ON HOMOGENEOUS ULTRAMETRIC SPACES

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Abstract. A metric space \( M \) is homogeneous if every isometry between finite subsets extends to a surjective isometry defined on the whole space. We show that if \( M \) is an ultrametric space, it suffices that isometries defined on singletons extend, i.e., that the group of isometries of \( M \) acts transitively. We derive this fact from a result expressing that the arity of the group of isometries of an ultrametric space is at most 2. An illustration of this result with the notion of spectral homogeneity is given. With this, we show that the Cauchy completion of a homogeneous ultrametric space is homogeneous. We present several constructions of homogeneous ultrametric spaces, particularly the countable homogeneous ultrametric space, universal for rational distances, and its Cauchy completion. From a general embeddability result, we prove that every ultrametric space is embeddable into a homogeneous ultrametric space with the same set of distances values and we also derive three embeddability results due respectively to F. Delon, A. Lemin and V. Lemin, and V. Feinberg. Looking at ultrametric spaces as 2-structures, we observe that the nerve of an ultrametric space is the tree of its robust modules, thus providing insight into possible further structure results.

INTRODUCTION AND DESCRIPTION OF THE RESULTS

A relational structure \( R \) is homogeneous if every isomorphism between finite induced substructures of \( R \) extends to an automorphism of the whole structure \( R \) itself. Introduced by Fraïssé [9] and Jónsson [13], homogeneous structures are now playing a fundamental role in Model Theory. They occur in other parts of mathematics as well, e.g. in the theory of groups and in the theory of metric spaces. A prominent homogeneous metric space is the Urysohn space, a separable complete metric space, in which every finite metric space is isometrically embeddable, and which is homogeneous in the sense that every isometry between finite subsets extends to an isometry of the whole space onto itself. Due to the contributions of Pestov [20] and of Kechris, Pestov and Todorcevic [14], this space has recently attracted some attention in the theory of infinite dimensional topological dynamics in connection with the study of extremely amenable groups. Subsequently, some additional research on homogeneous metric spaces has developed.

The study of indivisible metric spaces [3], [4], [5] led us to consider homogeneous ultrametric spaces. We noticed in [3] that for countable ultrametric spaces, the fact...
that the isometry group acts transitively ensures that the space is homogeneous. In this paper, we show that the countability condition is unnecessary (Corollary 2). We introduce the notion of spectral homogeneity and give a characterization of spaces with that property (Theorem 3). From this characterization it follows that the Cauchy completion of a homogeneous ultrametric space is homogeneous (Theorem 4). We describe several homogeneous ultrametric spaces, including the countable homogeneous ultrametric spaces and their Cauchy completion.

The spectrum $\text{Spec}(M)$ of an ultrametric space $M$ is the set of values taken by the distance. The degree of a closed ball $B$ of $M$ is the number $s_M(B)$ of sons of $B$, that is the number of open balls within $B$ of the same radius as $B$ (see Definition 2). The degree sequence of $M$ is the cardinal function $s_M$ which associates to each $r \in \text{Spec}(M)_* := \text{Spec}(M) \setminus \{0\}$ the supremum $s_M(r)$ of $s_M(B)$ where $B$ has diameter $r$. Given a subset $V$ of the non-negative reals which contains 0, we look at the collection $\mathcal{M}_s(V)$ of ultrametric spaces whose spectrum is $V$ and cardinal function $s$. The class $\mathcal{M}_s(V)$ contains a strictly increasing sequence of length $\omega_1$ of homogeneous ultrametric spaces provided that $V$ contains the non-negative rational (Theorem 2). Furthermore, every ultrametric space $M \in \mathcal{M}_s(V)$ can be isometrically embedded into the space $M_s(\text{Well}(V))$ made of functions $f \in \Pi_{r \in V}, s(r)$ whose support supp$(f)$ is dually well ordered, the distance $d$ being defined by $d(f,g) := \max\{r \in V : f(r) \neq g(r)\}$, and the function $s$ equals to $s_M$ (Theorem 3). Spaces of that form were characterized, by Feinberg [8], as homogeneous and $T$-complete ultrametric spaces (spaces in which every chain of non-empty balls has a non-empty intersection). With our embedding result, we give an other proof of Feinberg’s result (with a slight improvement). We obtain the existence of a space of density at most $\kappa^{s_0}$ (namely $M_s(\text{Well}(\mathbb{R}^*_+))$) in which every ultrametric space $M$ with density at most $\kappa$ is isometrically embeddable (a result due to A. and V. Lemin [16]). We show that the subspace $M_s(\text{Fin}(V))$ of $M_s(\text{Well}(V))$ made of functions with finite support can be isometrically embedded in every ultrametric spaces $M$ such that $s(r) \leq s_M(B)$ for every closed ball $B$ with radius $r$ (a result essentially due to F. Delon [6]). If $V$ is countable and $s(r)$ is countable for each $r$ then it follows from Theorem 2.7 of [3] (as well as [19]) that up to isometry $M_s(\text{Fin}(V))$ is the unique countable homogeneous ultrametric space with degree sequence $s$. Hence, countable homogeneous spaces are characterized by their spectrum and a cardinal function. Since the Cauchy completion of $M_s(\text{Fin}(V))$ is also homogeneous, then provided that $s$ is countable, this Cauchy completion is the unique separable and Cauchy complete ultrametric space with degree sequence $s$. But, it is unlikely that there is a simple characterization of homogeneous ultrametric spaces of arbitrary cardinalities. Yet we conclude this paper by establishing a link with the theory of 2-structures; we observe that the nerve of an ultrametric space is the tree of its robust modules (Proposition 4), thus providing a possible direction for further structural characterizations.

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1This result was announced in [4], see page 1465 line 6 with reference to a preliminary version of this paper. We derive this fact from a result expressing that the arity of the group of isometries of an ultrametric space is at most 2. The same conclusion for Polish ultrametric spaces was presented by Maciej Malicki in his lecture to Toposym, Prague 2011, and published in [17] (see 3) of Corollary 4.2 p. 1672; his proof is quite different.
1. Basic notions on ultrametric spaces

We recall the following notions. Let \( M := (M, d) \) be a metric space, where \( d \) is the distance function on \( M \). Let \( A \) be a subset of \( M \); we denote by \( d_{\restriction A} \) the restriction of \( d \) to \( A \times A \) and by \( M_{\restriction A} \) the metric space \( (A, d_{\restriction A}) \), which we call the metric subspace of \( M \) induced on \( A \); the diameter of \( A \) is \( \delta(A) := \text{Sup}\{d(x, y) : x, y \in A\} \). If \( x \in M \), the distance from \( x \) to \( A \) is \( d(x, A) := \text{Inf}\{d(x, y) : y \in A\} \). Let \( a \in M \); for \( r \in \mathbb{R}^+ \), the open, resp. closed, ball of center \( a \), radius \( r \) is the set \( B(a, r) := \{x \in M : d(a, x) < r\} \), resp. \( \hat{B}(a, r) := \{x \in M : d(a, x) \leq r\} \). In the sequel, the term ball means an open or a closed ball. When needed, we denote by \( \text{Ball}(M) \) the collection of balls of \( M \). A ball is non-trivial if it has more than one element. Two balls, possibly in different metric spaces, have the same kind if they have the same diameter which is attained in both or in none. Let \( M := (M, d) \) and \( M' := (M', d') \) be two metric spaces. A map \( \varphi : M \to M' \) is an isometry from \( M \) into \( M' \), or an embedding, if

\[
(1) \quad d'(\varphi(x), \varphi(y)) = d(x, y) \quad \text{for all} \quad x, y \in M
\]

This is an isometry from \( M \) onto \( M' \) if it is surjective. In particular we denote by \( \text{Iso}(M) \) the group of surjective isometries of \( M \) onto \( M' \). For brevity, and if this causes no confusion, we will say that a map from a subset \( A \) of \( M \) to a subset \( A' \) of \( M' \) is an isometry from \( A \) to \( A' \) if this is an isometry from \( M_{\restriction A} \) onto \( M'_{\restriction A'} \). We say that \( M \) is isometrically embeddable into \( M' \) if there is an isometry from \( M \) into \( M' \). If in addition there is no isometry from \( M' \) into \( M \) then \( M \) is strictly isometrically embeddable into \( M' \).

Four other notions will be of importance:

**Definitions 1.** Let \( a \in M \), the spectrum of \( a \) is the set

\[
\text{Spec}(M, a) := \{d(a, x) : x \in M\}.
\]

The multispectrum of \( M \) is the set

\[
\text{MSpec}(M) := \{\text{Spec}(M, a) : a \in M\}.
\]

The spectrum of \( M \) is the set

\[
\text{Spec}(M) := \bigcup \text{MSpec}(M) = \{d(x, y) : x, y \in M\}.
\]

The nerve of \( M \) is the set

\[
\text{Nerv}(M) := \{\hat{B}(a, r) : a \in M, r \in \text{Spec}(M, a)\}.
\]

A metric space is ultrametric if it satisfies the strong triangle inequality \( d(x, z) \leq \text{Max}\{d(x, y), d(y, z)\} \). Note that a space is ultrametric if and only if \( d(x, y) \geq d(y, z) \geq d(x, z) \) implies \( d(x, y) = d(y, z) \). Alternatively, all triangles are isosceles, the two equal sides being the largest. In an ultrametric space, balls (open or closed, as defined above) are both open and closed with respect to the metric topology. The essential property of ultrametric spaces, which follows trivially from the definition, is that balls are either disjoint or comparable with respect to the subset relation. Observe also that the diameter \( \delta(A) \) of a subset \( A \) of an ultrametric space is equal, for any \( a \in A \), to \( \sup\{d(a, x) : x \in A\} \). It turns out that an ultrametric space can be recovered from the pair made of \( (\text{Nerv}(M); \geq) \) and the diameter function \( \delta \). Ordered by the reverse of the subset relation, the nerve of an ultrametric space is a tree, more specifically a "ramified meet tree" in which every element is below a maximal element; the diameter function is a strictly decreasing map from the tree.
in the non-negative reals, which is zero on the maximal elements of the tree. For the exact statement of this characterization, which we will not need here, see [15] and also [3].

**Definition 2.** Let \( M := (M, d) \) be an ultrametric space, \( B \in \text{Nerv}(M) \) and \( r := \delta(B) \). If \( r > 0 \), a son of \( B \) is any open ball of radius \( r \) which is a subset of \( B \). We denote by \( \text{Sons}(B) \) the set of sons of \( B \) and by \( s_M(B) \) the cardinality of the set \( \text{Sons}(B) \).

For every \( r \in \text{Spec}(M) \setminus \{0\} \) we denote by \( s_M(r) \) the supremum of the cardinality \( s_M(B) \) where \( B \) is any member of \( \text{Nerv}(M) \) with diameter \( r \). In the sequel, we identify \( s_M(B) \) and \( s_M(r) \) with ordinals (more precisely with initial ordinals).

Note that the sons of \( B \) form a partition of \( B \). Also, note that they do not need to belong to \( \text{Nerv}(M) \). These are the immediate successors in the tree of strong modules of \( M \) (see Section 3) hence the terminology we use.

**Examples 1.** Let \( \mathbb{R}^+ \) be the set of non-negative reals. The map \( d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \), defined by \( d(x, y) = 0 \) if \( x = y \) and \( d(x, y) = \max\{x, y\} \) otherwise, is an ultrametric distance. It follows that every subset \( V \) of \( \mathbb{R}^+ \) containing \( 0 \) is the spectrum of some ultrametric space.

- Let \( K := 2^\omega \) be the set of all \( 0-1 \) valued \( \omega \)-sequences. Let \( d : K \times K \to \mathbb{R}^+ \) given by \( d(x, y) := 0 \) if \( x = y \) and \( d(x, y) := \frac{1}{\mu(x, y) + 1} \) otherwise, where \( \mu(x, y) := \min\{n \in \omega : x(n) \neq y(n)\} \). It is easily checked that \( K := (K, d) \) is an ultrametric space in which every non-trivial element of the nerve has two sons, which are again elements of the nerve. Let \( C \) be the subspace of \( K \) containing all \( \omega \)-sequences with finite support, that is \( 1 \)'s appearing only finitely often. Members of the nerves from both spaces having non zero radius identify to the binary tree of height \( \omega \). Clearly, \( K \) is the Cauchy completion of \( C \).

