On the 0-dimensional cusps of the Kähler moduli of a $K^3$ surface

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Abstract

Let $S$ be a complex algebraic $K^3$ surface. It is proved that the 0-dimensional cusps of the Kähler moduli of $S$ are in one-to-one correspondence with the twisted Fourier-Mukai partners of $S$. As a result, a counting formula for the 0-dimensional cusps of the Kähler moduli is obtained. Applications to rational maps between $K^3$ surfaces are given. When the Picard number of $S$ is 1, the bijective correspondence is calculated explicitly by using the Fricke modular curve.

1 Introduction

Let $S$ be an algebraic $K^3$ surface over the field of complex numbers. In [11] we considered a certain orthogonal modular variety $\mathcal{K}(S) = \Gamma(S)^+ \backslash \Omega^+_{NS(S)}$ and studied the Fourier-Mukai (FM) partners of $S$ in connection with a compactification of $\mathcal{K}(S)$. It is a classical result of Baily-Borel [1] that $\mathcal{K}(S)$ can be compactified to be a normal projective variety, by adjoining certain boundary components. The boundary components of $\mathcal{K}(S)$ are called cusps of $\mathcal{K}(S)$. One of the results of [11] is that the isomorphism classes of the FM partners of $S$ are in bijective correspondence with certain 0-dimensional cusps of $\mathcal{K}(S)$, called standard cusps. However, a general 0-dimensional cusp of $\mathcal{K}(S)$ is not necessarily standard. The purpose of this paper is to study the geometric counterparts of the non-standard 0-dimensional cusps of $\mathcal{K}(S)$.

As we shall explain below, a non-standard cusp corresponds to a twisted Fourier-Mukai partner of $S$. The notion of twisted FM partners, which was introduced by Căldăraru [4], generalizes that of FM partners. By definition, a twisted FM partner of $S$ is a twisted $K^3$ surface $(S', \alpha')$ such that there is an exact equivalence $D^b(S', \alpha') \simeq D^b(S)$ between the derived categories. As well as the FM partners, the twisted FM partners of $S$ can be regarded as certain geometric realizations of the category $D^b(S)$. They are useful for dealing with non-fine moduli spaces of sheaves on $S$ (see [5], [8]).

Our main result is the following.

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Theorem 1.1 (Theorem 3.7). Let $\text{FM}^d(S)$ be the set of isomorphism classes of the twisted Fourier-Mukai partners $(S', \alpha')$ of $S$ with $\text{ord}(\alpha') = d$. Let $C^d(S)$ be the set of 0-dimensional cusps of the Baily-Borel compactification of the modular variety $K(S)$ of divisibility $d$. Then there exists a canonical bijection

$$\text{FM}^d(S) \simeq C^d(S).$$

In particular, we have

$$\sum_d \#\text{FM}^d(S) = \#\{ \text{the 0-dimensional cusps of } K(S) \}.$$

The existence of a map $C^d(S) \to \text{FM}^d(S)$ is observed in [11]. In the present paper we study the correspondence in more detail and establish its bijectivity. In fact, a more general result than Theorem 1.1 will be proved. That is, for a twisted $K3$ surface $(S, \alpha)$ we establish a relation between the 0-dimensional cusps of the Kähler moduli of $(S, \alpha)$ and the twisted FM partners of $(S, \alpha)$. If we put $d = 1$ in Theorem 1.1, we recover the correspondence between the untwisted FM partners and the standard cusps in [11]. Twisted FM partners appear naturally if we consider all 0-dimensional cusps of $K(S)$.

The modular variety $K(S)$ is called Kähler moduli because it is an analogue of Kähler moduli of a Calabi-Yau 3-fold. The analogy was investigated extensively by Dolgachev [6]. In an effort to formulate mirror symmetry for $K3$ surfaces, Dolgachev defined for a $K3$ surface $S$ (satisfying suitable conditions) a moduli space $M'$ of certain “mirror K3 surfaces” of $S$, and observed that $M'$ is a modular variety uniformized by the Hermitian symmetric domain $\Omega_{SL^+}^+(S)$ associated to $S$. Our Kähler moduli $K(S)$ is almost isomorphic to Dolgachev’s mirror moduli space $M'$, in the sense that the arithmetic group $\Gamma(S)^+$ defining $K(S)$ contains the arithmetic group defining $M'$ as a finite-index subgroup. When the Hodge structure of $S$ is generic, $K(S)$ is indeed isomorphic to $M'$. We note that $K(S)$ is naturally dominated by the complexified Kähler cone of $S$ (see [6]). On the other hand, Bridgeland gave an intrinsic construction of $K(S)$ by proving that $K(S)$ contains a natural quotient space of the space of stability conditions on the category $D^b(S)$ as a Zariski open set (see [2]).

An observation for the mirror picture of [6] can be drawn from Theorem 1.1. In the formulation of [6], there is an ambiguity of the choice of mirror family, which depends on the choice of 0-dimensional cusp of a moduli space $M$ to which $S$ belongs. The moduli space $M$ can be identified with the Kähler moduli of a generic member $S'$ of a mirror moduli space of $S$. Theorem 1.1 applied to $S'$, suggests that the ambiguity of the choice of mirror family of $S$ comes from the existence of twisted FM partners of $S'$.

There is an application of Theorem 1.1 to the modular variety $K(S)$. The twisted Fourier-Mukai number $\#\text{FM}^d(S)$ is non-zero for only finitely many $d$, and each $\#\text{FM}^d(S)$ is a finite number (see [5]). A formula for the number $\#\text{FM}^d(S)$ is given in [12]. Roughly speaking, the number $\#\text{FM}^d(S)$ is expressed as a sum of certain ‘masses’ of the genera of some Lorentzian lattices.
Combining those formulae with Theorem 1.1, we obtain a counting formula for the 0-dimensional cusps of the modular variety $K(S)$ (Theorem 3.8). Via [6], it allows us to count the 0-dimensional cusps of the moduli spaces of certain lattice-polarized $K3$ surfaces. This formula for the cusps is a natural generalization of Scattone’s formula [17].

When some description of the set of cusps is available, we obtain a classification of the twisted FM partners by Theorem 1.1. We shall give such classification for certain elliptic $K3$ surfaces.

**Theorem 1.2** (Theorem 4.2). Let $S$ be a $K3$ surface whose Néron-Severi lattice $NS(S)$ contains the hyperbolic plane $U$. Then for each twisted Fourier-Mukai partner $(S',\alpha')$ of $S$ there exists an elliptic fibration $f : S \to \mathbb{P}^1$ such that $(S',\alpha')$ is isomorphic to the relative Jacobian of $f$.

See Definition 4.1 for the definition of relative Jacobian. In Kodaira’s terminology, the elliptic surface underlying the relative Jacobian is the basic elliptic surface associated to the original fibration. The assumption that $NS(S)$ contains $U$ admits a geometric interpretation that $S$ has the structure of an elliptic surface with a section. For example, this assumption is satisfied if the Picard number $\rho(S)$ is larger than or equal to 13.

**Theorem 1.2** yields applications to rational maps between $K3$ surfaces.

**Corollary 1.3** (Proposition 4.7). Let $S_+$ and $S_-$ be $K3$ surfaces with $\rho(S_+) \geq 13$. Let $T(S_\pm)$ be the transcendental lattice of $S_\pm$. Then there exists a Hodge isometry $T(S_+) \otimes \mathbb{Q} \simeq T(S_-) \otimes \mathbb{Q}$ if and only if there exist a $K3$ surface $S_0$ and rational maps $S_0 \dashrightarrow S_+, S_0 \dashrightarrow S_-$ of square degrees.

In other words, $S_+$ and $S_-$ are isogenous in the sense of Mukai [13] if and only if they are dominated by a common $K3$ surface by rational maps of square degrees.

When $\rho(S) = 1$, we calculate the correspondence in Theorem 1.1 explicitly. Dolgachev [6] showed that the Kähler moduli $K(S)$ for such $S$ is isomorphic to the Fricke modular curve, which is the quotient of the congruence modular curve $\Gamma_0(n)\backslash \mathbb{H}$ by an involution. Here $2n$ is the degree of the $K3$ surface $S$. It is easy to describe the cusps of the Fricke curve, because the theory of elliptic modular curves is available. On the other hand, a set of representatives of $FM^d(S)$ is given in [12] by certain moduli spaces of sheaves on $S$, twisted by natural obstruction classes. Then we have an explicit correspondence between the cusps of the Fricke curve and certain moduli spaces of sheaves on $S$. An advantage of considering the Fricke curve is that we have a complete description of its cusps. Thus, given two arbitrary primitive isotropic Mukai vectors, one can decide immediately whether the associated moduli spaces are isomorphic or not.

As we noted above, there is a derived-categorical construction of the Kähler moduli $K(S)$ by using stability conditions due to Bridgeland. Then the following question arises naturally from Theorem 1.1.

**Question 1.4.** Can one perform the Baily-Borel compactification of $K(S)$ by studying degenerations of stability conditions in various large volume limits?
The rest of the paper is structured as follows. In Sect. 2.1 we recall some lattice theory. In Sect. 2.2 we study lattice-theoretic properties of the twisted Mukai lattice of a twisted $K3$ surface. In Sect. 3 we prove Theorem 1.1. In Sect. 4 we derive Theorem 1.2 and Corollary 1.3. In Sect. 5.1 we exhibit the isomorphism between the Fricke modular curve and the Kähler moduli of a $K3$ surface of Picard number 1. In Sect. 5.2 we calculate the correspondence between the cusps of the Fricke modular curve and certain moduli spaces of sheaves.

