NEW APPLICATIONS OF ARAK’S INEQUALITIES TO THE LITTLEWOOD–OFFORD PROBLEM

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Abstract. Let $X_1, \ldots, X_n$ be independent identically distributed random variables. In this paper we study the behavior of concentration functions of weighted sums $\sum_{k=1}^n X_k a_k$ with respect to the arithmetic structure of coefficients $a_k$ in the context of the Littlewood–Offord problem. In recent papers of Eliseeva, Götze and Zaitsev, we discussed the relations between the inverse principles stated by Nguyen, Tao and Vu and similar principles formulated by Arak in his papers from the 1980’s. In this paper, we will derive some more general and more precise consequences of Arak’s inequalities providing new results in the context of the Littlewood–Offord problem.

1. Introduction

The concentration function of an $\mathbb{R}^d$-valued vector $Y$ with distribution $F = \mathcal{L}(Y)$ is defined by

$$Q(F, \tau) = \sup_{x \in \mathbb{R}^d} P(Y \in x + \tau B), \quad \tau \geq 0,$$

where $B = \{ x \in \mathbb{R}^d : \|x\| \leq 1/2 \}$ denotes the centered Euclidean ball of radius 1/2.

Let $X, X_1, \ldots, X_n$ be independent identically distributed (i.i.d.) random variables. Let $a = (a_1, \ldots, a_n) \neq 0$, where $a_k = (a_k^1, \ldots, a_k^d) \in \mathbb{R}^d$, $k = 1, \ldots, n$. Starting with seminal papers of Littlewood and Offord [13] and Erdős [5], the behavior of the concentration functions of the weighted sums $S_a = \sum_{k=1}^n X_k a_k$ has been intensively studied. Denote by $F_a = \mathcal{L}(S_a)$ the distribution of $S_a$. We refer to [9] for a discussion of the history of the problem.

Several years ago, Tao and Vu [19] and Nguyen and Vu [14] proposed the so-called ‘inverse principles’ in the Littlewood–Offord problem (see Section 2). In the papers of Götze, Eliseeva and Zaitsev [8] and [9], we discussed the relations between these inverse principles and similar principles formulated by Arak (see [1]–[3]) in his papers from the 1980’s. In the one-dimensional case, Arak has found a connection of the concentration function of the sum with the arithmetic structure of supports of distributions of independent random variables.
for arbitrary distributions of summands. Using these results, he has solved an old problem stated by Kolmogorov [12].

In the present paper, we show that a consequence of Arak’s inequalities provides results in the Littlewood–Offord problem of greater generality and improved precision compared to those proved in [8] and [9]. Moreover, using the results of Tao and Vu [18], we are able to describe the approximating sets much more precisely.

Let us introduce first the necessary notations. Below \(\mathbb{N}\) and \(\mathbb{N}_0\) will denote the sets of all positive and non-negative integers respectively. The symbol \(c\) will be used for absolute positive constants. Note that \(c\) can be different in different (or even in the same) formulas. We will write \(A \ll B\) if \(A \leq cB\). Furthermore, we will use the notation \(A \asymp B\) if \(A \ll B\) and \(B \ll A\). If the corresponding constant depends on, say, \(r\), we write \(A \ll rB\) and \(A \asymp rB\).

If \(\xi = (\xi_1, \ldots, \xi_d)\) is a random vector with distribution \(F = L(\xi)\), we denote \(F(j) = L(\xi_j), j = 1, \ldots, d\). Let \(\hat{F}(t) = \mathbb{E}\exp(i \langle t, \xi \rangle), t \in \mathbb{R}^d\), be the characteristic function of the distribution \(F\). Here \(\langle \cdot, \cdot \rangle\) is the inner product in \(\mathbb{R}^d\).

For \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) we denote \(\|x\|^2 = \langle x, x \rangle = x_1^2 + \cdots + x_n^2\) and \(|x| = \max_j |x_j|\). Let \(E_a\) be the distribution concentrated at a point \(a \in \mathbb{R}^n\). We denote by \([B]_\tau\) the closed \(\tau\)-neighborhood of a set \(B\) in the sense of the norm \(|\cdot|\). Products and powers of measures will be understood in the sense of convolution. Thus, we write \(F^n\) for the \(n\)-fold convolution of a measure \(F\). While a distribution \(F\) is infinitely divisible, \(F^\lambda, \lambda \geq 0\), is the infinitely divisible distribution with characteristic function \(\hat{F}(t)\). For a finite set \(K\), we denote by \(|K|\) the number of elements \(x \in K\). The symbol \(\times\) is used for the direct product of sets. We write \(O(\cdot)\) if the involved constants depend on the parameters named “constants” in the formulations, but not on \(n\).

The elementary properties of concentration functions are well studied (see, for instance, [3, 11, 16]). In particular, it is clear that

\[
Q(F, \mu) \leq (1 + |\mu/\lambda|)^d Q(F, \lambda), \quad \text{for any } \mu, \lambda > 0,
\]

where \(|x|\) is the largest integer \(k\) that satisfies the inequality \(k \leq x\). Hence,

\[
Q(F, c\lambda) \asymp_d Q(F, \lambda).
\]

Estimating the concentration functions in the Littlewood–Offord problem, it is useful to reduce the problem to the estimation of concentration functions of some symmetric infinitely divisible distributions. The corresponding statement is contained in Lemma [14] below.

Introduce the distribution \(H\) with the characteristic function

\[
\hat{H}(t) = \exp \left(-\frac{1}{2} \sum_{k=1}^n \left(1 - \cos \langle t, a_k \rangle \right)\right).
\]

It is clear that \(H\) is a symmetric infinitely divisible distribution. Therefore, its characteristic function is positive for all \(t \in \mathbb{R}^d\).
Let $\tilde{X} = X_1 - X_2$ be the symmetrized random vector, where $X_1$ and $X_2$ are i.i.d. vectors involved in the definition of $S_a$. In the sequel we use the notation $G = L(\tilde{X})$. For $\delta \geq 0$, we denote

$$p(\delta) = G\{\{z : |z| > \delta\}\}. \tag{4}$$

Below we will use the condition

$$G\{\{x \in \mathbb{R} : C_1 < |x| < C_2\}\} \geq C_3, \tag{5}$$

where the values of $C_1, C_2, C_3$ will be specified in the formulations below.

**Lemma 1.** For any $\kappa, \tau > 0$, we have

$$Q(F_a, \tau) \ll_d Q(H^p(\tau/\kappa), \kappa). \tag{6}$$

According to (1), Lemma 1 implies the following inequality.

**Corollary 1.** For any $\kappa, \tau, \delta > 0$, we have

$$Q(F_a, \tau) \ll_d (1 + \lfloor \kappa/\delta \rfloor)^d Q(H^p(\tau/\kappa), \delta). \tag{7}$$

Note that, in the case $\delta = \kappa$, Corollary 1 turns into Lemma 1. Sometimes, it is useful to be free in the choice of $\delta$ in (7). In a recent paper of Eliseeva and Zaitsev [4], a more general statement than Lemma 1 is obtained. It gives useful bounds if $p(\tau/\kappa)$ is small, even if $p(\tau/\kappa) = 0$. The proof of Lemma 1 is given in [9]. It is rather elementary and is based on known properties of concentration functions. We should note that $H^\lambda, \lambda \geq 0$, is a symmetric infinitely divisible distribution with the Lévy spectral measure $M_\lambda = \frac{1}{4} M^*$, where

$$M^* = \sum_{k=1}^n (E_{a_k} + E_{-a_k}).$$

Passing to the limit $\tau \to 0$ in (6), we obtain the following statement (see Zaitsev [21] for details).

**Lemma 2.** The inequality

$$Q(F_a, 0) \ll_d Q(H^p(0), 0) = H^p(0)\{0\} \tag{8}$$

holds.

Note that the case where $p(0) = 0$ is trivial, since then $Q(F_a, 0) = Q(H^p(0), 0) = 1$ for any $a$. Therefore we assume below that $p(0) > 0$.

The following definition is given in Tao and Vu [18] (see also [13], [15], [17], and [19]).

Let $r \in \mathbb{N}_0$ be a non-negative integer, $L = (L_1, \ldots, L_r)$ be a $r$-tuple of positive reals, and $g = (g_1, \ldots, g_r)$ be a $r$-tuple of elements of $\mathbb{R}^d$. The triplet $P = (L, g, r)$ is called symmetric ’generalized arithmetic progression’ (GAP) in $\mathbb{R}^d$. Here $r$ is the rank, $L_1, \ldots, L_r$ are the dimensions and $g_1, \ldots, g_r$ are the generators of the GAP $P$. We define the image

$$\text{Image}(P) \subset \mathbb{R}^d$$

of $P$ to be the set

$$\text{Image}(P) = \{m_1 g_1 + \cdots + m_r g_r : -L_j \leq m_j \leq L_j, m_j \in \mathbb{Z} \text{ for all } j = 1, \ldots, r\}.$$
For $t > 0$ we denote the dilate $P^t$ of $P$ as the symmetric GAP $P^t = (tL, g, r)$ with

$$\text{Image}(P^t) = \{ m_1 g_1 + \cdots + m_r g_r : -tL_j \leq m_j \leq tL_j, \ m_j \in \mathbb{Z} \ \text{for all} \ j = 1, \ldots, r \}.$$  

We define the size of $P$ to be $\text{size}(P) = |\text{Image}(P)|$.