- Let \( \kappa \) be a cardinal number at least equal to 2, identified to the set of ordinal numbers of cardinality strictly less than \( \kappa \) and let \( L_\kappa \) be the set of all functions \( f \) from \( \mathbb{Q}^*_+ \), the positive rationals, to \( \mathbb{R}_+ \), defined by \( f(x) = 0 \) for all \( x \in \mathbb{Q}^*_+ \) larger than some real \( f(f) \). Let \( d : [L_\kappa]^2 \to \mathbb{R}_+ \) given by \( d(f, g) := 0 \) if \( f = g \) and \( d(f, g) := \sup\{r \in \mathbb{Q}^*_+ : f(r) \neq g(r)\} \) otherwise.

It is easily checked that \( L_\kappa := (L_\kappa, d) \) is an ultrametric space in which every non-trivial element of the nerve has continuum many sons and no son of an element of the nerve is an element of the nerve. Furthermore, as shown by A. and V. Lemin [16], every ultrametric space \( M \) of weight at most \( \kappa \) can be embedded isometrically in \( L_\kappa \).

It is easy to see that for each of these last two examples, the group of isometries acts transitively, hence from Corollary 3 below, these ultrametric spaces are homogeneous. In Section 3.1 we will define more general constructions and also give an alternative proof of Lemin’s result.

The following facts for an ultrametric space \( M := (M, d) \) and ball \( B \) of \( M \) can easily be verified, in the order they are stated.

**Fact 1.**

1. If \( x, y \in B \) and \( z \in M \setminus B \) then \( d(x, z) = d(y, z) \).

2. If \( \varphi \) is a function of \( M \) to \( M \) which induces an isometry of \( B \) into \( \varphi(B) \) and also an isometry of \( M \setminus B \) onto \( M \setminus \varphi(B) \), and if further this latter isometry can be isometrically extended to some element \( x \), then \( \varphi \) is isometric.

3. If \( B \in \text{Nerv}(M) \) and \( x, y \in B \) so that the son of \( B \) containing \( x \) is different from the son of \( B \) containing \( y \) then \( d(x, y) = \delta(B) \). If \( x \in M \) and \( r \in \mathbb{R}_+ \) and
Spec(M, x) \setminus \{0\} then δ(\hat{B}(x, r)) = r and Sons(\hat{B}(x, r)) contains at least two different elements and the distance between elements in different sons of \(\hat{B}(x, r)\) is r.

(4) Let ψ be a bijection of M, B, B' ∈ Nerv(M) with δ(B) = δ(B') and f be a bijection of Sons(B) to Sons(B'). If for every A ∈ Sons(B) the function ψ_A is an isometry of A to f(A) and ψ_{1,M-B} extends isometrically to some element x of B, then ψ is an isometry.

(5) Let B and B' be two balls in Nerv(M) with δ(B) = δ(B') and x ∈ B, x' ∈ B'. If there exists an element ϕ ∈ Iso(M) with ϕ(x) = x' then ϕ[B] = B' and ϕ induces a bijection of Sons(B) to Sons(B'), and hence s_M(B) = s_M(B').

2. Isometries and Extensions

Here is our first result:

**Theorem 1.** Let M := (M, d) be an ultrametric space, n be a non negative integer and F, F' be two n-element subsets of M. A map ϕ: F → F' extends to a surjective isometry \(\overline{\phi}\) of M if and only if:

1. ϕ is an isometry from \(M_{1,F}\) onto \(M_{1,F'}\).
2. For each x ∈ F the function \(\phi_{1,x}\) extends to a surjective isometry of M.

**Proof.** Trivially, Properties (1) and (2) follow from the existence of \(\overline{\phi}\). For the converse, we argue by induction on n. For n = 0, the identity map extends the empty one. For n = 1, the hypothesis and the conclusion are the same. Suppose n ≥ 2. Pick a ∈ F and set r := d(a, F \setminus \{a\}) and \(B_0 := B(a, r)\) and \(B := \hat{B}(a, r)\), and pick b ∈ F \setminus \{a\} with d(a, b) = r = δ(B).

By induction assumption, there are two automorphisms of M denoted by \(\phi_{\{a\}}\) and \(\phi_{F \setminus \{a\}}\) extending \(\phi_{1,a}\) and \(\phi_{1,F \setminus \{a\}}\) respectively. Let

- \(a'' := \phi_{F \setminus \{a\}}^{-1} \circ \phi_{\{a\}}(a)\),
- \(B'_0 := \phi_{\{a\}}[B_0]\),
- \(B''_0 := \phi_{F \setminus \{a\}}[B'_0]\).

Because \(\phi_{\{a\}}(a) = \phi_{\{a\}}(a)\) and \(\phi_{F \setminus \{a\}}(b) = \phi(b)\) we get

\[d(a'', b) = d(\phi_{F \setminus \{a\}}^{-1} \circ \phi_{\{a\}}(a), \phi_{F \setminus \{a\}}^{-1} \circ \phi_{\{a\}}(b)) = d(\phi(a), \phi(b)) = d(a, b) = r\]

and hence from \(d(a, b) = r\) and \(d(a'', b) = r\), we have that \(d(a, a'') \leq r\) which in turn implies that \(B_0\) and \(B''_0\) are two sons of B which are equal or disjoint. We define \(\overline{\phi}: M \to M\) via the following conditions:

- \(\phi_{F \setminus \{a\}}\) on \(M \setminus (B_0 \cup B''_0)\),
- \(\phi_{\{a\}}\) on \(B_0\),
- \(\phi_{F \setminus \{a\}}^{-1} \circ \phi_{\{a\}}\) on \(B''_0\) if \(B''_0 \neq B_0\).

If \(B_0 = B''_0\) then \(\overline{\phi}\) induces an isometry of \(M \setminus B_0\) and an isometry of B, hence is an isometry of M onto M according to Fact [1] Item (2) with \(a''\) used for the element x in Item (2).

If \(B_0 \neq B''_0\) let \(B' := \phi_{F \setminus \{a\}}[B]\) and f be the function of Sons(B) to Sons(B') with \(f(A) = \phi_{F \setminus \{a\}}[A]\) for every son A ∈ B \setminus (B_0 ∪ B''_0) of B and \(f(B_0) = B'_0 = \phi_{\{a\}}[B_0]\) = \(\phi_{F \setminus \{a\}}[B'_0]\) and \(f(B''_0) = \phi_{F \setminus \{a\}}[B''_0]\). Hence it follows from Fact [1] Item (4), again with \(a''\) used for the element x, that \(\overline{\phi}\) is an isometry of M onto M.
Corollary 1. Let $M := (M,d)$ be an ultrametric space and $n$ be a non negative integer and $F,F'$ be two $n$-element subsets of $M$. A map $\varphi : F \to F'$ extends to a surjective isometry $\bar{\varphi}$ of $M$ if and only if for each $x,y \in F$ the function $\varphi_{\{x,y\}}$ extends to a surjective isometry of $M$.

Proof. The “only if” part is obvious; for the “if” part, observe that $\varphi$ is an isometry of $M|_F$ onto $M|_{F'}$. \qed

Corollary 1 describes a property of the action on $M$ of $\text{Iso}(M)$; in the terminology of $\text{[1]}$, it expresses that $\text{Iso}(M)$ has arity at most 2.

According to the terminology of Fraïssé $\text{[10]}$, a metric space $M$ is homogeneous if every isometry $f$ whose domain and range are finite subsets of $M$ extends to a surjective isometry of $M$ onto $M$.

The following is a straightforward corollary to Theorem 1.

Corollary 2. An ultrametric space $M$ is homogeneous if and only if $\text{Iso}(M)$ acts transitively on $M$.

We did not find mention of this result in the literature. We proved it in $\text{[3]}$ for countable ultrametric spaces, as a step in our characterization of countable homogeneous ultrametric spaces. It is not true that every isometry $\varphi : B \to A$ of a subset of a homogeneous ultrametric space $A$ extends to an isometry defined on $A$. Indeed, if $B$ is the image of $A$ by some isometry, its inverse will not extend. One may ask whether $\varphi$ extends if it is an isometry of $B$ onto itself (the question was asked to J. Melleray and communicated to us $\text{[18]}$). The answer is negative, even if $A$ is separable.

Example 2. Let $\omega^\omega$ be the set of integer valued $\omega$-sequence and $\omega^{[\omega]}$ be the subset of those with finite support. Let $A$ be one of these two sets equipped with the distance $d$ defined in Example 1. Set $A_{00} := \{x \in A : x(0) = x(1) = 0\}$, $A_{01} := \{x \in A : x(0) = 0, x(1) \geq 1\}$, $A_1 := \{x \in A : x(0) = 1\}$, $A_{2} := \{x \in A : x(0) \geq 2\}$. Let $B := A_{01} \cup A_1 \cup A_{2}$ and $\varphi : B \to B$ which is the identity on $A_{2}$, and is the bijection of order two from $A_{01}$ onto $A_1$ defined by $\varphi((0,i,\ldots)) = (1,i-1,\ldots))$. It is easy to check that $\varphi$ is an isometry from $B$ onto itself (note that the distance between any elements $x \in A_{2}$, $y \in A_{01}$ and $z \in A_1$ is 1. But, if $t$ is an element of $A_{00}$, there is no way to extend $\varphi$ to $t$. Indeed, pick $y \in A_{01}$. If $\varphi$ extends to $t$, then the image $t'$ of $t$ satisfies $d(t',\varphi(y)) = d(t,y) = \frac{1}{2}$. Hence, $t' \in A_1$ and the extension of $\varphi$ is not one to one.

We conclude this section with an open problem.

Problem 1. Corollary 2 does not hold for ordinary metric spaces. There are several metric spaces which are not homogeneous while their automorphisms group is transitive (a simple example is the product $\mathbb{Z} \times \mathbb{Z}$ equipped with the sup-distance $d((x,y),(x',y')) = \sup\{d(x,x'),d(y,y')\}$). The notion of ultrametric space extends to metric with values into a join-semilattice $V$ with a least element 0. We ask for which $V$, ultrametric spaces with values in $V$ whose automorphism group is transitive are homogeneous?.

We may note that the answer is trivial if $V$ has two elements, and positive also if $V$ is the 4-element Boolean algebra.
3. Examples of homogeneous ultrametric spaces

Corollary 2 leaves us with the problem of deciding under which conditions \( \text{Iso}(M) \) acts transitively on an ultrametric space \( M \), that is to characterize the ultrametric spaces \( M := (M, d) \) for which for all elements \( x, x' \in M \) there exists \( \varphi \in \text{Iso}(M) \) with \( \varphi(x) = x' \). In the following subsection we consider some necessary conditions.

3.1. Transitivity conditions.

**Definition 3.** An ultrametric space \( M := (M, d) \) has property \( h \) if it has properties \( h_1 \) and \( h_2 \) below, that is:

\[
\begin{align*}
\text{h}_1 & : \text{Spec}(M, x) = \text{Spec}(M, x') \text{ for all } x, x' \in M. \\
\text{h}_2 & : s_M(B) = s_M(B'), \text{ that is } |\text{Sons}(B)| = |\text{Sons}(B')|, \text{ for all } B, B' \in \text{Nerv}(M) \text{ with } \delta(B) = \delta(B').
\end{align*}
\]

Clearly, \( M \) has property \( h_1 \) if and only if \( \text{Spec}(M, x) = \text{Spec}(M) \) for every \( x \in M \). If \( M \) has property \( h \), we have \( s_M(B) = s_M(r) \) for every \( B \in \text{Nerv}(M) \) with diameter \( r \). In this case, we call the map \( s_M \) the *degree sequence* of \( M \). If \( M \) is homogeneous then, trivially, it satisfies property \( h_1 \). According to Item (5) of Fact 1 it satisfies property \( h_2 \), hence:

**Lemma 1.** Every homogeneous ultrametric space satisfies property \( h \).

Let \( V \) be a subset of \( \mathbb{R}^+ \) containing \( \{0\} \). Let \( V_* := V \setminus \{0\} \). As we will see (in (3) of Proposition 1), every map \( s \) which associate to each \( r \in V_* \) some cardinal number \( s(r) \geq 2 \) is the degree sequence \( s_M \) of some homogeneous ultrametric space \( M \). Looking at the collection \( M_s(V) \) of ultrametric spaces having spectrum \( V \) and degree sequence \( s \), we note that if \( V \) is countable and \( s(r) \) is countable for each \( r \) then it follows from Theorem 2.7 of [3] that up to isometry there is a unique countable ultrametric space \( M_s(\text{Fin}(V)) \) with degree sequence \( s \) and furthermore this ultrametric space is homogeneous. Hence, for countable spaces, property \( h \) is a sufficient condition for transitivity. If one considers uncountable spaces then property \( h \) is not in general a sufficient condition for transitivity. Examples show that topological conditions have to be satisfied as well. Observing that the Cauchy completion of \( M_s(\text{Fin}(V)) \) is also homogeneous, we show that, up to isometry, this is the unique separable and Cauchy complete ultrametric space with degree sequence \( s \). But Cauchy completeness with property \( h \) is not in general a sufficient condition for transitivity. It is unlikely that there be a simple characterization of ultrametric spaces \( M \) for which \( \text{Iso}(M) \) is transitive.

