The concept of free group based on braid group

M K M Nasution\textsuperscript{1*}, E Herawati\textsuperscript{2}, E Rosmaini\textsuperscript{2}

\textsuperscript{1}Teknologi Informasi, Fasilkom-TI, Universitas Sumatera Utara, Padang Bulan 20155 USU, Medan, Indonesia
\textsuperscript{2}Matematika, FMIPA, Universitas Sumatera Utara, Padang Bulan 20155 USU Medan Indonesia

E-mail: *mahyuddin@usu.ac.id

Abstract. Although free groups have long been present as part of algebra, and it steady has its implications for each implementation target, especially to determine invariant. However, the decrease in free groups of webbing requires different meanings of braid. This paper reveals a concept to form a free group of braid groups.

1. Introduction

As one of applications of group theory in modeling structure [1], the braid group is a model about an oldest works of human [2], and this give implication that group theory only touches the theory by which braid group becomes a learning to understand to abstract algebraic through the concept of geometry in woven form [3].

Although the braid group is a tool that can be used to understand modern mathematics, but it can also be used to understand a structure by giving meaning to the arrangement of each woven. Therefore, through the understanding of the woven structure can be recognized its better meaning, mainly about invariant of shapes. This paper intends to describe the concept of free group in braid group for giving mean to braids.

2. Review and Motivation

The braid group based on three woven forms is as follows [4].

**Definition 1.** The braid group is a set of braids \( A_n = \{ A_{n,k} | k = 1, 2, \ldots \} \) with deformation operation against triple \( \sigma^0, \sigma_i, \sigma_i^{-1} \) meet two conditions

\[
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad 1 \leq i < j < n, \quad j - i > 1.
\]  

and

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 1,
\]  

Geometrically, the deformation states that each woven in a braid has a unique position. Two woven forms such as in triple [5], like Figure 1, have

\[
\sigma_i = \begin{pmatrix}
\tau_v & \tau_i & \tau_{i+1} \\
\tau_v & \tau_{i+1} & \tau_{i+1}
\end{pmatrix}
\]
Figure 1. Triple of woven forms in braid

Figure 2. Defining free set for triple of woven forms

and

\[
\sigma_i^{-1} = \begin{pmatrix}
t_v & t_i & t_{i+1}^{-1} & t_i \\
t_v & t_i & t_{i+1}^{-1} & t_i \\
\end{pmatrix} \tag{4}
\]

where \( t \) is a curve being traversed by a closed path such that \( t_v \) for \( v \neq i, i + 1 \), while \( t_i \) and \( t_{i+1} \) based on two woven forms \( \sigma_i \) and \( \sigma_{i+1} \), respectively. Based on this (Artin) concept, free group is exists. Let \( X \) is a set of symbols to define a word in \( X \). If \( X = \{x_i|i = 1, \ldots, n\} \), then the word in \( X \) be a production on \( Y \) where \( Y = \{x_i, x_i^{-1}\} \), and we define free group as follows.

**Definition 2.** The free group \( F_X \) is a construction that free generated by set \( X \), and an arbitrary group is stated free by \( X \) if the group isomorphic to \( F_X \).

3. **An Approach**

To define words in based on free set \( X = \{x_i|i = 1, \ldots, n\} \), we assumpt that there are a path from one point \( y_0 \) to a loop \( g \). Each \( i \)-string of braid, we define \( x_i \) on the string by using arrow from left to right. By following \( x_i \), a loop is given an arrow to through the string traversing, then loop through the traversed string, and continued through the string traversing and last through the traversed string [2]. Therefore, for \( \sigma^0 \) the loop without arrow, for \( \sigma_i \) has a direction of loop that is opposite to the direction of the loop for \( \sigma_i^{-1} \), see Figure 2.

Based on defining free set, each generator of \( \sigma_i, \sigma_i^{-1} \) and \( \sigma^0 \), we can define words as follows.

