SOLVING THE ODD PERFECT NUMBER PROBLEM:
SOME OLD AND NEW APPROACHES

A Thesis
Presented to
The Faculty of the Mathematics Department
College of Science
De La Salle University - Manila

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Mathematics

by
Jose Arnaldo B. Dris

August 2008
Acknowledgments

The author of this thesis wishes to express his heartfelt gratitude to the following:

• The Mathematics Department, DLSU-Manila
• The Commission on Higher Education - Center of Excellence
• Dr. Severino V. Gervacio
• Dr. Leonor Aquino-Ruivivar
• Dr. Fidel Nemenzo
• Ms. Sonia Y. Tan
• Dr. Blessilda Raposa
• Dr. Ederlina Nocon
• Ms. Gladis Habijan
• Dr. Isagani B. Jos
• Dr. Arlene Pascasio
• Dr. Jose Tristan Reyes
• Dr. Yvette F. Lim
• Mr. Frumencio Co
• Dr. Julius Basilla
• Dr. Rizaldi Nocon
• Dr. John McCleary
• Dr. Carl Pomerance
• Dr. Douglas Iannucci
• Dr. Judy Holdener
• Prof. Richard P. Brent
• Prof. Richard F. Ryan
• Ms. Laura Czarnecki
• Mr. William Stanton
• Mr. James Riggs
• Mr. Tim Anderton
• Mr. Dan Staley
• Mr. William Lipp
• Mr. Tim Roberts
• Mr. Rigor Ponsones
• Mr. Gareth Paglinawan

• Mr. Christopher Thomas Cruz
• Ms. Michele Tan
• Mr. Mark Anthony Garcia
• Mr. John Ruero
• Mr. Vincent Chuaseco
• Mrs. Abigail Arcilla
• Mr. Mark John Hermano
Table of Contents

Title Page i

Acknowledgments ii

Table of Contents iv

List of Notations vi

Abstract vii

1 The Problem and Its Background 1

1.1 Introduction 1

1.1.1 Statement of the Problem 5

1.1.2 Review of Related Literature 8

2 Preliminary Concepts 14

2.1 Concepts from Elementary Number Theory 14

2.2 The Abundancy Index 25

2.3 Even Perfect Numbers 28

2.4 Odd Perfect Numbers 32
3 OPN Solution Attempts 1:

Some Old Approaches

3.1 Increasing the Lower Bound for $\omega(N)$ ................................. 36
3.2 Increasing the Lower Bound for an OPN $N$ ................................. 41
3.3 Congruence Conditions for an OPN $N$ ................................. 48
3.4 Some Interesting Results on Perfect Numbers ......................... 53
   3.4.1 Nonexistence of Consecutive Perfect Numbers ................. 53
   3.4.2 OPNs are Not Divisible by 105 ................................. 55
   3.4.3 OPNs as Sums of Two Squares ................................. 56

4 OPN Solution Attempts 2:

Some New Approaches

4.1 Abundancy Outlaws and Related Concepts ................................. 60
   4.1.1 Friendly and Solitary Numbers ................................ 60
   4.1.2 Abundancy Indices and Outlaws ................................. 63
   4.1.3 OPNs, Abundancy Outlaws and the Fraction $\frac{p+2}{p}$ .... 72
4.2 Bounds for the Prime Factors of OPNs ................................. 80
   4.2.1 Results on OPNs .................................................. 81
   4.2.2 Algorithmic Implementation of Factor Chains ................. 82
   4.2.3 Explicit Double-Sided Bounds for the Prime Factors ........ 91
   4.2.4 Relationships Between OPN Components ........................ 98
4.3 “Counting” the Number of OPNs ................................. 113

5 Analysis and Recommendations ................................. 117
**List of Notations**

\( \mathbb{N}, \mathbb{Z}^+ \) the set of all natural numbers/positive integers

\( a \mid b \) \( a \) divides \( b \), \( a \) is a divisor/factor of \( b \), \( b \) is a multiple of \( a \)

\( P^\alpha \| N \) \( P^\alpha \) is the largest power of \( P \) that divides \( N \), i.e. \( P^\alpha \mid N \) but \( P^{\alpha+1} \nmid N \)

\( a \equiv b \pmod{n} \) \( a \) is congruent to \( b \) modulo \( n \)

\( \gcd(a, b) \) the greatest common divisor of \( a \) and \( b \)

\( d(n) \) the number of positive divisors of \( n \)

\( \sigma(n) \) the sum of the positive divisors of \( n \)

\( \phi(n) \) the number of positive integers less than or equal to \( n \) which are also relatively prime to \( n \)

\( \omega(n) \) the number of distinct primes that divide \( n \)

\( \Omega(n) \) the number of primes that divide \( n \), counting multiplicities

\( \sigma_{-1}(n), I(n) \) the abundancy index of \( n \), i.e. the sum of the reciprocals of all the positive divisors of \( n \)

\( \prod_{i=1}^r p_i^{\alpha_i} \) the product \( p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r} \)

\( \sum_{j=0}^{r} q_i^j \) the sum \( 1 + q_i + q_i^2 + q_i^3 + \ldots + q_i^{s-1} + q_i^s \)
Abstract

A perfect number is a positive integer $N$ such that the sum of all the positive divisors of $N$ equals $2N$, denoted by $\sigma(N) = 2N$. The question of the existence of odd perfect numbers (OPNs) is one of the longest unsolved problems of number theory. This thesis presents some of the old as well as new approaches to solving the OPN Problem. In particular, a conjecture predicting an injective and surjective mapping $X = \frac{\sigma(p^k)}{p^k}$, $Y = \frac{\sigma(m^2)}{m^2}$ between OPNs $N = p^k m^2$ (with Euler factor $p^k$) and rational points on the hyperbolic arc $XY = 2$ with $1 < X < 1.25 < 1.6 < Y < 2$ and $2.85 < X + Y < 3$, is disproved. Various results on the abundancy index and solitary numbers are used in the disproof. Numerical evidence against the said conjecture will likewise be discussed. We will show that if an OPN $N$ has the form above, then $p^k < \frac{2}{3} m^2$ follows from [15]. We will also attempt to prove a conjectured improvement of this last result to $p^k < m$ by observing that $\frac{\sigma(p^k)}{p^k} \neq 1$ and $\frac{\sigma(p^k)}{m^2} \neq \frac{\sigma(m)}{p^k}$ in all cases. Lastly, we also prove the following generalization: If $N = \prod_{i=1}^{r} p_i^{\alpha_i}$ is the canonical factorization of an OPN $N$, then $\sigma(p_i^{\alpha_i}) \leq \frac{2}{3} \frac{N}{p_i^{\alpha_i}}$ for all $i$. This gives rise to the inequality $N^{2-r} \leq (\frac{1}{3})(\frac{2}{3})^{r-1}$, which is true for all $r$, where $r = \omega(N)$ is the number of distinct prime factors of $N$. 
Chapter 1

The Problem and Its Background

1.1 Introduction

Number theory is that branch of pure mathematics concerned with the properties of integers. It consists of various results and open problems that are easily understood, even by non-mathematicians. More generally, the field has developed to a point where it could now tackle wider classes of problems that arise naturally from the study of integers.

A particular example of an unsolved problem in number theory, one which has long captured the interest of both amateur and professional mathematicians, is to determine whether an odd perfect number exists. A positive integer \( n \) which is equal to the sum of its positive proper divisors (that is, excluding \( n \) itself) is called a perfect number. The smallest known example of a perfect number is 6 since \( 1 + 2 + 3 = 6 \), where 1, 2 and 3 are the positive proper divisors of 6. The Pythagoreans considered such numbers to possess mystical properties, thus calling them perfect numbers.
The sum of all positive divisors of a positive integer $n$ is called the sigma function of $n$, denoted by $\sigma(n)$. The definition of a perfect number is thus equivalent to determining those $n$ for which $\sigma(n) - n = n$. We formalize this definition as follows:

**Definition 1.1.1.** A positive integer $N$ is perfect if $\sigma(N) = 2N$.

For example, $\sigma(6) = 1 + 2 + 3 + 6 = 12 = 2(6)$.

Even though the Greeks knew of only four perfect numbers, it was Euclid who proved that if the sum $1 + 2 + 2^2 + \ldots + 2^{p-2} + 2^{p-1} = 2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect. Consider the sum $1 + 2 = 3$ as an example. Since 3 is prime, then $2^{2-1}(2^2 - 1) = 2(3) = 6$ is perfect.

Euler subsequently proved the following theorem about (even) perfect numbers:

**Theorem 1.1.1.** Every even perfect number is of the form $N = 2^{p-1}(2^p - 1)$, where $P$ and $2^P - 1$ are primes.

It is easy to prove that if $C$ is composite, then $2^C - 1$ is also composite. However, if $P$ is prime, it does not necessarily follow that $2^P - 1$ is also prime. (Consider the case $P = 11 : 2^{11} - 1 = 2047 = 23 \cdot 89$, which is composite.) Primes of the form $2^P - 1$ are called *Mersenne primes*, after the French monk and mathematician Marin Mersenne. In view of Theorem 1.1.1, the problem of searching for even perfect numbers is thus reduced to looking for Mersenne primes,
since the theorem essentially says that the even perfect numbers are in one-to-one correspondence with the Mersenne primes.

As of this writing, 44 perfect numbers are known [70], the last few of which have been found with the aid of high-speed computers. Following is a list of all the known exponents \( p \) for which \( M_p = 2^p - 1 \) is prime, along with other pertinent information (note that \( P_p \) refers to the perfect number \( N = 2^{p-1}(2^p - 1) \)):

| number | \( p \) (exponent) | digits in \( M_p \) | digits in \( P_p \) | year | discoverer |
|--------|---------------------|---------------------|---------------------|------|------------|
| 1      | 2                   | 1                   | 1                   |      |            |
| 2      | 3                   | 1                   | 2                   |      |            |
| 3      | 5                   | 2                   | 3                   |      |            |
| 4      | 7                   | 3                   | 4                   |      |            |
| 5      | 13                  | 4                   | 8                   | 1456 | anonymous |
| 6      | 17                  | 6                   | 10                  | 1588 | Cataldi    |
| 7      | 19                  | 6                   | 12                  | 1588 | Cataldi    |
| 8      | 31                  | 10                  | 19                  | 1772 | Euler      |
| 9      | 61                  | 19                  | 37                  | 1883 | Pervushin  |
| 10     | 89                  | 27                  | 54                  | 1911 | Powers     |
| 11     | 107                 | 33                  | 65                  | 1914 | Powers     |
| 12     | 127                 | 39                  | 77                  | 1876 | Lucas      |
| 13     | 521                 | 157                 | 314                 | 1952 | Robinson   |
| 14     | 607                 | 183                 | 366                 | 1952 | Robinson   |
| 15     | 1279                | 386                 | 770                 | 1952 | Robinson   |
| number | $p$ (exponent) | digits in $M_p$ | digits in $P_p$ | year | discoverer                  |
|--------|----------------|-----------------|-----------------|------|-----------------------------|
| 16     | 2203           | 664             | 1327            | 1952 | Robinson                    |
| 17     | 2281           | 687             | 1373            | 1952 | Robinson                    |
| 18     | 3217           | 969             | 1937            | 1957 | Riesel                      |
| 19     | 4253           | 1281            | 2561            | 1961 | Hurwitz                     |
| 20     | 4423           | 1332            | 2663            | 1961 | Hurwitz                     |
| 21     | 9689           | 2917            | 5834            | 1963 | Gillies                     |
| 22     | 9941           | 2993            | 5985            | 1963 | Gillies                     |
| 23     | 11213          | 3376            | 6751            | 1963 | Gillies                     |
| 24     | 19937          | 6002            | 12003           | 1971 | Tuckerman                   |
| 25     | 21701          | 6533            | 13066           | 1978 | Noll, Nickel                |
| 26     | 23209          | 6987            | 13973           | 1979 | Noll                        |
| 27     | 44497          | 13395           | 26790           | 1979 | Nelson, Slowinski           |
| 28     | 86243          | 25962           | 51924           | 1982 | Slowinski                   |
| 29     | 110503         | 33265           | 66530           | 1988 | Colquitt, Welsh             |
| 30     | 132049         | 39751           | 79502           | 1983 | Slowinski                   |
| 31     | 216091         | 65050           | 130100          | 1985 | Slowinski                   |
| 32     | 756839         | 227832          | 455663          | 1992 | Slowinski, Gage et al.      |
| 33     | 859433         | 258716          | 517430          | 1994 | Slowinski, Gage             |
| 34     | 1257787        | 378632          | 757263          | 1996 | Slowinski, Gage             |
| 35     | 1398269        | 420921          | 841842          | 1996 | Armengaud, Woltman, et al.  |
Question marks (??) were used instead of a number for the the last of the Mersenne primes because it will not be known if there are other Mersenne primes in between these until a check and double check is done by the Great Internet Mersenne Prime Search (GIMPS) \[70\] or other similar coordinated computing projects.

Note that all of the known perfect numbers are even.

1.1.1 Statement of the Problem

Our primary object of interest would be odd perfect numbers, though some of the results that would be discussed apply to even perfect numbers as well. It is unknown whether there are any odd perfect numbers. Various results have been obtained, but none has helped to locate one or otherwise resolve the question of their existence.
This thesis explores some of the old as well as new approaches used in trying to solve the Odd Perfect Number (OPN) Problem, namely:

• the use of the abundancy index to derive conditions for the existence of odd perfect numbers, in such instances as:
  
  – bounding the prime factors of an OPN;
  
  – determining whether a particular rational number may be an abundancy index of a positive integer;
  
  – increasing the lower bound for the number of distinct prime factors, \( \omega(N) \), that an OPN \( N \) must have;

• proving the inequality \( p^k < \frac{2}{3}m^2 \) where \( N = p^k m^2 \) is an OPN with \( p^k \) the Euler factor of \( N \), \( p \equiv k \equiv 1 \pmod{4} \), and \( \gcd(p, m) = 1 \);

• attempting to prove the conjectured improvement of the result \( p^k < \frac{2}{3}m^2 \) to \( p^k < m \) by observing that \( \frac{\sigma(p^k)}{m} \neq 1 \) and \( \frac{\sigma(m)}{p^k} \neq \frac{\sigma(p^k)}{m} \) apply in all cases;

• generalizing the result \( p^k < \frac{2}{3}m^2 \) to: if \( N = \prod_{i=1}^{r} p_i^{\alpha_i} \) is the canonical factorization of an OPN \( N \), then \( \sigma(p_i^{\alpha_i}) \leq \frac{2}{3} \frac{N}{p_i^{\alpha_i}} \) for all \( i \);

• showing that \( N^{2-r} \leq \left( \frac{1}{3} \right) \left( \frac{2}{3} \right)^{r-1} \) follows from \( \sigma(p_i^{\alpha_i}) \leq \frac{2}{3} \frac{N}{p_i^{\alpha_i}} \) for all \( i \), where \( r = \omega(N) \) is the number of distinct prime factors of an OPN \( N \);

• disproving the conjectured injectivity and surjectivity of the mapping
  
  \[ X = \frac{\sigma(p^k)}{p^k}, \quad Y = \frac{\sigma(m^2)}{m^2} \]

  between OPNs \( N = p^k m^2 \) (with Euler factor \( p^k \)) and rational points on the hyperbolic arc \( XY = 2 \) with

  \[ 1 < X < 1.25 < 1.6 < Y < 2 < 2.85 < X + Y < 3; \]
• a host of other interesting results on perfect numbers, including:
  
  – establishing that two consecutive positive integers cannot be both perfect;
  
  – the proof that an OPN is a sum of two squares.

Specifically, this thesis, among other things, aims to present (partial) expositions of the following articles/notes that deal with some of the above concerns:

• “Abundancy 'Outlaws' of the Form $\frac{\sigma(N)+4}{N}$” by W. G. Stanton, which was a joint research project with J. A. Holdener at Kenyon College, Gambier, OH (2007);

• “Conditions Equivalent to the Existence of Odd Perfect Numbers” by J. A. Holdener, which appeared in Mathematics Magazine 79(5) (2006);

• “Bounding the Prime Factors of Odd Perfect Numbers”, an undergraduate paper of C. Greathouse at Miami University (2005);

• “Hunting Odd Perfect Numbers: Quarks or Snarks?” by J. McCleary of Vassar College, Poughkeepsie, NY, which consists of lecture notes first presented as a seminar to students of Union College, Schenectady, NY (2001);

• “Consecutive Perfect Numbers (actually, the Lack Thereof!”) by J. Riggs, which was an undergraduate research project with J. A. Holdener at Kenyon College (1998).
1.1.2 Review of Related Literature

Benjamin Peirce was the first to prove (in 1832) that an OPN $N$ must have at least four distinct prime factors (denoted $\omega(N) \geq 4$) \textsuperscript{53}. Seemingly unaware of this result, James Joseph Sylvester published a paper on the same result in 1888, hoping that research along these lines would pave the way for a general proof of the nonexistence of an OPN \textsuperscript{68}. Later that same year, Sylvester established $\omega(N) \geq 5$ \textsuperscript{69}. This marked the beginning of the modern era of research on OPNs. It was only in 1925 that Gradstein was able to improve Sylvester’s result to $\omega(N) \geq 6$ \textsuperscript{19}. In the early 1970s, Robbins and Pomerance independently established that $\omega(N) \geq 7$ \textsuperscript{55}. Then, Chein demonstrated that $\omega(N) \geq 8$ in his 1979 doctoral thesis \textsuperscript{6}, which was verified independently by Hagis in a 1980 200-page manuscript \textsuperscript{23}. Recently, in a 2006 preprint titled “Odd perfect numbers have at least nine distinct prime factors”, Pace Nielsen was able to prove that $\omega(N) \geq 9$, and if $N_2$ is an OPN which is not divisible by 3, that $\omega(N_2) \geq 12$. “The proof ultimately avoids previous computational results for odd perfect numbers.” \textsuperscript{51}

Sylvester also showed (later in the year 1888) that an OPN cannot be divisible by 105 \textsuperscript{69}, and Servais (in the same year) proved that the least prime divisor of a perfect number with $r$ distinct prime factors is bounded above by $r + 1$ \textsuperscript{62}. Grün \textsuperscript{22}, Cohen and Hendy \textsuperscript{9}, and McDaniel \textsuperscript{49} established improvements and extensions to this last result later on.
Dickson showed in 1913 that there can only be finitely many OPNs with \( r \) distinct prime factors, for a given positive integer \( r \) \[16\]. Kanold then generalized Dickson’s theorem in 1956 to include any positive integer \( n \) satisfying \( \frac{\sigma(n)}{n} = \frac{a}{b} \), where \( a \) and \( b \) are positive integers and \( b \neq 0 \). \[13\]

(Call a number \( n \) non-deficient if \( \sigma(n) \geq 2 \). Dickson called a number "primitive non-deficient" provided that it is not a multiple of a smaller non-deficient number.) Mathematicians considered Dickson’s approach (i.e. first delineating all of the finitely many primitive odd non-deficient numbers associated with a particular \( r \)-value and then determining which among them are equal to the sum of their positive proper divisors) to the OPN question to be impractical for most values of \( r \), making it necessary to explore alternative approaches to examining the possible structure of an OPN. Pomerance suggested the following class of theorems in 1974: An OPN is divisible by \( j \) distinct primes > \( N \) \[55\]. Kanold was successful with \( j = 1, N = 60 \) in 1949 and used only elementary techniques \[11\]. In 1973, with the aid of computation, Hagis and McDaniel improved Kanold’s finding to \( j = 1, N = 11200 \) \[25\]. This was pushed to \( j = 1, N = 100110 \) by the same authors in 1975 \[26\]. Pomerance showed that \( j = 2, N = 138 \) in the same year. \[56\]

In a recent preprint (titled “Odd perfect numbers have a prime factor exceeding \( 10^8 \)” that appeared 2006, authors Takeshi Goto and Yasuo Ohno report that the largest prime factor of an OPN exceeds \( 10^8 \) \[18\]. It betters the previous bound of \( 10^7 \) established by Jenkins in 2003 \[39\]. New bounds for the second and third largest prime factors of an OPN were given by Iannucci in 1999 \[36\] and 2000
when he announced that they are larger than $10^4$ and $10^2$, respectively.

Mathematicians also began considering estimates on the overall magnitude of an OPN by imposing lower bounds. Turcaninov obtained the classical lower bound of $2 \cdot 10^6$ in 1908 [5]. The following table summarizes the development of ever-higher bounds for the smallest possible odd perfect number:

| Author                  | Bound     |
|-------------------------|-----------|
| Kanold (1957)           | $10^{20}$ |
| Tuckerman (1973)        | $10^{36}$ |
| Hagis (1973)            | $10^{50}$ |
| Brent and Cohen (1989)  | $10^{160}$|
| Brent et al. (1991)     | $10^{300}$|

There is a project underway at [http://www.oddperfect.org](http://www.oddperfect.org) (organized by William Lipp) seeking to extend the bound beyond $10^{300}$. A proof for $10^{500}$ is expected very soon, as all the remaining factorizations required to show this are considered “easy”, by Lipp’s standards. [21]

It would also be possible to derive upper bounds on the overall size of an OPN in terms of the number of its distinct prime factors. Heath-Brown was able to show, in 1994, that if $n$ is an odd number with $\sigma(n) = an$, then $n < (4d)^{4r}$, where $d$ is the denominator of $a$ and $r$ is the number of distinct prime factors of $n$ [30]. Specifically, this means that for an OPN $n$, $n < 4^{4r}$ which sharpens Pomerance’s previous estimate of $n < (4r)^{(4r)^2r^2}$ in 1977 [56]. Referring to his own finding, Heath-Brown remarked that it still is too big to be of practical value.
Nonetheless, it is to be noted that if it is viewed alongside the lower bound of $10^{300}$ given by Brent et al. \cite{4}, then Sylvester’s 1888 result that $\omega(n) \geq 5$ could then be demonstrated by no longer than a footnote. \footnote{In 1999, Cook enhanced Heath-Brown’s result for an OPN with $r$ distinct prime factors to $n < D^{4r}$ where $D = (195)^{1/7} \approx 2.124$ \cite{11}. In 2003, Pace Nielsen refined Cook’s bound to $n < 2^{4r}$ \cite{50}.}

In 1999, Cook enhanced Heath-Brown’s result for an OPN with $r$ distinct prime factors to $n < D^{4r}$ where $D = (195)^{1/7} \approx 2.124$ \cite{11}. In 2003, Pace Nielsen refined Cook’s bound to $n < 2^{4r}$ \cite{50}.

Addressing the OPN question from a congruence perspective on the allowable exponents for the non-Euler prime factors, Steuerwald showed in 1937 that if

$$n = p^\alpha q_1^{2\beta_1} q_2^{2\beta_2} \cdots q_s^{2\beta_s}$$

was an OPN where $p, q_1, q_2, \ldots, q_s$ are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod{4}$, then not all of the $\beta_i$’s can equal 1 \cite{65}. Further, Kanold discovered in 1941 that it is neither possible for all $\beta_i$’s to equal 2 nor for one of the $\beta_i$’s to be equal to 2 while all the rest are equal to 1 \cite{40}. Hagis and McDaniel proved in 1972 that not all the $\beta_i$’s can be equal to 3 \cite{24}. Then in 1985, Cohen and Williams summarized all previous work done on this area by eliminating various possibilities for the $\beta_i$’s, on the assumption that either some or all of the $\beta_i$’s are the same \cite{10}.

In 2003, Iannucci and Sorli placed restrictions on the $\beta_i$’s in order to show that $3$ cannot divide an OPN if, for all $i$, $\beta_i \equiv 1 \pmod{3}$ or $\beta_i \equiv 2 \pmod{5}$. They also provided a slightly different analysis by giving a lower bound of 37 on the total number of prime divisors (counting multiplicities) that an OPN must have.
(i.e. they proved that if \( n = p^\alpha \prod_{i=1}^{s} q_i^{2\beta_i} \) is an OPN, then \( \Omega(n) = \alpha + 2 \sum_{i=1}^{s} \beta_i \geq 37 \) \[38\]. This was extended by Hare later in the year 2003 to \( \Omega(n) \geq 47 \) \[28\]. In 2005, Hare submitted the preprint titled “New techniques for bounds on the total number of prime factors of an odd perfect number” to the journal Mathematics of Computation for publication, where he announced a proof for \( \Omega(n) \geq 75 \) \[29\].

In order to successfully search for perfect numbers, it was found necessary to consider a rather interesting quantity called the abundancy index or abundancy ratio of \( n \), defined to be the quotient \( \frac{\sigma(n)}{n} \). Obviously, a number \( n \) is perfect if and only if its abundancy index is 2. Numbers for which this ratio is greater than (less than) 2 are called abundant (deficient) numbers.

It can be shown that the abundancy index takes on arbitrarily large values. Also, we can make the abundancy index to be as close to 1 as we please because \( \frac{\sigma(p)}{p} = \frac{p+1}{p} \) for all primes \( p \). In fact, Laatsch showed in 1986 that the set of abundancy indices \( \frac{\sigma(n)}{n} \) for \( n > 1 \) is dense in the interval \((1, \infty)\) \[45\]. (Let \( I(n) = \frac{\sigma(n)}{n} \), and call a rational number greater than 1 an abundancy outlaw if it fails to be in the image of the function \( I \).) Interestingly, Weiner proved that the set of abundancy outlaws is also dense in \((1, \infty)\)! \[73\] It appears then that the implicit scenarios for abundancy indices and outlaws are both complex and interesting.

In 2006, Cruz \[12\] completed his M. S. thesis titled “Searching for Odd Perfect Numbers” which contained an exposition of the results of Heath-Brown
Cruz also proposed a hypothesis that may lead to a disproof of the existence of OPNs.
Chapter 2

Preliminary Concepts

The concept of divisibility plays a central role in that branch of pure mathematics called the theory of numbers. Indeed, mathematicians have used divisibility and the concept of unique factorization to establish deep algebraic results in number theory and related fields where it is applied. In this chapter, we survey some basic concepts from elementary number theory, and use these ideas to derive the possible forms for even and odd perfect numbers.

2.1 Concepts from Elementary Number Theory

For a better understanding of the topics presented in this thesis, we recall the following concepts.

Definition 2.1.1. An integer $n$ is said to be divisible by a nonzero integer $m$, denoted by $m \mid n$, if there exists some integer $k$ such that $n = km$. The notation $m \nmid n$ is used to indicate that $n$ is not divisible by $m$.

For example, 143 is divisible by 11 since $143 = 11 \cdot 13$. In this case, we also say that 11 and 13 are divisors/factors of 143, and that 143 is a multiple of 11.
(and of 13). On the other hand, 143 is not divisible by 3 since we will not be able to find an integer \( k \) that will make the equation \( 3k = 143 \) true.

If \( n \) is divisible by \( m \), then we also say that \( m \) divides \( n \).

We list down several properties of divisibility in Theorem 2.1.1.

**Theorem 2.1.1.** For integers \( k, l, m, \) and \( n \), the following are true:

- \( n \mid 0, 1 \mid n, \) and \( n \mid n \). (Any integer is a divisor of 0, 1 is a divisor of any integer, and any integer has itself as a divisor.)

- \( m \mid 1 \) if and only if \( m = \pm 1 \). (The only divisors of 1 are itself and \(-1\).)

- If \( k \mid m \) and \( l \mid n \), then \( kl \mid mn \). (Note that this statement is one-sided.)

- If \( k \mid l \) and \( l \mid m \), then \( k \mid m \). (This means that divisibility is transitive.)

- \( m \mid n \) and \( n \mid m \) if and only if \( m = \pm n \). (Two integers which divide each other can only differ by a factor of \( \pm 1 \).)

- If \( m \mid n \) and \( n \neq 0 \), then \( |m| \leq |n| \). (If the multiple of an integer is nonzero, then the multiple has bigger absolute value than the integer.)

