Ground state of nonlinear Schrödinger systems with saturable nonlinearity

Tai-Chia Lin*, Milivoj R. Bešić†, Milan S. Petrović‡ Goong Chen§

August 31, 2012

Abstract

We prove the existence of ground state in a multidimensional nonlinear Schrödinger model of paraxial beam propagation in isotropic local media with saturable nonlinearity. Such ground states exist in the form of bright counterpropagating solitons. From the proof, a general threshold condition on the beam coupling constant for the existence of such fundamental solitons follows.

1 Introduction

The existence of solitary waves in nonlinear evolution partial differential equations has been the major concern since the beginnings of the field [1, 2]. The existence of ground state or fundamental soliton in one-dimensional nonlinear Schrödinger (NLS) equation was never much of a concern – their existence and stability followed from the inverse scattering theory [1]. However, in more than one dimension this was, and still is, an open question [3]. In multidimensional cases there exists no mathematically rigorous theory that would guarantee their existence and uniqueness – let alone stability in propagation.

In this paper we prove the existence of ground state in the form of counterpropagating solitons in an isotropic local saturable NLS model of beam propagation [4]. This model represents a physically relevant description of soliton generation, following from the theory of photorefractive effect in crystals that respond to light by changing their index of refraction.

*Department of Mathematics & Mathematics Division, National Center for Theoretical Sciences at Taipei, National Taiwan University, Taipei, 10617, Taiwan. Email: tclin@math.ntu.edu.tw
†Texas A&M University at Qatar, P.O. Box 23874, Doha, Qatar.
‡Texas A&M University at Qatar, P.O. Box 23874, Doha, Qatar, Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia
§Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA.
2 The model

We consider the following dimensionless NLS system:

\[
\begin{align*}
  iF_z + \Delta F + \Gamma E_0 F &= 0, \\
  -iB_z + \Delta B + \Gamma E_0 B &= 0, \\
  \partial_t E_0 + E_0 &= -\frac{I_0}{1+I_0}, \\
  I_0 &= |F|^2 + |B|^2,
\end{align*}
\]

(2.1)

where \( F \) and \( B \) are the slowly-varying envelopes of the forward and backward propagating beams, \( z (0 < z < L) \) is the propagation coordinate, \( x = (x_1, x_2) \in \mathbb{R}^2 \) are the transverse coordinates, and \( \Delta = \sum_{j=1}^{2} \partial_{x_j}^2 \) is the transverse Laplacian. Furthermore, \( \Gamma \) is the beam coupling constant, \( E_0 \) the homogenous part of the space charge field generated in the photorefractive crystal, and \( I_0 \) is the beam intensity, expressed in terms of the background intensity \( I_b \). The time independent solution of (2.1) must satisfy \( \partial_t E_0 = 0 \), i.e. \( E_0 = -\frac{I_0}{1+I_0} = -\frac{|F|^2+|B|^2}{1+|F|^2+|B|^2} \). We consider such a situation. Then (2.1) becomes

\[
\begin{align*}
  iF_z + \Delta F - \Gamma \frac{|F|^2+|B|^2}{1+|F|^2+|B|^2} F &= 0, \\
  -iB_z + \Delta B - \Gamma \frac{|F|^2+|B|^2}{1+|F|^2+|B|^2} B &= 0.
\end{align*}
\]

(2.2)

Here, we assume the following boundary and initial conditions.

Boundary conditions: \( F(x, z) \, , \, B(x, z) \to 0 \) as \( |x| \to \infty \), for \( 0 \, z < L \).

Initial conditions: \( F(x, 0) = F_0(x) \, , \, B(x, L) = B_L(x) \) are given.

Equations (2.2) can be written as \( iF_z = \frac{\delta E[B,F]}{\delta F} \), \( -iB_z = \frac{\delta E[B,F]}{\delta B} \), where

\[
E[B,F] = \int_{\mathbb{R}^2} \frac{1}{2} (|\nabla B|^2 + |\nabla F|^2) + \frac{1}{2} \Gamma [ |B|^2 + |F|^2 - \ln(1 + |B|^2 + |F|^2) ]
\]

is the energy functional. The integral (here and elsewhere) is taken across the whole transverse plane. One basic conservation law of the system (2.2) is the power conservation, given by

\[
\int_{\mathbb{R}^2} |B|^2 + |F|^2 = \int_{\mathbb{R}^2} |B_0|^2 + |F_0|^2 \quad \text{for} \quad 0 < z < L.
\]

(2.3)

To get the standing wave profiles of the system (2.2), we set

\[
F(x, z) = e^{i\lambda z} u(x) \, , \, B(x, z) = e^{-i\lambda z} v(x),
\]

and then the system (2.2) can be transformed into the following system:

\[
\begin{align*}
  \Delta \downarrow u - \Gamma \frac{u^2 + v^2}{1 + u^2 + v^2} u &= \lambda u, \\
  \Delta \downarrow v - \Gamma \frac{u^2 + v^2}{1 + u^2 + v^2} v &= \lambda v,
\end{align*}
\]

(2.4)

so that the condition (2.3) also can be normalized as follows:

\[
P[u,v] = \int_{\mathbb{R}^2} (u^2 + v^2) = 1 \quad \text{(normalization)}.
\]

(2.5)
Here, $\lambda$ is the propagation constant, which can be regarded as the chemical potential in other physical settings; mathematically it is the Lagrange multiplier following from condition (2.5).

The system of equations (2.4) has been solved numerically in [5], but with an external lattice potential $I_g$ included instead of the uniform background intensity $I_b = 1$. Fundamental counterpropagating solitons have been obtained and a threshold condition determined. However, these results have been numerical, without rigorous proofs to substantiate their existence.

