An Information-state based Approach to the Optimal Output Feedback Control of Nonlinear Systems

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Abstract—This paper develops a data-based approach to the closed-loop output feedback control of nonlinear dynamical systems with a partial nonlinear observation model. We propose an “information-state” based approach to rigorously transform the partially observed problem into a fully observed problem where the information-state consists of the past several observations and control inputs. We further show the equivalence of the transformed and the initial partially observed optimal control problems and provide the conditions to solve for the deterministic optimal solution. We develop a data-based generalization of the iterative Linear Quadratic Regulator (iLQR) to partially-observed systems using a local linear time-varying model of the information-state dynamics approximated by an Autoregressive-moving-average (ARMA) model, that is generated using only the input-output data. This open-loop trajectory optimization solution is then used to design a local feedback control law, and the composite law then provides an optimum solution to the partially observed feedback design problem. The efficacy of the developed method is shown by controlling complex high dimensional nonlinear dynamical systems in the presence of model and sensing uncertainty.

I. INTRODUCTION

The problem of optimal control for nonlinear systems with partial observation finds several applications and is still considered an unsolved problem [1]. It is computationally intractable for complex high-order systems due to the ‘curse of dimensionality’ associated with solving dynamic programming [2]. The problem becomes more challenging when only some of the states are available for measurement, i.e., under partial state observation, which tends to be the case for most of the problems. One approach to solving partially observed problems is to use a feedback policy that acts on the current output, known as output feedback control [3, Ch.8]. The dynamic output feedback is a generalization of the static output feedback and also encompasses the approach where a nonlinear observer, such as a high gain observer, extended Kalman filter, or a particle filter, is used to reconstruct the full state of the system, which is then used for the full state feedback control [3], [4]. However, because of the estimation error, output feedback control is often suboptimal compared to full state feedback control, can become very challenging for complex nonlinear systems, and the separation principle may also not hold [3], [4]. Additionally, the problem becomes even more formidable when the model of the system is unknown [2].

In this work, we propose a data-based approach for learning to optimally control complex partially observed nonlinear dynamical systems. The primary idea is to convert the partially observed optimal control problem to a “fully-observed” problem described using the information-state [2, Ch.5] and to show that the problem described using the information-state is equivalent to the original problem. Then, we use a data-based approach to solve the information state optimal control problem.

There has been a significant body of work in the field of learning to control unknown dynamical systems through Approximate Dynamic Programming (ADP) and Reinforcement Learning (RL) techniques [5], [6]. These approaches have been successfully used in many areas like playing games [6], locomotion [7], and robotic hand manipulation [8]. However, despite excellent performance on several tasks, reinforcement learning (RL) is still considered very data-intensive, with typically a really large training time. Approaches that consider information state to solve partially observed problems are limited to either linear system models [9], [10] or finite state and action spaces [11]. There has been little to no work on solving these complex nonlinear problems using output feedback or partial observation models.

In order to find the optimal solution, the paper introduces a generalization of the iLQR algorithm [12] that can handle partially observed problems. The standard iLQR is a “local” trajectory-based method, similar to Differential Dynamic Programming (DDP), but only uses first-order dynamics information as opposed to second-order derivatives of the system dynamics needed in DDP [13]. This work suitably generalizes the iLQR method to partially observed systems in a systematic fashion and provides rigorous theoretical justifications for the same. The proposed approach iteratively generates Linear Time-Varying (LTV) state-space models, represented in the information-state, to obtain an optimized nominal information space trajectory. These LTV models are constructed using Autoregressive–Moving-Average (ARMA) models [14] along a nominal trajectory with the input-output perturbation data (rollouts of the system).

The proposed approach is a generalization of the so-called decoupled data-based control (D2C) approach [15] for designing a feedback controller, to partially observed systems. In a recent paper, we proposed an information-state-based system identification using an ARMA model, to generate a local closed-loop feedback controller [16].

Contributions: This paper builds on the reference [16] as follows: 1) it provides the theoretical justification of the information-state approach to partially observed problems (Section III), 2) it shows that the information-state-based optimal control problem is equivalent to the true partially
observed optimal control problem and proves a minimum principle for such problems (Sec. IV), 3) it derives the characteristic equations for solving dynamic programming and shows that there is a unique solution under some assumptions (Sec. IV-A), 4) the minimum principle allows us to generalize the iLQR approach to partially observed problems using the LTV-ARMA identification approach to solve the open-loop nonlinear optimization problem along with the local linear feedback (Sec. V), which is shown to be highly efficient (20-100x) compared to the gradient descent approach used in [16], and 5) it allows us to generate an optimal closed-loop feedback design under noise by augmenting the feedback law with a Kalman filter in the information-state (Sec. VI).

The rest of the paper is detailed as follows: Section II provides the optimal control problem formulation for the nonlinear system. Section III provides the results to transform a nonlinear partially observed problem into a fully observed problem in an information-state and then provides the equivalence of the optimal control problems along with the conditions to solve for the deterministic global optimal solution. Section IV then gives the framework to solve the problem using information-state-iLQR where the LTV-ARMA model is developed in a data-based fashion. Section V gives the details of the POD2C closed-loop feedback control algorithm. Finally, empirical results are shown for the partially observed control of complex robotic systems in the presence of process and sensor noise.

