Linear system security— detection and correction of adversarial attacks in the noise-free case

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Abstract—We address the problem of attack detection and attack correction for multi-output discrete-time linear time-invariant systems under sensor attack. More specifically, we focus on the situation where adversarial attack signals are added to some of the system’s output signals. A ‘security index’ is defined to characterize the vulnerability of a system against such sensor attacks. Methods to compute the security index are presented as are algorithms to detect and correct for sensor attacks. The results are illustrated by examples involving multiple sensors.

I. INTRODUCTION

In today’s society, the physical infrastructure in support of critical services such as water and energy can be described as a cyber-physical system. The end-to-end service can therefore be affected not only by the loss of functionality of the physical assets (breaking of wires or pipes, loss of motors or sensors, due to, for example, accidental breakage or maintenance outages or due to a natural disaster) but also through loss of functionality in the cyber assets (like loss of bandwidth in communication channels, message loss, or a virus in a computer operating system).

In this paper, the particular cyber attack scenario where sensor signals may be corrupted by additive signals is considered. In the case where such signals are injected based on knowledge of a model of the system’s behaviour, it no longer suffices to treat these external signals as mere disturbances or noise, as in fact they can be used to control the behaviour. For instance, it is conceivable that a modern autonomous vehicle may be hijacked using sensor spoofing [26].

This issue has captured the attention of the control community. Several attack detection methods have been proposed in the literature, such as [18], [11], [16]. Correction methods have also been discussed in the literature, such as [7], [23], [17], [4].

In this paper, the case of discrete-time, linear and time-invariant (LTI) systems where output signals may be compromised is considered. Attack signals are modeled as signals added to the output signals. To assist in the analysis, the notion of the ‘security index’ of a system is introduced. This is analogous to the notion of the ‘minimum distance’ used in coding theory. The main tools used in this paper are kernel representations of systems and the setting is the behavioral approach [19]. The security index is a quantitative representation-free measure of the vulnerability of a system to sensor attacks. It speaks to the detectability and correctability of attack signals.

Unlike much of the work in this area, our starting point in this paper is a kernel representation (see for example [19 Ch2.5]) rather than a state space representation. Reasons for this are: 1. Many well-established theorems based on kernel representations can be applied when we are discussing a system using a behavioral approach. 2. Every kernel representation can be brought into state space form (see subsection VII.4) and every observable system in state space form can be transformed into a kernel representation. 3. The implementation of systems in kernel representation can be done straightforwardly using shift operators.

Previous works involving systems under sensor attacks include [4], [7], [11], [23], [8]. The work of [7] focuses on reconstructing the state value using a relaxed optimization program to approximate an NP-hard $l_0$-norm optimization problem. We demonstrate that our formulation simplifies the approach. There is a consensus of our result with [4] that the output signals are only guaranteed to be reconstructible if a certain upper bound on the number of attacked sensors is met. We reformulate the assumptions of [4] in terms of the security index and derive methods for detection as well as correction. Other related works are [25] which focuses on the establishment of the models for various attack signals; and [10] which focuses on the security of power networks.

The focus of our paper is on the development of a conceptual approach to attack detection and correction. Unlike e.g. [4], [15], [6], we restrict ourselves to a noise-free environment to enable the reader to understand the essence of the proposed methods. These methods then serve as a starting point for further research on the noisy case.

The outline of this paper is as follows. Section II presents some notation used in the paper. Section III describes the system and the problem statements, while
Section IV defines the security index, attack detectability and attack correctability. A method of computing the security index in terms of a kernel representation is given in Section V. Section VI presents the Kronecker-Hermite canonical form representation. Section VII gives an attack detection method and uses the Kronecker-Hermite canonical form to design attack correction methods, first for the maximally secure case, then for the general case. The theory is illustrated in Section VIII by results and simulations for two discrete-time LTI system examples that involve multiple sensors. Finally, conclusions and future work directions are presented in Section IX. This paper builds on preliminary work by two of its authors [2], [3], and the main new result is the attack correction method.

II. NOTATION

- Let \( \mathbb{Z}^+ = \{0, 1, \ldots\} \) and \( \mathbb{R} := (-\infty, \infty) \).
- The \( N \)-dimensional signal \( y = (y_1, \ldots, y_N)^T \) is denoted as \( y : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \).
- An \( N \times N \) identity matrix is denoted by \( I_N \).
- The support of a signal is denoted by \( \text{supp}(y) := \{ i \in \{1, \ldots, N\} : y_i \neq 0 \} \).
- The weight of a signal \( y \) is denoted by \( \|y\| := |\text{supp}(y)| \), i.e., the number of components of \( y \) that are non-zero signals.
- If \( J \) is a subset of \( \{1, \ldots, N\} \) then its complement set is denoted by \( J^c \).
- The shift operator \( \sigma \) is defined as \( \sigma y(t) := y(t+1) \).
- The degree of a polynomial \( a(\xi) \) is denoted by \( \deg a(\xi) \).
- The greatest common divisor of two polynomials \( a(\xi) \) and \( b(\xi) \) is denoted by \( \text{GCD}(a(\xi), b(\xi)) \).

III. PROBLEM STATEMENT

Consider a linear time-invariant (LTI) system \( \Sigma \) in its kernel representation as follows
\[
\Sigma : \quad R(\sigma) y = 0, \tag{1}
\]
where \( y : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \) is the sensor output signal of the system \( \Sigma \) and \( R(\xi) \) is a real polynomial matrix of full rank, meaning that the system’s behaviour is autonomous with no free variables. The size of \( R(\xi) \) is \( N \times N \).

