BORROMEO SurgEY EQUIVALENCE OF SPIN 3-MANIFOLDS WITH BOUNDARY

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ABSTRACT. Matveev introduced Borromean surgery on 3-manifolds, and proved that the equivalence relation on closed, oriented 3-manifolds generated by Borromean surgeries is characterized by the first homology group and the torsion linking pairing. Massuyeau generalized this result to closed, spin 3-manifolds, and the second author to compact, oriented 3-manifolds with boundary.

In this paper we give a partial generalization of these results to compact, spin 3-manifolds with boundary.

1. INTRODUCTION

Matveev [5] introduced an equivalence relation on 3-manifolds generated by Borromean surgeries. This surgery transformation removes a genus 3 handlebody from a 3-manifold and glues it back in a nontrivial, but homologically trivial way. Thus, Borromean surgeries preserve the homology groups of 3-manifolds, and moreover the torsion linking pairings. Matveev gave the following characterization of this equivalence relation.

Theorem 1.1 (Matveev [5]). Two closed, oriented 3-manifolds $M$ and $M'$ are related by a sequence of Borromean surgeries if and only if there is an isomorphism $f: H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ inducing isomorphism on the torsion linking pairings.

Massuyeau [4] showed that Borromean surgery induces a natural correspondence on spin structures, and thus can be regarded as a surgery move on spin 3-manifolds. He generalized Theorem 1.1 as follows.

Theorem 1.2 (Massuyeau [4]). Two closed spin 3-manifolds $M$ and $M'$ are related by a sequence of Borromean surgeries if and only if there is an isomorphism $f: H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ inducing isomorphism on the torsion linking pairings, and the Rochlin invariants of $M$ and $M'$ are congruent modulo 8.

In a paper in preparation [3], the second author generalizes Matveev’s theorem to compact 3-manifolds with boundary (see Theorem 2.2 below).

In the present paper, we attempt to generalize the above results to compact spin 3-manifolds with boundary.

After defining the necessary ingredients in Sections 2 and 3, our main result is stated in Theorem 3.6.

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Figure 1. A $Y$-clasper.

Figure 2. How to replace a $Y$-clasper with a 6-component framed link. Here the framings of the three inner components are zero and the framings of the three outer components are determined by the annuli in the $Y$-clasper.

2. $Y$-surgery on 3-manifolds

Unless otherwise specified, we will make the following assumptions in the rest of the paper. All manifolds are compact and oriented. Moreover, all 3-manifolds are connected. All homeomorphisms are orientation-preserving. The (co)homology groups with coefficient group unspecified are assumed to be with coefficients in $\mathbb{Z}$.

2.1. $Y$-surgery and $Y$-equivalence. Borromean surgery is equivalent to $Y$-surgery used in the theory of finite type 3-manifold invariants in the sense of Goussarov and the second author [1, 2].

A $Y$-clasper in a 3-manifold $M$ is a connected surface (of genus 0, with 4 boundary components) embedded in $M$, which is decomposed into one disk, three bands and three annuli as depicted in Figure 1. We associate to a $Y$-clasper $G$ in $M$ a 6-component framed link $L_G$ contained in a regular neighborhood of $G$ in $M$ as depicted in Figure 2. Surgery along the $Y$-clasper $G$ is defined to be surgery along the framed link $L_G$. The result $M_{L_G}$ from $M$ of surgery along $L_G$ is called the result of surgery along the $Y$-clasper $G$ and is denoted by $M_G$.

By $Y$-surgery we mean surgery along a $Y$-clasper. Thus, we say that a 3-manifold $M'$ is obtained from another 3-manifold $M$ by a $Y$-surgery if there is a $Y$-clasper $G$ in $M$ such that the result of surgery, $M_G$, is homeomorphic to $M'$. It is well-known that this relation is symmetric, i.e., if $M'$ is obtained from $M$ by a $Y$-surgery then, conversely, $M$ can be obtained from $M'$ by a $Y$-surgery.