- If \( k \mid m \) and \( k \mid n \), then \( k \mid (am + bn) \) for any integers \( a \) and \( b \). (If an integer divides two other integers, then the first integer divides any linear combination of the second and the third.)
A very useful concept in the theory of numbers is that of the GCF or GCD of two integers.

**Definition 2.1.2.** Let $m$ and $n$ be any given integers such that at least one of them is not zero. The greatest common divisor of $m$ and $n$, denoted by $\gcd(m, n)$, is the positive integer $k$ which satisfies the following properties:

- $k \mid m$ and $k \mid n$; and

- If $j \mid m$ and $j \mid n$, then $j \mid k$.

**Example 2.1.1.** The positive divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36. For 81 they are 1, 3, 9, 27 and 81. Thus, the positive divisors common to 36 and 81 are 1, 3 and 9. Since 9 is the largest among the common divisors of 36 and 81, then $\gcd(36, 81) = 9$.

Another concept of great utility is that of two integers being relatively prime.

**Definition 2.1.3.** Let $m$ and $n$ be any integers. If $\gcd(m, n) = 1$, then $m$ and $n$ are said to be relatively prime, or coprime.

**Example 2.1.2.** Any two consecutive integers (like 17 and 16, or 8 and 9) are relatively prime. Note that the two consecutive integers are of opposite parity (i.e. one is odd, the other is even). If two integers are of opposite parity, but not consecutive, it does not necessarily follow that they are relatively prime. (See Example 2.1.1)

It turns out that the concept of divisibility can be used to partition the set of positive integers into three classes: the unit 1, primes and composites.
Definition 2.1.4. An integer $P > 1$ is called a *prime number*, simply a *prime*, if it has no more positive divisors other than 1 and $P$. An integer greater than 1 is called a *composite number*, simply a *composite*, if it is not a prime.

There are only 23 primes in the range from 1 to 100, as compared to 76 composites in the same range. Some examples of primes in this range include 2, 7, 23, 31, 41, and 47. The composites in the same range include all the larger multiples of the aforementioned primes, as well as product combinations of two or more primes from the range 1 to 10. (Note that we get 100 by multiplying the two composites 10 and 10.) We casually remark that the integer 2 is the only even prime. The integer 1, by definition, is neither prime nor composite. We shall casually call 1 the *unit*.

From the preceding discussion, we see that the set of prime numbers is *not closed* with respect to multiplication, in the sense that multiplying two prime numbers gives you a composite. On the other hand, the set of composite numbers is closed under multiplication. On further thought, one can show that both sets are not closed under addition. (It suffices to consider the counterexamples $2 + 7 = 9$ and $4 + 9 = 13$.)

If $P^\alpha$ is the largest power of a prime $P$ that divides an integer $N$, i.e. $P^\alpha | N$ but $P^{\alpha+1} \nmid N$, then this is denoted by $P^\alpha || N$.

We now list down several important properties of prime numbers as they relate to divisibility.
Theorem 2.1.2. If $P$ is a prime and $P | mn$, then either $P | m$ or $P | n$.

Corollary 2.1.1. If $P$ is a prime number and $P | m_1m_2 \cdots m_n$, then $P | m_i$ for some $i, 1 \leq i \leq n$.

Corollary 2.1.2. If $P, Q_1, Q_2, \ldots, Q_n$ are all primes and $P | Q_1Q_2 \cdots Q_n$, then $P = Q_i$ for some $i, 1 \leq i \leq n$.

All roads now lead to the Fundamental Theorem of Arithmetic.

Theorem 2.1.3. Fundamental Theorem of Arithmetic

Every positive integer $N > 1$ can be represented uniquely as a product of primes, apart from the order in which the factors occur.

The “lexicographic representation” of a positive integer as a product of primes may be achieved via what is called the canonical factorization.

Corollary 2.1.3. Any positive integer $N > 1$ can be written uniquely in the canonical factorization

$$N = P_1^{\alpha_1}P_2^{\alpha_2} \cdots P_r^{\alpha_r} = \prod_{i=1}^{r} P_i^{\alpha_i}$$

where, for $i = 1, 2, \ldots, r$, each $\alpha_i$ is a positive integer and each $P_i$ is a prime, with $P_1 < P_2 < \ldots < P_r$.

We illustrate these with some examples.

Example 2.1.3. The canonical factorization of the integer 36 is $36 = 2^2 \cdot 3^2$. Meanwhile, the canonical factorization for the integer 1024 is $1024 = 2^{10}$, while for 2145 it is $2145 = 3^1 \cdot 5^1 \cdot 11^1 \cdot 13^1$, written simply as $2145 = 3 \cdot 5 \cdot 11 \cdot 13$. 
Functions which are defined for all positive integers \( n \) are called *arithmetic functions*, or *number-theoretic functions*, or *numerical functions*. Specifically, a *number-theoretic function* \( f \) is one whose domain is the positive integers and whose range is a subset of the complex numbers.

We now define three important number-theoretic functions.

**Definition 2.1.5.** Let \( n \) be a positive integer. Define the number-theoretic functions \( d(n), \sigma(n), \phi(n) \) as follows:

\[
d(n) = \text{the number of positive divisors of } n,
\]

\[
\sigma(n) = \text{the sum of the positive divisors of } n,
\]

\[
\phi(n) = \text{the number of positive integers at most } n \text{ which are also relatively prime to } n.
\]

It would be good to illustrate with some examples.

**Example 2.1.4.** Consider the positive integer \( n = 28 \). Since the positive divisors of 28 are 1, 2, 4, 7, 14 and 28, then by definition:

\[
d(28) = 6
\]

and

\[
\sigma(28) = \sum_{d|28} d = 1 + 2 + 4 + 7 + 14 + 28 = 56.
\]

Note that the following list contains all the positive integers less than or equal to \( n = 28 \) which are also relatively prime to \( n \): \( L = \{1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27\} \).

By definition, \( \phi(28) = 12 \).
For the first few integers,

\[ d(1) = 1 \quad d(2) = 2 \quad d(3) = 2 \quad d(4) = 3 \quad d(5) = 2 \quad d(6) = 4 \]

while

\[ \sigma(1) = 1 \quad \sigma(2) = 3 \quad \sigma(3) = 4 \quad \sigma(4) = 7 \quad \sigma(5) = 6 \quad \sigma(6) = 12 \]

and

\[ \phi(1) = 1 \quad \phi(2) = 1 \quad \phi(3) = 2 \quad \phi(4) = 2 \quad \phi(5) = 4 \quad \phi(6) = 3. \]

Note that the functions \( d(n), \sigma(n), \) and \( \phi(n) \) are not monotonic, and their functional values at \( n = 1 \) is also 1. Also, for at least the first 3 primes \( p = 2, 3 \) and 5, \( d(p) = 2, \sigma(p) = p + 1, \) and \( \phi(p) = p - 1. \)

We shall now introduce the notion of a multiplicative number-theoretic function.

**Definition 2.1.6.** A function \( F \) defined on \( \mathbb{N} \) is said to be multiplicative if for all \( m, n \in \mathbb{N} \) such that \( \gcd(m, n) = 1 \), we have

\[ F(mn) = F(m)F(n). \]

**Example 2.1.5.** Let the function \( F \) be defined by \( F(n) = n^k \) where \( k \) is a fixed positive integer. Then \( F(mn) = (mn)^k = m^k n^k = F(m)F(n) \). We have therefore shown that \( F \) is multiplicative. Moreover, the condition \( \gcd(m, n) = 1 \) is not even required for the series of equalities above to hold. We call \( F \) in this example a totally multiplicative function.
It turns out that the three number-theoretic functions we introduced in Definition 2.1.5 provide us with more examples of multiplicative functions.

**Theorem 2.1.4.** The functions $d, \sigma$ and $\phi$ are multiplicative functions.

Multiplicative functions are completely determined by their values at prime powers. Given a positive integer $n$’s canonical factorization

$$n = \prod_{i=1}^{r} P_i^{\alpha_i},$$

then if $F$ is a multiplicative function, we have

$$F(n) = \prod_{i=1}^{r} F(P_i^{\alpha_i}).$$

This last assertion follows from the fact that prime powers derived from the canonical factorization of $n$ are pairwise relatively prime.

The next theorem follows from Definition 2.1.5 and Theorem 2.1.4 as well.

**Theorem 2.1.5.** If $n = \prod_{i=1}^{r} P_i^{\alpha_i}$ is the canonical factorization of $n > 1$, then

$$d(n) = \prod_{i=1}^{r} (\alpha_i + 1),$$

$$\sigma(n) = \prod_{i=1}^{r} \sigma(P_i^{\alpha_i}) = \prod_{i=1}^{r} \left( \frac{P_i^{\alpha_i+1} - 1}{P_i - 1} \right),$$

$$\phi(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{P_i} \right).$$
We illustrate with several examples, continuing from Example 2.1.3.

**Example 2.1.6.** The integer $36 = 2^2 \cdot 3^2$ has

$$d(36) = (2 + 1)(2 + 1) = 3 \cdot 3 = 9$$

and

$$
\sigma(36) = \left(\frac{2^3-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right) = 7 \cdot 13 = 91
$$

and $\phi(36) = 36(1 - \frac{1}{2})(1 - \frac{1}{3}) = 12$, while for the integer $1024 = 2^{10}$ one has

$$d(1024) = 10 + 1 = 11, \quad \sigma(1024) = \frac{2^{11}-1}{2-1} = 2047 = 23 \cdot 89$$

and $\phi(1024) = 1024(1 - \frac{1}{2}) = 512 = 2^5$.

Lastly, we have for the integer $2145 = 3 \cdot 5 \cdot 11 \cdot 13$ the following:

$$d(2145) = (1 + 1)(1 + 1)(1 + 1)(1 + 1) = 2^4 = 16$$

and

$$
\sigma(2145) = (3 + 1)(5 + 1)(11 + 1)(13 + 1) = 4032
$$

while

$$\phi(2145) = 2145(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{11})(1 - \frac{1}{13}) = 960.$$

The following corollary follows immediately from Theorem 2.1.5:

**Corollary 2.1.4.** Let $P$ be a prime number and $k$ a fixed positive integer. Then

$$d(P^k) = k + 1, \quad \sigma(P^k) = \sum_{i=0}^{k} P^i = \frac{P^{k+1} - 1}{P - 1}$$

and

$$\phi(P^k) = P^k(1 - \frac{1}{P}) = P^{k-1}(P - 1).$$

Note from Corollary 2.1.4 that for odd prime powers, the number and sum of divisors may or may not be prime, while $\phi(P^k)$ is always composite for $k > 1$. 
Divisibility gives rise to an equivalence relation on the set of integers, defined by the *congruence relation*.

**Definition 2.1.7.** Let $m$ be a fixed positive integer. Two integers $A$ and $B$ are said to be congruent modulo $m$, written as $A \equiv B \pmod{m}$, if $m \mid (A - B)$; that is, provided that $A - B = km$ for some integer $k$. When $m \nmid (A - B)$, we say that $A$ is incongruent to $B$ modulo $m$, and we denote this by $A \not\equiv B \pmod{m}$.

**Example 2.1.7.** Let us take $m = 3$. We can see that

$$14 \equiv 5 \pmod{3}, \quad -9 \equiv 0 \pmod{3}, \quad \text{and} \quad 35 \equiv -7 \pmod{3},$$

because $14 - 5 = 3 \cdot 3$, $-9 - 0 = (-3) \cdot 3$, and $35 - (-7) = 14 \cdot 3$.

On the other hand, $1200 \not\equiv 2 \pmod{3}$ because 3 does not divide $1200 - 2 = 1198$.

We now introduce two more additional number-theoretic functions.

**Definition 2.1.8.** Let $n$ be a positive integer. Then $\omega(n)$ is the number of distinct prime factors of $n$, i.e. $\omega(n) = \sum_{P_i \mid n} 1$ where each $P_i$ is prime. Furthermore, $\Omega(n)$ is the number of primes that divide $n$, counting multiplicities. That is, if $n$ has canonical factorization $n = \prod_{i=1}^{r} P_i^{\alpha_i}$, then

$$\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r = \sum_{i=1}^{r} \alpha_i = \sum_{P_i \mid |N} \alpha.$$

**Example 2.1.8.** Let us consider $n = 36 = 2^2 \cdot 3^2$. Since it has two distinct prime factors (namely 2 and 3), we have $\omega(36) = 2$. On the other hand, $\Omega(36) = 4$ since
its total number of prime factors, counting multiplicities, is four. For \( m = 1024 = 2^{10} \), we have \( \omega(1024) = 1 \) and \( \Omega(1024) = 10 \), while for \( k = 2145 = 3 \cdot 5 \cdot 11 \cdot 13 \), the functions have values \( \omega(2145) = 4 \) and \( \Omega(2145) = 4 \).

Notice in Example 2.1.8 that the number of distinct prime factors is less than or equal to the total number of prime factors (counting multiplicities). In general, it is true that \( \Omega(n) \geq \omega(n) \) for all positive integers \( n \).

In number theory, \textit{asymptotic density} or \textit{natural density} is one of the possibilities to measure how large is a subset of the set of natural numbers \( \mathbb{N} \). Intuitively, we feel that there are “more” odd numbers than perfect squares; however, the set of odd numbers is not in fact “bigger” than the set of perfect squares: both sets are infinite and countable and can therefore be put in one-to-one correspondence. Clearly, we need a better way to formalize our intuitive notion.

Let \( A \) be a subset of the set of natural numbers \( \mathbb{N} \). If we pick randomly a number from the set \( \{1, 2, \ldots, n\} \), then the probability that it belongs to \( A \) is the ratio of the number of elements in the set \( A \cap \{1, 2, \ldots, n\} \) and \( n \). If this probability tends to some limit as \( n \) tends to infinity, then we call this limit the \textit{asymptotic density} of \( A \). We see that this notion can be understood as a kind of probability of choosing a number from the set \( A \). Indeed, the asymptotic density (as well as some other types of densities) is studied in \textbf{probabilistic number theory}. 
We formalize our definition of asymptotic density or simply density in what follows:

**Definition 2.1.9.** A sequence \(a_1, a_2, \ldots, a_n\) with the \(a_j\) positive integers and \(a_j < a_{j+1}\) for all \(j\), has natural density or asymptotic density \(\alpha\), where \(0 \leq \alpha \leq 1\), if the proportion of natural numbers included as some \(a_j\) is asymptotic to \(\alpha\). More formally, if we define the counting function \(A(x)\) as the number of \(a_j\)'s with \(a_j < x\) then we require that \(A(x) \sim \alpha x\) as \(x \to +\infty\).

## 2.2 The Abundancy Index

As discussed in the literature review, the search for perfect numbers led mathematicians to consider the rather interesting quantity called the **abundancy index/ratio**.

**Definition 2.2.1.** The abundancy index/ratio of a given positive integer \(n\) is defined as \(I(n) = \frac{\sigma(n)}{n}\).

**Example 2.2.1.** 
\[
I(36) = \frac{\sigma(36)}{36} = \frac{91}{36},
\]
while
\[
I(1024) = \frac{\sigma(1024)}{1024} = \frac{2047}{1024}
\]
and
\[
I(2145) = \frac{\sigma(2145)}{2145} = \frac{4032}{2145}.
\]

We note that the abundancy index is also a multiplicative number-theoretic function because \(\sigma\) is multiplicative.
Looking back at Definition 1.1.1, it is clear that a number \( N \) is perfect if and only if its abundancy index \( I(N) \) is 2. It is somewhat interesting to consider the cases when \( I(N) \neq 2 \).

**Definition 2.2.2.** If the abundancy index \( I(N) < 2 \), then \( N \) is said to be *deficient*, while for \( I(N) > 2 \), \( N \) is said to be *abundant*.

**Example 2.2.2.** Referring to Example 2.2.1, 36 is abundant since \( I(36) = \frac{91}{36} > 2 \) while 1024 and 2145 are deficient since \( I(1024) = \frac{2047}{1024} < 2 \) and \( I(2145) = \frac{4032}{2145} < 2 \).

**Remark 2.2.1.** Giardus Ruffus conjectured in 1521 that most odd numbers are deficient. In 1975, C. W. Anderson [1] proved that this is indeed the case by showing that the density of odd deficient numbers is at least \( \frac{48-3\pi^2}{32-\pi^2} \approx 0.831 \). On the other hand, Marc Deléglise (in 1998 [13]) gave the bounds \( 0.2474 < A(2) < 0.2480 \) for the density \( A(2) \) of abundant integers. Kanold (1954) [42] showed that the density of odd perfect numbers is 0.

We list down several important lemmas describing useful properties of the abundancy index.

**Lemma 2.2.1.** \( \frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} \)

**Proof.** Straightforward:

\[
\frac{\sigma(n)}{n} = \frac{1}{n} \sum_{d|n} d = \frac{1}{n} \sum_{d|n} \frac{n}{d} = \sum_{d|n} \frac{1}{d}.
\]

**Lemma 2.2.2.** If \( m \mid n \) then \( \frac{\sigma(m)}{m} \leq \frac{\sigma(n)}{n} \), with equality occurring if and only if \( m = n \).
Essentially, Lemma 2.2.2 says that any (nontrivial) multiple of a perfect number is abundant and every (nontrivial) divisor of a perfect number is deficient.

**Lemma 2.2.3.** *The abundancy index takes on arbitrarily large values.*

*Proof.* Consider the number $n!$. By Lemma 2.2.1, we have $\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq \sum_{i=1}^{n} \frac{1}{i}$. Since the last quantity is a partial sum of a harmonic series which diverges to infinity, $\frac{\sigma(n!)}{n!}$ can be made as large as we please. □

**Lemma 2.2.4.** For any prime power $P^\alpha$, the following inequalities hold:

$$1 < \frac{P + 1}{P} < \frac{\sigma(P^\alpha)}{P^\alpha} < \frac{P}{P - 1}.$$ 

The proof of Lemma 2.2.4 follows directly from Corollary 2.1.4.

Certainly, we can find abundancy indices arbitrarily close to 1 because $I(p) = \frac{p + 1}{p}$ for all primes $p$. By Lemma 2.2.3 and since the abundancy index of a positive integer is a rational number, one would then desire to know the “distribution” of these ratios in the interval $(1, \infty)$. The next few results summarize much of what is known about the “distribution” of these ratios.

**Theorem 2.2.1.** *(Laatsch)* The set of abundancy indices $I(n)$ for $n > 1$ is dense in the interval $(1, \infty)$.

However, not all of the rationals from the interval $(1, \infty)$ are abundancy indices of some integer. This is due to the following lemma from Weiner:

**Lemma 2.2.5.** *(Weiner)* If $\gcd(m, n) = 1$ and $n < m < \sigma(n)$, then $\frac{m}{n}$ is not the abundancy index of any integer.
Proof. Suppose \( \frac{m}{n} = \frac{\sigma(k)}{k} \) for some integer \( k \). Then \( km = n\sigma(k) \) which implies that \( n \mid km \), and so \( n \mid k \) since \( m \) and \( n \) are coprime. Hence, by Lemma 2.2.2 we have

\[
\frac{\sigma(n)}{n} \leq \frac{\sigma(k)}{k} = \frac{m}{n}
\]

which yields \( \sigma(n) \leq m \) - a contradiction to the initial assumption that \( m < \sigma(n) \).

It is now natural to define the notion of an abundancy outlaw.

**Definition 2.2.3.** A rational number greater than 1 is said to be an abundancy outlaw if it fails to be in the range of the function \( I(n) \).

One can use the previous lemmas to establish an equally interesting theorem about the distribution of abundancy outlaws.

**Theorem 2.2.2.** (Weiner, Ryan) The set of abundancy outlaws is dense in the interval \((1, \infty)\).

Upon inspecting the results of Theorems 2.2.1 and 2.2.2 it appears that the scenario for abundancy indices and outlaws is both complex and interesting. We shall take a closer look into the nature of abundancy outlaws in Chapter 4.

### 2.3 Even Perfect Numbers

The Greek mathematician *Euclid* was the first to categorize the perfect numbers. He noticed that the first four perfect numbers have the very specific
forms:

\[ 6 = 2^1(1 + 2) = 2 \cdot 3 \]
\[ 28 = 2^2(1 + 2 + 2^2) = 4 \cdot 7 \]
\[ 496 = 2^4(1 + 2 + 2^2 + 2^3 + 2^4) = 16 \cdot 31 \]
\[ 8128 = 2^6(1 + 2 + 2^2 + \ldots + 2^6) = 64 \cdot 127. \]

Notice that the numbers 90 = 2\(^3\)(1 + 2 + 2\(^2\) + 2\(^3\)) = 8 \cdot 15 and 2016 = 2\(^5\)(1 + 2 + 2\(^2\) + \ldots + 2\(^5\)) = 32 \cdot 63 are missing from this list. Euclid pointed out that this is because 15 = 3 \cdot 5 and 63 = 3\(^2\) \cdot 7 are both composite, whereas the numbers 3, 7, 31 and 127 are all prime.

According to Book IX, proposition 36 of Euclid’s *Elements*: “If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect.” [52]

This observation is stated in a slightly more compact form as follows:

**Theorem 2.3.1.** (Euclid) If \(2^n - 1\) is prime, then \(N = 2^{n-1}(2^n - 1)\) is perfect.

**Proof.** Clearly the only prime factors of \(N\) are \(2^n - 1\) and 2. Since \(2^n - 1\) occurs as a single prime, we have simply that \(\sigma(2^n - 1) = 1 + (2^n - 1) = 2^n\), and thus

\[
\sigma(N) = \sigma(2^{n-1})\sigma(2^n - 1) = \left(\frac{2^n - 1}{2 - 1}\right)2^n = 2^n(2^n - 1) = 2N.
\]

Therefore, \(N\) is perfect. \(\square\)
The task of finding perfect numbers, then, is intimately linked with finding primes of the form $2^n - 1$. Such numbers are referred to as Mersenne primes, after the 17th-century monk Marin Mersenne, a contemporary of Descartes, Fermat, and Pascal. He investigated these unique primes as early as 1644. Mersenne knew that $2^n - 1$ is prime for $n = 2, 3, 5, 11, 13, 17, \text{ and } 19$ - and, more brilliantly, conjectured the cases $n = 31, 67, 127, 257$. It took almost two hundred years to test these numbers.

There is one important criterion used to determine the primality of Mersenne numbers:

**Lemma 2.3.1.** (*Cataldi-Fermat*) *If* $2^n - 1$ *is prime, then* $n$ *itself is prime.*

**Proof.** Consider the factorization of $x^n - 1 = (x - 1)(x^{n-1} + \ldots + x + 1)$. Suppose $n = rs$, where $r, s > 1$. Then $2^n - 1 = (2^r)^s - 1 = (2^r - 1)((2^r)^{s-1} + \ldots + 2^r + 1)$, so that $(2^r - 1) | (2^n - 1)$ which is prime, a contradiction. \hfill $\Box$

Note that the converse of Lemma 2.3.1 is not true - the number $2^{11} - 1$ which is equal to $2047 = 23 \cdot 89$ is composite, yet $11$ is prime, for instance.

Should all perfect numbers be of Euclid’s type? Leonard Euler, in a posthumous paper, proved that every even perfect number is of this type. \[48\]

**Theorem 2.3.2.** (*Euler*) *If* $N$ *is an even perfect number, then* $N$ *can be written in the form* $N = 2^{n-1}(2^n - 1)$, *where* $2^n - 1$ *is prime.*

**Proof.** Let $N = 2^{n-1}m$ be perfect, where $m$ is odd; since $2$ does not divide $m$, it is relatively prime to $2^{n-1}$, and
\[ \sigma(N) = \sigma(2^{n-1}m) = \sigma(2^{n-1})\sigma(m) = (2^{n-1})\sigma(m) = (2^n - 1)\sigma(m). \]

\(N\) is perfect so \(\sigma(N) = 2N = 2(2^{n-1}m) = 2^n m\), and with the above,

\[ 2^n m = (2^n - 1)\sigma(m). \]

Since \(2^n - 1\) is odd, \((2^n - 1) \mid m\), so we can write \(m = (2^n - 1)k\).

Now \((2^n - 1)\sigma(m) = 2^n(2^n - 1)k\), which implies \(\sigma(m) = 2^n k = (2^n - 1)k + k = m + k\).

But \(k \mid m\) so \(\sigma(m) = m + k\) means \(m\) has only two (2) divisors, which further implies that \(k = 1\). Therefore, \(\sigma(m) = m + 1\) and \(m\) is prime. Since \((2^n - 1) \mid m\), \(2^n - 1 = m\). Consequently, \(N = 2^{n-1}(2^n - 1)\) where \(2^n - 1\) is prime. \(\square\)

Even perfect numbers have a number of nice little properties. We list down several of them here, and state them without proof [71]:

- If \(N\) is an even perfect number, then \(N\) is triangular.
- If \(N = 2^{n-1}(2^n - 1)\) is perfect then \(N = 1^3 + 3^3 + \ldots + (2^{n-1} - 1)^3\).
- If \(N = 2^{n-1}(2^n - 1)\) is perfect and \(N\) is written in base 2, then it has \(2n - 1\) digits, the first \(n\) of which are unity and the last \(n - 1\) are zero.
- Every even perfect number ends in either 6 or 8.
- (Wantzel) The iterative sum of the digits (i.e. digital root) of an even perfect number (other than 6) is one.

Today 44 perfect numbers are known, \(2^{88}(2^{89} - 1)\) being the last to be discovered by hand calculations in 1911 (although not the largest found by hand calculations), all others being found using a computer. In fact computers have led to a revival of interest in the discovery of Mersenne primes, and therefore of perfect numbers. At the moment the largest known Mersenne prime is \(2^{32582657} - 1\). It was
discovered in September of 2006 and this, the 44th such prime to be discovered, contains more than 9.8 million digits. Worth noting is the fact that although this is the 44th to be discovered, it may not correspond to the 44th perfect number as not all smaller cases have been ruled out.

\section{Odd Perfect Numbers}

The Euclid-Euler theorem from Section 2.3 takes care of the even perfect numbers. What about the \textit{odd perfect numbers}?

Euler also tried to make some headway on the problem of whether odd perfect numbers existed. He proved that any odd perfect number $N$ had to have the form

$$N = (4m + 1)^{4k+1}b^2$$

where $4m + 1$ is prime and $\gcd(4m + 1, b) = 1$.

In Section 2.3, we have followed some of the progress of finding even perfect numbers but there were also attempts to show that an odd perfect number could not exist. The main thrust of progress here has been to show the minimum number of distinct prime factors that an odd perfect number must have. Sylvester worked on this problem and wrote:

\ldots the existence of [an OPN] - its escape, so to say, from the complex web of conditions which hem it in on all sides - would be little short of a miracle.
(The reader is referred to Section 1.1.2 of this thesis for a survey of the most recent conditions which an OPN must satisfy, if any exists.)

We give a proof of Euler’s characterization of OPNs here [48]:

**Theorem 2.4.1. (Euler)** Let $N$ be an OPN. Then the prime factorization of $N$ takes the form $N = q^{4e+1}p_1^{2a_1} \cdots p_r^{2a_r}$, where $q \equiv 1 \pmod{4}$.

**Proof.** Let $N = l_1^{e_1}l_2^{e_2} \cdots l_s^{e_s}$ for some primes $l_1, l_2, \ldots, l_s$. Since $N$ is odd, all $l_i$ are odd. Finally, $\sigma(N) = 2N$. Since $\sigma(N) = \sigma(l_1^{e_1}l_2^{e_2} \cdots l_s^{e_s}) = \sigma(l_1^{e_1})\sigma(l_2^{e_2})\cdots\sigma(l_s^{e_s})$, we take a look at $\sigma(l^e) = 1 + l + l^2 + \ldots + l^e$, a sum of $e + 1$ odd numbers. This is odd only if $e$ is even. Since $\sigma(l_1^{e_1}l_2^{e_2} \cdots l_s^{e_s}) = \sigma(l_1^{e_1})\sigma(l_2^{e_2})\cdots\sigma(l_s^{e_s}) = 2l_1^{e_1}l_2^{e_2} \cdots l_s^{e_s}$, we can only get one factor of 2. So the $e_i$ are even, all except one, say $e_1$. So $N = l_1^{e_1}p_1^{2a_1} \cdots p_r^{2a_r}$.

