SEMICLASSICAL ANALYSIS FOR LOGARITHMIC SCHRÖDINGER EQUATIONS WITH DECAYING POTENTIALS

XIAOMING AN

Abstract. In this paper, we consider the following logarithmic Schrödinger equation
\[-\varepsilon^2 \Delta u + V(x)u = u \log u^2 \text{ in } \mathbb{R}^N,\]
where \(\varepsilon > 0, N \geq 1, V(x) \in C(\mathbb{R}^N, \mathbb{R})\) is a continuous potential. We use variational methods and a new truncated skill to show the problem has a positive solution concentrating at a local minimum of \(V\) if \(\varepsilon \in (0, \varepsilon_0)\) for some \(\varepsilon_0 > 0\) is a small constant. All decay rates of \(V\) are admissible in this work.

Key words: Logarithmic Schrödinger equation; variational; truncated skill; decay rates; concentration.

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1. Introduction

We study the following Schrödinger equation with logarithmic nonlinear term:
\[-\varepsilon^2 \Delta u + V(x)u = u \log u^2, \ x \in \mathbb{R}^N, \tag{1.1}\]
where \(\varepsilon > 0, N \in \mathbb{N}, V(x) \in C(\mathbb{R}^N, \mathbb{R})\) is a continuous potential. Problem (1.1) admits applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation (see [29] and the references therein for more details).

The general form of equation (1.1) is
\[-\varepsilon^2 \Delta u + V(x)u = f(u), \ x \in \mathbb{R}^N, \tag{1.2}\]
which comes from the study of standing waves \(\psi(x, t) = e^{iEt/\varepsilon}i(x)\) of the following nonlinear Schrödinger equation:
\[i \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + (V(x) + E)\psi - f(\psi).\]

In recent decades, under some assumptions on the nonlinear term \(f\) and potential term \(V\), a lot of valuable work have been done on (1.2), showing that (1.2) has a family of solutions \(u_\varepsilon\) concentrating at local minimum(minima) or non-degenerate critical point(s) of \(V\) as \(\varepsilon \to 0\), see [2, 3, 7, 11, 17, 18, 21, 23, 24, 27] and the references therein for example.

The results for (1.1) with \(\varepsilon = 1\) and \(V \equiv \text{const.}\) are not so many as that for equation (1.2) with \(f(t)/t = o(t)\) as \(t \to 0\). The main reason is that the natural Euler-Lagrange
functional corresponding to (1.1)

\[ I(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda + 1)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx \]  

(1.3)
is not \(C^1\). Indeed, there exists \(u \in H^1(\mathbb{R}^N)\) such that \(\int_{\mathbb{R}^N} u^2 \log u^2 = -\infty\). Due to this loss of smoothness, in order to study the existence of solutions, to the best of our knowledge, at least four approaches were used so far in the literature. One is constructing a suitable Banach space \(B\) in which the functional \(I : B \to \mathbb{R}\) is \(C^1\), see [12] for example. The second way is penalizing the nonlinear term around the origin and then try to obtain a priori estimates to get a nontrivial solution at the limit, see [20]. However, the drawback of the first two ways is that the Palais-Smale condition cannot be obtained. The third way is, in order to get the \((P.S.)\) condition, restricting the functional on \(H^1_{rad}(\mathbb{R}^N)\) and regarding \(I\) as a merely lower semicontinuous and by applying the nonsmooth critical point theory of [15], see [14] for example. The fourth way is, respecting to the case that no radial restriction, decomposing the functional \(I\) into the sum of a \(C^1\) functional and a convex l.s.c (short for lower semicontinuous hereafter) functional and using the mountain pass Theorem 3.2 [26] for convex l.s.c functionals to find a critical point, see [25].

The semiclassical study for (1.1), i.e., the existence and concentration of (1.1) when \(\varepsilon \to 0\), are also very few. In [1], assuming that there exists an open bounded set \(\Omega\) such that

\[ \inf_{x \in \partial \Omega} (V(x) + 1) > \inf_{x \in \Omega} (V(x) + 1) = V_0 = \inf_{x \in \mathbb{R}^N} (V(x) + 1) \]
it used the fourth way stated above and the penalized idea in [18] to prove that (1.1) has a family of positive solutions \((u_\varepsilon)_{0 < \varepsilon < \varepsilon_0}\) concentrating at a local minimum of \(V\) in \(\Omega\) as \(\varepsilon \to 0\).

To the best of our knowledge, there is no semiclassical result for (1.1) when the local minimum of \(V\) is not global or \(V\) is vanishing. Our work of the present paper is considering (1.1) with \(V\) vanishing at infinity or is compactly supported. Assuming that \(V\) is continuous and satisfies

\((V_1)\) \(V(x) + 1 \geq 0 \quad x \in \mathbb{R}^N\)

and

\((V_2)\) there exists two bounded open sets \(\Lambda, U\) with smooth boundaries \(\partial \Lambda, \partial U\), such that

\[ 0 < \inf_{\Lambda} V(x) < \inf_{U \setminus \Lambda} V(x), \]

(1.4)
we have our main result:

