Noncommutativity vs gauge symmetry

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Abstract

In many aspects the most complicated foliated manifolds are those with nonvanishing Godbillon-Vey class. We argue that they probably do not appear in physics and that is due to gauge symmetry which prevents the foliation from becoming “too wild”; that means that the foliation does not develop resilient leaves which, at least in codim-1, by Duminy’s theorem are responsible for the nontriviality of the GV-class.

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1 Introduction and Motivation

Modern geometry is divided into two big areas: differential and algebraic geometry. The former studies structures called (algebraic) varieties, the later studies structures called manifolds. Both these structures are spaces which locally—but not necessarily globally—”look like” Euclidean spaces (using local coordinate charts); both varieties and manifolds share one common important property: the space of coordinate functions form a commutative ring (varieties) or algebra (manifolds). During the last 20 years or so a

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new geometry appeared which unifies and generalises the above two: it is called noncommutative geometry and it studies structures whose coordinate functions form a noncommutative ring or algebra. As its main creator, the French mathematician Alain Connes says, the motivation came directly from quantum physics (but also from Index and K-Theory): as it is very well known, the difference between classical and quantum physics lies on Planck’s constant which from the mathematical point of view measures precisely the failure of commutativity between position and momentum. We should also add that one of the main advantages of noncommutative geometry is that it offers the rigorous mathematical framework to study fractals and chaotic dynamical systems (as far as measure theory, analysis and topology are concerned).

Despite their increased level of difficulty and complexity, noncommutative spaces have already been proved very useful in physical applications, and their use is bound to increase during the years to come, an obvious fact if one only reflects back on the motivation for their construction to begin with; perhaps the first example historically of the appearance of noncommutative spaces in physical applications is the study of the Integral Quantum Hall Effect (IQHE in short) on square lattices in a uniform irrational magnetic field, primarily through the work of Jean Bellissard (see [5]). By a noncommutative space we mean a space whose algebra of coordinate functions (or its analogue) is noncommutative. The existence of the uniform magnetic field $B$ turns the Brillouin zone from a commutative to a noncommutative 2-torus $T^2_\theta$ with irrational slope $\theta$, where $\theta$ is directly related to the uniform magnetic field $B$. Essentially the same ideas were carried over to the M-Theory context in the classic article due to Connes-Douglas-Schwarz (CDS in short) [1], and subsequently by Seiberg and Witten). From the CDS article we learnt that M-Theory admits additional compactifications onto noncommutative tori which are higher dimensional versions of the noncommutative 2-torus. In that M-Theory context the role of the uniform magnetic field which is responsible for noncommutativity was played by the constant 3-form field $C$ of $D = 11$ supergravity. One can study the noncommutative tori either algebraically using their corresponding noncommutative algebras of coordinate functions or geometrically using the so-called Kronecker foliations on the tori. However these noncommutative tori in certain aspects which will hopefully become clearer below, are among the “less noncommutative spaces” available. We would like to see if other “more noncommutative” spaces may play any role in physics. That amounts to investigating what
may happen as long as noncommutativity is concerned if the magnetic field (or the 3-from \( C \)) is no longer uniform. In other words we shall try to give (at least a partial) answer to the following question:

*How much noncommutativity can we have in physics?*

In the present article we shall argue that *very noncommutative spaces*, examples of which are foliations with nonvanishing GV-class, probably do not appear in physics due to gauge symmetry; but spaces with “medium or little noncommutativity” may very well appear and have significant consequences. Our answer is perhaps not totally unexpected, after all Planck’s constant is indeed very small, hence we have little noncommutativity in nature. Our reasoning however will be different in this article.