**Notation.** All varieties are assumed to be algebraic varieties over the field of complex numbers. In particular, a $K3$ surface means a nonsingular complex algebraic $K3$ surface. For a $K3$ surface $S$, we denote by $NS(S)$ (resp. $T(S)$) the Néron-Severi (resp. transcendental) lattice of $S$. The Picard number $\rho(S)$ is the rank of $NS(S)$. Let
\[
\tilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) + H^2(S, \mathbb{Z}) + H^4(S, \mathbb{Z}),
\]
\[
\tilde{NS}(S) := H^0(S, \mathbb{Z}) + NS(S) + H^4(S, \mathbb{Z}),
\]
which are endowed with the Mukai pairing. The hyperbolic plane $U$ is the lattice $\mathbb{Z}e + \mathbb{Z}f$, $(e,e) = (f,f) = 0, (e,f) = 1$. We identify the lattice $H^0(S, \mathbb{Z}) + H^4(S, \mathbb{Z})$ endowed with the Mukai pairing with the hyperbolic plane $U$ by the identifications $(1,0,0) = e, (0,0,-1) = f$. Write
\[
\Lambda_{K3} := E_8^2 \oplus U^3, \quad \tilde{\Lambda}_{K3} := E_8^2 \oplus U^4.
\]

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2 Twisted Mukai lattices

2.1 Preliminaries from lattice theory

By an even lattice, we mean a free $\mathbb{Z}$-module $L$ of finite rank endowed with a non-degenerate symmetric bilinear form $L \times L \rightarrow \mathbb{Z}$ satisfying $(l,l) \in 2\mathbb{Z}$ for all $l \in L$. For a field $K$, the quadratic space $L \otimes K$ is denoted by $L_K$. For a vector $l \in L$ we define the divisibility $\text{div}(l)$ of $l$ as the positive generator of the ideal $(l,L) \subset \mathbb{Z}$. The set of the primitive isotropic vectors $l \in L$ with $\text{div}(l) = d$ is denoted by $I^d(L)$. To an even lattice $L$ we can associate a finite Abelian group $D_L := L^\vee/L$ and a quadratic form $q_L : D_L \rightarrow \mathbb{Q}/2\mathbb{Z}$ defined by $q_L(x) \equiv (x,x) \mod 2\mathbb{Z}, x \in D_L$. Then $(D_L, q_L)$ is called the discriminant form of $L$. We have a natural homomorphism $r_L : O(L) \rightarrow O(D_L)$, whose kernel is denoted by $O(L)_0$. The following facts due to Nikulin [14] are well-known. For later use, we indicate a proof.
Proposition 2.1 ([14]). Let $L$ be an even unimodular lattice and let $M$ be a primitive sublattice of $L$ with the orthogonal complement $M^\perp$.

1. There exists a natural isometry $\lambda_L : (D_M, q_M) \simeq (D_{M^\perp}, -q_{M^\perp})$.
2. For two isometries $\gamma_M \in O(M)$ and $\gamma_{M^\perp} \in O(M^\perp)$, $\gamma_M \oplus \gamma_{M^\perp}$ extends to the isometry of $L$ if and only if $r_M(\gamma_M) = \lambda_L^{-1} \circ r_{M^\perp}(\gamma_{M^\perp}) \circ \lambda_L$.

Proof. (1) The unimodularity of $r$ only if $\lambda$.

2. Let $\gamma_M \in O(M)$ and $\gamma_{M^\perp} \in O(M^\perp)$, $\gamma_M \oplus \gamma_{M^\perp}$ extends to the isometry of $L$ if and only if $r_M(\gamma_M) = \lambda_L^{-1} \circ r_{M^\perp}(\gamma_{M^\perp}) \circ \lambda_L$.

Let $L$ be an even lattice of sign($L$) = $(2, \text{rk}(L) - 2)$ and let $\Gamma \subset O(L)$ be a subgroup containing $\{\pm \text{id}\}$. Denote by $\Omega_L$ the set of the oriented positive-definite two-planes in $L$ of sign($L$), which carries a complex structure via the isomorphism

$$\Omega_L \simeq \left\{ \mathbb{C} \omega \in \mathbb{P}(L) \bigg| (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \right\}.$$ 

Then $\Omega_L$ has two connected components. A choice of a component of $\Omega_L$, say $\Omega_L^+$, is equivalent to a choice of an orientation for a positive-definite two-plane, and is called an orientation of $L$. Let $\Gamma^+ \subset \Gamma$ be the subgroup of the orientation-preserving isometries in $\Gamma$. The quotient space $\Gamma^+/\Omega_L^+$ admits the Baily-Borel compactification $\overline{\Gamma^+ \setminus \Omega_L^+}$, which turns out to be a normal projective variety ([11]). See also [17]. The set of the 0-dimensional cusps of $\Gamma^+ \setminus \Omega_L^+$ is canonically identified with the set

$$\bigcup_d \Gamma^+ \setminus I^d(L).$$

### 2.2 Twisted Mukai lattices

The twisted Mukai lattice of a twisted $K3$ surface was defined by Huybrechts in [7] and is studied in [8]. Here we develop lattice-theoretical properties of twisted Mukai lattices. The results of this section will be used in Section 3.

Let $S$ be a $K3$ surface. The Brauer group $\text{Br}(S)$ of $S$ is the group of the torsion elements of $H^2(\text{O}_S^\vee)$. Via the exponential sequence, we have

$$\text{Br}(S) \simeq H^2(S, \mathbb{Q})/(\text{NS}(S) \otimes \mathbb{Q} + H^2(S, \mathbb{Z})).$$ (1)

For example, let $\rho(S) = 20$. Then $\text{Br}(S)$ is the group of the finite-order points of the elliptic curve $H^2(\text{O}_S^\vee)/\text{T}(S)^\vee$.

A class $B \in H^2(S, \mathbb{Q})$ is called a (rational) B-field lift of a Brauer element $\alpha \in \text{Br}(S)$ if $B$ maps to $\alpha$ in the isomorphism ([11]). For an element $\alpha \in \text{Br}(S)$ we can find a B-field lift of $\alpha$ from $\frac{1}{d}H^2(S, \mathbb{Z})$, where $d = \text{ord}(\alpha)$. By considering the intersection pairings of the B-field lifts with $T(S)$, we also have

$$\text{Br}(S) \simeq \text{Hom}(T(S), \mathbb{Q}/\mathbb{Z}).$$ (2)
Lemma 2.2. Let $\omega_S \in H^2(S, \mathbb{C})$ be a period of $S$ and let $B \in H^2(S, \mathbb{Q})$ be a B-field lift of $\alpha$. By definition, the twisted Mukai lattice $\widetilde{H}(S, B, \mathbb{Z})$ of $(S, \alpha)$ and $B$ is the lattice $\widetilde{H}(S, \mathbb{Z})$ equipped with the twisted period $e^B(\omega_S) = (1, B, \frac{1}{2}(B, B)) \wedge (0, \omega_S, 0)$. Set

$$\widetilde{NS}(S, B) := e^B(\omega_S)\perp \widetilde{H}(S, B, \mathbb{Z}),$$

$$T(S, B) := \widetilde{NS}(S, B)\perp \widetilde{H}(S, B, \mathbb{Z}).$$

We have a Hodge isometry $e^B : T(S, \alpha) \simeq T(S, B)$, as each class $l \in T(S)$ is of pure degree 2 [9]. Since $\widetilde{NS}(S, B)_{\mathbb{Q}} = e^B(\widetilde{NS}(S)_{\mathbb{Q}})$, the orientation of $\widetilde{NS}(S)$ induces that of $\widetilde{NS}(S, B)$. That is, $e^B(\mathbb{R}(1, 0, -1) \oplus \mathbb{R}(0, l, 0))$ is of positive orientation for an ample class $l \in \widetilde{NS}(S)$. For another B-field lift $B' \in H^2(S, \mathbb{Q})$ of $\alpha$, we can write $B' = B + B_1 + B_2$ with $B_1 \in H^2(S, \mathbb{Z})$ and $B_2 \in \widetilde{NS}(S)_{\mathbb{Q}}$. Then we have an orientation-preserving Hodge isometry $e^{B_1} : \widetilde{H}(S, B, \mathbb{Z}) \xrightarrow{\cong} \widetilde{H}(S, B + B_1, \mathbb{Z}) = \widetilde{H}(S, B', \mathbb{Z})$ defined by the wedge product with the cohomology class $(1, B_1, \frac{1}{2}(B_1, B_1))$, which fixes the vector $(0, 0, 1)$.

In the remainder of this section, we fix a twisted $K3$ surface $(S, \alpha)$ and a B-field lift $B$ of $\alpha$. The basic ideas of the following Lemma 2.2 and Lemma 2.3 are present in [13].

Lemma 2.2 ([12]). Let $\lambda : (D_{\widetilde{NS}(S, B)}, q) \simeq (D_{T(S, B)}, -q)$ be the natural isometry. Then we have a Hodge isometry $e^B : T(S) \xrightarrow{\cong} \langle T(S, B), \lambda((0, 0, -\frac{1}{d})) \rangle \subset T(S, B)^\vee$, \hspace{1cm} (3)

where $d = \text{ord}(\alpha)$. The twisting $\alpha : T(S) \to \mathbb{Z} / d\mathbb{Z}$ is given by the homomorphism

$$T(S) \xrightarrow{\alpha} \langle T(S, B), \lambda((0, 0, -\frac{1}{d})) \rangle / T(S, B) \simeq \langle \lambda((0, 0, -\frac{1}{d})) \rangle \simeq \mathbb{Z} / d\mathbb{Z}. \hspace{1cm} (4)$$

Proof. For a transcendental class $l \in T(S)$ with $\alpha(l) = \bar{l} \in \mathbb{Z} / d\mathbb{Z}$, we have $e^B(l) = (0, l, \frac{1}{d} + k)$ with $k \in \mathbb{Z}$. Since $e^B((0, 0, -\frac{1}{d})) = e^B((0, 0, -\frac{1}{d})) \in \widetilde{H}(S, B, \mathbb{Z})$, we have $\lambda((0, 0, -\frac{1}{d})) = e^B((0, 0, -\frac{1}{d}))$ by the definition of $\lambda$ (Proposition 2.1). Thus we obtain $e^B(T(S)) = \langle T(S, B), \lambda((0, 0, -\frac{1}{d})) \rangle$. The image of $l$ by $\bar{l}$ is $\bar{l}$ and the image of $T(S, \alpha)$ by $\bar{l}$ is 0, so the second claim is proved.

Lemma 2.3. The divisibility of the primitive isotropic vector $(0, 0, 1) \in \widetilde{NS}(S, B)$ is equal to $d = \text{ord}(\alpha)$.
Proposition 2.4. When \( d \) is chosen from \( \frac{1}{d} H^2(S, \mathbb{Z}) \), \( d = \text{ord}(\alpha) \), we have the following orientation-preserving isometry:

\[
e^B \circ \kappa : \widetilde{NS}(S) \cong \left\langle N\widetilde{S}(S, B), (0, 0, -\frac{1}{d}) \right\rangle.
\]

Proof. Write \( \widetilde{M} := \left\langle \widetilde{NS}(S, B), (0, 0, -\frac{1}{d}) \right\rangle \). From the equalities \( e^B((0, 0, -\frac{1}{d})) = (0, 0, -\frac{1}{d}) \), \( e^B((d, 0, 0)) = (d, dB, \frac{1}{d}(dB, B)) \), \( e^B((0, l, 0)) = (0, l, (l, B)) \), we obtain the inclusion \( e^B \circ \kappa(\widetilde{NS}(S)) \subseteq \tilde{M} \). Since

\[
\det \tilde{M} = d^{-2} \cdot \det \widetilde{NS}(S, B) = d^{-2} \cdot \det T(S, \alpha) = \det \widetilde{NS}(S),
\]

we have \( e^B \circ \kappa(\widetilde{NS}(S)) = \tilde{M} \). \( \Box \)

After choosing a B-field lift \( B \in \frac{1}{d} H^2(S, \mathbb{Z}) \) of \( \alpha \), we write

\[
\tilde{M} := \left\langle N\widetilde{S}(S, B), (0, 0, -\frac{1}{d}) \right\rangle, \quad \tilde{T} := \left\langle T(S, B), \lambda(0, 0, -\frac{1}{d}) \right\rangle,
\]

where \( \lambda : D\widetilde{NS}(S, B) \cong D_{T(S, B)} \) is the natural isomorphism. Both \( \tilde{M} \) and \( \tilde{T} \) are even lattices. From \( \lambda \) we obtain the isometry \( \lambda : (D\tilde{M}, q) \cong (D\tilde{T}, -q) \). Denote by \( \lambda_0 : D\widetilde{NS}(S) \cong D_T(S) \) the natural isomorphism.

