SYMPLECTIC ACTION AROUND LOOPS IN Ham(M)

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Abstract. Let Ham(M) be the group of Hamiltonian symplectomorphisms of a quantizable, compact, symplectic manifold (M, ω). We prove the existence of an action integral around loops in Ham(M), and determine the value of this action integral on particular loops when the manifold is a coadjoint orbit.

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1. Introduction

Given a compact, symplectic manifold (M, ω), the group of Hamiltonian symplectomorphisms [16] [17] of (M, ω) is denoted by Ham(M). If ψ = {ψt}t∈[0,1] is a loop in Ham(M) at id and x is a point of M, then the closed curve in M {ψt(x)}t is nullhomologous [16] [15]. If S is a 2-chain in M whose boundary is this curve, one can consider

\[ \kappa_x(ψ) := \exp \left( 2\pi i \int_S \omega - 2\pi i \int_0^1 f_t(ψ_t(x)) dt \right), \]  \tag{1.1} 

where \( f_t \) is the normalized time-dependent Hamiltonian associated to ψ. Throughout this article we will assume that (M, ω) is quantizable [24], that is, ω defines an integral cohomology class in M, then the right hand side of (1.1) is independent of the 2-chain S. \( \kappa_x(ψ) \) is in fact the U(1)-valued action integral around the curve \( \{ψ_t(x)\}_t \) [16] [22]. In [21] we proved that \( \kappa_x(ψ) \) is independent of the point x; the proof is based in an analysis of some properties of the prequantization representation [20]. Here we give a new proof of this property in the context of the gauge transformations of a prequantum bundle [24]. The idea of the proof is the following: Since M is quantizable, there is a prequantum bundle over M, that is, a Hermitian line bundle \( π : L → M \) with a connection D, such that the curvature of D is \(-2\pi iω\). The time-dependent Hamiltonian \( f_t \) determines the corresponding Hamiltonian vector fields \( X_t \), and by \( X^L_t \) is denoted the horizontal lift of \( X_t \). From \( f_t \) one can define a vertical vector field \( W_{f_t} \) whose value at \( p \) is determined by the curve \( \{p \cdot \exp(2\pi itf(π(p)))\}_t \). Then \( \{Z_t := X^L_t - W_{f_t}\} \) generates a family \( F_t \) of diffeomorphisms of L which preserve the connection. That is, loops in Ham(M) lift to preserving-connection isotopies of a prequantum bundle. We prove that \( F_1 \) is the gauge transformation defined by the map \( x ∈ M → \kappa_x(ψ) ∈ U(1) \). It follows from this result that the map \( \kappa_x(ψ) \) is constant. From the independence of \( \kappa_x(ψ) \) of x one deduces that \( \kappa(ψ) \) depends only on the homotopy class of ψ. So we obtain

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a representation $\kappa$ of the group $\pi_1(\text{Ham}(M))$, which can be called the action integral representation. The existence of this representation has been proved in other contexts by Seidel [19] and Schwarz [18].

The isotopy $F_t$ of $L$, lifting of the loop $\psi$ in $\text{Ham}(M)$, allows us to assign to each section $\tau$ of $L$ a family $\tau_t$ of sections defined by $\tau_t = F_t \circ \tau \circ \psi_t^{-1}$. One can consider the correspondence $\tau \to \tau_t$ as a “transport” of sections of $L$ along $\psi$. On the other hand, in the prequantization of the manifold $M$ one constructs for each Hamiltonian vector field $X$ an operator $P_X$ (the corresponding prequantization operator [20]), which acts on the space $\Gamma(L)$ of sections of $L$. If $X_t$ is the family of vector fields that determines the isotopy $\psi$, we prove that

$$\frac{d\tau_t}{dt} = P_{X_t}(\tau_t), \quad \tau_0 = \tau$$

is the differential equation of the transport along $\psi$. In this context $\kappa(\psi)$ is the holonomy of this transport.

If $M$ is a Hamiltonian $G$-space [6] one can consider loops $\varphi$ in $\text{Ham}(M)$ defined by the $G$-action. If a family $g_t$ of elements in $G$ determines a loop $\varphi$ in $\text{Ham}(M)$, the independence of $\kappa_\varphi(\varphi)$ from $x$ allows us to express $\kappa(\varphi)$ in terms of the vertices of moment map. This fact, when the group $G$ is a torus $T$, implies the existence of a moment map for the $T$-action whose vertices are integer lattice points of $t^*$. In a Hamiltonian $G$-space to each $A \in \mathfrak{g}$ corresponds a vector field $X_A$ in $M$. The map $\mathcal{P}$, which assigns to each $A \in \mathfrak{g}$ the operator $\mathcal{P}_{X_A}$, is in fact a Lie algebra representation. The $G$-action is called pre-quantizable [7], if there is an action $\rho$ of $G$ on $L$ that induces $\mathcal{P}$. Guillemin and Sternberg proved the above property about the vertices of a moment map under the additional hypothesis that the $G$-action is pre-quantizable [7, § 8, Corollary 1].

In the particular case, when the $G$-action is pre-quantizable, we will express $\kappa(\varphi)$ in terms of the character of the restriction of $\rho$ to invariant finite-dimensional subspaces of $\Gamma(L)$. More precisely, the action $\rho$ induces a representation $\nu$ of $G$ on the space $\Gamma(L)$. On the other hand, if $G$ is a compact group there are $G$-invariant almost complex structures on $M$. Let $I$ be such an almost complex structure and $Q_I$ the subspace of $\Gamma(L)$ consisting of the $I$-polarized sections. Using that $\kappa(\varphi)$ is the holonomy of the transport along $\varphi$, $\kappa(\varphi)$ can be expressed in terms of the character of $\nu$ restricted to $Q_I$ (Theorem 16).

The coadjoint orbits of a compact Lie group $G$ are particular cases of Hamiltonian $G$-spaces. If $\eta \in \mathfrak{g}^*$ is integral, i.e. there is a character $\Lambda$ of the stabilizer subgroup $G_\eta$ of $\eta$ whose derivative is $2\pi i \eta$, then the action of $G$ on the orbit of $\eta$, $O_\eta$, is pre-quantizable. Now a prequantum bundle $L$ is the bundle on $G/G_\eta$ determined by $\Lambda$. So $\Gamma(L)$ can be identified with the space of $\Lambda$-equivariant functions on $G$. In $O_\eta$ one can consider loops $\psi$ of Hamiltonian symplectomorphisms generated by vector fields associated to elements of $\mathfrak{g}$; it turns out that the values of $\kappa$ on these particular loops are related with the character $\Lambda$. More precisely, if $\psi$ is the loop generated by a family $A_t$ of elements of $\mathfrak{g}$, from the action of $G$ on the $\Lambda$-equivariant functions it is easy to prove that $\kappa(\psi) = \Lambda(h_1)$, where $h_t$ is the solution to Lax equation $h_t h_t^{-1} = A_t$ (Theorem 20). On the other hand, according to the Kostant version of Borel-Weil theorem [13], to each integral orbit $O_\eta$ corresponds an irreducible representation $\pi$ of $G$, whose highest weight is determined by $\eta$. If $G$ is a semisimple Lie group, the choice of a maximal torus contained in $G_\eta$ permits us to define a $G$-invariant complex structure $I$ on $G/G_\eta = O_\eta$. This complex
structure, in turn, determines a holomorphic structure on $L$, and $Q_t$ is just the space $H^0(L)$ of holomorphic sections of $L$. When $G_\eta$ is itself a maximal torus, the Borel-Weil theorem allows us to characterize the restriction of $\nu$ to $H^0(L)$ in terms of its highest weight. It turns out that this restriction of $\nu$ is the representation $\pi^*$, dual of $\pi$. It follows from Theorem 16 that the invariant $\kappa(\psi)$ for the closed isotopy considered above is equal to $\chi(\pi^*)(h_1)/\dim \pi$, where $\chi(\pi^*)$ is the character of $\pi^*$ (Theorem 21). This result permits us to calculate $\kappa(\psi)$ using the Weyl’s character formula.

