COHOMOLOGY WITH TWISTED ONE-DIMENSIONAL
COEFFICIENTS FOR CONGRUENCE SUBGROUPS OF SL(4, Z)
AND GALOIS REPRESENTATIONS

AVNER ASH, PAUL E. GUNNELLS, AND MARK MCCONNELL

Abstract. We extend the computations in [AGM02, AGM08, AGM10] to
find the cohomology in degree five of a congruence subgroup \( \Gamma \) of SL(4, Z)
with coefficients in a field \( K \), twisted by a nebentype character \( \eta \), along with
the action of the Hecke algebra. This is the top cuspidal degree. In practice
we take \( K = F \), a finite field of large characteristic, as a proxy for \( \mathbb{C} \). For
each Hecke eigenclass found, we produce a Galois representation that appears
to be attached to it. Our computations show that in every case this Galois
representation is the only one that could be attached to it. The existence of
the attached Galois representations agrees with a theorem of Scholze [Sch15]
and sheds light on the Borel-Serre boundary for \( \Gamma \).

The computations require serious modifications to our previous algorithms
to accommodate the twisted coefficients. Nontrivial coefficients add a layer
of complication to our data structures, and new possibilities arise that must
be taken into account in the Galois Finder, the code that finds the Galois
representations. We have improved the Galois Finder so that it reports when
the attached Galois representation is uniquely determined by our data.

1. Introduction

1.1. This paper is the next step in our series of papers [AGM02, AGM08, AGM10]
devoted to the computation of the cohomology of congruence subgroups \( \Gamma \subseteq
SL(4, \mathbb{Z}) \) with complex vector spaces as coefficient modules, together with the
action of the Hecke operators on the cohomology. In this paper the coefficient modules
are twists of \( \mathbb{C} \) by a nebentype character \( \eta \). The earlier papers only considered
constant coefficients, \( \eta = 1 \).

We say that a representation \( \rho \) of the absolute Galois group of \( \mathbb{Q} \) is attached
to a Hecke eigenclass \( z \) if for almost all primes \( \ell \), the characteristic polynomial
of \( \rho(\text{Frob}_\ell) \) is equal to the Hecke polynomial at \( \ell \). If we only verify the equality
computationally for a finite set of \( \ell \), we say that \( \rho \) “appears” to be attached to
\( z \). In this paper, besides computing the Hecke operators for small \( \ell \), we find, for
each Hecke eigenclass, a Galois representation that appears to be attached to it.

1991 Mathematics Subject Classification. Primary 11F75; Secondary 11F67, 20J06, 20E42.
Key words and phrases. Cohomology of arithmetic groups, Galois representations, Voronoi
complex, Steinberg module, modular symbols.

AA wishes to thank the National Science Foundation for support of this research through
NSF grant DMS-0455240, and also the NSA through grants H98230-09-1-0050 and H98230-13-
1-0261. This manuscript is submitted for publication with the understanding that the United
States government is authorized to produce and distribute reprints. PG wishes to thank the
National Science Foundation for support of this research through the NSF grants DMS-0801214
and DMS-1101640.
These Galois representations are uniquely determined, in a sense to be explained in Section 1.2.

At the moment, computations of the cohomology of a congruence subgroup of $\text{SL}(n, \mathbb{Z})$ as a Hecke module are only feasible when $n$ either equals the virtual cohomological dimension (vcd) of $\text{SL}(n, \mathbb{Z})$ or is one less. In the latter case we can use Gunnells’ algorithm for computing Hecke operators [Gun00]. When $n = 4$, the vcd is 6. We compute in degree 5 rather than degree 6 because $H^5(\Gamma, \mathbb{C})$ supports cuspidal cohomology.

The next paper in this series is planned to deal with higher-dimensional twisted coefficients. This may allow us to test the generalizations of Serre’s conjecture [ADP02], [Her09] in cases of interest. The work in this paper is an important step towards implementing this longer range goal. In addition, the current paper already reveals some new phenomena that go beyond our previous investigations.

In order to avoid the inaccuracy of floating point numbers in our huge linear algebra computations, we use a finite field $F = \mathbb{F}_{p^r}$ as a proxy for $\mathbb{C}$. If $p > 5$ and there is no $p$-torsion in the $\mathbb{Z}$-cohomology, then the $\mathbb{C}$- and mod $p$-betti numbers coincide. We generally use primes that have four or five decimal digits. Both $p$ and the degree $r$ are chosen to meet certain criteria. We choose $p$ so that the exponent of $\left(\mathbb{Z}/N\right)^\times$ divides $p - 1$. This makes the group of characters $\left(\mathbb{Z}/N\right)^\times \to \mathbb{F}_p^\times$ isomorphic to the group of characters $\left(\mathbb{Z}/N\right)^\times \to \mathbb{C}^\times$. We choose $r$ to ensure that the various Hecke eigenvalues are $\mathbb{F}$-rational (see Section 4). Then for any $\mathbb{F}^\times$-valued character $\eta$ of $\left(\mathbb{Z}/N\right)^\times$, we define $\mathbb{F}_\eta$ to be the one-dimensional vector space $\mathbb{F}$ regarded a $\Gamma_0(N)$-module with action via the nebentype character $\eta$ (Section 2.5).

Let $\Gamma$ be a subgroup of finite index in $\text{SL}_4(\mathbb{Z})$. To compute cohomology, we use the sharply complex $\text{Sh}_\bullet$, defined in Section 2. There is an isomorphism of Hecke modules

$$H^5(\Gamma, M) \approx H_1(\Gamma, \text{Sh}_\bullet \otimes_\mathbb{Z} M),$$

where $M$ is any module on which the orders of the finite subgroups in $\text{SL}_4(\mathbb{Z})$ are invertible; this condition is satisfied for us since we take $M = \mathbb{F}_\eta$ to have characteristic $> 5$. In essence, the method we use to compute $H_1(\Gamma, \text{Sh}_\bullet \otimes_\mathbb{Z} \mathbb{F}_\eta)$ as a Hecke module is the same as in our papers cited above. However, we rewrote the data structures completely to accommodate twisted coefficients. The algorithm for the Hecke operators was also modified. Since we want in our next papers to compute with twisted coefficients that are $\mathbb{F}_{p^r}$-vector spaces of dimension greater than one, we made the modifications to accommodate such general coefficient modules. The modifications proved to be somewhat tricky. We explain them in Sections 2 and 3.

1.2. Cohomology with twisted coefficients is interesting because it gives new examples of Scholze’s theorem (recalled in 1.4) and because we can use it to test the Serre-type conjectures mentioned above in new cases. Our data also raise new questions about the geometry and number theory of the boundary $B_\Gamma$ of the Borel-Serre compactification $\overline{\mathcal{X}}/\Gamma$ of the locally symmetric space for $\Gamma$. Namely, which classes in $H^*(\overline{\mathcal{X}}/\Gamma, V)$ restrict to nonzero classes in $H^*(B_\Gamma, V)$, and why?

To interpret the cohomology properly, we search for Galois representations attached to Hecke eigenclasses, using a computer program, the Galois Finder, described in Section 4. This program is a modification of the one we used in [AGM15]. We look for reducible Galois representations of degree 4. For constituents, we search through all the 1-dimensional representations corresponding to Dirichlet characters. We also search through all the 2-dimensional representations coming from classical...
holomorphic modular forms of weights 2, 3, and 4, as explained in Section 5. Because we look at nontrivial nebentype characters, classical modular forms of weight 3 can occur. (With constant coefficients, only modular forms of weight 2 and 4 made an appearance.) We also consider symmetric squares of these Galois representations. Besides these, for the largest level we consider in this paper, \( N = 41 \), we needed to employ, as a 3-dimensional constituent, a Galois representation which is attached to a cuspidal automorphic representation with fixed vector under the congruence subgroup \( \Gamma_1(41) \) of \( \text{SL}(3, \mathbb{Z}) \) and which is not a symmetric square lift from \( \text{GL}_2 \). (In \cite{AGM08, AGM10} we computed for prime levels up to \( N = 211 \) and trivial nebentype characters; there we found other examples of 3-dimensional constituents that are not symmetric squares, and examples of irreducible 4-dimensional representations attached to Siegel modular forms.)

We always find a unique Galois representation, in the following sense. For a given Hecke eigenclass \( z \), we can compute the Hecke operators only for a few primes \( \ell \). (Throughout the paper, \( \ell \nmid pN \).) We always find exactly one Galois representation of the type we search for that is apparently attached to \( z \). For a more explicit description of this uniqueness and the procedure used to check it, see Section 4.2.

Our computations are complete for all composite \( N \leq 28 \) and all prime \( N \leq 41 \).

All the Galois representations found for levels \( N < 41 \), whether with trivial or non-trivial \( \eta \), have as constituents only Dirichlet characters or representations attached to classical modular forms or their symmetric squares.

It remains unclear why certain combinations of characters and cusp forms appear in our data and others do not. We know of no sufficiently explicit computation of the cohomology of the Borel-Serre boundary for congruence subgroups \( \text{SL}(4, \mathbb{Z}) \), and of the Eisenstein lifting problem for it, that would explain our findings.

The existence of apparently attached Galois representations helps to corroborate the correctness of our computations. It is very unlikely that apparently attached Galois representations could be found if the computed Hecke eigenvalues were random collections of numbers computed erroneously.

If we could compute enough Hecke operators, then, using Scholze’s theorem and the method of Faltings-Serre, we could prove that the apparently attached Galois representations we find are truly attached. But the computational cost of finding the Hecke eigenvalues at primes \( \ell \) greater than 17 or so is too great, as will be discussed in the next section.

1.3. The size of the level \( N \) and primes \( \ell \) in the Hecke operators \( T(\ell, k) \) which we compute is limited by computer speed and memory size. \( N \) is limited by the size of the memory and by the speed, because the numbers of rows and columns of the matrices which compute the sharbly homology grow like \( N^3 \). The speed limits the Hecke operators, because the number of single cosets in \( T(\ell, k) \) grows like \( \ell^3 \) for \( k = 1, 3 \) and like \( \ell^4 \) for \( k = 2 \). One new idea in this paper is that, for large \( \ell \), we compute \( T(\ell, 1) \) but not \( T(\ell, 2) \) or \( T(\ell, 3) \). This lets us avoid the \( O(\ell^4) \) part of the computation, while still letting us eliminate some spurious Galois representations. See Section 4.1.

1.4. The work of Peter Scholze \cite{Sch15} proves the existence of attached Galois representations for Hecke eigenclasses in the cohomology of congruence subgroups of \( \text{GL}(n, \mathbb{Z}) \). This result is conditional on stabilization of the twisted trace formula.
Scholze’s results are easily extended using standard spectral sequences to coefficient modules which are finite-dimensional $\mathbb{F}_p$-vector spaces on which the Hecke semigroup of matrices acts via reduction modulo some integer $M$. See [HLTT13] for earlier results of the same kind for characteristic 0 coefficients.

These results of Scholze allow us to view our cohomology computations as opening a view onto the world of Galois representations. Each of the Hecke eigenclasses we find has an attached Galois representation, and our computations allow us to investigate exactly which Galois representations occur, and for which levels and nebentypes. In this way we also obtain evidence for the Serre-type conjectures mentioned above.

1.5. Here is a guide to the paper. In Section 2 we recall the definitions of the Steinberg module, the sharbly complex, and the concept of attached Galois representation. In Section 3 we briefly describe how the sharbly homology is calculated as a Hecke module, with reference to our earlier papers for details, and with the modifications needed to deal with $\mathbb{F}_p$-coefficients. In Section 4 we describe our Galois Finder and how it was modified from [AGM15]. Section 5 offers an interpretation of our results, including heuristics. Section 6 contains the tables of our results.

1.6. Acknowledgments. We thank Darrin Doud, who verified the existence of the Hecke eigenclass for $\text{SL}_3$ at level $N = 41$ that we describe in Section 4.3. We thank David Rohrlich for helpful correspondence.

2. The Sharbly complex, Hecke operators, and Galois representations

2.1. Let $n \geq 2$. Let $\mathbb{Q}^n$ denote the space of $n$-dimensional column vectors.

**Definition 2.2.** The Sharbly complex $Sh_\bullet$ is the complex of left $\mathbb{Z}\Gamma(n, \mathbb{Q})$-modules defined as follows. As an abelian group, $Sh_k$ is generated by symbols $[v_1, \ldots, v_{n+k}]$, where the $v_i$ are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations:

(i) $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^n[v_1, \ldots, v_{n+k}]$ for all permutations $\sigma$;

(ii) $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of $\mathbb{Q}^n$; and

(iii) $[v_1, \ldots, v_{n+k}] - [av_1, v_2, \ldots, v_{n+k}]$ for all $a \in \mathbb{Q}^\times$.

The element $g \in \text{GL}(n, \mathbb{Q})$ acts on $Sh_\bullet$ by $g[v_1, \ldots, v_{n+k}] = [gv_1, \ldots, gv_{n+k}]$. The boundary map $\partial_k : Sh_k \to Sh_{k-1}$ is

$$\partial_k([v_1, \ldots, v_{n+k}]) = \sum_{i=1}^{n+k} (-1)^i [v_1, \ldots, \widehat{v_i}, \ldots, v_{n+k}],$$

where as usual $\widehat{v_i}$ means to delete $v_i$.

All these objects depend on $n$, which we suppress from the notation, since we will later work only with $n = 4$.

The sharbly complex

$$\cdots \to Sh_i \to Sh_{i-1} \to \cdots \to Sh_1 \to Sh_0$$

is an exact sequence of $\text{GL}(n, \mathbb{Q})$-modules. We may define the Steinberg module $St$ as the cokernel of $\partial_1 : Sh_1 \to Sh_0$ (cf. [AGMT12 Theorem 5]).

Let $\Gamma$ be a congruence subgroup of $\text{SL}(n, \mathbb{Z})$. 

