On the smallest singular value in the class of unit lower triangular matrices with entries in \([-a, a]\)

Abstract: Given a real number \(a \geq 1\), let \(K_n(a)\) be the set of all \(n \times n\) unit lower triangular matrices with each element in the interval \([-a, a]\). Denoting by \(\lambda_n(\cdot)\) the smallest eigenvalue of a given matrix, let \(c_n(a) = \min \{\lambda_n(YY^T) : Y \in K_n(a)\}\). Then \(\sqrt{c_n(a)}\) is the smallest singular value in \(K_n(a)\). We find all minimizing matrices. Moreover, we study the asymptotic behavior of \(c_n(a)\) as \(n \to \infty\). Finally, replacing \([-a, a]\) with \([a, b]\), \(a < b\), we present an open question: Can our results be generalized in this extension?

Keywords: real symmetric matrix, unit lower triangular matrix, GCD and LCM matrix, smallest eigenvalue, smallest singular value

MSC: 11C20, 15A18, 15A42, 15B36

1 Introduction

Let \(S = \{x_1, x_2, \ldots, x_n\}\) be a set of distinct positive integers and let \((x_i, y_i)\) and \([x_i, y_i]\) denote the greatest common divisor and the least common multiple of \(x_i\) and \(y_i\), respectively. The \(n \times n\) matrices \((S) = ((x_i, x_j))\) and \([S] = ((x_i, y_j))\) are called the GCD matrix and the LCM matrix on \(S\), respectively. Many results on these matrices, their various generalizations and relatives have been published in the literature. For general accounts, see [2–4, 6, 8, 12] and the references therein.

One of the richest topics in the study of GCD and LCM matrices is their spectral properties. In this frame, Hong and Loewy [6] studied the asymptotic behavior of the eigenvalues of power GCD matrices and introduced a new parameter to present a lower bound for the smallest eigenvalue of the power GCD matrix. They defined the numbers \(c_n\) depending only on \(n\) as follows:

\[
c_n = \min \{\lambda_n(YY^T) : Y \in K_n\},
\]

where \(\lambda_n(\cdot)\) is the smallest eigenvalue and \(K_n\) is the set of all \(n \times n\) nonsingular lower triangular (0, 1)-matrices. Then, in the light of their MATLAB calculations for \(n = 2, 3, \ldots, 7\), Ilmonen, Haukkanen and Merikoski [8] conjectured that \(c_n(a) = \lambda_n(Y_0 Y_0^T)\), where \(Y_0 = (y_{ij}) \in K_n\) is defined by \(y_{ij} = \frac{1}{1-i/j}\) if \(i > j\). Verifying the truth of the conjecture for \(n = 8\) and \(9\) in [1], the author of the present paper, Keskin, Yıldız and Demirbükęen [4] proved the conjecture and realized that there is only one matrix \(Y \in K_n\) for which \(c_n\) is attained. Therefore, it was conjectured that if \(c_n = \lambda_n(YY^T)\) for \(Y \in K_n\), then \(Y = Y_0\) [4, Conjecture 3.1]. Recently, Loewy [11] has proved the conjecture.

Due to the importance of the study of spectral properties of GCD and related matrices, some authors tried to find bounds for \(c_n\). In this frame, assuming the truth of the conjecture, Mattila [12] obtained the following
lower bound
\[ c_n \geq \left( \frac{48}{n^4 + 56n^2 + 48n} \right)^{\frac{1}{2}} \text{ for even } n, \]
\[ c_n \geq \left( \frac{48}{n^4 + 50n^2 + 48n - 51} \right)^{\frac{1}{2}} \text{ for odd } n. \]

Beside this, Merikoski, who contributed to the paper [4], improved Mattila’s lower bound as follows:
\[ c_n \geq \frac{2}{2F_n F_{n+1} + 1 + (-1)^n}, \]
where \( F_n \) is the \( n \)th Fibonacci number. In addition to these bounds, in [2] the inequality
\[ c_n \geq \left( n \sum_{k=1}^{n} \mu^2(k) \right)^{-1}, \]
where \( \mu \) is the Möbius function, was proposed for \( c_n \) by using a different method. Kaarnioja [9] improved the lower bound further and showed that
\[ c_n \geq \left( \frac{1}{25} \varphi^{-4n} + \frac{3 + (-1)^n}{25} \varphi^{-2n} - \frac{2}{5\sqrt{5}} n\varphi^{-2n} + \frac{13(-1)^n - 33}{50} + n + \frac{3 + (-1)^n}{25} \varphi^{2n} + \frac{2}{5\sqrt{5}} n\varphi^{2n} + \frac{1}{25} \varphi^{4n} \right)^{-\frac{1}{2}}, \]
where \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio. He also conjectured asymptotically that
\[ c_n \sim \frac{5}{\varphi^{2n}}, \]
which was proved by Loewy [11].

