WELL-POSEDNESS OF WEAK SOLUTION FOR A NONLINEAR POROELASTICITY MODEL

ZHIHAO GE† AND WENLONG HE‡

Abstract. In this paper, we study the existence and uniqueness of weak solution of a nonlinear poroelasticity model widely used in many fields such as geophysics, biomechanics, civil engineering, chemical engineering, materials science, and so on. To better describe the process of deformation and diffusion underlying in the original model, we firstly reformulate the nonlinear poroelasticity by a multiphysics approach which transforms the nonlinear fluid-solid coupling problem to a fluid-fluid coupling problem. Then, we adopt the similar technique of proving the well-posedness of nonlinear Stokes equations to prove the existence and uniqueness of weak solution of a nonlinear poroelasticity model. And we strictly prove the growth, coercivity and monotonicity of the nonlinear stress-strain relation, give the energy estimates and use Schauder’s fixed point theorem to show the existence and uniqueness of weak solution of the nonlinear poroelasticity model. Besides, we prove that the weak solution of nonlinear poroelasticity model converges to the nonlinear Biot’s consolidation model as the constrained specific storage coefficient trends to zero. Finally, we draw a conclusion to summary the main results of this paper.

Key words. Nonlinear poroelasticity; Multiphysics approach; Nonlinear Stokes equations; Schauder’s fixed point theorem.

AMS subject classifications. 35A01, 35B45, 86A25,

1. Introduction. In recent years, poroelasticity model is widely used in various fields such as geophysics, biomechanics, civil engineering, chemical engineering, materials science and so on, one can refer to [1, 4, 6, 11, 12, 15, 18, 19, 23]. Especially, in modern materials science, porous materials such as polymers and metal foams are of great significance in lightweight design and aircraft industry, one can refer to [4, 14, 23] and so on. The poroelasticity model is classified into linear poroelasticity model and nonlinear poroelasticity model according to the linear or nonlinear constitutive relation (cf. [7]). For linear poroelasticity, Schowalter provides the analysis of well-posedness of weak solution to a linear poroelasticity model in [21]. Phillips and Wheeler propose and analyze a continuous-in-time linear poroelasticity model in [20]. Besides, Feng, Ge and Li in [8, 9], propose a multiphysics approach to reformulate the linear poroelasticity model to a fluid-fluid coupled system, which reveals the underlying deformation

†LAST UPDATE: December 24, 2021
†School of Mathematics and Statistics, Henan University, Kaifeng 475004, P.R. China (zhihaoge@henu.edu.cn). The work of this author was supported by the National Natural Science Foundation of China under grant No.11971150.
‡School of Mathematics and Statistics, Henan University, Kaifeng 475004, P.R. China.
and diffusion processes of the original model. In this paper, following the idea of [9], we deal with the nonlinear poroelasticity model with the constitutive relation \( \tilde{\sigma}(u) = \mu \tilde{\varepsilon}(u) + \lambda tr(\tilde{\varepsilon}(u))I \), where the deformed Green strain tensor is \( \tilde{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla^T u + 2\nabla^T u \nabla u) \). Using the Cauchy-Schwarz inequality, Korn’s inequality and other inequalities (see Section 4), we prove the growth, coercivity and monotonicity of \( N(\nabla u) \) (see (2.8)), then we give the energy estimates and use Schauder’s fixed point theorem to show the existence and uniqueness of weak solution of the nonlinear poroelasticity model. Besides, we prove that the weak solution of the nonlinear poroelasticity model converges to a nonlinear Biot’s consolidation model as the constrained specific storage coefficient trends to zero. To the best of our knowledge, it is the first time to prove the existence and uniqueness of weak solution based on a multiphysics approach without any assumption on the nonlinear stress-strain relation. Moreover, we find out that the multiphysics approach is key to propose a stable numerical method for the nonlinear poroelasticity model, and we will present the main results about numerical method for the nonlinear poroelasticity model in the future work.

The remainder of this article is organized as follows. In Section 2, we reformulate the original model based on a multiphysics approach to a fluid-fluid coupling system. In Section 3, we give the definition of weak solution to the original model and the reformulated model. In Section 4, we prove the growth, coercivity and monotonicity based on a multiphysics approach without any assumption on the nonlinear stress-strain relation, and we use the energy estimates and Schauder’s fixed point theorem to prove the well-posedness of weak solution of the nonlinear poroelasticity model. Besides, we prove that the nonlinear poroelasticity model converges to the nonlinear Biot’s consolidation model as the constrained specific storage coefficient trends to zero. Finally, we draw a conclusion to summary the main results of this paper.

2. PDE model and multiphysics approach. In this paper, we consider the following quasi-static poroelasticity model (for the linear case, one can refer to [8,9,20]):

\begin{align}
(2.1) \quad -\text{div} \tilde{\sigma}(u) + \alpha \nabla p &= f \quad \text{in } \Omega_T := \Omega \times (0, T) \subset \mathbb{R}^d \times (0, T), \\
(2.2) \quad (c_0 p + \alpha \text{div} u)_t + \text{div} v_f &= \phi \quad \text{in } \Omega_T,
\end{align}

where

\begin{align}
(2.3) \quad \tilde{\sigma}(u) &= \mu \tilde{\varepsilon}(u) + \lambda tr(\tilde{\varepsilon}(u))I, \quad \tilde{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla^T u + 2\nabla^T u \nabla u),
\end{align}
\[ (2.4) \quad v_f := -\frac{K}{\mu_f}(\nabla p - \rho_f g). \]

Here \( \mathbf{u} \) denotes the displacement vector of the solid and \( p \) denotes the pressure of the solvent. \( \mathbf{I} \) denotes the \( d \times d \) identity matrix and \( \tilde{\varepsilon}(\mathbf{u}) \) is known as the deformed Green strain tensor. \( f \) is the body force. The permeability tensor \( K = K(x) \) is assumed to be symmetric and uniformly positive definite in the sense that there exists positive constants \( K_1 \) and \( K_2 \) such that \( K_1|\zeta|^2 \leq K(x)\zeta \cdot \zeta \leq K_2|\zeta|^2 \) for a.e. \( x \in \Omega \) and \( \zeta \in \mathbb{R}^d \); the solvent viscosity \( \mu_f \), Biot-Willis constant \( \alpha \), and the constrained specific storage coefficient \( c_0 \). In addition, \( \tilde{\sigma}(\mathbf{u}) \) is called the (effective) stress tensor. \( v_f \) is the volumetric solvent flux and (2.4) is called the well-known Darcy’s law.

To close the above system, we set the following boundary and initial conditions in this paper:

\[ (2.5) \quad \tilde{\sigma}(\mathbf{u}, p)\mathbf{n} = \tilde{\sigma}(\mathbf{u})\mathbf{n} - \alpha p\mathbf{n} = f_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T), \]
\[ (2.6) \quad v_f \cdot \mathbf{n} = -\frac{K}{\mu_f}(\nabla p - \rho_f g) \cdot \mathbf{n} = \phi_1 \quad \text{on } \partial \Omega_T, \]
\[ (2.7) \quad \mathbf{u} = \mathbf{u}_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}. \]

Introduce new variables

\[ q := \text{div} \mathbf{u}, \quad \eta := c_0 p + \alpha q, \quad \xi := \alpha p - \lambda q. \]

Denote

\[ \mathcal{N}(\nabla \mathbf{u}) = \tilde{\sigma}(\mathbf{u}) - \lambda \text{div} \mathbf{u} \mathbf{I}, \]

then we have

\[ (2.8) \quad \mathcal{N}(\nabla \mathbf{u}) = \mu \varepsilon(\mathbf{u}) + \mu \nabla^T \mathbf{u} \nabla \mathbf{u} + \lambda \|
abla \mathbf{u}\|^2 I. \]

Due to the fact of \( (\nabla^T \mathbf{u} \nabla \mathbf{v}, \text{rot} \mathbf{v}) = 0 \), \( (\|
abla \mathbf{u}\|^2 I, \text{rot} \mathbf{v}) = 0 \), so we have

\[ (\mathcal{N}(\nabla \mathbf{u}), \nabla \mathbf{v}) = (\mathcal{N}(\nabla \mathbf{u}), \varepsilon(\mathbf{v})), \]

where \( \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla^T \mathbf{u} + \nabla \mathbf{u}) \).