The $Y$-equivalence is the equivalence relation on 3-manifolds generated by $Y$-surgeries.
2.2. \(\Sigma\)-bordered 3-manifolds. Throughout the paper, we fix a closed surface \(\Sigma\), which may have arbitrary finite number of components. In this paper, we consider 3-manifolds whose boundaries are parameterized by \(\Sigma\).

A \(\Sigma\)-bordered 3-manifold is a pair \((M, \phi)\) of a compact, connected 3-manifold \(M\) and a homeomorphism \(\phi: \Sigma \cong \partial M\).

Two \(\Sigma\)-bordered 3-manifolds \((M, \phi)\) and \((M', \phi')\) are said to be homeomorphic if there is a homeomorphism \(\Phi: M \cong M'\) such that \((\Phi|\partial M) \circ \phi = \phi'\).

2.3. \(Y\)-equivalence for \(\Sigma\)-bordered 3-manifolds. The notions of \(Y\)-surgery and \(Y\)-equivalence extend to \(\Sigma\)-bordered 3-manifolds in a natural way.

For a \(\Sigma\)-bordered 3-manifold \((M, \phi)\) (or a homology isomorphism \(\phi\)) and a \(\Sigma\)-clasper \(G\) in \(M\), the result of surgery \(M_G\) has an obvious boundary parameterization \(\phi_G: \Sigma \cong \partial M_G\) induced by \(\phi\). Thus surgery along a \(\Sigma\)-clasper \(G\) in a \(\Sigma\)-bordered 3-manifold \((M, \phi)\) yields a \(\Sigma\)-bordered 3-manifold \((M, \phi_G) := (M_G, \phi_G)\). Two \(\Sigma\)-bordered 3-manifolds \((M, \phi)\) and \((M', \phi')\) are said to be related by a \(\Sigma\)-clasper \(G\) in \(M\) such that \((M, \phi_G)\) is homeomorphic to \((M', \phi')\). The \(\Sigma\)-equivalence on \(\Sigma\)-bordered 3-manifolds is generated by \(\Sigma\)-surgeries.

The following well known characterization of the \(\Sigma\)-equivalence is useful.

**Lemma 2.1.** Two \(\Sigma\)-bordered 3-manifolds \((M, \phi)\) and \((M', \phi')\) are \(\Sigma\)-equivalent if and only if there are finitely many, mutually disjoint \(\Sigma\)-claspers \(G_1, \ldots, G_n\) \((n \geq 0)\) in \(M\) such that the result of surgery, \((M, \phi)G_1, \ldots, G_n\) is homeomorphic to \((M', \phi')\).

2.4. Homology isomorphisms between compact 3-manifolds. Let \((M, \phi)\) and \((M', \phi')\) be \(\Sigma\)-bordered 3-manifolds. Set

\[\delta := \phi' \circ \phi^{-1}: \partial M \cong \partial M'.\]

A homology isomorphism\(^1\) from \((M, \phi)\) to \((M', \phi')\) (or a homology isomorphism from \(M\) to \(M'\) along \(\delta\)) is an isomorphism \(f = (f_i, \mathcal{I}_i)\) of the homology exact sequences of pairs \((M, \partial M)\) and \((M', \partial M')\)

\[\ldots \rightarrow H_i(\partial M) \xrightarrow{\delta} H_i(M) \xrightarrow{f_i} H_i(M, \partial M) \xrightarrow{\mathcal{I}_i} H_{i-1}(\partial M) \rightarrow \ldots\]

satisfying the following properties:

(i) \(f_0([pt]) = [pt]\);

(ii) \(f_i\) and \(\mathcal{I}_i\) are compatible with the intersection forms, i.e., for \(i = 0, 1, 2, 3\), the square commutes:

\[
\begin{array}{ccc}
H_i(M) \times H_{3-i}(M, \partial M) & \xrightarrow{\langle, \rangle_M} & \mathbb{Z} \\
\downarrow \quad f_i \quad \downarrow \mathcal{I}_i \\
H_i(M') \times H_{3-i}(M', \partial M') & \xrightarrow{\langle, \rangle_{M'}} & \mathbb{Z}
\end{array}
\]

Here \(\langle, \rangle_M\) and \(\langle, \rangle_{M'}\) denote the intersection forms.

