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Unconditional no-hidden-variables theorem

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Abstract Recently, [arXiv:0810.3134] is accepted and published. We present ultimate version of no-hidden-variables theorem. We derive a proposition concerning the quantum theory under the existence of the Bloch sphere in a single spin-1/2 system. The existence of a single classical probability space for measurement outcome within the formalism of von Neumann’s projective measurement does not coexist with the proposition concerning the quantum theory. We have to give up the existence of such a classical probability space for measurement outcome in the two-dimensional Hilbert space formalism of the quantum theory. The quantum theory does not accept a hidden-variable interpretation in the two-dimensional space.

Keywords The quantum theory · Hidden-variable theory

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1 Introduction

Recently, [1] is accepted and published. As a famous physical theory, the quantum theory (cf. 2[85]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [7], a hidden-variable interpretation of the quantum theory has been an attractive topic of research [34]. There are two main approaches to study the hidden-variable interpretation of the quantum theory. One is the Bell-EPR theorem [8]. This theorem says that the quantum predictions violate the inequality following from the EPR-locality condition in the Hilbert space formalism of the quantum theory. The EPR-locality condition

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tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (KS theorem) [9]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The quantum theory does not accept the KS type of hidden-variable theory. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [10, 11] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [12, 13, 14, 15, 16]).

Mermin considers the Bell-EPR theorem in a multipartite state. He derives multipartite Bell inequality [17]. The quantum predictions by n-partite GHZ state violate the Bell-Mermin inequality by an amount that grows exponentially with n. And, several multipartite Bell inequalities are reported [18, 19, 20, 21, 22, 23, 24, 25, 26]. They also say that the quantum predictions violate local hidden-variable theories by an amount that grows exponentially with n.

As for the KS theorem, it is begun to research the validity of the KS theorem by using inequalities (see Refs. [27, 28, 29, 30]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [31]. The KS theorem is related to the algebraic structure of a set of quantum operators. The KS theorem is independent of a quantum state under study. Nagata derives an inequality [30] as tests for the validity of the KS theorem. The quantum predictions violate the Nagata inequality when the system is in an uncorrelated state. The uncorrelated state is defined in Ref. [32]. The quantum predictions by n-partite uncorrelated state violate the Nagata inequality by an amount that grows exponentially with n.

Leggett-type nonlocal hidden-variable theory [33] is experimentally investigated [34, 35, 36]. The experiments report that the quantum theory does not accept Leggett-type nonlocal hidden-variable theory. These experiments are done in the four-dimensional space (two parties) in order to study nonlocality of hidden-variable theories. We ask: Can the quantum theory accept a hidden-variable interpretation in the two-dimensional space (only one party)? The answer is “No” whereas Kochen and Specker explicitly constructed [9] such a hidden-variable interpretation in the two-dimensional space formalism of the quantum theory.

Here we aim to show alternative and ultimate version of no-hidden-variables theorem. A pure spin-1/2 state (a quantum state with the two-dimensional space) constructs our theorem. Our theorem says that a hidden-variable interpretation of the quantum theory from the two-dimensional Hilbert space formalism of the quantum theory is impossible. The quantum theory does not accept a hidden-variable interpretation in the two-dimensional space.

In what follows, we derive a proposition concerning the quantum theory under the existence of the Bloch sphere in a spin-1/2 system. The existence of a classical probability space for measurement outcome within the formalism of von Neumann’s projective measurement does not coexist with the proposition concerning the quantum theory under the existence of the Bloch
sphere. We consider a single classical probability space. A single classical probability space is enough to investigate a hidden-variable interpretation of the quantum theory. We can consider the direct product of many spaces \((\Omega_1 \times \Omega_2 \times \Omega_3 \times \cdots)\) as a single space.

2 Notation and preparations

Throughout this paper, we assume von Neumann’s projective measurement. Throughout this paper, we confine ourselves to the two-dimensional space. Let \(R\) denote the reals where \(\pm \infty \not\in R\). We assume that every eigenvalue in this paper lies in \(R\). We assume that every Hermitian operator is associated with a unique observable (see Ref. [4]). We do not need to distinguish between them in this paper.

