SINGULAR INTEGRAL OPERATORS ON VARIABLE
LEBESGUE SPACES WITH RADIAL OSCILLATING WEIGHTS

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To the memory of Igor Borisovich Simonenko (1935–2008)

Abstract. We prove a Fredholm criterion for operators in the Banach algebra
of singular integral operators with matrix piecewise continuous coefficients
acting on a variable Lebesgue space with a radial oscillating weight over a
logarithmic Carleson curve. The local spectra of these operators are massive
and have a shape of spiralic horns depending on the value of the variable
exponent, the spirality indices of the curve, and the Matuszewska-Orlicz indices
of the weight at each point. These results extend (partially) the results of
A. Böttcher, Yu. Karlovich, and V. Rabinovich for standard Lebesgue spaces
to the case of variable Lebesgue spaces.

1. Introduction

Let $X$ be a Banach space and $B(X)$ be the Banach algebra of all bounded linear
operators on $X$. An operator $A \in B(X)$ is said to be $n$-normal (resp. $d$-normal) if
its image $\text{Im} A$ is closed in $X$ and the defect number $n(A; X) := \dim \ker A$ (resp.
$d(A; X) := \dim \ker A^*$) is finite. An operator $A$ is said to be semi-Fredholm on $X$ if
it is $n$-normal or $d$-normal. Finally, $A$ is said to be Fredholm if it is simultaneously
$n$-normal and $d$-normal. Let $N$ be a positive integer. We denote by $X_N$ the direct
sum of $N$ copies of $X$ with the norm

$$\|f\| = \|(f_1, \ldots, f_N)\| := (\|f_1\|^2 + \cdots + \|f_N\|^2)^{1/2}.$$ 

Let $\Gamma$ be a Jordan curve, that is, a curve that is homeomorphic to a circle. We
suppose that $\Gamma$ is rectifiable. We equip $\Gamma$ with Lebesgue length measure $|d\tau|$ and
the counter-clockwise orientation. The Cauchy singular integral of $f \in L^1(\Gamma)$ is
defined by

$$(Sf)(t) := \lim_{R \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),$$

where $\Gamma(t, R) := \{\tau \in \Gamma : |\tau - t| < R\}$ for $R > 0$. David [11] (see also [4,
Theorem 4.17]) proved that the Cauchy singular integral generates the bounded
operator $S$ on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if $\Gamma$ is a Carleson
(Ahlfors-David regular) curve, that is,

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{||\Gamma(t, R)||}{R} < \infty,$$

2000 Mathematics Subject Classification. Primary 47B35; Secondary 45E05, 46E30, 47A68.

Key words and phrases. Variable Lebesgue space, Carleson curve, variable exponent, radial
oscillating weight, Matuszewska-Orlicz indices, submultiplicative function.

The author is partially supported by the grant FCT/FEDER/POCTI/MAT/59972/2004.
where $|\Omega|$ denotes the measure of a measurable set $\Omega \subset \Gamma$. We can write $\tau - t = |\tau - t|e^{i \arg(\tau-t)}$ for $\tau \in \Gamma \setminus \{t\}$, and the argument can be chosen so that it is continuous on $\Gamma \setminus \{t\}$. It is known [41 Theorem 1.10] that for an arbitrary Carleson curve the estimate
\[ \arg(\tau - t) = O(-\log |\tau - t|) \quad (\tau \to t) \]
holds for every $t \in \Gamma$. One says that a Carleson curve $\Gamma$ satisfies the logarithmic whirl condition at $t \in \Gamma$ if
\[ \arg(\tau - t) = -\delta(t) \log |\tau - t| + O(1) \quad (\tau \to t) \]
with some $\delta(t) \in \mathbb{R}$. Notice that all piecewise smooth curves satisfy this condition at each point and, moreover, $\delta(t) \equiv 0$. For more information along these lines, see [3, 4, Chap. 1, 5].

A measurable function $w : \Gamma \to [0, \infty]$ is referred to as a weight function or simply a weight if $0 < w(\tau) < \infty$ for almost all $\tau \in \Gamma$. Suppose $p : \Gamma \to (1, \infty)$ is a continuous function. Denote by $L^{p(\cdot)}(\Gamma, w)$ the set of all measurable complex-valued functions $f$ on $\Gamma$ such that
\[ \int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)}d\tau < \infty \]
for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm
\[ \|f\|_{p(\cdot), w} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(\tau)w(\tau)/\lambda|^{p(\tau)}d\tau \leq 1 \right\}. \]
If $p$ is constant, then $L^{p(\cdot)}(\Gamma, w)$ is nothing else than the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma, w)$ as a weighted generalized Lebesgue space with variable exponent or simply as a weighted variable Lebesgue space. This is a special case of Musielak-Orlicz spaces [40] (see also [31]). Nakano [41] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, w)$ are referred to as weighted Nakano spaces. Since $\Gamma$ is compact, we have
\[ 1 < \min_{t \in \Gamma} p(t), \quad \max_{t \in \Gamma} p(t) < \infty. \]
Therefore, if $w \in L^{p(\cdot)}(\Gamma)$ and $1/w \in L^{q(\cdot)}(\Gamma)$, where
\[ q(t) := p(t)/(p(t) - 1) \]
is the conjugate exponent for $p$, then $L^{p(\cdot)}(\Gamma, w)$ is reflexive and its Banach dual can be identified with $L^{q(\cdot)}(\Gamma, 1/w)$ (see e.g. [30] Section 13, [31] Corollary 2.7 and also [21] Section 2).

Following [29] Section 2.3, denote by $W$ the class of all continuous functions $\varrho : [0, |\Gamma|] \to [0, \infty)$ such that $\varrho(0) = 0$, $\varrho(x) > 0$ if $0 < x \leq |\Gamma|$, and $\varrho$ is almost increasing, that is, there is a universal constant $C > 0$ such that $\varrho(x) \leq C \varrho(y)$ whenever $x \leq y$. Further, let $\mathbb{W}$ be the set of all functions $\varrho : [0, |\Gamma|] \to [0, \infty]$ such that $x^\alpha \varrho(x) \in W$ and $x^\beta / \varrho(x) \in W$ for some $\alpha, \beta \in \mathbb{R}$. Clearly, the functions $\varrho(x) = x^\gamma$ belong to $W$ for all $\gamma \in \mathbb{R}$. For $\varrho \in \mathbb{W}$, put
\[ \Phi_\varrho^0(x) := \limsup_{y \to 0} \frac{\varrho(xy)}{\varrho(y)}, \quad x \in (0, \infty). \]
Since \( \varrho \in \mathbb{W} \), one can show (see Subsection 2.2) that the limits

\[
m(\varrho) := \lim_{x \to 0} \frac{\log \Phi_\varrho(x)}{\log x}, \quad M(\varrho) := \lim_{x \to \infty} \frac{\log \Phi_\varrho(x)}{\log x}
\]

exist and \(-\infty < m(\varrho) \leq M(\varrho) < +\infty\). These numbers were defined under some extra assumptions on \( \varrho \) by Matuszewska and Orlicz [38, 39] (see also [36] and Chapter 11]). We refer to extra assumptions on \( M(\varrho) \) as the lower (resp. upper) Matuszewska-Orlicz index of \( \varrho \). For \( \varrho(x) = x^\gamma \) one has \( m(\varrho) = M(\varrho) = \gamma \). Examples of functions \( \varrho \in \mathbb{W} \) with \( m(\varrho) < M(\varrho) \) can be found, for instance, in [1, 37, p. 93, p. 46, Section 2].

Fix pairwise distinct points \( t_1, \ldots, t_n \in \Gamma \) and functions \( w_1, \ldots, w_n \in \mathbb{W} \). Consider the following weight

\[
w(t) := \prod_{k=1}^{n} w_k(|t-t_k|), \quad t \in \Gamma.
\]

Each function \( w_k(|t-t_k|) \) is a radial oscillating weight. This is a natural generalization of so-called Khvedelidze weights \( w(t) = \prod_{k=1}^{n} |t-t_k|^{\lambda_k} \), where \( \lambda_k \in \mathbb{R} \) (see, e.g., [4, Section 2.2], [14, Section 1.4], [26], [28]).

**Theorem 1.1.** Suppose \( \Gamma \) is a Carleson Jordan curve and \( p : \Gamma \to (1, \infty) \) is a continuous function satisfying

\[
\|p(\tau) - p(t)\| \leq -A_p/\log |\tau - t| \quad \text{whenever} \quad |\tau - t| \leq 1/2,
\]

where \( A_p \) is a positive constant depending only on \( \Gamma \). Let \( w_1, \ldots, w_n \in \mathbb{W} \) and the weight \( w \) be given by (1.3). The Cauchy singular integral operator \( S \) is bounded on \( L^p(\Gamma, w) \) if and only if

\[
0 < 1/p(t_k) + m(w_k), \quad 1/p(t_k) + M(w_k) < 1 \quad \text{for all} \quad k \in \{1, \ldots, n\}.
\]

For \( w(t) = \prod_{k=1}^{n} |t-t_k|^{\lambda_k} \), (1.5) reads as \( 0 < 1/p(t_k) + \lambda_k < 1 \) for all \( k \in \{1, \ldots, n\} \) and for these weights Theorem 1.1 is obtained in [28]. The sufficiency portion of Theorem 1.1 in the form stated above is obtained by V. Kokilashvili, N. Samko, and S. Samko [29, Theorem 4.3]. The necessity portion is a new result.

It will be proved in Section 1.1.

We define by \( PC(\Gamma) \) the set of all \( a \in L^{\infty}(\Gamma) \) for which the one-sided limits

\[
a(\tau \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)
\]

exist and finite at each point \( t \in \Gamma \); here \( \tau \to t - 0 \) means that \( \tau \) approaches \( t \) following the orientation of \( \Gamma \), while \( \tau \to t + 0 \) means that \( \tau \) goes to \( t \) in the opposite direction. Functions in \( PC(\Gamma) \) are called piecewise continuous functions.

The operator \( S \) is defined on \( L^p(\Gamma, w) \) elementwise. We let stand \( PC_{N \times N}(\Gamma) \) for the algebra of all \( N \times N \) matrix functions with entries in \( PC(\Gamma) \). Writing the elements of \( L^p_{N}(\Gamma, w) \) as columns, we can define the multiplication operator \( aI \) for \( a \in PC_{N \times N}(\Gamma) \) as multiplication by the matrix function \( a \). Let \( \text{alg}(S, PC, L^p_{N}(\Gamma, w)) \) denote the smallest closed subalgebra of \( B(L^p_{N}(\Gamma, w)) \) containing the operator \( S \) and the set \( \{aI : a \in PC_{N \times N}(\Gamma)\} \).

