UPPER TAILS FOR EDGE EIGENVALUES OF RANDOM GRAPHS

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Abstract. In this note we prove a precise large deviation principle for the largest and second largest eigenvalues of a sparse Erdős-Rényi graph. Our arguments rely on various recent breakthroughs in the study of mean field approximations for large deviations of low complexity non-linear functions of independent Bernoulli variables and solutions of the associated entropic variational problems.

1. Introduction

Establishing large deviations for edge eigenvalues (and other spectral functionals) of random matrices is an important problem, which in most cases is widely open. Even in the basic example of self-adjoint Wigner matrices with i.i.d. entries, the question is well-understood only in a few special cases. These include the integrable Gaussian models namely the GOE or GUE where the entries are i.i.d. real and complex gaussians respectively, crucially relying on the exact joint density of eigenvalues available for these models enabling the usage of Coulomb gas methods. This can be used to exactly compute the rate function for the large deviation principle (LDP) for the empirical spectral measure \[3\] as well as the spectral norm \[2\]. For results in the related setting of Gaussian covariance or Wishart matrices see \[17, 29\]. However, for the non-gaussian cases, the results have been few and far between. Bordenave and Caputo in \[9\] studied LDP for empirical spectral measure for Wigner matrices with stretch exponential tails while the corresponding results for the spectral norm were obtained in \[20\]. Very recently, Guionnet and Husson \[22\] established an LDP for the largest eigenvalue Rademacher matrices and more generally sub-Gaussian Wigner matrices.

In this note, we consider the upper tail large deviation problem for the extremal/edge eigenvalues of adjacency matrices of sparse random graphs. To this end, let \(G_{n,p}\) be the Erdős-Rényi random graph on \(n\) vertices with edge probability \(p\). Denote by \(\lambda_1(G_{n,p}) \geq \lambda_2(G_{n,p}) \geq \ldots \geq \lambda_n(G_{n,p})\) the eigenvalues of the adjacency matrix \(A(G_{n,p})\) of \(G_{n,p}\), arranged in non-increasing order. Here, we particularly focus on the largest and the second largest eigenvalues of \(G_{n,p}\), namely \(\lambda_1(G_{n,p})\) and \(\lambda_2(G_{n,p})\). The typical behaviors of the edge eigenvalues of \(G_{n,p}\) are known to great precision, for instance, with high probability \(\lambda_1(G_{n,p}) = (1 + o(1))np\) and \(\lambda_2(G_{n,p}) = (1 + o(1))\sqrt{np}\), for \(p\) not too sparse (see, for example, \[19\] and the references therein). General concentration inequalities for the edge eigenvalues of \(G_{n,p}\) are also well-known \[1\] (see \[27\] for a recent improvement for the largest eigenvalue).

The study of large deviations of random graphs, in spite of being a fairly recent topic, have witnessed a series of breakthroughs over the last decade starting with the seminal work of Chatterjee and Varadhan \[14\] who considered subgraph counts in \(G_{n,p}\) in the dense regime, that is, \(p \in (0, 1)\) is fixed. As a consequence in \[15\], they proved a large deviations principle for the entire spectrum of \(G_{n,p}\) thought of as a countable ordered sequence, with the topology of coordinate wise convergence, at scale \(np\). The problem turned out to be much more challenging in the sparse regime, where \(p = p(n) \to 0\). This was first addressed in a systematic way in the tour-de-force work of Chatterjee and Dembo \[13\]. Here, the authors explored the more general problem of approximating the partition
function for a general Gibbs measure on the hypercube with the so-called ‘mean-field’ variational principle; and came up with a notion of complexity of the gradient of the Hamiltonian (of the Gibbs measure) along with additional smoothness properties under which the above approximation holds. This as a direct consequence reduced the large deviations for subgraph counts in $\mathcal{G}_{n,p}$ to a ‘mean-field’ entropic variational problem in certain regimes of the sparsity parameter $p$. Thereafter, Eldan [18] obtained an improved set of conditions under which the above reduction holds, approximating the Gibbs measure, up to lower order entropy, by a product measure. Similar results for Gibbs measures beyond the hypercube were obtained by [6, 30], and recently by Austin [5] for very general product spaces. Furthermore, using different approaches, the large deviation behavior for several spectral and geometric functionals under almost optimal sparsity assumptions were established independently and simultaneously by Cook and Dembo [16] and Augeri [4]. In particular, Augeri [4] establishes the validity of the mean-field variational problem for the upper tail of triangles in $\mathcal{G}(n,p)$, down to the sparsity threshold $n^{-\frac{2}{3}}$.

Almost in parallel, fortunately, many of the associated entropic variational problems in the space of weighted graphs have been precisely analyzed in a separate series of papers: starting with the upper tail for triangle (more generally any regular subgraph) counts in the dense regime [26], followed by upper tails of clique counts in the sparse case [7], followed by upper tails of general subgraph counts in sparse case [31]. The corresponding problem for the lower tail was studied in [31], where several questions remain open. The analogous problem for the number the arithmetic progressions in a random set was recently studied in [8].

In this work, building on the above recent results of both kinds, we pin down the exact upper tail rate function for certain regimes of large deviations for $\lambda_1(\mathcal{G}_{n,p})$, the operator norm of the centered adjacency matrix $A(\mathcal{G}_{n,p}) - p11'$, and the second largest eigenvalue $\lambda_2(\mathcal{G}_{n,p})$, in the sparse setting. The proof relies on the connection between the number of cycles in a graph and the spectral moments of the adjacency matrix, and borrows techniques from [7] about the solution of the associated variational problems for large deviations of subgraph counts. As will be evident, in a sense made precise in this note, the large deviation for the extremal eigenvalues are dictated by low rank deformations. We leave several interesting questions open for further research.

1.1. Statement of Results. Let $\mathcal{G}_n$ denote the set of weighted undirected graphs on $n$ vertices with edge weights in $[0, 1]$, that is, if $A(G)$ is the adjacency matrix of $G$ then

$$\mathcal{G}_n = \{G_n : A(G_n) = (a_{ij})_{1\leq i, j \leq n}, 0 \leq a_{ij} \leq 1, a_{ij} = a_{ji}, a_{ii} = 0 \text{ for all } i, j\}.$$ 

For $G_n \in \mathcal{G}_n$, denote by $\lambda_1(G_n) \geq \lambda_2(G_n) \geq \cdots \lambda_n(G_n)$ the eigenvalues of $G_n$ in non-increasing order. Moreover, for $G_n \in \mathcal{G}_n$, $I_p(G_n)$ is the entropy relative to $p$, that is,

$$I_p(G) := \sum_{1 \leq i < j \leq n} I_p(a_{ij}) \quad \text{where} \quad I_p(x) := x \log \frac{x}{p} + (1 - x) \log \frac{1 - x}{1 - p}.$$

It is well-known that $\mathbb{E} \lambda_1(\mathcal{G}_{n,p}) = (1 + o(1))np$, for $p \gg n^{-1} \text{ (up to polynomial factors in log } n)$, Recently, Cook and Dembo [16, Proposition 1.13] established the upper tail large deviations for $\lambda_1(\mathcal{G}_{n,p})$, in terms of an entropic variational problem.$^2$  

$^1$Note that $A(\mathcal{G}_{n,p}) - p11'$ is the centered adjacency matrix up to a translation by $pI$, where $I$ is the identity matrix. Since this deterministically shifts every eigenvalue by $p$ which is asymptotically negligible, we will continue this abuse of terminology.

