Portfolio optimisation with mixture vector autoregressive models

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Abstract

Obtaining reliable estimates of conditional covariance matrices is an important task of heteroskedastic multivariate time series. In portfolio optimisation and financial risk management, it is crucial to provide measures of uncertainty and risk as accurately as possible. We propose using mixture vector autoregressive (MVAR) models for portfolio optimisation. Combining a mixture of distributions that depend on the recent history of the process, MVAR models can accommodate asymmetry, multimodality, heteroskedasticity and cross-correlation in multivariate time series data. For mixtures of Normal components, we exploit a property of the multivariate Normal distribution to obtain explicit formulas of conditional predictive distributions of returns on a portfolio of assets. After showing how the method works, we perform a comparison with other relevant multivariate time series models on real stock return data.

Keywords: Forecasting; Heteroskedasticity; Mixture vector autoregressive model; MVAR model; Portfolio.

Declarations of interest: none.
1 Introduction

Financial and econometric data often presents the feature of heteroskedasticity. For multivariate time series, this implies that the covariance matrix of an observation at a given time point depends upon the recent history of the process. This may be due to changes in the volatility of a single series, as well as in the cross-correlations between any two series of interest. As a result, one cannot trust sample estimates of the (unconditional) covariance matrix, or linear time series models to build reliable predictions about the future. Therefore, obtaining reliable estimates of covariance matrices remains an important challenge in portfolio optimisation and financial risk management which use, for instance, modern portfolio theory (Markowitz 1952). Bollerslev et al. (1988) and Engle and Kroner (1995) pioneered in the attempt to model conditional covariance matrices of predictors for multivariate time series with multivariate GARCH models, using different parametrisations known respectively as VEC and BEKK. Engle (2002) extended the idea of multivariate GARCH to the so-called Dynamic Conditional Correlation models, in which each element of the time-dependent covariance matrix of the data is modelled to follow a GARCH process. Such models have computational advantages over multivariate GARCH models in that the number of parameters to be estimated in the correlation process is independent of the number of series to be correlated, by use of common parameters across all correlations to be estimated.

Since then, much work has been done to develop multivariate GARCH models for portfolio optimisation. In particular, many attempts have been made in combining GARCH and factor models, with the aim of dimensionality reduction when modelling large portfolios or panel data. These models rely on the assumption that financial returns are described by a small number of underlying common variables, or factors, which can be used to model the data more parsimoniously. Although all equal in concept, different approaches used different assumptions on such factors, and different techniques are used to derive them. For instance, Alexander (2000) uses a principal components analysis in which factors are assumed to follow independent GARCH processes, whereas Van der Weide (2002) considers the case in which factors are not orthogonal. Finally, Santos and Moura (2014) introduced the dynamic factor GARCH model with time-varying factor loadings.

We propose using a mixture vector autoregressive (MVAR) model (Fong et al. 2007) for portfolio optimisation. MVAR models are the multivariate extension of the mixture autoregressive (MAR) model by Wong and Li (2000). Combining predictive distributions which depend on the recent history of the process, MVAR models can accommodate asymmetry, multimodality, heteroskedasticity.
and cross-correlation in multivariate time series data. Theoretical properties of MVAR were explored for the case of a multivariate Gaussian mixture in Fong et al. (2007) and Kalliovirta et al. (2016).

Financial returns are typically assumed to be uncorrelated or weakly correlated. The stationary region of the parameters of MAR and MVAR models contains the uncorrelated case, which allows these properties to be achieved smoothly as part of the estimation process.

Using the Gaussian MVAR model assumption, we are able to fully specify conditional predictive distributions for future observations. We will show how it is possible to combine modern portfolio theory (Markowitz 1952) and the assumption of Gaussian mixture vector autoregressive model for portfolio optimisation. Under this model assumption, we will also estimate the risk associated with the forecast. Finally, we will compare the performance of our method with that of the dynamic conditional correlation model by Engle (2002) and the vector autoregressive model (VAR).

2 The mixture vector autoregressive model

Mixture vector autoregressive models or MVAR (Fong et al. 2007) are the multivariate extension of Mixture Autoregressive Models (Wong and Li 2000).

