Counterfactual Optimism: 
Rate Optimal Regret for Stochastic Contextual MDPs

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Abstract
We present the UC₃RL algorithm for regret minimization in Stochastic Contextual MDPs (CMDPs). The algorithm operates under the minimal assumptions of realizable function class, and access to offline least squares and log loss regression oracles. Our algorithm is efficient (assuming efficient offline regression oracles) and enjoys an \(\tilde{O}(H^3 \sqrt{T|S||A| \log(|F|/\delta) + \log(|P|/\delta)})\) regret guarantee, with \(T\) being the number of episodes, \(S\) the state space, \(A\) the action space, \(H\) the horizon, and \(P\) and \(F\) are finite function classes, used to approximate the context-dependent dynamics and rewards, respectively. To the best of our knowledge, our algorithm is the first efficient and rate-optimal regret minimization algorithm for CMDPs, which operates under the general offline function approximation setting.

1 Introduction
Reinforcement Learning (RL) is a field of machine learning that pertains to sequential decision making under uncertainty. A Markov decision process (MDP) is the basic mathematical model at the heart of RL theory, and has been studied extensively. An agent repeatedly interacts with an MDP by observing its state \(s \in S\) and choosing an action \(a \in A\). Subsequently, the process transitions to a new state \(s'\) and the agent observes an instantaneous return, indicating the quality of their choice. MDPs have the power to characterize many real-life applications including: advertising, healthcare, games, robotics and more, where at each episode an agent interacts with the environment with the goal of maximizing her return. (See, e.g., Sutton and Barto [18] and Mannor et al. [13].)

However, in many modern applications there is additional side information that affects the environment. We refer to that information as the context. A naive way to handle the context is to extend the state space of the environment to include it. This approach has the disadvantage of increasing the state space, and hence the complexity of learning and the representation of a policy. Contextual MDPs (CMDPs), on the other hand, keep a small state space, and regard the context as additional side-information, observed by the agent at the start of each epoch. Additionally, there exists a mapping from each context to an MDP, and the optimal policy given a context is the optimal policy in the related MDP (Hallak et al. [8]). A classic example of a context is a user’s information (age, interests, etc.), which can deeply influence decision making yet remains static throughout the episode. This makes CMDPs an excellent model for recommendation systems.

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In recent years, regret bounds for CMDPs have been derived under various assumptions regarding the CMDP or the function approximation oracles. A significantly distinctive feature between works is whether they assume access to online or offline oracles. In both settings we have a function class \( \mathcal{F} = \{ f : X \to Y \} \), a loss \( \ell : Y \times Y \to \mathbb{R} \), and a dataset \( \{(x_i, y_i)\}_{i=1}^n \). An offline oracle observes the entire data and needs to find \( f_n \in \arg \min_{f \in \mathcal{F}} \sum_{i=1}^n \ell(f(x_i), y_i) \). On the other hand, an online oracle makes a sequence of predictions \( f_1, \ldots, f_n \) where \( f_i \) can depend on data up to \( i-1 \), and its goal is to minimize regret, given by \( \sum_{i=1}^n \ell(f(x_i), y_i) - \ell(f^*(x_i), y_i) \). It is well-known that the offline problem can be easier to solve than the online problem. Moreover, practical applications, e.g., deep learning, typically work in the offline regime.

Modi and Tewari [14] obtained \( \bar{O}(\sqrt{T}) \) regret for a generalized linear model (GLM). Foster et al. [7] obtain \( \bar{O}(\sqrt{T}) \) regret for general function approximation and adversarially chosen contexts, assuming access to a strong online estimation oracle. It thus remained open whether, for stochastic contexts, we can restrict the access to offline oracles. Recently, Levy and Mansour [12] gave an \( \bar{O}(\sqrt{T}/p_{min}) \) regret algorithm for stochastic contexts using offline least squares regression, where \( p_{min} \) is a minimum reachability parameter of the CMDP. This parameter can be arbitrarily small and for general CMDPs leads to an \( \bar{O}(T^{3/4}) \) regret guarantee. In this work we obviate the minimum reachability assumption and give the first \( \bar{O}(\sqrt{T}) \) regret algorithm for stochastic contexts using an offline oracle.

Contributions. We present the UC3RL algorithm, and prove that its regret is bounded as \( \bar{O}\left(H^3 \sqrt{T||S||A}(\log(|\mathcal{F}|/\delta) + \log(|\mathcal{P}|/\delta))\right) \) with probability at least \( 1 - \delta \), where \( S \) is the state space, \( A \) the action space, \( H \) the horizon, and \( \mathcal{P} \) and \( \mathcal{F} \) are finite function classes used to approximate the context-dependent dynamics and rewards, respectively. Our algorithm builds on the “optimistic in expectation” approach of Levy and Mansour [12] but modifies it with a log-loss oracle for the dynamics approximation and carefully chosen reward bonuses. A key technical tool that enables our result is a multiplicative change of measure inequality for the value function, which may be of separate interest. To the best of our knowledge, UC3RL is the first rate-optimal regret algorithm that only assumes access to offline regression oracles for dynamics and reward approximation.

Additional Related Work. Hallak et al. [8] were the first to study regret guarantees in the CMDP model. However, they assume a small context space, and their regret is linear in its size. Jiang et al. [9] present OLIVE, a sample efficient algorithm for learning Contextual Decision Processes (CDP) under the low Bellman rank assumption. In contrast, we do not make any assumptions on the Bellman rank. Modi et al. [15] present generalization bounds for learning smooth CMDPs and finite contextual linear combinations of MDPs. Modi and Tewari [14] present a regret bound of \( \bar{O}(\sqrt{T}) \) for CMDPs under a Generalized Linear Model (GLM) assumption. Our function approximation framework is more general than smooth CMDPs or GLM.

Foster et al. [7] present the Estimation to Decision (E2D) meta algorithm and apply it to obtain \( \bar{O}(\sqrt{T}) \) regret for Contextual RL. However, they assume access to an online estimation oracle and their bounds scale with its regret. In contrast, we use substantially weaker offline regression oracles. It is not clear to us whether their Inverse Gap Minimization (IGM) technique can be applied to CMDPs with offline regression oracles.

Levy and Mansour [11] study the sample complexity of learning CMDPs using function approximation, and provide the first general and efficient reduction from CMDPs to offline supervised learning. However, their sample complexity scales as \( \epsilon^{-8} \) and thus they cannot achieve the optimal \( \sqrt{T} \) rate for the regret. Levy and Mansour [12], previously mentioned here in relation to upper
bounds, also showed an $\Omega(\sqrt{TH|S||A|\log(|G|/|S|)/\log(|A|)})$ regret lower bound for the general setting of offline function approximation with $|G|$, the size of the function class.

More broadly, CMDPs are a natural extension of the extensively studied Contextual Multi-Armed Bandit (CMAB) model. CMABs augment the Multi-Arm Bandit (MAB) model with a context that determines the rewards [10, 17]. Agarwal et al. [2] use an optimization oracle to derive an optimal regret bound that depends on the size of the policy class they compete against. Regression based approaches appear in [1, 4, 5, 6, 16]. Most closely related to our work, Xu and Zeevi [19] present the first optimistic algorithm for CMAB. They assume access to a least-squares regression oracle and achieve $\tilde{O}(\sqrt{TH|A|\log|F|})$ regret, where $F$ is a finite and realizable function class, used to approximate the rewards. Extending their techniques to CMDPs necessitates accounting for the context-dependent dynamics whose interplay with the rewards significantly complicates decision making. Indeed, this is the main challenge both here and in [12].

2 Preliminaries

2.1 Episodic Loop-Free Markov Decision Process (MDP)

An MDP is defined by a tuple $(S, A, P, r, s_0, H)$, where $S$ and $A$ are finite sets describing the state and action spaces respectively; $s_0 \in S$ is the unique start state; $H \in \mathbb{N}$ is the horizon; $P : S \times S \times A \to [0, 1]$ defines the probability of transitioning to state $s'$ given that we start at state $s$ and perform action $a$; and $r(s, a)$ is the expected reward of performing action $a$ at state $s$.

An episode is a sequence of $H$ interactions where at step $h$, if the environment is at state $s_h$ and the agent performs action $a_h$ then (regardless of past history) the environment transitions to state $s_{h+1} \sim P(\cdot|s_h, a_h)$ and the agent receives reward $R(s_h, a_h) \in [0, 1]$, sampled independently from a distribution $D_{s_h, a_h}$ that satisfies $r(s_h, a_h) = \mathbb{E}_{D_{s_h, a_h}}[R(s_h, a_h)]$.

