BERNSTEIN-ZELEVINSKY DERIVATIVES, BRANCHING RULES AND HECKE ALGEBRAS

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Abstract. Let $G$ be a split reductive group over a $p$-adic field $F$. Let $B$ be a Borel subgroup and $U$ the maximal unipotent subgroup of $B$. Let $\psi$ be a Whittaker character of $U$. We describe the Iwahori-Hecke algebra action on the Gelfand-Graev representation $(\text{ind}_U^G\psi)^I$ by an explicit projective module. As a consequence, for $G = GL(n,F)$, we define and describe Bernstein-Zelevinsky derivatives of representations generated by $I$-fixed vectors in terms of the corresponding Iwahori-Hecke algebra modules. Furthermore, using Lusztig's reductions, we show that the Bernstein-Zelevinsky derivatives can be determined using graded Hecke algebras.

We give two applications of our study. Firstly, we compute the Bernstein-Zelevinsky derivatives of generalized Speh modules, which recovers a result of Lapid-Mínguez and Tadić. Secondly, we give a realization of the Iwahori-Hecke algebra action on some generic representations of $GL(n+1,F)$, restricted to $GL(n,F)$, which is further used to verify a conjecture on an Ext-branching problem of D. Prasad for a class of examples.

1. Introduction

1.1. Bernstein-Zelevinsky derivatives were first introduced and studied in [BZ] and [Ze] and are important for the classification of simple representations of $GL(n,F)$. The derivatives have other applications in representation theory such as branching rules [Pr] and study of $L$-functions.

One goal of this paper is to formulate a functor for Hecke algebras that corresponds to the Bernstein-Zelevinsky derivative and show that the Bernstein-Zelevinsky derivatives can be determined from the corresponding functor. The functor thus provides a framework to understand some problems from the Hecke algebra approach. As an application of our study, we compute the Bernstein-Zelevinsky derivatives of generalized Speh modules, which does not use the determinantal formula of Tadić [Ta] and Lapid-Mínguez [LM] or Kazhdan-Lusztig polynomials [Ze2, CG].

Another consequence of our study attempts to understand the branching problem for the pair $(GL(n+1,F), GL(n,F))$. The Hom-branching problem has been studied extensively, see for example [Pr1, GP, Pr2, CGP, AGRS]. The Ext-branching problems were first initiated and studied by Dipendra Prasad [Pr3]. Another result in this paper is to give a description of the localized Hecke algebra action on some generic representations of $GL(n+1,F)$, considered as representations of $GL(n,F)$, which is used to verify a conjecture of Prasad on Ext-multiplicity for some cases including all spherical generic representations of $GL(n,F)$. 

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1.2. Main results. Let $F$ be a $p$-adic field with the residual field of order $q$. Let $G$ be a split reductive reductive group over $F$. Let $T$ be a maximal split torus in $G$ and let $W$ be the Weyl group. We fix a Chevalley-Steinberg pinning of $G$, and emphasize that the data introduced here depends on the choice of the pinning. Precise definitions are in Section 2. Let $B = TU$ be a Borel subgroup with the maximal unipotent subgroup $U$ and $I$ be an Iwahori subgroup of $G$. The Iwahori-Hecke algebra $H$ is the convolution algebra of $I$-bi-invariant compactly supported functions on $G$. It contains a finite subalgebra $H_W$ of functions supported on the hyperspecial maximal compact subgroup determined by the pinning. As the notation indicates, $H_W$ has a basis $T_w$ of characteristic functions of double cosets parameterized by the Weyl group. The algebra $H_W$ has a one dimensional representation $\text{sgn}$, $T_w \mapsto (-1)^{l(w)}$, where $l$ is a length function on $W$. A prominent role in this paper is played by the element

$$S = \sum_W (-1/q)^{l(w)}T_w \in H_W.$$ 

If $\sigma$ is an $H_W$-module, then $S(\sigma)$ is the $\text{sgn}$-isotypic subspace of $\sigma$. We shall informally call $S$ a sign projector.

Let $\psi$ be a Whittaker character of $U$. Perhaps the most important result in this paper is a description of the space $\text{ind}^G_U \psi$ in terms of Hecke algebra actions:

**Theorem 1.1.** (Corollary 2.5) As $H$-modules, $(\text{ind}^G_U \psi)^I$ is isomorphic to $H \otimes H_W \text{sgn}$.

Bushnell and Henniart [BH] have studied Bernstein components of $\text{ind}^G_U \psi$ and have shown, among other things, that each component is a finitely generated $G$-module. Our result is therefore a refinement of theirs, for the particular component. The use of $H \otimes H_W \text{sgn}$ is independently inspired from the study of [Ch] and [Sa]. We remark that the occurrence of $\text{sgn}$ for representations admitting Whittaker models appeared in the study of Barbasch-Moy [BM]. Our Corollary 2.5 strengthens their result to the category of smooth representations.

**Theorem 1.1** plays an important role in the formulation of the Bernstein-Zelevinsky derivatives in the language of Hecke algebras. Let $G_n = GL(n, F)$ and $\pi$ a smooth representation of $G_n$. The $i$-th Bernstein-Zelevinsky derivative of $\pi$ is a $G_{n-i}$-representation, denoted $\pi^{(i)}$, obtained by applying a twisted Jacquet functor on $\pi$, in which the Whittaker character is involved (see Section 3.2 for the detailed formulation).

Let $I_n$ denote the Iwahori subgroup of $G_n$ and $H_n$ the Iwahori-Hecke algebra. The Weyl group of $G_n$ is isomorphic to the group $S_n$ of all permutation matrices. Let $S_n \in H_{S_n}$ be the sign projector. For every $i = 1, \ldots, n-1$, $H_{n-i} \otimes H_i$ is the Iwahori-Hecke algebra of a Levi subgroup of $G_n$. Using Bernstein’s generators and relations $H_{n-i} \otimes H_i$ can be viewed as a subalgebra of $H_n$. In particular, the map $h \mapsto h \otimes 1$ realizes $H_{n-i}$ as a subalgebra of $H_n$. Let $S_i^n$ be the image in $H_n$ of $1 \otimes S_i$, where $S_i$ is the sign projector in $H_i$. For every $H_n$-module $\sigma$.

\begin{equation}
\text{BZ}_i(\sigma) := S_i^n(\sigma)
\end{equation}

is naturally an $H_{n-i}$-module.
Theorem 1.2. (Theorem 3.3) Let $\pi$ be a smooth representation of $G_n$. Let $\mathcal{B}_i$ be the functor defined in (1.1). There is a natural isomorphism of $\mathcal{H}_{n-i}$-modules

$$(\pi^{(i)})^{I_{n-1}} \cong \mathcal{B}_i(\pi^{I_n}).$$

One then can similarly formulate the Bernstein-Zelevinsky derivative for graded Hecke algebras. We check in Section 4 that Bernstein-Zelevinsky derivatives between affine Hecke algebras and graded Hecke algebras agree under the Lusztig’s reductions. A reason for formulating the Bernstein-Zelevinsky derivatives for the graded Hecke algebra is that the theory of the symmetric group is relatively easier to apply. In particular, we use the Littlewood-Richardson rule for computing the Bernstein-Zelevinsky derivatives of generalized Speh representations. For the detailed notations, one refers to Section 5.

Corollary 1.3. (Corollary 5.2) Let $\pi$ be a generalized Speh representation of $GL(n,F)$ associated to a partition $\bar{n}$ of $n$. Then the $i$-th Bernstein-Zelevinsky derivatives $\pi^{(i)}$ is the direct sum of generalized Speh representations corresponding to the partitions obtained by removing $i$ boxes from $\bar{n}$ but at most one in each row, such that the resulting diagram is still a Young diagram.

The generalized Speh modules correspond to the single $S_n$-type Hecke algebra modules studied by Barbasch-Moy [BM2] and Ciubotaru-Moy [CM]. Because of the simple type structure, their Bernstein-Zelevinsky derivatives can be computed from the theory of symmetric groups.

We remark that Corollary 5.2 is independently proved by Lapid-Mínguez [LM] (following a suggestion of Tadić) and their result also covers a larger class which they call ladder representations.

We now turn to another direction of our study on branching problems for the pair $(GL(n+1,F), GL(n,F))$. A useful tool in studying that problem is the Bernstein-Zelevinsky geometric lemma. More precisely, the geometric lemma says that a smooth representation $\pi$ of $GL(n+1,F)$ restricted to the mirabolic subgroup $E_n$ admits a finite $E_n$-filtration such that the successive quotients can be described in terms of certain induction functors and twisted Jacquet functors (see Theorem 6.1 for the details). We shall call those successive quotients to be the Bernstein-Zelevinsky composition factors. Whittaker characters and Bernstein-Zelevinsky derivatives are involved in defining the functors and hence, in principle, Theorem 1.1 and Theorem 1.2 can be applied to study the Bernstein-Zelevinsky composition factors.

When restricting $\pi$ from $GL(n+1,F)$ to $GL(n,F)$ we shall only consider the Bernstein component (for $GL(n,F)$) of $\pi$ generated by the Iwahori-fixed vectors. Hence the formulation of our results necessitates additional notation involving the Iwahori-Hecke algebra $\mathcal{H}_n$. Let $Z_n$ be the center of $\mathcal{H}_n$ and let $\mathcal{J}$ be a maximal ideal in $Z_n$. Abusing language, representations annihilated by $\mathcal{J}$ will be said to have the central character $\mathcal{J}$. Let $\hat{Z}_n$ be the $\mathcal{J}$-adic completion of $Z_n$. To study the Bernstein-Zelevinsky composition factors, it is easier to deal with their $\mathcal{J}$-adic completions. Let $\hat{\mathcal{H}}_n = \hat{Z}_n \otimes Z \mathcal{H}_n$. The $\mathcal{J}$-adic completion
of an $H_n$-module $\chi$ is the $\hat{H}_n$-module $\hat{\chi} = \hat{\mathbb{Z}}_n \otimes_{\mathbb{Z}_n} \chi$. For a finite-dimensional $H_n$-module $\chi$, the $J$-adic completion is simply the summand of $\chi$ annihilated by a power of $J$.

It is hard to compute the $J$-adic completion of $\pi$ in general. However, some classes of examples of $\pi$, which we call locally nice representations at $J$, have a simple description of the completion. The structure will be explained in Theorem 1.4 below. See Definition 6.4 and Example 6.5 for the term locally nice. We know some immediate examples. For example, if there is unique isomorphism class of irreducible representations annihilated by $J$, then any generic representation $\pi$ of $GL(n+1,F)$ is locally nice at $J$. As another extreme, the Steinberg representation of $GL(n+1,F)$ is locally nice at every central character of $H_n$ (see Theorem 6.12 and Corollary 6.13).

Now we state another consequence of our study:

**Theorem 1.4.** (Theorem 6.8) Let $\pi$ be an irreducible generic representation of $GL(n+1,F)$ and let $I_n$ be the Iwahori subgroup of $GL(n,F)$. Regard $(\pi|_{GL(n,F)})^{I_n}$ as an $H_n$-module. Let $J$ be a maximal ideal in $\mathbb{Z}_n$. Suppose $\pi$ is locally nice at $J$ (see Definition 6.4 and Example 6.5). Then the $J$-adic completion of $(\pi|_{GL(n,F)})^{I_n}$ is isomorphic to $\hat{H}_n \otimes_{H_n} \text{sgn}$ and hence is projective in the category of $H_n$-modules.

For some comments on the proof of Theorem 1.4 see the paragraphs before Theorem 6.8.

One may think that locally nice representations have the simplest local structure. The complication of the local structure of a restricted generic representation starts to increase outside this class, and hence deeper understanding of the structure is needed. Also, determining the central characters at which a generic representation is locally nice is an interesting problem.

As a consequence, we obtain sufficient structural information to verify a conjecture of D. Prasad for those locally nice representations. We first recall the conjecture:

**Conjecture 1.5** (Prasad). [Pr3, Conjecture 1] Let $\pi_1$ be an irreducible generic representation of $GL(n+1,F)$ and let $\pi_2$ be an irreducible generic representation of $GL(n,F)$. Then

$$\text{Ext}^i_{GL_n(F)}(\pi_1, \pi_2) = 0$$

for all $i \geq 1$. (Here $\text{Ext}^i_{GL(n,F)}$ is taken in the category of smooth representations of $GL(n,F)$.)

**Corollary 1.6.** (Corollary 6.9) Let $\pi_2$ be an irreducible generic representation of $GL(n,F)$ with Iwahori fixed vectors with the central character $J$. Suppose $\pi_1$ is an irreducible representation of $GL(n+1,F)$ which is locally nice at $J$. Then

$$\text{Ext}^i_{GL(n,F)}(\pi_1, \pi_2) = 0$$

for all $i \geq 1$. 
The cases we considered in Corollary 1.6 can be seen as the simplest ones in the sense of Theorem 1.4 but still cover some cases that cannot be merely deduced from the Bernstein-Zelevinsky composition factors using the Frobenius reciprocity, central character considerations and the Euler-Poincaré pairing.

