Status of quarkonia - like negative and positive parity states in a relativistic confinement scheme

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Abstract Properties of quarkonia - like states in the charm and bottom sector have been studied in the framework of relativistic Dirac formalism with a linear confinement potential. We have computed the mass spectroscopy and decay properties (vector decay constant and leptonic decay width) of several quarkonia - like states. Present study is also intended to identify some of the unexplained states as mixed P-wave and mixed S-D wave states of Charmonia and Bottomonia. The results indicate that the X(4140) state can be an admixture of two P states of charmonium. And the charmonium like states X(4630) and X(4660) are the admixed state of S-D waves. Similarly, the X(10610) state recently reported by Belle II can be a mixed P - states of bottomonium. In the relativistic framework we have computed vector decay constant and the leptonic decay width for S wave charmonium and bottomonium. The leptonic decay width for the $J^{PC} = 1^{--}$ mixed states are also predicted. Further, both the masses and the leptonic decay width are considered for the identification of the quarkonia-like states.

Keywords Heavy quarkonia · Relativistic quark model · Mesons

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1 Introduction

In recent years remarkable experimental progress has been achieved in the investigation of charmonium-like and bottomonium-like states. The latest experimental results on heavy flavour hadrons have gained renewed interest in heavy flavor Physics [1,2] to understand the properties of strongly interacting hadrons. Conditions seemed to be very different for spectra above and below flavor threshold region. In the region above open charm threshold, number of charmonium-like states (the so-called ”X Y Z” states) have been discovered with unusual properties. These states might be exotic states, mesonic molecules or multi quark states [1].

Most of these unknown states do not fit in the standard charmonium and bottomonium spectra [3,4]. All the narrow charmonium states below the open-charm threshold have been observed experimentally and their mass spectrum can be well described by potential models [5]. We have sufficient knowledge of $\eta_c(1S)$ and $\eta_c(2S)$. The BESIII/BEPCII facility in Beijing, has shed more light on these spin-singlet states by collecting a new record of $\psi(3686)$ decays in electron-positron annihilations [6]. Recently, BESIII showed that the Y(4260) is split up in to two resonant states: one with a mass of 4222.0 ± 3.1 ± 1.4 MeV/c$^2$ and the other with a mass of 4320.0 ± 10.4 ± 7.0 MeV/c$^2$ in their cross section measurement of $e^+e^- \rightarrow J/\psi$ for center of mass energies from $\sqrt{s} = 3.77$ to 4.60 GeV [7]. Large amount of data on charmonium and bottomonium production is available at RHIC [8,9,10,11,12] and at the LHC [13,14,15,16,17,18,19] significantly extending our understanding of quarkonium production in deconfined matter [20]. To understand all these, we have to go beyond the conventional quark or quark anti-quark bound systems. There are various issues related to higher excited states which are still to be resolved. In this context, phenomenological models either non-relativistic quark model (NRQM) or the relativistic quark model have been developed to study the properties of heavy mesons (Charmonium and Bottomonium) [21,22,23].
1.1 $J^P = 1^-$ States

In ISR (Initial-State Radiation) process, BaBar observed peaks near 4300 $MeV/c^2$ in the $\pi^+\pi^- J/\psi$ and $\pi^+\pi^- \psi'$ channels. The partial widths for these two decay channels are larger than that required to observe charmonium states. As these states are produced via the ISR process, they have $J^P = 1^-$. Some of these $1^-$ states are listed in Table 1.

| Exp. State | Exp. mass (MeV) | $J^P$ | Process (mode) | Experiment |
|------------|-----------------|-------|----------------|------------|
| Y(4008)   | 4008$^{+123}_{-9}$ | $1^-$ | $e^+e^- \to \gamma(\pi^+\pi^- J/\psi)$ | Belle [26] |
| $\psi(4160)$ | 4191$^{+5}_{-6}$ | $1^-$ | $e^+e^- \to \eta J/\psi$ | Belle [27] |
| Y(4220)   | 4222.0$^{+3.1}_{-1.4}$ | $1^-$ | $e^+e^- \to \gamma(\pi^+\pi^- J/\psi)$ | BESIII [4] |
| Y(4260)   | 4263$^{+9}_{-3}$ | $1^-$ | $e^+e^- \to (\pi^+\pi^- J/\psi)$ | BABAR [28,29] CLEO [30], Belle [26] |
| Y(4330)   | 4320.0$^{+10.4}_{-7.0}$ | $1^-$ | $e^+e^- \to (\pi^+\pi^- J/\psi)$ | BESIII [7] |
| X(4630)   | 4631$^{+13}_{-1}$ | $1^-$ | $e^+e^- \to (\pi^+\pi^- J/\psi)$ | CLEO [31] |
| Y(4660)   | 4664$^{+12}_{-1}$ | $1^-$ | $e^+e^- \to (\pi^+\pi^- J/\psi)$ | CLEO [30,31] |
| Y$B_{\pi}$(10888) | 10888.4$^{+3.0}_{-1.0}$ | $1^-$ | $e^+e^- \to \gamma(\pi^+\pi^- J/\psi)$ | Belle [33] |
| X(10610)  | 10609$^{+4.0}_{-1}$ | $1^+$ | $e^+e^- \to T(2S)/T(3S)/T(4S)/T(5S)$ | Belle [34] |
| $h_c(1P)$ | 3525.41$^{+0.16}_{-0.14}$ | $1^+$ | $\psi(2S) \to \gamma \nu \bar{\nu}$ | CLEO [35,36] |
| $h_c(2P)$ | 10259.8$^{+1.5}_{-1.2}$ | $1^+$ | $\gamma(5S) \to \pi^+\pi^-$ | Belle [35,36] |
| X(3940)   | 3942$^{+5}_{-6}$ | $2^-$ | $e^+e^- \to J/\psi X$ | Belle [37] |
| X(4020)   | 4025.5$^{+2.9}_{-3.1}$ | $2^+$ | $e^+e^- \to (D^+ D^0)\pi^0$ | BESIII [7] |
| X(4410)   | 4413$^{+2.9}_{-1.2}$ | $2^+$ | $B^+ \to J/\psi K^+$ | CDF [38] |
| X(4550)   | 4350.6$^{+4.6}_{-5.1}$ | $2^+$ | $e^+e^- \to e^+e^- J/\psi$ | BELL [2] |

Table 1: Experimental status of some of the negative parity and positive parity quarkonia - like states.
resolved negative parity and positive parity states. The summary and conclusion of the present study is presented in Sec.V.