In some engineering literature, Lamé constant \( \mu \) is also called the shear modulus and denoted by \( G \), and \( B := \lambda + \frac{2}{3}G \) is called the bulk modulus. \( \lambda, \mu \) and \( B \) are computed from the Young’s modulus \( E \) and the Poisson ratio \( \nu \) by the following formulas

\[ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad B = \frac{E}{3(1-2\nu)}. \]
It is easy to check that
\[ p = \kappa_1 \xi + \kappa_2 \eta, \quad q = \kappa_1 \eta - \kappa_3 \xi, \]
where \( \kappa_1 = \frac{\alpha}{\alpha^2 + \lambda c_0}, \kappa_2 = \frac{\lambda}{\alpha^2 + \lambda c_0}, \kappa_3 = \frac{c_0}{\alpha^2 + \lambda c_0}. \)

Then the problem (2.1)-(2.4) can be rewritten as
\[ -\text{div} \mathcal{N} (\nabla u) + \nabla \xi = f \quad \text{in } \Omega_T, \]
\[ \kappa_3 \xi + \text{div} u = \kappa_1 \eta \quad \text{in } \Omega_T, \]
\[ \eta_t - \frac{1}{\mu_f} \text{div} [K(\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f g)] = \phi \quad \text{in } \Omega_T. \]

The boundary and initial conditions (2.5)-(2.7) can be rewritten as
\[ \tilde{\sigma}(u) n - \alpha (\kappa_1 \xi + \kappa_2 \eta) n = f_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T), \]
\[ -\frac{K}{\mu_f} (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f g) \cdot n = \phi_1 \quad \text{on } \partial \Omega_T, \]
\[ u = u_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}. \]

**Remark 2.1.** It is now clear that \((u, \xi)\) satisfies a generalized nonlinear Stokes problem for a given \(\eta\), and \(\eta\) satisfies a diffusion problem for a given \(\xi\). Thus, This new formulation reveals the underlying deformation and diffusion multiphysics process which occurs in the poroelastic material.

**3. Definition of weak solution.** In this paper, \(\Omega \subset \mathbb{R}^d (d = 1, 2, 3)\) denotes a bounded polygonal domain with the boundary \(\partial \Omega\). The standard function space notation is adopted in this paper, their precise definitions can be found in [2,3,22]. In particular, \((\cdot, \cdot)\) and \(\langle \cdot, \cdot \rangle\) denote respectively the standard \(L^2(\Omega)\) and \(L^2(\partial \Omega)\) inner products. For any Banach space \(B\), we let \(B^d\) denote its dual space. In particular, we use \(\langle \cdot, \cdot \rangle_{\text{dual}}\) to denote the dual product on \(H^1(\Omega) \times H^1(\Omega)\), and \(\| \cdot \|_{L^p(\Omega)} \) is a shorthand notation for \(\| \cdot \|_{L^p((0,T);B)}\).

We also introduce the function spaces
\[ L^2_0(\Omega) := \{ q \in L^2(\Omega); \langle q, 1 \rangle = 0 \}, \quad X := H^1(\Omega). \]

From [22], it is well known that the following inf-sup condition holds in the space \(X \times L^2_0(\Omega)\):
\[ \sup_{v \in X} \frac{(\text{div} \varphi, \varphi)}{\|v\|_{H^1(\Omega)}} \geq \alpha_0 \|\varphi\|_{L^2(\Omega)} \quad \forall \varphi \in L^2_0(\Omega), \quad \alpha_0 > 0. \]
Let
\[ \mathbf{RM} := \{ \mathbf{r} := a + b \times x; a, b, x \in \mathbb{R}^d \} \]
denote the space of infinitesimal rigid motions. It is well known [2, 13, 22] that \( \mathbf{RM} \) is the kernel of the strain operator \( \varepsilon \), that is, \( \mathbf{r} \in \mathbf{RM} \) if and only if \( \varepsilon(\mathbf{r}) = 0 \). Hence, we have
\[ \varepsilon(\mathbf{r}) = 0, \quad \text{div } \mathbf{r} = 0 \quad \forall \mathbf{r} \in \mathbf{RM}. \]

Let \( L^2_\perp(\partial \Omega) \) and \( H^1_\perp(\Omega) \) denote respectively the subspaces of \( L^2(\partial \Omega) \) and \( H^1(\Omega) \) which are orthogonal to \( \mathbf{RM} \), that is,
\[ H^1_\perp(\Omega) := \{ v \in H^1(\Omega); (v, \mathbf{r}) = 0 \forall \mathbf{r} \in \mathbf{RM} \}, \]
\[ L^2_\perp(\partial \Omega) := \{ g \in L^2(\partial \Omega); (g, \mathbf{r}) = 0 \forall \mathbf{r} \in \mathbf{RM} \}. \]

It is well known [5] that there exists a constant \( c_1 > 0 \) such that
\[ \inf_{\mathbf{r} \in \mathbf{RM}} \| v + \mathbf{r} \|_{L^2(\Omega)} \leq c_1 \| \varepsilon(v) \|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \]

From [9], we know that for each \( v \in H^1_\perp(\Omega) \) there holds the following alternative version of the inf-sup condition
\[ \sup_{v \in H^1_\perp(\Omega)} \frac{(\text{div } v, \varphi)}{\| v \|_{H^1(\Omega)}} \geq \alpha_1 \| \varphi \|_{L^2(\Omega)} \quad \forall \varphi \in L^2_0(\Omega), \quad \alpha_1 > 0. \]

For convenience, we assume that \( f, f_1, \phi \) and \( \phi_1 \) all are independent of \( t \) in the remaining of the paper. We note that all the results of this paper can be easily extended to the case of time-dependent source functions.

**Definition 3.1.** Let \( u_0 \in H^1(\Omega) \), \( f \in L^2(\Omega) \), \( f_1 \in L^2(\partial \Omega) \), \( p_0 \in L^2(\Omega) \), \( \phi \in L^2(\Omega) \), and \( \phi_1 \in L^2(\partial \Omega) \). Assume \( c_0 > 0 \) and \((f, v) + (f_1, v) = 0\) for any \( v \in \mathbf{RM} \). Given \( T > 0 \), a tuple \((u, p)\) with
\[ u \in L^\infty(0, T; H^1_\perp(\Omega)), \quad p \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \]
\[ p_t, (\text{div } u)_t \in L^2(0, T; H^1(\Omega)^\prime) \]
is called a weak solution to the problem (2.1)–(2.7), if there hold for almost every \( t \in [0, T] \)
\[ (N(\nabla u), \varepsilon(v)) + \lambda (\text{div } u, \text{div } v) - \alpha (p, \text{div } v) \]
\[ = (f, v) + (f_1, v) \quad \forall v \in H^1(\Omega), \]
\[ ((c_0 p + \text{adiv } u)_t, \varphi)_{\text{dual}} + \frac{1}{\mu_f} (K(\nabla p - \rho_f g), \nabla \varphi) \]
\[ = (\phi, \varphi) + \langle \phi_1, \varphi \rangle \quad \forall \varphi \in H^1(\Omega), \]
\[(3.7) \quad u(0) = u_0, \quad p(0) = p_0.\]

Similarly, we can define the weak solution to the problem (2.11)-(2.13) as follows:

**Definition 3.2.** Let \( u_0 \in H^1(\Omega), f \in L^2(\Omega), f_1 \in L^2(\partial \Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega), \) and \( \phi_1 \in L^2(\partial \Omega). \) Assume \( c_0 > 0 \) and \( (f, v) + \langle f_1, v \rangle = 0 \) for any \( v \in RM. \)

Given \( T > 0, \) a 5-tuple \((u, \xi, \eta, p, q)\) with

\[ u \in L^\infty(0, T; H^1(\Omega)), \quad \xi \in L^\infty(0, T; L^2(\Omega)), \]
\[ \eta \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)'), \quad q \in L^\infty(0, T; L^2(\Omega)), \]
\[ p \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \]

is called a weak solution to the problem (2.11)-(2.13), if there hold for almost every \( t \in [0, T] \)

\[ (N(\nabla u), \varepsilon(v)) - (\xi, \text{div} v) = (f, v) + \langle f_1, v \rangle \quad \forall v \in H^1(\Omega), \]
\[ (3.8) \]
\[ \kappa_3 (\xi, \varphi) + (\text{div} u, \varphi) = \kappa_1 (\eta, \varphi) \quad \forall \varphi \in L^2(\Omega), \]
\[ (3.9) \]
\[ \eta_t \psi_{\text{dual}} + \frac{1}{\mu_f} (K(\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f g), \nabla \psi) \]
\[ = (\phi, \psi) + \langle \phi_1, \psi \rangle \quad \forall \psi \in H^1(\Omega), \]
\[ (3.10) \]
\[ p := \kappa_1 \xi + \kappa_2 \eta, \quad q := \kappa_1 \eta - \kappa_3 \xi, \]
\[ (3.11) \]
\[ \eta(0) = \eta_0 := c_0 p_0 + \alpha q_0, \]

where \( q_0 := \text{div} u_0, \) \( u_0 \) and \( p_0 \) are same as in Definition (3.1).

