LOW REGULARITY LOCAL WELL-POSEDNESS FOR THE YANG-MILLS EQUATION IN LORENZ GAUGE

HARTMUT PECHER

FAKULTÄT FÜR MATHEMATIK UND NATURWISSENSCHAFTEN
BERGISCHE UNIVERSITÄT WUPPERTAL
GAUSSSTR. 20
42119 WUPPERTAL
GERMANY
E-MAIL PECHER@MATH.UNI-WUPPERTAL.DE

ABSTRACT. We prove that the Yang-Mills equation in Lorenz gauge in the 
(n+1)-dimensional case is locally well-posed for data of the gauge potential 
in $H^s$ and the curvature in $H^r$, where $s > n^2 - \frac{7}{8}$, $r > n^2 - \frac{7}{4}$, if $n \geq 4$, and 
$s > \frac{3}{4}$, $r > -\frac{1}{8}$, if $n = 3$. The proof is based on the fundamental results of
Klainerman-Selberg [KS] and on the null structure of most of the nonlinear 
terms detected by Selberg-Tesfahun [ST] and Tesfahun [Te].

1. Introduction

Let $G$ be the Lie group $SO(n, \mathbb{R})$ (the group of orthogonal matrices of determinant 1) or $SU(n, \mathbb{C})$ (the group of unitary matrices of determinant 1) and $g$ its Lie algebra $so(n, \mathbb{R})$ (the algebra of trace-free skew symmetric matrices) or $su(n, \mathbb{C})$ (the algebra of trace-free skew hermitian matrices) with Lie bracket $[X, Y] = XY - YX$ (the matrix commutator). For given $A_\alpha : \mathbb{R}^{1+n} \to g$ we define the 
curvature $F = F[A]$ by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$

where $\alpha, \beta \in \{0, 1, ..., n\}$ and $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$. 

Then the Yang-Mills system is given by

$$D^\alpha F_{\alpha\beta} = 0$$

in Minkowski space $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n$, where $n \geq 3$, with metric $\text{diag}(-1, 1, ..., 1)$. Greek indices run over $\{0, 1, ..., n\}$, Latin indices over $\{1, ..., n\}$, and the usual summation convention is used. We use the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$, where we write 
$(x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (2) we obtain the Gauss-law constraint

$$\partial^\beta F_{\beta 0} + [A^\beta, F_{\beta 0}] = 0.$$

The total energy for YM, at time $t$, is given by

$$\mathcal{E}(t) = \sum_{0 \leq \alpha, \beta \leq n} \int_{\mathbb{R}^n} |F_{\alpha\beta}(t, x)|^2 \, dx,$$

and is conserved for a smooth solution decaying sufficiently fast at spatial infinity, 
i.e.,

$$\mathcal{E}(t) = \mathcal{E}(0).$$

2010 Mathematics Subject Classification: 35Q40, 35L70
Key words and phrases: Yang-Mills, local well-posedness, Lorenz gauge
The system is gauge invariant. Given a sufficiently smooth function \( U : \mathbb{R}^{1+n} \rightarrow \mathcal{G} \) we define the gauge transformation \( T \) by \( TA_0 = A'_0 \) \( T(A_1, ..., A_n) = (A'_1, ..., A'_n) \), where
\[
A_\alpha \mapsto A'_\alpha = U A_\alpha U^{-1} - (\partial_\alpha U) U^{-1}.
\]
It is well-known that if \((A_0, ..., A_n)\) satisfies (1), (2) so does \((A'_0, ..., A'_n)\).

Hence we may impose a gauge condition. We exclusively study the Lorentz gauge \( \partial^0 A_\alpha = 0 \). Other convenient gauges are the Coulomb gauge \( \partial^j A_j = 0 \) and the temporal gauge \( A_0 = 0 \). It is well-known that for the low regularity well-posedness problem for the Yang-Mills equation a null structure for some of the nonlinear terms plays a crucial role. This was first detected by Klainerman and Machedon [KM], who proved global well-posedness in the case of three space dimensions in temporal and in Coulomb gauge in energy space. The corresponding result in Lorentz gauge, where the Yang-Mills equation can be formulated as a system of nonlinear wave equations, was shown by Selberg and Tesfahun [ST], who discovered that also in this case some of the nonlinearities have a null structure. This allows to rely on some of the methods that were previously used for the Lorentz gauge. Tesfahun [ST1] improved the local well-posedness result to data without finite energy. Sterbenz [St] considered also the four-dimensional case in Lorenz gauge and large data are also new. Crucial for this result are on one hand the methods developed in the papers by Selberg-Tesfahun [ST] and Tesfahun [Te], especially their detection of the null structure in most - unfortunately not all - the critical nonlinear terms. On the other hand we have to consider a more sophisticated solution space, where we rely on the methods by Klainerman and Selberg [KS] for a model problem for Yang-Mills, which ignores
the gauge condition. We modify their solution space appropriately and show that its main properties are preserved. We were unable to come down to the critical value $s = \frac{n}{2} - 1$, which is prevented mainly by one of the nonlinear terms, for which no null structure is known and which leads to the estimate \[ ST \] and \[ Te \] used solution spaces of wave-Sobolev type $H^{s,b}$, which are closely related to the Bourgain-Klainerman-Machedon spaces $X^{s,b}$, for which a convenient atlas of bilinear estimates was proven in \[ AFS \] and \[ AFSH \] in dimension $n \leq 3$ and stated in arbitrary dimension. We give a proof for a special case in $n \geq 4$ and also rely on a paper by Lee and Vargas \[ LV \], who obtain $L^p_t L^q_x$ estimates for products of solutions of the wave equation. If one uses solution spaces of $H^{s,b}$-type it seems to be impossible to obtain our results, because some of the bilinear estimates which we need simply fail. For details we refer to the remark preceding the appendix. Therefore it is necessary to modify the solution spaces appropriately.

In chapter 2 we recall the reformulation of the Yang-Mills equation as a system of nonlinear wave equations and state our main theorem (Theorem 2.1 and Corollary 2.1). We also fix some notation. Chapter 3 contains the bilinear estimates in wave-Sobolev spaces. Moreover we define the solution spaces and state its fundamental properties. We reduce the local well-posedness problem to a suitable set of nonlinearities in Proposition 3.16, where we complete rely on \[ KS \]. In chapter 4 we formulate the Yang-Mills equations in final form using the whole null structure and the necessary nonlinear estimates as in \[ Te \]. We also review some well-known properties of the standard null forms and the additional one detected in \[ Te \]. In chapter 4 and 5 we prove the estimates for the nonlinearities for $n \geq 4$ and $n = 3$, respectively.

Acknowledgment: I thank Axel Grünrock who pointed out the paper by Lee-Vargas \[ LV \] to me.

2. Main results

Expanding (2) in terms of the gauge potentials $\{A_\alpha\}$, we obtain:
\[
\Box A_\beta = \partial_\beta \partial^\alpha A_\alpha - [\partial^\alpha A_\alpha, A_\beta] - [A^\alpha, \partial^\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}].
\tag{3}
\]
If we now impose the Lorenz gauge condition, the system (3) reduces to the non-linear wave equation
\[
\Box A_\beta = -[A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}].
\tag{4}
\]
In addition, regardless of the choice of gauge, $F$ satisfies the wave equation
\[
\Box F_{\beta\gamma} = -[A^\alpha, \partial_\alpha F_{\beta\gamma}] - \partial^\alpha [A_\alpha, F_{\beta\gamma}] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] - 2[F^\alpha_{\beta\gamma}, F_{\gamma\alpha}].
\tag{5}
\]
Indeed, this will follow if we apply $D^\alpha$ to the Bianchi identity
\[
D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0
\]
and simplify the resulting expression using the commutation identity
\[
D_\alpha D_\beta X - D_\beta D_\alpha X = [F_{\alpha\beta}, X]
\]
and (2) \[ ST \].

Expanding the second and fourth terms in (5), and also imposing the Lorenz gauge, yields
\[
\Box F_{\beta\gamma} = -2[A^\alpha, \partial_\alpha F_{\beta\gamma}] + 2[\partial_\gamma A^\alpha, \partial_\alpha A_\beta] - 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] + 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] + 2[\partial^\alpha A_\gamma, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] + 2[F_{\alpha\beta}, [A^\alpha, A_\gamma]] - 2[F_{\alpha\gamma}, [A^\alpha, A_\beta]] - 2[[A^\alpha, A_\beta], [A_\alpha, A_\gamma]].
\tag{6}
\]
Note on the other hand by expanding the last term in the right hand side of (1), we obtain
\[ \Box A_\beta = -2 [A^\alpha, \partial_\alpha A_\beta] + [A^\alpha, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, A_\beta]]. \] (7)

We want to solve the system (6)-(7) simultaneously for \( A \) and \( F \). So to pose the Cauchy problem for this system, we consider initial data for (6) and (7) has a unique solution
\[ \dot{A}(0) = a, \quad \partial_t A(0) = \dot{a}, \quad F(0) = f, \quad \partial_t F(0) = \dot{f}. \] (8)

In fact, the initial data for \( F \) can be determined from \((a, \dot{a})\) as follows:
\[
\begin{align*}
\dot{f}_{ij} &= \partial_i a_j - \partial_j a_i + [a_i, a_j], \\
\dot{f}_{0i} &= \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\
\dot{f}_{ij} &= \partial_j \dot{a}_i - \partial_i \dot{a}_j + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\
\dot{f}_{00} &= \partial^j f_{ij} + [a^i, f_{0i}]
\end{align*}
\] (9)

where the first three expressions come from (1) whereas the last one comes from (2) with \( \beta = i \).

Note that the Lorenz gauge condition \( \partial^\alpha A_\alpha = 0 \) and (2) with \( \beta = 0 \) impose the constraints
\[ \dot{a}_0 = \partial^0 a_i, \quad \partial^i f_{00} + [a^i, f_{00}] = 0. \] (10)

Now we formulate our main theorem.

**Theorem 2.1.** If \( n \geq 4 \), assume that \( s \) and \( r \) satisfy the following conditions:
\[
\begin{align*}
s &> \frac{n}{2} - \frac{7}{8}, \quad r > \frac{n}{2} - \frac{7}{4}, \quad r < s < r + 1, \\
3r - 2s &> \frac{n}{2} - \frac{7}{2}, \\
2r - s &> -\frac{3}{4} \quad (\text{if } n = 4), \\
2s - r &> \frac{n}{2}, \\
3s - 2r &> \frac{n}{2} + \frac{1}{2}.
\end{align*}
\]

If \( n = 3 \), assume:
\[
\begin{align*}
s &> \frac{3}{4}, \\
r &> -\frac{1}{8}, \\
2r - s &> -1, \\
3s - 2r &> 2
\end{align*}
\]

Given initial data \((a, \dot{a}) \in H^s \times H^{s-1}, (f, \dot{f}) \in H^r \times H^{r-1}\), there exists a time \( T \geq 0 \), \( T = T(\|a\|_{H^s}, \|\dot{a}\|_{H^{s-1}}, \|f\|_{H^r}, \|\dot{f}\|_{H^{r-1}}) \), such that the Cauchy problem (6), (7) has a unique solution \( A \in F^r_2 \), \( F \in G^r_T \) (these spaces are defined in Def. [10]). This solution has the regularity
\[ A \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), \quad F \in C^0([0, T], H^r) \cap C^1([0, T], H^{r-1}). \]

**Remark:** Case \( n \geq 4 \). 1. The assumptions on \( s \) and \( r \) imply \( 6s - \frac{3}{2}n > 3r > 2s + \frac{n}{2} - \frac{7}{2} \), which can only be fulfilled, if \( s > \frac{n}{2} - \frac{7}{8} \), and therefore \( r > \frac{n}{2} - \frac{7}{4} \). One easily checks that the choice \( s = \frac{n}{2} - \frac{7}{8} + \epsilon \), \( r = \frac{n}{2} - \frac{7}{4} + \epsilon \) satisfies our assumptions, if \( \epsilon > 0 \) is small enough.

2. The following estimate is automatically fulfilled \( 2r - 2 > \frac{7}{2} - 3 \), because \( 2r - s = 3r - 2s - r > \frac{7}{2} \).

**Case \( n = 3 \).** 1. The assumptions on \( s \) and \( r \) imply \( r > \frac{5}{4} - \frac{1}{2} \), which implies \( r > -\frac{1}{8} \), if \( s > \frac{3}{4} \). One easily checks that the choice \( s = \frac{3}{4} + \epsilon \), \( r = -\frac{1}{8} + \epsilon \) satisfies our assumptions, if \( \epsilon > 0 \) is small enough.

2. The following conditions are automatically fulfilled
\[ 3r - 2s > -2, \quad 4r - s > -2, \quad 4s - r > 3, \]
Preliminaries

3. Preliminaries

The Strichartz type estimates for the wave equation are given in the next proposition.

Proposition 3.1. If \( n \geq 2 \) and

\[
2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{2}{q} \leq (n-1) \left( \frac{1}{2} - \frac{1}{r} \right),
\]

then the following estimate holds:

\[
\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{H^{\frac{n}{2} - rac{1}{q} + rac{1}{q}}}.
\]

especially in the case \( n \geq 4 \):

\[
\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{H^{\frac{n}{2} - \frac{1}{q} + rac{1}{q}}}.
\]

for \( \frac{2(n-1)}{n-4} \leq r < \infty \).
Proof. This is the Strichartz type estimate, which can be found for e.g. in [GV], Prop. 2.1, combined with the transfer principle.

An immediate consequence is the following modified Strichartz estimate.

**Proposition 3.2.** If \( n \geq 4 \), \( 2 \leq r \leq \frac{2(n-1)}{n-3} \), one has the estimate

\[
\|u\|_{L_t^rL_x^\frac{n}{r}} \lesssim \|u\|_{H^\frac{3n}{2(n-1)}(\mathbb{R}^n)} \lesssim \|u\|_{H^\frac{2n+1}{2n} r}. 
\]

**Proof.** The last estimate is trivial. For the first one we interpolate the trivial identity \( \|u\|_{L_t^2L_x^\frac{n}{2}} = \|u\|_{H^{0.0}} \) with the estimate

\[
\|u\|_{L_t^rL_x^\frac{n}{r}} \lesssim \|u\|_{H^\frac{2n+1}{2n} r}, 
\]

which holds by Prop. 3.1.

**Proposition 3.3.** If \( n = 3 \), \( 2 \leq r < \infty \), the following estimate holds:

\[
\|u\|_{L_t^{2+\epsilon}L_x^2} \lesssim \|u\|_{H^{1-\frac{3\epsilon}{2}+\epsilon}},
\]

for \( 0 < \epsilon \leq \frac{4}{3} \).

**Proof.** We use the following special case of Prop. 3.1

\[
\|u\|_{L_t^{2+\epsilon}L_x^2} \lesssim \|u\|_{H^{1-\frac{3\epsilon}{2}+\epsilon}},
\]

for arbitrary \( \epsilon_1, \epsilon_2 > 0 \), where we choose \( q = 2 + \epsilon_1 \), \( r = \frac{2(2+\epsilon_1)}{2+\epsilon_1} \), so that \( \frac{3}{2} - \frac{2}{r} - \frac{1}{q} = \frac{2}{2+\epsilon_1} < 1 \). Now we interpolate this inequality with the trivial identity \( \|u\|_{L_t^2L_x^2} = \|u\|_{H^{0.0}} \). We choose the interpolation parameter \( \theta \) by \( \frac{1}{r} = \theta \frac{\epsilon_1}{2+\epsilon_1} + (1-\theta) \frac{1}{2} \iff \theta = (2+\epsilon_1) \frac{1}{2} - \frac{1}{2} \) and require moreover \( \frac{1}{2q} = \frac{\theta}{2+\epsilon_1} + \frac{1}{2q} \), so that \( \epsilon_1 = 2 \left( \frac{\theta}{2} - \frac{1}{2} \right). \) This implies \( \theta = (2+\epsilon_1) \frac{1}{2} - \frac{1}{2} = 1 - \frac{1}{r} + \frac{1}{r} + \epsilon \leq 1 \) for \( \epsilon \leq \frac{4}{3} \), and \( \frac{\theta}{2} + \theta \epsilon_2 = \frac{1}{2} \left( 1 - \frac{1}{r} \right) + \frac{\theta}{2+\epsilon_1} + \theta \epsilon_2 < \frac{1}{q} \left( 1 - \frac{1}{r} \right) + \frac{1}{q} \) for sufficiently small \( \epsilon_2 > 0 \). Thus we obtain by interpolation

\[
\|u\|_{L_t^{2+\epsilon}L_x^2} \lesssim \|u\|_{H^0.0} \lesssim \|u\|_{H^{1-\frac{3\epsilon}{2}+\epsilon}}.
\]

The following product estimates for wave-Sobolev spaces are special cases of the very convenient much more general atlas by [AFS].

**Proposition 3.4.** Let \( n = 3 \) and

1. Assume \( s_0 + s_1 + s_2 > 1 \), \( s_0 + s_1 + s_2 + s_1 + s_2 > \frac{3}{2} \), \( s_0 + s_1 \geq 0 \), \( s_0 + s_2 \geq 0 \), \( s_1 + s_2 \geq 0 \).

The following estimate holds:

\[
\|uv\|_{H^{-s_0.0}} \lesssim \|u\|_{H^{s_1.\frac{1}{2}}} \|v\|_{H^{s_2.\frac{1}{2}}}. 
\]

2. Assume \( s_0 + s_1 + s_2 > 1 \), \( s_0 + s_1 \geq 0 \), \( s_0 + s_2 \geq 0 \), \( s_1 + s_2 \geq 0 \).

The following estimate holds:

\[
\|uv\|_{H^{-s_0.\frac{1}{2}+}} \lesssim \|u\|_{H^{s_1.\frac{1}{2}+}} \|v\|_{H^{s_2.\frac{1}{2}+}}. 
\]

The following proposition was proven by [KT].
Proposition 3.5. Let $n \geq 2$, and let $(q, r)$ satisfy:

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{2}{q} \leq \frac{n - 1}{\left(\frac{1}{2} - \frac{1}{r}\right)}$$

Assume that

$$0 < \sigma < n - \frac{2n}{r} \cdot \frac{4}{q},$$

$$s_1, s_2 < \frac{n}{2} - \frac{n}{r} - \frac{1}{q},$$

$$s_1 + s_2 + \sigma = n - \frac{2n}{r} - \frac{2}{q},$$

then

$$\|D^{-\sigma}(uv)\|_{L_t^{q/2}L_x^{r/2}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$ 

The following product estimate for wave-Sobolev spaces is a special case of the very convenient much more general atlas formulated by [AFS] in arbitrary dimension, but proven only in the case $1 \leq n \leq 3$. ([AFS] and [AFS1]). Therefore we have to give a proof.

Proposition 3.6. Assume $n \geq 4$ and

$$s_0 + s_1 + s_2 > \frac{n - 1}{2}, \quad (s_0 + s_1 + s_2) + s_1 + s_2 > \frac{n}{2}, \quad s_0 + s_1 \geq 0, \quad s_0 + s_2 \geq 0, \quad s_1 + s_2 \geq 0.$$ 

The following estimate holds:

$$\|uv\|_{H^{-s_0, 0}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$ 

Proof. We have to prove

$$I := \int \hat{u}_1(\xi_1, \tau_1) \hat{u}_2(\xi_2, \tau_2) \hat{u}_0(\xi_0, \tau_0) \lesssim \|u_1\|_{L_{s_1}} \|u_2\|_{L_{s_2}}.$$ 

Here $*$ denotes integration over $\xi_0 + \xi_1 + \xi_2 = 0$ and $\tau_0 + \tau_1 + \tau_2 = 0$. Remark, that we may assume that the Fourier transforms are nonnegative. We consider different regions.

1. If $|\xi_0| \sim |\xi_1| \gtrsim |\xi_2|$ and $s_2 \geq 0$, we obtain

$$I \sim \int \hat{u}_1(\xi_1, \tau_1) \hat{u}_2(\xi_2, \tau_2) \hat{u}_0(\xi_0, \tau_0).$$ 

Thus we have to show

$$\|uv\|_{L_{s_1}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}.$$ 

By Prop. 3.5 we obtain

$$\|uv\|_{L_{s_1}^{2}} \lesssim \|u\|_{L_{s_1}^{2}} \|v\|_{L_{s_2}^{2}} \|v\|_{L_{s_2}^{2}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \|v\|_{L_{s_2}}.$$ 

and also

$$\|uv\|_{L_{s_1}^{2}} \lesssim \|u\|_{L_{s_1}^{2}} \|v\|_{L_{s_2}^{2}} \|v\|_{H^{s_2}} \|v\|_{L_{s_2}^{2}}.$$ 

Bilinear interpolation gives for $0 \leq \theta \leq 1$:

$$\|uv\|_{L_{s_1}^{2}} \lesssim \|u\|_{H^{\frac{n-1}{2}+\theta}} \|v\|_{H^{n+\theta}} \|v\|_{H^{\frac{n-1}{2}+\theta}},$$ 

so that

$$\|uv\|_{L_{s_1}^{2}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \|v\|_{H^{s_2}} \|v\|_{H^{s_2}}.$$ 

if $s_0 + s_1 + s_2 > \frac{n-1}{2}$ and $s_1 + s_0 \geq 0$.

