MOST LAPLACIAN EIGENVALUES OF A TREE ARE SMALL

DAVID P. JACOBS, ELISMAR OLIVEIRA, AND VILMAR TREVISAN

ABSTRACT. We show that the number of Laplacian eigenvalues less than the average degree $2 - \frac{2}{n}$ of a tree having $n$ vertices is at least $\left\lceil \frac{n}{2} \right\rceil$.

1. Introduction

For a graph $G$ of order $n$, the Laplacian matrix of $G$ is $L = D - A$, where $A$ is the adjacency matrix and $D$ is the diagonal degree matrix. The eigenvalues of $L$, which lie in the interval $[0, n]$, are called Laplacian eigenvalues of $G$. Studying the distribution of Laplacian eigenvalues of graphs is a natural and relevant problem. It is relevant due to the many applications related to Laplacian matrices (see, for example \cite{16}). We believe it is also a hard problem because little is known about how the Laplacian eigenvalues are distributed in the interval $[0, n]$. Even though there exist results that bound the number of Laplacian eigenvalues in subintervals of $[0, n]$ (see for example \cite{7, 8, 11, 12, 15, 21} and the references therein) there lacks a finer understanding of the distribution. For instance, it is not known whether smaller eigenvalues outnumber the larger ones, and little known about how eigenvalues are clustered around a point \cite{10}.

We first consider the question What is a large Laplacian eigenvalue? A reasonable measure is to compare this eigenvalue with the average of all eigenvalues. Since the average of the Laplacian eigenvalues equals the average degree $d_n = \frac{2m}{n}$ of $G$, we say that a Laplacian eigenvalue is large if it is greater than or equal to the average degree, and small otherwise. Inspired by this idea, the paper \cite{5} introduces the spectral parameter $\sigma(G)$ which counts the number of Laplacian eigenvalues greater than or equal to $d_n$.

There is evidence that $\sigma(G)$ plays an important role in defining structural properties of a graph $G$. For example, it is related to the clique number $\omega$ of $G$ (the number of vertices of the largest induced complete subgraph of $G$) and it also gives insight about the Laplacian energy of a graph \cite{5} \cite{19}. Moreover, several structural properties of a graph are related to $\sigma$ (see, for example \cite{4} \cite{5}).

In this paper, we are interested in the distribution of Laplacian eigenvalues of trees. More precisely, we want to investigate $\sigma(T)$. If $I$ is a real interval, we let $m_G(I)$ denote the number of Laplacian eigenvalues of $G$ in $I$, counting multiplicities. Given a tree of order $n$, its average vertex degree is $d_n = 2 - \frac{2}{n}$. In 2011 it was conjectured that in any tree, at least half of the Laplacian eigenvalues were less than this average \cite{20}. The purpose of this paper is to prove the conjecture, which we state as the following result.

**Theorem 1.1.** For any tree $T$ of order $n$, $m_T[0, d_n) \geq \left\lceil \frac{n}{2} \right\rceil$.

1991 Mathematics Subject Classification. 05C50, 05C05, 15A18.

Key words and phrases. Laplacian eigenvalue; tree; eigenvalue distribution.
Since there are \( n \) Laplacian eigenvalues bounded by \( n \), this is equivalent to
\[
\sigma(T) = m_T[d_n, n] \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]
This property does not hold for general graphs, and complete graphs \( K_n \) provide a counter-example. It is known that rational Laplacian eigenvalues of graphs are integers, and when \( n \geq 3 \), \( d_n \) is not integer so it suffices to show \( m_T(d_n, n) \leq \left\lfloor \frac{n}{2} \right\rfloor \) for all trees \( T \).

The proof of this conjecture is, perhaps surprisingly, difficult. There have been a few attempts to solve it and we summarize now some partial results. First, computation has verified that Theorem 1.1 holds for all trees of order \( n \leq 20 \) (see the experiments by J. Carvalho [3]). In the paper [20], where the conjecture was set forward, it was proved that diameter 3 trees and caterpillars satisfy the conjecture and in [14] the theorem was shown to hold for all trees of diameter four. It is known [9, 2] that \( m_T[0, 2) \geq \left\lceil \frac{n}{2} \right\rceil \) for all trees \( T \). If there were never eigenvalues in \( (2 - \frac{2}{n}, 2) \), then Theorem 1.1 would easily follow from this result. However, there can exist eigenvalues in this interval.

By [3] we assume \( n \geq 8 \). For each \( n \geq 8 \), our proof makes use of four prototype trees which satisfy the theorem. If any tree can be transformed to a prototype tree in a way that does not decrease the number of eigenvalues above the average degree, then Theorem 1.1 holds.

The paper’s remainder is organized as follows. In the next section, we present a notation that is crucial for our strategy to prove our result. To simplify the representation of trees we introduce a concatenation operator that is executed on suns and paths to form generalized pendant paths. We also define a summation operator and starlike vertices which are based on these operators. The main tool to prove Theorem 1.1 is the algorithm Diagonalize that counts the number of Laplacian eigenvalues of a tree in any interval [2]. In Section 3 this algorithm is described along with a procedure called Transform, a high-level proof strategy. In Section 4 we show how to use the Diagonalize algorithm to transform a tree in a way that does not decrease the number of eigenvalues above average degree, which we call a proper transformation. In Section 5 we define prototype trees, which are close to stars and depend only on the congruence \( n \equiv \alpha \pmod{4} \). We prove that these prototype trees satisfy the conjecture, however they are extreme examples in which equality is achieved in Theorem 1.1. In Section 6 describe a procedure ReduceStarVertex that does much of the structural transformation. This is based on the proper transformations and since it is not obvious that Transform halts we also prove its correctness in Section 6. All trees get reduced to some small cases that are eventually transformed into prototype trees in Section 7. A complete example showing a tree properly transformed into a prototype tree is given in Section 8.

2. Notation

Consider a tree \( T \) with \( n \) vertices, whose average degree is \( d_n = 2 - \frac{2}{n} \). Recall that \( m_T[0, d_n) \) is the number Laplacian eigenvalues of \( T \) which are smaller than \( d_n \), and \( \sigma(T) = m_T(d_n, n] \), denotes the number of eigenvalues which are larger than \( d_n \). We will use the fact that \( m_T[0, d_n) + \sigma(T) = n \).
The concept of a pendant path is well known, but important here. Let \( u \) be a vertex of a tree \( T \) with degree \( \geq 3 \). Suppose that \( P_q = u \ u_1 \ldots u_q \) (\( q \geq 1 \)) is a path in \( T \) whose internal vertices \( u_1, \ldots, u_{q-1} \) all have degree 2 in \( T \), and where \( u_q \) is a leaf. Then we say that \( P_q \) is a pendant path of length \( q \) attached at \( u \). The following is known [17].

**Lemma 2.1.** Any tree that is not a path has at least one vertex \( u \), with \( \deg(u) \geq 3 \), having (at least) two pendant paths.

Let \( T \) be a tree with \( n \) vertices, and let \( u \) be a vertex of degree at least \( \ell + 1 \) of \( T \) having \( \ell \geq 1 \) pendant paths attached at \( u \). We denote the sum of pendant paths attached at \( u \) by \( P(u) = P_{q_1} \oplus \cdots \oplus P_{q_k} \), as illustrated in Figure 1. The number of edges in each path is denoted by \( \#P_q = q \).

![Figure 1. Vertex \( u \) with \( P(u) = P_1 \oplus 2P_2 \oplus P_4 \oplus P_5 \)](image)

Pendant paths of length 2 are key to our strategy. Essentially, we transform any tree \( T \) to a tree \( T' \) having only \( P_2 \)'s as pendant paths. A subgraph obtained by a vertex \( u \) attached to \( r \geq 1 \) paths of length 2, is called a sun with \( r \) rays and denoted by \( S_r \). Hence, if a vertex \( u \) of degree at least \( r + 1 \) has \( r \geq 1 \) pendant paths of length 2, say \( P(u) = rP_2 \), we will write \( P(u) = S_r \).

Consider a path between \( u \) and \( v \), where \( v \) has \( r \) pendant \( P_2 \) paths, that is \( P(v) = S_r \). To simplify the representation, we use the concatenation symbol and write \( P_q * S_r \).

Given a tree \( T \), we recall that a branch at a vertex \( u \) is a maximal subtree where \( u \) is a leaf. Let \( v \) be a vertex where \( P(v) = S_r \). If a vertex \( u \) of degree \( \geq 3 \) is connected to \( v \) by a path of length \( q \geq 0 \), we observe that \( P(u) = P_q * S_r \) is a branch at \( u \), which we call a generalized pendant path at \( u \), abbreviated by \( gpp \).

To further simplify the graphic part of the representation, we will use a black square ■ to represent a pendant sun \( S_r \) attached to a vertex, and a single edge to represent the entire path \( P_q \), omitting the \( r \) pendant \( P_2 \)'s and the \( q \) vertices. We will refer to this as the \((P_q, S_r)\) representation of this generalized pendant path \( P_q * S_r \), as shown in Figure 2.

We can consider \( q = 0 \) for paths \( P_q \) of length 0, as well as \( r = 0 \) for no pendant \( S_r \). However, we do not allow both \( r = q = 0 \) simultaneously. As we will see in Algorithm \texttt{InitiateRepresentation}, it is always possible to use the \((P_q, S_r)\) representation to represent every pendant path in a given tree. We call this a \((P_q, S_r)\) representation of \( T \). We adopt the following convention for representing paths.

\[
\begin{align*}
P_1 &= P_1 * S_0 \\
P_2 &= P_0 * S_1 \\
P_q &= P_{q-2} * S_1 \text{ for } q \geq 3.
\end{align*}
\]
This convention provides a particular \((P_q, S_r)\) representation of paths in a given tree in which \(r\) is always equal to 0 or 1. We observe that there are many other feasible choices, including the one where, if a vertex \(u\) has \(r \geq 2\) pendant \(P_2\) attached and no other pendant paths, we write \(P(u) = r P_2 = P_0 \ast S_r\).

Consider the fairly large tree \(\overline{T}\) in Figure 3 that will be used throughout the paper. Using the special symbol to represent pendant \(S_r\), we obtain, in Figure 19 (left), its \((P_q, S_r)\) representation. All trees are considered to be in \((P_q, S_r)\) representation.