For technical convenience and without loss of generality, we assume that the state space and accompanying transition probabilities have a loop-free (or layered) structure. Concretely, we assume that the state space can be decomposed into $H + 1$ disjoint subsets (layers) $S_0, S_1, \ldots, S_{H-1}, S_H$ such that transitions are only possible between consecutive layers, i.e., for $h' \neq h + 1$ we have $P(s_{h'}|s_h, a) = 0$ for all $s_{h'} \in S_{h'}, s_h \in S_h, a \in A$. In addition, $S_H = \{s_H\}$, meaning there is a unique final state with reward 0. We note that this assumption can always be satisfied by increasing the size of the state space by a factor of $H$.

A deterministic stationary policy $\pi : S \to A$ is a mapping from states to actions. Given a policy $\pi$ and MDP $M = (S, A, P, r, s_0, H)$, the $h \in [H - 1]$ stage value function of a state $s \in S_h$ is defined as

$$V_{M,h}^\pi(s) = \mathbb{E}_{\pi,M} \left[ \sum_{k=h}^{H-1} r(s_k, a_k) \bigg| s_h = s \right].$$

For brevity, when $h = 0$ we denote $V_{M,0}^\pi(s_0) := V_M^\pi(s_0)$, which is the expected cumulative reward under policy $\pi$ and its measure of performance. Let $\pi_M^* \in \arg\max_\pi \{V_M^\pi(s_0)\}$ be the optimal policy for MDP $M$. It is well known that such a policy is optimal even among the class of stochastic and history dependent policies (see, e.g., [18]).

2.2 Problem Setup: Stochastic Contextual Markov Decision Process (CMDP)

A CMDP is a tuple $(C, S, A, M)$ where $C \subseteq \mathbb{R}^d$ is the context space, $S$ the state space and $A$ the action space. The mapping $M$ maps a context $c \in C$ to an MDP $M(c) = (S, A, P^c, r^c, s_0, H)$, where $r^c(s, a) = \mathbb{E}[R^c(s, a)|c, s, a]$, $R^c(s, a) \sim D_{c,s,a}$. We assume that $R^c(s, a) \in [0, 1]$. 

We consider a stochastic CMDP, meaning, the context is stochastic. Formally, we assume that there is an unknown distribution \( D \) over the context space \( C \), and for each episode a context is sampled i.i.d. from \( D \). For mathematical convenience, we assume the context space \( C \) is finite (but potentially huge). Our results naturally extend to an infinite context space.

A deterministic context-dependent policy \( \pi = (\pi(c; \cdot) : S \to A)_{c \in C} \) maps a context \( c \in C \) to a policy \( \pi(c; \cdot) : S \to A \). Let \( \Pi_C \) denote the class of all deterministic context-dependent policies.

**Interaction protocol.** The interaction between the agent and the environment is defined as follows. In each episode \( t = 1, 2, ..., T \) the agent:

1. Observes context \( c_t \sim D \);
2. Chooses a policy \( \pi_t \) (based on \( c_t \) and the observed history);
3. Observes trajectory \( \sigma^t = (c_t, s^t_0, a^t_0, r^t_0, \ldots, s^t_{H-1}, a^t_{H-1}, r^t_{H-1}, s^t_H) \) generated by playing \( \pi_t \) in \( M(c_t) \).

Our goal is to minimize the regret, defined as

\[
\text{Regret}_T(\text{Algorithm}) := \sum_{t=1}^{T} V^{\pi^*_t(c_\cdot)}_{M(c_t)}(s_0) - V^{\pi_t(c_\cdot)}_{M(c_t)}(s_0),
\]

where \( \pi^* \in \Pi_C \) is an optimal context-dependent policy.

### 2.3 Assumptions

We note that, without further assumptions, the regret may scale linearly in the size of the context space (see, e.g., Hallak et al. [8]). Even worse, if the context space contains more than \( T \) contexts, and the distribution over the contexts is uniform, the regret may scale linearly in \( T \). We overcome this limitation by imposing the following function approximation assumptions, which extend similar notions in the Contextual Bandits literature (see, e.g., [1, 5, 6, 16]) to CMDPs.

**Realizable Reward function approximation.** Our algorithm gets as input a finite function class \( F \subseteq C \times S \times A \to [0, 1] \) such that there exists \( f^* \in F \) that satisfies \( f^*_c(s, a) = r^*_c(s, a) \) for all \( c \in C \) and \( (s, a) \in S \times A \).

**Realizable Dynamics function approximation.** Our algorithm gets as input a finite function class \( P \subseteq S \times (S \times A \times C) \to [0, 1] \) such that \( P^* \in P \), and every function \( P \in P \) represents valid transition probabilities, i.e., satisfies \( \sum_{s' \in S} P(s'|s, a, c) = 1 \) for all \( c \in C \) and \( (s, a) \in S \times A \). For convenience, we denote \( P(s'|s, a, c) = P^c(s'|s, a) \), for all \( P \in P \).

**Offline regression oracles.** Given a data set \( D = \{(c_i, s_i, a_i, s'_i, r_i)\}_{i=1}^n \), we assume access to offline oracles that solve the optimization problems

\[
\hat{f} \in \arg \min_{f \in F} \sum_{i=1}^{n} (f(c_i, s_i, a_i) - r_i)^2, \quad \text{(Least Squares Regression (LSR))}
\]

\[
\hat{P} \in \arg \min_{P \in P} \sum_{i=1}^{n} \log \frac{1}{P^{c_i}(s'_i|s_i, a_i)}, \quad \text{(Log Loss Regression (LLR))}
\]
Notice that the above problems can always be solved by iterating over the function class, however, since we consider strongly convex loss functions, there are function classes where these optimization problems can be solved efficiently. One particular example is the class of linear functions.

3 Algorithm and Main Result

We present Upper Counterfactual Confidence for Contextual Reinforcement Learning (UC³RL), given in Algorithm 1. At each episode $t$, the algorithm estimates the reward and dynamics using the regression oracles. It then constructs an optimistic CMDP using reward bonuses and plays its optimal policy. The reward bonuses are inspired by the notion of counterfactual confidence, suggested by [19] for CMABs. The original idea was to calculate the confidence bounds using the counterfactual actions of past policies given the current context. [12] adapted this approach to CMDPs using the minimum reachability assumption, without which, it becomes crucial to also consider counterfactual states. Notice that the states are stochastically generated by the MDP in response to the agent’s played actions. This makes counterfactual state computation impossible without access to the true dynamics. Instead we consider the counterfactual probabilities of a state action pair, and estimate this quantity using the estimated dynamics. These probabilities are typically referred to as occupancy measures (Zimin and Neu [20]). Concretely, for any non-contextual policy $\pi$ and dynamics $P$, let $q_h(s,a|\pi,P)$ denote the probability of reaching state $s \in S$ and performing action $a \in A$ at time $h \in [H]$ of an episode generated using policy $\pi$ and dynamics $P$. Note that, given $\pi$ and $P$, the occupancy measure of any state-action pair can be computed efficiently, using a standard planning algorithm. Finally, since our bonuses are based on past context-dependent policies, we first have to compute $\pi_k(a_t|\cdot)$ for all $k \in [t-1]$, which is the purpose of our internal loop.

**Algorithm 1 Upper Counterfactual Confidence for Contextual Reinforcement Learning (UC³RL)**

1: **inputs:** MDP parameters: $S, A, s_0, H$. Confidence $\delta > 0$ and tuning parameters $\beta_r, \beta_P$.
2: for round $t = 1, \ldots, T$ do
3:   compute $\hat{f}_t \in \arg \min_{f \in F} \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} (f(c_i,s_h,a_h) - r_h)^2$ using the LSR oracle.
4:   compute $\hat{P}_t \in \arg \min_{P \in P} \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P^c(s_{h+1}|s_h,a_h)} \right)$ using the LLR oracle.
5:   observe a fresh context $c_t \sim \mathcal{D}$
6: for $k = 1, 2, \ldots, t$ do
7:   compute for all $(s,a) \in S \times A$:
8:     $\hat{r}^c_k(s,a) = \hat{f}_k(c_t,s,a) + \min \left\{ 1, \frac{\beta_r/2}{1 + \sum_{i=1}^{k-1} q_h(s_h,a_h|\pi(c_t;\cdot),\hat{P}^c_k)} \right\}$
9:     $\hat{P}^c_k(s,a) = \arg \max_{s \in S} \sum_{a \in A} \hat{r}^c_k(s,a)$ using a planning algorithm.
10: play $\pi_t(c_t;\cdot)$ and observe a trajectory $\sigma^t = (c_t, s_0^t, a_0^t, r_0^t, s_1^t, \ldots, s_{H-1}^t, a_{H-1}^t, r_{H-1}^t, s_H^t)$.

