TRANSVERSE MOMENTUM DISTRIBUTION IN THE $B$ MESONS IN THE HEAVY-QUARK LIMIT: THE WANDZURA-WILCZEK PART

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In the heavy-quark limit, the valence Fock-state components in the $B$ mesons are described by a set of two light-cone wavefunctions. We show that these two wavefunctions obey simple coupled differential equations, which are based on the equations of motion in the Heavy Quark Effective Theory (HQET), and the analytic solutions for them are obtained. The results generalize the recently obtained longitudinal-momentum distribution in the Wandzura-Wilczek approximation by including the transverse momenta. We find that the transverse momentum distribution depends on the longitudinal momentum of the constituents, and that the wavefunctions damp very slowly for large transverse separation between quark and antiquark.

Keywords: $B$ meson; light-cone wavefunctions; transverse momentum; heavy quark effective theory.

Along with the progress in both theory and experiment, $B$ physics becomes one of the most active research areas in high energy physics. Many $B$ meson exclusive decay processes turn out to be calculable systematically in the frameworks of newly developed factorization formalisms, the so-called pQCD approach \textsuperscript{1,2} and QCD Factorization approach.\textsuperscript{3,4,5} In all of the calculations based on the factorization approaches, the light-cone distribution amplitudes of the participating mesons, which express the nonperturbative long-distance contributions in the factorized amplitudes,\textsuperscript{6,7} play an important role in making reliable predictions. The light-cone distribution amplitudes describe the probability amplitudes to find particular partons with definite light-cone momentum fraction in a meson, and thus are process-independent quant-
tity. It is well-known that, for the light mesons, the model-independent framework to construct the light-cone distribution amplitudes is established for leading and higher twists as well.\textsuperscript{8,9} However, unfortunately, the distribution amplitudes for the \( B \) mesons are not well-known at present and they provide a major source of uncertainty in the calculations of the decay rates.

Recently,\textsuperscript{10} we have presented the first systematic study for the \( B \) meson light-cone distribution amplitudes, and derived explicit forms for the quark-antiquark distribution amplitudes, which exactly satisfy the constraints coming from the equations of motion and heavy-quark symmetry. We have found that the “Wandzura-Wilczek-type” contributions which correspond to the valence quark distributions, are determined uniquely in analytic form in terms of \( \Lambda \), a fundamental mass parameter of Heavy Quark Effective Theory (HQET).\textsuperscript{11,12} We have also shown that both leading- and higher-twist distributions receive the contributions from the multiparticle states with additional dynamical gluons, and derived the exact integral representations for these contributions.

By definition, the light-cone distribution amplitudes are given by the light-cone (Bethe-Salpeter) wavefunctions at zero transverse separation of the constituents. Thus the previous results of Ref.\textsuperscript{10} have been obtained for the configuration in which the quark and antiquark are separated by exactly light-like distance. The information on transverse momentum distribution has been integrated out. However, the light-cone wavefunctions with transverse momentum dependence are necessary for computing the power corrections to the exclusive amplitudes, and also for estimating the transition form factors for \( B \to D, B \to \pi \), etc, which constitute another type of long-distance contributions appearing in the factorization approaches for the exclusive \( B \) meson decays.

In this Letter, we extend the analysis of Ref.\textsuperscript{10} to include the transverse momentum effects. In particular, we derive explicit analytic formulae for a complete set of the \( B \) meson light-cone wavefunctions within the Wandzura-Wilczek approximation.\textsuperscript{10} It should be noted that the Wandzura-Wilczek approximation is not equivalent to the free field approximation. The leading Fock-states, which correspond to the twist-2 contribution in the case of light meson wavefunctions, carries the effect from the QCD interaction.\textsuperscript{8,9} We also estimate the effects neglected in this “valence” approximation; we derive the exact result for the first moment of the transverse momentum squared \( k_T^2 \) in terms of the full light-cone wavefunctions, which include the higher Fock-states with additional dynamical, nonperturbative gluons.\textsuperscript{a}

The light-cone wavefunctions are related to the usual Bethe-Salpeter wavefunctions at equal light-cone time \( z^+ = (z_0 + z^3)/\sqrt{2} \). The quark-antiquark light-cone wavefunctions \( \psi_\pm(t, z^2) \) of the \( B \) mesons in the heavy-quark limit are defined by the vacuum-to-meson matrix element of nonlocal operators, following Refs.\textsuperscript{13,14}:

\[
\langle 0 | \bar{q}(z) \Gamma h_v(0) | B(p) \rangle
\]

\textsuperscript{a}Radiative corrections due to the virtual gluons and/or quark-antiquark pairs can be included as the renormalization scale-dependence of the wavefunctions, which is governed by the renormalization group equations for the nonlocal operators in Eq. (1) below. The discussion of this point is beyond the scope of this work.

