Proof of Security of a High-Capacity Quantum Key Distribution Protocol

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We prove the security of a high-capacity quantum key distribution protocol over noisy channels. By using entanglement purification protocol, we construct a modified version of the protocol in which we separate it into two consecutive stages. We prove their securities respectively and hence the security of the whole protocol.

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I. INTRODUCTION

Security has been one of the most important concerns ever since people began to communicate. Many classical cryptographic protocols are based on the computational infeasibility of some mathematical problems such as the factorization of large composite numbers, which will be solved in a quantum computer using Shor’s algorithm[1]. Quantum cryptography, on the other hand, relies on the principles of quantum mechanics, especially the uncertainty principle and the no-cloning theorem[2], therefore is provably secure. The quantum cryptography has captured more and more attention. It is mainly used in establishing secrete key between two parties. The scheme is as follows: two participants, commonly called Alice and Bob, share a quantum channel. They transmit quantum states (generally refer to as qubits) which encode classical information, and measure the qubits in certain bases to get key code. Since interference introduced by the eavesdropper, commonly known as Eve, will disturb quantum states and can be detected, Alice and Bob will share their messages while leaking little information to Eve. Many quantum key distribution protocols (QKD) have been proposed since the first protocol[3] published by Bennet and Brassard, for instance in protocols in Refs. [4, 5, 6, 7, 8, 9, 10, 11].

Security and efficiency are two principal factors in QKD protocols. Since both the channel noise and eavesdropping can disturb the quantum states, a good definition of security requires a distinct differentiation between them. One parameter of security serves this purpose. Denoted as the tolerable bit error rate, it is defined as follows: below this threshold, a QKD is secure by using quantum error correction and privacy amplification. The fast development of QKD protocols requires explicit proof of their securities over noisy channels. Mayers’s[12] and Bham’s[12] work reach this end by complex calculation. A different approach, by using entanglement purification protocols (EPP) which can purify the EPR pairs by sacrificing some of them[14], was proposed[15]. After that, many proofs with this basic idea have been given[8, 16, 17, 18].

Efficiency is also an important parameter of QKD protocols. In this paper, we will concentrate on a theoretically high-capacity QKD protocol[2]. Its efficiency is achieved by adopting EPR pair which encodes 2 classical bits.

The high-capacity protocol transmits an EPR sequence in two steps, one particle sequence at a time, thus is protected against eavesdropping in ideal quantum channel (that is, channel without noise). In order to make it error-resistant in noisy quantum channels, we apply the method of Shor and Preskill[16] and add an EPP to the protocol to construct a modified version so as to prove its security. Our paper is organized as follows: in section II we specify the notations used in this paper. In section III we briefly review the high-capacity protocol and outline the proof of security in ideal situations. In section IV we add security check procedures to the original protocol and prove the first stage of the modified version is secure. In section V we prove the security of the second stage of the modified high-capacity protocol. Then we prove that this modified version is equivalent to the original one and give a brief summary in section VI.

II. NOTATION

The notations used here are mostly the same as that of the high-capacity protocol[2] and that of the Shor and Preskill’s[16].

Bell bases are the four maximally entangled states:

\[|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \]  

(1)

We use \(|\Phi^{+}\rangle, |\Phi^{-}\rangle, |\Psi^{+}\rangle\) and \(|\Psi^{-}\rangle\) to represent 00, 01, 10 and 11 respectively.

The three Pauli matrices are:

\[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(2)

They are used in the error checking process.

The Hadamard transform, \(H\), is of the form:

\[H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]  

(3)
It interchanges the basis $|0\rangle$, $|1\rangle$ and $|+\rangle$, $|-\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. In what we will see in section IV after randomly performs the Hadamard gate, bit flip errors and phase flip errors would be uncorrelated.

We introduce the Calderbank-Shor-Steane (CSS) code in the entanglement purification process, a CSS code is defined as follows: $C_1$ and $C_2$ are two classical binary codes which satisfy

$$\{0\} \subset C_2 \subset C_1 \subset F_2^n \quad (4)$$

where $F_2^n$ is the binary space of $n$ bits. $C_1$ and $C_2^\perp$ can correct up to $t$ bit errors. A basis of the CSS code can be built as follows, for $v \in C_1$, define the vector

$$v \rightarrow \frac{1}{|C_2|^{1/2}} \sum_{\omega \in C_2} |v + \omega\rangle \quad (5)$$

Notice that when $v_1 - v_2 \in C_2$, they give the same code. So the CSS code corresponds to the coset of $C_2$ in $C_1$. Let $H_1$ and $H_2$ be the parity check matrix for the code $C_1$ and $C_2^\perp$ respectively. Then by measuring $\sigma^r_z$ for each row $r \in H_1$ and $\sigma^x_r$ for each row $r \in H_2$, the CSS code can correct up to $t$ bit errors.

