Frequency range and explicit expressions for negative permittivity and permeability for an isotropic medium formed by a lattice of perfectly conducting $\Omega$ particles

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Abstract

An analytical model is presented for a rectangular lattice of isotropic scatterers with electric and magnetic resonances. Each isotropic scatterer is formed by putting appropriately 6 Ω-shaped perfectly conducting particles on the faces of a cubic unit cell. A self-consistent dispersion equation is derived and then used to calculate correctly the effective permittivity and permeability in the frequency band where the lattice can be homogenized. The frequency range in which both the effective permittivity and permeability are negative corresponds to the mini-band of backward waves within the resonant band of the individual isotropic scatterer.

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1 Introduction

In 1967 Veselago considered an isotropic continuous medium with both negative permittivity and negative permeability and proved that a negative refraction would occur when a monochromatic wave impinges on the surface of such a medium [1]. As a result he obtained theoretically a quasi-lens (the lens does not focus parallel rays) from a slab of such a hypothetic material. In the Veselago theory this quasi-lens must form an image of any source without distortion (since each interface is a parallel plate). The phase of the radiated field at the focal point should be the same as the phase of the source dipole. These results are the consequences of backward waves in such a medium. The phase velocity and the Pointing vector are in opposite direction for a backward wave. In 2000 Pendry claimed that the Veselago quasi-lens must possess an extraordinary property to amplify and focus the evanescent waves in the spatial spectrum of a point source and therefore this lens must be perfect [2]. Since then much attention has been attracted to this research area both theoretically and experimentally (see e.g. [3]–[7]). Such negative media do not exist in nature but have already got different names in the literature such as backward-wave media, the Veselago media, left-handed media and negative meta-materials. Since it is not clear yet how to make such media in the optical range of frequencies, the experimental efforts have been concentrated on creating negative meta-materials at microwave frequencies. A structure allowing the propagation of backward waves within a certain frequency band has been suggested in [4] and then experimentally studied in [3], where the negative refraction phenomenon was primarily observed. However, the structure of [4] (a combination of two lattices: a lattice of infinitely long parallel wires and a lattice of the so-called split-ring resonators) is not isotropic. This structure can be described with negative permittivity and permeability only if the propagation direction is orthogonal to the axes of wires [5].

To the best of our knowledge, the only isotropic medium with possible negative permittivity and permeability (within a certain frequency band) that has been reported is formed by many isotropic cubic cells of Ω particles [6]. Each isotropic cubic cell is made by putting 6 Ω-shaped perfectly conducting particles
on its faces (as shown in Fig. 1). Fig. 1 shows how to make an isotropic unit cell by putting successively 3 pairs of Ω-particles on the opposite faces of a cubic unit cell in an appropriate way. Then each cubic cell can be described approximately as an isotropic resonant scatterer. The frequencies at which both the effective permittivity and permeability of such a composite medium are negative are within the resonant band of each individual isotropic scatterer. Note that conducting Ω-particles were first suggested in [8] for creating the bianisotropic composites.

In the present paper we consider a negative meta-material formed by a rectangular lattice of isotropic cubic unit cells of Ω particles (as shown in Fig. 1). We introduce an analytical model to describe the dispersion properties of the lattice and calculate correctly the effective permittivity $\epsilon_{\text{eff}}(\omega)$ and permeability $\mu_{\text{eff}}(\omega)$ in the frequency band where the lattice can be homogenized. We compare the results obtained by the present model with the previous results obtained by the Maxwell Garnett model, and show that the latter cannot give reliable values for either the real parts or the imaginary parts of the effective material parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$.

From the negative dispersion of the lattice at low frequencies we obtain the frequency range of backward waves within the resonant band of the isotropic scatterers. In this frequency range the phase velocity is in an opposite direction of the group velocity. Our calculations show that this backward wave range is exactly the same as the frequency range within which both $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are negative.

The case when the lattice is formed by many parallel bianisotropic Ω-particles is also considered in an appendix. It is shown that the bianisotropy of the lattice particles reduces the possibility for the existence of backward waves.

2 Polarizability of an isotropic cubic unit cell

For bianisotropic particles (a Ω-particle is a special case), one has the following four dyadic polarizability matrices relating the electric dipole moment $\mathbf{p}$ and
the magnetic dipole moment \( m \) with the local electric and magnetic fields:

\[
p = \overline{\alpha}_{ee} \cdot \mathbf{E}^{loc} + \overline{\alpha}_{em} \cdot \mathbf{H}^{loc},
\]

\[
m = \overline{\alpha}_{me} \cdot \mathbf{E}^{loc} + \overline{\alpha}_{mm} \cdot \mathbf{H}^{loc}.
\]

For reciprocal particles, one has \( \overline{\alpha}_{me} = -\overline{\alpha}^{T}_{me} \).

