Electric and Magnetic Polarizabilities
of Pointlike Spin-1/2 Particles

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Abstract—The electric and magnetic polarizabilities of pointlike spin-1/2 particles with an anomalous magnetic moment (AMM) are calculated by the transformation of an initial Hamiltonian into the Foldy–Wouthuysen (FW) representation. Corresponding results for spin-1/2 and spin-1 particles are compared.

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In this work, the author determines—by the Foldy–Wouthuysen (FW) transformation—the electric and magnetic polarizabilities of pointlike spin-1/2 particles that have an anomalous magnetic moment (AMM).

The unique properties of FW representation [1] make it a very convenient tool for transition to quasiclassical approximation and finding the classical limit of relativistic quantum mechanics. Even for relativistic particles in an external field, the operators in this representation are completely analogous to the corresponding operators of nonrelativistic quantum mechanics. In particular, the localization operator (the Newton–Wigner operator) and the momentum operator are equal to $r$ and $p = -i\hbar \nabla$, and the polarization operator for the spin-1/2 particles is expressed through Dirac matrix $\gamma_0$. In other representations, these operators are defined by considerably more cumbersome formulae (see [1, 3]). Apart from the simple and unambiguous form of operators that correspond to the classical observed ones, the most significant merit of FW representation is the restoration of the probabilistic interpretation of the wave function. Since, as was said above, it is in the FW representation that the Newton–Wigner operator, which characterizes the location of a particle’s geometric center, is equal to radius-vector $r$, the squared wave function modulus determines the probability density of the particle’s location at the point with the given radius vector. We should note that, in the FW representation, the Hamiltonian and all operators are diagonal over two spinors (block-diagonal). Using this representation eliminates the possibility of ambiguous solutions for the problem of finding the classical limit of relativistic quantum mechanics [1, 4].

The most essential parameters that characterize the particles and the nuclei are scalar electric and magnetic polarizabilities. Their contribution to the Hamiltonian in the FW representation is determined by the expression

$$\Delta \mathcal{H}_{FW} = -\frac{1}{2} \alpha E^2 - \frac{1}{2} \beta B^2.$$  \hspace{1cm} (1)

We consider here a case of stationary and homogeneous electric ($E$) and magnetic ($B$) fields and use the following system of units: $\hbar = 1$ and $c = 1$.

While in the classical physics a particle may have nonzero polarizabilities only provided that it has an internal structure, in quantum mechanics even point-like objects appear to have nonzero polarizabilities. These parameters can be defined by the FW transformation of the initial Dirac–Pauli equation [5] for particles with an AMM followed by transition to the classical limit. Solving this problem necessitates, however, the calculation of external-field quadratic summands. This procedure requires caution, since different techniques yield different results (see [6] and the references cited there). Correct results are obtained by the Eriksen method [7]. It is convenient to divide the initial Dirac–Pauli Hamiltonian [5] into the even and the odd summands that commute and anticommute with Dirac matrix $\beta$, respectively,

$$\mathcal{H}_D = \beta m + \mathcal{C} + \mathcal{C}_2,$$  \hspace{1cm} (2)

Here,

$$\mathcal{C} = e\Phi - \mu' \Pi \cdot B, \quad \mathcal{C}_2 = c\alpha \cdot \pi + i\mu' \gamma \cdot E,$$  \hspace{1cm} (3)

where $\mu'$ is the AMM. We use the conventional notation [8] for the Dirac matrices.

The expansion in $1/m$ powers of the Hamiltonian in the FW representation obtained by the Eriksen method is presented in [6, 9, 10]. In the case under consideration, the summands proportional to the fourth and higher powers of the reciprocal mass can be
neglected. In this case, the Hamiltonian in the FW representation is defined according to the equation

\[ \mathcal{H}_{FW} = \beta \left( m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \right) + e\Phi + \frac{1}{2m} \left( \mu_0 + \mu' \right) \]

\[ + \frac{\beta}{16m^3} \left\{ \mathcal{C}, \left[ \mathcal{C}, \mathcal{F} \right] \right\}, \]

(4)

where \( \mathcal{F} = \mathcal{E} - i\hbar/\partial t \). For the stationary problem in question, \( \mathcal{F} = \mathcal{E} \).

Calculation according to Eqs. (3) and (4) yields the following expression:

\[ \mathcal{H}_{FW} = \beta \left( m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} \right) + e\Phi + \frac{1}{2m} \left( \mu_0 + \mu' \right) \]

\[ \times \left( 2\Sigma \cdot [\pi \times E] - \nabla \cdot E \right) - \left( \mu_0 + \mu' \right) \mathbf{B} \cdot \mathbf{B} \]

\[ + \frac{\mu_0}{4m^3} \left\{ \mathbf{F} \cdot \mathbf{B}, \mathbf{B} \right\} + \frac{\beta}{2m} \left( \mu_0 + \mu' \right) \mu' \frac{E^2}{2m^2} - \beta \frac{\mu_0^2}{2m^2} B^2, \]

(5)

where \( \mu_0 = e/(2m) \) is the Dirac magnetic moment.

We should note that the last summand in Eq. (4) does not make any contribution to the electric and magnetic polarizabilities. Calculations by the method proposed in Foldy and Wouthuysen’s original work [1] and by other iteration methods (see [10, 11] and the references cited there) lead to a different form of this summand and, as a consequence, do not yield the correct expression for electric polarizability. A comparison of Eqs. (1) and (5) shows that the scalar electric and magnetic polarizabilities of the pointlike particles that have an AMM have the form

\[ \alpha_s = -\frac{(\mu_0 + \mu')\mu'}{m} = \frac{e^2 (g-2)}{16m^3}, \]

\[ \beta_s = \frac{\mu_0}{m} = \frac{e^2}{4m^3}. \]

(6)

Matrix \( \beta \) can be omitted since, in the FW representation, the lower spinor equals zero.

A comparison of polarizabilities of pointlike spin-1/2 and -1 particles is important. The spin-1 particles are characterized by not only scalar but also tensor polarizabilities calculated in [12]. The scalar polarizabilities of such particles are defined by the expressions [12]

\[ \alpha_s = \frac{e^2 (g-1)^2}{m^3}, \quad \beta_s = 0. \]

(7)

Thus, the scalar magnetic polarizability of the spin-1 particles equals zero and the scalar electric polarizability is nonzero for particles with not only an anomalous but also normal \( (g = 2) \) magnetic moment. These properties differ from the corresponding properties of the spin-1/2 particles.

We should note that the scalar polarizabilities of pointlike spin-0 particles are equal to zero (see [13]).

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