THE LIOUVILLE THEORY AND $SL(2, R)$ SYMMETRY *

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Abstract

The Liouville action emerges as the effective action of 2-d gravity in the process of path integral quantization of the bosonic string. It yields a measure of the violation of classical symmetries of the theory at the quantum level. Certain aspects of the residual $SL(2, R)$ invariance of the gauge-fixed Liouville theory are discussed.

1. Polyakov's path integral and the Liouville action

String moves trough a D-dimensional spacetime $X_D$, sweeping out a two-dimensional surface $\Sigma$. In the Polyakov version, the classical action of the bosonic string takes the form

$$I[x^\mu, g_{\alpha\beta}] = \int d^2 \xi \sqrt{g}g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}. \tag{1}$$

The worldsheet metric $g_{\alpha\beta}$ as well as $x^\mu$ are treated as independent dynamical variables, while the metric of spacetime $G_{\mu\nu}$ is fixed (we shall assume that $G_{\mu\nu} = \eta_{\mu\nu}$). The action (1) is classically equivalent to the Nambu action. The equivalence of the two formulations at the quantum level is a more complicated question. The Polyakov action is more convenient for the covariant path integral quantization.

In order to be able to apply the mathematical theory of Riemann surfaces we shall work with the Euclidean worldsheet, which is obtained by the Wick rotation from the original Minkowskian worldsheet. After all the calculations are completed, one may return to the Minkowskian theory by analytic continuation.

The functional integral. We shall limit our discussion to the usual perturbative approach. Thus, a scattering process for $n$ strings in the approximation of $l$ loops is described by a two-dimensional manifold (surface) of genus $\gamma = l$ and with $b = n$ boundary components. The scattering amplitude is assumed to be of the form [1-3]

$$A_{\gamma}(p_1, p_2, \ldots, p_n) = \int_{\Sigma_{\gamma,n}} Dg_{\alpha\beta} Dx^\mu e^{-I[x,g]} V(p_1) V(p_2) \ldots V(p_n), \tag{2}$$

where $V(p_i)$ are vertex operators assigned to the boundaries of $\Sigma$, and $\int Dx^\mu$ is the integration over all continuous maps $x^\mu(\xi)$ from $\Sigma$ to $X_D$. The complete amplitude for a given process is obtained by summing over all $\gamma$’s in Eq.(2).

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The amplitude with \( n = 0 \) (no external string states), describing the vacuum-to-vacuum transition,

\[
Z = \sum_{\gamma} \int_{\Sigma_{\gamma}} Dg_{\alpha\beta} Dx^\mu e^{-I[x,g]},
\]

will be the main subject of our discussion.

We shall concentrate on the closed, oriented bosonic string theory. We shall also restrict the surfaces in (3a) to be compact, closed and oriented (and connected, as we are interested in the connected part of the amplitude). Finally, we shall discuss mainly the contribution of genus zero surfaces, whose description is technically the simplest:

\[
Z_0 = \int_{\Sigma_0} Dg_{\alpha\beta} Dx^\mu e^{-I[x,g]}.
\]

Several mathematical theorems related to the geometry of two-dimensional surfaces are given in Appendix A.

The measure \( Dg_{\alpha\beta} \). The classical string action is invariant under Weyl rescalings and diffeomorphisms. As a consequence, the functional integral \( Z_0 \) is highly divergent. The elimination of this divergence demands a careful treatment of the measure, as in the case of gauge theories (the Faddeev-Popov construction).

The integration over all metrics in Eq.(3b) has the following meaning. All genus zero manifolds \( \Sigma_0 \) are topologically equivalent to each other, but different as metric spaces \((\Sigma_0, g)\). With the topology of \( \Sigma_0 \) fixed, the set of all metric spaces \((\Sigma_0, g)\) is determined by the set of all admissible metrics on \( \Sigma_0 \):

\[
\mathcal{M}_0(g) = \{ g | g \text{ is a metric on } \Sigma_0 \}.
\]

The measure \( Dg_{\alpha\beta} \) can be determined by introducing the metric \( G \) on \( \mathcal{M}_0(g) \).

Example: In an n-dimensional Riemannian space with metric \( ds^2 = G_{ab} dy^a dy^b \), the measure is defined as \( \sqrt{G} dy^1 \ldots dy^n \).

Let \( \delta g \) be a tangent vector at a point \( g \in \mathcal{M}_0(g) \) with components \( \delta g_{\alpha\beta}(\xi) \). The scalar product

\[
(\delta g^{(1)}, \delta g^{(2)}) = \int d^2 \xi \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} \delta g_{\alpha\beta}(\xi) \delta g_{\gamma\delta}(\xi)
\]

defines a natural metric in the tangent space \( T_0(g) \). This metric is invariant under reparametrizations, but not under Weyl rescalings. It defines the measure \( Dg_{\alpha\beta} \).

All genus zero metric spaces \((\Sigma_0, g)\) are conformally equivalent, i.e. any two metrics are related to each other by the local diffeomorphisms and Weyl rescalings only. This follows from \( T4 \) of Appendix A, and holds only for \( \gamma = 0 \) (if \( \gamma \geq 1 \) one has also global diffeomorphisms, and the discrete groups \( \Gamma_\gamma \) are nontrivial). Therefore,

choosing a particular metric \( \hat{g} \) in \( \mathcal{M}_0(g) \), an arbitrary metric \( g \) can be obtained from \( \hat{g} \) by Weyl rescalings and local diffeomorphisms only.
To be specific we can choose $\hat{g}$ to be conformally flat, i.e.

$$\hat{g}_{\alpha\beta} = e^{\hat{\phi}} \delta_{\alpha\beta}.$$  

This choice can always be made, as we know from T2 (there are no essential changes in the discussion which follows if $\hat{g}$ is chosen differently). The set of metrics that are connected to $\hat{g}$ by Weil rescalings defines the Weyl slice in $\mathcal{M}_0(g)$; if $\hat{g}$ is conformally flat, the corresponding Weyl slice is called the conformally flat gauge slice. It is easy to see that Weyl transformations

$$\delta g_{\alpha\beta} = \delta \phi g_{\alpha\beta}$$  

(5)

slide the conformally flat metric $\hat{g}_{\alpha\beta}$ along the conformally flat gauge slice.