(iii) \(f_1\) and \(\mathcal{I}_1\) are compatible with the torsion linking pairings, i.e., the square commutes:

\[
\begin{array}{ccc}
\text{Tors } H_1(M) \times \text{Tors } H_1(M, \partial M) & \xrightarrow{\text{Tors } f_1 \times \text{Tors } \mathcal{I}_1} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \quad \text{Tors } \mathcal{I}_1 \quad \downarrow \text{Tors } f_1 \\
\text{Tors } H_1(M') \times \text{Tors } H_1(M', \partial M') & \xrightarrow{\text{Tors } \langle, \rangle_{M'}} & \mathbb{Q}/\mathbb{Z}.
\end{array}
\]

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\(^1\)In [4], this is called "full enhanced homology isomorphism". In this paper, we call it "homology isomorphism" for simplicity.
Here Tors denotes torsion part, and $\tau_M$ denotes the torsion linking pairing of $M$.

The classification of compact 3-manifolds up to $Y$-equivalence is given by the following result.

**Theorem 2.2** *(3)*. Let $\Sigma$ be a closed surface, and let $(M, \phi)$ and $(M', \phi')$ be two $\Sigma$-bordered 3-manifolds. Then the following conditions are equivalent.

1. $(M, \phi)$ and $(M', \phi')$ are $Y$-equivalent.
2. There is a homology isomorphism from $(M, \phi)$ to $(M', \phi')$.

For closed 3-manifolds, Theorem 2.2 is equivalent to Matveev’s theorem (Theorem 1.1).

**3. Y-surgery on spin 3-manifolds**

**3.1. Spin structures.** For an oriented manifold $M$ with vanishing second Stiefel-Whitney class, let $\text{Spin}(M)$ denote the set of spin structures on $M$.

It is well known that $\text{Spin}(M)$ is affine over $H^1(M; \mathbb{Z}_2)$, i.e., acted by $H^1(M; \mathbb{Z}_2)$ freely and transitively

$$\text{Spin}(M) \times H^1(M; \mathbb{Z}_2) \to \text{Spin}(M), \quad (s, c) \mapsto s + c.$$ 

An embedding $f : M' \hookrightarrow M$ of a manifold $M'$ into $M$ induces a map

$$i^* : \text{Spin}(M) \to \text{Spin}(M').$$

If $i$ is an inclusion map, $i^*(s), s \in \text{Spin}(M)$, is denoted also by $s|_{M'}$.

**3.2. Y-surgery and spin structures.** Let $G$ be a $Y$-clasper in a 3-manifold $M$. Let $N(G)$ be a regular neighborhood of $G$ in $M$. Note that the result of surgery, $M_G$, can be identified with the manifold

$$(M \setminus \text{int} N(G)) \cup_{\partial N(G)} N(G)_G$$

obtained by gluing $M \setminus \text{int} N(G)$ with $N(G)_G$ along $\partial N(G)$.

As is proved by Massuyeau [4], for a spin structure $s \in \text{Spin}(M)$, there is a unique spin structure $s_G$ on $M_G$ such that

$$s_G|_{M \setminus \text{int} N(G)} = s|_{M \setminus \text{int} N(G)}.$$ 

This gives a bijection

$$\text{Spin}(M) \xrightarrow{\sim} \text{Spin}(M_G), \quad s \mapsto s_G.$$ 

The spin 3-manifold $(M_G, s_G)$ is called the result of surgery on the spin 3-manifold $(M, s)$ along $G$.

As in Section 2.1 the $Y$-equivalence on spin 3-manifolds is the equivalence relation generated by $Y$-surgery.

**3.3. Twisting a spin structure along an orientable surface.** Let $(M, s)$ be a spin 3-manifold possibly with boundary, and let $T$ be an orientable surface properly embedded in $M$. Then we can twist the spin structure $s$ along $T$. More precisely, we can define a new spin structure $s \ast T = s + [T] \in \text{Spin}(M)$, where $[T] \in H^1(M; \mathbb{Z}_2)$ is the Poincaré dual of $[T] \in H_2(M, \partial M; \mathbb{Z}_2)$. (One can consider similar operation when $T$ is non-orientable, but we do not need it in this paper.)