The paper is organized as follows. The second Section is concerned with the new proof of the independence of $\kappa$ from the point $x$. The existence of a lifting for $\kappa$ to an $\mathbb{R}$-valued map is proved as well. In Section 3 we study the map $\kappa$ when the manifold is a Hamiltonian $G$-space. In the case that $G$ is a torus, the existence of a moment map such that its vertices are integer lattice points is proved. In Section 4 we study the invariant $\kappa(\psi)$ in an integral coadjoint orbit of a compact Lie group $G$. Finally we apply some results of this Section to two particular cases. In the first one we determine the value of $\kappa(\psi)$ for a closed isotopy $\psi$ in a general flag manifold, when the isotopy is generated by the action of the corresponding unitary group. In [21] we determined the value $\kappa(\psi)$ for a closed Hamiltonian flow $\psi$ in $S^2$, by direct calculation. Here we recover this number twice; by applying Theorem 20, and using Weyl’s character formula.

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2. Action integral around isotopies

We denote by $(M,\omega)$ a compact, connected, symplectic, $2n$-dimensional manifold. We will suppose that $(M,\omega)$ is quantizable, so there exists a Hermitian line bundle $\pi : L \to M$ (a prequantum bundle) with a connection, such that its curvature is $-2\pi i\omega$, that is, the first Chern class $c_1(L) = [\omega]$.

$L^\times = L - \{\text{zero section}\}$ is the corresponding $\mathbb{C}^\times$-principal bundle. If $c \in \mathbb{C} = \text{Lie}(\mathbb{C}^\times)$ we denote by $W_c$ the vertical vector field on $L^\times$ whose value at $p$ is defined by the curve $\{p \cdot e^{2\pi ic}\}_t$. If $f$ is a function on $M$ by $W_f$ is denoted the vector field on $L^\times$ given by $W_f(p) = W_L(\pi(p))(p)$. As it is well-known each section $\lambda$ of $L$ determines an equivariant map $\lambda^\sharp : L^\times \to \mathbb{C}$ by the formula $\chi(\pi(p)) = p \cdot \lambda^\sharp(p)$, and one has the known relations (see [20] [12])

$$ (D_X\lambda)^\sharp = X^\sharp(\lambda^\sharp) = D_X\pi^\sharp, \quad D_X\lambda = (\lambda^*\alpha)(X)\lambda, \quad (2.1) $$

where $\alpha$ is the connection in the principal bundle $L^\times$. It is also easy to prove that

$$ \lambda_*(Y) = Y^\sharp(\lambda(x)) + W_c(\lambda(x)), \quad (2.2) $$

with $c = (2\pi i)^{-1}(\lambda^*\alpha)(Y)$.

On the other hand, if $X$ is the Hamiltonian vector field on $M$ determined by de function $f$ (i.e. $\iota_X\omega = -df$), then the Lie derivative $\mathcal{L}_Z\alpha = 0$, for $Z = X^\sharp - W_f$ (see [20, page 56]). Hence $X^\sharp - W_f$ defines a 1-parameter subgroup of diffeomorphisms of $L^\times$ which preserve the connection $\alpha$.

Let $\{\psi_t | t \in [0, 1]\}$ be the Hamiltonian isotopy associated to the time-dependent Hamiltonian $f_t$. This isotopy is generated by a unique family of vector fields $X_t$ by
the relations
\[ \frac{d\psi_t(x)}{dt} = X_t(\psi_t(x)), \quad \psi_0 = \text{id}, \]
with \( \iota_{X_t} \omega = -df_t \). The Hamiltonian \( f_t \) is determined by \( \{\psi_t\} \) up to an additive constant. In the general case this constant will be fixed imposing that \( \int_M f_t \omega^n = 0 \).

The family \( \{X_t^f - W_{f_t}\}_t \) of vector fields on \( L^x \) determines a set \( \{F_t\}_t \) of preserving connection diffeomorphisms of \( L^x \), such that \( \pi \circ F_t = \psi_t \circ \pi \). We call the isotopy \( F_t \) the lift of \( \psi_t \).

Given \( \tau \) a local section of \( L^x \), one defines \( \tau_t := F_t \circ \tau \circ \psi_t^{-1} \). It is easy to prove the following relation between the corresponding equivariant maps
\[ \tau_t^\psi = \tau^\psi \circ F_t^{-1}. \quad (2.3) \]

For each \( t \) we have the map \( \tau \in \Gamma(L^x) \mapsto \tau_t \in \Gamma(L^x) \). This family of maps can be regarded as a transport \( T_\psi \) along the isotopy \( \{\psi_t\} \).

First of all we will determine the differential equation that defines the transport \( T_\psi \). Given the point \( x \in M \) and the section \( \tau \) of \( L^x \), the tangent vector to the curve \( \{F_t(\tau(x))\}_t \) in \( L^x \) at the point \( q_t = F_t(\tau(x)) \) is, by definition of \( F_t \),
\[ Z_t(q_t) := X_t^f(q_t) - W_{f_t}(q_t). \quad (2.4) \]

On the other hand \( F_t(\tau(x)) = \tau_t(x_t) \), with \( x_t := \psi_t(x) \); hence
\[ Z_t(q_t) = \tau_t(\tau_t(x_t)) + W_a(q_t), \quad (2.5) \]
with
\[ 2\pi i a \cdot q_t = \left. \frac{d \tau_t(x_t)}{d \epsilon} \right|_{\epsilon = t}. \quad (2.6) \]
The formula (2.2) applied to \( \tau_t \) gives
\[ \tau_t(\tau_t(x_t)) = X_t^f(\tau_t(x_t)) + W_a(\tau_t(x_t)), \quad (2.7) \]
where \( c = (2\pi i)^{-1}(\tau_t^*\alpha)(X_t(x_t)) \). It follows from (2.4), (2.5) and (2.7)
\[ -W_{f_t}(q_t) = W_a(q_t) + W_a(q_t). \quad (2.8) \]
Using the identification of \( T_{q_t} L_{x_t} \) y \( L_{x_t} \), where \( L_{x_t} \) is the fiber of \( L \) over \( x_t \), one obtains from (2.8)
\[ -2\pi i f_t(x_t) \cdot q_t = (\tau_t^*\alpha)(X_t(x_t)) \cdot q_t + 2\pi i a \cdot q_t. \]
By (2.1) the first term on the right hand side is \( (D_{X_t} \tau_t)(x_t) \). Hence by (2.6)
\[ \left( \frac{d \tau_t}{d \epsilon} \right)(x_t) = -(D_{X_t} \tau_t)(x_t) - 2\pi i f_t(x_t) \cdot \tau_t(x_t). \]
As the point \( x \) is arbitrary we have proved the following Proposition that gives the differential equation for the transport \( T_\psi \) along \( \psi \)

**Proposition 1.** The family of sections \( \tau_t \) defined by \( \tau_t = F_t \circ \tau \circ \psi_t^{-1} \) is the solution to the equation
\[ \frac{d \tau_t}{d t} = -D_{X_t} \tau_t - 2\pi i f_t \tau_t, \quad \tau_0 = \tau. \quad (2.9) \]
Let $X$ be the Hamiltonian vector field on $M$ associated to the function $f$. This vector field determines the operator $\mathcal{P}_X := -D_X - 2\pi i f$, which acts on the space $\Gamma(L)$. So

$$\frac{d\tau_t}{dt} = \mathcal{P}_X(\tau_t), \quad \tau_0 = \tau,$$

is the differential equation for the transport $T_\tau$.