The MVAR model with \( g \) Gaussian components, and an \( m \) dimensional observation vector \( Y_t \) is defined as

\[
F(Y_t | \mathcal{F}_{t-1}) = \sum_{k=1}^{g} \pi_k \Phi \left( \Omega_k^{-1/2} \left( Y_t - \Theta_{k0} - \sum_{i=1}^{p_k} \Theta_{ki} Y_{t-i} \right) \right)
\]

(1)

where

- \( Y_t \) is a \( m \times 1 \) data vector at time \( t \).
- \( \pi = (\pi_1, \ldots, \pi_g) \) are the mixing weights, such that \( \pi_i > 0 \) for \( i = 1, \ldots, g \), and \( \sum_{i=1}^{g} \pi_i = 1 \).
- \( \Omega_k \) is the covariance matrix of component \( k \).
- \( p_k, k = 1, \ldots, g \) is the autoregressive order of component \( k \). We denote \( p = \max(p_k) \).
- \( \Theta_{k0} \) is a \( m \times 1 \) intercept vector for component \( k \), and \( \Theta_{k1}, \ldots, \Theta_{kp_k} \) are \( m \times m \) matrices of autoregressive parameters. If \( p_k < p \), then \( \Theta_{kl} = 0_{m} \) for \( p_k < l \leq p \), where \( 0_m \) is the zero-matrix of size \( m \times m \).
- \( \Phi(\cdot) \) is the CDF of the standard multivariate Normal distribution, and \( \phi(\cdot) \) is the corresponding pdf.
• Assuming start at \( t = 1 \), (1) holds for \( t > p \).

Regularity conditions and parameter estimation by EM algorithm are discussed in Fong et al. (2007) and Kalliovirta et al. (2016).

MVAR may be seen as an alternative to multivariate GARCH when the data presents heteroskedasticity and time-dependent correlation matrices, while also accounting for possible multimodality and asymmetry in the distribution.

For parameter estimation, we recur to the missing data formulation. Suppose that a \( m \)-variate time series \( \{ Y_t \} \) of length \( n \) follows a MVAR process. Let \( Z = (Z_1, \ldots, Z_n) \) be an unobserved allocation random variable, where \( Z_t \) is a \( g \)-dimensional vector with component \( k \) equal to 1 if \( Y_t \) comes from the \( k^{th} \) component, and 0 otherwise, and such that exactly one element of \( Z_t \) is equal to 1.

Following notation from Fong et al. (2007), let \( \hat{\Theta}_k = [\Theta_{k0}, \Theta_{k1}, \ldots, \Theta_{kp_k}] \) and \( X_{tk} = (1, Y_{t-1}^T, \ldots, Y_{t-p_k}^T)^T \). In addition, let \( \vartheta \) denote the complete set of parameters. Parameter estimates are then obtained by EM-algorithm (Dempster et al. 1977) with the following steps:

**E-step**

\[
\tau_{tk} = E[Z_{tk} \mid Y_t, \vartheta] = \frac{\pi_k \Phi \left( \frac{1}{2} \left( Y_t - \Theta_{k0} - \sum_{i=1}^{p_k} \Theta_{ki} Y_{t-i} \right) \right)}{\sum_{l=1}^q \pi_l \Phi \left( \frac{1}{2} \left( Y_t - \Theta_{l0} - \sum_{i=1}^{p_l} \Theta_{li} Y_{t-i} \right) \right)}
\]  

(2)

**M-step**

\[
\hat{\pi}_k = \frac{1}{n-p} \sum_{t=p+1}^n \tau_{tk}
\]

\[
\hat{\Theta}_k = \left( \sum_{t=p+1}^n \tau_{tk} X_{tk} X_{tk}^T \right)^{-1} \left( \sum_{t=p+1}^n \tau_{tk} X_{tk} Y_t^T \right)
\]

\[
\hat{\Omega}_k = \frac{\sum_{t=p+1}^n \tau_{tk} e_{tk} e_{tk}^T}{\sum_{t=p+1}^n \tau_{tk}}
\]

(3)

where \( e_{tk} = Y_t - \Theta_{k0} - \sum_{i=1}^{p_k} \Theta_{ki} Y_{t-i} \).