Definition III.1 (Behaviour of the system \( \Sigma \)). The behaviour of the system \( \Sigma \) is defined as the set given by
\[
\mathcal{B} = \{ y : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \mid R(\sigma) y = 0 \}. \tag{2}
\]

Consider a class \( \mathcal{A} \) of attack signals \( \eta : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \). A corrupted output signal is \( x = y + \eta \). Here we denote the resulting system by \( \Sigma_A \), more specifically we have the following definition.

Definition III.2 (Behaviour of the system \( \Sigma_A \)). The behaviour of the corrupted system \( \Sigma_A \) is defined as the set of possible received signals
\[
\mathcal{B}_A = \{ r : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \mid r = y + \eta, \text{ where } y \in \mathcal{B}, \eta \in \mathcal{A} \}. \tag{3}
\]

Definition III.3 (Attack detectability). A non-zero attack signal \( \eta \in \mathcal{A} \) is detectable if \( \eta \notin \mathcal{B} \).

Definition III.4 (Attack correctability). A non-zero attack signal \( \eta \in \mathcal{A} \) is correctable if for all \( \eta' \neq \eta \), the following is satisfied
\[
\eta' \in \mathcal{A} \Rightarrow \eta - \eta' \notin \mathcal{B}. \tag{4}
\]

Our objectives in this paper are to first determine the feasibility of attack detection/correction and then to give an attack detection as well as correction method. We show that these methods are guaranteed to produce the correct outcome under certain assumptions about the attack set \( \mathcal{A} \).

IV. ATTACK DETECTION/CORRECTION FEASIBILITY

In this section we first address the vulnerability of a system given by (1) against attacks on its sensor outputs \( y \). We then introduce a concept that is central to this paper called the security index \( \delta(\Sigma) \) of the system \( \Sigma \), and then we state conditions to achieve attack detectability and correctability. These conditions are stated in terms of \( \delta(\Sigma) \). The definitions and results of this section can also be found in [2], [3].

Definition IV.1. The security index of the system \( \Sigma \) is defined as
\[
\delta(\Sigma) := \min_{\eta \neq 0, \eta \in \mathcal{B}} \|\eta\|. \tag{5}
\]

Theorem IV.1. (Attack detection capability of the system) Let \( \mathcal{A} = \{ \eta : \mathbb{Z}^+ \rightarrow \mathbb{R}^N \mid \|\eta\| < \delta(\Sigma) \text{ and } \eta \neq 0 \} \). All attack signals \( \eta \in \mathcal{A} \) are detectable.

Proof. For any attack signal \( \eta \) from \( \mathcal{A} \) we must have \( \eta \notin \mathcal{B} \) because of Definition IV.1. According to Definition III.3 \( \eta \) is then detectable and this completes the proof.

Because of the above theorem, the security index \( \delta(\Sigma) \) of a system \( \Sigma \) can be viewed as the minimum number of sensors that have to be attacked in order to implement an undetectable attack. For example, if a system \( \Sigma \) with \( N = 7 \) sensors has security index \( \delta(\Sigma) = 5 \), then at least 5 of the 7 sensors have to be attacked to achieve an undetectable attack. In accordance with [2], [3] we call a system \( \Sigma \) with \( N \) outputs maximally secure if its security index equals \( \delta(\Sigma) = N \).
Theorem IV.2. (Attack correction capability of the system) Let $\mathcal{A} = \{\eta : \mathbb{Z}_+ \to \mathbb{R}^N \mid \|\eta\| < \delta(\Sigma)/2 \text{ and } \eta \neq 0\}$. All attack signals $\eta \in \mathcal{A}$ are correctable.

Proof. Consider an attack signal $\eta$ from $\mathcal{A}$. If there exists another non-zero $\eta'$ with $\|\eta'\| < \delta(\Sigma)/2$ and $\eta' \neq \eta$, then $\|\eta - \eta'\| \leq \|\eta\| + \|\eta'\| < \delta(\Sigma)$ which implies $\eta - \eta' \not\in \mathcal{B}$. According to Definition III.4, $\eta$ is then correctable and this completes the proof. \qed

It follows from the above theorem that the value of $\delta(\Sigma)/2$ can be viewed as the minimum number of sensors that have to be attacked in order to implement an uncorrectable attack. For example, if a system $\Sigma$ with $N = 7$ sensors has security index $\delta(\Sigma) = 5$, then at least 3 of the 7 sensors have to be attacked to achieve an uncorrectable attack.

Remark IV.3. Regarding the detection and correction of an attack signal $\eta$, there are three different situations:

Case 1: $\|\eta\| < \delta(\Sigma)/2$, i.e., the attack signal is both detectable and correctable. This means that one can detect the existence of $\eta$ and there exists a unique correction for this attack signal.

Case 2: $\delta(\Sigma)/2 \leq \|\eta\| < \delta(\Sigma)$, i.e., the attack signal is detectable but not correctable. This means that one can detect the existence of $\eta$ but cannot guarantee the existence of a unique correction for this attack signal.

Case 3: $\delta(\Sigma) \leq \|\eta\|$, i.e., the attack signal is neither detectable nor correctable. This means that one cannot guarantee that the attack signal can be detected, also one cannot guarantee the existence of a unique correction for this attack signal.

V. Computation of the security index

In this section, we seek to express the security index $\delta(\Sigma)$ of a LTI system $\Sigma$ in terms of its kernel representation [1]. The following preliminaries on polynomial matrices are required.