II. PROBLEM FORMULATION

Consider a nonlinear discrete-time dynamical system:

\[ x_{k+1} = f(x_k) + g(x_k)u_k, \quad z_k = h(x_k), \]

where \( x_k \in \mathbb{R}^n_x \) is the state, \( u_k \in \mathbb{R}^n_u \) is the control input and \( z_k \in \mathbb{R}^n_z \) is the output of the system defined \( \forall k \geq 0 \). The objective of this work is to find the optimal control inputs \( \{u_0, u_1, \ldots, u_{N-1} \} \) that minimizes the cost

\[ J(x_0) = \sum_{k=0}^{N-1} c(z_k, u_k) + c_N(z_N), \]

subject to the system model in Eq. (1), where \( c(z_k, u_k) \) denotes a running incremental cost and \( c_N(z_N) \) denotes a terminal cost function. The goal is to find an output feedback policy, i.e., a policy that only has access to the observations \( z_k \), such that the cost above is minimized from an unobserved initial state \( x_0 \).

III. INFORMATION-STATE BASED CONTROL PROBLEM

In this section, we introduce the information state and discuss how to transform the original optimal control problem in Eq. (2) to the information-state domain.

Let \( f^n(x_{k-q}; u_{k-q}, u_{k-q+1}, \ldots, u_{k-q+n-1}) \) denote the map from the state at time \( k-q \), \( x_{k-q} \), to the state \( x_{k-q+n} \) at time \( k-q+n \). Given the initial state and inputs from time \( k-q \) to time \( k-1 \), i.e., \( \{x_{k-q}; u_{k-q}, u_{k-q+1}, \ldots, u_{k-1} \} \), we can write the following expressions for the observations \( \{z_{k-q}, z_{k-q+1}, \ldots, z_k \} \) as:

\[ z_{k-q} = h(x_{k-q}), \]
\[ z_{k-q+1} = h(f^1(x_{k-q}; u_{k-q})), \]
\[ \vdots \]
\[ z_k = h(f^\delta(x_{k-q}; u_{k-q}, u_{k-q+1}, \ldots, u_{k-1})). \]

In the partially observed problem, one can find the underlying state \( x_{k-q} \), which is the solution to the above set of nonlinear equations. Let \( Z_k^q = [z_{k-q}^T, z_{k-q+1}^T, \ldots, z_k^T]^T \), \( U_k^q = [u_{k-q}^T, u_{k-q+1}^T, \ldots, u_{k-1}^T]^T \) and \( H_k^q(x_{k-q}) = [h(x_{k-q})^T, \ldots, h(f^\delta(x_{k-q}; u_{k-q}, u_{k-q+1}, \ldots, u_{k-1}))]^T \).

Assumption 1. Observability: Assume that there exists a finite \( q \), such that for all \( q \geq q_0 \), Eq. 3 has a unique solution for \( x_{k-q} \), regardless of \( (Z_k^q, U_k^q) \).

This assumption essentially means that the initial state can be unambiguously reconstructed from a finite history of measurements and control inputs. However, note that this condition needs only be satisfied in principle, and we do not ever explicitly try to solve this equation in our control synthesis. Due to the implicit function theorem and the observability assumption, we can write \( x_{k-q} \) as some unique function \( f(\cdot, \cdot) \) of past measurements and inputs as:

\[ x_{k-q} = \tilde{f}(Z_k^q, U_k^q). \]

Remark III.1. Note that typically for a partially observed problem, one would use a (nonlinear) observer to estimate the state, which would then be used to specify the control action assuming the estimated state to be the true state (certainty equivalence). Owing to the observability assumption 1, given the first \( q \) inputs and outputs, one can, in principle, exactly reconstruct the initial state, and thus predict the state evolution exactly after \( q \) steps, thereby mapping back to the fully observed problem. So, we shall assume that the first \( q \) inputs are specified in the partially observed control formulation we discuss later in this section. Finally, note that a typical nonlinear observer cannot perfectly reconstruct the state in a finite number of steps.

Next, let us write the state at current time \( x_k \) as:

\[ x_k = f^q(x_{k-q}; u_{k-q}, u_{k-q+1}, \ldots, u_{k-1}), \]

which can be written again by substituting for \( x_{k-q} \) from Eq. (4) in some unique functional form, based on observability assumption and implicit function theorem, as:

\[ x_k = \Psi(Z_k^q, U_k^q). \]

Let us now finally define the “Information-State”.

Definition 1. Information-state. The information-state [2, Ch.5], [17, Ch.6] of the system in Eq. (1) of order \( \delta \) (at time \( k \)) is defined as \( Z_k^q = [z_{k-q}^T, z_{k-q+1}^T, \ldots, z_{k-q+n}^T, u_{k-q}^T, \ldots, u_{k-1}^T]^T \in \mathbb{R}^n \), where \( n = (q+1)n_z + qn_u \).

The above definition allows us to write:

\[ x_k = \Psi(Z_k^q). \]

Further, due to the implicit function theorem, if the dynamics and the observation functions \( f, g, \) and \( h \) are \( \mathcal{C}^\infty \), so is the map \( \Psi \). The above development can be summarized as:
Lemma III.1. Given Assumption 1, there exists a unique function $\Psi(\cdot)$, such that the state at time $k$, $x_k = \Psi(Z^q_k)$. In particular, if $f(\cdot)$ and $h(\cdot)$ are $C^k$, so is the function $\Psi$.

The above result shows that the state at time $k$ is some nonlinear map of the observation and the control inputs at the previous two “q” time steps.

The remainder of this section shows how to transform the optimal control problem into the information-state domain. Consider the observation of the system as: $z_k = h(x_k) = h(f(x_{k-1}) + g(x_{k-1})u_{k-1})$. Using the relationship, $x_{k-1} = \Psi(Z^q_{k-1})$, the observation can be written in the form, $z_k = h(Z^q_{k-1}, u_{k-1})$, and further the equation can be written in terms of Information-state as: $Z^q_{k} = H(Z^q_{k-1}, u_{k-1})$.

This can be summarized in the following result.