Note that twisting along a closed surface preserves the restriction of the spin structure to the boundary.
**Proposition 3.1.** If $T$ is a closed, orientable surface in a spin 3-manifold $(M, s)$, then $(M, s * T)$ is $Y$-equivalent to $(M, s)$.

**Proof.** We may assume that $T$ is connected, since the general case follows from this special case.

Take a bicollar neighborhood $T \times [-1, 2] \subset M$. Set $T_2 = T \times \{2\} \subset M$. Let $c$ be a simple closed curve in $T$ bounding a disk in $T$. Let $A$ denote a bicollar neighborhood of $c$ in $T$. Let $D$ and $T'$ be the two components of $T \setminus \text{int} A$, where $D$ is a disk. Set

$$V_0 = A \times [-1, 1], \quad V_1 = (A \cup D) \times [-1, 1], \quad V_2 = (A \cup T') \times [-1, 1],$$

$$M_i = M \setminus \text{int} V_i, \quad i = 0, 1, 2.$$  

Note that $M_1, M_2 \subset M_0$. For $i = 0, 1, 2$, let $s_i = s|_{M_i} \in \text{Spin}(M_i)$.

Let $K = (c, +1)$ denote the framed knot in $M$ whose underlying knot is $c$ and the framing is $+1$. Let $M_K$ denote the result of surgery along $K$, which may be regarded as the manifold $M_0 \cup_0 (V_0)_K$ obtained from $M_0$ and the result of surgery $(V_0)_K$ by glueing along their boundaries in the natural way. We may regard $M_0, M_1$ and $M_2$ as submanifolds of $M_K$.

Note that $V_1$ and $(V_1)_K$ are 3-balls. Hence there is a unique spin structure $s_K \in \text{Spin}(M_K)$ such that $(s_K)|_{M_1} = s_1$. We have the spin homeomorphism $(M, s) \cong (M_K, s_K)$.

We have

$$s_K|_{M_0} = s_0 * D = s_0 * T_2.$$ 

Hence we have

$$s_K|_{M_2} = s_2 * T_2.$$ 

It suffices to prove that $(M_K, s_K)$ is $Y$-equivalent to $(M, s * T_2) = (M, s * T)$. Since the framed knot $K$ is null-homologous in $V_2$ and $+1$-framed, $(V_2)_K$ is $Y$-equivalent to $V_2$ in a way respecting the boundary [5]. This $Y$-equivalence extends to $Y$-equivalence of $M_K$ and $M$. This $Y$-equivalence implies the desired $Y$-equivalence of $(M_K, s_K)$ and $(M, s * T_2)$ since we have

$$s_K|_{M_2} = s_2 * T_2 = (s * T_2)|_{M_2},$$ 

and since the maps

$$\text{Spin}(M_K) \to \text{Spin}(M_2), \quad \text{Spin}(M) \to \text{Spin}(M_2)$$

induced by inclusions are injective.

\[\square\]

3.4. $(\Sigma, s_\Sigma)$-**bordered spin 3-manifolds.** We fix a spin structure $s_\Sigma \in \text{Spin}(\Sigma)$. In the following we consider $Y$-equivalence of spin 3-manifolds with boundary parameterized by the spin surface $(\Sigma, s_\Sigma)$.

A $(\Sigma, s_\Sigma)$-**bordered spin 3-manifold** is a triple $(M, \phi, s)$ consisting of a $(\Sigma)$-bordered 3-manifold $(M, \phi)$ and a spin structure $s \in \text{Spin}(M)$ such that $\phi^*(s) = s_\Sigma$.

Clearly, surgery along a $Y$-clasper in $M$ preserves the spin structure on the boundary of $M$. Hence a $Y$-surgery on a $(\Sigma, s_\Sigma)$-bordered spin 3-manifold yields another $(\Sigma, s_\Sigma)$-bordered spin 3-manifold.

