let us compute the quasi–classical limit (3.6) of the element $D_k$. The first step is to consider $D_k$ as an element of the Iwahori–Hecke algebra $H_n(q)$. In other words, we have to add to (3.3) and (3.4) the new relations

$$g_i^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})g_i + 1, \quad 1 \leq i \leq n - 1. \quad (3.7)$$
The next step is to consider the limit

$$\lim_{q \to 1} \frac{D_k - 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$ 

For this goal it is enough to compute $\frac{dD_k}{dq}|_{q=1}$. This can be done using the formula

$$\frac{dg_i}{dq} = \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})g_i + 2}{2q(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}.$$ 

Thus,

$$\frac{dD_k}{dq} = \sum_{i=2}^{k-1} \left\{ g_{k-1} \cdots g_{i+1} \left( \frac{dg_i}{dq} \right) g_{i-1}g_2g_1g_2 \cdots g_{k-1} \right\}$$

$$+ g_{k-1} \cdots g_2 \left( \frac{d}{dq} g_1^2 \right) g_2 \cdots g_{k-1}$$

$$\equiv \sum_{i=2}^{k-1} \left\{ \frac{1}{2}s_{k-1} \cdots s_{i+1}s_is_{i+1} \cdots s_{k-1} + \frac{1}{2}s_{k-1} \cdots s_{i+1}s_is_{i+1} \cdots s_{k-1} \right\}$$

$$+ s_{k-1} \cdots s_2s_1s_2 \cdots s_{k-1}$$

$$= \sum_{i=1}^{k-1} (i, k) = d_k.$$ 

Finally, let us explain a connection between the multiplicative Jucys–Murphy elements $D_k$ and the pure braid group $P_n$. More precisely, let us consider the infinitesimal deformations of the pure braid group $P_n$, and the multiplicative Jucys–Murphy elements $D_k$. To start, it is convenient to remind a few definitions.

By definition, the pure braid group $P_n$ is a kernel of the natural homomorphism $B_n \to S_n$, $g_i \mapsto s_i = (i, i + 1)$, where $s_i = (i, i + 1) \in S_n$, $1 \leq i \leq n - 1$, denote the transposition that interchanges $i$ and $i + 1$, and fixes all other elements of $[1, n]$. We have an exact sequence:

$$1 \to P_n \to B_n \to S_n \to 1.$$
It is well-known (see, e.g. [B]), that $P_n$ is generated by the following elements
\[ g_{ij} = (g_{j-1}g_{j-2} \cdots g_{i+1})g_i^2(g_{j-1}g_{j-2} \cdots g_{i+1})^{-1}, \quad 1 \leq i < j \leq n, \]
subject the following relations:
\begin{align*}
g_{ij}g_{kl} &= g_{kl}g_{ij}, \quad \text{if all } i, j, k, l \text{ are distinct,} \\
g_{ij}g_{ik}g_{jk} &= g_{ik}g_{jk}g_{ij}, \quad g_{ij}g_{ik}g_{jk} = g_{ik}g_{jk}g_{ij}, \quad \text{if } 1 \leq i < j < k \leq n, \\
g_{ik}g_{jk}g_{ji}g_{ij} &= g_{jk}g_{ji}g_{ij}g_{ik}, \quad \text{if } 1 \leq i < j < k < l \leq n.
\end{align*}
(3.8)–(3.10)

One can show using only the relations (3.8)–(3.10) that the elements
\[ D_k = g_{1,k}g_{2,k} \cdots g_{k-1,k} \in P_n, \quad 2 \leq k \leq n, \]
are mutually commute.

Now let us consider the infinitesimal deformation $g_{ij} \mapsto 1 + \epsilon X_{ij}$, of the pure braid group $P_n$. It is easy to see that the coefficients of $\epsilon^2$ on both sides of relations (3.8)–(3.10) coincide with the defining relations (2.1)–(2.2) for the braid algebra $B_n$, and the element $D_k$ transforms to the Jucys–Murphy elements $d_k$ (more precisely, $D_k = 1 + \epsilon d_k + o(\epsilon^2)$).

## 4 Quadratic Algebra $\mathcal{G}_n$

In the recent paper of the author and S. Fomin [FK] the quadratic algebra $\mathcal{E}_n^t$ was introduced. This algebra is closely related to the theory of quantum Schubert polynomials [FK], and their multiparameter deformation [K]. In this Section we introduce the quadratic algebra $\mathcal{G}_n$, which is a further generation of the quadratic algebra $\mathcal{E}_n^t$.

**Definition 4.1** Define the algebra $\mathcal{G}_n$ (of type $A_{n-1}$) as the quadratic algebra (say, over $\mathbb{Z}$) with generators $[ij], \quad 1 \leq i < j \leq n$, which satisfy the following relations
\begin{align*}
[ij][jk] &= [jk][ik] + [ik][ij], \\
[jk][ij] &= [ik][jk] + [ij][ik], \quad \text{for } i < j < k; \\
[ij][kl] &= [kl][ij], \\
\text{whenever } \{i, j\} \cap \{k, l\} = \emptyset, \quad i < j, \quad \text{and } k < l.
\end{align*}
(4.1)–(4.2)

The quadratic algebra $\mathcal{E}_n^t$, [FK], is the quotient of the algebra $\mathcal{G}_n$ by the two–side ideal generated by the relations
\[ [ij]^2 - t_{ij} = 0, \]
where the \( t_{ij} \), for \( 1 \leq i < j \leq n \), are a set of commuting parameters.

From (2.1) follows that the generators \([ij]\) satisfy the classical Yang–Baxter equation (CYBE)
\[
[[ij],[ik]] + [[ik],[jk]] = 0, \quad i < j < k,
\]
where the external brackets stand for the usual commutator: \([a,b] = ab - ba\).

**Theorem 4.2** The Hilbert series of the algebra \( G_n \) is given by
\[
H(G_n; t) = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}.
\]

Theorem 4.2 is a corollary of the following result which gives a description of an additive basis in the algebra \( G_n \):

**Theorem 4.3** A linear basis of the algebra \( G_n \) is given by the following set of non-commutative monomials
\[
Z_2 \cdot Z_3 \cdot \ldots \cdot Z_n,
\]
where
\[
Z_2 = \{ \phi, [12] \}
\]
\[
\ldots
\]
\[
Z_k = \{ \phi \} \cup \{ [i_1 k] \cdot \ldots \cdot [i_l k] \mid 1 \leq i_j < k, \ 1 \leq j \leq l \}, \quad 2 \leq k \leq n.
\]
A monomial in \([ij]\)'s is called to be in *normal form*, if it belongs to the set \(\{Z_n\}\).

A proof of Theorem 4.2 is similar to a proof of Corollary 2.5 given by X.-S. Lin, [L], Theorem 2.3. By this reason we omit a proof.

