THE SCHUR-WIELANDT THEORY FOR CENTRAL S-RINGS

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Abstract. Two basic results on the S-rings over an abelian group are the Schur theorem on multipliers and the Wielandt theorem on primitive S-rings over groups with a cyclic Sylow subgroup. None of these theorems is directly generalized to the non-abelian case. Nevertheless, we prove that they are true for the central S-rings, i.e., for those which are contained in the center of the group ring of the underlying group (such S-rings naturally arise in the supercharacter theory). We also generalize the concept of a B-group introduced by Wielandt, and show that any Camina group is a generalized B-group whereas with few exceptions, no simple group is of this type.

1. Introduction

A Schur ring or S-ring over a finite group $G$ can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free $\mathbb{Z}$-module spanned by a partition of $G$ closed under taking inverse and containing the identity $e$ of $G$ as a class (see Section 2 for details). The S-ring theory was initiated by Schur [13] and then developed by Wielandt [14] who wrote in [15] that S-rings provide one “of three major tools” to study a group action.

Until recently, the focus was on studying S-rings over abelian groups and the main applications of this theory were connected with algebraic combinatorics problems [12]. However, as it was observed in [9], the supercharacter theory developed to study group representations, is nothing else than the theory of commutative S-rings of a special form that we call here central.

Definition 1.1. An S-ring over a group $G$ is said to be central if it is contained in the center $\mathbb{Z}(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$.

An example of such a ring is obtained from any permutation group $K$ such that $G \leq \text{Sym}(G)$, where $\text{Inn}(G)$ is the inner automorphism group of $G$; the corresponding partition of $G$ is formed by the orbits of the stabilizer of $e$ in $K$. In the special case when $K = \text{Sym}(G)$, this produces the trivial central S-ring $Ge + \mathbb{Z}G$, where $G$ is the sum of all elements of $G$. On the other hand, if $K = G \text{Inn}(G)$, the orbits are the conjugacy classes of $G$; this shows that $\mathbb{Z}(\mathbb{Z}G)$ is a central S-ring. In particular, any S-ring over an abelian group is central. The main goal of the present paper is to extend the basic results on S-rings from abelian case to the central one.

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The two other tools are the representation theory and the method of invariant relations.
The Schur theorem on multipliers is a fundamental statement in the theory of S-rings over abelian groups. To explain it, given an integer \( m \) coprime to \( |G| \), we define a permutation on the elements of the group \( G \) by
\[
\sigma_m : G \rightarrow G, \quad x \mapsto x^m.
\]
It permutes also the conjugacy classes of \( G \), and so induces a linear isomorphism of the ring \( \mathbb{Z}(\mathcal{Z}G) \). If the group \( G \) is abelian, then \( \sigma_m \in \text{Aut}(G) \), \( \mathbb{Z}(\mathcal{Z}G) = \mathbb{Z}G \) and the Schur theorem on multipliers states that \( \sigma_m \) is a Cayley automorphism of every S-ring over \( G \). Our first result shows that in the nonabelian case, \( \sigma_m \) is still an automorphism (but not a Cayley one) of any central S-ring over \( G \).

**Theorem 1.2.** Let \( \mathcal{A} \) be a central S-ring over a group \( G \), and let \( m \) be an integer coprime to \( |G| \). Then \( \sigma_m(\mathcal{A}) = \mathcal{A} \) and \( \sigma_m|_A \in \text{Aut}(\mathcal{A}) \).

Based on this result for the abelian case, Wielandt generalized the Schur theorem on primitive groups having a regular cyclic subgroup. In fact, the Wielandt proof shows that if \( G \) is an abelian group of composite order that has a cyclic Sylow subgroup, then no proper S-ring over \( G \) is primitive.\(^2\) The following statement establishes “a central version” of the Wielandt theorem.

**Theorem 1.3.** Let \( \mathcal{A} \) be a nontrivial central S-ring over a group \( G \) of composite order. Suppose that \( G \) has a normal cyclic Sylow \( p \)-subgroup. Then \( \mathcal{A} \) is imprimitive.