**Remark 3.1.** It should be pointed out that the only reason for introducing the space \( H^1_\perp(\Omega) \) in the above two definitions is that the boundary condition (2.5) is a pure “Neumann condition”. If it is replaced by a pure Dirichlet condition or by a mixed Dirichlet-Neumann condition, there is no need to introduce this space. Thus, from the analysis point of view, the pure Neumann condition case is the most difficult case.

**4. Existence and uniqueness of weak solution.** The proof of next two lemmas about the stress-strain relation are required to obtain a well-posed weak solution of the nonlinear poroelasticity problem.

**Lemma 4.1.** There exist positive constants \( C_1, C_2 \) and \( C_4 \) such that

\[ ||N(\nabla u)||_{L^2(\Omega_T)} \leq C_1 ||\varepsilon(u)||_{L^2(\Omega_T)}, \]
\[ (4.1) \]
\[ (N(\nabla(u)), \varepsilon(u)) \geq C_2 ||\varepsilon(u)||^2_{L^2(\Omega)}, \]
\[ (4.2) \]
\[(4.3) \quad (N'(\nabla(u)) - N'(\nabla(v)), \varepsilon(u) - \varepsilon(v)) \geq C_4 \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)}^2.\]

**Proof.** Firstly, we know that \(\|\nabla u\|_{L^2(\Omega)}\) is bounded from [25], i.e. \(M \leq \|\nabla u\|_{L^2(\Omega)} \leq N\). So we can get \(M' \leq \|\nabla u\|_F \leq N'\). Using the Cauchy-Schwarz inequality and Korn’s inequality, we have

\[
\|N(\nabla(u))\|_{L^2(\Omega)} = \left\| \mu \varepsilon(u) + \mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I \right\|_{L^2(\Omega)}
\]

\[
\leq \|\mu \varepsilon(u)\|_{L^2(\Omega)} + \|\mu \nabla^T u \nabla u\|_{L^2(\Omega)} + \|\lambda \|\nabla u\|_F^2 I\|_{L^2(\Omega)}
\]

\[
\leq \mu \|\nabla u\|_{L^2(\Omega)} + \mu \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|\nabla u\|_F^2 d
\]

\[
\leq \left(\mu + \frac{\lambda d N'}{M}\right) \|\nabla u\|_{L^2(\Omega)}
\]

\[(4.4) \quad \leq c_2 \left(\mu + \frac{\lambda d N'}{M}\right) \|\varepsilon(u)\|_{L^2(\Omega)}.\]

Taking \(C_1 = c_2 \left(\mu + \frac{\lambda d N'}{M}\right)\) in (4.4), we see that (4.1) holds.

To prove (4.2), we use the fact of \(\frac{1}{2}[(a + b, a + b) - (a, a) - (b, b)] = (a, b)\) and the inequality \(\|x + y\|_{L^2(\Omega)}^2 \geq (\|x\|_{L^2(\Omega)} - \|y\|_{L^2(\Omega)})^2 \geq \|x\|_{L^2(\Omega)}^2 - \|y\|_{L^2(\Omega)}^2\) if and only if \(\|x\|_{L^2(\Omega)} \leq \|y\|_{L^2(\Omega)}\) to get

\[
(N(\nabla(u)), \varepsilon(u)) = (\mu \varepsilon(u) + \mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I, \varepsilon(u))
\]

\[
= (\mu \varepsilon(u), \varepsilon(u)) + (\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I, \varepsilon(u))
\]

\[
= \mu \|\varepsilon(u)\|_{L^2(\Omega)}^2 + \frac{1}{2} \left[\left(\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I + \varepsilon(u), \mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I + \varepsilon(u)\right)\right]
\]

\[
- \left(\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I, \mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I\right) - (\varepsilon(u), \varepsilon(u))\]

\[
= \mu \|\varepsilon(u)\|_{L^2(\Omega)}^2 + \frac{1}{2} \left[\|\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I + \varepsilon(u)\|_{L^2(\Omega)}^2ight]
\]

\[
- \left[\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I\right]_{L^2(\Omega)} + \frac{1}{2} \left[\|\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I\|_{L^2(\Omega)}^2ight]
\]

\[
\geq \mu \|\varepsilon(u)\|_{L^2(\Omega)}^2 + \frac{1}{2} \left[\|\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I\|_{L^2(\Omega)}^2ight]
\]

\[
- \left[2c_2(\mu N + \frac{\lambda d N'}{M}) - 1\right] \|\varepsilon(u)\|_{L^2(\Omega)}^2
\]

\[
- \left[\mu \nabla^T u \nabla u + \lambda \|\nabla u\|_F^2 I\|_{L^2(\Omega)}^2 - \|\varepsilon(u)\|_{L^2(\Omega)}^2\right]
\]

\[(4.5) \quad = \left(\mu - c_2(\mu N + \frac{\lambda d N'}{M})\right) \|\varepsilon(u)\|_{L^2(\Omega)}^2.\]
Due to
\[ \left\| \mu \nabla^T u \nabla u + \lambda \left\| \nabla u \right\|_{F}^2 I \right\|_{L^2(\Omega)} \leq \mu \left\| \nabla u \right\|_{L^2(\Omega)}^2 + \lambda d N^2 \]
\[ \leq (\mu N + \frac{\lambda d N^2}{M}) \left\| \nabla u \right\|_{L^2(\Omega)} \leq c_2 (\mu N + \frac{\lambda d N^2}{M}) \left\| \varepsilon(u) \right\|_{L^2(\Omega)}, \]
and taking \( C_2 = \mu - c_2 (\mu N + \frac{\lambda d N^2}{M}) > 0 \) in (4.5), we see that (4.2) holds.

The proof of (4.3) is similar to (4.2), in fact, we obtain
\[ (N(\nabla(u)) - N(\nabla(v)), \varepsilon(u) - \varepsilon(v)) \]
\[ = \mu (\varepsilon(u) - \varepsilon(v), \varepsilon(u) - \varepsilon(v)) \]
\[ + \left( \mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda \left\| \nabla u \right\|_{F}^2 I - \lambda \left\| \nabla v \right\|_{F}^2 I, \varepsilon(u) - \varepsilon(v) \right) \]
\[ = \mu \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)}^2 \]
\[ + \frac{1}{2} \left[ \left\| \mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda \left\| \nabla u \right\|_{F}^2 I - \lambda \left\| \nabla v \right\|_{F}^2 I, \varepsilon(u) - \varepsilon(v) \right] \]
\[ = \mu \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)}^2 \]
\[ + \frac{1}{2} \left[ \left\| \mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda \left\| \nabla u \right\|_{F}^2 I - \lambda \left\| \nabla v \right\|_{F}^2 I, \varepsilon(u) - \varepsilon(v) \right] \]
\[ \geq \mu \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)}^2 \]
\[ \geq \mu \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)}^2 \]
\[ + \frac{1}{2} \left[ \left\| \mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda \left\| \nabla u \right\|_{F}^2 I - \lambda \left\| \nabla v \right\|_{F}^2 I \right] \]
\[ \leq \left( \mu - 2c_2 (2\mu N + \frac{N^2 - M^2}{N - M}) \right) \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)} \]
\[ \leq \left( \mu - 2c_2 (2\mu N + \frac{N^2 - M^2}{N - M}) \right) \left\| \varepsilon(u) - \varepsilon(v) \right\|_{L^2(\Omega)}. \]