Proposition 2.5. Let \( B \in \frac{1}{d} H^2(S, \mathbb{Z}) \), \( d = \text{ord}(\alpha) \). The following diagram commutes.

\[
\begin{array}{ccc}
D_{\widetilde{NS}(S)} & \xrightarrow{e^B \circ \kappa} & D\tilde{M} \\
\downarrow \lambda_0 & & \downarrow \lambda \\
D_T(S) & \xrightarrow{e^B} & D\tilde{T}.
\end{array}
\]
Each isometry $\gamma \in \Gamma(S,B)$ is an orientation-preserving isometry in $\Gamma(S,B)$, where $\Gamma(S,B)$ is the group of isomorphism classes of the twisted Fourier-Mukai partners $(S',\alpha')$ of $(S,\alpha)$ with $\operatorname{ord}(\alpha') = d$.

Definition 3.1. We define the group $\Gamma(S, B) := O(\tilde{N}S(S, B))$ by

$$\Gamma(S, B) := r_{\tilde{N}S(S, B)}^{-1}(\lambda \circ r_T(S, B)(O_{\text{Hodge}}(T(S, B)))),$$

where $\lambda : O(D_T(S, B)) \simeq O(D_{\tilde{N}S(S, B)})$ is the isomorphism induced from the isometry $<D_{\tilde{N}S(S, B)}, q > \simeq <D_T(S, B), -q>$, and $O_{\text{Hodge}}(T(S, B))$ is the group of the Hodge isometries of $T(S, B)$. There are obvious inclusions

$$\{\pm \text{id}\} \times O(\tilde{N}S(S, B))_0 \subset \Gamma(S, B) \subset O(\tilde{N}S(S, B)).$$

Each isometry $\gamma \in \Gamma(S, B)$ can be extended to a Hodge isometry $\tilde{H}(S, B, \mathbb{Z}) \simeq \tilde{H}(S, B, \mathbb{Z})$. Recall that $\Gamma(S, B)^+$ is the subgroup of $\Gamma(S, B)$ consisting of the orientation-preserving isometries in $\Gamma(S, B)$. Then we can form the modular variety

$$K(S, \alpha) := \Gamma(S, B)^+ \backslash \Omega_{\tilde{N}S(S, B)}^+,$$ (8)

Proof. Every element of $D_{\tilde{N}S(S)}$ can be represented by a vector in $NS(S)^\vee$. Thus we take a vector $x \in NS(S)^\vee$ and prove the commutativity for the element $[x] \in D_{NS(S)} \simeq D_{\tilde{N}S(S)}$. Choose a vector $y \in T(S)^\vee$ representing $\lambda_0([x]) \in DT(S)$. By the definition of $\lambda_0$, we have $x + y \in H^2(S, \mathbb{Z})$. Since $e^B(x + y) \in H^2(S, \mathbb{Z}) \oplus \mathbb{Z}(0, 0, 1/d)$, there is an integer $k \in \mathbb{Z}$ such that

$$e^B(x) + (0, 0, k/d) + e^B(y) \in H^2(S, \mathbb{Z}) \subset \tilde{H}(S, B, \mathbb{Z}).$$

By Lemma 2.2 and Proposition 2.4, we have $e^B(y) \in T^\vee$ and $e^B(x) \in M^\vee$. As $(0, 0, k/d)$ is an integral vector in $M$, we obtain $e^B(x) \equiv e^B(x) + (0, 0, k/d) \in D_M$. It follows that

$$\tilde{\lambda}(e^B \circ \kappa([x])) = \tilde{\lambda}(e^B(x)) = \tilde{\lambda}(e^B(x) + (0, 0, k/d)) = [e^B(y)] = e^B \circ \lambda_0([x]).$$

3 Twisted Fourier-Mukai partners and 0-dimensional cusps

Let $(S, \alpha)$ be a twisted $K3$ surface with a B-field lift $B \in H^2(S, \mathbb{Q})$. A twisted $K3$ surface $(S', \alpha')$ is called a twisted Fourier-Mukai (FM) partner of $(S, \alpha)$ if there is an exact equivalence $D^b(S, \alpha) \simeq D^b(S', \alpha')$. Let $FM^d(S, \alpha)$ be the set of isomorphism classes of the twisted FM partners $(S', \alpha')$ of $(S, \alpha)$ with $\operatorname{ord}(\alpha') = d$.
the isomorphism class of which does not depend on the choices of the lift $B$ of $\alpha$. The set of 0-dimensional cusps of the Baily-Borel compactification of $\mathcal{K}(S, \alpha)$ is identified with the set $\bigcup_d \mathcal{C}^d(S, \alpha)$, where

$$\mathcal{C}^d(S, \alpha) := \Gamma(S, B)^+ \backslash I^d(\widetilde{NS}(S, B)).$$

When $\alpha = 1$, we write $\mathcal{K}(S) := \mathcal{K}(S, 1)$ and $\mathcal{C}^d(S) := \mathcal{C}^d(S, 1)$. Then we have

$$\mathcal{K}(S) = \Gamma(S)^+ \backslash \Omega^+_{\widetilde{NS}(S)}, \quad \text{where} \quad \Gamma(S) := \Gamma(S, 0).$$

Let $l \in I^d(\widetilde{NS}(S, B))$ be a primitive isotropic vector. By using the surjectivity of the period map, we shall construct a twisted $K3$ surface $(S_l, \alpha_l)$ from the cusp $[l] \in \mathcal{C}^d(S, \alpha)$ as follows. Let $\lambda : (D_{\widetilde{NS}(S, B)}, q) \simeq (D_{T(S, B)}, -q)$ be the natural isometry. Firstly we consider the extended even lattices

$$\tilde{M}_1 := \left\langle \widetilde{NS}(S, B), \frac{l}{d} \right\rangle, \quad T_1 := \left\langle T(S, B), \lambda\left(\frac{l}{d}\right) \right\rangle,$$

and a homomorphism

$$\alpha_l : T_1 \rightarrow T_1 / T(S, B) \simeq \left\langle \lambda\left(\frac{l}{d}\right) \right\rangle \simeq \mathbb{Z} / d\mathbb{Z}, \quad \lambda\left(\frac{l}{d}\right) \mapsto \bar{1}.$$

The lattice $\tilde{M}_1$ has the orientation induced from that of $\widetilde{NS}(S, B)$. Since $\frac{l}{d} \in I^1(\tilde{M}_1)$, there is an embedding $\varphi : U \rightarrow \tilde{M}_1$ satisfying $\varphi(f) = \frac{l}{d}$. The orthogonal complement $M_{\varphi} := \varphi(U)^\perp \cap \tilde{M}_1$ is of sign $(M_{\varphi}) = (1, \rho(S) - 1)$. We choose the connected component $M_{\varphi}^+$ of the open set $\{v \in (M_{\varphi})_R \mid (v, v) > 0\}$ so that $\mathbb{R} \varphi(e + f) + \mathbb{R} \varphi$ is of positive orientation for all $v \in M_{\varphi}^+$. From the isometry $\bar{\lambda} : (D_{\tilde{M}_1}, q) \simeq (D_{T_1}, -q)$, we obtain an embedding

$$\tilde{M}_1 \oplus \bar{T}_1 \leftarrow \Lambda_{K3}$$

with both $\tilde{M}_1$ and $\bar{T}_1$ embedded primitively. Then the orthogonal complement $\Lambda_{\varphi} := \varphi(U)^\perp \cap \Lambda_{K3}$ is isometric to the $K3$ lattice $\Lambda_{K3}$ and is endowed with a period by the sublattice $T_1 \subset \Lambda_{\varphi}$. By the surjectivity of the period map (see [20]), there exist a $K3$ surface $S_l$ and a Hodge isometry $\Phi : H^2(S_l, \mathbb{Z}) \simeq \Lambda_{\varphi}$ mapping the positive cone of $NS(S_l)$ to the cone $M_{\varphi}^+$. Pulling back the homomorphism $\alpha_l$ by $\Phi$, we obtain the twisted $K3$ surface $(S_l, \alpha_l)$. The following lemma can be proved similarly as Lemma 3.2.1 of [11], which treats the case $\alpha = 1$.

**Lemma 3.2.** The isomorphism class of the twisted $K3$ surface $(S_l, \alpha_l)$ is uniquely determined by the cusp $[l] \in \mathcal{C}^d(S, \alpha)$.

When $S$ is untwisted, the lattice $NS(S_l)$ is isogenus to an overlattice of $NS(S)$ with cyclic quotient.

Let $(S', \alpha')$ be a given twisted $K3$ surface with a B-field lift $B'$ such that there is an orientation-preserving Hodge isometry $\Phi : \tilde{H}(S', B', \mathbb{Z}) \simeq \tilde{H}(S, B, \mathbb{Z})$. Then we have $(S', \alpha') \in \text{FM}^d(S, \alpha)$ by Huybrechts-Stellari’s theorem ([9]), where $d = \text{ord}(\alpha')$. 

9
Proposition 3.3. In the above situation, we define the primitive isotropic vector \( l \in I^d(\NS(S,B)) \) by \( l = \Phi((0,0,-1)) \). Then we have the isomorphism \((S_l,\alpha_l) \simeq (S',\alpha')\).