(i) A word from \( y_0 \) to \( g \) based on generator \( \sigma_i \) is \( y_0 = g(\sigma_i) \) such that

\[
\sigma_i = x_ix_{i+1}x_i^{-1}x_{i+1}^{-1} \tag{5}
\]

Let \( 1 = x_ix_{i+1}x_i^{-1}x_{i+1}^{-1} \), we obtain

\[
x_{i+1} = x_ix_{i+1}x_i^{-1} \tag{6}
\]

(ii) A word from \( y_0 \) to \( g \) based on generator \( \sigma_i^{-1} \) is \( y_0 = g(\sigma_i^{-1}) \) such that

\[
\sigma_i^{-1} = x_{i+1}x_i^{-1}x_{i+1} \tag{7}
\]
or for $1 = x_{i+1} x_i x_{i+1}^{-1} x_i^{-1}$, $x_i = x_{i+1} x_i x_{i+1}^{-1}$, $x_i x_{i+1} = x_{i+1} x_i$, and then

$$x_{i+1} = x_i^{-1} x_{i+1} x_i$$

(8)

(iii) A word from $y_0$ to $g$ based on generator $\sigma^0$ is $y_0 = g(x_i x_{i+1} x_i^{-1} x_{i+1}^{-1}) = g(x_i^{-1}) = g(1)$ or $y_0 = g(x_{i+1} x_i x_{i+1}^{-1} x_i^{-1}) = g(1)$ such that

$$\sigma^0 = 1,$$

(9)

the symbol 1 as empty word.

The power of $x_i$ or $x_{i+1}$ is $-1$ for denoting the invers of $x_i$ or $x_{i+1}$ as consequence of a result of traversing with opposite directions.

Based on the construction of $\sigma_i$ and $\sigma_i^{-1}$, the composition of $\sigma_i \sigma_i^{-1}$ can be stated as follows

$$\sigma_i \sigma_i^{-1} = (x_i x_{i+1} x_i^{-1} x_{i+1}^{-1})(x_i x_{i+1} x_i^{-1} x_{i+1}^{-1}) = x_i x_{i+1} x_i^{-1} x_{i+1}^{-1} x_i x_{i+1} x_i^{-1} x_{i+1}^{-1} = 1$$

(10)

In general, it can be said that the composition of the word based on the free set for any generator $\sigma_i$ is by emphasizing the index of $\sigma_i$ as representing the $i$-string [6]. However, by substituting Eq. (8) to Eq. (6), we obtain

$$x_i^{-1} x_{i+1} x_i = x_i x_{i+1} x_i^{-1}$$

(11)

4. Example

In the $n$-ordered braid group $A_n$ there is $n$ generators, we can create words corresponding to the index of the generator. For example, let $A_{6,1} = \sigma_3^{-1} \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_3^{-1} \sigma_5 \sigma_2$ is a braid of 6-ordered braid group $A_6$ (see Figure 3), then $A_{6,1}$ has an arrangement of words as follows

![Figure 3. Generating the words for a braid $A_{6,1} \in A_6$](image-url)
\[\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 = (x_4 x_3 x_4^{-1} x_3^{-1}) (x_2 x_3 x_2^{-1} x_3^{-1}) (x_4 x_5 x_4^{-1} x_5^{-1}) (x_2 x_1 x_2^{-1} x_1^{-1}) (x_4 x_3 x_4^{-1} x_3^{-1}) (x_5 x_6 x_5^{-1} x_6^{-1}) (x_2 x_3 x_2^{-1} x_3^{-1}) = x_4 x_3 x_4^{-1} x_3^{-1} x_2 x_3 x_2^{-1} x_3^{-1} x_4 x_3 x_4^{-1} x_3^{-1} x_2 x_3 x_2^{-1} x_3^{-1} x_4 x_3 x_4^{-1} x_3^{-1} (x_2 x_1 x_2^{-1} x_1^{-1}) \]

(12)