For a cubic unit cell of \( \Omega \) particles as shown in Fig. 1, the situation is quite different. The bianisotropy cancels out since e.g. each pair of opposite \( \Omega \) particles shown in Fig. 1 have equal absolute value but opposite sign for \( \overline{\alpha}_{em} \). The polarizabilities \( \overline{\alpha}_{ee} \) and \( \overline{\alpha}_{mm} \) of a cubic unit cell of 6 \( \Omega \)-particles (see Fig. 1) are the sums of the polarizabilities \( \overline{\alpha}^{\Omega}_{ee} \) and \( \overline{\alpha}^{\Omega}_{mm} \) for individual particles [6]. The influence of the mutual coupling of the particles is small and the resonant frequency for such a cubic unit cell of 6 \( \Omega \)-particles has only a small shift from that for an isolated \( \Omega \)-particle. The resonant excitation by an electric field directed along the arms of \( \Omega \) particles is mainly due to the presence of the arms. The resonant behavior retains for the total electric polarization of the two opposite omega particles. Within the resonant band, the quasi-static polarizability component \( a^{\Omega \Omega}_{ee} \) (which describes the response of the particle to an electric field normal to the direction of the arms; see Fig. 1(a)) of the \( \Omega \) particle is small as compared to \( a^{\Omega \Omega}_{ee} \). Such a cubic unit cell can be therefore considered as an isotropic scatterer and thus one has \( a^{\Omega \Omega}_{ee} = a^{\Omega \Omega}_{yy} = a^{\Omega \Omega}_{zz} = a_{ee} \) and \( a^{\Omega \Omega}_{mm} = a^{\Omega \Omega}_{yy} = a^{\Omega \Omega}_{zz} = a_{mm} \) [6]. One can easily show that [6]

\[ a_{ee} \approx 2a^{\Omega \Omega}_{ee}, \quad a_{mm} \approx 2a^{\Omega \Omega}_{mm}. \]

The expressions for the parameters \( a^{\Omega \Omega}_{ee} \) and \( a^{\Omega \Omega}_{mm} \) have been given in [9]. In the present paper we consider a material formed by a rectangular lattice of such scatterers (see Fig. 1(b)).

The electric and magnetic polarizabilities for a cubic cell of \( \Omega \)-particles are calculated (cf. [6]) and shown in Fig. 2 as a numerical example. The geometric parameters for the \( \Omega \)-particles are chosen as \( r = 1.5 \) mm, \( w = 0.4 \) mm, \( h = 0.2 \) mm and \( l = 2 \) mm (see Fig. 1(a)). The size of the cubic unit cell is 4 mm. The relative permittivity for the background medium is chosen as \( \varepsilon_{b} = 1.5 \) (the permeability of the background medium is assumed be the same as the one for
vacuum in the present paper). Resonances of $a_{ee}$ and $a_{mm}$ (frequencies at which the real parts of these parameters become zero) occur at the frequency of 8.05 GHz and 8.14 GHz, respectively. We have chosen the time dependence $e^{j\omega t}$, and thus the imaginary parts of these parameters are negative. Below we consider a material formed by a rectangular lattice of such isotropic scatterers (see Fig. 1(b)).

3 Wrong results predicted by the Maxwell Garnett model for the regular lattice

In [6], the well-known Maxwell Garnett model was used and the following expressions for the material parameters were obtained,

$$\epsilon_{\text{eff}} = \epsilon_b + \frac{1}{F} \left( \frac{N a_{ee}}{\epsilon_0} - \frac{N^2 a_{ee} a_{mm}}{3 \epsilon_0 \epsilon_b \mu_0} \right), \quad (4)$$

$$\mu_{\text{eff}} = 1 + \frac{1}{F} \left( \frac{N a_{mm}}{\mu_0} - \frac{N^2 a_{ee} a_{mm}}{3 \epsilon_0 \epsilon_b \mu_0} \right), \quad (5)$$

where

$$F = 1 - \frac{N a_{ee}}{3 \epsilon_0} - \frac{N a_{mm}}{3 \mu_0} - \frac{N^2 a_{ee} a_{mm}}{9 \epsilon_0 \epsilon_b \mu_0}.$$ 

In the Maxwell Garnett model, one only needs to know the scatterer polarizabilities and the density $N$ of the scatterers. Thus the obtained result for a regular lattice of scatterers will be the same as that for a random distribution of scatterers if the density of the scatterers is the same. For the case considered in the present paper (i.e., a rectangular lattice of isotropic cubic unit cells as shown in Fig. 1(b) with periods $d_x$, $d_y$, $d_z$ along the Cartesian axes), one has $N = 1/d_x d_y d_z$. The frequency dependencies of $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ predicted by the Maxwell Garnett model are shown in Fig. 3 when $d_x = d_y = d_z = d = 8$ mm (i.e. $N = (1/8^3) \cdot 10^{-9}$ m$^{-3}$). From this figure one sees that both $\text{Re}(\mu_{\text{eff}})$ and $\text{Re}(\epsilon_{\text{eff}})$ are negative within a certain frequency band (8.34–8.62 GHz). However, within this frequency band the imaginary parts of $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ have high values (about $-4 \sim -6$). The radiation losses are so large that the obtained negative effective permittivity $\epsilon_{\text{eff}}$ and permeability $\mu_{\text{eff}}$ have no practical importance for applications.
The above results predicted by the Maxwell Garnett model (in [6] for a random arrangement of cubic unit cells) are not correct for the present case of the regular lattice. Within the resonant band (7.6-8.6 GHz) of each Ω-particle, the wavelength ($\lambda = 28 - 33$ mm) in the background medium ($\epsilon_b = 1.5$) is much larger than the distance $d$ (8 mm) between the isotropic cubic scatterers, and this distance is larger than the size (4 mm) of each scatterer. From this point of view one may expect that the homogenization is possible. However, the results predicted by the Maxwell Garnett model are inconsistent near the resonance of the scatterer. The Maxwell Garnett model is based on the Clausius-Mossotti relations for the electric and magnetic fields. These relations for cubic lattices of static dipoles have been found as good approximations in the works of Sivukhin [10] (for an infinite lattice and for a half-space) and McPhedran [11] (for arrays of finite sizes). The Clausius-Mossotti relations were also confirmed to be accurate enough for cubic lattices of dipoles in a time-harmonic case [12] under the condition that the wavelength is large compared to the lattice period $d$. However, the resonant case is not considered in these works. In the resonant case, the Maxwell Garnett model may give very high absolute values for the real parts of $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ within the resonant band, and this would lead to a dramatic shortening of the wavelength in the homogenized medium. The wavelength in the homogenized medium becomes comparable with the particle distance $d$ and thus the Maxwell Garnett model becomes contradictory at some frequencies.

