Characteristic polynomials and finitely dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$

Tianyi Jiang, Shoumin Liu

Abstract

In this paper, we obtain a general formula for the characteristic polynomial of a finitely dimensional representation of Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and the form for these characteristic polynomials, and prove there is one to one correspondence between representations and their characteristic polynomials. We define a product on these characteristic polynomials, endowing them with a monoid structure.

1 Introduction

The determinant and eigenvalues of a square matrix are classical topics in linear algebra. The interaction of several matrices is an important subject in multilinear algebra. We generalize the characteristic polynomial $\det(ZI - A)$ of a square matrix $A$ for several matrices, which means that we study the polynomial $\det(z_0 I + z_1 A_1 + \cdots + z_m A_m)$ for $n \times n$ matrices $A_1, A_2, \ldots, A_m$. In [12], Yang defines the projective spectrums through the multiparameters pencil $z_1 A_1 + \cdots + z_m A_m$, and there are many fruitful results in [1], [5] and [6]. For a representation $\pi$ of a finite group $G = \{g_i\}_{i=1}^n$, the characteristic polynomial

$$Q_{\pi}(z_0, z_1, \cdots, z_n) = \det(z_0 I + \sum_{i=1}^n z_i \pi(g_i))$$

is well studied in [3] and [4], where they prove that the characteristic polynomial of $\pi$ is irreducible when $\pi$ is an irreducible representation of $G$. It is quite natural to consider similar topics for Lie algebras of finite dimension. In [2], [9] and [7], the characteristic polynomial $f_{\pi}(z_0, z_1, z_2, z_3) = \det(z_0 I + z_1 \mathfrak{sl}(2, \mathbb{C})$
\[
\det(z_0 I + z_1 \phi(h) + z_2 \phi(e_1) + z_3 \phi(e_2)) \text{ in } \mathfrak{sl}(2, \mathbb{C}) \text{ on its irreducible representation } \pi \text{ of dimension } m + 1 \text{ is obtained, which is}
\]
\[
f_\pi = \begin{cases} 
z_0 \prod_{l=1}^{m/2} (z_0^2 - 4l^2(z_1^2 + z_2z_3)) & 2 \mid m. \\
\prod_{l=0}^{(m-1)/2} (z_0^2 - (2l + 1)^2(z_1^2 + z_2z_3)) & 2 \nmid m.
\end{cases}
\]

The finitely dimensional representations of \( \mathfrak{sl}(2, \mathbb{C}) \) are well studied, so it is natural to consider their characteristic polynomials, and describe all these polynomials and the corresponding relation between them. Since the representations of \( \mathfrak{sl}(2, \mathbb{C}) \) can form a monoidal category, there must be some algebraic structure on their characteristic polynomials, hence it is possible to endow these polynomials with an algebraic structure.

Our paper is sketched as the following. In Section 2, we present a theorem which can allow us to replace the canonical basis of \( \mathfrak{sl}(2, \mathbb{C}) \) under conjugation automorphisms. Apply the conclusion from Section 2, we present a new approach to prove the Hu and Zhang’s conjecture in Section 3. In Section 4, we show a clear formula of the characteristic polynomial for any finitely dimensional representation of \( \mathfrak{sl}(2, \mathbb{C}) \), and prove there is one to one correspondence between finitely dimensional representations and their characteristic polynomials, and we also present the forms of all characteristic polynomials. By the tensor product of representations, in Section 5 we define the resolution product of two characteristic polynomials, and we show there is a monoid structure on characteristic polynomials under the resolution product. In Section 6, we apply our results to the adjoint representation of \( \mathfrak{sl}(n, \mathbb{C}) \) restricted to \( \mathfrak{sl}(2, \mathbb{C}) \) to explore some results about the characteristic polynomial of \( \mathfrak{sl}(n, \mathbb{C}) \).