E-step and M-step are repeated recursively until convergence to maximum likelihood estimates of the parameters.
First and second order stationarity conditions are discussed by Saikkonen (2007), see also Boshnakov (2011) for the univariate case. Let

\[
A_k = \begin{bmatrix}
\Theta_{k1} & \Theta_{k2} & \ldots & \Theta_{kp-1} & \Theta_{kp} \\
I_m & 0_m & \ldots & 0_m & 0_m \\
0_m & I_m & \ldots & 0_m & 0_m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_m & 0_m & \ldots & I_m & 0_m
\end{bmatrix}, \quad k = 1, \ldots, g \tag{4}
\]

where \(I_m\) and \(0_m\) are respectively the identity matrix and the zero matrix of size \(m \times m\). A necessary and sufficient condition for the MVAR model to be stationary is that the eigenvalues of \(\sum_{k=1}^g \pi_k A_k \otimes A_k\) are smaller than 1 in modulus. A MVAR model that satisfies this condition is said to be Stable. In practice, to assess stability of the fitted model, parameters are replaced by their estimates.

### 2.1 Prediction with mixture vector autoregressive models

In the context of mixture models, density forecasts are often more attractive than point predictors and prediction intervals. This is because the qualitative features of a predictive distribution, such as multiple modes or skewness, are more intuitive and useful than simply a forecast and the associated prediction interval. In addition, when the predictive distribution is available, prediction intervals can easily be obtained by extracting the quantiles of interest (Boshnakov 2009, Lawless and Fredette 2005). Therefore, we here present derivation of full predictive distributions for MVAR models, which will be used throughout the analysis.

By model assumption, the one step ahead conditional predictive distribution at time \(t\) is fully specified, and it is that of (1) where, for notational convenience, we replace \(t\) with \(t + 1\), i.e.

\[
F(Y_{t+1} \mid F_t) = \sum_{k=1}^g \pi_k \Phi \left( \Omega_k^{-1/2} \left( Y_{t+1} - \Theta_{k0} - \sum_{i=1}^{p_k} \Theta_{ki} Y_{t+1-i} \right) \right).
\]

Thus, the conditional distribution of the one step ahead predictor is a mixture of \(g\) Gaussian components and it depends on previous observations. In particular, the conditional covariance matrix depends on previous values of the process, a defining property of heteroskedasticity. To obtain the conditional mean and the
covariance matrix let $\mu_{t+1,k} = \Theta_{k0} + \sum_{i=1}^{P_k} \Theta_{ki} Y_{t+1-i}$, for $k = 1, \ldots, g$. Then

$$E[Y_{t+1} | F_t] = \sum_{k=1}^{g} \pi_k \mu_{t+1,k} = \mu_{t+1}$$

$$Cov(Y_{t+1} | F_t) = \sum_{k=1}^{g} \pi_k \Omega_k + \sum_{k=1}^{g} \pi_k (\mu_{t+1,k} - \mu_{t+1}) (\mu_{t+1,k} - \mu_{t+1})^T$$

$$= \sum_{k=1}^{g} \pi_k \Omega_k + \sum_{k=1}^{g} \pi_k \mu_{t+1,k} \mu_{t+1,k}^T - \mu_{t+1} \mu_{t+1}^T$$

(5)

Using a method analogous to that of Boshnakov (2009), we can derive the conditional distribution for the two-step ahead predictor as a mixture of $g^2$ Gaussian components:

$$F(Y_{t+2} | F_t) = \sum_{k=1}^{g} \sum_{l=1}^{g} \pi_k \pi_l \Phi \left( \Psi_{kl}^{-1/2} (Y_{t+2} - \mu_{kl}) \right)$$

(6)

where, for each $k, l = 1, \ldots, g$,

$$\mu_{kl} = \Theta_{k0} + \Theta_{kl} \Theta_{l0} + \sum_{i=1}^{p-1} (\Theta_{k,i+1} + \Theta_{kl} \Theta_{li}) Y_{t-1-i} + \Theta_{kl} \Theta_{l0} Y_{t-1-p}$$

$$\Psi_{kl} = \Omega_k + \Theta_{kl} \Omega_l \Theta_{kl}^T$$

Note that, in general, $\mu_{kl} \neq \mu_{lk}$ and $\Psi_{kl} \neq \Psi_{lk}$. Expectation and covariance matrix of this predictor are:

$$E[Y_{t+2} | F_t] = \sum_{k=1}^{g} \sum_{l=1}^{g} \pi_k \pi_l \mu_{kl} = \mu_{t+2}$$

$$Cov(Y_{t+2} | F_t) = \sum_{k=1}^{g} \pi_k \pi_l \Psi_{kl} + \sum_{k=1}^{g} \sum_{l=1}^{g} \pi_k \pi_l \mu_{kl} \mu_{kl}^T - \mu_{t+2} \mu_{t+2}^T$$

(7)

Full derivation of (7), as well as proof of the conditional distribution of $Y_{t+2}$, is available in Appendix B. By recursing this procedure, we could derive a full distribution for any horizon $h$. However, the number of components in the mixture increases to $g^h$ when $h$ increases and therefore simulation methods may be preferred for approximate computation of predictive densities for larger horizons.