$^2$ We write $f \lesssim g$ to denote $f = O(g)$, $f \sim g$ means $f = (1 + o(1))g$, and $f \ll g$ means $f = o(g)$.
Theorem 1.1 (Cook and Dembo [16]). For any \( t > 1 \) fixed and \( n^{-\frac{1}{2}} \ll p \leq \frac{1}{2} \),
\[- \log \mathbb{P}(\lambda_1(G_{n,p}) \geq tnp) = (1 + o(1))\phi_1(n,p,(1 + o(1))t),\]
where
\[\phi_1(n,p,t) := \inf \{ I_p(G_n) : G_n \in \mathcal{G}_n \text{ with } \lambda_1(G_n) \geq tnp \}. \quad (1.1)\]

Here, we complement this result by solving the variational problem \( \phi_1(n,p,t) \), in the \textit{sparse regime}, that is, \( p \ll 1 \), which combined with the result above gives the precise upper tail asymptotics for \( \lambda_1(G_{n,p}) \):

Theorem 1.2. For any \( \delta > 0 \) fixed and \( n^{-\frac{1}{2}} \ll p \ll 1 \),
\[\lim_{n \to \infty} - \log \mathbb{P}(\lambda_1(G_{n,p}) \geq (1 + \delta)np) = \min \left\{ \frac{(1 + \delta)^2}{2}, \delta(1 + \delta) \right\}. \quad (1.2)\]

The proof of the Theorem is presented in Section 2. This is done in two steps: For the lower bound on the variational problem, our strategy is to bound the largest eigenvalue \( \lambda_1(G_{n,p}) \) with the Schatten \( s \)-norm of \( G_{n,p} \), where \( s \) is even.\(^3\) Then, using the correspondence between the Schatten \( s \)-norm and the homomorphism density of the \( s \)-cycle in \( G_n \), and analyzing the large \( s \) behavior of the corresponding variational problem for \( s \)-cycles, derived in [7], the lower bound follows. The matching upper bound is attained by planting in \( G_{n,p} \) either a \textit{clique} (a set of vertices connected between themselves), or an \textit{anti-clique} (a set of vertices connected every other vertex in the graph) of the required size.

Remark 1.1. Note that Theorem 1.2 above holds only in the sparse regime, \( p \ll 1 \). The dense regime, that is, \( p \in (0, 1) \) is fixed, falls in the purview of graphon theory [23], and the large deviation principle of \( \lambda_1(G_{n,p}) \) is a consequence of the general framework of Chatterjee and Varadhan [14, 15]. The associated variational problem was analyzed by Lubetzky and Zhao [25]. They obtained the exact region of \textit{replica symmetry}, that is, the set of values of \((p,t)\) for which the constant function uniquely minimizes the associated variational problem [25, Theorem 1.2].

Next, we study the large deviations of the second largest eigenvalue of \( G_{n,p} \). To this end, we begin by considering the operator norm of the centered adjacency matrix, that is, \( A(G_{n,p}) - p11^t \), a problem of independent interest.\(^4\) The following mean field approximation was proved in [16].

Theorem 1.3 (Cook and Dembo [16]). For \( \frac{\log n}{n} \lesssim p \leq \frac{1}{2} \) and \( t \gg \sqrt{n} \),
\[- \log \mathbb{P}(\|A(G_{n,p}) - p11^t\|_{\text{op}} \geq t) = (1 + o(1))\phi_2(n,p,t + o(t)),\]
where
\[\phi_2(n,p,t) := \inf \{ I_p(G_n) : G_n \in \mathcal{G}_n \text{ with } \|A(G_n) - p11^t\|_{\text{op}} \geq t \}. \quad (1.3)\]

Remark 1.2. Note that the above result gives the upper tail asymptotics for \( \|A(G_{n,p}) - p11^t\|_{\text{op}} \) for deviations growing faster than \( \sqrt{n} \). On the other hand, recall that the typical value of \( \|A(G_{n,p}) - p11^t\|_{\text{op}} \) is \( (1 + o(1))\sqrt{n}p \). As alluded to earlier, recently, Guionnet and Husson [22] established a large deviations principle for the largest eigenvalue of \( n \)-dimensional Wigner matrices, rescaled by \( \frac{1}{\sqrt{n}} \), whose independent, standardized entries have uniformly sub-Gaussian moment generating

\(^3\)For a \( n \times n \) symmetric matrix \( X \), the Schatten \( s \)-norm is defined as \( \|X\|_s := (\sum_{i=1}^{n} \lambda_i^s)^{\frac{1}{s}} \), where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( X \).

\(^4\)Recall that for a \( n \times n \) symmetric matrix \( A \), the operator norm is defined as: \( \|A\|_{\text{op}} = \sup_{\|x\| = 1} \|Ax\| \).
functions (which allow for Rademacher entries). Observe that even though $A(\mathcal{G}_{n,p}) - \mathbb{E}(A(\mathcal{G}_{n,p}))$ is a Wigner matrix, such uniform sub-Gaussian domination does not apply in the case when $p \ll 1$. Therefore, establishing the large deviations for $||A(\mathcal{G}_{n,p}) - p11'||_{op}$ in the scale $\sqrt{n}$ in the sparse regime ($p \ll 1$), where the mean-field variational problem (1.3) is expected to fail, remains open.

As in Theorem 1.2, we can complement Theorem 1.3 by solving the variational problem $\phi_2(n, p, t)$, in the sparse regime, proving the precise upper tail asymptotics for $||A(\mathcal{G}_{n,p}) - p11'||_{op}$, for deviations growing faster than $\sqrt{n}$.

**Theorem 1.4.** For $n^{-\frac{1}{2}} \ll p \ll 1$ and $\delta = O(1)$ such that $\delta np \gg n^{\frac{1}{2}}$,

$$\phi_2(n, p, \delta np) := (1 + o(1))\frac{1}{2}\delta^2 n^2 p^2 \log(1/p).$$

This implies, under the same conditions on $p$ and $\delta$ as above,

$$\lim_{n \to \infty} \frac{-\log \mathbb{P}(||A(\mathcal{G}_{n,p}) - p11'||_{op} \geq \delta np)}{\frac{1}{2}\delta^2 n^2 p^2 \log(1/p)} = 1.$$  

The proof of the Theorem is presented in Section 3. For the lower bound, we follow a similar strategy as before to bound $||A(\mathcal{G}_{n,p}) - p11'||_{op}$ with the corresponding Schatten $s$-norm, which in this case relates to the ‘signed’ homomorphism density of the $s$-cycle in $G_n$ (refer to Section 3.2 for the precise definition). The proof then follows from the analysis of the variational problem for ‘signed’ homomorphism densities, a result which might be of independent interest (Proposition 3.1). In this case, the matching upper bound is attained by planting in $\mathcal{G}_{n,p}$ a clique of the required size.

**Remark 1.3.** The conditions in above theorem allows $n^{-\frac{1}{2}} \ll \delta np \ll np$, covering all possible deviations of the operator norm from $\sqrt{n}$ to $O(np)$. Furthermore, although not explicitly stated, the arguments in [16] also can be used to show that the mean field variational formula dictates the large deviation behavior for the Schatten $s$-norm of $A(\mathcal{G}_{n,p}) - p11'$ in some regime of the parameter space and hence our analysis can be used to pin down the exact rate function in this case as well.