Lemma III.2. Under Assumption 1, the systems $x_{k+1} = f(x_k) + g(x_k)u_k$, $z_k = h(x_k)$ and $Z^q_{k+1} = H(Z^q_k, u_k)$ have the same input-output response after $k \geq q$, regardless of the “unknown” initial condition $x_0$.

Recall $\bar{q}$ is the minimum number of observations required to satisfy Assumption 1. Hence, the information state model needs at least $\bar{q}$ measurements and inputs before it can start prediction, as these are required to form the initial information state. We assume that the first $\bar{q}$ control inputs and resulting observations are specified to us, which implies that the state $x_{\bar{q}}$ is known given the observability assumption. Then, we may reformulate the original partially-observed control problem as the following fully observed problem starting at the initial state $x_{\bar{q}}$:

$$J^*_x = \min_{u_k} \sum_{k=\bar{q}}^{N-1} c(z_k, u_k) + c_N(z_N), \quad (5a)$$

subject to: $x_{k+1} = f(x_k) + g(x_k)u_k$, given $x_\bar{q}$.

Further, we pose the following “fully observed” optimal control problem in terms of the Information-State assuming that the same initial $\bar{q}$ inputs are applied as above:

$$J^*_z = \min_{u_k} \sum_{k=\bar{q}}^{N-1} c(z_k, u_k) + c_N(z_N), \quad (6a)$$

subject to: $Z^q_{k+1} = H(Z^q_k, u_k)$, given $Z^q_{\bar{q}}$.

Hence, we have transformed our optimal control problem to the information-state domain in Eq. (6).

A. Optimality of the Information-State Control Problem

This section discusses the equivalence of solving the optimal control problem using the information-state dynamics and the partially observed state-space dynamics and proves that transforming the problem to the information-state leads to no loss in optimality after a finite initial transience. The equivalence of the solution to the two problems is established in the following result.

Theorem III.1. Given the initial information-state $Z^q_{\bar{q}}$ or equivalently the state $x_{\bar{q}}$, the solution to the optimal control problem (Eq. (5)) is identical to the solution to the optimal information state control problem (Eq. (6)).

Proof. Consider the two problems in Eqs. (5) and (6). To show the equivalence between the two problems, one needs to show: (i) the initial conditions are equivalent; (ii) the input-output response of the state-space system and information-state system are equal; (iii) the optimal feedback policies of both the systems generate the same control inputs.

Using Lemma III.1, the initial condition for the systems are related by $x_{\bar{q}} = \Psi(Z^q_{\bar{q}})$, where the information-state $Z^q_k = [z^T_1, \ldots, z^T_{\bar{q}}, u^T_{\bar{q}-1}, \ldots, u^T_0]^T$. Further, Lemma III.2 shows they have the same input-output response.

To show that the feedback policies generate the same control inputs, let the optimal control policy mapping the state to the control at time $k$ for the state-space and information-space system be $\pi^*_x(\cdot)$, and $\pi^*_z(\cdot)$, respectively. The policy for the true full-state feedback $\pi^*_x(\cdot)$ can be applied to the information-state system, by using the transformation from Lemma III.1, $x_k = \Psi(Z^q_k)$, which gives the control input to be $u_k = \pi^*_z(\Psi(Z^q_k))$. Then, because of optimality of the policy $\pi^*_z(\cdot)$ for the information-state system, the input-output equivalence of the systems in Eqs. (5b) and (6b), and the same cost functions, we get: $J^*_x \leq J^*_z$. Similarly, the optimal information-state policy $\pi^*_z(\cdot)$, can be applied to the true state-space model by taking the past inputs and outputs as the argument to the policy. Because of optimality of $\pi^*_x$ for the state space system, we get: $J^*_x \leq J^*_z$. Owing to the above two inequalities, $J^*_x = J^*_z$. Hence, the optimal feedback on the information-state is identical to the optimal control for the underlying state-space system.

Remark III.2. Notice that the two problems (Eq. (5) and Eq. (6)) mention the initial state to be defined at $k = \bar{q}$ and assume that the initial few control inputs $\{u_0, u_1, \ldots, u_{\bar{q}-1}\}$ are specified. Intuitively, one can see that these inputs are ambiguous as they are needed to get enough information to reconstruct the initial state/ form the initial information state and should not be included in the optimization. Thus, it is advisable to take small perturbations to keep the system near the initial state till the state can be reconstructed at $k = \bar{q}$. Nonetheless, given that $\bar{q} \ll T$, we can expect that this initial transient will not affect the total cost significantly.

IV. A Minimum Principle for the Information-State Based Control Problem

In this section, we develop a minimum principle for the information-state optimal control problem. We make the following assumption for the problem given in Eq. (6) and show that the information-state dynamics is control-affine.

Assumption 2. The incremental cost in Eq. (6) is quadratic in the control: $c(z_k, u_k) = l(z_k) + \frac{1}{2}u_k^T R u_k$, where $R$ is positive definite.

Proposition IV.1. Given the underlying system dynamics is affine in control with the form: $\dot{x} = f(x) + g(x)u$, the
information-state model is affine in control:

\[
Z_k^q = F(Z_{k-1}^q) + G(Z_{k-1}^q)u_{k-1}.
\]  

(7)

**Proof.** We write the system dynamics with a forward Euler approximation for a small discretization time: \(x_{k+1} = x_k + f(x_k)\Delta t + g(x_k)u_k\Delta t\), and the observation model as: \(z_k = h(x_k) = h(x_{k-1} + dx_{k}), \) where \(dx_k = f(x_k)\Delta t + g(x_k)u_{k-1}\Delta t\). Given that the time discretization is sufficiently small, the observation model can be written using a first order Taylor expansion as:

\[
z_k = h(x_{k-1} + dx_k) = h(x_{k-1}) + \frac{\partial h}{\partial x} \bigg|_{x_{k-1}} dx_k,
\]

