EFFICIENCY AND COMPLEXITY OF HYPERPLANE ARRANGEMENTS

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Abstract. For a projective hyperplane arrangement, we study sufficient conditions in terms of combinatorial data for ESV-calculability of the monodromy eigenspaces of the first Milnor fiber cohomology for eigenvalues of order \( m > 1 \). This can be reduced to the line arrangement case by Artin’s theorem. These sufficient conditions are often unsatisfied if efficiency or complexity of the combinatorics of arrangement is high. In order to measure these, we introduce the notions of \( m \)-efficiency and \( m \)-complexity for \( m \geq 3 \). The former is defined to be the number of points with multiplicity divisible by \( m \) lying on one line in average. In many cases, one of the above sufficient conditions is satisfied if it is at most 2, although there are certain exceptional cases, especially when \( m = 3 \). The \( m \)-complexity is defined to be the maximal number of edges containing one vertex of the associated \( m \)-graph. We can show that one of the sufficient condition holds if it is at most \( (m + 1)/2 \).

Introduction

Let \( X \subset \mathbb{P}^{n-1} \) be a reduced projective hyperplane arrangement with \( f \) a defining polynomial of \( n \) variables. To calculate a monodromy eigenspace of its first Milnor fiber cohomology \( H^1(F_f, \mathbb{C})_{\lambda} \) for eigenvalue \( \lambda \) of order \( m > 1 \), we may assume \( n = 3 \), since the calculation is reduced to the case \( n = 3 \) using an iterated general hyperplane cut together with the Artin vanishing theorem. So we assume \( n = 3 \) in this paper, and \( X \) will be denoted by \( L \). (Note that the dimension of the second Milnor fiber cohomology is then easily determined from the first using the Euler number of \( \mathbb{P}^2 \setminus L \).

We may also assume \( \frac{d}{m} \in \mathbb{Z} \) with \( m := \text{ord} \lambda \), since the eigenspace \( H^j(F_f, \mathbb{C})_{\lambda} \) vanishes unless \( \frac{d}{m} \in \mathbb{Z} \). These eigenspaces can be calculated by using the corresponding Aomoto complex if some condition coming from [ESV] is satisfied, see [BDS], [BSY], [Sa1], etc. Combining this with the theory of cyclotomic polynomials, we can get the following (see (1.2) below).

**Theorem 1.** Let \( L \subset \mathbb{P}^2 \) be a reduced line arrangement defined by a homogeneous polynomial \( f \) of 3 variables with degree \( d \). Then the eigenspace \( H^1(F_f, \mathbb{C})_{\lambda} \) with eigenvalue \( \lambda \) of order \( m > 1 \) is ESV-calculable, that is, it can be calculated by the corresponding Aomoto complex, if there is a subset \( J \subset \{1, \ldots, d\} \) with \( |J| = \frac{d}{m} \), and one of the following two conditions is satisfied:

(a) \( L^{[>3]} \cap L^{[m]} \cap L^{[k]} \subset L^{[>km]} \quad (\forall k \geq 2) \),

(b) \( L^{[>3]} \cap L^{[km]} \subset L^{[>k]} \quad (\forall k \geq 1) \).

Here \( L_J := \bigcup_{i \in J} L_i \) with \( L_i \) (\( i \in \{1, \ldots, d\} \)) the irreducible components of \( L \), and

\[
L^{[k]} := \{ P \in L \mid \text{mult}_P L = k \}, \quad L^{[>k]} := \{ P \in L \mid \text{mult}_P L \geq k \},
\]

\[
L^{(k)} := \{ P \in L \mid \text{mult}_P L \in (k) \} \quad (k \in \mathbb{Z}_{>0}),
\]

with \( (k) := k\mathbb{Z} \) (similarly for \( L_J^{[k]} \), etc.)

Note that conditions (a), (b) are essentially equivalent to the condition in [ESV]. In the case \( m = 2 \), they are equivalent to each other replacing \( J \) with its complement.

As a corollary of Theorem 1, we get the following.
Corollary 1. In the notation of Theorem 1, the eigenspace $H^1(F_J, \mathbb{C})_{\lambda}$ with $\text{ord} \lambda = m$ is ESV-calculable if one of the following two conditions is satisfied:

(a) $L^{[\geq 3]} \cap L^{[m]} \cap \text{Sing} L_J = \emptyset$,

(b) $\exists P \in L$ with $d \leq \text{mult}_P L \notin m\mathbb{Z}$.

Remark 1. Condition (a) implies (a), and (a)$''$ implies (a)$'$. (For the latter assertion, we can take $J$ so that all the $L_i$ ($i \in J$) pass through $P$, since $|J| = \frac{d}{m}$.)

Remark 2. Condition (a) is equivalent to (a)$'$ if $L^{[m]} = L^{[m]}$. So condition (a) is a rather strong condition in this case (especially when $m$ is small).

Condition (a)$''$ in Corollary 1 means that we may have the ESV-calculability easily when $L$ contains a point of very big multiplicity.

The above sufficient conditions are often unsatisfied if complexity or efficiency of the combinatorics of arrangement is rather high. In order to measure these, we introduce the notions of $m$-complexity and $m$-efficiency for $m \geq 3$. (The case $m = 2$ cannot be treated similarly.) The $m$-complexity is defined by

$$C_{L,m} := \max_P n'_P \quad (m \geq 3).$$

Here $P$ runs over points of $L^{[m]}$, and $n'_P$ is the number of lines in $L$ containing $P$ and another point of $L^{[m]}$. This can be defined also as the maximal number of edges containing one vertex of the associated $m$-graph (which will be defined after Problem 1 below). We have the following (see (1.3) below).

