Figure 1
Figure 2
Figure 3
Figure 6
Figure 7
Figure 8
Figure 9
Figure 10
Figure 11
Figure 12
Figure 13
Figure 14
Figure 15
Figure 16
The STRUCTURE of BRANCHING in ANOSOV FLOWS of 3-MANIFOLDS

Sérgio R. Fenley
Mathematical Sciences Research Institute and
University of California, Berkeley

1 Introduction

In this article we study the topological structure of the lifts to the universal of the stable and unstable foliations of 3-dimensional Anosov flows. In particular we consider the case when these foliations do not have Hausdorff leaf space. We completely determine the structure of the set of non separated leaves from a given leaf in one of these foliations. As a consequence of this suspensions are characterized, up to topological conjugacy, as the only 3-dimensional Anosov flows without freely homotopic closed orbits. Furthermore the structure of branching is related to the topology of the manifold: if there are infinitely many leaves not separated from each other, then there is an incompressible torus transverse to the flow. Transitivity is not assumed for these results. Finally, if the manifold has negatively curved fundamental group we derive some important properties of the limit sets of leaves in the universal cover.

This article deals with a powerful technique for analysing Anosov flows in dimension 3, namely the study of the topological structure of the (weak) stable and unstable foliations when lifted to the universal cover. This technique was introduced in a remarkable paper of Verjovsky [Ve] in order to study codimension one Anosov flows. If the lifted (say) stable foliation has Hausdorff leaf space, then it is homeomorphic to the set of real numbers and we say that the stable foliation in the manifold is $\mathbb{R}$-covered. When both foliations are $\mathbb{R}$-covered the flow is said to be $\mathbb{R}$-covered. Two early uses of this technique were: (1) Ghys [Gh] proved that an Anosov flow in a Seifert fibered space is $\mathbb{R}$-covered. This was an essential step in showing that the flow is, up to finite covers, topologically conjugate to a geodesic flow in the unit tangent bundle of a closed surface of negative curvature (briefly, a geodesic flow). (2) If the fundamental group of the manifold is solvable then the $\mathbb{R}$-covered property, proved by Barbot [Ba1, Ba2], is again an essential step in Plante’s proof [Pl2, Pl3] that the flow is topologically conjugated to

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§1. Introduction

a suspension of an Anosov diffeomorphism of the torus (a suspension). In fact this last result holds for any codimension one Anosov flow.

More recently, a lot of information has been gained by analysing not just the individual leaf spaces, but rather the joint topological structure of the stable and unstable foliations. Using this and Dehn surgery on closed orbits of suspensions or geodesic flows [Fr, Go], a large family of examples was constructed where every closed orbit of the flow is freely homotopic to infinitely many other closed orbits [Fe3]. This never happens for suspensions or geodesic flows, and was thought to be impossible for any Anosov flow.

In addition the topological study gives information about metric properties of flow lines: We say that a flow is quasigeodesic if flow lines are uniformly efficient (up to a bounded multiplicative distortion) in measuring distances in relative homotopy classes. Suspensions and geodesic flows are always quasigeodesic and there are many quasigeodesic “pseudo-Anosov” flows in hyperbolic 3-manifolds [Ca-Th, Mos]. The Dehn surgery construction mentioned above produces a large family of Anosov flows in hyperbolic manifolds which are not quasigeodesic.

Barbot [Ba3, Ba4] also used this topological theory to study Anosov flows and proved the following remarkable result: Assume that there is a Seifert fibered piece of the torus decomposition of the manifold [Jo, Ja-Sh] and suppose that the corresponding fiber is not freely homotopic to a closed orbit of the flow. First isotopically adjust the boundary tori to be as transverse to the flow as possible [Ba3]. Then the flow in that piece is topologically conjugate to a (generalized) geodesic flow on the unit tangent bundle of a compact surface with boundary. If the manifold is a graph manifold and all fibers satisfy the condition above, then the flow is up to topological conjugacy obtained by Dehn surgery on finitely many closed orbits of a geodesic flow [Ba5]. Using this Barbot [Ba4, Ba5] has obtained the first known examples of graph manifolds which are neither torus bundles over the circle, nor Seifert fibered spaces and which do not admit Anosov flows.

The results above are in great part due to a complete characterization of the possible joint topological structures of $\mathbb{R}$-covered Anosov flows [Ba2, Fe3]. On the other hand very little is known about the non $\mathbb{R}$-covered case, for the simple reason that their structure is not understood at all. The purpose of this article is to start a systematic study of Anosov flows which are not $\mathbb{R}$-covered, where we then say the lifted foliations have branching.

It is easy to show that intransitivity implies that the flow is not $\mathbb{R}$-covered [So, Ba1] and for many years there was a great effort in trying to prove that these two properties are equivalent [Ve, Gh, Fe3, Ba2]. However in a
surprising development Bonatti-Langevin [Bo-La] have recently constructed a transitive, non $\mathbb{R}$-covered Anosov flow in dimension 3. Their example has an embedded torus transverse to the flow.

This leads us to two basic and very important questions concerning branching: (1) when can branching occur and (2) what are the possible structures of branching in Anosov flows of 3-manifolds. In this article we give a complete answer to the second question. We then show that the structure of branching is strongly related to dynamics of the flow, the topology of the manifold and the metric behavior of the stable and unstable foliations.

Let then $\Phi$ be an Anosov flow in $M^3$ with two dimensional stable and unstable foliations $\mathcal{F}^s, \mathcal{F}^u$. Let $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$ be the respective lifts to the universal cover $\tilde{M}$. Let $\mathcal{H}^s$ and $\mathcal{H}^u$ denote the leaf spaces of $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ respectively. If $\mathcal{F}^s$ is not $\mathbb{R}$-covered, then $\mathcal{H}^s$ is not Hausdorff. The branching leaves of $\tilde{\mathcal{F}}^s$ correspond to the non Hausdorff points in $\mathcal{H}^s$. Two leaves $F \neq F'$ of $\tilde{\mathcal{F}}^s$ form a branching pair if the corresponding points in $\mathcal{H}^s$ are not separated from each other. This is equivalent to saying that $F, F'$ do not have disjoint saturated neighborhoods in $\tilde{M}$, where a saturated neighborhood of $\tilde{\mathcal{F}}^s$ is an open set which is a union of leaves of $\tilde{\mathcal{F}}^s$. Similarly for $\tilde{\mathcal{F}}^u$.

Since the universal cover is simply connected, $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are always transversely orientable and an orientation is chosen. Then there is a notion of branching in the positive or negative directions. The first non trivial result about the structure of branching is the following [Fe5]: Suppose the flow is transitive. If there is branching in the positive direction of (say) the stable foliation then this foliation also has branching in the negative direction. This is a result about the “global” structure of branching.

We analyse the “local” structure of branching. For general foliations the branching of the lifted foliations can be very complicated [Im]. We will show here that branching in Anosov foliations is of a simple type which is very rigid. For simplicity many theorems are stated for $\tilde{\mathcal{F}}^s$ but they work equally well for $\tilde{\mathcal{F}}^u$.

A leaf of $\tilde{\mathcal{F}}^s$ or $\tilde{\mathcal{F}}^u$ is said to be periodic if it is left invariant by a non trivial covering translation of the universal cover. Equivalently, its image in $M$ contains a closed orbit of $\Phi$. We first show that branching puts a restriction in the type of the leaf.

**Theorem A** Let $\Phi$ be an Anosov flow in $M^3$. If $F$ is a branching leaf of $\tilde{\mathcal{F}}^s$, then $F$ is periodic.

Theorem A should be interpreted as a rigidity result in the sense that
periodic leaves are “rigid” while non periodic leaves are non rigid. This is
best seen in the manifold: if the stable leaf (in the manifold) is periodic,
then it contains a closed orbit of $\Phi$ and every orbit in the leaf is forward
asymptotic to this closed orbit. The nearby returns are in the same local
stable leaf. In case the leaf is not periodic the forward orbits limit in orbits
in the manifold, but the nearby returns are always in distinct local stable
leaves. This means that when lifted to the universal cover one can perturb
slightly the local structure, which will then produce a contradiction.

Our next goal is to understand the structure of the set $\mathcal{E}$ of non separated
leaves from a given leaf $F$ of (say) $\tilde{\mathcal{F}}^s$. There is a natural order in $\mathcal{E}$ given
by: if $E, L \in \mathcal{E}$ then we say that $E < L$ if there are $G, H \in \tilde{\mathcal{F}}^u$ with
$G \cap E \neq \emptyset, H \cap L \neq \emptyset$ and $G$ is in the back of $H$, see fig. 1. It is easy to see
that this is a total order in $\mathcal{E}$. Using this we can say that a branching leaf
$D$ is between $E$ and $L$ if $E < D < L$.

Figure 1: The set of non separated leaves from $F \in \tilde{\mathcal{F}}^s$. $D$ is between $E$ and $L$.

One measure of the complexity of branching is the number of branching
leaves between any $E, L \in \mathcal{E}$. A priori there could be infinitely many in
between branching leaves producing a very complicated structure. We prove:

**Theorem B** Let $\Phi$ be an Anosov flow in $M^3$. Let $F$ be a branching leaf
of $\tilde{\mathcal{F}}^s$ and $\mathcal{E}$ be the set of non separated leaves from $F$ with the total order
defined above. Then either

1. $\mathcal{E}$ is finite, hence order isomorphic to $\{1, 2, ..., n\}$ or,
2. $\mathcal{E}$ if infinite and order isomorphic to the set of integers $\mathbb{Z}$.

In particular given any $E, L \in \mathcal{E}$, there are only finitely many branching
leaves between them.

As in the case of theorem A, there is a rigidity proof of this result.
However it is quite long and complicated. Our tactic will be to first show:
Theorem C Let $\Phi$ be an Anosov flow in $M^3$ and let $(F, L)$ be a branching pair of $\tilde{F}^s$. Let $g$ be a non trivial covering translation with $g(F) = F$ and so that $g$ preserves transversal orientations to $\tilde{F}^s, \tilde{F}^u$. Then $g(L) = L$.

Using the important idea of lozenges (see definition in section 3) and a key result from [Fe4], theorem $B$ is an easy consequence of theorem $C$, except that to rule out the case that $E$ is order isomorphic to the natural numbers $\mathbb{N}$ we need theorem $E$ below. Section 4 contains a more detailed description of the set $E$.

Theorem $C$ implies that $\pi(F)$ and $\pi(L)$ contain freely homotopic closed orbits of the flow $\Phi$, which highlights the pervasiveness of freely homotopic orbits. This shows that the topological structure of the foliations is intimately related to the dynamics of the flow:

Corollary D Let $\Phi$ be an Anosov flow in $M^3$. Then $\Phi$ is topologically conjugate to a suspension of an Anosov diffeomorphism of the torus if and only if there are no freely homotopic closed orbits of $\Phi$ (including non trivial free homotopies of a closed orbit to itself).

This result does not assume that $\Phi$ is not $R$-covered. Another consequence of theorem $C$ is the following:

Theorem E Let $\Phi$ be a non $R$-covered Anosov flow in $M^3$. Then up to the action of covering translations, there are finitely many branching leaves in $\tilde{F}^s$. Equivalently there are finitely many distinguished closed orbits of $\Phi$ in $M$ so that their stable leaves lift to branching leaves in the universal cover.

It is very important to stress here that in the above results we do not assume that the flow is transitive nor is there any assumption on the manifold. Consequently these results are the most general possible. We also remark that theorems $A, B, C$ and $E$ were previously proved under the assumption that $M$ has negatively curved fundamental group and furthermore that the flow is quasigeodesic [Fe4]. This last hypothesis is a very strong assumption. The above results use only the topological structure of the lifted foliations and have no metric assumption.

We also show that the structure of branching is strongly related to the topology of the ambient manifold. We say that there is *infinite branching* if there are infinitely many leaves which are not separated from each other, otherwise we say that the branching is finite. An easy corollary of theorem $E$ is the following:
Corollary F  Let $\Phi$ be an Anosov flow in $M^3$ orientable, atoroidal. Then infinite branching cannot occur.

Even though the proof of corollary F is easy, it depends on a deep result of Gabai, namely the general torus theorem [Ga] which in turn depends on the solution of the Seifert fibered conjecture. In addition the proof uses the characterization of Anosov flows in Seifert fibered 3-manifolds [Gh]. In section 5 we study product regions (see definition in section 5) and then prove the following stronger result, using only the study of the topological structure of $\tilde{F}^s, \tilde{F}^u$:

Theorem G  Let $\Phi$ be an Anosov flow in $M^3$ orientable so that there is infinite branching in $\tilde{F}^s$. Then there is infinite branching in $\tilde{F}^u$. Furthermore there is an embedded torus $T$ transverse to $\Phi$, hence $T$ is incompressible.

We remark that infinite branching does occur, for example in the Bonatti-Langevin flow. Furthermore we show that finite (but non trivial) branching also occurs for a large class of Anosov flows, for example in the flows constructed by Franks and Williams [Fr-Wi].

Finally we apply these results to the case when $M$ has negatively curved fundamental group. Then $\tilde{M}$ is compactified with a sphere at infinity $S^2_\infty$. Furthermore the intrinsic geometry of a leaf $F$ of $\tilde{F}^s$ or $\tilde{F}^u$ is always negatively curved in the large so there is an intrinsic ideal boundary $\partial_\infty F$. In these manifolds it is fundamental to understand asymptotic behavior of sets in $\tilde{M}$ [Th1, Th2], [Mor], [Bon]. We say that $\tilde{\Phi}$ has the continuous extension property if the embedding $\varphi : F \to \tilde{M}$ extends continuously to $\varphi : F \cup \partial_\infty F \to \tilde{M} \cup S^2_\infty$, for any leaf $F$ in $\tilde{F}^s$ or $\tilde{F}^u$. This relates the foliation to the geometry in the large of the universal cover. This property can be defined for any Reebless codimension 1 foliation in such manifolds and it is true for fibrations [Ca-Th] and many depth one foliations [Fe1]. Recall that the limit set of $B$ is the set of accumulation points of $B$ in $S^2_\infty$. In this article we use the structure of branching to analyse limit sets of leaves when the continuous extension property holds.

Theorem H  Let $\Phi$ be an Anosov flow in $M^3$ with $\pi_1(M)$ negatively curved. Suppose that $\tilde{\Phi}$ has the continuous extension property. If $\tilde{\Phi}$ is not $\mathbb{R}$-covered then the limit set of any leaf $F$ of $\tilde{F}^s$ or $\tilde{F}^u$ is a Sierpinski curve, that is, the complement of a countable, dense union of open disks in $S^2_\infty$. If $\tilde{\Phi}$ is $\mathbb{R}$-covered then the limit set of any $F \in \tilde{F}^s \cup \tilde{F}^u$ is $S^2_\infty$, regardless of whether the continuous extension property holds or not [Fe2].
many $R$-covered examples [Fe3].

Bonatti and Langevin's example of a transitive, non $R$-covered Anosov flow in dimension 3, was generalized by Brunella [Br] who produced many examples by Dehn surgery on geodesic flows. The tool used to show that these flows are not $R$-covered was the existence of a transverse torus; hence all such examples were not in hyperbolic 3-manifolds. The main open conjecture in this theory was whether $M$ being hyperbolic would imply that the flow $\Phi$ is $R$-covered. In this article we answer this conjecture in the negative:

**Theorem I** There is a large class of transitive, non $R$-covered Anosov flows where the underlying 3-manifold is hyperbolic. This includes all Anosov flows in non orientable, hyperbolic 3-manifolds.