2 Conjugation automorphisms on \( \mathfrak{sl}(2, \mathbb{C}) \)

The Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) is a basic object in Lie theory, and it is well known that it has a canonical basis

\[
h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

with their Lie bracket relations:

\[
[e_1, e_2] = h, \quad [h, e_1] = 2e_1, \quad [h, e_2] = -2e_2.
\]

By the Lie algebra automorphisms of \( \mathfrak{sl}(2, \mathbb{C}) \), there are many possible choices for this kind of basis, the following theorem implies more choices for them.
Theorem 2.1. Suppose that we have a matrix \( h' \in \mathfrak{sl}(2, \mathbb{C}) \) with \( \det(h) = -1 \), then there exists a \( 2 \times 2 \) invertible matrix \( A \), such that \( AhA^{-1} = h' \). Let \( Ae_1A^{-1} = e'_1 \), \( Ae_2A^{-1} = e'_2 \). Therefore, \( h' \), \( e'_1 \), \( e'_2 \) satisfy the relations (2.1) by replacing \( h \), \( e_1 \), \( e_2 \), respectively.

Proof. Suppose that the eigenvalues of \( h' \) are \( \lambda_1, \lambda_2 \). Since we have \( h' \in \mathfrak{sl}(2, \mathbb{C}) \) with \( \det(h) = -1 \), it follows that

\[
\lambda_1 + \lambda_2 = 0, \quad \lambda_1 \lambda_2 = -1,
\]

therefore, we have \( \{\lambda_1, \lambda_2\} = \{\pm 1\} \). So there is a \( 2 \times 2 \) invertible matrix \( A \), such that \( AhA^{-1} = h' \). Easily, we can check that the conjugation map,

\[
\sigma_A : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{sl}(2, \mathbb{C}), \quad x \mapsto AxA^{-1}
\]

is a Lie algebra automorphism on \( \mathfrak{sl}(2, \mathbb{C}) \). Hence the three elements \( h' \), \( e'_1 \), \( e'_2 \) satisfy the relations (2.1) by replacing \( h \), \( e_1 \), \( e_2 \), respectively. \( \square \)

3 Hu-Zhang’s conjecture by conjugation automorphisms

Remark 3.1. It is well known that \( \mathfrak{sl}(2, \mathbb{C}) \) has an irreducible representation \( \phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(V) \), where \( V \) is a complex vector space of dimension \( m + 1 \) with maximal eigenvalue (highest weight) \( m \) for \( h \) and \( m \geq 0 \). And the linear space \( V \) has a basis \( v_0, v_1, \ldots, v_m \), with \( \phi(h)(v_i) = (m - 2i)v_i \). When we replace \( h \) by \( h' \) in Theorem 2.1, the matrix \( \phi(h') \) still has eigenvalues \(-m, -m + 2, \ldots, m - 2, m\), since the Lie algebra can be defined by generators and relations([10, Chapter 18]).

Definition 3.2. The polynomial

\[
f_\phi(z_0, z_1, z_2, z_3) = \det(z_0I + z_1\phi(h) + z_2\phi(e_1) + z_3\phi(e_2))
\]

is called the characteristic polynomial of \( \mathfrak{sl}(2, \mathbb{C}) \) on the representation \( \phi \), where \( \varphi \) is a finite representation of \( \mathfrak{sl}(2, \mathbb{C}) \).

In [9], Hu and Zhang present a conjecture that

\[
f_m(z_0, z_1) = f_\phi(z_0, z_1, 1, 1) = \prod_{i=1}^{m} \left(z_0 - (m - 2i)\sqrt{z_1^2 + 1}\right).
\]

The formula plays an important role in showing (1.1) in [2]. It is proved by showing the eigenvalues and eigenvectors directly in [2], or by computing the eigenvalues of tridiagonal matrices in [7]. Here we present a proof by avoiding these computations.
Proof. Let
\[ h' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e_1 + e_2. \]

Then we can see that \( h' \) satisfies the condition in Theorem 2.1, then we can find the \( e'_1 \) and \( e'_2 \) for Theorem 2.1 where
\[ e'_1 = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}, \quad e'_2 = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix}. \]