2. If $|\xi_0| \sim |\xi_2| \gtrsim |\xi_1|$ and $s_1 \geq 0$, we obtain similarly

$$\|uv\|_{L_{s_1}^{2}} \lesssim \|u\|_{H^{s_2}} \|v\|_{H^{s_2}} \|v\|_{H^{s_2}}.$$ 


if $s_0 + s_1 + s_2 > \frac{n-1}{2}$ and $s_2 + s_0 \geq 0$.
3. If $|\xi_1| \geq |\xi_2|$ , $s_0 \leq 0$ and $s_2 \geq 0$, we have $|\xi_0| \lesssim |\xi_1|$ , so that $\langle \xi_0 \rangle^{-s_0} \lesssim \langle \xi_1 \rangle^{-s_0}$ and we obviously obtain the same result as in 1.
4. If $|\xi_1| \leq |\xi_2|$ , $s_0 \leq 0$ and $s_2 \geq 0$ , we obtain the same result as in 2.
5. If $|\xi_0| \sim |\xi_1| \gtrsim |\xi_2|$ and $s_0 \leq 0$ we obtain

$$I \lesssim \int \left( \frac{\hat{u}_1(\xi_1, \tau_1)}{|\xi_1|^n + |\xi_2| + |\xi_0|} \right) \frac{\hat{u}_2(\xi_2, \tau_2)}{|\xi_2|^n + |\xi_0|} \hat{\mu}_0(\xi_0, \tau_0) \lesssim \|u_1\|_{L^n_{\xi_1}} \|u_2\|_{L^2_{\tau_2}},$$

because under our assumption $s_0 + s_1 + s_2 > \frac{n-1}{2}$ we obtain by Prop. 3.1

$$\|uv\|_{L^2_{\xi_1}} \leq \|u\|_{L^\infty_{\xi_1}} \|v\|_{L^\infty_{\xi_1}} \lesssim \|u\|_{H^{n+1} + s_2} + \|v\|_{H^{n+1} + s_2}.$$

6. If $|\xi_1| \geq |\xi_2|$ , $s_2 \leq 0$ and $s_0 \leq 0$ , or
7. If $|\xi_0| \sim |\xi_1| \geq |\xi_2|$ and $s_1 \leq 0$ , or
8. If $|\xi_1| \leq |\xi_2|$ , $s_1 \leq 0$ and $s_0 \leq 0$ , the same argument applies.

Thus we are done, if $s_0 \leq 0$ , and also, if $s_0 \geq 0$ , and $|\xi_0| \sim |\xi_2| \geq |\xi_1|$ or $|\xi_0| \sim |\xi_1| \geq |\xi_2|$.

It remains to consider the following case: $|\xi_0| \ll |\xi_1| \sim |\xi_2|$ and $s_0 > 0$ . We apply Prop. 3.6 which gives

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}},$$

under the conditions $0 < s_0 < \frac{n}{2} - 1$ , $s_0 + s_1 + s_2 = \frac{n}{2} - 1$ and $s_1, s_2 < \frac{n}{2} - 1$ .

The last condition is not necessary in our case $|\xi_1| \sim |\xi_2|$ . Remark that this implies $s_1 + s_2 > \frac{n}{2}$ , so that $s_0 + s_1 + s_2 + s_1 + s_2 > \frac{n}{2}$ . The second condition can now be replaced by $s_0 + s_1 + s_2 \geq \frac{n}{2} - \frac{s_0}{2}$ , because we consider inhomogeneous spaces.

Finally we consider the case $|\xi_0| \ll |\xi_1| \sim |\xi_2|$ and $s_0 \geq \frac{n}{2} - 1$ . If $s_0 > \frac{n}{2}$ and $s_1 + s_2 \geq 0$ we obtain the claimed estimate by Sobolev

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}}.$$

We now interpolate the special case

$$\|uv\|_{H^{-\frac{n}{2}+}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}},$$

with the following estimate

$$\|uv\|_{H^{-\frac{n}{2}+}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}},$$

which follows from Prop. 3.5 . We obtain

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}},$$

where $s_0 = (1 - \theta) \frac{n}{2} - \theta (1 - \frac{n}{2}) = \frac{n}{2} - \theta \leq 0 \leq \theta \leq 1$ , $k = \frac{n}{2} - \frac{n}{2}$ . Using our assumption $(s_0 + s_1 + s_2) + s_1 + s_2 > \frac{n}{2}$ \( \Leftrightarrow \frac{n}{2} - s_0 < 2(s_1 + s_2) \), we obtain $0 \leq k < \frac{n - 2s_0}{n - 2}$ . Because $\xi_1 \sim \xi_2$ , we obtain

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}}$$

for $(s_0 + s_1 + s_2) + s_1 + s_2 > \frac{n}{2}$ and $s_1 + s_2 \geq 0$ .

\( \square \)

**Corollary 3.1.** Under the assumptions of Prop. 3.6

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{1\frac{n_0}{2}+}} \|v\|_{H^{2\frac{n}{2}+}}.$$

**Proof.** This follows by bilinear interpolation of the estimate of Prop. 3.6 with the estimate

$$\|uv\|_{H^{n_0, 0}} \lesssim \|u\|_{H^{n\frac{n_0}{2}+}} \|v\|_{H^{n\frac{n}{2}+}},$$

where, say, $N > \frac{n}{2}$ , which follows by Sobolev apart from the special case $s_1 = -s_2$ , in which we interpolate with the estimate

$$\|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{n\frac{n_0}{2}+}} \|v\|_{H^{n\frac{n}{2}+}}.$$
in order to save the condition \( s_1 = -s_2 \).

\[ \square \]

**Corollary 3.2.** If \( s_1 > \frac{n-1}{2} \), \( 2s_1 + s_0 > \frac{n}{2} \), \( s_1 + s_0 \geq 0 \), \( 0 \leq \epsilon < \frac{1}{2} \) the following estimate holds

\[ \|uv\|_{H^{-n,\alpha}} \lesssim \|u\|_{H^{s_1 + \epsilon, 0}} \|v\|_{H^{-n_0, \frac{s_1}{2} + \epsilon}}. \]

**Proof.** By Prop. 3.6 we have

\[ \|uv\|_{H^{-n_0, 0}} \lesssim \|u\|_{H^{s_1 + \epsilon, 0}} \|v\|_{H^{-n_0, \frac{s_1}{2} + \epsilon}}. \]

Moreover for \( N > \frac{n}{2} \) we obtain by Sobolev

\[ \|uv\|_{H^{-n_0, \frac{s_1}{2} + \epsilon}} \lesssim \|u\|_{H^{N, 0}} \|v\|_{H^{-n_0, \frac{s_1}{2} + \epsilon}}. \]

The result follows by interpolation. \( \square \)

The following multiplication law is well-known:

**Proposition 3.7. (Sobolev multiplication law)** Let \( n \geq 2 \), \( s_0, s_1, s_2 \in \mathbb{R} \). Assume \( s_0 + s_1 + s_2 \geq \frac{n}{2} \), \( s_0 + s_1 \geq 0 \), \( s_0 + s_2 \geq 0 \), \( s_1 + s_2 \geq 0 \). Then the following product estimate holds:

\[ \|uv\|_{H^{-n}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}. \]

We also need the following bilinear estimates in \( H^{s,b} \)-spaces, which follow as a special case from the results stated in [AFS] and [AFS1], but proven in these papers only for \( n = 2 \) and \( n = 3 \). We postpone the proof to the appendix.

**Proposition 3.8.** Let \( n \geq 2 \). Assume \( s_0 + s_1 + s_2 \geq \frac{n}{2} + \epsilon \), \( s_1 + s_2 \geq \frac{1}{2} \), \( s_0 + s_1 \geq 0 \), \( s_0 + s_2 \geq 0 \). Then the following estimates hold for \( \epsilon > 0 \) sufficiently small:

\[ \|uv\|_{H^{-n_0, \frac{s_1}{2} + \epsilon}} \lesssim \|u\|_{H^{s_1 + \epsilon, 0}} \|v\|_{H^{-n_2, \frac{s_2}{2} + \epsilon}}, \quad (12) \]

\[ \|uv\|_{H^{-n_0, \frac{s_2}{2} - 2\epsilon}} \lesssim \|u\|_{H^{s_1, \frac{s_2}{2} - 2\epsilon}} \|v\|_{H^{-n_2, \frac{s_2}{2} + \epsilon}}. \quad (13) \]

Next we formulate a special case of the fundamental estimates for the \( L^q_t L^p_x \)-norm of the product of solutions of the wave equation due to Lee-Vargas [LV].

**Proposition 3.9.** Assume \( n \geq 4 \) and

\[ \square u = \square v = 0 \]

in \( \mathbb{R}^n \times \mathbb{R} \). The estimate

\[ \|uv\|_{L^q_t L^p_x} \lesssim \|(u(0))\|_{H^{n_1}} + \|(\partial_t u)(0)\|_{H^{n_2}} \|v(0)\|_{H^{n_2}} + \|(\partial_t v)(0)\|_{H^{n_2-1}} \]

holds, provided \( 1 < q \leq 2 \) and

\[ \alpha_1 + \alpha_2 = \frac{n}{2} - \frac{1}{q}, \quad (14) \]

\[ \frac{1}{q} < \frac{n-1}{4}, \quad (15) \]

\[ \alpha_1, \alpha_2 < \frac{n}{2} + \frac{1}{2} - \frac{2}{q}. \quad (16) \]

**Proof.** This follows by easy calculations from [LV], Theorem 1.1. \( \square \)

**Corollary 3.3.** If (14), (15) and (16) are satisfied for some \( 1 < q \leq 2 \) and additionally \( \alpha_1, \alpha_2 \geq 0 \), the following estimate holds

\[ \|uv\|_{H^{n_1, \frac{s_1}{2} + \epsilon}} \lesssim \|u\|_{H^{s_1, 0}} \|v\|_{H^{n_2, \frac{s_2}{2} + \epsilon}}. \]
Definition 3.2. Our solution spaces are defined as follows:

1. in the case $n \leq L^2$ Fourier transform. Observe that Proposition 3.10. given by [KS], starting with a Hölder-type estimate.

Proof. The proposition implies

$$
\|e^{\pm itD}f e^{\pm itD}g\|_{L^1_tL^2_x} \lesssim \|f\|_{H^{\alpha}} \|g\|_{H^{\beta}},
$$

so that the claimed estimate follows by the transfer principle combined with the estimate $\|uv\|_{H^{\alpha+\beta}} \lesssim \|uv\|_{L^1_tL^2_x}$.

We now come to the definition of the solution spaces, which are very similar to the spaces introduced by [KS]. We prepare this by defining a modification of the standard $L^1_tL^2_x$-spaces.

Definition 3.1. If $1 \leq q, r \leq \infty$, $u \in S'$ and $\hat{u}$ is a tempered function, set

$$
\|u\|_{L^q_tL^r_x} = \sup \left\{ \int_{\mathbb{R}^{1+n}} |\hat{u}(\tau, \xi)|\hat{v}(\tau, \xi) d\tau d\xi : v \in \mathcal{S}, \hat{v} \geq 0, \|v\|_{L^q_tL^r_x} = 1 \right\},
$$

where $1 = \frac{1}{q} + \frac{1}{r}$ and $1 = \frac{1}{r} + \frac{1}{\alpha}$. Let $L^q_tL^r_x$ be the corresponding subspace of $S'$.

This is a translation invariant norm and it only depends on the size of the Fourier transform. Observe that $L^q_tL^r_x = L^2_tL^2_x$ and

$$
\|u\|_{L^q_tL^r_x} \leq \|u\|_{L^1_tL^2_x} \quad \text{whenever} \quad \hat{u} \geq 0.
$$

Definition 3.2. Our solution spaces are defined as follows:

1. in the case $n \geq 4$

$$
\|u\|_{F^\infty} := \|A_+ u\|_{H^{\infty-1/4,+}} + \|A_+^{3/2} \Lambda^{\infty+3/2} A_+^{3} u\|_{L^4_tL^4_x},
$$

$$
\|v\|_{G^\infty} := \|A_+ v\|_{H^{\infty-1/4,+}} ,
$$

2. in the case $n = 3$

$$
\|u\|_{F^\infty} := \|A_+ u\|_{H^{\infty-1/4,+}} + \|A_+^{3/2} \Lambda^{\infty+3/2} A_+^{3} u\|_{L^4_tL^4_x},
$$

$$
\|v\|_{G^\infty} := \|A_+ v\|_{H^{\infty-1/4,+}} ,
$$

where $\epsilon > 0$ is sufficiently small. $F^\infty_T$ and $G^\infty_T$ denotes the restriction to the time interval $[0, T]$.

This is a Banach space ([KS], Prop. 4.2).

Next we recall some fundamental properties of the $L^q_tL^r_x$-spaces, which were given by [KS], starting with a Hölder-type estimate.

Proposition 3.10. Suppose $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, where the $q$'s and $r$'s all belong to $[1, \infty]$. Then

$$
\|uv\|_{L^q_tL^r_x} \leq \|u\|_{L^q_tL^r_x} \|v\|_{L^q_tL^r_x},
$$

for all $v$ with $\hat{v} \geq 0$.

The following duality argument holds.

Proposition 3.11. Let $1 \leq a, b, q, r \leq \infty$.

(a) If

$$
\|G\|_{L^a_tL^b_x} \lesssim \|A_+ A_\beta G\|_{L^a_tL^b_x},
$$

for all $G$, then

$$
\|F\|_{L^1_tL^2_x} \lesssim \|A_\alpha A_\beta F\|_{L^1_tL^2_x},
$$

for all $F$.

(b) If ([7]) holds for all $G$ with $\hat{G} \geq 0$, then

$$
\|F\|_{L^1_tL^2_x} \lesssim \|A_\alpha A_\beta F\|_{L^1_tL^2_x},
$$

for all $F$. 

Proof. [KS], Proposition 4.5. □

The next proposition shows that a Sobolev type embedding also carries over to the \( L^1 L^r \)-spaces.

**Proposition 3.12.** Let \( 1 \leq a, b, q, r \leq \infty \), \( \alpha, \beta \in \mathbb{R} \). If
\[
\| \Lambda^\alpha \Lambda^\beta u \|_{L^q L^r} \lesssim \| u \|_{L^q L^r}
\]
for all \( u \) with \( \hat{u} \geq 0 \), then
\[
\| \Lambda^\alpha \Lambda^\beta u \|_{L^1 q L^r} \lesssim \| u \|_{L^q L^r}.
\]

**Proof.** [KS], Cor. 4.6. □

The following result is also fundamental for the proof of our main theorem.

**Proposition 3.13.** Let \( n \geq 4 \). If \( \frac{2(n-1)}{n-3} \leq r < \infty \), \( s = \frac{\alpha}{p} - \frac{\beta}{p} - \frac{1}{2} \) and \( \theta > \frac{1}{2} \), then
\[
\| u \|_{L^1_q L^r} \lesssim \| \Lambda^\alpha \Lambda^\beta u \|_{L^1_q L^2}.
\]

**Proof.** [KS], Lemma 4.8. □

A similar result is also true in the case \( n = 3 \). We prepare this by the following proposition.

**Proposition 3.14.** If \( n = 3 \), \( 1 < p \leq q \leq 2 \), \( \frac{1}{q} + \frac{1}{q'} = 1 \) and \( s = \frac{p}{2} - 2 + \frac{1}{q} \), then
\[
\| \Lambda^{-s} \Lambda^{-\frac{1}{2}} u \|_{L^p L^q} \lesssim \| u \|_{L^q L^q}
\]
for all \( u \) with \( \hat{u} \geq 0 \).

**Proof.** We adapt the proof of [KS], Prop. 4.7 in space dimension \( n = 4 \) to the case \( n = 3 \). Let \( U := \Lambda^{-s} \Lambda^{-\frac{1}{2}} u \). Using the estimate
\[
\| U \|_{L^p L^q} \lesssim \| \int \hat{U}(\tau, \xi) d\tau \|_{L^q}
\]
for \( \hat{U} \geq 0 \) we reduce the claimed estimate to
\[
\int \int \langle \xi \rangle^{-s} \langle |\tau| - |\xi| \rangle^{-\frac{1}{2}} \hat{u}(\tau, \xi) f(\xi) d\tau d\xi \lesssim \| u \|_{L^q L^q} \| f \|_{L^q}
\]
for all \( f \) with inverse Fourier transform \( \hat{f} \geq 0 \). Define
\[
\tilde{v} \equiv \langle \xi \rangle^{-s} \langle |\tau - |\xi| \rangle^{-\frac{1}{2}} \hat{f}(\xi).
\]
Then the left hand side of (18) is bounded by
\[
\int \int u(v_+ + v_-) d\tau d\xi \lesssim \| u \|_{L^q L^q} (\| v_+ \|_{L^q L^q} + \| v_- \|_{L^q L^q}),
\]
where \( \frac{p}{p} + \frac{1}{q} = 1 \). It remains to show
\[
\| v_\pm \|_{L^q L^q} \lesssim \| f \|_{L^q}.
\]
Defining \( \tilde{\xi}(\tau) := \langle \tau \rangle^{-\frac{1}{2}} \) we obtain
\[
\tilde{v}_\pm (\tau, \xi) = \langle \xi \rangle^{-s} \tilde{\xi}(\tau \mp |\xi|) \hat{f}(\xi) = \langle \xi \rangle^{-s} \int c(t) e^{it(\tau \mp |\xi|)} dt \hat{f}(\xi)
\]
\[
= \int c(t) e^{it\tau} (\langle \xi \rangle^{-s} e^{it|\xi|} |f(\xi)|) dt,
\]
so that
\[
v_\pm (t, x) = c(t) \Lambda^{-s} e^{xt} D f(x).
\]
Proof. Proposition 3.16. for the nonlinearities. It is also essentially contained in the paper by [KS].

Assume that $u \in L^2(R)$, and $v \in L^2(R)$. Then

\[ \|u\|_{L_t^2 L_x^p} \lesssim \|\Lambda^{-\frac{s}{q}} f\|_{L_t^2 L_x^{p'}} \lesssim \|f\|_{L_t^2}, \]

where we applied Strichartz’ estimate (Prop. 3.11) under the assumption $2 \leq q \leq \infty$, $2 \leq p' < \infty$.

Corollary 3.4. Under the assumptions of Prop. 3.14:

\[ \|u\|_{L_t^2 L_x^{p'}} \lesssim \|\Lambda^s \Lambda^\frac{s}{q} u\|_{L_t^2 L_x^p}, \]

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. This follows from Prop. 3.14 by use of Prop. 3.11.

Corollary 3.5. If $n = 3$, $1 < q \leq 2$, $s = \frac{2}{q} - 1$, then

\[ \|\Lambda^{-s} \Lambda^\frac{s}{q} u\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^2 L_x^2} \]

for all $u$ with $\hat{u} \geq 0$.

Proof. In the special case $p = q$ Prop. 3.14 gives

\[ \|\Lambda^{-s} \Lambda^\frac{s}{q} u\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^2 L_x^2} \]

for $1 < q \leq 2$, $s = \frac{2}{q} - 2 + \frac{1}{q} = \frac{2}{q} - 1$. Interpolation with the trivial identity $\|u\|_{L_t^2 L_x^2} = \|u\|_{L_t^2 L_x^2}$ gives the result.

Proposition 3.15. If $n = 3$, $1 < q \leq 2$, $\frac{1}{q} + \frac{1}{q} = 1$ and $s = \frac{2}{q} - 1$ the following estimate holds

\[ \|u\|_{L_t^2 L_x^{p'}} \lesssim \|\Lambda^s \Lambda^\frac{s}{q} u\|_{L_t^2 L_x^2} \]

Proof. This follows from Corollary 3.5 by use of Prop. 3.11.

We also need an elementary estimate which is used as a tool for replacing $H^{s,-\frac{1}{2}+}$-norms by $H^{s,-\frac{1}{2}}$-norms.

Lemma 3.1. Let $\alpha, \beta \geq 0$. Then

\[ \Lambda^{-\beta}(uv) \lesssim \Lambda^{-\alpha-\beta}(\Lambda^\alpha u \Lambda^\beta v), \]

\[ \Lambda^{-\beta}(uv) \lesssim \Lambda^{-\alpha-\beta}(u \Lambda^\alpha v) + u \Lambda^{-\beta}v \]

for all $u$ and $v$ with $\hat{u}, \hat{v} \geq 0$.

Proof. [KS], Lemma 8.10.

Finally, we formulate the fundamental theorem which allows to reduce the local well-posedness for a system of nonlinear wave equations to suitable estimates for the nonlinearities. It is also essentially contained in the paper by [KS].

Proposition 3.16. Let $u_0 \in H^s$, $u_1 \in H^{s-1}$, $v_0 \in H^r$, $v_1 \in H^{r-1}$ be given. Assume that

\[ \|\Lambda^{-1} \Lambda^{-1} M(u, \partial u, v, \partial v)\|_{F^s} \leq \omega_1(\|u\|_{F^s}, \|v\|_{G^r}), \]

\[ \|\Lambda^{-1} \Lambda^{-1} N(u, \partial u, v, \partial v)\|_{G^r} \leq \omega_2(\|u\|_{F^s}, \|v\|_{G^r}), \]

where
and
\[\|A^{-1}_+ A^{-1}_-(\mathcal{M}(u, \partial u, v, \partial v))\|_{F^s} + \|A^{-1}_+ A^{-1}_-(\mathcal{N}(u, \partial u, v, \partial v))\|_{G^r} \leq \omega\|u\|_{F^s}, \|u'\|_{F^s}, \|v\|_{G^r}, \|v'\|_{G^r}(\|u - u'\|_{F^s} + \|v - v'\|_{G^r}),\]

where \(\omega, \omega_1, \omega_2\) are continuous functions with \(\omega(0, 0, 0, 0) = \omega_1(0, 0) = \omega_2(0, 0) = 0\).

**Proof.** This is proved by the contraction mapping principle provided the solution space fulfills suitable assumptions. The case of a single equation \(\Box u = \mathcal{M}(u, \partial u, v, \partial v)\) and the solution space \(X^s\) given by the norm \(\|u\|_{X^s} = \|A_+ u\|_{H^{s+1}} + \|A_+ A_+^2 u\|_{L^2_t L^2_x}, \gamma > 0\) small, was proven by [KS], Theorems 5.4 and 5.5, Propositions 5.6 and 5.7. Our case is a straightforward modification of their results, thus we omit the proof. We just remark that, if \(n = 3\), the only modification in the case of our solution space is the following estimate in the proof of [KS], Prop. 5.6:

\[\|A^{\frac{1}{2}+2\epsilon} L^{-\frac{1}{2}+\frac{1}{2}+3\epsilon} L^{\frac{1}{2}} u\|_{L_t^4 L_x^{14}} \lesssim \|A^{\frac{1}{2}+2\epsilon} L^{-\frac{1}{2}+\frac{1}{2}+3\epsilon} L^{\frac{1}{2}} u\|_{L_t^4 L_x^{14}} \lesssim \|A_+ A_+^{\frac{1}{2}+5\epsilon} L^{\frac{1}{2}+5\epsilon} u\|_{L_t^4 L_x^{14}} \lesssim \|A_+ A_+^{s-1} L^{s-1} u\|_{L_t^4 L_x^{14}}.\]

The first estimate follows from Corollary [3.3] and the last estimate holds by our assumption \(s > \frac{3}{4}\). \(\Box\)

4. **Reformulation of the problem and null structure**

The reformulation of the Yang-Mills equations and the reduction of our main theorem to nonlinear estimates is completely taken over from Tesfahun [Te] (cf. also the fundamental paper by Selberg and Tesfahun [ST]).

