Confinement of Slave-Particles in $U(1)$ Gauge Theories of Strongly-Interacting Electrons

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We show that slave particles are always confined in $U(1)$ gauge theories of interacting electron systems. Consequently, the low-lying degrees of freedom are different from the slave particles. This is done by constructing a dual formulation of the slave-particle representation in which the no-double occupancy constraint becomes linear and, hence, soluble. Spin-charge separation, if it occurs, is due to the existence of solitons with fractional quantum numbers.

Introduction. A number of attempts to understand the dynamics of strongly-correlated electrons have employed a ‘slave particle’ decomposition of the electron operator, such as

$$c_{i\sigma} = b_i^\dagger f_{i\sigma}$$

The slave boson operator, $b_i$, is supposed to carry the charge of the electron while the fermion $f_{i\sigma}$ carries the spin. It is sometimes supposed that these particles can be liberated at low energies, with spin-charge separation as an upshot.

In order for this to occur, the $U(1)$ gauge symmetry

$$b_i \to e^{i\theta_i} b_i, \quad f_{i\sigma} \to e^{i\theta_{i\sigma}} f_{i\sigma}$$

must be broken. This $U(1)$, it should be emphasized, is not the electromagnetic $U(1)$. It is due to the redundancy inherent in the slave particle description. It will be broken if the gauge field which parametrizes fluctuations about the broken symmetry state is not confining. Such a gauge theory is said to be in its Coulomb phase. In this paper, we suggest that this cannot happen. Since the gauge symmetry is an exact local symmetry of the model it can never be broken. This follows from Elitzur’s theorem. Global symmetries can be broken since the effect of an infinitesimal symmetry-breaking field on long-wavelength fluctuations is extensive; in the infinite-volume limit, it can inhibit the symmetry-restoring fluctuations. Local symmetries cannot be broken since they can be restored by purely local (pure gauge) fluctuations. The effects of an infinitesimal symmetry-breaking field are infinitesimal, in contrast to the global case; in this respect, gauge theories are similar to 1D systems.

Consequently, perturbation theory about an assumed broken-symmetry state is not valid. The slave-particle gauge theory is infinitely strongly-coupled – i.e. there is no kinetic energy for the gauge field. In this paper, we will discuss a way of analyzing the strongly-coupled theory and ramifications for the issue of spin-charge separation in strongly-correlated electron systems.

$U(1)$ Gauge Theory Formulation of the $t-J$ Model. It is often convenient to reformulate models of strongly-interacting electrons in terms of redundant auxiliary degrees of freedom which satisfy constraints reducing their enlarged Hilbert space to the physical one. Consider, for instance, the Hubbard model in the limit that the on-site repulsion is much larger than the hopping matrix element, $U >> t$. At energy and temperature scales much less than $U$, we may replace this model by the $t-J$ model

$$H = -t \sum_{<i,j>} \left( f_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.} \right) + J \sum_{<i,j>} \left( S_i \cdot S_j - \frac{1}{4} n_i n_j \right)$$

with $J = t^2/4U$ together with the constraint $n_i \leq 1$. The constraint is an extreme form of strong interactions; it makes the physics of the $t-J$ model opaque. By contrast, the residual interaction, $J$, is small.

We can rewrite this Hamiltonian using the slave particle representation. A slave boson $b_i^\dagger |0\rangle$ represents a vacant site, while an auxiliary fermion $f_{i\sigma}^\dagger |0\rangle$ represents a site occupied by an electron of spin $\sigma$. The no-double-occupancy constraint $n_i \leq 1$ now reads

$$b_i^\dagger b_i + f_{i\sigma}^\dagger f_{i\sigma} = 1$$

It restricts the Hilbert space of slave bosons and auxiliary fermions to the physical subspace which only has the above three states per site. Note that the constraint is now an equality rather than an inequality, thereby allowing a Lagrange multiplier formulation.