(8)

Thus, we can write the observation \(z_k\) as some function of \(x_{k-1}\) and \(u_{k-1}\) as: \(z_k = F(x_{k-1}) + G(x_{k-1})u_{k-1}\). Further substituting for \(x_{k-1}\) as \(x_{k-1} = \Psi(Z_{k-1}^q)\) from above, we can write the \(z_k\) in terms of information-state as:

\[
z_k = F(\Psi(Z_{k-1}^q)) + G(\Psi(Z_{k-1}^q))u_{k-1},
\]

(9a)

and finally, the entire equation can be trivially written in terms of \(Z_k^q\) in a control affine form as in Eq. (7). \(\square\)

Using the above results and considering the time-index \(k = 0\) to be the first time index the information-state in constructed, the problem in Eq. (6) can be rewritten as

\[
J = \min_{u_k} \sum_{k=0}^{N-1} (l(z_k) + \frac{1}{2} u_k^T R u_k) + c_N(z_N),
\]

(9a)

s.t. \(Z_{k+1} = F(Z_k) + G(Z_k)u_k,\) given \(Z_0\).

(9b)

The superscript \(q\) in \(Z^q\) and subscript \(Z\) in \(J_Z\) are ignored for the sake of convenience. The following result gives the necessary conditions for the solution \(\{\hat{Z}_k, \bar{u}_k\}\) to the above fully observed problem in Eq. (9) to satisfy.

**Theorem IV.1.** Let the cost functions \(l(\cdot), c_N(\cdot)\), the drift \(F(\cdot)\) and the input influence function \(G(\cdot)\) be \(C^2\), i.e., twice continuously differentiable. The minimum of the open-loop problem (Eq. (9)) starting at some initial information-state \(Z_0\) satisfies:

\[
\bar{u}_k = -R^{-1}\bar{G}(\hat{Z}_k)^T G_{k+1},
\]

(10)

\[
\dot{\bar{Z}}_{k+1} = F(\hat{Z}_k) - G(\hat{Z}_k)R^{-1}G(\hat{Z}_k)^T G_{k+1},
\]

(11)

\[
G_k = L_k^Z + A_k^T Z_k + K_{k+1}^T R_k + B_k^T B_{k+1} - K_{k+1}^T (R_k + B_k^T B_{k+1} B_k) K_k,
\]

(12)

\[
P_k = A_k^T P_{k+1} A_k + L_k^Z Z_k + \sum_{i=1}^{n} [F_{k,i}^Z + \sum_{j=1}^{n} \Gamma_{k,i}^{j,z} \bar{u}_k^j] G_{k+1}^z + K_{k+1}^T (R_k + B_k^T B_{k+1} B_k) K_k,
\]

(13)

\[
K_k = - (R_k + B_k^T P_{k+1} B_k)^{-1} \left[ \sum_{i=1}^{n} \bar{Z}_{k,i} \sum_{j=1}^{n} \Gamma_{k,i}^{j,z} \bar{u}_k^j \bar{G}_{k+1}^z + B_k^T P_{k+1} A_k \right] + B_k^T P_{k+1} A_k
\]

where \(A_k = \bar{F}_k^Z + \sum_{i=1}^{n} \bar{G}_{k,i}^z \bar{u}_k^j, \) \(B_k = \bar{G}(\hat{Z}_k),\) and \(\{\hat{Z}_k\}\) represents the optimal nominal trajectory, \(\bar{L}_k^Z = \nabla Z_l|_{\hat{Z}_k}, \) \(G_k = \nabla Z_l|_{\hat{Z}_k},\) with terminal condition \(G_N = \nabla Z L C^N|_{\hat{Z}_N}, \) \(P_k = \nabla Z L J_k|_{\hat{Z}_k},\) with terminal condition \(P_N = \nabla Z L C N|_{\hat{Z}_N}, \) \(\bar{u}_k = [\bar{u}_1^T \cdots \bar{u}_k^T]^T,\) the control influence matrix: \(G = [G(\dot{Z}) \cdots G^{n}(\dot{Z})],\) and \(\Gamma_{k}^{j,z}\) represents the control influence vector and optimal control vector corresponding to the \(j\)th input. Finally, \(\bar{F}_k^Z = \nabla Z F|_{\hat{Z}_k} \) and \(\bar{\Gamma}_{k,i}^{j,z} = \nabla Z F|_{\hat{Z}_k} \) gives the Jacobians of the system dynamics and \(\bar{F}_{k,i}^Z \) and \(\bar{\Gamma}_{k,i}^{j,z} \) gives the Hessians of the system dynamics along the nominal trajectory.

**Proof.** Equations (10) to (12) are the standard necessary conditions in optimal control for nonlinear systems [3]. The derivation for equations \(P_k\) and \(K_k\) is given in [18]. \(\square\)

**A. Global Optimum for the Nonlinear Problem**

We proceed to show that the problem in Eq. (9) has a unique minimum under some assumptions. The problem in Eq. (9) can equivalently be written as the following Dynamic Programming problem:

\[
J_k(Z_k) = \min_{u_k} \left[ (l(z_k) + \frac{1}{2} u_k^T R u_k) + J_{k+1}(Z_{k+1}) \right],
\]

(14)

where \(J_N(Z_N) = c_N(z_N)\).

The minimizing control \(u_k\) for the above problem satisfies the first-order necessary condition \(\frac{\partial J_k}{\partial u_k} = 0\), and given the current state is \(Z_k\), the control is given by

\[
\bar{u}_k = -R^{-1}\bar{G}(\hat{Z}_k)^T \frac{\partial J_{k+1}}{\partial Z_{k+1}} \bar{Z}_{k+1} = -R^{-1}\bar{G}(\hat{Z}_k)^T G_{k+1},
\]

(15)

where, \(\bar{Z}_{k+1} = F(\hat{Z}_k) - G(\hat{Z}_k)R^{-1}G(\hat{Z}_k)^T G_{k+1}\).