Definition 2.3. Let $M$ be a left $\Gamma$-module. The sharbly homology of $\Gamma$ with coefficients in $M$ is $H_\ell(\Gamma, Sh_\bullet \otimes \mathbb{Z} M)$, where $\Gamma$ acts diagonally on the tensor product.

If $(\Gamma, S)$ is a Hecke pair in $GL(n, \mathbb{Q})$ and $M$ is a left $S$-module, the Hecke algebra $H(\Gamma, S)$ acts on the sharbly homology, since $S$ acts (diagonally) on $Sh_\bullet \otimes \mathbb{Z} M$ and because the sharbly homology is the homology of the complex $H_\ell(\Gamma, Sh_\bullet \otimes \mathbb{Z} M)$.

The following theorem is proved in [AGM11].

Theorem 2.4. For any $\Gamma \subset GL(n, \mathbb{Z})$ and any coefficient module $M$ in which all the torsion primes of $\Gamma$ are invertible, there is a natural isomorphism of Hecke modules

$$H_\ell(\Gamma, Sh_\bullet \otimes \mathbb{Z} M) \to H^{(n)}(\ell, i)(\Gamma, M)$$

for all $i$.

2.5. We now define the $\Gamma$ and $\Gamma$-modules used in this paper.

Definition 2.6. Let $\Gamma_0(N)$ be the subgroup of matrices in $SL(n, \mathbb{Z})$ whose bottom row is congruent to $(0, \ldots, 0, * )$ modulo $N$.

Let $\mathbb{F} = \mathbb{F}_p^r$ be a finite field of characteristic $p$. Let $\eta : (\mathbb{Z}/N)^{\times} \to \mathbb{F}^{\times}$ be a character, which we will call the nebentype (even if it is trivial, although in that case we will sometimes speak of the “trivial character”.) In practice, $p$ will be a prime of four or five decimal digits. We will always choose $p$ so that the exponent of $(\mathbb{Z}/N)^{\times}$ divides $p - 1$. Hence $\eta$ takes values in $\mathbb{F}_p^{\times}$.

Define $S_{pN}$ to be the subsemigroup of integral matrices in $GL(n, \mathbb{Q})$ satisfying the same congruence condition as $\Gamma_0(N)$ and having positive determinant relatively prime to $pN$. Let $H(pN)$, the anemic Hecke algebra, be the $\mathbb{Z}$-algebra of double cosets $\Gamma_0(N)S_{pN}\Gamma_0(N)$. Then $H(pN)$ is a commutative algebra that acts on the cohomology and homology of $\Gamma_0(N)$ with coefficients in any $S_{pN}$-module. In particular, $H(pN)$ contains all double cosets of the form $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$, where $\ell$ is a prime not dividing $pN$, $0 \leq k \leq n$, and $D(\ell, k)$ is the diagonal matrix with the first $n - k$ diagonal entries equal to 1 and the last $k$ diagonal entries equal to $\ell$. These double cosets generate $H(pN)$ (cf. [Shi94, Thm. 3.20]). When we consider the double coset generated by $D(\ell, k)$ as a Hecke operator, we call it $T(\ell, k)$.

Write $\mathbb{F}_q$ for the $S_{pN}$-module where a matrix $s \in S_{pN}$ acts on $\mathbb{F}$ via $\eta(s_{nn})$, where $s_{nn}$ is the * in the bottom row congruent to $(0, \ldots, 0, * )$ mod $N$.

Definition 2.7. Let $V$ be an $\mathbb{F}[H(pN)]$-module. Suppose that $v \in V$ is a simultaneous eigenvector for all $T(\ell, k)$ and that $T(\ell, k)v = a(\ell, k)v$ with $a(\ell, k) \in \mathbb{F}$ for all prime $\ell \nmid pN$ and $0 \leq k \leq n$. If

$$\rho : G_Q \to GL(n, \mathbb{F})$$

is a continuous representation of $G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside $pN$, and if

$$\sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k = \det(I - \rho(\text{Frob}_v)X)$$

for all $\ell \nmid pN$, then we say that $\rho$ is attached to $v$.

Here, Frob refers to an arithmetic Frobenius element, so that if $\varepsilon$ is the cyclotomic character, we have $\varepsilon(\text{Frob}_\ell) = \ell$.

The polynomial in (1) is called the Hecke polynomial for $v$ and $\ell$.

As explained in the introduction, we have the following special case of a theorem of Scholze (conditional on stabilization of the twisted trace formula):

TWISTED 1-DIM COHOMOLOGY FOR SL(4) AND GALOIS REPRESENTATIONS 5
are confident \( \rho \) holds for a finite range of \( \ell \) which we have computed, a range large enough that we are confident \( \rho \) really is attached to \( v \).

3. Computing homology and the Hecke action

Following Theorem 2.4, we compute the Hecke operators acting on sharbly cycles that are supported on Voronoi sharblyes. Theorem 13 of [AGM12] guarantees that the packets of Hecke eigenvalues we compute do occur on eigenclasses in \( H_1(\Gamma_0(N), Sh_\bullet \otimes \mathbb{F}_p) \approx H^5(\Gamma, \mathbb{F}_p) \). In this section, we define the Voronoi sharblyes, recall results from [AGM11, AGM12], and explain how the results are modified to work with \( \mathbb{F}_p \)-coefficients.

3.1. The sharbly complex is not finitely generated as a \( \mathbb{Z} \mathbb{SL}(n, \mathbb{Z}) \)-module, which makes it difficult to use in practice to compute homology. To get a finite complex to compute \( H_1 \), we use the Voronoi complex. We refer to [AGM12 Section 5] for any unexplained notation in what follows.

Let \( X_0^0 \subset \mathbb{R}^{n(n+1)/2} \) be the convex cone of positive-definite real quadratic forms in \( n \) variables. This has a partial (Satake) compactification \( (X_0^0)^* \) obtained by adjoining rational boundary components, and the compactification is itself a convex cone. The space \( (X_0^0)^* \) can be partitioned into cones \( \sigma = \sigma(x_1, \ldots, x_m) \), called Voronoi cones, where the \( x_i \) are contained in certain subsets of nonzero vectors from \( \mathbb{Z}^n \). We write elements of \( \mathbb{Z}^n \) as column vectors, as we did in Section 2 for \( \mathbb{Q}^n \). The cones are built as follows. Each nonzero \( x_i \in \mathbb{Z}^n \) determines a rank-one quadratic form \( q(x_i) = x_i x_i^t \in (X_0^0)^* \). Let \( \Pi \) be the closed convex hull of the points \( \{q(x) \mid x \in \mathbb{Z}^n, x \neq 0\} \). Then each of the proper faces of \( \Pi \) is a polytope, and the \( \sigma \) are the cones on these polytopes. The indexing sets are constructed in the obvious way: if \( \sigma \) is the cone on \( F \subset \Pi \), and \( F \) has distinct vertices \( q(x_1), \ldots, q(x_m) \), then the indexing set is \( \{\pm x_1, \ldots, \pm x_m\} \). We let \( \Sigma \) denote the set of all Voronoi cones.

Let \( X_n^* \) be the quotient of \( (X_0^0)^* \) by homotheties. The images of the Voronoi cones are cells in \( X_n^* \). Let \( \mathbb{Z} V_\bullet \) be the oriented chain complex on these cells, graded by dimension. Let \( \mathbb{Z} \partial V_\bullet \) be the subcomplex generated by those cells that do not meet the interior of \( X_n^* \) (i.e., do not meet the image in \( X_n^* \) of the positive-definite cone). The Voronoi complex is then defined to be \( V_\bullet = \mathbb{Z} V_\bullet / \mathbb{Z} \partial V_\bullet \). For our purposes, it is convenient to reindex \( V_\bullet \) by introducing the complex \( \mathcal{W}_\bullet \), where \( \mathcal{W}_k = V_{n+k-1} \). The results of [AGM11, AGM12] show that, if \( n \leq 4 \), both \( \mathcal{W}_\bullet \) and \( Sh_\bullet \) give resolutions of the Steinberg module. In particular, let \( \Gamma = \Gamma_0(N) \). If \( M \) is a \( \mathbb{Z}[\Gamma] \)-module such that the order of all torsion elements in \( \Gamma \) is invertible, then \( H_4(\Gamma, \mathcal{W}_\bullet \otimes \mathbb{Z} M) \approx H_4(\Gamma, Sh_\bullet \otimes \mathbb{Z} M) \), and furthermore by Borel–Serre duality these are isomorphic (after reindexing) to \( H^4(\Gamma, M) \). These two complexes can be related as follows when \( n = 4 \): every Voronoi cell in \( X_4^* \) of dimension \( \leq 5 \) is a simplex. Thus for \( 0 \leq k \leq 2 \), we can define a map of \( \mathbb{Z}[\mathbb{SL}(4, \mathbb{Z})] \)-modules

\[
\theta_k : \mathcal{W}_k \to Sh_k
\]

that takes the Voronoi cell \( \sigma(v_1, \ldots, v_{k+4}) \) to \( \theta_k((v_1, \ldots, v_{k+4})) := [v_1, \ldots, v_{k+4}] \). This allows us to realize Voronoi cycles in these degrees in the sharbly complex.

Theorem 2.8. Let \( N \geq 1 \). Let \( v \) be a Hecke eigenclass in \( H^5(\Gamma_0(N), \mathbb{F}_p) \). Then there is attached to \( v \) a continuous representation unramified outside \( pN \):

\[
\rho : G_\mathbb{Q} \to \text{GL}(n, \mathbb{F}_p).
\]

Echoing Definition 2.7 we say that \( \rho \) is apparently attached to \( v \) if condition (1) holds for a finite range of \( \ell \) which we have computed, a range large enough that we are confident \( \rho \) really is attached to \( v \).
The image of $\theta_k$ is the set of Voronoi sharblies in degree $k$. Then $H_1(\Gamma, W_\bullet \otimes_\Z M) \cong H_1(\Gamma, Sh_\bullet \otimes_\Z M)$ by \cite[Corollary 12]{AGM12}.

3.2. We now explain concretely how we compute $H_1(\Gamma, W_\bullet \otimes_\Z \F_p)$. We have a body of code in Sage \cite{Dev17} for these computations. The code supports $G$-modules $M$, that is, representations of $G$. Here $G$ is a finite group, or a matrix group like $\Gamma_0(N)$ or $\text{SL}(p,n)$. The module $M$ has finite dimension over its base ring. The base ring is $\F_p$, $\Q$, or $\Z$ in this project, though it could be more general. Morphisms of $G$-modules are supported, as are kernel, cokernel, image, direct sum, and tensor products of $G$-modules. When $H$ is a finite-index subgroup of $G$, we support $\text{Res}_H^G$, $\text{Ind}_H^G$, and $\text{Coind}_H^G$ of $G$-modules, functorially.

The program takes as input the values of $N$, $p$, and the nebentype, which is a one-dimensional representation $\eta$ of $\text{Sp}_N$ with coefficients in $\F_p$. (The extension from $\F_p$ to $\F_{p^e}$ comes later, in the Galois Finder.) The nebentype is essentially a Dirichlet character $(\Z/N)\times \rightarrow \F_p^\times$. Sage makes it automatic to enumerate the Dirichlet characters.

The complex $W_\bullet$ has only finitely many classes of Voronoi cells modulo $\text{SL}(n, \Z)$ \cite{Vor08}. When $n = 4$, there are 18 classes. In fact, to compute $H_1$ we only need $W_0$, $W_1$, and $W_2$, so our code truncates away the rest of $W_\bullet$ for efficiency.

For each class of cells modulo $\text{SL}(n, \Z)$, the code maintains a standard representative cell $\sigma$ as listed in \cite{McC91}. The stabilizer $G_\sigma$ of $\sigma$ in $\text{SL}(n, \Z)$ acts on $\sigma$ with orientation character $Z_\sigma$. The code stores $G_\sigma$ and $Z_\sigma$.

Fix right coset representatives $r$, $r'$, ... for $\Gamma_0(N)\backslash \text{SL}(n, \Z)$ once and for all. Since $\Gamma_0(N)$ has finite index in $\text{SL}(n, \Z)$, the complex $W_\bullet$ has only finitely many classes of cells modulo $\Gamma_0(N)$. For each class modulo $\Gamma_0(N)$, we may choose a representative cell $\sigma_1 = r\sigma$, where $\sigma$ is one of the representative cells modulo $\text{SL}(n, \Z)$, and $r$ is one of the standard coset representatives. An awkward fact is that, for two different coset representatives $r$, $r'$, the cells $r\sigma$ and $r'\sigma$ may be in the same $\Gamma_0(N)$-orbit. This occurs when $r^{-1}r'$ is in the stabilizer $G_\sigma \subset \text{SL}(n, \Z)$. For computation we must choose $r$ or $r'$, not both; say we choose $r$. A class $\text{CellOrbitStructure}$ in our code handles these details. $\sigma_1$ itself may have a non-trivial stabilizer $G_{\sigma_1} \subset \Gamma_0(N)$; the $\text{CellOrbitStructure}$ takes care of these stabilizers $G_{\sigma_1}$ and how their orientation characters $Z_{\sigma_1}$ interact with the orientation characters $Z_\sigma$ of $G_\sigma$.

Equation (2) below presents a problem we need to solve repeatedly during the homology calculation. Suppose we are given a cell $\tau \in W_\bullet$, with $\tau = g\sigma$ for $\sigma$ a standard cell and for $g \in \text{SL}(n, \Z)$. Then $g = \gamma_1 r'$ for some coset representative $r'$ and $\gamma_1 \in \Gamma_0(N)$. Since we chose $r$ instead of $r'$, we have $\gamma r' = rg_\sigma$ for some stabilizer element $g_\sigma \in G_\sigma$ and some $\gamma \in \Gamma_0(N)$. Thus

$$g = (\gamma_1 \gamma^{-1}) r g_\sigma.$$  

The problem is, given $g$ and $\sigma$, to solve for $\gamma_1$, $\gamma$, $r$, $g_\sigma$, and to compute the orientation characters. The $\text{CellOrbitStructure}$ has a method $\text{decompose}$ that solves (2).