So \( h = e'_1 + e'_2 \), and \( z_1 h + e_1 + e_2 = z_1(e'_1 + e'_2) + h' \). Up to some base change on \( V \), \( h' \), \( e'_1 \), and \( e'_2 \) have the same matrices as \( h \), \( e_1 \), and \( e_2 \), respectively. Therefore, it follows that
\[ f_\phi(z_0, z_1, z_2, z_3) = \det(z_0 I + z_1 \phi(h') + z_2 \phi(e'_1) + z_3 \phi(e'_2)) \quad (3.1) \]

Hence we have
\[
\begin{align*}
f_m(z_0, z_1) &= f_\phi(z_0, z_1, 1, 1) \\
&= \det(z_0 I + z_1 \phi(h) + \phi(e_1) + \phi(e_2)) \\
&= \det(z_0 I + z_1 \phi(h') + \phi(e'_1) + \phi(e'_2)) \\
&= \det(z_0 I + z_1 \phi(e_1 + e_2) + \phi(h)) \\
&= \det(z_0 I + z_1 \phi(e_1) + z_1 \phi(e_2) + \phi(h)) \\
&= f_\phi(z_0, 1, z_1, z_1) \\
&= \det(z_0 I + \phi(h + z_1(e_1 + e_2))).
\end{align*}
\]

We have
\[ h + z_1(e_1 + e_2) = \begin{pmatrix} 1 & z_1 \\ z_1 & -1 \end{pmatrix}. \]

It follows that \( \det(h + z_1(e_1 + e_2)) = -1 - z_1^2 \). When \( z_1^2 \neq -1 \), we can check that
\[ h'' = \frac{h + z_1(e_1 + e_2)}{\sqrt{1 + z_1^2}} \]
satisfies the Theorem 2.1. Therefore, by Theorem 2.1 and Remark 3.1, we can have that \( \phi(h'') \) has eigenvalues \(-m, -m+2, \ldots, m\), which implies that \( h + z_1(e_1 + e_2) \) has eigenvalues \(-\sqrt{1 + z_1^2}m, -\sqrt{1 + z_1^2}(m-2), \ldots, \sqrt{1 + z_1^2}m\). Hence when we consider the \( z_0 \) and \( z_1 \) are complex numbers, and \( z_1^2 \neq -1 \), it follows that
\[ f_m(z_0, z_1) = \prod_{i=0}^{m} \left( z_0 + (m - 2i) \sqrt{1 + z_1^2} \right), \]
or
\[ f_m(z_0, z_1) = \begin{cases} z_0 \prod_{l=1}^{m/2} (z_0^2 - 4l^2(1 + z_1^2)) & 2 \mid m. \\ \prod_{l=0}^{(m-1)/2} (z_0^2 - (2l + 1)^2(1 + z_1^2)) & 2 \not\mid m. \end{cases} \] (3.2)

It is known that \( C^2 \setminus \{ z_1^2 = -1 \} \) is dense in \( C^2 \), so the conjecture holds by the continuity of polynomial functions.

Remark 3.3. In the proof, we can obtain the conclusion without so many deformations, by considering the eigenvalues of
\[ z_1 h + (e_1 + e_2) = \begin{pmatrix} z_1 & 1 \\ 1 & -z_1 \end{pmatrix}. \]
When we apply the formula (3.1), we can get one formula
\[ f_\phi(z_0, z_1, 1, 1) = f_\phi(z_0, 1, z_1, z_1), \]
which can show more symmetric relations among the variables \( z_1, z_2 \) and \( z_3 \).

