Abstract. We examine the relations between different capacities in the setting of a metric measure space. First, we prove a comparability result for the Riesz $(\beta, p)$-capacity and the relative Hajlasz $(\beta, p)$-capacity, for $1 < p < \infty$ and $0 < \beta \leq 1$, under a suitable kernel estimate related to the Riesz potential. Then we show that in geodesic spaces the corresponding capacity density conditions are equivalent even without assuming the kernel estimate. In the last part of the paper, we compare the relative Hajlasz $(1, p)$-capacity to the relative variational $p$-capacity.

1. Introduction

It is well known [24, 30] that the classical Newtonian capacity of a subset of the Euclidean space $\mathbb{R}^n$ can be characterized as the minimum of an energy functional in terms of the weak gradient on the Sobolev space $W^{1,2}(\mathbb{R}^n)$. This characterization has led to extensions of the notion of capacity in several directions and, in particular, to the development of a theory of capacities related to Riesz and Bessel potentials; see [3]. In many cases the different definitions lead to comparable capacities, see e.g. [1, 2, 3, 4, 34]. For instance, the Riesz $(\beta, p)$-capacity $R_{\beta,p}$ is comparable to the classical Newtonian capacity when $\beta = 1$ and $p = 2$.

In this paper, we consider various capacities in the setting of a metric measure space $X$, and our main goal is to clarify the connections between these different notions. Analogues of Riesz potentials in metric measure spaces have been studied for instance in [13, 15, 16, 28, 29], and with these it is possible to consider the corresponding Riesz capacities, see e.g. [32] and Section 3. On the other hand, generalizations of the (weak) gradients to the metric setting lead to variants of capacities defined in terms of Sobolev functions, such as the Newtonian capacity. In particular, we will use a metric space version of the relative variational $p$-capacity $\text{cap}_p(F, \Omega)$ defined via $p$-weak upper gradients as in [35, 5]; see Section 7. Here $\Omega \subset X$ is a bounded open set and $F \subset \Omega$ is a closed set. Another possibility is to use the so-called Hajlasz gradients. These were first defined in [11], and like Riesz potentials, they are of non-local nature, whereas $p$-weak upper gradients are local. The Hajlasz gradient approach to capacities in metric spaces has been used for instance in [22, 19, 23]. Following [8], we consider the relative Hajlasz $(\beta, p)$-capacity $\text{cap}_{\beta,p}(F, \Omega)$, where $F$ and $\Omega$ are as above; see Section 3.

After preliminary definitions and results in Sections 2 and 3, we begin the comparisons of capacities in Section 4, where we study the Riesz and Hajlasz capacities of sets relative to balls. In the special case of an Ahlfors regular metric space, the main result of that section (Corollary 4.5) reads as follows; see Remark 4.6.

**Theorem 1.1.** Let $1 < p < \infty$, $0 < \beta < 1$, and $0 < Q < \infty$ be such that $Q > \beta p$. Assume that $X$ is a complete and connected metric space equipped with an Ahlfors $Q$-regular measure. Then there is a constant $C > 0$ such that

$$C^{-1} R_{\beta,p}(E \cap B(x,r)) \leq \text{cap}_{\beta,p}(E \cap \overline{B(x,r)}, B(x,2r)) \leq C R_{\beta,p}(E \cap \overline{B(x,r)}) \quad (1)$$

whenever $E \subset X$ is a closed set, $x \in E$ and $0 < r < (1/8) \text{diam}(X)$.
More generally, the lower bound in (1) holds under much weaker assumptions, while in the upper bound the Ahlfors regularity assumption can be replaced with a reverse doubling condition and an explicit “kernel estimate”. See Theorems 4.2 and 4.4, respectively, for details.

From the point of view of analysis on general metric spaces, the assumptions for the upper bound in (1) are still rather restrictive. In Section 6, we take another approach to the comparison of capacities under different assumptions. Namely, instead of a direct comparison between the Hajlasz and Riesz capacities, we study the equivalence of the corresponding capacity density conditions. For example, a closed set $E \subset X$ satisfies the Riesz $(\beta, p)$-capacity density condition, if there is a constant $c > 0$ such that

$$R_{\beta, p}(E \cap B(x, r)) \geq c R_{\beta, p}(B(x, r))$$

for all $x \in E$ and all $0 < r < (1/8) \text{diam}(E)$. The definitions of density conditions for other capacities are similar, see Definitions 6.1 and 7.3.

The origins of the (Riesz) capacity density conditions come from [27], where the Riesz $(\beta, p)$-capacity $R_{\beta, p}$ was used in Euclidean spaces to define a fatness condition. The main result of [27] states that in $\mathbb{R}^n$ the Riesz capacity density condition is open ended on $\beta$ and on $p$, that is, if a set $E \subset \mathbb{R}^n$ satisfies the Riesz $(\beta, p)$-capacity density condition, then $E$ also satisfies the condition for some $q < p$ and $\alpha < \beta$. This result has found numerous applications in potential theory, in the study of Hardy inequalities and in partial differential equations, see for instance [20, 21, 31].

Capacity density conditions have been generalized also to metric spaces, and often these conditions turn out to be open-ended (self-improving) as well. For instance, the $p$-capacity density condition, defined in terms of the relative variational $p$-capacity for $1 < p < \infty$, has been shown to be open-ended in [7, 26] under the “standard assumptions” on analysis on metric spaces (see Section 7). On the other hand, for the Hajlasz $(\beta, p)$-capacity density condition, the open-endedness on both $\beta$ and $p$ was proven recently in [8] in complete geodesic spaces.

In Section 6, we show the equivalence of Riesz and Hajlasz $(\beta, p)$-capacity density conditions in a complete geodesic space, see Theorem 6.4. As a tool we use another density condition, given in terms of Hausdorff content of codimension $q$, where $0 < q < \beta p$. Under a suitable reverse doubling condition for the space $X$, such a Hausdorff content density condition for a set $E \subset X$ implies that $E$ satisfies the Riesz $(\beta, p)$-capacity density condition; this follows from the results in Section 5. On the other hand, in [8] it was shown that in complete geodesic spaces the Hajlasz $(\beta, p)$-capacity density condition is equivalent to the Hausdorff content density condition of codimension $q$, for some $0 < q < \beta p$. Taking also into account the lower bound in (1), which is valid under minimal assumptions, this leads to the equivalence of Riesz and Hajlasz $(\beta, p)$-capacity density conditions; see Section 6 for details.

Since the Hajlasz capacity density conditions are known to be open-ended by [8], we obtain as a particular consequence of the above equivalence result that also the Riesz $(\beta, p)$-capacity density condition is open-ended in complete geodesic spaces, see Corollary 6.5. This is an extension of the main result of [27] to metric spaces.

Finally, in Section 7 we show that the relative variational $p$-capacity $\text{cap}_{p}(F, \Omega)$ and the Hajlasz $(1, p)$-capacity $\text{cap}_{1, p}(F, \Omega)$ are comparable, for $1 < p < \infty$, assuming that the metric space $X$ supports a $q$-Poincaré inequality for some $1 \leq q < p$. Therefore we conclude that under suitable assumptions also the density conditions for the variational $p$-capacity, Hajlasz $(1, p)$-capacity, and Riesz $(1, p)$-capacity are equivalent; see Theorem 7.8.

2. Preliminaries

2.1. Metric spaces. Throughout the paper we assume that $X = (X, d, \mu)$ is a metric measure space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that $0 < \mu(B) <$
∞ for all balls $B \subset X$, each of which is an open set of the form

$$B = B(x, r) = \{ y \in X : d(y, x) < r \}$$

with $x \in X$ and $r > 0$. Under these assumptions the space $X$ is separable, see [5, Proposition 1.6]. We also assume that $\#X \geq 2$ and that the measure $\mu$ is doubling, that is, there is a constant $c_\mu > 1$, called the doubling constant of $\mu$, such that

$$\mu(2B) \leq c_\mu \mu(B)$$

(2)

for all balls $B = B(x, r)$ in $X$. Here we use for $0 < t < \infty$ the notation $tB = B(x, tr)$.

If $X$ is connected, then the doubling measure $\mu$ satisfies also the reverse doubling condition, that is, there is a constant $0 < c_R = C(c_\mu) < 1$ such that

$$\mu(B(x, r/2)) \leq c_R \mu(B(x, r))$$

(3)

for all $x \in X$ and all $0 < r < \text{diam}(X)/2$; see for instance [5, Lemma 3.7]. Iteration of (3) shows that there exist an exponent $\sigma > 0$ and a constant $c_\sigma > 0$, both depending on $c_\mu$ only, such that the quantitative reverse doubling condition

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq c_\sigma \left(\frac{r}{R}\right)^\sigma$$

(4)

holds in a connected space $X$ for all $x \in X$ and all $0 < r < R \leq 2 \text{diam}(X)$.

The space $X$ is called Ahlfors $Q$-regular, for $Q > 0$, if there is a constant $c_Q \geq 1$ such that

$$c_Q^{-1}r^Q \leq \mu(B(x, r)) \leq c_Qr^Q$$

(5)

for all $x \in X$ and all $0 < r < \text{diam}(X)$. Note that (4) holds in an Ahlfors $Q$-regular space with $\sigma = Q$.

By a curve we mean a nonconstant, rectifiable, continuous mapping from a compact interval of $\mathbb{R}$ to $X$; we tacitly assume that all curves are parametrized by their arc-length. We say that $X$ is a geodesic space, if every pair of points in $X$ can be joined by a curve whose length is equal to the distance between the two points.

2.2. Hölder functions and Hajłasz gradients. Let $A \subset X$. We say that $u : A \to \mathbb{R}$ is a $\beta$-Hölder function, with an exponent $0 < \beta \leq 1$ and a constant $0 \leq \kappa < \infty$, if

$$|u(x) - u(y)| \leq \kappa d(x, y)^\beta,$$

for all $x, y \in A$.

The set of all $\beta$-Hölder functions $u : A \to \mathbb{R}$ is denoted by $\text{Lip}_\beta(A)$. The 1-Hölder functions are also called Lipschitz functions, and we write $\text{Lip}(A) = \text{Lip}_1(A)$.

The definition of the Hajłasz capacities will be based on the following Hajłasz $\beta$-gradients (see Definition 3.1).

**Definition 2.1.** For each function $u : X \to \mathbb{R}$, we let $\mathcal{D}_H^\beta(u)$ be the (possibly empty) family of all measurable functions $g : X \to [0, \infty]$ such that

$$|u(x) - u(y)| \leq d(x, y)^\beta (g(x) + g(y))$$

(6)

almost everywhere, that is, there exists an exceptional set $N = N(g) \subset X$ for which $\mu(N) = 0$ and inequality (6) holds for every $x, y \in X \setminus N$. A function $g \in \mathcal{D}_H^\beta(u)$ is called a Hajłasz $\beta$-gradient of the function $u$.

