Spin-state estimation using the Stern-Gerlach experiment

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We propose a state estimation scheme for spins, using a modified setup of the Stern-Gerlach experiment, in which a beam of neutral spin-1/2 point particles interacts with a quadrupolar magnetic field. The proposed linear inversion estimation procedure, based on a quadrantal intensity detector, requires a suitable initial spatial state of the beam. The statistical characterization of the estimator of the initial spin state allows us not only to associate an error to the estimated parameters, but also to define a measure for comparing estimation procedures corresponding to different Stern-Gerlach setups.

I. INTRODUCTION

The well-known Stern-Gerlach experiment [1], in which a beam of neutral particles with definite magnetic dipole moment is made to interact with an external magnetic field, has been widely used to measure the spin projection of the particles along the direction of the field [2–5]. Early theoretical inquiries on the mechanism responsible for the spin alignment along (or against) the magnetic field treated the system classically, considering spins as tiny magnets [6]. Only after a long hiatus a complete quantum mechanical account of the experiment was given. The first quantum mechanical models, as well as most of the classical treatments, considered an inhomogeneous magnetic field with a gradient only in the direction of a large reference field [7, 8]. In this approximation, the Stern-Gerlach setup is an ideal spin measurement apparatus [9]. Although real magnetic fields must have a second gradient component to satisfy Gauss’s law, this component can be ignored by considering very short interaction times with the magnetic field [10]. More complete semiclassical [11] and quantum [12, 13] descriptions of the experiment have shown that the Stern-Gerlach setup is not an ideal spin meter: the magnetic field inhomogeneities cause beam deflections that are not determined only by the spin projections of the particles. However, the presence of a strong reference field still allows the correct estimation of the initial spin projection along its direction.

Although the purpose of the Stern-Gerlach setup is to measure a spin projection, it is tempting to question if setup modifications can provide more information. It was found that this is, indeed, the case. The ideas of initial state estimation [14, 15] were employed by Weigert [16] to show that a finite number of Stern-Gerlach measurements was sufficient to reconstruct the initial spin density matrix of the beam. In the case of a pure state of spin-1/2 particles, projection measurements along two directions differing only by an infinitesimal rotation, and a projection measurement in the direction perpendicular to the other two, enable state reconstruction. De Muynck [17] considered a different modification of the Stern-Gerlach setup. He used a setup lacking the reference magnetic field to obtain information about two of the components of the Bloch vector that defines the initial spin state of the beam. This result can be regarded as a demonstration that a large reference field somewhat limits the information that can be obtained about the spin state of the particles of the beam. The quadrupolar magnetic field used by De Muynck can also be used to perform a projection measurement of a spin component. Indeed, as shown by Garraway and Stenholm [18] in their study of the Stern-Gerlach experiment for free electrons, a carefully chosen initial spatial wave function is split by a quadrupolar field.

Thus, quadrupolar magnetic fields have been shown to allow either the estimation of two spin components or the projective measurement of one of these components. However, could quadrupolar fields allow the estimation of the whole initial spin state? In a naive view, the quadrupolar field can be seen as two apparatuses which try to measure two orthogonal components of the initial spin state. In such circumstances, it has been shown [19] that it is almost always possible to estimate the whole state. This heuristic analysis points to a positive answer to the previous question. In fact, as shown in this work (Sec. III), it is possible to estimate the initial (pure or mixed) spin state of a beam of neutral spin-1/2 particles, using linear inversion [20], when the initial spatial state is chosen to be an elongated Gaussian.

Our results are obtained by a combination of numerical and analytical methods. The time evolution corresponding to the Hamiltonian of the modified Stern-Gerlach setup (Sec. II) is numerically performed using the Suzuki-Trotter method (Sec. IV). The error of the proposed state estimation scheme is quantified by the logarithmic error \( \Delta(G, s) \) of the scheme, defined in Sec. V. Although the logarithmic error greatly varies from one set of parameters to another, and also depends on the initial state, it is reasonably low in some regions on the parameter space which are within reach of current exper-
II. DESCRIPTION OF THE MODEL

Consider the Stern-Gerlach setup shown in Fig. 1. A beam of neutral spin-1/2 particles of mass $m$ and magnetic dipole moment $\mu$ is prepared in a particular factorized initial state, $\rho(0) = R(0)\rho_S(0)$, where $R$ describes the spatial state and $\rho_S$ the spin state of the particles. The spatial state is considered to be completely defined, while the spin state is taken to be unknown. The particles of the beam are identical, indistinguishable, independent and far enough apart from each other, that any interaction between them can be ignored. After preparation, the particles are sent through an inhomogeneous magnetic field $\vec{B}$, generated by a magnet of length $L$, which deflects the beam. The magnetic field is assumed to have components only on the plane $(x, z)$, perpendicular to the propagation direction of the beam. Border effects are ignored. After the interaction with the magnetic field, the beam might evolve freely for some time before finally being detected on a screen.

The Hamiltonian for each particle of the beam in the presence of the magnetic field, as indicated by the superscript $(m)$, is

$$H^{(m)} = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \mu \vec{S} \cdot \vec{B}(x, z) = H_x^{(m)} + H_z^{(m)},$$

where the subscripts indicate the dependence on the spatial coordinates. The corresponding time evolution operator $U^{(m)}(t, 0) = \exp(-iH^{(m)}t/\hbar)$, can be written as $U^{(m)}(t, 0) = U_x^{(m)}(t, 0)U_z^{(m)}(t, 0)$, where $U_x^{(m)}(t, 0)$ and $U_y^{(m)}(t, 0)$ are the time evolution operators corresponding to $H_x^{(m)}$ and $H_y^{(m)}$, respectively.

