Hyper-stresses in $k$-Jet Field Theories

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Abstract For high-order continuum mechanics and classical field theories configurations are modeled as sections of general fiber bundles and generalized velocities are modeled as variations thereof. Smooth stress fields are considered and it is shown that three distinct mathematical stress objects play the roles of the traditional stress tensor of continuum mechanics in Euclidean spaces. These objects are referred to as the variational hyper-stress, the traction hyper-stress and the non-holonomic stress. The properties of these three stress objects and the relations between them are studied.

Keywords Continuum mechanics · Classical field theories · Fiber bundle · Hyper-stress · High order continuum mechanics

Mathematics Subject Classification 74A10 · 70S10 · 53Z05 · 58A32

1 Introduction

This manuscript is concerned with hyper-stresses for theories formulated on general fiber bundles over differentiable manifolds. Although the study of higher order continuum mechanics entered the focus of attention early in the second half of the 20th century (e.g., [16, 17, 30, 31]) it is still the subject of current research from both theoretical (e.g., [6–8, 19, 21]) and practical (e.g., [1–3, 22]) interests.

In parallel, analogous questions engaged the mathematical physics literature considering classical field theories (e.g., [4, 5, 9, 10, 12]), although the variational approach has been
much more pronounced. While in continuum mechanics fields are usually defined on a three-dimensional body manifold or the three-dimensional physical space, classical field theories are mostly formulated as sections of fiber bundles over a space-time manifold.

Following Walter Noll’s [18], it is by now accepted that the body object of continuum mechanics should be modeled as a manifold devoid of a metric or a connection. Furthermore, fields defined on the body, such as various order parameters or internal degrees of freedom, cannot always be expected to be valued in Euclidean spaces. Thus, formulations in the setting of general manifolds are relevant in continuum mechanics as much as they are for physical field theories.

While traditionally, \( k \)-th order hyper-stresses emerge in continuum mechanics as derivatives of a strain energy density with respect to derivatives of order \( k \) of the configuration, in [24, 27] their existence follows from a representation theorem for forces which are conjugate to the space of \( C^k \)-velocity fields. This setting has the advantage that it applies to the geometry of general manifolds and to hyper-stresses that may be as irregular as Borel measures. It is noted that for a theory of order \( k \) on a differentiable manifold, the derivatives of a particular order \( 0 < r \leq k \), do not give rise to an invariant geometric object. Rather, one has to use the \( k \)-jet of the generalized velocity field combining all derivatives of order less or equal to \( k \) into a single invariant object. (See [23] as a general reference on jet bundles.)

Among the peculiarities of continuum mechanics on manifolds, even in the smooth case considered in this paper and \( k = 1 \), we emphasize the distinction between variational stresses and traction stresses [25]. For classical continuum mechanics in a Euclidean space, the same mathematical object determines the traction on subbodies—the traction stress—and acts on velocity gradients to produce power—the variational stress, in our terminology. Yet, for the general geometric setting, two distinct mathematical objects play these roles. While the variational stress determines the traction stress, there is a class of traction stresses for each variational stress field. A unique variational stress density is determined when a traction stress and a body force density are given.

As we show below, it turns out that for high order theories, a variational hyper-stress does not determine a unique traction hyper-stress. In fact, a traction hyper-stress field together with the body hyper-force encode more information than a variational hyper-stress field. Now, a third object is needed, which we refer to as the non-holonomic stress field. While the non-holonomic stress has been introduced in [26] for the case \( k = 2 \) as a generalization of the variational stress enabling one to apply integral transformations, here we show that non-holonomic stresses are the objects that determine traction hyper-stresses as defined in [29].

Specifically, let \( W \) denote the vector bundle over the body \( \mathcal{X} \) where generalized velocity field assume their values and let \( J^k W \) denote it \( k \)-the jet bundle. Then, a variational hyper-stress density is a section \( S \) of the vector bundle

\[
L\left( J^k W, \bigwedge^n T^* \mathcal{X} \right),
\]

so that for a vector field \( w \), \( S \cdot j^k w \) is the power density that the hyper-stress expends for the \( k \)-th jet of the generalized velocity. The traction hyper-stress, \( \sigma \), determines the hyper-traction, \( t_{\mathcal{X}} \)—a section of \( L(J^{k-1} W, \bigwedge^{n-1} T^* \partial \mathcal{X}) \)—on the boundary of an arbitrary subbody \( \mathcal{X} \). Thus, \( \sigma \) is a section of

\[
L\left( J^{k-1} W, \bigwedge^{n-1} T^* \mathcal{X} \right)
\]

and the corresponding hyper-traction is determined by the condition that \( t_{\mathcal{X}} \cdot j^{k-1} w \) is the \((n - 1)\)-form obtained by restricting \( \sigma \cdot j^{k-1} w \) to \( T \partial \mathcal{X} \). The non-holonomic stress density,
$P$, is a section of

$$L\left(J^1(J^{k-1}W), \bigwedge^n T^* X \right).$$  \hfill (1.3)

The term “non-holonomic” stress has been chosen in order to reflect the fact that it acts on the first jet of sections $A$ of $J^{k-1}W$. A section of $J^{k-1}W$ need not be holonomic in the sense that it need not be given as $j^{k-1}w$, for some vector field $w$. Hence the distinction between the non-holonomic stresses and the variational stresses.

In order to introduce the terminology and exhibit the various roles played by these three distinct mathematical objects in the simplest terms, consider the case where the body $\mathcal{R}$ is identified with its current configuration in $\mathbb{R}^3$. Let $w^j$ be the components of a velocity field and let $w_{,i_1 \cdots i_k}$ be its $k$-th partial derivatives. The components of the variational stress of highest order $S_{i_1 \cdots i_k}^j$ act on the derivatives of the velocity in the form

$$I = \int_{\mathcal{R}} S_{i_1 \cdots i_k}^j w_{,i_1 \cdots i_k}^j \, dV.$$  \hfill (1.4)

Since the partial derivatives are symmetric relative to permutations of the indices $i_1 \cdots i_k$, the components of the variational stress satisfy the same symmetry conditions. We remark that this symmetry condition for the components of the variational stress emerges naturally in the hyper-elastic case where the components of the variational stress are given in terms of a potential function $\varphi$ and the symmetric components of the $k$-th partial derivatives of the configuration $\kappa_{i_1 \cdots i_k}$ by

$$S_{i_1 \cdots i_k}^j = \frac{\partial \varphi}{\partial \kappa_{i_1 \cdots i_k}^j}.$$  \hfill (1.5)

Returning to the expression for the action of the variational stress, one has

$$I = \int_{\mathcal{R}} \left[ (S_{i_1 \cdots i_k}^j w_{,i_1 \cdots i_k-1}^j)_{i_k} - S_{i_1 \cdots i_k-1}^j w_{,i_1 \cdots i_k-1}^j \right] dV,$$

$$= \int_{\partial \mathcal{R}} S_{i_1 \cdots i_k}^j n_{i_k} w_{,i_1 \cdots i_k-1}^j \, dA - \int_{\mathcal{R}} S_{i_1 \cdots i_k}^j w_{,i_1 \cdots i_k-1}^j \, dV,$$  \hfill (1.6)

where $n$ is the unit normal to the boundary. We will refer to a tensor such as $t_{i_1 \cdots i_k-1}^j = S_{i_1 \cdots i_k}^j n_{i_k}$ as a hyper-traction and to $b_{i_1 \cdots i_k-1}^j = S_{i_1 \cdots i_k}^j$ as a body hyper-force. Thus, in the expression

$$t_{i_1 \cdots i_k-1}^j = S_{i_1 \cdots i_k}^j n_{i_k},$$  \hfill (1.7)

the variational hyper-stress $S$ acts as a traction hyper-stress by determining the hypertraction on the boundary. That is, a traction hyper-stress $\sigma_{i_1 \cdots i_k}^j$ determines hyper-tractions by

$$t_{i_1 \cdots i_k-1}^j = \sigma_{i_1 \cdots i_k}^j n_{i_k}.$$  \hfill (1.8)

Nevertheless, if the role of the traction hyper-stress is exhibited in the last equation, there is no a-priori reason to assume that $\sigma_{i_1 \cdots i_k}^j$ is symmetric relative to permutations of all indices $i_1 \cdots i_k$ and not merely $i_1 \cdots i_k-1$. In other words a traction hyper-stress need not be determined by a variational hyper-stress but rather, by a more general object to which we refer as the non-holonomic stress.
One could postulate that traction hyper-stresses are indeed symmetric relative to all the indices $i_1 \cdots i_k$. However, it turns out that this condition cannot be formulated invariantly in the general geometric case. Indeed, on general fiber bundles, a traction hyper-stress field together with a body hyper-force determine a unique non-holonomic hyper-stress field. A non-holonomic hyper-stress field does not act on $k$-jets of velocity fields but rather on the first jets of sections of the $(k-1)$-jet bundle. Such sections of the $(k-1)$-jet bundle need not be holonomic—compatible—hence the terminology. It follows that non-holonomic hyper-stress fields obey less restrictive symmetry conditions in comparison with variational stresses. Conversely, a non-holonomic hyper-stress field determines a unique traction hyper-stress field. Furthermore, a non-holonomic hyper-stress determines a unique variational stress that may be used to compute the force on each subbody. However, this variational hyper-stress does not encode enough information as to determine the traction hyper-stress on each subbody. We find it somewhat intriguing that for higher-order continuum mechanics—originally based on the variational approach—it turns out that the generalization of the Cauchy construction plays such a crucial role.

