String stable integral control of vehicle platoons with disturbances

Guilherme Fróes Silva a, Alejandro Donaire b, Aaron McFadyen a, Jason Ford a

a School of Electrical Engineering and Robotics, Queensland University of Technology, 2 George St, 4000, QLD, Australia
b School of Engineering, University of Newcastle, University Drive, 2308, NSW, Australia

Abstract

This paper presents string stable controllers with disturbance rejection properties for vehicle platoons. Through the addition of integral action and a coordinate change, sufficient smoothness conditions on the closed loop system are established that ensure the proposed controller is string stable in the presence of time-varying disturbances, and is able to reject constant disturbances. Error bounds from desired platoon configuration are also developed. Further, a suitable controller structure is introduced, and an example is provided that achieves the required smoothness conditions and is examined in simulation studies.

Key words: Multi-agent systems; String stability; Vehicle platoons;

1 Introduction

Systems composed by multiple interacting agents are common in natural, social and engineering systems, being studied in a variety of fields, including cooperative systems [1,15,6], intelligent transportation system [9,20], and aerial vehicles coordination [5,21]. Due to the potential advantages of increasing traffic throughput and reducing fuel consumption [19], actively grouping vehicles as platoons has received recent attention, see [3] and references within. Generally, each platoon agent is coordinated to maintain a desired distance, or time headway, to its neighbours [24]. In this paper, we proposed a controller with integral action intended to compensate for constant disturbances, while ensuring system string stability properties in the presence of time varying disturbances.

Research on vehicle platoons and intelligent transportation systems dates back to 1960s [17], which introduced the idea of controlling vehicle positions based on sensors and/or communication data. In addition to the usual stability properties, it was realised it is desirable for multi-vehicles platoons to have additional stability properties [22]: a platoon system is termed string stable when disturbances and initial conditions are attenuated along the string, regardless of the string length [26,4].

The communication structure, spacing policy, and vehicle dynamics of platoons all impact their behaviour and stability properties, see [11,25] and references within. The communication structure can, among other topologies, be unidirectional [21,23], where information travels only one way in the string, and bidirectional [24,16,14], in which information travels both ways along the string. Furthermore, bidirectional strings can be divided into symmetric or asymmetric strings, the latter when different coupling is chosen between preceding and following vehicles [14,18]. Spacing policies, in turn, dictate the regulation of an agent speed as a function of the distance to other agents [8]. They affect the system stability and agent behaviour [24,2], where it is shown that strings that use only relative spacing information with constant spacing policy, and vehicles with two integrators in the open loop, are always $L_2$ string unstable for any linear controller. Finally, vehicle dynamics are generally linear [7,10,21,23], allowing the use of transfer functions and $H_\infty$ system norm string stability, which in turn guarantees $L_2$ string stability (not useful for large strings), or nonlinear [16,12,18], which, although harder to design, provides better performance.

In the weaker $L_2$ string stability setting, rather than $L_\infty$ string stability of [26], it has been shown that symmetry in position coupling combined with asymmetry in
velocity coupling in heterogeneous, asymmetric, bidirectional strings improves platoon performance [14]. Sufficient conditions were developed in [18] under which a controller ensures disturbance string stability (DSS) [4] in the sense that the response of the state is bounded by a function of the initial condition errors and input disturbances. Unfortunately, in the presence of non zero mean disturbances the platoon may drift to a non zero offset from the desired reference positions.

The key contribution of this paper is to propose a controller with integral action that provides string stability for a bidirectional platoon of vehicles with constant spacing policy in the presence of time varying disturbances. Further, the proposed controller can completely reject constant disturbances. We illustrate a method for selecting a suitable controller, and examine the performance of our proposed controller in simulation studies.

We present the system dynamics and string stability definitions in Section 2. The controller design and introduction of the sufficient conditions for string stability are discussed in Section 3. We present a numerical example to illustrate the control system performance in Section 4. Finally, we present some conclusions in Section 5.

2 Problem Formulation

We consider an interconnected system composed of $N \geq 1$ agents whose dynamics can be described as follows

$$\begin{align*}
\dot{x}_{1i} &= f_{1i}(x_{1i}, x_{2i}) \\
\dot{x}_{2i} &= f_{2i}(x_{1i}, x_{2i}) + u_i + d_i,
\end{align*}$$

for all $i = \{1, \ldots, N\}$, with $x_{1i}, x_{2i} \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$, and $d_i \in \mathbb{R}^n$, where $n \geq 1$. The state vector of the $i$th vehicle is $x_i = [x_{1i}^T, x_{2i}^T]^T$. The smooth functions $f_{1,2i}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describe the system dynamics $f_i(x_i) = [f_{1i}^T, f_{2i}^T]^T$. The control input of the $i$th agent is $u_i$ and $d_i = w_i(t) + w_i$ represents its disturbances, where $w_i(t)$ and $w_i$ are the time varying and constant component of force disturbances. We define a virtual agent as the reference state $x_0$.