**Examples 3.** (1) In the space \( K \) of Examples 1, let \( 0 \) be the constant 0-sequence \( 1 \) the constant 1-sequence, and \( 0' \) the sequence \((0, 1, 0, 0, 0, 0, \ldots)\). Then \( d(0, 0') = \frac{1}{2} \) and \( d(0, 1) = d(0', 1) = 1 \). Let \( K' \) be obtained from \( K \) by removing the point \( 0' \) from \( K \). Then each member of \( \text{Nerv}(K') \) still has exactly two sons, and \( \text{Spec}(K', x) = \text{Spec}(K', y) \) for all points \( x, y \) of \( K' \). Hence \( K' \) satisfies Property \( h \). But no element \( \varphi \) of \( \text{Iso}(K') \) maps \( 0 \) to \( 1 \) because \( \varphi \) would have to map the ball \( \hat{B}(0, \frac{1}{2}) \) of \( K' \) onto the ball \( \hat{B}(1, \frac{1}{2}) \) of \( K' \) which is impossible because the ball \( \hat{B}(0, \frac{1}{2}) \) is not Cauchy complete whereas the ball \( \hat{B}(1, \frac{1}{2}) \) is Cauchy complete. Note that this example also shows that requiring in addition to Property \( h \) that balls of the same diameter have the
same cardinality still is not a sufficient condition. Of course a similar example can be constructed starting with the ultrametric space $L_\kappa$ of Examples 4.

(2) The construction above will not work in the case of the countable ultrametric space $s$ because no ball of $s$ is Cauchy complete; of course it would also contradict Theorem 2.7 of [8]. But for a more direct argument, using the same elements $0, 0'$ and 1 for $s$ as for $K$ above, we remove $0'$ from $C$ to obtain the ultrametric space $C'$. Then we construct an isometry of the ball $B(0, \frac{1}{2})$ of $C'$ onto the ball $B(1, \frac{1}{2})$ of $C'$ by a simple back and forth argument. The crucial step for this argument is to realize that every partial isometry of the ball $B(0, \frac{1}{2})$ of $C'$ onto the ball $B(1, \frac{1}{2})$ of $C'$ can be extended to include any element of $B(0, \frac{1}{2})$ or of $B(1, \frac{1}{2})$. During this construction the missing limits of Cauchy sequences are simply rearranged.

(3) An example of Cauchy complete space satisfying property $h$ which is not homogeneous is the following. Let $V := V_0 \cup V_1 \cup \{0\}$, where $V_i := \{\frac{1}{n+i} + \frac{r}{2} : n \in \mathbb{N}\}$ for $i \in \{0, 1\}$. Let $M := 2^{(V_0 \cup V_1)}$ be the set of $0 - 1$ maps $f$ defined on $V_0 \cup V_1$ and $d : M \times M \to V$ be given by $d(f, g) := 0$ if $f = g$ and $d(f, g) := \text{Max}\{x \in V_0 \cup V_1 : f(x) \neq g(x)\}$. The space $M := (M, d)$ is Cauchy complete and homogeneous. Its restriction $M|_X$ where $X := \{f \in M : f(x) = 0 \text{ for some } x \in V\}$ is Cauchy complete but not homogeneous.

3.2. Constructions of homogeneous ultrametric spaces. Let $V$ be a subset of $\mathbb{R}^+$ which contains 0 and let $V_\kappa := V \setminus \{0\}$. Let $s$ be a function which associates a cardinal number $s(r) \geq 2$ to each $r \in V_\kappa$. Let $V_\kappa := \Pi_{r \in V_\kappa} s(r)$. Viewing each $s(r)$ as an ordinal number, let 0 be its least element and let also 0 be the element of $V_\kappa$ which takes value 0 in $s(r)$ for each $r \in V_\kappa$. For $f, g \in V_\kappa$, we set $\Delta(f, g) := \{r \in V_\kappa : f(r) \neq g(r)\}$. For $f \in V_\kappa$, the support of $f$ is the set $\text{supp}(f) := \{r \in V_\kappa : f(r) \neq 0\}$. Let $\sigma := (\sigma_r)_{r \in V_\kappa}$, where each $\sigma_r$ is a permutation of $s(r)$. If $f \in V_\kappa$, set $\overline{\sigma}(f) := (\sigma_r(f(r)))_{r \in V_\kappa}$. This defines a permutation $\overline{\sigma}$ of $V_\kappa$. It satisfies:

**Fact 2.** $\Delta(\overline{\sigma}(f), \overline{\sigma}(g)) = \Delta(f, g)$ for every $f, g \in V_\kappa$.

**Fact 3.** $\Delta(f, g) \cup \Delta(f, h) \subseteq \Delta(h, g) \cup \Delta(f, h)$ for every $f, g, h \in V_\kappa$.

Let $\mathcal{J}$ be a collection of subsets of $V_\kappa$. Set $V_\kappa(\mathcal{J}) := \{f \in V_\kappa : \text{supp}(f) \in \mathcal{J}\}$. For $f, g \in V_\kappa$, we set $f \equiv_\mathcal{J} g$ if $\Delta(f, g) \in \mathcal{J}$.

**Lemma 2.** If $\mathcal{J}$ is an initial segment of subsets of $V_\kappa$ (that is closed under downward inclusion), then $\mathcal{J}$ is an ideal of subsets of $V_\kappa$ if and only if $\equiv_\mathcal{J}$ is an equivalence relation on $V_\kappa$.

**Proof.** Suppose that $\mathcal{J}$ is an ideal of subsets of $V_\kappa$. Being non empty, it contains the empty set, hence $f \equiv_\mathcal{J} f$ for every $f$. Suppose that $f \equiv_\mathcal{J} g \equiv_\mathcal{J} h$. According to Fact 3 we have $\Delta(f, h) \subseteq \Delta(f, g) \cup \Delta(h, g)$. Since $\mathcal{J}$ is an ideal, $\Delta(f, h) \cup \Delta(h, g) \in \mathcal{J}$ and hence $\Delta(f, h) \in \mathcal{J}$, proving $f \equiv_\mathcal{J} h$. The symmetry being obvious, it follows that $\equiv_\mathcal{J}$ is an equivalence relation.

Conversely, suppose that this is an equivalence relation. Let $X, Y \in \mathcal{J}$, we show that $X \cup Y \in \mathcal{J}$. The set $X \cup Y$ is the support of its characteristic function $\chi_{X \cup Y}$ which belong to $V_\kappa$. Hence it suffices to show that $0 \equiv_\mathcal{J} \chi_{X \cup Y}$. The characteristic functions $\chi_X, \chi_Y$ of $X \setminus Y$ and $\chi_Y$ of $Y$ belong to $V_\kappa$. Their supports are $X \setminus Y$ and $Y$ which belong to $\mathcal{J}$. Hence $0 \equiv_\mathcal{J} \chi_X \equiv_\mathcal{J} \chi_{X \cup Y}$. Thus $0 \equiv_\mathcal{J} \chi_{X \cup Y}$ as required. \qed
For each subset \( X \) of \( \mathbb{R} \) we set \( \mu(X) := +\infty \) if \( X \) is not bounded above in \( \mathbb{R}^+ \) and \( \mu(X) := \sup_{x \in X} (x) \) otherwise. Let \( f, g \in V_s \). We set \( d(f, g) := \mu(\Delta(f, g)) \).

Let \( \text{Bound}(V) \) be the collections of subsets \( X \) of \( V_s \) which are bounded above in \( \mathbb{R}^+ \). Since \( \text{Bound}(V) \) is an ideal, \( \equiv_{\text{Bound}(V)} \) is an equivalence relation and on each equivalence class, \( d \) induces an ultrametric distance. If we extend the definition of ultrametric to functions taking infinite values then the pair \( (V_s, d) \) is an ultrametric space. The group \( \text{Iso}(M_s) \) of isometries of \( M_s \) is transitive. Indeed, let \( f, g \in V_s \), and \( \sigma := (\sigma_r)_{r \in V_s \setminus \{0\}} \) where \( \sigma_r \) is the transposition of \( s(r) \) which exchanges \( f(r) \) and \( g(r) \). Then \( \sigma(f) = g \) and according to Fact 2 \( \sigma \) is an isometry of \( M_s \).

Trivially, every isometry \( u \) of \( M_s \) preserves the equivalence relation hence the image \( C' \) of an equivalence class \( C \) is an equivalence class and thus \( u \) is an isometry of \( M_s \). Since \( \text{Iso}(M_s) \) is transitive, the spaces induced on two different equivalence classes are isometric, and for each equivalence class \( C \), \( \text{Iso}(M_s \upharpoonright C) \) is transitive. From Corollary 2 \( M_s \upharpoonright C \) is homogeneous. The equivalence class of \( 0 \) is the set \( V_s(\text{Bound}(V)) \) and the space induced on it is \( M_s(\text{Bound}(V)) \).

These facts extend to various ideals of subsets of \( V_s \).

If \( J \) is a collection of subsets of \( V_s \), we set \( M_s(J) := M_s|V_s(J) \). We denote by \( \text{Fin}(V) \) the collection of finite subsets of \( V_s \), and by \( \text{Well}(V) \) the collection of subsets of \( V_s \) which are dually well ordered. If \( \alpha \) is a countable ordinal, we denote by \( \text{Well}_\alpha(V) \) the subset of \( \text{Well}(V) \) made of subsets \( A \) such that \( \alpha \) is not embeddable into \( (A, \geq) \). With this definition, \( \text{Fin}(V) = \text{Well}_0(V) \). If \( \alpha \) is an indecomposable ordinal \( (i.e. \alpha = \omega^\beta \) for some ordinal \( \beta \) ), then \( \text{Well}_\alpha(V) \) is an ideal of subsets of \( V_s \).

The following proposition provides several examples of homogeneous ultrametric spaces.

**Proposition 1.** Let \( J \) be a subset of \( \text{Bound}(V) \). Then:

1. \( M_s(J) \) is an ultrametric space.
2. Let \( J := \{ A \in \text{Bound}(V) : [a, +\infty[ \cap A \in J \text{ for all } a \in \mathbb{R}^+ \setminus \{0\} \} \). If \( J \) is an initial segment of \( \mathcal{P}(V_s) \), then \( M_s(J) \) is the Cauchy completion of \( M_s(J) \).
3. If \( J \) is an ideal of subsets of \( V_s \) then \( M_s(J) \) is a homogeneous ultrametric space and \( \text{Spec}(M_s(J)) = J' := \{ \text{Sup}(W) : W \in J \} \). If \( \text{Fin}(V) \subseteq J \subseteq \text{Well}(V) \) then its degree sequence is \( s \).

**Proof.** Item 1 The space \( M_s(J) \) is the restriction to \( V_s(J) \) of the ultrametric space \( M_s(\text{Bound}(V)) \).

Item 2 (a) \( V_s(J) \) is topologically dense in \( M_s(J) \). Let \( f \in V_s(J) \). Let \( (a_n)_{n\in\mathbb{N}} \) be a strictly decreasing sequence of reals converging to zero. For each \( n \in \mathbb{N} \), select \( f_n \in V_s \) such that \( f_n(x) = f(x) \) for \( x \in A_n := [a_n, +\infty[ \cap \text{supp}(f) \) and \( f_n(x) = 0 \) for \( x \in V \setminus (\text{supp}(f) \cup \{0\}) \). Clearly, \( \text{supp}(f_n) = A_n \in J \), thus \( f_n \in V_s(J) \). Clearly, \( f = \lim_{n \to +\infty} f_n \), hence \( f \) belongs to the topological closure of \( V_s(J) \) in \( M_s(J) \).

