Factor Two Discrepancy of Hawking Radiation Temperature

Tadas K. Nakamura

Fukui Prefectural University, 4-1-1 Eiheiji, Fukui 910-0095, Japan

Abstract

The possibility of an alternative way to formulate the Hawking radiation in a static Schwarzschild spacetime has been explored. To calculate the Hawking radiation, there can be two possible choices of the spacetime wedge pairs in the Krucal-Szekeres coordinates. One is the wedge pair consists of exterior spacetime of a black hole and the exterior spacetime of a white hole, and the other is that of exterior and interior spacetimes of one black hole. The radiation from the former is the Hawking’s original one. Though the latter has been often regarded as the same phenomena as the former, the result here suggests it is not; its radiation has a temperature twice as high as the Hawking temperature.

Key words:
PACS: 04.62.+v, 04.70.Dy

1 Introduction

Right after the Hawking’s original paper [1], attempts have been made to derive the Hawking radiation from a vacuum in a static Schwarzschild spacetime [2,3,5,6]. The choice of the correct vacuum is crucial in such attempts. It is well known that the definition of a vacuum is not unique in curved spacetimes, and there is no general prescription to define a natural vacuum. Therefore we have to determine somehow which vacuum is actually realized, depending on each problem. The difficulty to find such a vacuum in a Schwarzschild spacetime comes from its bifurcating Killing horizons. The horizons divide the extended Schwarzschild spacetime into four spacetime wedges, and we have several choices of wedges or wedge pairs to define the vacuum.

Email address: tadas@fpu.ac.jp (Tadas K. Nakamura).
URL: http://mira.bio.fpu.ac.jp/tadas/ (Tadas K. Nakamura).
Unruh \cite{4} compared two distinct definitions of vacua, called $\eta$ definition and $\xi$ definition in his paper. The vacuum in $\xi$ definition is often referred as the Hartle-Hawking vacuum, which is calculated from analytic functions across the exterior spacetime of a black hole and the exterior spacetime of a white hole. On the other hand, it is also possible to define a vacuum state on the exterior wedge of a black hole alone, which is the $\eta$ definition; the vacuum with this definition is often called Boulware vacuum. Unruh \cite{4} considered the Hartle-Hawking vacuum is preferable because Boulware vacuum has singularity on physical quantities such as energy. This point has been examined extensively by several authors \cite{5,6}, and it was shown that under some basic assumptions, the Hartle-Hawking vacuum is the only mathematically reasonable one in a wide class of spacetimes with Killing horizons \cite{7}.

The legitimacy of the Hartle-Hawking vacuum shown in the above mentioned papers is, however, based on the particle number (or a conserved quantity along the Killing field, in general) defined on a Cauchy surface in the Kruscal-Szekeres coordinates. The vacua are calculated with analytic functions across the two exterior spacetime wedges; one is of a black hole and the other is of a white hole. Therefore, what have shown is that the Hartle-Hawking vacuum is the natural vacuum among the vacua across these two exterior spacetime wedges.

In the present paper we explore another possibility of vacuum that spans across the interior and exterior spacetime wedges of a black hole; we will call this R-F vacuum hereafter. This is the vacuum defined with the particle number on a surface with $t = \text{constant}$. There have been several papers \cite{8,9,10} that regard the R-F vacuum as the source of the Hawking radiation, however, it seems its difference from the Hartle-Hawking vacuum was not well recognized; most of papers consider the radiation from the R-F vacuum is the same phenomena as that from the Hartle-Hawking vacuum. However, our calculation shows the temperature of the radiation from the R-F vacuum is twice as large as the one with the Hartle-Hawking vacuum, which means the R-F vacuum is not identical to the Hartle-Hawking vacuum.

