On the Hermite expansions of functions from Hardy class

by

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Abstract. Considering functions \( f \) on \( \mathbb{R}^n \) for which both \( f \) and \( \hat{f} \) are bounded by the Gaussian \( e^{-\frac{1}{2a}|x|^2}, 0 < a < 1 \) we show that their Fourier-Hermite coefficients have exponential decay. Optimal decay is obtained for \( O(n) \)–finite functions thus extending the one dimensional result of Vemuri [11].

1. Introduction

Consider the normalised Hermite functions \( \Phi_\alpha, \alpha \in \mathbb{N}^n \) on \( \mathbb{R}^n \) which are eigenfunctions of the Hermite operator \( H = -\Delta + |x|^2 \) with eigenvalues \( (2|\alpha| + n) \). They form an orthonormal basis for \( L^2(\mathbb{R}^n) \) so that every \( f \in L^2(\mathbb{R}^n) \) has the expansion

\[
f = \sum_{\alpha \in \mathbb{N}^n} (f, \Phi_\alpha) \Phi_\alpha.
\]

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When the Hermite coefficients of \( f \) has exponential decay, say \(|(f, \Phi_\alpha)| \leq Ce^{-(2|\alpha|+n)t}\), for some \( t > 0 \), then by Mehler's formula (see [8]) it can be easily shown that \( f \) satisfies the estimate
\[
|f(x)| \leq Ce^{-\frac{1}{2}\tanh(t)|x|^2}.
\]
As \( \Phi_\alpha \) are also eigenfunctions of the Fourier transform with eigenvalues \((-i)^{|\alpha|}\), it follows that \(|(\hat{f}, \Phi_\alpha)| \leq Ce^{-(2|\alpha|+n)t}\) and hence \( \hat{f} \) also satisfies the same estimate as \( f \).

However, it is possible to prove better estimates for \( f \) and \( \hat{f} \). The assumption on \((f, \Phi_\alpha)\) together with the asymptotic properties of holomorphically extended Hermite functions lead us to the fact that \( f \) extends to \( \mathbb{C}^n \) as an entire function and satisfies
\[
|f(x+iy)| \leq C_m(1 + |x|^2 + |y|^2)^{-m}e^{-\frac{1}{2}\tanh(2s)|x|^2 + \frac{1}{2}\coth(2s)|y|^2}
\]
for every \( m \in \mathbb{N} \) and \( 0 < s < t \). And a similar estimate holds for \( \hat{f} \) as well. Indeed, under the assumption on \((f, \Phi_\alpha)\) the entire function \( f(z) \) belongs to the Hermite Bergman space \( \mathcal{H}_s(\mathbb{C}^n) \) consisting of entire functions which are square integrable with respect to the weight function
\[
U_s(x, y) = e^{\tanh(2s)|x|^2 - \coth(2s)|y|^2}
\]
for every \( s < t \) and hence as shown in [7] the functions \( f(z) \) and \( \hat{f}(z) \) both satisfy the above estimate.

Suppose we only know that \( f \) and \( \hat{f} \) are bounded on \( \mathbb{R}^n \) by the Gaussian \( e^{-\frac{1}{2}\tanh(2t)|x|^2} \). We would like to know if these conditions in turn imply some exponential decay of the Hermite coefficients of \( f \). It will be so if we can prove that \( f(z) \) satisfies
\[
|f(x+iy)|^2 \leq Ce^{-\tanh(2s)|x|^2 + \coth(2s)|y|^2}
\]
for some \( s > 0 \). Under the assumption on \( f \) and \( \hat{f} \) it is clear, from the Fourier inversion formula, that \( f \) extends to \( \mathbb{C}^n \) as an entire function which satisfies
\[
|f(x+iy)| \leq Ce^{\frac{1}{2}\coth(2s)|y|^2}.
\]
But a priori it is not at all clear if \( f(x+iy) \) has any decay in \( x \). In this article we address the problem of estimating \( f \) on \( \mathbb{C}^n \).
This problem has connections with a classical theorem of Hardy [5] proved in 1933 which says that a function \( f \) and its Fourier transform \( \hat{f} \) both cannot have arbitrary Gaussian decay. The precise statement is as follows. For a function \( f \in L^1(\mathbb{R}^n) \), let

\[
\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx
\]

be its Fourier transform. Suppose

\[
|f(x)| \leq Ce^{-a|x|^2}, \quad |\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}
\]

for some positive constants \( a \) and \( b \). Then \( f = 0 \) when \( ab > 1/4 \) and \( f(x) = Ce^{-a|x|^2} \) when \( ab = 1/4 \). Moreover, there are infinitely many linearly independent functions satisfying both conditions when \( ab < 1/4 \). Examples of such functions are provided by the Hermite functions \( \Phi_\alpha \).

Hardy’s theorem has received considerable attention over the last fifteen years or so as can be seen from the large number of papers written on the theorem, see e.g. the monograph [10] and the references therein. However, all the works so far have treated only the case \( ab \geq 1/4 \) in various set-ups. The case \( ab < 1/4 \) did not receive any closer study until recently where in [11] Vemuri has looked at functions satisfying Hardy conditions with \( a = b < 1/2 \). By a very clever use of Bargmann transform he has proved the following characterisation of such functions.

**Theorem 1.1.** Suppose \( f \in L^1(\mathbb{R}) \) satisfies the conditions

\[
|f(x)| \leq Ce^{-\frac{a}{2}x^2}, \quad |\hat{f}(\xi)| \leq Ce^{-\frac{b}{4}\xi^2}
\]

for some \( 0 < a < 1 \). Then the Fourier-Hermite coefficients of \( f \) satisfy \( |(f, \Phi_k)| \leq C(2k + 1)^{-\frac{1}{2}}e^{-\left(2k+1\right)t/2} \), where \( t \) is determined by the condition \( a = \tanh(2t) \).

In [11] the author has considered functions of one variable only. A natural question is whether a similar result is true for functions on \( \mathbb{R}^n \). The proof in [11], like many other proofs of Hardy-type theorems, depends on Phragmen-Lindelof maximum principle which is essentially
a theorem in one complex variables. If we consider functions $f$ which are tensor products of one dimensional functions, then an analogue of Theorem 1.1 follows easily. More generally, the arguments in [11] can be used to prove the following result. We state the result in terms of the Hermite projection operators $P_k$ which are defined by

$$P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha) \Phi_\alpha$$

for any $f \in L^2(\mathbb{R}^n)$. We refer to [8] for more about Hermite expansions.