Theorem 2. Condition (b) holds if

(c) $C_{L,m} \leq \lfloor m/2 \rfloor = \lfloor (m+1)/2 \rfloor \quad (m \geq 3),$

with $[\alpha] := \min \{ k \in \mathbb{Z} \mid k \geq \alpha \}$ for $\alpha \in \mathbb{R}$.

If $m = 3$, condition (c) is equivalent to that its $m$-graph is strictly unsaturated (since $\lfloor m/2 \rfloor = m-1$), see the definition after Problem 1 below. This bound is sharp for $m = 3$, since there are many ESV-non-calculable examples with $C_{L,m} = 3$, see for instance Examples (3.1) and (3.2) below. It is unclear for $m > 4$, see Remark (1.3) (iii) for $m = 4$.

The $m$-efficiency is the sum of the numbers of local irreducible components of $L$ passing through the points of $L^{[m]}$, which is divided by $d$, that is,

$$E_{L,m} := \sum_{k \geq 1} |L^{[km]}| \frac{km}{d} \quad (m \geq 3).$$

This measures how many points of $L^{[m]}$ lie on one line in average (since each point of $L^{[km]}$ contributes to $km$ lines). This number can become rather large as $\frac{d}{m}$ increases, see Examples (3.3) and (3.6) below. The number is closely related to condition (b) by the following.

Remark 3. Condition (b) is trivially satisfied if $E_{L,m} \leq 1$, that is, if $\sum_k |L^{[km]}| k \leq \frac{d}{m} = |J|$ (for $m \geq 3$).

It is rather surprising that it is not easy to improve this bound without adding some hypotheses as in Problem 1 below. In most cases, condition (b) is satisfied if $E_{L,m} \leq 2$. There are, however, certain exceptional cases, especially when $m = 3$. We have the following.

Problem 1. Is condition (b) always satisfied when $E_{L,m} \leq 2$ with $m \geq 4$, assuming the $m$-graph of $L$ is connected, and is strictly unsaturated?

Here the $m$-graph of $L$ can be described as follows: Its vertices are identified with the points of $L^{[m]}$, and there is an edge between two vertices if and only if there is a line in $L$ containing them, see (2.1) below for a more precise definition. It is a weighted graph, where a point belonging to $L^{[km]}$ has weight $k$. The $m$-graph is called reduced if the weight is 1 for
any vertex, or equivalently, if \(L^{(m)} = L^{[m]}\). In this case, \(L\) is called \(m\)-reduced. The \(m\)-graph is called strictly unsaturated if the number of edges containing each vertex is at most \(m-1\), or equivalently, if \(C_{L,m} \leq m-1\). (Here we do not assume that the \(m\)-graph is reduced.)

**Remark 4.** In the case \(m=3\), we have a positive answer to Problem 1 assuming only its last hypothesis (without assuming \(E_{L,m} \leq 2\), etc.), since the latter condition is equivalent to the hypothesis of Theorem 2 (that is, \(C_{L,m} \leq \lfloor m/2 \rfloor\)). Note that there are examples such that \(E_{L,3} > 2\), its 3-graph is reduced, and the last two hypotheses in Problem 1 are satisfied, see Remark (1.3) (ii) below.

**Remark 5.** When \(E_{L,m}\) is more than 2, it is usually difficult to satisfy condition (a) or (b). However, the situation does not seem quite simple as is seen below.

(i) If \(f = (x^n-ya^n)(x^n-z\bar{a}^n)\) for \(m=3\), we have \(d_3 = a\), \(E_{L,3} = a+1\) or \(a\), and \(H^1(F_f, \mathbb{C})_\lambda\) for ord \(\lambda = 3\) is either ESV-non-calculable or calculable, all depending on whether \(a \in 3\mathbb{Z}\) or not, see Example (3.1) below.

(ii) For \(m = 3\), there are two line arrangements such that \(E_{L,3} = 3\) for both, but \(H^1(F_f, \mathbb{C})_\lambda\) with ord \(\lambda = 3\) is either ESV-calculable or not, see Example (3.2) below.

(iii) For \(m = 3\), there is a family of line arrangements parametrized by \(a \in \mathbb{Z}_{>3}\) such that \(E_{L,3} = (a+3)/2\) if \(a \in 3\mathbb{Z}\), and \((a+1)/2\) otherwise, where \(d=3a\). It is rather surprising that \(H^1(F_f, \mathbb{C})_\lambda\) with ord \(\lambda = 3\) is ESV-calculable for any \(a > 3\).

(iv) In the Hessian arrangement case, we have \(E_{L,4} = d_4 = 3\), and \(H^1(F_f, \mathbb{C})_\lambda\) for ord \(\lambda = 4\) is ESV-calculable, see Example (3.4) below.

(v) For every \(m \geq 3\), there is a projective line arrangement such that \(E_{L,m} = 3\), \(d_m = m\), and \(H^1(F_f, \mathbb{C})_\lambda\) with ord \(\lambda = m\) is ESV-calculable if and only if \(m\) is odd, see Example (3.5) below.

(vi) In the case of a general hyperplane section of the reflection arrangement of type \(G_3\), we have \(E_{L,3} = 15\), \(E_{L,6} = 3\), \(d = 60\), and \(H^1(F_f, \mathbb{C})_\lambda\) for ord \(\lambda = 3\) or 6 is never ESV-calculable, see [BDY], [Sa2] and Example (3.6) below.