The standard null forms are given by
\[
\begin{align*}
Q_0(u, v) &= \partial_\alpha u \partial^\alpha v = -\partial_i u \partial_i v + \partial_i u \partial_i^1 v, \\
Q_{\alpha \beta}(u, v) &= \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.
\end{align*}
\]

For \(g\)-valued \(u, v\), define a commutator version of null forms by
\[
\begin{align*}
Q_0[u, v] &= [\partial_\alpha u, \partial^\alpha v] = Q_0(u, v) - Q_0(v, u), \\
Q_{\alpha \beta}[u, v] &= [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = Q_{\alpha \beta}(u, v) + Q_{\alpha \beta}(v, u).
\end{align*}
\]

Note the identity
\[
[\partial_\alpha u, \partial_\beta v] = \frac{1}{2} [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = \frac{1}{2} Q_{\alpha \beta}[u, v].
\]

Define
\[
Q[u, v] = -\frac{1}{2} Q_{jk} [\Lambda^{-1}(R^i u^k - R^k u^i), v] - Q_{0j} [R^i u_{0j}, v],
\]

where \(R_i = \Lambda^{-1} \partial_i\) is the Riesz transform.

We follow Tesfahun [Te] in the following generalizing his 3-dimensional results to arbitrary dimension \(n \geq 3\).
We split the spatial part $\mathbf{A} = (A_1, \ldots, A_n)$ of the potential into divergence-free and curl-free parts and a smoother part:

$$\mathbf{A} = A^\text{df} + A^\text{cf} + (\nabla)^{-2} \mathbf{A},$$

(23)

where

$$\begin{align*}
(A^\text{df})^j &= R^k(R_j A_k - R_k A_j), \\
(A^\text{cf})^j &= -R_j R_k A^k.
\end{align*}$$

**Lemma 4.1.** (cf. [16], Lemma 1) In the Lorenz gauge we have the identities

$$[A^\alpha, \partial_\alpha \phi] = Q[\Lambda^{-1} A^\alpha, \phi] + [\Lambda^{-2} A^\alpha, \partial_\alpha \phi],$$

(24)

$$[\partial_t A^\alpha, \partial_\alpha \phi] = Q_0[A^\alpha, \phi].$$

(25)

**Proof.** Writing

$$A^\alpha \partial_\alpha \phi = (-A_0 \partial_t \phi + A^\text{cf} \cdot \nabla \phi) + A^\text{df} \cdot \nabla \phi + \Lambda^{-2} \mathbf{A} \cdot \nabla \phi$$

one easily checks using the Lorenz gauge $\partial_t A_0 = \Lambda R_k A^k$:

$$\begin{align*}
A^\text{cf} \cdot \nabla \phi &= -R_j R_k A^k \partial^j \phi = -\partial_t \Lambda^{-1} R_j A_0 \partial^j \phi \\
A_0 \partial_t \phi &= -\Lambda^{-2} \partial_j \partial^j A_0 \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi \\
&= -\partial_j (\Lambda^{-1} R^j A_0) \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi,
\end{align*}$$

so that

$$-A_0 \partial_t \phi + A^\text{cf} \cdot \nabla \phi = -Q_0 (\Lambda^{-1} R^j A_0, \phi) - \Lambda^{-2} A_0 \partial_t \phi.$$

Next

$$\begin{align*}
A^\text{df} \cdot \nabla \phi &= R^k(R_j A_k - R_k A_j) \partial^j \phi \\
&= \Lambda^{-2} \partial^k \partial_j A_k \partial^j \phi + A_j \partial^j \phi \\
&= -\frac{1}{2} (\Lambda^{-2}(\partial_j \partial^j A_k - \partial_k \partial^j A_j) \partial^j \phi - \Lambda^{-2}(\partial^k \partial_j A_k - \partial_k \partial^j A_j) \partial^j \phi) \\
&= -\frac{1}{2} (\partial_j \Lambda^{-1}(R^j A_k - R_k A^j) \partial^j \phi - \partial^k \Lambda^{-1}(R_j A_k - R_k A_j) \partial^j \phi) \\
&= \frac{1}{2} Q_{jk}(\Lambda^{-1} (R^j A^k - R^k A^j), \phi).
\end{align*}$$

This leads to [24]. For [25] we use the Lorenz gauge to obtain

$$[\partial_t A^\alpha, \partial_\alpha \phi] = [-\partial_t A_0, \partial_\alpha \phi] + [\partial_t A^j, \partial_\alpha \phi] = -[\partial_t A^j, \partial_\alpha \phi] + [\partial_t A^j, \partial_\alpha \phi] = Q_0[A^\alpha, \phi].$$

□

**Lemma 4.2.** (cf. [16], Lemma 2) In the Lorenz gauge the following identity holds:

$$[A^\alpha, \partial_\beta A_\alpha] = \sum_{i=1}^4 \Gamma^i_\beta (A, \partial A, F, \partial F),$$

where

$$\begin{align*}
\Gamma^1_\beta &= -[A_0, \partial_\beta A_0] + [\Lambda^{-1} R_j (\partial_\alpha A_0), \Lambda^{-1} R^j \partial_\beta (\partial_\alpha A_0)], \\
\Gamma^2_\beta &= \sum_{i,j} Q_{ij} [\Lambda^{-1} R_k A^k, \Lambda^{-1} R_j \partial_\beta A_i] + \sum_{i,j} Q_{ij} [\Lambda^{-1} R_k \partial_\beta A^k, \Lambda^{-1} R_j A_i], \\
\Gamma^3_\beta &= \sum_j \left( [\Lambda^{-1} R^j F_{ji}, \Lambda^{-1} R^k \partial_\beta F_{jk}] + [\Lambda^{-1} R^j F_{ji}, \Lambda^{-1} \partial_\beta R^k [A_k, A_j]] \\
&\quad + [\Lambda^{-1} R^i [A_k, A_j], \Lambda^{-1} \partial_\beta R^k [A_k, A_j]] \right), \\
\Gamma^4_\beta &= [\Lambda^{-2} \mathbf{A}, \partial_\beta \mathbf{A}] + [\mathbf{A}, \Lambda^{-2} \partial_\beta \mathbf{A}].
\end{align*}$$
Proof. We write
\[ [A^\alpha, \partial_\beta A_\alpha] = -[A_0, \partial_\beta A_0] + [A^f, \partial_\beta A^f] + \sum_{i=1}^{4} \Gamma^i_\beta (A, \partial A, F, \partial F), \]
where
\[ \Gamma^1_\beta = -[A_0, \partial_\beta A_0] + [A^{cf}, \partial_\beta A^{cf}], \]
\[ \Gamma^2_\beta = [A^{cf}, \partial_\beta A^{df}], \]
\[ \Gamma^3_\beta = [A^{df}, \partial_\beta A^{df}] \]
and \( \Gamma^i_\beta \) as above.

For \( \Gamma^1_\beta \) we use \( \partial_t A_0 = \Lambda R_k A^k \) and obtain
\[ -A_0 \partial_\beta A_0 + A^{cf} \partial_\beta A^{cf} = -A_0 \partial_\beta A_0 + R_j R_k A^k \partial_\beta R^j R_k A^k \]
which gives the result. Concerning \( \Gamma^2_\beta \) we obtain
\[ A^{cf} \partial_\beta A^{df} = -R^i (R_k A^k) \partial_\beta R^i (R_j A_i - R_i A_j) \]
\[ = -R^i (R_k A^k) R^i (\partial_\beta R_i A_j) + R^i (R_k A^k) R^i (\partial_\beta R_j A_i) \]
\[ = \sum_{i,j} Q_{ij} (\Lambda^{-1} R_k A^k, \Lambda^{-1} R_i \partial_\beta A_i), \]
which gives the claimed result. For \( \Gamma^3_\beta \) we use
\[ F_{ji} := \partial_j A_i - \partial_i A_j + [A_j, A_i], \]
so that
\[ (A^{df})_j = R^i (R_j A_i - R_i A_j) = \Lambda^{-1} R^i F_{ji} + R^i (A_i A_j - A_j A_i). \]
This implies
\[ (A^{df})_j \partial_\beta A^{df} \]
\[ = \sum_j \Lambda^{-1} R^i F_{ji} \Lambda^{-1} R^k \partial_\beta F_{jk} + \Lambda^{-1} R^i F_{ji} \Lambda^{-1} \partial_\beta R^k (A_k A_j - A_j A_k) \]
\[ + \Lambda^{-1} R^i (A_i A_j - A_j A_i) \Lambda^{-1} \partial_\beta R^k F_{jk} \]
\[ + \Lambda^{-1} R^i (A_i A_j - A_j A_i) \Lambda^{-1} \partial_\beta R^k (A_k A_j - A_j A_k) \].
Thus we obtain the claimed result. □

Now we refer to Tesfahun [12], who showed that the system (3), (7) in Lorenz gauge can be written in the following form by use of Lemma 1.1, 21 and 22 for \( \Box_1 \) and Lemma 1.2 and Lemma 1.2 for \( \Box_2 \):
\[ \Box_1 A_\beta = M_\beta (A, \partial_t A, F, \partial_t F), \]
\[ \Box_2 F_{\beta \gamma} = N_{\beta \gamma} (A, \partial_t A, F, \partial_t F), \]
where
\[ M_\beta (A, \partial_t A, F, \partial_t F) = -2Q [\Lambda^{-1} A, A_\beta] + \sum_{i=1}^{4} \Gamma^i_\beta (A, \partial A, F, \partial F) - 2[\Lambda^{-2} A^\alpha, \partial_\alpha A_\beta] \]
\[ - [A^\alpha, [A_\alpha, A_\beta]], \]
\[
N_{ij}(A, \partial_t A, F, \partial_t F) = -2Q[\Lambda^{-1}A, F_{ij}] + 2Q[\Lambda^{-1}\partial_j A, A_i] - 2Q[\Lambda^{-1}\partial_i A, A_j] \\
+ 2Q_0[A_i, A_j] + Q_{ij}[A^0, A_0] - 2[\Lambda^{-2}A^0, \partial_j F_{ij}] \\
+ 2[\Lambda^{-2}\partial_j A^0, \partial_i A_j] - 2[\Lambda^{-2}\partial_i A^0, \partial_j A_j] \\
- [A^0, [A_i, F_{ij}]] + 2[\mathcal{F}_{\alpha i}, [A^0, A_j]] - 2[\mathcal{F}_{0j}, [A^0, A_i]] \\
- 2[[A^0, A_i], [A_0, A_j]],
\]

\[
N_{0i}(A, \partial_t A, F, \partial_t F) = -2Q[\Lambda^{-1}A, F_{0i}] + 2Q[\Lambda^{-1}\partial_i A, A_0] - 2Q_0[A^0, A_i] \\
+ 2Q_0[A_0, A_i] + Q_{0i}[A^0, A_0] - 2[\Lambda^{-2}A^0, \partial_i F_{0i}] \\
+ 2[\Lambda^{-2}\partial_i A^0, \partial_0 A_0] - [A^0, [A_0, F_{0i}]] + 2[\mathcal{F}_{50}, [A^0, A_i]] \\
- 2[\mathcal{F}_{0i}, [A^0, A_0]] - 2[[A^0, A_0], [A_0, A_i]]
\]

where \(\Gamma^i_\delta\) are defined in Lemma 12.

Now, looking at the terms in \(\mathcal{M}\) and \(\mathcal{N}\) and noting the fact that the Riesz transforms \(R_i\) are bounded in the spaces involved, the estimates in Proposition 3.16 reduce to proving (we remark, that due to the multilinear character of the nonlinearity the estimates for the difference can be treated exactly like the other estimates).

1. the corresponding estimates for the null forms \(Q_{ij} : Q_0\) and \(Q \in \{Q_{0i}, Q_j\}\) :

\[
\frac{\|A^{-1}\Lambda^{-1}Q[\Lambda^{-1}A, A]\|_{F'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (27)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}Q_{ij}[\Lambda^{-1}A, \Lambda^{-1}\partial A]\|_{F'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (28)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}Q[\Lambda^{-1}A, F]\|_{G'}}{\|A\|_{F'}\|F\|_{G'}} \lesssim (29)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}Q[A, A]\|_{G'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (30)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}Q_0[A, A]\|_{G'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (31)
\]

the following estimate for \(\Gamma^1\) and other bilinear terms

\[
\frac{\|A^{-1}\Lambda^{-1}\Gamma^1(A, \partial A)\|_{F'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (32)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A, A^{-2}\partial A)\|_{F'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (33)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}A, \Lambda^{-2}\partial A)\|_{F'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (34)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}F, \Lambda^{-1}\partial F)\|_{G'}}{\|F\|_{G'}\|F\|_{G'}} \lesssim (35)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}A, \partial F)\|_{G'}}{\|A\|_{F'}\|G\|_{G'}} \lesssim (36)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}A, \partial A)\|_{G'}}{\|A\|_{F'}\|A\|_{F'}} \lesssim (37)
\]

and

2. the following trilinear and quadrilinear estimates:

\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}F, \Lambda^{-1}\partial(AA))\|_{F'}}{\|F\|_{X', \frac{1}{2}, +}\|A\|_{F'}\|A\|_{F'}} \lesssim (38)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}\partial F, \Lambda^{-1}(AA))\|_{F'}}{\|F\|_{X', \frac{1}{2}, +}\|A\|_{F'}\|A\|_{F'}} \lesssim (39)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A^{-1}(AA), \Lambda^{-1}\partial(AA))\|_{F'}}{\|A\|_{F'}\|A\|_{F'}\|A\|_{F'}} \lesssim (40)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A, A, A)\|_{F'}}{\|A\|_{F'}\|A\|_{F'}\|A\|_{F'}} \lesssim (41)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A, A, F)\|_{G'}}{\|A\|_{F'}\|A\|_{F'}\|F\|_{G'}} \lesssim (42)
\]
\[
\frac{\|A^{-1}\Lambda^{-1}\Pi(A, A, A)\|_{G'}}{\|A\|_{F'}\|A\|_{F'}\|A\|_{F'}} \lesssim (43)
\]

where \(\Pi(\cdot, \cdot)\) denotes a multilinear operator in its arguments.
The matrix commutator null forms are linear combinations of the ordinary ones, in view of (23). Since the matrix structure plays no role in the estimates under consideration, we reduce (27) to estimates to the ordinary null forms for $\mathbb{C}$-valued functions $u$ and $v$ (as in (19)).

The null forms above satisfy the following estimates.

Lemma 4.3. The following estimates hold for $0 \leq \alpha \leq 1$ and $Q = Q_0$ or $Q = Q_{ij}$:

\[
Q_0(u, v) \leq D_+^{1-\alpha} D_-^{1-\alpha} (D_+^a u D_-^b v) + (D_+ D_-^{1-\alpha} u) (D_-^a v) + (D_+^a u) (D_+ D_-^{1-\alpha} v)
\]  

(44)

\[
Q_0(u, v) \leq D_+^{1-\alpha} (D_+ u D_-^a v) + D_-^{1-\alpha} (D_+^a u D_+ v) + (D_+ D_-^{1-\alpha} u) (D_-^a v) + (D_+^a u) (D_+ D_-^{1-\alpha} v)
\]  

(45)

\[
Q(u, v) \leq D_+^{1-2\alpha} D_-^{1-2\alpha} (D_+^a u D_-^b v) + D_+^{1-2\alpha} (D_+^a u D_-^b v) + D_-^{1-2\alpha} (D_+^a u D_-^b v)
\]  

(46)

\[
Q(u, v) \leq D_+^{1-2\alpha} D_-^{1-2\alpha} (D_+^a u D_-^b v) + D_+^{1-2\alpha} (D_+^a u D_-^b v) + (D_+ D_-^{1-2\alpha} u) (D_-^a v) + (D_+^a u) (D_+ D_-^{1-2\alpha} v)
\]  

(47)

\[
Q(u, v) \leq D_+^{1-2\alpha} D_-^{1-2\alpha} (D_+^a u D_-^b v) + D_+^{1-2\alpha} (D_+^a u D_-^b v) + (D_+ D_-^{1-2\alpha} u) (D_-^a v) + (D_+^a u) (D_+ D_-^{1-2\alpha} v)
\]  

(48)

Proof. (44) is Lemma 7.6 in [KS], and (45) follows immediately from [KMBT], Prop. 1. (46) follows by interpolating the estimate for the symbol (47) follows by (46) with its trivial bound (48) and (49) follow by the fractional Leibniz rule for $\Lambda_+$ and $D_+$ from (44) and (47), respectively.

Next we consider the term $\Gamma_1^1$. We may ignore its matrix form and treat

\[
\Gamma_1^1(A_0, \partial_t A_0) = -A_0(\partial_t A_0) + \Lambda^{-1} R_j(\partial_t A_0) \Lambda^{-1} R^i \partial_i(\partial_k A_0)
\]

for $k = 1, \ldots, n$ and

\[
\Gamma_0^1(A_0, \partial^i A_i) = -A_0(\partial_i A_0) + \Lambda^{-1} R_j(\partial_i A_0) \Lambda^{-1} R^k \partial_k(\partial_j A_0)
\]

(49)

(50)

where we used the Lorenz gauge $\partial_0 A_0 = \partial^i A_i$ in the last line in order to eliminate one time derivative. Thus we have to consider

\[
\Gamma^1(u, v) = -uv + \Lambda^{-1} R_j(\partial_t u) \Lambda^{-1} R^i \partial_i(v)
\]

(49)

where $u = A_0$ and $v = \partial^i A_i$ or $v = \partial_t A_0$.

The proof of the following theorem was essentially given by Tesfahun [TS]. In fact the detection of this null structure was the main progress of his paper over Selberg-Tesfahun [ST].

Lemma 4.4. The following estimates hold:

\[
\Gamma^1(u, v) \leq \Gamma_1^1(u, v) + \Gamma_2^1(u, v) + (\Lambda^{-2} u) v + u(\Lambda^{-2} v)
\]

(49)

\[
\Gamma^1(u, v) \leq uv + \Gamma_2^1(u, v)
\]

(50)
Proof. The symbol $I$ is controlled by $\Gamma$ for $0$

Thus we obtain (49) and using the trivial bound

We assume in the following that the Fourier transforms

The norms involved in the desired estimates do only depend on the size of the Fourier

We have

Now we estimate

where $\angle(\xi, \eta)$ denotes the angle between $\xi$ and $\eta$. We have

and

for $0 \leq \epsilon \leq \frac{1}{4}$ by [KMBT]. Proof of proposition 1. Thus the operator belonging to the symbol $I$ is controlled by $\Gamma^1_1(u, v) + (\Lambda^{-2} u) v + u(\Lambda^{-2}) v$. Moreover

This is the symbol of $Q_0(\Lambda^{-1} u, \Lambda^{-1} v)$, which is controlled by $\Gamma^1_2(u, v)$ by (14).

Thus we obtain (49) and using the trivial bound $|I| \lesssim 1$ also (50). Finally, (53)

follows by the fractional Leibniz rule for $D_+$ from (52).

5. Proof of the nonlinear estimates in the case $n \geq 4$

Important remark: We assume in the following that the Fourier transforms of $u$ and $v$ are nonnegative. This means no loss of the generality, because the norms involved in the desired estimates do only depend on the size of the Fourier transforms.

Proof of (57). We recall (15) for $\alpha = \epsilon$:

$$Q_0(u, v) \preceq D^1_{+} \epsilon (D_{-} u D_{+} v) + D^1_{+} \epsilon (D_{+} u D_{-} v) + (D_{+} D_{-}^1 u)(D_{+} v) + (D_{+} u)(D_{-} D_{-}^1 v).$$
Thus we have to show the following estimates and remark that we only have to consider the first and third term, because the other terms are equivalent by symmetry.

1. For the first term it suffices to show
\[
\|\Lambda_+^{-1} \Lambda_+^{-1+\epsilon} \Lambda_+^{-1} u + D_+ \Lambda_+^{-1+\epsilon} (D_+ u D_+^2 v)\|_{H^n} \lesssim \|\Lambda_+^{-1} u\|_{H^0}^{\frac{1}{2} + \epsilon} \|\Lambda_+^{-1} v\|_{H^n}^{\frac{1}{2} + \epsilon}.
\]
This follows from
\[
\|uv\|_{H^{-1+\epsilon}} \lesssim \|u\|_{H^{-1+\epsilon}} \|v\|_{H^{-1+\epsilon}},
\]
which is a consequence of Prop. 3.8 if \(2s - r - \epsilon > \frac{3}{2}\) and \(s \geq r\). This is fulfilled for a sufficiently small \(\epsilon > 0\) under our assumptions.

2. For the third term we show
\[
\|\Lambda_+^{-1} \Lambda_+^{-1+\epsilon} \Lambda_+^{-1} u D_+ D_+^{-1+\epsilon} u D_+ D_+^2 v\|_{H^n} \lesssim \|\Lambda_+^{-1} u\|_{H^0}^{\frac{1}{2} + \epsilon} \|\Lambda_+^{-1} v\|_{H^n}^{\frac{1}{2} + \epsilon}.
\]
This follows from
\[
\|uv\|_{H^{-1+\epsilon}} \lesssim \|u\|_{H^{-1+\epsilon}} \|v\|_{H^{-1+\epsilon}},
\]
which is a consequence of Prop. 3.8 as in 1. under the same assumptions. \(\square\)

**Proof of (30).** We use (47). By symmetry we only have to consider the first two terms.

1. For the first term it suffices to show
\[
\|\Lambda_+^{-1} \Lambda_+^{-1+\epsilon} \Lambda_+^{-1} u + \Lambda_+^{\frac{1}{2} - 2\epsilon} \Lambda_+^{\frac{1}{2} - 2\epsilon} (\Lambda_+^{\frac{1}{2} + 2\epsilon} u \Lambda_+^{\frac{1}{2} + 2\epsilon})\|_{H^n} \lesssim \|\Lambda_+^{-1} u\|_{H^0}^{\frac{1}{2} + \epsilon} \|\Lambda_+^{-1} v\|_{H^n}^{\frac{1}{2} + \epsilon}.
\]
This follows from
\[
\|uv\|_{H^{-1+\epsilon}} \lesssim \|u\|_{H^{-1+\epsilon}} \|v\|_{H^{-1+\epsilon}},
\]
which is a consequence of Prop. 3.6 with parameters \(s_0 = \frac{1}{2} - r + 2\epsilon\), \(s_1 = s_2 = s - \frac{1}{2} - 2\epsilon\), so that \(s_0 + s_1 + s_2 > \frac{5}{2}\), if \(2s - r > \frac{3}{2}\), and \(s_0 + s_1 + s_2 + s_1 + s_2 > \frac{7}{2}\), which holds under our assumptions.