The following is our main result for Algorithm 1.
Theorem 1 (UC$^3$RL regret bound). For any $T > 1$ and $\delta \in (0, 1)$, suppose we run Algorithm 1 with parameters $\beta_r = H \sqrt{\frac{112T \log(18T^4H|F||P|/\delta^2)}{|S||A|\log(T+1)}}$ and $\beta_P = H \sqrt{\frac{157T \log(3T^2|P|/\delta)}{|S||A|\log(T+1)}}$. Then, with probability at least $1 - \delta$, $\text{Regret}_T(\text{UC}^3\text{RL}) \leq \tilde{O}\left(H^3 \sqrt{T|S||A|(\log(|F|/\delta) + \log(|P|/\delta))}\right)$.

We remark that using covering numbers analysis, our result naturally extends to infinite function classes as well as context spaces. In addition, when comparing our regret upper bound to the lower bound of Levy and Mansour [11], there is an apparent gap of $H^{2.5}$ factor, which we leave for future research.

Computational efficiency of UC$^3$RL. The algorithm calls each oracle $T$ times, making it oracle-efficient. Aside from simple arithmetic operations, each of the $T(T+1)/2$ iterations of the internal loop call one MDP planning procedure and calculate the related occupancy measure. Both of these can be implemented efficiently using dynamic programming. Overall, excluding the oracle’s computation time, the run-time complexity of our algorithm is in $\text{poly}(T, |S|, |A|, H)$. Hence, if both the LSR and LLR oracles are computationally efficient then UC$^3$RL is also computationally efficient.

4 Analysis
Our analysis consists of four main steps:

(i) Establish expected regret of the square and log loss regression oracles;

(ii) Construct confidence bounds over the expected value of any context-dependent policy for both dynamics and rewards;

(iii) Define the optimistic approximated CMDP and establish optimism lemmas;

(iv) Combine the above to derive a high probability regret bound.

In what follows, we present the main claims of our analysis, deferring the proofs to Appendix A. Before beginning, we discuss some of the challenges and present a key technical result.

A Key Technical Challenge
Our goal is to derive computable and reliable confidence bounds over the expected value of any policy. The difficulty is that the offline regression oracles have regret guarantees only with respect to the trajectories’ distributions, which are related to the dynamics $P^\star$. Hence, a main technical challenges is to translate the oracle’s regret to a guarantee with respect to the estimated dynamics $\hat{P}_t$. Notice that the confidence bounds are computable only if stated in terms of $\hat{P}_t$. Following ideas from [7], we solve this issue using a multiplicative value change of measure that is based on the Hellinger distance.

Definition 2 (Squared Hellinger Distance). For any two distributions $P, Q$ over a discrete support $X$ we define the Squared Hellinger Distance as

$$D^2_H(P, Q) := \sum_{x \in X} \left(\sqrt{P(x)} - \sqrt{Q(x)}\right)^2.$$
The following change of measure lemma allows us to measure the value difference caused by the use of approximated transition probabilities in terms of the expected cumulative Hellinger distance (see proof in Appendix A.1).

**Lemma 3 (Value change of measure).** Let \( r : S \times A \rightarrow [0, 1] \) be a bounded expected rewards function. Let \( P_* \) and \( \hat{P} \) denote two dynamics and consider the MDPs \( M = (S, A, P_*, r, s_0, H) \) and \( \hat{M} = (S, A, \hat{P}, r, s_0, H) \). Then, for any policy \( \pi \) we have

\[
V_{\pi}^* M(s) \leq 3V_{\pi}^* \hat{M}(s) + 9H^2 \mathbb{E}_{P_*, \pi} \left[ \sum_{h=0}^{H-1} D_H^2 (\hat{P}(\cdot|s_h, a_h), P_*(\cdot|s_h, a_h)) \right] s_0 = s.
\]

Notice that this result is vacuous when the reward function is not very small. However, it is significantly tighter than standard results when the rewards are small. For example, consider the reward \( r_c(s, a) = (\hat{f}_t(c, s, a) - f_*(c, s, a))^2 \), which is the squared reward approximation error.

Letting \( \hat{M} = (S, A\hat{P}, r_c, s_0, H) \) and \( M = (S, AP_*, r_c, s_0, H) \), Lemma 3 implies that the expected reward approximation error with respect to \( \hat{P} \) is at most a constant multiple of the expected reward and dynamics approximation errors with respect to \( P_* \). In contrast, a standard change of measure replaces the squared Hellinger distance with Total Variation (TV), whose cumulative error scales as \( \sqrt{T} \) rather than the smaller \( \log T \) of the Hellinger distance.

**Step 1: Establishing Oracle Guarantees**

The regret guarantees of the least-squares oracle were established in Lemma C.10 of [12], stated here as Lemma 14. The following corollary bounds the cumulative expected least-squares loss of the sequence of the oracle’s predictions (see proof in Appendix A.2).

**Corollary 4 (Reward approximation bound).** Let \( \hat{f}_t \in \mathcal{F} \) be the least squares minimizer in Algorithm 1. For any \( \delta \in (0, 1) \) it holds that with probability at least \( 1 - \delta \) we have

\[
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi_i(c::)}, P_*^* \left[ \sum_{i=0}^{H-1} \left( \hat{f}_t(c, s_h, a_h) - f_*(c, s_h, a_h) \right)^2 \right] s_0 = s \right] \leq 68H \log(2T^3|\mathcal{F}|/\delta),
\]

simultaneously, for all \( t \geq 1 \).

Next, we analyse the expected regret of the dynamic’s log-loss oracle in terms of the Hellinger distance. The following result is a straightforward application of Lemma A.14 in [7] (see proof in Appendix A.2).

**Corollary 5 (Dynamics approximation bound).** Let \( \hat{P}_t \in \mathcal{P} \) be the log loss minimizer in Algorithm 1. For any \( \delta \in (0, 1) \) it holds that with probability at least \( 1 - \delta \) we have

\[
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi_i(c::)}, P_*^* \left[ \sum_{h=0}^{H-1} D_H^2 (P_*^*(\cdot|s_h, a_h), \hat{P}_t^*(\cdot|s_h, a_h)) \right] s_0 = s \right] \leq 2H \log(TH|\mathcal{P}|/\delta),
\]

simultaneously, for all \( t \geq 1 \).
Step 2: Constructing Confidence Bounds

Let $\pi_t$ denote the context-dependent policy played at round $t$. For any state-action pair $(s, a) \in S \times A$, context $c \in C$, and round $t \geq 1$, we define the reward bonuses

$$b^R_t(c, s, a) := \min \left\{ \frac{\beta_r}{2}, \frac{1}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h|\pi_i(c, \cdot), \tilde{P}^c_i)} \right\},$$

$$b^P_t(c, s, a) := \min \left\{ H, \frac{\beta_p H}{2}, \frac{1}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h|\pi_i(c, \cdot), \tilde{P}^c_i)} \right\}. \tag{1}$$

We use these bonuses in our optimistic construction to account for the approximation errors in the rewards and dynamics respectively. Next, for any context $c \in C$ and functions $f \in \mathcal{F}$, $P \in \mathcal{P}$ we define the MDP $\mathcal{M}(f, P)(c) = (S, A, P^c, f(c, \cdot, \cdot), s_0, H)$. The following results derive confidence bounds for the dynamics and rewards approximation in terms of the reward bonuses and approximation errors (see proofs in Appendix A.3).

**Lemma 6 (Confidence bound for rewards approximation w.r.t the approximated dynamics).** Let $P_*$ and $f_*$ be the true context dependent dynamics and rewards. Let $\hat{P}_t$ and $\hat{f}_t$ be the approximated context-dependent dynamics and rewards at round $t$. Then, for any $t \geq 1$, and context-dependent policy $\pi \in \Pi_C$ we have

$$\mathbb{E}_c \left[ V^{\pi(c; \cdot)}_{\mathcal{M}(f_*, \hat{P}_t)}(c)(s_0) - V^{\pi(c; \cdot)}_{\mathcal{M}(\hat{f}_t, \hat{P}_t)}(c)(s_0) \right] \leq \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c, \cdot), \hat{P}_t^c) b^R_t(c, s_h, a_h) \right]$$

$$+ \frac{3}{2\beta_r} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi_i(c, \cdot), P^c_*} \left[ \sum_{h=0}^{H-1} \left( f_*(c_i, s_h^i, a_h^i) - \hat{f}_t(c_i, s_h^i, a_h^i) \right)^2 \right] s_0 \right]$$

$$+ \frac{9H^2}{2\beta_r} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi_i(c, \cdot), P^c_*} \left[ \sum_{h=0}^{H-1} D^2_H(P^c_*(\cdot|s_h, a_h), \hat{P}_t^c(\cdot|s_h, a_h)) \right] s_0 \right]$$