Finally, we obtain as a corollary the upper tail rate function for $\lambda_2(\mathcal{G}_{n,p})$ in a certain regime. The probability upper bound relies on eigenvalue interlacing and the previous result while the lower bound relies on a well known variational representation of the second largest eigenvalue.

**Corollary 1.5.** For $n^{-\frac{1}{2}} \ll p \ll 1$ and $\delta < 1$ such that $\delta np \gg n^{\frac{1}{2}}$,

$$\lim_{n \to \infty} \frac{-\log \mathbb{P}(\lambda_2(\mathcal{G}_{n,p}) \geq \delta np)}{\frac{1}{2}\delta^2 n^2 p^2 \log(1/p)} = 1.$$  

Note that the above result is stated only for $\delta < 1$ causing $\lambda_2(\mathcal{G}_{n,p}) < np$ which is known to be the typical value of $\lambda_1(\mathcal{G}_{n,p})$. The reason behind the above restriction will be clear from the proof, which is presented in Section 4, and also the discussion on open problems in Section 5.

2. Proof of Theorem 1.2

By Theorem 1.1, to show (1.2), it suffices to prove

$$\lim_{n \to \infty} \frac{\phi_1(n, p, (1 + \delta))}{n^2 p^2 \log(1/p)} = \min \left\{ \frac{(1 + \delta)^2}{2}, \delta(1 + \delta) \right\}.$$  

The proof of the lower bound, which involves bounding the largest eigenvalue with the homomorphism density cycles, is given below in Section 2.1. The upper bound constructions are presented in Section 2.2.
2.1. Proof of the Lower Bound in (2.1). For any fixed graph $H$ and $G_n \in G_n$,

\[ t(H, G_n) := \frac{1}{n|V(H)|} \sum_{1 \leq i_1, \ldots, i_k \leq \Theta} \prod_{(x, y) \in E(H)} a_{x,y} \]

denotes the density of (labeled) copies of $H$ in $G_n$. The variational problem for the upper tail large deviations of $t(H, G_{n,p})$ is given by the following variational problem (see [16, Corollary 1.6] and [7] for details):

\[ \phi(C_s, n, p, t) := \inf \{ I_p(G_n) : G_n \in G_n \text{ with } t(C_s, G_{n,p}) \geq t\delta^s p^s \}. \]  

(2.2)

The solution of this variational problem in the sparse regime was established by Bhattacharya et al. [7]:

**Theorem 2.1.** [7] For any $t > 1$ and $n^{1/2} \ll p \ll 1$, the solution to the variational problem (2.2) satisfies

\[ \lim_{n \to \infty} \frac{\phi(C_s, n, p, t)}{n^2 p^2 \log(1/p)} = \min \left\{ \theta, \frac{1}{2} (t - 1) \frac{\delta^s}{2} \right\}, \]

where $\theta = \theta(C_s, t)$ is the unique positive solution to $PC_s(\theta) = t$, with $PC_s(\cdot)$ is the independence polynomial of $C_s$.\(^5\)

Let $G_n \in G_n$ be a weighted graph satisfying $\lambda_1(G_n) \geq (1 + \delta)np$. We can now use the above result to obtain a lower bound on (2.1), using the following simple inequality: Then, for $s \geq 1$ even,

\[ t(C_s, G_n) = \frac{1}{n^s} \sum_{j=1}^{\Theta} \lambda_j^n(G_n) \geq \left( \frac{\lambda_1(G_n)}{n} \right)^s \geq (1 + \delta)^s p^s, \]

since $\lambda_1(G_n) \geq 0$. This implies,

\[ \phi_1(n, p, (1 + \delta)) \geq \phi(C_s, n, p, (1 + \delta)^s). \]

Therefore, by Theorem 2.1, it follows that, for any $s$ even,

\[ \lim_{n \to \infty} \frac{\phi_1(n, p, (1 + \delta))}{n^2 p^2 \log(1/p)} \geq \lim_{n \to \infty} \frac{\phi(C_s, n, p, (1 + \delta)^s)}{n^2 p^2 \log(1/p)} = \min \left\{ \theta, \frac{1}{2} ((1 + \delta)^s - 1) \frac{\delta^s}{2} \right\}. \]  

(2.3)

where $\bar{\theta} = \bar{\theta}(C_s, \delta)$ is the unique positive solution to $PC_s(\bar{\theta}) = (1 + \delta)^s$, with $PC_s(\cdot)$ as defined in (2.4).

Therefore, to complete the proof of the lower bound in (2.1), it suffices to understand the asymptotics of (2.3), as $s \to \infty$. In turns out that the independence polynomial for cycles can be calculated in closed form in terms of Chebyshev polynomials, using the recursion\(^6\)

\[ PC_1(x) = PC_{s-1}(x) + xPC_{s-2}(x), \quad PC_2(x) = 2x + 1, \quad PC_3(x) = 3x + 1. \]

Then using the closely related recursion for Chebyshev’s polynomials gives,

\[ PC_s(x) = \frac{1}{2^{s-1}} \sum_{a=0}^{\lfloor s/2 \rfloor} \left( \begin{array}{c} s \\ 2a \end{array} \right) (1 + 4x)^a, \]

---

5 The independence polynomial of a graph $H$ is defined to be $P_H(x) := \sum_k i_H(k)x^k$, where $i_H(k)$ is the number of $k$-element independent sets in $H$.

6 By the definition of the independence polynomial, for any graph $H$ and vertex $v$ in it, $P_H(x) = P_{H_1}(x) + xP_{H_2}(x)$, where $H_1$ is obtained from $H$ by deleting $v$ and $H_2$ is obtained from $H$ by deleting $v$ and all its neighbors.
which for even $s$ simplifies to
\[ P_{C_s}(x) = \left[ \frac{1}{2}(\sqrt{1+4x} + 1) \right]^s + \left[ \frac{1}{2}(\sqrt{1+4x} - 1) \right]^s. \] (2.4)

Using this the equation $P_{C_s}(\bar{\theta}) = (1 + \delta)^s$ can be written as:
\[ (1 + \sqrt{1+4\bar{\theta}})^s \left[ 1 + \left( \frac{\sqrt{1+4\bar{\theta}} - 1}{\sqrt{1+4\bar{\theta}} + 1} \right)^s \right] = (2(1 + \delta))^s. \]

This shows $\eta := \lim_{s \to \infty} \bar{\theta}(C_s, \delta)$ must satisfy the equation $1 + \sqrt{1 + 4\eta} = 2(1 + \delta)$, which solves to $\eta = \delta(1 + \delta)$. Therefore, taking limit as $s \to \infty$, in (2.9) gives
\[ \liminf_{n \to \infty} \frac{\phi_1(n, p, (1 + \delta))}{n^2 p^2 \log(1/p)} \geq \min \left\{ \frac{(1 + \delta)^2}{2}, \delta(1 + \delta) \right\}, \] (2.5)

using $\lim_{s \to \infty} ((1 + \delta)^s - 1)^2 = (1 + \delta)^2$. This completes the proof of the lower bound.

### 2.2. Proof of the Upper Bound in (2.1)

The proof of the upper bound proceeds by verifying that the minimum entropy configurations are attained by planting a clique or an anti-clique of the required size in the Erdős Rényi graph.

