Quantum mechanical description of measurement and the basic properties of state transformation

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In this paper, without any priori assumption about the post-measurement state of system, we will examine how this state is restricted by assuming each of these following assumptions. First, by using this reasonable assumption that two successive measurements should be describable as one measurement. Second, by assuming the impossibility of faster than light signaling, ”No-signaling condition”. However, only by using these assumptions it is not possible to obtain the usual projection postulate. Instead, by means of a simple lemma, we will show that the density operator of system after a measurement is a linear function of the density operator determined by the usual post-measurement postulate. Furthermore we will show this linear function has a Kraus representation. Finally, we will discuss about the physical meaning of this consequence.

I. INTRODUCTION

A measurement on a system, most generally, is a process which produces different macroscopic outcomes; the only restriction is that initially measuring apparatus should have no correlations with the system being measured which can affect the outcomes. Any process with this property can be regarded as one measurement. Quantum mechanics claims that for every measurement there exists a set of positive operators like \{F_\mu\}, \sum_\mu F_\mu = I, such that the probability of occurrence of outcome \mu is calculated by the trace rule i.e. \( p(\mu) = tr(\rho F_\mu) \), where \( \rho \) is the density operator of the system under consideration \(^1\). So, to calculate the probabilities we do not need to know anything more about the underlying mechanism of measurement.

On the other hand, a measurement in itself always contains different time evolutions which finally correlate the state of the quantum system with the value of a macroscopic classical variable. Hence the need for consistency of quantum mechanical description of measurement with the time evolution rules may impose some constraints on the possible time evolutions. Usually to derive basic properties of quantum dynamics one considers this reasonable assumption that two successive time evolution should be describable as one time evolution \(^2\). Now similarly, according to the concept of measurement, it seems reasonable to assume a process including a time evolution which is followed by a measurement, as one measurement; so the total process should be describable as one measurement.

Let us first see how this property can be deduced from linearity of time evolution. Suppose the state of a system after time evolution is described by \( E(\rho) \) where \( E \) is a linear, positive and trace preserving map. Generally, every linear map on the space of linear operators can be represented by

\[
E(\rho) = \sum_i N_i \rho M_i, \tag{1}
\]

where \( N_i \) and \( M_i \) are suitable operators. For trace preserving maps we have \( \sum_i M_i N_i = I \). Suppose after this time evolution we perform a measurement described by the set \{F_\mu\} such that \( \sum_\mu F_\mu = I \). So the probability of outcome \( \mu \) is

\[
p(\mu) = tr(\sum_i N_\mu M_i M_\mu) = tr(\rho \sum_i M_i F_\mu N_i). \tag{2}
\]

Let \( F'_\mu \) be

\[
F'_\mu = \sum_i M_i F_\mu N_i. \tag{3}
\]

For an arbitrary \( \rho \) the probability \( p(\mu) \) is positive; so according to Eq. (2), \( F'_\mu \) is a positive operator; also it is clear that \( \sum_\mu F'_\mu = \sum_\mu F_\mu = I \) and \( p(\mu) = tr(\rho F'_\mu) \). The probability of different outcomes in this process are thus the same with a new measurement which is described by the set \{F'_\mu\}. For completeness, to show the whole process is equivalent to one new measurement, we should show that the state of system after the whole process is the same with the post-measurement state of this new measurement. We postpone this work until section (III).

Now imagine another process which includes two successive measurements on the system. Someone can regard the outcomes of the first measurement and the second ones together as the outcomes of one new measurement \(^3\); so it seems reasonable assumption that the total process should be describable as one measurement. It is straightforward to see that because of the usual post-measurement rule this property holds in quantum mechanics \(^1\).

In this manner possible state transformations, including changes by time evolutions or imposed by measurements are such that these reasonable properties hold, without any problem in definition of the measurement process. On the other hand, for consistency of these reasonable requirements with the trace rule, the possible state transformations should satisfy some constraints. For example, if cloning was possible, by cloning the state of system and performing measurement on the copies, one could perform measurements not obeying the trace rule. But the impossibility of cloning is a consequence of the linearity of time evolution \(^5\). In the present letter we will investigate these necessary constraints on the possible state transformation of system.