**Theorem 1.1.** Let \(V\) satisfy \((V_1)\) and \((V_2)\) and assume moreover that \(N \geq 3\) if \(\lim_{|x| \to \infty} \inf(V(x) + 1)|x|^2 = 0\). Then there exists a \(\varepsilon_0 > 0\) such that (1.1) has a positive solution \(u_\varepsilon\) if \(\varepsilon \in (0, \varepsilon_0)\). Moreover, \(u_\varepsilon\) has a global maximum point \(x_\varepsilon\) satisfying \(\lim_{\varepsilon \to 0} V(x_\varepsilon) = \inf_{x \in \Lambda} V(x)\) and

\[ u_\varepsilon(x) \leq \begin{cases} \tilde{C}e^{-\varepsilon_0 - \frac{|x - x_\varepsilon|}{\varepsilon}}, & \text{if } \lim_{|x| \to \infty} \inf(V(x) + 1)|x|^{2\sigma} > 0 \text{ with } 0 \leq \sigma < 1, \\ \tilde{C}e^{-\varepsilon_0 - 2 - \frac{N-2}{|x - x_\varepsilon|^{N-2}}}, & \text{if } \lim_{|x| \to \infty} \inf(V(x) + 1)|x|^2 \geq 0, \end{cases} \]
where $\tilde{c}_\sigma > 0$ is positive constant depending only on $\sigma$ and $\tilde{C} > 0$ is a positive constant.

The difficulties in proving Theorem 1.1 mainly lie in the following two aspects. Firstly, as we said before, the natural Euler-Lagrange functional corresponding to (1.1),

$$I_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + (V(x) + 1)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx,$$

is not smooth, which makes all the methods for equation (1.2) with $f(t)/t = o(t)$ as $t \to 0$ failed. Thanks to the Mountain Pass Theorem 3.2 in [26], we expect to find a critical point of $I_\varepsilon$ by decomposing $I_\varepsilon$ into the sum of a $C^1$ functional and a convex l.s.c functional. However, the expectation that the concentration of should occurs at a local minimum of $V$ in $\Lambda$ makes it necessary to truncate the nonlinear term outside $\Lambda$. In [1], under the global minimum assumption on $V$, i.e.,

$$\inf_{x \in \partial \Lambda} (V(x) + 1) > \inf_{x \in \Lambda} (V(x) + 1) = V_0 = \inf_{x \in \mathbb{R}^N} (V(x) + 1),$$

the two difficulties above was overcome by using the penalized idea in [18] and the decomposed skill in [25]. Differently, the potential $V$ here may be vanishing at infinity, which makes us have to truncate the logarithmic term with a new function. Moreover, the penalized methods in those papers [8, 22] which deal with problem (1.2) with vanishing potential are also failed, since here the nonsmoothness of $I_\varepsilon$ is caused by the fact that $\lim_{t \to 0^+} \frac{t \log t^2}{t} = -\infty$. In this paper, we use the characteristic function $\chi_{\mathbb{R}^N \setminus \Lambda}$ to truncate the nonlinear term, see (2.1) below for more details. This truncation makes us need not to decompose the logarithmic term as that in [1, 25] and the proofs of some properties such as $(PS)_c$ condition and mountain pass geometry of the penalized functional easier to understand. Moreover, the truncation makes the construction of super-solution easier than that in [8, 22], see the last part of Section 3 below for more details.

**Plan of the paper.** In Section 2, we obtain the penalized problem by truncating the logarithmic term in (1.1) outside by $\chi_{\mathbb{R}^N \setminus \Lambda} u$, then we use the Mountain Pass Theorem 3.2 in [26] to obtain a penalized solution $u_\varepsilon$. In Section 3, we study the concentration of $u_\varepsilon$ and use it to linearize the penalized equation in Section 2. At the last of Section 3, we construct suitable supersolution for the linearized equation to show the asymptotic behaviour of $u_\varepsilon$, from which we know $u_\varepsilon$ solves the origin problem.
2. Variational setting and penalized problem

According to different decay rates of $V$, we define the Hilbert space $\mathcal{D}_{V,\varepsilon}^1(\mathbb{R}^N)$ as

$$
\mathcal{D}_{V,\varepsilon}^1(\mathbb{R}^N) = \left\{ \nabla u \in L^2(\mathbb{R}^N) : (V(x) + 1)|u|^2 \in L^1(\mathbb{R}^N) \right\},
$$

where $\mathcal{D}_{0}^1(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$
\|u\|_{\varepsilon}^2 = \int_{\mathbb{R}^N} (\varepsilon^2|\nabla u|^2 + V(x)|u|^2),
$$

with the norm

$$
\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.
$$

We want to use the Mountain Pass Theorem [26, Theorem 3.2] to find a solution for (1.1). Considering the vanishing of $V$ and the concentration should occur in $\Lambda$, we modified the nonlinear term as follows. Define

$$
G_1(x,s) = \chi_\Lambda(x)s_+^2 \log s_+^2 \quad \text{and} \quad G_2(x,s) = \chi_{\mathbb{R}^N \setminus \Lambda}(x) \int_{0}^{s_+} \max\{4t_+, -2t_+ (1 + \log t_+^2)\} dt.
$$