A classical problem for interconnected systems is to have the string stability property, which will ensure for example that disturbances are not amplified when they propagate along the string while maintaining a desired configuration. We express the control objective requiring that the state $x_i$ should converge to the desired configuration $x_i^*$, where $x_i^*$ verifies $\dot{x}_i^* = f_i(x_i^*)$, as well as it is a solution of the system in the absence of disturbances.

We present now the formal definition of string stability proposed by Swaroop and Hedrick [26].

Definition 1 (Lyapunov String Stability) Consider the system (1) without disturbances, i.e. $d_i = 0$. Then, the origin $x_i^* = 0$ of (1) is string stable, if for any given $\eta > 0$, there exists a $\delta > 0$ such that $|x_i(t_0)|_\infty < \delta \Rightarrow \sup_i \|x_i(t)\|_\infty \leq \eta$.

Besselink and Johansson proposed in [4] the concept of disturbance string stability to capture the effects of external disturbances. The definition is an extension of classical string stability expressed in term of class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions.

Definition 2 (Disturbance String Stability) Consider the dynamics (1) in the absence of disturbances, i.e. $d_i = 0$, and assume that $x_i^*$ is the desired configuration. Then, the system is said to be disturbance string stable if there exists a $\mathcal{K}$ function $\gamma$ and a $\mathcal{KL}$ function $\lambda$ such that, for any disturbance $d_i$ and initial conditions the estimate

$$\sup_i |x_i(t) - x_i^*(t)|_2 \leq \gamma \left( \sup_i |x_i(0) - x_i^*(0)|_2, t \right) + \lambda \left( \sup_i \|d_i(t)\|_\infty \right)$$

is verified for all $t > 0$.

Notice that a key point in the string stability definition is that the estimate of the state error norm is independent of the number of agents, $N$.

The problem is to design an integral controller $u_i$ for the $i$th agent of a bidirectionally interconnected system (1) that ensures disturbance string stability and has the form

$$\begin{align*}
\dot{u}_i &= h_{i,i-1}(t, x_i, x_{i-1}) + \varepsilon_i h_{i,i+1}(t, x_i, x_{i+1}) \\
&\quad + h^0_i(t, x_i, x_0) + k_i, \\
\dot{\xi}_i &= g_{i,i-1}(t, x_i, x_{i-1}) + \varepsilon_i g_{i,i+1}(t, x_i, x_{i+1}) \\
&\quad + g^0_i(t, x_i, x_0)
\end{align*}$$

where the smooth functions $h_{i,j}: \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ represent the coupling functions between neighbour vehicles $i$ and $j$, while $h^0_i: \mathbb{R}_+ \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ is the coupling of vehicle $i$ with the reference state $x_0$ and $\varepsilon_i \in [0, 1]$ is the symmetry constant for vehicle $i$, weighting its coupling with the following vehicle. The controller state $\xi_i$ in $\mathbb{R}^n$ with dynamics (4) allows for integral action to compensate for disturbances. The constant $k$ is the gain of the integral action and the smooth functions $g_{i,j}: \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ shape the integral action dynamics.

3 Controller Design

The control objective is to drive the states of the system (1) to the desired configuration $x_i^*$, while satisfying the estimate (2), that is the controller ensures disturbance string stability.
3.1 Sufficient Conditions for String Stability

For the system (1) in closed loop with the static controller (3) and \( k = 0 \) (i.e., no integral action), the following sufficient conditions for string stability were presented in [18]:

C1 \( h_{i,i-1}(t, x_i^*, x_{i-1}^*) = 0, \ h_{i,i+1}(t, x_i^*, x_{i+1}^*) = 0, \) and \( h_i^0(t, x_i^*, x_0) = 0; \)

C2 for some \( c \neq 0 \) and \( b > 0 \)

\[
\mu_2 \left( \frac{\partial f_i(x_i)}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \right) \leq -c^2,
\]

\[
\max \left\{ \left\| \frac{\partial h_{i,i-1}}{\partial x_i} \right\|_2, \left\| \frac{\partial h_{i,i+1}}{\partial x_i} \right\|_2 \right\} \leq b,
\]

for all \( x_i, x_{i-1}, x_{i+1} \in \mathbb{R}^{2n}; \)

C3 \( \varepsilon_i < \frac{c^2}{b} - 1. \)

The following proposition formalise the result in [18].