It is easy to check that
\[ \left\| \mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda \left\| \nabla u \right\|_{F}^2 I - \lambda \left\| \nabla v \right\|_{F}^2 I \right\|_{L^2(\Omega)} \]
\[
\|\mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v\|_{L^2(\Omega)} + \|\lambda\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla v\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla v\|_{F_1^2} \\
= \|\mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v - \mu \nabla^T u \nabla v + \mu \nabla^T v \nabla v\|_{L^2(\Omega)} \\
+ \|\lambda\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla v\|_{F_1^2} \\
\leq \mu N \|\nabla u - \nabla v\|_{L^2(\Omega)} + \mu N \|\nabla u - \nabla v\|_{L^2(\Omega)} + \lambda(N^2 - M^2) \|\mu\|_{F_1^2} \|\mu\|_{F_1^2} \\
\leq c_2(2\mu N + \frac{N^2 - M^2}{N - M}) \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)}.
\]

Taking \(C_4 = \mu - 2c_2(2\mu N + \frac{N^2 - M^2}{N - M}) > 0\), then we get (4.3). The proof is complete. \(\square\)

**Lemma 4.2.** There exists a positive constant \(C_3\) such that

\[
(4.7)\quad \|\nabla(Nu) - \nabla(v)\|_{L^2(\Omega)} \leq C_3 \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)}.
\]

**Proof.** Using the Cauchy-Schwarz inequality and Korn’s inequality, we have

\[
\|\nabla(Nu) - \nabla(v)\|_{L^2(\Omega)} \\
= \|\mu \nabla^T u \nabla u - \mu \nabla^T v \nabla v + \lambda\|\nabla u\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla u\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla v\|_{F_1^2} \|\mu\|_{F_1^2} + \lambda\|\nabla v\|_{F_1^2} \\
\leq \mu N \|\nabla u - \nabla v\|_{L^2(\Omega)} + \mu N \|\nabla u - \nabla v\|_{L^2(\Omega)} + \lambda(N^2 - M^2) \|\mu\|_{F_1^2} \|\mu\|_{F_1^2} \\
+ \mu \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)} \\
= (2\mu N + \frac{N^2 - M^2}{M - N}) \|\nabla u - \nabla v\|_{L^2(\Omega)} + \mu \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)} \\
\leq c_2(2\mu N + \frac{N^2 - M^2}{M - N}) \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)} + \mu \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)} \\
= \left(\mu + c_2(2\mu N + \frac{N^2 - M^2}{M - N})\right) \|\varepsilon(u) - \varepsilon(v)\|_{L^2(\Omega)}.
\]

Taking \(C_3 = \mu + c_2(2\mu N + \frac{N^2 - M^2}{M - N}) > 0\), then (4.7) holds. The proof is complete. \(\square\)
Lemma 4.3. Every weak solution \((u, p)\) of the problem (3.5)–(3.7) satisfies the following energy law:

\[
E(t) + \int_0^t (N(\nabla u), \varepsilon(u)) \, ds + \frac{1}{\mu_f} \int_0^t (K(\nabla p - \rho_f g), \nabla p) \, ds \\
- \int_0^t (\phi, p) \, ds - \int_0^t (\phi_1, p) \, ds = E(0)
\]

for all \(t \in [0, T]\), where

\[
E(t) := \frac{1}{2} \left[ \lambda \| \text{div} u(t) \|^2_{L^2(\Omega)} + c_0 \| p(t) \|^2_{L^2(\Omega)} - 2(f, u(t)) - 2(f_1, u(t)) \right].
\]

Moreover, there holds

\[
\| (c_0 p + \alpha \text{div} u)_t \|_{L^2(0, T; H^1(\Omega)')} \leq \frac{1}{\mu_f} \| K \nabla p - \rho_f g \|_{L^2(\Omega_T)} + \| \phi \|_{L^2(\Omega_T)} + \| \phi_1 \|_{H^1(\Omega_T)} < \infty.
\]

Proof. We only consider the case of \(u_t \in L^2((0, T); L^2(\Omega))\), the general case can be converted into this case using the Steklov average technique (cf. [16, Chapter 2]). Setting \(\varphi = p\) in (3.6) and \(\psi = u_t\) in (3.5) yields for a.e. \(t \in [0, T]\)

\[
\mu(N(\nabla u), \varepsilon(u_t)) + \lambda(\text{div} u, \text{div} u_t) - \alpha(p, \text{div} u_t) = (f, u_t) + (f_1, u_t),
\]

\[
((c_0 p + \alpha \text{div} u)_t, p(t))_{\text{dual}} + \frac{1}{\mu_f} (K(\nabla p - \rho_f g), \nabla p) = (\phi, p) + (\phi_1, p).
\]

Adding the above two equations and integrating the sum in \(t\) over the interval \((0, s)\) for any \(s \in (0, T]\), we have

\[
E(s) + \frac{1}{\mu_f} \int_0^s (K(\nabla p - \rho_f g), \nabla p) \, dt - \int_0^s (\phi, p) \, dt - \int_0^s (\phi_1, p) \, dt = E(0).
\]

Here we have used the fact that \(f\) and \(f_1\) are independent of \(t\). Hence, (4.8) holds. (4.10) follows immediately from (4.8) and (3.6). The proof is complete. \(\Box\)

Likewise, the weak solution of (3.8)–(3.12) satisfy a similar energy law which is a rewritten version of (4.8) in the new variables.

Lemma 4.4. Every weak solution \((u, \xi, \eta)\) of the problem (3.8)–(3.12) satisfies the following energy law

\[
J(t) + \int_0^t (N(\nabla u), \varepsilon(u_t)) \, ds + \frac{1}{\mu_f} \int_0^t (K(\nabla p - \rho_f g), \nabla p) \, ds \\
- \int_0^t (\phi, p) \, ds - \int_0^t (\phi_1, p) \, ds = J(0)
\]
for all \( t \in [0, T] \), where

\[(4.15) \quad J(t) := \frac{1}{2} \left[ \kappa_2 \| \eta(t) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(t) \|_{L^2(\Omega)}^2 - 2 \langle f, u(t) \rangle - 2 \langle f_1, u(t) \rangle \right].\]

Moreover, there holds

\[(4.16) \quad \| \eta \|_{L^2(0,T;H^1(\Omega))} \leq \frac{1}{\mu_f} \| K \nabla p - \rho_f g \|_{L^2(\Omega_T)} + \| \phi \|_{L^2(\Omega_T)} + \| \phi_1 \|_{L^2(\partial \Omega_T)} < \infty.\]

Proof. We only consider the case of \( u_i \in L^2(0,T;L^2(\Omega)) \). Setting \( v = u_i \) in (3.8), differentiating (3.9) with respect to \( t \) followed by taking \( \varphi = \xi \), and setting \( \psi = p = \kappa_1 \xi + \kappa_2 \eta \) in (3.10), we have

\[(4.17) \quad (\nabla (\nabla u), \xi(u_i)) - (\xi, \text{div} u_i) = (f, u_i) + \langle f_1, u_i \rangle \quad \forall v \in H^1(\Omega),\]

\[(4.18) \quad \kappa_3 (\xi, \xi) + (\text{div} u_i, \xi) = \kappa_1 (\eta, \xi) \quad \forall \xi \in L^2(\Omega),\]

\[(4.19) \quad (\eta, p)_{\text{dual}} + \frac{1}{\mu_f} (K (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f g), \nabla p)
= (\phi, p) + \langle \phi_1, p \rangle \quad \forall \psi \in H^1(\Omega).\]

Adding the resulting equations and integrating in \( t \), we see that (4.14) holds. The inequality (4.16) follows immediately from (3.11) and (4.14). The proof is complete.

The above energy law immediately implies the following solution estimates.