A factorized initial spatial state of the particles, $R(0) = R_{xz}(0)R_y(0)$, allows the separation of the dynamics along $y$, the longitudinal coordinate. Since this dynamics corresponds to a free evolution, the time spent on the magnetic field region, $\tau$, can be approximated by $\tau = mL/\hbar k_0y$, where it has been assumed that the momentum distribution in the $y$ coordinate is strongly peaked around $\hbar k_0y$. Further influence of the time evolution in the longitudinal coordinate can be ignored, and so our study reduces to a two-dimensional problem. The remaining part of the initial spatial state, $R_{xz}(0)$, is assumed to be of the form $R_{xz}(0) = |\psi_{xz}\rangle \langle \psi_{xz}|$ where

$$\langle x, z|\psi_{xz}\rangle = \sqrt{\frac{1}{2\pi\sigma_0^2}} \exp\left[-\frac{x^2}{4\sigma^2} + \frac{z^2}{4\sigma^2}\right].$$

The transit time, $\tau$, and the dispersion of the initial spatial state in the $x$-direction, $\sigma$, are used as natural scales to define the dimensionless quantities $\tilde{t} = t/\tau$, $\tilde{x} = x/\sigma$, $\tilde{z} = z/\sigma$, $\tilde{p}_x = \sigma p_x/\hbar$, $\tilde{p}_z = \sigma p_z/\hbar$ and $\tilde{H}_x^{(m)} = \tau H_x^{(m)}/\hbar$. With these definitions, the equation of motion for the time evolution operator $U_{xz}^{(m)}(\tilde{t}, 0)$ becomes

$$\frac{dU_{xz}^{(m)}(\tilde{t}, 0)}{d\tilde{t}} = \tilde{H}_x^{(m)} U_{xz}^{(m)}(\tilde{t}, 0),$$

where

$$\tilde{H}_x^{(m)} = g_2 \left( \tilde{p}_x^2 + \tilde{p}_z^2 \right) + g_1 \left( \tilde{x}\sigma_1 - \tilde{z}\sigma_3 \right),$$

$$g_1 = \mu b\sigma/2\hbar, g_2 = \hbar\tau/2m\sigma^2$$

and $\{\sigma_i\}, i = 1, 2, 3$, stand for the Pauli spin operators. For this model, we have considered a quadrupolar magnetic field of the form

$$\vec{B}(\tilde{x}, \tilde{z}) = -b\sigma \left( \tilde{x}\hat{i} - \tilde{z}\hat{k} \right).$$

In the scaled coordinates, the initial Gaussian wavefunction reads

$$\langle \tilde{x}, \tilde{z}|\tilde{\psi}_{xz}\rangle = \sqrt{\frac{1}{2\pi\lambda}} \exp\left[-\frac{\tilde{x}^2}{4\lambda^2} + \frac{(\tilde{z}/\lambda)^2}{4}\right],$$

where $\lambda = \sigma^*/\sigma$.

The free evolution of the beam just after the interaction with the magnetic field and before being detected at time $T$ is represented by the operator $U_{xz}^{(f)}(T, 1) = \exp[-i\tilde{H}_x^{(f)}(T - 1)]$, where $\tilde{H}_x^{(f)} = g_2 \left( \tilde{p}_x^2 + \tilde{p}_z^2 \right)$. By combining the free and magnetic parts of the time evolution, we obtain the time evolution operator for the complete
Stern-Gerlach setup, \(U_{\bar{z}z}(T,0) = U_{\bar{z}z}^{(f)}(T,1)U_{\bar{z}z}^{(m)}(1,0)\). The final state of the particles of the beam just before detection will then be
\[
\rho(T) = U_{\bar{z}z}(T,0) (R_{xz}(0)\rho_S(0)) U_{\bar{z}z}^\dagger(T,0).
\]
From here on, we will drop the bar on top of the dimensionless variables to unclutter the notation. The Stern-Gerlach setup described in this work can be characterized by a set \(G = \{g_1, g_2, \lambda, T\}\) of parameters: \(g_1\) and \(g_2\), associated with the quadrupole field and the kinetic energy, respectively; \(\lambda\), which measures the elongation of the initial spatial wavefunction; and \(T\), the time interval of free-evolution after interaction with the magnetic field. Hence, in our model, the spatial intensity distribution on the screen
\[
\begin{aligned}
\langle x,z| &\rho(T)|x,z\rangle = \langle x,z|\rho_S(0)|x,z\rangle e^{-i\frac{\pi}{2}H_{\Omega_z}T} e^{-i\frac{\pi}{2}H_{\Omega_x}T} \\
&\propto \int dxdz \left[ U_{\bar{z}z}^{(f)}(x,0) U_{\bar{z}z}^{(m)}(x,1) \right] \langle x,z|\rho(0)|x,z\rangle e^{-i\frac{\pi}{2}H_{\Omega_z}T} e^{-i\frac{\pi}{2}H_{\Omega_x}T}.
\end{aligned}
\]

In the next section, we show that information about the complete initial spin state of the beam is indeed contained in the spatial intensity distribution of the beam. We then investigate how to use this intensity measurements to estimate the initial spin state of the particles.

### III. STATE ESTIMATION

It is intuitively reasonable that spin state estimation using measurements of the spatial intensity distribution should be possible under fairly general conditions. However, a state estimation scheme using a Stern-Gerlach setup have not yet been reported because initial spatial states rotationally invariant around the propagation direction of the beam are usually considered; \(\lambda = 1\) in Eq. (4). For a pure initial spin state \(|\chi\rangle\), this particular initial spatial state leads to a final state of the particles of the form
\[
\rho(T) = \frac{1}{2} \left[ U_{\bar{z}z}^{(f)}(T,1) \right] \langle \psi(1) | U_{\bar{z}z}^{(f)}(T,1) \rangle^\dagger,
\]
where
\[
\langle x,z|\psi(1)\rangle = \sum_{\mu=0}^3 \phi_\mu(x,z)\sigma_\mu |\chi\rangle,
\]
\(\phi_1(x,z) = x\phi(x,z)\), \(\phi_2(x,z) = -z\phi(x,z)\) and \(\phi_2(x,z) = 0\) (see appendix A). Here, \(\sigma_\mu\) denotes the identity operator for the spin degrees of freedom. The particular form of Eq. (6) is obtained by expanding the time evolution operator as \(U_{\bar{z}z}^{(m)}(1,0) = \sum_{\mu=0}^3 A_\mu \sigma_\mu\) and applying it to the initial state \(|\psi_{zz}\rangle\). The coefficients \(\{A_\mu\}\) are defined only over the Hilbert space corresponding to the spatial degrees of freedom of the particles, which we will denote as \(\mathcal{H}_{zz}\). In terms of \(\{A_\mu\}\), the functions \(\phi_\mu(x,z)\) read \(\phi_\mu(x,z) = \langle x,z|A_\mu|\psi_{zz}\rangle\), and the conditions \((zA_1 + xA_3)|\psi_{zz}\rangle = 0\) and \(A_2|\psi_{zz}\rangle = 0\) are satisfied.