1.3. We give some comments on other Bernstein components. We expect that results in this paper hold for other Bernstein components with a suitable reformulation with the use the theory of types by Bushnell-Kutzko [BK, BK2]. However, our approach in Section 2 cannot be adapted directly to other Bernstein components.

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2. Iwahori-fixed vectors for the Gelfand-Graev representation

Let $G$ be a Chevalley group over a $p$-adic field $F$. Let $\mathcal{O}$ be the ring of integers of $F$, let $\varpi$ be the uniformizer of $F$ and let $\mathfrak{p}$ be the maximal ideal in $\mathcal{O}$. Let $q = \text{card}(\mathcal{O}/\mathfrak{p})$. Let $B = TU$ be a Borel subgroup with a maximal unipotent subgroup $U$ and a torus $T$. The torus $T$ determines a root system $R$ and $U$ a set of simple roots $\Pi$ and positive roots $R^+$ for $R$. Let $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. We fix a Chevalley-Steinberg pinning of $G$. In particular, for every $\alpha \in R$, we have a one-parameter subgroup in $G$ whose elements are denoted by $x_\alpha(t)$, where $t \in F$. The group $U$ is generated by $x_\alpha(t)$ for $\alpha \in R^+$. For $\alpha \in R$, let $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$. We let $\hat{s}_\alpha = w_\alpha(1)$, where $s_\alpha$ is a reflection associated to $\alpha \in \Pi$. For a choice of reduced expression of $w = s_{\alpha_1} \ldots s_{\alpha_r} \in W$, we let $\hat{w} = w_{\alpha_1}(1) \ldots w_{\alpha_r}(1)$. It is a representative of $w$ and, for $\alpha \in R$, $\hat{w}x_\alpha(t)^{-1} = x_{w(\alpha)}(ct)$ for some $c \in \mathcal{O}^\times$.

Let $P$ be a closed subgroup of $G$. Let $(\pi, X)$ be a smooth representation of $P$. Denote by $\text{Ind}_P^G \pi$ the normalized induction. Denote by $\text{ind}_P^G \pi$ the normalized compact induction. Denote by $\pi$ the smooth dual of $\pi$.

If $P = MN$ is a parabolic subgroup with the Levi subgroup $M$ and the unipotent radical $N$, denote by $\pi_N$ the normalized Jacquet module of $\pi$.

Let $\psi$ be an additive character of $F$ with conductor $\mathfrak{p}$. Fix a Whittaker character $\psi$ of $U$ such that

$$\psi(\Pi_{\alpha \in R^+} x_\alpha(t_\alpha)) = \overline{\psi} \left( \sum_{\alpha \in \Pi} t_\alpha \right).$$

Let $V = \text{ind}_P^G \psi$. It is the space of smooth functions $f$ on $G$ satisfying

1. $f$ is compactly supported modulo $U$, and
2. $f(ug) = \psi(u)f(g)$ for all $g \in G$, and $u \in U$.

Let $B = TU$ be the Borel subgroup opposite to $B$, i.e. $\bar{U}$ is generated by $x_\alpha(t)$ for all $\alpha \in R^-$. Let $l : W \to \mathbb{Z}$ be the length function on $W$. Let $V^l$ be the subspace of $V$
consisting of all functions in $V$ supported in the union of cells $X_w = UwT\bar{U}$ for all $w \in W$ such that $l(w) \leq r$. Note that each $V_r$ is a $\bar{B}$-submodule of $V$.

**Lemma 2.1.** The inclusion $V_0 \subseteq V$ induces an isomorphism of $T$-modules $(V_0)_T \cong V_T$.

**Proof.** For every $w \in W$, let $V_w$ be the space of smooth functions $f$ on $X_w$ such that $f(ux) = \psi(u)f(x)$ for all $u \in U$ and $x \in X_w$, and such that the support of $f$ is contained in $UwT_f\bar{U}_f$ where $T_f$ is a compact subset of $T$ and $\bar{U}_f$ a compact subset of $\bar{U}$, both depending on $f$. For $r \geq 1$ we have an exact sequence

$$0 \to V_{r-1} \to V_r \to \bigoplus_{l(w)=r} V_w$$

obtained by restricting functions $f \in V_r$ to $X_w$ for $l(w) = r$. Each $V_w$ is an $\bar{U}$-module under the action by right translations. For $\bar{u} \in \bar{U}$, let $R(\bar{u})$ denote the right translation action.

Claim: $(V_w)_T = 0$, if $l(w) > 0$. Proof: If $l(w) > 0$, then there exists an open compact subgroup $\bar{U}_c$ of $\bar{U}$ such that

$$\int_{\bar{U} \cap \bar{U}_c \bar{U}_c^{-1}} \psi(u) \, du = 0.$$ 

Let $f \in V_w$, and assume that $f$ is supported in $UwT_f\bar{U}_f$ where $T_f$ is a compact subset of $T$ and $\bar{U}_f$ a compact subset of $\bar{U}$. We can enlarge $\bar{U}_f$ so that it is a subgroup of $\bar{U}$ and, for every $t \in T_f$, $t\bar{U}_ft^{-1}$ contains $\bar{U}_c$. It is a simple check that

$$\int_{\bar{U}_f} R(\bar{u})(f) \, d\bar{u} = 0.$$ 

This proves the claim.

By the exactness of the Jacquet functor, the claim implies that the inclusion $V_0 \subset V$ of $B$-modules gives an isomorphism $(V_0)_T \cong V_T$ of $T$-modules. \hfill $\square$

**Proposition 2.2.** *(also see [Sa, Theorem 1]*) There exists an isomorphism of $T$-modules

$$\Phi : V_T \to C_c^\infty(T)$$

**Proof.** By Lemma 2.1 it suffices to construct an isomorphism of $T$-modules between $(V_0)_T$ and $C_c^\infty(T)$. An element in $V_0$ is a function supported on the open cell $UT\bar{U}$ and the restriction to $T\bar{U}$ gives a bijection between $V_0$ and compactly supported functions on $T\bar{U}$. Fix an invariant measure on $\bar{U}$ such that the measure of $\bar{U} \cap I$ is 1. (This is a natural normalization coming from the pinning.) It is easy to check that the map from $V_0$ to $C_c^\infty(T)$ defined by

$$f \mapsto f_U(t) = \int_{\bar{U}} f(t\bar{u}) \, d\bar{u}$$

descends to an isomorphism of $(V_0)_T$ and $C_c^\infty(T)$. This gives $\Phi$. \hfill $\square$
2.1. **Iwahori-Hecke algebra action.** The choice of Chevalley-Steinberg pinning gives a structure to $G$ of a group scheme over $\mathcal{O}$ such that $G(\mathcal{O})$ is a hyperspecial maximal compact subgroup. Let $I$ be the Iwahori subgroup of $G$ which is the inverse image of $\bar{B}(\mathcal{O}/p)$ under the map $G(\mathcal{O}) \rightarrow G(\mathcal{O}/p)$. Let $\mathcal{H} = C_c(I \setminus G/I)$ be the convolution algebra of compactly supported $I$-bi-invariant functions on $G$. The double cosets are parameterized by an extended affine Weyl group $W = N_G(T)/T(\mathcal{O})$. For $w \in W$, let $T_w$ be the characteristic function of the double coset $IwI$. We shall normalize the measure on $G$ such that $T_1$ is an identity element, equivalently, the volume $\text{vol}(I) = 1$.

Recall that $q = \text{card}(\mathcal{O}/p)$. Define the length function $l : W \rightarrow \mathbb{Z}$ such that

$$q^l(w) = |IwI : I| = |I : (I \cap w^{-1}Iw)|$$

Then we have $T_{w_1}T_{w_2} = T_{w_1w_2}$ if $l(w_1w_2) = l(w_1) + l(w_2)$ and $(T_s - q)(T_s + 1) = 0$ for $l(s) = 1$.

Let $X = \text{Hom}(G_m, T)$ be the co-character lattice. Then $T \cong X \otimes_{\mathbb{Z}} F^\times$, and $X$ can be considered a subgroup of $T$ by the homomorphism $x \mapsto \hat{x} = x \otimes \varpi^{-1}$. (Note the inverse!) This homomorphism gives a bijection $X \cong T/T(\mathcal{O})$. It extends to an isomorphism between a semi-direct product of $X$ and $W$ and $W_{ex}$ by mapping $w \in W$ to its representative $\bar{w} \in N_G(T)$ defined earlier. Let $\langle \cdot, \cdot \rangle$ be the natural pairing between the co-character and character lattices. Let

$$X_{\text{dom}} = \{ x \in X : \langle x, \alpha \rangle \geq 0 \}.$$ 

Any element $x \in X$ can be written as a linear combination as $x = y - z$ for $y, z \in X_{\text{dom}}$. Following from Bernstein, let $\theta_x = q^{-l(y) - l(z)}/2 Ty T^{-1}$. Let $A$ be the commutative subalgebra of $\mathcal{H}$ generated by $\theta_x$ for $x \in X$. The algebra $A$ is isomorphic to the group algebra $\mathbb{C}[X]$, by the isomorphism $x \mapsto \theta_x$.

For a smooth representation $(\pi, E)$ of $G$, denote by $E^I$ or, abusing notation by $\pi^I$ if the vector space $E$ is not specified, the subspace of $I$-fixed vectors of $\pi$. The space $\pi^I$ is equipped with a $\mathcal{H}$-module structure by convolution.

Let $I_T = I \cap T = T(\mathcal{O})$. For any $T$-module, the subspace of $I_T$-fixed vectors is a module for $T/T(\mathcal{O}) \cong X$. Thus, it is a $\mathbb{C}[X]$-module. We have the following theorem, due to Borel, Casselman, Matsumoto and Bernstein [Bo]:

**Theorem 2.3.** Let $(\pi, E)$ be a smooth $G$-module. As $A \cong \mathbb{C}[X]$-modules

$$E^I \cong E^I_{U^T}$$

The isomorphism map is defined from the natural map from $E$ to $E_{U^T}$.

We shall apply this result to $V = \text{ind}_G^I(\psi)$. By Proposition 2.3 we $V_U \cong C_c^\infty(T)$. Note that $C_c^\infty(T)^{I_T} \cong C_c(T/I_T) \cong \mathbb{C}[X]$. Let $\text{ch}_{I_T} \in C_c^\infty(T)$ be the characteristic function of $I_T$.

Under the isomorphism $C_c^\infty(T)^{I_T} \cong \mathbb{C}[X]$, the function $\text{ch}_{I_T}$ corresponds to $1 \in \mathbb{C}[X]$. Thus it is a generator of this $\mathbb{C}[X]$-module. We shall now describe a corresponding generator in $V^I$ in the following lemma.

**Lemma 2.4.** Let $\text{ch}_{I_T}^{\psi}$ be a function on $G$, supported on $U \cdot (I \cap \bar{B})$ such that $\text{ch}_{I_T}^{\psi}(ui) = \psi(u)$ for all $u \in U$ and $i \in \bar{B} \cap I$. Then
(1) \( \text{ch}^\psi_I \in V_0 \),
(2) \( \text{ch}^\psi_I \in V^I \),
(3) \( T_w \cdot \text{ch}^\psi_I = (-1)^{l(w)} \text{ch}^\psi_I \), for \( w \in W \), and
(4) \( \Psi(\text{ch}^\psi_I) = \text{ch}_{I^\tau} \), where \( \Psi \) is the isomorphism of \( V^I \) and \( V^{I^\tau} \).