Substituting this control in Eq. (14) gives:

\[
J_k(Z_k) = l(z_k) + \frac{1}{2} G_{k+1}^T G(\hat{Z}_k)R^{-1}G(\hat{Z}_k)^T G_{k+1} + J_{k+1}(F(\hat{Z}_k) - G(\hat{Z}_k)R^{-1}G(\hat{Z}_k)^T G_{k+1}).
\]

(16)

Taking the partial derivative of Eq. (16) with respect to \(Z_k\) at \(\hat{Z}_k\) gives rise to the co-state equation:

\[
G_k = L_k^Z + A_k^T Z_k + (\bar{F}_k^Z - \bar{G}(\hat{Z}_k)^T G_{k+1})^T G_{k+1},
\]

(17)

(Note: In Eq. (17), \(\bar{G}(\hat{Z}_k)^T G_{k+1}\) would be a tensor for the vector case, but we abuse the notation for the sake of illustration.) Since Eq. (14) is a terminal value problem, we start from the terminal state. For a given terminal condition \(\{\hat{Z}_N, G_N = \frac{\partial J_N}{\partial Z_N}|_{\hat{Z}_N}\},\) Eqs. (15) and (17) can be solved backward in time to find the optimal solution. Hence, Eqs. (15) and (17) are the characteristic equations to the solution to the dynamic programming problem in Eq. (14). But, they need not have a unique backward evolution.

**Remark IV.1.** Equations (15), (17) are the discrete-time analog to the characteristic ODEs one obtains for the Hamilton-Jacobi-Bellman equation using the method of characteristics [19].

**Assumption 3.** Given \(\bar{Z}_{k+1}, \bar{G}_{k+1},\) Eq. (15) has a unique solution for \(\bar{Z}_k : \bar{Z}_k = F(\bar{Z}_{k+1}, \bar{G}_{k+1})\), where \(F(\cdot, \cdot)\) is an
Lemma IV.1. Let the functions, $\mathcal{F}, \mathcal{G}$ be continuously differentiable. If the underlying discretization time for the systems in Eqs. (15) and (17) are sufficiently small, i.e., $\Delta t \to 0$, then, there is a unique map from $\{\bar{Z}_{k+1}, \bar{G}_{k+1}\}$ to $\{\bar{Z}_{k}, \bar{G}_{k}\}$ and vice-versa.

Proof. Since $\Delta t \to 0$, the implicit function theorem shows that, in the neighborhood of $\{\bar{Z}_{k+1}, \bar{G}_{k+1}\}$, there exists a unique $\{\bar{Z}_{k}, \bar{G}_{k}\}$, that satisfies Eqs. (15) and (17). The same argument applies to the forward map. \hfill \Box

Lemma IV.2. If the control influence matrix $\mathcal{G}(\cdot)$ is state independent, then, there is a unique map from $\{\bar{Z}_{k+1}, \bar{G}_{k+1}\}$ to $\{\bar{Z}_{k}, \bar{G}_{k}\}$ and vice-versa.

Proof. Given $\{\bar{Z}_{k+1}, \bar{G}_{k+1}\}$, $\bar{u}_k$ is exactly determined from Eq. (10) without the knowledge of $\bar{Z}_k$, since $\mathcal{G}(\cdot)$ is state independent. Since the state reached by a dynamical system $\bar{Z}_{k+1}$, and the control action $\bar{u}_k$ taken to reach there are known, the state it originated from $\bar{Z}_{k}$ is unique. Also, Eq. (17) is a linear equation for state independent $\mathcal{G}$; hence $\bar{G}_{k}$ is unique. The uniqueness of the forward map can be similarly shown. \hfill \Box

Now, we show Eqs. (15) and (17) have a unique backward evolution and also show that trajectories originating from different terminal conditions do not cross.

Lemma IV.3. Under Assumption 3, given terminal conditions $\{\bar{Z}_N, \bar{G}_N = c_N^\mathcal{F}(\bar{Z}_N)\}$, the state and costate will have a unique evolution backward in time.

Proof. Let us consider at $k = N − 1$. Since $\{\bar{Z}_N, \bar{G}_N\}$ are given, $\bar{Z}_{N−1}$ is unique due to Assumption 3. Then, it is straightforward to see that $\bar{G}_{N−1}$ is unique from Eq. (17). The same argument is applicable at every time step $k$. \hfill \Box

Lemma IV.4. Under Assumption 3, trajectories originating from different terminal conditions do not cross in the $\{Z, G\}$ space for all time.

Proof. The only scenario the trajectories might cross is in between the time steps. The state of the dynamical system, in practice, will be continuously evolving in between time steps, while the costate will be constant (see Fig. 1). If trajectories have to intersect, they should start with the same costate, but different states. At the crossing point, they will have the same state, and a dynamical system cannot have different evolutions from the same state and costate. \hfill \Box
that satisfies the necessary conditions. However, this is, by
definition, a solution that is found by satisfying the Minimum
Principle. Therefore, owing to the development above, the
co-state \( G_0 = \phi_0(Z_0) \) is uniquely determined by the initial
state \( Z_0 \), and a solution that satisfies the minimum principle
is necessarily unique. Moreover, since this solution is the
unique solution to the DP problem (Eq. (14)) with initial
state \( Z_0 \), it is also the global optimum.

\[ \text{(V.1)} \]

V. OPEN-LOOP TRAJECTORY DESIGN USING PARTIALLY
OBSERVED DATA BASED iLQR (POD-iLQR)

The ILQR algorithm, in theory, owing to the minimum
principle, can be generalized to iteratively solve the nonlinear
information-state optimal control problem from section II.