For the case of constant \( p \in (1, \infty) \), Khvedelidze weights \( w \), and piecewise Lyapunov curves \( \Gamma \), the algebra \( \text{alg}(S, PC, L^p_{N}(\Gamma, w)) \) was well understood in 1970–1980’s (see e.g. [13, 14, 33, 49, 50]). In the earlier 1990’s, Spitkovsky [51] discovered
that local spectra of singular integral operators on Lebesgue spaces with Muckenhoupt weights on smooth curves have a shape of horns bounded by two circular arcs depending on the so-called indices of powerlikeness of the weight. In the middle of 1990’s, Böttcher, Yu. Karlovich, and Rabinovich further observed that these horns metamorphose to spiralic horns bounded by logarithmic double spirals \[3, 6, 42\] if one passes from nice curves to Carleson curves satisfying (1.1). These spiralic horns become even more interesting creatures (so-called “general leaves”) in case of arbitrary Carleson curves \[4, 5\]. The shape of a general leaf depends on the spirality indices of the curve and the indices of powerlikeness of the weight. The Fredholm theory for the algebra \(\text{alg}(S, PC, L_N^p(\Gamma, w))\) under the most general conditions on the curve \(\Gamma\) and the weight \(w\) is constructed by Böttcher and Yu. Karlovich and is presented in the monograph \[4\] (although, we advise to start the study of this theory from the nice survey \[5\]).

Partially, the results of \[4\] are extended to Orlicz spaces \[17\], and further to rearrangement-invariant spaces \[18\], and rearrangement-invariant spaces with Muckenhoupt weights \[20\]. The results of these papers resemble those of \[4\] with one important difference: the number \(1/p\) is replaced by the Boyd indices \(\alpha_X, \beta_X\) of the rearrangement-invariant space. Since the Boyd indices may be different, one more factor (a general space) leads to massive local spectra (for standard Lebesgue spaces these factors were general weights and general curves). Local spectra of singular integral operators can be massive also on weighted Hölder spaces on Carleson curves \[43\].

The study of singular integral operators (SIOs) with discontinuous coefficients on weighted variable Lebesgue spaces on sufficiently nice curves was started in \[21, 23, 27, 30\]. A Fredholm criterion for an arbitrary operator in \(\text{alg}(S, PC, L_N^{p(t)}(\Gamma, w))\) for Carleson curves satisfying (1.1) and Khvedelidze weights is proved in \[22\]. Local spectra in that paper have a shape of logarithmic double spirals and cannot be massive. Under the same assumptions on the curve and the weight, it is proved in \[24\] that every semi-Fredholm operator in the algebra \(\text{alg}(S, PC, L_N^{p(t)}(\Gamma, w))\) is Fredholm.

In this paper we extend the results of \[22, 23, 24\] to the case of Carleson curves satisfying (1.1) and weights of the form (1.3) satisfying the conditions of Theorem (1.1). Local spectra of SIOs corresponding to the points \(t \not \in \{t_1, \ldots, t_n\}\) are logarithmic double spirals depending on \(\delta(t)\) and \(1/p(t)\). These spirals blow up to spiralic horns at \(t = t_k\) if \(k \in \{1, \ldots, n\}\) and \(m(w_k) < M(w_k)\). These spiralic horns are bounded by two logarithmic double spirals depending on \(\delta(t_k)\) and on \(1/p(t_k)+m(w_k)\) and \(1/p(t_k)+M(w_k)\), respectively. Up to our knowledge, this paper is the first work, where massive local spectra of singular integral operators appear in the setting of weighted variable Lebesgue spaces. These results resemble those of \[3, 6\] for weighted standard Lebesgue spaces, although the weights considered in \[3, 6\] are more general than in the present paper.

The paper is organized as follows. In Section 2 we collect all necessary information on indices of submultiplicative functions associated with curves and weights. We prove that the indices of powerlikeness (see \[4\] Chap. 3) of the weight (1.3) at \(t_k\) coincide with the Matuszewska-Orlicz indices of \(w_k\). Section 3 contains standard results on singular integral operators with \(L_\infty\) coefficients: the necessary condition for Fredholmness, the local principle of Simonenko type, and the theorem on a
Wiener-Hopf factorization. In Section 4, we prove a Fredholm criterion for a singular integral operator $aP + Q$, where $a \in PC(\Gamma)$ and $P := (I + S)/2$, $Q := (I - S)/2$. In a sense, the main result of Section 4 is the heart of the paper. It follows from a more general necessary condition for the Fredholmness of $aP + Q$ (see [21, Theorem 8.1]) and a sufficient condition for the Fredholmness of $aP + Q$, whose proof is based on the local principle of Simonenko type and the Wiener-Hopf factorization of a local representative of $a$. In fact, [21, Theorem 8.1] together with the results of Section 2 imply the necessity of the conditions (1.5) for the boundedness of the operator $S$ on $L^p(\Gamma, w)$. Section 5 contains the Allan-Douglas local principle and the two projections theorem. These results are our main tools in the construction of a symbols calculus for the algebra $alg(S, PC, L^p(\Gamma, w))$ on the basis of the main result of Section 4. In Section 6, following the well known scheme (see e.g. [3], [4, Chap. 8] and also [17, 20, 22, 23]), we prove a Fredholm criterion for an arbitrary operator $A \in alg(S, PC, L^p(\Gamma, w))$. Finally, we collect some remarks on index formulas and semi-Fredholm operators in the algebra $alg(S, PC, L^p(\Gamma, w))$ and discuss open problems.

2. Submultiplicative functions and their indices

2.1. Submultiplicative functions. Following [4, Section 1.4], we say a function $\Phi : (0, \infty) \rightarrow (0, \infty)$ is regular if it is bounded in an open neighborhood of 1. A function $\Phi : (0, \infty) \rightarrow (0, \infty)$ is said to be submultiplicative if

$$\Phi(xy) \leq \Phi(x)\Phi(y) \quad \text{for all} \quad x, y \in (0, \infty).$$

It is easy to show that if $\Phi$ is regular and submultiplicative, then $\Phi$ is bounded away from zero in some open neighborhood of 1. Moreover, in this case $\Phi(x)$ is finite for all $x \in (0, \infty)$. Given a regular and submultiplicative function $\Phi : (0, \infty) \rightarrow (0, \infty)$, one defines

$$\alpha(\Phi) := \sup_{x \in (0, 1)} \log \frac{\Phi(x)}{\log x}, \quad \beta(\Phi) := \inf_{x \in (1, \infty)} \log \frac{\Phi(x)}{\log x}. $$

Clearly, $-\infty < \alpha(\Phi)$ and $\beta(\Phi) < \infty$.

Theorem 2.1 (see [4], Theorem 1.13 or [32], Chap. 2, Theorem 1.3). If a function $\Phi : (0, \infty) \rightarrow (0, \infty)$ is regular and submultiplicative, then

$$\alpha(\Phi) = \lim_{x \rightarrow 0} \log \frac{\Phi(x)}{\log x}, \quad \beta(\Phi) = \lim_{x \rightarrow \infty} \log \frac{\Phi(x)}{\log x}$$

and $-\infty < \alpha(\Phi) \leq \beta(\Phi) < +\infty$.

The quantities $\alpha(\Phi)$ and $\beta(\Phi)$ are called the lower and upper indices of the regular and submultiplicative function $\Phi$, respectively.

2.2. Matuszewska-Orlicz indices.

Lemma 2.2. Let $A > 0$ and $\varphi : (0, A] \rightarrow (0, \infty)$ be a continuous function. Then for every $B \in (0, A]$ the function

$$\Phi_{\varphi, B}(x) := \begin{cases} 
\sup_{0 < y \leq B} \frac{\varphi(xy)}{\varphi(y)} & \text{if} \quad x \in (0, 1], \\
\sup_{0 < y \leq B} \frac{\varphi(y)}{\varphi(x^{-1}y)} & \text{if} \quad x \in (1, \infty)
\end{cases}$$

(2.1)
is submultiplicative.

This statement is proved similarly to [4] Lemma 1.15.

**Lemma 2.3.** Let \( A > 0 \) and \( \varrho : (0, A] \to (0, \infty) \) be a continuous function. If for some \( B \in (0, A] \) the function \( \Phi_{\varrho,B} \) given by (2.1) is regular, then

\[
\Phi_{\varrho}(x) := \limsup_{y \to 0} \frac{\varrho(xy)}{\varrho(y)}
\]

is regular and submultiplicative and both functions have the same lower and upper indices

\[
\alpha(\Phi_{\varrho,B}) = \alpha(\Phi_{\varrho}) =: m(\varrho), \quad \beta(\Phi_{\varrho,B}) = \beta(\Phi_{\varrho}) =: M(\varrho).
\]

This statement is proved by analogy with [9, Lemma 2(a)] (see also [2, Theorem 8.18] and [4, Lemma 1.16]).

**Lemma 2.4.** If \( \varrho \in \mathcal{W} \), then for every \( B \in (0, \|\Gamma\|) \) the function \( \Phi_{\varrho,B} \) is regular.

**Proof.** Since \( \varrho \in \mathcal{W} \), there exist \( a, b \in \mathbb{R} \) such that the functions \( x^a \varrho(x) \) and \( x^b \varrho(x) \) are almost increasing on \((0, \|\Gamma\|)\), that is, there exist positive constants \( c_a, c_b \) such that \( x^a \varrho(x) \leq c_a y^a \varrho(y) \) and \( x^b \varrho(x) \leq c_b y^b \varrho(y) \) whenever \( x \leq y \) and \( x, y \in (0, \|\Gamma\|) \). Suppose \( x \in (0, 1] \) and \( y \in (0, \|\Gamma\|) \). Then

\[
\frac{\varrho(xy)}{\varrho(y)} = \frac{(xy)^a \varrho(xy)}{(xy)^a \varrho(y)} \leq \frac{c_a y^a \varrho(y)}{(xy)^a \varrho(y)} = \frac{c_a}{x^a},
\]

Hence

\[
\Phi_{\varrho,B}(x) \leq \Phi_{\varrho,\|\Gamma\|}(x) \leq c_a x^{-a}, \quad x \in (0, 1].
\]

Similarly,

\[
\Phi_{\varrho,B}(x) \leq \Phi_{\varrho,\|\Gamma\|}(x) \leq c_b x^b, \quad x \in (1, \infty).
\]

Thus, \( \Phi_{\varrho,B} \) is regular for every \( B \in (0, \|\Gamma\|) \). \( \square \)

From the above results and Theorem 2.1 we conclude that if \( \varrho \in \mathcal{W} \), then its Matuszewska-Orlicz indices are well defined by (1.2).