**Proposition 3 (Sufficient Conditions for DSS)**

Assume that the coupling functions \( h_{i,j} \) in (3) are designed such that conditions C1, C2, and C3 are satisfied. Then, the trajectories of the system (1) in closed loop with the controller (3) with no integral action \( (k = 0) \) satisfy

\[
\sup_i |x_i(t) - x_i^*(t)|_2 \leq e^{-c^2 t} \sup_i |x_i(0) - x_i^*(0)|_2 + \frac{1 - e^{-c^2 t}}{c^2} \sup_i \|d_i(t)\|_{\infty}
\]

where \( c^2 = c^2 - b(1 + \max_i \varepsilon_i), \) which ensures DSS.

**PROOF.** It follows directly from [18, Theorem 1]. \( \square \)

The sufficient conditions C1, C2, and C3 in Proposition 3 provide a tool to select controllers that ensure string stability of an interconnected system. The proposition also ensures that the states are ultimately bounded for bounded disturbances. However, under the classical scenario of constant disturbances, when a platoon of vehicles hit a sloping road for instance, the states will not converge to the desired values and the state error will not vanish. Moreover, the error will be bounded by the maximum value of the disturbance weighted by an increasing function of time. This behaviour is undesirable and the controller should be able to compensate, at least, for constant disturbances. We propose to design a controller with the addition of integral action capable of rejecting constant disturbances and preserving the string stability property.

3.2 String Stable Controller with Constant Disturbance Rejection

We augment the system (1) with the state \( \xi_i \in \mathbb{R}^n \) and consider the controller (3)-(4). Then, the dynamics of the closed loop can be written as follows

\[
\dot{z}_i = \phi_i(z_i) + v_i + m_i,
\]

where \( z_i = [x_i^T \ \xi_i^T]^T \) is the state vector and \( \xi_i = \xi_i + k^{-1} \bar{w}_i. \) Also, we define \( v_i = [0_{1 \times n} \ \nu_{i,i}^T \ \nu_{i,i+1}^T]^T, \) where \( 0_{a \times b} \) is the \( a \times b \) matrix of zeros and \( a, b \in \mathbb{N}, \) with \( \nu_{x,i} = h_{i,i-1} + \varepsilon_i h_{i,i+1} + h_i^0 + k \xi_i \) and \( \nu_{x,i} = g_i e_i + \varepsilon_i g_{i,i+1} + g_i^0, \) and the vector \( m_i = [0_{1 \times n} \ \nu_{i,i}^T \ 0_{1 \times n} \ \nu_{i,i+1}^T]^T \) contains the time varying disturbance. The system dynamics is defined by \( \phi_i : \mathbb{R}^{3n} \to \mathbb{R}^{3n}. \) Then, the reference for the new state vector becomes \( z_i^* = [x_i^T \ 0_{1 \times n} \ \nu_{i,i}^T]^T. \) Note that we dropped the function dependencies on the states to simplify the notation.

We aim at designing \( v_i \) such that the closed loop system (7) is disturbance string stable. Moreover, if conditions C1, C2, and C3 (applied to (7)) are satisfied, the inclusion of \( \xi_i \in \mathbb{R}^n \) and the change of coordinates lead the controller to reject constant disturbances whilst guaranteeing disturbance string stability. However, directly satisfying those conditions is generally a difficult task.

Similar to [18], we consider a heterogeneous car platoon system where \( \phi_i(z_i) = F z_i \) and we design a controller \( v_i \) that meets the control objectives. The system dynamics can be written as follows

\[
\dot{z}_i = F z_i + v_i + m_i,
\]

with \( F = [0_{3n \times n} \ [I_n \ 0_{n \times 2n}]^T \ 0_{3n \times n}], \) where \( I_n \) is the \( n \)-by-\( n \) identity matrix. It is also convenient to write \( v_i = H_i e_i + \varepsilon_i H_i e_{i+1} + H^0_i \) with \( H_i e_{i-1} = [0_{1 \times n} \ H_i e_{i-1} \ g_i e_{i+1}]^T, \) \( H_i e_{i+1} = [0_{1 \times n} \ H_i e_{i+1} \ g_i e_{i+1}]^T, \) and \( H^0_i = [0_{1 \times n} \ h_i^0 + k \xi_i^T \ g_i]^T. \)