(b) The space \( M_s(J) \) is Cauchy complete. Let \( f_n \in \mathbb{N} \) be a Cauchy sequence in \( M_s(J) \). Let \( (a_n)_{n\in\mathbb{N}} \) be a strictly decreasing sequence of reals converging to zero. Since \( (f_n)_{n\in\mathbb{N}} \) is a Cauchy sequence, there is a strictly increasing sequence of integers \( (n_k)_{k\in\mathbb{N}} \) such that the distance \( d \) on \( M_s(J) \) satisfies \( d(f_{n_k}, f_m) < a_k \) for all \( k \) and \( m \) such that \( m \geq n_k \). Let \( f \) be defined by \( f(x) := f_{n_k}(x) \) for \( x \in V \setminus [a_k, a_{k-1}[ \) for all \( k \in \mathbb{N} \) (with the convention that \( a_{-1} := +\infty \)). Then \( \text{supp}(f) \in J \), that is \( [a, +\infty[ \cap \text{supp}(f) \in J \) for all \( a \in \mathbb{R}^+ \setminus \{0\} \). Indeed let \( a \in \mathbb{R}^+ \setminus \{0\} \), and let \( k \in \mathbb{N} \).
such that \( a_k \leq a \). We further claim that \( f \) and \( f_{nk} \) coincide on \([a_k, +\infty[\). For this, let \( m \geq n_k \). For every \( k' \leq k \) we have \( d(f_{nk'}, f_m) = \sup(\Delta(f_{nk'}, f_m)) < a_{k'} \) meaning that \( f_{nk'} \) and \( f_m \) coincide on \([a_{k'}, +\infty[\). Due to its definition, \( f \) coincide with \( f_m \) on \([a_k, +\infty[\), proving our claim. Now, since \( \text{supp}(f_{nk}) \in J, [a, +\infty[ \cap \text{supp}(f_{nk}) \in J \). Hence, our claim ensures that \([a, +\infty[ \cap \text{supp}(f) \in J \), as required.

Item 3 By Corollary 2 in order to prove that \( M_\sigma(J) \) is homogeneous it suffices to prove that the group \( \text{Iso}(M_\sigma(J)) \) acts transitively on \( M_\sigma(J) \). This verification is similar to that previously done in the case of \( J = \text{Bound}(V) \).

Now let \( B \in \text{Nerv}(M_\sigma) \) with \( r = \delta(B) \neq 0 \). For every \( x \in B \), set \( x_r \) for the restriction of \( x \) to \( V_\sigma \cap ]0, +\infty[ \). Note that two elements \( x, x' \) of \( B \) belong to two distinct sons if and only \( x_r = x'_r \) and \( x(r) \neq x'(r) \). Since \( x(s(r)) = x'(s(r)) \), we have \( s_{M_\sigma}(B) \leq s(r) \).

Let \( y := x_r \) for some \( x \in B \). Since \( J \) is an initial segment, the element \( \bar{y} \) of \( V_\sigma \) which coincide with \( y \) on \( V_\sigma \cap ]0, +\infty[ \) and is 0 everywhere else belongs to \( V_\sigma(J) \). Since \( J \) is an ideal containing \( \text{Fin}(V) \), the elements of \( V_\sigma \) which coincide with \( \bar{y} \) outside \( r \) and take any value belonging to \( s(r) \) are in \( V_\sigma(J) \). Hence \( s_{M_\sigma}(B) = s(r) \). □

**Lemma 3.** Let \( V \) be a subset of \( \mathbb{R}^+ \) containing \( 0 \) such that \( V_\sigma \) contains a subset of type \( \beta^* \), the dual of \( \beta \). Then \( M_\sigma \text{(Well}_\beta(V)) \) is strictly isometrically embeddable into \( M_\sigma \text{(Well}_\beta(V)) \).

**Proof.** Let \( \gamma \) be the order type of \( V_\sigma \), dually ordered and \( (r_\alpha)_{\alpha < \gamma} \) be an enumeration of \( V_\sigma \) respecting the dual order. We define \( f \) by induction. Let \( \alpha < \gamma \). We suppose \( f(r_\beta) \) defined for every \( \beta < \alpha \). Since \( s(r_\alpha) \geq 2 \), the set \( B_\alpha \) of \( g \in M_\sigma \text{(Well}_\beta(V)) \) such that \( g(r_\beta) = f(r_\beta) \) for all \( \beta < \alpha \) contains two elements \( g_{a,0} \) and \( g_{a,1} \) such that \( g_{a,0}(r_\beta) \neq g_{a,1}(r_\beta) \). Since \( d_M(g_{a,0}, g_{a,1}) = r_\alpha \), \( d_M(f(r_\beta), g_{a,1}) = r_\alpha \), hence \( f(r_\beta) = g_{a,1}(r_\beta) \). One of these two values is distinct from 0. Take \( f(r_\alpha) := g_{a,1}(r_\alpha) \), where \( i_\alpha \equiv \min \{0, 1\} \) such that \( f(r_\alpha) = g_{a,i_\alpha}(r_\alpha) \). This is obvious: \( f \) and \( g_{a,i_\alpha} \) coincide up to \( r_\alpha \); since \( f \) is an isometry, \( \varphi(f) \) and \( \varphi(g_{a,i_\alpha}) \) coincide up to \( r_\alpha \). □

**Corollary 3.** Let \( \alpha \) and \( \beta \) be two countable ordinals with \( \alpha < \beta \) and \( V \) be a subset of \( \mathbb{R}^+ \) containing \( 0 \) such that \( V_\sigma \) contains a subset of type \( \beta^* \), the dual of \( \beta \). Then \( M_\sigma \text{(Well}_\alpha(V)) \) is strictly isometrically embeddable into \( M_\sigma \text{(Well}_\beta(V)) \).

**Proof.** Since \( \text{Well}_\alpha(V) \subseteq \text{Well}_\beta(V) \), we conclude that \( M_\sigma \text{(Well}_\alpha(V)) \) is an isometric subset of \( M_\sigma \text{(Well}_\beta(V)) \). Suppose that there is an isometric embedding of \( M_\sigma \text{(Well}_\alpha(V)) \) into \( M_\sigma \text{(Well}_\beta(V)) \). Let \( V' \) be a subset of \( V \) such that \( V_\sigma \) has order type \( \beta^* \). Let \( s' := s'_{V'} \). The subset of \( M_\sigma \text{(Well}_\beta(V)) \) made of \( f \) such that \( \text{supp}(f) \subseteq V' \) coincide with \( \text{Well}(V') \), hence \( M_\sigma \text{(Well}_\beta(V')) \) is an isometric subset of \( M_\sigma \text{(Well}_\beta(V)) \) and thus isometrically embeds into \( M_\sigma \text{(Well}_\alpha(V)) \). This is impossible. Indeed, let \( \varphi \) be an embedding. Let \( \theta \) be defined by setting \( \theta(f) := \varphi(f)|_{V'} \) for every \( f \in \text{Well}(V') \). Then \( \theta \) is an isometric embedding of \( M_\sigma \text{(Well}_\beta(V')) \) into \( M_\sigma \text{(Well}_\alpha(V')) \) (indeed, let \( f, g \in \text{Well}(V') \); since \( \varphi \) is an isometry, \( d(f, g) = d(\varphi(f), \varphi(g)) \), since \( d(f, g) \in V' \), \( d(\varphi(f), \varphi(g)) = d(\varphi(f)|_{V'}, \varphi(g)|_{V'}) \)). According to Lemma 3 there is some \( f \in M_\sigma \text{(Well}_\beta(V')) \) such that \( \text{supp}(\theta(f)) = V_\sigma \), contradicting the fact that \( \theta(f) \) must belong to \( \text{Well}_\alpha(V') \). □

**Theorem 2.** Let \( V \) be a subset of \( \mathbb{R}^+ \) containing \( \mathbb{Q}^+ \) and \( s \) be a cardinal function with domain \( V_\sigma \) and such that \( s(r) \geq 2 \) for every \( r \in V_\sigma \). Then there is a strictly
increasing $\omega_1$-sequence of homogeneous ultrametric spaces with spectrum $V$ and cardinal function $s$.

Proof. For each ordinal $\gamma$, let $M_{\gamma} := M_{\gamma}(\text{Well}_{\omega \gamma}(V))$. Since $\omega^\gamma$ is indecomposable, $V_{\omega^\gamma}$ is an ideal of subsets of $V_{\omega}$. According to Proposition \ref{proposition1}, $M_{\gamma}$ is homogeneous with spectrum $V$ and degree sequence $\sim$. Clearly $M_{\gamma}$ is embeddable into $M_{\gamma'}$ whenever $\gamma < \gamma'$. Since $V$ contains $\mathbb{Q}^{\omega_1}$, we may apply Corollary \ref{corollary3} hence this embedding is strict. The sequence $(M_{\gamma})_{\gamma < \omega_1}$ has the claimed property. \hfill \Box

3.3. Cauchy completion. Let $M := (M, d)$ be an ultrametric space, $X$ be a subset of $M$ and $x \in M$, we set $\text{Spec}(M |_{x}, x) := \{d(x, y) : y \in X\}$.

Lemma 4. Let $M := (M, d)$ be an ultrametric space and $X$ be a topologically dense subset of $M$. Then:

(1) $\text{Spec}(M, x) = \text{Spec}(M |_{x}, x) \cup \{0\}$ for all $x \in M$. In particular $\text{Spec}(M |_{x}, x) = \text{Spec}(M, x)$ for all $x \in X$.

(2) $\text{Spec}(M |_{x}, x) = \text{Spec}(M)$.

(3) For every non trivial $B \in \text{Ball}(M)$, $B \cap x \notin \text{Ball}(M |_{x}, x)$, $B$ is the topological adherence in $M$ of $B \cap X$ and has the same kind as $B \cap X$.

Proof. (1) Clearly, $\text{Spec}(M |_{x}, x) \cup \{0\} \subseteq \text{Spec}(M, x)$. Conversely, let $r \in \text{Spec}(M, x)$. If $r = 0$, then there is nothing to prove. Otherwise, pick $y \in M$ such that $d(x, y) = r$. Since $X$ is dense, there is some $x' \in B(y, r) \cap X$. From the strong triangle inequality, $d(x, x') = r$, hence $r \in \text{Spec}(M |_{x}, x)$ as claimed.

(2) This assertion follows directly from (1). It is due to A. Lemm \ref{lemma1}.

(3) Let $B \in \text{Ball}(M)$ be a non trivial ball. Let $r := \delta(B)$. Then, there is some $x \in M$ such that $B$ is either $B(x, r)$ or $\partial B(x, r)$. Since $X$ is dense and $r \neq 0$, we may pick some $x' \in B(x, r) \cap X$. Due to the strong triangle inequality, we have $B(x', r) = B(x, r)$ and $\partial B(x', r) = \partial B(x, r)$ from which follows that $B \cap X \in \text{Ball}(M |_{x}, x)$. The fact that $B$ is the topological adherence of $B \cap X$ follows directly from the density of $X$ when $B = B(x, r)$. When $B = \partial B(x, r)$ add the observation that by the strong triangle inequality $B(y, r') \subseteq B$ for every $y \in B$, $0 < r' \leq r$. Since $B$ is the topological adherence of $B \cap X$, these two balls have diameter $r$. By assertion (1), $r$ is attained in both or in none. Thus, these balls have the same kind. \hfill \Box

Proposition 2. Let $M := (M, d)$ be an ultrametric space and $X$ be a topologically dense subset of $M$. Then $M |_{X}$ satisfies property $h_1$, resp. property $h_2$, if and only if $M$ does. Moreover, when property $h$ is satisfied, the two spaces have the same degree sequence.

Proof. Suppose that $M$ satisfies property $h_1$. Then by (1) and (2) of Lemma \ref{lemma4} we have $\text{Spec}(M |_{X}, x) = \text{Spec}(M, x) = \text{Spec}(M |_{X})$ for every $x \in X$, amounting to the fact that $M |_{X}$ satisfies property $h_1$. For the converse, it suffices to prove that $\text{Spec}(M, x) = \text{Spec}(M)$ for every $x \in M$. Let $x \in M$. According to (2) of Lemma \ref{lemma4} it suffices to prove that $\text{Spec}(M |_{X}) \subseteq \text{Spec}(M, x)$. Let $r \in \text{Spec}(M |_{X})$. If $r = 0$ there is nothing to prove. Suppose $r \neq 0$. Since $X$ is dense, we may find $x' \in B(x, r) \cap X$. Since $M |_{X}$ satisfies property $h_1$, $r \in \text{Spec}(M |_{X}, x')$. Let $y \in X$ such that $d(x', y) = r$. Due to the strong triangle inequality, $d(x, y) = r$, hence $r \in \text{Spec}(M, x)$ as claimed.

We prove now that $M$ satisfies property $h_2$ if and only if $M |_{X}$ satisfies property $h_2$. From (3) of Lemma \ref{lemma4} we deduce that if $B \in \text{Nerv}(M)$ then $B \cap X \in \text{Nerv}(M |_{X})$ and $s_M(B) = s_{M |_{X}}(B \cap X)$. In particular, if $M |_{X}$ has property $h_2$ then $M$ too. For
Hence

Clearly

\[ r \phi \]

is an ideal, \( M \).

Proposition 2,

ultrametric space with degree sequence

Proof.

the converse, observe that if \( B' \in \text{Nerv}(M_{1X}) \), then for \( r = \delta(B') \) and \( x \in B' \), one has \( B := B(x, r) \in \text{Nerv}(M) \). Apply then the previous argument. \( \square \)

Corollary 4. The Cauchy completion of an ultrametric space satisfying property \( h_1 \), resp. property \( h \), satisfies property \( h_1 \), resp. property \( h \).