2 Theoretical framework

One of the most successful way to construct the quarkonium system is to solve Dirac equation for the quark and anti quark in a confinement potential. For the present study we have considered the confinement through a linear potential. The form of the model potential is expressed as,

\[ V(r) = \frac{1}{2}(1 + \gamma_0)(\lambda r^{1.0} + V_0) \]  

(1)

Where, \( \lambda \) is the strength of the confinement part of the potential \[43\]. \( V_0 \) is a constant negative potential depth \[44,45,46\].

The wave function which satisfy Dirac equation with a general potential is given by \[47,48\],

\[ (\alpha \cdot p + m_Q)\psi_q(r) = [E_q - V(r)\gamma_0] \psi_q(r), \]  

(2)

\[ [\gamma^0E_q - \alpha \cdot p - m_q - V(r)] \psi_q(r) = 0, \]  

(3)

where

\[ \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} ; \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \]  

(4)

\( V(r) \) is a potential which consist scalar + vector part. Main feature to use scalar plus vector potential is that it is applicable for the bound states of both mesons and baryons \[43\].

The solution of Dirac equation can be written as two component (positive and negative energies in the zeroth order) form as \[48,44,45,46\],

\[ \psi_{nlj}^+(r) = \begin{pmatrix} \psi_{nlj}^{(+)} \\ \psi_{nlj}^{(-)} \end{pmatrix} \]  

(5)

where

\[ \psi_{nlj}^{(+)}(r) = N_{nlj} \left( \frac{i g(r)/r}{(\sigma \cdot \hat{r})f(r)/r} \right) Y_{lm}(\hat{r}) \]  

(6)

\[ \psi_{nlj}^{(-)}(r) = N_{nlj} \left( \frac{i (\sigma \cdot \hat{r})f(r)/r}{g(r)/r} \right) (-1)^{l+m_j} Y_{lm}(\hat{r}) \]  

(7)

and \( N_{nlj} \) is the overall normalization constant \[48,44,45,46\]. The normalized spin angular part is expressed as

\[ Y_{lm}(\hat{r}) = \sum_{m_s} \sum_{m_s} \frac{1}{l^m} Y_{lm}^m Y_{lm}^{m_s} \]  

(8)

Here the spinor \( \chi_{\frac{1}{2}m_s} \) are eigenfunctions of the spin operators \[48,44,45,46\],

\[ \chi_{\frac{1}{2}+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi_{\frac{1}{2}-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  

(9)

The reduced radial part \( g(r) \) and \( f(r) \) of the Dirac spinor \( \psi_{nlj}(r) \) are the solutions of the equations given by \[48,44,45,46\],

\[ \frac{d^2g(r)}{dr^2} + \left[ (E_D + m_q)[E_D - m_q - V(r)] - \frac{\kappa(\kappa + 1)}{r^2} \right] g(r) = 0 \]  

(10)

and

\[ \frac{d^2f(r)}{dr^2} + \left[ (E_D + m_q)[E_D - m_q - V(r)] - \frac{\kappa(\kappa - 1)}{r^2} \right] f(r) = 0 \]  

(11)

it is appropreate to define a new quantum number \( \kappa \) \[48,44,45,46\] as,

\[ \kappa = \begin{cases} -(l + 1) = -(j + \frac{1}{2}) & \text{for } j = l + \frac{1}{2} \\ l = (j + \frac{1}{2}) & \text{for } j = l - \frac{1}{2} \end{cases} \]  

(12)

On converting these equation into dimensionless form \[47,41,45,46\] as,

\[ \frac{d^2g(\rho)}{d\rho^2} + \left[ \epsilon - \rho^{1.0} - \frac{\kappa(\kappa + 1)}{\rho^2} \right] g(\rho) = 0 \]  

(13)

\[ \frac{d^2f(\rho)}{d\rho^2} + \left[ \epsilon - \rho^{1.0} - \frac{\kappa(\kappa - 1)}{\rho^2} \right] f(\rho) = 0 \]  

(14)

where \( \rho = \frac{r}{r_0} \) is a dimensionless variable with suitably chosen scale factor \( r_0 = \frac{r}{(E+\epsilon)\lambda} \) and corresponding energy eigen value is given by \[41,45,46\],

\[ \epsilon = (E_D - m_q - V_0)(m_q + E_D)^{\frac{1}{2}} \]  

(15)

The solution of \( f(\rho) \) and \( g(\rho) \) are normalized to get \[41,45,46\],

\[ \int_0^\infty [f^2(\rho) + g^2(\rho)] d\rho = 1 \]  

(16)

Now the wave function for quarkonium system can be constructed by using positive and negative energy solutions of Dirac equation. Mass of particular Quark-Anti quark system can be written as \[41,45,46\],

\[ M_{QQ} = E_D^Q + E_D^Q - E_{cm} \]  

(17)

here, \( E_{cm} \) in general can be state dependent which we absorb in our potential parameter \( V_0 \). Thus, making \( V_0 \) as state dependent.

In this calculations, we incorporate additionally, the j-j coupling, spin-orbit and tensor interactions of confined one gluon exchange potential (COGEP) \[43,44\].
The spin-orbit term has been split into symmetric $\langle 1 Q \rangle^T = \frac{1}{3} \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial}{\partial \hat{r}} \right)$ and anti-symmetric $\langle 1 Q \rangle^T = \frac{2}{3} \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial}{\partial \hat{r}} \right)$ for charmonium and bottomonium, respectively.

The running strong coupling constant $\alpha_s$ is computed as

$$\alpha_s = \frac{4\pi}{(11 - \frac{2}{3} n_f) \ln \left( \frac{E_{QCD}^2}{\Lambda_{QCD}^2} \right)}$$

with $n_f = 3$ and $\Lambda_{QCD} = 0.250$ GeV for charmonium and $n_f = 4$ and $\Lambda_{QCD} = 0.156$ GeV for bottomonium. In Eqs. (20) the spin-orbit term has been split into symmetric ($\sigma_Q + \sigma_{\bar{Q}}$) and anti-symmetric ($\sigma_Q - \sigma_{\bar{Q}}$) terms.