**Proposition 4 (Transformed Sufficient Conditions)**

Consider the system (8) in closed loop with the controller (3)-(4). Let \( T_i \) be the transformation matrix, with the coupling constant matrices \( \alpha_i \in \mathbb{R}^{n \times n} \) and \( \beta_i \in \mathbb{R}^{n \times n}, \)

\[
T_i \triangleq \begin{bmatrix} I_n & \alpha_i & 0_{n \times n} \\ 0_{n \times n} & I_n & \beta_i \\ 0_{n \times n} & 0_{n \times n} & I_n \end{bmatrix}.
\]

Assume that the following sufficient conditions are satisfied,
\( C^1 \) \( H_{i,i-1}(t, z_i^{*-1}, z_i^*) = 0 \), \( H_{i,i+1}(t, z_i^{*-1}, z_i^*) = 0 \), and \( H_0(t, z_i^*, z_0) = 0 \).

\( C^2 \) for some \( c \neq 0 \) and \( b > 0 \)

\[ \mu_2(J_{i,i}) \leq -c^2, \]
\[ \max \{ \|J_{i,i-1}\|_2, \|J_{i,i+1}\|_2 \} \leq b, \] (10)
for all \( z_i, z_{i-1}, z_{i+1} \in \mathbb{R}^{3n} \);

\( C^3 \) \( \varepsilon_i < \frac{c^2}{b} - 1 \).

where the Jacobian \( J \) of the closed-loop system (8), is given by the matrices \( J_{i,i}(\alpha_i, \beta_i, z_i, z_{i-1}, z_{i+1}, \varepsilon_i) \), \( J_{i,i-1}(\alpha_i, \beta_i, z_i, z_{i-1}) \) and \( J_{i,i+1}(\alpha_i, \beta_i, z_i, z_{i+1}) \) below

\[ J_{i,i} = T_i F_i T_i^{-1} + \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha_i & \alpha_i \beta_i \\ 0 & -\beta_i \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha_i & \alpha_i \beta_i \\ 0 & -\beta_i \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} \] (11)

\[ J_{i,i+1} = \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha_i & \alpha_i \beta_i \\ 0 & -\beta_i \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ 0 & 1 \end{bmatrix} , \]

where \( \frac{\partial \bar{v}_i}{\partial z_i} = \frac{\partial T_i H_{i+1}}{\partial z_{i+1}} \) and \( \bar{v}_i = T_i v_i \).

Define \( \tilde{z}_i = T_i z_i \) and \( \tilde{\bar{v}}_i = T_i \bar{v}_i \). Then, the following properties hold true.

(i) The dynamics of the transformed system are

\[ \dot{\tilde{z}}_i = T_i F_i T_i^{-1} \tilde{z}_i + \tilde{v}_i + \tilde{\bar{v}}_i \] (12)

with the new state vector \( \tilde{z}_i \) defined as

\[ \tilde{z}_i = \begin{bmatrix} x_i + \alpha_i x_{i+1} \\ x_{i+1} + \beta_i x_i \end{bmatrix} = \begin{bmatrix} z_i^1 \\ z_i^2 \\ z_i^3 \end{bmatrix} , \] (13)

The transformed unperturbed closed-loop dynamics are

\[ \dot{\tilde{z}}_i = T_i F_i T_i^{-1} \tilde{z}_i + T_i [H_{i,i-1}(t, T_i^{-1} \tilde{z}_i, T_i^{-1} \tilde{z}_{i-1}) + \varepsilon_i H_{i,i+1}(t, T_i^{-1} \tilde{z}_i, T_i^{-1} \tilde{z}_{i+1}) + H_0(t, T_i^{-1} \tilde{z}_i, z_0)] \] (14)

and the desired configuration in terms of the new state is \( \tilde{z}_i^* = T_i z_i^* \).

(ii) The following estimate is satisfied

\[ \sup_i |\tilde{z}_i(t) - \tilde{z}_i^*(t)|_2 \leq e^{-c^2} \sup_i |\tilde{z}_i(0) - \tilde{z}_i^*(0)|_2 + \frac{1 - e^{-c^2}}{c^2} \sup_i \|\tilde{\bar{v}}_i(t)\|_\infty \] (15)

PROOF. First note that (13) follows from simple algebra and (12) follows by substituting \( z_i = T_i^{-1} \tilde{z}_i \) into (8), which proves property (i).

To prove property (ii), we show that the conditions \( C^1 \), \( C^2 \) and \( C^3 \) are verified if and only if the conditions \( C^1 \), \( C^2 \) and \( C^3 \) are verified for the dynamics (14). Thus, by Proposition 3, the closed loop system in coordinates \( \tilde{z}_i \) is DSS.