The following nonlocal generalization of the Leibniz rule is taken from [8, Theorem 3.4], see also [12]. The nonlocality is reflected by the appearance of the two global terms $\|\psi\|_\infty$ and $\kappa$ in the statement below.
Lemma 2.2. Let $0 < \beta \leq 1$. Assume that $u: X \to \mathbb{R}$ is a bounded $\beta$-Hölder function and $\psi: X \to \mathbb{R}$ is a bounded $\beta$-Hölder function with a constant $\kappa \geq 0$. Then $u\psi: X \to \mathbb{R}$ is a $\beta$-Hölder function and

$$(g_u\|\psi\|_{\infty} + \kappa|u|)\chi_{\{\psi \neq 0\}} \in D^\beta_H(u\psi)$$

for all $g_u \in D^\beta_H(u)$. Here $\{\psi \neq 0\} = \{y \in X : \psi(y) \neq 0\}$.

The following $(\beta, p, p)$-Poincaré inequality can be proved in a similar way as Theorem 5.15 in [15]. This inequality relates the $\beta$-Hajlasz gradient to the given measure. We want to emphasize that no additional assumptions on the measure are needed here.

Theorem 2.3. Let $1 \leq p < \infty$ and $0 < \beta \leq 1$, and assume that $u \in \text{Lip}_\beta(X)$ and $g \in D^\beta_H(u)$. Then

$$\int_B |u(x) - u_B|^p \, d\mu(x) \leq 2^p \text{diam}(B)^{\beta p} \int_B g(x)^p \, d\mu(x),$$

whenever $B \subset X$ is a ball.

Here we use the notation

$$u_B = \frac{1}{\mu(B)} \int_B u(y) \, d\mu(y)$$

for the integral average of $u \in L^1(B)$ over a ball $B \subset X$.

2.3. Riesz potentials. Riesz potentials appear frequently in potential analysis, see for instance [3, 24]. For a nonnegative measurable function $f$ on a metric space $X$, the Riesz potential of order $\beta > 0$ in $X$ can be defined by the expression

$$I_\beta f(x) = \int_X \frac{f(y)d(x, y)^\beta}{\mu(B(x, d(x, y)))} \, d\mu(y), \quad x \in X;$$

see, for instance, [13, 15, 16].

We prove two auxiliary lemmata on relations between the Riesz potential $I_\beta f$ and the non-centered Hardy–Littlewood maximal function $Mf$. If $f: X \to \mathbb{R}$ is a measurable function, then

$$Mf(x) = \sup_B \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y), \quad x \in X,$$

where the supremum is taken over all balls $B \subset X$ such that $x \in B$. The sublinear operator $M$ is bounded on $L^s(X)$ for $1 < s \leq \infty$, see [5, Theorem 3.13].

The first lemma is a variant of a well known Euclidean estimate, see [3, Lemma 3.1.1].

Lemma 2.4. Let $\beta > 0$ and let $f$ be a nonnegative measurable function on $X$. Then

$$\int_{B(z,r)} \frac{f(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) \leq C(c, \beta) r^\beta Mf(z)$$

for all $z \in X$ and all $r > 0$. 
Proof. Define \( r_i = 2^{-i} r, \ i = 0, 1, 2, \ldots \). Using the doubling property of \( \mu \) and the definition of the maximal function, we obtain
\[
\int_{B(z,r)} \frac{f(y)d(z,y)^\beta}{\mu(B(z,d(z,y)))} \, d\mu(y) = \sum_{i=0}^{\infty} \int_{B(z,r_i) \setminus B(z,r_{i+1})} \frac{f(y)d(z,y)^\beta}{\mu(B(z,d(z,y)))} \, d\mu(y)
\]
\[
\leq \sum_{i=0}^{\infty} \int_{B(z,r_i) \setminus B(z,r_{i+1})} \frac{f(y)r_i^\beta}{\mu(B(z,r_{i+1}))} \, d\mu(y)
\]
\[
\leq c_\mu \sum_{i=0}^{\infty} \int_{B(z,r_i)} f(y) \, d\mu(y)
\]
\[
\leq c_\mu r^\beta Mf(z) \sum_{i=0}^{\infty} 2^{-i\beta} = C(c_\mu, \beta)r^\beta Mf(z). \quad \square
\]

The second lemma shows that the maximal function \( Mf \) can be (essentially) used as a Hajlasz \( \beta \)-gradient of \( I_\beta f \). The method of proof is standard, we refer to \([10, 14]\), and it requires the validity of the rather technical kernel estimate (9). However, such estimate (or a suitable variant) holds, for instance, if the measure is \( Q \)-uniform or Ahlfors \( Q \)-regular, see Example 2.6 and Remark 2.7.

**Lemma 2.5.** Let \( 0 < \beta < \eta \) and let \( f \) be a nonnegative measurable function on \( X \) such that \( I_\beta f \) is finite everywhere in \( X \). Assume that there exists \( C_K > 0 \) such that for all \( w, y \in X, w \neq y \), we have
\[
\left| \frac{d(w,z)^\beta}{\mu(B(w,d(w,z)))} - \frac{d(y,z)^\beta}{\mu(B(y,d(y,z)))} \right| \leq C_K \frac{d(w,y)^\eta}{d(w,z)^{\eta-\beta} \mu(B(w,d(w,z)))}
\]
for all \( z \in X \setminus B(w,2d(w,y)) \). Then there is a constant \( C_1 = C(c_\mu, \beta, \eta, C_K) > 0 \) such that
\[
|I_\beta f(w) - I_\beta f(y)| \leq C_1 d(w,y)^\beta (Mf(w) + Mf(y))
\]
for every \( w, y \in X \). In particular, it follows that \( C_1 Mf \) is a Hajlasz \( \beta \)-gradient of \( I_\beta f \).

Proof. Let \( w, y \in X \) with \( w \neq y \). Then,
\[
|I_\beta f(w) - I_\beta f(y)| = \left| \int_X \frac{f(z)d(w,z)^\beta}{\mu(B(w,d(w,z)))} \, d\mu(z) - \int_X \frac{f(z)d(y,z)^\beta}{\mu(B(y,d(y,z)))} \, d\mu(z) \right|
\]
\[
\leq \int_{B(w,2d(w,y))} \frac{f(z)d(w,z)^\beta}{\mu(B(w,d(w,z)))} \, d\mu(z) + \int_{B(w,2d(w,y))} \frac{f(z)d(y,z)^\beta}{\mu(B(y,d(y,z)))} \, d\mu(z)
\]
\[
+ \int_{X \setminus B(w,2d(w,y))} f(z) \left| \frac{d(w,z)^\beta}{\mu(B(w,d(w,z)))} - \frac{d(y,z)^\beta}{\mu(B(y,d(y,z)))} \right| \, d\mu(z).
\]

By Lemma 2.4, we have
\[
\int_{B(w,2d(w,y))} \frac{f(z)d(w,z)^\beta}{\mu(B(w,d(w,z)))} \, d\mu(z) \leq C(c_\mu, \beta)d(w,y)^\beta Mf(w)
\]
and
\[
\int_{B(w,2d(w,y))} \frac{f(z)d(y,z)^\beta}{\mu(B(y,d(y,z)))} \, d\mu(z) \leq \int_{B(y,3d(w,y))} \frac{f(z)d(y,z)^\beta}{\mu(B(y,d(y,z)))} \, d\mu(z)
\]
\[
\leq C(c_\mu, \beta)d(w,y)^\beta Mf(y).
\]
On the other hand, the assumed kernel estimate \((9)\) gives
\[
\left| \int_{X \setminus B(w, 2d(w, y))} f(z) \frac{d(w, z)^\beta}{\mu(B(w, d(w, z)))} - \frac{d(y, z)^\beta}{\mu(B(y, d(y, z)))} \right| \, d\mu(z) \\
\leq c_K d(w, y)^\eta \int_{X \setminus B(w, 2d(w, y))} \frac{f(z) d(w, z)^{\beta - \eta}}{\mu(B(w, d(w, z)))} \, d\mu(z) .
\]

Write \(r_j = 2^{j+1}d(w, y)\) for every \(j = 0, 1, \ldots\). Since \(\eta > \beta\), we have
\[
\int_{X \setminus B(w, 2d(w, y))} \frac{f(z) d(w, z)^{\beta - \eta}}{\mu(B(w, d(w, z)))} \, d\mu(z) \leq \sum_{j=0}^\infty (r_j)^{\beta - \eta} \int_{B(w, r_j)} f(z) \, d\mu(z) \\
\leq c_\mu (\sum_{j=0}^\infty (r_j)^{\beta - \eta}) \int_{B(w, r_j)} f(z) \, d\mu(z) \\
\leq C(c_\mu, \beta, \eta) d(w, y)^{\beta - \eta} Mf(w) .
\]

By combining the estimates above, we obtain
\[
|I_\beta f(w) - I_\beta f(y)| \leq C(c_\mu, \beta, \eta, c_K) d(w, y)^\beta (Mf(w) + Mf(y))
\]
for every \(w, y \in X\). That is, we have \(C_1 Mf \in \mathcal{D}_H^\beta(I_\beta f)\) with \(C_1 = C(c_\mu, \beta, \eta, c_K)\). \(\square\)

**Example 2.6.** Assume that \(\mu\) is a \(Q\)-uniform measure in \(X\) for some \(Q > 0\), that is, there exists a constant \(C_1 > 0\) such that
\[
\mu(B(x, r)) = C_1 r^Q ,
\]
for all \(x \in X\) and all \(r > 0\).

If \(0 < \beta < \min\{Q, 1\}\), then the kernel estimate \((9)\) holds with \(\beta < \eta = 1\). To show this, we proceed as in [9, Lemma 4.2]. By the mean-value theorem, for every \(s, t > 0\) there exists \(0 < \theta < 1\) such that
\[
|s^{\beta - Q} - t^{\beta - Q}| \leq (Q - \beta)(1 - \theta) s + \theta t|^{\beta - Q - 1}|s - t| .
\]

Fix \(w, y \in X\), \(w \neq y\), and \(z \in X \setminus B(w, 2d(w, y))\). By \(Q\)-uniformity of \(\mu\) and inequality \((11)\), we have
\[
\left| \frac{d(w, z)^\beta}{\mu(B(w, d(w, z)))} - \frac{d(y, z)^\beta}{\mu(B(y, d(y, z)))} \right| = \frac{1}{C_1} \left| d(w, z)^{\beta - Q} - d(y, z)^{\beta - Q} \right| \\
\leq C(C_1, Q, \beta) \left| \frac{d(w, z) - d(y, z)}{d(w, z)^{Q - \beta + 1}} \right| \\
\leq C(C_1, Q, \beta) \frac{d(w, y)}{d(w, z)^{1 - \beta} \mu(B(w, d(w, z)))} .
\]

As an example, the \(n\)-dimensional Lebesgue measure in \(\mathbb{R}^n\) is \(n\)-uniform. Uniform measures were first studied in [33].