**Theorem 1.2.** Suppose $|f(x)| \leq C e^{-\frac{a}{2}|x|^2}$ and for any $j = 1, 2, ..., n$, $|\mathcal{F}_j f(\xi)| \leq C e^{-\frac{a}{2}|\xi|^2}$ where $\mathcal{F}_j f$ is the partial Fourier transform of $f$ in any set of $j$ variables. Then we have the estimates $\|P_k f\|_2 \leq C(2k + n)^{n-1} e^{-(2k+n)t/2}$ where $a = \tanh(2t)$.

There are strong reasons to believe that the result is true for all functions satisfying the Hardy conditions. However, at present we do not know how to prove the result. Nevertheless, we have the following slightly weaker result.

**Theorem 1.3.** Suppose $f \in L^1(\mathbb{R}^n)$ satisfies the estimates

$$|f(x)| \leq C e^{-\frac{1}{2a}|x|^2}, \quad |\hat{f}(x)| \leq C e^{-\frac{1}{2a}|x|^2}$$

for some $0 < a < 1$. Then $\|P_k f\|_2 \leq C(2k + n)^{\frac{n-1}{4}} e^{-(2k+n)s/2}$ where $s$ is determined by the condition $\tanh(2s) = a/2$.

We prove this theorem in Section 4 by relating the Hermite projections $P_k f$ with the Fourier-Wigner transform $V(f, f)$ and appealing to a version of Hardy’s theorem for Hankel transform. Since the Fourier transform of a radial function reduces to a Hankel transform, Theorem 1.1 can be shown to be true for all radial functions. More generally, we can prove the same for all $O(n)$-finite functions in $L^2(\mathbb{R}^n)$. In other words, Theorem 1.1 remains true for all functions whose restrictions to the unit sphere $S^{n-1}$ have only finitely many terms in their spherical harmonic expansions.
Theorem 1.4. Suppose $f \in L^1(\mathbb{R}^n)$ satisfies the same conditions as in the previous theorem. If we further assume that $f$ is $O(n)-$finite, then $\|P_k f\|_2 \leq C(2k + n)^{\frac{n}{2}} e^{-(2k+n)t/2}$ where $a = \tanh(2t)$.

We prove this theorem in Section 5 by studying a vector valued Bargmann transform. Let us define the Hardy class $H(a), 0 < a < 1$ as the set of all functions $f$ satisfying the Hardy conditions in Theorem 1.3. We are interested in estimating the Hermite coefficients of $f$ from $H(a)$. This problem has been completely solved in the one dimensional case by Vemuri [11]. A work closely related to this article is the paper by Janssen and Eijndhoven [6] where they have studied growth of Hermite coefficients in one dimension. Here we treat the higher dimensional case. It would also be interesting to find the precise relation between Hardy conditions and the membership in Hermite-Bergman spaces $\mathcal{H}(\mathbb{C}^n)$.

2. Preliminaries

In this section we set up the notations and collect relevant results about Hermite functions, Fourier-Wigner and Hankel transforms. We closely follow the notations used in [9] and [3] and we refer to the same for the proofs and any unexplained terminology. Writing down the Hermite expansion of $f \in L^2(\mathbb{R}^n)$ as $f = \sum_{k=0}^{\infty} P_k f$, the Plancherel theorem reads as $\|f\|_2^2 = \sum_{k=0}^{\infty} \|P_k f\|_2^2$.

The Fourier-Wigner transform of two functions $f, g \in L^2(\mathbb{R}^n)$ is a function on $\mathbb{C}^n$ defined by

$$V(f,g)(x + iy) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \frac{1}{2} x \cdot y)} f(\xi + y) g(\xi) d\xi.$$  

We make use of the identity (see [8])

$$\int_{\mathbb{C}^n} V(f_1, g_1)(z) \overline{V(f_2, g_2)(z)} dz = (f_1, f_2)(g_2, g_1).$$

for any $f_i, g_i \in L^2(\mathbb{R}^n)$. 
The special Hermite functions $\Phi_{\alpha,\beta} = V(\Phi_\alpha, \Phi_\beta)$ form an orthonormal basis for $L^2(\mathbb{C}^n)$. We observe that

$$\sum_{|\alpha|=k} |(f, \Phi_\alpha)|^2 = \int_{\mathbb{C}^n} V(f, f)(z) \sum_{|\alpha|=k} \overline{\Phi_{\alpha\alpha}}(z) \, dz.$$ 

If we let $\varphi_k^{n-1}(z)$ stands for the Laguerre function $L_k^{(n-1)}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$ then we know that

$$\sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(z) = (2\pi)^{-n/2} \varphi_k^{n-1}(z)$$

and therefore we get the useful relation

$$\|P_k f\|_2^2 = (2\pi)^{-n/2} \int_{\mathbb{C}^n} V(f, f)(z) \varphi_k^{n-1}(z) \, dz$$

which will be used in Section 4. The same idea has been used in [6] in the study of growth of Hermite coefficients.

In Section 5 we will make use of a Hecke-Bochner type formula for the Hermite projection operators. Let $P$ be a harmonic polynomial which is homogeneous of degree $m$, called a solid harmonic. It is well known that if $f$ is radial, then the Fourier transform of $fP$ is again of the same form, viz. $\hat{f}P = FP$ where $F$ is given by a Hankel transform. A similar result is true for the Hermite projections. Let $L_k^\delta$ stand for Laguerre polynomials of type $\delta$ which are defined by the generating function identity

$$\sum_{k=0}^{\infty} L_k^\delta(x)e^{-\frac{1}{2}x}r^k = (1 - r)^{-\delta-1}e^{-\frac{1}{2}\frac{\delta+1}{1-r}x}$$

for $|r| < 1, x > 0$. Define

$$R_k^\delta(f) = 2\frac{\Gamma(k + 1)}{\Gamma(k + \delta + 1)} \int_0^\infty f(s)\psi_k^\delta(s)s^{2\delta+1}ds$$

where the Laguerre functions $\psi_k^\delta$ are defined by

$$\psi_k^\delta(s) = L_k^\delta(s^2)e^{-\frac{1}{2}s^2}.$$
**Proposition 2.1.** Suppose \( f \in L^2(\mathbb{R}^n) \) be such that \( f = gP \) where \( g \) is radial and \( P \) is a solid harmonic of degree \( m \). Then \( P_j f = 0 \) unless \( j = 2k + m \) in which case

\[
P_{2k+m}f(x) = R_k^{n/2+m-1}(g)P(x)\psi_k^{m+1-1}(|x|).
\]

The restrictions of solid harmonics to \( S^{n-1} \) are called spherical harmonics. Let \( \{Y_{mj} : 1 \leq j \leq d_m, m \in \mathbb{N}\} \) be an orthonormal basis for \( L^2(S^{n-1}) \) consisting of spherical harmonics. Given \( f \in L^2(\mathbb{R}^n) \) we have the expansion

\[
f(r\omega) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} f_{mj}(r)Y_{mj}(\omega)
\]

where \( f_{mj} \) are the spherical harmonic coefficients of \( f \) defined by

\[
f_{mj}(r) = \int_{S^{n-1}} f(r\eta)Y_{mj}(\eta) \, d\eta.
\]

The above proposition leads to the formula

\[
P_{2k}f(x) = \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} R_{k-m}^{2m+1}(\tilde{f}_{2m,j}) \psi_{k-m}^{2m+1-1}(r) r^{2m}Y_{2m,j}(\omega).
\]

where \( \tilde{f}_{m,j}(r) = r^{-m} f_{mj}(r) \). A similar formula can be written for \( P_{2k+1}f \) as well. The functions \( \psi_k^\delta \) form an orthogonal system in \( L^2(\mathbb{R}^+, r^{2\delta+1} \, dr) \) and suitably normalised they form an orthonormal basis.

