Integrability and scattering of the boson field theory on a lattice

Manuel Campos∗, Esperanza López and Germán Sierra

Instituto de Física Teórica, Universidad Autónoma de Madrid, Cantoblanco, Madrid, Spain

E-mail: manuel.campos@uam.es

Received 20 September 2020, revised 4 December 2020
Accepted for publication 22 December 2020
Published 12 January 2021

Abstract
A free boson on a lattice is the simplest field theory one can think of. Its partition function can be easily computed in momentum space. However, this straightforward solution hides its integrability properties. Here, we use the methods of exactly solvable models, that are currently applied to spin systems, to a massless and massive free boson on a 2D lattice. The Boltzmann weights of the model are shown to satisfy the Yang–Baxter equation with a uniformization given by trigonometric functions in the massless case, and Jacobi elliptic functions in the massive case. We diagonalize the row-to-row transfer matrix, derive the conserved quantities, and implement the quantum inverse scattering method. Finally, we construct two factorized scattering S matrix models for continuous degrees of freedom using trigonometric and elliptic functions. These results place the free boson model in 2D in the same position as the rest of the models that are exactly solvable à la Yang–Baxter, offering possible applications in quantum computation.

Keywords: integrable systems, boson field theory, field theory on a lattice, factorizable scattering theory

(Some figures may appear in colour only in the online journal)

1. Introduction

Exactly solvable models in statistical mechanics and condensed matter physics have played a key role in the study of low-dimensional many body systems [1–7]. Together with field theoretical techniques, such as conformal field theory [8–10], and numerical methods based on tensor networks [11–13], they have led to a precise description of non perturbative phenomena as the fractionalization of the spin in antiferromagnetic spin chains and the spin-charge separation in one dimensional metals. Exactly solvable models have also appeared in the AdS/CFT duality

∗Author to whom any correspondence should be addressed.
in the form of spin chain Hamiltonians that describe the dilation operator of the $N = 4$ super Yang–Mills theory [14–16]. More recently, the algebraic Bethe ansatz has been formulated using tensor networks that allow for the application of novel numerical techniques and possible extensions to 2D [17, 18]. The list of exactly solved models is rather large: Ising, Potts, XX, XXZ and XYZ spin chains, Hubbard, $t - J$, etc. They are all characterized by Hamiltonians that commute with an infinite number of conserved quantities in involution. These operators can be derived from the Boltzmann weights of the corresponding partition functions that satisfy the Yang–Baxter equation (YBE).

The aim of this paper is to study the integrability of a free boson on a two dimensional square lattice. This model is solvable by elementary techniques like Fourier analysis if there is translational invariance. However, as far as we know, its integrability has not been studied using the tools of exactly solvable models like the Bethe ansatz or the quantum inverse scattering method (QISM) that rely on the YBE. In the models mentioned above the local degrees of freedom are discrete, e.g. spin in the Ising or XXZ models, fermions in the Hubbard model, etc. In the boson model we have to deal with continuous degrees of freedom given by the real values of the scalar field. Despite of this fact, we shall show that the techniques mentioned above can be applied directly obtaining new knowledge about this fundamental model in statistical mechanics and quantum field theory.

Another topic that we address in this paper is the construction of factorized scattering models using the Boltzmann weights of the free boson on a lattice. The former models describe the elastic scattering of particles, typically solitons, in a relativistic quantum field theory with an infinite number of conserved quantities. The scattering of these particles can be factorized into the product of two-particle scattering amplitudes that, for consistency, satisfy the YBE. It turns out that some solutions of the YBE can be used as Boltzmann weights of a statistical mechanical model or, alternatively, as scattering $S$ matrices in a relativistic quantum field theory with the appropriate identifications of variables. Well known examples of this dual application are the six-vertex model versus the sine-Gordon model, and the Baxter’s eight-vertex model [19] versus the Zamolodchikov’s elliptic sine-Gordon model [20]. We shall show that the Boltzmann weights of the boson model can be promoted to scattering $S$ matrices with the special feature that the particles carry a continuous degree of freedom, unlike the more common models where it is discrete. We construct two $S$ matrix models, one using the trigonometric functions, and another using Jacobi elliptic functions. Interestingly, they are similar to those proposed by Mussardo and Penati for the elliptic version of the sinh-Gordon model [21].

The paper is organized as follows. In section 2 we show that the $R$-matrix associated to the discretized free boson theory satisfies the YBE, both for the massless and massive cases. The row-to-row transfer matrix is explicitly constructed in section 3. From the diagonalization of the transfer matrix we recover the spectrum of the theory and obtain the expectation values of a tower of mutually commuting charges. Section 4 is devoted to the QISM. We find that the $R$-matrix formally coincides with the euclidean propagator of a harmonic oscillator. Using this result, we propose an operator expression for the conserved charges. A relativistic $S$-matrix satisfying the axioms of factorized scattering theory is constructed in section 5. Section 6 contains our conclusions. The paper ends with several appendices where technical details avoided in the main body are presented.

2. The boson field theory model

We consider a free scalar of mass $m_0$ living on a 2D lattice with periodic boundary conditions. The euclidean partition function of the model is
\[ Z = \int d\phi_j \prod_{ij} e^{-\frac{1}{2} \sum a_i a_j \left[ \frac{\phi_j - \phi_{j+1}}{a_i^2} + \frac{\phi_j - \phi_{j+1}}{a_j^2} + m_0^2 \phi_j^2 \right]}. \] (1)

where \( a_x \) and \( a_\tau \) denote the lattice spacings in the spatial and euclidean time directions, and \( \phi_j \in \mathbb{R} \). The interactions described by (1) are pairwise between the variables at neighbour lattice sites. We shall reformulate this partition function as that of a vertex model in statistical mechanics. The variables will live on the edges and the interactions take place at the vertices of a lattice whose orientation is 45° degrees rotated with respect to the original one.

This model was studied in reference [22] using the tensor network renormalization that combines renormalization group ideas with quantum information techniques. The analysis was carried out for an isotropic lattice with \( a_x = a_\tau = 1 \). The continuum limit of (1) is the standard partition function of a massive boson in 2 euclidean dimensions.

We want to express \( R_{12} \) as the evolution operator in euclidean time \( t \) of a Hamiltonian \( H_{12} \) acting on the Hilbert space \( L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \).

### 2.1. YBE: massless case

We will revisit the integrability properties of the discretized boson model using the standard techniques of exactly solvable models. In this section we shall focus on the massless case and show that it satisfies the YBE. The Boltzmann weights described in (2) allow to define a map \( R : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \), where \( L_2(\mathbb{R}) \) is the Hilbert space of squared integrable functions on the real line \( \mathbb{R} \). This operator is known as \( R \)-matrix when they act on finite dimensional spaces, and we shall keep the same notation. \( R \)-matrices depend on a variable that parameterizes a one-dimensional family of models, \( R \equiv R(c) \), and which is crucial to formulate the YBE. In order to study the free boson on a lattice from this point of view, we need to identify a variable playing such a role.

Typically \( R \)-matrices trivialize for some value of \( c \), which we will take to be \( c = 0 \). Namely \( R(0) = I \). This motivates the simple choice \( c = \frac{a_x}{a_\tau} \) and the definition

\[ R_{x_1y_1}^{x_2y_2}(c) = \frac{1}{2\pi c} e^{-\frac{1}{2c^2} \left[ 4(x_1 - y_1)^2 + 4(x_2 - y_2)^2 + 4(x_1 - x_2)^2 + 4(y_1 - y_2)^2 \right]}. \] (3)

The field variables in the vertex (2) have been renamed as \( x_{1,2} \) and \( y_{1,2} \) for simplicity. A normalization factor has been added such that

\[ \lim_{c \to 0} R_{x_1y_1}^{x_2y_2}(c) = \delta(x_1 - y_1)\delta(x_2 - y_2), \] (4)

since \( \frac{1}{\sqrt{2\pi c^2}} e^{-\frac{x^2}{2c^2}} \) approaches a delta function as \( c \) vanishes. In order to avoid confusion we will always use boldface letters to refer to operators and regular letters to describe their components.
The $R$-matrix components can be represented graphically as

\[
\begin{align*}
&\begin{array}{c}
y_1 \\
x_1 \\
x_2 \\
y_2
\end{array} \\
&\begin{array}{c}
y_3 \\
x_3 \\
x_4 \\
y_4
\end{array}
\end{align*}
\]

with the internal lines in the green shaded vertex corresponding to the four terms in the exponent of (3). The vertical lines represent the terms multiplied by $c^{-1}$, and the dotted horizontal ones those multiplied by $c$. In the extreme anisotropic limit (4), the links associated to the horizontal lines disappear.

Exactly solvable vertex models in statistical mechanics are those whose Boltzmann weights satisfy the YBE. This equation guarantees the existence of commuting row-to-row transfer matrices whose expansion in a variable generates an infinite number of conserved quantities. The YBE for the boson model reads

\[
(R(c_3) \otimes I)(I \otimes R(c_2))(R(c_1) \otimes I) = (I \otimes R(c_1))(R(c_2) \otimes I)(I \otimes R(c_3)).
\]

Its graphical representation is

\[
\begin{align*}
&\begin{array}{c}
y_1 \\
x_1 \\
x_2 \\
y_2
\end{array} \\
&\begin{array}{c}
y_3 \\
x_3 \\
x_4 \\
y_4
\end{array}
\end{align*} = \begin{align*}
&\begin{array}{c}
y_3 \\
x_3 \\
x_4 \\
y_4
\end{array} \\
&\begin{array}{c}
y_1 \\
x_1 \\
x_2 \\
y_2
\end{array}
\end{align*}
\]

It reads in components

\[
\int d\vec{z} R_{xy}^{z_1 z_2} (c_1) R_{xy}^{z_2 z_3} (c_2) R_{xy}^{z_3 z_1} (c_3) = \int d\vec{z} R_{xy}^{z_2 z_1} (c_3) R_{xy}^{z_1 z_2} (c_2) R_{xy}^{z_2 z_1} (c_1),
\]

where $\vec{z} = (z_1, z_2, z_3)$ and each integration runs over $\mathbb{R}$. The integration replaces the sum over a finite set of variables in the standard spin models.

We shall next determine the conditions that the parameters $c_i = 1, 2, 3$ must verify in order to fulfill (8). The weights (3) satisfy

\[
R_{xy}^{z_1 z_2} (c) = R_{yx}^{z_2 z_1} (c),
\]

which is equivalent to the invariance of the vertex (5) under a $180^\circ$ rotation. This discrete symmetry implies that the $rhs$ of the YBE equals the $lhs$ with the roles of $x$ and $y$ variables exchanged. Therefore it is enough to require that the $lhs$ of (8) defines a matrix invariant under $x_i \leftrightarrow y_i$. The $lhs$ of (8) can be rewritten as

\[
\int d\vec{z} e^{-\frac{1}{2}(\vec{x}, \vec{y}) M (\vec{z}, \vec{y})}, \quad M = \begin{pmatrix}
M_{xx} & M_{xy} & M_{xz} \\
M_{yx} & M_{yy} & M_{yz} \\
M_{zx} & M_{zy} & M_{zz}
\end{pmatrix}.
\]
with \( \vec{x} = (x_1, x_2, x_3) \) and \( \vec{y} = (y_1, y_2, y_3) \), and \( M \) a symmetric matrix whose entries are blocks of size 3 \( \times \) 3. The explicit expression of \( M \) is given in appendix A. The \( z \)-integrations can be easily performed, obtaining
\[
e^{-\frac{1}{2} \left( \vec{x}^2 + \vec{y}^2 \right)} N = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} - \begin{pmatrix} M_{xx} \\ M_{xy} \end{pmatrix} M_{yy}^{-1} \begin{pmatrix} M_{xx} & M_{yx} \end{pmatrix} . \quad (11)
\]
The YBE translates into requiring \( N_{xx} = N_{yy} \) and \( N_{xy} = N_{yx} \). In appendix A we show that these conditions hold if and only if
\[
c_2 = \frac{c_1 + c_3}{1 - c_1 c_3} . \quad (12)
\]
The relation between the parameters \( c_i \) is uniformized by the function \( c(u) = \tan(u) \), becoming equivalent to
\[
c_1 = c(u), \quad c_2 = c(u + v), \quad c_3 = c(v) . \quad (13)
\]
The physical interpretation of \( R(u) \) as the Boltzmann weights of the massless boson requires \( c \) to be positive, which is crucial for the regularity of the large field limit and the convergence of the integration (10) over the internal \( z \)'s variables. Therefore, the set where \( u \) can take values has to be restricted. We choose it to be the interval \( [0, \frac{\pi}{2}] \), such that in the limit of small \( c \) we have \( c \approx u \). This sort of restriction does not arise in other models like the 6 vertex where the parameter \( u \) can take any complex value.