Based on number of generators that appear, that is \(\sigma_1^{-1} = x_2 x_1 x_2^{-1} x_1^{-1}\), \(\sigma_2 = x_2 x_3 x_2^{-1} x_3^{-1}\), \(\sigma_3 = x_4 x_3 x_4^{-1} x_3^{-1}\), \(\sigma_4 = x_4 x_3 x_4^{-1} x_5^{-1}\), \(\sigma_5 = x_5 x_6 x_5^{-1} x_6^{-1}\) we have

\[x_1 = x_2 x_1 x_2^{-1}\]  
\[x_2 = x_3^{-1} x_2 x_3\]  
\[x_3 = x_4 x_3 x_4^{-1}\]  
\[x_4 = x_5 x_4 x_5\]  
\[x_5 = x_6^{-1} x_5 x_6\]

(13)

By interpreting the word in Eq. (13) be \(x_2 = x_1 x_2 x_1^{-1}\) and by substituting Eq. (14), i.e. \(x_3^{-1} x_2 x_3 = x_1 x_2 x_1^{-1}\) and we have \(x_3 = x_2 x_3 x_2^{-1} x_3^{-1}\). Similarly, based on Eq. (15) \(x_4 x_3 x_4^{-1} = x_4^{-1} x_3 x_4^{-1} x_4^{-1} x_1\) and we obtain \(x_4 = x_3 x_4^{-1} x_2 x_4^{-1} x_1\). Based on Eq. (16) \(x_5 x_4 x_5^{-1} = x_2 x_3 x_2^{-1} x_1 x_4 x_3^{-1}\) and \(x_5 = x_4 x_5 x_4^{-1} x_3 x_5 x_4^{-1} x_2 x_4 x_3^{-1}\) and then based on Eq. (17) \(x_6 x_5 x_6^{-1} = x_4 x_5 x_4^{-1} x_3 x_5 x_4^{-1} x_2 x_4 x_3^{-1}\) and finally there is a word

\[x_6 = x_5^{-1} x_6 x_4^{-1} x_5 x_2^{-1} x_3 x_4^{-1} x_2^{-1} x_1 x_4 x_3^{-1}\]

(18)

By interpreting the generator arrangement: First we collaborate Eq. (14) and Eq. (15), \(x_4 x_3 x_4^{-1} = x_2 x_3 x_2^{-1}\) and we obtain \(x_4 = x_2 x_3 x_2^{-1} x_4 x_3^{-1}\); Second, with Eq. (16), then \(x_5 x_4 x_5^{-1} = x_3 x_4 x_3^{-1} x_4 x_5 x_4^{-1}\) and we obtain the word for collaborating to Eq. (13), i.e. \(x_2 = x_4 x_3 x_2^{-1} x_4 x_3 x_2^{-1}\); Third \(x_3 = x_5^{-1} x_4^{-1} x_4^{-1} x_5 x_2 x_3^{-1}\); Fourth \(x_4 = x_5^{-1} x_4^{-1} x_5 x_2 x_3 x_2^{-1} x_2 x_1 x_4\), and then \(x_5 x_4 x_5^{-1} = x_5^{-1} x_4^{-1} x_5 x_2 x_3 x_2^{-1} x_2 x_1 x_4\), by exchanging positions of \(x_i\) we obtain \(x_3 = x_4^{-1} x_5 x_3 x_4^{-1} x_2 x_1 x_4\). Fifth \(x_6^{-1} x_5 x_6 = x_4^{-1} x_5 x_2 x_3 x_4^{-1} x_2 x_1 x_4 x_3^{-1} x_5 x_6^{-1} x_5 x_2 x_3 x_4^{-1} x_2 x_1 x_4\), for \(x_2\), we obtain \(x_6 x_5 x_6 x_4 x_3 x_4^{-1} x_3 x_4^{-1} x_2 x_1 x_4\) such that \(x_4 x_5 x_4^{-1} x_3^{-1} x_6^{-1} x_5 x_2 x_3 x_1^{-1}\). By substituting all equations (Eqs. (13)-(17)), \(A_{6,1}\) has a word as follows

\[x_3^{-1} x_2^{-1} x_3 x_1 x_4 x_3 x_4^{-1} x_2^{-1} x_3^{-1} x_2^{-1} x_4 x_3^{-1} x_5 x_2 x_3 x_1^{-1}\]

(19)

as resume of Eq. (12) in accordance with the approach taken through the Eq. (11). It is possible to reduce the length of a word other than construction a new word based on substitution Eqs. (14)-(17) alternately to Eq. (13). Therefore, the composition of two braids will produce a new braid and also produce a new word because it comes from more than two words.

