SPATIAL CLT FOR THE SUPERCritical ORNSTEIN-UHLENBECK SUPERPROCESS

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Abstract. In this paper we consider a superprocess being a measure-valued diffusion corresponding to the equation

$$u_t = Lu + \alpha u - \beta u^2,$$

where $L$ is the infinitesimal operator of the Ornstein-Uhlenbeck process and $\beta > 0$, $\alpha > 0$. The latter condition implies that the process is supercritical, i.e. its total mass grows exponentially. This system is known to fulfill a law of large numbers. In the paper we prove the corresponding central limit theorem. The limit and the CLT normalization fall into three qualitatively different classes. In what we call the small growth rate case the situation resembles the classical CLT. The weak limit is Gaussian and the normalization is the square root of the size of the system. In the critical case the limit is still Gaussian, however the normalization requires an additional term. Finally, when the growth rate is large the situation is completely different. The limit is no longer Gaussian, the normalization is substantially larger than the classical one and the convergence holds in probability. These different regimes arise as a result of “competition” between spatial smoothing due to the particles’ movement and the system’s growth which is local.

We prove also that the spatial fluctuations are asymptotically independent of the fluctuations of the total mass of the process (which is a continuous State Branching Processes).

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1. Introduction

1.1. Model. Let $\{P_t\}_{t \geq 0}$ be the semigroup of the Ornstein-Uhlenbeck process in $\mathbb{R}^d$ i.e. the one with the infinitesimal operator

$$L := \frac{1}{2} \sigma^2 \Delta - \mu x \circ \nabla, \quad \sigma > 0, \mu > 0,$$

where $\circ$ denotes the standard scalar product. Abusing the notation we will denote both the invariant distribution (of $L$) and its density with the same symbol, namely

$$\varphi(x) = \left(\frac{\mu}{\pi \sigma^2}\right)^{d/2} \exp \left(-\frac{\mu}{\sigma^2} \|x\|^2\right).$$

In this paper we will study the behavior of the superprocess $\{X_t\}_{t \geq 0}$ with semigroup $\mathcal{P}$ and the branching mechanism $\psi$ given by

$$\psi(\lambda) = -\alpha \lambda + \beta \lambda^2, \quad \alpha \in \mathbb{R}, \beta > 0.$$

Its behavior is described by a Markovian evolution; that is, for each $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ being the set of finite, compactly supported measures, by $\mathbb{P}_\nu$ we denote the the law of $X$ with initial condition $\nu$. Let $f \in \text{bp}(\mathbb{R}^d)$ (bounded, positive and measurable functions on $\mathbb{R}^d$). The following equation defines the evolution (see [8, 9] for more details).

$$-\log \mathbb{E}_\nu(e^{-(f,X_t)}) = \int_{\mathbb{R}^d} u_f(x,t)\nu(dx), \quad t \geq 0,$$

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where \( u_f(x,t) \) is the unique non-negative solution to the integral equation

\[
(1.5) \quad u_f(x,t) = \mathcal{P}_t f(x) - \int_0^t \mathcal{P}_s [\psi(u_f(\cdot,t-s))] (x) ds.
\]

In the paper we are interested only in the supercritical case, i.e. the case when the total mass of the superprocess grows (exponentially). This is ensured by the condition

\[\alpha > 0,\]

which we assume in the whole paper. The above definition is rather abstract but superprocess have a natural interpretation as the short life time and high density diffusion limit of a branching particle systems (see e.g. Introduction of [10] and Remark [8]). In our case the branching particle counterpart is the Ornstein-Uhlenbeck branching system considered in [1, 2]. We will comment on this connection later on in Remark [5].

1.2. Results. The expectation of the total mass of the system grows exponentially fast at a rate \( \alpha \). Furthermore, the system fulfills the law of large numbers [10, Theorem 1], i.e. for a bounded continuous function \( f \) we have

\[\lim_{t \to +\infty} e^{-\alpha t} \langle X_t, f \rangle = \langle \varphi, f \rangle V_\infty, \quad \text{in probability},\]

where \( V_\infty \) is a random variable, to be defined later (let us note that the results of [10] holds for a larger class of branching diffusions). The goal of our paper is to prove the corresponding central limit theorem. The second order behavior depends qualitatively on the sign of \( \alpha - 2\mu \). Roughly speaking this condition reflects the interplay of two antagonistic forces, the growth which is local and makes the system more coarse and the smoothing introduced by the spatial evolution corresponding to the OU-process. We will now describe this behavior in more detail. The main object of our interest are the spatial fluctuations given by:

\[F_t^{-1}(\langle X_t, f \rangle - \langle \varphi, f \rangle)\]

where \( F_t \) is some norming, not necessarily deterministic, and \( |X_t| := \langle X_t, 1 \rangle \) is the total mass of the system. We will describe the situation on the set where the process is not extinguished \( Ext^c \) (to be defined later). The results split into three classes

**Small growth rate** \( \alpha < 2\mu \): Our main result is contained in Theorem [3]. In this case the “movement part prevails” and the result resembles the standard CLT. The normalization is given by \( F_t = |X_t|^{1/2} \) (which is of order \( e^{-\alpha t/2} \)). Moreover, we obtain the limit which is Gaussian (though its variance is given by a complicated formula) and does not depend on the starting configuration.

**Critical growth rate** \( \alpha = 2\mu \): Our main result is contained in Theorem [4]. In this case “the growth prevails”. The behavior of the fluctuations slightly diverge from the classical setting. The normalization is bigger: \( F_t = t^{1/2} |X_t|^{1/2} \). The limit still does not depend on the starting condition and is Gaussian but its variance depends on the derivatives of \( f \). To explain this we notice that the growth is so fast that the fluctuations are not smoothed by the motion and become essentially local. In consequence they give rise to a “spatial white noise” and larger normalization is required.

**Large growth rate** \( \alpha > 2\mu \): Our main result is contained in Theorem [5]. In this case not only does the growth “prevail” but also “the motion fails to make any smoothing”. The normalization is even bigger: \( F_t = e^{(\alpha - \mu)t} \) and we have \( \alpha - \mu > \alpha/2 \). The limit is no longer Gaussian, it is given by \( \langle f, \text{grad} \varphi \rangle \circ H_\infty \) (where \( H_\infty \) is the limit of a certain martingale). What is perhaps surprising, the limit holds in probability. The first term, \( \langle f, \text{grad} \varphi \rangle \), means that like in the critical situation the growth is fast enough to produce some sort of a white noise. Even more, it is so fast that the limit depends on the starting condition and in fact, up to some extent, the system “remembers its whole evolution”, which is encoded in \( H_\infty \).

In either case we prove also that the spatial fluctuations become independent of fluctuations of the total mass as time increases.
1.3. Comments. This paper is a superprocess counterpart of [1]. Namely, $X$ can be defined as a limit of the Ornstein-Uhlenbeck particle systems considered in [1] (see Remark 5). It turns out that qualitatively the first and second order behavior are very similar (the reader aware of [1] easily notices that the list above is very similar to the one in [1]). Related problems for branching particle systems were considered also in [2, 4] however we are not familiar of any results in this direction concerning superprocesses. For general information about superprocesses we refer to books [9, 11, 12] and also to the references given in Introduction of [1].

Let us now comment on the methodology. Our proofs hinge on the backbone representation obtained recently in [5] which allows us to reuse many concepts developed in [1]. This together with analytical estimation of the behavior of $\mathcal{P}$ is enough to prove our results. A more detailed description of the proof startegy is in Remark 15.

The article is organized as follows. The next section presents notation and basic fact required further. Section 3 is devoted to presentation of results. Proofs are deferred to Sections 4-7 and the Appendix.

2. Preliminaries and notation

Let us first recall the notions which appeared in Introduction. $\mathcal{P}$ is the semigroup corresponding to (1.1). $\mathcal{M}_F$ is the space of finite, compactly supported measures and $bp(\mathbb{R}^d)$ is the space of bounded, positive and measurable functions on $\mathbb{R}^d$.

We use $\langle f, \nu \rangle := \int_{\mathbb{R}^d} f(x) \nu(dx)$. We denote the total mass of the measure $\nu$ by $|\nu| := \langle 1, \nu \rangle$.

By $x \lesssim y$ we will denote the fact that $x \leq Cy$ for a certain constant $C > 0$ (which exact value is not relevant to following calculations).

We will use $\mathcal{C} = \mathcal{C}(\mathbb{R}^d) := \{ f : \mathbb{R}^d \rightarrow \mathbb{R} : f \text{ is continuous and } \exists n \text{ such that } |f(x)|/\|x\|^n \rightarrow 0 \text{ as } \|x\| \rightarrow +\infty \}$, that is the space of continuous functions which grow at most polynomally. To shorten the notation we introduce $\{P_t^\alpha\}_{t \geq 0}$ by

$$P_t^\alpha f(x) := e^{at}\mathcal{P}_t f(x).$$

We may rewrite equation on $u_f$ as

$$(2.1) \quad u_f(x, t) = P_t^\alpha f(x) - \beta \int_0^t P_s^\alpha [u_f(\cdot, t-s)^2](x) ds.\tag{2.1}$$

By $Ext$ we denote the event that the process is extinguished, i.e.

$$(2.2) \quad Ext := \left\{ \lim_{t \rightarrow +\infty} |X_t| = 0 \right\}.\tag{2.2}$$

We denote also process $\{V_t\}_{t \geq 0}$ by

$$(2.3) \quad V_t := e^{-\alpha t}|X_t|,\tag{2.3}$$

The gather the basic facts of process $V$ which will be useful in the formulation of the main results

**Fact 1.** Let $\{X_t\}_{t \geq 0}$ be the OU-superprocess starting from $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ and let $\{V_t\}_{t \geq 0}$ be defined according to (2.3). Then $V$ with it natural filtration is a (positive) martingale. It converges

$$(2.4) \quad V_\infty := \lim_{t \rightarrow +\infty} V_t, \quad \text{a.s. and in } L^2.\tag{2.4}$$

Moreover, $\{V_\infty = 0\} = Ext$, $\mathbb{P}_\nu(Ext) = \exp \left\{ -|\nu|^2 \right\}$ and the law of $V_\infty$ can be described by

$$(2.5) \quad V_\infty = \sum_{i=1}^N E_i,\tag{2.5}$$
where $N$ is a Poisson random variable with parameter $|\nu|^2_\beta$ and $E_1, E_2, \ldots$ is an i.i.d sequence of exponential random variables with parameter $\alpha/\beta$, which are independent of $N$. We also have

\begin{equation}
\sigma_V^2 := \text{Var}(V_\infty) := \frac{2\beta}{\alpha} |\nu|.
\end{equation}

The proof of the fact is delegated to Appendix.

**Remark 2.** Although representation (2.5) may seem a little peculiar at the moment it will appear naturally once we introduce the backbone decomposition in Section 4.1.

### 3. Results

In this section we present the results of our paper. We split the section into three parts corresponding to each case listed in Introduction.

#### 3.1. Slow growth $\alpha < 2\mu$

Let us denote $\bar{f}(x) := f(x) - \langle \varphi, f \rangle$ and

\begin{equation}
\sigma^2_f := \frac{\beta}{\alpha} \int_0^\infty e^{-\alpha s} \langle \varphi, 2\beta \left( P_s^\alpha \bar{f}(\cdot) \right) ^2 - 2\beta \left( P_s^{-\alpha} \bar{f}(\cdot) \right) ^2 + 4\alpha\beta u(\cdot, s) \rangle ds,
\end{equation}

where $u(x, s) = \int_0^s P_{s-u}^{-\alpha} \left( \left( P_u^{-\alpha} \bar{f}(\cdot) \right) ^2 \right) (x) du$.

The main result of this section is

**Theorem 3.** Let $\{X_t\}_{t \geq 0}$ be the OU-superprocess starting from $\nu \in \mathcal{M}_F(\mathbb{R}^d)$. Let us assume $\alpha < 2\mu$ and $f \in \mathcal{C}(\mathbb{R}^d)$. Then $\sigma_f < +\infty$ and conditionally on set $\text{Ext}^c$ there is the convergence

\begin{equation}
\left( e^{-\alpha t} |X_t|, \frac{|X_t| - e^{\alpha t} V_\infty}{\sqrt{|X_t|}}, \frac{(X_t, f) - |X_t| \langle f, \varphi \rangle}{\sqrt{|X_t|}} \right) \to^d (\tilde{V}_\infty, G_1, G_2), \quad \text{as } t \to +\infty,
\end{equation}

where $G_1 \sim \mathcal{N}(0, \frac{2\beta}{\alpha}), G_2 \sim \mathcal{N}(0, \sigma^2_f)$ and $\tilde{V}_\infty$ is $V_\infty$ conditioned on $\text{Ext}^c$. Moreover, the variables $\tilde{V}_\infty, G_1, G_2$ are independent.

The proofs corresponding to this case are delegated to Section 5.

#### 3.2. Critical growth $\alpha = 2\mu$

We denote

\begin{equation}
\sigma^2_f := 2\frac{\beta^2}{\alpha} \int_{\mathbb{R}^d} (x \circ (\text{grad} f, \varphi))^2 \varphi(x) dx,
\end{equation}

where $\circ$ is the standard scalar product. The main result of this section is

**Theorem 4.** Let $\{X_t\}_{t \geq 0}$ be the OU-superprocess starting from $\nu \in \mathcal{M}_F(\mathbb{R}^d)$. Let us assume $\alpha = 2\mu$ and $f \in \mathcal{C}(\mathbb{R}^d)$. Then $\sigma_f^2 < +\infty$ and conditionally on set $\text{Ext}^c$ there is the convergence

\begin{equation}
\left( e^{-\alpha t} |X_t|, \frac{|X_t| - e^{\alpha t} V_\infty}{\sqrt{|X_t|}}, \frac{(X_t, f) - |X_t| \langle f, \varphi \rangle}{t^{1/2} |X_t|} \right) \to^d (\tilde{V}_\infty, G_1, G_2), \quad \text{as } t \to +\infty,
\end{equation}

where $G_1 \sim \mathcal{N}(0, \frac{2\beta}{\alpha}), G_2 \sim \mathcal{N}(0, \sigma^2_f)$ and $\tilde{V}_\infty$ is $V_\infty$ conditioned on $\text{Ext}^c$. Moreover, the variables $\tilde{V}_\infty, G_1, G_2$ are independent.