### 2.2. YBE: massive case

We introduce the following modification of the \( R \)-matrix
\[
R_{x_1x_2}^{y_1y_2}(c, \tilde{m}) = R_{x_1x_2}^{y_1y_2}(c) e^{-\frac{1}{2} \tilde{m}^2 \left( x_1^2 + y_1^2 + x_2^2 + y_2^2 \right)} . \quad (14)
\]
corresponding to the massive deformation of the free boson model. The parameter \( \tilde{m} \) is the boson mass measured in lattice units
\[
\tilde{m} = a, m_0 . \quad (15)
\]
The simple assumption of keeping \( \tilde{m} \) constant while the lattice anisotropy \( c \) varies, fails to satisfy the YBE. Therefore we will allow it to be a general function of \( c \), and determine it by imposing
\[
\int d\vec{\xi} R_{x_1x_2}^{y_1y_2}(c_1, \tilde{m}_1) R_{x_1x_2}^{y_1y_2}(c_2, \tilde{m}_2) R_{x_1x_2}^{y_1y_2}(c_3, \tilde{m}_3) \]
\[
= \int d\vec{\xi} R_{x_1x_2}^{y_1y_2}(c_1, \tilde{m}_1) R_{x_1x_2}^{y_1y_2}(c_2, \tilde{m}_2) R_{x_1x_2}^{y_1y_2}(c_3, \tilde{m}_3) . \quad (16)
\]
The \( R \)-matrix (14) is invariant under the 180\(^\circ\) rotation (9), implying that the treatment of the previous section also applies to the massive deformation. Following the same steps (see appendix A), we can show that the YBE holds if and only if
\[
\frac{c_1 \tilde{m}_1^2 - c_2 \tilde{m}_2^2}{c_1 \tilde{m}_1^2 - c_3 \tilde{m}_3^2} = \frac{c_1 - c_1 c_2 c_3}{c_1 - c_1}, \quad \frac{c_3 \tilde{m}_3^2 - c_2 \tilde{m}_2^2}{c_3 \tilde{m}_3^2 - c_1 \tilde{m}_1^2} = \frac{c_1 - c_1 c_2 c_3}{c_1 - c_1} , \quad (17)
\]
Figure 1. Left: family of lattice models described by (19) with fixed elliptic parameter \( \mu \).

Right: dependence of the lattice steps \( a_\tau \) and \( a_x \) on the uniformization parameter, under the assumption that the boson mass \( \tilde{m}_0 \) is constant along the family. \( a_x \) is obtained from (15) and \( a_\tau = ca_x \).

\[
\tilde{m}_2^2 = \frac{(1 - c_1 c_3)^2}{c_1 c_3} \left[ 1 - \frac{1}{c_2} \left( \frac{c_1 + c_3}{1 - c_1 c_3} \right)^2 \right].
\] (18)

The requirement that each \( \tilde{m}_i \) be independent of the variables \( c_{j \neq i} \) is far from obvious in view of (18). There is however a two parameter family of solutions which generalizes those of the massless case by promoting trigonometric to elliptic functions. Let us define

\[
c(u, \mu) = \sqrt{\mu_1} \frac{\text{sn}(u, \mu)}{\text{cn}(u, \mu) \text{dn}(u, \mu)}, \quad \tilde{m}(u, \mu) = \sqrt{\frac{4\mu}{\mu_1}} \text{cn}(u, \mu),
\] (19)

where \( \mu_1 = 1 - \mu \) and \( \text{sn}(u, \mu), \text{cn}(u, \mu), \text{dn}(u, \mu) \) are Jacobi elliptic functions of argument \( u \) and parameter \( \mu \) [23]. Equations (17) and (18) are satisfied by

\[
c_1 = c(u, \mu), \quad c_2 = c(u + v, \mu), \quad c_3 = c(v, \mu),
\] (20)

\[
\tilde{m}_1 = \tilde{m}(u, \mu), \quad \tilde{m}_2 = \tilde{m}(u + v, \mu), \quad \tilde{m}_3 = \tilde{m}(v, \mu).
\] (21)

The massless uniformization is recovered at vanishing elliptic parameter \( \mu \). As in that case, regularity of the \( R \)-matrix in the large field limit forces \( u \) to take values on a restricted interval. Asking again for a linear relation between \( u \) and \( c \) when \( c \) is small, we choose \( u \in [0, K(\mu)] \) with

\[
K(\mu) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mu \sin^2 \theta}}
\] (22)

the complete elliptic integral of the first kind. Since \( K(0) = \frac{\pi}{2} \), we conclude that the YBE is compatible with a smooth deformation away from the massless case.

Figure 1 shows the parameter space of the discretized bosonic theory together with the one-dimensional family of models selected by the YBE. Attributing the variation of \( \tilde{m} \) entirely to the lattice step \( a_x \) included in its definition (15), the following consistent picture is obtained. The boson mass keeps constant along the family of models (19). The dependence of the spatial and temporal lattice steps on the uniformization parameter is then easily obtained, and is plotted in figure 1. Although \( c \) diverges as \( u \to K(\mu) \), both \( a_\tau \) and \( a_x \) remain finite. Moreover for \( u \) larger than \( \frac{K(\mu)}{2} \), the point at which the lattice becomes isotropic, the roles of the spatial and
euclidean time directions get exchanged. We identify the boson mass as, \( m_0 \propto \sqrt{\frac{\mu}{\mu_1}} \). Positive \( m_0 \) requires \( \mu \in (0, 1) \), with \( \mu \to 1 \) corresponding to the infinite mass limit.

### 3. The row-to-row transfer matrix

The partition function \( Z \) of a vertex model on an \( L \times N \) lattice can be written as \( Z = \text{tr} T^N \), where \( T \) is the product of Boltzmann weights along a row with their horizontal variables identified and summed over, including the first and the last ones. This defines the so-called row-to-row transfer matrix that for the massive boson model is the map \( T(u) : \mathcal{H} \to \mathcal{H} \) depicted as

\[
T(u) = \begin{pmatrix}
\ldots & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\end{pmatrix},
\]

where \( \mathcal{H} = L_2(\mathbb{R})^\otimes L \) is the lattice Hilbert space and the same variable \( u \) characterizes every vertex.

The matrix elements of the transfer matrix read

\[
\langle \vec{y}|T(u)|\vec{x} \rangle \equiv T(\vec{x}, \vec{y}; u) = \frac{1}{(2\pi)^L} \int d\vec{z} e^{-\frac{i}{2L} \vec{x} \cdot \vec{z} \cdot M(\vec{x}, \vec{y})},
\]

where \( \vec{x} \) are the bottom variables, \( \vec{y} \) the top ones and \( \vec{z} \) the variables associated to the horizontal line. \( M \) is a symmetric matrix analogous to (10) but whose blocks are of dimension \( L \times L \) (see appendix B). The integration on the \( z \)-variables can be easily done, leading to an explicit expression for the transfer matrix

\[
T(\vec{x}, \vec{y}; u) = \frac{1}{(4\pi ac^2)^L} e^{-\frac{i}{2L} \vec{x} \cdot \vec{y} \cdot \left( N_{xx} + N_{yy} \right)},
\]

where we have introduced the convenient combination \( a = \frac{1}{c} + c + \tilde{m}^2 \), and \( N \) is again a symmetric matrix

\[
N_{xx} = N_{yy} = \left( a - \frac{1+c^4}{2ac^2} \right) - \frac{1}{2a} \left( S + S^T \right) \equiv N_1,
\]

\[
N_{xy} = \frac{1}{a} \left[ \frac{1+c^4}{2ac^2} \right] \left( S + c^4 S^T \right) \equiv N_2,
\]

with the shift matrix \( S_{ij} = \delta_{i,j+1} \) and \( L + 1 \equiv 1 \). We have defined the matrices \( N_1 \) and \( N_2 \) for later convenience.

In order to better understand how the tower of conserved charges emerges from the expansion of the transfer matrix, it is convenient to rewrite its components as a product of two factors, \( T = T_p T_q \), with

\[
T_p(\vec{x}, \vec{y}; u) = \frac{1}{(4\pi ac^2)^L} e^{-\frac{i}{2L} \sum_{i=1}^L (x_i - y_i)^2},
\]

\[
T_q(\vec{x}, \vec{y}; u) = e^{-\frac{i}{2L} \sum_{i=1}^L \left[ \frac{6(\mu_1^2 + \mu^2)}{\mu_1} + 2(\mu_1 - \mu)^2 + (\mu_1 - \mu_4)^2 + (\mu_1 - \mu_4)^2 + c^2(\mu_1 - \mu_4)^2 \right]}.
\]
The function $T_p$ tends to a delta distribution as $u \to 0$. At small $u$, $T_p$ can be expanded in terms of derivatives of a delta function (see appendix B)

$$T_p(\vec{x}, \vec{y}, u) = \prod_{i=1}^{L} \left( \sum_{k=0}^{\infty} \frac{(ac)^k}{k!} \delta^{2(k)}(x_i - y_{i+1}) \right), \quad (30)$$

which as $T_q$, has a well defined expansion around $u = 0$. At leading order only the $k = 0$ term in (30) contributes to the transfer matrix. Hence it reduces to a cyclic permutation

$$e^{ip}|x_1, x_2, \ldots, x_L\rangle = |x_L, x_1, \ldots, x_{L-1}\rangle, \quad (31)$$

where $a_i^{-1}P$ is the lattice momentum, the first conserved charge derived from the transfer matrix expansion. A graphical derivation of the identification $T(0) = e^{ip}$ can be obtained from (23), since at $u$ or equivalently $c = 0$, the dotted links inside the vertices vanish.

At higher orders $T_p$ contributes with powers of the bosonic field, and its discretized spatial derivatives to the corresponding conserved charge, while $T_p$ provides powers of the canonical momentum. The canonical commutations of the bosonic field $x$ and its conjugate momentum $\pi$ in the continuum, become in the discretized model

$$[x(z), \pi(z')] = i\hbar(z - z') \rightarrow [x_i, \pi_j] = ia_i^{-1}\delta_{ij}. \quad (32)$$

Therefore we can represent $\pi_i = -ia_i^{-1}\partial_{x_i}$, where

$$\langle \vec{y}|\partial_{x_i}|\vec{x}\rangle = \delta(x_i - y_i)\prod_{j\neq i}\delta(x_j - y_j). \quad (33)$$

Using this, it is immediate to obtain the next to leading contribution to the transfer matrix

$$-2\sqrt{\mu}ue^{ip}\hbar \cdot H = \frac{1}{\mu} \sum_{i=1}^{L} \left( a_i^{-2}\pi_i^2 + (x_i - x_{i+1})^2 + m_0^2\pi_i^2 \right), \quad (34)$$

where $a_i^{-2}H$ is the discretized free boson Hamiltonian and $m_0^2 = \frac{4\mu}{\hbar^2}$, in agreement with (19). If we interpret $a_i$ as a $u$-dependent parameter, as done in figure 1, we should select here its value at vanishing $u$. Obtaining the higher conserved charges along these lines is possible but cumbersome. Below we will follow an alternative strategy.