We are now ready to characterize separable, Cauchy complete and homogeneous ultrametric spaces.

Theorem 3. Let \( M := (M, d) \) be an ultrametric space. The following properties are equivalent:

(i) \( M \) is separable, Cauchy complete and homogeneous.

(ii) \( M \) is isometric to the Cauchy completion of a space of the form \( M_s(\text{Fin}(V)) \), where \( V \) is a countable subset of \( \mathbb{R}^+ \) and \( s \) is a map from \( V \) into \( \omega + 1 \setminus 2 \).

Proof. (i) \( \Rightarrow \) (ii). Let \( X \) be a countable dense subset of \( M \). Then according to Proposition \( \text{[2]} \), \( M_{1X} \) satisfies property \( h \). Let \( V \) be its spectrum and \( s \) be its degree sequence. According to Theorem 2.7 of \( \text{[3]} \), up to isometry there is a unique countable ultrametric space with degree sequence \( s \), namely \( M_s(\text{Fin}(V)) \). Hence \( M_{1X} \) is isometric to \( M_s(\text{Fin}(V)) \). Since \( M \) is Cauchy complete it is isometric to the Cauchy completion of \( M_s(\text{Fin}(V)) \).

(ii) \( \Rightarrow \) (i). Trivially \( M \) is separable and Cauchy complete. Let \( J := \text{Fin}(V) \). According to Proposition \( \text{[4]} \) the Cauchy completion of \( M_s(J) \) is \( M_s(J) \) and since \( J \) is an ideal, \( M_s(J) \) is homogeneous. This conclusion also follows from Theorem \( \text{[11]} \) below.

3.4. Embedding of an ultrametric space into a homogeneous one.

Theorem 4. Every ultrametric space \( M \) is embeddable into \( M_s(\text{Well}(V)) \) where \( V = \text{Spec}(M) \) and \( s = s_M \).

Proof. The proof is in two steps. First step. We define an embedding \( \varphi \) of \( M \) into an ultrametric space of the form \( M_{s'}(\text{Well}(V)) \) where \( s' \) defined on \( V_* \) takes ordinal values. At the end of the process, it will appear that \( s'(r) = s(r) \) if \( s(r) \) is finite and \( s'(r) \) is an ordinal with the same cardinality as \( s(r) \). Let \( \kappa := |M| \) and \( (a_\alpha)_{\alpha \in \kappa} \) be a transfinite enumeration of the elements of \( M \). We define an embedding \( \varphi \) from \( M \) into some \( M_{s'}(\text{Well}(V)) \). The definition is by induction on \( \alpha \). We set \( \varphi(a_0) = 0 \), the constant function equal to 0 everywhere. Let \( \alpha > 0 \). We suppose that \( \varphi \) is an isometry on the set \( A_\alpha := \{a_\beta : \beta < \alpha \} \). Our aim is to choose \( \varphi(a_\alpha) \in M_{s'}(\text{Well}(V)) \) in such a way that \( \varphi \) becomes an isometry on \( A_{\alpha+1} := A_\alpha \cup \{a_\alpha\} \). For \( r \in V \) we will define \( \varphi(a_\alpha)(r) \) depending how \( r \) compares to \( r_\alpha := d(a_\alpha, A_\alpha) := \inf\{d(a_\alpha, a_\beta) : \beta < \alpha \} \).

- If \( r < r_\alpha \), we set \( \varphi(a_\alpha) = 0 \).
- If \( r > r_\alpha \), then \( B_\alpha := B(a_\alpha, r) \cap A_\alpha \neq \emptyset \). We set \( \varphi(a_\alpha)(r) = \varphi(a_\beta)(r) \) where \( a_\beta \in B_\alpha \). This value is independent of the choice of \( a_\beta \). Indeed, for every \( a_\beta, a_\beta' \in B_\alpha \), \( \varphi(a_\beta) \) and \( \varphi(a_\beta') \) coincide on the interval \([r, \infty) \cap V^* \). Indeed, let \( s := d(a_\beta, a_\beta') \). Clearly \( s < r \). Since \( \varphi \) is an isometry on \( A_\alpha \), we have \( d(a_\beta, a_\beta') = d(\varphi(a_\beta), \varphi(a_\beta')) \). Hence \( d(\varphi(a_\beta), \varphi(a_\beta')) \leq |s| \in V_* \) and in particular on \([r, \infty) \cap V^* \).
- Suppose that \( r := r_\alpha \). There are two cases. Firstly, the distance from \( a_\alpha \) to \( A_\alpha \) is not attained. In this case, if \( r_\alpha \notin V_* \), \( \varphi(a_\alpha) \) is entirely defined already. If
We verify that \( \varphi \) has a dually well ordered support. By construction \( \text{supp}(\varphi(a_\alpha)) \subseteq [r_\alpha, \infty) \) and for each \( r \in V_* \cap [r_\alpha, \infty) \), \( \varphi(a_\alpha) \) coincide with some \( \varphi(a_\beta) \) on \( [r, \infty) \). Since from the induction hypothesis the support of \( \varphi(\alpha) \) is dually well founded, the support of \( \varphi(a_\alpha) \) is dually well founded.

Next we verify that \( \varphi \) is an isometry on \( A_{\alpha+1} \). Let \( \beta < \alpha \) and \( r := d(a_\alpha, a_\beta) \). Suppose that \( r > r_\alpha \). Let \( a_\beta \in A_\alpha \cap B(a_\alpha, r) \). We have \( d(a_\beta, a_\beta') = r \) and \( d(a_\alpha, a_\beta') < r \). By construction, \( \varphi(a_\alpha) \) coincides with \( \varphi(a_\beta') \) on \( [r, \infty) \). Also, since the supports of our \( \varphi(a_\beta) \)’s are dually well ordered, \( r \) is the largest index such that \( \varphi(a_\beta)(r) \neq \varphi(a_\beta')(r) \). Thus \( \varphi(a_\beta)(r) \neq \varphi(a_\alpha)(r) \) hence \( d(\varphi(a_\alpha), \varphi(a_\beta)) = r \). Suppose that \( r = r_\alpha \). By construction, \( \varphi(a_\alpha) \) coincides with \( \varphi(a_\beta) \) on \( [r_\alpha, \infty) \) and \( \varphi(a_\alpha)(r_\alpha) \neq \varphi(a_\beta)(r_\alpha) \). Hence \( d(\varphi(a_\alpha), \varphi(a_\beta)) = r_\alpha \).

The map \( \varphi \) is an isometry. A priori, the image is not in \( M_s(\text{Well}(V)) \), but we only need a further modification. To do this, given \( \psi \) an embedding of an ultrametric space \( M \) into a space of the form \( M_s(\text{Well}(V)) \) and \( B \in \text{Nerv}(M) \) with \( r := \delta(B) \neq 0 \), we set \( \tilde{\psi}(B) := \{ \psi(a)(r) : a \in B \} \).

**Claim 1.** For every \( B \in \text{Nerv}(M) \) with \( \delta(B) = r \neq 0 \), the set \( \tilde{\psi}(B) \) is an initial segment of \( s'(r) \) with cardinality \( s_M(B) \). Furthermore, if \( \gamma \in \tilde{\psi}(B) \) and \( \alpha \) is the least ordinal such that \( a_\alpha \in B \) and \( \varphi(a_\alpha)(r) = \gamma \) then \( \varphi(a_\alpha)(r') = 0 \) for every \( r' < r \) with \( r' \in V_* \).

**Proof of Claim [1]** The fact that \( |\tilde{\psi}(B)| = s_M(B) \) is immediate. Indeed, \( a_\alpha \) and \( a_{\alpha'} \) belong to the same son of \( B \) iff \( d(a_\alpha, a_{\alpha'}) < r \) which is equivalent to \( \varphi(a_\alpha)(r') = \varphi(a_{\alpha'})(r') \) for every \( r' \geq r \). Since \( d(a_\alpha, a_{\alpha'}) \leq r \), \( d(a_\alpha, a_{\alpha'}) < r \) is equivalent to \( \varphi(a_\alpha)(r') = \varphi(a_{\alpha'})(r') \). We verify that \( \tilde{\psi}(B) \) is an initial segment of \( s'(r) \). First, \( 0 \in \tilde{\psi}(B) \). Indeed, let \( \alpha \) be minimum such that \( a_\alpha \in B \). If \( \alpha = 0 \) then since \( \varphi(a_0)(0) = 0 \) and \( 0 \in \tilde{\psi}(B) \) as required. If \( \alpha \neq 0 \) then \( d(A_\alpha, a_\alpha) > r \) indeed, if \( d(a_\beta, a_\alpha) \leq r \) for some \( \beta < \alpha \) then \( a_\beta \in B \) contradicting the minimality of \( \alpha \). Since \( d(A_\alpha, a_\alpha) > r \) we have by construction \( \varphi(a_\alpha)(r) = 0 \). Hence \( 0 \in \varphi(\tilde{B}) \). Now, let \( \gamma \in \tilde{\psi}(B) \). We may suppose that for every \( \gamma' \in \tilde{\psi}(B) \) with \( \gamma' < \gamma \), the interval \( [\gamma', \gamma'] \) is included into \( \tilde{\psi}(B) \). Let \( \alpha \) be minimum such that \( a_\alpha \in B \) and \( \varphi(a_\alpha)(r) = \gamma \). Then \( r = r_\alpha \). This is because by construction, \( \varphi(a_\alpha)(r') = 0 \) for \( r' < r_\alpha \), we have \( r_\alpha \leq r \). If \( r_\alpha < r \) then \( \varphi(a_\alpha)(r) = \varphi(a_\beta)(r) \) for some \( \beta < \alpha \), contradicting our choice of \( \alpha \). Since \( \varphi(a_\alpha)(r) = \gamma \neq 0 \), the distance \( r_\alpha \) is attained. According to our definition \( \varphi(a_\alpha)(r_\alpha) = \min(s'(r) \setminus C_\alpha) \), where \( C_\alpha := \{ \varphi(a_\beta)(r_\alpha) : a_\beta \in \tilde{B}(a_\alpha, r_\alpha) \cap A_\alpha \} \). According to our hypothesis \( \varphi(a_\beta)(r_\alpha) \subseteq \tilde{\psi}(B) \) for every \( \beta \), hence \( C_\alpha \) is an initial segment of \( s'(r) \). Since \( \gamma \) is the least element of \( s'(r) \setminus C_\alpha \), \( (\gamma, \gamma) \subseteq \tilde{\psi}(B) \). It follows that \( \tilde{\psi}(B) \) is an initial segment of \( s'(r) \) and that the second part of our claim is satisfied.

For each \( B \in \text{Nerv}(M) \), with \( \delta(B) \neq 0 \) let \( s_B \) be a bijective map from \( \tilde{\psi}(B) \) onto \( s_M(B) \) viewed as an ordinal and such that \( 0 \) is mapped on \( 0 \).
For each $\alpha < \kappa$, and $r \in V_\kappa$, set $\psi(a_\alpha)(r) := 0$ if no member of Nerv$(M)$ containing $a_\alpha$ has diameter $r$, otherwise set $\psi(a_\alpha)(r) := s_B(\varphi(a_\alpha)(r))$ where $B$ is the unique member of Nerv$(M)$ which contains $a_\alpha$ and has diameter $r$.

Since for each $r \in V_\kappa$, $\psi(a_\alpha)(r) \in s_B$, $\psi(a_\alpha) \in V_\kappa(Well(V))$. This allows to define a map $\psi$ from $M$ into $M_{s}(Well(V))$.

**Claim 2.** The map $\psi$ is an isometry of $M$ into $M_{s}(Well(V))$.

**Proof of Claim 2.** Let $\beta < \alpha$ and $r := d(a_\alpha, a_\beta)$. Let $B := \hat{B}(a_\alpha, r)$. Since $a_\alpha, a_\beta \in M$, $B \in$ Nerv$(M)$, hence $\psi(a_\alpha)(r) = s_B(\varphi(a_\alpha)(r))$ and $\psi(a_\beta)(r) = s_B(\varphi(a_\beta)(r))$. Since $\varphi$ is an isometry, $\varphi(a_\alpha)(r) \neq \varphi(a_\beta)(r)$ and since $s_B$ is one-to-one, $\psi(a_\alpha)(r) \neq \psi(a_\beta)(r)$. Let $r' > r$ and $B \in$ Nerv$(M)$ with $\delta(B) = r'$. If $B$ contains $a_\alpha$ it contains $a_\beta$ and conversely. If no such $B$ contains $a_\alpha$ then $\psi(a_\alpha) = 0$ and similarly $\psi(a_\beta) = 0$. Suppose some $B$ contains $a_\alpha$ and $a_\beta$. We have $\psi(a_\alpha)(r') := s_B(\varphi(a_\alpha)(r'))$ and $\psi(a_\beta)(r') := s_B(\varphi(a_\beta)(r'))$. Since $r' > r$ $\varphi(a_\alpha)(r') = \varphi(a_\beta)(r')$. Hence $\psi(a_\alpha)(r') = \psi(a_\beta)(r')$. It follows that $d(\psi(a_\alpha), \psi(a_\beta)) = r$. Hence $\psi$ is an isometry. \qed

With this claim, the proof of Theorem 4 is complete. \hfill \Box

The embedding $\psi$ has an extra property that we will use in the proof of Theorem 5 below. We give it now.