### 3. Bargmann transform and Hardy’s theorem

For the convenience of the readers we briefly recall the argument used by Vemuri [11] in proving Theorem 1.1. As we have already mentioned we will be using variants of the same arguments, so it will help fixing the ideas. Recall that the Bargmann transform \( B \) defined by

\[
Bf(z) = e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) \, e^{-\frac{1}{4}|x|^2} e^{z\cdot x} \, dx
\]

for \( z \in \mathbb{C}^n \) is an isometric isomorphism from \( L^2(\mathbb{R}^n) \) onto the Fock space consisting of entire functions on \( \mathbb{C}^n \) that are square integrable with respect to the Gaussian \( e^{-\frac{1}{4}|z|^2} \), see Bargmann [1]. It takes the Hermite
functions $\Phi_\alpha$ onto the monomials $\zeta_\alpha(z) = \left(\frac{2^\alpha \pi^{\alpha/2}}{\pi_\alpha}ight)^{1/2} z^\alpha$. Moreover, it has the interesting property that $B\hat f(z) = (2\pi)^{\frac{n}{2}} Bf(-iz)$.

If $f$ satisfies the Gaussian estimate $f(x) = O(e^{-\frac{1}{2}a|x|^2})$ then from the definition of $B$ it follows that

$$|Bf(w)| \leq C(1+a)^{-\frac{n}{2}} \exp \left(\frac{u^2 + \mu v^2}{4}\right)$$

where $\mu = \frac{1-a}{1+a}$, $w = u + iv$ and $u^2 = \sum_{j=1}^{n} u_j^2$ etc. The relation $B\hat f(z) = (2\pi)^{\frac{n}{2}} Bf(-iz)$ then leads to

$$|Bf(w)| \leq C(1+a)^{-\frac{n}{2}} \exp \left(\frac{u^2 + \mu v^2}{4}\right).$$

When $n = 1$ and taking $w = re^{i\theta}$ we get

$$|Bf(w)| \leq C(1+a)^{-\frac{1}{2}} \exp \left(\frac{(\mu + (1-\mu) \sin^2 \theta)r^2}{4}\right)$$

and

$$|Bf(w)| \leq C(1+a)^{-\frac{1}{2}} \exp \left(\frac{(\mu + (1-\mu) \cos^2 \theta)r^2}{4}\right).$$

A Phragmen-Lindelof argument then leads to the estimate

$$|Bf(w)| \leq C(1+a)^{-\frac{1}{2}} \exp \left(\frac{\sqrt{\mu}}{4} r^2\right).$$

If $c_k$ are the Taylor coefficients of $Bf$ then Cauchy’s estimates lead to

$$|c_k| \leq C(1+a)^{-\frac{1}{2}} \exp \left(\frac{\sqrt{\mu}}{4} r^2\right) r^{-k}$$

and optimizing with respect to $r$ we can get

$$|c_k| \leq C(1+a)^{-\frac{1}{2}} \left(\frac{e^{\sqrt{\mu}}}{2k}\right)^{k/2}.$$

Since $c_k$ are related to the Hermite coefficients of $f$, we get a slightly weaker form of Theorem 1.1. For the argument leading to Theorem 1.1 we refer to [11]. In the $n$-dimensional case it is possible to use the same arguments to prove Theorem 1.2 under the extra assumptions made in the hypothesis. We leave the details to the reader.
4. Hardy’s theorem for Hankel transform and a proof of Theorem 1.3

For any \( \delta > -\frac{1}{2} \) we define the Hankel transform \( H_\delta \) on \( L^1(\mathbb{R}^+, r^{2\delta + 1}dr) \) by

\[
H_\delta f(r) = \int_0^\infty f(s) \frac{J_\delta(rs)}{(rs)\delta} s^{2\delta + 1} ds.
\]

It is well known that \( H_\delta \) extends to \( L^2(\mathbb{R}^+, r^{2\delta + 1}dr) \) as a unitary operator and the inversion formula is given by

\[
f(r) = \int_0^\infty H_\delta f(s) \frac{J_\delta(rs)}{(rs)\delta} s^{2\delta + 1} ds
\]

for all \( f \) for which \( H_\delta f \) is integrable with respect to \( s^{2\delta + 1}ds \). Moreover, it is known that \( H_\delta \psi_k^\delta = (-1)^k \psi_k^\delta \). We will make use of this fact in what follows.

An analogue of Hardy’s theorem (i.e. the case \( ab \geq 1/4 \)) is known for the Hankel transform as well. We now prove an analogue of Theorem 1.1 for the Hankel transform.

**Theorem 4.1.** Let \( f \in L^1(\mathbb{R}^+, r^{2\delta + 1}dr) \) be such that both \( f \) and \( H_\delta f \) satisfy the Hardy condition with \( a = \tanh(2\delta) \). Then the Laguerre coefficients of \( f \) satisfy the following estimate:

\[
|\langle f, \psi_k^\delta \rangle| \leq C(1 + a)^{-\delta} (4k + 2\delta + 1)^\delta e^{-2tk}.
\]

The proof of this theorem is similar to that of Theorem 1.1 given in [11]. We just need to replace the Bargmann transform by another transform adapted to the Hankel transform. We now proceed to define this transform which we denote by \( U_\delta \). For \( f \in L^2(\mathbb{R}^+, r^{2\delta + 1} dr) \) we let

\[
U_\delta f(w) = e^{\frac{w^2}{4}} \int_0^\infty f(s) \frac{J_\delta(iws)}{(iws)^\delta} e^{-\frac{1}{2}(w^2+s^2)} s^{2\delta + 1} ds.
\]