We have $2 \mid \sigma(l_1^{e_1})$ but $4 \nmid \sigma(l_1^{e_1})$. Since $l_1$ is odd, $e_1$ is odd. Now, modulo 4, we see that either $l_1 \equiv 1 \pmod{4}$ or $l_1 \equiv -1 \pmod{4}$. But if $l_1 \equiv -1 \pmod{4}$, then

$$\sigma(l_1^{e_1}) = 1 + l_1 + l_1^2 + l_1^3 + \ldots + l_1^{e_1-1} + l_1^{e_1} \equiv 1 + (-1) + 1 + (-1) + \ldots + 1 + (-1) \equiv 0 \pmod{4},$$

which is clearly a contradiction since $4 \nmid \sigma(l_1^{e_1})$. Thus, $l_1 \equiv 1 \pmod{4}$. Now, $\sigma(l_1^{e_1}) = 1 + l_1 + l_1^2 + l_1^3 + \ldots + l_1^{e_1-1} + l_1^{e_1} \equiv 1 + 1 + 1 + 1 + \ldots + 1 + 1 \equiv e_1 + 1 \pmod{4}$. Since $e_1$ is odd, either $e_1 + 1 \equiv 0 \pmod{4}$ or $e_1 + 1 \equiv 2 \pmod{4}$. If $e_1 + 1 \equiv 0 \pmod{4}$, then $4 \mid \sigma(l_1^{e_1})$ which is again a contradiction.
So $e_1 + 1 \equiv 2 \pmod{4} \iff e_1 + 1 = 4e + 2$, that is, $e_1 = 4e + 1$. Consequently,

$N = q^{4e+1}p_1^{2a_1} \cdots p_r^{2a_r}$, for $q \equiv 1 \pmod{4}$.

We call $q$ in Theorem 2.4.1 the special/Euler prime of $N$, while $q^{4e+1}$ will be called the Euler’s factor of $N$.

Interestingly, it is possible to show that no two consecutive integers can be both perfect, using the Euclid-Euler theorem on the form of even perfect numbers from Section 2.3 and Euler’s characterization of odd perfect numbers in this section. We shall give a discussion of the proof of this interesting result in Chapter 3. Moreover, we shall give there the (easy) proof of the fact that an odd perfect number must be a sum of two squares. We also give congruence conditions for the existence of odd perfect numbers in the next chapter. For the most part, the proofs will be elementary, requiring only an intermediate grasp of college algebra and the concepts introduced in this chapter.
Chapter 3

OPN Solution Attempts 1:
Some Old Approaches

In the previous chapters, we derived the possible forms of even and odd perfect numbers, and also surveyed the most recent results on the conditions necessitated by the existence of OPNs. In this chapter, we introduce the reader to the flavor of the mathematical techniques used to formulate theorems about OPNs by researchers who lived prior to the 21st century.

The following are some of the traditional attempts made by mathematicians (both amateur and professional) to prove or disprove the OPN Conjecture in the pre-21st century:

- Increasing the lower bound for the number of distinct prime factors, $\omega(N)$, that an OPN $N$ must have;
- Increasing the lower bound for the magnitude of the smallest possible OPN, if one exists;
- Deriving congruence conditions for the existence of OPNs.
All these itemized approaches attempt to derive a contradiction amongst the stringent conditions that an OPN must satisfy.

3.1 Increasing the Lower Bound for \( \omega(N) \)

Recall that, from Lemma 2.2.4, we have the following strict inequality for the abundancy index of a prime power:

\[
\frac{\sigma(p^\alpha)}{p^\alpha} < \frac{p}{p-1}.
\]

This gives rise to the following lemma:

**Lemma 3.1.1.** If \( N \) is a perfect number with canonical factorization \( N = \prod_{i=1}^{\omega(N)} p_i^{\alpha_i} \), then

\[
2 < \prod_{i=1}^{\omega(N)} \frac{p_i}{p_i - 1} = \prod_{i=1}^{\omega(N)} \left(1 + \frac{1}{p_i - 1}\right).
\]

**Proof.** This follows as an immediate consequence of Lemma 2.2.4 and the definition of perfect numbers. \( \square \)

Note that Lemma 3.1.1 applies to both even and odd perfect numbers.

The following is another very useful lemma:

**Lemma 3.1.2.** If \( N \) is a perfect number with canonical factorization \( N = \prod_{i=1}^{\omega(N)} p_i^{\alpha_i} \), then

\[
2 \geq \prod_{i=1}^{\omega(N)} \left(1 + \frac{1}{p_i} + \ldots + \frac{1}{p_i^{\beta_i}}\right),
\]

where \( 0 \leq \beta_i \leq \alpha_i \forall i \).
\textbf{Proof.} This immediately follows from the definition of perfect numbers and the fact that the abundancy index for prime powers is an increasing function of the exponents. \hfill \Box

When considering OPNs, Lemmas 3.1.1 and 3.1.2 are very useful because they can yield lower bounds for $\omega(N)$. Indeed, it was in using these lemmas (together with some ingenuity) that pre-21st century mathematicians were able to successfully obtain ever-increasing lower bounds for the number of distinct prime factors of an OPN $N$.

We now prove the classical result: "An OPN must have at least three distinct prime factors".

\textbf{Theorem 3.1.1.} If $N$ is an OPN, then $\omega(N) \geq 3$.

\textbf{Proof.} Let $N$ be an OPN. Since prime powers are deficient, $\omega(N) \geq 2$. Suppose $\omega(N) = 2$. Since $N$ is odd, $P_1 \geq 3$ and $P_2 \geq 5$ where $N = P_1^{\alpha_1}P_2^{\alpha_2}$ is the canonical factorization of $N$. Using Lemma 3.1.1

$$2 < \prod_{i=1}^{\omega(N)} \frac{P_i}{P_i - 1} = \prod_{i=1}^{2} \frac{P_i}{P_i - 1}$$

$$= \left( \frac{P_1}{P_1 - 1} \right) \left( \frac{P_2}{P_2 - 1} \right) = \frac{1}{1 - \frac{1}{P_1}} \cdot \frac{1}{1 - \frac{1}{P_2}} \leq \frac{1}{1 - \frac{1}{3}} \cdot \frac{1}{1 - \frac{1}{5}}$$

$$= \frac{3}{4} \cdot \frac{5}{3} = \frac{15}{12} = \frac{15}{8} = 1.875 < 2$$

Thus, the assumption $\omega(N) = 2$ for an OPN $N$ has resulted to the contradiction $2 < 2$. This contradiction shows that $\omega(N) \geq 3$. \hfill \Box
More work is required to improve the result of Theorem 3.1.1 to \( \omega(N) \geq 4 \), if we are to use a similar method.

**Theorem 3.1.2.** If \( N \) is an OPN, then \( \omega(N) \geq 4 \).

**Proof.** Let \( N \) be an OPN. By Theorem 3.1.1, \( \omega(N) \geq 3 \). Assume \( \omega(N) = 3 \). Since \( N \) is odd, \( P_1 \geq 3 \), \( P_2 \geq 5 \), and \( P_3 \geq 7 \). Let \( N = P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} \) be the canonical factorization of \( N \). Using Lemma 3.1.1:

\[
2 < \prod_{i=1}^{\omega(N)} \frac{P_i}{P_i - 1} = \prod_{i=1}^{3} \frac{P_i}{P_i - 1}
\]

\[
= \left( \frac{P_1}{P_1 - 1} \right) \left( \frac{P_2}{P_2 - 1} \right) \left( \frac{P_3}{P_3 - 1} \right) = \frac{1}{1 - \frac{1}{P_1}} \frac{1}{1 - \frac{1}{P_2}} \frac{1}{1 - \frac{1}{P_3}} \leq \frac{1}{1 - \frac{1}{3}} \frac{1}{1 - \frac{1}{5}} \frac{1}{1 - \frac{1}{7}}
\]

\[
= \frac{2}{3} \frac{2}{5} \frac{2}{7} = \frac{35}{105} = \frac{105}{48} = \frac{35}{16} = 2.1875
\]

whence we do not arrive at a contradiction. Now, suppose \( P_1 \geq 5 \), \( P_2 \geq 7 \), and \( P_3 \geq 11 \). Using Lemma 3.1.1 again, we get \( 2 < \left( \frac{2}{3} \right) \left( \frac{7}{6} \right) \left( \frac{11}{10} \right) = \frac{231}{120} = \frac{77}{40} \) (by use of Lemma 3.1.1) which is again a contradiction. Consequently, we know that \( P_1 = 3 \). Then, with this additional information about \( P_1 \), if we assume that \( P_2 \geq 7 \), we arrive at \( 2 < \left( \frac{3}{2} \right) \left( \frac{7}{6} \right) \left( \frac{11}{10} \right) = \frac{231}{120} = \frac{77}{40} \) (by use of Lemma 3.1.1) which is again a contradiction. Hence, we also know that \( P_2 = 5 \). Thus, \( P_3 \geq 7 \). Furthermore, using Lemma 3.1.1 again, the following inequality must be true:

\[
2 < \frac{35}{24} \frac{P_3}{P_3 - 1} \iff \frac{16}{15} < \frac{P_3}{P_3 - 1}.
\]

Solving this last inequality, we get \( P_3 < 16 \). This inequality, together with the fact that \( P_3 \) is prime, gives us 3 possible cases to consider:
Case 1: \( N = 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3} \)

Since \( N \) is odd, 4 cannot divide \( \sigma(N) = \sigma(3^{\alpha_1})\sigma(5^{\alpha_2})\sigma(7^{\alpha_3}) = 2N = 2 \cdot 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3} \).

In particular, \( 4 \nmid \sigma(3^{\alpha_1}) \) and \( 4 \nmid \sigma(7^{\alpha_3}) \). But \( \sigma(3^{\alpha_1}) = 4 \) for \( \alpha_1 = 1 \) and \( \sigma(7^{\alpha_3}) = 8 \) for \( \alpha_3 = 1 \). Consequently, \( \alpha_1 \geq 2 \) and \( \alpha_3 \geq 2 \). Now, by using Lemma 3.1.2 with \( \beta_1 = 2, \beta_2 = 1 \) and \( \beta_3 = 2 \), we have:

\[
2 \geq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7} + \frac{1}{7^2}\right) = \frac{494}{245},
\]

which is a contradiction. Hence, there is no OPN of the form \( N = 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3} \).

Case 2: \( N = 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3} \)

A. Using Lemma 3.1.1 with \( \alpha_2 = 1 \) gives \( 2 < \frac{3 \sigma(5^{\alpha_2})}{2 \cdot 5^{\alpha_2}} \cdot \frac{11}{10} = \frac{3 \cdot 6 \cdot 11}{2 \cdot 5 \cdot 10} = \frac{99}{50} \), which is a contradiction.

B. Let \( \alpha_2 \geq 2 \). For \( \alpha_1 = 1, 2, 3 \), we have \( \sigma(3^{\alpha_1}) = 4, 13, 40 \). Since \( 4 \nmid \sigma(N) \), \( \alpha_1 \neq 1 \) and \( \alpha_1 \neq 3 \). Also, \( 13 \nmid \sigma(N) = 2N = 2 \cdot 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3} \) and thus, \( 13 \nmid \sigma(3^{\alpha_1}) \). Hence, \( \alpha_1 \neq 2 \), which implies that \( \alpha_1 \geq 4 \). Similarly, \( \sigma(11^{\alpha_3}) = 12 \) for \( \alpha_3 = 1 \), which contradicts the fact that \( 4 \nmid \sigma(11^{\alpha_3}) \). Thus, \( \alpha_3 \geq 2 \). Now, using Lemma 3.1.2 with \( \beta_1 = 4, \beta_2 = 2 \) and \( \beta_3 = 2 \), we get:

\[
2 \geq \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2}\right) \left(1 + \frac{1}{11} + \frac{1}{11^2}\right) = \frac{4123}{2025},
\]

which is a contradiction. Hence, there is no OPN of the form \( N = 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3} \).

Case 3: \( N = 3^{\alpha_1}5^{\alpha_2}13^{\alpha_3} \)

A. Using Lemma 3.1.1 with \( \alpha_2 = 1 \) gives \( 2 < \frac{3 \sigma(5^{\alpha_2})}{2 \cdot 5^{\alpha_2}} \cdot \frac{13}{12} = \frac{3 \cdot 6 \cdot 13}{2 \cdot 5 \cdot 12} = \frac{39}{20} \), which is a contradiction.
B. Let $\alpha_2 \geq 2$. Similar to what we got from Case 2B, we have $\alpha_1 \neq 1, 3$.

1. $\alpha_1 = 2$

Using Lemma 3.1.1 we have: $2 < \frac{\sigma(3^{\alpha_1})}{3^{\alpha_1}} \frac{5}{4} \frac{13}{12} = \frac{13 \cdot 5}{9 \cdot 4} \frac{13}{12} = \frac{845}{432}$, which is a contradiction.

2. $\alpha_1 \geq 4$

Suppose $\alpha_3 = 1$. Then $\sigma(13^{\alpha_3}) = 14 = 2 \cdot 7$, yet $7 \nmid \sigma(N) = 2N = 2 \cdot 3^{\alpha_1} 5^{\alpha_2} 13^{\alpha_3}$.

Therefore, $7 \nmid \sigma(13^{\alpha_3})$, and then $\alpha_3 \geq 2$. Now, by using Lemma 3.1.2 with $\beta_1 = 4$, $\beta_2 = 2$ and $\beta_3 = 2$, we get:

$$2 \geq \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2}\right) \left(1 + \frac{1}{13} + \frac{1}{13^2}\right) = \frac{228811}{114075},$$

which is again a contradiction. Hence, there is no OPN of the form $N = 3^{\alpha_1} 5^{\alpha_2} 13^{\alpha_3}$.

Therefore, there is no OPN with exactly three (3) distinct prime factors, i.e. an OPN must have at least four (4) distinct prime factors.

**Remark 3.1.1.** Using more recent findings on an upper bound for OPNs by Nielsen [50] and on a lower bound by Brent et al. [4], it is possible to extend the results in this section. Thus, $10^{300} < N < 2^{\omega(N)}$, and this gives

$$\omega(N) > \left(\frac{2 + \log(3) - \log(\log(2))}{\log(4)}\right) > 4.9804,$$

which implies that $\omega(N) \geq 5$ since $\omega(N)$ should be an integer. (There is a project underway at [http://www.oddperfect.org](http://www.oddperfect.org), organized by William Lipp, which hopes to extend the lower bound for OPNs to $10^{500}$, or $10^{600}$ even.) Indeed, even more recently (2006), Nielsen [51] was able to show that $\omega(N) \geq 9$, “ultimately [avoiding] previous computational results for [OPNs]”.
3.2 Increasing the Lower Bound for an OPN $N$

We begin with a very useful inequality (which we shall not prove here) that can yield our desired estimates for a lower bound on OPNs:

**Lemma 3.2.1. The Arithmetic Mean-Geometric Mean Inequality**

Let $\{X_i\}$ be a sequence of (not necessarily distinct) positive real numbers. Then the following inequality must be true:

$$\frac{1}{n} \sum_{i=1}^{n} X_i \geq \left( \prod_{i=1}^{n} X_i \right)^{\frac{1}{n}}.$$  

Equality holds if and only if all of the $X_i$'s are equal.

We now derive a crude lower bound for an OPN $N = \prod_{i=1}^{\omega(N)} P_i^{\alpha_i}$ in terms of the $\alpha_i$'s:

**Lemma 3.2.2.** Let $N = \prod_{i=1}^{\omega(N)} P_i^{\alpha_i}$ be an OPN. Then

$$N > \left( \prod_{i=1}^{\omega(N)} \frac{\alpha_i + 1}{2} \right)^{2}.$$  

Proof. $\sigma(P_i^{\alpha_i}) = \sum_{j=0}^{\alpha_i} P_i^j$. Applying Lemma 3.2.1 and noting that prime powers of the $P_i$'s are distinct, we have for each $i$:

$$\sum_{j=0}^{\alpha_i} P_i^j > (\alpha_i + 1) \prod_{j=0}^{\alpha_i} P_i^{\alpha_i} \frac{j}{\alpha_i + 1} = (\alpha_i + 1) P_i^{\sum_{j=0}^{\alpha_i} \frac{j}{\alpha_i + 1}}.$$
\[
\frac{\alpha_i (\alpha_i + 1)}{\alpha_i (\alpha_i + 1) + 1} = (\alpha_i + 1) P_i \frac{\alpha_i}{2(\alpha_i + 1)} = (\alpha_i + 1) P_i \frac{\alpha_i}{2}.
\]

Consequently, by multiplying across all \(i\):

\[
\prod_{i=1}^{\omega(N)} \sigma(P_i^{\alpha_i}) > \prod_{i=1}^{\omega(N)} (\alpha_i + 1) P_i \frac{\alpha_i}{2}. \quad \text{But} \quad \sqrt{N} = \prod_{i=1}^{\omega(N)} P_i \frac{\alpha_i}{2}, \quad \text{which means that}
\]

\[
N = \frac{\prod_{i=1}^{\omega(N)} \sigma(P_i^{\alpha_i})}{\prod_{i=1}^{\omega(N)} (\alpha_i + 1) P_i \frac{\alpha_i}{2}} \geq \frac{\prod_{i=1}^{\omega(N)} \sigma(P_i^{\alpha_i})}{\prod_{i=1}^{\omega(N)} P_i \frac{\alpha_i}{2}} = \sqrt{N} \prod_{i=1}^{\omega(N)} (\alpha_i + 1) \frac{\alpha_i}{2}.
\]

Solving this last inequality for \(N\) gives us the desired result. \(\square\)

**Remark 3.2.1.** In the canonical factorization \(N = \prod_{i=1}^{\omega(N)} P_i^{\alpha_i}\) of an OPN \(N\), since \(\alpha_i \geq 1\) for all \(i\), from Lemma 3.2.2 we have the crude lower bound \(N > 2^{2\omega(N)-2}\), which, together with Nielsen’s lower bound of \(\omega(N) \geq 9\) for the number of distinct prime factors of \(N\), yields the lower bound \(N > 2^{16} = 65536\) for the magnitude of the smallest possible OPN. This lower bound can, of course, be improved. Indeed, Brent, et al. \cite{4} in 1991 showed that it must be the case that \(N > 10^{300}\).

**Remark 3.2.2.** Note that nowhere in the proof of Lemma 3.2.2 did we use the fact that \(N\) is odd. Hence, Lemma 3.2.2 applies to even perfect numbers as well.

We can make use of the results on the lower bound for the number of distinct prime factors (latest result is at 9 by Nielsen), lower bound for the smallest prime factor (currently at 3 - mathematicians are still unable to rule out the possibility that an OPN may be divisible by 3), and the nature of the exponents (the special/Euler prime has the sole odd exponent while the rest of the primes have even exponents) to derive a larger lower bound for an OPN.
Since all, except for one, of the distinct prime factors of an OPN $N = \prod_{i=1}^{\omega(N)} P_i^{\alpha_i}$ have even exponents, then $\alpha_i \geq 2$ for all but one $i$, say $i = j$, for which $\alpha_j \geq 1$. (Note that $\alpha_j$ is the exponent of the special or Euler prime.) Thereupon, we have the following improvements to the results in Remark 3.2.1:

$$\prod_{i=1}^{\omega(N)} (\alpha_i + 1) \geq 3^{\omega(N) - 1} \cdot 2$$

$$N > \left( \frac{\prod_{i=1}^{\omega(N)} (\alpha_i + 1)}{2} \right)^2 \geq 3^{2\omega(N) - 2}$$

$\omega(N) \geq 9 \quad [Nielsen] \Rightarrow N > 3^{16} = 43046721$

Note that there is approximately a 655.84% improvement in the magnitude of the bound thus obtained for the smallest possible OPN as compared to the previous crude lower bound of $2^{16}$. The novelty of the approach of Lemma 3.2.2 can be realized if we consider the fact that we did NOT need to check any of the odd numbers below 43046721 to see if they could be perfect.

**Remark 3.2.3.** We casually remark that the lower bound of $3^{16}$ obtained for an OPN here improves on the classical bound of $2 \cdot 10^6$ obtained by Turcaninov in 1908. However, at that time, the best-known bound for the number of distinct prime factors of an OPN $N$ was $\omega(N) \geq 5$, which was shown to be true by Sylvester in 1888, whereas we used the bound $\omega(N) \geq 9$ by Nielsen (2006) here.

As mentioned in Remark 3.2.3, A. Turcaninov showed in 1908 that no odd number less than two million can be perfect. The figure $2 \cdot 10^6$ is generally accepted as the minimum in standard texts. Nonetheless, it is easy to show by means of
well-known proofs that the smallest possible OPN must be greater than ten billion (i.e. $10^{10}$).

Let the prime factorization of an OPN $N_0$ be given by

$$N_0 = P_1^{a_1}Q_1^{b_1}Q_2^{b_2} \cdots Q_m^{b_m}$$

where $a_1$ is odd and $b_1, b_2, \ldots, b_m$ are even. The following conditions must hold:

- Euler had shown that $P_1 \equiv a_1 \equiv 1 \pmod{4}$.
- Sylvester demonstrated that it must be the case that $m \geq 4$.
- Steuerwald proved that $b_1 = b_2 = \cdots = b_m = 2$ is not possible.
- Brauer extended the last result to $b_i \neq 4$, when $b_1 = b_2 = \cdots = b_{i-1} = b_{i+1} = \cdots = b_m = 2$.
- Sylvester also showed that $105 = 3 \cdot 5 \cdot 7$ does not divide $N_0$. (We give a proof of this result in Section 3.4.)

The only numbers less than ten billion which satisfy all these itemized conditions are $3^6 \cdot 5^2 \cdot 11^2 \cdot 13^2 \cdot 17$ and $3^6 \cdot 5^2 \cdot 11^2 \cdot 13 \cdot 17^2$. One can verify that each of these two is abundant by directly computing the sum of its divisors.

We end this section with a copy of an email correspondence between the author and Richard Brent, one of the three co-authors of the 1991 paper which showed that an OPN $N$ must be bigger than $10^{300}$. The significance of the email lies with the fact that the author of this thesis was able to show that $N > p^k\sigma(p^k)$
where $k$ is unrestricted (i.e. $k$ can be even or odd). In fact, the author was able to show the slightly stronger statement $N \geq \frac{3}{2} p^k \sigma(p^k)$ with $k$ unrestricted. (Although Dr. Brent did not mention it in his reply, this result could give a higher lower bound for OPNs.) We shall give a proof of this last result in Theorem 4.2.5. For now, let us take a look at the author’s email inquiry and how Dr. Brent responded to it:
From: Richard Brent [rpbrent@gmail.com]
Sent: Tuesday, December 04, 2007 9:38 AM
To: Jose Dris
Subject: Re: Inquiry regarding the lower bound of \(10^{300}\) that you obtained for odd perfect numbers

Dear Arnie,

On 22/11/2007, Jose Dris (Jose.Dris@safeway.com) wrote:

Hi Dr. Brent,

I am Arnie Dris, a candidate for the degree of MS in Mathematics at DLSU here at Manila, Philippines. I am currently in the process of writing up a thesis on odd perfect numbers, and I came across your lower bound of \(10^{300}\) for odd perfect numbers that you obtained together with two co-authors.

In your paper, you stated that you used the simple observation that 
\[ N > p^k \sigma(p^k) \] where \(p^k||N\) and \(k\) is even. I would just like to ask whether this observation was used in ALL cases that you have considered, thereby proving that \(N > 10^{300}\) in each case?

It was a long time ago, but as I recall we used that observation in most cases. There were a few “hard cases” where we could not compute the sigma function
\(\sigma(p^k)\) because we did not know the complete factorisation that is needed to do this, e.g. \(\sigma(3221^{42})\) was a 148-digit composite number that we could not factorise at the time (it may have been factored since then). In such cases we had to use a more complicated method. See the “Proof of Theorem 1” in the paper.

You can get the paper online at

http://www.maths.anu.edu.au/~brent/pub/pub116.html

and there’s also a link there to the computer-generated “proof tree”.

William Lipp has a project to extend the bound \(10^{300}\) by much the same method but with more factorisations (since computers are faster now and algorithms such as the number field sieve are available). He hopes to go at least to \(10^{400}\) and possibly further.

Regards,

Richard Brent

–

Prof R. P. Brent, ARC Federation Fellow
MSI, ANU, Canberra, ACT 0200, Australia
rpbrent@gmail.com

http://www.maths.anu.edu.au/~brent/

phone: +61-4-18104021
3.3 Congruence Conditions for an OPN $N$

In this section, we revisit a theorem of Jacques Touchard dating back from 1953. Touchard proved that any odd perfect number must have the form $12m + 1$ or $36m + 9$. His proof relied on the fact that the numbers $\sigma(k)$ satisfy

$$\frac{n^2(n - 1)}{12} \sigma(n) = \sum_{k=1}^{n-1} \left[ 5k(n - k) - n^2 \right] \sigma(k) \sigma(n - k),$$

a recursion relation derived by Balth. van der Pol in 1951 using a nonlinear partial differential equation. We give here Judy Holdener’s proof of the same result in 2002, which is much shorter and more elementary than Touchard’s proof. The proof was inspired by Francis Coghlan’s solution to Problem #10711 published in the American Mathematical Monthly in 2001 regarding the nonexistence of two consecutive perfect numbers.

First, we briefly spell out a lemma on a congruence condition for an OPN $N$:

**Lemma 3.3.1.** If $N \equiv 5 \pmod{6}$, then $N$ is not perfect.

**Proof.** Assume that $N \equiv 5 \pmod{6}$. Then $N$ is of the form $6k + 5 = 3(2k + 1) + 2$, so $N \equiv 2 \pmod{3}$. Since all squares are congruent to 1 modulo 3, $N$ is not a square. Further, note that for any divisor $d$ of $N$, $N = d \cdot \left( \frac{N}{d} \right) \equiv 2 \equiv -1 \pmod{3}$ implies that either $d \equiv -1 \pmod{3}$ and $\frac{N}{d} \equiv 1 \pmod{3}$, or $d \equiv 1 \pmod{3}$ and $\frac{N}{d} \equiv -1 \pmod{3}$. Either way, $d + \left( \frac{N}{d} \right) \equiv 0 \pmod{3}$, and

$$\sigma(N) = \sum_{d|N, d < \sqrt{N}} \left( d + \frac{N}{d} \right) \equiv 0 \pmod{3}.$$
Therefore, \( \sigma(N) \equiv 0 \pmod{3} \) while \( 2N \equiv 4 \equiv 1 \pmod{3} \). These computations show that \( N = 6k + 5 \) cannot be perfect. \( \square \)

Using a similar argument, we can also show Euler’s result that any OPN is congruent to 1 modulo 4. For suppose otherwise that \( N \equiv 3 \pmod{4} \). Then, again, \( N \) is not a square and

\[
\sigma(N) = \sum_{d \mid N, d < \sqrt{N}} \left( d + \frac{N}{d} \right) \equiv 0 \pmod{4}.
\]

Hence, \( \sigma(N) \equiv 0 \pmod{4} \), while \( 2N \equiv 6 \equiv 2 \pmod{4} \).

Lemma 3.3.1 generalizes immediately to the following Corollary:

**Corollary 3.3.1.** If \( M \) is a number satisfying \( M \equiv 2 \pmod{3} \), then \( M \) is not perfect.

**Proof.** Note that Lemma 3.3.1 takes care of the case when \( M \) is odd. We now show that the statement is true for even \( M \).