Following \[14\], a finite group \( G \) is called a B-group if every primitive group containing a regular subgroup isomorphic to \( G \) is 2-transitive. It should be remarked that most of the B-groups \( G \) mentioned in \[14\] satisfy a priori a stronger condition: no nontrivial S-ring over \( G \) is primitive.\(^3\) In this sense, the following definition seems to be quite natural. In what follows, we say that a central S-ring over \( G \) is proper if it lies strictly between \( \mathbb{Z}(\mathcal{Z}G) \) and the trivial S-ring over \( G \).

**Definition 1.4.** A group \( G \) is called a generalized B-group if no proper central S-ring over \( G \) is primitive.

Clearly, every B-group is also a generalized one. The converse statement is not true; see Subsection \[5.2\]. A nontrivial example of a generalized B-group is given in Theorem \[15\]. The following statement gives a family of generalized B-groups; we don’t know whether they are B-groups. Below, under a Camina group, we mean a group \( G \) that has a proper nontrivial normal subgroup \( H \) such that each \( H \)-coset distinct from \( H \) is contained in a conjugacy class of \( G \) (in other terms, \( (G, H) \) is a Camina pair).\(^1\)

**Theorem 1.5.** Any Camina group is a generalized B-group.

The class of the Camina groups includes, in particular, all Frobenius and extra-special groups; see \[3\]. Thus, by Theorem \[15\] we obtain the following statement.

**Corollary 1.6.** Any Frobenius or extra-special group is a generalized B-group. \( \blacksquare \)

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\(^2\)The primitivity concept in S-ring theory plays the same role as the simplicity in group theory.

\(^3\)To simplify the presentation, we use the term “Camina group” not only in the case where \( (G, G') \) is a Camina pair.
The last result of the present paper shows that with a few possible exceptions, no simple group is a generalized B-group. The proof is based on the Schur theorem on multipliers and the characterization of rational simple groups given in [7].

**Theorem 1.7.** A generalized B-group $G$ is not simple unless $|G| \leq 3$, or $G \cong \text{Sp}(6,2)$ or $\text{O}^+(8,2)'$.

For the reader convenience, we collect the basic facts on S-rings in Section 2. The proofs of Theorems 1.2 and 1.3 are contained in Sections 3 and 4, respectively. The results concerning generalized B-groups are in Section 5.

**Notation.**

As usual, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the ring of integers and the fields of rationals and complex numbers, respectively.

The identity of a group $G$ is denoted by $e$; the set of non-identity elements in $G$ is denoted by $G#$. The set of conjugacy classes of $G$ is denoted by $\text{Cla}(G)$. Let $X \subseteq G$. The subgroup of $G$ generated by $X$ is denoted by $\langle X \rangle$; we also set $\text{rad}(X) = \{g \in G : gX = Xg = X\}$. The element $\sum_{x \in X} x$ of the group ring $\mathbb{Z}G$ is denoted by $X$.

For an integer $m$, we set $X(m) = \{x^m : x \in X\}$ and $X^{(m)} = X^{(m)}$.

The group of all permutations of the elements of $G$ is denoted by $\text{Sym}(G)$. The additive and multiplicative groups of the ring $\mathbb{Z}/(n)$ are denoted by $\mathbb{Z}_n$ and $\mathbb{Z}_n^\times$, respectively.

2. Preliminaries

Let $G$ be a finite group. A subring $\mathcal{A}$ of the group ring $\mathbb{Z}G$ is called a Schur ring (S-ring, for short) over $G$ if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of $G$ such that

(S1) $\{e\} \in \mathcal{S}$,

(S2) $X \in \mathcal{S} \Rightarrow X^{-1} \in \mathcal{S}$,

(S3) $\mathcal{A} = \text{Span}\{X : X \in \mathcal{S}\}$.

In particular, for all $X, Y, Z \in \mathcal{S}$ there is a nonnegative integer $c_{X,Y}^Z$ such that

$$XY = \sum_{Z \in \mathcal{S}} c_{X,Y}^Z Z,$$

these integers are the structure constants of $\mathcal{A}$ with respect to the linear base $\{X : X \in \mathcal{S}\}$. The number $\text{rk}(\mathcal{A}) = |\mathcal{S}|$ is called the rank of $\mathcal{A}$.