**Proof.** (1) is obvious. (2) follows from the decomposition \( I = (I \cap U) \cdot (I \cap B) \) and the fact that \( \psi \) is trivial on \( I \cap U \). For (3) it suffices to check the equation for \( T_{s_\alpha} \), where \( s_\alpha \) is the reflection corresponding to a simple root \( \alpha \). Using the decomposition \( G = UW_{ex}I \) (see [HKP]) we need to compute \( T_{s_\alpha} \cdot \text{ch}^\psi_I(w) \) for every \( w \in W_{ex} \):

\[
T_{s_\alpha} \cdot \text{ch}^\psi_I(w) = \int_{g \in I^{s_\alpha}} \text{ch}^\psi_I(wg) dg = \sum_{t \in O/\mathfrak{p}} \text{ch}^\psi_I(wx_\alpha(t)w_\alpha(1)).
\]

Let \( w \) be the projection of \( w \) in \( W \). We need the following version of Bruhat lemma, recall that \( \alpha \) is a simple root:

\[
U_w I s_\alpha I = \begin{cases} U w s_\alpha I & \text{if } w(\alpha) < 0 \text{ and } \\ U w s_\alpha I \cup U w I & \text{if } w(\alpha) > 0. \end{cases}
\]

Hence \( T_{s_\alpha} \cdot \text{ch}^\psi_I(w) = 0 \) if \( w \neq s_\alpha, 1 \). Assume now that \( w = s_\alpha \), and represent it by \( w_\alpha(1)^{-1} = w_\alpha(1) \). Then

\[
\sum_{t \in O/\mathfrak{p}} \text{ch}^\psi_I(w_\alpha(-1)x_\alpha(t)w_\alpha(1)) = \sum_{t \in O/\mathfrak{p}} \text{ch}^\psi_I(x_\alpha(-t)) = \sum_{t \in O/\mathfrak{p}} \psi(t) = 0.
\]

If \( w = 1 \), then \( \text{ch}^\psi_I(x_\alpha(t)w_\alpha(1)) = 0 \) unless \( t \in O^X \). If \( t \in O^X \) then the relation

\[
x_\alpha(t)w_\alpha(1) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ -1 & t \end{pmatrix} \equiv x_\alpha(t^{-1}) \pmod I
\]

and the invariance properties of \( \text{ch}^\psi_I \) give

\[
\text{ch}^\psi_I(x_\alpha(t)w_\alpha(1)) = \psi(t^{-1}).
\]

Summing up over \( t \in (O/\mathfrak{p})^X \) yields \(-1 \). This completes (3). (4) is trivial. \( \Box \)

Let \( \mathcal{H}_W \) be the finite subalgebra of \( \mathcal{H} \) generated by \( T_w \) for \( w \in W \). Let \( \text{sgn} \) denote the one-dimensional representation of \( \mathcal{H}_W \) on \( \mathbb{C} \) where \( T_w \) acts by \((-1)^{l(w)}\). Let \( \pi \) be a smooth representation of \( G \), so \( \pi^I \) is an \( \mathcal{H} \)-module. We have the following, tautological, Frobenius reciprocity

\[
\text{Hom}_\mathcal{H}(\mathcal{H} \otimes \mathcal{H}_W \text{sgn}, \pi^I) = \text{Hom}_{\mathcal{H}_W}(\text{sgn}, \pi^I),
\]

where an element \( A' \in \text{Hom}_{\mathcal{H}_W}(\text{sgn}, \pi^I) \) corresponds to \( A \in \text{Hom}_\mathcal{H}(\mathcal{H} \otimes \mathcal{H}_W \text{sgn}, \pi^I) \) defined by \( A(h \otimes 1) = \pi(h)(A'(1)) \), for all \( h \in H \).

**Corollary 2.5.**  
(1) \( V^I \) is a free \( \mathcal{A} \)-module generated by \( \text{ch}^\psi_I \).

(2) \( V^I \) is isomorphic to \( \mathcal{H} \otimes_{\mathcal{H}_W} \text{sgn} \).

**Proof.** (1) follows from Lemma 2.3 (4) and the discussion preceding the lemma. (2) By Lemma 2.3 (3) we have an element in \( \text{Hom}_{\mathcal{H}_W}(\text{sgn}, \pi^I) \) given by \( 1 \mapsto \text{ch}^\psi_I \), which, by Frobenius reciprocity, furnish a map from \( \mathcal{H} \otimes_{\mathcal{H}_W} \text{sgn} \) to \( V^I \). Now (2) follows from (1) since \( \mathcal{H} \otimes_{\mathcal{H}_W} \text{sgn} \) is a free \( \mathcal{A} \)-module generated by \( 1 \otimes 1 \). \( \Box \)
Let
\[ S = \sum_{w \in W} (-1/q)^{(w)} T_w \in \mathcal{H}_W. \]
If \( \pi \) is a smooth representation of \( G \) then \( S \), acting on \( \pi \), projects on the subspace of \( \pi^f \) consisting of elements on which \( T_w \) act by \((-1)^{(w)}\) for all \( w \in W \). Let \( S(\pi) \) denote that subspace.

Let \( \tilde{\pi} \) be the smooth dual of \( \pi \). If \( \pi \) is generated by \( \pi^f \), its Iwahori-fixed vectors, then so is \( \tilde{\pi} \). We have canonical isomorphisms \( \tilde{\pi}^f \cong (\pi^*)^f \cong (\pi^f)^* \) where * denotes the linear dual. In particular, \( S(\tilde{\pi}) \cong S(\pi)^* \). The following is a strengthening, to the category of smooth representations, of a genericity criteria due to Barbasch-Moy [BM] for representations generated by Iwahori-fixed vectors.

**Corollary 2.6.** Let \( \pi \) be a smooth representation of \( G \) generated by \( I \)-fixed vectors. The canonical map \( S(\pi) \to \pi_{U,\psi} \) obtained by composing the inclusion of \( S(\pi) \) into \( \pi \) and the projection of \( \pi \) onto \( \pi_{U,\psi} \) is a bijection.

**Proof.** It suffices to prove that the dual map \( (\pi_{U,\psi})^* \to S(\pi)^* \) is a bijection. We have the following natural isomorphisms:

\[
(\pi_{U,\psi})^* \cong \text{Hom}_G(\pi, \text{Ind}_{U}^G \psi) \\
\cong \text{Hom}_G(\text{Ind}_{U}^G \psi, \tilde{\pi}) \quad \text{(taking dual)} \\
\cong \text{Hom}_{\mathcal{H}}(\mathcal{H} \otimes \mathcal{H}_W \text{sgn}, \tilde{\pi}^f) \quad \text{(by Corollary 2.5)} \\
\cong \text{Hom}_{\mathcal{H}_W}(\text{sgn}, \tilde{\pi}^f) \quad \text{(by Frobenius reciprocity)} \\
\cong S(\pi)^*.
\]

It remains to show that this sequence of isomorphisms realizes the dual map \( (\pi_{U,\psi})^* \to S(\pi)^* \). To that end, let \( \ell \in (\pi_{U,\psi})^* \). For every \( v \in \pi \), let \( f_v(g) = \ell(\pi(g)v) \in \text{Ind}_{U}^G \psi \). Note that \( f_v(1) = \ell(v) \). So \( \ell \) defines \( A \in \text{Hom}_G(\pi, \text{Ind}_{U}^G \psi) \) by \( A(v) = f_v \), for all \( v \in \pi \), and this realizes the first isomorphism above. The map \( A \) defines \( \tilde{A} \in \text{Hom}_G(\text{ind}_{U}^G \tilde{\psi}, \tilde{\pi}) \) where, for every \( f \in \text{ind}_{U}^G \tilde{\psi}, \tilde{A}(f) \) is an element in \( \tilde{\pi} \) given by

\[ \tilde{A}(f)(v) = \int_{U \setminus G} f \cdot f_v \ dg \]

for all \( v \in \pi \). This realizes the second isomorphism. The third isomorphism is given by the identification of
\( \text{ind}_{U}^G(\tilde{\psi})^f \) and \( \mathcal{H}_n \otimes_{\mathcal{H}_W} \mathbb{C} \) where \( \text{ch}_{\tilde{\psi}} \) corresponds to \( 1 \otimes 1 \). The fourth isomorphism gives an element in \( \text{Hom}_{\mathcal{H}_W}(\text{sgn}, \tilde{\pi}^f) \) defined by \( 1 \mapsto \tilde{A}(\text{ch}_{\tilde{\psi}}) \). Thus, starting from \( \ell \in (\pi_{U,\psi})^* \) we have arrived to \( \tilde{A}(\text{ch}_{\tilde{\psi}}) \in S(\pi)^* \) given by

\[ \tilde{A}(\text{ch}_{\tilde{\psi}})(v) = \int_{U \setminus G} \text{ch}_{\tilde{\psi}} \cdot f_v \ dg, \]

for all \( v \in S(\pi) \). Since the measure on \( U \setminus G \) is fixed so that \( U \cap I \setminus I \) has volume 1, the integral is equal to \( f_v(1) \) and this is equal to \( \ell(v) \), as desired.

\[ \square \]
3. Bernstein-Zelevinsky derivatives for affine Hecke algebras

In this section, we specify to $GL(n,F)$. Set $G_n = GL(n,F)$. Let $U_n$ be the unipotent subgroup of $G_n$ consisting of upper triangular matrices and let $\mathcal{U}_n$ be the opposite unipotent subgroup of $G_n$ consisting of lower triangular matrices. Let $D_n$ be the subgroup of diagonal matrices. The group of co-character and character lattices can be naturally identified with $X = \mathbb{Z}^n$. The choice of $U_n$ determines the set of positive roots. Under these identifications the half-sum of all roots is $\rho = ((n-1)/2, \ldots, (1-n)/2)$. Let $S_n$ be the group of all permutations matrices in $G_n$. Let $I_n$ be the Iwahori subgroup determined from the Borel subgroup $D_n \mathcal{U}_n$ and let $\mathcal{H}_n = C_c(I_n \backslash G_n/I_n)$ (see notations in Section 2.1). Inside $\mathcal{H}_n$ we have a finite dimensional subalgebra $\mathcal{H}_{S_n}$ consisting of functions supported on $GL(n,o)$. Let $T_w$ be the characteristic function of $I_n w I_n$. Then $\mathcal{H}_{S_n}$ is spanned by $T_w$ for $w \in S_n$. Let $x = (m_1, \ldots, m_n) \in X$ such that $m_1 \geq \ldots \geq m_n$ i.e. $x$ is dominant. Let $\hat{x}$ be the diagonal matrices whose diagonal entries are $\varpi^{m_1}, \ldots, \varpi^{m_n}$. Let

$$\theta_x = q^{-\langle x, \rho \rangle} \text{ch}_{I_n \hat{x} I_n}.$$ 

Let $\mathcal{A}_n$ be the commutative subalgebra in $\mathcal{H}_n$ generated by $\theta_x$ and their inverses, for $x$ dominant. It is isomorphic to the group algebra $\mathbb{C}[X]$. The algebra $\mathcal{H}_n$ is generated by $\mathcal{H}_{S_n}$ and $\mathcal{A}_n$ modulo Bernstein’s relations.

3.1. Jacquet functor. We fix $i$ for the rest of this section. Let $P = MN$ be a parabolic subgroup containing $D_n U_n$ where $N$ is the unipotent subgroup, and the Levi subgroup $M \cong G_{n-i} \times G_i$ sitting in $G_n$ via the embedding

$$(g_{n-i}, g_i) \mapsto \begin{pmatrix} g_{n-i} & 0 \\ 0 & g_i \end{pmatrix}.$$ 

Let $I_M = I_n \cap M$. Let $\mathcal{H}_M = C_c(I_M \backslash M/I_M)$ be the convolution algebra of compactly supported $I_M$-bi-invariant functions on $M$. For every $w \in S_{n-i} \times S_i$ let $T_w^M \in \mathcal{H}_M$ be the characteristic function of $I_M w I_M$. Let $\rho_M$ be the half-sum of all roots in $M$. Let $x \in X$ be dominant, and set

$$\theta_x^M = q^{-\langle x, \rho_M \rangle} \text{ch}_{I_M \hat{x} I_M}.$$ 

Let $\mathcal{A}_M$ be a commutative subalgebra in $\mathcal{H}_M$ generated by $\theta_x$ and their inverses, for $x$ dominant. The following is a consequence of Bernstein’s relations for $\mathcal{H}_M$ and $\mathcal{H}_n$.

**Theorem 3.1.** The map $i_M(T^M_w) = T_w$, for $w \in S_{n-i} \times S_i$, and $i_M(\theta_x^M) = \theta_x$, for $x \in X$, defines an injective homomorphism of $\mathcal{H}_M$ and $\mathcal{H}_n$.

In particular, any $\mathcal{H}_n$-module $\sigma$ can be viewed as an $\mathcal{H}_M$-module by precomposing by $i_M$. The resulting $\mathcal{H}_M$-module will be denoted by $\text{res}_{\mathcal{H}_M}^{\mathcal{H}_n}(\sigma)$.

**Proposition 3.2.** Let $\pi$ be a smooth representation of $G$. The canonical isomorphism of linear spaces $p_N: \pi^I_n \rightarrow (\pi_N)^I_M$ gives a canonical isomorphism of $\mathcal{H}_M$-modules

$$\text{res}_{\mathcal{H}_M}^{\mathcal{H}_n}(\pi^I_n) \cong (\pi_N)^I_M.$$ 

**Proof.** This is proved by checking, by an explicit computation, that $p_N \circ T_w = T^M_w \circ p_N$, for $w \in S_{n-i} \times S_i$, and $p_N \circ \theta_x = \theta^M_x \circ p_N$, for dominant $x \in X$. \qed
3.2. Bernstein-Zelevinsky derivatives. We continue with the same setup. Let $U_i$ be the subgroup of $M$ consisting of matrices of the form
\[
\begin{pmatrix}
I_{n-i} & 0 \\
0 & u
\end{pmatrix},
\]
where $u$ is a strictly upper-triangular matrix in $G_i$. The character $\overline{\psi}$ of conductor $p$ defines a Whittaker character $\psi$ of $U_i$
\[
\psi(u) = \sum_{j=n-i+1}^{n-1} \overline{\psi}(u_{j,j+1})
\]
where $u_{j,j+1}$ refers to the matrix entries. Let $\sigma$ be a smooth $M$-module. Let $\sigma_{U_i,\psi}$ be the space of $\psi$-twisted $U_i$-coinvariants. It is naturally a $G_{n-i}$-module. If $\pi$ is a smooth $G$-module, the $i$-th Bernstein-Zelevinski derivative of $\pi$ is defined by
\[
\pi^{(i)} = (\pi_N)_{U_i,\psi}
\]
Thus the $i$-th Bernstein-Zelevinski derivative is a functor from the category of smooth $G_n$-modules to the category of smooth $G_{n-i}$-modules.