The Hamiltonian can now be written

$$H = -t \sum_{<i,j>} f_{i\sigma}^\dagger f_{j\sigma} b_i^\dagger b_j + \text{h.c.} - 2J \sum_{<i,j>} f_{i\sigma}^\dagger f_{j\sigma} f_{j\alpha}^\dagger f_{i\alpha}$$

$$+ \sum_i a_i^2 \left( b_i^\dagger b_i + f_{i\sigma}^\dagger f_{i\sigma} - 1 \right)$$

where $a_i^2$ is the Lagrange multiplier which enforces the constraint. Following, we now decouple the quartic terms with the aid of a Hubbard-Stratonovich field $\chi_{ij}$.

$$H = \sum_{<i,j>} \left( J |\chi_{ij}|^2 - J \chi_{ij} f_{i\sigma}^\dagger f_{j\sigma} + \text{h.c.} \right)$$
When $\chi_{ij}$ acquires an expectation value, the symmetry (2) is broken. We will assume that the magnitude of $\chi_{ij}$ is fixed and study fluctuations of the phase
\[ \chi_{ij,j+k} = \chi e^{ia_k(j)} \] (7)

When these phase fluctuations are large, the symmetry is restored. In the continuum limit, we can write the corresponding Lagrangian as (8a):
\[
L = f^\dagger_\sigma (\partial_\tau - a_0) f_\sigma + f^\dagger_\sigma \left( i\nabla - \vec{a} \right)^2 f_\sigma - a_0 \rho_0 \\
+ b^\dagger (\partial_\tau - a_0 - A_0) b + b^\dagger \left( i\nabla - \vec{a} - \vec{A} \right)^2 b + |b|^4 
\] (8)

where $m_s \sim 1/J$, $m_c \sim 1/t$, and $\rho_0 = 1/a^d$, where $a$ is the lattice spacing. In this Lagrangian, we have explicitly written the coupling to the external electromagnetic field. We have coupled it to the slave bosons, but this is purely a matter of taste. We could have coupled $A$ to the fermions instead; the physics would be the same thanks to the constraint. This arbitrariness is a reflection of the fact that the slave particles are confined, as we will see later. We may take $A_0$ to include the chemical potential; by varying it, we can change the electron density and, hence, the fermion and boson densities. The gauge symmetry of this model,

\[ b(x) \rightarrow e^{i\theta(x)} b(x) \]
\[ f_\sigma(x) \rightarrow e^{i\theta(x)} f_\sigma(x) \]
\[ a_\mu(x) \rightarrow a_\mu(x) - \partial_\mu \theta(x) \] (9)

reflects the redundancy of the slave-particle description.

Notice that there is no kinetic term $f^\dagger_\mu f_\mu$ for the gauge field $a_\mu$. Hence, this is a theory of spin-1/2 fermions and spinless bosons interacting with a gauge field at infinite coupling. The time component of the gauge field, $a_0$, simply enforces the no-double occupancy constraint, thereby reducing the redundancy of the slave-boson representation. The spatial components, $a_\mu$, simply restore the broken gauge symmetry (2). Since the gauge field is at infinite coupling, it is necessarily a confining gauge field: all physical states are gauge singlets and the symmetry (2) is restored. In particular, the slave bosons, $b$, and the fermions, $f_\sigma$, are not part of the physical spectrum. Nevertheless, they have been treated as quasiparticles weakly-coupled to a gauge field (2). A safer way of proceeding is by solving the constraints which follow from integrating out the gauge field $a_\mu$. In this paper, we show how this can be done.

**Slave Particle Confinement in 1D.** The basic strategy can be demonstrated in the 1 + 1-dimensional case. For simplicity, we consider the case of 1/2-filling, at which there are no slave bosons. Using bosonization, we rewrite the fermions, $f_\sigma$, in terms of bosonic fields $\phi_f$ and $\phi_s$:
\[
J_{R,L,\sigma} = e^{i\sigma} \phi_f^{R,L} + e^{ia} \phi_s^{R,L} 
\] (10)

where $f_\sigma = e^{-ikF} f_{R,\sigma} + e^{ikF} f_{L,\sigma}$. There are two bosonic fields, $\phi_f$ and $\phi_s$, because the fermions $f_\sigma$ carry two quantum numbers, fermion number and charge. The gauge field, $a_\mu$, couples to the fermion number. The fermions are neutral, so they are not coupled to the electromagnetic gauge field. The Lagrangian can now be rewritten in terms of $\phi_f = \phi_f^0 + \phi_f^L$ and the dual scalar field, $\tilde{\phi}_f$, defined by
\[
\partial_\mu \tilde{\phi}_f = \epsilon_{\mu \nu} (\partial_\nu \phi_f - a_\nu) 
\] (11)