Let $\mathcal{A}'$ be an S-ring over a group $G'$. Under a Cayley isomorphism from $\mathcal{A}$ to $\mathcal{A}'$, we mean a group isomorphism $f : G \to G'$ such that $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$. This is a special case of the ordinary isomorphism; by definition, it is a bijection $f : G \to G'$ that induces a ring isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ taking $X$ to $X'$ for all $X \in \mathcal{S}$, where $X' = X^f$.

The classes of the partition $\mathcal{S}$ are called the basic sets of the S-ring $\mathcal{A}$. Any union of them is called an $\mathcal{A}$-set. Thus, $X \subseteq G$ is an $\mathcal{A}$-set if and only if $X \in \mathcal{A}$. The set of all $\mathcal{A}$-sets is closed with respect to taking inverse and product. Any subgroup of $G$ that is an $\mathcal{A}$-set, is called an $\mathcal{A}$-subgroup of $G$ or $\mathcal{A}$-group. With each $\mathcal{A}$-set $X$, one can naturally associate two $\mathcal{A}$-groups, namely $\langle X \rangle$ and $\text{rad}(X)$ (see Notation).

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4In fact, we do not know whether two simple groups from Theorem 1.7 are generalized B-groups.
The S-ring \( A \) is called \textit{primitive} if the only \( A \)-groups are \( e \) and \( G \), otherwise this ring is called \textit{imprimitive}.

We will use the following statement proved in [14, Proposition 22.3]. Below for a function \( f : \mathbb{Z} \to \mathbb{Z} \) and an element \( \xi = \sum g_{a}g \) of the ring \( \mathbb{Z}G \), we set \( f[\xi] = \sum g_{a}f(a_{g})g \).

**Lemma 2.1.** Let \( A \) be an S-ring, \( f : \mathbb{Z} \to \mathbb{Z} \) an arbitrary function and \( \xi \in A \). Then \( f[\xi] \in A \).

The important special case is when \( f(a) = 1 \) or 0 depending on whether \( a \neq 0 \) or \( a = 0 \). Then \( f[\xi] = X \), where \( X \) is the support of \( \xi \), and we refer to Lemma 2.1 as to the Schur-Wielandt principle.

3. The Schur theorem on multipliers

**Proof of Theorem 1.2.** Since obviously \( \sigma_{mm'} = \sigma_{m}\sigma_{m'} \) for all \( m \) and \( m' \), without loss of generality, we can assume that \( m \) is a prime. We need the following auxiliary lemma.

**Lemma 3.1.** Let \( X \in \text{Cla}(G) \) and \( p \) an arbitrary prime. Then

\[
X^{p} = \sum_{Y \in \text{Cla}(G)} a_{Y}Y
\]

for some nonnegative integers \( a_{Y} \)'s. Moreover,

\[
a_{Y}|Y| = \begin{cases} |X| \ (\text{mod } p), & \text{if } Y = X^{(p)}, \\ 0 \ (\text{mod } p), & \text{if } Y \neq X^{(p)}. \end{cases}
\]

**Proof.** The first statement follows from the fact that \( \mathbb{Z}(\mathbb{Z}G) \) is an S-ring. To prove the second one, set

\[
T_{Y} = \{(x_{1}, \ldots, x_{p}) \in X^{p} : x_{1} \cdots x_{p} \in Y\}.
\]

Clearly, \((T_{Y})^{G} = T_{Y}\). Since also \((x_{1}x_{2} \cdots x_{p})x_{1} = x_{2} \cdots x_{p}x_{1}\), the set \( T_{Y} \) is invariant with respect to the cyclic shift \( \pi : (x_{1}, x_{2}, \ldots, x_{p}) \mapsto (x_{2}, \ldots, x_{p}, x_{1}) \). Moreover, since \( p \) is prime, we have

\[|(x_{1}, \ldots, x_{p})^{(\pi)}| = 1 \text{ or } p.\]

However, \(|(x_{1}, \ldots, x_{p})^{(\pi)}| = 1\) if and only if \( x_{1} = \cdots = x_{p} \), which is possible only if \( Y = X^{(p)} \); in the latter case, the group \( \langle \pi \rangle \) has exactly \( |X| \) orbits of the form \( \{(x, \ldots, x)\}, x \in X \). Taking into account that \( T_{Y} \) is a disjoint union of \( \langle \pi \rangle \)-orbits, we have

\[|T_{Y}| = |X|\delta + pu\]

where \( \delta = \delta_{Y,X^{(p)}} \) is the Kronecker delta and \( u \) is the number of the \( \langle \pi \rangle \)-orbits of size \( p \). Thus, the required statement follows because \( a_{Y}|Y| = |T_{Y}| \).