In a forthcoming paper [Fe8] we use the results of this article to study incompressible tori in 3-manifolds supporting Anosov flows. It is of great interest to find, in the isotopy class of the torus, the best position with respect to the flow [Ba3, Ba4]. We prove:

**Theorem** ([Fe8]) Let $\Phi$ be an Anosov flow in $M^3$ and let $T$ an incompressible torus in $M$. Suppose that no loop in $T$ is freely homotopic to a closed orbit of $\Phi$. Then $\Phi$ is topologically conjugate to a suspension Anosov flow. Furthermore $T$ is isotopic to a torus transverse to $\Phi$.

The article is organized as follows: in the next section we develop background material. In section 3 we prove theorem A and in the following section we prove theorem C and immediately derive theorems B and E and corollaries D and F. Section 5 studies product regions, which is then applied to a more detailed analysis of infinite branching and the construction of a transverse torus in section 6. In the final section we study the continuous extension property.

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## 2 Background

Let $\Phi_t : M \to M$ be a nonsingular flow in a closed Riemannian manifold $M$. The flow $\Phi$ is Anosov if there is a continuous decomposition of the tangent bundle $TM$ as a Whitney sum $TM = E^0 \oplus E^s \oplus E^u$ of $D\Phi_t$ invariant subbundles and there are constants $\mu_0 \geq 1, \mu_1 > 0$ so that:
(i) $E^0$ is one dimensional and tangent to the flow,
(ii) $||D\Phi_t(v)|| \leq \mu_0 e^{-\mu_1 t}||v||$ for any $v \in E^s$, $t \geq 0$,
(iii) $||D\Phi_{-t}(v)|| \leq \mu_0 e^{-\mu_1 t}||v||$ for any $v \in E^u$, $t \geq 0$.

In this article we restrict to $M$ of dimension 3. Then $E^s, E^u$ are one dimensional and integrate to one dimensional foliations $F^{ss}, F^{uu}$ called the strong stable and strong unstable foliations of the flow. Furthermore, the bundles $E^0 \oplus E^s$ and $E^0 \oplus E^u$ are also integrable [An] producing 2-dimensional foliations $F^s, F^u$ which are the stable and unstable foliations of the flow.

The flow is said to be orientable when both $F^s, F^u$ are transversely orientable. We remark that there is always a regular cover of order $\leq 4$ where the lifted $F^s$ and $F^u$ are transversely orientable. Whenever possible we will lift to such a cover.

The leaves of $F^s, F^u$ are either topological planes, annuli or Möebius bands. The last two correspond exactly to leaves containing closed orbits of $\Phi$. There is at most one closed orbit in a leaf of $F^s$, in which case all other orbits are forward asymptotic to it. Similarly for $F^u$.

The foliation $F^s$ is Reebless, so Novikov’s theorem [No] implies that given any closed orbit $\gamma$ of $\Phi$, $\gamma^n$ is not null homotopic for any $n \neq 0$.

Let $\pi : \bar{M} \to M$ be the universal covering space of $M$. This notation will be fixed throughout the article. The Anosov foliations $F^s, F^u$ lift to foliations $\bar{F}^s, \bar{F}^u$ in $\bar{M}$. The leaves of $\bar{F}^s, \bar{F}^u$ are topological planes, so $\bar{M}$ is homeomorphic to $R^3$ [Pa]. Therefore $\bar{M}$ is irreducible that is every embedded sphere in $\bar{M}$ bounds a 3-ball. The induced flow in $\bar{M}$ is denoted by $\bar{\Phi}$.

Let $\mathcal{O}$ be the orbit space of $\bar{\Phi}$ obtained by collapsing flow lines to points and let $\Theta : \bar{M} \to \mathcal{O}$ be the projection map. A fundamental property which will be repeatedly used here is that $\mathcal{O}$ is Hausdorff and hence homeomorphic to $R^2$ [Fe3]. This is a significant simplification since now much of the analysis can be done in dimension 2 instead of dimension 3. We stress that $\mathcal{O}$ is only a topological object. There is no natural metric in $\mathcal{O}$ since the flow direction contracts and expands distances in $\bar{M}$. The foliations $\bar{F}^s, \bar{F}^u$ induce two transverse 1-dimensional foliations in $\mathcal{O}$, which will also be denoted by $\tilde{F}^s, \tilde{F}^u$. By an abuse of notation we will many times identify sets in $\bar{M}$ or orbits of $\bar{\Phi}$ to their respective images in $\mathcal{O}$.

The fundamental group $\pi_1(M)$ is isomorphic to the set of covering translations of $\bar{M}$. We will usually assume one such identification is fixed. Given a covering translation $g$, we will also denote by $g$ its action on $\mathcal{O}$.

Let $W^s(x)$ be the leaf of $F^s$ containing $x$ and similarly define $W^u(x)$, $W^{ss}(x), W^{uu}(x), \tilde{W}^s(x), \tilde{W}^u(x), \tilde{W}^{ss}(x)$ and $\tilde{W}^{uu}(x)$. In the same way if $\alpha$ is an orbit of $\Phi$ we define $W^s(\alpha)$, etc..
3 Periodic branching leaves

The following definitions will be useful. If $L$ is a leaf of $\tilde{F}^s$ or $\tilde{F}^u$, then a *half leaf* of $L$ is a connected component $A$ of $L - \gamma$, where $\gamma$ is any full orbit in $L$. The closed half leaf is $\tilde{A} = A \cup \gamma$ and its boundary is $\partial A = \gamma$. If $L$ is a leaf of $\tilde{F}^s$ or $\tilde{F}^u$ then a flow band $B$ defined by orbits $\alpha \neq \beta$ in $L$ is the connected component of $L - \{\alpha, \beta\}$ which is not a half leaf of $L$. The closed flow band associated to it is $\tilde{B} = B \cup \{\alpha, \beta\}$ and its boundary is $\partial B = \{\alpha, \beta\}$.

Since $\tilde{M}$ is simply connected, $\tilde{F}^s$ and $\tilde{F}^u$ are transversely orientable. Choose one such orientation, assumed to agree with the lifts of the transversal orientations to $F^s, F^u$ if any of these is transversely oriented. Notice that in general, covering translations may not preserve transversal orientations.

For $p \in \tilde{M}$, let $\tilde{W}^s_+(p)$ be the half leaf of $\tilde{W}^s(p)$ defined by the orbit $\tilde{\Phi}_R(p)$ and the positive transversal orientation to $\tilde{F}^u$ at $p$. This is also called a *positive half leaf* of $\tilde{W}^s(p)$. Similarly define $\tilde{W}^s_-(p)$, a *negative half leaf* and also define $\tilde{W}^u_+(p)$ and $\tilde{W}^u_-(p)$.

A fundamental property for us is that any leaf $L$ in $\tilde{F}^s$ or $\tilde{F}^u$ separates $\tilde{M}$. This is a consequence of $\tilde{M}$ being simply connected. The *front* of $L$ is the component of $\tilde{M} - L$ defined by the positive transversal orientation to $L$. Similarly define the *back* of $L$. For $p \in \tilde{M}$ let $\tilde{W}^s_{\pm}(p) = \tilde{W}^s_{\pm}(p) \cap \tilde{W}^s(p)$. In the same way define $\tilde{W}^u_{\pm}(p)$, $\tilde{W}^u_{\pm}(p)$ and $\tilde{W}^u_{\pm}(p)$.

If $F \in \tilde{F}^s$ and $G \in \tilde{F}^u$ then $F$ and $G$ intersect in at most one orbit, since two intersections would force a tangency of $\tilde{F}^s$ and $\tilde{F}^u$. This is easiest seen in $O$, as $\tilde{F}^s$ and $\tilde{F}^u$ are then 1-dimensional foliations of the plane.

We say that leaves $F, L \in \tilde{F}^s$ and $G, H \in \tilde{F}^u$ form a *rectangle* if $F$ intersects both $G$ and $H$ and so does $L$, see fig. 2 a. We also say that $E$ intersects $G$ between $F$ and $L$ if $E \cap G$ is contained in the flow band in $G$ defined by $G \cap F$ and $G \cap L$. Then it is easy to prove [Fe5] that if $E \in \tilde{F}^s$ intersects $G$ between $F$ and $L$ then $E$ also intersects $H$ between $F$ and $L$. This means that there is a product structure of $\tilde{F}^s$ and $\tilde{F}^u$ in the region bounded by $F, L, G$ and $H$.

The following two definitions will be essential for all results in this article.
§3. Periodic branching leaves

Definition 3.1 Given \( p \in \tilde{M} \) (or \( p \in \mathcal{O} \)), let

\[ \mathcal{J}^u_+(p) = \{ F \in \tilde{F}^s \mid F \cap \tilde{W}^u_+(p) \neq \emptyset \}, \]

an open subset of \( \mathcal{H}^s \). Notice that the leaf \( \tilde{W}^s(p) \notin \mathcal{J}^u_+(p) \). Similarly define \( \mathcal{J}^u_- (p), \mathcal{J}^s_+(p) \) and \( \mathcal{J}^s_- (p) \).

Definition 3.2 Two leaves \( F, G, F \in \tilde{F}^s \) and \( G \in \tilde{F}^u \), form a perfect fit if \( F \cap G = \emptyset \) and there are half leaves \( F_1 \) of \( F \) and \( G_1 \) of \( G \) and also flow bands \( L_1 \subset L \in \tilde{F}^s \) and \( H_1 \subset H \in \tilde{F}^u \), (see figure 2 b) so that:

\[
\begin{align*}
\mathcal{L}_1 \cap \mathcal{G}_1 &= \partial L_1 \cap \partial G_1, \\
\mathcal{L}_1 \cap \mathcal{H}_1 &= \partial L_1 \cap \partial H_1, \\
\mathcal{H}_1 \cap \mathcal{F}_1 &= \partial H_1 \cap \partial F_1, \\
\end{align*}
\]

\[ \forall S \in \tilde{F}^u, \quad S \cap L_1 \neq \emptyset \iff S \cap F_1 \neq \emptyset \quad \text{and} \]

\[ \forall E \in \tilde{F}^s, \quad E \cap G_1 \neq \emptyset \iff E \cap H_1 \neq \emptyset. \]

Figure 2: a. Rectangles, b. Perfect fits in the universal cover.

Notice that the flow bands \( L_1, H_1 \) (or the leaves \( L, H \)) are not uniquely determined given the perfect fit \( (F, G) \). We will also say that \( F \) and \( G \) are asymptotic in the sense that if we consider stable leaves near \( F \) and on the side containing \( G \) they will intersect \( G \) and vice versa. Perfect fits produce “ideal” rectangles, in the sense that even though \( F \) and \( G \) do not intersect,
there is a product structure (of $\tilde{F}^s$ and $\tilde{F}^u$) in the interior of the region bounded by $F, L, G$ and $H$.

It is easy to show [Fe5] that there is at most one leaf $G \in \tilde{F}^u$ making a perfect fit with a given half leaf of $F \in \tilde{F}^s$ and in a given side of $F$. Therefore a perfect fit is a detectable property in $M$. This means that if $(L, G)$ forms a perfect fit and $g$ is any orientation preserving covering translation with $g(L) = L$, then $g(G) = G$. The last assertion follows from uniqueness of perfect fits and the fact that, as $g$ acts by homeomorphisms in the leaf spaces, it takes perfect fits to perfect fits.

If $p, q$ are in the same strong stable leaf let $[p, q]_s$ denote the closed segment in that leaf from $p$ to $q$ and let $(p, q)_s$ be the corresponding open segment. Similarly define $[p, q]_u$ and $(p, q)_u$.

We say that $J^s_+(p)$ and $J^s_+(q)$ are comparable and will denote this by $J^s_+(p) \sim J^s_+(q)$, if one of them is contained in the other. Then we write $J^s_+(p) < J^s_+(q)$ if the former is strictly contained in the latter. Similarly define $\leq$, $>$ and $\geq$. The symbol $\not\sim$ means not comparable.

We also say that an orbit $\gamma$ of $\tilde{\Phi}$ is periodic if it is left invariant by a non trivial covering translation.

**Theorem 3.3** Let $\Phi$ be an Anosov flow in $M^3$ and let $F$ be a branching leaf of $\tilde{F}^s$. Then there is a non trivial covering translation $g$ with $g(F) = F$, that is, $F$ is periodic.

**Proof of 3.3:** By taking a finite cover if necessary, we may assume that $\Phi$ is orientable. Let $L \in \tilde{F}^s$, $L \neq F$, so that $F, L$ form a branching pair of $\tilde{F}^s$. Assume without loss of generality that $F$ and $L$ are not separated on their negative sides, that is they are associated to branching of $\tilde{F}^s$ in the positive direction (positive branching).

Let $w_0 \in F$, $w' \in L$. Since $F$ and $L$ are not separated in their negative sides there are $y_0 \in \tilde{W}^u_-(w_0)$ ($y_0$ sufficiently near $w_0$) and $x_0 \in \tilde{W}^u(w') \cap \tilde{W}^{ss}(y_0)$ so that if $r_0 = \tilde{W}^u_+(x_0) \cap L$, then for any $E \in \tilde{F}^s$,

$$E \cap (y_0, w_0)_u \neq \emptyset \iff E \cap (x_0, r_0)_u \neq \emptyset. \quad (*)$$

This fact, which follows from the separation property of leaves of $\tilde{F}^s$, will often be implicitly used.

By switching $F$ and $L$ if necessary we may assume that $\tilde{W}^u(x_0)$ is in the front of $\tilde{W}^{ss}(y_0)$. Our first goal will be to find unique leaves associated to the branching which form perfect fits with $F$ and $L$. 
As there are \( z \in [y_0, x_0]_s \) with \( \tilde{W}^u(z) \cap F = \emptyset \) (for instance \( z = x_0 \)), let \( p_0 \) be the closest point to \( y_0 \) in \([y_0, x_0]_s\) so that \( \tilde{W}^u(p_0) \cap F = \emptyset \).

**Lemma 3.4** The leaves \( F \) and \( \tilde{W}^u(p_0) \) form a perfect fit.

**Proof of 3.4:** For candidates of flow bands let \( A = \tilde{\Phi}_R((y_0, w_0)_a) \) and \( B = \tilde{\Phi}_R((y_0, p_0)_s) \). Then \( \overline{A} \cap \overline{B} = \tilde{\Phi}_R(y_0) \), \( \overline{A} \cap F = \tilde{\Phi}_R(w_0) \) and \( \overline{B} \cap \tilde{W}^u(p_0) = \tilde{\Phi}_R(p_0) \).

Let \( E \in \tilde{F}^s \) with \( E \cap A \neq \emptyset \). Then \( E \cap \tilde{W}^u(x_0) \neq \emptyset \). Since \( \tilde{W}^u(p_0) \) separates \( M \) it follows that \( E \cap \tilde{W}^u(p_0) \neq \emptyset \). As \( E \) is in front of \( \tilde{W}^s(y_0) \) then \( E \cap \tilde{W}^u_+(p_0) \neq \emptyset \).

\[
\begin{align*}
\text{Figure 3: Branching in } \tilde{F}^s.
\end{align*}
\]

Conversely let \( E \in \tilde{F}^s \) with \( E \cap \tilde{W}^u_+(p_0) \neq \emptyset \). Suppose that \( E \cap A = \emptyset \). Since \( \tilde{W}^u(p_0) \cap F = \emptyset \), then the front of \( E \) is disjoint from the front of \( F \). For any \( z \in \tilde{W}^s(y_0) \) near enough \( p_0 \), \( \tilde{W}^u(z) \cap E \neq \emptyset \). As \( E \) is in the back of \( F \), it follows that \( \tilde{W}^u(z) \cap F = \emptyset \). This contradicts the choice of \( p_0 \). We conclude that \( E \cap A \neq \emptyset \Leftrightarrow E \cap \tilde{W}^u_+(p_0) \neq \emptyset \).

