ALL REAL EIGENVALUES OF SYMMETRIC TENSORS

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Abstract. This paper studies how to compute all real eigenvalues of a symmetric tensor. As is well known, the largest or smallest eigenvalue can be found by solving a polynomial optimization problem, while the other middle eigenvalues can not. We propose a new approach for computing all real eigenvalues sequentially, from the largest to the smallest. It uses Jacobian SDP relaxations in polynomial optimization. We show that each eigenvalue can be computed by solving a finite hierarchy of semidefinite relaxations. Numerical experiments are presented to show how to do this.

1. Introduction

Let \( \mathbb{R} \) be the real field, and let \( m \) and \( n \) be positive integers. An \( n \)-dimensional tensor of order \( m \) is an array indexed by integer tuple \((i_1, \ldots, i_m)\) with \( 1 \leq i_j \leq n \) \((j = 1, \ldots, m)\). Let \( T^m(\mathbb{R}^n) \) denote the space of all such real tensors. A tensor \( A \in T^m(\mathbb{R}^n) \) is indexed as

\[
A = (A_{i_1, \ldots, i_m})_{1 \leq i_1, \ldots, i_m \leq n}.
\]

The tensor \( A \) is symmetric if each entry \( A_{i_1, \ldots, i_m} \) is invariant with respect to all permutations of \((i_1, \ldots, i_m)\). Let \( S^m(\mathbb{R}^n) \) be the space of all symmetric tensors in \( T^m(\mathbb{R}^n) \). For \( A \in S^m(\mathbb{R}^n) \), we denote the polynomial

\[
A_x^m := \sum_{1 \leq i_1, \ldots, i_m \leq n} A_{i_1, \ldots, i_m} x_{i_1} \cdots x_{i_m}.
\]

Clearly, \( A_x^m \) is a form (i.e., a homogenous polynomial) of degree \( m \) in \( x := (x_1, \ldots, x_n) \). For a positive integer \( k \leq m \), denote

\[
x^{[k]} := ((x_1)^k, \ldots, (x_n)^k).
\]

Define \( A_x^k \) to be the symmetric tensor in \( S^{m-k}(\mathbb{R}^n) \) such that

\[
(A_x^k)_{i_1, \ldots, i_{m-k} : j_1, \ldots, j_k} := \sum_{1 \leq j_1, \ldots, j_k \leq n} A_{i_1, \ldots, i_{m-k}, j_1, \ldots, j_k} x_{j_1} \cdots x_{j_k}.
\]

So, \( A_x^{m-1} \) is an \( n \)-dimensional vector.

An important property of symmetric tensors is their eigenvalues. Eigenvalues of tensors are introduced in Qi [26] and Lim [19]. Unlike matrices, there are various definitions of eigenvalues for tensors. Useful ones include H-eigenvalues, Z-eigenvalues [26], and D-eigenvalues [30]. Eigenvalues of symmetric tensors have applications in signal processing [28], diffusion tensor imaging (DTI) [4, 30, 31], automatic control [20], etc. The tensor eigenvalue problem is an important subject
of multi-linear algebra. We refer to [13] [27] for introductions to tensors and their applications.

Since there are various definitions of eigenvalues, we here give a unified approach to define them. It is a variation of the approach introduced in [3] [19] [26]. Let \( \mathbb{C} \) be the complex field.

**Definition 1.1.** Let \( A \in \mathbb{S}^m(\mathbb{R}^n) \) and \( B \in \mathbb{S}^{m'}(\mathbb{R}^n) \) be two symmetric tensors (their orders \( m, m' \) are not necessarily equal). A number \( \lambda \in \mathbb{C} \) is a \( B \)-eigenvalue of \( A \) if there exists \( u \in \mathbb{C}^n \) such that

\[
A u^{m-1} = \lambda B u^{m'-1}, \quad B u^{m'} = 1.
\]

Such \( u \) is called a \( B \)-eigenvector associated with \( \lambda \), and such \( (\lambda, u) \) is called a \( B \)-eigenpair.

For cleanness of the paper, when the tensor \( B \) is clear in the context, \( B \)-eigenvalues (resp., \( B \)-eigenvectors, \( B \)-eigenpairs) are just simply called eigenvalues (resp., eigenvectors, eigenpairs). When a \( B \)-eigenvalue \( \lambda \) is real, the associated \( B \)-eigenvector \( u \) is not necessarily real. For convenience, we say that a \( B \)-eigenvalue \( \lambda \) is real if there exists a real nonzero vector \( u \) satisfying (1.1). By the largest (resp., smallest) \( B \)-eigenvalue, we mean the largest (resp., smallest) real \( B \)-eigenvalue. In the paper, we only discuss how to compute real \( B \)-eigenvalues.

The following special cases of \( B \)-eigenvalues are well known.

- When \( m = m' \) and \( B \) is the identity tensor (i.e., \( B x^m = x_1^m + \cdots + x_n^m \)), the \( B \)-eigenvalues are just the H-eigenvalues (cf. [26]). When \( m \) is even, a number \( \lambda \) is a real H-eigenvalue of \( A \) if there exists \( u \in \mathbb{R}^n \) such that

\[
A u^{m-1} = \lambda u^{m-1}, \quad u_1^m + \cdots + u_n^m = 1.
\]

Such \( (\lambda, u) \) is called an H-eigenpair.

- When \( m' = 2 \) and \( B \) is such that \( B x^2 = x_1^2 + \cdots + x_n^2 \), the \( B \)-eigenvalues are just the Z-eigenvalues (cf. [26]). Equivalently, a number \( \lambda \) is a real Z-eigenvalue if there exists \( u \in \mathbb{R}^n \) such that

\[
A u^{m-1} = \lambda u, \quad u_1^2 + \cdots + u_n^2 = 1.
\]

Such \( (\lambda, u) \) is called a Z-eigenpair.

- Let \( D \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix. When \( m' = 2 \) and \( B \) is such that \( B x^2 = x^T D x \), the \( B \)-eigenvalues are just the D-eigenvalues (cf. [30]). Equivalently, a pair \( \lambda \) is a real D-eigenvalue if there exists \( u \in \mathbb{R}^n \) such that

\[
A u^{m-1} = \lambda D u, \quad u^T D u = 1.
\]

Such \( (\lambda, u) \) is called a D-eigenpair.

The problem of computing eigenvalues of higher order tensors (i.e., \( m \geq 3 \)) is NP-hard (cf. [14]). Recently, there exists much work for computing the largest (or smallest) eigenvalues of symmetric tensors. Qi et al. [29] proposed an elimination method for computing the largest Z-eigenvalue when \( (n, m) = (2, 3) \). Hu et al. [12] proposed a sequence of semidefinite relaxations for computing extreme Z-eigenvalues. Kolda et al. [14] proposed a shifted power method for computing Z-eigenvalues. Zhang et al. [35] proposed a modified power method. Han [7] proposed an unconstrained optimization method for even order symmetric tensors. Hao et al. [8] proposed a sequential subspace projection method for computing extreme Z-eigenvalues.
The existing methods are mostly for computing the largest or smallest eigenvalues. However, there are very few methods for computing the other middle eigenvalues. Computing the second or other largest eigenvalues for symmetric tensors is also an important problem in some applications. In DTI [4, 30, 31], the three largest eigenvalues of a diffusion tensor describe the diffusion coefficient in different directions. As shown by Qi et al. [18], the second largest Z-eigenvalue for the characteristic tensor of a hypergraph can be used to get a lower bound for its bipartition width.