In the present case (see Fig. 3) the absolute values of real parts of $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are so high within the narrow band 8.37-8.40 GHz where the Maxwell Garnett model becomes contradictory. Outside this narrow band $|\text{Re}(\epsilon_{\text{eff}})|$ and $|\text{Re}(\mu_{\text{eff}})|$ are not very high. However, the inconsistency of the Maxwell Garnett model for the present case of regular lattice is evident over the whole resonant band of Ω-particles (8.1-8.5 GHz). In Fig. 3 one can see that the imaginary parts of $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are rather significant at those frequencies. On the other hand, we know that $\text{Im}(\mu_{\text{eff}})$ and $\text{Im}(\epsilon_{\text{eff}})$ must be identically zeros physically if the lossless scatterers are arranged in a regular lattice. The cancelation of the radiation resistances of the scatterers (due to their electromagnetic interaction)
is obvious for an infinite regular array and thus there is no radiation loss at all (see e.g. [13]).

In the present paper, we introduce another homogenization model which gives correctly zero value for the imaginary parts of $\epsilon_{eff}$ and $\mu_{eff}$. A comparison in Section 5 will show that the real parts of $\epsilon_{eff}$ and $\mu_{eff}$ predicted by the Maxwell Garnett model are also wrong within the resonant band of the scatterers.

4 Dispersion equation for the lattice of the isotropic scatterers

Consider the rectangular lattice of the isotropic scatterers (shown in Fig. 1) with periods $d_x, d_y$ along the $x$ and $y$ axes and $d_z = d$ along the $z$ axis. In the present case, the electric dipole moment is orthogonal to the magnetic dipole moment for each scatterer. Let $p$ and $m$ denote the dipole moments of a reference scatterer located at the origin. Thus one has

$$ p = a_{ee}E^{loc}, \quad (6) $$
$$ m = a_{mm}H^{loc}. \quad (7) $$

We need to find the eigenmodes for such a lattice of isotropic scatterers. Assume the wave propagates along the $z$-axis. Let $\beta$ denote the propagation constant of an eigenmode. The first Brillouin zone is the interval $0 \leq \beta d \leq \pi$. Then $p(n_x,n_y,n_z)$ and $m(n_x,n_y,n_z)$ (the electric and magnetic dipole moments of a scatterer located at a lattice node with coordinates $x = n_x d_x, y = n_y d_y, z = n_z d$) can be expressed through $p$ and $m$ (for the reference scatterer) as

$$ p(n_x,n_y,n_z) = pe^{-jn_z\beta d}, \quad (8) $$
$$ m(n_x,n_y,n_z) = me^{-jn_z\beta d}. \quad (9) $$

Since eigenwaves are linearly polarized in the present case, we can assume without loss of generality that $p = px_0$ and $m = my_0$ (see also Fig. 1). Then the local electric and magnetic fields are directed along the $x$ and $y$ directions, respectively (the local field has the same polarization as the eigenmode). This
allows the following scalar expressions for the local field amplitudes in terms of \( p \) and \( m \):

\[
E_{\text{loc}} = Ap + Bm, \tag{10}
\]

\[
H_{\text{loc}} = Cm + Dp, \tag{11}
\]

where \( A, B, C, D \) are the so-called interaction factors of the lattice, which depend only on the lattice geometry, the frequency \( \omega \) and the propagation factor \( \beta \) (they do not depend on the polarizabilities of the scatterers). From the reciprocity it follows that \( B = D \). From the duality it follows that \( C = A/\eta^2 \), where \( \eta = \sqrt{\mu_0/\epsilon_0} \) is the wave impedance of the background medium.

From Eqs. (6), (7), (10) and (11), one obtains the following two important relations,

\[
p(1 - a_{ee}A) = a_{ee}Dm, \tag{12}
\]

\[
m(1 - a_{mm}A/\eta^2) = a_{mm}Dp. \tag{13}
\]

Denote the dimensionless ratio \( \eta p/m \) as \( \alpha \), then we have

\[
\alpha = \eta p/m = \frac{(\frac{\eta^2}{a_{mm}} - \frac{A}{\eta})}{D}. \tag{14}
\]

This formula will be used in the next section to find the effective material parameters of the lattice at low frequencies.