**The Clique Construction:** For $s \geq 1$, denote by $G_{cl}(s)$ the clique graph, with adjacency matrix $A(G_{cl}(s)) = ((a_{ij}))$, which has zeros on the diagonal and
\[ a_{ij} = \begin{cases} 1 & \text{if } 1 \leq i \neq j \leq [s] + 1 \\ p & \text{otherwise.} \end{cases} \] (2.6)

(Note that this corresponds to planting a clique on the set of vertices $\{1, 2, 3, \ldots, [s] + 1\}$ in the random graph $G_{n,p}$.) Next, define
\[ v = \left( \frac{1}{\sqrt{[s] + 1}}, \ldots, \frac{1}{\sqrt{[s] + 1}}, 0, \ldots, 0 \right)^{\text{\# of times}}, \] (2.7)

Now, let $z := 1 + \delta$, and choosing $s = znp$ gives, (since $v^t v = 1$)
\[ \lambda_1(G_{cl}(znp)) \geq v^t A(G_{cl}(znp)) v = \sum_{1 \leq i \neq j \leq [znp] + 1} a_{ij} v_i v_j \]
\[ = \frac{1}{[znp] + 1} \sum_{1 \leq i \neq j \leq [znp] + 1} a_{ij} \]
\[ = [znp] \geq znp = (1 + \delta)np. \] (2.8)

This shows,
\[ \phi_1(n, p, (1 + \delta)) \leq I_p(G_{cl}(znp) \leq (1 + o(1)) \frac{1}{2} (1 + \delta)^2 n^2 p^2 \log(1/p). \] (2.9)
The Anti-clique Construction: Given \( z \geq 0 \), \( G_{acl}(z) \) be weighted graph which has an anti-clique on the vertices \( \{1, 2, 3, \ldots, \lfloor zn^2 \rfloor \} \) and \( p \) between other every pair of vertices, that is, \( A(G_{acl}(z)) = ((a_{ij})) \), which has zeros on the diagonal and
\[
a_{ij} = \begin{cases} 
1 & \text{if } 1 \leq i \leq \lfloor zn^2 \rfloor \text{ and } 1 \leq j \leq n \\
p & \text{otherwise.}
\end{cases}
\]

Now, fix \( z \) to be chosen later. Choose
\[
v = K \left( 1, 1, \ldots, 1, (1 + \delta)p, (1 + \delta)p, \ldots, (1 + \delta)p \right) ^{\prime},
\]
where \( K = \frac{1}{\sqrt{\lfloor zn^2 \rfloor + (n-\lfloor zn^2 \rfloor)(1+\delta)^2p^2}} \). This ensures \( v^\prime v = 1 \). Then
\[
\lambda_1(G_{acl}(z)) \geq v^\prime A(G_{acl}(z)) v
\]
\[
\geq 2K^2(1 + \delta)p[\lfloor zn^2 \rfloor (n - \lfloor zn^2 \rfloor)] + K^2(1 + \delta)^2p^3(n - \lfloor zn^2 \rfloor)(n - \lfloor zn^2 \rfloor - 1)
\]
\[
= K^2 n^2 p^3 (2(1 + \delta)z(1 - zp^2) + (1 + \delta)^2(1 - zp^2)^2) + o(np)
\]
\[
= (1 + \delta)np \left[ \frac{2z(1 - zp^2) + (1 + \delta)(1 - zp^2)^2}{z + (1 + \delta)^2(1 - zp^2)} \right] + o(np)
\]
\[
\geq np(1 + \delta),
\]
by choosing \( z \) appropriately. It is easy to check that any such choice of \( z \) must satisfy \( z = (1 + o(1))\delta(1 + \delta) \). This shows
\[
\phi_1(n, p, (1 + \delta)) \leq I_p(G_{acl}(z)) \leq (1 + o(1))\delta(1 + \delta)n^2p^2 \log(1/p).
\]
Combining (2.9) and (2.10) gives the desired upper bound
\[
\phi_1(n, p, (1 + \delta)) \leq (1 + o(1)) \min \left\{ \frac{(1 + \delta)^2}{2}, \delta(1 + \delta) \right\} n^2p^2 \log(1/p),
\]
completing the proof.

3. Proof of Theorem 1.4

Note that it suffices to prove (1.4). The proof of the upper bound is presented below in Section 3.1, and the proof of the lower bound is given in Section 3.2.

3.1. Proof of the Upper Bound in (1.4). To establish an upper bound on (1.4), it suffices to exhibit a weighted graph \( G_n \) with \( \|A(G_n) - p11^\prime\|_{op} \geq \delta np \), which has entropy \( (1 + o(1))\frac{1}{2}\delta^2n^2p^2 \log(1/p) \). For this, consider the clique graph \( G_{cl}(s) \) with adjacency matrix \( A(G_{cl}(s)) = ((a_{ij})) \) as in (2.6), with \( s = \frac{\delta np + 2p}{1-p} = (1 + o(1))\delta np \) (since \( \delta n \geq \delta np \to \infty \)). This ensures, \( [s] - p([s] + 1) \geq \delta np \).

Then choosing \( v \) as in (2.7), gives
\[
\|A(G_n) - p11^\prime\|_{op} \geq v^\prime (A(G_n) - p11^\prime) v = \sum_{1 \leq i \neq j \leq [s] + 1} a_{ij}(G_n) v_i v_j - p([s] + 1)
\]
\[
= \frac{1}{[s] + 1} \sum_{1 \leq i \neq j \leq [s] + 1} a_{ij}(G_n) - p([s] + 1)
\]
\[
= [s] - p([s] + 1) \geq \delta np.
\]
This shows (recall (1.3)), \( \phi_2(n, p, \delta np) \leq I_p(G_n) \leq (1 + o(1))\frac{1}{2}\delta^2n^2p^2 \log(1/p) \).
3.2. Proof of the Lower Bound in (1.4). For the lower bound, it is convenient to analyze a continuous version of the variational problem (2.2). To describe the continuous analogue of the variational problem, it is convenient to invoke the language of graph limit theory \cite{10, 11, 23, 24}. Define a signed graphon as a symmetric measurable function $W : [0, 1]^2 \to [-1, 1]$ (where symmetric means $W(x, y) = W(y, x)$). Every graphon $W$ defines an operator $T_W : L^2[-1, 1] \to L^2[-1, 1]$, by

$$ (T_W f)(x) = \int_0^1 W(x, y) f(y) dy. \quad (3.1) $$

$T_W$ is a Hilbert-Schmidt operator, which is compact and has a discrete spectrum, that is, a countable multiset of non-zero real eigenvalues. Denote by $\|W\|_{op}$ the operator norm of $T_W$.

In the continuous variational problem, the edge-weighted graph $G_n$ of (2.2), is replaced by a non-negative signed graphon $W$, that is, $W \in [0, 1]$, for all $x, y \in [0, 1]$ ($G_n$ should be viewed as a discrete approximation of a non-negative signed graphon.) Denote the space of all non-negative signed graphons (hereafter, simply referred to as graphons) by $\mathcal{W}$. Finally, defining $E[f(W)] := \int_{[0,1]^2} f(W(x, y)) dx dy$, we get the continuous analogue of (2.2).

**Definition 3.1** (Graphon variational problem). For $t > 0$ and $0 < p < 1$, let

$$ \phi_2(p, t) := \inf \left\{ \frac{1}{2} E[I_p(W)) : W \in \mathcal{W} \text{ with } \|W - pJ\|_{op} \geq t \right\}, \quad (3.2) $$

where $J := 1$, is the constant graphon 1.