To this end, suppose \( M \equiv 2 \pmod{3} \) is even. We show here that \( M \) cannot be perfect. Suppose to the contrary that \( M \) is even perfect. Then by the Euclid-Euler Theorem, \( M = 2^{p-1}(2^p - 1) \) for some prime number \( p \). If \( p = 2 \), then \( M = 6 \) which is divisible by 3. Assume \( p \geq 3 \). Then \( p \) is an odd prime. Thus, \( 2^p \equiv (-1)^p \equiv (-1)^{p-1}(-1) \equiv 1 \cdot (-1) \equiv -1 \equiv 2 \pmod{3} \). Therefore, \( 2^p - 1 \equiv 1 \pmod{3} \). Also, \( 2^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod{3} \) since \( p \) is assumed to be an odd prime.
Thus, if $M$ is to be even perfect, either $M \equiv 0 \pmod{3}$ (which occurs only when $p = 2$ and $M = 6$) or $M \equiv 1 \pmod{3}$ (when $p$ is a prime $\geq 3$). Consequently, if $M \equiv 2 \pmod{3}$ is even, it cannot be perfect. Lemma 3.3.1 says that this is also true when $M$ is odd. Hence, we have the general result: If a number $M$ satisfies $M \equiv 2 \pmod{3}$, then $M$ cannot be perfect. \qed

We will use the following formulation of the Chinese Remainder Theorem to prove the next major result:

**Theorem 3.3.1. Chinese Remainder Theorem**

Suppose $n_1, n_2, \ldots, n_k$ are integers which are pairwise relatively prime (or coprime). Then, for any given integers $a_1, a_2, \ldots, a_k$, there exists an integer $x$ solving the system of simultaneous congruences

\[
\begin{align*}
    x &\equiv a_1 \pmod{n_1} \\
    x &\equiv a_2 \pmod{n_2} \\
    &\vdots \\
    x &\equiv a_k \pmod{n_k}
\end{align*}
\]

Furthermore, all solutions $x$ to this system are congruent modulo the product $n = n_1 n_2 \cdots n_k$. Hence $x \equiv y \pmod{n_i}$ for all $1 \leq i \leq k$, if and only if $x \equiv y \pmod{n}$.

Sometimes, the simultaneous congruences can be solved even if the $n_i$’s are not pairwise coprime. A solution $x$ exists if and only if $a_i \equiv a_j \pmod{\gcd(n_i, n_j)}$ for all $i$ and $j$. All solutions $x$ are then congruent modulo the least common multiple of the $n_i$. 
We can now use Lemma 3.3.1 to prove Touchard’s theorem.

**Theorem 3.3.2.** *(Touchard)* An OPN must have the form $12m + 1$ or $36m + 9$.

**Proof.** Let $N$ be an OPN and apply Lemma 3.3.1. Any number of the form $6k + 5$ cannot be perfect, so $N$ must be of the form $6k + 1$ or $6k + 3$. But from Euler’s result, we know that $N$ is of the form $4j + 1$. Hence either $N = 6k + 1$ and $N = 4j + 1$, or $N = 6k + 3$ and $N = 4j + 1$. We now attempt to solve these two sets of simultaneous equations for $N$, thereby deriving congruence conditions for $N$:

**Case 1:** $N = 6k + 1$ and $N = 4j + 1$. This means that $N - 1 = 6k = 4j = \text{LCM}(4, 6)m = 12m$ (by the Chinese Remainder Theorem) where $k = 2m$ and $j = 3m$, which implies that $N = 12m + 1$.

**Case 2:** $N = 6k + 3$ and $N = 4j + 1$. This means that $N + 3 = 6k + 6 = 4j + 4 = 6(k + 1) = 4(j + 1) = \text{LCM}(4, 6)m = 12p$ (by the Chinese Remainder Theorem) where $k + 1 = 2p$ and $j + 1 = 3p$, which implies that $N = 12p - 3$. On setting $p = m_0 + 1$, we get $N = 12m_0 + 9$.

Finally, note that in Case 2, if $N = 12m_0 + 9$ and $3 \nmid m_0$, then $\sigma(N) = \sigma(3(4m_0 + 3)) = \sigma(3)\sigma(4m_0 + 3) = 4\sigma(4m_0 + 3)$. With this, we have $\sigma(N) \equiv 0 \pmod{4}$, while $2N = 2(12m_0 + 9) = 24m_0 + 18 = 4(6m_0 + 4) + 2 \equiv 2 \pmod{4}$. Therefore, $N$ cannot be perfect if $3 \nmid m_0$ in Case 2, and we conclude that $3 \mid m_0$ in this case, and on setting $m_0 = 3m$, we get $N = 12m_0 + 9 = 12(3m) + 9 = 36m + 9$. □
Remark 3.3.1. We emphasize that Touchard’s theorem is really simple. Holder’s proof as presented here is indeed elementary; it does not make use of the concept of unique factorization nor of sigma multiplicativity (other than in showing that 3 divides \( m \) when \( 12m + 9 \) is perfect). Touchard’s result emerges after summing divisors in pairs, and this can always be done because perfect numbers are never squares.

In January of 2008, Tim Roberts made a post at

http://www.unsolvedproblems.org/UP/Solutions.htm

where he outlined an improvement to Theorem 3.3.2.

Theorem 3.3.3. (Roberts) Let \( N \) be an OPN. Then either one of the following three congruences must hold:

- \( N \equiv 1 \) (mod 12).
- \( N \equiv 117 \) (mod 468).
- \( N \equiv 81 \) (mod 324).

Proof. Let \( N \) be an OPN. We note that, if \( 3 \mid N \), then \( 3^k \mid N \), where \( k \) is even (Euler). If \( k = 0 \), then by Theorem 3.3.2 \( N \equiv 1 \) (mod 12). Also, by the factor chain approach, if \( N \) is an OPN and a factor of \( N \) is \( 3^k \), then \( N \) is also divisible by \( \sigma(3^k) = 1 + 3 + 3^2 + \ldots + 3^k \). If \( k = 2 \), then again by Theorem 3.3.2 \( N \equiv 9 \) (mod 36). Further, since \( N \) is an OPN, we know that \( \sigma(3^2) = 1 + 3 + 3^2 = 13 \) divides \( N \). Hence, \( N \equiv 0 \) (mod 13). From the Chinese Remainder Theorem, we can deduce that \( N \equiv 117 \) (mod 468). If \( k > 2 \), then \( N \) is divisible by \( 3^4 = 81 \).
Thus, (again by Theorem 3.3.2) $N$ must satisfy both $N \equiv 9 \pmod{36}$ and $N \equiv 0 \pmod{81}$. Again, from the Chinese Remainder Theorem, we can deduce that $N \equiv 81 \pmod{324}$. Thus, if $N$ is an OPN, then either $N \equiv 1 \pmod{12}$, $N \equiv 117 \pmod{468}$ or $N \equiv 81 \pmod{324}$. \qed

It is, of course, similarly possible to further refine the last of these results, by separately considering even values of $k$ bigger than 4.

### 3.4 Some Interesting Results on Perfect Numbers

We conclude this chapter with the following (interesting) results on perfect numbers (with emphasis on OPNs):

- No two consecutive integers can be both perfect.
- An odd perfect number cannot be divisible by 105.
- An odd perfect number must be a sum of two squares.

#### 3.4.1 Nonexistence of Consecutive Perfect Numbers

From Corollary 3.3.1 we see that a number $M$ (odd or even) satisfying $M \equiv 2 \pmod{3}$ cannot be perfect. We make use of this observation to prove that no two consecutive integers can be both perfect. (This was shown to be true by James Riggs and Judy Holdener through a joint undergraduate research project in 1998.)
First, suppose \( N \) is an OPN. By Euler’s characterization of OPNs, \( N \equiv 1 \pmod{4} \). We claim that \( N + 1 \) cannot be an even perfect number. Observe that \( N + 1 \equiv 2 \pmod{4} \) means that \( 2 \mid (N + 1) \) but \( 4 \nmid (N + 1) \). The only even perfect number of the form \( N + 1 = 2^{p-1}(2^p - 1) \) satisfying these two conditions is the one for \( p = 2 \), i.e. \( N + 1 = 6 \). But this means that, by assumption, \( N = 5 \) must be an OPN, contradicting the fact that \( N = 5 \) is deficient.

Next, we also claim that \( N - 1 \) cannot be (even) perfect, if \( N \) is an OPN, where \( N - 1 \equiv 0 \pmod{4} \). From the discussion of the proof of Corollary 3.3.1 since \( N - 1 \equiv 0 \pmod{4} \) it follows that \( N - 1 = 2^{p-1}(2^p - 1) \) for some primes \( p \) and \( 2^p - 1 \) with \( p \geq 3 \). Thus, \( N - 1 \equiv 1 \pmod{3} \), which implies that \( N \equiv 2 \pmod{3} \). But our original assumption was that \( N \) is an OPN, contradicting the criterion in Corollary 3.3.1. Consequently, this means that \( N - 1 \) is not perfect in this case.

We have shown in the preceding paragraphs that, if \( N \) is an OPN of the form \( 4m + 1 \), then it cannot be true that \( N - 1 \) or \( N + 1 \) are also (even) perfect. To fully prove the assertion in the title of this section, we need to show that \( N - 1 \) and \( N + 1 \) cannot be OPNs if \( N \) is an even perfect number.

To this end, suppose \( N \) is even perfect, that is, \( N = 2^{p-1}(2^p - 1) \) for some primes \( p \) and \( 2^p - 1 \). If \( p = 2 \), then \( N = 6 \), and clearly, \( N - 1 = 5 \) and \( N + 1 = 7 \) are not perfect since they are both primes (and are therefore deficient).
Now let $p$ be a prime which is at least 3. Then $N \equiv 0 \pmod{4}$, whence it follows that $N - 1 \equiv 3 \pmod{4}$ and $N - 1$ cannot be an OPN by Euler’s characterization. Also, from the proof of Corollary 3.3.1 note that if $N$ is an even perfect number with $p \geq 3$, then $N \equiv 1 \pmod{3}$, which implies that $N + 1 \equiv 2 \pmod{3}$, which further means that $N + 1$ cannot be perfect by the criterion in Corollary 3.3.1.

### 3.4.2 OPNs are Not Divisible by 105

Mathematicians have been unable, so far, to eliminate the possibility that an odd perfect number may be divisible by 3. However, by use of Lemma 3.1.2, we can show that an odd number divisible by 3, 5 and 7 cannot be perfect. (This was proved by Sylvester in 1888.)

To this end, suppose that $N$ is an OPN that is divisible by 3, 5, and 7. Then $N$ takes the form $N = 3^a 5^b 7^c n$. Suppose that 5 is the special/Euler prime of $N$, so that $b \geq 1$. By Euler’s characterization of OPNs, $a$ and $c$ must be even, so we may take $a \geq 2$ and $c \geq 2$. Without loss of generality, since $\omega(N) \geq 9$ (see Remark 3.1.1) we may safely assume that $n > 11^2$. Then $I(n) > 1$, and consequently, by use of Lemma 3.1.2, we have:

$$2 \geq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7} + \frac{1}{7^2}\right) I(n)$$

$$> \frac{13657}{9549} \cdot 1 = \frac{4446}{2205} > 2.0163 > 2$$

resulting in the contradiction $2 > 2$. 

If, in turn, we assume that 5 is not the special/Euler prime of $N$ (so that $b \geq 2$), then without loss of generality, since $n$ must contain the special/Euler prime and $\omega(N) \geq 9$, we can assume that $n > 13$. In this case, it is still true that $I(n) > 1$ (also that $a \geq 2$ and $c \geq 2$). Hence, by use of Lemma 3.1.2, we obtain:

$$
2 \geq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2}\right) \left(1 + \frac{1}{7} + \frac{1}{7^2}\right) I(n)
$$

$$
> \frac{133157}{92549} \cdot 1 = \frac{22971}{11025} > 2.0835 > 2
$$

resulting, again, in the contradiction $2 > 2$.

We are therefore led to conclude that an OPN cannot be divisible by $3 \cdot 5 \cdot 7 = 105$.

### 3.4.3 OPNs as Sums of Two Squares

We borrow heavily the following preliminary material from [http://en.wikipedia.org/wiki/Proofs_of_Fermat’s_theorem_on_sums_of_two_squares](http://en.wikipedia.org/wiki/Proofs_of_Fermat’s_theorem_on_sums_of_two_squares). This is in view of the fact that the special/Euler prime $p$ of an OPN $N = p^k m^2$ satisfies $p \equiv 1 \pmod{4}$.

Fermat’s theorem on sums of two squares states that an odd prime $p$ can be expressed as $p = x^2 + y^2$ with $x$ and $y$ integers *if and only if* $p \equiv 1 \pmod{4}$. It was originally announced by Fermat in 1640, but he gave no proof. The *only if* clause is trivial: the squares modulo 4 are 0 and 1, so $x^2 + y^2$ is congruent to 0, 1, or 2 modulo 4. Since $p$ is assumed to be odd, this means that it must be congruent to 1 modulo 4.
Euler succeeded in proving Fermat’s theorem on sums of two squares in 1747, when he was forty years old. He communicated this in a letter to Goldbach dated 6 May 1747. The proof relies on infinite descent, and proceeds in five steps; we state the first step from that proof below as we will be using it in the next paragraph:

- The product of two numbers, each of which is a sum of two squares, is itself a sum of two squares.

Given an OPN $N = p^km^2$, since $p \equiv 1 \pmod{4}$, by Fermat’s theorem we can write $p$ as a sum of two squares. By Euler’s first step above, $p^k$ can likewise be expressed as a sum of two squares, $p^k$ being the product of $k$ $p$’s. Hence, we can write $p^k = r^2 + s^2$ for some positive integers $r$ and $s$. Multiplying both sides of the last equation by $m^2$, we get $N = p^km^2 = m^2(r^2 + s^2) = (mr)^2 + (ms)^2$. Hence, an odd perfect number may be expressed as a sum of two squares.

**Remark 3.4.1.** Let $\theta(n)$ be the number of integers $k \leq n$ that can be expressed as $k = a^2 + b^2$, where $a$ and $b$ are integers. Does the limit $\lim_{n \to \infty} \frac{\theta(n)}{n}$ exist and what is its value? Numerical computations suggest that it exists. $	heta(n)$ is approximately $\left(\frac{3}{4}\right) \cdot \frac{n}{\sqrt{\log(n)}}$, so that the limit exists and equals zero. More precisely, the number of integers less than $n$ that are sums of two squares behaves like $K \cdot \frac{n}{\sqrt{\log(n)}}$ where $K$ is the Landau-Ramanujan constant. Dave Hare from the Maple group has computed 10000 digits of $K$. The first digits are $K \approx 0.764223653...$. See [http://www.mathsoft.com/asolve/constant/lr/lr.html](http://www.mathsoft.com/asolve/constant/lr/lr.html) for more information. Therefore the sums of two squares have density 0. Since OPNs are expressible as sums of two squares, then OPNs have density 0, too.
In the next chapter, we shall take a closer look into the nature of abundancy outlaws (which were first described in Section 2.2). We shall also describe a systematic procedure on how to bound the prime factors of an OPN $N$, using the latest current knowledge on $N$ as well as some novel results. Lastly, we shall discuss some of the original results of the author pertaining to inequalities between the components of an OPN.
Chapter 4

OPN Solution Attempts 2:
Some New Approaches

In Chapter 3, we saw how increasing the lower bounds for \( \omega(N) \) (the number of distinct prime factors of an OPN \( N \)) and \( N \) itself could potentially prove or disprove the OPN Conjecture. We also saw how the concept of divisibility may be used to derive congruence conditions for \( N \).

Here, we shall take a closer look into the following new approaches for attempting to solve the OPN Problem:

- What are abundancy outlaws? How are they related to abundancy indices? How could one use the concept of abundancy outlaws to (potentially) disprove the OPN Conjecture?

- How can one bound the prime factors of an OPN \( N \)? Is there a systematic procedure on how to do this? (We discuss the author's results on the relationships between the components of \( N \) in Subsection 4.2.4.)
Can we use the abundancy index concept to “count” the number of OPNs? (We answer this question in the negative for a particular case.)

The reader is advised to review Section 2.2 of this thesis prior to commencing a study of this chapter.

4.1 Abundancy Outlaws and Related Concepts

Modern treatments of problems involving the abundancy index have been concerned with two fundamental questions:

I. Given a rational number $\frac{a}{b}$, does there exist some positive integer $x$ such that
   \[ I(x) = \frac{\sigma(x)}{x} = \frac{a}{b}? \]
II. When does the equation $I(x) = \frac{a}{b}$ have exactly one solution for $x$?

We give various answers to these two questions in the three subsections that follow.

4.1.1 Friendly and Solitary Numbers

If $x$ is the unique solution of $I(x) = \frac{a}{b}$ (for a given rational number $\frac{a}{b}$) then $x$ is called a solitary number. On the other hand, if $x$ is one of at least two solutions of $I(x) = \frac{a}{b}$ (for a given rational number $\frac{a}{b}$) then $x$ is called a friendly number. We formalize these two concepts in the following definition:

**Definition 4.1.1.** Let $x$ and $y$ be distinct positive integers. If $x$ and $y$ satisfy the equation $I(x) = I(y)$ then $(x, y)$ is called a friendly pair. Each member of
the pair is called a *friendly number*. A number which is not friendly is called a *solitary number*.

We illustrate these concepts with several examples.

**Example 4.1.1.** Clearly, if \( a \) and \( b \) are perfect numbers with \( a \neq b \) (i.e. \( \sigma(a) = 2a, \sigma(b) = 2b \)), then \((a, b)\) is a friendly pair.

**Example 4.1.2.** We claim that, given a positive integer \( n \) satisfying \( \gcd(n, 42) = 1 \), \((6n, 28n)\) is a friendly pair. To prove this, note that \( \gcd(n, 42) = 1 \) means that \( \gcd(n, 2) = \gcd(n, 3) = \gcd(n, 7) = 1 \). Let us now compute \( I(6n) \) and \( I(28n) \) separately. Since \( \gcd(n, 2) = \gcd(n, 3) = 1 \), then \( \gcd(n, 6) = 1 \) and \( I(6n) = I(6)I(n) = 2I(n) \) since 6 is a perfect number. Similarly, since \( \gcd(n, 2) = \gcd(n, 7) = 1 \), then \( \gcd(n, 28) = 1 \) and \( I(28n) = I(28)I(n) = 2I(n) \) since 28 is a perfect number. These computations show that \( I(6n) = I(28n) = 2I(n) \) whenever \( \gcd(n, 42) = 1 \), and therefore \((6n, 28n)\) is a friendly pair for such \( n \).

**Remark 4.1.1.** Since there exist infinitely many positive integers \( n \) satisfying \( \gcd(n, 42) = 1 \), Example 4.1.2 shows that there exist infinitely many friendly numbers.

**Example 4.1.3.** M. G. Greening showed in 1977 that numbers \( n \) such that \( \gcd(n, \sigma(n)) = 1 \), are solitary. For example, the numbers 1 through 5 are all solitary by virtue of Greening’s criterion. There are 53 numbers less than 100, which are known to be solitary, but there are some numbers, such as 10, 14, 15, and 20 for which we cannot decide “solitude”. (This is because it is, in general, difficult to determine whether a particular number is solitary, since the only tool that we
have so far to make such determination, namely Greening’s result, is sufficient but not necessary. In the other direction, if any numbers up to 372 (other than those listed in the Online Encyclopedia of Integer Sequences) are friendly, then the smallest corresponding values of the friendly pairs are $> 10^{30}$ [31].) Also, we remark that there exist numbers such as $n = 18, 45, 48$ and 52 which are solitary but for which $\gcd(n, \sigma(n)) \neq 1$.

We give here a proof of Greening’s criterion. Suppose that a number $n$ with $\gcd(n, \sigma(n)) = 1$ is not solitary. Then $n$ is friendly, i.e. there exists some number $x \neq n$ such that $I(n) = I(x)$ . This is equivalent to $x\sigma(n) = n\sigma(x)$, which implies that, since $\gcd(n, \sigma(n)) = 1$, $n \mid x$ or $x$ is a multiple of $n$. Thus any friend of $n$ must be a (nontrivial) multiple of it (since $n \neq x$). Hence we can write $x = mn$ where $m \geq 2$. Write $m = jk$ with $\gcd(j, n) = 1$ and $\gcd(k, n) > 1$. Then by virtue of Lemma 2.2.2, $I(x) > I(kn)$ since $kn$ is a factor of $x$ (unless $j = 1$), which implies that $I(x) > I(n)$ (since $k > 1$ and this follows from $\gcd(k, n) > 1$). This is a contradiction. If $j = 1$, then we have $x = kn$ with $\gcd(k, n) > 1$. Again, by virtue of Lemma 2.2.2 and similar considerations as before, $I(x) > I(n)$ (unless $k = 1$, but this cannot happen since $jk = m \geq 2$) which is again a contradiction. Thus, numbers $n$ with $\gcd(n, \sigma(n)) = 1$ are solitary.

**Example 4.1.4.** We claim that primes and powers of primes are solitary. It suffices to show that $p^k$ and $\sigma(p^k)$ are relatively prime. To this end, consider the equation $(p-1)\sigma(p^k) = p^{k+1} - 1$. Since this can be rewritten as $(1-p)\sigma(p^k) + p \cdot p^k = 1$, then we have $\gcd(p^k, \sigma(p^k)) = 1$. By Greening’s criterion, $p^k$ is solitary. Thus, primes and powers of primes are solitary.
Remark 4.1.2. Since there are infinitely many primes (first proved by Euclid), and therefore infinitely many prime powers, Example 4.1.4 shows that there are infinitely many solitary numbers.

Remark 4.1.3. While not much is known about the nature of solitary numbers, we do know that the density of friendly numbers is positive, first shown by Erdős [17]: The number of solutions of $I(a) = I(b)$ satisfying $a < b \leq x$ equals $Cx + o(x)$, where $C > 0$ is a constant (in fact, $C \geq \frac{8}{147}$ [2]). In 1996, Carl Pomerance told Dean Hickerson that he could prove that the solitary numbers have positive density, disproving a conjecture by Anderson and Hickerson in 1977. However, this proof seems not to ever have been published.

4.1.2 Abundancy Indices and Outlaws

On the other hand, rational numbers $\frac{a}{b}$ for which $I(x) = \frac{a}{b}$ has no solution for $x$ are called abundancy outlaws. (Recall Definition 2.2.3.) Of course, those rationals $\frac{a}{b}$ for which $I(x) = \frac{a}{b}$ has at least one solution for $x$ are called abundancy indices.

It is best to illustrate with some examples.

Example 4.1.5. At once, Lemma 2.2.5 reveals a class of abundancy outlaws. Since that lemma says that $\frac{m}{n}$ is an outlaw when $1 < \frac{m}{n} < \frac{\sigma(n)}{n}$ (with $\gcd(m, n) = 1$), then we have the class $\frac{\sigma(N) - t}{N}$ of outlaws (with $t \geq 1$). (We shall show later that, under certain conditions, $\frac{\sigma(N) + t}{N}$ is also an abundancy outlaw.)
Example 4.1.6. Let $a, b, c$ be positive integers, and let $p$ be a prime such that $\gcd(a, p) = 1$, $b = p^c$ (so that $b$ is a prime power, and $\gcd(a, b) = 1$), and $a \geq \sigma(b)$.

Suppose we want to find a positive integer, $n$, such that $\frac{\sigma(n)}{n} = \frac{a}{b}$. (That is, we want to determine if $\frac{a}{b}$ is an abundancy index or not.) This problem is equivalent to the problem of finding positive integers $m, k$ such that:

- $n = mp^k, k \geq c$, and $\gcd(m, p) = 1$ (or equivalently, $\gcd(m, b) = 1$)
- $\frac{\sigma(m)}{m} = \frac{ap^{k-c}}{\sigma(p^k)}$.

We will formally state this result as a lemma later (where we will then present a proof), but for now let us see how we may apply this result towards showing that the fraction $\frac{7}{2}$ is an abundancy index. (Indeed, we are then able to construct an explicit $n$ satisfying $\frac{\sigma(n)}{n} = \frac{a}{b}$ for a given $\frac{a}{b}$.)

We now attempt to find a positive integer $n$ such that $\frac{\sigma(n)}{n} = \frac{7}{2}$. This problem, is equivalent to the problem of finding positive integers $m, k$ such that:

1.1 $n = 2^k m$, where $\gcd(2, m) = 1$ (that is, $k$ is the largest power of 2 to divide $n$)
1.2 $\frac{\sigma(m)}{m} = \frac{7 \cdot 2^{k-1}}{\sigma(2^k)}$.

We will now check different values of $k$, attempting each time to find $m$ satisfying these conditions subject to the choice of $k$. For each $k$, we will proceed until one of the following happens:

- We find $m$ satisfying (1.1) and (1.2). In this case, $n = 2^k m$ is a solution to our problem.
We prove that there is no \( m \) satisfying (1.1) and (1.2). In this case, there is no solution to our problem of the form \( n = 2^k m \), where \( \gcd(2, m) = 1 \).

The problem becomes impractical to pursue. Often a given value of \( k \) will leave us with a problem which either cannot be solved with this method, or which is too complicated to be solved in a reasonable amount of time.

We start with \( k = 1 \). Then our conditions are:

\[
\begin{align*}
2.1 \quad & n = 2m, \text{ and } \gcd(m, 2) = 1 \\
2.2 \quad & \frac{\sigma(m)}{m} = \frac{7 \cdot 2^{1-1}}{\sigma(2^1)} = \frac{7}{3}.
\end{align*}
\]

Thus, our problem is to find \( m \) satisfying (2.1) and (2.2). Let us carry the process one step further for the case \( k = 1 \). To do this, we will treat \( m \) in the same manner in which we initially treated \( n \). Our goal is to find \( m_1, k_1 \) such that:

\[
\begin{align*}
3.1 \quad & m = 3^{k_1} m_1, \text{ and } \gcd(m_1, 3) = 1. \text{ (Note that since } m_1 \mid m, \text{ and } \gcd(m, 2) = 1, \\
& \text{ we actually need } \gcd(m_1, 6) = 1.) \\
3.2 \quad & \frac{\sigma(m_1)}{m_1} = \frac{7 \cdot 3^{k_1-1}}{\sigma(3^{k_1})}.
\end{align*}
\]

Let us now check the case of \( k = 1, k_1 = 1 \). Our goal is to find \( m_1 \) such that:

\[
\begin{align*}
4.1 \quad & m = 3m_1, \text{ and } \gcd(m_1, 6) = 1 \\
4.2 \quad & \frac{\sigma(m_1)}{m_1} = \frac{7}{4}.
\end{align*}
\]
Thus, we want to find some positive integer $m_1$ such that $\gcd(m_1, 6) = 1$ and $\frac{\sigma(m_1)}{m_1} = \frac{7}{4}$. However, if $\frac{\sigma(m_1)}{m_1} = \frac{7}{4}$, then $4 \mid m_1$ (since $\gcd(4, 7) = 1$). If $4 \mid m_1$, then $\gcd(m_1, 6) \neq 1$, a contradiction. Therefore, there is no such $m_1$.

Our method has shown that, in the case of $k = 1$, $k_1 = 1$, there is no positive integer $n$ which solves our original problem. In particular, $n$ is not of the form $n = 2^1 m = 2^1 (3^1 (m_1)) = 6m_1$, where $\gcd(m_1, 6) = 1$. (Another way to say this is that $6$ is not a unitary divisor of any solution to our problem.)

(Note that this does not prove the nonexistence of a solution to our problem; it only disproves the existence of a solution of the form given in the last paragraph. In order to disprove the existence of a solution of any given problem, we have to show that no solution exists for any value of $k$. Here we have not even eliminated the case of $k = 1$, but only the special case where $k_1 = 1$.)