## 2 Foliations and why they play a key role in noncommutative geometry

Here we recall some basic facts about foliated manifolds: let \( M \) be an \( m \)-dim smooth, closed and oriented manifold. A codim-\( q \) foliation \( F \) on \( M \) is given by a codim-\( q \) integrable subbundle \( F \) of the tangent bundle \( TM \) of \( M \). “Integrable” means that the tangential vector fields to the foliation, namely the smooth sections of the vector bundle \( F \) over \( M \), form a Lie subalgebra of the Lie algebra of smooth sections of \( TM \). An equivalent local definition is given by a nonsingular decomposable \( q \)-form \( \omega \) on \( M \) satisfying the integrability condition

\[
\omega \wedge d\omega = 0
\]  

What this does in effect is that it gives a decomposition of \( M \) into a disjoint union of codim-\( q \) (and hence of dimension \((m - q)\)) immersed and connected submanifolds of \( M \) called leaves. The tangent spaces over the leaves are defined precisely by the vector fields which are annihilated by \( \omega \). Basic examples of foliations are Cartesian products and the total space of fibre bundles (the fibres are the leaves). Two important differences between foliations and fibre bundles are:

1. the topology of each leaf may vary (some leaves may be compact but some others may not; the fundamental groups also vary) whereas in a fibre
bundle all fibres “look the same” as the typical fibre.

2. the relative geometry of neighbouring leaves is far more complicated because they may “spiral” over each other without intersecting.

It is known that foliated manifolds (more precisely the spaces of leaves of foliated manifolds) provide an excellent list of examples of noncommutative spaces (see [5]). Connes has given a recipe how to construct a corresponding algebra, more precisely a $C^*$-algebra, for every foliation using the holonomy groupoid of the foliation. The holonomy groupoid conceptually is a suitable generalisation of the holonomy of the connection on a vector bundle.

A noncommutative space is a space whose algebra of coordinate functions is noncommutative. Thus the corresponding algebra of a foliation can be thought of as the space of coordinate functions on its topologically ill-behaved space of leaves. We say that foliations provide an excellent list of examples of noncommutative spaces because although not every noncommutative algebra can be realised as the corresponding algebra to a foliation, it is known that all three types of factors can occur as corresponding algebras to foliations: Type I occurs at Reeb foliations, Type II at Kronecker foliations and Type III at foliations with nonvanishing Godbillon-Vey class. So it is fair to say that foliations constitute the “back bone” of noncommutative spaces. It is understood that the source of the noncommutativity at the corresponding $C^*$-algebra level is the holonomy of the foliation. This encodes all the information about the foliation: it contains information about the topology of the leaves (primarily their fundamental groups) but it also contains information about their relative geometry. It is perhaps illuminating to compare with the holonomy of a connection around a loop on a vector bundle: as it is well known this depends on the homotopy class of the loop, thus we see the fundamental group playing a role; but it also depends on the connection, ie how we parallel transport vectors along neighbouring fibres, so it depends on the relative geometry of the fibres. In the foliation case we “parallel transport” transversals by sliding them along loops on the leaves.

The most noncommutative spaces arise from foliations with non-vanishing GV-class. We want to argue here that at least these most extreme cases probably do not appear in physics. The reason, as was more or less suspected in [10], is that gauge symmetry “tames” the foliation and prevents it from becoming “very wild”. Let us emphasise here that we are talking about foliations with nonvanishing GV-class and not about Type III factors in general.
Now we shall briefly define the Godbillon-Vey (GV for short) class: our codim-$q$ foliation is given by a $q$-form $\omega$ satisfying the integrability condition. By Frobenius’ theorem the integrability condition is equivalent to

$$d\omega = a \wedge \omega$$

for another 1-form $a$. Then the GV-class is the real $(2q + 1)$ de Rham cohomology class $a \wedge (da)^q$. The geometric interpretation of $a$ is the following: since $F$ is a codim-$q$ integrable subbundle of the tangent bundle $TM$ of $M$, then its transverse bundle $Q := TM/F$ is of dimension $q$ and hence its $q$-th exterior power $\Lambda^q Q$ is a line bundle. Then $a$ is a (Bott) connection on $\Lambda^q Q$ with curvature $da$.