Remark V.1. In our recent arxiv paper [18], we showed
the iLQR algorithm converges to the global optimal solution
under some mild assumptions on cost functions (quadratic
in control) and system dynamics (affine in control). Thus, the
iLQR generalization to the proposed information-state based
formulation would also result in convergence to the global
optimal solution under the same assumptions.

Information-state based generalized iLQR algorithm:

Forward Pass - Given a nominal control sequence \( \{ u_k \}^{N-1}_{k=0} \)
and initial information-state vector \( Z_0 \), the state is prop-
gated in time in accordance with the dynamics \( \dot{Z}_{k+1} = \mathcal{H}(Z_k, u_k) \). Thus, we get the nominal information-state tra-
jectory \( (Z_k, \bar{u}_k) \).

Local LTV System - Next, we find the corresponding
local LTV system around the trajectory \( (Z_k, \bar{u}_k) \), which

\( \delta Z_{k+1} = A_k \delta Z_k + B_k \delta u_k \), where \( A_k \) and \( B_k \)
are the linearization of the information-state dynamics around the nominal trajectory and \( \delta Z_k = (\delta z_k, \delta z_{k-1}, \ldots, \delta z_{k-q}, \delta u_{k-1}, \ldots, \delta u_{k-q}) \), where \( \delta z_k = z_k - \bar{z}_k \) and \( \delta u_k = u_k - \bar{u}_k \) are the deviation from the
nominal observation at time \( k \).

Backward pass - Given the LTV parameters about the
nominal system, the ILQR algorithm computes a local
optimal control as:

\[ \delta u_k = \kappa_k \delta z_k \text{, where } \kappa_k = (R + B_k^T V_{k+1} B_k)^{-1} (B_k^T V_{k+1} A_k), \]

where \( V_k = I_k^Z + A_k^T V_{k+1} A_k - A_k^T V_{k+1} B_k (R + B_k^T V_{k+1} B_k)^{-1} \)

\( \cdot (B_k^T V_{k+1} A_k + R) \).

Notice that there always exists an exact fit for the ARMA
model for any \( q \geq q \), if there exists a number \( \hat{q} \) for which the matrix \( \Omega^q \) is full column rank. Using this result, we can
write linearized models in terms of ARMA parameters at
each step along the nominal trajectory.

Next, we show how to find the ARMA parameters from
a suitable least squares problem. We perturb the linear
system about the nominal trajectory and use the standard
least square method to estimate the system parameters \( a_{-i} \)
and \( \beta_{-i} \) for \( i = 1, \ldots, q \), from:

\[ \delta z^{(j)}_k = \alpha_{k-1} \delta z^{(j)}_{k-1} + \cdots + \alpha_{k-q} \delta z^{(j)}_{k-q} + \beta_{k-1} \delta u^{(j)}_{k-1} + \cdots + \beta_{k-q} \delta u^{(j)}_{k-q}, \]

where \( \delta u^{(j)}_k \sim \mathcal{N}(0, \sigma I) \) is the control input perturbation at step \( k \)
for the \( j \)-th simulation. The final linear relation between input-
output perturbation and system parameters can be written as

\[ z = \kappa z \text{, where } \kappa = \begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\n0 \\
0 \\
\end{array} \]

and can be solved using the standard least square method
as:

\( \alpha_{k-1} \cdots \alpha_{k-q} \beta_{k-1} \cdots \beta_{k-q} = \mathcal{X}^{\dagger} \), where \( \mathcal{X}^{\dagger} \)
denotes the pseudo-inverse of some matrix \( \mathcal{X} \). Please check
[16] for more details.
After identifying the system parameters $\alpha_{k-1}, \cdots, \alpha_{k-q}$ and $\beta_{k-1}, \cdots, \beta_{k-q}$ for $k = \{0, \cdots, N\}$, we write the perturbation LTV system in the information-state as given in Eq. (20), which is written with the observation model as:

$$\delta Z_k = A_{k-1} \delta Z_{k-1} + B_{k-1} \delta u_{k-1} + D_{k-1} w_{k-1},$$

where $w_k$ and $v_k$ are the process and measurement noise. The final control input to the system is $u = \ddot{Z}_k - K_k \delta \dot{Z}_k$, where $K_k$ is the time varying feedback gain and $\ddot{Z}_k$ is the estimate of the information-state which is calculated as $\delta \ddot{Z}_k = A_{k-1} \delta \dot{Z}_{k-1} + B_{k-1} \delta u_{k-1} + L_k (\delta \gamma_k - A_{k-1} \delta \dot{Z}_{k-1} - B_{k-1} \delta u_{k-1})$, where $L_k$ is the observer gain.

### VI. Partially-Observed Decoupled Data-based Control (POD2C) Algorithm

We now propose an extension to the so-called decoupled data-based control (D2C) algorithm [15], [21]. The D2C algorithm is a highly data-efficient Reinforcement Learning (RL) method that has shown to be much superior to state-of-the-art RL algorithms in terms of data efficiency and training stability while having better performance. The POD2C algorithm proposes a 2-step procedure to approximate the solution for the control problem in the presence of process as well as sensor noise. We first solve a noiseless open-loop optimization problem to find an optimal control sequence, $\hat{u}^*_k$ via the POD-iLQR scheme. Then, an LQG controller is synthesized for the LTV perturbation model described in the information-state: $\delta Z_k = A_{k-1} \delta Z_{k-1} + B_{k-1} \delta u_{k-1} + D_{k-1} w_{k-1}$ and measurement model: $\delta \gamma_k = \delta Z_k + v_k$, where $w_k$ and $v_k$ represent the process and measurement noise. The final control input to the system is $u_k = \hat{u}^*_k - K_k \delta \dot{Z}_k$, where $K_k$ is the time varying feedback gain and $\dot{Z}_k$ is the estimate of the information-state which is calculated as $\delta \dot{Z}_k = A_{k-1} \delta Z_{k-1} + B_{k-1} \delta u_{k-1} + L_k (\delta \gamma_k - A_{k-1} \delta \dot{Z}_{k-1} - B_{k-1} \delta u_{k-1})$, where $L_k$ is the observer gain.