The local diffeomorphisms $\text{Diff}_0$ have the form

$$\delta g_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha.$$  

For $\gamma = 0$ all diffeomorphisms are connected to the identity. These transformations can be decomposed into the traceless and trace part

$$\delta g_{\alpha\beta} = \delta^D g_{\alpha\beta} + (\nabla \cdot \epsilon) g_{\alpha\beta},$$  

$$\delta^D g_{\alpha\beta} \equiv \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - g_{\alpha\beta} (\nabla \cdot \epsilon) \equiv (P_1 \epsilon)_{\alpha\beta}.$$  

The trace part is tangent to the Weyl slice containing $\hat{g}$. Combining $\text{Diff}_0$ with Weyl rescalings one obtains

$$\delta g_{\alpha\beta} = \delta^W g_{\alpha\beta} + \delta^D g_{\alpha\beta},$$  

$$\delta^W g_{\alpha\beta} \equiv (\delta \phi + \nabla \cdot \epsilon) g_{\alpha\beta} \equiv \delta \lambda g_{\alpha\beta}.$$  

(6)

The tangent vectors $\delta^W g$ and $\delta^D g$ are orthogonal with respect to the scalar product (4); they span the tangent space to $\mathcal{M}_0(g)$ at the point $\hat{g}$ (no other transformations of metric are admissible in the $\gamma = 0$ case).

Our intention is to construct the measure $Dg_{\alpha\beta}$ in terms of parameters $(\delta \phi, \epsilon)$ associated with Weyl rescalings and $\text{Diff}_0$, which take us from $\hat{g}$ to $g$. The norm of the tangent vector $\delta g$, defined in Eq.(6), can be calculated by using the scalar product (4):

$$\|\delta g\|^2 = (\delta g, \delta g) = \int d^2 \xi \sqrt{g} (\delta \lambda)^2 + (P_1 \epsilon)_{\alpha\beta} (P_1 \epsilon)^{\alpha\beta}. $$

Note that this norm is reparametrization invariant but not Weil invariant, which will result in the loss of Weil invariance of the functional integral (quantum theory). After introducing norms in the tangent spaces of scalars and vectors as

$$\|\delta \lambda\|^2 \equiv \int d^2 \xi \sqrt{g} (\delta \lambda)^2,$$  

$$\|\delta v\|^2 \equiv \int d^2 \xi \sqrt{g} g_{\alpha\beta} \delta v^\alpha \delta v^\beta,$$

we can write

$$\|\delta g\|^2 = \|\delta \lambda\|^2 + \|P_1 \epsilon\|^2.$$  

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Since \(\|P_1\epsilon\|^2 = (P_1\epsilon, P_1\epsilon) = (\epsilon, P_1^+P_1\epsilon)\), we could conclude that the measure \(Dg_{\alpha\beta}\) is given by
\[
Dg_{\alpha\beta} = \sqrt{G}D\phi D\epsilon^\alpha = \left[\det(P_1^+P_1)\right]^{1/2}D\phi D\epsilon^\alpha.
\]
(7a)

However, this is not true since \(P_1\) has zero modes. Indeed, the equation
\[
(P_1\epsilon) \equiv \nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha - g_{\alpha\beta}(\nabla \cdot \epsilon) = 0
\]
is the conformal Killing equation whose nontrivial solutions are the Killing vectors \(\epsilon^K_\alpha\). The diffeomorphisms generated by \(\epsilon^K_\alpha\) have the same effect as a Weyl transformation. Therefore, the measure takes the form
\[
Dg_{\alpha\beta} = \left[\det'(P_1^+P_1)\right]^{1/2}D\phi D\epsilon^\alpha
\]
(7b)

where prime means that the zero modes have been deleted, and the functional integral (3b) becomes
\[
Z_0 = N \int D\phi D\epsilon^\alpha D\epsilon D^\mu \left[\det'(P_1^+P_1)\right]^{1/2} e^{-I[x,g]},
\]
(8)
where \(g_{\alpha\beta} = g_{\alpha\beta}(\phi, \epsilon^\alpha)\).

We want to extract the volume \(V(\text{Diff}_0f_0)\), which differs from \(V'(\text{Diff}_0f_0) = \int D\epsilon\) by the contribution of the Killing vectors, \(V^K(\text{Diff}_0f_0)\). Since the two types of the diffeomorphisms are mutualy orthogonal, we have \(V = V' \cdot V^K\). The number of independent Killing vectors in the case \(\gamma = 0\) is 6 (the number of independent parameters of \(SL(2, C)\), the conformal group of the compactified complex plane \(\hat{C}\)). Introducing a basis \(e_a(a = 1, \ldots, 6)\), we can write \(V^K = \left[\det(e_a, e_b)\right]^{1/2}\). After that the extraction of \(V\) from the integral for \(Z_0\) leads to
\[
Z_0 = N' \int D\phi D\epsilon D\epsilon D^\mu \left[\frac{\det'(P_1^+P_1)}{\det(e_a, e_b)}\right]^{1/2} e^{-I[x,g]}.
\]
(9)

The integration over \(g_{\alpha\beta}\) is now performed only over the Weyl slice containing \(\hat{g}_{\alpha\beta}\).

The result (9) is the first important step in the quantization procedure, corresponding to fixing the reparametrization invariance of the functional integral.