**Claim 3.** Let $B \in$ Nerv$(M)$ with $r := \delta(B) \neq 0$. Then $\tilde{\psi}(B) = s_M(B)$. Further, if $\zeta \in s_M(B)$ and $\alpha$ is the least ordinal such that $a_\alpha \in B$ and $\psi(a_\alpha)(r) = \zeta$, then $\psi(a_\alpha)(r') = 0$ for every $r' < r$ with $r' \in V_\kappa$.

**Proof of Claim 3.** Let $B \in$ Nerv$(M)$ and $r := \delta(B) \neq 0$. By definition, $\tilde{\psi}(B) = \{\psi(x)(r) : x \in B\}$, and hence $\tilde{\psi}(B) = s_B(\hat{\varphi}(B))$. Since $s_B$ is a bijective map of $\hat{\varphi}(B)$ onto $s_M(B)$, we have $\tilde{\psi}(B) = s_M(B)$, as claimed.

Now let $\zeta \in s_M(B)$ and $\alpha$ is the least ordinal such that $a_\alpha \in B$ and $\psi(a_\alpha)(r) = \zeta$. Then choose $\gamma \in B$ such that $\psi(\gamma) = \zeta$, and thus $\alpha$ is minimum such that $\varphi(a_\alpha)(r) = \gamma$; hence Claim 4 applies and we have $\varphi(a_\alpha)(r') = 0$ for every $r' < r$ with $r' \in V_\kappa$. Since $s_B$ maps 0 to 0, $\psi(a_\alpha)(r') = s_B(\varphi(a_\alpha)(r')) = 0$ for every $r' < r$ with $r' \in V_\kappa$. \hfill \Box

Since $M_{s}(Well(V))$ is homogeneous with spectrum $V$ we obtain:

**Theorem 5.** Every ultrametric space embeds isometrically into a homogeneous ultrametric space with the same spectrum.

An ultrametric space is $T$-complete if the intersection of every chain of non-empty balls is non-empty.

**Fact 4.** An ultrametric is $T$-complete iff the intersection of every chain of members of the nerve is non-empty.

For that it suffices to observe that if $B$ and $B'$ are two non-empty balls with $B' \subset B$, there is a member $D \in$ Nerv$(M)$ such that $B' \subset D \subset B$ (namely $D := \hat{B}(x, d(x, y))$ where $x \in B'$ and $y \in B \setminus B'$).

**Theorem 6.** The following properties are equivalent for an ultrametric space $M$:

(i) $M$ is $T$-complete and homogeneous;
(ii) $M$ is $T$-complete and satisfies condition $h$;
(iii) $M$ is isometric to some $M_{s}(Well(V))$. 
Proof. (i) ⇒ (ii). This part follows from Lemma [1]

(iii) ⇒ (i). This follows by applying Proposition [1]

(ii) ⇒ (iii). Let $\kappa := |M|$, $(a_n)_{\alpha<\kappa}$ be a transfinite enumeration of the elements of $M$ and $\psi : M \to M_s(Well(V))$ given by Theorem [1]. We prove that $\psi$ is surjective. We prove that for each ordinal $\alpha$, every $f \in V_s(Well(V))$ whose type of $A := \text{supp}(f)$ equipped with the dual order of $V$, $\alpha$ there is some $x \in M$ such that $\psi(x) = f$.

The proof is by induction on $\alpha$. We consider two cases.

**Case 1.** $\alpha$ is a limit ordinal. If $\alpha = 0$ then $f = 0$. Hence, $a_0$ will do. Suppose $\alpha > 0$. Since $\alpha$ is denumerable and limit, there is some strictly descending sequence $(a_n)_{n<\omega}$ of elements of $A$ which is coinitial, that is, every element $r \in A$ dominates some $r_n$. For each non-negative integer $n$ the type of the opposite order on $V$, $a_n < a_m$ is strictly smaller than $\alpha$. The map $f_n$ which coincides with $f$ on $a_n$ and takes the value 0 elsewhere belongs to $M_s(Well(V))$. Hence, according to the induction hypothesis, there is some $x_n \in M$ such that $\psi(x_n) = f_n$. Let $n < m < \omega$ then $d(x_n,x_m) = d(f_n,f_m) = d(f_n,f) = r_n$. Hence, the sequence of balls $B_n := \hat{B}_M(x_n,r_n)$ is decreasing. Since $M$ is $T$-complete, $\hat{B} := \bigcap_{n<\omega} B_n$ is non-empty. Let $\alpha$ be the least ordinal such that $a_\alpha \in B_\omega$. We claim that $\psi(a_\alpha) = f$, that is $\psi(a_\alpha)(r) = f(r)$ for each $r \in \hat{B}$. Let $r \in \hat{B}$. Subcase 1. $r \leq r_n$ for all $n < \omega$. Then $B := B(a_\alpha,r) \subseteq \hat{B}_\omega$ (indeed, let $n < \omega$. Since $a_\alpha \in B_n = \hat{B}_M(x_n,r_n)$ and $r \leq r_n$, we have $B(a_\alpha,r) \subseteq B_n$). Since $M$ satisfies condition $\text{h}_1$, $\hat{B}(a_\alpha,r) \in \text{Nerv}(M)$. Due to our choice of $\alpha$, this is the least ordinal such that $a_\alpha \in B$. According to the proof of Claim [1] in the proof of Theorem [1], $\varphi(a_\alpha)(r) = 0$, hence $\psi(a_\alpha)(r) = 0$ and since $f(r) = 0$, $\psi(a_\alpha)(r) = f(r)$.

Subcase 2. $r_n < r$ for some $n$. Since $a_\alpha \in B_n$, we have $d(a_\alpha,x_n) \leq r_n < r$. Since $\psi$ is an isometry, $d(\psi(a_\alpha),\psi(x_n)) < r$ hence $\psi(a_\alpha)(r) = \psi(x_n)(r)$. Since $\psi(x_n) = f_n$ and $f_n(r) = f(r)$, we have $\psi(a_\alpha)(r) = f(r)$. This proves our claim.

**Case 2.** $\alpha$ is a successor ordinal: $\alpha = \alpha' + 1$. Let $r$ be the least element of $A$ and $A' := A \setminus \{r\}$. The map $f' := f \upharpoonright A'$ which coincides with $f$ on $A'$ and takes the value 0 elsewhere belongs to $M_s(Well(V))$. Hence, according to the induction hypothesis, there is some $x' \in M$ such that $\psi(x') = f'$. Let $B := \hat{B}_M(x',r)$ and $\zeta := f(r)$. We claim that there is some $a \in B$ such that $\psi(a)(r) = \zeta$. Indeed, since $\zeta \in \text{sup}(r)$, it suffices to observe that $\psi(B) = \text{sup}(r)$ (this is immediate: since $M$ satisfies Property $\text{h}_1$, $B \in \text{Nerv}(M)$ and $\delta(B) = r$ and since $M$ satisfies Property $\text{h}_2$, $s_M(B) = s(r)$. According to Claim [3], $\psi(B) = s_M(B)$. The result follows). Now, let $\alpha$ be minimum such that $a_\alpha \in B$ and $\psi(a_\alpha)(r) = \zeta$. We claim that $\psi(a_\alpha) = f$. Indeed, according to Claim [3] if $r' < r$ and $r' \in V_\alpha$ then $\psi(a_\alpha)(r') = 0$. Since $f(r') = 0$, we have $\psi(a_\alpha)(r') = f(r')$. If $r' = r$ we have $\psi(a_\alpha)(r') = \zeta = f(r)$. Now, if $r < r'$ then $\psi(a_\alpha)(r') = f(r') = f(r)$ since $d(\psi(a_\alpha),f') = d(a_\alpha,x') \leq r$. This proves our claim.

The proof of the equivalence between (i) and (ii) in the result above is due to Feinberg [8]. The following result is essentially Proposition 33, p. 416 of [6].

**Theorem 7.** The ultrametric space $M_s(\text{Fin}(V))$ is embeddable into every ultrametric space $M$ with property $\text{h}$ such that $\text{Spec}(M) \supseteq V$ and $s_M \geq s$.

**Proof.** Let $V' := \text{Spec}(M)$ and $s' := s_M$. Let $\psi$ be an embedding from $M$ into $M_s(Well(V'))$ given by the proof of Theorem [1]. It suffices to prove that $V_s(\text{Fin}(V))$ is included into the range of $\psi$ and hence that for each non-negative integer $n$, every $f \in V_s(Well(V))$ with $|\text{supp}(f)| = n$ there is some $x \in M$ such that $\psi(x) = f$. The
proof is by induction on \( n \) and exactly the same as in Case 2 of the proof of Theorem 6.

Let us recall that the \textit{weight} of a topological space \( X \) is the minimum cardinality \( w(X) \) of a base (a set \( B \) of open sets of \( X \) such each open set of \( X \) is the union of members of \( B \)), the \( \pi \)-weight is the minimum cardinality \( \pi w(X) \) of a set \( B \) of non-empty open sets such that every non-empty open set contains some member of \( B \); the \textit{density} of \( X \) is the minimum cardinality \( d(X) \) of a dense set. As it is well known, \( w(X) = \pi w(X) = d(X) \) if \( X \) is metrizable. In particular, density, \( \pi \)-weight and weight coincide on ultrametric spaces.

Note that for an ultrametric \( M \) the following inequality holds:

\[
(2) \quad \text{Max}\{ |\Spec(M)|, \Sup \{ s_M(r) : r \in \Spec_*(M) \} \} \leq d(M).
\]

From Theorem 1 and inequality (2) we get:

**Theorem 8.** Every ultrametric space \( M \) with spectrum included into \( V \) and density at most \( \kappa \) is isometrically embeddable into \( M_\kappa(\text{Well}) \).

With the fact that \( d(M_\kappa(\text{Well})) \leq \kappa^{\omega_0} \) proved below we get the result of A. and V. Lemin [10].

**Fact 5.** Let \( V \) be an infinite subset of \( \mathbb{R}^+ \) containing 0.

1. \( d(M_\kappa(\text{Well})) = |M_\kappa(\text{Well})| = \kappa^{\omega_0} \) if \( V_* \) contains a subset of type \( 1 + \omega^* \)
2. \( d(M_\kappa(\text{Well})) = \text{Max}\{ \kappa, |V_*| \} \) and \( |M_\kappa(\text{Well})| = \kappa^{\omega_0} \) if \( V_* \) contains a subset of type \( \omega^* \) but no subset of type \( 1 + \omega^* \).
3. \( d(M_\kappa(\text{Well})) = |M_\kappa(\text{Well})| = \text{Max}\{ \kappa, |V_*| \} \) if \( V_* \) is well founded.