First notice that \( C^1 \) is equivalent to \( C^1 \) for the transformed closed loop dynamics (14). Hence, condition \( C^1 \) is satisfied if and only if \( C^1 \) is satisfied. Now, to compute \( C^2 \), we differentiate the transformed closed loop dynamics (12), and obtain the Jacobian \( J \in \mathbb{R}^{3n \times 3n} \) defined by the matrices \( J_{i,i} \in \mathbb{R}^{3n \times 3n} \), \( J_{i,i-1} \in \mathbb{R}^{3n \times 3n} \) and \( J_{i,i+1} \in \mathbb{R}^{3n \times 3n} \) defined below

\[ J_{i,i} = \frac{\partial (T_i F_i T_i^{-1})}{\partial z_i} + \frac{\partial \bar{v}_i}{\partial z_i} , \quad J_{i,i \pm 1} = \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} , \] (16)

which can be expressed in terms of the state vector \( z_i \) of (8), as \( \partial (T_i F_i T_i^{-1}) / \partial z_i = T_i F_i T_i^{-1} \) and

\[ \frac{\partial \bar{v}_i}{\partial z_i} = \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \\ \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} & \frac{\partial \bar{v}_i}{\partial z_i} \end{bmatrix} \]

\[ \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} = \begin{bmatrix} \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} & \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} & \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} \\ \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} & \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} & \frac{\partial T_i H_{i \pm 1}}{\partial z_{i \pm 1}} \end{bmatrix} \]

By solving the partial derivatives above, we obtain the Jacobian matrix \( J \) in (11), which is written in terms of the state vector \( z_i \), obtaining also condition \( C^2 \). This implies that \( C^2 \) is satisfied if and only if \( C^2 \) is satisfied. Condition \( C^3 \) follows directly from \( C^3 \). Finally, the inequality (15) results by application of Proposition 3, which concludes the proof. \( \square \)

It is important to note that to use the sufficient conditions \( C^1 \), \( C^2 \), and \( C^3 \), it is not necessary to compute the transformed system, but only the Jacobian \( J \).
and the transformation matrix $T_i$. Also, the change of coordinates and the transformation allows us to find controllers independent of the constant disturbance $\bar{w}_i$. Now we show that if the transformed system (12) is disturbance string stable, so is the system (8).

**Proposition 5 (DSS of the Augmented System)**
Considering the system (8) in closed loop with the controller (3)-(4), assume that the functions $H_{ij}$ are such that the conditions $C1^*$, $C2^*$, and $C3^*$ are satisfied, then

$$ \sup_i |z(t) - z^*(t)|_2 \leq K e^{-\frac{c^2}{2}t} \sup_i |z(0) - z^*(0)|_2 $$

$$ + K \frac{1 - e^{-\frac{c^2}{2}t}}{c^2} \sup_i \|w_i(t)\|_{\infty} $$

(17)\]

where $c^2 = c^2 - b(1 + \max_i \varepsilon_i)$ and $K = \frac{\max_i \{\sigma_{\max}(T_i)\}}{\min_i \{\sigma_{\min}(T_i)\}}$.

**PROOF.** As it is assumed that the conditions in Proposition 4 are satisfied, then (15) holds true. We also define $A_i \triangleq T_i^T T_i$ and note the minimum and maximum singular value of the matrix $T_i$ by $\sigma_{\min}(T_i)$ and $\sigma_{\max}(T_i)$, respectively. We also define $\sigma(T_i) = \sqrt{\lambda(A_i)}$, where $\lambda(A_i)$ are the eigenvalues of $A_i$, and we use it to obtain the following bound of the quadratic form

$$ \sigma \sup_i |z_i(t) - z_i^*(t)|_2 \leq \sup_i |\bar{z}_i(t) - \bar{z}_i^*(t)|_2 $$

$$ \sup_i |\bar{z}_i(0) - \bar{z}_i^*(0)|_2 \leq \bar{\sigma} \sup_i |z_i(0) - z_i^*(0)|_2 $$

$$ \sup_i \|\bar{m}_i(t)\|_{\infty} \leq \bar{\sigma} \|m_i(t)\|_{\infty} $$

(18)

where $\bar{\sigma} = \max_i \{\sigma_{\max}(T_i)\}$ and $\bar{\sigma} = \min_i \{\sigma_{\min}(T_i)\}$.