### 3.1. Spectrum

In this section we diagonalize the transfer matrix of the free boson lattice theory in a way that is reminiscent to the coordinate Bethe ansatz for spin systems. Since the Hamiltonian commutes with the transfer matrix, the eigenstates of the former are also eigenstates of the latter. A natural ansatz for the eigenstates, inspired by those of a harmonic oscillator, is

$$|\Psi\rangle = \int d\vec{x} f_n(\vec{x})e^{-\frac{1}{4}iK\pi^T}\langle \vec{x}\rangle,$$

with $K$ a symmetric matrix and $f_n$ a polynomial of degree $n$ in $x_i$. Straightforward manipulations, detailed in appendix B, show that the eigenstate condition $T(u)|\Psi\rangle = \Lambda|\Psi\rangle$ implies

$$\frac{1}{(4\pi ac)^2}\int d\vec{x} e^{-\frac{1}{4}iK(N_1 + K)^{-1}}f_n(\vec{x} - \vec{y}N_1^T(N_1 + K)^{-1}) = \Lambda f_n(\vec{y}), \quad (36)$$
The matrices $N_1, N_2$ have been defined in (26) and (27). Recall that, although they depend on $u$, $K$, the function $f_n$ should be independent of the uniformization parameter.

The matrices $N_1$ and $N_2$ commute because they are linear combinations of the identity, the shift matrix and its transpose. Assuming that $K$ also commutes with them, and using that the $a$ can be rewritten as

$$a = \sqrt{\left(c + \frac{1}{c}\right)^2 + m_0^2},$$

we obtain

$$K^2 = N_1^2 - N_2^2 = (m_0^2 + 2)I - S - S^T.$$  

This expression is indeed consistent with the previous assumption and satisfies the required independence of $u$. The eigenvalues of the shift matrix $S$ are roots of unity of order $L$. Hence the eigenvalues of $K$ are

$$\omega_k = \sqrt{m_0^2 + 4 \sin^2 \frac{p_k}{2}}, \quad k = 0, \ldots, L - 1.$$  

with $p_k = \frac{2\pi k}{L}$. The state (35) with $f_0$ a constant is the ground state of the lattice model. Indeed, the energy of a bosonic eigenmode with momentum $a_k^{-1}p_k$ and mass $m_0$ is $a_k^{-1}\omega_k$. Notice that, although $K^2$ only has entries on the diagonal and one step above or below it, its square root is a non-local matrix. The eigenvalue of the transfer matrix on the ground state is

$$\Lambda_0 = \frac{1}{(2ac^2)^{\frac{1}{2}} \sqrt{\det(N_1 + K)}} = \frac{1}{c^L \prod_k (a + \omega_k)}.$$  

Excited states are obtained when the function $f_n(\vec{x})$ in (4) is non-trivial. The obvious choice for this function are the Hermite polynomials

$$f_n(\vec{x}, \vec{v}) = \rho e^{i k v^T \vec{x}} \left( \frac{\partial}{\partial \vec{x}} \right)^n e^{-\vec{x}^T \vec{v} \vec{x}},$$

with $\rho$ a normalization constant and $\vec{v}$ a vector to be determined. Substituting the above ansatz into the eigenstate condition (36), we obtain (see appendix B)

$$\Lambda_0 f_n(\vec{y}; -\vec{v}(N_1 + K)^{-1}N_2) = \Lambda f_n(\vec{y}; \vec{v}).$$

Fulfilling equation (36) requires $\vec{v} = \vec{v}_k$ to be an eigenvector of $(N_1 + K)^{-1}N_2$. While the eigenvalues of $N_1$ and $K$ only depend on $\omega_k$, those of $N_2$ are functions of $p_k$. The corresponding eigenvalue of the transfer matrix is

$$\Lambda_{k,n} = e^{i n p_k} \left( 1 + c^2 e^{-i p_k} \right)^{2n} \Lambda_0.$$  

The ansatz (42) can be generalized by allowing for directional derivatives associated to different eigenvectors $\vec{v}_k$. In this way it can describe states with an arbitrary number of excitations of different momenta.
3.2. Conserved charges

The eigenvalue of the transfer matrix on a general state is

\[ \Lambda = e^{ipL} \prod_{k=0}^{L-1} \left( 1 + c^2 e^{-ip_k} \right)^{2n_k} / (c(a + \omega_k))^{2n_k+1}, \]  

(45)

where \( n_k \) is the number of excitations with momentum \( p_k \) and \( p = \sum_k p_k n_k \). Consistently with (31) and (34), the exponential prefactor on the rhs is the eigenvalue of the lattice shift operator, \( e^{P} \).

We will obtain the complete tower of conserved charges from the expansion of \( \log \Lambda \) around \( u = 0 \). The following equality holds (see appendix C)

\[ \log \Lambda = ip - \sum_{k=0}^{L-1} \left( n_k + \frac{1}{2} \right) \log \frac{a + \omega_k}{a - \omega_k} + \sum_{k=0}^{L-1} n_k \log \frac{1 + c^2 e^{-ip_k}}{1 + c^2 e^{ip_k}}. \]  

(46)

The second term on the rhs only gives rise to odd powers of \( u \). The third term, which turns out to be independent of the boson mass, generates even powers. Therefore they contribute to different sets of conserved charges. A simple basis for the conserved charges can be derived from (46). We will use \( c \) as expansion parameter for the second term and \( a^{-1} \) for the third. This just amounts to a linear redefinition of the charges derived using \( u \) as expansion parameter.

Hence we define

\[ \log \Lambda = ip - \sum_{l=0}^{\infty} a^{-(2l+1)} \langle Q_{2l+1} \rangle + i \sum_{l=1}^{\infty} c^{2l} \langle Q_{2l} \rangle. \]  

(47)

Equating with (46), we obtain

\[ \langle Q_{2l+1} \rangle = \frac{2}{2l+1} \sum_{k=0}^{L-1} \left( n_k + \frac{1}{2} \right) \omega_k^{2l+1}, \quad \langle Q_{2l} \rangle = \frac{(-1)^l}{l} \sum_{k=1}^{L-1} 2n_k \sin(p_k l). \]  

(48)

The first odd charge is \( Q_1 = 2H \), in agreement with (34). The vacuum expectation values of the conserved charges derived from the transfer matrix expansion are linear in the occupation numbers \( n_k \). This is expected since \( n_k \) form a basis of conserved quantities of the free boson theory. However, while the occupation numbers can not be derived from a local charge, we will see in the next section that \( Q_l \) are (quasi) local. Although the sums in (47) have infinite terms, only a set of \( L \) charges can be linearly independent.

4. Operator form of the R matrices

An alternative way to derive the conserved quantities of an integrable model is to use the QISM [25]. An advantage of this method is its applicability to inhomogeneous situations, contrary to the construction (23). In the previous section we have derived the expectation values of the tower of conserved charges. Here we will use the QISM to obtain their operator expression.

The \( R \)-matrices of the boson model are linear maps \( R : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \)

\[ R_{12}^{y_1 y_2} = \langle y_1, y_2 | R_{12} | x_1, x_2 \rangle, \]  

(49)

with \( |x_1, x_2 \rangle \) a complete basis of the Hilbert space on two lattice sites. We shall consider \( \mathbb{R} \otimes \mathbb{R} \) as the real space for the motion of two particles with coordinates \( x_1 \) and \( x_2 \). We want to express
\( \mathbf{R}_{12} \) as the evolution operator in euclidean time \( t \) of a Hamiltonian \( \mathbf{H}_{12} \)

\[ \mathbf{R}_{12} = e^{-i\mathbf{H}_{12}t}. \]  

(50)

The basic result we use is the euclidean propagator of a harmonic oscillator with mass \( m \) and angular frequency \( \omega \) [24]

\[ G_{m\omega}(x, x', t) = \left( \frac{m\omega}{2\pi \sinh(\omega t)} \right)^{1/2} e^{-\frac{m\omega(x^2 + x'^2 \sinh(\omega t) - 2x'x)}{2\sinh(\omega t)}}. \]  

(51)

In the limit \( \omega \to 0 \), it reduces to the propagator of a free particle of mass \( m \).

Let us define the center of mass and relative coordinates of the two particle system

\[ X = \frac{x_1 + x_2}{2}, \quad x = x_1 - x_2. \]  

(52)

The \( R \)-matrix (3) factorizes its dependence on center of mass and relative coordinates as

\[ R^{(1)\gamma_2}_{x_1, x_2} = e^{-\frac{i}{2}(X - \gamma)^2 + \left(\frac{1}{2} + \frac{1}{2}\right)(x^2 + x'^2) - \frac{i}{2}xy}. \]  

(53)

Comparing with (51) we make the identification

\[ R^{(1)\gamma_2}_{x_1, x_2} = G_{M,0}(X, Y, 1)G_{m\omega}(x, y, 1). \]  

(54)

The exponents of both expressions coincide provided that

\[ M = \frac{2}{c}, \quad m\omega \coth \omega = \frac{1}{2c} + c, \quad \frac{m\omega}{\sinh \omega} = \frac{1}{2c}. \]  

(55)

This also ensures agreement between the normalization factors of the \( R \)-matrix and the propagators. The parameters \( m\omega \) and \( \omega \) are related to the variable \( u = \arctanh(c) \) as

\[ m\omega = \frac{1}{\cos u}, \quad \omega = 2 \arcsinh(\tan u). \]  

(56)

The Hamiltonian \( \mathbf{H}_{12} \), corresponding to the Green function (54), is given by the sum of the free particle and harmonic oscillator Hamiltonians

\[ \mathbf{H}_{12} = \frac{p_x^2}{2M} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2 x^2. \]  

(57)

where \( p_x = p_1 + p_2 \) and \( p_z = \frac{i}{\hbar}(p_1 - p_2) \) are the center of mass and relative momenta of the particles. Expanding at small \( u \) we obtain

\[ \mathbf{H}_{12} = \hbar \mathbf{h}_{12} + O(u^3), \quad \mathbf{h}_{12} = \frac{p_1^2 + p_2^2}{2} + (x_1 - x_2)^2. \]  

(58)

The previous derivation can be repeated for the massive model. We summarize the main results.

The \( R \)-matrix (14) corresponds to two harmonic oscillators with masses \( M \) and \( m \), and angular frequencies \( \Omega \) and \( \omega \) given in terms of the variable \( u \) as

\[ M\Omega = 4 \sqrt{\frac{\mu}{\mu_1}} \frac{\text{cn}(u, \mu)}{\text{dn}(u, \mu)}, \quad \Omega = 2 \arcsinh \left( \sqrt{\frac{\text{sn}(u, \mu)}{\text{dn}(u, \mu)}} \right), \]  

\[ m\omega = \frac{1}{\sqrt{\mu_1}} \frac{\text{dn}(u, \mu)}{\text{cn}(u, \mu)}, \quad \omega = 2 \arcsinh \left( \frac{\text{sn}(u, \mu)}{\text{cn}(u, \mu)} \right). \]  

(59)

(60)
Relations (56) are recovered in the massless limit $\mu \to 0$. The expansion of the Hamiltonian of the two oscillators around $u = 0$ is
\[
H_{12} = \sqrt{\mu} h_{12} + O(u^3), \quad h_{12} = \frac{p_1^2 + p_2^2}{2} + m_0^2 \frac{x_1^2 + x_2^2}{2} + (x_1 - x_2)^2. \tag{61}
\]

4.1. Transfer matrix

To implement the QISM we use a new $\mathcal{R}$ matrix defined as [25]
\[
\mathcal{R}_{12} = \mathbf{P}_{12} \mathbf{R}_{12}, \tag{62}
\]
where $\mathbf{P}_{12}$ is a permutation, such that in components we have $\mathcal{R}_{12}^{(1,2)} = \mathcal{R}_{12}^{(2,1)}$. In integrable spin chain models, like the XXZ, these $\mathcal{R}$-matrices can be derived from a universal $\mathcal{R}$ matrix using the representation theory of affine quantum groups [4, 26]. It would be interesting to construct the quantum group underlying the $\mathcal{R}$-matrices of the boson model, specially in the massive case. However, this is not the purpose of this work. Using uniformization variables, the YBE equation expressed in terms of the $\mathcal{R}$-matrices takes the form
\[
\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v). \tag{63}
\]
In the QISM one introduces an auxiliary space $V_a$ and local quantum spaces $V_j (j = 1, \ldots, L)$, for the action of the operator
\[
\mathcal{R}_{a0}(u) : V_a \otimes V_j \to V_a \otimes V_j, \quad j = 1, \ldots, L. \tag{64}
\]
The transfer matrix depicted in (23), can be rewritten in terms of these operators as
\[
T(u) = \text{tr}_a (\mathcal{R}_{ad}(u)\mathcal{R}_{ad-1}(u) \cdots \mathcal{R}_{a2}(u)\mathcal{R}_{a1}(u)) . \tag{65}
\]
In this way we obtain an explicit operator expression for the transfer matrix. We will recover now the first conserved charges derived in the previous section from its small $u$ expansion.