Noting that $G_2(x,s) \geq 0$ and is convex since $G_2(x,s)$ is nondecreasing on $s$ for all $x \in \mathbb{R}^N \setminus \Lambda$, hence the functional $G^2 : \mathcal{D}_{V,\varepsilon}^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$
G^2(u) = \frac{1}{2} \int_{\mathbb{R}^N} G_2(x,u)
$$

is convex and l.s.c by Fatou’s Lemma. The boundedness of $\Lambda$ implies the functional $G^1 : \mathcal{D}_{V,\varepsilon}^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$
G^1(u) = \frac{1}{2} \int_{\mathbb{R}^N} G_1(x,u)
$$

is $C^1$. Hence, the functional $J_\varepsilon(u) = \Phi_\varepsilon(u) + \Psi(u)$ with $\Phi_\varepsilon(u) = \frac{1}{2} \|u\|^2_{V,\varepsilon} - G^1(u)$ and $\Psi = G^2(u)$ has the form stated in [26]. As a result, we can still use the Mountain Pass Theorem 3.2 in [26] to find a critical point for $J_\varepsilon$, although $J_\varepsilon$ is not $C^1$. We first stated some necessary definitions corresponding to those functionals has the form of $J_\varepsilon$.

**Definition 2.1.** Let $E$ be a Banach space, $E'$ be the dual space of $E$ and $\langle \cdot, \cdot \rangle$ be the duality paring between $E'$ and $E$. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and $\Psi$ is convex and l.s.c. We have the following definitions:

(i) A critical point of $J$ is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$
\langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \forall v \in E.
$$
(ii) A Palais-Smale sequence at level $c$ for $J$ is a sequence $(u_n) \subset E$ such that $J(u_n) \to c$ and there is a numerical sequence $\sigma_n \to 0^+$ with $$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\sigma_n \|v - u_n\|, \quad \forall v \in E.$$ 

(iii) The functional $J$ satisfies the Palais-Smale condition at level $c$ $(PS)_c$ condition if all Palais-Smale sequence at level $c$ has a convergent subsequence.

To use Theorem 3.2 in [26], we need to prove $J_\varepsilon$ satisfies the $(PS)_c$ condition (iii) above.

**Proposition 2.2.** $J_\varepsilon$ satisfies $(PS)_c$ condition, i.e., each sequence $(u_n) \subset D_{V,\varepsilon}^1(\mathbb{R}^N)$ with \( \lim_{n \to \infty} J_\varepsilon \to c \) and

$$J_\varepsilon(u_n) \geq \Phi(u_n) - \frac{1}{2} \|u_n\|^2_{V,\varepsilon} - \frac{1}{2} \int_A |u_n|^2 \log |u_n|^2. \quad (2.2)$$

The logarithmic inequality in [1, Lemma 3.2] which comes from [16, pg 153] says that there exists two positive constants $A, B$ such that

$$\int_{\mathbb{R}^N} |u|^2 \log |u|^2 \leq A + B \log \|u\|^2_{H^1(\mathbb{R}^N)} \quad \forall u \in H^1(\mathbb{R}^N).$$

Hence, letting $\eta \in C^\infty_c(U)$ be a function satisfying $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\overline{\Omega}$ and defining $v_n(x) = u_n(\varepsilon x) \eta(\varepsilon x)$, we have $v_n(x) \in H^1(\mathbb{R}^N)$ and then

$$\int_{\mathbb{R}^N} |v_n|^2 \log |v_n|^2 \leq A + B \log \|v_n\|^2_{H^1(\mathbb{R}^N)}$$

$$= A + B \log \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 + \inf_U (V(x) + 1)|v_n|^2 \right)$$

$$\leq A + B \log \left( \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \varepsilon^2 (|\nabla \eta|^2 |u_n|^2 + 2 \eta u_n \nabla \eta \nabla u_n + |\nabla u_n|^2 \eta^2 + (V(x) + 1)|u_n|^2) \right)$$

$$\leq A + B \log \left( \frac{C}{\varepsilon^N} \|u_n\|^2_{V,\varepsilon} \right),$$

which implies

$$\int_{\mathbb{R}^N} |\eta u_n|^2 \log |\eta u_n|^2 \leq \varepsilon^N \left[ A + B \log \left( \frac{C}{\varepsilon^N} \|u_n\|^2_{V,\varepsilon} \right) \right] \quad (2.3)$$
Thus, by (2.3), we have
\[ u \text{ and for each } J \]
and the fact that Lemma 2.3.
we immediately have:
\[ \int \Lambda |u_n|^2 \log |u_n|^2 \leq \varepsilon N \left[ A + B \log \left( \frac{C}{\varepsilon N} \|u_n\|^2_{V,\varepsilon} \right) \right] + \frac{1}{\varepsilon} |U \setminus \Lambda|. \]
Returning back to (2.2), we then have
\[ c + o_n(1) \geq \frac{1}{2} \|u_n\|^2_{V,\varepsilon} - \varepsilon N \left[ A + B \log \left( \frac{C}{\varepsilon N} \|u_n\|^2_{V,\varepsilon} \right) \right] - \frac{1}{\varepsilon} |U \setminus \Lambda|, \]
which and an easy analysis shows the boundedness of \((u_n)\). Going if necessary to a subsequence, we assume that \(u_n \rightharpoonup u\) weakly in \(D^1_{V,\varepsilon}\).

