ON GENERALISATIONS OF LOSEV-MANIN MODULI SPACES FOR CLASSICAL ROOT SYSTEMS

VICTOR BATYREV AND MARK BLUME

In memory of Eckart Viehweg

Abstract. Losev and Manin introduced fine moduli spaces $\mathcal{L}_n$ of stable $n$-pointed chains of projective lines. The moduli space $\mathcal{L}_{n+1}$ is isomorphic to the toric variety $X(A_n)$ associated with the root system $A_n$, which is part of a general construction to associate with a root system $R$ of rank $n$ an $n$-dimensional smooth projective toric variety $X(R)$. In this paper we investigate generalisations of the Losev-Manin moduli spaces for the other families of classical root systems.

Introduction

In [LM00] Losev and Manin introduced fine moduli spaces $\mathcal{L}_n$ of stable $n$-pointed chains of projective lines. These Losev-Manin moduli spaces are similar to the moduli spaces $\mathcal{M}_{0,n+2}$, but whereas $\mathcal{M}_{0,n+2}$ parametrises trees of projective lines with $n + 2$ marked points that are not allowed to coincide, the moduli space $\mathcal{L}_n$ parametrises chains of projective lines with two poles and $n$ marked points that may coincide.

The Losev-Manin moduli space $\mathcal{L}_{n+1}$ has the structure of an $n$-dimensional smooth projective toric variety such that the boundary divisors parametrising reducible curves correspond to the torus invariant divisors; it coincides with the toric variety $X(A_n)$ associated with the root system $A_n$. This is part of a general construction to associate with a root system $R$ of rank $n$ an $n$-dimensional smooth projective toric variety $X(R)$ ([Kis5], [Pr90]). In the introduction to [LM00] the authors asked about generalisations of the moduli spaces $\mathcal{L}_n$ for the other families of classical root systems. In the present paper we address this problem.

Concerning the family of root systems of type $B$ we present a variant of the Losev-Manin moduli problem by considering chains of projective lines of odd length with an involution permuting the two poles having one marked point $s_0$ invariant under the involution and $n$ pairs of marked points $s_i^\pm$ that are interchanged by the involution. We show that these pointed curves admit a fine moduli space $\mathcal{L}_n^{0,\pm}$ which is isomorphic to the toric variety $X(B_n)$ such that the boundary divisors of the moduli space get identified with the torus invariant divisors.

It is well known that for the Losev-Manin moduli spaces, as for the moduli spaces $\mathcal{M}_{0,n}$, the universal curve over $\mathcal{T}_{n+1}$ is the next moduli space $\mathcal{T}_{n+2}$ together with a natural forgetful morphism $\mathcal{T}_{n+2} \to \mathcal{T}_{n+1}$. In [BB11] we developed functorial properties of the toric varieties $X(R)$ with respect to maps of root systems and observed that this morphism $\mathcal{T}_{n+2} = X(A_{n+1}) \to \mathcal{T}_{n+1} = X(A_n)$ is induced by the inclusion of root systems $A_n \to A_{n+1}$. Furthermore, the $n+1$ sections $X(A_n) \to X(A_{n+1})$ come from projections of root systems $A_{n+1} \to A_n$ along the $n+1$ additional pairs of opposite roots in $A_{n+1}$ not contained in $A_n$.

The second author was supported by DFG-Schwerpunkt 1388 Darstellungstheorie.
All this generalises to the family of root systems of type $B$: the morphism $X(B_{n+1}) \rightarrow X(B_n)$ coming from the inclusion of root systems $B_n \rightarrow B_{n+1}$ is flat and its fibres have the structure of chains of projective lines of odd length. The $2n+1$ additional pairs of opposite roots in $B_{n+1}$ give $2n+1$ sections. There is a symmetry of $B_{n+1}$ fixing $B_n$ which induces an involution $I$ of $X(B_{n+1})$ over $X(B_n)$ such that the sections are grouped into $n$ pairs of sections $s_i^\pm$ interchanged by the involution and one section $s_0$ invariant under the involution. We show that $X(B_{n+1}) \rightarrow X(B_n)$ together with these sections and the involution $I$ forms the universal family over the fine moduli space $\overline{\mathcal{L}}_{n}^{0,\pm} = X(B_n)$. On the other hand, we will see that the toric varieties $X(R_n)$ for $R = C, D$ do not form fine moduli spaces of pointed reduced curves having $X(R_{n+1}) \rightarrow X(R_n)$ as universal family.

In the case of root systems of type $C$ the morphism $X(C_{n+1}) \rightarrow X(C_n)$ is flat with one-dimensional fibres having the structure of $2n$-pointed chains of projective lines of odd length with involution except that over a certain torus invariant divisor nonreduced components occur. On the one hand we can consider families of pointed curves as in the $B_n$-case but without the section $s_0$ and thereby allowing an additional involution as isomorphism. This gives rise to a toric Deligne-Mumford stack $\mathcal{X}(C_n)$ which is an orbifold having the toric variety $X(C_n)$ as coarse moduli space with stacky points over the divisor determined by the nonreduced fibres. On the other hand we can describe $X(C_n)$ as a fine moduli space $\overline{\mathcal{L}}_{n}^{+1}$ of $2n$-pointed chains of projective lines of odd and even length with involution with each of the marked points corresponding to a pair of opposite roots in $C_{n+1}\backslash C_n$ that defines a projection $C_{n+1} \rightarrow C_n$. The universal family arises from $X(C_{n+1}) \rightarrow X(C_n)$ by contracting the nonreduced components in the fibres.

In the case of the remaining family of root systems of type $D$ the morphism $X(D_{n+1}) \rightarrow X(D_n)$ is not flat. There are $2$-dimensional fibres that occur over closures of certain torus orbits of codimension $2$, over the other points as fibres we have $2n$-pointed chains of projective lines with involution.

We observe that in the cases of all families of root systems $R = A, B, C, D$ the torus fixed points of $X(R_n)$ correspond to pointed curves having the form of the Dynkin diagram for the root system $R_{n+1}$.

**Outline of the paper.** In the first sections 1–5 we deal with the case of root systems of type $B$. In section 1 we formulate a moduli problem of $(2n + 1)$-pointed chains of projective lines called $B_n$-curves, which is a variant of the Losev-Manin moduli problem. In section 2 we collect some facts about the toric varieties $X(B_n)$ associated with root systems of type $B$. Section 3 is about the morphism $X(B_{n+1}) \rightarrow X(B_n)$, which, together with its sections and the involution, forms a flat family of $B_n$-curves, and in section 4 we prove that the toric variety $\overline{\mathcal{L}}_{n}^{0,\pm} = X(B_n)$ is a fine moduli space of $B_n$-curves with universal family $X(B_{n+1}) \rightarrow X(B_n)$. To show that the moduli functor of $B_n$-curves is isomorphic to the functor of the toric variety $X(B_n)$ we use the description of the functor of toric varieties associated with root systems given in [BB11, 1.3]; our proof is a variation of our new proof of the respective statement for root systems of type $A$ given in [BB11, 3.3]. In section 5 we present some results on the (co)homology of the spaces $\overline{\mathcal{L}}_{n}^{0,\pm} = X(B_n)$, giving descriptions similar to the case of the Losev-Manin moduli spaces $\overline{\mathcal{L}}_{n+1} = X(A_n)$.

In the remaining sections 6 and 7 the cases of the root systems of type $C$ and $D$ are investigated.
1. Pointed chains of projective lines with involution

**Definition 1.1.** A chain of projective lines of length $m$ over an algebraically closed field $K$ is a projective curve $C = C_1 \cup \ldots \cup C_m$ over $K$ such that every irreducible component $C_j$ of $C$ is a projective line with poles $p_j^-$, $p_j^+$ and these components intersect as follows: different components $C_i$ and $C_j$ intersect only if $|i - j| = 1$. For $p_1^- \in C_1$ and $p_m^+ \in C_m$ we write $s_-$ and $s_+$. Two chains of projective lines $(C, s_-, s_+)$ and $(C', s'_-, s'_+)$ are called isomorphic if there is an isomorphism $\varphi: C \to C'$ such that $\varphi(s_-) = s'_-$, $\varphi(s_+) = s'_+$. 

**Definition 1.2.** A chain of projective lines with involution $(C, I, s_-, s_+)$ is a chain of projective lines together with an isomorphism $I: C \to C$ such that $I^2 = id_C$ and $I(s_-) = s_+$. In this case we use the following notation: if the chain has odd length denote by $(C_0, p_0^-, p_0^+)$ the central component; denote by $(C_j, p^-_j, p^+_j)$, $(C_{j-1}, p^-_{j-1}, p^+_{j-1})$, the pairs of $I$-conjugate components (i.e. $I(C_j) = C_{j-1}$, $I(p_j^-) = p_{j-1}^+$, $I(p_j^+) = p_{j-1}^-$) such that $p_j^- = p_{j+1}^+$, $p_{j-1}^- = p_{j+1}^+$ and in case of odd length $p_0^+ = p_0^-$ whereas in case of even length $p_{m+1}^+ = p_1^-$. In particular, we have $s_- = p_{m-1}^-$, $s_+ = p_m^+$ if the chain has length $2m$ or $2m + 1$. Two chains of projective lines with involution $(C, I, s_-, s_+)$ and $(C', I', s'_-, s'_+)$ are called isomorphic if there is an isomorphism of chains of projective lines $\varphi: (C, s_-, s_+) \to (C', s'_-, s'_+)$ such that $\varphi \circ I = I' \circ \varphi$.

In the following we are concerned with certain compactifications of the algebraic torus $(G_m)^n$ parametrising $n$ pairs of points of the form $(z, \frac{1}{z})$ in $(G_m, 1) \subset (\mathbb{P}^1, 0, \infty, 1)$, i.e. pairs of points which are interchanged by the involution of $\mathbb{P}^1$ that fixes the point 1 and interchanges the two poles 0 and $\infty$. These compactifications, which will be associated with root systems, parametrise isomorphism classes of certain pointed chains of projective lines with an involution. We now define the type of pointed curve which will be relevant in the case of root systems of type $B$.

**Definition 1.3.** A $(2n+1)$-pointed chain of projective lines with involution $(C, I, s_-, s_+, s_0, s_1^+, \ldots, s_n^+)$ is a chain of projective lines with involution $(C, I, s_-, s_+)$ of odd length together with (possibly coinciding) marked points $s_0, s_1^\pm, \ldots, s_n^\pm$ in $C$ different from the poles such that $I(s_0) = s_0$, $I(s_i^+) = s_i^-$. Two $(2n+1)$-pointed chains of projective lines with involution $(C, I, s_-, s_+, s_0, s_1^+, \ldots, s_n^+)$ and $(C', I', s'_-, s'_+, s'_0, s'_1^+, \ldots, s'_n^+)$ are called isomorphic if there is an isomorphism $\varphi: (C, I, s_-, s_+) \to (C', I', s'_-, s'_+)$ of the underlying chains of projective lines with involution such that $\varphi(s_0) = s'_0$, $\varphi(s_i^+) = s_i'^-$. A $(2n+1)$-pointed chain of projective lines with involution $(C, I, s_-, s_+, s_0, s_1^+, \ldots, s_n^+)$ is called stable if each component of $C$ contains at least one of the points $s_0, s_j^\pm$. A $B_n$-curve over an algebraically closed field $K$ is a stable $(2n+1)$-pointed chain of projective lines over $K$.
Definition 1.4. Let $Y$ be a scheme. A $B_n$-curve over $Y$ is a collection $(\pi: C \to Y, I, s_-, s_+, s_0, s_0^\pm, \ldots, s_n^\pm)$, where $C$ is a scheme, $\pi$ is a flat proper morphism of schemes, $I: C \to C$ an involution over $Y$ and $s_-, s_+, s_0, s_0^\pm, \ldots, s_n^\pm: Y \to C$ are sections such that for any geometric point $y$ of $Y$ the collection $(C_y, I_y, (s_-)_y, (s_+)_y, (s_0)_y, (s_0^\pm)_y, \ldots, (s_n^\pm)_y)$ is a $B_n$-curve over $y$. An isomorphism of $B_n$-curves over $Y$ is an isomorphism of $Y$-schemes that is compatible with the involution and the sections. We define the moduli functor of $B_n$-curves as the functor

$$\mathcal{F}_n^{0, \pm}: \text{ (schemes) } \to \text{ (sets) }$$

that associates to a scheme $Y$ the set of isomorphism classes of $B_n$-curves over $Y$ and to a morphism of schemes the map obtained by pulling back $B_n$-curves.

We will show in section 4 that a fine moduli space of $B_n$-curves $\mathcal{F}_n^{0, \pm}$ exists and that it is isomorphic to the toric variety associated with the root system $B_n$.

2. Toric varieties $X(B_n)$

For a root system $R$ of rank $n$ we have the $n$-dimensional smooth projective toric variety $X(R)$ associated with the fan that consists of the Weyl chambers of the root system and their faces ([Kis85], [Pr90], see also [BB11, 1.1]). Here we consider the particular case of root systems of type $B$.