The main goal of this paper is to compute all real eigenvalues of a symmetric tensor. For $A \in S^m(\mathbb{R}^n)$, $B \in S^m'(\mathbb{R}^n)$, it holds that
\[
\nabla A x^m = m A x^{m-1}, \quad \nabla B x^{m'} = m' B x^{m'-1}.
\]
Here, the symbol $\nabla$ denotes the gradient in $x$. Thus, (1.1) is equivalent to
\[
\frac{1}{m} \nabla A u^m = \frac{1}{m'} \lambda \nabla B u^{m'}.
\]
Then, $(\lambda, u)$ is a $B$-eigenpair if and only if $u$ is a critical point of the problem
\[
\max A x^m \text{ s.t. } B x^{m'} = 1. \tag{1.2}
\]
Moreover, the critical value associated with $u$ is $\lambda$, because
\[
u^T \nabla A u^m = m A u^m, \quad \nu^T \nabla B u^{m'} = m' B u^{m'}.
\]
This shows that $(\lambda, u)$ is a $B$-eigenpair if and only if $u$ is a critical point of (1.2) with the critical value $\lambda$. The polynomial optimization problem (1.2) has finitely many critical values (cf. [24]). That is, every symmetric tensor $A$ has finitely many $B$-eigenvalues. We order them monotonically as $\lambda_1 > \lambda_2 > \cdots > \lambda_K$. For convenience, denote $\lambda_{\max} := \lambda_1$ and $\lambda_{\min} := \lambda_K$.

In this paper, we study how to compute all real eigenvalues. Mathematically, this is equivalent to finding all the real critical values of (1.2), which is a polynomial optimization problem. The semidefinite relaxation method by Lasserre [15] can be applied to get the largest or smallest eigenvalue. To get other middle eigenvalues, we need to use new techniques. Recently, Nie [24] proposed a method for computing the hierarchy of local minimums in polynomial optimization, which uses the Jacobian SDP relaxation method from [22]. We mainly follow the approach in [24] to compute all real eigenvalues sequentially. Indeed, by this approach, each real eigenvalue can be obtained by solving a finite hierarchy of semidefinite relaxations. This is an attractive property that most other numerical methods do not have.

The paper is organized as follows. In §2 we present some preliminaries in polynomial optimization. In §3 we propose semidefinite relaxations for computing all real eigenvalues sequentially. In §4 we show extensive numerical examples to show how to compute all real eigenvalues.

2. Preliminaries

This section reviews some basics in polynomial optimization. We refer to [5, 16, 17] for details.

Denote by $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ the ring of polynomials in $x := (x_1, \ldots, x_n)$ and with real coefficients. For a degree $d$, $\mathbb{R}[x]_d$ denotes the space of all polynomials in $\mathbb{R}[x]$ whose degrees are at most $d$. The dimension of the space $\mathbb{R}[x]_d$ is $\binom{n+d}{d}$. An ideal of $\mathbb{R}[x]$ is a subset $J$ of $\mathbb{R}[x]$ such that $J \cdot \mathbb{R}[x] \subseteq J$ and $J + J \subseteq J$. For a tuple
The matrix $L$ denotes the symmetric matrix such that
\[ L = \phi_1 \cdot \mathbb{R}[x] + \cdots + \phi_r \cdot \mathbb{R}[x] \]
and is denoted by $I(\phi)$. The set
\[ I_k(\phi) := \phi_1 \mathbb{R}[x]_{k - \deg(\phi_1)} + \cdots + \phi_r \mathbb{R}[x]_{k - \deg(\phi_r)} \]
is called the $k$-th truncated ideal generated by $\phi$. Clearly, the union $\cup_{k \in \mathbb{N}} I_k(\phi)$ is equal to the ideal $I(\phi)$.

A polynomial $\sigma \in \mathbb{R}[x]$ is called a sum of squares (SOS) if there exist $p_1, \ldots, p_k \in \mathbb{R}[x]$ such that $\sigma = p_1^2 + \cdots + p_k^2$. Let $\Sigma[x]$ be the set of all SOS polynomials and
\[ \Sigma[x]_m := \Sigma[x] \cap \mathbb{R}[x]_m. \]

Both $\Sigma[x]$ and $\Sigma[x]_m$ are convex cones. As is well known, each SOS polynomial is nonnegative everywhere, while the reverse is not necessarily true. We refer to [32] for a survey on SOS and nonnegative polynomials. Let $\psi := (\psi_1, \ldots, \psi_t)$ be a tuple of polynomials in $\mathbb{R}[x]$. The set
\[ Q_N(\psi) := \Sigma[x]_{2N} + \psi_1 \Sigma[x]_{2N - \deg(\psi_1)} + \cdots + \psi_t \Sigma[x]_{2N - \deg(\psi_t)} \]
is called the $N$-th quadratic module generated by $\psi$. The union
\[ Q(\psi) := \cup_{N \geq 0} Q_N(\psi) \]
is called the quadratic module generated by $\psi$.

Let $\mathbb{N}$ be the set of nonnegative integers. For $x := (x_1, \ldots, x_n)$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$, denote $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For $d \in \mathbb{N}$, denote
\[ \mathbb{N}^d := \{ \alpha \in \mathbb{N}^n : |\alpha| \leq d \}. \]
The space dual to $\mathbb{R}[x]_d$ is the set of all truncated multisequences (tms') of degree $d$, which is denoted by $\mathbb{R}^{\mathbb{N}^d}$. A vector $y$ in $\mathbb{R}^{\mathbb{N}^d}$ is indexed by $\alpha \in \mathbb{N}^d$, i.e.,
\[ y = (y_\alpha)_{\alpha \in \mathbb{N}^d}. \]

Each $y \in \mathbb{R}^{\mathbb{N}^d}$ defines the linear functional $L_y$ acting on $\mathbb{R}[x]_d$ as
\[ L_y(x^\alpha) = y_\alpha \quad \forall \alpha \in \mathbb{N}^d. \]

Let $q \in \mathbb{R}[x]$. For each $y \in \mathbb{R}^{\mathbb{N}^d}$, the function $L_y(qp^2)$ is a quadratic form in $vec(p)$, the coefficient vector of polynomial $p$ with $\deg(qp^2) \leq 2k$. Let $L_q^{(k)}(y)$ be the symmetric matrix such that
\[ L_q^{(k)}(y) = vec(p)^T \left( L_q^{(k)}(y) \right) vec(p). \]
The matrix $L_q^{(k)}(y)$ is called the $k$-th localizing matrix of $q$ generated by $y$. It is linear in $y$. For instance, when $n = 2$, $k = 2$ and $q = 1 - x_1^2 - x_2^2$, we have
\[
L_{1 - x_1^2 - x_2^2}^{(2)}(y) = \begin{pmatrix} 1 & y_{10} & y_{10} & y_{10} & y_{10} \\ y_{10} & y_{12} & y_{12} & y_{12} & y_{12} \\ y_{10} & y_{10} & y_{10} & y_{10} & y_{10} \\ y_{10} & y_{10} & y_{10} & y_{10} & y_{10} \\ y_{10} & y_{10} & y_{10} & y_{10} & y_{10} \end{pmatrix}.
\]
When $q = 1$ (i.e., the constant one polynomial), $L_q^{(k)}(y)$ is called the $k$-th moment matrix generated by $y$, and it is denoted as $M_k(y)$. For instance, when $n = 2$ and
3. Semidefinite relaxations for real eigenvalues

In this section, we show how to compute all real eigenvalues sequentially. The Jacobian SDP relaxation technique in [22] is a useful tool for this purpose.