Let us move on to $k = 2$. Our goal is to find $m$ such that:

5.1 $n = 2^2 m = 4m$, with $\gcd(m, 2) = 1$

5.2 $\frac{\sigma(m)}{m} = \frac{7 \cdot 2^{2-1}}{\sigma(2^2)} = \frac{14}{7} = 2$.

Here, we must find $m$ such that $m$ is odd and $\frac{\sigma(m)}{m} = 2$; that is, $m$ must be an odd perfect number. If $m$ is an odd perfect number, then $n = 4m$ is a solution to our problem. This is not especially helpful in our search for a solution, so we will move on to another case.
Here we will skip the cases $k = 3$ and $k = 4$, because they are not especially interesting compared to the next case we will deal with.

Consider $k = 5$. Our goal now is to find $m$ such that:

\[
\begin{align*}
6.1 \quad & n = 2^5m = 32m, \text{ and } \gcd(m, 2) = 1 \\
6.2 \quad & \frac{\sigma(m)}{m} = \frac{7 \cdot 2^4}{\sigma(2^5)} = \frac{112}{63} = \frac{16}{9}.
\end{align*}
\]

We can now apply our method to $m$. Keep in mind that the process will be slightly different this time, since the denominator of $\frac{16}{9}$ is a prime power, not just a prime. Here, $b = 9 = 3^2$, so we will use $p = 3$ and $c = 2$ (as they are used in the beginning of this example). Our goal is to find positive integers $m_1, k_1$ such that:

\[
\begin{align*}
7.1 \quad & m = m_1 \cdot 3^{k_1}, k_1 \geq 2, \text{ and } \gcd(m_1, 3) = 1. \text{ (Note that } m_1 \mid m, \text{ and} \\
& \gcd(m, 2) = 1, \text{ so } \gcd(m_1, 6) = 1. \\
7.2 \quad & \frac{\sigma(m_1)}{m_1} = \frac{16 \cdot 3^{k_1-2}}{\sigma(3^{k_1})}.
\end{align*}
\]

We will consider two of the possible cases here: $k_1 = 2$ and $k_1 = 3$.

First, let $k_1 = 2$. We get

\[
\frac{\sigma(m_1)}{m_1} = \frac{16}{\sigma(3^2)} = \frac{16}{13}.
\]

Carrying the process one step further, we will search for such an $m_1$. We must find positive integers $m_2, k_2$ such that:
8.1 $m_1 = m_2 \cdot 13^{k_2}$, and $\gcd(m_2, 13) = 1$. (Note that $m_2 \mid m_1$, so $\gcd(m_2, 6) = \gcd(m_2, 13) = 1$.)

8.2 $\frac{\sigma(m_2)}{m_2} = \frac{16 \cdot 13^{k_2 - 1}}{\sigma(13^{k_2})}$.

Let $k_2 = 1$. Then $m_1 = 13m_2$, and

$$\frac{\sigma(m_2)}{m_2} = \frac{16}{14} = \frac{8}{7}.$$}

Let $m_2 = 7$. Then $m_2$ satisfies both (8.1) and (8.2) of the case $k = 5, k_1 = 2, k_2 = 1$. This gives us a solution; all we have to do now is work backwards until we get $n$.

First, $m_1 = 13m_2 = 13 \cdot 7$. Next, $m = 3^{k_1}m_1 = 13 \cdot 7 \cdot 3^2$. Finally, $n = 2^5m = 2^5 \cdot 3^2 \cdot 7 \cdot 13 = 26208$.

Thus, we have found a solution to the problem $\frac{\sigma(n)}{n} = \frac{7}{2}$.

Now we will consider the case $k_1 = 3$, which will give us one more solution:

$$\frac{\sigma(m_1)}{m_1} = \frac{16 \cdot 3}{\sigma(3^3)} = \frac{48}{40} = \frac{6}{5}.$$  

That is, we need to find $m_1$ such that $\frac{\sigma(m_1)}{m_1} = \frac{6}{5}$, and $\gcd(m_1, 6) = 1$. If we let $m_1 = 5$, then we have solved this problem, and have thus discovered another solution of the problem $\frac{\sigma(n)}{n} = \frac{7}{2}$, this time for the case $k = 5, k_1 = 3$. Again, we work backwards until we get $n$.

First, $m = 3^{k_1}m_1 = 3^3 \cdot 5$. Next, $n = 2^5m = 2^5 \cdot 3^3 \cdot 5 = 4320$.  

This is a second solution to the problem \( \frac{\sigma(n)}{n} = \frac{7}{2} \).

**Remark 4.1.4.** Example 4.1.6 shows that \( \frac{7}{2} \) is an abundancy index. Also, since the equation \( I(n) = \frac{7}{2} \) has at least two solutions, namely \( n_1 = 4320 \) and \( n_2 = 26208 \), this implies that \( n_1 \) and \( n_2 \) here are friendly.

Prior to discussing the proof of the lemma outlined in Example 4.1.6, we review some known properties of the abundancy index:

**Lemma 4.1.1. Properties of the Abundancy Index**

- If \( a = \prod_{j=1}^{k} p_j^{n_j} \), where \( p_1, p_2, p_3, \ldots, p_k \) are distinct primes, \( k \) is a positive integer, and the integral exponents \( n_1, n_2, n_3, \ldots, n_k \) are nonnegative, then
  \[
  I(a) = \prod_{j=1}^{k} \frac{p_j^{n_j+1} - 1}{p_j^{n_j}(p_j - 1)},
  \]
  where \( I \) is multiplicative.

- If \( p \) is prime then the least upper bound for the sequence \( \{I(p^n)\}_{n=0}^{\infty} \) is \( \frac{p}{p - 1} \).

- If \( a_1 | a \) and \( a_1 > 0 \), then \( I(a) \geq I(a_1) \).

- Obviously, if \( a_1 | a \) and \( I(a) = I(a_1) \), then \( a = a_1 \).

- If \( \gcd(a, \sigma(a)) = 1 \), then the unique solution to \( I(x) = I(a) \) is \( x = a \).

**Proof.** Only the last assertion is not so obvious. Note that \( I(x) = I(a) \) is equivalent to \( a \cdot \sigma(x) = x \cdot \sigma(a) \). If \( a \) and \( \sigma(a) \) are coprime, then \( a | x \) (so that \( a \) divides every solution). By the fourth result of Lemma 4.1.1 \( x = a \) is the sole solution. \( \blacksquare \)
We discuss some known properties that can help us decide whether a particular fraction $\frac{r}{s}$ is an abundancy index:

**Lemma 4.1.2.** When is $\frac{r}{s}$ an abundancy index?

For which rational numbers $\frac{r}{s}$ will

$$I(x) = \frac{r}{s} \quad (*)$$

have at least one solution? In order for $(*)$ to have solutions, $\frac{r}{s}$ must be greater than or equal to one. If $\frac{r}{s} = 1$, then by Lemma 4.1.1, $x = 1$ is the unique solution. Throughout the rest of this lemma, it will be assumed that $r$ and $s$ represent given positive integers which are relatively prime, and that $r > s$. Let us now state some known results:

- Note that $s$ must divide (every solution for) $x$ in $(*)$. This property is easy to observe since $I(x) = \frac{r}{s}$ implies that $s \cdot \sigma(x) = r \cdot x$, and $\gcd(r, s) = 1$.

- If a solution to $(*)$ exists, then $r \geq \sigma(s)$ (since $\frac{r}{s} = I(x) \geq I(s) = \frac{\sigma(s)}{s}$).

- If $I(a) > \frac{r}{s}$, then $(*)$ has no solution which is divisible by $a$. Additionally, if $s$ is divisible by $a$, then $(*)$ has no solution.

- $\{I(b) : b \in \mathbb{Z}^+\}$ is dense in the interval $(1, \infty)$.

- The set of values $\frac{r}{s}$ for which $(*)$ has no solution is also dense in $(1, \infty)$.

**Proof.** Here, we prove the third assertion. Suppose $(*)$ has a solution which is divisible by $a \in \mathbb{Z}^+$. Then $a \mid x$, and by the third result in Lemma 4.1.1, $I(x) \geq I(a)$. But then, by assumption $I(a) > \frac{r}{s}$, which implies that $I(a) > \frac{r}{s} = I(x) \geq I(a)$, resulting in the contradiction $I(a) > I(a)$. Thus, $(*)$ has no solution.
which is divisible by $a$. The second statement follows from this and the first result in Lemma 4.1.2 (i.e. $s$ must divide every solution for $x$).

We now state and prove the following lemma (taken from [47]) which was used in Example 4.1.6.

**Lemma 4.1.3. (Ludwick)** Let $a, b, c \in \mathbb{Z}^+$, and let $p$ be a prime such that $\gcd(a, p) = 1$, $b = p^c$ (so that $b$ is a prime power, and $\gcd(a, b) = 1$), and $a \geq \sigma(b)$. Suppose we want to find a positive integer, $n$, such that $\frac{\sigma(n)}{n} = \frac{a}{b}$. This problem is equivalent to the problem of finding positive integers $m, k$ such that:

- $n = mp^k, k \geq c$, and $\gcd(m, p) = 1$ (or equivalently, $\gcd(m, b) = 1$)

  $\frac{\sigma(m)}{m} = \frac{ap^{k-c}}{\sigma(p^k)}$.

**Proof.** First, we will show that if we can find $n \in \mathbb{Z}^+$ satisfying $\frac{\sigma(n)}{n} = \frac{a}{b}$, then we can find $m, k \in \mathbb{Z}^+$ satisfying the two itemized conditions above.

Suppose we have $n \in \mathbb{Z}^+$ such that $\frac{\sigma(n)}{n} = \frac{a}{b}$. Then by Lemma 4.1.2, $b \mid n$; that is, $p^c \mid n$. Since $b$ is a prime power, there is some $k \in \mathbb{Z}^+$ such that $n = mp^k$, with $\gcd(m, b) = 1$. Here, $k$ is the largest power of $p$ that divides $n$, so clearly $k \geq c$. This satisfies the first condition. Now, we have

$$\frac{\sigma(n)}{n} = \frac{a}{b} = \frac{a}{p^c}$$

and we also have

$$\frac{\sigma(n)}{n} = \frac{\sigma(p^k) \cdot \sigma(m)}{p^k \cdot m}.$$

Thus,
\[ \frac{a}{p^c} = \frac{\sigma(p^k) \sigma(m)}{p^k m}, \]

and so
\[ \frac{\sigma(m)}{m} = \frac{ap^{k-c}}{\sigma(p^k)}. \]

This satisfies the second condition. Therefore, solving \( \frac{\sigma(n)}{n} = \frac{a}{b} \) for \( n \) gives us \( m, k \in \mathbb{Z}^+ \) satisfying the two conditions.

Conversely, we will show that if we can find \( m, k \in \mathbb{Z}^+ \) satisfying the two conditions, then we can find \( n \in \mathbb{Z}^+ \) such that \( \frac{\sigma(n)}{n} = \frac{a}{b} \).

Suppose we have \( m, k \in \mathbb{Z}^+ \) satisfying the two conditions. Let \( n = mp^k \).

Then,
\[ \frac{\sigma(n)}{n} = \frac{\sigma(m) \sigma(p^k)}{m p^k} = \frac{ap^{k-c} \sigma(p^k)}{\sigma(p^k) p^k} = \frac{ap^{k-c}}{p^c} = \frac{a}{b}. \]

\( \square \)

### 4.1.3 OPNs, Abundancy Outlaws and the Fraction \( \frac{p^2 + 2}{p} \)

After defining the abundancy index and exploring various known properties, we briefly discuss some related concepts. Positive integers having integer-valued abundancy indices are said to be *multiperfect numbers*. One is the only odd multiperfect that has been discovered. Richard Ryan hopes that his “study of the abundancy index will lead to the discovery of other odd multiperfects”, or to the proof of their nonexistence. Since the abundancy index of a number \( n \) can be thought of as a measure of its perfection (i. if \( I(n) < 2 \) then \( n \) is deficient; ii. if \( I(n) = 2 \) then \( n \) is perfect; and iii. if \( I(n) > 2 \) then \( n \) is abundant), it is
fitting to consider it a very useful tool in gaining a better understanding of perfect numbers. In fact, Judy Holdener [32] proved the following theorem which provides conditions equivalent to the existence of an OPN:

**Theorem 4.1.1.** There exists an odd perfect number if and only if there exist positive integers $p$, $n$ and $\alpha$ such that $p \equiv \alpha \equiv 1 \pmod{4}$, where $p$ is a prime not dividing $n$, and $I(n) = \frac{2p^\alpha(p - 1)}{p^{\alpha + 1} - 1}$.

**Proof.** By Euler’s characterization of an OPN $N = p^\alpha m^2$, it must be true that $p$ is a prime satisfying $\gcd(p, m) = 1$ and $p \equiv \alpha \equiv 1 \pmod{4}$. Hence $\sigma(N) = \sigma(p^\alpha m^2) = \sigma(p^\alpha)\sigma(m^2) = 2p^\alpha m^2$, and

$$I(m^2) = \frac{\sigma(m^2)}{m^2} = \frac{2p^\alpha}{\sigma(p^\alpha)} = \frac{2p^\alpha(p - 1)}{p^{\alpha + 1} - 1}.$$ 

This proves the forward direction of the theorem.

Conversely, assume there is a positive integer $n$ such that $I(n) = \frac{2p^\alpha(p - 1)}{p^{\alpha + 1} - 1}$, where $p \equiv \alpha \equiv 1 \pmod{4}$ and $p$ is a prime with $p \nmid n$. Then

$$I(n \cdot p^\alpha) = I(n) \cdot I(p^\alpha) = \frac{2p^\alpha(p - 1)}{p^{\alpha + 1} - 1} \cdot \frac{p^{\alpha + 1} - 1}{p^\alpha(p - 1)} = 2.$$ 

So $n \cdot p^\alpha$ is a perfect number.

Next, we claim that $n \cdot p^\alpha$ cannot be even. Suppose to the contrary that $n \cdot p^\alpha$ is even. Then it would have the Euclid-Euler form for even perfect numbers:

$$n \cdot p^\alpha = 2^{m-1}(2^m - 1)$$

where $2^m - 1$ is prime. Since $2^m - 1$ is the only odd prime factor on the RHS, $p^\alpha = p^1 = 2^m - 1$. But $p \equiv 1 \pmod{4}$ and $2^m - 1 \equiv 3 \pmod{4}$ (because $m$ must
be at least 2 in order for $2^m - 1$ to be prime). This is clearly a contradiction, and thus $n \cdot p^\alpha$ is not even. Consequently, $n \cdot p^\alpha$ is an OPN.

By Theorem 4.1.1 it follows that if one could find an integer $n$ having abundancy index equal to $\frac{5}{3}$ (which occurs as a special case of the theorem, specifically for $p = 5$ and $\alpha = 1$), then one would be able to produce an odd perfect number. Here we then realize the usefulness of characterizing fractions in $(1, \infty)$ that are abundancy outlaws. (Recall Definition 2.2.3)

Let us now consider the sequence of rational numbers in $(1, \infty)$. (Note that, since the number 1 is solitary and $I(1) = 1$, the equation $I(x) = 1$ has the lone solution $x = 1$.) For each numerator $a > 1$, we list the fractions $\frac{a}{b}$ with $\gcd(a, b) = 1$, so that the denominators $1 \leq b < a$ appear in increasing order:

$\frac{2}{1}, \frac{3}{1}, \frac{3}{2}, \frac{4}{1}, \frac{4}{3}, \frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, \frac{6}{1}, \frac{6}{5}, \frac{7}{1}, \frac{7}{2}, \frac{7}{3}, \frac{7}{4}, \frac{7}{5}, \frac{7}{6}, \cdots$

It is intuitive that each term in this sequence must be either an abundancy index or an abundancy outlaw, but it is, in general, difficult to determine the status of a given fraction. We may thus partition the sequence into three (3) categories: (I) those fractions that are known to be abundancy indices, (II) those that are known to be abundancy outlaws, and (III) those whose abundancy index/outlaw status is unknown. We wish to capture outlaws from the third category, thereby increasing the size of the second category. Since fractions of the form $\frac{\sigma(N) - t}{N}$ for $t \geq 1$ belong to the first category (by Lemma 2.2.5 and Example 4.1.5), it is tempting to consider fractions of the form $\frac{\sigma(N) + t}{N}$. Judy Holdener and William Stanton proved in 2007 [34] that, under certain conditions, $\frac{\sigma(N) + t}{N}$ is
an abundancy outlaw. They noted that their original interest in such fractions stemmed from their interest in the fraction \(\frac{5}{3} = \frac{\sigma(3) + 1}{3}\). Unfortunately, the results they obtained do not allow them to say anything about fractions of the form \(\frac{\sigma(p) + 1}{p} = \frac{p + 2}{p}\). Such elusive fractions remain in category three.

Equivalently, we may ask: Does there exist an odd number \(s \in \mathbb{Z}^+\) (with \(s > 1\)) such that \(I(x) = \frac{s + 2}{s}\) has at least one solution? The answer to this question is unknown, but we can go ahead and discuss some properties.

By Lemma \(4.1.2\), \(s \mid x\). Assume \(s\) is an odd composite; then \(\sigma(s) \geq 1 + s + d\), where \(d\) is a divisor of \(s\) satisfying \(1 < d < s\). Since \(s\) is odd, \(d \geq 3\), which means that \(\sigma(s) \geq s + 4 > s + 2\), or \(I(s) > \frac{s + 2}{s}\), a contradiction. Hence, \(s\) must be prime. If \(1 < c < s\), then \(\gcd(c, s) = 1\) (since \(s\) is prime), and we have:

\[
I(cs) = I(c) \cdot I(s) \geq \frac{c + 1}{c} \cdot \frac{s + 1}{s} = \left(\frac{s + 1}{s}\right)^2 \geq \frac{s + 2}{s}.
\]

Thus, \(x\) does not have a factor between 1 and \(s\). Moreover, \(x\) is a perfect square; otherwise \(\sigma(x)\) would have a factor of 2 that cannot be “canceled” since the denominator, \(x\), is odd. (For the same reason, whenever \(r\) and \(s\) are both odd, any odd solution to \(I(x) = \frac{r}{s}\) must be a perfect square.) We also claim that \((s + 2) \nmid x\) and we prove this by showing that \(I(s^2(s + 2)^2) > \frac{s + 2}{s}\):

\[
I\left(s^2(s + 2)^2\right) \geq 1 + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s + 2} + \frac{1}{(s + 2)^2} + \frac{1}{s(s + 2)} + \frac{1}{s^2(s + 2)} + \frac{1}{s(s + 2)^2} + \frac{1}{s^2(s + 2)^2}
\]

\[
= \frac{s^2 + s + 1}{s^2} + \frac{1}{s + 2} \left[1 + \frac{1}{s + 2} + \frac{1}{s} + \frac{1}{s(s + 2)} \left(\frac{1}{s} + \frac{1}{s + 2} + \frac{1}{s(s + 2)}\right)\right]
\]

\[
= \frac{s^2 + s + 1}{s^2} + \frac{s^2 + 4s + 2}{s(s + 2)^2} + \frac{2s + 3}{s^2(s + 2)^2}
\]
\[
\frac{s^2 + s + 1}{s^2(s + 2)^2} + \frac{s + 2}{s^2(s + 2)^2} + \frac{3}{s^2(s + 2)^2} = \frac{(s + 1)^2(s + 2)^2 + 3}{s^2(s + 2)^2} = \left(\frac{s + 1}{s}\right)^2 + \frac{3}{s^2(s + 2)^2} > \frac{s^2 + 2s + 1}{s^2} > \frac{s + 2}{s}
\]

Using some of the principles in the last paragraph, Richard Ryan wrote a simple computer program which verified that \( I(x) = \frac{s + 2}{s} \) has no solution less than \( 10^{16} \) (when \( s \in \mathbb{Z}^+ \) is odd with \( s > 1 \))\[58\].

We now state (without proof) the conditions obtained by Holdener and Stanton \[34\] in order for the fraction \( \frac{\sigma(N) + t}{N} \) to be an abundancy outlaw.

**Theorem 4.1.2.** For a positive integer \( t \), let \( \frac{\sigma(N) + t}{N} \) be a fraction in lowest terms, and let \( N = \prod_{i=1}^{n} p_i^{k_i} \) for primes \( p_1, p_2, \ldots, p_n \). If there exists a positive integer \( j \leq n \) such that \( p_j < \frac{1}{t} \sigma\left(\frac{N}{p_j^{k_j}}\right) \) and \( \sigma(p_j^{k_j}) \) has a divisor \( D > 1 \) such that at least one of the following is true:

1. \( I(p_j^{k_j})I(D) > \frac{\sigma(N) + t}{N} \) and \( \gcd(D, t) = 1 \)
2. \( \gcd(D, Nt) = 1 \)

then \( \frac{\sigma(N) + t}{N} \) is an abundancy outlaw.

The following are some sequences of abundancy outlaws which can be constructed from Theorem 4.1.2.
• For all natural numbers \( m \) and nonnegative integers \( n \), and for all odd primes \( p \) such that \( \gcd(p, \sigma(2^m)) = 1 \), the fraction \( \frac{\sigma(2^m p^{2n+1}) + 1}{2^m p^{2n+1}} \) is an abundancy outlaw.

• For all primes \( p > 3 \), \( \frac{\sigma(2p) + 1}{2p} \) is an abundancy outlaw. If \( p = 2 \) or \( p = 3 \) then \( \frac{\sigma(2p) + 1}{2p} \) is an abundancy index.

• If \( N \) is an even perfect number, \( \frac{\sigma(2N) + 1}{2N} \) is an abundancy outlaw.

• Let \( M \) be an odd natural number, and let \( p, \alpha, \) and \( t \) be odd natural numbers such that \( p \mid M \) and \( p < \frac{1}{t} \sigma(M) \). Then, if \( \frac{\sigma(p^\alpha M) + t}{p^\alpha M} \) is in lowest terms, \( \frac{\sigma(p^\alpha M) + t}{p^\alpha M} \) is an abundancy outlaw.

• For primes \( p \) and \( q \), with \( 3 < q, p < q \), and \( \gcd(p, q + 2) = \gcd(q, p + 2) = 1 \), \( \frac{\sigma(pq) + 1}{pq} \) is an abundancy outlaw.

**Remark 4.1.5.** The last assertion in the preceding paragraph produces outlaws with ease. We illustrate this using odd primes \( p \) and \( q \) satisfying \( 3 < p < q \) and \( q \equiv 1 \pmod{p} \). It follows that \( p \nmid (q + 2) \) and \( q \nmid (p + 2) \). By Dirichlet’s theorem on arithmetic progressions of primes, we are assured of the existence of an infinite sequence of primes \( q \) satisfying \( q \equiv 1 \pmod{p} \). Thus, there is an infinite class of abundancy outlaws corresponding to each odd prime \( p > 3 \).

Judy Holdener, with Laura Czarnecki, also obtained the following results in the summer of 2007 \cite{35}: 
Theorem 4.1.3. If $\frac{a}{b}$ is a fraction greater than 1 in reduced form, $\frac{a}{b} = I(N)$ for some $N \in \mathbb{N}$, and $b$ has a divisor $D = \prod_{i=1}^{n} p_i^{k_i}$ such that $I(p_iD) > \frac{a}{b}$ for all $1 \leq i \leq n$, then $\frac{D}{\sigma(D)} \frac{a}{b}$ is an abundancy index as well.

Proof. Suppose that $I(N) = \frac{a}{b}$ for some $N \in \mathbb{N}$. Then by Lemma 4.1.2, $b \mid N$ since $\gcd(a, b) = 1$. Because $I(p_iD) > \frac{a}{b}$ for all $1 \leq i \leq n$, we know that it is impossible that $p_iD \mid N$ by the contrapositive of the third result in Lemma 4.1.1. However, we know (by Lemma 4.1.2) that $D \mid N$, so $p_i^{k_i+1} \nmid N$. Thus we may write $N = (\prod_{i=1}^{n} p_i^{k_i}) \cdot r = Dr$, where $\gcd(p_i, r) = 1$ for all $1 \leq i \leq n$, that is, $\gcd(D, r) = 1$. Then, since $\sigma(N)$ is multiplicative, we may write $\frac{a}{b} = \frac{\sigma(N)}{N} = \frac{\sigma(D) \sigma(r)}{D} r$. Therefore, $I(r) = \frac{\sigma(r)}{r} = \frac{a}{b} \frac{D}{\sigma(D)}$. Thus, if $I(N) = \frac{a}{b}$ for some $N \in \mathbb{N}$, then $I(r) = \frac{a}{b} \frac{D}{\sigma(D)}$ for some $r \in \mathbb{N}$.

Corollary 4.1.1. Let $m, n, t \in \mathbb{N}$. If $\frac{\sigma(mn) + \sigma(m)t}{mn}$ is in reduced form with $m = \prod_{i=1}^{l} p_i^{k_i}$ and $I(p_im) > \frac{\sigma(mn) + \sigma(m)t}{mn}$ for all $1 \leq i \leq l$, then $\frac{\sigma(n) + t}{n}$ is an abundancy index if $\frac{\sigma(mn) + \sigma(m)t}{mn}$ is an index.

Proof. The proof is very similar to that of Theorem 4.1.3. We only need to observe that, under the assumptions given in this corollary, $\gcd(m, n) = 1$.

Remark 4.1.6. If, in Corollary 4.1.1, we have $t = 1$ and $n = p$ for some prime $p$, then the corollary tells us that if $\frac{\sigma(mp) + \sigma(m)}{mp}$ is an abundancy index, then $\frac{\sigma(p) + 1}{p}$ is as well. The fractions $\frac{27}{14}, \frac{39}{22}, \frac{45}{26}$ and $\frac{57}{34}$ all illustrate this fact. If we could determine that these are indeed indices, then we could say that $\frac{9}{7}, \frac{13}{11}, \frac{15}{13}$.
and \( \frac{19}{17} \), all of the form \( \frac{\sigma(p) + 1}{p} \), are indices as well. Fractions of the form \( \frac{\sigma(p) + 1}{p} \) continue to elude characterization as indices or outlaws! This is significant because Paul Weiner [73] proved that if there exists an integer \( N \) with abundancy \( \frac{5}{3} \), then \( 5N \) is an odd perfect number. The fraction \( \frac{5}{3} \) is of the form \( \frac{\sigma(p) + 1}{p} \) for \( p = 3 \).

**Theorem 4.1.4.** If \( p > q > 2 \) are primes satisfying \( p > q^2 - q - 1 \), then \( \frac{\sigma(qp) + q - 1}{qp} \) is an abundancy outlaw.