Let $E$ be an $n$-dimensional Euclidean space with basis $u_1, \ldots, u_n$. The root system $B_n$ in $E$ consists of the following $2n^2$ roots:

$$\pm u_i \text{ for } i \in \{1, \ldots, n\}; \quad \pm(u_i + u_j), \pm(u_i - u_j) \text{ for } i, j \in \{1, \ldots, n\}, i < j.$$

The root lattice $M(B_n) \cong \mathbb{Z}^n$ of the root system $B_n$ is the lattice in $E$ generated by $u_1, \ldots, u_n$. The following is a set of simple roots:

$$u_i \pm u_j, \text{ for } i, j \in \{1, \ldots, n\}, i \neq j.$$

The Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \times S_n$ acts by $u_i \mapsto \pm u_i$ and by permuting the $u_i$. So there are $2^n n!$ sets of simple roots, these are of the form $\varepsilon_1 u_1 - \varepsilon_2 u_2, \varepsilon_2 u_2 - \varepsilon_3 u_3, \ldots, \varepsilon_{n-1} u_{n-1}, \varepsilon_n u_n, \varepsilon_n u_n$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_1, \ldots, \varepsilon_n$. For later use we list linear relations between positive roots of $B_n$.

Lemma 2.1. Let $B_n^+$ be the set of positive roots of $B_n$ corresponding to the set of simple roots $u_i - u_j, u_i - u_j, \ldots, u_n - u_n$ and put $\beta_{ij} = u_i - u_j, \gamma_{ij} = u_i + u_j$ for $i, j \in \{1, \ldots, n\}, i \neq j$. Then $B_n^+ = \{u_1, \ldots, u_n\} \cup \{\beta_{ij} \mid i < j\} \cup \{\gamma_{ij} \mid i \neq j\}$ and the triplets of positive roots $\alpha, \beta, \gamma \in B_n^+$ satisfying $\alpha + \beta = \gamma$ are the following:

$$\begin{align*}
\beta_{ij} + u_j &= u_i \quad (i, j \in \{1, \ldots, n\}, i < j) \\
u_i + u_j &= \gamma_{ij} \quad (i, j \in \{1, \ldots, n\}, i \neq j) \\
\beta_{ij} + \beta_{jk} &= \beta_{ik} \quad (i, j, k \in \{1, \ldots, n\}, i < j < k) \\
\beta_{ij} + \gamma_{jk} &= \gamma_{ik} \quad (i, j, k \in \{1, \ldots, n\}, i < j, k \neq i, j)
\end{align*}$$
Let $N(B_n)$ be the lattice dual to the root lattice $M(B_n)$ and $v_1, \ldots, v_n$ the basis of $N(B_n)$ dual to $u_1, \ldots, u_n$. The fan $\Sigma(B_n)$ is defined as the fan of Weyl chambers in $N(B_n)$, i.e. its maximal cones are the Weyl chambers $\sigma = S' = \{ \alpha \in S : \langle \alpha, u \rangle \geq 0 \}$ for sets of simple roots $S$ of the root system $B_n$ and all cones arise as faces of these. For the set of simple roots $S = \{u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n\}$ has the dual basis $v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_n$ of $N(B_n)$, the Weyl chamber $\sigma_S$ is equal to $(\langle v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_n \rangle_{Q \geq 0})$. All Weyl chambers, i.e. all maximal cones of the fan $\Sigma(B_n)$, arise as translates of $\sigma_S$ under the action of the Weyl group on $N(B_n)\mathbb{Q}$, thus they are generated by sets of elements of the form $\varepsilon_i v_i, \varepsilon_i v_i + \varepsilon_j v_j, \ldots, \varepsilon_i v_i + \ldots + \varepsilon_n v_n$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_i \in \{\pm 1\}$. The fan $\Sigma(B_n)$ has $3^n - 1$ one-dimensional cones generated by the elements of the form $\varepsilon_i v_i + \ldots + \varepsilon_k v_k$ for $k \in \{1, \ldots, n\}$. These are via $v_B := \sum_{\varepsilon_i \in B} \varepsilon_i v_i \leftrightarrow B$ in bijection with the set $B$ of all subsets $\emptyset \neq B \subseteq \{\pm 1, \ldots, \pm n\}$ such that $B \cap \{i, -i\} \neq \{i, -i\}$ for $i = 1, \ldots, n$. The one-dimensional cones for a family of such sets $B^{(1)}, \ldots, B^{(n)}$ form a higher dimensional cone whenever they can be ordered such that $B^{(i_1)} \subseteq \cdots \subseteq B^{(i_n)}$.

We have the $n$-dimensional smooth projective toric variety $X(B_n)$ associated with the fan $\Sigma(B_n)$ with respect to the lattice $N(B_n)$. As usual, any element $u \in M(B_n)$ defines a character of the open dense torus $T(B_n)$ (resp. a rational function on $X(B_n)$) denoted by $x^u$. The toric variety $X(B_n)$ has the following covering by affine spaces. For any set $S$ of simple roots we have the maximal cone $\sigma_S = S'$ and the chart $U_S = \text{Spec} \mathbb{Z}[\sigma_S \cap M(B_n)] \cong \mathbb{A}^n$, for example if $S = \{u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n\}$ then $\mathbb{Z}[\sigma_S \cap M(B_n)] = \mathbb{Z}[\frac{x_1}{x_2}, \ldots, \frac{x_{n-1}}{x_n}, x_n]$. The Weyl group $W(B_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts on $X(B_n)$, it permutes these affine charts.

By [BB11, 1.2] the closures of torus orbits in $X(B_n)$ are isomorphic to products $X(B_n) \times X(A_{n_2}) \times \ldots \times X(A_{n_k})$. The torus invariant divisor for the one-dimensional cone generated by $\varepsilon_i v_i + \ldots + \varepsilon_n v_n$ is isomorphic to $X(B_{n-k}) \times X(A_{k-1})$, in particular for $\varepsilon_i v_i + \ldots + \varepsilon_n v_n$ we have a divisor isomorphic to $X(A_{n-1})$.

3. The universal curve

We construct a $B_n$-curve over $X(B_n)$, which later turns out to be the universal curve over the moduli space of $B_n$-curves, by using functorial properties of the toric varieties associated with root systems developed in [BB11, 1.2]. We fix the following notations for roots of $B_n$ and $B_{n+1}$: $\beta_{ij} = u_i - u_j, \gamma_{ij} = u_i + u_j$ for $i, j \in \{1, \ldots, n\}, i \neq j$ and $\alpha_i^+ = u_{n+1} + u_i, \alpha_i^- = u_{n+1} - u_i$ for $i \in \{1, \ldots, n\}$.

Construction 3.1. (The universal $B_n$-curve.) Consider the root subsystem $B_n \subset B_{n+1}$ consisting of the roots in the subspace spanned by $u_1, \ldots, u_n$. The inclusion of root systems $B_n \subset B_{n+1}$ determines a proper surjective morphism $X(B_{n+1}) \rightarrow X(B_n)$.

There are $2n+1$ additional pairs of opposite roots, the pairs $\pm \alpha_i^+$ and $\pm \alpha_i^-$ for $i \in \{1, \ldots, n\}$ and the pair $\pm u_{n+1}$. Each of these defines a projection onto the root subsystem $B_n \subset B_{n+1}$ in the sense of [BB11, 1.2], thus the pairs $\pm \alpha_i^+$ and $\pm \alpha_i^-$.
define sections \( s_i^+, s_i^- : X(B_n) \to X(B_{n+1}) \) and an additional section \( s_0 : X(B_n) \to X(B_{n+1}) \) is given by the projection with kernel generated by \( u_{n+1} \).

Further, we have two sections \( s_\pm : X(B_n) \to X(B_{n+1}) \) which are inclusions of \( X(B_n) \) into \( X(B_{n+1}) \) as torus invariant divisors (cf. [BB11, Prop. 1.9, Rem. 1.11]) corresponding to the one-dimensional cones of the fan \( \Sigma(B_{n+1}) \) generated by \( \pm u_{n+1} \).

Locally, the image of \( s_- \) (resp. \( s_+ \)) is given by the equations \( x^{-\alpha_i^\pm} = 0 \), \( x^{-u_{n+1}} = 0 \) (resp. \( x^{\alpha_i^\pm} = 0 \), \( x^{u_{n+1}} = 0 \)) on the affine charts of \( X(B_{n+1}) \) corresponding to the sets of positive roots containing \( -\alpha_i^\pm, -u_{n+1} \) (resp. \( \alpha_i^\pm, u_{n+1} \)).

There is an involution \( I \) of \( X(B_{n+1}) \) over \( X(B_n) \) corresponding to the involution of \( B_{n+1} \) which fixes \( B_n \subset B_{n+1} \) determined by the linear map \( u_i \mapsto u_i \) for \( i \in \{1, \ldots, n\} \) and \( u_{n+1} \mapsto -u_{n+1} \). This element of the Weyl group \( W(B_{n+1}) \) is the reflection determined by the root \( \pm u_{n+1} \). The section \( s_0 \) is invariant under \( I \), whereas for each \( i \in \{1, \ldots, n\} \) the sections \( s_i^+ \) and \( s_i^- \) and also \( s_- \) and \( s_+ \) are interchanged.

**Proposition 3.2.** The collection \( (X(B_{n+1}) \to X(B_n), I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm) \) of construction 3.1 is a \( B_n \)-curve over \( X(B_n) \).

**Proof.** The morphism \( X(B_{n+1}) \to X(B_n) \) is proper. We can show that it is flat by considering the covering of \( X(B_{n+1}) \) and \( X(B_n) \) by affine toric charts similar as in the case of root systems of type \( A \) (see [BB11, Prop. 3.7]).

That any fibre is a \( B_n \)-curve follows from the results below. Remark 3.3 describes the universal curve in terms of equations, proposition 3.7 shows that such equations define a \( B_n \)-curve. It only remains to show that any \( B_n \)-data arises as in proposition 3.7 from a \( B_n \)-curve. This is done in lemma 3.8. \( \square \)

**Definition 3.3.** We call the object \( (X(B_{n+1}) \to X(B_n), I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm) \) of construction 3.1 the universal \( B_n \)-curve over \( X(B_n) \).

**Example 3.4.** We picture the universal curve \( X(B_2) \) over \( X(B_1) \cong \mathbb{P}^1 \) with the sections \( s_-, s_+, s_0, s_1^\pm \). The generic fibre is a \( \mathbb{P}^1 \), whereas the fibres over the two torus fixed points \( x^{-u} = 0 \) and \( x^u = 0 \) of \( X(B_1) \) are chains consisting of three projective lines.

This universal curve is constructed using the root system \( B_2 \) with its root subsystem \( B_1 = \{ \pm u_1 \} \) and the corresponding map of fans \( \Sigma(B_2) \to \Sigma(B_1) \).
The sections part of the universal data on \( z \) equations in \( t \) (1)

Therein, \( X(B_{n+1}) \) is the closed subvariety given by the equations corresponding to root subsystems of type \( A_2 \) in \( B_{n+1} \) which are not contained in \( B_n \). We choose the set of positive roots \( B_{n+1}^+ \) corresponding to the set of simple roots \( \{ u_{n+1} - u_1, u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_n \} \). Then \( B_{n+1}^+ \backslash B_n^+ = \{ u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm \} \) and we can write these equations as follows

\[
(1) \quad t_\beta z_{\alpha_2} z_{-\alpha_1} = t_{-\beta} z_{-\alpha_2} z_{\alpha_1} \quad \text{for } \alpha_1, \alpha_2 \in \{ u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm \} \text{ such that } \beta = \alpha_1 - \alpha_2 \text{ is a root of } B_n
\]

where \( t_{\pm \beta} \) are the homogeneous coordinates of \( \mathbb{P}^1_{\{ \pm \beta \}} \) (consider \( X(B_n) \) as embedded in \( P(B_n) \)) or equivalently the two generating sections of the line bundle \( \mathcal{L}_{\{ \pm \beta \}} \) being part of the universal data on \( X(B_n) \) as defined in [BB11, 1.3].

The sections \( s^\pm: X(B_n) \to X(B_{n+1}) \) for \( i \in \{ 1, \ldots, n \} \) are given by the additional equations \( z_{\alpha_i^\pm} = z_{-\alpha_i^\pm} \), the section \( s_0 \) by \( z_{u_{n+1}} = z_{-u_{n+1}} \). The sections \( s_- \) (resp. \( s_+ \))
are given by $z_{-u_{n+1}} = 0$, $z_{-\alpha_i^+} = 0$ for $i = 1, \ldots, n$ (resp. $z_{u_{n+1}} = 0$, $z_{\alpha_i^-} = 0$ for $i = 1, \ldots, n$).

**Example 3.6.** The universal $B_1$-curve $X(B_2) \subset (\mathbb{P}^1_{\{\pm \alpha_1^+\}} \times \mathbb{P}^1_{\{\pm \alpha_1^-\}} \times \mathbb{P}^1_{\{\pm u_2\}})_{X(B_1)}$ over $X(B_1)$ is given by the homogeneous equations

$$t_1 z_2 z_{-\alpha_1^+} = t_2 z_2 z_{-\alpha_1^-}, \quad t_1 z_{-\alpha_1^-} z_{-u_2} = t_2 z_{-\alpha_1^-} z_2$$

where $(\mathcal{L}_{\{u_2\}}, \{(t_1, t_2)\})$ is the universal $B_1$-data on $X(B_1) \cong \mathbb{P}^1$. We picture the $B_1$-curves defined by these equations for $(t_1 : t_2) = (0 : 1), (a : b), (1 : 0)$. If $(t_1 : t_2) = (a : b) \neq (0 : 1), (1 : 0)$ we have a projective line, we draw its projection onto $\mathbb{P}^1_{\{\pm u_2\}}$ and write the sections in terms of the homogeneous coordinates $z_2, z_{-u_2}$.