To prove the next Proposition we use the following Lemma; its proof is straightforward.

**Lemma 2.** Let $H$ be a diffeomorphism of $L$ over the diffeomorphism $\varphi$ of $M$. If $H$ preserves the connection, then

$$H_*(Y^\sharp(p)) = (\varphi_*(Y))^\sharp(H(p)),$$

for $Y \in T_x M$ and $p \in \pi^{-1}(x)$. And

$$H_*(W_f(p)) = W_{f \circ \varphi^{-1}}(H(p)),$$

for any function $f$ defined on $M$.

**Proposition 3.** Let $\tau_t$ be the solution to (2.9), and $Y$ a vector field on $M$. Then $D_y \tau_t = D_Y \tau$, for $Y_t = \psi_{\tau_t}(Y)$.

**Proof.** As $F_t$ preserves the connection, Lemma 2 is applicable

$$Y^\sharp_\tau(\tau^\sharp_t) = (\psi_{\tau_t}(Y))^\sharp(\tau^\sharp_t) = F_{\tau_t}(Y^\sharp_\tau(\tau^\sharp_t)).$$

Taking into account (2.3)

$$Y^\sharp_\tau(\tau^\sharp_t) = F_{\tau_t}(Y^\sharp_\tau(\tau^\sharp_t)), \quad Y^\sharp_\tau = Y^\sharp_\tau \circ F_{\tau_t}^{-1}.$$

From (2.1) it follows $D_y \tau_t = D_Y \tau$. \hfill \Box

Let $q$ be a point of $L^\times$ with $\pi(q) = x$, and $\tau$ a local section of $L^\times$ such that $x$ belongs to the domain of $\tau$. As $F_t(q)$ is in the fiber of $L$ over $x_t := \psi_t(x)$, then

$$F_t(q) = \tau(x_t) \cdot m_t, \quad (2.10)$$

with $m_t \in \mathbb{C}^\times$. The curve $\{F_t(q)\}_{t}$ defines at the point $q_t = F_t(q)$ the vector (2.4). On the other hand the curve $\tau(x_t) \cdot m_t$ defines also the vector

$$Z_t(q_t) = (R_{m_t})_\tau(\tau_*(X_t(x_t))) + W_c(q_t), \quad (2.11)$$

where $c = (2\pi i)^{-1}(m_t/m_t)$ and $R_d$ is the right multiplication by $d \in \mathbb{C}^\times$ in $L^\times$. As $R_{d*}$ preserves the horizontal and vertical components, by (2.2)

$$Z_t(q_t) = X_t^\sharp(q_t) + W_d(q_t) + W_c(q_t), \quad (2.12)$$

with $d = (2\pi i)^{-1}(\tau^*\alpha)(X_t(x_t))$. It follows from the equality of (2.12) with (2.4)

$$\frac{\dot{m}_t}{m_t} = - (\tau^*\alpha)(X_t(x_t)) - 2\pi i f_t(x_t).$$

That is,

$$m_t = \exp\left(-\int_0^t (\tau^*\alpha)(X_u(x_u)) \, du - 2\pi i \int_0^t f_u(x_u) \, du\right) \quad (2.13)$$
If the isotopy $\psi_t$ is closed, that is, $\psi_1 = \text{id}$, then the curve $\psi_t(x)$ is nullhomologous [16, page 334]. Let $S$ be any 2-chain in $M$ whose boundary is the closed curve $\{\psi_t(x)\}_t$. As the curvature of the connection is $-2\pi i \omega$, by the Stokes theorem

$$m_1 = \exp \left(2\pi i \int_S \omega - 2\pi i \int_0^1 f_t(\psi_t(x)) dt \right) \quad (2.14)$$

One obtains from (2.10), (2.14) and the definition (1.1) of $\kappa_x(\psi)$

$$F_1(q) = q \cdot \kappa_{\pi(q)}(\psi). \quad (2.15)$$

Therefore $F_1$ is the gauge transformation determined by the map $\kappa_-(\psi) : M \to U(1)$. As $F_1$ preserves the connection $\alpha$, this implies $d\kappa_-(\psi) = 0$. Hence the map $\kappa_-(\psi)$ is constant, and we have

**Theorem 4.** If $\{\psi_t\}_{t \in [0,1]}$ is a closed Hamiltonian isotopy in a quantizable manifold, the action integral along the curve $\{\psi_t(x) \mid t \in [0,1]\}$ is independent of the point $x \in M$.

As we said in Introduction a different proof of this result is given in [19] (see also [18, Section 3]).

From (2.15) and (2.3) it follows $\tau^X = \kappa(\psi) \tau^X$; that is,

**Corollary 5.** $\kappa(\psi)$ is the holonomy of the transport $T_\psi$.

As we said $L$ denote the space of loops in $\text{Ham}(M)$ based at $\text{id}$. If $\psi, \xi \in L$ and we denote by $\psi \ast \xi$ the usual product of paths, it is immediate to check that

$$\kappa(\psi \ast \xi) = \kappa(\psi)\kappa(\xi). \quad (2.16)$$

Next we will study the behavior of $\kappa(\psi)$ under $C^1$-deformations of $\psi$. We consider the derivative of $\kappa(\psi^s)$ with respect to the parameter $s$ in a deformation $\psi^s$ of $\psi$.

That is, $\psi^s = \{\psi^s_t \mid t \in [0,1]\}$ is an isotopy with $\psi^s_0 = \psi^s_1 = \text{id}$ generated by the time-dependent Hamiltonian $f^s_t$; furthermore we assume that $\psi^0 = \psi$. By $\{X^s_t\}_t$ is denoted the family of Hamiltonian vector fields defined by $\{f^s_t\}_t$.

For $x \in M$ we put $\sigma^s(t) := \psi^s_t(x)$, so $\{\sigma^s(t) \mid t \in [0,1]\}$ is a closed curve and then

$$\kappa(\psi^s) = \exp \left(2\pi i \int_{S^s} \omega - 2\pi i \int_0^1 f^s_t(\sigma^s(t)) \right),$$

where $S^s$ is a surface bounded by the curve $\sigma^s$. We set

$$X_t := X^0_t, \quad f_t := f^s_t, \quad \sigma(t) = \sigma^s(t).$$

The variation of $\sigma^s(t)$ with $s$ permits to define the vector fields $Y_t$; that is,

$$Y_t(\sigma^s(t)) := \frac{\partial}{\partial s} \sigma^s(t). \quad (2.17)$$

For an “infinitesimal” $s$ the curves $\sigma^t$, with $t \in [0, s]$ determine the “lateral surface” $J$ of one “wedge” whose base and cover are the surfaces $S$ and $S^s$ respectively. The ordered pairs of vectors $(X_t(\sigma(t)), Y_t(\sigma(t)))$ fix an orientation on $J$, which in turn determines an orientation on the closed surface $T = S \cup J \cup S^s$. If we assume that $S$ and $S^s$ are oriented by means of the orientations of curves $\sigma$ and $\sigma^s$, from the fixed orientation on $T$ it follows $T = J - S + S^s$.