**Remark 2.7.** In an Ahlfors \(Q\)-regular space \(X\), with \(Q > 0\), the following version of the Riesz potential is often used:
\[
\mathcal{I}_\beta f(x) = \int_X f(y) \frac{d\mu(y)}{d(x, y)^{Q - \beta}} , \quad x \in X ,
\]
see [9, 10] and references therein. By the \(Q\)-regularity, \(\mathcal{I}_\beta f\) is pointwise comparable with the potential \(I_\beta f\), that is, there exists a constant \(C > 0\) such that
\[
C^{-1} \mathcal{I}_\beta f(x) \leq I_\beta f(x) \leq C \mathcal{I}_\beta f(x) ,
\]
for every \(x \in X\).
If $X$ is $Q$-regular and $0 < \beta < \min\{Q, 1\}$, then one can obtain results analogous to Lemma 2.4 and Lemma 2.5 for $I_\beta f$. Observe that the analogue of (9) reads as
\[
\left| \frac{1}{d(w, z)^Q - \beta} - \frac{1}{d(y, z)^Q - \beta} \right| \leq c_K \frac{d(w, y)^\beta}{d(w, z)^{Q - \beta + \eta}}, \quad w, y \in X, \quad z \in X \setminus B(w, 2d(w, y)).
\]
This automatically holds for $0 < \beta < 1 = \eta$, see the reasoning in Example 2.6 for a proof.

3. Hajłasz and Riesz Capacities

Let $\Omega \subset X$ be a bounded open set and let $F \subset \Omega$. One of the main objects in this paper is a capacity of $F$ relative to $\Omega$ defined via $\beta$-Hajłasz gradients. The following definition is from [8]. In the case $\beta = 1$, such capacities were earlier considered in [23].

**Definition 3.1.** Let $1 \leq p < \infty$, $0 < \beta \leq 1$, and let $\Omega \subset X$ be a bounded open set. The variational Hajłasz $(\beta, p)$-capacity of a closed subset $F \subset \Omega$ is
\[
\text{cap}_{\beta, p}(F, \Omega) = \inf_{u} \inf_{g} \int_{X} g(x)^p d\mu(x),
\]
where the infimums are taken over all $u \in \text{Lip}_\beta(X)$, with $u \geq 1$ in $F$, $u = 0$ in $X \setminus \Omega$, and all $g \in D^\beta_H(u)$. If there are no such functions $u$, we set $\text{cap}_{\beta, p}(F, \Omega) = \infty$.

**Remark 3.2.** If $u \in \text{Lip}_\beta(X)$ and $g \in D^\beta_H(u)$, then $v = \max\{0, \min\{u, 1\}\}$ is a $\beta$-Hölder function and $g \in D^\beta_H(v)$. Hence, we may also assume $0 \leq u \leq 1$ in Definition 3.1.

The following lemma shows that in connected spaces capacities of balls are comparable to suitably scaled measures of balls.

**Lemma 3.3.** Assume that the metric space $X$ is connected, and let $1 \leq p < \infty$ and $0 < \beta \leq 1$. Then there is a constant $C = C(c\mu, p) > 0$ such that
\[
C^{-1} r^{-\beta p} \mu(B(x, r)) \leq \text{cap}_{\beta, p}(\overline{B(x, r)}, B(x, 2r)) \leq C r^{-\beta p} \mu(B(x, r)) \tag{13}
\]
for all $x \in X$ and all $0 < r < (1/8) \text{diam}(X)$. Moreover, the second inequality in (13) holds even if $X$ is not connected.

**Proof.** The first inequality in (13) is a consequence of [8, Example 4.5], where connectivity of $X$ is used. In order to show the second inequality in (13), we let $x \in X$ and $0 < r < (1/8) \text{diam}(X)$. Define
\[
u(y) = \max\left\{ 0, 1 - r^{-\beta} \text{dist}(y, B(x, r))^\beta \right\}, \quad y \in X.
\]
Then $u \in \text{Lip}_\beta(X)$, with a constant $r^{-\beta}$, and therefore $g = r^{-\beta} x_{B(x, 2r)} \in D^\beta_H(u)$ by [8, Lemma 3.3]. Since $u = 1$ in $\overline{B(x, r)}$ and $u = 0$ in $X \setminus B(x, 2r)$, we conclude that
\[
\text{cap}_{\beta, p}(\overline{B(x, r)}, B(x, 2r)) \leq \int_{X} g(x)^p d\mu(x) \leq r^{-\beta p} \mu(B(x, 2r)) \leq c\mu r^{-\beta p} \mu(B(x, r)). \quad \square
\]

Another capacity that we use in this paper is the Riesz $(\beta, p)$-capacity. Such capacities are well known in Euclidean spaces; we refer to [3] and [27], and references therein. In metric spaces, variants of Riesz capacities have been studied for instance in [32].

**Definition 3.4.** Let $F \subset X$, $\beta > 0$ and $p \geq 1$. The Riesz $(\beta, p)$-capacity of $F$ is
\[
R_{\beta, p}(F) = \inf \{ \| f \|^p_p : f \geq 0 \text{ and } I_{\beta} f \geq 1 \text{ on } F \},
\]
where $\| f \|^p_p = \| f \|^p_{L^p(X)}$ is the Lebesgue $p$-norm of $f$ on $X$.

See Lemma 5.5 for an analogue of Lemma 3.3 for the Riesz $(\beta, p)$-capacities.
4. Comparability of Riesz and Hajlasz capacities

In this section we show that, under certain geometric hypotheses on the metric and measure, the capacities defined in terms of Riesz potentials and Hajlasz gradients are comparable. One direction of the comparison, given in Theorem 4.2, holds under much weaker assumptions than the other one in Theorem 4.4. In particular, in the latter the kernel estimate (9) is assumed, together with a reverse doubling condition. Recall also that we assume throughout the paper that the measure μ is doubling, with a constant c_μ.

The following chaining lemma, which will be applied in the proof of Theorem 4.2, is a straightforward modification of a result in [13, p. 30]; hence we omit the proof.

Lemma 4.1. Assume that the metric space X is connected. Then there exists a constant M such that for all y ∈ X and all 0 < ρ < R < (3/8) diam(X), there exists k = k(μ, y, ρ, R) ∈ N and balls B_0, ..., B_k satisfying the following properties:

(i) B_0 ⊂ X \ B(y, R) and B_k ⊂ B(y, ρ),
(ii) M^{-1} diam(B_i) ≤ d(y, B_i) ≤ M diam(B_i) for all i = 0, 1, 2, ..., k,
(iii) there is a ball R_i ⊂ B_i \ B_{i+1} such that B_i ∪ B_{i+1} ⊂ MR_i for all i = 0, 1, 2, ..., k − 1,
(iv) No point of X belongs to more than M balls B_i.

The first main result of this section gives an upper bound for the Riesz capacity in terms of the Hajlasz capacity in connected metric spaces.

Theorem 4.2. Assume that X is connected and let 1 ≤ p < ∞ and 0 < β ≤ 1. Moreover, let E ⊂ X be a closed set, x ∈ E and 0 < r < (1/8) diam(X). Then

\[ \text{cap}_{β,p} \left( E ∩ \overline{B(x,r)}, B(x,2r) \right) ≥ C(μ, p)R_{β,p}(E ∩ \overline{B(x,r)}) . \] (14)

Proof. Let u ∈ Lip_p(X) be such that 0 ≤ u ≤ 1, u = 1 in E ∩ \overline{B(x,r)} and u = 0 outside of B(x,2r). Denote by κ the β-Hölder constant of u in X and let g ∈ D^β_H(u). Theorem 2.3 implies that the Hajlasz (β, 1, 1)-Poincaré inequality

\[ \frac{1}{|B|} \int_B |u - u_B| \, dμ ≤ 2 \text{diam}(B)^β \int_B g \, dμ \] (15)

holds for all balls B ⊂ X. We claim that (15), together with the facts that u = 0 outside of B(x,2r) and r < (1/8) diam(X), give the following inequality

\[ |u(y)| ≤ C(μ)I_β g(y) , \] (16)

for every y ∈ \overline{B(x,r)}. We postpone the proof of (16) and finish the proof of the theorem while assuming (16). It follows from the properties of u and (16) that I_β(C(μ)g) ≥ 1 in E ∩ \overline{B(x,r)}. Thus

\[ C(μ, p) \int_X g^p \, dμ = ||C(μ)g||_p^p ≥ R_{β,p}(E ∩ \overline{B(x,r)}) . \]

Taking infimum over all u and g ∈ D^β_H(u), we get

\[ \text{cap}_{β,p} \left( E ∩ \overline{B(x,r)}, B(x,2r) \right) ≥ C(μ, p)R_{β,p}(E ∩ \overline{B(x,r)}) . \] (17)

For convenience of the reader, we give the proof of (16), which is contained in [28]. Let y ∈ \overline{B(x,r)}, R = 3r < (3/8) diam(X) and 0 < ρ < R, and let B_0, B_1, ..., B_k be the corresponding family of balls given by Lemma 4.1.
Notice that since $B_0 \subset X \setminus B(y, 3r) \subset X \setminus B(x, 2r)$, we have $u = 0$ on $B_0$. Then, we can write

$$|u(y)| = |u(y) - u_{B_0}| \leq \sum_{i=0}^{k-1} |u_{B_{i+1}} - u_{B_i}| + |u(y) - u_{B_k}|$$

$$\leq \sum_{i=0}^{k-1} (|u_{B_{i+1}} - u_{R_i}| + |u_{B_i} - u_{R_i}|) + \kappa \rho^\beta$$

$$\leq C(c_{\mu}, M) \sum_{i=0}^{k} \int_{B_i} |u - u_{B_i}| \, d\mu + \kappa \rho^\beta.$$

Applying the Poincaré inequality (15), we get

$$|u(y)| \leq C(c_{\mu}, M) \sum_{i=0}^{k} \text{diam}(B_i)^\beta \int_{B_i} g \, d\mu + \kappa \rho^\beta$$

$$= C(c_{\mu}, M) \sum_{i=0}^{k} \int_{B_i} \frac{g(z) \text{diam}(B_i)^\beta}{\mu(B_i)} \, d\mu(z) + \kappa \rho^\beta$$

$$\leq C(c_{\mu}, M) \sum_{i=0}^{k} \int_{B_i} \frac{g(z) d(y, z)^\beta}{\mu(B(y, d(y, z)))} \, d\mu(z) + \kappa \rho^\beta$$

$$\leq C(c_{\mu}, M) I_\beta g(y) + \kappa \rho^\beta,$$

and inequality (16) follows by letting $\rho \to 0$. $\square$

Under additional assumptions on $X$ inequality (14) can be reversed, see Theorem 4.4. Before a precise formulation of this reverse estimate, we prove an auxiliary lemma.