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and is subject to the on-shell constraint, \( \not{p} v = 0 \), the light-cone projection of the 4-momentum of the \( B \) meson with mass \( M \). \( h_v(x) \) denotes the effective \( b \)-quark field, \( b(x) = \exp(-i m_b v \cdot x) h_v(x) \), and is subject to the on-shell constraint, \( \not{p} h_v = h_v \). \( \Gamma \) is a generic Dirac matrix and, here and in the following, the path-ordered gauge factors are implied in between the constituent fields. \( f_B \) is the decay constant defined as usually as

\[
\langle 0| \bar{q}(0) \gamma^\mu \gamma_5 h_v(0) | B(p) \rangle = i f_B M v^\mu ,
\]

so that \( \bar{\psi}_\pm(t = 0, z^2 = 0) = 1 \). Eq. (1) is the most general parameterization compatible with Lorentz invariance and the heavy-quark limit.

Note that in the definition (1) the separation \( z^\mu = (0, z^-, z^T) \) between quark and antiquark is not restricted on the light-cone, \( z^2 = -z_T^2 \), unlike the corresponding definition of the distribution amplitudes.\(^\text{10,15,16}\) Thus the distribution amplitudes \( \bar{\phi}_\pm(t) \) in the notation of Refs.\(^\text{10,15,16}\) are given by the light-cone limit of the above wavefunctions as \( \bar{\phi}_\pm(t) = \bar{\psi}_\pm(t, z^2) \bigg|_{z^2 \to 0} \).

We also introduce the Fourier transforms with respect to the longitudinal separation \( t \) by

\[
\bar{\psi}_\pm(t, z^2) = \int d\omega \ e^{-i\omega t} \bar{\psi}_\pm(\omega, z^2) .
\]

Here \( \omega v^+ \) has the meaning of the light-cone projection \( k^+ \) of the light-antiquark momentum in the \( B \) meson.

We now exploit the constrains from the equations of motion. The procedure here is completely in parallel with that of our previous work, so we refer the readers to Ref.\(^\text{10}\) for the detail. The matrix elements of the exact operator identities (Eqs. (3), (4) of Ref.\(^\text{10}\)) from the equations of motion yield a system of four differential equations:

\[
\begin{align*}
\omega \frac{\partial \psi_-}{\partial \omega} & + z^2 \left( \frac{\partial \psi_+}{\partial z^2} - \frac{\partial \psi_-}{\partial z^2} \right) + \psi_+ = 0 , \\
\omega \left( \frac{\partial \psi_+}{\partial \omega} - \frac{\partial \psi_-}{\partial \omega} \right) & + 4 \frac{\partial^2 \psi_+}{\partial \omega^2} \frac{\partial \psi_+}{\partial z^2} + 2 (\psi_+ - \psi_-) = 0 , \\
(\omega - \Lambda) \frac{\partial \psi_+}{\partial \omega} & + 2 \frac{\partial^2 \psi_+}{\partial \omega^2} \frac{\partial \psi_+}{\partial z^2} + \frac{1}{2} (3 \psi_+ - \psi_-) = 0 , \\
(\omega - \Lambda) \left( \frac{\partial \psi_+}{\partial \omega} - \frac{\partial \psi_-}{\partial \omega} \right) & + 2 \frac{\partial^2 \psi_+}{\partial \omega^2} \left( \frac{\partial \psi_+}{\partial z^2} - \frac{\partial \psi_-}{\partial z^2} \right) + 2 (\psi_+ - \psi_-) = 0 .
\end{align*}
\]

Here,

\[
\Lambda = M - m_b = \frac{i v \cdot \partial \langle 0| \bar{q} \Gamma h_v | B(p) \rangle}{\langle 0| \bar{q} \Gamma h_v | B(p) \rangle}
\]

is the usual “effective mass” of meson states in the HQET,\(^\text{12,17}\) and the shorthand notation \( \psi_\pm \equiv \psi_\pm(\omega, z^2) \) is adopted. These Eqs. (4), (5), (6), and (7) correspond
to Eqs. (7), (8), (10), and (11) in Ref. 10, respectively, but are given in the “ω-representation” instead of the “t-representation” via Eq. (3). Further differences compared with Ref. 10 are on two aspects. Firstly, the light-cone limit is not taken in order to explore the transverse momentum distribution, so that the terms proportional to \( z^2 \) appear in the above Eq. (4). Secondly, we restrict our interest within the two-particle Fock states neglecting the contribution from the quark-antiquark-gluon three-particle operators.