3.3. Bernstein-Zelevinsky derivative for $H_n$. Note that we have a canonical isomorphism $H_{n-i} \otimes H_i \cong H_M$ of the spaces of functions on $G_{n-i} \times G_i \cong M$. Composing with the injection $i_M : H_M \to H_n$, we have a homomorphism
\[
m : H_{n-i} \otimes H_i \to H_n.
\]
More concretely, we have the following formulae that will be of practical purpose later:
\[
m(T_w \otimes 1) \mapsto T_{\bar{w}} , \quad \text{for } w \in S_{n-i}, \text{ where } \bar{w} = w \times 1 \in S_{n-i} \times S_i, \ m(\theta_x \otimes 1) \mapsto \theta_x, \quad \text{where } x \in \mathbb{Z}^{n-i} \text{ is a viewed as an element of } \mathbb{Z}^n \text{ by adding } 0's \text{ at the end, and } m(1 \otimes T_w) \mapsto T_{\bar{w}}, \quad \text{for } w \in S_i, \text{ where } \bar{w} = 1 \times w \in S_{n-i} \times S_i, \text{ and } m(1 \otimes \theta_x) \mapsto \theta_x, \quad \text{where } x \in \mathbb{Z}^i \text{ is a viewed as an element of } \mathbb{Z}^n \text{ by adding } 0's \text{ in front.}
\]
Abusing notation, we shall identify $H_{n-i}$ and $m(H_{n-i} \otimes 1)$. Let $S_i \in H_i$ be the sign projector. Let $S_i^n = m(1 \otimes S_i)$. Let $\sigma$ be an $H_n$-module. The $i$-th Bernstein-Zelevinski derivative of $\sigma$ is the natural $H_{n-i}$-module
\[
\text{BZ}_i(\sigma) := S_i^n(\sigma).
\]
Let $\pi$ be a smooth $G_n$-module, generated by $I_n$-fixed vectors. Then the smooth $M$-module $\pi_N$ is generated by $I_M$-fixed vectors. It is easy to see that $\pi_N$, viewed purely as a $G_{n-i}$-module, is generated by its $I_{n-i}$-fixed vectors. Thus the $i$-th Bernstein-Zelevinski derivative $\pi^{(i)}$, being a quotient of $\pi_N$, is also generated by its $I_{n-i}$-fixed vectors. It follows that $\pi^{(i)}$ is determined by the corresponding $H_{n-i}$-module $(\pi^{(i)}|_{I_{n-i}})$. Now note that $(\pi^{(i)}|_{I_{n-i}})$ is a quotient of $\pi^{I_{n-i}}$, while $\text{BZ}_i(\pi^{I_n})$ is a submodule of $\pi^{I_{n-i}}$. Hence we have a canonical map $\text{BZ}_i(\pi^{I_n}) \to (\pi^{(i)}|_{I_{n-i}})$.

**Theorem 3.3.** Let $\pi$ be a smooth representation of $G_n$ generated by $I_n$-fixed vectors. The canonical map $\text{BZ}_i(\pi^{I_n}) \to (\pi^{(i)}|_{I_{n-i}})$ is an isomorphism of $H_{n-i}$-modules.

**Proof.** The proof of this theorem will occupy the rest of this section.
Lemma 3.4. Let $\sigma$ be a smooth $M$-module generated by its $I_M$-fixed vectors. Then the canonical map $S(\sigma)^{I_{n-i}} \to \{\sigma_{U_i, \psi}\}^{I_{n-i}}$ is an isomorphism of $\mathcal{H}_{n-i}$-modules.

Proof. The canonical map is a homomorphism of $\mathcal{H}_{n-i}$-modules, so it suffices to check that it is an isomorphism of vector spaces. Note that $\sigma^{I_{n-i}}$ is generated by its $I_i$-fixed vectors as a $G_i$-module. Hence Corollary 2.6 applied to $G_i$, implies the lemma. \qed

We now need the following observation. Let $\sigma$ be a smooth $M$-module. Then $\mathcal{H}_{n-i}$ and $\mathcal{H}_i$ both act on $\sigma^{I_M}$. The resulting tensor product action of $\mathcal{H}_{n-i} \otimes \mathcal{H}_i$ on $\sigma^{I_M}$ and the action of $\mathcal{H}_M$ are compatible with respect to the canonical isomorphism $\mathcal{H}_{n-i} \otimes \mathcal{H}_i \cong \mathcal{H}_M$.

Lemma 3.5. Let $\pi$ be a smooth $G_n$-module generated by its $I_n$-fixed vectors. The isomorphism $\pi^{I_n} \cong (\pi_N)^{I_M}$ induces an isomorphism $S(\pi)^{I_n} \cong S(\pi_N)^{I_{n-i}}$ of $\mathcal{H}_{n-i}$-modules.

The theorem is a simple combination of the two lemmas, using $\sigma = \pi_N$, in the first. \qed

4. Bernstein-Zelevinsky derivatives and Lusztig reductions

4.1. Affine Hecke algebras. We shall state the definition of an affine Hecke algebra in a greater generality which will be needed in the following subsections.

Let $(X, R, X^\vee, R^\vee)$ be a root datum where $R$ is a reduced root system and $X$ a $\mathbb{Z}$-lattice containing $R$. Let $W$ be the Weyl group of $R$. Let $Q \subseteq X$ be the root lattice and let $W_{\text{aff}} = Q \rtimes W$ be the affine Weyl algebra. Fix a set of simple roots $\Pi$. The choice of $\Pi$ determines a set $S_{\text{aff}}$ of simple affine reflections. Let $W_{\text{ex}}$ be the semidirect product $X \rtimes W$ (extended affine Weyl group). Let $Y \subseteq X$ be the sub lattice perpendicular to $R^\vee$. Then $W_{\text{ex}}/Y$ acts on a Coxeter complex and this action defines a length function $l : W_{\text{aff}} \to \mathbb{Z}$ such that $l(s) = 1$ for all $s \in S_{\text{aff}}$.

Definition 4.1. The affine Hecke algebra $\mathcal{H} := \mathcal{H}(X, R, \Pi, q)$ associated to the datum is defined to be a complex associative algebra generated by the elements $\{T_w : w \in W_{\text{ex}}\}$ subject to the relations

1. $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$,
2. $(T_s + 1)(T_s - q) = 0$ for $s \in S_{\text{aff}}$.

Denote by $\mathcal{H}_W$ the finite subalgebra of $\mathcal{H}$ generated by $T_w \ (w \in W)$. The algebra $\mathcal{H}$ has a large commutative subalgebra $A \cong \mathbb{C}[X]$, which depends on the choice of simple roots $\Pi$. We have an isomorphism of vector spaces $\mathcal{H} \cong A \otimes_{\mathbb{C}} \mathcal{H}_W$. Let $T = \text{Hom}(X, \mathbb{C}^x)$. The center $Z$ of $\mathcal{H}$ is isomorphic to $\mathbb{C}[X]^W$. Hence central characters of $\mathcal{H}$ are parameterized by $W$-orbits in $T$. We shall denote by $W_t$ the $W$-orbit of $t \in T$. Let $J_{W_t}$ be the corresponding maximal ideal in $Z$. For a finite-dimensional $\mathcal{H}$-module $\chi$, denote $\chi_{[W_t]}$ to be the subspace of $\chi$ annihilated by a power of $J_{W_t}$. Then

$$\chi \cong \bigoplus_{W_t \in T/W} \chi_{[W_t]}.$$
Let \( X_n = X_n^r = \bigoplus_{k=1}^{r} \mathbb{Z} \) be a \( \mathbb{Z} \)-lattice. Set \( \alpha_{k,l} = \epsilon_k - \epsilon_l \) (\( k \neq l \)) and also set \( \alpha_k = \alpha_{k,k+1} \) (\( k = 1, \ldots, n \)). Let \( R_n = R_n^r = \{ \epsilon_k - \epsilon_l : l \neq k \} \) be a root system of type \( A_{n-1} \). Let \( \Pi_n = \{ \epsilon_i - \epsilon_{i+1} : i = 1, \ldots, n-1 \} \). The Iwahori-Hecke algebra \( H_n \) of \( GL(n) \) (from Section 3) is isomorphic to \( H(X_n, R_n, \Pi_n, q) \).

### 4.2. Lusztig’s first reduction theorem

We shall use a variation of Lusztig’s reduction in [OS, Section 2] for the affine Hecke algebra \( H_n \) (also see [BM]), proofs are from [Lu, Section 8]. Let \( T_n = \text{Hom}(X_n, C^r) \). Any \( t \in T_n \) is identified with an \( n \)-tuple \( (z_1, \ldots, z_n) \) of non-zero complex numbers where \( z_i \) is the value of \( t \) at \( \epsilon_i \). Let \( T_r = \text{Hom}(X_n, \mathbb{R}^>0) \) and \( T_{un} = \text{Hom}(X_n, S^1) \). Any \( t \in T_n \) has a polar decomposition \( t = vu \) where \( v \in T_r \) and \( u \in T_{un} \). Write \( x(u) \) for the value of \( u \) at \( x \in X_n \). Hence \( u = (z_1, \ldots, z_n) \) where \( z_k = \epsilon_k(u) \). Without loss of generality we can permute the entries of \( u \) such that, for a partition \( n = (n_1, \ldots, n_m) \) of \( n \), \( z_1 = \ldots = z_{n_1} \neq z_{n_1+1} = \ldots \) etc. Let

\[
R_n = \{ \alpha \in R_n : \alpha(u) = 1 \}.
\]

It is a root subsystem of \( R_n \) which, as the notation indicates, depends on the partition \( n \). It is isomorphic to the product \( R_{n_1} \times \ldots \times R_{n_m} \). Let \( S_n \cong S_{n_1} \times \ldots \times S_{n_m} \) be its Weyl group. Let \( \Pi_n \) be the set of simple roots in \( R_n \) determined by \( R_n^+ = R_n^\alpha \cap R_n \). Let \( H_n := H(X_n, R_n, \Pi_n, q) \cong H_{n_1} \otimes \ldots \otimes H_{n_m} \), the associated affine Hecke algebra (see Definition 4.1). This is a Hecke algebra corresponding to the Levi subgroup \( M = G_{n_1} \times \ldots \times G_{n_m} \).

Let \( Z_n = A_n^{Z_n} \) be the center of \( H_n \). Let \( J_{S_n,t} \) be an ideal in \( Z_n \) corresponding to the central character \( S_n \cdot t \). Let \( \sigma \) be a finite-dimensional \( H_n \)-module annihilated by a power of \( J_{S_n,t} \). Then \( i(\sigma) = H_n \otimes_{H_n} \sigma \) is annihilated by a power of \( J_{S_n,t} \).

**Theorem 4.2.** The functor \( i \) defines an equivalence between the category of finite-dimensional \( H_n \)-modules annihilated by a power of \( J_{S_n,t} \) and the category of finite-dimensional \( H_n \)-modules annihilated by a power of \( J_{S_n,t} \).

**Proof.** Let \( \hat{Z}_n \) (depending on \( S_n \cdot t \)) be the \( J_{S_n,t} \)-adic completion of \( Z_n \). Let \( \hat{A}_n = \hat{Z}_n \otimes_{Z_n} A_n \).

Let \( \hat{H}_n = \hat{Z}_n \otimes_{Z_n} H_n \). By the Chinese Remainder Theorem for a commutative ring, we have a decomposition

\[
\hat{A}_n = \bigoplus_{\nu \in S_n \cdot t} \hat{A}_\nu,
\]

where \( \hat{A}_\nu \) is obtained by localizing \( \hat{A}_n \) at \( \nu \). For any \( \nu \in S_n \cdot t \), let \( 1_\nu \) be the unit element in \( \hat{A}_\nu \). We also regard \( 1_\nu \) as an element in \( \hat{A}_n \).

We define a similar formal completion of \( H_n \). Let \( \hat{Z}_n \) be the \( J_{S_n,t} \)-adic completion of \( Z_n \). Let \( \hat{A}_n = \hat{Z}_n \otimes_{Z_n} A_n \).

Let \( \hat{H}_n = \hat{Z}_n \otimes_{Z_n} H_n \). We have a decomposition

\[
\hat{A}_n = \bigoplus_{\nu \in S_n \cdot t} \hat{A}_\nu.
\]

Let \( 1_n = \sum_{\nu \in S_n \cdot t} 1_\nu \). Note that \( 1_n \) is in \( \hat{Z}_n \) and \( \hat{A}_n = 1_n \cdot \hat{A}_n = 1_n \cdot 1_n \).

Let \( \pi \) be an \( H_n \)-module annihilated by a power of \( J_{S_n,t} \). Then \( \pi \) is naturally an \( \hat{H}_n \)-module, and \( \sigma = 1_n \cdot \pi \) an \( \hat{H}_n \)-module, where \( 1_n \cdot \hat{H}_n = 1_n \cdot \hat{H}_n \cdot 1_n \). Following Lusztig’s arguments in [Lu, Section 8], \( \hat{H}_n \cong \hat{H}_n \). Hence by identifying \( n \hat{H}_n \cong \hat{H}_n \), we have a functor
\[ r(\pi) = 1_n \cdot \pi \] from the category of finite-dimensional \( \mathcal{H}_n \)-module annihilated by a power of \( \mathcal{J}_{S_n,t} \) to the category of finite-dimensional \( \mathcal{H}_n \)-modules annihilated by a power of \( \mathcal{J}_{S_n,t} \).