From Eqs. (12) and (13) one obtains

\[
\left( \frac{1}{a_{ee}} - A \right) \left( \frac{\eta^2}{a_{mm}} - A \right) = \eta^2 D. \tag{15}
\]

The relation

\[
\text{Im} \left( \frac{1}{a_{ee}} \right) = \frac{\omega^3(\epsilon_0\mu_0)^{\frac{3}{2}}}{6\pi\epsilon_0} \tag{16}
\]

was first obtained in [13] for a particle re-radiating the light and later was reproduced as the consequence of the energy conservation for a lossless dipole scatterer (see e.g. [14]). For lossless magnetic scatterers, one has the following corresponding form,

\[
\text{Im} \left( \frac{1}{a_{mm}} \right) = \frac{\omega^3(\epsilon_0\mu_0)^{\frac{3}{2}}}{6\pi\mu_0}. \tag{16}
\]
For the interaction factor $A$, an explicit approximation has been found in [14]:

$$A = \frac{\eta \omega}{2d_x d_y} \left( q_0 + \frac{\sin kd}{\cos kd - \cos \beta d} \right) + j \frac{k^3}{6\pi \epsilon_0 \epsilon_b},$$  \hspace{1cm} (17)

where $k = \omega \sqrt{\epsilon_0 \mu_0 \epsilon_b}$ is the wavenumber in the background medium, and $q_0$ is the real part of the dimensionless interaction factor of a 2D grid of dipoles with periods $d_x, d_y$. $q_0$ has the following explicit expression when $d_x = d_y = a$ [14],

$$q_0 = \frac{1}{2} \left( \frac{\cos k a s}{k a s} - \sin k a s \right),$$  \hspace{1cm} (18)

where $s \approx 1/1.4380 = 0.6954$. Relation (17) is almost exact when $d \gg d_x, d_y$, and has only a small error when $d = d_x = d_y$ [14].

Since $\text{Im} A = \text{Im} \left( \frac{1}{a_{ee}} \right) = \text{Im} \left( \frac{\eta^2}{a_{mm}} \right)$, the left-hand side of (15) is real. Thus we have

$$\text{Re} \left( \frac{1}{a_{ee}} - A \right) \text{Re} \left( \frac{\eta^2}{a_{mm}} - A \right) = \eta^2 D.$$  \hspace{1cm} (19)

Below we show that $D$ is also real and we derive its explicit expression. From the definition of $D$, one has

$$H^\text{loc}_y(x = 0, y = 0, z = 0) = \sum_{n_x} \sum_{n_y} \sum_{n_z} H_y(n_x d_x, n_y d_y, n_z d) \equiv D p,$$  \hspace{1cm} (20)

where $H_y(n_x d_x, n_y d_y, n_z d)$ is the $y$-component of the magnetic field at the origin produced by the $x$-polarized electric dipole with dipole moment $p(n_x, n_y, n_z) = p(n_x, n_y, n_z)\mathbf{x}_0$. From the reciprocity we know that $H_y(n_x d_x, n_y d_y, n_z d)$ is equal to the field at the point $(x = n_x d_x, y = n_y d_y, z = n_z d)$ produced by the reference dipole $p$. In Eq. (20) the summation is over all integers $(n_x, n_y, n_z)$ from $-\infty$ to $\infty$ except $n_x = n_y = n_z = 0$. Here we use the following plane-wave representation for the magnetic field produced by an electric dipole $p = p\mathbf{x}_0$ located at the origin [16]:

$$H_y(x, y, z) = \frac{j \omega}{8\pi^2} \Psi(z) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(q_x x + q_y y) - j^2q_z^2 - q_z^2} dq_x dq_y,$$  \hspace{1cm} (21)

where $\Psi(z) = +1$ for $z > 0$, $\Psi(z) = -1$ for $z < 0$ and $\Psi(z) = 0$ for $z = 0$. Substituting Eqs. (21) and (8) into definition (20) and changing the order of
the summation over \( n_z \) and the integration over \( q_x \) and \( q_y \), one obtains

\[
D = \frac{2j\omega}{8\pi^2} \sum_{n_x} \sum_{n_y} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(q_x x + q_y y)} \sum_{n_z=1}^{\infty} e^{-jn_z \beta d + jn_z d \sqrt{k^2 - q_x^2 - q_y^2}} dq_x dq_y.
\]

Summarizing the geometrical series and using known explicit formulas for the integration over \( q_x \) and \( q_y \), one can obtain the following final result for \( D \),

\[
D = -\frac{\omega}{2d_x d_y} \sum_{n_x} \sum_{n_y} \frac{\sin \beta d}{\cos \sqrt{k^2 - \left(\frac{2\pi n_x}{d_x}\right)^2 - \left(\frac{2\pi n_y}{d_y}\right)^2} \ d - \cos \beta d}.
\]  

(22)

Numerical calculations have shown that the contribution from the terms with \( n_x, n_y = \pm 1, \pm 2, \ldots \) becomes significant only if \( d_x, d_y \gg d \). These terms correspond to the effect of the evanescent Floquet spatial harmonics generated by the 2D grids of dipoles with \( n_z = \pm 1, \pm 2, \ldots \) (the reference dipole is located in the 2D grid with \( n_z = 0 \)). If we consider the case \( d_x = d_y = a \leq d \), then we can neglect these terms and take into account only the plane-wave interaction between the 2D grids of dipoles. For such a case, one has the following real-valued expression for \( D \),

\[
D = -\frac{\omega}{2d_x d_y} \frac{\sin \beta d}{\cos \beta d}.
\]  

(23)

Note that relation (17) was derived in [14] under the same condition. Formulas (17) and (23) are both approximate but self-consistent. Therefore, these relations lead to the real-valued dispersion equation (19).