We say a vertex \(u\) is a starlike vertex if it has degree \(\geq 3\) and has at least two generalized pendant paths attached to it. This definition depends on the particular \((P_q, S_r)\) representation, as well as the graph itself. For example, even though they represent the same subgraph, a vertex \(u\) with \(P(u) = 3(P_0 \ast S_1)\) has three generalized pendant paths (hence a starlike vertex), whereas \(P(u) = P_0 \ast S_3\) has a single generalized pendant path and therefore is not starlike. Let \(T\) be a tree and let \(u\) be a starlike vertex of \(T\) having \(\ell \geq 1\) generalized pendant paths attached at \(u\). According to our \((P_q, S_r)\) representation, we have \(P(u) = P_{q_1} \ast S_{r_1} \oplus \cdots \oplus P_{q_\ell} \ast S_{r_{\ell}}\). Let us call the weight of \(u\) and denote by \(w(u)\), the sum of the number of vertices of the generalized pendant paths, that is

\[
w(u) = \sum_{i=1}^{\ell} \#(P_{q_i} \ast S_{r_i}) = \sum_{i=1}^{\ell} (q_i + 2r_i).
\]

For example in ascending order of weight, the vertices \(u_1, u_2, u_3, u_4\) and \(u_5\) in the tree of Figure 19 are all its starlike vertices and have weights respectively, 2, 3, 4, 6 and 8.
Algorithm Transform(T)
input: a tree T with n ≥ 8 vertices.
output: a tree T’ with n vertices and σ(T) ≤ σ(T’)
initialize: T:=InitiateRepresentation(T).
order starlike vertices of T as u₁,...,uₖ by weight, w(u₁) ≤ ⋯ ≤ w(uₖ).
while k ≥ 3 do
loop invariant: w₁(u) ≤ 2⌊n/4⌋
T’ := ReduceStarVertex(T, u₁).
order starlike vertices of T as u₁,...,uₖ by weight, w(u₁) ≤ ⋯ ≤ w(uₖ).
end loop
return T’

Figure 4. Transforming T.

3. Strategy and algorithmic tools

A proper transformation is defined as an operation on a tree T which gives a new tree T’, with the same number of vertices, that does not decrease the number of eigenvalues above the average degree, that is

σ(T) ≤ σ(T’).

A proper transformation requires n ≥ 8, however as explained earlier, trees of smaller size have been checked by computation.

Our strategy to prove that, for any tree T of order n, σ(T) ≤ ⌊n/2⌋ is to make successive transformations on T to obtain a prototype tree T’. From the fact that we use proper transformations, and that σ(T’) ≤ ⌊n/2⌋, it follows that σ(T) ≤ σ(T’), proving Theorem 1.1. More precisely, for any tree with a given order n, we properly reduce it to a prototype tree that depends only on the congruence n ≡ α (mod 4). We refer to Section 5 for the definition of the prototype trees and for the proof they satisfy Theorem 1.1.

We describe a high level algorithm Transform to do the transformation, shown in Figure 4. The initialization procedure InitiateRepresentation(T) puts the tree T into a (P₉, S₉) representation, as illustrated in Figure 19. It may be formally described by the pseudo code of Figure 5.

The next step is the identification and ordering of all k starlike vertices of T. Recall that the weight of a starlike vertex u is the total number of vertices hanging at u. The main parameters of our transformation algorithm is the number k of starlike vertices and their weights.

The heart of our algorithm is the procedure ReduceStarVertex (T, u₁). It takes the tree T and its starlike vertex of minimum weight u₁ as arguments, and properly transforms the generalized pendant paths at u₁ into a single generalized pendant path. More precisely, P(u) = Pₗ₁ ∗ S₁ ⊕ ⋯ ⊕ Pₗₖ ∗ Sₖ is replaced by Pₗ ∗ Sₙ, for certain values of q and r. We will prove that if k ≥ 3, then there is always a starlike vertex u with w(u) ≤ 2⌊n/4⌋. Moreover, we will see that this do not increase the number of starlike vertices, but increases the minimum weight. This guarantees the algorithm stops.
InitiateRepresentation\((T)\)

input: a tree \(T\) with \(n\) vertices.

output: a \((P_q, S_r)\) representation of \(T\).

Identify all \(j\) pendant paths \(P_{q_1}, \ldots, P_{q_j}\) of \(T\).

for \(i\) from 1 to \(j\) do
    if \(q_i = 1\) then replace \(P_{q_i}\) with gpp \(P_1 \ast S_0\);
    if \(q_i = 2\) then replace \(P_{q_i}\) with gpp \(P_0 \ast S_1\);
    if \(q_i \geq 3\) then replace \(P_{q_i}\) with gpp \(P_{q_i-2} \ast S_1\);
end do

return \(T\).

**Figure 5.** Procedure \(\text{InitiateRepresentation}\).

While the \((P_q, S_r)\) representation of a tree changes during the transformation, the following invariants are preserved by \(\text{ReduceStarVertex}\). Leaves and only leaves are square representing \(S_r\), and pendant paths appear to have length one, representing \(P_q\), \(q \geq 0\), and where \(r + q \geq 1\). That is, gpps are on the tree’s extremities.

For trees with a small number of starlike vertices, we have a different strategy. In fact, if a tree \(T\) of order \(n \geq 8\) has \(k = 0, 1\) or 2 starlike vertices, we prove in Section 7 that \(T\) can be properly transformed into \(T_\alpha\), where \(n \equiv \alpha \pmod{4}\). These results, along with the main procedure \(\text{ReduceStarVertex}(T, u_1)\), are going to be described later (in Sections 6 and 7), as they need some technical analysis.

We finally review our main tool to prove the conjecture. It is the algorithm reproduced in Figure 6. For any tree \(T\) with \(n\) vertices, it produces a diagonal matrix \(D\) that is congruent to the matrix \(L + xI_n\), where \(L\) is the Laplacian matrix of \(T\).

This algorithm, presented first in [6], is the Laplacian matrix version of the adjacency matrix algorithm [13] that has been useful in many applications of spectral graph theory (see, for example, the recent ordering by the spectral radius [18, 11] of certain trees). One can show that

**Lemma 3.1.** The number of eigenvalues of \(T\) less (greater) than \(x\) is exactly the number of negative (positive) diagonal elements produced by \(\text{Diagonalize}(T, -x)\).

**Example 3.2.** We illustrate here how the algorithm may be executed on the tree itself. Considering the star with \(n > 2\) vertices \(K_{1, n-1}\) with \(n - 1\) leaves and a the center vertex of degree \(n - 1\). The tree can be rooted anywhere, but we choose the center vertex as the root. A vertex \(v\) is initialized with \(\text{deg}(v) + x\). When \(x = -2 + \frac{2}{n}\), the initial values at the leaves is \(-1 + \frac{2}{n} < 0\) and the initial value at the root is \(n - 3 + \frac{2}{n}\). The values at the leaves are kept, while the value at the root changes to

\[
 n - 3 + \frac{2}{n} - \frac{n - 1}{-1 + \frac{2}{n}} > 0.
\]

By Lemma 3.1 it implies there are \(n - 1\) Laplacian eigenvalues smaller than the average degree and a single eigenvalues above it.
Algorithm Diagonalize\( (T, x) \)

initialize \( d(v) := \text{deg}(v) + x \) for all vertices \( v \)

for \( k = 1 \) to \( n \)

if \( v_k \) is a leaf then continue

else if \( d(c) \neq 0 \) for all children \( c \) of \( v_k \) then

\[ d(v_k) := d(v_k) - \sum \frac{1}{d(c)}, \text{ summing over children} \]

else

select one child \( v_j \) of \( v_k \) for which \( d(v_j) = 0 \)

\[ d(v_k) := -\frac{1}{2}, \quad d(v_j) := 2 \]

if \( v_k \) has a parent \( v_l \), remove the edge \( v_k v_l \).

end loop

Figure 6. \( L + xI_n \) diagonalization

The technique we use to prove that for any tree \( T \) with \( n \) vertices, the number of Laplacian eigenvalues greater than \( 2 - \frac{2}{n} \) is at most \( \lfloor \frac{n}{2} \rfloor \), is the analysis of the signs on \( T \) after applying the algorithm \( \text{Diagonalize}(T, -2 + \frac{2}{n}) \).

4. Proper Transformations

In this section we present a few local transformations which are performed on a tree \( T \) that preserve the number of vertices and do not decrease the number of eigenvalues above the average degree. Such proper transformations are local and for this reason we can translate this property in terms of elementary rational recursions.

We start by analyzing the signs of the vertices after applying \( \text{Diagonalize}(T, -2 + \frac{2}{n}) \) on a tree having \( r \) pendant \( P_2 \)'s attached to a path as in Figure 7. We assume \( n \geq 8 \) is the number of vertices in \( T \) and \( 0 \leq r \leq \lfloor \frac{n}{2} \rfloor \) is the number of pendant paths \( P_2 \). In fact, using our notation established above, this is a generalized pendant path \( P_q \ast S_r \).

We use a white dots to indicate vertices where \( \text{Diagonalize}(T, -2 + \frac{2}{n}) \) produces a negative value, black dots indicate a positive value and light gray where we do not know the precise sign.

Figure 7. A tree end with a generalized pendant path \( P_q \ast S_r \).

Applying the algorithm to the Laplacian matrix to locate \( x = d_n \) we obtain, in each extremal vertex of the pendant path \( P_2 \) the value

\[ x_1 = 1 - d_n = -1 + \frac{2}{n} < 0 \]
and the next value is
\[ x_2 = 2 - d_n - \frac{1}{x_1} = \frac{2}{n} - \frac{1}{x_1} > 1. \]
For completeness, we may consider the recurrence relation

\[
\begin{aligned}
  x_1 &= -1 + \frac{2}{n} \\
  x_{j+1} &= \frac{2}{n} - \frac{1}{x_j}
\end{aligned}
\]
From these values we proceed processing the vertices of the path \( P_q \) in Figure 7 obtaining
\[ b_1 = r + 1 - d_n - \frac{r}{x_2} = x_1 + r \left( 1 - \frac{1}{x_2} \right), \]
and the rest of the values on the path are given by the recursion

\[
\begin{aligned}
  b_1 &= x_1 + r \left( 1 - \frac{1}{x_2} \right) \\
  b_{j+1} &= \frac{2}{n} - \frac{1}{b_j}
\end{aligned}
\]
for \( n \in [8, \infty) \). As the sequence \( b \) depends on \( r \) we denote \( b_j := b_j(r) \) for \( r \geq 0 \). We observe that if \( r = 0 \), then \( b_j = x_j \) and if \( r = 1 \), then \( b_j = x_{j+2} \). We can summarize the main properties of these two sequences in the following lemma.