**Remark 4.4** It is clear that \( H(Z_k; t) = \frac{1}{1 - (k-1)t} \), and
\[
H(G_n; t) = \prod_{k=2}^{n} H(Z_k; t).
\]

**Corollary 4.5** Let \( G'_n \) be a commutative version of the algebra \( G_n \), i.e. let us assume that the relations (4.2) are valid for all \( i, j, k, l \). Then
\[
H(G'_n; t) = (1 - t)^{-\begin{pmatrix} n \\ 2 \end{pmatrix}}.
\]
5 Dunkl elements

The Dunkl elements $\theta_j$, for $j = 1, \ldots, n$, in the quadratic algebra $G_n$ are defined by

$$\theta_j = -\sum_{1 \leq i < j} [ij] + \sum_{j < k \leq n} [jk].$$

(5.1)

For the quadratic algebra $E^t_n$ the Dunkl elements (5.1) coincide with those introduced in [FK], Section 5. The following result is well–known (cf., e.g. [Ki], Theorem 1.4, or [FK], Lemma 5.1)

**Lemma 5.1** The classical Yang–Baxter equation (4.3), together with the commutativity relation (4.2), imply that the Dunkl elements $\theta_j$ commute pairwise.

There exists an obvious relation between Dunkl’s elements $\theta_1, \ldots, \theta_n$ in the quadratic algebra $G_n$, namely, $\theta_1 + \cdots + \theta_n = 0$. The result below shows that in the algebra $G_n$ the Dunkl elements $\theta_1, \ldots, \theta_{n-1}$ are algebraically independent (cf. Theorem 7.3).

**Theorem 5.2** Let $\mathcal{H}_n$ be a commutative subalgebra of $G_n$ generated by the Dunkl elements $\theta_j$, $j = 1, \ldots, n$. Then a map $\theta_i \rightarrow x_i$, $1 \leq i \leq n-1$, defines an isomorphism

$$\mathcal{H}_n \cong \mathbb{Z}[x_1, \ldots, x_{n-1}],$$

**Problem 5.3** To describe the two–side ideals $\mathcal{R} \subset G_n$ such that

$$H(\mathcal{H}_n/\mathcal{R} \cap \mathcal{H}_n; t) = [n]! := \prod_{j=1}^{n} \frac{1-t^j}{1-t}.$$  

(5.2)

The examples of two–side ideals in the algebra $G_n$ with the property (5.2) will be given in Sections 7 and 8. These examples are related to the quantum cohomology ring of the flag variety [FK], and the multiparameter deformation of Schubert polynomials [K].

**Example 5.4** Let us take $n = 3$. In the algebra $G_3$ we have the following relations

$$\begin{align*}
\theta_1 + \theta_2 + \theta_3 &= 0, \\
\theta_1 \theta_2 + \theta_1 \theta_3 + \theta_2 \theta_3 + [12]^2 + [13]^2 + [23]^2 &= 0, \\
\theta_1 \theta_2 \theta_3 + [12]^2 \theta_3 + \theta_1 [23]^2 - [13]^2 \theta_1 - \theta_3 [13]^2 &= 0, \\
[\theta_2, [23]^2] &= [\theta_1, [13]^2],
\end{align*}$$

where $[a, b] = ab - ba$ is the usual commutator.
6 Dunkl and Jucys–Murphy elements

Let us start with the definition of the extended affine braid group \( \tilde{B}_n \).

**Definition 6.1** The extended affine braid group \( \tilde{B}_n \) is a group with generators 
\[ g_0, g_1, \ldots, g_{n-1}, w \]
which satisfy the following relations
\[ g_i g_j = g_j g_i, \quad |i - j| \geq 2, \quad 0 \leq i, j \leq n - 1, \quad (6.1) \]
\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 0 \leq i \leq n - 1, \quad (6.2) \]
\[ wg_i = g_{i-1} w, \quad 0 \leq i \leq n - 1, \quad (6.3) \]
with indices understood as elements of \( \mathbb{Z}/n\mathbb{Z} \).

It follows from (6.3) that \( w^n \) is a central element.

There exists a canonical homomorphism \( \pi \) from the extended affine braid group \( \tilde{B}_n \) to the classical braid group \( B_n \). On the generators \( g_i \) and \( w \) the homomorphism \( \pi \) is given by the following rules
\[ \pi(g_i) = g_i, \quad 1 \leq i \leq n - 1, \quad (6.4) \]
\[ \pi(w) = g_{n-1} g_{n-2} \cdots g_2 g_1. \]

It follows from (6.3) that
\[ \pi(g_0) = \pi(w) g_1 \pi(w)^{-1} = g_{n-1} g_{n-2} \cdots g_2 g_1 g_2^{-1} \cdots g_{n-2} g_{n-1}^{-1}. \]

Now we are going to define the Dunkl elements \( Y_1, \ldots, Y_n \), and the dual Dunkl elements \( Y_1^*, \ldots, Y_n^* \) in the extended affine braid group.

**Definition 6.2** For each \( i = 1, \ldots, n \), we define (cf. [KN]; [Ch1]) the Dunkl and dual Dunkl elements \( Y_i \) and \( Y_i^* \) respectively, by the following formulae
\[ Y_i = g_i g_{i+1} \cdots g_{n-1} w g_1^{-1} \cdots g_{i-1}^{-1}, \quad (6.5) \]
\[ Y_i^* = g_1^{-1} g_{i+1}^{-1} \cdots g_{n-1}^{-1} w g_1 \cdots g_{i-1}. \quad (6.6) \]

Note that \( Y_1 = g_1 \cdots g_{n-1} w, \ Y_n = w g_1^{-1} \cdots g_{n-1}^{-1}, \ Y_1^* = w g_1 \cdots g_{n-1} \) and \( Y_n^* = g_1^{-1} \cdots g_{n-1}^{-1} w \).

The Dunkl elements satisfy the following commutation relations with \( g_1, \ldots, g_{n-1} \):
\[ g_i Y_{i+1} g_i = Y_i, \quad g_i Y_j = Y_j g_i \ (j \neq i, i + 1), \quad (6.7) \]
\[ g_i Y_i^* g_i = Y_{i+1}^*, \quad g_i Y_j^* = Y_j g_i \ (j \neq i, i + 1). \quad (6.8) \]
It is clear from our construction, that the image in the braid group $B_n$ of the dual Dunkl element $Y_k^*$, $2 \leq k \leq n$, under the canonical homomorphism $\pi : \tilde{B}_n \to B_n$ coincides with the multiplicative Jucys–Murphy element $D_k$, namely

$$\pi(Y_k^*) = g_k^{-1}g_k^{-1}\cdots g_n^{-1}\pi(w)g_1\cdots g_{k-1}$$

$$= g_{k-1}\cdots g_2g_1^2g_2\cdots g_{k-1}D_k, \quad 2 \leq k \leq n,$$

and $\pi(Y_1^*) = 1$.