As $\omega$ satisfies the integrality condition

$$-\int_S \omega + \int_{S^s} \omega = -\int_J \omega \text{ (modulo Z).} \quad (2.18)$$
Moreover
\[ \int_J \omega = s \int_0^1 \omega(X_t(\sigma(t)), Y_t(\sigma(t))) dt + O(s^2). \] (2.19)

On the other hand, for a given \( t \in [0, 1] \)
\[ \left( \frac{d}{ds} f^*_t(\sigma^s(t)) \right)_{|s=0} = \left( \frac{\partial}{\partial s} f^*_t(\sigma(t)) \right)_{|s=0} + Y_t(\sigma(t))(f_t). \] (2.20)

We set
\[ \dot{f}_t(x) := \left( \frac{\partial}{\partial s} f^*_t(x) \right)_{|s=0}. \]

As \( \iota_X \omega = -df_t \), from (2.20) it follows
\[ \frac{d}{ds} \bigg|_{s=0} \int_0^1 \dot{f}_t(\sigma(t)) dt = \int_0^1 \dot{f}_t(\sigma(t)) dt - \int_0^1 \omega(X_t(\sigma(t)), Y_t(\sigma(t))) dt. \] (2.21)

By (2.18), (2.19) and (2.21)
\[ \kappa(\psi^s) - \kappa(\psi) = -2\pi i \kappa(\psi) \int_0^1 \dot{f}_t(\sigma(t)) dt + O(s^2), \]
and finally
\[ \left( \frac{d}{ds} \kappa(\psi^s) \right)_{|s=0} = -2\pi i \kappa(\psi) \int_0^1 \dot{f}_t(\psi_t(x)) dt. \]

We have proved the following Proposition

**Proposition 6.** If \( \psi^s \) is the deformation of \( \psi \) defined by the family \( f^*_t \) of time-dependent Hamiltonians, then
\[ \frac{1}{2\pi i \kappa(\psi)} \left( \frac{d}{ds} \kappa(\psi^s) \right)_{|s=0} = -\int_0^1 \dot{f}_t(\psi_t(x)) dt, \]
x being any point of \( M \).

**Proposition 7.** \( \kappa \) defines a group homomorphism \( \kappa : \pi_1(\Ham(M)) \to U(1) \).

**Proof.** With the above notations \( \int_M f^*_t \omega^n = 0 \) for any \( s \). Then
\[ \int_M (\dot{f}_t \circ \psi_t) \omega^n = 0. \]

By Proposition 6
\[ \int_0^1 \dot{f}_t(\psi_t(x)) dt \]
is independent of the point \( x \in M \). Hence
\[ \left( \frac{d}{ds} \kappa(\psi^s) \right)_{|s=0} \int_M \omega^n = -2\pi i \kappa(\psi) \int_0^1 dt \int_M (\dot{f}_t \circ \psi_t) \omega^n = 0. \]

So \( \kappa(\psi) \) depends only on the homotopy class \([\psi] \in \pi_1(\Ham(M))\). From (2.16) we conclude that \( \kappa \) is a group homomorphism. \( \square \)

As a consequence of Proposition 7 it makes sense to define the action integral around an element \([\psi] \in \pi_1(\Ham(M))\) as \( \kappa_x(\psi) \), for \( x \) an arbitrary point of \( M \).
3. Hamiltonian $G$-spaces.

Let $G$ be a compact, connected Lie group which acts on the left on the quantizable manifold $(M,\omega)$ by symplectomorphisms. We assume that this action is Hamiltonian, and that $\Phi : M \to \mathfrak{g}^*$ is a moment map for this action. That is, $M$ is a Hamiltonian $G$-space [6].

Given $A \in \mathfrak{g}$, we denote by $X_A$, the vector field on $M$ generated by $A$. Then $\langle \mathfrak{g} \Phi(Y), A \rangle = \omega(Y, X_A)$, for any vector field $Y$ on $M$. The $A$-component of $\Phi$ will be denoted by $f_A$; that is, $f_A(x) = \langle \Phi(x), A \rangle$. Hence

$$\iota_{X_A} \omega = -df_A \quad \text{and} \quad \{f_A, f_B\} := \omega(X_B, X_A) = f_{[A,B]}.$$ (3.1)

Now the prequantization map $P$ restricted to $\mathfrak{g}$ is $A \in \mathfrak{g} \mapsto P_A = -D_{X_A} - 2\pi i f_A \in \text{End}(\Gamma(L)),$ (3.2)

and using (3.1) it is straightforward to prove the following Proposition

**Proposition 8.** The map $P$ is a Lie algebra homomorphism.

Given a family $\{g_t\}_{t \in [0,1]}$ of elements of $G$ with $g_0 = e$, it determines a Hamiltonian isotopy $\varphi_t : M \to M$ by

$$\varphi_t(x) = g_t \cdot x.$$ (3.3)

It is easy to prove the following Proposition that gives the vector fields determined by the isotopy $\varphi_t$.

**Proposition 9.** The isotopy (3.3) is defined by the equations

$$\frac{d\varphi_t}{dt} = X_{A_t} \circ \varphi_t, \quad \varphi_0 = \text{id},$$

where $A_t$ is the element of $\mathfrak{g}$ given by $A_t := \dot{g}_tg_t^{-1}$.

If $x_0 \in M$ is a fixed point for the $G$-action, then $\Phi(x_0)$ is called a vertex of $\Phi$ [7]. If $\varphi$ is a closed isotopy we will express the $U(1)$-valued action integral $\kappa(\varphi)$ in terms of the vertices of $\Phi$.

**Theorem 10.** Let $(M,\omega)$ be a compact, quantizable $G$-space. If $x_0$ is a fixed point for the $G$-action and if the isotopy $\varphi_t$ defined by (3.3) is closed (i.e. $\varphi_1 = \text{id}$), then

$$\kappa(\varphi) = \exp(-2\pi i \langle \Phi(x_0), \int_0^1 A_t dt \rangle),$$

with $A_t = \dot{g}_tg_t^{-1}$.

**Proof.** In this case the curve $\{\varphi_t(x_0)\}$ is a point, so integral of $\omega$ in (1.1) vanishes. On the other hand, by Proposition 9 the corresponding time-dependent Hamiltonian is $f_{A_t} = \langle \Phi, A_t \rangle$. \[\Box\]

Now we assume that $G = T$ is the $r$-torus.

**Theorem 11.** Let $(M,\omega)$ be a compact, quantizable manifold equipped with a Hamiltonian action of the $r$-torus $T$. If $v$ and $v'$ are vertices of a moment map $\Phi$ for the $T$-action, then $v - v'$ belongs to the lattice $\mathbb{Z}^r$ of $t^*$. 

Proof. Let $x_0$ and $x_1$ be fixed points with $\Phi(x_0) = v$ and $\Phi(x_1) = v'$. If $A$ is a vector of $t^*$ with integer coordinates, then

$$\{\phi_t(x) = e^{2\pi itA} \cdot x\}_{t \in [0,1]}$$

is a closed isotopy of $(M, \omega)$. Hence, by the independence of $\kappa_x(\phi)$ from the point $x$ and Theorem 10, we conclude

$$\exp(4\pi^2 \langle \Phi(x_0), A \rangle) = \exp(4\pi^2 \langle \Phi(x_1), A \rangle).$$

That is, $\langle v - v', A \rangle = 0$. This relation holds for any $A \in t^* \cap \mathbb{Z}^r$. Hence $v - v' \in \mathbb{Z}^r$. \qed

Corollary 12. Under the hypotheses of Theorem 11, there is a moment map $\Phi$ for the $T$-action such that the vertices of $\Phi$ are integer lattice points.

Proof. If $\Phi$ is a moment map and $x_0$ is a fixed point, we set $v := \Phi(x_0)$. We define the map $\Phi := \Phi - v$, which is also a moment map, since $T$ is Abelian. Hence 0 is a vertex of $\Phi$. By Theorem 11, for any vertex $v'$ of $\Phi$, $v' = v - 0 \in \mathbb{Z}^r$. \qed

In particular, if the toric manifold associated to a Delzant polytope $\Delta$ is quantizable, then the vertices of $\Delta$ are integer lattice points. So we recover a well-known result relative to Delzant spaces [2] [6].