A square polynomial matrix is called unimodular if it has a polynomial inverse and a non-square polynomial matrix is called left (right) unimodular if it has a polynomial left (right) inverse. Two polynomial matrices $R(\xi)$ and $Q(\xi)$ of the same size are called left unimodularly equivalent if there exists a unimodular matrix $U(\xi)$ such that $Q(\xi) = U(\xi)R(\xi)$. In the next two definitions $\mathcal{J}$ is assumed to be a subset of $\{1, \ldots, N\}$.

Definition V.1. Define $R_\mathcal{J}(\xi)$ as an $N \times |\mathcal{J}|$ matrix that consists of the $i$-th columns of $R(\xi)$ where $i \in \mathcal{J}$.

Definition V.2. Define $y_\mathcal{J}$ as the signal that consists of the $i$-th components of $y$ where $i \in \mathcal{J}$; thus $y_\mathcal{J}$ is the signal that consists of the $i$-th components of $y$ where $i \not\in \mathcal{J}$.

The following theorem is a reformulation of a result in [2], [3].

Theorem V.1 (Security index calculation). Consider a system $\Sigma$ whose behavior $\mathcal{B}$ is non-zero and given by $[1]$, where $R(\xi)$ has full rank. Then

$$\delta(\Sigma) = L + 1,$$

where $L$ is the largest integer such that for any subset $\mathcal{J} \subseteq \{1, \ldots, N\}$ of cardinality $L + 1$ such that $R_\mathcal{J}(\xi)$ is not left unimodular.

Proof. Clearly, there exists a subset $\mathcal{J} \subseteq \{1, \ldots, N\}$ of cardinality $L + 1$ such that $R_\mathcal{J}(\xi)$ is not left unimodular. Thus there exists a nonzero signal $y^*$ that satisfies $R_\mathcal{J}(\sigma)y^* = 0$. Now let $y : \mathbb{Z}_+ \to \mathbb{R}^N$ be the signal satisfying $y_\mathcal{J} = y^*$ and $y_{\mathcal{J}^c} = 0$. Then $y \in \mathcal{B}$ and $\|y\| = \|y^*\| \leq L + 1$. This implies that

$$\delta(\Sigma) \leq L + 1.$$ (7)

To prove that also $\delta(\Sigma) \geq L + 1$, let $y$ be a signal in $\mathcal{B}$ of weight $\delta(\Sigma)$. Define $\mathcal{J} \subset \{1, 2, \ldots, N\}$ as the set of cardinality $\delta(\Sigma)$ for which $y_{\mathcal{J}^c} = 0$. Then $R_{\mathcal{J}^c}(\xi)y_{\mathcal{J}^c} = 0$ and because $y_{\mathcal{J}^c} \not= 0$ it follows that $R_{\mathcal{J}^c}(\xi)$ is not left unimodular. This implies that $L < \delta(\Sigma)$. Because of (7) it follows that equation (7) holds. \qed

Corollary V.2. ([2] Cor. IV.6) The system $\Sigma$ in [1] is maximally secure if and only if all $N \times (N - 1)$ submatrices of $R(\xi)$ are left unimodular.

VI. Canonical kernel representation

When we describe a system’s behavior $\mathcal{B}$ using a kernel representation $R(\sigma)y = 0$, the polynomial matrix $R(\xi)$ is not unique. In this section, we recall results from the literature around equivalent kernel representations for $\mathcal{B}$. We then use this to single out a canonical form of $R(\xi)$ which is vital for our results on attack correction.

Theorem VI.1. (e.g. [13], Theorem 3.9) Consider two systems $\Sigma$ and $\Sigma'$ whose behaviors $\mathcal{B}$ and $\mathcal{B}'$ are given by $\mathcal{B} = \{y | R(\sigma)y = 0\}$ and $\mathcal{B}' = \{y' | R'(\sigma)y' = 0\}$, respectively. Assume that $R(\xi)$ and $R'(\xi)$ are square matrices of the same size, then $\mathcal{B} = \mathcal{B}'$ if and only if $R(\xi)$ and $R'(\xi)$ are left unimodularly equivalent.

Minimal lag kernel representations (i.e., where the row degrees of $R(\xi)$ are minimal, namely the system’s observability indices) are relevant to many problems in systems and control. However, it turns out that minimal lag representations do not lend themselves well to the
design of attack correction methods. Instead, we find that a different canonical form needs to be used, namely the Kronecker-Hermite form. We recall this theory in the next theorem and more generally in Theorem VII.3. In the next section we will see that it serves as an important tool for our attack correction method.

Theorem VI.2. (Kronecker-Hermite canonical kernel representation of a maximally secure system) Let $R(\xi)$ be a $N \times N$ polynomial matrix whose determinant is non-zero. Assume that all $N \times (N-1)$ submatrices of $R(\xi)$ are left unimodular. Then there exists a unimodular matrix $U(\xi)$ such that

$$U(\xi)R(\xi) = \begin{bmatrix} \mathbb{I}_{N-1} & -c_1(\xi) & \cdots & -c_N(\xi) \\ 0 & -c_2(\xi) & & \vdots \\ & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a(\xi) \end{bmatrix}, \quad (8)$$

where $c_j(\xi)$ is coprime with $a(\xi)$ and $\deg c_j(\xi) < \deg a(\xi)$ for all $j \in \{1, \ldots, N-1\}$.