This temperature discrepancy of factor two was first reported by Akhmedov et al. \cite{12}. In relatively recent years, an approach based on the tunneling effect has been extensively investigated (\cite{11,12} and references therein). Akhmedov et al. \cite{12} have carefully examined the integration contour in the tunneling calculation, and concluded the resulting radiation temperature is twice as large as the Hawking’s original value. The view point with tunneling effect may not physically well founded, however, it can be reinterpreted in the context of canonical quantization in the R-F wedge pair when back reaction of the particles to the metric is neglected. Then what calculated by Akhmedov et al. \cite{12} (or other papers with tunneling picture) coincides with the radiation from the R-F vacuum in the present paper. Therefore, the factor two discrepancy
reported by Akhmedov et al. [12] can be explained by the difference of vacua (R-F or Hartle-Hawking) from which the radiation comes from.

We first examine the case of Unruh effect in the next section. Though the R-F vacuum has little physical significance in the Minkowski spacetime, its radiation is mathematically simpler and can be a good example for this type of vacua in other spacetimes. Its results are readily applied to the Schwarzschild spacetime since the structure near the bifurcating horizons are the same in both spacetimes. The application to the Schwarzschild spacetime is then sketched in the following section. A brief discussion on the validity of our approach is provided in the last section of this paper.

2 Radiation from Minkowski Vacua

2.1 Canonical Quantization with Horizons

Suppose two dimensional coordinate system $(\eta, \xi)$ with the following metric:

$$ds^2 = A(\xi) \, d\eta^2 - B(\xi)^{-1} \, d\xi^2. \quad (1)$$

The wave equation of a massless scalar particle may be written as

$$\frac{1}{A} \frac{\partial^2}{\partial \eta^2} \phi - \sqrt{\frac{B}{A}} \frac{\partial}{\partial \xi} \sqrt{AB} \frac{\partial}{\partial \xi} \phi = 0. \quad (2)$$

Now let us assume there is one and only one point where $A(\xi)B(\xi) = 0$. We choose $\xi$ coordinate such that $\xi = 0$ at that point; we call $\xi > 0$ region “positive side” and $\xi < 0$ region “negative side”.

Separating the variables, we write eigenfunctions with respect to $\xi$ on the positive/negative side as

$$u^P_k = \begin{cases} u^P_k & (\xi \geq 0) \\ 0 & (\xi < 0) \end{cases}, \quad u^N_k = \begin{cases} 0 & (\xi \geq 0) \\ u^N_k & (\xi < 0) \end{cases}. \quad (3)$$

We assume the inner product $\langle \cdots \rangle$ is properly defined from the metric, and $u^{p,N}_k$ are normalized as

$$\langle u^P_k, u^P_{k'} \rangle = \langle u^N_k, u^N_{k'} \rangle = \delta_{kk'}. \quad (4)$$
We can construct a solution $e^{-ik\eta}U_k(\xi)$ that satisfies Eq (2) over $-\infty < \xi < \infty$ across $\xi = 0$ in the following:

$$U_k = \begin{cases} 
\theta_k^P u_k^P & (\xi \geq 0) \\
\theta_k^N u_k^N & (\xi < 0) 
\end{cases} \quad (5)$$

The wave equation Eq (2) is satisfied when we adjust the coefficients $\theta_k^{P,N}$ so that $U_k(\xi)$ becomes analytic across $\xi = 0$; at the same time $\theta_k^{P,N}$ should satisfy the normalization condition $\langle U_k, U_k \rangle = 1$. Then we can expand a wave solution $\phi$ as

$$\phi(\eta, \xi) = \sum_k a(\eta) U_k(\xi)$$

Decomposing $a(\eta)$ into the positive and negative frequency modes and calculating Boglubov coefficients from $\theta_k^{P,N}$, we can obtain the spectrum of Hawking/Unruh radiation.

### 2.2 R-L (Right-Left) vacuum

This vacuum corresponds to the Hartle-Hawking vacuum in an extended Schwarzschild spacetime; we call this type of vacuum R-L vacuum in this paper. Obviously this vacuum is the usual vacuum realized in a flat spacetime.