From these considerations, the corresponding spatial intensity distribution, given by
\[
I(x,z) = \sum_{\mu,\nu=0}^3 \langle x,z|U_{zz}^{(f)}|\phi_\mu \phi_\nu^* U_{zz}^{(f)}|x,z\rangle \langle \chi|\sigma_\nu \sigma_\mu|\chi\rangle,
\]
does not depend on the expectation value of \(\sigma_2\), because the corresponding contribution to the intensity,
\[
i \langle x,z|U_{zz}^{(f)}(\phi_1 \phi_3^* - \phi_3 \phi_1^*)U_{zz}^{(f)}|x,z\rangle,
\]
identically vanishes. Since this expectation value corresponds to the second component of the Bloch vector that defines the initial spin state, \(\rho_S = |\chi\rangle \langle \chi| = \frac{1}{2}(s_0 \sigma_0 + s_1 \sigma_1 + s_2 \sigma_2 + s_3 \sigma_3)\), no spatial intensity measurement can encode any information about the complete initial spin state when \(\lambda = 1\). Consequently, the complete estimation of the initial spin state of the beam from measurements of its spatial intensity distribution requires us to assume that \(\lambda\) is different from unity.

For a general final state \(\rho(T)\), the intensity at a point \((x,z)\) on the screen can be written as
\[
I(x,z) = \text{Tr} \left[ \rho(T)|Q_{zz}\rangle \langle Q_{zz}| \rho(T) \right] = \text{Tr} \left[ \rho(T)|x,z\rangle \langle x,z|\sigma_0\right].
\]
Using, for example, a maximum likelihood estimation procedure [6], it is possible to reconstruct the initial spin state using the intensity at every point of the screen, that is, using the whole set of operators \(Q = \{Q_{xx}\}\). However, we will deal with an approach that employs the minimum number of operators necessary to obtain a complete estimation of the initial spin state. Given that any spin state can be defined by three real parameters, namely the three components of the Bloch vector, the set \(Q = \{Q_k\}\) must contain at least four elements [20]. We arbitrarily choose operators \(\{Q_k\}\) to represent the spatial intensity measurements over the four regions \(\Omega_k\), \(k = 1, 2, 3, 4\), shown in Fig. 2. Each one of these operators will then take the form
\[
Q_k = \int_{\Omega_k} |x,z\rangle \langle x,z| \sigma_0 dxdz.
\]
The probabilities $p_k(T)$ can be written in terms of the time-dependent operators $\hat{Q}_k = U_{xz}^\dagger(T,0)\hat{Q}_k U_{xz}(T,0)$ as

$$p_k(T) = \text{Tr} \left[ R_{xz}(0) \rho_S(0) \hat{Q}_k(T) \right].$$

This new set of operators $\hat{Q} = \{\hat{Q}_k\}$, which is also a POVM, represents the time evolution and the detection of the beam in the Stern-Gerlach setup.

As in the case of a pure initial spin state, the operator $\rho_S(0)$ can be written as an expansion of Pauli spin operators,

$$\rho_S(0) = \frac{1}{2} \sum_{\mu=0}^3 s_\mu \sigma_\mu,$$

where the real coefficients $\{s_\mu\}$ are the components of the Bloch vector that defines the spin state. In a similar way, operators $\{\hat{Q}_k\}$ can be expanded as

$$\hat{Q}_k(T) = \sum_{\mu=0}^3 m_{k\mu}(T) \sigma_\mu,$$

where the coefficients $m_{k\mu}(T)$ are Hermitian operators defined over the Hilbert space for the spatial degrees of freedom $\mathcal{H}_{xz}$. As hinted before, the estimation of the initial spin state consists in estimating the set of parameters $\{s_\mu\}$.

Inserting Eqs. (10) and (11) into Eq. (9), the probabilities $\{p_k\}$ can be expressed as

$$p_k(T) = \frac{1}{2} \sum_{\mu=0}^3 s_\mu \text{Tr} \left[ (m_{k\mu}(T) R_{xz}(0)) (\sigma_\mu \sigma_\nu) \right]$$

$$= \frac{1}{2} \sum_{\mu=0}^3 s_\mu \text{Tr}_{xz} \left[ m_{k\mu}(T) R_{xz}(0) \right] \text{Tr}_{S} \left[ \sigma_\mu \sigma_\nu \right]$$

$$= \sum_{\mu=0}^3 s_\mu \text{Tr}_{xz} \left[ m_{k\mu}(T) R_{xz}(0) \right],$$

where the symbols $\text{Tr}_{xz}(\cdot)$ and $\text{Tr}_S(\cdot)$ indicate the partial traces over the spatial and spin degrees of freedom, respectively. Defining the vectors $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\mathbf{s} = (s_0, s_1, s_2, s_3)$, Eq. (12) takes the following simple matrix form:

$$\mathbf{p}(T) = \mathbf{M}(T) \mathbf{s},$$

where the components of the measurement matrix $\mathbf{M}(T)$ are defined as $M_{k\mu}(T) = \text{Tr}_{xz} \left[ m_{k\mu}(T) R_{xz}(0) \right]$. It suffices to invert matrix $\mathbf{M}(T)$ to construct an estimator for the parameters $\mathbf{s}$:

$$\mathbf{s} = \mathbf{M}^{-1}(T) \mathbf{p}.$$

The caron (’) indicates that the quantity is an estimator, instead of a parameter. In a more general setup, when the set $\mathcal{Q}$ contains more than four elements and the measurement matrix becomes non-square, Eq. (13) holds if $\mathbf{M}^{-1}(T)$ is interpreted as the Moore-Penrose inverse.

For $R_{xz}(0) = |\psi_{xz}\rangle \langle \psi_{xz}|$, the components of the measurement matrix can be recast as

$$M_{k\mu}(T) = \langle \psi_{xz}| m_{k\mu}(T) \psi_{xz} \rangle.$$

Inserting the expansion of the time evolution operator in terms of Pauli spin operators, $U_{xz}(T,0) = \sum_{\alpha=0}^3 A_\alpha(T) \sigma_\alpha$, and employing the original definition operators $\{\hat{Q}_k\}$, we obtain

$$M_{k\mu}(T) = \sum_{\alpha,\beta=0}^3 d_{\alpha\beta} \int \phi_\alpha(x,z,T) \phi_\beta^*(x,z,T) dx dz,$$

where $\phi_\alpha(x,z,T) = \langle x,z|A_\alpha(T)|\psi_{xz}\rangle$ and $d_{\alpha\beta} = \text{Tr}_{S} \left[ \sigma_\alpha \sigma_\beta \sigma_\mu \right] / 2$. This form of the components $M_{k\mu}(T)$ will prove useful for the numerical computation of the measurement matrix, which is the subject of the next section.