**The standard Faddeev-Popov.** It is illuminating to derive the expression (8) for \(Z_0\) by using the standard Faddeev-Popov approach. We want to fix the reparametrization symmetry in the functional integral (3b) by using the gauge condition
\[
g_{\alpha\beta} = e^\phi \hat{g}_{\alpha\beta} \equiv g^w_{\alpha\beta}.
\]
To this end we define the Faddeev-Popov determinant by
\[
\int D\phi D\epsilon \delta(g_{\alpha\beta} - e^\phi \hat{g}_{\alpha\beta}) \Delta_{FP} = 1.
\]
Inserting this into Eq.(3b) for $Z_0$ yields

$$Z_0 = N \int D\phi D\epsilon^i Dg_{\alpha\beta} Dx^\mu \Delta_{FP} \delta(g_{\alpha\beta} - e^{\phi} \hat{g}_{\alpha\beta}) e^{-I[x,g]}.$$ 

The integration over $Dg_{\alpha\beta}$ is now easily performed, leading to the replacement $g_{\alpha\beta} \to e^{\phi} \hat{g}$ in the classical action. Writing the general gauge transformation of $g_{\alpha\beta}$ as

$$\delta g_{\alpha\beta} = (P_1 \epsilon')_{\alpha\beta} + \delta \lambda g_{\alpha\beta},$$
$$\delta \lambda \equiv \delta \phi + \nabla \cdot \epsilon',$$

the Faddeev-Popov determinant is defined by

$$\Delta_{FP} = \frac{\partial(\delta \lambda, P_1 \epsilon')}{\partial(\delta \phi, \epsilon')} = \text{det} \begin{pmatrix} 1 & * \\ 0 & P_1 \end{pmatrix} = \text{det}' P_1 = [\text{det}'(P_1^+ P_1)]^{1/2},$$

which leads directly to Eq.(8).

**The measure $Dx^\mu$.** Now we return to Eq.(9) and do the $x^\mu$ integration. The measure $Dx^\mu$ is defined by the metric in the tangent space spanned by the vectors $\delta x = (\delta x^\mu)$,

$$\|\delta x\|^2 = \int d^2 \xi \sqrt{g} \delta x^\mu \delta x^\nu G_{\mu\nu},$$

which is invariant under reparametrizations but not under Weyl rescalings. When a tangent vector $\delta x_0$ is $\xi$-independent we have

$$\|\delta x_0\|^2 = \delta x_0^\mu \delta x_0^\nu G_{\mu\nu} \int d^2 \xi \sqrt{g},$$

so that

$$Dx_0^\mu = \left( \int d^2 \xi \sqrt{g} \right)^{D/2} dx_0^1 \ldots dx_0^D.$$ 

By using the usual decomposition

$$x^\mu = x_0^\mu + x'^\mu,$$

the integration over $x^\mu$ in (9) leads to

$$\int Dx^\mu e^{-I[x,g_w]} = \int Dx_0^\mu \int Dx'^\mu \exp \left( - \int d^2 \xi \sqrt{g_w} g_\alpha^\beta \partial_\alpha x'^\mu \partial_\beta x'^\nu G_{\mu\nu} \right)$$
$$= \int Dx_0^\mu \int Dx'^\mu \exp \left( \int d^2 \xi \sqrt{g_w} x'^\mu \Delta_{g_w} x'^\nu G_{\mu\nu} \right)$$
$$= V_D \left( \int d^2 \xi \sqrt{g_w} \right)^{D/2} \left[ \text{det}'(\Delta_{g_w}) \right]^{-D/2}.$$
Here, $V_D = \int dx_0^1 \ldots dx_0^D$ is the volume of the spacetime $M_D$ and $\Delta_{g_w}$ is the covariant d’Alambertian (the Laplace-Beltrami operator),

$$\Delta_{g_w} = -\frac{1}{\sqrt{g_w}} \partial_\alpha \sqrt{g_w} g_w^{\alpha \beta} \partial_\beta \equiv -\Box_w.$$

Putting all together one easily finds

$$Z_0 = N \int D\phi \left[ \frac{\det ' (P_1^+ P_1)}{\det (e_a, e_b)} \right]^{1/2} \left[ \frac{\det ' (\Delta_{g_w})}{\int d^2 \xi \sqrt{g_w}} \right]^{-D/2}.$$  \hspace{1cm} (11)

The operator $\Delta_{g_w}$ is not Weyl invariant, and $\det ' (\Delta_{g_w})$ depends on the Weyl factor $\phi$. Thus, although the classical action is Weyl invariant, the functional integral is not, which leads to the appearance of the conformal anomaly.

**Conformal anomaly.** In the previous equation $g_w^{\alpha \beta}$ is the metric on the Weyl slice. If we choose the conformally flat slice, one can prove the relation

$$\ln \left[ \frac{\det ' (\Delta_{g_w})}{\int d^2 \xi \sqrt{g_w}} \right] - \ln \left[ \frac{\det ' (\Delta_{\hat{g}})}{\int d^2 \xi \sqrt{\hat{g}}} \right] = \frac{1}{24\pi} \Gamma[\phi],$$

$$\Gamma[\phi] = \int d^2 \xi \left[ \frac{1}{2} \delta^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi + \mu^2 (e^\phi - 1) \right],$$

so that

$$\left[ \frac{\det ' (\Delta_{g_w})}{\int d^2 \xi \sqrt{g_w}} \right]^{-D/2} = e^{\frac{D/2}{24\pi} \Gamma[\phi]}.$$  \hspace{1cm} (12b)

The calculation of the Faddeev-Popov determinant yields the result of the same form with $D \to -26$, leading to the final result

$$Z_0 = N \int D\phi \exp \left[ -\kappa \Gamma[\phi] \right], \quad \kappa \equiv \frac{26 - D}{48\pi}.$$  \hspace{1cm} (13)

The expression $\Gamma[\phi]$ characterizes the conformal anomaly of the theory. It yields a measure of the violation of the conformal symmetry at the quantum level. The action $\Gamma(\phi)$ defines the *Liouville theory*.

We are now going to make several comments.

1. In the process of calculating the final result (13) we had to perform some renormalizations. They can be done in the standard manner if the original Polyakov action is modified by adding two counterterms,

$$I[x^\mu, g_{\alpha \beta}] = \int d^2 \xi \sqrt{g} \Gamma^\alpha_\beta \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu \nu} + \mu_0^2 \int d^2 \xi \sqrt{g} + \lambda_0 \int d^2 \xi \sqrt{g} R,$$

where $\mu_0$ and $\lambda_0$ are the bare, cutoff-dependent coupling constants. By the global Gauss-Bonnet theorem the last term is a topological invariant and can be pulled out of the
functional integral as a factor \( \exp[4\pi\lambda_0(2-\gamma)] \), showing that a sum over \( \gamma \) is a perturbative expansion.