**Proof.** Suppose that \( I(N) = \frac{\sigma(qp) + q - 1}{qp} \) for some \( N \in \mathbb{N} \). Because \( \sigma(qp) + q - 1 = (q + 1)(p + 1) + q - 1 = qp + 2q + p \) and \( p > q > 2 \) are primes, we know that \( \text{gcd}(\sigma(qp) + q - 1, qp) = 1 \), and by Lemma 4.1.2 we have \( qp \mid N \). If \( q^2p \mid N \), then \( \frac{\sigma(qp) + q - 1}{qp} \geq \frac{\sigma(q^2p)}{q^2p} \). Simplifying, this becomes

\[
\frac{(q + 1)(p + 1) + q - 1}{qp} \geq \frac{(q^2 + q + 1)(p + 1)}{q^2p},
\]

and then \( q^2 - q - 1 \geq p \). This contradicts our hypothesis that \( p > q^2 - q - 1 \), and therefore \( q^2p \nmid N \). Consequently, we may write \( N = qK \), where \( \text{gcd}(q, K) = 1 \). Then, we have \( I(N) = I(q)I(K) \), which gives \( I(K) = \frac{I(N)}{I(q)} = \frac{\sigma(qp) + q - 1}{qp} \).

\[
\frac{q}{q + 1} = \frac{p \cdot \frac{q + 1}{2} + q}{p \cdot \frac{q + 1}{2}}.
\]

On setting \( m = p \cdot \frac{q + 1}{2} + q \) and \( n = p \cdot \frac{q + 1}{2} \) and observing

that \( n < m < \sigma(n) \) with \( \text{gcd}(m, n) = 1 \), then by Lemma 2.2.5 (and Example 4.1.5), \( \frac{m}{n} = \frac{\sigma(qp) + q - 1}{qp} \cdot \frac{q}{q + 1} \) is an abundancy outlaw. But this contradicts the fact that \( I(K) = \frac{I(N)}{I(q)} = \frac{\sigma(qp) + q - 1}{qp} \cdot \frac{q}{q + 1} \) for some \( K \in \mathbb{N} \), whence it follows that \( I(N) \neq \frac{\sigma(qp) + q - 1}{qp} \) for all \( N \in \mathbb{N} \). Consequently, \( \frac{\sigma(qp) + q - 1}{qp} \) must be an abundancy outlaw under the conditions specified for the primes \( p \) and \( q \). □
These results allow us to move a few more fractions in \((1, \infty)\) from the set of infinitely many fractions that we are unable to classify (category III), into the infinite set of fractions that are abundancy outlaws (category II). Furthermore, we can see that certain fractions are linked to others in important ways: determining the status of a given fraction can lead to the classification of new abundancy outlaws and indices. If the converse of Theorem 4.1.3 could be proved, then we would be able to divide certain fractions into equivalence classes of sorts, that is, sets of fractions with the same abundancy index/outlaw status. However, the question of the existence of an OPN (e.g. the status of the fraction \(\frac{\sigma(p) + 1}{p}\) for an odd prime \(p\)) remains as elusive as ever.

### 4.2 Bounds for the Prime Factors of OPNs

In this section, bounds for each of the distinct prime factors of an OPN \(N\) are derived, drawing heavily from existing works. We do this using cases based on the total number of distinct prime factors of \(N\) (i.e. \(\Omega(N)\)). We also study further results in the field and give examples of various techniques used, including an in-depth and detailed discussion of the factor chain approach. We give in Subsection 4.2.3 explicit double-sided bounds for each of the prime factors of an OPN \(N\) with \(\omega(N) = 9\). We end the section with a discussion of the author’s results on the relationships between the components of an OPN \(N\).
4.2.1 Results on OPNs

Let \( N = q_1^{a_1} q_2^{a_2} \cdots q_t^{a_t} \) be the canonical factorization of an OPN \( N \) (i.e. \( q_1, q_2, \ldots, q_t \) are distinct primes with \( q_1 < q_2 < \ldots < q_t \) and \( t = \omega(N) \)). Then the following statements are true:

- \( q_t \geq 100000007 \) from Goto and Ohno, improving on Jenkins
- \( q_{t-1} \geq 10007 \) from Iannucci, improving on Pomerance
- \( q_{t-2} \geq 101 \) from Iannucci
- \( q_i < 2^{2^{i-1}} (t - i + 1) \) for \( 2 \leq i \leq 6 \) from Kishore
- \( q_1 < \frac{2^{t+6}}{3} \) from Grün
- \( q_k^{a_k} > 10^{20} \) for some \( k \) from Cohen, improving on Muskat
- \( N > 10^{300} \) from Brent, et. al., improving on Brent and Cohen (A search is currently on in [http://www.oddperfect.org](http://www.oddperfect.org) to prove that \( N > 10^{500} \).
- \( N \equiv 1 \pmod{12}, N \equiv 81 \pmod{324} \) or \( N \equiv 117 \pmod{468} \) from Roberts, improving on Touchard and Holdener
- \( N \leq 2^{4^t} \) from Nielsen, improving on Cook
- \( \sum_{i=1}^{t} a_i = \Omega(N) \geq 75 \) from Hare, improving on Iannucci and Sorli
- \( t \geq 9 \) from Nielsen, improving on Hagis and Chein
- \( t \geq 12 \) if \( q_1 \geq 5 \) from Nielsen, improving on both Hagis and Kishore
- \( t \geq 17 \) if \( q_1 \geq 7 \) from Greathouse, improving on Norton
- \( t \geq 29 \) if \( q_1 \geq 11 \) from Greathouse, improving on Norton
Suryanarayana and Hagis [27] showed that, in all cases, \(0.596 < \sum_{p|N} \frac{1}{p} < 0.694\). Their paper gives more precise bounds when \(N\) is divisible by 3 or 5 (or both). Cohen [7] also gave more strict ranges for the same sum, including an argument that such bounds are unlikely to be improved upon significantly.

We use the preceding facts about OPNs to derive explicit double-sided bounds for the prime factors of an OPN \(N\) with \(\omega(N) = 9\), in Subsection 4.2.3.

### 4.2.2 Algorithmic Implementation of Factor Chains

(We borrow heavily the following material from [63].)

In the discussion that follows, we will let \(N\) denote an OPN, assuming one exists, with the prime decomposition

\[
N = \prod_{i=1}^{u} p_i^{a_i} \cdot \prod_{i=1}^{v} q_i^{b_i} \cdot \prod_{i=1}^{w} r_i^{c_i} = \lambda \cdot \mu \cdot \nu
\]

which we interpret as follows: each \(p_i^{a_i}\) is a known component of \(N\), each \(q_i\) is a known prime factor of \(N\) but the exponent \(b_i\) is unknown, and each prime factor \(r_i\) of \(N\) and exponent \(c_i\) are unknown. By “known”, we mean either explicitly postulated or the consequence of such an assumption. Any of \(u, v, w\) may be zero, in which case we set \(\lambda, \mu, \nu\), respectively, equal to 1. We also let \(\bar{m}\) denote a proper divisor of \(m\) (except \(\bar{1} = 1\)).

We can now illustrate the factor chain approach via an algorithmic implementation that can be used to test a given lower bound for \(\omega(N)\). We assume that \(N\) is an OPN with \(\omega(N) = t\) distinct prime factors. In brief, the algorithm
may be described as a progressive sieve, or "coin-sorter", in which the sieve gets finer and finer, so that eventually nothing is allowed through. We shall use the terminology of graph theory to describe the branching process. Since $3 \mid N$ if $t \leq 11$, for our present purposes the even powers of 3 are the roots of the trees. If $3^2$ is an exact divisor of $N$, then, since $\sigma(N) = 2N$, $\sigma(3^2) = 13$ is a divisor of $N$, and so the children of the root $3^2$ are labelled with different powers of 13. The first of these is $13^1$, meaning that we assume that 13 is an exact divisor of $N$ (and hence that 13 is the special prime), the second $13^2$, then $13^4$, $13^5$, ... Each of these possibilities leads to further factorizations and further subtrees. Having terminated all these, by methods to be described, we then assume that $3^4$ is an exact divisor, beginning the second tree, and we continue in this manner. Only prime powers as allowed in Subsection 4.2.1 are considered, and notice is taken of whether the special prime has been specified earlier in any path. These powers are called Eulerian.

We distinguish between initial components, which label the nodes and comprise initial primes and initial exponents, and consequent primes, which arise within a tree through factorization. It is necessary to maintain a count of the total number of distinct initial and consequent primes as they arise within a path, and we let $k$ be this number.

Often, more than one new prime will arise from a single factorization. All are included in the count, within $k$, and, whenever further branching is required, the smallest available consequent prime is used as the new initial prime. This preferred strategy will give the greatest increase in $I(\lambda \bar{\mu})$. On the other hand
a strategy of selecting the largest available consequent prime will usually give a significant increase in $k$.

To show that $t \geq \omega$, say, we build on earlier results which have presumably shown that $t \geq \omega - 1$, and we suppose that $t = \omega - 1$. (The reader may want to review Section 3.1 at this point.) If, within any path, we have $k > \omega - 1$, then there is clearly a contradiction, and that path is terminated. This is one of a number of possible contradictions that may arise and which terminate a path. The result will be proved when every path in every tree has been terminated with a contradiction (unless an OPN has been found). The different possible contradictions are indicated with upper case letters.

In the contradiction just mentioned, we have too Many distinct prime factors of $N$: this is Contradiction $M1$. If there are too Many occurrences of a single prime this is Contradiction $M2$; that is, within a path an initial prime has occurred as a consequent prime more times than the initial exponent. (So counts must also be maintained within each path of the occurrences of each initial prime as a consequent prime.)

If $k = \omega - 3$ but none of these $k$ primes exceeds 100, then Iannucci’s result must be (about to be) violated: this is Contradiction $P3$. If $k = \omega - 2$ and none of these primes exceeds $10^4$, then again, Iannucci’s result is violated: Contradiction $P2$. Or if, in this case, one exceeds $10^4$ but no other exceeds 100, then this is another version of Contradiction $P3$. If $k = \omega - 1$ and none of these primes exceeds $10^8$, then Goto/Ohno’s result is violated: Contradiction $P1$. In this case,
there are the following further possibilities: one prime exceeds $10^8$ but no other
exceeds $10^4$, or one exceeds $10^8$, another exceeds $10^4$, but no other exceeds 100.
These are other versions of contradictions $P2$ and $P3$, respectively. These, and
some of the other forms of contradiction below, require only counts or comparisons,
and no calculations.

At the outset, a number $B$ is chosen, then the number of subtrees with a
given initial prime $p$ is bounded by taking as initial components Eulerian powers $p^a$
with $p^{a+1} \leq B$. If possible, these trees are continued by factorizing $\sigma(p^a)$. When
$a$ becomes so large that $p^{a+1} > B$, which may occur with $a = 0$, then we write $q^b$
for $p^a$ and we have one more subtree with this initial prime; it is distinguished by
writing its initial component as $q^\infty$. This tree must be continued differently. In
the first place, the smallest available consequent prime, which is not already an
initial prime, is used to begin a new subtree. If no such primes are available, then
we opt to use the procedure that follows.

The product of the $u$ initial components $p^a$ within a path is the number
$\lambda$. Those initial primes $q$ with exponents $\infty$, and all consequent primes which
are not initial primes, are the $v$ prime factors of $\mu$. If $k < \omega - 1$ then there are
$w = \omega - k - 1$ remaining prime factors of $N$, still to be found or postulated.
These are the prime factors $r$ of $\nu$. The numbers $u, v, w$ are not fixed; they vary
as the path develops, for example, by taking a consequent prime as another initial
prime.
If factorization can no longer be used to provide further prime factors of
N, so, in particular, there are no consequent primes which are not initial primes,
then the following result (with proof omitted) is used:

**Lemma 4.2.1.** Suppose \( w \geq 1 \), and assume \( r_1 < r_2 < \cdots < r_w \). Then

\[
\frac{I(\lambda \bar{\mu} r_1^{\beta_1-1})}{2 - I(\lambda \mu)} \leq r
\]

for \( r = r_1 \), with strict inequality if \( v \geq 1 \) or \( w \geq 2 \). Further, if \( I(\lambda \mu^\infty) < 2 \), then

\[
r < \frac{2 + I(\lambda \mu^\infty)(w - 1)}{2 - I(\lambda \mu^\infty)}
\]

for \( r = r_1 \).

Here, \( \bar{\mu} \) is taken to be the product of powers \( q^{\beta} \), where \( q \mid \mu \) and \( \beta \) is given as follows. Let \( b_0 = \min \{ b : q^{b+1} > B \} \). If \( b_0 = 0 \), then we proceed in a manner to be described later. Otherwise, let

\[
\beta = \begin{cases} 
  b_0, & \text{if } b_0 \text{ is even } (b_0 > 0), \\
  b_0 + 1, & \text{if } b_0 \text{ is odd,}
\end{cases}
\]

with one possible exception. If \( \pi \) is the special prime, \( \pi \nmid \lambda \) and the set \( Q_1 = \{ q : q \equiv b_0 \equiv 1 \pmod{4} \} \) is nonempty, then take \( \beta = b_0 \) for \( q = \min Q_1 \). Values of \( I(p^a) \) and \( I(q^\beta) \) must be maintained, along with their product. This is the value of \( I(\lambda \bar{\mu}) \) to be used in the result mentioned in this paragraph. We shall refer to Lemma 4.2.1 as **Lemma X**.

**Lemma X** is used to provide an interval, the primes within which are considered in turn as possible divisors of \( \nu \). If there are No primes within the interval
that have not been otherwise considered, then this is Contradiction $N$. New primes within the interval are taken in increasing order, giving still further factors of $N$ either through factorization or through further applications of Lemma $X$. There will be occasions when no new primes arise through factorization, all being used earlier in the same path. Then again Lemma $X$ is used to provide further possible prime factors of $N$ (or, if $k = \omega - 1$, we may have found an OPN). This lemma specifically supplies the smallest possible candidate for the remaining primes; a still Smaller prime subsequently arising through factorization gives us Contradiction $S$.

We also denote by $q$ any consequent prime which is not an initial prime, and, for such primes, we let $Q_2 = \{q : q \equiv 1 \, (\text{mod} \, 4)\}$. Then, for such primes, we let $\beta = 2$ with the possible exception that, considering all primes $q$, we let $\beta = b_0$ or 1, as relevant, for $q = \min (Q_1 \cup Q_2)$, if this set is nonempty. Again, the value of $I(\lambda \bar{\mu})$, defined as before, must be maintained. If this value exceeds 2, we have an Abundant divisor of $N$, and the path is terminated: Contradiction $A$. This may well occur with $k < \omega - 1$. Values of $I(q^\infty)$ must also be maintained. These, multiplied with the values of $I(p^a)$, give values of $I(\lambda \mu^\infty)$. If this is less than 2 and $k = \omega - 1$ then, for all possible values of the exponents $b$, the postulated number $N$ is Deficient: Contradiction $D$.

Contradictions $A$ and $D$ are in fact contradictions of the following lemma, which we shall refer to as Lemma $Y$:

Lemma 4.2.2. For any OPN $N = \lambda \mu \nu$, as given as before, we have $I(\lambda \bar{\mu}) \leq 2 \leq I(\lambda \nu \mu^\infty)$. Both inequalities are strict if $v > 0$; the left-hand inequality is strict if
$w > 0$.

If, on the other hand, we have a postulated set of prime powers $p^a$ and $q^b$, for which $I(\lambda \bar{\mu}) \leq 2 \leq I(\lambda \mu^\infty)$, then the main inequality in Lemma $Y$ is satisfied and we have candidates for an OPN. If $v = w = 0$, so that we are talking only of known powers $p^a$, then their product is an OPN. Our sieving principle arises when $v > 0$.

In every such case where we have a set of prime powers satisfying the main inequality of Lemma $Y$, with $v > 0$, we increase the value of $B$ and investigate that set more closely. With the larger value of $B$, some prime powers shift from $\mu$ to $\lambda$, and allow further factorization, often resulting quickly in Contradiction $M1$ or $S$. The value of $I(\bar{\mu})$ increases, so the interval given by Lemma $X$ shortens, and hopefully the case which led to our increasing $B$ is no longer exceptional, or Contradiction $A$ or $D$ may be enforced. In that case, we revert to the earlier value of $B$ and continue from where we are. Alternatively, it may be necessary to increase $B$ still further, and later perhaps further again. When $w = 0$, since $I(\bar{\mu}) \to I(\mu^\infty)$ as $B \to \infty$, such cases must eventually be dispensed with, one way or the other.

We summarize the various contradictions in the following table:
| A | There is an Abundant divisor. |
|---|-------------------------------|
| D | The number is Deficient.    |
| M1 | There are too Many prime factors. |
| M2 | A single prime has occurred too Many times (an excess of that prime). |
| N | There is no New prime within the given interval. |
| P1 | There is no Prime factor exceeding $10^8$. |
| P2 | There is at most one Prime factor exceeding $10^4$. |
| P3 | There are at most two Prime factors exceeding 100. |
| S | There is a prime Smaller than the purportedly smallest remaining prime. |
| Π | None of the primes can be the special prime. |

One of these, Contradiction Π, was not discussed previously. Within any path with $k = \omega - 1$, if $\pi$ is not implicit in an initial component and if there is no prime $q \equiv 1 \pmod{4}$, then Contradiction Π may be invoked.

**Remark 4.2.1.** Notice that the algorithm as presented in this subsection could be programmed directly to run on a high-speed computer (even desktop PCs). Prior experience with such algorithmic programs, however, has shown that it can take months (or years even), to check and/or test a particular value for $t = \omega(N)$. Current computer architecture limits our capability to carry out these tasks at a reasonable amount of time.

**Example 4.2.1.** *Sigma chains* (otherwise known as *factor chains*) are an easily automated system for proving facts about OPNs. Each line of the proof starts with a prime factor known or assumed to divide an OPN $N$, along with its exponent.
Since $\sigma$ is multiplicative, knowledge of this prime power leads to knowledge of other prime powers of $N$. If an impossibility arises (see below), that chain of the proof is terminated and the next possibility is considered.

The following is the start of a proof that no OPN has a component less than $10^{30}$. It would take many thousands of pages to complete this proof (and this has currently not been completed); this merely serves as an example of how one constructs such proofs. (The best-known result in this direction is that of Cohen [8]: An OPN has a component bigger than $10^{20}$.)

The factor chains are terminated (the succeeding line is not to be indented further than the preceding) if it fails in one of the following ways:

**xs**: The indicated prime appears more times than it is allowed. (e.g. If the chain assumes that $3^6 \| N$ then a chain with 7 or more factors of 3 is terminated.)

**overabundant**: The abundancy of the prime factors already exceeds 2, so regardless of the other factors, $N$ will fail to be perfect.

The factorizations of the largest half-dozen composites are due to the WIMS (WWWInteractive Multipurpose Server) Factoris at wims.unice.fr. (This proof is taken from [20].)

\[
\begin{align*}
3^6 &> 1093 \\
1093 &> 2 \times 547 \\
547^2 &> 3 \times 163 \times 613 \\
163^2 &> 3 \times 7 \times 19 \times 67 \\
7^2 &> 3 \times 19 \\
19^2 &> 3 \times 127 \\
127^2 &> 3 \times 5419 \\
5419^2 &> 3 \times 31 \times 313 \times 1009 \\
31^2 &> 3 \times 3 \times 331 \\
31^3 &> 5 \times 11 \times 17 \times 351 \overabundant
\end{align*}
\]
\[31^6 > 917087137\]
\[917087137^2 > 9140488177676943907 = 3x\cdot 43 \cdot \ldots\]
\[917087137^4 > 70736313097541065394066657400343621 = 31747594185191 \cdot 2228 \cdot 9731\]
\[31747594185191^2 > 1007 \cdot \ldots \cdot 1673 = 2671 \cdot \ldots\]
\[2671^2 > 3x\cdot 7 \cdot 19 \cdot 31 \cdot 57\]
\[2671^4 > 5 \cdot 11^2 \cdot 571 \cdot 147399551 \text{ overabundant}\]
\[2671^5 > 1277 \cdot 2660238405785894351\]
\[2660238405785894351^2 > 8180 \ldots 1201 = 3x\cdot \ldots\]
\[2660238405785894351^4 > 66928167 \ldots 72862401\]
\[= 5 \cdot 11 \cdot 27362961781 \text{ overabundant}\]
\[2660238405785894351^6 > 5475369 \ldots \]
\[\ldots 5453601 = 7x\cdot 2027 \cdot 1013 \ldots 28007 \cdot \ldots\]
\[31747594185191^4 > 10158820370273998808700619554107312558108423204146361\]
\[= 5 \cdot 11 \cdot 27581 \text{ overabundant}\]
\[31747594185191^6 > 1023917 \ldots 0882641 = 29 \cdot 68279 \cdot 17581747 \cdot \ldots\]
\[29^2 > 13 \cdot 67\]
\[13^2 > 3x\cdot 61\]
\[13^4 > 30941\]

4.2.3 Explicit Double-Sided Bounds for the Prime Factors

The results from Subsection 4.2.1 give some restrictions on the magnitude of the prime factors of an OPN \(N\). For instance, we saw from Subsection 3.5.2 that \(N\) is not divisible by \(3 \cdot 5 \cdot 7 = 105\). Consequently, it must be true that the third smallest prime factor \(q_3 \geq 11\). To further derive bounds for the other prime factors, we will use some of the many published results on OPNs, a compendium of which has been presented in Subsection 4.2.1.

For the largest prime factors of an OPN, Iannucci and Jenkins have worked to find lower bounds. The largest three factors must be at least \(100000007, 10007,\) and \(101\). Goto and Ohno verified that the largest factor \(q_t\) must be at least \(100000007\) using an extension of the methods of Jenkins.
Nielsen, improving the bound of Hagis and Kishore, showed that if an OPN is not divisible by 3, it must have at least 12 distinct prime factors. Nielsen also showed that a general odd perfect number, if it exists, must have at least 9 distinct prime factors. Therefore, for $9 \leq \omega(N) \leq 11$, we have $q_1 = 3$.

A result by Grün (and perhaps, independently too, by Perisastri) will be useful for our purposes later: $q_1 < \frac{2}{3}t + 2$. Results similar to those previously mentioned reduce the practicality of Grün’s findings. In fact, a paper by Norton published about two years after supersedes Grün’s inequality, except that Norton’s method is slightly more computationally intensive.

With an application of Goto/Ohno’s and Iannucci’s results, we can modestly improve on these bounds with an otherwise straightforward utilization of the abundancy index function:

Let $q_1 \geq 7$. Now, suppose that $t = \omega(N) = 17$. Using the lower bounds indicated for the three largest prime factors of $N$ as before, we have:

$$
q_1 \geq 7
$$
$$
q_2 \geq 11
$$
$$
q_3 \geq 13
$$
$$
q_4 \geq 17
$$
$$
q_5 \geq 19
$$
$$
q_6 \geq 23
$$
$$
q_7 \geq 29
$$
$$
q_8 \geq 31
$$
$$
q_9 \geq 37
$$
\[ q_{10} \geq 41 \]
\[ q_{11} \geq 43 \]
\[ q_{12} \geq 47 \]
\[ q_{13} \geq 53 \]
\[ q_{14} \geq 59 \]
\[ q_{15} \geq 101 \]
\[ q_{16} \geq 10007 \]
\[ q_{17} \geq 10000007 \]

Recall from Lemma 3.1.1 that
\[ 2 = \frac{\sigma(N)}{N} < \prod_{i=1}^{t} \left( \frac{q_i}{q_i - 1} \right) \] Also, note that
\[ q_i \geq a_i \text{ for all } i \] implies that
\[ \frac{q_i}{q_i - 1} \leq \frac{a_i}{a_i - 1} \text{ for all } i, \]
so that we have
\[ 2 = \frac{\sigma(N)}{N} < \prod_{i=1}^{t} \left( \frac{q_i}{q_i - 1} \right) \leq \prod_{i=1}^{t} \left( \frac{a_i}{a_i - 1} \right). \]

The numerator of the rightmost fraction is approximately
\[ 6.4778249375254265314282935191886 \cdot 10^{33}, \] while the denominator is approximately
\[ 3.2172767985350308489460711424 \cdot 10^{33}, \] which gives a ratio of approximately
\[ 2.0134496790810999533771971435786, \] which is larger than 2. No contradiction at this point.

As before, let \( q_1 \geq 7 \), but now suppose that \( t = \omega(N) = 16 \). Proceeding similarly as before, we have:
\[ q_1 \geq 7 \]
\[ q_2 \geq 11 \]
Again, we have

\[ 2 = \frac{\sigma(N)}{N} < \prod_{i=1}^{t} \left( \frac{q_i}{q_i - 1} \right) \leq \prod_{i=1}^{t} \left( \frac{a_i}{a_i - 1} \right). \]

where \( q_i \geq a_i \) for each \( i \). Our computations show that:

\[
\prod_{i=1}^{t} \left( \frac{a_i}{a_i - 1} \right) = \frac{1.0979364300890553443098802574896 \cdot 10^{32}}{55470289629914324981828812800000}
= 1.979323413339626660318209208061 < 2.
\]

This results in the contradiction \( 2 < 2 \). We therefore conclude that \( t \geq 17 \) if \( q_1 \geq 7 \).

We may likewise prove, using the same method, that \( t \geq 29 \) if \( q_1 \geq 11 \).

A result of great utility here is an earlier work of Kishore [44], where he proves that
\[ q_i < 2^{2i-1}(t - i + 1) \text{ for } 2 \leq i \leq 6. \]

These results (by Grün/Periasastri and Kishore) allow us to give explicitly reduced bounds for the lowest six (6) prime factors for an OPN \( N \) with a given number \( t = \omega(N) \) of distinct prime factors. For example, an OPN with nine (9) distinct divisors has \( q_1 \leq 7 \) (the smallest prime strictly less than \( \frac{2 \cdot 9}{3} + 2 = 8 \)), \( q_2 \leq 31 \) (the smallest prime less than \( 2^{21} \cdot (9 - 2 + 1) = 32 \)), \( q_3 \leq 109 \), \( q_4 < 2^8 \cdot 6 = 3 \cdot 2^9 \), \( q_5 < 5 \cdot 2^{16} \), and \( q_6 < 2^{32} \cdot 4 = 2^{34} \).

By using the abundancy index function, we can further reduce the bound for \( q_2 \). If \( q_2 \geq 13 \), then

\[
2 = I(N) \leq I(3^5 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 101 \cdot 10007 \cdot 100000007) < \\
3 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 101 \cdot 10007 \cdot 100000007 = \frac{849216193914429412851}{426034693082080051200} = 1.9933029110162608467608441119731 < 2
\]

so that \( q_2 \leq 11 \).

We may also try reducing the bound for \( q_3 \). Proceeding in the same manner as before, if \( q_1 \geq 3 \), \( q_2 \geq 5 \) and \( q_3 \geq 53 \), then

\[
2 = I(N) \leq I(3^3 \cdot 5^3 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 3^5 \cdot 101 \cdot 10007 \cdot 100000007) < \\
3 \cdot 5 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 3^5 \cdot 101 \cdot 10007 \cdot 100000007 = \frac{1937532883237423387515}{9850131125407832064000} = 1.9670122749188491347643364823597 < 2
\]

so that \( q_3 \leq 47 \).

Notice that a major problem with the abundancy index function is that it is not capable of determining upper bounds on the prime factors of an OPN beyond
the smallest three (except in some special cases). This is due to the fact that the
first three primes could be 3, 5, and 11, in which case the sigma bounds would
allow an arbitrary number of additional prime divisors, but require no more.

In a preprint published in the electronic journal INTEGERS in 2003, Nielsen
improved on Cook’s bound by showing that \( N < 2^{4t} \). Since \( q_i < N \) for all \( i \),
Nielsen’s bound is an implicit upper limit on \( q_t \). If \( q_t \) is the special (or Euler)
prime factor with an exponent of 1 and the other \( q_i \)’s are small, then we can
say little else about \( q_t \). We can, however, give tighter limits for the other prime
factors.

Since only one of \( q_t, q_{t-1} \) can be the special prime, at least one exponent
is even. Consequently, \( q_t q_{t-1}^2 < N < 2^{4t} \), so that \( q_{t-1} < 2^{3 \cdot 4^t} \). Likewise, for
\( 1 \leq i \leq t \), we have \( q_i < 2^{2(t-i) + 1} \). This range can be limited further by
considering the other prime factors. Hare proved that there are at least 75 total
primes (not distinct), so we may take the other primes to be as small as possible
and raise the smallest prime to the appropriate power (and the others to the 2nd
power). By using such a method, we may be able to reduce the bound by perhaps
a million, depending on \( t = \omega(N) \).

