Scars of the Wigner function.

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We propose a picture of Wigner function scars as a sequence of concentric rings along a two-dimensional surface inside a periodic orbit. This is verified for a two-dimensional plane that contains a classical orbit of a Hamiltonian system with two degrees of freedom. The orbit is hyperbolic and the classical Hamiltonian is “softly chaotic” at the energies considered. The stationary wave functions are the familiar mixture of scarred and random waves, but the spectral average of the Wigner functions in part of the plane is nearly that of a harmonic oscillator and individual states are also remarkably regular. These results are interpreted in terms of the semiclassical picture of chords and centres, which leads to a qualitative explanation of the interference effects that are manifest in the other region of the plane. The qualitative picture is robust with respect to a canonical transformation that bends the orbit plane.

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Sixteen years have passed since Heller [1] detected scars of periodic orbits in individual eigenfunctions of chaotic systems. Explanations in terms of wave packets [1] or the semiclassical Green function [2,3] do predict an enhancement of intensity near the projection of a Bohr-quantized periodic orbit. However, such theories apply to a collective superposition of states near this quantization condition. In spite of some quite striking visual evidence, the issue of scarring for individual states, is confused by the fact that different branches of the same periodic orbit may overlap so that their contributions interfere in the position space and this must be superposed on the expected random wave background for the chaotic state. This has led to the need to make quantitative assessments of scarring strength [4]. We here show that the phase space picture can provide sharper qualitative evidence of the influence of a periodic orbit on its Wigner functions.

The old conjecture of Berry and Voros [3] that the Wigner functions for eigenstates of chaotic Hamiltonians are concentrated on the corresponding classical energy shells is widely disseminated [5], as it is compatible with Schnirelman’s theorem [6]. Nonetheless, no such restriction arises in the more recent semiclassical theory for the mixture of these states over a narrow energy window, i.e.

the spectral Wigner function $[10,11]$. Indeed, the computations presented in this letter show that individual Wigner functions, as well as their energy average can oscillate with large amplitudes deep inside the energy shell.

At points $x = (q, p)$ inside the energy shell, the semiclassical spectral Wigner function is $[10,11]$: $W(x, E, \varepsilon) = \sum_j A_j(x, E) e^{-\varepsilon t_j/\hbar} \cos \left[ \frac{S_j(x, E)}{\hbar} + \gamma_j \right].$ (1)

The sum is over all the trajectory segments on the $E$-shell with endpoints, $x_{j\pm}$, centered on $x$, as sketched in Fig.1. The action is merely the symplectic area, $S_j = \oint q \cdot dq$, for the circuit taken along the orbit from $x_{j-}$ to $x_{j+}$ and closed by the chord $-\xi_j(x)$. The time of traversal for the stretch along the trajectory is $t_j$ and $\varepsilon$ is the width of the energy window over which we average individual Wigner functions $W_n(x)$, i.e.,

$W(x, E, \varepsilon) = (2\pi\hbar)^{D/2} \sum_n \frac{\varepsilon/\pi}{(E - E_n)^2 + \varepsilon^2} W_n(x),$ (2)

where $D$ is the dimension of the phase space $(q, p)$. We shall not be concerned with the phase space $(q, p)$. We shall not be concerned with the Maslov phase $\gamma_j$, nor with the amplitude $A_j(x, E)$, except to note that these are purely classical quantities that vary smoothly with $x$ inside the shell, as compared with the high frequency oscillations of the cosine factor.

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The highly oscillatory nature of the spectral Wigner function inside the energy shell is mostly washed out on projection, i.e. for true probability densities. However, the oscillations are unavoidable in the study of interference effects.

The energy shell itself is a caustic, because the chord $\xi \to 0$ as $x$ approaches the shell. In this limit, the contributing trajectories either shrink to a point, or they are very close to a periodic orbit. The modification of $\xi$ leads to the Berry theory of scars [1], further refined in [13], for evaluation points $x$ close to the energy shell.

The point here is that large open segments of periodic orbits can be constructed by adding multiple windings to a primitive, small segment of a periodic orbit. If conditions for phase coherence, to be discussed, are satisfied, we can then obtain a scar deep inside the energy shell. This scar is located at the two-dimensional central surface constructed by the centers of all the chords with endpoints on the periodic orbit. For the case where the phase space dimension $D = 2$, the central surface is the phase space region enclosed by the convex hull of the orbit that here is the energy shell $^{1}$. For $D \geq 4$, the two-dimensional central surfaces are still bounded by one-dimensional periodic orbits that lie within the higher dimensional energy shell. In the simplest case where the orbit is plane and convex, the central surface coincides with the part of the plane within the orbit just as in the orbit is plane and convex, the central surface coincides everywhere with the invariant surface formed by the energy shell.