This paper has two main goals. The first is to expand the existing results on the process of finding all matrices for which \( \sqrt{c_n} \) is attained. The second is to determine the asymptotic behaviour of \( c_n \) in a larger class of matrices. Given a real number \( a \geq 1 \), let \( K_n(a) \) denote the set of all \( n \times n \) unit lower triangular matrices whose each element under diagonal is in the interval \([-a, a]\). We note that all diagonal entries of a unit lower triangular matrix are \( 1 \). Let
\[ c_n(a) = \min \left\{ \lambda_n(Y Y^T) : Y \in K_n(a) \right\}, \tag{1} \]
where \( \lambda_n(\cdot) \) is the smallest eigenvalue. It is clear that \( \sqrt{c_n(a)} \) is the smallest singular value in \( K_n(a) \). Let \( Y_0 = (y^0_{ij}) \in K_n(a) \) be defined by
\[ y^0_{ij} = (-1)^{|i-j|+1} a \tag{2} \]
for \( i > j \). In Section 2, we present a sharp upper bound for the absolute values of entries of \( Y^{-1} \) and \( (Y Y^T)^{-1} \), where \( Y \in K_n(a) \), and obtain all the entries of \( Y_0^{-1} \) and \( (Y_0 Y_0^T)^{-1} \) in terms of \( a \). In Section 3, we prove that \( c_n(a) = \lambda_n(Y_0 Y_0^T) \). We also show that if \( c_n(a) = \lambda_n(Y Y^T) \) for \( Y \in K_n(a) \) then \( Y = E Y_0 E \), where \( E = \text{diag}(e_1, e_2, \ldots, e_n) \) with \( e_i = \pm 1 \) for \( i = 1, 2, \ldots, n \). Moreover, we determine the asymptotic behavior of \( c_n(a) \). In other words, we show that \( c_n(a) \sim \frac{(a+2)^2}{(a+1)^2} \) as \( n \to \infty \). Finally, we present concluding remarks in Section 4, including some open problems related to a certain generalization of the results on \( c_n \) and \( c_n(a) \).

\section{2 Preliminaries}

In order to find the bound mentioned in the introduction, we present five lemmas here. In the first lemma, we obtain a recurrence relation for the entries of inverses of matrices in \( K_n(a) \).

\textbf{Lemma 1.} Let \( Y \in K_n(a) \), \( W = (w_{ij}) = Y - I \) and \( Y^{-1} = (b_{ij}) \). Then,
\[ b_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ -\sum_{k=j}^{i-1} w_{ik} b_{kj} & \text{if } i > j. \end{cases} \]
Proof. See [4, Lemma 2.2]. The set of matrices is different, but the proof works also here.

**Lemma 2.** [5] Let $Y$ and $Y^{-1}$ be as in Lemma 1. Then,

$$|b_{ij}| \leq a(a + 1)^{i-j-1}$$

for $1 \leq j < i \leq n$.

**Proof.** Let $W = (w_{ij})$ be as in Lemma 1. Let $j = 1, 2, \ldots, n-1$. We prove by induction on $t = 1, 2, \ldots, n-j$ that $|b_{j,t,j}| \leq a(a + 1)^{t-1}$. By Lemma 1, we have $|b_{j+1,j}| = |w_{j+1,j}|$, where $w_{j+1,j} \in [-a, a]$. Thus, $|b_{j+1,j}| \leq a$. Now assume that the inequality $|b_{j+t,j}| \leq a(a + 1)^{t-1}$ holds for each $t = 1, 2, \ldots, k-1$. We show that $|b_{j+k,j}| \leq a(a + 1)^{k-1}$. By Lemma 1 and the induction hypothesis, we have

$$|b_{j+k,j}| = \left| \sum_{i=j}^{j+k-1} w_{j+k,i} b_{ij} \right| \leq \sum_{i=j}^{j+k-1} |w_{j+k,i}| |b_{ij}| \leq a \sum_{i=j}^{j+k-1} |b_{ij}| \leq a \left( 1 + \sum_{i=j+1}^{j+k-1} a(a + 1)^{i-j-1} \right) = a(a + 1)^{k-1}.$$ 

The induction principle completes the proof.

