NOTE ON THE ASYMPTOTIC STABILITY OF STATIONARY SOLUTIONS OF THE INVISCID INCOMPRESSIBLE POROUS MEDIUM EQUATION

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ABSTRACT. We initiate the study of stability of solutions of the 2D inviscid incompressible porous medium equation (IPM). We begin by classifying all stationary solutions of the inviscid IPM under mild conditions. We then prove some linear stability results. We then study solutions of the IPM equation which are sufficiently regular perturbations of linearly stable steady states. We first prove that all global solutions (even with large data) which are initially regular perturbations of linearly stable steady states must weakly converge to a steady state. We then prove that sufficiently regular perturbations which are also small must be globally regular and strongly converge to a steady state. The mechanism behind the stability is stratification as opposed to previous stability results based on dispersion and/or mixing [1]. More or less, we prove that stratified stationary solutions which do not go against gravity are asymptotically stable.

1. Introduction

1.1. Well-posedness for active scalar equations. The question of well-posedness for active scalar equations is one which has attracted much attention in the last several years. One of the breakthroughs in the study of active scalar equations was the proof(s) of well-posedness for the 2-D critically viscous Surface Quasi-Geostrophic (SQG) equations. A proof of wellposedness was discovered by Kiselev, Nazarov, and Volberg using their “modulus of continuity method” [19]. Around the same time, Caffarelli and Vasseur proved the well-posedness for SQG by extending De-Giorgi’s method for non-linear elliptic equations to non-local equations [3]. Another proof was discovered by Constantin and Vicol using their “Nonlinear Maximum Principle” [10].

All of these proofs have yielded the PDE community a wide variety of methods to approach fluid equations with critical dissipation. In a sense, the results of [19], [3], and [3] allow one to say that virtually any active scalar equation with critical dissipation and divergence-free velocity field is well-posed. In the supercritical case, however, the situation is very different. At the time of the writing of this paper, there are little or no results on the global existence of strong solutions to “supercritical” active scalar equations. For the inviscid SQG equation, for example, the question of global existence vs. finite time blow-up of strong solutions is currently wide open with the exception of local results or blow-up criteria [9], [24], [18].

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1.2. The question of long-time behavior. Yet another major gap in our understanding of fluid models lies in questions related to the long-time behavior of solutions of models we know to be globally well-posed. A good example of this is the 2D Euler equations. On the one hand, there is the landmark result of Kiselev and Sverak [20] on the growth of solutions of the 2D Euler equations in a bounded domain. The intuition behind Kiselev and Sverak’s construction is that a certain singular steady state of the Euler equations is stable. This stable steady state destabilizes smooth solutions which are close to it in some sense and lead to double exponential growth. On the other end of the spectrum is the recent resolution of the stability of the Couette flow for 2D Euler by Bedrossian and Masmoudi [1]. Bedrossian and Masmoudi’s main result is that a certain linearly stable steady state of the Euler equations is \textit{asymptotically stable}. The mechanism behind the stability is that the linearized Euler equations around the Couette flow exhibit certain damping properties. Bedrossian and Masmoudi are able to take advantage of these damping properties to prove that “very smooth” perturbations of the Couette flow which are small in a particular Gevrey class converge back to a shear flow stationary solution of the Euler equations. A related result in plasma physics was achieved earlier by Mouhot and Villani [22] and then simplified and strengthened by Bedrossian, Masmoudi, and Mouhot [2]. One can say that, for the result of [1], the main idea is that \textit{mixing can be a stabilizing force}.

In this article, the main idea is that \textit{stratification can be a stabilizing force}. One can imagine that a fluid with density that is proportional to depth (i.e. that the density of the fluid increases the deeper you go into the fluid) is, in some sense, “stable.” On the other hand, if the density is inversely proportional to depth, then one would imagine that such a scenario is unstable—in fact this is where one sees the so-called Rayleigh-Darcy convection even in the presence of viscosity [11]. We are concerned mainly with the stable case. The question we wish to ask here is whether (in the absence of viscosity), one is able to establish non-linear stability results for (1.1)-(1.3). We will show that this is indeed the case. In fact, we will be able to prove that smooth perturbations of stratified stable solutions are stable for all time in Sobolev spaces.

We emphasize that this seems to be the first construction of a non-trivial global smooth solution for the inviscid IPM equation.