Next we show that \((u_n)\) has a convergent subsequence. Since \(u_n \rightharpoonup u \in D^1_{V,\varepsilon}\), by the boundedness of \(\Lambda\) and the fact that \(|G_1(x,s)| \leq C \|t\|^{\frac{5}{2}} + C_2 |t|^{\frac{3}{2}}\), where \(C_1, C_2 > 0\) are two positive constants, we have
\[ G_1'(x, u_n) u_n \rightarrow G_1'(x, u). \]
Note that \(J'_\varepsilon(u_n) \varphi = o_n(1) \|\varphi\|_{V,\varepsilon}\) for all \(\varphi \in C^\infty_c(\mathbb{R}^N)\), we deduce that \(J'_\varepsilon(u) \varphi = 0\) for all \(\varphi \in C^\infty_c(\mathbb{R}^N)\), and so, \(J'_\varepsilon(u) = 0\). Combining with \(J'_\varepsilon(u_n) u_n = o_n(1) \|u_n\|\), we have
\[ \|u_n\|^2_{V,\varepsilon} + \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda} G_2'(x, u_n) u_n = \|u\|^2_{V,\varepsilon} + \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda} G_2'(x, u) u + o_n(1). \]
Finally, by \(\|u\| \leq \liminf_{n \rightarrow \infty} \|u_n\|^2_{V,\varepsilon}\) and Fatou’s Lemma, we get \(u_n \rightarrow u\) strongly in \(D^1_{V,\varepsilon}\). This completes the proof. \(\square\)

Obviously, there exists \(\rho > 0\) such that
\[ J_\varepsilon(u) \geq \Phi_\varepsilon(u) \geq \frac{1}{2} \|u\|^2_{V,\varepsilon} - C \|u\|^p_{V,\varepsilon} > 0 \text{ for all } u \text{ with } \|u\|^p_{V,\varepsilon} = \rho, \]
and for each \(u \in C^\infty_c(\Lambda) \setminus \{0\}\), it holds
\[ J_\varepsilon(su) \rightarrow -\infty \text{ as } s \rightarrow +\infty, \]
i.e., \(J_\varepsilon\) owns mountain pass geometry. Thus by Proposition 2.2 and Theorem 3.2 in [26], we immediately have:

**Lemma 2.3.** The mountain pass value
\[ c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\gamma(t)) \]
is positive and can be achieved by a positive function \(u_\varepsilon\) which is a critical point of \(J_\varepsilon\) and solves the following penalized problem
\[ -\varepsilon^2 \Delta u_\varepsilon + (V(x) + 1) u = G_1'(x, u_+) - G_2'(x, u_+). \quad (2.4) \]
Proof. By Theorem 3.2 in [26], $c_\varepsilon$ is a critical value, i.e., there exists $u_\varepsilon \in D(J_\varepsilon) = \{ v \in D_{V,\varepsilon}^1 : J_\varepsilon(v) < +\infty \}$ with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ such that

$$\langle \Phi'_\varepsilon(u_\varepsilon), v - u_\varepsilon \rangle + \Psi(v) - \Psi(u_\varepsilon) \geq 0 \quad \forall v \in D_{V,\varepsilon}^1.$$ 

In particular, letting $t > 0$ and $v = u_\varepsilon + t\varphi$ with $\varphi \in C^\infty_c(\mathbb{R}^N)$, we have

$$\langle \Phi'_\varepsilon(u_\varepsilon), \varphi \rangle + \frac{\Psi(u_\varepsilon + t\varphi) - \Psi(u_\varepsilon)}{t} \geq 0 \quad \forall t > 0$$

and then

$$\langle J'_\varepsilon(u_\varepsilon), \varphi \rangle = \langle \Phi'_\varepsilon(u_\varepsilon), \varphi \rangle + \int_{\mathbb{R}^N} G'_2(x, u_\varepsilon) \varphi \geq 0.$$ 

Rearranging $\varphi = -\psi$, we eventually have

$$\langle J'_\varepsilon(u_\varepsilon), \varphi \rangle = 0 \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N),$$

which implies (2.4).

Letting $(u_\varepsilon)_\perp$ be a test function to (2.4), we find $u_\varepsilon \geq 0$. Finally, by the standard regularity assertion in [19], we conclude that $u_\varepsilon$ is positive. \hfill \Box

3. Energy estimation and Concentration

In this section we will prove the concentration phenomenon of $u_\varepsilon$ via energy estimation. From the concentration we can prove that

$$- (1 + \log |u_\varepsilon|^2) \geq 2 \quad \forall x \in \mathbb{R}^N \setminus \Lambda,$$

which and (2.4) indicate $u_\varepsilon$ solves the origin problem (1.1).