In fact, $\text{Image}(P)$ is the image of an integer box $B = \{(m_1, \ldots, m_r) \in \mathbb{Z}^r : -L_j \leq m_j \leq L_j \}$ under the linear map

$$\Phi : (m_1, \ldots, m_r) \in \mathbb{Z}^r \rightarrow m_1 g_1 + \cdots + m_r g_r.$$  

We say that $P$ is proper if this map is one to one, or, equivalently, if

$$\text{size}(P) = r \prod_{j=1}^r (2 \lfloor L_j \rfloor + 1). \ (9)$$  

The right-hand side of (9) is denoted by $\text{Vol}(P)$. It is called the volume of $P$. For non-proper GAPs, we have, of course,

$$\text{size}(P) < r \prod_{j=1}^r (2 \lfloor L_j \rfloor + 1). \ (10)$$

For $t > 0$, we say that $P$ is $t$-proper if $P^t$ is proper. It is infinitely proper if it is $t$-proper for any $t > 0$. In general, for $t > 0$, we have

$$\text{size}(P^t) \leq \prod_{j=1}^r (2 \lfloor tL_j \rfloor + 1). \ (11)$$

**Remark 1.** In the case $r = 0$ the vectors $L$ and $g$ have no elements and the image of the GAP $P$ consists of the unique zero vector $0 \in \mathbb{R}^d$.

**Remark 2.** Symmetric GAPs are defined not only by their images (sets of points in $\mathbb{R}^d$ admitting the representation $m_1 g_1 + \cdots + m_r g_r$, where $-L_j \leq m_j \leq L_j, \ m_j \in \mathbb{Z}$, for $1 \leq j \leq r$, see [13]). The definition includes the generators $g_1, \ldots, g_r \in \mathbb{R}^d$ and the dimensions $L_1, \ldots, L_r \in \mathbb{R}$. Different symmetric GAPs can have the same image. For example, if $L_j < 1$, then their generators $g_j$ are not used in constructing the image of $P$. However, the image of $P^t$ depends on $g_j$ if $tL_j \geq 1$. Obviously, by definition, the GAPs $P$ and $P^t$ have the same generators and the same rank.

Recall that a convex body in the $r$-dimensional Euclidean space $\mathbb{R}^r$ is a compact convex set with non-empty interior.

**Lemma 3.** Let $V$ be a convex symmetric body in $\mathbb{R}^r$, and let $\Lambda$ be a lattice in $\mathbb{R}^r$. Then there exists a symmetric, infinitely proper GAP $P$ in $\Lambda$ with rank $l \leq r$ such that we have

$$\text{Image}(P) \subset V \cap \Lambda \subset \text{Image}(P^{(c_1r)^{r/2}}) \ (12)$$

with an absolute constant $c_1 \geq 1$. Moreover, the generators $g_j$ of $P$, for $1 \leq j \leq l$, are contained in the symmetric body $lV$. 
The main part of Lemma 3 is contained in Theorem 1.6 of \cite{18}. The last statement of this Lemma follows from \cite{17} Theorem 3.34. The basis \(g_1, \ldots, g_l \in \mathbb{R}^r\) is sometimes called Mahler basis for the sublattice of \(\Lambda\) spanned on \(\Lambda \cap V\).

**Corollary 2.** Under the conditions of Lemma 3

\[
\text{size}(P(c_1 r)^{3r/2}) \leq \left(2 (c_1 r)^{3r/2} + 1\right)^r |V \cap \Lambda|.
\]

**Proof of Corollary 2** Using Lemma 3 and relations (9) and (11), we obtain

\[
\text{size}(P(c_1 r)^{3r/2}) \leq \prod_{j=1}^r \left(\left\lfloor 2 \left(\frac{c_1 r}{2} L_j \right)^{3r/2} + 1\right\rfloor + 1\right)
\leq \left(2 \left(\frac{c_1 r}{2} + 1\right)^r \text{size}(P) \leq \left(2 \left(\frac{c_1 r}{2} + 1\right)^r |V \cap \Lambda|.
\]

Here the numbers \(L_j\) are the dimensions of \(P\). We used that \(\left\lfloor 2^t L \right\rfloor + 1 \leq (2^t + 1) \left(\left\lfloor 2^t L \right\rfloor + 1\right)\), for \(L, t > 0\). \(\square\)

The following Lemma 4 shows that symmetric progressions are contained in proper progressions. It can be found in \cite{18} Theorem 1.9, see also \cite{17} Theorem 3.40 and \cite{10} Theorem 2.1.

**Lemma 4.** Let \(P\) be a symmetric GAP in \(\mathbb{R}\), and let \(t \geq 1\). Then there exists a \(t\)-proper symmetric GAP \(Q\) with \(\text{rank}(Q) \leq r = \text{rank}(P)\), \(\text{Image}(P) \subset \text{Image}(Q)\), and

\[
\text{size}(P) \leq \text{size}(Q) \leq (2 t)^r 6^r^2 \text{size}(P).
\]

We start now to formulate Theorem 1 which is a one-dimensional Arak type result, see \cite{2}. Let us introduce the necessary notation.

Let \(r \in \mathbb{N}_0\), \(m \in \mathbb{N}\) be fixed, let \(h\) be an arbitrary \(r\)-dimensional vector, and let \(V\) be an arbitrary closed symmetric convex subset of \(\mathbb{R}^r\) containing not more than \(m\) points with integer coordinates. We define \(\mathcal{K}_{r,m}\) as the collection of all sets of the form

\[
K = \{ \langle \nu, h \rangle : \nu \in \mathbb{Z}^r \cap V \} \subset \mathbb{R}.
\]

We shall call such sets CGAPs (‘convex generalized arithmetic progressions’, see \cite{10}), by analogy with the notion of GAPs.

Here the number \(r\) is the rank and \(|\mathbb{Z}^r \cap V|\) is the size of a CGAP in the class \(\mathcal{K}_{r,m}\). It seems natural to call a CGAP from \(\mathcal{K}_{r,m}\) proper if all points \(\{ \langle \nu, h \rangle : \nu \in \mathbb{Z}^r \}\) are disjoint.

For any Borel measure \(W\) on \(\mathbb{R}\) and \(\tau \geq 0\) we define \(\beta_{r,m}(W, \tau)\) by

\[
\beta_{r,m}(W, \tau) = \inf_{K \in \mathcal{K}_{r,m}} W\{ R \setminus [K]_\tau \}.
\]

We now introduce a class of \(d\)-dimensional CGAPs \(\mathcal{K}_{r,m}^{(d)}\) which consists of all sets of the form \(K = \times_{j=1}^d K_j\), where \(K_j \in \mathcal{K}_{r_j,m_j}\), \(r = (r_1, \ldots, r_d) \in \mathbb{N}_0^d\), \(m = (m_1, \ldots, m_d) \in \mathbb{N}^d\). We call \(R = r_1 + \cdots + r_d\) the rank and \(|\mathbb{Z}^{r_1} \cap V_1| \cdot \cdots \cdot |\mathbb{Z}^{r_d} \cap V_d|\) the size of \(K\). Here \(V_j \subset \mathbb{R}^{r_j}\) are symmetric convex subsets from the representation (16) for \(K_j\).
Remark 3. In the case $r = 0$ the class $K_{r,m} = K_{0,m}$ consists of the one set $\{0\}$ having zero as the unique element.

The following result is a particular case of Theorem 4.3 of Chapter II in [3].

**Theorem 1.** Let $D$ be a one-dimensional infinitely divisible distribution with characteristic function of the form $\exp(\alpha (\hat{W}(t) - 1))$, $t \in \mathbb{R}$, where $\alpha > 0$ and $W$ is a probability distribution. Let $\tau \geq 0$, $r \in \mathbb{N}_0$, $m \in \mathbb{N}$. Then

$$Q(D, \tau) \leq c_2^{r+1}\left(\frac{1}{m \sqrt{\alpha \beta_{r,m}(W, \tau)}} + \frac{(r+1)^{5r/2}}{(\alpha \beta_{r,m}(W, \tau))^{(r+1)/2}}\right),$$

where $c_2$ is an absolute constant.

**Remark 4.** Arak [2] did not assume that the set $V$ is closed in the definition (16). It is easy to see however that this does not change the formulation of Theorem 1.

Arak [2] proved an analogue of Theorem 1 for sums of i.i.d. random variables (see Theorem 4.2 of Chapter II in [3]). He used this theorem in the proof of the following remarkable result:

*There exists a universal constant $C$ such that for any one-dimensional probability distribution $F$ and for any positive integer $n$ there exists an infinitely divisible distribution $D_n$ such that*

$$\rho(F^n, D_n) \leq C n^{-2/3},$$

*where $\rho(\cdot, \cdot)$ is the classical Kolmogorov’s uniform distance between corresponding distribution functions.*

This gives the definitive solution of an old problem stated by Kolmogorov [12] in the 1950’s (see [3] for the history of this problem).

Estimation of concentration functions is the main tool for the bound of $\rho(F^n, D_n)$. Moreover, Arak’s inequalities play a crucial role in constructing the approximating distribution $D_n$. Standard distributions, such as Gaussian, stable, accompanying compound Poisson laws, won’t suffice to ensure the bound for arbitrary $F$ and $n$. Roughly speaking, the main idea is that either the concentration function of $F^n$ is relatively small (and the standard approximation is good enough) or the support of the distribution of the summands has a simple arithmetical structure. In the latter case, this structure is used for constructing the distribution $D_n$.

The investigations of Arak in [1] and [2] were motivated by the ideas of Freiman [7] on the structural theory of set addition. These ideas were used by Nguyen and Vu [14] and [15] as well. The proof of Theorem 1 is based on Esséen’s inequality [6] estimating the concentration function by an integral of the modulus of characteristic function. The important tools used are the Parseval equality and the following obvious inequality for characteristic functions of one-dimensional distributions $U$:

$$|\hat{U}(t + h) - \hat{U}(t)|^2 \leq 2 (1 - \text{Re} \hat{U}(h)), \quad \text{for all } t, h \in \mathbb{R}.$$
Corollary 11 Lemma 2 and Theorem 11 imply the following Theorem 2.