1.3. The inviscid IPM equation. The system we study is the inviscid IPM equation:

\begin{align}
\frac{\mu}{\kappa} u &= -\nabla p - (0, g \rho), \\
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\text{div}(v) &= 0
\end{align}

\(u\) is the velocity of the fluid, \(p\) is the pressure, \(\mu\) is the dynamic viscosity, \(\kappa\) is the permeability of the isotropic medium, \(\rho\) is the liquid density and \(g\) is the
gravitational acceleration. When these equations are studied on a bounded domain, we assume that \( u \) is satisfies the no-slip boundary condition:

\[
u \cdot n = 0,
\]
on the boundary of the domain where \( n \) is the normal to the boundary.

Our goal here is to study the (non-linear) stability of exact solutions to this system. For simplicity, let’s take \( \mu = \kappa \) and \( g = -1 \).

We note that this system can also be written as follows:

\[
\partial_t \rho + u \cdot \nabla \rho = 0 \\
u = R_1 R_\perp \rho
\]

where \( R_\perp = (R_2, -R_1) \) and \( R_1, R_2 \) are the Riesz transforms and this is exactly the same as the SQG system except that \( u = R_\perp \rho \) in that case.

1.4. The steady states. The kinds of exact solutions we are interested in are: \( \rho = f(y), u = 0, \) and \( p = 0 \). It is trivial to check that these are stationary solutions for any \( C^1 \) function, \( f \).

In fact, under mild assumptions, these are the only stationary solutions.

**Lemma 1.1.** Let \( \rho \) be a \( C^1 \) stationary solution of the inviscid IPM equation on a bounded domain \( \Omega \) or in \( \mathbb{R}^2 \). If we are studying the equation on \( \mathbb{R}^2 \), we also assume that \( |\rho(x, y)| \lesssim \frac{1}{x^2 + y^2 + 1} \). Then, \( \rho \) is a function of \( y \) only and, in particular, \( u \equiv 0 \).

**Proof.** We know that

\[
u \cdot \nabla \rho = 0
\]

This means that

\[
\int u \cdot \nabla \rho y = 0.
\]

Due to the assumption on the decay of \( \rho \) in the \( \mathbb{R}^2 \) case or the no-slip condition on \( u \) in the bounded domain case and using the divergence-free condition we see:

\[
\int u \cdot \nabla y \rho = 0.
\]

This implies, on the one hand, that:

\[
\int u_2 \rho = 0
\]

On the other hand, if we take the equation

\[
u + \nabla p = (0, \rho)
\]

and dot with \( u \) we see (after an integration by parts and using the boundary conditions)

\[
\int u_2 \rho = \int |u|^2.
\]
Thus, \( u \equiv 0 \). As a result, \( \rho_x \equiv 0 \) and the lemma is proved.

Although, as far as active scalar equations go, this system of equations shares many similarities with the surface quasi-geostrophic equation and the 2D Euler equations, the IPM system has a very simple structure of stationary solutions. It turns out, in addition, that the linearized IPM equation around any one of these steady states has a special structure which allows us to deduce (in a relatively easy fashion) results on the long time behavior of perturbations of stationary solutions.

Recall that \( \rho = f(y) \) and \( u = 0 \) are the stationary solutions of this system. Now suppose that we perturb the data a little bit: \( \rho = f(y) + \tilde{\rho} \) and \( v = \tilde{v} \). Then we see:

\[ \tilde{v} = -\nabla p + (0, \tilde{\rho}) \]

and

\[ \partial_t \tilde{\rho} + (\tilde{v}) \cdot \nabla (f(y) + \tilde{\rho}) = 0. \]

Now we rewrite \( \tilde{v} \) as \( v \) and \( \tilde{\rho} \) as \( \rho \) and simplify:

\[ v = -\nabla p + (0, \rho) \]

\[ \partial_t \rho + v_2 f'(y) + v \cdot \nabla \rho = 0 \]

It is clear that under the no-slip boundary condition \( u \cdot n = 0 \) and under the assumption that the domain is simply connected, we may pass to the stream function formulation where we define \( u = \nabla \Psi \):

\[ \Delta \Psi = -\partial_x \rho \]

and \( \psi = 0 \) on the boundary of the domain.