The proofs corresponding to this case are delegated to Section 7.
3.3. Fast growth $\alpha > 2\mu$.
Let us define a process $\{H_t\}_{t \geq 0}$ by
\[ H_t := e^{-(\alpha - \mu)t} \int_{\mathbb{R}^d} x X_t(dx). \]

**Fact 5.** The process $H$ is a martingale and under assumption $\alpha > 2\mu$ it is $L^2$-bounded.

From the fact it follows that in the setting of this section the following limit (both a.s. and $L^2$) is well-defined and a.s. finite
\[ H_\infty := \lim_{t \to +\infty} H_t. \]

Let us note that $H_\infty$ depends on the starting condition $\nu \in \mathcal{M}_F(\mathbb{R}^d)$, describing this dependence would be notationally cumbersome at this point, hence is deferred to Theorem 22.

The main result of this section is

**Theorem 6.** Let $\{X_t\}_{t \geq 0}$ be the OU-superprocess starting from $\nu \in \mathcal{M}_F(\mathbb{R}^d)$. Let us assume $\alpha > 2\mu$ and $f \in C(\mathbb{R}^d)$. Then conditionally on the set of non-extinction $\text{Ext}_c^\nu$ there is the convergence
\[ (e^{-\alpha t} |X_t|, |X_t| - e^{\alpha V_\infty}, (X_t, f) - |X_t| \langle f, \phi \rangle) \to^{d} (\tilde{V}_\infty, G, \langle \text{grad} f, \varphi \rangle \circ H_\infty), \]
where $G \sim \mathcal{N}(0, \frac{2\beta}{\alpha})$, variables $\tilde{H}_\infty, \tilde{V}_\infty$ are respectively $H_\infty, V_\infty$ conditioned on $\text{Ext}_c^\nu$ and $(\tilde{V}_\infty, \tilde{H}_\infty), G$ are independent. Moreover
\[ (e^{-\alpha t} |X_t|, \frac{(X_t, f) - |X_t| \langle f, \varphi \rangle}{\exp((\alpha - \mu)t)}) \to (V_\infty, \langle \text{grad} f, \varphi \rangle \circ H_\infty), \text{ in probability.} \]

The proofs corresponding to this case are delegated to Section 6.

3.4. Discussion and remarks.
As a corollary to the above statements we obtain a weak law of large numbers for a slightly larger class of the test functions, compared to [10, Theorem 1].

**Theorem 7.** Let $\{X_t\}_{t \geq 0}$ be the OU-superprocess starting from $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ and $f \in C(\mathbb{R}^d)$ then
\[ \lim_{t \to +\infty} e^{-\alpha t} (X_t, f) = \langle \varphi, t \rangle V_\infty, \text{ in probability.} \]

**Remark 8.** As mentioned in the Introduction the results are closely related to the ones in [1]. This follows naturally by the fact that $X$ can be defined in terms of the OU branching systems considered therein. This construction can be described as follows. In the $n$-th approximation each particles carries mass $1/n$ and lives for an exponential time with parameter $1/n$. During this time it executes a random movement according to the Ornstein-Uhlenbeck process with the infinitesimal operator $L$. When it dies the particle is replaced by a random number of offspring. The mean of this number is $1 + \alpha/\mu$, while the variance is $2\beta$.

The particle viewpoint gives more intuition. Having this in mind it is easier to understand the discussion in the introduction, moreover some further heuristics are given in [1, Remark 3.3, Remark 3.7, Remark 3.11].

**Remark 9.** The law of $H_\infty$ is an unresolved problem. It can be proved however that it is not Gaussian. Using Fact 22 we will see that $H_\infty$ is closely related to the corresponding limit for the Ornstein-Uhlenbeck branching process. For further discussion we refer the reader to [1, Remark 3.12].

**Remark 10.** We suspect that the convergence in (3.5) is in fact almost sure.

**Remark 11.** In our paper, for the sake of simplicity, we choose to work with the branching mechanism [13]. Using our methods it should be straightforward to prove the results for any branching mechanism which admits the fourth moment (i.e. $\psi^{(4)}(0) < +\infty$).

We conjecute also that the results are valid for any branching mechanism with the second moment. An interesting question would be to go beyond this assumption. It is natural to expect different normalization and convergence to some stable random variable.
Remark 12. The other remarks of [1] are also relevant to our case. For the sake of brevity we only mention that the most important extensions of the present paper will be to study superprocesses with general diffusion, instead for the Ornstein-Uhlenbeck process and the case of superprocesses with non-homogenous branching rates. This will be by no means a trivial task, a short explanation of the forthcoming difficulties is given in [1, Remark 3.16].

4. Proof Preliminaries

In this section we gather all auxiliary facts used in the proofs of results presented in Section 3. The proof itself are contained separately in Sections 5-7.

4.1. Backbone construction. Following recent developments, e.g. [3], we present a so-called backbone construction of a supercritical superprocess. The main idea is to define a backbone, being a supercritical branching particle system, which is dressed with subcritical superprocesses. This allows, up to some extent, to treat a superprocess as a discrete object. Such property makes things easier. On the conceptual level the proofs presented in the paper owes much to the proofs for the branching particle system in [1]. Our strategy is to take a proof of [1] and control the dressing behavior in a suitable way. This is the main technical difficulty of the paper. We will comment once again about the strategy after presenting decomposition (4.7).

To make this paper self-contained we will now briefly present the aspects of the backbone construction which are relevant to our paper. Much of the text below is “borrowed” from [3, Section 2.4]. We refer to this paper a reader interested in a more general and detailed description. Let us recall that we assume that the branching mechanism is given by (1.3) and that $\alpha > 0$. Let $\lambda^*$ be the largest root of $\psi(\lambda) = 0$, i.e.

$$\lambda^* = \frac{\alpha}{\beta}.$$ 

We denote $\psi^*(\lambda) := \psi(\lambda + \lambda^*)$ and check that

$$\psi^*(\lambda) = \beta(\lambda + \frac{\alpha}{\beta})^2 - \alpha(\lambda + \frac{\alpha}{\beta}) = \beta\lambda^2 + 2\alpha\lambda + \frac{\alpha^2}{\beta} - \alpha\lambda - \frac{\alpha^2}{\beta} = \alpha\lambda + \beta\lambda^2.$$ 

The superprocess construction presented in Section 2 is also valid for $\psi^*$ being the branching mechanism. This superprocess is subcritical i.e. its total mass decays exponentially fast. We will refer to it using additional superscript $\ast$, e.g. $E^\ast_\nu(1.3)$.

We calculate the branching law of the prolific backbone (see [3, Section 2.4])

$$F(s) = \frac{1}{\lambda^*} \psi(\lambda^*(1-s)) = (-\alpha(1-s) + \alpha(1-s)^2) = \alpha(-1 + s + 1 - 2s + s^2) = \alpha(s^2 - s).$$

Let $\mathcal{M}_a(\mathbb{R}^d) \subset \mathcal{M}_F(\mathbb{R}^d)$ be the space of finite atomic measures on $\mathbb{R}^d$. We shall write $\{z_t\}_{t \geq 0}$ for a branching $\mathcal{F}$-motion whose total mass has generator given by (1.2). Hence $z$ is the $\mathcal{M}_a(\mathbb{R}^d)$-valued Markov process in which individuals from the moment of birth, live for an independent and exponentially distributed period of time with parameter $\psi'(\lambda^*) = \alpha$ during which they execute an Ornstein-Uhlenbeck diffusion issued from their position of birth and at death they give birth at the same position to two offspring. Let us stress that the backbone process does not suffer from extinction. In our case the process $z$ is nothing else than the Ornstein-Uhlenbeck branching process (studied in [1]). We shall also refer to $z$ as the backbone (the name will become self-explanatory soon). The initial configuration of $z$ is denoted by $\gamma \in \mathcal{M}_a(\mathbb{R}^d)$. Moreover, when referring to individuals in $z$ we may use the classical Ulam-Harris notation, see for example [13, p. 290]. The only feature that we really need of the Ulam-Harris notation is that the individuals are uniquely identifiable amongst $\mathcal{T}$, the set labels of individuals realized in $z$. For each individual $u \in \mathcal{T}$ we shall write $\tau_u$ and $\sigma_u$ for its birth and death times respectively and $\{z_u(r) : r \in [\tau_u, \sigma_u]\}$ for its spatial trajectory.

\footnote{The author thanks Andreas Kyprianou for letting to use the parts of description in [3]. We note that our notation is mainly consistent with the one in [3]. The only notable exception is of $\alpha$ in the branching mechanism function $\psi$. We prefer to assume that $\alpha > 0$ and put $-\alpha$ in (1.3) instead of the form in [3, Section 2.1].}
Definition 13. For $\nu \in \mathcal{M}_b(\mathbb{R}^d)$ and $\nu' \in \mathcal{M}_F(\mathbb{R}^d)$ let $z$ be the Ornstein-Uhlenbeck branching process with initial configuration $\gamma$ and $\bar{X}$ an independent copy of $X$ under $\mathbb{P}^*_\nu$ (that is with the subcritical branching mechanism function $\psi^*$. Then we define a $\mathcal{M}_F(\mathbb{R}^d)$-valued stochastic process $\{\Lambda_t\}_{t \geq 0}$ by

$$\Lambda = \bar{X} + I^*,$$

where the processes $\{I^*_t\}_{t \geq 0}$ is independent of $\bar{X}$. Moreover, this process is described path-wise as follows (we note that the construction in [5] contains more ingredients as it covers a larger class of branching diffusions).

Continuous immigration: The process $I^*_t$ is measure-valued on $\mathbb{R}^d$ such that

$$I^*_t := \sum_{u \in T} \sum_{t \wedge \tau_u < r \leq t \wedge \sigma_u} X^\ast_{t-r},$$

where, given $z$, independently for each $u \in T$ such that $\tau_u < t$, the processes $X^\ast_{t-r}$ are countable in number and correspond to $X$-valued, Poissonian immigration along the space-time trajectory $\{(z_u(r), r) : r \in (\tau_u, t \wedge \sigma_u)\}$ with rate $2\beta dr \times d\nu^*_z(r)$. To complete the definition we need to explain measures $\{N^*_x, x \in \mathbb{R}^d\}$. They are associated with the laws $\{P^*_x, x \in \mathbb{R}^d\}$ defined on the same measurable space, namely

$$N^*_x(1 - e^{-(f,X)}) = -\log E^*_{z_x}(e^{-(f,X)}),$$

for all $f \in \text{bp}(\mathbb{R}^d)$ and $t \geq 0$. Such measures are formally defined and explored in detail in [8].

Intuitively speaking, the branching property implies that $P^*_x$ is an infinitely divisible measure on the path space of $X$, $\mathcal{X} := \mathcal{M}(\mathbb{R}^d) \times [0, \infty)$, and (4.4) is a ‘Lévy-Khinchine’ formula in which $N^*_x$ plays the role of its ‘Lévy measure’. In this sense, $N^*_x$ can be considered as the ‘rate’ at which superprocesses ‘with zero initial mass’ contribute to a unit mass at position $x$.

Moreover, we denote the law of $\Lambda$ by $\mathbb{P}_{\nu \times \gamma}$.

We will now present the main result concerning the backbone construction. First we randomize the law of $\mathbb{P}_{\nu \times \gamma}$ for $\nu \in \mathcal{M}_F(\mathbb{R}^d)$ by replacing the deterministic choice of $\gamma$ with a Poisson random measure having intensity $\lambda^*|\nu|$. We denote the resulting law by $\mathbb{P}_\nu$. We have [6, Theorem 2]

Theorem 14. For any $\nu \in \mathcal{M}_F(\mathbb{R}^d)$, the process $(\Lambda, \mathbb{P}_\nu)$ is Markovian and has the same law as $(X, \mathbb{P}_\nu)$.

The construction above states that the superprocess can be seen as immigration of a mass, which further will be called dressing, in a Poissonian fashion along the backbone $z$. The backbone is a supercritical process contrary to the fact that the immigrating mass follows the superprocess dynamics with subcritical mechanism $\psi^*$. This, up to some degree, means that the backbone is what really matters and the superprocess can be regarded as a discrete entity. Following the fact $z$ is the Ornstein-Uhlenbeck branching process is that many of proof techniques from [1] can be reused.

The backbone cannot die thus the set of extinction has a particularly simple description for $(\Lambda, \mathbb{P}_\nu)$. Namely,

$$\text{Ext} = \{z_0 = 0\},$$

that is, the extinction holds only when no backbone particle ever appeared.

We denote two families of processes. The first $\{D^*_t\}_{t \geq 0}$ where $s \geq 0$ is given by

$$D^*_t := \sum_{u \in T} \sum_{t \wedge \tau_u < r \leq t \wedge \sigma_u} X^\ast_{t-r+s} 1_{r \leq s}.$$

More intuitively speaking this process describes the evolution of the dressing which appeared in the system before time $s$. The complementary process is defined in terms of the backbone. Namely, given
the $i$-th particle of $z_t$, by $\{\Gamma_{t}^{i,s}\}_{t \geq 0}$ we denote process
\begin{equation}
\Gamma_{t}^{i,s} := \sum_{u \in T^{i,s}} \sum_{t \wedge \tau_u < r \leq t \wedge \sigma_u} X_{t-r+s}^{(1,u,r)},
\end{equation}
where $T^{i,s}$ is a (random) tree stemming from $i$-th particle. Intuitively $\Gamma_{t}^{i,s}$ is a “sub-superprocess” stemming from the $i$-th prolific individual at time $s$. Let us recall (4.3) and fix $s \geq 0$, for any $t \geq s$ we have
\begin{equation}
X_{t} = X_{t}^{0} + D_{t-s}^{n} + \sum_{i=1}^{\lfloor z_{t} \rfloor} \Gamma_{t-s}^{i,s}.
\end{equation}
Now we come back to the description of the proof strategy.