**Remark 6.** In general, non-vanishing Milnor fiber cohomology groups with non-unipotent monodromy are closely related to the resonance varieties which are defined as the non-vanishing loci of the cohomology of the corresponding Aomoto complexes. There is a close relation between resonance varieties and multinets. Moreover it is known that there are no \(k\)-multinets for \(k > 4\), see [FY], [LY], [PY], [Yu1], [Yu2]. It is, however, rather unclear whether these can imply the vanishing of \(H^1(F_f, \mathbb{C})_\lambda\) in the ESV-calculable case when the order \(m\) of \(\lambda\) is more than 4, see (2.5) below.

In Section 1 we recall some basics of Aomoto complexes, and prove Theorems 1 and 2. In Section 2 we explain why the assumptions are needed in Problem 1. In Section 3 we calculate some examples.

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### 1. ESV-calculation

In this section we recall some basics of Aomoto complexes, and prove Theorems 1 and 2.

**1.1. Aomoto complexes.** Let \(L \subset Y := \mathbb{P}^{n-1}\) be a reduced projective line arrangement of degree \(d\). Let \(\mathcal{S}(L)\) be the intersection poset consisting of any intersections of the irreducible components \(L_i\) of \(L\) (\(i \in \{1, \ldots, d\}\)). This contains the ambient space \(Y\), but not the empty set. Set \(U := Y \setminus L\). By [Br], [OS], there is an isomorphism of \(\mathbb{C}\)-algebras

\[
A^*_\mathcal{S}(L) \simarrow H^*(U, \mathbb{C}),
\]

(1.1.1)
where \( A_{S(L)}^\bullet \) is a quotient algebra of the exterior algebra \( \bigwedge^\bullet (\bigoplus_{i=1}^{d-1} \mathbb{C} e_i) \) divided by an ideal \( \mathcal{I} \), and is called the Orlik-Solomon algebra. Note that the induced affine arrangement on \( \mathbb{C}^{n-1} = \mathbb{P}^{n-1} \setminus L_d \) is used here so that the \( e_i \) are identified with \( dg_i/g_i \), where \( g_i \) is a linear function with a constant term defining \( L_i \setminus L_d \subset \mathbb{C}^{n-1} \). Moreover, the ideal \( \mathcal{I} \) is determined by the combinatorial data; see [loc. cit.]

From now on, assume \( n = 3 \) for simplicity. Let \( \alpha_i \in \mathbb{C} \) \((i \in \{1, \ldots, d\})\) satisfying the following conditions:

\[
\begin{align*}
\alpha_i & \notin \mathbb{Z}_{>0}, \quad \sum_{i=1}^{d} \alpha_i = 0, \\
\alpha_P := \sum_{i \in P} \alpha_i & \notin \mathbb{Z}_{>0} \quad (\forall P \in \mathcal{L}^{[3]}),
\end{align*}
\]

where \( \mathcal{L}^{[3]} \) is as in the introduction. Set

\[
\omega = \sum_{i=1}^{d} \alpha_i e_i \in A_{S(L)}^1.
\]

This defines a complex \((A_{S(L)}^\bullet, \omega \wedge)\), called the Aomoto complex associated to \( \omega \).

Since the \( e_i \) are identified with \( dg_i/g_i \), we get also a regular singular connection \( \nabla^\omega \) on \( \mathcal{O}_U \) such that

\[
\nabla^\omega h = dh + h \omega \quad (h \in \mathcal{O}_U).
\]

The main theorem of [ESV] asserts that if condition (1.1.2) is satisfied, then we have the isomorphisms

\[
H^j(A_{S(L)}^\bullet, \omega \wedge) \cong H^j_{DR}(U, (\mathcal{O}_U, \nabla^\omega)) \quad (j \in \mathbb{Z}).
\]

In this case we say that the de Rham cohomology of \( \nabla^\omega \) is ESV-calculable.

**1.2. Proof of Theorem 1.** It is well-known (see [Di1], [CS], [BS], etc.) that the monodromy eigenspace \( H^j(F_j, \mathbb{C})_\lambda \) vanishes unless \( \lambda^d = 1 \), and there are local systems \( \mathcal{L}^{(k)} \) of rank 1 on \( U \) \((k = 0, \ldots, d-1)\) such that

\[
H^j(F_j, \mathbb{C})_\lambda = H^j(U, \mathcal{L}^{(k)}) \quad (\lambda = \exp(-2\pi ik/d)).
\]

Moreover the monodromy around any irreducible component of \( L \) is given by multiplication by \( \lambda^{-1} = \exp(2\pi ik/d) \). In particular, \( \mathcal{L}^{(0)} = \mathbb{C}_U \) so that \( H^j(F_j, \mathbb{C})_1 = H^j(U, \mathbb{C}) \).

It is also known that the dimensions of \( H^j(F_j, \mathbb{Q})_\lambda \) (and \( H^j(F_j, \mathbb{C})_\lambda \)) are independent of \( \lambda \in \mu_d \); such that \( \text{ord} \lambda = m \) for a fixed \( m \). Indeed, these \( \lambda \) are called primitive roots of unity of order \( m \), and are conjugate to each other under the action of the Galois group of \( \mathbb{Q}/\mathbb{Q} \) (by the irreducibility of cyclotomic polynomials). In particular, we may assume \( \lambda = \exp(\pm 2\pi i/m) \) in (1.2.1), that is,

\[
\frac{k}{d} = \frac{1}{m} \quad \text{or} \quad 1 - \frac{1}{m}.
\]

Here one can take a convenient choice as one likes.

