Quantum Hall effect in carbon nanotubes

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We investigate the effects of a transverse magnetic field on the transport properties of carbon nanotubes, making use of a long-wavelength description in terms of Dirac fermion fields. For values of the magnetic length smaller than the nanotube radius, we observe that the electronic states organize into incipient Landau subbands, with a highly degenerate level at zero energy. We show that only the states in dispersive branches, localized at the flanks of the nanotube, are able to transport current in the longitudinal direction. This is at the origin of the quantization of the Hall conductivity, that turns out to be given by even multiples of $2e^2/h$. We also analyze the effects of the electron-electron interaction, showing that the magnetic field induces a suppression of the electronic correlations, reflected in particular in the enhancement of the tunneling density of states near the Fermi level.

Two-dimensional carbon compounds with $sp^2$ bonding have attracted recently much attention, due to the experimental observation of a number of novel electronic properties. It has been possible to measure the transport properties of a single layer of graphite (so-called graphene), providing evidence that the quasiparticles have a conical dispersion around discrete Fermi points\textsuperscript{[1, 2]}. This seems to be at the origin of remarkable features of the resistivity as well as of the Hall conductivity. Carbon nanotubes can be also considered as the result of wrapping up the graphene sheet, leading to systems where the transport seems to be ballistic under certain conditions\textsuperscript{[8]}, as a consequence of the suppression of the scattering between different low-energy subbands\textsuperscript{[4, 5]}.

The metallic carbon nanotubes and the graphene sheet have in common that their low-energy electronic dispersion is governed by a massless Dirac equation, around each of the two Fermi points of the undoped systems\textsuperscript{[6, 7, 8]}. The low-energy electronic states can be encoded into a couple of Dirac fermion fields, and the appearance of an additional pseudo-spin quantum number gives rise to important effects in the spectrum. This new degree of freedom has allowed us to understand, for instance, the degeneracy of the molecular orbitals in the fullerenes\textsuperscript{[9]}, the quantization rule of the Hall conductivity in graphene\textsuperscript{[10, 11]}, or the properties of the polarizability in carbon nanotubes\textsuperscript{[12]}.

In this paper we investigate the effects of a transverse magnetic field on the transport properties of the carbon nanotubes, making use of the description of the electronic states in terms of Dirac fermion fields. While it is well-known the effect of a magnetic field parallel to the tube axis, the transport properties of carbon nanotubes under a transverse magnetic field are less understood\textsuperscript{[13]}. We will see that carbon nanotubes of sufficiently large radius may also have a quantum Hall regime, with a quantized Hall conductivity $\sigma_{xy}$. In the case of graphene, it has been shown that $\sigma_{xy}$ has plateaus at odd multiples of $2e^2/h$\textsuperscript{[10, 12]} as a consequence of the peculiar Dirac spectrum of 2D graphene\textsuperscript{[10, 11]}. We will find that the different topology of the carbon nanotubes leads instead to a quantization in even steps of the quantity $2e^2/h$.

For the Hall regime to arise, the diameter of the nanotubes has to be larger than the magnetic length $l = \sqrt{\hbar c/eB}$, since this is the typical size of the localized states in Landau levels. Suitable conditions can be already found in thick multi-walled nanotubes for magnetic fields $B \gtrsim 1$ T, which correspond to magnetic lengths $l \lesssim 30$ nm. We will see that, for $l < R$, the eigenstates of the carbon nanotube organize into incipient Landau subbands, with a highly degenerate level at zero energy. The branches with linear dispersion correspond to states localized at the flanks of the nanotube, carrying quantized currents which are responsible for the conductivity along the longitudinal dimension of the nanotube.

In order to establish a relation with the quantum Hall effect in graphene, it is convenient to set up an approach focusing on the features of the states over distances much larger than the C-C distance $a$. The low-energy band structure can be obtained in graphene by taking a continuum limit in which the momenta are much smaller than the inverse lattice spacing $1/a$\textsuperscript{[9, 14, 15]}. In the case of carbon nanotubes under magnetic field, a sensible continuum limit requires also that $l \gg a$, so that lattice effects can be disregarded. In that limit, we obtain a simple field theory of Dirac spinors coupled to the magnetic field, allowing us to investigate different features of the quantum Hall effect in the tubular geometry.