The permutation operators satisfy $O_{a0} \mathbf{P}_{aj} = \mathbf{P}_{aj} O_{a0}$, for any two-site operator $O$. Applying this property repeatedly allows to bring all permutation operators in the transfer matrix to the left of the $R$-matrices obtaining
\[
T = e^{P} \text{tr}_a (\mathbf{P}_{aL} \mathbf{R}_{L} \mathbf{R}_{L-1} \cdots \mathbf{R}_{23} \mathbf{R}_{12} \mathbf{R}_{a1}) . \tag{66}
\]
We have dropped the explicit dependence on $u$ in order to simplify the notation. Observe that the auxiliary space appears twice on the rhs, in the operators $\mathbf{P}_{aL}$ and $\mathbf{R}_{a1}$. To trace over $V_a$, we decompose $\mathbf{R}_{\ell k}$ as a sum of operators acting in the spaces $V_j$ and $V_{k'}$
\[
\mathbf{R}_{\ell k} = \sum_{\ell'} \mathbf{r}_{\ell k}^{(\ell')} \otimes \mathbf{r}_{\ell' k}^{(\ell')}. \tag{67}
\]
The transfer matrix is finally given by
\[
T = e^{P} \sum_{\ell} \mathbf{r}_{+L}^{(\ell)} \mathbf{R}_{L-1} \mathbf{R}_{L-2} \cdots \mathbf{R}_{23} \mathbf{R}_{12} \mathbf{r}_{-L}^{(\ell)}. \tag{68}
\]
This expression can also be written as
\[
T = e^{P} \sum_{L_{1}, \ldots, L_{L}} \mathbf{v}_{L_{1}}^{(L_{1}-1)} \mathbf{v}_{L-1}^{(L_{1}-2)} \cdots \mathbf{v}_{2}^{(L_{1})} \mathbf{v}_{1}^{(L_{1})}. \tag{69}
\]
with \( V_{ij}^{(l)} = r_{i}^{(l)} \otimes r_{j}^{(l)} \), showing explicitly the cyclicity of (68).

To find the lowest order conserved quantities we consider the operator \( R_{12} \) given in (50) with \( t = 1 \), and substitute the expansion of \( H_{12} \) derived in the previous subsection. Plugging the expression so obtained into (68) gives

\[
\log T = i\mathbf{p} - 2\sqrt{\mu_{1} u} H + i\mu_{1} u^{2} Q_{2} + O(u^{3}).
\]

Identifying \( p_{i} = a_{i} \pi_{i} \), we recover the Hamiltonian of the lattice model

\[
H = \frac{1}{2} \sum_{i=1}^{L} h_{i,i+1} = \frac{1}{2} \sum_{i=1}^{L} \left( p_{i}^{2} + (x_{i} - x_{i+1})^{2} + \tilde{m}_{0}^{2} x_{i}^{2} \right).
\]

The charge \( Q_{2} \) is given by

\[
Q_{2} = \frac{1}{2} \sum_{j=1}^{L} \left[ h_{j,i+1}, h_{j+1,i+2} \right] = \sum_{i=1}^{L} p_{i}(x_{i+1} - x_{i+1}).
\]

Using this insight, we propose the following operator expression for the even conserved charges of the discretized boson theory

\[
Q_{2j} = \frac{(-1)^{j}}{l-a_{i} \bar{\pi}} \left( S^{i} - S^{j} \right) \bar{x},
\]

with \( l \geq 1 \). For the odd charges we have

\[
Q_{2j+1} = \frac{1}{2l+1} \left( a_{i}^{2} \bar{\pi} K^{2j+1} \bar{x} + \bar{x} K^{2j+2} \bar{x} \right),
\]

with \( l \geq 0 \) and \( K^{2} \) given in (39). In appendix C we show that these charges indeed commute and their expectation values agree with (48).

Using that \( S^{i} = 1 \), it is immediate to see that there are \( \left[ \frac{L}{2} \right] + 1 \) linearly independent odd charges (74) and \( \left[ \frac{L+1}{2} \right] \) even charges (73), where \( [\cdot] \) denotes retaining the integer part. This makes a set \( L \) quasi local operators that, avoiding \( \mathbf{P} \), can be used as a basis of the conserved charges of the discretized bosonic theory. The higher the charge, the farther the sites it couples. Even charges couples sites \( i, i \pm l \) and odd charges couples sites up to \( i, i \pm (l+1) \). In the continuum limit, long range effects translate into multiple spatial derivatives.

We observe that the operator \( Q_{2} \) provides a method to compute the momentum of the bosonic excitations, that of course agrees with the one obtained using the lattice shift operator \( e^{\phi} \). However, unlike \( e^{\phi} \), the operator \( Q_{2} \) has the advantage of being local. Finally, it is worth noticing that for a massless bosonic theory in the continuum limit

\[
H \to \frac{1}{2} \int dx \left( (\partial_{i} \phi)^{2} + (\partial_{i} \phi)^{2} \right) = \frac{2\pi}{L}(L_{0} + L_{0}),
\]

\[
Q_{2} \to \int dx \partial_{i} \phi \partial_{i} \phi = \frac{2\pi}{L}(L_{0} - L_{0}),
\]

where \( L_{0} \) and \( L_{0} \) are the holomorphic and antiholomorphic Virasoro operators of the \( c = 1 \) CFT of the massless boson [27]. These equations imply that \( H + Q_{2} \) and \( H - Q_{2} \) provide a local lattice version of the Virasoro operators \( L_{0} \) and \( L_{0} \).
5. Scattering S matrix

In a free boson field theory one expects that the scattering theory should be trivial, as it is indeed the case. What we propose in this section is a physical interpretation of the $R$-matrix, not as Boltzmann weights like in previous sections, but as scattering amplitudes. An example of a double use of $R$-matrices is the 6 vertex model in statistical mechanics and the sine-Gordon model in quantum field theory. In the former model the entries of the $R$-matrix are labelled by the spins $\pm 1/2$ lying at a common vertex of a lattice, while in the latter model the labels are the topological charges $1(-1)$ of the incoming and outgoing solitons (antisolitons) interacting on a line. This example shows that the physical meaning of the entries of an $R$-matrix can be very different and depends on the context. So far we have interpreted the $R$-matrix as Boltzmann weights of a free boson field on a lattice. We shall now regard $R$ as the $S$-matrix for relativistic particles carrying a continuous degree of freedom, somehow, to the topological charge in the sine-Gordon model although it is not quantized. To go from the $R$-matrix to the $S$-matrix we perform a Wick rotation replacing $c \in \mathbb{R}$ by $ic$ in (3), obtaining essentially a pure phase. We shall show below this rotated matrices can be used to formulate a consistent relativistic scattering theory, a result that is far from obvious.

Let us briefly review the factorized $S$-matrix theory. This theory describes the scattering of a set of particles $\{A_i\}_{i=1}^N$ in a relativistic quantum field theory [4, 7, 28]. If the particles are massive, their energy and momentum are parameterized in terms of their rapidity $\theta \in \mathbb{R}$ as $(p_0^i, p_1^i) = m_i (\cosh \theta, \sinh \theta)$, where $m_i$ is the mass of the particle $A_i$. The two particle scattering process between incoming and outgoing asymptotic states is given by

$$S|A_i(\theta_1), A_j(\theta_2)\rangle_{\text{in}} = \sum_{j,k} S_{ij}^{kl}(\theta_{12})|A_k(\theta_2), A_\ell(\theta_1)\rangle_{\text{out}},$$

(77)

where, by relativistic invariance, the matrix $S$ only depends on the difference of rapidities $\theta_{12} = \theta_1 - \theta_2 > 0$. Factorization guarantees that the two particle amplitude (77) completely determines all possible scattering processes. It requires for consistency

$$\sum_{p_1} S_{12}^{i_1 p_1}(\theta_{12})S_{p_2 i_2}^{j_1}(\theta_{13})S_{p_{13} j_2}^{i_2}(\theta_{23}) = \sum_{p_2} S_{12}^{i_2 p_2}(\theta_{23})S_{p_{23} i_2}^{j_1}(\theta_{13})S_{p_{13} j_2}^{i_2}(\theta_{12}),$$

(78)

which is equivalent to the YBE.

Rapidities can be allowed to take complex values within the so called physical strip, $\text{Im} \theta \in (0, \pi)$. With this extension $\theta$ uniformizes the branch cuts both in the $s$ and $t$-scattering channels, which map respectively to $\text{Im} \theta = 0$ and $\text{Im} \theta = \pi$. Hence the amplitude $S_{ij}^{kl}(\theta)$ must be a meromorphic function whose only singularities are poles in the imaginary axis of the physical strip, associated to the eventual appearance of a bound state [7]. Besides (78), the $S$-matrix has to fulfill the following axioms

(a) Normalization : $\lim_{\theta \to 0} S(\theta) = 1$ (79)

(b) Unitarity : $S(\theta)S(-\theta) = 1$ (80)

(c) Realanalyticity : $S^*(\theta) = S(-\theta^*)$ (81)

(d) Crossingsymmetry : $S_{ij}^{kl}(\theta) = S_{kl}^{ji}(i\pi - \theta)$.

(82)
Condition (a) means that no scattering takes place when the relative velocities of the two particles vanishes. Condition (b) is obtained applying (77) twice. Conditions (b) and (c) imply physical unitarity $S^{\dagger}(\theta)S(\theta) = 1$. Condition (d) relates the scattering channel $A_i \times A_j \rightarrow A_k \times A_l$ to the crossed channel $A_j \times A_i \rightarrow A_l \times A_k$, where the bar denotes the corresponding antiparticle.

5.1. The trigonometric S matrix

We make the following change of variables from the parameter $u$ employed in the massless statistical mechanics model to a rapidity variable $\theta$, which implements the Wick rotation of the Boltzmann weights

$$u = \frac{\theta}{2i} \implies c(\theta) = -i \tanh \frac{\theta}{2}. \quad (83)$$

Replacing this into (3), and allowing for a $\theta$-dependent proportionality constant, we define $S(\theta) = g(\theta)R(\theta)$ with

$$R_{x_1x_2y_1y_2}(\theta) = \frac{i}{2\pi} \coth \frac{\theta}{2} \left[ (x_1-y_1)^2 + (x_2-y_2)^2 \right]^\frac{1}{2} \tanh \frac{1}{2} \left[ (x_1-x_2)^2 + (y_1-y_2)^2 \right]. \quad (84)$$

This ensures that the factorization condition (78) is satisfied. We want to stress the very different meaning of $u$ in the statistical mechanics model, where it parameterizes the lattice anisotropy, and $\theta$ in the scattering theory, which is a dynamical variable parameterizing the dispersion relations $(p^0, p^1) = m(\cosh \theta, \sinh \theta)$.