**Lemma 3.** [5] Let $Y_0$ be as in (2) and $Y^{-1}_0 = (c_{ij})$. Then

$$c_{ij} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ (-1)^{i-j}a(a + 1)^{i-j-1} & \text{if } i > j. \end{cases}$$

**Proof.** Let $W_0 = (w_{ij}^0) = Y_0 - I$. Then

$$w_{ij}^0 = \begin{cases} 0 & \text{if } i \leq j, \\ (-1)^{i-j-1}a & \text{otherwise.} \end{cases}$$

Since $Y_0 \in K_n(a)$, we have by Lemma 1

$$c_{ij} = -\sum_{k=j}^{i-1} w_{ik}^0 c_{kj}$$

for $i > j$.

Next, we show by induction on $t = i-j$ that

$$c_{ij} = (-1)^{i-j}a(a + 1)^{i-j-1}, \quad i > j. \tag{3}$$

If $t = 1$, then

$$c_{j+1,j} = -w_{j+1,j}^0 = -a.$$ 

Now assume that (3) holds for $t = 1, 2, \ldots, k-1$. Then

$$c_{j+k,j} = -\sum_{s=j}^{j+k-1} w_{j+k,s}^0 c_{sj} = (-1)^{k}a + \sum_{s=j+1}^{j+k-1} (-1)^{j+k+s}a(-1)^{i-j}a(a + 1)^{i-j-1} = (-1)^{k}a(a + 1)^{k-1}.$$ 

The induction principle completes the proof.
In the following, we denote by $(\cdot)_{ij}$ the $ij$-th entry of a given matrix.

**Lemma 4.** [5] Let $Y_0$ be as in (2) and $Z_0 = Y_0 Y_0^T$. Then,

$$ (Z_0^{-1})_{ij} = \begin{cases} 
1 + \frac{a^2}{(a+1)^2 - 1} \left( (a + 1)^{2(n-i)} - 1 \right) & \text{if } i = j, \\
(1)^{i-j} \left( a(a + 1)^{i-j-1} + \frac{a^2}{(a+1)^2 - 1} (a + 1)^{i-j} \left( (a + 1)^{2(n-\max(i,j))} - 1 \right) \right) & \text{if } i \neq j. 
\end{cases} $$

*Proof.* By Lemma 3, we have

$$ (Z_0^{-1})_{ii} = 1 + \sum_{k=i+1}^{n} a^2 (a + 1)^{2(k-i-1)} = 1 + \frac{a^2}{(a+1)^2 - 1} (a + 1)^{2(n-i)} - 1 \tag{4} $$

for all $i = 1, 2, \ldots, n$ and

$$ (Z_0^{-1})_{ij} = c_{ij} + \sum_{t=j+1}^{n} c_{t} c_{ij} $$

$$ = (-1)^{i-j} \left( a(a + 1)^{i-j-1} + \sum_{t=j+1}^{n} a^2 (a + 1)^{2(i-j-2)} \right) $$

$$ = (-1)^{i-j} \left( a(a + 1)^{i-j-1} + a^2 (a + 1)^{i-j} \left( (a + 1)^{2(n-j)} - 1 \right) \right) \tag{5} $$

for $1 \leq i < j \leq n$. Since $Z_0$ is symmetric, the claim follows. \hfill $\square$

**Lemma 5.** [5] Let $Z = YY^T, Y \in K_n(a)$, and let $Z_0$ be as in Lemma 4. Then $|Z^{-1}| \leq |Z_0^{-1}|$, that is $|(Z^{-1})_{ij}| \leq |(Z_0^{-1})_{ij}|$ for all $1 \leq i, j \leq n$.