IV. NUMERICAL CALCULATION OF THE MEASUREMENT MATRIX $\mathbf{M}(T)$

Using Eq. (15), it is easily seen that the calculation of the measurement matrix is reduced to the calculation of
the integrals
\[ \Phi^{(k)}_{\alpha\beta} = \int_{\Omega} \phi_\alpha(x, z, T) \phi_\beta(x, z, T) \, dx \, dz. \tag{16} \]

The symmetry \( (\Phi^{(k)}_{ij})^* = \Phi^{(k)}_{ji} \) reduces the number of independent integrals from sixteen to ten, for each \( k \). We will compute the functions \( \phi_\alpha(x, z, T) \) and the integrals \( \Phi^{(k)}_{ij} \) using the method described below. Though we will assume a pure initial spin state of the beam for the description of the method, the results are also valid for mixed initial spin states.

The state of the particles at time \( T \), the time of detection, is \( |\psi(T)\rangle = U_{zz}^{(f)}(T, t) U_{zz}^{(m)}(t, 0) |\psi_{zz}\rangle |\chi\rangle \), where \( U_{zz}^{(f)}(T, t) \) and \( U_{zz}^{(m)}(t, 0) \) are the evolution operators, free and in presence of the magnetic field, respectively. Particles are assumed to enter and to exit the magnetic field region at times \( t = 0 \) and \( t = 1 \), respectively.

Let us begin with the evolution in the magnetic field region. Since \( h_1 = g_2 (p_x^2 + p_z^2) \) and \( h_m = g_1 (x_s - z \sigma_3) \) do not commute, it is important to find analytic expression for the unitary operator \( U_{zz}^{(m)}(t, 0) \) in the evolution operators, free and in presence of the magnetic field, respectively. Particles are assumed to enter and to exit the magnetic field region at times \( t = 0 \) and \( t = 1 \), respectively.

At each time step, three evolution operators are applied. For the application of the first one, it is convenient to expand the state and the evolution operator as

\[ |\psi(t_n)\rangle = e^{-i\delta t \frac{\delta z}{N_z}} |\psi(t_{n-1})\rangle, \tag{18} \]

where \( \delta t = t/N_t \), \( |\psi(t_0)\rangle = |\psi_{xx}\rangle |\chi\rangle \) and \( |\psi(t_{N_t})\rangle = |\psi(t)\rangle \). The index \( n \) runs from 0 to \( N_t \).

The second operator to be applied at each time step is \( e^{-i\delta t/2} |\psi(t_n)\rangle \), and \( e^{-i\delta t/2} |\psi(t_n)\rangle \) then turn into \( (N_x + 1) \times (N_z + 1) \) arrays. Accordingly, the multiplicative application of discretized operators over discretized states becomes a Hadamard (element-wise) product between arrays of the same size and the transformation from position to momentum representation becomes a fast Fourier transform.

Having found the arrays \( \Phi_\mu(T) \), the corresponding values of \( \Phi^{(k)}_{\mu
u}(T) \) can be computed as

\[ \Phi^{(k)}_{\mu
u} = \sum_{(x_i, z_j) \in \Omega_k} \mu \delta_{\nu}(x_i, z_j) \Phi^*_\mu(x_i, z_j) \delta x \delta z, \tag{19} \]

where the sum extends over the pairs \( (x_i, z_j) \) belonging to the region \( \Omega_k \). Once we know the values \( \Phi^{(k)}_{\mu
u} \), the calculation of \( M(T) \) directly follows from equation \( \tag{15} \).

V. ERROR OF THE ESTIMATION

In principle, relation \( \hat{S} = M^{-1}(T) \hat{S} \) allows the estimation of all the parameters that define \( \rho_S(0) \), including the normalization condition \( s_0 = Tr \rho_S(0) \). However, it is necessary to evaluate how reliable the estimation of these parameters can actually be. It is expected, for
example, that the estimation of $s_2$ becomes increasingly difficult as $\lambda$ approaches unity. A suitable state estimation thus requires a proper choice of the setup parameters $G = \{g_1, g_2, \lambda, T\}$. To investigate this problem, we will quantify the error of the estimation and analyze its dependence on the setup parameters.

To do this, we must first write the estimator of the initial spin state, $\hat{s}$, in terms of the outcomes of the measurement process. These outcomes correspond to the numbers of particles $n_k$ detected at region $k$ after $N = \sum_{k=1}^{4} n_k$ runs of the experiment. The values $n = \{n_1, n_2, n_3, n_4\}$ constitute a set of random variables whose probability distribution is a multinomial distribution of the form [21]

$$P(n|p) = \frac{N!}{n_1! n_2! n_3! n_4!} [p_k(T)]^{n_k}.$$  \hfill (20)

For fixed setup parameters, the values $p_k(T)$, given by equation (9), represent the fixed probability for each particle to be detected at region $k$.

Experimentally, the values $p = \{p_k(T)\}$ are the unknown parameters that define $P(n|p)$, and must be estimated from the measurement results. A maximum-likelihood, unbiased estimator for each $p_k(T)$ is given by [22]

$$\hat{p}_k = \frac{n_k}{N}. \hfill (21)$$

Notice that we expressed estimator $\hat{s}$ in terms of $\hat{p} = \{\hat{p}_k\}$, namely, we wrote in Eq. (14) $\hat{s} = M^{-1}(T)\hat{p}$. Consequently, in terms of the measurement results, the estimator for each one of the parameters defining the initial spin state of the beam takes the form

$$\hat{s}_\mu = \frac{1}{N} \sum_{k=1}^{4} \left[ M^{-1}(T) \right]_{\mu k} n_k. \hfill (22)$$

The performance of the estimator $\hat{s}$ is statistically characterized by its covariance matrix Cov $(\hat{s}, \hat{s})$, because its bias $E[\hat{s} - s]$, the expectation value of $\hat{s} - s$ taken with respect to distribution [20], vanishes. Estimator $\hat{s}$ is unbiased because it is a linear combination of the unbiased estimators $\hat{p}_k = n_k/N$. All components of the covariance matrix, Cov $(\hat{s}_\mu, \hat{s}_\nu) = E[\hat{s}_\mu \hat{s}_\nu] - E[\hat{s}_\nu] E[\hat{s}_\mu]$ [23], are statistically meaningful; however, the diagonal elements (i.e. the variances) are more useful, because they are associated with the error of the estimated parameters $s_\mu = E[\hat{s}_\mu] \pm \sqrt{\text{Cov}(\hat{s}_\mu, \hat{s}_\mu)}$.