Over the two torus fixed points of $X(B_1)$ the curve is a chain of three projective lines in $\mathbb{P}^1_{\{\pm \alpha_1^+\}} \times \mathbb{P}^1_{\{\pm \alpha_1^-\}} \times \mathbb{P}^1_{\{\pm u_2\}}$.

![Diagram](image)

By remark [3.5] the universal $B_n$-curve over $X(B_n)$ can be embedded into a product $P^1(B_{n+1}/B_n)_{X(B_n)} \cong (\mathbb{P}^1)_{2n+1}$ and the embedded curve is given by equations [(1)] determined by the universal $B_n$-data. We show that any $B_n$-curve $C$ over a field can be embedded into a product $(\mathbb{P}^1)_{2n+1}$ and extract $B_n$-data such that $C$ is described by the same equations as the universal curve.

We fix the following notation: given a $B_n$-curve $(C \to Y, I, s_-, s_+, s_0, s_1^+, \ldots, s_n^*)$ we associate with the sections $s_0, s_1^-, s_1^+$ the roots $u_{n+1}, \alpha_i^-, \alpha_i^+$ of $B_{n+1} \setminus B_n^* = \{u_{n+1}, \alpha_i^-, \alpha_i^+\}$ (cf. remark [3.5] and construction [3.1]); we will write $\alpha_s$ for the positive root associated with the section $s$ and conversely $s_{\alpha}$ for the section associated with the root.

**Proposition 3.7.** Let $(C, I, s_-, s_+, s_0, s_1^+, \ldots, s_n^*)$ be a $B_n$-curve over a field. For any $s \in \{s_0, s_1^+, \ldots, s_n^*\}$ let $z_{\alpha_+}, z_{-\alpha_-} \in H^0(C, \mathcal{O}_C(s))$ be a basis of $H^0(C, \mathcal{O}_C(s))$ such that $z_{-\alpha_+}(s_-) = 0$, $z_{-\alpha_-}(s_+) = 0$, $z_{-\alpha_+}(s) = z_{\alpha_-}(s) \neq 0$ (cf. remark [3.5] for this choice). We will write $P^1_{\{\pm \alpha_s\}}$ for $P^1_{\{\pm \alpha_s\}}(H^0(C, \mathcal{O}_C(s)))$. Then by

$$(t_\beta : t_-) = (z_{-\alpha_+}(s_1) : z_{\alpha_-}(s_1))$$

if $\beta = \alpha_1 - \alpha_2$ is a root of $B_n$ and $\alpha_1, \alpha_2$ are roots corresponding to distinct marked points $s_1, s_2 \in \{s_0, s_1^+, \ldots, s_n^*\}$, we can define $B_n$-data $(t_\beta : t_-)_{\{\pm \beta\} \subseteq B_n}$ and the morphism

$$C \to \prod_{\beta} P^1_{\{\pm \alpha_s\}} \times \prod_{\beta} P^1_{\{\pm \alpha_s\}} \times P^1_{\{\pm u_{n+1}\}} = P(B_{n+1}/B_n)$$
is an isomorphism onto the closed subvariety $C' \subseteq P(B_{n+1}/B_n)$ determined by the homogeneous equations

\[ t_\beta z_\alpha z_{-\alpha} = t_{-\beta} z_- \alpha z_\alpha \quad \text{for} \quad \alpha, \alpha_2 \in \{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} \]

such that $\beta = \alpha_1 - \alpha_2$ is a root of $B_n$.

Furthermore, $C'$ together with the marked points $s_i^0, s_i^\pm$ defined by the additional equations $z_{u_{n+1}} = z_{-u_{n+1}}$ resp. $z_{\alpha_i^\pm} = z_{-\alpha_i^\pm}$, the poles $s_i^\pm$ defined by $z_{-u_{n+1}} = 0$, $z_{\alpha_i^\pm} = 0$ ($i = 1, \ldots, n$) and the involution $I'$ given by $P_{\{\pm \alpha_i\}} \leftrightarrow P_{\{\pm \alpha_i^\pm\}}$, $z_{\alpha_i^\pm} \leftrightarrow z_{-\alpha_i^\pm}$ and $P_{\{\pm u_{n+1}\}} \leftrightarrow P_{\{\pm u_{n+1}\}}$, $z_{u_{n+1}} \leftrightarrow z_{-u_{n+1}}$

is a $B_n$-curve and $(C, I, s_-, s_+, s_0, s_1^\pm, \ldots, s_n^\pm) \rightarrow (C', I', s'_-, s'_+, s'_0, s'_1^\pm, \ldots, s'_n^\pm)$ an isomorphism of $B_n$-curves.

**Proof.** The data $(t_\beta : t_{-\beta})$ is defined as position of a marked point $s_1$ relative to another marked point $s_2$ of $C$ if $\beta = \alpha_1 - \alpha_2$. We also write $s_1/s_2$ for this data.

We have the following cases:

\[
\begin{align*}
\beta_{ij} &= \alpha_i^+ - \alpha_j^+ = \alpha_j^- - \alpha_i^- \\
\gamma_{ij} &= \alpha_i^+ - \alpha_j^- = \alpha_j^+ - \alpha_i^- \\
u_i &= \alpha_i^+ - u_{n+1} = u_{n+1} - \alpha_i^- \\

\end{align*}
\]

Note that because of the symmetry of the $B_n$-curve with respect to the involution $I$ we have for the corresponding data $s_i^+/s_j^- = s_j^-/s_i^+, s_i^+/s_j^+ = s_j^+/s_i^-, s_i^+/s_0 = s_0/s_i^-$, so the data $(t_\beta : t_{-\beta})_{\{\pm \beta\} \subseteq B_n}$ is well defined.

The rest of the proof is similar to the proof of [BB11, Prop. 3.12]. To check that $(t_\beta : t_{-\beta})_{\{\pm \beta\} \subseteq B_n}$ is $B_n$-data, we have to check the equations $t_\beta t_\gamma - t_{-\beta} t_{-\gamma}$ for the linear relations $\beta + \gamma = \delta$ given in lemma 2.7 these equations can be written in the form $s_1/s_2 \cdot s_2/s_3 = s_1/s_3$ for some sections $s_1, s_2, s_3$.

We will continue to use the notations $s'/s = (t_\beta : t_{-\beta})$ for $\beta = \alpha_i^+ - \alpha_s$, we have $s_-/s = (0 : 1)$ and $s_+/s = (1 : 0)$ (points $s'_-, s_-, s_+$ with respect to the coordinates $(z_{-\alpha_s} : z_{\alpha_s})$).

**Lemma 3.8.** Any $B_n$-data over a field arises as $B_n$-data extracted from a $B_n$-curve by the method of proposition 3.7.

**Proof.** Let $(t_\beta : t_{-\beta})_{\{\pm \beta\} \subseteq B_n}$ be $B_n$-data over a field.

We can define an ordering $\prec$ on the set of positive roots $\{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} = B_{n+1}^+ \backslash B_n^+$: for distinct $\alpha, \alpha'$ define $\alpha' \prec \alpha$ (resp. $\alpha' \preceq \alpha$) if $(t_\beta : t_{-\beta}) = (0 : 1)$ (resp. $(t_\beta : t_{-\beta}) \neq (1 : 0)$) for $\beta = \alpha' - \alpha$. This defines a decomposition $\{u_{n+1}, \alpha_1^\pm, \ldots, \alpha_n^\pm\} = P_{-m} \sqcup \ldots \sqcup P_m$ into nonempty equivalence classes such that $\alpha' \prec \alpha \iff \alpha' \in P_k, \alpha \in P_k$ for $k' < k$. We have the symmetries $u_{n+1} \prec \alpha_i^\pm \prec u_{n+1}$ and $\alpha^\pm_{i_1} \prec \alpha^\pm_{i_2} \iff \alpha_{i_2}^\pm \prec \alpha_{i_1}^\pm$ and these symmetries imply $u_{n+1} \in P_0$ and $\alpha_i^+ \in P_k \iff \alpha_i^- \in P_{-k}$.

Now it is easy to construct a $B_n$-curve such that the $B_n$-data extracted from it by the method of proposition 3.7 is the given $B_n$-data by taking a chain of projective lines of length $2m + 1$ with involution $(C, I, s_-, s_+)$ (see definition 1.2) and choosing suitable marked points satisfying $s_0 \in C_0$ and $s_i^\pm \in C_k \iff \alpha_i^\pm \in P_k$. \qed
Let $C$ be a $B_n$-curve over a field. It decomposes into irreducible components $C = C_m \cup \ldots \cup C_m$ with $s_- \in C_m$, $s_+ \in C_m$. The decomposition

\[
\{0, \pm 1, \ldots, \pm n\} = P_m \sqcup \ldots \sqcup P_m
\]
such that $0 \in P_0$ and $ei \in P_k \iff s_i^+ \in C_k$, we will call the combinatorial type of the $B_n$-curve (or of the corresponding $B_n$-data) over a field. We will also write this in the form $s_i^+ \ldots s_i^+$ with the sections for the different sets $P_k$ separated by the symbol $\mid$ starting on the left with $P_{-m}$. Considering the fibres of the universal $B_n$-curve resp. the universal $B_n$-data, these combinatorial types determine a stratification of $X(B_n)$ which coincides with the stratification of this toric variety into torus orbits.

**Proposition 3.9.** Over the torus orbit in $X(B_n)$ corresponding to the one-dimensional cone generated by $\varepsilon_{i_1}v_{i_1} + \ldots + \varepsilon_{i_k}v_{i_k}$ we have the combinatorial type

\[
s_{i_1}^\varepsilon \ldots s_{i_k}^\varepsilon |s_0 s_{i_k+1}^\pm \ldots s_{i_k}^\pm |s_0 s_{i_k+1}^\pm \ldots s_{i_1}^\pm.
\]

**Proof.** The universal $B_n$-data over each point of the closure of the orbit corresponding to a generator of a one-dimensional cone generated by $v$ has the property $(\beta : t_{-\beta}) = (0 : 1)$ if $(\beta, v) > 0$ (see [BB11, Rem. 1.21]). For $v = \varepsilon_{i_1}v_{i_1} + \ldots + \varepsilon_{i_k}v_{i_k}$ this in particular implies $s_{i_k}^\varepsilon / s_0 = s_0 / s_{i_k}^\varepsilon = (t_{\varepsilon_{i_k}u_{i_k}} : t_{-\varepsilon_{i_k}u_{i_k}} = (0 : 1) = s_- / s_0$.

\[\square\]

4. MODULI SPACE OF $B_n$-CURVES

In this section we show that there is a fine moduli space of $B_n$-curves $\mathbb{T}^n_{\mathbb{P}^1}$ which is isomorphic to the toric variety $X(B_n)$ by constructing an isomorphism between the moduli functor of $B_n$-curves and the functor of $X(B_n)$. For the second functor we use the description in [BB11, 1.3] in terms of $B_n$-data.

To relate $B_n$-curves to $B_n$-data we consider an embedding of arbitrary $B_n$-curves over a scheme $Y$ into a product $(\mathbb{P}^1)^{2n+1}$ that generalises the embedding in proposition 3.7 to the relative situation. The main tool are the following contraction morphisms (cf. [BB11, 3.3]): for a subset $\{s_1, \ldots, s_l\}$ of the sections of a pointed chain of projective lines $C$ there is a line bundle $\mathcal{O}_C(s_1 + \ldots + s_l)$ on $C$ and a morphism $C \rightarrow C_{\{s_1,\ldots,s_l\}} \subseteq \mathbb{P}Y(\pi_\ast \mathcal{O}_C(s_1 + \ldots + s_l))$ such that the morphisms $C_y \rightarrow (C_{\{s_1,\ldots,s_l\}})_y$ on the fibres are isomorphisms on the components containing one of the sections $s_l(y)$ and contract all other components (see [BB11, Constr. 3.15]).

We will make use of the particular cases of contraction with respect to one section onto a $\mathbb{P}^1$-bundle, with respect to two sections onto an $A_1$-curve and with respect to three sections onto an $A_2$-curve; we will apply [BB11, Constr. 3.16; Lemma 3.17 and 3.18].