The following lemma shows how we can obtain a lower bound on the discrete variational problem using the graphon variational problem:

**Lemma 3.1.** Under the assumptions of Theorem 1.4, $\phi_2(p, (1 + o(1))\delta p) \leq \frac{1}{\delta p^2} \phi_2(n, p, \delta np)$.

**Proof.** Given weighted graph $G_n \in \mathcal{G}_n$ with adjacency matrix $(a_{ij})_{1 \leq i, j \leq n}$, form two graphons $W^{G_n}$ and $\tilde{W}^{G_n}$ as follows: Divide $[0, 1]$ into $n$ equal-length intervals $I_1, I_2, \ldots, I_n$, that is, $I_i = \left(\frac{i-1}{n}, \frac{i}{n}\right]$, and set (1) $\tilde{W}^{G_n}(x, y) = a_{ij}$ if $x \in I_i, y \in I_j$ and $i \neq j$, and $\tilde{W}^{G_n}(x, y) = 0$ if $x, y \in I_i$ for some $i$, and (2) $W^{G_n}(x, y) = a_{ij}$ if $x \in I_i, y \in I_j$ and $i \neq j$, and $W^{G_n}(x, y) = p$ if $x, y \in I_i$ for some $i$.

Note that the operator norm of $\tilde{W}^{G_n} - pJ$ restricted to act on the set of step functions that are constant on the intervals $\{I_i\}_{i=1}^n$ is exactly $\frac{1}{n} \|A(G_{n,p}) - pI_n\|_{op}$. This implies,

$$ \|A(G_{n,p}) - pI_n\|_{op} \leq \frac{1}{\delta p^2} \|\tilde{W}^{G_n} - pJ\|_{op} $$

$$ \leq n \|W^{G_n} - pJ\|_{op} + n \|W^{G_n} - \tilde{W}^{G_n}\|_{op} $$

$$ \leq n \|W^{G_n} - pJ\|_{op} + p, $$

where final inequality follows by observing that $\tilde{W}^{G_n} - W^{G_n}$ is the graphon which is $p$ on $I_i \times I_i$, for $1 \leq i \leq n$ and zero otherwise, and checking that $\|\tilde{W}^{G_n} - W^{G_n}\|_{op} = p/n$.

This shows, if $\|A(G_{n,p}) - pI_n\|_{op} \geq \delta np$, then $\|W^{G_n} - pJ\|_{op} \geq \delta p - p/n = (1 + o(1))\delta p$ (since $\delta n \geq \delta np \to \infty$). The result follows by noting that $\frac{1}{2} I_p(W^{G_n}) = \frac{1}{n^2} I_p(G_n)$ (diagonal entries contribute 0 to $I_p(W^{G_n}))$. \hfill $\square$

The proof of the lower bound in (1.4) is a consequence of the following proposition and the above lemma:

**Proposition 3.1.** For $p \ll 1$ and $\delta = O(1)$, $\phi_2(p, \delta p) \geq (1 - o(1))\frac{1}{2} \delta^2 p^2 \log(1/p)$. 

Proof of Proposition 3.1: As in the proof of the lower bound in Theorem 1.2, our strategy is to bound the operator norm of \( \|W - pJ\|_{op} \) with the Schatten-norm, which in this case counts the density of signed cycles in the graphon \( W \). To this end, for any fixed graph \( H \) and \( W \in \mathcal{W} \), define the signed density of \( H \) as:

\[
t_+(H, W) := \int_{[0,1]^{|V(H)|}} \prod_{(i,j) \in E(H)} (W(x_i, x_j) - p) \prod_{i=1}^{|V(H)|} dx_i,
\]

and consider the variational problem

\[
\phi_+(H, p, t) := \inf \left\{ \frac{1}{2} \mathbb{E}[I_p(W)] : \text{with } W \in \mathcal{W} \text{ such that } t_+(H, W) \geq t \right\}.
\]  

The inequality \( ||W - pJ||^s_{op} \leq t_+(C_s, W) \), for \( s \) even (see [23, Chapter 7]), then implies

\[
\phi_2(p, \delta p) \geq \phi_+(C_s, p, \delta^s p^s),
\]

that is, to prove Proposition 3.1 it suffices to show that, for \( s > 0 \) even,

\[
\phi_+(C_s, p, \delta^s p^s) \geq (1 - o(1)) \frac{1}{2} \delta^2 p^2 \log(1/p),
\]

for \( p, \delta \) as in Proposition 3.1. The rest of this section is devoted to the proof of (3.4). Throughout, we will assume that \( s > 0 \) is even.

A key first step in the proof of (3.4) which will allow us to rely on the approach in [7] is to show that the the minimization in \( \phi_+(C_s, p, \delta^s p^s) \) can be restricted to the set of all \( W \), such that \( W - p \), in non-negative.

Lemma 3.2. Let \( W = p + U \), where \( -p \leq U \leq 1 - p \). Then,

\[
\phi_+(C_s, p, \delta^s p^s) \geq \inf \left\{ \frac{1}{2} \mathbb{E}[I_p(p + U)] : \text{with } 0 \leq U \leq 1 - p \text{ and } t(C_s, U) \geq \delta^s p^s \right\}.
\]

Proof. For any signed graphon \( U \), let \(|U|\) denote the graphon obtained by taking the absolute value of \( U \). Then clearly, \( t_+(C_s, p + |U|) \geq t_+(C_s, p + U) \). Then we can conclude using the following lemma which guarantees \( I_p(p + x) \geq I_p(p + |x|) \) in the regime of our interest. \( \Box \)

Lemma 3.3. For \( p \leq \frac{1}{2} \) and \( 0 \leq x \leq p \), \( I_p(p - x) \geq I_p(p + x) \).

Proof. Define \( F_p(x) := I_p(p - x) - I_p(p + x) \). It is easy to check that

\[
F'_p(x) = \log \left( \frac{p - x - 1}{p - 1} \right) - \log \left( \frac{p - x}{p} \right) - \log \left( \frac{p + x}{p} \right) + \log \left( \frac{p + x - 1}{p - 1} \right).
\]

Now, using \( F'_p(0) = 0 \) and \( F''_p(x) = \frac{2(1-2p)x}{(1-p+x)(p-x)(1-p-x)(p+x)} \geq 0 \), for \( p \leq \frac{1}{2} \), implies \( F''_p(\cdot) \) is non-decreasing, that is, \( F'_p(x) \geq F'_p(0) = 0 \). This, in turn implies that \( F_p(\cdot) \) is non-decreasing, that is, \( F_p(x) \geq F_p(0) = 0 \).

The inequality in (3.5) shows that to prove (3.4) it suffices to show:

\[
\inf \left\{ \frac{1}{2} \mathbb{E}[I_p(p + U)] : \text{with } 0 \leq U \leq 1 - p \text{ and } t(C_s, U) \geq \delta^s p^s \right\} \geq (1 - o(1)) \frac{1}{2} \delta^2 p^2 \log(1/p).
\]

Moreover, by the upper bound construction in Section 3.1, it suffices consider the minimum in the LHS above over \( U \) such that \( 0 \leq U \leq 1 - p \) and \( \mathbb{E}[I_p(p + U)] \leq \delta^2 p^2 I_p(1) \). Now, if \( \delta \) was fixed, the solution to the variational problem in (3.6) would follow by invoking by Theorem (2.1) above. However, for our purposes we need to allow for \( \delta \) going to zero with \( n \) as well. It turns out that the arguments in [7] can be extended without much modification to yield what we need. We include the arguments with all the necessary details below.
Proof of \((3.6)\) via Adaptive Degree Thresholding: For \(b \in (0, 1]\), define the set \(B_b\) of points \(x\) with high normalized degree \(d(x)\) in \(U\) by

\[
B_b = B_b(U) := \{ x: d_U(x) \geq b \}, \quad d(x) = d_U(x) := \int_0^1 U(x, y)\, dy. \tag{3.7}
\]

Hereafter, the dependence on \(U\) will be dropped from \(B_b(U)\) and \(d_U(x)\), whenever the graphon \(U\) is clear from the context.