The action of $G$ on $M$ is said to be pre-quantizable if there is a global action of $G$ on the prequantum bundle $L$ which induces the action (3.2) of $g$ on the space $\Gamma(L)$ [7] (In [8] one says that the prequantum data $(L, D)$ are $G$-invariant). The thesis of Corollary 12 has been proved in [7, § 8, Corollary 1] under the additional assumption that the $G$-action is pre-quantizable.

Henceforth in this Section we assume the existence of this lift of the $G$-action to $L$. We denote by $\rho(g)$ (resp. $\varphi_g$) the diffeomorphism of $L$ (resp. $M$) associated to $g \in G$. So $\pi \circ \rho(g) = \varphi_g \circ \pi$. We denote by $v$ the representation of $G$ on $\Gamma(L)$ induced by $\rho$. That is, for $g \in G$ and $\tau \in \Gamma(L)$

$$v(g)(\tau) = \rho(g) \circ \tau \circ \varphi_g^{-1}. \quad (3.4)$$

Moreover, given a curve $\{g_t\}_{t \in [0,1]}$ in $G$ starting at $e$, the fact that $\rho$ induces $\mathcal{P}$ implies that

$$\left. \frac{d}{dt} \right|_{t=0} v(g_t)\tau = \mathcal{P} \Delta \tau, \quad (3.5)$$

where $A \in \mathfrak{g}$ is the derivative $g_t$ at $t = 0$.

As we said in Section 2, the Hamiltonian isotopy $\varphi_t = \varphi_{g_t}$ admits a lift to an isotopy $F_t$ of $L^\times$. By Proposition 9 the family of vector fields

$$Z_t := X^{A_t}_{A_t} - W_{f_t},$$

where $A_t := \dot{g}_t g_t^{-1}$ and $f_t = \langle \Phi, A_t \rangle$, generates the isotopy $F_t$.

It is proper to ask if the diffeomorphisms $F_t$ and $\rho(g_t)$ are related. We will prove that $F_t = \rho(g_t)$.

Lemma 13. Let $h_t$ be a curve in $G$ with $h_0 = e$ and $\dot{h}(0) = B \in \mathfrak{g}$, and let $p$ be a point in $L^\times$. Then

$$\left. \frac{d}{dt} \rho(h_t)(p) \right|_{t=0} = X^B_H(p) - W_{f_t}(p).$$
Proof. We will determine the vector tangent to \( \{ \rho(h_t)p \} \), at \( t = 0 \). Let \( \tau \) be a section such that \( \tau(x) = p \), then
\[
\frac{d}{dt} \rho(h_t)\tau(h_t^{-1}x) \bigg|_{t=0} = \frac{d}{dt} \rho(h_t)p \bigg|_{t=0} + \tau_*(-X_B(x)). \tag{3.6}
\]
It follows from (2.2)
\[
\tau_*(X_B(x)) = X^\sharp_B(\tau(x)) + (\tau^*\alpha)(X_B)\tau(x) = X^\sharp_B(\tau(x)) + (D_{X_B}\tau)(x). \tag{3.7}
\]
The left hand side in (3.6) is equal to \((P_A\tau)(x)\) by (3.5) and (3.4). Therefore (3.6) and (3.7) give rise to
\[
\frac{d}{dt} \rho(h_t)(p) \bigg|_{t=0} = X^\sharp_B(p) - W_{f_\theta}(p).
\]
\[\square\]

Proposition 14. The isotopy \( \rho(g_t) \) is generated by the family of vector fields \( Z_t \).

Proof. Given \( t \), we put \( h_t := g_t g_t^{-1} \); then \( \dot{h}_t = A_t \). Since \( \rho \) is a representation of \( G \), by Lemma 13
\[
\frac{d}{dt} \rho(g_t)(p) \bigg|_{t'=t} = \frac{d}{dt} \rho(g_t^{-1}(p)) \bigg|_{t'=t} = X^\sharp_{A_t}(p) - W_{f_\theta}(p).
\]
That is,
\[
\frac{d}{dt} \rho(g_t) = Z_t \circ \rho(g_t).
\]
\[\square\]

The following Corollary asserts that the transport along \( \varphi \) is defined by the representation \( \upsilon \).

Corollary 15. If \( g_t \) is a family of elements in \( G \) with \( g_0 = e \) and \( A_t := \dot{g}_t g_t^{-1} \), then the solution to
\[
\frac{d}{dt} \tau_t = \mathcal{P}_{A_t}(\tau_t), \quad \tau_0 = \tau \tag{3.8}
\]
is \( \tau_t = \upsilon(g_t)(\tau) \). In particular, \( \tau_1 \) depends only on the endpoint \( g_1 \) of the curve \( g_t \).

Proof. The statement follows from Proposition 1 together with Proposition 14 and (3.4).

Now we will express the \( U(1) \)-action integral \( \kappa(\varphi) \) in terms of the representation \( \upsilon \). We will construct finite-dimensional \( \upsilon \)-invariant subspaces of \( \Gamma(L) \), and \( \kappa(\varphi) \) will be related with the characters of the restrictions of \( \upsilon \) to these subspaces.

An almost complex structure \( J \) on \( M \) is called compatible with \( \omega \) if \( \omega(J, J) = \omega(\ldots) \) and \( \omega(\ldots, J) \) is positive definite. Since \( G \) is a compact group one can construct \( G \)-invariant compatible almost complex structures on \( M \). Let \( I \) be such an almost complex structure. A section \( \tau \) of \( L \) is said to be \( I \)-polarized if \( D_X\tau = 0 \), for any vector field \( X \) on \( M \) of type \((0,1)\) relative to \( I \). We set \( Q_I \) for the space of \( I \)-polarized sections.
With $\Omega^{0,k}(L)$ we denote the space of $(0,k)$-forms on $M$ with values in $L$. The operator $\bar{D} = (1 + iI)D$ extends in the usual manner to an operator

$$\bar{D} : \Omega^{0,k}(L) \to \Omega^{0,k+1}(L).$$

Although $(\Omega^{0,*}(L), \bar{D})$ is not a complex, the Riemannian metric $\omega(\cdot, \cdot)$ on $M$ and the Hermitian metric on $L$ determine the adjoint operator $\bar{D}^* = \bar{D}^{\ast}$ of $\bar{D}$; moreover the operator

$$\bar{D} + \bar{D}^* : \Omega^{0,\text{even}}(L) \to \Omega^{0,\text{odd}}(L)$$

is elliptic (see [6, page 75]). Since

$$\text{dim} Q_I = \text{Ker}((\bar{D} + \bar{D}^*)|_{\Omega^{0,0}}),$$

then $Q_I$ is a finite dimensional vector space.

Given $g \in G$, $g$ is the endpoint of a curve $g_t$ in $G$ with $g_0 = e$. By Corollary 15 $v(g)(\tau) = \tau_1$, where $\tau_1$ is the solution of (3.8). If $Y$ is a vector of type $(0,1)$ with respect to $I$, by the $G$-invariance of $I$ the vector $\varphi_{t*}(Y)$ is of type $(0,1)$ as well, $\varphi_t$ being the diffeomorphism defined in (3.3). This fact together with Proposition 3 imply that $\tau_1 \in Q_I$, if $\tau$ is $I$-polarized. Hence $Q_I$ is a $v$-invariant subspace of $\Gamma(L)$. We set $v_I$ for the restriction of $v$ to $Q_I$.

If the isotopy defined in (3.3) is closed, from Corollary 5 and Corollary 15 it follows $v_I(g_f)(\tau) = \kappa(\varphi)\tau$. Hence the character $\chi(v_I)$ at the point $g_1$ equals $\kappa(\varphi) \text{dim} Q_I$. One has the following Theorem

**Theorem 16.** If $M$ is a Hamiltonian $G$-space and the $G$-action is pre-quantizable, then the action integral $\kappa(\varphi)$ around the closed isotopy $\varphi_t(x) = g_t \cdot x$ is equal to

$$\frac{\chi(v_I)(g_1)}{\text{dim} Q_I},$$

where $I$ is any $G$-invariant, compatible almost complex structure on $(M, \omega)$.