The local system \( \mathcal{L}^{(k)} \) is then isomorphic to the solution local system of the connection \( \nabla^\omega \) on \( \mathcal{O}_U \) in (1.1) by setting

\[
\alpha_i = \begin{cases} 
1 - \frac{1}{m} & (i \in J), \\
-\frac{1}{m} & (i \notin J),
\end{cases} \quad \text{or} \quad \alpha_i = \begin{cases} 
-1 + \frac{1}{m} & (i \in J), \\
\frac{1}{m} & (i \notin J),
\end{cases}
\]

where \( J \subset I := \{1, \ldots, d\} \) is a subset with \( |J| = \frac{d}{m} \). (Note that a local system of rank 1 on the complement of a curve in \( \mathbb{P}^2 \) is determined by the local monodromies around the smooth points of the curve, taking the tensor product of a local system with the inverse of another, since the singular points of the curve have codimension 2 in \( \mathbb{P}^2 \), and \( \mathbb{P}^2 \) contains a simply connected Zariski-open subset \( \mathbb{C}^2 \).)
In order to get the isomorphism (1.1.5) (that is, the $\lambda$-eigenspace of the Milnor fiber cohomology is ESV-calculable), the last condition of (1.1.2) should be satisfied for the $\alpha_i$ defined by one of the equalities in (1.2.3). Setting
\[ J_P := \{ i \in J \mid L_i \ni P \}, \quad I_P := \{ i \in I \mid L_i \ni P \}, \]
the conditions are expressed respectively by the following inequalities for $P \in L[(m)] \cap L[\geq 3]$
\[ m |J_P| \leq |I_P|, \quad m |J_P| \geq |I_P|. \]
Let $k_P, k'_P \in \mathbb{Z}_{\geq 0}$ with $P \in L_j^{[k_P]} \cap L_j^{[k'_P m]}$, that is, $|J_P| = k_P$, $|I_P| = k'_P m$. The above conditions are then equivalent to conditions (a) and (b) respectively. This finishes the proof of Theorem 1.

1.3. Proof of Theorem 2. For $P \in L[(m)]$, let $n_P$ be the number of lines $L_i \in L$ such that
\[ L_i \cap L[(m)] = \{ P \}. \]
Let $k_P \in \mathbb{Z}_{\geq 0}$ such that $P \in L[k_P m]$. Condition (c) means that
\[ (1.3.1) \quad n'_P = k_P m - n \leq \lceil m/2 \rceil, \quad n \geq k_P m - (m+1)/2, \]
where $\lceil m/2 \rceil = \lceil (m+1)/2 \rceil$. By Remark 1.3 (i) below, the assertion can be reduced to the $m$-reduced case, and we may assume $L$ is $m$-reduced.

In this case, it is enough to show the following.

(1.3.2) For any $m$-reduced line arrangement $L$ with condition (c) satisfied, there are $r$ lines $L_i \in L$ ($i \in \{1, \ldots, r\}$) with $mr \leq d$ and $L[m] \subset \bigcup_{i=1}^{r} L_i$, where $d := \deg L$.

Here we do not assume $d/m \in \mathbb{Z}$.

We proceed by induction on $|L[m]|$. The assertion holds in the case $|L[m]| = 1$, since $d \geq m$. Assume $|L[m]| \geq 2$. Take any line $L_1 \subset L$ such that $|L_1 \cap L[m]| \geq 2$. (If there is no such line, the assertion is reduced to the case $|L[m]| = 1$.) Let $L'' \subset L$ be the union of $L_1$ and the lines $L_i \subset L$ such that
\[ L_1 \supset L_i \cap L[m] \neq \emptyset. \]
Let $L' \subset L$ be the union of the other lines so that $L = L' \cup L''$ and $\dim L' \cap L'' = 0$. By condition (1.3.1) (with $k_P = 1$), we have
\[ (1.3.4) \quad d'' := \deg L'' \geq 1 + 2(m -(m+1)/2) = m. \]
Hence $d'' \geq mr''$ with $r'' := 1$. By inductive hypothesis, there are $r'$ lines in $L'$ covering $L'[m]$ with $d' := \deg L' \geq mr'$. By (1.3.3) we have $L'[m] = L[m] \setminus L_1$. Hence there are $r$ lines in $L$ covering $L[m]$ with $r := r' + r''$. Theorem 2 then follows, since $d = d' + d''$.

Remark 1.3 (i). For a line arrangement $L$ with $C_{L,m} \leq m$, we have the decomposition $L = L' \cup L''$ such that $L$ is $m$-reduced (that is, $L'[m] = L'[m]$, $L''[km] = L''[km]$ ($\forall k \in \mathbb{Z}_{>0}$), and $|L'' \cap L''[m]| = 1$ for any line $L''_i \subset L'$. This $L'$ is called the $m$-reduced arrangement associated to $L$. It is easy to see that condition (a) (resp. (b)) is satisfied for $L$ if and only if it is satisfied for $L'$.