Let $\sigma_1$ run through all the representatives $r\sigma$ of the classes of cells modulo $\Gamma_0(N)$. During the homology computation, we need, for each $\sigma_1$, to restrict the nebentype $\eta$ to the finite stabilizer group $G_{\sigma_1}$, and to tensor the restriction with the orientation character of $G_{\sigma_1}$. This tensor product $\eta_{\sigma_1} : G_{\sigma_1} \rightarrow \F_p^\times$ is called
the local representation for $\sigma_1$. The CellOrbitStructure keeps track of the local representations.

As we explain in [AGM11, AGM12], $H_*(\Gamma, W_{k} \otimes \mathbb{Z} \mathcal{F}_{\eta})$ is computed by a spectral sequence. The columns are indexed by $k$, and the $j$-th row is the direct sum of the homology groups $H_j(G_{\sigma_1}, \eta_{\sigma_1})$. Since the torsion in $\Gamma_0(N)$ has order prime to $p$, all the homology groups $H_j$ vanish for $j > 0$. The $E^1$ term has only one row, whose entry in the $k$-th box is the module of co-invariants

$$E^1_{k,0} = H_0(\Gamma, W_{k} \otimes \mathbb{Z} \mathcal{F}_{\eta}) = W_{k} \otimes \mathbb{Z} \mathcal{F}_{\eta}.$$ 

As $\sigma_1$ runs through representatives of the cells modulo $\Gamma_0(N)$, the co-invariant module $W_{k} \otimes \mathbb{Z} \mathcal{F}_{\eta}$ breaks up as a direct sum:

$$E^1_{k,0} = \bigoplus_{\sigma_1 \text{ of degree } k} H_0(G_{\sigma_1}, \eta_{\sigma_1}).$$

Each summand $H_0(G_{\sigma_1}, \eta_{\sigma_1})$ is the module of co-invariants for the local representation $\eta_{\sigma_1}$. It is isomorphic to $\mathbb{F}$ if $\eta_{\sigma_1}$ is a trivial representation, and is zero otherwise.

The $E^2_{k,0}$ of the spectral sequence is isomorphic to $H_k(W_{\bullet} \otimes \mathbb{Z} \mathcal{F}_{\eta})$. This is computed using the differential $\partial_k$ that is the tensor product with $\eta$ of the differential $\partial_k$ on sharblies in Section [22]. $\partial_k$ is constructed in Sage as a sparse matrix of size $\dim E^1_{k,0} \times \dim E^1_{k-1,0}$. As before, we are computing $H_1$, so we only compute $\partial_2$ and $\partial_1$.

We illustrate the sizes of these matrices with the example of $N = 41$, $p = 21881$, and trivial nebentype. Here $\partial_2$ is $24590 \times 7100$, and $\partial_1$ is $7100 \times 746$. (This is small compared to [AGM10], where, for $N = 211$ and trivial nebentype, $\partial_2$ was about four million by one million. We did not compute the Hecke operators in [AGM10].)

We write the matrices $\partial_2$ and $\partial_1$ to disk, partly as insurance in case of a computer crash during a long run. The next step is to choose a basis $\{x_i\}$ of the homology, $\ker(\partial_1)/\im(\partial_2)$. We choose the basis using Sheafhom, a package written by one of us (MM) in Common Lisp and described in [AGM10]. Sheafhom performs homology calculations by row- and column-reducing large sparse matrices while saving the change-of-basis matrices to disk. It works with base rings $\mathbb{F}_p$ as well as $\mathbb{Z}$. If $y$ is a cycle in the homology, Sheafhom can express it as a linear combination of the homology basis, $y = \sum c_i x_i$, using only a small amount of RAM.

3.3. To compute the Hecke operators, we use the basis $\{x_i\}$ we found for the homology group $H_1(W_{\bullet} \otimes \mathbb{Z} \mathcal{F}_{\eta})$. We identify the $x_i$ with elements $y_i = \theta_{1,*}(x_i) \in H_1(Sh_{\bullet} \otimes \mathbb{Z} \mathcal{F}_{\eta})$. Let $T$ be a Hecke operator. Using the algorithm of Gunnells mentioned in the introduction, we compute each Hecke translate $Ty_i$ and then find a sharby cycle $z_i$ such that $z_i = Ty_i$ in $H_1(Sh_{\bullet} \otimes \mathbb{Z} \mathcal{F}_{\eta})$ and such that $z_i$ is in the image of the map $\theta_{1,*}$. The inverse images $\theta_{1,*}^{-1}(z_i)$ can be written as linear combinations $\sum c_i x_i$ as in the previous paragraph. This gives a matrix representing the action of $T$. From this matrix we can find eigenclasses and eigenvalues.

4. Finding Attached Galois Representations

From now on we set $n = 4$. 
Suppose we have a finite-dimensional $F$-vector space $V$ together with an action of the Hecke operators from $\mathcal{H}(pN)$. We now describe how we find Galois representations that are apparently attached to Hecke eigenvectors $v$ in $V$. Our Galois Finder program is part of our Sage code.

4.1. As in Section 4.3, we compute the action on $V = H_1(\Gamma_0(N), W_k \otimes_{\mathbb{Z}} F_\eta)$ of the Hecke operators $T(\ell, k)$ for $k = 1, 2, 3$ and for $\ell$ ranging through a set

$$L = \{\ell \mid \ell \text{ prime, } \ell \leq \ell_0, \ell \not\in \{pN\}\}.$$ 

The upper bound $\ell_0$ depends on the level $N$ and the nebentype $\eta$, because sometimes we need more $\ell$ to find a unique Galois representation. $\ell_0$ is never less than 5, and is occasionally as high as 17. For the larger $\ell$, as we have mentioned, we sometimes compute only $T(\ell, 1)$ and not $T(\ell, k)$ for $k = 2, 3$. The tables in Section 4.1 list which operators we computed. $T(\ell, 0)$ is always the identity and $T(\ell, 4)$ is $\eta(\ell)$ times the identity. To check the work, we always verify that our Hecke operators commute pairwise.

For a given level $N$, we look for reducible Galois representations apparently attached to a given packet of Hecke eigenvalues. For some $N$ there are irreducible Galois representations attached to certain packets, but our data does not extend to such large $N$.

Some constituents of the Galois representations we are looking for are 1-dimensional, coming from Dirichlet characters mod $N$ taking values in the cyclotomic field $K_\mathbb{Q} = \mathbb{Q}(\zeta_N)$, with $\zeta_N \in \mathbb{C}$ a primitive $N$-th root of unity. Others are 2-dimensional, coming from newforms of level dividing $N$ and weights 2, 3, or 4. Using Sage, we compute the newforms of level $N_1$ and character $\psi$, for all $N_1 \mid N$, all Dirichlet characters $\psi$ mod $N_1$, and all weights 2, 3, and 4 (see Section 4.3 for full details). Other constituents are 3-dimensional, coming either from symmetric squares of 2-dimensional representations or from GL(3)-homology classes which are not symmetric squares (which in our data occurs only for $N = 41$).

Consider the fields of definition $K_1, K_2, \ldots$ of the newforms we have listed, together with $K_0$. The Galois Finder will be computing, not in $F_p$, but in the residue class fields for the primes $\mathfrak{p}$ over $p$ in the different $K_i$. We define $r$ to be the smallest integer so that all these residue class fields embed in $F = \mathbb{F}_{p^r}$. The field $F$ is recorded at the top of each table as $F = GF(p^r)$.

Computation in $\mathbb{F}_{p^r}$ slows down when $r$ becomes large. We would have liked to choose $p$ so that it splits completely in all the $K_i$, meaning $r = 1$. As we have mentioned, we always choose $p$ so that it splits completely in $K_0$. But for $N$ in the 20s and higher, the fields $K_1, K_2$ become large enough that we cannot choose a four or five digit $p$ that splits completely everywhere. Instead, we choose $p$ so that $r$ will be as small as possible.

The Hecke operators $T(\ell, k)$ we compute are all semisimple. We do not know how to prove that this would always be the case.

For each $T(\ell, k)$ that we compute, we decompose $V$ into eigenspaces under that operator. In principle, the eigenvalues $\alpha(\ell, k)$ of $T(\ell, k)$ might lie in an extension field of $\mathbb{F}$, but we always observe that they lie in $\mathbb{F}$.

After decomposing $V$ into eigenspaces, we take the common refinement of the decompositions. Let $E$ have the form $\bigcap_{(\ell, k)} E_{\ell,k}$, where $E_{\ell,k}$ is any one of the eigenspaces for the operator at $(\ell, k)$, and the intersection is over all $\ell \in L$ and all $k$ we have computed. We find all the non-zero $E$ of this form. They are the
simultaneous eigenspaces. $V$ is the direct sum of the $E$. By construction, the Hecke eigenvalues $a(\ell, k)$ are constant on each $E$ and characterize it. The function $(\ell, k) \mapsto a(\ell, k)$ is the Hecke eigenpacket of $E$.

We distinguish two kinds of multiplicity for $E$. We define the Hecke multiplicity of $E$ to be $\dim_E E$. A second kind, the Galois multiplicity, is defined in Section 4.6.

To a simultaneous eigenspace $E$ we attach a family of polynomials. The polynomial system $F(E)$ is the mapping that sends $\ell \in L$ to the Hecke polynomial with eigenvalues $a(\ell, k)$ defined in (1), or to a partial Hecke polynomial which we now explain. For small $\ell$, we can compute the Hecke eigenvalues $a(\ell, k)$ for all $k = 0, \ldots, 4$, so we know the whole Hecke polynomial (1); call this a full Hecke polynomial. For larger $\ell$, computing $T(\ell, 2)$ would be too slow. In this case, we compute $T(\ell, 1)$, and we only know that the Hecke polynomial is $1 - a(\ell, 1)X + O(X^2)$, where $O(X^2)$ means some undetermined linear combination of $X^2, X^3,$ and $X^4$. We call the latter a partial Hecke polynomial. A partial Hecke polynomial is implemented in Sage as an element of the quotient ring $\mathbb{F}[X]/(X^2)$. As a whole, $F(E)$ contains one or more full polynomials, all of degree 4, and zero or more partial polynomials, whose degree is undefined. We say $\deg F(E) = 4$.

4.2. We use known Galois representations $\rho$ unramified outside $pN$, taking values in $\text{GL}(m, \mathbb{F})$ for $m = 1$ or 2. These are the Galois representations coming from Dirichlet characters and newforms as described roughly in Section 4.1, and to be described in full detail in Section 4.3. We also use the symmetric squares of the $\rho$ coming from newforms; these take values in $\text{GL}(m, \mathbb{F})$ for $m = 3$. The characteristic polynomial of Frobenius for each of these representations is known and is of degree $m$ for each $\ell \nmid pN$. In the language above, they are all full polynomials. In our code, we define the polynomial system $F(\rho)$ to be the mapping that sends $\ell \in L$ to the characteristic polynomial of Frobenius for $\rho$ at $\ell$. We say $\deg F(\rho) = m$.

Let us describe how we conjecturally attach a sum of $\rho$’s to a simultaneous eigenspace $E$. On polynomial systems, it is natural to define $F(\rho_1 \oplus \cdots \oplus \rho_t) = \prod_{i=1}^t F(\rho_i)$, a product of polynomial systems. We can also define quotients, but we must be careful about the partial Hecke polynomials, as we now explain. Let $F_1$ and $F_2$ be two polynomial systems with the same $L$. Say that $F_1$ divides $F_2$ if, for each $\ell \in L$, the polynomial at $\ell$ for $F_1$ divides the polynomial at $\ell$ for $F_2$. Implicit in this definition is that $\deg F_1 \leq \deg F_2$. When one polynomial system divides another, define the quotient system in the obvious way. The degree of the quotient system is $\deg F_2 - \deg F_1$. For some $\ell$ we will be dividing a partial Hecke polynomial by a full Hecke polynomial, but we never use a partial polynomial as a divisor. Dividing a partial Hecke polynomial $f_1(x) \pmod{X^2}$ by a full Hecke polynomial $f_2(x)$ is well defined because $f_2(x)$ always has constant term 1, hence $f_2(x)$ projects via $\mathbb{F}[X] \to \mathbb{F}[X]/(X^2)$ to a unit $\mathbb{F}[X]/(X^2)$: the inverse of $1 - aX$ in $\mathbb{F}[X]/(X^2)$ is $1 + aX$. Note that we can divide indefinitely in $\mathbb{F}[X]/(X^2)$, because $1/(1 - aX)^\nu$ exists for arbitrarily large $\nu$. The reason we keep track of the degree of a polynomial system is that, although we could divide into the partial polynomials indefinitely, we will stop dividing by $F_1$ as soon as the full polynomials of the quotient reach degree 0.

For a given $E$, we make a list $\mathcal{R}$ of all the $\rho$ for which $F(\rho)$ divides $F(E)$. Then we run through all possible finite subsets of $\mathcal{R}$, say $\{\rho_1, \ldots, \rho_t\}$, and we make a list $\mathcal{R}'$ of all the direct sums $\rho_1 \oplus \cdots \oplus \rho_t$ for which $F(\rho_1 \oplus \cdots \oplus \rho_t) = F(E)$. We always find that $\mathcal{R}'$ is non-empty. If $\mathcal{R}'$ has two or more elements, we take more primes $\ell$,
add them to $L$, compute the Hecke operators $T(\ell, k)$ (or at least $T(\ell, 1)$), and refine the eigenspaces $E$ for the new operators if necessary. We almost always find we can take enough $\ell$ to make $R'$ have exactly one element. In the “minor” exceptions to this statement, recounted in the next paragraph, it is still true that the Galois representations $\rho_1 \oplus \cdots \oplus \rho_t$ for the elements in $R'$ are isomorphic to each other. Therefore in every case we can discover the uniqueness of the Galois representation among those our Finder looks through that seems to be attached to any given Hecke eigenspace we have computed. We assert this uniqueness even though our data is rather limited, i.e., $L$ is not that large. Of course, by Chebotarev Density the truly attached Galois representation is unique, up to semisimplification.