### 3 Portfolio optimisation with MVAR models

Suppose that a multivariate time series $\{Y_t\}$ of asset returns is observed, and it is believed that the underlying generating process is MVAR. From Section 2.1,
conditional distributions of the 1 and 2 step predictors are fully specified, and can be estimated by plugging parameter estimates into the relevant equations.

Now, let \( w \) denote the weights of a portfolio built with assets \( \{ Y_t \} \) (allowing short selling), and let \( R_{t+1} = w^T Y_{t+1} \) be the portfolio return at time \( t+1 \). Intuitively, because our model consists of a mixture of multivariate normal components, we can apply the property in (18) to conclude that the conditional distribution of \( R_{t+1} \) is also (univariate) mixture normal, with corresponding mixing weights \( \pi \) from the fitted multivariate model. By model assumption in fact, at each time \( t+1 \) an observation \( Y_{t+1} \) is assumed to be generated from one of \( g \) components of the mixture. Consequently, \( R_{t+1} \) is obtained by applying (18) to the selected component. Recursing this for all \( g \) components the result is itself a mixture distribution for \( R_{t+1} \).

In terms of MVAR model parameters we write:

\[
F(R_{t+1} | \mathcal{F}_t) = \sum_{k=1}^{g} \pi_k \Phi \left( \frac{R_{t+1} - w^T \mu_{t+1,k}}{\sqrt{w^T \Omega_{k} w}} \right)
\]  

(8)

Conditional mean and variance of \( R_{t+1} \) are:

\[
E[R_{t+1} | \mathcal{F}_t] = \sum_{k=1}^{g} \pi_k (w^T \mu_{t+1,k}) = \sum_{k=1}^{g} \pi_k \mu_{t+1,k}^* = \mu^*
\]

\[
\text{Var}(R_{t+1} | \mathcal{F}_t) = \sum_{k=1}^{g} \pi_k (w^T \Omega_{k} w) + \sum_{k=1}^{g} \pi_k (\mu_{t+1,k}^*)^2 - (\mu^*)^2
\]  

(9)

Modern portfolio theory (Markowitz 1952) gives us a way to calculate weights \( w^* \) to construct the most efficient portfolio for a given return, and to calculate the efficient portfolio of assets with the minimum possible variance. A portfolio with target return \( \mu \) is said to be an efficient portfolio when the variance associated with it is the lowest amongst all portfolios of the same assets having that same target return. The minimum variance portfolio is the efficient portfolio with the lowest possible variance of all efficient portfolios of the same assets. For the remainder of the analysis, we will denote efficient portfolios with the subscript EFF, and minimum variance portfolios with the subscript MVP. We now see how modern portfolio theory can be used to predict future observations assuming a MVAR model.

For the MVAR case, let \( E[Y_{t+1} | \mathcal{F}_t] = \mu_{t+1} \) and \( \text{Cov}(Y_{t+1} | \mathcal{F}_t) = \Omega_{t+1} \). In addition, let

\[
A = \mathbb{I} \Omega_{t+1}^{-1} \mu_{t+1}, \quad B = \mu_{t+1} \Omega_{t+1}^{-1} \mu_{t+1}, \quad C = \mathbb{I} \Omega_{t+1}^{-1} \mathbb{I}, \quad D = CB - A^2
\]  

(10)

where \( \mathbb{I} \) is a vector of 1s of the same length as \( \mu_{t+1} \).
It can be proved that optimal weights for an efficient portfolio of these assets and target return $\mu_{\text{EFF}}$ are

$$w_{\text{EFF}} = \frac{1}{D} \left( B\Omega_{t+1}^{-1} - A\Omega_{t+1}^{-1} \mu_{t+1} + \mu^* (C\Omega_{t+1}^{-1} \mu_{t+1} - A\Omega_{t+1}^{-1} \mu_{t+1}) \right)$$  \hspace{1cm} (11)$$

and the variance of such portfolio can be calculated equivalently as $\text{Var}(R_{t+1} | F_{t-1})$ (MVAR model assumption) or $w^T\Omega_{t}w$ (the variance of an efficient portfolio of assets) since

$$w_{\text{EFF}}^T \Omega_{t+1} w_{\text{EFF}} = \sum_{k=1}^{g} \pi_k \left( w_{\text{EFF}}^T \Omega_k w_{\text{EFF}} \right) + \sum_{k=1}^{g} \pi_k \left( w_{\text{EFF}}^T \mu_{t+1,k} \right)^2$$

$$- \left[ \sum_{k=1}^{g} \pi_k \left( w_{\text{EFF}}^T \mu_{t+1,k} \right) \right]^2 = \text{Var}(R_{t+1} | F_{t})$$  \hspace{1cm} (12)$$