We illustrate the long-wavelength limit in the case of zig-zag nanotubes. These have a unit cell with length $3a$, containing four transverse arrays of $N$ carbon atoms at different longitudinal positions $x_i$, $i = 1, \ldots, 4$. We introduce the Fourier transform of the electron operator $c(x_i, n)$ with respect to the position $n = 1, 2, \ldots, N$ in the
transverse section
\[ c(x, n) \approx \sum_p d_p(l; z) e^{i2\pi np/N} \]  
(1)

where \( l \in Z \) runs over the different cells. The index \( p \) labels the different 1D subbands, \( p = 0, 1, \ldots, N - 1 \). Their dispersion can be obtained from the diagonalization of a 1D system with four orbitals per unit cell. This leads in general to massive subbands with parabolic dispersion, with a gap \( 2\Delta_p = 2t[1 - 2\cos((\pi p)/N)] \), where \( t = 3 \) eV is the tight-binding element between two nearest-neighbor carbon atoms. The dispersive branches can be decoupled from the high-energy branches that appear near the top of the spectrum. It turns out that the low-energy dispersion corresponds to a reduced 2-component spinor, with a hamiltonian depending on the subband index \( p \) and the longitudinal momentum \( k \)

\[ \mathcal{H}_p^{\theta} = \delta_{p,p'} \begin{pmatrix} \hbar v_F k + \Delta_p & \hbar v_F k - h \Delta_p \\ h c \Delta_p & -h \Delta_p \end{pmatrix} \]  
(2)

where the Fermi velocity is \( v_F = 3ta/2\hbar \).

In the presence of a transverse magnetic field, the electron field picks up in general a factor \( e^{i\phi} \) when transported between nearest-neighbor sites of the carbon lattice, with \( \phi \propto a(e/hc)BR \sin(2\pi n/|N|) \). In the continuum limit, characterized by \( (e/hc)BRa \ll 1 \), we can linearize in the strength of the magnetic field. This introduces an interaction which is nondiagonal in the space of the different subbands, and that can be represented by the operator

\[ \Delta \mathcal{H}_{p,p'} = \delta_{p,p'\pm1} \begin{pmatrix} \pm iv_F e^{BR/2} & 0 \\ -iv_F e^{BR/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  
(3)

The total hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \Delta \mathcal{H} \) can be more easily expressed when acting on the Dirac spinor \( \Psi(k; \theta) \) depending on the angular variable \( \theta \) around the tube.

In this basis there are in fact two different sectors describing states about the two angular momenta \( \pm \hbar \) with vanishing gap, \( \Delta_{\pm\hbar} \approx 0 \). The hamiltonian is in either sector

\[ \mathcal{H} = \begin{pmatrix} \hbar v_F k + iv_F e^{BR/2} \sin(\theta) & -i(\hbar v_F/a) \partial_\theta \\ -i(\hbar v_F/a) \partial_\theta & -\hbar v_F k - \hbar v_F e^{BR/2} \sin(\theta) \end{pmatrix} \]  
(4)

where the periodic modulation matches with the orientation of a magnetic field normal to the nanotube surface at \( \theta = 0 \). This hamiltonian resembles that for a nanotube in a transverse electric field [12], although it poses the additional difficulty that the perturbation \( \Delta \mathcal{H} \) hybridizes positive and negative energy eigenstates of \( \mathcal{H}_0 \).

Expression (4) corresponds actually to the Dirac hamiltonian with the usual prescription for the coupling to the vector potential. We have checked that, as expected, the eigenstates of (4) provide a good approximation to the low-energy band structure of the carbon nanotubes for \( aR/l^2 \ll 1 \).