Let $\mathcal{A} \in S^m(\mathbb{R}^n)$ and $\mathcal{B} \in S^m(\mathbb{R}^n)$. For convenience, denote $f(x) := \mathcal{A}x^m$ and $g(x) := \mathcal{B}x^m - 1$. Then (1.2) is the same as

$$\max \ f(x) \ s.t. \ g(x) = 0.$$  (3.1)

In the introduction, we have seen that $(\lambda, u)$ is a $\mathcal{B}$-eigenpair if and only if $\lambda$ is a critical value of (3.1), and $u$ is an associated critical point. The problem (3.1) always has finitely many critical values (cf. [24]). So, $\mathcal{A}$ has finitely many real eigenvalues. We order them monotonically as

$$\lambda_1 > \lambda_2 > \cdots > \lambda_K,$$

where $K$ is the total number of distinct real eigenvalues. Denote

$$\mathcal{W} := \{x \in \mathbb{R}^n \mid \text{rank}[\nabla f(x) \ \nabla g(x)] \leq 1\}.$$  

Clearly, if $(\lambda, u)$ is a $\mathcal{B}$-eigenpair of $\mathcal{A}$, then $u \in \mathcal{W}$. Suppose $g(x) = 0$ is a smooth real hypersurface, i.e., $\nabla g(x) \neq 0$ for all real points on $g(x) = 0$. It follows from Definition 1.1 that any $u \in \mathcal{W}$ satisfying $g(u) = 0$ is a $\mathcal{B}$-eigenvector of $\mathcal{A}$ associated with the eigenvalue $\lambda = f(u)$. For the frequently used Z-eigenvalues (i.e., $g(x) = x^T x - 1$) and H-eigenvalues (i.e., $g(x) = x_1^n + \cdots + x_n^n - 1$), the hypersurface $g(x) = 0$ is smooth. The description of the set $\mathcal{W}$ does not use the Lagrange multiplier. This is an advantage in computations. Clearly, a point $u$ belongs to $\mathcal{W}$ if and only if

$$f_{x_i}(u)g_{x_j}(u) - f_{x_j}(u)g_{x_i}(u) = 0 \quad (1 \leq i < j \leq n),$$

where $f_{x_i} = \frac{\partial}{\partial x_i} f(x)$ and $g_{x_i} = \frac{\partial}{\partial x_i} g(x)$. There are totally \binom{n}{2} equations. Indeed, the number of defining equations for $\mathcal{W}$ can be dropped to $2n - 3$ (cf. Bruns and Vetter [11] Chap. 5). It suffices to use the following $2n - 3$ equations (cf. [11][22]):

$$h_r := \sum_{i+j=r+2} (f_{x_i}g_{x_j} - f_{x_j}g_{x_i}) = 0 \quad (r = 1, \cdots, 2n - 3).$$  (3.2)

For convenience, let $h_{2n-2} := g$ and

$$h := (h_1, \ldots, h_{2n-2}).$$  (3.3)

Clearly, (3.3) is equivalent to the maximization problem

$$\max \ f(x) \ s.t. \ h_r(x) = 0 \quad (r = 1, \ldots, 2n - 2).$$  (3.4)

Assume the real hypersurface $g(x) = 0$ is smooth. Then, a point $u$ is feasible for (3.4) if and only if $u$ is a critical point of (3.1), i.e., $u$ is a $\mathcal{B}$-eigenvector. This implies that the objective value of (3.4) at any feasible point is a $\mathcal{B}$-eigenvalue of $\mathcal{A}$. Thus, the feasible objective values are $\lambda_1, \ldots, \lambda_K$.
In the following subsections, we show how to compute the eigenvalues sequentially. That is, we compute \( \lambda_1 \) first, then \( \lambda_2 \) second, and then \( \lambda_3, \ldots \) if they exist.

### 3.1. The largest eigenvalue

The largest eigenvalue \( \lambda_1 \) is the maximal value of problem (3.4). Write the polynomial \( f(x) = Ax^m \) as

\[
f(x) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| = m} f_\alpha x^\alpha.
\]

For a polynomial \( y \in \mathbb{R}^{2N} \) with degree \( 2N \geq m \), denote

\[
\langle f, y \rangle := \sum_{\alpha \in \mathbb{N}^n: |\alpha| = m} f_\alpha y^\alpha.
\]

Clearly, \( \langle f, y \rangle \) is a linear function in \( y \). Denote

\[
N_0 := \lceil (m + m' - 2)/2 \rceil.
\]

Lasserre’s hierarchy of semidefinite relaxations (cf. [15]) for solving (3.4) is

\[
\begin{align*}
\rho_N^{(1)} := \max & \quad \langle f, y \rangle \\
\text{s.t.} & \quad L_{h_r}^{(N)}(y) = 0 \quad (r = 1, \cdots, 2n - 2), \\
& \quad M_N(y) \succeq 0.
\end{align*}
\]

Let \( h \) be the tuple as in (3.3). The dual problem of (3.5) is then

\[
\eta_N^{(1)} := \min \gamma \quad s.t. \quad \gamma - f \in I_{2N}(h) + \Sigma[x]_{2N}.
\]

It can be shown that the optimal values \( \rho_N^{(1)}, \eta_N^{(1)} \) are upper bounds for \( \lambda_1 \). Both sequences \( \{\rho_N^{(1)}\} \) and \( \{\eta_N^{(1)}\} \) are monotonically decreasing. That is

\[
\rho_1^{(1)} \geq \rho_2^{(1)} \geq \cdots \geq \rho_N^{(1)} \geq \cdots \geq \lambda_1,
\]

\[
\eta_1^{(1)} \geq \eta_2^{(1)} \geq \cdots \geq \eta_N^{(1)} \geq \cdots \geq \lambda_1.
\]

By the weak duality, we also have

\[
\rho_N^{(1)} \leq \eta_N^{(1)} \quad (N = N_0, N_0 + 1, \ldots).
\]

In fact, they both have the nice property of converging to \( \lambda_1 \) in finitely many steps, i.e., \( \rho_N^{(1)} = \eta_N^{(1)} = \lambda_1 \) for all \( N \) big enough.

**Theorem 3.1.** Let \( A \in \mathbb{S}^m(\mathbb{R}^n) \) and \( B \in \mathbb{S}^{m'}(\mathbb{R}^n) \). Suppose the real hypersurface \( Bx^{m'} = 1 \) is smooth. Let \( \lambda_1 \) be the largest \( B \)-eigenvalue of \( A \). Then, we have the following properties:

(i) It holds that \( \rho_N^{(1)} = \eta_N^{(1)} = \lambda_1 \) for all \( N \) big enough.