2.3. **Spirality indices of Carleson curves.** Fix \( t \in \Gamma \) and put

\[
d_t := \max_{\tau \in \Gamma} |\tau - t|.
\]

Suppose \( \psi : \Gamma \setminus \{t\} \to (0, \infty) \) is a continuous function and consider

\[
F_{\psi,t}(R_1, R_2) := \max_{\tau \in \Gamma, |\tau - t| = R_1} \psi(\tau) / \min_{\tau \in \Gamma, |\tau - t| = R_2} \psi(\tau), \quad R_1, R_2 \in (0, d_t).
\]

By [4] Lemma 1.15, the function

\[
(W_t \psi)(x) := \begin{cases} 
\sup_{0 < R \leq d_t} F_{\psi,t}(x R, R) & \text{if } x \in (0, 1], \\
\sup_{0 < R \leq d_t} F_{\psi,t}(R, x^{-1} R) & \text{if } x \in (1, \infty)
\end{cases}
\]

is submultiplicative. For \( t \in \Gamma \), we have,

\[
\tau - t = |\tau - t| \arg(\tau - t), \quad \tau \in \Gamma \setminus \{t\},
\]

and the argument \( \arg(\tau - t) \) may be chosen to be continuous on \( \Gamma \setminus \{t\} \). Consider

\[
\eta_t(\tau) := e^{-\arg(\tau - t)}.
\]
Lemma 2.5 (see [4], Theorem 1.18). If \( \Gamma \) is a Carleson Jordan curve, then for every \( t \in \Gamma \) the function \( W_t \eta_t \) is regular and submultiplicative. If, in addition, it satisfies (1.1) at some point \( t \in \Gamma \), then
\[
\alpha(W_t \eta_t) = \beta(W_t \eta_t) = \delta(t).
\]

The numbers \( \alpha(W_t \eta_t) \) and \( \beta(W_t \eta_t) \) are called the lower and upper spirality indices of \( \Gamma \) at \( t \), respectively. If \( \Gamma \) is a piecewise smooth curve, then \( \alpha(W_t \eta_t) = \beta(W_t \eta_t) = 0 \) for all \( t \in \Gamma \) (non-spiral curves). Carleson curves satisfying (1.1) behave like perturbed logarithmic spirals. Examples of Carleson curves which have arbitrary prescribed and distinct spirality indices are given in [4, Section 1.6].

2.4. Indices of powerlikeness of continuous nonvanishing weights. Let \( \psi \) be a weight on \( \Gamma \) such that \( \log \psi \in L^1(\Gamma(t,R)) \) for every \( R \in (0,d_t] \). Put
\[
H_{\psi,t}(R_1,R_2) := \frac{\exp\left(\frac{1}{\Gamma(t,R)} \int_{\Gamma(t,R)} \log \psi(\tau)|d\tau| \right)}{\exp\left(\frac{1}{\Gamma(t,R)} \int_{\Gamma(t,R)} \log \psi(\tau)|d\tau| \right)}, \quad R_1, R_2 \in (0,d_t].
\]

Consider the function
\[
(V_t^0 \psi)(x) := \limsup_{R \to 0} H_{\psi,t}(xR,R), \quad x \in (0,\infty).
\]

Lemma 2.6. Let \( \Gamma \) be a Carleson Jordan curve. Suppose \( \psi : \Gamma \to [0,\infty) \) is a weight which is continuous and nonvanishing on \( \Gamma \setminus \{t\} \). If \( W_t \psi \) is regular, then \( V_t^0 \psi \) is regular and submultiplicative and
\[
\alpha(W_t \psi) = \alpha(V_t^0 \psi), \quad \beta(W_t \psi) = \beta(V_t^0 \psi).
\]

This statement follows from [4, Theorem 3.3(c) and Lemma 3.16].

The numbers \( \alpha(V_t^0 \psi) \) and \( \beta(V_t^0 \psi) \) are called the lower and upper indices of powerlikeness of \( \psi \) at \( t \in \Gamma \), respectively. This terminology can be explained by the simple fact that for the power weight \( \psi(\tau) := |\tau - t|^\lambda \) its indices of powerlikeness coincide and are equal to \( \lambda \). Examples of Muckenhoupt weights with distinct indices of powerlikeness are given in [4, Proposition 3.25 and Examples 3.26–3.27].

2.5. Indices of powerlikeness of radial oscillating weights.

Lemma 2.7. Let \( \Gamma \) be a Carleson Jordan curve and \( \varrho \in W \). Suppose
\[
\psi_t(\tau) := \varrho(|\tau - t|) \text{ for } \tau \in \Gamma \setminus \{t\}.
\]

Then the functions \( W_t \psi_t \) and \( V_t^0 \psi_t \) are regular and submultiplicative and
\[
\alpha(\varrho) = \alpha(W_t \psi_t) = \alpha(V_t^0 \psi_t), \quad \beta(\varrho) = \beta(W_t \psi_t) = \beta(V_t^0 \psi_t).
\]

Proof. By Lemmas 2.2, 2.3 the functions \( \Phi_{\varrho,d_t} \) and \( \Phi_{\varrho}^0 \) are regular and submultiplicative and
\[
\alpha(\Phi_{\varrho,d_t}) = \alpha(\Phi_{\varrho}^0) = \alpha(\varrho), \quad \beta(\Phi_{\varrho,d_t}) = \beta(\Phi_{\varrho}^0) = \beta(\varrho).
\]

If \( x \in (0,1] \) and \( 0 < R \leq d_t \), then
\[
F_{\psi_t,\varrho}(xR,R) := \max_{\tau \in \Gamma,|\tau - t| = xR} \frac{\varrho(|\tau - t|)}{\varrho(R)} = \min_{\tau \in \Gamma,|\tau - t| = R} \frac{\varrho(|\tau - t|)}{\varrho(R)}.
\]
if \( x \in [1, \infty) \) and \( 0 < R \leq d_t \), then
\[
F_{\psi,t}(R, x^{-1}R) = \frac{\max_{\tau \in \Gamma, |\tau-t|=R} \theta(|\tau-t|)}{\min_{\tau \in \Gamma, |\tau-t|=x^{-1}R} \theta(|\tau-t|)} = \frac{\varrho(R)}{\varrho(x^{-1}R)}.
\]

From (2.3) and (2.4) it follows that
\[
(W_t \psi_t)(x) = \Phi_{\varrho,d_t}(x), \quad x \in (0, \infty).
\]
Hence \( W_t \psi_t \) is regular because \( \Phi_{\varrho,d_t} \) is so and
\[
\alpha(W_t \psi_t) = \alpha(\Phi_{\varrho,d_t}), \quad \beta(W_t \psi_t) = \beta(\Phi_{\varrho,d_t}).
\]
Combining (2.2), (2.3) and Lemma 2.6 we arrive at the desired statement. \( \square \)

**Theorem 2.8.** Suppose \( \Gamma \) is a Carleson Jordan curve. If \( w_1, \ldots, w_n \in \mathbb{W} \) and \( w(\tau) = \prod_{k=1}^{n} w_k(|\tau-t_k|) \), then for every \( t \in \Gamma \) the function \( V_t^0 w \) is regular and submultiplicative and
\[
\alpha(V_t^0 w) = m(w_k), \quad \beta(V_t^0 w) = M(w_k) \quad \text{for} \quad k \in \{1, \ldots, n\},
\]
\[
\alpha(V_t^0 w) = 0, \quad \beta(V_t^0 w) = 0 \quad \text{for} \quad t \in \Gamma \setminus \{t_1, \ldots, t_n\}.
\]

*Proof.* If \( t \notin \{t_1, \ldots, t_n\} \), then there exists \( D_t > 0 \) such that the portion \( \Gamma(t, D_t) \) does not contain any point \( t_1, \ldots, t_n \). Since the weight \( w \) may vanish or go to infinity only at \( t_1, \ldots, t_n \) and \( w_k(|\tau-t_k|) \) are continuous on \( \Gamma \setminus \{t_k\} \), we can conclude that there exist constants \( c_t, C_t \in (0, \infty) \) such that \( c_t \leq w(\tau) \leq C_t \) for all \( \tau \in \Gamma(t, D_t) \). Assume that \( x \in (0, \infty) \) and \( xR, R \in (0, D_t) \). Then
\[
e^{c_t-C_t} \leq H_{w,t}(xR, R) \leq e^{C_t-c_t}.
\]
Hence
\[
e^{c_t-C_t} \leq (V_t^0 w)(x) \leq e^{C_t-c_t}, \quad x \in (0, \infty),
\]
which implies that \( V_t^0 w \) is regular. By [4] Theorem 3.3(c), the function \( V_t^0 w \) is submultiplicative. It is easy to see that \( \alpha(V_t^0 w) = \beta(V_t^0 w) = 0 \).

If \( k \in \{1, \ldots, n\} \), then there exists \( D_k > 0 \) such that the portion \( \Gamma(t_k, D_k) \) does not contain any point of \( \{t_1, \ldots, t_n\} \setminus \{t_k\} \). As before, there exists positive constants \( c_k, C_k \) such that
\[
c_k \leq \frac{w(\tau)}{w_k(|\tau-t_k|)} \leq C_k \quad \text{for all} \quad \tau \in \Gamma(t_k, D_k).
\]
If \( x \in (0, \infty) \) and \( xR, R \in (0, D_k) \), then taking into account that
\[
H_{w,t_k}(xR, R) = H_{w_k(|\tau-t_k|), t_k}(xR, R) \cdot H_{w/w_k(|\tau-t_k|), t_k}(xR, R),
\]
we get
\[
e^{c_k-C_k} H_{w_k(|\tau-t_k|), t_k}(xR, R) \leq H_{w,t_k}(xR, R) \leq e^{c_k-C_k} H_{w_k(|\tau-t_k|), t_k}(xR, R).
\]
Therefore, for all \( x \in (0, \infty) \),
\[
e^{c_k-C_k} (V_t^0 w_k(|\tau-t_k|))(x) \leq (V_t^0 w)(x) \leq e^{c_k-C_k} (V_t^0 w_k(|\tau-t_k|))(x).
\]
By Lemma 2.7, the function \( V_t^0 w_k(|\tau-t_k|) \) is regular and
\[
m(w_k) = \alpha(V_t^0 w_k(|\tau-t_k|)), \quad M(w_k) = \beta(V_t^0 w_k(|\tau-t_k|)).
\]
Since the function $V^0_{t_k} w_k(|\tau - t_k|)$ is regular, from (2.6) it follows that $V^0_{t_k} w$ is also regular and

$$\alpha(V^0_{t_k} w_k(|\tau - t_k|)) = \alpha(V^0_{t_k} w), \quad \beta(V^0_{t_k} w_k(|\tau - t_k|)) = \beta(V^0_{t_k} w).$$

Combining (2.7) and (2.8), we finally arrive at the equalities $\alpha(V^0_{t_k} w) = m(w_k)$ and $\beta(V^0_{t_k} w) = M(w_k).$

2.6. **Indicator functions.** Let $\Gamma$ be a rectifiable Jordan curve and $p : \Gamma \to (1, \infty)$ be a continuous function. We say that a weight $w : \Gamma \to [0, \infty]$ belongs to $A_{p(\cdot)}(\Gamma)$ if

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{1}{R} \| w \chi_{\Gamma(t,R)} \|_{p(\cdot)} \| w^{-1} \chi_{\Gamma(t,R)} \|_{q(\cdot)} < \infty.$$ 

If $p = \text{const} \in (1, \infty),$ then this class coincides with the well known Muckenhoupt class. From the Hölder inequality for $L^p(\cdot)(\Gamma)$ (see e.g. [31, Theorem 2.1]) it follows that if $w \in A_{p(\cdot)}(\Gamma),$ then $\Gamma$ is a Carleson curve.