Section 2 introduces the notation and some of the terminology used in the sequel, mainly, that corresponding to symmetric tensors. Section 3 reviews the fundamentals of continuum mechanics and field theories on $k$-jet bundles of fiber bundles. In particular, variational hyper-stress densities are introduced. Section 4 outlines higher-order theories and introduces traction hyper-stresses using a framework which generalizes the classical Cauchy approach. Section 5 considers non-holonomic stresses and their relations with traction stresses. Next, the variational stress induced by a non-holonomic stress is presented in Sect. 6. Finally, in Sect. 7, we describe the geometric setting in which elastic constitutive relations are formulated.

2 Notation and Preliminaries

Multi-index notation, as used in this manuscript, will help make the algebraic expressions below more compact. In this section we introduce the notation adopted here, with particular application to symmetric tensors. (For additional details, see [29].)

2.1 Multi-indices

A collection of indices $i_1 \cdots i_k$, $i_r = 1, \ldots, n$ will be represented as a multi-index $I$ and we will write $|I| = k$, the length of the multi-index. Multi-indices will be denoted by upper-case roman letters and the associated indices will be denoted by the corresponding lower-case letters. In what follows, we will use the summation convention for repeated indices as well as repeated multi-indices. Whenever the syntax is violated, e.g., when a multi-index appears more than twice in a term, it is understood that summation does not apply.

A multi-index $I$ induces a sequence $(I_1, \ldots, I_n)$ in which $I_r$ is the number of times the index $r$ appears in the sequence $i_1 \cdots i_k$. Thus, $|I| = \sum_r I_r$. Multi-indices may be concatenated naturally such that $|IJ| = |I| + |J|$. Additional convenient notation is introduced by

\[ I! := I_1! \cdots I_n!, \quad \delta_I^J := \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k}. \tag{2.1} \]

Greek letters will be used for strictly increasing multi-indices used in the representation of alternating tensors and forms, e.g.,

\[ \omega = \omega_{\lambda_1 \cdots \lambda_{|\lambda|}} \, dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_{|\lambda|}}. \tag{2.2} \]
To simplify the notation, the local volume element induced by a coordinate system will be denoted by $dx$, i.e.,
\[ dx = dx^1 \wedge \cdots \wedge dx^n. \] (2.3)

2.2 Permutations

We denote by $\mathcal{P}_l$ the group of permutations of $(1, \ldots, l)$. For a multi-index $I = i_1 \cdots i_l$ and a permutation $p \in \mathcal{P}_l$,
\[ p(I) = I \circ p = i_{p(1)} \cdots i_{p(l)}. \] (2.4)
An $l$-permutation $p$ acts on an $l$-dimensional array by $p(T)_I = T_{p(I)}$.

We observe that for a multi-index $I$, of all $|I|!$ permutations $p(I)$, there are $|I|!$ permutations that leave $I$ invariant. Thus, there are $|I|!/l!$ elements in the collection, $\mathcal{P}_I$, containing permutations that give distinct multi-indices $J = p(I)$.

We will also use the notation
\[ |\varepsilon|^I_J = \begin{cases} 1, & \text{if } I \text{ may be obtained by a permutation of } J, \\ 0, & \text{otherwise}. \end{cases} \] (2.5)

2.3 Symmetric Tensors

We will view an $l$-tensor as an $l$-multilinear mapping, an element of $L^l(V, W) \simeq \left( \bigotimes^l V^* \right) \otimes W$ for two vector spaces $V$ and $W$. In order to simplify the notation, for the rest of this section, we will use $W = \mathbb{R}$. For other finite dimensional vector spaces, the extension is straightforward. A permutation $p$ acts on an $l$-tensor $T$ by
\[ p(T)(v_1, \ldots, v_l) = T(v_{p(1)}, \ldots, v_{p(l)}) \] (2.6)
which implies that the array of $p(T)$ is indeed $p(T)_I = T_{p(I)}$. A symmetric tensor $T$ satisfies the condition
\[ p(T) = T, \quad \text{or equivalently, } T_{p(I)} = T_I, \] (2.7)
for all permutations $p \in \mathcal{P}_l$. A symmetric array is uniquely determined by the elements of the form $T_I$ where we use the convention that multi-indices denoted by bold faced characters are non-decreasing, that is $i_1 \leq \cdots \leq i_l$, and alternatively, when a multi-index appears inside angle brackets, it is implied that it is permuted so that it is non-decreasing. When the summation convention should be applied to such multi-indices, it is understood that the summation is carried out only for non-decreasing occurrences.

For an $n$-dimensional vector space $V$, the subspace of symmetric tensors, $L^l_S(V, \mathbb{R})$, has the dimension (see [14]),
\[ \dim L^l_S(V, \mathbb{R}) = \frac{(n + l - 1)!}{(n - 1)!l!}. \] (2.8)

2.4 Symmetrization

We will use the symmetrization operator
\[ \mathcal{S}: L^l(V, W) \rightarrow L^l_S(V, \mathbb{R}), \quad \text{whereby, } \mathcal{S}(T) = \frac{1}{l!} \sum_{p \in \mathcal{P}_l} p(T). \] (2.9)
which implies that the array of the symmetrized tensor is the symmetrized array, \( i.e., \)
\[
T(I) := S(T) I = \sum_{p \in \mathcal{P}} T_p(I) .
\] (2.10)

The symmetrized tensor product is \( T \odot R = S(T \otimes R) \). Using the notation
\[
e^I := \bigotimes^j e^I = e^{i_1} \otimes \cdots \otimes e^{i_l}, \quad e^{(I)} := \bigodot^j e^I = e^{i_1} \odot \cdots \odot e^{i_l},
\] (2.11)
a basis for the space of symmetric tensors may be formed by the elements \( \{e^{(I)}\} \) (where we observe the \( I \) multi-indices are non-decreasing). Thus, one has the identification
\[
L^I_s(V, \mathbb{R}) \simeq \bigodot^I V^*. \] (2.12)

Consider the inclusion
\[
\iota_S: \bigodot^I V^* \rightarrow \bigotimes^I V^*. \] (2.13)

For a symmetric tensor \( T \), one has
\[
\iota_S(T) = T = T_I e^I = T_I e^{(I)},
\]
\[
= \sum_I \sum_{p \in \mathcal{P}} T_{p(I)} e^{p(I)},
\]
\[
= \sum_I \frac{|I|!}{I!} T_I e^{(I)} \quad (\text{no sum}). \] (2.14)

If we want the components of \( T \) relative to the basis in \( \bigodot^I V^* \) to be equal to the corresponding components of the array of \( \iota_S(T) = T \), we should use the modified basis
\[
\overleftarrow{e}^{(I)} := \frac{|I|!}{I!} e^{(I)}, \quad \text{so that} \quad T = T_I \overleftarrow{e}^{(I)}. \] (2.15)

Similarly, one can easily show that for the primal basis \( \{e_i\} \),
\[
\overrightarrow{e}^{(I)}(e_{(J)}) = \delta^I_J, \quad e^{(I)}(\overrightarrow{e}^{(J)}) = \delta^I_J, \] (2.16)
and so \( \{\overleftarrow{e}^{(I)}\} \) may serve as the dual basis of \( \{e_{(J)}\} \) and vice versa.

For a basis of the tangent space to a manifold induced by the coordinates \( (x^i) \), we will use the basis consisting of the elements
\[
\partial_I := \partial_{i_1} \odot \cdots \partial_{i_l}, \quad \partial_i := \frac{\partial}{\partial x^i},
\] (2.17)
for the space of symmetric tensors, while the dual basis consists of the elements
\[
\overleftarrow{dx}^I := \frac{|I|!}{I!} dx^{i_1} \odot \cdots \odot dx^{i_l}. \] (2.18)

It is noted that in both cases, only non-decreasing indices are used.
It follows that $\mathcal{O}^l V^*$ may be identified with $(\mathcal{O}^l V)^*$. In analogy with the above, for $R = R^l e_{(I)} \in \mathcal{O}^l V$, $T = T_J \overrightarrow{e^J}$, $T(R) = T_I R^l$, and

$$T_J R^l = \sum_I \frac{|I|!}{I!} T_I R^l \quad \text{(no sum on } I) \quad (2.19)$$

So setting,

$$\overrightarrow{R}^I = \frac{|I|!}{I!} R^I, \quad \overrightarrow{R}^J = \frac{J!}{|J|!} R^J, \quad (2.20)$$

one has

$$T_J R^l = T_I \overrightarrow{R}^I, \quad T_J \overrightarrow{R}^J = T_I R^l = T(R). \quad (2.21)$$

The last expression may be interpreted as a statement that the components $\overrightarrow{R}^J$ represent the inclusion of the symmetric tensor $R$ in the space of all tensors.

We will refer to elements of $(\mathcal{O}^{l-1} V^*) \otimes V^*$ as *almost symmetric* tensors. It can be shown [14] that for an almost symmetric tensor $T$,

$$\mathcal{F}(T) = \frac{1}{I} \sum_{p \in \tilde{\mathcal{P}}_I} p(T), \quad (2.22)$$

where $\tilde{\mathcal{P}}_I$ contains permutations $p$ such that $p(l) = 1, \ldots, l$, and $p(1) < \cdots < p(l-1)$, that is, $p$ switches $l$ with another number and then orders the first $l-1$ numbers.

For a symmetric array $w_I$ and an array $T^J$, $|I| = |J|$, we will encounter below expressions such as

$$T^I w_I = T^{(I)} w_I, \quad \text{and} \quad T^{IJ} w_{JJ} = T^{(JJ)} w_{JJ}. \quad (2.23)$$

We wish to find the relation between these two expressions. Note that each pair $J j$, determines by ordering a unique $I$ represented by $(J_1, \ldots, J_j + 1, \ldots, J_n)$.