**Lemma 4.1.** Consider the above defined sequences \( x_j \) and \( b_j \) then

a) \(-1 < x_1 < 0 \) and \( x_2 > 1 \);

b) \( x_1 \leq b_1(r) \), for all \( r \), with strict inequality when \( r \geq 1 \).

c) If \( r = 0 \) then \( b_1(r) = x_1 \) and for \( r = 1 \) we have \( b_1(r) = x_3 \);

d) The map \( r \to b_1(r) \) is linear, strictly increasing from \( \mathbb{N} \to [x_1, \infty) \);

e) \( b_1(r + 1) - b_1(r) = 1 - \frac{1}{x_2} \) for all \( r \).

**Proof.** All five claims are straightforward computations. For example, for (c) we know that \( x_1 = -1 + \frac{2}{n} \), \( x_2 = \frac{2}{n} - \frac{1}{x_1} \), \( b_1(0) = x_1 + 0(1 - \frac{1}{x_2}) = x_1 \) and \( b_1(1) = x_1 + (1 - \frac{1}{x_2}) = \frac{2}{n} - \frac{1}{x_2} = x_3 \).

Our first concern is the dependence of the initial condition \( b_1(r) \) with respect to \( r \).

**Lemma 4.2.** Let \( n \geq 8 \). If \( 0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor \) then \( b_1(r) < 0 \).

**Proof.** Assume \( n \geq 8 \). Recall that \( b_1(r) = x_1 + r(1 - \frac{1}{x_2}) \). On the non-negative reals, define the linear function into \( \mathbb{R} \)
\[ g(r) = x_1 + r \left( 1 - \frac{1}{x_2} \right). \]
Then \( g(0) = x_1 < 0 \). Since \( g'(r) = (1 - \frac{1}{x_2}) > 0 \), it is increasing. By continuity there is a unique point \( g(r_0) = 0 \). Solving for \( r_0 \) we have \( r_0 = \frac{-x_1}{1 - \frac{1}{x_2}} > 0 \). Since \( x_1 = -\frac{n-2}{n} \) and \( x_2 = \frac{n^2+2n-4}{n^2-2n} \), \( r_0 \) depends rationally on \( n \)
\[ r_0 = \frac{(n-2)(n^2+2n-4)}{4n(n-1)}. \]
Therefore \( b_1(r) < 0 \) if and only if \( r \leq \lfloor r_0 \rfloor \). We claim that \( \frac{n}{4} < r_0 \) for \( n \geq 7 \). Indeed, the inequality \( \frac{n}{4} < r_0 \) can be simplified to \( 0 < n^2 - 8n + 8 \) whose largest root is \( 4 + 2\sqrt{2} \approx 6.8 \). \( \square \)

The proof of Lemma 4.2 shows that the inequality holds when \( n \geq 7 \), however it will be more convenient to assume \( n \geq 8 \).

**Corollary 4.3.** Let \( T \) be a tree with \( n \geq 8 \) vertices and \( u \) a vertex having \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor \) pendant \( P_2 \)'s and a path \( P_q \) with \( q \geq 2 \). Then \( b_2(r) > 0 \).

**Proof.** We know from Lemma 4.2 that \( b_1(r) < 0 \) for \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor \) and by the recurrence relation \( b_2(r) = \frac{2}{n} - \frac{1}{b_1(r)} \) we conclude that \( b_2(r) > 0 \). \( \square \)

We observe that these results do not account for the sign associated with the vertex where the gpp \( P_q \ast S_r \) is attached. We now present the first proper transformation that we use.

**Proposition 4.4.** (Star-up transform) Let \( u \) be a vertex that is not a leaf of a tree \( T \) with \( n \geq 8 \) vertices. If \( u \) has a path \( P_q \), \( q \geq 2 \) connecting \( u \) to a vertex that has exactly \( 0 \leq r \leq \lfloor \frac{n}{4} \rfloor - 1 \) pendant \( P_2 \), and no other pendant path, then the transformation \( P_q \ast S_r \rightarrow P_{q-2} \ast S_{r+1} \)

is proper.

**Figure 8.** Star-up transform

**Proof.** We consider the transformation on a tree \( T \), as illustrated in Figure 8. This takes one pendant path \( P_2 \) at the vertex \( u \) connected to the sun \( S_r \) formed by \( r P_2 \) paths, and produces a new tree \( T' \) with a sun \( S_{r+1} \) attached at \( u \). We consider \( u \) as the root of \( T \), meaning that it is going to be the last vertex to be processed.

The signs at the vertices of the branch of \( T' \) not containing \( S_{r+1} \) are the same as the signs in \( T \). By Lemma 4.2 and Corollary 4.3 we know that \( b_1 < 0 \) and \( b_2 > 0 \). Hence after applying the transformation, there are exactly the same number of negative signs in \( P(u) = P_2 \ast S_r \) of \( T \) as in \( P(u) = P_0 \ast S_{r+1} \) of \( T' \). Hence, to prove that the transformation is proper, we need to compare \( f_T(u) \) and \( f_{T'}(u) \), the values obtained by the application of the algorithm Diagonalize \( (T, -2 + \frac{2}{n}) \) and Diagonalize \( (T', -2 + \frac{2}{n}) \).
Applying the algorithm we obtain
\[
\begin{align*}
    f_T(u) &= \deg_T(u) - d_n - \xi - \frac{1}{b_2(r)} \\
    f_T(u) &= \deg_T(u) + r - d_n - \xi - \frac{r + 1}{x_2} \\
    &= \deg_T(u) + r(1 - \frac{1}{x_2}) - d_n - \xi - \frac{1}{x_2},
\end{align*}
\]
where \(\xi\) is the processing of the part not affected by the transformation. Therefore, using the value of \(b_1(r)\) and \(b_2(r)\) in (2) and \(x_2\) in (1)
\[
f_T'(u) - f_T(u) = \frac{1}{b_2(r)} - \frac{1}{x_2} + r(1 - \frac{1}{x_2}) \\
    = \frac{1}{b_2(r)} - \frac{1}{x_2} - x_1 + x_1 + r(1 - \frac{1}{x_2}) \\
    = \frac{1}{b_2(r)} - \frac{1}{x_2} - x_1 + b_1(r) \\
    = (b_1(r) + \frac{1}{b_2(r)}) - (x_1 + \frac{1}{x_2}) \\
    = \frac{1}{\frac{2}{n} - b_1(r)} - (x_1 + \frac{1}{\frac{2}{n} - x_1}).
\]
We need to show that (3) non-negative for \(0 \leq r \leq \lfloor \frac{n}{4} \rfloor - 1\). Consider the function
\[
g(t) = t + \frac{1}{\frac{2}{n} - t} - (x_1 + \frac{1}{\frac{2}{n} - x_1}).
\]
We see that \(g(x_1) = 0\). By Lemma 4.1 and Lemma 4.2 one has \(-1 < x_1 \leq b_1(r) < 0\). We need to show that and \(g(t)\) is positive on the interval \((x_1, 0)\). Now, \(g'(t) = 1 + \frac{1}{(\frac{2}{n} - t)^2} > 0\) so \(g(t)\) is strictly increasing and, hence \(g(t) > 0\) in the desired interval. This shows that for each \(r \leq \lfloor \frac{n}{4} \rfloor - 1\), one has \(g(b_1(r)) \geq 0\), and therefore \(f_T(u) \leq f_T'(u)\). Therefore, the number of positive values produced by \textbf{Diagonalize}(\(T', -2 + \frac{2}{n}\)) either equals or exceeds by one, the number of positive values in \textbf{Diagonalize}(\(T', -2 + \frac{2}{n}\)), showing the Star-up transformation is proper. \(\square\)

\textbf{Figure 9.} Repeated applications of Star-up
Example 4.5. We consider the proper transformation of a tree $T$ with a path $P_9$ into a new tree $T'$ with the replacement of $P_9$ by $P_1 * S_4$, by successive applications of the Star-up transform. See Figure 9 for an illustration.

We now present the second proper transformation that we will use.

![Diagram]

**Figure 10. Star-down.**

Proposition 4.6. (Star-down transform) Consider a transformation on a tree $T$ that takes one pendant path $P_1$ on a vertex $u$ connected to a sun $S_r$ and another pendant path $P_2$ on the vertex $u$ and produces a new tree $T'$ with a sun $S_{r+1}$ attached in $P_1$. If $0 \leq r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1$ then

$$P_1 * S_r \oplus P_2 \rightarrow P_1 \oplus S_{r+1}$$

is a proper transformation.

Proof. We root both trees at $u$. We apply the diagonalization algorithm on $T$ and $T'$ and notice that the number of positive vertices remains unchanged except at the vertex $u$ (as $r \leq \left\lfloor \frac{n}{4} \right\rfloor - 1$ we know that $b_1(r) < 0$). Therefore we must compare $f_T(u)$ and $f_{T'}(u)$. We will have $\sigma(T) \leq \sigma(T')$ if $f_T(u) \leq f_{T'}(u)$. Applying the algorithm we obtain

$$f_T(u) = \deg_T(u) - d_n - \xi - \frac{1}{b_1(r)} - \frac{1}{x_2}$$

$$f_{T'}(u) = \deg_{T'}(u) - 1 - d_n - \xi - \frac{1}{b_1(r + 1)}$$

where $\xi$ is the processing of the part not affected by the transform. Using Lemma 4.1, part e

$$f_{T'}(u) - f_T(u) = -\left(1 - \frac{1}{x_2}\right) + \left(\frac{1}{b_1(r)} - \frac{1}{b_1(r + 1)}\right)$$

$$= \left(1 - \frac{1}{x_2}\right) \left(-1 + \frac{1}{b_1(r)b_1(r + 1)}\right)$$

(4)

Using the properties in Lemma 4.1 and Lemma 4.2 $-1 < b_1(r) < b_1(r + 1) < 0$ and $x_2 > 1$, and so (4) is positive. This completes the proof. \qed

The third transformation we use is the following.
Proposition 4.7. (Star-star transform) Suppose $u$ has two generalized pendant paths $P_{q_1} \ast S_{r_1}$ and $P_{q_2} \ast S_{r_2}$, where $q_1, q_2 \in \{0, 1\}$. The following transformations are proper:

a) If $q_1 = q_2 = 0$, for any $r_1', r_2'$ such that $r_1' + r_2' = r_1 + r_2$;

\[
(P_0 \ast S_{r_1}) \oplus (P_0 \ast S_{r_2}) \rightarrow (P_0 \ast S_{r_1'}) \oplus (P_0 \ast S_{r_2'}),
\]

b) If $q_1 = q_2 = 1$ and $0 \leq r_1, r_2 \leq \lfloor \frac{n}{4} \rfloor$;

\[
(P_1 \ast S_{r_1}) \oplus (P_1 \ast S_{r_2}) \rightarrow \begin{cases} 
P_2 \ast S_{r_1} + r_2 & \text{if } r_1 + r_2 \leq \lfloor \frac{n}{4} \rfloor \\
P_0 \ast S_{r_1 + r_2 - \lfloor \frac{n}{4} \rfloor} \oplus (P_2 \ast S_{\lfloor \frac{n}{4} \rfloor}) & \text{if } r_1 + r_2 > \lfloor \frac{n}{4} \rfloor
\end{cases}
\]

c) If $q_1 = 1$, $q_2 = 0$ and $0 \leq r_1, r_2 \leq \lfloor \frac{n}{4} \rfloor$;

\[
(P_1 \ast S_{r_1}) \oplus (P_0 \ast S_{r_2}) \rightarrow \begin{cases} 
P_1 \ast S_{r_1} + r_2 & \text{if } r_1 + r_2 \leq \lfloor \frac{n}{4} \rfloor \\
P_0 \ast S_{r_1 + r_2 - \lfloor \frac{n}{4} \rfloor} \oplus (P_1 \ast S_{\lfloor \frac{n}{4} \rfloor}) & \text{if } r_1 + r_2 > \lfloor \frac{n}{4} \rfloor
\end{cases}
\]

Proof. The case $q_1 = 0$ and $q_2 = 0$ (see Figure 11) is actually a formal rearrangement. We do not change the graph, only perform a different partition of the $r_1 + r_2$ original...
$P_2$s attached to $u$. In this case it is not necessary to consider the sign of any vertices. It is a trivial Star-star transform, shown in Figure 11.