**Theorem 2.9.** Let $\Gamma$ be a rectifiable Jordan curve and let $p : \Gamma \to (1, \infty)$ be a continuous function. If $w : \Gamma \to [0, \infty]$ is an arbitrary weight such that the operator $S$ is bounded on $L^p(\cdot)(\Gamma, w),$ then $w \in A_{p(\cdot)}(\Gamma).$

This statement can be proved by analogy with [17] Theorem 4.3, [18] Theorem 3.2 (see also [1] Theorem 4.8). If $p = \text{const} \in (1, \infty),$ then $w \in A_p(\Gamma)$ is also sufficient for the boundedness of $S$ on the weighted Lebesgue space $L^p(\Gamma, w)$ (see e.g. [1] Theorem 4.15).

**Lemma 2.10.** Let $\Gamma$ be a rectifiable Jordan curve, let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (1.3), and let $w : \Gamma \to [0, \infty]$ be an arbitrary weight. Suppose $w \in A_{p(\cdot)}(\Gamma).$ Then

(a) for every $t \in \Gamma$ and $x \in \mathbb{R},$ the function $V^0_t (\eta^x w)$ is regular and submultiplicative;

(b) for every $t \in \Gamma,$

$$0 \leq 1/p(t) + \alpha(V^0_t w), \quad 1/p(t) + \beta(V^0_t w) \leq 1.$$ 

Part (a) follows from [21] Lemmas 4.5, 5.8, and 5.13. Part (b) is the consequence of [21] Lemma 4.9 and Theorem 5.9.

Under the conditions of Lemma 2.10, the indicator functions of the pair $(\Gamma, w)$ at every $t \in \Gamma$ are well defined by

$$\alpha_t(x) := \alpha(V^0_t (\eta^x w)), \quad \beta_t(x) := \beta(V^0_t (\eta^x w))$$

because of Theorem 2.1.

**Lemma 2.11.** Let $\Gamma$ be a Carleson Jordan curve satisfying the logarithmic whirl condition (1.1) at every $t \in \Gamma,$ let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (1.3), and let $w : \Gamma \to [0, \infty]$ be an arbitrary weight. If $w \in A_{p(\cdot)}(\Gamma),$ then for every $t \in \Gamma$ and $x \in \mathbb{R},$

$$\alpha_t(x) = \alpha(V^0_t w) + \delta(t)x, \quad \beta_t(x) = \beta(V^0_t w) + \delta(t)x.$$ 

This result follows from Lemma 2.5 and [21] Lemma 5.8, Corollary 5.16(b).
3. Singular integral operators with $L^\infty$ coefficients

3.1. General necessary condition for Fredholmness. In this section we will suppose that $\Gamma$ is a Carleson Jordan curve, $p : \Gamma \to (1, \infty)$ is a continuous function, and $w : \Gamma \to [0, \infty]$ is an arbitrary weight (not necessarily of the form (1.3)) such that $S$ is bounded on $L^{p(\cdot)}(\Gamma, w)$. Under these assumptions,

$$ P := (I + S)/2, \quad Q := (I - S)/2 $$

are bounded projections on $L^{p(\cdot)}(\Gamma, w)$ (see [21, Lemma 6.4]). The operators of the form $aP + Q$, where $a \in L^{\infty}(\Gamma)$, are called singular integral operators (SIOs).

**Theorem 3.1.** Suppose $a \in L^{\infty}(\Gamma)$. If $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$, then $a^{-1} \in L^{\infty}(\Gamma)$.

This result follows from [21, Theorem 6.11].

3.2. The local principle of Simonenko type. Two functions $a, b \in L^{\infty}(\Gamma)$ are said to be locally equivalent at a point $t \in \Gamma$ if

$$ \inf \{ ||(a - b)c||_{\infty} : c \in C(\Gamma), \ c(t) = 1 \} = 0. $$

**Theorem 3.2.** Suppose $a \in L^{\infty}(\Gamma)$ and for each $t \in \Gamma$ there exists a function $a_t \in L^{\infty}(\Gamma)$ which is locally equivalent to $a$ at $t$. If the operators $a_t P + Q$ are Fredholm on $L^{p(\cdot)}(\Gamma, w)$ for all $t \in \Gamma$, then $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$.

For weighted Lebesgue spaces this theorem is known as Simonenko's local principle [49]. It follows from [21, Theorem 6.13].

3.3. Wiener-Hopf factorization. The curve $\Gamma$ divides the complex plane $\mathbb{C}$ into the bounded simply connected domain $D^+$ and the unbounded domain $D^-$. Without loss of generality we assume that $0 \in D^+$. We say that a function $a \in L^{\infty}(\Gamma)$ admits a Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma, w)$ if $1/a \in L^{\infty}(\Gamma)$ and $a$ can be written in the form

$$ a(t) = a_-(t)t^\kappa a_+(t) \quad \text{a.e. on } \Gamma, $$

where $\kappa \in \mathbb{Z}$, and the factors $a_\pm$ enjoy the following properties:

(i) $a_- \in QL^{p(\cdot)}(\Gamma, w) + \mathbb{C}$, \quad $1/a_- \in QL^{p(\cdot)}(\Gamma, 1/w) + \mathbb{C}$,

$$ a_+ \in PL^{p(\cdot)}(\Gamma, 1/w), \quad 1/a_+ \in PL^{p(\cdot)}(\Gamma, w), $$

(ii) the operator $(1/a_+)Sa_+I$ is bounded on $L^{p(\cdot)}(\Gamma, w)$.

One can prove that the number $\kappa$ is uniquely determined.

**Theorem 3.3.** A function $a \in L^{\infty}(\Gamma)$ admits a Wiener-Hopf factorization (3.1) on $L^{p(\cdot)}(\Gamma, w)$ if and only if the operator $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$. If $aP + Q$ is Fredholm, then its index is equal to $-\kappa$.

This theorem goes back to Simonenko [47, 49]. For more about this topic we refer to [3, Section 6.12], [8, Section 5.5], [13, Section 8.3] and also to [10, 35] in the case of weighted Lebesgue spaces. Theorem 3.3 follows from [21, Theorem 6.14].
4. Singular integral operators with $PC$ coefficients

4.1. Necessary condition for Fredholmness in case of arbitrary weights.

**Theorem 4.1 (see [21], Theorem 8.1).** Let $\Gamma$ be a Carleson Jordan curve and let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (1.3). Suppose $w : \Gamma \to [0, \infty]$ is an arbitrary weight such that the operator $S$ is bounded on $L^p(\Gamma, w)$. If the operator $aP + Q$, where $a \in PC(\Gamma)$, is Fredholm on $L^p(\Gamma, w)$, then $a(t \pm 0) \neq 0$ and

$$-rac{1}{2\pi} \arg \frac{a(t - 0)}{a(t + 0)} + \frac{1}{p(t)} + \theta \alpha_t \left( \frac{1}{2\pi} \log \left| \frac{a(t - 0)}{a(t + 0)} \right| \right) + (1 - \theta) \beta_t \left( \frac{1}{2\pi} \log \left| \frac{a(t - 0)}{a(t + 0)} \right| \right)$$

is not an integer number for all $t \in \Gamma$ and all $\theta \in [0, 1]$.

Notice that from Theorem 2.9 and Lemma 2.10(b) it follows that the indicator functions $\alpha_t$ and $\beta_t$ are well defined for all $t \in \Gamma$.

For standard Lebesgue spaces with Muckenhoupt weights the converse to Theorem 4.1 is also true (see [4], Proposition 7.3).

**Corollary 4.2.** Let $\Gamma$ be a Carleson Jordan curve and let $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (1.4). If $w : \Gamma \to [0, \infty]$ is an arbitrary weight such that the operator $S$ is bounded on $L^p(\Gamma, w)$, then

$$0 < 1/p(t) + \alpha(V_t^0 w), \quad 1/p(t) + \beta(V_t^0 w) < 1$$

for every $t \in \Gamma$.

**Proof.** From Theorem 2.9 and Lemma 2.10(b) it follows that

$$0 \leq 1/p(t) + \alpha(V_t^0 w), \quad 1/p(t) + \beta(V_t^0 w) \leq 1.$$  

On the other hand, if we take $a = 1$, then $aP + Q = I$ is obviously invertible on $L^p(\Gamma, w)$. By Theorem 4.1

$$1/p(t) + \theta \alpha(V_t^0 w) + (1 - \theta) \beta(V_t^0 w) \notin \mathbb{Z}$$

for all $t \in \Gamma$ and all $\theta \in [0, 1]$. Combining (4.2) and (4.3) with $\theta = 0$ and $\theta = 1$, we arrive at (1.1). $\square$

**Corollary 4.3.** Let $\Gamma$ be a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ be a continuous function satisfying (1.3). Suppose a weight $w$ is given by (1.3), where $w_1, \ldots, w_n \in \mathcal{W}$. If the Cauchy singular integral operator $S$ is bounded on the space $L^p(\Gamma, w)$, then the condition (1.5) is fulfilled.

This result follows from Corollary 4.2 and Theorem 2.8. It gives the necessity portion of Theorem 4.1.

4.2. Wiener-Hopf factorization of local representatives. Fix $t \in \Gamma$. For a function $a \in PC(\Gamma)$ such that $a^{-1} \in L^\infty(\Gamma)$, we construct a “canonical” function $g_{t, \gamma}$ which is locally equivalent to $a$ at the point $t \in \Gamma$. The interior and the exterior of the unit circle can be conformally mapped onto $D^+$ and $D^-$ of $\Gamma$, respectively, so that the point $1$ is mapped to $t$, and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let $\Lambda_0$ and $\Lambda_\infty$ denote the images of $[0, 1]$ and $[1, \infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins $0$ to $\infty$ and meets $\Gamma$ at exactly one point, namely $t$. Let $\arg z$ be a continuous branch of argument in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbb{C}$, define the function $z^\gamma := |z|^e^{\gamma \arg z}$, where $z \in \mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, $z^\gamma$ is an analytic function in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of $z^\gamma$ to $\Gamma \setminus \{t\}$ will be denoted by $g_{t, \gamma}$. [12]
Obviously, \( g_{t, \gamma} \) is continuous and nonzero on \( \Gamma \setminus \{t\} \). Since \( a(t \pm 0) \neq 0 \), we can define \( \gamma_t = \gamma \in \mathbb{C} \) by the formulas

\[
\begin{align*}
\text{Re} \gamma_t &:= \frac{1}{2\pi} \arg \frac{a(t - 0)}{a(t + 0)}, \\
\text{Im} \gamma_t &:= -\frac{1}{2\pi} \log \left| \frac{a(t - 0)}{a(t + 0)} \right|,
\end{align*}
\]

where we can take any value of \( \arg(a(t - 0)/a(t + 0)) \), which implies that any two choices of \( \text{Re} \gamma_t \) differ by an integer only. Clearly, there is a constant \( c_t \in \mathbb{C} \setminus \{0\} \) such that \( a(t \pm 0) = c_t g_{t, \gamma_t}(t \pm 0) \), which means that \( a \) is locally equivalent to \( c_t g_{t, \gamma_t} \) at the point \( t \in \Gamma \).