Remark 1.3 (ii). For $m = 3$, there are many examples such that the 3-graph is connected, reduced, and strictly unsaturated with $E_{L,3} > 2$. For instance, take sufficiently general $n$ lines $L_i \in \mathbb{P}^2$ ($i \in \{1, \ldots, n\}$) such that their union $L' := \bigcup_{i=1}^{n} L_i$ is a divisor with normal crossings. For each singular point of $L'$, choose a sufficiently general line passing through it so that its intersection with Sing $L'$ consists only of this singular point and we have $L[(3)] = L[3] = \text{Sing } L'$, where $L$ is the union of these lines and $L'$. Then
\[ |L[(3)]| = |L[3]| = n(n-1)/2, \quad d := \deg L = n(n+1)/2. \]
Hence
\[ E_{L,3} = \frac{3(n-1)}{(n+1)} > 2 \quad \text{if} \quad n \geq 6. \]

**Remark 1.3** (iii). Put \( m' := m-1 \). For a line arrangement \( L' \) with \( L'^{(m)} = \emptyset \), there is a line arrangement \( L \) whose \( m \)-graph is the same as the \( m' \)-graph of \( L' \) and
\[ d = d' + \sum_{k \geq 1} |L'^{km'}|k, \]
with \( d := \deg L, \ d' := \deg L' \), by adding sufficiently general \( k \) lines passing through each point of \( L'^{km'} \) for \( k \geq 1 \). We have
\[ \frac{d}{m} = \left( \frac{d'}{m'} + \sum_{k \geq 1} |L'^{km'}|k \right) / m = \left( \frac{d'}{m'} (m' + E_{L',m'}) / m. \right. \]
This means that \( d/m \) is close to \( d'/m' \) if \( E_{L',m'} \) is close to 1. This can be used to show that Theorem 2 is sharp for \( m = 4 \). Here we use the construction in (2.3) below, and consider the union of \( \gamma_j(L) (j \in \{1, \ldots, m\}) \), where the \( \gamma_j \in \text{Aut}(\mathbb{P}^2) \) fix some appropriate smooth point of \( L \), and are sufficiently general among the automorphisms satisfying this condition.

### 2. Necessity of the hypotheses in Problem 1.

In this section we explain why the assumptions are needed in Problem 1.

#### 2.1. \( m \)-graphs.
For a line arrangement \( L \subset \mathbb{P}^2 \) and \( m \in \mathbb{Z}_{\geq 3} \), the \( m \)-graph is defined as follows: Its vertices are identified with the points of \( L^{(m)} \). Its edges correspond to lines in \( L \) containing at least two points of \( L^{(m)} \), and are expressed by closed connected smooth real curves whose both ends are vertices. An edge may contain vertices in its interior so that the vertices contained in it are identified with the points of \( L^{(m)} \) contained in the corresponding line in \( L \). We assume that two edges containing a same vertex have different limit tangent lines. (Note that an intersection of edges is not always a vertex of the graph unless a vertex is marked at the intersection point. If two edges intersect at a point which is not a vertex, we consider that they do not intersect as if the graph is in \( \mathbb{R}^3 \).)

This is a weighted graph, where a vertex has weight \( k \) if its corresponding point belongs to \( L^{km} \). We can decide whether condition (a) or (b) is satisfied by looking at this graph. Note that the number of lines in \( L \) containing only a given vertex is the difference between the weight of the vertex multiplied by \( m \) and the number of edges containing the vertex. The weights are omitted if these are always 1, that is, if \( L \) is \( m \)-reduced.

Let \( f = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3) \). This is a special case of Example (3.1) below, and is one of the simplest ESV-non-calculable examples. Its 3-graph can be written as follows (here we use piecewise smooth curves whose tangent lines vary continuously by technical reasons):

![Graph](image)

This is related to the restriction of \( L \) to the affine space \( \mathbb{C}^2 \subset \mathbb{P}^2 \) defined by the equation \((x^3 - y^3)(x^3 - 1)(y^3 - 1) = 0 \). The nine vertices written in the left lower part correspond to \((\zeta^i, \zeta^j) \in \mathbb{C}^2 \ (i, j) \in (\mathbb{Z}/3\mathbb{Z})^2 \),
with \( \zeta = e^{2\pi \sqrt{-1}/3} \). Among the remaining three vertices, \( P, Q \) correspond to the points at infinity, and \( R \) corresponds to the origin of \( \mathbb{C}^2 \). (One can find similar pictures in the literature. Note that \( \text{Sing } L = L^{[3]} \) in this case.)

### 2.2. Disconnected m-graph case.

Assume \( L = L' \cup L'' \) with \( d', d'' \) divisible by \( m \), \( d = d' + d'' \) (where \( d' := \deg L' \), etc.), and moreover
\[
L^{[m]} = L'^{[m]} \cup L''^{[m]},
\]
then
\[
E_{L,m} = \frac{d'}{d} E_{L',m} + \frac{d''}{d} E_{L'',m},
\]
(in particular, \( E_{L',m} > E_{L'',m} \) if \( E_{L',m} > E_{L'',m} \)). This implies a negative answer to Problem 1 in the case the \( m \)-graph is disconnected. (Note that \( E_{L'',m} = 1 \) if \( L'' \) is non-essential, that is, if \( d'' = km \) and \( L''^{[km]} \neq \emptyset \) with \( k \in \mathbb{Z}_{>0} \)).

### 2.3. Subarrangements of m-star type, I.

For any line arrangement \( L' \), any \( k_0 \in \mathbb{Z}_{>0} \), and any line \( L_0 \subset L' \) containing \( P \in L'^{[km]} \) with \( k_1 \in \mathbb{Z}_{>0} \), we can construct a line arrangement \( L = L' \cup L'' \) such that \( L'' \) has an \( m \)-graph of \( m \)-star type and the following conditions are satisfied:

(i) The intersection \( L' \cap L'' \) coincides with the union of lines in \( L' \) containing \( P \).