4.3. There are some exceptions to the statement that $R'$ has exactly one element. The minor exceptions are found in the tables in Section 6.1 at

- $N = 24, \eta = \chi_{24,0}\chi_{24,1}\chi_{24,2}$, representations with $\sigma_{24,2c}$;
- $N = 28, \eta = \chi_{28,0}\chi_{28,1}$, representations with $\sigma_{28,2c}$.

Here, one Galois representation with a symmetric square in it happens to coincide with one representation without a symmetric square. We checked by computer that the Hecke polynomials match for all $\ell < 1000, \ell \nmid N$. In these two cases, the symmetric square is of a “dihedral” Galois representation, so that its symmetric square is reducible. These are in fact the same four-dimensional Galois representation.

The major exceptions occurred at level $N = 41$ and the nebentype $\eta = \chi_{41}^{10}$ whose image has order 4. Here $\dim V = 8$, splitting into eight $E$'s of dimension 1, and $R'$ was empty for two out of the eight $E$. Darrin Doud, upon our request, using computer programs he developed, found an autochthonous form for $\text{SL}_3$. Specifically, he found a three-dimensional Galois representation $\delta$ attached to a cohomology class $z$ for a congruence subgroup of $\text{SL}(3, \mathbb{Z})$ and with coefficients in $F_\eta$, which is not a lift from any lower-rank group. Of the two four-dimensional representations that we could not identify using characters and cusp forms, one proved to be $1 \oplus \varepsilon\delta$, and the other $\varepsilon^3 \oplus \delta$. This strongly suggests that these simultaneous eigenspaces are different Eisenstein lifts of $z$ from parabolic subgroups of type $(3, 1)$.

4.4. The Galois representation we find could be an impostor. There could be, for example, some irreducible four-dimensional $\rho$ that gives conjugate matrices to ours when evaluated at Frob$_\ell$ for the few $\ell$ we can compute and which is the truly attached one. However, this seems very unlikely.

For $N$ larger than 41, there will be truly attached $\rho$ that our Galois Finder has not been designed to find. This happened for trivial nebentype character in our previous papers [AGM08, AGM10], where we found lifts of forms from $\text{GSp}(4)$. This does not happen in this paper because we are not taking $N$ big enough. We would be delighted to find an apparently attached Galois representation that is irreducible and not essentially self-dual. Such a representation would not be a lift from $\text{GSp}(4)$ or from any proper reductive subgroup of $\text{SL}(4)$. However, that has not happened to date.

4.5. We now describe in detail the list of Galois representations $\rho$ which our Galois Finder was programmed to use.
We only look at Galois representations whose conductor divides \(N\), since these are the ones we expect to be constituents of Galois representations attached to our Hecke eigenclasses of level \(N\).

We begin with Dirichlet characters \(\chi\) with values in \(\mathbb{F}\), which we identify with one-dimensional Galois representations as usual. We take all the characters of conductor \(N_1\) for all \(N_1 | N\). Sage’s class \texttt{DirichletGroup} enumerates the \(\chi\) automatically. The characteristic polynomial of Frobenius at \(\ell\) for \(\chi\) is \(1 + \chi(\ell)X\), for all \(\ell \notdivides pN\). Each \(\chi\) can be lifted to characteristic zero, since \(p \equiv 1 \pmod{N}\).

Another one-dimensional character is the cyclotomic character \(\varepsilon\). We look at \(\varepsilon^w\) for \(w = 0, 1, 2, 3\), because this is predicted by the generalizations of Serre’s conjecture for mod \(p\) Galois representations [AS86, ADP02]. These \(w\) would also be the Hodge numbers of the motives conjecturally attached to our homology eigenclasses. Our standard list \(L_1\) of one-dimensional characters is \(\chi \otimes \varepsilon^w\), for all the \(\chi\) just described and for all \(w = 0, 1, 2, 3\).

After the Dirichlet characters, we put into the list the Galois representations \(\rho\) coming from newforms for certain congruence subgroups of \(\text{SL}(2, \mathbb{Z})\). We emphasize that these are classical cusp forms in characteristic zero, even though the \(\rho\) take values in characteristic \(p\). The characteristic polynomials of Frobenius for the cusp forms are naturally defined over number fields, so, as we describe which cusp forms we use, we must also describe how we reduce to get Galois representations defined over \(\mathbb{F}\).

Let \(N_1 | N\). Let \(\mathbb{Q}(\zeta_{N_1})\) be the field of \(N_1\)-th roots of unity. Let \(\psi\) be any Dirichlet character of conductor \(N_1\) taking values in \(\mathbb{C}^\times\). The Galois group \(\text{Gal}(\mathbb{Q}(\zeta_{N_1})/\mathbb{Q})\) acts on the \(\psi\)’s by acting on their values; we take only one \(\psi\) from each Galois orbit, since the others give Galois-conjugate representations. Let \(f\) be a newform of weight 2, 3, or 4 for \(\Gamma_1(N_1)\) with nebentype character \(\psi\). The coefficients of the \(q\)-expansion of \(f\) generate a number field \(K_f\), with ring of integers \(\mathcal{O}_{K_f}\). (This field was called \(K_1\) in Section 4.2.) Let \(\mathfrak{P}\) be a prime of \(K_f\) over \(p\). If \(\mathbb{F}\) is of high enough degree over \(\mathbb{F}_p\), then the finite field \(\mathcal{O}_{K_f}/\mathfrak{P}\) will have an embedding \(\alpha_{\mathfrak{P}}\) into \(\mathbb{F}\). As we have mentioned, the extension field \(\mathbb{F}\) of \(\mathbb{F}_p\) is chosen so that all these embeddings will exist. Therefore, the pair \((f, \mathfrak{P})\) gives rise to a Galois representation \(\rho\) into \(\text{GL}(2, \mathbb{F})\), by reduction mod \(\mathfrak{P}\) composed with \(\alpha_{\mathfrak{P}}\). For any \(\ell \notdivides pN\), the characteristic polynomial of Frobenius is \(1 - \alpha_{\mathfrak{P}}(a_\ell)X + X^2\), where \(a_\ell\) is the \(\ell\)-th coefficient in the \(q\)-expansion of \(f\). If we chose a different prime \(\mathfrak{P}\), we would get a Galois-conjugate representation.

We make a list \(L_2\) containing the representation \(\rho\) for \((f, \mathfrak{P})\), for all \(N_1 | N\) and all newforms \(f\) of weight 2, 3, or 4 for \(\Gamma_1(N_1)\) and all nebentypes \(\psi\). Sage’s class \texttt{CuspForms}, with its method \texttt{newforms}, makes this automatic.

We take all the \(\rho\) in \(L_2\), and tensor them in all possible ways with the one-dimensional representations from the list \(L_1\) of Dirichlet characters and cyclotomic character powers. This list of tensor products is our final list \(L_2\) of two-dimensional Galois representations.

Our list of three-dimensional Galois representations is the list of symmetric squares of \(\rho \in L_2\), tensored in all possible ways with \(L_1\).

In the results, our Galois representations have a term \(\text{Sym}^2(\sigma)\) for a cusp form \(\sigma\) for three levels, the prime levels \(N = 29, 37,\) and 41. It all three cases, it occurs only when \(\eta\) is a quadratic character. This is because the symmetric square of the
Galois representation attached to a cusp form with quadratic nebentype can have prime level.

We define the Hodge-Tate (HT) numbers for $\rho$ as follows. For an element $\chi \otimes \varepsilon^w \in \mathcal{L}_1$, there is a list of one $\varepsilon$ power, $[w]$. To a representation coming from a newform $\rho$ of weight $k$, there is a list of two $\varepsilon$ powers, $[0, k - 1]$. For $\chi \otimes \varepsilon^w \otimes \rho$, the list is $[w, w+k-1]$. For direct sums of representations, the lists are concatenated. For the four-dimensional Galois representations we find to fit our data, we always observe that the list is $[0, 1, 2, 3]$ after sorting. This is what we expected, based on the Serre-type conjectures and the conjectural HT numbers. This gives us a check on our computations. See also Section 5.6.

Another check on our computations comes from considering the relationship between the nebentype character and the determinant of the apparently attached Galois representation. For example, consider a Galois representation between the nebentype character and the determinant of the apparently attached Galois level.

As we have indicated, the Galois groups Gal($\chi \otimes \varepsilon^w$, $\rho$) acting on our lists $\mathcal{L}_1$ and $\mathcal{L}_2^0$. Sometimes a cohomology group will contain Hecke eigenspaces $E(1), \ldots, E(g)$ which seem to be attached to Galois representations $\rho(1), \ldots, \rho(g)$ where $\rho(1), \ldots, \rho(g)$ is an orbit under the Galois action. We define the Galois multiplicity of each of $E(1), \ldots, E(g)$ to be $g$ in this case. In the tables in Section 6.1 we only list one of the $E(i)$, and we indicate the Galois multiplicity in the first column.

In the table in Section 6.1 for level $N = 23$ and $\eta = 1$, for example, we read that $H^5$ is the direct sum of five one-dimensional Hecke eigenspaces (lines). The first and second lines are Galois conjugate, the third and fourth are Galois conjugate, and the fifth is fixed by Galois. The Galois multiplicities are therefore 2, 2, and 1. Lines one through four are for the cusp form $\sigma_{1,2a}$, which is defined over a number field with Galois group $G$ of order 2. Because $G$ fixes the trivial nebentype $\eta = 1$, it acts on the cohomology. It interchanges the pairs of lines 1–2 and 3–4. The fifth line is for the cusp form $\sigma_{23,4}$, which is defined over $\mathbb{Q}$. Hence Galois acts trivially on the fifth line.

5. Observed regularities in the data and heuristics

This section details the regularities we observed in the tables below. When we have a reasonable heuristic explanation of a pattern, we give it. Converting any of these regularities or heuristics to theorems would require a finer analysis of the Borel-Serre boundary than is presently available and a greater expertise with Eisenstein series than we possess.

In this section, we let $\Gamma_0(a, b)$ denote the subgroup of $\text{GL}(a, \mathbb{Z})$ where the bottom row is congruent to $(0, \ldots, 0, \ast)$ modulo $b$. Thus $\Gamma_0(N) = \Gamma_0(4, N) \cap \text{SL}(4, \mathbb{Z})$ in our notation. We shall refer to a Hecke eigenclass in $H^5(\Gamma_0(N), \mathbb{F}_\eta)$ by the letter $z$ and to its attached Galois representation by $\rho$.

One pattern mentioned in the previous section is that the determinant of $\rho$ always equals $\varepsilon^0 \eta$. This is a tautology from the definition of attachment.

Another pattern we observe is that $\rho$ must be odd. In other words, the eigenvalues of $\rho(c)$ are $+1, -1, +1, -1$, where $c$ denotes complex conjugation. This must be the case, as follows from a theorem of Caraiani and LeHung [CLH16].
The Serre spectral sequence of this fibration degenerates at $E_L$ put a homology class on each block of $L_i$. In Section 5.4, where $k$ type of $P$ L components corresponding to the homology classes on the blocks of boundary homology, with attached Galois representations that are reducible, with complement in the associate class of $P$ coefficients in $H_j$ to a $P$ as in Figure 1. Each diagram represents a standard parabolic subgroup conjugate to a $P$ that gives rise to some kind of boundary homology.

One question is why the weights $2, 3, 4$ occur for the 2-dimensional irreducible components of the Galois representations. Heuristically, the observed weights can be explained in terms of the homology of the Borel-Serre boundary. This is outlined in detail in our first paper [AGM02], to which we refer the reader. Another question is why the exponents of the powers of $\varepsilon$ that occur as factors of the 1-dimensional components are always contained in the set $\{0, 1, 2, 3\}$, and what is the relationship between these exponents and the other components. The heuristic for this comes from deep (conjectural) connections between Hodge-Tate numbers of Galois representations and the coefficients of the cohomology classes to which they are attached. Finally, why do we sometimes observe that the multiplicity of a Hecke eigenspace is $3$, whereas usually it is $1$? This has to do with oldforms versus newforms. We now explain these answers in more detail.

Let $\Gamma = \Gamma_0(N)$. Let $B_\Gamma$ be the Borel-Serre boundary of the locally symmetric space $X_\Gamma = \Gamma\backslash \text{SL}(4, \mathbb{R})/\text{SO}(4)$. Then $B$ is the union of faces $F(P)$, where $P$ runs over a set of representatives of $\Gamma$-orbits of parabolic subgroups $P$ of $\text{GL}(4, \mathbb{Q})$. It is simpler to discuss homology rather than cohomology; this changes nothing qualitatively about the Hecke eigenvalues and attached Galois representations. The injection $B_\Gamma \to X_\Gamma \cup B_\Gamma$ induces a map on homology, for any coefficient system $M$:

$$H_5(B_\Gamma, M) \to H_5(X_\Gamma \cup B_\Gamma, M) = H_5(\Gamma, M).$$

The boundary homology is the image of this map. In this paper, every class we computed appears to be in the boundary homology.

For each parabolic subgroup $P$, let $P = LU$, where $L$ is a Levi component of $P$ and $U$ is the unipotent radical of $P$. Let $\pi : P \to P/U$ be the projection. The image of $\pi$ is isomorphic to $L$ and is a product of “blocks” $\text{GL}(n_i, \mathbb{Q})$, where $\sum n_i = 4$. If $P$ is conjugate to a standard parabolic subgroup (i.e., one containing the upper triangular matrices), the block sizes down the diagonal can be recorded as $(n_1, \ldots, n_{k+2})$. We call this tuple the “type” of $P$. The nonnegative integer $k$ equals the codimension of $F(P)$ in $B_\Gamma$. (If there is more than one parabolic subgroup in the associate class of $P$ we choose the type of one of them to be the type of $P$. The ambiguity has no importance for us.) Below, $k = 0$ except in Section 5.4 where $k = 1$.