### 4. The invariant $\kappa(\psi)$ in a coadjoint orbit

Let $G$ be a compact Lie group, and we consider the coadjoint action of $G$ on $\mathfrak{g}^*$ defined by

$$(g \cdot \eta)(A) = \eta(g^{-1} \cdot A),$$

for $g \in G$, $\eta \in \mathfrak{g}^*$, $A \in \mathfrak{g}$ and $g \cdot A = \text{Ad}_g A$ (see [11] [24]).

If $X_A$ is the vector field on $\mathfrak{g}^*$ determined by $A$, the map $l_g : \mu \in \mathfrak{g}^* \mapsto g \cdot \mu \in \mathfrak{g}^*$ satisfies

$$(l_g)_\ast(X_A(\mu)) = X_{g \cdot A}(g \cdot \mu).$$

(4.1)

Given $\eta \in \mathfrak{g}^*$, by $O_\eta =: \mathcal{O}$ will be denoted the orbit of $\eta$ under the coadjoint action of $G$. On $\mathcal{O}$ one can consider the 2-form $\omega$ determined by

$$\omega_\nu(X_A(\nu), X_B(\nu)) = \nu([A, B]).$$

(4.2)

This 2-form determines a symplectic structure on $\mathcal{O}$, and the action of $G$ preserves $\omega$. For each $A \in \mathfrak{g}$ one defines the function $h_A \in C^\infty(\mathcal{O})$ by $h_A(\nu) = \nu(A)$, and for this function holds the formula

$$\iota_{X_A} \omega = dh_A.$$

(4.3)
The orbit $O$ can be identified with $G/G_\eta$, where $G_\eta$ is the subgroup of isotropy of $\eta$. The Lie algebra of this subgroup is

$$\mathfrak{g}_\eta = \{ A \in \mathfrak{g} \mid [\eta([A, B])] = 0, \text{ for every } B \in \mathfrak{g}\}$$

One says that the linear functional

$$\lambda : C \in \mathfrak{g}_\eta \mapsto 2\pi i\eta(C) \in i\mathbb{R}$$

is integral iff there is a character $\Lambda : G_\eta \to U(1)$ whose derivative is the functional (4.4) (see [14]). Henceforth we assume the existence of such a character $\Lambda$. We will prove that the orbit $O$ possesses a $G$-invariant prequantization. A prequantum bundle $L$ over $O = G/G_\eta$ is defined by $L = G \times_\Lambda \mathbb{C} = (G \times \mathbb{C})/\sim$, with $(g, z) \sim (gb^{-1}, \Lambda(b)z)$, for $b \in G_\eta$.

Each section $\sigma$ of $L$ determines a $\Lambda$-equivariant function $s : G \to \mathbb{C}$ by the relation

$$\sigma(gG_\eta) = [g, s(g)].$$

(4.5)

The $\mathbb{C}^\times$-principal bundle associated to $L$ is $L^\times = L - \{\text{zero section}\}$. The lift $\sigma^\sharp : L^\times \to \mathbb{C}$ of the section $\sigma$ and its corresponding $\Lambda$-equivariant function $s$ are related by the formula

$$s(g) = \sigma^\sharp([g, z])z.$$  

(4.6)

If $v$ denotes the element $[e, 1] \in L^\times$, then $T_v(L^\times) \simeq (\mathfrak{g} \oplus \mathbb{C})/f_v$, with

$$f_v = \{(B, -2\pi i\eta(B)) \mid B \in \mathfrak{g}_\eta\}.$$  

The connection form $\Omega$ on $L^\times$ is constructed in [14] p.198. The form $\Omega$ can be written $\Omega = (\theta, d)$, where $\theta$ is the left invariant form on $G$ whose value at $e$ is $\eta$, and $d \in \text{Hom}_\mathbb{C}(\mathbb{C}, \mathbb{C})$ is defined by $d(z) = (2\pi i)^{-1}z$. It is clear that $\Omega_v$ vanishes on $f_v$ and that it defines an element of $T_v^*(L^\times)$.

We denote by $\mathcal{E}_\Lambda$ the space of $\Lambda$-equivariant functions on $G$. The identification $\Gamma(L) \simeq \mathcal{E}_\Lambda$ allows us to translate the action $\mathcal{P}$ defined in (3.2) to a representation of $\mathfrak{g}$ on $\mathcal{E}_\Lambda$.

**Theorem 17.** The action $\mathcal{P}$ on $\mathcal{E}_\Lambda$ is given by $\mathcal{P}_A(s) = -R_A(s)$, where $R_A$ is the right invariant vector field on $G$ determined by $A$.

**Proof.** Let $\sigma$ be a section of $L$, by (4.3) $\mathcal{P}_A(\sigma) = -D_{X_A} \sigma + 2\pi i h_A \sigma$. We will determine the lift $(\mathcal{P}_A(\sigma))^\sharp$.

The vector $X_A(g \cdot \eta) \in T_{g\eta}(O)$ is defined by the curve $u \mapsto e^{uA}g \cdot \eta$ in $O$. A lift of this curve at the point $[g, z] \in L^\times$ will be a curve of the form $\gamma(u) = [e^{uA}g, zu]$, with $zu = ze^{ux}$. The vector tangent to $\gamma$ at $[g, z]$ is $\dot{\gamma}(0) = [R_A(g), x]$, where $R_A(g)$ is the value at $g$ of the right invariant vector field in $G$ defined by $A$.

The condition $\Omega(\dot{\gamma}(0)) = 0$ implies

$$x = -2\pi i\eta(g^{-1} \cdot A).$$

(4.7)

Therefore the horizontal lift of $X_A(g \cdot \eta)$ is

$$X_A^\sharp([g, z]) = [R_A(g), -2\pi i\eta(g^{-1} \cdot A)],$$

and by (4.6) the action of $X_A^\sharp([g, z])$ on the function $\sigma^\sharp$ can expressed in terms of $s$

$$X_A^\sharp([g, z])(\sigma^\sharp) = \frac{d}{du} \bigg|_{u=0} \left( \frac{s(e^{uA}g)}{ze^{ux}} \right) = \frac{R_A(g)(s)}{z} - \frac{xs(g)}{z}.$$
Since $X^1_A(\sigma^t) = (D_X, \sigma)^2$, from (4.7) and (4.6) it turns out that the equivariant function associated to $D_X\sigma$ is

$$g \in G \mapsto R_A(g)(s) + 2\pi i\eta(g^{-1} \cdot A)s(g) \in \mathbb{C}. \quad (4.8)$$

Obviously the equivariant function defined by the section $h_A\sigma$ is the function $\lambda_{A}\sigma$, where $\lambda_{A}(g) = h_{A}(gG_{\eta}) = (g \cdot \eta)(A) = \eta(g^{-1} \cdot A)$. It follows from (4.8) that the equivariant function which corresponds to $-D_X\sigma + 2\pi i\hbar_A\sigma$ is $-R_A(s)$. □

**Corollary 18.** The action $P$ on $\mathcal{E}_{\mathcal{A}}$, is induced by the action

$$\nu : (b, s) \in G \times \mathcal{E}_{\mathcal{A}} \mapsto s \circ L_{b^{-1}} \in \mathcal{E}_{\mathcal{A}},$$

where $L_c$ is left multiplication by $c$ in the group $G$.