It can be shown that, for intermediate values of the magnetic length \( l \approx R \), the spectrum starts to develop a flat Landau level at zero energy, similar to that in the case of graphene. According to the mentioned doubling of the subbands, there is a four-fold degeneracy for each momentum \( k \) in the flat level. This extends into dispersive branches with particle- and hole-like character, at each side in momentum space. A typical band structure is represented in Fig. 1. It can be checked that the energy levels at \( k = 0 \) follow the quantization rule \( \varepsilon_n \propto \sqrt{n} \), which is peculiar of graphene [13]. The plot corresponds to a zig-zag nanotube, but it can be shown that the shape of the band structure remains the same for other geometries, with the two valleys at zero energy (that appear superposed in Fig. 1) expanding in general around the two Fermi points of the system at \( B = 0 \).

A remarkable point is that each eigenfunction of (4) is in general localized around a certain value of the angular variable \( \theta \). The zero-energy states at \( k = 0 \), for instance, have gaussian wave functions localized at \( \theta = 0 \) or \( \theta = \pi \). For positive (negative) longitudinal momentum, the zero-energy states are localized at angles between 0 and \( \pi/2 \) (\( -\pi/2 \)), or between \( \pi \) and \( \pi/2 \) (\( -\pi/2 \)), depending on the subband chosen. Quite interestingly, the states in the dispersive branches have gaussian wave functions centered about \( \pi/2 \) (for a right branch) or \( -\pi/2 \) (for a left branch).

The localization of the states in the dispersive branches at the flanks of the tube suggests that, despite having no boundary, the carbon nanotube may support edge states in similar fashion as in systems with planar geometry. To check this fact, one may compute the current flowing in the longitudinal direction for the different states. For the Dirac spinors, the definition of the current follows from the continuity equation

\[ \partial_\theta (\Psi_R^+ \Psi_R + \Psi_L^+ \Psi_L) = v_F \partial_\theta (\Psi_R^+ \Psi_R - \Psi_L^+ \Psi_L) \]  
(5)

FIG. 1: Band structure of a zig-zag nanotube in transverse magnetic field, for a radius \( R \approx 20 \) nm and field strength \( B = 20 \) T. This choice corresponds to \( aR/l^2 \approx 0.1 \) and \( R/l \approx 3.5 \). Energy is in units of \( \hbar/v_F \) and momentum is in units of \( \hbar/v_F \).
where $\Psi_R$ and $\Psi_L$ are the two spinor components, in the basis used to write the Dirac hamiltonian $H$. We have computed the integral over $\theta$ of the current

$$j = \Psi_R^+ \Psi_R - \Psi_L^+ \Psi_L$$

for the different eigenstates. The results for the lowest energy subbands are represented in Fig. 2. It turns out that, in general, the states corresponding to the flat part of the Landau level do not carry any current in the longitudinal direction, while the states in the dispersive branches saturate quickly the unit of current as the dispersion approaches a constant slope.

The quantization of the current for the states in the dispersive branches opens the possibility to observe the quantization of the Hall conductivity in thick carbon nanotubes. In general, the quantization of the current is more accurate for smaller curvature of the dispersive branches. It happens moreover that, when the Fermi level crosses one of the bumps with parabolic dispersion shown in Fig. 1, the two contributions to the current from the respective Fermi points go in the opposite direction and tend to cancel one each other. Envisaging an experiment where a potential difference is applied between the two flanks of a thick nanotube, we obtain that the current in the longitudinal direction is given approximately by the excess (or defect) of filled states in the right dispersive branches, with respect to those in the left dispersive branches. Making the parallel of the arguments applied for planar geometries, we conclude that the Hall conductivity $\sigma_{xy}$ must follow an approximate quantization rule, with a prefactor given by the spin degeneracy and the doubling of the subbands shown in Fig. 1.

$$\sigma_{xy} = \frac{e^2}{h} n$$

(7)

Opposite to what happens in the case of graphene, we observe that the Hall conductivity is quantized in even steps of $2e^2/h$. It can be shown that this feature is a consequence of the vanishing net flux traversing the nanotube surface. We have checked that cutting the nanotube along the longitudinal direction at $\theta = \pi/2$ does not lead to an appreciable change in the band structure, while cutting it at $\theta = 0$ or $\pi$ produces a sudden distortion of the subbands. In the latter case, states that were before at zero energy form two subbands dispersing towards higher energies, and two other dispersing downwards. In these conditions, only one valley is left at zero energy, evidencing the transition to the odd-integer quantization of the Hall conductivity of graphene.