For foliations one also has the important notion of topological entropy (see [7] or [14]) which roughly is another measure of how wild (or “how noncommutative”) the quotient space of leaves is. Roughly speaking topological entropy contains information only about the relative geometry of the leaves and not about their topology. So it encodes some of the information contained in the holonomy groupoid. A Corollary to a deep theorem due to Gerard Duminy relates topological entropy with the Godbillon-Vey class for codim-1 foliations: if the GV-class is nonzero then the topological entropy is also nonzero or equivalently if the topological entropy vanishes, then so does the GV-class. However if the GV-class vanishes, then the topological entropy may or may not vanish, one does not know. That means that somehow the topological entropy of a foliation is a more delicate notion than the GV-class.

3 A list of Foliations in increasing order of complexity

We said that the most noncommutative spaces arise from foliations with nonvanishing GV-class. At the other extreme of the spectrum we have the less noncommutative spaces: fibre bundles (we ignore the completely trivial example of Cartesian products). Fibre bundles have corresponding algebras which are noncommutative but they are strongly Morita equivalent to commutative algebras and their topological entropy vanishes, as does their
GV-class. Nevertheless fibre bundles may have non trivial holonomy, for example for a principal $G$-bundle where $G$ is a Lie group, the holonomy groupoid is a subgroup of the Lie group $G$ itself.

The next more noncommutative spaces are foliations defined by closed forms. (The integrability condition is trivially satisfied by closed forms). As was exhibited in [10], fibre bundles with compact base manifold constitute particular examples of foliations defined by closed forms. Foliations defined by closed forms have always vanishing GV-class but their topological entropy vanishes only in codim-1 case. Moreover in codim-1, foliations with trivial holonomy are homoeomorphic to foliations defined by closed forms but we cannot deduce that a foliation defined by a closed form has trivial holonomy.

The next more noncommutative spaces to foliations defined by closed forms are the taut foliations. We shall define them in the codim-1 case: A codim-1 foliation $F$ on an $m$-dim manifold $M$ is called (topologically) taut if there exists an $S^1$ intersecting transversely all leaves. A codim-1 foliation is called (geometrically) taut if there exists a metric for which all leaves are minimal submanifolds (ie have mean curvature zero). It is in fact a theorem to prove that these two definitions are equivalent. These definitions can be generalised for foliations of codim greater than 1.

For taut foliations we have Rumler’s criterion: for each taut $(m-1)$-dim foliation there exists an $(m-1)$-form which is $F$-closed and transverse to $F$. The second condition means that the form is nonsingular when restricted on every leaf. The condition that the form is $F$-closed weakens the condition of it being closed: in general for a $p$-dim foliation $F$, a $p$-form $\eta$ is called $F$-closed if $d\eta = 0$ whenever at least $p$ vectors are tangent to $F$.

Moreover from the Hilsum-Skandalis theorem (see [12]) we learn that the corresponding algebra of foliations with a complete transversal (a complete transversal is a transversal intersecting all leaves) simplifies drastically (for taut foliations the complete transversal is just $S^1$): it is Morita equivalent to the algebra of the restriction of the holonomy groupoid to only the transversal itself. Taut foliations may prove a key ingredient in defining a Noncommutative Floer Homology for closed, oriented and connected 3-manifolds which are not necessarily homology 3-spheres (see [11]).

Now the noncommutative spaces (noncommutative torus $T^2_\theta$) appearing
in physics literature both in the Integral Quantum Hall Effect as well as in the CDS article are foliations defined by closed forms, in fact constant forms. Their corresponding algebras are Type II since they are essentially algebras associated to Kronecker foliations on the torus defined by constant differential forms. In the IQHE we have a codim-1 foliation on a 2-torus defined by a constant 1-form (the “slope” \( \theta \) of the Kronecker flow) which is essentially determined by the uniform magnetic field. Even the noncommutative 2-torus \( T^2_\theta \) which has irrational slope \( \theta \) is in fact homotopic to the commutative one \( T^2 \) (usual torus or with rational slope \( \theta \)), thus having cyclic (co)homology and K-Theory isomorphic to the de Rham cohomology and K-Theory respectively of the ordinary (commutative) 2-torus and hence from the point of view of noncommutative topology it is a rather trivial example. The only difference between the K-Theories of \( T^2_\theta \) and \( T^2 \) is the order of the Abelian groups.