The maximum performance of the state estimation scheme, corresponding to the minimum values that the variances can take, will be characterized by the inverse of its Fisher information matrix, $J(s)$. This characterization is possible because any unbiased estimator of $s$ satisfies the Cramér-Rao inequality [23]

$$\text{Cov}(\hat{s}, \hat{s}) - J^{-1}(s) \geq 0. \hfill (23)$$

The inequality means that the difference between matrices is positive semidefinite. The information matrix $J(s)$, which does not depend on the construction of the estimators of parameters $\{s_\mu\}$, imposes a lower bound on the covariance matrix. This lower bound is attained by efficient estimators. Estimator $\hat{p}_k = n_k/N$ is asymptotically efficient, because it is a maximum-likelihood estimator. Since this property is maintained under linear transformations [23], also $\hat{s}$ is asymptotically efficient; that is, the lower bound will be achieved for a sufficiently large number of particles $N$,

$$\lim_{N \to \infty} \text{Cov}(\hat{s}, \hat{s}) = J^{-1}(s). \hfill (24)$$

This limit further justifies the use of the information matrix for the quantification of the goodness of the estimation procedure.

The information matrix $J(s)$ linearly grows with $N$, the number of runs of the experiment, as can be shown by direct calculation. The components of the information matrix for parameters $s$ are calculated from the definition

$$J_{\mu \nu}(s) = -\mathbb{E} \left[ \left( \frac{\partial^2 P(n|p)}{\partial s_\mu \partial s_\nu} \right) \right], \hfill (25)$$

where $l(n|p) = \ln P(n|p)$ is the log-likelihood function. By expressing the partial derivatives in Eq. (25) in terms of parameters $p_k$ instead of parameters $s_\mu$, we can see that

$$J_{\mu \nu}(s) = \sum_{kl} J_{kl}(p) \frac{\partial p_k}{\partial s_\mu} \frac{\partial p_l}{\partial s_\nu}, \hfill (26)$$

where $J_{kl}(p)$ are the components of the information matrix for parameters $p$. Given the fact that $J_{kl}(p) = N \delta_{kl}/p_k$ and $\partial p_k/\partial s_\mu = M_{k\mu}(T)$, the information matrix can be computed as $J(s) = N [M(T)]^T \text{diag}(M(T)s)^{-1} M(T) = NF(s)$, where $\text{diag}(M(T)s)$ is a diagonal matrix whose elements are the probabilities $(p_k(T))$, written as functions of $s$. Matrix $F^{-1}(s)$ cannot be generally written as $M^{-1}(T) \text{diag}(M(T)s) [M^{-1}(T)]^T$, since $M(T)$ is generally non-square [23].

To eliminate the dependence of the information matrix on the number of runs of the experiment, we consider the scaled information matrix $F(s) = J(s)/N$. Since $J^{-1}(s)$ decreases at a rate $N^{-1}$, we can achieve a desired value for the variances by choosing a large but adequate number of particles. However, the choice of $N$ will be strongly limited by how large the diagonal elements of $F^{-1}(s)$ are. For this reason, we will ignore the explicit presence of the number of particles and define the error of the estimation procedure as a function of these diagonal elements.

To quantify the quality of the estimation, we define the logarithmic error

$$\Delta(G, s) = \log_{10} \left[ \text{tr}(F^{-1}(s)) \right], \hfill (27)$$

where we use the symbol $\text{tr}(\cdot)$ to distinguish the trace of matrix from the trace of an operator, indicated by $\text{Tr}(\cdot)$. The logarithmic error depends not only on the
state parameters $s_i$, but also on the set of parameters $G = \{g_1, g_2, \lambda, T\}$ of the experimental setup. This dependence comes from both the measurement matrix and the probabilities of $p_k(T)$. The logarithmic scale is useful for large variances, like those that are expected for values of $\lambda$ around unity.

To study the performance of estimator (22), $\bar{s}_\mu = \frac{1}{N} \sum_{k=1}^{4} \left[ M^{-1}(T) \right]_{\mu k} n_k$, we will assume that the initial spin state is normalized, so $s_0 = Tr_\rho(\rho_0) = 1$. As a result, the scaled information matrix, with elements $F_{ij} = \sum_k \frac{1}{p_k} \frac{d p_k}{d s_i} \frac{d p_k}{d s_j}$, becomes a $3 \times 3$ matrix, which can be written as

$$ F(s) = \left[ \bar{M}(T) \right]^T \left[ \text{diag}(\bar{M}(T)s) \right]^{-1} \bar{M}(T). \quad (28) $$

The $4 \times 3$ matrix $\bar{M}(T)$ is equal to the original $M(T)$ without its first column (because the derivatives with respect to $s_0$ are not taken into account).

Since the logarithmic error depends on seven parameters, a relatively large parameter space, we need to focus on a sensible parameter subspace. We will consider initial pure spaces, which can be parametrized by the angles $\theta \in [0, \pi]$ and $\phi \in (0, 2\pi]$, where $s_1 = \sin \theta \cos \phi$, $s_2 = \sin \theta \sin \phi$ and $s_3 = \cos \theta$. We will assume no free evolution after the beam interacts with the magnetic field; that is, $T = 1$. In the usual setup of the Stern-Gerlach experiment, the additional free evolution helps to clearly split the beam, guaranteeing a projective measurement of the spin component in that direction. Here, no beam separation is expected; therefore, this free evolution is not necessary. However, the influence of the parameter $T$ will be considered at the end of this section. In previous studies [12, 13], the deflection of the beam in the usual experimental setup was found to be sizable when the product $g_1 g_2$ exceeds unity. We will consider values of $g_1 \in [1.0, 5.0]$ and $g_2 \in (0, 4.0]$. These values for $g_1$ and $g_2$, similar to those used in these studies, are far from the usual approximation where $g_1 \gg g_2$ [12].

Since the exploration of the reduced parameter space would be quite time consuming, we consider the variation of $\Delta(G, s)$ as a function of $g_1$ and $g_2$ for different values of $\lambda$ and a fixed initial spin state, as shown in Fig. 3. The error for chosen initial state, defined by the values $\theta = 1.91$ and $\phi = 4.78$, is maximum in a setup where $g_1 \gg g_2$ [12].