This implies that \( v_2 = R \rho \) where \( R = \partial_{xx} (-\Delta)^{-1} \). Here, \((-\Delta)^{-1}\) is the inverse of the Dirichlet Laplacian.

Thus, our equation reads:

\[ \partial_t \rho + f'(y) R \rho + v \cdot \nabla \rho = 0 \]

What is interesting about this equation is that \( R \) is a negative operator.

This structure will allow us to prove the following result:

**Theorem 1.2. (Main Result on \( \mathbb{R}^2 \))**

Let \( f \) be such that there exists \( K > 0 \) such that \( f'(y) \geq K, \ f'''(y) \leq 0 \) for all \( y \in \mathbb{R} \). Suppose further that \( f' \in W^{4,\infty} \).

Then the stationary solution \( \bar{\rho} = f(y) \) of the IPM equation is asymptotically stable in \( H^4 \). In other words, there exists \( \epsilon > 0 \) (depending only on \( K \)) such that if we solve IPM with initial data \( \rho_0 + \bar{\rho} \) with \( |\rho_0|_{H^4} \leq \epsilon_0 \leq \epsilon \) then the solution \( \tilde{\rho} \) satisfies the following:

\[ (1) |\tilde{\rho}(t) - f|_{H^4} \leq M \epsilon \ \forall \ t > 0 \]
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\[ (2) |u|_{H^3} \leq \epsilon e^{-cKt} \]

\[ (3) |\tilde{\rho}(t)|_{L^2} \to 0, \ t \to \infty \text{ strongly in } H^{4-\delta}, \ \forall \ \delta > 0 \]

for some \( c \) small, \( M \) large depending on \( f \) only, and for some \( G \) which is a function of \( y \) only.

**Theorem 1.3. (Main Result on } \mathbb{T}^2 \text{)**

Let \( f \) be such that there exists \( K > 0 \) such that \( f' \geq K, |f'''|_{L^\infty} < 2\pi^2 K \). Suppose further that \( f' \in W^{4,\infty} \).

Then the stationary solution \( \tilde{\rho} = f(y) \) of the IPM equation is asymptotically stable in \( H^4 \). In other words, there exists \( \epsilon > 0 \) (depending only on \( K \)) such that if we solve IPM with initial data \( \rho_0 + \tilde{\rho} \) with \( |\rho_0|_{H^4} \leq \epsilon_0 \leq \epsilon \) then the solution \( \tilde{\rho} \) satisfies the following:

\[ (1) |\tilde{\rho}(t) - f|_{H^4} \leq M\epsilon \ \forall \ t > 0 \]

\[ (2) |u|_{H^3} \leq \epsilon e^{-ct} \]

\[ (3) |\tilde{\rho}(t) - G|_{L^2} \to 0, \ t \to \infty \text{ strongly in } H^{4-\delta}, \ \forall \ \delta > 0 \]

for some \( c \) small, \( M \) large depending upon \( f \) only, and for some \( G \) which is a function of \( y \) only.

**Remark 1.4.** We note here that \( \tilde{\rho} \) will never belong to \( H^4 \) (or even \( L^2 \)) in the whole space case. However, if we perturb \( \tilde{\rho} \) by an \( H^s \) function the perturbation will remain \( H^s \) for all time (unless the solution blows up in finite time). Similarly, \( \tilde{\rho} \) may not be periodic but in some cases (e.g. \( \tilde{\rho}(y) = y \)) we may perturb it by a periodic function and once more the perturbation will remain periodic. Note also that in the \( \mathbb{R}^2 \) case, the perturbation \( \tilde{\rho} \to 0; \) this is because there is no function of \( y \) only which belongs to \( L^2(\mathbb{R}^2) \).

Note that an easy consequence of the theorem is that the \( x \)– derivatives of \( \rho \) decay exponentially fast whereas the \( y \) derivatives need not decay at all. Indeed, if we perturb the stationary solution by a function of \( y \) only then there should be no decay! This is part of the difficulty. However, if we analyze the nonlinear term:

\[ u \cdot \nabla \rho = (-R_1 R_2 \rho, R_1^2 \rho) \cdot \nabla \rho, \]

then we will notice that each term of the nonlinearity contains two \( x \)-derivatives. This is the idea behind controlling the nonlinear term.