Proof. It follows from e.g. Theorem B.1.1 in [19] that there exists a unimodular matrix $U_0(\xi)$ such that $U_0(\xi)R(\xi)$ is an upper triangular polynomial matrix, say $R_0(\xi)$, written as

$$R_0(\xi) = \begin{bmatrix} a_1(\xi) & b_{12}(\xi) & \cdots & b_{1(N-1)}(\xi) & b_{1N}(\xi) \\ 0 & a_2(\xi) & \cdots & b_{2(N-1)}(\xi) & b_{2N}(\xi) \\ & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{N-1}(\xi) & b_{(N-1)N}(\xi) \\ 0 & 0 & \cdots & 0 & a_N(\xi) \end{bmatrix}.$$  

Since all $N \times (N-1)$ submatrices of $R(\xi)$ are left unimodular, it follows that in particular the matrix formed by the first $N-1$ columns is left unimodular. This implies that all its diagonal elements are nonzero constants. Without restrictions, $R_0(\xi)$ can then be written as

$$R_0(\xi) = \begin{bmatrix} 1 & b_{12}(\xi) & \cdots & b_{1(N-1)}(\xi) & b_{1N}(\xi) \\ 0 & 1 & \cdots & b_{2(N-1)}(\xi) & b_{2N}(\xi) \\ & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{(N-1)N}(\xi) \\ 0 & 0 & \cdots & 0 & a_N(\xi) \end{bmatrix}. \quad (9)$$

It has been shown in e.g. [9 Theorem 2.40], [3 Theorem 7.5] or [11] that there exists a unimodular matrix $U_1(\xi)$ such that $U_1(\xi)R_0(\xi)$ is in Kronecker-Hermite canonical form as in (9) with $\deg a(\xi) > \deg c_j(\xi)$.

Finally, according to Corollary VII.2 the matrix in (8) has the property that all its $N \times (N-1)$ submatrices are left unimodular. It can be easily checked that this implies that $c_j(\xi)$ is coprime with $a(\xi)$, i.e., GCD$(c_j(\xi), a(\xi)) = 1$ for all $j \in \{1, \ldots, N-1\}$, and this completes the proof. \hfill $\Box$

VII. ATTACK DETECTION AND CORRECTION ALGORITHMS

In this section, we propose methods to achieve attack detection and correction. Recall (1) and (3), the output generated by the attacked system $\Sigma_A$ is $r = y + \eta$ where $y$ is the attack-free sensor output of the system $\Sigma$. In linear coding theory, syndrome computation is an effective method for error detection, see e.g. [20 Ch. 7]. Similarly we will work with a signal that we call the “residual signal”, defined as $s = R(\sigma)r$ to perform attack detection (Section VII-A). To achieve attack correction (Sections VII-B and VII-C), we use a majority vote rule reminiscent of decoding techniques such as in [21].

A. Attack detection

For attack detection, we propose Algorithm 1 below.

Algorithm 1 Attack detection

1: procedure $(R(\xi), r, \eta)$
   \hspace{1cm} \triangleright Given $R(\xi)$ and $r$, detect whether $\eta$ is the zero signal.
2: \hspace{1cm} Calculate $s = R(\sigma)r$.
3: \hspace{1cm} if $s = 0$ then decide no attack, i.e., $\eta = 0$.
4: \hspace{1cm} else decide attack occurred, i.e., $\eta \neq 0$.
5: \hspace{1cm} end if
6: end procedure

Theorem VII.1 (Attack detection). Consider a system given by (1). Let $r = y + \eta$ be a received signal with $y \in B$. Then the residual signal $s = 0$ if and only if $\eta \in B$. Thus Algorithm 1 gives the correct result if $\eta$ is detectable.

Proof. This follows straightforwardly from

$$s = R(\sigma)r = R(\sigma)(y + \eta) = R(\sigma)y + R(\sigma)\eta = R(\sigma)\eta. \quad \Box$$

B. Attack correction for a maximally secure system

In this section we show how the Kronecker-Hermite canonical form kernel representation of the previous section can be used to perform attack correction for a maximally secure system. Without loss of generality, we
assume that $R(\xi)$ is in the Kronecker-Hermite canonical form \(^8\). Thus the system is given by
\[
\begin{bmatrix} \mathbb{I}_N \\ 0 \end{bmatrix} y = \begin{bmatrix} c_1(\xi) \\ \vdots \\ c_{N-1}(\xi) \\ 1 \end{bmatrix} y_N. \tag{10}
\]
Before defining our method of attack correction, we need the following definitions and computations:
- Define polynomials $p_j(\xi)$ and $q_j(\xi)$ satisfying
\[
[p_j(\xi) \quad q_j(\xi)] \begin{bmatrix} c_j(\xi) \\ a(\xi) \end{bmatrix} = 1, \forall j \in \{1, \ldots, N-1\}. \tag{11}
\]
Note that the existence of $p_j(\xi)$ and $q_j(\xi)$ follows from the fact that $\gcd(c_j(\xi), a(\xi)) = 1$; the Extended GCD Algorithm (e.g. Ch 4.2 in [24]) can be used to find $p_j(\xi)$ and $q_j(\xi)$.
- The majority vote function over a set of signals \(\{v_1, v_2, \ldots, v_L\}\), denoted by $\text{Maj}\{v_1, v_2, \ldots, v_L\}$, is defined to be the most frequently occurring signal in the set of $v_j$’s.