It is clear that \( U_\delta f \) extends to \( \mathbb{C} \) as an even entire function of \( w \). Moreover, the generating function identity

\[
\sum_{k=0}^\infty \frac{L_k^\delta(x)}{\Gamma(k + \delta + 1)} w^k = e^{w(xw)^{-\frac{1}{2}}} J_\delta(2(xw)^{\frac{1}{2}})
\]




satisfied by the Laguerre polynomials can be rewritten as
\[
\sum_{k=0}^{\infty} \frac{(-1)^k 2^{-2k}}{\Gamma(k + \delta + 1)} \psi^\delta_k(r) \ w^{2k} = 2^k e^{\frac{-w^2}{4}} \frac{J_\delta(irw)}{(irw)^\delta} e^{-\frac{1}{2}(r^2+w^2)}.
\]
In view of this we have
\[
U_\delta f(w) = 2^{-\delta} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-2k}}{\Gamma(k + \delta + 1)} (f, \psi^\delta_k) \ w^{2k}.
\]
This shows that the transformation \( U_\delta \) takes the Laguerre functions \( \psi^\delta_k \) onto constant multiples of the monomials \( w^{2k} \). We also have the relation \( U_\delta H_\delta f(w) = U_\delta f(-iw) \) which follows from the fact that \( H_\delta \psi^\delta_k = (-1)^k \psi^\delta_k \).

The image of \( L^2(\mathbb{R}^+, r^{2\delta+1} \ dr) \) under the transform \( U_\delta \) is known to be a weighted Bargmann space, see [2]. Indeed, if we let
\[
h(w) = \frac{2\delta}{\pi} \left( \frac{|w|^2}{2} \right)^{2\delta+1} K_{\delta+\frac{1}{2}} \left( \frac{|w|^2}{2} \right)
\]
where
\[
K_\delta(z) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma(\delta + \frac{1}{2})} \int_0^\infty e^{-t}\left(1 + \frac{t}{2z}\right)^{-\delta-\frac{1}{2}} dt
\]
then the image is precisely the Hilbert space of even entire functions that are square integrable with respect to \( h(w)dw \) (see Cholewinski [2]). As \( h(w) \) is radial it is clear that \( w^{2k} \) form an orthogonal system with respect to \( h(w)dw \). Moreover, it can be shown that (see [2])
\[
\int_C |w|^{4k} h(w) dw = 2^{1+2\delta} 2^{4k} \Gamma(k + 1) \Gamma(k + \delta + 1).
\]
Thus, if we let
\[
\zeta_k(w) = (-1)^k \left( 2^{1+2\delta+4k} \Gamma(k + 1) \Gamma(k + \delta + 1) \right)^{-\frac{1}{2}} w^{2k}
\]
then \( \zeta_k \) form an orthonormal basis for the image of \( L^2(\mathbb{R}^+, r^{2\delta+1} \ dr) \) under \( U_\delta \) and \( U_\delta \left( \psi^\delta_k \right)(w) = \zeta_k(w) \).

We can now proceed as in Vemuri [11] with \( U_\delta \) playing the role of the Bargmann transform to prove Theorem 4.1.

We now use Hardy’s theorem for the Hankel transform to prove Theorem 1.3. Let \( \mathcal{F}_s \) stands for the symplectic Fourier transform. We
need the following estimate on the Fourier-Wigner transform $V(f,f)$ when $f$ and $\hat{f}$ satisfy Hardy conditions.

**Proposition 4.2.** Let $f \in L^2(\mathbb{R}^n)$ satisfies $|f(x)| \leq Ce^{-a|x|^2}$ and $|\hat{f}(\xi)| \leq Ce^{-a|\xi|^2}$ for some $C > 0$ and $a > 0$. Then $|V(f,f)(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|z|^2}$ and $|\mathcal{F}_s V(f,f)(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|z|^2}$, where $C_n > 0$ depends only on $C$ and $n$.

**Proof.** By definition, for $z = x + iy \in \mathbb{C}^n$

$$V(f,f)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x,\xi + \frac{1}{2}x,y)} f(x + y) \overline{f(\xi)} \, d\xi.$$  

An easy calculation using Fourier inversion shows that

$$V(\hat{f},\hat{f})(z) = (2\pi)^n V(f,f)(iz).$$

From the definition it follows that $|V(f,f)(z)|$ is bounded by

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |f(x + y)||\overline{f(\xi)}| \, d\xi \leq C \int_{\mathbb{R}^n} e^{-a|\xi + y|^2} e^{-a|\xi|^2} \, d\xi.$$  

The last integral is equal to

$$Ce^{-\frac{1}{2}a|y|^2} \int_{\mathbb{R}^n} e^{-2a|\xi + \frac{1}{2}y|^2} \, d\xi = C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|y|^2}.$$  

Replacing $f$ by $\hat{f}$, we also get

$$|V(\hat{f},\hat{f})(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|y|^2}.$$  

And thus the relation $V(f,f)(z) = (2\pi)^{-n} V(\hat{f},\hat{f})(-iz)$ gives

$$|V(f,f)(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|x|^2}.$$  

Combining these two, we get

$$|V(f,f)(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|z|^2}.$$  

The above calculation together with the relation $\mathcal{F}_s V(f,f)(z) = (8\pi)^{\frac{n}{2}} V(f,\hat{f})(z)$ implies that

$$|\mathcal{F}_s V(f,f)(z)| \leq C_n a^{-\frac{n}{2}} e^{-\frac{1}{2}a|z|^2}.$$  

This completes the proof of the proposition. □
In view of the expression for the norm of $P_k$ in terms of the Laguerre coefficients of $V(f, f)$, in order to prove Theorem 1.3 we only need to prove the following result.

**Theorem 4.3.** Suppose $f \in L^2(\mathbb{R}^n)$ is such that $|V(f, f)(z)| \leq Ce^{-\frac{a}{4}|z|^2}$ and $|\mathfrak{F}_s V(f, f)(z)| \leq Ce^{-\frac{a}{4}|z|^2}$. Then

$$|\int_{C^n} V(f, f)(z)\varphi^{n-1}_k(z)dz| \leq C(2k + n)^{n-1}e^{-(2k+n)s}$$

where $s$ is determined by $a/2 = \tanh(2s)$.