Hence, by using (18) in (15) and noting that $\sup_i \|m_i(t)\|_{\infty} = \sup_i \|w_i(t)\|_{\infty}$, we obtain

$$ \sup_i |z_i(t) - z_i^*(t)|_2 \leq \frac{\bar{\sigma}}{\bar{\sigma}} e^{-\frac{c^2}{2}t} \sup_i |z_i(0) - z_i^*(0)|_2 $$

$$ + \frac{\bar{\sigma}}{\bar{\sigma}} \frac{1 - e^{-\frac{c^2}{2}t}}{c^2} \sup_i \|w_i(t)\|_{\infty} $$

from where we obtain (17) by setting $K = \frac{\bar{\sigma}}{\bar{\sigma}}$.

Proposition 4 shows DSS of the closed loop dynamics (8), which includes the integral action. Now, we will show DSS holds for the original states of the system (1).

**Corollary 6 (DSS of the Original System)** Consider the system (1) in closed loop with the controller (3)-(4). If the functions $H_{ij}$ are such that the conditions $C1^*$,

$$ C2^*, \text{ and } C3^* \text{ are satisfied, then the state trajectories satisfy} $$

$$ \sup_i |x_i(t) - x_i^*(t)|_2 \leq K e^{-\frac{c^2}{2}t} \sup_i |x_i(0) - x_i^*(0)|_2 $$

$$ + K e^{-\frac{c^2}{2}t} \sup_i |\xi_i(0) + k^{-1} \bar{w}_i|_2 $$

$$ + K \frac{1 - e^{-\frac{c^2}{2}t}}{c^2} \sup_i \|w_i(t)\|_{\infty} $$

(19)

where $c^2 = c^2 - b(1 + \max_i \varepsilon_i)$ and $K = \frac{\max_i \{\sigma_{\max}(T_i)\}}{\min_i \{\sigma_{\min}(T_i)\}}$.

**PROOF.** First note that $z_i = [x_i^T \xi_i T_i]^T$ implies that

$$ \sup_i |x_i(t) - x_i^*(t)|_2 \leq \sup_i |z_i(t) - z_i^*(t)|_2. $$

(20)

As all the assumptions of Proposition 5 are satisfied, inequality (17) holds. Thus, using (17) in (20), we obtain

$$ \sup_i |x_i(t) - x_i^*(t)|_2 \leq \sup_i |z_i(t) - z_i^*(t)|_2 $$

$$ \leq K e^{-\frac{c^2}{2}t} \sup_i |z_i(0) - z_i^*(0)|_2 $$

$$ + K \frac{1 - e^{-\frac{c^2}{2}t}}{c^2} \sup_i \|w_i(t)\|_{\infty}. $$

(21)

Also, using the triangle inequality, we can write $\sup_i |z_i(0) - z_i^*(0)|_2 \leq \sup_i |x_i(0) - x_i^*(0)|_2 + \sup_i |\xi_i(0)|_2$. Then from (21) and the fact that $\xi_i(0) = \xi_i(0) + k^{-1} \bar{w}_i$, we obtain (19), which completes the proof.

Corollary 6 proves that the state error of the original states of the system (1) are bounded by the initial state and initial integral action deviation from their respective equilibria and by the infinite norm of the time-variant disturbance. The bound (19) is comparable to the bound in Corollary 1 of [18], which is

$$ \sup_i |x_i(t) - x_i^*(t)|_2 \leq K e^{-\frac{c^2}{2}t} \sup_i |x_i(0) - x_i^*(0)|_2 $$

$$ + K \frac{1 - e^{-\frac{c^2}{2}t}}{c^2} \sup_i \|\bar{w}_i + w_i(t)\|_{\infty}. $$

(22)

Also, the integral action ensures that when the agents are subject to constant disturbances, the states converge to their desired values.
4 Numerical Experiment/Results

In this section, we consider \( x_i = [q_i, \dot{q}_i]^T \), where \( q_i, \dot{q}_i \in \mathbb{R} \) are the position and speed of vehicle \( i \) in a vehicle platoon, whose closed-loop dynamics is described by system (1), with \( n = 1, f_{1i} = \dot{q}_i \) and \( f_{2j} = 0 \), and controller (3)-(4), with the coupling functions below

\[
\begin{align*}
    h_{i,i-1} &= h_i^p(q_{i-1} - q_i - \delta_{i,i-1}) + K_i^v(q_{i-1} - \dot{q}_i) \\
    h_{i,i+1} &= h_i^p(q_{i+1} - q_i + \delta_{i,i+1}) + K_i^v(q_{i+1} - \dot{q}_i) \\
    h_i^0 &= K_i^{p0}(q_0 - q_i - \delta_{i,0}) + K_i^{v0}(\dot{q}_0 - \dot{q}_i)
\end{align*}
\]