3.1. Limiting problem. The limiting equation corresponding to (1.1) is

$$- \Delta u + \lambda u = u \log |u|^2,$$

where $\lambda > -1$. Its Euler-Lagrange functional is

$$\mathcal{L}_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (\lambda + 1)|u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2.$$ 

In [25], it was proved that the limiting problem (3.2) has a least energy solution $U_\lambda$ with

$$\mathcal{L}_\lambda(U_\lambda) = C_\lambda := \inf_{\varphi \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} \mathcal{L}_\lambda(t\varphi) = \inf_{\varphi \in C^\infty_c(\mathbb{R}^N) \setminus \{0\}, \varphi \geq 0} \max_{t > 0} \mathcal{L}_\lambda(t\varphi).$$

For $C_\lambda$, we have

Proposition 3.1. the function $C_\lambda : (-1, +\infty) \to (0, +\infty)$ is continuous and increasing.
Proof. Let $-1 < \lambda < \lambda' < +\infty$ and $U_{\lambda'}$ be as above. An easy analysis shows that the following function $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$f(t) = L_{\lambda}(tU_{\lambda'}) = L_{\lambda'}(tU_{\lambda'}) + \frac{t^2}{2}(\lambda - \lambda') \int_{\mathbb{R}^N} |U_{\lambda'}|^2$$

has a unique maximum point $t' \in (0, +\infty)$, from which we have

$$C_{\lambda} \leq \max_{t > 0} f(t) = L_{\lambda'}(t'U_{\lambda'}) + \frac{(t')^2}{2}(\lambda - \lambda') \int_{\mathbb{R}^N} |U_{\lambda'}|^2$$

$$< C_{\lambda'}.$$

Similarly, it holds

$$C_{\lambda'} \leq C_{\lambda} + \frac{\bar{t}^2}{2}(\lambda' - \lambda) \int_{\mathbb{R}^N} |U_{\lambda}|^2$$

for some unique $\bar{t} \in (0, +\infty)$. Then $C_{\lambda}$ is increasing and continuous. \hfill \square

By the analysis above, we have the following upper bound of $c_{\varepsilon}$.

**Proposition 3.2.** It holds

$$\limsup_{\varepsilon \to 0} \frac{c_{\varepsilon}}{\varepsilon N} \leq \min_{x \in \Lambda} C_V(x).$$

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$, $\varphi \geq 0$ and define for each $x_0 \in \Lambda$

$$\varphi_{\varepsilon}(x) = \varphi\left(\frac{x - x_0}{\varepsilon}\right).$$

Obviously, $\text{supp}\varphi_{\varepsilon} \subset \Lambda$ for small $\varepsilon$ and $\gamma_{\varepsilon}(t) = tT_0 \varphi_{\varepsilon} \in \Gamma_{\varepsilon}$ for some $T_0$ large enough. Then we have

$$c_{\varepsilon} \leq \frac{\max_{t \in [0,1]} J_{\varepsilon}(\gamma_{\varepsilon}(t))}{\varepsilon N} \leq L_{V(x_0)}(t\varphi) + o_\varepsilon(1)$$

and

$$\limsup_{\varepsilon \to 0} \frac{c_{\varepsilon}}{\varepsilon N} \leq \inf_{\varphi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}} L_{V(x_0)}(t\varphi) = C_V(x_0),$$

which implies our conclusion. \hfill \square

Next, we give the lower bounds of solutions of (2.4).

**Proposition 3.3.** Let $(u_{\varepsilon_n})$ with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ be a family of solutions of (2.4). If for each $k \in \mathbb{N}$, there exists $k$ families of points $\{(x^i_{\varepsilon_n}) : 1 \leq i \leq k\}$ with $\lim_{n \to \infty} x^i_{\varepsilon_n} = x^i_*$ such that

$$\liminf_{n \to \infty} \|u_{\varepsilon_n}\|_{L^\infty(B_{\varepsilon_n\rho}(x^i_{\varepsilon_n}))} > 0, \quad 1 \leq i \leq k,$$

$$\liminf_{n \to \infty} \frac{|x^i_{\varepsilon_n} - x^j_{\varepsilon_n}|}{\varepsilon_n} = +\infty, \quad 1 \leq i \neq j \leq k$$

for some $\rho > 0$, then

$$\liminf_{n \to \infty} \frac{c_{\varepsilon_n}}{\varepsilon_n^N} \geq 0.$$
and

\[ \limsup_{n \to \infty} \frac{J_{\epsilon_n}(u_{\epsilon_n})}{\epsilon_n^N} < +\infty, \]

then

\[ \liminf_{n \to \infty} \frac{J_{\epsilon_n}(u_{\epsilon_n})}{\epsilon_n^N} \geq \sum_{i=1}^k C_{V(x_i^*)}. \]

**Proof.** Fixing a \( 1 \leq i \leq k \) and rescaling the function \( u_{\epsilon_n} \) as \( v_n^i(x) = u(\epsilon_n x + x_{\epsilon_n}^i), \ x \in \mathbb{R}^N \), we have by the estimate before that

\[ \sup_n \int_{\mathbb{R}^N} (|\nabla v_n^i|^2 + V_n(x)|v_n^i|^2) < +\infty, \]

where \( V_n^i(\cdot) = V(\epsilon_n x + x_{\epsilon_n}^i) \). Obviously, \( v_n^i \) satisfies

\[ -\Delta v_n^i + V_n(x)v_n^i = G_1^i(\epsilon_n x + x_{\epsilon_n}^i, v_n^i) - G_2'(\epsilon_n x + x_{\epsilon_n}^i, v_n^i) \]

(3.3)