To obtain the ground state of (2.2) rigorously, we consider the following energy minimization problem:

$$\mu_\Gamma = \inf \left\{ E[u, v] : u, v \in H^1(\mathbb{R}^2), P[u, v] = 1 \right\},$$

(2.6)

where the energy functional now is

$$E[u, v] = \int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2) + \Gamma [u^2 + v^2 - \ln (1 + u^2 + v^2)].$$

The main issue with the problem (2.6) is whether the value $\mu_\Gamma$ (depending on the coupling constant $\Gamma$) can be achieved at a minimizer called the ground state solution of (2.2). It is obvious that $E[u, v] \geq -|\Gamma|$ for $u, v \in H^1(\mathbb{R}^2)$, $I(u, v) = 1$, which implies the value $\mu_\Gamma \geq -|\Gamma| > -\infty$.

Note that for the NLS equations with power nonlinearity, the infimum energy may not exist for some power magnitudes (see [7]). Here, as $\Gamma < 0$, i.e., the self-focusing case, the potential energy of the saturable nonlinearity

$$\int_{\mathbb{R}^2} \Gamma \left[ \rho^2 - \ln (1 + \rho^2) \right]$$

may compete with the kinetic energy

$$\int_{\mathbb{R}^2} |\nabla \rho|^2,$$

so the magnitude of $\Gamma$ affects the existence of the ground state solution. The main results may be stated as follows:

**Theorem 2.1.** Let $T_0$ be the following positive constant:

$$T_0 = \inf_{w \in H^1(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} [w^2 - \ln (1 + w^2)]}$$

(i) If $\Gamma > -T_0$, then $\mu_\Gamma$ can not be attained by a minimizer i.e. there is no ground state solution.

(ii) If $\Gamma < -T_0$, then $\mu_\Gamma < 0$ and there exists a ground state solution which is radially symmetric and is denoted by $(u, v) = \rho(r) (\cos \phi, \sin \phi)$, where $\phi \in \mathbb{R}$ is an arbitrary constant and $\rho = \rho(r)$ is the energy minimizer of the following problem:

$$\text{Minimize } H[\rho] \text{ over } \rho \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho^2 = 1,$$

(2.7)

where

$$H[\rho] = \int_{\mathbb{R}^2} |\nabla \rho|^2 + \Gamma \left[ \rho^2 - \ln (1 + \rho^2) \right].$$

(2.8)
The physical meaning is as follows: Theorem 2.1 indicates that ground states only can behave like bright solitary waves in saturable photorefractive media. The constant $-T_0$ is the threshold for the existence of ground state solutions, which may be changed under the effect of external intensity $I_b$ (see Theorem 2.1 (ii)).

Remark 2.2. By Schwartz symmetrization, it is obvious that the minimizer $\rho$ of the problem (2.4) must be radially symmetric and its Euler-Lagrange equation may be expressed as follows:

$$\begin{cases}
\rho'' + \frac{1}{r} \rho' - \Gamma \frac{\rho^3}{1 + \rho^2} = \lambda \rho \text{ for } r > 0, \\
\rho'(0) = 0, \rho(0) > 0,
\end{cases}$$

where $\lambda$ is the Lagrange multiplier of the problem (2.4). We shall show that $\lambda > 0$ and the minimizer $\rho$ is a positive and monotone decreasing function which decays to zero exponentially as the variable $r$ goes to infinity.

3 Proof of Theorem 2.1

The proof is divided in a number of steps. The following lemma is crucial in proving Theorem 2.1.

Lemma 3.1. The value $\mu_\Gamma$ defined in (2.6) satisfies

$$\mu_\Gamma = \inf \left\{ H[\rho] : \rho \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho^2 = 1 \right\}.$$

Proof. Let $u = \rho \cos \phi$ and $v = \rho \sin \phi$, where both $\rho$ and $\phi$ are $H^1$ functions. Then

$$|\nabla u|^2 + |\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \phi|^2$$

and $u^2 + v^2 = \rho^2$, and hence the proof of Lemma 3.1 is obvious.

3.1 Proof of Theorem 2.1 (i)

Proposition 3.2. Let

$$T_0 = \inf_{w \in H^1(\mathbb{R}^2) : \|w\|_2 = 1} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} [w^2 - \ln(1 + w^2)]}.$$

Then $T_0 > 0$.

To prove Proposition 3.2 we need

Claim A1. $\sup_{s > 0} \frac{s - \ln(1+s)}{s^2} = \frac{1}{2}$.

Proof. Let $h(s) = \frac{s - \ln(1+s)}{s^2}$ for $s > 0$. Then $h'(s) = -\frac{2 + s}{s^2(1+s)} + \frac{2}{s^2} \ln(1+s)$ and $(s^3 h'(s))' = -\frac{s^2}{(1+s)^2} < 0$ for $s > 0$, which implies $h'(s) < 0$ for $s > 0$, i.e. $h$ is a monotone decreasing function for $s > 0$. Note that $s^3 h'(s) = 0$ at $s = 0$. On the other hand, by direct calculation, $\lim_{s \to 0^+} h(s) = \frac{1}{2}$ and hence the proof is complete.
By Claim A1, \( w^2 - \ln (1 + w^2) \leq \frac{1}{2} w^4 \), so \( \int_{\mathbb{R}^2} [w^2 - \ln (1 + w^2)] \leq \frac{1}{2} \int_{\mathbb{R}^2} w^4 \) for \( w \in H^1(\mathbb{R}^2), \|w\|_2 = 1 \), which implies

\[
T_0 \geq \inf_{w \in H^1(\mathbb{R}^2), \|w\|_2 = 1} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\frac{1}{2} \|w\|_4^4}.
\]

On the other hand, by (2.2.5) in [6],

\[
\inf_{w \in H^1(\mathbb{R}^2), \|w\|_2 = 1} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\frac{1}{2} \|w\|_4^4} = 2S_{2,4} > 0,
\]

where \( S_{2,4} \) is the Sobolev constant. Therefore, \( T_0 \geq 2S_{2,4} > 0 \) and we complete the proof of Proposition 3.2.