where \( \delta_{i,j} \) is the desired spacing, while \( \delta_{i,0} = \sum_{j=1}^i \delta_{j,j-1} \) is the distance to the reference \( x_0 \). The functions that shape the integral action dynamics are

\[
\begin{align*}
    g_{i,i-1} &= g_i^p(q_{i-1} - q_i - \delta_{i,i-1}) + G_i^v(q_{i-1} - \dot{q}_i) \\
    g_{i,i+1} &= g_i^p(q_{i+1} - q_i + \delta_{i,i+1}) + G_i^v(q_{i+1} - \dot{q}_i) \\
    g_i^0 &= G_i^{p0}(q_0 - q_i - \delta_{i,0}) + G_i^{v0}(\dot{q}_0 - \dot{q}_i)
\end{align*}
\]

where we selected \( h_i^p(x) = K_i^p \tanh(K_i^{p2}x) \) and \( g_i^p(x) = G_i^p \tanh(G_i^{p2}x) \). The controller gains are \( K_{1i}, K_{2i}, K_i^v, K_i^{p0}, K_i^{v0}, k, G_{1i}, G_{2i}, G_i^v, G_i^{p0}, \) and \( G_i^{v0} \). We compute the partial derivatives in (11) and using CVX, a package for specifying and solving convex programs [13], find the controller gains that satisfy the conditions \( \text{C1}^*, \text{C2}^* \) and \( \text{C3}^* \) so that (17) holds. The values of the controller gains used in the simulation are \( \alpha_i = 0.3, \beta_i = -0.4, \varepsilon_i = 1, K_i^{p1} = K_i^{p2} = 0.1187, K_i^v = 0.0121, K_i^{p0} = 0.6, K_i^{v0} = 0.6, k = 0.25, G_i^{p1} = G_i^{p2} = 0.1, G_i^v = 0.01, G_i^{p0} = 0.2881, \) and \( G_i^{v0} = 0.342 \).

We consider a string of \( N = 25 \) vehicles, even though the estimates hold for any \( N \), and compare the controllers obtained using Corollary 1 in [18] and Proposition 5, which we noted \( C_1 \) and \( C_2 \) respectively. The initial conditions are \( x_i(0) = [q_0(0) - \delta_{i,0} + \rho_i, 0] \), the inter-vehicle spacing is \( \delta_{i,i-1} = \delta_{i+1,i} = 10 \) m and the reference speed \( \dot{q}_0 = 20 \) m/s. Also, we set the time-varying disturbance as \( w_i(t) = r_isin(t)exp(-0.1t) \) and the constant disturbance is \( \bar{w}_i = (5 + r_i) \) m/s², where \( r_i \) is uniformly randomly generated in the interval \([-1,1]\). Figure 1 shows the state error norm together with the bounds (22) and (19), for comparison. As expected, since the time-varying disturbances vanish, the state error norm converges to a constant for the controller \( C_1 \). The controller \( C_2 \) shows better performance as it compensates for the constant disturbance and the state error norm is suppressed. Also, note that the bounds of Corollary 1 in [18] grows rapidly above the bound (19). Figure 2 shows the deviation of the inter-vehicle distances, that is \( \varepsilon_{i,i-1} = q_{i-1} - q_i - \delta_{i,i-1} \) from the desired value. We note that the controller \( C_1 \) cannot compensate for disturbances and the inter-vehicle distances do not converge to the desired values, but controller \( C_2 \) does whilst ensuring a reasonable inter-vehicle distances during the transient. Figure 3 shows that control signals are smooth and within reasonable values. Figure 4 shows the speed of the vehicles, converging to the reference velocity, and the state of the integral action, converging to a value proportional to the constant disturbance.
Fig. 4. Time histories of the vehicle velocity (top) and integral state (bottom).

5 Conclusion

In this paper we presented modified sufficient conditions which guarantee a linear, asymmetric, bidirectional, interconnected system under (possibly) nonlinear control to be disturbance string stable. Under these conditions, an integral controller will ensure the state errors are bounded by functions of the initial conditions and the time-variant (zero mean) disturbance, whilst rejecting constant (non-zero mean) disturbance due to the addition of integral action. Future work will focus on developing sufficient conditions for controllers that use local information without global knowledge of reference signals.