Similar things hold for the CDS article: there one has the D=11 supergravity real 3-form potential \( C \) which is also assumed to be constant (and hence again closed) and that gives rise to higher dimensional Kronecker foliations on the compactified tori.

Now the topological entropy of the Kronecker foliation is zero, so the foliations appearing in physics literature up to now are the most trivial noncommutative spaces. And since they are defined by constant forms (which are therefore closed), they also have vanishing GV-class.

It was the notion of topological entropy of foliations and its possible relation with physical entropy which served as our motivation for [10]: it is known that string theory gives in some cases an explanation of the microscopic origin of the black hole entropy. The compactified dimensions play an important role in this argument due to Horowitz, Strominger and Vafa back in 1996 (see [2]). From the CDS article we had some important new input, that M-theory admits additional compactifications to (foliated) noncommutative tori. So it is interesting to see what will happen if we assume that the compactified dimensions form not simply a noncommutative torus but a noncommutative torus with nonvanishing topological entropy. That might imply some modification to the Beckenstein-Hawking area entropy formula for black holes (that’s by the string theory origin of black hole entropy). To guarantee that the foliation has nonvanishing topological entropy, by Duminy’s theorem, one may assume that the foliation has nonvanishing
 GV-class (this is a sufficient but not necessary condition as we explained above). A first answer to this question based on [9] which still needs further improvement was given in [10].

4 Foliations with nonvanishing GV-class vs gauge symmetry

The starting point is to try to see if foliations with nonvanishing GV-class can occur as D=11 supergravity solutions following the D=11 supergravity interpretation in the CDS article.

Let us recall the bosonic part of the D=11 supergravity Lagrangian density (we follow [6]):

\[
L_{11} = \frac{1}{2k_{11}^2} (R - \frac{1}{2.4!} G \wedge \ast G) - \frac{1}{12k_{11}^2} \frac{1}{3!4!^2} C \wedge G \wedge G + \text{"fermions"} \quad (3)
\]

where by definition \( G := dC \). Our bosonic fields are the metric \( g_{MN} \) with scalar curvature \( R \) (all capital letters appearing as subscripts take the values 0, 1, ..., 10) whose field equations are analogous to Einstein’s equations with electromagnetic field in D=11:

\[
R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{12} (G_{MSPQ} G_{N}^{SPQ} - \frac{1}{8} g_{MN} G_{STXY} G^{STXY}) \quad (4)
\]

and the real 3-form \( C \) whose equations of motion are:

\[
d \ast G + \frac{1}{2} G \wedge G = 0 \quad (5)
\]

where "\( \ast \)" denotes the Hodge dual. For simplicity we set all fermionic fields equal to zero. If now we assume that \( C \) defines a codim-3 foliation namely \( C \wedge G = 0 \), that means that the Chern-Simons term in D=11 supergravity action \( C \wedge G \wedge G \) vanishes and hence we are left with the equation of motion for \( C \):

\[
d \ast dC = 0 \quad (6)
\]
The key question then is if such solutions exist, namely we want to see if there exist real 3-forms $C$ satisfying the Euler-Lagrange equation 6 above but at the same time they define a codim-3 foliation with nonvanishing GV-class, namely $C$ must also satisfy

$$G := dC = \theta \wedge C$$  \hspace{1cm} (7)$$

The above equation 7 is equivalent to $C \wedge G = 0$ by the Frobenius theorem. The Hodge star in equation 6 refers to some metric which satisfies Einstein’s equations. The 1-form $\theta$ is the 1-form which appears in the definition of the corresponding GV-class which in this case will be $\theta \wedge (d\theta)^3$. [This 1-form $\theta$ can be seen as a (partial) Bott connection on the transverse bundle]. If we assume that the ambient 11-manifold is closed, namely compact without boundary, then the equation of motion for $C$ is equivalent to $C$ being closed; hence the GV-class will vanish (we assume that $C$ is nonzero).