We notice that by Corollary 4.3, the case $q_1 = 1$ and $q_2 = 1$ (see Figure 12) is clearly proper because the number of positive signs (black vertices) increases by one. Therefore, it does not matter what happens in $u$, the transformation is proper.

The case $q_1 = 1$ and $q_2 = 0$ (see Figure 13) is actually a particular application of several Star-down transformations (Proposition 4.6). In each step, a $P_2$ from the $P_0*S_{r_2}$ is brought down to the gpp $P_1*S_{r_1}$. Thus the Star-star transformation is proper. □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.pdf}
\caption{An example of a 1-0 Star-star transform. On top $\left\lfloor \frac{n}{4} \right\rfloor = 6$. On bottom $\left\lfloor \frac{n}{4} \right\rfloor = 9$.}
\end{figure}

5. Equality for prototype trees

In Section 6 we present a reduction procedure that properly transforms several gpp’s attached to a starlike vertex to a single gpp, leading to an eventual decrease on the number of starlike vertices of a tree. Then, in Section 7 we show that a tree $T$ of order $n$ with $k = 0, 1$ or $2$ starlike vertices can be properly transformed into a prototype tree $T_\alpha$, where $n \equiv \alpha \pmod{4}$. These prototype trees $T_\alpha$ are defined as follows.

**Definition 5.1.** Let $r \geq 2$ and $\alpha \in \{0, 1, 2, 3\}$. For $n = 4r + \alpha \geq 8$, define the tree $T_\alpha$ of order $n$ given in Figure 14.

If $T$ is properly transformed to $T_\alpha$, that is $\sigma(T) \leq \sigma(T_\alpha)$, in order to prove the conjecture is true, it will remain to prove that $\sigma(T_\alpha) \leq \left\lfloor \frac{n}{2} \right\rfloor$. As a warm up application of the results of the previous section, we prove here this result. In fact, we prove that $T_\alpha$ satisfy the equality in Theorem 1.1 that is $\sigma(T_\alpha) = \left\lfloor \frac{n}{2} \right\rfloor$. 
Theorem 5.2. For \( r \geq 2 \), \( \alpha \in \{0, 1, 2, 3\} \), let \( n = 4r + \alpha \), and \( d_n = 2 - \frac{\alpha}{n} \). Then

(a) \( m_{T_0}([0, d_n]) = 2r \);
(b) \( m_{T_1}([0, d_n]) = 2r + 1 \);
(c) \( m_{T_2}([0, d_n]) = 2r + 1 \);
(d) \( m_{T_3}([0, d_n]) = 2r + 2 \).

In particular \( m_{T_\alpha}([0, d_n]) = \left\lceil \frac{n}{2} \right\rceil \), or equivalently, \( \sigma(T_\alpha) = \left\lfloor \frac{n}{2} \right\rfloor \).

Proof. We first observe that in all cases \( \frac{n}{4} = r + \frac{\alpha}{4} \), and as \( 0 \leq \alpha/4 < 1 \), we have \( r = \left\lfloor \frac{n}{4} \right\rfloor \), so therefore when applying Diagonalize\( (T_\alpha, -2 + \frac{\alpha}{n}) \), the value \( b_1 < 0 \), by Lemma 4.2. We used Maple to compute \( f(u) \) for each \( \alpha \), as shown below.

(a) We observe that \( T_0 = P(u) = P_0 \ast S_{r-1} \oplus P_1 \ast S_r \), and \( n = 4r \). In \( P_0 \ast S_{r-1} \) there are \( r-1 \) negative signs and \( r-1 \) positive signs. Due to the observation above, in \( P_1 \ast S_r \) there are \( r+1 \) negative signs and \( r \) positive. To show equality we need to show that the value at the vertex \( u \) is positive. Then

\[
f(u) = \deg u - d_n - (r-1) \left( \frac{1}{x_2} - \frac{1}{b_1} \right) \\
= r - (2 - \frac{2}{4r}) - \left( \frac{r-1}{x_2} - \frac{1}{b_1} \right) \\
= r - 2 + \frac{1}{2r} - \frac{r-1}{x_2} - \frac{1}{b_1}.
\]
One can write $x_2 = \frac{4r^2 + 2r - 1}{(2r)(2r - 1)}$ and $b_1 = \frac{-2r^2 + 4r - 1}{2r(4r^2 + 2r - 1)}$. Substituting into (5) we obtain

$$f(u) = \frac{1}{2} \frac{64r^6 + 64r^5 - 36r^4 + 36r^3 - 32r^2 + 10r - 1}{r (4r^2 + 2r - 1) (2r^2 - 4r + 1)}.$$ 

A plot of $f(u)$ shows it is positive for all $r \geq 2$. This proves that $\sigma(T_0) = \lfloor \frac{n}{2} \rfloor$.

(b) We observe that $T_1 = P(u) = P_0 \ast S_r \oplus P_0 \ast S_r$, and $n = 4r + 1$. In $P_0 \ast S_r$ there are $r$ negative and $r$ positive signs, accounting for a total of $2r$ positive and $2r$ negative signs. We must show $f(u)$ is negative. Indeed,

$$f(u) = \deg u - d_n - 2r \frac{1}{x_2} = 2r - (2 - \frac{2}{n}) - \frac{2r}{x_2},$$

Letting $n = 4r + 1$, one can show $x_2 = \frac{16r^2 + 16r - 1}{16r^2 - 1}$. Substituting into the right side of (6) we obtain

$$f(u) = -8 \frac{r (12r - 1)}{(4r + 1) (16r^2 + 16r - 1)},$$

which is negative for $r \geq 2$.

(c) $T_2 = P(u) = P_0 \ast S_r \oplus P_1 \ast S_r$. There are $2r + 1$ negative signs and $r$ positive signs except at the vertex $u$. Here $n = 4r + 2$.

$$f(u) = r + 1 - d_n - \frac{r}{x_2} - \frac{1}{b_1}$$

Expressing $n$, $x_2$ and $b_1$ in terms of $r$, and substituting in $f(u)$, leads to

$$f(u) = \frac{64r^6 + 256r^5 + 348r^4 + 260r^3 + 95r^2 + 16r + 1}{(2r + 1) (4r^2 + 6r + 1) r (6r + 1)}$$

which is clearly positive proving that $\sigma(T_2) = \lfloor \frac{n}{2} \rfloor$.

(d) Proceeding similarly as in the previous case, $T_3 = P(u) = P_1 \ast S_r \oplus P_1 \ast S_r$, we obtain $2r + 2$ negative signs and $2r$ positive signs. To show equality we need to show that the value at $u$ is positive. Here $n = 4r + 3$, and

$$f(u) = 2 - d_n - \frac{2}{b_1}$$

Writing $n$ and $b_1$ in terms of $r$, and substituting into $f(u)$, yields

$$f(u) = \frac{4(128r^4 + 448r^3 + 576r^2 + 302r + 55)}{(4r + 3) (64r^2 + 52r + 11)}$$

which is clearly positive showing that $\sigma(T_3) = 2r + 1 = \lfloor \frac{n}{2} \rfloor$. 

\qed
6. Reduction: Starlike vertices

In some cases a starlike vertex \( u \) can be properly transformed in such a way that its generalized pendant paths are reduced to a single generalized pendant path. This is possible when the weight of \( u \) is small, more precisely when \( w(u) \leq 2\lfloor n/4 \rfloor \).

**Theorem 6.1.** Consider a tree \( T \) in \((P_q, S_r)\) representation and \( u \) a starlike vertex with \( \ell \geq 2 \) generalized pendant paths, or \( P(u) = P_{q_1} * S_{r_1} \oplus \cdots \oplus P_{q_\ell} * S_{r_\ell} \). If \( w(u) \leq 2\lfloor n/4 \rfloor \) then we can properly transform \( T \) to \( T' \) obtaining

\[
P_{q_1} * S_{r_1} \oplus \cdots \oplus P_{q_\ell} * S_{r_\ell} \Rightarrow P_{q'} * S_{r'}
\]

where

\[
\begin{align*}
q' &\equiv \sum q_i \pmod{2} \in \{0, 1\} \\
r' &= \frac{w(u) - q'}{2} \leq \lfloor n/4 \rfloor
\end{align*}
\]

**Proof.** The proof is by induction on \( \ell \). For \( \ell = 2 \) assume

\[
P(u) = P_{q_1} * S_{r_1} \oplus P_{q_2} * S_{r_2},
\]

where \( w(u) \leq 2\lfloor n/4 \rfloor \). Then we may write

\[
w(u) = q_1 + 2r_1 + q_2 + 2r_2 = \alpha_1 + \alpha_2 + 2(k_1 + k_2 + r_1 + r_2)
\]

where \( \alpha_i \equiv q_i \pmod{2} \) and \( k_i = \lfloor n/4 \rfloor \) for \( i = 1, 2 \). Since \( r_1 + k_1 < \frac{w(u)}{2} \leq \lfloor n/4 \rfloor \), by Proposition 4.4 the Star-up transform

\[
P_{q_1} * S_{r_1} \Rightarrow P_{q_1-2} * S_{r_1+1}
\]

can be performed \( k_1 \) times. Hence, the transformation

\[
P_{q_1} * S_{r_1} \Rightarrow P_{\alpha_1} * S_{r_1+k_1}
\]

is proper, and similarly,

\[
P_{q_2} * S_{r_2} \Rightarrow P_{\alpha_2} * S_{r_2+k_2}
\]

is proper. As \( \alpha_1, \alpha_2 \in \{0, 1\} \) by Proposition 4.7 a Star-star operation can be performed since \( r_1 + k_1 + r_2 + k_2 \leq \lfloor n/4 \rfloor \) producing

\[
P_{\alpha_1} * S_{r_1+k_1} \oplus P_{\alpha_2} * S_{r_2+k_2} \Rightarrow P_{\alpha_1 + \alpha_2} * S_{r_1+k_1+r_2+k_2}.
\]

If \( q' = \alpha_1 + \alpha_2 \in \{0, 1\} \) we are done, otherwise, if \( \alpha_1 + \alpha_2 = 2 \) we can perform an additional Star-up transformation

\[
P_{\alpha_1 + \alpha_2} * S_{r_1+k_1+r_2+k_2} \Rightarrow P_0 * S_{r_1+k_1+r_2+k_2+1}
\]

because \( 2(r_1 + k_1 + r_2 + k_2 + 1) = w(u) \leq 2\lfloor n/4 \rfloor \), and so \( r_1 + k_1 + r_2 + k_2 \leq \lfloor n/4 \rfloor - 1 \).