(ii) Suppose \( \lambda_1 \) has finitely many eigenvectors on \( Bx^{m'} = 1 \). If \( N \) is large enough, then, for every optimizer \( y^* \) of (3.5), there exists an integer \( t \leq N \) such that

\[
\text{rank } M_{t-\lambda_0}(y^*) = \text{rank } M_t(y^*).
\]

**Proof.** Note that \(-\lambda_1\) is the minimum value of

\[
\min \quad -f(x) \quad s.t. \quad g(x) = 0.
\]

The polynomials \( h_1, \ldots, h_{2n-3} \) are constructed by using Jacobian SDP relaxations in [22]. The relaxations (3.2), (3.4), (3.5)–(3.6) are specializations of the semidefinite relaxation...
Remark 3.2. The second and other largest eigenvalues. Suppose the $k$-th largest $\mathcal{B}$-eigenvalue $\lambda_k$ of $\mathcal{A}$ is known. We want to compute the $(k+1)$-th largest eigenvalue $\lambda_{k+1}$ if it exists. Let $\delta$ be such that
\[
0 < \delta < \lambda_k - \lambda_{k+1}.
\]
Consider the optimization problem
\[
\begin{align*}
\max & \quad f(x) \\
\text{s.t.} & \quad h_r(x) = 0 \ (r = 1, \ldots, 2n - 2), \\
& \quad f(x) \leq \lambda_k - \delta.
\end{align*}
\]
When (3.8) is satisfied, the optimal value of (3.9) is $\lambda_{k+1}$. Lasserre’s hierarchy of semidefinite relaxations for solving (3.9) is ($N = N_0, N_0 + 1, \ldots$)
\[
\begin{align*}
\rho_{N}^{(k+1)} := \max & \quad \langle f, y \rangle \\
\text{s.t.} & \quad L_{N}^{(r)}(y) = 0 \ (r = 1, \ldots, 2n - 2), \\
& \quad L_{\lambda_k - \delta - f}^{(N)}(y) \geq 0, \quad M_N(y) \geq 0.
\end{align*}
\]
Its dual problem is then
\[
\eta_{N}^{*(k+1)} := \min \gamma \quad \text{s.t.} \quad \gamma - f \in L_N(h) + Q_N(\lambda_k - \delta - f).
\]
Semidefinite relaxations (3.10), (3.11) have the following properties:

**Theorem 3.3.** Suppose the real hypersurface $\mathcal{B}x^{m'} = 1$ is smooth. Let $\lambda_k$ (resp., $\lambda_{k+1}$) be the $k$-th (resp., $(k+1)$-th) largest $\mathcal{B}$-eigenvalue of $\mathcal{A}$. For all $\delta$ satisfying (3.5), we have the following properties:
(i) For all $N$ big enough, we have $\rho_N^{(k+1)} = \eta_N^{(k+1)} = \lambda_{k+1}$.

(ii) Suppose $\lambda_{k+1}$ has finitely many eigenvectors on $Bx^{m'} = 1$. If $N$ is large enough, then for every optimizer $y^\ast$ of (3.10), there exists an integer $t \leq N$ such that (3.7) holds.

Proof. Note that $-\lambda_k$ is the $k$-th smallest critical value of

$$\min_{x} -f(x) \quad \text{s.t.} \quad g(x) = 0.$$  

The polynomials $h_1, \ldots, h_{2n-3}$ are constructed by using Jacobian SDP relaxations in [22]. The semidefinite relaxations (3.9)-(3.11) are specializations of (4.9)-(4.11) in [24]. Thus, the items (i)-(ii) can be obtained by Theorem 4.3 of [24].

Remark 3.4. The finite convergence of $\rho_N^{(k+1)}$ and $\eta_N^{(k+1)}$ to $\lambda_{k+1}$ can be identified by checking the rank condition (3.7). If it is satisfied, we can get all $B$-eigenvectors associated with $\lambda_{k+1}$. When the semidefinite relaxations (3.10) and (3.11) are solved by primal-dual interior point methods, we typically can get all $B$-eigenvectors, provided there are finitely many ones. However, if $\lambda_{k+1}$ has infinitely many $B$-eigenvectors lying on $Bx^{m'} = 1$, (3.7) is typically not satisfied. We refer to Remark 3.2.

In practice, we usually do not know whether $\lambda_{k+1}$ exists or not. Even if it exists, we do not know how small $\delta$ should be chosen to satisfy (3.8). Interestingly, this issue can be fixed by solving the optimization problem

$$\begin{cases}
\chi_k := \min_{x} & f(x) \\
\text{s.t.} & h_r(x) = 0 (r = 1, \ldots, 2n-2), \\
 & f(x) \geq \lambda_k - \delta.
\end{cases}$$

(3.12)

The existence of $\lambda_{k+1}$ and the relation (3.8) can be checked as follows.

Proposition 3.5. Suppose the real hypersurface $Bx^{m'} = 1$ is smooth. Let $\lambda_k$ (resp., $\lambda_{\min}$) be the $k$-th largest (resp., smallest) $B$-eigenvalue of $A$. For all $\delta > 0$, we have the following properties:

(i) The relaxation (3.10) is infeasible for some $N$ if and only if $\lambda_k - \delta < \lambda_{\min}$.

(ii) If $\chi_k = \lambda_k$ and $\lambda_{k+1}$ exists, then $\lambda_{k+1} < \lambda_k - \delta$, i.e., (3.8) holds.

(iii) If $\chi_k = \lambda_k$ and (3.10) is infeasible for some $N$, then $\lambda_{\min} = \lambda_k$.

Proof. (i) This can be implied by Theorem 4.3 (i) of [24].

(ii) Clearly, $\chi_k$ is the smallest $B$-eigenvalue $\geq \lambda_k - \delta$. If $\lambda_{k+1}$ exists and $\chi_k = \lambda_k$, we must have $\lambda_{k+1} < \lambda_k - \delta$.

(iii) From (i), we know $\lambda_k - \delta < \lambda_{\min}$. If otherwise $\lambda_{\min} < \lambda_k$, then $\lambda_{k+1}$ exists and $\lambda_{k+1} < \lambda_k - \delta$ by (ii). This results in the contradiction $\lambda_{k+1} < \lambda_{\min}$. So, $\lambda_{\min} = \lambda_k$.  

For numerical reasons, the number $\delta > 0$ can not be too small. A typical value like 0.05 is preferable in computations. The problem (3.12) is also a polynomial optimization problem. Similar semidefinite relaxations like (3.10)-(3.11) can be constructed to solve it. The hierarchy of such relaxations can also be shown to have finite convergence (cf. [24]). Thus, the optimal value $\chi_k$ of (3.12) can be computed by solving its semidefinite relaxations. If $\chi_k = \lambda_k$, then (3.8) holds; otherwise, decrease the value $\delta$ as $\delta := \delta/5$ and solve (3.12) again. After repeating this process for several times, we can always get $\chi_k = \lambda_k$, and the resulting $\delta$ satisfies (3.8).
We would like to point out that some variations of eigenvalue problems can also be solved by using similar semidefinite relaxations. The largest eigenvalue in an interval \([a, b]\) is the optimal value of the problem

\[
\begin{align*}
\max & \quad f(x) \\
\text{s.t.} & \quad h_r(x) = 0 \ (r = 1, \ldots, 2n - 2), \\
& \quad a \leq f(x) \leq b.
\end{align*}
\]

If in advance we know there exists an eigenvector \(u\) for \(\lambda_{k+1}\) lying in some region, say, defined by some polynomial inequalities \(p_1(x) \geq 0, \ldots, p_s(x) \geq 0\), then we can get such \(u\) by solving the optimization problem

\[
\begin{align*}
\max & \quad f(x) \\
\text{s.t.} & \quad h_r(x) = 0 \ (r = 1, \ldots, 2n - 2), \\
& \quad f(x) \leq \lambda_k - \delta, \\
& \quad p_1(x) \geq 0, \ldots, p_s(x) \geq 0.
\end{align*}
\]

Similar semidefinite relaxations like (3.10)-(3.11) can be constructed to solve such polynomial optimization problems, and we can get the desired eigenpairs.