### III. REVIEW OF THE HIGH-CAPACITY PROTOCOL

The key point of the high-capacity protocol is that it uses each EPR pair to encode 2 bits of key code. By building an ordered EPR sequence and sending each half in two steps from Alice to Bob, Alice and Bob can protect their communication against most eavesdropping attacks. Here we outline the high-capacity protocol.

**Protocol 1: Theoretical efficient high-capacity Protocol**

1. Alice produces an ordered $N$ EPR pair sequence: $[(P_1(1), P_1(2)), (P_2(1), P_2(2)), \ldots, (P_1(1), P_1(2)), \ldots, (P_N(1), P_N(2))]$.

2. Then Alice takes one particle from each EPR pair to form two ordered EPR partner particle sequences: $[P_1(1), P_2(1), \ldots, P_N(1)]$ and $[P_1(2), P_2(2), \ldots, P_N(2)]$. Alice sends to Bob one ordered EPR partner particle sequence: $[P_1(2), P_2(2), \ldots, P_N(2)]$.

3. After Bob receiving the ordered EPR partner particle sequence, randomly he chooses a sufficiently large subset of his sequence and performs measurement on the particles in the subset randomly in of the two measuring-basis $\sigma_z$ or $\sigma_x$. The result of this measurement will be either 0 or 1. Bob stores the rest of the particles of his EPR particle sequence.

4. Then Bob tells Alice through a classical channel his reception of the particle sequence and the particles that he has chosen to measure in a certain direction.

5. (First eavesdropping check.) After hearing from Bob, Alice then performs measurement on the partner subset of those particles whose partner has been measured by Bob. They then publicly compare their results of these measurements to check eavesdropping.

6. If they are certain that there is no eavesdropping, then Alice sends Bob the remaining EPR particle sequence: $[P_1(1), P_2(1), \ldots, P_N(1)]$.

7. After Bob receives these $N$ particles, he first drops the particles that have been measured, then takes one particle from each particle sequence in order and performs Bell-basis measurement on them. He records the results of the measurements.

8. (Second eavesdropping check.) Alice and Bob choose a sufficiently large subset of these Bell-basis measurement results to determine if the QKD is successful. If the error rate in this check is below a certain threshold, then the results are taken as raw key.

Unlike many other QKD protocols, in which an EPR pair is used to encode one key bit, the high-capacity protocol here encodes two bits, thus the capacity is doubled.

As discussed in the original paper, this protocol is secure in quantum channels without noise.

### IV. THE FIRST STAGE OF THE MODIFIED HIGH-CAPACITY PROTOCOL

From the review of the high-capacity protocol, we see that it is an efficient protocol, but a critical questions remains: how can this protocol protect against quantum channel noise? In other words, how can Bob distinguish between channel noise and eavesdropping, and can correct bit errors when necessary?

In this section, we add entanglement purification steps to the first stage of the high-capacity protocol and prove its security over noisy channels.

**Protocol 2: Secure high-capacity Protocol (stage 1)**

1. Alice produces an ordered $N$ EPR pair sequence according to a quaternary string $a$. More specifically, she creates $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$ or $|\Psi^-\rangle$ when the corresponding $a_i = 0, 1, 2$ or 3. Thus build $[(P_1(1), P_1(2)), (P_2(1), P_2(2)), \ldots, (P_1(1), P_1(2)), \ldots, (P_N(1), P_N(2))]$.

2. Then Alice takes one particle from each EPR pair to form two ordered EPR partner particle sequences: $[P_1(1), P_2(1), \ldots, P_N(1)]$ and $[P_1(2), P_2(2), \ldots, P_N(2)]$. She retains the first sequence.

3. Alice randomly chooses the bases of the second half of the EPR pairs. More specifically, Alice selects a random $N$ binary string $b$ and applies Hadamard
transformation on $P_i(2)$ when the corresponding $i$th bit is 1.

4. Alice sends the second EPR sequence to Bob.

5. Bob receives the $N$ qubits and publicly announces the reception.

6. Alice randomly chooses $n(n < N)$ bits as checking bits and leaves the rest $N - n$ bits unchanged.

7. Alice then tells Bob the bit string $b$ and which $n$ bits are checking bits.

8. Bob performs Hadamard transformation on the qubits where the corresponding components of $a$ are 1. They measure the $n$ checking qubits in $|0\rangle, |1\rangle$ bases, if too many outcomes disagree, they abort the protocol.