Let now \( R \in \tilde{F}^u \) with \( R \cap B \neq \emptyset \). If \( R \cap F = \emptyset \), then \( z = R \cap [y_0, p_0]_s \) is closer to \( y_0 \) (in \( \tilde{W}^s(y_0) \)) than \( p_0 \), contradiction. Hence \( R \cap F \neq \emptyset \), in particular \( R \cap \tilde{W}^u_+(w_0) \neq \emptyset \).

Conversely suppose that \( R \cap \tilde{W}^u_+(w_0) \neq \emptyset \). Let \( F^* \in \tilde{F}^s \) be close enough to \( F \) so that \( F^* \cap R \neq \emptyset \), \( F^* \cap \tilde{W}^u(y_0) \neq \emptyset \) and \( F^* \cap \tilde{W}^u(x_0) \neq \emptyset \). Then \( \tilde{W}^u(y_0), \tilde{W}^u(x_0), \tilde{W}^s(y_0) \) and \( F^* \) form a rectangle. Since and \( R \cap F^* \neq \emptyset \) is between \( F^* \cap \tilde{W}^u(y_0) \) and \( F^* \cap \tilde{W}^u(x_0) \) then \( R \cap \tilde{W}^s(y_0) \neq \emptyset \). As \( R \) is in front of \( \tilde{W}^u(y_0) \) then \( R \cap \tilde{W}^u_+(y_0) \neq \emptyset \). Since \( R \cap F \neq \emptyset \) then \( R \) is in the back of \( \tilde{W}^u(p_0) \). Therefore \( R \cap B \neq \emptyset \). This finishes the proof of the lemma.
Continuation of the proof of theorem 3.3

In the same way there is a unique \( q_0 \in [y_0, x_0] \), with \( \tilde{W}^u(q_0) \) and \( L \) forming a perfect fit. By uniqueness of perfect fits, the leaves \( W^u(p_0), \tilde{W}^u(q_0) \) depend only on \( F \) and \( L \). If follows from \((*)\) and lemma 3.4, that given \( E \in \tilde{F}^s, E \cap \tilde{W}^u_+(p_0) \neq \emptyset \Leftrightarrow E \cap \tilde{W}^u_+(q_0) \neq \emptyset \). Equivalently \( J^u_+(p_0) = J^u_+(q_0) \).

**Case 1.** \( p_0 = q_0 \).

Let \( G = W^u(p_0) = \tilde{W}^u(q_0) \). If \( G \) is periodic there is \( g \neq \text{id} \) with \( g(G) = G \). By uniqueness of perfect fits and preserving of transversal orientations it follows that \( g(F) = F \) and we are done. So we may assume that \( G \) is not periodic.

Let \( c_0 = \pi(p_0) \). Since \( G \) is not periodic, \( \Phi R(c_0) \) is not a closed orbit, nor is it backwards asymptotic to a closed orbit. Let \( c \) be a negative limit point of \( \Phi R(c_0) \) and let \( c_i = \Phi t_i(c_0), t_i \to -\infty \), with \( c_i \to c \). If \( c_i \) and \( c_j \) are in the same local unstable leaf near \( c \), then there is a closed path in \( W^u(c_i) \) consisting of the flow segment from \( c_i \) to \( c_j \) and then a small strong unstable segment from \( c_j \) to \( c_i \) in the local unstable leaf through \( c_j \). This path is not null homotopic in \( W^u(c_i) \), hence \( W^u(c_i) \) contains a closed orbit, contradiction to our assumption. This is the key fact used in the proof of the theorem and it will imply that non periodic leaves in the universal cover are not rigid.

Lift \( c_i \) to \( p_i \in \tilde{M} \) with \( p_i \to p \) and \( \pi(p) = c \). Then \( p_i = g_i(\tilde{\Phi}_{t_i}(p_0)) \), where \( g_i \) are covering translations. By the above argument \( W^s(p_i) \neq W^s(p_k) \) for any \( i \neq k \). This is the non rigidity we are looking for.

Let \( F_i = g_i(F), L_i = g_i(L), A_i = g_i(A), B_i = g^i(B) \) and \( G_i = g_i(G) \). Let \( y_i = g_i(\tilde{\Phi}_{t_i}(y_0)) \) and let \( x_i = g_i(\tilde{\Phi}_{t_i}(x_0)) \). Up to subsequence assume that all \( p_i \) and \( p \) are near enough, in a product neighborhood of \( \tilde{F}^u \) of diameter \(<1 \). Assume also that for all \( i, k \),

\[
I(\tilde{\Phi}_{t_i}([y_0, p_i]_s)) > 1 \quad \text{and} \quad I(\tilde{\Phi}_{t_k}([p_0, x_0]_s)) > 1. \quad (**)
\]

Choose \( i, k \) so that \( p_i \) is in the back of \( \tilde{W}^u(p_k) \), see fig. 4. Since \( d(p_i, p_k) < < 1 \) it follows that \( \tilde{W}^s(p_k) \cap \tilde{W}^u(p_i) \neq \emptyset \) and \( \tilde{W}^s_+(p_i) \cap \tilde{W}^u(p_k) \neq \emptyset \). By (**), this implies that \( y_k \) is in the back of \( \tilde{W}^u(p_i) \) and \( x_i \) is in the front of \( \tilde{W}^u(p_k) \), see fig. 4. Hence \( \tilde{W}^u(y_k) \) is in the back of \( \tilde{W}^u(p_i) \). Then \( \tilde{W}^u(p_i) \cap B_k \neq \emptyset \), hence \( \tilde{W}^u(p_i) \cap F_k \neq \emptyset \). As \( L_i \) makes a perfect fit with
§3. Periodic branching leaves

Figure 4: Rigidity of branching leaves: the adjacent case

\( \widetilde{W}^u(p_i) \), this implies that \( L_i \) is in front of \( F_k \), hence \( L_i \) is in the back of \( \widetilde{W}^u(p_k) \).

On the other hand, \( L_i \cap \widetilde{W}^u(x_i) \neq \emptyset \). Since \( \widetilde{W}^u(x_i) \) is in front of \( \widetilde{W}^u(p_k) \) then \( \widetilde{W}^u(p_k) \cap \tilde{\Phi}_R([p_i, x_i]_s) \neq \emptyset \). As \( L_i \) and \( \widetilde{W}^u(p_i) \) form a perfect fit, this implies that \( \widetilde{W}^u(p_k) \cap L_i \neq \emptyset \). This contradicts the previous paragraph.

This shows that if \( p_0 = q_0 \), then \( G \) is periodic, left invariant by \( g \), hence \( F \) and \( L \) are periodic and both left invariant under \( g \).

Remarks: (1) If we apply the argument above when \( G \) is periodic, we get \( \widetilde{W}^s(p_i) = \widetilde{W}^s(p_k) \) for all \( i, k \). There is no small perturbation of the local picture, which is then rigid. This will imply that the whole set of non separated leaves from \( F \) is very rigid.

(2) It is tempting to try the following “intuitive” approach to the above proof: as \( \pi(\widetilde{W}^u(p_0)) \) is not compact in \( M \), there are always translates \( S_1 \) and \( S_2 \) of \( \widetilde{W}^u(p_0) \) and points \( u_i \in S_i \) arbitrarily near each other. However there is no control of the rest of the picture. For instance we do not know a priori what happens to the respective stable lengths. This is the reason why we fixed an orbit \( \tilde{\Phi}_R(\pi(p_0)) \) and flowed backwards in order to insure that stable lengths are as big as we want.

Case 2 \( p_0 \neq q_0 \).

We use the same notation as in case 1. As \( q_0 \neq p_0 \), let \( q_i = g_i(\tilde{\Phi}_{t_i}(q_0)) \).
diff

Figure 5: Rigidity of branching: the separated case.

Choose \( i, k \) with \( p_i \) in the back of \( \bar{W}^u(p_k) \). As in case 1, \( \bar{W}^u_+(p_i) \cap F_k \neq \emptyset \). There is no a priori contradiction because now \( L_i \) does not form a perfect fit with \( \bar{W}^u(p_i) \), and in fact \( L_i \) is probably in the front of \( \bar{W}^u(p_k) \). Let

\[
e_1 = \bar{W}^u(p_k) \cap \bar{W}^{ss}_+(p_i), \quad e_2 = \bar{W}^u(p_i) \cap \bar{W}^{ss}_+(p_k).
\]

Then \( J_+^u(p_k) < J_+^u(e_2) \) and by the local product structure of \( \bar{F}^s, \bar{F}^u \) near \( p \), it follows that \( J_+^u(p_i) > J_+^u(e_1) \), see fig. 5. Choose \( E \in J_+^u(p_i) - J_+^u(e_1) \).

By the above considerations it is clear that \( E \cap \bar{W}^u(p_k) = \emptyset \). But

\[
J_+^u(q_i) = J_+^u(g_i(\bar{\Phi}_{t_i}(q_0))) = g_i(J_+^u(\bar{\Phi}_{t_i}(q_0))) = g_i(J_+^u(\bar{\Phi}_{t_i}(p_0))) = J_+^u(p_i),
\]

hence \( E \in J_+^u(q_i) \). As a result \( E \cap \bar{W}^u_+(q_i) \neq \emptyset \). But now \( \bar{W}^u_+(q_i) \) is in the front of \( \bar{W}^u(p_k) \). Since \( \bar{W}^u(p_k) \) separates \( \tilde{M} \), then \( E \cap \bar{W}^u(p_k) \neq \emptyset \), contradiction. As before we conclude that \( G \) is periodic, left invariant by \( g \neq id \), so \( F \) is also left invariant by \( g \).

\[3.3\]

Caution: The same argument shows that \( L \) and \( \bar{W}^u(q_0) \) are also periodic. We do not know at this point that the same covering translation leaves invariant both \( F \) and \( L \). This is a much stronger fact.
4 Branching structure

In this section we show that if $F$ and $L$ are not separated, then not only are they periodic, but there is a common covering translation leaving both of them invariant. As a result, branching forces a non trivial free homotopy between closed orbits of $\Phi$ in $M$ and this gives the topological characterization of suspensions. Furthermore we will show that $F$ and $L$ are connected by a finite sequence of lozenges, as defined below. This completely determines the structure of the set of non separated leaves from $F$. As a consequence we show there are only finitely many branching leaves up to covering translations. This in turn implies that if there is infinite branching then there is an incompressible torus in $M$.

**Definition 4.1** **Lozenges** - Let $p, q \in \tilde{M}$, $p \notin \tilde{W}^s(q)$, $p \notin \tilde{W}^u(q)$. Let $H_p$ be the half leaf of $\tilde{W}^u(p)$ defined by $\tilde{\Phi}(p)$ and contained in the same side of $\tilde{W}^s(p)$ as $q$. Let $L_p$ be the similarly defined half leaf of $\tilde{W}^s(p)$ and in the same fashion define $H_q, L_q$. Then $p, q$ form a lozenge, fig. 6, a if $H_p, L_q$ and $H_q, L_p$ respectively form perfect fits.

Figure 6: a. A lozenge, b. A chain of adjacent lozenges.

We say that $p, q$ (or $\tilde{\Phi}(p), \tilde{\Phi}(q)$) are corners of the lozenge. If the lozenge with corner $p$ is contained in the back of $\tilde{W}^s(p)$ then $p$ is a corner of type $(+, \ast)$, otherwise it is of type $(-, \ast)$. Similarly using $\tilde{W}^u(p)$ define types $(\ast, +), (\ast, -)$. The sides of the lozenge are $H_p, L_p, H_q$ and $L_q$. Since given any four leaves there is at most one lozenge defined by them we will also
say the full leaves are the sides of the lozenge. Notice that if \( p \) is a corner of type \((-,-)\) then \( J^u_+(p) = J^u_-(q), J^s_+(p) = J^s_-(q) \) and similarly for the other cases.

Two lozenges are adjacent if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 6 b. A chain of lozenges is a collection \( \{B_i\}, 1 \leq i \leq n, \) so that \( B_i \) and \( B_{i+1} \) share a corner. Consecutive lozenges may be adjacent or not.

The following theorem will be essential for the results in this section:

**Theorem 4.2** (Fe4) Let \( \Phi \) be an Anosov flow in \( M^3 \). Suppose that \( F_i, i = 0, 1 \) are leaves of \( \tilde{F}^s \) for which there is a non trivial covering translation \( g \) with \( g(F_i) = F_i, i = 0, 1. \) Let \( \alpha_i, i = 0, 1 \) be the periodic orbits of \( \tilde{\Phi} \) in \( F_i \) so that \( g(\alpha_i) = \alpha_i. \) Then \( \alpha_0 \) and \( \alpha_1 \) are connected by a finite chain of lozenges \( \{B_i\}, 1 \leq i \leq n \) and \( g \) leaves invariant each lozenge \( B_i \) as well as their corners.

Furthermore there is a unique chain that is minimal, in the sense that any other chain from \( \alpha_0 \) to \( \alpha_1 \) contains this chain [Fe7]. Given any chain \( \mathcal{B} = \{B_i\}, 1 \leq i \leq n \) from \( \alpha_0 \) to \( \alpha_1 \), let \( \gamma_0 = \alpha_0 \) and inductively define \( \gamma_i, i > 0 \) to be the remaining corner of \( B_i. \) The minimal chain from \( \alpha_0 \) to \( \alpha_1 \) is defined by: \( B_{i+1} \) is on the same side of \( \tilde{W}^s(\gamma_i) \) and \( \tilde{W}^u(\gamma_i) \) that \( \alpha_1 \) is.

A closed orbit of \( \Phi \) traversed once is called an indivisible closed orbit. The following result will be often used in this article:

**Theorem 4.3** (Fe7) Let \( \Phi \) be an orientable Anosov flow in \( M^3 \). If \( \gamma \) is an indivisible closed orbit of \( \Phi, \) then \( \gamma \) represents an indivisible element in \( \pi_1(M). \) Equivalently if \( g^n(F) = F, \) where \( F \in \tilde{F}^s \cup \tilde{F}^u, \) \( g \) is a covering translation and \( n \neq 0, \) then \( g(F) = F. \)

There is a related result if \( \Phi \) is not assumed to be orientable.