For \( t \in \Gamma \) and \( \gamma \in \mathbb{C} \), consider the weight

\[
\varphi_{t, \gamma}(\tau) := |(\tau - t)^\gamma|, \quad t \in \Gamma \setminus \{t\}.
\]

From [21] Lemma 7.1 we get the following.

**Lemma 4.4.** Let \( \Gamma \) be a Carleson Jordan curve and let \( \varphi : \Gamma \rightarrow (1, \infty) \) be a continuous function. Suppose \( w : \Gamma \rightarrow [0, \infty) \) is an arbitrary weight such that the operator \( S \) is bounded on \( L^p(\Gamma, w) \). If, for some \( k \in \mathbb{Z} \) and \( \gamma \in \mathbb{C} \), the operator \( \varphi_{t, k} S \varphi_{t, \gamma - k} I \) is bounded on \( L^p(\Gamma, w) \), then the function \( g_{t, \gamma} \) admits a Wiener-Hopf factorization on \( L^p(\Gamma, w) \).

### 4.3. Fredholm criterion

Now we are in a position to prove one of the main results of the paper.

**Theorem 4.5.** Let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition \([14]\) at each point \( t \in \Gamma \) and \( \varphi : \Gamma \rightarrow (1, \infty) \) be a continuous function satisfying \([14]\). Suppose a weight \( w \) is given by \([13]\), where for \( w_1, \ldots, w_n \in \mathbb{W} \) the condition \([15]\) is fulfilled. The operator \( aP + Q \), where \( a \in PC(\Gamma) \), is Fredholm on the weighted variable Lebesgue space \( L^p(\Gamma, w) \) if and only if \( a(t \pm 0) \neq 0 \) and

\[
\begin{align*}
\text{Re} \gamma_t &:= \frac{1}{2\pi} \arg \frac{a(t - 0)}{a(t + 0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t - 0)}{a(t + 0)} \right| + \frac{1}{p(t)} + \theta \mu_t + (1 - \theta)\nu_t \notin \mathbb{Z}
\end{align*}
\]

for all \( t \in \Gamma \) and all \( \theta \in [0, 1] \), where

\[
\begin{align*}
\mu_t &= m(w_k), \quad \nu_t = M(w_k) \quad \text{for} \quad k \in \{1, \ldots, n\}, \\
\mu_t &= 0, \quad \nu_t = 0 \quad \text{for} \quad t \in \Gamma \setminus \{t_1, \ldots, t_n\}.
\end{align*}
\]

**Proof.** The proof is developed by analogy with the proof of [22] Theorem 3.3.

**Necessity.** By Theorem \( [12] \) the operator \( S \) is bounded on \( L^p(\Gamma, w) \). Then from Theorem \( [2.9] \) it follows that \( w \in A_p(\Gamma) \). Therefore, by Lemma \( [2.10] \) a), the function \( V^p_t(\eta w) \) is regular and submultiplicative for all \( t \in \Gamma \) and all \( x \in \mathbb{R} \). Since \( \Gamma \) satisfies \([1.1]\) for all \( t \in \Gamma \), from Lemma \( [2.11] \) and Theorem \( [2.8] \) it follows that the indices of \( V^p_t(\eta w) \) (that is, the indicator functions \( \alpha, \beta \) of the pair \( (\Gamma, w) \)) at \( t \in \Gamma \) are calculated by

\[
\begin{align*}
\alpha_k(x) &= m(w_k) + \delta(t)x, \quad \beta_k(x) = M(w_k) + \delta(t)x \quad (x \in \mathbb{R}), \\
\alpha_t(x) &= \delta(t)x, \quad \beta_t(x) = \delta(t)x \quad (x \in \mathbb{R}),
\end{align*}
\]

for \( k \in \{1, \ldots, n\} \) and

\[
\begin{align*}
\alpha_t(x) &= \delta(t)x, \quad \beta_t(x) = \delta(t)x \quad (x \in \mathbb{R})
\end{align*}
\]

for \( t \in \Gamma \setminus \{t_1, \ldots, t_n\} \). Theorem \( [1.1] \) and \([4.7] - [4.8]\) imply \([4.5] \) and \( a(t \pm 0) \neq 0 \) for all \( t \in \Gamma \) and all \( \theta \in [0, 1] \).

**Sufficiency.** If \( aP + Q \) is Fredholm, then, by Theorem \( [3.1] \) \( a(t \pm 0) \neq 0 \) for all \( t \in \Gamma \). Fix an arbitrary \( t \in \Gamma \). Choose \( \gamma = \gamma_t \in \mathbb{C} \) as in \([14]\). Then the function \( a \) is
From (4.5) it follows that there exists an $s_t$ locally equivalent to $c_t < \gamma_t$ for all $\theta \in [0, 1]$. In particular, if $\theta = 1$, then
\begin{equation}
0 < s_t - \text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t + 1/p(t) + \theta \mu_t + (1 - \theta) \nu_t < 1
\end{equation}
for all $\theta \in [0, 1]$. If $\theta = 0$, then
\begin{equation}
0 < 1/p(t) + (s_t - \text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t) + \mu_t;
\end{equation}
if $\theta = 0$, then
\begin{equation}
1/p(t) + (s_t - \text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t) + \nu_t < 1.
\end{equation}
Let $\psi_t(x) := x^{s_t} e^{-\text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t}$ for $x \in (0, |\Gamma|]$ and $\tilde{w}(\tau) := \psi_t(|\tau - t|)w(\tau), \quad \tau \in \Gamma$.

Clearly, the weight $\tilde{w}$ is of the form (1.3).

If $t \in \Gamma \setminus \{t_1, \ldots, t_n\}$, then obviously
\begin{equation}
m(\psi_t) = M(\psi_t) = s_t - \text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t.
\end{equation}
From (4.9)–(4.11) we get
\begin{equation}
0 < 1/p(t) + m(\psi_t) \leq 1/p(t) + M(\psi_t) < 1.
\end{equation}
Combining these inequalities with Theorem 1.1 we conclude that the operator $S$ is bounded on $L^{p(\cdot)}(\Gamma, \tilde{w})$.

If $t = t_k$ for some $k \in \{1, \ldots, n\}$, then one can easily show that
\begin{align}
m(\psi_{t_k} w_k) &= s_{t_k} - \text{Re} \gamma_{t_k} - \delta(t_k) \text{Im} \gamma_{t_k} + m(w_k), \\
M(\psi_{t_k} w_k) &= s_{t_k} - \text{Re} \gamma_{t_k} - \delta(t_k) \text{Im} \gamma_{t_k} + M(w_k).
\end{align}
From (4.9)–(4.10) and (4.12)–(4.13) we obtain
\begin{equation}
0 < 1/p(t_k) + m(\psi_{t_k} w_k) \leq 1/p(t_k) + M(\psi_{t_k} w_k) < 1.
\end{equation}
These inequalities and Theorem 1.1 yield the boundedness of the operator $S$ on $L^{p(\cdot)}(\Gamma, \tilde{w})$.

In view of the logarithmic whirl condition (1.1) we have
\begin{align}
\varphi_{t, s_t - \gamma_t}(\tau) &= |\tau - t|^{s_t - \text{Re} \gamma_t} e^{\text{Im} \gamma_t \arg(\tau - t)} \\
&= |\tau - t|^{s_t - \text{Re} \gamma_t} e^{-\text{Im} \gamma_t (\delta(t) \log |\tau - t| + O(1))} \\
&= |\tau - t|^{s_t - \text{Re} \gamma_t - \delta(t) \text{Im} \gamma_t} e^{-\text{Im} \gamma_t O(1)}
\end{align}
as $\tau \to t$. Therefore the operator $\varphi_{t, s_t - \gamma_t} S \varphi_{t, \gamma_t - s_t} I$ is bounded on $L^{p(\cdot)}(\Gamma, w)$. Then, by Lemma 4.4, the function $g_{t, \gamma_t}$ admits a Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma, w)$. Due to Theorem 3.3, the operator $g_{t, \gamma_t} P + Q$ is Fredholm. Then the operator $c_t g_{t, \gamma_t} P + Q$ is Fredholm, too. Since the function $c_t g_{t, \gamma_t}$ is locally equivalent to the function $a$ at every point $t \in \Gamma$, the operator $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma, w)$ in view of Theorem 3.2.

Remark 4.6. In the case of standard Lebesgue spaces a complete description of the set of all $\gamma \in \mathbb{C}$ such that the operator $S$ is bounded on $L^p(\Gamma, \varphi_{t, \gamma} w)$ is known in terms of the indicator functions $\alpha_t$ and $\beta_t$ (see [4] Sections 3.6–3.7). This description allowed the authors of [3] to consider the case of Carleson curves that may not satisfy (1.1) and may have distinct spirality indices $\alpha(W_t \eta_t)$ and $\beta(W_t \eta_t)$. However, the only result on the boundedness of $S$ on weighted variable Lebesgue spaces in our disposal is Theorem 1.1 which is not applicable to weights of the form
Theorem 4.3. The sufficiency part follows from Theorem 4.5.

Proof. The idea of the proof is borrowed from [4, Proposition 7.16] (see also [24, only if (4.5) is sufficiently small.

\[ \psi \]

Theorem 1.1 to the weight \( \psi_{t, \lambda}(\tau) := |\tau - t|^\lambda \) for some \( \lambda \in \mathbb{R} \) and one can apply Theorem 1.1 to the weight \( \psi_{t, \lambda}(\tau) \). Therefore, to treat the case of arbitrary Carleson curves by this method, a more general boundedness result than Theorem 1.1 is needed.

4.4. Criterion for the closedness of the image.

Theorem 4.7. Let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition (1.1) at each point \( t \in \Gamma \) and \( p: \Gamma \to (1, \infty) \) be a continuous function satisfying (1.3). Suppose a weight \( w \) is given by (1.3), where for \( w_1, \ldots, w_n \in \mathbb{W} \) the condition (1.5) is fulfilled. Suppose \( a \in PC(\Gamma) \) has finitely many jumps and \( a(t \pm 0) \neq 0 \) for all \( t \in \Gamma \). Then the image of \( aP + Q \) is closed in \( L^{p(\cdot)}(\Gamma, w) \) if and only if (4.5) holds for all \( t \in \Gamma \).

Proof. The idea of the proof is borrowed from [4, Proposition 7.16] (see also [24, Theorem 4.3]). The sufficiency part follows from Theorem 1.5.