Algorithm 2 Attack correction for a maximally secure system given by (10)

1: \textbf{procedure} \(a(\xi), c_1(\xi), \ldots, c_{N-1}(\xi), r, \hat{y}\) \>
\>
2: \textbf{Given} \(a(\xi), c_j(\xi)\)’s and \(r\), compute \(\hat{y}\).
3: \textbf{Calculate}
\[
\hat{y}_N = \text{Maj}\{p_1(\sigma)r_1, p_2(\sigma)r_2, \ldots, p_{N-1}(\sigma)r_{N-1}, r_N\},
\tag{12}
\]
where \(p_j(\xi)\) is defined as in (11).
4: \textbf{return} \(\hat{y} = [\hat{y}_1 \quad \hat{y}_2 \quad \ldots \quad \hat{y}_N]\).
5: \textbf{end procedure}

Theorem VII.2. Consider a maximally secure system \(\Sigma\) given by (1). Let the received signal \(r\) be input to Algorithm 2. Assume that \(r = y + \eta\) with \(y \in \mathcal{B}\) and \(\|\eta\| < N/2\). Then the output \(\hat{y}\) of Algorithm 2 equals \(y\).

\textbf{Proof.} We have
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} y_j = \begin{bmatrix} c_j(\sigma) \\ a(\sigma) \end{bmatrix} y_N,
\]
so that it follows from (11) that \(p_j(\sigma)y_j = y_N\) for \(j = 1, 2, \ldots, N\) (here we define \(p_N(\xi) \equiv 1\)). Now
\[
p_j(\sigma)r_j = p_j(\sigma)(y_j + \eta_j) = y_N + p_j(\sigma)\eta_j
\]
for \(j = 1, 2, \ldots, N\). Since \(\|\eta\| < N/2\) it follows that (12) computes \(\hat{y}_N = y_N\). Consequently \(y_j\) can be found from \(y_j = c_j(\sigma)y_N\) for \(j = 1, 2, \ldots, N - 1\) and this proves the theorem.

\[\square\]

Note that the results in this subsection require the system to be maximally secure. The next subsection deals with the general case where systems are not necessarily maximally secure.

C. Attack correction for the general case

Theorem VII.3. (Kronecker-Hermite canonical kernel representation of \(R(\xi)\)—general case) Let \(R(\xi)\) be a \(N \times N\) polynomial matrix whose determinant is nonzero. Let \(L\) be the largest integer such that, for any subset \(J \subseteq \{1, \ldots, N\}\) of cardinality \(L\), the \(N \times L\) matrix \(R_J(\xi)\) is left unimodular. Then there exists a unimodular matrix \(U(\xi)\) such that
\[
U(\xi)R(\xi) = \begin{bmatrix} \mathbb{I}_L & -M_1(\xi) \\ 0 & \ddots & & -M_L(\xi) \\ \vdots & & \ddots & \ddots \\ 0 & \ldots & 0 & \mathbb{I}_L \end{bmatrix} D(\xi). \tag{13}
\]
where \(D(\xi)\) is an upper triangular matrix and the degree of the diagonal entries of \(D(\xi)\) denoted as \(\deg d_{ii}(\xi)\) for \(i \in \{1, \ldots, N - L\}\), is strictly the highest within the corresponding column of (13).

\textbf{Proof.} The proof follows the same reasoning as the proof of Theorem VII.2 but replacing \(N - 1\) by \(L\).

Combining Theorem V.1 and Theorem VII.3 it follows that the signals \(y\) in the behavior of a system \(\Sigma\) with security index \(\delta\) are given by the following representation
\[
\begin{bmatrix} \mathbb{I}_{\delta-1} & 0 \\ 0 & \mathbb{I}_{N-\delta+1} \end{bmatrix} y = \begin{bmatrix} M_1(\sigma) \\ \vdots \\ \mathbb{I}_{N-\delta+1} \end{bmatrix} D(\sigma) \ell, \tag{14}
\]
where the signal \(\ell\) is an auxiliary signal that can be interpreted as a “state signal” that drives the system’s behavior. In the representation (14) the signal \(\ell\) simply coincides with the last \(N - \delta + 1\) components of \(y\).

In fact, the above representation (14) is a special case of a more general representation [27], [14], given as
\[
\begin{bmatrix} \mathbb{I}_N \\ 0 \end{bmatrix} y = \begin{bmatrix} M(\sigma) \\ D(\sigma) \end{bmatrix} \ell, \tag{15}
\]
where \(\ell : \mathbb{Z}_+ \rightarrow \mathbb{R}^m\), \(M(\xi)\) is a \(N \times m\) polynomial matrix and \(D(\xi)\) is a \(m \times m\) polynomial matrix, for some integer \(m\). We make the observability assumption that the \((N + m) \times m\) polynomial matrix \(\begin{bmatrix} M(\xi) \\ D(\xi) \end{bmatrix}\) is left unimodular, noting that this clearly holds for the above
representation (14). Note that the representation (8) of the previous subsection is a special case of (15), namely

\[
M(\xi) := \begin{bmatrix} c_1(\xi) \\ \vdots \\ c_{N-1}(\xi) \\ 1 \end{bmatrix}
\quad \text{and} \quad D(\xi) := a(\xi).
\]

As a first step towards attack correction in the general case, we express the system’s security index in terms of the polynomial matrices \( M(\xi) \) and \( D(\xi) \) of the general representation (15).

**Theorem VII.4.** Consider a system \( \Sigma \) whose behavior \( B \) is nonzero and given by (15). Then

\[
\delta(\Sigma) = N + 1 - \bar{L},
\]

where \( \bar{L} \) is the smallest integer such that for any subset \( \mathcal{J} \subseteq \{1, \ldots, N\} \) of cardinality \( \bar{L} - 1 \) such that \( [M_J(\xi) D(\xi)] \) is not left unimodular.