**Proof.** As $\varphi^{n-1}_k(z)$ is radial, recalling the definition of $\psi^{n-1}_k$ the integral we want to estimate reduces to $2^n \int_0^\infty F(\sqrt{2}s)\psi^{n-1}_k(r)r^{2n-1}dr$

where

$$F(r) = \int_{S^{2n-1}} V(f, f)(r\omega)d\omega$$

which clearly satisfies the estimate $|F(r)| \leq Ce^{-\frac{a}{4}r^2}$. If we can show that the function $G(r) = F(\sqrt{s}r)$ satisfies the estimate $|H_{n-1}G(r)| \leq Ce^{-\frac{a}{4}s^2}$, then we can appeal to Theorem 4.1 to get the required estimate. Since $\varphi^{n-1}_k$ is an eigenfunction of the symplectic Fourier transform

$$\int_{C^n} V(f, f)(z)\varphi^{n-1}_k(z)dz = (-1)^k \int_{C^n} \mathfrak{F}_s V(f, f)(z)\varphi^{n-1}_k(z)dz.$$

We now perform the following calculations:

$$\int_{S^{2n-1}} \mathfrak{F}_s V(f, f)(\sqrt{2}r\omega) d\omega = \int_{S^{2n-1}} \overline{V(f, f)}\left(-\frac{i}{2}\sqrt{2}r\omega\right) d\omega$$

$$= \int_{S^{2n-1}} \overline{V(f, f)}\left(\frac{r}{\sqrt{2}}\omega\right) d\omega$$

$$= \int_0^\infty F(s)\frac{J_{n-1}(\frac{s}{\sqrt{2}s})}{(\frac{s}{\sqrt{2}s})^{n-1}}s^{2n-1}ds$$

$$= 2^n \int_0^\infty F(\sqrt{2}s)\frac{J_{n-1}(rs)}{(rs)^{n-1}}s^{2n-1}ds$$

$$= 2^n \int_0^\infty G(s)\frac{J_{n-1}(rs)}{(rs)^{n-1}}s^{2n-1}ds$$

$$= 2^n (H_{n-1}G)(r)$$

which proves our claim on $H_{n-1}G(r)$. \qed
If we can improve the estimates in Proposition 4.2 to 
\[ |V(f, f)(z)| \leq Ce^{-\frac{1}{2}|z|^2} \] and 
\[ |\mathfrak{H}_s V(f, f)(z)| \leq Ce^{-\frac{1}{2}|z|^2} \] then we could prove Theorem 4.3 with \( \tanh(2s) = a \). In fact, one needs only
\[ |\int_{S^{2n-1}} V(f, f)(r\omega) d\omega| \leq Ce^{-\frac{1}{2}ar^2} \]
and a similar estimate for \( \mathfrak{H}_s V(f, f) \) which is good enough to improve Theorem 1.3. But there are some limitations on the decay of Fourier-Wigner transform due to the uncertainty principle proved in [4]. The following example shows that improving the estimates in Proposition 4.2 is not always possible which means that the proof via Fourier-Wigner transform is not robust enough to lead to Theorem 1.3.

**Example 4.1.** Let \( a = \frac{1}{\sqrt{2}} \) and consider the function \( f \in L^2(\mathbb{R}^2) \), defined by
\[ f(x_1, x_2) = e^{-\frac{a}{2}(x_1^2 + x_2^2 + 2ix_1x_2)}. \]

An easy calculation (using \( a = \frac{1}{2a} \)) shows that
\[ \hat{f}(\xi, \eta) = \frac{\pi \sqrt{2}}{a} e^{-\frac{1}{4a}(\xi^2 + \eta^2 - 2i\xi\eta)} = 2\pi e^{-\frac{a}{2}(\xi^2 + \eta^2 - 2i\xi\eta)} \]
and with \( z = (x_1 + iy_1, x_2 + iy_2) \)
\[ V(f, f)(z) = \frac{1}{2a} e^{-\frac{a}{2}|z|^2} e^{\frac{i}{2}(x_1y_2 - x_2y_1)}. \]

From the above expression for \( V(f, f) \), it is clear that the estimate
\[ |V(f, f)(z)| \leq Ce^{-\frac{a}{4}|z|^2} \]
is not valid.

For any \( 0 < b < \frac{1}{2} \), let us write
\[ R_b = \{ \omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in S^3 \mid \omega_1 \omega_4 > b, \omega_2 \omega_3 > 0 \}. \]
Clearly, $R_b$ is a subset of $S^3$ of positive measure. Let us denote its measure by $|R_b|$. Then,

$$\int_{S^3} V(f, f)(r\omega)d\omega = \frac{1}{2a} e^{-\frac{a}{2}r^2} \int_{S^3} e^{\frac{1}{2}r^2(\omega_1^2 + \omega_2^2 + \omega_3^2)} d\omega \geq \frac{1}{2a} e^{-\frac{a}{2}r^2} \int_{R_b} e^{\frac{1}{2}r^2(\omega_1^2 + \omega_2^2 + \omega_3^2)} d\omega \geq \frac{1}{2a} |R_b| e^{-\frac{a}{2}r^2} e^{\frac{b}{2}r^2} = \frac{1}{2a} |R_b| e^{-\frac{a}{2}r^2} e^{\frac{1}{2}(b - \frac{a}{2})r^2}$$

Since $a = \frac{1}{\sqrt{2}}$, one can choose $0 < b < \frac{1}{2}$ such that $b - \frac{a}{2} > 0$. Therefore, $\int_{S^3} V(f, f)(r\omega)d\omega$ can not be bounded by $e^{-\frac{a}{2}r^2}$.

However we can show that $\|P_kf\|_2$ has the required decay. Indeed, for any $0 < r < 1$, we have

$$\sum_{k=0}^{\infty} r^k \|P_kf\|_2^2 = (2\pi)^{-1} \int_{\mathbb{C}^2} V(f, f)(z) \left( \sum_{k=0}^{\infty} r^k \varphi_k(z) \right) dz$$

and hence using the generating function identity

$$\sum_{k=0}^{\infty} r^k \varphi_k(z) = (1 - r)^{-1} e^{-\frac{1}{4} \frac{1+r}{1-r} |z|^2}$$

and the explicit expression for $V(f, f)$ we can calculate that

$$\sum_{k=0}^{\infty} r^k \|P_kf\|_2^2 = \frac{8\pi a(1 - r)^{-2}}{(2a + \frac{1+r}{1-r}) (2a + \frac{1+r}{1-r} - \frac{1}{(2a + \frac{1+r}{1-r})})}.$$

Writing $\mu = \frac{1-r}{1+a}$ and simplifying, the above takes the form

$$\sum_{k=0}^{\infty} r^k \|P_kf\|_2^2 = \frac{2\pi}{1+a} \sum_{k=0}^{\infty} \mu^k r^{2k}.$$

Comparing coefficients of $r^k$ we see that $P_{2k+1}f = 0$ and $\|P_{2k}f\|_2^2 = \frac{2\pi}{1+a} \mu^k$ which is the expected decay.
5. A vector valued Bargmann transform and a proof of Theorem 1.4

In order to prove Theorem 1.4 we need to study a vector valued Bargmann transform. For functions \( f \) from \( L^2(\mathbb{R}^n) \) consider
\[
Bf(z, \omega) = e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) \ e^{-\frac{1}{2}|x|^2} e^{zx.\omega} dx
\]
where \( z \in \mathbb{C} \) and \( \omega \in S^{n-1} \). We think of \( Bf \) as an entire function of the one complex variable taking values in the vector space \( L^2(S^{n-1}) \).