**Proposition 3.3.** Suppose \( \Gamma \in (-T_0, 0) \) i.e. \( 0 > \Gamma > -T_0 \). Then the value \( \mu_\Gamma \) can not attain a minimizer such that \( \mu_\Gamma \leq 0 \).

**Proof.** We may prove by contradiction. Suppose there exists \( u \) a minimizer of the value \( \mu_\Gamma \) such that \( \mu_\Gamma \leq 0 \). Then \( \int_{\mathbb{R}^d} |\nabla u|^2 + \Gamma \int_{\mathbb{R}^d} [u^2 - \ln (1 + u^2)] \leq 0 \) and \( \|u\|_2 = 1 \). Hence

\[
\int_{\mathbb{R}^d} |\nabla u|^2 \leq -\Gamma \int_{\mathbb{R}^d} [u^2 - \ln (1 + u^2)] = \frac{-\Gamma}{T_0} \left\{ T_0 \int_{\mathbb{R}^d} [u^2 - \ln (1 + u^2)] \right\} \leq \frac{-\Gamma}{T_0} \int_{\mathbb{R}^d} |\nabla u|^2,
\]

which implies \( u \equiv 0 \), since \( 0 < \frac{-\Gamma}{T_0} < 1 \). However, \( u \equiv 0 \) contradicts \( \|u\|_2 = 1 \). Therefore, we have completed the proof. \( \square \)

**Proposition 3.4.** Suppose \( \Gamma \in (-T_0, 0) \), i.e. \( 0 > \Gamma > -T_0 \). Then the value \( \mu_\Gamma \) can not attain a minimizer such that \( \mu_\Gamma > 0 \).

**Proof.** It can be proved by contradiction. Suppose there exists a minimizer \( u \) of the value \( \mu_\Gamma \) such that \( \mu_\Gamma > 0 \). Then \( u \) satisfies the Euler-Lagrange equation of the problem

\[
\mu_\Gamma = \inf \left\{ H[\rho] : \rho \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho^2 = 1 \right\}
\]

given by

\[
\Delta u - \Gamma \frac{u^3}{1 + u^2} = \lambda u \quad \text{for} \quad x \in \mathbb{R}^2
\]

(3.1)

with \( \|u\|_2 = 1 \) and \( u(x) \to 0 \) as \( |x| \to \infty \), where \( \lambda \) is the Lagrange multiplier. Multiplying equation (3.1) by \( x \cdot \nabla u \) and integrating over \( \mathbb{R}^2 \), we may derive the Pohozaev identity as follows:

\[
\lambda = -\Gamma \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)].
\]
The derivation is quite standard, so we omit the details here. On the other hand, multiplying equation (3.1) by \( u \) and integrating over \( \mathbb{R}^2 \), gives

\[ \lambda = - \int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma \int_{\mathbb{R}^2} \frac{u^4}{1 + u^2}. \]  

(3.3)

Here we have used integration by parts. Suppose \( \mu_\Gamma > 0 \). Then it is obvious that

\[ \Gamma \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)] > - \int_{\mathbb{R}^2} |\nabla u|^2. \]

Hence (3.3) implies

\[ \lambda < \Gamma \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)] - \Gamma \int_{\mathbb{R}^2} \frac{u^4}{1 + u^2}. \]  

(3.4)

Combining with (3.2) and (3.4), we have

\[ -2\Gamma \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)] < -\Gamma \int_{\mathbb{R}^2} \frac{u^4}{1 + u^2}, \]

which is equivalent to

\[ 2 \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)] < \int_{\mathbb{R}^2} \frac{u^4}{1 + u^2}, \]  

(3.5)

since \( \Gamma < 0 \). Let

\[ F(s) = 2[s - \ln (1 + s)] - \frac{s^2}{1 + s} \text{ for } s \geq 0. \]

Then \( F(0) = 0 \) and

\[ F'(s) = \frac{s^2}{(1 + s)^2} > 0 \text{ for } s > 0, \]

and hence \( F(s) \geq 0 \) for \( s \geq 0 \), i.e.

\[ 2[s - \ln (1 + s)] \geq \frac{s^2}{1 + s} \text{ for } s \geq 0. \]

Therefore, replacing \( s \) by \( u^2 \), \( 2 \int_{\mathbb{R}^2} [u^2 - \ln (1 + u^2)] \geq \int_{\mathbb{R}^2} \frac{u^4}{1 + u^2} \), which contradicts (3.3) and we have completed the proof.