(ii) There is \( Q \in L_0 \cap L''^{[km]} \) such that \( |L'' \cap L''^{[km]}| = 2 \) for any line \( L_i'' \subset L'' \) with \( Q \in L_i'' \).

(iii) \( L'^{[m]} \cup L''^{[m]} = L^{[m]} \), \( L'^{[m]} \cap L''^{[m]} = \{P\} \), \( |L''^{[m]}| = k_0 m + 1 \).

(iv) \( \deg L'' = \sum_{k \geq 1} |L''^{[km]}| km - k_0 m \).

Put \( d := \deg L \), \( d' := \deg L' \), \( d'' := \deg L'' \). We have
\[
d - d' = d'' - k_1 m = k_{L''} m, \quad \tilde{E}_{L'',m} = (k_{L''} + k_0 + k_1) m/d'',
\]
with
\[
k_{L''} := \sum_{k \geq 1} |L''^{[km]}| k - (k_0 + k_1).
\]
Setting
\[
\tilde{E}_{L'',m} := (k_{L''} + k_0) m/(d - d') = 1 + k_0/k_{L''},
\]
we then get that
\[
E_{L,m} = \frac{d'}{d} E_{L',m} + \frac{d - d'}{d} \tilde{E}_{L'',m}.
\]
Here \( E_{L,m} \) can be arbitrarily close to 1, since we can make \( k_{L''} \) arbitrarily large. This implies a negative answer to Problem 1 unless we assume the \( m \) is strictly unsaturated. (Note that if there is \( J \subset \{1, \ldots, d\} \) such that condition (a) (resp. (b)) is satisfied, then \( |J \cap \{d'+1, \ldots, d\}| \) must be at most (resp. at least) \( (d - d')/m = k_{L''} \).)

### 2.4. Subarrangements of m-star type, II.

In the case \( m = 3 \), it is not difficult to construct an arrangement having a subarrangement with an \( m \)-graph of \( m \)-star type and such that condition (b) is not satisfied although the \( m \)-efficiency is at most 2. For instance, consider a line arrangement \( L' \) of degree 8 whose 3-graph is as follows:

\[
\begin{array}{c}
\text{The number of edges is 7, and there is a line } L'_0 \text{ in } L' \text{ which does not contribute to an edge. This line passes through the leftest vertex.}
\end{array}
\]

Let \( L = \bigcup_{i=1}^{3} L^{(i)} \), where \( L^{(i)} \) (\( i = 1, 2, 3 \)) is as above (with \( \deg L^{(i)} = 8 \)) and \( L^{(i)} \cap L^{(j)} \) is discrete (\( i \neq j \)). Assume
\[
\bigcap_{i=1}^{3} L^{(i)} = \{P\} \not\subset \bigcup_{i=1}^{3} \text{Sing } L^{(i)}, \quad L^{[3]} = \bigcup_{i=1}^{3} L^{(i)[3]} \cup \{P\},
\]
where \( L^{(i)}_0 \subset L^{(i)} \) is the unique line containing only one point of multiplicity 3 (and not contributing to an edge). Then \( \deg L = 24 \), \( E_{L,3} = 2 \), and conditions (a), (b) are both unsatisfied.
It is also possible to consider the case $L = \bigcup_{i=1}^{6} L^{(i)}$ where $L^{(i)}$ ($i \in \{1, \ldots, 4\}$) is as above (with $|L^{(i)}| = 8$, $|L^{(3)}| = 6$, $|L^{(4,3)}| = 4$, $|L^{(6)}| = 1$, $L^{(i)} \cap L^{(j)}$ is discrete ($i \neq j$), and $\cap_{i=1}^{4} L^{(i)} \cap L^{(5)} \cap L^{(6)} = \{P\} \not\subset \bigcup_{i=1}^{5} \text{Sing} L^{(i)}$, $L^{(3,3)} = \bigcup_{i=1}^{5} L^{(i)} \cap \{P\}$.

Here $L^{(5)}$ is as in Example (3.1) below with $a = 2$. Then $\deg L = 39$, $|L^{[3]}| = 24$, $|L^{[6]}| = 1$, and $E_{L,3} = 2$. Conditions (a) and (b) are both unsatisfied.

One can replace $L^{(5)}$ in the above example by a non-essential arrangement of degree $3a$ ($a \in \mathbb{Z}_{>2}$) so that $|L^{(5)}| = 3a$, $L^{(3,3)} \not\subset \emptyset$. In this case we have

$$\deg L = 3(a+11), \quad |L^{[3]}| = 20, \quad |L^{[6]}| = 1, \quad |L^{[3a]}| = 1,$$

$$E_{L,3} = (a+22)/(a+11).$$

We can verify that conditions (a) and (b) are both unsatisfied.

### 2.5. Relation with non-existence of $k$-multinets for $k > 4$

It is known that there is a close relation between resonance varieties, pencils, and multinets, and there are no $k$-multinets for $k > 4$, see [FY], [LY], [PY], [Yu1], [Yu2]. However, it seems unclear whether these imply the vanishing of $H^1(F_j, \mathbb{C})_{\lambda}$ with $m := \text{ord } \lambda > 4$ in the ESV-calculable case.