*Proof.* Let $Y^{-1}$ be as in Lemma 1. Then, by Lemmas 2 and 4,

$$ |(Z^{-1})_{ii}| = \left| \sum_{k=1}^{n} b_{ki}^2 \right| $$

$$ = \sum_{k=1}^{n} |b_{ki}|^2 $$

$$ \leq 1 + \sum_{k=i+1}^{n} a^2 (a + 1)^{2(k-i-1)} \tag{6} $$

for all $i = 1, 2, \ldots, n$. Let $i < j$. Again, by Lemmas 2 and 4,

$$ |(Z^{-1})_{ij}| = \left| \sum_{t=1}^{n} b_{ti}b_{tj} \right| $$

$$ \leq \sum_{t=1}^{n} |b_{ti}||b_{tj}| $$

$$ = \left| b_{ij} \right| + \sum_{t=j+1}^{n} |b_{ti}||b_{tj}| $$

$$ \leq a(a + 1)^{i-j-1} + \sum_{t=j+1}^{n} a^2 (a + 1)^{2(i-j-2)} \tag{7} $$
Since $Z^{-1}$ and $Z_0^{-1}$ are symmetric, we obtain $|Z^{-1}| \leq |Z_0^{-1}|$.

\section{Main Results}

\begin{theorem}
Let $Y_0$ be as in (2). Then $c_n(a) = \lambda_n(Y_0 Y_0^T)$. Conversely, if $Y \in K_n(a)$ and $\lambda_n(Y Y^T) = c_n(a)$, then $Y = E Y_0 E$, where $E = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ for $i = 1, 2, \ldots, n$.
\end{theorem}

\begin{proof}
For the first claim, we proceed as in the proof [4] of the Ilmonen-Haukkanen-Merikoski conjecture. Let $Z_0 = Y_0 Y_0^T$. Then, by (4) and (5), $\text{tr}Z_0^k = \text{tr}|Z_0^{-1}|^k$ for all positive integers $k$. By Newton's identities [10], one can easily see that $Z_0^{-1}$ and $|Z_0^{-1}|$ have the same characteristic polynomial. Since they are real symmetric matrices, their spectral radii are equal, that is, $\rho(|Z_0^{-1}|) = \rho(Z_0^{-1})$. Let $Y \in K_n(a)$ and $Z = YY^T$. By Lemma 5 and [7, Theorem 8.1.18], we obtain the following inequalities

$$
\rho(Z^{-1}) \leq \rho(|Z^{-1}|) \leq \rho(|Z_0^{-1}|) = \rho(Z_0^{-1}).
$$

Since $Z^{-1}$ and $Z_0^{-1}$ are positive definite, we have

$$
\rho(Z^{-1}) = \lambda_1(Z^{-1}) \quad \text{and} \quad \rho(Z_0^{-1}) = \lambda_1(Z_0^{-1}),
$$

where $\lambda_1(\cdot)$ is the largest eigenvalue. Hence, by (8),

$$
\lambda_n(Z) = \frac{1}{\lambda_1(Z^{-1})} \geq \frac{1}{\lambda_1(Z_0^{-1})} = \lambda_n(Z_0),
$$

and the claim follows.

To show that the second claim, let $Y \in K_n(a)$ satisfy $c_n(a) = \lambda_n(Z)$, $Z = YY^T$. Then $\lambda_n(Z) = \lambda_n(Z_0)$ and further $\rho(Z^{-1}) = \rho(Z_0^{-1})$, which implies by (8) that $\rho(|Z^{-1}|) = \rho(|Z_0^{-1}|)$. Now, by the same reasoning as in the proof of [11, Theorem 3.1], we have

$$
|Z^{-1}| = |Z_0^{-1}|.
$$

Let $Y^{-1} = (b_{ij})$ and $Y_0^{-1} = (c_{ij})$. Since $Y^{-1}$ is a lower triangular matrix, inequalities (6) and (7), and equality (9) together imply that

$$
\left| \sum_{k=1}^{n} b_{ki} b_{kj} \right| = \sum_{k=1}^{n} |b_{ki}||b_{kj}|,
$$

for $1 \leq i < j \leq n$. Here, all the summands are nonzero. Note that all the summands on the left-hand side of the last equality have the same argument. Furthermore, we have by Lemmas 2 and 3 that

$$
b_{ij} = \varepsilon_{i,j} c_{ij},
$$

where $\varepsilon_{i,j} = \pm 1$ and $\varepsilon_{i,i} = 1$ for all $1 \leq i, j \leq n$. Then,