The proof of this theorem is based on the linear decay coming from

\[ \rho_t = R(\rho) \]

and “careful” energy estimates which capitalize on the fact that the nonlinear term has two \( x \)-derivatives.
1.5. **Comparison with other results.** The idea of taking a non-linear equation where global well-posedness is unknown and proving global well-posedness of a perturbation of the equation is not new. Very recently, this was done for the MHD equation by Lin and Zhang [21] and has also been done for complex fluids in other contexts as well ([7],[23]).

We emphasize here that the kind of damping that the linearized equation gives us:

\[ g_t = R(g) \]

with \( R = R^2_1 \) does not seem to have been considered in other works. Indeed, this sort of linearized equation gives damping only in the \( x \)-direction which may not be strong enough, in general, to control non-linear terms. In our case, however, due to the structure of the non-linearity in the IPM equation (the fact that each term in the non-linearity has two \( x \)-derivatives), we are able to control the non-linearity well enough to prove decay of the velocity field. This anisotropy was also present in the work of Bedrossian and Masmoudi [1].

In the class of smooth solutions, this seems to be the first global well-posedness result for a supercritical and inviscid active scalar equation. Another notable result in the class of vortex-patch type solutions is that of Hmidi and Hassainia [16]. They prove global existence and uniqueness of a certain kind of “periodically-rotating” vortex patch for a class of supercritical active scalar equations. This was subsequently improved to a broader class of models (including the SQG equation) by Cordoba et al. [4]. In the class of weaker solutions, Isett and Vicol [17] were able to use convex integration to construct global weak solutions to the IPM equation which are of class \( C^{\frac{1}{2},2}_{t,x} \). Unfortunately, as is established by Isett and Vicol, these solutions are highly non-unique. A slightly different class of results were recently attained by a number of authors ([5],[8],[6]) on the Muskat problem which can be seen as the “free boundary problem” for the IPM equations.

We close by mentioning that in [17],[13], and [15],[14] the authors indicate a difference between active scalars where the operator relating the velocity field and the advected quantity (\( u \) and \( \rho \) in our case), has an even Fourier symbol or an odd Fourier symbol. We are taking advantage of the fact that, for the IPM equation, this symbol is even and one component of it has a sign. Such a thing can never happen if the symbol is odd; however, in [12] the authors establish certain dispersive properties of equations of the form \( f_t = R_1(f) \) which can also act as a stabilizing force. It is possible that using this sort of dispersion, one can say something about stationary solutions for active scalar equations with odd symbol.

1.6. **Organization of the Paper.** In the next section we will study some properties of stationary solutions as well as properties of the linear equation

\[ \partial_t \rho = -f'(y)R\rho. \]

In the final section we will prove Theorem 1.1. We note that we do not write out the proof of Theorem 1.2 because it is exactly the same as the proof of Theorem 1.1.
2. Linearized equation and $L^2$ convergence

We begin by proving the following stability theorem under certain monotonicity conditions on the stable solution:

**Proposition 2.1.** If $f' \geq K$ and $f''' \leq 0$ then

\begin{equation}
- (f'(y)Rg, g) \geq K|R_1g|_{L^2}^2 \quad \forall g \in L^2
\end{equation}

If $f' \geq K$ and $|f''(x)|_{L^\infty} < 2\pi^2 K$, then:

\begin{equation}
(f'(y)R(g), g) \geq \left( K - \frac{|f''(x)|_{L^\infty}}{2\pi^2} \right) |Rg|_{L^2}^2, \quad \forall g \in L^2(T^2).
\end{equation}

Here, $f'''$ denotes the positive part of the third derivative of $f$.

**Remark 2.2.** Note that the relaxation of the assumptions on $f$ given in the second part of the proposition can be made if the functions $g$ are restricted to a domain where the Poincaré inequality holds.

**Proof.** We want to study

\[ (f'(y)Rg, g). \]

Therefore, we let $g = -\Delta \phi$.