To complete the induction, assume the theorem is true for \( \ell \). To obtain the correctness for \( \ell + 1 \), first transform

\[
P_{q_1} * S_{r_1} \oplus \cdots \oplus P_{q_\ell} * S_{r_\ell} \Rightarrow P_{q'} * S_{r'},
\]

and then repeat the proof for \( \ell = 2 \) on \( P_{q'} * S_{r'} \oplus P_{q_{\ell+1}} * S_{r_{\ell+1}} \).

\[\square\]
ReduceStarVertex($T, u$)

**input:** a tree $T$ with $n$ vertices  
  a starlike vertex $u$ with $P(u) = P_{q_1} \ast S_{r_1} + \cdots + P_{q_\ell} \ast S_{r_\ell}$.  
  precondition $w(u) \leq 2\lfloor \frac{n}{4} \rfloor$

**output:** a tree $T'$ where the gpps at $u$ are replaced by a single gpp.

Let $w = w(u)$  
Compute $q' = \sum q_i \pmod{2}$.  
Compute $r' = \frac{w - q'}{2}$.  
Replace in $T$ all gpps at $u$ with $P(u) = P_{q'} \ast S_{r'}$, forming $T'$.  
if $\deg_T(u) = 1$ then  
  find $v$, the nearest vertex to $u$ having $\deg_T(v) > 2$  
  remove $u$ and path to $v$  
create gpp $P(v) = P_{q''} \ast (P_q \ast S_{r'}) = P_{q'' + q'} \ast S_{r'}$, where $d(u, v) = q''$
return $T'$.

**Figure 15.** Procedure ReduceStarVertex.

Theorem 6.1 above justifies the introduction of our Procedure ReduceStarVertex shown in Figure 15 which replaces all the generalized pendant paths at a starlike vertex by a single gpp. It also labels the new generalized pendant path with $P_{q'} \ast S_{r'}$ calculated in the procedure. If $u$ is a leaf in the new tree, we remove it and collapse the path to the nearest vertex $v$ whose degree is greater than two, creating the gpp $P_{q'' + q'} \ast S_{r'}$ at $v$ where $d(u, v) = q''$. If the procedure were applied to $u_2$ in the tree of Figure 19(left), for example, we would first form $P_1 \ast S_1$ and then the path of length one would be collapsed creating the gpp $P_2 \ast S_1$.

As this example illustrates, a vertex is no longer starlike or is removed after the transformation. Hence the number of starlikes may reduce by one. However a new starlike vertex may be created. As an example, consider applying ReduceStarVertex at vertex $u_1$ of Figure 19(left). Initially $P(u_1) = P_1 \ast S_0 + P_1 \ast S_0$, having weight 2. After applying the procedure we obtain $P(u_1) = P_1 \ast S_1$. Now $u_1$ is no longer starlike, but its neighbor $v$ now has $P(v) = P_1 \ast S_1 + P_1 \ast S_0$ with weight 4. In general, this happens when the body of the if-statement in ReduceStarVertex is executed, and the vertex $v$ already had a single gpp. The if-statement is not always executed as Figures 21 and 22 show. A full example of the reduction procedure is executed at the end of this paper in Section 8.

We claim that ReduceStarVertex will always be called with a vertex $u$ satisfying the precondition $w(u) \leq 2\lfloor \frac{n}{4} \rfloor$. Note it is only called when there are at least three starlike vertices, and is passed the one with smallest weight. It is easy to see that if the number of starlike vertices is $k \geq 3$, at least one of them has weight $w \leq 2\lfloor \frac{n}{4} \rfloor$. For suppose by contradiction there were three starlike vertices with weight $2\lfloor \frac{n}{4} \rfloor$. The sum of their weights is bounded by $n$, thus $6\lfloor \frac{n}{4} \rfloor \leq n$. But this is impossible when $n \geq 4$.

The correctness of algorithm Transform depends on not only showing each procedure is correct, but also showing that it halts. Clearly it halts if and only if its while loop halts. Being a local transformation, ReduceStarVertex only operates on a starlike vertex $u$, and can not create more than one new starlike vertex since $u$ is adjacent (or has a path) to only one other vertex. Hence the number of starlike vertices does not
increase by the application of ReduceStarVertex. Moreover, once a new starlike vertex is created, its weight includes the weight of \( u \) and, hence, the total weight of the starlike vertices increase. Therefore Transform must stop, as the total weight is bounded by \( n \).

7. Reduction: Small number of starlike vertices

In this section we show that a tree with fewer than three starlike vertices can be properly transformed into a prototype tree. We introduce a new notation for the entire tree. Here

\[ T = v + X \oplus Y \]

denotes a tree in which \( v \) is the root, \( X \) and \( Y \) are gpps attached to \( v \). Using this notation, \( u + P_{q'} \ast S_{r'} \oplus P_{q''} \ast S_{r''} \) and \( u + P_a \ast S_{r} \oplus P_b \ast S_{r'} \) are different representations of the same tree, provided \( q' + q'' = a + b \).

Given a tree in \( (P_q, S_r) \) representation, recall that a starlike vertex is a vertex \( u \) of degree \( \geq 3 \) having at least two generalized pendant paths \( P_q \ast S_r \). In this section we deal with trees that have up to 2 starlike vertices. We always assume that \( n = 4 \left\lceil \frac{4}{3} \right\rceil + \alpha \) and \( \alpha = 0, 1, 2, 3 \). Let \( k \geq 0 \) be the number of starlike vertices in this representation. We first handle the case \( k = 0 \).

**Lemma 7.1.** \( T \) has a starlike vertex iff it has a vertex \( v \), deg\((v)\) \( \geq 3 \), not in a gpp.

**Proof.** The ‘only if’ is trivial. Conversely, assume \( T \) has a vertex \( v \), deg\((v)\) \( \geq 3 \), not part of a gpp. Note that trees in \( (P_q, S_r) \) representation have squares for leaves, and pendant paths are represented by each gpp, and appear to have length one, although these paths represent a path \( P_q \). Indeed this property exists at initialization and is maintained by ReduceStarVertex. Consider the tree \( \tilde{T} \) of \( T \) obtained by converting the square leaves to normal vertices and removing all labels. Since \( q + r \geq 1 \), \( \tilde{T} \) is a subtree of \( T \), and its pendant paths of length one correspond to gpps in \( T \). This tree also has a vertex of degree greater than 2, so it is not a path. By Lemma 2.1 it has a vertex \( u \) having degree at least 3 with at least two pendant paths. This must be a starlike vertex in \( T \). \( \square \)

From Lemma 7.1 if \( T \) does not have a starlike vertex, then it is a path with a gpp on each end, and we may write

\[ T = u + P(u) = u + P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2}. \]

An illustration of a general form of this tree is given in Figure 16.

![Figure 16. The case k = 0 starlike vertices.](image-url)
Theorem 7.2. Let \( T \) be a tree of order \( n \) having no starlike vertices. Then \( T \) can be properly transformed into \( T_\alpha \), \( \alpha \in \{0, 1, 2, 3\} \) according to \( n \equiv \alpha \pmod{4} \).

Proof. Assuming that \( n = 4r + \alpha \) and \( r = \left\lfloor \frac{n}{4} \right\rfloor \), we divide our proof in two cases.

Case 1: \( r_1, r_2 \geq r \). It follows that \( w(P_{q_1} S_{r_1} \oplus P_{q_2} S_{r_2}) + 1 = n = 4r + \alpha \) or
\[
q_1 + 2r_1 + q_2 + 2r_2 + 1 = 4r + \alpha.
\]

Using the assumption of Case 1, we conclude that \( q_1 + q_2 \leq \alpha - 1 \). Let us analyze the possibilities of \( \alpha \).

Note that \( \alpha = 0 \) is impossible because \( q_1 + q_2 \geq 0 \). When \( \alpha = 1 \), we must have \( q_1 = q_2 = 0 \), and then \( T \) is already \( T_1 \).

Consider next \( \alpha = 2 \). This implies that \( q_1 + q_2 \leq 1 \). A possible solution is \( q_1 = q_2 = 0 \). This is not feasible, as otherwise \( 1 + 2r_1 + 2r_2 = n \equiv 1 \pmod{2} \) but \( n \) is even when \( \alpha = 2 \). The other possible solution is \( q_1 = 0 \) and \( q_2 = 1 \), leading to \( 1 + 1 + 2r_1 + 2r_2 = n = 4r + \alpha = 4r + 2 \), or \( 2r = r_1 + r_2 \). As \( r_1, r_2 \geq r \), we have \( r_1 = r_2 = r \). We conclude that \( T = u + P_0 S_r \oplus P_1 S_r = T_2 \), as we claimed. We notice that \( q_1 = 1 \) and \( q_2 = 0 \) also leads to \( T_2 \) by symmetry.