3.5. **Gluing of $(\Sigma, s_\Sigma)$-bordered spin 3-manifolds.** Let $(M, \phi, s)$ and $(M', \phi', s')$ be two $(\Sigma, s_\Sigma)$-bordered spin 3-manifolds. Let $M'' = (-M) \cup_{\phi, \phi'} M'$ be the closed 3-manifold obtained from $-M$ (the orientation reversal of $M$) and $M'$ by gluing their boundaries along $\phi' \circ \phi^{-1}$.

By a gluing of $s$ and $s'$, we mean a spin structure $s'' \in \text{Spin}(M'')$ satisfying

$$s''|_{-M} = s, \quad s''|_{M'} = s'.$$
If $\Sigma$ is empty or connected, then $s''$ is uniquely determined by $s$ and $s'$. Otherwise, $s''$ is not unique.

The spin manifold $(M'', s'')$ is called a gluing of $(M, \phi, s)$ and $(M', \phi', s')$.

**Proposition 3.2.** All the gluings of two $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds $(M, \phi, s)$ and $(M', \phi', s')$ are mutually Y-equivalent.

**Proof.** If $\Sigma$ has at most one boundary component, then there is nothing to prove since there is only one gluing of $(M, \phi, s)$ and $(M', \phi', s')$.

Suppose $\Sigma$ has components $\Sigma_1, \ldots, \Sigma_n$ with $n \geq 2$. For $i = 2, \ldots, n$, choose a framed knot $K_i$ in $M'' = (-M) \cup_{\phi, \phi'} M'$ which transversely intersects each of $\Sigma_1$ and $\Sigma_i$ by exactly one point and is disjoint from the other components of $\Sigma$. There are $2^{n-1}$ gluings $s''_{e_2, \ldots, e_n} \in \text{Spin}(M'')$ of $s$ and $s'$ for $e_2, \ldots, e_n \in \{0, 1\}$, where for $i = 2, \ldots, n$ the framed knot $K_i$ is even framed with respect to $s''_{e_2, \ldots, e_n}$ if $e_i = 0$, and odd framed otherwise. Moreover, we have

$$ s''_{e_2, \ldots, e_n} = s''_{0, \ldots, 0} \bigcup_{2 \leq i \leq n, e_i = 1} \Sigma_i. $$

Hence, by Proposition 3.1, $(M'', s'')$ and $(M'', s''_{0, \ldots, 0})$ are Y-equivalent. \(\square\)

3.6. **Rochlin invariant mod 8 of pairs of $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds.** Let $(M, \phi, s)$ and $(M', \phi', s')$ be two $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds. Set

$$ R_S((M, \phi, s), (M', \phi', s')) := (R(M'', s'')) \mod 8 \in \mathbb{Z}_8, $$

where $M'' = (-M) \cup_{\phi, \phi'} M'$ as before and $s'' \in \text{Spin}(M'')$ is any gluing of $s$ and $s'$. Proposition 3.2 and Theorem 1.2 imply that (1) is well defined.

**Lemma 3.3.** The invariant $R_S((M, \phi, s), (M', \phi', s'))$ depends only on the Y-equivalence classes of $(M, \phi, s)$ and $(M', \phi', s')$.

**Proof.** Suppose that $(M_1, \phi_1, s_1)$ is Y-equivalent to $(M_2, \phi_2, s_2)$ and that $(M'_i, \phi'_i, s'_i)$ is Y-equivalent to $(M''_i, \phi''_i, s''_i)$ for $i = 1, 2$. Then $(M''_1, s''_1)$ and $(M''_2, s''_2)$ are Y-equivalent. Hence we have

$$ R_S((M_1, \phi_1, s_1), (M'_1, \phi'_1, s'_1)) = (R(M''_1, s''_1)) \mod 8 = (R(M''_2, s''_2)) \mod 8 = R_S((M_2, \phi_2, s_2), (M'_2, \phi'_2, s'_2)). $$

\(\square\)

3.7. **Main results.** Now we state the main result of the present paper, which gives a characterization of Y-equivalence of $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds in terms of homology isomorphism and the Rochlin invariant mod 8.