Using the Frobenius reciprocity, intertwining operators (see [Lu] Lemma 8.9(a)) and the fact that \( 1 = \sum_{t' \in S_n} 1_{t'} \), we obtain a natural isomorphism from \( i \circ r(\pi) \) to \( \pi \). Using intertwining operators (see [Lu] Lemma 8.9(a)) and the fact that \( 1_n \cdot 1_{t'} = 0 \) if \( t' \not\in S_n t \), we obtain \( r \circ i \cong \text{Id} \). Hence \( i \) defines an equivalence of categories. □

4.3. First reduction for the Bernstein-Zelevinsky derivatives. We keep using notations from the previous subsection. In particular, we fixed \( t = vu \in T_n \), and we have a canonical isomorphism \( \mathcal{H}_n \cong \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_m} \), where \( n = (n_1, \ldots, n_m) \) is a partition of \( n \), arising from \( u \).

Fix an integer \( i \leq n \). For each \( m \)-tuple \( i = (i_1, \ldots, i_m) \) of integers, such that \( i_1 + \ldots + i_m = i \) and \( 0 \leq i_k \leq n_k \) (\( k = 1, \ldots, m \)), define another \( m \)-tuple \( n - i = (n_1 - i_1, \ldots, n_m - i_m) \). Each pair \((n_k - i_k, i_k)\) gives rise to an embedding \( \mathcal{H}_{n_k - i_k} \otimes \mathcal{H}_{i_k} \subseteq \mathcal{H}_{n_k} \), as in Section 4.3 and these combine to give an embedding

\[ \mathcal{H}_{n-i} \otimes \mathcal{H}_i \subseteq \mathcal{H}_n \]

where \( \mathcal{H}_i \cong \mathcal{H}_{i_1} \otimes \ldots \otimes \mathcal{H}_{i_m} \) etc. (Note, if \( i_k = 0 \), then the corresponding factor is the trivial algebra \( \mathbb{C} \).) Abusing notation, we shall identify \( \mathcal{H}_{n-i} \) with its image in \( \mathcal{H}_n \) via the map \( h \mapsto h \otimes 1 \). Let \( S_i \in \mathcal{H}_i \) be the sign projector in \( \mathcal{H}_i \), and let \( S_i^\pi \) be the image of \( 1 \otimes S_i \) in \( \mathcal{H}_n \). Let \( \sigma \) be an \( \mathcal{H}_n \)-module. Then \( S_i^\pi(\sigma) \) is naturally an \( \mathcal{H}_{n-i} \)-module. Thus we have a functor

\[ \mathbb{BZ}_i^\pi(\sigma) := S_i^\pi(\sigma) \]

from the category of \( \mathcal{H}_n \)-modules to the category of \( \mathcal{H}_{n-i} \)-modules.

Observe that \( \mathcal{H}_{n-i} \) is a Levi subalgebra of \( \mathcal{H}_n \) and \( \mathcal{H}_i \) is a Levi subalgebra of \( \mathcal{H}_i \). We are now ready to state the first reduction result.

**Theorem 4.3.** Let \( \pi \) be a finite-dimensional \( \mathcal{H}_n \)-module annihilated by a power of \( \mathcal{J}_{S_n,t} \). Let \( \sigma \) be a finite-dimensional \( \mathcal{H}_n \)-module annihilated by a power of \( \mathcal{J}_{S_n,t} \) such that \( \pi \cong i(\sigma) \) (see Theorem 4.2). Then there is an isomorphism

\[ \mathbb{BZ}_i(\pi) \cong \bigoplus_i \mathcal{H}_{n-i} \otimes \mathcal{H}_{n-i} \mathbb{BZ}_i^\pi(\sigma) \]

where the sum is taken over all \( m \)-tuple of integers \( i = (i_1, \ldots, i_m) \) satisfying \( i_1 + \ldots + i_m = i \) and \( 0 \leq i_k \leq n_k \) (\( k = 1, \ldots, m \)).

**Proof.** By using the Mackey theorem for affine Hecke algebras (see e.g. [K] Section 3.5) for a similar setting), we have

\[ \text{res}^{\mathcal{H}_n}_{\mathcal{H}_{n-i} \otimes \mathcal{H}_i}(\mathcal{H}_n \otimes \mathcal{H}_n \sigma) \cong \bigoplus_i (\mathcal{H}_{n-i} \otimes \mathcal{H}_i) \otimes (\mathcal{H}_{n-i} \otimes \mathcal{H}_i) \left( \text{res}^{\mathcal{H}_n}_{\mathcal{H}_{n-i} \otimes \mathcal{H}_i}(\mathcal{H}_n \otimes \mathcal{H}_n \sigma) \right) \]

where the sum is over \( i \) as in the statement of the theorem. We remark that the Mackey Theorem asserts that the composition factors of \( \text{res}^{\mathcal{H}_n}_{\mathcal{H}_{n-i} \otimes \mathcal{H}_i}(\mathcal{H}_n \otimes \mathcal{H}_n \sigma) \) are of the form in the left hand side of the above isomorphism. Those composition factors are indeed direct.
summands since the $\mathcal{H}_{n-i} \otimes \mathcal{H}_i$-central characters of those composition factors are distinct. Furthermore, using the Frobenius reciprocity, we have

\begin{equation}
S^\pi_i((\mathcal{H}_{n-i} \otimes \mathcal{H}_i) \otimes (\mathcal{H}_{n-i} \otimes \mathcal{H}_i)) \sigma \cong \mathcal{H}_{n-i} \otimes \mathcal{H}_{n-i} S^\pi_i(\sigma).
\end{equation}

Combining (4.3) and (4.5), we obtain (4.3). □

Remark 4.4. When $\pi \cong i(\sigma)$ is an irreducible $\mathcal{H}_n$-module, then $\sigma \cong \sigma_1 \boxtimes \ldots \boxtimes \sigma_m$ for some irreducible $\mathcal{H}_{i_k}$-modules $\sigma_k$. In this case,

$$
\text{BZ}_i^\pi(\sigma) \cong \text{BZ}_{i_1}(\sigma_1) \boxtimes \ldots \boxtimes \text{BZ}_{i_m}(\sigma_m).
$$

From this viewpoint, Theorem 4.3 can be seen as a Leibniz rule.

4.4. Graded affine Hecke algebras. We shall now need the affine graded Hecke algebra attached to the root datum $(X, R, X^\vee, R^\vee)$. Let $V = X \otimes \mathbb{C}$.

**Definition 4.5.** [Lu] Section 4] The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(V, R, \Pi, \log q)$ is an associative algebra with an unit over $\mathbb{C}$ generated by the symbols $\{t_w : w \in W\}$ and $\{f_v : v \in V\}$ satisfying the following relations:

1. The map $w \mapsto t_w$ from $\mathbb{C}[W] = \bigoplus_{w \in W} \mathbb{C}w \to \mathbb{H}$ is an algebra injection,
2. The map $v \mapsto f_v$ from $S(V) \to \mathbb{H}$ is an algebra injection, where $S(V)$ is the polynomial ring for $V$,
3. writing $v$ for $f_v$ from now on, for $\alpha \in \Pi$ and $v \in V$,

$$
v t_{s_{\alpha}} - t_{s_{\alpha}} v = \log q \cdot \langle v, \alpha^\vee \rangle.
$$

In particular, $\mathbb{H} \cong S(V) \otimes \mathbb{C}[W]$ as vector spaces. We also set $\mathbb{A} = S(V)$, the graded algebra analogue of $\mathcal{A}$. Let $Z = \mathbb{A}^W$ be the center of $\mathbb{H}$. Let $V^* = \text{Hom}(X, \mathbb{C})$. The central characters of irreducible representations are parameterized by $W$-orbits in $V^*$. If $\zeta \in V^*$, let $W\zeta$ denote the corresponding orbit an the central character. Let $\mathbb{I}_{W\zeta} \subset Z$ be the corresponding maximal ideal.

4.5. Lusztig’s second reduction theorem. Let $\mathcal{H} = \mathcal{H}(X, R, \Pi, q)$ be the affine Hecke algebra defined in Section 4.1 and $\mathcal{A} \cong \mathbb{C}[X]$ the commutative sub algebra. Let $\theta_x \in \mathcal{A}$ correspond to $x \in X$. Let $Z \cong \mathbb{C}[X]^W$ be the center of $\mathcal{H}$. Let $F$ be the quotient field of $\mathcal{A}$. Let $\mathcal{H}_F \cong \mathcal{H}_W \otimes F$ with the algebraic structure naturally extending $\mathcal{H}$.

Following Lusztig [Lu] Section 5], for $\alpha \in \Pi$, define $\tau_{s_{\alpha}} \in \mathcal{H}_F$ by

$$
\tau_{s_{\alpha}} + 1 = (T_{s_{\alpha}} + 1) \mathcal{G}(\alpha)^{-1},
$$

where

$$
\mathcal{G}(\alpha) = \frac{\theta_{\alpha} q - 1}{\theta_{\alpha} - 1} \in F.
$$

It is shown in [Lu] Section 5] that the map from $W$ to the units of $\mathcal{H}_F$ defined by $s_{\alpha} \mapsto \tau_{s_{\alpha}}$ is an injective group homomorphism.

On the graded Hecke algebra side, let $\mathbb{H} = \mathbb{H}(V, R, \Pi, \log q)$ be as in Definition 4.5. Let $\mathcal{F}$ be the quotient field of $\mathbb{A}$ and let $Z$ be the center of $\mathbb{H}$. Let $\mathbb{H}_F \cong \mathbb{H}_W \otimes \mathcal{F}$ with the algebraic structure naturally extending $\mathbb{H}$. For $\alpha \in \Pi$, define $\tau_{s_{\alpha}} \in \mathbb{H}_F$ by

$$
\tau_{s_{\alpha}} + 1 = (t_{s_{\alpha}} + 1) g(\alpha)^{-1},
$$
where
\[ g(\alpha) = \frac{\alpha + \log q}{\alpha} \in \mathbb{F}. \]
As in the affine case, the map from \( W \) to the units of \( \mathbb{H}_F \) defined by \( s_\alpha \mapsto \tau_{s_\alpha} \) is an injective group homomorphism.

Any \( \zeta \in \mathbb{V}^* \) defines \( t \in T = \text{Hom}(X, \mathbb{C}^*) \) by \( x(t) = e^{\xi(\zeta)} \), for all \( x \in X \). We shall express this relationship by \( t = \exp(\zeta) \). We shall say that \( \zeta \) is real for the root system \( R \) if \( \alpha(\zeta) \in \mathbb{R} \) for all \( \alpha \in R \). Then \( t = \exp(\zeta) \) satisfies \( \alpha(t) > 0 \), for all \( \alpha \in R \). Conversely, every such \( t \) arises in this fashion, from a real \( \zeta \). Let \( \hat{\mathbb{Z}} \) be the \( \mathcal{J}_{W_1} \)-adic completion of \( \mathbb{Z} \) and let \( \hat{\mathbb{Z}} \) be the \( \mathbb{J}_{W_1^*} \)-adic completion of \( \mathbb{Z} \). Let \( \hat{\mathbb{H}} = \hat{\mathbb{Z}} \otimes \mathcal{H} \) and let \( \hat{\mathbb{H}} = \hat{\mathbb{Z}} \otimes \mathcal{H} \). Let \( \hat{\mathbb{H}}_F = \hat{\mathbb{Z}} \otimes \mathcal{H}_F \) and let \( \hat{\mathbb{H}}_F = \hat{\mathbb{Z}} \otimes \mathcal{H}_F \). Let \( \hat{\mathcal{A}} = \hat{\mathbb{Z}} \otimes \mathcal{A} \) and let \( \hat{\mathcal{A}} = \hat{\mathbb{Z}} \otimes \mathcal{A} \). Let \( \hat{\mathcal{J}}_{W_1} = \hat{\mathbb{Z}} \otimes \mathcal{J}_{W_1} \) and let \( \hat{\mathcal{J}}_{W_1^*} = \hat{\mathbb{Z}} \otimes \mathcal{J}_{W_1^*} \).

**Theorem 4.6.** [Lu, Theorem 9.3, Section 9.6] Recall that we are assuming that \( \zeta \in \mathbb{V}^* \) is real for the root system \( R \).

1. There is an isomorphism denoted \( j \) between \( \hat{\mathbb{H}}_F \) and \( \hat{\mathbb{H}}_F \) determined by \( j(\tau_{s_\alpha}) = \tau_{s_\alpha} \), \( j(\tau_x) = e^x \).

2. The above map also induces isomorphisms between \( \hat{\mathbb{Z}} \) and \( \hat{\mathbb{Z}} \), between \( \hat{\mathcal{A}} \) and \( \hat{\mathcal{A}} \) and between \( \mathcal{H} \) and \( \hat{\mathcal{H}} \).