**Lemma 3.4.** Let \(U\) be a graphon satisfying\(^7\)

\[
\mathbb{E}[I_p(p + U)] \lesssim \delta^2 p^2 I_p(1). \tag{3.8}
\]

Then

\[
\mathbb{E}[U] \lesssim \delta p^2 \sqrt{\log(1/p)}, \tag{3.9}
\]

and

\[
\mathbb{E}[U^2] \lesssim \delta^2 p^2, \tag{3.10}
\]

and furthermore \(B_b = \{ x: d(x) \geq b \}\), with \(\sqrt{p \log(1/p)} \ll b \ll 1\), satisfies

\[
\mu(B_b) \lesssim \frac{\delta^2 p^2}{b}, \tag{3.11}
\]

where \(\mu\) denotes the Lebesgue measure, and, writing \(\overline{B}_b := [0, 1]\setminus B_b,\)

\[
\int_{B_b} d(x)^2\, dx \lesssim \delta^2 p^2 b. \tag{3.12}
\]

The proof of the above is postponed to the appendix. However with these estimates, we are now ready to establish the desired bound in \((3.6)\). Define

\[
\theta_b := (\delta p)^{-2} \int_{B_b \times \overline{B}_b} U(x, y)^2\, dx\, dy \quad \text{and} \quad \eta_b := (\delta p)^{-2} \int_{\overline{B}_b \times B_b} U(x, y)^2\, dx\, dy. \tag{3.13}
\]

Moreover, for a non-negative integer \(s > 0\), define

\[
W(z|C_s) := \prod_{i=1}^s W(x_i, x_{i+1}), \tag{3.14}
\]

where \(z = (x_v)_{v \in [s]}\) is clear from context and \(x_{s+1} := x_1.\)\(^8\) At this point we will rely on the following key generalization of Hölder’s inequality \([21, \text{Theorem} 2.1]\) (closely related to the Brascamp–Lieb inequalities \([12]\)) used in \([7, 25, 26]\).

**Theorem 3.2** (Generalized Hölder’s inequality). Let \(\mu_1, \mu_2, \ldots, \mu_n\) be probability measures on \(\Omega_1, \ldots, \Omega_n\) resp., and let \(\mu = \prod_{i=1}^n \mu_i.\) Let \(A_1, \ldots, A_m\) be non-empty subsets of \([n] = \{1, \ldots, n\}\) and for \(A \subseteq [n]\) put \(\mu_A = \prod_{j \in A} \mu_j\) and \(\Omega_A = \prod_{j \in A} \Omega_j.\) Let \(f_i \in L^{p_i}(\Omega_A, \mu_A)\) for each \(i \in [m],\) and further suppose that \(\sum_{i: A_i \ni j}(1/p_i) \leq 1\) for all \(j \in [n].\) Then

\[
\int \prod_{i=1}^m |f_i|\, d\mu \leq \prod_{i=1}^m \left( \int |f_i|^{p_i}\, d\mu_{A_i} \right)^{1/p_i}. \]

\(^7\)More precisely, the statement is that for every constant \(C > 0\) there is some constant \(C' > 0\) such that if \((3.8)\) holds with constant hidden \(C,\) then \((3.9)\)–\((3.12)\) all hold with hidden constant \(C'.\)

\(^8\)If the domain of an integral is omitted, then it is assumed to be \([0, 1]\) for every \(x_v.\)
Lemma 3.5. Let $U$ a graphon satisfying (3.8). For any $\sqrt{p \log(1/p)} \ll b_0 \ll 1$, there exists some $b$ with $b_0 \leq b \ll 1$ such that

$$\int U(x|C_s)1\{x_1 \in B_b, x_3 \in \overline{B}_b\} \, dx = o(\delta^s p^s).$$

Proof. Fix $\varepsilon \geq 0$ (which can be made arbitrarily small). It suffices to show that one can find $b$ (depending on $\varepsilon$, $L$ and $U$) with $b_0 \leq b \ll 1$ such that

$$\int U(x|C_s)1\{x_1 \in B_{b'}, x_3 \in \overline{B}_{b''}\} \, dx \leq (1 + o(1)) \varepsilon \delta^s p^s$$

uniformly for all $3 \leq s \leq L$. By removing the vertex labeled 1 and then applying Theorem 3.2 followed by Lemma 3.4, we have, for any $p \ll b', b'' \ll 1$,

$$\int U(x|C_s)1\{x_1 \in B_{b'}, x_3 \in \overline{B}_{b''}\} \, dx \leq \mu(B_{b'}) \int 1\{x_3 \in \overline{B}_{b''}\} U(x_2, x_3)U(x_3, x_4) \cdots U(x_{s-1}, x_s) \, dx$$

(recall (3.14))

$$\leq \mu(B_{b'}) \int 1\{x_3 \in \overline{B}_{b''}\} d(x_3)U(x_3, x_4) \cdots U(x_{s-1}, x_s) \, dx$$

(integrating over $x_2$)

$$\leq \mu(B_{b'}) \left( \int_{\overline{B}_{b''}} d(x_3)^2 \, dx_3 \right)^{1/2} \mathbb{E}[U^2]^{(s-3)/2}$$

(by Theorem 3.2)

$$\leq \frac{\delta^2 p^{2/3}}{b'} (\delta^2 p^{2/3} b')^{1/2} (\delta p)^{s-3}$$

(by Lemma 3.4)

$$\leq \frac{\sqrt{b'}}{b'} \delta^s p^s.$$  

(3.17)

Fix some $s$ for now. By (3.15), there is some constant $C$ such that $t(C_s, U) \leq C \delta^s p^s$. Let $M := \lceil C/\varepsilon \rceil$. Since it never hurts to make $b_0$ larger, assume that $p \ll b_0 \ll 1$. For any sequence $b_0 < b_1 < \cdots < b_M = o(1)$ with $b_{i-1} \leq b_i^2$ for each $i \leq M$, there is some $1 \leq i \leq M$ such that

$$\int U(x|C_s)1\{x_1 \in B_{b_i}, x_3 \in B_{b_{i-1}} \setminus B_{b_i}\} \, dx \leq \varepsilon \delta^s p^s,$$
since otherwise the sum of these integrals over $1 \leq i \leq M$ (note that the sets $B_{b_i - 1}\setminus B_{b_i}$, $1 \leq i \leq M$, are disjoint) would violate $t(C_s, U) \leq C\delta^3 p^s$. Combining the above estimate with (3.17) applied with $b' = b_1$ and $b'' = b_{i-1}$ (so that $\sqrt{b''/b'} = o(1)$), we see that (3.16) holds with $b = b_i$, completing the proof.