We have adopted the form of the confined gluon propagators which are given by

$$D_0(r) = \left( \frac{\alpha_1}{r} + \alpha_2 \right) \exp(-r^2 \alpha_0^2/2)$$

![Table 2](image1.png)

**Table 2** Model Parameters fitted in our model for the Charmonium and bottomonium systems.

| System Parameters | $b\bar{b}$ | $c\bar{c}$ |
|-------------------|-----------|-----------|
| Quark mass (in GeV)$^2$ | $m_{b\bar{b}} = 4.67$ | $m_{c\bar{c}} = 1.27$ |
| $V_0$(GeV) | $44.24$ | $40.14$ |
| Potential strength ($\lambda$) (GeV$^2$) | $0.18$ | $0.084$ |

![Table 3](image2.png)

**Table 3** S-wave mass spectrum for $b\bar{b}$ and $c\bar{c}$ bound states (in MeV).

| nL | State | Present | Experimental $^2$ | $^3$ | $^4$ | $^5$ | $^6$ | $^7$ |
|----|-------|---------|-------------------|-----|-----|-----|-----|-----|
| 1S | $1^3S_1$ | 9460.99 | 9460.30 ± 0.26 | 9460.43 | 9460.38 | 9460 | 9608 |
|    | $1^1S_0$ | 9399.7 | 9399.0 ± 2.3 | 9392.38 | 9392.91 | 9392 | 9607 |
| 2S | $2^3S_1$ | 10024.1 | 10023.26 ± 0.31 | 10023.8 | 10023.3 | 10015 | 10023.3 |
|    | $2^1S_0$ | 9999.3 | ... | 9990.88 | 9987.42 | 9990 | ... |
| 3S | $3^3S_1$ | 10356.2 | 10355.2 ± 0.5 | 10345.8 | 10364.2 | 10343 | 10353.3 |
|    | $3^1S_0$ | 10325.3 | ... | 10323.4 | 10333.9 | 10326 | ... |
| 4S | $4^3S_1$ | 10576.2 | 10579.4 ± 1.2 | 10575.2 | 10636.4 | 10597 | 10580 |
|    | $4^1S_0$ | 10554.4 | ... | 10558.3 | 10609.4 | 10584 | ... |
| 5S | $5^3S_1$ | 10758.5 | ... | 10755.4 | ... | 10811 | 10865 |
|    | $5^1S_0$ | 10738.4 | ... | 10741.4 | ... | 10800 | ... |

![Image](image3.png)
and

\[ D_1(r) = \frac{2}{r} \exp(-r^2c_1^2/2) \]  

where \( \alpha_1 = 1.035, \alpha_2 = 0.3977, c_0 = 0.3418 \) GeV, \( c_1 = 0.4123 \) GeV, \( \gamma = 0.8639 \) are the fitted parameter as in [49]. Other model parameters employed in the present calculation are listed in Table 2.

The hyperfine splittings of ground and radial excitation of the bottomonium and charmonium are important for the study of the radiative transition amplitudes. The high precision experimental data have provided accurate description of the hyperfine and fine structure interactions of quarkonia. The hyperfine splitting for S-wave and the ratio of spin orbit splitting for P-wave are given by equation (25) and (26) respectively.

\[ \Delta M_{hf}(nS) = M(n^3S_1) - M(n^1S_0) \]  

and

\[ R = \frac{M(3P_2)}{M(3P_1)} - \frac{M(3P_1)}{M(3P_0)} \]  

Table 4  P-wave mass spectrum for \( \bar{b}b \) and \( \bar{c}c \) bound states(in MeV).

| nl \( \) State | Present | Experimental [2] | [50] | [51] | [52] | [53] |
|---------------|---------|------------------|-----|-----|-----|-----|
| 1P 1\( ^3P_2 \) | 9912.3 | 9912.21 ± 0.26   | 9907.89 | 9912.3 | 9921 | 9812 |
| 1P 1\( ^3P_1 \) | 9901.8 | 9892.78 ± 0.26   | 9887.63 | 9904.7 | 9903 | 9812 |
| 1P 0\( ^3P_0 \) | 9889.2 | 9859.44 ± 0.42   | 9862.29 | 9861.39 | 9864 | 9811 |
| 1P 1\( ^1P_1 \) | 9854.1 | 9899.3 ± 0.8     | 9896.07 | 9899.93 | 9909 | 9812 |
| 2P 2\( ^3P_2 \) | 10259.9 | 10266.5 ± 0.22   | 10267.65 | 10271.2 | 10264 | 10044 |
| 2P 1\( ^3P_1 \) | 10258.9 | 10255.46 ± 0.22  | 10255.74 | 10254.8 | 10249 | 10043 |
| 2P 2\( ^1P_2 \) | 10234.7 | 10232.50 ± 0.40  | 10240.85 | 10230.5 | 10220 | 10042 |
| 2P 1\( ^1P_1 \) | 10264.9 | ...              | ... | 10260.70 | 10261.8 | 10254 | 10043 |
| 3P 3\( ^3P_3 \) | 10516.9 | ...              | 10516.28 | ... | 10528 | 10272 |
| 3P 1\( ^3P_1 \) | 10568.8 | 10512.1 ± 2.3    | 10507.24 | ... | 10515 | 10271 |
| 3P 2\( ^3P_2 \) | 10497.6 | ...              | 10497.07 | ... | 10490 | 10270 |
| 3P 1\( ^1P_1 \) | 10540.2 | ...              | 10511.30 | ... | 10519 | ... |
| 4P 4\( ^3P_4 \) | 10707.0 | ...              | ... | ... | ... | ... |
| 4P 2\( ^3P_2 \) | 10706.5 | ...              | ... | ... | ... | ... |
| 4P 0\( ^3P_0 \) | 10703.8 | ...              | ... | ... | ... | ... |
| 4P 1\( ^1P_1 \) | 10704.6 | ...              | ... | ... | ... | ... |

The computed S - wave, P - wave and D - wave mass spectra of bottomonium and charmonium are tabulated in Table 3, 4 and 5. The corresponding energy level diagram are shown in Fig 1 and 2 respectively. The hyperfine splitting for S-wave and the spin orbit splitting ratio for P-wave are tabulated in Table 6 and 7.