The stabilizer \( \mathcal{T}(F) \) of a leaf \( F \) of \( \tilde{F}^s \) (or \( \tilde{F}^u \)) is the subgroup of \( \pi_1(M) \) of those \( g \) with \( g(F) = F. \) If \( \pi(F) \) does not contain a periodic orbit, then \( \mathcal{T}(F) \) is trivial. Otherwise let \( \gamma \) be the indivisible closed orbit in \( \pi(F). \) Then \( \mathcal{T}(F) \) is infinite cyclic and it has a generator conjugate to \([\gamma]\) in \( \pi_1(M). \)

The main technical result in this section is the following:

**Theorem 4.4** Let \( \Phi \) be an Anosov flow in \( M^3. \) Suppose that \( F, L \) form a branching pair of \( \tilde{F}^s. \) Let \( g \) be a non trivial covering translation with \( g(F) = F, \) so that \( g \) preserves transversal orientations to \( \tilde{F}^s, \tilde{F}^u. \) Then \( g(L) = L. \) Similarly for \( \tilde{F}^u. \)
§4. Branching structure

Proof of 4.4: Up to a finite cover assume that Φ is orientable. Since $g$ preserves transversal orientations, then $g$ is still a covering translation of the finite cover. Without loss of generality suppose that $F$ and $L$ are not separated on their negatives sides, corresponding to positive branching. Finally we may assume that $g$ generates $T(F)$.

As in theorem 3.3 there are unique leaves $G, H \in \tilde{F}^u$ making perfect fits with $F$ and $L$ respectively and so that: $G$ separates $F$ from $L$ and so does $H$. Let $p \in G$ so that $\tilde{W}^{ss}(p)$ intersects $H$ and let $q = \tilde{W}^{ss}(p) \cap H$. Recall from the proof of theorem 3.3, that $\mathcal{J}^u_+(p) = \mathcal{J}^u_+(q)$.

Since $g$ preserves transversal orientations then $g(G) = G$. Our goal is to show that $g(L) = L$. Suppose then that $g(L) \neq L$, hence by the same argument $g(H) \neq H$. Let $\gamma \subset G$ be the periodic orbit of $\tilde{\Phi}$ in $G$, so $g(\gamma) = \gamma$.

Claim 1 - There is $R \in \tilde{F}^u$ in the back of $L$ making a perfect fit with a positive half leaf of $L$, hence $R$ is in the front of $H$.

We may assume that $p \in \tilde{W}^u_+(\gamma)$. Let $E = \tilde{W}^s(p)$. By taking $g^{-1}$ if necessary assume that $g(E)$ is in front of $E$. Hence $g(E) \in \mathcal{J}^u_+(p)$, therefore $g(E) \in \mathcal{J}^u_+(q)$. Then $H \cap g(E) \neq \emptyset$. There are 2 cases:

1. $g(H)$ is in front of $H$, see fig. 7.

Let $e' = \tilde{W}^{ss}(g(p)) \cap H$. Since $g(p) \in \tilde{W}^u_+(p)$ then $\mathcal{J}^u_+(g(p)) = \mathcal{J}^u_+(e')$. But also $\mathcal{J}^u_+(g(p)) = \mathcal{J}^u_+(g(q))$, so $\mathcal{J}^u_+(g(q)) = \mathcal{J}^u_+(e')$, where $g(q) \in g(H)$ and $e' \in H$. Since $L$ makes a perfect fit with $H$ and $g(L)$ makes a perfect fit with $g(H)$ this shows that $g(L)$ is not separated from $L$.

Figure 7: Iterating non invariant leaves.
As in the proof of theorem 3.3, there is a unique $e_0 \in [e', g(q)]$, with $\tilde{W}^u(e_0)$ making a perfect fit with $L$ and $L$ in the back of $\tilde{W}^u(e_0)$. In this case let $R = \tilde{W}^u(e_0)$.

(2) Suppose now that $g(H)$ is in the back of $H$.

Notice that $E, g(E), H$ and $G$ form a rectangle. Since $g(H) \cap g(E) \neq \emptyset$ and $g(H)$ is between $G$ and $H$ it follows that $g(H) \cap E \neq \emptyset$ and $g(H) \cap E$ is an orbit in $E$ between $E \cap G$ and $E \cap H$.

In this case let $c = g(H) \cap \tilde{W}^s(p)$. Then $c \in (p, q)$. Since $J^u_+(p) = J^u_+(q)$ and $L \notin J^u_+(q)$, then $g(L) \notin J^u_+(q)$, so $g(L) \cap H = \emptyset$. Hence $g(L)$ is in the back of $H$. As in case (1), it follows that $L$ and $g(L)$ form a branching pair.

Let $c_2 \in (c, q)$ with $\tilde{W}^u(c_2)$ making a perfect fit with $g(L)$ and with $g(L)$ in the back of $\tilde{W}^u(c_2)$. Then $R = g(\tilde{W}^u(c_2))$ makes a perfect fit with $L$ and $L$ is in the back of $\tilde{W}^u(c_2)$. This finishes the proof of claim 1.

By theorem 3.3, $L$ is periodic and let $\alpha^*$ be the indivisible periodic orbit in $L$. Let $h$ a generator of $\mathcal{T}(H)$. Since $\Phi$ is orientable, $h(H) = H, h(R) = R$. Let $\alpha$ be the periodic orbit in $H$. Therefore $L$ and $H$ are 2 of the sides of a lozenge $N_1$ with other sides in $\tilde{W}^s(\alpha)$ and $\tilde{W}^u(\alpha^*)$, that is $\alpha$ and $\alpha^*$ are the corners of the lozenge. In the same way $L$ and $R$ are the 2 sides of a lozenge $N_2$. The lozenges are adjacent and intersect the stable leaf $E$. Let $\mathcal{N} = N_1 \cup N_2$.

We now show that $F$ also makes a perfect fit with $U \in F^u, U \neq G$ and $F$ in the front of $U$, hence $G$ is in the front of $U$, see fig. 8. If $h(G) = G$ then since $g$ generates $\mathcal{T}(G)$, it follows that $h = g^n$ for some $n \in \mathbb{Z}$. Hence $g^n(H) = H$. Theorem 4.3 then implies that $g(H) = H$ contrary to assumption. It follows that $h(G) \neq G$. Using claim 1 with the roles of $F, L$ exchanged, we produce the required $U \in F^u$. Furthermore there are two adjacent lozenges $D_1$ and $D_2$ with (some) sides in $U, F, G$. Let $\mathcal{D}$ be their union. Both lozenges intersect a stable leaf which we may assume is $E$.

From now on the proof goes roughly as follows: We will show that $\tilde{W}^s(\gamma)$ intersects of $\tilde{W}^u(\alpha)$ and similarly that $\tilde{W}^s(\alpha)$ intersects $\tilde{W}^u(\gamma)$. Producing a contradiction.

By taking $g^{-1}$ if necessary suppose that $g(H)$ is in the back of $H$. Let $H_i = g^i(H)$. Then as in case (2) of the claim, $H_{i+1}$ is in the back of $H_i$, and for all $i \geq 0, H_i \cap E \neq \emptyset$. Furthermore $H_i$ is always in front of $G$. This implies that $H_i \rightarrow S$ with $S \cap E \neq \emptyset$ (and maybe $H_i$ also converges to other leaves of $F^u$).
Let $\mathcal{A}_i$ be the front of $H_i$ and let $\mathcal{A} = \cup_{i \in \mathbb{N}} \mathcal{A}_i$. Then $g(\mathcal{A}_i) = \mathcal{A}_{i+1}$ so $g(\mathcal{A}) = \mathcal{A}$ and consequently $g(\partial \mathcal{A}) = \partial \mathcal{A}$. Since $S \not\subseteq \mathcal{A}$ it follows that $\partial \mathcal{A}$ is a non empty union of unstable leaves and furthermore $S \subset \partial \mathcal{A}$. Notice that $S$ is the unique leaf which is either equal to $G$ or separates $G$ from $\mathcal{A}$. In the second case since $g(\mathcal{A}) = \mathcal{A}$ and $g(\mathcal{S}) = \mathcal{S}$ it follows that $g(S) = S$. If $S$ is the unique leaf which is either equal to $G$ or separates $G$ from $\mathcal{A}$. In the second case since $g(\mathcal{A}) = \mathcal{A}$ and $g(\mathcal{S}) = \mathcal{S}$ it follows that $g(S) = S$. Then there is an orbit $\beta$ of $\tilde{\Phi}$ in $S$ with $g(\beta) = \beta$.

Suppose not. Let $r \in \beta$ and $r' \in \gamma$. Notice that $g^i(q) \in H_i$. If $g^i(q)$ is in front of $\tilde{W}^s(r)$ then $\tilde{W}^s(g^i(q))$ is in front of $\tilde{W}^s(r)$, contradiction to $\tilde{W}^s(g^i(q)) = \tilde{W}^s(g^i(p))$ being in the back of $\tilde{W}^s(r)$. Otherwise $\tilde{W}^s(r) \in J^u_+(g^i(q))$, implying $\tilde{W}^s(r) \in J^u_+(g^i(p))$ also a contradiction. This proves claim 2.

Consequently $\gamma$ is in front of $\tilde{W}^s(\beta)$ and $\gamma$, $\beta$ are connected by and even number of adjacent lozenges. Therefore $J^u_+(r) = J^u_+(r')$. Since $R_i$ separates $H_i$ from $H_{i-1}$ for all $i$, it follows that $\tilde{W}^s(\beta) \cap R_i \neq \emptyset$, for all $i$ big enough. Since $g(\tilde{W}^s(\beta)) = \tilde{W}^s(\beta)$ this shows that $\tilde{W}^s(\beta) \cap H \neq \emptyset$ and similarly $\tilde{W}^s(\beta) \cap R \neq \emptyset$. Therefore $\tilde{W}^s(\beta) \cap G \neq \emptyset$ and as a result $\tilde{W}^s(\beta)$ intersects $\tilde{W}^u_+(\alpha)$. 

**Claim 2** - For all $i$, $B_i$ is in the front of $\tilde{W}^s(\beta)$. In particular $\gamma$ is in front of $\tilde{W}^s(\beta)$.
§4. Branching structure

Conclusion: There is an orbit $\beta$ of $\Phi$ with $g(\beta) = \beta$, $\hat{W}^s(\beta) \cap \hat{W}^u_+(\alpha) \neq \emptyset$ and $\hat{W}^s(\beta) \cap R \neq \emptyset$.

Figure 9: Impossible intersection of leaves: a. Case $\delta = \alpha$, b. Case $\delta \neq \alpha$.

Notice that there is $Z \in \hat{F}^s$ making a perfect fit with $Y = \hat{W}^u(\beta)$ so that $Z$ is in the back of $Y$ and $Z$ and $L$ are not separated, see fig. 9 a. Hence $Z, L$ satisfy the hypothesis of the theorem. As in claim 1 there is $X \in \hat{F}^u$, $X \neq Y$, $X$ making a perfect fit with $Z$ and intersecting $E$, see fig. 9 a. Therefore the same arguments done before work with $G$ replaced by $Y$, that is the argument works with $\beta \subset Y$ and $\alpha \subset H$.

Now switch the roles of $Y$ and $H$ and apply the same argument as above to find an orbit $\delta$ of $\Phi$ with $h(\delta) = \delta$ and $\hat{W}^s(\delta) \cap \hat{W}^u_+(\beta) \neq \emptyset$, $\hat{W}^s(\delta) \cap X \neq \emptyset$. In addition $\delta$ is connected to $\alpha$ by an even chain of lozenges all intersecting a common stable leaf. Hence if $u \in \delta, u' \in \alpha$, then $\hat{J}^u_+(u) = \hat{J}^u_+(u')$.

If $\delta = \alpha$ this produces an immediate contradiction since $\hat{W}^s(\beta)$ intersects $\hat{W}^s_+(\alpha)$ and $\hat{W}^s(\alpha)$ intersects $\hat{W}^s_+(\beta)$, see fig 9, a.

Suppose that $\delta \neq \alpha$. As $\hat{W}^s(\delta) \cap \hat{W}^u_+(\beta) \neq \emptyset$, then $\hat{W}^s(\beta)$ is in the back of $\hat{W}^s(\delta)$. In particular $\hat{W}^s(\beta) \not\subset \hat{J}^u_+(u)$. Hence $\hat{W}^s(\beta) \not\subset \hat{J}^u_+(u')$, a contradiction to the fact that $\hat{W}^s(\beta)$ intersects $\hat{W}^u_+(\alpha)$, see fig. 9, b.

This contradiction implies that $g(H) = H$. Hence $g(L) = L$ as desired.

Corollary 4.5 Let $\Phi$ be an Anosov flow in $M^3$. Suppose $\Phi$ has branching and $F, L \in \hat{F}^s$ are not separated. Then $F$ and $L$ are connected by an even
chain of lozenges, all intersected by a common stable leaf. In particular there are only finitely many branching leaves between $F$ and $L$.

Proof of 4.5: Up to finite cover we may assume that $\Phi$ is orientable. Suppose that $F, L$ are not separated in their negative sides. Let $g \neq id$ be a covering translation with $g(F) = F$. By the previous theorem $g(L) = L$. Let $\gamma$ and $\delta$ be the respective periodic orbits in $F$ and $L$. Furthermore suppose $\tilde{W}^u(\gamma)$ is in the back of $\tilde{W}^u(\delta)$.

By theorem 4.2, $\gamma$ and $\delta$ are connected by a finite chain of lozenges. Let $B = \{B_i\}, 1 \leq i \leq n$, be the minimal chain from $\delta$ to $\gamma$. Since $\delta$ is in the back of $\tilde{W}^s(\gamma)$ and in the front of $\tilde{W}^u(\gamma)$ it follows that $\gamma$ is the $(+, -)$ corner of $B_1$. Let $\gamma_1$ be the $(-, +)$ corner of $B_1$. Then $\delta$ is in front of $\tilde{W}^s(\gamma_1)$ and in front of $\tilde{W}^u(\gamma_1)$, hence $B_2$ has $(-, -)$ corner $\gamma_1$ and let $\gamma_2$ be the $(+, +)$ corner of $B_2$. If $\gamma_2 = \delta$ we are done. Otherwise $\tilde{W}^s(\gamma_2)$ is not separated from $F$ hence not separated from $L$. Induction produces $\gamma_4, ..., \gamma_{2k} = \delta$ (hence $n = 2k$). Clearly the $\tilde{W}^s(\gamma_{2i}), 1 \leq i \leq k$ are non separated from each other.

Figure 10: The correct picture of in between branching.

Conversely suppose that $E \in \hat{F}^s$ is not separated from $F, L$ and is between $F$ and $L$. Let $B_k, k \in \mathbb{N}$, be a sequence of stable leaves so that $B_k \to F$ as $k \to \infty$. As $E$ is not separated from $F$, $B_k \to E$ in $\mathcal{H}^s$ when $k \to \infty$. But since $F$ and $L$ are connected by a finite chain of lozenges, then for $k$ big all $B_k$ intersect the interior of these lozenges. Therefore the only possible leaves in the limit of $B_k$ which are between $F$ and $L$ are those in the stable boundary of the lozenges $B_i$. This completely characterizes such leaves and hence there are finitely many in between leaves.
§4. Branching structure

An \( R \)-covered Anosov flow can only have one of two topological types (up to isotopy in \( \tilde{M} \)) for the joint structure of \( \tilde{F}^s, \tilde{F}^u \) [Fe3]. They are characterized by:

1. Any leaf of \( \tilde{F}^s \) intersects every leaf of \( \tilde{F}^u \) and vice versa. This is the called the product type.

2. There is a leaf of \( \tilde{F}^s \) which does not intersect every leaf of \( \tilde{F}^u \). This is the skewed type, see detailed definition in [Fe3].

Suspensions have product type and geodesic flows have skewed type.

**Corollary 4.6** Let \( \Phi \) be an Anosov flow in \( M^3 \). Then \( \Phi \) is topologically conjugate to a suspension of an Anosov diffeomorphism of the torus if and only if there are no free homotopies between closed orbits of \( \Phi \) (including non trivial free homotopies from an orbit to itself).