We have thus given explicit formulations (through Grün/Perisastri and Kishore)
for the upper limits on the smallest 6 prime divisors, which are augmented with
sigma conditions for the lowest 3. The higher prime divisors are likewise restricted
(through Nielsen), though not as tightly.
Going beyond the results previously mentioned is not easy, considering the fact that both Iannucci and Jenkins used proofs based on the divisibility of cyclotomic polynomials $F_p(x)$ to find lower bounds for the highest prime factors, and that topic is quite hard to follow, to my knowledge. The method used in most modern proofs is that of factor/sigma chains. Consider an OPN $N$ with a component $5^{4k+1}$. (By component, we mean either a prime power that divides $N$, or simply a factor of $N$, which may not necessarily be a prime power.) We know that $\sigma(5) \mid \sigma(5^{4k+1})$ for all positive integers $k$, so we may conclude that $\sigma(5) \mid 2N$. But $\sigma(5) = 6$, so this is an indication that $3 \mid N$. This illustrates how knowledge of a particular prime power leads to knowledge of other prime powers, for an OPN, by virtue of the fact that the $\sigma$ function is multiplicative.

We summarize the results in this subsection (which are taken from Greathouse [20]) as follows:

\begin{align*}
\omega(N) &= 9 \\
100000007 &\leq q_9 < \frac{24^9}{4^9} \\
1007 &\leq q_8 < \frac{283}{4^9} \\
101 &\leq q_7 < \frac{2}{5} \\
23 &\leq q_6 \leq 17179869143 \\
19 &\leq q_5 \leq 327673 \\
13 &\leq q_4 \leq 1531 \\
11 &\leq q_3 \leq 47 \\
5 &\leq q_2 \leq 11 \\
3 &\leq q_1 \leq 3
\end{align*}

**Remark 4.2.2.** These bounds, together with the algorithm presented in Subsection 4.2.2, can (potentially) prove the conjecture that $\omega(N) \geq 10$, for a general OPN $N$. 
In the next subsection, we shall discuss some of the author’s own results on the relationships between the components of an OPN \( N \).

### 4.2.4 Relationships Between OPN Components

Throughout this subsection, we let \( N = p^k m^2 \) be an OPN with special/Euler prime \( p \) with \( p \equiv k \equiv 1 \pmod{4} \) and gcd\((p, m) = 1\). (Recall from Section 2.4 that \( p^k \) is called the Euler’s factor of the OPN \( N \).) It will also be useful later to consider the canonical factorization \( N = \prod_{i=1}^{\omega(N)} p_i^{\alpha_i} \), where \( p_1 < p_2 < \cdots < p_t \), \( t = \omega(N) \) and \( \alpha_i > 0 \) for all \( i \).

We begin with some numerical results:

**Lemma 4.2.3.** \( 1 < \frac{\sigma(p^k)}{p^k} < \frac{5}{4} < \frac{8}{5} < \frac{\sigma(m^2)}{m^2} < 2 \)

**Proof.**

\[
\frac{\sigma(p^k)}{p^k} = \frac{p^k + p^{k-1} + \cdots + p^2 + p + 1}{p^k} = 1 + \frac{1}{p} + \left(\frac{1}{p}\right)^2 + \cdots + \left(\frac{1}{p}\right)^{k-1} + \left(\frac{1}{p}\right)^k \geq 1 + \frac{1}{p}
\]

In particular, \( \frac{\sigma(p^k)}{p^k} > 1 \).

\[
\frac{\sigma(p^k)}{p^k} = \frac{p^{k+1} - 1}{p^k (p - 1)} < \frac{p^{k+1}}{p^k (p - 1)} = \frac{p}{p - 1} = \frac{1}{1 - \frac{1}{p}}
\]

But since \( p^k m^2 \) is an OPN, \( p \) is prime and \( p \equiv 1 \pmod{4} \). This implies that \( p \geq 5 \), from which it follows that \( 1 - \frac{1}{p} \geq 1 - \frac{1}{5} = \frac{4}{5} \). Thus, \( \frac{p}{p - 1} = \frac{1}{1 - \frac{1}{p}} \leq \frac{5}{4} \), and we have \( \frac{\sigma(p^k)}{p^k} < \frac{5}{4} \). Note that, for \( k \geq 1 \) and prime \( p \), we have:
Also, since \( p^k m^2 \) is an OPN, \( \left[ \frac{\sigma(p^k)}{p^k} \right] \left[ \frac{\sigma(m^2)}{m^2} \right] = 2 \), which implies that 
\[
\frac{\sigma(p^k)}{p^k} = \frac{2m^2}{\sigma(m^2)}. 
\]
But \( 1 < \frac{\sigma(p^k)}{p^k} = \frac{2m^2}{\sigma(m^2)} < \frac{5}{4} \). Consequently, we have 
\[
\frac{4}{5} < \frac{\sigma(m^2)}{2m^2} < 1, \text{ and thus, } \frac{8}{5} < \frac{\sigma(m^2)}{m^2} < 2. 
\]

**Corollary 4.2.1.** If \( N = p^k m^2 \) is an OPN with Euler’s factor \( p^k \), then 
\[
\frac{p+1}{p} \leq \frac{\sigma(p^k)}{p^k} < \frac{p}{p-1} < \frac{2(p-1)}{p} < \frac{\sigma(m^2)}{m^2} < \frac{2p}{p+1} 
\]

**Proof.** The proof is similar to that for Lemma 4.2.3. We give here a proof of the inequality in the middle. Suppose to the contrary that 
\[
\frac{p}{p-1} \geq \frac{2(p-1)}{p}. 
\]
Since \( p \geq 5 > 0, p^2 \geq 2(p-1)^2 \). This implies that \( p^2 - 4p + 2 \leq 0 \). This last inequality is a contradiction since it implies that \( p(p-4) + 2 \leq 0 \), whereas \( p \geq 5 \) implies that \( p(p-4) + 2 \geq 7 \). 

In what follows, we set \( X = \frac{\sigma(p^k)}{p^k} \) and \( Y = \frac{\sigma(m^2)}{m^2} \).

**Lemma 4.2.4.** \( \frac{57}{20} < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} < 3 \)

**Proof.** By Lemma 4.2.3, \( 1 < X < \frac{5}{4} < \frac{8}{5} < Y < 2 \). Consider \((X-1)(Y-1)\). This quantity is positive because \( 1 < X < Y \). Thus, 
\[
(X-1)(Y-1) = XY - (X+Y) + 1 > 0, 
\]
which implies that \( X + Y < XY + 1 \). But \( XY = 2 \). Thus, \( X + Y < 3 \). Now, consider \( \left( X - \frac{5}{4} \right) \left( Y - \frac{5}{4} \right) \). This quantity is negative because \( X < \frac{5}{4} < Y \). Thus,
\[
\left( X - \frac{5}{4} \right) \left( Y - \frac{5}{4} \right) = XY - \frac{5}{4} (X + Y) + \frac{25}{16} < 0,
\]
which implies that \( \frac{5}{4} (X + Y) > XY + \frac{25}{16} \). But again, \( XY = 2 \). Consequently, \( \frac{5}{4} (X + Y) > \frac{57}{16} \), and hence \( X + Y > \frac{57}{20} \). \( \square \)

**Corollary 4.2.2.** \( \frac{3p^2 - 4p + 2}{p(p - 1)} < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} \leq \frac{3p^2 + 2p + 1}{p(p + 1)} \)

**Proof.** From Corollary [4.2.1]

\[
\frac{p + 1}{p} \leq X < \frac{p}{p - 1} < \frac{2(p - 1)}{p} < Y \leq \frac{2p}{p + 1}
\]

Consider \( \left( X - \frac{p + 1}{p} \right) \left( Y - \frac{p + 1}{p} \right) \). This quantity is nonnegative because \( \frac{p + 1}{p} \leq X < Y \). Thus,

\[
\left( X - \frac{p + 1}{p} \right) \left( Y - \frac{p + 1}{p} \right) = XY - \frac{p + 1}{p} (X + Y) + \frac{(p + 1)^2}{p^2} \geq 0
\]
which implies that \( \frac{p + 1}{p} (X + Y) \leq 2 + \frac{p^2 + 2p + 1}{p^2} = \frac{3p^2 + 2p + 1}{p^2} \). Consequently, \( X + Y \leq \frac{3p^2 + 2p + 1}{p(p + 1)} \). Now, consider \( \left( X - \frac{p}{p - 1} \right) \left( Y - \frac{p}{p - 1} \right) \). This quantity is negative because \( X < \frac{p}{p - 1} < Y \). Thus,

\[
\left( X - \frac{p}{p - 1} \right) \left( Y - \frac{p}{p - 1} \right) = XY - \frac{p}{p - 1} (X + Y) + \frac{p^2}{(p - 1)^2} < 0
\]
which implies that \( \frac{p}{p - 1} (X + Y) > 2 + \frac{p^2}{p^2 - 2p + 1} = \frac{3p^2 - 4p + 2}{(p - 1)^2} \). Consequently, \( \frac{3p^2 - 4p + 2}{p(p - 1)} < X + Y \).

Finally, we need to check that, indeed,

\[
\frac{3p^2 - 4p + 2}{p(p - 1)} = 3 - \frac{p - 2}{p(p - 1)} < 3 - \frac{p - 1}{p(p + 1)} = \frac{3p^2 + 2p + 1}{p(p + 1)}
\]
Suppose to the contrary that
\[
\frac{3p^2 - 4p + 2}{p(p-1)} \geq \frac{3p^2 + 2p + 1}{p(p+1)}.
\]

This last inequality implies that \(3 - \frac{p - 2}{p(p-1)} \geq 3 - \frac{p - 1}{p(p+1)}\), or equivalently, 
\[
\frac{p - 1}{p(p+1)} \geq \frac{p - 2}{p(p-1)}.
\]
Since \(p \geq 5 > 0\), we have \((p - 1)^2 \geq (p + 1)(p - 2)\), or equivalently, \(p^2 - 2p + 1 \geq p^2 - p - 2\), resulting in the contradiction \(p \leq 3\).

Hence, we have
\[
\frac{3p^2 - 4p + 2}{p(p-1)} < X + Y \leq \frac{3p^2 + 2p + 1}{p(p+1)},
\]
and we are done. \(\square\)

If we attempt to improve the results of Corollary 4.2.2 using Corollary 4.2.1, we get the following result:

**Theorem 4.2.1.** The series of inequalities

\[
L(p) < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} \leq U(p)
\]

with

\[
L(p) = \frac{3p^2 - 4p + 2}{p(p-1)}
\]

and

\[
U(p) = \frac{3p^2 + 2p + 1}{p(p+1)}
\]

is best possible, for a given Euler prime \(p \equiv 1 \pmod{4}\) of an OPN \(N = p^km^2\).

**Proof.** From Corollary 4.2.1, we have:

\[
\frac{p + 1}{p} \leq X < \frac{p}{p - 1},
\]
and

\[ \frac{2(p-1)}{p} < Y \leq \frac{2p}{p+1}. \]

We remark that such bounds for \( X \) and \( Y \) are best possible by observing that \( k \equiv 1 \pmod{4} \) implies \( k \geq 1 \). Adding the left-hand and right-hand inequalities give rise to:

\[ \frac{3p-1}{p} < X + Y < \frac{p(3p-1)}{(p+1)(p-1)} \]

Comparing this last result with that of Corollary 4.2.2, the result immediately follows if we observe that

\[ \max \left\{ \frac{3p^2 - 4p + 2}{p(p-1)}, \frac{3p-1}{p} \right\} = \frac{3p^2 - 4p + 2}{p(p-1)} \]

and

\[ \min \left\{ \frac{3p^2 + 2p + 1}{p(p+1)}, \frac{p(3p-1)}{(p+1)(p-1)} \right\} = \frac{3p^2 + 2p + 1}{p(p+1)}, \]

with both results true when \( p \geq 5 \) (specifically when \( p \) is a prime with \( p \equiv 1 \pmod{4} \)).

The reader might be tempted to try to improve on the bounds in Lemma 4.2.4 using Theorem 4.2.1, but such efforts are rendered futile by the following theorem:

**Theorem 4.2.2.** The bounds in Lemma 4.2.4 are best possible.

**Proof.** It suffices to get the minimum value for \( L(p) \) and the maximum value for \( U(p) \) in the interval \([5, \infty)\), or if either one cannot be obtained, the greatest lower bound for \( L(p) \) and the least upper bound for \( U(p) \) for the same interval would likewise be useful for our purposes here.
From basic calculus, we get the first derivatives of $L(p), U(p)$ and determine their signs in the interval $[5, \infty)$:

$$L'(p) = \frac{p(p - 4) + 2}{p^2(p - 1)^2} > 0$$

and

$$U'(p) = \frac{p(p - 2) - 1}{p^2(p + 1)^2} > 0$$

which means that $L(p), U(p)$ are increasing functions of $p$ on the interval $[5, \infty)$. Hence, $L(p)$ attains its minimum value on that interval at $L(5) = \frac{57}{20}$, while $U(p)$ has no maximum value on the same interval, but has a least upper bound of $\lim_{p \to \infty} U(p) = 3$.

This confirms our earlier findings that

$$\frac{57}{20} < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} < 3,$$

with the further result that such bounds are best possible.

\[\square\]

**Remark 4.2.3.** Let

$$f(p, k) = X + Y = \frac{\sigma(p^k)}{p^k} + \frac{2p^k}{\sigma(p^k)}.$$

Using Mathematica, we get the partial derivative:

$$\frac{\partial}{\partial p} f(p, k) = \frac{p^{-1-k} (k - kp + p(-1 + p^k)) (-1 + p^k(2p + p^k(2 + (-4 + p)p)))}{(-1 + p)^2(-1 + p^{1+k})^2}$$

which is certainly positive for prime $p \equiv 1 \pmod{4}$ and $k$ a fixed positive integer satisfying $k \equiv 1 \pmod{4}$. This means that $f(p, k) = X + Y$ is a strictly monotonic increasing function of $p$, for such primes $p$ and fixed integer $k$. Lastly, $\lim_{p \to \infty} X + Y = 3.$
Remark 4.2.4. Why did we bother to focus on improving the bounds for \(X+Y\) in the first place? This is because Joshua Zelinsky, in response to one of the author’s posts at the Math Forum \(\text{http://www.mathforum.org/kb/message.jspa?messageID=414071\#tstart=0}\), said that “[he does not] know if this would be directly useful for proving that no [OPNs] exist, [although he is not] in general aware of any sharp bounds [for \(X+Y\)].

Given that there are odd primitive abundant numbers \(n\) of the form \(n = P^K M^2\) with \(P\) and \(K\) congruent to \(1\) modulo \(4\) and \(\gcd(P, M) = 1\), [he] would be surprised if one could substantially improve on these bounds. Any further improvement of the lower bound would be equivalent to showing that there are no [OPNs] of the form \(5m^2\) which would be a very major result. Any improvement on the upper bound of \(3\) would have similar implications for all arbitrarily large primes and thus [he thinks] would be a very major result. [He’s] therefore highly curious as to what [the author had] done.” In particular, by using Mathematica, if one would be able to prove that \(\frac{43}{15} < X + Y\), then this would imply that \(p > 5\) and we arrive at Zelinsky’s result that “there are no OPNs of the form \(5m^2\)”. Likewise, if one would be able to derive an upper bound for \(X + Y\) smaller than \(3\), say \(2.9995\), so that \(X + Y < 2.9995\), then this would imply that \(p \leq 1999\), confirming Zelinsky’s last assertion.

Our first hint at one of the relationships between the components of an OPN is given by the following result:

Lemma 4.2.5. If \(N = p^k m^2\) is an OPN, then \(p^k \neq m^2\).

Proof. We will give four (4) proofs of this same result, to illustrate the possible approaches to proving similar lemmas:
- If \( p^k = m^2 \), then necessarily \( \omega(p^k) = \omega(m^2) \). But \( \omega(p^k) = 1 < 8 \leq \omega(m^2) \), where the last inequality is due to Nielsen.

- Suppose \( p^k = m^2 \). This can be rewritten as \( p \cdot p^{k-1} = m^2 \), which implies that \( p | m^2 \) since \( p \) is a prime. This contradicts \( \gcd(p, m) = 1 \).

- Assume \( p^k = m^2 \). Then \( N = p^{2k} \) is an OPN. This contradicts the fact that prime powers are deficient.

- Let \( p^k = m^2 \). As before, \( N = p^{2k} \) is an OPN. This implies that \( \sigma(N) = \sigma(p^{2k}) = 1 + p + p^2 + \ldots + p^{2k-1} + p^{2k} \equiv (2k+1) \pmod{4} \equiv 3 \pmod{4} \) (since \( p \equiv k \equiv 1 \pmod{4} \)). But, since \( N \) is an OPN, \( \sigma(N) = 2N \). The parity of LHS and RHS of the equation do not match, a contradiction.

\[
\square
\]

By Lemma 4.2.5, either \( p^k < m^2 \) or \( p^k > m^2 \).

We now assign the values of the following fractions to the indicated variables, for ease of use later on:

\[
\rho_1 = \frac{\sigma(p^k)}{p^k} \\
\rho_2 = \frac{\sigma(p^k)}{m^2} \\
\mu_1 = \frac{\sigma(m^2)}{m^2} \\
\mu_2 = \frac{\sigma(m^2)}{p^k}
\]

From Lemma 4.2.3 we have \( 1 < \rho_1 < \frac{5}{4} < \frac{8}{5} < \mu_1 < 2 \). Note that \( \rho_1 \mu_1 = \rho_2 \mu_2 = 2 \). Also, from Lemma 4.2.5 we get \( \rho_1 \neq \rho_2 \) and \( \mu_1 \neq \mu_2 \).
The following lemma is the basis for the assertion that “Squares cannot be perfect”, and will be extremely useful here:

**Lemma 4.2.6.** Let $A$ be a positive integer. Then $\sigma(A^2)$ is odd.

*Proof.* Let $A = \prod_{j=1}^{R} q_i^{\beta_i}$ be the canonical factorization of $A$, where $R = \omega(A)$. Then

$$\sigma(A^2) = \sigma(\prod_{j=1}^{R} q_i^{2\beta_i}) = \prod_{j=1}^{R} \sigma(q_i^{2\beta_i}),$$

since $\sigma$ is multiplicative. But

$$\prod_{j=1}^{R} \sigma(q_i^{2\beta_i}) = \prod_{j=1}^{R} (1 + q_i + q_i^2 + \ldots + q_i^{2\beta_i-1} + q_i^{2\beta_i}).$$

But this last product is odd regardless of whether the $q_i$’s are odd or even, i.e. regardless of whether $A$ is odd or even. Consequently, $\sigma(A^2)$ is odd. \qed

A reasoning similar to the proof for Lemma 4.2.6 gives us the following lemma:

**Lemma 4.2.7.** Let $N = p^k m^2$ be an OPN with Euler’s factor $p^k$. Then

$$\sigma(p^k) \neq \sigma(m^2).$$

*Proof.* By Lemma 4.2.6, $\sigma(m^2)$ is odd. If we could show that $\sigma(p^k)$ is even, then we are done. To this end, notice that

$$\sigma(p^k) = 1 + p + p^2 + \ldots + p^k \equiv (k + 1) \pmod{4}$$

since $p \equiv 1 \pmod{4}$. But $k \equiv 1 \pmod{4}$. This means that $\sigma(p^k) \equiv 2 \pmod{4}$, i.e. $\sigma(p^k)$ is divisible by 2 but not by 4. \qed
From Lemma 4.2.7, we get at once the following: 
\[ \rho_1 = \frac{\sigma(p^k)}{p^k} \neq \frac{\sigma(m^2)}{p^k} = \mu_2 \]
and 
\[ \rho_2 = \frac{\sigma(m^2)}{m^2} = \mu_1. \]
Also, by a simple parity comparison, we get:
\[ \rho_2 = \frac{\sigma(p^k)}{m^2} \neq \frac{\sigma(m^2)}{p^k} = \mu_2. \]
Lastly, \( \rho_2 \neq 1 \) and \( \mu_2 \neq 2. \)

From the equation \( \sigma(p^k)\sigma(m^2) = 2p^km^2 \) and Example 4.1.4, we know that \( p^k \mid \sigma(m^2). \) This means that \( \mu_2 \in \mathbb{Z}^+. \) Suppose that \( \mu_2 = 1. \) This implies that \( \rho_2 = 2, \) and therefore, \( \sigma(m^2) = p^k \) and \( \sigma(p^k) = 2m^2. \) However, according to the paper titled “Some New Results on Odd Perfect Numbers” by G. G. Dandapat, J. L. Hunsucker and Carl Pomerance: No OPN satisfies \( \sigma(p^k) = 2m^2, \sigma(m^2) = p^k. \)

This result implies that \( \rho_2 \neq 2 \) and \( \mu_2 \neq 1. \) But \( \rho_2 \neq 1 \) and \( \mu_2 \neq 2. \) Since \( \mu_2 \in \mathbb{Z}^+, \) we then have \( \mu_2 \geq 3. \) (Note that \( \mu_2 \) must be odd.) Consequently, we have the series of inequalities:

\[ 0 < \rho_2 \leq \frac{2}{3} < 1 < \rho_1 < \frac{5}{4} < \frac{8}{5} < \mu_1 < 2 < 3 \leq \mu_2. \]

In particular, we get the inequalities \( p^k < \sigma(p^k) \leq \frac{2}{3}m^2 \) and \( \frac{\sigma(p^k)}{\sigma(m^2)} = \frac{\rho_1 + \rho_2}{\mu_1 + \mu_2} < \frac{5}{12}. \)

Recall that, from Lemma 4.2.3, \( \frac{57}{20} < \rho_1 + \mu_1 < 3. \) Consider \( (\rho_2 - 3)(\mu_2 - 3). \) This quantity is nonpositive because \( \rho_2 < 3 \leq \mu_2. \) But \( (\rho_2 - 3)(\mu_2 - 3) \leq 0 \) implies that \( \rho_2 \mu_2 - 3(\rho_2 + \mu_2) + 9 \leq 0, \) which means that \( 11 \leq 3(\rho_2 + \mu_2) \) since \( \rho_2 \mu_2 = 2. \) Consequently, we have the series of inequalities \( \frac{57}{20} < \rho_1 + \mu_1 < 3 < \frac{11}{3} \leq \rho_2 + \mu_2. \) In particular, \( \rho_1 + \mu_1 \neq \rho_2 + \mu_2. \)

We summarize our results from the preceding paragraphs in the theorem that follows:
Theorem 4.2.3.

\[ 0 < \rho_2 \leq \frac{2}{3} < 1 < \rho_1 < \frac{5}{4} < \frac{8}{5} < \mu_1 < 2 < 3 \leq \mu_2 \]

and

\[ \frac{57}{20} < \rho_1 + \mu_1 < 3 \leq \frac{11}{3} \leq \rho_2 + \mu_2 \]

Remark 4.2.5. We remark that the results here were motivated by the initial finding that \( I(p^k) < I(m^2) \), i.e. \( \rho_1 < \mu_1 \). We prove here too that \( I(p^k) < I(m) \). We start with: For all positive integers \( a \) and \( b \), \( \sigma(ab) \leq \sigma(a)\sigma(b) \) with equality occurring if and only if \( \gcd(a,b) = 1 \). (For a proof, we refer the interested reader to standard graduate textbooks in number theory.) It is evident from this statement that for any positive integer \( x > 1 \), \( I(x^2) < (I(x))^2 \). In particular, from Lemma 4.2.3, we have \( \frac{8}{5} < I(m^2) < (I(m))^2 \), which implies that \( \frac{2\sqrt{10}}{5} < I(m) \).

But \( I(p^k) < 1.25 \) (again from Lemma 4.2.3), and \( \frac{2\sqrt{10}}{5} \approx 1.26491106406735 \). Consequently, \( I(p^k) < I(m) \). (Note that \( \gcd(p,m) = 1 \).) This should motivate the succeeding discussion, which attempts to improve the result \( p^k < \frac{2}{3}m^2 \) to \( p^k < m \), where again \( N = p^km^2 \) is an OPN with Euler’s factor \( p^k \).

We now attempt to obtain the improvement mentioned in Remark 4.2.5. Since proper factors of a perfect number are deficient, we have \( I(m) < 2 \). Consequently, we have the bounds \( \frac{2\sqrt{10}}{5} < I(m) < 2 \). Likewise, since \( p^km \) is a proper factor of \( N \) which is perfect, and from Lemma 4.2.3 we have \( 1 < I(p^k) \), hence the following must be true:

Lemma 4.2.8. Let \( N = p^km^2 \) be an OPN with Euler’s factor \( p^k \). Then

\[ \frac{2\sqrt{10}}{5} < I(p^k)I(m) = \frac{\sigma(p^k)\sigma(m)}{m} < p^k < 2. \]
Now, as before, let

\[ \rho_1 = \frac{\sigma(p^k)}{p^k} \]
\[ \rho_3 = \frac{\sigma(p^k)}{m} \]
\[ \mu_3 = \frac{m}{\sigma(m)} \]
\[ \mu_4 = \frac{m}{\sigma(m)} \]

From the preceding results, we get

\[ 1 + \frac{2\sqrt{10}}{5} < \rho_1 + \mu_3 < 3, \]
where \( 1 + \frac{2\sqrt{10}}{5} \approx 2.26491 \) and the rightmost inequality is obtained via a method similar to that used for Lemma 4.2.4. Also, from Lemma 4.2.8 and Lemma 3.2.1 (Arithmetic Mean-Geometric Mean Inequality), we get the lower bound

\[ \frac{2\sqrt{1000}}{5} < \rho_3 + \mu_4, \]
where \( \frac{2\sqrt{1000}}{5} \approx 2.24937. \)

We observe that \( \rho_3 \neq 1 \) since \( \sigma(p^k) \equiv 2 \pmod{4} \) while \( m \equiv 1 \pmod{2} \) since \( N \) is an OPN. Hence, we need to consider two (2) separate cases:

**Case 1:** \( \rho_3 < 1 \). Multiplying both sides of this inequality by \( \mu_4 > 0 \), we get \( \rho_3 \mu_4 < \mu_4 \). But from Lemma 4.2.8, we have \( \frac{2\sqrt{10}}{5} < \rho_3 \mu_4 \). Therefore: \( \rho_3 < 1 < 1.26491106406735 \approx \frac{2\sqrt{10}}{5} < \mu_4 \). (In particular, \( \rho_3 = \frac{\sigma(p^k)}{m} \neq \frac{\sigma(m)}{p^k} = \mu_4 \).) Note that, from Lemma 4.2.3, \( \frac{p^k}{m} < \rho_3 < 1 \), which implies that \( p^k < m \).

**Case 2:** \( 1 < \rho_3 \). Similar to what we did in Case 1, we get from Lemma 4.2.8 \( \mu_4 < 2 \). Note that, from Lemma 4.2.3, \( 1 < \rho_3 < \frac{(5/4)p^k}{m} \), which implies that \( p^k > \frac{4}{5}m \).
We now claim that $\rho_3 \neq \mu_4$ also holds in Case 2. For suppose to the contrary that $1 < \rho_3 = \mu_4 < 2$. Since these two ratios are rational numbers between two consecutive integers, this implies that $m \nmid \sigma(p^k)$ and $p^k \nmid \sigma(m)$. But $\rho_3 = \mu_4 \Rightarrow p^k \sigma(p^k) = m \sigma(m)$, which, together with $\gcd(p, m) = \gcd(p^k, m) = 1$, results to a contradiction. This proves our claim. Hence, we need to consider two (2) further subcases:

Subcase 2.1: $1 < \rho_3 < \mu_4 < 2$

\[
1 < \frac{\sigma(p^k)}{m} < \frac{\sigma(m)}{p^k} < 2 \Rightarrow p^k \sigma(p^k) < m \sigma(m) \Rightarrow \frac{p^k \sigma(p^k)}{(p^k m)^2} < \frac{m \sigma(m)}{(p^k m)^2}
\]

\[
\frac{1}{m^2} \frac{\sigma(p^k)}{p^k} < \frac{1}{p^{2k}} \frac{\sigma(m)}{m} \Rightarrow \frac{1}{m^2} < \frac{1}{p^{2k}} \frac{\sigma(p^k)}{p^k} \frac{1}{p^{2k}} \frac{\sigma(m)}{m} < \frac{2}{p^{2k}}
\]

\[
p^{2k} < 2m^2 \Rightarrow p^k < \sqrt{2}m
\]

Thus, for Subcase 2.1, we have: $\frac{4}{5}m < p^k < \sqrt{2}m$. (Note that it may still be possible to prove either $p^k \sigma(p^k) < m \sigma(m)$ and that we can derive the upper bound $\rho_3 + \mu_4 < 3$.