**Proposition 3.5.** Suppose \( \Gamma \geq 0 \). Then \( \mu_\Gamma = 0 \) can not attain a minimizer.

*Proof.* Let \( w \in H^1(\mathbb{R}^2) \) and \( ||w||_2 = 1 \). For \( \delta > 0 \), let \( w_\delta(x) = \delta w(\delta x) \) for \( x \in \mathbb{R}^2 \). Then \( ||w_\delta||_2 = ||w||_2 = 1 \) and \( \int_{\mathbb{R}^2} |\nabla w_\delta(x)|^2 = \delta^2 \int_{\mathbb{R}^2} |\nabla w(y)|^2 \, dy \). Moreover, by Claim A1, \( \int_{\mathbb{R}^2} [w_\delta^2 - \ln (1 + w_\delta^2)] \leq \frac{1}{2} \int_{\mathbb{R}^2} w_\delta^4 = \frac{4}{3} \delta^2 \int_{\mathbb{R}^2} w^4 \, dy \). Hence \( H[w_\delta] = O(\delta^2) \) tends to zero as \( \delta \) goes to zero. This implies that \( \mu_\Gamma = 0 \). On the other hand, since \( \Gamma \geq 0 \), it is obvious that \( \mu_\Gamma = 0 \) can not attain a minimizer. Therefore, the proof is completed.

Combining Propositions 3.2-3.5, completes the proof of Theorem 2.1 (i).
3.2 Proof of Theorem 2.1 (ii)

For the proof of Theorem 2.1 (ii), we firstly consider the following problem:

\[ \mu_{\Gamma, \varepsilon} = \inf_{u \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \cap P_\varepsilon[u] = 1} H_\varepsilon[u], \]

where

\[ H_\varepsilon[u] = \int_{B_{\frac{1}{\varepsilon}}} |\nabla u|^2 + \Gamma [u^2 - \ln (1 + u^2)] \]

and \( P_\varepsilon[u] = \int_{B_{\frac{1}{\varepsilon}}} u^2 \) for \( \varepsilon > 0 \) and \( u \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \). Hereafter, \( B_{\frac{1}{\varepsilon}} \) is the ball with radius \( \frac{1}{\varepsilon} \) and center at the origin.

**Lemma 3.6.** Assume \( \Gamma < -T_0 < 0 \). Then

(i) For \( \varepsilon > 0 \), \( \mu_{\Gamma, \varepsilon} \) can be achieved by a minimizing \( u_\varepsilon = u_\varepsilon(r) > 0 \) with radial symmetry.

(ii) For \( \varepsilon > 0 \) sufficiently small, \( \mu_{\Gamma, \varepsilon} \leq -c_0 \), where \( c_0 \) is a positive constant independent of \( \varepsilon \).

**Proof.** Fix \( \varepsilon > 0 \) arbitrarily. Since \( 0 \leq s^2 - \ln (1 + s^2) \leq C s^2 \) for \( s \in \mathbb{R} \), where \( C \) is a positive constant independent of \( s \), then \( H_\varepsilon[u] \geq \Gamma C \int_{B_{\frac{1}{\varepsilon}}} u^2 = \Gamma C P_\varepsilon[u] = \Gamma C \) for \( u \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \) and \( P_\varepsilon[u] = 1 \). Hence the value \( \mu_{\Gamma, \varepsilon} = \inf_{u \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \cap P_\varepsilon[u] = 1} H_\varepsilon[u] \) exists. Let \( \{v_k\}_{k=1}^\infty \) be a minimizing sequence of the value \( \mu_{\Gamma, \varepsilon} \). Without loss of generality, we may assume \( H_\varepsilon[v_k] \downarrow \mu_{\Gamma, \varepsilon} \) as \( k \to \infty \).

Note that each \( v_k \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \) and \( P_\varepsilon[v_k] = 1 \). Apply symmetry rearrangement on each \( v_k \), say \( v_k^* \in H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \) and \( P_\varepsilon[v_k^*] = P_\varepsilon[v_k] = 1 \). Note that \( \int_{B_{\frac{1}{\varepsilon}}} |\nabla v_k^*|^2 \leq \int_{B_{\frac{1}{\varepsilon}}} |\nabla v_k|^2 \) and \( \int_{B_{\frac{1}{\varepsilon}}} [(v_k^*)^2 - \ln (1 + (v_k^*)^2)] = \int_{B_{\frac{1}{\varepsilon}}} [(v_k^2)^2 - \ln (1 + (v_k^2)^2)] \). Then \( \mu_{\Gamma, \varepsilon} \leq H_\varepsilon[v_k^*] \leq H_\varepsilon[v_k] \) for all \( k \). Hence we may replace \( v_k \) by \( v_k^* \) and regard \( v_k \)'s as functions with radial symmetry. On the other hand, since \( H_\varepsilon[v_k] \downarrow \mu_{\Gamma, \varepsilon} \) as \( k \to \infty \) and \( 0 \leq s^2 - \ln (1 + s^2) \leq C s^2 \) for \( s \in \mathbb{R} \), then

\[ \int_{B_{\frac{1}{\varepsilon}}} |\nabla v_k|^2 \leq C_1, \]

where \( C_1 \) is a positive constant independent of \( k \). Hence Sobolev embedding gives \( v_k \to u_\varepsilon \) weakly in \( H_0^1 \left( B_{\frac{1}{\varepsilon}} \right) \) as \( k \to \infty \) (up to a subsequence). Moreover, by the Sobolev compact embedding, \( v_k \to u_\varepsilon \) in \( L^2 \left( B_{\frac{1}{\varepsilon}} \right) \) as \( k \to \infty \). Consequently, \( P_\varepsilon[u_\varepsilon] = \lim_{k \to \infty} P_\varepsilon[v_k] = 1 \) and
Lemma 3.8. Here we have used the fact that 

$$\lim_{k \to \infty} H_{\varepsilon} [v_k] = \mu_{\Gamma, \varepsilon}.$$  

This implies that \( u_{\varepsilon} \) is the minimizer of the value \( \mu_{\Gamma, \varepsilon} \). Here, we have used Fatou’s Lemma. Moreover, since each \( v_k \) is radially symmetric and \( v_k \to u_{\varepsilon} \) in \( L^2 \left( B_{\frac{1}{\varepsilon}} \right) \) as \( k \to \infty \), then \( u_{\varepsilon} \) is radially symmetric. This completes the proof of Lemma 3.6 (i).