The coexistence of a magnetic field and a Hall voltage, $V_H$, leads to an interesting realization of the setup discussed in Ref. The Hall voltage leads to an electric field at the top and bottom regions of the nanotube, $E \approx V_H/R$. Due to the 1D nature of the nanotube, this field will not be screened. The effect of this field on the Landau levels can be analyzed by noticing that it can be made to vanish by a suitable Lorentz transformation, leading to a new effective Hamiltonian with a reduced magnetic field $V_H/R$. Hence, one obtains that the new Landau levels are determined by the magnetic length $l = l(1 - \beta^2)^{-1/4}$, where $\beta = E/((v_F/c)B)$. The system undergoes a phase transition, similar to dielectric breakdown, for $\beta = 1$. In the case of a nanotube, this transition takes place when $(V_H/R)/[(v_F/c)B] = 1$. This corresponds to a current flowing along the nanotube:

$$I_c = \sigma_{xy}V_H \sim 4n\frac{e^2}{h} FR$$

(8)

The transition at higher bias currents leads to a significant change in the electronic wavefunction, and to the suppression of the gaps between Landau levels. It seems likely that, beyond this transition, the Hall conductance will no longer be quantized.

The existence of extended states along the flanks of the nanotube gives also the clue to understand the effects of the electron-electron interaction in the presence of the magnetic field. At $B = 0$, the transport properties are dictated by the so-called Luttinger liquid behavior, which is a reflection of the repulsive interaction in the nanotubes. When the magnetic field is switched on, however, the Coulomb interaction renormalizes in a different manner the Fermi velocity as well as the compressibility for the extended states near the Fermi level, with the consequent modification of the transport properties.

Focusing on the case where the Fermi level is right above (or below) the plateau at zero energy, the extended states must correspond to the outermost dispersive branches in the band structure of Fig. 1. In the 1D notation of the different interaction channels, the Fermi velocity is renormalized by the so-called $g_4$ coupling, given by the matrix element of the Coulomb interaction.
interaction for electrons near the same Fermi point. This quantity can be computed using the eigenfunctions of the Dirac Hamiltonian. At the flanks of the nanotube, their wave functions have gaussian shape, with a width that is proportional to $\sqrt{l}$. Thus, the coupling $g_4$ gets larger values for increasing magnetic field, as a consequence of the strong Coulomb repulsion between the currents localized at a given flank of the nanotube. The matrix element computed instead for electrons near the Fermi level with opposite longitudinal momenta gives the so-called $g_2$ coupling. The weak Coulomb repulsion between currents localized at antipodal points in the nanotube leads to relative small values of $g_2$. This enters in the expression of the Luttinger liquid parameter $K$ which governs the low-energy transport properties. The variation of $K$ upon switching on the magnetic field, represented in Fig. 3 translates into a sharp decrease of the exponent $\alpha$ for the the power-law behavior of the tunneling density of states, according to the relation $\alpha = (K + 1/K - 2)/8$. The behavior of the exponent, plotted in Fig. 3 provides a signature of the suppression of the electronic correlations in the presence of a transverse magnetic field, which seems to have been observed in the measures of $\alpha$ in multi-walled nanotubes.

To summarize, we have shown that, for thick carbon nanotubes in a transverse magnetic field, the transport properties are governed by the states localized at the flanks of the nanotube, which carry quantized currents in the longitudinal direction. By placing a potential difference at opposite sides of a nanotube section, it should be possible to observe steps in the Hall conductivity at even multiples of $2e^2/h$. Furthermore, the small overlap between states with currents flowing in opposite direction must lead to the absence of significant backscattering interactions. This means that the effects of the $e-e$ interactions have to reflect in a simple renormalization of physical parameters, and that there should be good perspectives to observe a robust chiral liquid at the flanks of the nanotube, by means for instance of scanning tunneling microscopy.

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