Let us continue the proof of the theorem. Since \( p := m \) is coprime to \( |G| \), the mapping

\[
X \mapsto X^{(p)}, \quad X \in \text{Cla}(G),
\]

is a bijection. It induces a linear isomorphism of the ring \( \mathbb{Z}(\mathbb{Z}) \); the image of the element \( \xi \) under this isomorphism is denoted by \( \xi^{(p)} \). From Lemma 3.1 it follows
that $X^p = X^{(p)} \pmod{p}$; here, we make use of the fact that $|X^{(p)}| = |X|$ for all $X$. Therefore,

$$\xi^{(p)} = f[\xi^p] \quad \text{for all} \quad \xi \in \mathbb{Z}(\mathbb{Z}G).$$

where $f(a)$ is the remainder in the division of $a$ by $p$.

To prove the first part of the theorem, let $X \in S(A)$. Then $X$ is a union of some classes $X_i \in \text{Cl}(G)$, $i \in I$. Thus, by (1), we have

$$\sigma_p(X) = \sum_{i \in I} X_i^{(p)} = \sum_{i \in I} f[X_i^p] = f[\sum_{i \in I} X_i^p] = f[(\sum_{i \in I} X_i)^p] = f[X^p].$$

However, by Lemma 2.1 the right-hand side belongs to $A$. Therefore $\sigma_p(X) \in A$ and $X^{(p)}$ is an $A$-set. Moreover, suppose that it contains a proper basic set $Y$. By the Dirichlet Theorem, one can find a prime $p'$ such that $pp' = 1 \pmod{n}$, where $n = |G|$. Now, the above argument shows that $Y^{(p')}$ is a proper $A$-subset of $X$. Thus $X^{(p)} \in S(A)$ and so $\sigma_p(A) = A$.

To prove the second part of the theorem, it suffices to verify that $\sigma_m$ induces a ring isomorphism of $\mathbb{Z}(\mathbb{Z}G)$: then it is, obviously, an $S$-ring isomorphism of $\mathbb{Z}(\mathbb{Z}G)$ that takes $A$ to itself, and hence it is an isomorphism of $A$, as required. To do this, without loss of generality, we can assume that $p > 2n$ (for otherwise, by the Legendre theorem, there exists a prime $q > 2n$ such that $q = p \pmod{n}$, and then, obviously, $\xi^{(q)} = \xi^{(p)}$ for all $\xi \in A$). We have to prove that

$$\xi^{(p)} = e_{X^{(p)}Y^{(p)}} = e_{X^{(p)}Y}^{(p)}$$

for all $X,Y,Z \in \text{Cl}(G)$, where the numbers in the both sides are the structure constants of the $S$-ring $\mathbb{Z}(\mathbb{Z}G)$. Since this ring is commutative and $p$ is prime, formula (2) implies that

$$X^{(p)}Y^{(p)} = X^{(p)}Y^{(p)} = (X^{(p)}Y)^{p} = \left(\sum_{\mathbb{Z}} c_{X^{(p)}YZ}^{(p)}\right) = \sum_{\mathbb{Z}} c_{X^{(p)}YZ}^{(p)} \pmod{p}.$$ 

Thus the relation (3) is true modulo $p$. Since $p > 2n$, we are done.