Substituting (17) and (23) into (19), one obtains

\[
\cos^2 \beta d (1 + \gamma_1 \gamma_2) - \cos \beta d (2 \gamma_1 \gamma_2 \cos kd - (\gamma_1 + \gamma_2) \sin kd)
\]

\[-(1 - \gamma_1 \gamma_2) \cos^2 kd - (\gamma_1 + \gamma_2) \sin kd \cos kd = 0,
\]  

\[
(24)
\]

where

\[
\gamma_1 = \text{Re} \left( \eta \frac{a_{ee}}{\eta \omega} \right) 2d_x d_y / \eta \omega, \quad \gamma_2 = \text{Re} \left( \eta^2 m \right) 2d_x d_y / \eta \omega.
\]

The dispersion relation (24) has two roots for a fixed frequency. One of the roots satisfies the following condition (obtained from Eqs. (12) and (13))

\[
\left( \frac{\eta^2 m}{\eta a_{ee}} - A \right) \frac{\eta D}{\left( \frac{\eta a_{ee}}{\eta} - A \right)} = 0.
\]  

11
This root is $\beta$ that we show in our dispersion plots below. The other root does not satisfy the above condition and thus is spurious.

Fig. 4 shows the dispersion for the same cubic lattice of isotropic scatterers as used for Fig. 3. From this figure one can see that there is a band (between 8.14 and 8.37 GHz) in which the group velocity is negative. This is the backward-wave band. Since the Poynting vector in a lossless medium must be directed along the group velocity, this band corresponds to the case when the energy transports oppositely to the phase. Of course, this situation would be trivial for some high-frequency dispersion branches. For a lattice of dipoles, negative dispersion is inherent for every even dispersion branch $(\pi(2n+1) < kd < \pi 2(n + 1)\pi$, where $n = 0, 1, 2 \ldots$). However, only in the first frequency zone (where $0 < kd < \pi$) the lattice period is smaller than $\lambda/2$ (half of the wavelength in the background medium) and thus the homogenization of the lattice is possible. Only if the lattice of scatterers can be homogenized, the material parameters $\epsilon_{eff}$ and $\mu_{eff}$ can be introduced and the lattice of scatterers can be interpreted as a continuous medium. In the Veselago theory [1] the backward wave in a continuous medium should correspond to a case when both $\epsilon_{eff}$ and $\mu_{eff}$ are negative. The negative dispersion in the homogenized structure implies the backward wave and therefore our $\epsilon_{eff}$ and $\mu_{eff}$ should be negative if the Veselago theory is applicable for dispersive media.

Fig. 5 gives an enlarged view of Fig. 4 for the frequency dependence of $Re\beta$ and $Im\beta$ in the frequency range 7-10 GHz. From this figure one sees that there are two stopbands within the resonant band of each scatterer (cf. Fig. 2). The lower one (stopband 1) is very narrow (8.06-8.14 GHz), and the higher (stopband 2) is wider (8.37-8.52 GHz). Between these two stopbands there is the band of backward waves. Stopband 2 is a conventional lattice stopband which has $Re\beta = 0$. Stopband 1 corresponds to the so-called complex mode which is known in the theory for lattices of infinite wires (see e.g. [17],[18]). This complex mode is decaying and the real part of the propagation factor is identically equal to $\pi/d$. In our case, the relation $Re\beta = \pi/d$ reflects the fact that the directions of the dipole moments of the scatterers are alternating along the propagation axis (i.e., two adjacent isotropic scatterers have opposite polarizations).
Substituting relations (17) and (23) into Eq. (14), one obtains the following explicit formula for the parameter $\alpha$,

$$\alpha(\omega) = \left[ q_0 - \frac{2d_x d_y \eta}{\omega} \Re \left( \frac{1}{a_{mm}} \right) \right] \frac{(\cos kd - \cos \beta d) + \sin kd}{\sin \beta d}. \quad (25)$$

Here $a_{mm}$ is a function of $\omega$ for given geometrical parameters of scatterers and given $\epsilon_b$. Since the frequency dependence of $\beta(\omega)$ is known from the solution of the dispersion equation (24), the frequency dependence of $\alpha(\omega)$ can be calculated from the above equation. In the next section we will use this frequency dependence of $\alpha(\omega)$ to find the explicit expressions for the effective material parameters $\epsilon_{eff}(\omega)$ and $\mu_{eff}(\omega)$ of the homogenized lattice.

Our calculations show that $\alpha$ is also a crucial parameter for the frequency bounds of the backward-wave range. The central frequencies of both stopbands are frequencies at which $\alpha(\omega)$ changes the sign. It is positive below 8.1 GHz, negative in 8.1-8.45 GHz (backward-wave range) and positive again above 8.45 GHz. Both stopbands are centered by the 2 zero-points of $\alpha(\omega)$. Of course, both stopbands are within the resonant band of each isotropic scatterer.