**Lemma 6.3** The Dunkl elements $Y_k$ (resp. the dual Dunkl elements $Y_k^*$) commute pairwise.

Let us illustrate the main idea of the proof of Lemma 6.3 on some example. Let us take $n = 5$ and prove that $Y_2^*Y_3^* = Y_3^*Y_2^*$. Indeed,

$$Y_2^*Y_3^* = g_2^{-1}g_3^{-1}g_4^{-1}wg_1g_3^{-1}g_4^{-1}w_1g_2g_4^{-1}g_3^{-1}g_4g_0w^2 = g_2^{-1}g_3^{-1}g_4^{-1}g_0g_2^{-1}g_3^{-1}g_4g_0w^2$$

Similarly,

$$Y_3^*Y_2^* = g_3^{-1}g_4^{-1}wg_1g_3^{-1}g_4^{-1}w_1g_1g_4^{-1}g_3^{-1}g_4g_0w^2.$$

It is easy to see that in $\tilde{B}_5$ we have

$$g_3^{-1}g_2^{-1}g_3^{-1}g_4g_0g_4 = g_3^{-1}g_2^{-1}g_4^{-1}g_3^{-1}g_0g_4.$$

Now let us study the quasi–classical limit of the Dunkl element $Y_k$. The first step is to consider $Y_k$ as an element of the extended affine Hecke algebra $H(\tilde{W})$, [KN], Section 2; [Ch1], [Ch2]. In other words, we have to add to the relations (6.1)–(6.3) the new ones

$$g_i^2 = (t - 1)g_i + t, \quad 0 \leq i \leq n \quad (6.9)$$

where $t$ is a new parameter. The next step is to consider a representation of the extended affine Hecke algebra $H(\tilde{W})$ in the algebra $D_{q,x}[W]$ of $q$–difference operators with permutations, see, for example, [Ch1], [KN]. Follow [KN], we define the elements $T_i$, $i = 0, 1, \ldots, n - 1$, in $D_{q,x}[W]$ by

$$T_i = t + \frac{x_{i+1} - tx_i}{x_{i+1} - x_i}(s_i - 1), \quad i = 1, \ldots, n - 1,$$

$$T_0 = t + \frac{x_1 - tqx_n}{x_1 - qx_n}(s_0 - 1).$$

Here $s_1, \ldots, s_{n-1}$ are the standard generators of the symmetric group $S_n$, and $s_0 = w_1w^{-1}$, where $w = s_{n-1}s_{n-2}\cdots s_1\tau_1$, and $\tau_1 = T_{q,x_1}$ is the $q$–shift operator:

$$(\tau_1f)(x_1, \ldots, x_n) = f(qx_1, x_2, \ldots, x_n).$$
One can check that the elements $T_i, 0 \leq i \leq n - 1$, and $w$, satisfy the relations (6.1)–(6.3) and (6.9), and define a representation of the extended affine Hecke algebra $H(W)$ in the algebra $D_{q,x}[W]$ of $q$–difference operators with permutations. In this representation, the Dunkl element $Y_k$ (resp. $Y_k^*$), $1 \leq k \leq n$, up to a power of $t$ coincides (see [Ch1], [Ch2], [KN]) with the Dunkl–Cherednik operator $Y_k$:

$$Y_k = t^{-n+2k-1}T_iT_{i+1}\ldots T_{n-1}wt^{-1}_1\ldots T_{k-1},$$  

(6.10)

$$Y_k^* = t^{n-2k+1}T_{k+1}^{-1}\ldots T_{n-1}^{-1}wT_1\ldots T_{k-1}.$$  

(6.11)

In order to understand better the quasi–classical behavior of the Dunkl–Cherednik operator $Y_k$, it is convenient to rewrite slightly the formulae (6.10)–(6.11). Namely, let us define (1 ≤ $i < j$ ≤ n) the following operators acting on the ring of polynomials $Z[t, t^{-1}][x_1, \ldots, x_n]$

$$T_{ij} = 1 + \frac{(1 - t^{-1})x_j}{x_i - x_j}(1 - s_{ij}) = t^{-1}T_{ij}s_{ij},$$

where $s_{ij}$ is the exchange operator for variables $x_i, x_j$. Then

$$Y_i = T_{i,i+1}T_{i,i+2}\ldots T_{i,n}s_{i}s_{i-1}\ldots T_{1,i}^{-1},$$  

(6.12)

where $\tau_i = s_is_{i+1}\ldots s_{n-1}ws_1\ldots s_{i-1}$ is the $q$–shift operator: $\tau_i = \tau_{x_i} = T_{q,x_i}$. Finally, in the quasi–classical limit $q \to 1$ with rescaling $t = q^\beta$, we obtain from (6.12) that

$$D_j := \lim_{q \to 1} \frac{1 - Y_j}{1 - q} = x_j\frac{\partial}{\partial x_j} + \beta \sum_{i<j} x_j \partial_{ij} - \beta \sum_{k>j} \partial_{jk}x_k.$$  

(6.13)

Note that these Dunkl operators $D_k$ commute with each other, i.e. $[D_i, D_j] = 0$.

More generally, let us define the affine extension of the algebra $\mathcal{G}_n$:

**Definition 6.4** Define the algebra $\tilde{\mathcal{G}}_n$ as the algebra (say over $Z[q, q^{-1}]$) with generators

$$[ij], \ 0 \leq i < j \leq n; \ x_i, \ 1 \leq i \leq n, \text{ and } w,$$

which satisfy the relations (4.1) and (4.2) for $i, j, k, l \in [1, n]$, and the following additional relations

$$x_ix_j = x_jx_i, \ 1 \leq i, j \leq n,$$  

(6.14)

$$x_i[a] = [a]x_i, \text{ if } i \neq a, b,$$  

(6.15)

$$x_j[ij] = [ij]x_j + 1, \ x_i[ij] = [ij]x_i - 1 \ 1 \leq i < j \leq n,$$  

(6.16)

$$w^i = q^{\delta_{i,1}}x_{i-1}w,$$  

(6.17)

$$w[ij] = q^{-\delta_{i,0}}[i - 1, j - 1]w,$$  

(6.18)

with indices understood as elements of $Z/nZ$. 

14
It is clear from (6.16), that
\[ [x_i + x_j, [ij]] = 0. \] (6.19)

We define the Dunkl elements \( \tilde{\theta}_j \) in the algebra \( \tilde{\mathcal{G}}_n \) by the following rule
\[ \tilde{\theta}_j = x_j + \theta_j, \quad 1 \leq j \leq n. \] (6.20)

The following result is well-known, see, e.g. [Ki]:

**Lemma 6.5** The classical Yang–Baxter equation (4.3), together with commutativity relation (4.2) and relation (6.19), imply that the Dunkl elements \( \tilde{\theta}_j = x_j + \theta_j \) commute pairwise.

**Proof.** If \( i < j \), then
\[ [\tilde{\theta}_i, \tilde{\theta}_j] = [x_i, x_j] + [x_i, \theta_j] + [\theta_i, x_j] + [\theta_i, \theta_j] = [x_i + x_j, [ij]] = 0. \]

Let us introduce the algebra \( \tilde{\mathcal{G}}_n^0 \) which is the quotient of the algebra \( \tilde{\mathcal{G}}_n \) by the two–side ideal generated by the relation
\[ [ij]^2 = 0, \quad 1 \leq i < j \leq n. \] (6.21)

**Definition 6.6** For \( 1 \leq i < j \leq n \) let us define an element \( T_{ij} \in \tilde{\mathcal{G}}_n^0 \) by the following formula (cf. [KN])
\[ T_{ij} = t - (x_j - tx_i)[ij]. \] (6.22)

It follows from (6.16) and (6.21), that
\[ T_{ij}^2 = (t - 1)T_{ij} + t. \]

Our next goal is to check that the elements \( \tilde{T}_i := T_{i,i+1}, 1 \leq i \leq n - 1 \), satisfy the Coxeter relations.