Assume this eigenspace does not vanish. Then $L$ must support a $k$-multinet. Hence there is a partition

$$I := \{1, \ldots, d\} = \bigsqcup_{j=1}^{k} I_{j}$$

together with multiplicities $m_{i} \in \mathbb{Z}_{>0}$ ($i \in I$) such that $\text{GCD}(m_{i}) = 1$ and in the notation of (1.1) we have

$$(2.5.1) \quad \omega = \sum_{j=1}^{k} c_{j} \eta^{(j)} \quad \text{with} \quad \eta^{(j)} = \sum_{i \in I_{j}} n_{i} m_{i} e_{i},$$

with $c_{j} \in \mathbb{C}$ ($j \in \{1, \ldots, k\}$) satisfying $\sum_{j=1}^{k} c_{j} = 0$, see loc. cit. Here $\omega$ is defined by (1.1.3) and (1.2.3). The equality (2.5.1) then implies that $c_{j} \in \mathbb{Q}$, and $J$ is compatible with the partition of $I$, that is, $J$ is a union of $I_{j}$ ($j \in K$) for some subset $K \subset \{1, \ldots, k\}$ (using the positivity: $m_{i} \in \mathbb{Z}_{>0}$). We then get that $m_{i}$ for $i \in I_{j}$ depends only on $j$ (denoted by $m_{j}$) using (1.2.3). However, it is unclear whether the $m_{j}$ are independent of $j$. (Consider, for instance, the case

$$k = 3, \quad c_{1} = 2, \quad c_{2} = c_{3} = -1, \quad |I_{1}| = b, \quad |I_{2}| = |I_{3}| = ab,$$

and $m_{i} = a$ if $i \in I_{1}$, and 1 otherwise, where $a, b \in \mathbb{Z}_{>0}$.)

### 3. Examples

In this section we calculate some examples.

**Example 3.1.** For any integer $a \geq 2$, let

$$f = (x^{a} - y^{a})(x^{a} - z^{a})(y^{a} - z^{a}),$$

with $d = 3a$. This is a reflection arrangement of type $G(a, a, 3)$, see [OT] p. 280]. If $a = 2$, this is the simplest example of non-vanishing Milnor fiber cohomology with non-unipotent monodromy. We have $L^{[k]} = \emptyset$ ($k \neq 3, a$), and

$$|L^{[3]}| = a^{2}, \quad |L^{[a]}| = 3 \quad (a \neq 3), \quad |L^{[3]}| = 12 \quad (a = 3).$$

Hence

$$\frac{4}{3} = a, \quad E_{L,3} = \begin{cases} \frac{a+1}{3} & (a \in 3\mathbb{Z}), \\ a & (a \notin 3\mathbb{Z}) \end{cases},$$

$$\frac{4}{a} = 3, \quad E_{L,a} = 1 \quad (a \neq 3).$$
If \( a \notin 3\mathbb{Z} \), we see that \( H^1(F_f, \mathbb{C})_\lambda \) for ord \( \lambda = 3 \) is ESV-calculable, where \( J \) can be given by any of the three factors of \( f \). In the other case, it is not ESV-calculable. Its dimension is either 2 or 1, depending on whether \( a \in 3\mathbb{Z} \) or not, see [Di3]. Note that it is 1 in the ESV-calculable case, since we should have a 3-net, see [BDS], etc.

**Example 3.2.** Let

\[
\begin{align*}
  f_1 &= xyz(y+2z)(x-y)(x-y+z)(x+y+z)(x+y+2z)(x-2y-z), \\
  f_2 &= xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(2x+3y+3z).
\end{align*}
\]

The pictures of the restrictions of the arrangements to \( \mathbb{C}^2 = \mathbb{P}^2 \setminus \{z = 0\} \) are as below. Note that the line at infinity belongs to \( L \), and there are three points of multiplicity 3 at infinity.

\[\text{The first one is a 3-net which is one of the second simplest examples for non-vanishing Milnor fiber cohomology with non-unipotent monodromy. The second one is one of the simplest ESV-non-calculable examples, see also [CS]. In both cases we have } d = 9, \text{ and } |L^{[3]}| = |L^{[3]}| = 9. \text{ Hence } E_{L,3} = \frac{d}{3} = 3.\]

One can verify that \( H^1(F_f, \mathbb{C})_\lambda \) with ord \( \lambda = 3 \) is always ESV-calculable independently of whether \( a \in 3\mathbb{Z} \) or not. Indeed, if \( a \notin 3\mathbb{Z} \), we can take \( J \) corresponding to any one of the three factors of the above factorization. (This is trivial.) In the case \( a = 3a' \), we can take \( J \) corresponding to the polynomial

\[
g = \prod_{i=0}^{a'-1} (x-i) \prod_{j=0}^{a'-1} (y-j) \prod_{k=2a'}^{a'-1} (x+y-kz).
\]

The picture in the case \( a = 6 \) is as below.

\[\text{The picture in the case } a = 6 \text{ is as below.} \]
Notice that we lose the ESV-calculability in the case $a \in 3\mathbb{Z}$, if the last product in the definition of $f$ is taken over $k \in \{1, \ldots, a\}$ instead of $k \in \{0, \ldots, a-1\}$.