It can be easily shown that nonlinear modifications of time evolution, as given in \(^6\), leads to the faster than light communication \(^7\). Furthermore assuming the impossibility of
faster than light signaling, "no-signaling condition", one can obtain the basic properties of time evolutions, namely linearity and complete positivity [8, 9]. Following this argument, we will also see that the post-measurement state rule can be obtained using the "no-signaling condition."

Before following these ideas we should remark an important point about the effects of initial correlations with the environment. In the absence of initial correlations density operator of a system after time evolution is a function of its present density operator [10]. But in general the evolution of an open system may be affected by its initial correlations with environment like entanglement between the system and its environment or dependency of the state of environment to the state of system. In this situation the density operator of system after evolution is not necessarily a function of its present density operator. [11] contains an example of these cases which is corrected in [12]. As a simple example suppose the interaction between a system and its environment is governed by a unitary like $U$ such that

$$U |\psi1\rangle = |1\rangle_{sys}|0\rangle_{env}, \quad U |\psi2\rangle = |0\rangle_{sys}|0\rangle_{env},$$

where

$$|\psi1\rangle = \frac{|1\rangle_{sys}|0\rangle_{env} + |0\rangle_{sys}|1\rangle_{env}}{\sqrt{2}}$$

and

$$|\psi2\rangle = \frac{|1\rangle_{sys}|0\rangle_{env} - |0\rangle_{sys}|1\rangle_{env}}{\sqrt{2}}$$

For either of these states, the density operator of the system is

$$\rho_{sys} = \frac{|0\rangle_{sys}\langle 0| + |1\rangle_{sys}\langle 1|}{2}$$

and the density operator of environment is

$$\rho_{env} = \frac{|0\rangle_{env}\langle 0| + |1\rangle_{env}\langle 1|}{2}.$$ 

After the evolution, density operator of the system in two cases are different. In the case of $|\psi1\rangle$ density operator evolves to $|0\rangle_{sys}\langle 0|$, but in the case of $|\psi2\rangle$ it evolves to $|1\rangle_{sys}\langle 1|$. Initially $\rho_{sys}$ and $\rho_{env}$ in two cases is the same, and the only difference is the difference between correlations. Hence it is clear that the effects of initial correlation should arise in our argument; Furthermore we can deduce that the linearity of time evolution and other dynamical characteristics of quantum mechanics are not the immediate consequence of its statics rules; Indeed they holds only in a special condition.

**II. TIME EVOLUTION**

Imagine a time evolution of a system which may include interactions with the environment. After this evolution we perform a measurement on the system. As it has been already mentioned, we should be able to assume the whole process as one measurement. When we perform a measurement on the system the measurement apparatus should have no such initial correlations correlations which can produce observable effects. This is a necessary condition in every measurement. Hence to regard the whole process as one measurement, it is necessary that in the initial time evolution system have no effective initial correlations with the environment; because in this picture the environment can be regarded as a part of measuring apparatus. In this manner this necessary restriction on the time evolutions arise in our argument in a natural way.

Suppose with this time evolution $|\psi\rangle \rightarrow |\phi\rangle$ evolves to $E(|\psi\rangle|\psi\rangle)$. A priori we make no assumption about the dynamics of pure states, for example it might be described by a nonlinear equation. Let initially system be in $|\psi_i\rangle$ with the probability $p_i$, so $\rho_{in} = \sum_i p_i |\psi_i\rangle\langle \psi_i|$, is the initial density operator of the ensemble under consideration. After evolution system is in $E(|\psi_i\rangle\langle \psi_i|)$ with the probability $p_i$, thus it is described by $\rho'_i = \sum_i p_i E(|\psi_i\rangle\langle \psi_i|)$. Then we perform a measurement on the system which is described by the set $\{F_\mu\}$. So the probability of occurrence of outcome $\mu$ is $\sum_i p_i tr(E(|\psi_i\rangle\langle \psi_i|)F_\mu)$. But there should exist another set of positive operators like $\{F'_\mu\}$, $\sum_\mu F'_\mu = 1$, which describes the whole process as one measurement such that $p(\mu) = tr(\rho_{in} F'_\mu)$. Hence