Consider a particular $I$ represented by $(I_1, \ldots, I_n)$. Let

$$\{I\} = \{r \mid I_r > 0\}, \quad c(I) = \text{cardinality}\{I\}, \quad (2.24)$$

*i.e.*, $c(I)$ is the number of indices that appear in $I$ one time or more. Since a pair $J j$ determines a unique $I$, we may also write $c(J j)$. Let $j \in \{I\}$, then,

$$J j, \quad \text{with } J = (I_1, \ldots, I_j - 1, \ldots, I_n) \quad (2.25)$$

will give $I$ upon rearranging. Thus, for each $I$ there are exactly $c(I)$ pairs $J j$ that may be rearranged to give $I$. It follows that

$$T^{JJ} w_{JJ} = \sum_{j \in \{I\}} T^{JJ} w_{JJ} = c(I) T^I w_I \quad \text{(no sum on } I). \quad (2.26)$$

Define the arrays $\overrightarrow{T}^J J$ and $\overrightarrow{T}^I$ by

$$\overrightarrow{T}^J J = \frac{1}{c(J j)} T^{JJ}, \quad \text{and} \quad \overrightarrow{T}^I = c(I) T^I. \quad (2.27)$$
It is concluded that

\[ T^J w_J = T^I w_I, \quad \text{and} \quad T^J w_J = T^I w_I. \]  \hspace{1cm} (2.28)

### 2.5 Jets

For a fiber bundle \( \xi : Y \rightarrow X \), we will denote by \( \xi^k : J^k(X, Y) \rightarrow X \) the corresponding \( k \)-jet bundle of sections of \( \xi \). When no ambiguity may occur, we will often use the simpler notation \( \xi^k : J^k Y \rightarrow X \). One has the additional natural projections \( \xi^k : J^k(X, Y) \rightarrow J^l(X, Y), l < k \), and in particular \( \xi^k_0 : J^k(X, Y) \rightarrow Y \), [23].

Let \( \eta : W \rightarrow X \) be a vector bundle and let \( \eta^k : J^k W \rightarrow X \) be the corresponding \( k \)-jet bundle. For \( l < k \), the \( l \)-vertical subbundle of the jet bundle is defined by

\[ V^l J^k W = \ker \eta^k = \{ \dot{A} \in J^k W \mid \eta^k_l (\dot{A}) = 0 \}. \]  \hspace{1cm} (2.29)

Evidently, \( V^l J^k W \) is a vector subbundle of the jet bundle and we denote the inclusion by

\[ \iota_{V^l} : V^l J^k W \rightarrow J^k W. \]  \hspace{1cm} (2.30)

### 2.6 Pullback of Forms

Here we use “\#” to indicate the pullback of a form by duality, rather than using “\(*\)”. The latter is used to indicate the pullback of a form viewed as a section of the pullback bundle.

### 3 Continuum Mechanics and Field Theories on \( k \)-Jet Bundles

The fundamental object we consider is a fiber bundle

\[ \xi : Y \rightarrow X \]  \hspace{1cm} (3.1)

where \( X \) is an \( n \)-dimensional orientable manifold and the typical fiber is an \( m \)-dimensional manifold. For the sake of simplicity, it is assumed that \( X \) is compact. This fiber bundle may have the following interpretations.

In Lagrangian continuum mechanics, it is usually assumed that the total space \( Y \) is trivial, \( i.e., \)

\[ Y = X \times \mathcal{S} \]  \hspace{1cm} (3.2)

where the manifold \( X \) is interpreted as the body manifold and the manifold \( \mathcal{S} \) is interpreted as the space manifold. In this case, a section \( \kappa : X \rightarrow Y \) may be identified with a mapping \( \mathcal{S} \rightarrow \mathcal{S} \). Such a mapping is interpreted as a configuration of the body in the physical space.

In Eulerian continuum mechanics, \( X \) is interpreted as the physical space and the fibers of \( Y \) are interpreted as the possible values that fields over the space may have. For classical field theories, \( X \) is interpreted as space-time. (See [13] for further motivation, references and examples.)
3.1 The Configuration Space

A configuration of the system, or a field, is a section $\kappa : \mathcal{X} \to \mathcal{Y}$. In accordance with the paradigm of analytical mechanics, a central role is played by the configuration space of the system, an infinite dimensional object in the case of a field theory. We are motivated by the case of continuum mechanics where one requires traditionally that configurations of a body in space be embeddings, and the fact that the subset of embeddings is open in the manifold of all $C^k$-mappings between two manifolds, for $k \geq 1$. (See [11, 15].) Thus, for $k \geq 1$, we consider the collection of $C^k$-sections of $\xi$, equipped with the $C^k$-topology, and define the configuration space $\mathcal{Q}$ to be an open subset of the manifold of sections, i.e.,

$$\mathcal{Q} \subset C^k(\xi) := C^k(X, Y),$$

(3.3)

Note that by the notation $C^k(X, Y)$ we refer only to sections of the fiber bundle rather than all mappings $X \to Y$.

3.2 Generalized Velocities

Generalized velocities (virtual velocities, virtual displacements) at the configuration $\kappa$, or variations of the configuration $\kappa$, are elements of the tangent space $T_\kappa \mathcal{Q}$. Since $\mathcal{Q}$ is assumed to be open,

$$T_\kappa \mathcal{Q} = T_\kappa C^k(\mathcal{X}, \mathcal{Y}).$$

(3.4)

Furthermore, let

$$V_\xi = V_{\mathcal{Y}} := \text{Kernel } T_\xi$$

(3.5)

be the vertical subbundle of $T \mathcal{Y}$. Then (see [20, p. 51]),

$$T_\kappa \mathcal{Q} = T_\kappa C^k(\mathcal{X}, \mathcal{Y}) \simeq C^k(\mathcal{X}, \kappa^*V_{\mathcal{Y}}),$$

(3.6)

where $\kappa^*V_{\mathcal{Y}}$ denotes the pullback of the vertical bundle onto $\mathcal{X}$. Thus, every generalized velocity may be identified uniquely with a $C^k$-section

$$w : \mathcal{X} \longrightarrow \kappa^*V_{\mathcal{Y}},$$

(3.7)

in accordance with the traditional interpretation in continuum mechanics. (It is noted that in [20], $V_{\mathcal{Y}}$ is denoted by $TF(\mathcal{Y})$ and $\kappa^*V_{\mathcal{Y}}$ is denoted by $T_\kappa \mathcal{Y}$.)

Thus,

$$TC^k(\mathcal{X}, \mathcal{Y}) \simeq C^k(\mathcal{X}, V_{\mathcal{Y}})$$

(3.8)

and a section $w$ represents an element of $T_\kappa \mathcal{Q}$ if

$$\tau_{\mathcal{Y}}|_{V_{\mathcal{Y}}} \circ w = \kappa,$$

(3.9)

where $\tau_{\mathcal{Y}} : T\mathcal{Y} \longrightarrow \mathcal{Y}$ is the natural tangent bundle projection. In the sequel, in order to simplify the notation, we will denote the vector bundle

$$\xi \circ \tau_{\mathcal{Y}}|_{V_{\mathcal{Y}}} : \kappa^*V_{\mathcal{Y}} \longrightarrow \mathcal{X} \text{ by } \pi : W \longrightarrow \mathcal{X}.$$

(3.10)

Using local coordinates $(x^i, y^a)$ in $\mathcal{Y}$, an element of $T\mathcal{Y}$ is represented locally in the form $\dot{x}^i \partial_i + \dot{y}^a \partial_a$. For an element of $V\mathcal{Y}$, $\dot{x}^i = 0$, and so, an element of $\kappa^*V_{\mathcal{Y}} = W$ is represented...
locally in the form \( \dot{y}^a \kappa^* \partial_a \), which we may write also simply as \( \dot{y}^a e_a \), \( e_a := \kappa^* \partial_a \). The dual bases will be denoted accordingly. However, in what follows we often identify \( \kappa^* \partial_a \) with \( \partial_a \) in the notation.

### 3.3 Generalized Forces and Their Representations by Variational Hyper-stresses

A generalized force at a configuration \( \kappa \) is an element of \( T^*_\kappa \mathcal{Q} \). As such, a force \( F \) at \( \kappa \) is a continuous linear functional in \( C^k(\mathcal{Q}, \kappa^* V \mathcal{Y})^* \). Henceforth, we will use the concise notation and will write

\[
T^*_\kappa \mathcal{Q} = C^k(\kappa^* V \mathcal{Y}) = C^k(\kappa^* V \mathcal{Y})^* = C^k(W)^*, \quad F \in C^k(\kappa^* V \mathcal{Y})^* = C^k(W)^*.
\] (3.11)

We observe that the jet extension mapping

\[
j^k : C^k(W) \longrightarrow C^0(J^kW), \quad w \mapsto j^k w,
\] (3.12)

is an embedding. (Local representations of the various variables and mappings are provided below.) That is, if an atlas is used in order to define compatible Banach space structures on \( C^k(W) \) and \( C^0(J^kW) \), then \( j^k \) is an isometry (see [20]). Note that \( j^k \) is not surjective as non-holonomic sections are not included in the image. It follows from the Hahn-Banach theorem that its dual mapping,

\[
j^{k*} : C^0(J^kW)^* \longrightarrow C^k(W)^*,
\] (3.13)

is surjective. One concludes that for a given force \( F \), there is some element \( \varsigma \in C^0(\mathcal{Q}, J^kW)^* \) such that

\[
F = j^{k*} \varsigma \quad \text{or} \quad F(w) = \varsigma(j^k w),
\] (3.14)

for every \( w \in C^k(W) \). In their form, (3.14) generalize the equation of equilibrium and the expression for the principle of virtual work of continuum mechanics. Specifically, it follows that the power expended by the stress on the jet of a velocity field is equal to the power expended by the force for that velocity field. We obtained equations that are usually considered as representing laws of mechanics from a mathematical representation theorem. It is noted, however, that the stress object contains terms that are not included in the standard treatment. For example, we did not require (nor can we require without an additional geometric structure) that the power vanishes for some particular classes of vector fields (e.g., translations and rotations). In addition, we have no notions such as “internal force” and “external forces”.