From Eqs. (5) and (6), we obtain

\[
(\omega - 2\bar{\Lambda}) \frac{\partial \psi_+}{\partial \omega} + \omega \frac{\partial \psi_-}{\partial \omega} + \psi_+ + \psi_- = 0. \tag{9}
\]

This can be integrated with boundary conditions \( \psi_\pm(\omega, z^2) = 0 \) for \( \omega < 0 \) or \( \omega \to \infty \) as

\[
(\omega - 2\bar{\Lambda}) \psi_+ + \omega \psi_- = 0. \tag{10}
\]

A system of Eqs. (4) and (10) has been solved for \( z^2 = 0 \) and completely determined the wavefunctions in the light-cone limit. 10 The results are

\[
\psi_+(\omega, z^2 = 0) = \phi_+(\omega) = \frac{\omega}{2\bar{\Lambda}^2} \theta(\omega) \theta(2\bar{\Lambda} - \omega), \tag{11}
\]

\[
\psi_-(\omega, z^2 = 0) = \phi_-(\omega) = \frac{2\bar{\Lambda} - \omega}{2\bar{\Lambda}^2} \theta(\omega) \theta(2\bar{\Lambda} - \omega). \tag{12}
\]

These solutions for \( z^2 = 0 \) serve as “boundary conditions” to solve Eqs. (4)-(7) for \( z^2 \neq 0 \). From Eqs. (4) and (10) for \( z^2 \neq 0 \), we find

\[
\psi_\pm(\omega, z^2) = \phi_\pm(\omega) \xi(z^2\omega(2\bar{\Lambda} - \omega)), \tag{13}
\]

where \( \xi(x) \) is some function of a single variable \( x \), and satisfies \( \xi(0) = 1 \) due to Eqs. (11), (12). The functional form of \( \xi(x) \) can be easily determined from a remaining differential equation, e.g., (6), which was useless in the light-cone limit 10 (Eq. (7) gives the result identical to Eq. (6)). We obtain \( \xi(x) = J_0(\sqrt{-x}) \), where \( J_0 \) is a (regular) Bessel function, so that the analytic solution for the coupled differential equations (4)-(7) is given by

\[
\psi_\pm(\omega, -z^2_T) = \phi_\pm(\omega) J_0(\omega(2\bar{\Lambda} - \omega)) \frac{1}{\sqrt{\omega(2\bar{\Lambda} - \omega)}}. \tag{14}
\]

These are the light-cone wavefunctions for the transverse separation \( z_T \) between quark and antiquark.

For the momentum-space wavefunctions \( \psi_\pm(\omega, k_T) \) defined by

\[
\tilde{\psi}_\pm(t, -z^2_T) = \int d\omega d^2k_T \ e^{-i\omega t + i\mathbf{k}_T \cdot \mathbf{z}_T} \psi_\pm(\omega, k_T), \tag{15}
\]

our solution gives

\[
\psi_+(\omega, k_T) = \frac{\omega}{2\pi \bar{\Lambda}^2} \theta(\omega) \theta(2\bar{\Lambda} - \omega) \delta(k_T^2 - \omega(2\bar{\Lambda} - \omega)), \tag{16}
\]

\[
\psi_-(\omega, k_T) = \frac{2\bar{\Lambda} - \omega}{2\pi \bar{\Lambda}^2} \theta(\omega) \theta(2\bar{\Lambda} - \omega) \delta(k_T^2 - \omega(2\bar{\Lambda} - \omega)). \tag{17}
\]
The results (16) and (17) give exact description of the valence Fock components of the $B$ meson wavefunctions in the heavy-quark limit, and represent their transverse momentum dependence explicitly. These results show that the dynamics within the two-particle Fock states is determined solely in terms of a single nonperturbative parameter $\bar{\Lambda}$.

Up to now, the transverse momentum distributions in the $B$ mesons have been completely unknown, so that various models have been used in the literature. Sometimes the dependence of the light-cone wavefunctions on the transverse separation is simply neglected, such as $\psi(\omega, -z_T^2) = \psi(\omega, 0)$; clearly, this in general contradicts our results (14). Another frequently used models assume complete separation (factorization) between the longitudinal and transverse momentum-dependence in the wavefunctions, such as $\psi(\omega, k_T) = \phi(\omega) \tau(k_T)$ (see e.g. Refs.13,2,19). One typical example of such models is given by

$$\psi^{KLS}(\omega, k_T) = N\omega^2(1 - \omega)^2 \exp\left(-\frac{\omega^2}{2\omega_0^2}\right) \times \exp\left(-\frac{k_T^2}{2K^2}\right),$$

(18)

where $N$ is the normalization constant and $\omega_0 = 0.3\text{GeV}$, $K = 0.4\text{GeV}$. Our results (16) and (17) show that the dependence on transverse and longitudinal momenta is strongly correlated through the combination $k_T^2/[(\omega(2\Lambda - \omega))]$, therefore the “factorization models” are not justified.\textsuperscript{c} We further note that many models assume\textsuperscript{13,2,19} Gaussian distribution for the $k_T$-dependence as in Eq. (18). These models show strong dumping at large $|z_T|$ as $\sim \exp(-K^2z_T^2/2)$. In contrast to this asymptotic behavior, our wavefunctions (14) have slow-damping with oscillatory behavior as $\psi(\omega, -z_T^2) \sim \cos(\sqrt{\omega(2\Lambda - \omega)} - \pi/4)/\sqrt{|z_T|}$.