**Conjecture 3.4.** Let $(M, \phi, s)$ and $(M', \phi', s')$ be two $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds. Then the following conditions are equivalent.

1. $(M, \phi, s)$ and $(M', \phi', s')$ are Y-equivalent.
2. There is a homology isomorphism from $(M, \phi)$ to $(M', \phi')$, and we have

$$ R_S((M, \phi, s), (M', \phi', s')) = 0 \mod 8. $$

It follows from Theorem 2.2 that Conjecture 3.4 is equivalent to the following.

**Conjecture 3.5.** Let $(M, \phi, s)$ and $(M', \phi', s')$ be two $(\Sigma, s_{\Sigma})$-bordered spin 3-manifolds. Then the following conditions are equivalent.

1. $(M, \phi, s)$ and $(M', \phi', s')$ are Y-equivalent.
2. $(M, \phi)$ and $(M', \phi')$ are Y-equivalent, and we have

$$ R_S((M, \phi, s), (M', \phi', s')) = 0 \mod 8. $$
The following theorem says that Conjecture 3.5 holds when $H_1(M;\mathbb{Z})$ has no 2-torsion. The proof of this result does not use definitions and results given in [4], which is not available when we are writing the present paper.

**Theorem 3.6.** In the setting of Conjecture 3.5, (1) implies (2). Moreover, if $H_1(M;\mathbb{Z})$ has no 2-torsion, then

(2) $(M, \phi)$ and $(M', \phi')$ are $Y$-equivalent.

implies (1).

4. Proof of Theorem 3.6

4.1. Proof of (1) ⇒ (2). Suppose that (1) of Theorem 3.5 holds. Then, clearly, $(M, \phi)$ and $(M', \phi')$ are $Y$-equivalent. We have to prove $R(M'', s'') \equiv 0 \pmod{8}$, where $(M'', s'')$ is a gluing of $(M, \phi, s)$ and $(M', \phi', s')$.

Since $(M, \phi, s)$ and $(M', \phi', s')$ are $Y$-equivalent, Lemma 3.3 implies that $(M'', s'')$ is $Y$-equivalent to a gluing $(M_0'', s_0'')$ of $(M, s)$ and itself.

Consider the 4-manifold $C$ which is the quotient of the cylinder $M \times [0, 1]$ by the equivalence relation $(x, t) \sim (x, t')$ for $x \in \partial M$ and $t, t' \in [0, 1]$. Then we may naturally identify $M_0''$ with $\partial C$. The 4-manifold $C$ has a spin structure $s_C$ induced by the spin structure $s \times s_{[0,1]} \in \text{Spin}(M \times [0,1])$, where $s_{[0,1]}$ is the unique spin structure of $[0,1]$. We have

$$R(C, s_C) \equiv R(M \times [0,1], s \times s_{[0,1]}) \equiv \sigma(M \times [0,1]) = 0 \pmod{16}.$$ 

Since both $s''_0$ and $s_C$ are gluings of $(M, s)$ and itself, Proposition 3.2 implies that $(M_0'', s_0'')$ and $(C, s_C)$ are $Y$-equivalent. Hence, by Theorem 1.2 we have

$$R(M'', s'') \equiv R(M_0'', s_0'') \equiv R(C, s_C) \equiv 0 \pmod{8}.$$ 

4.2. Proof of (2') ⇒ (1) when $H_1(M;\mathbb{Z})$ has no 2-torsion. We assume that $H_1(M;\mathbb{Z})$ has no 2-torsion.

We divide the proof into three cases:

- $M$ is a $\mathbb{Z}_2$-homology handlebody, i.e., $\partial M$ is connected and $H_1(M, \partial M; \mathbb{Z}_2) = 0$.
- $M$ has non-empty boundary.
- $M$ is closed.

4.2.1. Case where $M$ is a $\mathbb{Z}_2$-homology handlebody. Since $\text{Spin}(M) \rightarrow \text{Spin}(\Sigma)$ and $\text{Spin}(M') \rightarrow \text{Spin}(\Sigma)$ are injective, $Y$-equivalence of $(M, \phi)$ and $(M', \phi')$ implies $Y$-equivalence of $(M, \phi, s)$ and $(M', \phi', s')$.

