COMPLETE MINIMAL SUBMANIFOLDS
OF COMPACT LIE GROUPS

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ABSTRACT. We give a new method for manufacturing complete minimal sub-
manifolds of compact Lie groups and their homogeneous quotient spaces. For this we make use of harmonic morphisms and basic representation theory of Lie
groups. We then apply our method to construct many examples of compact
minimal submanifolds of the special unitary groups.

1. INTRODUCTION

In this paper we introduce a new method for constructing complete minimal
submanifolds of compact Lie groups and their homogeneous quotient spaces. Our most important ingredients are harmonic morphisms and some
basic representation theory of compact Lie groups.

Complex-valued harmonic morphisms on Riemannian manifolds are har-
monic functions which additionally satisfy the non-linear condition of hori-
zontal conformality. They are in general difficult to find but in several im-
portant cases they can be constructed via the so-called eigenfamily method
described below. The elements of such a family are eigenfunctions of the
Laplace-Beltrami operator diagonalizing a bilinear operator associated with
the non-linear horizontal conformality condition. This is where representa-
tion theory comes into play.

We apply our method to the standard representations of the simple Lie
groups $\text{SU}(n)$, $\text{SO}(n)$ and $\text{Sp}(n)$. This yields well-known eigenfamilies on
these space, already constructed in [7].

Then we focus our attention on the representation $\mathfrak{su}(n)$ of the special
unitary group $\text{SU}(n)$. This gives many new eigenfamilies on $\text{SU}(n)$ and its
various homogeneous quotient spaces as flag manifolds. In the last section
we then construct an interesting continuous family of minimal submanifolds
of $\text{SU}(n)$, all of codimension 2:

Theorem 1.1. Let $H$ be an $n \times n$ complex matrix which has $n$ different
eigenvalues. Then the compact subset

$$M = \{ z = (z_1, \ldots, z_n) \in \text{SU}(n) | \ z_1^t H \bar{z}_2 = 0 \}$$

of the special unitary group is a minimal submanifold of codimension two.
For an introduction to representation theory we highly recommend the excellent text [4].

2. HARMONIC MORPHISMS

Let $M$ and $N$ be two manifolds of dimension $m$ and $n$, respectively. Then a Riemannian metric $g$ on $M$ gives rise to the notion of a Laplacian on $(M,g)$ and real-valued harmonic functions $f : (M,g) \rightarrow \mathbb{R}$. This can be generalized to the concept of a harmonic map $\phi : (M,g) \rightarrow (N,h)$ between Riemannian manifolds being a solution to a semi-linear system of partial differential equations, see [2].

**Definition 2.1.** [3],[8] A map $\phi : (M,g) \rightarrow (N,h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \rightarrow \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, the composition $f \circ \phi : \phi^{-1}(U) \rightarrow \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between Riemannian manifolds is due to Fuglede and Ishihara. For the definition of horizontal (weak) conformality we refer to [2].

**Theorem 2.2.** [3],[8] A map $\phi : (M,g) \rightarrow (N,h)$ between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

The next result of Baird and Eells gives the theory of harmonic morphisms a strong geometric flavour. It shows that when the codomain $N$ is a surface the conditions characterizing harmonic morphisms are independent of conformal changes of the metric on $N$.

**Theorem 2.3.** [1] Let $\phi : (M,g) \rightarrow (N^2,h)$ be a horizontally conformal submersion from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if $\phi$ has minimal fibres.

The following result is very useful when dealing with harmonic morphism from Lie groups and their homogeneous quotient spaces.

**Proposition 2.4.** [6] Let $(M,g)$, $(\hat{M},\hat{g})$ and $(N,h)$ be Riemannian manifolds. Furthermore, let $\phi : (M,g) \rightarrow (N,h)$ be a map, $\pi : (\hat{M},\hat{g}) \rightarrow (M,g)$ be a surjective harmonic morphism and $\hat{\phi} : (\hat{M},\hat{g}) \rightarrow (N,h)$ be the composition $\hat{\phi} = \phi \circ \pi$. Then $\phi$ is a harmonic morphism if and only if $\hat{\phi}$ is a harmonic morphism.

For the general theory of harmonic morphisms, we refer to the standard reference [2] and the regularly updated on-line bibliography [5].

3. THE METHOD OF EIGENFAMILIES

Let $\phi, \psi : (M,g) \rightarrow \mathbb{C}$ be functions on a Riemannian manifold. Then the metric $g$ induces the complex-valued Laplacian $\tau(\phi)$ and the gradient $\nabla \phi$ with values in the complexified tangent bundle $T^C M$ of $M$. We extend the
metric $g$ to be complex bilinear on $T^CM$ and define the symmetric bilinear operator $\kappa$ by

$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$

Two functions $\phi, \psi : M \to \mathbb{C}$ are said to be orthogonal if $\kappa(\phi, \psi) = 0$. With this notation, the harmonicity and horizontal conformality of $\phi : (M, g) \to \mathbb{C}$ take the following form

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$

**Definition 3.1.** [7] Let $(M, g)$ be a Riemannian manifold. Then a set

$$\mathcal{E} = \{\phi_i : M \to \mathbb{C} \mid i \in I\}$$

of complex-valued functions is said to be an eigenfamily for $M$ if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$\tau(\phi) = \lambda \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \phi \psi.$$

The next result is a reformulation of Theorem 2.5 of [7]. It shows that an eigenfamily for a Riemannian manifold can be used to produce a large variety of harmonic morphisms.

