T-STABILITIES FOR A WEIGHTED PROJECTIVE LINE

SHIQUAN RUAN AND XINTIAN WANG

Abstract. The present paper focuses on the study of t-stabilities on a triangulated category in the sense of Gorodentsev, Kuleshov and Rudakov. We give an equivalent description for the finest t-stability on a piecewise hereditary triangulated category and, describe the semistable subcategories and final HN triangles for (exceptional) coherent sheaves in \(D^b(\text{coh} \mathcal{X})\), which is the bounded derived category of coherent sheaves on the weighted projective line \(\mathcal{X}\) of weight type (2). Furthermore, we show the existence of a t-exceptional triple for \(D^b(\text{coh} \mathcal{X})\). As an application, we obtain a result of Dimitrov–Katzarkov which states that each stability condition \(\sigma\) in the sense of Bridgeland admits a \(\sigma\)-exceptional triple for the acyclic triangular quiver \(Q\). Note that this implies the connectedness of the space of stability conditions associated to \(Q\).

1. Introduction

The notion of a stability condition on a triangulated category was first introduced by Bridgeland in \[2\]. The motivation comes from the study of Dirichlet branes in string theory in physics, and especially from Douglas’s work on II-stability. The main result in \[2\] states that the space \(\text{Stab}(\mathcal{C})\) of all locally finite stability conditions on an essentially small triangulated category \(\mathcal{C}\) is a complex manifold, with a natural right action by the group \(\widetilde{\text{GL}}^+(2, \mathbb{R})\) of the universal covering space of the group of rank two matrices with positive determinant, and a left action by the group \(\text{Aut}(\mathcal{C})\) of exact autoequivalences of \(\mathcal{C}\). This space carries an interesting geometric and topological structure which reflects the properties of \(\mathcal{C}\). Moreover, stability conditions are related to many mathematical subjects, such as Donaldson-Thomas (DT) theory, homological mirror symmetry theory and so on.

Obviously, one would like to be able to compute the spaces \(\text{Stab}(\mathcal{C})\) of stability conditions in some interesting examples, such as the bounded derived category \(\mathcal{C} = D^b(\text{coh} \mathcal{X})\) of coherent sheaves over smooth projective varieties \(\mathcal{X}\), or the bounded derived category \(\mathcal{C} = D^b(\text{mod} kQ)\) of finite dimensional modules over the path algebra \(kQ\) for a quiver \(Q\). For \(\mathcal{C} = D^b(\text{coh} \mathcal{X})\), Bridgeland \[2\] first dealt with the elliptic curve case; and Macri \[13\] considered any smooth projective curve over \(\mathbb{C}\) of positive genus, showing that the action of \(\widetilde{\text{GL}}^+(2, \mathbb{R})\) on the subspace of locally finite numerical stability conditions is free and transitive, moreover, he also investigated the case of projective line \(\mathbb{P}^1\) (by using exceptional sequences), which was first dealt by Okada \[15\]. For \(\mathcal{C} = D^b(\text{mod} kQ)\), many people have worked for \(Q\) of Dynkin type, c.f. \[3\] \[16\] \[17\]; for \(n\)-Kronecker quiver \(Q\) with \(n \geq 2\), Macri proved in \[13\] that \(\text{Stab}(\mathcal{C})\) is a connected and simply connected 2-dimensional complex

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Date: April 27, 2018.
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2010 Mathematics Subject Classification. 18E10, 18E30, 16G20, 14F05, 16G70.
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Key words and phrases. t-stability, stability condition, weighted projective line, exceptional sequence.
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manifold, and Dimitrov–Karzarkov described this manifold in more details in [7]. Recently, Dimitrov and Katzarkov [5, 6] worked on the acyclic triangular quiver $Q$, showing that $\text{Stab}(C)$ is connected and contractible.

In [13], Macri gave a procedure generating stability conditions from exceptional sequences. More precisely, he gave a natural way to associate to a complete Ext-exceptional sequence a heart of a bounded t-structure and then a family of stability conditions which have this one as heart. In this way, one can define a collection of open connected subsets of $\text{Stab}(C)$ of maximal dimension, parametrized by the orbits of the action of the braid group on exceptional sequences. This method provides a new way to investigate the stability conditions via exceptional sequences. Basing on this idea, Dimitrov and Katzarkov in [5] defined a $\sigma$-exceptional sequence for a given stability condition $\sigma$, and proved that for the acyclic triangular quiver $Q$, there exists a $\sigma$-exceptional triple for each stability condition $\sigma$ on $C = \mathbb{D}^b(\text{coh} kQ)$. We remark that this implies the connectedness of the space $\text{Stab}(C)$ by the transitivity of the complete exceptional triples.

The concept of a t-stability in a triangulated category $C$ was first introduced by Gorodentsev–Kuleshov–Rudakov in [9], which is a generalization of Bridgeland’s stability condition. They established the relations between t-stabilities and bounded t-structures on $C$. Indeed, they achieved a classification of the bounded t-structures for $C = \mathbb{D}^b(\text{coh} \mathbb{P}^1)$.

In the present paper we study the finest t-stabilities on $C$ and apply them to the study of stability conditions. We give a sufficient and necessary condition to determine when a t-stability is finest for $C$ piecewise hereditary. Moreover, for the bounded derived category $\mathbb{D}^b(\text{coh} X)$ of coherent sheaves on the weighted projective line $X$ of weight type (2), we describe the semistable subcategories and the final HN triangles for finest t-stabilities in details. After introducing the notion of a t-exceptional sequence on $C$, we show the existence of a t-exceptional triple for $\mathbb{D}^b(\text{coh} X)$. Note that there is an equivalence between $\mathbb{D}^b(\text{coh} X)$ and the bounded derived category $\mathbb{D}^b(\text{mod} kQ)$ for the acyclic triangular quiver $Q$. We obtain that each stability condition $\sigma$ on $\mathbb{D}^b(\text{mod} kQ)$ admits a $\sigma$-exceptional triple, which implies the connectedness of the space $\text{Stab}(\mathbb{D}^b(\text{mod} kQ))$ and was first shown by Dimitrov–Katzarkov in [5].

The paper is organized as follows. In Section 2 we briefly introduce the category $\text{coh} X$ of coherent sheaves on a weighted projective line $X$, and recall the definition and basic results for exceptional sequences in a triangulated category $C$. Section 3 is the main part of this paper. We first recall the definition of a (finest) t-stability on $C$, and give a sufficient and necessary condition for a t-stability to be finest. Moreover, for the bounded derived category $\mathbb{D}^b(\text{coh} X)$ of coherent sheaves of weight type (2), we describe the semistable subcategories as well as the final HN triangles for coherent sheaves. Furthermore, by introducing the notion of a t-exceptional sequence in $C$, we prove that each finest t-stability admits a t-exceptional triple for $\mathbb{D}^b(\text{coh} X)$. Basing on the results of t-stabilities, we investigate stability conditions on $C$ in the sense of Bridgeland in Section 4. We obtain that each stability condition $\sigma$ admits a $\sigma$-exceptional triple for the acyclic triangular quiver $Q$, which was first shown by Dimitrov–Katzarkov [5].
Throughout this paper we always assume that $\mathcal{C}$ is an essentially small triangulated category, that is, the isomorphism classes of objects in $\mathcal{C}$ form a set. Given a set $\mathcal{S}$ of objects in $\mathcal{C}$, we write $\langle \mathcal{S} \rangle$ for the smallest strictly full extension-closed subcategory of $\mathcal{C}$ that contains all the objects in $\mathcal{S}$, and write $\text{Tr}(\mathcal{S})$ for the minimal full triangulated subcategory containing $\mathcal{S}$ which is closed under isomorphisms. For an object $E \in \mathcal{C}$, we use the simple notation $E^n$ to denote the direct sum of $n$ copies of $E$. For $E,F \in \mathcal{C}$, we simply write $\text{Hom}(E,F) = \text{Hom}_\mathcal{C}(E,F)$ and $\text{Ext}^n(E,F) = \text{Hom}_\mathcal{C}(E,F[n])$.

2. Preliminaries

2.1. Coherent sheaves on a weighted projective line. Following [3], a weighted projective line $\mathbb{X} = \mathbb{X}_k$ over a field $k$ is given by a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers, and a sequence $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of distinct closed points (of degree 1) in the projective line $\mathbb{P}^1 := \mathbb{P}^1_k$ which can be normalized as $\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1$. More precisely, let $\mathbb{L} = \mathbb{L}(\mathbf{p})$ be the rank one abelian group with generators $\vec{x}_1, \ldots, \vec{x}_t$ and the relations

$$p_1 \vec{x}_1 = \cdots = p_t \vec{x}_t =: \vec{c},$$

where $\vec{c}$ is called the canonical element of $\mathbb{L}$. Each element $\vec{x} \in \mathbb{L}$ has the normal form $\vec{x} = \sum_{i=1}^t l_i \vec{x}_i + l\vec{c}$ with $0 \leq l_i \leq p_i - 1$ and $l \in \mathbb{Z}$. Denote by $S$ the commutative algebra

$$S = S(\mathbf{p}, \mathbf{\lambda}) = k[X_1, \ldots, X_t]/I := k[x_1, \ldots, x_t],$$

where $I = (f_3, \ldots, f_t)$ is the ideal generated by $f_i = X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}$ for $3 \leq i \leq t$. Then $S$ is $\mathbb{L}$-graded by setting

$$\text{deg}(x_i) = \vec{x}_i \quad \text{for} \ 1 \leq i \leq t.$$

Finally, the weighted projective line associated with $\mathbf{p}$ and $\mathbf{\lambda}$ is defined to be

$$\mathbb{X} = \mathbb{X}_k = \text{Spec}^h S,$$

the spectrum of $\mathbb{L}$-graded homogeneous prime ideals of $S$.

The category of coherent sheaves on $\mathbb{X}$ is defined to be the quotient category

$$\text{coh} \mathbb{X} = \text{mod}^{\mathbb{L}} S/\text{mod}_0^{\mathbb{L}} S,$$

where $\text{mod}^{\mathbb{L}} S$ is the category of finitely generated $\mathbb{L}$-graded $S$-modules, while $\text{mod}_0^{\mathbb{L}} S$ is the Serre subcategory of finite length $\mathbb{L}$-graded $S$-modules. The grading shift gives the twist $E(\vec{x})$ for every sheaf $E$ and $\vec{x} \in \mathbb{L}$.

Moreover, $\text{coh} \mathbb{X}$ is a hereditary abelian category with Serre duality of the form

$$D \text{Ext}^1(X,Y) \cong \text{Hom}(Y, X(\vec{\omega})), \quad (2.1)$$

where $D = \text{Hom}_k(-, k)$, and $\vec{\omega} := (t - 2)\vec{c} - \sum_{i=1}^t \vec{x}_i \in \mathbb{L}$, called the dualizing element. This implies the existence of almost split sequences in $\text{coh} \mathbb{X}$ with the Auslander–Reiten translation $\tau$ given by the grading shift with $\vec{\omega}$.