3 Decay constants and Leptonic Decay width of \( 1^- \) quarkonia

The leptonic decay width is a tool to understand the compactness of the mesonic system. We know that leptonic decay width of \( J/\Psi \) is reasonably predicted by the phenomenological model. At the same time heavy
is expressed through meson wave function short range one gluon exchange interaction between quarkonium states are precisely most sensitive to the

$$\sqrt{q^2 + m^2} \sqrt{3}$$ is the color factor.

Where, $E = \sqrt{q^2 + m^2}$ and $\sqrt{3}$ is the color factor. M is the mass of vector state. The leptonic decay width is expressed as [64]:

$$\Gamma(V \rightarrow e^+e^-) = \frac{4}{3} \pi \alpha^2 e_q^2 \frac{f_V}{M}$$

(28)

The computed decay constants and leptonic decay widths in the case of charmonium and bottomonium states are presented in Table 8 and Table 9 respectively. The ratio of $\Gamma_{\pi^0(nS)}$ for bottomonium and charmonium states are listed in Table 12. Along with the mass predictions, the

### Table 5 D - wave spectrum for $b\bar{b}$ and $c\bar{c}$ bound states (in MeV).

| nL  | State  | Present | Experimental [2] | [50] | [51] | [52] | [53] |
|-----|--------|---------|-------------------|------|------|------|------|
| 1D  | $1^3D_1$ | 10140.4 | ...               | 10176.68 | 10163.1 | 10157.3 | 10157.3 | 9980 |
|     | $1^3D_2$ | 10138.7 | 10163.7 ± 1.4     | 10162.26 | 10157.3 | 10153 | 9980 |
|     | $1^3D_1$ | 10136.0 | ...               | 10147.31 | ... | 10146 | 9980 |
|     | $1^3D_2$ | 10068.2 | ...               | 10166.00 | 10158.6 | 10153 | 9980 |
| 2D  | $2^1D_1$ | 10398.7 | ...               | 10447.09 | 10455.7 | 10436 | 10175 |
|     | $2^1D_2$ | 10397.1 | ...               | 10455.72 | 10450.3 | 10432 | 10174 |
|     | $2^1D_3$ | 10385.7 | ...               | 10427.59 | ... | 10425 | 10174 |
|     | $2^3D_1$ | 10381.6 | ...               | 10440 | 10451.4 | 10432 | 10174 |
|     | $2^3D_2$ | 1036.0 | ...               | 10651.86 | ... | ... | ... |
|     | $3^3D_1$ | 10620.9 | ...               | 10644.62 | ... | ... | ... |
|     | $3^3D_2$ | 10619.3 | ...               | 10637.12 | ... | ... | ... |
|     | $3^3D_3$ | 10618.6 | ...               | 10646.50 | ... | ... | ... |
| 4D  | $4^3D_3$ | 10820.9 | ...               | 10816.93 | ... | ... | ... |
|     | $4^3D_2$ | 10819.3 | ...               | 10811.09 | ... | ... | ... |
|     | $4^3D_1$ | 10816.9 | ...               | 10805.03 | ... | ... | ... |
|     | $4^1D_2$ | 10768.8 | ...               | 10812.60 | ... | ... | ... |
|     | $5^3D_3$ | 11005.2 | ...               | 10955.6 | ... | ... | ... |
|     | $5^3D_2$ | 11003.7 | ...               | 10950.7 | ... | ... | ... |
|     | $5^3D_1$ | 11001.4 | ...               | 10945.6 | ... | ... | ... |
|     | $5^3D_2$ | 10956.7 | ...               | 10952.0 | ... | ... | ... |

In relativistic quark model, the vector decay constant is expressed through meson wave function $f(\bar{q})$ in momentum space as given by [63]:

$$f_V = \frac{2\sqrt{3}}{M} \int d^4q \left( \frac{m + E}{E} - \frac{\bar{q}^2}{3E^2} \right) f(\bar{q})$$

(27)
Table 6 The hyperfine splitting (in MeV) for S-wave bottomonium and charmonium.

| State (ns) | Hyperfine splitting | Present | Experimental [2] | [23] | [50] |
|-----------|---------------------|---------|------------------|-----|-----|
| bottomonium |                     |         |                  |     |     |
| 1S        | $\Delta M(1S) = M(1^{3}S_{1}) - M(1^{1}S_{0})$ | 70.2 | 70 | 60 | 68 |
| 2S        | $\Delta M(2S) = M(2^{3}S_{1}) - M(2^{1}S_{0})$ | 24.8 (~ 25) | 24 | 30 | 33 |
| 3S        | $\Delta M(3S) = M(3^{3}S_{1}) - M(3^{1}S_{0})$ | 30.9 (~ 31) | ... | 27 | 22 |
| 4S        | $\Delta M(4S) = M(4^{3}S_{1}) - M(4^{1}S_{0})$ | 21.8 (~ 22) | ... | 26 | 17 |
| 5S        | $\Delta M(5S) = M(5^{3}S_{1}) - M(5^{1}S_{0})$ | 20.1 | ... | ... | 14 |

| Charmonium |                     |         |                  |     |     |
| State (ns) | Hyperfine splitting | Present | Experimental [2] | [56] | NRp model [56] |
|------------|---------------------|---------|------------------|-----|----------------|
| 1S         | $\Delta M(1S) = M(1^{3}S_{1}) - M(1^{1}S_{0})$ | 118.9 (~ 119) | 116 | 114 | 108 |
| 2S         | $\Delta M(2S) = M(2^{3}S_{1}) - M(2^{1}S_{0})$ | 53.9 (~ 54) | 49 | 41 | 42 |
| 3S         | $\Delta M(3S) = M(3^{3}S_{1}) - M(3^{1}S_{0})$ | 31.6 | ... | 25 | 29 |
| 4S         | $\Delta M(4S) = M(4^{3}S_{1}) - M(4^{1}S_{0})$ | 4.3 | 3 | ... | ... |
| 5S         | $\Delta M(5S) = M(5^{3}S_{1}) - M(5^{1}S_{0})$ | 2.3 | ... | ... | ... |

Table 7 The ratios of spin orbit splitting (R) for P wave bottomonium and charmonium.