### VII. Empirical Results

We use a black box MuJoCo simulator [22] to design the nominal trajectory and closed-loop feedback policy for the system with initial configurations given in Figure 3.

**Cart-Pole:** The classic robotics model [16].

**15-link Swimmer:** The system has 17 DOF and together with their rates and controls are torques on the 14 joints. The 15-link swimmer requires only angular positions of every other joint and the fish needs only the angles of the fin and tail joints. The $T_2D_1$ arm is a soft-robotic arm where the node-positions of 8 (out of 25) node points were used.

**Fish:** The fish has 13 DOF and 6 control channels with torques on the fin and tail joints.

**T2D1 Robotic Arm:** The tensegrity model consists of 33 bars and 46 strings with controls as tensions in the strings.

### 1D Viscous Burgers PDE: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$. We take external control inputs on the boundaries corresponding to blowing/suction such that $u(0, t) = U_1(t); u(L, t) = U_2(t)$. Given an initial velocity profile $u(x, 0) = \sin(\pi(x - 1))$, the control task is to reach the goal state $u(x, T) = -0.50$ for $x = [0 : 0.02 : 2]$ with two observers added at both ends of the boundary.

**Allen-Cahn PDE:** $\frac{\partial \phi}{\partial t} = -M(\frac{\partial F}{\partial \phi} - \gamma \nabla^2 \phi)$. We adopt the following general form of energy density function $F: F(\phi; T; h) = \phi^4 + T \phi^2 + h \phi$, where $\phi(x, t) \in [-1, 1]$ is the order parameter, and $T(x, t)$ and $h(x, t)$ are external control inputs. A periodic boundary condition is applied, and the control scheme is such that all grid points converging to the same goal state $\phi_k = 1$ or $-1$ are fed the same control inputs. Given a uniform initial condition $\phi_0 = 0$, the control task is to reach the custom goal state as shown in Fig. 6.

![Fig. 3: Models simulated in MuJoCo in their initial and final states.](image-url)
![Fig. 4: Averaged episodic reward vs measurement noise level for fixed 10% process noise with LQG as closed-loop feedback](image1)

![Fig. 5: Averaged episodic reward vs process noise level for fixed 10% measurement noise with LQG as closed-loop feedback](image2)

![Fig. 6: High dimensional, complex, and partially observed nature of the problem governed by Allen-Cahn PDE (2500 states, 16 outputs)](image3)

### TABLE I: Comparison results between Gradient Descent [16] and POD-iLQR (*Gradient descent failed to converge*)

| System        | Grad. Des. Total Time (sec.) | POD-iLQR Total Time (sec.) | Iteration Number | q | States | Outputs |
|---------------|-----------------------------|----------------------------|------------------|---|--------|---------|
| Cart-pole     | 15                          | 0.8                        | 40               | 2 | 4      | 2       |
| Swimmer       | 376/36.72                   | 2110.0                     | 100              | 5 | 34     | 10      |
| Fish          | *                           | 1720.5                     | 200              | 2 | 27     | 11      |
| Robotic Arm   | *                           | 355.8                      | 40               | 3 | 150    | 24      |
| Burgers Eqn.  | 276/3.4                     | 38.2                       | 10               | 2 | 100    | 2       |
| Allen-Cahn Eqn. | 21078/18                  | 47.78                      | 10               | 2 | 2500   | 16      |

as they have higher non-linearity due to the fluid-structure interaction in the swimming motion. However, the training is far more time-efficient compared with the first-order gradient descent method used in our prior work [16]. The T2D1 arm system also takes less time and iterations to converge despite the high dimensionality of the model and limited outputs. The iteration numbers given in Table I show smooth and efficient convergence, even for systems with high non-linearity and high dimensionality. The results are obtained using Matlab and MuJoCo on a Ryzen 3700 PC. The most time-consuming procedure is running simulations to collect data for fitting the ARMA model, which is in serial for now but can be in parallel as the rollouts are independent of each other and further improve the time efficiency and have the potential for real-time operation.

**Robustness to measurement and process noise:** Figure 4 shows the plots for the episodic cost with the variation in the measurement noise. The figure compares the open-loop and the closed-loop control policies under different measurement noise levels, while the process noise standard deviation is set to 10% of the maximal nominal control. The measurement noise level on the x-axis is the percentage of the measurement noise standard deviation w.r.t. the maximal nominal measurement. Note that both the measurement and process noise is added as zero-mean Gaussian i.i.d. noise to all measurement and control channels at each step.

Note that the closed-loop policy can successfully finish the task with smaller noise levels than what is indicated by the black threshold lines in the figure. Under the open-loop policy, the process noise drives the model off the nominal trajectory and results in high episodic cost, while the closed-loop feedback can help the system stay close to the nominal trajectory. This can be seen from the figure as the episodic cost mean and variance of the closed-loop policy is much smaller than the open-loop policy on the entire tested noise range, although both policies fail the task when the process noise becomes larger than what the black threshold line indicates. The threshold values of process and measurement noise are empirically calculated at which the system will go too far away from the nominal trajectory and the final goal can not be achieved. The above analysis regarding the performance of control policy under noise
proves that the LQG closed-loop feedback wrapped around the nominal trajectory makes the full closed-loop policy robust to measurement and process noise.