It is known that $\text{coh} \mathbb{X}$ admits a splitting torsion pair $(\text{coh}_0 \mathbb{X}, \text{vect} \mathbb{X})$, where $\text{coh}_0 \mathbb{X}$ and $\text{vect} \mathbb{X}$ are full subcategories of torsion sheaves and vector bundles, respectively. The free
module $S$ yields a structure sheaf $O \in \text{vect } \mathcal{X}$, and each object in vect $\mathcal{X}$ has a finite filtration by line bundles, that is, sheaves of the form $O(\vec{x})$. Moreover, for any $\vec{x}, \vec{y} \in \mathbb{L}$ we have
\[
\text{Hom}(O(\vec{x}), O(\vec{y})) \cong S_{\vec{y} - \vec{x}}.
\]
The subcategory $\text{coh}_0 \mathcal{X}$ admits ordinary simple sheaves $S_\lambda$ for each $\lambda \in \mathbb{H}_k := \mathbb{P}_k^1 \setminus \{\lambda_1, \ldots, \lambda_t\}$ and exceptional simple sheaves $S_{i,j}$ for $1 \leq i \leq t$ and $0 \leq j \leq p_i - 1$. For any line bundle $L$, $S_\lambda$ is determined by the exact sequence
\[
0 \to L \xrightarrow{x_2 - \lambda x^2_1} L(\vec{c}) \to S_\lambda \to 0.
\]
If we denote by $S_{i,L}$ the unique exceptional simple sheaf satisfying that $\text{Hom}(L, S_{i,L}) \neq 0$, then $S_{i,L}$ fits into the following exact sequences
\[
0 \to L(-\vec{x}_i) \xrightarrow{x_i} L \to S_{i,L} \to 0.
\]
Moreover, the nonzero extensions between these simple sheaves are given by
\[
\text{Ext}^1(S_\lambda, S_\mu) \cong k(\lambda), \quad \text{Ext}^1(S_{i,j}, S_{i,j'}) \cong k \quad \text{for } j' \equiv j - 1 \pmod{p_i},
\]
where $k(\lambda)$ denotes the finite extension of $k$ with $[k(\lambda) : k]$ the degree of $\lambda$. For each simple sheaf $S$ and $n \geq 1$, there is a unique sheaf $S^{(n)}$ with length $n$ and top $S$, which is uniserial. Indeed, the sheaves $S^{(n)}$ form a complete set of indecomposable objects in $\text{coh}_0 \mathcal{X}$. For convenience, we also use the notation $S_{i,t}$ for $1 \leq i \leq t$ and $t \in \mathbb{Z}$ to denote the simple sheaf $S_{i,j}$ with $j \equiv l \pmod{p_i}$ and $0 \leq j \leq p_i - 1$.

The following result establishes a close relation between the weighted projective lines and the canonical algebras $\Lambda(\mathbf{p}, \lambda)$ introduced by Ringel:

**Proposition 2.1.** ([K Prop. 4.1]) There exists a canonical tilting sheaf $T_{\text{can}} = \bigoplus_{\vec{x} \in \mathcal{X}} O(\vec{x})$ in $\text{coh } \mathcal{X}$ with endomorphism algebra isomorphic to the canonical algebra $\Lambda := \Lambda(\mathbf{p}, \lambda)$. In particular, there is a derived equivalence $D^b(\text{coh } \mathcal{X}) \cong D^b(\text{mod } \Lambda)$.

### 2.2. Exceptional sequence.

In this subsection we recall some basic results on exceptional sequences on a triangulated category $\mathcal{C}$.

**Definition 2.2.** An object $E$ in $\mathcal{C}$ is called exceptional if $\text{Hom}(E, E[n]) \cong \delta_{n,0}k$. An ordered collection of exceptional objects $(E_0, E_1, \cdots, E_n)$ is called an exceptional sequence if $\text{Hom}^*(E_j, E_i) = 0$ for $i < j$; and it is further called an Ext-exceptional sequence if $\text{Hom}^{\leq 0}(E_i, E_j) = 0$ for $i < j$.

Let $(E, F)$ be an exceptional pair. Recall that the left mutation $\mathcal{L}_E(F)$ and right mutation $\mathcal{R}_F(E)$ are defined by the following distinguished triangles (see for example [13]):
\[
\mathcal{L}_E(F) \to \text{Hom}^*(E, F) \otimes E \to F,
\]
\[
E \to \text{DHom}^*(E, F) \otimes F \to \mathcal{R}_F(E);
\]
where $D = \text{Hom}_k(-, k), V[l] \otimes E$ (with $V$ a vector space) denotes an object isomorphic to the direct sum of $\text{dim } V$ copies of the object $E[l]$. 


Lemma 2.3. Let \((E_1, E_2, \cdots, E_n)\) be an Ext-exceptional sequence in \(\mathcal{C}\) with \(\text{Ext}^i(E_i, E_{i+1}) \neq 0\) for some \(1 \leq i < n\), and \(j \in \mathbb{Z}\). Then \((E_1, \cdots, E_{i-1}, \mathcal{R}_{E_{i+1}}(E_i)[-j])\) and \((\mathcal{L}_{E_i}(E_{i+1})[j], E_{i+2}, \cdots, E_n)\) are Ext-exceptional sequences.

Proof. We only prove for the right mutation case, the proof for the left mutation one is similar. Obviously, it suffices to show that for any \(1 \leq i < n\), and \(j \in \mathbb{Z}\), \(\text{Ext}^i(E_i, E_{i+1})\) is an Ext-exceptional pair.

By the definition of right mutation we know that \((E_i, \mathcal{R}_{E_{i+1}}(E_i)[-j])\) is an exceptional pair. Observe that \(\text{Hom}^* (E_i, E_{i+1}) = \text{Hom}(E_i, E_{i+1})\). By applying \(\text{Hom}^0(E_i, -)\) to the triangle

\[
E_i \to \text{DHom}(E_i, E_{i+1}) \otimes E_{i+1} \to \mathcal{R}_{E_{i+1}}(E_i)
\]

we obtain that \(\text{Hom}^0(E_i, \mathcal{R}_{E_{i+1}}(E_i)[-j]) = 0\). Hence, \((E_i, \mathcal{R}_{E_{i+1}}(E_i)[-j])\) is Ext-exceptional.

The following result is well-known for the triangulated category \(D^b(\text{coh}\ X)\):

Lemma 2.4. ([14 Lem. 3.2.4]) For any exceptional objects \(E, F\) in \(D^b(\text{coh}\ X)\), there exists at most one integer \(n\), such that \(\text{Hom}(E, F[n]) \neq 0\).

In this paper we mainly focus on the bounded derived category \(D^b(\text{coh}\ X)\) for the weighted projective line \(X\) of weight type (2). For this special case, we can say more on exceptional sequences:

Proposition 2.5. Assume \(X\) has weight type (2). Up to degree shift, all the exceptional pairs in \(\text{coh}\ X\) are given by

\[
(\mathcal{O}, \mathcal{O}(\overline{c})), (\mathcal{O}, \mathcal{O}(\overline{x})), (\mathcal{O}, \mathcal{O}(\overline{x}_1)), (\mathcal{O}, S_{1,0}), (S_{1,1}, \mathcal{O});
\]

and all the complete exceptional sequences in \(\text{coh}\ X\) have the following forms

\[
(\mathcal{O}, \mathcal{O}(\overline{c}), S_{1,0}), (\mathcal{O}, \mathcal{O}(\overline{x}_1), \mathcal{O}(\overline{c})), (\mathcal{O}, S_{1,0}, \mathcal{O}(\overline{x}_1)), (S_{1,1}, \mathcal{O}, \mathcal{O}(\overline{c})).
\]

Proof. The first assertion follows from the facts \(\text{Hom}(\mathcal{O}(\overline{x}), \mathcal{O}(\overline{y})) \cong S_{\overline{y}-\overline{x}}\) and \(\text{Hom}(\mathcal{O}, S_{1,i}) \cong \delta_{0,k}\) for \(i = 0, 1\) and the Serre duality (2.1). The second one is an immediate consequence. □

Remark 2.6. Let \((E_0, E_1, \cdots, E_n)\) be an exceptional sequence on \(\text{coh}\ X\) for any weighted projective line \(X\). It is known that \((E_0[k_0], E_1[k_1], \cdots, E_n[k_n])\) is again an exceptional sequence for any \(k_i \in \mathbb{Z}\). Moreover, for \(1 \leq i, j \leq n\), there exists at most one integer \(k_{ij}\) satisfying \(\text{Ext}^1(E_i, E_j[k_{ij}]) \neq 0\). In case that \(X\) is of weight type (2), there are no Hom-orthogonal exceptional pairs by Proposition 2.3. Hence, such \(k_{ij}\) always exists. In this case, set \(\widetilde{E}_0 = E_0, \widetilde{E}_1 = E_1[k_0], \widetilde{E}_2 = E_2[k_0 + k_1]\). Then \((\widetilde{E}_0, \widetilde{E}_1, \widetilde{E}_2)\) is a complete exceptional sequence on \(D^b(\text{coh}\ X)\) satisfying \(\widetilde{E}_0 \in \text{coh}\ X\), \(\text{Ext}^1(\widetilde{E}_0, \widetilde{E}_1) \neq 0\) and \(\text{Ext}^1(\widetilde{E}_1, \widetilde{E}_2) \neq 0\). Therefore, each Ext-exceptional sequence in \(D^b(\text{coh}\ X)\) has the form

\[
(\widetilde{E}_0[k_0], \widetilde{E}_1[k_1], \widetilde{E}_2[k_2]) \quad \text{with} \quad k_0 \geq k_1 \geq k_2.
\]

Example 2.7. Assume \(X\) has weight type (2). The exceptional sequences \((\widetilde{E}_0, \widetilde{E}_1, \widetilde{E}_2)\) defined as above can be explicitly listed as follows:

(i) \((E_0, E_1, E_2) = (\mathcal{O}, \mathcal{O}(\overline{x}), \mathcal{O}(\overline{c})) \Rightarrow (\widetilde{E}_0, \widetilde{E}_1, \widetilde{E}_2) = (\mathcal{O}, \mathcal{O}(\overline{x}_1)[-1], \mathcal{O}(\overline{c})[-2]);\)
Finally, we introduce the notion of t-exceptional triples and prove their existence.

In this section, we first recall the definition and some basic results of (finest) t-stability for a triangulated category $\mathcal{C}$ in the sense of Gorodentsev–Kuleshov–Rudakov, and describe a sufficient and necessary condition for a t-stability to be finest. Furthermore, for the bounded derived category of coherent sheaves on the weighted projective line of weight type (2), we describe the semistable subcategories, as well as the final HN triangles for certain coherent sheaves. Finally, we introduce the notion of t-exceptional triples and prove their existence.

3. Finest t-stability

3.1. t-stability

**Definition 3.1.** ([9 Def. 3.1]) Let $\Phi$ be a linearly ordered set and $\Pi_\varphi \subset \mathcal{C}$ be a strictly full extension-closed non-empty subcategory for each $\varphi \in \Phi$. The pair $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ is called a t-stability if

(i) the grading shift functor $X \mapsto X[1]$ acts on $\Phi$ as a non-decreasing automorphism, that is, there is a bijection $\tau_\Phi \in \text{Aut } \Phi$ such that $\Pi_{\tau_\Phi(\varphi)} = \Pi_{\varphi}[1]$ and $\tau_\Phi(\varphi) \geq \varphi$ for all $\varphi$;

(ii) $\text{Hom}^{\leq 0}(\Pi_\psi, \Pi_\varphi) = 0$ for all $\psi > \varphi$ in $\Phi$;

(iii) for each non-zero object $X \in \mathcal{C}$, there exists a sequence of triangles

\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n = X,
\]

where $A_j \in \Pi_{\varphi_j}$ with $\varphi_i > \varphi_j$, $\forall 1 \leq i < j \leq n$. \hspace{1cm} (3.1)

It has been shown in [9] that the decomposition (3.1) for each $X$ is unique up to isomorphism, which is known as the Harder-Narasimhan filtration (HN filtration for short) of $X$. Define $\varphi^-(X) := \varphi_n$ and $\varphi^+(X) := \varphi_1$. Then $X \in \Pi_\varphi$ if and only if $\varphi^-(X) = \varphi^+(X) = \varphi := \varphi(X)$. The categories $\Pi_\varphi$ are called the semistable subcategories of the t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$. Note that each $\Pi_\varphi$ is closed under extensions and direct summands, but, in general, not abelian. The nonzero objects in $\Pi_\varphi$ are said to be semistable of phase $\varphi$, while the minimal objects are said to be stable. For any interval $I \subseteq \Phi$, $\Pi_I$ is defined to be the extension-closed subcategory of $\mathcal{C}$ generated by the subcategories $\Pi_{\varphi}$, $\varphi \in I$. 
Proposition 3.2. ([9 Cor. 5.2]) Let \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) be a t-stability on \(C\). Then each \(\varphi \in \Phi\) determines two t-structures \(A_\varphi, B_\varphi\) such that
\[
A^\geq_\varphi = (\Pi_\psi \mid \psi \leq \tau_\Phi(\varphi)), \quad A^\leq_\varphi = (\Pi_\psi \mid \psi > \varphi);
B^\geq_\varphi = (\Pi_\psi \mid \psi < \tau_\Phi(\varphi)), \quad B^\leq_\varphi = (\Pi_\psi \mid \psi \geq \varphi).
\]
Moreover, the corresponding hearts are given by \(\Pi_{(\varphi, \tau_\Phi(\varphi))}\) and \(\Pi_{[\varphi, \tau_\Phi(\varphi)]}\), respectively.