We associate with the sections $s_0, s_i^+$ the roots $u_{n+1}, \alpha_i^\pm$ as we did before proposition 3.7. For a $B_n$-curve $(C \rightarrow Y, I, s_-, s_+, s_1^\pm, \ldots, s_n^\pm)$ we denote the contraction morphisms with respect to one section $s_0, s_i^+$ resp. $s_i^+$ by $p_0 : C \rightarrow (\mathbb{P}^1_{\{\pm u_{n+1}\}})_Y$; $\alpha_i^+ : C \rightarrow (\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$ resp. $p_i^+ : C \rightarrow (\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$, where $(\mathbb{P}^1_{\{\pm u_{n+1}\}})_Y$, $(\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$ resp. $(\mathbb{P}^1_{\{\pm \alpha_i^+\}})_Y$ is a copy of $\mathbb{P}^1_Y$ with homogeneous coordinates $z_{u_{n+1}}, z_{-u_{n+1}}$ resp.
In each of these cases we can write $s_{\alpha_i^+, \alpha_{-i}^-}$ resp. $s_{\alpha_i^+, \alpha_{-i}^-}$ such that in these coordinates $s_{-i}, s_{+i}, s_{0}$ resp. $s_{-i}, s_{+i}, s_{-i}$ resp. $s_{+i}, s_{-i}, s_{+i}$ become the $(1 : 0), (0 : 1), (1 : 1)$-section of $P^1_i$.

**Theorem 4.1.** There exists a fine moduli space $\mathcal{T}_{n}^{0,\pm}$ of $B_n$-curves isomorphic to the toric variety $X(B_n)$ with universal family $X(B_{n+1}) \to X(B_n)$.

**Proof.** We show that the moduli functor of $B_n$-curves $\mathcal{T}_{n}^{0,\pm}$ as defined in section [I] is isomorphic to the functor $F_{B_n}$ of the toric variety $X(B_n)$ as described in [BB11 1.3].

Let $Y$ be a scheme. For $B_n$-data on $Y$ we construct a $B_n$-curve $C$ over $Y$ via equations in $P(B_{n+1}/B_n)_Y$ as in remark 3.5 with the given $B_n$-data on $Y$ replacing the universal $B_n$-data on $X(B_n)$. This is a $B_n$-curve: any $B_n$-data is pull-back of the universal $B_n$-data on $X(B_n)$, so the constructed curve is pull-back of the universal $B_n$-curve over $X(B_n)$.

In the other direction, given a $B_n$-curve over $Y$ we extract $B_n$-data. For each pair of distinct sections $s_1, s_2 \in \{s_0, s_1^+, \ldots, s_n^+\}$ we have a contraction morphism $C \to C_{\{s_1, s_2\}}$ onto an $A_1$-curve over $Y$. From $(C_{\{s_1, s_2\}}, s_1, s_2)$ we extract $A_1$-data $(L_{\{1, 2\}}, \{t_{12}, t_{21}\})$ as in [BB11 Constr. 3.16]: we put $L_{\{\pm\}} := L_{\{1, 2\}}, t_{\beta} := t_{12}$, $t_{-\beta} := t_{21}$ for $\beta = \alpha_1 - \alpha_2$ (then $(t_{\beta}, t_{-\beta})$ measures the position of $s_1$ relative to $s_2$, we write this as $s_1/s_2$). We have the following cases:

- $\beta_{ij} = \alpha^+_i - \alpha^-_j = \alpha_i - \alpha_j$
- $\gamma_{ij} = \alpha^+_i - \alpha^-_j = \alpha^+_j - \alpha^-_i$
- $u_i = \alpha^+_i - u_{n+1} = u_{n+1} - \alpha^-_i$

Because of the symmetry of the $B_n$-curve with respect to the involution we have for the corresponding data $s^+_i/s^+_j = s^-_j/s^-_i, s^+_i/s^-_j = s^+_j/s^-_i, s^+_i/s_0 = s_0/s^-_i$, so the data $(L_{\{\pm\}}, \{t_{\beta}, t_{-\beta}\})_{\{\pm\}} \subseteq B_n$ is well defined.

We show that the data obtained this way is $B_n$-data. Let $\beta, \gamma, \delta$ be positive roots of $B_n$ such that $\beta + \gamma = \delta$. We have to verify that the collection $\{(L_{\{\pm\}}, \{t_{\beta}, t_{-\beta}\}), (L_{\{\gamma\}}, \{t_{\gamma}, t_{-\gamma}\}), (L_{\{\delta\}}, \{t_{\delta}, t_{-\delta}\})\}$ satisfies $t_{\beta}t_{\gamma}t_{-\delta} = t_{-\beta}t_{-\gamma}t_{\delta}$, which means that it is $A_2$-data. By lemma [2.1] we have the following cases:

- $\beta_{ij} + u_j = u_i$ (i, j ∈ {1, ..., n}, i < j)
- $u_i + u_j = \gamma_{ij}$ (i, j ∈ {1, ..., n}, i ≠ j)
- $\beta_{ij} + \beta_{jk} = \beta_{ik}$ (i, j, k ∈ {1, ..., n}, i < j < k)
- $\beta_{ij} + \gamma_{jk} = \gamma_{ik}$ (i, j, k ∈ {1, ..., n}, i < j, k ≠ i, j)

In each of these cases we can write $\beta = \alpha_{s_1} - \alpha_{s_2}, \gamma = \alpha_{s_2} - \alpha_{s_3}$ for three distinct sections $s_1, s_2, s_3 \in \{s_0, s_1^+, \ldots, s_n^+\}$. Then these equations can be interpreted as relations between the relative positions of pairs of sections in a set of three sections, we write this as $s_1/s_2, s_2/s_3 = s_1/s_3$:

- $\beta_{ij} + u_j = u_i, \beta_{ij} = \alpha^+_i - \alpha^-_j, u_j = \alpha^+_j - u_{n+1}, s^+_i/s^-_j/s_0 = s^+_i/s_0$
- $u_i + u_j = \gamma_{ij}, u_i = \alpha^+_i - u_{n+1}, u_j = u_{n+1} - \alpha^-_j, s^+_i/s^-_j/s^-_j = s^+_i/s^-_j$
- $\beta_{ij} + \beta_{jk} = \beta_{ik}, \beta_{ij} = \alpha^+_i - \alpha^-_j, \beta_{jk} = \alpha^+_j - \alpha^-_k, s^+_i/s^+_j/s^-_k = s^+_i/s^-_k$
- $\beta_{ij} + \gamma_{jk} = \gamma_{ik}, \beta_{ij} = \alpha^+_i - \alpha^-_j, \gamma_{jk} = \alpha^+_j - \alpha^-_k, s^+_i/s^+_j/s^-_k = s^+_i/s^-_k$

We have a contraction morphism $C \to C_{\{s_1, s_2, s_3\}}$ over $Y$ onto an $A_2$-curve $C_{\{s_1, s_2, s_3\}}$ over $Y$. The data $\{(L_{\{\pm\}}, \{t_{\beta}, t_{-\beta}\}), (L_{\{\gamma\}}, \{t_{\gamma}, t_{-\gamma}\})\}$.
\((\mathcal{L}_{[\pm \delta]}, \{l_5, l_{-5}\})\) coincides with the data extracted from this \(A_2\)-curve and is \(A_2\)-data by [BB11] Lemma 3.18.

Both constructions commute with base-change and thus define morphisms of functors \(F_{B_n} \to \mathcal{L}_{n}^0, \pm\) and \(\mathcal{L}_{n}^0, \pm \to F_{B_n}\). As in the proof of [BB11] Thm. 3.19 one shows that they are inverse to each other.

**Remark 4.2.** The moduli space \(\mathcal{L}_{n}^0, \pm\) embeds naturally into \(\mathcal{L}_{2n+1}\). A morphism \(\mathcal{L}_{n}^0, \pm \to \mathcal{L}_{2n+1}\) is given by considering a \(B_n\)-curve with sections \(s^+, s^-\) as in the proof of [BB11], Lemma 3.18. Here, as in the case of the toric varieties \(X(A_n)\), all primitive collections consist of two elements corresponding to non-comparable sets \(B, B' \in \mathcal{B}\). As usual the integral cohomology is torsion free and confined to the even degrees and standard methods from toric geometry (see e.g. [Dan] (10.8)) give:

**Proposition 5.1.** For the cohomology ring of the toric variety \(X(B_n)\) over the complex numbers we have

\[
H^*(X(B_n), \mathbb{Z}) \cong \mathbb{Z}[l_B : B \in \mathcal{B}]/(R_1 + R_2)
\]

where \(R_1\) is the ideal generated by the elements \(r_i = \sum_{i \in B} l_B - \sum_{-i \in B} l_B\) for \(i = 1, \ldots, n\) and \(R_2\) the ideal generated by the elements \(r_{B,B'} = l_B l_{B'}\) for \(B, B' \in \mathcal{B}\) such that \(B \not\subseteq B', B' \not\subseteq B\).

We proceed by determining the Betti numbers and the Poincaré polynomial and obtain the following closed formula which is an analogue to [LM00] (2.3).

**Proposition 5.2.** Let \(p_{X(B_n)}(t) = \sum_{i=0}^{n} \beta_2i(X(B_n))t^i\) be the Poincaré polynomial of \(X(B_n)\) with \(\beta_2i(X(B_n)) = \text{rk}\, H^{2i}(X(B_n), \mathbb{Z})\) the Betti numbers. Then we have

\[
\sum_{n=0}^{\infty} \frac{p_{X(B_n)}(t)}{n!} y^n = e^{(t-1) - l - e^{2g(t-1)}} \in \mathbb{Z}[t][[y]]
\]

**Proof.** We have \(p_{X(B_n)}(t) = \sum_{m=0}^{n} d_m(B_n)(t-1)^{n-m}\) (see [Ful] p. 92 or [Dan] (10.8)); this can be shown in different ways, one possibility is by counting points over finite fields as in [LM00] with \(d_m(B_n) = \text{number of } (n-m)\text{-dim. torus orbits of } X(B_n) =\)
number of $m$-dim. cones of $\Sigma(B_n)$. Inserting this into $\sum_{n=0}^{\infty} \frac{p_X(B_n)(t)}{n!} y^n$ and inter-
changing summation by $n$ and $m$, we get
\[
\sum_{n=0}^{\infty} \frac{p_X(B_n)(t)}{n!} y^n = \sum_{m=0}^{\infty} \frac{1}{(t-1)^m} \sum_{n=m}^{\infty} \frac{d_m(B_n)}{n!} (t - 1)^n y^n
\]
The number $d_m(B_n)$ can be calculated as
\[
\frac{1}{m!} d_m(B_n) = \sum_{(a_0, a_1, \ldots, a_m)} \frac{y^{a_0}}{a_0!} \cdots \frac{y^{a_m}}{a_m!}
\]
where the sum runs over sequences $a_0 \in \mathbb{Z}_{\geq 0}$, $a_1 \in \mathbb{Z}_{>0}$, \ldots, $a_m \in \mathbb{Z}_{>0}$ such that\sum_i a_i = n (note that any family $B^{(m)} \subset \cdots \subset B^{(1)}$ of elements of $B$ corresponding to an $m$-dimensional cone of $\Sigma(B_n)$ determines such a partition by $a_m = |B^{(m)}|$, $a_{m-1} = |B^{(m-1)}| - |B^{(m)}|$, \ldots, $a_0 = n - |B^{(1)}|$, in addition we have orderings and signs). Making use of the fact that $\frac{1}{m!} d_m(B_n)$ coincides with the coefficient of $x^n$ in the power series $e^x(e^{2x} - 1)^m$, we obtain
\[
\sum_{n=0}^{\infty} \frac{p_X(B_n)(t)}{n!} y^n = e^{yt-1} \sum_{m=0}^{\infty} \frac{1}{(t-1)^m} (e^{2yt-1} - 1)^m
\]
which yields the result.

In particular we have $\chi(X(B_n)) = 2^m n!$ (this reflects the fact that we have $2^n n!$ maximal cones), $\beta_2(X(B_n)) = 3^n - n - 1$ (corresponding to the fact that we have $3^n - 1$ one-dimensional cones) and for the first Poincaré polynomials
\[
\begin{align*}
p_X(B_1)(t) &= t + 1, \\
p_X(B_2)(t) &= t^2 + 6t + 1, \\
p_X(B_3)(t) &= t^3 + 23t^2 + 23t + 1 \\
p_X(B_4)(t) &= t^4 + 76t^3 + 230t^2 + 76t + 1 \\
p_X(B_5)(t) &= t^5 + 237t^4 + 1682t^3 + 1682t^2 + 237t + 1 \\
p_X(B_6)(t) &= t^6 + 722t^5 + 10543t^4 + 23548t^3 + 10543t^2 + 722t + 1
\end{align*}
\]
The ring $\mathbb{Z}[l_B : B \in \mathcal{B}] / R_2$ is the Stanley-Reisner ring for the triangulation of the $(n - 1)$-dimensional sphere determined by the fan $\Sigma(B_n)$. It is a Cohen-Macaulay ring and the elements $r_1, \ldots, r_n$ that generate $R_1$ form a regular sequence. The calculation of the Poincaré polynomial of a toric variety in $[\text{Dan}, (10.8)]$ in terms of the numbers of cones of dimension $d = 1, \ldots, n$ only depends on the Hilbert-Poincaré series of the Stanley-Reisner ring of the fan and the fact that the quotient by an ideal generated by a regular sequence is taken. In $[\text{Re01}]$ a ring has been defined by taking the same Stanley-Reisner ring (over a field) but instead of $R_1$ an ideal generated by a different regular sequence, so by construction this ring has the same Poincaré polynomial as the cohomology ring of $X(B_n)$.