Fixing \( R > 0 \), we have by continuity that

\[ \limsup_{n \to \infty} \|v_n^i\|^2_{H^1(B_R)} \leq \limsup_n (\inf_{\Lambda_n} \int_{\mathbb{R}^N} (|\nabla v_n^i|^2 + V_n(x)|v_n^i|^2) < +\infty, \]

which says that \((v_n^i)\) is bounded in \( H^1_{\text{loc}}(\mathbb{R}^N) \) and then by diagonal argument, we can assume without loss of generality that \( v_n^i \to v_*^i \) weakly in \( H^1_{\text{loc}} \) as \( n \to \infty \). By

\[ \|v_*^i\|^2_{H^1(B_R)} \leq \liminf_{n \to \infty} \|v_n^i\|^2_{H^1(B_R)} < +\infty, \]

we have \( v_*^i \in H^1(\mathbb{R}^N) \)

The smoothness of \( \Lambda \) makes the set \( \Lambda^i_n = \{ x : \epsilon_n x + x_{\epsilon_n}^i \in \Lambda \} \) converges to a set \( \Lambda^i_* \in \{\emptyset, H, \mathbb{R}^N \} \) as \( n \to \infty \), where \( H \) is a half plane, by which we have

\[ \int_{\mathbb{R}^N} \left( G_1^i(\epsilon x + x_{\epsilon_n}^i, v_n^i)\varphi - G_2'\epsilon x + x_{\epsilon_n}^i, v_n^i)\varphi \right) \]

\[ \to \int_{\mathbb{R}^N} \chi_{\Lambda_*^i} (1 + \log |v_*^i|^2)v_*^i \varphi - \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda_*^i} \max\{2v_*^i, -(1 + \log |v_*^i|^2)v_*^i\} \varphi \]

for all \( \varphi \in C_{c}^\infty(\mathbb{R}^N) \) as \( n \to \infty \). Then we conclude that \( v_*^i \) satisfies the following equation:

\[ -\Delta v_*^i + (V(x_*^i) + 1)v_*^i \]

\[ = \chi_{\Lambda_*^i}(x)v_*^i(1 + \log |v_*^i|^2) - \chi_{\mathbb{R}^N \setminus \Lambda_*^i} \max\{2v_*^i, -(1 + \log |v_*^i|^2)v_*^i\} \] in \( \mathbb{R}^N \). (3.4)

By the similar regularity argument in [19], we have

\[ \|v_*^i\|_{L^\infty(B_{\rho}(x_*^i))} = \lim_{n \to \infty} \|v_n^i\|_{L^\infty(B_{\rho}(x_{\epsilon_n}^i))} = \lim_{n \to \infty} \|u_n^i\|_{L^\infty(B_{\rho}(x_{\epsilon_n}^i))} > 0, \]

which implies \( v_*^i \) is nontrivial. The Euler-Lagrange functional corresponding to (3.4) is

\[ J_*^i(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x_*^i) + 1)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} \chi_{\Lambda_*^i} |u|^2 \log |u|^2 \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda_*^i} \max\{2|u|^2, -|u|^2 \log |u|^2\}, \]
which implies

$$J^i_*(v^i_*) = \max_{t>0} J^i_*(tv^i_*) \geq \max_{t>0} \mathcal{L}_{V(x^i_*)}(tv^i_*) \geq C_{V(x^i_*)}.$$  

Then, after rescaling, we have

$$
\liminf_{n \to \infty} \frac{1}{2\varepsilon_n^N} \int_{B_{\varepsilon_n R}(x^i_{\varepsilon_n})} \left( (\varepsilon_n^2 |\nabla u_n|^2 + (V(x) + 1)|u_n|^2) - \chi_{N\Delta} |u_n|^2 \log |u_n|^2 
\right.
\left. + \chi_{R \setminus \Delta} \max\{2|u_n|^2, -|u_n|^2 \log |u_n|^2\} \right) 
\geq J^i_*(v^i_*) + o_R(1) \geq C_{V(x^i_*)} + o_R(1). 
$$

Now let us estimate the energy outside $\bigcup_{i=1}^k B_{\varepsilon_n}(x^i_{\varepsilon_n})$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be a smooth function taking value 0 on $B_{\frac{2}{5}R} \cup B_{\frac{3}{4}R}$ and 1 on $B_{R \setminus B_{\frac{2}{5}R}}$. Texting (3.3) against with $\varphi v^i_{\varepsilon_n}$, by the convergence of $v^i_n$, we find

$$
\int_{\mathbb{R}^N} (|\nabla v^i_n|^2 + (V^i_n(x) + 1)|v^i_n|^2) \varphi \leq C \int_{B_{\frac{2}{5}R \setminus B_{\frac{2}{5}R}}^c} (|v^i_n|^2 + |v^i_n|^p) = o_R(1) (3.5)
$$

for some $2 < p < 2^* = +\infty$ if $N = 1, 2$ and $\frac{2N}{N-2}$ if $N \geq 3$. Choose $\eta$ as another cut-off function with $\eta \equiv 0$ in $B_{\frac{2}{5}R}$ and $\eta \equiv 1$ on $B^c_{\frac{2}{5}R}$ and define

$$
\eta_n(\cdot) = \prod_{i=1}^k \eta \left( \cdot - \frac{x^i_{\varepsilon_n}}{\varepsilon_n} \right), \quad \varphi_n(\cdot) = \eta_n(\cdot) u_{\varepsilon_n}.
$$