Lemma 3.3. Let \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) be a t-stability on \(C\). Assume there exists some \(\varphi_0 \in \Phi\) such that all the objects in the triangle \(A \to X \to B\) are nonzero in \(\Pi_{[\varphi_0, \tau_\Phi(\varphi_0)]}\), then
\[
\varphi^+(A) \leq \varphi^+(X) \quad \text{and} \quad \varphi^-(X) \leq \varphi^-(B).
\]

3.2. Finest t-stability. We recall the definition of a partial order for t-stabilities given in [9].

Definition 3.4. Let \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}), (\Psi, \{P_\psi\}_{\psi \in \Psi})\) be t-stabilities on \(C\) and let the grading shift functor act on \(\Phi, \Psi\) by automorphisms \(\tau_\Phi, \tau_\Psi\) respectively. We say that the t-stability \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) is finer than \((\Psi, \{P_\psi\}_{\psi \in \Psi})\) if there exists a surjective map \(r : \Phi \to \Psi\) such that
\[
(i) \ r \tau_\Phi = \tau_\Psi r;
(ii) \ \varphi' > \varphi'' \ implies \ r(\varphi') \geq r(\varphi'');
(iii) \ for \ any \ \psi \in \Psi, \ P_\psi = (\Pi_\varphi \mid \varphi \in r^{-1}(\psi)).
\]

Minimal elements with respect to this partial order will be called the finest t-stabilities. In this subsection we will give an equivalent description of finest t-stabilities for a triangulated category \(C\) which is piecewise hereditary. That is, we assume that there exists a hereditary abelian category \(H\) such that there is a triangulated equivalence \(C \cong D^b(H)\) in the rest of this subsection.

Lemma 3.5. Let \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) be a finest t-stability on \(C\). Then \(\tau_\Phi\) is strictly increasing.

Proof. Suppose there exists \(\psi \in \Phi\) such that \(\tau_\Phi(\psi) = \psi\). Define \(\Pi_{\psi_i} = \Pi_\psi \cap (H[i])\) for all \(i \in \mathbb{Z}\). Then \(\Pi_{\psi} = \bigcup_{i \in \mathbb{Z}} \Pi_{\psi_i}\). Set
\[
\Psi := (\Phi \setminus \{\psi\}) \cup \{\psi_i \mid i \in \mathbb{Z}\}.
\]
Define an automorphism \(\tau_\Psi\) on \(\Psi\) by setting
\[
\tau_\Psi(\psi_i) = \psi_{i+1}, \forall \ i \in \mathbb{Z}; \ \tau_\Psi(\varphi) = \tau_\Phi(\varphi), \ \forall \varphi \in \Phi \setminus \{\psi\}.
\]
We define a linear order on \(\Psi\) by keeping the order relations in \(\Phi \setminus \{\psi\}\) and adding new order relations
\[
\varphi_1 < \psi_i < \psi_{i+1} < \varphi_2 \quad (\forall \ i \in \mathbb{Z})
\]
whenever \(\varphi_1 < \psi < \varphi_2\) with \(\varphi_1, \varphi_2 \in \Phi \setminus \{\psi\}\). Then \((\Psi, \{\Pi_\varphi\}_{\varphi \in \Psi})\) is a t-stability which is strictly finer than \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\), a contradiction. \(\square\)

Lemma 3.6. Let \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) be a t-stability on \(C\). If there exist some \(\psi \in \Phi\) and \(X, Y \in \Pi_\psi\) such that \(\text{Hom}\ (X, Y) = 0\), then \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) is not finest.
Proof. By Lemma 3.5 we can assume that $\tau_\Phi$ is strictly increasing. Denote by $\mathcal{A} := \Pi_{(\tau_\Phi^{-1}(\psi),\psi]}$. By Proposition 3.2 $\mathcal{A}$ is abelian. Let

$$\mathcal{F} = \{Z \in \mathcal{A} \mid \text{Hom}(X,Z) = 0\}; \quad \mathcal{T} = \{W \in \mathcal{A} \mid \text{Hom}(W,Z) = 0, \forall Z \in \mathcal{F}\}.$$ 

Then $(\mathcal{T}, \mathcal{F})$ forms a torsion pair in $\mathcal{A}$. By definition, $Y \in \mathcal{F}$ and $X \in \mathcal{T}$. Moreover, we have $\Pi_{(\tau_\Phi^{-1}(\psi),\psi]} \subseteq \mathcal{F}$, which ensures that $\mathcal{T} \subseteq \Pi_\psi$. Set

$$\Psi := (\Phi \setminus \{\tau_\Phi^n(\psi) \mid n \in \mathbb{Z}\}) \bigcup \{\psi_{1,n}, \psi_{2,n} \mid n \in \mathbb{Z}\}.$$ 

Define an automorphism $\tau_\Psi$ on $\Psi$ by setting

$$\tau_\Psi(\psi_{i,n}) = \psi_{i,n+1}, i = 1, 2; \quad \tau_\Psi(\varphi) = \tau_\Phi(\varphi), \forall \varphi \in \Phi \setminus \{\tau_\Phi^n(\psi) \mid n \in \mathbb{Z}\}.$$ 

We define a linear order on $\Psi$ by keeping the order relations in $\Phi \setminus \{\tau_\Phi^n(\psi) \mid n \in \mathbb{Z}\}$ and adding new order relations

$$\psi_{1,n} < \psi_{2,n} < \psi_{1,n+1} \quad (\forall n \in \mathbb{Z})$$

and

$$\tau_\Psi^n(\varphi_1) < \psi_{1,n} < \psi_{2,n} < \tau_\Psi^n(\varphi_2)$$

whenever $\varphi_1 < \psi < \varphi_2$ with $\varphi_1, \varphi_2 \in \Phi \setminus \{\tau_\Phi^n(\psi) \mid n \in \mathbb{Z}\}$.

Define $\Pi_{\psi_{1,0}} = \Pi_\psi \cap \mathcal{F}$, $\Pi_{\psi_{2,0}} = \Pi_\psi \cap \mathcal{T}$, and set $\Pi_{\psi_{i,n}} = \Pi_{\psi_{i,0}}[n]$ for $n \in \mathbb{Z}$, $i = 1, 2$. We claim that $\Pi_{\psi_{1,0}}, \Pi_{\psi_{2,0}}$ satisfy the following conditions:

(i) $\text{Hom}(\Pi_{\psi_{2,0}}, \Pi_{\psi_{1,0}}) = 0$;

(ii) For each element $Z \in \Pi_\psi$, there exists a unique exact sequence $0 \to Z_2 \to Z \to Z_1 \to 0$, where $Z_i \in \Pi_{\psi_{i,0}}$ for $i = 1, 2$.

In fact, since $(\mathcal{T}, \mathcal{F})$ is a torsion pair, the statement (i) follows immediately from the fact $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$. For the second statement, note that $Z$ has a unique decomposition $0 \to Z_2 \to Z \to Z_1 \to 0$ with $Z_2 \in \mathcal{T}$ and $Z_1 \in \mathcal{F}$. Thus, it suffices to show that $Z_1 \in \Pi_\psi$. This follows from the fact that $\text{Hom}(Z, \Pi_\varphi) = 0$ for any $\varphi \in (\tau_\Phi^{-1}(\psi),\psi)$. Thus, $(\Psi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ is a t-stability on $C$, which is strictly finer than $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$. We are done.

The following result gives an equivalent description of the finest t-stabilities.

**Theorem 3.7.** A t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ on $C$ is finest if and only if for any $\varphi \in \Phi$ and $X, Y \in \Pi_\varphi$, $\text{Hom}(X, Y) \neq 0 \neq \text{Hom}(Y, X)$.

**Proof.** The “if” part follows from [9, Prop. 5.5], while the “only if” part follows from Lemma 3.6. \hfill \Box

3.3. Semistable subcategories. From now onwards, let $X$ be the weighted projective line over $k$ of weight type (2) and $\text{coh} X$ be the category of coherent sheaves on $X$. Let $\mathcal{D} = D^b(\text{coh} X)$ be the bounded derived category of $\text{coh} X$. Similar to the case of the projective line $\mathbb{P}^1$, each t-stability in $\mathcal{D}$ can be refined to a finest one. In the following, we always fix a finest t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ on $\mathcal{D}$.

The following result is an immediate consequence of Theorem 3.7.
Corollary 3.8. If $S_{1,0}, S_{1,1}$ are semistable, then $\varphi(S_{1,0}) \neq \varphi(S_{1,1})$.

Proof. This follows from Theorem 3.7 and $\text{Hom}(S_{1,0}, S_{1,1}) = 0 = \text{Hom}(S_{1,1}, S_{1,0})$.  

Corollary 3.9. At most one of $S^{(2)}_{1,0}$ or $S^{(2)}_{1,1}$ is semistable.

Proof. Suppose that both of $S^{(2)}_{1,0}, S^{(2)}_{1,1}$ are semistable. Since $\text{Hom}(S^{(2)}_{1,0}, S^{(2)}_{1,1}) \neq 0$ $\neq \text{Hom}(S^{(2)}_{1,1}, S^{(2)}_{1,0})$, we obtain that $\varphi(S^{(2)}_{1,0}) = \varphi(S^{(2)}_{1,1})$. Now consider the Auslander–Reiten sequence

$$0 \to S^{(2)}_{1,i} \to S^{(2)}_{1,i+1} \oplus S_{1,i} \to S^{(2)}_{1,i+1} \to 0$$

for $i = 0, 1$. Thus, $S_{1,0}, S_{1,1}$ are semistable and $\varphi(S_{1,0}) = \varphi(S^{(2)}_{1,0}) = \varphi(S_{1,1})$, contradicting to Corollary 3.8.  

Lemma 3.10. Let $S$ be the torsion sheaf $S_{\lambda}$ for some ordinary point $\lambda \in \mathbb{H}_k$ or $S^{(2)}_{1,i}$ for $i = 0, 1$. Then $S \in \Pi_\varphi$ if and only if $S(n) \in \Pi_\varphi$ for any $n$.

Proof. Assume that $S \in \Pi_\varphi$, then by $S(n) \in \langle S \rangle$ we know that $S(n) \in \Pi_\varphi$. Now assume $S(n) \in \Pi_\varphi$ for some $n$. Consider the exact sequence $0 \to S^{(n)} \to S \oplus S^{(2n-1)} \to S^{(n)} \to 0$. Since $\Pi_\varphi$ is closed under extensions and direct summands, we conclude that $S \in \Pi_\varphi$.  

Lemma 3.11. For any $n \in \mathbb{N}$ and $i = 0, 1$, $S^{(2n+1)}_{1,i}$ is not semistable.

Proof. Suppose there exist $n \in \mathbb{N}$, $i = 0$ or $1$, such that $S^{(2n+1)}_{1,i} \in \Pi_\psi$ for some $\psi \in \Phi$. From the exact sequence $0 \to S^{(2n+1)}_{1,i} \to S^{(2n-1)}_{1,i} \oplus S^{(2n+3)}_{1,i} \to S^{(2n+1)}_{1,i} \to 0$ and by induction, we get $S^{(2n-1)}_{1,i+1} \in \Pi_\psi$ for any $m \in \mathbb{N}$. It follows that $S^{(2)}_{1,i}$ and $S^{(2)}_{1,i+1}$ are both semistable. Note that there are two sequences of non-zero morphism $S^{(3)}_{1,i+1} \to S^{(2)}_{1,i+1} \to S^{(2)}_{1,i+1}$ and $S^{(2)}_{1,i+1} \to S^{(2)}_{1,i} \to S^{(3)}_{1,i+1}$. Hence $\varphi(S_{1,i}) = \varphi(S^{(2)}_{1,i}) = \psi$. Now from the exact sequence $0 \to S^{(2)}_{1,i+1} \to S_{1,i+1} \oplus S^{(3)}_{1,i} \to S_{1,i} \to 0$ we obtain that $\varphi(S_{1,i+1}) = \psi = \varphi(S_{1,i})$, a contradiction to Lemma 3.8.  

The following is a characterization for semistable subcategories for $\mathcal{D}$.

Theorem 3.12. Each semistable subcategory of $\mathcal{D}$ has the form $\langle E[j] \rangle$, where $j \in \mathbb{Z}$ and $E$ is a coherent sheaf satisfying that $\text{End}(E)$ is a division algebra.