5 Homogenization of the lattice of the isotropic scatterers

Consider the frequency band $0 < kd \leq \pi$ within which homogenization is possible and effective parameters $\epsilon_{eff}$ and $\mu_{eff}$ can be, perhaps, introduced. To determine the frequency dependence of the two parameters $\epsilon_{eff}(\omega)$ and $\mu_{eff}(\omega)$, we need to find two relations (involving the propagation constant $\beta(\omega)$) between these two parameters. The first relation can be obtained by fitting the value $\beta/k$ with the effective refraction index $n_{eff} = \sqrt{\epsilon_{eff} \mu_{eff}}$, i.e., one has

$$\mu_{eff} = \frac{\beta^2}{k^2 \epsilon_{eff}}. \quad (26)$$

Note that $k$ is proportional to $\omega$ (i.e., $k = \omega \sqrt{\epsilon_0 \mu_0 \epsilon_b}$) and $\beta = \beta(\omega)$ is given by the dispersion curve calculated in the previous section.

To find another relation, we define the following averaged (over a cubic unit
cell) electric and magnetic polarizations per unit volume:

\[ P = \frac{p}{V}, \quad M = \frac{m}{V}, \]

where \( V = d^3 \) (in this section we assume that the lattice periods are the same in all the three directions, i.e., \( d_x = d_y = d_z = d \); otherwise the lattice will be anisotropic). In the same way we can introduce the following averaged electric and magnetic fields,

\[ < E > = \frac{1}{V} \int_E V, \quad < H > = \frac{1}{V} \int_H V. \]

Since the material considered here is isotropic, its wave impedance \( Z \) can be expressed as the ratio \( < E > / < H > \), i.e.,

\[ \frac{< E >}{< H >} = Z. \quad (27) \]

If we use the conventional expression \( Z = \sqrt{\frac{\mu_0 \mu_{eff}}{\epsilon_0 \epsilon_{eff}}} \), the square root can be chosen either positive (in the range of forward waves) or negative (in the range of backward waves; the ratio \( \mu_{eff}/\epsilon_{eff} \) is always positive). For a backward wave in a continuous isotropic medium, the vectors \( < E > = Ex_0 \) and \( < H > = Hy_0 \) form a left-hand triad with the wave vector \( K = \beta z_0 \) [1], and thus the wave impedance should be negative. Substituting (26) into \( Z = \sqrt{\frac{\mu_0 \mu_{eff}}{\epsilon_0 \epsilon_{eff}}} \), one obtains

\[ Z = \frac{\beta}{\epsilon_0 k \epsilon_{eff}}. \quad (28) \]

The above expression is more convenient for use since it gives automatically a negative impedance for a backward wave (when \( \epsilon_{eff} < 0 \)).

From the definitions for the effective material parameters, one has

\[ \epsilon_0 \epsilon_{eff} < E > = \epsilon_0 \epsilon_b < E > + P, \quad \mu_0 \mu_{eff} < H > = \mu_0 < H > + M. \]

Using Eqs. (27) and (28) and taking into account the above relations, one can express the dimensionless parameter \( \alpha = \eta P/M \) (determined by Eq. (25)) as

\[ \alpha = \frac{\eta P}{M} = \frac{\eta}{\mu_0} \frac{\epsilon_0 (\epsilon_{eff} - \epsilon_b) < E >}{\mu_0 (\mu_{eff} - 1) < H >} = \frac{\epsilon_0 (\epsilon_{eff} - \epsilon_b) Z}{\mu_0 (\mu_{eff} - 1) \sqrt{\epsilon_b (\mu_{eff} - 1) k \epsilon_{eff}}}. \quad (29) \]
The above equation gives the second required relation between the two effective parameters.

Finally, from the two relations (26) and (29) we obtain the following expression for determining $\epsilon_{\text{eff}}(\omega)$ from the dispersion relation $\beta(\omega)$,

$$
\epsilon_{\text{eff}}(\omega) = \frac{\beta^2}{k^2} + \frac{\beta}{k\alpha \sqrt{\epsilon_b}} \sqrt{1 + \frac{\beta}{k\alpha \sqrt{\epsilon_b}}}.
$$

(30)

Note that $k = \omega \sqrt{\epsilon_0 \mu_0 \epsilon_b}$ and $\alpha = \alpha(\omega)$ is given by formula (25). After $\epsilon_{\text{eff}}(\omega)$ is calculated, $\mu_{\text{eff}}(\omega)$ is then calculated from Eq. (26).

Fig. 6 shows the effective permittivity and permeability for the same lattice of isotropic scatterers as used before in Figs. 3–5. From this figure one sees that both $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are negative in the band of backward waves. This is consistent to the Veselago theory.

