Generalized power functions for one-dimensional supersymmetric quantum mechanics

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Abstract

Complex-valued functions defined on a finite interval $[a, b]$ generalizing power functions of the type $(x - x_0)^n$ for $n \geq 0$ are studied. These functions called $\Phi$-generalized powers, $\Phi$ being a given nonzero complex-valued function on the interval, were considered to construct a general solution representation of the Sturm-Liouville equation in terms of the spectral parameter \cite{17,22}. The $\Phi$-generalized powers can be considered as a natural basis functions for the one-dimensional supersymmetric quantum mechanics systems taking $\Phi = \psi_0^2$, where the function $\psi_0(x)$ is the ground state wave function of one of the supersymmetric scalar Hamiltonians. Several properties are obtained such as $\Phi$-symmetric conjugate and antisymmetry of the $\Phi$-generalized powers, a supersymmetric binomial identity for these functions, a supersymmetric Pythagorean elliptic (hyperbolic) identity involving four $\Phi$-trigonometric ($\Phi$-hyperbolic) functions as well as a supersymmetric Taylor series expressed in terms of the $\Phi$-derivatives. These generalized derivatives imply nonconstant coefficients homogeneous linear ordinary differential equations of order $n$ for which a fundamental set of solutions is given in terms of the $\Phi$-generalized powers. Finally, we present a general solution of the stationary Schrödinger equation in terms of geometric series where the Volterra compositions of the first type is considered.

Keywords: Stationary Schrödinger equation, supersymmetric quantum mechanics, spectral parameter power series, generalized Taylor series, Volterra composition

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1 Introduction

Spectral parameter power series (SPPS) method was first introduced in 2008 [17] and subsequently generalized in 2010 from a different approach [22]. This method has been widely used in the last ten years. One of the reasons is that SPPS has implications from both a mathematical and a physical point of view. Indeed, as illustrated in the following theorem, SPPS consists of a representation for the solutions of the Sturm-Liouville equation as a spectral parameter. The range of applications is therefore considerable.

Theorem 1 [22] Assume that on a finite real interval \([a, b]\), equation \((pu')' + qu_0 = 0\) possesses a particular solution \(u_0\) such that \(u_0^2 r\) and \(1/(u_0^2 p)\) are continuous on \([a, b]\). Then the general solution of \((pu')' + qu = \lambda ru\) on \((a, b)\) has the form

\[ u = c_1 u_1 + c_2 u_2 \]  

where \(c_1\) and \(c_2\) are arbitrary complex constants,

\[ u_1 = u_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \tilde{X}^{(2k)} \quad \text{and} \quad u_2 = u_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} X^{(2k+1)} \]

with \(\tilde{X}^{(n)}\) and \(X^{(n)}\) being defined by the recursive relations

\[ \begin{align*}
\tilde{X}^{(0)} &= 1, & X^{(0)} &= 1, \\
X^{(n)}(x_0, x) &= \begin{cases} \\
\int_{x_0}^x X^{(n-1)}(x_0, \xi) \frac{1}{u_0^2(\xi)p(\xi)} d\xi, & n \text{ odd}, \\
\int_{x_0}^x X^{(n-1)}(x_0, \xi) u_0^2(\xi)r(\xi) d\xi, & n \text{ even}, 
\end{cases} \\
\tilde{X}^{(n)}(x_0, x) &= \begin{cases} \\
\int_{x_0}^x \tilde{X}^{(n-1)}(x_0, \xi) u_0^2(\xi)r(\xi) d\xi, & n \text{ odd}, \\
\int_{x_0}^x \tilde{X}^{(n-1)}(x_0, \xi) \frac{1}{u_0^2(\xi)p(\xi)} d\xi, & n \text{ even}, 
\end{cases}
\end{align*} \]

where \(x_0\) is an arbitrary point in \([a, b]\) such that \(p\) is continuous at \(x_0\) and \(p(x_0) \neq 0\). The two series in \((2)\) converge uniformly on \([a, b]\).

From the mathematical point of view, SPPS was considered for the study of the transmutation operators (see [1, 4, 8, 26, 27]) in [19, 20, 23, 24], for the Sturm-Liouville or the Hill equations [3, 10, 14, 15, 28] as well as for the study of perturbed Bessel equations [6, 25].

From the physical point of view, SPPS method was used for acoustic [30], the heat transfer problem [16], the one-dimensional quantum scattering [29] as well as optical fibers [5]. However, SPPS was first derived using pseudo-analytic function theory in [17] and it has has been shown that this theory
is closely related to the two-dimensional supersymmetric quantum mechanics (SUSYQM) \[2\].

In recent years, SPPS has also proved to be very efficient numerically, see for instance \[3, 5, 6, 10, 12, 13, 14, 16, 17, 19, 20, 22, 24, 25\]. Nevertheless, in this paper we study the generalized power functions \(X^{(n)}, \breve{X}^{(n)}\) appearing in the SPPS method from the analytical point of view. Indeed, despite the extensive use of these functions in the mathematical and the physical literature in recent years and their fundamental role in the SPPS method, these generalized power functions have not been studied much analytically; this is the main objective of this paper.

In Section 2, we present an overview of the one-dimensional SUSYQM. The purpose of this section is mainly to illustrate the fundamental role played by the ground state wave function \(\psi_0(x)\) in these systems. Indeed, we show that despite the function \(1/\psi_0(x)\) cannot be considered as an eigenfunction, the symmetry \(\psi_0(x) \rightarrow 1/\psi_0(x)\) acts as a permutation of the bosonic and the fermionic parts of the SUSYQM system. In Section 3, considering this symmetry the \(\Phi\)-generalized power functions are formally introduced in agreement with the SPPS method (Theorem 1). The principle of alternation \(\Phi(x)\) and \(1/\Phi(x)\) in the definition of the \(\Phi\)-generalized power functions \(X^{(n)}, \breve{X}^{(n)}\) appears according to the parity of \(n\), where \(\Phi(x) = \psi_0^2(x)\) for the SUSYQM system. Many properties of the \(\Phi\)-generalized power functions are given: 1) the symmetries of the variables permutation of these functions which, among other things, will prevent many numerical calculations for the SPPS method, 2) a binomial identity and 3) a generalized trigonometric Pythagorean identity. Generalized Taylor series are developed in Section 4 for the real-valued functions \(\Phi(x)\). Section 5 is devoted to the relations between the Volterra composition of the first type and the generalized power functions. These relationships allow us to construct a general solution representation of the stationary Schrödinger equation in terms of the Volterra composition on a finite interval.

2 One-dimensional supersymmetric quantum mechanics

2.1 General results and overview

Let us consider the simplest one-dimensional SUSYQM system characterised by the existence of charge operators \(Q, Q^\dagger\) that obey the algebra \[33\]

\[
\{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q = \mathcal{H}, \quad Q^2 = 0, \quad Q^{12} = 0.
\]  

From these equations it is easy to see that

\[
[Q, \mathcal{H}] = 0, \quad [Q^\dagger, \mathcal{H}] = 0.
\]
The algebra defined in (4) can be achieved by considering the matrices

\[ Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \tag{5} \]

where \( A \) is a linear operator and \( A^\dagger \) is the adjoint. Equations (4),(5) lead to the supersymmetric Hamiltonian

\[ \mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix}, \quad \text{where} \ \mathcal{H}_1 = A^\dagger A \quad \text{and} \quad \mathcal{H}_2 = AA^\dagger. \]

These scalar Hamiltonians \( \mathcal{H}_1, \mathcal{H}_2 \) are both positive semi-definite operators with eigenvalues greater than or equal to zero.

Let us choose the ground state energy of \( \mathcal{H}_1 \) to be zero. This can be achieved by shifting the potential with the ground state energy, this will result in a shifted energy spectrum, but it will have no other effect on quantum system. Hence, considering the nodeless ground state wave function \( \psi_0^{(1)}(x) \) belonging to the Hamiltonian \( \mathcal{H}_1 \), the Schrödinger equation

\[ \mathcal{H}_1 \psi_0^{(1)}(x) = -\frac{d^2}{dx^2} \psi_0^{(1)}(x) + V_1(x) \psi_0^{(1)}(x) = 0 \]

gives us the possibility to obtain the potential \( V_1 \) in terms of the ground state wave function \( \psi_0^{(1)} \):

\[ V_1(x) = \frac{[\psi_0^{(1)}(x)]''}{\psi_0^{(1)}(x)}. \]

It is now simple to factorize the Hamiltonian \( \mathcal{H}_1 \) as \( A^\dagger A \) for

\[ A = \frac{d}{dx} + W(x) \quad \text{and} \quad A^\dagger = -\frac{d}{dx} + W(x), \tag{6} \]

where the function \( W(x) \) is generally referred to as the superpotential. Hence, by using the operators \( A \) and \( A^\dagger \) it is possible to write \( V_1(x) \) in terms of the superpotential:

\[ V_1(x) = W^2(x) - W'(x). \tag{7} \]

One solution for \( W(x) \) in terms of the ground state wave function of \( \mathcal{H}_1 \) is

\[ W(x) = -\frac{[\psi_0^{(1)}(x)]'}{\psi_0^{(1)}(x)}. \tag{8} \]

This solution is obtained by recognizing that once we satisfy \( A\psi_0^{(1)} = 0 \), i.e.