Finally consider \( \alpha = 3 \). This implies that \( q_1 + q_2 \leq 2 \) and the possibilities are
\[
\begin{align*}
(i) \quad & q_1 = q_2 = 0, \\
(ii) \quad & q_1 = q_2 = 1, \\
(iii) \quad & q_1 = 0 \text{ and } q_2 = 1, \\
(iv) \quad & q_1 = 0 \text{ and } q_2 = 2.
\end{align*}
\]

Because \( 1 + 1 + 2r_1 + 2r_2 = n \) is even we see that (iii) is not feasible. If (ii) happens, we are faced to \( 1 + q_1 + 2r_1 + q_2 + 2r_2 = 3 + 2r_1 + 2r_2 = n = 4r + 3 \), or \( r_1 + r_2 = 4r \). And as before, \( r_1 = r_2 = 2r \) or \( T = u + P_1 S_r \oplus P_1 S_r = T_3 \), as claimed. If (iv) happens, a counting argument similar to (ii) shows that \( T = u + P_0 S_r \oplus P_2 S_r \). If we root \( T \) at the middle vertex of \( P_2 \) rather than \( u \) it has the form \( P_3 \).

We consider now the case (i) \( q_1 = q_2 = 0 \). As \( n = 4r + 3 \), it implies \( r_1 + r_2 = 2r + 1, r_1 = r \) and \( r_2 = r + 1 \) because \( r_1, r_2 \geq r \). We consider the transformation described in Figure 17 which transforms a tree \( T_* = v + P_0 S_r \oplus P_0 S_{r+1} \) with \( n = 4r + 3 \) vertices into the tree \( T_3 = u + P_1 S_r \oplus P_1 S_r \). Using the Diagonalize algorithm in both trees we have the knowledge of all signs except that of \( v \) in \( T_* \) (we know the signs in \( T_3 \) from Theorem 5.2).

Let us call \( f_{T_*}(v) \) the value obtained by the Diagonalize algorithm

\[
f_{T_*}(v) = (2r + 1) - (2 - \frac{2}{n}) - (2r + 1) \frac{1}{x_2} = -2 + \frac{2}{n} + (2r + 1) \left( 1 - \frac{1}{x_2} \right)
\]

Figure 17. On the left the tree \( T_* \) on the right side the tree \( T_3 \).
where $\deg_T(v) = 2r + 1$. Analyzing the correspondence $r \rightarrow f_T(v)$, using Maple, we expressed $x_2$ in terms of $r$, obtaining rational function

$$-4 \frac{24r^2 + 22r + 5}{(4r + 3)(16r^2 + 32r + 11)}$$

which is always negative for $r \geq 2$, thus $f_T(v) < 0$ and we paint this vertex as white. Counting the black vertices in $T$, we see that $\sigma(T) = 2r + 1$ and we already know that $\sigma(T_3) = 2r + 1$. Therefore this transformation is proper.

**Case 2:** $r_1 < r$ or $r_2 < r$. Let us assume, say, that $r_1 < r$. If $q_1 + q_2 \geq 2$ we can rewrite $T$ as

$$v + P_{q_1+q_2} \ast S_{r_1} \oplus P_0 \ast S_{r_2}.$$ 

Then, using a Star-up, we may transform $T \rightarrow v + P_{q_1+q_2-2} \ast S_{r_1+1} \oplus P_0 \ast S_{r_2}$. More generally, performing several Star-up transformations we may assume that $T$ is transformed into

$$T' = v + P_{q'_1} \ast S_{r'_1} \oplus P_{q'_2} \ast S_{r'_2}$$

with either

(a) $r'_1, r'_2 \geq r$ or
(b) $r'_1 < r$ or $r'_2 < r$ and $q'_1 + q'_2 \leq 1$.

If (a) happens the result follows by the previous Case 1. We may suppose then $r'_1 < r$ and $q'_1 + q'_2 \leq 1$. The possibilities are the following.

- $q'_1 = q'_2 = 0$, which corresponds to $T' = v + P_0 \ast S_{r'_1} \oplus P_0 \ast S_{r'_2}$ and $n = 1 + 2r'_1 + 2r'_2$ is odd. Thus $n = 4r + 1$ or $n = 4r + 3$, corresponding to $\alpha = 1$ or $\alpha = 3$. If $\alpha = 1$, then $4r + 1 = 1 + 2r'_1 + 2r'_2$ which is equivalent to $r'_1 + r'_2 = 2r$. A Star-star transformation produces the tree $v + P_0 \ast S_r \oplus P_0 \ast S_r = T_1$. If $\alpha = 3$, then $4r + 3 = 1 + 2r'_1 + 2r'_2$ which is equivalent to $r'_1 + r'_2 = 2r + 1$. A Star-star transformation produces a tree $v + P_0 \ast S_r \oplus P_0 \ast S_{r+1} = T_r$. As we saw in Case 1 above, this tree can be properly transformed into $T_3$.

- $q'_1 = 0$ and $q'_2 = 1$, which corresponds to $T' = v + P_0 \ast S_{r'_1} \oplus P_1 \ast S_{r'_2}$ and so $n = 1 + 2r'_1 + 1 + 2r'_2 = 2 + 2r'_1 + 2r'_2$ is even, implying $n = 4r$ or $n = 4r + 2$, which corresponds to $\alpha = 0$ or $\alpha = 2$. If $\alpha = 0$, then $4r = 2 + 2r'_1 + 2r'_2$ which is equivalent to $r'_1 + r'_2 = 2r - 1$. Using the $0 - 1$ Star-star transformation we can obtain $T_0 = v + P_0 \ast S_{r-1} \oplus P_1 \ast S_r$. If $\alpha = 2$, then $4r + 2 = 1 + 2r'_1 + 1 + 2r'_2$ which is equivalent to $r'_1 + r'_2 = 2r$. Using the $0 - 1$ Star-star transformation we can obtain $T_2 = v + P_0 \ast S_r \oplus P_1 \ast S_r$. 

We now turn our attention to a tree $T$ with a single starlike vertex. By Lemma 7.1, there is exactly one vertex of degree $\geq 3$ not in a gpp. Let us nominate $u$ to be this starlike vertex, then $T$ has the form

$$T = u + P(u) = u + P_{q_1} \ast S_{r_1} \oplus \cdots \oplus P_{q_L} \ast S_{r_L},$$

for $L \geq 3$. An example is given in Figure 18 (left).

**Theorem 7.3.** Let $T$ be a tree of order $n$ having a single starlike vertex. Then $T$ can be properly transformed into $T_\alpha$, $\alpha \in \{0, 1, 2, 3\}$ according to $n \equiv \alpha \pmod{4}$. 

implying other gpps. This implies that meaning that particular tree. We have α in L − 1. Case 1: l would have n possible. Let us analyze then the possible values of l transform contradicting the fact that we have made these transformations as much as possible.

In this proof we will use ReduceStarVertex to simply combine several gpps into a single gpp, but we will not execute the if-statement inside it.

Without loss of generality assume the gpps are ordered in non-decreasing \( r_i \). Since \( L \geq 3 \) there are three positive indices in \( \{L - 2, L - 1, L\} \). We claim that there exists an index \( l_0 \in \{L - 2, L - 1, L\} \) such that \( r_i \leq r - 1 \) if and only if \( i \leq l_0 \). Moreover \( q_i \in \{0, 1\} \) when \( i \leq l_0 \). To see this, we observe that if \( r_i \geq r \) for \( L - 2, L - 1 \) and \( L \), we would have \( n \geq 6r \), a contradiction. And if for some \( i \leq l_0 \) \( q_i \geq 2 \) we could use Star-up transform contradicting the fact that we have made these transformations as much as possible. Let us analyze then the possible values of \( l_0 \).

**Case 1:** \( l_0 = L - 2 \). Here \( r_{L-1}, r_L \geq r \), meaning that these two gpps have weight at least \( 4r \).

Taking into account the root \( u \), there are at least \( 4r + 1 \) vertices. This means that \( \alpha = 0 \) is impossible. If \( \alpha = 1 \), we would have \( n = 4r + 1 \), leaving nothing for the first \( L - 2 \) gpps, an impossibility.

If \( \alpha = 2 \), we have \( n = 4r + 2 \) and there is a single vertex to be distributed to the other gpps. This implies that \( \mathcal{T} = u + P_1 * S_0 \oplus P_0 * S_r \oplus P_0 * S_r \). Let \( v \) be the vertex in \( P_1 \) having degree one. By using Star-down \( r \) times, we can move \( r \) of the \( P_2 \)’s to \( v \), and the tree is properly transformed into \( u + P_1 * S_r \oplus P_0 * S_r \), as we want.

Finally, if \( \alpha = 3 \), we have \( n = 4r + 3 \), which gives more freedom for the choice of the particular tree. We have

\[
\begin{align*}
n = 4r + 3 & = 1 + w(P_{q_1} * S_{r_1} \oplus \ldots \oplus P_{q_0} * S_{r_0}) + q_{L-1} + 2r_{L-1} + q_L + 2r_L \\
& \geq 1 + 1 + q_{L-1} + 2r_{L-1} + q_L + 2r_L \\
& \geq 2 + q_{L-1} + q_L + 4r,
\end{align*}
\]

implying \( 1 \geq q_{L-1} + q_L \). The possible cases choices for \( q_{L-1} \) and \( q_L \) are \( q_{L-1} = q_L = 0 \), implying

\[
\begin{align*}
n = 4r + 3 & = 1 + w(P_{q_1} * S_{r_1} \oplus \ldots \oplus P_{q_0} * S_{r_0}) + 2(r_{L-1} + r_L) \\
& \geq 1 + w(P_{q_1} * S_{r_1} \oplus \ldots \oplus P_{q_0} * S_{r_0}) + 4r,
\end{align*}
\]

meaning that \( w(P_{q_1} * S_{r_1} \oplus \ldots \oplus P_{q_0} * S_{r_0}) \leq 2 \), leading to the following possibilities:

(i) \( P_{q_1} * S_{r_1} \oplus \ldots \oplus P_{q_0} * S_{r_0} = P_1 * S_0 \)
(ii) $P_{q_1} \ast S_{r_1} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}} = P_0 \ast S_1$

(iii) $P_{q_1} \ast S_{r_1} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}} = P_1 \ast S_0 \oplus P_1 \ast S_0$.

Subcase (i) is impossible because $n$ is odd. If (ii) happens then the result follows from the analysis of the previous case, namely using $r$ Star-down transformations we can achieve $T_3$.

If case (iii) happens, then $T = u + P_1 \ast S_0 \oplus P_1 \ast S_0 \oplus P_0 \ast S_{r_{L-1}} \oplus P_0 \ast S_{r_L}$ and by counting the number of vertices since $r_{L-1} - r_L \geq r$, we must have $r_{L-1} = r_L = r$. Now, for each pair $P_1 \ast S_0 \oplus P_0 \ast S_r$ we apply Star-down transforming into $P_1 \ast S_r$, transforming $T$ into $u + P_1 \ast S_r \oplus P_1 \ast S_r = T_3$, as we wish.