**Proof of 4.6:** If \( \Phi \) is not \( R \)-covered, theorem 4.4 shows that there are \( F_0 \neq F_1 \in \tilde{F}^s \) and \( g \) a nontrivial covering translation with \( g(F_i) = F_i \). Let \( \alpha_i \) be the periodic orbit in \( F_i \). Then \( g(\alpha_i) = \alpha_i \). Therefore \( \pi(\alpha_0), \pi(\alpha_1) \) are closed orbits of \( \Phi \) (they may be the same orbit) which are non trivially freely homotopic to each other.

If \( \Phi \) is \( R \)-covered and has product type, then by theorem 2.8 of [Ba2] (see announcement in [So]) \( \Phi \) is topologically conjugate to a suspension. Otherwise \( \Phi \) has skewed type and theorem 3.4 of [Fe3] produces many non trivial free homotopies between closed orbits of \( \Phi \).

Given 2 adjacent lozenges \( B_1 \) and \( B_2 \) the pivot of their union is the common corner of \( B_1 \) and \( B_2 \).

**Corollary 4.7** Let \( \Phi \) be an Anosov flow in \( M^3 \). Then up to covering translations there are only finitely many branching leaves.

**Proof of 4.7:** Suppose there are infinitely many inequivalent stable branching leaves, where the associated branching is in the positive direction. Given any two non separated leaves \( F, L \) let \( \gamma, \alpha \) be the respective periodic orbits which are connected by a chain of lozenges. For any two adjacent lozenges, the pivot is uniquely determined, furthermore the pivots are always periodic orbits.
§4. Branching structure

Hence there are infinitely many inequivalent periodic pivots \( p_i, i \in \mathbb{N} \). Since \( \pi(p_i) \) accumulates in \( M \), assume up to covering translations that all \( p_i \) are in a very small product neighborhood of \( p \in \tilde{M} \), so let \( i \neq k \) with

\[
\tilde{W}^u(p_i) \cap \tilde{W}^s(p_k) \neq \emptyset \quad \text{and} \quad \tilde{W}^s(p_i) \cap \tilde{W}^u(p_k) \neq \emptyset.
\]

An argument exactly like case 1 of theorem 3.3 shows this is impossible.

4.7

We can now completely characterize the structure of the set of non separated leaves:

**Corollary 4.8** Let \( \Phi \) be an Anosov flow in \( M^3 \). Let \( F \) be a branching leaf of \( \tilde{F}^s \) and \( E \) be the set of non separated leaves from \( F \). Given \( E, L \in E \) we say that \( E < L \) in \( E \) if there are \( G, H \in \tilde{F}^u \), with \( G \cap E \neq \emptyset, H \cap L \neq \emptyset \) and \( G \) in the back of \( H \). Then either

1. \( E \) is finite, hence order isomorphic to \( \{1, 2, \ldots, n\} \) or,
2. \( E \) is infinite and order isomorphic to the set of integers \( \mathbb{Z} \).

In particular given any \( E, L \in E \), there are only finitely many branching leaves between them.

**Proof of 4.8:** Up to finite cover if necessary assume that \( \Phi \) is orientable. Let \( E \) be the set of non separated leaves from \( E \in \tilde{F}^s \). If \( E \) is finite, the result is immediate, so assume it is infinite. Suppose all leaves in \( E \) are not separated on their negative sides.

By corollary 4.7 there are \( E' \neq E^* \in E \) and \( f \) a covering translation with \( f(E') = E^* \). Assume that \( E' < E^* \) in the ordering of \( E \). Theorem 4.2 implies that \( E', E^* \) are connected by a finite chain with positive stable boundaries in \( E_0 = E', E_1, \ldots, E_n = f(E_0) = E^* \in \tilde{F}^s \). Clearly \( E_i < E_j \) if \( i < j \). Since \( E_0 \) is not separated from \( E_n \), then \( f(E_0) = E_n \) is not separated from \( f(E_n) \). This produces \( E_{n+1}, \ldots, E_{2n} = f(E_n) \), a sequence of non separated leaves. Using \( f^i, i \in \mathbb{Z} \), one constructs a sequence \( \{E_i\}_{i \in \mathbb{Z}} \subset E \) of non separated leaves.

Let now \( E \in E \). Then \( E \) and \( E_0 \) are not separated, hence connected by a finite chain of adjacent lozenges all intersecting a common stable leaf. Notice that the lozenges in the chain are completely determined by a corner plus a direction. On the other hand, starting from \( E_0 \) and in any direction from \( E_0 \) (in \( E \)) there are infinitely many adjacent lozenges intersecting a common stable leaf. This implies that \( E \) will be eventually achieved by lozenges in
\( \mathcal{E} \), that is \( E = E_i \) for some \( i \in \mathbb{Z} \). Hence \( \mathcal{E} = \{ E_i \}_{i \in \mathbb{Z}} \). Clearly the order induced above shows that \( E_i < E_j \) if \( i < j \). Hence \( \mathcal{E} \) is order isomorphic to \( \mathbb{Z} \) as desired.

\( \text{4.8} \)

Notice that any covering translation \( f \) conjugates the stabilizers of \( F \) and \( f(F) \) that is \( f \circ (\mathcal{T}(F)) \circ f^{-1} = \mathcal{T}(f(F)) \). Therefore conjugation by \( f \) takes a generator of \( \mathcal{T}(F) \) to a generator of \( \mathcal{T}(f(F)) \).

**Corollary 4.9** Let \( \Phi \) be an Anosov flow in \( M^3 \), orientable. If \( F_i, i \in \mathbb{N} \subset \tilde{\mathcal{F}}^s \) is an infinite collection of non separated leaves of \( \tilde{\mathcal{F}}^s \), then \( M \) has an incompressible torus.

**Proof of 4.9:** As \( M \) is orientable, then if necessary lift to a double cover \( M_2 \) where both \( \mathcal{F}^s \) and \( \mathcal{F}^u \) are transversely orientable. The structure of \( \tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u \) is the same. By corollary 4.7 there is a covering translation \( f \) of \( M_2 \) with \( f(F_i) = F_j \) and \( i \neq j \).

Let \( g \neq id \) be a generator of the stabilizer of \( F_i \) in \( \pi_1(M_2) \). Then \( fgf^{-1} \) is a generator of \( \mathcal{T}(F_j) \). Theorem 4.4 implies that \( g(F_j) = F_j \). By theorem 4.3, \( g \) is indivisible in \( \pi_1(M_2) \), hence \( g \) is also a generator of \( \mathcal{T}(F_j) \). This implies that either \( fgf^{-1} = g \) or \( fgf^{-1} = g^{-1} \).

In the first case \( f \) and \( g \) generate an abelian subgroup of \( \pi_1(M_2) \). If \( f^n g^m = 1 \), then \( f^n g^m(F_i) = F_i \) hence \( f^n(F_i) = F_i \). If \( n \neq 0 \) theorem 4.3 implies that \( f(F_i) = F_i \), contradiction to \( F_i \neq F_j \). Hence \( n = 0 \). Since no multiple of a closed orbit is null homotopic, then \( g^m = id \) implies that \( m = 0 \) also. Hence there is a \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup of \( \pi_1(M_2) \).

If \( fgf^{-1} = g^{-1} \), then \( f^2 \) and \( g \) generate an abelian subgroup of \( \pi_1(M_2) \) and the same argument produces \( \mathbb{Z} \oplus \mathbb{Z} < \pi_1(M_2) \). Therefore there is a \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup of \( \pi_1(M) \). By the torus theorem [Ga] (which uses \( M \) being orientable), either \( M \) is a Seifert fibered space or there is an embedded incompressible torus. In the first case, Ghys [Gh] proved that \( \Phi \) is up to finite covers, topologically conjugate to a geodesic flow. But then \( \Phi \) would be \( \mathbb{R} \)-covered, contrary to hypothesis. Hence \( M \) is toroidal as desired.

\( \text{4.9} \)
5 Product regions

Definition 5.1 A positive unstable product region $P$ of $\tilde{\Phi}$ is a region in $\tilde{M}$ defined by an open strong stable segment $\eta \subset F \in \tilde{F}^s$ (or by a flow band $\tilde{\Phi}R(\eta)$) so that

$$\forall p, q \in \eta, \quad \mathcal{J}^u_+(p) = \mathcal{J}^u_+(q).$$

Then $P = \bigcup_{p \in \eta} \tilde{W}^u_+(p)$.

The segment $\eta$ (which may be infinite) is called a base segment for the product region. Similarly define negative unstable product regions and stable product regions.

The main property of product regions is the following: for any $F \in \tilde{F}^s$, $G \in \tilde{F}^u$ so that (i) $F \cap P \neq \emptyset$ and (ii) $G \cap P \neq \emptyset$, then $F \cap G \neq \emptyset$. To see why this is true, notice first that (ii) implies that $\emptyset \neq G \cap \eta = p$. By (i) let $q \in \eta$ with $F \cap \tilde{W}^u_+(q) \neq \emptyset$. Then $F \in \mathcal{J}^u_+(q)$ hence $F \in \mathcal{J}^u_+(p)$, that is $F \cap G \neq \emptyset$. This is the reason for the terminology product region.

The purpose of this section is to show that the existence of product regions implies that the flow is $\mathbb{R}$-covered. The main difficulty is that we will not assume that $\Phi$ is transitive. With the additional hypothesis of transitivity the proof of this fact is simple and was done in [Fe5].

Given $e > 0$ and $z \in \tilde{M}$, let $\sigma^*_e(z)$ be the segment in $\tilde{W}^{ss}(z)$ centered at $z$ and with length $e$.

Theorem 5.2 Let $\Phi$ be an Anosov flow in $M^3$. If there is a product region in $\tilde{M}$ then $\Phi$ is $\mathbb{R}$-covered. Furthermore any leaf of $\tilde{F}^s$ intersects every leaf of $\tilde{F}^u$ and vice versa. As a result $\Phi$ is topologically conjugate to a suspension Anosov flow.

Proof of 5.2: By lifting to a finite cover if necessary suppose that $\Phi$ is orientable. Assume that there is a positive unstable product region defined by $\eta \subset \tilde{W}^{ss}(y_1)$. The proof will be achieved by producing bigger and bigger product regions in $\tilde{M}$, which eventually fill all of $\tilde{M}$. This will show there is a product structure in $\tilde{M}$ and hence that the flow is $\mathbb{R}$-covered.

If $\Omega$ is the nonwandering set of $\Phi$ then $\tilde{W}^s(\Omega) = \tilde{M}$ [Pu-Sh]. Since the periodic orbits are dense in $\Omega$ [Sm, Pu-Sh] it follows that the set of annular leaves of $\Phi$ forms a dense subset of $\tilde{M}$. Therefore there is a periodic orbit $\gamma$ of $\tilde{\Phi}$ so that if $p \in \gamma$, then $\tilde{W}^u_+(p) \cap \eta \neq \emptyset$. If $e > 0$ is small enough then for any $z \in \sigma^*_e(p)$, $\tilde{W}^u(z) \cap \eta \neq \emptyset$. Hence $\sigma^*_e(p)$ is the defining segment of a product region.
§5. Product regions

Let $g$ be a generator of $T(\tilde{W}^{s}(\gamma))$. For any $y_2 \in \tilde{W}^{ss}(p)$, $y_2$ near enough $p$, then $y_2 \in \sigma^{s}_c(p)$, hence $J^u_+(y_2) = J^u_+(p)$. Since $g(\tilde{W}^{u}_+(p)) = \tilde{W}^{u}_+(p)$, then

$$J^u_+(g^i(y_2)) = g^i(J^u_+(y_2)) = g^i(J^u_+(p)) = J^u_+(p), \quad \forall i \in \mathbb{Z},$$

Consequently for any $y_3 \in \tilde{W}^{s}(p)$ it follows that $J^u_+(y_3) = J^u_+(p)$. Let

$$\mathcal{A} = \bigcup_{y_2 \in \tilde{W}^{ss}(p)} \tilde{W}^{u}_+(y_2).$$

Then $\mathcal{A}$ is a product region with an infinite basis segment $\tilde{W}^{ss}(p)$.

We now prove that the front of $\tilde{W}^{s}(p)$ is exactly the set $\mathcal{A}$. This shows that there is a product structure of $\mathcal{F}^s, \mathcal{F}^u$ in the front of $\tilde{W}^{s}(p)$.

Lemma 5.3 $\partial \mathcal{A} = \tilde{W}^{s}(p)$.

Proof of 5.3: Let $a \in \partial \mathcal{A}$. Suppose $a \notin \tilde{W}^{s}(p)$. There are $a_i \in \mathcal{A}$ with $a_i \to a$. Let $b_i = \tilde{W}^{u}(a_i) \cap \tilde{W}^{ss}(p)$. Without loss of generality we may assume $b_i \in \tilde{W}^{ss}_+(p)$. Then $\tilde{W}^{u}(b_i) \to \tilde{W}^{u}(a)$ and maybe other leaves too. Notice that $a$ is in front of $\tilde{W}^{ss}(p)$ as all $a_i$ are.

Suppose that $\tilde{W}^{u}(a) \cap \tilde{W}^{s}(p) \neq \emptyset$. As $a$ is in front of $\tilde{W}^{s}(p)$ it would follow that $a \in \mathcal{A}$. Hence $\tilde{W}^{u}(a)$ is contained in the front of $\tilde{W}^{s}(p)$, in particular $\tilde{W}^{u}(a) \cap \mathcal{A} = \emptyset$.

Claim - $b_i \to \infty$ in $\tilde{W}^{ss}(p)$.

Otherwise assume up to subsequence that $b_i \to b_0 \in \tilde{W}^{ss}(p)$. Since $a_i \to a$ and $\tilde{W}^{u}(a) \cap \tilde{W}^{ss}(p) = \emptyset$, then $\tilde{W}^{s}(a), \tilde{W}^{u}(b_0)$ form a branching pair of $\mathcal{F}^u$. For $i$ big enough $\tilde{W}^{s}(a) \cap \tilde{W}^{u}(b_i) \neq \emptyset$. Hence

$$\tilde{W}^{s}(a) \subset J^u_+(b_i) = J^u_+(p) = J^u_+(b_0).$$

Hence $\tilde{W}^{s}(a) \cap \tilde{W}^{u}(b) \neq \emptyset$, which is a contradiction to $\tilde{W}^{u}(a), \tilde{W}^{u}(b)$ being non separated. The claim follows.

Since $\tilde{W}^{u}(b_i) \to \tilde{W}^{u}(a)$; in fact $\tilde{W}^{u}_+(b_i) \to \tilde{W}^{u}(a)$ and also $\tilde{W}^{u}(a) \cap \mathcal{A} = \emptyset$ then $\tilde{W}^{u}(a) \subset \partial \mathcal{A}$. Given $c \in \tilde{W}^{u}(a)$, choose $c'$ near $c$ with $c' \in \tilde{W}^{s}(c)$ and $c' \in \mathcal{A}$. Since $\tilde{W}^{s}(c') \cap \mathcal{A} \neq \emptyset$ then $\tilde{W}^{s}(c') \cap \tilde{W}^{uu}(p) \neq \emptyset$. As a result for any $c \in \partial \mathcal{A}$ with $c \notin \tilde{W}^{s}(p)$, then $\tilde{W}^{s}(c) \cap \tilde{W}^{uu}(p) \neq \emptyset$.