Completing the Proof of the Lower Bound. Denote by $[s]_+$ be the subset of even numbers in $[s] := \{1, 2, \ldots, s\}$. Then by Lemma 3.5, for any $\sqrt{p}\log(1/p) \ll b_0 \ll 1$, there exists some $b$ with $b_0 \leq b \ll 1$ such that

$$t(C_s, U) = \Gamma_1 + \Gamma_2 + \Gamma_3 + o(\delta^s p^s),$$

(3.18)

where

$$\Gamma_1 = \int U(x|C_s) \left\{ \forall v \in [s]_+: x_v \in B_b \right\} dx, \quad \Gamma_2 = \int U(x|C_s) \left\{ \forall u \notin [s]_+: x_u \in B_b \right\} dx,$$

and

$$\Gamma_3 = \int U(x|C_s) \left\{ \forall v \in V(H): x_v \in B_b \right\} dx.$$

(Note that $\Gamma_1$ and $\Gamma_2$ quantifies the contribution to $t(C_s, U)$ from embeddings with alternate vertices in $B_b$ and $B_b^C$, and $\Gamma_3$ quantifies the contribution from embeddings with all vertices in $B_b^C$. All other embeddings have contribution $o(\delta^s p^s)$ by Lemma 3.5.) Now, recalling (3.13), by Theorem 3.2,

$$\Gamma_1 \leq \theta_b^{3/2} \delta^s p^s, \quad \Gamma_2 \leq \theta_b^{3/2} \delta^s p^s, \quad \text{and} \quad \Gamma_3 \leq \eta_b^{3/2} \delta^s p^s.$$

Therefore, $t(C_s, U) \geq \delta^s p^s$ and (3.18) implies $2\theta_b^{3/2} + \eta_b^{3/2} \geq 1 - o(1)$. Thus, by (3.13) and Corollary 6.1,

$$E[I_p(p + U)] \geq (\theta_b + \frac{1}{2} \eta_b - o(1))\delta^2 p^2 \log(1/p)$$

$$\geq (1 - o(1)) \min_{2x^2 + y^2 \geq 1} (x + \frac{1}{2} y)\delta^2 p^2 \log(1/p)$$

$$= (1 - o(1)) \frac{1}{2} \delta^2 p^2 \log(1/p),$$

where the last step uses the lemma below. (The minimum is attained at $x = 0$, since the value corresponding to $y = 0$ is always larger.)

**Lemma 3.6.** Let $f, g$ be convex nondecreasing functions on $[0, \infty)$ and let $a > 0$. The minimum of $x + y$ over the region $\{x, y \geq 0: f(x) + g(y) \geq a\}$ is attained at either $x = 0$ or $y = 0$.

**Proof.** By convexity, if $\gamma = \frac{a}{x+y}$ then $f(x) \leq \gamma f(0) + (1-\gamma)f(x+y)$ and $g(y) \leq (1-\gamma)g(0)+\gamma g(x+y)$, so

$$f(x) + g(y) \leq \gamma[f(0) + g(x+y)] + (1-\gamma)[f(x+y) + g(0)]$$

$$\leq \max\{f(0) + g(x+y), f(x+y) + g(0)\}.$$

This shows that for a fixed value of $x+y$, $f(x) + g(y)$ is maximized at $x = 0$ or $y = 0$. The claim then follows. \qed
4. Proof of Corollary 1.5

The upper bound in Corollary 1.5 is an immediate consequence of Theorem 1.4 and the following simple consequence of Weyl’s interlacing inequality (see [28]):

\[ \lambda_2(A(G_{n,p})) \leq ||A(G_{n,p}) - pI||_{op}. \]

The lower bound follows by planting a clique of appropriate size in the random graph. Given \( s \geq 0 \), let \( C_s \) be the event that the vertices \( \{1, 2, 3, \ldots, [s] + 1\} \) forms a clique in \( G_{n,p} \). Note that

\[
\mathbb{P}(\lambda_2(G_{n,p}) \geq \delta np) \geq \mathbb{P}(\lambda_2(G_{n,p}) \geq \delta np | C_{\delta np})
\]

\[= \mathbb{P}(\lambda_2(G_{n,p}) \geq \delta np | C_{\delta np}) \cdot \mathbb{P}(C_{\delta np}) \]

\[= \frac{1}{s} e^{-(1+o(1)) \frac{1}{8} s^2 n^2 p^2 \log(1/p)}, \]

using \( \mathbb{P}(C_{\delta np}) = p^{(\delta np + 1)} \) and the lemma below. This completes the proof of the lower bound.

**Lemma 4.1.** Under the assumption of Corollary 1.5, \( \mathbb{P}(\lambda_2(G_{n,p}) \geq \delta np | C_{\delta np}) \geq \frac{1}{8} \).

**Proof.** Throughout the proof, \( v \) will denote the vector in (2.8) with \( s = \delta np \). Moreover, \( \overline{v} = \frac{v}{n} \mathbf{1} \) will denote the projection of \( v \) on \( \mathbf{1} \), and \( \hat{v} = v - \overline{v} \), which is a vector orthogonal to \( \mathbf{1} \).

Next, recall the following well-known variational representation of the second eigenvalue of the adjacency matrix \( A(G_{n,p}) = ((a_{ij})) \):

\[
\lambda_2(G_{n,p}) = \sup_{V: \dim(V) = 2} \left\{ \inf_{w \in V} \frac{w'A(G_{n,p})w}{w'w} \right\}. \tag{4.1}
\]

Therefore, to prove the lemma, it suffices to show that given \( C_{\delta np} \), the following hold with probability at least \( \frac{1}{8} \):

\[
\frac{1'A(G_{n,p})\mathbf{1}}{1'\mathbf{1}} \geq (1 + o(1)) np \quad \text{and} \quad \frac{\hat{v}'A(G_{n,p})\hat{v}}{\hat{v}'\hat{v}} \geq \delta np (1 + o(1)),
\]

since then replacing \( \delta \) by \( \delta (1+o(1)) \) above, along with the assumption that \( \delta < 1 \) ensures, \( \lambda_2(G_{n,p}) \geq \delta np \) given \( C_{\delta (1+o(1)) np} \), with probability at least \( \frac{1}{8} \), by taking \( V \) to be the 2-dimensional subspace generated by \( \mathbf{1} \) and \( v \) in (4.1). For the first inequality, observe that given \( C_{\delta np} \),

\[
1'A(G_{n,p})\mathbf{1} \geq \sum_{[s] + 2 \leq i \neq j \leq n} a_{ij} \sim \text{Bin}(n - [s] - 1, p).
\]

Now, using \( \mathbb{P} (\text{Bin}(n - [s] - 1, p) - (n - [s] - 1)p \geq 0) \geq \frac{1}{4} \), shows \( 1'A(G_{n,p})\mathbf{1} \geq n(n - [s] - 1)p = (1 + o(1))n^2 p \), with probability at least \( \frac{1}{4} \).