**Lemma 4.3.** Let $1 \leq p < \infty$ and $0 < \beta \leq 1$, and assume that $\mu$ satisfies the quantitative reverse doubling condition (4) for some exponent $\sigma > \beta p$. Let $x \in X$, $0 < r < (1/8) \text{diam}(X)$ and let $f \in L^p(X)$ be a nonnegative function such that

$$\int_{X \setminus B(z, r/5)} \frac{f(y) d(y, z)^\beta}{\mu(B(z, d(y, z)))} \, d\mu(y) \geq \frac{1}{2} \quad (18)$$

for some $z \in \overline{B(x, r)}$. Then there exists a constant $C = C(c_{\mu}, c_\sigma, \beta, \sigma, p)$ such that

$$\|f\|_{L^p(X)} \geq C r^{-\beta p} \mu(B(x, r)).$$

**Proof.** Throughout the proof, we assume that $X$ is bounded. The case when $X$ is unbounded is treated in a similar way, but the essential difference is that then the sums below will have infinitely many terms. Write $r_k = 2^k r/5$ for every $k = 0, 1, 2, \ldots, K+1$, where $K \in \mathbb{N}$ is chosen such that $r_K \leq \text{diam}(X) < 2r_K$. Note that $X = B(z, r_{K+1})$ and recall that $z \in \overline{B(x, r)}$. Hence,
by the doubling condition for the measure \( \mu \) and by (18), we have
\[
\frac{1}{2} \mu(B(x,r))^{\frac{1}{p}} \leq \frac{c_{\mu}^4}{2} \mu(B(z,r/5))^{\frac{1}{p}} = c_{\mu}^4 \mu(B(z,r/5))^{\frac{1}{p}} \int_{X \setminus B(z,r/5)} \frac{f(y)d(z,y)^{\beta}}{\mu(B(z,d(z,y)))} d\mu(y) \\
\leq c_{\mu}^4 \sum_{k=0}^{K} \mu(B(z,r/5))^{\frac{1}{p}} \int_{B(z,rk+1) \setminus B(z,rk)} \frac{f(y)d(z,y)^{\beta}}{\mu(B(z,d(z,y)))} d\mu(y) \\
\leq c_{\mu}^4 \sum_{k=0}^{K} \mu(B(z,r/5))^{\frac{1}{p}} (rk_{k+1})^{\beta} \int_{B(z,rk+1)} f(y) d\mu(y) \\
\leq c_{\mu}^5 \sum_{k=0}^{K} \mu(B(z,r/5))^{\frac{1}{p}} (rk_{k+1})^{\beta} \left( \int_{B(z,rk+1)} f(y)^p d\mu(y) \right)^{\frac{1}{p}}.
\]
By the reverse doubling condition (4), for every \( k = 0, 1, 2 \ldots, K \), we get
\[
\frac{\mu(B(z,r/5))^{\frac{1}{p}} (rk_{k+1})^{\beta}}{\mu(B(z,rk+1))^{\frac{1}{p}}} = \frac{\mu(B(z,r/5))^{\frac{1}{p}} (rk_{k+1})^{\beta - \frac{\beta}{p}}}{\mu(B(z,rk+1))^{\frac{1}{p}}} \leq c_{\sigma}(r/5)^{\frac{\beta}{p}} (rk_{k+1})^{\beta - \frac{\beta}{p}},
\]
since \( rk_{k+1} \leq 2 \text{diam}(X) \). Hence, we can continue to estimate as follows
\[
\frac{1}{2} \mu(B(x,r))^{\frac{1}{p}} \leq c_{\mu}^5 \sum_{k=0}^{K} \mu(B(z,r/5))^{\frac{1}{p}} (rk_{k+1})^{\beta} \left( \int_{B(z,rk+1)} f(y)^p d\mu(y) \right)^{\frac{1}{p}} \\
\leq c_{\mu}^5 c_{\sigma} \sum_{k=0}^{K} \frac{(r/5)^{\frac{\beta}{p}} (rk_{k+1})^{\beta - \frac{\beta}{p}}}{\mu(B(z,rk+1))^{\frac{1}{p}}} \left( \int_{B(z,rk+1)} f(y)^p d\mu(y) \right)^{\frac{1}{p}} \\
\leq c_{\mu}^5 c_{\sigma} \left( \int_{X} f(y)^p d\mu(y) \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} (r/5)^{(\beta - \frac{\beta}{p})(k+1)}.
\]
Since \( \sigma > \beta p \), the geometric series converges. By combining the above estimates, we get
\[
\mu(B(x,r))^{\frac{1}{p}} \leq C(c_{\mu}, c_{\sigma}, \beta, \sigma, p) r^{\beta} \left( \int_{X} f(y)^p d\mu(y) \right)^{\frac{1}{p}}.
\]
The estimate \( \|f\|_{L^p(X)} \geq C r^{-\beta p} \mu(B(x,r)) \) follows after simplification. \( \Box \)

The second main result of this section is a converse of Theorem 4.2. Observe that assumption (19) is the same as the kernel estimate (9) that appears in Lemma 2.5.

**Theorem 4.4.** Assume that the metric space \( X \) is complete. Let \( 1 < p < \infty \) and \( 0 < \beta \leq 1 \), and assume that \( \mu \) satisfies the quantitative reverse doubling condition (4) for some exponent \( \sigma > \beta p \). In addition, assume that there exist \( \eta > \beta \) and \( c_K > 0 \) such that for all \( w, y \in X \), \( w \neq y \), we have
\[
\left| \frac{d(w, z)^{\beta}}{\mu(B(w,d(w,z)))} - \frac{d(y, z)^{\beta}}{\mu(B(y,d(y,z)))} \right| \leq c_K \frac{d(w, y)^{\eta}}{d(w, z)^{\eta-\beta} \mu(B(w,d(w,z)))}
\] (19)
for all \( z \in X \setminus B(w, 2d(w,y)) \). Then there exists a constant \( C = C(c_K, c_{\mu}, c_{\sigma}, \beta, \eta, \sigma, p) \) such that
\[
\text{cap}_{\beta, p} \left( E \cap \overline{B(x,r)}, B(x,2r) \right) \leq C R_{\beta, p}(E \cap \overline{B(x,r)})
\] (20)
whenever \( E \subset X \) is a closed set, \( x \in E \) and \( 0 < r < (1/8) \text{diam}(X) \).

**Proof.** We may assume that the Riesz \((\beta, p)\)-capacity on the right-hand side of (20) is finite. Fix a nonnegative function \( f \in L^p(X) \) such that \( I_\beta f \geq 1 \) on \( F = E \cap \overline{B(x, r)} \). It suffices to show that

\[
\text{cap}_{\beta, p}(F, B(x, 2r)) \leq C(c_K, c_\mu, c_\sigma, \beta, \eta, \sigma, p) \int_X f(y)^p \, d\mu(y) \tag{21}
\]

since then the claim (20) follows by taking infimum over all \( f \) as above. In order to prove inequality (21), we consider several cases.

First we assume that

\[
\int_{X \setminus B(z, r/5)} \frac{f(y) d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) \geq \frac{1}{2} \tag{22}
\]

for some \( z \in F \subset \overline{B(x, r)} \). Then, by Lemma 4.3, we have

\[
r^{-\beta p} \mu(B(x, r)) \leq C(c_\mu, c_\sigma, \beta, \sigma, p) \int_X f(y)^p \, d\mu(y).
\]

By monotonicity and Lemma 3.3, we get

\[
\text{cap}_{\beta, p}(F, B(x, 2r)) \leq \text{cap}_{\beta, p}(\overline{B(x, r)}, B(x, 2r)) \leq C(c_\mu, c_\sigma, \beta, \sigma, p) \int_X f(y)^p \, d\mu(y), \tag{23}
\]

and so (21) holds in this case.

Next we consider the case when (22) does not hold for any \( z \in F \). Since \( I_\beta f \geq 1 \) on \( F \), we have

\[
\int_{B(z, r/5)} \frac{f(y) d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) = I_\beta f(z) - \int_{X \setminus B(z, r/5)} \frac{f(y) d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) \geq \frac{1}{2}, \tag{24}
\]

for every \( z \in F \).

Assume first that \( f \) is bounded in \( X \). Let

\[
\psi(z) = \max \{0, 1 - r^{-\beta} d(z, B(x, r))^\beta\}
\]

for every \( z \in X \). Then \( 0 \leq \psi \leq 1, \psi = 0 \) in \( X \setminus B(x, 2r) \), \( \psi = 1 \) in \( \overline{B(x, r)} \), and \( \psi \) is a \( \beta \)-Hölder function in \( X \) with a constant \( r^{-\beta} \). Define \( h = 2f X_{B(x, 2r)} \) and

\[
v(z) = I_\beta h(z) \psi(z), \quad z \in X. \tag{25}
\]

By inequality (24), for every \( z \in F = E \cap \overline{B(x, r)} \) we have

\[
1 \leq \int_X \frac{2f(y) X_{B(x, 2r)}(y) d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) = I_\beta (2f X_{B(x, 2r)})(z) = I_\beta h(z).
\]

Thus \( \psi \geq 1 \) in \( F = E \cap \overline{B(x, r)} \) and \( v = 0 \) in \( X \setminus B(x, 2r) \). By using Lemma 2.4 and properties of \( h \), we see that \( I_\beta h \) is finite everywhere in \( X \). Since \( M h \) is bounded in \( X \), Lemma 2.5 implies that \( I_\beta h \in \text{Lip}_B(X) \) and \( C_1 M h \in \mathcal{D}_H^\beta(I_\beta h) \), where \( C_1 = C(c_\mu, \beta, \eta, c_K) > 0 \).

Let \( M = \sup_{z \in B(x, 2r)} I_\beta h(z) \) and \( u = \min \{I_\beta h, M\} \). Then \( u \) is a bounded \( \beta \)-Hölder function that coincides with \( I_\beta h \) on \( B(x, 2r) \); thus \( v = u \psi \) in \( X \). We also have that \( C_1 M h \in \mathcal{D}_H^\beta(u) \). Lemma 2.2 implies that the function \( v \) is \( \beta \)-Hölder in \( X \) and

\[
g_v = (C_1 M h ||\psi||_\infty + r^{-\beta} I_\beta h) \chi_{\{\psi \neq 0\}} = (C_1 M h ||\psi||_\infty + r^{-\beta} u) \chi_{\{\psi \neq 0\}} \in \mathcal{D}_H^\beta(v).
\]
Thus the pair $v$ and $g_v$ is admissible for testing the Hajlasz $(\beta, p)$-capacity of $F$ relative to $B(x, 2r)$ and
\[
\text{cap}_{\beta, p}(F, B(x, 2r)) \leq \int_X g_v(z)^p \, d\mu(z)
\leq 2^p C_1^p \int_{B(x, 2r)} (Mh(z))^p \, d\mu(z) + 2^p r^{-\beta p} \int_{B(x, 2r)} (I_\beta h(z))^p \, d\mu(z).
\]
Since $h \leq 2f$, we have $Mh \leq 2Mf$. By Lemma 2.4, we also have
\[
I_\beta h(z) = 2 \int_{B(x, 2r)} \frac{f(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y)
\leq 2 \int_{B(z, 4r)} \frac{f(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) \leq C(c_\mu, \beta)r^\beta Mf(z)
\]
for every $z \in B(x, 2r)$. Hence, we see that
\[
\text{cap}_{\beta, p}(F, B(x, 2r)) \leq C(C_1, p, c_\mu, \beta) \int_X (Mf(z))^p \, d\mu(z) \leq C(C_1, p, c_\mu, \beta) \int_X f(z)^p \, d\mu(z).
\]
The last step follows from the Hardy–Littlewood maximal theorem since $p > 1$, see [5, Theorem 3.13]. We have shown that inequality (21) holds.