**Comparison with a Direct RL Method:** In a direct RL method such as DDPG [7], deep neural networks are used to represent the closed-loop control policy. In general, Direct RL methods require full state observation and it is not clear how to generalize them to partially observed problems. Nonetheless, we run the DDPG method on the fish example with the same information-state as found by POD2C. After training for approximately 20 hours, the DDPG algorithm failed to converge and the fish could not swim to the target.

**VIII. Conclusions**

The paper proposed an optimal information-state approach to transform partially observed problems into fully observed optimal nonlinear control problems. We showed that the information-state based formulation is equivalent to the original system formulation with the partial nonlinear observation model. The paper further provides the conditions to meet the minimum principle in the information-state formulation. We then provide the algorithm that generated the optimal nominal trajectory using iLQR with partial state observations. Finally, the paper presented a decoupled data-based approach to control complex robotic systems by designing the closed-loop feedback law with partial state observations. Empirical results are shown for complex robotic systems, including challenging cases of hard-to-model soft contact constraints, dynamic fluid-structure interactions, and partial differential equations, under motion as well as sensing uncertainty. In our opinion, the POD2C approach is highly efficient for RL in partially observed problems, however, questions regarding optimality under noise shall be explored in future work.

**References**

[1] A. M. Annaswamy, K. H. Johansson, G. J. Pappas et al., “Control for societal-scale challenges: Road map 2030,” IEEE Control Systems Society Publication, 2023.

[2] D. P. Bertsekas, Dynamic Programming and Optimal Control, 2nd ed. Athena Scientific, 2000, vol. I.

[3] F. Lewis, D. Vrabie, and V. Syrmos, Optimal Control. Wiley, 2012.

[4] H. Khalil, “Adaptive output feedback control of nonlinear systems represented by input-output models,” IEEE Transactions on Automatic Control, vol. 41, no. 2, pp. 177–188, 1996.

[5] V. Mnih, K. Kavukcuoglu, D. Silver, A. A. Rusu, J. Veness, M. G. Bellemare, A. Graves, M. Riedmiller, A. K. Fidjeland, G. Ostrovski et al., “Human-level control through deep reinforcement learning,” Nature, vol. 518, no. 7540, p. 529, 2015.

[6] D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre et al., “Mastering the game of go with deep neural networks and tree search,” nature, vol. 529, no. 7587, p. 484, 2016.

[7] T. P. Lillicrap, J. J. Hunt, A. Pritzel, N. Heess, T. Erez, Y. Tassa, D. Silver, and D. Wierstra, “Continuous control with deep reinforcement learning,” arXiv preprint arXiv:1509.02971, 2015.

[8] S. Levine, C. Finn, T. Darrell, and P. Abbeel, “End-to-end training of deep visuomotor policies,” The Journal of Machine Learning Research, vol. 17, no. 1, pp. 1334–1373, 2016.

[9] F. L. Lewis and K. G. Vamvoudakis, “Reinforcement learning for partially observable dynamic processes: Adaptive dynamic programming using measured output data,” IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), vol. 41, no. 1, pp. 14–25, 2011.

[10] T. Sadamoto and A. Chakraborty, “Fast real-time reinforcement learning for partially-observable large-scale systems,” IEEE Transactions on Artificial Intelligence, vol. 1, no. 3, pp. 206–218, 2020.

[11] J. Subramanian, A. Sinha, R. Seraj, and A. Mahajan, “Approximate information state for approximate planning and reinforcement learning in partially observed systems,” J. Machine Learning Research, vol. 23, no. 1, 2022.

[12] W. Li and E. Todorov, “Iterative linearization methods for approximately optimal control and estimation of non-linear stochastic system,” International Journal of Control, vol. 80, no. 9, pp. 1439–1453, 2007.

[13] D. Jacobsen and D. Mayne, Differential Dynamic Programming. Elsevier, 1970.

[14] G. E. Box, G. M. Jenkins, and G. C. Reinsel, Time series analysis: forecasting and control. John Wiley & Sons, 2011, vol. 734.

[15] R. Wang, K. S. Parunandi, D. Yu, D. Kalathi, and S. Chakravorty, “Decoupled data-based approach for learning to control nonlinear dynamical systems,” IEEE Transactions on Automatic Control, 2021.

[16] R. Wang”, R. Goyal”, S. Chakravorty, and R. E. Skelton, “Data-based control of partially-observed robotic systems,” in IEEE International Conference on Robotics and Automation (ICRA), Xi’an China, 2021.

[17] P. Kumar and P. Varaiya, “Stochastic systems: estimation, identification and adaptive control,” 1986.

[18] R. Wang, K. S. Parunandi, A. Sharma, R. Goyal, and S. Chakravorty, “On the search for feedback in reinforcement learning,” 2022, arXiv 2002.09478.

[19] M. N. G. Mohamed, S. Chakravorty, R. Goyal, and R. Wang, “On the feedback law in stochastic optimal nonlinear control,” American Control Conference (ACC), pp. 970–975, 2022, arXiv:2004.01041.

[20] M. N. G. Mohamed, R. Goyal, S. Chakravorty, and R. Wang, “The information-state based approach to linear system identification,” in American Control Conference (ACC), 2023, pp. 301–306.

[21] D. Yu, M. Rafieisakhaei, and S. Chakravorty, “Stochastic Feedback Control of Systems with Unknown Nonlinear Dynamics,” in IEEE Conference on Decision and Control(CDC), 2017.

[22] E. Todorov, T. Erez, and Y. Tassa, “MuJoCo: A physics engine for model-based control,” in IEEE/RSJ International Conference on Intelligent Robots and Systems. IEEE, 2012, pp. 5026–5033.