$$+ \frac{H}{2\beta_r}.$$

**Lemma 7 (Confidence bound for dynamics approximation w.r.t the true rewards $f_*$).** Let $P_*$ and $f_*$ be the true context dependent dynamics and rewards. Let $\hat{P}_t$ be the approximated context-dependent dynamics at round $t$. Then, for any $t \geq 1$, and context-dependent policy $\pi \in \Pi_C$ we have

$$\mathbb{E}_c \left[ V^{\pi(c; \cdot)}_{\mathcal{M}(f_*, \hat{P}_t)}(c)(s_0) - V^{\pi(c; \cdot)}_{\mathcal{M}(f_*, \hat{P}_t)}(c)(s_0) \right] \leq \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c, \cdot), \hat{P}_t^c) b^P_t(c, s_h, a_h) \right]$$

$$+ \frac{78H^3}{\beta_p} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi_i(c, \cdot), P^c_*} \left[ \sum_{h=0}^{H-1} D^2_H(P^c_*(\cdot|s_h, a_h), \hat{P}_t^c(\cdot|s_h, a_h)) \right] s_0 \right]$$

$$+ \frac{H^2}{2\beta_p}.$$

Now, applying the high probability approximation bounds in Corollaries 4 and 5 to Lemmas 6 and 7, we obtain the following high probability confidence bound on the value approximation (see proof in Appendix A.3).
Corollary 8. Under the terms of Lemmas 6 and 7, the following holds with probability at least \(1 - 2\delta/3\) simultaneously for all \(t \geq 1\) and \(\pi \in \Pi_C:\)
\[
\mathbb{E}_c \left[ V_{\mathcal{M}(f^*, P^*)}(c)(s_0) - V_{\mathcal{M}(f^*, P^*)}(c)(s_0) \right] \leq \frac{112 H^3}{\beta r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2) + \frac{157H^4}{\beta_P} \log(3TH|\mathcal{P}|/\delta)
\]
\[
+ \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_t)(b^R_t(c, s_h, a_h) + b^P_t(c, s_h, a_h)) \right].
\]

Step 3: Establishing Optimism Lemmas

Following the results of step 2, we define the optimistic-in-expectation context-dependent reward function at every round \(t \geq 1\) as
\[
\hat{r}_t(c, s, a) := \hat{f}_t(c, s, a) + b^R_t(c, s, a) + b^P_t(c, s, a),
\]
where the bonuses are defined in Eq. (1), and we note that \(\hat{r}_t(c, s, a) \in [0, H + 2]\) for all \((c, s, a) \in \mathcal{C} \times S \times A\). The approximated optimistic-in-expectation CMDP at round \(t\) is defined as \((C, S, A, \hat{\mathcal{M}}_t)\) where for any context \(c \in \mathcal{C}\) we define \(\hat{\mathcal{M}}_t(c) := (S, A, \hat{\mathcal{P}}_t, \hat{r}_t, s_0, H)\). We also recall that \(\mathcal{M}(c) = \mathcal{M}(f^*, P^*)\) is the true CMDP. For brevity, we also define the contextual potential of policy \(\pi\) at round \(t\) as
\[
\phi_t(\pi) := \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_t) \right].
\]

The following result shows that the cumulative potential of the played policies is small. The proof is an immediate consequence of Lemma 18, a standard algebraic argument.

Lemma 9 (Cumulative contextual potential upper bound). The following holds.
\[
\sum_{t=1}^{T} \phi_t(\pi_t) = \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_t) \right] \leq 2H|\mathcal{S}||\mathcal{A}| \log(T + 1).
\]

The following Lemmas establish the properties of the optimistic CMDP, i.e., it upper bounds the performance of any policy yet is not significantly larger than the true value on the sequence of played policies (see proofs in Appendix A.4).

Lemma 10 (Optimism in expectation). Let \(\pi^*_t\) be an optimal context-dependent policy for \(\mathcal{M}\). Suppose that \(H > 10\). Under the good event of Corollary 8, we have that for any \(t \geq 1\)
\[
\mathbb{E}_c \left[ V_{\mathcal{M}(c)}(s_0) - V_{\hat{\mathcal{M}}_t(c)}(s_0) \right] \leq \mathbb{E}_c \left[ V_{\mathcal{M}(c)}(s_0) \right] + \frac{157H^4}{\beta_P} \log(3TH|\mathcal{F}||\mathcal{P}|/\delta) + \frac{12H^3}{\beta r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2).
\]

Lemma 11 (The cost of optimism). Under the good event of Corollary 8, we have that for \(t \geq 1\)
\[
\mathbb{E}_c \left[ V_{\mathcal{M}(c)}(s_0) - V_{\hat{\mathcal{M}}_t(c)}(s_0) \right] \leq \mathbb{E}_c \left[ V_{\mathcal{M}(c)}(s_0) \right] + (\beta_r + H\beta_P)\phi_t(\pi_t)
\]
\[
+ \frac{157H^4}{\beta_P} \log(3TH|\mathcal{F}|/\delta) + \frac{12H^3}{\beta r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2).
\]
Step 4: Deriving the Regret Bound

Using the above results, we derive Theorem 1 as follows.

**Proof of Theorem 1.** We prove a regret bound under the following good events. The first event is that of Corollary 8, which occurs with probability at least $1 - 2\delta/3$. The second event is that

\[
\sum_{t=1}^{T} V^{T}_{M(c_t)}(s_0) - V^{T}_{M(c_t)}(s_0) \leq \sum_{t=1}^{T} \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] - \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] + H \sqrt{T \log(6/\delta)}.
\]  

(3)

By Azuma’s inequality (where the filtration is the histories \(\{\mathbb{H}_t\}_{t=1}^{T}\)), the above holds with probability at least \(1 - \delta/3\). Taking a union bound, the good event holds with probability at least \(1 - \delta\). Hence, assume the good events hold, and consider the following derivation.

\[
\sum_{t=1}^{T} \left( \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] - \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] \right)
\]

\[
= \sum_{t=1}^{T} \left( \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] - \mathbb{E}_c \left[ V^{T}_{\hat{M}(c_t)}(s_0) \right] \right) + \sum_{t=1}^{T} \left( \mathbb{E}_c \left[ V^{T}_{\hat{M}(c_t)}(s_0) \right] - \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] \right)
\]

\[
\leq \sum_{t=1}^{T} (\beta_r + H\beta_P)\phi_t(\pi_t) + 2T \left( \frac{157H^4}{\beta_P} \log(3TH|\mathcal{P}|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2) \right)
\]

(Lemmas 10 and 11)

\[
\leq 2H|S||A|(\beta_r + H\beta_P)\log(T + 1) + 2T \left( \frac{157H^4}{\beta_P} \log(3TH|\mathcal{P}|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2) \right)
\]

(Lemma 9)

\[
= 4H^2 \sqrt{112T|S||A|\log(T + 1) \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2)}
\]

\[+ 4H^3 \sqrt{157T|S||A|\log(T + 1) \log(3TH|\mathcal{P}|/\delta)}
\]

\[
\leq 93H^3 \sqrt{T|S||A|\log(T + 1) \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2)}.
\]

Finally, we get that

\[
\text{Regret}_T(UC^3RL) = \sum_{t=1}^{T} V^{T}_{M(c_t)}(s_0) - V^{T}_{M(c_t)}(s_0)
\]

\[
\leq \sum_{t=1}^{T} \left( \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] - \mathbb{E}_c \left[ V^{T}_{M(c_t)}(s_0) \right] \right) + H \sqrt{T \log(3/\delta)} \quad \text{(Eq. (3))}
\]

\[
\leq 94H^3 \sqrt{T|S||A|\log(T + 1) \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2)}
\]

\[
= \tilde{O} \left( H^3 \sqrt{T|S||A| \log(|\mathcal{F}|/\delta) + \log(|\mathcal{P}|/\delta)} \right).
\]

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References

[1] A. Agarwal, M. Dudík, S. Kale, J. Langford, and R. Schapire. Contextual bandit learning with predictable rewards. In Artificial Intelligence and Statistics, pages 19–26. PMLR, 2012.

[2] A. Agarwal, D. Hsu, S. Kale, J. Langford, L. Li, and R. Schapire. Taming the monster: A fast and simple algorithm for contextual bandits. In International Conference on Machine Learning, pages 1638–1646. PMLR, 2014.

[3] Y. Efroni, L. Shani, A. Rosenberg, and S. Mannor. Optimistic policy optimization with bandit feedback. arXiv preprint arXiv:2002.08243, 2020.

[4] D. Foster and A. Rakhlin. Beyond ucb: Optimal and efficient contextual bandits with regression oracles. In International Conference on Machine Learning, pages 3199–3210. PMLR, 2020.