**Proof.** Clearly, there exists a subset \( \mathcal{J} \subseteq \{1, \ldots, N\} \) of cardinality \( \bar{L} - 1 \) such that \( [M_J(\xi) D(\xi)] \) is not left unimodular. Thus there exists a nonzero signal \( \ell^* \) that satisfies

\[
[M_J(\sigma) D(\sigma)] \ell^* = 0.
\]

Now consider the signal \( y \) defined as

\[
y := M(\sigma) \ell^*.
\]

Clearly \( ||y|| \leq N - (\bar{L} - 1) = N - \bar{L} + 1 \). This implies that

\[
\delta(\Sigma) \leq N - \bar{L} + 1.
\]

To prove that also \( \delta(\Sigma) \geq N - \bar{L} + 1 \), let \( y^* \) be a signal in \( B \) of weight \( \delta(\Sigma) \). Thus there exists a nonzero signal \( \ell^* \) such that

\[
\begin{bmatrix} 1_N \\ \eta \end{bmatrix} y^* = [M(\sigma) D(\sigma)] \ell^*.
\]

Define \( \bar{J} \subset \{1, 2, \ldots, N\} \) as the set of cardinality \( \delta(\Sigma) \) for which \( y^*_\bar{J} = 0 \). Then \([M_{\bar{J}}(\sigma) D(\sigma)] \ell^* = 0\) and because \( \ell^* \neq 0 \), it follows that \([M_{\bar{J}}(\sigma) D(\sigma)]\) is not left unimodular. This implies that \( \bar{L} \geq N - \delta(\Sigma) + 1 \). Because of (17), it follows that equation (16) holds.

Before defining our method of attack correction, we need several preliminary computations. Let \( \mathcal{J} \) be a subset of \( \{1, \ldots, N\} \) of cardinality \( N + 1 - \delta \). Suppose that the matrix \([M_\mathcal{J}(\xi) D(\xi)]\) is left unimodular. Define polynomial matrices \( P_\mathcal{J}(\xi) \) and \( Q_\mathcal{J}(\xi) \) such that

\[
[M_\mathcal{J}(\xi) D(\xi)] = \begin{bmatrix} P_\mathcal{J}(\xi) & Q_\mathcal{J}(\xi) \end{bmatrix} = I_{N+1-\delta}.
\]

**Algorithm 3** Attack correction for general system given by (15)

1: procedure \((M(\xi), D(\xi), \delta, r, y)\)

\[ \triangleright \text{Given } M(\xi), D(\xi), \delta \text{ and } r, \text{ compute } \hat{y}. \]

2: Calculate

\[
\hat{\ell} = \text{Maj}\{P^J(\sigma)r_J\},
\]

where the majority vote is taken over all subsets \( J \) of cardinality \( N + 1 - \delta \) and \( P_J(\xi) \) is defined as in (19).

3: \[ \hat{y} = M(\sigma)\hat{\ell}. \]

4: return \( \hat{y} \).

5: end procedure

In Algorithm 3 we can interpret each computation \( P^J(\sigma)r_J \) as an exact observer (see proof of Theorem VII.5) that produces an estimated signal \( \hat{\ell} \) from the received signal \( r \). The algorithm directs us to take a majority vote of all such observer outcomes and to declare this signal to be the correct signal \( \ell \). As we will see below, this algorithm and the next theorem are the main results of this paper. We note that the next theorem requires a nontrivial proof, reminiscent of majority vote proofs in the classical coding literature, such as [21].

**Theorem VII.5.** Consider a system \( \Sigma \) given by (1); denote its security index by \( \delta \). Let the received signal \( r \) be input to Algorithm 3 Assume that \( r = y + \eta \) with \( y \in B \) and \( ||\eta|| < \delta/2 \). Then the output \( \hat{y} \) of Algorithm 3 equals \( y \).

**Proof.** Let’s denote \( ||\eta|| = \text{number of attacked sensors by } t \), thus \( 2t < \delta \). Let \( J \) be a subset of cardinality \( N + 1 - \delta \) from the set of unattacked sensors. Then

\[
\begin{bmatrix} 1_N \\
0 \end{bmatrix} r_J = \begin{bmatrix} 1_N + 1-\delta \\
0 \end{bmatrix} y_J = \begin{bmatrix} M_J(\sigma) \\ D(\sigma) \end{bmatrix} \ell, \tag{21}
\]

where \( \ell \) is the correct signal. We first show that (20) is well defined. Because of Theorem VII.4, the matrix \([M_\mathcal{J}(\xi) D(\xi)]\) is left unimodular, so that matrices \( P_\mathcal{J}(\xi) \) and \( Q_\mathcal{J}(\xi) \) can be found such that (19) holds. Using (19), it follows from (21) that \( \hat{\ell} = P^J(\sigma)r_J \) equals the correct signal \( \ell \). There are \( N_{N+1-\delta} \) ways to choose a subset \( J \) of cardinality \( N + 1 - \delta \) from the set of unattacked sensors. Each of these choices leads to \( \hat{\ell} \) as the correct signal \( \ell \).