We emphasize that these scars of the Wigner function that correspond to a very recognizable oscillatory pattern have special localization properties in phase space of dimension $D \geq 4$. Previous explorations of phase space representations of eigenstates have been too blunt to discern these fine features. The work of Feingold et al. [14] involves a projection from a three-dimensional section, which is fine for the Husimi function, but washes away the details of an oscillatory Wigner function. On the other hand, Agam and Fishman [15] restrict their investigation to the neighbourhood of the orbit and, hence, to the energy shell. This is just the edge of the central surface.

In this letter we tested our prediction for the simplest case of an unstable periodic orbit lying in a two dimensional plane. Accidentally this plane is a symmetry plane, however, we have checked that the enhanced amplitude of the spectral Wigner function accompanies the distortion of the orbit due to a nonlinear canonical transformation.

Since the Wigner function is not invariant in this case, we thus obtain an essentially new system, in which the new central surface is not a symmetry surface. Thus, our scarring effect cannot be attributed to a special property of the symmetry plane. This case is essentially different from recent studies of higher dimensional systems where the scars arise over marginally stable invariant planes that are indeed special [12].

The chord structure for segments of a periodic orbit is sketched in Fig. 2. This is similar to a system with $D = 2$, for which Berry [13] showed that there is only one chord for most points $x$. However, we must now distinguish between the chords $\xi_{in} = -\xi_{out}$ and the sum in $\xi$ includes each different winding of the periodic orbit for primitive orbit segments, “in” and “out”, associated with the two respective chords. Evidently all these “in” and “out” contributions build up if the energy is close to one of the values for which this periodic orbit is Bohr-quantized. We show in [16] that the condition for these two contributions to be in phase is the same as that for a maximal contribution to the Gutzwiller trace formula [7]. Therefore, near the quantization energy, the sum of all the contributions in $\xi$, owing to chords in the periodic orbit, can be approximated by an expression analogous to Berry’s simplest semiclassical approximation [13,17] for the Wigner function in $D = 2$ systems, but now evaluated over points $x$ in the two-dimensional central surface. Hence, in this case, we have an enhanced contribution to the spectral Wigner function with the phase $S_{in}(x)/\hbar$ and the Maslov phase $\gamma = 0$. The successive contours $S_{in}(x) = \text{constant}$ thus determine rings of constant phase along the central surface. Therefore, we predict rings of positive and negative amplitude superimposed on the background of contributions from uncorrelated trajectory segments that also contribute to the spectral Wigner function. The edge of this system of rings along the orbit itself is the object of the Berry theory [3], though it does not deal with multiple windings.

We have tested this prediction for the coupled nonlinear oscillator ($D = 4$)

$$H(q, p) = \frac{(p_1^2 + p_2^2)}{2} + 0.05 q_1^2 + \left( q_2 - \frac{q_1^2}{2} \right)^2 , \quad (3)$$

sometimes referred to as the “Nelson Hamiltonian”. The classical periodic orbits together with the topology of their families in the energy vs. period plot has been studied in [18] and the wave functions and their scars were discussed in [19]. Evidently the plane $q_1 = p_1 = 0$ is classically invariant, so we obtain an isolated periodic orbit on this plane for each energy (the vertical family). This

$^{1}$Note that if the orbit is convex the central surface is just the phase space region inside it.
is then, the simplest case where the central surface is flat - the central plane -. The restriction of the Hamiltonian to this plane defines a harmonic oscillator, so there is a single chord with endpoints in the periodic orbit (except for sign) for each point \((p_2, q_2)\) inside the orbit, except at \(q_2 = p_2 = 0\).

We computed the wave functions \(\langle q|n\rangle\) for the eigenstates of \(\mathcal{H}\) within the range of energies \(0.821 \leq E \leq 0.836\) taking \(\hbar = 0.05\) and using a basis of eigenfunctions \(\phi_{n_1}^{(q_1)}(q_1) \phi_{n_2}^{(q_2)}(q_2 - q_1^2/2)\), in the same way as in [13]. The Wigner function for each state was then calculated directly by the double integral over \(Q = (Q_1, Q_2)\):

\[
W_n(q,p) = \frac{1}{(2\pi \hbar)^2} \int dQ \langle q + Q/2|n\rangle \langle n| q - Q/2 \rangle e^{-i\frac{p\cdot Q}{\hbar}}.
\]

(4)

In Fig.3 we display the average over Wigner functions for the energy window chosen, surrounding the 11’th Bohr energy for the vertical orbit that has Maslov function for each point \((p_2, q_2)\) inside the orbit, except at \(q_2 = p_2 = 0\).