4.2.2. Case where $\partial M$ is non-empty. We will use the following result.

**Lemma 4.1.** Let $M$ be a 3-manifold with boundary such that $H_1(M;\mathbb{Z})$ has no 2-torsion. Then $M$ can be obtained from a $\mathbb{Z}_2$-homology handlebody $V$ by attaching 2-handles $h_1, \ldots, h_n$ (with $n \geq 0$) along simple closed curves $c_1, \ldots, c_n$ in $\partial V$ in such a way that each $c_i$ is null-homologous (over $\mathbb{Z}$) in $V$.

**Proof.** $M$ can be obtained from a solid torus $V'$ of genus $g$ by attaching some 2-handles along simple closed curves $c'_1, \ldots, c'_k$ in $\partial V'$. After finitely many handle-slides, we can assume the following.

- There is a basis $x_1, \ldots, x_g$ of $H_1(V';\mathbb{Z})$ such that we have

$$[c_i] = \sum_{j=1}^{g} a_{i,j} x_j$$

for $i = 1, \ldots, k$, where the matrix $(a_{i,j})$ is diagonal (but not necessarily square), in the sense that $a_{i,j} = \delta_{i,j} d_i$. 


Clearly, $H_1(M; \mathbb{Z})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{Z}d_i$. By the assumption that $H_1(M; \mathbb{Z})$ has no 2-torsion, each $d_i$ is either odd or 0.

We may assume that, for some $n$, we have $d_1 = \cdots = d_n = 0$ and $d_{n+1}, \ldots, d_k$ are odd. The union $V := V' \cup h_{n+1}' \cup \cdots \cup h_k'$ is a $\mathbb{Z}_2$-homology handlebody. Setting $c_i = c_i', h_i = h_i'$ for $i = 1, \ldots, n$, we have the result.

Let $M$ be obtained as above from a $\mathbb{Z}_2$-homology handlebody $V$ by attaching $2$-handles $h_1, \ldots, h_n$ along disjoint simple closed curves $c_1, \ldots, c_n \subset \partial V$, $n = \text{rank } H_2(M; \mathbb{Z}) \geq 0$, such that $c_i$ is null-homologous in $M$ and such that $\partial M \setminus (c_1 \cup \cdots \cup c_n)$ is connected.

The proof is by induction on $n$. The case $n = 0$ is proved in Section 4.2.1.

Suppose $n > 0$.

Let $N = h_n = D^2 \times [0, 1] \subset M$ be one of the $2$-handles. Set
\[ A = \partial D^2 \times [0, 1] \subset \partial N, \]
\[ B = D^2 \times \{0, 1\} \subset \partial N, \]
\[ M_0 := \overline{M \setminus N} = V \cup h_1 \cup \cdots \cup h_{n-1} \subset M. \]

Thus, $M = M_0 \cup_A N$ is obtained from a $3$-manifold $M_0$ by attaching $N$ along an annulus $A \subset \partial M_0$.

Since $(M, \phi)$ and $(M', \phi')$ are $Y$-equivalent, it follows from Lemma 2.1 that there exists a disjoint family $\mathcal{G}$ of $Y$-claspers in $M$ and a homeomorphism $\Psi: (M_G, \phi_G) \xrightarrow{\cong} (M', \phi')$.

By isotoping $\mathcal{G}$ if necessary, we may assume that $\mathcal{G}$ is contained in the interior of $M_0$.

Set $\Sigma_0 := (\Sigma \setminus \text{int}(\phi^{-1}(B))) \cup A$. Then we have a $\Sigma_0$-bordered $3$-manifold $(M_0, \phi_0)$ where $\phi_0: \Sigma_0 \xrightarrow{\cong} \partial M_0$ is obtained by gluing $\phi|_{\Sigma \setminus \text{int}(\phi^{-1}(B))}$ and $\text{id}_A$.