Remark 15. The first two terms of (4.7) are subcritical superprocesses and as such are negligible when $t \gg s$. The third term is a sum of random variables indexed with the branching process $z$ to which some techniques similar to [1] can be applied.

The proofs in [1] relied on the existence of an explicit coupling of two Ornstein-Uhlenbeck processes [1, Fact 4.1]. In [1] this coupling can be “transferred” on the level of branching processes in a way that two coupled systems shared the same genealogical structure and the corresponding particles were coupled OU processes. The main advantage of such approach is that it makes proofs conceptually clear. However in this paper we decided not to follow this strategy. Reasons are twofold. Firstly, transferring coupling to the superprocess level is less obvious (although possible), secondly in the proofs below we use analytical notions which should be easier to use for more general diffusions. Let us note that beside these changes “the high level structure” of [1]’s proofs could be reused. The main idea taken from [1] is to study the system on two different time scales.

4.2. Ornstein-Uhlenbeck semigroup properties. Throughout the proofs we will denote
\[ \tilde{f} := f - \langle f, \varphi \rangle. \]
We will now summarize properties of the Ornstein-Uhlenbeck semigroup (with the infinitesimal operator (1.1)) which we will use later.

**Fact 16.** Let $f \in C(\mathbb{R}^{d})$ and $\tilde{f}$ is defined as above. Then there exist constants $C, n > 0$ such that for any $t \geq 0$ we have
\begin{equation}
|\mathcal{P}_{t}f(x)| \leq C (\|x\|^{n} e^{-\mu t} + 1).
\end{equation}
\begin{equation}
|\mathcal{P}_{t} \tilde{f}(x)| \leq C (1 + \|x\|^{n}) e^{-\mu t}, \quad \mathcal{P}_{t} \tilde{f}(0) \leq Ce^{-2\mu t}.
\end{equation}
We also have
\begin{equation}
\mathcal{P}_{t}id(x) = e^{-\mu t}id(x),
\end{equation}
where $id(x) = x$. Moreover
\[ \lim_{t \to +\infty} e^{\mu t} \mathcal{P}_{t} \tilde{f}(x) = x \circ (\text{grad} f, \varphi), \]
where $\circ$ is the standard scalar product and $\varphi$ is given by (1.2). Finally
\begin{equation}
|e^{\mu t} \mathcal{P}_{t} \tilde{f}(x) - x \circ (\text{grad} f, \varphi)| \leq C (1 + \|x\|^{n}) e^{-\mu t}.
\end{equation}

**Proof.** As explained in [1, (19)] we have $\mathcal{P}_{t}f(x) = E_{f}(xe^{-\mu t} + ou(t)G)$, where $G \sim \varphi$ and $ou(t) = \sqrt{1 - e^{-2\mu t}}$. Using the triangle inequality and the binomial expansion one gets
\[ \mathcal{P}_{t}f(x) = E_{f}(xe^{-\mu t} + ou(t)G) \leq cE\|xe^{-\mu t} + ou(t)G\|^{n} \leq E\left(\|xe^{-\mu t}\| + \|ou(t)G\|\right)^{n} \leq \sum_{i=0}^{n} \|xe^{-\mu t}\|^{i} E\|ou(t)G\|^{n-i}. \]
Now (4.8) follows simply by the fact that any moment of a Gaussian variable exists. The rest of the statements were proved in [1, Lemma 4.3].
4.3. Moments calculation. The main aim of this section is to calculate moments of \((\Gamma_t^i, s, f)\) where \(\Gamma_t^i, s\) is given by (1.5). We will utilize moments up to order 4 but, as it yields no additional cost, we make some calculations for an arbitrary order.

Before the calculations we recall the generalization of the chain rule, Faà di Bruno’s formula, which states that

\[
\frac{d^k}{dx^k} h(g(x)) = \sum_{m \in A_k} a_m \cdot h^{(m_1 + \cdots + m_k)}(g(x)) \cdot \prod_{j=1}^k \left( g(j)(x) \right)^{m_j},
\]

where \(a_{m_1, \ldots, m_n} := \frac{m!}{m_1!^1 \cdot m_2!^2 \cdot \cdots \cdot m_n!^n}\) and the sum is over the set \(A_k\) of all \(k\)-tuples of non-negative integers \(m = (m_1, \ldots, m_k)\) satisfying the constraint \(\sum_{j=1}^k j m_j = k\).

Let \(f \in \mathcal{B}_p(\mathbb{R}^d)\); we recall that \(u_f\) is the solution of (1.5). We introduce an additional parameter \(\theta > 0\) and denote

\[
\theta \mapsto u_{\theta f}(x, t) = \mathcal{P}_t(\theta f)(x) - \int_0^t \mathcal{P}_{t-s}[\psi(u_{\theta f}(.), s)](x)ds.
\]

It is obvious (as (1.5) has a unique non-negative solution) that

\[
u_{\theta f}(x, t) = 0.\]

Differentiating with respect to \(\theta\) and using (1.12) we get

\[
\frac{\partial}{\partial \theta} u_{\theta f}(x, t) = \mathcal{P}_t f(x) - \int_0^t \mathcal{P}_{t-s} \left[ \frac{\partial}{\partial \theta} u_{\theta f}(x, s) \psi'(u_{\theta f}(.), s) \right](x).
\]

\[
\frac{\partial^k}{\partial \theta^k} u_{\theta f}(x, t) = - \int_0^t \mathcal{P}_{t-s} \left[ \sum_{m \in A_k} a_m \psi^{(m_1 + \cdots + m_k)}(u_{\theta f}(x, s)) \cdot \prod_{j=1}^k \left( \frac{\partial^j}{\partial \theta^j} u_{\theta f}(x, s) \right)^{m_j} \right](x), \text{ for } k \geq 2.
\]

The above calculation is formal. To legalize them we fix \(T > 0\) and notice that by Fact 27 there exists \(\theta_T > 0\) such that \(\mathbb{E} e^{\theta(\xi - d)} < +\infty\) for any \(\theta \leq \theta_T\) and \(t \leq T\). By the standard properties of the Laplace transform we conclude that the derivatives \(\frac{\partial^k}{\partial \theta^k} u_{\theta f}(x, t)\) exist for \(\theta > -\theta_T\) and \(t \leq T\) and are continuous as function of \(\theta\). Further, one need to apply standard calculus tricks to conclude that the above formulas a valid for any \(t \leq T\) and \(\theta > -\theta_T\). Finally, for \(\theta = 0\) the formulas are valid for any \(t\).

We denote \(u_f^k(x, t) := \frac{\partial^k}{\partial \theta^k} u_{\theta f}(x, t)\bigg|_{\theta = 0}\). The above equations yield \(u_f^0(x, t) = 0\). Using (1.3) we obtain

\[
u_f^0(x, t) = \mathcal{P}_t f(x) + \alpha \int_0^t \mathcal{P}_{t-s} [u_f^s(.), s](x).
\]

\[
u_f^k(x, t) = - \int_0^t \mathcal{P}_{t-s} \left[ \sum_{m \in A_k} a_m \psi^{(m_1 + \cdots + m_k)}(0) \cdot \prod_{j=1}^k \left( u_f^j(x, s) \right)^{m_j} \right](x), \text{ for } k \geq 2.
\]

The first equation is solved by

(4.13)

\[
u_f^0(x, t) = \mathcal{P}_t^\alpha f(x).
\]

To treat the second one we denote \(B_k := A_k \setminus \{(0, \ldots, 0, 1)\}\) and notice that

\[
u_f^k(x, t) = - \int_0^t \mathcal{P}_{t-s} \left[ -\alpha u_f^k(x, s) + \sum_{m \in B_k} a_m \psi^{(m_1 + \cdots + m_k)}(0) \cdot \prod_{j=1}^k \left( u_f^j(x, s) \right)^{m_j} \right](x), \text{ for } k \geq 2.
\]

It is solved by

(4.14)

\[
u_f^k(x, t) = - \int_0^t \mathcal{P}_t^\alpha \left[ \sum_{m \in B_k} a_m \psi^{(m_1 + \cdots + m_k)}(0) \cdot \prod_{j=1}^k \left( u_f^j(x, s) \right)^{m_j} \right](x), \text{ for } k \geq 2.
\]
The above equations make it possible to calculate recursively any \( u^k_f \). Performing this operation for \( k \leq 4 \) (which is all we need in this paper) and taking into account the special form of \( \psi \) we get

\[
(4.15) \quad u^2_f(x,t) = -2\beta \int_0^t \mathcal{P}^\alpha_{t-s} \left( (\mathcal{P}^\alpha_s f(\cdot))^2 \right)(x) ds.
\]

\[
(4.16) \quad u^3_f(x,t) = -6\beta \int_0^t \mathcal{P}^\alpha_{t-s} \left[ u^1_f(\cdot,s) u^2_f(\cdot,s) \right](x),
\]

\[
(4.17) \quad u^4_f(x,t) = -\beta \int_0^t \mathcal{P}^\alpha_{t-s} \left[ 4u^1_f(\cdot,s)u^2_f(\cdot,s) + \left( u^2_f(\cdot,s) \right)^2 \right](x),
\]

The same calculation are valid for the superprocess with branching mechanism \( \psi^* \) given by (4.1) (which requires only changing \( \alpha \) to \( -\alpha \)). We will denote the quantities corresponding to the system with \( \psi^* \) using additional superscript \( * \).

Let us now prove some properties of \( u^k_f \).

**Fact 17.** Equations (4.13), (4.15)-(4.17) are well-defined for any \( f \in \mathcal{C}(\mathbb{R}^d) \). Moreover, given \( f \in \mathcal{C}(\mathbb{R}^d) \), there exist \( C, n > 0 \) such that

\[
(4.18) \quad |u^2_f(x,t)| \leq C e^{2\alpha t}(\|x\|^n + 1).
\]

\[
(4.19) \quad |u^4_f(x,t)| \leq C e^{-\alpha t}(\|x\|^n + 1).
\]

We have also

\[
(4.20) \quad |u^3_f(x,t)| \leq C e^{-\alpha t}(\|x\|^n + 1), \quad |u^4_f(x,t)| \leq C e^{-\alpha t}(\|x\|^n + 1).
\]

Let us note that potentially we can obtain better constants in some estimation. E.g. “\( n \)” in (4.19) could be smaller than in (4.20). However exact constants are not important in our proofs.

**Proof.** Formulas (4.13)-(4.17) were obtained for \( f \in \mathcal{B}p(\mathbb{R}^d) \) but can be “upgraded” to \( f \in \mathcal{C} \) by means of iterated application of standard integral-theoretic reasonings and Fact 16 (4.18) follows by Fact 16 and the following calculations

\[
|u^2_f(x,t)| \lesssim \int_0^t \mathcal{P}^\alpha_{t-s} \left[ 2e^{2\alpha s}(\|x\|^n + 1)^2 \right](x) \lesssim e^{\alpha t} \int_0^t e^{\alpha s}(\|x\|^{2n} e^{-\mu(t-s)} + 1) ds \lesssim e^{2\alpha t}(\|x\|^{2n} + 1),
\]

and in the same vein we obtain (4.19). In order to prove (4.20) we utilize (4.16) and Fact 16

\[
|u^3_f(x,t)| \lesssim \int_0^t \mathcal{P}^\alpha_{t-s} \left[ u^1_f(\cdot,s) u^2_f(\cdot,s) \right](x) ds \lesssim e^{-\alpha t} \int_0^t e^{\alpha s} \mathcal{P}^\alpha_{t-s} \left[ e^{-\alpha s}(1 + \|x\|^n) e^{-\alpha s}(1 + \|x\|^n) \right](x) ds \lesssim e^{-\alpha t}(1 + \|x\|^{n_1}),
\]

for some \( n_1 \geq 0 \). The case of \( u^4_f(x,t) \) follows similarly. By (4.17) we have

\[
|u^4_f(x,t)| \lesssim \int_0^t \mathcal{P}^\alpha_{t-s} \left[ u^1_f(\cdot,s) u^3_f(\cdot,s) + \left| u^3_f(\cdot,s) \right|^2 \right](x) ds \lesssim e^{-\alpha t} \int_0^t e^{\alpha s} \mathcal{P}^\alpha_{t-s} \left[ (1 + \|x\|^{3n}) \right](x) ds \lesssim e^{-\alpha t}(1 + \|x\|^{n_2}),
\]

for some \( n_2 \geq 0 \). \( \Box \)

Now let us recall to the backbone construction given in Definition 13. We fix \( f \in \mathcal{B}p(\mathbb{R}^d) \) and apply Theorem 1 with \( \mu = 0, \nu = \delta_x, \theta = \theta f, h = 0 \). This yields

\[
\mathbb{E} \left( e^{-\langle \theta f, \Lambda_t \rangle} \right) = V_{\theta f}(x,t),
\]
where \( V_{\theta f}(x,t) \) is the unique \([0,1]\)-valued solution to the integral equation
\[
V_{\theta f}(x,t) = 1 + \frac{\beta}{\alpha} \int_0^t \mathcal{P}_{t-s} \left[ \psi^* \left( \frac{\alpha}{\beta} V_{\theta f}(\cdot,s) + u_{\theta f}^*(\cdot,s) \right) - \psi^*(u_{\theta f}^*(\cdot,s)) \right] (x) \mathrm{d}s,
\]
we recall that \( u^* \) is the solution of \([13]\) with the branching mechanism \( \psi^* \). Obviously, we have
\[
V_0(x,t) = 1.
\]
Using \([4.12]\) we obtain (one has to justify the validity of the below calculations in the same spirit as Fact 18.