The $\mathbb{Z}$-module $\mathbb{Z}[l_B : B \in \mathcal{B}] / (R_1 + R_2)$ is generated by the classes of square-free monomials (see $[\text{Dan}, (10.7.1)]$). We can restrict to monomials each of which has only factors corresponding to one-dimensional faces of one maximal cone. Such a monomial $\prod_{i=1}^{m} l_{B^{(i)}}$ corresponds to an $m$-dimensional face of the respective maximal cone and on the other hand to a collection $B^{(m)} \subset \cdots \subset B^{(1)}$ of elements of $\mathcal{B}$. We denote the $\mathbb{Z}$-submodule of $\mathbb{Z}[l_B : B \in \mathcal{B}]$ generated by these monomials by $G$. There is the canonical isomorphism of $\mathbb{Z}$-modules $G / U \cong \mathbb{Z}[l_B : B \in \mathcal{B}] / (R_1 + R_2)$ where $U = (R_1 + R_2) \cap G$. As usual, the module $G / U$ can be identified with the
homology module \( H_* (X(B_n), \mathbb{Z}) \). The monomial \( \prod_{i=1}^{m} l_{B(i)} \) then corresponds to the class of the orbit closure for the cone determined by the collection \( B^{(m)} \subset \ldots \subset B^{(1)} \), in particular the monomials of \( G \) of degree \( m \) generate \( H_{2(n-m)} (X(B_n), \mathbb{Z}) \).

The maximal cones of the fan \( \Sigma(B_n) \) correspond to collections \( B^{(m)} \subset \ldots \subset B^{(1)} \) of elements of \( \mathcal{B} \) and these correspond to so called signed permutations, that is elements of the Weyl group \( W(B_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n =: S^+_n \). A signed permutation \( w \in S^+_n \) corresponds via \( (w(1), \ldots, w(n)) \) to a sequence of distinct elements in \( \{ \pm 1, \ldots, \pm n \} \) for any \( i \) not containing both \(-i\) and \( i\). For a collection \( B^{(m)} \subset \ldots \subset B^{(1)} \) of elements of \( \mathcal{B} \) the corresponding signed permutation \( \sigma \in S^+_n \) is given by \( \{ w(k) \} = B^{(k)} \setminus B^{(k+1)} \) for \( k = 1, \ldots, n \) (put \( B^{(n+1)} = \emptyset \)). The descent set of a signed permutation \( w \in S^+_n \) is the set (put \( w(0) = 0 \))

\[ \text{Desc}(w) = \{ k \in \{ 1, \ldots, n \} \mid w(k-1) > w(k) \} \]

For any \( w \in S^+_n \) we define a monomial in \( G \) by

\[ l^w = \prod_{k \in \text{Desc}(w)} l_{\{ w(k), \ldots, w(n) \}} \]

this way we have defined \( 2^n n! \) distinct monomials.

**Proposition 5.3.** The classes of the monomials \( l^w \) for \( w \in S^+_n \) form a basis of the homology module \( G/U = H_* (X(B_n), \mathbb{Z}) \). The module of relations \( U \) is generated by the elements

\[ r_{i,j}((B^{(h)})_h, k) = \left( \sum_{j \in B} l_B - \sum_{i \notin B} l_B \right) \prod_{h=1}^{m} l_{B^{(h)}} \]

(sums over sets \( B^{(k+1)} \subset B \subset B^{(k)} \) for collections \( B^{(m)} \subset \ldots \subset B^{(1)} \), \( m \geq 1 \) of elements of \( \mathcal{B} \) and \( k \in \{ 1, \ldots, m \} \), \( i, j \in B^{(k)} \setminus B^{(k+1)} \) (put \( B^{(m+1)} = \emptyset \), \( i \neq j \), and by the elements

\[ r_i((B^{(h)})_h) = \left( \sum_{i \in B} l_B - \sum_{-i \notin B} l_B \right) \prod_{h=1}^{m} l_{B^{(h)}} \]

(sums over sets \( B \in \mathcal{B} \) such that \( B^{(1)} \subset B \) if \( m \geq 1 \)) for collections \( B^{(m)} \subset \ldots \subset B^{(1)} \), \( m \geq 0 \) of elements of \( \mathcal{B} \) and \( i \in \{ 1, \ldots, n \} \) such that \(-i, i \notin B^{(1)} \) if \( m \geq 1 \).

**Proof.** We observe that the given relations are contained in \( U \). We have \( 2^n n! \) monomials \( l^w \), this number coincides with the rank of \( G/U \). Thus it remains to show that every monomial in \( G \) via the given relations is equivalent to a linear combination of the monomials \( l^w \).

For a monomial \( \prod_{k=1}^{m} l_{B^{(k)}} \) corresponding to a collection \( B^{(m)} \subset \ldots \subset B^{(1)} \), \( m \geq 1 \) we define the number \( d(\prod_{k=1}^{m} l_{B^{(k)}}) := |\{ k \in \{ 1, \ldots, m \} \mid \min P_{k-1} > \max P_k \}| \in \mathbb{Z}_{\geq 0} \) in terms of the associated partition \( P_m = B^{(m)}, P_{m-1} = B^{(m-1)} \setminus B^{(m)}, \ldots, P_1 = B^{(1)} \setminus B^{(2)}, P_0 = \{ 0, \pm 1, \ldots, \pm n \} \setminus \{ \pm i \mid i \in B^{(1)} \} \) or \( -i \in B^{(1)} \). The monomials \( y \in G \) satisfying \( d(y) = 0 \) are exactly the monomials of the form \( l^w \). We define the following ordering \( < \) of the monomials of \( G \): take the partition \( (P_k)_{k=0}^{m} \) associated with a monomial and consider the sequence that arises by taking the sets \( P_m, \ldots, P_1 \) in this order and by ordering the elements of each \( P_k \) according to their size, on these sequences we take the lexicographic order.
We show that every monomial in $G$ modulo $U$ is equivalent to a linear combination of the monomials $l^w$, $w \in S_n^\pm$ by showing that every monomial $y \in G$ with $d(y) > 0$ modulo a relation is equivalent to a linear combination of monomials $y'$ with $y < y'$. In fact, let $B^{(m)} \subseteq \ldots \subseteq B^{(1)}$, $m \geq 1$ be a collection of elements of $B$ (put $B^{(m+1)} := \emptyset$) with associated partition $(P_k)_{k=0,\ldots,m}$ such that the corresponding monomial $y = \prod_{k=1}^m l_{B^{(k)}}$ satisfies $d(y) > 0$. Take $k \in \{1,\ldots,m\}$ such that $i := \min P_{k-1} > \max P_k =: j$. If $k \in \{2,\ldots,m\}$ then

$$r_{i,j}((B^{(k)})_{h\neq k}, k - 1) = \left( \sum_{i \in B} l_B - \sum_{i \notin B} l_B \right) \prod_{h \neq k} l_{B^{(h)}}$$

(sums over sets $B$ such that $B^{(k+1)} \subseteq B \subsetneq B^{(k-1)}$) is a relation that contains $y$ as the unique monomial minimal with respect to $<$. If $k = 1$ then

$$r_{-j}((B^{(k)})_{h\neq 1}) = \left( \sum_{-j \in B} l_B - \sum_{j \in B} l_B \right) \prod_{h=2}^m l_{B^{(h)}}$$

(sums over sets $B \in B$ such that $B^{(2)} \subsetneq B$) is such a relation. \qed

The proposition implies that the Betti numbers of $X(B_n)$ coincide with the number of signed permutations with prescribed number of descents, for this see also [DL94, Section 4], [St94]. Our basis of the number of signed permutations with prescribed number of descents, for this see also [KL95], [Kl95] in the general case of toric varieties associated with root systems (see the following remark).

**Remark 5.4.** In [Kl85] a basis of the homology $H_\ast(X(R), \mathbb{Z})$ is constructed as follows. For a fixed set of simple roots $S \subset R$ and the corresponding Weyl chamber $\sigma_S = S^\vee$, consider for each $w \in W(R)$ the face $\sigma_w \subseteq w\sigma_S$ given as the intersection of those walls of $w\sigma_S$ that separate $\sigma_S$ and $w\sigma_S$, i.e., we have the intersection of $w\sigma_S$ with those subspaces $(w\alpha)^\perp$, $\alpha \in S$, for which $w\alpha$ is a negative root. The cycles corresponding to the family of cones $(\sigma_w)_{w \in W(R)}$ form a basis of $H_\ast(X(R), \mathbb{Z})$.

In our case we may choose the set of simple roots $S = \{u_n - u_{n-1}, \ldots, u_2 - u_1, u_1\} \subset B_n$, the corresponding Weyl chamber is generated by $v_n, v_{n-1} + v_n, \ldots, v_1 + \ldots + v_n$. Then for $w \in W(B_n) = B_n^+$ we have $w(u_k - u_{k-1})$ is negative $\iff w(k-1) > w(k)$ for $k \in \{2,\ldots, n\}$ and $w(u_1)$ is negative $\iff 0 > w(1)$. So, each root $\alpha \in S$ such that $w\alpha$ is negative corresponds to an element of $\text{Desc}(w)$. Since $(w(u_k - u_{k-1}))^\perp \cap w\sigma_S$ is generated by $\{w(v_n), \ldots, w(v_1 + \ldots + v_n)\} \setminus \{w(u_k + \ldots + v_n)\}$ and $(w(u_1))^\perp \cap w\sigma_S$ by $\{w(v_n), \ldots, w(v_2 + \ldots + v_n)\}$, it follows that $\sigma_w$ is generated by $\{v_{w(k), \ldots, w(n)} | k \notin \text{Desc}(w)\}$ and the class of the respective torus invariant cycle corresponds to the monomial $l^w$.

### 6. Root systems of type $C$

Consider an $n$-dimensional Euclidean space $E$ with basis $u_1, \ldots, u_n$. The root system $C_n$ in $E$ consists of the $2n^2$ roots:

$$\pm 2u_i \quad \text{for } i \in \{1, \ldots, n\}; \quad \pm(u_i + u_j), \pm(u_i - u_j) \quad \text{for } i, j \in \{1, \ldots, n\}, i < j.$$

The following is a set of simple roots:

$$u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, 2u_n.$$
Let $M(C_n)$ be the root lattice. The Weyl group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ acts by $u_i \mapsto \pm u_i$ and permuting the $u_i$. So there are $2^n n!$ sets of simple roots, these are of the form $\pm 1 u_1, \pm 2 u_2, \pm 2 u_3, \ldots, \pm (n-1) u_{n-1}, \pm n u_n$, $2^2 u_i u_i$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_1, \ldots, \varepsilon_n$.

The vector space $E^* \subset E$ dual to $E$ with basis $v_1, \ldots, v_n$ dual to $u_1, \ldots, u_n$ contains the lattice $N(C_n)$ dual to $M(C_n)$. To describe the fan $\Sigma(C_n)$ in the lattice $N(C_n)$ we describe a Weyl chamber. For the set of simple roots $S = \{u_1-u_2, u_2-u_3, \ldots, u_{n-1}-u_n, 2u_n\}$ has the dual basis $v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{n-1}, \frac{1}{2}(v_1 + \ldots + v_n)$ of $N(C_n)$, the Weyl chamber $\Sigma_S$ is equal to $\langle v_1, v_1 + v_2, \ldots, v_1 + \ldots + v_{n-1}, \frac{1}{2}(v_1 + \ldots + v_n) \rangle_{\mathbb{Q}_{\geq 0}}$. All Weyl chambers are generated by collections of elements of the form $\varepsilon_1 v_{i_1} + \varepsilon_2 v_{i_2}, \ldots, \frac{1}{2}(\varepsilon_1 v_{i_1} + \ldots + \varepsilon_n v_{i_n})$ for orderings $i_1, \ldots, i_n$ of the set $\{1, \ldots, n\}$ and signs $\varepsilon_i$. There are $3^n - 1$ one-dimensional cones generated by elements of the form $\varepsilon_1 v_{i_1} + \ldots + \varepsilon_k v_{i_k}$ for $k \in \{1, \ldots, n-1\}$ or of the form $\frac{1}{2}(\varepsilon_1 v_{1} + \ldots + \varepsilon_n v_n)$.

The torus invariant divisor for the one-dimensional cone generated by $\varepsilon_1 v_{i_1} + \ldots + \varepsilon_k v_{i_k}$ is isomorphic to $X(C_{n-k}) \times X(A_{k-1})$, that for $\frac{1}{2}(\varepsilon_1 v_{1} + \ldots + \varepsilon_n v_n)$ is isomorphic to $X(A_{n-1})$.

**$X(C_{n+1})$ over $X(C_n)$**. Consider the proper surjective morphism $X(C_{n+1}) \to X(C_n)$ induced by the root subsystem $C_n \subset C_{n+1}$ consisting of the roots in the subspace generated by $u_1, \ldots, u_n$. As in the B-case one shows that $X(C_{n+1})$ is flat over $X(C_n)$.