$$tr(\rho'_1 F_\mu) = tr(\rho_{in} F'_\mu). \quad (4)$$

Consider another ensemble of pure states described by $\rho_{in}$, in which system is in $|\phi\rangle\langle \phi|$ with the probability of $q_i$ such that $\sum_i q_i |\phi_i\rangle\langle \phi_i| = \rho_{in}$. After the time evolution this system is in $E(|\phi\rangle\langle \phi|)$ with the probability $q_i$, so after evolution this system is described by $\rho'_2 = \sum_i q_i E(|\phi\rangle\langle \phi|)$. Hence according to Eq. 4 the probability of occurrence of outcome $\mu$ is

$$tr(\rho'_2 F_\mu) = tr(\rho_{in} F'_\mu). \quad (5)$$

Comparing with Eq. 4 shows

$$tr(\rho'_1 F_\mu) = tr(\rho'_2 F_\mu). \quad (6)$$

This equality should hold for any positive operator like $F_\mu$, so we can deduce $\rho'_1 = \rho'_2$. Hence two systems which are initially described by the same density operator, after evolution still have the same density operator. Thus density operator of the system after evolution, $\rho'$, can be expressed as a function of its present density operator, $\rho_{in}$. We represent this fact by $\rho' = E(\rho_{in})$. So we have

$$tr(E(\rho)F_\mu) = tr(\rho_{in} F'_\mu). \quad (7)$$

Hence

$$tr(E(p_1 \rho_1 + p_2 \rho_2) F_\mu) = tr((p_1 \rho_1 + p_2 \rho_2) F'_\mu). \quad (8)$$

But we know

$$tr(E(\rho_1) F_\mu) = tr(\rho_1 F'_\mu), \quad tr(E(\rho_2) F_\mu) = tr(\rho_2 F'_\mu).$$

So

$$tr(E(p_1 \rho_1 + p_2 \rho_2) F_\mu) = tr(p_1 E(\rho_1) + p_2 E(\rho_2) F_\mu). \quad (9)$$
Because this equality should holds for any positive operator like \( F_\mu \) we can deduce

\[
E(p_1 \rho_1 + p_2 \rho_2) = p_1 E(\rho_1) + p_2 E(\rho_2).
\] (10)

Thus we conclude that time evolution is linear.

Note that we have made no specific assumption about the time evolution, except about initial correlations.

With a similar argument which have been used in [5], we will show the complete positivity of time evolution. First note that positivity and linearity does not imply complete positivity. For example the map of \( \rho \) to \( \rho^T \) is a positive and linear map but it is not completely positive [10]. In fact this evolution can be implemented by a suitable interaction and initial correlations between our system and its environment [11]. For proving complete positivity we again need to assume that there exists no effective initial correlation between our system and its environment. Suppose there exists an imaginary ancillary system which has no interaction with the outside. We can use our argument for the composite system, because it satisfies the necessary assumptions; hence the evolution of the composite system should be linear. Also we can obtain this result for both subsystems, because each subsystem has no interaction with the another. Suppose the state of the composite system is a product state like \( \rho_{sys} \otimes \rho_{anc} \). It is obvious that the state of two system after evolution should still remain a product state. So it will be \( E_{sys} \otimes E_{anc}(\rho_{sys} \otimes \rho_{anc}) \). But any state can be expanded in terms of product states. Hence for any state of the composite system time evolution should be described by \( E_{sys} \otimes E_{anc} \). On the other hand the ancillary system has no interaction with the outside, thus in principle its state can remain constant. So, \( E_{sys} \otimes I_{anc} \) is a legitimate evolution of the composite system. Therefore for an arbitrary density operator of the composite system, \( \rho_{total} \), \( E_{sys} \otimes I_{anc}(\rho_{total}) \) is a positive operator, whatever the dimension of the ancillary system. So, \( E_{sys} \) is a completely positive map and has Kraus representation. On the other hand, a linear, trace preserving and completely positive map always can be realized quantum mechanically, with a unitary time evolution on a larger Hilbert space [12]. So we have obtained all the theoretical restrictions on an arbitrary time evolution of system.