We denote \( V^k := \frac{\partial^k V_{\theta f}}{\partial \theta^k} \big|_{\theta = 0} \). The above equation yields
\[
V^k_f(x,t) = \frac{\beta}{\alpha} \int_0^t \mathcal{P}_{t-s} \left[ \sum_{m \in A_k} a_m \psi^*(m_1 + \ldots + m_k)(\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^*) \prod_{j=1}^k \left( -\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^* \right)^{m_j} - \sum_{m \in A_k} a_m \psi^*(m_1 + \ldots + m_k)(0) \prod_{j=1}^k (u_{\theta f}^*)^{m_j} \right] (x) \mathrm{d}s.
\]

We recall that \( B_k = A_k \setminus \{(0, \ldots, 0, 1)\} \) and \( u_0^* = 0, V_0 = 1 \). Moreover \( \psi^* \left( -\frac{\alpha}{\beta} \right) = -2\frac{\beta}{\alpha} + \alpha = -\alpha \) and \( \psi^* (0) = \alpha \). Therefore
\[
V^k_f(x,t) = \frac{\beta}{\alpha} \int_0^t \mathcal{P}_{t-s} \left[ -\alpha \left( -\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^* \right)^k + \sum_{m \in B_k} a_m \psi^*(m_1 + \ldots + m_k)(\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^*) \prod_{j=1}^k \left( -\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^* \right)^{m_j} - \alpha u_{\theta f}^* \right] (x) \mathrm{d}s.
\]

This equation is solved by
\[
(4.21) \quad V^k_f(x,t) = \frac{\beta}{\alpha} \int_0^t \mathcal{P}_{t-s} \left[ \sum_{m \in B_k} a_m \psi^*(m_1 + \ldots + m_k)(\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^*) \prod_{j=1}^k \left( -\frac{\alpha}{\beta} V_{\theta f} + u_{\theta f}^* \right)^{m_j} - 2\alpha u_{\theta f}^* - \sum_{m \in B_k} a_m \psi^*(m_1 + \ldots + m_k)(0) \prod_{j=1}^k (u_{\theta f}^*)^{m_j} \right] (x) \mathrm{d}s.
\]

We list now some properties of \( V^k_f \) used in the proofs below below.

**Fact 18.** For any \( f \in C(\mathbb{R}^d) \) we have
\[
(4.22) \quad E_{0 \times \Delta_x} (f, A_k)^k = (-1)^k V^k_f(x,t), \quad k \in \mathbb{N},
\]
\[
(4.23) \quad V^1_f(x,t) = -\frac{\beta}{\alpha} \mathcal{P}_t f(x) \sinh(\alpha t),
\]
The fact follows immediately by Corollary 19 and Laplace transform calculations as in previous sections.

Moreover, we have

\[ V_f^2(x, t) = \frac{\beta}{\alpha} \int_0^t P_{t-s}^{\alpha} \left[ 2 \beta (P_s^\alpha f(\cdot))^2 - 2 \beta \left( u_f^{\ast,1}(\cdot, s) \right)^2 - 2 \alpha u_f^{\ast,2}(\cdot, s) \right] (x) ds. \]

By \((4.13)\) it follows

\[ V_f^1(x, t) = -2\beta \int_0^t e^{\alpha(t-s)} P_{t-s} \left[ e^{-\alpha s} P_s f(\cdot) \right] (x) ds = -2\beta e^{\alpha t} P_t f(x) \int_0^t e^{-2\alpha s} ds = -2\beta \alpha P_t f(x) \sinh(\alpha t). \]

To prove \((4.24)\) we again use \((4.21)\)

\[ V_f^2(x, t) = \frac{\beta}{\alpha} \int_0^t P_{t-s}^{\alpha} \left[ 2 \beta \left( -\frac{\alpha}{\beta} V_f^1(\cdot, s) + u_f^{\ast,1}(\cdot, s) \right)^2 - 2 \beta \left( u_f^{\ast,1}(\cdot, s) \right)^2 - 2 \alpha u_f^{\ast,2}(\cdot, s) \right] (x) ds. \]

We notice that \(-\beta V_f^1(x, s) + u_f^{\ast,1}(x, s) = P_s^\alpha f(x)\) which is enough to prove \((4.24)\). \((4.25)\) holds by \((4.19), (4.8)\) and the following calculation

\[ V_f^2(x, t) \lesssim \int_0^t P_{t-s}^{\alpha} \left[ (e^{\alpha s} P_s f(\cdot))^2 + (e^{-\alpha s} P_s f(\cdot))^2 + |u_f^{\ast,2}(\cdot, s)| \right] (x) ds \lesssim e^{\alpha t} \int_0^t e^{\alpha s} P_{t-s}^{\alpha} \left[ (\|\cdot\|^2 + 1)^2 \right] (x) ds \lesssim e^{2\alpha t}(\|x\|^{2n} + 1). \]

\[ \square \]

4.4. Dressing behavior. Let us recall “the dressing process” \(\{D_t^\psi\}_{t \geq 0}\) defined by \((1.5)\). It is a superprocess which at time \(t = 0\) is distributed as \(X_s\) and then evolve according to subcritical dynamics with the branching mechanism \(\psi^\ast\). Using the Markov property in Theorem 14 we obtain

**Corollary 19.** Let \(f \in bp(\mathbb{R}^d)\) then

\[ E_\nu(e^{-(f, D_t^\psi)}) = E_\nu E_{X_s}(e^{-(f, X_s)}) = E_\nu \exp \left\{ -\int_{\mathbb{R}^d} u_f^\ast(x, t) X_s(dx) \right\} = \exp \left\{ -\int_{\mathbb{R}^d} v_f(x, s; t) \nu(dx) \right\}, \]

where \(u_f^\ast\) is defined as \((2.21)\) with \(\psi^\ast\) instead of \(\psi\) and \(v_f(x, s; t)\) is the solution of

\[ v_f(x, t; s) = P_t(u_f^\ast(\cdot, s))(x) - \beta \int_0^t P_u^{\alpha}[v_f(\cdot, t - u; s)](x) du. \]

**Fact 20.** Let \(f \in C(\mathbb{R}^d)\) then

\[ E_\nu(f, D_t^\psi) = e^{\alpha(s-t)} \int_{\mathbb{R}^d} P_t f(x) \nu(dx). \]

The fact follows immediately by Corollary 19 and Laplace transform calculations as in previous sections.
4.5. Martingales. We define two martingales \( \{W_t\}_{t \geq 0}, \{I_t\}_{t \geq 0} \) associated with the backbone process \( z \). Namely,
\[
W_t := e^{-\alpha t} |z_t|, \\
I_t := e^{-(\alpha - \mu) t} \sum_{i=1}^{\|z\|} z_t(i).
\]
They are closely related to \( V \) and \( H \) (defined by (2.3) and (3.3)). Let us assume that all of them are defined in terms of \( \Lambda \) (see Theorem (13)), so that they “live” in the same probability space.

**Fact 21.** \( W \) is an \( L^2 \)-bounded martingale. We denote its limit by \( W_\infty \). Moreover,
\[
(4.27) \quad V_\infty = \frac{\beta}{\alpha} W_\infty \quad \text{a.s.}
\]
\( I \) is a martingale, which for \( \alpha > 2\mu \) it is \( L^2 \)-bounded. Then we denote its limit by \( I_\infty \) and we have
\[
(4.28) \quad H_\infty = \frac{\beta}{\alpha} I_\infty \quad \text{a.s.}
\]

The proof uses some facts which are presented later on, hence it is postponed to Appendix.

We are now able to describe the relation of \( V_\infty, H_\infty \) and the dependence of the latter on the starting conditions. Let us first denote by \( \{J_t\}_{t \geq 1} \) an i.i.d. sequence of random variables distributed as \( I_\infty \) but with assumption that \( z_0 = \delta_0 \). Under assumption \( \alpha > 2\mu \) this exists, we refer to [1, Fact 3.8] where it is known under name \( H_\infty \) and to [1, Remark 3.12] for some information about its law.

**Theorem 22.** Let \( \{X_t\}_{t \geq 0} \) be the OU-superprocess starting from \( \nu \in \mathcal{M}_F(\mathbb{R}^d) \) and \( \alpha > 2\mu \). Let us define
\[
\check{H}_\infty := \frac{\beta}{\alpha} \left( \sum_{i=1}^{\nu} J_i + \sum_{i=1}^{\nu} x_i E_i \right), \quad \check{V}_\infty := \frac{\beta}{\alpha} \left( \sum_{i=1}^{\nu} x_i E_i \right),
\]
where \((x_1, x_2, \ldots, x_N)\) is a realization of a Poisson point process with intensity \( \nu \) and \( \{E_i\}_{i \geq 1} \) is an i.i.d. sequence of exponential random variables with parameter 1. Moreover, we assume that all defining objects are independent. Then
\[
(H_\infty, V_\infty) =^d (\check{H}_\infty, \check{V}_\infty).
\]

The proof follows easily using techniques of Fact 21 and [1, Fact 3.8], hence is skipped.

5. Proof of Theorem 3

We first gather properties of \( V_f^k \), extending the list in Fact 18 in the case when \( \alpha < 2\mu \). We recall that \( \check{f}(x) = f(x) - \langle f, \varphi \rangle \).

**Fact 23.** Let \( \alpha < 2\mu \) and \( f \in C(\mathbb{R}^d) \). Then for any \( x \in \mathbb{R}^d \) there is
\[
(5.1) \quad e^{-(\alpha/2)t} V_f^1(x, t) \to 0, \quad \text{as } t \to +\infty,
\]
\[
(5.2) \quad e^{-\alpha t} V_f^2(x, t) \to \sigma_f^2, \quad \text{as } t \to +\infty,
\]
as \( t \to +\infty \), where \( \sigma_f^2 \) is given by (3.1). We have \( \sigma_f^2 < +\infty \). Moreover, there exists \( l > 1/2 \) such that
\[
(5.3) \quad \sup_{\|x\| \leq t} |e^{-\alpha t} V_f^2(x, t) - \sigma_f^2| \to 0, \quad \text{as } t \to +\infty.
\]
Finally, there exist \( C, n > 0 \) such that
\[
(5.4) \quad |V_f^4(x, t)| \leq Ce^{2\alpha t}(\|x\|^n + 1).
\]
Proof. The proof of (5.1) follows easily by (4.23) and Fact 16 hence is skipped. To get (4.25) we use (4.24) and write

\[ e^{-\alpha t} V_f^2(x, t) = \frac{\beta}{\alpha} \int_0^t e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( \mathcal{P}_s^{\alpha} \hat{f}(\cdot) \right)^2 \right] (x) ds \]

\[ - 2\frac{\beta}{\alpha} \int_0^t e^{-\alpha s} \mathcal{P}_{t-s} \left[ \beta \left( u_j^{s,1}(\cdot, s) \right)^2 + \alpha u_j^{s,2}(\cdot, s) \right] (x) ds =: I_1(t) + I_2(t). \]

We have

\[ I_1(t) = \frac{\beta}{\alpha} \int_0^t \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (x) ds. \]

By (4.9) the integrand in the last expression can be estimated as follows

\[ \mathcal{P}_{t-s} \left[ \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (x) \preceq e^{(\alpha-2\mu)s} \mathcal{P}_{t-s} \left[ (1 + \| \cdot \|^2)^2 \right] (x). \]

Using (4.8) it can be checked that for any \( t \geq 0 \) we have \( \mathcal{P}_t \left[ (1 + \| \cdot \|^2)^2 \right] (x) \preceq (1 + \| x \|^{2n}). \) The dominated Lebesgue theorem implies that

\[ I_1(t) \to \frac{\beta}{\alpha} \int_0^\infty \langle \varphi, 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \rangle ds < +\infty. \]

A completely analogous argument, using (4.13) and (4.19), can be applied to treat \( I_2(t) \).

We will prove (5.3) with \( l = 1 \) in the formulation of the fact we used general \( l \) in order to keep it consistent with forthcoming Fact 24. It is enough to show

\[ \sup_{\| x \| \leq t} e^{-\alpha t} | V_f^2(x, t) - V_f^2(0, t) | \to 0. \]

We will prove this for the first term of (5.5) which is the hardest one and leave other terms to the reader. That is, we will show that

\[ \sup_{\| x \| \leq t} \int_0^t e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (x) ds - \int_0^t e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (0) ds \to 0. \]

By Fact 16 we have

\[ \sup_{\| x \| \leq t} \int_{t/2}^t e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (x) ds \preceq \sup_{\| x \| \leq t} \int_{t/2}^t e^{(\alpha-2\mu)s} \mathcal{P}_{t-s} \left[ (1 + \| x \|^2)^2 \right] (x) ds \]

\[ \preceq t^{2n} e^{((\alpha-2\mu)/2)t} \to 0. \]

It is well-known that \( \mathcal{P}_t \hat{f}(x) = \int_{\mathbb{R}^d} g_t(xe^{-\mu t} + y) \hat{f}(y) dy \), where \( g_t \) is the density of \( \mathcal{N}(0, (1 - e^{-2\mu t}) \) (see e.g. (19)). Therefore using Fact 16 and properties of the Gaussian density we obtain

\[ \text{(5.7)} \quad \sup_{\| x \| \leq t} \int_0^{t/2} e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (x) ds - \int_0^{t/2} e^{-\alpha s} \mathcal{P}_{t-s} \left[ 2\beta \left( e^{(\alpha/2)s} \mathcal{P}_s \hat{f}(\cdot) \right)^2 \right] (0) ds \]

\[ = 2\beta \sup_{\| x \| \leq t} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-\alpha s} \left( g_{t-s}(xe^{-\mu(t-s)} + y) - g_{t-s}(y) \right) \left( \mathcal{P}_s^{\alpha} \hat{f}(y) \right)^2 dy ds \]

\[ \preceq \sup_{\| x \| \leq t} \int_0^{t/2} \int_{\mathbb{R}^d} e^{-\alpha s} e^{-\mu(t-s)} \| x \| (\| x \| + t) g_{t-s}(y) \left( e^{(\alpha-\mu)s} (1 + \| x \|^2) \right)^2 dy ds \]

\[ \preceq t^2 \int_0^{t/2} e^{-\alpha s} e^{-\mu(t-s)} e^{(2(\alpha-\mu))s} ds \leq t^2 e^{-(\mu/2)t} \int_0^{t/2} e^{(\alpha-2\mu)s} ds \preceq t^2 e^{-(\mu/2)t} \to 0. \]
In order to prove (5.4) we apply the triangle inequality to (4.21)

\[ |\hat{V}_t^k(x,t)| \lesssim \sum_{m \in B_k} \int_0^t \mathcal{P}_{t-s}^\alpha \left[ \prod_{j=1}^k |V_j^i + u_j^i|^m \right] (x) ds \]

\[ + \int_0^t \mathcal{P}_{t-s}^\alpha \left[ u_j^i \right] (x) ds + \sum_{m \in B_k} \int_0^t \mathcal{P}_{t-s}^\alpha \left[ \prod_{j=1}^k |u_j^i|^m \right] (x) ds. \]