**Proof.** From inequality (2), we have \( \text{Max}\{ \kappa, |V_*| \} \leq d(M_\kappa(\text{Well})) \leq |V_\kappa(\text{Well})| \). Since \( V_* \) is a subset of \( \mathbb{R} \), each dually well ordered subsets is countable, hence its contribution to \( V_\kappa(\text{Well}) \) has cardinality at most \( \kappa^{\omega_0} \); since the number of these subsets is at most \( 2^{\omega_0} \), \( |V_\kappa(\text{Well})| \leq \kappa^{\omega_0} \cdot 2^{\omega_0} = \kappa^{\omega_0} \). If \( V \) is not well founded, \( \kappa^{\omega_0} \) is attained. If \( V \) is well founded, no infinite subset of \( V_* \) is dually well founded hence \( |V_\kappa(\text{Well})| \leq \kappa^{\omega} \cdot |V_*| = \text{Max}\{ \kappa, |V_*| \} \). In this case, we obtain the equalities in item 3. If \( V_* \) contains a subset of type \( \omega^* \) but no subset of type \( 1 + \omega^* \) then \( d(M_\kappa(\text{Well})) = \pi w(M_\kappa(\text{Well})) \leq |\text{Nerv}(M_\kappa(\text{Well})) \setminus M_\kappa(\text{Well})| \leq \text{Max}\{ \kappa, |V_*| \} \), hence Item 2 holds. If \( V_* \) contains a subset of type \( 1 + \omega^* \) then there are \( \kappa^{\omega_0} \) pairwise disjoint members of \( \text{Nerv}(M_\kappa(\text{Well})) \) with non-empty diameter, hence \( d(M_\kappa(\text{Well})) \geq \kappa^{\omega_0} \). The equality \( d(M_\kappa(\text{Well})) = \kappa^{\omega_0} \) follows. This proves that Item 1 holds.

\[
\square
\]

4. **Spectral homogeneity**

Examples show that cardinality conditions are not sufficient to imply homogeneity of an ultrametric space. In this section we introduce a necessary and sufficient condition for homogeneity (Theorem 10), from which we derive the fact that homogeneity is preserved under Cauchy completion (Theorem 11). These results are by-products of the study of more general ultrametric spaces, that we call spec-homogeneous ultrametric spaces.
Theorem 9. An ultrametric space is spec-homogeneous if and only if for any pair of similar balls $B, B'$ where

\[(5) \text{Spec}(M, x) = \text{Spec}(M, f(x)) \text{ for every } x \in \text{dom}(f).\]

The space $M$ is spec-homogeneous if every local spec-isometry of $M$ with finite domain extends to a surjective isometry of $M$.

Definitions 4. Let $M$ be a metric space. A local isometry of $M$ is an isometry from a metric subspace of $M$ onto another one. A local spectral-isometry, in brief a local spec-isometry, is a local isometry $f$ of $M$ such that:

\[(3) \text{Spec}(M, x) = \text{Spec}(M, f(x)) \text{ for every } x \in \text{dom}(f).\]

Definitions 5. Two balls $B, B'$ in Ball($M$) are similar whenever

1. $B$ and $B'$ have the same kind.
2. There are $x \in B$, $x' \in B'$ such that $\text{Spec}(M, x) = \text{Spec}(M, x')$.

Let $X \subseteq M$. The past of $X$ in $M$ is the set:

\[
\text{Past}(M, X) := \{\delta(B) : X \subseteq B \in \text{Nerv}(M)\}.
\]

Lemma 5. Two balls $B, B'$ of an ultrametric space $M$ are similar if and only if they have the same past and the multispectra of $M_{1B}$ and $M_{1B'}$ intersect.

Proof. Suppose that $B$ and $B'$ have the same past and the multispectra of $M_{1B}$ and $M_{1B'}$ intersect. Item (1) follows from the fact that the multispectra of $M_{1B}$ and $M_{1B'}$ intersect. Item (2) follows from Formula (4) below:

\[(4) \text{Spec}(M, y) = \text{Spec}(M_{1B}, y) \cup \text{Past}(M, B).\]

for every $B \in \text{Ball}(M)$ and $y \in B$.

The converse follows from Formula (5) below:

\[(5) \text{Spec}(M_{1B}, y) = \text{Spec}(M, y) \cap X\]

where $X := ]0, \delta(B)[$ if $\delta(B)$ is attained and $X := ]0, \delta(B)[$ otherwise. \qed

Here is our next result.

Theorem 9. An ultrametric space is spec-homogeneous if and only if for any pair of similar balls $B, B' \in \text{Ball}(M)$, the subspaces $M_{1B}$ and $M_{1B'}$ are isometric.

Proof. We argue first that the condition on balls is necessary. Let $B, B' \in \text{Ball}(M)$ which are similar. According to Definition 5 we have $\text{Spec}(M, x) = \text{Spec}(M, x')$ for some elements $x \in B$ and $x' \in B'$. Since $M$ is spec-homogeneous, there is some $\varphi \in \text{Iso}(M)$ such that $\varphi(x) = x'$. Let $r := \delta(B)$. Since $M$ is an ultrametric space, $B = \tilde{B}(x, r)$ if $\delta(B)$ is attained and $B = B(x, r)$ otherwise. Since $B$ and $B'$ are similar, $\delta(B') = r$ and $B' = \tilde{B}(x', r)$ if $\delta(B')$ is attained, $B' = B(x', r)$ otherwise. Clearly, $\varphi$ is an isometry from $M_{1B}$ onto $M_{1B'}$. This proves that the condition on balls is necessary.

Next we prove that the condition on balls suffices. According to Theorem 1 this amounts to prove that for every $x, x' \in M$ with the same spectrum there is an automorphism $\varphi$ which carries $x$ onto $x'$. Let $x, x' \in M$ such that $\text{Spec}(M, x) = \text{Spec}(M, x') := S$. For $r \in S$, set $D(x, r) := \tilde{B}(x, r) \setminus B(x, r)$ and similarly $D(x', r) := \tilde{B}(x, r) \setminus B(x, r)$. The existence of $\varphi$ amounts to the existence, for each $r \in S$, of an isometry $\varphi_r$ from $M_{1D(x, r)}$ onto $M_{1D(x', r)}$ (since, as it is easy to check, the map which sends $x$ to $x'$ and coincides with $\bigcup_{r \in S} \varphi_r$ on $\bigcup_{r \in S} D(x, r)$ is an isometry from $M$ into itself, this condition suffices; the converse is obvious). Again, the existence
of \( \varphi_r \) amounts to the existence of a bijective map \( \psi : \text{Sons}(\hat{B}(x, r)) \to \text{Sons}(\hat{B}(x', r)) \) such that:

\[
\psi(B(x, r)) = B(x', r)
\]

and

\[
M_{1D} \text{ is isometric to } M_{1\psi(D)}
\]

for every \( D \in \text{Sons}(\hat{B}(x, r)) \setminus \{B(x, r)\} \). Indeed, if for each \( D \in \text{Sons}(\hat{B}(x, r)) \setminus \{B(x, r)\} \) there is an isometry \( \varphi_D \) of \( M_D \) onto \( M_{\psi(D)} \), the map \( \varphi_r := \bigcup \{ \varphi_D : D \in \text{Sons}(\hat{B}(x, r)) \setminus \{B(x, r)\} \} \), is an isometry of \( M_{1D(x, r)} \) onto \( M_{1D(x', r)} \). The converse is obvious. In our case, the balls \( \hat{B}(x, r) \) and \( \hat{B}(x', r) \) being similar, by assumption there is an isometry \( \varphi \) of \( M_{1\hat{B}(x, r)} \) onto \( M_{1\hat{B}(x', r)} \). This isometry induces a bijective map \( \varphi_* \) from \( B := \text{Sons}(\hat{B}(x, r)) \) onto \( B' := \text{Sons}(\hat{B}(x', r)) \) such that \( M_{1D} \) is isometric to \( M_{1\varphi_*(D)} \) for every \( D \in B \). But, since there is no reason for which \( \varphi_*(B(x, r)) = B(x', r) \), some modification of \( \varphi_* \) is needed. Let \( A \) be the set of \( D \in B \) such that \( M_{1D} \) is isomorphic to \( M_{1B(x, r)} \) and let \( A' \) be the subset of \( B' \) defined in a similar way. Since \( x \) and \( x' \) have the same spectrum, \( B(x, r) \) and \( B(x', r) \) are similar, hence, according to our hypotheses, \( M_{1B(x, r)} \) and \( M_{1B(x', r)} \) are isometric. Since \( B(x, r) \) and \( B(x', r) \) belong to \( A \) and \( A' \) respectively, \( \varphi_* \) induces a bijection of \( A \) onto \( A' \). Set \( D' := \varphi_*(B(x, r)) \) and \( D'' := \varphi_*^{-1}(B(x', r)) \) and replace \( \varphi_* \) by the map \( \psi \) which send \( B(x, r) \) to \( B(x', r) \), \( D'' \) to \( D' \) and coincides with \( \varphi_* \) on the other members of \( B \). The map \( \psi \) satisfies Conditions (6) and (7) above. It yields the required \( \varphi_r \).

As a corollary, we obtain:

**Theorem 10.** An ultrametric space \( M \) is homogeneous if and only if

1. \( \text{Spec}(M, x) = \text{Spec}(M, x') \) for all \( x, x' \in M \).
2. Balls of \( M \) of the same kind are isometric.

From Theorem 11 we deduce:

**Theorem 11.** The Cauchy completion of an homogeneous ultrametric space is homogeneous.

**Proof.** Let \( M := (M, d) \) be a homogeneous ultrametric space and \( \overline{M} := (\overline{M}, \overline{d}) \) be its Cauchy completion. We show that \( \overline{M} \) satisfies the two conditions in Theorem 11. We may suppose that \( M \) is a dense subset of \( \overline{M} \) and that \( d \) is the restriction of \( \overline{d} \) to \( M \). Since \( M \) is dense in \( \overline{M} \), \( \text{Spec}(\overline{M}) = \text{Spec}(M) \), furthermore \( \text{Spec}(\overline{M}, x) = \text{Spec}(\overline{M}) \) for every \( x \in \overline{M} \) (Lemma 1) hence Condition (1) of Theorem 11 is satisfied. Let \( B_0 \) and \( B_1 \) be two balls in \( \overline{M} \) having the same kind. We may suppose that these two balls are non-trivial. Set \( B_i' := B_i \cap M \) for \( i < 2 \). According to Lemma 1, \( B_i' \) and \( B_1 \) have the same kind. Thus \( B_0' \) and \( B_1' \) have the same kind. Since \( \overline{M} \) is homogeneous, \( M_{1B_0'} \) and \( M_{1B_1'} \) are isometric. An isometry from \( M_{1B_0'} \) onto \( M_{1B_1'} \) extends uniquely to the Cauchy completions of these spaces. Since by Lemma 1, \( B_i' \) is dense in \( B_i \) and \( B_i \) is topologically closed in \( \overline{M} \) (no matter the kind of \( B_i \)), \( M_{1B_i} \) is the completion of \( M_{1B_i'} \). Hence the two balls \( B_0 \) and \( B_1 \) are isometric and Condition (2) of Theorem 11 is satisfied.
4.1. **Countable spec-homogeneous ultrametric spaces.** Theorem 9 leads to the following problem:

**Problem 2.** Is an ultrametric space \( M \) spec-homogeneous whenever for any pair of similar balls \( B, B' \in \text{Nerv}(M) \), the subspaces \( M_{1B} \) and \( M_{1B'} \) are isometric?

According to the next theorem below, a counterexample to Problem 2 must be uncountable. But first for convenience we label the desired property.

**Definition 6.** An ultrametric space \( M \) satisfies condition (A) if for every \( B, B' \in \text{Nerv}(M) \) the subspaces \( M_{1B} \) and \( M_{1B'} \) are isometric provided that \( B \) and \( B' \) are similar. It satisfies condition (B) if for every open balls \( B := B(x, r), B' := B(x', r) \) the subspaces \( M_{1B} \) and \( M_{1B'} \) have the same multispectrum provided that \( B \) and \( B' \) are similar.

**Theorem 12.** A countable ultrametric space is spec-homogeneous if and only if it satisfies condition (A).

The key ingredients of the proof of this result are the following notion and Lemma 6 below.

**Definition 7.** A metric space \( M \) satisfies the finite spec-extension property if every local-spec isometry with finite domain extends at any new element to a local spec-isometry.

We note that a spec-homogeneous metric space satisfies the finite spec-extension property. The converse holds if \( M \) is countable. This noticeable fact holds for larger classes of structures. The proof uses the back and forth method invented by Cantor (see [9], [12]).