2. For the second term we show
\[
\|\Lambda_+^{-1} \Lambda_+^{-1+\epsilon} \Lambda_+^{-1} u + \Lambda_+^{\frac{1}{2} - 2\epsilon} \Lambda_+^{\frac{1}{2} - 2\epsilon} (\Lambda_+^{\frac{1}{2} + 2\epsilon} u \Lambda_+^{\frac{1}{2} + 2\epsilon})\|_{H^n} \lesssim \|\Lambda_+^{-1} u\|_{H^0}^{\frac{1}{2} + \epsilon} \|\Lambda_+^{-1} v\|_{H^n}^{\frac{1}{2} + \epsilon}.
\]

Using \(\Lambda_+^{-1} u \lesssim \Lambda_+^{-1} u\) it suffices to show
\[
\|uv\|_{H^{-\frac{1}{2} + 2\epsilon}} \lesssim \|u\|_{H^{-\frac{1}{2} + 2\epsilon}} \|v\|_{H^{-\frac{1}{2} + 2\epsilon}},
\]
which is a consequence of Prop. 3.6 as in 1. under the assumptions \(2s - r > \frac{3}{2}\) and \(3s - 2r > \frac{3}{2}\), which hold under our assumptions. \(\square\)

**Proof of (35).** A. We start with the first part of the \(F^*\)-norm. As before it is easy to see that we can reduce to
\[
\|uv\|_{H^{-1+\epsilon}} \lesssim \|u\|_{H^{1+\epsilon}} \|v\|_{H^{1+\epsilon}}.
\]
Thus we have to show the following estimates
\[
\|uv\|_{H^{0+\epsilon} \lesssim \|u\|_{H^{1+\epsilon}} \|v\|_{H^{1+\epsilon}}, \quad \|uv\|_{H^{0+\epsilon} \lesssim \|u\|_{H^{1+\epsilon}} \|v\|_{H^{1+\epsilon}}.}
\]
1. If $2r - s > \frac{3}{2} - \frac{5}{2}$ both estimates are fulfilled by Proposition 3.6.

2. Assume now that $2r - s \leq \frac{3}{2} - \frac{5}{2}$ and that our assumption $2r - s > \frac{3}{2} - \frac{5}{2} - 3$ is fulfilled. We want to apply Corollary 3.3 with the parameters $\alpha_1 = r - s + 2$, $\alpha_2 = r$ for (53), and $\alpha_1 = r + 1$, $\alpha_2 = r - s + 1$ for (53). We check the conditions of that Corollary in either case.

- (14): The condition $\alpha_1 + \alpha_2 = 2r - s + 2 = \frac{2}{q} - \frac{3}{2}$ can be met with a suitable $1 < q \leq 2$ under our assumption $\frac{2}{q} - \frac{3}{2} \geq 2r - s > \frac{3}{2} - 3$.

- (15): Using this $q$ the condition $\frac{1}{q} < \frac{n - 1}{2}$ is equivalent to $2r - s > \frac{3}{2} - \frac{7}{2}$. If $n = 4$ this means $2r - s > -\frac{3}{4}$, whereas for $n \geq 5$ it is automatically fulfilled under our assumption $2r - s > \frac{3}{2} - 3$.

- (16): We need using (14) $r + 1 < \frac{2}{q} + \frac{1}{2} - \frac{3}{2} = \frac{2}{q} + \frac{1}{2} - n + 4 + 4r - 2s$, which is equivalent to our assumption $3r - s > \frac{3}{2} - \frac{5}{2}$.

B. For the second part of the $F^s$-norm we reduce to

$$\|\Lambda^{-\frac{1}{2}} \Lambda^{-\frac{1}{2} + 2r} \Lambda^{-\frac{s}{2} + 3r} \Lambda^{-\frac{1}{2} + t} (uv)\|_{L^2_t L^2_x} \lesssim \|\Lambda_+ u\|_{H^{s-\frac{1}{2} + t}} \|\Lambda_+ v\|_{H^{s-\frac{1}{2} + t}},$$

and further to

$$\|\Lambda^{-\frac{1}{2} + 7r} \Lambda^{-\frac{1}{2} - \epsilon} (uv)\|_{L^2_t L^2_x} \lesssim \|u\|_{H^{s+1} \frac{1}{2} + \epsilon} \|v\|_{H^{s+1} \frac{1}{2} + \epsilon},$$

By Prop. 3.13 we obtain

$$\|\Lambda^{-\frac{1}{2} + 7r} \Lambda^{-\frac{1}{2} - \epsilon} (uv)\|_{L^2_t L^2_x} \lesssim \|\Lambda^{-2 + 7r} (uv)\|_{L^2_t L^2_x},$$

which by the fractional Leibniz rule, Sobolev and Prop. 3.12 can be estimated as follows:

$$\|(\Lambda^{-2 + 7r} u) v\|_{L^2_t L^2_x} \lesssim \|\Lambda^{-2 + 7r} u\|_{L^2 L^2_x} \|v\|_{L^2 L^2_x} \lesssim \|u\|_{H^{s+1} \frac{1}{2} + \epsilon} \|v\|_{H^{s+1} \frac{1}{2} + \epsilon},$$

$$\|u \Lambda^{-2 + 7r} v\|_{L^2_t L^2_x} \lesssim \|u\|_{L_t^2 L_x^\infty} \|\Lambda^{-2 + 7r} v\|_{L^2 L_x^\infty} \lesssim \|u\|_{H^{s+1} \frac{1}{2} + \epsilon} \|v\|_{H^{s+1} \frac{1}{2} + \epsilon}.$$

□

Proof of (56). A. For the first part of the $F^s$-norm it is sufficient to show

$$\|\Gamma^1(u,v)\|_{H^{s-1} \frac{1}{2} + 2r} \lesssim \|u\|_{F^r} \|\Lambda_+ v\|_{H^{s-2} \frac{1}{2} + r}.$$  

(56)

We use Lemma 4.1

a. We first consider $\Gamma^1_1(u,v)$. By (53) it suffices to show the following estimates, all of which are consequences of Proposition 3.6 and Corollary 3.1.

$$\|D_3 \Lambda^{-1} u D_3^{\frac{1}{2} + 2r} \Lambda^{-1} v\|_{H^{s-1} 0} \lesssim \|\Lambda_+ u\|_{H^{s-1} \frac{1}{2} + \epsilon} \|\Lambda_+ v\|_{H^{s-2} \frac{1}{2} + \epsilon},$$

which follows from

$$\|uv\|_{H^{s-1} 0} \lesssim \|u\|_{H^{s-1} \frac{1}{2} + \epsilon} \|v\|_{H^{s-2} \frac{1}{2} + \epsilon}$$

and

$$\|uv\|_{H^{s-1} 0} \lesssim \|u\|_{H^{s-1} \frac{1}{2} + 2r} \|v\|_{H^{s-1} \frac{1}{2} + 2r},$$

$$\|uv\|_{H^{s-1} \frac{1}{2} + 2r} \lesssim \|u\|_{H^{s} 0} \|v\|_{H^{s-1} \frac{1}{2} + 2r},$$

$$\|uv\|_{H^{s-1} \frac{1}{2} + 2r} \lesssim \|u\|_{H^{s+1} \frac{1}{2} + 2r} \|v\|_{H^{s-1} 0}.$$
b. Assume that $u$ and $v$ have frequencies $\geq 1$ , so that $\Lambda^s u \sim D^s u$. In this case we use (19) and consider $\Gamma^1(u,v)$ . By (21) we may reduce estimates for the first and third terms on the right hand side to

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$

By the fractional Leibniz rule we have to show the following two estimates:

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|\Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{\frac{1}{2}-2\epsilon} u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}. \tag{57}$$

Both estimates follow from Corollary 3.1. The second term is reduced to the following estimate

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|\Lambda^{\frac{1}{2}-2\epsilon} u\|_{H^{s-1}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}. \tag{58}$$

By the fractional Leibniz rule we have to show the following two estimates:

b1.

$$\|(\Lambda^{s-\frac{1}{2}-2\epsilon})^u v\|_{H^{s+\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}},$$

which follows from Corollary 3.1.

b2.

$$\|u(\Lambda^{s-\frac{1}{2}-2\epsilon})^v\|_{H^{s\frac{1}{2}+2\epsilon}} \lesssim \|\Lambda^{s-\frac{1}{2}-2\epsilon} \Lambda^{s-\frac{1}{2}-2\epsilon} u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$

Using the definition of the $F^n$-norm this is implied by

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim (\|\Lambda^{s-\frac{1}{2}-2\epsilon} \Lambda^{s-\frac{1}{2}-2\epsilon} u\|_{L^2} L^{\infty} + \|\Lambda^{s-\frac{1}{2}-2\epsilon} u\|_{H^{s-2\epsilon}}) \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}. \tag{58}$$

By Lemma 3.1 we have for $\alpha, \beta > 0$:

$$\|uv\|_{H^{\alpha-\beta}} \lesssim \|\Lambda^\beta u\|_{H^{\alpha}} + \|\Lambda^\alpha u\|_{H^{\alpha}},$$

for $u, v$ with $\hat{u}, \hat{v} \geq 0$. Thus for the choice $\beta = \frac{1}{2} - 2\epsilon$ , $\alpha = \frac{1}{2} - \epsilon$ the estimate (58) reduces to the following two estimates:

b2.1.

$$\|(\Lambda^{s-\frac{1}{2})}^u v\|_{H^{s+\frac{1}{2}+2\epsilon}} \lesssim \|\Lambda^{s-\frac{1}{2}} u\|_{H^{s+\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$ 

This follows by Sobolev, because $s > \frac{1}{2} - 1$.

b2.2.

$$\|(\Lambda^{s-\frac{1}{2})}^u v\|_{H^{s+\frac{1}{2}+2\epsilon}} \lesssim \|\Lambda^{s-\frac{1}{2}} u\|_{L^2} L^{\infty} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$ 

This follows from Prop. 3.12 and Prop. 3.10.

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|uv\|_{L^2} \lesssim \|u\|_{L^2} \|v\|_{L^\infty} \lesssim \|\Lambda^{s-\frac{1}{2}} u\|_{L^2} L^{\infty} \|u\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$ 

c. Consider $(\Lambda^{s+\frac{1}{2})} u$ and $u(\Lambda^{-s})^v$. It suffices to show

$$\|(\Lambda^{-s})^v\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$ 

Both follow easily from Prop. 3.7 under our assumption $s > \frac{1}{2} - 1$.

d. Let us finally consider the case where the frequencies of $u$ or $v$ are $\leq 1$. We use (20) instead of (19). Because $\Gamma^1(u,v)$ has already been handled, we only have to consider $uv$. If $u$ has low frequencies we obtain by Sobolev’s multiplication law (5.7):

$$\|uv\|_{H^{s-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{s-\frac{1}{2}+2\epsilon}} \|v\|_{H^{s-\frac{1}{2}+2\epsilon}}.$$ 

Similarly we treat the case where $v$ has low frequencies.
B. Now we consider the second part of the $F^s$-norm. We want to show
\[
\|\Lambda_+^{-1} \Lambda_+^{-1 + \epsilon} \Lambda_+^{\frac{1}{3} + 2\epsilon} \Lambda_+^{\frac{1}{3} + 3\epsilon} \Lambda_+^\epsilon \|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\] (59)

We use Lemma 4.3.4.

1. We first consider $\Gamma_1(u, v)$.

The estimate for the first term on the right hand side of (52) reduces to
\[
\|\Lambda_+^{-1} \Lambda_+^{-1 + \epsilon} \Lambda_+^{\frac{1}{3} + 2\epsilon} \Lambda_+^{\frac{1}{3} + 3\epsilon} \Lambda_+^\epsilon \|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\]

and therefore we only have to prove
\[
\|\Lambda_+^{-\frac{1}{3} + 3\epsilon} (uv)\|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\] (60)

This follows from Proposition 3.3.5 with parameters $q = 2$, $r = 8\eta$, $\sigma = \frac{3}{2} - 3\epsilon$, $s_1 = \frac{3}{2} - \frac{5}{6} - \epsilon$ and $s_2 = \frac{3}{2} - \frac{1}{6} + 5\epsilon$. The claimed estimate follows, because $s_1 < s + \frac{1}{2} - 2\epsilon$ and $s_2 < s - \frac{1}{2} - 2\epsilon$ under our assumption $s > \frac{5}{6} - \frac{7}{6} \epsilon$.

2. For the second term we have to show
\[
\|\Lambda_+^{-\frac{1}{3} + 5\epsilon} \Lambda_+^{-\frac{1}{3} + \epsilon} \|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon, 0}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\]

which by Lemma 3.1 reduces to
\[
\|\Lambda_+^{-\frac{1}{3} + 5\epsilon} \Lambda_+^{-\frac{1}{3} + \epsilon} (uv)\|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon, 0}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\]

We obtain by Proposition 3.3.13
\[
\|\Lambda_+^{-\frac{1}{3} + 5\epsilon} \Lambda_+^{-\frac{1}{3} + \epsilon} (uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda_+^{-\frac{1}{2} - 2\epsilon} (uv)\|_{L^1_t L^2_x},
\]

so that by the fractional Leibniz rule this requires the following estimates:

2.1.
\[
\|u \Lambda_+^{-\frac{1}{2} - 2\epsilon, 0} \|_{L^1_t L^2_x} \lesssim \|u\|_{L^2_t L^4_x} \|\Lambda_+^{-\frac{1}{2} - 2\epsilon} \|_{L^2_t L^\infty_x}.
\]

By Sobolev the first factor is bounded by $\|u\|_{H^{\frac{1}{2} - 2\epsilon}}$ for $s > \frac{5}{6} - 1$. For the second factor we use Proposition 3.2.2 which gives
\[
\|v\|_{L^2_t L^4_x} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{6} \epsilon, \frac{1}{6} + \epsilon}},
\]

thus
\[
\|\Lambda_+^{-\frac{1}{2} - 2\epsilon} \|_{L^2_t L^\infty_x} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{6} \epsilon, \frac{1}{6} + \epsilon}} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{6} \epsilon, \frac{1}{6} + \epsilon}},
\]

because $s > \frac{5}{6} - \frac{7}{6} \epsilon$ and $n \geq 4$.

2.2.
\[
\|u \Lambda_+^{-\frac{1}{2} - 2\epsilon, 0} u\|_{L^1_t L^2_x} \lesssim \|u\|_{L^2_t L^4_x} \|\Lambda_+^{-\frac{1}{2} - 2\epsilon} \|_{L^2_t L^\infty_x} \|v\|_{L^2_t L^4_x}.
\]

The first factor is bounded by Sobolev by $\|u\|_{H^{\frac{1}{2} - 2\epsilon}}$ for $s > \frac{5}{6} - 1$ and the second factor in the case $n \geq 5$ by Proposition 3.1
\[
\|v\|_{L^2_t L^4_x} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{6} \epsilon, \frac{1}{6} + \epsilon}} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{6} \epsilon, \frac{1}{6} + \epsilon}},
\]

whereas in the case $n = 4$ we have to use (11), which implies
\[
\|v\|_{L^2_t L^4_x} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{4} \epsilon, \frac{1}{4} + \epsilon}} \lesssim \|v\|_{H^{\frac{1}{2} - \frac{1}{4} \epsilon, \frac{1}{4} + \epsilon}},
\]

under our assumption $s > \frac{5}{8}$.

3.
\[
\|\Lambda_+^{-\frac{1}{3} + 5\epsilon} \Lambda_+^{-\frac{1}{3} + \epsilon} (uv)\|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon, 0}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\]

By Lemma 3.3 this reduces to the following estimate:
\[
\|\Lambda_+^{-\frac{1}{3} + 5\epsilon} \Lambda_+^{-\frac{1}{3} + \epsilon} (uv)\|_{L^1_t L^2_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon, 0}} \|\Lambda_+ v\|_{H^{\frac{1}{2} + 2\epsilon}}.
\] (62)
We obtain by Proposition \ref{prop:3.13}
\[
\|A^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}-\epsilon}(uv)\|_{L^1_tL^2_x} \lesssim \|A^{-\frac{5}{2}-2+5\epsilon}(uv)\|_{L^1_tL^2_x},
\]
so that by Leibniz’ rule we argue as follows:

1. By Sobolev and Strichartz (Proposition \ref{prop:3.1}) we estimate
\[
\|\left(A^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(uv)\right)\|_{L^1_tL^2_x} \lesssim \|A^{-\frac{5}{2}-2+5\epsilon}u\|_{L^1_tL^2_x} \lesssim \|u\|_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} ||v||_{H^{\frac{3}{2}+2\epsilon,0}},
\]
so that the desired estimate follows for \(s > \frac{9}{2} - 1\).

2. Using Proposition \ref{prop:3.1} again we obtain
\[
\|uA^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(uv)\|_{L^1_tL^2_x} \lesssim \|u\|_{L^1_tL^2_x} \|A^{-\frac{5}{2}-2+5\epsilon}v\|_{L^1_tL^2_x} \lesssim \|u\|_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} ||v||_{H^{\frac{3}{2}+2\epsilon,0}},
\]
which is sufficient for \(s > \frac{9}{2} - 1\).

b1. In the case where \(v\) and \(v\) have frequencies \(\geq 1\) we use \eqref{eq:33} and consider \(G'\).

1. The estimate for the first term on the right hand side of \eqref{eq:64} reduces to \eqref{eq:65}.

2. The second term on the right hand side of \eqref{eq:64} reduces to
\[
\|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x} \lesssim \|u\|_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} ||\Lambda^{-\frac{1}{2}+3\epsilon}v||_{L^1_tL^2_x} ||v||_{H^{\frac{3}{2}+2\epsilon,0}}.
\]

By Lemma \ref{lem:3.1} this requires
\[
\|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x} \lesssim \|u\|_{H^{\frac{3}{2}+2\epsilon,0}} \|\Lambda^{-\frac{1}{2}+3\epsilon}v||_{L^1_tL^2_x} ||v||_{H^{\frac{3}{2}+2\epsilon,0}}.
\]

We start with Proposition \ref{prop:3.13}
\[
\|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x} \lesssim \|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x}
\]
and use the fractional Leibniz rule to reduce to

\[
||\left(A^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\right)\|_{L^1_tL^2_x} \lesssim \|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x}
\]

This is implied by Sobolev and a direct application of Proposition \ref{prop:3.1} which is possible for \(n \geq 5\), as one easily checks. In the case \(n = 4\) we use Prop. \ref{prop:3.12} which gives
\[
\|\Lambda^{-\frac{1}{2}+2\epsilon}v\|_{L^1_tL^2_x} \lesssim ||v||_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} \lesssim ||v||_{H^{\frac{3}{2}+2\epsilon,0}},
\]
under our assumption \(s > \frac{9}{2}\).

2.2. By use of Prop. \ref{prop:3.10} for the first step and Cor. \ref{cor:3.12} for the second step we obtain
\[
\|\left(A^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\right)\|_{L^1_tL^2_x} \lesssim \|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x}
\]
\[
\lesssim \|\Lambda^{-\frac{1}{2}+2\epsilon}u\|_{L^1_tL^2_x} \|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x}
\]
\[
\lesssim \|\Lambda^{-\frac{1}{2}+3\epsilon}u\|_{L^1_tL^2_x} ||v||_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} \lesssim \|\Lambda^{-\frac{1}{2}+3\epsilon}u\|_{L^1_tL^2_x} ||v||_{H^{\frac{3}{2}+2\epsilon,0}}.
\]

3. The estimate \eqref{eq:65} has to be replaced by
\[
\|\Lambda^{-\frac{1}{2}+3\epsilon}A^{-\frac{1}{2}+2\epsilon}(u\Lambda^{-\frac{1}{2}+2\epsilon}v)\|_{L^1_tL^2_x} \lesssim \|u\|_{H^{\frac{3}{2}+2\epsilon,\frac{1}{2}}} ||v||_{H^{\frac{3}{2}+2\epsilon,0}}.
\]
We argue similarly as for 2. We first apply (51) and reduce to the following estimates:

3.1. By Sobolev and Prop.3.1 we obtain

\[ \| (\Lambda + \frac{5}{2} + 3\epsilon) \Lambda^{-\frac{1}{2} + 2\epsilon} u (\Lambda^{-\frac{1}{2} + 2\epsilon} v) \|_{L^1_t L_x^2} \]
\[ \lesssim \| (\Lambda + \frac{5}{2} + 3\epsilon) \Lambda^{-\frac{1}{2} + 2\epsilon} u \|_{L^3_t L^\infty_x} \| (\Lambda + \frac{5}{2} + 2\epsilon) v \|_{L^2_t L^\infty_x} \]
\[ \lesssim \| u \|_{H^{s-1} + \frac{1}{2} + 2\epsilon} \| v \|_{H^{s-2} + 2\epsilon} . \]

3.2. Similarly

\[ \| (\Lambda + \frac{5}{2} + 2\epsilon) u (\Lambda + \frac{5}{2} + 3\epsilon) \Lambda^{-\frac{1}{2} + 2\epsilon} v \|_{L^1_t L_x^2} \]
\[ \lesssim \| (\Lambda + \frac{5}{2} + 2\epsilon) u \|_{L^3_t L^\infty_x} \| (\Lambda + \frac{5}{2} + 3\epsilon) \Lambda^{-\frac{1}{2} + 2\epsilon} v \|_{L^2_t L^\infty_x} \]
\[ \lesssim \| u \|_{H^{s-1} + \frac{1}{2} + 2\epsilon} \| v \|_{H^{s-2} + 2\epsilon} . \]

b2. Now we consider the case where \( u \) and/or \( v \) have frequency \( \leq 1 \) and use (50). It remains to consider the first term \( uv \). Thus it suffices to show

\[ \| \Lambda^{-\frac{1}{2} + \epsilon} u (\Lambda + \frac{5}{2}) \Lambda^{-\frac{1}{2} + 3\epsilon} \Lambda^{-\frac{1}{2}} (uv) \|_{L^1_t L_x^2} \lesssim \| u \|_{H^{s-1} + \epsilon} \| v \|_{H^{s-2} + \epsilon} . \]

We crudely estimate the left hand side by

\[ \| \Lambda^{-\frac{1}{2} + 5\epsilon} (uv) \|_{L^1_t L_x^2} \lesssim \| u \|_{H^{s+1} + \epsilon} \| v \|_{H^{s+1} + \epsilon} \]

by use of Proposition 3.3, where \( s_1 \) and \( s_2 \) have to fulfill \( s_1 + s_2 \geq n - \frac{5}{2} + 5\epsilon \) and \( s_1, s_2 < \frac{n}{2} - \frac{3}{2} \). If \( u \) has frequency \( \leq 1 \) choose \( s_1 = \frac{n}{2} - \frac{5}{2} - 6\epsilon, s_2 = \frac{n}{2} - \frac{3}{2} - 6\epsilon < s - 1 \). The value of \( s_1 \) is irrelevant in view of the low frequency assumption. If \( v \) has frequency \( \leq 1 \) choose \( s_1 = \frac{n}{2} - 1 < s, s_2 = \frac{n}{2} - \frac{3}{2} + 5\epsilon \), where the value of \( s_2 \) is irrelevant.

c. According to Lemma 3.4 we finally have to consider \( (\Lambda^{-2} u)v \) and \( u (\Lambda^{-2} v) \). We have to show

\[ \| \Lambda^{-1 + \epsilon} u^2 \Lambda^{\frac{1}{2} + 2\epsilon} \Lambda^{\frac{1}{2}} (u^2) \|_{L^1_t L_x^2} \lesssim \| u \|_{H^{s+2} + \epsilon} \| v \|_{H^{s-1} + \epsilon} . \]

We argue as in 4. choosing \( s_1 = \frac{n}{2} - \frac{5}{2} - \epsilon < s + 2 \) and \( s_2 = \frac{n}{2} - \frac{5}{2} + 6\epsilon < s - 1 \). In the same way we also obtain

\[ \| \Lambda^{-1 + \epsilon} u^2 \Lambda^{\frac{1}{2} + 2\epsilon} \Lambda^{\frac{1}{2}} (u^2) \|_{L^1_t L_x^2} \lesssim \| u \|_{H^{s+2} + \epsilon} \| v \|_{H^{s+1} + \epsilon} . \]

by the choice \( s_1 = \frac{n}{2} - 1 < s, s_2 = \frac{n}{2} - \frac{5}{2} + 5\epsilon < s + 1 \). The proof of (52) is now complete.