**Theorem 3.2.** Let $(M, g)$ be a Riemannian manifold and $\mathcal{E} = \{\phi_1, \ldots, \phi_n\}$ be a finite eigenfamily of complex-valued functions on $M$. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearly independent homogeneous polynomials of the same degree then $\phi : U \to \mathbb{C}P^1$ with

$$\phi = [P(\phi_1, \ldots, \phi_n), Q(\phi_1, \ldots, \phi_n)]$$

is a non-constant harmonic morphism on the open and dense subset

$$U = \{p \in M \mid P(\phi_1(p), \ldots, \phi_n(p)) \neq 0 \quad \text{or} \quad Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.$$

4. Complete Minimal Submanifolds

Let $(M, g)$ be a Riemannian manifold and $\mathcal{E} = \{\phi_1, \ldots, \phi_n\}$ be a finite eigenfamily of complex-valued functions on $M$. Let $P, Q : \mathbb{C}^n \to \mathbb{C}$ be linearly independent homogeneous polynomials of the same degree. For each non-zero $\xi = (\alpha, \beta) \in \mathbb{C}^2$ and $\xi = [\alpha, \beta] \in \mathbb{C}P^1$ we define $M_{\xi}$ by

$$M_{\xi} = \{p \in M \mid \beta \cdot P(\phi_1(p), \ldots, \phi_n(p)) - \alpha \cdot Q(\phi_1(p), \ldots, \phi_n(p)) = 0\}.$$

Then it is clear that $M_{\xi}$ is a complete subset of

$$M = \bigcup_{\xi \in \mathbb{C}P^1} M_{\xi}$$

and if $\xi_1, \xi_2 \in \mathbb{C}P^1$ are different then $M_{\xi_1} \cap M_{\xi_2} = Z$ where

$$Z = \{p \in M \mid P(\phi_1(p), \ldots, \phi_n(p)) = 0 \quad \text{and} \quad Q(\phi_1(p), \ldots, \phi_n(p)) = 0\}.$$

For the above situation we define the map $\Psi_{\xi} : M \to \mathbb{C}$ by

$$\Psi_{\xi}(p) = \beta \cdot P(\phi_1(p), \ldots, \phi_n(p)) - \alpha \cdot Q(\phi_1(p), \ldots, \phi_n(p)).$$
The implicit function theorem tells us that if \(0 \in \mathbb{C}\) is a regular value of \(\Psi \hat{\xi}\) then the inverse image \(M_\xi = \Psi^{-1}(\{0\})\) is a submanifold of \(M\) of codimension two. In that case it follows from Theorems \([3.2, 2.3]\) that the open and dense subset \(M_\xi \setminus Z\) of \(M_\xi\) is minimal in \(M\). Hence \(M_\xi\) is a complete minimal submanifold in \(M\). This gives us an attractive method for producing complete minimal submanifolds of Riemannian manifolds. In Section \([11]\) we apply this to the special unitary groups \(\text{SU}(n)\), after elaborating on the needed harmonic morphisms.

5. Some useful Representation Theory

Let \(G\) be a compact Lie group equipped with a bi-invariant inner product. Let \(g\) be the Lie algebra of \(G\) and fix a maximal torus \(T\) in \(G\) with Lie algebra \(t\). Denote by \(\Lambda_W\) the weight lattice in \((t^\mathbb{C})^*\) and fix a dominant Weyl chamber \(W \subset (t^\mathbb{C})^*\); for any \(\lambda \in \overline{W} \cap \Lambda_W\), denote by \(V_\lambda\) the irreducible representation of \(g^\mathbb{C}\) with highest weight \(\lambda\). Recall that this representation lifts to an irreducible representation of \(G\) if and only if \(\lambda\) is analytically integral.

According to the Peter-Weyl theorem we have an orthogonal decomposition

\[
L^2(G) = \bigoplus_\lambda M(V_\lambda),
\]

where the sum is taken over all analytically integral dominant weights of \(G\) and \(M(V_\lambda)\) denotes the space spanned by the matrix elements of the representation.

We fix a \(G\)-invariant Hermitian inner product on \(V_\lambda\). When the representation is of real (quaternionic) type we also fix an invariant symmetric (skew-symmetric) bi-linear form on \(V_\lambda\). Consider a function \(\phi : G \to \mathbb{C}\) of the form

\[
\phi(g) = q(ga, b) \quad (a, b \in V_\lambda),
\]

where \(q\) is any non-degenerate invariant bi-linear or Hermitian form on \(V_\lambda\). Such a function is an eigenfunction of the Laplacian on \(G\) since

\[
\tau(\phi(g)) = \sum_{X \in B} q(gX^2 a, b) = q(gCa, b),
\]

where \(B\) is an orthonormal basis for \(g\) and

\[
C = \sum_{X \in B} X^2
\]

is the Casimir element in the universal enveloping algebra of \(g^\mathbb{C}\). As the representation is irreducible, \(C\) acts on \(V_\lambda\) as the scalar

\[
\alpha_\lambda = - (|\lambda|^2 + 2 \langle \lambda, \delta \rangle),
\]

where \(\delta\) is half the sum of the positive roots, see Proposition 5.28 of \([9]\). Hence

\[
\tau(\phi) = \alpha_\lambda \phi.
\]
In particular, all the functions in $M(V_{\lambda})$ are eigenfunctions of the Laplacian, all with the same eigenvalue $\alpha_{\lambda}$.