\[ \psi_0^{(1)}(x) = N \exp \left( -\int_{-\infty}^{x} W(\xi) d\xi \right), \quad N \text{ a normalization constant}, \]

we automatically have a solution to \( \mathcal{H}_1 \psi_0^{(1)} = A^\dagger A\psi_0^{(1)} = 0 \).
Let us now consider the other scalar Hamiltonian $H_2 = AA^\dagger$ obtained by reversing the order of $A$ and $A^\dagger$ in $H_1$. A little simplification shows that the operator $H_2$ is in fact a Hamiltonian corresponding to a new potential $V_2(x)$:

$$H_2 = -\frac{d^2}{dx^2} + V_2(x), \quad V_2(x) = W^2(x) + W'(x). \quad (9)$$

The energy eigenvalues and the wave functions of the Hamiltonians $H_1$ and $H_2$ are related in the following way. For $H_1$ the Schrödinger equation is

$$H_1\psi_m^{(1)} = A^\dagger A\psi_m^{(1)} = E_m^{(1)}\psi_m^{(1)}$$

such that

$$H_2(A\psi_m^{(1)}) = AA^\dagger A\psi_m^{(1)} = E_m^{(1)}(A\psi_m^{(1)}).$$

Accordingly for some $n \geq 0$ we have $\psi_n^{(2)} = c_mA\psi_m^{(1)}$, where $c_m$ is a complex constant and $m \geq 1$. If we suppose that $\psi_m^{(1)}$ is normalized, then $\psi_n^{(2)}$ is normalized with $c_m = \left[E_m^{(1)}\right]^{-\frac{1}{2}}$. We find

$$H_2\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)} = E_n^{(2)}[E_m^{(1)}]^{-\frac{1}{2}} A\psi_m^{(1)}$$

and

$$H_2\psi_n^{(2)} = [E_m^{(1)}]^{-\frac{1}{2}} A A^\dagger A\psi_m^{(1)} = [E_m^{(1)}]^{\frac{1}{2}} A\psi_m^{(1)}$$

such that the difference of these two equations imply $E_n^{(2)} = E_m^{(1)}$. However, since $n \geq 0$ and $m \geq 1$, the lowest energy eigenvalue of $H_2$ is $E_n^{(2)} = E_0^{(2)} = E_0^{(1)}$, and in general we have $E_n^{(2)} = E_{n+1}^{(1)}$. Similar calculations can be made reversing Hamiltonians $H_1$ and $H_2$. Hence, for $n \geq 0$, we can summarize the results as

$$E_0^{(1)} = 0, \quad E_n^{(2)} = E_{n+1}^{(1)},$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A\psi_{n+1}^{(1)}, \quad \text{and} \quad \psi_n^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^\dagger\psi_n^{(2)}.$$

Operator $A$ ($A^\dagger$) not only converts an eigenfunction of $H_1$ ($H_2$) into an eigenfunction of $H_2$ ($H_1$) with the same energy, but it also destroys (creates) an extra node in the eigenfunction. Since the ground state wave function of $H_1$ is annihilated by the operator $A$, this state has no SUSY partner. Hence, from the eigenfunctions of $H_1$ we can determine the eigenfunctions of $H_2$ using the operator $A$, and vice versa. Writing now the eigenfunctions of the Hamiltonians $H_1$, $H_2$ in terms of a two-dimensional column vectors and applying the operators $Q, Q^\dagger$ to these vectors we find

$$Q \begin{pmatrix} \psi_n^{(1)} \\ 0 \end{pmatrix} = \sqrt{E_n^{(2)}} \begin{pmatrix} 0 \\ \psi_n^{(2)} \end{pmatrix}, \quad Q^\dagger \begin{pmatrix} 0 \\ \psi_n^{(2)} \end{pmatrix} = \sqrt{E_{n+1}^{(1)}} \begin{pmatrix} \psi_n^{(1)} \\ 0 \end{pmatrix}, \quad n \geq 0.$$

Hence, the application of operators $Q$ and $Q^\dagger$ can be considered as changing the character of the state from bosonic to fermionic, and vice versa.
2.2 The bosonic-fermionic permutation symmetry from the ground state wave function

Let us consider a symmetry of the supersymmetric Hamiltonian. We observe that from equations (7) and (9) the Hamiltonians \( H_1 \) and \( H_2 \) can be expressed in terms of the superpotential \( W(x) \) and its first derivative. In other words, from equation (8) the partner Hamiltonians \( H_1 \) and \( H_2 \) depend on the ground state wave function \( \psi_0^{(1)}(x) \) of \( H_1 \). Introducing now the operator \( R \psi_0^{(1)} \) which transforms \( \psi_0^{(1)}(x) \rightarrow 1/\psi_0^{(1)}(x) \) we find

\[
R \psi_0^{(1)}[\psi_0^{(1)}] = \tilde{\psi}_0^{(1)} = \frac{1}{\psi_0^{(1)}}, \quad R \psi_0^{(1)}[W] = \tilde{W} = -W,
\]

and

\[
R \psi_0^{(1)}[H_1] = \tilde{H}_1 = H_2, \quad R \psi_0^{(1)}[H_2] = \tilde{H}_2 = H_1,
\]

where the tilde notation will be used in what follows to represent the functions or the operators on which the operator \( R \psi_0^{(1)} \) is applied. Hence, the operator \( R \psi_0^{(1)} \) is just a permutation of the two scalar Hamiltonians \( H_1 \) and \( H_2 \) in the supersymmetric Hamiltonian \( H \).

In particular, we note that since \( A \psi_0^{(1)}(x) = 0 \), we have

\[
R \psi_0^{(1)}[A \psi_0^{(1)}] = \tilde{A} \tilde{\psi}_0^{(1)}(x) = -A^\dagger \left( 1/\psi_0^{(1)}(x) \right) = 0,
\]

i.e. \( H_2 \left( 1/\psi_0^{(1)}(x) \right) = 0 \). However, if we suppose that \( \psi_0^{(1)}(x) \) is normalized, then the function \( 1/\psi_0^{(1)}(x) \) cannot be normalized. Therefore, the function \( 1/\psi_0^{(1)}(x) \) is not considered as the ground state wave function of \( H_2 \); this implies that \( E_0^{(1)} \) is the only nondegenerate state of the supersymmetric Hamiltonian \( H \).

Despite that \( 1/\psi_0^{(1)}(x) \) is not a ground state wave function of \( H_2 \), the symmetry induced by \( R \psi_0^{(1)} \) remains for the partner Hamiltonians for all the eigenvalues \( E_0^{(2)} = E_{n+1}^{(1)} \) of the system for \( n \geq 0 \). Moreover, the SUSYQM system is completely determined by the ground state wave function of \( H_1 \).

In what follows, we will study complex-valued functions generalizing powers functions of the type \( (x-x_0)^n \) for \( n \geq 0 \), taking into account the permutation symmetry \( R \psi_0^{(1)} \) of the one-dimensional SUSYQM system.

3 Definition and properties of the \( \Phi \)-generalized power functions

In the frame work of this paper, in order to simplify the problem, we shall study the functions \( X^{(n)} \) and \( \tilde{X}^{(n)} \) appearing in the representation for the general solution of \( u'' + qu = \lambda u \), i.e. for \( p = r = 1 \) in Theorem 1. This leads us to the following definition.
Definition 2 Let $\Phi(x)$ be a given continuous nonzero complex-valued function on $[a, b]$ and $x_0 \in [a, b]$. The $\Phi$-power functions (or the generalized power functions) are defined iteratively by $X^{(0)}(x_0, x) \equiv 1$, $X^{(0)}(x_0, x) \equiv 1$ and

$$X^{(n)}(x_0, x) = n \int_{x_0}^{x} X^{(n-1)}(x_0, \xi) \left( \frac{1}{\Phi(\xi)} \right)^{(-1)^n} d\xi,$$

$$\tilde{X}^{(n)}(x_0, x) = n \int_{x_0}^{x} \tilde{X}^{(n-1)}(x_0, \xi) \left( \Phi(\xi) \right)^{(-1)^n} d\xi,$$

where $n > 0$. We say that the $\Phi$-power function $\tilde{X}^{(n)}$ is the $\Phi$-conjugate of $X^{(n)}$ (and conversely).

Remark 3 We observe that the $\Phi$-conjugaison of the generalized power functions are obtained from the transformation $R_{\Phi}[X^{(n)}] = \tilde{X}^{(n)}$.

Remark 4 In the case $\Phi \equiv 1$ we have $\tilde{X}^{(n)}(x_0, x) = X^{(n)}(x_0, x)$ and these functions are equal to $(x - x_0)^n$. In the general case, the $\Phi$-power functions $X^{(n)}$, $\tilde{X}^{(n)}$ are distinct and not necessarily polynomial functions.

When considering the general solution representation of the one-dimensional stationary Schrödinger equation $H_1 \psi(x) = -\frac{d^2}{dx^2} \psi(x) + V_1(x) \psi(x) = \lambda \psi(x)$ from the point of view of Theorem 1 i.e. $p = -1$, $r = 1$ and $q = V_1 = [\psi_0^{(1)}]^n / \psi_0^{(1)}$, then the generalized power functions $X_1^{(n)}$, $\tilde{X}_1^{(n)}$ associated with this Schrödinger equation in Theorem 1 are related to the generalized power functions $X^{(n)}$, $\tilde{X}^{(n)}$ in (10) by taking $\Phi = [\psi_0^{(1)}]^2$. Since $p = -1$ we have $X_1^{(n)} = (-1)^n X^{(n)}$ and $\tilde{X}_1^{(n)} = (-1)^{n+1} \tilde{X}^{(n)}$. Hence, from (1) and (2) the general solution representation of $H_1 \psi(x) = \lambda \psi(x)$ can be written in terms of the functions $X^{(n)}$, $\tilde{X}^{(n)}$ (up to a minus sign absorbed by the arbitrary complex constants $c_1$, $c_2$) defined in (10) by

$$\psi = c_1 \psi_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \tilde{X}^{(2k)} + c_2 \psi_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} X^{(2k+1)}.$$

Applying now operator $R_{\Phi}$ on equation $H_1 \psi(x) = \lambda \psi(x)$, we obtain the general solution representation $\tilde{\psi}(x)$ of the supersymmetric partner Schrödinger equation $H_2 \tilde{\psi}(x) = \lambda \tilde{\psi}(x)$, i.e.

$$\tilde{\psi} = c_1 \tilde{\psi}_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} X^{(2k)} + \tilde{c}_2 \psi_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \tilde{X}^{(2k+1)}.$$

Theorem 5 Let $X^{(n)}$ and $\tilde{X}^{(n)}$ be $\Phi$-generalized powers. These functions are $\Phi$-symmetric conjugate for even $n$ and antisymmetric for odd $n$, i.e.