Let $G = \tilde{W}^{u}(a)$ and let $F = \tilde{W}^{s}(p)$. Notice that $g(G) \neq G$. Otherwise there is an orbit $\delta$ of $\Phi$ in $G$ with $g(\delta) = \delta$. By the above $\tilde{W}^{s}(\delta) \cap \tilde{W}^{u}(p) \neq \emptyset$, a contradiction to both left invariant under $g$. 
5. Product regions

In fact this shows that $g^n(G) \neq g^m(G)$ for any $n \neq m \in \mathbb{Z}$. Let $G_k = g^k(G)$. Then $G_k \subset \partial A$ so the $G_k$ are not separated from each other. Therefore by theorem 4.4, $G_k$ contains a periodic orbit $\delta_k$ and there is an indivisible, non trivial covering translation $f$ with $f(G_k) = G_k$ for all $k \in \mathbb{Z}$. By the above $\emptyset \neq \hat{W}_{s}^u(\delta_k) \cap \hat{W}_{u}^s(p) = q_k$ for any $k \in \mathbb{Z}$. Assume that $q_k = g^k(q_0) \to p$ as $k \to +\infty$.

As $\hat{W}_{u}(b) \to G_0$, let $S \in \hat{F}$ with $S \cap \hat{W}_{s}^u(\delta_0) \neq \emptyset$ and $S \cap F \neq \emptyset$. Then $f(S) \cap \hat{W}_{s}^u(\delta_0) \neq \emptyset$ and we may assume that $f(S)$ is in front of $S$. As $g$ acts as an expansion in the set of orbits of $\hat{W}_s(p)$ then $g^j(S) \to G_0$ as $j \to +\infty$. Let $j$ with $g^j(S)$ in front of $f(S)$ and with $g^j(S) \cap \hat{W}_s^u(\delta_0) \neq \emptyset$, see fig. 11. Then $S, g^j(S), \hat{W}_{u}^s(\delta_0)$ and $\hat{W}_s(p)$ form a rectangle. As $f(S)$ intersects $\hat{W}_s^u(\delta_0)$ between $S \cap \hat{W}_s^u(\delta_0)$ and $g^j(S) \cap \hat{W}_s^u(\delta_0)$, it follows that $f(S) \cap F \neq \emptyset$. In particular $F$ and $f(F)$ both intersect the unstable leaf $f(S)$.

Figure 11: Boundaries of product regions.

If $f(F)$ is in the front of $F$, then as $q_k \to p$ when $k \to +\infty$, it follows that there is some $\hat{W}_s^u(q_k)$ which is in the back of $f(F)$, see fig. 11. This is a contradiction because $f$ leaves $\hat{W}_s^u(q_k)$ invariant. Similarly if $f(F)$ is in the back of $F$ then $f^{-1}(F)$ intersects $S$ and is in front of $F$ producing the same contradiction.

We conclude that $f(F) = F$. As a result $f = g^n$. But $f(G_k) = G_k \neq G_{k+n} = g^n(G_k)$, contradiction.

This shows that the hypothesis $\partial A \neq \hat{W}_s^u(p)$ is impossible, hence the lemma follows.

lipro
Continuation of the proof of theorem 5.2

Let $\alpha$ be a periodic orbit, $\alpha \neq \gamma$ with $\tilde{W}^u(\alpha) \cap \tilde{W}^s(\gamma) \neq \emptyset$. Let $q \in \alpha$. Assume that $q$ is in front of $\tilde{W}^s(p)$. Let $\mathcal{C} = \bigcup_{z \in \tilde{W}^s(q)} W^u_{\gamma}(z)$. Then $\mathcal{C}$ is a product region and as in lemma 5.3, $\partial \mathcal{C} = \tilde{W}^s(q)$. Since $\partial \mathcal{C} = \tilde{W}^s(p)$, it follows that $\mathcal{C} \subset \mathcal{A}$.

Let $h$ a generator of $\mathcal{T}(\tilde{W}^s(\alpha))$ so that $h$ acts as an expansion in the set of orbits of $\tilde{W}^u(\alpha)$. Since $\tilde{W}^u(q) \cap \partial \mathcal{A} \neq \emptyset$, then for any $i > 0$, $h^i(\mathcal{A})$ is a product region strictly bigger than $\mathcal{A}$ and $\partial h^i(\mathcal{A}) = h^i(\tilde{W}^s(p))$.

Therefore for any $z, y \in W^u(q)$ there is $i > 0$ so that $z, y \in h^i(\mathcal{A})$. Let $w = \tilde{W}^u(q) \cap h^i(\tilde{W}^s(p))$, so $z, y \in W^u_{\gamma}(w)$.

If $G \in \mathcal{F}^u$ and $G \in J^s_+(z)$ then $G$ intersects the front of $\tilde{W}^s(w)$. By the previous lemma the front of $\tilde{W}^s(w)$ is equal to $h^i(\mathcal{A})$. As $h^i(\mathcal{A})$ is a product region, then

$$G \cap h^i(\mathcal{A}) \neq \emptyset, \quad \tilde{W}^s(y) \cap h^i(\mathcal{A}) \neq \emptyset \quad \Rightarrow \quad G \cap \tilde{W}^s(y) \neq \emptyset.$$ 

Since $G$ is in front of $\tilde{W}^u(y)$ then $G \in J^s_+(y)$. By symmetry $J^s_+(z) = J^s_+(y)$.

It follows that $\tilde{W}^u(y)$ is then a basis segment of a positive stable product region $P_1$. By lemma 5.3, $\partial P_1 = \tilde{W}^u(q)$. Similarly $\tilde{W}^u(y)$ is also the basis segment of a negative stable product region $P_2$ and $\partial P_2 = \tilde{W}^u(q)$. Hence $M = P_1 \cup P_2$.

It follows from this analysis that for any $E \in \tilde{F}^s$, $E \cap \tilde{W}^u(q) \neq \emptyset$. Therefore $\mathcal{F}^s$ is $\mathcal{R}$-covered. Similarly for any $R \in \mathcal{F}^u$ if it in the front of $\tilde{W}^u(q)$, then $R \subset P_1$ hence $R \cap \tilde{W}^s(q) \neq \emptyset$, and similarly for $R$ in the back of $\tilde{W}^u(q)$. This shows that $\mathcal{F}^u$ is also $\mathcal{R}$-covered, hence that $\Phi$ is $\mathcal{R}$-covered.

Let now $E \in \tilde{F}^s, R \in \mathcal{F}^u$. Assume that $R$ is (say) in front of $\tilde{W}^u(q)$. Then $R \cap P_1 \neq \emptyset$ and $E \cap P_1 \neq \emptyset$, so $E \cap R \neq \emptyset$. Therefore any leaf of $\tilde{F}^s$ intersects every leaf of $\mathcal{F}^u$ and vice versa. Theorem 2.8 of [Ba2] implies that $\Phi$ is topologically conjugate to a suspension Anosov flow.

5.2

6 Infinite branching and transverse tori

In this section we show that, if infinite branching occurs, then a particular type of structure, called a scalloped region, occurs in $\tilde{M}$ (or $\mathcal{O}$) and there is an embedded torus transverse to the flow. We then show that there are many examples with only finite branching.
Theorem 6.1 Let $\Phi$ be an Anosov flow in $M^3$. It there is infinite branching in $F^s$, then there is associated infinite branching in $F^u$.

Proof of 6.1: Let $E = \{E_i\}_{i \in \mathbb{Z}}$ be an infinite, totally ordered collection of non-separated leaves. Assume they are not separated on their negative sides. Let $\gamma_i$ be the periodic orbit in $E_i$. Theorem 4.2 implies that for any $i$, $E_i$ forms part of the boundary of two lozenges: let $B_{2i-1}$ be the lozenge with $(+, +)$ corner $\gamma_i$ and let $B_{2i}$ be the lozenge with $(+, -)$ corner $\gamma_i$. Let $F_i \in F^s$ be the other leaf in the boundary of $B_{2i}$ and $B_{2i+1}$, where $B_{2i}$ and $B_{2i+1}$ are in front of $F_i$. Let $\zeta_i$ be the periodic orbit in $F_i$, see fig. 12. Then the $\{F_i\}_{i \in \mathbb{Z}} \subset F^s$ are all non-separated from each other on their positive sides. Furthermore all $B_i$ intersect a common stable leaf.

Figure 12: Chain of lozenges.

Notice that the sides of $B_{2i}$ are $W^s_+(\gamma_i), W^u_+(\gamma_i), W^s_-(\zeta_i)$ and $W^u_-(\zeta_i)$; while the sides of $B_{2i+1}$ are $W^s_+(\gamma_{i+1}), W^u_+(\gamma_{i+1}), W^s_-(\zeta_i)$ and $W^u_-(\zeta_i)$, see fig. 12. Let $L = \bigcup_{i \in \mathbb{Z}} B_i$. Then all of the following sets are equal:

$$J^u_-(\gamma_i), i \in \mathbb{Z}, \quad J^u_+(\zeta_j), j \in \mathbb{Z}.$$  

Let $C_i$ be the back of $W^u_-(\gamma_i)$ and let $C = \cup_{i \in \mathbb{N}} C_i$. The set $\mathcal{C}$ is a union of adjacent lozenges. Then for any $p, q \in W^u_-(\gamma_0)$ and any $i > 0$, $W^u_-(\gamma_i) \in \mathcal{J}_+(p) \cap \mathcal{J}_+(q)$. If $\mathcal{C} = \tilde{M}$, then the intersections of $W^u_-(\gamma_i)$ with $\tilde{W}^s_+(p)$ and $\tilde{W}^u_+(q)$ are escaping to infinity in these leaves. This implies that $\mathcal{J}_+(p) = \mathcal{J}_+(q)$. Therefore $\tilde{W}^u_+(\gamma_0)$ would be the basis segment of a positive stable product region in $\tilde{M}$. By theorem 5.2, $\Phi$ would be $\mathbb{R}$-covered contrary to hypothesis. Hence $\mathcal{C} = \tilde{M}$. This is the key fact which will produce a covering translation $f$ commuting with $g$. 

forw
Let then $p \in \partial C$, hence $\tilde{W}^u(p) \subset \partial C$. For all $i$ big enough $\tilde{W}^s(p) \cap \tilde{W}^u(\gamma_i) \neq \emptyset$. This implies that $\tilde{W}^s(p) \cap \tilde{W}^u(\gamma_i) \neq \emptyset$ for any $i \in \mathbb{Z}$. As a result $\tilde{W}^u(p) \subset \partial \mathcal{L}$.

Since $g(C) = C$, then $g^n(\tilde{W}^u(p)) \subset \partial \mathcal{L}$ for any $n \in \mathbb{Z}$. If $g^n(\tilde{W}^u(p)) = \tilde{W}^u(p)$ for some $n \neq 0$, let $\beta$ be the periodic orbit in $\tilde{W}^u(p)$. Then $g^n(\tilde{W}^s(\beta)) = \tilde{W}^s(\beta)$, $g^n(\tilde{W}^u(\gamma_i)) = \tilde{W}^u(\gamma_i)$ and $\tilde{W}^s(\beta) \cap \tilde{W}^u(\gamma_i) \neq \emptyset$, contradiction. Hence the leaves $g^n(\tilde{W}^u(p)), n \in \mathbb{Z}$ are all distinct and all non separated from each other on their negative sides. By theorem 4.4, $g^n(\tilde{W}^u(p))$ are all periodic and let $f$ be the indivisible covering translation leaving all invariant and acting as an expansion in the set of orbits in $\tilde{W}^u(p)$.

Notice that $g(\tilde{W}^s(p))$ is in front of $\tilde{W}^s(p)$. Let $H_0 = \tilde{W}^u(p), H_1, ..., H_n = g(\tilde{W}^u(p))$ be the chain of non separated leaves from $\tilde{W}^u(p)$ to $g(\tilde{W}^u(p))$. As in the argument above, one constructs $\{H_k\}_{k \in \mathbb{Z}}$, all in $\partial \mathcal{L}$. Let $\beta_k$ be the periodic orbits in $H_k$. Then $\beta_k$ is the corner of two lozenges $R_{2k-1}$ and $R_{2k}$. Then all $\mathcal{R}_k$ intersect a common unstable leaf.

Figure 13: A scalloped region in the universal cover.

Furthermore if $q \in \partial C$, then $\tilde{W}^u(q)$ is not separated from $H_0$, so $\tilde{W}^u(q)$ is one of $H_k$. Let $\{G_k\}_{k \in \mathbb{Z}}$ be the sequence of leaves which form the negative unstable boundary of the lozenges $\{\mathcal{R}_k\}_{k \in \mathbb{Z}}$. Then $f(G_k) = G_k$ for all $k$. 
Given $l \in \mathbb{Z}$ then for $j > 0$ big enough $\widetilde{W}_u^{n}(\gamma_j) \cap \widetilde{W}_s^{n}(\beta_l) \neq \emptyset$. Since all $\mathcal{J}_u^{n}(\beta_k), k \in \mathbb{Z}$ are equal as are all $\mathcal{J}_s^{n}(\gamma_i)$ this implies that for any $i, k \in \mathbb{Z}$, $\mathcal{B}_i \cap \mathcal{R}_k \neq \emptyset$. As $g(\mathcal{B}_i) = \mathcal{B}_i$ for any $i \in \mathbb{Z}$ and $g(\mathcal{R}_k) = \mathcal{R}_{k+n}$ for any $k \in \mathbb{Z}$, then for any $i \in \mathbb{Z}, \mathcal{B}_i \subset \cup_{k \in \mathbb{Z}} \mathcal{R}_k$.

In addition notice that $g^m(\mathcal{W}^s(\beta_0)) \to \cup_{i \in \mathbb{Z}} \mathcal{W}^s_i = \mathcal{E}$ as $m \to +\infty$. As $f(\mathcal{W}^s(\beta_k)) = \mathcal{W}^s(\beta_k), \forall k \in \mathbb{Z}$ then $f$ leaves invariant the set $\mathcal{E}$. Therefore there is $j \in \mathbb{N}$ so that $f(\mathcal{E}_i) = \mathcal{E}_{i+j}$ for all $i \in \mathbb{Z}$. Since $f(\mathcal{R}_k) = \mathcal{R}_k, \forall k \in \mathbb{Z}$, then the same argument as above implies that $\mathcal{R}_k \subset \cup_{i \in \mathbb{Z}} \mathcal{B}_i$ for any $k \in \mathbb{Z}$.

We conclude that

$$\mathcal{L} = \bigcup_{i \in \mathbb{Z}} \mathcal{B}_i = \bigcup_{k \in \mathbb{Z}} \mathcal{R}_k.$$ 

The region $\mathcal{L}$ is called a *scalloped* region, see fig. 13 and is uniquely associated to the infinite branching $\mathcal{E}$. Notice that $\mathcal{F}_s$ and $\mathcal{F}_u$ restrict to foliations with $\mathbb{R}$ leaf space in $\mathcal{L}$.

6. Infinite branching and transverse tori

**Theorem 6.2** Let $\Phi$ be an Anosov flow in $M^3$ orientable. If there is infinite branching in (say) $\mathcal{F}_s$ then there is an embedded torus transverse to $\Phi$.