The continuous nature of field variables in the boson model implies the presence of an infinite set of particles in the scattering theory, labelled by a continuum index $x$. The $R$-matrix (84) is easily seen to be compatible with the axioms (a)–(d), translating them into conditions on the function $g$. Using (4), normalization (a) holds provided

$$\lim_{\theta \to 0} g(\theta) = 1. \quad (85)$$

The unitarity condition (b) reads

$$\int \mathcal{D}S_{x_1x_2y_1y_2}(\theta)S_{x_1x_2y_1y_2}(-\theta) = \delta(x_1-z_1)\delta(x_2-z_2) \implies g(\theta)g(-\theta) = 1. \quad (86)$$

The integral is done assuming that $\theta$ is real. The latter assumption also guarantees the convergence of the integrals in (78). Real analyticity (c) implies

$$\left(S_{x_1x_2y_1y_2}(\theta)\right)^* = S_{x_1x_2y_1y_2}(-\theta^*) \implies g^*(\theta) = g(-\theta^*). \quad (87)$$

Crossing symmetry has been used to determine the proportionality constant in the linear relation between $u$ and $\theta$ (83). Then (d) holds if

$$S_{x_1x_2y_1y_2}(\theta) = S_{x_1x_2y_1y_2}(i\pi - \theta) \implies g(i\pi - \theta) = -g(\theta)\coth^2 \frac{\theta}{2}. \quad (88)$$

Notice that the counterpart of the physical strip in the statistical mechanical model, is the requirement that the real part of $u$ takes values in the interval $(0, \frac{\pi}{2})$. We have already encountered this restriction when requiring a regular large field limit of the $R$-matrix (3). Hence the
natural domains of dependence of the variables $\theta$ and $u$ map to each other, supporting the $S$-matrix interpretation of the discretized free boson.

Using the infinite product decomposition

$$\coth \frac{\theta}{2} = i \prod_{n=-\infty}^{\infty} \frac{n + \frac{1}{2} + \frac{\theta}{2\pi i}}{n - \frac{1}{2} - \frac{\theta}{2\pi i}},$$

we find the following solution of equations (85)–(88) (see appendix D)

$$g(\theta) = \lim_{M \to \infty} \prod_{n=-M}^{M} \left( \frac{\Gamma (n + \frac{\theta}{2\pi i}) \Gamma (n + \frac{1}{2} + \frac{\theta}{2\pi i})}{\Gamma (n - \frac{\theta}{2\pi i}) \Gamma (n + \frac{1}{2} - \frac{\theta}{2\pi i})} \right)^2. \quad (90)$$

The parameter $M$ and its limit is required for regularization. This is a meromorphic function with double poles at $\theta = \pi i, 3\pi i, \ldots$ and quadruple poles at $\theta = -2\pi i, -4\pi i, \ldots$. Hence there are no singularities on the physical strip except at its boundary $\theta = i\pi$. This double pole of $g$ conspires with the simple zero of $\coth \frac{\theta}{2}$ in (84), and the exponential term to give

$$S_{x_1,x_2}^{\mu_1}(i\pi) = \delta(x_1 - x_2)\delta(y_1 - y_2). \quad (91)$$

This equation also follows from the normalization condition $S(0) = 1$ and crossing symmetry. An integral representation of the $g$ function in the region $|\Im \theta| < \pi$ is given by (see appendix D)

$$g(\theta) = \exp \left( -i \int_0^\infty \frac{dt}{t} \frac{\sin(2\theta t)}{\cosh^2(\pi t)} \right). \quad (92)$$

This expression can be explicitly integrated, obtaining

$$g(\theta) = i \exp \left[ -\frac{2i}{\pi} \left( \theta \log \frac{1 - e^\theta}{1 + e^\theta} + \text{Li}_2(e^{\theta}) - \text{Li}_2(-e^{\theta}) \right) \right], \quad (93)$$

where $\text{Li}_2(z)$ is the polylogarithmic function. The quotient $\frac{\xi}{\pi} = ig \coth \frac{\theta}{2}$ is plotted in figure 2. This graphic renders crossing evident, since in terms of $\frac{\xi}{\pi}$ it just amounts to symmetry under the exchange $i\pi - \theta \leftrightarrow \theta$. Finally, we should mention that $g$ is not unique due to the so called CDD ambiguity. Namely, the scattering matrix can be multiplied by a meromorphic function satisfying (85)–(87) and invariant under crossing, which adds extra zeros and poles [28]. What we have obtained above is a minimal solution that does not contain CDD poles.

5.2. The elliptic $S$-matrix

Guided again by crossing we choose

$$u = \frac{K\theta}{\pi i}, \quad (94)$$

with $K \equiv K(\mu)$ defined in (22). Substituting this change of variables in (19) leads to

$$c(\theta, \mu) = \sqrt{\mu_1} \frac{\sin \left( \frac{K\theta}{\pi i}, \mu \right)}{\text{cn} \left( \frac{K\theta}{\pi i}, \mu \right) \text{dn} \left( \frac{K\theta}{\pi i}, \mu \right)}, \quad (95)$$

$$\tilde{m}(\theta, \mu) = \sqrt{\frac{4\mu}{\mu_1} \text{cn} \left( \frac{K\theta}{\pi i}, \mu \right)}.$$
It is easy to show that $c(\theta, \mu)$ becomes purely imaginary and $\tilde{m}(\theta, \mu)$ real, when $\theta \in \mathbb{R}$. We define again $S(\theta, \mu) = g(\theta, \mu)R(\theta, \mu)$. As in the previous section, the function $g$ will be determined by the $S$-matrix theory axioms. Equations (85)–(87) remain unaltered in the elliptic case. Equation (88), derived from crossing, also applies after replacing $\coth^2(\theta/2)$ by $-1/c^2(\theta, \mu)$.

The knowledge of the zeros and poles of the Jacobi elliptic functions allows us to rewrite

$$c(\theta, \mu) = \lim_{M \to \infty} \prod_{m=-M}^{M} \tan \left( \frac{\theta + m\tau}{2i} \right)$$

with $\tau = \frac{\pi K(\mu)}{K(\mu)}$. Using this equation one finds the solution (see appendix D)

$$g(\theta, \mu) = \lim_{M \to \infty} \prod_{n,m=-M}^{M} \left[ \frac{\Gamma(n + \frac{1}{2}(\theta - m\tau)) \Gamma(n + \frac{1}{2} - \frac{1}{2\tau}(\theta + m\tau))}{\Gamma(n - \frac{1}{2}\tau(\theta + m\tau)) \Gamma(n + \frac{1}{2} + \frac{1}{2\tau}(\theta - m\tau))} \right]^2.$$  

(97)

In the $\mu \to 0$ limit $\tau$ diverges and trivializes all contributions $m \neq 0$, reducing this expression to (90). Compared to it, $g(\theta, \mu)$ is also meromorphic with additional poles and zeros, but all of them lie outside the physical strip or at its boundary. Applying a discrete version of (92), equation (97) can be expressed as

$$g(\theta, \mu) = \exp \left[ -i \left( \frac{\pi \theta}{\tau} + \sum_{n=1}^{\infty} \frac{\sin(2\pi n/\tau)}{n \cosh^2(\pi^2 n/\tau)} \right) \right].$$

(98)

The main difference between the $\mu = 0$ and $\mu \neq 0$ $S$-matrices is the cyclic structure, with period $\tau$, introduced on the physical strip by the elliptic deformation. In figure 3 we have plotted the combination $\frac{\xi}{\tau}$ in the region $|\text{Re}(\theta)| \leq \frac{\xi}{2}$. Its structure indeed replicates that of figure 2.
In connection with the $S$-matrix described above, we would like to mention other models whose $S$-matrices are expressed in terms of Jacobi elliptic functions. Zamolodchikov constructed an $S$-matrix with a $\mathbb{Z}_4$ symmetry, which is doubly periodic in the rapidity and depends on two coupling constants, being one of them the modulus of the elliptic functions [20]. In the limit where the modulus vanishes one recovers the $S$-matrix of the sine-Gordon model with $O(2)$ symmetry. The model possess an infinite number of resonances related to the elliptic modulus. At high energies ($s \gg m^2$), the amplitudes and cross-sections are periodic in $\log s$, which is a characteristic feature of renormalizable quantum field theories with limit cycles [29]. This is the field theory interpretation proposed in [20] to correspond to the $S$-matrix. This reference also notices the formal relation between the elliptic $S$-matrix and the Baxter’s eight vertex model [19].

Another example of elliptic $S$-matrix was proposed by Mussardo and Penati that contains only one type of fundamental particle with $S$-matrix and $\mathbb{Z}_2$ symmetry [21]

$$S(\theta) = \frac{\text{sn} \left( \frac{K(\theta - i \pi a)}{\tau} \right) \text{cn} \left( \frac{K(\theta + i \pi a)}{\tau} \right) \text{dn} \left( \frac{K(\theta - i \pi a)}{\tau} \right)}{\text{sn} \left( \frac{K(\theta + i \pi a)}{\tau} \right) \text{cn} \left( \frac{K(\theta - i \pi a)}{\tau} \right) \text{dn} \left( \frac{K(\theta + i \pi a)}{\tau} \right)}$$

(99)

where $a$ is a coupling constant in the interval $[0, 1/2]$. This $S$ is periodic in $\theta$ also with period $\tau$ and correspondingly an infinite number of unstable resonances. In the limit where the elliptic modulus goes to zero, one recovers the $S$-matrix of the sinh-Gordon model. In reference [21], it is conjectured that the $S$-matrix (99) corresponds in the UV to a non unitary and irrational CFT. Notice that (99) can be written as $S(\theta) = \frac{\text{sn}(\theta - i \pi a)}{\text{sn}(\theta + i \pi a)}$. This relation suggests a possible deformation of our model with a parameter similar to $a$ in (99).

Finally, there are other models with continuous variables living on a circle whose Boltzmann weights satisfy the star triangle relation, and are expressed in terms of the elliptic gamma-function [30, 31].
5.3. The $S$-matrix in Fourier space

The two particle scattering equation (77) can be given an algebraic form due to Faddeev and Zamolodchikov in terms of operators $A_i(\theta)$ with $\theta$ real, whose action on the Hilbert space vacuum $|0\rangle$ creates a particle with rapidity $\theta$,

$$A_i(\theta)|0\rangle = |A_i(\theta)|.$$

(100)

The bosonic model has associated a continuous set of such operators $A_i(\theta)$. Equation (77) is equivalent to the exchange relation

$$A_{i_1}(\theta_1) A_{i_2}(\theta_2) = \int dy_1 dy_2 S_{i_1 i_2}^{(12)}(\theta_{12}) A_{i_3}(\theta_2) A_{i_4}(\theta_1).$$

(101)

Explicit realizations of field theory operators satisfying this relation are generally unknown. We next propose a partial realization of them, which in the massless limit corresponds to the vertex operators of the CFT describing a massless boson.