4 Characteristic polynomials of \( \mathfrak{sl}(2, \mathbb{C}) \)

In \cite{2}, the authors obtain the \( f_\phi(z_0, z_1, z_2, z_3) \) by proving the Hu’s conjecture. Now we consider a \( \mathfrak{sl}(2, \mathbb{C}) \) representation \( \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V) \), where \( V \) is a finite dimension complex vector space. It is known that \( \phi(h) \) is semisimple (diagonalizable) with integer eigenvalues. For each \( n \in \mathbb{Z} \), we use \( d_{n,\phi} \) denote the dimension of eigenvector space \( \phi(h) \) of the eigenvalue \( n \), namely
\[ d_{n,\phi} = \dim \{ v \in V \mid \phi(h)v = nv \}. \]

For the representation \( \phi \), the following theorem holds.

Theorem 4.1. Let \( \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V) \) be a representation of \( \mathfrak{sl}(2, \mathbb{C}) \), then the characteristic polynomial
\[ f_{\phi}(z_0, z_1, z_2, z_3) = z_0^{d_{1,\phi}} \prod_{n \geq 1} (z_0^2 - n^2(z_1^2 + z_2z_3))^{d_{n,\phi}}. \] (4.1)

Proof. By (1.1), it is easy to verify that the formula (4.1) holds for the finite dimensional irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \). By the semisimplicity of \( \mathfrak{sl}(2, \mathbb{C}) \), then we have the decomposition
\[ \phi \simeq \bigoplus_{t=1}^s \phi_t, \]
Where each $\phi_t$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. By the definition of characteristic polynomial, we have

$$f_{\phi}(z_0, z_1, z_2, z_3) = \prod_{t=1}^{s} f_{\phi_t}(z_0, z_1, z_2, z_3).$$

Furthermore, we have $d_{n,\phi} = \sum_{t=1}^{s} d_{n,\phi_t}$. It follows that

$$f_{\phi}(z_0, z_1, z_2, z_3) = \prod_{t=1}^{s} f_{\phi_t}(z_0, z_1, z_2, z_3) = \prod_{t=1}^{s} \left( z_0^{d_{0,\phi_t}} \prod_{n \geq 1} (z_0^2 - n^2(z_1^2 + z_2z_3))^{d_{n,\phi_t}} \right) = z_0^{d_{0,\phi}} \prod_{n \geq 1} (z_0^2 - n^2(z_1^2 + z_2z_3))^{d_{n,\phi}}.$$

On the other direction, the structure of a finite dimensional $\mathfrak{sl}(2, \mathbb{C})$-module is totally decided by its characteristic polynomial.

**Theorem 4.2.** Two finite dimensional representations $\phi$ and $\psi$ of $\mathfrak{sl}(2, \mathbb{C})$ are isomorphic if and only if they have the same characteristic polynomial.

**Proof.** The necessity is obvious. Suppose that the characteristic polynomial $f_{\phi}(z_0, z_1, z_2, z_3)$ is fixed, we will show that the representation $\phi$ can be reconstructed in a unique way. Because the polynomial ring $\mathbb{C}[z_0, z_1, z_2, z_3]$ is a unique factorization ring, by Theorem 4.1 we can suppose that

$$f_{\phi}(z_0, z_1, z_2, z_3) = z_0^{d_0} \prod_{n=1}^{N} (z_0^2 - n^2(z_1^2 + z_2z_3))^{d_n}, \quad (4.2)$$

with $d_N \geq 1$. Let $\varphi_m$ be the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $m + 1$ with highest weight $m$, for $m \geq 0$. Suppose that

$$\phi \simeq \bigoplus_{m \geq 0} l_{m} \varphi_m,$$

where $l_m$ is the multiplicity of $\varphi_m$ in $\phi$. By the equation (4.2), we see that

$$\phi \simeq \bigoplus_{m=0}^{N} l_{m} \varphi_m.$$
For Characteristic polynomials, it is known that

\[ f_\phi = \prod_{m=0}^{N} f^{l_m}_{\phi_m}. \]

Therefore by the equation (4.2), we see that

\[ l_N = d_N. \]

Therefore

\[ f_{l_N-1}^{m} = f^{l_m}_{\phi_m}, \]

thus the \( l_m \) for \( m \leq N - 1 \) can be decided by induction.