Then,

\[ (f'(y)Rg, g) = (f'(y)\phi_{xx}, \phi_{xx} + \phi_{yy}) \geq K|\phi_{xx}|_{L^2}^2 + (f'(y)\phi_{xx}, \phi_{yy}). \]

Now,

\[ (f'(y)\phi_{xx}, \phi_{yy}) = -(f'(y)\phi_x, \phi_{xyy}) = (f'(y)\phi_{xy}, \phi_{xy}) + (f''(y)\phi_x, \phi_x) \]

\[ = (f'(y)\phi_{xy}, \phi_{xy}) - \frac{1}{2}(f'''(y)\phi_x, \phi_x). \]

Putting all this together, and using that $f''' \leq 0$ we see:

\[ (f'(y)Rg, g) \geq K(|\phi_{xx}|^2 + |\phi_{xy}|^2) = K|\nabla \phi_x|^2 = K'|\Delta \phi_x|^2 = K|R_1g|^2. \]

This concludes the proof of the first part of Proposition 3.1. For the second part, we simply note that

\[ \int_{T^2} |\phi_x|^2 \leq \frac{1}{\pi^2} \int_{T^2} |\nabla \phi_x|^2 \]

\[ \square \]

Some examples of stationary solutions satisfying the assumptions of Proposition 3.1 are:

1. Linear: $\rho(y) = Ky + b, K > 0$
2. Linear plus periodic: $\rho(y) = Ky + \sin(y), K > \frac{3}{2}$.

From here on we restrict ourselves to the whole space case $\mathbb{R}^2$ while the case of $T^2$ is similar.
Proposition 2.3. Let \( \rho \) be a smooth solution of (1.6)-(1.7) in \( \mathbb{R}^2 \) with \( f \) satisfying the conditions of Proposition 2.1.

Then,

\[
|u|_{L^2} = |R_1 \rho|_{L^2} \leq |\rho_0|_{L^2} e^{-Kt}.
\]

Proof. We only prove the first inequality, the second being similar.

\[
\partial_t \rho + u \cdot \nabla \rho = -f'(y) R \rho.
\]

Now upon multiplying this equation by \( \rho \) and integrating in space we see that (because \( \text{div}(u) = 0 \)):

\[
\frac{1}{2} \partial_t |\rho|_{L^2}^2 = -(f'(y) R \rho, \rho).
\]

Upon applying Proposition 2.1 we see:

\[
\frac{1}{2} \partial_t |\rho|_{L^2}^2 \leq -K |R_1 \rho|_{L^2}^2.
\]

Now, integrate in time:

\[
|\rho|_{L^2}^2 \leq |\rho_0|_{L^2}^2 - 2K \int_0^t |R_1 \rho|_{L^2}^2.
\]

Now, using Plancherel’s identity, it is easy to see that \( |R_1 \rho|_{L^2} \leq |\rho|_{L^2} \).

Thus,

\[
|R_1 \rho|_{L^2}^2 \leq |\rho_0|_{L^2}^2 - 2K \int_0^t |R_1 \rho|_{L^2}^2.
\]

The proposition follows by Gronwall’s lemma.

In particular, the whole velocity field decays exponentially in \( L^2 \). Note that this is true for large and small data perturbations of \( f \). This implies that so long as the solution exists for all time (i.e. no blow-up occurs), every solution of (1.6)-(1.7) will converge weakly in \( L^2 \) to a stationary solution.

Corollary 2.4. Let \( \rho \) be any global solution of (1.6)-(1.7) on \( \mathbb{T}^2 \). Then, \( u \to 0 \) strongly in \( L^2 \). Moreover, \( \rho - \bar{\rho}(y) \to 0 \) weakly in \( L^2 \), where \( \bar{\rho} \) is the x-average of \( \rho \).

Proof. The convergence of \( u \) to 0 is a consequence of Proposition 2.3. The weak convergence of \( \rho \) is due to the following simple computation:

\[
\rho - \bar{\rho} = \sum_{n,k \neq 0} \rho_{n,k} e_{n,k}
\]

where \( \rho_{n,k} \) are the Fourier coefficients of \( \rho \) and \( e_{n,k} \) is the Fourier basis of \( \mathbb{T}^2 \).

The corollary will follow once we establish that \( \rho_{n,k} \to 0, t \to \infty \) for all \( n,k \) with \( k \neq 0 \).
However,
\[ |u|_{L^2} = \sum_{n,k} \frac{n^2}{n^2 + k^2} \rho_{n,k}(t) \lesssim e^{-Kt}. \]

As a result, \( \rho_{n,k} \to 0 \) as \( t \to \infty \) and the corollary is proven. In fact, strong convergence may be established so long as \( |\rho|_{H^\delta} \) grows sub-exponentially for some \( \delta > 0 \).

\[ \square \]

The last corollary implies that every global solution of the (1.6)-(1.7) must converge weakly back to a steady state of the IPM equation. Our main theorem, Theorem 1.1, states that solutions of (1.6)-(1.7) which are initially small in \( H^4 \) remain small for all time and converge strongly to a stationary solution of the IPM equation.