2. The calculation of the anomaly can be carried out for higher genus surfeses in a similar manner. In that case two metrics lying in the space of all admissible metrics \( \mathcal{M}_\gamma(g) \) are not always connected by local diffeomorphisms and Weyl rescalings. There exist additional degrees of freedom which label the conformal classes of metrics, called the Teichmüller parameters. The functional integral will include the summation over conformally inequivalent classes of metrics, i.e., the integration over the Teichmüller parameters. However, the anomaly is a local object, which means that its form does not depend on the genus. For higher genus it is convenient to use a Weyl slice which is not conformally flat. This slice contains a reference metric \( \hat{g} \), which is chosen so as to make the additional integration simple. There is no need to know the explicit form of \( \hat{g} \) for the calculation of the anomaly. The effective theory has the same form as in Eq.(13), with

\[
\Gamma[\phi] = \int d^2\xi \left[ \frac{1}{2} \hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \hat{R} \phi + \mu^2 \left( e^\phi - 1 \right) \right]. \tag{14}
\]

3. For the critical dimension \( D = 26 \) the \( \phi \)-dependence disappears from the functional integral, and the volume of the Weyl slice \( V_w = \int D\phi \) can be absorbed into the normalization. If \( D \neq 26 \) there is no Weyl symmetry at the quantum level (there is an anomaly), and the quantization of the theory demands a special care, as discussed by Bogojević and Sazdović at this meeting.

2. Residual \( SL(2, R) \) symmetries of the Liouville theory

By studying the Liouville theory in the light–cone gauge Polyakov discovered a residual \( SL(2, R) \) invariance [4-6], which is very important for understanding the structure of the theory. We shall now discuss certain aspects of this symmetry at the Lagrangean level [7-9].

**Conformal anomaly and the Liouville action.** Classical action for the bosonic string is invariant under 2-d reparametrizations and local Weyl rescalings. Quantization leads to the appearance of an anomaly, which means that not all classical symmetries are the symmetries of the quantum theory. We can use the reparametrization invariance and fix the metric to the conformally flat form

\[ g_{\alpha\beta}(x) = e^\phi \eta_{\alpha\beta} \]

(we are now working in the Minkowski space). After that, the integration over all metrics in the functional integral reduces to the integration over the conformally flat gauge slice. Then, after introducing the corresponding ghosts and using a convenient regularization, one finds that the anomaly has the form

\[
A[\phi, C] = \kappa \int d^2\xi C(\xi)(\Box \phi + \mu^2 e^\phi), \tag{15}
\]
where $C(\xi)$ is the Weyl ghost, and $\hat{\square} = \eta^{\alpha\beta} \partial_\alpha \partial_\beta$. The anomaly is related to the notion of the effective action by the relation
\[
\delta W[\phi] = A[\phi, \delta \phi],
\]
which after integration yields
\[
W[\phi, \eta] = \kappa \int d^2 \xi \left( \frac{1}{2} \phi \hat{\square} \phi + \mu^2 e^\phi \right).
\]
Note that the quantity $\Gamma[\phi]$, appearing in the functional $Z_0$ in Eq.(13), is given by $\Gamma[\phi] = W[\phi] - W[0]$.

The above expression for the effective action can be easily transformed into a covariant-looking form. By using the relations $\sqrt{-g}R = -\hat{\square} \phi$ and $\sqrt{-g} \hat{\square} = \hat{\square}$ valid in the conformally flat gauge, one finds
\[
W[g] = \kappa \int d^2 \xi \sqrt{-g} \left( \frac{1}{2} R \frac{1}{\hat{\square}} R + \mu^2 \right),
\]
which is a nonlocal expression, suitable for discussing the effective action on other gauge slices. The effective action can be written as a local functional by introducing an auxiliary field $F$:
\[
W[F, g] = \kappa \int d^2 \xi \sqrt{-g} \left( -\frac{1}{2} F \hat{\square} F + FR + \mu^2 \right).
\]
The elimination of $F$ with the help of its equation of motion leads back to (18a).

The Liouville theory is, up to a sign, defined by the effective action $W$,
\[
I_L(F, g) = -\int d^2 \xi \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha F \partial_\beta F + \frac{\alpha}{2} FR + \mu^2 \right],
\]
where the constant $\alpha$ is introduced for convenience (classically $\alpha = 2$), and we used $\kappa = 1$.

The light-cone gauge. The Liouville theory simplifies significantly in the light-cone gauge, defined by
\[
g_{++} = 1, \quad g_{--} = 0, \\
ds^2 = h(d\xi^+)^2 + 2d\xi^+ d\xi^-, 
\]
where the components of the metric tensor are given in the light-cone coordinates $\xi^\pm = (\xi^0 \pm \xi^1)/\sqrt{2}$ as
\[
g_{+-} \equiv \frac{1}{2}(g_{00} - g_{11}), \quad g_{-+} \equiv \frac{1}{2}(g_{00} + g_{11}) - g_{01}, \\
h \equiv g_{++} = \frac{1}{2}(g_{00} + g_{11}) + g_{01}.
\]

From the point of view of the functional integral, the use of this gauge means that we are integrating over the light-cone gauge slice. We expect that the amplitudes, defined by the functional integral, do not depend on the choice of the gauge slice (the gauge
independence of the S-matrix!\). Although the transition from the Weyl to the light-cone gauge slice is admissible, it would be very interesting to check this change of gauge in more details, as it is one of the basic consistency requirements on the theory.