**Example 3.4.** Let

$$f = xyz \prod_{i,j=0}^{2} (\zeta^i x + \zeta^j y + z) = xyz \prod_{k=0}^{2} (x^3 + y^3 + z^3 - 3\zeta^k xyz),$$

with $\zeta = e^{2\pi \sqrt{-1}/3}$. This is the Hessian arrangement, which is a unique example of 4-net, see [OT] Ex. 6.30 and p. 232, [FY] Ex. 3.5, [BDS] Remark 3.3 (iii), [Dj2] Theorem 8.19.

It is conjectured that there are no other 4-nets, see [Yu2]. (This arrangement seems to be confused with a different one in some paper which is not quoted in this paper.)

We have $d = 12$, and $|L^{[4]}| = 9$; more precisely

$$L^{[4]} = L^{[4]} = \{x^3 + y^3 + z^3 = xyz = 0\} \subset \mathbb{P}^2.$$

These imply that

$$E_{L, i} = \frac{d}{4} = 3.$$ 

Here $H^1(F_j, \mathbb{C})$ with ord $\lambda = 4$ is ESV-calculable, and its dimension is 2 (since it is a 4-net). The subset $J$ can be given by any factor in the last factorization of $f$.

**Example 3.5.** For $m \geq 3$, let

$$f = \prod_{i,j=0}^{m-1} (\zeta^i x + \zeta^j y + z),$$

with $\zeta := e^{2\pi \sqrt{-1}/m}$. (This is a subarrangement of the Hessian arrangement when $m = 3$.) We have $d = m^2$, $|L^{[m]}| = |L^{[m]}| = 3m$ with $L^{[m]} = L \cap \{xyz = 0\}$, and $\text{Sing } L = L^{[2]} \cup L^{[m]}$. (The last assertion can be reduced to the injectivity of the map

$$\Theta_i \times \Theta_j \ni (\lambda, \lambda') \mapsto \lambda/\lambda' \in \mathbb{C},$$

with $\Theta_i := \{\lambda \in \mathbb{C} \mid |\lambda + \zeta^i| = 1, \lambda \neq 0\}$, calculating the intersection points of lines in $L$, where we get that $x = -(\zeta^i - \zeta^j)/(\zeta^i - \zeta^j)$ (with $i, j$ fixed) after setting $y = 1$. Here we may assume $i = j = 0$ using the action of $\mu_m$ on $\mathbb{C}$. We have

$$\Theta_0 = \{re^{i\theta} \mid r = -2 \cos \theta, \theta \in \left(\frac{1}{2} \pi, \frac{3}{2} \pi\right)\}.$$ 

If $\lambda_1/\lambda'_1 = \lambda_2/\lambda'_2$, we have $\arg \lambda_1 - \arg \lambda'_1 = \arg \lambda_2 - \arg \lambda'_2$ (which is denoted by $\alpha \in (0, \pi)$). Setting $\theta = \arg \lambda'$ (so that $\arg \lambda = \theta + \alpha$), the assertion is then reduced to the injectivity of the map

$$\left(\frac{1}{2} \pi, \frac{3}{2} \pi - \alpha\right) \ni \theta \mapsto \cos(\theta + \alpha)/\cos \theta \in \mathbb{R},$$

where the image is equal to $\cos \alpha - \sin \alpha \tan \theta$. So the assertion follows.) We then get that

$$\frac{d}{m} = m, \quad E_{L, m} = 3.$$ 

In the $m$ even case, we can verify that $H^1(F_j, \mathbb{C})$ for ord $\lambda = m$ is not ESV-calculable. Indeed, assume $J$ is given by

$$J = \left\{ ([a_k], [b_k]) \in (\mathbb{Z}/m\mathbb{Z})^2 \mid k \in \mathbb{Z}/m\mathbb{Z}\right\},$$

with $a_k, b_k \in [0, m-1]$. Here $(i, j) \in (\mathbb{Z}/m\mathbb{Z})^2$ is identified with a line defined by the equation $\zeta^i x + \zeta^j y + z = 0$. In order to satisfy condition (a) or (b), we must have

$$\{a_k\} = \{b_k\} = \{a_k - b_k + m\delta_k\} = \{0, \ldots, m-1\},$$

taking the intersection with $\{y = 0\}$, $\{x = 0\}$, $\{z = 0\}$. Here $\delta_k = 1$ if $a_k < b_k$, and 0 otherwise. We then get that

$$\sum_{k=0}^{m-1} a_k = \sum_{k=0}^{m-1} b_k = \sum_{k=0}^{m-1} (a_k - b_k + m\delta_k) = (m-1)/2.$$ 

However, these imply that $\sum_k \delta_k = (m-1)/2 \in \mathbb{Z}$, which is a contradiction, since $m$ is even. So ESV-non-calculability follows.
In the $m$ odd case, it is easy to see that $H^1(F_f, \mathbb{C})_{\lambda}$ for ord $\lambda = m$ is ESV-calculable, since $J$ can be given by

$$J = \{(i, 2i) \in (\mathbb{Z}/m\mathbb{Z})^2 \mid i \in \mathbb{Z}/m\mathbb{Z}\}.$$ 

Note that the multiplication by 2 is an automorphism of $\mathbb{Z}/m\mathbb{Z}$.

**Example 3.6.** Consider a general hyperplane section of a reflection arrangement of type $G_{31}$. Here $d = 60$ and each line has 16 triple points and 3 points of multiplicity 6, see [BDY, OT, Sa2]. Hence

$$E_{L,3} = 19, \quad E_{L,6} = 3.$$ 

In this case $H^1(F_f, \mathbb{C})_{\lambda}$ with ord $\lambda = 3$ or 6 is never ESV-calculable, see loc. cit.

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