Lemma 4.5. There exists a positive constant \( \hat{C}_1 = \hat{C}_1(\| u_0 \|_{H^1(\Omega)}, \| p_0 \|_{L^2(\Omega)}, \| f \|_{L^2(\Omega)}, \| f_1 \|_{L^2(\partial \Omega)}, \| \phi \|_{L^2(\Omega)}, \| \phi_1 \|_{L^2(\partial \Omega)}) \) such that

\[(4.20) \quad \sqrt{C_2} \| \varepsilon(u) \|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\kappa_2} \| \eta \|_{L^\infty(0,T;L^2(\Omega))}
+ \sqrt{\kappa_3} \| \xi \|_{L^\infty(0,T;L^2(\Omega))} + \frac{\sqrt{K_1}}{\mu_f} \| \nabla p \|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_1,\]

\[(4.21) \quad \| u \|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_1, \quad \| p \|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_1 (\kappa_2^{1/2} + \kappa_1 \kappa_3^{-1/2}),\]

\[(4.22) \quad \| p \|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_1, \quad \| \xi \|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_1 \kappa_1^{-1} (1 + \kappa_2^4).\]

Proof. Taking \( \psi = p \) in (3.10) and integrating from 0 to \( t \), we have

\[(4.23) \quad \int_0^t (\eta, p)_{\text{dual}} ds + \int_0^t \frac{1}{\mu_f} (K (\nabla (\kappa_1 \xi + \kappa_2 \eta) - \rho_f g), \nabla p) ds
= \int_0^t ([\phi, p] + (\phi_1, p)] ds.\]
Taking \( v = u_t \) in (3.8) and \( \varphi = \xi \) in (3.9), we have

\[
\begin{align*}
(4.24) \quad \int_0^t (\eta_t, p)_{\text{dual}} \, ds &= \int_0^t (\eta_t, \kappa_1 \xi + \kappa_2 \eta)_{\text{dual}} \, ds \\
&= \int_0^t (\eta_t, \kappa_1 \xi)_{\text{dual}} \, ds + \int_0^t (\eta_t, \kappa_2 \eta)_{\text{dual}} \, ds \\
&= \frac{1}{2} \kappa_2 (\| \eta(t) \|_{L^2(\Omega)}^2 - \| \eta(0) \|_{L^2(\Omega)}^2) + \int_0^t (\eta_t, \kappa_1 \xi)_{\text{dual}} \, ds,
\end{align*}
\]

\[
(4.25) \quad \int_0^t (\eta_t, \kappa_1 \xi)_{\text{dual}} \, ds = \int_0^t (\kappa_1 \xi_t + \kappa_3, \xi)_{\text{dual}} ds \\
= \int_0^t (\nabla u_t, \xi)_{\text{dual}} ds + \frac{1}{2} \kappa_3 (\| \xi(t) \|_{L^2(\Omega)}^2 - \| \xi(0) \|_{L^2(\Omega)}^2) \\
= \int_0^t \left( \langle \nabla (\nabla u), \varepsilon (\xi) \rangle - (f, u_t) - (f_1, u_t) \right) ds \\
+ \frac{1}{2} \kappa_3 (\| \xi(t) \|_{L^2(\Omega)}^2 - \| \xi(0) \|_{L^2(\Omega)}^2).
\]

Substituting (4.24) and (4.25) into (4.23), we get

\[
\begin{align*}
&\int_0^t (\nabla (\nabla u), \varepsilon (\xi)) ds + \frac{1}{2} \left[ \kappa_2 \| \eta(t) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(t) \|_{L^2(\Omega)}^2 \right] \\
&+ \frac{1}{\mu_f} \int_0^t (K (\nabla p - \rho_f g), \nabla p) ds \\
&= \frac{1}{2} \left[ \kappa_2 \| \eta(0) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(0) \|_{L^2(\Omega)}^2 + 2 (f, u(t) - u(0)) + 2 (f_1, u(t) - u(0)) \right] \\
&+ \int_0^t (\phi, p) ds + \int_0^t \langle \phi_1, p \rangle ds.
\end{align*}
\]

Using (4.2), we have

\[
\begin{align*}
C_2 \int_0^t (\varepsilon (u, \varepsilon (u_t)) ds + \frac{1}{2} \left[ \kappa_2 \| \eta(t) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(t) \|_{L^2(\Omega)}^2 \right] \\
+ \frac{1}{\mu_f} \int_0^t (K (\nabla p - \rho_f g), \nabla p) ds \\
\leq \frac{1}{2} \left[ \kappa_2 \| \eta(0) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(0) \|_{L^2(\Omega)}^2 + 2 (f, u(t) - u(0)) + 2 (f_1, u(t) - u(0)) \right] \\
+ \int_0^t (\phi, p) ds + \int_0^t \langle \phi_1, p \rangle ds.
\end{align*}
\]

Hence, (4.20) holds. It’s easy to check that (4.21) holds from (4.20) and the relation \( p = \kappa_1 \xi + \kappa_2 \eta \). We note that (4.22) follows from (4.20), (3.3), the Poincaré inequality and (4.37) below, and the relation \( p = \kappa_1 \xi + \kappa_2 \eta \). The proof is complete. \( \square \)
Theorem 4.6. Suppose that $u_0$ and $p_0$ are sufficiently smooth, then there exist positive constants $\hat{C}_2 = \hat{C}_2(\hat{C}_1, \|\nabla p_0\|_{L^2(\Omega)})$ and $\hat{C}_3 = \hat{C}_3(\hat{C}_1, \hat{C}_2, \|u_0\|_{H^2(\Omega)}, \|p_0\|_{H^2(\Omega)})$ such that

\begin{equation}
\sqrt{C_2} \|\varepsilon(u_t)\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\kappa_2} \|\eta_t\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\kappa_3} \|\xi_t\|_{L^2(0,T;L^2(\Omega))} + \frac{K_1}{\mu_f} \|\nabla p\|_{L^\infty(0,T;L^2(\Omega))} \leq \hat{C}_2,
\end{equation}

\begin{equation}
\sqrt{C_2} \|\varepsilon(u_t)\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\kappa_2} \|\eta_t\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\kappa_3} \|\xi_t\|_{L^\infty(0,T;L^2(\Omega))} + \frac{K_1}{\mu_f} \|\nabla p_t\|_{L^2(0,T;L^2(\Omega))} \leq \hat{C}_3,
\end{equation}

\begin{equation}
\|\eta_t\|_{L^2(H^1(\Omega))} \leq \sqrt{\frac{K_2}{\mu_f}} \hat{C}_3.
\end{equation}

Proof. Differentiating (3.8) and (3.9) with respect to $t$, taking $v = u_t$ and $\varphi = \xi_t$ in (3.8) and (3.9) respectively, and adding the resulting equations, we have

\begin{equation}
\langle N_t(\nabla u), \varepsilon(u_t) \rangle = (q_t, \xi_t) = \kappa_1 (\eta_t, \xi_t) - \kappa_3 \|\xi_t\|^2_{L^2(\Omega)}.
\end{equation}

Setting $\psi = p_t = \kappa_1 \xi_t + \kappa_2 \eta_t$ in (3.10), we get

\begin{equation}
\kappa_1 (\eta_t, \xi_t) + \kappa_2 \|\eta_t\|^2_{L^2(\Omega)} + \frac{K}{2\mu_f} \frac{d}{dt} \|\nabla p - \rho_f g\|^2_{L^2(\Omega)} = \frac{d}{dt} \left[ \langle \phi, p \rangle + \langle \phi_1, p \rangle \right].
\end{equation}

Adding (4.29) and (4.30) and integrating in $t$ we get for $t \in [0, T]$, we have

\begin{align}
\frac{K}{2\mu_f} \|\nabla p(t) - \rho_f g\|^2_{L^2(\Omega)} + \int_0^T \left[ \langle N_t(\nabla u), \varepsilon(u_t) \rangle + \kappa_2 \|\eta_t\|^2_{L^2(\Omega)} + \kappa_3 \|\xi_t\|^2_{L^2(\Omega)} \right] \, ds
&= \frac{K}{2\mu_f} \|\nabla p_0 - \rho_f g\|^2_{L^2(\Omega)} + \langle \phi, p(t) - p_0 \rangle + \langle \phi_1, p(t) - p_0 \rangle,
\end{align}

which implies that (4.26) holds.