Letting $\varphi_n$ be a test function in (3.3), by (3.5) and the fact that $v^i_n \to v^i_*$ strongly in $L^q_{loc}(\mathbb{R}^N)$, $1 < q < 2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $+\infty$ if $N = 1, 2$, we have

$$
\frac{1}{2\varepsilon_n^N} \int_{\bigcup_{i=1}^k B_{\varepsilon_n R}(x^i_{\varepsilon_n})} (\varepsilon_n^2 |\nabla u_n|^2 + (V(x) + 1)|u_n|^2) 
- \frac{1}{2\varepsilon_n^N} \int_{\bigcup_{i=1}^k B_{\varepsilon_n R}(x^i_{\varepsilon_n})} (\varepsilon_n^2 |\nabla u_n|^2 + (V(x) + 1)|u_n|^2) 
\geq \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \chi \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x^i_{\varepsilon_n}) \right) (x - \eta_n(x))^2 (\varepsilon_n^2 |\nabla u_n|^2 + (V(x) + 1)|u_n|^2) 
- \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \eta_n + \frac{1}{2\varepsilon_n^N} \int_{\Delta} \eta_n |u_n|^2 (1 + \log |u_n|^2) 
- \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \eta_n \max\{2|u_n|^2, -|u_n|^2 (1 + \log |u_n|^2)\} 
- \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \eta_n \max\{2|u_n|^2, -|u_n|^2 (1 + \log |u_n|^2)\} 
- \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \eta_n \max\{2|u_n|^2, -|u_n|^2 (1 + \log |u_n|^2)\} 
$$
\[ + \frac{1}{2\varepsilon_n^N} \int \left( \bigcup_{k=1}^{N} B_{\varepsilon_n R}(x_{\varepsilon_n}) \right)^c \max \{2|u_n|^2, -|u_n|^2 \log |u_n|^2 \} \]

\[ \geq \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \left[ \chi_{\left( \bigcup_{k=1}^{N} B_{\varepsilon_n R}(x_{\varepsilon_n}) \right)}(x) - \eta_n(x) \right] (\varepsilon_n^2 |\nabla u_n|^2 + (V(x) + 1)|u_n|^2) \]

\[ + \frac{1}{2\varepsilon_n^N} \int_{\Lambda} \eta_n |u_n|^2 (1 + \log |u_n|^2) - \frac{1}{2\varepsilon_n^N} \int_{\left( \bigcup_{k=1}^{N} B_{\varepsilon_n R}(x_{\varepsilon_n}) \right)} \cap \Lambda \cdot |u_n|^2 \log |u_n|^2 \]

\[ - \int_{\mathbb{R}^N} \varepsilon_n^{2-N} u_n \nabla u_n \nabla \eta_n \]

\[ = o_R(1) - C \sum_{i=1}^{k} \int_{B_R \setminus B_{R/2}} (|v^i|^2 + |v^i|^p) \]

\[ = o_R(1), \]

where \( p \in (2, 2^*) \).

Finally, by the analysis above, we have

\[ \liminf_{n \to \infty} \frac{J_{\varepsilon_n}(u_{\varepsilon_n})}{\varepsilon_n^N} \geq \sum_{i=1}^{k} C_{V(x_i)} + o_R(1) \]

and the conclusion then follows by letting \( R \to \infty \).

Now we prove the concentration of \( u_\varepsilon \).

**Lemma 3.4.** Let \( \rho > 0 \) and \( u_\varepsilon \) be the penalized solution given by Lemma 2.3. There exists a family of points \( (x_\varepsilon) \subset \Lambda \) such that

\( (i) \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty B_{\varepsilon\rho}(x_\varepsilon)} > 0, \)

\( (ii) \lim_{\varepsilon \to 0} V(x_\varepsilon) = \inf_{\Lambda} V, \)

\( (iii) \lim_{R \to \infty} \|u_\varepsilon\|_{L^\infty(U \setminus B_{\varepsilon\rho}(x_\varepsilon))} = 0. \)

**Proof.** Easily, we have

\[ 0 < \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u_\varepsilon|^2 + (V(x) + 1)|u_\varepsilon|^2) \leq \int_{\Lambda \cap \{x : u_\varepsilon(x) > \varepsilon^{-1/2}\}} |u_\varepsilon|^2 (1 + \log |u_\varepsilon|^2), \]

which implies the set \( \Lambda \cap \{x : u_\varepsilon(x) > \varepsilon^{-1/2}\} \) has positive measure. Then by the similar regularity assertion in [19], there exists \( x_\varepsilon \in \Lambda \) such that

\[ u_\varepsilon(x_\varepsilon) = \sup_{x \in \Lambda} u_\varepsilon(x) \quad \text{and} \quad \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty B_{\varepsilon\rho}(x_\varepsilon)} > 0. \]

This proves (i).
For (ii), assuming without loss of generality that \( \lim_{\varepsilon \to 0} x_\varepsilon = x^* \), by the lower and upper bounds of \( u_\varepsilon \) in Propositions 3.2 and 3.3, we have
\[
\min_{x \in \Lambda} C_V(x) \geq \liminf_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} \geq C_V(x^*),
\]
which implies \( V(x^*) = \min_{x \in \Lambda} V(x) \).