Set
\[ M_0' := \Psi((M_0)_G) = M' \setminus \Psi(N) \subset M'. \]

We have a $\Sigma_0$-bordered $3$-manifold $(M_0', \phi_0')$, where $\phi_0': \Sigma_0 \xrightarrow{\cong} \partial M_0'$ is obtained by gluing $\phi'|_{\Sigma \setminus \text{int}(\phi^{-1}(B))}$ and $\Psi|_A: A \xrightarrow{\cong} \Psi(A)$.

We have a homeomorphism of $\Sigma_0$-bordered $3$-manifolds
\[ \Psi_0 := \Psi|_{M_0'}: ((M_0)_G, (\phi_0)_G) \xrightarrow{\cong} (M_0', \phi_0'). \]

Set $s_{\Sigma_0} = (\phi_0)^* (s|_{M_0}) \in \text{Spin}(\Sigma_0)$ and $s_{\Sigma_0}' = (\phi_0')^* (s'|_{M_0'}) \in \text{Spin}(\Sigma_0)$. Note that $s_{\Sigma_0}|_{\Sigma \setminus \text{int}(\phi^{-1}(B))} = s_{\Sigma_0}'|_{\Sigma \setminus \text{int}(\phi^{-1}(B))}$. Hence we have either
\begin{equation}
(2) \quad s_{\Sigma_0} = s_{\Sigma_0}'
\end{equation}
or
\begin{equation}
(3) \quad s_{\Sigma_0}' = s_{\Sigma_0} + [a]' \quad \text{and} \quad s_{\Sigma_0} \neq s_{\Sigma_0}',
\end{equation}
where $a = c_n = \partial D^2 \times \{1/2\} \subset A$ is the core of the annulus $A$, and $[a]' \in H^1(\Sigma_0; \mathbb{Z}_2)$ is the Poincaré dual to $[a] \in H_1(\Sigma_0; \mathbb{Z}_2)$.

Claim. We may assume (2).

Proof. If $a$ is separating in $\Sigma_0$, then we have (2).

Suppose that $a$ is non-separating in $\Sigma_0$, and that we have (3). Since $a$ is null-homologous in $\partial V \subset M_0$, it is so also in $M_0'$. Therefore, there is a connected, oriented surface $T_0$ properly embedded in $M_0'$ such that $\partial T_0 = a$. Set $D' = \Psi(D^2 \times \{1/2\})$, and $T' = T_0 \cup D'$, which is a connected, oriented, closed surface in $M'$. 

Set \( \hat{s}' := s' \ast T' \in \text{Spin}(M') \) and \( \hat{s}'_{\Sigma_0} = (\phi_0'\ast (\hat{s}'|_{M_0'}) \in \text{Spin}(\Sigma_0) \). By Proposition 3.1, it follows that \((M', s')\) and \((M', \hat{s}')\) are \(Y\)-equivalent. Thus, we may replace the spin manifold \((M', s')\) with \((M', \hat{s}')\). We have
\[
\hat{s}'_{\Sigma_0} = (\phi_0'\ast ((s' \ast T')|_{M_0'}) = (\phi_0')^*(s') + [a'] = s'_{\Sigma_0} + [a'] = s_{\Sigma_0}.
\]
Hence, we have only to consider the case where (2) holds.

We assume (2). Set \( s_0 = s|M_0 \in \text{Spin}(M_0) \) and \( s_0' = s'|_{M_0'} \in \text{Spin}(M_0') \). Then \((M_0, \phi_0, s_0)\) and \((M_0', \phi_0', s_0')\) are \((\Sigma_0, s_{\Sigma_0})\)-bordered spin 3-manifolds.

We can use the induction hypothesis to deduce that \((M_0, \phi_0, s_0)\) and \((M_0', \phi_0', s_0')\) are \(Y\)-equivalent, and hence so are \((M, \phi, s)\) and \((M', \phi', s')\).

4.2.3. Case where \( M \) is closed. This case is a special case of Theorem 1.2.

Alternatively, this case easily follows from the previous case by considering the punctures \( M \setminus \text{int} B^3 \) and \( M' \setminus \text{int} B^3 \).

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