The fact that \( j^k \) is not surjective, implies that the representation is not unique. This non-uniqueness of the stress representation of a given force is a generalization of the well-known static indeterminacy of continuum mechanics.

Since \( \varsigma \) is a linear functional on the space of continuous sections of the vector bundle \( \pi^k : J^kW \rightarrow \mathcal{Q} \), it may be represented by a measure \( \mu \) valued in \( (J^kW)^* \) in the form

\[
\varsigma(\dot{A}) = \int_{\mathcal{Q}} \dot{A} \cdot d\mu
\] (3.15)

for any section \( \dot{A} \) of \( \pi^k \).
For a chart in $\mathcal{D}$ with coordinates $(x^i)$ and a local basis $\{e_a\}$ for $W$, a section of $W$ is represented locally in the form

$$w = w^ae_a$$

and its jet is represented in the form

$$j^k w = w^ae_a^I, \quad 0 \leq |I| \leq k,$$

where

$$e_a^I := \frac{\partial}{\partial x^i} \otimes e_a.$$

Thus, a section $\dot{A}$ of the jet bundle is represented in the form

$$\dot{A} = \dot{A}_I^ae_a^I, \quad 0 \leq |I| \leq k.$$

Locally, $\mu$ is represented by a collection of real valued measures $\mu^I_a, |I| \leq k$ so that

$$\dot{A} \cdot d\mu = \dot{A}_I^a \cdot d\mu^I_a.$$

The representing elements $\zeta \in C^0(J^kW)^*$ are the hyper-stresses as implied by the principle of virtual work above. Furthermore, their role as a means for restriction of forces results from their representation by measures. Thus, a hyper-stress $\zeta$ represented by the measure $\mu$, induces for any $n$-dimensional submanifold $\mathcal{R}$, a force $F_{\mathcal{R}}$ by

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} j^k w \cdot d\mu.$$ (3.21)

Hence, while forces cannot be restricted to submanifolds, the importance of the representation by hyper-stresses follows from the fact that measures may be restricted to Borel sets.

### 3.4 Smooth Variational Hyper-stresses

In what follows, we consider a smooth configuration $\kappa$ and set $W = \kappa^*V\mathcal{Y}$. Furthermore, the focus of this work is placed on smooth force distributions and hyper-stresses, in other words, forces and hyperstresses represented by smooth densities. Consider the vector bundle of linear mappings

$$L\left(J^kW, \wedge^nT^*\mathcal{X}\right) \simeq (J^kW)^* \otimes \mathcal{X} \wedge^nT^*\mathcal{X}.$$ (3.22)

A smooth variational hyper-stress field is a smooth section $S$ of $L(J^kW, \wedge^nT^*\mathcal{X})$. Alternatively, a variational hyper-stress field may be viewed as a smooth vector bundle morphism

$$S : J^kW \longrightarrow \wedge^nT^*\mathcal{X},$$ (3.23)

where both bundles have $\mathcal{X}$ as the base manifold. A variational hyper-stress field $S$ induces a stress $\zeta$ by

$$\zeta(\dot{A}) = \int_{\mathcal{X}} S \cdot \dot{A}.$$ (3.24)
The integral on the right hand side makes sense as \( S \cdot \dot{A} \) is an \( n \)-form on \( \mathcal{R} \). In particular, the power that a force \( F \) expends for the generalized velocity \( w \) is given by

\[
F(w) = \varsigma(j^k w) = \int_{\mathcal{R}} S \cdot j^k w, \tag{3.25}
\]

and the restriction of the force, induced by the stress, to the subbody \( \mathcal{R} \) is given by

\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} S \cdot j^k w. \tag{3.26}
\]

To consider the local representation of the variational hyper-stress densities, we first note that as the dual basis to \( \{\frac{\partial}{\partial x^I}\} \) is \( \{\partial_I\} \), the dual basis corresponding to \( \{e^\alpha_I\} \) is \( \{e^\alpha_I := \partial_I \otimes e^\alpha\} \).

\[
(3.27)
\]

It follows that a stress density is represented locally in the form

\[
S^I_a e^\alpha_I \otimes dx, \quad 0 \leq |I| \leq k, \tag{3.28}
\]

and the action of the variational stress density on a section \( \dot{A} \) of the jet bundle is given by

\[
S \cdot \dot{A} = S^I_a \dot{A}^\alpha_I dx, \quad \text{in particular,} \quad S \cdot j^k w = S^I_a w^\alpha_I dx. \tag{3.29}
\]

Consider next the transformation rule for the components \( S^I_a \). Let \( x'^I(x') \) be a transformation of coordinates in a subset of \( \mathcal{R} \) and \( w'^\alpha = \dot{A}'^\alpha_I(x') w^{\alpha} \) be the coordinate transformation in \( W \). It is noted that by definition, the transformation rule for the components of elements of \( J^k W \), may be written in the form

\[
w'^\alpha_I = G'^\alpha_I \cdot w^{\alpha}_I, \quad |I| \leq |I'|. \tag{3.30}
\]

where \( G'^\alpha_I \) contains derivatives of the aforementioned transformations.

Using the transformation rules (3.30) and denoting the Jacobian determinant of the transformation by \( \mathcal{J} \), we may write

\[
S^I_a w^{\alpha}_I dx = S'^I_a w'^\alpha_I dx', \quad |I| \leq |I'|,
\]

\[
= S'^I_a G'^\alpha_I \cdot w^{\alpha}_I dx. \tag{3.31}
\]

The independence of the point values of the derivatives \( w^\alpha_I \) imply that the transformation rule for the components \( S^I_a \) is

\[
S^I_a = \mathcal{J} S'^I_a G'^\alpha_I, \quad |I| \leq |I'| \leq k. \tag{3.32}
\]

We conclude that the transformation of the components of the variational stress densities of order \( l \) involves all components of equal or higher order \( l \leq r \leq k \). In particular, the statement that all the components of order \( r > l \) vanish, is invariant. This, of course, gives meaning to the statement that a material is of order \( k \) and not any higher order.
4 Hyper-stresses, the Cauchy Approach

4.1 Body Hyper-forces

A body hyper-force density is an element of $L(J^{k-1}W, \bigwedge^n T^*X)$. A body hyper-force field is a section of $L(J^{k-1}W, \bigwedge^n T^*X)$. Given a body force field $b$ and a subbody $\mathcal{R} \subset X$, the total power of the body hyper-force is

$$F_b(\dot{A}) = \int_{\partial \mathcal{R}} b \cdot \dot{A},$$

(4.1)

for any section $\dot{A}$ of $J^{k-1}W$. In particular, for a section $w$ of $W$,

$$F_b(j^{k-1}w) = \int_{\partial \mathcal{R}} b \cdot j^{k-1}w.$$  

(4.2)

Using the notation introduced above, a body hyper-force field is represented locally in the form

$$b = b^J_\alpha e^\alpha_J \otimes dx, \quad 0 \leq |J| \leq k - 1.$$  

(4.3)

The action of a body force is $b \cdot \dot{A} = b^J_\alpha \dot{A}^\alpha_J dx$, and in particular, $b \cdot j^k w = b^J_\alpha w^\alpha_J dx, \quad 0 \leq |J| \leq k - 1$.  

(4.4)

Consider next the transformation rule for the components $b^J_\alpha$. Using (3.30), one has

$$b^J_\alpha w^\alpha_J dx = b'^J_\alpha' w'^\alpha_J' dx', \quad |J| \leq |J'| \leq k - 1,$$

$$= b'^J_\alpha' G^\alpha_J J^\alpha_{J'} w'^\alpha_J' dx' \mathcal{J}.$$  

(4.5)

The independence of the point values of the derivatives $w^\alpha_J$ imply that the transformation rule for the components $S^J_\alpha$ is

$$b^J_\alpha = \mathcal{J} b'^J_\alpha' G^\alpha_J J^\alpha_{J'}.$$

(4.6)

Similarly to the variational stress, we conclude that the transformation of the components of the body hyper-force densities of order $l$ involves all components of equal or higher order $l \leq r \leq k$. In particular, the statement that all the components of order $r > l$ vanish, is invariant.

Thus, for $k$-th order continuum mechanics, body hyper-forces are like variational hyper-stresses for continuum mechanics of order $k - 1$. All the analysis concerning variational hyper-stresses applies to body hyper-forces. It observed that in case a metric structure is provided, body hyper-forces may include densities of couples.

4.2 Hyper-traction Fields

Let $\mathcal{R} \subset X$ be a subbody. A surface hyper-force density or a hyper-traction on $\partial \mathcal{R}$ is an element of $L(J^{k-1}W, \bigwedge^{n-1} T^*\partial \mathcal{R})$. Here, and in what follows, we view $L(J^{k-1}W, \bigwedge^{n-1} T^*\partial \mathcal{R})$ as a vector bundle over $\partial \mathcal{R}$ and in our notation we omit the indication that $J^{k-1}W$ is restricted to $\partial \mathcal{R}$. A hyper-traction field on $\partial \mathcal{R}$ is a section of
Given a subbody \( \mathcal{R} \subset \mathcal{X} \) and a hyper-traction field \( t_{\mathcal{R}} \) and, the total power of the hyper-traction is the action of the functional

\[
F_t(\dot{A}) = \int_{\partial \mathcal{R}} t_{\mathcal{R}} \cdot \dot{A},
\]

for any section \( \dot{A} \) of \( (J^{k-1}W)|_{\partial \mathcal{R}} \). In particular, for a section \( w \) of \( W \),

\[
F_t(j^{k-1}w) = \int_{\partial \mathcal{R}} t_{\mathcal{R}} \cdot j^{k-1}w.
\]

It is noted that the jet of \( w \) is taken relative to \( \mathcal{X} \) and not \( \partial \mathcal{R} \).