In practice, $\mu_{t+1,k}$ and $\Omega_{t+1}$ are replaced with their estimates $\hat{\mu}_{t+1,k}$ and $\hat{\Omega}_{t+1}$.

Weights of the minimum variance portfolio of same assets $\{Y_t\}$, and corresponding return, are:

$$w_{\text{MVP}} = \frac{\Omega_{t+1}^{-1} \mu_{\text{MVP}}}{C} \quad \mu_{\text{MVP}} = A \quad \Omega_{\text{MVP}} = \frac{\Omega_{t+1}}{C}$$  \hspace{1cm} (13)$$

Conditional predictive distributions can also be calculated analytically for any $h \geq 2$. However, one must keep in mind that such predictive distribution would be a mixture of $g^h$ components, so that simulation methods may be preferred in some cases as $h$ increases.

Consider the case $h = 2$. The conditional predictive distribution $F(Y_{t+2} | F_t)$ for the MVAR model is a mixture of $g^2$ Gaussian components given in (7). Similarly to the case $h = 1$, we can derive the full conditional distribution of $R_{t+2}$, which is again a mixture of $g^2$ Gaussian components:

$$F(R_{t+2} | F_t) = \sum_{k,l=1}^{g} \pi_k \pi_l \Phi \left( \frac{R_{t+2} - w^{(2)} \mu_{kl}}{w^{(2)} \Psi_{kl} w^{(2)}} \right)$$  \hspace{1cm} (14)$$

where $w^{(2)}$ is the vector of optimal weights for this portfolio. Similarly to the case $h = 1$, one can now calculate $E[Y_{t+2} | F_t] = \mu_{t+2}$ and $\text{Cov}(Y_{t+2} | F_t) = \Omega_{t+2}$ and adapt (10), (11), (12) and (13) to obtain an efficient or minimum variance portfolio.
4 Simulation example

We simulate a series of size \( n = 500 \) of hypothetical stock returns from the 3 variate MVAR(2; 1, 1) process

\[
F(Y_t | F_{t-1}) = 0.75 \Phi \left( \frac{Y_t - v_1}{\Omega_1} \right) + 0.25 \Phi \left( \frac{Y_t - v_2}{\Omega_2} \right)
\]

where

\[
v_1 = \Theta_{10} + \Theta_{11} Y_{t-1} \quad v_2 = \Theta_{20} + \Theta_{21} Y_{t-1}
\]

and

\[
\Theta_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Theta_{11} = \begin{bmatrix} 0.5 & 0 & 0.4 \\ -0.3 & 0 & 0.5 \\ -0.6 & 0.5 & -0.3 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 1 & 0.5 & -0.4 \\ 0 & 0.8 & 4 \end{bmatrix}, \quad \Theta_{20} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Theta_{21} = \begin{bmatrix} -0.5 & 1 & -0.4 \\ 0.3 & 0 & -0.2 \\ 0 & -0.5 & 0.5 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 2 & -0.55 \\ 0 & -0.55 & 4 \end{bmatrix}
\]

The three univariate series can be seen in Figure 1, with their autocorrelation and cross-correlation plots in Figure 2. The data is very representative of what we should be looking for, in a real case scenario, to assume an underlying MVAR process. We notice in fact signs of heteroskedasticity in each of the series, and autocorrelations and cross-correlations significantly different from 0 at lags larger than 0. The latter is what separates MVAR from multivariate GARCH models, which assume the original series to be uncorrelated.