Proof. It suffices to prove the assertion when \( B' \in \frac{1}{d} H^2(S',\Z) \). Let

\[
\tilde{\lambda} : D_{\tilde{M}_{l'}} \simeq D_{T_l}, \quad \lambda' : D_{\NS(S',B')} \simeq D_{T(S', B')}, \quad \lambda'_0 : D_{\NS(S')} \simeq D_{T(S')},
\]

be the natural isomorphisms, and set

\[
\tilde{M}' := \langle \NS(S', B') , (0,0, -\frac{1}{d}) \rangle, \quad T' := \langle T(S', B') , \lambda'((0,0, -\frac{1}{d})) \rangle.
\]

We have a natural isometry \( \tilde{\lambda}' : (D_{\tilde{M}_{l'}}, q) \simeq (D_{T_l}, -q) \). Then \( \tilde{\Phi} \) induces an orientation-preserving isometry \( \tilde{M}' \simeq \tilde{M}_l \) and a Hodge isometry \( T' \simeq T_l \). By Lemma 2.2 and the commutative diagram

\[
\begin{array}{ccc}
T(S') & \xrightarrow{e^{B'}} & T'/T(S', B') \\
\Phi \circ e^{B'} & \downarrow \cong & \Phi \\
T_l & \xrightarrow{\Phi} & T_l/T(S, B) \\
\end{array}
\]

we obtain \( (\tilde{\Phi} \circ e^{B'})^* \alpha_l = \alpha' \).

On the other hand, the following diagram commutes by Proposition 2.5

\[
\begin{array}{ccc}
D_{\NS(S')} & \xrightarrow{e^{B'} \circ \kappa} & D_{\tilde{M}_l} \\
\lambda'_0 & \downarrow \tilde{\lambda} \\
D_{T(S')} & \xrightarrow{e^{B'}} & D_{T_l} \\
\end{array}
\]

It follows from Proposition 2.4 that the isometry

\[
(\tilde{\Phi} \circ e^{B'} \circ \kappa) \oplus (\tilde{\Phi} \circ e^{B'}) : \NS(S') \oplus T(S') \xrightarrow{\cong} \tilde{M}_l \oplus T_l
\]

extends to the orientation-preserving Hodge isometry

\[
\tilde{\Psi} : \tilde{H}(S', \Z) \xrightarrow{\cong} \tilde{\Lambda}_{K3}, \quad (0,0,-1) \mapsto \frac{l}{d}, \quad (\tilde{\Psi}|_{T(S')})^* \alpha_l = \alpha'.
\]

By considering \( \tilde{\Psi} : H^0(S', \Z) + H^4(S', \Z) \hookrightarrow \tilde{\Lambda}_{K3} \), we obtain an embedding \( \varphi : U \leftarrow \tilde{M}_l \subset \tilde{\Lambda}_{K3} \) with \( \varphi(f) = \frac{l}{d} \). Then we have a Hodge isometry \( \Psi : H^2(S', \Z) \simeq \Lambda_\varphi \) which maps the positive cone of \( \NS(S') \) to \( M_\varphi^+ \) and satisfies \( (\Psi|_{T(S')})^* \alpha_l = \alpha' \). By Lemma 3.2 we have \((S',\alpha') \simeq (S_l,\alpha_l)\). \(\square\)

Let \( \FM^d(S,\alpha)^+ \) be the subset of \( \FM^d(S,\alpha) \) consisting of those partners \((S',\alpha')\) such that there are orientation-preserving Hodge isometries \( \tilde{H}(S', B', \Z) \simeq \tilde{H}(S, B, \Z) \). It is conjectured by Huybrechts-Stellari (Conjecture 0.2 of [8]) that \( \FM^d(S,\alpha)^+ = \FM^d(S,\alpha) \). When \( \ord(\alpha) \leq 2 \), we actually have \( \FM^d(S,\alpha)^+ = \FM^d(S,\alpha) \) because the twisted Mukai lattice admits an orientation-reversing Hodge isometry (see [8]).
Proposition 3.4. The twisted K3 surface \((S_1, \alpha_1)\) constructed from the cusp \([l] \in C^d(S, \alpha)\) belongs to \(FM^d(S, \alpha)^+\).

Proof. We write \(l\) degreewise as \(l = (r, m, s)\).

Case (i). Assume that \(r < 0\). We can apply Yoshioka’s theorem ([19]). See also [9]) on the existence of moduli space of twisted sheaves with Mukai vector \(\Phi\). So we can find a twisted K3 surface \((M, \beta)\) and an orientation-preserving Hodge isometry \(\Phi: H(M, B', Z) \cong H(S, B, Z)\) with \(\Phi(0, 0, 1) = -l\), where \(B' \in H^2(M, \mathbb{Q})\) is a B-field lift of \(\beta\). By Proposition 3.5 we have \((S_1, \alpha_1) \cong (M, \beta) \in FM^d(S, \alpha)^+\).

Case (ii). When \(r > 0\), we have \((S_1, \alpha_1) \cong (\alpha_{-r}, m, s)\) \(\in FM^d(S, \alpha)^+\) by Lemma 3.2.

Case (iii). Finally assume that \(r = 0\). By the following lemma, there is a vector \(l' = (r', m', s') \in \Gamma(S, B)^+ \cdot l\) such that \(r' < 0\). Thus we have \((S_1, \alpha_1) \cong (\alpha_{r'}, s') \in FM^d(S, \alpha)^+\).

Lemma 3.5. For a primitive isotropic vector \(l_0 = (0, m_0, s_0) \in \tilde{NS}(S, B)\), there exists a vector \(l_1 = (r_1, m_1, s_1) \in O(\tilde{NS}(S, B))^+ \cdot l_0\) such that \(r_1 < 0\).

Proof. Set \(M := \{m \in NS(S) \mid (m, B) \in \mathbb{Z}\}\). Take a positive integer \(d'\) so that \(d'(1, B, \frac{1}{2}(B, B)) \in \tilde{NS}(S, B)\). We can find a vector \(v \in NS(S)\) satisfying the following conditions:

\[(m_0, v) < 0, \quad (v, v) = 0, \quad v \in d'M.\]

Define the isometry \(\varphi\) of \(\tilde{NS}(S, B)\) by the followings:

\[
(1, B, \frac{1}{2}(B, B)) \mapsto \left(1, B, \frac{1}{2}(B, B)\right),
\]

\[
(0, 0, 1) \mapsto (0, 0, 1) + (0, v, (v, B)),
\]

\[
(0, m, (m, B)) \mapsto (0, m, (m, B)) + (m, v) \cdot (1, B, \frac{1}{2}(B, B)), \quad m \in NS(S).
\]

Let \(M' := e^B(M)\) and \(N' := \langle d'(1, B, \frac{1}{2}(B, B)), (0, 0, 1)\rangle\). Then \(M' \oplus N'\) is a finite-index sublattice of \(\tilde{NS}(S, B)\). One can check that \(\varphi\) preserves \(M' \oplus N'\) and acts trivially on \(D_{M' \oplus N'}\). Hence \(\varphi\) preserves \(\tilde{NS}(S, B)\) and acts trivially on \(D_{\tilde{NS}(S, B)}\). The \(H^0(S, \mathbb{Z})\)-component of \(\varphi(l_0)\) is equal to \((m_0, v) < 0\).

Up to now we confirmed that the assignment \([l] \mapsto (S_1, \alpha_1)\) induces a well-defined map

\[C^d(S, \alpha) \rightarrow FM^d(S, \alpha)^+\]

which is surjective by Proposition 3.3. We prove the injectivity.

Proposition 3.6. Let \(l_1, l_2 \in I^d(\tilde{NS}(S, B))\) be two primitive isotropic vectors. If \((S_{l_1}, \alpha_{l_1}) \cong (S_{l_2}, \alpha_{l_2})\), then \(l_2 \in \Gamma(S, B)^+ \cdot l_1\).
Proof. From the proof of Proposition 3.4 we see the existence of twisted FM partners \((M_i, \beta_i), i = 1, 2\), and orientation-preserving Hodge isometries \(\tilde{H}(M_i, B_i, \mathbb{Z}) \simeq \tilde{H}(S, B, \mathbb{Z})\) mapping \((0, 0, -1)\) to \(I_i\). By the assumption we have \((M_1, \beta_1) \cong (M_2, \beta_2)\). Thus there is an effective Hodge isometry \(\Phi : H^2(M_2, \mathbb{Z}) \simeq H^2(M_1, \mathbb{Z})\) such that the twistings given by \(B_1\) and \(\Phi(B_2)\) coincide. So we obtain an orientation-preserving Hodge isometry \(\tilde{H}(M_2, B_2, \mathbb{Z}) \simeq \tilde{H}(M_1, B_1, \mathbb{Z})\) mapping \((0, 0, 1)\) to \((0, 0, 1)\). After all, we have an orientation-preserving Hodge isometry \(\tilde{H}(S, B, \mathbb{Z}) \simeq \tilde{H}(S, B, \mathbb{Z})\) mapping \(l_1\) to \(l_2\).

Thus we obtain the following theorem.

Theorem 3.7. For a twisted K3 surface \((S, \alpha)\), the assignment \([l] \mapsto (S_l, \alpha_l)\) induces a bijection between the sets
\[
C^d(S, \alpha) \simeq FM^d(S, \alpha)^+.
\]
In particular, for a K3 surface \(S\) we have a canonical bijection
\[
C^d(S) \simeq FM^d(S).
\]

In this way the set of 0-dimensional cusps of the Kähler moduli \(K(S)\) is identified with the set \(\bigcup_d FM^d(S)\) of isomorphism classes of the twisted FM partners of \(S\). For this correspondence, it is essential to distinguish two twisted K3 surfaces with a common underlying K3 surface by their twisting classes.

A formula expressing the number \#FM^d(S) is proved in [12]. Summing up those formulae over \(d \in \mathbb{N}\), we obtain a counting formula for the 0-dimensional cusps of the modular variety \(K(S)\). To exhibit the formula, we prepare some notation. Let \(I(D_{NS}(S))\) be the set of isotropic elements of \((D_{NS}(S), q)\). Each element \(x \in I(D_{NS}(S))\) gives rise to overlattices \(M_x, T_x\) of \(NS(S), T(S)\) respectively, and a homomorphism \(\alpha_x : T_x \to \mathbb{Z}/\text{ord}(\alpha)\mathbb{Z}\) with \(\text{Ker}(\alpha_x) = T(S)\). The lattice \(T_x\) inherits the period from \(T(S)\). Let
\[
O_{Hodge}(T_x, \alpha_x) := \{g \in O_{Hodge}(T_x), g^*\alpha_x = \alpha_x\}.
\]
For an even lattice \(L\), the genus of \(L\) is denoted by \(G(L)\). We define
\[
G_1(L) := \{L' \in G(L) \mid O(L')_0^+ \neq O(L)_0\}, \quad G_2(L) := G(L) - G_1(L).
\]
For a natural number \(d \in \mathbb{N}\), let \(\varepsilon(d) = 1\) if \(d \leq 2\), and \(\varepsilon(d) = 2\) if \(d \geq 3\).