A crucial point for the proof of (2) is the fact that
\[ \frac{e^\alpha q - 1}{e^\alpha - 1} : \frac{\alpha}{\alpha + \log q} \in \mathbb{F} \]
is holomorphic and nonvanishing at any \( \zeta' \in W_1 \), and hence is an invertible element in \( \hat{\mathcal{A}} \).

Now (2) gives the following isomorphisms:
\[ \mathcal{H}/\mathcal{J}_{W_1} \mathcal{H} \cong \hat{\mathcal{H}}/\hat{\mathcal{J}}_{W_1} \hat{\mathcal{H}} \cong \hat{\mathcal{H}}/\mathcal{J}_{W_1^*} \hat{\mathcal{H}} \cong \hat{\mathcal{H}}/\mathcal{J}_{W_1^*} \hat{\mathcal{H}} \]
and hence:

**Theorem 4.7.** [Lu, Section 10] Assume that \( \zeta \in \mathbb{V}^* \) is real. There is an equivalence of categories between the category of finite-dimensional \( \mathbb{H} \)-modules annihilated by a power of \( \mathbb{J}_{W_1} \) and the category of finite-dimensional \( \mathcal{H} \)-modules annihilated by a power of \( \mathcal{J}_{W_1} \), where \( t = \exp(\zeta) \).

Let \( \Lambda \) be the functor in Theorem 4.7. Explicitly, for a finite-dimensional \( \mathbb{H} \)-module annihilated by a power of \( \mathbb{J}_{W_1} \), \( \Lambda(\pi) \) is equal to \( \pi \), as linear spaces, but the \( \mathcal{H} \)-action on \( \pi \) is given by
\[ h \cdot \mathcal{H} \pi = j(h) \cdot \hat{\mathcal{H}} \pi, \]
where \( h \in \mathcal{H} \) and \( x \in \pi \). Note that the functor extends to the category of finite dimensional \( \mathbb{H} \)-modules that are sums of \( \mathbb{H} \)-modules, where each summand is annihilated by a power of \( \mathbb{J}_{W_1} \) for some real \( \zeta \).

**Proposition 4.8.** Recall the sign projector \( S = \sum_{w \in W} (-1/q)^{l(w)} T_w \) in \( \mathcal{H} \) and let \( s = \sum_{w \in W} (-1)^{l(w)} t_w \) be the corresponding sign projector in \( \mathbb{H} \). Then \( j(S) = a \cdot s \), where \( a \) is an invertible element in \( \hat{\mathcal{A}} \).
Proof. Let \( \alpha \in \Pi \). Firstly, by a direct computation, we have
\[
j(1 - q^{-1}T_{s_{\alpha}}) = j(G(-\alpha))^{-1}g(-\alpha)q^{-1}(1 - t_{s_{\alpha}}).
\]
Secondly,
\[
S = \left( \sum_{w \in W^{\Pi \setminus \{\alpha\}}} (-1/q)^{l(w)}T_w \right) (1 - q^{-1}T_{s_{\alpha}}),
\]
where \( W^{\Pi \setminus \{\alpha\}} \) is the set of minimal representatives of \( W/W^{\Pi \setminus \{\alpha\}} \) and \( W^{\Pi \setminus \{\alpha\}} \) is the parabolic subgroup associated to \( \Pi \setminus \{\alpha\} \). Therefore
\[
j(S) = j \left( \sum_{w \in W^{\Pi \setminus \{\alpha\}}} (-1/q)^{l(w)}T_w \right) j(G(-\alpha))^{-1}g(-\alpha)q^{-1}(1 - t_{s_{\alpha}}).
\]
Hence we have \( j(S)t_{s_{\alpha}} = -j(S) \). This shows that \( j(S) \in \widehat{\mathbb{H}} \cdot s \). Since \( \widehat{\mathbb{H}} \cdot s = \hat{A} \cdot s \), we have \( j(S) = a \cdot s \), for some \( a \in \hat{A} \). Using the same argument, for \( j^{-1} \), we obtain \( j^{-1}(s) = b \cdot S \) for some \( b \in \hat{A} \). Hence \( j(b)a = 1 \) and \( a \) is invertible.

We have the following corollary to Proposition 4.8.

**Corollary 4.9.** Let \( \pi \) be a finite dimensional \( \mathbb{H} \)-module annihilated by a power of \( J_{W_\zeta} \), where \( \zeta \in V^* \) is real. Identify \( \pi \) and \( \Lambda(\pi) \) as linear spaces. The multiplication by \( a \in \hat{A} \) (from Proposition 4.8) provides a natural isomorphism between the linear spaces \( s(\pi) \) and \( S(\Lambda(\pi)) \).

4.6. Bernstein-Zelevinsky derivatives for graded algebras. Let \( V_n = X_n \otimes \mathbb{C} \), and \( \mathbb{H}_n := \mathbb{H}(V_n, R_n, \Pi_n, \log q) \). For every \( i = 0, \ldots, n \), we have a Levi subalgebra \( \mathbb{H}_{n-i} \otimes \mathbb{H}_i \). Let \( s_i \in \mathbb{H}_i \) be the sign projector, and let \( s_i^\circ \in \mathbb{H}_n \) be the image of \( 1 \otimes s_i \) under the inclusion \( \mathbb{H}_{n-i} \otimes \mathbb{H}_i \subseteq \mathbb{H}_n \).

Let \( \pi \) be a finite dimensional representation of \( \mathbb{H}_n \). The \( i \)-the Bernstein-Zelevinsky derivative of \( \pi \) is the natural \( \mathbb{H}_{n-i} \)-module
\[
gBZ_i(\pi) := s_i^\circ(\pi).
\]
Write any \( \zeta \in V_n^* = \text{Hom}(X_n, \mathbb{C}) \) as an \( n \)-tuple \( (\zeta_1, \ldots, \zeta_n) \) where \( \zeta_i \) is the value of \( \zeta \) on the standard basis element \( e_i \in X_n \). In this case \( \zeta \) is real for \( R_n \) if and only if \( \zeta_k - \zeta_l \in \mathbb{R} \) for all \( 1 \leq k, l \leq n \).

**Theorem 4.10.** Assume that \( \zeta \in V_n^* \) is real for the root system \( R_n \), and \( \pi \) is a finite-dimensional \( \mathbb{H}_n \)-module annihilated by a power of \( J_{S_n \zeta} \). There is a natural isomorphism of \( \mathbb{H}_{n-i} \)-modules \( BZ_i(\Lambda(\pi)) \) and \( \Lambda(gBZ_i(\pi)) \).

**Proof.** Note that the functor \( \Lambda \) commutes with the restriction to Levi subalgebras, that is, we can either restrict to \( \mathbb{H}_{n-i} \otimes \mathbb{H}_i \) and then apply \( \Lambda \), or apply \( \Lambda \) and then restrict to \( \mathbb{H}_{n-i} \otimes \mathbb{H}_i \). Decompose \( \pi \) under the action of \( \mathbb{H}_i \)
\[
\pi = \bigoplus \pi[S_i \zeta^i]
\]
where $\pi_{S,\zeta'}$ is the summand annihilated by a power of $G_{S,\zeta'}$. Concretely, the sum runs over $S_\tau$-orbits of the $i$-tuples $\zeta'$ that appear as the tail end of the $n$-tuples in the $S_n$-orbit of $\zeta$. We have the corresponding decomposition for the action of $H_i$,

$$\Lambda(\pi) = \oplus \Lambda(\pi)_{[S_i,\nu]}$$

where $t' = \exp(\zeta')$. (The underlying vector spaces of $\pi_{S,\zeta'}$ and $\Lambda(\pi)_{[S_i,\nu]}$ are the same.) It follows that $\Lambda(\pi)_{[S_i,t']}$ and $\Lambda(\pi_{S,\zeta'})$ are isomorphic $H_{n-i} \otimes H_i$-modules. Recall that $S_i^n = 1 \otimes S_i$ and $s_i^n = 1 \otimes s_i$, where $S_i$ and $s_i$ are the sign projectors in $H_i$ and $H_i$, respectively. Now we have the following isomorphisms of $H_{n-i}$-modules

$$S_i^n(\Lambda(\pi)_{[S_i,\nu]}) \cong S_i^n(\Lambda(\pi_{S,\zeta'})) \cong \Lambda(s_i^n(\pi_{S,\zeta'}))$$

where the second is furnished by Corollary 4.9. This isomorphism is given by the action of an invertible element in $\mathbb{H}_i$ and therefore intertwines $H_{n-i}$-action. □

4.7. Second reduction for Bernstein-Zelevinsky derivatives. In this section, we transfer the problem of computing Bernstein-Zelevinsky derivatives $BZ^n_i$ in Theorem 4.3 to the corresponding problem for graded Hecke algebras. We retain the notations in Sections 4.2 and 4.3. In particular, $n = (n_1, \ldots, n_m)$ is a partition of $n$, and we have fixed $t \in T_n$ such that $\alpha(t) > 0$ for all $\alpha \in R_n$. Then there exists $\zeta \in V_n^*$, real for the root system $R_n$, such that $t = \exp(\zeta)$. Let

$$\mathbb{H}_n := \mathbb{H}(V_n, R_n, \Pi_n, \log q) \cong \mathbb{H}_{n_1} \otimes \ldots \otimes \mathbb{H}_{n_m}.$$

Let $i = (i_1, \ldots, i_m)$ be an $m$-tuple of integers such that $0 \leq i_k \leq n_k$ for all $k$ and $n - i = (n_1 - i_1, \ldots, n_m - i_m)$. Each pair $(n_k - i_k, i_k)$ gives rise to an embedding $\mathbb{H}_{n_k - i_k} \otimes \mathbb{H}_{i_k} \subseteq \mathbb{H}_{n_k}$, and these combine to give an embedding

$$\mathbb{H}_{n-i} \otimes \mathbb{H}_i \subseteq \mathbb{H}_n$$

where $\mathbb{H}_i \cong \mathbb{H}_{i_1} \otimes \ldots \otimes \mathbb{H}_{i_m}$ etc. Abusing notation, we shall identify $\mathbb{H}_{n-i}$ with its image in $\mathbb{H}_n$ via the map $h \mapsto h \otimes 1$. Let $s_i \in \mathbb{H}_i$ be the sign projector in $\mathbb{H}_i$, and let $s^n_i$ be the image of $1 \otimes s_i$ in $\mathbb{H}_n$. Let $\sigma$ be an $\mathbb{H}_n$-module. Then $s^n_i(\sigma)$ is naturally an $\mathbb{H}_{n-i}$-module. Thus we have a functor

$$gBZ^n_i(\sigma) := s^n_i(\sigma)$$

from the category of $\mathbb{H}_n$-modules to the category of $\mathbb{H}_{n-i}$-modules. The following is proved in the same way as Theorem 4.10.

Theorem 4.11. Let $\zeta \in V_n^*$ be real for the root system $R_n$. Let $\pi$ be a finite-dimensional $\mathbb{H}_n$-module annihilated by a power of $G_{S,\zeta}$. Then we have a natural isomorphism of $\mathbb{H}_{n-i}$-modules

$$(BZ^n_i(\Lambda(\pi)) \cong \Lambda(gBZ^n_i((\pi))).$$
5. Bernstein-Zelevinsky derivatives and filtrations

5.1. Speh modules. Speh representations of $p$-adic groups were studied extensively by Tadić as a part of studying the unitary dual. We recall the definition of (generalized) Speh representations. Let $\tilde{n}$ be a partition of $n$, write $\tilde{n}^t = (e_1, \ldots, e_f)$, $e_1 \geq \ldots \geq e_f$, where $t$ is the transpose. Let $\text{St}_{e_k}$ be the Steinberg representation of $GL(e_k, F)$ and let $\text{St}'_{e_k} = \nu^{-\log (\det(g))} \text{St}_{e_k}$ be a twist of $\text{St}_{e_k}$, where $\nu = |\det(g)|_F$. Let $P_n$ be the standard parabolic subgroup associated to the partition $\tilde{n}^t$. Let $\rho(g) = |\det(g)|^r_F$ for some complex number $r$. The unique quotient of the induced representation

$$\pi(\tilde{n}, \rho) = \text{Ind}_{P_n}^{GL(n, F)}(\rho \text{St}_{e_1} \boxtimes \rho \text{St}_{e_2}' \cdots \boxtimes \rho \text{St}_{e_f}')$$

is the generalized Speh representation associated to $(\tilde{n}, \rho)$. If $e_1 = e_2 = \ldots = e_f$ then $\pi_{\tilde{n}}$ is a Speh representation.