As before, one can easily verify that \( \hat{Bf}(z, \omega) = (2\pi)^{\frac{n}{2}} Bf(-iz, \omega) \).

We consider functions satisfying the conditions:
\[
\begin{align*}
(5.1) & \quad \left( \int_{S^{n-1}} |f(s\eta)|^2 \ d\eta \right)^{\frac{1}{2}} \leq C e^{-\frac{a}{2}s^2}, \\
(5.2) & \quad \left( \int_{S^{n-1}} |\hat{f}(s\eta)|^2 \ d\eta \right)^{\frac{1}{2}} \leq C e^{-\frac{4a}{2}s^2}
\end{align*}
\]
for some \( a > 0 \). The basic estimates on the Bargmann transforms of such functions are given below.

**Proposition 5.1.** If \( f \in L^2(\mathbb{R}^n) \) be such that \((5.1)\) and \((5.2)\) are valid for some \( a > 0 \). Then for every \( z = u + iv \in \mathbb{C} \),
\[
\begin{align*}
\int_{S^{n-1}} |Bf(z, \omega)|^2 \ d\omega & \leq C(1 + a)^{-n} \exp \left( \frac{u^2 + \mu v^2}{2} \right), \\
\int_{S^{n-1}} |Bf(z, \omega)|^2 \ d\omega & \leq C(1 + a)^{-n} \exp \left( \frac{u^2 + \mu v^2}{2} \right)
\end{align*}
\]
where \( \mu = \frac{1-a}{1+a} \) as before.

**Proof.** For \( z \in \mathbb{C} \) and \( \omega \in S^{n-1} \),
\[
Bf(z, \omega) = e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) \ e^{-\frac{1}{2}|x|^2} e^{zx.\omega} dx
\]
\[
= e^{-\frac{1}{4}z^2} \int_0^\infty \left( \int_{S^{n-1}} f(s\eta) \ e^{s\eta.\omega} \ d\eta \right) e^{-\frac{1}{2}s^2} s^{n-1} ds
\]
Thus we get the estimate
\[
|B f(z,\omega)| \leq |e^{-\frac{1}{4}z^2}| \int_0^\infty |T_{sz}f(\omega)| \ e^{-\frac{1}{2}s^2} s^{n-1} ds
\]
where
\[
T_{sz}f(\omega) = \int_{S^{n-1}} f(s\eta) \ e^{sz\eta\omega} \ d\eta
\]
If we write \( z = u + iv \), then
\[
|T_{sz}f(\omega)| \leq \int_{S^{n-1}} |f(s\eta)| \ e^{isu\omega} \ d\eta
\]
and consequently,
\[
\left( \int_{S^{n-1}} |T_{sz}f(\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq \left( \int_{S^{n-1}} |f(s\eta)|^2 \ d\eta \right)^\frac{1}{2} \int_{S^{n-1}} e^{isu\omega} \ d\eta
\]
\[
\leq C \ e^{-\frac{1}{2}s^2} \frac{J_{\frac{n}{2}-1}(isu)}{(isu)^{\frac{n}{2}-1}}
\]
Notice that \( \left( \int_{S^{n-1}} |B f(z,\omega)|^2 d\omega \right)^\frac{1}{2} \) is bounded by
\[
e^{-\frac{1}{4}(u^2-v^2)} \left( \int_{S^{n-1}} \left( \int_0^\infty |T_{sz}f(\omega)| \ e^{-\frac{1}{2}s^2} s^{n-1} ds \right)^2 d\omega \right)^{\frac{1}{2}}
\]
\[
\leq e^{-\frac{1}{4}(u^2-v^2)} \int_0^\infty \left( \int_{S^{n-1}} |T_{sz}f(\omega)|^2 d\omega \right)^{\frac{1}{2}} \ e^{-\frac{1}{2}s^2} s^{n-1} ds
\]
where the last inequality is achieved using Minkowski’s integral inequality.

Now, using the above estimates, we get
\[
\left( \int_{S^{n-1}} |B f(z,\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq C \ e^{-\frac{1}{4}(u^2-v^2)} \int_0^\infty e^{-\frac{1}{2}(1+a)s^2} \frac{J_{\frac{n}{2}-1}(isu)}{(isu)^{\frac{n}{2}-1}} s^{n-1} ds
\]
\[
\leq \frac{C}{(1+a)^{\frac{n}{2}}} \exp \left( -\frac{u^2-v^2}{4} \right) \exp \left( \frac{u^2}{2(1+a)} \right)
\]
\[
= \frac{C}{(1+a)^{\frac{n}{2}}} \exp \left( \frac{v^2 + \mu u^2}{4} \right).
\]
Replacing \( f \) by \( \hat{f} \) and using the fact that \( B\hat{f}(z,\omega) = (2\pi)^{\frac{n}{2}} B f(-iz,\omega) \), we also get
\[
\left( \int_{S^{n-1}} |B f(z,\omega)|^2 d\omega \right)^{\frac{1}{2}} \leq \frac{C}{(1+a)^{\frac{n}{2}}} \exp \left( \frac{v^2 + \mu u^2}{4} \right)
\]
Hence the proposition is proved.

\[ \square \]

**Theorem 5.2.** For a function \( f \) on \( \mathbb{R}^n \) satisfying the conditions (5.1) and (5.2) let \( Bf(z, \omega) = \sum_{k=0}^{\infty} d_k(\omega) z^k \) be the Taylor series expansion of the Bargmann transform. Then

\[
\int_{S^{n-1}} |d_k(\omega)|^2 \, d\omega \leq C 2^{-k} \frac{k^{-\frac{1}{2}}}{\Gamma(k+1)} \mu^\frac{k}{2}.
\]

**Proof.** The proof of this theorem can be reduced to the scalar valued case treated in [11]. Indeed, for any normalised \( g \in L^2(S^{n-1}) \) the scalar valued function

\[ F_g(z) = \int_{S^{n-1}} Bf(z, \omega)g(\omega) \, d\omega \]

is an entire function satisfying estimates stated in Proposition 5.1. The arguments in [11] lead to estimates for the integral

\[ \int_{S^{n-1}} d_k(\omega)g(\omega) \, d\omega. \]

Taking supremum over all such \( g \) we get the required estimates. \( \square \)

In order to apply the above estimates to prove Theorem we need the following result which shows that the \( L^2(S^{n-1}) \) norms of \( d_k(\omega) \) can be expressed in terms of Laguerre coefficients of the spherical harmonic components of \( f \) restricted to the unit sphere.