The automorphism of $C_{n+1}$ given as the reflection for the root $\pm u_{n+1}$ fixes $C_n \subset C_{n+1}$ and induces an involution $I$ of $X(C_{n+1})$ over $X(C_n)$. We have two sections $s_-, s_+$ defined as in the B-case. There are $2n+1$ additional pairs of opposite roots, the pairs $\pm \alpha_i^+ = \pm (u_{n+1} + u_i)$, $\pm \alpha_i^- = \pm (u_{n+1} - u_i)$ for $i \in \{1, \ldots, n\}$ and the pair $\pm 2u_{n+1}$. Any pair $\pm \alpha_i^+, \pm \alpha_i^-$ defines a projection onto the root subsystem $C_n \subset C_{n+1}$ in the sense of [BB11, 1.2], thus we have sections $s_i^+$ and $s_i^-$. The pair $\pm 2u_{n+1}$ does not define a projection of root systems $C_{n+1} \to C_n$, so it does not induce a section. However, we can consider the morphism $X(C_{n+1}) \to P^1_{\{\pm 2u_{n+1}\}}$, and the preimage of the point $(1 : 1)$. We denote this subscheme of $X(C_{n+1})$ by $S_0$; it is finite flat of degree 2 over $X(C_n)$ (see below), such a subscheme we will call a double-section.

If we consider $X(C_{n+1})$ and $X(C_n)$ as embedded $X(C_{n+1}) \subseteq P(C_{n+1})$, $X(C_n) \subseteq P(C_n)$, then the morphism $X(C_{n+1}) \to X(C_n)$ is induced by the projection onto the subproduct $P(C_{n+1}) \to P(C_n)$ and $X(C_{n+1})$ is given in $P(C_{n+1}/C_n)X(C_n) = (\prod_{i=1}^n P^1_{\{\pm \alpha_i^+\}} \times \prod_{i=1}^n P^1_{\{\pm \alpha_i^-\}} \times P^1_{\{\pm 2u_{n+1}\}})X(C_n)$ by the homogeneous equations involving the universal $C_n$-data on $X(C_n)$

\begin{align}
    z_{\alpha_i^-} z_{\alpha_i^+} z_{-2u_{n+1}} &= z_{-\alpha_i^-} z_{-\alpha_i^+} z_{2u_{n+1}}, & i \in \{1, \ldots, n\} \\
    t_\beta z_{\alpha_2} z_{-\alpha_1} &= t_{-\beta} z_{-\alpha_2} z_{\alpha_1}, & \alpha_1, \alpha_2 \in \{\alpha_1^+, \ldots, \alpha_n^+\}, \alpha_1 \neq \alpha_2, \beta = \alpha_1 - \alpha_2
\end{align}
**Example 6.1.** We picture the inclusion of root systems $C_1 \subset C_2$ and the map of fans $\Sigma(C_2) \to \Sigma(C_1)$.

The fibres of $X(C_{n+1}) \to X(C_n)$ can be studied for example using the above description in terms of equations or by employing the description of $X(C_n)$ as quotient of $X(B_n)$ (see below). We obtain the following result, in particular the fibres over a union of torus invariant divisors are not reduced.

**Proposition 6.2.** We define $D \subset X(C_n)$ to be the union of the torus invariant divisors corresponding to the one-dimensional cones of $\Sigma(C_n)$ generated by elements of the form $\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$. For the structure of the fibres of the morphism $X(C_{n+1}) \to X(C_n)$ together with the involution $I$, the sections $s^\pm_i$ and the double-section $S_0$, there are the following two situations.

Over $X(C_n) \setminus D$ the fibres are $B_n$-curves except that instead of the section $s_0$ we have a double-section $S_0$ which consists of the two fixed points under $I$. In this case the central component contains some of the sections $s^\pm_i$.

Over $D$ the fibres are $B_n$-curves except that the central component is nonreduced of the form $\mathbb{P}^1_{K[\varepsilon]/(\varepsilon^2)}$ with the double-section $S_0 \cong \text{Spec } K[\varepsilon]/(\varepsilon^2)$ concentrated in one point. The intersection of the central component with the other components locally is isomorphic to the subscheme in $X^2_K = \text{Spec } K[x,y]$ defined by the equation $x^2y = 0$. All sections $s^\pm_i$ are on the other components.

In both cases the combinatorial types over the torus invariant divisors, after the appropriate modifications, are given by the description in the $B$-case (prop. 3.9).

**$X(C_n)$ as quotient of $X(B_n)$.** We investigate the description of $X(C_n)$ as a quotient $X(B_n)/\mu_2$. On the moduli side this leads to a characterisation of $X(C_n)$ as the coarse moduli space of a toric Deligne-Mumford stack. For simplicity, in this part we will work over the field of complex numbers.

On the moduli space $\mathcal{L}^{0,\pm}_n$ of $B_n$-curves we have an involution $J$ that transforms a $B_n$-curve over a scheme $Y$ to the $B_n$-curve with the other fixed point section with respect to the involution $I$ as section $s_0$, i.e. we apply the automorphism of the central component that commutes with $I$ (see the following remark) to the section $s_0$. 
Remark 6.3. Let \((C, I, s_-, s_+)\) be a chain of projective lines with involution of odd length over \(\mathbb{C}\). Consider the central component \((C_0, p_0^-, p_0^+)\) which we identify with \((\mathbb{P}^1, 0, \infty)\) such that \(I|_{C_0}: x \mapsto \frac{1}{x}\). Then there are two automorphisms of \((C_0, p_0^-, p_0^+)\) that commute with \(I\), namely the identity and \(x \mapsto -x\), determined by the action on the fixed points \(\{1, -1\}\) of \(I|_{C_0}\).

Identifying \(\mathbb{T}_n^{0,\pm}\) with \(X(B_n)\), the involution \(J\) is given on the functor of \(B_n\)-data (see [BB11, 1.3]) by \((\mathcal{L}^{s_{\pm u_i}}_{\{\pm u_i\}}, \{t_u, t_{-u}\}) \mapsto (\mathcal{L}^{s_{\pm u_i}}_{\{\pm u_i\}}, \{t_u, -t_{-u}\})\) or equivalently \(f_{\pm u} \mapsto -f_{\pm u}\) on the part corresponding to the roots \(\pm u_1, \ldots, \pm u_n\), whereas the part corresponding to the other roots remains unchanged.

In both the \(C_n\)-case and the \(B_n\)-case we start with the same vector space \(E\) with basis \(u_1, \ldots, u_n\). The root lattice \(M(C_n)\) is a sublattice of the root lattice \(M(B_n)\) of index 2 and dually \(N(B_n) \subset N(C_n)\) of index 2, whereas the fan \(\Sigma(C_n)\) as a set of cones in \(N(C_n)_{\mathbb{Q}} = N(B_n)_{\mathbb{Q}}\) is the same as the fan \(\Sigma(B_n)\). Thus, the toric variety \(X(C_n)\) is the quotient of \(X(B_n)\) by the involution that maps \(x^{u_i} \mapsto -x^{u_i}\). This involution on \(X(B_n)\) coincides with the involution \(J\). Locally, we have quotients \(\mathbb{A}^n/\mu_2\) by the action of \(\mu_2\) that changes the sign of one coordinate of \(\mathbb{A}^n\). In particular, \(X(B_n)\) is flat over \(X(C_n)\) of degree 2. \(X(C_n)\) can be considered as the \(\mu_2\)-Hilbert scheme of \(X(B_n)\), then \(X(B_n) \to X(C_n)\) forms the universal family of \(\mu_2\)-clusters, the fibres over \(X(C_n) \setminus D\) consist of two points, the fibres over \(D\) are nonreduced \(\mu_2\)-clusters.

Concerning the double-section \(S_0 \subset X(C_{n+1})\) we obtain:

Lemma 6.4. The scheme \(S_0\) is isomorphic to \(X(B_n)\) over \(X(C_n)\).

Proof. Let \(\tilde{S}_0 \subset X(B_{n+1})\) be the fixed point subscheme of the involution \(I\) on \(X(B_{n+1})\). The scheme \(\tilde{S}_0\) over \(X(B_n)\) consists of two components \(s_0(X(B_n))\) and another copy of \(X(B_n)\) such that \(J: X(B_{n+1}) \to X(B_{n+1})\) restricts to an isomorphism between these components over \(J: X(B_n) \to X(B_n)\). The scheme \(S_0\) arises as quotient of \(\tilde{S}_0\) by \(J\), the section \(s_0: X(B_n) \to \tilde{S}_0\) determines an isomorphism \(X(B_n) \to S_0\) over \(X(C_n) = X(B_n)/\mu_2\).

We are led to the following type of curves to be parametrised by \(X(C_n)\).

Definition 6.5. (First definition of \(C_n\)-curves). A \(C_n\)-curve over a scheme \(Y\) is a collection \((\pi: C \to Y, I, s_-, s_+,..., s_{1\pm}, s_{n\pm})\) which arises from a \(B_n\)-curve over \(Y\) by omitting the section \(s_0\).

Equivalently, we can replace the section \(s_0\) of a \(B_n\)-curve \(C \to Y\) by the subscheme \(s_0(Y) \cup J(s_0(Y))\), which coincides with the fixed point subscheme of the involution \(I\) on \(C\). The section \(s_0\) selects one of the two components of this fixed point subscheme. Forgetting this information, the \(B_n\)-curves for points \(y, Jy\) in the moduli space \(\mathbb{T}_n^{0,\pm} = X(B_n)\) define \(C_n\)-curves related by an isomorphism of \(C_n\)-curves. If the central component contains sections \(s_{i\pm}\), then two nonisomorphic \(B_n\)-curves over a field give rise to isomorphic \(C_n\)-curves. If the central component does not contain a section \(s_{i\pm}\), then one \(B_n\)-curve corresponds to one \(C_n\)-curve, but \(C_n\)-curves of this type have an extra automorphism that interchanges the two fixed points of \(I\) (cf. remark [6.3]).
This functor of $C_n$-curves cannot be representable by a scheme. However, we can consider the stack of $C_n$-curves.

**Theorem 6.6.** The category of $C_n$-curves forms a Deligne-Mumford stack $\mathcal{X}(C_n)$ isomorphic to the quotient stack $[X(B_n)/\mu_2]$ with the group operation given by $J: X(B_n) \to X(B_n)$.

**Proof.** Let $\mathcal{X}(C_n)$ be the category of $C_n$-curves, i.e. an object of $\mathcal{X}(C_n)$ over a scheme $Y$ is a $C_n$-curve $C$ over $Y$, a morphism $(C \to Y) \to (C' \to Y')$ over $Y \to Y'$ is a cartesian diagram compatible with the involution $I$ and the sections. This is a category fibred in groupoids, we show that it is equivalent as a fibred category to the Deligne-Mumford stack $[X(B_n)/\mu_2]$.

An object of $[X(B_n)/\mu_2]$ over a scheme $Y$ is a $\mu_2$-torsor $T \to Y$ together with a $\mu_2$-equivariant map $T \to X(B_n)$. A morphism $(T \to Y, \alpha: T \to X(B_n)) \to (T' \to Y', \alpha': T' \to X(B_n))$ over $Y \to Y'$ is a cartesian diagram of $\mu_2$-torsors given by a morphism $\theta: T \to T'$ such that $\alpha' \circ \theta = \alpha$. We will use that the functor of $X(B_n)$ is isomorphic to the functor of $B_n$-curves and fix an isomorphism resp. a universal family over $X(B_n)$.

We define a morphism of fibred categories $\Phi: [X(B_n)/\mu_2] \to \mathcal{X}(C_n)$. For an object $(T \to Y, \alpha: T \to X(B_n))$ we have a $B_n$-curve $B \to T$ corresponding to the equivariant morphism $\alpha$ such that the action of $\mu_2$ on $T$ is given by interchanging the two possible choices of $s_0$. After forgetting the section $s_0$, the quotient of $B \to T$ by $\mu_2$ gives a $C_n$-curve $C \to Y$ using the canonical isomorphism $T/\mu_2 \cong Y$. For a morphism $(T \to Y, \alpha: T \to X(B_n)) \to (T' \to Y, \alpha': T' \to X(B_n))$ we obtain a cartesian diagram of $C_n$-curves $(C \to Y) \to (C' \to Y')$.

We define a morphism of fibred categories $\Psi: \mathcal{X}(C_n) \to [X(B_n)/\mu_2]$. Let $C \to Y$ be a $C_n$-curve over $Y$. Consider the fixed point subscheme $T \subset C$ under $I$, this is a $\mu_2$-torsor over $Y$. Let $B$ be the pull-back of the $C_n$-curve $C \to Y$, with the section $s_0$ defined as the diagonal of $T \times_Y T \subset B$ this is a $B_n$-curve and defines a $\mu_2$-equivariant morphism $\alpha: T \to X(B_n)$. A morphism $(C \to Y) \to (C' \to Y')$ given by $\gamma: C \to C'$ determines a cartesian diagram of $B_n$-curves by $\gamma \times \gamma: B = C \times_Y T \to B' = C' \times_Y T'$ over a cartesian diagram of $\mu_2$-torsors given by $\gamma: T \to T'$. So the diagram formed by $\gamma: T \to T'$ and $T, T' \to X(B_n)$ is commutative.