In order then to have some hope to find a solution of the equations of motion which also define a codim-3 foliation with nonvanishing GV-class, we should either add a boundary or go to the noncompact case.

We can simplify our discussion further: since D=11 supergravity is very similar to gravity coupled to electromagnetism in odd dimensions, we start with the simplest case: gravity coupled to electromagnetism in dimension 3. In this case we denote by $A$ the electromagnetic potential and its field strength is denoted $F := dA$. We have two options: either that the foliation is defined by the potential $A$ or by its field strength $F$. We start from the first: Maxwell’s equation reads:

$$d \star dA = 0$$  \hspace{1cm} (8)$$

Since we want $A$ to define a codim-1 foliation it also has to satisfy

$$A \wedge dA = 0$$  \hspace{1cm} (9)$$

where $F := dA = \theta \wedge A$ for another 1-form $\theta$, thus the electromagnetic potential $A$ defines a codim-1 foliation with nonvanishing GV-class $\theta \wedge d\theta$. As it is well-known by the Frobenius theorem the equation $A \wedge dA = 0$ is equivalent to $dA = \theta \wedge A$. 

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Finally the metric has to satisfy Einstein’s equations coupled to electromagnetism:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{12} (F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{8} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}) \] (10)

Again on a closed 3-manifold Maxwell’s equation is equivalent to \( A \) being closed hence the GV-class will vanish (we assume \( A \) is nonsingular). In order to hope to have nonvanishing GV-class (ie \( A \) not closed) one has either to add a boundary or consider noncompact 3-manifolds.

Now here is the key observation: equation 9 which says that the potential \( A \) defines a codim-1 foliation with nonvanishing GV-class is not gauge invariant. Physics is not sensitive in gauge transformations, namely we can replace \( A \) by \( \tilde{A} := A + d\phi \) where \( \phi \) is a zero form. A little calculation shows that although we started with a foliation defined by \( A \), namely \( A \wedge dA = 0 \), its gauge transform \( \tilde{A} \wedge d\tilde{A} = A \wedge dA + d\phi \wedge dA = d\phi \wedge dA \neq 0 \) is not zero in general. Hence the foliation structure can be completely destroyed by a gauge transformation!

Aside Note: The remaining term \( d\phi \wedge dA \) however is a total derivative which will not contribute to the action if the manifold has no boundary; can we perhaps do something here to save the day? We do not have a concrete suggestion but the following might be of relevance: as it is well-known \( A_\infty \)-algebras appear in the BV-formalism when the commutator of two BRST transformations does not close on shell. Some more evidence which made us think of \( A_\infty \)-algebras was from [13] where homotopy associative algebras appear in open string theory when the symplectic structure is lost, ie when the gauge invariant combination \( \Omega := B + F \) is no longer closed; this “looks similar” to a codim-2 foliation defined by a 2-form \( \Omega \) which is not closed, thus may have nonvanishing GV-class; physically this corresponds to a curved D-brane embedded in a curved background. More precisely in [13] the authors investigate the deformation of D-brane world-volumes in curved backgrounds. They calculate the leading corrections to the boundary conformal field theory involving the background fields, and in particular they study the correlation functions of the resulting system. This allowed them to obtain the world-volume deformation, identifying the open string metric and the noncommutative deformation parameter. The picture that unfolded was the following: when the gauge invariant combi-
nation $\Omega = B + F$ is constant one obtains the standard Moyal deformation of the brane world-volume. Similarly, when $d\Omega = 0$ one obtains the non-commutative Kontsevich deformation, physically corresponding to a curved brane in a flat background. When the background is curved, $d\Omega \neq 0$, they find that the relevant algebraic structure is still based on the Kontsevich expansion, which now defines a nonassociative star product with an $A_\infty$ homotopy associative algebraic structure. They then recovered, within this formalism, some known results of Matrix theory in curved backgrounds.