Concerning the $\kappa$-operator, let $\phi, \psi : G \to \mathbb{C}$ be given by $\phi(g) = q(ga, b)$, $\psi(g) = q(gu, v)$ where $a, b, u, v \in V_{\lambda}$. Then

$$\kappa(\phi(g), \psi(g)) = \sum_{X \in B} q(gxa, b)q(gxu, v).$$

Hence we have a good reason to consider the map

$$Q(a, b, c, d) = \sum_{X \in B} q(Xa, b)q(Xc, d).$$

This map is clearly $G$-invariant and its interpretation depends on the type of the form $q$ on $V_{\lambda}$:

1. If $q = \langle \cdot, \cdot \rangle$ is a Hermitian form, then $Q$ may be thought of as a self-adjoint $G$-equivariant endomorphism on $V_{\lambda} \otimes V_{\lambda}^*$ given by

$$\langle Q(a \otimes c), b \otimes d \rangle = Q(a, b, c, d).$$

Alternatively, we can think of $Q$ as a $G$-equivariant endomorphism on $V_{\lambda} \otimes V_{\lambda}^* \cong \text{End}(V_{\lambda})$, defined by

$$\langle Q(a \otimes b), c \otimes d \rangle = Q(a, b, c, d).$$

2. If $q = (\cdot, \cdot)$ is a symmetric bi-linear form (in which case the representation is of real type), then $Q$ may be thought of as a self-adjoint $G$-equivariant endomorphism on $\Lambda^2 V_{\lambda}$ given by

$$(Q(a \wedge b), c \wedge d) = Q(a, b, c, d).$$

Let $W$ denote either $V_{\lambda} \otimes V_{\lambda}$, $V_{\lambda} \otimes V_{\lambda}^*$ or $\Lambda^2 V_{\lambda}$. For any irreducible subrepresentation $W'$ of $W$, $Q$ restricts to a $G$-equivariant endomorphism on $W'$, which, by Shur’s Lemma, must be a multiple of the identity endomorphism on $W'$. Hence there is a scalar $\mu$, such that $Q = \mu q$ on $W'$.

6. The standard representation $\mathbb{C}^n$ of $\text{SU}(n)$

Consider the standard representation $\mathbb{C}^n$ of $\text{SU}(n)$, equipped with the standard Hermitian inner product $\langle \cdot, \cdot \rangle$, and $Q$ as a self-adjoint map

$$Q : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n.$$

Since

$$\mathbb{C}^n \otimes \mathbb{C}^n = \text{Sym}^2(\mathbb{C}^n) \oplus \Lambda^2 \mathbb{C}^n$$

and $\text{Sym}^2(\mathbb{C}^n)$ is irreducible, we restrict $Q$ to $\text{Sym}^2(\mathbb{C}^n)$. If $a, b \in \mathbb{C}^n$, we denote by $a \cdot b$ the image of $a \otimes b$ in $\text{Sym}^2(\mathbb{C}^n)$.

Following Shur’s Lemma this is a scalar multiple $\mu$ of the identity i.e.

$$\langle Q(a \cdot b), c \cdot d \rangle = \mu \langle a \cdot b, c \cdot d \rangle = \mu (\langle a, c \rangle \langle b, d \rangle + \langle a, d \rangle \langle b, c \rangle).$$

In particular, we have

$$\langle Q(a \cdot a), c \cdot d \rangle = 2\mu \langle a, c \rangle \langle a, d \rangle.$$

(6.1)
For $z \in \text{SU}(n)$, we let $\phi_c(z) = \langle za, c \rangle$. As a direct consequence of (6.1) we see that the $\kappa$-operator satisfies

$$
\kappa(\phi_c(z), \phi_d(z)) = \sum_{X \in B} \langle zXa, c \rangle \langle zXa, d \rangle \\
= \sum_{X \in B} \langle Xa, z^{-1}c \cdot z^{-1}d \rangle \\
= \langle Q(a \cdot a), z^{-1}c \cdot z^{-1}d \rangle \\
= \mu(a \cdot a, z^{-1}c \cdot z^{-1}d) \\
= \mu(\langle a, z^{-1}c \rangle \langle a, z^{-1}d \rangle + \langle a, z^{-1}c \rangle \langle a, z^{-1}d \rangle) \\
= 2\mu \langle za, c \rangle \langle za, d \rangle \\
= 2\mu \phi_c(z) \phi_d(z).
$$