There is an alternative way to prove the second part of Theorem 1.2. It is related to the action of the group $\mathbb{Z}_m^n$ on the set $\text{Irr}(A)$ of all irreducible $C$-characters of the $S$-ring $A$, where as before, we can assume that $A = \mathbb{Z}(\mathbb{Z}G)$. Let $\varepsilon$ be an $n$-th primitive complex root of unity. Then each $m \in \mathbb{Z}_m^n$ determines an automorphism $\tau_m$ of the cyclotomic field $\mathbb{Q}(\varepsilon)$, which sends $\varepsilon$ to $\varepsilon^m$. It follows that for any $\chi \in \text{Irr}(G)$, the function $\chi^{\tau_m}(g) := \left(\chi(g)\right)^{\tau_m}, g \in G$, is also an irreducible character of $G$ and

$$\chi^{\tau_m}(g) = \chi(g^m)$$

(see [11 Proposition 3.16]). The primitive idempotents of $A$ coincide with the central primitive idempotents of the group algebra $\mathbb{Q}(\varepsilon)[G]$ which, in turn, are in a one-to-one correspondence with the irreducible characters of $G$. More precisely, if $e_\chi$ is the idempotent corresponding to $\chi \in \text{Irr}(G)$, then

$$e_\chi = \frac{1}{|G|} \sum \chi(g) g^{-1}.$$ 

A direct computation shows that $\sigma_m(e_\chi) = e_{\chi^{\tau_m}}$. Thus, $\sigma_m$ permutes the primitive idempotents of $A$. This implies that $\sigma_m$ is an automorphism of $A$, as required. We
note that the above formula shows that there is a natural one-to-one correspondence between the $\text{Irr}(G)$ and $\text{Irr}(\mathcal{A})$. More precisely,

\[(4) \quad \text{Irr}(\mathcal{A}) = \{ \frac{1}{\chi(1)} \chi|_{\mathcal{A}} : \chi \in \text{Irr}(G) \}.
\]

Given a set $X \subseteq G$, denote by $\text{tr}(X)$ the union of the sets $X^{(m)}$, where $m$ runs over the integers coprime to $n = |G|$; it is called the trace of $X$. Let $\mathcal{A}$ be a central S-ring over $G$. Then from Theorem 1.2, it follows that $\text{tr}(X) \in \mathcal{A}$ for all $X \in \mathcal{S}(\mathcal{A})$. Therefore,

$$\text{tr}(\mathcal{A}) = \text{Span}\{\text{tr}(X) : X \in \mathcal{S}(\mathcal{A})\}$$

is a submodule of $\mathcal{A}$. It is easily seen that it consists of all fixed points of the natural action of the group $\{\sigma_m : (m, n) = 1\}$ on $\mathcal{A}$. Thus, $\text{tr}(\mathcal{A})$ is an S-ring, which is obviously central; it is called the rational closure of the S-ring $\mathcal{A}$. It should be noted that our definitions agreed with the relevant definitions in the abelian case.

The following statement immediately follows from the fact that $\text{tr}(H) = H$ for any group $H \leq G$.

**Proposition 3.2.** Let $\mathcal{A}$ be a central S-ring over $G$. Then $\mathcal{A}$ is primitive if and only if so is $\text{tr}(\mathcal{A})$.

We say that a central S-ring is rational if it coincides with its rational closure, or equivalently, if each of its basic sets is rational. The following statement justified the term “rational”.

**Theorem 3.3.** Let $\mathcal{A}$ be a central S-ring over a group $G$. Then it is rational if and only if $\pi(X) \in \mathbb{Q}$ for all $\pi \in \text{Irr}(\mathcal{A})$ and all $X \in \mathcal{S}(\mathcal{A})$.

**Proof.** Let $m$ be an integer coprime to $n = |G|$. Since any character $\pi \in \text{Irr}(\mathcal{A})$ is equal to the restriction to $\mathcal{A}$ of a suitable character $\chi \in \text{Irr}(G)$, from relation (4) it follows that

\[(5) \quad \pi(X)^{\tau_m} = \pi(X^{(m)}), \quad X \in \mathcal{S}(\mathcal{A}),
\]

where $\tau_m$ is the above defined automorphism of the field $\mathbb{Q}(\epsilon)$. If the S-ring $\mathcal{A}$ is rational, then the right-hand side of this equality does not depend on the choice of $m$. So the number $\pi(X)^{\tau}$ does not depend on the automorphism $\tau$ of $\mathbb{Q}(\epsilon)$. Thus, $\pi(X) \in \mathbb{Q}$.