3. The proof of Theorem 1.1

We focus on the \( \mathbb{R}^2 \) case as the \( \mathbb{T}^2 \) case is similar.

3.1. Energy Estimates.

Lemma 3.1. The following two estimates hold for \( s = 3, 4 \).

(3.1) \[ \partial_t |\rho|_{H^s}^2 \leq C |u|_{H^s}^2 |\rho|_{H^{s+1}} - \frac{K}{2} |u|_{H^s}^2 + C |u|_{H^{s-1}}^2. \]

(3.2) \[ \partial_t |\rho|_{H^{s+1}}^2 \leq C \left( |\nabla u|_{L^\infty} |\rho|_{H^{s+1}}^2 + |u|_{H^{s+1}}^2 |\rho|_{H^{s+1}} \right) - \frac{K}{2} |u|_{H^{s+1}}^2 + C |u|_{H^s}^2. \]

Proof. The usual method of using the Kato-Ponce inequality will only give us \( |\rho|_{H^s}^2 |u|_{H^s} \) on the right hand side of the energy inequality. We will need to carry out the energy estimates carefully to ensure that the estimate of the nonlinear term is quadratic in \( u \) and this is why we will lose one derivative in the estimate.

We first want to make the following simple observations:

1. \( |R_1 \rho|_{H^s} = |u|_{H^s} \).
2. \( |\partial_s \rho|_{H^s} = |R_1 \Lambda \rho|_{H^s} = |R_1 \rho|_{H^{s+1}} = |u|_{H^{s+1}} \).

We are interested in \( H^s \) estimates so we will focus on first controlling the nonlinear term \( u \cdot \nabla \rho \).

Step 1: The non-linear term

\[ (\partial^s (u \cdot \nabla \rho), \partial^s \rho) = \sum_{i=1}^s a_{i,s} \left( \partial^i u \cdot \nabla \partial^{s-i} \rho, \partial^s \rho \right). \]

So we must study

\[ \left( \partial^i u \cdot \nabla \partial^{s-i} \rho, \partial^s \rho \right), \]

for \( 1 \leq i \leq s \).
Using observation (1)-(2) above, it actually suffices to consider the term
\[ (\partial^i \psi_x \partial^s \rho_y \partial^s \rho), \]
where \( u = \nabla^\perp \psi \). Now, we want to distinguish between two kinds of terms, first the case where \( i = 1 \) and then the case where \( i \geq 2 \).

The case \( i = 1 \).

This means that we study \( (\partial \psi_x \partial^s \rho_y \partial^s \rho) \).

We will estimate this term in two different ways (to get (2.7) and (2.8)).

One can estimate directly:
\[ |(\partial \psi_x \partial^s \rho_y \partial^s \rho)| \lesssim |\nabla u|_{L^\infty} |\rho|^2_{H^s}. \]

One can also do the following
\[ (\partial \psi_x \partial^s \rho_y \partial^s \rho) = -(\partial \psi \partial^s \rho_y \partial^s \rho) - (\partial \psi \partial^s \rho_y \partial^s \rho_x). \]

We can now integrate by parts once more to get:
\[ |(\partial \psi_x \partial^s \rho_y \partial^s \rho)| \lesssim |\nabla u|_{L^\infty} |u|_{H^s} |\rho|_{H^{s+1}} \leq |u|^2_{H^s} |\rho|_{H^{s+1}}. \]

This concludes this case.

The case \( i \geq 2 \).

We will study \( (\partial^i \psi_x \partial^s \rho_y \partial^s \rho) \).