$$\tilde{X}^{(n)}(x_0, x) = X^{(n)}(x, x_0), \quad n \text{ even } \quad (11)$$
and
\[ \widetilde{X}^{(n)}(x_0, x) = -\tilde{X}^{(n)}(x, x_0), \quad n \text{ odd}, \]
\[ X^{(n)}(x_0, x) = -X^{(n)}(x, x_0), \quad n \text{ odd}. \]  

Proof. Let \( f_1(x), \ldots, f_n(x) \) be continuous complex-valued functions in \([a, b]\) such that \( f_i(x) \neq 0 \) for all \( x \in [a, b] \). We consider Chen’s iterated path integrals introduced in [7] and defined by
\[
\int_{x_0}^{x} f_1(\xi)d\xi \cdot \cdots \cdot f_n(\xi)d\xi := \int_{x_0}^{x} \left( \int_{x_0}^{\xi} f_1(\eta)d\eta \cdot \cdots \cdot f_{n-1}(\eta)d\eta \right) f_n(\xi)d\xi \quad \text{for } n > 1.
\]
When \( n = 0 \), set the integral to be 1. The following property was obtained by Chen for iterated integrals [7]:
\[ \int_{x_0}^{x} f_1(\xi)d\xi \cdot \cdots \cdot f_n(\xi)d\xi = (-1)^n \int_{x_0}^{x} f_n(\xi)d\xi \cdot \cdots \cdot f_1(\xi)d\xi. \]  

Now let us consider the particular case
\[ f_n = \left( \frac{1}{\Phi} \right)^{(-1)^n} \quad \text{for } n \geq 1. \]  

Then equation (13) becomes
\[ \frac{1}{n!} \widetilde{X}^{(n)}(x_0, x) = \begin{cases} 
\frac{-1}{n!} \tilde{X}^{(n)}(x_0, x), & n \text{ odd} \\
\frac{1}{n!} X^{(n)}(x_0, x), & n \text{ even},
\end{cases} \]
which demonstrate property (11) and property (12) for the \( \Phi \)-conjugate \( \tilde{X}^{(n)} \). Property (12) for the \( X^{(n)} \) is obtained by considering the case \( f_n(x) = \Phi^{(-1)^n} \) instead of functions (14). \( \blacksquare \)

Remark 6 Theorem 5 is particularly interesting when SPPS method is used for numerical analysis. Indeed, when a SPPS numerical calculations is performed both set of functions \( \tilde{X}^{(n)}(x_0, x) \), \( X^{(n)}(x_0, x) \) are calculated for some \( N > 0 \) and \( n = 1, \ldots, N \). For even numbers \( n \), this theorem gives us the possibility to obtain all the functions \( X^{(n)}(x_0, x) \) from the functions \( X^{(n)}(x, x_0) \). Consequently, numerical calculations of the SPPS could be improved by about 25 \% by using this result.

Remark 7 A corollary is obtained from Theorem 5 for \( n \geq 1 \) the \( \Phi \)-conjugate power functions are related by the formulas
\[ X^{(n)}(x_0, x) = n \int_{x_0}^{x} \frac{\tilde{X}^{(n-1)}(\xi, x)}{\Phi(\xi)} d\xi, \quad \text{n even} \]
\[ \tilde{X}^{(n)}(x_0, x) = n \int_{x_0}^{x} \Phi(\xi) X^{(n-1)}(\xi, x)d\xi, \quad \text{n odd}. \]
Indeed, considering first the even \( n \), from the definition we have
\[
\tilde{X}(n)(x, x_0) = n \int_x^{x_0} \frac{\tilde{X}(n-1)(x, \xi)}{\Phi(\xi)} d\xi = n \int_{x_0}^{x} \frac{\tilde{X}(n-1)(\xi, x)}{\Phi(\xi)} d\xi,
\]
where in the last equality is obtained by interchanging the order of integration and using Theorem 5. However, since \( n \) is even \( \tilde{X}(n)(x, x_0) = X(n)(x_0, x) \) again from Theorem 5. Hence, we obtain the the desired result when \( n \) is even. The second formula when \( n \) is odd is easily proven in a similar way.

**Theorem 8 (Binomial identity for even \( n \))** Let \( \Phi(x) \) be a given continuous nonzero complex-valued function on \([a, b]\) and \( x_0 \in [a, b] \). Then for a finite and an even \( n \geq 2 \) the \( \Phi \)-powers functions \((10)\) satisfy the following identity:
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} X(k) \tilde{X}(n-k) = 0
\]
or equivalently
\[
\tilde{X}(n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} X(k) \tilde{X}(n-k).
\]

**Proof.** By integrating by parts for an even \( n \geq 2 \) we have
\[
X(n)(x_0, x) = n \int_{x_0}^{x} X(n-1)(x_0, \xi) \Phi(\xi) d\xi
= n \left[ X(n-1) \tilde{X}(1) - (n-1) \int_{x_0}^{x} \tilde{X}(1) X(n-2) \frac{1}{\Phi} d\xi \right],
\]
where for simplicity the arguments of the functions have been omitted in the second line (and for the rest of the proof). In particular, we observe that when \( n = 2 \) we obtain \( X(2) = 2X(1)\tilde{X}(1) - \tilde{X}(2) \) which prove the identity in this case.

Now by integrating by parts again for \( n \geq 3 \), we find
\[
\int_{x_0}^{x} \tilde{X}(1) X(n-2) \frac{1}{\Phi} d\xi = \frac{1}{2} \tilde{X}(2) X(n-2) - \frac{1}{2} (n-2) \int_{x_0}^{x} \tilde{X}(2) X(n-3) \Phi d\xi
\]
such that
\[
X(n) = n \tilde{X}(1) X(n-1) - \frac{1}{2} n(n-1) \tilde{X}(2) X(n-2) + \frac{1}{2} n(n-1)(n-2) \int_{x_0}^{x} \tilde{X}(2) X(n-3) \Phi d\xi.
\]
Pursuing these calculations, i.e. applying \( n - 1 \) integration by parts for an even \( n \), we finally obtain
\[
X(n) = \sum_{k=1}^{n-1} (-1)^{k+1} \frac{n!}{k!(n-k)!} X(k) X(n-k) - n \int_{x_0}^{x} \tilde{X}(n-1) X(0) \frac{1}{\Phi} d\xi,
\]
where by definition the last term is just \( \tilde{X}(n) \) and this complete the proof. \( \blacksquare \)
Remark 9 Considering the same proof when $n$ is odd and finite, we obtain the trivial identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{(k)} X^{(n-k)} = 0.$$  

Remark 10 From the usual binomial expansion, Theorem 8 can be seen from a purely symbolic point of view as

$$(X - \bar{X})^{(n)} = 0 \quad \text{for even } n.$$

A corollary can be obtained from Theorem 8 to obtain a generalization of the elliptic and the hyperbolic Pythagorean identities. For that, let us define the $\Phi$-generalized sine and cosine functions as

$$C(x_0, x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} X^{(2j)}(x_0, x), \quad \tilde{C}(x_0, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \tilde{X}^{(2k)}(x_0, x),$$

$$S(x_0, x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} X^{(2j+1)}(x_0, x), \quad \tilde{S}(x_0, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \tilde{X}^{(2k+1)}(x_0, x),$$

and the $\Phi$-generalized hyperbolic functions as

$$Ch(x_0, x) = \sum_{j=0}^{\infty} \frac{X^{(2j)}(x_0, x)}{(2j)!}, \quad \tilde{Ch}(x_0, x) = \sum_{k=0}^{\infty} \frac{\tilde{X}^{(2k)}(x_0, x)}{(2k)!},$$

$$Sh(x_0, x) = \sum_{j=0}^{\infty} \frac{X^{(2j+1)}(x_0, x)}{(2j+1)!}, \quad \tilde{Sh}(x_0, x) = \sum_{k=0}^{\infty} \frac{\tilde{X}^{(2k+1)}(x_0, x)}{(2k+1)!}.$$  

These series are convergent. Indeed, from definition 2 we find

$$\frac{X^{(2j)}(x_0, x)}{(2j)!} = \int_{x_0}^{x} \Phi(\xi_1) \int_{x_0}^{\xi_1} \frac{1}{\Phi(\xi_2)} \int_{x_0}^{\xi_2} \Phi(\xi_3) \int_{x_0}^{\xi_3} \cdots \frac{1}{\Phi(\xi_{2j})} d\xi_{2j} \cdots d\xi_1$$

and

$$\frac{X^{(2j+1)}(x_0, x)}{(2j+1)!} = \int_{x_0}^{x} \frac{1}{\Phi(\xi)} \int_{x_0}^{\xi} \Phi(\xi_2) \int_{x_0}^{\xi_2} \frac{1}{\Phi(\xi_3)} \int_{x_0}^{\xi_3} \cdots \int_{x_0}^{\xi_{2j}} \frac{1}{\Phi(\xi_{2j+1})} d\xi_{2j+1} \cdots d\xi_1$$

such that on the interval $[a, b]$ we have

$$|X^{(2j)}(x_0, x)| \leq (\max |\Phi(x)|)^j \left(\max \left|\frac{1}{\Phi(x)}\right|\right)^j |b - a|^{2j} = c^j$$

and

$$|X^{(2j+1)}(x_0, x)| \leq (\max |\Phi(x)|)^j \left(\max \left|\frac{1}{\Phi(x)}\right|\right)^{j+1} |b - a|^{2j+1} = |b - a| \left(\max \left|\frac{1}{\Phi(x)}\right|\right)c^j$$

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for \( c = \left( \max \Phi(x) \right) \left( \max \frac{1}{\Phi(x)} \right) |b - a|^2 \). Hence, the series \( \sum_{j=0}^{\infty} \frac{c^j}{j!} < \infty \) and \( |b - a| \left( \max \frac{1}{\Phi(x)} \right) \sum_{j=0}^{\infty} \frac{c^j}{(2j+1)!} < \infty \) such that the \( \Phi \)-trigonometric functions and the \( \Phi \)-hyperbolic functions converge absolutely and uniformly from the Weierstrass M-test.