To show (4.27), first differentiating (3.8) one time with respect to $t$ and setting $v = u_{tt}$, differentiating (3.9) twice with respect to $t$ and setting $\varphi = \xi_t$, and adding the resulting equations, we get

\begin{equation}
\langle N_t(\nabla u), \varepsilon(u_{tt}) \rangle = (q_{tt}, \xi_t) = \kappa_1 (\eta_{tt}, \xi_t) - \frac{\kappa_3}{2} \frac{d}{dt} \|\xi_t\|^2_{L^2(\Omega)}.
\end{equation}

Secondly, differentiating (3.10) with respect $t$ one time and taking $\psi = p_t = \kappa_1 \xi_t + \kappa_2 \eta_t$, we get

\begin{equation}
\kappa_1 (\eta_{tt}, \xi_t) + \frac{\kappa_2}{2} \frac{d}{dt} \|\eta_t\|^2_{L^2(\Omega)} + \frac{K}{\mu_f} \|\nabla p_t\|^2_{L^2(\Omega)} = 0.
\end{equation}
Finally, adding (4.31)-(4.32) and integrating in $t$, we obtain

$$
2 \int_0^t \left( \mathcal{N_t}(\nabla \mathbf{u}), \varepsilon(\mathbf{u}_t) \right) ds + \kappa_2 \| \eta(t) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(t) \|_{L^2(\Omega)}^2 \\
+ \frac{2K}{\mu_f} \int_0^t \| \nabla p_t \|_{L^2(\Omega)} ds = \kappa_2 \| \eta(0) \|_{L^2(\Omega)}^2 + \kappa_3 \| \xi(0) \|_{L^2(\Omega)}^2,
$$

which implies that (4.27) holds. (4.28) follows immediately from the following inequality

$$
(\eta_t, \psi) = -\frac{1}{\mu_f} \left( K \nabla p_t, \nabla \psi \right) \leq \frac{K}{\mu_f} \| \nabla p_t \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)},
$$

(4.27) and the definition of the $H^1(\Omega)'$-norm. The proof is complete. \Box

**Remark 4.1.** The above estimates require $p_0 \in H^1(\Omega)$, $\mathbf{u}_t(0) \in L^2(\Omega)$, $\eta_t(0) \in L^2(\Omega)$ and $\xi_t(0) \in L^2(\Omega)$. The values of $\mathbf{u}_t(0)$, $\eta_t(0)$ and $\xi_t(0)$ can be computed using the PDEs as follows. It follows from (2.13) that $\eta_t(0)$ satisfies

$$
\eta_t(0) = \phi + \frac{1}{\mu_f} \text{div} \left[ K(\nabla p_0 - \rho_f g) \right].
$$

Hence, $\eta_t(0) \in L^2(\Omega)$ provided that $p_0 \in H^2(\Omega)$. To find $\mathbf{u}_t(0)$ and $\xi_t(0)$, differentiating (2.11) and (2.12) with respect to $t$ and setting $t = 0$, we get

$$
\begin{align*}
-\text{div} \mathcal{N}(\nabla \mathbf{u}_t(0)) + \nabla \xi_t(0) &= 0 \quad \text{in } \Omega, \\
\kappa_3 \xi_t(0) + \text{div} \mathbf{u}_t(0) &= \kappa_1 \eta_t(0) \quad \text{in } \Omega.
\end{align*}
$$

Hence, $\mathbf{u}_t(0)$ and $\xi_t(0)$ can be determined by solving the above generalized Stokes problem.

The next lemma shows that the weak solution of the problem (3.8)-(3.12) preserves some “invariant” quantities, it turns out that these “invariant” quantities play a vital role in the proof of existence and uniqueness of the weak solution to the reformulated fluid-fluid coupling system.

**Lemma 4.7.** Every weak solution $(\mathbf{u}, \xi, \eta, p, q)$ to the problem (3.8)-(3.12) satisfies the following energy laws

$$
\begin{align*}
C_q(t) &:= (\eta(\cdot, t), 1) = (\eta_0, 1) + [(\phi, 1) + \langle \phi_1, 1 \rangle] t, \quad t \geq 0, \\
C_\xi(t) &:= (\xi(\cdot, t), 1), \\
C_q(t) &:= (q(\cdot, t), 1) = \kappa_1 C_q(t) - \kappa_3 C_\xi(t), \\
C_p(t) &:= (p(\cdot, t), 1) = \kappa_1 C_\xi(t) + \kappa_2 C_\eta(t), \\
C_u(t) &:= \langle \mathbf{u}(\cdot, t) \cdot \mathbf{n}, 1 \rangle = C_q(t).
\end{align*}
$$
Proof. We first notice that (4.34) follows immediately from taking \( \psi \equiv 1 \) in (3.10). To prove (4.35), taking \( \mathbf{v} = \mathbf{x} \) in (3.8) and \( \varphi = 1 \) in (3.9), which are valid test functions, and using the identities \( \nabla \mathbf{x} = \mathbf{I} \), \( \text{div} \mathbf{x} = 0 \), and \( \varepsilon(\mathbf{x}) = \mathbf{I} \), we get
\[
\left( \mathcal{N}(\nabla \mathbf{u}), \mathbf{I} \right) = d(\xi, 1) + \langle \mathbf{f}, \mathbf{x} \rangle + \langle \mathbf{f}_1, \mathbf{x} \rangle,
\]
\[
(\text{div} \mathbf{u}, 1) = \kappa_1(\eta, 1) - \kappa_3(\xi, 1).
\]
It is easy to check that
\[
C_\xi(t) := (\xi(t), 1) = \frac{1}{d - \kappa_3}[\left( \mathcal{N}(\nabla \mathbf{u}), \mathbf{I} \right) + (\text{div} \mathbf{u}, 1) - \kappa_3 C_\eta(t) - (\mathbf{f}, \mathbf{x}) - \langle \mathbf{f}_1, \mathbf{x} \rangle],
\]
which implies that (4.35) holds.

Finally, since \( q = \kappa_1 \eta - \kappa_3 \xi \), \( q = \kappa_1 \xi + \kappa_2 \eta \), (4.36) and (4.37) follow from (4.34) and (4.35). (4.38) is an immediate consequence of \( q = \text{div} \mathbf{u} \) and the Gauss divergence theorem. The proof is complete. \( \square \)

With the help of the above lemmas, we can show the solvability of the problem (2.1)-(2.7).

Theorem 4.8. Let \( \mathbf{u}_0 \in H^1(\Omega), \mathbf{f} \in L^2(\Omega), f_1 \in L^2(\partial \Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega), \phi_1 \in L^2(\partial \Omega) \). Suppose \( c_0 > 0 \) and \( \langle \mathbf{f}, \mathbf{v} \rangle + \langle f_1, \mathbf{v} \rangle = 0 \) for any \( \mathbf{v} \in \mathbb{R}^M \).

Then there exists a unique weak solution to the problem (2.1)-(2.7) in the sense of Definition 3.1. Likewise, there exists a unique weak solution to the problem (2.11)-(2.16) in the sense of Definition 3.2.

Proof. We first prove the existence of solution of the problem (3.8)-(3.12). Given a function \( \mathbf{u} \in L^\infty(0, T; H^1_\perp(\Omega)) \), supposing that \( U \subset L^\infty(0, T; H^1_\perp(\Omega)) \) is a compact and convex subspace, defining \( g(t) := -\mu \nabla^T \mathbf{u} \nabla \mathbf{u} - \lambda \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 \mathbf{u}(0 \leq t \leq T) \), we have
\[
\left( \mu \nabla^T \mathbf{u} \nabla \mathbf{u} + \lambda \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 \right) \leq \mu \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 + \lambda dN^2
\]
\[
\leq (\mu N + \frac{\lambda dN^2}{M}) \| \nabla \mathbf{u} \|_{L^2(\Omega)}. \]

In the sight of (4.39), we see that \( g \in L^2(0, T; \mathbf{L}^2(\Omega)) \). Denote \( \mathbf{V} = L^\infty(0, T; \mathbf{H}^1_\perp(\Omega)), \mathbf{Y} = L^\infty(0, T; \mathbf{L}^2(\Omega)), \mathbf{W} = L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \). We consider the linear problem: find \( (\mathbf{w}, \xi, \eta) \in \mathbf{V} \times \mathbf{Y} \times \mathbf{W} \) satisfying
\[
\mu(\varepsilon(\mathbf{w}), \varepsilon(\mathbf{v})) - (\xi, \text{div} \mathbf{v}) = (g, \mathbf{v}) + (f_1, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),
\]
\[
\kappa_3(\xi, \varphi) + (\text{div} \mathbf{w}, \varphi) = \kappa_1(\eta, \varphi) \quad \forall \varphi \in \mathbf{L}^2(\Omega),
\]
\[
(d_\eta, \psi) + \frac{1}{\mu_f} (K(\nabla(\kappa_1 \xi + \kappa_2 \eta))
\]

\[ - \rho_f g, \nabla \psi) = (\phi, \psi) + (\phi_1, \psi), \quad \forall \psi \in H^1(\Omega). \]

As for the equations of (2.12)-(2.13), according to the theory of linear parabolic equations, we know that $\xi$ and $\eta$ can be uniquely determined by $w$, that is, $\exists \Phi$ and $\Psi$, s.t. $\xi = \Phi(w)$, $\eta = \Psi(w)$. Thus, the problem (4.40)-(4.42) is equivalent to the following problem

\begin{equation}
\begin{aligned}
\text{(4.43)} \quad & \text{Solve } w \in V \text{ such that} \\
& \mu(\varepsilon(w), \varepsilon(v)) + (\Phi(w), \text{div } v) = (g, v) + (f, v) + (f_1, v) \quad \forall v \in H^1(\Omega).
\end{aligned}
\end{equation}

Following the method of [10], we can prove that the solution of (4.43) uniquely exists, here we omit the details of proof.