Proof of (27) and (58). We have to prove

\[ \| \Lambda^{-1 + \epsilon} F(u, v) \|_{F^s} \lesssim \| u \|_{F^s} \| v \|_{H^{s-1} + \epsilon} . \]

We want to use (17).

A. The first part of the \( F^s \)-norm is handled as follows:

1. For the first term we reduce to the estimate

\[ \| uv \|_{H^{s-1} + \frac{1}{2} - 2\epsilon} \lesssim \| u \|_{H^{s} + \frac{1}{2} - 2\epsilon} \| v \|_{H^{s-1} + \frac{1}{2} - 2\epsilon}. \]

which follows by Prop. 3.6.

2. For the last term we reduce to

\[ \| u v \|_{H^{s-1} + \frac{1}{2} - 2\epsilon} \lesssim \| u \|_{H^{s} + \frac{1}{2} - 2\epsilon} \| v \|_{H^{s-1} + \frac{1}{2} - 2\epsilon}. \]
which holds by Prop. 3.5.6
3. For the second term we want to prove

\[ \|uv\|_{H^{\frac{1}{2}-\epsilon} - \frac{1}{2} + \epsilon} \lesssim \|\Lambda^{\frac{1}{2}-2\epsilon} \Lambda^{-\frac{1}{2}+2\epsilon} u\|_{F^s} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}, \]

which follows from

\[ \|uv\|_{H^{\frac{1}{2}-2\epsilon} - \frac{1}{2} + \epsilon} \lesssim (\|u\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} + \|\Lambda^{\frac{1}{2}+3\epsilon} u\|_{L_t^1 L_x^2}) \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

By Lemma 3.5.2 this reduces to the following two estimates:

\[ \|uv\|_{H^{\frac{1}{2}-2\epsilon} - \frac{1}{2} + \epsilon} \lesssim (\|u\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} + \|\Lambda^{\frac{1}{2}+\epsilon} u\|_{L_t^1 L_x^2}) \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}, \]

\[ \|uv\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} \lesssim \|u\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

The last estimate follows immediately from Prop. 3.7. The first estimate is handled by the fractional Leibniz rule:

3.1. By Sobolev we have

\[ \|u^\Lambda^{\frac{1}{2}-2\epsilon} v\|_{H^{\frac{1}{2}-\epsilon} - \frac{1}{2} + \epsilon} \lesssim \|u^\Lambda^{\frac{1}{2}-2\epsilon} v\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} v\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} u\|_{L_t^1 L_x^2} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

3.2. We obtain

\[ \|u^\Lambda^{\frac{1}{2}+2\epsilon} v\|_{H^{\frac{1}{2}-\epsilon} - \frac{1}{2} + \epsilon} \lesssim \|u\|_{H^{\frac{1}{2}-\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

by Prop. 3.6.6 with parameters \( s_0 = 1 - \frac{\epsilon}{2} \), \( s_1 = 0 \), \( s_2 = s - \frac{1}{2} - 2\epsilon \), so that

\( s_0 + s_1 + s_2 = s + \frac{1}{2} - \frac{3\epsilon}{2} \) \( n_2 \) \( 2s - \frac{13\epsilon}{4} > \frac{1}{4} \)

\( n \geq 4 \).

B. The second part of the \( F^s \)-norm is handled as follows:

1. The first term on the right hand side of (47) requires the estimate

\[ \|\Lambda^{\frac{1}{2}+2\epsilon} \Lambda^{-\frac{1}{2}+3\epsilon} A^{\frac{1}{2}-2\epsilon} \Lambda^{\frac{1}{2}+2\epsilon} u\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} u\|_{H^{\frac{1}{2}-\epsilon} + \frac{1}{2} + \epsilon}. \]

which reduces to

\[ \|\Lambda^{\frac{1}{2}+2\epsilon}(uv)\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 L_x^2} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

This follows by Prop. 3.5.9 with parameters \( q = 2 \), \( r = 8n \), \( \sigma = \frac{3}{2} - 3\epsilon \). This requires

\( s_1, s_2 < \frac{1}{2} - \frac{\epsilon}{2} \) and \( s_1 + s_2 \geq n - 2 + 3\epsilon \). We choose \( s_1 = \frac{1}{2} - \frac{\epsilon}{2} \), so \( s_2 = \frac{1}{2} - \frac{3\epsilon}{2} + 4\epsilon < s - \frac{1}{2} - 2\epsilon \).

2. The estimate for the second term reduces to

\[ \|\Lambda^{\frac{1}{2}+2\epsilon} \Lambda^{-\frac{1}{2}+3\epsilon}(uv)\|_{L_t^1 L_x^2} \lesssim \|\Lambda^{\frac{1}{2}+2\epsilon} u\|_{F^s} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

where we used Lemma 3.6. By Prop. 3.13 we obtain the following bound for the left hand side: \[ \|\Lambda^{\frac{1}{2}+2\epsilon}(uv)\|_{L_t^1 L_x^2} \]

Using the fractional Leibniz rule we estimate:

2.1.

\[ \|u\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} v\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} v\|_{L_t^1 L_x^2} \lesssim \|u\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]

2.2.

\[ \|\Lambda^{\frac{1}{2}+2\epsilon} u\|_{L_t^1 L_x^2} \lesssim \|u\|_{L_t^1 L_x^2} \|\Lambda^{\frac{1}{2}+2\epsilon} u\|_{L_t^1 L_x^2} \lesssim \|u\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon} \|v\|_{H^{\frac{1}{2}-2\epsilon} + \frac{1}{2} + \epsilon}. \]
where $\frac{1}{p} = \frac{1}{\nu} - \frac{2n}{n} - 4\epsilon$ and $\frac{1}{r} = \frac{2n}{n} - 4\epsilon$, so that by Sobolev $H^{s+\frac{1}{2} - 4\epsilon} \hookrightarrow H^{\frac{\nu}{2} - \frac{1}{2} + \epsilon}$, for $s > \frac{n}{2} - \frac{\nu}{2}$. For the last factor we may apply Prop. 3.1 directly in the case $n \geq 5$, which gives

$$
\|v\|_{L^2 L^\nu} \lesssim \|v\|_{H^{s+\frac{1}{2} - 4\epsilon}} \lesssim \|v\|_{H^{s+\frac{1}{2} - 4\epsilon}},
$$

whereas in the case $n = 4$ we obtain by Prop. 3.2:

$$
\|v\|_{L^2 L^\nu} \lesssim \|v\|_{H^{s+\frac{1}{2} - \frac{1}{4} + \epsilon}},
$$

as one easily checks.

3. The last term on the right hand side of (47) is reduced to

$$
\|u\|_{L^2 L^\nu} \lesssim \|u\|_{H^{s+\frac{1}{2} - 4\epsilon}},
$$

As before we obtain the following bound for the left hand side: $\|\Lambda^{s+\frac{1}{2} + 2\epsilon} (uv)\|_{L^2 L^\nu}$. Using the fractional Leibniz rule we estimate:

3.1. By Sobolev and Prop. 3.1 we obtain:

$$
\|(\Lambda^{s+\frac{1}{2} + 2\epsilon})v\|_{L^2 L^\nu} \lesssim \|(\Lambda^{s+\frac{1}{2} + 2\epsilon})u\|_{L^2 L^\nu} \|v\|_{L^2 L^\nu} \lesssim \|u\|_{H^{s+\frac{1}{2} - 4\epsilon}} \lesssim \|u\|_{H^{s+\frac{1}{2} - 4\epsilon}},
$$

where $\frac{1}{\nu} = \frac{1}{\nu} - \frac{2n}{n}$ and $\frac{1}{r} = \frac{2n}{n}$, so that by Sobolev $H^{s+\frac{1}{2} - 4\epsilon} \hookrightarrow L^p$ for $s > \frac{\nu}{2} - \frac{\nu}{2}$. For the first factor we may apply Prop. 3.1 which gives

$$
\|\Lambda^{s+\frac{1}{2} + 2\epsilon} u\|_{L^2 L^\nu} \lesssim \|u\|_{H^{s+\frac{1}{2} + 2\epsilon}},
$$

as one easily checks.

3.2. Similarly

$$
\|u(\Lambda^{s+\frac{1}{2} + 2\epsilon})\|_{L^2 L^\nu} \lesssim \|u\|_{L^2 L^\nu} \|\Lambda^{s+\frac{1}{2} + 2\epsilon}\|_{L^2 L^\nu} \lesssim \|u\|_{H^{s+\frac{1}{2} - 4\epsilon}} \lesssim \|u\|_{H^{s+\frac{1}{2} - 4\epsilon}},
$$

where $\frac{1}{\nu} = \frac{1}{\nu} - \frac{2n}{n}$ and $\frac{1}{r} = \frac{2n}{n}$, so that by Sobolev $H^{s+\frac{1}{2} - 4\epsilon} \hookrightarrow H^{s+\frac{1}{2} + 2\epsilon}$ for $s > \frac{\nu}{2} - \frac{\nu}{2}$. For the first factor we may apply Prop. 3.1 which gives

$$
\|u\|_{L^2 L^\nu} \lesssim \|u\|_{H^{s+\frac{1}{2} + 2\epsilon}},
$$

\[\square\]

**Proof of (29).** We may reduce to

$$
\|\Lambda^{s+\frac{1}{2}} Q(u, v)\|_{H^{s+\frac{1}{2} + \epsilon}} \lesssim \|A u\|_{L^p} \|\Lambda v\|_{H^{s+\frac{1}{2} + \epsilon}}.
$$

Now we use (18) with $\epsilon = 0$ for $Q(u, v)$ and estimate the six terms as follows:

1. The estimate for the first term is reduced to (using the trivial estimate $\Lambda^{s+\frac{1}{2}} u \lesssim \Lambda^{s+\frac{1}{2}} u$):

$$
\|uv\|_{H^{s+\frac{1}{2} + \epsilon}} \lesssim \|uv\|_{H^{s+\frac{1}{2} + \epsilon}},
$$

which follows from Prop. 3.6.

2. The estimate for the second term reduces to

$$
\|uv\|_{H^{s+\frac{1}{2} + \epsilon}} \lesssim \|uv\|_{H^{s+\frac{1}{2} + \epsilon}},
$$

which is a consequence of Cor. 3.2 under our assumption $2s + r > \frac{\nu}{2}$.

3. The third term requires

$$
\|uv\|_{H^{s+\frac{1}{2} + \epsilon}} \lesssim \|uv\|_{H^{s+\frac{1}{2} + \epsilon}},
$$

4. The forth term similarly reduces to

$$
\|uv\|_{H^{s+\frac{1}{2} + \epsilon}} \lesssim \|uv\|_{H^{s+\frac{1}{2} + \epsilon}},
$$
6. The estimate for the sixth term similarly follows from
\[ \|uv\|_{H^{1-r, \frac{1}{2} + 2\varepsilon}} \lesssim \|u\|_{H^{r, \frac{1}{2} + \varepsilon}} \|v\|_{H^{r, \frac{1}{2} + \varepsilon}}. \]
These three estimates follow from Cor. 5.1.

5. The fifth term is the most complicated one. It follows from
\[ \|uv\|_{H^{r, \frac{1}{2} + 2\varepsilon}} \lesssim (\|u\|_{H^{r, \frac{1}{2} + \varepsilon}} + \|A^{\frac{1}{2} + 5\varepsilon}u\|_{L^2_1 L^2_{3n}}) \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}. \]
By use of Lemma 4.3 we obtain
\[ \|uv\|_{H^{r, \frac{1}{2} + 2\varepsilon}} \lesssim (\|A^{\frac{1}{2} + 5\varepsilon}u\|_{L^2_1 L^2_{3n}} + \|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{H^{1-r, \frac{1}{2} + \varepsilon}}). \]
The last term is easily estimated by Sobolev for \( s > \frac{n}{2} - 1 \):
\[ \|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{H^{1-r, \frac{1}{2} + \varepsilon}} \lesssim \|u\|_{H^{r+1-2s, 0}} \|v\|_{H^{1-r, \frac{1}{2} + 2\varepsilon}}. \]
For the first term we want to show
\[ \|uv\|_{H^{r, \frac{1}{2} + 2\varepsilon}} \lesssim (\|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{L^2_1 L^2_{3n}} + \|u\|_{H^{r+1-2s, 0}}) \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}. \]  
(67)
Consider first the case \( r \geq \frac{3}{2} \). (67) is handled by the fractional Leibniz rule as follows:
1. \[ \|u\Lambda^{r-1}v\|_{H^{r, \frac{1}{2} + \varepsilon}} \lesssim \|u\Lambda^{r-1}v\|_{L^2 L^2} \lesssim \|u\|_{L^2_1 L^2_{3n}} \|\Lambda^{r-1}v\|_{L^2_{1n} L^2} \lesssim \|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{L^2_1 L^2_{3n}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}, \]
where we used Prop. 5.7.1 and Prop. 5.7.10.
2. We obtain the estimate
\[ \|\Lambda^{r-1}uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \lesssim \|uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}, \]
by Prop. 5.6 under our assumptions \( s > \frac{n}{2} - 1 \) and \( r \geq \frac{3}{2} \) in which case one easily checks the necessary conditions. Thus under these assumptions we proved (67).
Next we consider the case \( r \leq \frac{1}{2} \). (67) follows by duality from the estimate
\[ \|uv\|_{H^{1-r, \frac{1}{2} + \varepsilon}} \lesssim (\|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{L^2_1 L^2_{3n}} + \|u\|_{H^{r+1-2s, 0}}) \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}. \]
Using the fractional Leibniz rule we obtain
\[ \|\Lambda^{1-r}uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \lesssim \|uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}} \]
by Prop. 5.6 under the assumptions \( s > \frac{n}{2} - 1 \) and \( r \leq \frac{1}{2} \) (and \( s > \frac{n}{2} - r \)). Moreover similarly as (68):
\[ \|u\Lambda^{1-r}v\|_{H^{r, \frac{1}{2} + \varepsilon}} \lesssim \|\Lambda^{\frac{1}{2} + 2\varepsilon}u\|_{L^2_1 L^2_{3n}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}. \]
Thus we have proven (67) for \( r \leq \frac{1}{2} \) and \( s > \frac{n}{2} - 1 \) as well.
By interpolation we also obtain this estimate in the remaining case \( \frac{1}{2} < r < \frac{3}{2} \).

\[ \Box \]

Proof of (63) and (64). (63) reduces to the following estimates
\[ \|uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \lesssim \|uv\|_{H^{r, \frac{1}{2} + \varepsilon}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}, \]
which easily follows from Prop. 5.7.1 because \((1-s) + s + (s+1) = s + 2 > \frac{n}{2} + 1\), and
\[ \|\Lambda^{-\frac{1}{2} + 5\varepsilon} \Lambda^{-\frac{1}{2} + \varepsilon} (uv)\|_{L^2_1 L^2_{3n}} \lesssim \|\Lambda^{-\frac{1}{2} + 7\varepsilon} \Lambda^{-\frac{1}{2} + \varepsilon} (uv)\|_{L^2_1 L^2_{3n}} \]
\[ \lesssim \|\Lambda^{-\frac{1}{2} + 7\varepsilon} (uv)\|_{L^2_1 L^2_{3n}} \lesssim \|uv\|_{H^{1-r, \frac{1}{2} + \varepsilon}} \|v\|_{H^{1-r, \frac{1}{2} + \varepsilon}}, \]
where we used Prop. 5.7.1 and Prop. 5.7.1 because \((2 - \frac{n}{2} - 7\varepsilon) + s + (s+1) > \frac{n}{2} + 1\).
The estimate (64) is proven in exactly the same way.

\[ \Box \]
Proof of (38) and (39). (38) reduces to the following estimate
\[ \|uv\|_{H^{r-1,-\frac{1}{2}+\frac{1}{2}+s}} \lesssim \|u\|_{H^{r,\frac{1}{2}+s}} \|v\|_{H^{r-1,-\frac{1}{2+s}},} \]
which follows from Prop. 3.14 because \( s + 2 > \frac{n}{2} + 1 \). Similarly (39) can be proven. \( \square \)

Proof of (38). A. The estimate for the first part of the \( F^a \)-norm reduces to
\[ \|uvw\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r+1,-\frac{1}{2}+s}}. \]
We first assume \( s \leq \frac{n}{2} - \frac{1}{2} \). In a first step we apply Cor. 3.1 using our assumptions \( s > \frac{n}{2} - \frac{3}{4} \) and \( r > \frac{n}{2} - \frac{3}{4} \). By elementary calculations we obtain
\[ \|uvw\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|vw\|_{H^{r+1,-\frac{1}{2}+s}}. \]
For the second step we obtain by Prop. 3.6
\[ \|vw\|_{H^{2r,-\frac{1}{2}+\frac{1}{2}+s}} \lesssim \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]
If however \( s > \frac{n}{2} - \frac{1}{2} \) we obtain similarly by easy calculations
\[ \|uvw\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|vw\|_{H^{r,\frac{1}{2}+s}} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]

B. The estimate for the second part of the \( F^a \)-norm reduces to
\[ \|\Lambda^{-\frac{3}{4}+5s} \Lambda_{-\frac{1}{2}+s} (uvw)\|_{L^1_t L^2_x} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]
By Prop. 3.13 we obtain
\[ \|\Lambda^{-\frac{3}{4}+5s} \Lambda_{-\frac{1}{2}+s} (uvw)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{\frac{1}{4}-2+7s} \Lambda_{-\frac{1}{2}+s} (uvw)\|_{L^1_t L^2_x}, \]
which by the fractional Leibniz rule and symmetry in \( v \) and \( w \) is estimated as follows:
1. By Sobolev and Prop. 3.1 we obtain under our assumption \( s > \frac{n}{2} - \frac{7}{8} \) and \( r > \frac{n}{2} - \frac{3}{4} \):
\[ \|\Lambda^{\frac{3}{4}-2+7s} u vw\|_{L^1_t L^2_x} \lesssim \|\Lambda^{\frac{3}{4}-2+7s} u\|_{L^\infty_t L^{\frac{8}{3}}_x} \|v\|_{L^\infty_t L^{\frac{8}{3}}_x} \|w\|_{L^\infty_t L^{\frac{8}{3}}_x} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]
2. Similarly we also obtain
\[ \|u(\Lambda^{\frac{3}{4}-2+7s} v) w\|_{L^1_t L^2_x} \lesssim \|u\|_{L^1_t L^\infty_x} \|\Lambda^{\frac{3}{4}-2+7s} v\|_{L^\infty_t L^{\frac{8}{3}}_x} \|w\|_{L^1_t L^\infty_x} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]

Proof of (39). A. We may reduce to
\[ \|u\Lambda^{-1}(vw)\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r+1,-\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]
In the case \( \frac{n}{2} - \frac{7}{8} < s \leq \frac{n}{2} - \frac{1}{2} \) we use Cor. 3.1 and Prop. 3.6 and obtain
\[ \|u\Lambda^{-1}(vw)\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r,\frac{1}{2}+s}} \|\Lambda^{-1}(vw)\|_{H^{r+1,-\frac{1}{2}+s}} \lesssim \|u\|_{H^{r,\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]
In the case \( s > \frac{n}{2} - \frac{1}{2} \) we obtain similarly
\[ \|u\Lambda^{-1}(vw)\|_{H^{r,-1,-\frac{1}{2}+2s}} \lesssim \|u\|_{H^{r,\frac{1}{2}+s}} \|\Lambda^{-1}(vw)\|_{H^{r+1,0}} \lesssim \|u\|_{H^{r,\frac{1}{2}+s}} \|v\|_{H^{r,\frac{1}{2}+s}} \|w\|_{H^{r,\frac{1}{2}+s}}. \]

B. We have to control
\[ \|\Lambda^{-\frac{3}{4}+5s} \Lambda_{-\frac{1}{2}+s} (u\Lambda^{-1}(vw))\|_{L^1_t L^2_x} \lesssim \|\Lambda^{\frac{3}{4}-2+7s} (u\Lambda^{-1}(vw))\|_{L^1_t L^2_x}, \]
where we used Prop. 3.1. By the fractional Leibniz rule we have to consider two terms.