**Theorem 2.** Let \( \varkappa, \delta > 0, \tau \geq 0 \), and let \( X \) be a real random variable satisfying condition (5) with \( C_1 = \tau / \varkappa, C_2 = \infty \) and \( C_3 = p(\tau / \varkappa) > 0 \). Let \( d = 1, \ r \in \mathbb{N}_0, m \in \mathbb{N} \). Then

\[
Q(F_a, \tau) \leq c_3^{r+1} \left(1 + [\varkappa / \delta] \right) \left( \frac{1}{m \sqrt{p(\tau / \varkappa) \beta_{r,m}(M^*, \delta)}} + \frac{(r + 1)^{5r/2}}{(p(\tau / \varkappa) \beta_{r,m}(M^*, \delta))^{(r+1)/2}} \right), \quad \text{if } \tau > 0,
\]

and

\[
Q(F_a, 0) \leq c_3^{r+1} \left( \frac{1}{m \sqrt{p(0) \beta_{r,m}(M^*, 0)}} + \frac{(r + 1)^{5r/2}}{(p(0) \beta_{r,m}(M^*, 0))^{(r+1)/2}} \right), \quad \text{if } \tau = 0,
\]

where \( M^* = \sum_{k=1}^n (E_{a_k} + E_{-a_k}) \) and \( c_3 \) is an absolute constant.

In order to prove Theorem 2, it suffices to apply Corollary 11 Lemma 2 and Theorem 11 and to note that \( H^{p(\tau / \varkappa)} \) is an infinitely divisible distribution whose Lévy spectral measure is \( p(\tau / \varkappa) M^*/4 \). Introduce as well \( M = \sum_{k=1}^n E_{a_k} \). It is obvious that \( M \leq M^* \) and \( \beta_{r,m}(M, \delta) \leq \beta_{r,m}(M^*, \delta) \).

2. Results

The main results of the present paper are Theorems 3 and 4. Their proofs are based on Theorem 2. Theorems 3 and 4 have non-asymptotic character. They provide information about the arithmetic structure of \( a = (a_1, \ldots, a_n) \) without assumptions like \( q_j = Q(F_a^{(j)}, \tau) \geq n^{-A}, j = 1, \ldots, d \), which are imposed in Theorem 3 below. Theorems 3 and 4 are formulated for fixed \( n \) and the dependence of constants on parameters is given explicitly. No analogues of Theorems 3 and 4 follow from the asymptotical results of Nguyen, Tao and Vu [14], [15] and [19], see Theorem 8. The conditions of Theorem 5 are weaker than those used in the results of Nguyen, Tao and Vu. Theorem 5 was derived from Theorem 2 in the paper of Eliseeva, Götze and Zaitsev [9].

In the following we state Theorems 6 and 7 which are more general than Theorem 5. Theorems 6 and 7 will be deduced from Theorem 3. We conclude with a comparison of Theorems 6 and 7 with the results of Nguyen, Tao and Vu. Notice that in the asymptotic Theorems 5 and 8 where \( n \to \infty \), the elements \( a_j \) of \( a \) may depend on \( n \).

Finally, we state improved and generalized versions of Theorems 5 and 6 of [9], see Theorems 9 and 12 of the present paper.

**Theorem 3.** Let \( d = 1, \ a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \ p(0) > 0 \), and \( q = Q(F_a, \tau), \tau \geq 0 \). There exists positive absolute constants \( c_4, c_5 \) such that for any \( \varkappa > 0, \delta \geq 0 \), for any fixed \( r \in \mathbb{N}_0 \), and any \( n' \in \mathbb{N} \) satisfying the inequalities \( \delta \leq \max\{\varkappa, \tau\} \) and

\[
(2 c_4^{r+1} (r + 1)^{5r/2} \varkappa / q \delta)^{2/(r+1)} / p(\tau / \varkappa) \leq n' \leq n, \quad \text{if } \tau > 0,
\]
or
\[
(2c_4^{r+1} (r+1)^{5r/2}/q)^{2/(r+1)} / p(0) \leq n' \leq n, \quad \text{if } \tau = 0,
\]
there exist \( m \in \mathbb{N} \) and CGAPs \( K^*, K^{**} \subset \mathbb{R} \) having ranks \( \leq r \) and sizes \( \ll m \) and \( \ll c(r)m \) respectively and such that

1. At least \( n-2n' \) elements \( a_k \) of \( a \) are \( \delta \)-close to \( K^* \), that is, \( a_k \in [K^*]_\delta \) (this means that for these elements \( a_k \) there exist \( y_k \in K^* \) such that \( |a_k-y_k| \leq \delta \)).
2. The above number \( m \) satisfies the inequalities
\[
m \leq \frac{2c_4^{r+1} \nu}{q \delta \sqrt{p(\tau/\nu)} n'} + 1, \quad \text{if } \tau > 0,
\]
and
\[
m \leq \frac{2c_4^{r+1}}{q \sqrt{p(0)} n'} + 1, \quad \text{if } \tau = 0.
\]
3. The set \( K^* \) is contained in the image \( \overline{K} \) of a symmetric GAP \( \overline{P} \) which has rank \( 1 \leq r \), size \( \ll (c_5 r)^{3r^2/2m} \) and generators \( \overline{f}_j, j = 1, \ldots, \overline{l} \), satisfying inequality \( |\overline{f}_j| \leq 2r \|a\| / \sqrt{n'} \).
4. The set \( K^{**} \) is contained in the image \( \overline{K} \) of a proper symmetric GAP \( \overline{P} \) which has rank \( 1 \leq r \) and size \( \ll (c_6 r)^{15r^2/2m} \).
5. At least \( n-2n' \) elements of \( a \) are \( \delta \)-close to \( K^{**} \).
6. The set \( K^{**} \) is contained in the image \( \overline{K} \) of a proper symmetric GAP \( \overline{P} \) which has rank \( 1 \leq r \) and size \( \ll (c_7 r)^{21r^2/2m} \) and generators \( \overline{g}_j, j = 1, \ldots, \overline{l} \), satisfying the inequality \( |\overline{g}_j| \leq 2r \|a\| / \sqrt{n'} \).

The statement of Theorem 3 is rather cumbersome, but this is the price for its generality. The formulation may be simplified in particular cases, for example, for \( \nu = \delta \) or for \( \nu = \tau \).

The assertion of Theorem 3 is non-trivial for each fixed \( r \) starting with \( r = 0 \). In this case \( m = 1 \) and Theorem 3 gives a bound for the amount \( N \) of elements \( a_k \) of \( a \) which are outside of the interval \([-\delta, \delta]\) around zero. Namely,
\[
N \leq \left(2c_4 \nu / q \delta \right)^2 / p(\tau/\nu) + 1, \quad \text{if } \tau > 0,
\]
and
\[
N \leq \left(2c_4/q \right)^2 / p(0) + 1, \quad \text{if } \delta = \tau = 0.
\]
Comparing item 3 with items 4 and 6 of Theorem 3, we see that in item 3 the approximating GAP may be non-proper. However, the size of proper approximating GAPs is larger in items 4 and 6. Moreover, if \( \delta > 0 \), then it is obvious that by small perturbations of generators of a non-proper GAP \( \overline{P} \) with \( \text{Image}(\overline{P}) = \overline{K} \), we can construct a proper GAP \( \overline{P} \) with \( \text{Image}(\overline{P}) = \overline{K} \), with the size and generators satisfying the bounds of item 3 and such that \([\overline{K}]_\delta \subset [\overline{K}]_{2\delta} \). The set \([\overline{K}]_{2\delta} \) approximates the set of elements of \( a \) not worse than \([\overline{K}]_\delta \). Note that, according to (2), in the conditions of our results there is no essential difference between \( \delta \) and \( 2\delta \)-neighborhoods. Thus, in fact the statements of items 4–6 are useful in the case \( \delta = \tau = 0 \) only. Otherwise, the statement of item 3 is good enough.
Remark 5. Notice that Theorem 3 does not provide any information if there is no \( n' \) satisfying inequalities (21) or (22). In particular, if \( (2c_4r_j^{s+1}/q_{d_j}/q_{d_j})^{2/(r_j+1)} \geq n \) and \( \tau > 0 \). The same may be said if \( -2n' \leq 0 \). Similar remarks can be made about Theorems 4-12.

Theorem 3 is formulated for one-dimensional \( a_k, k = 1, \ldots, n \). However, it may be shown that Theorem 3 provides sufficiently rich arithmetic properties for the set \( a = (a_1, \ldots, a_n) \in (R^\delta)^n \) in the multivariate case as well (see Theorem 4 below). It suffices to apply Theorem 3 to the distributions \( F_{a(j)}, j = 1, \ldots, d \), where \( F_{a(j)} \) are distributions of coordinates of the vector \( S_a \).

Introduce the vectors \( a^{(j)} = (a_{1j}, \ldots, a_{nj}), j = 1, \ldots, d \). It is obvious that \( F_{a(j)} = F_{a(j)} \).

**Theorem 4.** Let \( d > 1, p(0) > 0, q_j = Q(F_{a(j)}, \tau_j), \tau_j \geq 0, j = 1, \ldots, d \). Below \( c_4-c_7 \) are positive absolute constants from Theorem 3. Suppose that \( a = (a_1, \ldots, a_n) \in (R^\delta)^n \) is a multi-subset of \( R^d \). Let \( \kappa_j > 0, \delta_j \geq 0, \tau_j \in N_0, \text{and } n_j \in N, j = 1, \ldots, d, \) satisfy inequalities \( \delta_j \leq \max\{\kappa_j, \tau_j\} \) and

\[
(2c_4r_j^{s+1}/q_{d_j}/q_{d_j})^{2/(r_j+1)}/p(\tau_j/\kappa_j) \leq n_j \leq n, \quad \text{if } \tau_j > 0,
\]

or

\[
(2c_4r_j^{s+1}/q_{d_j}/q_{d_j})^{2/(r_j+1)}/p(\tau_j/\kappa_j) \leq n_j \leq n, \quad \text{if } \tau_j = 0.
\]

Then, for each \( j = 1, \ldots, d \), there exist \( m_j \in N \) and CGAPs \( K_j^*, K_j^{**} \subset R \) having ranks \( \leq r_j \) and sizes \( \ll m_j \) and \( \ll c(r_j) m_j \) respectively and such that

1. At least \( n - 2n_j \) elements \( a_{kj} \) of \( a^{(j)} \) are \( \delta_j \)-close to \( K_j^* \), that is, \( a_{kj} \in [K_j^*]_{\delta_j} \) (this means that for these elements \( a_{kj} \) there exist \( y_{kj} \in K_j^* \) such that \( |a_{kj} - y_{kj}| \leq \delta_j \)).