Let us prove the necessity part. Assume that \( a(t \pm 0) \neq 0 \) for all \( t \in \Gamma \). Since the number of jumps, that is, the points \( t \in \Gamma \) at which \( a(t - 0) \neq a(t + 0) \), is finite, it is clear that

\[
- \frac{1}{2\pi} \sum \frac{a(t - 0)}{a(t + 0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t - 0)}{a(t + 0)} \right| + \frac{1}{1 + \varepsilon} \notin \mathbb{Z},
\]

\[
- \frac{1}{2\pi} \sum \frac{a(t + 0)}{a(t - 0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t + 0)}{a(t - 0)} \right| + \frac{1}{1 + \varepsilon} \notin \mathbb{Z}
\]

for all \( t \in \Gamma \) and all sufficiently small \( \varepsilon > 0 \). By Theorem 1.5, the operators \( aP + Q \) and \( a^{-1}P + Q \) are Fredholm on the Lebesgue space \( L^{1+\varepsilon}(\Gamma) \) whenever \( \varepsilon > 0 \) is sufficiently small.

If (1.5) holds, then there exists a number \( \varepsilon > 0 \) such that

\[
0 < (1/p(t_k) + m(w_k))(1 + \varepsilon) \leq (1/p(t_k) + M(w_k))(1 + \varepsilon) < 1
\]

for all \( k \in \{1, \ldots, n\} \). It is easy to see that

\[
m(w^{1+\varepsilon}_k) = (1 + \varepsilon)m(w_k), \quad M(w^{1+\varepsilon}_k) = (1 + \varepsilon)M(w_k).
\]

Hence, by Theorem 1.1, the operator \( S \) is bounded on \( L^{p(\cdot)/(1+\varepsilon)}(\Gamma, w^{1+\varepsilon}) \). Taking into account this observation, one can prove as in [24, Lemma 4.1] that there is an \( \varepsilon_0 > 0 \) such that

\[
L^{p(\cdot)}(\Gamma, w) \subset L^{1+\varepsilon_0}(\Gamma), \quad L^{q(\cdot)}(\Gamma, 1/w) \subset L^{1+\varepsilon_0}(\Gamma)
\]

and \( aP + Q, a^{-1}P + Q \) are Fredholm on \( L^{1+\varepsilon_0}(\Gamma) \). Then

\[
(4.14) \quad n(aP + Q; L^{p(\cdot)}(\Gamma, w)) \leq n(aP + Q; L^{1+\varepsilon_0}(\Gamma)) < \infty,
\]

and, by duality (see [24, Lemma 3.8]),

\[
(4.15) \quad d(aP + Q; L^{p(\cdot)}(\Gamma, w)) = n(a^{-1}P + Q; L^{q(\cdot)}(\Gamma, 1/w)) \leq n(a^{-1}P + Q; L^{1+\varepsilon_0}(\Gamma)) < \infty.
\]

If (1.5) does not hold, then \( aP + Q \) is not Fredholm on \( L^{p(\cdot)}(\Gamma, w) \) in view of Theorem 1.5. From this fact and (4.14)–(4.15) we conclude that the image of \( aP + Q \) is not closed in \( L^{p(\cdot)}(\Gamma, w) \), which contradicts the hypothesis. \( \square \)
5. Tools for the construction of the symbol calculus

5.1. The Allan-Douglas local principle. Let $B$ be a Banach algebra with identity. A subalgebra $Z$ of $B$ is said to be a central subalgebra if $zb = bz$ for all $z \in Z$ and all $b \in B$.

**Theorem 5.1** (see [8], Theorem 1.35(a)). Let $B$ be a Banach algebra with identity $e$ and let $Z$ be a closed central subalgebra of $B$ containing $e$. Let $M(Z)$ be the maximal ideal space of $Z$, and for $\omega \in M(Z)$, let $J_\omega$ refer to the smallest closed two-sided ideal of $B$ containing the ideal $\omega$. Then an element $b$ is invertible in $B$ if and only if $b + J_\omega$ is invertible in the quotient algebra $B/J_\omega$ for all $\omega \in M(Z)$.

The algebra $B/J_\omega$ is referred to as the local algebra of $B$ at $\omega \in M(Z)$ and the spectrum of $b + J_\omega$ in $B/J_\omega$ is called the local spectrum of $b$ at $\omega \in M(Z)$.

5.2. The two projections theorem. Recall that an element $p$ of a Banach algebra is called an idempotent (or, somewhat loosely, also a projection), if $p^2 = p$.

The following two projections theorem was obtained by Finck, Roch, Silbermann [12] and Gohberg, Krupnik [15] (see also [4, Section 8.3]).

**Theorem 5.2.** Let $B$ be a Banach algebra with identity $e$, let $C$ be a Banach subalgebra of $B$ which contains $e$ and is isomorphic to $\mathbb{C}^{N \times N}$, and let $p$ and $q$ be two idempotent elements in $B$ such that $cp = pc$ and $cq = qc$ for all $c \in C$. Let $A = \text{alg}(C, p, q)$ be the smallest closed subalgebra of $B$ containing $C, p, q$. Put

$$x = pqp + (e - p)(e - q)(e - p),$$

denote by $\text{sp} x$ the spectrum of $x$ in $B$, and suppose the points 0 and 1 are not isolated points of $\text{sp} x$. Then

(a) for each $z \in \text{sp} x$ the map $\sigma_z$ of $C \cup \{p, q\}$ into the algebra $\mathbb{C}^{2N \times 2N}$ of all complex $2N \times 2N$ matrices defined by

$$\sigma_z c = \begin{bmatrix} c & O \\ O & c \end{bmatrix}, \sigma_z p = \begin{bmatrix} E & O \\ O & O \end{bmatrix}, \sigma_z q = \begin{bmatrix} zE & \sqrt{z(1-z)}E \\ \sqrt{z(1-z)}E & (1-z)E \end{bmatrix},$$

where $c \in C$, $E$ and $O$ denote the $N \times N$ identity and zero matrices, respectively, and $\sqrt{z(1-z)}$ denotes any complex number whose square is $z(1-z)$, extends to a Banach algebra homomorphism

$$\sigma_z : A \to \mathbb{C}^{2N \times 2N};$$

(b) every element $a$ of the algebra $A$ is invertible in the algebra $B$ if and only if

$$\det \sigma_z a \neq 0 \quad \text{for all} \quad z \in \text{sp} x;$$

(c) the algebra $A$ is inverse closed in $B$ if and only if the spectrum of $x$ in $A$ coincides with the spectrum of $x$ in $B$.

A further generalization of the above result to the case of $n \geq 3$ projections is contained in [4, Section 8.4].
6. Algebra of singular integral operators

6.1. Operators of local type. In this section we will suppose that $\Gamma$ is a Carleson curve satisfying the logarithmic whirl condition (1.1) at each point $t \in \Gamma$, $p : \Gamma \to (1, \infty)$ is a continuous function satisfying (1.3), $w$ is a weight of the form (1.3) with $w_1, \ldots, w_n \in \mathbb{W}$ satisfying (1.3). Under these conditions the operator $S$ (defined elementwise) is bounded on $L^p_N(\Gamma, w)$, where $N \geq 1$ (see Theorem 1.1).

Let $\mathcal{B} := \mathcal{B}(L^p_N(\Gamma, w))$ be the Banach algebra of all bounded linear operators on $L^p_N(\Gamma, w)$ and let $\mathcal{K} := \mathcal{K}(L^p_N(\Gamma, w))$ be the closed ideal of all compact operators on $L^p_N(\Gamma, w)$. We will denote by $\mathcal{B}^\pi$ the Calkin algebra $\mathcal{B}/\mathcal{K}$ and by $A^\pi$ the coset $A + \mathcal{K}$ for any operator $A \in \mathcal{B}$. An operator $A \in \mathcal{B}$ is said to be of local type if $A \operatorname{diag}\{c, \ldots, c\}I - \operatorname{diag}\{c, \ldots, c\}A \in \mathcal{K}$ for every continuous function $c$ on $\Gamma$. This notion goes back to Simonenko [18, 50]. It is easy to see that the set $\mathcal{L}$ of all operators of local type is a Banach subalgebra of $\mathcal{B}$.

Lemma 6.1. We have
\begin{equation}
\mathcal{K} \subset \operatorname{alg}(S, PC, L^p_N(\Gamma, w)) \subset \mathcal{L}.
\end{equation}
Proof. Let $\operatorname{alg}(S, C, L^p_N(\Gamma, w))$ be the smallest closed subalgebra of $\mathcal{B}$ containing the operators of multiplication by continuous matrix functions and the operator $S$. Obviously,
\begin{equation}
\operatorname{alg}(S, C, L^p_N(\Gamma, w)) \subset \operatorname{alg}(S, PC, L^p_N(\Gamma, w)).
\end{equation}
On the other hand, by analogy with [17] Lemma 9.1 or [23] Lemma 5.1 one can show that
\begin{equation}
\mathcal{K} \subset \operatorname{alg}(S, C, L^p_N(\Gamma, w)).
\end{equation}
Combining (6.2) and (6.3), we arrive at the first embedding in (6.1).

If $a \in PC_{N \times N}(\Gamma)$ and $c \in C(\Gamma)$, then the operators $aI$ and $\operatorname{diag}\{c, \ldots, c\}I$ obviously commute. Then $aI$ is of local type. By [21] Lemma 6.5, the operator $\operatorname{diag}\{c, \ldots, c\}S - S \operatorname{diag}\{c, \ldots, c\}I$ is compact. Thus, $S$ is of local type, too. Since the generators of $\operatorname{alg}(S, PC, L^p_N(\Gamma, w))$ are of local type, each element of this algebra is of local type, which proves the second embedding in (6.1).

Lemma 6.2. An operator $A \in \mathcal{L}$ is Fredholm if and only if the coset $A^\pi$ is invertible in the quotient algebra $\mathcal{L}^\pi := \mathcal{L}/\mathcal{K}$.

The proof is straightforward.