In case $f$ is not bounded in $X$ and (24) holds for all $z \in F$, we write $f_k(y) = \min\{k, f(y)\}$ for every $y \in X$ and $k \in \mathbb{N}$. Observe that $f_k \to f$ almost everywhere in $X$, as $k \to \infty$. Hence, by Fatou’s lemma
\[
\frac{1}{2} \leq \int_{B(z, r/5)} \frac{f(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y) \leq \liminf_{k \to \infty} \int_{B(z, r/5)} \frac{f_k(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y)
\]
for every $z \in F$. We claim that there exists $k \in \mathbb{N}$ such that
\[
\frac{1}{3c_\mu} \leq \int_{B(z, r/5)} \frac{f_k(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} \, d\mu(y)
\]
for all $z \in F$. Assume that such a number $k$ does not exist. Then for every $k \in \mathbb{N}$ there exists $z_k \in F$ such that
\[
\frac{1}{3c_\mu} > \int_{B(z_k, r/5)} \frac{f_k(y)d(z_k, y)^\beta}{\mu(B(z_k, d(z_k, y)))} \, d\mu(y).
\]
Since $F \subset X$ is a closed and bounded set and $X$ is a complete doubling space, by [5, Proposition 3.1] we find that $F$ is compact. In particular, there exists a subsequence $(z_{k_m})_{m \in \mathbb{N}}$ of $(z_k)_{k \in \mathbb{N}}$ such that $\lim_{m \to \infty} z_{k_m} = z_0 \in F$. By Fatou’s lemma
\[
\frac{1}{3c_\mu} \geq \liminf_{m \to \infty} \int_{B(z_{k_m}, r/5)} \frac{f_{k_m}(y)d(z_{k_m}, y)^\beta}{\mu(B(z_{k_m}, d(z_{k_m}, y)))} \, d\mu(y) \geq \frac{1}{c_\mu} \int_{B(z_0, r/5)} \frac{f(y)d(z_0, y)^\beta}{\mu(B(z_0, d(z_0, y)))} \, d\mu(y).
\]
This is a contradiction with (24), since $z_0 \in F$. We have shown that there exists $k \in \mathbb{N}$ such that (26) holds for all $z \in F$. Using a test function $v$ as in (25) but with $h = 3c_\mu f_k \chi_{B(x, 2r)}$, we repeat the reasoning above and finish the proof.

Combining Theorems 4.2 and 4.4, we obtain the following result on the comparability of the Riesz and Hajlasz capacities.

**Corollary 4.5.** Assume that the metric space $X$ is complete and connected. Let $1 < p < \infty$ and $0 < \beta \leq 1$, and assume that $\mu$ satisfies the quantitative reverse doubling condition (4) for some exponent $\sigma > \beta p$ and that the kernel estimate (19) holds for some $\eta > \beta$ and $c_K > 0$. Then there is a constant $C = C(c_K, c_\mu, c_\sigma, \beta, \eta, p, \sigma)$ such that
\[
C^{-1} R_{\beta, p}(E \cap B(x, r)) \leq \text{cap}_{\beta, p}(E \cap \overline{B(x, r)}, B(x, 2r)) \leq C R_{\beta, p}(E \cap B(x, r)),
\]
(27)
whenever \( E \subset X \) is a closed set, \( x \in E \) and \( 0 < r < (1/8) \text{diam}(X) \).

**Remark 4.6.** In an Ahlfors \( Q \)-regular space \( X \) it is natural to consider capacities defined in terms of Riesz potentials \( I_\beta f \), see Remark 2.7. For these capacities, one can obtain the results analogous to Theorem 4.2 and Theorem 4.4. The proofs are similar and use the accordingly modified statements of Lemma 2.4, Lemma 2.5 and Lemma 4.3. In this setting with \( 0 < \beta < 1 \), the kernel estimate (19) is not required anymore, see Remark 2.7, and the quantitative reverse doubling condition (4) holds with \( \sigma = Q \). Hence, it follows that the corresponding Riesz \((\beta, p)\)-capacity \( I_\beta f \) (and by (12) also \( I_\beta f \)) is comparable to the Hajlasz \((\beta, p)\)-capacity in any complete and connected Ahlfors \( Q \)-regular space with \( 0 < \beta < 1 \) and \( Q > \beta p \). This proves Theorem 1.1.

## 5. Hausdorff Content Density Condition

In this section, we introduce a version of the Hausdorff content density condition, slightly different from the one that was used in [25]. This is an auxiliary condition that will be used in Section 6 to connect Riesz and Hajlasz capacity density conditions.

**Definition 5.1.** The \((\rho\text{-restricted})\) Hausdorff content of codimension \( q \geq 0 \) of a set \( F \subset X \) is defined by

\[
\mathcal{H}^{\mu,q}_\rho(F) = \inf \left\{ \sum_k \mu(B(x_k, r_k)) r_k^{-q} : F \subseteq \bigcup_k B(x_k, r_k) \text{ and } 0 < r_k \leq \rho \right\}.
\]

A closed set \( E \subset X \) satisfies the Hausdorff content density condition of codimension \( q \geq 0 \) if there is a constant \( c_0 > 0 \) such that

\[
\mathcal{H}^{\mu,q}_\rho(E \cap \overline{B(x, r)}) \geq c_0 \mathcal{H}^{\mu,q}_r(B(x, r))
\]

for all \( x \in E \) and all \( 0 < r < \text{diam}(E) \).

**Remark 5.2.** Let \( q \geq 0 \). Then it is easy to show that \( r^{-q} \mu(B(x, r)) = \mathcal{H}^{\mu,q}_r(B(x, r)) \) and that

\[
r^{-q} \mu(B(x, r)) \leq \mathcal{H}^{\mu,q}_r(B(x, r)) \leq C(c_\mu) r^{-q} \mu(B(x, r))
\]

for all \( x \in X \) and all \( r > 0 \). This can be seen as an analogue of Lemma 3.3 for the Hausdorff content. In particular, it follows that a closed set \( E \subset X \) satisfies the Hausdorff content density condition of codimension \( q \geq 0 \) if, and only if there is a constant \( c_1 > 0 \) such that

\[
\mathcal{H}^{\mu,q}_\rho(E \cap \overline{B(x, r)}) \geq c_1 r^{-q} \mu(B(x, r))
\]

for every \( x \in E \) and all \( 0 < r < \text{diam}(E) \). This equivalent condition was taken as the definition of the Hausdorff content density condition of codimension \( q \) in [8].

By definition, we have \( \mathcal{H}^{\mu,q}_{2\text{diam}(X)}(F) \leq \mathcal{H}^{\mu,q}_r(F) \) if \( 0 < r \leq 2 \text{diam}(X) \) and \( F \subset X \). Theorem 5.3 gives a condition under which this inequality can be reversed.

**Theorem 5.3.** Let \( q > 0 \). The following conditions are equivalent:

(i) There is a constant \( c_q > 0 \) such that the quantitative reverse doubling condition

\[
\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq c_q \left( \frac{r}{R} \right)^q
\]

holds for all \( x \in X \) and all \( 0 < r < R \leq 2 \text{diam}(X) \).

(ii) There is a constant \( C_2 > 0 \) such that

\[
\mathcal{H}^{\mu,q}_r(F) \leq C_2 \mathcal{H}^{\mu,q}_{2\text{diam}(X)}(F)
\]

whenever \( F \subset \overline{B(x, r)} \), with \( x \in X \) and \( r > 0 \).
(iii) There is a constant $C_3 > 0$ such that

$$r^{-q}\mu(B(x, r)) \leq C_3 \mathcal{H}^\mu_{2 \text{diam}(X)}(B(x, r))$$

for all $x \in X$ and all $r > 0$.

Proof. We assume that condition (i) holds and we prove condition (ii). For this purpose, we fix $F \subset B(x, r)$ with $x \in X$ and $r > 0$. Let $\{B(x_k, r_k)\}_{k \in I}$ be a finite or countable cover of $F$ such that $B(x_k, r_k) \cap F \neq \emptyset$ and $0 < r_k \leq 2 \text{diam}(X)$ for all $k \in I$. If $r_k \leq r$ for all $k \in I$, then

$$\mathcal{H}^\mu_{r}(F) \leq \sum_{k \in I} \mu(B(x_k, r_k))r_k^{-q}.$$ 

Next we assume that at least one of the radii $r_k$, $k \in I$, is not bounded by $r$. Without loss of generality, we may assume that $r_1 > r$. By the $5r$-covering lemma [5, Lemma 1.7], we obtain $N$ pairwise disjoint balls $B(y_i, r/5)$ such that $F \subset \bigcup_{i=1}^{N} B(y_i, r)$ and $y_i \in F$ for every $i = 1, \ldots, N$. The doubling condition implies that $N \leq C(c_\mu)$. By the assumed reverse doubling condition, followed by the doubling condition, we get

$$\mathcal{H}^\mu_{r}(F) \leq \sum_{i=1}^{N} \mu(B(y_i, r))r^{-q} \leq c_\mu \sum_{i=1}^{N} \mu(B(y_i, r_1))r_1^{-q} \leq N c_\mu^2 c_\mu \mu(B(x_1, r_1))r_1^{-q} \leq C(c_\mu, c_\mu) \sum_{k \in I} \mu(B(x_k, r_k))r_k^{-q}.$$ 

In any case, we see that

$$\mathcal{H}^\mu_{r}(F) \leq C(c_\mu, c_\mu) \sum_{k \in I} \mu(B(x_k, r_k))r_k^{-q},$$

and condition (ii) with a constant $C_2 = C(c_\mu, c_\mu) > 0$ follows by taking infimum over all covers $\{B(x_k, r_k)\}_{k \in I}$ of $F$ as above.

Condition (ii) implies condition (iii) by choosing $F = B(x, r)$ and using Remark 5.2. Finally, assume that condition (iii) holds. Let $x \in X$ and $0 < r < R \leq 2 \text{diam}(X)$. Then Remark 5.2 implies that

$$r^{-q}\mu(B(x, r)) \leq C_3 \mathcal{H}^\mu_{2 \text{diam}(X)}(B(x, r)) \leq C_3 \mathcal{H}^\mu_{2 \text{diam}(X)}(B(x, R)) \leq C_3 \mathcal{H}^\mu_{R}(B(x, R)) = C_3 R^{-q}\mu(B(x, R)).$$

Thus, condition (i) holds with a constant $c_\mu = C_3$. \hfill $\square$

Assuming (29) for some $q > 0$, then a closed set $E \subset X$ satisfies the Hausdorff content density condition of codimension $q \geq 0$ if and only if

$$\mathcal{H}^\mu_{2 \text{diam}(X)}(E \cap \overline{B(x, r)}) \geq C \mathcal{H}^\mu_{2 \text{diam}(X)}(\overline{B(x, r)})$$

for all $x \in E$ and all $0 < r < \text{diam}(E)$. This is a direct consequence of Theorem 5.3.

The next lemma is an extension of [17, Proposition 3.12] to metric spaces, with a new proof; see also [3, Corollary 5.1.14].

It gives a Riesz capacity estimate for sets satisfying the Hausdorff content density condition. This result and also its consequence in Lemma 5.5 will be applied in the proof of Theorem 6.4, which relates the density conditions corresponding to the Riesz capacity, the Hajlasz capacity and the Hausdorff content.