4.3 Smooth Force Functionals and Hyper-force Systems

We will say that a force functional on a subbody \( \mathcal{R} \) is smooth if it is induced by a body hyper-force field \( b \) and a hyper-traction field \( t_{\mathcal{R}} \) in the form

\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} b \cdot j^{k-1}w + \int_{\partial \mathcal{R}} t_{\mathcal{R}} \cdot j^{k-1}w
\]

for every virtual displacement \( w : \mathcal{X} \to W \). Henceforth, we will be interested only in smooth force functionals.

Let \( \{t_{\mathcal{R}}\} \) be a collection of sections \( t_{\mathcal{R}} \) of \( L(U|_{\partial \mathcal{R}}, \wedge^{n-1}T^*\partial \mathcal{R}) \) for all subbodies \( \mathcal{R} \subset \mathcal{X} \). We will refer to \( \{t_{\mathcal{R}}\} \) as a system of hyper-tractions. The collection \( (b, \{t_{\mathcal{R}}\}) \) will be referred to as a smooth system of hyper-forces, where each force functional in \( \{F_{\mathcal{R}}\} \) is represented as in the equation above.

4.4 Traction Hyper-stresses

A traction hyper-stress is an element

\[
\sigma_0 \in L\left(J^{k-1}W, \wedge^{n-1}T^*\mathcal{X}\right)
\]

and a traction hyper-stress field is a smooth section \( \sigma \) of \( L(J^{k-1}W, \wedge^{n-1}T^*\mathcal{X}) \).

Let \( \mathcal{R} \subset \mathcal{X} \) be an \( n \)-dimensional submanifold with boundary \( \partial \mathcal{R} \). The inclusion

\[
t_{\partial \mathcal{R}} : \partial \mathcal{R} \to \mathcal{X},
\]
induces the following diagram where \((\iota_{\partial \mathcal{R}})^* T \mathcal{X} \simeq T \mathcal{X}|_{\partial \mathcal{R}}\) is the pullback of the tangent bundle and \(\delta \iota_{\partial \mathcal{R}}\) is the induced vector bundle morphism over the boundary that makes the diagram commutative—an inclusion of tangent vectors.

The dual vector bundle morphism,

\[
\rho_{\partial \mathcal{R}} = (\delta \iota_{\partial \mathcal{R}})^*: \left( \bigwedge^{n-1} T^* \mathcal{X} \right)_{\partial \mathcal{R}} \simeq \bigwedge^{n-1} \left( \iota_{\partial \mathcal{R}}^* T \mathcal{X} \right)^* \longrightarrow \bigwedge^{n-1} T^* \partial \mathcal{R},
\]

restricts alternating tensors to vectors tangent to the boundary and it induces the pullback of forms.

Let \(z \in \partial \mathcal{R}\), and \(\sigma_0 \in L(J^{k-1} W, \bigwedge^{n-1} T^* \partial \mathcal{R})_z\). Then,

\[
t_0 = \rho_{\partial \mathcal{R}} \circ \sigma_0 \in L(J^{k-1} W, \bigwedge^{n-1} T^* \partial \mathcal{R})_z
\]

is a hyper-traction induced at \(z\) by \(\sigma_0\). Similarly, a traction hyper-stress field \(\sigma\) induces a hyper-traction field

\[
t = \rho_{\partial \mathcal{R}} \circ \sigma = \iota^* \rho_{\partial \mathcal{R}} \sigma.
\]

Thus, (4.17) is a generalization of the traditional Cauchy formula.

Let \(v \cdot \omega\) denote the contraction (inner product) of the form \(\omega\) with the tangent vector \(v\). As \(\{\partial_\alpha dx\}\) may serve as a basis for \(\bigwedge^{n-1} T^* \mathcal{X}\), in analogy with (3.28), a traction hyper-stress field may be represented in the form

\[
\sigma = \sigma^J_a e^\alpha_j \otimes (\partial_\alpha dx), \quad |J| \leq k - 1.
\]

For a section \(\dot{B}\) of \(J^{k-1} W\), one has therefore,

\[
\sigma \cdot \dot{B} = \sigma^J_a \dot{B}^j_x (\partial_\alpha dx),
\]

and in particular,

\[
\sigma \cdot j^{k-1} w = \sigma_a^J w^\alpha_j (\partial_\alpha dx), \quad |J| \leq k - 1.
\]

Consider next the transformation rule for the components \(\sigma^J_a\). Using the transformations (3.30) and the invariance of the action, we have

\[
\sigma^J_a w^\alpha_j \partial_\alpha dx = \sigma'^{J'}_{a'} w'^\alpha'_{j'} \partial'^{J'}_{\alpha'} dx', \quad |I| \leq |I'|,
\]

\[
= \sigma'^{J'}_{a'} G^I_{a'} w'^\alpha'_{j'} x'^I_i \partial_\alpha dx'.
\]
Thus,
\[ \sigma_a^I = \mathcal{J} \sigma_a^I' G_{I'a'}^I \chi_{x'}^I, \quad |I| \leq |I'|. \] (4.22)

The last equation is a generalization of the classical relation between the Cauchy stress and the first Piola-Kirchhoff stress, which as presented here, is just a transformation of coordinates of the right geometric object. Once again, the transformation of the components of the traction hyper-stress densities of order \( l \) involves all components of equal or higher order \( l \leq r \leq k - 1 \).

Finally, given a traction hyper-stress, \( \sigma \), and a body hyper-force, \( b \), one may write the force functional on each subbody as
\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} b \cdot j^{k-1} w + \int_{\partial \mathcal{R}} \sigma \cdot j^{k-1} w, \\
= \int_{\mathcal{R}} b \cdot j^{k-1} w + \int_{\mathcal{R}} d(\sigma \cdot j^{k-1} w). \tag{4.23}
\]

### 4.5 Cauchy’s Theorem and Traction Hyper-stresses

We review here Cauchy’s theorem on manifolds as in [25, 28]. The Cauchy theorem asserts that under certain boundedness and locality conditions, a system of forces on all subbodies induce a unique traction stress field. The theorem is formulated for a general vector bundle \( U \) and the terminology used is for the case of standard continuum mechanics, \( k = 1 \). We will use it later for the case \( U = J^{k-1} \mathcal{W} \).

It is assumed in the sequel that the collection of subbodies of \( \mathcal{X} \) includes \( n \)-dimensional chains and in particular simplices. Furthermore, it is assumed that a particular orientation has been chosen for the manifold \( \mathcal{X} \).

**Definition 1** We will say that a system \( \{ t_\mathcal{R} \} \) of tractions is **consistent** if the following conditions are satisfied.

1. **Boundedness.** There is a section \( \xi \) of \( L(J^1 U, \bigwedge^n T^* \mathcal{X}) \) such that for each \( \mathcal{R} \),
   \[
   \left| \int_{\partial \mathcal{R}} t_\mathcal{R}(u) \left|_{\partial \mathcal{R}} \right. \right| \leq \int_{\mathcal{R}} |\xi(j^1 u)|, \tag{4.24}
   \]
   for every smooth section \( u \) of \( U \).

2. **Cauchy’s postulate of locality.** Let \( v_1, \ldots, v_{n-1} \in T_x \mathcal{X} \) be a collection of tangent vectors at \( x \in \mathcal{X} \). Let \( \mathcal{R} \) be any subbody such that \( v_1, \ldots, v_{n-1} \in T_x \partial \mathcal{R} \), and the collection of vectors is positively oriented relative to the orientation of \( \partial \mathcal{R} \). Then, \( t_\mathcal{R}(u)(v_1, \ldots, v_{n-1}) \) is independent of \( \mathcal{R} \). Thus, there is an alternating multilinear mapping
   \[
   \Sigma_\xi : (T_x \mathcal{X})^{n-1} \rightarrow U_x^* \tag{4.25}
   \]
   such that
   \[
   t_\mathcal{R}(u)(v_1, \ldots, v_{n-1}) = \Sigma_\xi(v_1, \ldots, v_{n-1})(u), \quad \text{for all } u \in U_x. \tag{4.26}
   \]

3. **Regularity.** The mapping \( x \mapsto \Sigma_\xi \) is smooth.

A smooth system of forces \( (b, \{ t_\mathcal{R} \}) \) is said to be consistent if \( \{ t_\mathcal{R} \} \) is a consistent system of tractions.
Theorem 2 Let \( \{ t_\mathcal{R} \} \) be a consistent force system, then, there is a unique traction stress field \( \sigma \) such that for each subbody \( \mathcal{R} \), the surface force is given by the Cauchy formula

\[
t_{\mathcal{R}} = \rho_{\mathcal{R}} \circ \sigma.
\]  

(4.27)

Cauchy’s theorem, in the generalized setting outlined above, applies readily to traction hyper-stresses by setting \( U = J^{k-1}W \). Assuming that the system of hyper-tractions satisfies the conditions above, it follows from Cauchy’s theorem that there is a unique traction hyper-stress that induces it using (4.27).

5 Non-holonomic Variational Stresses

Unlike the simple case, \( k = 1 \), variational hyper-stresses do not determine uniquely traction hyper-stresses for higher order theories. Another mathematical object is needed—the non-holonomic stress.

5.1 The Exterior Jet and Non-holonomic Variational Stresses

Proposition 3 There is a natural linear, first order-differential operator, the exterior jet \( \partial \), taking sections of \( L(J^{k-1}W, \bigwedge^{n-1} T^* \mathcal{X}) \) into sections of \( L(J^1(J^{k-1}W), \bigwedge^n T^* \mathcal{X}) \), defined by the condition

\[
\partial \sigma \cdot j^1 \dot{A} = d(\sigma \cdot \dot{A})
\]  

(5.1)

for every section \( \dot{A} \) of \( J^{k-1}W \).