**Lemma 6.** Every ultrametric space satisfying condition (A) satisfies condition (B).

**Proof.** Let \( B := B(x, r), B' := B(x', r) \) be similar balls, and thus by Lemma 5 they have the same past and the multispectra of \( M_{1B} \) and \( M_{1B'} \) intersect. Let \( z \in B \). We claim that there is some \( z' \in B' \) with the same spectrum. We observe first that according to Definition 5 \( B \) and \( B' \) contain \( x_1 \) and \( x'_1 \) such that \( \text{Spec}(M, x_1) = \text{Spec}(M, x'_1) \). Set \( s := d(z, x_1), B_1 := \hat{B}(x_1, s) \) and \( B'_1 := \hat{B}(x'_1, s) \). We have \( s \in \text{Spec}(M, x_1) \) and, since \( \text{Spec}(M, x_1) = \text{Spec}(M, x'_1) \), \( s \in \text{Spec}(M, x'_1) \), proving that \( B_1, B'_1 \in \text{Nerv}(M) \). Clearly \( B_1 \) and \( B'_1 \) have the same past; hence, according to Condition (A), \( M_{1B_1} \) and \( M_{1B'_1} \) are isometric. It follows that \( B'_1 \) contains some element \( z' \) such that \( \text{Spec}(M_{1B_1}, z) = \text{Spec}(M_{1B'_1}, z') \). According to Lemma 5 \( \text{Spec}(M, z') = \text{Spec}(M, z) \). This proves our claim. We get the same conclusion if we exchange the roles of \( B \) and \( B' \). This suffices to prove the lemma. \( \square \)

4.2. Proof of Theorem 12.

**Proof.** Let \( M := (M, d) \) satisfying Condition (A). Since \( M \) is countable, it suffices to prove that \( M \) satisfies the finite spec-extension property. Let \( \varphi : F \to M \) be a local spec-isometry of \( M \) and let \( a \in M \setminus F \). If \( F \) is empty, then \( \varphi \) extends to \( \{a\} \) by the identity. Assume that \( F \) is non-empty, then set \( r := d(a, F), B := \hat{B}(a, r) \) and \( A := F \cap B \). Our goal is to find an element of \( M \) at distance \( r \) from every element of \( \varphi[A] \) and with the same spectrum as \( a \). We have \( B \in \text{Nerv}(M), \delta(B) = r \) and furthermore \( B = \hat{B}(a', r) \) for every \( a' \in A \). Since \( \text{Spec}(M, a') = \text{Spec}(M, \varphi(a')) \) for
every \( a' \in A \), the closed ball \( B' := B(\varphi(a'), r) \) belongs to \( \text{Nerv}(M) \) and is independent from the choice of \( a' \) in \( A \). We have to find a son of \( B' \) disjoint from \( \varphi[A] \) whose multispectrum contains the spectrum of \( a \). Since \( \varphi \) preserves spectra, \( B \) and \( B' \) have the same past. So by assumption \( M_{1B} \) and \( M_{1B'} \) are isometric. Let \( \psi : B \to B' \) be an isometry. Then \( \psi \) induces a function \( \psi_* \) from \( B := \text{Sons}(B) \) onto \( B' := \text{Sons}(B') \). In particular \( M_{1D} \) and \( M_{1\psi_*(D)} \) have the same multispectrum for every \( D \in B \).

Let \( A \) denote the set of sons of \( B \) that meet \( A \). Then \( \varphi \) yields an injection \( \varphi_* \) from \( A \) into \( B' \). Furthermore, for each \( D \in A \), \( D \) and \( \varphi_*(D) \) are isometric. Let \( \psi_* \) from \( B := \text{Sons}(B) \) onto \( B' := \text{Sons}(B') \). In particular \( M_{1D} \) and \( M_{1\psi_*(D)} \) have the same multispectrum for every \( D \in B \).

Let \( C \) be the son of \( B \) containing \( a \). Let \( A^+ \) denote the set of members \( D \) of \( A \) such that \( \text{Spec}(M_{1D}, d) = \text{Spec}(M_{1C}, d) \) for some \( d \in D \), and let \( A^- := A \setminus A^+ \). We claim that:

\[
\psi_*[A^+ \cup \{C\}] \notin \varphi_*[A]
\]

Indeed, first \( \psi_*[A^+ \cup \{C\}] \) is disjoint from \( \varphi_*[A^-] \) since the multispectrum of any member of \( \psi_*[A^+ \cup \{C\}] \) contains \( \text{Spec}(M_{1C}, a) \) while none of those of \( \varphi_*[A^-] \) does. Next, since \( C \) does not meet \( F \) and the maps \( \psi_* \) and \( \varphi_* \) are one-to-one, the size of \( \psi_*[A^+ \cup \{C\}] \) is larger than the size of \( \varphi_*[A^+] \). This proves that \( \psi_*[A^+ \cup \{C\}] \) is not included in \( \varphi_*[A] = \varphi_*[A^+] \cup \varphi_*[A^-] \), as claimed.

To conclude the proof of the theorem, observe that any member of \( \psi_*[A^+ \cup \{C\}] \setminus \varphi_*[A] \) contains an element \( a' \) with the same spectrum as \( a \) and at distance \( r \) from any element of \( \varphi[A] \).

\( \square \)

**Corollary 5.** A countable ultrametric space is spec-homogeneous provided that any two members of the nerve with the same diameter are isometric.

A construction of ultrametric spaces satisfying this condition is given in [4]. To each ultrametric space \( M \) we associate an ultrametric space, the **path extension** of \( M \), denoted by \( \text{Path}(M) \), whose elements, the **paths**, are special finite unions of chains in \( (\text{Nerv}(M), \leq) \).

We recall that if \( M := (M, d) \) is an ultrametric space and \( \alpha \in \mathbb{R}_+^* \cup \{+\infty\} \), the pair \( M_{\alpha} := (M, d \land \alpha) \), where \( d \land \alpha(x, y) := \min(d(x,y), \alpha) \), is an ultrametric space. We recall the following properties of the path extension (see Theorem 9 of [4]).

**Proposition 3.** For every ultrametric space \( M \), the path extension \( \text{Path}(M) \) of \( M \) satisfies the following properties:

1. \( \text{Path}(M) \) is an isometric extension of \( M \) with the same spectrum as \( M \) and \( |\text{Path}(M)| \leq |M| + \aleph_0 \).
2. \( \text{Path}(M)_{1B} \) is isometric to \( \text{Path}(M_{\delta(B)}) \) for every non-trivial \( B \in \text{Nerv}(\text{Path}(M)) \).

With Corollary 5 we obtain our final result:

**Theorem 13.** Every countable ultrametric space extends via a spec-isometry to a countable spec-homogeneous ultrametric space.

**Proof.** Let \( M \) be an ultrametric space. Set \( M' := \text{Path}(M) \). It follows from Item 2 of Proposition 3 that \( M_{1B} \) and \( M_{1B'} \) are isometric provided that \( B, B' \in \text{Nerv}(M') \) and \( \delta(B) = \delta(B') \). Thus, according to Corollary 5 \( M' \) is spec-homogeneous provided that it is countable. According to Item 1 of Proposition 3 this is the case if \( M \) is
Lemma 7. Let \( M \rightarrow M' \) be a set. A \( V \)-labelled \( 2 \)-structure (briefly, a \( 2 \)-structure) is a pair \((E, v)\) where \( v \) is a map from \( E \times E \setminus \Delta_E \) into \( V \), with \( \Delta_E := \{(x, x) : x \in E\} \) denoting the diagonal of \( E \).

A subset \( A \) of \( E \) is a module if \( v(x, y) = v(x, y') \) and \( v(y, x) = v(y', x) \) for every \( x \in E \setminus A \) and \( y, y' \in A \) (other names are autonomous sets, or intervals).

The whole set, the empty set and the singletons are modules. These are the trivial modules. We recall first the basic and well-known properties of modules.

Lemma 7. Let \((E, v)\) be a \( 2 \)-structure.

1. The intersection of a non-empty set of modules is a module (with \( \cap \emptyset = E \)).
2. The union of two modules that meet is a module, and more generally, the union of a set of modules is a module as soon as the meeting relation on that set is connected.
3. For two modules \( A \) and \( B \), if \( A \setminus B \) is non-empty, then \( B \setminus A \) is a module.

A module is strong if it is comparable (w.r.t. inclusion) to every module it meets.

Lemma 8. Let \((E, v)\) be a \( 2 \)-structure.

1. The intersection of any set of strong modules is a strong module.
2. The union of any directed set of strong modules is a strong module.

A module is robust if this is the least strong module containing two elements of \( V \). The robust modules of a binary structure \((E, v)\) form a tree under reverse inclusion. This tree is canonically endowed with an additional structure from which the given \( 2 \)-structure can be recovered (see [2]). First we call a \( 2 \)-structure \((E, v)\) is symmetric if \( v(y, x) = v(x, y) \) for every distinct \( x \) and \( y \) in \( E \). Further, we call a \( 2 \)-structure \((E, v)\) is hereditary decomposable if every induced substructure on at least three elements of \( E \) has a nontrivial module.

Now an ultrametric space \( M := (M, d) \) is obviously a symmetric \( 2 \)-structure with values in the set of positive reals. It is also hereditary decomposable since every ultrametric space with at least three elements has a nontrivial module. Indeed, if the distance assumes only one nonzero value then consider any pair of two distinct elements of \( M \), and if it assume at least two values \( r < s \) then consider the close ball \( B(x, r) \) for any \( x \in E \) such that there is a \( y \in E \) with \( d(x, y) = r \).

Thus since our interest lies in ultrametric spaces, we simply consider symmetric hereditary decomposable \( 2 \)-structures. This additional structure on the tree of robust modules is just a labelling of the nonsingleton nodes into \( V \); indeed to each nonsingleton robust module \( R \) there corresponds some \( v(R) \in V \) such that for any two distinct \( x \) and \( y \) in \( E \), \( v(x, y) = v(R) \), where \( R \) is the least strong module containing \( x \) and \( y \). Let us call the decomposition tree of a symmetric hereditary decomposable \( 2 \)-structure its tree of robust modules endowed with this node labelling into \( V \).
Proposition 4. The decomposition tree of an ultrametric space $M$ is equal to its nerve endowed with the diameter function.

This is the last statement of Corollary 3.

In the statement below, we allow balls of infinite radius (which are equal to $M$).

Lemma 9. Let $M := (M, d)$ be an ultrametric space. The least module including a subset $A$ of $M$ is the union of all open balls of radius $\delta(A)$ centered in $A$ (this is the set $\bigcup_{a \in A} B(a, \delta(A))$). In particular, if $A$ is unbounded, this module is $M$. As a consequence, $A$ is a module if and only if it is a union of open balls of radius $\delta(A)$.

Proof. Given $A \subseteq M$, let $A'$ denote the union of all open balls of radius $\delta(A)$ centered in $A$. We first check that $A'$ is included in any module $C$ of $M$ that includes $A$. Indeed given any $a' \in A'$, consider some $a \in A$ such that $d(a, a') < \delta(A)$. Then, recalling that $\delta(A) = \sup\{d(a, b) : b \in A\}$, consider some $b \in A$ such that $d(a, a') < d(a, b)$. It follows from the strong triangle inequality that $d(a, a') < d(a, b) = d(a', b)$. Thus $a'$, which distinguishes the two elements $a$ and $b$ of $C$, belongs to $C$. Then we check that $A'$ is a module. Consider $x \in M \setminus A'$ and $a'$ and $b'$ in $A'$. Then consider $a$ and $b$ in $A$ such that $d(a, a') < \delta(A)$ and $d(b, b') < \delta(A)$. Thus on the one hand $d(x, a) \geq \delta(A) \geq d(a, b)$ from which the strong triangle inequality yields $d(x, a) = d(x, b)$. On the other hand $d(x, a) \geq \delta(A) \geq d(a, a')$, hence $d(x, a) = d(x, a')$. Likewise $d(x, b) = d(x, b')$. Finally $d(x, a') = d(x, a) = d(x, b) = d(x, b')$. □

Corollary 6. Let $M := (M, d)$ be an ultrametric space.

1. The strong modules of $M$ are the balls (open or closed) of $M$, that is the sets $B(a, r)$ and $\hat{B}(a, r)$ for some $a$ and $r$, and $M$.
2. The robust modules are the closed balls attaining their diameter. These are the sets of the form $\hat{B}(a, d(a, b))$ for some $a, b \in M$.

Proof. (1) Let us first check that any ball $B$ of attained diameter $r$ is a strong module. It follows from the lemma that this is a module since $B = \bigcup_{a \in B} B(a, r)$. Now given a module $C$ meeting $B$, say with $a \in B \cap C$, so that $B = \hat{B}(a, r)$, let us check that $B$ and $C$ are comparable. If $\delta(C) > r$ then $C \subseteq B$. If $\delta(C) \leq r$ then $\hat{B}(a, r) \subseteq B(a, \delta(C))$ which is included in $C$, according to the lemma. Thus $B$ is a strong module. Finally observe that any open ball $B(a, r)$, as a union of $\bigcup_{s \in r} \hat{B}(a, s)$ of a chain of strong modules, is a strong module itself.

2. Consider a nonempty ball $B$. If it attains its diameter $r$, then given any $a$ and $b$ in $B$ with $d(a, b) = r$, $B$ is the least ball containing $a$ and $b$, hence the least strong module containing $a$ and $b$. In this case $B$ is robust. If it does not attain its diameter $r \leq \infty$, then for any $a$ and $b$ in $B$, the least ball containing $a$ and $b$ has diameter $d(a, b) < r$ and therefore is distinct from $B$. In this case $B$ cannot be robust. □

A 2-structure is said to be strong-modular complete if every nonempty chain of strong modules has a nonempty intersection. According to Corollary 4, we conclude that the strong-modular complete ultrametric spaces are the T-complete ultrametric spaces.
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