**Theorem 11 (Supersymmetric Pythagorean identities)** The \( \Phi \)-trigonometric functions satisfy the following generalized Pythagorean trigonometric identity

\[
C(x_0, x) \tilde{C}(x_0, x) + S(x_0, x) \tilde{S}(x_0, x) = 1.
\]

Moreover, the \( \Phi \)-hyperbolic functions satisfy the generalized hyperbolic identity

\[
Ch(x_0, x) \tilde{Ch}(x_0, x) - Sh(x_0, x) \tilde{Sh}(x_0, x) = 1.
\]

**Proof.** Let us consider first the Pythagorean trigonometric identity. From the infinite double sum property

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{k,j} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} c_{q,p-q}, \tag{15}
\]

we obtain

\[
C\tilde{C} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k}}{(2j)!(2k)!} X^{(2j)} \tilde{X}^{(2k)} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{(-1)^p}{(2q)!(2p-2q)!} X^{(2p-2q)} \tilde{X}^{(2q)}
\]

\[
= 1 + \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{(-1)^{p+1}}{(2q)!(2p+2-2q)!} X^{(2p+2-2q)} \tilde{X}^{(2q)}
\]

\[
= 1 + \sum_{p=0}^{\infty} \left( \sum_{q=0}^{p} \frac{(-1)^{p+1}}{(2q)!(2p+2-2q)!} X^{(2p+2-2q)} \tilde{X}^{(2q)} + \frac{(-1)^{p+1}}{(2p+2)!} \tilde{X}^{(2p+2)} \right).
\]

Using again (15) for \( S\tilde{S} \) we find

\[
S\tilde{S} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{(-1)^p}{(2q+1)!(2p+1-2q)!} X^{(2p+1-2q)} \tilde{X}^{(2q+1)}
\]
such that

\[ C\tilde{C} + S\tilde{S} = 1 + \sum_{p=0}^{\infty} \left( \sum_{q=0}^{p} \frac{(-1)^{p+1}}{(2q)!((2p + 2) - 2q)!} X^{(2p+2-2q)} \tilde{X}(2q) + \right. \]

\[ + \frac{(-1)^p}{(2q + 1)!(2p + 1 - 2q)!} X^{(2p+1-2q)} \tilde{X}(2q+1) + \frac{(-1)^{p+1}}{(2p + 2)!} \tilde{X}^{(2p+2)} \]

\[ = 1 + \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{(2p + 2)!} \left( \sum_{q=0}^{p} \frac{(2p + 2)!}{(2q)!(2p + 2 - 2q)!} X^{(2p+2-2q)} \tilde{X}(2q) + \right. \]

\[ - \frac{(2p + 2)!}{(2q + 1)!(2p + 2 - (2q + 1))!} X^{(2p+1-2q)} \tilde{X}(2q+1) + \tilde{X}^{(2p+2)} \]

\[ = 1 + \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{(2p + 2)!} \sum_{q=0}^{2p+2} (-1)^q \binom{2p + 2}{q} X^{(2p+2-q)} \tilde{X}(q). \]

However, since \(2p + 2\) is even and greater or equal than 2 the last summation over \(q\) is always zero from Theorem 8 and this complete the proof for the first identity.

The hyperbolic identity can be proved in a similar way. ■

The \(\Phi\)-trigonometric functions are illustrated in Figure 4 for the real-valued function \(\Phi(x) = (1 + x)^2\). The generalized Pythagorean trigonometric identity of Theorem 11 are then shown in Figure 2 for the same function \(\Phi(x)\). This case can be seen as a perturbation of the standard trigonometric functions, i.e. \(\Phi \equiv 1\) around \(x = 0\). In the same way, we illustrate the \(\Phi\)-trigonometric functions in Figures 3 and the generalized Pythagorean trigonometric identity in Figures 4 for the real-valued function \(\Phi(x) = \sqrt{\cosh x}\). In both case, we set \(x_0 = 0\). We limit ourselves to these two examples, however several other functions \(\Phi(x)\) have been considered.
Figure 1: Graphics of the $\Phi$-trigonometric functions for $\Phi = (1 + x)^2$ and $x_0 = 0$
Figure 2: Generalized Pythagorean trigonometric identity for $\Phi = (1 + x)^2$ and $x_0 = 0$ in the phase space of the $\Phi$-trigonometric functions
Figure 3: Graphics of the $\Phi$-trigonometric functions for $\Phi = \sqrt{\cosh x}$ and $x_0 = 0$
Figure 4: Generalized Pythagorean trigonometric identity for $\Phi = \sqrt{\cosh x}$ and $x_0 = 0$ in the phase space of the $\Phi$-trigonometric functions

In what follows the set of $\Phi$-power functions $\{X(2k)\}_{k=0}^{\infty} \cup \{\tilde{X}(2k+1)\}_{k=0}^{\infty}$ (as well as the $\Phi$-conjugate set $\{\tilde{X}(2k)\}_{k=0}^{\infty} \cup \{X(2k+1)\}_{k=0}^{\infty}$) will be of particular interest. For simplicity, we define the functions $Y_n(x_0, x)$, $\tilde{Y}_n(x_0, x)$ as

$$Y_n(x_0, x) := \begin{cases} \tilde{X}^{(n)}(x_0, x), & n \text{ odd}, \\ X^{(n)}(x_0, x), & n \text{ even}, \end{cases}$$

(16)

and

$$\tilde{Y}_n(x_0, x) := \begin{cases} X^{(n)}(x_0, x), & n \text{ odd}, \\ \tilde{X}^{(n)}(x_0, x), & n \text{ even}, \end{cases}$$

(17)

for $n \geq 0$.

It was shown that generalized powers posses the property of completeness in $L^2(a, b)$ [18, 21]. Indeed, for $x_0$, an arbitrary fixed point in $[a, b]$ it has been shown that when the point $x_0$ coincides with one of the extreme of the interval, then the infinite system of functions $\{X(2k)\}_{k=0}^{\infty}$ is complete in $L^2(a, b)$. Under the same condition, the completeness of $\{X(2k+1)\}_{k=0}^{\infty}$ is guaranteed in $L^2(a, b)$. Meanwhile, when $x_0$ is an interior point of the interval the system of functions $\{Y_k\}_{k=0}^{\infty}$ is complete in $L^2(a, b)$. For instance, in the case $\Phi \equiv 1$ and choosing $x_0 = 0$ on the interval $[0, 1]$, we find

$$X^{(0)}(x) = 1, \quad X^{(2)}(x) = x^2, \quad X^{(4)}(x) = x^4, \ldots$$

(18)

and

$$\tilde{X}^{(1)}(x) = x, \quad \tilde{X}^{(3)}(x) = x^3, \quad \tilde{X}^{(5)}(x) = x^5, \ldots$$

(19)

Both systems of monomials are complete in $L^2(0, 1)$ due to the Müntz theorem [9, p.275]. Hence, the system of functions $\{X^{(2n)}\}_{n=0}^{\infty}$ and $\{\tilde{X}^{(2n+1)}\}_{n=0}^{\infty}$ repre-
sent a direct generalization of the system of monomials (18), (19) if instead of \( \Phi \equiv 1 \) an arbitrary sufficiently smooth and nonvanishing function \( \Phi \) is chosen.

Let us now introduce a \( \Phi \)-derivative which will be used in what follows:

\[
D_h(x) = \Phi(x) \frac{d}{dx} h(x) \quad \text{and} \quad \tilde{D}_h(x) = \frac{1}{\Phi(x)} \frac{d}{dx} h(x),
\]

where \( h(x) \) is an arbitrary complex-valued function well defined on the interval of interest. For \( D(0)h(x) \equiv h(x) \) and \( \tilde{D}(0)h(x) \equiv h(x) \) and the higher-order \( \Phi \)-derivatives are defined alternately as

\[
D^{(k)}h(x) = D\left(\tilde{D}^{(k-1)}h(x)\right) \quad \text{and} \quad \tilde{D}^{(k)}h(x) = \tilde{D}\left(D^{(k-1)}h(x)\right).
\]

These definitions of generalized derivatives agree with many aspects with the standard derivative applied on functions of the type \( (x - x_0)^n \). For instance, we have that

\[
D^{(k)}X^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) X^{(n-k)}(x_0, x),
\]

(20)

\[
\tilde{D}^{(k)}\tilde{X}^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) \tilde{X}^{(n-k)}(x_0, x).
\]

(21)

On the other, when \( n, k \in \mathbb{Z}_{\geq 0} \) are not of the same parity and \( n > k \), we obtain

\[
\tilde{D}^{(k)}X^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) X^{(n-k)}(x_0, x),
\]

(22)

\[
D^{(k)}\tilde{X}^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) \tilde{X}^{(n-k)}(x_0, x).
\]

(23)

Proposition 12 When \( n, k \in \mathbb{Z}_{\geq 0} \) are of the same parity and \( n \geq k \), we have

\[
D^{(k)}X^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) X^{(n-k)}(x_0, x),
\]

(20)

\[
\tilde{D}^{(k)}\tilde{X}^{(n)}(x_0, x) = \left(\frac{n!}{(n - k)!}\right) \tilde{X}^{(n-k)}(x_0, x).
\]

(21)

Proof. We begin by considering induction on \( n \) for equation (20). The cases \( n = 0, 1 \) are trivially verified. Let us suppose that \( n \) is even and equation (20) is valid for \( 0 \leq k \leq n \) and \( k \) even. Then for \( 1 \leq k \leq n + 1 \) and \( k \) odd we obtain

\[
D^{(k)}X^{(n+1)}(x_0, x) = D\left(\tilde{D}^{(k-1)}X^{(n+1)}(x_0, x)\right) = D^{(k-1)}(DX^{(n+1)})
\]

(24)

\[
= D^{(k-1)}\left((n + 1)\Phi X^{(n)}(x_0) \frac{1}{\Phi}\right) = (n + 1) \left(\frac{n!}{(n + 1 - k)!}\right) X^{(n-k+1)},
\]

where the second equality in (24) is valid for \( k \) odd. This proves the first equation (20) when \( n \) is even. The case \( n \) odd (and \( k \) odd) of equation (20) is proved in a similar way as well as its \( \Phi \)-conjugate counterpart (21).