III. POST-MEASUREMENT STATE

Now by a similar argument like previous part we will derive the post-measurement state rule. Suppose by performing a measurement which is described by the set \( \{ F_\mu \} \), we observe result \( \mu \) with the probability \( tr(\rho F_\mu) \) and the system jumps to the state \( \rho_\mu \). A priori we make no assumption about this state; so it may be different for different ensembles of pure states which are described initially by the same density operator. Now performing another measurement which is described by the set \( \{ G_\nu \} \), we obtain result \( \nu \) with the probability \( tr(\rho_\mu G_\nu) \). So the probability of obtaining \( \mu \) in the first measurement and \( \nu \) in the second one is

\[
p(\mu, \nu) = tr(\rho F_\mu) \times tr(\rho_\mu G_\nu).
\] (11)

On the other hand, as it has been assumed, we can regard the total process as one measurement with outcomes \( (\mu, \nu) \). So there should exist a set of positive operators like \( \{ \Omega_{\mu \nu} \} \), \( \sum_{\mu \nu} \Omega_{\mu \nu} = I \), such that \( p(\mu, \nu) = tr(\rho \Omega_{\mu \nu}) \). So we can deduce

\[
tr(\rho \Omega_{\mu \nu}) = tr(\rho F_\mu) \times tr(\rho_\mu G_\nu).
\] (12)

Comparing this equation with Eq. (11), by the same argument which we have used in that case, we can show that \( B_\mu (\rho) = \rho_\mu tr(\rho F_\mu) \) and \( \rho_\mu \) should be the same for different ensemble described by one \( \rho \) and moreover \( B_\mu (\rho) \) should be a linear function of \( \rho \). Also by definition it is clear that

\[
tr(B_\mu (\rho)) = tr(\rho F_\mu).
\] (13)

In the appendix we will show that always there exists a linear, positive, trace preserving map like \( E_\mu \) such that

\[
B_\mu (\rho) = E_\mu(\sqrt{\rho F_\mu} \sqrt{\rho F_\mu}).
\] (14)

Hence according to the definition of \( B_\mu \) the state of system after obtaining result \( \mu \) is

\[
A_\mu (\rho) = E_\mu(\rho_\mu \sqrt{\rho F_\mu}) / tr(\rho F_\mu).
\] (15)

By a similar argument which we have used in the previous part, we can see that in the absence of effective initial correlations, if one consider an imaginary ancillary system, \( B_\mu \) should be replaced with \( B_\mu \otimes I_{anc} \). So \( B_\mu \) should be completely positive and thus, as we will see in the appendix, we can always choose \( E_\mu \) to be completely positive. Therefore \( E_\mu \), which is a linear, trace preserving and completely positive map, can be regarded as a time evolution of system for outcome \( \mu \). Regarding the concept of measurement this result seems reasonable, because in a measurement different outcomes may have different time evolutions. As we will see in the appendix, we can not specify \( E_\mu \) uniquely; but regarding Eq. (15), this ambiguity has no physical meaning. The necessity of quantum collapse can simply be deduced from this post-measurement rule.

We call a measurement in which all \( E_\mu \) are the identity maps an ideal measurement. Therefore we have shown that every real measurement is equivalent to an ideal measurement which is followed by different time evolutions for different outcomes. So theoretically, all different manners of measuring one physical property, are equivalent to the same ideal measurement followed by different time evolutions; these time evolution are dependent to the special manner of measuring that physical property. Note that in a real measurement this two parts, ideal measurement and the following time evolution, may be inseparable. For example in a Stern-Gerlach measurement the outcome beams because of their different spins obtain different phases in the magnetic field, which is equivalent to a unitary time evolution. Actually, in more realistic model different outcomes may experience different magnetic fields such that they will be no longer described by mutually orthogonal state vectors. As a better example suppose we are going to measure the energy of an excited atom in the
following manner. Returning an excited electron to its ground state releases a photon; measuring the frequency of this photon, we can measure the initial energy of the excited atom. So we can regard this process as a measurement on the atom. After measurement, the state of the atom is independent of its initial state. So this process can be regarded as an ideal energy measurement which projects the state of the atom to the energy eigenstates followed by time evolutions which transform all the energy eigenstates to the ground state. Note that regarding the real process, in neither of these examples we cannot separate one part of process as an ideal measurement and one part as a time evolution; but, as we have shown, it is possible theoretically. 