These calculations show that for any fixed non-zero element $a \in \mathbb{C}^n$ the set

$$
\mathcal{E}_a = \{ \phi_c : \text{SU}(n) \to \mathbb{C} \mid \phi_c(z) = \langle za, c \rangle, \ c \in \mathbb{C}^n \}
$$

is an eigenfamily on $\text{SU}(n)$. This was already constructed in Theorem 5.2 of [7] using a different approach. The stabilizer subgroup of $\text{SU}(n)$ fixing the element $a$ is isomorphic to $\text{SU}(n-1)$ so $\mathcal{E}_a$ induces an eigenfamily on the odd-dimensional sphere

$$
S^{2n-1} = \text{SU}(n)/\text{SU}(n-1).
$$

The induced local harmonic morphisms live on the complex projective space

$$
\mathbb{C}P^{n-1} = \text{SU}(n)/\text{S(U(1) \times U(n-1))}.
$$

They are clearly holomorphic with respect to the standard Kähler structure.

**Example 6.1.** Any non-zero element $a \in \mathbb{C}^2$ induces the following eigenfamily of complex valued functions

$$
\mathcal{E}_a = \{ \phi_c : \text{SU}(2) \to \mathbb{C} \mid \phi_c(z) = \langle za, c \rangle, \ c \in \mathbb{C}^2 \}.
$$

For linearly independent $c, d \in \mathbb{C}^2$ and non-zero $\hat{\xi} = (\alpha, \beta) \in \mathbb{C}^2$ define $\Psi_{\hat{\xi}} : \text{SU}(2) \to \mathbb{C}$ by

$$
\Psi_{\hat{\xi}} : z \mapsto \beta \langle za, c \rangle - \alpha \langle za, d \rangle.
$$

Then $Z = \{ z \in \text{SU}(2) \mid \langle za, c \rangle = 0 \text{ and } \langle za, d \rangle = 0 \}$ is empty so we have a globally defined harmonic morphism $\phi : \text{SU}(2) \to \mathbb{C}P^1$ given by

$$
\phi : z \mapsto [\langle za, c \rangle, \langle za, d \rangle].
$$

The fibres of this map are the well-known compact Hopf circles in $\text{SU}(2)$. 
7. The standard representation $\mathbb{C}^n$ of $\text{SO}(n)$

Consider the standard representation $\mathbb{C}^n$ of $\text{SO}(n)$, equipped with the standard bi-linear form $(\cdot, \cdot)$, and $Q$ as a self-adjoint map
\[ \mathbb{C}^n \otimes \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n. \]

Since
\[ \mathbb{C}^n \otimes \mathbb{C}^n = \text{Sym}^2(\mathbb{C}^n) \oplus \Lambda^2 \mathbb{C}^n \]
and $\Lambda^2 \mathbb{C}^n$ is irreducible, we restrict $Q$ to $\Lambda^2 \mathbb{C}^n$. Following Shur’s lemma this is a scalar multiple $\mu$ of the identity i.e.
\[ (Q(a \wedge b), c \wedge d) = \mu((a, c)(b, d) - (a, d)(b, c)). \]

In particular, we have
\[ (Q(a \wedge b), a \wedge d) = \mu((a, a)(b, d) - (a, d)(b, a)). \]

For a fixed isotropic element $a \in \mathbb{C}^n$ we now see that the $\kappa$-operator satisfies
\[ \kappa(\phi_b(x), \phi_d(x)) = \sum_X (xa, b)(xa, d) \]
\[ = \sum_X (Xa, x^{-1}b)(Xa, x^{-1}d) \]
\[ = (Q(a \wedge x^{-1}b), a \wedge x^{-1}d) \]
\[ = \mu(a \wedge x^{-1}b, a \wedge x^{-1}d) \]
\[ = \mu((a, a)(x^{-1}b, x^{-1}d) - (a, x^{-1}b)(a, x^{-1}d)) \]
\[ = -\mu \phi_b(x)\phi_d(x). \]

This shows that for a fixed isotropic element $a \in \mathbb{C}^n$ the following set of complex-valued functions is an eigenfamily on $\text{SO}(n)$
\[ E_a = \{ \phi_b : \text{SO}(n) \rightarrow \mathbb{C} \mid \phi_b(x) = (xa, b), \ b \in \mathbb{C}^n \}. \]

These are exactly those constructed in Theorem 4.3 of [7]. The stabilizer subgroup of $\text{SO}(n)$ fixing the isotropic $a \in \mathbb{C}^n$ is isomorphic to $\text{SO}(n - 2)$ so induced local harmonic morphisms live on the complex quadric
\[ Q_{n-2} = \text{SO}(n)/\left(\text{SO}(2) \times \text{SO}(n - 2)\right). \]

These maps are holomorphic with respect to the standard complex structure on $Q_{n-2}$ induced by the holomorphic embedding $Q_{n-2} \hookrightarrow \mathbb{C}P^{n-1}$.