**Proof of 6.2:** Assume first that $\Phi$ is orientable. We use the notation from the previous theorem. Let $\nu_{(i,k)} = \mathcal{W}_u^{n}(\gamma_i) \cap \mathcal{W}_s^{n}(\beta_k)$ an orbit of $\Phi$. Then there are $\mathbb{Z} \oplus \mathbb{Z}$ such orbits in $\mathcal{L}$. Recall that $g(\gamma_i) = \gamma_i, f(\beta_k) = \beta_k$, and $f$ acts as a contraction in the set of orbits in $\mathcal{W}_s^{n}(\beta_k)$ and likewise for the action $g$ in $\mathcal{W}_s^{n}(\gamma_i)$. Then there are $a, b \in \mathbb{N} - \{0\}$ so that:

$$f(\nu_{(0,0)}) = \nu_{(a,0)}, \quad \text{since} \quad f(\mathcal{W}_s^{n}(\beta_0)) = \mathcal{W}_s^{n}(\beta_0),$$

$$gf(\nu_{(0,0)}) = \nu_{(a,b)}, \quad \text{since} \quad g(\mathcal{W}_u^{n}(\gamma_0)) = \mathcal{W}_u^{n}(\gamma_0),$$

$$f^{-1}gf(\nu_{(0,0)}) = \nu_{(b,0)}, \quad \text{as} \quad f(\mathcal{W}_s^{n}(\beta_0)) = \mathcal{W}_s^{n}(\beta_0), \ f^{-1}(\mathcal{W}_u^{n}(\gamma_0)) = \mathcal{W}_u^{n}(\gamma_0),$$

and finally

$$g^{-1}f^{-1}gf(\nu_{(0,0)}) = \nu_{(0,0)}.$$ 

Since $\nu_{(0,0)} \subset \mathcal{W}_u^{n}(\gamma_0), \nu_{(0,0)}$ is not a periodic orbit of $\Phi$. Therefore the last equation above implies that $gf = fg$. Furthermore $f^ng^m = id$, clearly implies that $n = m = 0$ so $f, g$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$. Notice that this subgroup preserves $\mathcal{L}$ and hence also preserves $\partial \mathcal{L}$.
Let \( p \in \nu_{(0,0)} \). Let \( \xi_1 \) be an embedded arc in \( \tilde{W}^u(\beta_0) \) from \( p \in \nu_{(0,0)} \) to \( f(p) \in \nu_{(0,0)} \) transverse to \( \tilde{\Phi} \) and so that \( \pi(\xi_1) \) is a smooth closed curve in \( M \). Let \( \xi_2 \) be a similar arc from \( p \in \nu_{(0,0)} \) to \( g(p) \in \nu_{(0,0)} \) contained in \( \tilde{W}^u(\gamma_0) \). Since \( fg = gf \) then \( \xi = \xi_1 \ast f(\xi_2) \ast (g(\xi_1))^{-1} \ast (\xi_2)^{-1} \) is a closed loop in \( \tilde{M} \). As \( \tilde{W}^u(\gamma_0), \tilde{W}^u(\gamma_b), \tilde{W}^s(\beta_a) \) form a rectangle it is easy to produce an smooth embedded disk \( D_1 \) in \( \tilde{M} \), which is transverse to \( \tilde{\Phi} \) and so that \( \partial D_1 = \xi \).

After a small perturbation of \( D_1 \) near \( \partial D_1 \), we may assume that \( D = \pi(D_1) \) is a smooth closed surface transverse to \( \Phi \). A priori \( D \) is only an immersed surface. Again after a small perturbation of \( D \), we may assume that \( D \) is transverse to itself. Using cut and paste techniques [He, Ja], as explicit done by Fried [Fr], one can eliminate all triple points of intersection and double curves of intersection, transforming \( D \) into a union of embedded surfaces transverse to \( \Phi \).

Any such surface has induced stable and unstable foliations hence it has zero Euler characteristic. It is transverse to the flow, hence it is two sided in \( M \) and as \( M \) is orientable, then this transverse surface has to be a torus.

If \( \Phi \) is not orientable, the above proof can be applied to a double cover \( M' \) of \( M \) where the lifted flow is orientable. The image in \( M \) of the transverse torus in \( M' \) is an (immersed) torus in \( M \) and again cut and paste techniques yield the result.

Remarks: (1) With much more work, one can in fact show there is a transverse torus \( T \) intersecting exactly those orbits in \( \pi(L) \). This is done in detail by Barbot in the proof of theorem B in [Ba3] where the hypothesis are the existence of two commuting covering translations \( f, g \), so that both \( f \) and \( g \) are associated to (different free homotopy classes of) closed orbits of \( \Phi \) in \( M \).

(2) As mentioned earlier the Anosov flow constructed by Bonatti and Langevin has infinite branching. The scalloped region of this flow was explained in detail in [Fe5]. The Bonatti-Langevin flow is the simplest Anosov flow with infinite branching in the sense that there is only one orbit \( \gamma \) of \( \Phi \) which does not intersect the transverse torus \( T \). In this case all periodic orbits in \( \partial L \) are lifts of \( \gamma \). The picture in \( \tilde{M} \) is very symmetric.

We now prove that a large class of Anosov flows in dimension 3 have only finite branching. If the branching is finite we define its length to be the
number of non separated leaves.

**Theorem 6.3** Any Anosov flow obtained by the Franks-Williams construction [Fr-Wi] is not \( \mathbb{R} \)-covered and has only length 2 branching.

**Proof of 6.3:** First we recall the Franks-Williams construction [Fr-Wi]. Start with a suspension Anosov flow \( \Phi_0 \) in \( N \) and a closed orbit \( \gamma \). Modify the flow in a neighborhood of \( \gamma \) using Smale’s DA (derived from Anosov) construction [Sm, Wi], so that \( \gamma \) becomes an expanding orbit and 2 new hyperbolic orbits \( \gamma_1 \) and \( \gamma_2 \) parallel to \( \gamma \) are created, see fig. 14, a. This produces the new flow \( \Phi^* \) in \( N \).

One can do this in a way that the stable foliation is still preserved by the new flow. Now remove a solid torus neighborhood \( V \) of \( \gamma \) with a boundary torus \( T_1 \), transverse to the flow. This creates a manifold \( M_1 = N - V \) where the flow is incoming in the boundary. There is an induced stable foliation in \( \partial M_1 = T_1 \), which has two closed leaves and two Reeb components in between them. Using a time reversal of this flow construct \( M_2 \) with a boundary torus \( T_2 \) where the flow is outgoing and there is an induced unstable foliation in \( T_2 \). Finally glue \( T_1 \) to \( T_2 \) so that after gluing the stable and unstable foliations are transverse. Let \( T \) be the torus obtained by gluing \( T_1 \) to \( T_2 \). Franks and Williams show that such flows are Anosov and clearly intransitive since \( T \) is a separating torus. Hence the flows are not \( \mathbb{R} \)-covered.

By theorem 4.4, any branching of \( \tilde{F}^s \) and \( \tilde{F}^u \) produces freely homotopic closed orbits, so we first understand free homotopies. Let \( \alpha \) and \( \beta \) be freely homotopic closed orbits of \( \Phi \). and let \( \tau : A \to M \) be an annulus realizing the free homotopy. Assume that \( A \) is in general position and is transverse to \( T \). Notice that \( \partial A = \alpha \cup \beta \) is disjoint from \( T \). Then \( \tau^{-1}(T) \) is a union of closed curves in \( A \). We can eliminate all null homotopic components as follows: since \( T \) is transverse to \( \Phi \), it is incompressible [Fe4]. Then any null homotopic component of \( \tau^{-1}(T) \) also produces a null homotopic curve in \( T \).

Using cut and paste arguments [He, Ja] and the fact that \( M \) is irreducible we can eliminate this component by a homotopy of the annulus. We may then assume that \( \tau^{-1}(T) \) is a union of finitely many curves parallel to \( \partial A \).

Let now \( B_1 \) be the closure of a component of \( A - (\tau^{-1}(T)) \) containing a boundary component of \( A \). Let \( \partial_1 B \) be this boundary component (suppose that \( \tau(\partial_1 B) = \alpha \)) and let \( \partial_2 B = \partial B - \partial_1 B \). Assume that \( \tau(B_1) \subset M_1 \).

Notice that \( M_1 \) fibers over the circle with fiber \( F \) a torus minus a disk. Any closed orbit of \( \Phi \) in \( M_1 \) has non zero algebraic intersection with \( F \), hence the same is true for the other boundary of \( B \), that is \( \partial_2 B \) is not a multiple
Figure 14: a. DA construction, b. Induced foliations in a lift of the torus.

of the meridian. Reglue the solid torus $V$ as originally to recover $N$ and the DA flow $\Phi^*$ in $N$. Since $\partial_B$ is not a meridian, then $\partial_B$ is freely homotopic (in $V$) to $\gamma_n$, $n \neq 0$, hence freely homotopic to $\gamma_1$.

The DA construction is equivalent to splitting $\tilde{W}^u(\gamma)$ into two and blowing air in between the 2 sides [Wi], much in the sense of essential laminations [Ga-Oe]. In particular there is a topological semiconjugacy between $\Phi^*$ and $\Phi_0$. Hence free homotopies between closed orbits of $\Phi^*$ produce a free homotopy between two closed orbits of $\Phi_0$ in $N$. But any free homotopy in a suspension is trivial [Fe3]. Therefore $\alpha$ is either $\gamma_1$ or $\gamma_2$ and the free homotopy can be homotoped into $W^s(\gamma_1)$ (or into $W^s(\gamma_2)$).

Furthermore $M_1$ is acylindrical, that is, any properly immersed annulus can be homotoped into the boundary. This is due to Waldhausen (for a proof see [Jo]) and follows from the fact that $M_1$ is atoroidal (in fact it is hyperbolic [Th2]), $M_1$ not a Seifert fibered space and $\partial M_1$ is a single torus.

These two facts imply that the only non trivial free homotopies between closed orbits of $\Phi$ can always be homotoped into $T$. Notice that there are such free homotopies, since $\gamma_1$ is freely homotopic to $\gamma_2$ in $M_1$ and also there are two closed orbits of $\Phi$ in $M_2$ which are freely homotopic to each other and freely homotopic to $\gamma_1$. These orbits are associated to the 4 closed leaves of the induced stable and unstable foliations in $T$.

As a result of this, in order to understand branching in the universal cover all we need to do is understand the structure of $\tilde{F}^s, \tilde{F}^u$ induced in lifts of $T$. Since there are two closed leaves in $\tilde{F}^s \cap T$ and Reeb components in
7 Continuous extension of Anosov foliations

If $\Phi$ is an Anosov flow in $M^3$, Sullivan [Su] showed that the intrinsic geometry of leaves of $\tilde{F}^s$ and $\tilde{F}^u$ is negatively curved in the large as defined by Gromov [Gr]. This holds without any assumption on $M$. Then any leaf $F \in \tilde{F}^s \cup \tilde{F}^u$ has a canonical compactification with an intrinsic ideal boundary $\partial_\infty F$ [Gr].

We proved in [Fe2] that $\partial_\infty F$ is always homeomorphic to a circle. If $F \in \tilde{F}^s$ then the intrinsic ideal points correspond to the (distinct) negative limit points of flow lines in $F$ and to the common positive limit point of all flow lines [Fe3]. The intrinsic geometry of $F \in \tilde{F}^s$ resembles that of the hyperbolic plane $\mathbb{H}^2$ where the flow lines correspond to the geodesics in $\mathbb{H}^2$ which have a common limit point in the ideal boundary of $\mathbb{H}^2$, see fig. 15. Analogous results hold for $\tilde{F}^u$.

Figure 15: Intrinsic ideal points.

If $p \in F \in \tilde{F}^s$, we define $p_- \in \partial_\infty F$ to be the intrinsic negative limit point of the flow line through $p$, that is $p_- = \lim_{t \to -\infty} \tilde{\Phi}_t(p)$, where the limit
§7. Continuous extension of Anosov foliations

is taken in $F \cup \partial_{\infty} F$, see fig. 15. Similarly define $p_+$. For any $p, q \in F \in \bar{F}^s$, $p_+ = q_+ \in \partial_{\infty} F$ and this is also denoted by $F_+$. Furthermore if $p_i \in \bar{W}^{ss}(p)$ and $p_i \to \infty$ in $\bar{W}^{ss}(p)$, then $(p_i)_- \to p_+$ as points in $\partial_{\infty} F$ [Fe3]. This can be clearly seen in the model of the hyperbolic plane.

Notice that $p_- = q_-$ for any $p, q$ in the same flow line $\alpha$ of $\bar{\Phi}$, so this is also denoted by $(\alpha)_- \in \partial_{\infty} F$ and similarly $(\alpha)_+ = F_+$.

From now on we assume that $\pi_1(M^3)$ is negatively curved as defined by Gromov [Gr]. Gromov constructed a canonical compactification of $\bar{M}$ with an ideal boundary $\partial \bar{M}$. When $M$ is irreducible (always the case for us), Bestvina and Mess [Be-Me] showed that $\partial \bar{M}$ is homeomorphic to a sphere, denoted by $S^2_{\infty}$. Furthermore $\bar{M} \cup S^2_{\infty}$ is homeomorphic to a closed 3-ball.

Recall that the foliations $\bar{F}^s, \bar{F}^u$ are transversely oriented.

**Definition 7.1** The limit set of a subset $B$ of $\bar{M}$ is $\Lambda_B = \bar{B} \cap S^2_{\infty}$, where the closure is taken in $\bar{M} \cup S^2_{\infty}$. Given $F \in \bar{F}^s$ or $\bar{F}^u$ and $p \in S^2_{\infty} - \Lambda_F$, we say that $p$ is above $F$ if there is a neighborhood $U$ of $p$ in $\bar{M} \cup S^2_{\infty}$ so that $U \cap \bar{M}$ is in front of $F$. Otherwise we say that $p$ is below $F$. Given a connected component of $S^2_{\infty} - \Lambda_F$ either all of its points are above $F$ and we say this component is above $F$ or all points are below $F$ and we say the component is below $F$. Similarly for $G \in \bar{F}^u$.

**Proposition 7.2** Let $\Phi$ be an Anosov flow in $M^3$ with negatively curved $\pi_1(M)$. Either $\Lambda_F = S^2_{\infty}$ for every $F \in \bar{F}^s$; or for every $F \in \bar{F}^s$, $S^2_{\infty} - \Lambda_F$ has at least one connected component above $F$ and one component below $F$.

*Proof of 7.2:* Classical 3-dimensional topology [He, Ja] and Smale’s spectral decomposition theorem [Sm] imply that $\Phi$ is transitive [Fe4].

We may assume that $\bar{F}^s, \bar{F}^u$ are transversely orientable. Suppose there is $F \in \bar{F}^s$ with $\Lambda_F \neq \emptyset$. Assume that there is a component $Z$ of $S^2_{\infty} - \Lambda_F$ which is above $F$. Since stable leaves are dense in $M$, then for every $L \in \bar{F}^s$ there is a covering translation $g$ with $g(L)$ in the back of $F$ and so that $F, g(L)$ intersect a common unstable leaf. Then $Z \cap \Lambda_{g(L)} = \emptyset$ and since $Z$ is in front of $F$, it is also in front of $g(L)$. Therefore there is a component of $S^2_{\infty} - \Lambda_{g(L)}$ above $g(L)$. Translating by $g^{-1}$ we conclude that there is a component of $S^2_{\infty} - \Lambda_L$ above $L$.