Let $X_L$ denote the symmetric space of $L(\mathbb{R})$. Let $P_\Gamma = P \cap \Gamma$, $U_\Gamma = U \cap \Gamma$, and $L_\Gamma = \pi(P_\Gamma)$. The face $F(P)$ is a fibration with base $X_L/L_\Gamma$ and fiber $U(\mathbb{R})/U_\Gamma$.

The Serre spectral sequence of this fibration degenerates at $E^2$. Therefore, if we put a homology class on each block of $L_\Gamma$, whose degrees $i_1, \ldots, i_{k+2}$ add to $i$, with coefficients in $H_j(U_\Gamma, M)$, we obtain a class in $H_{i+j}(F(P), M)$. This class may or may not give rise to a nonzero class $H_{i+j+k}(B_\Gamma, M)$, depending on how it behaves in the Leray spectral sequence for the covering of $B_\Gamma$ by its faces. Finally, if there is a nonzero class in $H_{i+j+k}(B_\Gamma, M)$ obtained this way, it may or may not map to a nonzero class in $H_5(\Gamma, M)$. All this behaves Hecke-equivalently.

In this way we expect various kinds of homology Hecke eigenclasses in the boundary homology, with attached Galois representations that are reducible, with components corresponding to the homology classes on the blocks of $L$. We always have $i + j + k = 5$ because we computed $H_5(\Gamma, F_0)$.

For each type of $z$ it is convenient to have a schematic picture of the parameters, as in Figure 1. Each diagram represents a standard parabolic subgroup conjugate to a $P$ that gives rise to some kind of boundary homology.
For a $2 \times 2$ block $L'$ of $L$, we use the Eichler-Shimura theorem to interpret the homology of a congruence subgroup of $L'$ with coefficients in $\text{Sym}^g(\mathbb{F}_2^2) \otimes \eta$ in terms of classical modular forms of weight $g + 2$ and nebentype $\eta$. Therefore the corresponding component of $\rho$ will be attached to such a modular form. If we put on $L'$ a class in $H_0(\Gamma \cap L', M)$, the corresponding component of $\rho$ is observed always to be the sum of two consecutive powers of the cyclotomic character. If we take $H_1(\Gamma \cap L', M)$, we observe either a sum of two characters (corresponding to an Eisenstein series) or the Galois representation attached to a cusp form.

A general remark on nebentypes: different $\Gamma$-orbits of the same type of parabolic subgroups may result in different levels of the components of $L\Gamma$. If $N$ is composite, various nebentype characters can occur, but they will all have conductor dividing $N$.

In the following sections we give heuristics along the lines sketched above that account for all our data. We reiterate that additional heuristic schemes are possible and would be needed if we had been able to push our computations to much higher levels.

5.1. **GL(3) classes.** In this case (Figure 1(a)), $P$ is a $(1, 3)$-parabolic subgroup; $i_1 = 0, i_2 = 2, j = 3$. Note that $H_3(U\Gamma, \mathbb{F}_\eta)$ is a one-dimensional $L'$-module. We place a cuspidal homology class $w$ from $H_2(\Gamma_0(3, N), \mathbb{F}_\eta)$ on the second block. This class $w$ can be the symmetric square of a classical cusp form, or a class that is not a symmetric square. The latter occurs in our data only at level 41.

When $w$ is a symmetric square of the cusp form $s$, the level of $s$ equals $N$, the nebentype of $s$ equals the nebentype $\eta$, and $\eta$ is the quadratic character. This is necessary for a symmetric square at prime level $N$ to have the same level $N$ as the cusp form.

Writing the symmetric square of the Galois representation attached to $w$ as $\tau$, it always appears twice in our data, as $\rho = \varepsilon^0 \oplus \varepsilon \tau$ and $\rho = \varepsilon^3 \oplus \tau$. This is because there will be two relevant $\Gamma$-orbits of $P$, corresponding to block sizes $(1, 3)$ and $(3, 1)$ down the diagonal.

5.2. **Holomorphic cusp forms of weight 2.** In this case (Figure 1(b)), $P$ is a $(2, 2)$-parabolic subgroup; $i_1 = 0, i_2 = 1, j = 4$. Note that $H_4(U\Gamma, \mathbb{F}_\eta)$ is a one-dimensional $L'$-module. We place a cusp form $v$ on one of the two blocks. We observe that $v$ always has level $N$.

In our data, $\sigma$ always appears twice: once in $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \chi \sigma$ and once in $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \chi \sigma$, for some character $\chi$. This is because there will be two relevant $\Gamma$-orbits of $P$, both corresponding to block sizes $(2, 2)$; but in the second orbit, $v$ gets placed on the first block instead of the second block.
The auxiliary character \( \chi \) is the same in both expressions. We can and do always choose the ideal \( \mathfrak{P} \) so that \( \chi = 1 \). Sometimes these Galois representations appear with multiplicity 1, and sometimes with higher multiplicity, and we don’t know why the variability occurs.

5.3. Holomorphic cusp forms of weight 3. Cusp forms of odd weight can appear only if \( p = 2 \), as in [AGM15], or if odd nebentypes are available, as in the current paper.

In this case (Figure 1(c)), \( P \) is a \((2, 2)\)-parabolic subgroup; \( i_1 = 1, i_2 = 1, j = 3 \). Note that \( H_3(U_1, \mathbb{F}_q) \) restricted to either of the \( 2 \times 2 \)-blocks is a sum of two copies of the standard 2-dimensional GL(2)-representation. We place a cusp form \( v \) of weight 3 on one of the two blocks and an Eisenstein series \( u \) on the other block.

Let \( \sigma \) be the Galois representation attached to \( v \). We observe that \( \rho \) always has level strictly dividing \( N \) and always appears in our data four times as follows:

\[
\rho = \psi \varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon \rho
\]
\[
\rho = \varepsilon^0 \oplus \psi \varepsilon^2 \oplus \varepsilon \rho
\]
\[
\rho = \psi \varepsilon^1 \oplus \varepsilon^3 \oplus \sigma
\]
\[
\rho = \varepsilon^1 \oplus \psi \varepsilon^3 \oplus \sigma
\]

with the same character \( \psi \) all four times.

We only have three examples of this, at levels \( N = 24, 27 \) and 28. It doesn’t always occur even when \( N \) is composite and there is an appropriate \( v \) available. For example, there is a weight 3 cusp form of level 7 that contributes to \( N = 28 \) but it does not contribute when \( N = 14 \). We have no conjecture as to when a weight 3 cusp form appears for a given \((N, \eta)\).

5.4. Holomorphic cusp forms of weight 4. In this case (Figure 1(d)), \( P \) is a \((1, 2, 1)\)-parabolic subgroup; \( i_1 = 0, i_2 = 1, i_3 = 0, j = 3 \). Note that \( H_3(U_1, \mathbb{F}_q) \) contains an \( L' \)-submodule isomorphic to \( \text{Sym}^2 \) of the standard representation. We place a cusp form \( v \) of weight 4 on the second block.

Let \( \sigma \) be the Galois representation attached to \( v \). We observe that \( \rho = \varepsilon^1 \oplus \varepsilon^2 \oplus \sigma \) occurs only once in our data, if at all. It occurs if and only if the special value \( L(v, 1/2) \) of the \( L \)-function is 0. For the levels we have computed, this occurs only when \( \eta = 1 \). The level of \( v \) always divides \( N \) but need not equal \( N \).

5.5. Sums of 4 characters. See Figure 1(e). Here, as in (5.2), \( P \) is a \((2, 2)\)-parabolic subgroup; \( i_1 = 0, i_2 = 1, j = 4 \). We place an Eisenstein series \( e \) on one of the two blocks. Not surprisingly, \( e \) always has level dividing \( N \) and the two characters \( \psi \) and \( \chi \) associated with \( e \) have conductors dividing \( N \).

The following behavior is mysterious to us. If \( \eta \) factors nontrivially as \( \eta = \psi \chi \) then either all three of the following or none of the following occur:

\[
\rho = \psi \varepsilon^0 \oplus \chi \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3
\]
\[
\rho = \psi \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi \varepsilon^3
\]
\[
\rho = \varepsilon^0 \oplus \varepsilon^1 \oplus \psi \varepsilon^2 \oplus \chi \varepsilon^3
\]

For example, when \( N = 9 \) all three forms occur, and when \( N = 13 \) none of the three occur. Note that in a given triple of \( \rho \)'s, one of the characters multiplies even powers of \( \varepsilon \) and the other multiplies odd powers, which to some extent is explained because \( \rho \) must be odd. But why don’t we ever get \( \rho = \varepsilon^0 \oplus \chi \varepsilon^1 \oplus \psi \varepsilon^2 \oplus \varepsilon^3 \)? For some reason, the unadorned powers of \( \varepsilon \) are always consecutive. This is true of all the patterns observed above, except for those in (5.3). In the case of weight 3, the two 1-dimensional components of \( \rho \) are not both naked powers of \( \varepsilon \).
Note that \( \psi \) and \( \chi \) can trade places to get another triple, giving 6 \( \rho \)'s in total, for example, when \( n = 15 \). Factoring of \( \eta \) seems to be important here. For example, when \( \eta = 1 \) we never get \( \rho = \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \). However, for \( \text{GL}(n) \) with \( n \) larger than 4, the sum of consecutive powers of \( \varepsilon \) may be attached to a homology class, for example in the case of a Borel stable class \([\text{Ash92}, (4.2)]\).

5.6. **Powers of \( \varepsilon \).** Another observed pattern has to do with the powers of the cyclotomic character that appear in \( \rho \). Let us say that \( \varepsilon^i \) has HT (Hodge-Tate) number \( i \). We assign HT numbers \( 0, k - 1 \) to a Galois representation attached to a cusp form of weight \( k \). We assign HT numbers \( 0, k - 1, 2(k - 1) \) to a Galois representation attached to the symmetric square of a cusp form of weight \( k \). (In our data, we only see symmetric squares when \( k = 2 \).) We assign HT numbers \( 0, 1, 2 \) to a Galois representation attached to a GL(3)-homology eigenclass with trivial coefficients. If a Galois representation is tensored with \( \varepsilon^i \), then add \( i \) to each of its HT-numbers.

A folklore conjecture in the theory of arithmetic cohomology predicts that a Galois representation attached to a Hecke eigenclass in \( H^5(\Gamma, \mathbb{F}_\eta) \) should have HT numbers \( 0, 1, 2, 3 \). This is observed in all of our data.

5.7. **Hecke multiplicity 3.** We defined **Hecke multiplicity** in Section 4. In every case of our data, the Hecke multiplicity of the eigenspace for a system of Hecke eigenvalues equals either 1 or 3. As stated in Section 4 the Hecke operators we computed are always observed to be semisimple.

Hecke multiplicity 3 occurs in our data only when \( N \) is composite and the components of \( \rho \) have conductors strictly dividing \( N \). In our data, this happens for \( N = 18, 22, 26, 27, \) and 28. When \( N = 18 \) or 27 the relevant \( \rho \) is a sum of four characters, involving in two of the summands the quadratic character of conductor 3. In the other three cases, one of the components of \( \rho \) is attached to a cusp form of weight 2 and level \( N/2 \). We see no general rule as to why these cases of Hecke multiplicity 3 occur and not others that would be possible.

Here is a partial explanation of why the multiplicity is 3 rather than some other number when one of the components of \( \rho \) is attached to a cusp form \( \sigma \) of weight 2 and level \( N/2 \). First consider a parabolic subgroup of type \((2, 2)\) such that \( L_\Gamma \) is isomorphic to a subgroup of index two in \( \Gamma_0(2, N) \times \text{GL}(2, \mathbb{Z}) \). Since \( \sigma \) is an oldform for \( \Gamma_0(2, N) \), its system of Hecke eigenvalues contributes twice to the cohomology of \( L_\Gamma \), giving a Hecke multiplicity of 2, so far, for the system of Hecke eigenvalues to which \( \rho \) is attached. However, there is another \( \Gamma \)-orbit of parabolic subgroups of type \((2, 2)\) such that \( L_\Gamma \) is isomorphic to a subgroup of index two in \( \Gamma_0(2, N/2) \times \Gamma_0(2, 2) \). Here, \( \sigma \) is a newform for \( \Gamma_0(2, N/2) \), so its system of Hecke eigenvalues contributes once to the cohomology of \( L_\Gamma \), adding 1 to the Hecke multiplicity for the system of Hecke eigenvalues to which \( \rho \) is attached. The total is \( 2 + 1 = 3 \).

6. **Results**

6.1. The tables in this section present the main results of the paper.

The topmost box in each table gives the level \( N \), the nebentype \( \eta \), and the field \( \mathbb{F}_{p^e} = GF(p^e) \) that was our proxy for \( \mathbb{C} \). We only include one representative for each Galois orbit of nebentype characters. Next we list the Hecke operators
we computed. \( T_\ell \) means we computed \( T_{\ell,1}, T_{\ell,2}, \) and \( T_{\ell,3} \). Listing \( T_{\ell,1} \) means we computed only that part of \( T_\ell \).

The succeeding rows in each table give the Galois multiplicity (Section 4.1), the Hecke multiplicity (Section 4.1), and the Galois representation itself.

The characters \( \chi_N \) or \( \chi_{N,i} \) are a basis for the Dirichlet characters \((\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}_p\). They are listed in a separate table in Section 6.2. The cyclotomic character is denoted \( \varepsilon \).

The \( \sigma_{N,k} \) are classical cuspidal homomorphic newforms of level \( N \) and weight \( k \). They are listed in a separate table in Section 6.3. We use the same symbol \( \sigma_{N,k} \) to stand for the two-dimensional Galois representation attached to the cusp form of that name. When we have more than one cusp form for the same \( N \) and \( k \), we give them names like \( \sigma_{17,2a} \) and \( \sigma_{17,2b} \). The symmetric square of \( \sigma \) is denoted \( \text{Sym}^2(\sigma) \).