As an important special case, suppose the measurement process is such that we can choose $E_\mu$ to be a unitary evolution specified by a unitary operator like $U_\mu$. In this situation Eq. (15) becomes

$$A_\mu(\rho) = \frac{U_\mu(\sqrt{F_\mu} \rho \sqrt{F_\mu}) U_\mu^\dagger}{\text{tr}(F_\mu)}.$$  

According to the polar decomposition theorem [5], any operator like $M_\mu$ can be decomposed to $M_\mu = V_\mu \sqrt{M_\mu^\dagger M_\mu}$, where $V_\mu$ is a unitary operator. So, for a given set of operators like $\{M_\mu\}$ which satisfies $\sum_\mu M_\mu^\dagger M_\mu = I$, we can choose in Eq. (16) $U_\mu$ equal to $V_\mu$ and $F_\mu$ equal to $M_\mu^\dagger M_\mu$. In this manner we can deduce that, this set of operators describes a measurement such that the probability of outcome $\mu$ in this measurement is $\text{tr}(\rho M_\mu^\dagger M_\mu)$ and the post-measurement state of system is

$$A_\mu(\rho) = \frac{M_\mu \rho M_\mu^\dagger}{\text{tr}(M_\mu^\dagger M_\mu)}.$$  

In fact this is the description of the well-known generalised measurement [1]. But note that Eq. (15) for describing the post measurement state is more general and there exists measurements which their post-measurement state does not obey Eq. (17). In the above example for measuring the energy of an excited atom, it is straightforward to see that when the energy levels have degeneracy, the post-measurement state can not be described by Eq. (17).

Now suppose the state of system after two successive measurement is $A_{\mu\nu}(\rho)$. According to Eq. (15), one can easily show that $\text{tr}(\rho \Omega_{\mu\nu}) A_{\mu\nu}(\rho)$ is a linearly complete positive map. Also it is obvious that its trace is equal to $\text{tr}(\rho \Omega_{\mu\nu})$. So according to the appendix lemma, there exists always a completely positive and trace preserving map like $E_{\mu\nu}$ such that

$$A_{\mu\nu}(\rho) = \frac{E_{\mu\nu}(\sqrt{\Omega_{\mu\nu}} \rho \sqrt{\Omega_{\mu\nu}})}{\text{tr}(\Omega_{\mu\nu})},$$

which clearly obeys the post-measurement rule Eq. (15).

Therefore the total process is really equivalent with a measurement; i.e. there exists a measurement such that it has the same outcome state with the same probabilities. Because a time evolution can be regarded as a special measurement with one outcome, so any sequence of measurements and time evolution can be regarded as one measurement.

### IV. NO-SIGNALING CONDITION

In a recent paper [8], the authors by a simple argument based on the impossibility of faster than light signaling, the “no-signaling condition”, have derived the linearity and completeness of time evolution. Imagine an entangled pair which is shared between Alice and Bob. From the no-signaling condition we can deduce, performing a measurement on Alice’s system does not affect density operator of Bob’s one. Furthermore using this condition, without using the projection postulate, the authors have shown that by performing suitable measurements on Alice’s system, one can prepare all possible decomposition of the density operator of Bob’s system [8, 9]. Now under an arbitrary time evolution of Bob’s system, all of these ensembles should remain indistinguishable; otherwise this scheme can be used for faster than light signaling. In this manner authors have shown that time evolution should be linear. Also from the linearity and positivity of time evolution they have deduced the complete positivity.