**Theorem 5.3.** For \( f \in L^2(\mathbb{R}^n) \), if \( Bf(z, \omega) = \sum_{k=0}^{\infty} d_k(\omega) z^k \) is the Taylor series expansion, then for all \( k \geq 1 \),

\[
\int_{S^{n-1}} |d_{2k}(\omega)|^2 \, d\omega = 2^{-n-4k+2} \sum_{m=0}^{d_{2m}} \sum_{j=1}^{d_{2m}} \left| \frac{(f_{2m,j}, \psi_{k-m}^{\frac{n}{2}+2m-1})}{\Gamma\left(\frac{n}{2} + k + m\right)} \right|^2,
\]

\[
\int_{S^{n-1}} |d_{2k+1}(\omega)|^2 \, d\omega = 2^{-n-4k} \sum_{m=0}^{d_{2m+1}} \sum_{j=1}^{d_{2m+1}} \left| \frac{(f_{2m+1,j}, \psi_{k-m}^{\frac{n}{2}+2m})}{\Gamma\left(\frac{n}{2} + k + m + 1\right)} \right|^2.
\]

**Proof.** We know that for every \( z_1, z_2 \in \mathbb{C} \)

\[ e^{iz_1 \eta \omega} = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \frac{J_{\frac{n}{2}+m-1}(z_1 \bar{z}_2)}{(z_1 \bar{z}_2)^{\frac{n}{2}-1}} Y_{mj}(\eta) Y_{mj}(\omega). \]
which in particular implies that
\[
e^{sz_{\eta,\omega}} = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \frac{J_{\frac{n}{2}+m-1}(-isz)}{(-isz)^{\frac{n}{2}-1}} Y_{mj}(\eta) Y_{mj}(\omega)
\]
and thus
\[
\int_{S^{n-1}} f(s\eta) e^{sz_{\eta,\omega}} d\eta = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} f_{mj}(s) \frac{J_{\frac{n}{2}+m-1}(-isz)}{(-isz)^{\frac{n}{2}-1}} Y_{mj}(\omega).
\]
Now,
\[
Bf(z,\omega) = e^{-\frac{1}{4}z^2} \int_{\mathbb{R}^n} f(x) e^{-\frac{1}{2}||x||^2} e^{zx,\omega} dx
\]
\[
= e^{-\frac{1}{4}z^2} \int_{0}^{\infty} \int_{S^{n-1}} f(s\eta) e^{-\frac{1}{2}s^2} e^{sz_{\eta,\omega}} s^{n-1} ds d\eta
\]
\[
= e^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} \left( \int_{0}^{\infty} f_{mj}(s) \frac{J_{\frac{n}{2}+m-1}(-isz)}{(-isz)^{\frac{n}{2}-1}} e^{-\frac{1}{2}s^2} s^{n-1} ds \right) Y_{mj}(\omega)
\]
\[
= \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} (-iz)^m \left( U_{m} \tilde{f}_{mj}(z) \right) Y_{mj}(\omega)
\]
where for simplicity we have written $U_m$ in place of $U_{\frac{n}{2}+m-1}$.

If we write the power series expansion of $U_m \tilde{f}_{mj}$ as
\[
(U_m \tilde{f}_{mj})(z) = \sum_{k=0}^{\infty} b_{k,mj} z^{2k}
\]
then, as we saw in Section 4,
\[
b_{k,mj} = 2^{-(\frac{n}{2}+m-1)} \frac{(-1)^k 2^{-2k}}{\Gamma(\frac{n}{2}+k+m)} (\tilde{f}_{mj}, \psi_{k}^{\frac{n}{2}+m-1})
\]
Now, $d_k(\omega) = \frac{1}{2\pi i} \int_{\gamma} Bf(z,\omega) dz$ implies that
\[
d_{2k}(\omega) = (-1)^{\frac{n}{2}-1} \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} (-i)^m (b_{k-m, 2m,j}) Y_{2m,j}(\omega)
\]
which implies,
\[
\int_{S^{n-1}} |d_{2k}(\omega)|^2 d\omega = \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} |b_{k-m; 2m, j}|^2
\]
and therefore
\[
\int_{S^{n-1}} |d_{2k}(\omega)|^2 d\omega = 2^{-n-4k+2} \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} |(\tilde{f}_{2m,j} , \psi_{k-m}^{\frac{n}{2}+2m-1})|^2
\]
\frac{\Gamma(\frac{n}{2} + k + m)}{(\Gamma(\frac{n}{2} + k + m + 1))^2}
\]
A similar calculation shows that
\[
\int_{S^{n-1}} |d_{2k+1}(\omega)|^2 d\omega = 2^{-n-4k} \sum_{m=0}^{k} \sum_{j=1}^{d_{2m+1}} |(\tilde{f}_{2m+1,j} , \psi_{k-m}^{\frac{n}{2}+2m})|^2
\]
\frac{\Gamma(\frac{n}{2} + k + m + 1)}{(\Gamma(\frac{n}{2} + k + m + m + 1))^2}
\]
which completes the proof of the theorem.

Combining Theorems 5.2 and 5.3 we can prove Theorem 1.4. To see this, we first observe that the Hecke-Bochner formula for the Hermite projections lead to

**Proposition 5.4.**
\[
\|P_{2k} f\|_2^2 = 2 \sum_{m=0}^{k} \sum_{j=1}^{d_{2m}} \left( \Gamma(k-m+1) \Gamma(\frac{n}{2} + k + m) \right) \frac{|(\tilde{f}_{2m,j} , \psi_{k-m}^{\frac{n}{2}+2m-1})|^2}{(\Gamma(\frac{n}{2} + k + m))^2}
\]
\]
A similar expression holds for $P_{2k+1} f$ also.

We note the similarity between the expression for $\|P_{2k} f\|_2^2$ and the $L^2(S^{n-1})$ norms of $d_{2k}(\omega)$. We therefore, rewrite expression for $\|P_{2k} f\|_2^2$ as
\[
2 \sum_{m=0}^{k} 2^{2m} c(k, m) \left( 2^{-2k} \Gamma(2k+1) \frac{|(\tilde{f}_{2m,j} , \psi_{k-m}^{\frac{n}{2}+2m-1})|^2}{(\Gamma(\frac{n}{2} + k + m))^2} \right)
\]
where
\[
c(k, m) = \left( 2^{2(k-m)} \Gamma(k-m+1) \Gamma(\frac{n}{2} + k + m) \frac{\Gamma(2k+1)}{\Gamma(2k+1)} \right).
\]
Stirling’s formula for gamma functions show that $c(k, k) = O(k^{n/2-1})$.
In general, we have

**Lemma 5.5.** For any $0 \leq m \leq k$ we have $c(k, m) = O(k^{(n-1)/2})$.
PROOF. We show that for all \( k \geq 1 \) and \( 0 \leq m \leq k \),

\[
2^{2(k-m)} \frac{\Gamma(k - m + 1) \Gamma(k + m + \frac{n}{2})}{\Gamma(2k + 1)} = O(k^{\frac{n}{2} - \frac{1}{2}}).
\]