**Proposition 6.7** We have the following relation in the algebra \( \tilde{\mathcal{G}}_n^0 \):
\[ T_{ab}T_{bc}T_{ab} = T_{bc}T_{ab}T_{bc}, \quad a < b < c. \] (6.23)

**Proof.** Direct computation based on (6.16) and the following relation in the algebra \( \tilde{\mathcal{G}}_n^0 \):
\[ [ab][bc][ab] = [bc][ab][bc], \quad 1 \leq a < b < c \leq n. \]

As a corollary, we see that the elements \( w, \tilde{T}_0 = w \tilde{T}_1 w^{-1}, \tilde{T}_1, \ldots, \tilde{T}_{n-1} \) generate the representation of the extended affine Hecke algebra \( H(\tilde{W}) \). In this representation
the quasi–classical limit \( q \to 1 \) with rescaling \( t = q^\beta \) of the Dunkl element \( t^{-n+2j-1}Y_j \) is equal to
\[
x_j \frac{\partial}{\partial x_j} + \beta \sum_{i<j} x_j[i,j] - \beta \sum_{k>j} [jk]x_k.
\]

The element (6.24) does not belong to the algebra \( \tilde{\mathcal{G}}_n \), but its further extension \( \tilde{\tilde{\mathcal{G}}}_n \) which is an analog of the double affine Hecke algebra, introduced by I. Cherednik. It is an interesting problem to understand some connections between the Schubert calculus and the Dunkl–Cherednik operators.

**Conjecture 6.8** (Nonnegativity conjecture, cf. [FK], Conjecture 8.1) For any \( w \in S_n \), the Schubert polynomial \( S_w \) evaluated at the Dunkl elements \( \tilde{\theta}_1, \ldots, \tilde{\theta}_n \) belongs to the positive cone \( \tilde{\mathcal{G}}_n^+ \), where \( \tilde{\mathcal{G}}_n^+ \) is the cone of all nonnegative integer linear combinations of all (noncommutative in general) monomials in the generators \( x_i, 1 \leq i \leq n \), and \( [ij], 1 \leq i < j \leq n \), of the algebra \( \tilde{\mathcal{G}}_n^0 \).
7 Quadratic algebra $\mathcal{E}_n^t$

In this and the next sections we study some interesting quotients of the quadratic algebra $\mathcal{G}_n$.

**Definition 7.1** (cf. [FK], Section 15) Define the algebra $\mathcal{E}_n^t$ as the algebra over $\mathbb{Z}$ with generators $[ij], 1 \leq i < j \leq n$, which satisfy the relations (4.1), (4.2) and the additional relations

$$([ij], [kl]^2) = 0, \quad \text{for all } i, j, k, l. \quad (7.1)$$

The relations (7.1) mean that the squares $[kl]^2$ belong to the center of $\mathcal{E}_n^t$. Let us put $t_{ij} = [ij]^2$, and consider the ring of polynomials $\mathbb{Z}[t] := \mathbb{Z}[t_{ij}, 1 \leq i < j \leq n]$.

**Conjecture 7.2** ([FK]) The algebra $\mathcal{E}_n^t$ is a finite dimensional module over the ring of polynomials $\mathbb{Z}[t]$.

**Theorem 7.3** ([FK], Conjecture 15.1; [P]) Let $\mathcal{H}_n^t$ be a commutative subalgebra of $\mathcal{E}_n^t$ generated by the Dunkl elements $\theta_1, \ldots, \theta_n$. Then

$$\mathcal{H}_n^t \cong \mathbb{Z}[t][\theta_1, \ldots, \theta_n]/J,$$

where the ideal $J$ is generated by the generalized elementary functions

$$e_m(X_n|t) := \sum_l \sum_{1 \leq i_1 < \cdots < i_l \leq n \atop j_1 > i_1, \ldots, j_l > i_l} e_{m-2l}(X_{I_{I_{I_{\ldots}}}}) \prod_{k=1}^{l} t_{i_k j_k}, \quad (7.2)$$

where $i_1, \ldots, i_l, j_1, \ldots, j_l$ should be distinct elements of the set $\{1, \ldots, n\}$, and $X_{I_{I_{I_{\ldots}}}}$ denotes set of variables $x_a$, for which the subscript $a$ is neither one of the $I_k$ nor one of the $J_k$.

One can check that the relations (7.1) are equivalent to the following ones

$$(\theta_i, [kl]^2) = 0, \quad \text{for all } i, k, l.$$  

The relations (7.1) in the algebra $\mathcal{E}_n^t$ imply some nontrivial relations between monomials in the space $Z := Z_2 \cdot Z_3 \cdots \cdot Z_n$. We will describe part of these relations in Proposition 7.4 below, but before that, let us introduce some additional notations. For given $n \geq 3$ and $k, 1 \leq k < n$, let us denote by $\Lambda_{n,k}$ a set of all sequences of integer numbers $A = (a_1, \ldots, a_k)$, such that $a_1 + \cdots + a_k = n - 2$, and $a_i \in \mathbb{Z}_{>0}$. 

17
For each sequence \( A \) from the set \( \Lambda_{n,k} \) let us define a monomial \( [A] \) in the space \( Z \) as the following ordered product

\[
[A] = \prod_{j=1}^{k} a_j \prod_{l=1}^{n-k} \prod_{s=1}^{j} (a_s - 1), \quad n - k + j - 1.
\]

For example, for \( n = 7 \) and \( A = (2,1,2) \) we have

\[
[A] = [24][34][25][16][26].
\]

**Proposition 7.4**

\[
F_n := \sum_{i=1}^{n-1} [i,n][i+1,n] \cdots [n-1,n][1,n][2,n] \cdots [i,n]
\]

\[
= \sum_{k=2}^{n-1} (-1)^{n-k-1} (t_{kn} - t_{k-1,n}) \sum_{A \in \Lambda_{n,n-k}} [A].
\]

**Proof.** By induction, using the following formula

\[
\Phi_{n+1} \cdot [12] - [12] \cdot \Phi_{n+1} = F_{n+1}, \quad \text{where}
\]

\[
\Phi_n := [2,n] \cdots [n-1,n][2,n] + \cdots + [n-1,n][2,n] \cdots [n-2,n][n-1,n].
\]

Let us make the following simple but useful observation. The action of the symmetric group, namely, \( \omega([i_1,j_1] \cdots [i_k,j_k]) = [\omega(i_1)\omega(j_1)] \cdots [\omega(i_k)\omega(j_k)], \omega \in S_n, \) transforms every specific identity in \( G_n \) (resp. \( \mathcal{E}_n^t, \mathcal{E}_n^0 \)) into an orbit of identities. As an example, the identity (in \( \mathcal{E}_n^t \))

\[
[12][13][14][12]+[13][14][12][13]+[14][12][13][14] = (t_{34} - t_{24})[13][23] - (t_{24} - t_{14})[12][13]
\]

(a special case of Proposition 7.4) immediately implies a more general identity

\[
[ab][ac][ad][ab] + [ac][ad][ab][ac] + [ad][ab][ac][ad] = ([cd]^2 - [bd]^2)[ac][bc] - ([bd]^2 - [ad]^2)[ab][ac].
\]

**Proposition** below describes some interesting relations in the quadratic algebra \( \mathcal{E}_n^t \). They can be interpreted as the Pieri rules for the classical and quantum Schubert polynomials, \( [FK], [P] \), and for the multiparameter deformation of Schubert polynomials \( [K], [P] \).
Proposition 7.5 ([P]) Let $A$ be a subset in $[1, \ldots, n]$, and let us put $\theta_A = \prod_{i \in A} \theta_i$.