The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are isomorphic to the respective identities. In the case of $\Phi \circ \Psi$ the quotient of the pull-back of a $C_n$-curve $C \to Y$ to $T \subset C$ is canonically isomorphic to $C \to Y$. In the case of $\Psi \circ \Phi$ the quotient of a $B_n$-curve $B \to T$ over a $\mu_2$-torsor $T$ gives a $C_n$-curve $C \to Y$, together these form a cartesian square. The section $s_0: T \to B$ determines an inclusion $T \subset C$ as fixed point subscheme with respect to $I$. Applying the functor $\Psi$ we recover a $B_n$-curve canonically isomorphic to the original $B_n$-curve. □

**Corollary 6.7.** The toric variety $X(C_n)$ is a coarse moduli space of $C_n$-curves.

The stack $\mathcal{X}(C_n)$ is a toric Deligne-Mumford stack as introduced in [BCS03] (see also [FMN10]): we define the stacky fan $\Sigma(C_n)$ as the fan $\Sigma(C_n)$ in the lattice $N(C_n)$ with the difference that we choose on the rays generated by $\frac{1}{2}(\varepsilon_1v_1 + \ldots + \varepsilon_nv_n)$ the second lattice points $\varepsilon_1v_1 + \ldots + \varepsilon_nv_n$. In comparision to the fan $\Sigma(B_n)$ the
underlying lattice is finer and the toric DM stack associated with $\Sigma(C_n)$ coincides with the quotient stack $[X(B_n)/\mu_2]$.

**Corollary 6.8.** The stack $X(C_n)$ is isomorphic to the toric Deligne-Mumford stack associated with the stacky fan $\Sigma(C_n)$.

**Example 6.9.** The stacky fan $\Sigma(C_2)$ in the lattice $\mathbb{Z}_{2}^{1}v \cong \mathbb{Z}$ consists of the two cones $Q_{\geq 0}v, Q_{\leq 0}(-v)$. The associated toric DM stack is $X(C_2) \cong [\mathbb{P}^1/\mu_2]$ (cf. also [FMN10] example 7.31), it is an orbifold with two stacky points.

$X(C_n)$ as fine moduli space. We give a characterisation of $X(C_n)$ as a fine moduli space $T_{n}^{\pm}$ of $2n$-pointed chains of projective lines. Here the universal curve is not $X(C_{n+1}) \to X(C_n)$, however, the universal curve and the general notion of a $C_n$-curve are defined naturally in terms of the inclusion of root systems $C_n \to C_{n+1}$.

We have the root subsystem $C_n \subset C_{n+1}$ in the subspace generated by the roots $u_1, \ldots, u_n$. Take those pairs of opposite roots in $C_{n+1} \setminus C_n$ which define projections $C_{n+1} \to C_n$ in the sense of [BB11, 1.2]; these are $\pm \alpha_i^\pm$, $\pm \alpha_2^\pm, \ldots, \pm \alpha_n^\pm$ but not $\pm 2u_{n+1}$. To each of these pairs $\pm \alpha_i^\pm$ and $\pm \alpha_i^+$ we associate a section $s_i^-$ and $s_i^+$. The element of the Weyl group given as the reflection for the root $\pm 2u_{n+1}$ mapping $u_{n+1} \mapsto -u_{n+1}$ and $u_i \mapsto u_i$ for $i \in \{1, \ldots, n\}$ is an isomorphism of $C_{n+1}$ fixing $C_n \subset C_{n+1}$. It maps $\alpha_i^\mp \mapsto -\alpha_i^\mp$. This leads us to the following definition.

**Definition 6.10.** (Second definition of $C_n$-curves). A $C_n$-curve over an algebraically closed field $K$ is a chain of projective lines with involution of odd or even length with $2n$ (possibly coinciding) marked points $s_1^\pm, \ldots, s_n^\pm$ different from the poles, the involution interchanging $s_i^\pm \leftrightarrow s_i^\pm$, such that every component contains at least one of the points $s_i^\pm$. We define a $C_n$-curve over an arbitrary scheme, isomorphisms of $C_n$-curves and the moduli functor of $C_n$-curves in the same way as we did in the case of $B_n$-curves.

**Construction 6.11.** Let the subscheme

$$C(C_{n+1}/C_n) \subset (\prod_{i=1}^{n} \mathbb{P}^1_{\{\pm \alpha_i^-\}} \times \prod_{i=1}^{n} \mathbb{P}_1^1_{\{\pm \alpha_i^+\}})_{X(C_n)}$$

be defined by the equations $\mathfrak{D}$ using the universal $C_n$-data on $X(C_n)$. This morphism $C(C_{n+1}/C_n) \to X(C_n)$ has the sections $s_{-i}, s_{+i}, s_i^\pm$, where $s_{-i}$ is defined by $z_{-\alpha_i^+} = 0$ ($i = 1, \ldots, n$), $s_{+i}$ is defined by $z_{\alpha_i^-} = 0$ ($i = 1, \ldots, n$) and the sections $s_i^\pm$ by the equations $z_{\alpha_i^\pm} = z_{-\alpha_i^\pm}$. The involution maps $\mathbb{P}_1^1_{\{\pm \alpha_i^-\}} \leftrightarrow \mathbb{P}_1^1_{\{\pm \alpha_i^+\}}$.

**Remark 6.12.** The toric variety $C(C_{n+1}/C_n)$ arises from $X(C_{n+1})$ by contracting certain torus invariant prime divisors. The fibres of $X(C_{n+1}) \to X(C_n)$ over the divisors corresponding to the rays generated by elements of the form $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$ (forming $D$ in proposition 6.2) have a central component containing none of the sections $s_i^\pm$. In $X(C_{n+1})$ the support of the central components of the fibers over the divisor corresponding to $\frac{1}{2} (\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n)$ forms a torus invariant divisor which corresponds to the ray in $\Sigma(C_{n+1})$ generated by $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$ and is
isomorphic to $X(C_1) \times X(A_{n-1}) \cong \mathbb{P}^1 \times X(A_{n-1})$. We contract these divisors $\mathbb{P}^1 \times X(A_{n-1})$ to $X(A_{n-1})$ by omitting the rays in $\Sigma(C_{n+1})$ generated by elements of the form $\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n$, but retaining the two-dimensional cones $\langle \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n - v_{n+1}), \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n + v_{n+1}) \rangle_{Q \geq 0}$. On the fibers over $D$ the central components are contracted.

**Proposition 6.13.** The morphism $C(C_{n+1}/C_n) \to X(C_n)$ with the involution $I$ and the sections $s_-, s_+, s_1^\pm, \ldots, s_n^\pm$ is a $C_n$-curve. The combinatorial types of the fibres over the torus orbits corresponding to one-dimensional cones are as follows:

\[
\begin{align*}
\varepsilon_{i_1} v_{i_1} & \quad s_{i_1}^\pm s_{i_2}^\pm \ldots s_{i_n}^\pm s_{i_1}^{\varepsilon_1} \\
\varepsilon_{i_1} v_{i_1} + \varepsilon_{i_2} v_{i_2} & \quad s_{i_1}^\pm s_{i_2}^{\varepsilon_1} s_{i_3}^{\pm} \ldots s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1} \\
\vdots & \quad \vdots \\
\varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_{n-2}} v_{i_{n-2}} & \quad s_{i_1}^{\varepsilon_1} \ldots s_{i_{n-2}}^{\varepsilon_1} s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_1} s_{i_3}^{\varepsilon_1} \ldots s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1} \\
\varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_{n-1}} v_{i_{n-1}} & \quad s_{i_1}^{\varepsilon_1} \ldots s_{i_{n-1}}^{\varepsilon_1} s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_1} s_{i_3}^{\varepsilon_1} \ldots s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1} \\
\frac{1}{2}(\varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_n} v_{i_n}) & \quad \frac{1}{2} s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_1} s_{i_3}^{\varepsilon_1} \ldots s_{i_n}^{\varepsilon_1} s_{i_1}^{\varepsilon_1}
\end{align*}
\]

**Definition 6.14.** We call $C(C_{n+1}/C_n) \to X(C_n)$ together with the involution $I$ and the sections $s_-, s_+, s_1^\pm, \ldots, s_n^\pm$ the universal $C_n$-curve over $X(C_n)$.

By the same procedure as in the case of root systems of type $A$ and $B$ we can prove the following.

**Theorem 6.15.** There exists a fine moduli space $\overline{L}_n^\pm$ of $C_n$-curves isomorphic to the toric variety $X(C_n)$ with universal family $C(C_{n+1}/C_n) \to X(C_n)$.

**Remark 6.16.** There is a natural closed embedding of the moduli spaces $\overline{L}_n^\pm = X(C_n) \to \overline{L}_{2n} = X(A_{2n-1})$ determined by considering a $C_n$-curve with sections $s_1^-, \ldots, s_n^-, s_1^+, \ldots, s_n^+$ as an $A_{2n-1}$-curve with sections $s_1, \ldots, s_{2n}$. The toric morphism $X(C_n) \to X(A_{2n-1})$ is given by the projection of root systems $A_{2n-1} \to C_n$ induced by $\bigoplus_{i=1}^{2n} \mathbb{Z} u_i \to M(C_n)$, $u_i \mapsto u_i$, $u_{2n+1-i} \mapsto -u_i$ for $i = 1, \ldots, n$. The kernel in $M(A_{2n-1})$ is generated by $u_{2n+1-i} + u_i - u_{2n+1-j} - u_j$ for some fixed $j$ and $i \in \{1, \ldots, n\} \setminus \{j\}$. By employing this embedding we have an alternative approach to prove the above statements.

## 7. Root systems of type $D$

Consider for $n \geq 2$ an $n$-dimensional Euclidean space $E$ with basis $u_1, \ldots, u_n$. The root system $D_n$ in $E$ consists of the $2n(n-1)$ roots

\[\pm(u_i + u_j), \pm(u_i - u_j) \quad \text{for} \quad i, j \in \{1, \ldots, n\}, i < j.\]

The following is a set of simple roots:

\[u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_{n-1} + u_n.\]

The Weyl group $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ acts by $u_i \mapsto \varepsilon_i u_i$, where the $\varepsilon_i$ are signs such that $\prod_i \varepsilon_i = 1$, and by permuting the $u_i$. So there are $2^{n-1}n$! sets of simple roots, these are of the form $\varepsilon_1 u_1 - \varepsilon_2 u_2, \varepsilon_2 u_2 - \varepsilon_3 u_3, \ldots, \varepsilon_{n-1} u_{n-1} - \varepsilon_n u_n, \varepsilon_{n-1} u_{n-1} + \varepsilon_n u_n$. 

for orderings \(i_1, \ldots, i_n\) of the set \(\{1, \ldots, n\}\) and signs \(\varepsilon_1, \ldots, \varepsilon_n\) (note that \(\varepsilon_n = 1\) and \(\varepsilon_n = -1\) give the same set).

The vector space \(E^*\) dual to \(E\) with basis \(v_1, \ldots, v_n\) dual to \(u_1, \ldots, u_n\) contains the lattice \(N(D_n)\) dual to the root lattice \(M(D_n)\). To describe the fan \(\Sigma(D_n)\) in the lattice \(N(D_n)\) we determine a Weyl chamber. The set of simple roots \(u_1 - u_2, u_2 - u_3, \ldots, u_{n-1} - u_n, u_{n-1} + u_n\) has the dual basis \(v_1, v_2, \ldots, v_{n-2}, \frac{1}{2}(v_1 + \ldots + v_{n-1} - v_n), \frac{1}{2}(v_1 + \ldots + v_{n-1} + v_n)\) of \(N(D_n)\) which generates the corresponding Weyl chamber. There are \(3^n - n2^{n-1} - 1\) one-dimensional cones generated by elements of the form \(\sum_{i \in A} \varepsilon_i v_i\) for \(A \subset \{1, \ldots, n\}\), \(1 \leq |A| \leq n - 2\) or of the form \(\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1} + \varepsilon_n v_n)\), where the \(\varepsilon_i\) are signs.

The torus invariant divisor for the one-dimensional cone generated by \(\varepsilon_1 v_1 + \ldots + \varepsilon_k v_k, 1 \leq k \leq n - 2\) is isomorphic to \(X(D_{n-k}) \times X(A_{k-1})\), that for \(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-2} v_{n-2}\) is isomorphic to \(X(A_1) \times X(A_1) \times X(A_{n-3}) \cong X(D_2) \times X(A_{n-3})\) and that for \(\frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1})\) is isomorphic to \(X(A_{n-1})\) (see \[BB11, 1.2\]).

**\(X(D_{n+1})\) over \(X(D_n)\)**. Consider the proper surjective morphism \(X(D_{n+1}) \rightarrow X(D_n)\) induced by the root subsystem \(D_n \subset D_{n+1}\) consisting of the roots in the subspace generated by \(u_1, \ldots, u_n\). We have a projection of fans \(\Sigma(D_{n+1}) \rightarrow \Sigma(D_n)\) along the subspace generated by \(v_{n+1}\). The generic fibre is \(\mathbb{P}^1\). Note that the torus invariant divisor in \(X(D_{n+1})\) corresponding to \(v_1 + \ldots + v_{n-1}\) is lying over the closure of the torus orbit in \(X(D_n)\) of codimension 2 corresponding to the 2-dimensional cone generated by \(\frac{1}{2}(v_1 + \ldots + v_{n-1} + v_n), \frac{1}{2}(v_1 + \ldots + v_{n-1} - v_n)\); here (and on the translates under the Weyl group \(W(B_n)\)) we have fibres of dimension 2. This implies that the morphism \(X(D_{n+1}) \rightarrow X(D_n)\) is not flat.