Define $A : V \to V$ by $A[u] = w$. Similarly, it’s easy to know the following problem is equivalent to the problem (3.8)-(3.10).

\begin{equation}
\begin{aligned}
\text{(4.44)} \quad & \text{Solve } u \in V \text{ such that} \\
& (A(\nabla u), \varepsilon(v)) + (\Phi(u), \text{div } v) = (f, v) + (f_1, v) \quad \forall v \in H^1(\Omega).
\end{aligned}
\end{equation}

Next, we prove that $A$ is continuous. To do that, choose $u, \tilde{u}$ and define $w = A[u]$, $\tilde{w} = A[\tilde{u}]$ as above. Consequently $w$ verifies (4.40)-(4.42) and $\tilde{w}$ satisfies a similar identity for $\tilde{g} = -\mu \nabla^T \tilde{u} \nabla \tilde{u} - \lambda \| \nabla \tilde{u} \|^2_L$. $I$.

Using (3.3), (4.40) and the Young inequality, we have

\[
\begin{aligned}
& \mu \| \tilde{w} - w \|^2_{L^2(\Omega)} + c_1^2(\Phi(\tilde{w}) - \Phi(w), \tilde{w} - w) \\
& \leq c_1^2 \mu \| \varepsilon(\tilde{w}) - \varepsilon(w) \|^2_{L^2(\Omega)} + c_1^2(\Phi(\tilde{w}) - \Phi(w), \tilde{w} - w) \\
& = c_1^2(\tilde{g} - g, \tilde{w} - w) \\
& \leq c_1^2 \left[ \epsilon \| \tilde{w} - w \|^2_{L^2(\Omega)} + \frac{1}{\epsilon} \| \tilde{g} - g \|^2_{L^2(\Omega)} \right]
\end{aligned}
\]

by Poincaré inequality, where $c_1$ is a real positive constant in (3.3). Selecting $\epsilon > 0$ sufficiently small, we have

\[
\begin{aligned}
& c_1^2(\Phi(\tilde{w}) - \Phi(w), \tilde{w} - w) \leq C_f \| \tilde{g} - g \|^2_{L^2(\Omega)} \leq C_f \| \tilde{u} - u \|^2_{L^2(\Omega)}.
\end{aligned}
\]

where $C_f$ is a real positive number. We discover

\[
\begin{aligned}
& \| A[\tilde{u}] - A[u] \|^2_{L^2(\Omega)} = \| \tilde{w} - w \|^2_{L^2(\Omega)} \leq C_f \| \tilde{u} - u \|^2_{L^2(\Omega)}.
\end{aligned}
\]

Thus, we get

\[
\| A[\tilde{u}] - A[u] \|_{L^2(\Omega)} \leq \sqrt{C_f} \| \tilde{u} - u \|_{L^2(\Omega)}.
\]
Hence, the solution of the problem (3.8)-(3.12) is unique. The proof is complete.

Thus, using (4.47) and the initial value (4.47), we obtain

\[
\begin{align*}
N(\nabla u_1) - N(\nabla u_2), \varepsilon(v) - (\xi_1 - \xi_2, \nabla \cdot v) &= 0 \quad \forall v_h \in H^1(\Omega), \\
\kappa_3 (\xi_1 - \xi_2, \varphi) + (\nabla \cdot u_1 - \nabla \cdot u_2, \varphi) &= 0 \quad \forall \varphi_h \in L^2(\Omega).
\end{align*}
\]

Adding (4.45) and (4.46), letting \( v = u_1 - u_2, \varphi = \xi_1 - \xi_2, \) using (4.7), we have

\[
0 \leq C_3 \| \varepsilon(u) - \varepsilon(v) \|_{L^2(\Omega(t))}^2 + \kappa_3 \| \xi_1 - \xi_2 \|_{L^2(\Omega(t))}^2 = 0.
\]

Thus, using (4.47) and the initial value \( u_0, \) we obtain

\[
u_1 = u_2, \quad \xi_1 = \xi_2.
\]

Since \( p = \kappa_1 \xi + \kappa_2 \eta, \) \( q = \kappa_1 \eta - \kappa_3 \xi, \) so we have

\[
p_1 = p_2, \quad q_1 = q_2.
\]

Hence, the solution of the problem (3.8)-(3.12) is unique. The proof is complete.

We conclude this section by establishing a convergence result for the solution of the problem (2.11)-(2.13), (2.14)-(2.16) when the constrained specific storage coefficient \( c_0 \) tends to 0. Such a convergence result is useful and significant for that the poroelasticity model studied in this paper reduces into the nonlinear Biot’s consolidation model from soil mechanics [17, 20] and some polymer gels [10, 24].

**Theorem 4.9.** Let \( u_0 \in H^1(\Omega), f \in L^2(\Omega), f_1 \in L^2(\partial \Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega), \) and \( \phi_1 \in L^2(\partial \Omega). \) Suppose \( (f, v) + (f_1, v) = 0 \) for any \( v \in \mathbb{R}^M. \) Let \( (u_{c_0}, \eta_{c_0}, \xi_{c_0}) \) denote the unique weak solution to the problem (2.11)-(2.13). Then there exists \((u_*, \eta_*, \xi_*) \in L^\infty(0,T;H^1_0(\Omega)) \times L^\infty(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Omega))\) such that \((u_{c_0}, \eta_{c_0}, \xi_{c_0}) \) converges weakly to \((u_*, \eta_*, \xi_*) \) as \( c_0 \to 0. \)

**Proof.** It follows immediately from (4.26)-(4.28) and Korn’s inequality that

- \( u_{c_0} \) is uniformly bounded (in \( c_0 \)) in \( L^\infty(0,T;H^1_0(\Omega)); \)
- \( \sqrt{\kappa_3} \eta_{c_0} \) is uniformly bounded (in \( c_0 \)) in \( L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)^*); \)
- \( \sqrt{\kappa_3} \xi_{c_0} \) is uniformly bounded (in \( c_0 \)) in \( L^\infty(0,T;L^2(\Omega)); \)
\[ \xi_{c_0} \] is uniformly bounded (in \( c_0 \)) in \( L^2(\Omega) \).

On noting that \( \lim_{c_0 \to 0} \kappa_1 = \frac{1}{\alpha}, \lim_{c_0 \to 0} \kappa_2 = \frac{1}{\alpha^2} \) and \( \lim_{c_0 \to 0} \kappa_3 = 0 \), by the weak compactness of reflexive Banach spaces and Aubin-Lions Lemma (cf. [5]), we know that there exist \( (u_*, \eta_*, \xi_*) \in L^\infty(0, T; H^1_\perp(\Omega)) \times L^\infty(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)) \) and a subsequence of \( (u_{c_0}, \eta_{c_0}, \xi_{c_0}) \) (still denoted by the same notation) such that as \( c_0 \to 0 \) (a subsequence of \( c_0 \), to be exact)

- \( u_{c_0} \) converges to \( u_* \) weak \(*\) in \( L^\infty(0, T; H^1(\Omega)) \) and weakly in \( L^2(0, T; H^1(\Omega)) \);
- \( \sqrt{\kappa_2} \eta_{c_0} \) converges to \( \frac{\kappa}{\alpha} \eta_* \) weak \(*\) in \( L^\infty(0, T; L^2(\Omega)) \) and strongly in \( L^2(\Omega) \);
- \( \kappa_3 \xi_{c_0} \) converges to 0 strongly in \( L^2(\Omega) \);
- \( \xi_{c_0} \) converges to \( \xi_* \) weakly in \( L^2(\Omega) \).