Next, let’s consider a subset \( J^* \) of cardinality \( N + 1 - \delta \) that leads to a signal \( \hat{\ell} \) that is incorrect, say \( \ell^* \neq \ell \). Clearly \( J^* \) must involve one or more attacked sensors. Since \( J^* \) has only \( N + 1 - \delta \) elements, it follows that
all unattacked sensors that are involved in $\mathcal{J}^*$ fit into a certain subset, say $I$ of $N - \delta$ unattacked sensors. Now define the set $I^*$ as the union of $I$ and all $t$ attacked sensors. Then a set $\mathcal{J}$ of cardinality $N + 1 - \delta$ that is a subset of $I^*$ may lead to the same incorrect $\ell^*$. Let’s now consider a set $\mathcal{J}$ of cardinality $N + 1 - \delta$ that leads to the incorrect signal $\ell^*$ but that is not a subset of $I^*$. We note that $\mathcal{J}$ must then involve an unattacked sensor outside of $I$. Thus there exist more than $N - \delta$ unattacked sensors that lead to the same signal $\ell^*$. Since any set of $N - \delta + 1$ unattacked sensors lead to the correct signal $\ell$, it follows that $\ell^*$ must be the correct signal $\ell$, which is a contradiction. We conclude that $\mathcal{J}$ does not lead to $\ell^*$.

From this reasoning, it follows that there are at most $\binom{N - \delta + t}{N + 1 - \delta}$ choices of $\mathcal{J}$ that lead to the same incorrect $\ell^*$. Now recall from the first part of this proof that there are at least $\binom{N - t}{N + 1 - \delta}$ choices of $\mathcal{J}$ that lead to the correct signal $\ell$. Since $2t < \delta$ we have

$$\binom{N - t}{N + 1 - \delta} > \binom{N - \delta + t}{N + 1 - \delta},$$

so that the majority vote in Algorithm 3 leads to the correct $\ell$. As a result, $\hat{y} = M(\sigma)\ell = M(\sigma)\ell$ equals the correct signal $y$ and this proves the theorem.

**D. Comparison with the literature**

The attack correction method for the general case of Subsection VII-C is useful because the representation (15) is very general. For example, a special case of (15) is

$$M(\xi) := \begin{bmatrix} c_1(\xi) \\ \vdots \\ c_N(\xi) \end{bmatrix} \quad \text{and} \quad D(\xi) := a(\xi),$$

and we see that Lemma IV-8 of [4] then follows immediately from the above theorem.

Another special case is a state space representation—then $M(\xi)$ equals a constant $N \times n$ “output matrix” $C$ and $D(\xi) = \xi I_n - A$, where $A$ is a constant “state transition matrix”. As this is the type of representation that is considered in e.g. [4], [23], [22], [7], we are now in a position to compare our results with the literature in this area. In contrast to our representation-free definition of “security index”, much of the literature implicitly defines a similar notion in terms of the matrices $A$ and $C$ of the state space representation. For example [4] defines $M$-attack observability of a state space representation $(A, C)$ in terms of the matrices $A$ and $C$. In terms of our notion of security index $\delta(\Sigma)$, any observable state space representation $(A, C)$ is $[\delta(\Sigma)/2]$-attack observable. Alternatively, in the terminology of [23], [22], this is phrased as $(\delta(\Sigma) - 1)$-sparse attack observability.

Comparing our Algorithm 3 to the noise-free attack correction method of [4, Section III-A] and [22, Section III-A], we see that our Algorithm 3 is simpler with lower computational complexity, as it requires fewer observers and fewer observer comparisons.

**VIII. Examples**

We illustrate the workings of the proposed attack correction method with two examples. In the first example the $A$ matrix is in companion form so as to be able to illustrate our theory with very simple observers. The second example has an $A$ matrix that is not in companion form.

**A. Example 1**

Consider a discrete-time LTI system given by a state space representation $(A, C)$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix}, \quad C = I_3, \quad y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$  

Note that this system is marginally stable with three distinct eigenvalues: $\lambda_1 = 1 \angle 60^\circ$, $\lambda_2 = 1 \angle -60^\circ$ and $\lambda_3 = \frac{1}{2}$. Its Kronecker-Hermite canonical kernel representation can be computed as

$$R(\xi) = \begin{bmatrix} 1 & 0 & -6\xi^2 + 7\xi - 6 \\ 0 & 1 & -2\xi^2 + 3\xi - 3 \\ 0 & 0 & \xi^3 - \frac{3}{2}\xi^2 + \frac{3}{2}\xi - \frac{1}{2} \end{bmatrix}.$$  

Using Corollary VII.2 we find that the system is maximally secure, i.e., $\delta(\Sigma) = 3$. According to Theorems IV.1 and IV.2 the system can detect up to 2 sensor attacks and it can correct any single sensor attack. The value of observer $p_j(\xi)$s and corresponding $q_j(\xi)$s that satisfy equation (11) can be computed as follows:

$$p_1(\xi) = \xi^2, \quad q_1(\xi) = -6\xi - 2$$
$$p_2(\xi) = \xi, \quad q_2(\xi) = -2$$

Algorithm 2 yields the following attack correction outcomes:

$$\hat{y}_3 = \text{Maj}\{\sigma_1^2 r_1, \sigma r_2, r_3\}$$
$$\hat{y}_1 = (6\sigma^2 - 7\sigma + 6)\hat{y}_3$$
$$\hat{y}_2 = (2\sigma^2 - 3\sigma + 3)\hat{y}_3$$

Figure 1 shows the block diagram of the attack correction method.