The error for this state to be a pessimistic estimation of the typical error for other values of the parameters $g_1$, $g_2$ and $\lambda$. We choose values of $\lambda$ for which the error shows local minima. For this computation, the $x$ and $z$ coordinates were sampled over the interval $[-50, 50]$, the total number of samples in each direction was $N_x = N_z = 600$, and the total number

![Graph showing error of the estimation procedure, $\Delta(G, s)$, as function of parameters $g_1$ and $g_2$ for different values of parameter $\lambda$. The beam was assumed to be detected just at the end of the interaction with the magnetic field, that is at $T = 1$. The initial spin state of the beam was a pure state defined by its Bloch vector $s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, where $\theta = 1.91$ and $\phi = 4.78$.](image)
of temporal steps was $N_t = 600$.

Inspection of Fig. 3 shows regions of the parameter space where the error is large and others where it is (relative) low. To test our suspicion that the variance of $\sigma_2$ is responsible for large errors, we plot, in Fig. 4, the logarithmic error excluding this variance. We find setups where this is, indeed, the case, near and far from $\lambda = 1$. For example, for $g_1 = 2.2$, $g_2 = 2.4$, $\lambda = 1.1$, the variance of $\sigma_2$ is of the order of $10^{12}$; for $g_1 = 4.6$, $g_2 = 0.24$, $\lambda = 0.2$, the variance of $\sigma_2$ is of the order of $10^6$. We also find cases where the variance of $s_2$ is not the largest one (for $g_1 = 1.0$, $g_2 = 3.96$, $\lambda = 0.2$, the variances of $s_1$, $s_2$, and $s_3$ are 10.6, 60.8, and 119.0, respectively). From the point of view of the information matrix, large values for the variance associated to $s_2$ mean that the intensity distribution encodes very little information about this parameter. This is not only caused by the definition of the setup parameters, but also by the choice of the regions over which the intensity distribution is measured (quadrants, in this work).

The best regions to perform state estimation are those where the error remains low and stable under small, but not infinitesimal, changes of the parameters that define the experimental setup. For example, when $g_1 = 2.0$, $g_2 = 3.24$, $\lambda = 0.3$, the variance associated to $s_2$ is 55.3,

while those associated to $s_1$ and $s_3$ are 3.75 and 2.12, respectively.

FIG. 4. Error of the estimation procedure, $\Delta(G, s)$, as function of parameters $g_1$ and $g_2$ for different values of parameter $\lambda$, without including the variance corresponding to the estimation of $s_2$. The beam was assumed to be detected just at the end of the interaction with the magnetic field, that is at $T = 1.0$. The initial spin state of the beam was a pure state defined by the values $\theta = 1.91$ and $\phi = 4.78$.

FIG. 5. Error of the estimation procedure, $\Delta(G, s)$, as a function of the parameters defining a pure initial spin state, $\theta$ and $\phi$. The setup parameters used were $g_1 = 2.0$, $g_2 = 3.24$, $\lambda = 0.3$, and $T = 1.0$. 
For a given set of parameters $g_1$, $g_2$ and $\lambda$, the error of the estimation depends on the spin state to be estimated. However, if the difference between the lowest and the largest possible error remains sufficiently small, as in the example of Fig. 3, the error of the estimation procedure can be defined as the error associated to the state with the worst possible estimation.

To quantify the role of the free evolution on the estimation error, it is necessary to increase the $(x, z)$ region where the intensity distribution is calculated, because the wavefunction broadens. For this computation, the $x$ and $z$ coordinates were sampled over the interval $[-100, 100]$, and the total number of samples in each direction was increased to 650. In Fig. 6, we show examples of the influence of the parameter $T$ on the estimation error. Sometimes, the error monotonically grows with $T$ (for example, for $g_1 = 4.0$, $g_2 = 3.24$, and $\lambda = 0.3$); sometimes, it rapidly increases before decreasing again and reaching a stable value, lower than the one obtained just after the interaction with the magnetic field (for example, for $g_1 = 4.0$, $g_2 = 0.5$, and $\lambda = 0.3$). However, errors are larger than the minimum error found without free evolution. Even if it is not generally the case, having a setup where the time of detection ensures a lower value of the error of estimation could prove useful for experimental situations where optimal values for parameters $g_1$ and $g_2$ cannot be easily obtained. In these situations, one could choose an optimum value for $\lambda$ and/or other regions over which the intensity is evaluated, to lower the error as much as possible.

As a final remark, we would like to compare the variances obtained by our estimation procedure with the lowest possible values they can take. According to Watanabe [21], there are three sources of error in an estimation procedure: quantum fluctuations, errors in repeated identical measurements and errors from the estimation procedure. This last type of error is due to the definition of the estimators for the parameters, which, in turn, is derived from the definition of the observables and the experimental setup. If we rule out both the error of the measurement and the error in the design of the estimation procedure, the variance of the estimated parameters would be entirely determined by quantum fluctuations. In our case, these fluctuations correspond to the variances of observables $\sigma_1$, $\sigma_2$ and $\sigma_3$, taken with respect to the initial spin state and divided by the total number of particles detected [21]. These variances constitute the lowest possible value of the error in the estimation of the initial spin state. Taking into account that $\text{Var}(\sigma_\mu) = 1 - s_\mu^2$, the lowest possible value of the error for the estimation of an initial spin state would be $\Delta(G, s) = \log_{10} \{3 - \sum_{\mu=1}^{3} s_\mu^2\}$. This error would vary between $\log_{10}(2) \approx 0.3$, for pure states, and $\log_{10}(3) \approx 0.48$ for the maximally mixed state. If we only take into account the error associated to $s_1$ and $s_2$, we would have for both pure and mixed states $\Delta(G, s) > 0.3$.

As can be seen in Figs. 3 and 4, our estimation procedure does not attain the lower possible bound. In the explored region of parameters, only the estimations of parameters $s_1$ and $s_3$ are close to the optimal value of the error. However, in the case of a real experiment, the suitable choice of the number of particles can help to obtain reasonable values for the variances of all the parameters that define the initial spin state.

VI. CONCLUSIONS AND PERSPECTIVES

In this work we have shown how a modified setup of the Stern-Gerlach experiment can be used to estimate the initial spin state of a beam of neutral spin-1/2 particles. There are three modifications: the use of a magnetic field without a large reference component, the measurement of the spatial intensity distribution of the beam over at least four different regions of the plane of detection, and the suitable choice of the initial spatial state of the beam of particles.

Using a quantum-mechanical description of the experimental setup, we derived an estimation procedure by linear inversion for the parameters that define the initial spin state. Unless the initial spin state is rotationally invariant along the direction of propagation of the beam, all of the parameters that define the initial spin state can be estimated.