Theorem 3.8. For a pair \((x, M)\) such that \(x \in I(D_{NS}(S))\) and \(M \in G(M_x)\), we write
\[
\tau(x, M) := \#(O_{Hodge}(T_x, \alpha_x) \setminus O(D_{M_x})/O(M))\n\]
Then
\[
\# \bigcup_d C^d(S) = \sum_x \left\{ \sum_M \tau(x, M) + \varepsilon(\text{ord}(x)) \sum_{M'} \tau(x, M') \right\}.
\]
Here \(x \in O_{Hodge}(T(S)) \setminus I(D_{NS}(S)), M \in G_1(M_x),\) and \(M' \in G_2(M_x)\).

This counting formula for the cusps is an extension of a formula of Scattone [17], which in our situation is the formula for those K3 surfaces whose Néron-Severi lattices contain the hyperbolic plane \(U\). It is the appearance of the subtle arithmetic invariants \(G_i(M_x)\) and \(\tau(x, M)\) that our generalization brings.
4 Twisted Fourier-Mukai partners and relative Jacobians

4.1 Relative Jacobian

Let $f : S \to \mathbb{P}^1$ be an elliptic fibration on a K3 surface $S$ which does not necessarily admit a section, and let $l \in NS(S)$ be the class of its fibres. The vector $v := (0, l, 0)$ is a primitive isotropic vector in $\tilde{NS}(S)$. Let $J(S/\mathbb{P}^1)$ be the coarse moduli space of stable (with respect to a generic polarization) sheaves on $S$ with Mukai vector $v$. By a theorem of Yoshioka [19], $J(S/\mathbb{P}^1)$ is a K3 surface. Let $\beta \in Br(J(S/\mathbb{P}^1))$ be the obstruction to the existence of a universal $\beta$-twisted universal sheaf on $S \times J(S/\mathbb{P}^1)$ induces an equivalence $D^b(J(S/\mathbb{P}^1), \beta) \simeq D^b(S)$. If we denote by $B$ a B-field lift of $\beta$, this derived equivalence induces an orientation-preserving Hodge isometry $\tilde{H}(J(S/\mathbb{P}^1), B, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z})$ mapping $(0, 0, 1)$ to $v = (0, l, 0)$.

Theorem 4.2. Let $S$ be a K3 surface such that $NS(S)$ admits an embedding of the hyperbolic plane $U$. Then for each twisted Fourier-Mukai partner $(S', \alpha')$ of $S$ there exists an elliptic fibration $f : S \to \mathbb{P}^1$ such that $(S', \alpha')$ is isomorphic to the relative Jacobian $(J(S/\mathbb{P}^1), \beta)$ of $f$.

Proof. By Theorem 4.7 there exists a primitive isotropic vector $l \in I^d(\tilde{NS}(S))$ such that $(S', \alpha') \simeq (S_l, \alpha_l)$, where $d = \text{ord}(\alpha')$. Let

$$I^d(D_{\tilde{NS}(S)}) := \{ x \in D_{\tilde{NS}(S)} | q(x) \equiv 0, \text{ord}(x) = d \}. $$

Since $\tilde{NS}(S)$ contains $U \oplus U$, it follows from Proposition 4.1.3 of [17] that the map

$$\Gamma(S)^+ \setminus I^d(\tilde{NS}(S)) \longrightarrow r(\Gamma(S)^+) \setminus I^d(D_{\tilde{NS}(S)}), \quad l \mapsto \frac{l}{d} \quad (11)$$

is bijective. On the other hand, the map

$$I^d(NS(S)) \longrightarrow I^d(D_{NS(S)}) \simeq I^d(D_{\tilde{NS}(S)}), \quad l \mapsto \frac{l}{d}$$

is surjective by Lemma 4.1.1 of [17]. Thus there is a primitive isotropic vector $l' \in NS(S) \cap (\Gamma(S)^+ \cdot l)$. We have $(S', \alpha') \simeq (S_{l'}, \alpha_{l'})$. 



13
Next, we transform \( l' \) to a nef primitive isotropic vector \( l'' \in \text{NS}(S) \) by the actions of \( \{ \pm \text{id} \} \) and the reflections with respect to \((-2)\)-curves on \( S \), as in [16]. Since \( l'' \) is the fibre class of an elliptic fibration \( f : S \rightarrow \mathbb{P}^1 \), the twisted FM partner \((S'_{\nu}, \alpha_{\nu})\) is isomorphic to the relative Jacobian \((J(S/\mathbb{P}^1), \beta)\) of \( f \).

Hence we have \((S', \alpha') \cong (J(S/\mathbb{P}^1), \beta)\).

\[ \square \]

**Remark 1.** The bijectivity of the map (11) can also be deduced from its surjectivity, which can be proved in an elementary way, and the formula for \( \# \mathcal{C}^d(S) \) (see [12] Corollary 4.4).

The above proof might be regarded as an extension of the argument for elliptic fibrations in [16]. Theorem 4.2 is a classification theorem for twisted FM partners of \( S \). The lattice \( \text{NS}(S) \) contains \( U \) if and only if \( S \) admits an elliptic fibration with a section. When \( \rho(S) \geq 13 \), we can always embed \( U \) into \( \text{NS}(S) \) by Corollary 1.13.5 of [14]. Certain 2-elementary \( K3 \) surfaces give other series of examples satisfying the assumption.

What happens if the assumption that \( \text{NS}(S) \) contains \( U \) is weakened? To be precise, let \( S \) be a \( K3 \) surface admitting at least one elliptic fibration, and \((S', \alpha')\) be an arbitrary twisted FM partner of \( S \). Is \((S', \alpha')\) isomorphic to the relative Jacobian of some elliptic fibration on \( S' \)? More generally, is \((S', \alpha')\) isomorphic to the twisted moduli space of relative sheaves associated to a primitive Mukai vector \((0, \alpha, \beta)\)? Here \( l \) is the fibre class of some elliptic fibration on \( S \), \( \alpha \) is a natural number, and \( \beta \) is an integer coprime to \( \alpha \). The following is a negative example to this question.

**Example 4.3.** Let \( S \) be a \( K3 \) surface such that \( \text{NS}(S) \simeq U(d) = \langle \frac{d}{2}, \frac{d}{2} \rangle \) and \( O_{\text{Hodge}}(T(S)) = \{ \pm \text{id} \} \). The surface \( S \) admits exactly two elliptic fibrations, whose fibre classes are denoted by \( l \) and \( m \) respectively. For a pair \((\alpha, \beta)\) of coprime integers, consider the primitive isotropic vectors in \( \text{NS}(S) \) defined by

\[
\begin{align*}
v_{\alpha, \beta} &= (0, \alpha l, \beta), \\
v'_{\alpha, \beta} &= (0, \alpha m, \beta).
\end{align*}
\]

We put the assumption that the twisted FM partner associated to \( v_{\alpha, \beta} \) is un-twisted, which is exactly the case that \( \beta \) is coprime to \( d \). Take integers \( \gamma, \delta \) satisfying \( \alpha \gamma d + \beta \delta = 1 \) and define the isotropic vectors in \( \text{NS}(S) \) by

\[
\begin{align*}
u_{\alpha, \beta} &= (0, \alpha l, \beta), \\
l_{\alpha, \beta} &= (0, \delta l, -\gamma d), \\
m_{\alpha, \beta} &= (\alpha d, \beta m, 0).
\end{align*}
\]

One checks that

\[
\langle v_{\alpha, \beta}, u_{\alpha, \beta} \rangle \simeq U, \\
\langle l_{\alpha, \beta}, m_{\alpha, \beta} \rangle \simeq U(d), \\
\langle v_{\alpha, \beta}, u_{\alpha, \beta} \rangle \perp \langle l_{\alpha, \beta}, m_{\alpha, \beta} \rangle.
\]

Therefore we obtain an isometry \( \varphi_{\alpha, \beta} \in O(\text{NS}(S)) \) which satisfies \( \varphi_{\alpha, \beta}(v_{\alpha, \beta}) = (0, 0, 1) \) and acts on the discriminant group \( D_{\text{NS}(S)} = \langle \frac{1}{4}, \frac{1}{4} \rangle \) by \( \langle \frac{1}{4}, \frac{1}{4} \rangle \mapsto (\beta^2, \beta^{-1} \frac{d}{4}) \). Since any isometry \( \varphi \in O(\text{NS}(S)) \) fixing \( (0, 0, 1) \) must preserve the subset \( \langle \frac{1}{4}, \frac{1}{4} \rangle \cup (\frac{d}{4}) \) in \( D_{\text{NS}(S)} \), we see that any isometry \( \varphi \in O(\text{NS}(S)) \) satisfying \( \varphi(v_{\alpha, \beta}) = (0, 0, 1) \) must preserve the subset \( \langle \frac{1}{4}, \frac{d}{4} \rangle \) in \( D_{\text{NS}(S)} \). By symmetry, the same holds for the vector \( v'_{\alpha, \beta} \).
When $d$ can be divided by at least two primes, there exists an isometry $\varphi \in O(\tilde{NS}(S))$ which does not preserve the subset $\langle \frac{1}{d} \rangle \cup \langle \frac{2}{d} \rangle$ in $D_{NS(S)}$. If we put $v := \varphi((0, 0, 1))$, then $v$ is not $\Gamma(S)^+$-equivalent to $v_{\alpha, \beta}$ nor $v'_{\alpha, \beta}$ for any $(\alpha, \beta)$. In other words, the FM partner associated to $v$ cannot be realized as moduli space of relative sheaves.

Bridgeland and Maciocia ([3] Proposition 4.4) proved that for a minimal surface $S$ of non-zero Kodaira dimension which admits an elliptic fibration $f : S \rightarrow C$, every untwisted FM partner of $S$ is isomorphic to a moduli space of relative sheaves supported on the fibers of $f$. Example 4.3 shows that the analogous statement does no longer holds for a general elliptic $K3$ surface, even if one allows all elliptic fibrations on the surface.

Remark 2. If we do not restrict to untwisted $K3$ surfaces, we can give negative examples to the above question more handily by analyzing the surjective map $I_d(\tilde{NS}(S)) \rightarrow I_d(D_{NS(S)}), d > 1$.

4.2 Applications to rational maps

Among the twisted FM partners of a $K3$ surface $S$, the relative Jacobians are related rather directly to the geometry of $S$. We give an application of Theorem 4.2 to rational maps between $K3$ surfaces.

Lemma 4.4. Let $S$ be a $K3$ surface and assume that $NS(S)$ contains $U$. Then every twisted Fourier-Mukai partner $(S', \alpha')$ of $S$ admits a rational map $S \dashrightarrow S'$ of degree $\text{ord}(\alpha')^2$.