[5] D. Foster, A. Agarwal, M. Dudík, H. Luo, and R. Schapire. Practical contextual bandits with regression oracles. In International Conference on Machine Learning, pages 1539–1548. PMLR, 2018.

[6] D. J. Foster and A. Krishnamurthy. Efficient first-order contextual bandits: Prediction, allocation, and triangular discrimination. Advances in Neural Information Processing Systems, 34: 18907–18919, 2021.

[7] D. J. Foster, S. M. Kakade, J. Qian, and A. Rakhlin. The statistical complexity of interactive decision making. arXiv preprint arXiv:2112.13487, 2021.

[8] A. Hallak, D. Di Castro, and S. Mannor. Contextual markov decision processes. arXiv preprint arXiv:1502.02259, 2015.

[9] N. Jiang, A. Krishnamurthy, A. Agarwal, J. Langford, and R. E. Schapire. Contextual decision processes with low bellman rank are pac-learnable. In International Conference on Machine Learning, pages 1704–1713. PMLR, 2017.

[10] T. Lattimore and C. Szepesvári. Bandit Algorithms. Cambridge University Press, 2020.

[11] O. Levy and Y. Mansour. Learning efficiently function approximation for contextual MDP. arXiv preprint arXiv:2203.00995, 2022.

[12] O. Levy and Y. Mansour. Optimism in face of a context: Regret guarantees for stochastic contextual MDP. arXiv preprint arXiv:2207.11126. (To Appear in AAAI 2023), 2022.

[13] S. Mannor, Y. Mansour, and A. Tamar. Reinforcement Learning: Foundations. 2022. URL https://sites.google.com/view/rlfoundations/home.

[14] A. Modi and A. Tewari. No-regret exploration in contextual reinforcement learning. In Conference on Uncertainty in Artificial Intelligence, pages 829–838. PMLR, 2020.

[15] A. Modi, N. Jiang, S. Singh, and A. Tewari. Markov decision processes with continuous side information. In Algorithmic Learning Theory, pages 597–618. PMLR, 2018.

[16] D. Simchi-Levi and Y. Xu. Bypassing the monster: A faster and simpler optimal algorithm for contextual bandits under realizability. Mathematics of Operations Research, 2021.
[17] A. Slivkins. Introduction to multi-armed bandits. *Found. Trends Mach. Learn.*, 12(1-2):1–286, 2019.

[18] R. S. Sutton and A. G. Barto. *Reinforcement Learning: An Introduction*. The MIT Press, second edition, 2018.

[19] Y. Xu and A. Zeevi. Upper counterfactual confidence bounds: a new optimism principle for contextual bandits. *arXiv preprint arXiv:2007.07876*, 2020.

[20] A. Zimin and G. Neu. Online learning in episodic markovian decision processes by relative entropy policy search. *Advances in neural information processing systems*, 26, 2013.
A Proofs

A.1 Multiplicative Value Change of Measure

Recall the Hellinger distance given in Definition 2. The following change of measure result is due to [7].

Lemma 12 (Lemma A.11 in [7]). Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\mathcal{X}, \mathcal{F})$. For all $h: \mathcal{X} \to \mathbb{R}$ with $0 \leq h(X) \leq R$ almost surely under $\mathbb{P}$ and $\mathbb{Q}$, we have

\[
|\mathbb{E}_\mathbb{P}[h(X)] - \mathbb{E}_\mathbb{Q}[h(X)]| \leq \sqrt{2R(\mathbb{E}_\mathbb{P}[h(X)] + \mathbb{E}_\mathbb{Q}[h(X)])} \cdot D^2_H(\mathbb{P}, \mathbb{Q}).
\]

In particular,

\[
\mathbb{E}_\mathbb{P}[h(X)] \leq 3\mathbb{E}_\mathbb{Q}[h(X)] + 4RD^2_H(\mathbb{P}, \mathbb{Q}).
\]

Next, we need the following refinement of the previous result.

Corollary 13. For any $\beta \geq 1$,

\[
\mathbb{E}_\mathbb{P}[h(X)] \leq (1 + 1/\beta)\mathbb{E}_\mathbb{Q}[h(X)] + 3\beta RD^2_H(\mathbb{P}, \mathbb{Q}).
\]

Proof. Let $\eta \in (0, 1)$. Consider the following derivation.

\[
\mathbb{E}_\mathbb{P}[h(X)] - \mathbb{E}_\mathbb{Q}[h(X)] \leq \sqrt{2R(\mathbb{E}_\mathbb{P}[h(X)] + \mathbb{E}_\mathbb{Q}[h(X)])} \cdot D^2_H(\mathbb{P}, \mathbb{Q}) \\
\leq \eta(\mathbb{E}_\mathbb{P}[h(X)] + \mathbb{E}_\mathbb{Q}[h(X)]) + \frac{R}{2\eta} D^2_H(\mathbb{P}, \mathbb{Q}).
\]

The above implies

\[
\mathbb{E}_\mathbb{P}[h(X)] \leq \frac{1 + \eta}{1 - \eta} \mathbb{E}_\mathbb{Q}[h(X)] + \frac{R}{2\eta(1 - \eta)} D^2_H(\mathbb{P}, \mathbb{Q}) \\
= \left(1 + \frac{1}{\beta}\right) \mathbb{E}_\mathbb{Q}[h(X)] + 3R\frac{(2\beta + 1)^2}{2\beta} D^2_H(\mathbb{P}, \mathbb{Q}) \quad \text{(Plug } \eta = \frac{1}{2\beta + 1} \text{ for all } \beta \in (0, \infty).)
\]

\[
\leq \left(1 + \frac{1}{\beta}\right) \mathbb{E}_\mathbb{Q}[h(X)] + 3\beta RD^2_H(\mathbb{P}, \mathbb{Q}). \quad \text{(For any } \beta \geq 1)
\]

Lemma (restatement of Lemma 3). Let $r: S \times A \to [0, 1]$ be a bounded expected rewards function. Let $P_\ast$ and $\hat{P}$ denote two dynamics and consider the MDPs $M = (S, A, P_\ast, r, s_0, H)$ and $\hat{M} = (S, A, \hat{P}, r, s_0, H)$. Then, for any policy $\pi$ we have

\[
V^\pi_M(s) \leq 3V^\pi_{\hat{M}}(s) + 9H^2 \mathbb{E}_{P_\ast, \pi} \left[ \sum_{h=0}^{H-1} D^2_H(\hat{P}(|s_h, a_h), P_\ast(|s_h, a_h)) \right] s_0 = s.
\]

Proof. We first prove by backwards induction that for all $h \in [H - 1]$ the following holds.

\[
V^\pi_{\hat{M}, h}(s) \leq \left(1 + \frac{1}{H}\right)^{H-h} \left[ V^\pi_{\hat{M}, h}(s) + \mathbb{E}_{P_\ast, \pi} \left[ \sum_{h' = h}^{H-1} 3H^2 D^2_H(\hat{P}(|s_{h'}, a_{h'}), P_\ast(|s_{h'}, a_{h'})) \right] s_h = s \right].
\]

\[\Box\]
The base case, $h = H - 1$ is immediate since $V_{M,h}^\pi(s) = V_{M,h}^\pi(s)$. Now, we assume that the above holds for $h + 1$ and prove that it holds for $h$. To see this, we have that

$$V_{M,h}^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ r(s,a) + \mathbb{E}_{s', \sim \tilde{P}(|s,a)} \left[ V_{M,h+1}^\pi(s') \right] \right]$$

(By Bellman’s equations)

$$\leq \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ r(s,a) + \left( 1 + \frac{1}{H} \right) \mathbb{E}_{s', \sim \tilde{P}(|s,a)} \left[ V_{M,h+1}^\pi(s') \right] + 3H^2D_H^2(\tilde{P}(|s,a), P_* (|s,a)) \right]$$

(Corollary 13)

$$\leq \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ r(s,a) + 3H^2D_H^2(\tilde{P}(|s,a), P_* (|s,a)) \right]$$