The proof of Lemma 3.6 (ii) is given as follows: From the definition of \( T_0 \),

$$\forall \delta > 0, \exists w_{\delta} \in H^1 \left( \mathbb{R}^2 \right), \| w_{\delta} \|_2 = 1 \text{ such that } T_0 > \frac{\int_{\mathbb{R}^2} |\nabla w_{\delta}|^2}{\int_{\mathbb{R}^2} w_{\delta}^2 - \ln (1 + w_{\delta}^2)} - \delta.$$  

Let \( \delta = -\frac{\Gamma + T_0}{2} \). Then \( \delta > 0 \) and \( \Gamma < -T_0 - \delta \), i.e., \( -\Gamma > T_0 + \delta \) due to \( \Gamma < -T_0 \). Hence \( -\Gamma > \frac{\int_{\mathbb{R}^2} |\nabla w_{\delta}|^2}{\int_{\mathbb{R}^2} w_{\delta}^2 - \ln (1 + w_{\delta}^2)} \), i.e., \( H(w_{\delta}) < 0 \). Let \( w_{\delta, \varepsilon} = \frac{w_{\delta, \varepsilon}}{\| w_{\delta, \varepsilon} \|_2} \), where \( \varphi_{\varepsilon} \in C^\infty_0 (\mathbb{R}^2) \) is a cut-off function such that: \( \varphi_{\varepsilon} = 1 \) in \( B_{\frac{1}{\varepsilon} - 1} \), \( \varphi_{\varepsilon} = 0 \) in \( B_{\frac{1}{\varepsilon}} = \mathbb{R}^2 - B_{\frac{1}{\varepsilon}} \) and \( \| \varphi_{\varepsilon} \|_{C^1} = O(1) \), where \( O(1) \) is a bounded quantity. Then \( w_{\delta, \varepsilon} \in H^1 \left( B_{\frac{1}{\varepsilon}} \right) \) and \( \| w_{\delta, \varepsilon} \|_2 = 1 \). Note that \( \varepsilon \) and \( \delta \) are independent of each other. It is obvious that \( \int_{B_{\frac{1}{\varepsilon}}} w_{\delta, \varepsilon}^2 + |\nabla w_{\delta, \varepsilon}|^2 = o_\varepsilon(1) \), where \( B_{\frac{1}{\varepsilon} - 1} = \mathbb{R}^2 - B_{\frac{1}{\varepsilon} - 1} \) and \( o_\varepsilon(1) \) is a small quantity tending to zero as \( \varepsilon \) goes to zero. Moreover, \( \| w_{\delta, \varepsilon} \|_2 = \| w_{\delta} \|_2 + o_\varepsilon(1) = 1 + o_\varepsilon(1) \), \( \int_{\mathbb{R}^2} |\nabla (w_{\delta, \varepsilon})|^2 = \int_{\mathbb{R}^2} |\nabla w_{\delta}|^2 \varphi_{\varepsilon}^2 + \int_{\mathbb{R}^2} 2 (w_{\delta} \nabla w_{\delta}) \cdot (\varphi_{\varepsilon} \nabla \varphi_{\varepsilon}) + w_{\delta}^2 |\nabla \varphi_{\varepsilon}|^2 = \int_{\mathbb{R}^2} |\nabla w_{\delta}|^2 + o_\varepsilon(1) \), \( \| w_{\delta, \varepsilon} \|_2 = \| w_{\delta} \|_2 + o_\varepsilon(1) \leq -c_0 \) as \( \varepsilon \) gets sufficiently small, where \( c_0 = -\frac{1}{2} H[w_{\delta}] > 0 \) is a constant independent of \( \varepsilon \). Therefore, \( \mu_{\Gamma, \varepsilon} \leq H_{\varepsilon} [w_{\delta, \varepsilon}] \leq -c_0 \) and we complete the proof of Lemma 3.6 (ii).

\[ \square \]

Remark 3.7. From Lemma 3.6 (ii), we get \( \mu_{\Gamma} \leq \mu_{\Gamma, \varepsilon} \leq -c_0 < 0 \).

Lemma 3.8. Under the same hypothesis of Lemma 3.6, the minimizer \( u_{\varepsilon} \) satisfies \( \| u_{\varepsilon} \|_{H^1 \left( B_{\frac{1}{\varepsilon}} \right)} \leq K_0 \) for \( \varepsilon > 0 \) sufficiently small, where \( K_0 \) is a positive constant independent of \( \varepsilon \).

Proof. By Lemma 3.6 (ii) \( \mu_{\Gamma, \varepsilon} \leq -c_0 \), which implies

$$-c_0 \geq \int_{B_{\frac{1}{\varepsilon}}} |\nabla u_{\varepsilon}|^2 + \Gamma \int_{B_{\frac{1}{\varepsilon}}} u_{\varepsilon}^2 - \ln (1 + u_{\varepsilon}^2) \geq \int_{B_{\frac{1}{\varepsilon}}} |\nabla u_{\varepsilon}|^2 + \Gamma.$$  