$$
\left| \sum_{k=1}^{n} \varepsilon_{k,i} \varepsilon_{k,j} c_{ki} c_{kj} \right| = \sum_{k=1}^{n} |\varepsilon_{k,i} c_{ki}||c_{kj}| \iff \left| \sum_{k=1}^{n} \varepsilon_{k,i} c_{kj} c_{ki} c_{kj} \right| = \sum_{k=1}^{n} |c_{ki}||c_{kj}| \iff \varepsilon_{i,j} = \varepsilon_{i+1,j} \varepsilon_{i+2,j} \varepsilon_{i+3,j} \cdots \varepsilon_{n,i} \varepsilon_{n,j}
$$

for $1 \leq i < j \leq n$. This means that every $\varepsilon_{i,j}$ depends only $\varepsilon_{n,i}$ and $\varepsilon_{n,j}$. Therefore, setting $\varepsilon_i = \varepsilon_{n,i}$ for all $i = 1, 2, \ldots, n$, for the sake of brevity, we have

$$
b_{ij} = \varepsilon_i \varepsilon_j c_{ij}$$
or equivalently
\[ Y^{-1} = EY_0^{-1}E^{-1}, \]
where \( E = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \). It is clear that \( E = E_1E_2 \cdots E_n \), where each \( E_i \) \((i = 1, 2, \ldots, n)\) is an elementary matrix obtained from the \( n \times n \) identity matrix by multiplying its \( i \)th row by \( \varepsilon_i \). So, \( E = E^{-1} \). and therefore \( Y = EY_0E \). This concludes the proof. \( \square \)

**Theorem 2.** We have
\begin{equation}
\frac{1}{n - \frac{an}{a+2} + \frac{(a+1)^{n-2} - 1}{(a+2)^2}} \leq c_n(a) \leq \frac{1}{1 - \frac{an}{a+2} + \frac{(a+1)^{n-2} - 1}{(a+2)^2}},
\end{equation}
and consequently
\begin{equation}
\lim_{n \to \infty} c_n(a) = 1.
\end{equation}

**Proof.** To prove the theorem, we use a similar but not the same method to those of Theorem 2.1 in [11].

First, let \( Z_0 \) be as in Lemma 4 and \( H_n = I - Z_0^{-1} \). Now, add \( a + 1 \) multiple of the second column to the first column, and then add \( a + 1 \) multiple of the second row to the first row. Let \( M_n = (m_{ij}) \) be the matrix obtained by these two elementary operations from \( H_n \). It is clear that \( \det(H_n) = \det(M_n) \), \( m_{11} = a(a+2) \), \( m_{12} = m_{21} = a \) and \( m_{ij} = m_{j1} = 0 \) for \( j = 3, 4, \ldots, n \). By expanding \( \det(M_n) \), one can easily obtain the following recurrence relation for \( n \geq 3 \),
\[ \det(H_n) = a(a+2)\det(H_{n-1}) - a^2\det(H_{n-2}) \]
with the initial conditions \( \det(H_1) = 0 \) and \( \det(H_2) = -a^2 \).

Second, we show by induction on \( n \) that \( \det(H_{n+1}) < \det(H_n) \) for \( n > 1 \). It is clear that \( \det(H_2) < \det(H_1) = -a^2 \). Suppose \( \det(H_n) < \det(H_{n-1}) \). Then,
\[ \det(H_{n+1}) = a(a+2)\det(H_n) - a^2\det(H_{n-1}) < a(a+2)\det(H_n) - a^2\det(H_n) = 2a\det(H_n) < \det(H_n) \]
by the induction hypothesis and the assumption \( a \geq 1 \). Thus, we have \( \det(H_{n+1}) < \det(H_n) < 0 \) for all \( n > 1 \).

Next, we show that every eigenvalue of \( Z_0^{-1} \), except its spectral radius \( \rho(Z_0^{-1}) \), lies in the interval \([0, 1]\).