| State (nP) | Present | Experimental [2] | [23] | [50] |
|------------|---------|------------------|-----|-----|
| bottomonium |         |                  |     |     |
| 1P         | 0.83 | 0.60 | 0.80 | 0.80 |
| 2P         | 0.29 | 0.56 | 0.60 | 0.80 |
| 3P         | 0.72 | ... | 0.72 | 0.78 |
| 4P         | 0.18 | ... | ... | ... |

| Charmonium |         |                  |     |     |
| State (nP) | Present | Experimental [2] | Lattice QCD [56] | NRp model [56] |
|------------|---------|------------------|----------------|----------------|
| 1P         | 0.43 | 0.47 | 0.46 | 0.62 |
| 2P         | 0.25 | ... | ... | 0.64 |
| 3P         | 0.75 | ... | ... | ... |
| 4P         | 0.23 | ... | ... | ... |

Table 8 Vector Decay Constant($F_v$ in MeV) of the S - wave and D - wave Bottomonium and Charmonium states.

| Bottomonium |         | Experimental [2] | [57] | [58] | [59] |
|------------|---------|------------------|-----|-----|-----|
| State      | Present |                  |     |     |     |
| 1S         | 705.4 | 715 ± 5 | 831 | 665 | 867 | 1D | 208.3 |
| 2S         | 554.9 | 498 ± 8 | 566 | 475 | 673 | 2D | 181.3 |
| 3S         | 436.8 | 430 ± 4 | 507 | 418 | 595 | 3D | 151.4 |
| 4S         | 332.4 | 336 ± 18 | 481 | 388 | 549 | 4D | 135.4 |
| 5S         | 286.5 | ... | 458 | 367 | 516 | 5D | 113.1 |

| Charmonium |         | Experimental [2] | Lattice QCD [56] | [57] | [58] |
|------------|---------|------------------|----------------|-----|-----|
| State      | Present |                  |                  |     |     |
| 1S         | 419.9 | 416 ± 6 | 462 | 393 | 589 | 1D | 102.5 |
| 2S         | 285 | 304 ± 4 | 369 | 293 | 328 | 2D | 83.9 |
| 3S         | 218 | ... | 329 | 258 | 244 | 3D | 65.6 |
| 4S         | 165.7 | ... | 310 | ... | ... | 4D | 54.2 |
| 5S         | 106.2 | ... | 290 | ... | ... | 5D | 42.3 |
Table 9: Leptonic decay width (in keV) of the S-wave and D-wave Bottomonium and Charmonium states.

| Bottomonium | State | Present | Experimental | [2] | [50] | [51] | [43] | state | Our |
|-------------|-------|---------|---------------|-----|------|------|------|-------|-----|
| Bottomonium | 1S    | 1.30    | 1.34 ± 0.018  | 1.203 | 1.33 | 1.809 | 1D | 0.106 |
|             | 2S    | 0.76    | 0.612 ± 0.011 | 0.519 | 0.62 | 0.797 | 2D | 0.078 |
|             | 3S    | 0.45    | 0.443 ± 0.008 | 0.330 | 0.48 | 0.618 | 3D | 0.051 |
|             | 4S    | 0.26    | 0.272 ± 0.029 | 0.241 | 0.40 | 0.541 | 4D | 0.042 |
|             | 5S    | 0.18    | ...           | 0.19  | ... | ...   | 5D | 0.028 |

| Charmonium  | State | Present | Experimental | [2] | [51] | [61] | [62] | state | Our |
|-------------|-------|---------|---------------|-----|------|------|------|-------|-----|
| Charmonium  | 1S    | 5.63    | 5.55 ± 0.14   | 4.94 | 1.89 | 5.469 | 1D | 0.27 |
|             | 2S    | 2.19    | 2.48 ± 0.06   | 1.686 | 1.04 | 2.140 | 2D | 0.17 |
|             | 3S    | 1.20    | ...           | 0.959 | 0.77 | 0.796 | 3D | 0.099 |
|             | 4S    | 0.63    | ...           | 0.654 | 0.65 | 0.288 | 4D | 0.064 |
|             | 5S    | 0.24    | ...           | 0.489 | ... | ...   | 5D | 0.044 |

Table 10: Mixing angle and the leptonic decay widths of S - D wave admixture states.

| Experimental state | J P mixed state | % mixing of S-wave | Mass of mixed state (MeV) | Mixed state leptonic decay width (keV) |
|--------------------|-----------------|---------------------|---------------------------|---------------------------------------|
|                    | Our             | Experimental        | Our                       | Experimental                          |
| Charmonium like states |                 |                     |                           |                                       |
| Y(4008) 1−          | 23S1 and 23D1   | 8.6%                | 4008.4                    | 0.347 0.862 ± 0.241 (2)               |
| ψ(4160) 1−          | 33S1 and 33D1   | 39.2%               | 4191                      | 0.534 0.48 ± 0.22 keV (2)             |
| Y(4220) 1−          | 33S1 and 33D1   | 28.7%               | 4220.7                    | 0.417 NA                               |
| X(4260) 1−          | 33S1 and 33D1   | 14.41%              | 4260.5                    | 0.258 NA                               |
| X(4360) 1−          | 43S1 and 43D1   | 64.97%              | 4360                      | 0.431 < 0.57 eV (2)                    |
| X(4630) 1−          | 53S1 and 53D1   | 37.92%              | 4634                      | 0.117 NA                               |
| X(4660) 1−          | 53S1 and 53D1   | 34.37%              | 4645                      | 0.110 < 0.45 eV (2)                    |

X(10860) 1− 53S1 and 53D1 45.45% 10880.1 10891 ± 4 (2) 0.096 0.31 ± 0.07 keV (2)

Table 11: Masses of mixed P-wave +ve parity states.

| Experimental state | Mixed state configuration | Present(MeV) |
|--------------------|----------------------------|--------------|
| Charmonium like states |                            |              |
| X(3940) 3P1 and 3P1 |                            | 3939.06      |
| X(4020) 3P1 and 3P1 |                            | 4011.56      |
| X(4140) 3P1 and 3P1 |                            | 4121.33      |
| X(4350) 3P1 and 3P1 |                            | 4349.76      |

Bottornium like state

X(10610) 4P1 and 4P1 10595.63

leptonic decay widths are also important for the identification of the structures of quarkonia-like states.

4 Quarkonia-like states as mixed quarkonia states

It is known that many of the hadronic states which are observed and yet not clear about their structure can be the admixture of the nearby iso-parity states. In general, the mass of a mixed state \( M_{nL} \) can be
expressed in terms of the two mixing states ($nl$ and $n'l'$) as

$$M_{nL} = |a^2| \cdot M_{nl} + (1 - |a^2|)M_{n'l'}$$

(29)

Where, $|a^2| = \cos^2 \theta$ and $\theta$ is mixing angle. With the help of this equation we can obtain mixed state configuration and mixing angle [50].