Here we have used the fact that \( \int_{B_{\frac{1}{\varepsilon}}} u_{\varepsilon}^2 = 1 \). Thus

$$\| u_{\varepsilon} \|_{H^1 \left( B_{\frac{1}{\varepsilon}} \right)}^2 = \int_{B_{\frac{1}{\varepsilon}}} |\nabla u_{\varepsilon}|^2 + \int_{B_{\frac{1}{\varepsilon}}} u_{\varepsilon}^2 \leq 1 - c_0 - \Gamma$$  

and we have completed the proof. \[ \square \]
We may extend \( u_\varepsilon \) on the entire plane \( \mathbb{R}^2 \) by setting \( u_\varepsilon (r) = 0 \) for \( r > \frac{1}{\varepsilon} \). Note that each \( u_\varepsilon \) is radially symmetric. Then Lemma 3.8 gives \( \| u_\varepsilon \|_{H^1_r(\mathbb{R}^2)} \leq K_0 \), which implies

\[ u_\varepsilon \to U \text{ weakly in } H^1_r(\mathbb{R}^2) \tag{3.6} \]

as \( \varepsilon \) goes to zero (up to a subsequence). Furthermore, by the compact embedding of \( H^1_r \) radial functions to \( L^4_r \) functions (cf. Lions paper [7]), we have

\[ u_\varepsilon \to U \text{ in } L^4_r(\mathbb{R}^2) \tag{3.7} \]

as \( \varepsilon \) goes to zero (up to a subsequence). Now, we want to prove that \( U \) is nontrivial. By Claim A1, we get \( u_\varepsilon^2 - \ln (1 + u_\varepsilon^2) \leq \frac{1}{2} u_\varepsilon^2 \). Hence by \((3.4), (3.7)\) and Lemma 3.8 (ii),

\[ -c_0 \geq \operatorname{liminf}_{\varepsilon \to 0^+} \mu_{\Gamma, \varepsilon} = \operatorname{liminf}_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} | \nabla u_\varepsilon |^2 + \Gamma \left[ u_\varepsilon^2 - \ln (1 + u_\varepsilon^2) \right] , \tag{3.8} \]

\[ \geq \operatorname{liminf}_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} | \nabla u_\varepsilon |^2 + \frac{1}{2} \Gamma u_\varepsilon^4 \geq \int_{\mathbb{R}^2} | \nabla U |^2 + \frac{1}{2} \Gamma U^4 , \tag{3.9} \]

which implies that \( U \) is nontrivial. Here we have used again Fatou’s Lemma. Otherwise, \( U \equiv 0 \) and then \( 0 > -c_0 \geq 0 \), a contradiction.

To complete the proof, we have to prove that \( \| U \|_2 = 1 \). The idea is to use the concentration compactness. We shall prove that the class \( \{ u_\varepsilon \}_{\varepsilon > 0} \) is neither vanishing nor dichotomy as \( \varepsilon \to 0 \) by contradiction. Suppose \( \{ u_\varepsilon \}_{\varepsilon > 0} \) is vanishing i.e. \( u_\varepsilon \to 0 \) in \( L^2(\mathbb{R}^2) \) as \( \varepsilon \to 0 \). Then by Egoroff Theorem, \( u_\varepsilon \to 0 \) almost everywhere as \( \varepsilon \to 0 \) (up to a subsequence). On the other hand, by \((3.7)\) and Egoroff Theorem, \( u_\varepsilon \to U \) almost everywhere as \( \varepsilon \to 0 \) (up to a subsequence). Consequently, \( U \equiv 0 \) but \( U \) is nontrivial. This gives a contradiction, so the class \( \{ u_\varepsilon \}_{\varepsilon > 0} \) is not vanishing.

To show that \( \{ u_\varepsilon \}_{\varepsilon > 0} \) is not dichotomy as \( \varepsilon \to 0 \), we study the solution profile of \( u_\varepsilon \). The minimizer \( u_\varepsilon \) satisfies the following equation:

\[ \Delta u_\varepsilon - \Gamma \frac{u_\varepsilon^3}{1 + u_\varepsilon^2} = \lambda_\varepsilon u_\varepsilon \text{ in } B_{\frac{1}{\varepsilon}} \tag{3.10} \]

with the zero Dirichlet boundary condition \( u_\varepsilon = 0 \) on \( \partial B_{\frac{1}{\varepsilon}} \), where \( \lambda_\varepsilon \) is the associated Lagrange multiplier. Multiply equation \((3.10)\) by \( u_\varepsilon \) and integrate over \( B_{\frac{1}{\varepsilon}} \). Then using integration by parts and \( \int_{B_{\frac{1}{\varepsilon}}} u_\varepsilon^2 = 1 \), we get \( \lambda_\varepsilon = \lambda_\varepsilon \int_{B_{\frac{1}{\varepsilon}}} u_\varepsilon^2 = -\int_{B_{\frac{1}{\varepsilon}}} \left| \nabla u_\varepsilon \right|^2 - \Gamma \int_{B_{\frac{1}{\varepsilon}}} \frac{u_\varepsilon^4}{1 + u_\varepsilon^2} \). Hence by Lemma 3.8

\[ |\lambda_\varepsilon| \leq K_1 , \tag{3.11} \]

where \( K_1 \) is a positive constant independent of \( \varepsilon \).

Since \( u_\varepsilon = u_\varepsilon (r) \) is radially symmetric, equation \((3.10)\) and the zero Dirichlet boundary condition can be reduced to a boundary value problem of an ordinary differential equation as follows:

\[ \begin{cases} u''_\varepsilon + \frac{1}{r} u'_\varepsilon - \Gamma \frac{u_\varepsilon^3}{1 + u_\varepsilon^2} = \lambda_\varepsilon u_\varepsilon & \text{for } 0 < r < \frac{1}{\varepsilon} , \\ u'_\varepsilon (0) = 0 , \ u_\varepsilon \left( \frac{1}{\varepsilon} \right) = 0 . \end{cases} \tag{3.12} \]

From the energy comparison, we may set \( u_\varepsilon (r) \geq 0 \) for \( 0 < r < \frac{1}{\varepsilon} \).
Lemma 3.9. The minimizer \( u_\varepsilon = u_\varepsilon (r) \) is positive and monotone decreasing with \( r \).