The \( \text{SL}_3 \) representation \( \delta \) is defined in Section 4.3.

| Level \( N = 9 \), Nebentype \( \eta = 1 \), Field \( \mathbb{F} = GF(12379) \) | Field \( \mathbb{F} = GF(12379) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_9 \varepsilon^2 \oplus \chi_9 \varepsilon^3 \) | \( \chi_9 \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_9 \varepsilon^3 \) |
| \( \chi_9 \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \) |

| Level \( N = 11 \), Nebentype \( \eta = 1 \), Field \( \mathbb{F} = GF(4001^2) \) | Field \( \mathbb{F} = GF(4001^2) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{11,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{11,2} \) |
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{11,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{11,2} \) |

| Level \( N = 12 \), Nebentype \( \eta = \chi_{12,0} \chi_{12,1} \), Field \( \mathbb{F} = GF(5413^2) \) | Field \( \mathbb{F} = GF(5413^2) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{12,0} \varepsilon^2 \oplus \chi_{12,1} \varepsilon^3 \) | \( \chi_{12,0} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{12,1} \varepsilon^2 \oplus \chi_{12,0} \varepsilon^3 \) |
| \( \chi_{12,0} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{12,1} \varepsilon^2 \oplus \chi_{12,0} \varepsilon^3 \) | \( \chi_{12,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \) |
| \( \chi_{12,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \) | \( \chi_{12,1} \varepsilon^0 \oplus \chi_{12,0} \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3 \) |

| Level \( N = 13 \), Nebentype \( \eta = 1 \), Field \( \mathbb{F} = GF(12037) \) | Field \( \mathbb{F} = GF(12037) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,4} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,4} \) |
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,4} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,4} \) |

| Level \( N = 13 \), Nebentype \( \eta = \chi_{13} \), Field \( \mathbb{F} = GF(12037) \) | Field \( \mathbb{F} = GF(12037) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,2} \) |
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{13,2} \) |

| Level \( N = 14 \), Nebentype \( \eta = 1 \), Field \( \mathbb{F} = GF(12379^2) \) | Field \( \mathbb{F} = GF(12379^2) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{14,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{14,2} \) |
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{14,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{14,2} \) |

| Level \( N = 15 \), Nebentype \( \eta = 1 \), Field \( \mathbb{F} = GF(12037^2) \) | Field \( \mathbb{F} = GF(12037^2) \) |
|---|---|
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{15,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{15,2} \) |
| \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{15,2} \) | \( \varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{15,2} \) |
| Level $N = 15$. Nebentype $\eta = \chi_{15,0}\chi_{15,1}$. Field $F = GF(12037^2)$. | Level $N = 16$. Nebentype $\eta = 1$. Field $F = GF(4001^8)$. |
| Computed $T_2, T_7$. | Computed $T_3, T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \chi_{15,0}\varepsilon^2 + \chi_{15,1}\varepsilon^3$ |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \chi_{15,1}\varepsilon^2 + \chi_{15,0}\varepsilon^3$ |
| 1 1 | $\chi_{15,0}\varepsilon^0 + \varepsilon^1 + \varepsilon^2 + \chi_{15,1}\varepsilon^3$ |
| 1 1 | $\chi_{15,1}\varepsilon^0 + \varepsilon^1 + \varepsilon^2 + \chi_{15,0}\varepsilon^3$ |
| 1 1 | $\chi_{15,0}\varepsilon^0 + \chi_{15,1}\varepsilon^1 + \varepsilon^2 + \varepsilon^3$ |
| 1 1 | $\chi_{15,1}\varepsilon^0 + \chi_{15,0}\varepsilon^1 + \varepsilon^2 + \varepsilon^3$ |
| Level $N = 16$. Nebentype $\eta = \chi_{16,1}$. Field $F = GF(4001^9)$. | Level $N = 17$. Nebentype $\eta = 1$. Field $F = GF(16001^2)$. |
| Computed $T_3, T_5, T_7$. | Computed $T_2, T_3, T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{16,2}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{16,2}$ |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{17,2a}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{17,2a}$ |
| 1 1 | $\varepsilon^1 + \varepsilon^2 + \varepsilon^0\varepsilon_{17,4}$ |
| Level $N = 17$. Nebentype $\eta = \chi_{17,7}$. Field $F = GF(16001^2)$. | Level $N = 18$. Nebentype $\eta = 1$. Field $F = GF(3637^2)$. |
| Computed $T_2, T_3, T_5, T_7$. | Computed $T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{17,2b}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{17,2b}$ |
| Level $N = 18$. Nebentype $\eta = \chi_{18}$. Field $F = GF(3637^2)$. | Level $N = 18$. Nebentype $\eta = 1$. Field $F = GF(3637^2)$. |
| Computed $T_5, T_7, T_{11.1}$. | Computed $T_2, T_3, T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{18,2}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{18,2}$ |
| Level $N = 19$. Nebentype $\eta = 1$. Field $F = GF(3637^2)$. | Level $N = 19$. Nebentype $\eta = \chi_{19}^7$. Field $F = GF(3637^2)$. |
| Computed $T_2, T_3, T_5, T_7$. | Computed $T_2, T_3, T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{19,2a}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{19,2a}$ |
| 1 1 | $\varepsilon^1 + \varepsilon^2 + \varepsilon^0\varepsilon_{19,4}$ |
| Level $N = 19$. Nebentype $\eta = \chi_{19}^7$. Field $F = GF(3637^2)$. | Level $N = 19$. Nebentype $\eta = 1$. Field $F = GF(3637^2)$. |
| Computed $T_2, T_3, T_5, T_7$. | Computed $T_2, T_3, T_5, T_7$. |
| 1 1 | $\varepsilon^0 + \varepsilon^1 + \varepsilon^2\varepsilon_{19,2b}$ |
| 1 1 | $\varepsilon^2 + \varepsilon^3 + \varepsilon^0\varepsilon_{19,2b}$ |
| Level $N = 20$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(12037^{12})$. Computed $T_3, T_7, T_{11.1}, T_{13.1}$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \sigma_{20,2a}$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{20,2a}$ |

| Level $N = 20$. Nebentype $\eta = \chi_{20,0} \chi_{20,1}$. Field $\mathbb{F} = GF(12037^{12})$. Computed $T_3, T_7, T_{11.1}, T_{13.1}$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{20,0} \varepsilon^2 \oplus \chi_{20,1} \varepsilon^5$ |
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{20,1} \varepsilon^2 \oplus \chi_{20,0} \varepsilon^5$ |
| 1 1 | $\chi_{20,0} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{20,1} \varepsilon^5$ |
| 1 1 | $\chi_{20,1} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{20,0} \varepsilon^5$ |
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{20,1} \varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^5$ |
| 1 1 | $\chi_{20,1} \varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{20,0} \varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^5$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{20,2b}$ |

| Level $N = 21$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(12037^6)$. Computed $T_2, T_5$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \sigma_{21,2a}$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{21,2a}$ |
| 1 1 | $\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{21,4}$ |

| Level $N = 21$. Nebentype $\eta = \chi_{21,1}^2$. Field $\mathbb{F} = GF(12037^6)$. Computed $T_2, T_5$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \sigma_{21,2b}$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{21,2b}$ |

| Level $N = 21$. Nebentype $\eta = \chi_{21,0} \chi_{21,1}$. Field $\mathbb{F} = GF(12037^6)$. Computed $T_2, T_5, T_{11.1}, T_{13.1}$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,0} \varepsilon^2 \oplus \chi_{21,1} \varepsilon^5$ |
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,1} \varepsilon^2 \oplus \chi_{21,0} \varepsilon^5$ |
| 1 1 | $\chi_{21,0} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{21,1} \varepsilon^5$ |
| 1 1 | $\chi_{21,1} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{21,0} \varepsilon^5$ |
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,0} \varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^5$ |
| 1 1 | $\chi_{21,1} \varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,1} \varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^5$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{21,2c}$ |
| 1 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{21,2c}$ |

| Level $N = 21$. Nebentype $\eta = \chi_{21,0} \chi_{21,1}^3$. Field $\mathbb{F} = GF(12037^6)$. Computed $T_2, T_5$. |
|---|---|---|---|
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,0} \varepsilon^2 \oplus \chi_{21,1}^3 \varepsilon^3$ |
| 1 1 | $\varepsilon^0 \oplus \varepsilon^4 \oplus \chi_{21,1}^2 \varepsilon^2 \oplus \chi_{21,0} \varepsilon^3$ |
| 1 1 | $\chi_{21,0} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{21,1}^3 \varepsilon^3$ |
| 1 1 | $\chi_{21,1} \varepsilon^0 \oplus \varepsilon^4 \oplus \varepsilon^2 \oplus \chi_{21,0} \varepsilon^3$ |
| 1 1 | $\varepsilon^0 \oplus \chi_{21,0} \varepsilon^2 \oplus \varepsilon^4 \sigma_{7,3}$ |
| 1 1 | $\varepsilon^1 \oplus \chi_{21,0} \varepsilon^4 \oplus \varepsilon^0 \sigma_{7,3}$ |
| 1 1 | $\chi_{21,0} \varepsilon^0 \oplus \varepsilon^2 \oplus \varepsilon^4 \sigma_{7,3}$ |
| 1 1 | $\chi_{21,0} \varepsilon^1 \oplus \varepsilon^4 \oplus \varepsilon^0 \sigma_{7,3}$ |
| Level $N = 22$. Nebentype $\eta = 1$. Field $F = GF(16001^2)$. |  |
|---|---|
| Computed $T_3$, $T_5$, $T_7$. |  |
| 1 | 1 | $\varepsilon^\eta \oplus \varepsilon^3 \oplus \varepsilon^4 \sigma_{11,2}$ |
| 1 | 3 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{11,2}$ |
| 1 | 1 | $\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^\eta \sigma_{22,4}$ |

| Level $N = 22$. Nebentype $\eta = \chi_{22}^2$. Field $F = GF(16001^2)$. |  |
|---|---|
| Computed $T_3$, $T_5$, $T_7$. |  |
| 1 | 1 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^4 \sigma_{22,2}$ |
| 1 | 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{22,2}$ |

| Level $N = 23$. Nebentype $\eta = 1$. Field $F = GF(22067^{50})$. |  |
|---|---|
| Computed $T_2$, $T_3$, $T_5$, $T_7$. |  |
| 2 | 1 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^4 \sigma_{23,2a}$ |
| 2 | 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{23,2a}$ |
| 1 | 1 | $\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^\eta \sigma_{23,4}$ |

| Level $N = 23$. Nebentype $\eta = \chi_{23}^2$. Field $F = GF(22067^{50})$. |  |
|---|---|
| Computed $T_2$, $T_3$, $T_5$, $T_7$. |  |
| 1 | 1 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^4 \sigma_{23,2b}$ |
| 1 | 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{23,2b}$ |

| Level $N = 24$. Nebentype $\eta = 1$. Field $F = GF(12379^2)$. |  |
|---|---|
| Computed $T_5$, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. |  |
| 1 | 1 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^4 \sigma_{24,2a}$ |
| 1 | 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{24,2a}$ |

| Level $N = 24$. Nebentype $\eta = \chi_{24,1}$. Field $F = GF(12379^2)$. |  |
|---|---|
| Computed $T_5$, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. |  |
| 2 | 1 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^4 \sigma_{24,2b}$ |
| 2 | 1 | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^\eta \sigma_{24,2b}$ |

| Level $N = 24$. Nebentype $\eta = \chi_{24,0,24,2}$. Field $F = GF(12379^2)$. |  |
|---|---|
| Computed $T_5$, $T_{7,1}$, $T_{11,1}$, $T_{13,1}$. |  |
| 1 | 3 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \chi_{24,0,24,2} \varepsilon^2 \oplus \chi_{24,2} \varepsilon^4$ |
| 1 | 3 | $\varepsilon^\eta \oplus \varepsilon^1 \oplus \chi_{24,0,24,2} \varepsilon^2 \oplus \chi_{24,0} \varepsilon^4$ |
| 1 | 3 | $\chi_{24,0} \varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,2} \varepsilon^3$ |
| 1 | 3 | $\chi_{24,2} \varepsilon^\eta \oplus \varepsilon^1 \oplus \varepsilon^2 \oplus \chi_{24,0} \varepsilon^3$ |
| 1 | 3 | $\chi_{24,0} \varepsilon^\eta \oplus \chi_{24,2} \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$ |
| 1 | 3 | $\chi_{24,2} \varepsilon^\eta \oplus \chi_{24,0} \varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^3$ |
| Level $N = 24$. Nebentype $\eta = \chi_{24.0}\chi_{24.1}\chi_{24.2}$. Field $\mathbb{F} = GF(12379^2)$. Computed $T_5, T_{7,1}, T_{11,1}, T_{13,1}, T_{17,1}$. |
|---|
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{24.0}\chi_{24.1}\epsilon^2 \oplus \chi_{24.2}\epsilon^3$ |
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{24.2}\epsilon^2 \oplus \chi_{24.0}\chi_{24.1}\epsilon^3$ |
| 1 1 | $\chi_{24.0}\chi_{24.1}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \chi_{24.2}\epsilon^3$ |
| 1 1 | $\chi_{24.2}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \chi_{24.0}\chi_{24.1}\epsilon^3$ |
| 1 1 | $\chi_{24.0}\chi_{24.1}\epsilon^0 \oplus \chi_{24.2}\epsilon^1 \oplus \epsilon^2 \oplus \epsilon^3$ |
| 1 1 | $\chi_{24.2}\epsilon^0 \oplus \chi_{24.0}\chi_{24.1}\epsilon^1 \oplus \epsilon^2 \oplus \epsilon^3$ |
| 2 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \sigma_{24.2c}$ |
| 2 1 | $\epsilon^2 \oplus \epsilon^3 \oplus \epsilon^0 \sigma_{24.2c}$ |
| 1 1 | $\epsilon^0 \oplus \chi_{24.2}\epsilon^2 \oplus \epsilon^3 \sigma_{8,3}$ |
| 1 1 | $\epsilon^1 \oplus \chi_{24.2}\epsilon^2 \oplus \epsilon^3 \sigma_{8,3}$ |
| 1 1 | $\chi_{24.2}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \epsilon^3 \sigma_{8,3}$ |
| 1 1 | $\chi_{24.2}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \epsilon^3 \sigma_{8,3}$ |