The computed masses and their leptonic decay width of the S-D wave admixture states are presented in Table 12.

### Table 12 The ratios of $\Gamma_{c\bar{c} (nS)}$ for bottomonium and charmonium states.

| $\Gamma_{c\bar{c} (nS)}$  | Present | Experimental |
|-------------------------|---------|--------------|
| $\Gamma_{c\bar{c} (1S)}$ |         |              |
| $\Gamma_{c\bar{c} (1S)}$ | 0.58    | 0.46 0.50    |
| $\Gamma_{c\bar{c} (2S)}$ | 0.35    | 0.33 0.36    |
| $\Gamma_{c\bar{c} (3S)}$ | 0.20    | 0.20 0.29    |
| $\Gamma_{c\bar{c} (4S)}$ | 0.13    | ... 0.24     |

### Charmonium

| $\Gamma_{c\bar{c} (nS)}$  | Present | Experimental |
|-------------------------|---------|--------------|
| $\Gamma_{c\bar{c} (1S)}$ | 0.39    | 0.43 0.48    |
| $\Gamma_{c\bar{c} (2S)}$ | 0.21    | ... 0.32     |
| $\Gamma_{c\bar{c} (3S)}$ | 0.11    | ... 0.24     |
| $\Gamma_{c\bar{c} (4S)}$ | 0.04    | ... 0.19     |

![Fig. 1](mass_spectrum_bottomonium.png)

**Fig. 1** Mass spectrum of Bottomonium

![Fig. 2](mass_spectrum_charmonium.png)

**Fig. 2** Mass spectrum of Charmonium

In this context we consider the admixture of nearby P-waves for the predictions of some of the $1^+$ states and for other $1^-$ states we consider the S-D wave mixing [44][65][66]. The mixed P wave states can be expressed as [44][65][66].

$$|\alpha| = \sqrt{\frac{1}{3} |P_1|} + \sqrt{\frac{1}{3} |P_1|}$$

(30)
\[ |\beta| = -\sqrt{\frac{1}{3}M_3^3 P_1} + \sqrt{\frac{2}{3}M_1^1 P_1} \]  

(31)

Where, \(|\alpha|\), \(|\beta|\) are states having same parity. We can write the masses of these states in terms of the predicted masses of the pure P wave states (\(3^3 P_1\) and \(1^1 P_1\)) as \[M(|\alpha|) = \frac{2}{3} M(3^3 P_1) + \frac{1}{3} M(1^1 P_1)\]  

(32)

and \[M(|\beta|) = \frac{2}{3} M(1^1 P_1) + \frac{1}{3} M(3^3 P_1)\]  

(33)

The computed mixed P - wave states for the positive parity quarkonia - like states are listed in Table 10 and the experimental states which are close to these mixed states are also listed for comparison.

5 Result and discussion

In the framework of the Dirac relativistic quark model, we have studied the mass spectrum of bottomonium - like and Charmonium -like states. To obtain these mass spectra we have solved Dirac equations with a linear plus constant confinement potential. In our calculations, spin-dependent interactions are included to remove degeneracy of the states. The predicted bottomonium and charmonium spectra are in good agreement with the experimental data and other available theoretical data. We have also predicted the 4S and 5S states for charmonium and bottomonium and compared them with the available theoretical results. The predicted masses of the S-wave bottomonium states \(3^3 S_1\) (10356.2 MeV) and \(4^1 S_1\) (10576.2 MeV) are in accordance with experimental results as quoted in the particle data group (PDG 2016) [2]. The predicted masses of the S-wave charmonium states \(3^3 S_0\) (3990.8 MeV), \(4^1 S_0\) (4262.1 MeV) and \(5^1 S_0\) (4439.2 MeV) are in accordance with other model predictions [50, 51, 53]. The computed P-wave bottomonium states \(2^3 P_1\) (10265.9 MeV), \(2^1 P_1\) (10258.9 MeV) and \(2^3 P_0\) (10234.7 MeV) are in good agreement with experimental [2] results of 10268.65 ± 0.22 MeV, 10255.46 ± 0.22 MeV and 10232.50 ± 0.40 MeV respectively. For charmonium, the predicted P-wave mass of 3921.2 MeV for \(2^3 P_2\) state is in very good agreement with the available experimental result of 3927.2 ± 2.6 MeV [2]. The masses of 3P, 4P and 1D to 4D states and their fine structure splitting for bottomonium and charmonium are fairly in good agreement with the available experimental results and other theoretical predictions. The predicted states such as \(\chi_3(3P)\), \(\chi_3(4P)\), \(\Upsilon(3D)\), \(\Upsilon(4D)\) and \(\Upsilon(5D)\) of bottomonium and \(\chi_c(2P)\) to \(\chi_c(4P)\), 2D to 5D states of charmonium are not available experimentally.

In this paper, we have also examined the vector decay constant and leptonic decay widths for \(nS\) states of Bottomonium and charmonium within the relativistic framework. Decay widths of \(T(nS) \to e^+e^-\) and \(\psi(nS) \to e^+e^-\) are shown in Table 8. All the results for vector decay constant and leptonic decay width are calculated without QCD corrections. The calculated leptonic decay widths for \(\Upsilon(3S)\) and \(\Upsilon(4S)\) are in good agreement with the available experimental data, but the calculated values for \(\Upsilon(2S)\) is slightly higher than the experimental value. In the case of \(\psi(3S)\) state of charmonium, vector decay constant and leptonic decay width are slightly higher than the experimental result. Our results for \(f_{J\psi(1S)}\) and \(f_{\psi(2S)}\) are in good agreement with the lattice QCD results of 399 ± 4 GeV and 143 ± 81 MeV respectively [67]. It is observed that the leptonic decay widths decrease with radial excitations. From this we can conclude that the relativistic treatment is important for higher excited states.

To understand the structure of some of the newly found \(\{X, Y, Z\}\) quarkonia like - states with hidden charm and hidden bottom flavors, we consider here the possibilities for the mixing of \(3^3 P_1\) and \(1^1 P_1\) and of \(3^3 S_1\) and \(3^3 D_1\) iso-parity states. The calculated mixed states of \(3^3 P_1 - 1^1 P_1\) and \(3^3 S_1 - 3^3 D_1\) are listed in Table 10 and 11 respectively. The corresponding leptonic decay widths of the \(1^{++}\) admixture states are also listed in Table 10.