References

[1] M. Arcak. Passivity as a Design Tool for Group Coordination. IEEE Transactions on Automatic Control, 52(8):1380–1390, 2007.
[2] P. Barooah and J. P. Hespanha. Error amplification and disturbance propagation in vehicle strings with decentralized linear control. In The IEEE Conference on Decision and Control, and the European Control Conference, Seville, Spain, 2005.
[3] C. Bergenhem, H. Pettersson, E. Coelingh, C. Englund, S. Shladover, and S. Tsugawa. Overview of platooning systems. In The ITS World Congress, Vienna, Austria, 2012.
[4] B. Besselink and K. H. Johansson. String stability and a delay-based spacing policy for vehicle platoons subject to disturbances. IEEE Transactions on Automatic Control, 62(9):4376–4391, 2017.
[5] N. Cai, Y.-S. Zhong, and J.-X. Xi. Swarm stability of high-order linear time-invariant swarm systems. IET Control Theory & Applications, 5(2):402–408, 2011.
[6] Y.-Q. Chen and Z. Wang. Formation control: A review and a new consideration. In The IEEE International Conference on Intelligent Robots and Systems, pages 3181–3186, Edmonton, Canada, 2005.
[7] P. A. Cook. Conditions for string stability. Systems & Control Letters, 54(10):991–998, 2005.
[8] S. Darbha and K. R. Rajagopal. Intelligent cruise control systems and traffic flow stability. Transportation Research Part C: Emerging Technologies, 7(6):329–352, 1999.
[9] C. Canudas de Wit and B. Brogliato. Stability issues for vehicle platooning in automated highway systems. In The IEEE International Conference on Control Applications, Kohala Coast, HI, USA, 1999.
[10] J. Eyre, D. Yanakiev, and I. Kanellakopoulos. A simplified framework for string stability analysis of automated vehicles. Vehicle System Dynamics, 30(5):375–405, 1998.
[11] S. Feng, Y. Zhang, S. E. Li, Z. Cao, H. X. Liu, and L. Li. String stability for vehicular platoon control: Definitions and analysis methods. Annual Reviews in Control, 47:1–17, 2019.
[12] J. Ferguson, A. Donaire, S. Knorn, and R. H. Middleton. Decentralized control for $l_2$ weak string stability of vehicle platoon. In The IFAC World Congress, Toulouse, France, 2017.
[13] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 2.1. http://cvxr.com/cvx, March 2014.
[14] I. Herman, S. Knorn, and A. Ahlén. Disturbance scaling in bidirectional vehicle platoons with different asymmetry in position and velocity coupling. Automatica, 82(8):13–20, 2017.
[15] S. Knorn, Z. Chen, and R. H. Middleton. Overview: Collective Control of Multiagent Systems. IEEE Transactions on Control of Network Systems, 3(4):334–347, Dec 2016.
[16] S. Knorn, A. Donaire, J. C. Agiiero, and R. H. Middleton. Passivity-based control for multi-vehicle systems subject to string constraints. Automatica, 50(12):3224–3230, 2014.
[17] W. S. Levine and M. Athans. On the Optimal Error Regulation of a String of Moving Vehicles. IEEE Transactions on Automatic Control, 11(3):355–361, 1966.
[18] J. Monteil, G. Russo, and R. Shorten. On $L_\infty$ string stability of nonlinear bidirectional asymmetric heterogeneous platoon systems. Automatica, 105(7):198–205, 2019.
[19] G. J. L. Naus, R. P. A. Vugts, J. Ploeg, M. J. G. van de Molengraft, and M. Steinbuch. String-stable CACC design and experimental validation: A frequency-domain approach. IEEE Transactions on Vehicular Technology, 59(9):4268–4279, 2010.
[20] S. Öncü, N. van de Wouw, and H. Nijmeijer. Cooperative adaptive cruise control: Tradeoffs between control and network specifications. In The IEEE Conference on Intelligent Transportation Systems, Virginia, USA, 2011.
[21] A. Pant, P. Seiler, and K. Hedrick. Mesh stability of look-ahead interconnected systems. IEEE Transactions on Automatic Control, 47(2):403–407, 2002.
[22] L. Peppard. String stability of relative-motion PID vehicle control systems. IEEE Transactions on Automatic Control, 19(5):579–581, 1974.
[23] J. A. Rogge and D. Aeyels. Vehicle platoons through ring coupling. IEEE Transactions on Automatic Control, 53(6):1370–1377, 2008.
[24] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. IEEE Transactions on Automatic Control, 49(10):1835–1841, 2004.
[25] S. Strüder, M. M. Soron, and R. H. Middleton. From vehicular platoons to general networked systems: String stability and related concepts. Annual Reviews in Control, 44:157–172, 2017.
[26] D. Swaroop and J. K. Hedrick. String stability of interconnected systems. IEEE Transactions on Automatic Control, 41(3):349–357, 1996.