We investigate if a hidden-variable interpretation of the two-dimensional Hilbert space formalism of the quantum theory is possible. Let \(O\) be the space of Hermitian operators described in the two-dimensional Hilbert space. Let \(T\) be the space of density operators described in the Hilbert space. Namely, \(T = \{\psi|\psi \in \Omega \land \psi \geq 0 \land \text{Tr}[\psi] = 1\}\). We define the notation \(\theta\) which represents one result of quantum measurements. Suppose that the measurement of a Hermitian operator \(A\) for a system in the state \(\psi\) yields a value \(\theta(A) \in R\).

We consider the following propositions. We define \(\chi_\Delta(x), (x \in R)\) as the characteristic function. We define \(\Delta\) as any subset of the reals \(R\).

**Proposition:** BSF (the Born statistical formula),

\[
\text{Prob}(\Delta^\psi_{\theta(A)}) = \text{Tr}[\psi \chi_\Delta(A)].
\] (1)

The symbol \((\Delta)^\psi_{\theta(A)}\) denotes the following proposition: \(\theta(A)\) lies in \(\Delta\) if the system is in the state \(\psi\). The symbol “Prob” denotes the probability that the proposition \((\Delta)^\psi_{\theta(A)}\) holds.

We consider a classical probability space \((\Omega, \Sigma, \mu_\psi)\). \(\Omega\) is a nonempty space. \(\Sigma\) is a \(\sigma\)-algebra of subsets of \(\Omega\). \(\mu_\psi\) is a \(\sigma\)-additive normalized measure on \(\Sigma\) such that \(\mu_\psi(\Omega) = 1\). The subscript \(\psi\) means that the probability measure is determined uniquely when the quantum state \(\psi\) is specified.

We introduce measurable functions (classical random variables) onto \(\Omega\) \((f : \Omega \mapsto R)\). The measurable function is written as \(f_A(\omega)\) for an operator \(A \in O\). Here \(\omega \in \Omega\) is a hidden variable.

**Proposition:** HV (a hidden-variable interpretation of the quantum theory).

The measurable function \(f_A(\omega)\) exists for every Hermitian operator \(A\) in \(O\).

**Proposition:** D (the probability distribution rule),

\[
\mu_\psi(f_A^{-1}(\Delta)) = \text{Prob}(\Delta)^\psi_{\theta(A)}. \tag{2}
\]

The possible value of \(f_A(\omega)\) takes eigenvalues of \(A\) almost everywhere with respect to \(\mu_\psi\) in \(\Omega\) if we assign the truth value “1” for Proposition BSF, Proposition HV, and Proposition D, simultaneously. We have the following Lemma.
Lemma: Let \( S_A \) stand for the eigenvalues of the Hermitian operator \( A \). For every quantum state described in a Hilbert space, \[ \text{BSF} \land \text{HV} \land \text{D} \Rightarrow f_A(\omega) \in S_A, \quad (\mu_\psi - \text{a.e.}). \] (3)

We review the following:

**Lemma:** Let \( S_A \) stand for the eigenvalues of the Hermitian operator \( A \). If

\[
\text{Tr}[\psi A] := \sum_{y \in S_A} \text{Prob}(\{y\})^\psi_{\theta(A)} y,
\]

\[
E_\psi(A) := \int_{\omega \in \Omega} \mu_\psi(\text{d}\omega) f_A(\omega),
\]

then

\[ \text{BSF} \land \text{HV} \land \text{D} \Rightarrow \text{Tr}[\psi A] = E_\psi(A). \] (4)

**Proof:** Note \( \omega \in f_A^{-1}(\{y\}) \Leftrightarrow f_A(\omega) \in \{y\} \Leftrightarrow y = f_A(\omega), \)

\[
\int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(\text{d}\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} = 1,
\]

\[ y \neq y' \Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \phi. \] (5)

Hence we have

\[
\text{Tr}[\psi A] = \sum_{y \in S_A} \text{Prob}(\{y\})^\psi_{\theta(A)} y = \sum_{y \in \mathbb{R}} \text{Prob}(\{y\})^\psi_{\theta(A)} y
\]

\[
= \sum_{y \in \mathbb{R}} \mu_\psi(f_A^{-1}(\{y\})) y
\]

\[
= \sum_{y \in \mathbb{R}} \mu_\psi(f_A^{-1}(\{y\})) y \times \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(\text{d}\omega)}{\mu_\psi(f_A^{-1}(\{y\}))}
\]

\[
= \sum_{y \in \mathbb{R}} \int_{\omega \in f_A^{-1}(\{y\})} \mu_\psi(f_A^{-1}(\{y\})) \times \frac{\mu_\psi(\text{d}\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega)
\]

\[
= \int_{\omega \in \Omega} \mu_\psi(\text{d}\omega) f_A(\omega) = E_\psi(A). \] (6)

QED.