Let us define the Rindler coordinate system as

$$t = \xi \sinh a\eta, \quad x = \xi \cosh a\eta, \quad (6)$$

where $t$ and $x$ are ordinary time and space coordinates in the Minkowski spacetime. The range of the space coordinate $\xi$ spans $-\infty < \xi < \infty$ so that the above equation covers both R and L regions (referred as Wedge R and Wedge L hereafter) illustrated in Figure 1.
The inner product is defined as

\[ \langle \phi, \phi' \rangle = \int_{-\infty}^{\infty} \xi^{-1} \phi \phi'^* \, d\xi . \]  

(7)

and the eigenfunctions on Wedges R and L become

\[ u^R_k = \exp \left( \frac{ik}{a} \ln \xi \right), \quad u^L_k = \exp \left( -\frac{ik}{a} \ln |\xi| \right) . \]  

(8)

It should be noted that the norm of the above eigenfunctions calculated from Eq (7) diverges as usually we encounter in this type of calculations. In the following we assume an appropriate prescription, such as wave packet treatment, has been applied implicitly to avoid this difficulty.

We wish to obtain the solution across R and L Wedges in the form of \( U_k \) in Eq (5). To this end, we must choose \( \theta^P,N_k \) such that Eq (2) holds across the point of \( \xi = 0 \). If we choose arbitrary \( \theta^P,N_k \), then discontinuity occurs at \( \xi = 0 \), and the right hand side of Eq (2) will have a \( \delta \)-function shaped “source term” at \( \xi = 0 \). To avoid this we have to adjust \( \theta^P,N_k \) so that \( U_k \) becomes analytic across \( \xi = 0 \). Using the analytic continuation of the logarithmic function, \( \ln(-\xi) = i\pi + \ln \xi \), such coefficients \( \theta^P,N_k \) can be calculated in the following:

\[
\theta^P_k = \begin{cases} 
\frac{1}{\sqrt{1 - e^{-\pi k/a}}} & (k \geq 0) \\
\frac{e^{\pi k}}{\sqrt{1 - e^{\pi k/a}}} & (k < 0) 
\end{cases}, \quad \theta^N_k = \begin{cases} 
\frac{e^{-\pi k/a}}{\sqrt{1 - e^{-\pi k/a}}} & (k \geq 0) \\
\frac{1}{\sqrt{1 - e^{\pi k/a}}} & (k < 0) 
\end{cases} .
\]  

(9)
The Unruh radiation spectrum becomes

\[ P(k) \propto \frac{1}{2} \left( \theta_k^P + \theta_k^N + 1 \right) = \frac{1}{\exp(2\pi k/a) - 1} \]  

(10)

for \( k > 0 \) (see, eg., Birrel and Davies [13], p105 for details of calculation).

2.3 R-F (Right-Future) vacuum

The above calculation has been done with the solution that spans over Wedges of R and L in Figure 1. We perform the same calculations with the solution over Wedges R and F in this subsection. This does not have physical significance for a flat spacetime, however, the similar calculation becomes important in the Schwarzschild spacetime as we will see in the next section. We examine the R-F case with a flat spacetime because it has essentially the same but mathematically simpler spacetime structure.

We define the coordinates \((\eta', \xi')\) in Wedge F by

\[
t = \xi' \cosh a\eta', \quad x = \xi' \sinh a\eta',
\]

(11)

where \( \eta' \) and \( \xi' \) are real numbers, The eigenfunction with respect to \( \xi' \) in Wedge F becomes

\[
u^F_k = \exp \left( \frac{ik}{a} \ln \xi' \right).
\]

(12)

Let us recall that the key point in the previous calculation is in the process to construct the solution \( U_k \) that satisfies Eq (2) across the singular point of \( \xi = 0 \). There we utilized the analyticity of \( U_k \) as a function of \( \xi \) across the both wedges. The reason why this works is that the coordinate \( \xi \) is itself analytic across the wedges, in other words, we can express any point in both wedges with the same single expression of Eq (6).

We wish to take the same approach here, that is, to express points in Wedge F with Eq (11). This can be done by complexifying \( \eta \) and \( \xi \) as

\[
\eta = \frac{i\pi}{2} - \eta', \quad \xi = i\xi',
\]

(13)

then the complex numbers \((\eta, \xi)\) can express any points in Wedges F and R with Eq (6). Having done that, \( u^F_k \) may be expressed using logarithmic analytic
continuation \((\ln i\xi' = i\pi/2 + \ln \xi')\) as
\[
u_k^F = \exp \left( \frac{ik}{a} \ln \xi \right) = e^{\pi k/2a} \exp \left( \frac{ik}{a} \ln \xi' \right). \tag{14}
\]

Then the same calculation as in the R-L case gives the radiation spectrum as
\[
P(k) \propto \frac{1}{\exp(\pi k/a) - 1}, \tag{15}
\]
which has the temperature twice as large as in the R-L case.