Under the Borel-Casselman equivalence, generalized Speh representations correspond to $\mathcal{H}_n$-modules with single $\mathcal{H}_{S_n}$-type (see [BC], [BM3], [CM]). Since these $\mathcal{H}_n$-modules have real infinitesimal character, we can look at the corresponding modules for the graded algebra $\mathbb{H}_n$. They can be intrinsically constructed as follows. For $\kappa = -r \log q$, we have the following Jucys-Murphy elements: for $k = 2, \ldots, n$,

$$(5.6) \quad JM_k := -p(t_{s_{1,k}} + \cdots + t_{s_{k-1,k}}) + \kappa$$

and $JM_1 = \kappa$, where $p = \log q$. It is straightforward to check that the maps $e_k \mapsto JM_k$ and $t_w \mapsto t_w$ define an algebra homomorphism from $\mathbb{H}_n$ to $\mathbb{C}[S_n]$. Let $\sigma_{\tilde{n}}$ be the irreducible $\mathbb{C}[S_n]$-module corresponding to $\tilde{n}$. For example, the partition $(n)$ defines the trivial representation while $(1, \ldots, 1)$ defines the sign representation. Let $\sigma_{(\tilde{n}, \kappa)}$ be the $\mathbb{H}_n$-module pulled back from $\sigma_{\tilde{n}}$ via the map defined above, where $JM_k$ depends on $\kappa$. This is the generalized Speh module associated to $(\tilde{n}, \kappa)$. The module $\sigma_{(\tilde{n}, \kappa)}$ corresponds to $\pi(\tilde{n}, \kappa)$ under the Borel-Casselman equivalence and the Lusztig equivalence in Theorem 4.7.

Recall that $\text{gBZ}_i(\pi)$ is the $i$-the Bernstein-Zelevinsky derivative of an $\mathbb{H}_n$-module $\pi$.

**Lemma 5.1.** Let $\pi$ be the generalized Speh $\mathbb{H}_n$-module associated to the datum $(\tilde{n}, \kappa)$. Then $\text{gBZ}_i(\pi)$ is a direct sum of generalized Speh $\mathbb{H}_{n-i}$-modules. Moreover, $e_i$ acts by the constant $\kappa$ on each direct summand of $\text{gBZ}_i(\pi)$.

**Proof.** This follows from the construction of generalized Speh modules (see e.g. [5.6]) and the fact that the category of $\mathbb{C}[S_n]$-modules is semisimple. □

We now recover a result of Lapid-Mínguez (for the case of generalized Speh modules).

**Corollary 5.2.** Let $\pi$ be a generalized Speh representation of $GL(n, F)$ associated to $(\tilde{n}, \rho)$. Then $\pi^{(i)}$ is the direct sum of generalized Speh modules associated to $(\tilde{n}', \rho)$, where $\tilde{n}'$ runs for all the partitions obtained by removing $i$ boxes from $\tilde{n}$ with at most one in each row such that the resulting diagram is still a Young diagram.

**Proof.** Since $\Lambda(\sigma_{\tilde{n}, \kappa}) = \pi^{(i)}$, it suffices to compute $\text{gBZ}_i(\sigma_{\tilde{n}, \kappa})$ by Theorem 4.14. From the observation in Lemma 5.1 it suffices to determine the $\mathbb{C}[S_{n-i}]$-module structure of
gBZ$_i$(σ$_{n,i}$), and this follows from a special case of the Littlewood-Richardson rule (or the Pieri’s formula).

Generalized Speh modules form a subclass of ladder representations defined by Lapid-Mínguez [LM]. Bernstein-Zelevinsky derivatives of ladder representations are computed there using a determinantal formula of Tadić.

6. BRANCHING RULES AND LOCALLY NICE REPRESENTATIONS

6.1. Bernstein-Zelevinsky filtration. Let $E_n$ be the mirabolic subgroup of $GL(n+1, F)$ i.e. the subgroup of all matrices of the form \[
\begin{pmatrix}
g & v \\
0 & 1
\end{pmatrix},
\] where $g \in GL(n, F)$ and $v \in M_{n \times 1}$.

For $i = 1, \ldots, n + 1$ let

\[
R_i = \left\{ \begin{pmatrix} g & v \\ 0 & u \end{pmatrix} : g \in GL(n + 1 - i, F), v \in M_{n+1-i,i}, u \in U_i \right\}.
\]

We recall a result of Bernstein-Zelevinsky:

**Theorem 6.1.** Let $(\pi, X)$ be a smooth representation of $GL(n + 1, F)$. Then, as a representation of $E_n$, $\pi$ admits a filtration

\begin{equation}
0 = X_{n+1} \subset X_n \subset \ldots \subset X_1 \subset X_0 = X
\end{equation}

such that for $i = 1, \ldots, n + 1$

\[
X_{i-1}/X_i \cong \text{ind}^{E_n}_{R_i} (\pi^{(i)} \boxtimes \psi_i),
\]

$\pi^{(i)}$ is the $i$-th Bernstein-Zelevinsky derivative, and $\psi_i$ is the Whittaker character for $U_i$.

We abbreviate $G_n = GL(n, F)$ etc. Since $E_n = G_nR_i$, any element in $\text{ind}^{E_n}_{R_i} (\pi^{(i)} \boxtimes \psi_i)$ is determined by its restriction to $G_n$. Hence, for $i \geq 1$, the restriction of functions defines an isomorphism of $G_n$-modules,

\[
\text{ind}^{E_n}_{R_i} (\pi^{(i)} \boxtimes \psi_i) \cong \text{ind}^{G_n}_{Q_i}(\nu^{\frac{1}{2}} \pi^{(i)} \boxtimes \psi_{i-1}),
\]

where $Q_i = R_i \cap G_n$ and $\nu(g) = |\det(g)|_F$.

Let $P_i = M_iN_i$ be the maximal parabolic consisting of block upper triangular matrices in $G_n$ with the Levi factor $M_i = G_{n+1-i} \times G_{i-1}$ of block diagonal matrices. In particular, $P_i$ contains $Q_i$. Fix an embedding of $\mathcal{H}_{n+1-i} \otimes \mathcal{H}_{i-1}$ into $\mathcal{H}_n$ such that the restriction functor from the category of $\mathcal{H}_n$-modules to the category of $\mathcal{H}_{n+1-i} \otimes \mathcal{H}_{i-1}$-modules corresponds, in the category of representations of $G_n$ generated by Iwahori-fixed vectors, to the Jacquet functor with respect to the parabolic opposite to $P_i$. (Note that this is not the same embedding as in Section 3.) Now there are two ways to construct the right adjoint of the restriction functor. One way is tensoring by $\mathcal{H}_n$ and the other, by the second adjointness theorem of Bernstein, is the parabolic induction from $P_i$ to $G_n$. Hence, if $\sigma$ is a smooth representations of $M_i$, then, by the Yoneda lemma, we have a natural isomorphism of $\mathcal{H}_n$-modules

\[
\text{Ind}^{G_n}_{M_i}(\sigma)^{I_n} \cong \mathcal{H}_n \otimes (\mathcal{H}_{n+1-i} \otimes \mathcal{H}_{i-1}) (\sigma^{I_{M_i}}).
\]
Lemma 6.2. Let $P_{s_{i-1}}^{i-1} = H_{i-1} \otimes H_{s_{i-1}}$. The $H_n$-module $(\text{ind}_{Q_i}^{H_n}(\nu_{\frac{1}{2}} \pi^{(i)} \boxtimes \psi_{i-1}))^I_n$ is isomorphic to $H_n \otimes (H_{n+1} \otimes H_{s_{i-1}} ((\nu_{\frac{1}{2}} \pi^{(i)})^{I_{n+1}} \boxtimes P_{s_{i-1}}^{i-1})).$

Proof. By the transitivity of inductions, since $G_n \supset P_i \supset Q_i$, $\text{ind}_{Q_i}^{H_n}(\nu_{\frac{1}{2}} \pi^{(i)} \boxtimes \psi_{i-1}) \cong \text{Ind}_{M_i}^{H_n}(\nu_{\frac{1}{2}} \pi^{(i)} \boxtimes \text{ind}_{U_i}^{M_i} \psi_{i-1}).$

Lemma follows by taking Iwahori-fixed vectors and using Corollary 2.5. □

Lemma 6.2 implies the following:

Corollary 6.3. Let $\pi$ be an irreducible generic representation of $GL(n+1, F)$. Then $\pi^I_n$ is a finitely generated $H_n$-module.

6.2. Locally nice representations. We use the notations in Sections 3 and 4. This section does not directly use the realization of the Bernstein-Zelevinsky derivative via the Iwahori-Hecke algebras, but it is motivated by the Bernstein-Zelevinsky composition factors. The sign character plays a role in a number of places.

We first define a certain class of representations below. Since we only deal with Iwahori-fixed vector cases, it is more convenient to formulate the notions related to affine Hecke algebras.

Definition 6.4. Let $\pi_1$ be an irreducible generic representation of $GL(n+1, F)$. Let $J$ be a maximal ideal of $Z_n$. We say that $\pi_1$ is locally nice at $J$ if the only irreducible representation $\pi_2$ of $GL(n, F)$ (with Iwahori-fixed vectors) satisfying the conditions that

1. $\text{Hom}_{GL(n, F)}(\pi_1, \pi_2) \neq 0$, and
2. $\pi_2^I_n$ is annihilated by $J$,

is the unique irreducible generic representation annihilated by $J$.

Examples for Definition 6.4 are given below. Classifying locally nice representations is a Hom-restriction problem.

Example 6.5. Let $J$ be such that there exists only one isomorphism class of irreducible representations annihilated by $J$. This happens if the the irreducible generic representation of $GL(n, F)$ is also spherical, (see e.g. [BM], [Re]). Then any generic representation of $GL(n+1, F)$ is locally nice at $J$.

We state some results useful in proving Theorem 6.8.

Theorem 6.6. (see [Pr], [Pr3], [AGRS]) Let $\pi_1$ be an irreducible generic representation of $GL(n+1, F)$ and let $\pi_2$ be an irreducible generic representation of $GL(n, F)$. Then $\text{Hom}_{GL(n, F)}(\pi_1, \pi_2) = 1$

Lemma 6.7. Let $\pi$ be an irreducible generic representation of $GL(n+1, F)$. Then $\pi|_{GL(n, F)}$ contains $\text{ind}_{U_n}^{GL(n, F)} \psi_n$ as a submodule.
Proof. This follows from the Bernstein-Zelevinsky filtration (Theorem 6.1), definition of Bernstein-Zelevinsky derivatives (see Section 3.2) and the definition of a generic representation. □

Main ingredients of the proof of Theorem 6.8 below are the multiplicity one theorem above (Theorem 6.4, Definition 6.4 and Corollary 6.6). We remark that Theorem 6.8 is certainly not true without the condition of locally nicety.

**Theorem 6.8.** Let \( \pi \) be an irreducible generic representation of \( GL(n+1, F) \) and let \( I_n \) be the Iwahori subgroup of \( GL(n, F) \). Regard \( (\pi|_{GL(n, F)})^I_n \) as an \( H_n \)-module. Let \( J \) be a maximal ideal in \( \mathbb{Z}_n \). Let \( \mathbb{Z}_n \) be the \( J \)-adic completion of \( \mathbb{Z}_n \). Set \( \hat{H}_n = \mathbb{Z}_n \otimes_{\mathbb{Z}_n} H_n \).

Suppose \( \pi \) is locally nice at \( J \) (see Definition 6.4). Then \( \hat{Z}_n \otimes_{\mathbb{Z}_n} (\pi|_{GL(n, F)})^I_n \) is isomorphic to \( \hat{H}_n \otimes_{\mathcal{H}_{s_n}} \text{sgn} \) and hence is projective in the category of \( \hat{H}_n \)-modules.

**Proof.** For simplicity, set \( \chi = (\pi|_{GL(n, F)})^I_n \), and let \( \hat{\chi} = \hat{Z}_n \otimes_{\mathbb{Z}_n} \chi \). Let \( \hat{J} = \hat{Z}_n \otimes_{\mathbb{Z}_n} J \). First of all, by Corollary 6.3, \( \hat{\chi} \) is a finitely generated \( \hat{H}_n \)-module. We divide the proof into several steps.

**Step 1:** Let \( \hat{\chi}' \) be the \( \hat{H}_n \)-submodule of \( \hat{\chi} \) generated by \( S_n(\hat{\chi}) \), where \( S_n \) is the sign projector.

**Claim:** \( \hat{\chi}' = \hat{\chi} \).

**Proof of the claim:** The key idea is to use Definition 6.4. Let \( \nu = \hat{\chi}/\hat{\chi}' \). Consider \( \nu \) as a \( \hat{Z}_n \)-module. A quotient of a finitely generated module is finitely generated and furthermore \( \hat{H}_n \) is finitely generated as \( \hat{Z}_n \)-module. Hence by the transitivity of finitely generatedness, \( \nu \) is a finitely-generated \( \hat{Z}_n \)-module. Suppose \( \nu \neq 0 \). This implies \( \nu \otimes \hat{J} \nu \neq 0 \) (Nakayama’s Lemma). Now \( \nu \otimes \hat{J} \nu \) descends to an \( \hat{H}_n/\hat{J} \hat{H}_n \)-module, which is finitely generated. Hence \( \nu \otimes \hat{J} \nu \) is also finite-dimensional (and non-zero). Thus there exists a (non-zero) irreducible \( \hat{H}_n \)-quotient, say \( \nu' \), of \( \nu \otimes \hat{J} \nu \). However from our construction, \( \nu' \) does not contain a sign representation and hence \( \nu' \) is not generic (Corollary 2.6). This contradicts that \( \text{Hom}_{\hat{H}_n}(\chi, \nu') = 0 \) by our assumption that \( \pi \) is locally nice at \( J \).