Upon integrating by parts, we see:
\[ (\partial^i \psi_x \partial^s \rho_y \partial^s \rho) = -(\partial^i \psi \partial^s \rho_y \partial^s \rho) - (\partial^i \psi \partial^s \rho_y \partial^s \rho_x) = I + II. \]

Now, it is clear that
\[ |I| \lesssim |u|^2_{H^s} |\rho|_{H^s}. \]

Moreover, by writing:
\[ II = (\partial (\partial^i \psi \partial^s \rho_y), \partial^s \rho_x) \]
and noting that \( i \leq 2 \) we see:
\[ |II| \lesssim |u|^2_{H^s} |\rho|_{H^s}. \]

Step 2: The linear term

We are interested in studying
\[ (\partial^s (f'(y) R \rho), \partial^s \rho). \]

This term is written as:
\[ (\partial^s (f'(y) R \rho), \partial^s \rho) = (f'(y) \partial^s R \rho, \partial^s \rho) + III \geq K |u|^2_{H^s} - |III| \]
where $III$ consists of terms of the form $(\partial^{s_1} f'(y) \partial^{s_2} R \rho, \partial^{s} \rho)$ with $s_1 + s_2 = s$ and $s_1 \geq 1$. It is easy to see that

$$|III| \leq \frac{K}{2} |u|^{2}_{H^{s}} + C |\rho|^{2}_{H^{s-1}},$$

where $C$ depends upon $K$ and $|f'|_{W^{4,\infty}}$.

This concludes the proof of (2.5)-(2.6).  

\[\square\]

3.2. Bootstrap. Suppose that $|\rho|_{H^{4}} \leq 4\epsilon$ on an interval $[0, T]$. Then, one can choose large constants $C_s$ such that:

\begin{equation}
(3.3) \quad \partial_{t} \sum_{s=0}^{3} C_s |\rho|_{H^{s}} \leq C |u|^{2}_{H^{3}} |\rho|_{H^{4}} - \frac{K}{2} |u|^{2}_{H^{3}}.
\end{equation}

Since $|\rho|_{H^{4}} \leq 4\epsilon$ on $[0, T]$ we have that

$$\partial_{t} \sum_{s=0}^{3} C_s |\rho|^{2}_{H^{s}} \leq -\frac{K}{4} |u|^{2}_{H^{3}}.$$  

We claim that if $|\rho_0|_{H^{4}} \leq \frac{4\epsilon}{\pi}$ for some $M$ large and fixed, and if $|\rho|_{H^{4}} \leq 4\epsilon$ on $[0, T]$, then $|\rho|_{H^{4}} \leq 2\epsilon$ on $[0, T]$. If the claim is true, a simple continuity argument will imply that $|\rho|_{H^{4}} \leq 2\epsilon$ whenever $|\rho_0|_{H^{4}} \leq \frac{4\epsilon}{\pi}$.

Indeed, application of Gronwall’s lemma implies that

$$|u|_{H^3} \leq C |\rho_0|_{H^3} e^{-\frac{K}{4} t}$$

for some large and fixed $C$ depending only upon $f$.

Now, using (3.2),

$$\partial_{t} |\rho|^{2}_{H^{4}} \leq C |u|^{2}_{H^{3}} |\rho|^{2}_{H^{4}} + C |u|^{2}_{H^{3}}.$$  

This inequality coupled with the exponential decay of $u$ imply that $|\rho|_{H^{4}} \leq C |\rho_0|_{H^{4}}$. Now take $|\rho_0|_{H^{4}} \leq \frac{2\epsilon}{C}$ and we see that $|\rho|_{H^{4}} \leq 2\epsilon$.

This concludes the proof of Theorem 1.1. \[\square\]

4. The 3D Case

In this section we investigate the corresponding IPM system in 3D:

\begin{align}
(4.1) \quad u &= \nabla p + (0, 0, \rho), \\
(4.2) \quad \partial_{t} \rho + u \cdot \nabla \rho &= 0.
\end{align}

Proposition 4.1. The only stationary solutions of (4.1)-(4.2) satisfying:

$$|u(x, y, z)| \lesssim \frac{1}{1+x^2+y^2+z^2}$$

are such that $\rho$ is a function of $z$ only and $u \equiv 0$.  

\[\square\]
The proof is exactly the same as the proof in the 2D case except we multiply (4.2) by $z$ instead of $y$.

We focus upon the stationary solution $\rho(z) = z$. The general conditions on the stationary solution given in Theorems 1.1 and 1.2 can be done in 3d as well. Since we have already completed the proof of this in 2d we will not reproduce it in 3d except in the case $\rho(z) = z$ for the sake of completeness.