Let us first define the Fourier transform of the Faddeev-Zamolodchikov operators

$$\mathcal{A}_q(\theta) = \int dx e^{i q x} A_i(\theta), \quad q \in \mathbb{R},$$

(102)

where the integral runs over the real line. Recalling that $x$ represents the scalar field $\phi$, we interpret $q$ as a charge associated to the symmetry $\phi \to \phi + \text{constant}$ of the massless $S$-matrix. We shall find below further support of this interpretation. The operators $\mathcal{A}_q(\theta)$ satisfy the following exchange relation derived from (101)

$$\mathcal{A}_{q_1}(\theta_1) \mathcal{A}_{q_2}(\theta_2) = \int dp_1 dp_2 \tilde{S}_{q_1 q_2}^{(12)}(\theta_{12}) \mathcal{A}_{p_1}(\theta_2) \mathcal{A}_{p_2}(\theta_1),$$

(103)

where $\tilde{S}_{q_1 q_2}^{(12)}$ is the Fourier transform of $S_{i_1 i_2}^{(12)}$. Diagonalizing the quadratic form in the exponent of the $R$-matrix (14) before Fourier transforming, we easily obtain (see appendix D)

$$\tilde{S}_{q_1 q_2}^{(p_1 p_2)} = \frac{g \mu_1}{8 \pi c \sqrt{\beta}} \exp \left[ -\frac{1}{4} \left( \frac{1}{c^2 m^2} (q_1+q_2-p_1-p_2)^2 + \frac{c}{4 + c^2 m^2} (q_1+q_2+p_1+p_2)^2 \right) 
+ \frac{1}{4 (4 + m^2)} (q_1-q_2-p_1+p_2)^2 + \frac{c}{4 c^2 (4 + m^2)} (q_1-q_2+p_1-p_2)^2 \right].$$

(104)

In the trigonometric limit $\bar{m}$ vanishes and the first term in the exponential gives rise to a delta function

$$\tilde{S}_{q_1 q_2}^{(p_1 p_2)} = \delta(q_1+q_2-p_1-p_2) i g \ \text{coth} \ \frac{\theta}{2} \left( \frac{\sinh \theta}{8 \pi i} \right)^{1/2} \times \exp \left[ i \left( \frac{\theta}{2} (q_1+q_2)^2 - \frac{\theta}{2} (q_1-p_1)^2 \right) \right. 
+ \left. \frac{i}{2} \sinh \left( q_1-p_2 \right)^2 \right].$$

(105)

It implies that the scattering preserves the total charge, that is $q_1 + q_2 = p_1 + p_2$. This property is due to the invariance of the trigonometric $S$-matrix under the shift of all the
variables. Moreover, in the limit of large rapidity \( \theta \to \pm \infty \) a new delta function emerges. The combination \( ig \coth \frac{\theta}{2} \) at the same time tends to 1, as seen in figure 2, obtaining

\[
\lim_{\theta \to \pm \infty} \tilde{S}^{(p_1 p_2)}_{q_1 q_2}(\theta) = \delta(q_1 - p_2) \delta(q_2 - p_1) e^{\pm i q_1 q_2}. \tag{106}
\]

The exchange equation (103) reduces then to

\[
\tilde{A}_{q_1}(\theta_1) \tilde{A}_{q_2}(\theta_2) \to e^{\pm i q_1 q_2} \tilde{A}_{p_2}(\theta_2) \tilde{A}_{p_1}(\theta_1), \quad \theta_{12} \to \pm \infty. \tag{107}
\]

This expression is similar to the braiding of chiral and antichiral vertex operators \( e^{\pm i q_1 q_2} \) and \( e^{\pm i p_1 p_2} \) in the \( c = 1 \) CFT of a massless boson [27]. The result above suggests the existence of an explicit form of \( \tilde{A}_q(\theta) \) interpolating between the chiral and antichiral vertex operators for generic rapidity.

In the elliptic case the rapidity becomes cyclic with period \( \tau \). At the boundaries of the cycle, \( \theta = \pm \frac{\tau}{2} \), we have

\[
\tilde{S}^{(p_1 p_2)}_{q_1 q_2} \left( \pm \frac{\tau}{2} \right) = (1 + \sqrt{\mu}) e^{\pm i \sqrt{\mu}(q_1 + p_2)(q_2 + p_1)} \times \left( \frac{\sqrt{\mu_1}}{8\pi \sqrt{\mu}} e^{\pm i \sqrt{\mu_1}(q_1 - p_2)(q_2 - p_1)} \right). \tag{108}
\]

Writing the exponent of last term in parenthesis as the difference of two squares, we observe that it defines a gaussian distribution centered around \( q_1 = p_2 \) and \( q_2 = p_1 \) with breadth \( \frac{8\sqrt{\mu}}{\sqrt{\mu_1}} \). This is consistent with the periodic structure in the real direction of the rapidity, which acts as an effective UV cutoff and implies that there is no limit in the theory where the mass completely decouples. In the limit \( \mu \to 0 \) the gaussian distribution tends to a product of delta functions, recovering (106).

6. Conclusions

In this paper we have applied the theory of exactly solvable models to a massless and massive boson living in a two dimensional square lattice. We have shown that the Boltzmann weights satisfy the YBE, with the difference property in the rapidity variable, using a parameterization in terms of trigonometric functions in the massless model, and Jacobi elliptic functions in the massive model. In the former case, the partition function is invariant under the shift of the scalar field \( \phi \to \phi + \text{constant} \), while in the latter case it has the \( \mathbb{Z}_2 \) invariance \( \phi \to -\phi \).

These properties are reminiscent of the Boltzmann weights of the 6 vertex and 8 vertex models. We have calculated the eigenvalues and eigenvectors of the row-to-row transfer matrix, and the corresponding conserved quantities, that were also obtained using the QISM. In the massless case the connection with the \( c = 1 \) CFT was established. Finally, starting from the Boltzmann weights of the statistical mechanics model, we have proposed a scattering theory for massive particles with a continuous degree of freedom that satisfy all the standard axioms. We conjecture that the trigonometric \( S \)-matrix corresponds in the UV to a relevant perturbation of the \( c = 1 \) massless CFT. The field theory associated to the elliptic solution is more difficult to interpret due to the presence of a UV cutoff related to the modulus of the elliptic solutions.

A possibility, along the lines of the Zamolodchikov elliptic \( S \)-matrix model with \( \mathbb{Z}_4 \) symmetry, is that this field theory has limit cycles.

In this work we have considered a discretized free field theory, but the method can be in principle applied to lattice versions of integrable field theories. From another viewpoint, the results presented in this work could have applications to encode quantum information in continuous degrees of freedom [32], and the design of quantum circuits that enjoy an analog of crossing symmetry. Further investigations are required to clarify all these issues.
Acknowledgments

We would like to thank Francisco Alcaraz, Giuseppe Mussardo, Adrián Franco-Rubio, Paul Pierce and Guifrè Vidal for discussions. This work has been financed by the grants PGC2018-095862-B-C21, QUITEMAD + S2013/ICE-2801, SEV-2016-0597 of the ‘Centro de Excelencia Severo Ochoa’ Programme and the CSIC Research Platform on Quantum Technologies PTI-001.

Appendix A

The massless YBE can be constructed from (3) and (6). The explicit form of this equation is given by the matrix defined in (10). Its off-diagonal blocks are

\[
M_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_2 & 0 & 0 \end{pmatrix}, \quad M_{xz} = \begin{pmatrix} 0 & -c_1^{-1} & 0 \\ -c_1^{-1} & 0 & 0 \\ -c_2 & 0 & 0 \end{pmatrix}, \quad M_{x} = \begin{pmatrix} 0 & 0 & -c_2 \\ 0 & 0 & -c_1 \\ 0 & -c_1^{-1} & 0 \end{pmatrix},
\]

and \( M_{ij} = M_{ji} \). The diagonal blocks are

\[
M_{xx} = \begin{pmatrix} c_1 + c_1^{-1} & -c_1 & 0 \\ -c_1 & c_1 + c_1^{-1} & 0 \\ 0 & 0 & c_2 + c_2^{-1} \end{pmatrix}, \quad M_{yy} = \begin{pmatrix} c_2 + c_2^{-1} & 0 & 0 \\ 0 & c_3 + c_3^{-1} & -c_3 \\ 0 & -c_3 & c_3 + c_3^{-1} \end{pmatrix},
\]

\[
M_{zz} = M_{xx} + M_{yy} + M_{xy} + M_{yx}.
\]

When the integration over the \( z \) variables in (10) is done, we obtain the matrix \( N \) defined in (11). Its blocks are given by

\[
N_{xx} = M_{xx} - M_{x} M_{zz}^{-1} M_{zx},
\]

\[
N_{yy} = M_{yy} - M_{y} M_{zz}^{-1} M_{zy},
\]

\[
N_{xy} = N_{yx} = M_{xy} - M_{x} M_{zz}^{-1} M_{zy}.
\]

The massless YBE is then reduced to conditions \( N_{xx} = N_{yy} \) and \( N_{xy} = N_{yx} \). This system of equations has a unique solution which is given by

\[
c_2 = \frac{c_1 + c_3}{1 - c_1 c_3}.
\]

In the massive case, the analogues of (110) are given by

\[
\tilde{M}_{xx} = M_{xx} + \frac{1}{2} \begin{pmatrix} c_1 \tilde{m}_1^2 & 0 & 0 \\ 0 & c_1 \tilde{m}_1^2 & 0 \\ 0 & 0 & c_2 \tilde{m}_2^2 \end{pmatrix},
\]

\[
\tilde{M}_{yy} = M_{yy} + \frac{1}{2} \begin{pmatrix} c_2 \tilde{m}_2^2 & 0 & 0 \\ 0 & c_3 \tilde{m}_3^2 & 0 \\ 0 & 0 & c_3 \tilde{m}_3^2 \end{pmatrix}.
\]
\[
\tilde{M}_{zz} = M_{zz} + \frac{1}{2} \begin{pmatrix}
    c_1 \tilde{m}_1^2 + c_2 \tilde{m}_2^2 & 0 & 0 \\
    0 & c_1 \tilde{m}_1^2 + c_3 \tilde{m}_3^2 & 0 \\
    0 & 0 & c_2 \tilde{m}_2^2 + c_3 \tilde{m}_3^2
\end{pmatrix}.
\] (117)

The other blocks remain identical \( \tilde{M}_{ij} = M_{ij}, i \neq j \). After the integration of the \( z \) variables is performed, the resulting blocks are described by an equation analogous to (112)–(114) but replacing \( M_{ij} \rightarrow \tilde{M}_{ij} \). In this case, conditions \( N_{xx} = N_{yy} \) and \( N_{xy} = N_{yx} \) form a system of equations which has the solution given by (17) and (18).

**Appendix B**

The expectation values of the coordinate transfer matrix are codified in (24) in terms of a matrix \( M \) analogous to the ones used in the appendix A. Its blocks are

\[
M_{xx} = M_{yy} = \frac{1}{2} M_{zz} = a 1, \quad a = \frac{1}{c} + c \frac{\tilde{m}}{2},
\]

\[
M_{xy} = 0, \quad M_{xz} = -\frac{1}{c} S - c 1, \quad M_{yz} = -\frac{1}{c} 1 - c S,
\] (118)

where the shift matrix is defined as \( S_{ij} = \delta_{i,j+1} \), with \( L + 1 \equiv 1 \). When the integration of the \( z \) variables is done, we obtain a matrix \( N \) with the same structure as (112)–(114). The computation of this matrix is straightforward, its explicit form is shown in (26) and (27).

The elements of the transfer matrix can be written as the product of two factors, \( T_p \) and \( T_q \), defined in (28) and (29). When the lattice anisotropy \( c \) is small, \( T_p \) becomes a sharply picked Gaussian which can be expanded in terms of delta functions. Let \( f(x) \) be an analytic function, then we can expand

\[
\frac{1}{\sqrt{\pi c}} \int dx \ e^{-x^2/c} f(x) = \frac{1}{\sqrt{\pi c}} \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} \int dx x^{2k} e^{-x^2/c}.
\] (120)

The integrals can be easily performed, and the derivatives of the function \( f \) reexpressed as follows

\[
f^{2k}(0) = \int dx \delta^{2k}(x) f(x).
\] (121)

Substituting this in (120) we obtain

\[
\frac{1}{\sqrt{\pi c}} \int dx e^{-x^2/c} f(x) = \sum_{k=0}^{\infty} \frac{c^k}{k! 4^k} \int dx \delta^{2k}(x) f(x),
\] (122)

which in the limit \( c \to 0 \) implies the desired expansion

\[
\frac{1}{\sqrt{\pi c}} e^{-x^2/c} = \delta(x) + \frac{c}{4} \delta''(x) + \mathcal{O}(c^2).
\] (123)

Finally, we will clarify some technical details on the diagonalization of the transfer matrix performed in section 3. Using the ansatz

\[
|\Psi\rangle = \int \tilde{d}x f_n(x) e^{-\frac{1}{2} \tilde{k} x^T \tilde{k}} |x\rangle,
\] (124)
and the transfer matrix elements (25)–(27), the eigenstate condition $T(u) \mid \Psi \rangle = \Lambda \mid \Psi \rangle$ becomes

$$
\frac{1}{(4\pi ac^2)^2} \int \tilde{d}\bar{x} f_n(\bar{x}) e^{-\frac{i}{2}(\bar{K}+N_1)\vec{v}^T - \frac{i}{2}yN_2\vec{v}^T} = \Lambda f_n(y)e^{-\frac{i}{2}yK\vec{v}^T}.
$$