In these variables, it is:
\[
L = \frac{1}{8\pi} (\partial_\mu \phi_f)^2 + \frac{1}{8\pi} (\partial_\mu \tilde{\phi}_f)^2 + a_\mu \partial_\mu \tilde{\phi}_f 
\] (12)

Notice that the constraint is linear in the dual bosonized representation. Hence, we can simply solve it,
\[
\partial_\mu \tilde{\phi}_f = 0 
\] (13)

thereby passing to a Lagrangian which only contains physical, gauge-neutral variables:
\[
L = \frac{1}{8\pi} (\partial_\mu \phi_f)^2 
\] (14)

The solitons of this Lagrangian are spinons, neutral spin-1/2 excitations created by $e^{ia} \phi_f^{R,L}$. They are clearly not the same as the $f_\sigma$’s, despite having the same spin and charge quantum numbers, as may be seen by comparison with (1). The former are physical, gauge-invariant excitations, while the latter are not part of the physical spectrum. A similar result was found by Mudry and Fradkin [12] for the $1 + 1 - D$ $t - J$ model.

The key step in this analysis was the use of a bosonized representation. Since the constraint is linear in this representation, it can be solved, thereby leaving only the physical degrees of freedom in the Lagrangian. The purpose of this paper is to do this in 2 + 1-dimensions, a task to which we turn in the next section.

**Slave Particles in 2D.** In two-dimensions, we can use boson-vortex duality [14-18] to represent the slave bosons, $b$. In order to represent the auxiliary fermions $f_\sigma$, we use the construction of [13]. We represent the auxiliary $f_\sigma$ as auxiliary bosons interacting with a $U(1)$ Chern-Simons gauge field which attaches flux to $S_z$. A dual representation is then used for these auxiliary bosons. We omit the details of this construction for $f_\sigma$; they are given in [13], where this construction is used for electrons themselves. The resulting dual Lagrangian is
\[
L_{\text{dual}} = \sum_{\alpha} L_{GL}(\Phi^{\alpha}, \frac{1}{2}(a^{\alpha}_\mu \pm a^{\alpha}_\mu)) + L_{GL}(\Phi^c, a^{\alpha}_\mu) 
\]
where

\[ J_\mu^{\rho,\sigma} = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^\sigma, \quad J_\mu^\rho = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^\rho \]

are, respectively, the auxiliary fermion number current, the \( S_2 \) current, and the slave boson number current. The last of these is equal to the charge current. The up- and down-spin fermion currents are given by

\[ J_\mu^{\uparrow} = (J_\mu^\rho \pm J_\mu^\sigma) / 2. \]

\( \Phi^\dagger, \Phi^\dagger, \Phi^\dagger, \Phi^\dagger \) and \( \Phi^\dagger, \Phi^\dagger, \Phi^\dagger, \Phi^\dagger \) annihilate vortices in these currents. \( \mathcal{L}_{GL} \) is given by:

\[ \mathcal{L}_{GL}(\Phi^\sigma, a_\mu^\sigma) = \frac{1}{2}(i\partial_\alpha - a_\mu^\alpha)\Phi^\sigma|2 + V(\Phi^\sigma) + \frac{1}{2}(f_\mu^{\rho\sigma})^2. \]

and a similar expression for \( \mathcal{L}_{GL}(\Phi^\alpha, a_\mu^\alpha + a_\mu^\sigma)/2 \). The “potential” can be expanded as \( V(\Phi) = r|\Phi|^2 + u|\Phi|^4 + \ldots \)

The constraint now reads:

\[ \epsilon_{\mu\nu\lambda} \partial_\nu (a_\mu^\alpha + a_\mu^\sigma) - \rho_\alpha \delta_\mu \alpha = 0 \]

The solution of the constraint is

\[ a_\mu^\sigma \equiv -a_\mu^\alpha + \rho_\alpha \epsilon_{\lambda j} x_j \]

where the \( \equiv \) sign means equal up to a gauge transformation. The physics of this equation is simple: the auxiliary fermion number and slave boson number currents are not independent; they are equal and opposite.