**Lemma 5.4.** Let $1 < p < \infty$ and $0 < \beta \leq 1$, and assume that $\mu$ satisfies the quantitative reverse doubling condition (4) for some exponent $\sigma > \beta p$. Moreover, let $E \subset X$ be a closed set that satisfies the Hausdorff content density condition (28) for some $0 < q < \beta p$. Then there is a constant $C = C(c_1, c_\mu, c_\sigma, \sigma, \beta, q, p, \beta) > 0$ such that

$$R_{\beta, p}(E \cap \overline{B(x, r)}) \geq C r^{-\beta p}\mu(B(x, r)).$$
for all $x \in E$ and all $0 < r < (1/8) \text{diam}(E)$. Here the constant $c_1$ is defined in (30).

Proof. By Remark 5.2, there is a constant $c_1 > 0$ such that

$$
\mathcal{H}^{\mu,q}_r(E \cap B(x,r)) \geq c_1 r^{-q} \mu(B(x,r)),
$$

(30)

for all $x \in E$ and all $0 < r < \text{diam}(E)$.

Fix $x \in E$ and $0 < r < (1/8) \text{diam}(E)$, and write $F = E \cap \overline{B(x,r)}$. We may assume that $R_{\beta,p}(F) < \infty$. Let $f \in L^p(X)$ be a nonnegative test function for the capacity $R_{\beta,p}(F)$. Then, in particular,

$$
1 \leq I_\delta f(z) = \int_X \frac{f(y)d(z,y)^\beta}{\mu(B(z,d(z,y)))} d\mu(y)
$$

(31)

for every $z \in F$. We consider two cases.

First, assume that

$$
\frac{1}{2} \leq \int_{B(z,r/5)} f(y)d(z,y)^\beta \mu(B(z,d(z,y))) d\mu(y)
$$

for every $z \in F$. In this case, we proceed as in the proof of [32, Theorem 4.3]. Namely, we fix $z \in F$ and define $r_i = 2^{-i} r/5$, $i = 0, 1, 2, \ldots$. Estimating as in Lemma 2.4, we obtain

$$
1 \leq 2 \int_{B(z,r/5)} \frac{f(y)d(z,y)^\beta}{\mu(B(z,d(z,y)))} d\mu(y) \leq 2c_\mu \sum_{i=0}^{\infty} r_i^\beta \int_{B(z,r_i)} f(y) d\mu(y)
$$

$$
\leq 2c_\mu \sum_{i=0}^{\infty} \frac{r_i^\beta}{\mu(B(z,r_i))^{1/p}} \left( \int_{B(z,r_i)} f(y)^p d\mu(y) \right)^{1/p}.
$$

For every $\delta > 0$, there is a constant $C(\delta) > 0$ such that

$$
1 = C(\delta) \sum_{i=0}^{\infty} 2^{-i\delta} = C(\delta) 5^\delta \sum_{i=0}^{\infty} (r_i/r)^\delta = C(\delta) (r/5)^{-\delta} \sum_{i=0}^{\infty} r_i^\delta.
$$

Choose $\delta = \beta - q/p > 0$, and combine two previous expressions to obtain the inequality

$$
\sum_{i=0}^{\infty} r_i^{\beta - q/p} \leq C(c_\mu, \beta, q, p) r^{\beta - q/p} \sum_{i=0}^{\infty} \frac{r_i^\beta}{\mu(B(z,r_i))^{1/p}} \left( \int_{B(z,r_i)} f(y)^p d\mu(y) \right)^{1/p},
$$

(32)

which is valid for every $z \in F$.

It follows from (32) that, for each $z \in F$, there is a ball $B(z,r_{i_z})$ such that $r_{i_z} \leq r/5$ and

$$
\mu(B(z,r_{i_z})) r_{i_z}^{-q} \leq C(c_\mu, \beta, q, p) r^{\beta - q - p} \int_{B(z,r_{i_z})} f(y)^p d\mu(y).
$$

Using the $5r$-covering lemma [5, Lemma 1.7], we obtain countably many points $z_k \in F$, $k \in I$, such that the corresponding balls $B(z_k,r_k) = B(z_k,r_{i_k})$ are pairwise disjoint and $F \subseteq \bigcup_{k \in I} B(z_k, 5r_k)$. Recalling definition (28) of the Hausdorff content of codimension $q$, and using the doubling property of measure $\mu$ and the fact that $\{B(z_k,r_k)\}_{k \in I}$ is a family of pairwise disjoint balls, we obtain

$$
\mathcal{H}_r^{\mu,q}(F) \leq \sum_{k \in I} \mu(B(z_k, 5r_k))/(5r_k)^{-q} \leq c_\mu^3 \sum_{k \in I} \mu(B(z_k,r_k)) r_k^{-q}
$$

$$
\leq C(c_\mu, \beta, q, p) r^{\beta - q - p} \sum_{k \in I} \int_{B(z_k,r_k)} f(y)^p d\mu(y) \leq C(c_\mu, \beta, q, p) r^{\beta - q - p} \int_X f(y)^p d\mu(y).
$$

Applying (30), we get

$$
r^{-q} \mu(B(x,r)) \leq c_1^{-1} \mathcal{H}_r^{\mu,q}(F) \leq C(c_1, c_\mu, \beta, q, p) r^{\beta - q - p} \int_X f(y)^p d\mu(y).
$$
Simplifying the exponents yields

\[ r^{-\beta_p}\mu(B(x, r)) \leq C(c_1, c_\mu, \beta, q, p) \int_X f(y)^p d\mu(y). \] (33)

In the second case, we assume that there exists \( z \in F \) such that

\[ \int_{B(z, r/5)} \frac{f(y)d(z, y)^\beta}{\mu(B(z, d(z, y)))} d\mu(y) < \frac{1}{2}. \]

By inequality (31) and Lemma 4.3 we have

\[ r^{-\beta_p}\mu(B(x, r)) \leq C(c_1, c_\mu, \beta, \sigma, p) \int_X f(y)^p d\mu(y). \] (34)

From (33) and (34) it follows that

\[ r^{-\beta_p}\mu(B(x, r)) \leq C(c_1, c_\mu, \beta, p) \int_X f(y)^p d\mu(y) \]

for all nonnegative test functions \( f \in L^p(X) \) for the capacity \( R_{\beta, p}(F) \). By taking infimum over all such functions, we obtain

\[ r^{-\beta_p}\mu(B(x, r)) \leq C(c_1, c_\mu, \sigma, \beta, p)R_{\beta, p}(E \cap \overline{B(x, r)}) \]

for all \( x \in E \) and all \( 0 < r < \text{diam}(E)/8 \), as desired. \( \square \)

Lemma 5.4 leads to an analogue of Lemma 3.3 for the Riesz \((\beta, p)\)-capacities.

**Lemma 5.5.** Assume that \( X \) is connected. Let \( 1 < p < \infty \) and \( 0 < \beta \leq 1 \), and assume that \( \mu \) satisfies the quantitative reverse doubling condition (4) for some exponent \( \sigma > \beta p \). Then there is a constant \( C = C(c_\mu, c_\sigma, \beta, \sigma, p) > 0 \) such that

\[ C^{-1}r^{-\beta_p}\mu(B(x, r)) \leq R_{\beta, p}(\overline{B(x, r)}) \leq Cr^{-\beta_p}\mu(B(x, r)) \]

for all \( x \in X \) and all \( 0 < r < (1/8) \text{diam}(X) \).

**Proof.** Observe that \( X \) itself is a closed set that satisfies the Hausdorff content density condition (28) for every \( q \geq 0 \). Hence, the first inequality follows from Lemma 5.4. On the other hand, by Theorem 4.2 and inequality (13), we obtain

\[ R_{\beta, p}(\overline{B(x, r)}) \leq C \text{cap}_{\beta, p}(\overline{B(x, r)}, B(x, 2r)) \leq C r^{-\beta_p}\mu(B(x, r)). \]

The second inequality follows. \( \square \)

We finish this section with an example showing that the quantitative reverse doubling condition (4) for some exponent \( \sigma > \beta p \) cannot be removed from Lemma 5.5.

**Example 5.6.** Consider \( X = \mathbb{R}^n \) equipped with the Euclidean distance \( d \) and the \( n \)-dimensional Lebesgue measure which we denote by \( \mu \). Fix \( 1 < p < \infty \) and \( 0 < \beta \leq 1 \) such that \( \beta p > n \). Observe that the quantitative reverse doubling condition (4) does not hold in \((\mathbb{R}^n, d, \mu)\) for any \( \sigma > \beta p \), since \( \beta p > n \). Define \( A_j = B(0, 3j) \setminus B(0, 2j) \) for all \( j = 1, 2, \ldots \). Let \( z \in B(0, 1) \). Then

\[ \begin{align*}
I_\beta \left( j^{-\beta} \chi_{A_j} \right)(z) &= j^{-\beta} \int_{A_j} \frac{d(z, y)^\beta}{\mu(B(z, d(z, y)))} d\mu(y) \\
&\geq j^{-\beta} \int_{A_j} \frac{1}{\mu(B(0, 5j))} d\mu(y) = C(n) > 0.
\end{align*} \]

It follows that functions \( f_j = C(n)^{-1} j^{-\beta} \chi_{A_j} \) are admissible test functions for the Riesz \((\beta, p)\)-capacity of \( \overline{B(0, 1)} \) for every \( j = 1, 2, \ldots \), and therefore

\[ R_{\beta, p}(\overline{B(0, 1)}) \leq \| f_j \|_p^p = C(n, p) j^{-\beta_p} \mu(A_j) \leq C(n, p) j^{n-\beta_p} \]
for all \( j = 1, 2, \ldots \). By taking \( j \to \infty \), we see that \( R_{\beta,p}(B(0,1)) = 0 \).

### 6. Equivalence of density conditions

In Corollary 4.5 we showed that the Riesz and Hajlasz \((\beta, p)\)-capacities are comparable in a complete and connected space \( X \) if \( \mu \) satisfies a suitable reverse doubling condition and the kernel estimate (19) holds. Next, we consider the corresponding capacity density conditions and show their equivalence in Theorem 6.4. The kernel estimate is no longer needed, but instead we need to assume that the space \( X \) is geodesic since that is required in Theorem 6.2.

The Hajlasz capacity density condition is known to be self-improving (open-ended) by the results in [8], and hence the same also holds for the Riesz capacity density condition. This is stated explicitly in Corollary 6.5.

**Definition 6.1.** A closed set \( E \subset X \) satisfies the Hajlasz \( (\beta, p)\)-capacity density condition, for \( 1 \leq p < \infty \) and \( 0 < \beta \leq 1 \), if there is a constant \( c_0 > 0 \) such that

\[
\text{cap}_{\beta,p}(E \cap B(x,r), B(x,2r)) \geq c_0 \text{cap}_{\beta,p}(B(x,r), B(x,2r))
\]  

(35)

for all \( x \in E \) and all \( 0 < r < (1/8) \text{diam}(E) \).