Foliations are only invariant under multiplications of $A$ by a nowhere vanishing function $f$ so it seems that gauge invariance forbids foliations with nonvanishing GV-class to play any role in physics even if they can exist as solutions of the equations of motion. Thus we see that foliations with nonvanishing GV-class are very delicate objects.

However if the foliation is defined by a closed 1-form $A$, i.e. $dA = 0$ (which is also an electromagnetic potential), thus having vanishing GV-class, then if we gauge transform $A$ to $\tilde{A} = A + d\phi$, then since $\tilde{A}$ is also closed the foliation structure remains since $\tilde{A} \wedge d\tilde{A} = 0$ and the new foliation defined by the closed 1-form $\tilde{A}$ has again vanishing GV-class. Hence foliations defined by closed forms (and consequently have vanishing GV-class) are very rigid structures with respect to gauge transformations.

So to sum-up: If we start from a foliation defined by the electromagnetic potential $A$ which is a closed 1-form and thus has vanishing GV-class and we gauge transform $A$, we get another foliation defined by another closed 1-form and thus has vanishing GV-class too; so in this case a gauge transformation does not change the GV-class. Yet if we start from a foliation with nonvanishing GV-class and we gauge transform it, this transformation will not only change the GV-class but it may destroy the foliation structure completely. This very different behaviour under gauge transformations between foliations defined by closed forms (which form a particular family of foliations with vanishing GV-class) and foliations with nonvanishing GV-class was surprising. Note that in codim-1 case a foliation defined by a closed 1-form has zero topological entropy as well.

There seems to be another alternative however, namely to assume that the foliation is defined by the field $F := dA$ (or its dual), in this case we shall have a codim-2 foliation (provided $F$ is decomposable and nonsingular).
But now $F$ is closed due to Bianchi identity, hence the codim-2 foliation now will have again vanishing GV-class. (Equivalently the derivative of the dual field vanishes due to Maxwell's equations). This picture is consistent with IQHE where the slope of the noncommutative 2-torus comes from a uniform magnetic field in $z$-coordinate (this is a component of $F$, not $A$). However the important difference here is that a codim-2 foliation defined by a closed 2-form, although it will have zero GV-class, it may have nonzero topological entropy. Hence we might have interesting phenomena appearing even when the Bianchi identity holds (which would mean vanishing GV-class). The problem is that we do not have a way to detect the appearance of the topological entropy in codimensions greater than 1. Duminy's criterion which uses the GV-class, although not absolutely satisfactory since it is not an if and only if statement, applies only to the codim-1 case.

There is also yet another setting, that of Seiberg-Witten (or monopole) equations in $N=2$ SUSY Yang-Mills theory as modified by Kronheimer and Mrowka (see [3]) for a 4-manifold with boundary. The boundary 3-manifold may have a contact structure. Contact structures are “cousins” to foliations and moreover taut foliations correspond to tight contact structures. This possibility needs further study to see if one can get codim-1 foliations with nonvanishing GV-class in that set-up. One of the interesting points in that article is a correspondence between tight contact structures and taut foliations on the boundary and symplectic structures on the bulk.

5 Remarks:

Let us make some remarks:

1. The existence of the GV-class for a codim-$q$ foliation $q \geq 1$ follows from Bott's vanishing theorem for the Pontrjagin classes in degree $k > 2q$ of the transverse bundle of the foliation. One roughly can think of the GV-class as something like the “corresponding (Abelian) Chern-Simons” form in the following way: Bott's theorem says that given a smooth closed $m$-manifold $M$ with tangent bundle $TM$, if a codim-$q$ subbundle $F$ of $TM$ is integrable then the Pontrjagin classes of the transverse bundle $Q := TM/F$ in degree $k > 2q$ must vanish. Hence supposing $F$ defines indeed a codim-$q$ foliation, the first vanishing Pontrjagin class of its transverse bundle will be in degree $(2q + 2)$. So the GV-class which is a real $(2q + 1)$-form can be thought of
as an “Abelian Chern-Simons” form whose exterior derivative will give the Pontrjagin class in degree \((2q + 2)\). This however vanishes by Bott’s theorem and so the GV-class is indeed a cohomology class (ie it is closed).