2. \( m_j \) satisfies inequality \( m_j \leq w_j \), where

\[
w_j = \frac{2c_4r_j^{s+1}/q_{d_j}/q_{d_j}}{\sqrt{p(\tau_j/\kappa_j)} n_j} + 1, \quad \text{if } \tau_j > 0,
\]

or

\[
w_j = \frac{2c_4r_j^{s+1}/q_{d_j}/q_{d_j}}{\sqrt{p(0)} n_j} + 1, \quad \text{if } \tau_j = 0.
\]

3. The set \( K_j^* \) is contained in the image \( \overline{K}_j \) of a symmetric GAP \( \overline{P}_j \) which has rank \( \overline{l}_j \leq r_j \), size \( \ll (c_5 r_j)^{3r_j/2} m_j \) and generators \( \overline{a}^{(j)}_p, p = 1, \ldots, \overline{l}_j \), satisfying inequality \( \overline{a}^{(j)}_p \leq 2r_j \|a^{(j)}\| \sqrt{n_j} \).

4. The set \( K_j^{**} \) is contained in the image \( \overline{K}_j \) of a proper symmetric GAP \( \overline{P}_j \) which has rank \( \overline{l}_j \leq r_j \) and size \( \ll (c_6 r_j)^{15r_j/2} m_j \).

5. At least \( n - 2n_j \) elements of \( a^{(j)} \) are \( \delta_j \)-close to \( K_j^{**} \).
6. The set \( K^*_j \) is contained in the image \( \tilde{K}_j \) of a proper symmetric GAP \( \widetilde{P}_j \) which has rank \( \tilde{l}_j \leq r_j \), size \( \ll (c_7 r_j)^{21 r_j^2/2} m_j \) and generators \( \tilde{g}_p^{(j)} \), \( p = 1, \ldots, \tilde{l}_j \), satisfying inequality \( |\tilde{g}_p^{(j)}| \leq 2 r_j \|a^{(j)}\|/\sqrt{n_j} \).

7. The multi-vector \( \alpha \) is well approximated by the \( d \)-dimensional CGAPs \( K^* = \times_{j=1}^d K_j^* \) and \( K^{**} = \times_{j=1}^d K_j^{**} \), by the image \( K = \times_{j=1}^d K_j \) of a symmetric GAP \( P \), and by the image \( \tilde{K} = \times_{j=1}^d \tilde{K}_j \) of proper symmetric GAPs \( \tilde{P} \) and \( \tilde{P} \) of ranks \( \leq R = r_1 + \cdots + r_d \). At least \( n - 2 \sum_{j=1}^d n_j' \) elements of a belong to each of the sets \( \times_{j=1}^d [K_j^*]_\delta , \times_{j=1}^d [K_j]_\delta , \times_{j=1}^d [\tilde{K}_j]_\delta , \times_{j=1}^d [\tilde{K}_j]_\delta \). Furthermore,

\[
|\times_{j=1}^d K_j^*| \ll_d \prod_{j=1}^d m_j \leq \prod_{j=1}^d w_j \tag{29}
\]

\[
|\times_{j=1}^d K_j| \ll_d \prod_{j=1}^d (c_5 r_j)^{3 r_j^2/2} m_j \leq \prod_{j=1}^d (c_5 r_j)^{3 r_j^2/2} w_j \tag{30}
\]

\[
|\times_{j=1}^d \tilde{K}_j| \ll_d \prod_{j=1}^d (c_6 r_j)^{15 r_j^2/2} m_j \leq \prod_{j=1}^d (c_6 r_j)^{15 r_j^2/2} w_j \tag{31}
\]

and

\[
|\times_{j=1}^d \tilde{K}_j| \ll_d \prod_{j=1}^d (c_7 r_j)^{21 r_j^2/2} m_j \leq \prod_{j=1}^d (c_7 r_j)^{21 r_j^2/2} w_j \tag{32}
\]

8. Let \( l_j \) be a short notation for \( \tilde{l}_j, \tilde{l}_j, l_j \), the number of generators of \( P_j, \tilde{P}_j, \tilde{P}_j \) respectively. The generators \( \overline{g}_s, \overline{g}_s, \overline{g}_s \in \mathbb{R}^d, s = 1, \ldots, l_1 + \cdots + l_d \), of the GAPs \( \overline{P}, \overline{P}, \tilde{P} \) respectively have only one non-zero coordinate each. Denote

\[
s_0 = 0 \quad \text{and} \quad s_k = \sum_{j=1}^k l_j, \quad k = 1, \ldots, d.
\]

For \( s_{k-1} < s \leq s_k \), the generators \( \overline{g}_s, \overline{g}_s, \overline{g}_s \) are non-zero in the \( k \)-th coordinates only and these coordinates are equal to the sequence of generators \( \overline{g}_1^{(k)}, \ldots, \overline{g}_{l_k}^{(k)}; \overline{g}_1^{(k)}, \ldots, \overline{g}_{l_k}^{(k)}; \overline{g}_1^{(k)}, \ldots, \overline{g}_{l_k}^{(k)} \) of the GAPs \( \overline{P}_k, \overline{P}_k, \tilde{P}_k \) respectively satisfying inequality

\[
\max\{|\overline{g}_p^{(k)}|, |\tilde{g}_p^{(k)}|\} \leq 2 r_k \|a^{(k)}\|/\sqrt{n_k}, \quad p = 1, \ldots, l_k.
\]

The following Theorem 5 was obtained in [23] with the use of Theorem 2 (see [23] Theorem 3 and Proposition 1).

**Theorem 5.** Let \( d \geq 1, 0 < \varepsilon \leq 1, 0 < \theta \leq 1, A > 0, B > 0, C_3 > 0 \) be constants and \( \tau_n \geq 0 \) be a parameter that may depend on \( n \). Let \( X \) be a real random variable satisfying condition [23] with \( C_1 = 1, C_2 = \infty \) and \( C_3 \leq p(1) \). Suppose that \( a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n \)
is a multi-subset of $\mathbb{R}^d$ such that $q_j = Q(F_a^{(j)}, \tau_n) \geq n^{-A}$, $j = 1, \ldots, d$, where $F_a^{(j)}$ are distributions of coordinates of the vector $S_a$. Let $p_n$ denote a non-random sequence satisfying $n^{-B} \leq p_n \leq 1$ and let $\delta_n = \tau_n p_n$. Then, for any number $n'$ such that $\varepsilon n^\theta \leq n' \leq n$, there exists a proper symmetric GAP $P$ such that

1. At least $n - \varepsilon n'$ elements $a_j$ of $a$ are $\delta_n$-close to the image $K$ of the GAP $P$ in the norm $| \cdot |$.

2. $P$ has small rank $R = O(1)$, and small size

$$\text{size}(P) = |K| \leq \prod_{j=1}^d \max \left\{ O\left(q_j^{-1} \rho_n^{-1}(n')^{-1/2}\right), 1 \right\}. \quad (33)$$

**Remark 6.** In the first version of the preprint of the paper [9], the GAP $K$ may be non-proper in Theorem 5. In order to get the properness of $K$ we have moreover used arguments from a paper of Tao and Vu [18].

Theorems 6 and 7 below are consequences of Theorem 3.

**Theorem 6.** Let $b_n > 0$, $n = 1, 2, \ldots, d$, be a (depending on $n$) sequence of non-random parameters tending to infinity as $n \to \infty$. Let $A, \theta, \varepsilon_1, \varepsilon_2 > 0$ be constants, and $p(0) > 0$. Suppose that $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ is a multi-subset of $\mathbb{R}^d$ such that $q_j = Q(F_a^{(j)}, 0) \geq \varepsilon_1 b_n^{-A}, j = 1, \ldots, d$, where $F_a^{(j)}$ are distributions of coordinates of the vector $S_a$. Then, for any numbers $n'_j$ such that $\varepsilon_2 b_n^\theta \leq n'_j \leq n, j = 1, \ldots, d$, there exists a proper symmetric GAP $P$ such that

1. At least $n - 2 \sum_{j=1}^d n'_j$ elements of $a$ belong to the image $K$ of the GAP $P$.

2. $P$ has small rank $L \leq R = r_1 + \cdots + r_d = O(1)$, and small size

$$\text{size}(P) = |K| \leq \prod_{j=1}^d \max \left\{ O\left(q_j^{-1} (n'_j)^{-1/2}\right), 1 \right\}. \quad (34)$$

3. Moreover, $K = \times_{j=1}^d K_j$, where $K_j$ are images of one-dimensional symmetric GAPs $P_j$ of rank $l_j \leq r_j$, and the generators $g_s, s = 1, \ldots, L = l_1 + \cdots + l_d$, of the GAP $P$ of rank $L$ have only one non-zero coordinate each. Denote

$$s_0 = 0 \quad \text{and} \quad s_k = \sum_{j=1}^k l_j, \quad k = 1, \ldots, d.$$ 

For $s_{k-1} < s \leq s_k$, the generators $g_s$ are non-zero in the $k$-th coordinates only and these coordinates are equal to the sequence of generators $g_{1}^{(k)}, \ldots, g_{r_k}^{(k)}$ of the GAPs $P_k$, satisfying the inequality $|g_p^{(k)}| \leq 2 r_k \|a^{(k)}\| / \sqrt{n_k}, \quad p = 1, \ldots, l_k.$
Theorem 7. Let $\theta, A, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ and $B, D \geq 0$ be constants, and $\theta > D$. Let $b_n, \kappa_n, \delta_n, \tau_n, \rho_n > 0$, $n = 1, 2, \ldots$, be depending on $n$ non-random parameters satisfying the relations $p(\tau_n/\kappa_n) \geq \varepsilon_3 b_n^D$, $\varepsilon_4 b_n^B \leq \rho_n = \delta_n/\kappa_n \leq 1$, $\delta_n \leq \max\{\kappa_n, \tau_n\}$, for all $n \in \mathbb{N}$, and $b_n \to \infty$ as $n \to \infty$. Suppose that $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ is a multi-subset of $\mathbb{R}^d$ such that $q_j = Q(F_a^{(j)}, \tau_n) \geq \varepsilon_1 b_n^{-A}$, $j = 1, \ldots, d$, $n = 1, 2, \ldots$, where $F_a^{(j)}$ are distributions of coordinates of the vector $S_n$. Then, for any numbers $n_j'$ such that $\varepsilon_2 b_n^D \leq n_j' \leq n$, $j = 1, \ldots, d$, there exists a proper symmetric GAP $P$ such that

1. At least $n - 2 \sum_{j=1}^d n_j'$ elements of $a$ are $\delta_n$-close to the image $K$ of the GAP $P$ in the norm $| \cdot |$.