(125)

In order to perform the integration over the $x$ variables, we make the shift $\bar{x} \to \bar{x} - \bar{y}N_2^{-1}(N_1 + K)^{-1}$ and obtain

$$
\frac{1}{(4\pi ac^2)^2} \int \tilde{d}\bar{x} f_n(\bar{x} - \bar{y}N_2^{-1}(N_1 + K)^{-1}) e^{-\frac{i}{2}(\bar{K}+N_1)\vec{v}^T} = \Lambda f_n(y)e^{-\frac{i}{2}yK\vec{v}^T},
$$

(126)

where $\bar{K} = K - N_1 + N_2(N_1 + K)^{-1}N_2$. This relation implies the following two conditions

$$
K - N_1 + N_2(N_1 + K)^{-1}N_2 = 0,
$$

(127)

$$
\int \tilde{d}\bar{x} f_n(\bar{x} - \bar{y}N_2^{-1}(N_1 + K)^{-1}) e^{-\frac{i}{2}(\bar{K}+N_1)\vec{v}^T} = (4\pi ac^2)^2 \Lambda f_n(y).
$$

(128)

The first equation leads to the solution (39) for $K$. The second allows to obtain both the functions $f_n$ and the eigenvalues $\Lambda$. We make an ansatz for $f_n$ based on the Hermite polynomials

$$
f_n(\bar{x}; \vec{v}) = \rho e^{iK\vec{v}^T} \left( \vec{v}, \frac{\partial}{\partial \bar{x}} \right)^n e^{-i\bar{x}\vec{v}^T},
$$

(129)

with $\rho$ a normalization constant and $\vec{v}$ a vector to be determined. This expression is inserted in (128) and the integration variables shifted $\bar{x} \to \bar{x} - 2\bar{y}N_2^{-1}K$ to rewrite the $lhs$ as

$$
\rho e^{iK\vec{v}^T} \int \tilde{d}\bar{x} e^{-\frac{i}{2}N_1\vec{v}^T} \left( \vec{v}, \frac{\partial}{\partial \bar{x}} \right)^n e^{-i\bar{x}\vec{v}^T},
$$

(130)

where $\bar{z} = \bar{x} - \bar{y}N_2^{-1}(N_1 + K)$. Using now that $\partial_z f(\bar{z}) = -(N_1 + K)^{-1}N_2 \cdot \partial_z f(\bar{z})$, the derivative can be brought outside the integration. We can then easily perform the gaussian integration, reducing (128) to

$$
\Lambda_0 \rho e^{iK\vec{v}^T} \left( -\vec{v}(N_1 + K)^{-1}N_2 \cdot \frac{\partial}{\partial y} \right)^n e^{-i\bar{y}\vec{v}^T} = \Lambda f_n(y; \vec{v}).
$$

(131)

The constant $\Lambda_0$ is the eigenvalue of the transfer matrix on the ground state, $f_0 = 1$, given in (41). This equation is satisfied provided $\vec{v}$ is an eigenstate of the matrix $-(N_1 + K)^{-1}N_2$, whose eigenvalues are

$$
\lambda = e^{i\kappa(1 + c^2 e^{-\pi p})^2},
$$

(132)

with $p = \frac{2L}{T}$ and $k = 0, \ldots, L - 1$. Then $\Lambda = \Lambda_0 \lambda^n$.

**Appendix C**

The model we are considering has an infinite tower of conserved charges derived from the expansion of the transfer matrix. We proposed in (73) and (74) the following operator form for the charges

$$
Q_{2a} = \frac{(-1)^a}{a} a_t \tilde{\pi} \left( S_n^\alpha - S_n^\beta \right) \tilde{x}, \quad Q_{2a+1} = \frac{1}{2a+1} \left( a_t^2 \tilde{\pi} K^2 \tilde{\pi} + \tilde{x} K^{2a+2} \tilde{x} \right).
$$

(133)
The aim of this appendix is to check that they indeed commute among themselves, and that their expectation values agree with (48).

The vanishing commutation between even charges can be easily derived from

\[
[\vec{\pi}A\vec{x}, \vec{\pi}B\vec{x}] = i\vec{\pi}[A, B]\vec{x},
\]

identifying \( A = S^e - S^o \) and \( B = S^o - S^e \), and using that \( S \) and its transpose commute. Between odd charges from

\[
[\vec{\pi}A\vec{x}, \vec{\pi}B\vec{x}] = i\vec{\pi}A(B + B^T)\vec{x} + i\vec{\pi}A(B + B^T)\vec{x},
\]

identifying \( A = K^{2a+2} \) and \( B = K^{2b} \), and using that \( K \) is symmetric. Between even and odd charges from

\[
[\vec{\pi}A\vec{x}, \vec{\pi}B\vec{x}] = i\vec{\pi}B(A + A^T)\vec{x}, \quad [\vec{\pi}A\vec{x}, \vec{\pi}B\vec{x}] = i\vec{\pi}(A + A^T)B\vec{x}
\]

identifying \( A = K^2 \) and \( B = S^e - S^o \) and using that, since \( K \) and \( S \) commute, \((A + A^T)B = B(A + A^T)\) is an antisymmetric matrix.

The expectation values of the tower of conserved charges can be derived expanding the eigenvalues of the transfer matrix. In order to avoid powers of lower charges contributing to higher ones, we consider the logarithm of a generic eigenvalue

\[
\log \Lambda = ip - \sum_{k=1}^{L-1} (2n_k + 1) \log c(a + \omega_k) + \sum_{k=1}^{L-1} 2n_k \log (1 + c^2 e^{-i\omega_k}),
\]

The third term on the \( rhs \) only contains even powers of the uniformization parameter \( u \), while the second contributes to both even and odd powers. We can rewrite

\[
\log (c(a + \omega_k)) = \frac{1}{2} \left[ \log c^2(a^2 - \omega_k^2) + \log \frac{a + \omega_k}{a - \omega_k} \right].
\]

Using (38) and (40), we can see that

\[
c^2(a^2 - \omega_k)^2 = (1 + c^2 e^{i\omega_k})(1 + c^2 e^{-i\omega_k}).
\]

Substituting we have

\[
\log \Lambda = ip - \sum_{k=0}^{L-1} \left( n_k + \frac{1}{2} \right) \log \frac{a + \omega_k}{a - \omega_k} + \sum_{k=0}^{L-1} n_k \log \frac{1 + c^2 e^{-i\omega_k}}{1 + c^2 e^{i\omega_k}} - \log(1 - c^2)^2.
\]

The second term only contains now odd powers of \( u \) and the third even ones. The second and third terms contribute therefore to different conserved charges. Using in \( a^{-1} \) as expansion parameter for the second term and \( c^2 \) for the third, we derive the following tower of conserved charges

\[
\langle Q_{2a+1} \rangle = \frac{2}{2a+1} \sum_{k=0}^{L-1} \left( n_k + \frac{1}{2} \right) \omega_k^{2a+1}, \quad \langle Q_{2a} \rangle = \frac{(-1)^a}{a} \sum_{k=1}^{L-1} 2n_k \sin(p_k a).
\]
Notice that \( \langle Q^2_a \rangle \), with \( a \) a multiple of \( L \), is trivially zero. There are however additional contributions to the eigenvalues of the transfer matrix at orders \( c^{2L} \), coming from the last term in (140). These contributions are independent of the occupation numbers. They are also independent from momenta and energies, contrary to the vacuum piece of the odd charges vacuum expectation value. In terms of operators they must be associated with the identity, and hence are of no relevance to the integrability structure. For this reason we have dropped them in the main text, equation (46).

Finally we will show that the expectation values of the operators (133) agree with (141). For simplicity, the check will be performed only over one-particle states. For them (129) reduces to

\[
f_1(\vec{x}; \vec{v}) = -2\rho \vec{v} K \vec{x}, \tag{142}
\]

where \( \vec{v} \) is an eigenvector \((N_1 + K)^{-1}N_2\), or equivalently, of the shift matrix \( S \). Hence \( \vec{v} = \vec{v}_k \), with \((v_k)_i = \frac{1}{\sqrt{L}} e^{i p_k i}.\) Using (33), for the even charges we have

\[
\langle Q^2_a \rangle_{1k} = -4i \rho^2 \int d\vec{x} d\vec{y} (\vec{v}_k K \vec{x}) (2\vec{v}_k K \vec{x} + (\vec{v}_k K \vec{x}) \text{tr} K^{2a+1}) e^{-i \vec{x} K \vec{y}}. \tag{143}
\]

with \( A = \frac{(-1)^{p_k}}{a} (S^{2a+1} - S^a) \). This simplifies to

\[
\langle Q^2_a \rangle_{1k} = 4i \rho^2 \int d\vec{x} (\vec{v}_k K \vec{x}) (\vec{v}_k K \vec{x} - (\vec{v}_k K \vec{x}) \text{tr} K \vec{x} - \vec{x}) e^{-i \vec{x} K \vec{y}}. \tag{144}
\]

Upon integration, the second term in the parenthesis gives a vanishing contribution and we obtain

\[
\langle Q^2_a \rangle_{1k} = -\frac{i \rho^2}{2a+1} \int d\vec{x} (\vec{v}_k K \vec{x}) (\vec{v}_k K \vec{x}) e^{-i \vec{x} K \vec{y}}. \tag{145}
\]

where we have used that \( \vec{v}_k \) is also an eigenvector of \( K \). Along the same lines, the odd charges lead to

\[
\langle Q^2_{a+1} \rangle_{1k} = 4i \rho^2 \int d\vec{x} (\vec{v}_k K \vec{x}) (2\vec{v}_k K^{2a+1} \vec{x} + (\vec{v}_k K \vec{x}) \text{tr} K^{2a+1}) e^{-i \vec{x} K \vec{y}}. \tag{146}
\]

The second term in the parenthesis is the vacuum contribution to the vacuum expectation value of the odd charges. Integrating we get

\[
\langle Q^2_{a+1} \rangle_{1k} = \frac{2}{2a+1} \left( \frac{1}{2} \omega_k^{2a+1} + \frac{1}{2} \sum_{l=1}^{L-1} \omega_k^{2a+1} \right). \tag{147}
\]

**Appendix D**

In this appendix we find a solution \( g(\theta) \) of equation (88). The method will be later generalized to the elliptic model. In what follows we shall use the variable

\[
z = \frac{\theta}{2\pi i}. \tag{148}
\]
We first investigate the periodicity properties of $g(z)$. Combining the unitarity (86) and crossing symmetry relations (88) one gets

$$g(z + 1) = \tan^4(\pi z) g(z),$$

(149)

that implies that $g(z)$ is not a periodic function in $z$ (or periodic in $\theta$ with period $2\pi i$). Some $S$-matrices exhibit this periodicity that is equivalent to the double sheet structure in the $s$-plane, where $s$ is the Mandelstam variable [21]. In our case, the lack of periodicity implies that the $S(s)$ has an infinite number of sheets in the $s$ complex plane.

Let us define the function

$$h(z) = \lim_{M \to \infty} \prod_{n=-M}^{M} h_n(z), \quad h_n(z) = \frac{\Gamma(n + z) \Gamma(n + \frac{1}{2} - z)}{\Gamma(n - z) \Gamma(n + \frac{1}{2} + z)},$$

(150)

where $h_n(z)$ satisfies

$$h_n \left( \frac{1}{2} - z \right) = \frac{n - \frac{1}{2} + z}{n - z} h_n(z).$$

(151)

Using

$$\tan(\pi z) = \lim_{M \to \infty} \prod_{n=-M}^{M} \frac{n - z}{n - \frac{1}{2} + z},$$

(152)

we find that

$$h \left( \frac{1}{2} - z \right) = \frac{h(z)}{\tan(\pi z)} \to h^2 \left( \frac{1}{2} - z \right) = \frac{h^2(z)}{\tan^2(\pi z)},$$

(153)

which gives a solution of the crossing symmetry relation satisfied by $g(z)$.

$$g(z) = h^2(z) = \lim_{M \to \infty} \prod_{n=-M}^{M} \left( \frac{\Gamma(n + z) \Gamma(n + \frac{1}{2} - z)}{\Gamma(n - z) \Gamma(n + \frac{1}{2} + z)} \right)^2,$$

(154)

and the equations (85)–(88). The product form of (154) is a regularization of the function $g(z)$. If we naively replace the limit by an infinite product then (154) would be invariant under the replacement $z \to z + 1$, in contradiction with (149).