Proof. Suppose \( \lambda_\varepsilon > 0 \). Then equation (3.10) implies

\[
\Delta u_\varepsilon - \lambda_\varepsilon u_\varepsilon = \Gamma \frac{u_\varepsilon^3}{1 + u_\varepsilon^2} \leq 0 \text{ in } B_1^1 \varepsilon
\]

with the zero Dirichlet boundary condition \( u_\varepsilon = 0 \) on \( \partial B_1^1 \varepsilon \). Hence by the strong maximum principle, \( u_\varepsilon > 0 \) in \( B_1^1 \varepsilon \) and

\[
u_\varepsilon (r) = \min_{x \in \partial B_r^1} u_\varepsilon (x) = \min_{x \in B_s} u_\varepsilon (x) = u_\varepsilon (s)
\]

for \( 0 < r < s < \frac{1}{\varepsilon} \) since \( u_\varepsilon = u_\varepsilon (r) \) is radically symmetric. This shows that \( u_\varepsilon = u_\varepsilon (r) \) is positive and monotone decreasing with \( r \). Similarly, if \( \lambda_\varepsilon \leq 0 \), then equation (3.10) implies

\[
\Delta u_\varepsilon = \left( \lambda_\varepsilon + \Gamma \frac{u_\varepsilon^2}{1 + u_\varepsilon^2} \right) u_\varepsilon \leq 0 \text{ in } B_1^1 \varepsilon
\]

with the zero Dirichlet boundary condition \( u_\varepsilon = 0 \) on \( \partial B_1^1 \varepsilon \). Thus using the strong maximum principle again, we may prove that \( u_\varepsilon = u_\varepsilon (r) \) is positive and monotone decreasing with \( r \), and thus complete the proof of Lemma 3.9. \( \square \)

From Lemma 3.9, \( u_\varepsilon \) can not be splitted into two parts as \( \varepsilon \to 0+ \), and hence \( \{ u_\varepsilon \}_{\varepsilon > 0} \) can not be dichotomy as \( \varepsilon \to 0+ \). Therefore, by the concentration compactness Theorem, \( u_\varepsilon \to U \) in \( L^2 (\mathbb{R}^2) \) as \( \varepsilon \to 0+ \) (up to a subsequence), which implies \( \| U \|_2 = 1 \) since \( \| u_\varepsilon \|_2 = 1 \).

Now we claim that the limit function \( U \) satisfies

\[
\Delta U - \Gamma \frac{U^3}{1 + U^2} = \lambda_0 U \text{ in } \mathbb{R}^2,
\]

\( U = U (r) \geq 0 \) is radially symmetric, and \( \lim_{r \to \infty} U (r) = 0 \), where \( \lambda_0 \) is the limit of \( \lambda_\varepsilon \)’s (up to a subsequence) since (3.11) implies

\[
\lambda_\varepsilon \to \lambda_0 \text{ as } \varepsilon \to 0+ \text{ (up to a subsequence)}.
\]

(3.14)

Let \( \phi \in C^\infty_c (\mathbb{R}^2) \) be any test function. Since \( u_\varepsilon \) satisfies (3.10), then

\[
\int_{\mathbb{R}^2} \nabla u_\varepsilon \cdot \nabla \phi + \Gamma \int_{\mathbb{R}^2} \frac{u_\varepsilon^3}{1 + u_\varepsilon^2} \phi = -\lambda_\varepsilon \int_{\mathbb{R}^2} u_\varepsilon \phi.
\]

(3.15)

Hence (3.6) and (3.7) give

\[
\int_{\mathbb{R}^2} \nabla u_\varepsilon \cdot \nabla \phi \to \int_{\mathbb{R}^2} \nabla U \cdot \nabla \phi,
\]

\[
\int_{\mathbb{R}^2} u_\varepsilon \phi \to \int_{\mathbb{R}^2} U \phi,
\]

and

\[
\int_{\mathbb{R}^2} \frac{u_\varepsilon^3}{1 + u_\varepsilon^2} \phi = \int_{\mathbb{R}^2} u_\varepsilon \phi - \int_{\mathbb{R}^2} \frac{u_\varepsilon \phi}{1 + u_\varepsilon^2}.
\]

10
\[
\begin{align*}
&= \int_{\mathbb{R}^2} u_\varepsilon \phi - \int_{\mathbb{R}^2} \frac{\phi}{1 + u_\varepsilon^2} (u_\varepsilon - U) - \int_{\mathbb{R}^2} \frac{\phi}{1 + u_\varepsilon^2} U \\
&= \int_{\mathbb{R}^2} u_\varepsilon \phi - \int_{\mathbb{R}^2} \frac{\phi}{1 + u_\varepsilon^2} (u_\varepsilon - U) \\
&\quad - \int_{\mathbb{R}^2} \frac{1}{1 + U^2} U - \int_{\mathbb{R}^2} (U - u_\varepsilon) \frac{U + u_\varepsilon}{(1 + u_\varepsilon^2) (1 + U^2)} U \phi \\
&\quad \to \int_{\mathbb{R}^2} U \phi - \int_{\mathbb{R}^2} \frac{U}{1 + U^2} U = \int_{\mathbb{R}^2} \frac{U^3}{1 + U^2} \phi.
\end{align*}
\]