The quality of the estimation of the initial spin state was quantified by the logarithm of the sum of the variances of the parameters which characterize the state (Bloch vector components). This measure allowed us to compare the errors associated to different experimental
setups and to find the typical values of the variances that can be obtained with the use of the estimation procedure. Although these variances do not generally attain the lower limit imposed by quantum fluctuations, they can take reasonably low values when the number of particles of the beam is large enough.

An optimization of the error of estimation could reveal possible experimental setups that attain variances that are closer to the lower bound imposed by quantum fluctuations. This goal could also be achieved by modifying the measurement of the beam, for instance, by defining optimal regions for the measurement of the intensity distribution. Another possible approach to this problem would be the implementation of a estimation procedure that uses intensity measurements on every point of the detection screen.

Although the intention of this work was not to propose a real experimental setup that achieves the estimation of the initial spin state, it is interesting to discuss a possible set of experimental parameters compatible with the values of \( g_1 \) and \( g_2 \) that we chose for the quantification of the estimation error. In terms of the real experimental parameters, \( g_1 = \mu B \sigma \tau / 2h \) and \( g_2 = h \tau / 2 m \sigma^2 \). We will assume that the particles of the beam are neutrons, in this way we fix the values of \( \mu \) and \( m \) to \( \mu = 0.97 \times 10^{-26} \text{J/T} \) and \( m = 1.67 \times 10^{-27} \text{kg} \). Usual field gradients in Stern-Gerlach experiments vary between 1 T/m and 100 T/m [2] [3] [4]. If the neutrons are slow enough, a large gradient is not necessary, so it is reasonable to assume that \( b \sim 1 \text{T/m} \). In these same experiments, the length of the magnet is usually close to 1 m; we will take this value as a reasonable length for the magnet. Experiments with cold neutrons report average beam speeds between 400 m/s and 600 m/s [5]. Assuming these speeds, the time of interaction with the magnetic field would vary between \( \tau \sim 1.7 \text{ms} \) and \( \tau \sim 2.5 \text{ms} \). By taking these values for \( b \) and \( \tau \), and considering the conditions over \( g_1 \) and \( g_2 \) that were used to calculate the error of the estimation, \( \sigma \) would vary between \( \sigma \sim 5 \mu \text{m} \) and \( \sigma \sim 10 \mu \text{m} \).

We consider the values for speeds, field gradients, and other physical quantities discussed on the previous paragraph, to be adequate for an experimental implementation of the estimation procedure. Although actual experimental results might significantly differ from our numerical results, due to the idealizations we have made in the model Hamiltonian (like neglecting the variation of the magnetic field along the direction of the beam), we would expect state estimation to be possible.

### Appendix A: Intensity measurements for \( \lambda = 1 \)

In this appendix we show that the spatial intensity distribution of the beam does not encode information about the parameter \( s_2 \) when \( \lambda = 1 \).

The Hamiltonian \( H_{zz} \) is, in polar coordinates

\[
H_{zz} = g_2 \left( p_r^2 + \frac{L_y^2}{r^2} \right) + g_1 r \left( \cos \theta \sigma_1 - \sin \theta \sigma_3 \right), \tag{A1}
\]

where \( p_r \) is the radial momentum, \( L_y \) the angular momentum in the \( y \) direction, \( x = r \cos \theta \), and \( z = r \sin \theta \). The initial spatial state, expressed in the same coordinates, is

\[
\langle r, \theta | \psi_{zz} \rangle = \sqrt{1 \over 2 \pi \lambda} e^{-r^2 \over 2 \lambda^2} \exp \left[ \left( \lambda^2 - 1 \right) r^2 \sin^2 \theta \over 4 \lambda^2 \right]. \tag{A2}
\]

While the state of the beam at time of detection is \( \rho(T) = U_{zz}(T,0) | \psi_{zz} \rangle \langle \psi_{zz} | U_{zz}^\dagger(T,0) \), the evolution operator can be factorized as \( U_{zz}(T,0) = U_{zz}^{(f)}(T,1) U_{zz}^{(m)}(1,0) \). By expanding \( U_{zz}^{(m)}(1,0) \) as \( U_{zz}^{(m)}(1,0) = \sum_{\alpha=0}^3 \alpha \sigma_\alpha \), and the initial spin state as \( | \psi_{zz} \rangle = (1/2) \sum_{\mu=0}^1 \mu \sigma_\mu \), we find

\[
\rho(T) = \frac{1}{2} \sum_{\alpha,\beta,\mu=0}^3 \sigma_\alpha \sigma_\beta | \phi_\alpha(T) \rangle \langle \phi_\beta(T) | \sigma_\mu, \tag{A3}
\]

where \( | \phi_\alpha(T) \rangle = U_{zz}^{(f)}(T,1) A_\alpha | \psi_{zz} \rangle \).

We expand operator \( U_{zz}^{(m)}(1,0) \) in a power series of the Hamiltonian \( H_{zz} \), \( U_{zz}^{(m)}(1,0) = \sum_{k=0}^{\infty} (-i)^k H_{zz}^k \). We also expand each power of the Hamiltonian as \( H_{zz}^k = \sum_{\alpha=0}^3 \sum_{\alpha=0}^3 (-i)^k h_\alpha^{(k)} \sigma_\alpha \), where \( \{ h_\alpha^{(k)} \} \) are spatial Hermitian operators. In this way, \( U_{zz}(1,0) = \sum_{k=0}^{\infty} (-i)^k h_\alpha^{(k)} \sigma_\alpha \).

By direct comparison, \( A_\alpha \) is found to be

\[
A_\alpha = \sum_{k=0}^{\infty} (-i)^k h_\alpha^{(k)}. \tag{A4}
\]