In braid group, the composition of words based on the composition of braids. Thus, the composition between words of braids is not commutative but can be associative such as the braids composition. Next, we will express through the example about inverse of \(A_{6,1}\).

Inverse of \(A_{6,1}\) (see Fig. 4), i.e. \(A_{6,1}^{-1} \in A_6\) has an arrangement of words as follows

\[\sigma_2^{-1} \sigma_4^{-1} \sigma_3 \sigma_1^{-1} \sigma_3 \sigma_1^{-1} \sigma_4^{-1} \sigma_2^{-1} \sigma_3 = (x_3 x_2 x_3^{-1} x_2^{-1})(x_6 x_5 x_6^{-1} x_5^{-1})(x_3 x_4 x_3^{-1} x_4^{-1})(x_1 x_2 x_1^{-1} x_2^{-1}) (x_3 x_2 x_3^{-1} x_2^{-1}) (x_3 x_4 x_3^{-1} x_4^{-1})(x_3 x_4 x_3^{-1} x_4^{-1}) = x_3 x_2 x_3^{-1} x_2^{-1} x_6 x_5 x_6^{-1} x_5^{-1} x_3 x_4 x_3^{-1} x_4^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_5 x_4 x_5^{-1} x_1 x_3 x_2 x_3^{-1} x_2^{-1} x_3 x_4 x_3^{-1} x_4^{-1}\]

(20)
Figure 4. Generating the words for a braid $A_{6,1}^{-1} \in A_6$

So, in line with $A_{6,1}A_{6,1}^{-1} = (\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_4\sigma_1^{-1}\sigma_3^{-1}\sigma_5\sigma_2^{-1}) (\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_4^{-1}\sigma_2^{-1}\sigma_3^{-1}) = 1$ we can conclude that

$$(x_4 x_3 x_4^{-1} x_2 x_3 x_2^{-1} x_3^{-1} x_4 x_5 x_4^{-1} x_5^{-1} x_2 x_1 x_2^{-1} x_1^{-1} x_4 x_3 x_4^{-1} x_3^{-1} x_5 x_6 x_5^{-1} x_6^{-1} x_2 x_3 x_2^{-1} x_3^{-1}) (x_3 x_2 x_3^{-1} x_2^{-1} x_6 x_5 x_6^{-1} x_5^{-1} x_3 x_4 x_3^{-1} x_4^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_5 x_4 x_5^{-1} x_4^{-1} x_3 x_2 x_3^{-1} x_2^{-1} x_3 x_4 x_3^{-1} x_4^{-1}) = 1 \text{ or empty word.}
$$

Similarly, by involving the same approach $A_{6,1}^{-1}$ has the word being the inverse of the word in Eq. (19). Thus, if each generator of braid has a word, then set of words of braids be a group that free generated by symbol $x_i$.

This proves that the Definition 1 can be changed to reveal the meaning of each woven structure in the braid groups, that is Eq. (1) be

$$x_i x_{i+1} x_i^{-1} x_{i+1}^{-1} x_j x_{j+1} x_j^{-1} x_{j+1}^{-1} = x_j x_{j+1} x_j^{-1} x_{j+1}^{-1} x_i x_{i+1} x_i^{-1} x_{i+1}^{-1} \quad (21)$$

where $|i - j| \geq 2$. Therefore, this concept forms an isomorphic as expressed in Definition 2.

5. Conclusion

We have disclosed the meaning of a braid through an approach, by which the meaning that produces the word in accordance with the braid. Of course, the implications of the braid group form a free group, which gives other meaning to the braid. The future work will prove a free group based on the braid group.

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