To summarize, in this work we have derived a system of differential equations for the $B$ meson light-cone wavefunctions including the transverse degrees of freedom, and obtained the exact solution within the valence Fock states. The differential equations are derived from the exact equations of motion of QCD in the heavy-quark limit. The heavy-quark symmetry plays an essential role as in the case of the light-cone limit.\textsuperscript{10,15,16} Heavy-quark spin symmetry reduces the number of independent wavefunctions drastically, so that the valence Fock-state components in the $B$ mesons are described by only two light-cone wavefunctions. As a result, a system of four differential equations from the equations of motion becomes a “complete set” to determine these two wavefunctions, and this enables us to obtain the exact solution with full account of the $k_T$-dependence. Also due to the power of heavy-quark symmetry, our final results are given in simple analytic formulae involving one single nonperturbative parameter $\bar{\Lambda}$. Heavy-quark symmetry also guarantees that the solution in the present paper provides complete description of the light-cone valence Fock wavefunctions for the $B^*$ mesons and also for the $D, D^*$ mesons in the heavy-quark limit.

\textsuperscript{b}In this model, the two independent wavefunctions $\psi_{\pm}$ of the $B$ mesons are set equal to each other, $\psi^{KLS} = \psi^{KLS} = \psi^{KLS}$.

\textsuperscript{c}The “non-factorization” in the light-cone wavefunctions for the light-mesons has been discussed in Ref.20, where the coupling between transverse and longitudinal momenta through the variable $k_T^2/[u(1 - u)]$, with $u$ the momentum fraction of the light quark, has been demonstrated.
Finally a comment is in order concerning the error induced by the Wandzura-Wilczek approximation. From the study of the $B$ meson distribution amplitudes in the light-cone limit, there has been indication that, in the heavy-light quark systems, the higher Fock states could play important roles even in the leading twist level.\textsuperscript{10} This would suggest that the shape of the wavefunctions as function of momenta and their quantitative role in the phenomenological applications would be modified when including the higher Fock states. For example, inspecting the $t \to 0$ limit of Eqs. (8) and (11) of Ref.\textsuperscript{10}, one immediately obtains the exact result for the first moment of $k_T^2$ as

$$\int d\omega d^2 k_T k_T^2 \psi_{\pm}^{(\text{tot})}(\omega, k_T) = 4 \left. \frac{\partial \tilde{\psi}_{\pm}^{(\text{tot})}(t = 0, z^2)}{\partial z^2} \right|_{z^2 \to 0} = \frac{2}{3} \left( \bar{\Lambda}^2 + \lambda_E^2 + \lambda_H^2 \right),$$

where $\psi_{\pm}^{(\text{tot})}$ denote the total wavefunctions including the higher Fock contributions $\psi_{\pm}^{(hF)}$, and $\lambda_E$ and $\lambda_H$ denote the reduced matrix elements of relevant quark-antiquark-gluon operators in the notation of Ref.\textsuperscript{10}, representing the chromoelectric and chromomagnetic fields in the $B$ meson rest frame, respectively. The first term $\frac{2}{3} \bar{\Lambda}^2$ in the RHS of Eq. (19) coincides with the moment of our solution $\psi_{\pm}$, Eqs. (16) and (17), in the Wandzura-Wilczek approximation, while other terms $\frac{2}{3} (\lambda_E^2 + \lambda_H^2)$ come from the higher Fock contributions $\psi_{\pm}^{(hF)}$. The result (19), combined with an estimate $\lambda_E^2/\bar{\Lambda}^2 = 0.36 \pm 0.02, \lambda_H^2/\bar{\Lambda}^2 = 0.60 \pm 0.23$ by QCD sum rules,\textsuperscript{15} suggests that the higher Fock contributions might considerably broaden the transverse momentum distribution. This point can be studied in a systematic way, and more sophisticated wavefunctions $\psi_{\pm}^{(\text{tot})}$ will be discussed in detail in a separate publication.\textsuperscript{21} However, qualitative features revealed in this paper, like non-factorization of longitudinal and transverse directions, “slow-damping” for transverse directions, etc., will be unaltered by the effects of multi-particle states, and helpful in elucidating QCD factorization theorems.

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