(Induction hypothesis)

$$+ \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ \left( 1 + \frac{1}{H} \right)^{H-h} \mathbb{E}_{s' \sim P_* (|s,a)} \left[ V_{M,h+1}^\pi(s') \right] \right]$$

$$+ \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ \left( 1 + \frac{1}{H} \right)^{H-h} \sum_{h' = h+1}^{H-1} 3H^2D_H^2(\tilde{P}(|s_{h'}, a_{h'}), P_* (|s_{h'}, a_{h'})) \bigg| s_{h+1} = s' \right]$$

$$\leq \left( 1 + \frac{1}{H} \right)^{H-h} \mathbb{E}_{a \sim \pi(\cdot|s)} \left[ r(s,a) + \mathbb{E}_{s' \sim P_* (|s,a)} \left[ V_{M,h+1}^\pi(s') \right] \right]$$

$$+ \left( 1 + \frac{1}{H} \right)^{H-h} \mathbb{E}_{P_* , \pi} \left[ \sum_{h' = h}^{H-1} 3H^2D_H^2(\tilde{P}(|s_{h'}, a_{h'}), P_* (|s_{h'}, a_{h'})) \bigg| s = s' \right]$$

$$= \left( 1 + \frac{1}{H} \right)^{H-h} \left[ V_{M,h}^\pi(s) + \mathbb{E}_{P_* , \pi} \left[ \sum_{h' = h}^{H-1} 3H^2D_H^2(\tilde{P}(|s_{h'}, a_{h'}), P_* (|s_{h'}, a_{h'})) \bigg| s = s' \right] \right],$$

(By Bellman’s equations)

as desired. Plugging in $h = 0$ and using that $(1 + \frac{1}{H})^H \leq 3$ concludes the proof. \hfill \blacksquare

### A.2 Oracle Bounds (Step 1)

**Reward oracle.**

**Lemma 14 (Lemma C.10 [12]).** For any $\delta \in (0, 1)$, with probability at least $1 - \delta$ we have

$$\sum_{i=1}^{t-1} \mathbb{E} \left[ \sum_{h=0}^{H-1} (f_i(c_i, s_h^i, a_h^i) - f_*(c_i, s_h^i, a_h^i))^2 \right]^{\frac{1}{2}} \leq 68H \log(2|F| H^3/\delta) + 2 \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} (f_i(c_i, s_h^i, a_h^i) - f_*(c_i, s_h^i, a_h^i))^2$$

simultaneously, for all $t \geq 2$ and any fixed sequence of functions $f_2, f_3, \ldots \in F$.

**Corollary (restatement of Corollary 4).** Let $\hat{f}_t \in F$ be the least squares minimizer in Algorithm 1. For any $\delta \in (0, 1)$ it holds that with probability at least $1 - \delta$ we have

$$\mathbb{E}_{c} \left[ \sum_{i=1}^{t-1} \mathbb{E}_{c_i, P_* , \pi} \left[ \sum_{h=0}^{H-1} (\hat{f}_t(c, s_h, a_h) - f_*(c, s_h, a_h))^2 \bigg| s_0 \right] \right] \leq 68H \log(2T^3|F|/\delta),$$
simultaneously, for all \( t \geq 1 \).

**Proof.** Recall that for all \( t \geq 2 \), \( \hat{f}_t \) is the least square minimizer at round \( t \). Hence, by our assumption that \( f_* \in \mathcal{F} \)

\[
\sum_{i=1}^{t-1} \sum_{h=0}^{H-1} (\hat{f}_t(c_i, s_h^i, a_h^i) - r_h^i)^2 - (f_*(c_i, s_h^i, a_h^i) - r_h^i)^2 \leq 0.
\]

Thus the corollary immediately follows by Lemma 14.

**Dynamics oracle.** Recall the Hellinger distance given in Definition 2. The following lemma by [7] upper bounds the expected cumulative Hellinger Distance in terms of the log-loss.

**Lemma 15 (Lemma A.14 from [7]).** Consider a sequence of \( \{0,1\} \)-valued random variables \((\mathbb{I}_t)_{t \leq T}\) where \( \mathbb{I}_t \) is \( P^{(t-1)} \)-measurable. For any \( \delta \in (0,1) \) we have that with probability at least \( 1 - \delta \),

\[
\sum_{t=1}^{T} \mathbb{E}_t \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} D^2_H (\hat{g}^{(t)}(x^{(t)}), g^{(t)}(x^{(t)})) \mathbb{I}_t \right] \leq \sum_{t=1}^{T} \left( \log(\hat{g}^{(t)}) - \log(g^{(t)}) \right) \mathbb{I}_t + 2 \log(1/\delta).
\]

Using the above lemma, we bound the expected cumulative Hellinger distance between the approximated and true dynamics, by the actual regret of the log-loss regression oracle (and constant terms), with high probability.

**Lemma 16 (Transition to log-loss oracle regret).** For any \( \delta \in (0,1) \) it holds that with probability at least \( 1 - \delta \) we have

\[
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi(c_i):P_c^i} \left[ \sum_{h=0}^{H-1} D^2_H (P^c_*(\cdot|s_h, a_h), P^c(\cdot|s_h, a_h)) \right] \right] \leq \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P_c^i(s_{h+1}^i|s_h^i, a_h^i)} \right) - \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P_c^i(s_{h+1}^i|s_h^i, a_h^i)} \right) + 2H \log(TH|\mathcal{P}|/\delta).
\]

simultaneously, for all \( t \geq 1 \) and \( P \in \mathcal{P} \).

**Proof.** Fix some \( t \geq 1 \) and \( P \in \mathcal{P} \). We have with probability at least \( 1 - \frac{\delta}{T|\mathcal{P}|} \) that

\[
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_{\pi(c_i):P_c^i} \left[ \sum_{h=0}^{H-1} D^2_H (P^c_*(\cdot|s_h, a_h), P^c(\cdot|s_h, a_h)) \right] \right] = \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \mathbb{E}_c \left[ \mathbb{E}_{\pi(c_i):P_c^i} \left[ D^2_H (P^c_*(\cdot|s_h, a_h), P^c(\cdot|s_h, a_h)) \right] \right] \leq \sum_{h=0}^{H-1} \left( \sum_{i=1}^{t-1} \log \left( \frac{P_c^i(s_{h+1}^i|s_h^i, a_h^i)}{P_c^i(s_{h+1}^i|s_h^i, a_h^i)} \right) \right) + 2 \log(TH|\mathcal{P}|/\delta) \quad \text{(By Lemma A.14 from [7])}
\]

\[
= \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P_c^i(s_{h+1}^i|s_h^i, a_h^i)} \right) - \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P_c^i(s_{h+1}^i|s_h^i, a_h^i)} \right) + 2H \log(TH|\mathcal{P}|/\delta)
\]

The filtration used in (i) is over the history up to time \( t \), \( \mathbb{H}_{t-1} = (\sigma^1, \ldots, \sigma^{t-1}) \). Now, by taking a union bound over every \( t = 1, \ldots, T \) and \( P \in \mathcal{P} \), we obtain the lemma.

\[\blacksquare\]
Corollary (restatement of Corollary 5). Let $\hat{P}_t \in \mathcal{P}$ be the log loss minimizer in Algorithm 1. For any $\delta \in (0,1)$ it holds that with probability at least $1 - \delta$ we have

$$
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \pi_i(c^i) \frac{d}{\beta_t} \left[ H \sum_{h=0}^{H-1} \left. D^2_H(P^c_s(-|s_h, a_h), \hat{P}^c_t(-|s_h, a_h)) \right| s_0 \right] \right] \leq 2H \log(TH|\mathcal{P}|/\delta),
$$

simultaneously, for all $t \geq 1$.

Proof. By our assumption that $P_s \in \mathcal{P}$, and $\hat{P}_t$ is the log loss minimizer at time $t$, it holds that

$$
\sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{\hat{P}^c_t(s^i_h|s_h^i, a_h^i)} \right) - \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \log \left( \frac{1}{P^c_s(s^i_h|s_h^i, a_h^i)} \right) \leq 0.
$$

Thus, the corollary immediately follows by Lemma 16.

A.3 Confidence Bounds (Step 2)

In the following analysis, we use an occupancy measures-based representation of the value function. Recall the definition of the occupancy measures (Zimin and Neu [20]). For any non-contextual policy $\pi$ and dynamics $P$, let $q_h(s, a|\pi, P)$ denote the probability of reaching state $s \in S$ and performing action $a \in A$ at time $h \in [H]$ of an episode generated using policy $\pi$ and dynamics $P$.

Using this notation, the value function of any policy $\pi$ with respect to the MDP $(S, A, P, r, s_0, H)$ at any layer $h$ and state $s \in S_h$ can be represented as follows.

$$
V^\pi_{M, h}(s) = \sum_{k=h}^{H-1} q_h(s_k, a_k|\pi, P) \cdot r(s_k, a_k).
$$

Thus, the following is an immediate corollary of Lemma 3.