Now with the similar argument which they have used for the time evolution, we will show that our results about the post-measurement rule can be derived using the no-signaling condition of our assumption. We use this consequence of [8, 9] that by using the no-signaling condition and without using the projection postulate, it can be shown that Alice by performing suitable measurements can prepare every possible ensemble realization of the density operator of Bob’s system. Suppose one of these ensembles is prepared. Now Bob performs a measurement which is described by $\{F_\mu\}$ such that result $\mu$ is obtained with the probability $\text{tr}(\rho F_\mu)$ and system jumps to the state $\rho_\mu$. A priori we make no assumption about this state; so it may be different for different ensembles of pure states which are described initially by the same density operator. Now Bob performs another measurement which is described by $\{G_\nu\}$ and obtains result $\nu$ with the probability $\text{tr}(\rho G_\nu)$; But we know that this probability should be the same for different ensembles of pure states which are described by one density operator; otherwise, if it was different from one to another, Alice by performing different measurements on her own system and preparing different decompositions of the density operator of Bob’s system could send faster than light signals. But here $G_\nu$ is an arbitrary positive operator; hence $\rho_\nu$ should be the same for different preparations of a density operator and can be expressed as a function of $\rho$, like $A_\mu(\rho)$. Now suppose the system is in $\rho_1$ with the probability $p_1$ and in $\rho_2$ with the probability $p_2$. Bob performs a measurement which is described by $\{F_\mu\}$. For the state $\rho_1$ result $\mu$ is obtained with the probability $\text{tr}(\rho_1 F_\mu)$ and the system jumps to $A_\mu(\rho_1)$; for the state $\rho_2$ result $\mu$ is obtained with the probability $\text{tr}(\rho_2 F_\mu)$ and then the system jumps to $A_\mu(\rho_2)$. So, for the ensemble under consideration, we obtain result $\mu$ with the probability $p_1 \text{tr}(\rho_1 F_\mu) A_\mu(\rho_1) + p_2 \text{tr}(\rho_2 F_\mu) A_\mu(\rho_2)$ and after obtaining this result the state of system is

$$\frac{p_1 \text{tr}(\rho_1 F_\mu) A_\mu(\rho_1)}{p_1 \text{tr}(\rho_1 F_\mu) + p_2 \text{tr}(\rho_2 F_\mu)} + \frac{p_2 \text{tr}(\rho_2 F_\mu) A_\mu(\rho_2)}{p_1 \text{tr}(\rho_1 F_\mu) + p_2 \text{tr}(\rho_2 F_\mu)}.$$
On the other hand, initially the system is described by $p_1 \rho_1 + p_2 \rho_2$, hence after obtaining result $\mu$ the system should jump to $A_\mu(p_1 \rho_1 + p_2 \rho_2)$. Equating $A_\mu(p_1 \rho_1 + p_2 \rho_2)$ with expression $\rho_\mu$, one can easily show that $tr(\rho F_\mu) A_\mu(\rho)$ is a linear function of $\rho$. Now we can easily follow all the arguments which results Eqs. (14,15).

V. DISCUSSION AND CONCLUSION

As we have already mentioned, most generally any process which produces different outcomes is a measurement on the system; the only necessary condition is the lack of effective initial correlations. For example consider an experimentalist who performs different measurements and then after his observations produce an outcome. Now the whole instruments and the experimentalist altogether can be regarded as the measuring apparatus and the whole process can be regarded as one measurement. The only condition is that there should exist no initial correlation which can affect the outcomes. For example the initial state of the system should be necessarily unknown to the experimentalist. On the other hand, if he know anything about the state of system, the total process can not be regarded as one measurement. Indeed knowing this information, he can produce outcomes such that their probabilities do not obey the usual trace rule.

We have seen that the quantum mechanical description of measurement and especially the rule for the outcome’s probabilities in quantum mechanics are such that this property holds in the theory if and only if the outcomes of a measurement be described by Eq. (15). This has been driven with the help of a simple lemma. The necessity of quantum collapse is a consequence of this equation. As we have seen, theoretically all different manners of measuring one physical property are equivalent with a special ideal measurement, associated to the physical property, followed by time evolutions which depends to the special manner of measuring. Also we have seen how the description of the generalized measurements can be obtained as a special case. We have mentioned an example which its post-measurement state cannot be described by the post-measurement state rule of generalized measurements, but can be described by Eq. (15).

At the end we have seen that from the impossibility of faster than light signaling one can also derive this post-measurement state rule. In this manner we have completed the main purpose of these discussions to derive fundamental properties of quantum transformations from the no-signaling condition and the usual trace rule.