6.2. Localization. From Lemma 6.1 we deduce that the quotient algebras
\[ \operatorname{alg}^\pi(S, PC, L^p_N(\Gamma, w)) := \operatorname{alg}(S, PC, L^p_N(\Gamma, w))/\mathcal{K} \]
and $\mathcal{L}^\pi := \mathcal{L}/\mathcal{K}$ are well defined. We will study the invertibility of an element $A^\pi$ of $\operatorname{alg}^\pi(S, PC, L^p_N(\Gamma, w))$ in the larger algebra $\mathcal{L}^\pi$ by using Theorem 5.1. To this end, consider
\[ Z^\pi := \{ (\operatorname{diag}\{c, \ldots, c\}I)^\pi : c \in C(\Gamma) \} \]
From the definition of $\mathcal{L}$ it follows that $Z^\pi$ is a central subalgebra of $\mathcal{L}^\pi$. The maximal ideal space $M(Z^\pi)$ of $Z^\pi$ may be identified with the curve $\Gamma$ via the Gelfand map $\mathcal{G}$ given by
\[ \mathcal{G} : Z^\pi \to C(\Gamma), \quad (\mathcal{G}(\operatorname{diag}\{c, \ldots, c\}I)^\pi)(t) = c(t) \quad (t \in \Gamma). \]
In accordance with Theorem 5.1 for every \( t \in \Gamma \) we define \( \mathcal{J}_t \subset \mathcal{L}^\pi \) as the smallest closed two-sided ideal of \( \mathcal{L}^\pi \) containing the set
\[
\{(\text{diag}\{c, \ldots, c\})^\pi : c \in C(\Gamma), c(t) = 0\}.
\]

Consider a function \( \chi_t \in PC(\Gamma) \) which is continuous on \( \Gamma \setminus \{t\} \) and satisfies \( \chi_t(t-0) = 0 \) and \( \chi_t(t+0) = 1 \). For \( a \in PC_{N\times N}(\Gamma) \) define \( a_i \in PC_{N\times N}(\Gamma) \) by
\[
a_i := a(t-0)(1 - \chi_t) + a(t+0)\chi_t.
\]

Clearly \( (aI)^\pi - (a_1I)^\pi \in \mathcal{J}_t \). Hence, for any operator \( A \in \text{alg}(S, PC, L^p_N(\Gamma, \omega)) \), the coset \( A^\pi + \mathcal{J}_t \) belongs to the smallest closed subalgebra \( \mathcal{A}_t \) of \( \mathcal{L}^\pi / \mathcal{J}_t \) containing the cosets
\[
p := P^\pi + \mathcal{J}_t, \quad q := (\text{diag}\{\chi_1, \ldots, \chi_t\})^\pi + \mathcal{J}_t
\]
and the algebra
\[
\mathcal{C} := \{(cI)^\pi + \mathcal{J}_t : c \in C^{N\times N}\}.
\]
The latter algebra is obviously isomorphic to \( C^{N\times N} \), so \( \mathcal{C} \) and \( C^{N\times N} \) can (and will) be identified with each other. It is easy to see that
\[
p^2 = p, \quad q^2 = q, \quad pc = cp, \quad qc = cq
\]
for all \( c \in \mathcal{C} \). To apply Theorem 5.2 to the algebras \( \mathcal{L}^\pi / \mathcal{J}_t \) and \( \mathcal{A}_t = \text{alg}(\mathcal{C}, p, q) \), we need to identify the spectrum of the element
\[
x := pqp + (e - p)(e - q)(e - p)
\]
\[
= (P \text{diag}\{\chi_1, \ldots, \chi_t\} P + Q \text{diag}\{1 - \chi_1, \ldots, 1 - \chi_t\} Q)^\pi + \mathcal{J}_t
\]
in the algebra \( \mathcal{L}^\pi / \mathcal{J}_t \).

6.3. Spiralic horns. Given two real numbers \( a, b \) satisfying \( 0 < a \leq b < 1 \), two complex numbers \( z_1, z_2 \), and a real number \( \delta \), we define the spiralic horn between \( z_1 \) and \( z_2 \) as the set
\[
S(z_1, z_2; \delta; a, b) := \{z_1, z_2\} \cup \left\{ u \in \mathbb{C} \setminus \{z_1, z_2\} : \frac{1}{2\pi} \arg \frac{u - z_1}{u - z_2} - \delta \log \left| \frac{u - z_1}{u - z_2} \right| \in [a, b] + \mathbb{Z} \right\}.
\]
If \( z_1 = z_2 \), then \( S(z_1, z_2; \delta; a, b) \) degenerates to the point \( z_1 \). Assume \( z_1 \neq z_2 \). Then the set \( S(z_1, z_2; 0; 1/2, 1/2) \) is the segment between \( z_1 \) and \( z_2 \). If \( a \neq 1/2 \), then \( S(z_1, z_2; 0; a, a) \) is the circular arc between \( z_1 \) and \( z_2 \). If \( a < 1/2 \) (resp. \( a > 1/2 \)), then one sees the straight line between \( z_1 \) and \( z_2 \) under the angle \( 2\pi a \) (resp. \( 2\pi(1 - a) \)) and running through the arc from \( z_1 \) to \( z_2 \) this straight line is located at the left-hand side (resp. right-hand side). The importance of these arcs in the Fredholm theory of singular integral operators was first observed by Widom [52], they were exploited intensively by Gohberg and Krupnik [13, 14 Chap. 9].

If \( z_1 \neq z_2 \) and \( a < b \), then \( S(z_1, z_2; 0; a, b) \) is an ordinary horn bounded by the circular arcs \( S(z_1, z_2; 0; a, a) \) and \( S(z_1, z_2; 0; b, b) \). If \( \delta \neq 0 \), then \( S(z_1, z_2; \delta; a, a) \) is a logarithmic double spiral. If \( a < b \), then \( S(z_1, z_2; \delta; a, b) \) is the union of logarithmic double spirals
\[
S(z_1, z_2; \delta; a, b) = \bigcup_{\lambda \in [a, b]} S(z_1, z_2; \delta; \lambda, \lambda).
\]
Hence this set is the closed set between two logarithmic double spirals. It was introduced in [3] (see also [4] Sections 7.3–7.6 and [6, 5]).
6.4. The local spectrum. Now we are ready to identify the spectrum of \( x \) in the algebra \( B = \mathcal{L}^e / \mathcal{J}_t \).

**Lemma 6.3.** Let \( \chi_t \in PC(\Gamma) \) be a continuous function on \( \Gamma \setminus \{ t \} \) such that
\[
\chi_t(t-0) = 0, \quad \chi_t(t+0) = 1, \quad \chi_t(\Gamma \setminus \{ t \}) \cap \mathcal{S}(0,1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t) = \emptyset,
\]
where \( \mu_t \) and \( \nu_t \) are defined by (6.3). Then the spectrum of the element \( x \) given by (6.8) in the algebra \( B = \mathcal{L}^e / \mathcal{J}_t \) coincides with \( \mathcal{S}(0,1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t) \).

**Proof.** From Theorem 3.3 we immediately get that the set of all \( \lambda \in \mathbb{C} \) such that the operator \( (\chi_t - \lambda)P + Q \) is not Fredholm on \( L^{p(.)}(\Gamma, w) \) coincides with \( \mathcal{S}(0,1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t) \). After this observation the proof can be developed by a literal repetition of the proof of \([17]\) Lemma 9.4] with the Boyd indices \( \alpha_M \) and \( \beta_M \) of an Orlicz space considered there replaced by the numbers \( 1/p(t) + \mu_t \) and \( 1/p(t) + \nu_t \), respectively. \( \square \)

6.5. Construction of the symbol calculus. Now we are in a position to prove the main result of the paper.

**Theorem 6.4.** Let \( N \in \mathbb{N} \), let \( \Gamma \) be a Carleson Jordan curve satisfying the logarithmic whirl condition (1.1) at every point \( t \in \Gamma \), let \( p : \Gamma \to (1, \infty) \) be a continuous function satisfying (1.4), and let \( w : \Gamma \to [0, \infty] \) be a radial oscillating weight given by (1.3) with \( w_1, \ldots, w_n \in \mathbb{W} \) satisfying (1.3). Put
\[
\mu_{t_k} = m(w_k), \quad \nu_{t_k} = M(w_k) \quad \text{for} \quad k \in \{1, \ldots, n\},
\]
\[
\mu_t = 0, \quad \nu_t = 0 \quad \text{for} \quad t \in \Gamma \setminus \{ t_1, \ldots, t_n \}
\]
and define the “spiralic horn bundle” by
\[
\mathcal{M} := \bigcup_{t \in \Gamma} \left( \{ t \} \times \mathcal{S}(0,1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t) \right).
\]

(a) For each point \( (t, z) \in \mathcal{M} \), the map
\[
\sigma_{t,z} : \{ S \} \cup \{ aI : a \in PC_{N \times N}(\Gamma) \} \to \mathbb{C}^{2N \times 2N}
\]
given by
\[
\sigma_{t,z}(S) = \begin{bmatrix} E & O \\ O & -E \end{bmatrix}, \quad \sigma_{t,z}(aI) =
\]
\[
\begin{bmatrix}
(a(t+0)z + a(t-0)(1-z)) & (a(t+0) - a(t-0)) \sqrt{z(1-z)} \\
(a(t+0) - a(t-0)) \sqrt{z(1-z)} & a(t+0)(1-z) + a(t-0)z
\end{bmatrix},
\]
where \( E \) and \( O \) denote the \( N \times N \) identity and zero matrices, respectively, and \( \sqrt{z(1-z)} \) denotes any complex number whose square is \( z(1-z) \), extends to a Banach algebra homomorphism
\[
\sigma_{t,z} : \text{alg}(S, PC, L^{p(.)}_N(\Gamma, w)) \to \mathbb{C}^{2N \times 2N}
\]
with the property that \( \sigma_{t,z}(K) \) is the \( 2N \times 2N \) zero matrix whenever \( K \) is a compact operator on \( L^{p(.)}_N(\Gamma, w) \).

(b) An operator \( A \in \text{alg}(S, PC, L^{p(.)}_N(\Gamma, w)) \) is Fredholm on \( L^{p(.)}_N(\Gamma, w) \) if and only if
\[
\det \sigma_{t,z}(A) \neq 0 \quad \text{for all} \quad (t, z) \in \mathcal{M}.
\]
(c) The quotient algebra $\text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$ is inverse closed in the Calkin algebra $B^\pi$, that is, if a coset $A^\pi \in \text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$ is invertible in $B^\pi$, then $(A^\pi)^{-1} \in \text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$.

Proof. The idea of the proof is borrowed from [4, Section 8.5]. Here we follow the proof of [22, Theorem 5.1] and [23, Theorem 5.4]. Since the latter two sources may not be readily available, we give a self-contained proof.

Fix $t \in \Gamma$ and choose a function $\chi_t \in PC(\Gamma)$ as in the assumptions of Lemma 6.3. From (6.7) and Lemma 6.3 we deduce that the algebras $L^\pi/\mathcal{J}_t$ and $A_t = \text{alg}(\mathcal{C}, p, q)$, where $p, q$, and $\mathcal{C}$ are given by (6.3) and (6.4), respectively, satisfy all the assumptions of Theorem 5.2.