If $q_{L-1} = 1$ and $q_L = 0$, proceeding as before, we obtain that the only possibility for $P_{q_1} \ast S_{r_1} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}$ is $P_1 \ast S_0$. This means that $T = u + P_1 \ast S_0 \oplus P_0 \ast S_{r_{L-1}} \oplus P_1 \ast S_{r_L}$. As $r_{L-1} + r_L = 2r$ and $r_{L-1}, r_L \geq r$ and we must have $r_{L-1} = r_L = r$. Apply Star-star to the first two summands getting $P_1 \ast S_r$. Now $T = T_3$.

**Case 2:** $l_0 = L - 1$. Here only $r_L \geq r$, meaning that the gpp $P_{q_L} \ast S_{r_L}$ has weight at least $2r$. Taking into account the root $u$, there are at least $2r + 1$ vertices. The whole tree is $T = u + P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{L-1}} \ast S_{r_{L-1}} \oplus P_{q_L} \ast S_{r_L}$ whose weight is $n$. We analyze the possible values of $\alpha$.

If $\alpha = 0$, then

$$n = 4r = 1 + w(P_1 \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{L-1}} \ast S_{r_{L-1}}) + w(P_{q_L} \ast S_{r_L})$$

$$\geq 1 + w(P_1 \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) + 2r,$$

meaning that

$$w(P_1 \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) \leq 2r - 1.$$  

We then may apply **ReduceStarVertex** transforming it into a single gpp $P_{q'} \ast S_{r'}$ and

$$T = u + P_{q'} \ast S_{r'} \oplus P_{q_L} \ast S_{r_L}.$$  

Now $T$ has no starlike vertices and then by Theorem [7.2] we can transform $T$ into $T_0$.

If $\alpha = 1$, proceeding as in the previous paragraph, we see that

$$w(P_1 \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) \leq 2r.$$  

Hence we can still apply **ReduceStarVertex**, so that the tree has a single gpp. Now the tree has no starlike vertices and by Theorem [7.2] $T$ is properly transformed into $T_1$.

For $\alpha = 2$, the similar analysis allows one to conclude that

$$w(P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) \leq 2r + 1.$$  

If the left of (8) is $\leq 2r$, **ReduceStarVertex** can be applied and, as before, Theorem [7.2] shows that $T$ is properly transformed into $T_2$. Therefore we may assume equality occurs in (8). This means that $q_L = 0$ and $r_L = r$ and so

$$T = u + (P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) \oplus P_0 \ast S_r.$$  

As $L \geq 3$, we see that $l_0 \geq 2$. Moreover, there exists at least an index $i \in \{1, \ldots, l_0\}$ such that $q_i$ is 1 for, otherwise $n = 4r + 2$ would be odd. Consider the gpps

$$(P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \ldots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) - P_{q_i} \ast S_{r_i}.$$
Since their weight $\leq 2r$, we can apply $\text{ReduceStarVertex}$ transforming $T$ into

$$u + P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus P_0 \ast S_r.$$

As $w(P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2}) = 2r + 1$, it follows that $r_i + r' = r$. Since $q_i = 1$, by parity we must have $q' = 0$. Therefore we apply Star-star transformations to convert $P_1 \ast S_{r_1} \oplus P_0 \ast S_{r'}$ into $P_1 \ast S_r$ reducing $T$ to $u + P_1 \ast S_r \oplus P_0 \ast S_r = T_2$.

For $\alpha = 3$, we have $n = 4r + 3$

$$= 1 + w(P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \cdots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}) + q_L + 2r_L$$

Now, we look at $P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \cdots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}$ and claim that there exists at least one gpp $P_{q_i} \ast S_{r_j}$ with $r_j \geq 1$ for, otherwise all the gpps have the form $P_1 \ast S_0$ and then we can take two of them and transform into $P_0 \ast S_1$. Consider the tree

$$u + P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \cdots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}.$$

Its weight of the last tree is $\leq 2r$. Hence we can apply $\text{ReduceStarVertex}$ transforming it into a single gpp $P_{q_i} \ast S_{r_j}$ and $T = u + P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \cdots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}$, which by Theorem 7.2 may be transformed into $T_3$.

If $q_i \geq 2$, it is easy to show that $w(P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1}) \leq 2r$ and then, by applying $\text{ReduceStarVertex}$, we transform it into a gpp and $T = u + P_{q_1} \ast S_{r_1} \oplus P_{q_2} \ast S_{r_2} \oplus \cdots \oplus P_{q_{i_0}} \ast S_{r_{i_0}}$, which by Theorem 7.2 can be transformed into $T_3$.

If $q_i = 1$ then $w(P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1}) \leq 2r + 1$ and, as before, we only need to consider when $w(P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1}) = 2r + 1$. From this we conclude that $r_L = r$ and $T = u + P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1} \oplus P_1 \ast S_r$. Since $q_j + 2r_j + q' + 2r' = 2r + 1$, and $q_j, q' \in \{0, 1\}$, by parity we conclude that $q_j + q' = 1$ and so $r_j + r' = r$. Thus we can apply Star-star transforming $P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1}$ into $P_1 \ast S_r$ and $T = u + P_1 \ast S_r \oplus P_1 \ast S_r = T_3$.

If $q_i = 0$, then $T = u + P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1} \oplus P_0 \ast S_{r_L}$, where $1 \leq r_j \leq r - 1$, $r_L \geq r$, and $q_j, q' \in \{0, 1\}$. We transform this into $u + P_{q_i} \ast S_{r_j} \oplus P_{q_1} \ast S_{r_1} \oplus P_1 \ast S_{r_L-(r-r_j)}$, using Star-star. Since $n$ is odd, either $q_j = q' = 0$ or $q_j = q' = 1$. In the first case $T = u + P_0 \ast S_{r-r'+r_L-(r-r_j)} = u + P_0 \ast S_{r_j+r'+r-L}$. From $n = 4r + 3 = 1 + 2(r_j + r' + r_L)$, we deduce that $r_j + r' + r_L = 2r + 1$, so this is the tree $T_4$ in Figure 17 that was transformed into $T_3$. If $q_j = q' = 1$, the summands are $P_1 \ast S_r \oplus P_1 \ast S_{r'} \oplus P_0 \ast S_{r-L-(r-r_j)}$. It is easy to see that $r = r' + r_L - (r - r_j)$, so $P_1 \ast S_{r'} \oplus P_0 \ast S_{r_L-(r-r_j)} \Rightarrow P_1 \ast S_r$, obtaining $T_3$.

**Case 3:** $l_0 = L$. In this case we have

$$T = u + P(u) = u + P_{q_1} \ast S_{r_1} \oplus \cdots \oplus P_{q_L} \ast S_{r_L},$$

with

$$q_i \in \{0, 1\} \text{ and } r_i \leq r - 1, \text{ for all } i = 1, \ldots, L.$$

We claim that any tree having $L \geq 3$ gpps and satisfying the conditions in (9) can be reduced to a prototype tree. Our proof is by induction on $L$. Note that the weight of any pair $P_{q_1} \ast S_{r_1} \oplus P_{q_1} \ast S_{r_1}$ of gpps is at most $2r$. When $L = 3$ we apply $\text{ReduceStarVertex}$ to any pair of gpps transforming the tree to one having only two gpps. By Theorem 7.2 the result follows. Now assume $L > 3$ and that all trees of having fewer gpps and
satisfying \((9)\) can be transformed to a prototype tree. Apply \texttt{ReduceStarVertex} to any pair of gpps, obtaining a new gpp \(P_q' \ast S_r'\). This reduces the number of the number of gpps to \(L - 1\). If \(r' \leq r - 1\) we are done by induction. On the other hand if \(r' \geq r\) then the tree may be transformed by Case 2. \hfill \square

We finally handle the case of trees having two starlike vertices.

**Theorem 7.4.** Let \(T\) be a tree of order \(n\) having two starlike vertices. Then \(T\) is properly transformed into \(T_n\), \(\alpha \in \{0, 1, 2, 3\}\) according to \(n \equiv \alpha \pmod{4}\).

**Proof.** Let \(u\) and \(v\) be the two distinct starlike vertices of \(T\). If the weight of both \(u\) and \(v\) exceed \(2r\), we see that \(n \geq 1 + 2 + 2r + 1 + 2r + 1 = 4r + 5 = n = 4r + \alpha\). Thus at least \(u\) or \(v\) has weight \(\leq 2r\). Let us say \(u\) is such a vertex and we can use \texttt{ReduceStarVertex} to reduce all the gpps’s attached at \(u\) to single gpp. After this reduction, we have two possibilities. Either the obtained gpp is attached to \(v\) and the transformed \(T\) has a single starlike vertex, in which case the result follows by Theorem 7.3. Or we still have two starlike vertices and the one (say) \(u'\), is such that \(w(u') \geq w(u) + 1\). We repeat the process to vertices \(u'\) and \(v\). As the weight of the starlike vertices strictly increase, we observe that the procedure must terminate with a single starlike vertex, otherwise both vertices would have weight larger than \(2r\), an impossibility. \hfill \square

### 8. A COMPLETE EXAMPLE

Consider the tree \(T\) given by Figure 19. This tree has \(n = 53\) vertices. Since \(53 \equiv 1 \pmod{4}\) we are going to properly transform it into the prototype tree \(T_1\). We notice that, in this case, \(r = \left\lfloor \frac{n}{4} \right\rfloor = 13\). Thus we can always apply \texttt{ReduceStarVertex}(\(T, u\)) provided that \(w(u) \leq 26\).

![Figure 19](image-url)

The initialization step of algorithm \texttt{Transform} is to apply the procedure \texttt{InitiateRepresentation} to the tree \(T\) of Figure 19 obtaining the \((P_q, S_r)\) representation illustrated in Figure 19 (left). The second step is to identify the starlike vertices. There are five, \(u_1, u_2, u_3, u_4, u_5\) in increasing order of weights 2, 3, 4, 6, 8. As prescribed by the Algorithm \texttt{Transform}(\(T\)), for \(k = 5 \geq 3\) we should apply \texttt{ReduceStarVertex}(\(T, u_1\)), which is \(P(u_1) = P_1 \ast S_0 \oplus P_1 \ast S_0\). After \texttt{ReduceStarVertex}(\(T, u_1\)), the gpps at \(u_1\) are reduced to \(P(u_1) = P_0 \ast S_1\). However \(q\) gets incremented by one in the if-statement of \texttt{ReduceStarVertex} so the new gpp is \(P_1 \ast S_1\). The starlike vertex \(u_1\) is
eliminated producing a new starlike vertex with weight 4 which is labeled as \( u_2 \), where 
\[
P(u_2) = P_1 \ast S_0 \oplus P_1 \ast S_1.
\]
The previous \( u_2 \) is labeled as the new \( u_1 \). This is illustrated in Figure 19 (right).