**Step 2:** Since \( \hat{\chi} \) is finitely generated and \( \hat{\chi}' = \hat{\chi} \) (from the proved claim), there exists a finite set of elements \( x_1, \ldots, x_r \) in \( S_n(\hat{\chi}) \) which generates \( \hat{\chi} \). Assume that \( r \) is the smallest possible. From our choices of generators \( x_1, \ldots, x_r \), we have a surjective map

\[
\Psi : \bigoplus_{k=1}^{r} \hat{H}_n \otimes_{\mathcal{H}_{s_n}} \text{sgn} \rightarrow \hat{\chi}
\]

given by \( (0, \ldots, 1 \otimes 1, \ldots, 0) \mapsto x_k \), where \( 1 \otimes 1 \) is in the \( k \)-th summand of \( \bigoplus_{k=1}^{r} \hat{H}_n \otimes_{\mathcal{H}_{s_n}} \text{sgn} \).

Let\[
\mathcal{P}_l : \bigoplus_{k=1}^{r} \hat{H}_n \otimes_{\mathcal{H}_{s_n}} \text{sgn} \rightarrow \hat{H} \otimes_{\mathcal{H}_{s_n}} \text{sgn}
\]
be the projection onto the \( l \)-th factor. The minimality of \( r \) implies the following claim.

**Claim:** \( \mathcal{P}_l(\ker \Psi) \neq \hat{H}_n \otimes_{\mathcal{H}_{s_n}} \text{sgn} \) for all \( l \).

**Claim:** \( r \leq 1 \).
Proof of the claim: Let
\[ A_t := \mathcal{P}_t \left( \ker \Psi + \bigoplus_{k=1}^r \hat{\mathcal{J}} \left( \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \right) \right). \]

If \( A_t = \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \) then \( \mathcal{P}_t(\ker \Psi) = \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \), by Nakayama’s lemma, and this contradicts the previous claim. Thus \( \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn}/A_t \) is non-zero and moreover finite-dimensional. Let \( \nu_t \) be an irreducible quotient. By the Frobenius reciprocity, \( \nu_t \) contains \( \text{sgn} \) and hence is the unique generic representation \( \chi_{\text{gen}} \) annihilated by \( \mathcal{J} \). Hence this defines a map, denoted \( f_t \), from \( \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \) to \( \nu_t \). Now we define a map \( F_t : \bigoplus_{k=1}^r \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \to \nu_t \) by \( F_t = f_t \circ \mathcal{P}_t \). From our construction, \( F_t(\ker \Psi) = 0 \) and hence descends to a map from \( \chi \) to \( \chi_{\text{gen}} \). Note that \( F_t \) are linearly independent, hence \( \text{Hom}_{\mathcal{H}_n}(\chi, \chi_{\text{gen}}) \geq r \). Theorem 6.6 proves the claim.

Step 3: We have shown that \( \hat{\chi} \) is isomorphic to \( (\mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn})/\ker \Psi \). It remains to prove \( \ker \Psi = 0 \). Suppose not. Let \( a \otimes 1 \in \ker \Psi \) for some non-zero \( a \in \hat{\mathcal{Z}}_n \otimes \mathcal{A}_n \). By Corollary 2.5 and Lemma 6.7, \( \mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn} \) embeds into \( (\mathcal{H}_n \otimes \mathcal{H}_{s_n} \text{sgn})/\ker \Psi \), say the element \( 1 \otimes 1 \) is mapped to an element represented by \( a' \otimes 1 \) for some \( a' \in \hat{\mathcal{Z}}_n \otimes \mathcal{A}_n \). Now \( a \otimes 1 \neq 0 \) is mapped to an element represented by \( aa' \otimes 1 \), but this one is in \( \ker \Psi \). This is a contradiction.

Theorem 6.8 provides a simple conceptual explanation to Conjecture 1.5 for those locally nice representations. Our cases cover some that cannot be merely deduced from the composition factors of Bernstein-Zelevinsky filtrations and the Euler-Poincaré pairing. Moreover, as mentioned before, Theorem 6.8 does not hold in general and thus a proof for a general Ext-multiplicity result will require detailed understanding of structure or an alternate approach.

Corollary 6.9. Let \( \pi_2 \) be an irreducible generic representation of \( \text{GL}(n, F) \) with Iwahori-fixed vectors annihilated by a maximal ideal \( \mathcal{J} \) in \( \mathcal{Z}_n \). Suppose \( \pi_1 \) is an irreducible generic representation of \( \text{GL}(n+1, F) \) locally nice at \( \mathcal{J} \). Then
\[ \text{Ext}^i_{\text{GL}(n,F)}(\pi_1, \pi_2) = 0 \]
for all \( i \geq 1 \).

Proof. Corollary follows from Theorem 6.8 using
\[ \text{Ext}^i_{\mathcal{H}_n}(\pi_1|_{\text{GL}(n,F)}^{I_n}, \pi_2^{I_n}) \cong \text{Ext}^i_{\hat{\mathcal{H}_n}}((\pi_1|_{\text{GL}(n,F)}^{I_n}, (\pi_2)^{I_n}). \]

6.3. Branching rule for the Steinberg representation. This section employs similar strategy as in Section 2 to compute the \( \mathcal{H}_n \)-structure of the Steinberg representation of \( \text{GL}(n+1) \). We work firstly with a general split reductive group \( G \).

Let \( St \) be the Steinberg representation of \( G \). We use the notation from Section 2. In particular, \( B \) is the Borel subgroup of \( G \), \( \hat{U} \) the unipotent radical of \( B \), the Borel opposite to \( B \), and \( X_w = Bu\hat{U} \) are the Bruhat cells. Write \( X = B\hat{U} \) for the open cell. For any subset \( J \) of simple roots \( \Pi \), let \( P_J \) be the standard parabolic subgroup associated to \( J \) (and containing \( B \)). In particular, \( P_0 = B \). Let \( C_c^{\infty}(P_J \setminus G) \) be the space of compactly


supported smooth $P_J$-invariant functions on $G$. We use the following realization of the Steinberg representation:

$$\text{St} = C_c^\infty(B \setminus G)/\sum_{\emptyset \neq J \subseteq \Pi} C_c^\infty(P_J \setminus G).$$

Thus we have a $\tilde{B}$-equivariant map $\Omega : C_c^\infty(B \setminus X) \to \text{St}$ given as the composition of natural maps

$$(6.8) \quad C_c^\infty(B \setminus X) \to C_c^\infty(B \setminus G) \to \text{St}.$$

**Proposition 6.10.** The map $\Omega$ is a $\tilde{B}$-equivariant isomorphism of $C_c^\infty(B \setminus X)$ and $\text{St}$.

**Proof.** Let $\mathbb{C}[W]$ denote the space of functions on $W$. Consider it a $W$-module for the action by right translations. For every simple root $\alpha$, let $W_\alpha = \{1, s_\alpha\}$. Then $\mathbb{C}[W_\alpha \setminus W]$ is a submodule of $\mathbb{C}[W]$ consisting of left $W_\alpha$-invariant functions. For injectivity we need the following lemma.

**Lemma 6.11.** Let $\delta \in \mathbb{C}[W]$ be the delta function corresponding to the identity element. Then $\delta$ cannot be written as a linear combination of elements in $\mathbb{C}[W_\alpha \setminus W]$ where $\alpha$ runs over all simple roots.

**Proof.** Functions in $\mathbb{C}[W_\alpha \setminus W]$ are perpendicular to the sign character. Hence any linear combination of such functions is also perpendicular to the sign character. But $\delta$ is not, hence lemma. \qed

We can now prove injectivity of $\Omega$. Let $f \in C_c^\infty(B \setminus X)$ be in the kernel of $\Omega$. Then there exist $f_\alpha \in C_c^\infty(P_\alpha \setminus G)$ such that $f = \sum_{\alpha \in \Pi} f_\alpha$. For every $\tilde{u} \in \tilde{U}$, the function $w \mapsto f_\alpha(w\tilde{u})$ is in $\mathbb{C}[W_\alpha \setminus W]$. On the other hand, $w \mapsto f(w\tilde{u})$ is a multiple of $\delta$. Lemma implies that $f(\tilde{u}) = 0$.

For surjectivity, let $V_r \subseteq C_c^\infty(B \setminus G)$ be the subspace of functions supported on the union of the Bruhat cells $X_w$ for $w \in W$ such that $l(w) \leq r$. Let $V_w = C_c^\infty(B \setminus X_w)$. Then, if $r > 1$, we have an exact sequence

$$0 \to V_{r-1} \to V_r \to \bigoplus_{l(w)=r} V_w \to 0.$$ 

Let $v \in \text{St}$ be the image of $f \in V_r$. We need to show that $\bar{v}$ is the image of some $f' \in V_{r-1}$. For every $w$ such that $l(w) = r$, pick $f_w \in V_r$ supported on $X_{w'}$ for $l(w') < r$ and $X_w$. Then $f - \sum_{l(w)=r} f_w \in V_{r-1}$. Since $r > 1$, for every $w$ such that $l(w) = r$, there exists a simple root $\alpha$ such that $l(s_\alpha w) = r - 1$. The group $G$ has a cell decomposition as a union of $Y_w = P_\alpha w$ where $w$ runs over all $w \in W$ such that $l(s_\alpha w) = l(w) - 1$. Note that $B \setminus X_w = P_\alpha \setminus Y_w$ for such $w$. Going back to our fixed $w$ such that $l(w) = r$, there exists a function $h_w \in C_c^\infty(P_\alpha \setminus G)$ such that the support of $h_w$ is on $Y_w$ and larger orbits, and $h_w = f_w$ on $B \setminus X_w = P_\alpha \setminus Y_w$. The support of $h_w$, viewed as an element of $C_c^\infty(B \setminus G)$, is contained in $X_w$ and the union of $X_{w'}$ such that $l(w') < l(w)$. Hence $f' = f - \sum_{l(w)=r} h_w \in V_{r-1}$ and $f'$ has the image $\bar{v}$ in $\text{St}$. Hence $\Omega$ is surjective. \qed
Let $\chi_I$ be the characteristic function of $B(\mathcal{U} \cap I)$. Since $I = (B \cap I)(\mathcal{U} \cap I)$, it is an $I$-fixed element in $C^\infty_c(B \setminus G)$. Hence $v_0 = \Omega(\chi_I)$ spans the line of $I$-fixed vectors in $\text{St}$.

We now specialize to $GL(n)$.

**Theorem 6.12.** Let $\text{St}_{n+1}$ be the Steinberg representation of $GL(n+1)$ and $v_0 = \Omega(\chi_{I_{n+1}})$ the non-zero $I_{n+1}$-fixed vector. The $\mathcal{H}_n$-module $\text{St}_{n+1}^{n}$ is generated by $v_0$ and isomorphic to $\mathcal{H}_n \otimes_{\mathcal{H}_{S_n}} \text{sgn}$. In particular, it is projective.

**Proof.** Note that $\mathcal{H}_{S_n}$ acts on $v_0$ as the sign character. Let $D_n \cong (F^\times)^n$ be the group of diagonal matrices and $B_n = D_n U_n$ be the Borel group of upper triangular matrices in $GL(n)$. Pick $A_n \subseteq \mathcal{H}_n$, isomorphic to the group algebra of the lattice $D_n/(D_n \cap I_n)$, such that the Jacquet functor with respect to $\mathcal{U}_n$ corresponds to the restriction to $A_n$. It suffices to show that $\text{St}_{n+1}^{I_{n+1}}$ is freely generated by $v_0$ as an $A_n$-module. This will be checked by passing to the Jacquet module with respect to $\mathcal{U}_n$. We have a decomposition $\mathcal{U}_{n+1} = \mathcal{U}_n \tilde{V}_n$ where

$$\tilde{V}_n = \left\{ \left( \begin{array}{cc} I_{n \times n} & 0 \\ I_{n \times 1} & 1 \end{array} \right) : v = (a_1, \ldots, a_n) \in M_{n \times 1} \right\} \cong F^n.$$ 

This identification and Proposition 5.10 which says that $\text{St}_{n+1} \cong C_c^\infty(\tilde{U}_{n+1})$, imply that there is an isomorphism of $D_n$-representations

$$\Phi : (\text{St}_{n+1})_{\tilde{U}_n} \cong C_c^\infty(F^n).$$

Furthermore, $\Phi(v_0)$ is the characteristic function of $\mathcal{O}^n \subset F^n$. The theorem follows from the observation that $D_n/(D_n \cap I_n)$-translates of the characteristic function of $\mathcal{O}^n$ form a basis of $C_c^\infty(F^n)^{(D_n/(I_n))}$.

**Corollary 6.13.** The Steinberg representation $\text{St}_{n+1}$ of $GL(n+1, F)$ is locally nice at every central character of $\mathcal{H}_n$.

Note that Theorem 6.12 for $\text{St}_{n+1}$ can be recovered directly from Theorem 6.12.

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