8. The standard representation $\mathbb{C}^{2n}$ of $\text{Sp}(n)$

Consider the standard representation $\mathbb{C}^{2n}$ of $\text{Sp}(n)$ equipped with the standard Hermitian inner product $(\cdot, \cdot)$. The form $Q$ defines a self-adjoint map $Q : \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ by
\[ \langle Q(a \otimes c), b \otimes d \rangle = Q(a, b, c, d) \]

Since
\[ \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} = \text{Sym}^2(\mathbb{C}^{2n}) \oplus \Lambda^2 \mathbb{C}^{2n} \]
and $\text{Sym}^2 \mathbb{C}^{2n}$ is irreducible, we consider the restriction of $Q$ to $\text{Sym}^2 \mathbb{C}^{2n}$. Following Shur’s lemma this is a scalar multiple $\mu$ of the identity i.e.

$$\langle Q(a \cdot c), b \cdot d \rangle = \mu(a \cdot c, b \cdot d) = \mu(\langle a, b \rangle \langle c, d \rangle + \langle a, d \rangle \langle c, b \rangle).$$

In particular, we have

$$Q(a, b, a, d) = \langle Q(a^2), b \cdot d \rangle = 2\mu(\langle a, b \rangle \langle a, d \rangle).$$

As a direct consequence we see that the $\kappa$-operator satisfies

$$\kappa(\phi_c(q), \phi_d(q)) = \sum_{X \in B} \langle qXa, c \rangle \langle qXa, d \rangle$$

$$= \sum_{X \in B} \langle Xa, q^{-1}c \rangle \langle Xa, q^{-1}d \rangle$$

$$= Q(a, q^{-1}c, a, q^{-1}d)$$

$$= 2\mu(\langle a, q^{-1}c \rangle \langle a, q^{-1}d \rangle)$$

$$= 2\mu \phi_c(q) \phi_d(q).$$

This shows that for any fixed element $a \in \mathbb{C}^{2n}$ the following set of complex-valued functions is an eigenfamily on $\text{Sp}(n)$

$$\mathcal{E}_a = \{ \phi_c : \text{Sp}(n) \to \mathbb{C} | \phi_c(q) = \langle qa, c \rangle, c \in \mathbb{C}^{2n} \}.$$  

These are exactly those constructed in Theorem 6.2 in [7]. For a given element $a \in \mathbb{C}^{2n}$ the stabilizer subgroup of $\text{Sp}(n)$ fixing $a$ is isomorphic to $\text{Sp}(n - 1)$ so the induced local harmonic morphisms live on the sphere

$$S^{4n-1} = \text{Sp}(n)/\text{Sp}(n - 1).$$

### 9. The dual representation $(\mathbb{C}^n)^*$ of $\text{SU}(n)$

To the standard representation $\mathbb{C}^n$ of $\text{SU}(n)$ we have the dual representation $(\mathbb{C}^n)^*$ given by

$$\mathbb{C}^n \ni b \mapsto \langle \cdot, b \rangle \in (\mathbb{C}^n)^*.$$  

A calculation, similar to that above, shows that for any fixed non-zero element $a \in \mathbb{C}^{n}$ the set

$$\mathcal{E}_a^* = \{ \phi_c : \text{SU}(n) \to \mathbb{C} | \phi_c(z) = \langle c, za \rangle, c \in \mathbb{C}^n \}$$

is an eigenfamily on $\text{SU}(n)$. The stabilizer subgroup of $\text{SU}(n)$ fixing $a$ is isomorphic to $\text{SU}(n - 1)$ so $\mathcal{E}_a^*$ induces an eigenfamily on $S^{2n-1}$ and the induced local harmonic morphisms live on $\mathbb{C}P^{n-1}$. They are clearly anti-holomorphic with respect to the standard Kähler structure.
10. The representation $\mathfrak{sl}_n(\mathbb{C})$ of $\text{SU}(n)$

As representations of $\text{SU}(n)$, we can identify the tensor product $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ with $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ by declaring $a \otimes b$ to correspond to the linear map $v \mapsto \langle v, b \rangle a \in \mathbb{C}^n$.

The representation $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ of $\text{SU}(n)$ corresponds then to the standard adjoint representation of $\text{SU}(n)$ on $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ i.e. $z \cdot A = zAz^{-1}$. The standard invariant inner product

$$\langle A, B \rangle = \text{trace}(A \cdot B^*)$$

on $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ then translates to the product

$$\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle d, b \rangle$$

on $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ and they are obviously invariant under $\text{SU}(n)$. Furthermore,

$$ (a \otimes b)^* = b \otimes a. $$

The representation $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ of $\text{SU}(n)$ decomposes into irreducible subrepresentations

$$ \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) = \text{span}\{I\} \oplus \mathfrak{sl}_n(\mathbb{C}) $$

where $\mathfrak{sl}_n(\mathbb{C})$ are the trace-free endomorphisms. For $A, B \in \mathfrak{sl}_n(\mathbb{C})$ we define

$$ \phi(z) = \langle z \cdot A, B \rangle = \langle zAz^{-1}, B \rangle \quad (z \in \text{SU}(n)). $$

Since the representation is irreducible, we know that $\phi$ is an eigenfunction of the Laplacian.