Note that this constraint does not commute with the Hamiltonian (in the terminology introduced by Dirac, it is a second-class constraint). Hence, we must also impose the condition which follows from the commutator of the Hamiltonian with the constraint. Equivalently, the constraint removes all of \( a_\mu^\alpha \) except for a pure gauge degree of freedom. Before choosing a gauge and eliminating this degree of freedom, we must impose the equation which follows from its variation. For illustrative purposes, we will solve this equation under the assumption that the \( Z_2 \) symmetry

\[ \Phi_{\uparrow,\downarrow} \rightarrow -\Phi_{\uparrow,\downarrow}, \]

is unbroken. In the \( Z_2 \)-broken case, a similar but slightly different solution is available. We introduce the fields \( \Phi_\rho \) and \( \Phi_\sigma \), following [13]:

\[ \Phi_\rho = \Phi_{\uparrow}\Phi_{\downarrow}; \quad \Phi_\sigma = \Phi_{\uparrow}^\dagger \Phi_{\downarrow} \]

which are the appropriate degrees of freedom when the \( Z_2 \) symmetry is unbroken. The effective Lagrangian for \( Z_2 \)-symmetric phases is:

\[ \mathcal{L}_{eff} = \mathcal{L}_\rho + \mathcal{L}_\sigma + \mathcal{L}_{GL}(\Phi^\dagger, a_\mu^\rho) + \mathcal{L}_{int} + \mathcal{L}_{con}, \]

with an auxiliary fermion number sector, \( \mathcal{L}_\rho = \mathcal{L}_{GL}(\Phi^\rho, a_\mu^\rho) \) and a spin sector,

\[ \mathcal{L}_\sigma = \mathcal{L}_{GL}(\Phi^\sigma, a_\mu^\sigma) + i \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu^\rho \partial_\nu a_\lambda^\sigma \]

\( \mathcal{L}_{int} \) contains (subleading) interactions between the charge and spin sectors. \( \mathcal{L}_{con} \) is the constraint. The equation which follows from the commutator of the constraint with the Hamiltonian now takes the form:

\[ \text{Im} (\Phi^{\dagger}(\partial_\mu - ia_\mu^\rho)\Phi^\sigma) + \text{Im} (\Phi^{\dagger}(\partial_\mu - ia_\mu^\sigma)\Phi^\rho) = 0 \]

This may be solved by taking \( \Phi_\rho = \Phi^{\dagger} \). The effective Lagrangian which results from solving the constraints is:

\[ \mathcal{L}_{eff} = \frac{1}{2} |(\partial_\mu - ia_\mu^\rho)\Phi_\rho|^2 + r_\rho |\Phi_\rho|^2 + u_\rho |\Phi_\rho|^4 \]

\[ + \frac{1}{2} (f_\mu^{\rho\rho})^2 + \frac{1}{2\pi} A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda^\rho \]

\[ + \frac{1}{2} (f_\mu^{\rho\sigma})^2 + 2 \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} a_\mu^\rho \partial_\nu a_\lambda^\sigma + \mathcal{L}_{int} \]

This is the main result of this paper: after solving the constraint, we are left with an effective action of the generic form introduced in [13]. A similar conclusion has recently been reached by D.-H. Lee [19]. This Lagrangian contains no fields which transform under the gauge symmetry \( \{ \} \), so the slave particles are not part of the theory. As discussed in [13], the possibility of spin-charge separation depends on the existence of fractional quantum number solitons in this Lagrangian (see also [20]).