1. By Sobolev and Prop. 5.1 we obtain

\[
\| (\Lambda_{\|}^{\frac{1}{2}+\epsilon} - \lambda v) w_{\|} L^1_t L^2_x \| \leq \| \Lambda_{\|}^{\frac{1}{2}+\epsilon} \| \| w_{\|} L^1_t L^2_x \| \leq C x \epsilon \| w_{\|} H^{\frac{1}{2}+\epsilon} \| H^{\frac{1}{2}+\epsilon} ,
\]

2. We have to estimate \( \| u \Lambda_{\|}^{\frac{1}{2}+\epsilon} (vw) \| L^1_t L^2_x \), which in the case \( n \geq 6 \) by symmetry and the fractional Leibniz rule reduces to \( \| u \Lambda_{\|}^{\frac{1}{2}+\epsilon} v \| L^1_t L^2_x \). By Sobolev and Prop. 5.1 we obtain

\[
\| u \Lambda_{\|}^{\frac{1}{2}+\epsilon} v \| L^1_t L^2_x \| \leq \| u \| L^2_t L^\infty_x \| \Lambda_{\|}^{\frac{1}{2}+\epsilon} \| L^2_t L^\infty_x \| w \| L^2_t L^\infty_x \| \leq C x \epsilon \| w \| H^{\frac{1}{2}+\epsilon} ,
\]

If \( 4 \leq n \leq 5 \) we obtain by the same means

\[
\| u \Lambda_{\|}^{\frac{1}{2}+\epsilon} (vw) \| L^1_t L^2_x \| \leq \| u \| L^4_t L^{\frac{6}{5}}_x \| \Lambda_{\|}^{\frac{1}{2}+\epsilon} \| L^4_t L^{\frac{6}{5}}_x \| vw \| L^2_t L^\infty_x \| \leq C x \epsilon \| vw \| H^{\frac{1}{2}+\epsilon} .
\]

**Proof of (41).**

A. We have to show the estimate

\[
\| uvw \| H^{\frac{1}{2}+\epsilon} \leq \| u \| H^{\frac{1}{2}+\epsilon} \| v \| H^{\frac{1}{2}+\epsilon} \| w \| H^{\frac{1}{2}+\epsilon} .
\]

It is sufficient to consider the (minimal) value \( s = \frac{n}{2} - 1 + \epsilon \), which immediately implies this for any larger \( s \). This follows from Cor. 5.1 and Prop. 5.6:

\[
\| uvw \| H^{\frac{n}{2}+\epsilon} \leq \| u \| H^{\frac{n}{2}+\epsilon} \| v \| H^{\frac{n}{2}+\epsilon} \| w \| H^{\frac{n}{2}+\epsilon} .
\]

B. For the second part of the \( F^s \)-norm we use Prop. 5.13 and obtain

\[
\| \Lambda_{\|}^{\frac{1}{2}+\epsilon} \| L^1_t L^2_x \| \leq \| \Lambda_{\|}^{\frac{1}{2}+\epsilon} (uvw) \| L^1_t L^2_x \| .
\]

Now by symmetry we only have to estimate

\[
\| (\Lambda_{\|}^{\frac{1}{2}+\epsilon} u) w_{\|} L^1_t L^2_x \| \leq \| \Lambda_{\|}^{\frac{1}{2}+\epsilon} u \| L^2_t L^\infty_x \| v \| L^2_t L^\infty_x \| w \| L^2_t L^\infty_x \| \leq C x \epsilon \| w \| H^{\frac{1}{2}+\epsilon} .
\]

**Proof of (42).**

It suffices to show

\[
\| uvw \| H^{\frac{1}{2}+\epsilon} \leq \| u \| H^{\frac{1}{2}+\epsilon} \| v \| H^{\frac{1}{2}+\epsilon} \| w \| H^{\frac{1}{2}+\epsilon} ,
\]

which follows by Cor. 5.1 and Prop. 5.6 in the case \( r = \frac{n}{2} - \frac{1}{2} \) from

\[
\| uvw \| H^{\frac{1}{2}+\epsilon} \leq \| u \| H^{\frac{1}{2}+\epsilon} \| v \| H^{\frac{1}{2}+\epsilon} \| w \| H^{\frac{1}{2}+\epsilon} ,
\]

whereas in the case \( r > \frac{n}{2} - \frac{1}{2} \) we obtain

\[
\| uvw \| H^{\frac{1}{2}+\epsilon} \leq \| u \| H^{\frac{1}{2}+\epsilon} \| v \| H^{\frac{1}{2}+\epsilon} \| w \| H^{\frac{1}{2}+\epsilon} ,
\]

using our assumption \( 2s - r > \frac{n}{2} \).
Proof of (40). A. For the first part of the $F^s$-norm we have to show

$$\|\Lambda^{-1}(uv)wz\|_{H^{s-1,-\frac{3}{2}+2\varepsilon}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|w\|_{H^{s+\frac{3}{2}}} + \|z\|_{H^{s+\frac{3}{2}}}. $$

It suffices to consider the minimal value $s = \frac{9}{2} - \frac{3}{2} + \varepsilon$, which by Cor. 3.11 and Prop. 3.8 can be estimated as follows:

$$\|\Lambda^{-1}(uv)wz\|_{H^{s+\frac{3}{2}}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|wz\|_{H^{s+\frac{3}{2}}} + \|z\|_{H^{s+\frac{3}{2}}}.$$ 

By Prop. 3.13 and the fractional Leibniz rule we obtain

$$\|\Lambda^{-1}(uv)wz\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} + \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} + \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x}.$$

1. In the case $n \geq 6$ we reduce the estimate for the first term by symmetry to

$$\|\Lambda^{-1}(uv)wz\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x}.$$

by Sobolev and Prop. 3.5 whereas for $4 \leq n \leq 5$ we similarly obtain

$$\|\Lambda^{-1}(uv)wz\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x}.$$ 

2. The estimate for the second term is as follows

$$\|\Lambda^{-1}(uv)wz\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x} \lesssim \|\Lambda^{-1}(uv)\|_{L^1_t L^2_x}.$$ 

where we used Sobolev and Strichartz as before.

Proof of (43). We have to show

$$\|uvwz\|_{H^{s-1,-\frac{3}{2}+2\varepsilon}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|w\|_{H^{s+\frac{3}{2}}} + \|z\|_{H^{s+\frac{3}{2}}}. $$

By our assumption $2s - r > \frac{9}{2}$ the left hand side is bounded by the term

$$\|uvwz\|_{H^{s-1,-\frac{3}{2}+2\varepsilon}}.$$ 

It suffices to prove the remaining estimate for the (minimal) value $s = \frac{9}{2} - \frac{3}{2} + \varepsilon$. By Cor. 3.11 and Prop. 3.7 we obtain

$$\|uvwz\|_{H^{s+\frac{3}{2}}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|wz\|_{H^{s+\frac{3}{2}}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|wz\|_{H^{s+\frac{3}{2}}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|wz\|_{H^{s+\frac{3}{2}}} \lesssim \|u\|_{H^{s+\frac{3}{2}}} + \|v\|_{H^{s+\frac{3}{2}}} + \|wz\|_{H^{s+\frac{3}{2}}}. $$

□
6. Proof of the nonlinear estimates in the case $n = 3$

Proof of (30). We recall (15) for $\alpha = \epsilon$:

$$Q_0(u, v) \lesssim D_1^{-\epsilon}(D_+ u D_+ v) + D_1^{-\epsilon}(D_+ u D_+ v)$$

Thus we have to show the following estimates and remark that we only have to consider the first and third term, because the last two terms are equivalent by symmetry.

1. For the first term it suffices to show

$$\|A_s^{-1}A^{-1+\epsilon}A^{-1}A_+ D_1^{-\epsilon}(D_+ u D_+ v)\|_{H^0_{\frac{1}{2}+s}} \lesssim \|A_s^{-1}A_+ u\|_{H^0_{\frac{1}{2}+s}} \|A_s^{-1}A_+ v\|_{H^0_{\frac{1}{2}+s}}$$

This follows from

$$\|uv\|_{H^{-\frac{1}{2}}+s} \lesssim \|u\|_{H^{-\frac{1}{2}}+s} \|v\|_{H^{-\frac{1}{2}}+s},$$

which is a consequence of Prop. 3.8 under our conditions $s \geq r$ and $2s - r > \frac{3}{2}$. Here we also need the assumption $s > \frac{5}{2}$.

2. For the second term we show

$$\|A_s^{-1}A^{-1+\epsilon}A^{-1}A_+ (D_+ D_1^{-\epsilon} u D_+ v)\|_{H^0_{\frac{1}{2}+s}} \lesssim \|A_s^{-1}A_+ u\|_{H^0_{\frac{1}{2}+s}} \|A_s^{-1}A_+ v\|_{H^0_{\frac{1}{2}+s}}$$

Thus it suffices to show

$$\|uv\|_{H^{-\frac{1}{2}}+s} \lesssim \|u\|_{H^{-\frac{1}{2}}+s} \|v\|_{H^{-\frac{1}{2}}+s},$$

which by duality is equivalent to

$$\|uv\|_{H^{-\frac{1}{2}}-s} \lesssim \|u\|_{H^{-\frac{1}{2}}-s} \|v\|_{H^{-\frac{1}{2}}-s},$$

This is a consequence of Prop. 3.8 as in 1. under the same assumptions.

Proof of (31). We use (17). Thus we have to show the following estimates and remark that we only have to consider the first two terms, because the last two terms are equivalent by symmetry.

1. For the first term it suffices to show

$$\|A_s^{-1}A^{-1+\epsilon}A^{-1}A_+ \frac{1}{2}A^{-2\epsilon} \Lambda_+^2 - 2\epsilon (A_+^{\frac{1}{2}+2\epsilon} u A_+^{\frac{1}{2}+2\epsilon} v)\|_{H^0_{\frac{1}{2}+s}} \lesssim \|A_s^{-1}A_+ u\|_{H^0_{\frac{1}{2}+s}} \|A_s^{-1}A_+ v\|_{H^0_{\frac{1}{2}+s}}$$

This follows from

$$\|uv\|_{H^{-\frac{1}{2}}-2s_0} \lesssim \|u\|_{H^{-\frac{1}{2}}-2s_0} \|v\|_{H^{-\frac{1}{2}}-2s_0},$$

which is a consequence of Prop. 3.3 with parameters $s_0 = \frac{1}{2} - r + 2\epsilon$, $s_1 = s_2 = s - \frac{1}{2} - 2\epsilon$, so that $s_0 + s_1 + s_2 > 1$, if $2s - r > \frac{1}{2}$, and $s_0 + s_1 + s_2 + s_1 > \frac{1}{2}$, if $4s - r > 3$, which holds under our assumptions.

2. For the second term we show

$$\|A_s^{-1}A^{-1+\epsilon}A^{-1}A_+ \frac{1}{2}A^{-2\epsilon} (A_+^{\frac{1}{2}+2\epsilon} A_+^{\frac{1}{2}+2\epsilon} u A_+^{\frac{1}{2}+2\epsilon} v)\|_{H^0_{\frac{1}{2}+s}} \lesssim \|A_s^{-1}A_+ u\|_{H^0_{\frac{1}{2}+s}} \|A_s^{-1}A_+ v\|_{H^0_{\frac{1}{2}+s}}$$

Using $A_+^s u \lesssim A_+^s u$ it suffices to show

$$\|uv\|_{H^{-\frac{1}{2}}-2s_0} \lesssim \|u\|_{H^{-\frac{1}{2}}-2s_0} \|v\|_{H^{-\frac{1}{2}}-2s_0},$$

which is a consequence of Prop. 3.3 as in 1., if $2s - r > \frac{3}{2}$ and $3s - 2r > 2$, which holds under our assumptions. 

\[\Box\]
Proof of (62). A. We start with the first part of the $F^s$-norm. As before it is easy to see that we can reduce to
\[ \|uv\|_{H^{-1,-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}. \]
This is a consequence of Prop. (3.3) One easily checks that it can be applied under the conditions $s \leq r + 1, 2r - s > -1, 4r - s > -2$ and $3r - 2s > -2$, all of which are satisfied under our assumptions.

B. For the second part of the $F^s$-norm we reduce to
\[ \|\Lambda^{-1}\Lambda^{-1+\epsilon}A_{x}^{\frac{1}{2}+2\epsilon}A_{-\frac{1}{2}-\frac{1}{2}+3\epsilon}A_{z}^{\frac{1}{2}}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|A_{+}u\|_{H^{r+\frac{1}{2}+\epsilon}} \|A_{+}v\|_{H^{r-\frac{1}{2}+\epsilon}}. \]
and further to
\[ \|A_{-}^{-\frac{1}{2}+\epsilon}A_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}. \]
By Prop. (3.3) we obtain
\[ \|A_{-}^{-\frac{1}{2}+\epsilon}A_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|A_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}}. \]
Next we show for a suitable $r_2$ the estimate
\[ \|A_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}, \]
which by duality is equivalent to
\[ \|uv\|_{H^{r+\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}. \]
Using the fractional Leibniz rule we have to consider two terms:
\[ \|\Lambda_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|\Lambda_{-}^{-\frac{1}{2}+\epsilon}u\|_{L^{2}_{t}L^{2}_{x}} \|v\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}, \]
where we choose $\frac{1}{q} = \frac{1}{2} - \frac{1}{2} + \epsilon$ and $\frac{1}{p} = \frac{1}{2} - \frac{1}{2} + \epsilon$, so that by Sobolev $H^{r+\frac{1}{2}} \hookrightarrow L^{q},$ and
\[ \|u\Lambda_{-}^{-\frac{1}{2}+\epsilon}w\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|u\|_{L^{1}_{t}L^{1}_{x}} \|w\|_{H^{r+\frac{1}{2}}}, \]
where $\frac{1}{q} = \frac{1}{2} - \frac{1}{2} + \epsilon$, $\frac{1}{p} = \frac{1}{2} - \frac{1}{2} + \epsilon$, so that by Sobolev $H^{r+\frac{1}{2}} \hookrightarrow L^{q}$ and $H^{r+\frac{1}{2}} \hookrightarrow L^{r+\frac{1}{2}}.$
Thus we obtain by Prop. (3.3):
\[ \|A_{-}^{-\frac{1}{2}+\epsilon}(uv)\|_{L^{2}_{t}L^{2}_{x}} \lesssim \|u\|_{L^{2}_{t}H^{r+\frac{1}{2}+\epsilon}} \|v\|_{L^{2}_{t}H^{r-\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r+\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r+\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r+\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}+\epsilon}} \|v\|_{H^{r+\frac{1}{2}+\epsilon}}. \]
for $r > -\frac{1}{2}. \quad \square$

Proof of (62). A. For the first part of the $F^s$-norm it is sufficient to show
\[ \|\Gamma^{1}(u,v)\|_{H^{-1,-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{F^{r}} \|A_{+}v\|_{H^{r-\frac{1}{2}+\epsilon}}, \tag{69} \]
for the minimal value $s = \frac{3}{2}+$, because the estimate for any $s > \frac{3}{2}$ follows immediately. We use Lemma (1.3) a. We first consider $\Gamma_{1}^{1}(u,v)$. By (62) it suffices to show the following estimates, all of which are consequences of Proposition (3.3):
\[ \|uv\|_{H^{r+\frac{1}{2}-2\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}-2\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}} ; \]
\[ \|uv\|_{H^{r+\frac{1}{2}-2\epsilon}} \lesssim \|u\|_{H^{r+\frac{1}{2}-2\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}} ; \]
\[ \|uv\|_{H^{r-\frac{1}{2}+\epsilon}} \lesssim \|u\|_{H^{r-\frac{1}{2}+\epsilon}} \|v\|_{H^{r-\frac{1}{2}+\epsilon}}. \]
b. Assume that $u$ and $v$ have frequencies $\geq 1$, so that $\Lambda_+^{\alpha} u \sim D_+^{\alpha} u$. In this case we use (10) and consider $\Gamma_1^2(u,v)$. By (51) we may reduce the estimates for the first and third term on the right hand side to

$$||uv||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}} \lesssim ||u||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}} ||v||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}}.$$  

Both estimates follow from Proposition 3.3. The second term is reduced to the following estimate

$$||uv||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}} \lesssim ||\Lambda_{-}^{\frac{1}{2}+2s} u||_{F^s} ||v||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}}.$$  

By the fractional Leibniz rule we have to show the following two estimates:

b1. $$||(\Lambda^{s-\frac{1}{2}+2s})u||_{H^{0,0}} \lesssim ||u||_{H^{s-\frac{1}{2},0}} ||v||_{H^{s,0}}.$$  

which follows from Proposition 3.3. Here we need our assumption $s > \frac{3}{4}$. 

b2. $$(\Lambda^s-\frac{1}{2}+2s)v \lesssim ||u(\Lambda^{s-\frac{1}{2}+2s})v||_{L^1_t L^2_x} \lesssim ||u||_{L^1_t L^2_x} ||(\Lambda^{s-\frac{1}{2}+2s})v||_{L^\infty_t L^2_x} \lesssim ||\Lambda^{s-\frac{1}{2}+2s}u||_{F^s} ||v||_{H^{s-\frac{1}{2},-\frac{1}{2}+2s}},$$

where we used Prop. 3.12 and Prop. 3.10 in order to replace $L^2_t L^2_x$-norms by $L^qTL^q_x$-norms.

c. Consider $(\Lambda^{-2}u) v$ and $u(\Lambda^{-2})v$. It suffices to show

$$||((\Lambda^{-2})v)||_{H^{s-1,-\frac{1}{2}+2s}} \lesssim ||u||_{H^{s+\frac{1}{2},0}} ||v||_{H^{s+\frac{1}{2},0}},$$

$$||((\Lambda^{-2}v)||_{H^{s-1,-\frac{1}{2}+2s}} \lesssim ||u||_{H^{s+\frac{1}{2},0}} ||v||_{H^{s+\frac{1}{2},0}}.$$  

Both follow easily from Prop. 3.7 under our assumption $s > \frac{1}{4}$.

d. Let us now consider the case where the frequencies of $u$ or $v$ are $\leq 1$. We use (51) instead of (10). Because $\Gamma_2^1(u,v)$ has already been handled, we only have to consider $uv$. If $u$ has low frequencies we obtain by Prop. 3.7:

$$||uv||_{H^{s-1,-\frac{1}{2}+2s}} \lesssim ||u||_{H^{s+\frac{1}{2},0}} ||v||_{H^{s+\frac{1}{2},0}}.$$  

Similarly we treat the case where $v$ has low frequencies.

B. Now we consider the second part of the $F^s$-norm. We want to show

$$||\Lambda_{+}^{-1}\Lambda_{-}^{-1+\epsilon} \Lambda_{+}^{\frac{1}{2}+2s} \Lambda_{-}^{-\frac{1}{2} + 3\epsilon} \Lambda_{+}^{2} \Gamma_1(u,v)||_{L^1_t L^2_x} \lesssim ||u||_{F^s} ||\Lambda_{+} v||_{H^{s-\frac{1}{2},+\frac{1}{2}}}. \quad (71)$$  

a. We first consider $\Gamma_1^2(u,v)$ and use (72).

1. The estimate for the first term on the right hand side reduces to

$$||\Lambda_{+}^{-1}\Lambda_{-}^{-1+\epsilon} \Lambda_{+}^{\frac{1}{2}+2s} \Lambda_{-}^{-\frac{1}{2} + 3\epsilon} \Lambda_{+}^{2} \Gamma_1(u,v)||_{L^1_t L^2_x} \lesssim ||\Lambda_{+} u||_{H^{s-1+\epsilon,0}} ||\Lambda_{+} v||_{H^{s-\frac{1}{2},0}}$$

and therefore we only have to prove

$$||\Lambda_{+}^{\frac{1}{2}+3\epsilon} (uv)||_{L^1_t L^2_x} \lesssim ||u||_{H^{s+\frac{1}{2},0}} ||\Lambda_{+} v||_{H^{s-\frac{1}{2},0}} \quad (72)$$
We obtain by Proposition 3.15. We need

\[ D_+ D_+^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u D_+^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v = D_+^{\frac{3}{2}} (D_+^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u D_+^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v) \]

and

\[ \Lambda^{-1} \Lambda^{-1 + \epsilon} \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v) \|_{L_t^{\frac{13}{14}} L_x^{14}} \]

which by duality is equivalent to

\[ \| \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1 + \epsilon} \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v) \|_{L_t^{\frac{13}{14}} L_x^{14}} \]

The claimed estimate follows, because \( s_1 < s + \frac{\epsilon}{2} - 2\epsilon \) and \( s_2 < s - \frac{\epsilon}{2} - 2\epsilon \) under our assumption \( s > \frac{\epsilon}{2} \).

2. The second term is modified by partial integration:

\[ D_+ D_+^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u D_+^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v = D_+^{\frac{3}{2}} (D_+^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u D_+^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v) \]

2.1. We have to prove

\[ \| \Lambda^{-1} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| \Lambda u \|_{H^{s_1 + \frac{\epsilon}{2}}} \| \Lambda \Lambda \|_{H^{s_2 - \frac{\epsilon}{2}}} \]

We use \( \Lambda^{2\epsilon} u \leq \Lambda^{2\epsilon} u \) and reduce to

\[ \| \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| u \|_{H^{s_1 + \frac{\epsilon}{2}}} \| v \|_{H^{s_2 - \frac{\epsilon}{2}}} \]

We obtain by Proposition 3.15

\[ \| \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \]

We now show that

\[ \| uv \|_{H^{s_1 + \frac{\epsilon}{2}}} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]

which by duality is equivalent to

\[ \| uv \|_{H^{s_1 + \frac{\epsilon}{2}}} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]

By the fractional Leibniz rule

\[ \| uv \|_{H^{s_1 + \frac{\epsilon}{2}}} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]

where we choose \( \frac{1}{q} = \frac{1}{2} - \frac{1}{2\epsilon} + \frac{\epsilon}{2} \) and \( \frac{1}{p} = \frac{1}{2} - \frac{1}{2\epsilon} - \frac{\epsilon}{2} \), so that by Sobolev \( H^{s_1 + \frac{\epsilon}{2}} \subset L^q \) and \( H^{s_2 + \frac{\epsilon}{2}} \subset L^p \). Thus we obtain

\[ \| uv \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| u \|_{H^{s_1 + \frac{\epsilon}{2}}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]

for \( s > \frac{\epsilon}{2} \), where we used Prop. 3.3.