For this, fix \( k \in \mathbb{N} \) and consider \( c(k, m) \). When \( m = k \), \( c(k, k) = \frac{\Gamma(2k + \frac{n}{2})}{\Gamma(2k + 1)} \) and by Stirling’s formula, for large \( k \) it behaves like

\[
\frac{(2k + \frac{n}{2} - 1)^{2k + \frac{n}{2} - \frac{1}{2}} e^{-(2k + \frac{n}{2} - 1)}}{(2k)^{2k + \frac{n}{2}} e^{-2k}} = \frac{(1 + \frac{n}{2k})^{2k} (2k + \frac{n}{2} - 1)^{\frac{n}{2} - \frac{1}{2}} e^{-\left(\frac{n}{2} - 1\right)}}{(2k)^{\frac{n}{2}}}.
\]

As \( (1 + \frac{n}{2k})^{2k} \leq e^{\frac{n}{2} - 1}, c(k, k) = O\left(k^{\frac{n}{2} - 1}\right) \).

Now for \( 0 \leq m < k \), consider

\[
c(k, m) = 2^{2(k-m)} \frac{\Gamma(k - m + 1) \Gamma(k + m + \frac{n}{2})}{\Gamma(2k + 1)}
\sim 2^{2(k-m)} \frac{(k-m)^{k-m+\frac{1}{2}} e^{-(k-m)}(k + m + \frac{n}{2} - 1)^{k+m+\frac{n}{2} - \frac{1}{2}} e^{-(k+m+\frac{n}{2} - 1)}}{(2k)^{2k+\frac{n}{2}} e^{-2k}}
\]

\[
= \frac{e^{-\left(\frac{n}{2} - 1\right)}}{\sqrt{2}} \frac{(2-2m)(k-m)^{k-m+\frac{1}{2}} (k + m + \frac{n}{2} - 1)^{k+m+\frac{n}{2} - m - \frac{1}{2}}}{k^{2k+\frac{n}{2}}}
\]

\[
\leq C_n e^{-\left(\frac{n}{2} - 1\right)} 2^{-2m} \left(1 - \frac{m}{k}\right)^k \left(1 + \frac{m + \frac{n}{2} - 1}{k}\right)^k \left\{\frac{1 + \frac{m + \frac{n}{2} - 1}{k}}{1 - \frac{m}{k}}\right\}^m k^{\frac{n}{2} - \frac{1}{2}}
\]

\[
\leq C_n e^{-\left(\frac{n}{2} - 1\right)} 2^{-2m} e^{-m e^{m + \frac{n}{2} - 1}} \left\{\frac{1 + \frac{m + \frac{n}{2} - 1}{k}}{1 - \frac{m}{k}}\right\}^m k^{\frac{n}{2} - \frac{1}{2}}
\]

\[
= C_n 2^{-2m} \left\{\frac{1 + \frac{m + \frac{n}{2} - 1}{k}}{1 - \frac{m}{k}}\right\}^m k^{\frac{n}{2} - \frac{1}{2}}
\]

But, \( 2^{-2m} \left\{\frac{1 + \frac{m + \frac{n}{2} - 1}{k}}{1 - \frac{m}{k}}\right\}^m \leq 1 \) if and only if \( 1 + \frac{m + \frac{n}{2} - 1}{k} \leq 4(1 - \frac{m}{k}) \) which happens precisely when \( m \leq \frac{1}{5}(3k - \frac{n}{2} + 1) \). Since for sufficiently large \( k \), \( \frac{1}{5}(3k - \frac{n}{2} + 1) \geq \frac{[k+1]}{5} \), it follows that for \( 0 \leq m \leq \frac{[k+1]}{2} \), \( c(k, m) \leq C_n k^{\frac{n}{2} - \frac{1}{2}} \). Now consider \( \frac{[k+1]}{2} < m < k \).
In this case
\[
c(k, m) \leq C_n e^{-\left(\frac{k}{2}-1\right)2^{-2m}} \left(1 - \frac{m}{k}\right)^{k-m} \left(\frac{k + m + \frac{n}{2} - 1}{k}\right)^{k+m} \frac{n}{k^2 - \frac{1}{2}}
\]
\[
\leq C_n e^{-\left(\frac{k}{2}-1\right)2^{-2m}} \left(1 - \frac{m}{k}\right)^{k-m} \left(1 + \frac{m}{k} - 1\right)^{k+m} \frac{n}{k^2 - \frac{1}{2}}
\]
\[
= C_n e^{-\left(\frac{k}{2}-1\right)2^{-2m}} \left(1 - \frac{m}{k}\right)^{k-m} \left(1 + \frac{m}{k} - 1\right)^{k+m} \frac{n}{k^2 - \frac{1}{2}}
\]
\[
\leq C_n e^{-\left(\frac{k}{2}-1\right)2^{-2m}} \left(\frac{1}{2}\right)^{k-m} 2^{k+m} e^{\frac{n}{2} - 1} k^{\frac{n}{2} - \frac{1}{2}}
\]
\[
= C_n k^{\frac{n}{2} - \frac{1}{2}}
\]

In the second last step, we use the fact that \(m > \left[\frac{k+1}{2}\right]\) implies that \(1 - \frac{m}{k} < -\frac{1}{2}\).

The extra factor of \(2^{2m}\) in the expression for \(\|P_{2k}f\|_2^2\) suggests that we consider the operator \(T\) defined by

\[
Tf(r\omega) = \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \left(\sum_{j=1}^{d_m} f_{mj}(r) Y_{mj}(\omega)\right).
\]

It is then clear that when \(f\) satisfies the Hardy conditions \(Tf\) satisfies

\[
\int_{L^2(S^{n-1})} |Tf(r\omega)|^2 d\omega \leq Ce^{-\frac{1}{2}ar^2}.
\]

A similar estimate is true for \(\widehat{Tf}\) as well.

**Theorem 5.6.** Suppose \(f\) satisfies the Hardy conditions with \(a = \tanh(2t)\). Then for any \(k = 0, 1, 2, \ldots\) we have

\[
\|P_k(Tf)\|_2 \leq C(2k + n)^{\frac{n-2}{2}} e^{-(2k+n)t/2}.
\]

The theorem follows by using the estimates obtained in Theorem 5.2 along with the above lemma. Theorem 1.4 follows as a corollary to Theorem 5.6 since for such functions \(\|P_kf\|_2 \leq C\|P_k(Tf)\|_2\), where \(C\) depends on the number of spherical harmonic coefficients present in the expansion of \(f\).
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