To study the $\kappa$-operator, we fix an orthonormal basis $\mathcal{B}$ of $\mathfrak{su}(n)$ and consider the map

$$ Q(A, B, C, D) = \sum_{X \in \mathcal{B}} \langle [X, A], B \rangle \langle [X, C], D \rangle \quad (A, B, C, D \in \mathfrak{sl}_n(\mathbb{C})). $$

If $\phi(z) = \langle z \cdot A, B \rangle$ and $\psi(z) = \langle z \cdot C, D \rangle$, then we have

$$ \kappa(\phi(z), \psi(z)) = Q(z \cdot A, B, z \cdot C, D). $$

Now, it follows easily that

$$ \langle [A, B], C \rangle = \langle A, [C, B^*] \rangle. $$

Furthermore, since $\mathfrak{sl}_n(\mathbb{C})$ is the complexification of $\mathfrak{su}(n)$, $\mathcal{B}$ is a Hermitian basis of $\mathfrak{sl}_n(\mathbb{C})$. Hence

$$ Q(A, B, C, D) = \sum_{X \in \mathcal{B}} \langle [X, A], B \rangle \langle [X, C], D \rangle $$

$$ = \sum_{X \in \mathcal{B}} \langle X, [B, A^*] \rangle \langle X, [D, C^*] \rangle $$

$$ = \sum_{X \in \mathcal{B}} \langle X, [B, A^*] \rangle \langle [D^*, C], X \rangle $$

$$ = \langle [D^*, C], [B, A^*] \rangle. $$
Let us now write $A = a \otimes b$, $B = c \otimes d$, $C = e \otimes f$, $D = g \otimes h$ and assume that $a \perp b$, $e \perp f$, which also ensures that $A, C$ are trace-free. Then
\[
\langle [D^*, C], [B, A^*] \rangle = \langle [h \otimes g, e \otimes f], [c \otimes d, b \otimes a] \rangle = \langle (e, g) h \otimes f - \langle h, f \rangle e \otimes g, (b, d) c \otimes a - \langle c, a \rangle b \otimes d \rangle.
\]

Further $A = C$ i.e. $a = e$, $b = f$ give $Q(A, B, A, D) = -2 \langle A, B \rangle \langle A, D \rangle$, hence
\[
\kappa(\phi(z), \psi(z)) = Q(z \cdot A, B, z \cdot A, D) = -2 \phi(z) \psi(z).
\]

In fact, we see that by fixing $a, b \in \mathbb{C}^n$ with $a \perp b$, then
\[
\{\langle z \cdot (a \otimes b), c \otimes d \rangle \mid c, d \in \mathbb{C}^n\} = \{\langle z \cdot a, c \rangle \langle d \cdot b \rangle \mid c, d \in \mathbb{C}^n\}
\]
is an eigenfamily. As a direct consequence we have the following far going generalization of Example 2.3 of [3].

**Example 10.1.** Equip $\mathbb{C}^n$ with its standard Hermitian scalar product and let $\{e_1, e_2, \ldots, e_n\}$ be some orthonormal basis. Then define the complex-valued functions $\phi_{ik} : SU(n) \to \mathbb{C}$ by
\[
\phi_{ik}(z) = \langle z \cdot e_1, e_i \rangle \langle e_k, z \cdot e_2 \rangle = \langle z_1, e_i \rangle \langle e_k, z_2 \rangle = z_{i1} \bar{z}_{k2}.
\]

It then follows from the above calculations that these functions generate the following complex $n^2$-dimensional eigenfamily on $SU(n)$
\[
\mathcal{E} = \{\phi_A : SU(n) \to \mathbb{C} \mid \phi_A(z) = z_1^t A \bar{z}_2, A \in \mathbb{C}^{n \times n}\}.
\]

Let $K \cong S(D \times U(n - 2))$ be the stabilizer subgroup of $SU(n)$ fixing $e_1 \otimes e_2^*$. Here $D \cong U(1)$ denotes the diagonal of $U(2)$. The set $\mathcal{E}$ induces an eigenfamily on the homogeneous quotient space $SU(n)/S(D \times U(n - 2))$ and the harmonic morphisms, induced by this, actually live on the flagmanifold
\[
F = SU(n)/S(U(1) \times U(1) \times U(n - 2)).
\]

They are not holomorphic with respect to the standard Kähler structure on $F$ induced by the standard Kähler structure on the complex Grassmannian of 2-planes in $\mathbb{C}^n$ via the homogeneous projection map
\[
F \to SU(n)/S(U(2) \times U(n - 2)).
\]

Here we have an alternative proof of the construction in Example 10.1 in the spirit of the [7].
Proof. Let \( \{e_1, e_2, \ldots, e_n \} \) be some orthonormal basis for \( \mathbb{C}^n \) and define the complex-valued functions \( \phi_i, \psi_k : \text{SU}(n) \to \mathbb{C} \) on the special unitary group by
\[
\phi_i(z) = \langle z \cdot e_1, e_i \rangle = z_i, \quad \psi_k(z) = \langle e_k, z \cdot e_2 \rangle = \bar{z}_k.
\]
It then follows from above that
\[
\mathcal{E}_1 = \{ \phi_i \mid i = 1, 2, \ldots, n \} \quad \text{and} \quad \mathcal{E}_2 = \{ \psi_k \mid k = 1, 2, \ldots, n \}
\]
are two eigenfamilies and simple calculation shows that \( \kappa(\phi_i, \psi_k) = 0 \). As a direct consequence of Lemma A.1 in [7] we see that
\[
\mathcal{E} = \{ \phi_i \psi_k \mid i, k = 1, 2, \ldots, n \}
\]
is also an eigenfamily of \( \text{SU}(n) \).
\[ \square \]

With the following example we now generalize further.