### 3.3. Getting all real eigenpairs.

We can compute all the real \(B\)-eigenvalues sequentially as follows. First, we compute the largest one \(\lambda_1\) by solving the hierarchy of semidefinite relaxations (3.5)-(3.6). As shown in Theorem 3.1, this hierarchy converges in finitely many steps. After getting \(\lambda_1\), we solve the hierarchy of (3.10)-(3.11) for \(k = 1\). If \(\chi_1 = \lambda_1\) and (3.10) is infeasible for some \(N\), then \(\lambda_1\) is the smallest eigenvalue. If \(\chi_1 = \lambda_1\) and (3.10) is feasible for all \(N\), then \(\rho_N^{(2)} = \lambda_2\), for \(N\) big enough. Repeating this procedure, we can get \(\lambda_3, \lambda_4, \ldots\) if they exist, or we get the smallest eigenvalue and stop.

As above, we get the following algorithm.

**Algorithm 3.6.** Compute all the real \(B\)-eigenvalues of a symmetric tensor \(A\).

1. **Step 0:** Choose a small positive value \(\delta_0\) (e.g., 0.05). Let \(k = 1\).
2. **Step 1:** Solve the hierarchy of (3.10) and get the largest eigenvalue \(\lambda_1\).
3. **Step 2:** Let \(\delta = \delta_0\) and solve the optimal value \(\chi_k\) of (3.12). If \(\chi_k = \lambda_k\), then go to Step 3; If \(\chi_k < \lambda_k\), let \(\delta := \min(\delta/5, \lambda_k - \chi_k)\), and compute \(\chi_k\). Repeat this process until (3.11) holds.
4. **Step 3:** Solve the hierarchy of (3.17). If (3.17) is infeasible for some order \(N\), then \(\lambda_k\) is the smallest eigenvalue and stop. Otherwise, we can get the next largest eigenvalue \(\lambda_{k+1}\).
5. **Step 4:** Let \(k := k + 1\) and go to Step 2.

In Step 2, if \(\chi_k < \lambda_k\), we should expect \(\delta < \lambda_k - \chi_k\). This is why we update \(\delta\) as the minimum of \(\delta/5\) and \(\lambda_k - \chi_k\).

### 4. Numerical experiments

In this section, we present numerical experiments to show how to compute all real eigenvalues. The computations are implemented in a Thinkpad W520 Laptop, with an Intel® dual core CPU at 2.20GHz \(\times\) 2 and 8GB of RAM, in a Windows 7 operating system. We use the software Matlab 2013a and GloptiPoly 3 [10] to solve the semidefinite relaxations for polynomial optimization problems. In the display of numerical results, we only show four decimal digits.
By the definition of \( B \)-eigenvalues as in [1.1], \((\lambda, u)\) is an eigenpair if and only if \((-1)^{m-m'}\lambda, -u)\) is an eigenpair. For \( H \)-eigenvalues \((m = m')\), the \( H \)-eigenvectors always appear in \( \pm \) pairs; so we only list \( H \)-eigenvectors \( u \) satisfying \( \Sigma_i u_i \geq 0 \). For \( Z \)-eigenvalues \((m' = 2)\), when \( m \) is even, the \( Z \)-eigenvectors appear in \( \pm \) pairs, and so we only list those \( u \) satisfying \( \Sigma_i u_i \geq 0 \); when \( m \) is odd, \((\lambda, u)\) is a \( Z \)-eigenpair if and only if \((-\lambda, -u)\) is a \( Z \)-eigenpair, and so they appear in \( \pm \) pairs.

If the rank condition \((3.7)\) is satisfied, then we can get the \( B \)-eigenvalue \( \lambda_k \) and \( \ell := \text{rank} M_4(y^*) \) associated \( B \)-eigenvectors. When primal-dual interior point methods are applied to solve the semidefinite relaxations and \((3.7)\) holds, generally all \( B \)-eigenvectors associated to \( \lambda_k \) can be obtained. We refer to Remarks 3.2 and 3.3. In our numerical experiments, the SDP solver \texttt{SeDuMi} \cite{Bou98} is called by the software \texttt{GloptiPoly 3}. The solver \texttt{SeDuMi} is based on primal-dual interior point methods. So, when the rank condition \((3.7)\) is satisfied, we typically get all \( B \)-eigenvectors of \( \lambda_k \). In such cases, the multiplicities of computed eigenvalues are also known. In the display of our numerical results, we use the notation \( \lambda^{(\ell)} \) to mean that \( \ell \) distinct \( B \)-eigenvectors (modulo scaling) are found for the eigenvalue \( \lambda \).

When \( \lambda_k \) has infinitely many \( B \)-eigenvectors on \( B x^{m'} = 1 \), the rank condition \((3.7)\) is typically not satisfied. When \( \lambda_k \) has finitely many \( B \)-eigenvectors on \( B x^{m'} = 1 \) but the number is big, the order \( N \) for \((3.7)\) to hold might be very high, and the semidefinite relaxations would be very expensive to solve. Sometimes, the semidefinite relaxations can not be solved very accurately if there are ill-conditioning issues in numerical computations. For such cases, it is often very hard to get \( B \)-eigenvectors. Here we propose a practical approach. Let \( \epsilon > 0 \) be small such that \( \lambda_k \) is a unique \( B \)-eigenvalue of \( A \) in the interval \([\lambda_k - \epsilon, \lambda_k + \epsilon]\). Choose a generic vector \( c \in \mathbb{R}^n \) and then solve the problem

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad h_r(x) = 0 (r = 1, \ldots, 2n - 2), \\
& \quad \lambda_k - \epsilon \leq Ax^m \leq \lambda_k + \epsilon.
\end{align*}
\]

When \( c \) is generic, \((4.1)\) has a unique minimizer, which is a \( B \)-eigenvector corresponding to \( \lambda_k \). We can construct semidefinite relaxations like \((3.10)\) for solving \((4.1)\). In practice, when \((3.7)\) is not satisfied, we can often get a \( B \)-eigenvector by solving \((4.1)\). On the other hand, we can only get one, but not all, \( B \)-eigenvector corresponding to \( \lambda_k \). In this section, we use the superscript \((*)\) to mean that a \( B \)-eigenvector is computed by solving \((4.1)\).