Now, we briefly summarize the structure of some of the newly found quarkonia - like states based on our results of the masses and the leptonic decay widths.

- The \(X(3940)\) has been observed in the \(D^* \bar{D}\) channel with \(J^P\) value \(1^+\) but not in the \(D\bar{D}\) decay mode [69]. Looking in to its parity, the possible identification of this state could be one of the charmonium-like states. However the predicted 2P states are lower than this mass while the 3P states are slightly higher than this mass. Based on our analysis of mixed states, we predict \(X(3940)\) as \(1^+\) state with admixture of \(2^3 P_1\) and \(2^1 P_1\) charmonia.

- Similarly, we predict \(X(4020)\) as an admixture of \(2^3 P_1\) and \(3^3 P_1\) state with \(J^P = 1^+\) having mass 4011.56 MeV.

- The Particle Data Group has renamed \(Y(4140)\) to \(X(4140)\) [2]. Many attempts are made to study this state. According to ref [70] this state might be a candidate of tetra quark state but because of unknown value
of \( J^P \), its status is still not confirmed. According to our analysis it does not fit in to the admixture of P-states. But it fits well with the pure charmonium \( 3^3P_1 \) state where by predicting its \( J^P \) value as \( 0^+ \).

- The X(4350) state whose \( J^P \) value is not known experimentally is predicted as an admixture of \( 4^3P_1 \) and \( 3^3P_1 \) states having mass 4349.76 MeV or it might be pure charmonium \( 4^3P_1 \) state with \( J^P \) value \( 1^+ \). Similarly, the X(10610) is also found to be the admixture of \( 4^3P_1 \) and \( 3^3P_1 \) states of bottomonium having mass 10595.63 MeV with \( J^P \) equal to \( 1^+ \).

- For the better identification of some of the \( 1^- \) states, we have considered both the predicted masses and their leptonic decay widths in accordance with our previous study \[50\] as the successive mass differences \([n-1S] - nS\) as well as the leptonic decay widths of \( 1^- \) states of quarkonia have shown to follow a specific decreasing pattern, characteristic of the bound state.

- Accordingly, our predicted mass for \( \psi(3S) \) state as 4022.4 MeV and its leptonic decay width as 1.21 keV do not follow with the experimental mass of \( \psi(3770) \) state and its experimental leptonic decay width of 0.262 \( \pm \) 0.018 keV \[2\]. But according to our predicted mass of \( 1^3D_1 \) state as 3745.3 MeV and its computed leptonic decay width of 0.27 keV indicate that \( \psi(3770) \) is right candidate for a pure charmonium \( 1^3D_1 \) state.

- In 2013, Belle Collaboration updated the analysis of \( e^+e^- \rightarrow J/\psi \pi^+\pi^- \) with a 967 \( fb^{-1} \) data sample and shows the invariant mass distribution of \( J/\psi \pi^+\pi^- \), the distribution is fitted with two coherent resonances. The existence of \( Y(4008) \) state was further confirmed \[74\] in consistent with Belle’s previous results \[26\]. There are some theoretical approaches to understand the structure of \( Y(4008) \) after Belle’s confirmation \[26\]. X. Liu discussed some possibilities for the \( Y(4008) \), including both the (3S) charmonium states and the \( D^*D^* \) molecular state. For both possibilities, he found the branching ratio of \( Y(4008) \rightarrow J/\psi \pi^+\pi^- \) is comparable with that of \( Y(4008) \rightarrow J/\psi \pi^+\pi^- \) \[72\]. Li and Chao studied the higher charmonium states in the non-relativistic screened potential model, and they predicted the \( Y(4008) \) as the (3S) charmonium state \[75\]. Chen, Ye, and Zhang found that the \( Y(4008) \) is difficult to identify \( 3^3S_1 \) pure charmonium \[74\]. The \( Y(4008) \) was studied by Maiani, Piccinini, Polosa, and Riquer in their type-II diquark-antidiquark model, and interpreted as tetraquark state \[73\] and they have also assigned \( Y(4360) \) as the first radial excitations of \( Y(4008) \) tetraquark state. In \[76\], Zhou, Deng, and Ping also predicted the \( Y(4008) \) as a tetraquark state \( c\bar{c}q\bar{q} \) with \( J^{PC} = 1^- \) and \( n^{2S+1}L_J = 1^1P_1 \). They have used a color flux-tube model with a four-body confinement potential to interpret the status of \( Y(4008) \). According to Dian-Yong Chen, the Fano-like interference induces an extra broad structure in \( Y(4008) \rightarrow \pi^+\pi^-j/\psi \) as a companion peak to \( Y(4260) \) and also it explained why \( Y(4008) \), \( Y(4260) \) and \( Y(4360) \) are absent in the experimental data of the R value scan \[77\] and they have concluded that appearance of \( Y(4008) \) peak is due to the interference of \( \psi(4160)/\psi(4415) \) with the continuum of \( Y(4008) \rightarrow \pi^+\pi^-J/\psi \) \[77\]. However, very recently BESIII could not confirm the existence of \( Y(4008) \) \[7\]. In this context, \( Y(4008) \) is still a controversial state.

In the present study, the predicted mass of \( Y(4008) \) is found to be close to \( 2^3D_1 \) charmonium state with just 8.6% mixing with the \( 2^3S_1 \) charmonium state. However, the computed leptonic decay width of the admixture state (0.347 keV) is much lower than the experimentally reported value of 0.86 keV \[72\]. Thus by considering both the mass and the leptonic decay width together, it is difficult to confirm or understand the structure of \( Y(4008) \). We look forward more refined experimental data for better understanding of this state.

- According to the present study, the state \( \psi(4160) \) is found to be as an admixture of \( 3^3P_1 \) (60.8%) and \( 3^3S_1 \) (39.2%) states with its leptonic decay width as 0.534 keV which is in accordance with the experimental result of 0.48 \( \pm \) 0.22 keV \[2\].