From the algorithm in the Proof of Theorem 4.2, it is easy to obtain the following description for the characteristic polynomials of all finite dimensional representations of \( \mathfrak{sl}(2, \mathbb{C}) \).

\[ \text{Corollary 4.3.} \quad \text{A polynomial} \quad f(z_0, z_1, z_2, z_3) \in \mathbb{C}[z_0, z_1, z_2, z_3] \quad \text{is a characteristic polynomial of a finite dimensional representation of} \quad \mathfrak{sl}(2, \mathbb{C}) \quad \text{if and only if} \]

\[ f_{\phi}(z_0, z_1, z_2, z_3) = z_0^d_0 \prod_{n=1}^{N} (z_0^2 - n^2(z_1^2 + z_2 z_3))^{d_n} \]

with \( d_n \geq d_{n+2} \), for any \( n \in \mathbb{N} \).

\[ \text{5} \quad \text{A monoid structure on} \quad \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \]

We denote characteristic polynomials of all finite dimensional representations of \( \mathfrak{sl}(2, \mathbb{C}) \) by \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) in the Section 4, which is a subset of \( \mathbb{C}[z_0, z_1, z_2, z_3] \). In this section, a monoid structure on \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) will be defined.

\[ \text{Definition 5.1.} \quad \text{Let} \quad f_\phi, \ f_\psi \quad \text{be two polynomials in} \quad \text{CP}_{\mathfrak{sl}(2, \mathbb{C})}. \quad \text{Considering} \]

\[ z_0 = z_0 - 0 \sqrt{z_1^2 + z_2 z_3} \quad \text{in Theorem 4.2} \quad \text{for two representations} \quad \phi \quad \text{and} \quad \psi, \quad \text{we can write} \]

\[ f_\phi = \prod_{i=1}^{n} \left(z_0 + \alpha_i \sqrt{z_1^2 + z_2 z_3}\right), \]

\[ f_\psi = \prod_{j=1}^{m} \left(z_0 + \beta_j \sqrt{z_1^2 + z_2 z_3}\right), \]

with \( \alpha_i, \ i = 1, \ldots, n \) being all the eigenvalues of \( \phi(h) \) and \( \beta_j, \ j = 1, \ldots, m \) being all the eigenvalues of \( \psi(h) \). Define \( f_\phi \ast f_\psi \in \mathbb{C}[\sqrt{z_1^2 + z_2 z_3}, z_0, z_1, z_2, z_3] \) by

\[ f_\phi \ast f_\psi = \prod_{i,j=1}^{n,m} \left(z_0 + (\alpha_i + \beta_j) \sqrt{z_1^2 + z_2 z_3}\right) \quad (5.1) \]

and we call \( f_\phi \ast f_\psi \) the resolution product of \( f_\phi \) and \( f_\psi \).
Proposition 5.2. Let \( f_\phi, f_\psi \) be two polynomials in \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) as in the Definition 5.1. It follows that
\[
f_\phi \ast f_\psi = f_{\phi \otimes \psi}.
\]