In Figure 19 (right) we still have \( k = 5 \geq 3 \) so we apply \texttt{ReduceStarVertex}(\( T, u_1 \)). After \texttt{ReduceStarVertex}(\( T, u_1 \)), the starlike vertex \( u_1 \) is transformed to \( P(u_1) = P_1 \ast S_1 \).

Now, in Figure 20 (right), we have \( k = 4 \geq 3 \) and then after applying \texttt{ReduceStarVertex}(\( T, u_1 \)), we obtain a new starlike vertex \( P_1 \ast S_2 \oplus P_2 \ast S_1 \), whose weight is 9. Hence, the number of starlike vertices remains the same and the remaining starlike vertices are relabeled according to its weights, as in Figure 20 (right).

![Figure 20.](image)

Now, in Figure 20 (right), we have \( k = 4 \) and then after applying \texttt{ReduceStarVertex}(\( T, u_1 \)), the ggp \( P_1 \ast S_2 \) is created. The starlike vertex \( u_1 \) is eliminated while a new starlike vertex is created having weight equal 7. Then it is relabeled as \( u_2 \) again and \( u_2 \) is relabeled as \( u_1 \). Only \( u_3 \) and \( u_3 \) remain unchanged, as Figure 21 (left) illustrates.

We still have \( k = 4 \) and apply \texttt{ReduceStarVertex}(\( T, u_1 \)), creating the ggp \( P_2 \ast S_3 \). The starlike vertex \( u_1 \) is eliminated and a new starlike vertex is created. Counting the weights we label the new starlike vertices as \( u_1, u_2, u_3 \) and \( u_4 \) with weights 7, 8, 9 and 17, as in Figure 21 (right).

![Figure 21.](image)

In Figure 21 (right), \( k \) is four. Applying \texttt{ReduceStarVertex}(\( T, u_1 \)), eliminates the starlike vertex \( u_1 \). The remaining three starlike vertices will be labeled as \( u_1, u_2 \) and \( u_3 \) with weights 8, 9 and 17, as in Figure 22 (left).
We now have $k = 3$ then we still apply $\text{ReduceStarVertex}(T, u_1)$. The starlike vertex $u_1$ is eliminated but a new starlike vertex is created as $P(u_2) = P_1*S_3 \oplus P_1*S_4$. Counting the weights we label new starlike vertices $u_1$, $u_2$ and $u_3$ with weights 9, 16 and 17. This is illustrated in Figure 22 (right).

In Figure 22 (right), we have $k = 3$. After $\text{ReduceStarVertex}(T, u_1)$, the starlike vertex $u_1$ is just eliminated and the remaining two starlike vertices will be labeled as $u$ and $v$ with weights 16 and 17, as Figure 23 (left) illustrates.

Since $k = 2$, according to Theorem 7.4 we should apply $\text{ReduceStarVertex}(T, u)$, because $w(u) = 16 \leq 2r = 26$. After $\text{ReduceStarVertex}(T, u)$, the starlike vertex $u$ is eliminated but a new starlike vertex is created. Counting the weights, the remaining two starlike vertices will be labeled as $u$ and $v$ with weights 17 and 20, Figure 23 (right).

In Figure 23 (right), we have $k = 2$, and by the proof of Theorem 7.4, because $w(u) = 17 \leq 2r = 26$, we should apply $\text{ReduceStarVertex}(T, u)$. After $\text{ReduceStarVertex}(T, u)$, the starlike vertex $u$ is eliminated but a new starlike vertex is created. Counting the weights, the remaining two starlike vertices will be labeled as $u$ and $v$ with weights 20 and 27, respectively, as in Figure 24 (left).
We still have $k = 2$ so we should apply $\text{ReduceStarVertex}(T, u)$, according to Theorem 7.4 because $w(u) = 20 \leq 26$. After $\text{ReduceStarVertex}(T, u)$, the starlike vertex $u$ is eliminated but a new starlike vertex is created. The remaining two starlike vertices will be labeled as $u$ and $v$ with weights 24 and 27, as illustrated in Figure 24 (right).

In Figure 24 (right), $k$ remains at two, so we should apply $\text{ReduceStarVertex}(T, u)$, according to Theorem 7.4 because $w(u) = 24 \leq 2r = 26$. After this transformation the starlike vertex $u$ is eliminated. There remains only one starlike vertex which will be labeled $u$ and $T = u + P_2 * S_8 \oplus P_1 * S_4 \oplus P_1 * S_{12}$, which is illustrated in Figure 25 (left).

We now have $k = 1$. Then, by Theorem 7.3, we should apply the transformation Star-up where it is possible. In this case $P_2 * S_8 \rightarrow P_0 * S_9$. Therefore, $T$ is transformed to $u + P_0 * S_9 \oplus P_1 * S_4 \oplus P_1 * S_{12}$. Here, $L = 3$ and $\ell_0 = 3$ because $r = 13$. As prescribed by Case 3 of Theorem 7.3 we perform $\text{ReduceStarVertex}$ on two summands, $P_0 * S_9 \oplus P_1 * S_4 \rightarrow P_1 * S_{13}$, as Figure 25 (right) illustrates.

Finally, we have $T = u + P_1 * S_{13} \oplus P_1 * S_{12}$, and $k = 0$ meaning that there are no starlike vertices anymore. As $r_2 < r$ we are in Case 2 of Theorem 7.2. Recall that we change the root from $u$ to the vertex $v$ obtaining $T' = v + P_0 * S_{13} \oplus P_2 * S_{12}$. Then we apply Star-up in $P_2 * S_{12} \rightarrow P_0 * S_{13}$. We reached our goal. That is, $T_1 = v + P_0 * S_{13} \oplus P_0 * S_{13}$ as expected. These final transformations are illustrated in Figure 26.

**Acknowledgments**

VT acknowledges partial support of CNPq grants 409746/2016-9 and 303334/2016-9, CAPES under project MATHAmSud 88881.143281/2017-01 and FAPERGS under project PqG 17/2551-0001.

**References**

[1] Belardo, F., Oliveira, E. R., and Trevisan, V. Spectral ordering of trees with small index. *Linear Algebra and its Applications* 575 (2019), 250 – 272.
[2] Braga, R. O., Rodrigues, V. M., and Trevisan, V. On the distribution of Laplacian eigenvalues of trees. *Discrete Math.* 313, 21 (2013), 2382–2389.

[3] Carvalho, J. B. Staring at the spectrum of graphs. http://www2.mat.ufrgs.br/~carvalho/pesquisa/interativa/graphenergy/graphspec.php, 2009.

[4] Das, K. C., Gutman, I., and Mojallal, S. A. On Laplacian energy in terms of graph invariants. *Applied Mathematics and Computation* 268 (2016), 83 – 92.

[5] Das, K. C., Mojallal, S. A., and Trevisan, V. Distribution of Laplacian eigenvalues of graphs. *Linear Algebra and its Applications* 508, 1 (2016), 48 – 61.

[6] Fritscher, E., Hoppen, C., Rocha, I., and Trevisan, V. On the sum of the Laplacian eigenvalues of a tree. *Linear Algebra and its Applications* 435 (2011), 371–399.

[7] Grone, R., and Merris, R. The Laplacian spectrum of a graph. II. *SIAM J. Discrete Math.* 7, 2 (1994), 221–229.

[8] Grone, R., Merris, T., and Sunder, V. The Laplacian spectrum of a graph. *SIAM Journal on Matrix Analysis and Applications* 11, 2 (1990), 218–238.

[9] Guo, J.-M. The 4th Laplacian eigenvalue of a tree. *J. Graph Theory* 54, 1 (2007), 51–57.

[10] Guo, J.-M. On limit points of Laplacian spectral radii of graphs. *Linear Algebra Appl.* 429, 7 (2008), 1705–1718.

[11] Guo, J. M., Wu, X. L., Zhang, J. M., and Fang, K. F. On the distribution of Laplacian eigenvalues of a graph. *Acta Mathematica Sinica, English Series* 27, 11 (Oct 2011), 2259.

[12] Hedetniemi, S. T., Jacobs, D. P., and Trevisan, V. Domination number and Laplacian eigenvalue distribution. *European J. Combin.* 53 (2016), 66–71.

[13] Jacobs, D. P., and Trevisan, V. Locating the eigenvalues of trees. *Linear Algebra Appl.* 434, 1 (2011), 81–88.

[14] Jacobs, D. P., and Trevisan, V. A conjecture on Laplacian eigenvalues of trees. In *Graph Theory – Favorite Conjectures and Open Problems*, R. Gera, T. Haynes, and S. Hedetniemi, Eds., vol. II of *Problem Books in Mathematics*. Springer Verlag, 2017.

[15] Merris, R. The number of eigenvalues greater than two in the Laplacian spectrum of a graph. *Portugaliae mathematica* 48, 3 (1991), 345–349.

[16] Mohar, B. Laplace eigenvalues of graphs a survey. *Discrete Mathematics* 109, 1 (1992), 171 – 183.

[17] Mohar, B. On the Laplacian coefficients of acyclic graphs. *Linear Algebra and its Applications* 422, 2 (2007), 736 – 741.

[18] Oliveira, E., Stevanović, D., and V. Trevisan. Spectral radius ordering of starlike trees. *Linear and Multilinear Algebra* (2018).

[19] Pirzada, S., and Ganie, H. A. On the Laplacian eigenvalues of a graph and Laplacian energy. *Linear Algebra and its Applications* 486 (2015), 454 – 468.

[20] Trevisan, V., Carvalho, J. B., Del Vecchio, R. R., and Vinagre, C. T. M. Laplacian energy of diameter 3 trees. *Appl. Math. Lett.* 24, 6 (2011), 918–923.

[21] Zhou, L., Zhou, B., and Du, Z. On the number of Laplacian eigenvalues of trees smaller than two. *Taiwanese J. Math.* 19, 1 (2015), 65–75.

School of Computing, Clemson University, Clemson, USA

E-mail address: dpj@clemson.edu

Instituto de Matemática e Estatística, UFRGS, Porto Alegre, Brazil

E-mail address: elismar.oliveira@ufrgs.br

Instituto de Matemática e Estatística, UFRGS, Porto Alegre, Brazil

E-mail address: trevisan@mat.ufrgs.br