- The state \( Y(4260) \) was observed by the BaBar Collaboration in the \( J/\psi \pi^+\pi^- \) channel in the initial state radiation (ISR) process \[25\]. It was confirmed by CLEOc \[30\], Belle \[26\] and an additional analysis done by BaBar \[78\], with mass values varying in different analyses. The decay modes of the \( Y(4260) \) into \( J/\psi \) and other charmonium states indicate the presence of a \( c\bar{c} \) content. From PDG \[2\], the masses of some radial excitations, \( \psi(2S) \) and \( \psi(1P) \) are well established but masses of \( \psi(3S), \psi(4S), \psi(1P), \psi(2P) \) and \( \psi(1D) \) still need more experimental investigation. Some theoretical interpretations for the \( Y(4260) \) are: hybrid mesons (mixing of \( c\bar{c} \) and \( c\bar{c}q\bar{q} \)) \[79-81\], tetraquark state \[82\], hydrocharmonium \[83,84\], hadronic molecules of \( DD(2420)+c\bar{c}c\bar{c} \) \[85\], \( \omega_{X_{c0}} \) \[86\] etc. For the \( \omega_{X_{c0}} \) molecule, the predicted leptonic decay width is only about 23 eV \[89\]. According to Felipe J. Llanes-Estrad, Y(4260) was proposed to be a conventional charmonium \( \psi(4S) \) state and also estimated its leptonic decay width as 0.2 - 0.35 keV \[87\]. Wen Qin, Si-Run Xue and Qiang Zhao have predicted the upper limit of the \( Y(4260) \) leptonic
decay width about 500 eV \[88\]. The LQCD also predicts the leptonic decay width as \(<\) 40 eV for a hybrid charmonium state \[90\].

According to new results from BES III \[7\], Y(4260) is not a simple peak. This measurement of the \(e^+e^- \rightarrow \pi^+\pi^- J/\psi\) cross section was done by using both a small number of high-statistical data points and a large number of low-statistics data points \[7\]. They found resonance Y(4260) is described as combination of two peaks Y(4220) and Y(4330) \[7\]. However, the structure and interpretations of Y(4220) and Y(4330) are not yet understood. Recently, X. Y. Gao, C. P. Shen and C. Z. Yuan have predicted the value of leptonic decay width for Y(4220) can be as large as 200 eV or even higher based on current information \[88\]. So the peaks observed by BESIII will provide more information about their structure. Thus the states, Y(4260), Y(4220) and Y(4330) have opened up new challenges in the charm sector.

According to latest PDG 2016 \[2\], the earlier state Y(4260) is now renamed as X(4260). We have analysed the status of X(4260), Y(4220) and Y(4330) states.

According to the present study Y(4220) state do not fit to be a pure charmonium state but fit to be an admixture of \(3^3 D_1\) (71.3 \%) and \(3^1 S_1\) (28.7 \%) states with its estimated leptonic decay width as 0.417 keV.

The second resonance reported by BES III, Y(4330) with mass 4326.8 \pm 10 MeV is close to our predicted \(3^3 D_1\) state having mass 4300.6 MeV and its predicted leptonic decay width as 0.099 keV.

If we now consider X(4260) as pure \(\psi(4S)\) state with the predicted mass equal to 4266.4 MeV then its leptonic decay width is predicted as 0.63 keV which is higher than the upper limit of 0.500 keV \[88\]. And if we consider X(4260) as the mixed states of Y(4220) and Y(4330) with a mixing probability of 0.67 : 0.33, then its leptonic decay width has to be 0.258 keV. The recent experimental measurements of BESIII \[91,92\] suggests comparatively very small leptonic decay width for X(4260) \[88\]. Thus, X(4260) state can neither be identified as a pure state nor a mixed state. X(4260) state might be exotic state or hadronic molecular state. We require more experimental data for the confirmation of X(4260) state.

- Another controversial state is Y(4360) having \(J^P\) value \(1^-\) has now been renamed as X(4360) \[2\]. This state might be a diquark-antidiquark type tetraquark state or it may be a mixed S-D wave charmonium states \[93\]. Present analysis suggests it to be a mixed \(4^3 S_1\) and \(4^3 D_1\) charmonium state with leptonic decay width of 0.431 keV.

- X(4630) is compatible with our \(1^-\) mixed charmonium state with admixture of \(5^3 S_1\) and \(5^3 D_1\) states having mass 4634 MeV and its predicted leptonic decay width as 0.117 keV.

- Structure of Y(4660) is also interesting because this state was neither observed in \(e^+e^- \rightarrow \gamma ISR p^+ p^- J/\psi\) process, nor in the mass distributions of a \(cc\) in the final stage of \(e^-e^+\) collision experiment \[94\]. Other theoretical approaches suggested it to be a molecular like structure. According to latest PDG \[2\], Y(4660) has been renamed as X(4660) and its \(J^P = 1^-\) value suggests that it might be an admixture of \(5^3 S_1\) and \(5^3 D_1\) state and we have predicted its leptonic decay width to be 0.110 keV. However, the status of these states is still a mystery and to resolve this we need more experimental results on their leptonic decay widths.

- The Particle Data Group has renamed \(Y_0(10888)\) to \(\Upsilon(10860)\) \[2\]. In our present study by considering both its mass and leptonic decay width, we find it very difficult to assign it as the \(b\bar{b}, 5S\) state. Even if we consider it as an admixture of \(5^3 S_1\) and \(5^3 D_1\) state, its leptonic decay width estimated to be equal to 0.096 keV which is much lower than the experimentally reported value of \(0.31 \pm 0.07 \text{ keV} \[2\]. So, the status of \(Y_0(10888)\) or \(\Upsilon(10860)\) as a conventional bottomonium state or an admixture of S-D states is doubted. More refined experimental observations of \(\Upsilon(10860)\) can throw more light towards the understanding of this state.

6 Summary

In the present paper we have proposed a quark model for hadrons. The approach is attractive due to its simplicity in applications to quarkonia and exotic hadrons. In our model for meson spectroscopy we have solved Dirac equation to obtain the binding energy for individual quark/antiquark. Further the masses of the bound state is computed by adding the binding energies of the quark and antiquark with the addition of a center of mass correction.

In the last few years many states have been observed at B-factories (BaBar, Belle and CLEO), at proton-proton colliders (ATLAS, CMS, CDF, D0, LHCb) and also at \(\tau\)-charm facilities (CLEO-c, BES3) in the heavy quarkonium sector. These charmonium-like and bottomonium-like states have provided new challenges for theorists as
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