Next, we find an expression for $\log g(z)$ using the formula [23]

$$\log \Gamma(x) = \left( x - \frac{1}{2} \right) \log x - x - \frac{1}{2} \log(2x) + \int_{0}^{\infty} \frac{dt}{t} e^{-t} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right), \quad \text{Re } x > 0.$$

(155)

This equation can be applied in (154) to the terms where $n > 0$ but not for those where $n < 0$. However, the latter terms can be transformed into the former ones using the relations

$$\Gamma(z + 1) = z \Gamma(z), \quad \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},$$

(156)

and

$$\log \Gamma(x) = \left( x - \frac{1}{2} \right) \log x - x - \frac{1}{2} \log(2x) + \int_{0}^{\infty} \frac{dt}{t} e^{-t} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right), \quad \text{Re } x > 0.$$
that allow us to write
\[
\prod_{n=-M}^{M} h_n(z) = (-1)^{M+1} \frac{\Gamma \left( \frac{1}{2} - z \right) \Gamma (M + 1 + z)}{\Gamma (\frac{1}{2} + z) \Gamma (M + 1 - z)} \prod_{n=1}^{M} h_n^2(z). \tag{157}
\]

Performing the sum over \( n \) and taking the limit \( M \to \infty \) yields
\[
\log g(z) = \int_{0}^{\infty} \frac{dt}{t} \frac{\sinh(tz)}{\cosh^2(t/4)}, \quad |\text{Re } z| < \frac{1}{2}. \tag{158}
\]

An alternative expression is obtained taking the derivative respect to \( z \)
\[
\frac{d}{dz} \log g(z) = \int_{0}^{\infty} \frac{dt}{t} \frac{\cosh(tz)}{\sinh(t/4)} = \frac{8\pi z}{\sinh(2\pi z)}, \tag{159}
\]
and integrating back
\[
\log g(z) = \frac{i\pi}{2} + 4z \log \frac{1 - e^{2\pi i z}}{1 + e^{2\pi i z}} + \frac{2i}{\pi} \left( \text{Li}_2(-e^{2\pi i z}) - \text{Li}_2(e^{2\pi i z}) \right), \tag{160}
\]
where \( \text{Li}_2(x) \) is a particular case of the polylogarithmic function defined by the analytic extension of the series
\[
\text{Li}_i(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^i}, \quad |z| < 1, \tag{161}
\]
to the complex plane. For \( z = 1 \) it becomes the Riemann zeta function, that is \( \text{Li}_i(1) = \zeta(i) \).

In the massive model, the function \( g(z, \mu) \), can be found employing the previous results. Indeed, the crossing symmetry relation satisfied by \( g(z, \mu) \) reads
\[
g \left( \frac{1}{2} - z, \mu \right) = \frac{1}{c(z, \mu)} g(z, \mu), \tag{162}
\]
where
\[
c(z, \mu) = \sqrt{\frac{\text{sn}(2Kz)}{\text{cn}(2Kz) \text{dn}(2Kz)}}. \tag{163}
\]
with \( K = K(\mu) \) defined in (22). In the trigonometric case it becomes \( c(z, 0) = \tan(\tau z) \). The function (163) satisfies \( c(z, \mu) = -c(z \pm \frac{i\pi}{2\mu}, \mu) \), where \( \tau = \frac{K'}{K} \) and \( K' = K(\mu_1) \). It is related to \( c(z, 0) \) as
\[
c(z, \mu) = \lim_{M \to \infty} \prod_{n=-M}^{M} \tan \left[ \pi \left( z + \frac{i\pi \tau}{2} \right) \right]. \tag{164}
\]

This yields a solution for \( g(z, \mu) \) in terms of trigonometric function (154)
\[
g(z, \mu) = \lim_{M \to \infty} \prod_{n=-M}^{M} g \left( z + \frac{i\pi \tau}{2} \right), \tag{165}
\]
that using (154) becomes
\[
g(z, \mu) = \lim_{M \to \infty} \prod_{n=-M}^{M} \prod_{m=-M}^{M} \left[ \frac{\Gamma \left( n + \frac{im\tau}{2\mu} + z \right) \Gamma \left( n + \frac{im\tau}{2\mu} - z \right) \Gamma \left( n + \frac{i\pi}{2} + \frac{im\tau}{2\mu} - z \right) \Gamma \left( n + \frac{i\pi}{2} + \frac{im\tau}{2\mu} + z \right)}{\Gamma \left( n + \frac{i\pi}{2\mu} + z \right) \Gamma \left( n + \frac{i\pi}{2\mu} - z \right)} \right]^2. \tag{166}
\]
Figure 4. Plot of the real and imaginary parts of the eigenvalues \( \lambda_i \) for \( \mu = 0.3 \) and \( \text{Re} z = 0.05 \).

Similarly, using (158) we get

\[
\log g(z, \mu) = \lim_{M \to \infty} \sum_{m=1}^{M} \log g \left( z + \frac{im \pi}{2} \right)
= \frac{2\pi^2}{r} \tau + \sum_{n=1}^{\infty} \frac{1}{n} \sinh(4n \pi^2 z / r) \cosh(2n \pi^2 r / \tau),
\]

| Re \( z \) | < \( \frac{1}{2} \).

(167)

Using this expression, we see that \( g \) has the same periodicity along imaginary \( z \) as \( c \), namely \( g(z, \mu) = -g(z \pm \frac{im \pi}{2}, \mu) \). Hence the prefactor in the elliptic \( S \)-matrix

\[
S_{y_1y_2}^{x_1x_2} = \frac{g}{2 \pi c} e^{\sum_{i=1}^{4} \lambda_i e^i},
\]

(169)

where

\[
e_1 = \frac{1}{2} (x_1 + x_2 + y_1 + y_2), \quad \lambda_1 = \frac{c}{4} \tilde{m}^2,
\]

(170)

\[
e_2 = \frac{1}{2} (x_1 + x_2 - y_1 - y_2), \quad \lambda_2 = \frac{1}{c} + \frac{c}{4} \tilde{m}^2,
\]

(171)

\[
e_3 = \frac{1}{2} (x_1 - x_2 + y_1 - y_2), \quad \lambda_3 = c + \frac{c}{4} \tilde{m}^2,
\]

(172)

\[
e_4 = \frac{1}{2} (x_1 - x_2 - y_1 + y_2), \quad \lambda_4 = \frac{1}{c} + c + \frac{c}{4} \tilde{m}^2.
\]

(173)

The real and imaginary part of \( \lambda_i \) is plotted in figure 4 along a cycle \( |\text{Im} z| \leq \frac{\tau}{2} \). In spite of the manifest \( \frac{\tau}{2} \) periodicity of the eigenvalues, an effective cyclicity with half the period emerges by combining \( z \to z \pm \frac{im \pi}{2} \) with the exchange \( \lambda_1 \leftrightarrow \lambda_4 \) and \( \lambda_2 \leftrightarrow \lambda_3 \). This exchange just amounts to a change of sign in the continuous labels \( x_2 \) and \( y_1 \).
Consistency requires that the massive $S$-matrix has a well behaved limit when $|e_i| \to \infty$. This implies that the eigenvalues $\lambda_i$ must always have a non-negative real part. Figure 4 shows that this is indeed the case, and provides a non-trivial test since $c$ alone fails to fulfill this requirement on half the $\pi$ cycle.

ORCID iDs

Manuel Campos © https://orcid.org/0000-0002-5533-3393

References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[2] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[3] Gaudin M 1995 *Modèles Exactement Résolus* (France: Les éditions de Physique)
[4] Gómez C, Ruiz-Altaba M and Sierra G 1996 *Quantum Groups in Two-Dimensional Physics* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[5] Sutherland B 2004 *Beautiful Models: 70 Years of Exactly Solved Quantum Many-body Problems* (Singapore: World Scientific)
[6] Dukelsky J, Pittel S and Sierra G 2004 Colloquium: exactly solvable Richardson–Gaudin models for many-body quantum systems Rev. Mod. Phys. 76 643
[7] Mussardo G 2010 *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics* (Oxford: Oxford University Press)
[8] Tsvelik A M 1995 *Quantum Field Theory in Condensed Matter Physics* (Cambridge: Cambridge University Press)
[9] Gogolin A O, Nersesyan A A and Tsvelik A 1998 *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
[10] Giamarchi T 2004 *Quantum Physics in One Dimension* (Oxford: Clarendon)
[11] White S R 1992 Density matrix formulation for quantum renormalization groups Phys. Rev. Lett. 69 2863
[12] Schollwöck U 2005 The density-matrix renormalization group Rev. Mod. Phys. 77 259
[13] Orús R 2019 Tensor networks for complex quantum systems Nat. Rev. Phys. 1 538
[14] Minahan J A and Zarembo K 2003 The Bethe ansatz for script $N$ script $= 4$ super Yang–Mills J. High Energy Phys. JHEP03(2003)013
[15] Besert N and Staudacher M 2003 The SYM integrable super spin chain Nucl. Phys. B 670 439
[16] Hernández R, López E, Periáñez A and Sierra G 2005 Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings J. High Energy Phys. JHEP06(2005)011
[17] Murg V, Korepin V E and Verstraete F 2012 The algebraic Bethe ansatz and tensor networks Phys. Rev. B 86 045125
[18] Chong Y Q, Murg V, Korepin V and Verstraete F 2015 The nested algebraic Bethe ansatz for the supersymmetric t-J model and tensor networks Phys. Rev. B 91 195132
[19] Baxter R J 1972 Partition function of the eight-vertex lattice model Ann. Phys., NY 70 193
[20] Zamolodchikov A B 1979 $Z_4$-symmetric factorized $S$-matrix in two space-time dimensions Commun. Math. Phys. 69 165
[21] Mussardo G and Penati S 2000 A quantum field theory with infinite resonance states Nucl. Phys. B 567 454
[22] Campos M, Sierra G and López E 2019 Tensor renormalization group in bosonic field theory Phys. Rev. B 100 195106
[23] Abramowitz M and Stegun I 1972 *Handbook of Mathematical Functions* (New York: Dover)
[24] Galindo A and Pascual P 1990 *Quantum Mechanics I* (Theoretical and Mathematical Physics) (Berlin: Springer) Texts and Monographs in Physics
[25] Faddeev L D 1995 How algebraic Bethe ansatz works for integrable model Symétries Quantiques: Proc. of Les Houches School of Physics, Session LXIV (North-Holland 1998) pp 149–219
[26] Drinfeld V G Quantum groups 1987 Proc. of the Int. Congress of Mathematics (Berkeley 1986) ed A M Gleason vol 1 (Providence, RI: American Mathematical Society) p 798
[27] Di Francesco P, Mathieu P and Sénéchal D 1997 Conformal Field Theory (Berlin: Springer)
[28] Zamolodchikov A B and Zamolodchikov A B 1979 Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models Ann. Phys., NY 120 253
[29] Wilson K G 1971 Renormalization group and strong interactions Phys. Rev. D 3 1818
[30] Bazhanov V V and Sergeev S M 2012 A master solution of the quantum Yang–Baxter equation and classical discrete integrable equations Adv. Theor. Math. Phys. 16 65
[31] Bazhanov V V and Sergeev S M 2012 Elliptic gamma-function and multi-spin solutions of the Yang–Baxter equation Nucl. Phys. B 856 475
[32] Gottesman D, Kitaev A and Preskill J 2001 Encoding a qubit in an oscillator Phys. Rev. A 64 012310