### 3 Schwarzschild coordinates

Now let us move on to the Schwarzschild coordinate system whose metric is given by
\[
ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{16}
\]
where symbols have conventional meaning. The Rindler coordinates \((\eta, \xi)\) correspond to \((t, r - 2M)\) in the above Schwarzschild coordinates. Wedges R and F in Figure 1 correspond to the interior and exterior spacetimes of a black hole respectively, and Wedges L and P correspond to the “white hole” in the extended Schwarzschild coordinates. Since the spacetime structure near Killing horizons are the same in the Rindler and Schwarzschild coordinates (see, e.g., [14], p128), the arguments in the previous subsections are valid for the Schwarzschild coordinates with \(a = 1/4M\). We briefly sketch in the following the procedure of analytic continuation that leads us to this result.

The solutions to the wave equation Eq (2) have the form of
\[
u_k^{R,L} \propto \exp ik[\xi + 2M + 2M \ln(\xi/2M)] \tag{17}
\]
where \(\xi = r - 2M\). The coordinates \((t, \xi)\) naturally covers Wedge R with \(\xi > 0\) and Wedge L with \(\xi < 0\). Using \(\ln(-\xi/2M) = \ln(\xi/2M) + i\pi\), we obtain the amplitude discontinuity as \(\exp(-\pi Mk)\). This means the radiation temperature is twice as large as the Hawking’s prediction.

Continuation from Wedge R to Wedge L is not that straightforward. We notice in the above calculation the discontinuity comes from the analytic continuation of the logarithmic function, therefore we have to find an appropriate way to
analytically extend the logarithmic function from Wedge R to Wedge L. To this end, we express the Schwarzschild coordinates in Wedge L as \((t', \xi')\) and examine the analytic relation of \(\ln \xi\) in Wedge R and \(\ln \xi'\) in Wedge L. The following Kruskal Szekeres coordinates \((U, V)\) are analytical across Wedges R and L,

\[
UV = -2M\xi \exp\left(\frac{\xi}{2M} + 1\right) \tag{18}
\]

\[
|U/V| = \exp\left(\frac{t}{2M}\right),
\]

therefore, expressing \(\xi\) by \(U\) and \(V\) gives the analyticity across the Wedges. Near \(\xi = 0\) we can approximate

\[
V - U \simeq \sqrt{2M\xi}. \tag{19}
\]

at \(t = 0\). In Wedge R this is equivalent to

\[
\ln(V - U) = \frac{1}{2} \ln 2M\xi, \tag{20}
\]

since \(V - U > 0\); its analytic continuation to the region of \(V - U < 0\) (Wedge L) is

\[
\ln(V - U) = \pi i + \ln |V - U|. \tag{21}
\]

Therefore the analytic continuation of the logarithmic function from Wedge R to Wedge L may be written as

\[
\ln \xi' = 2\pi i + \ln \xi. \tag{22}
\]

The above expression inserted into Eq (17) results in the amplitude jump of \(\exp(-2\pi Mk)\). The rest of the calculation is the same as in the Rindler case, and we obtain the temperature predicted in the Hawking’s original paper [1].

4 Concluding Remarks

The present paper has explored the possibility of an alternative vacuum to obtain Hawking radiation; the difference is in the choice of the wedge pair to define the vacuum. Most of past papers with a static Schwarzschild spacetime \([4, 2, 3, 5, 6, 7]\) were based on the vacuum across the exterior spacetime of a black
hole and the exterior space of a white hole, which is referred as R-L vacuum in the present paper. On the other hand, several attempts have made to calculate the radiation with the vacuum across the interior and exterior spacetimes of a black hole (R-F vacuum) [8,9,10,11,12]. To the author’s knowledge, the difference of these two vacua is not well recognized so far, and the radiation form R-L and R-F vacua are regarded as the same phenomena. The result of the present paper suggests these two are distinct; the temperature of the radiation from the R-F vacuum is twice as large as that of R-L vacuum.