Equations (22) and (23) are proven in a similar way. ■
Remark 13 Despite these results, other combinations of \( \Phi \)-derivatives applied on the \( \Phi \)-power functions do not give such simple results. For instance, we have
\[
D^{(2)} X^{(3)} = 6X^{(1)} - 6 \left( \frac{\Phi'}{\Phi} \right) X^{(2)}.
\]

Remark 14 Considering the \( \Phi \)-derivatives of the generalized trigonometric and the hyperbolic functions, we easily show that
\[
DS(x_0, x) = C(x_0, x), \quad DC(x_0, x) = -S(x_0, x), \quad D[S\varphi(x_0, x)] = C\varphi(x_0, x) \text{ and } D[C\varphi(x_0, x)] = S\varphi(x_0, x).
\]

4 Supersymmetric Taylor series from the \( \Phi \)-power functions

In this section, and only in this section, we suppose that the function \( \Phi(x) \) is a real-valued function on \([a, b]\). We will treat the approximation of functions by generalized Taylor series using \( \Phi \)-generalized powers. Taylor series have already been considered from the iterated integrals \( X^{(n)} \), \( \bar{X}^{(n)} \) for the Sturm-Liouville equation, see [21]; here we consider a more general setting and a different approach.

In what follows we shall have frequent occasions to use Wronskian so that it will be convenient to introduce some notations. The functions \( f_1(x), \ldots, f_n(x) \) being of class \( C^n \), we set
\[
W[f_1(x), f_2(x), \ldots, f_n(x)] = \begin{vmatrix}
  f_1(x) & f_2(x) & \cdots & f_n(x) \\
  f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(n)}_1(x) & f^{(n)}_2(x) & \cdots & f^{(n)}_n(x)
\end{vmatrix}.
\]

Moreover, since in what follows we will pay special attention to the sets
\( \{Y_0(x_0, x), Y_1(x_0, x), \ldots, Y_n(x_0, x)\} \) and \( \{\tilde{Y}_0(x_0, x), \tilde{Y}_1(x_0, x), \ldots, \tilde{Y}_n(x_0, x)\} \), we set as a particular case
\[
W_n(x) = W[Y_0(x_0, x), Y_1(x_0, x), \ldots, Y_n(x_0, x)]
\]
and
\[
\tilde{W}_n(x) = W[\tilde{Y}_0(x_0, x), \tilde{Y}_1(x_0, x), \ldots, \tilde{Y}_n(x_0, x)].
\]

Proposition 15 Let \( x_0 \) be a given point in the interval \((a, b)\) of \( \Phi \) and the two sets of functions \( S_n = \{Y_0(x_0, x), \ldots, Y_n(x_0, x)\} \), \( \tilde{S}_n = \{\tilde{Y}_0(x_0, x), \ldots, \tilde{Y}_n(x_0, x)\} \) are of class \( C^n(a, b) \). Then the sets of functions \( S_n \) and \( \tilde{S}_n \) are linearly independant.

Proof. Let us first consider the Wronskian matrix at the point \( x = x_0 \), i.e. calculate the matrix \( Y^{(i)}(x_0, x_0) \) for \( 0 \leq i, j \leq n \). We have that \( Y_j(x_0, x_0) = \delta_{i,0} \), where \( \delta_{i,k} \) is the Kronecker delta. The first line and column of the Wronskian matrix being known, we now consider the strictly upper diagonal terms \( Y^{(j)}_k(x_0, x) \), where \( 1 \leq j < k \leq n \). Considering the derivatives, these terms can
be expanded as a summation of the functions $Y_r(x_0, x)$, $\tilde{Y}_r(x_0, x)$ for $1 \leq r \leq k - 1$. Hence, evaluated at $x = x_0$ we find $\tilde{Y}_k^{(j)}(x_0, x) = 0$ and the Wronskian matrix is lower triangular. By similar calculations we find that Wronskian matrix $\tilde{Y}_j^{(i)}(x_0, x_0)$ of the functions $\tilde{S}_n$ at $x = x_0$ is lower triangular.

On the diagonal of the Wronskian matrices we obtain

$$Y_m^{(m)}(x_0, x_0) = \begin{cases} m! \Phi(x_0), & m \text{ odd}, \\ m!, & m \text{ even}, \end{cases}$$  \hspace{1cm} (25)$$

and

$$\tilde{Y}_m^{(m)}(x_0, x_0) = \begin{cases} m! \Phi^{-1}(x_0), & m \text{ odd}, \\ m!, & m \text{ even}, \end{cases}$$  \hspace{1cm} (26)$$

for $0 \leq m \leq n$. This can be shown by induction on $n$. Indeed, equations (25), (26) are satisfied for $n = 0, 1$. For an even $n > 0$ suppose that equations (25), (26) are valid for $0 \leq m \leq n - 1$. We have

$$Y_n^{(n)}(x_0, x_0) = \frac{d^{n-1}}{dx^{n-1}} Y_n = \frac{d^{n-1}}{dx^{n-1}} \left(n \Phi \tilde{Y}_{n-1}^{(n-1)} \right) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \Phi(n-1-k) \tilde{Y}_{n-1}^{(k)}.$$

As has been shown below, the terms $\tilde{Y}_{n-1}^{(k)}$ evaluated at $x = x_0$ are zero for $k = 0, \ldots, n - 2$ such that

$$Y_n^{(n)}(x_0, x_0) = n \Phi(x_0) \tilde{Y}_{n-1}^{(n-1)}(x_0, x_0) = n \Phi(x_0)(n-1)! \Phi^{-1}(x_0) = n!.$$

The case $n$ odd and greater than 1 can be shown in a similar way to complete the proof of (25). Equation (26) is also proved by induction in a similar way.

Hence, the Wronskians at $x = x_0$ is given by

$$W_n(x_0) = \prod_{k=0}^{n} \tilde{Y}_k^{(k)}(x_0, x_0) = \begin{cases} \alpha_n \left(\sqrt{\Phi(x_0)}\right)^{n+1}, & n \text{ odd}, \\ \alpha_n \left(\sqrt{\Phi(x_0)}\right)^n, & n \text{ even}, \end{cases}$$

and

$$\tilde{W}_n(x_0) = \begin{cases} \alpha_n \left(\frac{1}{\sqrt{\Phi(x_0)}}\right)^{n+1}, & n \text{ odd}, \\ \alpha_n \left(\frac{1}{\sqrt{\Phi(x_0)}}\right)^n, & n \text{ even}, \end{cases}$$

where

$$\alpha_n := \prod_{k=0}^{n} k!.$$

Now since the set of $n + 1$ functions of $S_n$ ($\tilde{S}_n$) are $n$ times differentiable over the interval $(a, b)$ with $W_n(x_0) \neq 0$ ($\tilde{W}_n(x_0) \neq 0$) for some $x_0 \in (a, b)$, the functions of $S_n$ ($\tilde{S}_n$) are linearly independent from the Abel’s identity. \qed
Proposition 16 Let \( x_0 \) be a given point in the interval \([a, b]\) of \( \Phi \). Then the fundamental set of solutions for the ordinary differential equation \( D^{(n+1)}y(x) = 0 \) of order \( n + 1 \) is

\[
\begin{align*}
S_n &= \{Y_0(x_0, x), \ldots, Y_n(x_0, x)\}, \quad n \text{ odd}, \\
\tilde{S}_n &= \{\tilde{Y}_0(x_0, x), \ldots, \tilde{Y}_n(x_0, x)\}, \quad n \text{ even}.
\end{align*}
\]

Moreover, the fundamental set of solutions for the ordinary differential equation \( D^{(n+1)}y(x) = 0 \) of order \( n + 1 \) is

\[
\begin{align*}
\tilde{S}_n &= \{\tilde{Y}_0(x_0, x), \ldots, \tilde{Y}_n(x_0, x)\}, \quad n \text{ odd}, \\
S_n &= \{Y_0(x_0, x), \ldots, Y_n(x_0, x)\}, \quad n \text{ even}.
\end{align*}
\]

Proof. Let us first consider the case \( n \) odd. For \( 0 \leq k \leq n \) we obtain

\[
D^{(n+1)}Y_k = \begin{cases} \\
D^{(n+1-k)}(\tilde{D}^{(k)}X^{(k)}) = D^{(n+1-k)}k! = 0, & k \text{ odd}, \\
D^{(n+1-k)}(D^{(k)}X^{(k)}) = D^{(n+1-k)}k! = 0, & k \text{ even},
\end{cases}
\]

where proposition [12] was used. Similarly, we show that \( \tilde{D}^{(n+1)}Y_k = 0 \) when \( n \) is odd and \( 0 \leq k \leq n \).

For \( n \) even and \( 0 \leq k \leq n \) we have

\[
D^{(n+1)}\tilde{Y}_k = \begin{cases} D^{(n+1-k)}(D^{(k)}X^{(k)}) = D^{(n+1-k)}k! = 0, & k \text{ odd}, \\
D^{(n+1-k)}(\tilde{D}^{(k)}X^{(k)}) = D^{(n+1-k)}k! = 0, & k \text{ even}.
\end{cases}
\]

In the same way, we find that \( \tilde{D}^{(n+1)}Y_k = 0 \) when \( n \) is even and \( 0 \leq k \leq n \).