After finding the inverse metric

\[ g^{++} = 0, \quad g^{+-} = g^{-+} = 1, \quad g^{--} = -h, \]

the calculation of the metric connection \( \Gamma^\alpha_{\beta\gamma} \) yields the following nonvanishing components:

\[ \Gamma^{++}_{++} = -\frac{1}{2} \partial_+ h, \quad \Gamma^{--}_{--} = \frac{1}{2} (\partial_+ h + h \partial_-), \]

where \( \partial_\pm = (\partial_0 \pm \partial_1)/\sqrt{2} \). The curvature components and the Laplacian are of the simple form

\[ R^{++} = \frac{1}{2} h \partial_+^2 h, \quad R^{-+} = \frac{1}{2} \partial_-^2 h, \quad R = \partial_+^2 h, \]

\[ \square = \partial_-(2 \partial_+ - h \partial_-). \]

In the light-cone gauge the Liouville action becomes

\[ \hat{I}_L = \int d^2 \xi \left[ -\partial_+ F \partial_- F + \frac{1}{2} h (\partial_- F)^2 - \frac{1}{2} \alpha F \partial_-^2 h - \mu^2 \right]. \quad (21) \]

The equation of motion for \( F \) takes the form

\[ \square F = (\alpha/2) R, \quad \Rightarrow \quad \partial_- (2 \partial_+ - h \partial_-) F = (\alpha/2) \partial_-^2 h. \quad (22) \]

The equations of motion for \( g_{\alpha\beta} \) can be obtained from the energy-momentum (EM) tensor. By using the relation

\[ \delta \int d^2 \xi \sqrt{-g} F R = - \int d^2 \xi \sqrt{-g} \delta g^{\alpha\beta} (\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2) F, \]

one easily finds

\[ T_{\alpha\beta} = -\frac{1}{2} \partial_\alpha F \partial_\beta F + \frac{\alpha}{2} \left( \nabla_\alpha \nabla_\beta - g_{\alpha\beta} \nabla^2 \right) F + \frac{1}{2} g_{\alpha\beta} \left( \frac{1}{2} g^{\gamma\delta} \partial_\gamma F \partial_\delta F + \mu^2 \right). \]

The traceless part of \( T_{\alpha\beta} \), \( \tilde{T}_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \), takes the form

\[ \tilde{T}_{--} = -\frac{1}{2} (\partial_- F)^2 + \frac{\alpha}{2} \partial_-^2 F = T_{--}, \]

\[ \tilde{T}_{+-} = \frac{1}{2} h T_{--}, \]

\[ \tilde{T}_{++} = \frac{1}{4} h^2 T_{--} + (\alpha^2 / 4) \frac{1}{8} \left[ (\partial_- h)^2 - 2 h \partial_-^2 h + 4 \partial_- \partial_+ h \right]. \quad (23) \]

Here, we used \( T = -\frac{1}{2} (\alpha^2 \partial_-^2 h - 2 \mu^2) \), and eliminated \( F \) by using its equation of motion. The relation

\[ \nabla_+ T_{--} + h \nabla_- T_{--} = \frac{1}{2} (\alpha/2)^2 \partial_-^3 h \]
leads to the important result
\[ \partial^3_- h = 0, \quad (24) \]

**The Polyakov formulation.** To simplify the forthcoming discussion we shall work with \( \alpha / 2 = 1 \). The equation of motion for \( F \) can be solved by looking at the simpler equation
\[ (2\partial_+ - h\partial_-)F = \partial_- h. \quad (25) \]
If \( f \) is a solution of the homogenous equation,
\[ (2\partial_+ - h\partial_-)f = 0, \quad (26a) \]
then one can check that
\[ F_0 = \ln \partial_- f \quad (26b) \]
is a particular solution of the complete equation (25), and therefore a solution of (22).

The action (21) can be written in the form
\[ \hat{I}_L = -\int d^2\xi \left( \frac{1}{2} \partial^2_- h F_0 + \mu^2 \right). \quad (27a) \]
After going over to the \( f \) variable, we find
\[ \hat{I}_L[f] = -\int d^2\xi \left[ -\frac{1}{2} \partial_- \left( \frac{2\partial_+ f}{\partial_- f} \right) \partial_- \ln \partial_- f + \mu^2 \right] \]
\[ = \int d^2\xi \left[ \frac{(\partial_- \partial_+ f) (\partial^2_- f)}{(\partial_- f)^2} - \frac{(\partial^2_- f)(\partial_+ f)}{(\partial_- f)^3} - \mu^2 \right]. \quad (27b) \]

It is interesting to note that by varying this action over \( f \) and after partial integrations \((\delta f = 0 \text{ at the boundary})\) one obtains the relation
\[ -2\partial_+ \left( \frac{\partial^2_- F_0}{\partial_- f} \right) + \partial_- \left( \frac{h\partial^2_- f_0 - \partial^2_- h}{\partial_- f} \right) = 0, \]
which is easily transformed into the simple form (24).

**SL(2,R) symmetry.** An interesting feature of the gauge-fixed Liouville action is the presence of a residual symmetry. Let us show that the transformation
\[ \delta_0 f = \epsilon^- \partial_- f \quad (28a) \]
is an invariance of the action (21) if the parameter \( \epsilon^- \) satisfies certain conditions. The above transformation implies
\[ \delta_0 h = 2\partial_+ \epsilon^- - h\partial_- \epsilon^- + \epsilon^- \partial_- h, \]
\[ \delta_0 F = \epsilon^- \partial_- F + \frac{\alpha}{2} \partial_- \epsilon^-, \quad (28b) \]
Then, a direct calculation yields

$$\delta \hat{I}_L = \int d^2 \xi h \partial_3 \varepsilon^-.$$ 

Therefore, the transformation (28) is an invariance of the action provided $\varepsilon^-(\xi^+, \xi^-)$ satisfies the condition

$$\partial_3 \varepsilon^- = 0. \quad (29)$$

An interesting interpretation of the nature of this symmetry was given by Dass and Summitra [8]. The classical string action is invariant under local reparametrizations and Weyl rescalings:

$$\delta_0 g_{--} = 2 \partial_- \varepsilon^- g_{--} + 2 \partial_- \varepsilon^+ g_{+-} + \varepsilon \cdot \partial g_{--} + \lambda g_{--},$$

$$\delta_0 g_{+-} = \partial_+ \varepsilon^+ g_{+-} + \partial_+ \varepsilon^- g_{--} + \partial_- \varepsilon^+ g_{++} + \partial_- \varepsilon^- g_{+-+-} + \varepsilon \cdot \partial g_{+-} + \lambda g_{+-},$$

$$\delta_0 g_{++} = 2 \partial_+ \varepsilon^- g_{+-} + 2 \partial_+ \varepsilon^+ g_{++} + \varepsilon \cdot \partial g_{++} + \lambda g_{++}.$$ 