For the second inequality, using \( \hat{v}'\hat{v} \leq v'v = 1 \) gives,

\[
\frac{\hat{v}'A(G_{n,p})\hat{v}}{\hat{v}'\hat{v}} \geq \hat{v}'A(G_{n,p})\hat{v} = v'A(G_{n,p})v - 2\overline{v}'A(G_{n,p})v + \overline{v}'A(G_{n,p})\overline{v}
\]

\[\geq v'A(G_{n,p})v - 2\overline{v}'A(G_{n,p})v \]

\[\geq \delta np - 2\overline{v}'A(G_{n,p})v, \tag{4.2}\]

using \( v'A(G_{n,p})v \geq \delta np \), as \( s = \delta np \) (as in (2.8)). Now, it is easy to verify that \( n \{ \overline{v}'A(G_{n,p})v \} \) is the number of edges incident on the vertices labeled \( 1, 2, 3, \ldots, [s] + 1 \), in the graph \( G_{n,p} \). Therefore, on the event \( C_{\delta np} \),

\[
n\{ \overline{v}'A(G_{n,p})v \} \leq [s]([s] + 1) + \sum_{1 \leq i \leq [s] + 1} \sum_{11 \leq j \leq n} a_{ij}
\]
Finally, by Markov’s inequality, the second term above is less than
\[ 4E \sum_{1 \leq i \leq [s]+1} \sum_{1 \leq j \leq n} a_{ij} = O(ns) = O(\delta n^2 p^2), \]
with probability at least \( \frac{3}{4} \). Now, combining (4.2) and (4.3) it follows that \( \frac{\hat{\mu}^t A(\mathcal{G}_{n,p}) \hat{\mu}}{\hat{\mu}^t \hat{\mu}} \geq (1+o(1))\delta np \) with probability at least \( \frac{3}{4} \). This completes the proof of the lemma. \( \square \)

5. Discussions and Future directions

The results in this paper were concerned with the analysis of variational problems for the upper tails of edge eigenvalues of \( \mathcal{G}_{n,p} \). The corresponding problem for the lower tail is also interesting, which for the case of the largest eigenvalue is well understood. It follows from results in Lubetzky and Zhao [25, Proposition 3.9] for the dense case and, more generally, from Cook and Dembo [16, Theorem 1.21], that the lower tail problem for \( \lambda_1(\mathcal{G}_{n,p}) \) exhibits replica symmetry, that is, the corresponding variational problem is minimized by the constant function. More precisely, it is known that, for \( \frac{\log n}{n} \ll p \leq \frac{1}{2} \) and \( 0 < q < p \) (such that \( s := q/p \in (0, 1) \) is fixed),
\[ P(\lambda_1(\mathcal{G}_{n,p}) \leq q(n-1)) = e^{-\left(1+o(1))\left(\frac{q}{p}\right)I_p(q).} \]

Finally, this work leaves many questions open, especially for \( \lambda_2(\mathcal{G}_{n,p}) \) and other smaller edge eigenvalues. We list below a few of them.

1. Extending Corollary 1.5 to the case \( \delta > 1 \). The key issue one faces is that \( \lambda_2(\mathcal{G}_{n,p}) > np \) automatically guarantees that \( \lambda_1(\mathcal{G}_{n,p}) > np \) as well, and in particular the Perron-Frobenius eigenvector will now be not close to the constant vector. Hence, one cannot automatically transfer the knowledge about the operator norm of \( A(\mathcal{G}_{n,p}) - p11' \) to the second eigenvalue of \( A(\mathcal{G}_{n,p}) \).

2. Establishing a joint large deviation for cycle homomorphism densities of different sizes, and using these, or otherwise, obtain a joint LDP for \( \lambda_1(\mathcal{G}_{n,p}) \) and \( \lambda_2(\mathcal{G}_{n,p}) \), or of various spectral moments.

3. Compute large deviation probabilities for the \( k \)-th largest eigenvalue \( \lambda_k(\mathcal{G}_{n,p}) \). Does the upper tail large deviation behavior of \( \lambda_k(\mathcal{G}_{n,p}) \) agree with the probability of planting \( k \) small cliques of appropriate sizes (up to negligible factors in the exponential scale) in some regime of the parameter space?

6. Appendix: Proof of Lemma 3.4

In this section, we prove Lemma 3.4 which follows by a straightforward modification of [7, Lemma 4.2], adapted to include the case \( \delta \to 0 \), as \( n \to \infty \). We begin by recalling the following estimates for \( I_p(x) \) from [26]. The \( \sim \) notation below is with respect to limits as \( p \to 0 \).

**Lemma 6.1** ([26, Lemma 3.3]). If \( 0 \leq x \ll p \), then \( I_p(p+x) \sim \frac{x^2}{2p} \), whereas when \( p \ll x \leq 1-p \) we have \( I_p(p+x) \sim x \log(x/p) \).

**Lemma 6.2** ([26, Lemma 3.4]). There is some constant \( p_0 > 0 \) such that for every \( 0 < p \leq p_0 \),
\[ I_p(p+x) \geq (x/b)^2 I_p(p+b) \quad \text{for any } 0 \leq x \leq b \leq 1-p - \log(1-p). \]

**Corollary 6.1** ([26, Corollary 3.5]). There is some constant \( p_0 > 0 \) such that for every \( 0 < p \leq p_0 \),
\[ I_p(p+x) \geq x^2 I_p(1-1/\log(1/p)) \sim x^2 I_p(1) \quad \text{for any } 0 \leq x \leq 1-p. \]
Proof of Lemma 3.4: From Lemma 6.1 we have $I_p(p + \delta p^2 \sqrt{\log(1/p)}) \sim \frac{1}{2} \delta^2 p^2 I_p(1)$ for any $p \leq p_0$ (since $\delta p^2 \sqrt{\log(1/p)} \lesssim p^2 \sqrt{\log(1/p)} \ll p$). However, $I_p(p + \mathbb{E}[U]) \leq \mathbb{E}[I_p(p + U)] \lesssim \delta^2 p^2 I_p(1)$ by the convexity of $I_p(\cdot)$ and (3.8). Therefore, by the monotonicity of $I_p(p + x)$ for $x \geq 0$ we obtain the upper bound $\mathbb{E}[U] \lesssim \delta p^2 \sqrt{\log(1/p)}$, proving (3.9). Furthermore, using (3.8) and Corollary 6.1 we obtain $\mathbb{E}[U^2] \lesssim \mathbb{E}[I_p(p + U)]/I_p(1) \lesssim \delta^2 p^2$, proving (3.10).

To prove (3.11), we notice that by the convexity of $I_p(\cdot)$, for any $b \gg p$,

$$\mathbb{E}[I_p(p + U)] = \int_{[0,1]^2} I_p(p + U(x, y)) \, dx dy \geq \int_0^1 I_p(p + d_U(x)) \, dx \geq \mu(B_b) I_p(p + b).$$

It follows from Lemma 6.1 (combined with (3.8)) that for any $p \ll b \leq 1 - p$,

$$\mu(B_b) \leq \frac{\mathbb{E}[I_p(p + U)]}{I_p(p + b)} \lesssim \frac{\delta^2 p^2 I_p(1)}{b \log(b/p)} \lesssim \frac{\delta^2 p^2}{b}, \quad (6.1)$$

proving (3.11). Furthermore to prove (3.12), by the convexity of $I_p(x)$ and Lemma 6.2,

$$\mathbb{E}[I_p(p + U)] \geq \int_{\mathbb{B}_b} I_p(p + d(x)) \, dx \geq I_p(p + b) \int_{\mathbb{B}_b} (d(x)/b)^2 \, dx.$$

Combining these, we get

$$\int_{\mathbb{B}_b} d(x)^2 \, dx \leq \frac{b^2 \mathbb{E}[I_p(p + U)]}{I_p(p + b)} \lesssim \delta^2 p^2 b,$$

where in the final inequality we use the bound recorded in (6.1), proving (3.12). □

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