By (4.20) we can see that terms containing \( u_j^i \) will not contribute (in fact they will be of order \((1 + \|x\|^n)\) for some \( n \in \mathbb{N} \), which is strictly smaller then the whole expression). Therefore we skip these terms. For \( k = 3 \) the first sum reduces to (see also (4.16) for similar calculations)

\[ \int_0^t \mathcal{P}_{t-s}^\alpha \left[ |V_j^i|^2 \right] (x) ds \leq e^{\alpha t} \int_0^t \mathcal{P}_{t-s} e^{-\alpha s} \left[ |V_j^i|^2 e^{(\alpha - \mu)s} (1 + \| \cdot \|^n) \right] (x) ds \]

\[ \leq e^{\alpha t} \int_0^t \mathcal{P}_{t-s} e^{-\mu s} \left[ e^{\alpha s} (1 + \| \cdot \|^{2n}) (1 + \| \cdot \|^{n}) \right] (x) ds \leq e^{(3\alpha/2)t} (1 + \| \cdot \|^{3n}), \]

where we used (4.23), (5.6), and \( \alpha < 2\mu \). In this way we have proved that \( |\hat{V}_t^3(x,t)| \leq e^{(3\alpha/2)t} (1 + \| \cdot \|^{n}). \)

Now we proceed to \( k = 4 \). Again the second and third term of (5.8) can be skipped. The sum in the first term reduces to (see also (4.17) for similar calculations)

\[ \int_0^t \mathcal{P}_{t-s}^\alpha \left[ |V_j^i|^2 \right] (x) ds + \int_0^t \mathcal{P}_{t-s}^\alpha \left[ |V_j^i|^3 \right] (x) ds \]

\[ \leq \int_0^t \mathcal{P}_{t-s}^\alpha \left[ e^{(3\alpha/2)s} (1 + \| \cdot \|^{3n}) e^{(\alpha - \mu)s} (1 + \| \cdot \|^{3n}) \right] (x) ds \]

\[ + \int_0^t \mathcal{P}_{t-s}^\alpha \left[ e^{\alpha s} (1 + \| \cdot \|^{2n})^2 \right] (x) ds \leq C e^{2\alpha t} (1 + \| \cdot \|^{4n}), \]

where we used the results proved above, Fact (16) and \( \alpha < 2\mu \). \( \square \)

We now proceed to the proof of Theorem 3. In the proof we show a weak convergence hence we will work with \( \Lambda \) which has the backbone representation as described in Definition (13). We start with the following random vector

\[ Z_1(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{-(\alpha/2)t}(|\Lambda_t| - e^{\alpha t} V_\infty), e^{-(\alpha/2)t}(|\Lambda_t| - \hat{V}_t^i) \right). \]

Let \( n \in \mathbb{N} \) to be fixed later; the limit of \( Z_1(nt) \) is obviously the same as the on of \( Z_1(t) \). By Fact (1) we have \( V_{nt} - V_t \to 0 \) a.s. Therefore the limit of \( Z_1(t) \) coincides with the limit of

\[ Z_2(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{-(\alpha/2)t}(|\Lambda_t| - e^{\alpha t} V_\infty), e^{-(\alpha/2)t}(|\Lambda_{nt}| - \hat{V}_t^i) \right). \]

It is well known, by the so-called branching property, that the evolution of the total mass of the system after time \( nt \) is the same as it was split into \( |\Lambda_{nt}| \) of systems \( \{ \Lambda_t^i \}_{t \geq 0} \) having initial mass 1 and one system \( \hat{\Lambda} \) of size \( |\Lambda_{nt}| - |\Lambda_{nt}| \). For each \( i \) we can define a corresponding martingale by formula (2.3) and denote its limit by \( V_{\infty}^{(i)} \). Moreover the limit of martingale of the system \( \hat{\Lambda} \) is denoted by \( \hat{V}_\infty \). Obviously

\[ V_\infty = e^{-\alpha t} \left( \sum_{i=1}^{||\Lambda_{nt}||} V_{\infty}^{(i)} + \hat{V}_\infty \right). \]
Therefore, we have
\[
e^{-(\alpha/2)t}(|A_{nt}| - e^{\alpha t}V_\infty) = d \sum_{i=1}^{[\alpha_1]} (1 - V_\infty^t) + e^{-(\alpha/2)t}(|A_{nt}| - |A_{nt}| - \tilde{V}_\infty).
\]

One easily sees that the second term converges to 0 (in probability) hence is negligible in our analysis.

Now we perform arguably the most crucial step of the proof i.e. we use \(4.7\)

\[
e^{-(\alpha/2)t} \langle A_{nt}, \tilde{f} \rangle = e^{-(\alpha/2)t} \langle \tilde{X}_{nt}, \tilde{f} \rangle + e^{-(\alpha/2)t} \langle D_{(n-1)t}^t, \tilde{f} \rangle + e^{-(\alpha/2)t} \sum_{i=1}^{[z_i]} (\Gamma_{i,t} - \tilde{f}).
\]

Using \(4.26\) and Fact \(16\) we obtain

\[
E \alpha e^{-(\alpha/2)t} |\langle D_{(n-1)t}^t, \tilde{f} \rangle| \leq e^{-(\alpha/2)t}E \alpha |\langle D_{(n-1)t}^t, \tilde{f} \rangle| = e^{-(\alpha/2)t}e^{-(\alpha/2)t} \int_{\mathbb{R}^d} P_t \tilde{f}(x)\nu(dx) \to 0,
\]

if \(n \geq 2\) is large enough. Analogously one can prove that \(E \alpha e^{-(\alpha/2)t} |\langle \tilde{X}_{nt}, \tilde{f} \rangle| \to 0\). Using the facts proved above we conclude that the limit of \(Z_3(t)\) is the same as the one of

\[
Z_4(t) := \left( e^{-\alpha t} |A_t|, e^{(\alpha/2)t} \sum_{i=1}^{[\alpha_1]} (1 - V_\infty^t), e^{-(\alpha/2)t} \sum_{i=1}^{[z_i]} (\Gamma_{i,t} - \tilde{f}) \right).
\]

We denote \(Z_{n.i} := e^{-(n-1)/2}t |\langle i,t \rangle|\), \(E \alpha e^{-(\alpha/2)t} |\langle D_{(n-1)t}^t, \tilde{f} \rangle| \leq e^{-(\alpha/2)t}e^{-(\alpha/2)t} \int_{\mathbb{R}^d} P_t \tilde{f}(x)\nu(dx) \to 0\), if \(n \geq 2\) is large enough. Analogously one can prove that \(E \alpha e^{-(\alpha/2)t} |\langle \tilde{X}_{nt}, \tilde{f} \rangle| \to 0\). Using the facts proved above we conclude that the limit of \(Z_3(t)\) is the same as the one of

\[
Z_5(t) := \left( e^{-\alpha t} |A_t|, e^{(\alpha/2)t} \sum_{i=1}^{[\alpha_1]} (1 - V_\infty^t), e^{-(\alpha/2)t} \sum_{i=1}^{[z_i]} (Z_{n,i} - Z_{n.i}) \right).
\]

Let \(l\) be the same as in \(23\). We check that

\[
E \sum_{i=1}^{[z_i]} |Z_{n,i} - Z_{n,i}| 1_{\{||z_t(i)|| \geq t\}^t} = e^{\alpha t}E |Z_{n,1} - Z_{1,1}||1_{\{||z_t(1)|| \geq t\}}
\]

\[
\leq e^{\alpha t}E |Z_{1,1} - Z_{1,1}|^t \mathbb{P} (||z_t(1)|| \geq t^t) \leq e^{\alpha t}e^{-ct^t}E |Z_{1,1}|^2 \to 0 \quad \text{as} \quad t \to +\infty.
\]

To justify the above convergence it is enough to notice that by Fact \(18\) we have \(E |Z_{1,1}|^2 \leq e^{ct^t}\), for some \(c > 0\). By these and Fact \(24\) the limit of \(Z_3(t)\) is the same as the one of

\[
Z_6(t) := \left( e^{-\alpha t} |z_t|, e^{(\alpha/2)t} \sum_{i=1}^{[\alpha_1]} (1 - V_\infty^t), e^{(\alpha/2)t} \sum_{i=1}^{[z_i]} (Z_{n,i} - Z_{n,i}) 1_{\{||z_t(i)|| \geq t\}} \right).
\]

We denote now

\[
Z_6(t) := \left( e^{-\alpha t} |z_t|, |A_{nt}|^{-1/2} \sum_{i=1}^{[\alpha_1]} (1 - V_\infty^t), |z_t|^{-1/2} \sum_{i=1}^{[z_i]} (Z_{n,i} - Z_{n,i}) 1_{\{||z_t(i)|| \geq t\}} \right).
\]

Moreover, we consider the above quantity conditionally on the event \(\{||A_{nt}| \geq t\}\). We will denote the corresponding expectation by \(E'_\nu\) and write
Using (5.3) it is easy to check that
\[
\tilde{\chi} := \exp \left\{ i \beta \int_0^t e^{-\alpha t} |z_t| + i \theta_2 |\Lambda_{nt}|^{-1/2} \sum_{i=1}^{||\Lambda_{nt}||} (1 - V_i^r) \right\}.
\]

Let us denote by \( h \) the characteristic function of \((1 - V_i^r)\). Using the conditional expectation we obtain
\[
\chi_1(\theta_1, \theta_2, \theta_3; t) := \mathbb{E}_\nu \exp \left\{ i \beta \int_0^t e^{-\alpha t} |z_t| + i \theta_2 |\Lambda_{nt}|^{-1/2} \sum_{i=1}^{||\Lambda_{nt}||} (Z^{n,i}_t - z^{n,i}_t) 1_{||z_t(i)|| < t'} \right\}.
\]

By the central limit theorem we know that \( h(\theta_2/\sqrt{n})^n \to e^{-\theta^2_2/(2\sigma^2)} \), where \( \sigma_V = \frac{2\beta}{\alpha} \) (see [2.6]). Let us now denote
\[
\chi_2(\theta_1, \theta_2, \theta_3; t) := \mathbb{E}_\nu \exp \left\{ i \beta \int_0^t e^{-\alpha t} |z_t| + i \theta_2 |z_t|^{-1/2} \sum_{i=1}^{||z_t||} (Z^{n,i}_t - z^{n,i}_t) 1_{||z_t(i)|| < t'} \right\} e^{-\theta^2_2/(2\sigma^2)}.
\]

We check that
\[
|\chi_1(\theta_1, \theta_2, \theta_3; t) - \chi_2(\theta_1, \theta_2, \theta_3; t)| \leq \mathbb{E}_\nu \left\| h \left( \frac{\theta_2}{||\Lambda_{nt}||} \right)^{1/2} \right\| |X_{nt}|^{-1/2} - e^{-\theta^2_2/(2\sigma^2)} \to 0, \quad \text{as } t \to +\infty.
\]

Let us now notice that \( \mathbb{P}_\nu(\varepsilon \leq |X_{nt}| \leq t) \to 0 \) for any \( \varepsilon > 0 \). Therefore \( 1_{|X_{nt}| > t} \to 1_{\text{extc}} \). We denote the expectation conditional on \( \text{extc} \) by \( \tilde{\mathbb{E}}_\nu \). Further we write
\[
\chi_3(\theta_1, \theta_2, \theta_3; t) := \tilde{\mathbb{E}}_\nu \exp \left\{ i \beta \int_0^t e^{-\alpha t} |z_t| + i \theta_2 |z_t|^{-1/2} \sum_{i=1}^{||z_t||} (Z^{n,i}_t - z^{n,i}_t) 1_{||z_t(i)|| < t'} \right\} e^{-\theta^2_2/(2\sigma^2)}.
\]

One easily checks that its limit is the same as the one of \( \chi_2 \).