**Example 10.2.** Equip \( \mathbb{C}^n \) with its standard Hermitian scalar product and let \( \{e_1, e_2, \ldots, e_n \} \) be some orthonormal basis. Then define the complex-valued functions \( \phi_{ijkl} : \text{SU}(n) \to \mathbb{C} \) by
\[
\phi_{ijkl}(z) = \langle z \cdot e_j, e_i \rangle \langle e_k, z \cdot e_l \rangle = z_{ij} \bar{z}_{kl}, \quad \text{where} \quad i, j, k, l = 1, \ldots, n.
\]
It then follows from the above calculations that, for each \( s \in \mathbb{Z}^+ \) with \( 2s \leq n \), we have an eigenfamily \( \mathcal{E}_s \) on \( \text{SU}(n) \) given by
\[
\mathcal{E}_s = \left\{ \sum_{r=1}^{s} z^t_{2r-1} A_r \bar{z}_{2r} \mid A_r \in \mathbb{C}^{n \times n} \text{ and } r = 1, 2, \ldots, s \right\},
\]
which clearly is a complex vector space of dimension \( n^{2s} \). Let
\[
K \cong S(D \times \cdots \times D \times U(n-2s))
\]
be the stabilizer subgroup of \( \text{SU}(n) \) fixing
\[
e_1 \otimes e_2^*, \ldots, e_{2s-1} \otimes e_{2s}^*.
\]
Here \( D \cong U(1) \) denotes the diagonal of \( U(2) \). The set \( \mathcal{E}_s \) induces an eigenfamily on the homogeneous quotient space \( \text{SU}(n) / S(D \times U(n-2)) \) and the harmonic morphisms, induces by this, actually live on the flagmanifold
\[
\text{SU}(n) / S(U(1) \times \cdots \times U(1)) \times U(n-2))
\]

11. **Compact minimal submanifolds in \( \text{SU}(n) \)**

In this section we use the previous constructions to produce smooth closed minimal submanifolds of \( \text{SU}(n) \). If \( z \) is a matrix in \( \text{SU}(n) \) then we denote its columns by \( z_1, \ldots, z_n \) and write \( z = (z_1, \ldots, z_n) \).

**Theorem 11.1.** Let \( H \) be an \( n \times n \) complex matrix which has \( n \) different eigenvalues. Then the compact subset
\[
M = \{(z_1, \ldots, z_n) \in \text{SU}(n) \mid z^t_1 H \bar{z}_2 = 0 \}
\]
of the special unitary group is a minimal submanifold of codimension two.
Proof. We begin by showing that \( M \) is smooth. To this effect, we let

\[
\Phi(z) = z_1 H z_2 = \langle \text{Ad}_z(H) e_1, e_2 \rangle = \langle \text{Ad}_z(H), e_2 \otimes e_1 \rangle = \langle H, z^{-1}(e_2 \otimes e_1) \rangle.
\]

Note that if \( e_2 \otimes e_1 \) represents the endomorphism \( A \), then

\[
z^{-1}(e_2 \otimes e_1) = z^{-1}(e_2) \otimes z^{-1}(e_1) = z^{-1} A z.
\]

This implies that the gradient of \( \Phi \) satisfies

\[
\nabla \Phi(z) = \sum_{X \in B} \langle [X, H], z^{-1}(e_2 \otimes e_1) \rangle X,
\]

where \( B \) is any orthonormal basis for the Lie algebra \( \mathfrak{su}(n) \). Our goal is thus to find a vector \( X \) in \( \mathfrak{gl}_n(\mathbb{C}) \) such that

\[
\langle [X, H], z^{-1}(e_2 \otimes e_1) \rangle \neq 0.
\]

Since \( H \) has \( n \) distinct eigenvalues, there exists a diagonal matrix \( D \) and a matrix \( P \in \mathfrak{GL}_n(\mathbb{C}) \) such that

\[
H = PDP^{-1}.
\]

We let \( \{\epsilon_1, \ldots, \epsilon_n\} \) be a basis with respect to which \( D \) is diagonal and we denote its eigenvalues by \( \lambda_1, \ldots, \lambda_n \) i.e. we have

\[
D = \sum_{i=1}^{n} \lambda_i \epsilon_i \otimes \epsilon_i.
\]

Thus, if \( X \in \mathfrak{sl}_n(\mathbb{C}) \) then \( [X, H] = P[P^{-1}XP, D]P^{-1} \) and

\[
\langle [X, H], z^{-1}(e_2 \otimes e_1) \rangle = \langle P[P^{-1}XP, D]P^{-1}, z^{-1}(e_2 \otimes e_1) \rangle
\]

\[
= \text{trace}(P[P^{-1}XP, D]P^{-1}(z^{-1}(e_2 \otimes e_1)^*))
\]

\[
= \text{trace}([P^{-1}XP, D]P^{-1}(z^{-1}(e_2 \otimes e_1)^*)P)
\]

\[
= \langle [P^{-1}XP, D], P^*z^{-1}(e_2 \otimes e_1)(P^{-1})^* \rangle
\]

\[
= \langle [P^{-1}XP, D], P^*z^{-1}(e_2 \otimes e_1)(P^*)^{-1} \rangle.
\]