It follows from Lemma 3.3 that a closed set \( E \) in a connected metric space \( X \) satisfies the Hajlasz \( (\beta, p)\)-capacity density condition, for \( 1 \leq p < \infty \) and \( 0 < \beta \leq 1 \), if and only if there is a constant \( c_1 > 0 \) such that

\[
\text{cap}_{\beta,p}(E \cap B(x,r), B(x,2r)) \geq c_1 r^{-\beta p} \mu(B(x,r))
\]  

(36)

for all \( x \in E \) and all \( 0 < r < (1/8) \text{diam}(E) \). This condition was taken as the definition of the Hajlasz \( (\beta, p)\)-capacity density condition in [8].

The following result from [8, Theorem 9.5] shows that in complete geodesic metric spaces the Hajlasz capacity density condition is equivalent to a Hausdorff content density condition.

**Theorem 6.2.** Assume that \( X \) is a complete geodesic space. Let \( 1 < p < \infty \) and \( 0 < \beta \leq 1 \), and let \( E \subset X \) be a closed set. Then the following conditions are equivalent

(i) \( E \) satisfies the Hajlasz \( (\beta, p)\)-capacity density condition (35).

(ii) \( E \) satisfies the Hausdorff content density condition (28) for some \( 0 < q < \beta p \).

Theorem 6.2 and Lemma 5.4 are the main ingredients in the proof of Theorem 6.4, which is one of our main results and adds the following Riesz capacity density condition to the list of conditions in Theorem 6.2. See also [8, Theorem 9.5] for many more equivalent conditions, and Theorem 7.8 for the case \( \beta = 1 \).

**Definition 6.3.** A closed set \( E \subset X \) satisfies the Riesz \( (\beta, p)\)-capacity density condition, for \( 1 \leq p < \infty \) and \( \beta > 0 \), if there is a constant \( c_0 > 0 \) such that

\[
R_{\beta,p}(E \cap B(x,r)) \geq c_0 R_{\beta,p}(B(x,r))
\]  

(37)

for all \( x \in E \) and all \( 0 < r < (1/8) \text{diam}(E) \).

**Theorem 6.4.** Assume that \( X \) is a complete geodesic space. Let \( 1 < p < \infty \) and \( 0 < \beta \leq 1 \), and assume that \( \mu \) satisfies the quantitative reverse doubling condition (4) for some exponent \( \sigma > \beta p \). Then the following conditions are equivalent for a closed set \( E \subset X \):

(i) \( E \) satisfies the Riesz \( (\beta, p)\)-capacity density condition (37).

(ii) \( E \) satisfies the Hajlasz \( (\beta, p)\)-capacity density condition (35).

(iii) \( E \) satisfies the Hausdorff content density condition (28) for some \( 0 < q < \beta p \).

**Proof.** Since \( X \) is geodesic, it is in particular connected. To show the implication from (i) to (ii) we assume that \( E \) satisfies the Riesz \( (\beta, p)\)-capacity density condition (37). By using this condition, Theorem 4.2 and Lemma 5.5, we find that inequality (36) holds for all \( x \in E \) and
all $0 < r < (1/8) \text{diam}(E)$. By Lemma 3.3, this is equivalent to the Hajłasz $(\beta,p)$-capacity density condition.

The implication from (ii) to (iii) is a part of Theorem 6.2, and the implication from (iii) to (i) follows by Lemma 5.4 and Lemma 5.5. □

From Theorem 6.4 we obtain as a corollary that the Riesz capacity density condition is self-improving. This can be regarded as an extension of Lewis’ result [27] to the setting of complete geodesic spaces. For the proof of the corollary, it suffices to notice that condition (iii) in Theorem 6.4 remains valid even if we modify both $\beta$ and $p$ slightly. In the Euclidean case, this approach to the self-improvement of Riesz capacity density condition, which is largely based on the results in [8], is completely different from the original argument of Lewis.

**Corollary 6.5.** Assume that $X$ is a complete geodesic space. Let $1 < p < \infty$ and $0 < \beta \leq 1$, and assume that $\mu$ satisfies the quantitative reverse doubling condition (4) for some exponent $\sigma > \beta p$. If a closed set $E \subset X$ satisfies the Riesz $(\beta,p)$-capacity density condition, then there exists $0 < \delta < \min\{\beta, p - 1\}$ such that $E$ satisfies the Riesz $(\gamma, s)$-capacity density condition whenever $\beta - \delta < \gamma \leq 1$, $p - \delta < s < \infty$ and $\sigma > \gamma s$.

### 7. Comparability of Hajłasz $(1,p)$-capacity and $p$-capacity

In this last section of the paper, we add into the considerations one more notion of capacity, the variational $p$-capacity. The definition of this capacity is given in terms of $p$-weak upper gradients, and we begin by recalling relevant preliminaries. The main result of this section is Theorem 7.5, which shows the comparability of the variational $p$-capacity and the $(1,p)$-Hajłasz capacity under suitable assumptions. The comparability of the capacities implies also the equivalence of the corresponding density conditions, stated at the end of the section in Corollary 7.7, which in turn can be combined with the results in Section 6 to give further lists of equivalent conditions in the case $\beta = 1$; see Theorems 7.8 and 7.9.

Let $u$ be a real valued function on $X$. A Borel function $g \geq 0$ on $X$ is an upper gradient of $u$ if for all curves $\gamma$ (see Section 2.1) joining any two points $x, y \in X$ we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds,$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g \, ds = \infty$ otherwise. In addition, a measurable function $g \geq 0$ on $X$ is a $p$-weak upper gradient of $u$, for $1 \leq p < \infty$, if inequality (38) holds for $p$-almost every curve $\gamma$ joining arbitrary points $x$ and $y$ in $X$. That is, there exists a nonnegative Borel function $\rho \in L^p_{\text{loc}}(X)$ such that $\int_\gamma \rho \, ds = \infty$ whenever (38) does not hold for the curve $\gamma$. Here $\rho \in L^p_{\text{loc}}(X)$ means that for each $x \in X$ there exists $r_x > 0$ such that $\rho \in L^p(B(x, r_x))$. We refer to [5] for more information on $p$-weak upper gradients.

We say that the space $X$ supports a $p$-Poincaré inequality, for $1 \leq p < \infty$, if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions $u$ on $X$, and all $p$-weak upper gradients $g$ of $u$ we have

$$\int_B |u - u_B| \, d\mu \leq C_P \text{ diam}(B) \left(\int_{\lambda B} g^p \, d\mu\right)^{1/p}.$$  \hfill (39)

Here $u_B$ is the integral average of $u$ over $B$ as in (7), and the left-hand side of (39) is interpreted as $\infty$ whenever $u_B$ is not defined. We refer to [5, Chapter 4] for further details. The doubling condition (2) and the Poincaré inequality are the standard assumptions on analysis on metric spaces based on (weak) upper gradients; recall that we assume throughout that $\mu$ is doubling with a constant $C_\mu$. 
Definition 7.1. Let $1 \leq p < \infty$ and let $\Omega \subset X$ be a bounded open set. The variational $p$-capacity of a closed subset $F \subset \Omega$ is

$$\text{cap}_p(F, \Omega) = \inf_u \inf_g \int_\Omega g(x)^p \, d\mu(x),$$  \hspace{1cm} (40)$$

where the infimums are taken over all $u \in \text{Lip}(X)$, with $u \geq 1$ in $F$, $u = 0$ in $X \setminus \Omega$, and all $p$-weak upper gradients $g$ of $u$. If there are no such functions $u$, we set $\text{cap}_p(F, \Omega) = \infty$.

**Remark 7.2.** If $\text{cap}_p(F, \Omega) < \infty$, then the infimum in (40) can be restricted to functions $u \in \text{Lip}(X)$ satisfying $\chi_F \leq u \leq \chi_\Omega$ and to $p$-weak upper gradients $g$ of $u$ such that $g = g\chi_\Omega \in L^p(X)$. For a proof we refer to [26, Remark 2.1].

**Definition 7.3.** A closed set $E \subset X$ satisfies the $p$-capacity density condition, for $1 \leq p < \infty$, if there is a constant $c_0 > 0$ such that

$$\text{cap}_p(E \cap \overline{B(x, r)}, B(x, 2r)) \geq c_0 \text{cap}_p(B(x, r), B(x, 2r))$$  \hspace{1cm} (41)$$

for all $x \in E$ and all $0 < r < (1/8) \text{diam}(E)$.

**Remark 7.4.** If $X$ supports a $p$-Poincaré inequality for some $1 \leq p < \infty$, then there is a close connection between the $p$-capacity and the measure of balls, similar to Lemma 3.3. Namely, there is a constant $C > 0$ such that for all balls $B = B(x, r)$, with $0 < r < (1/8) \text{diam}(X)$, and for all closed sets $F \subset \Omega$ we have

$$\frac{\mu(F)}{C r^p} \leq \text{cap}_p(F, 2B) \leq \frac{c_\mu \mu(B)}{r^p}.$$  \hspace{1cm} (42)$$

For a proof of this fact see, for instance, [5, Proposition 6.16]. In particular, it holds for all balls $B = B(x, r)$ with $0 < r < (1/8) \text{diam}(X)$ that

$$\frac{\mu(B)}{C r^p} \leq \text{cap}_p(\overline{B}, 2B) \leq \frac{c_\mu \mu(B)}{r^p}.$$  \hspace{1cm} (43)$$

As a consequence, if $E \subset X$ is a closed set and $X$ supports a $p$-Poincaré inequality, then $E$ satisfies the $p$-capacity density condition (41) if and only if there is a constant $c_1 > 0$ such that

$$\text{cap}_p(E \cap \overline{B(x, r)}, B(x, 2r)) \geq c_1 r^{-p} \mu(B(x, r))$$

for all $x \in E$ and all $0 < r < (1/8) \text{diam}(E)$.

The following Theorem 7.5 says that for $1 < p < \infty$ the variational $p$-capacity and $(1, p)$-Hajłasz capacity are comparable in the appropriate geometrical setting. In the proof we use the noncentered maximal function $Mf$, which is defined by (8).

**Theorem 7.5.** Let $1 < p < \infty$. Assume that $\Omega \subset X$ is a bounded open set and $F \subset \Omega$ is a closed set. Then

$$\text{cap}_p(F, \Omega) \leq 4^p \text{cap}_{1,p}(F, \Omega).$$  \hspace{1cm} (44)$$

Moreover, if $X$ supports a $q$-Poincaré inequality for some $1 \leq q < p$, with constants $C_p$ and $\lambda \geq 1$, then there is a constant $C = C(C_p, c_\mu, p, q)$ such that

$$\text{cap}_{1,p}(F, \Omega) \leq C \text{cap}_p(F, \Omega).$$  \hspace{1cm} (45)$$

**Proof.** There are test functions for $\text{cap}_p(F, \Omega)$ if and only if there are test functions for $\text{cap}_{1,p}(F, \Omega)$. Namely, the existence of test functions is in both cases characterized by the inequality $\text{dist}(F, X \setminus \Omega) > 0$. Without loss of generality, we may assume that this inequality holds since otherwise both capacities are equal to $\infty$.