**Discussion.** We have seen how spin-charge separation does not result from the deconfinement of slave particles. Spin-charge separation, if it occurs, is due to the existence of solitons with fractional quantum numbers. Slave particles are always confined by a \( U(1) \) gauge field which is at infinite coupling since its purpose is to impose a constraint reducing the Hilbert space to the physical one. Attempts to treat the gauge field perturbatively fail to impose the constraint and therefore lead, incorrectly, to the conclusion that the slave particles can be deconfined. The fallacy can be seen by considering:

\[ S = \int d^4 x \left( \overline{\psi} \gamma^\mu (i\partial_\mu - a_\mu^\rho) \psi_\rho + \overline{\chi} \gamma^\mu (i\partial_\mu - a_\mu^\sigma) \chi \right) \]

\( \psi_\rho \) carries spin but no charge, while \( \chi \) carries charge but no spin. \( A_\mu \) is the electromagnetic field while \( a_\mu^\rho \) imposes the constraint \( J_\mu = \overline{\psi}\gamma_\mu \psi_\rho + \overline{\chi}\gamma_\mu \chi = 0 \). This constraint holds at every point in space, at any scale at which we choose to probe the system. On the other hand, one might imagine that one can use an RG transformation to produce a low-energy effective field theory of the form:

\[ S = \int d^4 x \left( \overline{\psi} \gamma^\mu (i\partial_\mu - a_\mu^\rho) \psi_\rho \right) \]
\begin{equation}
\chi \gamma^\mu (i \partial_\mu - a_\mu - A_\mu) + \frac{1}{2e^2} f^2 \rho^\mu
\end{equation}

since integrating out the fermions would appear to generate such a term. One can imagine that such a theory will have a Coulomb phase, in which the fermions are weakly-coupled. (Note, however, that it is believed that, even for this model with finite coupling, the Coulomb phase can occur only for \( D \geq 3 \).) However, this line of reasoning is incorrect. An RG transformation should integrate out the short-distance fluctuations of both the fermions and the gauge fields. Since the fermions and gauge fields are infinitely strongly-coupled at short-distances (i.e. in the bare action), this procedure cannot be done perturbatively. The strong-coupling expansion should be used instead, and it leads to the conclusion that the fermions are confined. It is permissible, at least formally, to integrate out the \( \chi \) field alone, to derive an effective action for the gauge fields and \( \psi_\alpha \):

\begin{equation}
S = \int d^dx \left( \bar{\psi}_\alpha \gamma^\mu \left( i \partial_\mu - a_\mu \right) \psi_\alpha + \frac{1}{2e^2} \mathcal{L}_M(a + A) \right)
= \int d^dx \left( \bar{\psi}_\sigma \gamma^\mu \left( i \partial_\mu - a'_\mu + A_\mu \right) \psi_\sigma + \frac{1}{2e^2} \mathcal{L}_M(a') \right)
\end{equation}

where \( \mathcal{L}_M \) is the Maxwell Lagrangian and \( a'_\mu = a_\mu + A_\mu \). We are left with a Lagrangian with a matter field which carries both spin and charge, regardless of whether or not \( a'_\mu \) is confining.

In principle, there is another strongly-coupled phase which is possible – the Higgs phase – in which, again, there are no massless gauge bosons and no free slave particles. In fact \([22] \), the Higgs and confining phases are not distinct if the Higgs field has gauge charge 1. A condensate of slave bosons would be such a phase; it is a Fermi liquid phase with spin-charge confinement. If the Higgs field has higher charge – such as a composite Higgs formed by a pair of auxiliary fermions – then there is a distinct Higgs phase, but it is still true that there are no massless gauge bosons and no free slave particles. According to Wen \([24] \), the short-ranged RVB state \([25,20] \) belongs to such a phase; it is superficially similar to the phase obtained by condensing \( \Phi_\rho \) and \( \Phi_\sigma \) in \([26] \). However, as reflected in the phase diagram of Fradkin and Shenker \([22] \), the strong-coupling expansion is exact in a model with infinite gauge coupling; it implies that such a phase will not occur.

Finite-temperature could give us a window into physics within the confinement length; in a gauge theory with matter fields, this would occur via a crossover \([27] \). However, the confinement scale is the lattice scale, so this would not occur within the physically relevant temperature regime.

We circumvented the difficulties associated with a strongly-coupled gauge field by using a dual representation of the slave particle currents; in this dual representation, the constraint is linear and, hence, soluble. The approach used here might prove fruitful in the analysis of other models with a slave-particle formulation.

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