Proof. Suppose that \( \{v_i\}_{i=1}^n \) is a basis of representation \( \phi \) such that
\[
\phi(h)(v_i) = \alpha_i v_i,
\]
and \( \{w_j\}_{j=1}^m \) is a basis of representation \( \psi \) such that
\[
\psi(h)(w_j) = \beta_j w_j.
\]
Let \( \lambda_{ni} = \alpha_i \) and \( \lambda_{wj} = \beta_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). It is known that \( \{v_i \otimes w_j\}_{i=1,j=1}^{n,m} \) is a basis of \( \phi \otimes \psi \). Furthermore, we have
\[
\phi \otimes \psi(h)(v_i \otimes w_j) = \phi(h)(v_i) \otimes \psi(h)(w_j) = (\alpha_i + \beta_j)(v_i \otimes w_j).
\]
Let \( \lambda_{ni\otimes wj} = \alpha_i + \beta_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). By \( \lambda_{ni}, \lambda_{wj} \) and \( \lambda_{ni\otimes wj} \), we can rewrite the \( f_\phi, f_\psi \) and \( f_{\phi \otimes \psi} \) as the follows,
\[
f_\phi = \prod_{i=1}^n \left(z_0 + \lambda_{ni} \sqrt{z_1^2 + z_2 z_3}\right),
\]
\[
f_\psi = \prod_{j=1}^m \left(z_0 + \lambda_{wj} \sqrt{z_1^2 + z_2 z_3}\right),
\]
\[
f_{\phi \otimes \psi} = \prod_{i,j=1}^{n,m} \left(z_0 + \lambda_{ni\otimes wj} \sqrt{z_1^2 + z_2 z_3}\right)
\]
\[
= \prod_{i,j=1}^{n,m} \left(z_0 + (\alpha_i + \beta_j) \sqrt{z_1^2 + z_2 z_3}\right) = f_\phi \ast f_\psi.
\]
\[\square\]

Theorem 5.3. The set \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) is a commutative monoid under the resolution product with the unit element \( z_0 \).

Proof. By Proposition 5.2, the set \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) is closed under the resolution product. For three representations \( \phi, \psi, \varphi \) of \( \mathfrak{sl}(2, \mathbb{C}) \), we have \( \phi \otimes (\psi \otimes \varphi) \simeq (\phi \otimes \psi) \otimes \varphi \). By Proposition 5.2, it follows that
\[
f_\phi \ast (f_\psi \ast f_\varphi) = f_{\phi \otimes (\psi \otimes \varphi)} = f_{(\phi \otimes \psi) \otimes \varphi} = (f_\phi \ast f_\psi) \ast f_\varphi.
\]
Let \( \varphi_0 \) denote one dimensional trivial representation of \( \mathfrak{sl}(2, \mathbb{C}) \), then \( f_{\varphi_0} = z_0 \). For each representation \( \phi \) of \( \mathfrak{sl}(2, \mathbb{C}) \), we have \( \phi \simeq \phi \otimes \varphi_0 \simeq \varphi_0 \otimes \phi \), so it follows that \( f_{\phi} = f_{\phi} \ast z_0 = z_0 \ast f_{\phi} \). Similarly, the equation \( f_{\phi} \ast f_\psi = f_\psi \ast f_{\phi} \) holds for \( \phi \otimes \psi \simeq \psi \otimes \phi \). \[\square\]
Let \( \varphi_m \) be the irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \) of dimension \( m + 1 \) with highest weight \( m \), for \( m \geq 0 \). For \( 0 \leq n \leq m \), it is well known that

\[
\varphi_m \otimes \varphi_n \cong \bigoplus_{k=0}^{n} \varphi_{m-n+2k}.
\]

By Proposition 5.2, the corollary below holds.

**Corollary 5.4.** For \( 0 \leq n \leq m \), we have

\[
f_{\varphi_m} \ast f_{\varphi_n} = \prod_{k=0}^{n} f_{\varphi_{m-n+2k}}.
\]

**Remark 5.5.** Under the tensor product, the category of finitely dimensional representations of \( \mathfrak{sl}(2, \mathbb{C}) \) is a monoidal category, hence it is natural that the set \( \text{CP}_{\mathfrak{sl}(2, \mathbb{C})} \) should have a monoid structure. Considering that the \( \det(A \otimes I_m - I_n \otimes B) \) is the resolution of characteristic polynomials of \( A \) and \( B \), where \( A \) is a \( n \times n \) matrix, and \( B \) is a \( m \times m \) matrix, we name the product resolution product. There are some definitions of resolution or determinant of multivariable polynomials in algebraic geometry, so there might be some necessary research for their relations from the view of algebraic geometry.