However in this picture, strictly speaking, the GV-class really refers to the transverse bundle and not to the foliation itself. Then the question is: up to what extent is the behaviour of the foliation determined by its transverse bundle?

The GV-class is not the most natural object to study in order to deduce results about the foliation itself because primarily it is some information about the transverse bundle. This is so because as it is discussed in [4], given a smooth closed \(m\)-manifold \(M\), the functor from codim-\(q\) Haefliger structures on \(M\) (foliations are particular examples of Haefliger structures) to \(GL(q; \mathbb{R})\)-bundles over \(M\) which assigns the transverse bundle to any codim-\(q\) Haefliger structure is essentially defined by the derivative of the \(\Gamma_q\)-cocycle, the Jacobian of the local diffeomorphisms. This functor is not an equivalence of categories: it is neither surjective (due to Bott’s result not every \(GL(q; \mathbb{R})\)-bundle can occur as the normal bundle of some codim-\(q\) Haefliger structure since at least it has to satisfy Bott’s theorem, ie it must have vanishing Pontrjagin classes in degree \(k > 2q\)). Nor is it injective since there are different Haefliger structures with the same transverse bundle. So the lesson we have learnt is that studying foliations using secondary classes of their transverse bundle is like studying functions by their 1st derivatives and this is an approximation, one loses information.

The noncommutative geometry approach to study foliations is probably a tool which is more “sensitive”, hence it gives a better approximation; it is more delicate but more complicated because one uses the holonomy groupoid of the foliation itself, which is essentially the Haefliger cocycle itself: one starts with the holonomy groupoid, then one takes the vector space of half-densities, equips it with a convolution product and with an involution and then completes it to a \(C^*\)-algebra (or even more to a Hopf algebra) and then one studies its cyclic cohomology. However passing from the holonomy groupoid to the \(C^*\)-algebra (even the reduced) amounts to loss of information again. At the K-Theory level this is probably not too bad since the Baum-Connes conjecture is true in many cases. Finally we get the transverse fundamental cyclic cocycle (abreviated to “tfcc”, see [5]) which for a codim-\(q\) foliation belongs to the \(q\)-th cyclic cohomology group of
the corresponding algebra of the foliation. The reason why we believe that the noncommutative geometry approximation (Connes’ approximation) is better than the Bott approximation is that at least in the codim-1 case the derivative of the tfcc is the GV-class as explained in Connes’ book. Hence the tfcc is a more delicate object than the GV-class. That makes one to suspect that there may be a relation between the tfcc and the vanishing or not of the topological entropy of a foliation in a “necessary and sufficient” fashion which would improve considerably Duminy’s result.

2. Our current level of understanding for codim-1 foliations is the following (we assume compact manifolds): (fibre bundles) ⊂ (foliations defined by closed forms) ⊂ (foliations with zero topological entropy) ⊂ (foliations with zero GV-class). In higher codimensions one only has the following relations: (fibre bundles) ⊂ (foliations defined by closed forms) ⊂ (foliations with zero GV-class); the topological entropy may still be defined but we know nothing about when it is vanishing. It would be desirable to introduce the tfcc into the picture as well.

So an important difference between the codim-1 and codim greater than 1 cases is the following: in codim-1 a foliation defined by a closed form has zero GV-class as well as topological entropy yet for codim greater than 1, foliations defined by closed forms have zero GV-class but may have non-zero topological entropy.

3. In general one can study foliations using two approaches: either using differential topology methods or operator algebraic tools. If one follows the first approach one meets notions like the GV-class and topological entropy. The most complicated foliations using this language are those with nonvanishing GV-class. Probably they do not appear in physical applications. The second less complicated case is foliations with nonvanishing topological entropy. We would like to see if these appear in physical applications. Unfortunately we cannot see that clearly at this stage because we do not have a satisfactory criterion which will indicate the existence of topological entropy. Duminy’s theorem is a criterion only for the codim-1 case and even then not as satisfactory as one might wish since it is based on the GV-class in the way described above.

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