Proof. Combining Lemma 3.10 with Lemma 3.11 the possible simple objects (up to shift) for each semistable category are line bundles or simple sheaves or $S^{(2)}_{1,i}$ for $i = 0, 1$. We observe that these are all the sheaves in $\text{coh} \mathcal{X}$ whose endomorphism algebras are division algebras. Note that for any two such sheaves $X, Y$ and any integers $m, n$, we have $\text{Hom}(X[m], Y[n]) = 0$ or $\text{Hom}(Y[n], X[m]) = 0$. Then by Theorem 3.7 $X[m]$ and $Y[n]$ have different phases. Hence, each semistable subcategory contains a unique simple object. We are done.  

3.4. Final HN triangles. For any object $X \in D^b(\text{coh} \mathcal{X})$, we use $\Delta_X$ to denote the final HN triangle $E_{n-1} \to X \to A_n$ in the HN filtration 3.11 and call $A_n$ the final HN factor of $X$. In this subsection, we will investigate the possible forms of $\Delta_X$ for indecomposable $X$. If $X$ itself is semistable, then $\Delta_X$ has trivial form. For this reason, we always assume that $X$ is not semistable and $\Delta_X$ has non-trivial form in the following.


**Lemma 3.13.** Let \( Y \to X \to Z \) be the final HN triangle of \( X \). Then there does not exist a semistable object \( W \) satisfying \( \text{Hom}(Y, W) \neq 0 \) and \( \varphi(W) < \varphi(Z) \).

**Proof.** Suppose there exists such a semistable object \( W \). Then \( \varphi^-(Y) \leq \varphi(W) < \varphi(Z) \), which is a contradiction. \( \square \)

**Lemma 3.14.** Assume \( Z_1, Z_2 \) are semistable with \( \varphi(Z_1) < \varphi(Z_2) \). If \( \text{Hom}(X, Z_i) \neq 0 \) for \( i = 1, 2 \), then \( Z_2 \) is not the HN factor of \( X \).

**Proof.** Suppose that \( \Delta_X \) has the form \( Y \to X \to Z_2^k \) for some \( k \in \mathbb{N} \). It follows that \( \varphi^-(Y) > \varphi(Z_2) > \varphi(Z_1) \). Hence, \( \text{Hom}(Y, Z_1) = 0 = \text{Hom}(Z_2, Z_1) \), which implies that \( \text{Hom}(X, Z_1) = 0 \), a contradiction. \( \square \)

The following result is very useful in a triangulated category, but we could not find a proof in the literature. We include the proof suggested by Nan Gao.

**Lemma 3.15.** Let \( f : X \to Y \) be a morphism in a hereditary abelian category \( \mathcal{H} \). Then \( f \) fits into the following triangle in the bounded derived category \( D^b(\mathcal{H}) \):

\[
X \xrightarrow{f} Y \to \text{coker}(f) \oplus \text{ker}(f)[1].
\]

**Proof.** Consider the following commutative diagram in \( \mathcal{H} \):

\[
\begin{array}{ccc}
0 & \to & \text{ker}(f) \\
& \downarrow \pi & \downarrow \iota \\
0 & \to & X & \xrightarrow{f} & Y & \to & \text{coker}(f) \\
& & \downarrow \text{Im}(f) & & & & \\
& & X & \xrightarrow{f} & Y & \to & \text{coker}(f) \oplus \text{ker}(f)[1]
\end{array}
\]

By the Octahedral Axiom, we obtain the following commutative diagram of triangles:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \text{Im}(f) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
& \text{cone}(f) & \\
\end{array}
\quad
\begin{array}{ccc}
\text{coker}(f) & \xrightarrow{\iota} & \text{coker}(f) \\
\downarrow & & \downarrow \\
& \text{coker}(f) & \xrightarrow{\iota} \text{coker}(f)[1].
\end{array}
\]

Since \( \mathcal{H} \) is hereditary, we obtain that \( \text{Hom}(\text{coker}(f), \text{ker}(f)[2]) = 0 \). It follows that \( \text{cone}(f) \cong \text{coker}(f) \oplus \text{ker}(f)[1] \). We are done. \( \square \)

**Lemma 3.16.** Let \( X \) be an indecomposable coherent sheaf with \( \Delta_X \) of the form \( Y \to X \xrightarrow{f} Z \). Then \( Z \notin \langle S_\lambda \rangle \) for any \( \lambda \in \mathbb{H}_k \).

**Proof.** Suppose that \( Z \in \langle S_\lambda \rangle \) for some \( \lambda \in \mathbb{H}_k \). Consider the exact sequence in \( \text{coh} X \):

\[
0 \to \text{ker}(f) \to X \xrightarrow{f} Z \to \text{coker}(f) \to 0.
\]

It follows from Lemma 3.15 that \( Y = \text{ker}(f) \oplus \text{coker}(f)[-1] \). If \( X \) is a line bundle, then \( \text{ker}(f) \) is a line bundle, hence \( \text{Hom}(\text{ker}(f), Z) \neq 0 \), a contradiction. If \( X \) is a torsion sheaf,
then $X \in \langle S \rangle$, it follows that $X$ is semistable, again a contradiction. This finishes the proof. \hfill \Box

**Lemma 3.17.** Let $X$ be an indecomposable object in $\mathcal{D}$. Then the final HN factor of $X$ has the form $E^n[i]$, where $j \in \mathbb{Z}, n \in \mathbb{N}$, $E$ is a line bundle or $E \in \{S_{1,i}, S_{1,i}^{(2m)} | i = 0, 1; m \in \mathbb{N}\}$. Moreover, if $E \in \langle S_{1,i}^{(2)} \rangle$ for $i = 0$ or 1, then $n = 1$.

**Proof.** Without loss of generality, we assume that $X \in \text{coh}X$ and $\Delta_X$ has the form $Y \rightarrow X \overset{f}{\rightarrow} Z$. By Theorem 3.12 and Lemma 3.16, the final HN factor $Z$ of $X$ lies in $\langle E[j] \rangle$, where $j = 0$ or 1, $E$ is a line bundle or $E \in \{S_{1,i}, S_{1,i}^{(2)} | i = 0, 1\}$. If $E$ is a line bundle or $S_{1,i}, i = 0, 1$, then we are done. Otherwise, $E = S_{1,i}^{(2)}$ for $i = 0$ or 1. Similar to the proof of Lemma 3.16, we know that $X \in \langle S_{1,0}, S_{1,1} \rangle$. It follows that $\text{Hom}(X, S_{1,i}^{(2)}) \neq 0$ if and only if $\text{Hom}(X, S_{1,i}^{(2)}[1]) \neq 0$. By Lemma 3.14, $S_{1,i}^{(2)}[1]$ is not the final HN factor of $X$. Hence $Z \in \langle S_{1,i}^{(2)} \rangle$. We claim that $Z$ is indecomposable. Indeed, consider the following exact sequence in $\text{coh}X$:

$$0 \rightarrow \ker(f) \rightarrow X \overset{f}{\rightarrow} Z \rightarrow \text{coker}(f) \rightarrow 0.$$ 

Then by Lemma 3.15 we have $Y \cong \ker(f) \oplus \text{coker}(f)[-1]$. It follows that $\ker(f), \text{coker}(f) \in \langle S_{1,i} \rangle$ by $\text{Hom}(Y, Z) = 0$. Moreover, $X$ is indecomposable implies $\ker(f) \in \{0, S_{1,i}\}$. Hence $\text{Im}(f)$ is also indecomposable, which ensures $\text{coker}(f) \in \{0, S_{1,i}\}$. Therefore, $Z$ is indecomposable, i.e., $n = 1$. \hfill \Box

**Theorem 3.18.** For any $X \in \mathcal{D}$ with $\Delta_X : Y \overset{g}{\rightarrow} X \overset{f}{\rightarrow} Z$, where $Z \in \langle E[j] \rangle$ for some $j \in \mathbb{Z}$,

(i) if $E = S_{1,i}^{(2)}$ for $i = 0$ or 1, then $f$ has the form $X \rightarrow S_{1,i}^{(2m)}[j]$ for some $m \geq 1$.

(ii) if $E$ is exceptional, then $f$ has the form $X \overset{\text{ev}}{\rightarrow} \text{Hom}(X, E[j]) \otimes E[j]$.

**Proof.** By Theorem 3.12, the final HN factor has the form $Z = E[j]^n$, where $j \in \mathbb{Z}, n \in \mathbb{N}$ and $E$ is a coherent sheaf satisfying that $\text{End}(E)$ is a division algebra. If $E = S_{1,i}^{(2)}$ for $i = 0$ or 1, then $Z$ is indecomposable by Lemma 3.17 we are done. If else, $E$ is exceptional. Assume that $f = (f_1, f_2, \cdots, f_n)^t$ with $f_i \in \text{Hom}(X, E[j])$. We claim that $n = \dim \text{Hom}(X, E[j])$ and $f_1, f_2, \cdots, f_n$ form a basis of $\text{Hom}(X, E[j])$. In fact, if $n < \dim \text{Hom}(X, E[j])$, then there exists a map $f_0 \in \text{Hom}(X, E[j])$ which is not a linear combination of $\{f_1, f_2, \cdots, f_n\}$. Now consider the following diagram:

$$
\begin{array}{ccc}
Y & \overset{g}{\rightarrow} & X^{(f_1, f_2, \cdots, f_n)^t} \\
\downarrow & & \downarrow f_0 \\
E[j] & \rightarrow & E[j]
\end{array}
$$

Since $E$ is exceptional, we have $\text{Hom}(E[j], E[j]) \cong k$. Thus $f_0$ can not factor through $(f_1, f_2, \cdots, f_n)^t$. It follows that $f_0 \circ g \neq 0$, hence $\text{Hom}(Y, E[j]) \neq 0$. This contradicts the definition of HN-filtration. Now suppose $n > \dim \text{Hom}(X, E[j])$, or $n = \dim \text{Hom}(X, E[j])$ with $\{f_1, f_2, \cdots, f_n\}$ linearly dependent, we obtain that the triangle $Y \overset{g}{\rightarrow} X^{(f_1, f_2, \cdots, f_n)^t}$
$E[j]^n \to Y[1]$ is split and $E[j]$ is a direct summand of $Y[1]$. It follows that $\text{Hom} (Y, E[j - 1]) \neq 0$, a contradiction. We are done. \hfill \Box

In the following we describe the explicit forms of the final HN triangles for special coherent sheaves.

**Proposition 3.19.** Let $L$ be a line bundle. Then $\Delta_L$ has one of the following forms:

(i) $L((k + 1)c)[k][-1] \to L \to L(kc)^{k+1}$, $k > 0$;
(ii) $L(-kc)^{k+1} \to L \to L(-(k + 1)c)[1]$, $k > 0$;
(iii) $L((k + 1)c + \bar{x}_1)[k][-1] \oplus S_{1,L(x_1)}[1] \to L \to L(kc + \bar{x}_1)^{k+1}$, $k \geq 0$;
(iv) $L(-kc) \oplus L(-kc + \bar{x}_1)^{k} \to L \to L(-kc - \bar{x}_1)^{k}[1]$, $k > 0$;
(v) $L(-\bar{x}_1) \to L \to S_{1,L}$.