(a) By Theorem 5.2(a), for every $z \in \mathcal{S}(0, 1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t)$, the map

$$\sigma_{t, z} = \sigma_z \circ \pi_t : \text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, w)) \to \mathbb{C}^{2N \times 2N},$$

where $\sigma_z$ is given in Theorem 5.2(a) and $\pi_t$ acts by the rule $A \mapsto A^\pi + \mathcal{J}_t$ for every $A \in \text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, w))$, is a well defined Banach algebra homomorphism. It is easy to check that

$$\sigma_{t, z}(S) = 2\sigma_z p - \sigma_z e = \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}.$$  

If $a \in PC_{N \times N}(\Gamma)$, then from (6.1) and $(aI - a_t I)^\pi \in \mathcal{J}_t$ it follows that

$$\sigma_{t, z}(aI) = \sigma_{t, z}(a_t I) = \sigma_z(a(t - 0))\sigma_z(e - q) + \sigma_z(a(t + 0))\sigma_z q = \begin{bmatrix} a(t + 0)z + a(t - 0)(1 - z) & (a(t + 0) - a(t - 0))\sqrt{z(1 - z)} \\ (a(t + 0) - a(t - 0))\sqrt{z(1 - z)} & a(t + 0)(1 - z) + a(t - 0)z \end{bmatrix}.$$  

From Lemma 6.1 it follows that $\pi_t(K)$ is correctly defined for every $K \in \mathcal{K}$ and $\pi_t(K) = \mathcal{J}_t$. Hence

$$\sigma_{t, z}(K) = \sigma_z(0) = \begin{bmatrix} O & 0 \\ 0 & O \end{bmatrix}.$$  

Part (a) is proved.

(b) From Lemma 6.2 it follows that the Fredholmness of an operator $A$ from $\text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, w))$ is equivalent to the invertibility of $A^\pi \in L^\pi$. By Theorem 5.1 the latter is equivalent to the invertibility of $\pi_t(A) = A^\pi + \mathcal{J}_t$ in $L^\pi/\mathcal{J}_t$ for every $t \in \Gamma$. By Theorem 5.2(b), this is equivalent to

$$\det \sigma_{t, z}(A) = \det \sigma_z \pi_t(A) \neq 0 \quad \text{for all} \quad (t, z) \in \mathcal{M}.$$  

Part (b) is proved.

(c) Since the compact simply connected set $\mathcal{S}(0, 1; \delta(t); 1/p(t) + \mu_t, 1/p(t) + \nu_t)$ does not separate the complex plane, it follows that the spectra of the element $x$ given by (6.8) in the algebras $L^\pi/\mathcal{J}_t$ and $A_t$ coincide. Suppose $A$ belongs to $\text{alg}(S, PC, L^{p(\cdot)}_N(\Gamma, w))$. If $A^\pi$ is invertible in $B^\pi$, then (6.9) is fulfilled. Consequently, by Theorem 5.2(b), (c), $\pi_t(A) = A^\pi + \mathcal{J}_t$ is invertible in $A_t$ for every $t \in \Gamma$. Applying the Allan-Douglas local principle (Theorem 5.1) to the algebra $\text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$, its central subalgebra $Z^\pi$ and the ideals $\mathcal{J}_t$, we obtain that $A^\pi$ is invertible in $\text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$. Thus, $\text{alg}^\pi(S, PC, L^{p(\cdot)}_N(\Gamma, w))$ is inverse closed in $B^\pi$.  

□
For constant $p$, Khvedelidze weights, and piecewise Lyapunov curves, Theorem 6.4 was obtained in [13] by a different method. For constant $p$, arbitrary Muckenhoupt weights, and arbitrary Carleson curves, an analogue of this result is contained in [4, Section 8.5]. In the case of variable exponent $p : \Gamma \to (1, \infty)$ and Khvedelidze weights, an earlier version of this theorem for Lyapunov curves or Radon curves without cusps is in [23, Theorem 5.4] and for Carleson curves satisfying (1.1) is in [22, Theorem 5.1].

6.6. Final remarks.

Remark 6.5 (On index formulas). We do not consider formulas for the index of an operator in the algebra $\text{alg}(S, PC, L^p_N((\cdot), w))$ since the approach to the study of Banach algebras of SIOs based on the Allan-Douglas local principle and the two projections theorem does not allow us to get formulas for the index of an arbitrary operator in the Banach algebra of SIOs with piecewise continuous coefficients. These formulas can be obtained similarly to the classical situation considered by Golberg and Krupnik [13] (see also [4, Chap. 10]). For reflexive Orlicz spaces over Carleson curves with logarithmic whirl points this was done by the author [19]. In the case of variable Lebesgue spaces with radial oscillating weights weights over Carleson curves with logarithmic whirl points, the index formulas are almost the same as in [19]. It is only necessary to replace the Boyd indices $\alpha_M$ and $\beta_M$ of an Orlicz space $L^M$ by the numbers $1/p(t_k) + m(w_k)$ and $1/p(t_k) + M(w_k)$ (at the nodes $t_k$ of the weight), respectively, or both Boyd indices by $1/p(t)$ (at all other points $t$) in corresponding index formulas.

Remark 6.6 (On semi-Fredholm operators). Since one has Theorem 4.7 at hands, by using ideas of Spitkovsky [51], one can prove that if $a, b \in PC_{N \times N}(\Gamma)$, then $aP + bQ$ is semi-Fredholm if and only if it is Fredholm. With the help of the linear dilation procedure (see e.g. [13]) this fact can be extended to the operators of the form $\sum_j \prod_i (a_{ij}P + b_{ij}Q)$ with $a_{ij}, b_{ij} \in PC_{N \times N}(\Gamma)$. Since the property of an operator to be Fredholm or semi-Fredholm is stable under small perturbations, we finally arrive at the following result. Its proof, given for Khvedelidze weights in [24], works also for weights considered in this paper.

Theorem 6.7. Suppose that all the hypotheses of Theorem 6.4 are fulfilled. If an operator in the algebra $\text{alg}(S, PC, L^p_N((\cdot), w))$ is semi-Fredholm, then it is Fredholm.

Remark 6.8 (On coefficients beyond $PC(\Gamma)$). The approach of this paper is based on the Wiener-Hopf factorization, local principles, and the two projections theorem. Note that there is also another approach to study SIOs on standard Lebesgue spaces $L^p(\Gamma, w)$ with slowly oscillating (i.e. not arbitrary!) Muckenhoupt weights over slowly oscillating (i.e. not all!) Carleson curves. It is based on the technique of pseudodifferential operators and limit operators developed by Rabinovich and his coauthors (see e.g. [22] and [6, 7]). This approach allows one to study SIOs not only with piecewise continuous coefficients, but also with slowly oscillating coefficients. Notice that a satisfactory theory of pseudodifferential operators in the setting of variable Lebesgue spaces did not exist a couple of years ago. Very recently Rabinovich and S. Samko [44] have started to develop such a theory, however they do not consider in that paper SIOs on curves.

Remark 6.9 (On more general variable exponents). It is well known that the boundedness of $S$ is guaranteed as soon as a suitable maximal function is bounded.
on $L^p(\Gamma, w)$. Lerner \[24\], among other things, observed that if $p(x) = \alpha + \sin(\log \log(1/|x|)) \chi_E(x)$, where $\alpha > 2$ is some constant and $\chi_E$ is the characteristic function of the ball $E := \{x \in \mathbb{R}^n : |x| \leq 1/e\}$, then the Hardy-Littlewood maximal function is bounded on $L^p(\mathbb{R}^n)$. Clearly, the exponent $p$ in this example is discontinuous at the origin, so it does not satisfy (an $\mathbb{R}^n$ analog of) the condition (1.4). This exponent belongs to the class of pointwise multipliers for $BMO$ (the space of functions of bounded mean oscillation). Kapanadze and Kopaliani \[16\] proved that if $p$ belongs to $VMO^{1/\log 1}$, a weighted space of functions with vanishing mean oscillation, then the Hardy-Littlewood maximal function is bounded on variable Lebesgue space $L^p(\Omega)$ over a bounded domain $\Omega$. Thus, I believe that necessary and sufficient conditions for the boundedness of the Cauchy singular integral operator (and other singular integrals and maximal functions) on weighted variable Lebesgue spaces $L^p(\Gamma, w)$ should be formulated in terms of integral means of the exponent $p$ (i.e., in $BMO$ terms), but not in pointwise terms like \[14\]. It is natural that the number $1/p(t)$ in \[15\] under eventual more general boundedness hypotheses for $S$ should be replaced by a pair of indices of a suitable submultiplicative function (see e.g. \[21\] Section 4.4). I believe that, in general, these indices may be different and the following is true.

**Conjecture 6.10.** Suppose $T$ is the unit circle and $a : T \to \mathbb{C}$ be a continuous on $T \setminus \{1\}$ function such that $a(1 - 0) = 0$ and $a(1 + 0) = 1$. There is a measurable function $p : \Gamma \to (1, \infty)$ such that the operator $S$ is bounded on $L^p(\mathbb{R}^n)$ and the essential spectrum

$$\sigma_{\text{ess}}(aP + Q) := \{\lambda \in \mathbb{C} : (aP + Q) - \lambda I \text{ is not Fredholm on } L^p(\mathbb{R}^n)\}$$

of the operator $aP + Q$ is massive, that is, it has a nonzero plane measure.

If $p : T \to (1, \infty)$ is a measurable function such that

\[6.10\]
$1 < \text{ess inf } p(\tau), \quad \text{ess sup } p(\tau) < \infty$

and $S$ is bounded on $L^p(\mathbb{T})$, then from \[21\] Theorem 7.3 it follows that

$$a(T \setminus \{1\}) \cup S(0, 1; 0; \alpha, \beta) \subset \sigma_{\text{ess}}(aP + Q)$$

where $\alpha$ and $\beta$ are the lower and upper indices of the following submultiplicative function

$$Q(x) := \limsup_{R \to 0} \frac{\|\chi_{\Delta(x)}\|_{p(\cdot)}\|\chi_{\Delta(x)}\|_{q(\cdot)}}{|\Delta(x)|}, \quad x \in (0, \infty),$$

where $\Delta(x) := \Gamma(1, R) \setminus \Gamma(1, R/2)$. Notice that if $p$ satisfies \[14\], then $\alpha = \beta = 1/p(1)$ by \[21\] Lemma 5.8. If $0 < \alpha < \beta < 1$, then the horn $S(0, 1; 0; \alpha, \beta)$ has a nonzero plane measure. Thus, to confirm Conjecture 6.10 it is sufficient to construct an exponent $p : T \to (1, \infty)$ such that \[6.10\] holds, $S$ is bounded on $L^p(\mathbb{T})$, and $0 < \alpha < \beta < 1$.

**Remark 6.11** (On arbitrary Carleson curves). Very recently the author \[25\] has obtained sufficient conditions for the boundedness of the Cauchy singular integral operator $S$ on variable Lebesgue spaces $L^p(\Gamma, w)$ with special weights $w(\tau) = |(\tau - t)\gamma|$, where $\gamma$ is a complex number, over arbitrary Carleson Jordan curves. This allows him to obtain analogues of Theorems 4.5 and 6.4 for the case of nonweighted variable Lebesgue spaces $L^p(\Gamma)$ over arbitrary Carleson curves (that is, without condition (1.1)). As in the present paper, local spectra of singular integral operators...
can be massive, but the reason for that is different. In the present paper this effect is due to oscillation of weights, but in the forthcoming paper [25] this effect is due to oscillation of arbitrary Carleson curves.

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