## 6 The adjoint representation of \( \mathfrak{sl}(2, \mathbb{C}) \) on \( \mathfrak{sl}(n, \mathbb{C}) \)

Let us recall the canonical basis of \( \mathfrak{sl}(n, \mathbb{C}) \) first. Suppose that \( e_{ij} \) is the complex \( n \times n \) matrix, with entry 1 at the \( i \)th row and \( j \)th column and 0 otherwise. The \( \mathfrak{sl}(n, \mathbb{C}) \) is the simple Lie algebra of type \( A_{n-1} \) with a basis \( h_i = e_{ii} - e_{i+1,i+1} \) for \( 1 \leq i \leq n-1 \), and \( e_{ij} \) for \( 1 \leq i \neq j \leq n \).

**Theorem 6.1.** Let \( \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C}) \) be the Lie algebra homomorphism defined by \( \phi(h) = h_1 \), \( \phi(e_1) = e_{12} \), \( \phi(e_2) = e_{21} \). Suppose that \( \text{ad} \circ \phi \) be the composition of \( \phi \) and the adjoint representation \( \text{ad} \) of \( \mathfrak{sl}(n, \mathbb{C}) \). Then

\[
f_{\text{ad} \circ \phi}(z_0, z_1, z_2, z_3) = z_0^{n^2-5n+6}(z_0^2 - (z_0^2 + z_2 z_3))^{2n-4}(z_0^2 - 4(z_1^2 + z_2 z_3)) \quad (6.1)
\]

**Proof.** By Theorem 4.1 we need to compute the eigenvalues of \( \text{ad} h_1 \) and their multiplicities.

We recall the root system \( \Phi \) of type \( A_{n-1} \). The set \( \Phi \) can be realized in \( \mathbb{R}^n \), with \( \Phi = \{ \epsilon_j - \epsilon_i \mid 1 \leq i \neq j \leq n \} \) with \( \{ \epsilon_i \}_{i=1}^{n} \) being the canonical orthonormal basis of \( \mathbb{R}^n \). For \( \text{ad} h_1 \), we have

\[
[h_1, h_i] = 0, \quad \text{for} \quad 1 \leq i \leq n-1
\]

\[
[h_1, e_{ij}] = (\epsilon_j - \epsilon_i, \epsilon_2 - \epsilon_1) e_{ij} \quad \text{for} \quad 1 \leq i \neq j \leq n,
\]

where \( (\epsilon_j - \epsilon_i, \epsilon_2 - \epsilon_1) \) is the canonical inner product of \( \epsilon_j - \epsilon_i \) and \( \epsilon_2 - \epsilon_1 \). Hence we get the eigenvalues of \( \text{ad} h_1 \) are \( 2, -2, 1, -1 \) and 0 with their multiplicities \( 1, 1, 2n-4, 2n-4 \) and \( n^2 - 5n + 6 \), respectively. Therefore, the theorem holds for Theorem 4.1. \( \square \)
In [11], the characteristic polynomial of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is defined as $Q_{\mathfrak{sl}(n, \mathbb{C})}(z)$, the determinant of the linear pencil $z_0I + z_1 \text{ad}h_1 + \cdots + z_{n-1} \text{ad}h_{n-1} + z_{12} \text{ad}e_{12} + \cdots + z_{21} \text{ad}e_{21} + \cdots + z_{n-1,n} \text{ad}e_{n,n-1}$. By the Theorem 6.1 the following holds.

**Corollary 6.2.** For the adjoint representation of $\mathfrak{sl}(n, \mathbb{C})$, Let

$$f_i(z_0, z_i, z_{i,i+1}, z_{i+1,i}) = \det(z_0I + z_i \text{ad}h_i + z_{i,i+1} \text{ad}e_{i,i+1} + z_{i+1,i} \text{ad}e_{i+1,i}).$$

Then it follows that

$$f_i(z_0, z_i, z_{i,i+1}, z_{i+1,i}) = f_{\text{ad}\circ\phi}(z_0, z_i, z_{i,i+1}, z_{i+1,i}).$$