Let us now consider a sequence of \( \{a_m\}_{m \geq 0}, \{p_m\}_{m \geq 0} \) such that each \( a_m \in \mathbb{N} \) and \( p_m \in \mathbb{R}^{d \times a_m} \). Moreover, we assume that \( a_m \sim e^{am} \) and \( \forall i \leq a_m \) we have \( \|p_m(i)\| \leq m^l \). We denote
\[
S_m := a_m^{-1/2} \sum_{i=1}^{a_m} (\tilde{Z}^{n,i}_m - \tilde{z}^{n,i}_m),
\]
where \( \tilde{Z}^{n,i}_m \) is defined as \( Z^{n,i}_t \), but with assumption that it starts off from position \( p_m(i) \), and \( \tilde{z}^{n,i}_m = \mathbb{E}Z^{n,i}_m \).

\[
\text{Var} \left( a_m^{-1/2} \sum_{i=1}^{a_m} (\tilde{Z}^{n,i}_m - \tilde{z}^{n,i}_m) \right) = a_m^{-1} \sum_{i=1}^{a_m} \text{Var}(\tilde{Z}^{n,i}_m - \tilde{z}^{n,i}_m) = a_m^{-1} \sum_{i=1}^{a_m} \mathbb{E}(\tilde{Z}^{n,i}_m)^2 - (\tilde{z}^{n,i}_m)^2 = a_m^{-1} \sum_{i=1}^{a_m} \mathbb{E}(\tilde{Z}^{n,i}_m)^2 - a_m^{-1} \sum_{i=1}^{a_m} (\tilde{z}^{n,i}_m)^2 = (\ast).
\]

By Fact 15 and Fact 16 we know that for some \( k > 0 \) we have
\[
|\tilde{z}^{n,i}_m| \lesssim e^{(n-1)/2} m |\mathcal{P}_{(n-1)m} f(p_m(i))| \lesssim m^k e^{(n-1)/2 - \mu m}.
\]

Therefore by assumption \( \alpha < 2\mu \) the second term of \( (\ast) \) disappears. By Fact 15 we have
\[
\mathbb{E}(Z^{n,i}_m)^2 = e^{-\alpha(n-1)m} \mathbb{E}(V_f p_m(i), (n-1)m).
\]

Using [8.3] it is easy to check that \( a_m^{-1} \sum_{i=1}^{a_m} \mathbb{E}(\tilde{Z}^{n,i}_m)^2 \to \sigma_f^2 \). Therefore, by the CLT, we know that
\[
S_m \to^d \mathcal{N}(0, \sigma_f^2),
\]
once we check the Lyapunov condition. To this end, we use (5.10) and (5.4), viz.,

\[ a_m^{-2} \sum_{i=1}^{a_m} \mathbb{E} [Z_{m,i}^n - z_{m,i}^n]^4 \lesssim a_m^{-2} \sum_{i=1}^{a_m} \mathbb{E} [z_{m,i}]^4 - k \mathbb{E} (Z_{m,i}^n)^k \lesssim a_m^{-1} + a_m^{-2} \sum_{i=1}^{a_m} 4 \mathbb{E} [Z_{m,i}^n]^k \]

\[ \lesssim a_m^{-1} + a_m^{-2} \sum_{i=1}^{a_m} 4 \mathbb{E} [Z_{m,i}^n]^k \lesssim a_m^{-1} + a_m^{-2} \sum_{i=1}^{a_m} (1 + \|p_m(i)\|)^n \lesssim a_m^{-1} m^kn \rightarrow 0. \]

Applying the above CLT to \(|z(t)|^{-1/2} \sum_{i=1}^{\|z(t)\|} (Z_{t,i}^n - z_{t,i}^n) 1(\|z_{t,i}\| < t)\) (and performing the same reasoning as in case of \(\chi_1\) and \(\chi_2\)) one can show that the limit of \(\chi_3\) is the same as the one of

\[ \chi_4(\theta_1, \theta_2, \theta_3; t) := \tilde{E}_\nu \exp \left\{ i \theta_1 \frac{\beta}{\alpha} e^{-\alpha t} |z_t| \right\} e^{-\theta_2^2/(2\sigma^2)} e^{-\theta_3^2/(2\sigma^2)}, \]

where \(\sigma^2\) is given by (5.11). Therefore, by Fact 21 we know that

\[ \chi_4(\theta_1, \theta_2, \theta_3; t) \rightarrow \left( \tilde{E}_\nu \exp \left\{ i \theta_1 \frac{\beta}{\alpha} W_\infty \right\} \right) e^{-\theta_2^2/(2\sigma^2)} e^{-\theta_3^2/(2\sigma^2)} = \left( \tilde{E}_\nu \exp \left\{ i \theta_1 V_\infty \right\} \right) e^{-\theta_2^2/(2\sigma^2)} e^{-\theta_3^2/(2\sigma^2)}. \]

Coming back step by step one sees that this proves that \(Z_0(t) \rightarrow^d \tilde{V}_\infty, G_1, G_2\), where the limit is as described in Theorem 6. This means that on \(Ext^\alpha\) we have convergence \(Z_0(t) \rightarrow^d (V_\infty, V_\infty^{-1/2} G_1, V_\infty^{-1/2} G_2)\). By standard arguments we also have \(Z_1(t) \rightarrow^d (V_\infty, V_\infty^{-1/2} G_1, V_\infty^{-1/2} G_2)\) which is equivalent to the thesis of the theorem.

6. PROOF OF THEOREM 6

We first gather properties of \(V^\alpha_f\), extending the list in Fact 18 in the case when \(\alpha > 2\mu\). We recall that \(\bar{f}(x) = f(x) - \langle f, \varphi \rangle\).

**Fact 24.** Let \(\alpha > 2\mu\) and \(f \in C(\mathbb{R}^d)\). Then, there exists \(C, n > 0\) such that

\[ e^{-2(\alpha-\mu n)} |V^\alpha_f(x, t)| \leq C(1 + \|x\|^n). \]

**Proof.** By (4.21), Fact 16 and using (1.19) we obtain

\[ |V^\alpha_f(x, t)| \lesssim \int_0^t \mathcal{P}_{t-s}^\alpha \left[ (\mathcal{P}_s^{\alpha \tilde{f}(\cdot)} \bar{f}(\cdot))^2 \right](x) ds + \int_0^t \mathcal{P}_{t-s}^\alpha \left[ (\mathcal{P}_s^{\alpha \bar{f}(\cdot)} \bar{f}(\cdot))^2 \right](x) ds \]

\[ + \int_0^t \mathcal{P}_{t-s}^\alpha \left[ u_f^{\alpha 2}(\cdot, s) \right](x) ds \lesssim \int_0^t e^{\alpha(t-s)} e^{2(\alpha-\mu)s} \mathcal{P}_{t-s} \left[ \left( 1 + \| \cdot \| n \right)^2 \right](x) ds \]

\[ + \int_0^t e^{\alpha(t-s)} e^{-2\alpha s} \mathcal{P}_{t-s} \left[ \left( 1 + \| \cdot \| n \right)^2 \right](x) ds \]

\[ + \int_0^t e^{2(\alpha-\mu)s} \mathcal{P}_{t-s} \left[ \left( 1 + \| \cdot \| 2n \right)^2 \right](x) ds \lesssim e^{2(\alpha-\mu)t} (1 + \|x\|^{2n}). \]

\[ \square \]

We assume that \(\Lambda\) has the backbone representation as described in Section 4.1. Although this is harmless when we prove the weak convergence in (5.24), some additional argument will be required in case of (5.3). This will be explained at the end of the proof. Our first aim is to prove the convergence of the spatial fluctuations. To this end we denote

\[ Y_1(t) := e^{-(\alpha-\mu)t} (\langle \Lambda_t, \tilde{f} \rangle - |\Lambda_t| \langle f, \varphi \rangle) = e^{-(\alpha-\mu)t} (\Lambda_t, \tilde{f}), \]

where as usual \(\tilde{f}(x) = f(x) - \langle f, \varphi \rangle\). We define

\[ Y_2(s, t) := e^{-(\alpha-\mu)t} \sum_{i=1}^{\|z_i\|} e^{-(\alpha-\mu)s} \langle \Gamma_{s,t}^{i \varphi}, \tilde{f} \rangle, \]
By (4.22) and using the same argument as in the previous proof (e.g. see (5.9)) one checks that

$$|Y_1(t + s) - Y_2(s, t)| \to 0 \quad \text{in probability},$$

if \(s, t \to +\infty\) and \(s \geq t\) (this is a sufficient condition). We write

$$Y_2(s, t) = e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor n \rfloor} (Z_s^{i,t} - z_s^{i,t}) + e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor n \rfloor} z_s^{i,t} =: Y_3(s, t) + Y_4(s, t),$$

where \(Z_s^{i,t} = e^{-(\alpha - \mu)s} \langle T_s^{i,t}, \tilde{f} \rangle\) and \(z_s^{i,t} = \mathbb{E} \left( Z_s^{i,t} | z_t \right)\). One checks that

\[
\mathbb{E} Y_3(s, t)^2 \leq e^{-2(\alpha - \mu)t} \mathbb{E} \left( \sum_{i=1}^{\lfloor n \rfloor} \left| z_t^{i} \right| \sum_{j=1}^{\lfloor n \rfloor} (Z_s^{i,t} - z_s^{i,t})(Z_s^{j,t} - z_s^{j,t}) \right) \leq e^{-2(\alpha - \mu)t} \mathbb{E} \left( \sum_{i=1}^{\lfloor n \rfloor} \sum_{j=1}^{\lfloor n \rfloor} \left| z_t^{i} \right| (Z_s^{i,t} - z_s^{i,t})(Z_s^{j,t} - z_s^{j,t}) \right),
\]

By (6.1) Fact 18 and Fact 19 we get \(\mathbb{E} \left( (Z_s^{i,t} - z_s^{i,t})^2 | z_t \right) \lesssim \|z_t(i)\|^n + 1\), for some \(n \geq 0\). And therefore by Fact 1 (23)], under assumption \(\alpha > 2\mu\), we obtain

$$\mathbb{E} Y_3(s, t)^2 \lesssim e^{-2(\alpha - \mu)t} \mathbb{E} \left( \sum_{i=1}^{\lfloor n \rfloor} \left| z_t^{i} \right| \right) \lesssim e^{-2(\alpha - \mu)t} e^{\alpha t} \to 0, \quad \text{as } t \to +\infty.$$

By (4.22) and (4.23) we have

\[
z_s^{i,t} = \frac{\beta}{\alpha} \left( 2e^{-\alpha s} \sinh(\alpha s) \right) \left( e^{\mu s} P_0 \tilde{f}(z_t(i)) \right) = \frac{\beta}{\alpha} \left( 2e^{-\alpha s} \sinh(\alpha s) \right) \left( e^{\mu s} P_0 \tilde{f}(z_t(i)) - z_t(i) \circ \langle \text{grad} f, \varphi \rangle \right)
\]

\[+ \frac{\beta}{\alpha} \left( 2e^{-\alpha s} \sinh(\alpha s) \right) (z_t(i) \circ \langle \text{grad} f, \varphi \rangle) = x_s^{i,t} + y_s^{i,t}.\]

This leads to

$$Y_4(s, t) = Y_5(s, t) + Y_6(s, t) := e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor n \rfloor} x_s^{i,t} + e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor n \rfloor} y_s^{i,t}.$$

By Fact 16 we have \(|x_s^{i,t}| \leq \tilde{C}(1 + \|z_t(i)\|^n)e^{-\mu s}\) for some \(n \geq 0\). Therefore

$$|Y_5(s, t)| \leq e^{-(\alpha - \mu)t} e^{-\mu s} \sum_{i=1}^{\lfloor n \rfloor} (1 + \|z_t(i)\|^n) = Ce^{\mu(t-s)} \to 0,$$
whenever if \( s, t \to +\infty \) and \( \frac{s}{t} \to +\infty \). To treat \( Y_\theta(s, t) \) we recall (3.3). Using Fact 21 we obtain

\[
Y_\theta(s, t) := \left( \frac{2\beta}{\alpha} e^{-\alpha s} \sinh(\alpha s) \right) e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor z_t \rfloor} (z_t(i) \circ (\grad f, \varphi))
= \left( \frac{2\beta}{\alpha} e^{-\alpha s} \sinh(\alpha s) \right) \left( (\grad f, \varphi) \circ e^{-(\alpha - \mu)t} \sum_{i=1}^{\lfloor z_t \rfloor} z_t(i) \right)
= \frac{\beta}{\alpha} (\grad f, \varphi) \circ I_\infty = (\grad f, \varphi) \circ H_\infty, \quad \text{a.s.}
\]
as \( s, t \to +\infty \). Reviewing the above steps one checks easily that

\[
Y_1(t) \to \langle \grad f, \varphi \rangle \circ H_\infty, \quad \text{in probability,}
\]
as \( t \to +\infty \). This was proved for a special (backbone) realization of the superprocess. Let us fix some other realization \( \tilde{X} \), denote \( Y_1(t) := e^{-(\alpha - \mu)t} \langle \tilde{X}_t, \tilde{f} \rangle \) and by \( H_\infty \) the limit of corresponding martingale (see (3.3)). From (6.3) we can conclude only that \( \tilde{Y}_1(t) \to \langle \grad f, \varphi \rangle \circ H_\infty \) in distribution. However, from (6.3) we know that \( (Y_1(t) - \langle \grad f, \varphi \rangle \circ H_\infty) \to 0 \) in probability. From this we conclude that \( (\tilde{Y}_1(t) - \langle \grad f, \varphi \rangle \circ \tilde{H}_\infty) \to 0 \) in distribution and hence also in probability. In this way (6.3) holds for any realization of the superprocess.

To conclude the proof we notice that to obtain the joint convergence in (3.4) one needs to use the same methods as in the proof of Theorem 4.

In this section we also present

**Proof. (of Fact 5)** The fact that \( H \) is a martingale follows directly from the Markov property of \( X \), (4.13) and (4.10), viz.,

\[
\mathbb{E}_\nu (X_t, id) = \int_{\mathbb{R}^d} u_{1d}^t(x, t) \nu(dx) = e^{\alpha t} \int_{\mathbb{R}^d} \mathcal{P}_t |id(x)| \nu(dx) = e^{(\alpha - \mu)t} \int_{\mathbb{R}^d} x \nu(dx),
\]
where \( id(x) = x \). By standard Laplace transform arguments one easily checks that \( H \) is \( L^2 \)-bounded if only \( e^{-2(\alpha - \mu)t} |u_{1d}^t| \leq c(x) \), for any \( t \geq 0 \) and some function \( c(x) \). We recall (4.15), Fact 16 and write

\[
e^{-2(\alpha - \mu)t} |u_{1d}^t(x, t)| \leq e^{-2(\alpha - \mu)t} \int_0^t \mathcal{P}_{t-s}^{\alpha} [e^{2(\alpha - \mu)s} t d.|(\cdot)|^2] (x) ds
= e^{-2(\alpha - \mu)t} \int_0^t e^{(\alpha - \mu)s} \mathcal{P}_{t-s}^{\alpha} [t d.|(\cdot)|^2] (x) ds \lesssim (1 + \|x\|^n).
\]

\[ \square \]

7. PROOF OF THEOREM 4

We first gather properties of \( V_f^k \), extending the list in Fact 18 in the case when \( \alpha = 2\mu \). We recall that \( \bar{f}(x) = f(x) - \langle f, \varphi \rangle \).