If $\Phi$ were $R$-covered, then $\Lambda_F = S^2_{\infty}$ [Fe2], contrary to assumption. Hence $\Phi$ is not $R$-covered and by transitivity, it follows that $\Phi$ has branching in the positive and negative directions [Fe5]. Let then $E, E' \in \bar{F}^s$ so that they
are not separated on their negative sides. By the above argument $S^2_\infty - \Lambda_E$ has a component $Z_0$ above $E$. Since $E$ is in the back of $E'$ and $E'$ is in the back of $E$ it follows that $Z_0 \cap \Lambda_{E'} = \emptyset$ and all points in $Z_0$ are below $E'$. Hence $S^2_\infty - \Lambda_E$ has a component below $E'$. Using the same argument as above we conclude that for every $L \in \tilde{F}$, there is a component of $S^2_\infty - \Lambda_L$ below $L$. This completes the proof.

7.2 We say that $\tilde{\Phi}$ has the continuous extension property if for any leaf $F \in \tilde{F}^s \cup \tilde{F}^u$, the embedding $\varphi_F : F \to \tilde{M}$, extends continuously to $\varphi_F : F \cup \partial_\infty F \to \tilde{M} \cup S^2_\infty$. This gives a continuous parametrization of the limit sets $\Lambda_F = \varphi_F(\partial_\infty F)$. This also implies that there is a continuous function

$$\eta_- : \tilde{M} \to S^2_\infty, \quad \eta_-(x) = \lim_{t \to -\infty} \tilde{\Phi}_t(x),$$

where the limit is computed in $\tilde{M} \cup S^2_\infty$. Since the function is constant along an orbit $\alpha$ of $\tilde{\Phi}$, this will also denote $\eta_-(\alpha)$. Furthermore for any $G \in \tilde{F}^u$, $\eta_-$ is a constant function in $G$ with value $\varphi_G(G_-)$. Similarly define $\eta_+ : \tilde{M} \to S^2_\infty$. The continuous extension property implies that for any $p \in F \in \tilde{F}^s$, $\Lambda_F = \varphi_F(\partial_\infty F) = \eta_-(\tilde{W}^{ss}(p)) \cup \eta_+(p)$.

In [Fe6] we study the continuous extension property for $\mathbb{R}$-covered flows.

**Theorem 7.3** Let $\Phi$ be an Anosov flow in $M^3$ with negatively curved $\pi_1(M)$. Suppose that $\Phi$ is not $\mathbb{R}$-covered and in addition that $\tilde{\Phi}$ has the continuous extension property. Then for any leaf $C \in \tilde{F}^s \cup \tilde{F}^u$, the limit set $\Lambda_C$ is a Sierpinski curve, that is the complement of a countable, dense union of open disks in the sphere $S^2_\infty$.

**Proof of 7.3:** We may assume that $\tilde{F}^s, \tilde{F}^u$ are transversely orientable. We first prove that $\Lambda_C \neq S^2_\infty$ and then use part of the proof of this fact to show that limit sets are Sierpinski curves. The first part is the same as the proof of theorem 5.5 of [Fe4]. In [Fe4] we used the hypothesis of quasigeodesic behavior of flow lines of $\tilde{\Phi}$ in order to describe the structure of branching of $\tilde{F}^s$ and $\tilde{F}^u$. In this article we obtained a description of branching without any hypothesis and this is what is needed for the proof of theorem 7.3.

Since $\Phi$ is transitive, $\tilde{F}^s$ has branching in the positive and negative directions. Using theorem 4.8 we produce $\Theta$, a union of two adjacent lozenges in $\tilde{M}$ (or $O$) intersecting a common stable leaf so that: (1) the boundary of $\Theta$ has unstable sides in $G, S \in \tilde{F}^u$, and stable sides in $E, F, L \in \tilde{F}^s$ (2)
§7. Continuous extension of Anosov foliations

E, L are not separated on their negative sides, (3) G is in the back of S and
(4) $E \cap G \neq \emptyset$, $L \cap S \neq \emptyset$, see fig. 16. By G we mean the half leaf in the
boundary of $\Theta$. Then $\pi(G)$ is dense in $M\lbrack Fe2\rbrack$.

limits

Figure 16: Sequence of lozenges.

Let $C \in \tilde{\mathcal{F}}^s$ be a leaf intersecting both G and S, hence C intersects $\Theta$. Choose a covering translation $g_1$ so that

$$g_1(G) \cap F \neq \emptyset, \quad g_1(G) \cap L \neq \emptyset.$$  

Since $g_1(F)$ makes a perfect fit with $g_1(G)$, then $g_1(F)$ is in the back of F. Since $g_1(L)$ makes a perfect fit with $g_1(E)$ then both are in the front of L.

Finally $g_1(S)$ is in the front of $g_1(G)$, in the back of S and intersects both L and F. Inductively choose covering translations $g_i$ so that $g_i(G)$ is in the back of S,

$$g_i(G) \cap F \neq \emptyset, \quad g_i(G) \cap L \neq \emptyset, \quad g_i(G) \to S \text{ as } i \to \infty,$$  

and $g_i(G)$ is in the front of $g_{i-1}(S)$, see fig. 16. Let $G_i = g_i(G)$ and similarly define $F_i, L_i, S_i$ and $E_i$.

Let $C_i = C \cap g_i(\Theta)$. For any flow line $\gamma \in F_i, \tilde{W}^u(\gamma)$ intersects $C_i$ and vice versa. Hence $\eta_-(C_i) = \eta_-(F_i)$. Let $q \in C \cap S$. By continuity of $\eta_-$, there is a neighborhood $Y$ of $q$ in $\tilde{M}$ so that $\eta_-(Y)$ is contained in a small neighborhood $Y'$ of $\eta_-(q)$ in $S^2_{\infty}$. As $C_i \cap \tilde{W}^{ss}(q) \to q$, then $\eta_-(C_i) \subset Y'$ for
i big enough. Therefore \( \eta_-(F_i) \subset Y' \) and as a result \( \Lambda_{F_i} \) is contained in the closure of \( Y' \) and is not \( S^2_\infty \).

We can now apply the previous proposition to deduce that for any \( L' \in \bar{F}^s \), there are components of \( S^2_\infty - \Lambda_{L'} \) above \( L' \) and components below \( L' \).

For each \( i \) let \( Z_i \) be a component of \( S^2_\infty - \Lambda_{F_i} \) below \( F_i \). Since \( C \) is in front of \( F_i \), \( Z_i \cap \Lambda_C = \emptyset \). Hence \( Z_i \) is contained in a component \( Z_i^* \) of \( S^2_\infty - \Lambda_C \) which is below \( C \). The argument above used to prove that \( \Lambda_C \neq S^2_\infty \) shows that \( \Lambda_{F_i} \subset \Lambda_C \), hence the component \( Z_i^* \) of \( S^2_\infty - \Lambda_C \) is equal to \( Z_i \).

For each \( i \), \( Z_i \) is below \( F_i \). In addition for each \( i \neq j \), \( F_i \) is in the front of \( F_j \) and \( F_j \) is in the front \( F_i \). This implies that \( Z_i \cap Z_j = \emptyset \). Hence \( \{Z_i\}, i \in \mathbb{N} \) is an infinite family of distinct components of \( S^2_\infty - \Lambda_C \) below \( C \).

Using branching of \( \bar{F}^s \) in the negative direction, one constructs countably many components of \( S^2_\infty - \Lambda_C \) above \( C \).

Since \( \Phi \) is transitive, then for any \( C' \in \bar{F}^s \) there is a covering translation \( f \) so that \( f(C') \cap \Theta \neq \emptyset \). Since \( S^2_\infty - \Lambda_{f(C')} \) has infinitely many components above and below \( f(C') \), translation by \( f^{-1} \) yields the same result for \( C' \).

Notice that these arguments also imply that for every leaf \( F' \in \bar{F}^s \) either \( F' \) is in the back of \( C \), hence \( \Lambda_{F'} \) misses at least all components of \( S^2_\infty - \Lambda_C \) above \( C \) for fixed \( C \); or \( F' \) is in front of \( C \) and \( \Lambda_{F'} \) misses all components of \( S^2_\infty - \Lambda_C \) below \( C \) for a fixed \( C \). In particular this implies that there is \( \epsilon > 0 \) so that every \( \Lambda_{F'} \) misses at least some disk of radius \( \epsilon \) in \( S^2_\infty \).

Suppose now that for some \( R \) in \( \bar{F}^s \), \( \Lambda_R \) has no empty interior. Let \( h \) be a covering translation with both fixed point in the interior of \( \Lambda_R \). By applying \( h^n \) for \( n \) big we get \( \Lambda_{h^n(R)} \) is almost all of \( S^2_\infty \) except for an arbitrarily small neighborhood of the attracting fixed point of \( h \). This contradicts the previous paragraph. This finishes the proof of the theorem.

\[ \text{Lemma 7.4} \]

Let \( \Phi \) be an Anosov flow in \( M^3 \), with \( \pi_1(M) \) negatively curved. Suppose that \( \tilde{\Phi} \) has the continuous extension property. If \( F \in \bar{F}^s \) is periodic and \( x \in F \) is in the periodic orbit of \( F \) then for any \( x_1 \in \bar{W}^{ss}(x) \), \( \eta_-(x_1) = \eta_-(x) \) implies that \( x_1 = x \). Furthermore \( \eta_-(x_1) \neq \eta_+(x) \).

**Proof of 7.4:** Let \( h \) be the generator of \( \mathcal{T}(F) \) associated to the closed orbit \( \pi(\Phi \gamma R(x)) \) traversed in the positive flow direction. Hence \( h \) acts as an expansion in the set of orbits of \( \Phi \) in \( F \). Suppose that \( \eta_-(x_1) = \eta_-(x) \), but \( x_1 \not\in \gamma = \Phi \gamma R(x) \). Let \( \alpha = \Phi \alpha R(x_1) \). Then
\[ \eta_-(\gamma) = \eta_-(\alpha) \Rightarrow \eta_-(h^n(\alpha)) = h^n(\eta_-(\alpha)) = h^n(\eta_-(\gamma)) = \eta_-(\gamma). \]

But

\[ \lim_{n \to +\infty} \eta_-(h^n(\alpha)) = \lim_{n \to +\infty} \varphi(h^n(\alpha)) = \varphi(F_+) = \eta_+(\gamma), \]

because the intrinsic negative limit points of \( h^n(\alpha) \) converge in \( \partial_{\infty} F \) to the positive limit point associated to \( \eta \) periodic orbit. Similarly and only if it intersects \( \sigma \) \( L \) we can assume this is a Jordan arc. Since periodic orbits are dense in \( \Lambda \), we can choose this to be a Jordan arc strictly bigger than \( \eta \). This proves the lemma.

7.4

We now prove a local property of the limit sets. Given \( L \in \tilde{\mathcal{F}}^s \), \( \Lambda_L \) is the image of \( \partial_{\infty} L \simeq S^1 \) under a continuous map, hence \( \Lambda_L \) is locally connected.

**Theorem 7.5** Let \( \Phi \) be an Anosov flow in \( M^3 \) with negatively curved fundamental group. Assume that \( \Phi \) has the continuous extension property. Given any \( L \in \mathcal{F}^s \) or \( \mathcal{F}^u \) and any \( p \in \Lambda_L \), then for each neighborhood \( U \) of \( p \in S^2_{\infty} \), \( \Lambda_L \cap U \) is neither a Jordan arc nor a Jordan curve.

**Proof of 7.5:** Since \( \Lambda_L \) is locally connected, we can choose \( U \) so that \( U \cap \Lambda_L \) is connected.

Assume that \( U \cap \Lambda_L \) is a Jordan arc or Jordan curve and let \( z \) be a relative interior point. Suppose that \( z \neq \eta_+(L) \). Then \( z = \eta_-(c') \) for some \( c' \in L \) and there is \( e' > 0 \) small enough so that the segment \( \sigma^s_{e'}(c') \subset \tilde{\mathcal{F}}^{ss}(c') \) (as defined in section 5), satisfies \( \eta_-(\sigma^s_{e'}(c')) \subset U \cap \Lambda_L \). As this is connected we can assume this is a Jordan arc. Since periodic orbits are dense in \( M \), choose \( e \) in a periodic orbit of \( \Phi \) near \( c' \) and let \( e > 0 \) so that the segment \( \sigma^s_{e}(c) \subset \tilde{\mathcal{F}}^{ss}(c) \) satisfies the following property: \( G \in \mathcal{F}^u \) intersects \( \sigma^s_{e}(c') \) if and only if it intersects \( \sigma^s_{e}(c) \). Then \( K = \eta_-(\sigma^s_{e}(c')) = \eta_-(\sigma^s_{e}(c)) \) is a Jordan arc. We may also assume that \( \eta_+(c) \notin K \). Let \( C = \tilde{\mathcal{F}}^{ss}(c) \).

Let \( g \) be the covering translation associated to the closed orbit \( \pi(\tilde{\Phi}(c)) \) of \( \Phi \) and assume that \( \eta_-(c) \) is the repelling fixed point of \( g \). Therefore \( g \) does not fix the endpoints of \( K \) and \( g(K) \) is a Jordan arc strictly bigger than \( K \) in both directions. Notice that \( g(K) \subset \Lambda_C \).

Then \( g^i(K) = \eta_-(g^i(\sigma^s_{e}(c))) \) is also a Jordan arc \( \forall i \in \mathbb{N} \). Express \( g^i(K) \) as the image of an embedding \( \tau_i : [-i, i] \to S^2_{\infty} \), so that if \( x \in [-i, i] \) and
\section*{8. Non $\mathbb{R}$-covered Anosov flows in hyperbolic 3-manifolds}

There is a large class of non $\mathbb{R}$-covered Anosov flows in hyperbolic 3-manifolds, including all Anosov flows in non orientable hyperbolic 3-manifolds.

\textbf{Proof of 8.1:} Theorem C of [Ba2] states that if $\Phi$ is an $\mathbb{R}$-covered Anosov flow in $M^3$, then either $\Phi$ is topologically conjugate to a suspension Anosov flow or the underlying manifold is orientable (notice that Barbot uses the term “product” instead of $\mathbb{R}$-covered). Since hyperbolic manifolds can never be the underlying manifolds of suspension Anosov flows, it suffices to produce Anosov flows in non orientable hyperbolic 3-manifolds.

Consider therefore the suspension of an orientation reversing Anosov diffeomorphism of the torus $T^2$. Let $M$ be the underlying manifold of the
suspension and let $\alpha$ be an orientation preserving closed orbit of the flow. As described by Goodman [Go] and Fried [Fr], one can do Dehn surgery along this orbit. Then $(n,1)$ Dehn surgery on $\alpha$ yields an Anosov flow in the surgered manifold $M_{(n,1)}$.

Notice now that $(M - \alpha)$ is irreducible, atoroidal and homeomorphic to the interior of a compact 3-manifold with boundary. By Thurston's hyperbolization theorem [Th2, Mor] it follows that $(M - \alpha)$ admits a complete hyperbolic structure of finite volume. By the hyperbolic Dehn surgery theorem [Th1], most Dehn fillings on $(M - \alpha)$ yield closed, hyperbolic manifolds. Since $M$ was non orientable, all of these manifolds are non orientable. Whenever the Dehn surgery coefficient is of the form $(n,1)$, the surgered manifold admits an Anosov flow. This produces infinitely many Anosov flows in non orientable hyperbolic 3-manifolds and finishes the proof.

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