Then in the algebra $E_t^n$ we have the following relation

$$e_m(\theta_A|t) = \sum [i_1j_1] \cdots [i_mj_m],$$

where the summation is taken over all sequences $I = \{i_1, \ldots, i_m\}$, $J = \{j_1, \ldots, j_m\}$ such that

i) $I \subset A$, $J \subset [1, n] \setminus A$,

ii) $i_1, \ldots, i_m$ are all distinct,

iii) $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n$.

Problem 7.6 Find a nontrivial faithful representation of the quadratic algebra $E_t^n$.

8 Quadratic algebra $E_0^n$

In this Section we study the properties of the algebra $E_t^n$ when all parameters $t$ are equal to zero. Hence, in this Section we assume that

$$[ij]^2 = 0, \quad 1 \leq i < j \leq n.$$  \hfill (8.1)

This algebra was introduced and studied at first in [FK]. It follows from Theorem 4.3 that any element of $E_0^n$ can be presented as $\mathbb{Z}$-linear combination of monomials $z$ of the following form

$$z = z_1z_2 \cdots z_n,$$  \hfill (8.2)

where $z_i \in Z_i$, and $(2 \leq k \leq n)$

$$Z_k := Z_{n,k} = \{\phi\} \cup \{[i_1k] \cdots [i_lk] \mid 1 \leq i_l < k, \ 1 \leq j \leq l\}. \hfill (8.3)$$

In the algebra $G_n$ the monomials in the set $Z_k$ are linearly independent (Theorem 4.3). However, in the quotient algebra $E_0^n$ some new relations between the monomials in $Z_k$ are appeared. The Proposition 8.1 below is a special case of Proposition 7.4 and gives the relations in subalgebra $Z_m \subset E_0^n$, but not only in the space $Z = Z_2 \cdot Z_3 \cdots Z_n$.

Proposition 8.1 (cf. [FK]) Let $a_1, \ldots, a_m$ be a sequence of pairwise distinct integer numbers, $1 \leq a_i < n$, $m \leq n$. Then

$$a_1a_2 \cdots a_na_1 + a_2a_3 \cdots a_1a_2 + \cdots + a_ma_1a_2 \cdots a_{m-1}a_m = 0.$$  \hfill (8.4)

Here we used an abbreviation $a_j = [a_jn]$. 

19
Proof. It is enough to prove that

\[ F_n := \sum_{i=1}^{n-1} [i, n][i+1, n] \cdots [n-1, n][1, n][2, n] \cdots [i, n] = 0. \]  

(8.5)

Let us prove (8.5) by induction. Thus we may assume that \( F_k = 0 \) for \( k \leq n \). Let us consider the following element in \( Z_{n+1} \):

\[ \Phi_n = [2, n+1] \cdots [n, n+1][2, n+1] + \cdots + [n, n+1][2, n+1] \cdots [n-1, n+1][n, n+1]. \]

By induction assumption, we have \( \Phi_n = 0 \). On the other hand, it is easy to check that

\[ \Phi_n \cdot [12] - [12] \cdot \Phi_n = F_{n+1}. \]  

(8.6)

The last equality shows that \( F_{n+1} = 0 \).

\[ \Phi_n = [2, n+1] \cdots [n, n+1][2, n+1] + \cdots + [n, n+1][2, n+1] \cdots [n-1, n+1][n, n+1]. \]

Example 8.2 If \( m = 3 \), the relation (8.4) has the following form \((a < b < n)\)

\[ aba + bab = 0, \text{ in the algebra } \mathcal{E}_n^0 \]  

(8.7)

Similarly, if \( a, b, c \) are distinct, \( < n \), then

\[ abca + bcab + cabc = 0, \text{ in the algebra } \mathcal{E}_n^0 \]  

(8.8)

We don’t know how to describe all relations between monomials from the space \( Z_k \) for fixed \( k \), but we can show that if \( k \leq 5 \), then all relations between monomials in \( Z_k \) follows from (8.4). This observation allows us to compute the dimensions of the homogeneous components \( \mathcal{E}_{n,k}^0 \) for \( 1 \leq k \leq 5 \), and also \( \dim \mathcal{E}_{n,6}^0 \). We summarize the results in the following

**Proposition 8.3**

- \( \dim \mathcal{E}_{n,1}^0 = \binom{n}{2} \),
- \( \dim \mathcal{E}_{n,2}^0 = 3 \binom{n}{4} + 4 \binom{n}{3} \),
- \( \dim \mathcal{E}_{n,3}^0 = 15 \binom{n}{6} + 40 \binom{n}{5} + 30 \binom{n}{4} + 3 \binom{n}{3} \),
- \( \dim \mathcal{E}_{n,4}^0 = 105 \binom{n}{8} + 420 \binom{n}{7} + 610 \binom{n}{6} + 366 \binom{n}{5} + 67 \binom{n}{4} + \binom{n}{3} \),
- \( \dim \mathcal{E}_{n,5}^0 = \)
• \( \dim E_{n,5}^0 = 945 \binom{n}{10} + 5040 \binom{n}{9} + 10780 \binom{n}{8} + 11571 \binom{n}{7} \)
\[ + 6285 \binom{n}{6} + 1480 \binom{n}{5} + 96 \binom{n}{4}, \]

• \( \dim E_{n,6}^0 = 10395 \binom{n}{12} + 69300 \binom{n}{11} + 195300 \binom{n}{10} + 299908 \binom{n}{9} \)
\[ + 268674 \binom{n}{8} + 138545 \binom{n}{7} + 37456 \binom{n}{6} + 4231 \binom{n}{5} \]
\[ + 106 \binom{n}{4}. \]

Conjecture 8.4

\[
\dim E_{n,k}^0 = (2^k - 1)!! \binom{n}{2k} + (2^k - 1)!! \frac{4(k - 1)}{3} \binom{n}{2k - 1} \quad (8.9)
\]
\[ + (2^k - 3)!! \frac{(k - 1)(k - 2)(16k - 3)}{9} \binom{n}{2k - 2} + \cdots. \]

Proof of Proposition 8.3. First of all, let us compute \( \dim Z_{n,k} \) for small values of \( k \). It is clear that \( \dim Z_{n,1} = n - 1 \) and \( \dim Z_{n,2} = (n - 1)(n - 2) \), because there are no nontrivial relations of degree < 3 in \( Z_n \). In degree 3 we have to take into account the relations (8.7).

\[
\dim Z_{n,3} = (n - 2) \dim Z_{n,2} - \binom{n - 1}{2} = \frac{(n - 1)(n - 2)(2n - 5)}{2}.
\]

Similarly, in degree 4 we have to consider both the relations (8.7) and (8.8). Thus,

\[
\dim Z_{n,4} = (n - 2) \dim Z_{n,3} - (n - 2) \binom{n - 1}{2} - 2 \binom{n - 1}{3}
\]
\[ = \frac{(n - 1)(n - 2)(n - 3)(3n - 7)}{3} = 24 \binom{n - 1}{4} + 10 \binom{n - 1}{3}. \]

Using the same method, we can compute

\[
\dim Z_{n,5} = 120 \binom{n - 1}{5} + 86 \binom{n - 1}{4} + 9 \binom{n - 1}{3}.
\]

When \( k \geq 6 \), some additional to (8.4) relations between monomials in \( Z_k \) will appear. We describe such new relations in degree 6.
**Lemma 8.5** Let $a, b, c, d$ be a set of distinct integer numbers. Then
\[
abcdca + bcdcab + cabadc + dcabda + cdcaba = abacdc + acdcba + bacdcb + cdabac + dabacd.
\] (8.10)

where $a := [an]$, $n \geq 5$.