2.2. We need

\[ \| \Lambda^{-1} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| u \|_{H^{s_1 + \frac{\epsilon}{2}}} \| v \|_{H^{s_2 - \frac{\epsilon}{2}}} \]

We obtain by Proposition 3.15

\[ \| \Lambda^{-1} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \leq \| \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-\frac{3}{2} - 2\epsilon} \Lambda^{\frac{3}{2}} \Lambda^{\frac{3}{2} - 2\epsilon} \Lambda^{-1} u \Lambda^{\frac{3}{2} + 2\epsilon} \Lambda^{-1} v \|_{L_t^{\frac{13}{14}} L_x^{14}} \]

We now show that

\[ \| uv \|_{H^{s_1 + \frac{\epsilon}{2}}} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]

which by duality is equivalent to

\[ \| uv \|_{H^{s_1 + \frac{\epsilon}{2}}} \leq \| u \|_{H^{s_1}} \| v \|_{H^{s_2 + \frac{\epsilon}{2}}} \]
By the fractional Leibniz rule
\[ \|uv\|_{L^2} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} u\|_{L^2} + \|u\Lambda^{-\frac{7}{2}\epsilon} v\|_{L^2} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} u\|_{L^2} + \|u\Lambda^{-\frac{7}{2}\epsilon} v\|_{L^2} \lesssim \|u\|_{H^s} \|\Lambda^{-\frac{7}{2}\epsilon} v\|_{L^2}, \]
where we choose \( \frac{1}{q} = \frac{1}{2} - \frac{1}{p} + \frac{7}{4} \epsilon \) and \( \frac{1}{p} = \frac{1}{2} - \frac{1}{q} + \frac{7}{4} \epsilon \), so that by Sobolev \( H^{\frac{7}{2}\epsilon} \hookrightarrow L^q \) and \( H^{1-\frac{7}{2}\epsilon} \hookrightarrow L^p \). Thus we obtain
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|u\|_{L^2} \|v\|_{L^2} \]
for \( s > \frac{7}{4} \) by Prop. 3.13.

3. We reduce to
\[ \|\Lambda^{-1-\frac{7}{2}\epsilon} \Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|u\|_{L^2} \|v\|_{L^2} \]
We obtain by Proposition 3.15
\[ \|\Lambda^{-1-\frac{7}{2}\epsilon} \Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2}. \]
Now for suitable \( p \) we obtain
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{H^{\frac{7}{2}\epsilon}} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{H^{\frac{7}{2}\epsilon}}, \]
which by duality is equivalent to
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{H^{-\frac{7}{2}\epsilon}} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{H^{-\frac{7}{2}\epsilon}}. \]
By the fractional Leibniz rule
\[ \|uv\|_{H^{\frac{7}{2}\epsilon}} \lesssim \|\Lambda^\frac{7}{2}\epsilon uv\|_{L^2} + \|u\Lambda^\frac{7}{2}\epsilon v\|_{L^2} \lesssim \|\Lambda^\frac{7}{2}\epsilon u\|_{L^2} \|v\|_{L^2} + \|u\|_{L^2} \|\Lambda^\frac{7}{2}\epsilon v\|_{L^2} \lesssim \|u\|_{H^{\epsilon}} \|v\|_{H^{\epsilon}}, \]
where we choose \( \frac{1}{q} = \frac{1}{2} - \frac{1}{p} + \frac{7}{4} \epsilon \) and \( \frac{1}{p} = \frac{1}{2} - \frac{1}{q} + \frac{7}{4} \epsilon \), so that by Sobolev \( H^{\frac{7}{2}\epsilon} \hookrightarrow L^q \), \( H^{1-\frac{7}{2}\epsilon} \hookrightarrow L^p \) and \( H^{\frac{7}{2}\epsilon} \hookrightarrow L^p \). Thus we obtain
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|u\|_{L^2} \|v\|_{L^2} \]
for \( s > \frac{7}{4} \) by Prop. 3.13.

b. If the frequencies of \( u \) or \( v \) are \( \geq 1 \), we use (41) and consider \( \Gamma_1 \) using (51). The estimate for the first term on the right hand side of (51) reduces to (72).

The second term on the right hand side of (51) reduces to
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \]
We start with Proposition 3.15
\[ \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \lesssim \|\Lambda^{-\frac{7}{2}\epsilon} (uv)\|_{L^1 L^2} \]
(75)
By the fractional Leibniz rule we have to consider two terms. 

2.1. 

\[ \|u(\Lambda^{\frac{1}{2}} - \frac{1}{2} + 5\epsilon v)\|_{L^1_t L^2_x} \lesssim \|u\|_{L^1_t L^6_x} \|v\|_{L^{14 + \epsilon}_t H^{\frac{1}{2} + 5\epsilon}_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2}} u\|_{L^1_t L^{14}_x} \|v\|_{H^{\frac{1}{2} + 5\epsilon}_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + 5\epsilon} v u\|_{L^1_t L^{14}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, \frac{1}{4} + \epsilon}_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + 5\epsilon} v u\|_{L^1_t L^{14}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, \frac{1}{4} + \epsilon}_x} \].

2.2. We obtain

\[ \|(\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u) v\|_{L^1_t L^2_x} \lesssim \|\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u\|_{L^1_t L^6_x} \|v\|_{L^{14 + \epsilon}_t L^{1}_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u\|_{L^1_t L^{14}_x} \|v\|_{H^{\frac{1}{2} + 5\epsilon}_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2} - 2\epsilon} \Lambda^{\frac{1}{2} + 5\epsilon} u\|_{L^1_t L^{14}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, \frac{1}{4} + \epsilon}_x} \].

where \( \frac{1}{2} = \frac{1}{14} - \frac{1}{31} + \frac{5}{6} \epsilon \) and \( \frac{1}{2} = \frac{1}{2} - \frac{1}{14} - \frac{5}{6} \epsilon \), so that by Sobolev \( H^{\frac{1}{2} - \frac{1}{4} + 5\epsilon, q} \Leftrightarrow H^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} \Leftrightarrow L^q \). The last estimate follows as in 2.1.

3. The last term on the right hand side of (71) requires

\[ \|\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} (uv)\|_{L^{14 + \epsilon}_t L^{1}_x} \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon, \frac{1}{4} + \epsilon}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, 0}_x}. \]  (76)

The left hand side is estimated using Prop. 5.15 by \( \|\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} (uv)\|_{L^1_t L^2_x} \). By the fractional Leibniz rule we have to treat two terms.

3.1. By Sobolev and Prop. 5.13 we obtain

\[ \|(\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u) v\|_{L^1_t L^2_x} \lesssim \|\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u\|_{L^1_t L^{14}_x} \|v\|_{L^1_t L^2_x} \]

\[ \lesssim \|\Lambda^{\frac{1}{2} - \frac{1}{4} + 5\epsilon} u\|_{H^{\frac{1}{2} + \epsilon}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, 0}_x} \]

\[ \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon + \frac{1}{2} + \epsilon}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, 0}_x}, \]

using \( H^{\frac{1}{2} - 2\epsilon} \Leftrightarrow H^{\frac{1}{2}} \Leftrightarrow L^2 \).

3.2. Similarly by Prop. 5.13

\[ \|u(\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} v)\|_{L^1_t L^2_x} \lesssim \|u\|_{L^1_t L^{14}_x} \|\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} v\|_{L^1_t L^2_x} \]

\[ \lesssim \|u\|_{L^1_t H^{\frac{1}{2} + \epsilon}_x} \|\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} v\|_{L^1_t L^2_x} \]

\[ \lesssim \|u\|_{H^{\frac{1}{2} + \frac{1}{2} + \epsilon}_x} \|v\|_{L^{14 + \epsilon}_t L^{1}_x} \]

\[ \lesssim \|u\|_{H^{\frac{1}{2} - 2\epsilon + \frac{1}{2} + \epsilon}_x} \|v\|_{H^{-\frac{1}{2} - 2\epsilon, 0}_x}. \]

4. We now consider the case where \( u \) and/or \( v \) have frequency \( \leq 1 \). In this case we use (71) instead of (70). Because \( \Gamma^1_2(u, v) \) is already handled we only have to estimate \( uv \). Thus it suffices to show

\[ \|\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} (uv)\|_{L^{14 + \epsilon}_t L^{1}_x} \lesssim \|u\|_{H^{\frac{1}{2} - \frac{1}{4} + \epsilon}_x} \|v\|_{H^{-\frac{1}{2} - \frac{1}{4} + \epsilon}_x}. \]

We crudely estimate the left hand side by

\[ \|\Lambda^{\frac{1}{2} - \frac{1}{2} + 5\epsilon} (uv)\|_{L^{14 + \epsilon}_t L^{1}_x} \lesssim \|u\|_{H^{\frac{1}{2} - \frac{1}{4} + \epsilon}_x} \|v\|_{H^{\frac{1}{2} - \frac{1}{4} + \epsilon}_x} \]

by use of Proposition 5.15 with parameters \( r = 28, q = \frac{28}{13} \), where \( s_1 \) and \( s_2 \) have to fulfill \( s_1 + s_2 = \frac{1}{2} \) and \( s_1, s_2 < \frac{1}{2} - \epsilon \). If \( u \) has frequency \( \leq 1 \) choose \( s_1 = \frac{1}{2} - \frac{1}{2} - \epsilon \), \( s_2 = -\frac{5}{14} + \epsilon < s - 1 \). The value of \( s_1 \) is irrelevant in view of the low frequency assumption. If \( v \) has frequency \( \leq 1 \) choose \( s_1 = \frac{1}{2} < s, s_2 = -\frac{1}{4} \), where the value
of $s_2$ is irrelevant.

(c) According to Lemma 4.4 we finally have to consider $(\Lambda^{-2} u) v$ and $u (\Lambda^{-2} v)$. We have to show

$$\|\Lambda^{-1} \frac{\partial}{\partial t} + 5 \varepsilon \Lambda_{-\frac{1}{2} + \epsilon} (\Lambda^{-2} u v)\|_{L^\frac{4}{3}} L^1_t \lesssim \|\Lambda^{-2} u\|_{H^{s_2 + 2}} \|v\|_{H^{s_1 + 1}}.$$  

We argue as in 4. choosing $s_1 = -\frac{7}{14} + \epsilon < s + 2$ and $s_2 = -\frac{5}{14} + \epsilon < s - 1$. In the same way we also obtain

$$\|\Lambda^{-1} \frac{\partial}{\partial t} + 5 \varepsilon \Lambda_{-\frac{1}{2} + \epsilon} (u (\Lambda^{-2} v))\|_{L^\frac{4}{3}} L^1_t \lesssim \|u\|_{H^{s_2 + 2}} \|\Lambda^{-2} v\|_{H^{s_1 + 1}}$$

by the choice $s_1 = \frac{5}{7} < s$, $s_2 = -\frac{1}{2} < s + 1$. The proof of (32) is now complete. \(\square\)

**Proof of (27) and (36).** It suffices to prove

$$\|\Lambda^{-1} \frac{\partial}{\partial t} + \varepsilon Q(u, v)\|_{F^s} \lesssim \|\Lambda u\|_{F^s} \|\Lambda_v v\|_{H^{s_1 + 1}}$$

for $s = \frac{5}{7}$, which immediately implies the case $s > \frac{5}{7}$. We want to use (17).

A. The first part of the $F^s$-norm is handled as follows:

1. For the first term we reduce to the estimate

$$\|uv\|_{H^{s_2 + 2}, 2, 0} \lesssim \|u\|_{H^{s_2 + 2}, 2} \|v\|_{H^{s_1 + 1}}$$

which follows by Prop. 3.4.

2. For the last term we reduce to

$$\|uv\|_{H^{s_2 + 2}, 2} \lesssim \|u\|_{H^{s_2 + 2}, 2} \|v\|_{H^{s_1 + 1}}$$

which also holds by Prop. 3.4.

3. For the second term we want to prove

$$\|uv\|_{H^{s_1 + 1}} \lesssim \|\Lambda^{-2} \Lambda_{-\frac{1}{2} + \epsilon} u\|_{F^s} \|v\|_{H^{s_1 + 1}}$$

Using the fractional Leibniz rule we obtain:

3.1. By Sobolev we have

$$\|u \Lambda^{-\frac{1}{2} - 2 \epsilon} v\|_{H^{s_2}, 2} \lesssim \|u \Lambda^{\frac{1}{2} - 2 \epsilon} v\|_{L^\infty_t L^\frac{4}{3}} \lesssim \|u\|_{L^\infty_t L^\frac{4}{3}} \|\Lambda^{\frac{1}{2} - 2 \epsilon} v\|_{L^\infty_t L^\frac{4}{3}} \lesssim \|u\|_{L^\infty_t L^\frac{4}{3}} \|v\|_{H^{s_1 + 1}}.$$  

3.2.

$$\|(A^{s_1 - \frac{1}{2} - 2 \epsilon} u) v\|_{H^{s_2}, 2} \lesssim \|A^{s_1 - \frac{1}{2} - 2 \epsilon} u\|_{L^\infty_t L^\frac{4}{3}} \|v\|_{L^\infty_t L^\frac{4}{3}} \lesssim \|A^{s_1 - \frac{1}{2} - 2 \epsilon} u\|_{L^\infty_t L^\frac{4}{3}} \|v\|_{H^{s_1 + 1}},$$

where $\frac{1}{7} = \frac{1}{2} - \frac{1}{14} + \epsilon$, $\frac{1}{7} = -\frac{2}{14} + \epsilon$ so that the Sobolev embeddings $H^{s_1 - \frac{1}{2} - 2 \epsilon} \hookrightarrow L^\infty$ and $H^{s_1 + 1} \hookrightarrow L^\frac{4}{3}$ hold.

B. The second part of the $F^s$-norm is handled as follows:

1. The first term on the right hand side of (17) requires the estimate

$$\|\Lambda_{-\frac{1}{2} + \epsilon} \Lambda_1^{-1 + \epsilon} \Lambda_{-\frac{1}{2} + \epsilon} (A^{s_1 - \frac{1}{2} - 2 \epsilon} u) A^{s_1 - \frac{1}{2} - 2 \epsilon} (\Lambda_1^{s_1 - 2 \epsilon} u + 2 u v)\|_{L^\frac{4}{3}} L^1_t$$

$$\lesssim \|\Lambda_{-\frac{1}{2} + \epsilon} \Lambda_1^{-1 + \epsilon} \Lambda_{-\frac{1}{2} + \epsilon} (v)\|_{H^{s_1 + 1}}.$$  

which reduces to
\[ \|A^{-\frac{5}{12} - \frac{5}{3} + 3\epsilon}(uv)\|_{L_t^{14} L_x^4} \lesssim \|u\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon}} \|v\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon - \frac{5}{4} - \epsilon}}. \]

This follows by Prop. 3.5 with parameters \( q = \frac{28}{13} \), \( r = 28 \), \( \sigma = \frac{11}{14} - 3\epsilon \). This requires \( s_1, s_2 < \frac{14}{15} \) and \( s_1 + s_2 \geq \frac{14}{15} \). We choose \( s_1 = \frac{14}{15} - \epsilon < s + \frac{2}{3} - 2\epsilon \), \( s_2 = \frac{1}{3} + \epsilon < s - \frac{5}{3} - 2\epsilon \).

2. The estimate for the second term reduces to
\[ \|A^{-\frac{5}{12} - \frac{5}{3} + 2\epsilon}(u)\|_{L_t^{14} L_x^4} \lesssim \|A^{-\frac{5}{12} - 2\epsilon}u\|_{L_t^{14} L_x^4}, \]
where we used Sobolev’s embedding \( H^{s - \frac{5}{12} - 2\epsilon} \hookrightarrow L^{\frac{8}{7}} \) for \( s > \frac{7}{12} \). Moreover
\[ \|u\|_{L_t^{14} L_x^4} \lesssim \|A^{-\frac{5}{12} - 2\epsilon}u\|_{L_t^{14} L_x^4}, \]
where we used Sobolev’s embedding \( H^{s - \frac{5}{12} - 2\epsilon} \hookrightarrow L^{\frac{8}{7}} \) for \( s > \frac{7}{12} \). Moreover
\[ \|u\|_{L_t^{14} L_x^4} \lesssim \|A^{-\frac{5}{12} - 2\epsilon}u\|_{L_t^{14} L_x^4}. \]

3. The last term on the right hand side of (17) is modified by partial integration:
\[ \Lambda^+_{\frac{5}{12} - 2\epsilon} (A^+_{\frac{5}{12} - 2\epsilon} u A^+_{\frac{5}{12} - 2\epsilon} v) = A_+ u A^+_{\frac{5}{12} - 2\epsilon} v + A^+_{\frac{5}{12} - 2\epsilon} u A_+ A^+_{\frac{5}{12} - 2\epsilon} v. \]

We first consider the second term and reduce to
\[ \|A^{-1 - \frac{5}{12} + 3\epsilon} \Lambda^+_{\frac{5}{12} - 2\epsilon} (uv)\|_{L_t^{14} L_x^4} \lesssim \|u\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon - \frac{5}{4} - \epsilon}} \|v\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon - \frac{5}{4} - \epsilon}}. \]
This is essentially identical with (23).

Finally we reduce the first term to
\[ \|A^{-1 - \frac{5}{12} + 3\epsilon} \Lambda^+_{\frac{5}{12} - 2\epsilon} (uv)\|_{L_t^{14} L_x^4} \lesssim \|u\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon}} \|v\|_{H^{\frac{1}{2} - \frac{5}{2} - 2\epsilon}}. \]

By Prop. 3.15 the left hand side is bounded by \( \|A^{-\frac{7}{12} + 4\epsilon} (uv)\|_{L_t^4 L_x^3} \). Next we prove for a suitable \( p \):
\[ \|uw\|_{H^{\frac{1}{2} + 4\epsilon}} \lesssim \|u\|_{H^{\frac{1}{2} - 3\epsilon}} \|v\|_{H^{\frac{1}{2} - 4\epsilon}}, \]
which by duality is equivalent to
\[ \|uw\|_{H^{\frac{1}{2} - 4\epsilon}} \lesssim \|u\|_{H^{\frac{1}{2} - 3\epsilon}} \|w\|_{H^{\frac{1}{2} - 4\epsilon}}. \]

We use the fractional Leibniz rule and obtain
\[ \|(\Lambda^+ uv)\|_{L_t^p} \lesssim \|\Lambda^+ u\|_{L_t^p} \|v\|_{L_t^3} \lesssim \|v\|_{H^{\frac{1}{2} - 4\epsilon}} \|w\|_{H^{\frac{1}{2} - 4\epsilon}}, \]
where \( \frac{1}{q} = \frac{1}{2} - \frac{1}{2} + \frac{4}{5} - \epsilon \), \( \frac{1}{p} = \frac{6}{7} + \frac{4}{5} - \epsilon \) and \( \frac{1}{p'} = \frac{1}{2} - \frac{4}{5} - \epsilon \), so that by Sobolev \( H^{\frac{1}{2} - 4\epsilon} \hookrightarrow L^q \). Moreover
\[ \|vA^+ w\|_{L_t^q} \lesssim \|v\|_{L_t^3} \|A^+ w\|_{L_t^3}. \]
where \( \frac{1}{r_1} = \frac{1}{2} - \frac{1}{4} \epsilon \), \( \frac{1}{r_2} = \frac{3}{4} + \frac{4}{5} \epsilon \), so that \( H^\frac{1}{r_1} \hookrightarrow L^{r_1} \) and \( H^{\frac{3}{4} - 4 \epsilon} \hookrightarrow H^{\frac{1}{r_2} + 4} \).

Therefore we obtain by Prop. 3.3:

\[
\| \Lambda^{\frac{3}{4} + 4 \epsilon}(uv) \|_{L^1_t L^2_x} \lesssim \| u \|_{L^2_t H^{\frac{3}{4} + 4 \epsilon}_x} \| v \|_{L^2_t H^{\frac{1}{2} + 4 \epsilon}_x} \\
\lesssim \| u \|_{H^{\frac{3}{4} + 2 \epsilon} + \frac{1}{2} + 4 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon}_x} \\
\lesssim \| u \|_{H^{\frac{3}{4} + 2 \epsilon} + \frac{1}{2} + 4 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon}_x},
\]

for \( s > \frac{5}{7} \), as one easily checks.

**Proof of (29).** We may reduce to

\[
\| \Lambda^{\frac{3}{4} - 1} \Lambda^{-1} A_+ Q(u, v) \|_{H^{\frac{1}{2} + 4 \epsilon}_x} \lesssim \| Au \|_{F_s} \| A_+ v \|_{H^{\frac{3}{4} - 1 + 4 \epsilon}_x}.
\]

Now we use (17) and estimate the three terms as follows:

1. The estimate for the first term is reduced to (using the trivial estimate \( \Lambda^{2 \epsilon} u \lesssim \Lambda^{2 \epsilon} u \)):

\[
\| u v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

which follows from Prop. 3.3 using our assumptions \( s > \frac{5}{7} \), \( r > \frac{1}{2} \).

2. The estimate for the last term reduces to

\[
\| u v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

which is also a consequence of Prop. 3.3 under our assumption \( 2s - r > \frac{5}{7} \).

3. The second term requires

\[
\| u v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

We first consider the case \( r \leq \frac{1}{2} - 2 \epsilon \). By duality we have to show

\[
\| u v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

By the fractional Leibniz rule we have to consider two terms.

3.1.

\[
\| (\Lambda^{\frac{3}{4} - r - 2 \epsilon} u) \|_{H^0, \frac{1}{2} - 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

which follows from Prop. 3.3 because \( s_0 + s_1 + s_2 = s - r + 1 - 4 \epsilon > \frac{7}{4} - \frac{1}{3} > \frac{5}{7} \) under our assumption \( 2s - r > \frac{5}{7} \) and \( r \leq \frac{1}{2} - 2 \epsilon \).