**Proof.** If $g_t = e^{tA} \in G$, then

$$\frac{dv_{g_t}(s)}{dt} \bigg|_{t=0} (g) = \frac{d}{dt} \bigg|_{t=0} s(e^{-tA}g) = -R_A(g)(s) = P_A(s)(g).$$

□

**Corollary 19.** The action of $G$ on $\mathcal{O}_{\eta}$ is pre-quantizable.

**Proof.** On $L$ we define the following representation of $G$, $\rho(g'([g, z]) = [g', g, z]$. For $g_t = e^{tA}$ and the section $\sigma$ of $L$

$$\frac{d}{dt} \bigg|_{t=0} \rho(g_t)\sigma(g_t^{-1}gG_{\eta}) = \frac{d}{dt} \bigg|_{t=0} [g, s(e^{-tA}g)] = [g, -R_A(g)(s)] = P_A(\sigma)(gG_{\eta}).$$

The Corollary follows from (3.4) and (3.5). □

Let $\{\psi_t : t \in [0, 1]\}$ be a closed Hamiltonian isotopy on $\mathcal{O}$. We also assume that the corresponding Hamiltonian vector fields are generated by elements of $\mathfrak{g}$; that is,

$$\frac{d}{dt} \psi_t(q) = X_{A_t}(\psi_t(q)), \text{ with } A_t \in \mathfrak{g}.$$

If $\sigma$ is a section of $L$, $\sigma_t$ will denote the solution to the equation

$$\frac{d\sigma_t}{dt} = P_{A_t}(\sigma_t), \quad \sigma_0 = \sigma. \quad (4.9)$$

By Theorem 17, equation (4.9) on the points $\{h_t\}_{t \in [0, 1]}$ of a curve in $G$ gives rise to

$$\dot{h}_t(h_t) = -R_{A_t}(h_t)(s_t), \quad (4.10)$$

for the corresponding equivariant functions. In particular, if $h_t$ is the curve such that $h_0 = e$ and $\dot{h}_t = R_{A_t}(h_t) \in T_{h_t}(G)$; in other words, $h_t$ satisfies the Lax equation $\dot{h}_t h_t^{-1} = A_t$, then

$$R_{A_t}(h_t)(s_t) = \frac{d}{dt} \bigg|_{t=0} s_t(h_u).$$

Using (4.10) one deduces

$$\dot{s}_t(h_t) + \dot{h}_t(s_t) = 0 \quad (4.11)$$
If we consider the function \( w : [0, 1] \to \mathbb{C} \) defined by \( w_t = s_t(h_1) \); by (4.11) \( w \) is constant. So \( s_1(h_1) = s_0(e) \). If \( h_1 \in G_\eta \), as \( s_1 \) is \( \Lambda \)-equivariant \( s_1(h_1) = \Lambda(h_1^{-1})s_1(e) \); so

\[
\sigma_1(eG_\eta) = \Lambda(h_1)\sigma_0(eG_\eta). \tag{4.12}
\]

The following Theorem, which gives the invariant \( \kappa(\psi) \) in terms of \( \Lambda \), is consequence of Corollary 5 and (4.12)

**Theorem 20.** If \( \{\psi_t\} \) is the closed Hamiltonian isotopy in \( \mathcal{O} \) generated by the vector fields \( \{ X_{\alpha} \} \), then \( \kappa(\psi) = \Lambda(h_1) \), where \( h_t \in G \) is the solution to \( h_t h_t^{-1} = A_t \), with \( h_0 = e \) and \( h_1 \in G_\eta \).

Let us assume that \( G \) is semisimple Lie group \([4]\), and let \( T \) a maximal torus with \( T \subset G_\eta \) (see \([9]\) p.166). One has the standard decomposition of \( g_\mathbb{C} = g \otimes_{\mathbb{R}} \mathbb{C} \) in direct sum of root spaces

\[
g_\mathbb{C} = h \oplus \sum \alpha \mathfrak{g}_\alpha,
\]

where \( h = t_\mathbb{C} \), and \( \alpha \) ranges over the set of roots.

We denote by \( \alpha^\vee \) the element of \( \{ \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \} \) such that \( \alpha(\alpha^\vee) = 2 \). On the other hand \( \eta \) extends in a natural way to \( g_\mathbb{C} \), and if \( Y \in \mathfrak{g}_\alpha \), then

\[
0 = \eta([\alpha^\vee, Y]) = 2\eta(Y).
\]

Hence vanishes on \( \sum \mathfrak{g}_\alpha \). If \( \eta(\alpha^\vee) \neq 0 \), for all root \( \alpha \), then \( \mathfrak{g}_\eta = t \); in this case \( \eta \) is said to be regular. Henceforth we assume that \( \eta \) is regular. Let \( P \) be the set of roots \( \alpha \) such that \( \eta(\alpha^\vee) < 0 \). Then the real counterpart of the above direct sum decomposition is

\[
g = t \oplus \sum_{\alpha \in P} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap g.
\]

We define \( \mathfrak{b} = h \oplus n \), where

\[
n = \sum_{\alpha \in P} \mathfrak{g}_\alpha.
\]

Then \( \mathfrak{b} \) is a Borel subalgebra of \( g_\mathbb{C} \), which corresponds to a Borel subgroup \( B \) of \( G \).

We have

\[
T_\eta(\mathcal{O}) = g/\mathfrak{g}_\eta = \sum_{\alpha \in P} \left( \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \right) \cap g.
\]

Hence

\[
T^\mathbb{C}_\eta(\mathcal{O}) = \sum_{\alpha \in P} \left( \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \right).
\]

One defines

\[
T^{0,1}_\eta \mathcal{O} := n,
\]

and

\[
T^{0,1}_{g^\eta} \mathcal{O} := \{ X_{g^\Lambda}(g \cdot \eta) \mid A \in n \}.
\]

If \( g_1 \cdot \eta = g_2 \cdot \eta \), then \( g_1^{-1}g_2 \in T \). As \( \mathfrak{g}_\alpha \) is an eigenspace for the action of \( T \), then \( g_1^{-1}g_2 \cdot A \in n \), if \( A \in n \). Therefore the spaces \( T^{0,1}_{g^\eta} \) are well-defined.

For \( A \in n \), one can define the vector field \( \mathcal{A} \) on \( \mathcal{O} \) by \( \mathcal{A}(g \cdot \eta) = X_{g^\Lambda}(g \cdot \eta) \). By (4.1) \( (\eta) \cdot \mathcal{A} = \mathcal{A} \), hence the above complex foliation defined on \( \mathcal{O} \) is \( G \)-invariant. Since the vector \( X_{g^\Lambda}(g \cdot \eta) \) is defined by the curve \( e^{tg^\Lambda}g \cdot \eta = ge^{t\mathcal{A}} \cdot \eta \), then the left invariant vector field \( L_\mathcal{A} \) on \( G/T \) is the field which corresponds to \( \mathcal{A} \), in the identification of \( G/T \) with \( \mathcal{O} \).
The vector spaces $T^{1,0}$ are defined in the obvious way. As $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}_C$, the decomposition $T^\mathbb{C}(\mathcal{O}) = T^{1,0} \oplus T^{0,1}$ define a complex structure $I$ on $\mathcal{O}$. This complex manifold can be identified with $G_C/B$.

Using the complex structure on $\mathcal{O} = G/T$ and the covariant derivative $D$ on the prequantum bundle $L = G \times_A \mathbb{C}$, it is possible to define a holomorphic structure in $L$. The section $\tau$ of $L$ is said to be holomorphic iff $D_Z \tau = 0$ for any vector field $Z$ of type $(0, 1)$. In this way $L$ can be regarded as a holomorphic line bundle over $G_C/B$, and with the notation of Section 3 $Q_I = H^0(G_C/B, L)$.