There are \(2n\) additional pairs of opposite roots, the pairs \(\pm \alpha_i^+ = \pm (u_{n+1} + u_i)\) and \(\pm \alpha_i^- = \pm (u_{n+1} - u_i)\) for \(i \in \{1, \ldots, n\}\). The projections along the subspaces generated by these do not define projections of root systems \(D_{n+1} \rightarrow D_n\) in the sense of \[BB11, 1.2\]: we have \(\alpha_i^+ - \alpha_i^- = 2u_i\), so the projection along the subspace generated by \(\alpha_i^+\) (resp. \(\alpha_i^-\)) maps \(\alpha_i^-\) (resp. \(\alpha_i^+\)) to \(2u_i\) which is not a multiple of a root of \(D_n\). Instead we can consider the preimages of \(1: 1 \in \mathbb{P}_{\{\pm \alpha_i^-\}}^1 \times \mathbb{P}_{\{\pm \alpha_i^+\}}^1\) with respect to the projections \(X(D_{n+1}) \rightarrow \mathbb{P}_{\{\pm \alpha_i^-\}}^1, \mathbb{P}_{\{\pm \alpha_i^+\}}^1\) determined by the inclusions of root systems \(\{\pm \alpha_i^-\}, \{\pm \alpha_i^+\} \subset D_{n+1}\), we denote these subschemes by \(s_i^-, s_i^+\). As in the \(B\) and \(C\)-case we have sections \(s, s^+\); further we have an involution \(I\) coming from the automorphism of \(D_{n+1}\) fixing \(D_n \subset D_{n+1}\) which maps \(u_{n+1} \mapsto -u_{n+1}\), \(u_i \mapsto u_i\) for \(i \in \{1, \ldots, n\}\) and is not an element of the Weyl group \(W(D_{n+1})\).

As in the other cases we can study \(X(D_{n+1})\) over \(X(D_n)\) via the embedding into \(P(D_{n+1}/D_n) X(D_n) = \left(\prod_{i=1}^n \mathbb{P}_{\{\pm \alpha_i\}}^1\right) \times \prod_{i=1}^n \mathbb{P}_{\{\pm \alpha_i\}}^1 X(D_n)\). The subscheme \(X(D_{n+1})\) is given by the homogeneous equations parametrised by the universal \(D_n\)-data
\[
\ell_{\beta} z_{\alpha_2}z_{-\alpha_1} = \ell_{-\beta} z_{-\alpha_2}z_{\alpha_1} \quad \text{for } \alpha_1, \alpha_2 \in \{\alpha_1^+, \ldots, \alpha_n^+\} \\
\text{such that } \beta = \alpha_1 - \alpha_2 \text{ is a root of } D_n
\]
We will see that over the complement of a closed subset of codimension 2 the fibres are chains of projective lines with sections \( s_i^+ \). Over these points we have a combinatorial type for a fibre resp. for the universal \( D_n \)-data as in the \( B \)-case (see proposition 3.9), we use the notation introduced there.

**Example 7.1.** \( X(D_3) \) over \( X(D_2) \).

The root system \( D_2 \) consists of the 4 roots \( \pm u_1 \pm u_2 \). It is contained in the root system \( D_3 \), this has the 8 additional roots \( \pm \alpha_1 = \pm (u_3 - u_1), \pm \alpha_2 = \pm (u_3 + u_1) \), \( \pm \alpha_2 = \pm (u_3 - u_2), \pm \alpha_2 = \pm (u_3 + u_2) \). Because of the isomorphism of root systems \( D_2 \cong A_1 \times A_1 \) we have \( X(D_2) \cong \mathbb{P}^1 \times \mathbb{P}^1 \). The fan \( \Sigma(D_2) \) has 4 one-dimensional cones generated by \( \frac{1}{2}(\pm v_1 \pm v_2) \). The fan \( \Sigma(D_3) \) has 14 one-dimensional cones, 6 of the form \( \pm v_i \) and 8 of the form \( \frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3) \). The projection \( \Sigma(D_3) \to \Sigma(D_2) \) maps the generator of the one-dimensional cone \( \frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_3 v_3) \) to the generator of the one-dimensional cone \( \frac{1}{2}(\varepsilon_1 v_1 + \varepsilon_2 v_2) \), the vector \( \pm v_i \) for \( i = 1, 2 \) is not mapped to a one-dimensional cone of \( D_2 \) but into the interior of the 2-dimensional cone \( \langle \pm v_i \pm v_j, \pm v_i - v_j \rangle \). In \( P(D_3/D_2) \times \Sigma(D_2) \) the subscheme \( X(D_3) \) is given by 4 equations parametrised by the universal \( D_2 \)-data on \( X(D_2) \). For each point we have a \( D_2 \)-data of the form \( (t_{\beta_{12}} : t_{-\beta_{12}}), (t_{\gamma_{12}} : t_{-\gamma_{12}}) \) where \( \beta_{12} = u_1 - u_2, \gamma_{12} = u_1 + u_2 \). Over the affine chart \( \text{Spec} \mathbb{Z}[\frac{1}{2} x_1 x_2] \) corresponding to the cone \( \langle \frac{1}{2}(v_1 - v_2), \frac{1}{2}(v_1 + v_2) \rangle \) for the set of simple roots \( \beta_{12}, \gamma_{12} \) this data has the property \( (t_{\beta_{12}} : t_{-\beta_{12}}) \neq (1 : 0), (t_{\gamma_{12}} : t_{-\gamma_{12}}) \neq (1 : 0) \) (see [BB11, Rem. 1.21]). We study the fibres of \( X(D_3) \to X(D_2) \) over this affine chart. Over the dense torus we have a \( \mathbb{P}^1 \), over the torus orbit corresponding to \( \frac{1}{2}(v_1 - v_2) \) (resp. \( \frac{1}{2}(v_1 + v_2) \)) we have chains of two \( \mathbb{P}^1 \) of combinatorial type \( s_1^+ s_2^- | s_1^- s_2^+ \) (resp. \( s_1^+ s_2^+ | s_1^- s_2^- \)). Over the torus fixed point corresponding to the cone \( \langle \frac{1}{2}(v_1 - v_2), \frac{1}{2}(v_1 + v_2) \rangle \) we have \( D_2 \)-data of the form \( (t_{\beta_{12}} : t_{-\beta_{12}}) = (0 : 1), (t_{\gamma_{12}} : t_{-\gamma_{12}}) = (0 : 1) \) and the fibre decomposes into three irreducible components \( \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \).

The general case can be studied using the same methods, see also the \( B_n \)-case and in particular proposition 3.9 here details will be left to the reader. We define \( Z \subset X(D_n) \) to be the union of the closures of torus orbits corresponding to the 2-dimensional cones of the form \( \langle \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1} + \varepsilon_n v_n), \frac{1}{2}(\varepsilon_1 v_1 + \ldots + \varepsilon_{n-1} v_{n-1} - \varepsilon_n v_n) \rangle \).
Proposition 7.2. Over $X(D_n) \setminus Z$ the fibres of the morphism $X(D_{n+1}) \to X(D_n)$ are chains of projective lines of odd or even length with sections $s_i^\pm$. The combinatorial type of the fibres over the torus orbits corresponding to one-dimensional cones are as follows:

$$
\begin{align*}
&\varepsilon_{i_1} v_{i_1} \\
&\varepsilon_{i_1} v_{i_1} + \varepsilon_{i_2} v_{i_2} \\
&\vdots \\
&\varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_{n-2}} v_{i_{n-2}} \\
&\square \left( \varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_{n-2}} v_{i_{n-2}} + \varepsilon_{i_{n-1}} v_{i_{n-1}} + v_{i_n} \right) \\
&\square \left( \varepsilon_{i_1} v_{i_1} + \ldots + \varepsilon_{i_{n-2}} v_{i_{n-2}} + \varepsilon_{i_{n-1}} v_{i_{n-1}} - \varepsilon_{i_n} v_{i_n} \right)
\end{align*}
$$

Over $Z$ the fibres are 2-dimensional and decompose into irreducible components isomorphic to $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ intersecting transversally. We have a central component $\mathbb{P}^1 \times \mathbb{P}^1$ with action of $I$ that interchanges two torus fixed points and leaves the other two fixed. Further, we have chains of $\mathbb{P}^1$ emanating from the two torus fixed points of $\mathbb{P}^1 \times \mathbb{P}^1$ interchanged by $I$ with the sections $s_\pm$ on the outer components. Concerning the subschemes $s_i^\pm$, each of them intersects only with one component, those intersecting with one of the $\mathbb{P}^1$ locally are sections, one pair $s_i^-$, $s_i^+$ intersects with $\mathbb{P}^1 \times \mathbb{P}^1$ as $(1:1) \times \mathbb{P}^1$, $\mathbb{P}^1 \times (1:1)$.

Remark 7.3. The combinatorial type of fibres of $X(R_{n+1})$ over the torus fixed points of $X(R_n)$ can be pictured in form of the Dynkin diagram of the root system $R_{n+1}$ such that a component $\mathbb{P}^1$ with one section corresponds to a vertex.

In the $A_n$-case (see [BB1]) we have a string starting with the section $s_{i_1}$ on the component containing $s_-$ and ending with the section $s_{i_{n+1}}$ on the central component containing $s_+$ in the form of the Dynkin diagram for the root system $A_{n+1}$:

$$
\begin{align*}
&\text{s}_{i_1} \quad \text{s}_{i_2} \quad \text{s}_{i_3} \quad \ldots \quad \text{s}_{i_{n+1}}
\end{align*}
$$

In the $B_n$-case, because of the involution $I$, it suffices to consider the central component containing the section $s_0$ and one of the two chains emanating from the central component. This forms a Dynkin diagram of type $B_{n+1}$:

$$
\begin{align*}
&\text{s}_0 \quad \text{s}_{i_1}^\varepsilon \quad \text{s}_{i_2}^\varepsilon \quad \ldots \quad \text{s}_{i_n}^\varepsilon
\end{align*}
$$

In the $C_n$-case we have the double-section $S_0$ replacing the section $s_0$:

$$
\begin{align*}
&\text{S}_0 \quad \text{s}_{i_1}^\varepsilon \quad \text{s}_{i_2}^\varepsilon \quad \ldots \quad \text{s}_{i_n}^\varepsilon
\end{align*}
$$

Finally, in the $D_n$-case we can take the torus invariant divisors in the central component $\mathbb{P}^1 \times \mathbb{P}^1$ and their intersection with the fibres of the schemes $s_i^\pm$. Together with the other components we have a Dynkin diagram of type $D_{n+1}$:

$$
\begin{align*}
&\text{s}_{i_1}^+ \quad \text{s}_{i_2}^\varepsilon \quad \text{s}_{i_3}^\varepsilon \quad \ldots \quad \text{s}_{i_n}^\varepsilon
\end{align*}
$$
References

[BB11] V. Batyrev, M. Blume, The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces, Tohoku Math. J. 63 (2011), arXiv:0911.3607.

[BCS04] L. Borisov, L. Chen, G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, J. Amer. Math. Soc. 18 (2004), 193–215, arXiv:math/0309229.

[Dan] V. Danilov, The Geometry of Toric Varieties, Russian Math. Surveys 33:2, 97–154, (1978).

[DL94] I. Dolgachev, V. Lunts, A character formula for the representation of a Weyl group in the cohomology of the associated toric variety, J. Algebra 168 (1994), 741–772.

[FMN10] B. Fantechi, E. Mann, F. Nironi, Smooth toric Deligne-Mumford stacks, J. reine angew. Math. 648 (2010), 201–244, arXiv:0708.1254.

[Ful] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies 131, Princeton University Press, 1993.

[Kl85] A. Klyachko, Orbits of a maximal torus on a flag space, Functional Anal. Appl. 19 (1985), 65–66.

[Kl95] A. Klyachko, Toric varieties and flag spaces, Trudy Mat. Inst. Steklov. 208 (1995), Teor. Chisel, Algebra i Algebr. Geom., 139–162.

[LM00] A. Losev, Yu. Manin, New Moduli Spaces of Pointed Curves and Pencils of Flat Connections, Michigan Math. J. 48 (2000), 443–472, arXiv:math/0001003.

[Pr90] C. Procesi, The toric variety associated to Weyl chambers, Mots, 153–161, Lang. Raison. Calc., Hermès, Paris, 1990.

[Re01] M. Readdy, The Yuri Manin Ring and Its $B_n$-Analogue, Advances in Applied Mathematics 26 (2001), 154–167.

[St94] J. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, Adv. Math. 106 (1994), 244–301.

Mathematisches Institut, Universität Tübingen,
auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail address: batyrev@everest.mathematik.uni-tuebingen.de

Mathematisches Institut, Universität Münster,
einsteinstrasse 62, 48149 Münster, Germany
E-mail address: mark.blume@uni-muenster.de