Proof Let \( \sigma \) be a traction hyper-stress field represented as in (4.18) and let a generic section \( \dot{A} \) of \( J^{k-1}W \) be given locally by \( \dot{A} = \dot{A}_J e_J^a \). One has

\[
d(\sigma \cdot \dot{A}) = (\sigma^J_{a,b} \dot{A}_J^a + \sigma_a^J J^a_{J,b}) dx, \quad |J| \leq k - 1.
\]  

(5.2)

It follows that \( d(\sigma \cdot \dot{A}) \) is an \( n \)-form that depends pointwise on the values of the representatives of \( \dot{A} \) and their first derivatives, i.e., it depends on \( j^1 \dot{A} \). Hence, there is a unique section \( \partial \sigma \) of \( L(J^1(J^{k-1}W), \bigwedge^n T^* \mathcal{X}) \), for which condition (5.1) holds. The value of the section \( \partial \sigma \) at a point \( x \in \mathcal{X} \) depends linearly on the first jet at \( x \) of the section \( \sigma : \mathcal{X} \to L(J^{k-1}W, \bigwedge^{n-1} T^* \mathcal{X}) \). Hence, \( \partial \) is a first order linear differential operator. \( \square \)

Definition 4 An element of \( L(J^1(J^{k-1}W), \bigwedge^n T^* \mathcal{X}) \) will be referred to as a non-holonomic (variational hyper-) stress. A section of \( L(J^1(J^{k-1}W), \bigwedge^n T^* \mathcal{X}) \) is a non-holonomic stress field.

Thus, \( \partial \sigma \) is a non-holonomic stress field. As noted, the terminology follows from the fact that an element of \( J^1(J^{k-1}W) \) need not be associated with the value of a \( k \)-th jet of a section of \( W \).
5.2 Local Representation of Non-holonomic Stresses and Exterior Jets

An element $\dot{B} \in J^1(J^{k-1}W)$ is represented locally in the form

$$\dot{B} = \dot{B}_a^a e_a^J + \dot{B}_a^J e_a^J \otimes dx^j,$$

(5.3)

where it is noted that while $\dot{B}_a^a$ and $\dot{B}_a^J$ are symmetric under permutations of $J$, $\dot{B}_a^J$ need not be symmetric under permutations of $J$. For a section $\dot{A}$ of $J^{k-1}W$, represented locally in the form $\dot{A} = \dot{A}_a^a e_a^J$, one has locally

$$j^1 \dot{A} = \dot{A}_a^a e_a^J + \dot{A}_a^J e_a^J \otimes dx^j,$$

(5.4)

and for a section $w$ of $W$,

$$j^1(j^{k-1}w) = w_a^a e_a^J + w_a^J e_a^J \otimes dx^j.$$

(5.5)

Evidently, $j^1(j^{k-1}w)$ represents a holonomic element of $J^1(J^{k-1}W)$ for which complete symmetry holds.

It follows that a non-holonomic stress should be locally of the form

$$P = (P_a^a e_a^J + P_a^J e_a^J \otimes \partial_j) \otimes dx.$$

(5.6)

Its action is given by

$$P(\dot{B}) = (P_a^a \dot{B}_a^a + P_a^J \dot{B}_a^J)dx, \quad P(j^1 \dot{A}) = (P_a^a \dot{A}_a^a + P_a^J \dot{A}_a^J)dx,$$

(5.7)

and

$$P(j^1(j^{k-1}w)) = (P_a^a w_a^a e_a^J + P_a^J w_a^J e_a^J)dx.$$

(5.8)

Given a traction hyper-stress field $\sigma$, the exterior jet field should be of the form

$$\partial \sigma = ((\partial \sigma)_a^J e_a^J + (\partial \sigma)_a^J e_a^J \otimes \partial_j) \otimes dx.$$

(5.9)

The definition of the exterior jet implies that for any section $\dot{A}$ of $J^{k-1}W$, $\partial \sigma \cdot j^1 \dot{A} = d(\sigma \cdot \dot{A})$, and so

$$(\partial \sigma)_a^J \dot{A}_a^a + (\partial \sigma)_a^J \dot{A}_a^J = \sigma_{a,J}^J \dot{A}_a^a + \sigma_{a,J}^J \dot{A}_a^J.$$

(5.10)

At any point $x$, the values of the components $\dot{A}_a^a$ and $\dot{A}_a^J$ are independent. Thus,

$$\partial \sigma = (\sigma_{a,J}^J e_a^J + \sigma_{a,J}^J e_a^J \otimes \partial_j) \otimes dx.$$

(5.11)

5.3 The Non-holonomic Stress Induced by a Consistent Hyper-force System

We now consider systems consisting of body hyper-forces and hyper-tractions as in (4.12). We will say that the hyper-force system is consistent if it satisfies the conditions of consistency of the generalized Cauchy theorem for the case $U = J^{k-1}W$. 

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**Proposition 5** A smooth consistent hyper-force system \((b, \{ t_R \})\) induces a unique non-holonomic stress field \(P\) which represents it by

\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} P \cdot j^1(j^{k-1}w).
\]  

**(Proof)** A consistent hyper-force system induces, by the generalized Cauchy theorem for the case \(U = J^{k-1}W\), a unique traction hyper-stress \(\sigma\). Thus, (4.23) applies, and using the definition of the exterior jet, we may rewrite it as

\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} b \cdot j^1(j^{k-1}w) + \partial \sigma \cdot j^1(j^{k-1}w). \tag{5.14}
\]

Observing (5.14), the value of the integrand at any \(x \in \mathcal{X}\) is an \(n\)-alternating tensor that is evidently a linear function of \(j^1(j^{k-1}w)(x)\). For a section \(A\) of \(J^{k-1}W\), set

\[
P \cdot j^1\dot{A} = b \cdot \dot{A} + \partial \sigma \cdot j^1\dot{A}. \tag{5.15}
\]

Thus, \(P\) is a section of \(L(J^1(J^{k-1}W), \wedge^n T^* \mathcal{X})\) such that

\[
F_{\mathcal{R}}(w) = \int_{\mathcal{R}} P \cdot j^1(j^{k-1}w). \tag{5.16}
\]

The local representation of \(P\) may be readily obtained from (5.12) as

\[
P = ((\sigma^J_{a,j} + b^J_a)e^a_j + \sigma^J_{a,j}e^a_j \otimes \partial_j) \otimes dx, \quad 0 \leq |J| \leq k-1, \tag{5.18}
\]

\[
P^J_a = \sigma^J_{a,j} + b^J_a, \quad P^J_a = \sigma^J_{a,j}. \tag{5.19}
\]

**Proposition 6** There is a natural mapping

\[
p_{\sigma} : L(J^1(J^{k-1}W), \wedge^n T^* \mathcal{X}) \longrightarrow L(J^{k-1}W, \wedge^{n-1} T^* \mathcal{X}), \tag{5.20}
\]

whereby a non-holonomic stress field \(P\) induces a unique traction hyper-stress \(\sigma\).

**(Proof)** The local expression for the action of \(P\) is

\[
P \cdot j^1\dot{A} = (P^J_a \dot{A}^a_j + P^J_a \dot{A}^a_j) \otimes dx, \quad 0 \leq |J| \leq k-1. \tag{5.21}
\]

Restricting \(P(x)\) to sections for which \(\dot{A}^a_j(x) = 0\) for all \(J, |J| \leq k-1, i.e., to elements of

\[
V^0 J^1(J^{k-1}W) = \text{Kernel } \pi^1_0, \tag{5.22}
\]
the action of \( P \) determines \( \sigma \) uniquely. In other words, the inclusion \( \iota_1^0 \) of the vertical bundle induces

\[
\iota_1^0 : L\left( J^1(J^{k-1}W), \bigwedge^n T^*\mathcal{X} \right) \longrightarrow L\left( VJ^1(J^{k-1}W), \bigwedge^n T^*\mathcal{X} \right),
\]

\[
\simeq L\left( (T\mathcal{X}, Jk^{-1}W)^* \otimes \bigwedge^n T^*\mathcal{X}, \right),
\]

\[
\simeq \left( J^{k-1}W \right)^* \otimes \bigwedge^n T^*\mathcal{X},
\]

\[
\simeq L\left( J^{k-1}W, \bigwedge^{n-1} T^*\mathcal{X} \right),
\]

(5.23)

where we used the natural isomorphism \( VJ^1W \simeq L(T\mathcal{X}, U) \) for any vector bundle \( U \) over \( \mathcal{X} \), and the isomorphism

\[
T\mathcal{X} \otimes \bigwedge^n T^*\mathcal{X} \longrightarrow \bigwedge^{n-1} T^*\mathcal{X}, \quad (v, \theta) \mapsto v \lrcorner \theta.
\]

(5.24)

Thus, locally,

\[
\sigma_{\alpha}^{Jj} = P_{\alpha}^{Jj}.
\]

(5.25)

Proposition 7 There is a natural linear differential operator \( \text{div} \) mapping jets of non-holonomic stress fields into body hyper-force fields. The differential operator \( \text{div} \) is defined by

\[
\text{div} \ P = \partial \circ p_\sigma (P) - P.
\]

(5.26)

The non-holonomic stress \( P \) represents the force system \((\mathbf{b}, \{t_I\})\) if an only if

\[
p_\sigma \circ P = \sigma, \quad \text{and} \quad \text{div} \ P + \mathbf{b} = 0.
\]

(5.27)

Proof To prove that \( \text{div} \ P \) is a section of \( L(J^{k-1}W, \bigwedge^n T^*\mathcal{X}) \), we use (5.25) and (5.12). Thus, locally, for \( 0 \leq |J| \leq k - 1 \),

\[
\text{div} \ P = \left( P_{\alpha}^{Jj} e^\alpha_j + P_{\alpha}^{Jj} e^\alpha_j \otimes \partial_j - P_{\alpha}^{Jj} e^\alpha_j - P_{\alpha}^{Jj} e^\alpha_j \otimes \partial_j \right) \otimes dx,
\]

\[
= \left( P_{\alpha}^{Jj} - P_{\alpha}^{Jj} \right) e^\alpha_j \otimes dx.
\]

(5.28)

In view of Propositions 5 and 6, it remains to show that \( \mathbf{b} = - \text{div} \ P \). This follows, however, from (5.15) and (5.26).