The probability measure \( \mu_\psi \) is chosen such that the following equation is valid if we assign the truth value “1” for Proposition BSF, Proposition HV, and Proposition D, simultaneously:

\[
\text{Tr}[\psi A] = \int_{\omega \in \Omega} \mu_\psi(\text{d}\omega) f_A(\omega)
\]

(7)

for every Hermitian operator \( A \) in \( \mathcal{O} \).
3 Unconditional no-hidden-variables theorem

We discuss main result of this paper. Assume a pure spin-1/2 state $\psi$ in the two-dimensional space. Let $\sigma$ be $(\sigma_x, \sigma_y, \sigma_z)$. $\sigma$ is the vector of Pauli operator. The measurements (observables) on a pure spin-1/2 state are parameterized by a unit vector $n$ in $\mathbb{R}^3$ (its direction along which the spin component is measured). Here, $\cdot$ is the scalar product in $\mathbb{R}^3$. We define three vectors in $\mathbb{R}^3$ as $\mathbf{x}^{(1)} := \mathbf{x}$, $\mathbf{x}^{(2)} := \mathbf{y}$, and $\mathbf{x}^{(3)} := \mathbf{z}$. They are the Cartesian axes relative to which spherical angles are measured. We write the unit vectors in a spherical coordinate system defined by $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ in the following way:

$$n := \sin \theta \cos \phi \mathbf{x}^{(1)} + \sin \theta \sin \phi \mathbf{x}^{(2)} + \cos \theta \mathbf{x}^{(3)}. \quad (8)$$

We define a quantum expectation value $E_{QM}$ as

$$E_{QM} := \text{Tr}[\psi \cdot \sigma]. \quad (9)$$

We derive a necessary condition for the quantum expectation value for the system in a pure spin-1/2 state $\psi$ given in (9). We derive the possible values of the scalar product $\int \Omega (E_{QM} \times E_{QM}) =: \|E_{QM}\|^2$. $E_{QM}$ is the quantum expectation value given in (9). We use decomposition (8). We introduce the usual measure $\int \Omega = \sin \theta d\theta d\phi$. We introduce simplified notations as

$$T_i = \text{Tr}[\psi \mathbf{x}^{(i)} \cdot \sigma] \quad (10)$$

and

$$(c^1, c^2, c^3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (11)$$

We have

$$\|E_{QM}\|^2 = \int \Omega \left( \sum_{i=1}^{3} T_i c^i \right)^2 = 4\pi/3 \sum_{i=1}^{3} T_i^2 \leq 4\pi/3, \quad (12)$$

where we use the orthogonality relation $\int \Omega c^\alpha_k c^\beta_k = (4\pi/3) \delta_{\alpha\beta}$. $\sum_{i=1}^{3} T_i^2$ is bounded as $\sum_{i=1}^{3} T_i^2 \leq 1$ if we assign the truth value “1” for a proposition of the quantum theory [The Bloch sphere exists.]. The reason of the condition (12) is the Bloch sphere

$$\sum_{i=1}^{3} (\text{Tr}[\psi \mathbf{x}^{(i)} \cdot \sigma])^2 \leq 1. \quad (13)$$

We derive a proposition concerning the quantum theory while assigning the truth value “1” for the quantum proposition [The Bloch sphere exists.] (in a spin-1/2 system). The proposition is $\|E_{QM}\|^2 \leq 4\pi/3$. This inequality is saturated iff $\psi$ is a pure spin-1/2 state. We derive the following proposition concerning the quantum theory while assigning the truth value “1” for the quantum proposition [The Bloch sphere exists.]

$$\|E_{QM}\|_{\text{max}} = 4\pi/3. \quad (14)$$
The symbol \( \|E_{QM}\|_{\text{max}}^2 \) is the maximal value of \( \|E_{QM}\|^2 \).