Proof. By Theorem 4.2 there exists an elliptic fibration $f : S \rightarrow \mathbb{P}^1$ such that $S'$ is isomorphic to the $K3$ surface $J(S/\mathbb{P}^1)$ underlying the relative Jacobian of $f$. Let $l \in NS(S)$ be the fibre class of $f$. The order of $\alpha'$ is equal to the divisibility $d$ of $l$ in $NS(S)$. It suffices to construct a rational map $S \dashrightarrow J(S/\mathbb{P}^1)$ of degree $d^2$. Take a line bundle $M$ on $S$ such that $(M.l) = d$. Let $U \subset \mathbb{P}^1$ be a Zariski open set such that $f : f^{-1}(U) \rightarrow U$ is smooth. We can define a morphism $f^{-1}(U) \rightarrow J(S/\mathbb{P}^1)$ by setting

$$P \mapsto O_{f^{-1}(f(P))}(dP) \otimes M^{-1}, \quad P \in f^{-1}(U),$$

the degree of which is obviously equal to $d^2$.

We provide an example of the rational map given in Lemma 4.4.

Example 4.5. Let $g : A \rightarrow E$ be an elliptic fibration on an Abelian surface $A$ where $E$ is an elliptic curve, and $f : \text{Km}(A) \rightarrow \mathbb{P}^1$ be the associated elliptic fibration on the Kummer surface $\text{Km}(A)$. We take a line bundle $M$ on $\text{Km}(A)$ such that the degree of $M$ on an $f$-fiber is equal to the divisibility $d$ of the fiber class of $f$ in $NS(\text{Km}(A))$. The pullback of $M$ by the rational quotient map $A \dashrightarrow \text{Km}(A)$ extends to a line bundle $L$ on $A$. The degree of $L$ on a $g$-fiber is also equal to $d$. Let $\bar{g} : B \rightarrow E$ be the elliptic surface associated to the relative Jacobian of $g$. The surface $B$ is isomorphic to the product of $E$ and
a $g$-fiber. We can use the line bundle $L$ to define a morphism $\varphi : A \to B$ by $P \mapsto \mathcal{O}_{g^{-1}(g(P))}(dP) \otimes L^{-1}$, where $P \in A$. We choose the identity point of $B$ so that $\varphi$ is a homomorphism between Abelian surfaces. Then the elliptic Kummer surface $\tilde{f} : \text{Km}(B) \to \mathbb{P}^{1}$ induced from $\tilde{g}$ is canonically isomorphic to the elliptic $K3$ surface associated to the relative Jacobian of $f$. Under this identification, the rational map $\psi : \text{Km}(A) \dashrightarrow \text{Km}(B)$ induced from the homomorphism $\varphi$ is given by the correspondence $[12]$ in Lemma 4.4.

We describe the elimination of the indeterminacy of $\psi$. Let $A_2, B_2$ be the sets of 2-division points of $A, B$ respectively, and $\tilde{A}$ be the blow-up of $A$ at $\varphi^{-1}(B_2)$. The set $T$ consists of $16d^2$ 2d-division points and contains $A_2$. The inverse morphism $-1_A$ of $A$ extends to an involution $\iota$ on $\tilde{A}$. We denote by $X$ the quotient surface $\tilde{A}/(\iota)$. The natural morphism $\pi : X \to \text{Km}(A)$ is the blow-up at the point set $(\varphi^{-1}(B_2)\setminus A_2)/(-1_A)$, and the finite morphism $\tilde{\psi} : X \to \text{Km}(B)$ induced from $\varphi$ is the elimination of the indeterminacy of $\psi$. The ramification divisor of $\tilde{\psi}$ is the exceptional divisor of $\pi$, namely the $8(d^2 - 1)$ $(1\text{-})$-curves. The ramification index of $\tilde{\psi}$ at each $(1\text{-})$-curve is 2. Let $\{2E^A_i + E^A_j + E^A_k + E^A_l\}_{i=1, \ldots, 4}$ and $\{2E^B_i + E^B_j + E^B_k + E^B_l\}_{i=1, \ldots, 4}$ be the four $I_0$ singular fibers of $f, \tilde{f}$ respectively. All components $E^A_{ij}, E^B_{ij}$ are $(2\text{-})$-curves, and the components $E^A_i, E^B_i$ are the quotients of fibers of $g, \tilde{g}$ respectively. The surface $X$ is obtained by blowing up $2(d^2 - 1)$ points on each curve $E^A_i, 1 \leq i \leq 4$. The branch curve of $\tilde{\psi}$ is contained in the curve $\bigcup_{i,j} E^B_{ij}$. To observe the branching further, we consider the case of odd $d$ and the case of even $d$ separately. Renumbering the $i$ if necessary, we may assume that the image of $E^A_i$ by $\tilde{\psi}$ is $E^B_i$. When $d$ is odd, the curve $\bigcup_{j=1}^4 E^A_{ij}$ is mapped by $\tilde{\psi}$ isomorphically to $\bigcup_{j=1}^4 E^B_{ij}$. We may assume that $\psi(E^A_{ij}) = E^B_{ij}$. Then for each $j$ the inverse image $\tilde{\psi}^{-1}(E^B_{ij})$ consists of the proper transform of $E^A_{ij}$ and $\frac{1}{2}(d^2 - 1)$ ramifying $(1\text{-})$-curves. When $d$ is even, after renumbering the $j$ for $E^B_{ij}$, we have $\psi(E^A_{ij}) = E^B_{ij}$ for all $j$. Then $\tilde{\psi}^{-1}(E^B_{ij})$ consists of the proper transform of $\bigcup_{j=1}^4 E^B_{ij}$ and $\frac{1}{2}(d^2 - 4)$ ramifying $(1\text{-})$-curves. On the other hand, for $2 \leq j \leq 4$, $\tilde{\psi}^{-1}(E^B_{ij})$ consists of $\frac{1}{2}d^2$ ramifying $(1\text{-})$-curves. We remark that when $d = 2$, $\psi$ is the Galois covering for the symplectic action of the group $\text{Ker}(\varphi) \simeq (\mathbb{Z}/2\mathbb{Z})^{\otimes 2}$ on $\text{Km}(A)$.

Following Mukai’s approach in [13], we deduce the next proposition.

**Proposition 4.6.** Let $S$ and $S'$ be $K3$ surfaces with $\rho(S) = \rho(S')$. Assume that $NS(S)$ contains $U$. If there is a Hodge embedding $T(S) \hookrightarrow T(S')$, then there exists a rational map $S \dashrightarrow S'$ of degree $[T(S') : T(S)]^2$.

**Proof.** We regard $T(S)$ as a finite-index sublattice of $T(S')$ via the Hodge embedding. By the invariant factor theorem, there exists a filtration

$$T(S) = T_0 \subset T_1 \subset \cdots \subset T_n = T(S')$$

such that $T_i / T_{i-1}$ are cyclic. By using similar arguments as in the construction of $S_i$ in Lemma 3.2 we can find for each $i$ a $K3$ surface $S_i$ such that $T(S_i)$ is
Hodge isometric to $T_i$ and that $NS(S_i)$ is isometric to an overlattice of $NS(S)$. By the assumption on $NS(S)$, the lattice $NS(S_i)$ admits an embedding of $U$. Since there exist Hodge embeddings $T(S_{i-1}) \hookrightarrow T(S_i)$ of cyclic quotients, we can find twistings $\alpha_i \in Br(S_i)$ such that $T(S_i, \alpha_i)$ are Hodge isometric to $T(S_{i-1})$. By [9], we have $D^b(S_i, \alpha_i) \simeq D^b(S_{i-1})$. Now the assertion follows from Lemma 4.4. \hfill \square

Two $K3$ surfaces $S_+$ and $S_-$ are isogenous (in the sense of Mukai [13]) if there exists an algebraic cycle $Z \in H^4(S_+ \times S_-, \mathbb{Q})$ such that the correspondence

$$
\Phi_Z : H^2(S_+, \mathbb{Q}) \to H^2(S_-, \mathbb{Q}), \quad l \mapsto (\pi_-)_*(Z \wedge \pi^+_l),
$$

is a Hodge isometry.

**Proposition 4.7.** Let $S_+$ and $S_-$ be $K3$ surfaces with $\rho(S_{\pm}) \geq 13$. The following two conditions are equivalent:

1. $S_+$ and $S_-$ are isogenous.
2. There exist a $K3$ surface $S_0$ and rational maps $S_0 \dashrightarrow S_+, S_0 \dashrightarrow S_-$ of square degrees.

**Proof.** By Mukai’s theorem (Corollary 1.10 of [13]), $S_+$ and $S_-$ are isogenous if and only if $T(S_+)_{\mathbb{Q}}$ and $T(S_-)_{\mathbb{Q}}$ are Hodge isometric.

(1) $\Rightarrow$ (2): We identify $T(S_+)_{\mathbb{Q}}$ and $T(S_-)_{\mathbb{Q}}$ by a Hodge isometry. Let $T_0 := T(S_+) \cap T(S_-)$, which is of finite index in both $T(S_+)$ and $T(S_-)$. The lattice $T_0$ is endowed with the period. Since $\text{rk}(T_0) \leq 9$, the lattice $T_0$ can be embedded primitively into the $K3$ lattice $\Lambda_{K3}$ by [14]. By the surjectivity of the period map, there exist a $K3$ surface $S_0$ and a Hodge isometry $T(S_0) \simeq T_0$. The lattice $NS(S_0)$ admits an embedding of $U$, because $\rho(S_0) \geq 13$. Hence the claim follows from Proposition 4.4.

(2) $\Rightarrow$ (1): Let $f_\pm : S_0 \dashrightarrow S_\pm$ be rational maps of degree $d_\pm^2$. We have Hodge embeddings $f_\pm^* : T(S_{\pm})/(d_\pm^2) \hookrightarrow T(S_0)$ of finite indices. Since the lattices $T(S_{\pm})/(d_\pm^2)$ can be embedded into the lattices $T(S_{\pm})$ by the multiplications by $d_\pm$, it follows that $T(S_{\pm})_{\mathbb{Q}}$ are Hodge isometric to $T(S_0)_{\mathbb{Q}}$. \hfill \square

By the proof, the $K3$ surfaces $S_+$ and $S_-$ are obtained from $S_0$ by taking certain relative Jacobians successively.

**Remark 3.** Inose [10] introduced another notion of isogeny for singular $K3$ surfaces, i.e. $K3$ surfaces with Picard number 20. Two singular $K3$ surfaces $S_+$ and $S_-$ are defined to be isogenous in the sense of Inose if one of the following three equivalent conditions is satisfied:

1. There exists a dominant rational map $S_+ \dashrightarrow S_-.$
2. There exists a dominant rational map $S_- \dashrightarrow S_+.$
3. There exists a Hodge isometry $T(S_+)_{\mathbb{Q}} \simeq T(S_-)_{\mathbb{Q}}$ for some natural number $n.$

17
For singular $K3$ surfaces (and also for $K3$ surfaces with Picard number 19) Inose’s notion of isogeny contains that of Mukai and is a direct analogue of the notion of isogeny for Abelian varieties. Unfortunately, the author does not know successful extension of Inose’s notion of isogeny to $K3$ surfaces with Picard number $\leq 16$. See also [15].