To demonstrate Proposition [15] it was sufficient to show that \( \tilde{W}_n(x_0) \neq 0 \) and \( \tilde{W}_n(x_0) \neq 0 \). Nevertheless, in the following it will be useful to obtain an explicit form of the Wronskians \( W_n(x) \) and \( \tilde{W}_n(x) \) for all \( x \) in the interval \([a, b]\). For that we use explicitly the Abel's identity. From Proposition [16] the set \( S_n = \{Y_0(x_0, x), \ldots, Y_n(x_0, x)\} \) is a fundamental set of solutions for the ordinary differential equations \( D^{(n+1)}y(x) = 0 \) with \( n \) odd and \( \tilde{D}^{(n+1)}y(x) = 0 \) with \( n \) even. By developing these equations in terms of the usual derivatives, i.e.

\[
D^{(n+1)}y(x) = a_{n+1}(x)y^{(n+1)}(x) + a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) = 0
\]

and

\[
\tilde{D}^{(n+1)}y(x) = \tilde{a}_{n+1}(x)y^{(n+1)}(x) + \tilde{a}_n(x)y^{(n)}(x) + \cdots + \tilde{a}_1(x)y'(x) = 0
\]

we find

\[
a_{n+1}(x) = 1, \quad a_n(x) = -\frac{n+1}{2} \frac{\Phi'(x)}{\Phi(x)}, \quad n \text{ odd},
\]

\[
\tilde{a}_{n+1}(x) = \frac{1}{\Phi(x)}, \quad \tilde{a}_n(x) = -\frac{n}{2} \frac{\Phi'(x)}{\Phi(x)^2}, \quad n \text{ even},
\]

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such that $W_n(x) = W_n(x_0) \exp \left( - \int_{x_0}^{x} \frac{a_n(\xi)}{a_{n+1}(\xi)} d\xi \right)$ by the Abel’s identity. Explicitly, we find

$$W_n(x) = \begin{cases} 
\alpha_n \left( \sqrt{\Phi(x)} \right)^{n+1}, & n \text{ odd}, \\
\alpha_n \left( \sqrt{\Phi(x)} \right)^n, & n \text{ even}.
\end{cases} \tag{29}$$

By similar calculation, the set $\tilde{S}_n = \{ \tilde{Y}_0(x_0, x), \ldots, \tilde{Y}_n(x_0, x) \}$ is a fundamental set of solutions for the ordinary differential equations $D^{(n+1)}y(x) = 0$ with $n$ even and $\tilde{D}^{(n+1)}y(x) = 0$ with $n$ odd. We find

$$\tilde{W}_n(x) = \begin{cases} 
\alpha_n \left( \frac{1}{\sqrt{\Phi(x)}} \right)^{n+1}, & n \text{ odd}, \\
\alpha_n \left( \frac{1}{\sqrt{\Phi(x)}} \right)^n, & n \text{ even}.
\end{cases} \tag{30}$$

The function $\tilde{X}^{(n)}/\Phi$ (resp. $X^{(n)}/\Phi$) is the function of Cauchy used in obtaining a particular solution of the nonhomogeneous equation $D^{(n+1)}y(x) = h(x)$ (resp. $\tilde{D}^{(n+1)}y(x) = h(x)$) from their solution of the corresponding homogeneous equation. The particular solutions $y_p(x)$ is given by

$$y_p(x) = \frac{1}{n!} \int_{x_0}^{x} \tilde{X}^{(n)}(\xi, x) \frac{h(\xi)}{\Phi(\xi)} d\xi$$

for equation $D^{(n+1)}y(x) = h(x)$ and

$$y_p(x) = \frac{1}{n!} \int_{x_0}^{x} X^{(n)}(\xi, x) \Phi(\xi) h(\xi) d\xi$$

for equation $\tilde{D}^{(n+1)}y(x) = h(x)$.

Indeed, we have

$$D^{(n+1)}y_p(x) = D\tilde{D}^{(n)}y_p(x) = D \left[ \frac{1}{n!} \int_{x_0}^{x} \left( \tilde{D}^{(n)}\tilde{X}^{(n)}(\xi, x) \right) \frac{h(\xi)}{\Phi(\xi)} d\xi \right]$$

$$= \int_{x_0}^{x} \frac{h(\xi)}{\Phi(\xi)} d\xi = h(x).$$

Similar calculations are made to show a particular solution of $\tilde{D}^{(n+1)}y(x) = h(x)$.

**Theorem 17 (Supersymmetric $\Phi$-Taylor series)** Let the functions $f(x), \gamma_0(x), \gamma_1(x), \ldots, \gamma_n(x)$ be real-valued functions of class $C^{n+1}$ in the interval $[a, b]$ and $a \leq x_0 \leq b$. Then

$$f(x) = \sum_{k=0}^{n} \frac{D_k f(x_0)}{k!} \gamma_k(x_0, x) + R_n(x), \tag{31}$$

21
where

\[ D_k f(x) = \begin{cases} \bar{D}^{(k)} f(x), & k \text{ odd,} \\ D^{(k)} f(x), & k \text{ even,} \end{cases} \quad \text{and} \quad R_n(x) = \frac{1}{n!} \int_{x_0}^{x} [\Phi(\xi)]^{(-1)^n} Y_n(\xi, x) D_{n+1} f(\xi) d\xi. \]

**Proof.** From Proposition 15 the Wronskian \( W_n(x) > 0 \) in the interval \([a, b]\). Now considering the result obtained by Widder [32, p.138], we have

\[ f(x) = \sum_{k=0}^{n} L_k f(x_0) g_k(x_0, x) + R_n(x), \quad (32) \]

where the functions \( g_k(x_0, x) \) are defined as \( g_0(x_0, x) = 1 \) and

\[ g_k(x_0, x) = \left( \frac{1}{W_k(x_0)} \right) \begin{vmatrix} 1 & Y_1(x_0, x_0) & \cdots & Y_k(x_0, x_0) \\ 0 & Y_1^{(1)}(x_0, x_0) & \cdots & Y_k^{(1)}(x_0, x_0) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Y_1^{(k-1)}(x_0, x_0) & \cdots & Y_k^{(k-1)}(x_0, x_0) \\ 1 & Y_1(x_0, x) & \cdots & Y_k(x_0, x) \end{vmatrix}, \quad (33) \]

while the operators \( L_k \) are defined as \( L_0 = \text{id} \) and

\[ L_k f(x) = \frac{W[Y_0(x_0, x), Y_1(x_0, x), \ldots, Y_{k-1}(x_0, x), f(x)]}{W_{k-1}(x)}. \quad (34) \]

However, these functions \( g_k \) and operators \( L_k \) defined in terms of a determinant can be expressed in a more convenient way in terms of functions \( \phi_k(x) \) given by

\[ \phi_0(x) = 1, \quad \phi_1(x) = \Phi(x), \quad \phi_i(x) = \frac{W_i(x)W_{i-2}(x)}{W_{i-1}(x)^2}, \quad 2 \leq i \leq n-1. \]

Indeed, from [32] we have

\[ g_k(x_0, x) = \frac{\phi_0(x) \cdots \phi_{k-1}(x)}{\phi_0(x_0) \cdots \phi_{k-1}(x_0)} \int_{x_0}^{x} \phi_1(\xi_1) \int_{x_0}^{\xi_1} \cdots \int_{x_0}^{\xi_{k-2}} \phi_k(\xi_k) d\xi_k d\xi_{k-1} \cdots d\xi_1 \]

and

\[ L_k f(x) = \phi_0(x) \phi_1(x) \cdots \phi_{k-1}(x) \frac{d}{dx} \frac{1}{\phi_{k-1}(x)} \frac{d}{dx} \frac{1}{\phi_{k-2}(x)} \cdots \frac{d}{dx} \frac{1}{\phi_1(x)} \frac{d}{dx} \phi_0(x) f(x). \]

In our case the functions \( \phi_i(x) \) are calculated from equation (29). We obtain \( \phi_0(x) = 1 \) and \( \phi_k(x) = k \left( \frac{1}{\Phi(x)} \right)^{(1)^k} \). It is now easy to identify the functions
$g_k(x_0, x)$ and the operators $L_k$ as

$$g_k(x_0, x) = \begin{cases} 
\frac{1}{k!} \frac{1}{\Phi(x_0)} \bar{X}^{(k)}(x_0, x), & k \text{ odd}, \\
\frac{1}{k!} \bar{X}^{(k)}(x_0, x), & k \text{ even}, 
\end{cases}$$

and

$$L_k f(x) = \begin{cases} 
\Phi(x) \bar{D}^{(k)} f(x), & k \text{ odd}, \\
D^{(k)} f(x), & k \text{ even}, 
\end{cases}$$

(35)

for $k \geq 0$.

Now considering the rest, we have that $R_n(x)$ is of the form

$$R_n(x) = \int_{x_0}^{x} g_n(\xi, x) L_{n+1} f(\xi) d\xi,$$

which is equivalent to write

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} [\Phi(\xi)]^{(-1)^n} \mathcal{V}_n(\xi, x) D_{n+1} f(\xi) d\xi.$$ 

The proof of Theorem [17] gives us the possibility to obtain formulas of the coefficients $a_k(x), \tilde{a}_k(x)$ for the development of the generalized derivatives $D^{(n)} y(x), \tilde{D}^{(n)} y(x)$ in terms of the usual derivatives, i.e.

$$D^{(n)} f(x) = \sum_{k=1}^{n} a_k(x) f^{(k)}(x) \quad \text{and} \quad \tilde{D}^{(n)} f(x) = \sum_{k=1}^{n} \tilde{a}_k(x) f^{(k)}(x).$$