Now, demanding that the above transformations do not change the light-cone gauge, we obtain the following conditions on the parameters $\varepsilon^+, \varepsilon^-$ and $\lambda$:

$$\delta_0 g_{--} = 2 \partial_- \varepsilon^+ = 0,$$

$$\delta_0 g_{+-} = \partial_+ \varepsilon^+ + \partial_+ \varepsilon^- + \lambda = 0.$$ 

These conditions yield

$$\varepsilon^+ = \varepsilon^+(\xi^+), \quad \lambda = -\partial_+ \varepsilon^+ - \partial_- \varepsilon^-,$$

whereafter the transformation law of $g_{++} = h$ becomes

$$\delta_0 h = (2 \partial_+ \varepsilon^- + \varepsilon^- \partial_+ h - h \partial_+ \varepsilon^-) + (2 \partial_+ \varepsilon^+ h + \varepsilon^+ \partial_+ h - h \partial_+ \varepsilon^+).$$

Finally, choosing $\varepsilon^+ = 0$, we obtain the correct form of the symmetry transformation for $h$. The symmetry (28) is thus seen to be a remnant of the original classical string symmetry in the light-cone gauge.

There is an apparent contradiction between the determination of $h$ as the $(++)$ component of the metric, and the fact that (28) is not the tensorial transformation law. This can be easily understood if we note that in the light-cone gauge $\det g_{\alpha \beta} = -1$, so that $h$ can be equally well considered as a tensor density. Indeed, the transformation law (28) is in agreement with this assumption.

**Exercise.** Derive the transformation law for $G_{++} \equiv (\sqrt{g} g_{++})$ under reparametrizations with $\varepsilon^-$. 

Let us now show that the residual symmetry is $SL(2, R)$. It is clear from Eq.(29) that $\varepsilon^-$ must be a polynomial in $\xi^-$:

$$\varepsilon^-(\xi^+, \xi^-) = \omega_-(\xi^+) + \xi^- \omega_0(\xi^+) + (\xi^-)^2 \omega_+(\xi^+). \quad (30)$$
Using this form of $\varepsilon^-$ the transformation law for $h$ can be rewritten in the form

$$
\delta_0 h = (\omega_- l^- + \omega_0 l^0 + \omega_+ l^+) h + 2 (\partial_+ \omega_- + \xi^- \partial_+ \omega_0 + (\xi^-)^2 \partial_+ \omega_+) ,
$$

$$
l^- \equiv \partial_- , \quad l^0 \equiv \xi^- \partial_- - 1 , \quad l^+ \equiv (\xi^-)^2 \partial_- - 2 \xi^- .
$$

(31)

The quantities $l^a$ are the generators of the $SL(2, R)$ algebra (see Appendix C):

$$
[l^0, l^-] = -l^-, \quad [l^0, l^+] = l^+, \quad [l^+, l^-] = -2l^0 .
$$

(32)

Thus, up to the $\partial_+ \omega_a$ terms, the transformation of $h$ is an $SL(2, R)$ transformation.

A more transparent way of seeing the nature of the transformation law of $h$ is the following. The solution of the equation of motion (24) for $h$ is a polynomial in $\xi^-:

$$
h(\xi^+, \xi^-) = J^+(\xi^+) - 2 \xi^- J^0(\xi^+) + (\xi^-)^2 J^-(\xi^-) .
$$

(33)

The transformation law for $J^a$ is given by

$$
\delta_0 J^+ = -2\omega_- J^0 - \omega_0 J^+ + 2\partial_+ \omega_-, \\
\delta_0 J^0 = \omega_+ J^+ - \omega_- J^- - \partial_+ \omega_0 , \\
\delta_0 J^- = \omega_0 J^- + 2\omega_+ J^0 + 2\partial_+ \omega_+ .
$$

(34a)

By using the results of Appendix C, one can rewrite the above relations in the form

$$
\delta_0 J^a = f^{abc} \omega_b g_{cd} J^d + 2g^{ab} \partial_+ \omega_b ,
$$

(34b)

where the $SL(2, R)$ character of $J^a$ and $\omega_b$ is explicit.

It is interesting to note that after using $T_{--} = 0$, the traceless EM tensor $\tilde{T}^{++}$ can be written in terms of the quantities $J^a$ as follows:

$$
\tilde{T}^{++}_y = \frac{1}{4} [2(J^0)^2 - J^+ J^- - J^- J^+] + (-\partial_+ J^0 + \xi^- \partial_+ J^-) .
$$

(35a)

Now, the equation $T = 0$ leads to $J^- = \mu^2$, and, consequently, to $\partial_+ J^- = 0$, so that the EM tensor can be written in the Sugawara form:

$$
\tilde{T}^{++}_y = \frac{1}{4} g_{ab} J^a J^b - \partial_+ J^0 .
$$

(35b)
3. Equivalence between conformal and light-cone gauge

We shall now consider a relation between the conformal and the light-cone gauge, and see how one can derive Polyakov’s action and the EM tensor in the light-cone gauge from the corresponding expressions in the conformal gauge [10-12].

1. In the conformal gauge the metric is defined by

\[ ds^2 = 2e^{\phi(x)}dx^+dx^-, \]

whereas in the light-cone gauge it takes the form

\[ ds^2 = h(y)dy^+dy^+ + 2dy^+dy^- . \]

It is easy to see that the coordinate transformation

\[ x^+ = y^+, \quad x^- = f(y^+, y^-), \quad (36a) \]

transforms the first metric into the second one provided the function \( f(y) \) satisfies the conditions

\[ h(y) = 2e^{\phi(x)}\partial_+ f(y), \quad 1 = e^{\phi(x)}\partial_- f(y). \quad (36b) \]

From here we obtain the relation

\[ h = 2\frac{\partial_+ f}{\partial_- f}, \quad (36c) \]

that has been extensively used by Polyakov in his study of the light-cone gauge dynamics.

We see that the function \( f(y) \) has a clear geometrical meaning. The above coordinate transformation will be used to relate certain results on the induced 2-d gravity in the conformal and the light-cone gauge, respectively.