\( \square \)

6 Induced Variational Stresses

One could naively assume that just as in the case \( k = 1 \), a variational hyper-stress, a section of \( L(J^kW, \bigwedge^n T^*\mathcal{X}) \) will induce a unique traction hyper-stress. This, however, is not the case. It is sufficient to observe that in the local expression for the variational hyper-stress \( S^I_{\alpha} e^\alpha_I \otimes dx, |I| \leq k \), the components \( S^I_{\alpha} \) are symmetric under permutations of \( I \). On the other hand, in the local expression for the traction hyper-stress \( \sigma_{\alpha}^{Jj} e^\alpha_j \otimes (\partial_j \otimes dx), |J| \leq k - 1 \), the
arrays $\sigma^J$, which are of the same order as $S^J$, need not be symmetric under all permutations of $J_j$. In other words, unlike the non-holonomic stress that is in one-to-one correspondence with a force system, the variational hyper-stress encodes less information and cannot be used to determine the hyper-traction on the boundaries of subbodies. The present section studies these issues with some detail.

6.1 The Variational Hyper-stress Induced by a Non-holonomic Stress

Let
\begin{equation}
\iota^k_{1,k-1} : J^k W \longrightarrow J^1(J^{k-1} W), \quad j^k w(x) \longmapsto j^1(j^{k-1} w)(x), \quad (6.1)
\end{equation}
be the natural inclusion of holonomic elements. Then, the dual mapping
\begin{equation}
\iota^{k*}_{1,k-1} : L(J^1(J^{k-1} W), \bigwedge^n T^* X) \longrightarrow L(J^k W, \bigwedge^n T^* X), \quad P \longmapsto P \circ \iota^k_{1,k-1}, \quad (6.2)
\end{equation}
assigns variational hyper-stresses to non-holonomic stresses. Let $S = \iota^{k*}_{1,k-1}(P)$. The local representations of the actions of $S$ and $P$ imply that for any section $w$ of $W$,
\begin{align*}
S^J_a w^a_I &= P^J_a w^a_L + \hat{P}^J_a w^a_L, \quad 0 \leq |I| \leq k, \quad 0 \leq |J| \leq k - 1, \\
&= P_a w^a + \hat{P}^L a w^a_L + \hat{P}^L a w^a_L + \hat{P}^K a w^a_K, \quad 0 \leq |L| \leq k - 2, \quad |K| = k, \\
&= P_a w^a + (\hat{P}^L a + \hat{P}^L a) w^a_L + \hat{P}^K a w^a_K, \quad 1 \leq |J| \leq k - 1, \quad |K| = k.
\end{align*}
As the relations above should hold for any point $x \in X$ and since compatibility imposes no restrictions on the point values of the various derivatives, we conclude that $S = \iota^{k*}_{1,k-1}(P)$ is represented by
\begin{align*}
S^K_a &= \hat{P}^K a, \quad |K| = k, \\
S^J_a &= P^J_a + \hat{P}^J a, \quad 1 \leq |J| < k, \\
S_a &= P_a. \quad (6.3)
\end{align*}

Remark 8 It is emphasized that without additional geometric structure there is no projection $J^1(J^{k-1} W) \rightarrow J^k W$, an inverse to the inclusion above. (See, for example [23, p. 169].) That is, a non-holonomic section does not determine a holonomic section. As a consequence, a variational hyper-stress cannot determine by duality a unique non-holonomic stress. Accepting this somewhat disappointing obstacle, one might consider the following alternative to the procedure of representation of forces by variational stresses in Sect. 3.3. Let the completely non-holonomic, iterated jet bundle
\begin{equation}
\hat{\pi}^k : \hat{J}^k W \longrightarrow X \quad (6.4)
\end{equation}
be defined by
\begin{equation}
\hat{J}^1 W = J^1 W, \quad \hat{J}^r W = J^1(\hat{J}^{r-1} W), \quad (6.5)
\end{equation}
and let
\begin{equation}
\hat{j}^k : C^k(W) \longrightarrow C^0(\hat{J}^k W) \quad (6.6)
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram illustrating the variational hyper-stress induced by a non-holonomic stress.}
\end{figure}
be the natural repeated jet extension. Then, the analog of the construction in Sect. 3.3 may be carried out replacing \( j^k \) by \( \hat{j}^k \). This will lead to representation of forces by completely non-holonomic stresses, elements \( \Pi \) of \( C^0(\hat{J}^k W)^* \), in the form

\[
F = \hat{j}^{k*}(\Pi).
\] (6.7)

6.2 The Reduced Exterior Jet

The reduced exterior jet of a traction hyper-stress \( \sigma \) is defined as

\[
d_k \sigma := \iota_{k-1} \cdot (\sigma \cdot j^k),
\] (6.8)

that is,

\[
d_k \sigma \cdot j^k w = \partial \sigma \cdot j^{k-1} w.
\] (6.9)

Using (6.3) and (5.12) one obtains immediately,

\[
\begin{align*}
(d_k \sigma)_K^\alpha &= \hat{\sigma}^{(K)}_\alpha, & |K| &= k, \\
(d_k \sigma)_J^\alpha &= \sigma_{a,j}^j + \hat{\sigma}^{(J)}_\alpha, & 1 \leq |J| &< k, \\
(d_k \sigma)_a^\alpha &= \sigma_{a,j}^j.
\end{align*}
\] (6.10)

Evidently, one can obtain these relations directly from (6.9).

6.3 The Variational Hyper-stress Induced by a Consistent Hyper-force System

Let \( (b, \{t_\mathcal{R}\}) \) be a consistent hyper-force system corresponding to the traction hyper-stress \( \sigma \). Then, the corresponding unique non-holonomic stress, as in (5.18), determines through (6.3) a unique variational hyper-stress \( S \) given locally by

\[
\begin{align*}
S_K^\alpha &= \hat{\sigma}^{(K)}_\alpha, & |K| &= k, \\
S_J^\alpha &= \sigma_{a,j}^j + b_J^\alpha + \hat{\sigma}^{(J)}_\alpha, & 1 \leq |J| &< k, \\
S_a^\alpha &= \sigma_{a,j}^j.
\end{align*}
\] (6.11)

As mentioned above, only the symmetrized components of \( \sigma \) determine \( S \). Conversely, specifying \( S \) and \( b \) is not sufficient in order to determine a unique traction hyper-stress.

One may be tempted to postulate a-priori that the components of \( \sigma \) be symmetric. However, as shown below, symmetry of the components of elements of \( L(J^{k-1} W, \wedge^{n-1} T^* \mathcal{R}) \) is not an invariant property.

Example 9 Consider the case where \( W = \mathcal{R}^\times \times \mathbb{R} \) and \( k - 1 = 3 \). Let \( x^{i'} = x^{i'}(x^i) \) be a local coordinate transformation. Then, \( w_{i'i} = w_{i'i}(x^i) \),

\[
\begin{align*}
w_{i'j'} &= w_{i'j'}^i x_i^{j'}, \\
w_{i'j'k'} &= w_{i'j'k'}^{ijk} x_i^j x_j^{k'} + w_{i'j'} x_i^{j'} x_{jk'} + w_{i'j'} x_i^{j'} x_{jk'} + w_{i'j'} x_i^{j'} x_{jk'} + w_{i'j'} x_i^{j'} x_{jk'} + w_{i'j'} x_i^{j'} x_{jk'}.
\end{align*}
\] (6.12)
Invariance of the action imposes the condition that
\[ \sigma^{I} w_{I} \partial_{I} \text{d}x = \sigma^{I'} w_{I} \partial_{I} \text{d}x', \quad |I| \leq 3, \]
\[ = \sigma^{I'} w_{I} x'_{I} \partial_{I} \text{d}x, \quad (6.13) \]
where \( \mathcal{J} \) is the Jacobian determinant. After substitution of the transformed derivatives, and in view of the arbitrariness of the values of \( w_{I} \), we obtain
\[ \sigma^{ijkl} / \mathcal{J} = \sigma^{ij'kl'} x_{i} x_{j'} x_{k} x_{l'}, \]
\[ \sigma^{iji} / \mathcal{J} = \sigma^{ij'j'k} x_{i} x_{j'} x_{k} x_{j'}, \]
\[ \sigma^{iij} / \mathcal{J} = \sigma^{ij'j'k} x_{i} x_{j'} x_{k} x_{j'}, \]
\[ \sigma^{ilj} / \mathcal{J} = \sigma^{ij'j'k} x_{i} x_{j'} x_{k} x_{j'}, \]
\[\sigma^{i} / \mathcal{J} = \sigma^{ii'}, \quad (6.14)\]

We note that in case \( \sigma^{ij'k'l'} \) is symmetric, so is \( \sigma^{ijkl} \). However, even if all the components \( \sigma^{ijl} \) are symmetric, this need not hold for \( \sigma^{ij} \) and \( \sigma^{il} \).

We conclude that symmetry of the components of a traction hyper-stress is not an invariant property. One does not encounter this issue in the formulations of continuum mechanics in a Euclidean space (or whenever a connection is given) because one assumes, invariantly in this case, that the hyper-stress contains only the components of the tensor of highest order. Then, as noted above, these components preserve their symmetry under a transformation of coordinates.

**Example 10** It is observed that for the case \( k = 2 \), so that \( \sigma^{ijl} = \sigma^{ijkl} = 0 \) in the last equation, \( \sigma^{i} \) is symmetric if and only if \( \sigma^{ij} \) is symmetric. Hence, for \( k = 2 \), one may impose the condition that \( \sigma^{ij} \) is symmetric. This, together with the condition \( b_{i} = 0 \), imply that the variational stress determines a unique traction hyper-stress and a unique body hyper-force.