3.2.

\[
\| u (\Lambda^{\frac{3}{4} - r - 2 \epsilon} u) \|_{H^0, \frac{1}{2} - 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| u \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon},
\]

Now we consider the case \( r \geq \frac{1}{2} - 2 \epsilon \), which is treated by the fractional Leibniz rule as follows:

\[
\| u v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \lesssim \| (\Lambda^{\frac{3}{4} - r - 2 \epsilon} u) \|_{H^0, \frac{1}{2} - 2 \epsilon} + \| u (\Lambda^{\frac{3}{4} - r - 2 \epsilon} u) \|_{H^0, \frac{1}{2} - 2 \epsilon} + \| u \|_{L^2_t H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 2 \epsilon} + \frac{1}{2} + 2 \epsilon}.
\]

Here we used Prop. 3.3 with parameters \( s_0 = s - r + 1 - 4 \epsilon \), \( s_1 = 0 \), \( s_2 = r - \frac{1}{2} - 2 \epsilon \), so that \( s_0 + s_1 + s_2 = s + \frac{1}{2} - 6 \epsilon > 1 \) and \( s_0 + s_1 + s_2 = s - r + 8 \epsilon > \frac{3}{4} + \frac{1}{2} - 4 \epsilon > \frac{7}{4} \), the latter by our assumption \( 2s - r > \frac{5}{7} \) and \( r \geq \frac{1}{2} - 2 \epsilon \).

**Proof of (33) and (34).** A. The first part of the \( F^s \)-norm in the case of (33) reduces to the following estimate

\[
\| u v \|_{H^{\frac{1}{2} + 1} + \frac{1}{2} + 2 \epsilon} \lesssim \| u \|_{H^{\frac{1}{2} + 1} + \frac{1}{2} + 2 \epsilon} \| v \|_{H^{\frac{1}{2} + 1} + \frac{1}{2} + 2 \epsilon},
\]

which easily follows from the Sobolev multiplication law (Prop. 3.7), because \( 1 - s ) + s + (s + 1) = s + 2 > \frac{5}{7} \), and (34) is treated in the same way.
B. In the case of (33) for the second part of the $F^s$-norm we use Prop. 3.15 and Prop. 3.7 to obtain:

$$
\|\Lambda^{-1/2+5\varepsilon} \Lambda^{-1/2-\varepsilon} (uv)\|_{L^1_t H^s_x L^2_x} \lesssim \|\Lambda^{1/2+7\varepsilon} (uv)\|_{L^1_t H^s_x L^2_x} \\
\lesssim \|u\|_{L^2_t H^s_x} \|v\|_{L^2_t H^s_x} \lesssim \|u\|_{H^s} \|v\|_{H^s}.
$$

In the same way in the case of (34) we obtain that the left hand side of the previous estimate is bounded by \(\|u\|_{H^{s+2}} \|v\|_{H^{s-1}}\), as desired.

\textbf{Proof of (36) and (37).} (36) reduces to the following estimate

$$
\|uv\|_{H^{s-1} - \frac{1}{2} + \varepsilon} \lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|v\|_{H^{s-1} - \frac{1}{2} + \varepsilon},
$$

which follows from Prop. 3.7, because \(s+2 > \frac{3}{2}\). Similarly (37) reduces to

$$
\|uv\|_{H^{s-1} - \frac{1}{2} + \varepsilon} \lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|v\|_{H^{s-1} - \frac{1}{2} + \varepsilon},
$$

which also holds by Sobolev, where we use our assumption \(2s - r + 1 > \frac{5}{2}\).

\textbf{Proof of (38).} A. The estimate for the first part of the $F^s$-norm reduces to

$$
\|uvw\|_{H^{s-1} - \frac{1}{2} + 2\varepsilon} \lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|vw\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|v\|_{H^{s} - \frac{1}{2} + \varepsilon} \|w\|_{H^{s} - \frac{1}{2} + \varepsilon},
$$

which follows from Prop. 3.4, where we use \(2r - s > -1\) and \(r \geq s - 1\) for the first step and \(2s - r > \frac{3}{2}\) for the second step.

B. The estimate for the second part of the $F^s$-norm reduces to

$$
\|\Lambda^{-1/2+5\varepsilon} \Lambda^{-1/2+\varepsilon} (uvw)\|_{\dot{L}^1_t H^s_x L^2_x} \lesssim \|\Lambda^{-1/2+7\varepsilon} (uvw)\|_{\dot{L}^1_t H^s_x L^2_x} \lesssim \|\Lambda^{1/2+5\varepsilon} (uvw)\|_{\dot{L}^1_t H^s_x L^2_x} \\
\lesssim \|u\|_{L^p_t H^{s+\varepsilon} L^2_x} \|v\|_{L^2_t L^{2+\frac{1}{2} - s} x} \lesssim \|u\|_{H^{s+\varepsilon} L^{2+\frac{1}{2} - s}} \|v\|_{L^2_t L^{2+\frac{1}{2} - s}} \|w\|_{L^2_t L^{2+\frac{1}{2} - s}},
$$

which we obtain for \(s > \frac{1}{2}\) and \(r > \frac{1}{2}\) by Prop. 5.13, Sobolev and Prop. 5.1, where \(p = \frac{3}{2} + 1 - 2\varepsilon\), \(q = \frac{1}{2} - \frac{3}{2}\), and \(\theta = \frac{3}{2} - \frac{3}{2}\), so that \(H^{s+\varepsilon} \subset L^2\) and \(H^{s+\varepsilon} \subset L^p\).

\textbf{Proof of (39).} A. We may reduce to

$$
\|u\Lambda^{-1}(vw)\|_{H^{s-1} - \frac{1}{2} + 2\varepsilon} \lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|\Lambda^{-1}(vw)\|_{H^{s-1} - \frac{1}{2} + \varepsilon} \\
\lesssim \|u\|_{H^{s+1} - \frac{1}{2} + \varepsilon} \|v\|_{H^{s} - \frac{1}{2} + \varepsilon} \|w\|_{H^{s} - \frac{1}{2} + \varepsilon},
$$

by our assumption that \(2r - s > -1\), where we use Prop. 3.4 twice.

B. For the second part of the $F^s$-norm we use Prop. 3.7 and obtain:

$$
\|\Lambda^{-1/2+5\varepsilon} \Lambda^{-1/2+\varepsilon} (u\Lambda^{-1}(vw))\|_{\dot{L}^1_t H^s_x L^2_x} \lesssim \|\Lambda^{-1/2+7\varepsilon} (u\Lambda^{-1}(vw))\|_{\dot{L}^1_t H^s_x L^2_x} \\
\lesssim \|u\|_{L^p_t H^{s+\varepsilon} L^2_x} \|\Lambda^{-1}(vw)\|_{L^1_t H^{s+\varepsilon} L^2_x} \lesssim \|u\|_{H^{s+\varepsilon} L^{2+\frac{1}{2} - s}} \|v\|_{H^{s} - \frac{1}{2} + \varepsilon} \|w\|_{H^{s} - \frac{1}{2} + \varepsilon}.
$$

We have to estimate \(\|\Lambda^{3/2}(vw)\|_{\dot{L}^1_t L^2_x}\), which by symmetry and the fractional Leibniz rule reduces to \(\|\Lambda^{3/2} (vw)\|_{\dot{L}^1_t L^2_x}\). By Prop. 3.1 we obtain

$$
\|\Lambda^{3/2} (vw)\|_{\dot{L}^1_t L^2_x} \lesssim \|\Lambda^{3/2} (v)\|_{\dot{L}^1_t L^2_x} \|w\|_{L^{2+\frac{3}{2}}_t L^{2+\frac{3}{2}}_x} \\
\lesssim \|v\|_{H^{s+\varepsilon} - \frac{1}{2} + \varepsilon} \|w\|_{H^{s+\varepsilon} - \frac{1}{2} + \varepsilon} \lesssim \|v\|_{H^{s+\varepsilon} - \frac{1}{2} + \varepsilon} \|w\|_{H^{s+\varepsilon} - \frac{1}{2} + \varepsilon}.
$$
Proof of (41). A. It suffices to consider the case $s = \frac{5}{7}$. We easily obtain the desired estimate by Prop. 3.3

$$\|uvw\|_{H^{\frac{5}{7} + 2\epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|w\|_{H^{\frac{5}{7} + \epsilon}}.$$  

B. For the second part of the $F^s$-norm we use Prop. 3.15 and obtain

$$\|\Lambda^{-1} \Lambda_{\frac{5}{7} + \epsilon} (uvw)\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|\Lambda^{-1} \Lambda_{\frac{5}{7} + \epsilon} (uvw)\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|u\|_{L^{2+}_{t} L^{2+}_{x}} \|v\|_{L^{2+}_{t} L^{2+}_{x}} \|w\|_{L^{2+}_{t} L^{2+}_{x}},$$

where $\frac{1}{p} = \frac{1}{7} + \frac{1}{7} - \frac{2}{7} \epsilon$, $\frac{1}{7} = \frac{1}{2} - \frac{1}{7} \epsilon$, $\frac{1}{7} = \frac{1}{2} - \frac{2}{7} \epsilon$, which implies by Sobolev $H^{\frac{5}{7}} \hookrightarrow L^q$, and Prop. 3.3 implies $\|v\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|v\|_{H^{\frac{5}{7}} + \epsilon}$.

Proof of (43). We obtain by Prop. 3.4

$$\|uvw\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|w\|_{H^{\frac{5}{7} + \epsilon}}.$$  

Proof of (44). A. For the first part of the $F^s$-norm we have to show

$$\|\Lambda^{-1} (uvw) z\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|w\|_{H^{\frac{5}{7} + \epsilon}} \|z\|_{H^{\frac{5}{7} + \epsilon}}.$$

It suffices to consider the minimal value $s = \frac{5}{7}$, which by Proposition 3.4 and Proposition 3.3 can be estimated as follows:

$$\|\Lambda^{-1} (uvw) z\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|\Lambda^{-1} (uvw)\|_{H^{\frac{5}{7} + \epsilon}} \|wz\|_{H^{\frac{5}{7}} + \epsilon} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|w\|_{H^{\frac{5}{7} + \epsilon}} \|z\|_{H^{\frac{5}{7} + \epsilon}}.$$  

B. We obtain by Prop. 3.15

$$\|\Lambda^{-1} \Lambda_{\frac{5}{7} + \epsilon} (\Lambda^{-1} (uvw) z)\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|\Lambda^{-1} \Lambda_{\frac{5}{7} + \epsilon} (\Lambda^{-1} (uvw))\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|\Lambda^{-1} \Lambda_{\frac{5}{7} + \epsilon} (uvw)\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|w\|_{H^{\frac{5}{7} + \epsilon}} \|z\|_{H^{\frac{5}{7} + \epsilon}}.$$  

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{7} - \frac{2}{7} \epsilon$, $\frac{1}{7} = \frac{1}{2} - \frac{1}{7} \epsilon$, $\frac{1}{7} = \frac{1}{2} - \frac{2}{7} \epsilon$, so that by Sobolev $H^{\frac{5}{7}} \hookrightarrow L^q$, and by Prop. 3.3 $\|w\|_{L^{1+}_{t} L^{2+}_{x}} \lesssim \|w\|_{H^{\frac{5}{7} + \epsilon}}$.

Proof of (45). We have to show

$$\|uvwz\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|wz\|_{H^{\frac{5}{7} + \epsilon}}.$$  

By our assumption $2s - r > \frac{3}{2}$ the left hand side is bounded by the term $\|uvwz\|_{H^{\frac{5}{7} + \epsilon}}$. It suffices to prove the remaining estimate for the (minimal) value $s = \frac{5}{7}$. By Proposition 3.4 and Prop. 3.3 we obtain

$$\|uvwz\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|wz\|_{H^{\frac{5}{7} + \epsilon}} \lesssim \|u\|_{H^{\frac{5}{7} + \epsilon}} \|v\|_{H^{\frac{5}{7} + \epsilon}} \|wz\|_{H^{\frac{5}{7} + \epsilon}} \|z\|_{H^{\frac{5}{7} + \epsilon}}.$$  

Remark: Assume $n = 3$. If one would try to use only $H^{\frac{5}{7} + \epsilon}$-spaces as solution spaces as Selberg-Tesfahun [ST], by modifying in Definition 3.2 the $F^s$-norm appropriately by cancelling its second term, our proof of (42) would fail, because the estimate

$$\|uvw\|_{H^{\frac{5}{7} - 2\epsilon} + \frac{1}{7} + 2\epsilon} \lesssim \|u\|_{H^{\frac{5}{7} - 2\epsilon} + \frac{1}{7} + 2\epsilon} \|v\|_{H^{\frac{5}{7} - 2\epsilon} + \frac{1}{7} + 2\epsilon} \|w\|_{H^{\frac{5}{7} - 2\epsilon} + \frac{1}{7} + 2\epsilon}$$

only holds for $s > 1$ (cf. [AFS]). This easily follows from [AFS], where necessary and sufficient conditions are given for such an estimate. The same remark applies
to the proof of (27) and (28) (cf. [61]). The proof of (29) also fails for \( s = \frac{d}{2} + \delta \), \( r = \frac{d}{2} + \delta \) for a sufficient small \( \delta > 0 \). Tesfahun [16] improved the result of [ST] by replacing \( H^{s, \frac{d}{2} + \epsilon} \) by \( H^{s, \frac{d}{2} + \epsilon} \) and could treat the case \( s = \frac{d}{2} + \delta \), \( r = \frac{d}{2} + \delta \), but in order to obtain our result \( s = \frac{d}{2} + \delta \), \( r = \frac{d}{2} + \delta \) a manipulation of the \( H^{s, \delta} \)-spaces seems not to be sufficient. The situation for dimensions \( n \geq 4 \) is similarly (cf. [60] and [77]).

7. Appendix: Proof of Proposition 3.8

The proof relies on the following fundamental result by Foschi and Klainerman.

**Proposition 7.1.** Let \( n \geq 2 \). Assume \( s_0 + s_1 + s_2 + b_0 = \frac{n-1}{2} \), \( b_0 < \frac{n-3}{2} \), \( s_0 < \frac{d}{2} \), \( s_1, s_2 \leq \frac{d}{2} - b_0 \), \( s_1 + s_2 \geq \frac{1}{2} \). Then the following estimate holds:

\[
\| D_{x}^{s_0} (uv) \|_{H^{-s_0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

**Proof.** This immediately follows from [FK], Theorem 1.1 by the transfer principle. \( \Box \)

**Proof of Proposition 3.8** We first prove (12) in the case \( s_0 + s_1 + s_2 = \frac{d}{2} + \epsilon \). By Proposition 3.7 we obtain

\[
\| D_{x}^{s_0} (uv) \|_{H^{-s_0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

In the case \( s_0, s_1, s_2 \geq 0 \) this immediately implies

\[
\| D_{x}^{s_0} (uv) \|_{H^{-s_0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

Combining this with the estimate

\[
\| uv \|_{H^{-s_0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}},
\]

which holds by Proposition 3.7, we obtain (12).

Next consider the case \( s_0 < 0 \). The estimate

\[
\| D_{x}^{s_0} ((D^{-s_0} u) v) \|_{H^{0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}
\]

is equivalent to

\[
\| D_{x}^{s_0} (uv) \|_{H^{0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}},
\]

which holds by Proposition 3.7. Using \( s_1 + s_0 \geq 0 \), \( s_2 \geq s_0 + s_2 \geq 0 \) this implies

\[
\| D_{x}^{s_0} (uv) \|_{H^{0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

By Proposition 3.7 we obtain

\[
\| uv \|_{H^{0, 0}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

so that

\[
\| uv \|_{H^{0, \frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}
\]

and therefore

\[
\| (\Lambda^{-s_0} u) v \|_{H^{0, \frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]

Similarly

\[
\| u (\Lambda^{-s_0} v) \|_{H^{0, \frac{1}{2}+\epsilon}} \lesssim \| u \|_{H^{s_1, \frac{1}{2}+\epsilon}} \| v \|_{H^{s_2, \frac{1}{2}+\epsilon}},
\]

so that (12) follows by the fractional Leibniz rule.

Next consider the case \( s_1 < 0 \) (in the same way the case \( s_2 < 0 \) can be treated). The estimate

\[
\| (D^{-s_1} v) w \|_{H^{0, -\frac{1}{2}+\epsilon}} \lesssim \| D_{x}^{\frac{1}{2}+\epsilon} w \|_{H^{0, 0}} \| w \|_{H^{s_2, \frac{1}{2}+\epsilon}}.
\]
is equivalent to
\[ \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|D_{\frac{1}{2}+\epsilon}v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
By duality this is equivalent to
\[ \|D_{\frac{1}{2}+\epsilon}v\|_{H^{-s_1-s_0,0}} \lesssim \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
which holds by Proposition 7.4. Using \( s_1 + s_0 \geq 0 \) and \( s_2 \geq 0 \) this immediately implies
\[ \|D_{\frac{1}{2}+\epsilon}v\|_{H^{-s_1-s_0,0}} \lesssim \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
Because the estimate
\[ \|uv\|_{H^{-s_1-s_0,0}} \lesssim \|u\|_{H^{0,\frac{1}{4}+\epsilon}} \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
holds by Proposition 3.7 we obtain
\[ \|uv\|_{H^{-s_1-s_0,0}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
By duality this is equivalent to
\[ \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
thus
\[ \|(A^{-s_1}v)w\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \tag{78} \]
Similarly the estimate
\[ \|v(D^{-s_1}w)\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|D_{-\frac{1}{2}+\epsilon}v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}} \]
is equivalent to
\[ \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|D_{-\frac{1}{2}+\epsilon}v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
By duality this is equivalent to
\[ \|D_{\frac{1}{2}+\epsilon}(uv)\|_{H^{s_1+s_0,0}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
which is valid by Proposition 3.7. Using \( s_0 \geq 0 \) and \( s_1 \geq 0 \) this immediately implies
\[ \|D_{\frac{1}{2}+\epsilon}(uv)\|_{H^{-s_1-s_0,0}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
which combined with Proposition 3.7 which gives the estimate
\[ \|uv\|_{H^{-s_1-s_0,0}} \lesssim \|u\|_{H^{s_1+s_0,0}} \|v\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
implies
\[ \|uv\|_{H^{-s_1-s_0,0}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
By duality this is equivalent to
\[ \|vw\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}, \]
and also
\[ \|v(A^{-s_1}w)\|_{H^{0,-\frac{1}{4}+\epsilon}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \tag{79} \]
\( \tag{78} \) and \( \tag{79} \) imply by the fractional Leibniz rule
\[ \|vw\|_{H^{-s_1-s_0,0}} \lesssim \|v\|_{H^{s_1+s_0,0}} \|w\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \]
This is by duality our desired estimate \( \tag{12} \).
Consider now the more general case \( s_0 + s_1 + s_2 \geq \frac{3}{2} + \epsilon \). It is not difficult to see that in the case of (homogeneous) \( H^{s,b} \)-spaces the inequality \( \tag{12} \) may be reduced to the special case considered before.
Moreover a simple application of Proposition 3.7 shows that
\[ \|uv\|_{H^{-s_1-s_0,0}} \lesssim \|u\|_{H^{s_1,0}} \|v\|_{H^{s_2,\frac{1}{4}+\epsilon}}. \tag{80} \]
Estimate (13) now follows by interpolation between (12) and (80).

References

[AFS] P. d'Ancona, D. Foschi, and S. Selberg: Atlas of products for wave-Sobolev spaces on $\mathbb{R}^{1+3}$. Trans. Amer. Math. Soc. 364, (2012), 31-63.

[AFS1] P. d'Ancona, D. Foschi, and S. Selberg: Null structure and almost optimal local well-posedness of the Maxwell-Dirac system. Amer. J. Math. 132 (2010), 771–839.

[FK] D. Foschi and S. Klainerman: Bilinear space-time estimates for homogeneous wave equations. Ann. Scient. Ec. Norm. Sup. 4e ser., 33 (2000), 211-274.

[GV] J. Ginibre and G. Velo: Generalized Strichartz inequalities for the wave equation. J. Functional Anal. 133 (1995), 60-68.

[KM] S. Klainerman and M. Machedon: Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$. Ann. Math. 142 (1995), 39-119.

[KMBT] S. Klainerman and M. Machedon (Appendices by J. Bougain and D. Tataru): Remark on Strichartz-type inequalities. Int. Math. Res. Notices 1996, no.5, 201-220.

[KS] S. Klainerman and S. Selberg: Bilinear estimates and applications to nonlinear wave equations. Communications in Contemporary Mathematics. 4 (2002) 223-295.

[KT] S. Klainerman and D. Tataru: On the optimal local regularity for the Yang-Mills equations in $\mathbb{R}^{4+1}$. Journal of the AMS 12 (1999), 93-116.

[KrSt] J. Krieger and J. Sterbenz: Global regularity for the Yang-Mills equations on high dimensional Minkowski space. Mem. AMS 223 (2013), No. 1047.

[Kr'T] J. Krieger and D. Tataru: Global well-posedness for the Yang-Mills equations in 4+1 dimensions. Small energy. Ann. of Math. 185 (2017), 831893.

[LV] S. Lee and A. Vargas: Sharp null form estimates for the wave equation. Amer. J. Math. 130 (2008), 1279-1326.

[O] S. Oh: Gauge choice for the Yang-Mills equations using the Yang-Mills heat flow and local well-posedness in $H^1$. J. Hyperbolic Differ. Equ. 11 (2014), 1108.

[O1] S. Oh: Finite energy global well-posedness of the Yang-Mills equations on $\mathbb{R}^{1+3}$: an approach using the Yang-Mills heat flow. Duke Math. J. 164 (2015), 16691732.

[P] H. Pecher: Local well-posedness for the (n+1)-dimensional Yang-Mills and Yang-Mills-Higgs system in temporal gauge. Nonl. Diff. Equ. Appl. 23 (2016), 23-40.

[ST1] S. Selberg and A. Tesfahun, Finite-energy global well-posedness of the Maxwell-Klein-Gordon system in Lorenz gauge. Comm. Partial Differential Equations 35 (2010), 1029–1057.

[ST] S. Selberg and A. Tesfahun: Null structure and local well-posedness in the energy class for the Yang-Mills equations in Lorenz gauge. Journal of the European Mathematical Society 18 (2016), 17291752.

[St] J. Sterbenz: Global regularity and scattering for general non-linear wave equations. II. (4+1) dimensional Yang-Mills equations in the Lorentz gauge. Amer. J. Math. 129 (2007), 611664.

[T] T. Tao: Local well-posedness of the Yang-Mills equation in the temporal gauge below the energy norm. J. Diff. Equ. 189 (2003), 366-382.

[Te] A. Tesfahun: Local well-posedness of Yang-Mills equations in Lorenz gauge below the energy norm. Nonlin. Diff. Equ. Appl. 22 (2015), 849-875.