Indeed, from (35) and its associate $\Phi$-conjugaison, we have

$$D^{(n)} f(x) = \begin{cases} 
\Phi(x) \bar{L}_n f(x), & n \text{ odd}, \\
L_n f(x), & n \text{ even}, 
\end{cases}$$

and

$$\tilde{D}^{(n)} f(x) = \begin{cases} 
\frac{1}{\Phi(x)} L_n f(x), & n \text{ odd}, \\
L_n f(x), & n \text{ even}. 
\end{cases}$$
Considering first the case $n > 0$ and even we find

$$D^{(n)} f(x) = \frac{1}{W_{n-1}(x)} \begin{vmatrix} \mathcal{Y}_0 & \mathcal{Y}_1 & \ldots & \mathcal{Y}_{n-1} & f \\ \mathcal{Y}'_0 & \mathcal{Y}'_1 & \ldots & \mathcal{Y}'_{n-1} & f' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{Y}^{(n)}_0 & \mathcal{Y}^{(n)}_1 & \ldots & \mathcal{Y}^{(n)}_{n-1} & f^{(n)} \end{vmatrix}$$

$$= \frac{1}{\alpha_{n-1} \Phi(x)^{\frac{n}{2}}} \begin{vmatrix} \mathcal{Y}'_1 & \mathcal{Y}'_2 & \ldots & \mathcal{Y}'_{n-1} & f' \\ \mathcal{Y}''_1 & \mathcal{Y}''_2 & \ldots & \mathcal{Y}''_{n-1} & f'' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{Y}^{(n-1)}_1 & \mathcal{Y}^{(n-1)}_2 & \ldots & \mathcal{Y}^{(n-1)}_{n-1} & f^{(n-1)} \end{vmatrix}$$

$$= \frac{1}{\alpha_{n-1} \Phi(x)^{\frac{n}{2}}} \sum_{k=1}^{n} \varepsilon_{i_1, \ldots, i_{n-1}, k} \begin{vmatrix} \mathcal{Y}^{(i_1)}_1 & \mathcal{Y}^{(i_1)}_2 & \ldots & \mathcal{Y}^{(i_1)}_{n-1} \\ \mathcal{Y}^{(i_2)}_1 & \mathcal{Y}^{(i_2)}_2 & \ldots & \mathcal{Y}^{(i_2)}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}^{(i_{n-1})}_1 & \mathcal{Y}^{(i_{n-1})}_2 & \ldots & \mathcal{Y}^{(i_{n-1})}_{n-1} \end{vmatrix} f^{(k)}(x)$$

where $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$ and $\varepsilon_{i_1 i_2 \cdots i_{n-1} k}$ represents the $n$-dimensional Levi-Cevita symbol. By a similar calculation for $n$ odd, we finally obtain the following results for the coefficients $a_k(x)$ (coefficients $\tilde{a}_k(x)$ are obtained by $\Phi$-conjugaison).

**Proposition 18** The $n$-order $\Phi$-generalized derivatives $D^{(n)} f(x)$ and $\tilde{D}^{(n)} f(x)$ can be expanded in terms of the usual derivatives $D^{(n)} f(x) = \sum_{k=1}^{n} a_k(x) f^{(k)}(x)$ and $\tilde{D}^{(n)} f(x) = \sum_{k=1}^{n} \tilde{a}_k(x) f^{(k)}(x)$, where

$$a_k(x) = \left\{ \begin{array}{ll} \frac{1}{\alpha_{n-1} \Phi(x)^{\frac{n+k}{2}}} \begin{vmatrix} \mathcal{Y}^{(i_1)}_1 & \mathcal{Y}^{(i_1)}_2 & \ldots & \mathcal{Y}^{(i_1)}_{n-1} \\ \mathcal{Y}^{(i_2)}_1 & \mathcal{Y}^{(i_2)}_2 & \ldots & \mathcal{Y}^{(i_2)}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}^{(i_{n-1})}_1 & \mathcal{Y}^{(i_{n-1})}_2 & \ldots & \mathcal{Y}^{(i_{n-1})}_{n-1} \end{vmatrix}, & n \text{ odd,} \\
\frac{1}{\alpha_{n-1} \Phi(x)^{\frac{n+k}{2}}} \begin{vmatrix} \mathcal{Y}^{(i_1)}_1 & \mathcal{Y}^{(i_1)}_2 & \ldots & \mathcal{Y}^{(i_1)}_{n-1} \\ \mathcal{Y}^{(i_2)}_1 & \mathcal{Y}^{(i_2)}_2 & \ldots & \mathcal{Y}^{(i_2)}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}^{(i_{n-1})}_1 & \mathcal{Y}^{(i_{n-1})}_2 & \ldots & \mathcal{Y}^{(i_{n-1})}_{n-1} \end{vmatrix}, & n \text{ even,} \end{array} \right.$$
Let us now introduce the following functions

\[ f \ast g \]

As shown in [31], the composition of the first type is associative and distributive. Moreover, if \( f \ast g = g \ast f \) then the two functions \( f \) and \( g \) are said to be permutable. The following notation will be used when the function \( f \) is composed with itself:

\[ f^{(n)} = (f^{(n-1)} \ast f), \quad n \geq 1. \]

Here \( f^{(0)}(x, y) = \delta(y - x) \) and \( f^{(1)}(x, y) = f(x, y) \) by definition, where \( \delta(y - x) \) is the Dirac delta function. Hence, it is easy to show that functions \( f^{(n)} \) and \( f^{(m)} \) are permutable for some positive integers \( m, n \).

Let us now introduce the following functions

\[ 1(x, y) \equiv 1 \quad \text{and} \quad \sigma(x, y) = \frac{\Phi(y)}{\Phi(x)}. \]
By direct calculations, we find that

\[(1 \star \sigma)(x_0, x) = \Phi(x)X^{(1)}(x_0, x), \quad (1 \star \sigma^{-1})(x_0, x) = \frac{1}{\Phi(x)}\tilde{X}^{(1)}(x_0, x)\]

\[(1 \star \sigma \star 1)(x_0, x) = \frac{X^{(2)}(x_0, x)}{2!}, \quad (1 \star \sigma^{-1} \star 1)(x_0, x) = \frac{\tilde{X}^{(2)}(x_0, x)}{2!}\]

\[(1 \star \sigma)^{(2)}(x_0, x) = \Phi(x)\frac{X^{(3)}(x_0, x)}{3!}, \quad (1 \star \sigma^{-1})^{(2)}(x_0, x) = \frac{1}{\Phi(x)}\tilde{X}^{(3)}(x_0, x)\]

Taking into account that \((1 \star \sigma) = \Phi X^{(1)}\) and \((1 \star \sigma^{-1}) = \frac{1}{\Phi} \tilde{X}^{(1)}\), we have for \(n \geq 1\)

\[\left(\Phi X^{(1)}\right)^{(n)} = \Phi \frac{X^{(2n-1)}}{(2n-1)!}, \quad \left(\frac{1}{\Phi} \tilde{X}^{(1)}\right)^{(n)} = \frac{1}{\Phi} \frac{\tilde{X}^{(2n-1)}}{(2n-1)!}\]

and

\[\left(\Phi X^{(1)}\right)^{(n)} \star 1 = \frac{X^{(2n)}}{(2n)!}, \quad \left(\frac{1}{\Phi} \tilde{X}^{(1)}\right)^{(n)} \star 1 = \frac{\tilde{X}^{(2n)}}{(2n)!}\]

Proposition 19 The following identities are valid for the Volterra composition of the first type for the \(\Phi\)-power functions:

\[\Phi X^{(2n-1)} \star X^{(2m)} = A_{n,m} X^{(2n+2m)}, \quad \frac{1}{\Phi} \tilde{X}^{(2n-1)} \star \tilde{X}^{(2m)} = A_{n,m} \tilde{X}^{(2n+2m)},\]

\[\Phi X^{(2n-1)} \star X^{(2m-1)} = B_{n,m} X^{(2n+2m-1)}, \quad \frac{1}{\Phi} \tilde{X}^{(2n-1)} \star \tilde{X}^{(2m-1)} = B_{n,m} \tilde{X}^{(2n+2m-1)},\]

where

\[A_{n,m} := \frac{(2n-1)!(2m)!}{(2n+2m)!}, \quad B_{n,m} := \frac{(2n-1)!(2m-1)!}{(2n+2m-1)!}, \quad n, m \geq 1.\]

Proof. Let us calculate the first identity. We have

\[\Phi X^{(2n-1)} \star X^{(2m)} = (2n-1)!\left(\Phi X^{(1)}\right)^{(n)} \star (2m)\left[\left(\Phi X^{(1)}\right)^{(m)} \star 1\right]\]

\[= (2n-1)!(2m)!\left[\left(\Phi X^{(1)}\right)^{(n+m)} \star 1\right]\]

\[= (2n-1)!(2m)! \frac{X^{(2n+2m)}}{(2n+2m)!}.\]

Other identities are calculated in a similar way. ■

In [31] Volterra has shown that given any analytic function \(\sum_{n=0}^{\infty} c_n z^n\), which converges within a certain circle and given any bounded function \(f(x, y)\),
then $\sum_{n=0}^{\infty} c_n f^{(n)}$ is convergent for all values of $f(x, y)$. In other words, there is an isomorphism between algebraic formulas which involve only addition and multiplication and those obtained from the algebraic formulas by replacing the powers of the variable with the powers by composition of the corresponding function $f$. It follows that if $F(z)$ and $G(z)$ are two analytic functions and $F(z)G(z) = K(z)$ then $F(f(1)) \ast G(f(1)) = K(f(1))$.

**Theorem 20** Assume that on a finite real interval $[a, b]$, equation $-\psi''(x) + V(x) \psi_0(x) = 0$ possesses a particular solution $\psi_0(x)$ such that $\psi_0^2(x)$ and $1/\psi_0^2(x)$ are continuous on $[a, b]$. Then the general solution of the Schrödinger equation $-\psi''(x) + V(x) \psi(x) = \lambda \psi(x)$ on $(a, b)$ has the form

$$
\psi(x) = c_1 \psi_0 \int_{x_0}^{x} \frac{1}{1 - \lambda \bar{\rho}(1)} d\xi + \frac{c_2}{\psi_0} \left( \frac{\rho(1)}{1 - \lambda \rho(1)} \right),
$$

(39)

where

$$
\rho(x_0, x) = \psi_0^2(x) \int_{x_0}^{x} \frac{d\xi}{\psi_0^2(\xi)}, \quad \bar{\rho}(x_0, x) = \frac{1}{\psi_0^2(x)} \int_{x_0}^{x} \psi_0^2(\xi) d\xi,
$$

c_1, c_2$ are two arbitrary complex constants and $x_0$ is an arbitrary fixed point in $[a, b]$.

The geometric series in (39) converge for every $x$ on the interval $[a, b]$.