Corollary 17. For any (non-contextual) policy $\pi$, two dynamics $P$ and $\hat{P}$, and rewards function $r$ that is bounded in $[0, 1]$ it holds that

$$
\sum_{h=0}^{H-1} \sum_{s \in S_h} \sum_{a \in A} q_h(s, a|\pi, \hat{P}) \cdot r(s, a) \leq 3 \sum_{h=0}^{H-1} \sum_{s \in S_h} \sum_{a \in A} q_h(s, a|\pi, P) \cdot r(s, a) + 9H^2 \sum_{h=0}^{H-1} \sum_{s \in S_h} \sum_{a \in A} q_h(s, a|\pi, P) \cdot D^2_H(P(-|s, a), \hat{P}(-|s, a)).
$$

We are now ready to prove the confidence bounds. Recall the reward bonuses $b^R_t, b^P_t$ defined in Eq. (1).

Lemma (restatement of Lemma 6). Let $P_s$ and $f_s$ be the true context dependent dynamics and rewards. Let $\hat{P}_t$ and $\hat{f}_t$ be the approximated context-dependent dynamics and rewards at round $t$. Then, for any $t \geq 1$, and context-dependent policy $\pi \in \Pi_c$ we have

$$
\mathbb{E}_c \left[ V^\pi_{M(f_s, \hat{P}_t)(c)}(s_0) - V^\pi_{M(f_s, \hat{P}_t)(c)}(s_0) \right] \leq \mathbb{E}_c \left[ H \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c^i), \hat{P}^c_t) b^R_t(c, s_h, a_h) \right] + \frac{3}{2\beta_t} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \pi_i(c^i) P^c_t \left[ H \sum_{h=0}^{H-1} \left( f_s(c^i, s^i_h, a^i_h) - \hat{f}_t(c^i, s^i_h, a^i_h) \right)^2 \right] \right].
$$
Proof. We have that

\[
E_c \left[ V^{\pi(c; \cdot)}_{\mathcal{M}(f_t, \hat{P}_t)}(s_0) \right] - E_c \left[ V^{\pi(c; \cdot)}_{\mathcal{M}(f_t, \hat{P}_t)}(s_0) \right]
\]

\[
= E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c) \cdot (\hat{f}_t(c, s_h, a_h) - f_t(c, s_h, a_h)) \right]
\]

(Explicit representation of the expectation using occupancy measure, Eq. (4))

\[
\leq E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c) \cdot |\hat{f}_t(c, s_h, a_h) - f_t(c, s_h, a_h)| \right]
\]

(triangle inequality)

\[
= \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c) \cdot |\hat{f}_t(c, s_h, a_h) - f_t(c, s_h, a_h)| \right\}
\]

(the average of numbers in [0, 1] is bounded by 1)

\[
\leq \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^c} \sum_{a_h \in A} \frac{\beta_r}{2} \frac{q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c)}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h|\pi_i(c; \cdot), \hat{P}_t^c)} \right\}
\]

\[
+ \frac{1}{2\beta_r} \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c) \left( f_t(c, s_h, a_h) - \hat{f}_t(c, s_h, a_h) \right)^2 \]

\[
\leq \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^c} \sum_{a_h \in A} \frac{\beta_r}{2} \frac{q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c)}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h|\pi_i(c; \cdot), \hat{P}_t^c)} \right\}
\]

\[
+ \frac{H}{2\beta_r} \sum_{h=0}^{H-1} \sum_{s_h \in S_h^c} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}_t^c) \left( f_t(c, s_h, a_h) - \hat{f}_t(c, s_h, a_h) \right)^2 \]

\[
\leq \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^c} \sum_{a_h \in A} \frac{\beta_r/2}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h|\pi_i(c; \cdot), \hat{P}_t^c)} \right\}
\]

\[
+ \frac{3}{2\beta_r} \sum_{h=0}^{H-1} \sum_{a_h \in A} \left( f_t(c, s_h, a_h) - \hat{f}_t(c, s_h, a_h) \right)^2 \left| s_0 \right| \]

(By Corollary 17)

\[
+ \frac{9H^2}{2\beta_r} \sum_{h=0}^{H-1} \sum_{a_h \in A} D^2_H(P^c_h|s_h, a_h), \hat{P}_t^c|s_h, a_h) \left| s_0 \right| + \frac{H}{2\beta_r},
\]

as stated.

Lemma (restatement of Lemma 7). Let \( P_* \) and \( f_* \) be the true context dependent dynamics and rewards. Let \( \hat{P}_t \) be the approximated context-dependent dynamics at round \( t \). Then, for any \( t \geq 1 \),
and context-dependent policy \( \pi \in \Pi_c \) we have

\[
\left| E_c \left[ V_{\mathcal{M}(f_s, P_s)}(c)(s_0) - V_{\mathcal{M}(f_s, \hat{P}_s)}(c)(s_0) \right] \right| \leq E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) b^P_t(c, s_h, a_h) \right] \\
+ \frac{78H^3}{\beta_P} E_c \left[ \sum_{l=1}^{t-1} \sum_{s_l \in S^T} \sum_{a_l \in A} D_{H}^2(P^c_t(\cdot | s_h, a_h), \hat{P}_c^t(\cdot | s_h, a_h)) \right] s_0 \right] \left[ H^2 \right] \\
+ \frac{H^2}{2\beta_P}.
\]

**Proof.** The following holds for any \( t \geq 1 \) and a context-dependent policy \( \pi \in \Pi_c \).

\[
E_c[V_{\mathcal{M}(f_s, P_s)}(c)(s_0) - V_{\mathcal{M}(f_s, \hat{P}_s)}(c)(s_0)] = E_c \left[ V_{\mathcal{M}(f_s, \hat{P}_s)}(c)(s_0) - V_{\mathcal{M}(f_s, P_s)}(c)(s_0) \right] \\
= E_c \left[ \pi(c; \cdot), \hat{P}_c^t \left[ \sum_{h=0}^{H-1} \sum_{s' \in S} (P^c(s' | s_h, a_h) - \hat{P}_c^t(s' | s_h, a_h)) \right] \right] \\
\text{(By the Value Difference Lemma 19)}
\]

\[
= E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) \left( \sum_{s' \in S} (P^c(s' | s_h, a_h) - \hat{P}_c^t(s' | s_h, a_h)) \right) \right] \\
\text{(Explicit representation of the expectation using occupancy measure, Eq. (4))}
\]

\[
\leq E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) \left( \sum_{s' \in S} (P^c(s' | s_h, a_h) - \hat{P}_c^t(s' | s_h, a_h)) \right) \right] \\
\text{(By triangle inequality)}
\]

\[
\leq H E_c \left[ \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^T} \sum_{a_h \in A} \left( \frac{\beta_P}{2} \frac{q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)}{1 + \sum_{l=1}^{t-1} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)} \right) \right\} \\
\text{(Since } f_s \in [0, 1], \text{ for all } h \in [H] \text{ and } s' \in S \text{ we have that } V_{\mathcal{M}(f_s, \hat{P}_s)}(s') \in [0, H])
\]

\[
\leq H E_c \left[ \sum_{h=0}^{H-1} \min \left\{ 1, \sum_{s_h \in S_h^T} \sum_{a_h \in A} \left( \frac{\beta_P}{2} \frac{q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)}{1 + \sum_{l=1}^{t-1} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)} \right) \right\} \\
\text{ (ab } \leq \frac{a^2}{2} + \frac{b^2}{2})
\]

\[
+ \frac{q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)}{2\beta_P} \left( 1 + \sum_{l=1}^{t-1} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) \right) \left( \sum_{s' \in S} \left( P^c(s' | s_h, a_h) - \hat{P}_c^t(s' | s_h, a_h) \right) \right)^2 \right\} \\
\leq E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) H \min \left\{ 1, \frac{\beta_P}{2} \frac{q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)}{1 + \sum_{l=1}^{t-1} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)} \right\} \right] + H^2 \frac{2}{2\beta_P}
\]

\[
+ \frac{H}{2\beta_P} E_c \left[ \sum_{l=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) \left( \sum_{s' \in S} \left( P^c(s' | s_h, a_h) - \hat{P}_c^t(s' | s_h, a_h) \right) \right)^2 \right] \\
\leq E_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h^T} \sum_{a_h \in A} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c) H \min \left\{ 1, \frac{\beta_P}{2} \frac{q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)}{1 + \sum_{l=1}^{t-1} q_h(s_h, a_h | \pi(c; \cdot), \hat{P}_c)} \right\} \right] + H^2 \frac{2}{2\beta_P}
\]

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\[ + \frac{3H}{2\beta p} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \left( \sum_{s' \in S} P_c^*(s' | s_h, a_h) - \hat{P}_t^*(s' | s_h, a_h) \right)^2 \right] \]
\[ + \frac{18H^3}{\beta p} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \left( \sum_{c'} D_H^2(P_c^* | s_h, a_h), \hat{P}_t^*(s | s_h, a_h) \right) \right] \leq \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \right] \leq 1 + \frac{\beta p / 2}{1 + \sum_{i=1}^{t-1} q_h(s_h, a_h)} \right] \right] + \frac{H^2}{2\beta p} \]
\[ + \frac{78H^3}{\beta p} \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \left( \sum_{c'} D_H^2(P_c^* | s_h, a_h), \hat{P}_t^*(s | s_h, a_h) \right) \right] \leq \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \left( \sum_{c'} b_i^R(c, s_h, a_h) + b_i^P(c, s_h, a_h) \right) \right]. \]