The central surface of the harmonic orbit considered in this work coincides with an invariant plane of the Hamiltonian \(\mathcal{H}\) which is a reflection symmetry plane of the system. This might induce to the conclusion that the symmetry is the actual responsible for the scarring observed. We show that this is not the case by applying a nonlinear canonical transformation, described in [16], deforming the central plane \(q_1 = p_1 = 0\) into a new invariant surface (Fig.3a). Now, the new central surface of the distorted periodic orbit no longer coincides everywhere with the invariant surface (Fig.3b) losing in this way any symmetry. We calculated the Wigner functions for this new system, over the central surface of the periodic orbit, for eigenstates within the same range of energies surrounding the 11’th Bohr level. Both the average and the individual Wigner functions, projected over the \((q_2, p_2)\) plane, display the same features as those shown in Fig.3 and Fig.4 respectively. Reference [16] also considers the thickness of scar surfaces.

Our main point is that individual trajectory segments do play a role in the spectral Wigner function. Indeed, phase coherent contributions from a periodic orbit can add up to scar in the form of concentric rings deep inside the energy shell. Even though the Berry-Voros hypothesis [13] ties in more intuitively with the concept of quantum ergodicity, it should be noted that the chord picture of Wigner functions does not contradict Shnirelman’s theorem and subsequent exact results [3], because the oscillations make negligible contributions to integrals with smooth functions.

It would seem that the structure of interfering chords, from which the spectral Wigner function is built up, could only generate a messy picture where scars would be even harder to recognize than in wave functions for which quantitative methods are needed. We have now shown that the converse can be true in the simplest case, i.e. that the relatively low dimension of the central surface can allow it to miss, over an appreciable region, the contribution of chords from other orbits. In this region we obtain a regular ring structure, not only for the energy average, but even for the pure Wigner functions shown in Fig.3. Though we cannot yet predict why the relative weight of this particular periodic orbit should vary so markedly among the states close to the Bohr-quantized
energy, we are now able to recognize in Fig. 4 a remarkable example of an individual scar.

FIG. 1. In general there are many orbit segments with endpoints \( x_1^- \) and \( x_1^+ \) centered on a given point \( x \). The circuit, closed by the chord \( -\xi = x_1^- - x_1^+ \), defines the phase of the semiclassical contribution to \( \xi^2 \).

FIG. 2. If the endpoints lie on a periodic orbit, we can add an infinite number of windings to the two shortest primitive segments, "in" and "out", divided by the chords \( \xi_{\text{in}} = -\xi_{\text{out}} \). All the points \( x \), centers of chords in the periodic orbit, form a two dimensional central surface. If the orbit is plane (as in the graph) the central surface is the part of the plane enclosed by the convex hull of the orbit.

FIG. 3. Averaged Wigner function on the \( (q_2, p_2) \) plane for an energy window containing eleven eigenstates centered on the 11'th Bohr level. The thick dark line is the superposition of each energy shell in the plane (periodic orbit) for energies in the range considered; the middle one corresponding to the Bohr-quantization energy.
FIG. 4. Some of the individual Wigner functions on the \((q_2, p_2)\) plane that were averaged in Fig. 3. The plots are globally normalized to point out the enhanced amplitude of the scar for the \(n = 305\) state. The eigenenergy of this state is at distance \(\approx 1.5\Delta\) from the Bohr energy \((\Delta\) the mean level spacing in the chosen average window). The black ellipse, in each graph, is the periodic orbit for the corresponding eigenenergy.

FIG. 5. The application of a particular nonlinear canonical transformation to the Hamiltonian \(\mathbf{a}\) deforms the invariant plane \(q_1 = p_1 = 0\) into the surface displayed in the graph \(\mathbf{a}\), that also has \(p_1 = 0\). The curve over this surface is the distortion of the original periodic orbit (ellipse displayed in the invariant plane). The graph \(\mathbf{b}\) shows the new central surface for the deformed periodic orbit.

FIG. 6. Two of the many symmetric periodic orbits that have chords centered on the \((q_2, p_2)\) plane. The plots are in position space, so that the momenta for the symmetric endpoints are displayed as vectors. All centers have \(q_2 > 0\). The thin line is an equipotential of \(\mathbf{b}\) for the energy of the 11’th Bohr level.

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