Because \( e_1 \) and \( e_2 \) are orthogonal in \( \mathbb{C}^n \), \( z^{-1}(e_1) \) and \( z^{-1}(e_2) \) are also orthogonal, hence the endomorphisms \( z^{-1}(e_2 \otimes e_1) \) and \( P^*z^{-1}(e_2 \otimes e_1)(P^*)^{-1} \) are nilpotent. On the other hand, \( P^*z^{-1}(e_2 \otimes e_1)(P^*)^{-1} \) is not identically zero, hence it is not diagonal and there exist \( i,j \) such that \( i \neq j \) and

\[
\langle \epsilon_i \otimes \epsilon_j, P^*z^{-1}(e_2 \otimes e_1)(P^*)^{-1} \rangle \neq 0.
\]

We let \( X_0 = P(\epsilon_i \otimes \epsilon_j)P^{-1} \) and obtain

\[
[X_0, D] = [\epsilon_i \otimes \epsilon_j, D] = (\lambda_i - \lambda_j)\epsilon_i \otimes \epsilon_j.
\]

Since \( \lambda_i \neq \lambda_j \) it follows that

\[
\langle [X_0, H], z^{-1}(e_2 \otimes e_1) \rangle \neq 0.
\]

Using Lemma \[11.2\] we derive that \( \nabla \Psi \) is non-zero, hence \( M \) is smooth.
To prove that $M$ is minimal, we choose two linearly independent $A, B \in \mathbb{C}^{n \times n}$ such that $H = A - B$. It then follows from Example 10.1 and Theorem 3.2 that the map $\phi : U \to \mathbb{C}P^1$ with
$$\phi(z) = [z_1^TA\bar{z}_2, z_1^TB\bar{z}_2]$$
is a non-constant harmonic morphism on the open and dense subset
$$U = \{z \in \text{SU}(n) | z_1^TA\bar{z}_2 \neq 0 \text{ or } z_1^TB\bar{z}_2 \neq 0\}.$$The fibre $\phi^{-1}([1,1])$ in $\text{SU}(n)$ is a dense open subset of $M$; it is moreover minimal in $\text{SU}(n)$ by Theorem 2.3. It follows that $M$ is minimal.

Lemma 11.2. Assume that the gradient $\nabla \Psi$ vanishes at a point $z \in \text{SU}(n)$. Then, for all $X \in \mathfrak{gl}(n, \mathbb{C})$
(11.5)$$\langle [X, H], z^{-1}(e_2 \otimes e_1) \rangle = 0.$$Proof. First we notice that if $X = \lambda I$ for some complex number $\lambda$, then $[X, H] = 0$ and (11.5) is automatically verified. So we can assume that $X \in \mathfrak{sl}(n, \mathbb{C})$. If $X$ belongs to $\mathfrak{su}(n)$, then equation (11.5) follows immediately from (11.1). To treat the general case, we notice that $\Psi$ is the restriction to $\text{SU}(n)$ of the holomorphic map $\tilde{\Psi} : \text{SL}(n, \mathbb{C}) \to \mathbb{C}$ with
$$z \mapsto \langle H, z^{-1}(e_2 \otimes e_1) \rangle.$$If $X \in \mathfrak{sl}(n, \mathbb{C})$, there exist $X_1, X_2$ in $\mathfrak{su}(n)$ such that $X = X_1 + iX_2$. Since $\tilde{\Psi}$ is holomorphic,
d$\tilde{\Psi}(X) = d\tilde{\Psi}(X_1) + id\tilde{\Psi}(X_2) = d\Psi(X_1) + id\Psi(X_2) = 0.$$

References
[1] P. Baird and J. Eells, A conservation law for harmonic maps, Geometry Symposium Utrecht 1980, Lecture Notes in Mathematics 894, 1-25, Springer (1981).
[2] P. Baird and J. C. Wood, Harmonic morphisms between Riemannian manifolds, London Math. Soc. Monogr. No. 29, Oxford Univ. Press (2003).
[3] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier 28 (1978), 107-144.
[4] W. Fulton, J. Harris, Representation theory. A first course, Graduate Texts in Mathematics 129. Springer 1991.
[5] S. Gudmundsson, The Bibliography of Harmonic Morphisms, http://www.matematik.lu.se/ matematiklu/personal/sigma/harmonic/bibliography.html
[6] S. Gudmundsson, Harmonic morphisms from complex projective spaces, Geom. Dedicata 53 (1994), 155-161.
[7] S. Gudmundsson and A. Sakovich, Harmonic morphisms from the classical compact semisimple Lie groups, Ann. Global Anal. Geom. 33 (2008), 343-356.
[8] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Soc. Japan 7 (1979), 345-370.
[9] T. Knapp, Lie Groups Beyond an Introduction, Birkhäuser (2002).