Assume now that $\pi(X) \in \mathbb{Q}$ for all $\pi \in \text{Irr}(\mathcal{A})$ and $X \in \mathcal{S}(\mathcal{A})$. Then from (5) it follows that $\pi(X) = \pi(X^{(m)})$ for all integers $m$ coprime to $n$ and all characters $\pi \in \text{Irr}(\mathcal{A})$. This implies that

\[X = \sum_{\pi \in \text{Irr}(\mathcal{A})} \pi(X)e_{\pi} = \sum_{\pi \in \text{Irr}(\mathcal{A})} \pi(X^{(m)})e_{\pi} = X^{(m)},\]

where $e_{\pi}$ is the primitive idempotent corresponding to the character $\pi$. Thus, the S-ring $\mathcal{A}$ is rational. 

4. **Proof of Theorem 1.3**

By the theorem hypothesis, $G$ has a normal Sylow $p$-subgroup $P \cong \mathbb{Z}_{p^n}$. So by the Schur-Zassenhaus theorem, $G = PK$, where $K$ is a Hall $p'$-subgroup of $G$. In what follows, we denote by $H$ the unique subgroup of $P$ of order $p$.

**Lemma 4.1.** Let $x \in G$ be such that $Hx \not\subseteq x^G$. Then $x \in C_G(P)$.

Proof. The element $x$ acts by conjugation as an automorphism of the cyclic group $P$. Therefore, there exists an integer $m$ coprime to $p$ such that $h^x = h^m$ for all $h \in P$. Rewriting this equality as $x^h = xh^{1-m}$, we obtain

$$x^G \supseteq x^P \supseteq P^{(1-m)}x.$$ 

Since $P^{(1-m)}$ is a subgroup of $P$ and $x^G \not\subseteq xH$, this implies that $P^{(1-m)} = e$. Thus, $x^h = x$ for all $h \in P$, which means that $x \in C_G(P)$. 

Suppose on the contrary that the $S$-ring $A$ is primitive. Take a nontrivial basic set $X$, which intersects $H$ nontrivially. Then $\langle X \rangle \neq H$; indeed, otherwise $\langle X \rangle = H$ by the primitivity of $A$ and $n = p$ is a prime in contrast to the hypothesis. This proves the second part of the following relations (the first one follows from the choice of $X$):

(6) $X \cap H \neq \emptyset$ and $X \setminus H \neq \emptyset$ and $\langle X \cap H \rangle \leq \text{rad}(X \setminus H)$.

To prove the third one, set $X_0 = \{x \in X : xH \not\subseteq X\}$. Then from Lemma 4.1 it follows that

(7) $(X_0)^{(p)} \subseteq P^{(p)} K \subseteq X$.

Moreover, it is easily seen that the sets $X_0$ and $X \setminus X_0$ are unions of some conjugacy classes of $G$. For these classes, we can refine Lemma 3.1 as follows.

Lemma 4.2. For any class $Y \in \text{Cla}(G)$, we have

$$Y^p = \begin{cases} \sum_{y \in S} Y^y (\mod p), & \text{if } Y \subseteq C_G(P), \\ 0 (\mod p), & \text{if } Y \not\subseteq C_G(P). \end{cases}$$

Proof. The group $C := C_G(P)$ is obviously normal in $G$. Therefore,

$Y \subseteq C$ or $Y \cap C = \emptyset$.

Suppose first that $Y \cap C = \emptyset$. Since $H \subseteq G$, we have $yHy = Hy$ for all $y \in Y$. Denote by $S$ a full system of representatives of the family $\{Hy : y \in Y\}$. Then, since $|H| = p$, we have

$$Y^p = \sum_{y \in S} H^y = H^p S^p \equiv 0 (\mod p),$$

as required. Let now $Y \subseteq C$. Then $Y$ is a normal subset of $C$, i.e. $Y^G = Y$. Since the group $C$ is a direct product of $P$ and $O_{p'}(C)$, each normal subset of $C$ is the disjoint union of $gY$, $g \in P$, where $Y_g$ is a normal subset of $O_{p'}(C)$. Now

(8) $Y^p = \sum_{g \in P} Y_{g}^g (\mod p)$.