Corollary (restatement of Corollary 8). Under the terms of Lemmas 6 and 7, the following holds with probability at least \(1 - 2\delta/3\) simultaneously for all \(t \geq 1\) and \(\pi \in \Pi_c\):
\[
\mathbb{E}_c \left[ V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) - V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) \right] \leq \frac{12H^3}{\beta p} \log \left(9 \mathcal{H}^4 |\mathcal{F}| |\mathcal{P}| / \delta^2 \right) + \frac{157H^4}{\beta p} \log \left(3 \mathcal{H}^2 |\mathcal{P}| / \delta \right)
\]
\[ + \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} q_h(s_h, a_h) \left( \sum_{c'} b_i^R(c, s_h, a_h) + b_i^P(c, s_h, a_h) \right) \right]. \]

Proof. We begin by taking a union bound on the events of Corollaries 4 and 5 to get that with probability at least \(1 - 2\delta/3\), simultaneously for all \(t \geq 1\)
\[
\mathbb{E}_c \left[ \sum_{i=1}^{t-1} \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \left( \hat{f}_t(c, s_h, a_h) - f_t(c, s_h, a_h) \right)^2 \right] \right] \leq 68H \log \left(6 \mathcal{H}^3 |\mathcal{F}| / \delta \right)
\]

Assuming this event holds, we get that for all \(t \geq 1\) and context-dependent policies \(\pi \in \Pi_c\).
\[
\mathbb{E}_c \left[ V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) - V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) \right] \leq \mathbb{E}_c \left[ V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) - V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) \right] + \mathbb{E}_c \left[ V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) - V^{\pi(c)}_{\mathcal{M}(f, \hat{P}_t)}(s_0) \right] \leq 2H \log \left(3 \mathcal{H}^2 |\mathcal{P}| / \delta \right). \]

(Lemmas 6 and 7)
\[ + \frac{3}{2\beta_r} \mathbb{E}_{c} \left[ \sum_{i=1}^{t-1} \pi_i(c_i), P^*_c \left[ \sum_{h=0}^{H-1} \left( f_{\pi}(c_i, s_h^i, a_h^i) - \hat{f}_{\pi}(c_i, s_h^i, a_h^i) \right)^2 \right] \right] \]

\[ + \left( \frac{9H^2}{2\beta_r} + \frac{78H^3}{\beta_P} \right) \mathbb{E}_c \left[ \sum_{i=1}^{t-1} \pi_i(c_i), P^*_c \left[ \sum_{h=0}^{H-1} D^2_{H}(P^c_{\pi}(-|s_h, a_h), P^c_{\pi}(-|s_h, a_h)) \right] \right] + \frac{H}{2\beta_r} + \frac{H^2}{2\beta_P} \]

\[ \leq \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}^c_{\pi}) \left( b^R_{\pi}(c, s_h, a_h) + b^P_{\pi}(c, s_h, a_h) \right) \right] \]

\[ + \frac{3H}{2\beta_r} 68H \log(6T^3|F|/\delta) + \left( \frac{9H^2}{2\beta_r} + \frac{78H^3}{\beta_P} \right) 2H \log(3TH|P|/\delta) + \frac{H}{2\beta_r} + \frac{H^2}{2\beta_P} \]

\[ \leq \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}^c_{\pi}) \left( b^R_{\pi}(c, s_h, a_h) + b^P_{\pi}(c, s_h, a_h) \right) \right] \]

\[ + \frac{157H^4}{\beta_P} \log(3TH|P|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|F||P|/\delta^2), \]

as stated.

A.4 Establishing Optimism Lemmas (Step 3)

Lemma (restate of Lemma 10). Let \( \pi_\star \) be an optimal context-dependent policy for \( M \).

Suppose that \( H > 10 \). Under the good event of Corollary 8, we have that for any \( t \geq 1 \)

\[ \mathbb{E}_c \left[ V_{\pi_\star(c)}(s_0) \right] \leq \mathbb{E}_c \left[ V_{\hat{M}_t(c)}(s_0) \right] + \frac{157H^4}{\beta_P} \log(3TH|P|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|F||P|/\delta^2). \]

Proof. Fix any round \( t \geq 1 \) consider the following derivation.

\[ \mathbb{E}_c \left[ V_{\pi_\star(c)}(s_0) \right] = \mathbb{E}_c \left[ V_{\hat{M}_t(c)}(s_0) \right] \]

\[ \leq \mathbb{E}_c \left[ V_{\pi_\star(c)}(s_0) \right] + \mathbb{E}_c \left[ \sum_{h=0}^{H-1} \sum_{s_h \in S_h} \sum_{a_h \in A} q_h(s_h, a_h|\pi(c; \cdot), \hat{P}^c_{\pi}) \left( b^R_{\pi}(c, s_h, a_h) + b^P_{\pi}(c, s_h, a_h) \right) \right] \]

\[ + \frac{157H^4}{\beta_P} \log(3TH|P|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|F||P|/\delta^2) \]

\[ = \mathbb{E}_c \left[ V_{\hat{M}_t(c)}(s_0) \right] + \frac{157H^4}{\beta_P} \log(3TH|P|/\delta) + \frac{112H^3}{\beta_r} \log(72T^4H|F||P|/\delta^2) \]

(Value representation in terms of occupancy measure, Eq. (4))

\[ \leq \mathbb{E}_c \left[ V_{\hat{M}_t(c)}(s_0) \right] + \frac{157H^4}{\beta_P} \log(3TH|P|/\delta) + \frac{112H^3}{\beta_r} \log(72T^4H|F||P|/\delta^2), \]

(Since \( \pi_t \) is an optimal context-dependent policy of \( \hat{M}_t \))

as the lemma states.

Lemma (restate of Lemma 11). Under the good event of Corollary 8, we have that for
where the last transition bounded the minimum in the definition of the reward bonuses, Eq. (1). Proof. Consider the following derivation. For all $t \geq 1$

$$E_c \left[ V_{\hat{M}(c)}(s_0) \right] \leq E_c \left[ V_{\hat{M}(c)}(s_0) \right] + (\beta_r + H\beta_P)\phi_t(\pi_t)$$

$$+ \frac{157H^4}{\beta_P} \log(3TH|\mathcal{P}|/\delta) + \frac{112H^3}{\beta_r} \log(18T^4H|\mathcal{F}||\mathcal{P}|/\delta^2).$$

(B) Auxiliary lemmas

Lemma 18. Let $S_t = \lambda + \sum_{k=1}^{t-1} x_t$. Suppose $x_t \in [0, \lambda]$ and, then

$$\sum_{t=1}^{T} \frac{x_t}{S_t} \leq 2 \log(T + 1).$$

Proof. The following holds.

$$\sum_{t=1}^{T} \frac{x_t}{S_t} = \sum_{t=1}^{T} \frac{S_{t+1} - S_t}{S_t}$$

$$= \sum_{t=1}^{T} \frac{S_{t+1}}{S_t} - 1$$

$$\leq 2 \sum_{t=1}^{T} \log \frac{S_{t+1}}{S_t}$$

$$\leq 2 \sum_{t=1}^{T} \log S_{t+1} - \log S_t$$

(1 $\leq \frac{S_{t+1}}{S_t} \leq 2$ since $x_t \leq \lambda$)
\[
= 2 \log \frac{S_{T+1}}{S_1}
\]
\[
\leq 2 \log(T + 1).
\]
(telescopic sum)

Lemma 19 (value-difference, Corollary 1 in [3]). Let \( M, M' \) be any \( H \)-finite horizon MDP. Then, for any two policies \( \pi, \pi' \) the following holds

\[
V_1^{\pi, M}(s) - V_1^{\pi', M'}(s) = \sum_{h=1}^{H-1} \mathbb{E} \left[ \langle Q_h^{\pi, M}(s_h, \cdot), \pi_h(\cdot|s_h) \rangle - \pi_h'(\cdot|s_h) \rangle s_1 = s, \pi', M' \right]
\]
\[
+ \sum_{h=1}^{H-1} \mathbb{E} \left[ c_h(s_h, a_h) - c_h'(s_h, a_h) + (p_h(\cdot|s_h, a_h) - p_h'(\cdot|s_h, a_h))V_{h+1}^{\pi, M}|s_h = s, \pi', M' \right].
\]