**Example 4.1.** \cite{Qi05} Consider the tensor \( A \in \mathbb{S}^4(\mathbb{R}^3) \) such that

\[
Ax^4 = x_1^4 + 2x_2^4 + 3x_3^4.
\]

It is a diagonal tensor (i.e., its entries \( A_{i_1,i_2,i_3} \) are all zeros except for \( i_1 = i_2 = i_3 \)). Its \( Z \)-eigenvalues were computed by Qi \cite{Qi05} Proposition 9). For this tensor, the optimization problem \((3.4)\) is

\[
\begin{align*}
\max & \quad x_1^4 + 2x_2^4 + 3x_3^4 \\
\text{s.t.} & \quad 2x_1x_2^2 - x_2x_1^4 = 0, \ 3x_1x_3^3 - x_3x_1^3 = 0, \\
& \quad 3x_2x_3^3 - 2x_3x_2^3 = 0, \ x_1^2 + x_2^2 + x_3^2 = 1.
\end{align*}
\]

Using Algorithm 3.6, we get all the real \( Z \)-eigenvalues and \( Z \)-eigenvectors, which are shown in Table 4.1. The computation takes about 9 seconds.
Table 4.1. Z-eigenpairs of the tensor in Example 4.1

| $k$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $\lambda_k$ | 3.0000 | 2.0000 | 1.2000 | 1.0000 | 0.7500 | 0.6667 | 0.5455 |
| $u_k$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 0.8660 | 0.8165 | 0.7386 |
|      | 0.0000 | 1.0000 | 0.7746 | 0.0000 | 0.0000 | ±0.5773 | ±0.5222 |
|      | 1.0000 | ±0.6324 | 0.0000 | ±0.5000 | 0.0001 | 0.4264 | 0.4264 |

Example 4.2. For the diagonal tensor $D \in S^5(\mathbb{R}^4)$ such that $Dx^5 = x_1^5 + 2x_2^5 - 3x_3^5 - 4x_4^5$, its orthogonal transformations have the same Z-eigenvalues as $D$ (cf. Qi [26, Theorem 7]). Consider $A \in S^4(\mathbb{R}^4)$ such that $Ax^5 = D(Px)^5$ where

$$P = (I - 2w_1w_1^T)(I - 2w_2w_2^T)(I - 2w_3w_3^T)$$

and $w_1, w_2, w_3$ are randomly generated unit vectors. Using Algorithm 3.6, we get all the real Z-eigenvalues. It takes about 263 seconds. The order $m = 5$ is odd, so the Z-eigenvalues of $A$ appear in ± pairs. Its nonnegative Z-eigenvalues are

4.0000, 3.0000, 2.0000, 1.2163, 1.0000, 0.9611, 0.8543, 0.6057, 0.5550, 0.5402, 0.4805, 0.3887, 0.3466, 0.3261, 0.2518.

For cleanness, the Z-eigenvalues are not shown.

Example 4.3. ([26, Example 3]) Consider the tensor $A \in S^4(\mathbb{R}^3)$ such that

$$Ax^3 = 2x_1^4 + 3x_2^4 + 5x_3^4 + 4ax_1^2x_2x_3,$$

where $a$ is a parameter. The polynomial optimization problem (3.4) is

$$\max \quad 2x_1^4 + 3x_2^4 + 5x_3^4 + 4ax_1^2x_2x_3$$

s.t. $x_1^{p-1}(3x_2^3 + ax_1^2x_3) - x_2^{p-1}(2x_1^3 + 2ax_1x_2x_3) = 0,$

$x_1^{p-1}(5x_3^3 + ax_1^2x_2) - x_3^{p-1}(2x_1^3 + 2ax_1x_2x_3) = 0,$

$x_2^{p-1}(5x_3^3 + ax_1^2x_2) - x_3^{p-1}(3x_2^3 + ax_1^2x_3) = 0,$

$x_1^p + x_2^p + x_3^p = 1,$

where $p = 2$ for Z-eigenvalues and $p = 4$ for H-eigenvalues. Using Algorithm 3.6, we get all the real Z and H eigenvalues, which are shown in Table 4.2. For each value of $a$, it takes a couple of seconds (from 5 to 20). For cleanness, the eigenvectors are not shown.

Example 4.4. ([26, Example 4]) Let $A \in S^4(\mathbb{R}^2)$ be the tensor such that

$$Ax^4 = 3x_1^4 + x_2^4 + 6ax_1^2x_2^2,$$

where $a$ is a parameter. As shown in [26], this tensor always has two Z-eigenvalues 3, 1. When $a < \frac{1}{6}$ or $a > 1$, $A$ has another double Z-eigenvalue

$$\frac{3(9a^3 - 6a^2 - 3a + 2)}{2(3a - 2)^2}.$$

For some values of $a$, the Z-eigenvalues are shown in Table 4.3. For each case of $a$, the computation takes about 1 second.
The bipartition width. The following is such an example. It is a cubic tensor of dimension six. Its Z-eigenvalues appear in non-negative Z-eigenvalues: \[ \lambda = 3 \].

Table 4.2. Z-eigenvalues and H-eigenvalues of the tensor in Example 4.3

| \( \alpha \) | Z-eigenvalues | H-eigenvalues |
| --- | --- | --- |
| 0 | 5.0000 3.0000 2.0000 1.8750(2) 1.2000(2) 0.9670(4) | 5.0000 3.0000 2.0000 1.8750(2) |
| 0.25 | 5.0000 3.0000 2.0000 1.8750(2) 1.4412(2) 1.2150(2) 1.0881(2) 0.8646(2) 0.7243(2) | 5.0000 3.0000 2.0000 1.8750(2) 1.6133(2) 0.4787(2) |
| 0.5 | 5.0000 3.0000 2.0000 1.8750(2) 1.4783(2) 1.2593(2) 1.2060(2) 0.7243(2) | 5.0000 3.0000 2.0000 1.8750(2) |
| 1 | 5.0000 3.0000 2.0000 1.8750(2) 1.6133(2) 0.4787(2) | 5.0000 3.0000 2.0000 1.8750(2) |
| 3 | 5.0000 3.0000 2.2147(2) 2.0000 1.8750(2) | 5.0000 3.0000 2.0000 1.8750(2) |

Table 4.3. Z-Eigenvalues of the tensor in Example 4.4

| \( a \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) |
| --- | --- | --- | --- | --- | --- | --- |
| -1 | 3.0000 | 1.0000 | -0.6000(2) | 3.0000 | 1.0000 | 0.7500(2) |
| 0.25 | 3.0000 | 1.0000 | 0.9750(2) | 3.0000 | 1.0000 | 0.7500(2) |
| 2 | 4.1250(2) | 3.0000 | 1.0000 | |

Example 4.5. (Example 3.5), [24] Example 3.4. Consider the tensor \( A \in S^3(\mathbb{R}^3) \) such that \( A_{111} = 0.2883, A_{112} = -0.0031, A_{113} = 0.1973, A_{122} = -0.2485, A_{123} = -0.2939, A_{133} = 0.3847, A_{122} = -0.2972, A_{123} = 0.1862, A_{133} = 0.0919, A_{222} = 0.1241, A_{223} = -0.3420, A_{223} = 0.2127, A_{333} = 0.2727, A_{333} = -0.3619, A_{333} = 0.8750, A_{333} = -0.3054. Using Algorithm 2.6, we get all the real Z-eigenvalues and Z-eigenvectors. They are shown in Table 4.4. The computation takes about 9 seconds.