2. $P$ has small rank $L \leq R = r_1 + \cdots + r_d = O(1)$, and small size

$$|K| \leq \prod_{j=1}^d \max \left\{ O(q_j^{-1} \rho_n^{-1} (n_j' p(\tau_n/\kappa_n))^{-1/2}), 1 \right\}. \quad (35)$$

3. Moreover, the properties of the generators $g_s$ of the GAP $P$ described in the item 3 of the formulation of Theorem 6 are still satisfied. In particular, the inequality $\|g_s\| \leq 2 r_k \|a^{(k)}\| / \sqrt{n_k}$ hold, for $s_{k-1} < s \leq s_k$.

Theorem 7 is more general than Theorem 5 where we restricted ourselves to the case $b_n = n$, $\kappa_n = \tau_n$, $n_j' = n'$ only. Theorem 7 provides new substantial information if, for instance, the ratio $\tau_n/\delta_n$ is large and if $p(\tau_n/\kappa_n)$ is not too small.

In applications of Theorem 7 it is sometimes useful to minimize the parameter $\delta_n$ responsible for the size of the neighborhood of the set $K$. Assume, for simplicity, that $\delta_n = \kappa_n$, for all $n \in \mathbb{N}$. Then the condition $p(\tau_n/\delta_n) = p(\tau_n/\kappa_n) \geq \varepsilon_3 b_n^{-D}$ is satisfied for larger values of $\tau_n/\delta_n$ if the function $p(x)$ decreases slowly as $x \to \infty$, that is, if the distribution $\mathcal{L}(\tilde{X})$ has heavy tails. Moreover, it is clear that for any function $f(n)$ tending to infinity as $n \to \infty$ there exists a distribution $\mathcal{L}(\tilde{X})$ such that $p(\tau_n/\delta_n) \geq \varepsilon_3 b_n^{-D}$ and $\tau_n/\delta_n \geq f(n)$, for sufficiently large $n$.

A discussion concerning the comparison of Theorem 5 with the results of Nguyen, Tao and Vu [14] [15] [19] [20] is given in [8] and [9].

A few years ago Tao and Vu [19] formulated in the discrete case (with $\tau_n = 0$) the so-called 'inverse principle', stating that

A set $a = (a_1, \ldots, a_n)$ with large small ball probability must have strong additive structure.

Here "large small ball probability" means that $Q(F_a, 0) = \max_x P\{S_n = x\} \geq n^{-A}$ with some constant $A > 0$. "Strong additive structure" means that a large part of vectors $a_1, \ldots, a_n$ belong to a GAP with bounded size.

Nguyen and Vu [14] have extended this inverse principle to the continuous case (with $\tau_n > 0$) proving, in particular, the following result.
Theorem 8. Let $X$ be a real random variable satisfying condition (3) with positive constants $C_1, C_2, C_3$. Let $0 < \varepsilon < 1$, $A > 0$ be constants and $\tau_n > 0$ be a parameter that may depend on $n$. Suppose that $a = (a_1, \ldots, a_n) \in (\mathbb{R}^d)^n$ is a multi-subset of $\mathbb{R}^d$ such that $q = Q(F_a, \tau_n) \geq n^{-A}$. Then, for any number $n'$ between $n^\varepsilon$ and $n$, there exists a symmetric proper GAP $P$ with image $K$ such that

1. At least $n - n'$ elements of $a$ are $\tau_n$-close to $K$.
2. $P$ has small rank $r = O(1)$, and small size $|K| \leq \max\{O(q^{-1}(n')^{-1/2}), 1\}$. (36)

3. There is a non-zero integer $p = O(\sqrt{n'})$ such that all generators $g_j$ of $P$ have the form $g_j = (g_{j1}, \ldots, g_{jd})$, where $g_{jk} = \|a\| \tau_n p_{jk} / p$ with $p_{jk} \in \mathbb{Z}$ and $p_{jk} = O(\tau_n^{-1} \sqrt{n'})$.

Theorem 8 allows us to derive a one-dimensional version of the first two statements of Theorem 8. Theorem 7 contains an analogue of the third one. Moreover, in Theorem 7 the generators of approximating GAPs have norms bounded from above by the quantities with $\sqrt{n'_k}$ in the denominator, in contrast with Theorem 8. Furthermore, the condition $Q(F_a, \tau_n) \geq n^{-A}$ of Theorem 8 implies the condition $Q(F_a^{(j)}, \tau_n) \geq n^{-A}$, $j = 1, \ldots, d$, of Theorem 5 since $Q(F_a^{(j)}, \tau_n) \geq Q(F_a, \tau_n)$. In addition, $C_1, C_2, C_3$ are fixed finite constants in Theorem 8 while $C_2 = \infty$ in Theorems 2 and $C_1 = \tau_n / \tau_m$, $C_3 = \varepsilon_3 b_n^{-p}$ in Theorem 7. Notice that $p(1)$ and $p(\tau_n / \tau_m)$ are involved in our Theorems 5 and 7 explicitly. Theorem 8 corresponds to the case $b_n = n$ in the more general Theorems 6 and 7 in which, however, number-theoretical properties of generators (as in Theorem 8 item 3) are not provided. We would like to emphasize the non-asymptotic character of Theorems 2, 3 and 4.

For technical reasons, we use in our results the quantity $2 n'$ instead of $n'$ for the number of points which can be not approximated. It is clear that this difference is not significant.

We have to say that there are some results from [14, 15, 19, 20] which do not follow from the results of Arak. In particular, we don’t consider distributions on general additive groups.

Sometimes, for $d > 1$, inequality (33) (with $\rho_n = 1$) or inequality (35) (with $\rho_n = 1$, $\kappa_n = \tau_n$, $b_n = n$) may be even stronger than inequality (36). For example, if the vector $S_a$ has independent coordinates (this may happen if each of the vectors $a_j$ has only one non-zero coordinate), then

$$q = Q(F_a, \tau) \asymp_d \prod_{j=1}^d q_j.$$ (37)

Assuming, for simplicity, that $q_1 = \cdots = q_d = n^{-2\alpha}$, for some constant $0 < \alpha < 1$, in this case we have $q \asymp_d q_1^d = n^{-2\alpha d}$. Applying Theorem 7 with $p(1) = 1/2$, $n'_1 = \cdots = n'_d = n^{2\alpha}$, we obtain the bound $|K| \ll_d n'^{d\alpha}$ under the conditions of Theorem 8 with $n' = 2 d n'_1$. Theorem 8 itself gives in this case the bound $|K| \ll_d n^{(2d-1)\alpha}$ only (which is worse for $d > 1$).
Clearly, Theorem 7 can provide stronger bounds than Theorem 8 even if relation (33) is replaced by some weaker conditions, for example, if
\[ q \ll_d \left( \prod_{j=1}^{d} q_j \right)^{\beta}, \quad \text{for some } \beta \geq 1/2. \] (38)

Conditions (37) or (38) are usually satisfied for non-degenerate distributions \( F_a \), for example, if \( F_a \) is close to a non-degenerate Gaussian distribution.

Theorem 8 will be stronger than Theorem 7 if
\[ q \asymp_d q_1 = \cdots = q_d. \] (39)

This is possible, for instance, if the distribution \( F_a \) is close to almost degenerate one which is concentrated in a neighborhood of a one-dimensional subspace. Degenerate distributions can turn, however, into non-degenerate ones after applying linear operators.

Note that we could derive a multivariate analogue of Theorem 8 from its one-dimensional version arguing precisely as in the proof of our Theorem 5. Then we get inequality (33) instead of (36).

Similarly as in [9], now we state analogues of Theorems 3–7 for GAPs of logarithmic rank and with special dimensions, all equal to 1. Theorems 9–12 below extend Theorems 5–7 in [9].