**Proof of Lemma 8.5.** Let us consider the difference
\[
[ac]F_4(a, d, c, b) - F_4(a, b, c, d)[ac],
\] (8.11)

where
\[
F_4(a, b, c, d) = [an][bn][cn][dn][an][bn] + [cn][dn][an][bn][cn][dn] + [dn][an][bn][cn][dn].
\]

First of all, the difference (8.11) is equal to zero. On the other hand, using the relations in the algebra $E_0 n$, one can transform the expression (8.11) to the difference between the left and the right hand sides of (8.10).

Using Lemma 8.5, we can compute
\[
\dim Z_{n,6} = 720 \left( \begin{array}{c} n-1 \\ 6 \end{array} \right) + 756 \left( \begin{array}{c} n-1 \\ 5 \end{array} \right) + 187 \left( \begin{array}{c} n-1 \\ 4 \end{array} \right) + 6 \left( \begin{array}{c} n-1 \\ 3 \end{array} \right).
\]

Finally, in order to compute the dimensions of homogeneous components $E^0_{n,k}$, $1 \leq k \leq 6$, we use an observation that monomials from the set $Z_2 \cdot Z_3 \cdot Z_4 \cdot Z_5 \cdot Z_6$ form a basis in $\bigoplus_{k=0}^6 E^0_{n,k}$. The proof of Proposition 8.3 is finished.

**Conjecture 8.6** (Factorization property)
\[
H(E^0_n; t) = H(Z_{n,2}; t)H(Z_{n,3}; t) \cdots H(Z_{n,n}; t).
\] (8.12)

**Example 8.7** If $n = 2$, then $Z_2 = \{\phi\} \cup \{12\}$, and $H(Z_2; t) = 1 + t$.

If $n = 3$, then $Z_3 = \{\phi\} \cup \{13, 23, 13 \cdot 23, 23 \cdot 13, 13 \cdot 23 \cdot 13\}$, and
\[
H(Z_3; t) = 1 + 2t + 2t^2 + t^3.
\]

Now, let us consider $n = 4$. Then ($a := [an]$)
\[
Z_{4,1} = \{1, 2, 3\},
Z_{4,2} = \{12, 13, 21, 31, 23, 32\},
Z_{4,3} = \{121, 123, 131, 132, 213, 231, 232, 312\},
Z_{4,4} = \{1213, 1231, 1232, 1312, 1321, 2131, 2312, 2321\},
Z_{4,5} = \{12131, 12132, 12312, 12321, 13121, 21321, 23121, 23213\},
Z_{4,6} = \{121312, 121321, 123121, 131213, 213213, 231213\},
Z_{4,7} = \{1213121, 1231213, 2131213\},
Z_{4,8} = \{12131213\}.
\]
Hence, $H(Z_4; t) = (1 + t)^2(1 + t + t^2)(1 + t^2)^2$, and

$$H(E^0_4; t) = (1 + t)^4(1 + t^2)^2(1 + t + t^2)^2.$$  

Formula for $H(E^0_4; t)$ was appeared in [FK] and confirmed by J.-E. Ross [Ro], who also computed the Hilbert polynomial $H(E^0_5; t)$.

9 Quadratic algebra $B^t_n$

In this Section we define a certain quotient $B^t_n$ of the braid algebra $B_n$, and formulate a conjecture that the algebras $B^t_n$ and $E^t_n$ have the same Hilbert polynomials.

**Definition 9.1** define the algebra $B^t_n$ as the algebra over $\mathbb{Q}$ with generators $X_{ij}$, $1 \leq i < j \leq n$, which satisfy the relations (2.1), (2.2) and the additional relations

$$[ij]^2 = t_{ij},$$  

where the $t_{ij}$ for $1 \leq i < j \leq n$, are a set of commuting parameters.

In the case when all parameters $t_{ij}$ are equal to zero, we denote the algebra $B^t_n$ by $B^0_n$.

**Conjecture 9.2** The algebra $B^t_n$ is a finite dimensional module over the ring of polynomials $\mathbb{Q}[t] := \mathbb{Q}[t_{ij}, 1 \leq i < j \leq n]$.

**Problem 9.3** To find all relations among the Jucys–Murphy elements $d_j$, $2 \leq j \leq n$, and find the Hilbert polynomial of commutative subalgebra $K^0_n$ of the algebra $B^0_n$ generated by the Jucys–Murphy elements $d_j$, $2 \leq j \leq n$.

**Example 9.4** Let us consider the algebra $B^0_3$. This is an algebra over $\mathbb{Q}$ with generators $X_{12}, X_{13}, X_{23}$ subject the following quadratic relations

$$X_{12}^2 = X_{13}^2 = X_{23}^2 = 0,$$  

$$X_{13}X_{12} = X_{12}X_{13} - X_{13}X_{23} + X_{23}X_{13},$$  

$$X_{23}X_{12} = X_{12}X_{23} - X_{23}X_{13} + X_{13}X_{23}.$$  

Using these relations, we can find the new relations in the algebra $B^0_3$, namely,

$$X_{23}X_{13}X_{23} - X_{13}X_{23}X_{13} + X_{12}X_{13}X_{23} - X_{12}X_{23}X_{13} = 0,$$  

$$X_{23}X_{13}X_{23}X_{13} + X_{12}X_{13}X_{23}X_{13} = 0,$$  

$$X_{13}X_{23}X_{13}X_{23} + X_{12}X_{23}X_{13}X_{23} = 0,$$  

$$X_{12}X_{23}X_{13}X_{23} - X_{12}X_{13}X_{23}X_{13} = 0.$$
Remark 9.5 To be more precise, from the relations \((9.2)–(9.4)\) follow only that
\[ 2 \cdot \text{LHS}(9.5) = 0 \]
We expect that the \(\mathbb{Z}\)-form of the quadratic algebra \(B_3^0\) has only 2-torsion.

It is easy to check from the relations \((9.2)–(9.8)\) that the Hilbert polynomial of the algebra \(B_3^0\) is equal to
\[ H(B_3^0; t) = 1 + 3t + 4t^2 + 3t^3 + t^4 = (1 + t)^2(1 + t + t^2) = H(\mathcal{E}_n^0; t). \]
Now, let us consider the commutative subalgebra \(K_3^0 := \mathbb{Q}[d_2, d_3]\), where \(d_2 = X_{12}\) and \(d_3 = X_{13} + X_{23}\). The Jucys–Murphy elements \(d_2\) and \(d_3\) satisfy the following relations
\[ d_2^2 = 0, \quad d_2d_3 = d_3d_2, \quad (d_2 + d_3)d_3^2 = 0, \]
and the elements \(1, d_2, d_3, d_2d_3, d_3^2, d_2d_3^2, d_3^3, d_2d_3^3\) form a linear basis in the algebra \(K_3^0\). Thus we have
\[ \dim K_3^0 = 8, \quad H(K_3^0; t) = (1 + t)^2(1 + t^2). \]
For general \(n\) we expect that
\[ H(K_n^0; t) = \prod_{k=1}^{n-1} \frac{1 - t^{2k}}{1 - t} = H(W(B_{n-1}); t), \quad (9.9) \]
where \(W(B_{n-1})\) is the Weyl group of type \(B_{n-1}\).