Table 4.4. Z-eigenvectors of the tensor in Example 4.5

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \lambda_2 \) | 0.8893 | 0.8160 | 0.5105 | 0.1613 | 0.2884 | 0.2638 | 0.2433 | 0.1733 | -0.0351 | -0.5629 | 1.0995 |
| \( \lambda_3 \) | 0.6672 | 0.2434 | -0.1027 | 0.8412 | -0.2634 | 0.4772 | 0.3598 | -0.4780 | 0.5159 | 0.2670 | 0.5443 |
| \( \lambda_4 \) | 0.7160 | 0.6099 | 0.4362 | 0.6616 | -0.1318 | 0.4425 | 0.8870 | 0.9895 | 0.0947 | -0.1088 | 0.3357 |
| \( \lambda_5 \) | 0.9073 | 0.2531 | 0.7997 | 0.6135 | 0.1250 | 0.1762 | -0.1796 | 0.9678 | -0.5915 | 0.7467 | 0.3043 |

Example 4.6. (Example 9.1) Consider the tensor \( A \in S^3(\mathbb{R}^6) \) such that \( A x^3 = x_1^3 + x_2^3 + 30 x_7^2 x_2 + \cdots + 30 x_7^2 x_6 \).

It is a cubic tensor of dimension six. Its Z-eigenvalues appear in ± pairs. In total, there are 19 non-negative Z-eigenvalues:

16.2345, 15.4552, 15.4298, 10.9710, 8.7347, 8.6596, 8.5979, 8.1888, 7.2165, 6.0000, 5.5674, 5.5688, 5.5218, 5.4817, 5.1402, 4.3358, 4.2464, 4.0225, 3.9992.

It takes about 10870 seconds to compute them. For cleanness, the Z-eigenvectors are not shown.

Characteristic tensors of hypergraphs have important applications, as shown in Li et al. [18]. The second largest Z-eigenvalue can be used to get a lower bound for the bipartition width. The following is such an example.
Example 4.7. ([18, Example 6.4]) Consider the tensor $A \in \mathbb{S}^4(\mathbb{R}^6)$ such that
\[
Ax^4 = (x_1 - x_2)^4 + (x_1 - x_3)^4 + (x_1 - x_4)^4 + (x_1 - x_5)^4 + (x_1 - x_6)^4 \\
+ (x_2 - x_3)^4 + (x_2 - x_4)^4 + (x_2 - x_5)^4 + (x_2 - x_6)^4 \\
+ (x_3 - x_4)^4 + (x_3 - x_5)^4 + (x_3 - x_6)^4 \\
+ (x_4 - x_5)^4 + (x_4 - x_6)^4 + (x_5 - x_6)^4.
\]
The polynomial $Ax^4$ is symmetric in $x$. Every permutation of a Z-eigenvector is also a Z-eigenvector. So, we can add extra conditions $x_1 \geq x_2 \geq \cdots \geq x_6$ to (3.4) and (3.5), while not changing eigenvalues. Then we solve the corresponding semidefinite relaxations. The tensor $A$ has five real Z-eigenvalues, which are respectively
\[
\lambda_1 = 0.0000, \quad \lambda_2 = -4.0000, \quad \lambda_3 = -4.5000, \quad \lambda_4 = -6.0000, \quad \lambda_5 = -7.2000.
\]
The Z-eigenvectors, whose entries are ordered monotonically decreasingly, are shown in Table 4.5. It takes about 44 seconds to get them. In the computation of $\lambda_3$, the rank condition (3.7) is not satisfied. We get one of its Z-eigenvectors by solving (4.4).

Example 4.8. ([34, Example 2]) Consider the tensor $A \in \mathbb{S}^4(\mathbb{R}^5)$ such that
\[
Ax^4 = (x_1 + x_2 + x_3 + x_4)^4 + (x_2 + x_3 + x_4 + x_5)^4.
\]
Using Algorithm 3.6, we get all the three real Z-eigenvalues of this tensor, which are respectively
\[
\lambda_1 = 24.5000, \quad \lambda_2 = 0.5000, \quad \lambda_3 = 0.0000.
\]
It takes about 37 seconds to get them. The Z-eigenvectors are shown in Table 4.6. There are infinitely many Z-eigenvectors for $\lambda_3$. In the computation of $\lambda_3$, the rank condition (3.7) is not satisfied. So, we solve (4.4) and get one of its Z-eigenvectors.

Example 4.9. ([2, Example 5.7]) Consider the cubic tensor $A \in \mathbb{S}^3(\mathbb{R}^3)$ such that
\[
Ax^3 = 2x_1^3 + 3x_1x_2^2 + 3x_1x_3^2.
\]
Using Algorithm 3.6, we get two real Z-eigenvalues, which are $\lambda_1 = 2$ and $\lambda_2 = -2$. Their Z-eigenvectors are $(1, 0, 0)$ and $(-1, 0, 0)$ respectively. It takes about 1 second to compute them.
Example 4.10. ([2] Example 5.8) Consider the tensor $A \in \text{S}^6(\mathbb{R}^3)$ such that
\[ Ax^6 = x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3, \]
which is the Motzkin polynomial. Since $Ax^6$ has only even powers in each of $x_1, x_2, x_3$, we can add the extra conditions $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ to (3.4) and (3.9), while not changing eigenvalues. Then we solve the corresponding semidefinite relaxations. The tensor $A$ has three real $H$-eigenvalues. Using Algorithm 3.6 we get all of them, which are respectively
\[ \lambda_1 = 1.0000, \ \lambda_2 = 0.0555, \ \lambda_3 = 0.0000. \]
The $H$-eigenvectors are shown in Table 4.7. It takes about 21 seconds.

| $k$ | $\lambda_k$ | $u_k^i$ |
|----|-------------|--------|
| 1  | 1.0000(4)   | (0.0000 0.0000 1.0000) |
| 2  | 0.0555(8)   | (0.4487 ±0.9823 ±0.6735) |
| 3  | 0.0000(6)   | (1.0000 0.0000 0.0000) |

Example 4.11. ([25] Example 3.5) Consider the tensor $A \in \text{S}^3(\mathbb{R}^n)$ such that
\[ A_{i,j,k} = \frac{(-1)^i}{i} + \frac{(-1)^j}{j} + \frac{(-1)^k}{k}. \]
For the case $n = 5$, we get all the real $Z$-eigenvalues, which are respectively
\[ \lambda_1 = 9.9779, \ \lambda_2 = 4.2876, \ \lambda_3 = 0.0000, \ \lambda_4 = -4.2876, \ \lambda_5 = -9.9779. \]
The computation takes about 120 seconds. The $Z$-eigenvectors of $\lambda_1, \lambda_2, \lambda_3$ are shown in Table 4.8. The $Z$-eigenvector of $\lambda_4$ (resp., $\lambda_5$) is just the negative of that of $\lambda_2$ (resp., $\lambda_1$). In the computation of $\lambda_3$, the rank condition (3.7) is not satisfied.

| $k$ | $\lambda_k$ | $u_k^i$ |
|----|-------------|--------|
| 1  | 9.9779      | (-0.7313 -0.1375 -0.4674 -0.2365 -0.4146) |
| 2  | 4.2876      | (-0.1859 0.7158 0.2149 0.5655 0.2950) |
| 3  | 0.0000(*)   | ( 0.5072 -0.0980 0.4280 -0.7344 -0.1028) |

We get a $Z$-eigenvector for $\lambda_3$ by solving (4.1).