Next, we conduct a simulation to illustrate the workings of the attack correction method for this example. First, we specify a single sensor attack signal on sensor 3, as illustrated in Figure 2. Here the attack signal is
Algorithm 2 indeed reconstructs the attacked 3rd output signal correctly in accordance with Theorem VII.2. The inequality at the first 4 time instants is caused by the latency in observers $p_j(\xi)$s and output regeneration shift operators $c_j(\xi)$, more specifically the latency equals $\max_i \deg p_i(\xi) + \max_j \deg c_j(\xi) = 4$.

**B. Example 2**

Consider a 6-output stable voltage-source converter [12] that is modelled as a discrete-time LTI system. After discretization by means of the zero-order hold method with sampling period $T_s = 200 \mu s$, the resulting $R(\xi)$ matrix is given by

$$R(\xi) = \xi I_6 - \exp(\bar{A}T_s),$$

where $\bar{A}$ is given by

$$\bar{A} = \begin{bmatrix}
-\frac{R_1}{L_1} & \omega_1 & 0 & 0 & -\frac{1}{L_1} & 0 \\
-\omega_1 & -\frac{R_1}{L_1} & 0 & 0 & -\frac{1}{L_1} & 0 \\
0 & 0 & -\frac{R_2}{L_2} & \omega_1 & \frac{1}{L_2} & 0 \\
0 & 0 & -\omega_1 & -\frac{R_2}{L_2} & \frac{1}{L_2} & 0 \\
\frac{1}{C_0} & 0 & -\frac{1}{C_0} & 0 & -\omega_1 & 0 \\
0 & \frac{1}{C_0} & 0 & -\omega_1 & -\omega_1 & 0 \\
\end{bmatrix},$$

with parameters shown in Table I.

| Parameter | Value |
|-----------|-------|
| $L_1$     | 4.3mH |
| $R_1$     | 67.3m\Omega |
| $L_2$     | 2.4mH |
| $R_2$     | 83.1m\Omega |
| $C_0$     | 18\mu F |
| $\omega_1$ | 100\pi \text{Rad/s} |
The Kronecker-Hermite canonical form of $R(\xi)$ can be computed as follows, where we denote $a \times 10^b$ as $aeb$.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0.74e2\xi^5 - 1.8e3\xi^4 + 2.9e3\xi^3 \\
0 & 1 & 0 & 0 & -2.9e3\xi^2 + 1.8e3\xi - 7.3e2 \\
0 & 0 & 1 & 0 & 9.4e5 - 2.7e2\xi^4 + 4.3e2\xi^3 \\
0 & 0 & 0 & 1 & -4.8e2\xi^2 + 2.9e2\xi - 1.4e2 \\
0 & 0 & 0 & 0 & 7.4e2\xi^5 - 1.8e3\xi^4 + 2.9e3\xi^3 \\
0 & 0 & 0 & 0 & -2.9e3\xi^2 + 1.8e3\xi - 7.3e2 \\
0 & 0 & 0 & 0 & 9.4e5 - 2.7e2\xi^4 + 4.3e2\xi^3 \\
0 & 0 & 0 & 0 & -4.8e2\xi^2 + 2.9e2\xi - 1.4e2 \\
0 & 0 & 0 & 0 & 4.7e5 - 3.2e4 + 3.3e3 \\
0 & 0 & 0 & 0 & -2.4e3 + 1.2e - 3.3 \\
0 & 0 & 0 & 0 & 3.6e4\xi^6 - 1.3e5\xi^5 + 2.3e5\xi^4 - 2.9e5\xi^3 + 2.3e5\xi^2 - 1.2e5\xi + 3.6e4
\end{bmatrix}$$

Using Corollary \[\underline{\underline{2}}\] we find that the system is maximally secure, i.e., $\delta(\Sigma) = 6$. Furthermore, we compute the observers $p_j(\xi)$s as

\begin{align*}
p_1(\xi) &= 1.3e2\xi^5 - 2.7e2\xi^4 + 2.2e2\xi^3 - 69\xi^2 - 88\xi + 78 \\
p_2(\xi) &= 4.1e - 2.5 - 19\xi^4 + 44\xi^3 - 65\xi^2 + 75\xi - 35 \\
p_3(\xi) &= -72\xi^5 + 1.5e2\xi^4 - 1.2e2\xi^3 + 39\xi^2 + 49\xi - 44 \\
p_4(\xi) &= -2.3e-3\xi^5 + 11\xi^4 - 24\xi^3 + 36\xi^2 - 42\xi + 19 \\
p_5(\xi) &= -4.7e5 + 3.2e4 - 3.3e3 + 2.4e2 - 1.2e + 3.3.
\end{align*}

According to Theorems \[\underline{\underline{1}}\] and \[\underline{\underline{2}}\] any attacks on maximally 5 of the outputs are guaranteed to be detected and attacks on maximally 2 of its outputs are guaranteed to be corrected. Note that for this example it is easy to construct attack scenarios that attack 3 or 4 of the outputs but can still be corrected by our method.

**IX. CONCLUSIONS AND FUTURE WORK**

In this paper, we proposed attack detection and correction methods for zero-input discrete LTI systems in the noise-free case. The purpose of this paper is to provide a proof of concept around the application of ideas from error control coding theory to handle attacks on LTI systems. We have shown how the interaction between the inner system dynamics and the sensor placements determines the vulnerability of the system against sensor attacks. We quantified this via the notion of a system security index. We presented detection and correction methods that exploit the known dynamics of the system. In these methods the security index plays a prominent role.

Future research directions are: build on the discussed results to develop attack detection and correction methods for different system models such as systems with disturbance and noise, multi-input multi-output systems, nonlinear systems and systems over finite alphabets.

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