**Proof.** We can apply the analogous argument by constructing a Lie algebra isomorphism $\phi_i$ from $\mathfrak{sl}(2, \mathbb{C})$ to the subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ generated by $h_i, e_{i,i+1}$ and $e_{i+1,i}$. By [11, Theorem 2.3], there is another way to prove this. From [11, Theorem 2.3], it is known that the characteristic polynomial $Q_{\mathfrak{sl}(n, \mathbb{C})}(z)$ is invariant under the Lie algebra automorphism. In the Lie algebra, the image of $\phi_i$ is corresponding to the simple root $\alpha_i = \epsilon_{i+1} - \epsilon_i$ as in the proof of Theorem 6.1, namely, $\phi(\mathfrak{sl}(2, \mathbb{C})) = \mathbb{C} \epsilon_{\alpha_i} + \mathbb{C} \epsilon_{-\alpha_i}$. Under the action Weyl group $W$ of type $A_{n-1}$ associated to $\mathfrak{sl}(n, \mathbb{C})$, all these roots are on the same orbit of $W$. So it implies that there exists a $\sigma \in W$, such that $\sigma(\alpha_i) = \alpha_j$ for $1 \leq i, j \leq n-1$. It can be interpreted as a Lie algebra automorphism of $\mathfrak{sl}(n, \mathbb{C})$ such that

$$\sigma(h_\alpha) = h_{\sigma(\alpha)}, \quad \sigma(e_\alpha) = e_{\sigma(\alpha)}.$$

Therefore, the conclusion follows just by considering the case in Theorem 6.1.

**Remark 6.3.** From [11, Theorem 2.3] and the proof of the Corollary 6.2, for a complex simple Lie algebra $L$, its characteristic polynomial is invariant under its Weyl group, hence the invariant theory of Weyl groups might be helpful to compute the characteristic polynomial.

References

[1] I. Chagouel, M. Stessin, K. Zhu, Geomtric spectral theory for compact operators, Trans. AMS, 368(2016), No.3, 1559–1582.

[2] Z. Chen, X. Chen, M. Ding, On the characteristic polynomial of $\mathfrak{sl}(2, \mathbb{C})$, 579(2019), 237-243.
[3] R. Dedekind, Gesammelte Mathematische Werke, Vol. II, Chelsea, New York, 1969.

[4] F. Frobenius, über vertauschbare Matrizen, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 1896, 601–614.

[5] R. Grigorchuk, Y. Yang, Joint spectrum and the infinite dihedral groups, Proc. Steklov Inst. Math. 297(1) (2017), 145–178.

[6] W. He, R. Yang, Projective spectrums and kernel bundle, Sci. China Math. 57(2014), 1–10.

[7] Z. Hu, Eigenvalues and eigenvectors of a class of irreducible, tridiagonal matrices, 619(2021), 328–337.

[8] Z. Hu, R. Yang, On the characteristic polynomials of multiparameter pencils, Linear Algebra Appl. 558 (2018), 250–263.

[9] Z. Hu, P. Zhang, Determinant and characteristic polynomials of Lie algebras, Linear Algebra Appl. 563 (2019), 426–439.

[10] J. Humphreys, Introduction to Lie algebras and representation theory, Spring-verlag, New York-Berlin, 1972.

[11] F. A. Key, R. Yang, Spectral invariants for finite dimensional Lie algebras, Linear Algebra Appl. 611 (2021), 148–170.

[12] R. Yang, Projective spectrum in Banach algebras, J. Topo. Anal. 1(2009), No.3, 289–306.

Tianyi Jiang
Email: jtyoo2021@163.com
School of Mathematics, Shandong University
Shanda Nanlu 27, Jinan,
Shandong Province, China
Postcode: 250100

Shoumin Liu
Email: s.liu@sdu.edu.cn
School of Mathematics, Shandong University
Shanda Nanlu 27, Jinan,
Shandong Province, China
Postcode: 250100