Example 4.12. ([25]) Consider the tensor $A \in \text{S}^4(\mathbb{R}^n)$ such that
\[ A_{i_1,...,i_4} = \sin(i_1 + i_2 + i_3 + i_4). \]
For the case $n = 5$, we get all the real $Z$-eigenvalues which are respectively
\[ \lambda_1 = 7.2595, \ \lambda_2 = 4.6408, \ \lambda_3 = 0.0000, \ \lambda_4 = -3.9204, \ \lambda_5 = -8.8463. \]
The $Z$-eigenvectors are shown in Table 4.9. It takes about 883 seconds. In the computation of $\lambda_3$, the rank condition (3.7) is not satisfied. We get a $Z$-eigenvector for $\lambda_3$ by solving (4.1).
Table 4.9. Z-eigenpairs of the tensor in Example 4.12

| $k$ | $\lambda_k$ | $u_k^T$ |
|-----|--------------|---------|
| 1   | 7.2595       | $(-0.2686, 0.6150, 0.3959, -0.1872, -0.5982)$ |
| 2   | 4.6408       | $(-0.5055, 0.1228, 0.6382, 0.5669, -0.0256)$ |
| 3   | 0.0000(*)    | $(0.5935, 0.3675, -0.1224, 0.5449, 0.4341)$ |
| 4   | -3.9204      | $(-0.1785, 0.4847, 0.7023, 0.2742, -0.4060)$ |
| 5   | -8.8463      | $(-0.5809, -0.3563, 0.1959, 0.5680, 0.4179)$ |

Example 4.13. Consider the tensor $A \in S^4(\mathbb{R}^n)$ such that

$$A_{i_1,\ldots,i_4} = \tan(i_1) + \tan(i_2) + \tan(i_3) + \tan(i_4).$$

For the case $n=6$, we get all the real Z-eigenvalues which are respectively

$$\lambda_1 = 45.5045, \quad \lambda_2 = 0.0057, \quad \lambda_3 = -133.2871.$$  

The Z-eigenvectors are displayed in Table 4.10. It takes about 797 seconds to compute them. In the computation of $\lambda_2$, the rank condition (3.7) is not satisfied. We get a Z-eigenvector for $\lambda_2$ by solving (4.1).

Example 4.14. Consider the tensor $A \in S^5(\mathbb{R}^n)$ such that

$$A_{i_1,\ldots,i_5} = \ln(i_1) + \cdots + \ln(i_5).$$

For the case $n=4$, we get all the real Z-eigenvalues which are respectively

$$\lambda_1 = 132.3070, \quad \lambda_2 = 0.7074, \quad \lambda_3 = 0.0000, \quad \lambda_4 = -0.7074, \quad \lambda_5 = -132.3070.$$  

The Z-eigenvectors of $\lambda_1, \lambda_2, \lambda_3$ are shown in Table 4.11. The Z-eigenvector of $\lambda_4$ (resp., $\lambda_5$) is just the negative of that of $\lambda_2$ (resp., $\lambda_1$). It takes about 1255 seconds to compute them. In the computation of $\lambda_3$, the rank condition (3.7) is not satisfied.

Table 4.10. Z-eigenpairs of the tensor in Example 4.13

| $k$ | $\lambda_k$ | $u_k^T$ |
|-----|--------------|---------|
| 1   | 45.5045      | $(0.6281, 0.0717, 0.3754, 0.5087, -0.1060, 0.3533)$ |
| 2   | 0.0057(*)    | $(0.5246, -0.0183, 0.2981, 0.2550, -0.5986, -0.4606)$ |
| 3   | -133.2871    | $(0.1936, 0.5222, 0.3429, 0.2287, 0.6272, 0.3559)$ |

get a Z-eigenvector for $\lambda_2$ by solving (4.1).

Example 4.15. (random tensors) An interesting question is the number of real Z-eigenvalues for the symmetric tensors. Cartwright and Sturmfels [2, Theorem 5.5] showed that every symmetric tensor $A$ of order $m$ and dimension $n$ has at most

$$M(m, n) := \frac{(m - 1)^n - 1}{m - 2}$$
distinct complex Z-eigenvalues. In [2], $(\lambda, u)$ and $((-1)^m \lambda, -u)$ are considered to be the same eigenpair. To be consistent with [2], for odd ordered tensors, we here only count their nonnegative Z-eigenvalues. Furthermore, they also showed that when $A$ is generic, $A$ has exactly $M$ distinct complex Z-eigenvalues. Clearly, $M(m,n)$ is an upper bound for the number of real Z-eigenvalues. But it might not be sharp for generic tensors. In this example, we explore possibilities of distributions of the numbers of real Z-eigenvalues. For each $(m,n)$, we generate 50 symmetric tensors randomly. Each symmetric tensor is generated as the symmetrization of a random nonsymmetric tensor $rand\{i_1, \ldots, i_m\}$ in Matlab. The number of their real Z-eigenvalues are shown in Table 4.12. The notation $k^{(\mu)}$ means that there are $\mu$ instances for which the number of real Z-eigenvalues equals to $k$. The table confirms that $M(m,n)$ is an upper bound for the numbers of real Z-eigenvalues. For the case that $(m,n) = (4,4)$ or $(5,4)$, the biggest number of real Z-eigenvalues is about one half of $M(m,n)$. For the other cases of $(m,n)$, the biggest number of real Z-eigenvalues is close to $M(m,n)$. Moreover, the numbers of real Z-eigenvalues are not evenly distributed. We do not know the reason for such distributions.

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### Table 4.12. Numbers of real Z-eigenvalues of random symmetric tensors

| $(m,n)$ | $M(m,n)$ | numbers of real Z-eigenvalues |
|--------|----------|-----------------------------|
| (3, 5) | 31 | 7, 9, 11$^2$, 12$^2$, 13$^2$, 15$^2$, 16, 17$^3$, 18$^2$, 19$^4$, 20$^4$, 21$^6$, 22, 23$^2$, 24, 28 |
| (3, 4) | 15 | 3, 5$^2$, 7$^2$, 9$^2$, 10, 11$^2$, 12, 13$^2$, 15 |
| (3, 3) | 7 | 1, 3$^2$, 5$^2$, 7$^2$ |
| (4, 4) | 40 | 8$^2$, 10$^2$, 12$^2$, 13, 14$^2$, 15, 16$^2$, 18$^2$, 20$^2$, 22$^2$, 23 |
| (4, 3) | 13 | 3$^2$, 5$^2$, 6, 7$^2$, 9$^2$, 11$^2$, 13 |
| (5, 4) | 85 | 15, 17$^2$, 18, 19, 20, 21$^2$, 22$^2$, 23$^2$, 24, 25$^2$, 26$^2$, 27$^2$, 28$^2$, 29$^2$, 30$^2$, 32$^2$, 33$^2$, 34, 37, 39, 41, 42 |
| (5, 3) | 21 | 5$^2$, 7$^2$, 9$^2$, 10, 11$^2$, 12, 13$^2$, 14, 15$^2$, 17$^2$ |
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