**Remark 7.** In Theorems 9–12, we use the convention 0/0=1.

**Theorem 9.** Let the conditions of Theorem 3 be satisfied, except for conditions (21) and (22). Then there exist an absolute positive constant \( c_8 \) and a GAP \( P \) of rank \( r \in \mathbb{N} \), of volume \( 3^r \), with generators \( g_k \in \mathbb{R}, \ k = 1, \ldots, r \), and such that its image \( K \subset \mathbb{R} \) has the form
\[ K = \left\{ \sum_{k=1}^{r} m_k g_k : m_k \in \{-1, 0, 1\}, \ \text{for } k = 1, \ldots, r \right\}. \] (40)

Moreover, in the case \( \tau > 0 \) we have
\[ r \leq c_8 \left( |\log q| + \log(\kappa/\delta) + 1 \right), \] (41)
and at least \( n-n' \) elements of \( a \) are \( \delta \)-close to \( K \), where \( n' \in \mathbb{N} \) and
\[ n' \leq c_8 \left( p(\tau/\kappa) \right)^{-1} \left( |\log q| + \log(\kappa/\delta) + 1 \right)^3. \] (42)

In the case \( \tau = 0 \) we have
\[ r \leq c_8 \left( |\log q| + 1 \right), \] (43)
and at least \( n-n' \) elements of \( a \) belong to \( K \), where
\[ n' \leq c_8 \left( p(0) \right)^{-1} \left( |\log q| + 1 \right)^3. \] (44)
Theorem 10. Let the conditions of Theorem 4 be satisfied, except for conditions (25), (26). Below \(c_8\) denotes the absolute positive constant from Theorem 9. Then, for each \(j = 1, \ldots, d\), there exists a GAP \(P_j \subset \mathbb{R}\) of rank \(r_j \in \mathbb{N}\), of volume \(3^{r_j}\), with generators \(g_k^{(j)} \in \mathbb{R}\), \(k = 1, \ldots, r_j\), and such that its image \(K_j \subset \mathbb{R}\) has the form

\[
K_j = \left\{ \sum_{k=1}^{r_j} m_k g_k^{(j)} : m_k \in \{-1, 0, 1\}, \text{ for } k = 1, \ldots, r_j \right\}. \tag{45}
\]

Moreover, in the case \(\tau_j > 0\) we have

\[
r_j \leq c_8 \left( |\log q_j| + \log(\zeta_j/\delta_j) + 1 \right), \tag{46}
\]

and at least \(n - n'_j\) elements of \(a^{(j)}\) are \(\delta_j\)-close to \(K_j\), where \(n'_j \in \mathbb{N}\) satisfy

\[
n'_j \leq c_8 \left( p(\tau_j/\zeta_j) \right)^{-1} \left( |\log q_j| + \log(\zeta_j/\delta_j) + 1 \right)^3. \tag{47}
\]

In the case \(\tau_j = 0\) we have

\[
r_j \leq c_8 \left( |\log q_j| + 1 \right), \tag{48}
\]

and at least \(n - n'_j\) elements of \(a^{(j)}\) belong to \(K_j\), where

\[
n'_j \leq c_8 \left( p(0) \right)^{-1} \left( |\log q_j| + 1 \right)^3. \tag{49}
\]

Define \(K = \times_{j=1}^d K_j\). Then the set \(K\) is the image of the \(d\)-dimensional GAP \(P\) with rank

\[
R = \sum_{j=1}^d r_j \leq c_8 \sum_{j=1}^d \left( |\log q_j| + \log(\zeta_j/\delta_j) + 1 \right), \tag{50}
\]

and such that at least \(n - \sum_{j=1}^d n'_j\) elements of \(a\) belong to the set \(\times_{j=1}^d [K_j]_{\delta_j}\). Here

\[
\sum_{j=1}^d n'_j \leq c_8 \sum_{j=1}^d \left( p(\tau_j/\zeta_j) \right)^{-1} \left( |\log q_j| + \log(\zeta_j/\delta_j) + 1 \right)^3. \tag{51}
\]

Furthermore, the set \(K\) can be represented as

\[
K = \left\{ \sum_{k=1}^R m_s g_s : m_s \in \{-1, 0, 1\}, \text{ for } s = 1, \ldots, R \right\}. \tag{52}
\]

Moreover, every vector \(g_s \in \mathbb{R}^d\), \(s = 1, \ldots, R\), has one non-zero coordinate only. Denote

\[
s_0 = 0 \quad \text{and} \quad s_j = \sum_{m=1}^j r_m, \quad j = 1, \ldots, d.
\]

For \(s_{j-1} < s \leq s_j\), the vectors \(g_s\) are non-zero in the \(j\)-th coordinates only and these coordinates are equal to the elements of the sequence \(g_1^{(j)}, \ldots, g_{r_j}^{(j)}\) from (45).
Theorem 11. Let $A > 0$ and $B \geq 0$ be constants. Let $b_n, \kappa_n, \delta_n, \tau_n > 0$, $n = 1, 2, \ldots$, be depending on $n$ non-random parameters satisfying the relations $b_n^B \leq \delta_n/\kappa_n \leq 1$, $\delta_n \leq \max\{\kappa_n, \tau_n\}$, for all $n \in \mathbb{N}$, and $b_n \to \infty$ as $n \to \infty$. Let $q_j = Q(F_a^{(j)}, \tau_n) \geq b_n^{-A}$, for $j = 1, \ldots, d$. Then, for each $j = 1, \ldots, d$, there exists a GAP $P_j \subset \mathbb{R}$ of rank $r_j$, of volume $3^{r_j}$, with generators $g_k^{(j)} \in \mathbb{R}$, $k = 1, \ldots, r_j$, and such that its image $K_j \subset \mathbb{R}$ has the form

$$K_j = \left\{ \sum_{k=1}^{r_j} m_k g_k^{(j)} : m_k \in \{-1, 0, 1\}, \text{ for } k = 1, \ldots, r_j \right\}. \quad (53)$$

Moreover, the set $K = \times_{j=1}^{d} K_j$ is the image of the $d$-dimensional GAP $P$ with rank

$$R = \sum_{j=1}^{d} r_j \ll d \left( (A + B) \log b_n + 1 \right), \quad (54)$$

and such that at least $n - n'$ elements of a belong to the set $\times_{j=1}^{d} [K_j]^{\delta_n}$. Here $n' \in \mathbb{N}$ and

$$n' \ll d \left( p(\tau_n/\kappa_n) \right)^{-1} \left( (A + B) \log b_n + 1 \right)^3. \quad (55)$$

Furthermore, the description of the set $K$ at the end of the formulation of Theorem 11 remains true.

Theorem 12. The statements of Theorems 10 and 11 hold when replacing $p(\tau_j/\kappa_j)$ or $p(\tau_n/\kappa_n)$ by $p(0)$ in the particular case, where the parameters $\tau_j$, $j = 1, \ldots, d$, or $\tau_n$, $n \in \mathbb{N}$, involved in the formulations of these theorems, are all zero.

In Theorems 5–7 in [9], we obtained particular cases of our Theorems 9–12, where $b_n = n$ and $\tau = \kappa$, $\tau_j = \kappa_j$, $j = 1, \ldots, d$, or $\tau_n = \kappa_n$, $n \in \mathbb{N}$.

In Theorems 9–12 the approximating GAP may be non-proper. We could try to get the results with proper GAPs as in Theorems 3–7, but then we will lose the nice representations for the images of GAPs, see (40), (45), (53). Moreover, the ranks of GAPs will be too large.

3. PROOF OF THEOREM 3

Proof of Theorem 3. Applying Theorem 2 with $\delta > 0$, we derive that, for $r \in \mathbb{N}_0$, $m \in \mathbb{N}$,

$$Q(F_a, \tau) \leq c_4^{r+1} \frac{\kappa}{\delta} \left( \frac{1}{m \sqrt{p(\tau/\kappa) \beta_{r,m}(M^*, \delta)}} + \frac{(r + 1)^{5r/2}}{(p(\tau/\kappa) \beta_{r,m}(M^*, \delta))^{(r+1)/2}} \right), \quad \text{if } \tau > 0, \quad (56)$$

and

$$Q(F_a, \tau) \leq c_4^{r+1} \left( \frac{1}{m \sqrt{p(0) \beta_{r,m}(M^*, \delta)}} + \frac{(r + 1)^{5r/2}}{(p(0) \beta_{r,m}(M^*, \delta))^{(r+1)/2}} \right), \quad \text{if } \tau = 0, \quad (57)$$

with $c_4 = 2c_3$, where $c_3$ is the constant from Theorem 2. We assert that the $c_4$ in (56)–(57) may be taken as the $c_4$ in Theorem 3.
Let $r \in \mathbb{N}_0$ be fixed and $\tau > 0$. Choose now a positive integer $m = \lfloor y \rfloor + 1$, where

$$y = \frac{2c_4^{r+1} \kappa}{q \delta \sqrt{p(\tau / \kappa)} n'} \leq m. \quad (58)$$

Assume that $\beta_{r,m}(M^*, \delta) > n'$. Recall that $n' \geq (2c_4^{r+1}(r+1)^{5r/2} \kappa / q \delta)^{2/(r+1)} / p(\tau / \kappa)$. Then, using (56) and our assumptions, we have

$$q < q/2 + q/2 = q. \quad (59)$$

This leads to a contradiction with the assumption $\beta_{r,m}(M^*, \delta) > n'$. Hence we conclude $\beta_{r,m}(M, \delta) \leq \beta_{r,m}(M^*, \delta) \leq n'$.

This means that at least $n - n'$ elements of $a$ are $\delta$-close to a CGAP $K \in K_{r,m}$ admitting representation (16), where $h$ is a r-dimensional vector, $V$ is a symmetric convex subset of $\mathbb{R}^r$ containing not more than $m$ points with integer coordinates. Now equality (58) implies the inequality

$$m \leq \frac{2c_4^{r+1} \kappa}{q \delta \sqrt{p(\tau / \kappa)} n'} + 1, \quad \text{if } \tau > 0. \quad (60)$$

Without loss of generality we can assume that the absolute values of $a_k$, $k = 1, \ldots, n$, are non-increasing: $|a_1| \geq \cdots \geq |a_n|$. Then, it is easy to see that $|a_n| \leq \cdots \leq |a_{n'+1}| \leq \|a\| / \sqrt{n'}$.

If $\delta > \|a\| / \sqrt{n'}$, we can take as $K^*$ the GAP having zero as the unique element. Then $a_k \in [K^*]_\delta$, $k = n'+1, \ldots, n$.

Let $\delta \leq \|a\| / \sqrt{n'}$ and

$$V^* = V \cap \{x \in \mathbb{R}^r : \langle x, h \rangle \leq 2 \|a\| / \sqrt{n'}\} \quad (61)$$

In this case we take

$$K^* = \{\langle \nu, h \rangle : \nu \in \mathbb{Z}^r \cap V^*\} \subset \mathbb{R}. \quad (62)$$

It is clear that $V^* \subset V$ is a symmetric convex subset of $\mathbb{R}^r$ and $K^* \in K_{r,m}$. Moreover,

$$K^* = K \cap [-2 \|a\| / \sqrt{n'}, 2 \|a\| / \sqrt{n'}]. \quad (63)$$

If $a_k \in [K]_\delta$ and $|a_k| \leq \|a\| / \sqrt{n'}$, then $a_k \in [K^*]_\delta$. Thus, only $n'$ elements of $a$, namely, $a_1, \ldots, a_{n'}$, may be contained in $[K]_\delta$ and not contained in $[K^*]_\delta$. Therefore, at least $n - 2n'$ elements of $a$ are $\delta$-close to the CGAP $K^* \in K_{r,m}$.