In light of the foregoing discussion we make the following

**Definition 11** A variational hyper-stress \( S \) is said to be consistent with a non-holonomic stress \( P \) if \( S = l_{k-1}(P) \).

**Proposition 12** If a variational hyper-stress \( S \) is consistent with a non-holonomic stress \( P \) representing a hyper-force system \( \{F_{R}\} = (b, \{\mathcal{I}_{R}\}) \), then, although \( b \) and \( \{\mathcal{I}_{R}\} \) cannot be determined by \( S \), the force functionals \( \{F_{R}\} \) can be determined by \( S \) in the form
\[ F_{R}(w) = \int_{\mathcal{R}} S \cdot j^{k} w. \quad (6.15) \]
In other words, while \( S \) represents the force system \( \{F_{R}\} \) as in Sects. 3.3, 3.4, it does not determine the corresponding body hyper-force and hyper-tractions.

**Proof** The assertion follows immediately from the fact that consistency implies that
\[ F_{\mathcal{R}}(w) = \int_{\mathcal{R}} P \cdot j^{k} (j^{k-1} w) = \int_{\mathcal{R}} S \cdot j^{k} w. \quad (6.16) \]
\[ \square \]
7 Spaces of Hyper-stresses and Constitutive Relations

Hyper-stresses have been defined above in the context of some particular configuration $\kappa$. To formulate constitutive relations, one has to identify the appropriate spaces to which hyper-stresses belong for all possible configurations. The present section is concerned with these issues.

7.1 The Space of Variational Hyper-stress Density Values

We first recall the following two natural isomorphisms. As fiber bundles over $\mathcal{X}$,

$$V J^k \mathcal{Y} \simeq J^k V \mathcal{Y}, \quad (7.1)$$

and for any section $\kappa : \mathcal{X} \to \mathcal{Y},$

$$(j^k \kappa)^* V J^k \mathcal{Y} \simeq J^k (\kappa^* V \mathcal{Y}) \quad (7.2)$$

(see [23, Theorem 4.4.1] and [20, Theorem 17.1, p. 82]).

For a configuration $\kappa$ and a point $x \in \mathcal{X}$, consider the value $S(x)$ of some variational hyper-stress field at the configuration $\kappa$. Using the isomorphism above and recalling (3.10), one has,

$$S(x) \in L \left( J^k (\kappa^* V \mathcal{Y}) \right),$$

$$= L \left( (j^k \kappa)^* V J^k \mathcal{Y} \right),$$

$$= L \left( (j^k \kappa)^* V J^k \mathcal{Y} \right),$$

$$= L \left( V J^k \mathcal{Y} \right)_{j^k \kappa(x)},$$

$$= L \left( V J^k \mathcal{Y}, \pi^k \Lambda^n T^* \mathcal{X} \right)_{j^k \kappa(x)},$$

$$= \left[ (j^k \kappa)^* L \left( V J^k \mathcal{Y}, \pi^k \Lambda^n T^* \mathcal{X} \right) \right]_{j^k \kappa(x)}.$$  \hfill (7.3)

One concludes that the values of variational hyper-stress fields may always be identified with elements of the vector bundle

$$\pi_\mathcal{S} : \mathcal{S} := L \left( V J^k \mathcal{Y}, \pi^k \Lambda^n T^* \mathcal{X} \right) \longrightarrow J^k \mathcal{Y}, \quad (7.4)$$

independently of the configuration under consideration. For any configuration $\kappa$, variational hyper-stress densities are valued in

$$(j^k \kappa)^* \mathcal{S} \simeq L \left( J^k (\kappa^* V \mathcal{Y}), \Lambda^n T^* \mathcal{X} \right). \quad (7.5)$$

Intuitively, sections of $J^k \mathcal{Y}$ may be thought of as “local configurations” and sections of $V J^k \mathcal{Y}$ may be conceived as variations thereof or as “local velocity fields”. Thus, following this line of thought, variational hyper-stresses may be thought of as “local forces”.

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7.2 The Space of Traction Hyper-stress Values

In analogy with Sect. 7.1, we identify here the space where traction hyper-stress fields assume their values, independently of a particular configuration. Thus, for a configuration \( \kappa \) and a point \( x \in \mathcal{X} \), we consider the value \( \sigma(x) \) of some traction hyper-stress field at the configuration \( \kappa \). One has,

\[
\sigma(x) \in L\left( (J^{k-1}W)_x, \left( \wedge^{n-1} T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( (J^{k-1}(\kappa^* V \mathcal{Y}))_x, \left( \wedge^{n-1} T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( \left( (j^{k-1}\kappa)^* V J^{k-1} \mathcal{Y} \right)_x, \left( \wedge^{n-1} T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( V J^{k-1} \mathcal{Y}, j^{k-1}(\kappa(x))^* \pi^{k-1} \left( \wedge^{n-1} T^* \mathcal{X} \right) \right).
\]

It is concluded that the values of traction hyper-stress fields may be identified with elements of the vector bundle

\[
\pi_{\Sigma} : \Sigma := L\left( V J^{k-1} \mathcal{Y}, \pi^{k-1} \wedge^{n-1} T^* \mathcal{X} \right) \longrightarrow J^{k-1} \mathcal{Y}. \tag{7.7}
\]

For a given configuration \( \kappa \), traction hyper-stresses are valued in

\[
(j^{k-1}\kappa)^* \Sigma \simeq L\left( (j^{k-1}\kappa)^* V J^{k-1} \mathcal{Y}, \wedge^{n-1} T^* \mathcal{X} \right). \tag{7.8}
\]

7.3 The Space of Values of Non-holonomic Stresses

Let

\[
A : \mathcal{X} \longrightarrow J^{k-1} \mathcal{Y} \tag{7.9}
\]

be a smooth section and let \( x \in \mathcal{X} \). Then, the value of a non-holonomic stress field \( P \) at \( x \) satisfies

\[
P(x) \in L\left( (J^1(J^{k-1}W))_x, \left( \wedge^n T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( (J^1(A^* V J^{k-1} \mathcal{Y}))_x, \left( \wedge^n T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( \left( (j^1A)^* V J^1(J^{k-1} \mathcal{Y}) \right)_x, \left( \wedge^n T^* \mathcal{X} \right)_x \right),
\]

\[
= L\left( V J^1(J^{k-1} \mathcal{Y}), j^1(A(x))^* \pi^{1*} \left( \wedge^n T^* \mathcal{X} \right) \right).
\]

\[
= L\left( V J^1(J^{k-1} \mathcal{Y}), \pi^{1*} \left( \wedge^n T^* \mathcal{X} \right) \right)_{j^1(A(x))},
\]

\[=
\left[ (j^1A)^* L\left( V J^1(J^{k-1} \mathcal{Y}), \pi^{1*} \left( \wedge^n T^* \mathcal{X} \right) \right) \right]_x. \tag{7.10}
\]
Here, we view \( \pi^{k-1} : J^{k-1}\mathcal{Y} \to \mathcal{X} \) as a fiber bundle over \( \mathcal{X} \), and so \( \pi^1 : J^1(J^{k-1}\mathcal{Y}) \to \mathcal{X} \) is the natural projection. Hence, values of non-holonomic hyper-stress fields may be viewed as elements of the vector bundle

\[
\pi_P : P := L(VJ^1(J^{k-1}\mathcal{Y}), \pi^1_*\bigwedge^n T^*\mathcal{X}) \to J^1(J^{k-1}\mathcal{Y}).
\] (7.11)

For a given section \( A : \mathcal{X} \to J^{k-1}\mathcal{Y} \), non-holonomic stresses at \( A \) are valued in

\[
(j^1 A)^*P = L(J^1(A^*VJ^{k-1}\mathcal{Y}), \bigwedge^n T^*\mathcal{X}).
\] (7.12)

### 7.4 Elastic Constitutive Relations

Once the spaces of hyper-stresses have been identified, one may introduce constitutive relations for the variational hyper-stresses and for the non-holonomic stresses.

An elastic constitutive relation for the variational hyper-stress is a section

\[
\Psi : J^k\mathcal{Y} \to \mathcal{S} = L(VJ^k\mathcal{Y}, \pi^k_*\bigwedge^n T^*\mathcal{X})
\] (7.13)

of \( \pi_S \). Thus, a constitutive relation assigns a value of a variational hyper-stress at a point \( x \) to a \( k \)-jet of a section at \( x \). In particular, for a section \( \kappa \),

\[
\Psi \circ j^k \kappa : \mathcal{X} \to \mathcal{S}
\] (7.14)

is identified with a variational hyper-stress field according to (7.3).

Similarly, an elastic constitutive relation for the non-holonomic stress is a section

\[
\Phi : J^1(J^{k-1}\mathcal{Y}) \to P = L(VJ^1(J^{k-1}\mathcal{Y}), \pi^1_*\bigwedge^n T^*\mathcal{X})
\] (7.15)

of \( \pi_P \). For a section \( A : \mathcal{X} \to J^{k-1}\mathcal{Y} \),

\[
\Phi \circ j^1 A : \mathcal{X} \to P
\] (7.16)

is identified with a non-holonomic stress field according to (7.10). In particular, for a section \( \kappa : \mathcal{X} \to \mathcal{Y} \), one has the induced non-holonomic stress field \( \Phi \circ j^1(j^{k-1}\kappa) \).

**Acknowledgements** Both authors are grateful to BIRS for sponsoring the Banff Workshop on Material Evolution, June 11–18, 2017, which led to this collaboration. R.S.’s work has been partially supported by H. Greenhill Chair for Theoretical and Applied Mechanics and the Pearlstone Center for Aeronautical Engineering Studies at Ben-Gurion University.

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