We begin with the proof of inequality (44). Let $u$ be a test function for $\text{cap}_{1,p}(F, \Omega)$, that is, $u \in \text{Lip}(X)$ with $u \geq 1$ in $F$ and $u = 0$ in $X \setminus \Omega$. Let $\kappa \geq 0$ be a Lipschitz constant of $u$ and let $g$ be a Hajłasz $1$-gradient of $u$. By redefining $g = \kappa$ in the exceptional set $N = N(g)$
of measure zero, we may assume that inequality (6) holds for all $x, y \in X$, with $\beta = 1$. Then arguing as in [35, Lemma 4.7] with the aid of [5, p. 20], we see that $4g$ is actually a $p$-weak upper gradient of $u$. Therefore,

$$\text{cap}_p(F, \Omega) \leq \int_{\Omega} (4g)^p \, d\mu \leq 4^p \int_X g^p \, d\mu.$$  

Since this holds for all $u$ and all Hajlasz 1-gradients $g$ of $u$, we conclude that (44) holds.

Next we prove estimate (45) under the assumption that $X$ supports a $q$-Poincaré inequality for some $1 \leq q < p$, with constants $C_p$ and $\lambda \geq 1$. Let $u \in \text{Lip}(X)$ be a test function for $\text{cap}_p(F, \Omega)$ and let $g$ be a $p$-weak upper gradient of $u$. Observe that $g$ is also a $q$-weak upper gradient of $u$, since $q < p$. By Remark 7.2, we may assume that $g = g \chi_\Omega$.

We claim that a constant multiple of $(Mg^q)^{1/q}$ is a 1-Hajlasz gradient of $u$, that is,

$$|u(x) - u(y)| \leq C(C_p, c_\mu) \, d(x, y) \left( (Mg^q(x))^{1/q} + (Mg^q(y))^{1/q} \right)$$

(46)

for all $x, y \in X$. To prove (46), we follow a chaining argument from [13, p. 13–14]. For this purpose, we let $x, y \in X$, $x \neq y$, and define $B_0^x = B_0^y = B(x, 2d(x, y))$ and

$$B_j^x = B(x, 2^{-j}d(x, y)), \quad B_j^y = B(y, 2^{-j}d(x, y)), \quad j \geq 1.$$  

Notice that the definition of $B_j^x = B_0^y$ is different from the other ones. Nevertheless, the balls are nested in the following way: $B_0^x \supset B_1^x \supset B_2^x \supset \cdots$ and $B_0^y \supset B_1^y \supset B_2^y \supset \cdots$. Moreover, we have $\mu(B_j^x) \leq c_\mu^2 \mu(B_j^x+1)$ and $\mu(B_j^y) \leq c_\mu^2 \mu(B_j^y+1)$ for all $j = 0, 1, \ldots$ by the doubling property (2). Using continuity of $u$ and the above properties, we have

$$|u(x) - u(y)| \leq |u(x) - u_{B_0^x}| + |u(y) - u_{B_0^y}|$$

$$\leq \sum_{j=0}^\infty |u_{B_{j+1}^x} - u_{B_j^x}| + \sum_{j=0}^\infty |u_{B_{j+1}^y} - u_{B_j^y}|$$

$$\leq \sum_{j=0}^\infty \int_{B_{j+1}^x} |u - u_{B_j^x}| \, d\mu + \sum_{j=0}^\infty \int_{B_{j+1}^y} |u - u_{B_j^y}| \, d\mu$$

$$\leq c_\mu^2 \sum_{j=0}^\infty \int_{B_j^x} |u - u_{B_j^x}| \, d\mu + c_\mu^3 \sum_{j=0}^\infty \int_{B_j^y} |u - u_{B_j^y}| \, d\mu.$$  

By applying the assumed $q$-Poincaré inequality to the pair $u$ and $g$, we obtain

$$|u(x) - u(y)|$$

$$\leq c_\mu^2 C_p \sum_{j=0}^\infty \text{diam}(B_j^x) \left( \int_{\lambda B_j^x} g^q \, d\mu \right)^{1/q} + c_\mu^3 C_p \sum_{j=0}^\infty \text{diam}(B_j^y) \left( \int_{\lambda B_j^y} g^q \, d\mu \right)^{1/q}$$

$$\leq c_\mu^2 C_p (Mg^q(x))^{1/q} \sum_{j=0}^\infty \text{diam}(B_j^x) + c_\mu^3 C_p (Mg^q(y))^{1/q} \sum_{j=0}^\infty \text{diam}(B_j^y)$$

$$\leq C(C_p, c_\mu) \, d(x, y) \left( (Mg^q(x))^{1/q} + (Mg^q(y))^{1/q} \right).$$

Here we use also the facts that $x \in \lambda B_j^x$ and $y \in \lambda B_j^y$ for all $j = 0, 1, \ldots$. This works even for the ball $\lambda B_0^y$ that is not centered at $y$, but contains $y$. This proves the claim (46).
Since $C(C_P, c_\mu)(M g^q)^{1/q} \in \mathcal{D}_q^\mu(u)$ and the maximal operator is bounded on $L^{p/q}(X)$, we obtain for the Hajl/\'asz $(1, p)$-capacity the estimate
\begin{align}
\text{cap}_{1,p}(F, \Omega) \leq C(C_P, c_\mu, p, q) \int_X (M g^q)^{p/q} \, d\mu \\
\leq C(C_P, c_\mu, p, q) \int_X g^p \, d\mu = C(C_P, c_\mu, p, q) \int_\Omega g^p \, d\mu,
\end{align}
where we also use the fact that $g = g\chi_\Omega$. Since estimate (47) holds for all test functions $u$ for $\text{cap}_{p}(F, \Omega)$ and all their $p$-weak upper gradients $g$ satisfying $g = g\chi_\Omega$, we conclude that (45) holds. This completes the proof.

Next we give an example which shows that the $q$-Poincar/é inequality assumption cannot be omitted from the second part of Theorem 7.5, i.e. for inequality (45) to hold. This example also shows that the $p$-Poincar/é inequality assumption cannot be omitted for the first inequality in (43) to hold.

**Example 7.6.** Fix $1 < p < q < \infty$ and let $w(x) = \text{dist}(x, \{1, -1\})^{q-1}$ for all $x \in \mathbb{R}$. We consider the metric measure space $X = \mathbb{R}$ equipped with the Euclidean distance $d$ and the weighted measure $\mu$ such that
$$\mu(A) = \int_A w(x) \, dx$$
for all Borel sets $A \subset \mathbb{R}$. By [21, Theorem 10.26] we see that $w$ belongs to the Muckenhoupt class $A_{q+\varepsilon}$ for all $\varepsilon > 0$. In particular, the measure $\mu$ is doubling. Fix $0 < \rho < 1$ and consider the Lipschitz test function
$$u(x) = \max \left\{ 0, 1 - \frac{\text{dist}(x, \{-1, 1\})}{\rho} \right\}, \quad x \in \mathbb{R}.$$ 

Then $|u'|$ is a $p$-weak upper gradient of $u$, by [5, Proposition 1.14], and so
$$\text{cap}_p(B(0, 1), B(0, 2)) \leq \int_{B(0,2)} |u'(x)|^p w(x) \, dx \leq 2\rho^{-p} \int_0^\rho t^{q-1} \, dt = \frac{2}{q} \rho^{q-p}.$$ 

By taking $\rho \to 0$, we find that
$$\text{cap}_p(B(0, 1), B(0, 2)) = 0.$$ 

Hence, the first inequality in (43) cannot hold. Nevertheless, Remark 3.3 implies that
$$\text{cap}_{1,p}(B(0, 1), B(0, 2)) \geq C \mu(B(0, 1)) > 0$$
and therefore inequality (45) cannot hold for any $C_2 > 0$. From [6, Theorem 2] and [21, Theorem 10.26] it follows that $(X, d, \mu)$ supports a $(q + \varepsilon)$-Poincar/é inequality for all $\varepsilon > 0$ and that $(X, d, \mu)$ does not support a $q$-Poincar/é inequality. In particular, $(X, d, \mu)$ does not support a $(p - \varepsilon)$-Poincar/é inequality for any $\varepsilon > 0$.

The equivalence of the variational $p$-capacity and the $(1, p)$-Hajl/\'asz capacity in Theorem 7.5 gives immediately also the equivalence of the corresponding density conditions, provided the space supports a suitable Poincar/é inequality.

**Corollary 7.7.** Let $1 \leq q < p < \infty$ and assume that $X$ supports a $q$-Poincar/é inequality. Then a closed set $E \subset X$ satisfies the Hajl/\'asz $(1, p)$-capacity density condition if and only if $E$ satisfies the $p$-capacity density condition.

By combining Corollary 7.7 and Theorem 6.4, we obtain the following characterization of the $p$-capacity density condition in terms of the Riesz capacity, Hajl/\'asz capacity and Hausdorff content density conditions.
Theorem 7.8. Let $1 < p < \infty$ and assume that $X$ is a complete geodesic space supporting a $p$-Poincaré inequality. In addition, assume that $\mu$ satisfies the quantitative reverse doubling condition (4) for some exponent $\sigma > p$. Then the following conditions are equivalent for a closed set $E \subset X$:

(i) $E$ satisfies the $p$-capacity density condition.
(ii) $E$ satisfies the Riesz $(1,p)$-capacity density condition.
(iii) $E$ satisfies the Hajlasz $(1,p)$-capacity density condition.
(iv) $E$ satisfies the Hausdorff content density condition (28) for some $0 < q < p$.

Proof. Since $X$ is complete and $\mu$ is doubling, the Keith–Zhong theorem [18] implies that $X$ supports an $s$-Poincaré inequality for some $1 < s < p$. The conditions (i) and (iii) are then equivalent by Corollary 7.7. On the other hand, the equivalence of the conditions (ii), (iii) and (iv) follows from Theorem 6.4. □

A similar argument using Theorem 6.2 instead of Theorem 6.4 gives the following result, which excludes the Riesz capacity density condition but does not assume the quantitative reverse doubling condition.

Theorem 7.9. Let $1 < p < \infty$ and assume that $X$ is a complete geodesic space supporting a $p$-Poincaré inequality. Then the following conditions are equivalent for a closed set $E \subset X$:

(i) $E$ satisfies the $p$-capacity density condition.
(ii) $E$ satisfies the Hajlasz $(1,p)$-capacity density condition.
(iii) $E$ satisfies the Hausdorff content density condition (28) for some $0 < q < p$.

Since the Hajlasz $(1,p)$-capacity density condition is self-improving with respect to $p$ in geodesic spaces, the above results imply the self-improvement also for the $p$-capacity density condition. The following corollary is known, see [7], but our method of proof, based on the results from [8], is new.

Corollary 7.10. Let $1 < p < \infty$ and assume that $X$ is a complete geodesic space supporting a $p$-Poincaré inequality. Let $E \subset X$ be a closed set satisfying the $p$-capacity density condition. Then $E$ satisfies the $q$-capacity density condition for some $1 < q < p$.

Proof. By the Keith–Zhong theorem [18], we see that $X$ supports an $s$-Poincaré inequality for some $1 < s < p$. Hence, the conclusion follows from Theorem 7.9. □

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