**Proof.** We set $\Phi = \psi_0^2$. Now, from Theorem 1 and equations (37), (38), the general solution of $-\psi''(x) + V(x) \psi = \lambda \psi(x)$ is given by

$$
\psi = c_1 \psi_0 \sum_{k=0}^{\infty} \lambda^k \bar{\mathcal{X}}^{(2k)}(2k)! + c_2 \psi_0 \sum_{k=0}^{\infty} \lambda^k \mathcal{X}^{(2k+1)}(2k+1)!
$$

$$
= c_1 \psi_0 \left\{ 1 + \sum_{k=1}^{\infty} \lambda^k \left[ \left( \frac{1}{\psi_0^2} \mathcal{X}^{(1)} \right)^{(k)} \ast 1 \right] \right\} + c_2 \psi_0 \sum_{k=0}^{\infty} \lambda^k \frac{1}{\psi_0^2} \mathcal{X}^{(1)}^{(k+1)}
$$

$$
= c_1 \psi_0 \left\{ 1 + \sum_{k=1}^{\infty} \lambda^k \left[ \bar{\rho}^{(k)} \ast 1 \right] \right\} + \frac{c_2}{\psi_0} \sum_{k=0}^{\infty} \lambda^k \rho^{(k+1)},
$$

(40)

where the last equality is valid since

$$
\frac{1}{\psi_0^2} \mathcal{X}^{(1)} = \frac{1}{\psi_0^2} \int_{x_0}^{x} \psi_0^2(\xi) d\xi = \bar{\rho}, \quad \psi_0^2 \mathcal{X}^{(1)} = \psi_0^2 \int_{x_0}^{x} \frac{d\xi}{\psi_0^2(\xi)} = \rho.
$$

Now, since $\bar{\rho}^{(0)} \ast 1 = \delta(x - x_0) \ast 1 = 1$ and the Volterra composition is associative, equation (40) becomes

$$
\psi = c_1 \psi_0 \left\{ \left( \sum_{k=0}^{\infty} \lambda^k \bar{\rho}^{(k)} \right) \ast 1 \right\} + \frac{c_2}{\psi_0} \sum_{k=0}^{\infty} \lambda^k \rho^{(k+1)}.
$$

(41)
Considering the geometric series
\[
\Omega(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k z^k = \frac{1}{1 - \lambda z},
\]
where for the usual multiplication the last equality is valid for $|\lambda z| < 1$. However, for the Volterra composition $\ast$ equation (41) becomes
\[
u = c_1 \psi_0 \left( \Omega(\lambda, \bar{\rho}^{(1)}) \ast 1 \right) + \frac{c_2}{\psi_0} \left( \rho^{(1)} \ast \Omega(\lambda, \rho^{(1)}) \right)
= c_1 \psi_0 \left( \frac{1}{1 - \lambda \bar{\rho}^{(1)}} \ast 1 \right) + \frac{c_2}{\psi_0} \left( \frac{\rho^{(1)}}{1 - \lambda \rho^{(1)}} \right)
\]
and the geometric series $\Omega(\lambda, \bar{\rho}^{(1)}), \Omega(\lambda, \rho^{(1)})$ converge for all values of $x_0, x$ and $\lambda$. Furthermore, by definition of the Volterra composition the first term of equation (42) is given by
\[
c_1 \psi_0 \left( \frac{1}{1 - \lambda \bar{\rho}^{(1)}} \ast 1 \right) = c_1 \psi_0 \int_{x_0}^{x} \frac{1}{1 - \lambda \bar{\rho}^{(1)}(x_0, \xi)} d\xi
\]
which complete the proof.

**Remark 21** The solution representation $\tilde{\psi}(x)$ of the supersymmetric partner Schrödinger equation can be obtained by applying operator $R_{\Phi}$ in Theorem 20, i.e.
\[
\tilde{\psi}(x) = \frac{c_1}{\psi_0} \int_{x_0}^{x} \frac{1}{1 - \lambda \rho^{(1)}} d\xi + \frac{c_2}{\psi_0} \left( \frac{\bar{\rho}^{(1)}}{1 - \lambda \bar{\rho}^{(1)}} \right).
\]

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References

[1] Heinrich G. W. Begehr and Robert P. Gilbert. Transformations, transmutations and kernel functions, volume 1,2. Longman Scientific & Technical, 1992.

[2] Alex Bilodeau and Sébastien Tremblay. On two-dimensional supersymmetric quantum mechanics, pseudoanalytic functions and transmutation operators. Journal of Physics A: Mathematical and Theoretical, 46(42):425302, 2013.

[3] Herminio Blancarte, Hugo M. Campos, and Kira V. Khmelnnytskaya. Spectral parameter power series method for discontinuous coefficients. Mathematical Methods in the Applied Sciences, 38(10):2000–2011, 2015.

[4] Robert W. Carroll. Transmutation theory and applications, volume 11. North-Holland, 1985.

[5] Raúl Castillo-Pérez, Vladislav V. Kravchenko, and Sergii M. Torba. Analysis of graded-index optical fibers by the spectral parameter power series method. Journal of Optics, 17(2):025607, 2015.

[6] Ral Castillo-Prez, Vladislav V. Kravchenko, and Sergii M. Torba. Spectral parameter power series for perturbed Bessel equations. Applied Mathematics and Computation, 220:676 – 694, 2013.

[7] Kuo-Tsai Chen. Iterated path integrals. Bulletin of the American Mathematical Society, 83(5):831–879, 1977.

[8] David L. Colton. Solution of boundary value problems by the method of integral operators, volume 11. Pitman, 1976.

[9] Philip J. Davis. Interpolation and Approximation. Dover Publications, 1975.

[10] Lynn Erbe, Raziye Mert, and Allan Peterson. Spectral parameter power series for Sturm-Liouville equations on time scales. Applied Mathematics and Computation, 218(14):7671 – 7678, 2012.

[11] E. Goursat. Cours d’analyse mathématique, Vol. 2: Théorie des fonctions analytiques; Équations différentielles; Équations aux dérivées partielles du premier ordre (Classic Reprint). Fb&c Limited, 2018.

[12] Zheng Han, Yaozhong Hu, and Chihoon Lee. Optimal pricing barriers in a regulated market using reflected diffusion processes. Quantitative Finance, 16(4):639–647, apr 2016.

[13] Kira V. Khmelnnytskaya, Vladislav V. Kravchenko, and Jesús A. Baldenebro-Obeso. Spectral parameter power series for fourth-order sturmliouville problems. Applied Mathematics and Computation, 219(8):3610 – 3624, 2012.
[14] Kira V. Khmelnytskaya, Vladislav V. Kravchenko, and Haret C. Rosu. Eigenvalue problems, spectral parameter power series, and modern applications. *Mathematical Methods in the Applied Sciences*, 2014.

[15] Kira V. Khmelnytskaya and Haret C. Rosu. Spectral parameter power series representation for hills discriminant. *Annals of Physics*, 325(11):2512 – 2521, 2010.

[16] Kira V. Khmelnytskaya and Ibrahim Serroukh. The heat transfer problem for inhomogeneous materials in photoacoustic applications and spectral parameter power series. *Mathematical Methods in the Applied Sciences*, 36(14):1878–1891, 2013.

[17] Vladislav V. Kravchenko. A representation for solutions of the Sturm-Liouville equation. *Complex Variables and Elliptic Equations*, 53(8):775–789, 2008.

[18] Vladislav V. Kravchenko. On the completeness of systems of recursive integrals. *Communications in Mathematical Analysis*, (3):172–176, 2011.

[19] Vladislav V. Kravchenko. Construction of a transmutation for the one-dimensional schrdinger operator and a representation for solutions. *Applied Mathematics and Computation*, 328:75 – 81, 2018.

[20] Vladislav V. Kravchenko, Samy Morelos, and Sergii M. Torba. Liouville transformation, analytic approximation of transmutation operators and solution of spectral problems. *Applied Mathematics and Computation*, 273:321 – 336, 2016.

[21] Vladislav V. Kravchenko, Samy Morelos, and Sébastien Tremblay. Complete systems of recursive integrals and taylor series for solutions of Sturm-Liouville equations. *Mathematical Methods in the Applied Sciences*, 35(6):704–715, 2011.

[22] Vladislav V Kravchenko and R Michael Porter. Spectral parameter power series for Sturm-Liouville problems. *Mathematical Methods in the Applied Sciences*, 33(4):459–468, 2010.

[23] Vladislav V. Kravchenko and Sergii M. Torba. *Transmutations and spectral parameter power series in eigenvalue problems*, pages 209–238. Springer Basel, Basel, 2013.

[24] Vladislav V. Kravchenko and Sergii M. Torba. Analytic approximation of transmutation operators and applications to highly accurate solution of spectral problems. *Journal of Computational and Applied Mathematics*, 275:1 – 26, 2015.

[25] Vladislav V. Kravchenko, Sergii M. Torba, and Ral Castillo-Prez. A neumann series of Bessel functions representation for solutions of perturbed Bessel equations. *Applicable Analysis*, 97(5):677–704, 2018.
[26] Boris M. Levitan. *Inverse Sturm-Liouville problems*. VNU Science Press, 1987.

[27] Vladimir Marchenko. *Sturm-Liouville Operators and Applications*. Birkhuser Basel, 1986.

[28] Michael R. Porter. On Sturm-Liouville equations with several spectral parameters. *Boletín de la Sociedad Matemática Mexicana*, 22(1):141–163, 2016.

[29] Vladimir Rabinovich and Francisco Urbano Altamirano. *Application of the SPPS method to the one-dimensional quantum scattering*, volume 17. Communications in Mathematical Analysis, 2007.

[30] Vladimir S. Rabinovich and Josué Hernández-Juárez. Method of the spectral parameter power series in problems of underwater acoustics of the stratified ocean. *Mathematical Methods in the Applied Sciences*, 38(10):1990–1999, 2015.

[31] Vito Volterra. *Theory of functionals and of integral and integro-differential equations*. Courier Corporation, 2005.

[32] D. V. Widder. A generalization of Taylor’s series. *Transactions of the American Mathematical Society*, 30(1):126–154, 1928.

[33] Edward Witten. Dynamical breaking of supersymmetry. *Nuclear Physics B*, 188(3):513 – 554, 1981.