2. The local form of the Liouville action is given by Eq.(19). To simplify the forthcoming expressions we shall work with \( \alpha/2 = 1. \) By using the equation of motion for \( F \) and going over to the conformal gauge,

\[ \Box F = R \quad \implies \quad \Box F = -\Box \phi, \]

the Liouville action takes the form

\[ I_L = -\int d^2x \left( \frac{\partial F}{\partial x^+} \frac{\partial F}{\partial x^-} + \mu^2 e^{\phi} \right). \quad (37) \]

In what follows, we shall use the notation \( F \equiv F(x), \) \( F' \equiv F(y), \) and assume that \( F \) is a scalar function, i.e. \( F' = F. \) By using the transformation rule (36) the action (37) goes over into the expression

\[ I_L = -\int d^2y(\partial_- f)[-\frac{1}{2}(2\partial_+ - h\partial_-)F'(\partial_- f)^{-1}\partial_- F' + \mu^2 e^{\phi}] \]

\[ = -\int d^2y [ -\frac{1}{2}\partial_+ h\partial_- F_0 + \mu^2 ], \quad (38) \]

13
which is the Liouville action in the light-cone gauge, as demonstrated in sect. 2.

3. The EM tensor corresponding to the Liouville action in the conformal gauge takes the form

\[ T^x_{++} = -\frac{1}{2} \left( \frac{\partial F}{\partial x^+} \right)^2 + \nabla_+^2 F, \]
\[ T^x_{--} = -\frac{1}{2} \left( \frac{\partial F}{\partial x^-} \right)^2 + \nabla_-^2 F, \]
\[ T^x_{+-} = -\nabla_+ \nabla_- F + \frac{1}{2} \mu^2 e^\phi, \]  
\[ \text{(39a)} \]

where

\[ \nabla_\pm^2 F = (\partial_\pm - \partial_\pm \phi)\partial_\pm F, \quad \nabla_+ \nabla_- F = \partial_+ \partial_- F, \]

and \( \partial_\pm F \) can be replaced by \( -\partial_\pm \phi \). The equation \( T^x_{+-} = 0 \) is the well known Liouville equation.

The transition to the light-cone gauge is realised with the help of the coordinate transformation (36),

\[ T_{\alpha\beta}(y) = \frac{\partial x^\gamma}{\partial y^\alpha} \frac{\partial x^\delta}{\partial y^\beta} T_{\gamma\delta}(x), \]

which leads to

\[ T^y_{++} = T^x_{++} + h(\partial_- f) T^x_{+-} + \frac{1}{4} h^2 (\partial_- f)^2 T^x_{--}, \]
\[ T^y_{--} = (\partial_- f)^2 T^x_{--}, \]
\[ T^y_{+-} = (\partial_- f) T^x_{+-} + \frac{1}{2} h (\partial_- f)^2 T^x_{--}. \]  
\[ \text{(40)} \]

The components \( T_{\alpha\beta}(x) \), when expressed in the light-cone gauge coordinates \( y \), take the form

\[ T^x_{++} = \frac{1}{8} [ (\partial_- h)^2 - 2h \partial_-^2 h ] - \frac{1}{2} \partial_+ \partial_- h, \]
\[ (\partial_- f)^2 T^x_{--} = -\partial_-^2 \ln \partial_- f + \frac{1}{2} (\partial_- \ln \partial_- f)^2 \equiv -\{ f, y^- \}, \]
\[ (\partial_- f) T^x_{+-} = -\frac{1}{2} \partial_-^2 h + \frac{1}{2} \mu^2, \]  
\[ \text{(41)} \]

where \( \{ f, y^- \} \) denotes the Schwarz derivative.

Introducing the traceless tensor \( \tilde{T}^y_{\alpha\beta} \), the above equation can be rewritten as

\[ \tilde{T}^y_{++} = T^y_{++} - \frac{1}{2} h T^y = T^x_{++} + \frac{1}{4} h^2 T^y_{--}, \]
\[ \tilde{T}^y_{--} = T^y_{--}, \quad \tilde{T}^y_{+-} = 0. \]  
\[ \text{(42)} \]

By using the solution of the equation of motion \( \{ f, y^- \} = 0 \),

\[ f(y^+, y^-) = \frac{a(y^+) y^- + b(y^+)}{c(y^+) y^- + d(y^+), \quad ad - bc = 1,} \]  
\[ \text{(43)} \]

(which is \( SL(2, R) \) invariant) the corresponding expression for \( h(y) \) takes the form (33), where the currents \( J^a \) are determined in terms of the coefficients \( a, b, c \) and \( d \). Then, a direct calculation of \( \tilde{T}^y_{++} \), with the help of the equation \( T^y_{+-} = 0 \), leads to the effective EM tensor as in Eq.(35a). Thus, the solution (43) correctly reveals the \( SL(2, R) \) structure of the theory.
4. Concluding remarks

We have seen that $SL(2, R)$ is the residual symmetry of the gauge-fixed Liouville action. However, $SL(2, R)$ is not the symmetry of the complete Liouville theory in the light-cone gauge [8,9], as the gauge-fixed action is not equivalent to the gauge-fixed theory. The essential features of the residual $SL(2, R)$ symmetry can be successfully treated in the Hamiltonian framework [13-14]. This is an important step in understanding the full quantum Liouville theory [15,16].

Appendix

A. Geometry of surfaces . Here we shall state several mathematical theorems (non-rigorously and without the proof) which will be very useful in discussing the functional integral $Z_0$ in Section 1.

**T1.** Any compact, oriented, closed and connected two-dimensional manifold without boundaries is topologically equivalent to a sphere with $\gamma$ handles. The genus $\gamma$ is related to the Euler characteristic $\chi$ by the relation (the global Gauss-Bonnet theorem)

$$
\frac{1}{4\pi} \int d^2 \xi \sqrt{g} R \equiv \chi = 2 - 2\gamma.
$$

**T2.** A manifold $\Sigma$ endowed with a metric $g = (g_{\alpha\beta})$ is called the metric space. Any metric on $\Sigma$ can be locally put into the conformally flat form

$$g_{\alpha\beta}(\xi) = e^{\phi(\xi)} \delta_{\alpha\beta},$$

by a suitable coordinate transformation.

By a Weyl rescaling we can, again locally, reduce the metric to the Euclidean form, $g_{\alpha\beta} = \delta_{\alpha\beta}$. For $\gamma \neq 1$ this cannot be done globally; indeed, globally flat metric implies $R = 0$ and, therefore, $\chi = 0$, which is possible only if $\gamma = 1$.