By Lemma 3 and Corollary 2 there exists a symmetric, infinitely proper GAP $P$ in $\mathbb{Z}^r$ with rank $\overline{t} \leq r$ such that we have

$$\text{Image}(P) \subset V^* \cap \mathbb{Z}^r \subset \text{Image}(P_0), \quad P_0 = P(c_1 r)^{3r/2}, \quad (64)$$

with absolute constant $c_1 \geq 1$ from the statement of Lemma 3 and

$$\text{size}(P_0) \leq (2(c_1 r)^{3r/2} + 1)^r |V^* \cap \mathbb{Z}^r| \leq (2(c_1 r)^{3r/2} + 1)^r |V \cap \mathbb{Z}^r| \leq (2(c_1 r)^{3r/2} + 1)^r m. \quad (65)$$

Moreover, the generators $g_j$ of $P_0$, for $1 \leq j \leq \overline{t}$, are contained in the symmetric body $\overline{IV}^*$.

Let $\phi : \mathbb{R}^r \to \mathbb{R}$ be a linear map defined by $\phi(y) = \langle y, h \rangle$, where $h \in \mathbb{R}^r$ is involved in the definition of $K$ and $K^*$. Define now the symmetric GAP $\overline{P}$ with $\text{Image}(\overline{P}) = \overline{K} = \{\phi(y) :
Then zero as the unique element. Then \( K^* \subset \overline{K} \subset \mathbb{R} \) and \( \overline{P} \) is a symmetric GAP of rank \( l \) and size \( \ll (c_5 r)^{3r^2/2} m \). The generators \( \overline{g}_j \) are contained in the set \( \{ \phi(y) : y \in \overline{N}^* \} \). Hence, they satisfy inequality \( |\overline{g}_j| \leq 2r \|a\|/\sqrt{m} \) (see (62) and (63)).

Applying Lemma 4 to the GAP \( \overline{P} \), we see that, for any \( t \geq 1 \), there exists a \( t \)-proper symmetric one-dimensional GAP \( \overline{P} \) with rank(\( \overline{P} \)) \( \leq \) rank(\( P \)), Image(\( \overline{P} \)) \( \subset \) Image(\( \overline{P} \)) \( \subset \mathbb{R} \), and

\[
\text{size}(\overline{P}) \leq \text{size}(\overline{P}) \leq (2t)^{r r^{6r^2}} \text{size}(\overline{P}) \ll (2t)^{r r^{6r^2}} (c_5 r)^{3r^2/2} m. \tag{66}
\]

The statement of item 4 follows from (66) if we take \( t = 1 \).

Let now \( t = (c_8 r)^{3r^2/2} \), where \( c_8 \) is a sufficiently large absolute constant such that \( c_8^{3r^2/2} \geq 2r^{5/2} c_1^{3r^2/2} \). Let \( \overline{P} = (\overline{L}, \overline{g}, k) \), that is,

\[
\overline{K} = \text{Image}(\overline{P}) = \{m_1 \overline{g}_1 + \cdots + m_k \overline{g}_k : -\overline{L}_j \leq m_j \leq \overline{L}_j, \ m_j \in \mathbb{Z}, \ 	ext{for all } j \text{ such that } 1 \leq j \leq k = \text{rank}(\overline{P}) \leq r \} \tag{67}
\]

with some generators \( \overline{g}_j \in \mathbb{R} \).

Let us prove items 5 and 6. If \( \delta > \|a\|/\sqrt{m} \), then we can take as \( K^{**} \) the GAP having zero as the unique element. Then \( a_k \in [K^{**}]_\delta, k = n' + 1, \ldots, n \).

Let \( \delta \leq \|a\|/\sqrt{m} \). It is easy to see that Image(\( \overline{P} \)) \( \subset \mathbb{R} \) is the image of the box \( B = \{ (m_1/\overline{L}_1, \ldots, m_k/\overline{L}_k) : (m_1, \ldots, m_k) \in \mathbb{Z}^k : -\overline{L}_j \leq m_j \leq \overline{L}_j \text{ for all } j = 1, \ldots, k \} \) under the linear map

\[
\Phi : (m_1/\overline{L}_1, \ldots, m_k/\overline{L}_k) \to m_1 \overline{g}_1 + \cdots + m_k \overline{g}_k.
\]

Let now

\[
W = \{ x = (x_1, \ldots, x_k) \in \mathbb{R}^k : |x_j| \leq 1 \text{ for all } j = 1, \ldots, k \} \tag{68}
\]

and

\[
W_1 = \{ tx \in \mathbb{R}^k : x \in W \} = \{ x \in \mathbb{R}^k : |x_j| \leq t \text{ for all } j = 1, \ldots, k \}.
\]

Let

\[
\Lambda = \{ (m_1/\overline{L}_1, \ldots, m_k/\overline{L}_k) : (m_1, \ldots, m_k) \in \mathbb{Z}^k \}.
\]

Obviously, \( \Lambda \) is a lattice in \( \mathbb{R}^k \). Let \( u = (u_1, \ldots, u_k) \in \mathbb{R}^k \), where \( u_j = \overline{L}_j \overline{g}_j, \text{ for } j = 1, \ldots, k \). Then

\[
\overline{K} = \text{Image}(\overline{P}) = \{ (\nu, u) : \nu \in \Lambda \cap W \}.
\]

Define

\[
W^* = W \cap \{ x \in \mathbb{R}^k : |\langle x, u \rangle| \leq 2 \|a\|/\sqrt{m} \} \tag{69}
\]

Now we define

\[
K^{**} = \{ (\nu, u) : \nu \in \Lambda \cap W^* \} \subset \mathbb{R}. \tag{70}
\]

It is clear that \( W^* \subset W \) is a symmetric convex subset of \( \mathbb{R}^k \) and \( K^{**} \) is a CGAP. Moreover,

\[
K^{**} = \overline{K} \cap [-2 \|a\|/\sqrt{m}, 2 \|a\|/\sqrt{m}]. \tag{71}
\]
If \( a_k \in [K]_\delta \) and \( |a_k| \leq \|a\|/\sqrt{n'} \), then \( a_k \in [K^{**}]_\delta \). Thus, only \( n' \) elements of \( a \), namely, \( a_1, \ldots, a_{n'} \), may be contained in \( [K]_\delta \setminus [K^{**}]_\delta \). Note that while counting approximated points, we have already taken into account that these elements may be not approximated. Therefore, at least \( n - 2n' \) elements of \( a \) are \( \delta \)-close to the CGAP \( K^{**} \).

By Lemma 3 and Corollary 2, there exists a symmetric, infinitely proper GAP \( R = (N, w, \tilde{l}) \) in \( \Lambda \) with rank \( \tilde{l} \leq k \leq r \) such that we have
\[
\text{Image}(R) \subset \Lambda \cap W^* \subset \text{Image}(R_0), \quad R_0 = R(c_1 r)^{3r/2},
\]
and
\[
\text{size}(R_0) \leq (2 (c_1 r)^{3r/2} + 1)^r |\Lambda \cap W^*| \leq (2 (c_1 r)^{3r/2} + 1)^r |\Lambda \cap W| \leq (2 (c_1 r)^{3r/2} + 1)^r (2 t)r^6 r^{6r} (c_5 r)^{3r^2/2} m. \tag{73}
\]

Moreover, the common generators \( w_j \) of \( R \) and \( R_0 \), for \( 1 \leq j \leq \tilde{l} \), are contained in the symmetric body \( \tilde{l}W^* \subset \tilde{l}W \subset rW \) and \( w_j \in \Lambda \). In particular, together with (68) this implies that \( \|w_j\| \leq r^{3/2} \).

The linear map \( \Phi : R^k \to R \) can be written as \( \Phi(y) = \langle y, u \rangle \), \( y \in R^k \), where \( u \in R^k \) is defined above. Define now the symmetric GAP \( \tilde{P} \) with \( \text{Image}(\tilde{P}) = \tilde{K} = \{ \Phi(y) : y \in \text{Image}(R_0) \} \) and with generators \( \tilde{g}_j = \Phi(w_j) \), \( j = 1, \ldots, \tilde{l} \). Obviously, \( K^{**} \subset \tilde{K} \subset R \) and \( \tilde{P} \) is a symmetric GAP of rank \( \tilde{l} \). The generators \( \tilde{g}_j \) are contained in the set \( \{ \Phi(y) : y \in \tilde{l}W^* \} \). Hence, they satisfy the inequality \( |\tilde{g}_j| \leq 2r \|a\|/\sqrt{n'} \) (see (70) and (71)).

Let \( \nu \in \text{Image}(R_0) \) and \( \tilde{l} = (c_1 r)^{3r/2} \). Then \( \nu \) has a unique representation in the form
\[
\nu = m_1 w_1 + \cdots + m_\tilde{l} w_\tilde{l},
\]
where \( \tilde{N}_j \leq m_j \leq \tilde{N}_j, \) \( m_j \in \mathbb{Z} \), for all \( j \) such that \( 1 \leq j \leq \tilde{l} = \text{rank}(R_0) \leq r \). If \( N_j \geq 1 \), then \( [N_j] \geq 1 \) and \( [N_j] w_j \in \text{Image}(R) \subset W \). Therefore, \( \tilde{N}_j \|w_j\| \leq 2 \|w_j\| \leq 2\sqrt{r} \). Thus, for all \( j = 1, \ldots, \tilde{l} \), we have \( N_j \|w_j\| \leq 2t^{3/2} \). Hence, \( \|\nu\| \leq 2t^{3/2} \leq t \) and \( \nu \in \Lambda \cap W_1 \).