Parameter estimates were calculated using the EM Algorithm with the formulas in (2) and (3). In order to perform out of sample prediction, data from \( Y_1 \) to \( Y_{498} \) were used for estimation, with \( Y_{499} \) and \( Y_{500} \) being left out as observations 1 and 2 time points in the future:

\[
\hat{\pi} = (0.7242, 0.2758)
\]

\[
\hat{\Theta}_{10} = \begin{bmatrix} -0.0022 \\ -0.0303 \\ 0.1276 \end{bmatrix}, \quad \hat{\Theta}_{11} = \begin{bmatrix} 0.4931 & -0.0339 & 0.4169 \\ -0.3156 & -0.0012 & 0.5078 \\ -0.6141 & 0.6007 & -0.3844 \end{bmatrix}, \quad \hat{\Omega}_1 = \begin{bmatrix} 0.9551 & 0.4783 & -0.2776 \\ 0.4783 & 1.9123 & 0.9736 \\ -0.2776 & 0.9736 & 3.9455 \end{bmatrix}, \quad \hat{\Theta}_{20} = \begin{bmatrix} 0.0338 \\ 0.5499 \\ -0.7580 \end{bmatrix}, \quad \hat{\Theta}_{21} = \begin{bmatrix} -0.4595 & 1.0124 & -0.4004 \\ 0.3343 & -0.1423 & -0.1551 \\ -0.1273 & -0.2336 & 0.6509 \end{bmatrix}, \quad \hat{\Omega}_2 = \begin{bmatrix} 0.8767 & 0.4794 & -0.3627 \\ 0.4794 & 2.9148 & -0.6576 \\ -0.3627 & -0.6576 & 9.8135 \end{bmatrix}
\]

We then calculated the one step ahead conditional mean and variance based
Figure 1: Simulated time series of stock returns Asset 1 (top left), Asset 2 (top right) and Asset 3 (bottom).

Figure 2: Autocorrelation and cross-correlation plots of the simulated time series data.
on parameter estimates:

\[
E[Y_{499} | F_{498}] = \hat{\mu}_{499} = \begin{bmatrix}
-0.1750 \\
-0.9655 \\
-1.4361
\end{bmatrix}
\]

\[
\text{Cov}(Y_{499} | F_{498}) = \hat{\Omega}_{499} = \begin{bmatrix}
1.3109 & -0.6080 & -0.0768 \\
-0.6080 & 5.3174 & -0.5642 \\
-0.0768 & -0.5642 & 5.9420
\end{bmatrix}
\]

Given \(\hat{\Omega}_{499}\), we can calculate the minimum variance portfolio, which is obtained for weights \(w_{\text{MVP}} = (0.6434, 0.2228, 0.1338)^T\). The corresponding expected return on this portfolio at \((t + 1) = 499\) is \(\mu_{\text{MVP}} = -0.5198\), with standard deviation \(\sigma_{\text{MVP}} = 0.8475\).

Suppose now that we wish to increase our return to \(\mu^* = 0\), i.e. no expected loss, at the cost of a larger variance. We can calculate weights to construct an efficient portfolio of these assets as seen in Section 3. We obtain:

\[
w_{\text{EFF}} = \begin{bmatrix}
1.1097 \\
0.0781 \\
-0.1878
\end{bmatrix}
\]

The interpretation of \(w_{\text{EFF}}\) is that the optimal portfolio yielding expected return of 0 is constructed by short-selling a small amount of Asset 3, and investing 110.97% and 7.81% of the initial capital (meanwhile increased by short selling) into Asset 1 and Asset 2 respectively. Notice that the target return is \(w_{\text{EFF}}\mu_t = \mu_{\text{EFF}} = 0\) as desired.

We can now calculate the quantities we need for the conditional predictive distribution of \(R_{499}\):

\[
\begin{align*}
\mu^*_1 &= w^*\mu_{499,1} = 0.2642 \\
\sigma^2_1 &= 1.4968 \approx (1.2235)^2 \\
\mu^*_2 &= w^*\mu_{499,2} = -0.6939 \\
\sigma^2_2 &= 1.6062 \approx (1.3025)^2
\end{align*}
\]

Therefore, the conditional distribution of \(R_{499} = w^T Y_{499}\) is

\[
F(R_{499} | F_{498}) = 0.7242 \times \Phi \left( \frac{R_{499} - 0.2642}{1.2235} \right) + 0.2758 \times \Phi \left( \frac{R_{499} - 0.6939}{1.3025} \right)
\]

The standard deviation associated to this portfolio is \(\sigma_{\text{EFF}} = 1.3173\) which as expected is larger than \(\sigma_{\text{MVP}}\). More importantly, we can use the distribution assumption on \(R_{499}\) to estimate risk measures. Figure 3 shows the conditional distribution of \(R_{499}\). The dot on the left hand side, highlighted with a dashed line, is the value at risk at 95% level. It was found that the value at risk at
such level is $-2.2039$, with expected shortfall of $-2.7912$. This means that an investor could expect a loss on this portfolio higher than $2.2039$ with probability $0.05$, and when this threshold is exceeded, the expected loss is of $2.7912$. The observed return is also shown in Figure 3 as a dot with dotted line. We notice that it lies on a region of high density of the predictive distribution.