For (iii), one will get a contradiction like
\[
\min_{x \in \Lambda} C_V(x) \geq \liminf_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} \geq C_V(x^*) + C_V(y^*)
\]
for some \( y^* \in \bar{U} \) by Proposition 3.3 if (iii) is not true. \( \square \)

3.2. Back to the origin problem. At last of this section, we use Lemma 3.4 to prove (3.1), which implies \( u_\varepsilon \) indeed is the solution of (1.1).

Noting that by the regular assertion in [19], we can assume that
\[
\sup_{\Lambda} u_\varepsilon(x) \leq C < \infty,
\]
(3.6)
where \( C \) is a positive constant. Hence by Lemma 3.4, we can linearize the penalized equation as follows.

**Proposition 3.5.** Let \( \varepsilon > 0 \) be small enough, \( x_\varepsilon \) be the point of Lemma 3.4. Then there exists \( R > 0 \) such that
\[
\begin{cases}
-\varepsilon^2 \Delta u_\varepsilon + (V(x) + 1)u_\varepsilon \leq 0, & \text{in } \mathbb{R}^N \setminus B_{\varepsilon R}(x_\varepsilon) \\
u_\varepsilon \leq C, & \text{in } B_{\varepsilon R}(x_\varepsilon)
\end{cases}
\]

**Proof.** For \( \varepsilon > 0 \) small enough, by Lemma 3.4, there exists \( R > 0 \) such that \( 1 + \log |u_\varepsilon|^2 \leq 0 \) for all \( x \in \mathbb{R}^N \setminus B_{\varepsilon R}(x_\varepsilon) \), the conclusion then follows by inserting (3.6) into (2.4). \( \square \)

Following, letting \( v_\varepsilon = u_\varepsilon(\varepsilon x + x_\varepsilon) \), we have
\[
\begin{cases}
-\Delta v_\varepsilon + (V(\varepsilon x + x_\varepsilon) + 1)v_\varepsilon \leq 0, & \text{in } \mathbb{R}^N \setminus B_R \\
v_\varepsilon \leq C, & \text{in } B_R.
\end{cases}
\]
(3.7)

Now we construct super-solutions to the linearized equation above with different decay rates of \( V(x) \).

**Case 1** \( \liminf_{|x| \to \infty} (V(x) + 1)|x|^{2\sigma} > 0 \) with \( \sigma \in [0, 1) \).

It was proved in [4] that for every \( m > 0 \), there exists \( \tilde{R} > 0 \) and \( \varepsilon_0 > 0 \) such that
\[
V(\varepsilon x + x_\varepsilon) + 1 \geq \frac{m}{|x|^{2\sigma}}, \text{ for all } |x| \geq \tilde{R} \text{ if } 0 < \varepsilon < \varepsilon_0,
\]
by which and (3.7), we conclude without loss of generality that
\[
\begin{cases}
-\Delta v_\varepsilon + \frac{m}{|x|^{2\sigma}} v_\varepsilon(x) \leq 0, & x \in \mathbb{R}^N \setminus B_{\tilde{R}} \\
v_\varepsilon(x) \leq C, & x \in B_{\tilde{R}} \\
v_\varepsilon(x) \to 0 \text{ as } |x| \to \infty.
\end{cases}
\]
Then by the results in [4] again, we have (3.7) is satisfied by $v_\varepsilon$. Then for every $|x| > \tilde{R}$, it holds
\[
v_\varepsilon(x) \leq \begin{cases} 
\tilde{C}_1 e^{-\frac{\sqrt{m}}{1-\sigma}|x|^{1-\sigma}}, & \text{if } 0 < \sigma < 1, \\
\tilde{C}_2 |x|^{-N - \sqrt{(N-2)^2 + 4m}} & \text{if } \sigma = 1,
\end{cases}
\]
where $\tilde{C}_i$, $i = 1, 2$, are suitably positive constants.

Case 2 \(\liminf_{|x| \to \infty} (V(x) + 1)|x|^2 \geq 0\). Note that in this case we require $N \geq 3$. It is easy to check that \(\frac{CR^{N-2}}{|x|^{N-2}}\) is a super solution to (3.7), hence
\[
v_\varepsilon(x) \leq \frac{CR^{N-2}}{|x|^{N-2}} \quad x \in \mathbb{R}^N \setminus B_R.
\]

Finally, returning back to $u_\varepsilon$, for every $x \in \mathbb{R}^N$, we have
\[
u_\varepsilon(x) \leq \begin{cases} 
\tilde{C} e^{-\frac{|x|}{\varepsilon}}, & \text{if in case 1} \\
\tilde{C} \varepsilon^{-N |x|^{-N}} & \text{if in case 2},
\end{cases}
\]
which implies (3.1). The proof of Theorem 1.1 is then completed.

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School of Mathematics and Statistics & Guizhou University of Finance and Economics, Guiyang, 550025, P. R. China

E-mail address: 651547603@qq.com