Moreover, since $Y_g$ is contained in the $p'$-subgroup $O_{p'}(C)$, by Lemma 3.1 we obtain

(9) $Y_{g^p} = Y_{g^p}^p (\mod p)$.

Thus, from (8) and (9), it follows that

$$Y^p = \sum_{g \in P} (g_{Y_g})^p = \sum_{g \in P} g^p Y_{g}^p \equiv \sum_{g \in P} g^p Y_{g}^p (\mod p)$$

as required.
To complete the proof of the third relation in (6), suppose on the contrary that the set $X_0$ is not empty. Then, if $X$ is the union of conjugacy classes $X_i$, $i \in I$, then by Lemma 4.2, we have

$$X^p = \left( \sum_{i \in I} X_i \right)^p = \sum_{i \in I} X_i^p = \sum_{i \in I_0} X_i^{(p)} \mod p,$$

where $I_0 = \{ i \in I : X_i \subseteq X_0 \}$. Moreover, by Lemma 4.1, given $x, y \in X_0$, the equality $x^p = y^p$ holds if and only if $y \in Hx$. Since also $1 \leq |H \cap X_0| \leq p - 1$ for all $x \in X_0$, the coefficient at $x^p \in G$ in the right-hand sum of (10) is between 1 and $p - 1$. Thus,

$$\xi := f[X^p]$$

is a non-zero element of the S-ring $A$, where $f$ is the function used in (11). By the Schur-Wielandt principle, this implies that the support $Y$ of the element $\xi$ is an $A$-set. Therefore, $(Y)$ is an $A$-subgroup of $G$. This subgroup is proper: $(Y) \neq G$ by (11) and $(Y) \neq e$, because $X_0 \neq \emptyset$. But this contradicts the primitivity of the S-ring $A$.

Thus, all the relations in (6) are true. To complete the proof, we make use of the following theorem on separating subgroup proved in [6].

**Theorem 4.3.** Let $A$ be an S-ring over a group $G$. Suppose that $X \in S(A)$ and $H \leq G$ satisfy relations (11). Then $X = \langle X \rangle \setminus \text{rad}(X)$ and $\text{rad}(X) \leq H \leq \langle X \rangle$.

Now, since $\text{rad}(X)$ and $\langle X \rangle$ are $A$-groups, the primitivity assumption implies that $\text{rad}(X) = e$ and $\langle X \rangle = G$. By Theorem 4.3, this implies that $X = G \setminus e$. This means that $\text{rk}(A) = 2$, i.e., the S-ring $A$ is trivial. Contradiction.

5. **Generalized B-groups**

5.1. **Proof of Theorem 1.5** Let $G$ be a Camina group. Then it has a normal subgroup $H$ such that $(G, H)$ is a Camina pair. Let $A$ be a proper central primitive S-ring over $G$. Take a set $X \in S(A)$ that contains a nonidentity element of $H$. It follows from the primitivity of $A$ that

$$\text{rad}(X) = e \quad \text{and} \quad \langle X \rangle = G.$$  

(11)

In particular, the first two relations in (10) hold. Next, the set $X$ is a union of some conjugacy classes of $G$ as the S-ring $A$ is central. By the definition of a Camina pair, we have

$$xH = Hx \subseteq X \setminus H$$

for all $x \in X \setminus H$. This proves the third relation in (10). Thus, $X = \langle X \rangle \setminus \text{rad}(X)$ by Theorem 4.3. By (11), this implies that $X = G \setminus e$ and hence $\text{rk}(A) = 2$. The latter means that the S-ring $A$ is not proper. Contradiction.
5.2. A generalized B-group, which is not a B-group. Let \( p > 3 \) be a prime congruent to 3 modulo 4, and let \( G \) be the extraspecial group of order \( p^3 \) and exponent \( p \). Then there exists a skew Hadamard difference set \( X \) in the group \( G \); see [3]. This exactly means that \( Y := X^{-1} \) is equal to \( G^# \setminus X \) and
\[
XY = |X|e + \frac{|X| - 1}{2}(X + Y).
\]
Therefore, the module \( A = \text{Span}\{e, X, Y\} \) is a subring of \( \mathbb{Z}G \) that satisfies the conditions (S1), (S2), and (S3) with \( S = \{e, X, Y\} \). Thus, \( A \) is an S-ring of rank 3 over \( G \). This S-ring is, obviously, primitive. Since it is also proper, \( G \) is not a B-group. On the other hand, it is a generalized B-group by Corollary 1.6.