**Proof.** According to Lemma 3.17 and using a similar proof for Lemma 3.16 we know that the possible final HN factors for $L$ are $S_{1,L}$, $L(\bar{x})$ or $L(-c - \bar{x})[1]$ for some $\bar{x} > 0$. Now it remains to show that the (co)cone of the evaluation maps have the desired forms. The first two triangles follow from the embedding $S^1_{1,L} = \text{coh} \mathbb{P}^1 \hookrightarrow \text{coh} X$ and [15, Lemma 3.1], and the last triangle is trivial. We only show (iii) and (iv). By applying $\text{Hom} (-, L(kc + \bar{x}_1))$ to the exact sequence $0 \to L \to L(\bar{x}_1) \to S_{1,L(x_1)} \to 0$, we get an isomorphism $\text{Hom} (L, L(kc + \bar{x}_1)) \cong \text{Hom} (L(\bar{x}_1), L(kc+\bar{x}_1))$. Then the statement (iii) follows from the following pullback diagram:

$$
\begin{array}{c}
L((k + 1)c + \bar{x}_1)[k][-1] \oplus S_{1,L(x_1)}[1] \to L \xrightarrow{ev} L(kc + \bar{x}_1)^{k+1} \\
\downarrow \downarrow \downarrow \\
L((k + 1)c + \bar{x}_1)[k][-1] \to L(\bar{x}_1) \xrightarrow{ev} L(kc + \bar{x}_1)^{k+1} \\
\downarrow \downarrow \\
S_{1,L(x_1)} \cong S_{1,L(\bar{x}_1)}.
\end{array}
$$

Similarly, the statement (iv) follows from the following pullback diagram:

$$
\begin{array}{c}
L(-kc) \oplus L(-kc + \bar{x}_1)^{k} \to L \xrightarrow{ev} L(-kc - \bar{x}_1)^{k}[1] \\
\downarrow \downarrow \downarrow \\
L(-kc + \bar{x}_1)^{k+1} \to L(\bar{x}_1) \xrightarrow{ev} L(-kc - \bar{x}_1)^{k}[1] \\
\downarrow \downarrow \\
S_{1,L(x_1)} \cong S_{1,L(\bar{x}_1)}.
\end{array}
$$

\hfill \Box

Let $L$ be a line bundle and let $Y \to X \to L[j]^n$ be the final HN triangle of $X$ for some $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $L(\bar{c})[j - 1]$ is a direct summand of $Y$, then we say that $\Delta_X$ is of type $(L, L(\bar{c}))$. It follows from the above proposition that each line bundle $L$ is of type $(L(\bar{x}), L(\bar{x} + \bar{c}))$ for some $\bar{x}$, or $\Delta_L$ has the form

$$L(-\bar{x}_1) \to L \to S_{1,L} \text{ or } S_{1,L(x_1)}[-1] \to L \to L(\bar{x}_1).$$
Proposition 3.20. The final HN triangle of $S_{1,i}, i = 0, 1$ has one of the following forms:

(i) $\mathcal{O}(k\vec{c} + i\vec{x}_1) \to S_{1,i} \to \mathcal{O}(k\vec{c} + (i - 1)\vec{x}_1)[1]$ for $k \in \mathbb{Z};$

(ii) $S_{1,i}^{(2)} \to S_{1,i} \to S_{1,i+1}[1];$

(iii) $S_{1,i+1}[-1] \to S_{1,i} \to S_{1,i+1}^{(2)}.$

Proof. By Lemma 3.17, all the possibilities of the final HN factor of $S_{1,i}$ are given by $S_{1,i+1}[1], S_{1,i+1}^{(2n)}$ for $n \geq 1,$ and $\mathcal{O}(k\vec{c} + (i - 1)\vec{x}_1)[1]$ for $k \in \mathbb{Z}.$ If the final HN factor is $\mathcal{O}(k\vec{c} + (i - 1)\vec{x}_1)[1]$ or $S_{1,i+1}[1],$ then it is easy to see that $\Delta_{S_{1,i}}$ has the form in (i),(ii) respectively. Now assume the final HN factor is $S_{1,i+1}^{(2n)},$ then $\Delta_{S_{1,i}}$ is given by $S_{1,i+1}^{(2n-1)}[-1] \to S_{1,i} \to S_{1,i+1}^{(2n)}.$ By $\text{Hom} (S_{1,i+1}^{(2n-1)}[-1], S_{1,i+1}^{(2n)}) = 0$ we obtain $n = 1,$ hence (iii) holds. □

Proposition 3.21. The final HN triangle of $S_{1,i}^{(2)}, i = 0, 1$ has one of the following forms:

(i) $S_{1,i+1} \to S_{1,i}^{(2)} \to S_{1,i};$

(ii) $S_{1,i+1} \oplus S_{1,i+1}[-1] \to S_{1,i}^{(2)} \to S_{1,i+1}^{(2)};$

(iii) $\mathcal{O}(k\vec{c} + \vec{c} + i\vec{x}_1) \to S_{1,i}^{(2)} \to \mathcal{O}(k\vec{c} + i\vec{x}_1)[1]$ for $k \in \mathbb{Z};$

(iv) $\mathcal{O}(k\vec{c} + i\vec{x}_1) \oplus S_{1,i+1} \to S_{1,i}^{(2)} \to \mathcal{O}(k\vec{c} + (i - 1)\vec{x}_1)[1]$ for $k \in \mathbb{Z}.$

Proof. The proof is similar as in the above corollary. Here we only treat the triangles (ii) and (iv), the other two are trivial. Note that $\dim \text{Hom} (S_{1,i}^{(2)}, S_{1,i+1}^{(2)}) = 1,$ and both of kernel and cokernel for any non-zero morphism in $\text{Hom} (S_{1,i}^{(2)}, S_{1,i+1}^{(2)})$ are given by $S_{1,i+1},$ then (ii) follows. The triangle (iv) follows from the following pullback diagram:

\[
\begin{array}{ccc}
\mathcal{O}((k - 1)\vec{c} + i\vec{x}_1) & \longrightarrow & \mathcal{O}(k\vec{c} + (i - 1)\vec{x}_1) \\
\mathcal{O}((k - 1)\vec{c} + i\vec{x}_1) & \longrightarrow & \mathcal{O}(k\vec{c} + i\vec{x}_1) \\
\downarrow & & \downarrow \\
S_{1,i}^{(2)} & \longrightarrow & S_{1,i+1}^{(2)} \\
\downarrow & & \downarrow \\
S_{1,i} & \longrightarrow & S_{1,i}.
\end{array}
\]

□

3.5. The distance $d(\Phi).$

Lemma 3.22. Let $m$ be the number of semistable line bundles (up to shift). Then $m \geq 2.$

Proof. Suppose that $m \leq 1,$ i.e. there is at most one semistable line bundle, say $L$ if it exists. It follows that $L(\vec{c})$ and $L(\vec{x}_1)$ are not semistable. By Proposition 3.19 we obtain the HN filtration of $L(\vec{c})$ as follows:

\[
\begin{array}{ccc}
L & \longrightarrow & L(\vec{x}_1) \\
\downarrow & & \downarrow \\
S_{1,L(\vec{x}_1)} & \longrightarrow & S_{1,L}.
\end{array}
\]

Then $\text{Hom} (L, S_{1,L}) \neq 0$ yields a contradiction. Hence $m \geq 2.$ □
Define the *distance* of the t-stability \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) by
\[
d(\Phi) = \min\{|\vec{y} - \vec{x}| : O(\vec{x}), O(\vec{y})\text{ are semistable}\}.
\]

It is well-defined since we have at least two different line bundles by the above lemma and \(L\) is a total order. Furthermore, we have the following result.

**Proposition 3.23.** \(d(\Phi) \leq c\).

*Proof.* Suppose that \(d(\Phi) = \vec{y} > c\). Let \(L\) be a semistable line bundle. Then \(L(\pm \vec{x}_1)\) and \(L(\pm c)\) are not semistable.

We claim that \(\Delta_{L(\vec{x}_1)}\) has the form \(L \rightarrow L(\vec{x}_1) \rightarrow S_{1,L(\vec{x}_1)}\). In fact, by Proposition 3.19 it suffices to show that \(\Delta_{L(\vec{x}_1)}\) is not of type \((L(\vec{z}), L(\vec{z} + c))\) for any \(\vec{z}\). Otherwise, we have \(\vec{z} \leq -c - \vec{x}_1\) or \(\vec{z} \geq c + \vec{x}_1\). We can always take \(W = L\) to deduce a contradiction to Lemma 3.13. This finishes the proof of the claim.

Similarly, one can show that \(\Delta_{L(c)}\) is not of type \((L(\vec{z}), L(\vec{z} + c))\) for any \(\vec{z}\). It follows that \(\Delta_{L(c)}\) is
\[
L(\vec{x}_1) \rightarrow L(c) \rightarrow S_{1,L} \text{ or } S_{1,L(\vec{x}_1)}[-1] \rightarrow L(c) \rightarrow L(c + \vec{x}_1).
\]
For the first case, the HN filtration of \(L(c)\) has the form \([3,2]\), which yields a contradiction. For the second case, since \(\text{Hom}(S_{1,L(\vec{x}_1)}[-1], L) \neq 0\) and \(\text{Hom}(L(c + \vec{x}_1)) \neq 0\), we take \(W = L\) to get a contradiction to Lemma 3.13. This finishes the proof.

Now we can say more on the final HN triangles for exceptional sheaves.

**Proposition 3.24.** Assume \(d(\Phi) = \vec{x}_1\) with \(L, L(\vec{x}_1)\) semistable for some line bundle \(L\). Then
\begin{enumerate}[(i)]  
  
  \item \(\Delta_{S_{1,L}}\) has the form \(L(0) \rightarrow S_{1,L} \rightarrow L(0 - \vec{x}_1)[1]\), where \(k = 0\) or \(1\);  
  
  \item \(\Delta_{S_{1,L(\vec{x}_1)}}\) has the form \(L(\vec{x}_1) \rightarrow S_{1,L(\vec{x}_1)} \rightarrow L[1]\);  
  
  \item for any line bundle \(L'\), if \(\Delta_{L'}\) is of type \((L(\vec{x}), L(\vec{x} + \vec{c}))\), then \(\vec{x} = 0\) or \(-\vec{x}_1\).
\end{enumerate}

*Proof.* (i) By Proposition 3.20, the final HN triangle of \(S_{1,L}\) has three possibilities. Firstly, we assume \(\Delta_{S_{1,L}}\) has the form \(L(k) \rightarrow S_{1,L} \rightarrow L(k - \vec{x}_1)[1]\) for some \(k \in \mathbb{Z}\). We use Lemma 3.13 to deduce contradictions for \(k \neq 0, 1\). More precisely, if \(k < 0\), then \(\text{Hom}(L(k), L) \neq 0\) and \(\text{Hom}(L, L(k - \vec{x}_1)[1]) \neq 0\), it follows that \(\varphi(L) < \varphi(L(k - \vec{x}_1)[1])\). By Lemma 3.13 we can take \(W = L(\vec{x}_1)[1]\) to get a contradiction. Similarly, if \(k > 1\), then we take \(W = L(\vec{x}_1)[1]\) to get a contradiction. Moreover, we can take \(W = L(\vec{x}_1)[1]\) if \(\Delta_{S_{1,L}}\) has the form \(S_{1,L}^{(2)} \rightarrow S_{1,L} \rightarrow S_{1,L(\vec{x}_1)}[1]\), and take \(W = L\) if \(\Delta_{S_{1,L}}\) has the form \(S_{1,L(\vec{x}_1)}[-1] \rightarrow S_{1,L} \rightarrow S_{1,L(\vec{x}_1)}^{(2)}\) to get contradictions. We are done.

The proofs for (ii) and (iii) are similar as for (1) by using Lemma 3.13. We omit the details.

**Proposition 3.25.** Assume \(d(\Phi) = c\) with \(L, L(c)\) semistable for some line bundle \(L\). Then
\begin{enumerate}[(i)]  
  
  \item \(\Delta_{S_{1,L}}\) has the form \(L(c) \rightarrow S_{1,L} \rightarrow L(\vec{x}_1)[1]\) or \(S_{1,L}^{(2)} \rightarrow S_{1,L} \rightarrow S_{1,L(\vec{x}_1)}[1]\);  
  
  \item \(\Delta_{S_{1,L(\vec{x}_1)}}\) has the form \(L(\vec{x}_1) \rightarrow S_{1,L(\vec{x}_1)} \rightarrow L[1]\) or \(S_{1,L}[-1] \rightarrow S_{1,L(\vec{x}_1)} \rightarrow S_{1,L}^{(2)}\);  
  
  \item for any line bundle \(L'\), if \(\Delta_{L'}\) is of type \((L(\vec{x}), L(\vec{x} + c))\), then \(\vec{x} = 0\).
\end{enumerate}
Proof. Assume $\Delta_{S_{1,L}}$ has the form $L(k\vec{c}) \to S_{1,L} \to L(k\vec{c} - \vec{x}_1)[1]$ for some $k \in \mathbb{Z}$. If $k \leq 0$, then $\text{Hom}(L(k\vec{c}), L(\vec{c})) \neq 0$ and $\text{Hom}(L(\vec{c}), L(k\vec{c} - \vec{x}_1)[1]) \neq 0$, by taking $W = L(\vec{c})$ we get a contradiction to Lemma 3.13. If $k > 1$, then we take $W = L[1]$ to get a contradiction. Hence $k = 1$ and then $\Delta_{S_{1,L}}$ has the form $L(\vec{c}) \to S_{1,L} \to L(\vec{x}_1)[1]$. Now assume $\Delta_{S_{1,L}}$ has the form $S_{1,L}(\vec{x}_1)[-1] \to S_{1,L} \to S_{S_{1,L}}^{(2)}(\vec{x}_1)$, then we can take $W = L$ to get a contradiction. This finishes the proof of (i) by Proposition 3.20. Similarly, we can prove (ii) and (iii). □

### 3.6. t-exceptional sequences.

In this subsection we will introduce the notion of a t-exceptional sequence in a triangulated category $\mathcal{C}$, and show the existence for $\mathcal{D} = D^b(\text{coh } \mathbb{X})$, where $\mathbb{X}$ is the weighted projective line of weight type (2).