Note that \(\left\| \frac{\phi}{1 + u_\varepsilon^2} \right\|_3 \leq \| \phi \|_3^4\) and \(\left\| \frac{U + u_\varepsilon}{(1 + u_\varepsilon^2) (1 + U^2)} U \phi \right\|_3 \leq \|U\|_4 \|\phi\|_2\) since

\[
\frac{U + u_\varepsilon}{(1 + u_\varepsilon^2) (1 + U^2)} \leq \frac{U}{(1 + u_\varepsilon^2) (1 + U^2)} + \frac{u_\varepsilon}{(1 + u_\varepsilon^2) (1 + U^2)} \leq \frac{1}{2} + \frac{1}{2} = 1.
\]

Thus (3.13) and (3.14) imply

\[
\int_{\mathbb{R}^2} \nabla U \cdot \nabla \phi + \Gamma \int_{\mathbb{R}^2} \frac{U^3}{1 + U^2} \phi = -\lambda_0 \int_{\mathbb{R}^2} U \phi
\]

and then \(U\) satisfies (3.13) and \(\lim_{|x| \to \infty} U(x) = 0\). Moreover, \(U = U(r) \geq 0\) is radially symmetric, since each \(u_\varepsilon\) is positive and radially symmetric. Therefore, the equation for \(U\) can be written as

\[
\begin{aligned}
U'' + \frac{1}{r} U' - \Gamma \frac{U^3}{1 + U^2} &= \lambda_0 U & \text{for } r > 0, \\
U'(0) = 0, \quad U(\infty) = 0.
\end{aligned}
\]

By the uniqueness of ordinary differential equations, we may assume \(U(0) > 0\).

**Lemma 3.10.** \(U(r) > 0\) for \(r \geq 0\).

**Proof.** We may prove by contradiction. Suppose there exists \(r_0 > 0\) a minimum point of \(U\) such that \(U(r_0) = 0\). Then \(U'(r_0) = U(r_0) = 0\). Hence by the uniqueness of ordinary differential equations, \(U \equiv 0\) contradicts \(\|U\|_2 = 1\) and we have completed the proof. \(\square\)

Due to \(\lim_{r \to \infty} U(r) = 0\), there exists \(R_1 > 0\) such that \(0 < U(r) \leq 1\) for \(r \geq R_1\). By equation (3.13), \(\Delta U = \left(\Gamma \frac{U^2}{1 + U^2} + \lambda_0\right) U \in L^1(B_{R_1})\), since \(\Gamma \frac{U^2}{1 + U^2} + \lambda_0 \in L^\infty\). Hence by the standard regularity theorem of Poisson equation, \(U \in W^{2,4}(B_{R_1})\) and then by the Sobolev embedding \(W^{2,4}(B_{R_1}) \subset L^\infty(B_{R_1})\), we obtain

**Lemma 3.11.** \(U(r) \leq K_2\) for \(r \geq 0\), where \(K_2\) is a positive constant.

Now we prove that \(\lambda_0\) is positive by contradiction. Suppose \(\lambda_0 \leq 0\). Then equation (3.13) and Lemma 3.10 imply

\[
\Delta U = \lambda_0 U + \Gamma \frac{U^3}{1 + U^2} \leq 0 \text{ in } \mathbb{R}^2.
\]

Hence by Lemma 3.11 and the Liouville Theorem, \(U\) must be a constant function, i.e. \(U \equiv 0\), which is impossible. Therefore, we conclude that
Lemma 3.12. \( \lambda_0 > 0 \).

Since \( \Gamma < 0 \), the equation (3.13) becomes

\[
\Delta U - \lambda_0 U = \Gamma \frac{U^3}{1 + U^2} \leq 0 \text{ in } \mathbb{R}^2.
\]

Hence, as for the proof of Lemma 3.9, we may use the strong maximum principle to prove that

Lemma 3.13. \( U = U(r) \) is monotone decreasing with \( r \).

In this manner, we have completed the proof of Theorem 2.1 (ii), and with this the complete proof of Theorem 2.1.

4 Conclusion

In this paper the existence of ground states in the form of counterpropagating solitons in a multidimensional nonlinear Schrödinger model of paraxial beam propagation in media with saturable nonlinearity has been proven. From the proof, a threshold condition on the beam coupling constant for the existence of such fundamental solitons followed.

This work has been supported by the Qatar National Research Fund project NPRP 09-462-1-074, by the Texas Norman Hackman Advanced Research Program Grant No. 010366-0149-2009 from the Texas Higher Education Coordinating Board, and by the grant from the National Science Council of Republic of China.

References

[1] S. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov, Theory of Solitons: The Inverse Scattering Method (Plenum Publishing, New York, 1984); M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations, and Inverse Scattering (Cambridge University Press, Cambridge, 1991).

[2] G.L. Lamb, Elements of soliton theory (Wiley, New York, 1980); N. N. Akhmediev and A. A. Ankiewicz, Solitons (Chapman and Hall, London, 1997); Y. S. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals (Academic, New York, 2003).

[3] C. Sulem and P. Sulem, The nonlinear Schroedinger equation: Self-Focusing and wave collapse (Springer-Verlag, Berlin-Heidelberg-New York, 1999).

[4] M.S. Petrović, M.R. Belić, C. Denz, and Yu.S. Kivshar, Laser Photonics Rev. 5, 214-233 (2011).

[5] D. Jović, R. Jovanović, S. Prvanović, M. Petrović, M. Belić, Opt. Materials 30 (2008) 1173-1176.

[6] E. H. Lieb and R. Seiringer, The stability of matter in quantum mechanics (Cambridge Univ. Press, New York, 2010).

[7] T. Cazenave and P.-L. Lions, Comm. Math. Phys. Volume 85, Number 4 (1982), 549-561.