We assign the truth value “1” for Proposition BSF, Proposition HV, and Proposition D, simultaneously. Then, the quantum expectation value given in (9) is

\[
E_{QM}^k = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{n_k}(\omega). \tag{15}
\]

The possible values of \( f_{n_k}(\omega) \) are \( \pm 1 \) (in \( \hbar/2 \) unit) almost everywhere with respect to \( \mu_\psi \) in \( \Omega \) if we assign the truth value “1” for Proposition BSF, Proposition HV, and Proposition D, simultaneously.

We derive a necessary condition for the quantum expectation value given in (15). Again, we derive the possible values of the scalar product \( \|E_{QM}\|^2 \) of the quantum expectation value. The quantum expectation value is \( E_{QM}^k \) given in (15). We have

\[
\|E_{QM}\|^2 = \int \Omega \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{n_k}(\omega) \times \int_{\omega' \in \Omega} \mu_\psi(d\omega') f_{n_k}(\omega') \right)
= \int \Omega \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) \int_{\omega' \in \Omega} \mu_\psi(d\omega') f_{n_k}(\omega) f_{n_k}(\omega') \right)
\leq \int \Omega \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) \int_{\omega' \in \Omega} \mu_\psi(d\omega') |f_{n_k}(\omega) f_{n_k}(\omega')| \right)
= \int \Omega \left( \int_{\omega \in \Omega} \mu_\psi(d\omega) \int_{\omega' \in \Omega} \mu_\psi(d\omega') \right) = 4\pi. \tag{16}
\]

The above inequality (16) is saturated since

\[
\begin{align*}
\{ \omega | \omega \in \Omega \land f_{n_k}(\omega) = 1, (\mu_\psi - a.e.) \} \\
= \{ \omega' | \omega' \in \Omega \land f_{n_k}(\omega') = 1, (\mu_\psi - a.e.) \}, \\
\{ \omega | \omega \in \Omega \land f_{n_k}(\omega) = -1, (\mu_\psi - a.e.) \} \\
= \{ \omega' | \omega' \in \Omega \land f_{n_k}(\omega') = -1, (\mu_\psi - a.e.) \}. \tag{17}
\end{align*}
\]

Hence we derive the following proposition if we assign the truth value “1” for Proposition BSF, Proposition HV, and Proposition D, simultaneously

\[
\|E_{QM}\|_{\text{max}}^2 = 4\pi. \tag{18}
\]

We do not assign the truth value “1” for two propositions (14) and (18), simultaneously, when the system is in a pure spin-1/2 state. We are in the contradiction when the system is in a pure spin-1/2 state.

We do not accept the following four propositions, simultaneously, when the system is in a pure spin-1/2 state:

1. Proposition BSF
2. Proposition HV
3. Proposition D
4. The Bloch sphere exists.
Suppose that the quantum theory is a set of propositions. Suppose that all quantum propositions are true. We have to give up either Proposition HV or Proposition D if we assign the truth value “1” for both Proposition BSF and [The Bloch sphere exists.]. We have to give up a hidden-variable interpretation of the two-dimensional Hilbert space formalism of the quantum theory.

It is that \( \|E_{QM}\|_{\text{max}}^2 = 4\pi/3 \) if we assign the truth value “1” for [The Bloch sphere exists.] when the system is in a pure spin-1/2 state. However, accepting Proposition BSF, Proposition HV, and Proposition D, the existence of a classical probability space of the results of von Neumann’s projective measurements assigns the truth value “1” for the different proposition \( \|E_{QM}\|_{\text{max}} = 4\pi \). We are in the contradiction when the system is in a pure spin-1/2 state. We have to give up, at least, one of propositions, Proposition BSF, Proposition HV, Proposition D, [The Bloch sphere exists.], and \( \psi \) is a spin-1/2 pure state.

4 Conclusions

In conclusion, we have presented alternative and ultimate version of no-hidden-variables theorem. The existence of a classical probability space for the results of von Neumann’s projective measurement has not coexisted with the existence of the Bloch sphere. There has not been a classical probability space for projective measurement outcome. Our result has been obtained in a quantum system which is in a pure spin-1/2 state in the two-dimensional space. The quantum theory has not accepted a hidden-variable interpretation in the two-dimensional space.

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