Since the GAP \( \tilde{P} \) is \( t \)-proper, all points of the form \( \langle \nu, u \rangle \), \( \nu \in \Lambda \cap W_1 \), are distinct. The same can be said about all points of the form \( \langle \nu, u \rangle \), \( \nu \in \text{Image}(R_0) \subset \Lambda \cap W_1 \). This implies that the GAP \( \tilde{P} \) is proper. The size of the GAP \( \tilde{P} \) coincides with that of \( R_0 \). It is estimated by the right-hand size of (73) which is \( \ll (c_7 r)^{21r^2/2} m \) with an absolute constant \( c_7 \). This completes the proof of Theorem 3 for \( \tau > 0 \).

The case \( \tau = 0 \) can be considered similarly while using (57) instead of (56). Theorem 3 is proved. □

4. Proof of Theorems 6 and 7

Proof of Theorem 6: First we will prove Theorem 6 for \( d = 1 \). Denote \( q = Q(F_2, 0) \). Let \( r = r(A, \theta) \in N_0 \) be the minimal non-negative integer such that \( A < \theta (r + 1)/2 \). Thus,
$r \leq 2A/\theta$ and $b_n^d < b_n^{\theta(r+1)/2}$ for all $b_n > 1$. Recall that $b_n \to \infty$ as $n \to \infty$. Assume without loss of generality that $n$ is so large that

$$
(2c_4^{r+1}(r+1)^{5/2}/q)^{2(r+1)}/p(0) \leq (2c_4^{r+1}(r+1)^{5/2}/\varepsilon_1^{-1} b_n^d)^{2(r+1)}/p(0)
$$

$$
\leq \varepsilon_2 b_n^d \leq n' = n_1' \leq n.
$$

(74)

It remains to apply Theorem 3 with $\tau = \delta = 0$.

If (74) is not satisfied, then $n' \leq n = O(1)$ and we can take as $K$ the set

$$
K(n') = \left\{ \sum_{k=n'+1}^n s_k a_k : s_k \in \{-1, 0, 1\}, \text{ for } k = 1, \ldots, r \right\}.
$$

(75)

Without loss of generality we can assume again that the absolute values of $|a_k|, k = 1, \ldots, n$, are non-increasing: $|a_1| \geq \cdots \geq |a_n|$. Clearly, $K(n')$ is the image of a GAP $P(n')$ of rank $n - n'$ and of size $3^{n-n'}$. The generators of $P(n')$ are $a_{n'+1}, \ldots, a_n$ satisfying $|a_n| \leq \cdots \leq |a_{n'+1}| \leq \|a\|/\sqrt{n'}$. At least $n - n'$ elements of $a$ (namely, $a_{n'+1}, \ldots, a_n$) belong to $K(n')$. Of course, the gap $P(n')$ may be non-proper. In order to find a proper gap $P$, one should proceed like as in the proof of Theorem 3 using Lemma 3 and Corollary 2. Thus Theorem 6 is proved for $d = 1$.

Let us now assume that $d > 1$. We apply Theorem 6 with $d = 1$ to the distributions of the coordinates of the vector $S_a$, taking the vector $a^{(i)} = (a_{ij}, \ldots, a_{nj})$ as vector $a, j = 1, \ldots, d$. Then, for each $a^{(j)}$, there exists a proper symmetric GAP $P_j$ with image $K_j \in K_{r, m_j}$, which satisfies the assertion of Theorem 6 that is:

1. At least $n - 2n'_j$ elements of $a^{(j)}$ are contained in $K_j$;
2. $P_j$ has small rank $l_j \leq r_j = O(1)$, and

$$
m_j \leq \max \left\{ O\left(q_j^{-1}(n'_j)^{-1/2}\right), 1 \right\}.
$$

(76)

3. The generators $g^{(j)}_1, \ldots, g^{(j)}_{l_j}$ of $P_j$ satisfy the inequality $|g^{(j)}_p| \leq 2r_j \|a^{(j)}\|/\sqrt{n_j'}$, for $p = 1, \ldots, l_j$.

Thus, the multi-vector $a$ is well approximated by the GAP $P$ with image $K = \times_{j=1}^d K_j$ of rank $l_1 + \cdots + l_d = L \leq R = r_1 + \cdots + r_d$. At least $n - 2\sum_{j=1}^d n'_j$ elements of $a$ are contained in $K$.

It is easy to see that $K \in K^{(d)}_{r, m}, r = (r_1, \ldots, r_d) \in \mathbb{N}_0^d, m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$, $m_j = \max_{j=1}^d \left\{ O\left(q_j^{-1}(n'_j)^{-1/2}\right), 1 \right\}$.

(77)

$\square$

Proof of Theorem 7 First we will prove Theorem 7 for $d = 1$. Denote $q = Q(F_a, \tau_n)$. Let $r = r(A, B, \theta, D)$ be the minimal positive integer such that $A + B < (\theta - D)(r + 1)/2$. Thus,
\( r \leq 2(A + B)/(\theta - D) \) and \( b_n^{A+B} < b_n^{(\theta-D)(r+1)/2} \) for all \( b_n > 1 \). Recall that \( b_n \to \infty \) as \( n \to \infty \).

Assume without loss of generality that \( n \) is large enough such that
\[
(2c_4^{r+1}(r+1)^{5r/2}\kappa_n/\nu_n)^{2/(r+1)}/p(\tau_n/\kappa_n) \leq \left(2c_4^{r+1}(r+1)^{5r/2}\varepsilon_1^{-1}\varepsilon_4^{-1}b_n^{A+B}\right)^{2/(r+1)}/\varepsilon_2 b_n^D
\]
\leq \varepsilon_2 b_n^D \leq n' \leq n.
(78)

It remains to apply Theorem 3. If (78) is not satisfied, then \( n = \mathcal{O}(1) \) and we can again take as \( K \) the set \( K(n') \) defined in (75). In order to find a proper gap \( P \), one should proceed as in the proof of Theorem 3. Thus Theorem 7 is proved for \( d = 1 \).

Let us now assume that \( d > 1 \). We apply Theorem 7 with \( d = 1 \) to the distributions of the coordinates of the vector \( S_n \), taking the vector \( a^{(j)} = (a_{1j}, \ldots, a_{nj}) \) as vector \( a \), \( j = 1, \ldots, d \). Then, for each \( a^{(j)} \), there exists a proper symmetric GAP \( P_j \) with image \( K_j \in K_{r_j,m_j} \), which satisfies the assertion of Theorem 7 that is:
1. At least \( n - 2n'_j \) elements of \( a^{(j)} \) are \( \delta_n \)-close to \( K_j \);
2. \( P_j \) has small rank \( l_j \leq \mathcal{O}(1) \), and
\[
m_j \leq \max \left\{ O \left(q_j^{-1}\rho_n^{-1}(n'_j p(\tau_n/\kappa_n))^{-1/2}\right), 1 \right\}.
(79)
\]
3. The generators \( g^{(j)} \) of \( P_j \), satisfy the inequality \( |g^{(j)}_p| \leq 2r_j \|a^{(j)}\|/\sqrt{n'_j} \), for \( p = 1, \ldots, l_j \).

Thus, the multi-vector \( a \) is well approximated by the GAP \( K = \times_{j=1}^d K_j \). It is easy to see that \( K \in K^{(d)}_{r,m}, r = (r_1, \ldots, r_d) \in \mathbb{N}^d, m = (m_1, \ldots, m_d) \in \mathbb{N}^d \),
\[
|\times_{j=1}^d K_j| \leq \prod_{j=1}^d m_j \leq \prod_{j=1}^d \max \left\{ O \left(q_j^{-1}\rho_n^{-1}(n'_j p(\tau_n/\kappa_n))^{-1/2}\right), 1 \right\}.
(80)
\]

Since at most \( 2n'_j \) elements of \( a^{(j)} \) are far from the GAPs \( K_j \), there are at least \( n - 2 \sum_{j=1}^d n'_j \) elements of \( a \) that are \( \delta_n \)-close to the GAP \( K \). In view of relation (80) and taking into account that \( K = \times_{j=1}^d K_j \), we obtain relation (35). □

**Remark 8.** Notice that, in Theorems 6 and 7, the ranks of \( P_j \) are actually the same for all \( j = 1, \ldots, d \). Moreover, in Theorems 6 and 7 we get explicit bounds for \( r_j \), for sufficiently large \( n \), namely: \( r_j \leq 2A/\theta \) and \( r_j \leq A/(A + B)/(\theta - D) \) respectively.

5. **Proofs of Theorems** 9-12

**Proof of Theorem** 9. By Corollary 1 we have
\[
q \ll \frac{\kappa}{\delta} Q, \quad \text{where} \quad Q = Q(H^{p(\tau/\kappa)}, \delta).
(81)
\]
Note that \( H^{p(\tau/\kappa)} \) is the symmetric infinitely divisible distribution with Lévy spectral measure \( \frac{p(\tau/\kappa)}{4} M^* \), where \( M^* = \sum_{k=1}^n (E_{a_k} + E_{-a_k}) \).
Applying Theorem 3.3 of Chapter II [3] (see Theorem 4 in [9]), we obtain that there exist \( r \in \mathbb{N}_0 \) and \( g_j \in \mathbb{R}, j = 1, \ldots, r \), such that

\[
r \ll |\log Q| + 1,
\]

(82)

and

\[
p(\tau/\kappa) M^* \{ \mathbb{R} \setminus [K]_0 \} \ll (|\log Q| + 1)^3,
\]

(83)

where \( K \) has the form (40). Recall that \( \delta \leq \kappa \). By (81),

\[
|\log Q| \ll |\log q| + \log \left( \kappa/\delta \right).
\]

(84)

Inequalities (82)–(84) together imply the statement of Theorem 9 in the case \( \tau > 0 \). The case \( \tau = 0 \) is a little bit easier. \( \square \)

Theorems 10–12 are direct consequences of Theorem 9.

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