Since the coefficients \( \{ h_\alpha^{(k)} \} \) are obtained from powers of the Hamiltonian, we can find recurrence relations between them for each order in the power series. By using relation \( H_{zz}^{(k+1)} = H_{zz} H_{zz}^{(k)} \), we find the following expressions for the computation of the coefficients at higher orders:

\[
h_0^{(k+1)} = g_2 P^2 h_0^{(k)} + g_1 r \left( \cos \theta h_1^{(k)} - \sin \theta h_3^{(k)} \right), \tag{A5}
\]

\[
h_1^{(k+1)} = g_2 P^2 h_1^{(k)} + g_1 r \left( \cos \theta h_0^{(k)} + i \sin \theta h_2^{(k)} \right), \tag{A6}
\]

\[
h_2^{(k+1)} = g_2 P^2 h_2^{(k)} - ig_1 r \left( \cos \theta h_3^{(k)} + \sin \theta h_1^{(k)} \right), \tag{A7}
\]

\[
h_3^{(k+1)} = g_2 P^2 h_3^{(k)} - g_1 r \left( \sin \theta h_0^{(k)} - i \cos \theta h_2^{(k)} \right), \tag{A8}
\]

where we have made the definition \( P^2 = p_r^2 + r^{-2} L_y^2 \). These relations are complemented by the initial conditions \( h_0^{(0)} = I_{zz}, h_1^{(0)} = h_2^{(0)} = h_3^{(0)} = 0 \), where \( I_{zz} \) is the identity operator over \( H_{zz} \).
When acting over the initial spatial state, coefficients \( \{ h^{(k)}_s \} \) satisfy the following relations for every order:

\[
\begin{align*}
    h^{(k)}_2 |\psi_{xx}\rangle &= (1 - \lambda^2) G^{(k)}_2 |\psi_{xx}\rangle, \\
    (\sin \theta h^{(k)}_1 + \cos \theta h^{(k)}_3) |\psi_{xx}\rangle &= (1 - \lambda^2) G^{(k)}_1 |\psi_{xx}\rangle, \\
    (\sin \theta L_y h^{(k)}_3 - \cos \theta L_y h^{(k)}_1) |\psi_{xx}\rangle &= (1 - \lambda^2) F^{(k)}_{13} |\psi_{xx}\rangle, \\
    L_y h^{(k)}_0 |\psi_{xx}\rangle &= (1 - \lambda^2) G^{(k)}_0 |\psi_{xx}\rangle.
\end{align*}
\]

Operators \( G^{(k)}_2, G^{(k)}_1, F^{(k)}_{13} \) and \( G^{(k)}_0 \) generally depend on \( r, \theta \) and \( \lambda \).

To prove these properties, we will proceed by induction. At first order, these properties are valid; there are two non-vanishing terms, \( F^{(1)}_{13} = \frac{i g x^2 \sin(2\theta)}{4\lambda^2} \) and

\[
G^{(1)}_0 = \frac{-i g x^2 \sin(2\theta)}{32\lambda^6} \left[ (\lambda^4 - 1) r^2 \cos(2\theta) + (\lambda^4 + 1) r^2 - 12\lambda^2(\lambda^2 + 1) \right].
\]

Assuming that all properties hold at order \( k \), we obtain the following expressions for the operators at order \( k+1 \):

\[
\begin{align*}
    G^{(k+1)}_2 &= g_2 P^2 G^{(k)}_2 - i g_1 r G^{(k)}_{13}, \\
    G^{(k+1)}_{13} &= g_2 P^2 C^{(k)}_{13} - g_2 r^2 (C^{(k)}_{13} + 2i F^{(k)}_{13}) + i g_1 r G^{(k)}_2, \\
    F^{(k+1)}_{13} &= g_2 (P^2 + 3r^2 - 2) C^{(k)}_{13} - g_1 r (C^{(k)}_0 + G^{(k)}_2) \\
    &+ 2i g_x r (L_y^2 - 1) G^{(k)}_{13}, \\
    G^{(k+1)}_0 &= g_2 P^2 G^{(k)}_0 - g_1 r (F^{(k)}_{13} - i C^{(k)}_{13}).
\end{align*}
\]

Therefore, relations (A9) to (A12) hold for every order.

We now use Eq. (A4) to express the previous properties in terms of operators \( \{ A_\alpha \} \):

\[
A_2 |\psi_{xx}\rangle = (1 - \lambda^2) G_2 |\psi_{xx}\rangle,
\]

\[
(zA_1 + xA_3) |\psi_{x\bar{z}}\rangle = (1 - \lambda^2) G_{13} |\psi_{x\bar{z}}\rangle,
\]

\[
(x L_y A_1 - zL_y A_3) |\psi_{xx}\rangle = (1 - \lambda^2) F_{13} |\psi_{xx}\rangle,
\]

\[
L_y A_0 |\psi_{xx}\rangle = (1 - \lambda^2) G_0 |\psi_{xx}\rangle,
\]

where \( G_2, G_{13}, F_{13} \) and \( G_0 \) are obtained from the corresponding series of operators \( \{ G^{(k)}_2 \}, \{ G^{(k)}_{13} \}, \{ F^{(k)}_{13} \} \) and \( \{ G^{(k)}_0 \} \), respectively.

Now we can explore the implications of having \( \lambda = 1 \). Eq. (A13) implies that \( |\psi_2(T)\rangle = 0 \). Eq. (A14), on the other hand, implies that \( A_2 |\psi_{xx}\rangle = x A_1 |\psi_{xx}\rangle \) and \( A_3 |\psi_{xx}\rangle = -z A_1 |\psi_{xx}\rangle \). Additionally, when combined with Eq. (A15), yields to the relation \( (p_x A_1 + p_x A_3) |\psi_{xx}\rangle = 0 \), which allows to see that

\[
\left( z U^{(f)}_{x\bar{z}}(T,1) A_1 + x U^{(f)}_{x\bar{z}}(T,1) A_3 \right) |\psi_{xx}\rangle = 0.
\]

This means that \( x \phi_3(T) = -z \phi_1(T) \), which, in turn, implies that \( |\phi_3(T)\rangle = |\phi_1(T)\rangle - |\phi_3(T)\rangle = 0 \).

These results have an enormous influence in the structure of the spatial intensity distribution of the beam. Remembering the expression \( I(x, z) = T r_s \langle (x, z) | r(T) | x, z \rangle \) and using Eq. (A3), we see that

\[
I(x, z) = \sum_{\alpha, \beta, \mu=0}^3 d_{\alpha\beta\mu} \phi_\alpha(x, z, T) \phi^\ast_\beta(x, z, T) s_\mu,
\]

where \( d_{\alpha\beta\mu} = T r_s (\sigma_\alpha \sigma_\mu \sigma_\beta) / 2 \) and functions \( \phi_\alpha(x, z, T) \) are calculated as \( \langle x, z | \phi_\alpha(T) \rangle \). For the intensity distribution to depend on \( s_2 \), the term \( 2 \text{Re} (\phi_3 \phi^\ast_2) - 2 \text{Im} (\phi_1 \phi^\ast_1) \) must be different from zero. However, when \( \lambda = 1 \), this term identically vanishes, and thus, the estimation of \( s_2 \) cannot be achieved by using intensity measurements over any region of the \( (x, z) \) plane.

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