Conjecture 9.6 The quadratic algebras \(B_n^0\) and \(\mathcal{E}_n^0\) have the same Hilbert polynomials
\[ H(B_n^0; t) = H(\mathcal{E}_n^0; t), \quad (9.10) \]
We can check \((9.10)\) for \(n = 3\) (Example \((9.4)\)) and \(n = 4\). In order to prove the equality \((9.10)\) for \(n = 4\), we use the same method as in Example \((8.7)\). But, instead of identity \((8.8)\) in the algebra \(\mathcal{E}_n^0\), we have to use the following 14-term relations in the algebra \(B_n^0\), \(n \geq 4\):
\[ X_{12}(X_{14}X_{34}X_{24} - X_{24}X_{14}X_{34} - X_{34}X_{14}X_{24} + X_{34}X_{24}X_{14}) = 0, \quad (9.11) \]
and
\[ X_{12}(X_{14}X_{34}X_{24} - X_{24}X_{14}X_{34} - X_{34}X_{14}X_{24} + X_{34}X_{24}X_{14}) = 0, \quad (9.12) \]
It is also natural to ask whether or not the quadratic algebra \(B_n^0\) has the Koszul property (see, e.g. [Ma] and references therein).
10 Quadratic algebra $\mathcal{A}_n^t$

In this section we consider the quadratic algebra $\mathcal{A}_n^t$ (denoted by $\mathcal{E}_n$ in [FK], Section 4.3) which is the commutative quotient of the algebra $\mathcal{E}_n^t$.

**Definition 10.1** Define the algebra $\mathcal{A}_n^t$ (of type $A_{n-1}$) as the quadratic algebra over $\mathbb{Z}$ with generators $[ij]$, $1 \leq i < j \leq n$, which satisfy the following relations

\[
[ij][kl] = [kl][ij], \quad \text{for all } i, j, k, l; \tag{10.1}
\]
\[
[ij][jk] = [ik][ij] + [ik'][jk], \quad i < j < k; \tag{10.2}
\]
\[
[ij]^2 = t_{ij}. \tag{10.3}
\]

Let us denote by $\mathcal{A}_n^0$ the specialization $t_{ij} = 0$, $1 \leq i < j \leq n$, of the algebra $\mathcal{A}_n^t$.

**Theorem 10.2** (cf. [A]) The Hilbert polynomial of the algebra $\mathcal{A}_n^0$ is given by

\[
H(\mathcal{A}_n^0; t) = (1 + t)(1 + 2t) \cdots (1 + (n-1)t). \tag{10.4}
\]

**Corollary 10.3** (cf. [A]) An additive basis of the algebra $\mathcal{A}_n^t$ is given by the following set of (commutative) monomials

\[
[j_1k_1] \cdots [j_lk_l],
\]

where $j_s < k_s$, $1 \leq s \leq l$, and $k_1 < k_2 < \cdots < k_l$.

The formula (10.4) was stated at first in [FK], Proposition 4.2.

**Problem 10.4** It is an interesting task to find the Koszul dual of the algebra $\mathcal{A}_n^t$.

Let us postpone a proof of Theorem 10.2 to the end of this Section, and start with a review of some results obtained by V.I. Arnold [A], I.M. Gelfand and A.N. Varchenko [GV].

It seems to be an interesting problem to understand the connections between the algebra $\mathcal{E}_n^0$ and the Orlik–Solomon algebra [OS] that corresponds to the Coxeter hyperplane arrangement (of type $A_{n-1}$). Let us remind that V.I. Arnold [A] described this algebra as the quotient of the exterior algebra of the vector space spanned by the $[ij]$ by the ideal generated by the relations (10.2). The “even analog” of the Orlik–Solomon algebra of a hyperplane arrangement was introduced and studied by I.M. Gelfand and A.N. Varchenko in [GV]. For the Coxeter hyperplane arrangement of type $A_{n-1}$, the results of Theorems 5 and 6 of [GV] can be formulated as follows.

Let $\mathcal{P}_n$ be the ring of integer–valued functions that are defined on the complement $M_C$ of the union of a finite set $C$ of hyperplanes in real, $n$–dimensional affine space, that have constant values on each connected component. The ring $\mathcal{P}_n$ has a
distinguish set of generators, namely, the Heaviside functions of hyperplanes: for each hyperplane \( f(x) = 0 \), the Heaviside function of the hyperplane \( f \) takes the value 1 from one side (where \( f(x) > 0 \)), and 0 from the other (where \( f(x) < 0 \)). Any element of the ring \( \mathcal{P}_n \) is a polynomial in the Heaviside functions. Now let us assume that

\[
\mathcal{M}_C = \mathbb{R}^n \setminus \bigcup_{1 \leq i < j \leq n} \{ z_i - z_j = 0 \}.
\]

Let \( X_{ij} \) be the Heaviside function of the hyperplane \( H_{ij} = \{ z_i - z_j = 0 \} \subset \mathbb{R}^n \), then ([GV], Theorems 5, 6, and 8)

- The Heaviside functions \( X_{ij}, 1 \leq i < j \leq n \), satisfy the following relations:
  \[
  X_{ij}^2 - X_{ij} = 0, \quad (10.5)
  
  X_{ij}(X_{ik} - 1)X_{jk} - (X_{jk} - 1)X_{ik}(X_{ij} - 1) = 0, \quad (10.6)
  \]
  for any \( 1 \leq i < j < k \leq n \).
- Let \( \mathcal{T}_n \) be an ideal in the ring of polynomials \( \mathbb{Z}[X_{ij}, 1 \leq i < j \leq n] \) generated by the left–hand sides of the relations (10.5) and (10.6). Then the natural homomorphism \( \mathbb{Z}[X_{ij}]/\mathcal{T}_n \to \mathcal{P}_n \) is an isomorphism.
- Let us define the degree of a function \( p \in \mathcal{P}_n \) as the minimum of the degrees of polynomials in \( \{ X_{ij} \} \) that represent \( p \), and denote by \( \mathcal{P}_k^n \) the subspace of functions degree not higher then \( k \). Then there exists a noncanonical linear map

\[
\pi_k : \mathcal{P}_n^k \to H^k(M_C, \mathbb{Z}),
\]

which induces an isomorphism (\( \mathcal{P}_n^{-1} = \phi \))

\[
\mathcal{P}_n^k/\mathcal{P}_n^{k-1} \cong H^k(M_C, \mathbb{Z}), \quad 0 \leq k \leq n.
\]

As a corollary, we have

\[
H(\mathcal{P}_n; t) := \sum_{k=0}^{n} t^k \dim \mathcal{P}_n^k/\mathcal{P}_n^{k-1} = \prod_{j=1}^{n-1} (1 + jt), \quad (10.7)
\]

Now we are ready to prove Theorem 10.2.