5.3. Simple groups. According to [7], a group \( G \) is said to be rational if the number \( \chi(g) \) is rational for all \( \chi \in \text{Irr}(G) \) and all \( g \in G \). Finite simple rational groups were characterized in Corollary B1 of that paper as follows: a noncyclic simple group \( G \) is rational if and only if \( G \cong \text{Sp}(6,2) \) or \( O^+(8,2)' \).

Proof of Theorem 1.7. Let \( G \) be a finite simple group other than \( \text{Sp}(6,2) \) or \( O^+(8,2)' \). Without loss of generality, we can assume that \( G \) is not cyclic. Then by the above characterization of rational groups, \( G \) is not rational and has two elements \( x \) and \( y \) of distinct orders. Then the orders of \( x^m \) and \( y^m \) are also distinct for all integers \( m \) coprime to \( |G| \). This implies that the order of any element of \( \text{tr}(x^G) \) does not equal the order of any element of \( \text{tr}(y^G) \). So,
\[
(12) \quad \text{tr}(x^G) \neq \text{tr}(y^G).
\]
Therefore, the rational closure \( \text{tr}(A) \) of the S-ring \( A = \mathbb{Z}(\mathbb{Z}G) \) is of rank at least 3. On the other hand, \( \text{tr}(A) \neq A \), for otherwise the irreducible characters of \( A \) are rational valued (Theorem 3.3) and then \( G \) is a rational group. Thus, \( \text{tr}(A) \) is a proper central S-ring. It is primitive because so is \( A \) (Proposition 3.2). Therefore \( G \) can not be a generalized B-group.

5.4. AS-free groups. According to [1], a transitive permutation group is called AS-free if it preserves no nontrivial symmetric association scheme. From Theorem 17 of that paper, it follows that given a nonabelian simple group \( G \), the permutation group on \( G \) defined by
\[
K = \langle G_{\text{right}}, \text{Aut}(G), \sigma \rangle,
\]
is AS-free, where \( G_{\text{right}} \) is the group of all right translations of \( G \) and \( \sigma \) is a permutation of \( G \) that takes \( g \) to \( g^{-1} \), \( g \in G \). It is easily seen that the orbits of the stabilizer of \( e \) in \( K \) are the basic sets of a central S-ring \( A \) over \( G \). Thus, using the above result one can get another proof that \( G \) is not a generalized B-group whenever \( A \neq \mathbb{Z}(\mathbb{Z}G) \). However, in general, the latter inequality is not true, e.g., for the group \( \text{Sp}(6,2) \).

5.5. Miscellaneous. Let \( G \) be a finite group having a relatively prime conjugacy class (examples of such groups can be found, e.g., in [1]). Denote by \( \mathcal{X} \) the association scheme of the permutation group \( G \text{Inn}(G) \leq \text{Sym}(G) \) (see also, [2] Theorem 7.2). Then one can see that \( \text{Cl}(G) \) forms a relatively prime equitable partition for \( \mathcal{X} \) in the sense of [10]. By Theorem 3.1 of that paper, any primitive fusion of the scheme \( \mathcal{X} \) must have rank 2. So, using the correspondence between the Cayley schemes and S-rings over \( G \), one can show that \( G \) is a generalized B-group.
Let $G = G_1 \times G_2$ where $G_1$ and $G_2$ are groups of the same order $n > 1$. Then $G$ is not a generalized B-group. Indeed, set $X_0 = \{(e_1, e_2)\}$ where $e_i$ is the identity of $G_i$, and

$$X_1 = e_1 \times (G_2)^\# \cup (G_1)^\# \times e_2.$$ 

Denote by $A$ the span of the set $\{X_i : i = 0, 1, 2\}$ where $X_2$ is the complement to $X_0 \cup X_1$ in $G$. Then $A$ is, obviously, a central S-ring of rank 3 over $G$. Since it is also primitive, we are done.

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