VI. APPENDIX

Lemma: Let $B$ be a linear and positive map on the space of linear operator which satisfies

$$tr(B(\rho)) = tr(\rho F),$$  \hspace{1cm} (20)

where $F$ is a positive operator. Then there exists a linear, positive and trace preserving map like $E$ such that

$$B(\rho) = E(\sqrt{F}\rho\sqrt{F}).$$  \hspace{1cm} (21)

Furthermore for a completely positive $B$, always $E$ can be chosen completely positive.

Proof: Suppose $\{|k\rangle\}$ are the eigenstates of $F$ with nonzero eigenvalues and $P_l$ is the projective operator to this subspace. Also suppose $\{|l\rangle\}$ are the eigenstates with zero eigenvalues and $P_{II} = I - P_l$ is the projective to this subspace. It is obvious that

$$B(\rho) = B(P_l \rho P_l) + B(P_l \rho P_{II} + P_{II} \rho P_l) + B(P_{II} \rho P_{II}).$$  \hspace{1cm} (22)

$B$ is a positive map so the last term in Eq. (22) should be a positive operator. Because $tr(B(\rho)) = tr(\rho F)$, this term is traceless; so it is always zero. Now we will show the second term in the right-hand side is also zero. Consider a vector like $|\psi\rangle = \alpha |k\rangle + \beta |l\rangle$ with real $\alpha, \beta$. In this situation

$$B(|\psi\rangle \langle \psi|) = \alpha^2 B(|k\rangle \langle k|) + \alpha \beta B(|k\rangle \langle l| + |l\rangle \langle k|).$$  \hspace{1cm} (23)

The left-hand side should be positive. In the other side although $B(|k\rangle \langle k|)$ is positive, the second term is not necessarily so. Thus there exists $\alpha, \beta$ which makes right-hand side non-positive, in contradiction with the left-hand side. So any term like $B(|k\rangle \langle l| + |l\rangle \langle k|)$ should be zero. Also one can repeat such an argument for another kind of terms like $B(|k\rangle \langle l| - i |l\rangle \langle k|)$ with the same result. Because the second term in the right-hand side of Eq. (22) is just a linear combination of these two kind of terms, it will vanishes; so we can conclude that

$$B(\rho) = B(P_l \rho P_l).$$  \hspace{1cm} (24)

Let $F^{-1}$ be an operator which satisfies $F^{-1} F = P_l$. Now suppose an arbitrary positive, trace preserving map like $E'$. We define $E$ to be

$$E(\rho) = B(\sqrt{F^{-1}} P_l \rho P_l \sqrt{F^{-1}}) + E'(P_{II} \rho P_{II}).$$  \hspace{1cm} (25)

Clearly it is a positive map; regarding Eq. (20), it is trace preserving. Also according to Eq. (24) it obviously satisfies Eq. (21). By choosing $E'$ to be completely positive, from complete positivity of $B$ one can deduce complete positivity of $E$.

VII. ACKNOWLEDGMENT

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[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] For example see J.J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley Publishing Company, rev. ed. 1994).

[3] Indeed $F_{\mu}$ corresponds to the Heisenberg picture while $F_{\mu}$ corresponds to the Schrodinger picture.

[4] More generally, any sequence of different time evolutions and measurements should be describable as one measurement.

[5] W.K. Wootters and W.H. Zurek, Nature (London) 299, 802 (1982).

[6] S. Weinberg, Ann. Phys. (N.Y.) 194, 336 (1989).

[7] N. Gisin, Phys. Lett. A 143, 1 (1990).

[8] C. Simon, V. Bužek, and N. Gisin, Phys. Rev. Lett. 87, 170405 (2001).

[9] C. Simon, V. Bužek, and N. Gisin, Phys. Rev. Lett. 90, 208902 (2003).

[10] J. Preskill, *Lecture Notes on Quantum Computation*, http://www.theory.caltech.edu/people/preskill/ph229/#lecture

[11] P. Štefancovič and V. Bužek, Phys Rev. A 64, 062106 (2001).

[12] P. Štefancovič and V. Bužek, Phys Rev. A 67, 029902(E) (2003).