Hence forth, the equation we will consider is the perturbed equation:

(4.3) \[ u = \nabla p + (0, 0, \rho), \]
(4.4) \[ \partial_t \rho + u \cdot \nabla \rho = -u_3, \]
which corresponds to studying solutions of (4.1)-(4.2) which are of the form $\rho = \tilde{\rho} + z$.

**Theorem 4.2.** There exists $\epsilon > 0$ such that for all $\rho_0 \in H^4$ with $|\rho_0| \leq \epsilon$ for which (4.3)-(4.4) has a global classical solution for which

1. $|u|_{H^3} \leq |\rho_0|_{H^4} e^{-ct}$,
2. $|\rho|_{H^4} \leq C|\rho_0|_{H^4}$,
3. $\rho \to G$ strongly in $H^{4-\delta}$, $\forall \delta > 0$

for some universal constants $c, C$ and some function $G$ of $z$ only.

**Remark 4.3.** It may strike the reader as odd that we take the same degree of regularity in 2d as we did in 3d. In fact the 2d regularity level we chose is not optimal. The optimal regularity in 2d would be to take $\rho_0$ in $H^{3+}$ and in 3d it would be to take $\rho_0$ in $H^{3.5+}$. Achieving these levels of regularity are simple if one uses the Kato-Ponce inequalities in the right way. We opted out of doing this for the sake of clarity.

The proof is once more based upon careful energy estimates and a bootstrap argument. We only give a sketch here.

Note that (similar to the 2d case)

\[-u_2 = R_1^2 \rho + R_2^2 \rho := L(\rho).\]

Notice that $L$ damps in $x$ and in $y$ but not in $z$. Notice further that the nonlinearity $u \cdot \nabla \rho$ can be written as follows:

\[ u \cdot \nabla \rho = R_1 R_3 \rho \rho_x + R_2 R_3 \rho \rho_y - (R_1^2 + R_2^2) \rho \rho_z \]

The crucial point is that each term in $u \cdot \rho$ has exactly two $x$ or $y$ derivatives. This will allow us to prove Lemma 3.1 in the 3D case and then apply the bootstrap argument. We leave the details to the reader.

5. **Better decay in the whole space**

We end this note by quantifying statement (3) in Theorem 1.2.

Consider the linear equation

(5.1) \[ \partial_t \rho = R_1^2 \rho \]
on $\mathbb{R}^2$. Since it is impossible for a (not identically zero) $\rho \in L^2(\mathbb{R}^2)$ to be a function of $y$ only, it is expected that solutions of the linear equation (5.1) actually decay. This cannot be said on $\mathbb{T}^2$, obviously.

**Proposition 5.1.** Let $\rho_0 \in L^2(\mathbb{R}^2)$. Then, the unique solution of (5.1) $\rho(t)$ with initial data $\rho_0$ satisfies:

$$
|\rho(t)|_{L^2} \leq (1 + |t|)^{-\frac{1}{4}}|\rho_0|_{L^2}
$$

**Proof.** Using the Fourier transform, we can solve (5.1) explicitly:

$$
|\rho(t)|_{L^2}^2 = \int_{\mathbb{R}^2} e^{-2\xi^2 \frac{t^2}{1 + \xi^2}} \hat{\rho}_0(\xi)^2 d\xi = \int_0^{2\pi} e^{-2\cos^2(\theta)t} \int_0^\infty \hat{\rho}_0(r, \theta)^2 r dr d\theta
$$

Now let $G(\theta) = \int_0^\infty \hat{\rho}_0(r, \theta)^2 r dr$. Then we see,

$$
|\rho(t)|_{L^2}^2 = \int_0^{2\pi} e^{-2\cos^2(\theta)t} G(\theta) d\theta.
$$

By the method of stationary phase in one dimension, we see:

$$
|\rho(t)|_{L^2}^2 \leq C \int_0^{2\pi} |G(\theta)| d\theta = C \frac{1}{(1 + |t|)^{\frac{1}{2}}} |\rho_0|_{L^2}^2
$$

□

**Corollary 5.2.** The decay rate in (3) of Theorem 1.2 is at least on the order of $t^{-\frac{1}{4}}$ as $t \to \infty$.

The proof of this is simple and we leave it to the reader.

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