**Fact 25.** Let \( \alpha = 2\mu \) and \( f \in C(\mathbb{R}^d) \). Then for any \( x \in \mathbb{R}^d \) there is

\[
t^{-1/2} e^{-(\alpha/2)t} V_f^1(x, t) \to 0, \quad \text{as } t \to +\infty,
\]
\[
t^{-1} e^{-\alpha t} V_f^2(x, t) \to \sigma_f^2, \quad \text{as } t \to +\infty,
\]
where \( \sigma_f^2 \) is given by (3.2). We have \( \sigma_f^2 < +\infty \). Moreover, there exists \( l > 0 \) such that

\[
\sup_{\|x\| \leq l} |t^{-1} e^{-\alpha t} V_f^2(x, t) - \sigma_f^2| \to 0, \quad \text{as } t \to +\infty.
\]
Finally there exists $C, n > 0$ such that
\[(7.4) \quad |V_j^s(x, t)| \leq Ct^2 e^{2\alpha t}(\|x\|^{2n} + 1).
\]

**Proof.** (7.1) follows by (4.23) and Fact 19. Indeed
\[t^{-1/2} e^{-(\alpha/2)t} V_j^1(x, t) = e^{-(\alpha/2)t} t^{-1/2} e^{\alpha t} |P_t \tilde{f}(x)| \leq t^{-1/2} e^{(\alpha/2)t} e^{-\mu t} (1 + \|x\|^{2n}) \to 0.
\]

Using (4.23) we have
\[t^{-1} e^{-\alpha t} V_j^2(x, t) = e^{-\alpha t} t^{-1} \beta \int_0^t P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 - 2\beta \left( \mathcal{P}_s^\alpha \tilde{f}(\cdot) \right)^2 - 2\alpha u_j^2(s, s) \right] (x) ds
\]
\[= t^{-1} \beta \int_0^t e^{-\alpha s} P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 - 2\beta \left( \mathcal{P}_s^\alpha \tilde{f}(\cdot) \right)^2 - 2\alpha u_j^2(s) \right] (x) ds
\]
\[= t^{-1} \beta \int_0^t P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 \right] (x) ds + t^{-1} \beta \int_0^t e^{-\alpha s} P_{t-s} \left[ -2\beta \left( \mathcal{P}_s^\alpha \tilde{f}(\cdot) \right)^2 - 2\alpha u_j^2(s) \right] (x) ds.
\]

By Fact 17 it is easy to check that the second term converges to 0. Using Fact 10 (and 4.11 in particular) we have
\[t^{-1} \beta \int_0^t P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 \right] (x) ds \leq t^{-1} \int_0^t P_{t-s} \left[ e^{-\mu s} (1 + \| \cdot \| n) (1 + \| \cdot \| n) \right] (x) ds \leq t^{-1} (1 + \|x\|^{2n}) \to 0.
\]

Using standard integral arguments one easily shows that $t^{-1} \beta \int_0^t P_{t-s} \left[ 2\beta \left( \cdot \circ (\text{grad} f, \varphi) \right)^2 \right] (x) ds \to \sigma_f^2$ and that $\sigma_f^2 < +\infty$.

Now in order to prove (7.4) it is enough to show
\[\sup_{\|x\| \leq t} t^{-1} e^{-\alpha t} |V_j^2(x, t) - V_j^2(0, t)| \to 0, \quad t \to +\infty.
\]

We will use the same decomposition as before and concentrate only on the first term (leaving other, simpler ones, to the reader), that is, we will show that
\[\sup_{\|x\| \leq t} t^{-1} \int_0^t e^{-\alpha s} P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 \right] (x) ds \to 0.
\]

By Fact 10
\[\sup_{\|x\| \leq t} t^{-1} \int_{t-t^{-1/2}}^t e^{-\alpha s} P_{t-s} \left[ 2\beta \left( \mathcal{P}_s \tilde{f}(\cdot) \right)^2 \right] (x) ds \leq \sup_{\|x\| \leq t} t^{-1} \int_{t-t^{-1/2}}^t e^{(\alpha-\mu)s} P_{t-s} \left[ ((1 + \| \cdot \| n)^2 \right] (x) ds \lesssim t^{-1/2} t^{1/2} \to 0,
\]
if only $l$ is small enough. Further we proceed as in (5.7) putting $t - t^{1/2}$ instead of $t/2$.

To prove (7.4) we will follow the same route as in the proof of Fact 23 let us recall (5.3). Analogously we omit terms $u_j^*$. For $k = 3$ we calculate
\[\int_0^t P_{t-s} \left[ \left| V_j^2(\cdot, s) \right|^2 \right] (x) ds \leq e^{\alpha t} \int_0^t P_{t-s} e^{-\alpha s} \left[ |V_j^2(\cdot, s)| e^{(\alpha-\mu)s} (1 + \| \cdot \| n) \right] (x) ds \lesssim e^{3\alpha/2} t (1 + \| \cdot \|^{3n}).
\]
We proceed to $k = 4$. Again following the framework of Fact 24 we write
\[
\int_0^t \mathcal{P}_t \alpha_s \left[ \left| V_f(\cdot, s)V_f(\cdot, s) \right| \right] (x) ds + \int_0^t \mathcal{P}_t \alpha_s \left[ \left| V_f^2(\cdot, s) \right| \right] (x) ds
\leq \int_0^t \mathcal{P}_t \alpha_s \left[ s e^{(3n/2)s} \left( 1 + \| \cdot \|^{3n} \right) e^{(\alpha - \mu)s} \left( 1 + \| \cdot \|^{3n} \right) \right] (x) ds
+ \int_0^t \mathcal{P}_t \alpha_s \left[ s e^{\alpha s} \left( 1 + \| \cdot \|^{2n} \right) \right] (x) ds \lesssim t^2 e^{2\alpha t} \left( 1 + \| \cdot \|^{3n} \right).
\]

Finally we are able to prove Theorem 4. We start with the following random vector
\[
Z_4(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{-(\alpha/2)t} (|\Lambda_t| - e^{\alpha t} V_\infty), t^{-1/2} e^{-(\alpha/2)t} (\Lambda_t, \tilde{f}) \right).
\]
Let $n \in \mathbb{N}$ to be fixed later; the limit of $Z_4(nt)$ is obviously the same as the one of $Z_4(t)$. Using Fact 4 we obtain that $V_{nt} - V_t \to 0$ a.s. Therefore the limit of $Z_4(t)$ coincides with the one of
\[
Z_4^n(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{-(\alpha/2)t} (|\Lambda_t| - e^{\alpha t} V_\infty), (nt)^{-1/2} e^{-(\alpha/2)t} (\Lambda_{nt}, \tilde{f}) \right).
\]
We denote $Z_4^{n,i} := \left( \frac{\alpha}{n} \right)^{1/2} \left((n - 1)t\right)^{-1/2} e^{-(n-1)(\alpha/2)t} (\tilde{I}_{(n-1)t}, \tilde{f})$ and $z_t^{n,i} := E \left( Z_t^{n,i} | z_t \right)$. We keep the notation consistent to the one in the proof of Theorem 3 but we add superscript $n$ as this parameter will vary. Using argument of the aforementioned proof we have that the limit of $Z_4^n$ is the same as the one of
\[
Z_4^n(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{(\alpha/2)t} \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} (1 - V_{i,t}^\prime), e^{-(\alpha/2)t} \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} (Z_t^{n,i} - z_t^{n,i}) + H_t^n \right),
\]
where $H_t^n := e^{-(\alpha/2)t} \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} z_t^{n,i}$. Applying 1.23 to $z_t^{n,i}$ we obtain
\[
H_t^n = 2 \frac{\beta}{\alpha} (nt)^{-1/2} e^{-((n-1)(\alpha/2)t) \sinh((n - 1)at) \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} \mathcal{P}_{(n-1)t} \tilde{f}(z_t(i)).
\]
Using 1.24 we get
\[
\mathbb{E} \left( H_t^n \right)^2 \lesssim (nt)^{-1} e^{\alpha(n-2)t} V_\infty^2 (x, t),
\]
where $g(x) := \mathcal{P}_{(n-1)t} \tilde{f}(x)$. By 1.53 we have $|u_t^{n,1}(x, s)| \lesssim |e^{-\alpha s} \mathcal{P}_{(n-1)t} \tilde{f}(x)| \lesssim e^{-\alpha s} e^{-\mu(n-1)(t+s)} (1 + \| x \|^{2n}) \lesssim e^{-\alpha s} e^{-\mu(n-1)t} (1 + \| x \|^{2n})$. As usual $u_t^{n,1}$ can be skipped. Further
\[
\int_0^t \mathcal{P}_t \alpha_s \left[ (\mathcal{P}_s \alpha g(\cdot))^2 \right] (x) ds = e^{\alpha t} \int_0^t e^{\alpha s} \mathcal{P}_{t-s} \left[ (\mathcal{P}_{s+(n-1)t} \tilde{f}(\cdot))^2 \right] (x) ds \lesssim e^{\alpha t} \int_0^t e^{\alpha s} \mathcal{P}_{t-s} \left[ e^{-\mu(s+(n-1)t)} (1 + \| \cdot \|^{2n})^2 \right] (x) ds
= e^{-\alpha(n-2)t} \int_0^t \mathcal{P}_{t-s} \left[ ((1 + \| \cdot \|^{2n})^2 \right] (x) ds \lesssim te^{-\alpha(n-2)t}.
\]
Therefore there exists a constant $C > 0$ such that
\[
(7.5) \quad \mathbb{E} \left( H_t^n \right)^2 \lesssim \frac{C}{n}.
\]
In fact also the reverse inequality holds, hence $H_t^n$ is not negligible (unlike in the proof of Theorem 3). We skip it for a moment and denote
\[
Z_4^n(t) := \left( e^{-\alpha t} |\Lambda_t|, e^{(\alpha/2)t} \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} (1 - V_{i,t}^\prime), e^{-(\alpha/2)t} \sum_{i=1}^{\lfloor |\Lambda_t| \rfloor} (Z_t^{n,i} - z_t^{n,i}) \right).
\]
Further, we consider
\[
I(t) := \mathbb{E} \left( e^{-(\alpha/2)t} \left( \sum_{i=1}^{|z_t|} (Z_t^{n,i} - z_t^{n,i}) 1\{\|z_t(i)\| \geq t\} \right) \right)^2
\]
\[
= e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{|z_t|} \sum_{j=1}^{|z_t|} \mathbb{E} \left( (Z_t^{n,i} - z_t^{n,i})(Z_t^{n,j} - z_t^{n,j}) 1\{\|z_t(i)\| \geq t\} 1\{\|z_t(j)\| \geq t\} \right) \right)
\]
\[
= e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{|z_t|} \mathbb{E} \left( (Z_t^{n,i} - z_t^{n,i})^2 1\{\|z_t(i)\| \geq t\} \right) \right) = e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{|z_t|} (Z_t^{n,i})^2 1\{\|z_t(i)\| \geq t\} \right)
\]
\[+ e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{|z_t|} (Z_t^{n,i})^2 1\{\|z_t(i)\| \geq t\} \right) = I_1(t) + I_2(t).
\]
Our aim is to prove that both $I_1(t) \to 0$, $I_2(t) \to 0$ and hence $I(t)$ is negligible. Let us recall (4.23), Fact (16) and write
\[
I_2(t) \lesssim t^{-1} e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{z_t} 1^{z_t-101} (e^{-(n-1)\alpha t} t^\beta (Z_t^{n,i})^2 1\{\|z_t(i)\| \geq t\}) \right)
\]
\[
\lesssim t^{-1} e^{-(n-1)\alpha t} \mathbb{E} \left( \sum_{i=1}^{z_t} (e^{-(n-1)\alpha t} t^\beta (\|z_t(i)\|)^k)^2 1\{\|z_t(i)\| \geq t\}) \right)
\]
\[
\lesssim t^{-1} e^{-\alpha t} \mathbb{E} \left( \sum_{i=1}^{z_t} (1 + \|z_t(i)\|)^{2k} 1\{\|z_t(i)\| \geq t\}) \right)
\]
\[= t^{-1} \mathbb{E} \left( (1 + \|z_t(i)\|)^{2k} 1\{\|z_t(i)\| \geq t\}) \right) \to 0.
\]
To estimate $I_1$, we recall (4.24) and write
\[
\mathbb{E} \left( (\Pi_{(n-1)t}^{\alpha} f(i))^2 \right) = \frac{\beta}{\alpha} \int_0^{(n-1)\alpha t} \mathcal{P}_{(n-1)t-s}^{\alpha} \left( 2\beta \left( \mathcal{P}_{(n-1)t-s}^{\alpha} f(i) \right)^2 - 2\beta \left( \mathcal{U}^{\alpha}_{(n-1)t-s} f(i) \right)^2 - 2\alpha \mathcal{U}^{\alpha}_{(n-1)t-s} f(i)^2 \right) (z_t(i)) ds.
\]
As usual it is enough to focus on the first term
\[
\frac{\beta}{\alpha} \int_0^{(n-1)\alpha t} \mathcal{P}_{(n-1)t-s}^{\alpha} \left( 2\beta \left( \mathcal{P}_{(n-1)t-s}^{\alpha} f(i) \right)^2 \right) (z_t(i)) ds
\]
\[\lesssim e^{-(n-1)\alpha t} \int_0^{(n-1)\alpha t} \mathcal{P}_{(n-1)t-s}^{\alpha} \left( e^{\alpha s} e^{-\alpha s} (1 + \|\cdot\|)^k \right)^2 \left( (z_t(i)) ds \right)
\]
\[\lesssim e^{-(n-1)\alpha t} \int_0^{(n-1)\alpha t} \mathcal{P}_{(n-1)t-s} \left( (1 + \|\cdot\|)^k \right)^2 \left( (z_t(i)) ds \right) \lesssim e^{(n-1)\alpha t} (n-1)t (1 + \|z_t(i)\|)^{2k}.
\]
Therefore
\[
\mathbb{E} \left( (Z_t^{n,i})^2 \right) = (nt)^{-1} e^{(n-1)\alpha t} \mathbb{E} \left( (\Pi_{(n-1)t}^{\alpha} f(i))^2 \right) \lesssim 1 + \|z_t(i)\|^{2k}.
\]
Using the fact that $z_t(1)$ is Gaussian with bounded variance we conclude
\[
I_1(t) \lesssim e^{-\alpha t} \sum_{i=1}^{z_t} (1 + \|z_t(i)\|)^{2k} 1\{\|z_t(i)\| \geq t\} = \mathbb{E} (1 + \|z_t(1)\|^{2k} 1\{\|z_t(1)\| \geq t\})
\]
\[\lesssim (\mathbb{E} (1 + \|z_t(1)\|^{4k}))^{1/2} \mathbb{E} (\|z_t(1)\| \geq t)^{1/2} \to 0.
\]
Therefore the limit of $Z^n_3(t)$ is the same as the one of
\[ Z^n_3(t) := \left( \frac{\beta}{\alpha} e^{-\alpha t} |z_t|, \| X_{nt} \| \right) \to \left( 1 - V^i_\infty, |z_t|^{-\alpha/2} \sum_{i=1}^{\| z_t \|} (Z^{n,i}_t - z^{n,i}_t) \mathbbm{1}_{\| z_t \| < t} \right). \]