In any of the two stopbands, the value $\beta$ contains non-zero imaginary part and thus Eqs. (30) and (26) give complex values for $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ at these frequencies. Since the structure is lossless, $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ must be real at the frequencies for which the homogenization is possible. Therefore, we can conclude that the homogenization can not be performed within the stopbands. That’s why in Fig. 6 no value is given for $\epsilon_{\text{eff}}$ or $\mu_{\text{eff}}$ in the stopbands. From Fig. 6 one sees that outside the two stopbands the absolute values of $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ are not very high. The wavelength in the homogenized medium (which is equal to $2\pi/\beta$) is not dramatically shortened within the passbands (in which $\beta d < \pi$). This indicates that the present homogenization model is self-consistent (unlike expressions (4) and (5) derived from the Maxwell Garnett model). By comparison Fig. 6 with Fig. 3, we can conclude that for a lattice of resonant particles the Maxwell Garnett model can not give reliable values for either the real parts or the imaginary parts for the effective material parameters ($\mu_{\text{eff}}$) and ($\epsilon_{\text{eff}}$). The Maxwell Garnett model can only predicts qualitatively the location of the frequency band where the effective permittivity and permeability have negative real parts.
6 Conclusion

In the present paper, we have presented an analytical model for an isotropic negative meta-material, which is formed by a rectangular lattice of isotropic scatterers with electric and magnetic resonances. Each isotropic scatterer is formed by putting 3 pairs of Ω-particles on the opposite faces of a cubic unit cell in an appropriate way. We have derived a self-consistent dispersion equation and studied the dispersion properties of the lattice. The obtained dispersion curves are then used to calculate correctly the effective permittivity and permeability in the frequency band where the lattice can be homogenized. It is interesting to see that the dispersion curves agree well with the Veselago theory which predicts backward waves when both permittivity and permeability are negative. The Veselago theory is non-complete (it was only for monochromatic waves and did not take into account the dispersion of the medium) and some questions are left open. It has been shown in the present paper that the frequency range in which both the effective permittivity and permeability are negative corresponds to the mini-band of negative dispersion of the lattice at low frequencies within the resonant band of the individual isotropic scatterer. It has been shown that the frequency range of backward waves is determined by the edges of the two stopbands at low frequencies. We have compared the results obtained by the present model with the previous results obtained by the Maxwell Garnett model, and shown that the latter can not give reliable values for either the real part or the imaginary part for any effective material parameter ($\mu_{eff}$ or $\varepsilon_{eff}$) of the lattice. For lattices of resonant scatterers, the Maxwell Garnett model can only predict approximately the location of the frequency band where the effective permittivity and permeability have negative real parts.

The case when the lattice is formed by parallel bianisotropic Ω-particles has also been studied (in the appendix). It has been shown that the bianisotropy of the lattice particles suppresses those terms in the dispersion equation that can allow a backward wave propagation and thus reduces the possibility for the existence of backward waves. Therefore, the present formulas for calculating correctly $\varepsilon_{eff}$ and $\mu_{eff}$ for the negative meta-material from resonant isotropic scatterers (the only isotropic negative meta-material that we know) is of impor-
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Appendix: The case of a lattice of bianisotropic particles

In this appendix, we present briefly our studies for the case when the lattice is formed by an infinite number of parallel Ω-particles. We also explain why the magnetoelectric coupling of these bianisotropic particles reduce the possibility for the existence of backward waves.

We use a simplified wire-and-loop model for each Ω-particle, in which we assume that the current flowing around the loop is uniform (i.e., the electric polarization of the loop is neglected).

If the particle is excited by a $x$-directed electric field $E^{\text{loc}}$ (see Fig. 1), then the current in the loop and the straight wire arms will be equal to the induced voltage in the straight wire portion divided by the impedance for the series connection of the loop impedance $Z_l$ and the straight wire impedance $Z_w$, i.e.,

$$I = \frac{\mathcal{E}}{Z_l + Z_w} = \frac{E^{\text{loc}}l}{Z_l + Z_w}.$$

Thus one obtains the following electric and magnetoelectric polarizabilities,

$$a_{ee} = \frac{H}{j\omega E^{\text{loc}}} = -j \frac{l^2}{Z_l + Z_w},$$

$$a_{me} = \frac{l\mu_0 S}{E^{\text{loc}}} = \frac{\mu_0 S l}{Z_l + Z_w},$$

where $S = \pi r^2$ is the loop area.

Now let the particle be excited by a $y$-directed local magnetic field $H^{\text{loc}}$. An electromotive force (i.e., voltage $\mathcal{E}$) is induced in the loop, and one has

$$I = \frac{\mathcal{E}}{Z_l + Z_w} = -j\omega\mu_0 S H^{\text{loc}}.$$

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Thus, one has electromagnetic and magnetic polarizabilities

\[
a_{em} \equiv \frac{H}{j\omega H_{loc}} = -\frac{\mu_0 l S}{Z_l + Z_w},
\]

\[
a_{mm} \equiv \frac{I\mu_0 S}{H_{loc}} = \frac{j\omega \mu_0^2 s^2}{Z_l + Z_w}.
\]

From the above expressions for the polarizabilities, the following identities can be obtained:

\[
a_{ee} a_{mm} = a_{em} a_{me}, \quad (A1)
\]

\[
a_{me} = -a_{em}. \quad (A2)
\]

The second identity is due to the reciprocity property of the particle. For such a bianisotropic particle (local electric field is \(x\)-polarized and magnetic field is \(y\)-polarized), we have (cf. Eqs. (6) and (7))

\[
p = a_{ee} E_{loc}^{\text{el}} + a_{em} H_{loc}^{\text{m}}, \quad (A3)
\]

\[
m = a_{me} E_{loc}^{\text{el}} + a_{mm} H_{loc}^{\text{m}}. \quad (A4)
\]