Subcase 2.2: $\mu_4 < \rho_3$, $1 < \rho_3$ and $\mu_4 < 2$

\[
\frac{\sigma(m)}{p^k} < \frac{\sigma(p^k)}{m} \Rightarrow m \sigma(m) < p^k \sigma(p^k) \Rightarrow \frac{m \sigma(m)}{(p^k m)^2} < \frac{p^k \sigma(p^k)}{(p^k m)^2}
\]
\[
\frac{1}{p^{2k}} \frac{\sigma(m)}{m} < \frac{1}{m^2} \frac{\sigma(p^k)}{p^k} < \frac{1}{m^2} \frac{\sigma(m)}{m}
\]
\[
\frac{1}{p^{2k}} < \frac{1}{m^2} \Rightarrow m^2 < p^{2k} \Rightarrow m < p^k
\]

It is here that the author’s original conjecture that \(p^k < m\) in all cases is disproved.

Note also the following improvements to the bounds for \(\rho_3\) and \(\mu_4\) in this subcase: \(\sqrt{\frac{1000}{5}} < \rho_3\) and \(\mu_4 < \sqrt{2}\).

At this point, the author would like to set the following goals to treat this Subcase 2.2 further:

- Obtain an upper bound for \(\rho_3\).
- Obtain a lower bound for \(\mu_4\).
- Obtain an upper bound for \(\rho_3 + \mu_4\).

We summarize our results in the following theorem:

**Theorem 4.2.4.** Let \(N = p^k m^2\) be an OPN with Euler’s factor \(p^k\). Then \(\rho_3 \neq \mu_4\), and the following statements hold:

- If \(\rho_3 < 1\), then \(p^k < m\).
- \(\rho_3 < m\), then \(\mu_4 < \rho_3\).
  - If \(\rho_3 < \mu_4\), then \(\frac{4}{5}m < p^k < \sqrt{2}m\).
  - If \(\mu_4 < \rho_3\), then \(m < p^k\).
We now state and prove here the generalization to $\rho_2 \leq \frac{2}{3}$ mentioned in Section 3.2:

**Theorem 4.2.5.** Let $N = \prod_{i=1}^{\omega(N)} p_i^{\alpha_i}$ be the canonical factorization of an OPN $N$, where $p_1 < p_2 < \cdots < p_t$ are primes, $t = \omega(N)$ and $\alpha_i > 0$ for all $i$. Then $\sigma(p_i^{\alpha_i}) \leq \frac{2N}{3p_i^{\alpha_i}}$ for all $i$.

**Proof.** Let $N = p_i^{\alpha_i}M$ for a particular $i$. Since $p_i^{\alpha_i} | N$ and $N$ is an OPN, then $\sigma(p_i^{\alpha_i})\sigma(M) = 2p_i^{\alpha_i}M$. From Example 4.1.4 we know that $p_i^{\alpha_i} | \sigma(M)$ and we have $\sigma(M) = hp_i^{\alpha_i}$ for some positive integer $h$. Assume $h = 1$. Then $\sigma(M) = p_i^{\alpha_i}$, forcing $\sigma(p_i^{\alpha_i}) = 2M$. Since $N$ is an OPN, $p_i$ is odd, whereupon we have an odd $\alpha_i$ by considering parity conditions from the last equation. But this means that $p_i^{\alpha_i}$ is the Euler’s factor of $N$, and we have $p_i^{\alpha_i} = p^{\beta}$ and $M = m^2$. Consequently, $\sigma(m^2) = \sigma(M) = p_i^{\alpha_i} = p^{\beta}$, which contradicts the fact that $\mu_2 \geq 3$. Now suppose that $h = 2$. Then we have the equations $\sigma(M) = 2p_i^{\alpha_i}$ and $\sigma(p_i^{\alpha_i}) = M$. (Note that, since $M$ is odd, $\alpha_i$ must be even.) Applying the $\sigma$ function to both sides of the last equation, we get $\sigma(\sigma(p_i^{\alpha_i})) = \sigma(M) = 2p_i^{\alpha_i}$, which means that $p_i^{\alpha_i}$ is an odd superperfect number. But Kanold [66] showed that odd superperfect numbers must be perfect squares (no contradiction at this point, since $\alpha_i$ is even), and Suryanarayana [67] showed in 1973 that “There is no odd super perfect number of the form $p^{2\alpha_n}$ (where $p$ is prime). Thus $h = \frac{\sigma(M)}{p_i^{\alpha_i}} \geq 3$, whereupon we have the result $\sigma(p_i^{\alpha_i}) \leq \frac{2}{3}M = \frac{2}{3} \frac{N}{p_i^{\alpha_i}}$ for the chosen $i$. Since $i$ was arbitrary, we have proved our claim in this theorem.

The following corollary is a direct consequence of Theorem 4.2.5.
Corollary 4.2.3. Let $N$ be an OPN with $r = \omega(N)$ distinct prime factors. Then

$$N^{2-r} \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{r-1}.$$ 

In the next section, we attempt to "count" the number of OPNs by trying to establish a bijective map between OPNs and points on a certain hyperbolic arc. We prove there that such a mapping (which is based on the concept of the abundancy index) is neither surjective nor injective, utilizing results on solitary numbers in the process.

4.3 “Counting” the Number of OPNs

In this section, we will be disproving the following conjecture:

Conjecture 4.3.1. For each $N = p^k m^2$ an OPN with $N > 10^{300}$, there corresponds exactly one ordered pair of rational numbers $\left(\frac{\sigma(p^k)}{p^k}, \frac{\sigma(m^2)}{m^2}\right)$ lying in the region $1 < \frac{\sigma(p^k)}{p^k} < \frac{5}{4}$, $\frac{8}{5} < \frac{\sigma(m^2)}{m^2} < 2$, and $\frac{57}{20} < \frac{\sigma(p^k)}{p^k} + \frac{\sigma(m^2)}{m^2} < 3$, and vice-versa.

We begin our disproof by observing that, for prime $p$, $\frac{\sigma(p^k)}{p^k}$ is a decreasing function of $p$ (for constant $k$) and is also an increasing function of $k$ (for constant $p$). These observations imply that $\frac{\sigma(p_1^{k_1})}{p_1^{k_1}} \neq \frac{\sigma(p_2^{k_2})}{p_2^{k_2}}$ for the following cases:

1. $p_1 \neq p_2$, $k_1 = k_2$ and
2. $p_1 = p_2$, $k_1 \neq k_2$. To show that the same inequality holds for the case (3) $p_1 \neq p_2$, $k_1 \neq k_2$, we proceed as follows: Suppose to the contrary that $\frac{\sigma(p_1^{k_1})}{p_1^{k_1}} = \frac{\sigma(p_2^{k_2})}{p_2^{k_2}}$ but $p_1 \neq p_2$, $k_1 \neq k_2$. Then

$$p_2^{k_2}(p_2 - 1)(p_1^{k_1+1} - 1) = p_1^{k_1}(p_1 - 1)(p_2^{k_2+1} - 1).$$
Since $p_1$ and $p_2$ are distinct (odd) primes, $\gcd(p_1, p_2) = 1$ which implies that

$$p_1^{k_1} | (p_2 - 1)(p_1^{k_1+1} - 1) \text{ and } p_2^{k_2} | (p_1 - 1)(p_2^{k_2+1} - 1).$$

Now, let us compute $\gcd(p^k, p^{k+1} - 1)$ using the Euclidean Algorithm:

$$p^{k+1} - 1 = p \cdot p^k - 1 = (p - 1) \cdot p^k + (p^k - 1)$$
$$p^k = 1 \cdot (p^k - 1) + 1 \rightarrow \text{last nonzero remainder}$$
$$p^k - 1 = (p^k - 1) \cdot 1 + 0$$

Consequently, $\gcd(p^k, p^{k+1} - 1) = 1$. Hence, we have that

$$p_1^{k_1} \mid (p_2 - 1) \text{ and } p_2^{k_2} \mid (p_1 - 1).$$

Since $p_1$ and $p_2$ are primes, this means that $p_1p_2 \mid (p_1 - 1)(p_2 - 1)$, which further implies that $p_1p_2 \leq (p_1 - 1)(p_2 - 1) = p_1p_2 - (p_1 + p_2) + 1$, resulting in the contradiction $p_1 + p_2 \leq 1$.

**Remark 4.3.1.** An alternative proof of the results in the preceding paragraph may be obtained by using the fact that prime powers are solitary (i.e., $I(p^k) = I(x)$ has the sole solution $x = p^k$).

Let $X = \frac{\sigma(p^k)}{p^k}$ and $Y = \frac{\sigma(m^2)}{m^2}$. It is straightforward to observe that, since the abundancy index is an arithmetic function, then for each $N = p^k m^2$ an OPN (with $N > 10^{300}$), there corresponds exactly one ordered pair of rational numbers $(X, Y)$ lying in the hyperbolic arc $XY = 2$ bounded as follows: $1 < X < 1.25$, $1.6 < Y < 2$, and $2.85 < X + Y < 3$. (Note that these bounds are the same ones obtained in Subsection 4.2.4.)
We now disprove the backward direction of Conjecture 4.3.1. We do this by showing that the mapping $X = \frac{\sigma(p^k)}{p^k}$ and $Y = \frac{\sigma(m^2)}{m^2}$ is neither surjective nor injective in the specified region.

**$(X, Y)$ is not surjective.** We prove this claim by producing a rational point $(X_0, Y_0)$ lying in the specified region, and which satisfies $X_0 \neq \frac{\sigma(p^k)}{p^k}$ for all primes $p$ and positive integers $k$. It suffices to consider the example $X_0 = \frac{\sigma(pq)}{pq}$ where $p$ and $q$ are primes satisfying $5 < p < q$. Notice that $1 < X_0 = \frac{(p+1)(q+1)}{pq} = \left(1 + \frac{1}{p}\right)\left(1 + \frac{1}{q}\right) \leq \frac{8}{7} \frac{12}{11} < 1.2468 < 1.25$. Now, by setting $Y_0 = \frac{2}{X_0}$, the other two inequalities for $Y_0$ and $X_0 + Y_0$ would follow. Thus, we now have a rational point $(X_0, Y_0)$ in the specified region, and which, by a brief inspection, satisfies $X_0 \neq \frac{\sigma(p^k)}{p^k}$ for all primes $p$ and positive integers $k$ (since prime powers are solitary). Consequently, the mapping defined in the backward direction of Conjecture 4.3.1 is not surjective.

**Remark 4.3.2.** Since the mapping is not onto, there are rational points in the specified region which do not correspond to any OPN.

**$(X, Y)$ is not injective.** It suffices to construct two distinct OPNs $N_1 = p_1^{k_1}m_1^2$ and $N_2 = p_2^{k_2}m_2^2$ (assuming at least two such numbers exist) that correspond to the same rational point $(X, Y)$. Since it cannot be the case that $p_1^{k_1} \neq p_2^{k_2}$, $m_1^2 = m_2^2$, we consider the scenario $p_1^{k_1} = p_2^{k_2}$, $m_1^2 \neq m_2^2$. Thus, we want to produce a pair $(m_1, m_2)$ satisfying $I(m_1^2) = I(m_2^2)$. (A computer check by a foreign professor using Maple produced no examples for this equation in the range $1 \leq m_1 < m_2 \leq 300000$. But then again, in pure mathematics, absence
of evidence is not evidence of absence.) Now, from the inequalities $p^k < m^2$ and $N = p^km^2 > 10^{300}$, we get $m^2 > 10^{150}$. A nonconstructive approach to finding a solution to $I(m_1^2) = I(m_2^2)$ would then be to consider $10^{150} < m_1^2 < m_2^2$ and Erdős’ result that "The number of solutions of $I(a) = I(b)$ satisfying $a < b \leq x$ equals $Cx + o(x)$ where $C \geq \frac{8}{147}$." ([17], [2]) (Note that $C$ here is the same as the (natural) density of friendly integers.) Given Erdős’ result then, this means that eventually, as $m_2 \to \infty$, there will be at least $10^{150} \frac{8}{147}$ solutions $(m_1, m_2)$ to $I(m_1^2) = I(m_2^2)$, a number which is obviously greater than 1. This finding, though nonconstructive, still proves that the mapping defined in the backward direction of Conjecture [4.3.1] is not injective.

Thus, we have failed our attempt to “count” the number of OPNs by showing that our conjectured correspondence is not actually a bijection. However, this should not prevent future researchers from conceptualizing other correspondences that may potentially be bijections. The author is nonetheless convinced that, ultimately, such bijections would have to make use of the concept of the abundancy index one way or another.
Chapter 5

Analysis and Recommendations

Up to this point, the OPN problem is still open. We have discussed some of the old as well as new approaches to solving this problem, with the factor chain approach taking center stage in mainstream OPN research. But it appears that, for the ultimate solution of the problem using such approaches, we need fresh ideas and what may be called a “paradigm shift”. For example, the index/outlaw status of the fraction $\frac{\sigma(p) + 1}{p}$, which may potentially disprove the OPN Conjecture, might probably require concepts from other disparate but related fields of mathematics, such as real analysis. But then again, this is just an educated guess.

Two open problems necessitating further investigations would come to mind, if the reader had read and fully understood the contents of this thesis:

- Prove or disprove: If $N = p^km^2$ is an OPN with Euler’s factor $p^k$, then $p^k < m$.

- There exists a bijection $(X, Y)$ from the set of OPNs to the hyperbolic arc $XY = 2$ lying in the region $1 < X < 2, 1 < Y < 2, 2\sqrt{2} < X + Y < 3$, and this bijection makes use of the concept of abundancy index to define
the mappings $X$ and $Y$.

The underlying motivation for these open problems have been described in sufficient detail in Chapter 4 and should set the mood for extensive investigation into their corresponding solutions.

Lastly, we warn future researchers who would be interested in pursuing this topic that while a multitude of evidence certainly suggests that no OPNs exist, neither heuristic nor density arguments but only a (mathematical) proof of their nonexistence could lay the problem to a “conclusive” rest. The author has tried, and although he failed like the others, eventually somebody will give a final end to this problem that had defied solution for more than three centuries.
Bibliography

[1] Anderson, C. W., *Advanced Problem 5967: Density of Odd Deficient Numbers*, American Mathematical Monthly, vol 82, no 10, (1975) pp 1018–1020

[2] Anderson, C. W., Hickerson, Dean, and Greening, M. G., *Advanced Problem 6020: Friendly Integers*, American Mathematical Monthly, vol 84, no 1, (1977) pp 65–66

[3] Bezverkhnyev, Slava, *Perfect Numbers and Abundancy Ratio*, Undergraduate project in a course in Analytic Number Theory (submitted April 3, 2003) at Carleton University, Ottawa, Canada, Available online: [http://www.slavab.com/EN/math.html](http://www.slavab.com/EN/math.html), Viewed: July 2007

[4] Brent, R. P., Cohen, G. L., and te Riele, H. J. J., *Improved techniques for lower bounds for odd perfect numbers*, Math. of Comp., vol 57, (1991) pp 857–868

[5] Burton, D. M., *Elementary Number Theory*, Third Edition, Wm. C. Brown Publishers (1994)

[6] Chein, E. Z., *An odd perfect number has at least eight prime factors*, Ph.D. Thesis, Pennsylvania State University, (1979)
[7] Cohen, G. L., *On odd perfect numbers. II. Multiperfect numbers and quasiperfect numbers*, Journal of the Australian Mathematical Society, vol 29, no 3, (1980) pp 369–384

[8] Cohen, G.L., *On the largest component of an odd perfect number*, Journal of the Australian Mathematical Society, vol 42, no 2, (1987) pp 280-286

[9] Cohen, G. L. and Hendy, M. D., *On odd multiperfect numbers*, Math. Chron., vol 10, (1981) pp 57–61

[10] Cohen, G. L. and Williams, R. J., *Extensions of some results concerning odd perfect numbers*, Fibonacci Quarterly, vol 23, (1985) pp 70–76

[11] Cook, R. J., *Bounds for odd perfect numbers*, CRM Proc. and Lect. Notes, Centres de Recherches Mathématiques, vol 19, (1999) pp 67–71

[12] Cruz, Christopher Thomas R., *Searching for Odd Perfect Numbers*, M. S. Thesis, De La Salle University, Manila, (2006)

[13] Curtiss, D. R., *On Kellogg’s Diophantine Problem*, American Mathematical Monthly, vol 29, no 10, (1922) pp 380–387

[14] Deléglise, Marc, *Bounds for the density of abundant integers*, Experiment. Math., vol 7, no 2, (1998) pp 137–143

[15] Dandapat, G. G., Hunsucker, J. L., and Pomerance, C., *Some New Results on Odd Perfect Numbers*, Pacific J. Math., vol 57, (1975) pp 359–364, Available online:
http://projecteuclid.org/Dienst/UI/1.0/Display/euclid.pjm/1102905990
Viewed: 2006

[16] Dickson, Leonard, *Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors*, Amer. J. Math., vol 35, (1913) pp 413–422

[17] Erdös, P., *Remarks on number theory, II, Some problems on the σ function*, Acta Arith., vol 5, (1959) pp 171–177

[18] Goto, Takeshi and Ohno, Yasuo, *Odd perfect numbers have a prime factor exceeding 10^8*, Preprint, 2006, Available online: http://www.ma.noda.tus.ac.jp/u/tg/perfect.html Viewed: June 2007

[19] Gradstein, I. S., *O nečetnych soveršenných čísłah*, Mat. Sb., vol 32, (1925) pp 476–510

[20] Greathouse, C., *Bounding the Prime Factors of Odd Perfect Numbers*, An undergraduate paper at Miami University, (2005)

[21] Greathouse, Charles and Weisstein, Eric W, *Odd Perfect Number*, From MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com/OddPerfectNumber.html

[22] Grün, O., *Über ungerade vollkommene Zahlen*, Math. Zeit., vol 55, (1952) pp 353–354

[23] Hagis Jr., Peter, *Outline of a proof that every odd perfect number has at least eight prime factors*, Math. Comp., (1980) pp 1027–1032
[24] Hagis Jr., Peter and McDaniel, Wayne, *A new result concerning the structure of odd perfect numbers*, Proc. Amer. Math. Soc., vol 32, (1972) pp 13–15

[25] Hagis Jr., Peter and McDaniel, Wayne, *On the largest prime divisor of an odd perfect number*, Math. of Comp., vol 27, (1973) pp 955–957

[26] Hagis Jr., Peter and McDaniel, Wayne, *On the largest prime divisor of an odd perfect number*, Math. of Comp., vol 29, (1975) pp 922–924

[27] Hagis Jr., Peter and Suryanarayana, D., *A theorem concerning odd perfect numbers*, Fibonacci Quarterly, vol 8, no 4, (1970) pp 337–346

[28] Hare, K. G. *More on the Total Number of Prime Factors of an Odd Perfect Number*, Math. Comput., (2003)

[29] Hare, K. G. *New Techniques for Bounds on the Total Number of Prime Factors of an Odd Perfect Number*, Math. Comput., vol 74, (2005) pp 1003–1008

[30] Heath-Brown, D. R., *Odd perfect numbers*, Math. Proc. Camb. Phil. Soc., vol 115, (1994) pp 191–196

[31] Hickerson, Dean (ed.), *A074902: Known Friendly Numbers*, From the On-Line Encyclopedia of Integer Sequences, Available online: [http://www.research.att.com/~njas/sequences/A074902](http://www.research.att.com/~njas/sequences/A074902), Viewed: 2008

[32] Holdener, J. A., *Conditions Equivalent to the Existence of Odd Perfect Numbers*, Mathematics Magazine, vol 79, no 5, (2006)
[33] Holdener, J. A. and Riggs, J., *Consecutive Perfect Numbers (actually, the Lack Thereof!)*, An undergraduate research project at Kenyon College, Gambier, OH (1998)

[34] Holdener, J. A. and Stanton, W. G., *Abundancy 'Outlaws' of the Form* \( \frac{\sigma(N) + t}{N} \), A joint research project at Kenyon College, Gambier, OH (2007)

[35] Holdener, J. A. and Czarnecki, L., *The Abundancy Index: Tracking Down Outlaws*, A joint research project at Kenyon College, Gambier, OH (2007)

[36] Iannucci, Douglas E., *The second largest prime divisor of an odd perfect number exceeds ten thousand*, Math. of Comp., vol 68, (1999) pp 1749–1760

[37] Iannucci, Douglas E., *The third largest divisor of an odd perfect number exceeds one hundred*, Math. of Comp., vol 69, (2000) pp 867–879

[38] Iannucci, Douglas E. and Sorli, R. M., *On the total number of prime factors of an odd perfect number*, Math. of Comp., vol 72, (2003) pp 2077–2084

[39] Jenkins, Paul M., *Odd perfect numbers have a prime factor exceeding 10^7*, Math. of Comp., vol 72, (2003) pp 1549–1554

[40] Kanold, H. J., *Untersuchungen über ungerade vollkommene Zahlen*, J. Reine Angew. Math, vol 183, (1941) pp 98–109

[41] Kanold, H. J., *Folgerungen aus dem vorkommen einer Gauss’schen primzahl in der primfactorenzerlegung einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math., vol 186, (1949) pp 25–29
[42] Kanold, H. J., Über die Dichten der Mengen der vollkommenen und der befreundeten Zahlen, Math. Z., vol 61, (1954) pp 180–185

[43] Kanold, H. J., Über einen Satz von L. E. Dickson, II, Math. Ann, vol 132, (1956) p 273

[44] Kishore, M. On Odd Perfect, Quasiperfect, and Odd Almost Perfect Numbers, Math. Comput., vol 36, (1981) pp 583–586

[45] Laatsch, R., Measuring the Abundancy of Integers, Mathematics Magazine, vol 59, (1986) pp 84–92

[46] Lipp, W., OddPerfect.Org, Available online: [http://www.oddperfect.org](http://www.oddperfect.org), Viewed: 2008

[47] Ludwick, Kurt, Analysis of the ratio $\frac{\sigma(n)}{n}$, Undergraduate honors thesis at Penn State University, PA, May, 1994, Available online: [http://www.math.temple.edu/~ludwick/thesis/thesisinfo.html](http://www.math.temple.edu/~ludwick/thesis/thesisinfo.html), Viewed: 2006

[48] Mc Cleary, J. (Vassar College, Poughkeepsie, NY), Hunting Odd Perfect Numbers: Quarks or Snarks?, Lecture notes first presented as a seminar to students of Union College, Schenectady, NY, (2001)

[49] McDaniel, Wayne, On odd multiply perfect numbers, Boll. Un. Mat. Ital., vol 2, (1970) pp 185–190

[50] Nielsen, Pace P., An upper bound for odd perfect numbers, Integers: Electronic Journal of Combinatorial Number Theory, vol 3, (2003) p A14
[51] Nielsen, Pace P., *Odd perfect numbers have at least nine distinct prime factors*, Mathematics of Computation, in press, 2006, arXiv:math.NT/0602485

[52] O'Connor, J. J. and Robertson, E. F. *History Topic: Perfect Numbers*, Available online: http://www-history.mcs.st-andrews.ac.uk/HistTopics/Perfect_numbers.html, Viewed: July 2007

[53] Peirce, Benjamin, *On perfect numbers*, New York Math. Diary, vol 2, no XIII, (1832) pp 267–277

[54] Periasastri, M., *A note on odd perfect numbers*, Mathematics Student, vol 26, (1958) pp 179–181

[55] Pomerance, Carl, *Odd perfect numbers are divisible by at least seven distinct primes*, Acta Arithmetica, vol XXV, (1974) pp 265–300

[56] Pomerance, Carl, *The second largest divisor of an odd perfect number*, Math. Of Comp., vol 29, (1975) pp 914–921

[57] Pomerance, Carl, *Multiply perfect numbers, Mersenne primes, and effective computability*, Math. Ann., vol 226, (1977), pp 195–226

[58] Ryan, R. F., *Results concerning uniqueness for $\frac{\sigma(x)}{x} = \frac{\sigma(p^nq^m)}{p^nq^m}$ and related topics*, International Mathematical Journal, vol 2, no 5, (2002) pp 497–514

[59] Ryan, R. F., *A Simpler Dense Proof Regarding the Abundancy Index*, Mathematics Magazine, vol 76, (2003) pp 299–301
[60] Sandor, Jozsef and Crstici, Borislav (eds.), *Perfect Numbers: Old and New Issues; Perspectives*, (2004), Handbook of Number Theory, vol II, pp 15–98, Dordrecht, The Netherlands: Kluwer Academic Publishers, Retrieved June 10, 2007, from Gale Virtual Reference Library via Thomson Gale: http://find.galegroup.com/gvrl/infomark.do?&contentSet=EBKS&type=retrieve&tabID=T001&prodId=GVRL&docId=CX268800009&source=gale&userGroupName=gvrlasia30&version=1.0

[61] Sandor, Jozsef, Mitrinovic, Dragoslav, and Crstici, Borislav (eds.), *Sum-of-Divisors Function, Generalizations, Analogues; Perfect Numbers and Related Problems*, (2006), Handbook of Number Theory, vol I, (2nd ed., pp 77–120), Dordrecht, The Netherlands: Springer, Retrieved June 10, 2007, from Gale Virtual Reference Library via Thomson Gale: http://find.galegroup.com/gvrl/infomark.do?&contentSet=EBKS&type=retrieve&tabID=T001&prodId=GVRL&docId=CX2594000011&source=gale&userGroupName=gvrlasia30&version=1.0

[62] Servais, C., *Sur les nombres parfaits*, Mathesis, vol 8, (1888) pp 92–93

[63] Sorli, Ronald M., *Algorithms in the Study of Multiperfect and Odd Perfect Numbers*, Ph.D. Thesis, University of Technology, Sydney, (2003)

[64] Starni, P., *On the Euler’s Factor of an Odd Perfect Number*, J. Number Theory, vol 37, no 3, (1991) pp 366–369

[65] Steuerwald, Rudolf, *Verscharfung einen notwendigen Bedeutung für die Existenz einen ungeraden vollenommenen Zahl*, Bayer. Akad. Wiss. Math. Natur., (1937) pp 69–72
[66] Suryanarayana, D., *Super Perfect Numbers*, Elem. Math., vol 24, (1969) pp 16–17

[67] Suryanarayana, D., *There Is No Odd Super Perfect Number of the Form $p^{2^\alpha}$*, Elem. Math., vol 24, (1973) pp 148–150

[68] Sylvester, James Joseph, *Sur les nombres parfaits*, Comptes Rendus, vol CVI, (1888) pp 403–405

[69] Sylvester, James Joseph, *Sur l'impossibilitie de l'existence d'un nombre parfait qui ne contient pas au 5 diviseurs premiers distincts*, Comptes Rendus, vol CVI, (1888) pp 522–526

[70] Various contributors, *Great Internet Mersenne Prime Search*, Available online: [http://www.mersenne.org/prime.htm](http://www.mersenne.org/prime.htm) Viewed: 2008

[71] Various contributors, *Perfect Number*, From Wikipedia - the free encyclopedia, [http://en.wikipedia.org/wiki/Perfect_number](http://en.wikipedia.org/wiki/Perfect_number)

[72] Voight, J., *Perfect Numbers: An Elementary Introduction*, Available online: [http://magma.maths.usyd.edu.au/~voight/notes/perfelem.pdf](http://magma.maths.usyd.edu.au/~voight/notes/perfelem.pdf) Viewed: 2006

[73] Weiner, P. A., *The Abundancy Ratio, A Measure of Perfection*, Mathematics Magazine, vol 73, (2000) pp 307–310