Further we proceed along the lines of the proof of Theorem 4 using Fact 25 instead of Fact 23. There are some minor changes. We describe two of them which are not completely obvious. [5,10] now writes as
\[ |z^n_m| \lesssim m^{-1/2} e^{((n-1)/2)m} \mathbbm{P}_1(m), \]
Decreasing $l$ if necessary, let us note that any other part of the proof works for any “small” $l$, we make $2kl - 1/2 < 0$ and therefore get the convergence to 0. The second change is that
\[ a_m^{-1} \sum_{i=1}^{a_m} \mathbbm{E} \left( \hat{Z}^{n,i}_m \right)^2 \to \left( \frac{n - 1}{n} \right) \sigma_f^2. \]

The additional term $(n - 1)/n$ stems from the corresponding term in definition of $Z_t^{n,i}$. Finally we arrive at
\[ (7.6) \quad Z^n_3(t) \to^d \left( V^i_\infty, \sqrt{V^i_\infty} G_1, \left( \frac{n - 1}{n} \right)^{1/2} \sqrt{V^i_\infty} G_2 \right), \quad \text{as } t \to +\infty, \]
conditionally on $Ext^\vee$, where $G_1, G_2$ are the same as in Theorem 6.

To finish the proof we need some topological notions. Let $\mu_1, \mu_2$ be two probability measures on $\mathbb{R}$, and Lip(1) be the space of continuous functions $\mathbb{R} \to [0,1]$ with the Lipschitz constant smaller or equal to 1. We define
\[ m(\mu_1, \mu_2) := \sup_{g \in \text{Lip}(1)} |\langle g, \mu_1 \rangle - \langle g, \mu_2 \rangle|. \]

It is well known that $m$ is a distance equivalent to weak convergence. One easily checks that when $\mu_1, \mu_2$ correspond to two random variables $X_1, X_2$ on the same probability space then we have
\[ m(\mu_1, \mu_2) \leq \|X_1 - X_2\|_1 \leq \sqrt{\|X_1 - X_2\|_2}. \]
We denote the law of of the triple the limit (7.6) by $L_n$ and the law of $(W, \sqrt{W} G_1, \sqrt{W} G_2)$ by $L_\infty$.

Let us fix $\ve > 0$. We choose $n$ such that $\sqrt{C/n} \leq \ve$, where $C$ is the same as in (7.5). Hence, for any $t$ we have $m(Z^n_3(t), Z^n_3(t)) \leq \ve$.

By the fact that $Z^n_3$ have the same limit there exist $T^n_1 > 0$ such that for any $t > T^n_1$ we have $m(Z^n_3(t), L_n) \leq \ve$. Analogously $Z^i_1$ has the same limit as $Z^n_3$ therefore there exists $T^n_2$ such that for any $t > T^n_2$ we have $m(Z^n_3(t), Z^n_3(t)) \leq \ve$. Applying the triangle inequality we obtain
\[ m(Z_1(t), L_\infty) \leq 3\ve, \]
if only $t \geq T^n_1 \lor T^n_2$. The proof is concluded since $\ve$ can be taken arbitrarily small.

**Appendix.**

*Proof. (of Fact 1)* Using (1.4) and (1.5) one easily checks that
\[ \mathbbm{E}_\nu(e^{-\theta |X_t|}) = \exp (-|\nu| v_\theta(t)), \]
where
\[ v_\theta(t) = \alpha v_\theta(t) - \beta v_\theta(t)^2, \quad v_\theta(0) = \theta. \]
This equation can be solved explicitly, viz.
\[ (7.7) \quad v_\theta(t) = \frac{e^{\alpha t} \alpha}{C_\theta + e^{\alpha \theta} \beta}, \quad C_\theta := \frac{\alpha - \beta \theta}{\theta}. \]
By direct calculations we obtain $\mathbbm{E}_\nu |X_t| = e^{\alpha t} |\nu|$. Hence, using the Markov property of $X$ we conclude that $V_t$ is a martingale.
Finally, \( z \) again by Fact 18 we have \( \psi \). Analogously using (4.13) (with \( \text{id} \)) one checks that
\[
-\frac{\alpha}{\beta} \frac{\epsilon_t}{\alpha} \to \frac{\alpha}{\alpha^{\theta_{1+\beta}} + \epsilon_t \beta} \to \frac{\alpha}{\alpha^{\theta_{1+\beta}} + \epsilon_t \beta} \quad \text{as } t \to +\infty.
\]

Now we check that \( \frac{\alpha}{\beta} \frac{\epsilon_t}{\alpha} \to \alpha/\beta \) as \( \theta \to +\infty \), hence \( \{ V_\infty = 0 \} = \text{Ext} \). One may verify representation (2.35) by direct calculations (although it is easier guessed from the backbone construction). In order to prove \( L^2 \) convergence one needs to apply (4.15) with \( f = 1 \).

**Proof.** (of Fact 21) The fact that \( W \) is a martingale is a well-known fact from the Galton-Watson theory (see e.g. [3]). The properties of \( I \) are proved in [1] in Section 3.4.3 (where it is under name \( H \)). Having a.s. convergence of \( H \) and \( I \) in order to prove (4.28) one needs to show that
\[
H_{nt} - \frac{\beta}{\alpha} I_t \to 0,
\]
in probability. We denote \( \text{id}(x) = x \) and fix some \( n > 2 \). Using (4.17) we obtain
\[
H_{nt} - \frac{\beta}{\alpha} I_t \equiv e^{-n(\alpha-\mu)t}(X_{nt}, \text{id}) + e^{-n(\alpha-\mu)t}(D_{(n-1)t}, \text{id}) + e^{-n(\alpha-\mu)t}(\Gamma_{(n-1)t}, \text{id}) - e^{-n(\alpha-\mu)t}\sum_{i=1}^{\infty} \frac{\beta}{\alpha} z_i(i)
\]
where \( Z_{i,t} := e^{-n(1-n)(\alpha-\mu)t}(\Gamma_{(n-1)t}, \text{id}) \) and \( z_{i,t} := \mathbb{E} (Z_{i,t} | z_i) \). By (4.20) we have
\[
\mathbb{E}_\nu |I_2(t)| \leq e^{-n(\alpha-\mu)t} \mathbb{E}(D_{(n-1)t}, \text{id}) \leq e^{-n(\alpha-\mu)t} e^{(2-n)t} \to 0.
\]
Analogously using (4.19) (with \( \psi^* \) as explained in Section 4.1) one can show that \( \mathbb{E}_\nu |I_1(t)| \to 0 \), at \( t \to +\infty \). In order to treat \( I_3 \) we perform the same calculations as for (4.2) and obtain
\[
\mathbb{E} I_2(t)^2 \equiv e^{-2(\alpha-\mu)t} \mathbb{E} \left( \sum_{i=1}^{\infty} \mathbb{E} \left( (Z_{i,t} - z_{i,t})^2 | z_i \right) \right).
\]
Using (4.10) and Fact 18 one checks that
\[
\| z_{i,t} \| \leq e^{-n(1-n)(\alpha-\mu)t} \sinh((n-1)\alpha t)e^{-(n-1)\mu t} \| z_i(i) \| \leq \| z_i(i) \|.
\]
Again by Fact 18 we have \( \mathbb{E} \left( (Z_{i,t}^2) | z_i \right) \leq (1 + \| z_i \|^2) \). Therefore, under assumption \( \alpha > 2\mu \) we get
\[
\mathbb{E} I_2(t)^2 \equiv e^{-2(\alpha-\mu)t} e^{\alpha t} = e^{-2(\alpha-\mu)t} \to 0.
\]
Finally, \( z_{i,t} - z_i(i) = \frac{\beta}{\alpha} e^{-2(\alpha-\mu)t} z_i(i) \) which implies that \( I_3(t) \to 0 \). These together yield that (4.28).

(4.28) can be proven using a very similar, but simpler, reasoning hence is skipped.

In the following proof we will need

**Lemma 26.** Let \( X \) be an a.s. positive random variable. If there exists an analytic function \( w : (-\infty, a) \to \mathbb{R}_+, \) \( a > 0 \) such that
\[
w(\theta) = \mathbb{E} e^{\theta X}, \quad \text{for } \theta \leq 0,
\]
then \( w(\theta) = \mathbb{E} e^{\theta X} \) holds on \((-\infty, a_0)\) for some \( a_0 > 0\).
Proof. There exists a sequence \( \{a_n\}_{n \geq 0} \) such that
\[ w(\theta) = \sum_{n \geq 0} \frac{a_n}{n!} \theta^n \]
and the series is absolutely convergent in some interval \([−\epsilon, \epsilon]\). For any \( \lambda \in (−\epsilon, 0) \) we have (we use the positivity here):
\[ w^{(n)}(\theta) = EX^n e^{\theta X}. \]
Passing to the limit (which is valid by the monotone Lebesgue theorem) we get
\[ a_n = w^{(n)}(0) = EX^n. \]
Therefore all the moments exists. Let us now take some \( \theta \in (0, \epsilon) \) obviously we have
\[ w(\theta) = \lim_{k \to +\infty} \sum_{n=0}^{k} \frac{a_n}{n!} \theta^n = \lim_{k \to +\infty} \sum_{n=0}^{k} \frac{EX^n}{n!} \theta^n = \sum_{n=0}^{\infty} \frac{EX^n}{n!} \theta^n = \sum_{n=0}^{\infty} \frac{EX^n}{n!} \theta^n = \sum_{n=0}^{\infty} \frac{EX^n}{n!} \theta^n = EX e^{\theta X}, \]
where we used the Fubini theorem. Extending the equality to the whole negative axis follows by a standard argument. □

Fact 27. Let \( T > 0 \), there exists \( \theta > 0 \) such that
\[ \mathbb{E}e^{\theta |X_t|} < +\infty, \]
for any \( t < T \).

Proof. We denote \( w(\theta, t) := \mathbb{E}e^{\theta |X_t|} \), which so far is well-defined, for \( \theta < 0 \). Using (7.7) it is easy to check that for any \( T > 0 \) there exists \( \epsilon > 0 \) the functions above are analytic on \((−\epsilon, \epsilon)\) (one has to ensure that the denominators are bounded away from 0). Now the conclusion holds by Lemma 26. □

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References

[1] R. Adamczak and P. Miłos. CLT for Ornstein-Uhlenbeck branching particle system. arXiv:1111.4559, 2011.
[2] R. Adamczak and P. Miłoś. U-statistics of Ornstein-Uhlenbeck branching particle system. arXiv:1111.4560, 2011.
[3] Krishna B. Athreya and Peter E. Ney. Branching processes. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.
[4] V. Bansaye, J.-F. Delmas, L. Marsalle, and V. C. Tran. Limit theorems for Markov processes indexed by continuous time Galton-Watson trees. arXiv:0911.1973, 2009.
[5] J. Berestycki, A. Kyprianou, and A. Murillo. The prolific backbone for supercritical superdiffusions. Stoch. Proc. Appl., 121(6):1315–1331, 2011.
[6] E. B. Dynkin. Superprocesses and partial differential equations. Ann. Probab., 21(3):1185–1262, 1993.
[7] E. B. Dynkin. Diffusions, Superdiffusions, and Partial Differential Equations. American Mathematical Society, 2002.
[8] E. B. Dynkin and S. E. Kuznetsov. N-measures for branching exit Markov systems and their applications to differential equations. Probab. Theory Related Fields, 130(1):135–150, 2004.
[9] Eugene B. Dynkin. An Introduction to Branching Measure-Valued Processes, volume 6 of CRM Monograph Series. American Mathematical Society, 1994.
[10] J. Englander and A. Winter. Law of large numbers for a class of superdiffusions. Ann. Inst. H. Poincaré, 42(2):171–185, 2006.
[11] A. Etheridge. An Introduction to Superprocesses. American Mathematical Society, 2000.
[12] J. F. Le Gall. Spatial Branching Processes, Random Snakes, and Partial Differential Equations. Birkhauser, 1999.
[13] R. Hardy and S.C. Harris. A spine approach to branching diffusions with applications to Lp-convergence of martingales. Séminaire de Probabilités XLII, 2009.