**Proof of Theorem 10.3.** Let us rewrite the relations (10.5) and (10.6) in slightly different form

\[
X_{ij}^2 = \beta X_{ij}, \quad 1 \leq i < j \leq n; \quad (10.8)
\]

\[
X_{ij}X_{jk} = X_{jk}X_{ik} + X_{ik}X_{ij} - \beta X_{ik}, \quad 1 \leq i < j < k \leq n, \quad (10.9)
\]

where \( \beta \) is an additional deformation parameter.

We define a deformation \( \mathcal{P}_{n,\beta} \) of the algebra \( \mathcal{P}_n \) as the quotient algebra

\[
\mathbb{Z}[X_{ij}]/\mathcal{T}_{n,\beta} = \mathcal{P}_{n,\beta},
\]
where $\mathcal{T}_{n,\beta}$ is an ideal in the commutative ring of polynomials $\mathbb{Z}[X_{ij}]$ generated by the relations (10.8) and (10.9).

It is clear that $\mathcal{P}_{n,\beta=0} \cong \mathcal{A}_n^0$, and $\mathcal{P}_{n,\beta}$ is a flat deformation of $\mathcal{A}_n^0$. Thus, we have

$$\prod_{j=1}^{n-1} (1 + jt) = H(\mathcal{P}_n; t) = H(\mathcal{P}_{n,\beta}; t) = H(\mathcal{P}_{n,\beta=0}; t) = H(\mathcal{A}_n^0; t).$$

Let us introduce a noncommutative version of the algebra $\mathcal{P}_{n,\beta}$.

**Definition 10.5** Define the algebra $\mathcal{L}_{n,\beta}$ (of type $A_{n-1}$) as the quadratic algebra over $\mathbb{Z}[\beta]$ with generators $[ij], 1 \leq i < j \leq n$, which satisfy the following relations

$$
[ij][jk] = [jk][ik] + [ik][ij] + \beta [ik],
$$

$$
[jk][ij] = [ik][jk] + [ij][ik] + \beta [ik], \quad \text{for } i < j < k; \tag{10.10}
$$

$$
[ij][kl] = [kl][ij],
$$

whenever $\{i,j\} \cap \{k,l\} = \emptyset, i < j$, and $k < l$. \tag{10.11}

The algebra $\mathcal{L}_{n,\beta}$ is a smooth deformation of the algebra $\mathcal{G}_n$, and has the same Hilbert series:

$$H(\mathcal{L}_{n,\beta}; t) := \sum_{k=0}^{\infty} t^k \dim \mathcal{L}_{n,\beta}^{(k)}/\mathcal{L}_{n,\beta}^{(k-1)} = H(\mathcal{G}_n; t).$$

We can define also the quotients $\mathcal{L}_{n,\beta}^0$, $\mathcal{L}_{n,\beta}^1$, and commutative subalgebras $\mathcal{H}_{n,\beta}$, $\mathcal{H}_{n,\beta}^0$ and $\mathcal{H}_{n,\beta}^1$ generated by the Dunkl elements $\theta_j$ (see (7.1)), $1 \leq j \leq n$.

It seems interesting to describe the relations in the commutative subalgebra $\mathcal{H}_{n,\beta}^1 \subset \mathcal{L}_{n,\beta}^1$. The work is in progress and we hope to present our results in the nearest future.
References

[A] Arnold V.I., *The cohomology ring of the pure braid group*, Mat. Zametki, 1969, v.5, 227-231;

[B] Brieskorn E., *Automorphic sets, and braids and singularities*, Contemporary Math., 1988, v.78, p.45-115;

[B-N] Bar–Natan D., *Bibliography of Vassiliev invariants*; This bibliography is available electronically at http://www.ma.huji.ac.il/~drorbn;

[Ch1] Cherednik I., *Double affine Hecke algebras and Macdonald conjectures*, Annals of Math., 1995, v.141, p.191-216;

[Ch2] Cherednik I., *Lectures on affine quantum Knizhnik–Zamolodchikov equations, quantum many body problems, Hecke algebras, and Macdonald theory*, Lectures given at the IIAS, Kyoto, 1996;

[Co] Cohen F.R., *Artin’s braid group, classical homotopy theory, and sundry other curiosities*, Contemporary Math., 1988, v.78, p.167-206;

[Dr1] Drinfeld V.G., *On quasi–Hopf algebras*, Leningrad Math. Journ., 1990, v.1, p.1419-1457;

[Dr2] Drinfeld V.G., *On quasi–triangular quasi–Hopf algebras and a group closely connected to \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)*, Leningrad Math. Journ., 1991, v.2, p.829-860;

[Du] Dunkl C.F., *Differential–difference operators associated to reflection groups*, Trans. Amer. Math. Soc., 1989, v.311, p.167-183;

[FK] Fomin S. and Kirillov A.N., *Quadratic algebras, Dunkl elements and Schubert calculus*, Prepr., 1997, AMSPPS #199703-05–001, 34p.;

[F] Funar L., *Vassiliev invariants I: braid groups and rational homotopy theory*, Prepr., 1995, q-alg/9510008, 24p.;

[GV] Gelfand I.M. and Varchenko A.N., *Heaviside functions of a configuration*, Funkts. Anal. Prilozhen., 1987, v.21, n.4. p.1-18;

[H] Heckman G.J., *A remark on the Dunkl differential-difference operators*, in: *Harmonic analysis on reductive groups*, W. Barker and P. Sally, eds., Birkhäuser, 1991, 181–191;

[J] Jucys A.A., *On the Young operators of symmetric groups*, Lithuanian Physics Journal, 1966, v.6, p.163-180;

[Ki] Kirillov A.A., Jr., *Lectures on affine Hecke algebras and Macdonald’s conjectures*, Prepr., 1995, 43p.;
[K] Kirillov A.N., *Cauchy identities for universal Schubert polynomials*, Prepr., 1997, q-alg/9703047, 16p.;

[KN] Kirillov A.N. and Noumi M., *Affine Hecke algebras and raising operators for Macdonald polynomials*, to appear in Duke Math. Journ.; q-alg/9605004, 1996, 35p.;

[Ko1] Kohno T., *Séries de Poincaré–Koszul associée aux groupes de tresses pures*, Invent. Math., 1985, v.82. p.57-75;

[Ko2] Kohno T., *Linear representations of braid groups and classical Yang–Baxter equations*, Contemporary Math., 1988, v.78, p.339-363;

[Ko3] Kohno T., *Vassiliev invariants and de Rham complex on the space of knots*, Contemporary Math.,1994, v.179, p.123-138;

[L] Lin X.-S., *Braid algebras, trace modules and Vassiliev invariants*, Prepr., 1993, 28p.;

[Ma] Manin Yu. I., *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montréal, 1988.

[Mu] Murphy G.E., *A new construction of Young’s seminormal representation of the symmetric groups*, J. of Algebra, 1981, v.69, p.287-297;

[OS] Orlik P. and Solomon L., *Combinatorics and topology of complements of hyperplanes*, Invent. Math., 1980, v.56, p.167-189;

[P] Postnikov A., *On Quantum Monk’s and Pieri’s Formulas*, Prepr., 1997, AM-SPPS #199703-05-002, 12p;

[Ra] Ram A., *Seminormal representations of Weyl groups and Iwahori–Hecke algebras*, 1995, to appear in Proc. London Math. Soc.;

[Ro] Roos J.-E., *Some non-Koszul algebras*, Prepr., 1997, 5p.