At the present we do not know which (R-F or R-L) vacuum should be realized around a black hole. The R-L vacuum requires the Kruscal extension, or a “white hole”, which is usually considered unphysical. On the other hand, the quantum construction for R-F vacuum may be questionable. In general, the procedure of quantization in a curved spacetime is based on a Cauchy surface with timelike normal vectors. However, if we choose the surface of \( t = \text{constant} \) in the Schwarzschild coordinates for a R-F vacuum, it is not a Cauchy surface because its normal vectors become spacelike in Wedge F.

There may be two approaches to avoid this problem of R-F vacuum. One is to extend the method of quantization to be able to utilize surfaces with spacelike normal vectors instead of Cauchy surfaces. Recently several papers have been published proposing a quantization method in which an arbitrary closed surface can play the role of the Cauchy surface [15,16,17]. This method may be applicable to the quantization for R-F vacuum. The other way is to stay in the conventional quantization with a Cauchy surface, but define the particle number on the surface with \( t = \text{constant} \). Then the surface to define the particle number can have spacelike normal vector within the conventional framework. Now the author of the present paper is working on this direction and the result will be reported in a forthcoming paper.

Before closing this paper, let us briefly take a look at the original derivation by Hawking [1]. The R-F vacuum means the ground state with respect to the total “energy” in the interior and exterior spacetimes. (Here “energy” means the conserved quantity that agrees with the usual energy in a flat spacetime at the region far away from the black hole.) Hawking [1] examined the process of a star collapse, assuming a vacuum state long before the star collapse remains unchanged long after the black hole formation. With this assumption, what realized at the later time is the Hartle-Hawking vacuum. It was the ground state at the initial time, but it is not after the black hole formation; the ground state at a later time is the R-F vacuum. The author of the present paper feels the state is likely to settle down to the ground state somehow long after the black hole formation, however, further investigation will be required to verify this point. The scope of the present paper is just to point out the possibility of R-F vacuum and cannot tell which vacuum is actually realized around black holes at the present.
Acknowledgment: The author would like to thank M. Maeno for suggestions and discussions.

References

[1] Hawking, S. W., Comm. Math. Phys. 43, 199 (1975).

[2] Hartle, J. B. and S. W. Hawking, Phys. Rev. D189, 13 (1976).

[3] Israel, W., Phys. Lett. A57, 107 (1976).

[4] Unruh, W.G., Phys. Rev. D14, 870 (1976).

[5] Fulling, S.A., J. Phys. A10, 917 (1977).

[6] Kay, S.B., Commun. Math. Phys. 100, 57 (1985).

[7] Kay, S.B and M. Wald, Phys. Reports 207 49 (1991).

[8] Unruh, W. G., Phys. Rev. D15 367 (1977).

[9] Fredenhagen, K., and R. Haag, Commun. Math. Phys. 127 273 (1990).

[10] Jacobson, T., gr-qc/0308048.

[11] Parikh, M. K. and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000), hep-th/9907001.

[12] Akhmedov E. T., V. Akhmedova, and D. Singleton, Phys. Lett. B642, 124 (2006), hep-th/0608098. Akhmedov E. T., V. Akhmedova, P. Terry, and D. Singleton, Int. J. Mod. Phys A22, 1705 (2007), hep-th/0605137.

[13] Birrell, N. D., and P. C. W. Davis, Quantum fields in curved space, Cambridge Univ. Prs. (1982).

[14] Wald, R. M., Quantum field theory in curved spacetime and black hole thermodynamics, Univ. Chicago Prs, (1994).

[15] Oeckl, R. Phys. Lett. B575 318 (2003).

[16] Conrady, F., C. Rovelli, Int. J. Mod. Phys. A19 4037 (2004), hep-th/0310246

[17] Doplicher, L., Phys. Rev. D70 64037 (2004), gr-qc/0505006.