| Level $N = 25$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(16001^{50})$. Computed $T_2, T_3$. |
|---|
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\epsilon^1 \oplus \epsilon^2 \oplus \epsilon^0 \sigma_{25,4}$ |

| Level $N = 25$. Nebentype $\eta = \chi_{25}$. Field $\mathbb{F} = GF(16001^{50})$. Computed $T_2, T_3$. |
|---|
| 2 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \sigma_{25,2a}$ |
| 2 1 | $\epsilon^2 \oplus \epsilon^3 \oplus \epsilon^0 \sigma_{25,2a}$ |

| Level $N = 25$. Nebentype $\eta = \chi_{25}$. Field $\mathbb{F} = GF(16001^{50})$. Computed $T_2, T_3$. |
|---|
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \sigma_{25,2b}$ |
| 1 1 | $\epsilon^2 \oplus \epsilon^3 \oplus \epsilon^0 \sigma_{25,2b}$ |

| Level $N = 25$. Nebentype $\eta = \chi_{25}$. Field $\mathbb{F} = GF(16001^{50})$. Computed $T_2, T_3$. |
|---|
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \epsilon^3$ |
| 1 1 | $\chi_{25}\epsilon^0 \oplus \chi_{25}\epsilon^1 \oplus \chi_{25}\epsilon^2 \oplus \chi_{25}\epsilon^3$ |
| 1 1 | $\epsilon^1 \oplus \epsilon^2 \oplus \epsilon^0 \sigma_{25,4}$ |

| Level $N = 26$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(12037^{2})$. Computed $T_3, T_5$. |
|---|
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \sigma_{26,2a}$ |
| 1 1 | $\epsilon^2 \oplus \epsilon^3 \oplus \epsilon^0 \sigma_{26,2a}$ |
| 1 1 | $\epsilon^0 \oplus \epsilon^1 \oplus \epsilon^2 \sigma_{26,2b}$ |
| 1 1 | $\epsilon^2 \oplus \epsilon^3 \oplus \epsilon^0 \sigma_{26,2b}$ |
| 1 3 | $\epsilon^1 \oplus \epsilon^2 \oplus \epsilon^0 \sigma_{13,4}$ |
| Level \(N = 26\). Nebentype \(\eta = \chi_{26}^3\). Field \(F = \text{GF}(12037^2)\) | Computed \(T_3, T_5\). |
|---|---|
| 1 | \(e^0 + e^1 + e^2\sigma_{13,2}\) |
| 1 | \(e^2 + e^3 + e^0\sigma_{13,2}\) |

| Level \(N = 26\). Nebentype \(\eta = \chi_{26}^3\). Field \(F = \text{GF}(12037^2)\) | Computed \(T_3, T_5\). |
|---|---|
| 1 | \(e^0 + e^1 + e^2\sigma_{26,2c}\) |
| 1 | \(e^2 + e^3 + e^0\sigma_{26,2c}\) |

| Level \(N = 26\). Nebentype \(\eta = \chi_{26}^3\). Field \(F = \text{GF}(12037^2)\) | Computed \(T_3, T_5\). |
|---|---|
| 2 | \(e^0 + e^1 + e^2\sigma_{26,2d}\) |
| 2 | \(e^2 + e^3 + e^0\sigma_{26,2d}\) |

| Level \(N = 27\). Nebentype \(\eta = 1\). Field \(F = \text{GF}(11863^b)\) | Computed \(T_2, T_5, T_{7,1}\). |
|---|---|
| 1 | \(e^0 + e^1 + \chi_{27}^9 e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\chi_{27}^9 e^0 + e^1 + e^2 + \chi_{27}^3 e^3\) |
| 1 | \(\epsilon^u + \chi_{27}^9 e^2 + \epsilon^1 \sigma_{27.2a}\) |
| 1 | \(\epsilon^u + \chi_{27}^9 e^2 + \epsilon^1 \sigma_{27.2a}\) |
| 1 | \(\epsilon^u + \chi_{27}^9 e^2 + \epsilon^1 \sigma_{27.2a}\) |

| Level \(N = 27\). Nebentype \(\eta = \chi_{27}^1\). Field \(F = \text{GF}(11863^b)\) | Computed \(T_2, T_5, T_{7,1}\). |
|---|---|
| 2 | \(e^0 + e^1 + e^2\sigma_{27.2b}\) |
| 2 | \(e^2 + e^3 + e^0\sigma_{27.2b}\) |

| Level \(N = 27\). Nebentype \(\eta = \chi_{27}^b\). Field \(F = \text{GF}(11863^b)\) | Computed \(T_2, T_5, T_{7,1}\). |
|---|---|
| 1 | \(e^0 + e^1 + \chi_{27}^{15} e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^1 + e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^1 + e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^1 + e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^1 + e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^1 + e^2 + \chi_{27}^9 e^3\) |
| 1 | \(\epsilon^0 + \chi_{27}^{15} e^2 + \epsilon^1 \sigma_{9.3}\) |
| 1 | \(\epsilon^0 + \chi_{27}^{15} e^2 + \epsilon^1 \sigma_{9.3}\) |
| 1 | \(\epsilon^0 + \chi_{27}^{15} e^2 + \epsilon^1 \sigma_{9.3}\) |
| 1 | \(\chi_{27}^{15} e^0 + e^2 + \chi_{27}^{15} e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^2 + \chi_{27}^{15} e^3\) |
| 1 | \(\chi_{27}^{15} e^0 + e^2 + \chi_{27}^{15} e^3\) |

| Level \(N = 28\). Nebentype \(\eta = 1\). Field \(F = \text{GF}(12379^{12})\) | Computed \(T_3, T_5, T_{11,1}, T_{13,1}\). |
|---|---|
| 1 | \(e^0 + e^1 + e^2\sigma_{14,2}\) |
| 1 | \(e^2 + e^3 + e^0\sigma_{14,2}\) |
| 1 | \(e^4 + e^5 + e^0\sigma_{14,2}\) |

| Level \(N = 28\). Nebentype \(\eta = \chi_{28,1}^1\). Field \(F = \text{GF}(12379^{12})\) | Computed \(T_3, T_5\). |
|---|---|
| 1 | \(e^0 + e^1 + e^2\sigma_{28,2a}\) |
| 1 | \(e^2 + e^3 + e^0\sigma_{28,2a}\) |

| Level \(N = 28\). Nebentype \(\eta = \chi_{28,1}^1\). Field \(F = \text{GF}(12379^{12})\) | Computed \(T_3, T_5\). |
|---|---|
| 1 | \(e^0 + e^1 + e^2\sigma_{28,2a}\) |
| 1 | \(e^2 + e^3 + e^0\sigma_{28,2a}\) |
### Level $N = 28$. Nebentype $\eta = \chi_{28,0,28,1}$. Field $F = GF(12379^{12})$.
Composed $T_3$, $T_5$.

| Level | Nebentype $\eta = \chi_{28,0,28,1}$ | Field $F = GF(12379^{12})$ |
|-------|-------------------------------------|----------------------------|
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$. |

### Level $N = 28$. Nebentype $\eta = \chi_{28,0,28,1}$. Field $F = GF(12379^{12})$.
Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$.

| Level | Nebentype $\eta = \chi_{28,0,28,1}$ | Field $F = GF(12379^{12})$ |
|-------|-------------------------------------|----------------------------|
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 1     | $\chi_{28,0,28,1} \varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \chi_{28,0,28,1} \varepsilon^2 \oplus \chi_{28,0,28,1} \varepsilon^3$ | Composed $T_3$, $T_5$, $T_{11,1}$, $T_{13,1}$. |

### Level $N = 29$. Nebentype $\eta = 1$. Field $F = GF(2297^6)$.
Composed $T_2$, $T_3$, $T_5$.

| Level | Nebentype $\eta = 1$ | Field $F = GF(2297^6)$ |
|-------|---------------------|-------------------------|
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2a}$ | Composed $T_2$, $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2a}$ | Composed $T_2$, $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2a}$ | Composed $T_2$, $T_3$, $T_5$. |

### Level $N = 29$. Nebentype $\eta = \chi_{29}^2$. Field $F = GF(2297^6)$.
Composed $T_2$, $T_3$, $T_5$.

| Level | Nebentype $\eta = \chi_{29}^2$ | Field $F = GF(2297^6)$ |
|-------|---------------------------------|-------------------------|
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2b}$ | Composed $T_2$, $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2b}$ | Composed $T_2$, $T_3$, $T_5$. |

### Level $N = 29$. Nebentype $\eta = \chi_{29}^2$. Field $F = GF(2297^6)$.
Composed $T_2$, $T_3$, $T_5$.

| Level | Nebentype $\eta = \chi_{29}^2$ | Field $F = GF(2297^6)$ |
|-------|---------------------------------|-------------------------|
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2c}$ | Composed $T_2$, $T_3$, $T_5$. |
| 1     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2c}$ | Composed $T_2$, $T_3$, $T_5$. |

### Level $N = 29$. Nebentype $\eta = \chi_{29}^2$. Field $F = GF(2297^6)$.
Composed $T_2$, $T_3$, $T_5$.

| Level | Nebentype $\eta = \chi_{29}^2$ | Field $F = GF(2297^6)$ |
|-------|---------------------------------|------------------------|
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2d}$ | Composed $T_2$, $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2d}$ | Composed $T_2$, $T_3$, $T_5$. |
| 2     | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{29,2d}$ | Composed $T_2$, $T_3$, $T_5$. |
| Level $N = 31$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(4201^{60})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $2 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2a}$ |
| $2 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2a}$ |
| $2 \mid 1$ | $\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{31,4}$ |

| Level $N = 31$. Nebentype $\eta = \chi_{31}^2$. Field $\mathbb{F} = GF(4201^{61})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $2 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2b}$ |
| $2 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2b}$ |

| Level $N = 31$. Nebentype $\eta = \chi_{31}^5$. Field $\mathbb{F} = GF(4201^{60})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2c}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2c}$ |

| Level $N = 31$. Nebentype $\eta = \chi_{31}^{11}$. Field $\mathbb{F} = GF(4201^{61})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $2 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{31,2d}$ |
| $2 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{31,2d}$ |

| Level $N = 37$. Nebentype $\eta = 1$. Field $\mathbb{F} = GF(3889^{24})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5, T_{7,1}, T_{13,1}$. |  |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2a}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2a}$ |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2b}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2b}$ |
| $4 \mid 1$ | $\varepsilon^1 \oplus \varepsilon^2 \oplus \varepsilon^0 \sigma_{37,4}$ |

| Level $N = 37$. Nebentype $\eta = \chi_{37}^{3}$. Field $\mathbb{F} = GF(3889^{24})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $3 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2c}$ |
| $3 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2c}$ |

| Level $N = 37$. Nebentype $\eta = \chi_{37}^{3}$. Field $\mathbb{F} = GF(3889^{24})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2d}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2d}$ |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2e}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2e}$ |

| Level $N = 37$. Nebentype $\eta = \chi_{37}^{b}$. Field $\mathbb{F} = GF(3889^{24})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $2 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2f}$ |
| $2 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2f}$ |

| Level $N = 37$. Nebentype $\eta = \chi_{37}^{12}$. Field $\mathbb{F} = GF(3889^{24})$. |  |
| --- | --- |
| Computed $T_2, T_3, T_5$. |  |
| $1 \mid 1$ | $\varepsilon^0 \oplus \varepsilon^1 \oplus \varepsilon^2 \sigma_{37,2g}$ |
| $1 \mid 1$ | $\varepsilon^2 \oplus \varepsilon^3 \oplus \varepsilon^0 \sigma_{37,2g}$ |
6.2. For each $N$, the next table specifies the basis that Sage chooses for the group of characters $(\mathbb{Z}/N\mathbb{Z})^\times \to F_p$. If there is one basis element, it is denoted $\chi_N$. If there is more than one, they are denoted $\chi_{N,0}$, $\chi_{N,1}$, etc. The order of $\chi$ is the smallest positive $n$ so that $\chi^n$ is trivial on $(\mathbb{Z}/N\mathbb{Z})^\times$. The parity is even if $\chi(-1) = +1$ and odd if $\chi(-1) = -1$.

| $\chi_{N,i}$ | $p$  | order | parity | definition |
|-------------|-----|-------|--------|------------|
| $\chi_7$    | 12037 | 6     | odd    | $3 \mapsto -1293$ |
| $\chi_9$    | 12379 | 6     | odd    | $2 \mapsto 5770$ |
| $\chi_{12,0}$| 5413  | 2     | odd    | $7 \mapsto -1$, $5 \mapsto 1$ |
| $\chi_{12,1}$| 5413  | 2     | odd    | $7 \mapsto 1$, $5 \mapsto -1$ |
| $\chi_{13}$ | 12037 | 12    | odd    | $2 \mapsto 4019$ |
6.3. In the following table we give the $q$-expansions of the holomorphic cusp forms that we observed in our computations. $S_k(N, \chi)$ denotes the space of weight $k$ cusp forms on $\Gamma_0(N)$ with character $\chi$. The notation $\sigma_{N,k}$ for individual cusp forms makes manifest the level $N$ and weight $k$. The $q$-expansions were computed using Sage [Dev17].