The homomorphism $\Lambda : T \to U(1)$ extends trivially to $B$, since $B$ is a semidirect product of $H = T_C$ and the nilpotent subgroup whose Lie algebra is $\mathfrak{n}$. And each section $\sigma$ of $L$ determines a function $s : G_C \to \mathbb{C}$ which is $\Lambda$-equivariant. On the other hand, given $A \in \mathfrak{n}$, the Proof of Theorem 17 shows that the equivariant function associated to $D_A \sigma$ is the map

$$g \in G_C \mapsto R_{g \cdot A}(g)s + 2\pi i \eta(g^{-1} g \cdot A)s(g) \in \mathbb{C}.$$ 

As $\eta$ vanishes on $\mathfrak{n}$ and the vectors $R_{g \cdot A}(g)$ and $L_A(g)$ are equal, the equivariant function associated to $D_A \sigma$ is $L_A(s)$. Therefore if $\sigma$ is holomorphic, then $L_{A}(s) = 0$ for any $A \in \mathfrak{n}$; that is, $s$ is a holomorphic function on $G_C$. So the space $H^0(G_C/B, L)$ is isomorphic to the space

$$\mathcal{E}_{\Lambda, I} := \{s : G_C \to \mathbb{C} \mid s \text{ is holomorphic and } \Lambda - \text{equivariant}\}.$$ 

The Borel-Weil Theorem asserts that the action of $G$ on the space $\mathcal{E}_{\Lambda, I}$ given by $g \star s = s \circ L_{g^{-1}}$ is an irreducible representation of $G$; more precisely the contragredient representation of that one whose highest weight is $-\lambda$ (see [3] pages 290, 300).

Denoting by $\pi$ the irreducible representation of $G$ whose highest weight is $-2\pi i \eta$ and by $\pi^*$ its dual, from Corollary 18 it follows that the restriction of $\nu$ to $\mathcal{E}_{\Lambda, I}$ is $\pi^*$. From Theorem 16 and Proposition 9 we deduce

**Theorem 21.** Let $\eta$ be an element of $\mathfrak{g}^*$, such that $2\pi i \eta$ is an integral character on $\mathfrak{g}_\eta$, and $G_\eta$ is a maximal torus of $G$. If $\{\psi_i\}$ is the closed Hamiltonian isotopy in $\mathcal{O}_\eta$ generated by the vector fields $\{X_{A_i}\}$, then

$$\kappa(\psi) = \frac{\chi(\pi^*)(h_i)}{\dim \pi},$$

where $h_i \in G$ is the solution to $\dot{h}_i h_i^{-1} = A_i$, $h_0 = e$, and $\pi$ is the representation of $G$ whose highest weight is $-2\pi i \eta$.

Now the character $\chi(\pi^*)$ and the dimension $\dim \pi$ can be determined by Weyl’s character formula [3], and so $\kappa(\psi)$.

**Examples**

1. **Action integral in flag manifolds.** Set $D = \operatorname{diag}(id_1, \ldots, id_n) \in \mathfrak{u}(n)$, with $d_j \in \mathbb{R}$. We denote by $p_1 < \cdots < p_k$ the distinct values of the $d_j$ and by $n_1, \ldots, n_k$ the corresponding multiplicities. $D$ determines an element $\eta \in \mathfrak{u}(n)^*$ by the relation $\eta(Y) = \operatorname{tr}(DY)$. The coadjoint orbit $\mathcal{O}_\eta$ is the flag manifold $U(n)/U(n_1) \times \cdots \times U(n_k)$. And for $Y = (B_1, \ldots, B_k) \in \mathfrak{g}_\eta = \bigoplus_j \mathfrak{u}(n_j)$,

$$\eta(Y) = \sum_{j=1}^k ip_j \operatorname{tr}(B_j).$$
The manifold $O_\eta$ depends only on the multiplicities $n_j$. However the symplectic form $\omega$, defined in (4.2) depends also on the $p_j$. The manifold $(O_\eta, \omega)$ admits a $U(n)$-invariant prequantization if $-2\pi p_j =: m_j \in \mathbb{Z}$, for $j = 1, \ldots, k$. In this case the character $\Lambda$ of $G_\eta = \prod_j U(n_j)$ defined by

$$\Lambda(A_1, \ldots, A_k) = \prod_{j=1}^k (\det(A_j))^{m_j}$$

has as derivative $2\pi i \eta$. Now the symplectic flag manifold $(O_\eta, \omega)$ is quantizable and it is determined by $(m_1, n_1; \ldots; m_k, n_k)$.

If $\{g_t \in U(n) \mid t \in [0, 1]\}$ defines a loop $\varphi$ in $\text{Ham}(O_\eta)$ by (3.3), then $g_1 g G_\eta = g G_\eta$, for every $g \in U(n)$. So $g_1$ is a multiple of the identity: $g_1 = z I_n$, with $|z| = 1$. By Theorem 20

$$\kappa(\varphi) = \Lambda(g_1) = \prod_{j=1}^k z^{m_j n_j}.$$ 

Thus we have

**Proposition 22.** The symplectic flag manifold

$$(O_\eta = U(n)/U(n_1) \times \cdots \times U(n_k), \omega)$$

determined by the integers $(m_1, n_1; \ldots; m_k, n_k)$ admits a $U(n)$-invariant prequantization. If $\varphi$ is the loop in $\text{Ham}(O_\eta)$ defined by a family $\{g_t \in U(n)\}$, with $g_0 = I_n$ and $g_1 = z I_n$, then

$$\kappa(\varphi) = z^a,$$

where $a = \sum_j m_j n_j$.

2. The invariant $\kappa$ of a Hamiltonian flow in $S^2$. For $G = SU(2)$, if $\eta: \begin{pmatrix} ai & w \\ -\overline{w} & -ai \end{pmatrix} \in su(2) \mapsto \frac{n a}{2\pi} \in \mathbb{R}$,

with $n \in \mathbb{Z}$, then the orbit $O_\eta = SU(2)/U(1) = S^2$ admits and $SU(2)$-invariant quantization and the corresponding character $\Lambda$ of $U(1)$ is $\Lambda(z) = z^n$.

Let $E$ be a matrix of $su(2)$, and we assume that $e^E = -\text{Id}$. If we denote by $\psi_t$ the symplectomorphism of $S^2$ given by

$$\psi_t(q) = \exp(tE) \cdot q,$$

then the family $\{\psi_t\}_{t \in [0, 1]}$ is a closed Hamiltonian flow on the orbit $O_\eta$. By Theorem 20

$$\kappa(\psi) = \Lambda(e^E) = \Lambda(-\text{Id}) = (-1)^n.$$ 

This result agrees with that one obtained in [21, Theorem 21] by direct calculation. This value is also obtained in [18, Example 3.6].

This value can also be deduced from Theorem 21. The Weyl’s character formula [3] is very simple for the group $SU(2)$: in this case, there is only one positive root $\alpha$ and the Weyl group has only two elements. We take for $\alpha$ the linear map defined by

$$\alpha(\text{diag}(ai, -ai)) = 2ai;$$
so \( \alpha^\vee = \text{diag}(1, -1) \). If \( n < 0 \), then \( -\lambda := -2\pi \eta \) is the highest weight of a representation \( \pi \) of \( SU(2) \). For \( t \in U(1), t^\lambda = t^{-n} \) and \( t^\alpha = t^2 \). Therefore (see [3])

\[
\dim \pi = -n + 1 \quad \text{and} \quad \chi_\pi(t) = \sum_{k=0}^{-n} t^{-n-2k}.
\]

Hence

\[
\chi_{\pi^*}(h_1) = \chi_\pi(-1) = (-n + 1)(-1)^n,
\]

and from (4.13) we again obtain the value \((-1)^n\) for \( \kappa(\psi) \).

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