**Definition 3.26.** Let $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ be a t-stability on a triangulated category $\mathcal{C}$. An exceptional sequence $E = (E_0, E_1, \cdots, E_n)$ is called a t-exceptional sequence if:

(i) $E_i$ is semistable, $i = 0, \cdots, n$;

(ii) $\text{Hom}(E_i, E_j) = 0$ for $0 \leq i < j \leq n$;

(iii) there exists $\varphi_0 \in \Phi$ such that $\varphi(E_i) \in (\varphi_0, \tau_\Phi(\varphi_0))$ for $0 \leq i \leq n$.

**Definition 3.27.** A t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ on $\mathcal{C}$ is said to be effective if for any two semistable objects $X, Y$, there exists $i \in \mathbb{Z}$ such that $\varphi(X[i]) \leq \varphi(Y) < \varphi(X[i + 1])$.

Now we give the main result of this section.

**Theorem 3.28.** Let $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ be an effective finest t-stability on $\mathcal{D} = D^b(\text{coh } \mathbb{X})$. Then there exists a t-exceptional triple.

Proof. By Proposition 3.23, we have $d(\Phi) \leq \vec{c}$. We consider the following two cases.

1. $d(\Phi) = \vec{c}$. Without loss of generality, we assume that $\mathcal{O}, \mathcal{O}(\vec{c})$ are semistable. Then $\mathcal{O}(\vec{x}_1)$ is not semistable. By Proposition 3.25 $\Delta_{\mathcal{O}(\vec{x}_1)}$ has the following two possible forms

   (i) $\mathcal{O} \to \mathcal{O}(\vec{x}_1) \to S_{1,1}$;

   (ii) $S_{1,0}[-1] \to \mathcal{O}(\vec{x}_1) \to \mathcal{O}(\vec{c})$.

   For Case (i), we obtain that $S_{1,1}$ is semistable and $\varphi(\mathcal{O}(\vec{c})) > \varphi(\mathcal{O}) > \varphi(S_{1,1})$. By the effectivity of t-stability, there exist unique $m$ and $n$ such that $\varphi(\mathcal{O}(\vec{c})[m])$ and $\varphi(S_{1,1}[n])$ belong to the interval $(\varphi(\mathcal{O}), \varphi(\mathcal{O}[1]))$. Clearly, $m \leq 0 < n$. Then by Remark 2.6 and Example 2.7 we conclude that $(S_{1,1}[n], \mathcal{O}[1], \mathcal{O}(\vec{c})[m])$ is a t-exceptional triple.

   For Case (ii), we claim that $S_{1,0}$ is semistable and then $\varphi(S_{1,0}[-1]) > \varphi(\mathcal{O}(\vec{c}))$. Indeed, if $S_{1,0}$ is not semistable, then by Proposition 3.25 $\Delta_{S_{1,0}}$ has the form $S_{1,0}^{(2)} \to S_{1,0} \to S_{1,1}[1]$. This implies that the last two triangles of the HN-filtration of $\mathcal{O}(\vec{x}_1)$ have the form

$$
\begin{array}{ccc}
S_{1,0}^{(2)}[-1] & \to & S_{1,0}[-1] \\
\downarrow & & \downarrow \\
S_{1,1} & & \mathcal{O}(\vec{c}) \\
\end{array}
$$

Then $\text{Hom}(S_{1,0}^{(2)}[-1], \mathcal{O}(\vec{c})) \neq 0$ yields a contradiction. By similar arguments as for Case (i), there exist unique $m$ and $n$ with $m, n \leq 0$ such that $\varphi(\mathcal{O}(\vec{c})[m])$ and $\varphi(S_{1,0}[n])$ belong to the interval $(\varphi(\mathcal{O}), \varphi(\mathcal{O}[1]))$. Then $\varphi(\mathcal{O}(\vec{c})[m]) < \varphi(S_{1,0}[n])$ or $\varphi(S_{1,0}[n]) < \varphi(\mathcal{O}(\vec{c})[m]) < \varphi(\mathcal{O}(\vec{c})[m]) < \varphi(S_{1,0}[n])$. □
\( \varphi(S_{1,0}[n + 1]) \). Recall that \( \varphi(S_{1,0}[-1]) > \varphi(\mathcal{O}(\bar{c})) \). It follows that \( n + 1 \leq m \leq 0 \) or \( n + 2 \leq m \leq 0 \) respectively. In both cases, \((\mathcal{O}[1], \mathcal{O}(\bar{c})[m], S_{1,0}[n])\) is a t-exceptional triple by Remark 2.6 and Example 2.7.

(2) \( d(\Phi) = \bar{x}_1 \). Without loss of generality, we assume that \( \mathcal{O}, \mathcal{O}(\bar{x}_1) \) are semistable. If \( S_{1,0} \) is semistable, then there exist unique \( m \) and \( n \) with \( m, n \leq 0 \) such that \( \varphi(\mathcal{O}(\bar{x}_1)[m]), \varphi(S_{1,0}[n]) \in (\varphi(\mathcal{O}), \varphi(\mathcal{O}[1])) \). Since \( \text{Hom}(S_{1,0}, \mathcal{O}(\bar{x}_1)[1]) \neq 0 \), we have \( \varphi(S_{1,0}) \leq \varphi(\mathcal{O}(\bar{x}_1)[1]) \). If \( \varphi(\mathcal{O}(\bar{x}_1)[m]) \leq \varphi(S_{1,0}[n]) \), then \( m \leq n \); and if \( \varphi(\mathcal{O}(\bar{x}_1)[m - 1]) < \varphi(S_{1,0}[n]) < \varphi(\mathcal{O}(\bar{x}_1)[m]) \), then \( m \leq n + 1 \). Therefore,

\[
\varphi(\mathcal{O}) < \varphi(\mathcal{O}(\bar{x}_1)[m]) < \varphi(S_{1,0}[n]) < \varphi(\mathcal{O}[1]), m \leq n \leq 0,
\]

or

\[
\varphi(\mathcal{O}) < \varphi(S_{1,0}[n]) < \varphi(\mathcal{O}(\bar{x}_1)[m]) < \varphi(\mathcal{O}[1]), m \leq n + 1 \leq 0.
\]

We conclude that \((\mathcal{O}, S_{1,0}[n - 1], \mathcal{O}(\bar{x}_1)[m - 1])\) or \((\mathcal{O}, S_{1,0}[n], \mathcal{O}(\bar{x}_1)[m - 1])\) is a t-exceptional triple respectively.

Now assume \( S_{1,0} \) is not semistable. Then \( \Delta_{S_{1,0}} \) has the following two possible forms

(i) \( \mathcal{O}(\bar{c}) \rightarrow S_{1,0} \rightarrow \mathcal{O}(\bar{x}_1)[1] \);

(ii) \( \mathcal{O} \rightarrow S_{1,0} \rightarrow \mathcal{O}(-\bar{x}_1)[1] \).

For Case (i), \( \mathcal{O}(\bar{c}) \) must be semistable with \( \varphi(\mathcal{O}(\bar{c})) > \varphi(\mathcal{O}(\bar{x}_1)[1]). \) Otherwise, \( \Delta_{\mathcal{O}(\bar{c})} \) is given by \( \mathcal{O}(\bar{x}_1) \oplus \mathcal{O} \rightarrow \mathcal{O}(\bar{c}) \rightarrow \mathcal{O}(-\bar{x}_1)[1]. \) Hence, the final two triangles of the HN-filtration of \( S_{1,0} \) have the form

\[
\begin{array}{ccc}
\mathcal{O}(\bar{x}_1) \oplus \mathcal{O} & \longrightarrow & \mathcal{O}(\bar{c}) \\
& & \longrightarrow \\
& & S_{1,0}
\end{array}
\]

Then \( \text{Hom}^{-1}(\mathcal{O}(\bar{x}_1), \mathcal{O}(\bar{x}_1)[1]) \neq 0 \) yields a contradiction. Therefore, there exist \( m, n \) such that

\[
\varphi(\mathcal{O}) < \varphi(\mathcal{O}(\bar{c})[n]) < \varphi(\mathcal{O}(\bar{x}_1)[m]) < \varphi(\mathcal{O}[1]), n + 2 \leq m \leq 0,
\]

or

\[
\varphi(\mathcal{O}) < \varphi(\mathcal{O}(\bar{x}_1)[m]) < \varphi(\mathcal{O}(\bar{c})[n]) < \varphi(\mathcal{O}[1]), n + 1 \leq m \leq 0.
\]

In both cases, \((\mathcal{O}[1], \mathcal{O}(\bar{x}_1)[m], \mathcal{O}(\bar{c})[n])\) is a t-exceptional triple.

For Case (ii), \( \mathcal{O}(-\bar{x}_1) \) is semistable with \( \varphi(\mathcal{O}) > \varphi(\mathcal{O}(-\bar{x}_1)[1]). \) Hence, there exist \( m, n \) such that

\[
(i) \varphi(\mathcal{O}(-\bar{x}_1)) < \varphi(\mathcal{O}[m]) < \varphi(\mathcal{O}(\bar{x}_1)[n]) < \varphi(\mathcal{O}(-\bar{x}_1)[1]), n \leq m \leq -1,
\]

or

\[
(ii) \varphi(\mathcal{O}(-\bar{x}_1)) < \varphi(\mathcal{O}(\bar{x}_1)[n]) < \varphi(\mathcal{O}[m]) < \varphi(\mathcal{O}(-\bar{x}_1)[1]), n + 1 \leq m \leq -1.
\]

In both cases, \((\mathcal{O}(-\bar{x}_1), \mathcal{O}[m], \mathcal{O}(\bar{x}_1)[n - \delta_{m,n}])\) is a t-exceptional triple.

\[ \square \]
4. Relations with Bridgeland’s stability conditions

In this section we use t-stabilities to investigate the stability conditions in the sense of Bridgeland. By Proposition 2.1 there is a derived equivalence between the category coh\(X\) of coherent sheaves over the weighted projective line \(X\) of weight type (2) and the module category \(\text{mod} kQ\) for the acyclic triangular quiver \(Q\). The aim of this section is to apply the results in the previous section to derive the existence of \(\sigma\)-exceptional triple for each stability condition \(\sigma\) on \(D^b(\text{mod} kQ)\), which is the main result in \([5]\) and implies the connectedness of the space of stability conditions on \(D^b(\text{mod} kQ)\).

We first recall the definition of stability conditions on a triangulated category \(\mathcal{C}\) introduced by Bridgeland \([2]\).

**Definition 4.1.** A stability condition \(\sigma = (Z, \mathcal{P})\) on \(\mathcal{C}\) consists of a group homomorphism \(Z: K_0(\mathcal{C}) \to \mathbb{C}\), called the central charge, and full additive subcategories \(\mathcal{P}(\phi)\) of \(\mathcal{C}\) for each \(\phi \in \mathbb{R}\), satisfying the following axioms:

- (i) if \(E \in \mathcal{P}(\phi)\), then \(Z(E) = m(E)\exp(i\pi\phi)\) for some \(m(E) \in \mathbb{R}_{>0}\);
- (ii) for all \(\phi \in \mathbb{R}\), \(\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]\);
- (iii) if \(\phi_1 > \phi_2\) and \(A_j \in \mathcal{P}(\phi_j)\), then \(\text{Hom}(A_1, A_2) = 0\);
- (iv) for each \(0 \neq E \in \mathcal{C}\), there are a finite sequence of real numbers \(\phi_1 > \phi_2 > \cdots > \phi_n\) and a sequence of triangles

\[
0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{n-1} \to E_n = E
\]

\[
\begin{array}{ccccccc}
A_1 & \to & & & & \to & A_n \\
\downarrow & & & & & & \downarrow \\
E_0 & \to & E_1 & \to & E_2 & \to & \cdots & \to & E_{n-1} & \to & E_n = E
\end{array}
\]

with \(A_j \in \mathcal{P}(\phi_j)\) for \(1 \leq j \leq n\).