Using (4.7), we deduce that \( \mathcal{N}(\nabla u_{c_0}) \) converges to \( \mathcal{N}(\nabla u_*) \) weak \(*\) in \( L^\infty(0, T; H^1_\perp(\Omega)) \) and weakly in \( L^2(0, T; H^1(\Omega)) \).

Then, setting \( c_0 \to 0 \) in (3.8)-(3.12) yields

\[
\begin{align*}
&\left(\mathcal{N}(\nabla u_*), \varepsilon(v)\right) - \left(\xi_*, \text{div } v\right) = \left(f, v\right) + \left(f_1, v\right) \quad \forall v \in H^1(\Omega), \\
&\left(\text{div } u_*, \varphi\right) = \frac{1}{\alpha} \left(\eta_*, \varphi\right) \quad \forall \varphi \in L^2(\Omega), \\
&\left((\eta_*)_\perp, \psi\right)_\perp + \frac{1}{\mu_f} \left(K(\nabla \eta_* - \rho_f g), \nabla \psi\right) \quad = \left(\phi, \psi\right) + \left(\phi_1, \psi\right) \quad \forall \psi \in H^1(\Omega), \\
p_* := \frac{1}{\alpha} \xi_* + \frac{\lambda}{\alpha^2} \eta_*, \quad q_* := \frac{1}{\alpha} \eta_* \quad \text{in } L^2(\Omega), \\
u_0(0) = u_0, \\
\eta_0(0) = \eta_0 := \alpha q_0.
\end{align*}
\]

That is,

\[
\begin{align*}
&\left(\mathcal{N}(\nabla u_*), \varepsilon(v)\right) - \left(\xi_*, \text{div } v\right) = \left(f, v\right) + \left(f_1, v\right) \quad \forall v \in H^1(\Omega), \\
&\left(\text{div } u_*, \varphi\right) = \left(q_*, \varphi\right) \quad \forall \varphi \in L^2(\Omega), \\
&\alpha \left((q_*)_\perp, \psi\right)_\perp + \frac{1}{\mu_f} \left(K(\nabla (\lambda \alpha^{-1} q_* + \alpha^{-1} \xi_* - \rho_f g), \nabla \psi\right) \quad = \left(\phi, \psi\right) + \left(\phi_1, \psi\right) \quad \forall \psi \in H^1(\Omega), \\
p_* := \frac{1}{\alpha} \left(\xi_* + \lambda q_*\right) \quad \text{in } L^2(\Omega), \\
q_0(0) = q_0 := \text{div } u_0.
\end{align*}
\]

Hence, \( (u_*, \eta_*, \xi_*) \) is a weak solution of the nonlinear Biot’s consolidation model (cf. [10,24]). Using Theorem 4.8, we conclude that the whole sequence \( (u_{c_0}, \eta_{c_0}, \xi_{c_0}) \) converges to \( (u_*, \eta_*, \xi_*) \) as \( c_0 \to 0 \) in the above sense. The proof is complete. \( \blacksquare \)
5. Conclusion. In this paper, we deal with the nonlinear poroelasticity model with the constitutive relation $\tilde{\sigma}(u) = \mu \tilde{\varepsilon}(u) + \lambda \text{tr}(\tilde{\varepsilon}(u))I$, where the deformed Green strain tensor is $\tilde{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla^T u + 2\nabla^T u \nabla u)$. To better describe the process of deformation and diffusion underlying in the original model, we firstly reformulate the nonlinear poroelasticity by a multiphysics approach which transforms the nonlinear fluid-solid coupling problem to a fluid-fluid coupling problem. Then, we adopt the similar technique of proving the well-posedness of nonlinear Stokes equations to prove the existence and uniqueness of weak solution of a nonlinear poroelasticity model. And we strictly prove the growth, coercivity and monoticity of the nonlinear stress-strain relation by using the Cauchy-Schwarz inequality and some other inequalities, give the energy estimates and use Schauder’s fixed point theorem to show the existence and uniqueness of weak solution of the nonlinear poroelasticity model. Besides, we prove that the weak solution of nonlinear poroelasticity model converges to the nonlinear Biot’s consolidation model as the constrained specific storage coefficient trends to zero. To the best of our knowledge, it is the first time to prove the existence and uniqueness of weak solution based on a multiphysics approach without any adding condition on the nonlinear stress-strain relation. Besides, we find out that the multiphysics approach is key to propose a stable numerical method for the nonlinear poroelasticity, and we will give the main results of some relative numerical method for the nonlinear poroelasticity in the future work.

REFERENCES

[1] M. Biot, Theory of elasticity and consolidation for a porous anisotropic media, Journal of Applied Physics, 1955, 26(2): 182-185.
[2] S. Brenner, L. Scott, The Mathematical Theory of Finite Element Methods, third edition, Springer, 2008.
[3] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[4] O. Coussy, Poromechanics, Wiley & Sons, England, 2004.
[5] R. Dautray, J. Lions, Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 1, Springer Verlag, 1990.
[6] M. Doi, S. Edwards, The Theory of Polymer Dynamics, Clarendon Press, Oxford, 1986.
[7] L. Evans, Partial Differential Equations, American Mathematical Society, 2016.
[8] X. Feng, Z. Ge, Y. Li, Multiphysics finite element methods for a poroelasticity model, arXiv:1411.7464, [math.NA], (2014).
[9] X. Feng, Z. Ge, Y. Li, Analysis of a multiphysics finite element method for a poroelasticity model, IMA Journal of Numerical Analysis, 2018, 38: 330-359.
[10] X. Feng, Y. He, Fully discrete finite element approximations of a polymer gel model, SIAM Journal Numerical Analysis, 2010, 48: 2186-2217.
[11] M. Ferronato, N. Castelletto, G. Gambolati, A fully coupled 3-D mixed finite element model
of Biot consolidation, Journal of Computational Physics, 2010, 229(12): 4813C4830.

[12] D. Gawin, P. Baggio, B. Schrefler, Coupled heat, water and gas flow in deformable porous media, International Journal for Numerical Methods in Fluids, 1995, 20: 969C978.

[13] V. Girault, P. Raviart, Finite Element Method for Navier-Stokes Equations: theory and algorithms, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[14] I. Hamley, Introduction to Soft Matter, John Wiley & Sons, 2007.

[15] J. Hudson, O. Stephansson, J. Andersson, C. Tsang, L. Ling, Coupled TCHCM Issues related to radioactive waste repository design and performance, International Journal of Rock Mechanics and Mining Sciences, 2001, 38: 143C161.

[16] O. Ladyženskaja, V. Solonnikov, N. Uráiceva, Linear and quasilinear equations of parabolic type, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, 1967.

[17] M. Murad, A. Loula, Improved accuracy in finite element analysis of Biot’s consolidation problem, Computer Methods in Applied Mechanics and Engineering, 1992, 95: 359-382.

[18] D. Nemec, J. Levec, Flow through packed bed reactors: 1. single-phase flow, Chemical Engineering Science, 2005, 60: 6947-6957.

[19] W. Pao, R. Lewis, I. Masters, A fully coupled hydro-thermo-poro-mechanical model for black oil reservoir simulation, International Journal for Numerical and Analytical Methods in Geomechanics, 2001, 25: 1229C1256.

[20] P. Phillips, M. Wheeler, A coupling of mixed and continuous Galerkin finite element methods for poroelasticity I: the continuous in time case, Computational Geosciences, 2007, 11: 131-144.

[21] R. Showalter, Diffusion in poro-elastic media, Journal of Mathematical Analysis and Applications, 2000, 251: 310-340.

[22] R. Temam, Navier-Stokes Equations, Studies in Mathematics and its Applications, Vol. 2, North-Holland, 1977.

[23] A. Vuong, L. Yoshihara, W. Wall, A general approach for modeling interacting flow through porous media under finite deformations, Computer Methods in Applied Mechanics and Engineering, 2015, 283: 1240C1259.

[24] T. Yamaue, M. Doi, Swelling dynamics of constrained thin-plate under an external force, Physical Review E, 2004, 70: 011401.

[25] Z. Zhu, Discussion of nonlinear strain, Advances in Mechanics, 1983, 13(3): 259-272.