In the rectangular lattice of parallel \(\Omega\)-particles, the electric dipole moment of each particle is directed along the \(x\)-axis and the magnetic dipole moment is directed along the \(y\)-axis when the eigenmode with \(E = E X_0\) and \(H = H Y_0\) propagates along the \(z\)-axis (the local field has the same polarization as the eigenmode). Substituting relations (10) and (11) into Eqs. (A3) and (A4), one obtains

\[
p(1 - a_{ee} A - a_{em} D) = m(a_{ee} D + a_{em} A), \quad (A5)
\]

\[
m(1 - a_{mm} \frac{A}{\eta^2} - a_{me} D) = p(a_{mm} D + a_{em} A). \quad (A6)
\]

The above two equations give

\[
(1 - a_{ee} A - a_{em} D)(1 - a_{mm} \frac{A}{\eta^2} - a_{me} D) = (a_{ee} D + a_{em} A)(a_{mm} D + a_{em} A).
\]

Since all terms containing \(D\) cancel out due to identities (A1) and (A2), one obtains

\[
1 - a_{ee} A - a_{mm} \frac{A}{\eta^2} = 0,
\]

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which can be rewritten in the following form

\[ \frac{1}{a_{ee} + \frac{a_{mm}}{\eta}} = A. \]  

\[ (A7) \]

For a bianisotropic particle, Eq. (16) is generalized to the following relation [19],

\[ \text{Im} \left( \frac{1}{a_{ee} + \frac{a_{mm}}{\eta^2}} \right) = \frac{\omega^3(\epsilon_0\mu_0)^{\frac{3}{2}}}{6\pi\epsilon_0\epsilon_b}. \]

\[ (A8) \]

Substituting Eq. (17) into Eq. (A7) and using Eq. (A8), one obtains the following dispersion relation,

\[ \cos \beta d = \cos kd + \frac{\sin \beta d}{X}, \]

\[ (A9) \]

where

\[ X = \frac{2d_x d_y \text{Re} \left( \frac{1}{a_{ee} + \frac{a_{mm}}{\eta^2}} - q_0 \right)}{\omega \eta}. \]

Eq. (A9) has the same form as the classical dispersion equation for a periodically loaded line [20]), which is also known as the Kronig-Pennie equation in the solid-state physics. In the known cases described by this equation (see e.g. [20]) the negative dispersion exists only within the even zones of frequencies \((2\pi n < kd < \pi(2n + 1)), n = 1, 2, 3, \ldots\). We did not find any physically realizable \(\Omega\)-particle for which the above dispersion relation would give a negative dispersion within the first frequency zone \((0 < kd < \pi)\).

In the above analysis, the model for the \(\Omega\)-particle is rough (since the electric polarization of the loop is neglected) and identity (A1) is approximate. We have studied the lattice of parallel \(\Omega\)-particles using a more accurate model (as used in [9]). This accurate model leads to another dispersion equation (different from Eq. (A9)) which is quadratic with respect to \(\cos \beta d\) and contains the electromagnetic interaction factor \(D\). Again, we did not find any physically realizable \(\Omega\)-particle for which this dispersion relation would give a negative dispersion within the first frequency zone \((0 < kd < \pi)\). The error in Eq. (A1) for real \(\Omega\)-particles is quite small and the terms containing \(D\) in this dispersion equation are relatively small. Neglecting these small terms and removing the spurious root, we obtained the same result as what Eq. (A9) leads to.
Therefore, we can conclude that the bianisotropy of the lattice particles suppresses those terms in the dispersion equation that can allow a backward wave propagation at low frequencies. The possibility for the existence of backward waves is thus reduced (it was reported with another reason in [21] that the bianisotropy is not good for the existence of backward waves).

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Figure captions

**Figure 1:**
(a) Geometry for a cubic unit cell of Ω-particles. For graphic clearness, only 1 pair of Ω-shaped perfectly conducting particles are shown on 2 opposite faces of the cubic unit cell. (b) A rectangular lattice of isotropic cubic unit cells of Ω-particles.

**Figure 2:**
Left: The real (solid lines) and imaginary (dashed lines) parts of the electric polarizability (left figure) and the magnetic polarizability (right figure) for an isotropic scatterer (a cubic unit cell of Ω-particles).
Figure 3: The frequency dependencies of $\varepsilon_{eff}$ and $\mu_{eff}$ predicted (incorrectly for the regular lattice) by the Maxwell Garnett model for the negative meta-material formed by a lattice of isotropic cubic unit cells of $\Omega$-particles (as shown in Fig. 1(b)). Re($\varepsilon_{eff}$) (solid line), Re($\mu_{eff}$) (dashed line), Im($\varepsilon_{eff}$) (circles), and Im($\mu_{eff}$) (crosses) are shown.

Figure 4: The dispersion (indicated by the circles) for the same lattice of isotropic resonant scatterers as used for Fig. 3. The thin lines indicates the dispersion of the
homogenous background medium.

Figure 5: An enlarged view of Fig. 4 for the frequency dependence of \( \text{Re} \beta \) (crosses) and \( \text{Im} \beta \) (circles) near the resonance. The thin line indicates the dispersion of the homogenous background medium.

Figure 6: The frequency dependencies of the effective permittivity \( \varepsilon_{\text{eff}} \) (circles) and the effective permeability \( \mu_{\text{eff}} \) (crosses) for the same lattice of isotropic resonant scatterers as considered in Figs. 3–5.