9. Alice and Bob applies a CSS code and makes $\sigma_2^{[r]}$ according to each row $r \in H_1$ and $\sigma_x^{[r']}$ according to each row $r' \in H_2$ to their EPR particles. They compute the syndromes and make corrections in order that they obtain $k(k < N - n)$ nearly perfect EPR pairs.

The above protocol is the first stage of the modified version of the high-capacity protocol. We employ CSS code here to purify the entangled states. Notice that the vector space $C_1$ and $C_2$ are orthogonal, so $\sigma_2^{[r]}$ and $\sigma_x^{[r']}$ commutes. The measurement computes the error syndrome for bit flip and phase flip respectively, then after the measurement, Alice and Bob can obtain $k$ perfect EPR pairs.

We next show that the bit and phase errors are uncorrelated. As noted by Lo and Chau 3, for a quantum channel, the error rate can be expressed by a density matrix $\text{diag}(a, b, c, d)$, in which $a, b, c$ and $d$ represent the probabilities of zero errors, bit flip errors, phase flip errors and both bit and phase flip errors. As noted in section 11, the Hadamard transformation interchanges the basis $|0\rangle, |1\rangle$ and $|+\rangle, |−\rangle$, so it changes bit flip error to phase flip error and vice versa. When $N$ is large enough, there is a high probability to have $\frac{N}{2} 1$ in the binary string $b$, so there are almost half Hadamard transforms operating on the EPR pairs. Averaging over the two cases of Identity and Hadamard, the effective density matrix shared by Alice and Bob after the operation is $\text{diag}(a, \frac{b}{2}, \frac{b}{2}, d)$.

From this matrix, we can see that the Hadamard transformation makes the bit flip error and the phase flip error with equal probability and uncorrelated.

Because this modified high-capacity protocol uses CSS code, it can successfully correct quantum state which differs from the input state (we denoted as $|\psi\rangle$) less than $t$ bit flip errors and $t$ phase flip errors. Since all the measurements in this protocol commute under the Bell states, we can use the classical method to calculate the fidelity of the purified $k$ EPR pairs. In the transmission, the error rates for the check bits and code bits are almost the same. And because Eve doesn’t know anything about the check bits and the code bits, her interference is also the same to these two sets. Then as $(N - n) \to \infty$, we can use classical probability theory to calculate the fidelity, which will yield $F(\rho, \psi) \geq 1 - 2^{-s}$.

Applying the Lemma 1 and 2 of Lo and Chau 4, we know that if Alice and Bob share a state with fidelity greater than $1 - 2^{-s}$ with input state $\rho$, then Eve’s mutual information with the key would be exponentially small, so the first stage of the modified version of the high-capacity protocol is safe.

V. THE SECOND STAGE OF THE MODIFIED HIGH-CAPACITY PROTOCOL

In this section, we prove that the second stage of the protocol is secure too. This stage can be seen as a repetition of the first stage, with a some variation as we list below.

Protocol 2: Secure high-capacity Protocol(stage 2)

10. Alice has one ordered half of $k$ perfect EPR particle sequence: $[P_1^1(1), P_2^1(1), \ldots, P_k^1(1)]$ and Bob has the corresponding sequence: $[P_1^2(2), P_2^2(2), \ldots, P_k^2(2)]$.

11. Alice randomly chooses the bases of her own halves of the EPR pairs. To be more specific, Alice selects a random $k$ binary string $b'$ and applies Hadamard gates on $P_i(1)$ when the corresponding $i$th bit is 1.

12. Alice sends her EPR sequence to Bob.

13. After receiving the sequence, Bob publicly announces this fact.

14. Alice then tells Bob the binary string $b'$.

15. Bob selects a sufficiently large subset among his EPR pairs and measures them according to $b'$, if too many results inconsistent, he aborts the communication.

16. Bob performs entanglement purification in the same way as in stage 1 to the remaining EPR pairs to obtain $m$ perfect EPR pairs.

17. Bob performs Bell-basis measurement to obtain the secret key.

The proof of security of the stage 2 of protocol 2 is similar to that of stage 1. So as discussed above, Eve can know nothing about the pure EPR pairs in Bob’s hand, then it is obvious that stage 2 is also unconditionally secure.

Combining the previous two stages together, we will see that by adding entanglement purification protocol to the high-capacity protocol will protect it against quantum channel noise.
VI. CONCLUSION AND FURTHER DISCUSSION

We have given the security proof of the high-capacity QKD protocol by using the Shor-Preskill method. The proof is divided into two stages. When the error rate is below the threshold value, 11% for the one-way communication protocol as is true in our case, Alice and Bob can still obtain a secure key by using entanglement purification.

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