The decompositions in axiom (iv) are uniquely defined up to isomorphism. Given a nonzero object \(0 \neq E \in \mathcal{C}\), define real numbers \(\phi^-(E) := \phi_1\) and \(\phi^+(E) := \phi_n\). Then \(E \in \mathcal{P}(\phi)\) if and only if \(\phi^-(E) = \phi^+(E) = \phi =: \phi(E)\). Each subcategory \(\mathcal{P}(\phi)\) is extension-closed and abelian. Its non-zero objects are said to be semistable of phase \(\phi\) with respect to \(\sigma\), and its minimal objects are called stable.

For each interval \(I \subseteq \mathbb{R}\), define \(\mathcal{P}(I) := \langle \mathcal{P}(\phi) | \phi \in I \rangle\) to be the extension-closed subcategory of \(\mathcal{C}\) generated by \(\mathcal{P}(\phi)\) for \(\phi \in I\). It has been shown by Bridgeland that for \(\phi \in \mathbb{R}\), both \(\mathcal{P}(\phi, \phi + 1]\) and \(\mathcal{P}[\phi, \phi + 1]\) are hearts of bounded t-structures on \(\mathcal{C}\).

In the following we always fix a stability condition \(\sigma = (Z, \mathcal{P})\) on \(\mathcal{C}\). Now we recall the definition of \(\sigma\)-exceptional sequence, which was first introduced in \([5]\).

**Definition 4.2.** An exceptional sequence \(\mathcal{E} = (E_0, E_1, \cdots, E_n)\) is called a \(\sigma\)-exceptional sequence if

- (i) \(E_i\) is semistable, \(i = 0, \cdots, n\);
- (ii) \(\text{Hom}^{\leq 0}(E_i, E_j) = 0, \forall 0 \leq i < j \leq n\);
- (iii) there exists \(\phi \in \mathbb{R}\) such that \(\phi(E_i) \in (\phi, \phi + 1]\) for each \(i\).

**Remark 4.3.** (1) The condition (iii) is equivalent to the following condition
- there exists $\phi \in \mathbb{R}$ such that $\phi(E_i) \in [\phi, \phi + 1)$ for each $i$.

(2) If the condition (iii) is replaced by the condition
- there exists $\phi \in \mathbb{R}$ such that $\phi(E_i) \in [\phi, \phi + 1]$ for each $i$,
then $\mathcal{E}$ is called a weakly $\sigma$-exceptional sequence.

For the later use, we give the following two general results.

**Lemma 4.4.** Let $X \xrightarrow{f} Y \rightarrow Z$ be a triangle in $\mathcal{C}$ with $X, Y \in \mathcal{P}(\phi)$ and $Z$ semistable. Then $\phi(Z) = \phi$ or $\phi + 1$.

**Proof.** Consider the following commutative diagram in $\mathcal{P}(\phi)$:

\[\begin{array}{ccccccccc}
0 & \xrightarrow{} & \text{ker}(f) & \xrightarrow{f} & X & \xrightarrow{\pi} & Y & \xrightarrow{\im(f)} & 0 \\
& & \downarrow{\pi} & \swarrow{i} & \downarrow{\im(f)} & & \downarrow{u} & & \downarrow{v} \\
& & \text{ker}(f) & & Y & & Z & & \text{coker}(f)
\end{array}\]

By the Octahedral Axiom, we obtain the following commutative diagram of triangles:

\[\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi} & \downarrow{i} & \downarrow{u} \\
\text{ker}(f)[1] & \xrightarrow{\im(f)} & \text{coker}(f)
\end{array}\]

Note that each term above is semistable and $\phi(\ker(f)) = \phi(\text{coker}(f)) = \phi$. Recall that $\text{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ for $\phi_1 > \phi_2$. It follows that $u = 0$ or $v = 0$, that is, $Z \cong \text{coker}(f)[1]$. Therefore, $\phi(Z) = \phi$ or $\phi + 1$. \qed

**Lemma 4.5.** Let $(E_1, E_2)$ be an exceptional pair with $\dim \text{Hom}(E_1, E_2[i]) = 1$ or 2 for some $i \in \mathbb{Z}$. If $\phi(E_1) = \phi(E_2[i]) = \phi$, then

(i) $\mathcal{R}_{E_2}(E_1)$ is semistable with phase $\phi$ or $\phi + 1$;

(ii) $\mathcal{L}_{E_1}(E_2)$ is semistable with phase $\phi - i$ or $\phi - i - 1$.

**Proof.** By assumption, the triangulated subcategory $\text{Tr}(E_1, E_2)$ is equivalent to the bounded derived category $D^b(\text{mod}P_l)$, where $l = \dim \text{Hom}(E_1, E_2[i])$ and $P_l$ is the $l$-Kronecker algebra, see for example [13, Section 3.3]. In the following we show that $\mathcal{R}_{E_2}(E_1)$ is semistable. Then by Lemma 4.4 it has phase $\phi$ or $\phi + 1$, so we finish the proof of (i). The proof for the second statement is similar.

Indeed, if $l = 1$, then $\text{Tr}(E_1, E_2)$ is triangulated equivalent to the bounded derived category of type $A_2$. Hence we have the Auslander–Reiten triangle $E_1 \rightarrow E_2[1] \rightarrow \mathcal{R}_{E_2}(E_1) \rightarrow E_1[1]$. If $\mathcal{R}_{E_2}(E_1)$ is not semistable, then it has the following HN-filtration $E_2[1] \rightarrow \mathcal{R}_{E_2}(E_1) \rightarrow E_1[1]$. But $\phi(E_2[1]) < \phi(E_1[1])$, a contradiction. Hence $\mathcal{R}_{E_2}(E_1)$ is semistable. If $l = 2$, then there is a triangulated equivalence $\text{Tr}(E_1, E_2) \cong \text{coh}(\mathbb{P}^1)$. If we view the equivalence as identity, we can assume $(E_1, E_2[i]) = (\mathcal{O}, \mathcal{O}(1))$ without loss of generality. Then by [15]...
Lemma 4.6. Assume \( L, L(\vec{c}) \in \mathcal{P}(\phi) \) for a line bundle \( L \). Then one of the following holds:

(i) there exists \( n > 0 \) such that \( \langle L(n\vec{c}), L(n\vec{c} + \vec{c})[-1] \rangle \subseteq \mathcal{P}(\phi) \);

(ii) there exists \( m > 0 \) such that \( \langle L(-m\vec{c} + \vec{c}), L(-m\vec{c})[1] \rangle \subseteq \mathcal{P}(\phi) \).

Consequently, \( \mathcal{P}(\phi) \) contains a subcategory of the form \( \text{mod}kP_2 \), where \( P_2 \) is the Kronecker quiver.

Proof. For any line bundle \( L' \), we have the following canonical triangle in \( \mathcal{D} \):

\[
\xi(L') : L'(-\vec{c}) \xrightarrow{(x_2,-x_1)^f} L' \oplus L' \xrightarrow{(x_1,x_2)} L'(-\vec{c})[1]
\]

By the assumption, \( L, L(\vec{c}) \in \mathcal{P}(\phi) \). We claim that one of the following statements holds:

- for any \( m > 0 \), \( L(-m\vec{c}) \in \mathcal{P}(\phi) \);

- there exists \( m > 0 \), such that \( L(-m\vec{c} + \vec{c}) \in \mathcal{P}(\phi) \) and \( L(-m\vec{c})[1] \in \mathcal{P}(\phi) \).

In fact, from the canonical triangle \( \xi(L) \) and by Lemma 4.5 we get \( L(-\vec{c}) \) is semistable with phase \( \phi \) or \( \phi - 1 \). If \( \phi(L(-\vec{c})) = \phi - 1 \), then the second statement holds by taking \( m = 1 \). If \( \phi(L(-\vec{c})) = \phi \), by considering the canonical triangle \( \xi(L(-\vec{c})) \) we get \( L(-2\vec{c}) \) is semistable with phase \( \phi \) or \( \phi - 1 \). Repeating the procedure gives the proof of the claim.

Dually, one of the following statements holds:

- for any \( n > 0 \), \( L(n\vec{c}) \in \mathcal{P}(\phi) \);

- there exists \( n > 0 \), such that \( L(n\vec{c}) \in \mathcal{P}(\phi) \) and \( L(n\vec{c} + \vec{c})[-1] \in \mathcal{P}(\phi) \).

Therefore, we conclude that one of the following holds:

(i) for any \( m \in \mathbb{Z} \), \( L(m\vec{c}) \in \mathcal{P}(\phi) \); in this case, we get a contradiction to the fact that \( \mathcal{P}(\phi) \) is of finite length;

(ii) there exists \( m > 0 \), such that \( L(-m\vec{c} + \vec{c}) \in \mathcal{P}(\phi) \) and \( L(-m\vec{c})[1] \in \mathcal{P}(\phi) \); in this case, \( \langle L(-m\vec{c} + \vec{c}), L(-m\vec{c})[1] \rangle \subseteq \mathcal{P}(\phi) \);

(iii) there exists \( n > 0 \), such that \( L(n\vec{c}) \in \mathcal{P}(\phi) \) and \( L(n\vec{c} + \vec{c})[-1] \in \mathcal{P}(\phi) \); in this case, \( \langle L(n\vec{c}), L(n\vec{c} + \vec{c})[-1] \rangle \subseteq \mathcal{P}(\phi) \).

\( \square \)

In the following we show the existence of a \( \sigma \)-exceptional triple in \( \mathcal{D} \) for certain special cases.

Lemma 4.7. Assume \( L, L(\vec{c}) \in \mathcal{P}(\phi) \) for a line bundle \( L \). If \( S_{1,L}[-i] \) or \( \tau S_{1,L}[i-1] \in \mathcal{P}(\phi) \) for some \( i > 0 \), then there exists a \( \sigma \)-exceptional triple.

Proof. Without loss of generality, we assume \( L = \mathcal{O} \) and prove the result under the assumption that \( S_{1,\mathcal{O}}[-i] \in \mathcal{P}(\phi) \). The proof under the assumption that \( \tau S_{1,\mathcal{O}}[-i-1] \in \mathcal{P}(\phi) \) can be treated similarly. By Lemma 4.6 there exists some \( n > 0 \), such that \( \langle \mathcal{O}(-n\vec{c})[1], \mathcal{O}(-n\vec{c} + \vec{c}) \rangle \subseteq \mathcal{P}(\phi) \).
\(\vec{c}\) or \((\mathcal{O}(n\vec{c}), \mathcal{O}(n\vec{c} + \vec{c})[-1])\) is an exceptional pair in \(\mathcal{P}(\phi)\). For the first case, we get \((\mathcal{O}(-n\vec{c})[1], \mathcal{O}(-n\vec{c} + \vec{c}), S_{1,0}[-i])\) is a \(\sigma\)-exceptional triple. For the second case, consider the exceptional triple \(\mathcal{E} := (\mathcal{O}(n\vec{c}), \mathcal{O}(n\vec{c} + \vec{c})[1], S_{1,0}[-i])\) in \(\mathcal{P}(\phi)\). If \(i > 1\), then \(\mathcal{E}\) is a \(\sigma\)-exceptional triple. If \(i = 1\), then from the triangle
\[
\mathcal{O}(n\vec{c} + \vec{x})[-1] \rightarrow \mathcal{O}(n\vec{c} + \vec{c})[1] \rightarrow S_{1,0}[-1]
\]
and by Lemma 4.5 we know that \(\phi(\mathcal{O}(n\vec{c} + \vec{x})) = \phi + 1\) or \(\phi\).