5 The case of Picard number 1

Let $S$ be a $K3$ surface with $NS(S) = \mathbb{Z}H$, $(H, H) = 2n > 0$. In this section we shall calculate for $S$ the correspondence between the twisted FM partners and the 0-dimensional cusps concretely. The set $FM^d(S)$ is calculated in [12] as a set of moduli spaces of sheaves on $S$ with explicit Mukai vectors, twisted by the obstruction classes. On the other hand, Dolgachev showed in [6] Theorem 7.1 that the group $O(\tilde{NS}(S))_{0}^+$ is isomorphic to the Fricke modular group, by direct calculations based on an isomorphism $PSO(1, 2) \simeq PSL_2(\mathbb{R})$. Hence the Kähler moduli $K(S)$ is isomorphic to the Fricke modular curve. We shall describe the groups $\Gamma(S)$ and $O(\tilde{NS}(S))$ via quaternion orders, which are nicely compatible with the discriminant form. The tube domain realization of $\Omega_{\tilde{NS}(S)}^+$ induces an explicit isomorphism between the modular curves.

5.1 Fricke modular curves

Let

$$\mathcal{O} := \left\{ \begin{pmatrix} a & 2b \\ 2nc & d \end{pmatrix} \in M_2(\mathbb{Q}) \mid a, b, c, d \in \mathbb{Z}, \ a + d \in 2\mathbb{Z} \right\},$$

and $\mathcal{O}_0 := \{ A \in \mathcal{O}, \ Tr(A) = 0 \}$. A natural decomposition $\mathcal{O} = \mathbb{Z}(\frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) \oplus \mathcal{O}_0$ holds, and $\mathcal{O}$ has a basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -2n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}. \quad (13)$$

The $\mathbb{Z}$-module $\mathcal{O}$ is of rank 4 and is closed under multiplication. In other words, $\mathcal{O}$ is an order.

We define a quadratic form on the $\mathbb{Q}$-vector space $V := \{ A \in M_2(\mathbb{Q}), \ Tr(A) = 0 \}$ by $(A, B) := -\frac{1}{2} Tr(AB)$, where $A, B \in V$.

**Lemma 5.1.** The lattice $\mathcal{O}_0 \subset V$ is isometric to $\Lambda(-2n)$, where $\Lambda = \tilde{NS}(S)^\vee$.

**Proof.** In fact, an isometry is given by the assignment

$$a \begin{pmatrix} 0 & 0 \\ -2n & 0 \end{pmatrix} + b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \mapsto (a, b \frac{H}{2n}, c) \in \tilde{NS}(S)^\vee. \quad (14)$$

$\square$
For a matrix $\gamma \in GL_2(\mathbb{Q})$ the adjoint action $\text{Ad}_\gamma$ on $V$ defined by $A \mapsto \gamma A \gamma^{-1}$, $A \in V$, is an isometry of $V$. Then we obtain the isomorphism

$$GL_2(\mathbb{Q})/\mathbb{Q}^\times \simeq SO(V), \quad \gamma \mapsto \text{Ad}_\gamma.$$  

The normalizer $N(\mathcal{O}_0) \subset GL_2(\mathbb{Q})$ of $\mathcal{O}_0$ in $V$ coincides with the normalizer $N(\mathcal{O}) \subset GL_2(\mathbb{Q})$ of $\mathcal{O}$ in $M_2(\mathbb{Q})$. Thus we obtain the isomorphisms

$$O(N \widetilde{S}(S)^\vee) \simeq O(\mathcal{O}_0) \simeq (N(\mathcal{O})/\mathbb{Q}^\times, -\text{id}). \quad (15)$$

To calculate the normalizer $N(\mathcal{O})$ we consider the following order $\mathcal{O}'$ called an Eichler order:

$$\mathcal{O}' := \left\{ \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \in M_2(\mathbb{Q}) \mid a, b, c, d \in \mathbb{Z} \right\}. \quad (16)$$

The $\mathbb{Z}$-module $2\mathcal{O}'$ is a submodule of $\mathcal{O}$ of index 2.

**Proposition 5.2.** We have $N(\mathcal{O}) = N(\mathcal{O}')$.

**Proof.** Let $\gamma \in N(\mathcal{O}')$. Since $\text{Ad}_\gamma((1\ 0), \gamma(0\ 1), \gamma(0\ 0)) \in \mathcal{O}'$, then we have $\text{Ad}_\gamma((2\ 0), \gamma(0\ 2), \gamma(0\ 0)) \in 2\mathcal{O}' \subset \mathcal{O}$. Thus $N(\mathcal{O}') \subset N(\mathcal{O})$.

Let $\gamma \in N(\mathcal{O})$. Then $\text{Ad}_\gamma(2\mathcal{O}')$ is an index 2 submodule of $\mathcal{O}$, and the $\mathbb{Z}$-module $\frac{1}{2}\text{Ad}_\gamma(2\mathcal{O}') = \text{Ad}_\gamma(\mathcal{O}')$ is closed under multiplication. Among $2^4 - 1 = 15$ index 2 submodules $L$ of $\mathcal{O}$, $2\mathcal{O}'$ is characterized by the property that $\frac{1}{2}L$ is closed under multiplication. Hence $\text{Ad}_\gamma(2\mathcal{O}') = 2\mathcal{O}'$ and so $\gamma \in N(\mathcal{O}')$.

We shall describe a structure of the normalizer $N(\mathcal{O}')$ following [18] and then calculate the action of $N(\mathcal{O}')$ on the discriminant group $D_\mathcal{O}(\mathcal{O})$. For $n > 1$ let $n = \prod_{i=1}^{\tau(n)} p_i^{e_i}$ be the prime decomposition of $n$, where $\tau(n)$ is the number of the prime divisors of $n$. Each element $\sigma \in (\mathbb{Z}/2\mathbb{Z})^{\tau(n)}$ corresponds to a pair of coprime natural numbers $(N_\sigma, M_\sigma)$ satisfying $N_\sigma M_\sigma = n$. Choose integers $a_\sigma, b_\sigma$ such that $a_\sigma N_\sigma - b_\sigma M_\sigma = 1$ and put

$$\gamma_\sigma := \begin{pmatrix} a_\sigma & b_\sigma \\ n & N_\sigma \end{pmatrix}. \quad (17)$$

We have $\det(\gamma_\sigma) = N_\sigma$. For the $\sigma$ with $(N_\sigma, M_\sigma) = (n, 1)$, we take $(a_\sigma, b_\sigma) = (0, -1)$ especially. Then $\gamma_\sigma = (1\ 0)\left( -\frac{1}{n} \frac{0}{1} \right)$.

**Proposition 5.3** (cf. [18]). Let $(\mathcal{O}')^\times$ be the unit group of $\mathcal{O}'$. For $n > 1$ we have $N(\mathcal{O}')/\mathbb{Q}^\times(\mathcal{O}')^\times \simeq (\mathbb{Z}/2\mathbb{Z})^{\tau(n)}$ with the quotient group $(\mathbb{Z}/2\mathbb{Z})^{\tau(n)}$ represented by the set $\{\gamma_\sigma\}_{\sigma}$. If $n = 1$, we have $N(\mathcal{O}') = N(M_2(\mathbb{Z})) = \mathbb{Q}^\times(\text{SL}_2(\mathbb{Z}), (\frac{1}{1} \frac{0}{1})).$

**Proof.** Let $n > 1$. For $i \leq \tau(n)$ we denote $g_i := \left( \begin{array}{c} 0 \ 1 \\ p_i \ 0 \end{array} \right)$. The two maximal orders $\mathcal{O}_i^+ := M_2(\mathbb{Z}_{p_i})$ and $\mathcal{O}_i^- := g_i \mathcal{O}_i^+ g_i^{-1}$ in $M_2(\mathbb{Q}_{p_i})$ satisfy $\mathcal{O}' \otimes \mathbb{Z}_{p_i} = \mathcal{O}_i^+ \cap \mathcal{O}_i^-$. It is known (see [18] Chapitre III) that we have a surjective homomorphism $N(\mathcal{O}') \rightarrow (\mathbb{Z}/2\mathbb{Z})^{\tau(n)}$ by looking for each $i$ whether the action preserves or exchanges $\mathcal{O}_i^\pm$, and that the kernel is given by $\mathbb{Q}^\times(\mathcal{O}')^\times$. Direct calculations show that $\gamma_\sigma$ exchanges $\mathcal{O}_i^\pm$ for those $i$ with $p_i | N_\sigma$ and preserves $\mathcal{O}_j^\pm$ for those $j$ with $p_j | N_\sigma$. \qed
Let \( \Gamma_0(n) := O' \cap SL_2(\mathbb{Z}) \) as usual. We have \( (O')^\times = \langle \Gamma_0(n), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \rangle \). By Proposition 5.3 we obtain a description of the full isometry group \( O(\text{NS}(S)^\vee) = O(\text{NS}(S)) \). To find its subgroup

\[
\Gamma(S)^+ = O(\text{NS}(S)) \cap \{ \pm \text{id} \} \subset O(\text{NS}(S)^\vee),
\]

we study the actions of \( \Gamma_0(n), (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \), and \( \{ \gamma_\sigma \} \) on the discriminant group \( D_{\text{NS}(S)} \). By the correspondence \([14]\), it suffices to observe the actions to \((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})\) modulo \( \mathbb{Z}(\begin{smallmatrix} -2n & 0 \\ 0 & 2n \end{smallmatrix}) \). Then \( \Gamma_0(n) \) and \( (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) \) act trivially on \( D_{\text{NS}(S)} \). Let \( n > 1 \). The isometry \( \gamma_\sigma \) acts on \( D_{\text{NS}(S)} \simeq \mathbb{Z}/2n\mathbb{Z} \) as multiplication by \( a_\sigma n_\sigma + b_\sigma M_\sigma \). For odd \( p_i \), \( \gamma_\sigma \) acts as \(-\text{id}\) (resp. \( \text{id}\)) on the component \( \mathbb{Z}/p_i^\varepsilon_\sigma \mathbb{Z} \) of \( \text{NS}(S) \) if \( p_i | n_\sigma \) (resp. \( p_i | M_\sigma \)). For \( p = 2 \), \( \gamma_\sigma \) acts as \(-\text{id}\) (resp. \( \text{id}\)) on the component \( \mathbb{Z}/2^{\varepsilon_1+1} \mathbb{Z} \) if \( 2 | n_\sigma \) (resp. \( 2 | M_\sigma \)). It is worth noting that \( \gamma_\sigma \) acts as \(-\text{id}\) on the component \( \mathbb{Z}/p_i^{\varepsilon_i + 1} \mathbb{Z} \) if and only if \( \sigma \) exchanges the orders \( O_i^\pm \).