The field of definition of a cusp form, if not specified, is the field generated by the coefficients we display. For instance, $q + 2q^2 + 55q^3 + \cdots$ has coefficients in $\mathbb{Q}(i)$. By $\zeta_m$ we mean a primitive $m$-th root of unity. When we must specify the field, it is in the line beginning with “over”.

| $\chi$ | $q$-expansion |
|--------|----------------|
| $\chi_{15,0}$ | $2$ | odd | $11 \mapsto -1$, $7 \mapsto 1$ |
| $\chi_{15,1}$ | $4$ | odd | $11 \mapsto 1$, $7 \mapsto 3417$ |
| $\chi_{16,0}$ | $2$ | odd | $15 \mapsto -1$, $5 \mapsto 1$ |
| $\chi_{16,1}$ | $4$ | even | $15 \mapsto 1$, $5 \mapsto -899$ |
| $\chi_{17}$ | $16$ | odd | $3 \mapsto 83$ |
| $\chi_{18}$ | $6$ | odd | $11 \mapsto -695$ |
| $\chi_{19}$ | $18$ | odd | $2 \mapsto -31$ |
| $\chi_{20,0}$ | $2$ | odd | $11 \mapsto -1$, $17 \mapsto 1$ |
| $\chi_{20,1}$ | $4$ | odd | $11 \mapsto 1$, $17 \mapsto 3417$ |
| $\chi_{21,0}$ | $2$ | odd | $8 \mapsto -1$, $10 \mapsto 1$ |
| $\chi_{21,1}$ | $6$ | odd | $8 \mapsto 1$, $10 \mapsto -1293$ |
| $\chi_{22}$ | $10$ | odd | $13 \mapsto 3018$ |
| $\chi_{23}$ | $22$ | odd | $5 \mapsto 7863$ |
| $\chi_{24,0}$ | $2$ | odd | $7 \mapsto -1$, $13 \mapsto 1$, $17 \mapsto 1$ |
| $\chi_{24,1}$ | $2$ | even | $7 \mapsto 1$, $13 \mapsto -1$, $17 \mapsto 1$ |
| $\chi_{24,2}$ | $2$ | odd | $7 \mapsto 1$, $13 \mapsto 1$, $17 \mapsto -1$ |
| $\chi_{25}$ | $20$ | odd | $2 \mapsto 7734$ |
| $\chi_{26}$ | $12$ | odd | $15 \mapsto 4019$ |
| $\chi_{27}$ | $18$ | odd | $2 \mapsto 5034$ |
| $\chi_{28,0}$ | $2$ | odd | $15 \mapsto -1$, $17 \mapsto 1$ |
| $\chi_{28,1}$ | $6$ | odd | $15 \mapsto 1$, $17 \mapsto 5770$ |
| $\chi_{29}$ | $28$ | odd | $2 \mapsto 1108$ |
| $\chi_{31}$ | $30$ | odd | $3 \mapsto -1970$ |
| $\chi_{37}$ | $36$ | odd | $2 \mapsto -1338$ |
| $\chi_{41}$ | $40$ | odd | $6 \mapsto -10354$ |

\[
\sigma_{7,3} = q - 3q^2 + 5q^3 - 7q^4 + O(q^5) \text{ in } S_5(7, \chi_2^2)
\]
\[
\sigma_{8,3} = q - 2q^2 - 2q^3 + 4q^4 + 4q^5 - 8q^6 - 5q^7 + 14q^8 + 8q^9 + O(q^{10}) \text{ in } S_9(8, \chi_{24,0} \chi_{24,1})
\]
\[
\sigma_{9,3} = q + (-\zeta_6 - 1)q^4 + (3\zeta_6 - 3)q^5 - \zeta_6 q^6 + O(q^7) \text{ in } S_5(9, \chi_6^{13})
\]
\[
\sigma_{11,2} = q - 2q^2 + 2q^3 + q^4 + 2q^5 - 2q^6 + O(q^7) \text{ in } S_2(11, 1)
\]
\[
\sigma_{13,2} = q + (\zeta_6 - 1)q^2 + (2\zeta_6 - 2)q^3 + \zeta_6 q^4 + O(q^5) \text{ in } S_2(13, \chi_6^{13})
\]
\[
\sigma_{13,4} = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 - 13q^7 + O(q^8) \text{ in } S_4(13, 1)
\]
\[
\sigma_{14,2} = q - q^2 - 2q^3 + q^4 + q^5 + O(q^6) \text{ in } S_2(14, 1)
\]
\[
\sigma_{15,2} = q - q^2 - q^3 - q^4 + q^5 + O(q^6) \text{ in } S_2(15, 1)
\]
\[
\sigma_{16,2} = q + (-i - 1)q^2 + (i - 1)q^3 + 2iq^4 + O(q^5) \text{ in } S_2(16, \chi_{16,1})
\]
\[
\sigma_{17,2a} = q - q^2 - q^3 - 2q^4 + 4q^5 + O(q^6) \text{ in } S_2(17, 1)
\]
\[
\sigma_{17,2b} = q + (-\zeta_8^2 + \zeta_8 - 1)q^2 + (\zeta_8 - \zeta_8^5 - \zeta_8 - 1)q^3 + O(q^4) \text{ in } S_2(17, \chi_{17}^2)
\]
\[ \sigma_{20,2d} = q + b_0 q^2 - b_0 q^1 - 3q^4 - 3q^3 + 5q^0 + 2q^7 + O(q^8) \text{ in } S_2(29, \chi_{39}^{14}) \]
over \(Q|b_0|/(b_0^3 + 5)\)

\[ \sigma_{31,2} = q + b_0 q^2 - (3b_0 - 8) q^3 + (-2b_0 - 7) q^4 + (4b_0 - 1) q^7 + O(q^8) \text{ in } S_2(29,1) \]
over \(Q|b_0|/(b_0^2 + 2b_0 - 1)\)

\[ \sigma_{31,2} = q + b_0 q^2 - 2b_0 q^3 + (b_0 - 1) q^4 + (-2b_0 - 2) q^0 + O(q^8) \text{ in } S_2(31,1) \]
over \(Q|b_0|/(b_0^2 - b_0 - 1)\)

\[ \sigma_{31,2} = q + b_0 q^2 + (3b_0 - 1 - 2q_3 + 2q_2 - 1) q^4 + O(q^8) \text{ in } S_2(31, \chi_{39}^{14}) \]
over \(Q|\zeta_3|)|b_0|/(b_0^3 + (\zeta_3^3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)

\[ \sigma_{31,2} = q + (\zeta_3^3 + \zeta_3 + \zeta_3) q^4 - \zeta_3^2 q^3 + (\zeta_3 + 1) q^4 + O(q^8) \text{ in } S_2(31, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + 2\zeta_3 + b_0 - 1)\)

\[ \sigma_{31,2} = q + b_0 q^2 + (\zeta_3 + 1 - b_0) q^4 + (3b_0 - 5) q^0 + O(q^8) \text{ in } S_3(31,1) \]
over \(Q|b_0|/(b_0^2 + 5b_0 + 2)\)

\[ \sigma_{37,2} = q - 2q^7 + 3q^5 + 3q^4 - 2q^2 + 6q^0 - q^1 + 6q^9 - 5q^{11} + O(q^{12}) \text{ in } S_2(37,1) \]

\[ \sigma_{37,2} = q + \zeta_3 q^2 - 2q^7 - q^5 - 2q^3 + q^1 - 2q^2 - 4q^{10} + O(q^{11}) \text{ in } S_2(37,1) \]

\[ \sigma_{37,2} = q + b_0 q^2 + (2\zeta_3^2 + 2\zeta_3 + 1) b_0 - 2\zeta_3^2 - \zeta_3^2 + \zeta_3 - 1) q^4 + O(q^8) \text{ in } S_2(37, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + 2\zeta_3 + b_0 - 1)\)

\[ \sigma_{37,2} = q + b_0 q^2 + (\zeta_3 + 1 - b_0) q^4 + (3b_0 - 5) q^0 + O(q^8) \text{ in } S_2(37, \chi_{39}^{14}) \]
over \(Q|b_0|/(b_0^2 + 5b_0 + 2)\)

\[ \sigma_{37,2} = q + \zeta_3 q^2 - 2q^7 - q^5 - 2q^3 + q^1 - 2q^2 - 4q^{10} + O(q^{11}) \text{ in } S_2(37, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + 2\zeta_3 + b_0 - 1)\)

\[ \sigma_{37,2} = q + b_0 q^2 + (\zeta_3 + 1 - b_0) q^4 + (3b_0 - 5) q^0 + O(q^8) \text{ in } S_2(37, \chi_{39}^{14}) \]
over \(Q|b_0|/(b_0^2 + 5b_0 + 2)\)

\[ \sigma_{37,2} = q - 2q^7 + 3q^5 + 3q^4 - 2q^2 + 6q^0 - q^1 + 6q^9 - 5q^{11} + O(q^{12}) \text{ in } S_2(37,1) \]
over \(Q|b_0|/(b_0^2 + 5b_0 + 2)\)

\[ \sigma_{41,2} = q + b_0 q^2 + (\zeta_5 + \zeta_5 + \zeta_5) q^4 - \zeta_5 q^3 + (\zeta_5 + \zeta_5 + \zeta_5 - 1) q^4 + O(q^8) \text{ in } S_2(41,1) \]
over \(Q|b_0|/(b_0^2 + 5b_0 - 5b_0 - 1)\)

\[ \sigma_{41,2} = q + b_0 q^2 + ((-\zeta_5 + \zeta_5 + \zeta_5 + \zeta_5) b_0 - 2\zeta_5 - \zeta_5 + \zeta_5 + \zeta_5) q^4 + O(q^8) \text{ in } S_2(41, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + (\zeta_3 + 2\zeta_3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)

\[ \sigma_{41,2} = q + b_0 q^2 + ((-\zeta_5 + \zeta_5 + \zeta_5 + \zeta_5) b_0 - 2\zeta_5 - \zeta_5 + \zeta_5 + \zeta_5) q^4 + O(q^8) \text{ in } S_2(41, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + (\zeta_3 + 2\zeta_3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)

\[ \sigma_{41,2} = q + b_0 q^2 + ((\zeta_5 + \zeta_5 + \zeta_5 + \zeta_5) b_0 + 2\zeta_5^6 - \zeta_5^4 + \zeta_5^2 + \zeta_5 + 3) q^4 + O(q^8) \text{ in } S_2(41, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + (\zeta_3 + 2\zeta_3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)

\[ \sigma_{41,2} = q + b_0 q^2 + ((\zeta_5 + \zeta_5 + \zeta_5 + \zeta_5) b_0 + 2\zeta_5^6 - \zeta_5^4 + \zeta_5^2 + \zeta_5 + 3) q^4 + O(q^8) \text{ in } S_2(41, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + (\zeta_3 + 2\zeta_3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)

\[ \sigma_{41,2} = q + b_0 q^2 + ((\zeta_5 + \zeta_5 + \zeta_5 + \zeta_5) b_0 + 2\zeta_5^6 - \zeta_5^4 + \zeta_5^2 + \zeta_5 + 3) q^4 + O(q^8) \text{ in } S_2(41, \chi_{39}^{14}) \]
over \(Q|\zeta_3)(b_0|/(b_0^3 + (\zeta_3 + 2\zeta_3 + 1) b_0 + 2\zeta_3^6 - \zeta_3^4 + \zeta_3^2 + \zeta_3 + 2)\)
over \( \mathbb{Q}[b_0]/(b_0^3 + 3b_0^2 - 5b_0 - 3) \)

**References**

[ADP02] Avner Ash, Darrin Doud, and David Pollack, *Galois representations with conjectural connections to arithmetic cohomology*, Duke Math. J. 112 (2002), no. 3, 521–579.

[AGM02] Avner Ash, Paul E. Gunnells, and Mark McConnell, *Cohomology of congruence subgroups of \( SL_4(\mathbb{Z}) \)*, J. Number Theory 94 (2002), no. 1, 181–212.

[AGM08] _____, *Cohomology of congruence subgroups of \( SL_4(\mathbb{Z}) \). II*, J. Number Theory 128 (2008), no. 8, 2263–2274.

[AGM10] _____, *Cohomology of congruence subgroups of \( SL_4(\mathbb{Z}) \). III*, Math. Comp. 79 (2010), no. 271, 1811–1831.

[AGM11] _____, *Torsion in the cohomology of congruence subgroups of \( SL(4,\mathbb{Z}) \) and Galois representations*, J. Algebra 325 (2011), 404–415.

[AGM12] _____, *Resolutions of the Steinberg module for \( GL(n) \)*, J. Algebra 349 (2012), 380–390.

[AGM15] _____, *Mod \( 2 \) homology for \( GL(4) \) and Galois representations*, J. Number Theory 146 (2015).

[AS86] Avner Ash and Glenn Stevens, *An analogue of Serre’s conjecture for Galois representations and Hecke eigenclasses in the mod \( p \) cohomology of \( GL(n,\mathbb{Z}) \)*, J. reine u. angew. Math. (1986), 192–220.

[Ash92] Avner Ash, *Galois representations attached to mod \( p \) cohomology of \( GL(n,\mathbb{Z}) \)*, Duke Math. J. 65 (1992), 235–255.

[CLH16] Ana Caraiani and Bao V. Le Hung, *On the image of complex conjugation in certain galois representations*, Compositio Math. 152 (2016), no. 7, 14761488.

[Dev17] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 8.0)*, 2017, www.sagemath.org.

[Gun00] Paul E. Gunnells, *Computing Hecke eigenvalues below the cohomological dimension*, Experiment. Math. 9 (2000), no. 3, 351–367.

[Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame \( n \)-dimensional Galois representations*, Duke Math. J. 149 (2009), no. 1, 37–116.

[HLTT13] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne, *On the rigid cohomology of certain Shimura varieties*.

[McC91] Mark McConnell, *Classical projective geometry and arithmetic groups*, Math. Annalen 290 (1991), 441–462.

[Sch15] Peter Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) 182 (2015), no. 3, 945–1066.

[Shi94] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[Vor08] Georges Voronoï, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques*, J. reine u. angew. Math. 133 (1908), 97–178.

Boston College, Chestnut Hill, MA 02445

E-mail address: Avner.Ash@bc.edu

University of Massachusetts Amherst, Amherst, MA 01003

E-mail address: gunnells@math.umass.edu

Princeton University, Princeton, New Jersey 08540

E-mail address: markwm@princeton.edu