- If \(\phi(\mathcal{O}(n\vec{c} + \vec{x})) = \phi + 1\), then \((\mathcal{O}(n\vec{c}), S_{1,0}[-1], \mathcal{O}(n\vec{c} + \vec{x})[-1])\) is a \(\sigma\)-exceptional triple.
- If \(\phi(\mathcal{O}(n\vec{c} + \vec{x})) = \phi\), then from the triangle
\[
\mathcal{O}(n\vec{c}) \rightarrow \mathcal{O}(n\vec{c} + \vec{x}) \rightarrow S_{1,1}
\]
we know that \(\phi(S_{1,1}) = \phi + 1\) or \(\phi\). It follows that \((\mathcal{O}(n\vec{c} + \vec{x}), S_{1,1}[-1], \mathcal{O}(n\vec{c} + \vec{c})[-1])\) or \((S_{1,1}, \mathcal{O}(n\vec{c}), \mathcal{O}(n\vec{c} + \vec{c})[-1])\) is a \(\sigma\)-exceptional triple.

\[\square\]

An exceptional triple \((E_0, E_1, E_2)\) is said to be of \(\sigma\)-type \((\phi_1, \phi_2, \phi_3)\) if \(E_i\) is \(\sigma\)-semistable and \(\phi(E_i) = \phi_i\) for \(1 \leq i \leq 3\).

**Lemma 4.8.** Let \((E_0, E_1, E_2)\) be an Ext-exceptional triple in \(\mathcal{D}\) of \(\sigma\)-type \((\phi + 1, \phi, \phi)\) or \((\phi + 1, \phi + 1, \phi)\). Then there is a \(\sigma\)-exceptional triple in \(\mathcal{D}\).

**Proof.** We only prove the existence of \(\sigma\)-exceptional triple if \((E_0, E_1, E_2)\) has \(\sigma\)-type \((\phi + 1, \phi, \phi)\). The proof for the other one is similar. By Proposition 2.3 \(\dim \text{Ext}^1(E_0, E_1) \leq 2\). If \(\text{Ext}^1(E_0, E_1) = 0\), then by Lemma 2.8 we have \(\text{Ext}^1(E_0, E_2) = 0\), thus \((E_0[-1], E_1, E_2)\) is a \(\sigma\)-exceptional triple. If \(\dim \text{Ext}^1(E_0, E_1) = 2\), then we can assume \((E_0, E_1) = (\mathcal{O}[1], \mathcal{O}(\vec{c}))\) without loss of generality. It follows that \(E_2 = S_{1,0}[-i]\) for some \(i > 0\), then we are done by Lemma 4.7. Hence we assume \(\dim \text{Ext}^1(E_0, E_1) = 1\) in the following. Denote by \(W_1 = \mathcal{L}_{E_0}(E_1)\), then there is a triangle \(W_1 \rightarrow E_0[-1] \rightarrow E_1 \rightarrow W_1[1]\). It follows that \(\phi(W_1) = \phi - 1\) or \(\phi\).

Assume \(\phi(W_1) = \phi - 1\), then \(\mathcal{E}_1 := (W_1[1], E_0[-1], E_2)\) is an exceptional triple in \(\mathcal{P}(\phi)\). By Remark 2.9 and Example 2.7 we know that \(\dim \text{Hom}(E_0[-1], E_2) \leq 1\). If \(\dim \text{Hom}(E_0[-1], E_2) = 0\), then \(\mathcal{E}_1\) is a \(\sigma\)-exceptional triple. If \(\dim \text{Hom}(E_0[-1], E_2) = 1\), then \(W_2 := \mathcal{L}_{E_0}(E_2)\) is semistable which fits into the triangle \(W_2 \rightarrow E_0[-1] \rightarrow E_2 \rightarrow W_2[1]\). Hence \(\phi(W_2) = \phi\) or \(\phi - 1\). It follows that \((W_1[1], W_2[1], E_0[-1])\) or \((W_1[1], E_2, W_2)\) is a \(\sigma\)-exceptional triple, since \(\text{Hom}(W_1[1], W_2[1]) \cong \text{Hom}(E_1, W_2[1]) = 0\) and \(\text{Hom}(W_1[1], E_2) = 0\).

Now assume \(\phi(W_1) = \phi\), then \(\mathcal{E}_2 := (E_1, W_1, E_2)\) is an exceptional triple in \(\mathcal{P}(\phi)\) with \(\dim \text{Hom}(W_1, E_2) \leq 2\). If \(\dim \text{Hom}(W_1, E_2) = 0\), then \(\mathcal{E}_2\) is a \(\sigma\)-exceptional triple. If \(\dim \text{Hom}(W_1, E_2) = 1\), then \(W_3 := \mathcal{L}_{W_1}(E_2)\) is semistable which fits into the triangle \(W_3 \rightarrow W_1 \rightarrow E_2 \rightarrow W_3[1]\). Hence \(\phi(W_3) = \phi\) or \(\phi - 1\). It follows that \((E_1, E_2, W_3)\) or \((E_1, W_3[1], W_1)\) is a \(\sigma\)-exceptional triple. If \(\dim \text{Hom}(W_1, E_2) = 2\), then we can assume \((W_1, E_2) = (\mathcal{O}, \mathcal{O}(\vec{c}))\) without loss of generality. It follows that \(E_1 = S_{1,1}[i - 1]\) for some \(i > 0\), then we are done by Lemma 4.7. This finishes the proof.  
\[\square\]
Proposition 4.9. If $\mathcal{D}$ admits a weakly $\sigma$-exceptional triple, then there is a $\sigma$-exceptional triple in $\mathcal{D}$.

Proof. Let $\mathcal{E} := (E_0, E_1, E_2)$ be a weakly $\sigma$-exceptional triple in $\mathcal{D}$ with $\phi(E_i) \in [\phi, \phi + 1]$ for some $\phi$. If $\phi(E_0) = \phi$ or $\phi(E_2) = \phi + 1$, then $(E_0[1], E_1, E_2)$ or $(E_0, E_1, E_2[-1])$ is a weakly $\sigma$-exceptional triple respectively. Therefore, it suffices to show the existence of a $\sigma$-exceptional triple under the assumptions $\phi(E_0) > \phi$ and $\phi(E_2) < \phi + 1$. Obviously, if $\phi(E_i) \in [\phi, \phi + 1)$ or $(\phi, \phi + 1]$ for each $0 \leq i \leq 2$, then by definition, $\mathcal{E}$ is a $\sigma$-exceptional triple, we are done. Now assume that there exists a permutation $s \in S_3$ such that $\phi(E_{s(0)}) = \phi$ and $\phi(E_{s(1)}) = \phi + 1$. Now if $\phi(E_{s(2)}) = \phi$ or $\phi + 1$, then by Lemma 4.8 we are done. Hence, we can further assume that $\phi(E_{s(2)}) := \psi \in (\phi, \phi + 1)$. Therefore, $\mathcal{E}$ has one of the following $\sigma$-type:

(i) $(\phi + 1, \phi, \psi)$; (ii) $(\psi, \phi + 1, \phi)$; (iii) $(\phi + 1, \psi, \phi)$.

For the first case, we get that $\mathcal{E}' := (E_0, E_1[1], E_2) \subseteq \mathcal{P}(\phi, \phi + 1)$. If $\text{Ext}^1(E_0, E_1) = 0$, then by definition $\mathcal{E}'$ is a $\sigma$-exceptional triple. If $\text{Ext}^1(E_0, E_1) \neq 0$, then $\dim \text{Ext}^1(E_0, E_1) = 1$ or 2 by Proposition 2.5.

Case 1: $\dim \text{Ext}^1(E_0, E_1) = 1$; then by Lemma 4.5 $\mathcal{R}_{E_1}(E_0)$ is semistable with phase $\phi + 1$ or $\phi + 2$. It follows that $(\mathcal{R}_{E_1}(E_0), E_0, E_2)$ or $(E_1[1], \mathcal{R}_{E_1}(E_0)[-1], E_2)$ is a $\sigma$-exceptional triple.

Case 2: $\dim \text{Ext}^1(E_0, E_1) = 2$; then we can assume $(E_0, E_1) = (\mathcal{O}[1], \mathcal{O}(\mathfrak{c}))$ without loss of generality. It follows that $E_2 = S_{1,0}[1-i]$ for some $i > 0$ and $\mathcal{O}, \mathcal{O}(\mathfrak{c}) \in \mathcal{P}(\phi)$. Then from Lemma 4.6, there exists $n > 0$ such that $\mathcal{O}(n\mathfrak{c} - \mathfrak{c}), \mathcal{O}(n\mathfrak{c})[-1] \subseteq \mathcal{P}(\phi)$, or there exists $m > 0$ such that $\mathcal{O}(-m\mathfrak{c} + \mathfrak{c}), \mathcal{O}(-m\mathfrak{c})[1] \subseteq \mathcal{P}(\phi)$. It follows that $\mathcal{O}(n\mathfrak{c} - \mathfrak{c}), \mathcal{O}(n\mathfrak{c})[-1], E_2)$ or $(\mathcal{O}(-m\mathfrak{c} + \mathfrak{c}), \mathcal{O}(-m\mathfrak{c})[1], E_2)$ is a $\sigma$-exceptional triple.

For the second case, $\mathcal{E}' := (E_0, E_1[-1], E_2) \subseteq \mathcal{P}[\phi, \phi + 1]$. The proof is dual. For the third case, if $\text{Hom}(E_1, E_2[1]) = 0$, then by Lemma 2.5 we get $\text{Hom}(E_0, E_2[1]) = 0$. It follows that $\mathcal{E}'$ is a $\sigma$-exceptional triple. Now assume that $\text{Hom}(E_1, E_2[1]) \neq 0$. Note that $(E_1, E_2) \subseteq \mathcal{P}[\phi, \phi + 1]$. We obtain from [5] Prop.3.15 that $\sigma$ induces a stability condition $\sigma' = (Z', \mathcal{P}')$ on $\text{Tr}(E_1, E_2)$ satisfying $\mathcal{P}'(t) = \mathcal{P}(t) \cap \text{Tr}(E_1, E_2)$ for any $t \in \mathbb{R}$. Then by [13] Prop.3.17, the right mutation $\mathcal{R}_{E_2}(E_1)$ is semistable with phase $\psi'$ in the interval $(\phi + 1, \phi + 2)$. Hence we get an exceptional triple $(E_0, E_2[1], \mathcal{R}_{E_2}(E_1)[-1])$, which has $\sigma$-type $(\phi + 1, \phi + 1, \psi' - 1)$, then by an argument as for the first case we can finish the proof. □

Now we can show that Theorem 3.28 implies the main theorem in [5] which is the key to prove that the space of stability conditions on $\mathcal{D}$ is connected.

Theorem 4.10. ([5] Thm. 10.1) Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$. Then there exists a $\sigma$-exceptional triple.

Proof. By the definition, $(\mathbb{R}, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}})$ is a $t$-stability on $\mathcal{D}$, which can be refined to a finest one, say, $(\Phi, \{\Pi_{\phi}\}_{\phi \in \Phi})$. Then there exists a surjective map $r : \Phi \rightarrow \mathbb{R}$ such that for each $\phi \in \Phi$, $r(\tau_{\phi}(\phi)) = r_{\mathbb{R}}(r(\phi)) = r(\phi) + 1$. Hence, for any $\phi, \phi' \in \Phi$, there exists $i \in \mathbb{Z}$ such that

$$r(\tau^{i}_{\phi}(\phi)) = r(\phi) + i \leq r(\phi') < r(\phi) + i + 1 = r(\tau^{i+1}_{\phi}(\phi)).$$
It follows that \( \tau^i_\phi(\varphi) \leq \varphi' < \tau^{i+1}_\phi(\varphi) \). Therefore, \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) is an effective finest t-stability. By Theorem 3.28, there exists a t-exceptional triple \((E_0, E_1, E_2)\) in \(\mathcal{D}\). Hence, there exists some \(\varphi_0 \in \Phi\) such that \(\varphi(E_i) \in (\varphi_0, \tau_\phi(\varphi_0))\) for each \(i\). It follows that \(E_i \in \mathcal{P}[r(\varphi_0), r(\varphi_0) + 1]\) for each \(i\). Then by Proposition 4.9, there exists a \(\sigma\)-exceptional triple on \(\mathcal{D}\). \(\square\)

Acknowledgements

The authors would like to thank Bangming Deng and Zhe Han for their valuable suggestions. We are also grateful to Nan Gao for providing the proof of Lemma 3.15.

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