In conclusion, we have

\[
\Gamma(S) \simeq \langle (\mathbb{Q}^\times \Gamma_0(n), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ n & 0 \end{smallmatrix}) )/\mathbb{Q}^\times, -\text{id}\rangle,
\]

and therefore

\[
\Gamma(S)^+ \simeq \langle (\mathbb{Q}^\times \Gamma_0(n), (\begin{smallmatrix} 0 & 1 \\ n & 0 \end{smallmatrix}))/\mathbb{Q}^\times, -\text{id}\rangle
\]

\[
\simeq \langle \Gamma_0(n), (\begin{smallmatrix} 0 & -\sqrt{\tau+1} \\ 1 & 0 \end{smallmatrix}) \rangle. \quad (17)
\]

The last isomorphism is induced by the projection to \( SL_2(\mathbb{R}) \). The group \( \Gamma_0(n)^+ := \langle \Gamma_0(n), (\begin{smallmatrix} 0 & -\sqrt{\tau+1} \\ 1 & 0 \end{smallmatrix}) \rangle \) is called the Fricke modular group.

Consider the tube domain realization

\[
\mathbb{H} \xrightarrow{\sim} \Omega^+_{\text{NS}(S)}, \quad \tau \mapsto \mathbb{C}(1, \tau, \tau^{2n}). \quad (18)
\]

By sending \( \infty \mapsto \mathbb{C}(0, 0, 1) \), the isomorphism \([18]\) extends to the isomorphism

\[
\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \xrightarrow{\sim} \left( \Omega^+_{\text{NS}(S)} \right)^* := \Omega^+_{\text{NS}(S)} \cup \bigcup_{\ell \in \text{NS}(S)} \mathcal{C}^\ell. \quad (19)
\]

Then the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma_0(n)^+ & \sim & \mathbb{H}^* \\
\downarrow & & \downarrow \\
\Gamma(S)^+ & \sim & \left( \Omega^+_{\text{NS}(S)} \right)^*.
\end{array}
\]

In this way we obtain the desired isomorphism.

**Proposition 5.4 (cf.\( [16] \)).** The tube domain realization \([15]\) induces the isomorphism

\[
\Gamma_0(n)^+ \backslash \mathbb{H} \xrightarrow{\sim} \Gamma(S)^+ \backslash \Omega^+_{\text{NS}(S)}, \quad (20)
\]

which extends naturally to the isomorphism between the compactifications of both sides.

The curve \( \Gamma_0(n)^+ \backslash \mathbb{H} \) is called the Fricke modular curve.
5.2 The cusps of the Fricke modular curve

As is well-known, the set of the $\Gamma_0(n)$-cusps is naturally identified with the following finite set:

$$\left\{ (k,e) \mid e \in \mathbb{Z}_{>0}, \ e|n, \ k \in (\mathbb{Z}/d\mathbb{Z})^\times \text{ where } d = (e, \frac{n}{e}) \right\}. \quad (21)$$

The correspondence is as follows. For a rational number $\frac{n}{x} \in \mathbb{Q}$ with $r$ and $s$ coprime, we put $e := (r, n)$ and $k := \alpha s \in (\mathbb{Z}/d\mathbb{Z})^\times$ where $\alpha \in (\mathbb{Z}/n\mathbb{Z})^\times$ is such that $\alpha e = r \in \mathbb{Z}/n\mathbb{Z}$. Conversely, for a pair $(k, e)$ in the set $(21)$ we associate a rational number $\frac{n}{x} \in Q$, where $k \in \mathbb{Z}$ is such that $k \equiv k \in \mathbb{Z}/d\mathbb{Z}$ and $(k, e) = 1$.

The Fricke involution \((\frac{0}{\sqrt{r}}, \frac{-\sqrt{r-1}}{0})\) acts on the set $(21)$ by $(k, e) \mapsto (-k, \frac{n}{e})$.

Hence the set of the $\Gamma_0(n)^+$-cusps is identified with the following finite set:

$$\bigcup\left\{ (k,e) \mid e \in \mathbb{Z}_{>0}, \ e|n, \ e > \frac{n}{e}, \ k \in (\mathbb{Z}/d\mathbb{Z})^\times \text{ where } d = (e, \frac{n}{e}) \right\}$$

Writing $e = dr$ and $\frac{n}{x} = ds$, we obtain the following description.

**Proposition 5.5.** The set of the $\Gamma_0(n)^+$-cusps is identified with the following set:

$$\mathcal{C}(n)^+ := \bigcup_{d^2<n} \left\{ (k,dr) \mid k \in (\mathbb{Z}/d\mathbb{Z})^\times, \ \frac{n}{d^2} = rs, \ (r,s) = 1, \ r > s \right\}$$

$$\bigcup \left\{ (k',d) \mid d^2 = n, \ k' \in (\mathbb{Z}/d\mathbb{Z})^\times / \{\pm \text{id}\} \right\}. \quad (22)$$

Here $d$ runs over $\mathbb{Z}_{>0}$.

On the other hand, the set $\text{FM}^d(S)$ is described in [12], which we recall now briefly. Let $d$ be a positive integer satisfying $d^2|n$. For each $k \in (\mathbb{Z}/d\mathbb{Z})^\times$ choose a natural number $\bar{k}$ such that $k \equiv k \in \mathbb{Z}/d\mathbb{Z}$ and $(k, 2n) = 1$. When $d^2 < n$, each element $\sigma \in (\mathbb{Z}/2\mathbb{Z})^{\times (d^2-n)}$ corresponds to a pair $(r_\sigma, s_\sigma)$ of coprime natural numbers satisfying $r_\sigma, s_\sigma = d^{-2}n$. Let $\Sigma_d \subset (\mathbb{Z}/2\mathbb{Z})^{\times (d^2-n)}$ be the subset consisting of those pairs $(r_\sigma, s_\sigma)$ satisfying $r_\sigma > s_\sigma$. When $d^2 = n$, consider the element $\sigma$ corresponding to the pair $(r_\sigma, s_\sigma) = (1, 1)$ and put $\Sigma_d := \{\sigma\}$.

For a pair $\sigma, k \in \Sigma_d \times (\mathbb{Z}/d\mathbb{Z})^\times$ we define

$$v_{\sigma,k} := (dr_\sigma, \bar{k}H, \bar{k}^2ds_\sigma) \in I^d(\overline{NS}(S)).$$

Let $(M_{\sigma,k}, \alpha_{\sigma,k})$ be the moduli space of stable sheaves on $S$ with Mukai vector $v_{\sigma,k}$, where $\alpha_{\sigma,k}$ is the obstruction to the existence of a universal sheaf $(3)$. We have the isomorphism

$$(M_{\sigma,k}, \alpha_{\sigma,k}) \simeq (S_{v_{\sigma,k}}, \alpha_{v_{\sigma,k}}),$$

where $(S_{v_{\sigma,k}}, \alpha_{v_{\sigma,k}})$ is the twisted $K3$ surface constructed from the cusp $[v_{\sigma,k}]$.

By calculating the Hodge structures of $M_{\sigma,k}$ and applying the formula for $\#\text{FM}^d(S)$, the following result is obtained in [12].
Proposition 5.6 \textsuperscript{(12)}. We have \( \text{FM}^d(S) = \emptyset \) unless \( d^2|n \). Let \( d^2|n \).

1. If \( d^2 < n \), then

\[
\text{FM}^d(S) = \left\{ (M_{\sigma,k}, \alpha_{\sigma,k}) \mid (\sigma,k) \in \Sigma_d \times \left( \mathbb{Z}/d\mathbb{Z} \right)^\times \right\}.
\]

2. Let \( d^2 = n \). Choose a set \( \{ j \} \subset \left( \mathbb{Z}/d\mathbb{Z} \right)^\times \) of representatives of the quotient set \( \left( \mathbb{Z}/d\mathbb{Z} \right)^\times /\{ \pm \text{id} \} \). Then

\[
\text{FM}^d(S) = \left\{ (M_{\sigma,k'}, \alpha_{\sigma,k'}) \mid (\sigma,k') \in \Sigma_d \times \{ j \} \right\}.
\]

Via the tube domain realization \textsuperscript{(19)}, we obtain the correspondence

\[
(M_{\sigma,k}, \alpha_{\sigma,k}) \longleftrightarrow v_{\sigma,k} \quad \xrightarrow{\text{via} \ (19)} \quad \frac{k}{dr_{\sigma}} \implies (k,dr_{\sigma}) \in \mathcal{C}(n)^+.
\] (23)

Comparing Proposition 5.5 and Proposition 5.6, we now observe that \( \bigcup_d \text{FM}^d(S) \) and \( \mathcal{C}(n)^+ \) correspond bijectively, as expected. For an arbitrary rational number \( \frac{a}{b} \in \mathbb{Q} \) with \( a \) coprime to \( b \), we can calculate the isomorphism class of the corresponding twisted FM partner with the aid of (22) and (23). In particular, we see that the K3 surface underlying the partner is determined only by the value \( [b] \in \mathbb{Z}/n\mathbb{Z} \). In terms of a primitive isotropic Mukai vector \( (r, kH, s) \), the K3 surface underlying the twisted moduli space is determined by the value

\[
[rk^{-1}] \in \mathbb{Z}/n\mathbb{Z}.
\]

Conversely, we may find the Mukai vectors \( v_{\sigma,k} \) by referring to the cusps of the Fricke modular curve. If we resort to Theorem 3.7, we reprove that the twisted moduli spaces \( (M_{\sigma,k}, \alpha_{\sigma,k}) \) represent \( \text{FM}^d(S) \), without calculating the twisted FM number of \( S \) nor the Hodge structures of the moduli spaces \( M_{\sigma,k} \).

In these ways, the classification of the twisted FM partners is simplified and is strengthened by using the Fricke modular curve.

Example 5.7. Let \( S \subset \mathbb{P}^5 \) be a generic K3 surface of degree 8 whose Picard group is generated by the class \( H := \mathcal{O}(1)|_S \). It is well-known that \( S \) is a \((2,2,2)\) complete intersection. The Kähler moduli \( \mathcal{K}(S) \simeq \Gamma_0(4)^+ \backslash \mathbb{H} \) has two cusps \([\infty] \) and \( \left[ \frac{1}{2} \right] \). The cusp \([\infty] \) corresponds to \( S \) itself, while the cusps \( \left[ \frac{1}{2} \right] \) corresponds to the twisted moduli space \((M,\alpha)\) associated with the Mukai vector \( (2, H, 2) \).

The underlying K3 surface \( M \) is of degree 2, and can be realized as a double covering of \( \mathbb{P}^2 \) ramified over a sextic.

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