**T3.** Any oriented two-dimensional metric space $(\Sigma, g)$ is a complex manifold.

**T4.** Let $\Sigma$ be a compact, oriented and closed two-dimensional manifold of genus $\gamma$, with a metric $g$. Then, there exists a (nonsingular) combination of diffeomorphisms (local + global !) and Weyl rescalings that transforms the metric $g$ into one of the following canonical forms (characterized by the constant curvature),

$$
R = +1, \quad ds^2 = \frac{|dz|^2}{(1 + |z|^2)^2}, \quad \gamma = 0, \\
R = 0, \quad ds^2 = |dz|^2, \quad \gamma = 1, \\
R = -1, \quad ds^2 = \frac{|dz|^2}{|Im z|^2}, \quad \gamma \geq 2,
$$

whereas the manifold $\Sigma$ goes over into

$$
\hat{C}, \quad \gamma = 0, \\
C/\Gamma_1, \quad \gamma = 1, \\
U/\Gamma_\gamma, \quad \gamma \geq 2.
$$
Here, $\hat{C}$ is the compactified complex plane (the Riemann sphere), $C$ is the complex plane, $U$ is the open upper half plane excluding $z = \infty$, while $\Gamma_\gamma$ are some discrete groups of conformal transformations, which leave the canonical metrics invariant. The notation $C/\Gamma_1$ means that the points in $C$ which are connected by $\Gamma_1$ should be identified, and similarly for $U/\Gamma_\gamma$.

One can show that there exists the fundamental region in $C (U)$ such that any point in $C (U)$ can be obtained from this fundamental region by applying an element of $\Gamma_\gamma$. Note that $\Gamma_0$ is trivial, i.e. the manifold $\Sigma_0$ can be obtained from the Riemann sphere $\hat{C}$ by using only Weyl rescalings and local diffeomorphisms.

Comment. The group of all diffeomorphisms $Diff$ consists of the local diffeomorphisms $Diff_0$ (which are conected to the identity) and the global ones (which are not).

T5. The number of the conformal Killing vectors $N'_\gamma$ of the metrics (6a) is given by the relation
\[
N'_0 = 6, \quad \gamma = 0,
N'_1 = 2, \quad \gamma = 1,
N'_\gamma = 0, \quad \gamma \geq 2.
\]

The group $\Gamma_\gamma$ is uniquely determined by a finite number $N_\gamma$ of real Teichmüller parameters. For all $\gamma$ we have a relation
\[
N_\gamma - N'_\gamma = 6\gamma - 6,
\]
which is known as the Riemann-Roch theorem.

B. General form of the effective action  The effective action (3) is calculated with respect to the flat background metric $\hat{g}_{\alpha \beta} = \eta_{\alpha \beta}$. We shall find out here its form with respect to an arbitrary background $\hat{g}_{\alpha \beta}$.

First we note that the anomaly can be written in the covariant form
\[
A = \kappa \int d^2 \xi C(\xi) \sqrt{-g} (-R(g) + \mu^2).
\]
This follows from Eq.(1) by noting that $\sqrt{-g}R(g) = -\Box \phi$ and $\sqrt{-g} = e^\phi$ in the conformally flat gauge. Now we can generalize the gauge condition to the form,
\[
g_{\alpha \beta} = e^\phi \hat{g}_{\alpha \beta},
\]
(which, in the functional integral, corresponds to integrating over an arbitrary Weyl slice in the space of all metrics) so that, after using the identity
\[
\sqrt{-g}R = \sqrt{-\hat{g}} (\hat{R} - \Box \phi),
\]
the anomaly becomes
\[
A[\phi, \hat{g}] = \kappa \int d^2 \xi C(\xi) \sqrt{-\hat{g}} (\Box \phi - \hat{R} + \mu^2 e^\phi).
\]
Integrating the relation (2) we obtain the effective action in the form

\[ W[\phi, \hat{g}] = \kappa \int d^2 \xi \sqrt{-\hat{g}} \left( \frac{1}{2} \phi \hat{\square} \phi - \phi \hat{R} + \mu^2 e^\phi \right), \]  

(B1)

which is a natural generalization of Eq.(3).

**C. The SL(2, R) algebra.** The basic matrix representation of the group \( SL(2, R) \) consists of all real matrices \( M \) of order two, with \( \det M = 1 \). The corresponding generators are real, traceless \( 2 \times 2 \) matrices. Let us choose a basis of generators

\[ T_2 = \frac{1}{2} \sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_0 = \frac{1}{2} i \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_1 = \frac{1}{2} \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

where \( \sigma_i \) are the Pauli matrices. The commutation rules

\[ [J_0, T_2] = T_1, \quad [J_0, T_1] = -T_2, \quad [T_1, T_2] = J_0, \]

define \( sl(2, R) \), the Lie algebra of \( SL(2, R) \). In the \((+-)\) basis

\[ G_\pm = J_0 \pm T_2, \quad G_0 = 2T_1, \]

the commutation relations take the form

\[ [G_0, G_+] = 2G_+, \quad [G_0, G_-] = -2G_-, \quad [G_+, G_-] = -G_0. \]  

(C1)

It is now easy to find the structure constants of the algebra,

\[ [G_a, G_b] \equiv f_{abc} G_c, \]

\[ f_{00} = 2, \quad f_{0-} = -2, \quad f_{+-} = -1, \]

and the related Cartan’s metric,

\[ g_{ab} \equiv -Tr(G_a G_b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (a, b = +, 0, -). \]  

(C2)

Since the metric \( g_{ab} \) is nonsingular, one can calculate its inverse \( g^{ab} \) and define completely antisymmetric structure constants:

\[ f^{abc} \equiv g^{ad} g^{be} f_{de}^c, \quad f^{+-} = 1. \]  

(C3)

The values of the structure constants and metric depend on the choice of basis. The quadratic invariant takes the correct form

\[ g^{ab} G_a G_b = 2(G_+ G_-) - \frac{1}{2} (G_0)^2 = 2 \left[ (J_0)^2 - (T_2)^2 - (T_1)^2 \right], \]
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