Let \( \lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_n^{(n)} \) be the eigenvalues of the \( n \times n \) matrix \( Z_0^{-1} \). It is clear that \( \lambda_n^{(n)} > 0 \) as \( Z_0^{-1} \) is positive definite. We show again by induction on \( n \) that \( \lambda_1^{(n)} > 1 \) and \( \lambda_2^{(n)} < 1 \). For \( n = 2 \),
\[ \lambda_1^{(2)} = \frac{a^2 + 2 + \sqrt{(a^2 + 2)^2 - 4}}{2} > 1 \quad \text{and} \quad \lambda_2^{(2)} = \frac{a^2 + 2 - \sqrt{(a^2 + 2)^2 - 4}}{2} < 1. \]
Assume \( \lambda_1^{(n-1)} > 1 \) and \( \lambda_2^{(n-1)} < 1 \). Then it follows from the Cauchy interlacing inequalities, see [7], that \( \lambda_1^{(n)} \leq \lambda_2^{(n-1)} \). Note that \( \det(I - Z_0^{-1}) = \det(H_0) < 0 \) from paragraph above. Moreover, the characteristic polynomial of \( Z_0^{-1} \), evaluated at \( x = 1 \) is negative. So, we obtain \( \lambda_2^{(n)} < 1 \). Thus, it follows that
\[ \rho(Z_0^{-1}) \leq \text{tr}(Z_0^{-1}) \leq \rho(Z_0^{-1}) + n - 1. \]
In addition to this, by (4), one can easily calculate the trace of \( Z_0^{-1} \) as
\[ \text{tr}(Z_0^{-1}) = \sum_{i=1}^{n} \left( 1 + \frac{a^2}{(a+1)^{2(n-i)} - 1} \right) = n - \frac{an}{a+2} + \frac{(a+1)^{2n} - 1}{(a+2)^2}. \]
So, it is obvious that \( \text{tr}(Z_0^{-1}) - n + 1 > 0 \). The first result given in (10) follows from the inequalities in (12) and the fact \( c_n(a) = \rho(Z_0^{-1})^{-1} \). For the second result, we restate the inequalities in (10) as
\[ \frac{1}{2(a+2)} \frac{n}{(a+1)^{n-1} - 1} + 1 \leq \frac{c_n(a)}{(a+2)^2} \leq \frac{1}{a+1} \cdot \frac{a}{a - a^2} + \frac{1}{(a+1)^{n-1} - 1}. \]
Thus, the claim in (11) follows. □

4 Concluding Remarks

In this paper, introducing a new constant $c_n(a)$, we have expanded the results on the finding all minimizing matrices of the constant $c_n$ and its asymptotic behaviour to a larger class of matrices. We do not reckon that our constant $c_n(a)$ could be used in eigenvalue estimation of GCD and related matrices as $c_n$ was used in the literature, see [3, 6, 8, 9, 11, 12]. However, it seems possible that the techniques of this paper could be applied to a larger class of matrices than those considered in this present paper.

Throughout the paper, we assumed that $a \geq 1$. Almost all of our results are valid for $a > 0$. However, in this case, it seems not easy to prove that $\det(H_n) < 0$ for $n > 1$, and the lower and upper bounds for $c_n(a)$ given in (10) for small values of $n$ do not provide good estimates. For example, when we compute the three terms in the inequalities in (10) for $a = 0.1$ and $n = 5$, we find the numbers

$$0.195186787128691, \ 0.657178123193002, \ 0.890236025695808,$$

respectively. Also

$$\frac{(a + 2)^2}{(a + 1)^{2n} - 1} = 2.76707191431876.$$ 

On the other hand, when $a \geq 1$, we have better lower and upper bounds for $c_n(a)$ even for small $a$ and $n$. Indeed, computing the three terms in (10) for $a = 3$ and $n = 5$, we find the numbers

$$0.0000238407438312075, \ 0.00002384095983146484, \ 0.0000238430175723040,$$

respectively. Moreover,

$$\frac{(a + 2)^2}{(a + 1)^{2n} - 1} = 0.0000238418806475455.$$

In addition to this, we believe that the results concerning $c_n(a)$ could be generalized to the interval $[a, b]$, where $a \leq 0$ and $b > 0$. Let $K_n[a, b]$ denote the set of all $n \times n$ unit lower triangular matrices with each element in the interval $[a, b]$. Let $c_n(a, b) = \min \left\{ \lambda_n(YY^T) : Y \in K_n[a, b] \right\}$, i.e., $\sqrt{c_n(a, b)}$ is the smallest singular value in $K_n[a, b]$. We end the paper by raising the following open problems: Can one determine such a matrix $Y$ in $K_n[a, b]$? Is it possible to determine the asymptotic behavior of $c_n(a, b)$?

Acknowledgement: The author would like to thank the anonymous referees for their valuable remarks and corrections which helped to improve the manuscript.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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