We can also estimate the conditional distribution of the two-step ahead predictor at $t = 498$, $F(R_{500} \mid F_{498})$. This is shown in Figure 4.

The minimum variance portfolio for a two-step ahead portfolio of assets is calculated with weights $w^{(2)}_{MVP} = (0.4367, 0.2822, 0.2811)$, with an expected return $\mu^{(2)}_{MVP} = -0.3918$, with $\sigma^{(2)}_{MVP} = 1.1784$, showing the increasing uncertainty as we attempt to predict further into the future.

Once again we consider building a portfolio of assets yielding expected return $\mu^* = 0$. Optimal weights for this portfolio are

$$w^{(2)}_{MVP} = \begin{bmatrix} -0.9404 \\ 1.5193 \\ 0.4211 \end{bmatrix}$$

From Figure 4, we notice how the density is now flatter, which is sign of a larger variability. In fact, the estimated standard deviation of $R_{500}$ is $3.5056$,
which is a significant increase. This also results in much larger estimated VaR = $-5.0207$ (in absolute value) at the same 95% level, with expected shortfall equal to $-7.4505$. Once again, the observed return (dotted line) is in a high density region of the predictive distribution.

Overall, we can be satisfied with the performance of our method in predicting portfolio returns.

5 Application to the US stock market

We consider a multivariate dataset of $m = 4$ stocks on the US stock market: Dell Technologies Inc. (DELL), Microsoft Corporation (MSFT), Intel Corporation (INTC), and International Business Machine Corporation (IBM). The data were obtained from Yahoo! Finance (https://finance.yahoo.com). The original time series include daily Adjusted Close Prices between January 2nd 2016 and January 29th 2020 (867 observations). For each series and $t = 2, \ldots, 867$, we calculated daily returns as $(\text{Price}_t - \text{Price}_{t-1})/\text{Price}_{t-1}$. The resulting series, displayed in Figure 5, includes 866 observations.

All four univariate series in Figure 5 appear to be heteroskedastic. Their histograms also show signs of heavy tails, which was confirmed by calculation of sample excess kurtosis (all significantly larger than 0). In addition, from a
preliminary analysis, it was noticed that the data presents autocorrelation at
least at lags 1 and 2, and cross-correlations at lags 0, 1 and 2 (see Figure 6).
Therefore, it is reasonable to consider a MVAR generating process for the data.

Several models were fitted. In terms of diagnostics, a MVAR(3; 3, 2, 1) was
chosen as best fit. Estimation was carried out on the first 864 observations,
omitting the last two for out-of-sample prediction.

Given parameter estimates and the one-step ahead predictive distribution
at $t = 864$, we calculate weights for the minimum variance portfolio built with
these assets, which yields a mean return of approximately 0.0024 (0.24%). The
standard deviation associated with this portfolio is $\sigma_{MVP} = 0.0092$.

Now, assume we would like to increase our mean return to 0.007 = 0.7%.
We can calculate optimal weights

$$w_{EFF} = (-0.5832, 0.9538, 0.1085, 0.5209)^T.$$  

Weights are interpreted as follows: an investor shall short-sell an amount of
around 0.58 times their initial capital in DELL stocks, and reinvest the new
total in the remaining three assets, with a major bet on MSFT and IBM. The
idea behind this is that it is believed that DELL stocks will decrease in value
between the present and the nearest future, and therefore one could short-sell to make a profit. On the other hand, it is believed that the remaining three assets will increase their value in the same time span, and in particular MSFT stocks. However, the standard deviation associated with this portfolio is $\sigma_{\text{EFF}} = 0.0139$, a slight increase compared to $\sigma_{\text{MVP}}$, considering the scale of the data.

For the latter portfolio, we calculate the one-step ahead conditional distribution of $R_{865} = \sum_{m=1}^{4} w_{m} y_{m,865}$ using parameter estimates from the MVAR model fitting:

$$F(R_{865} \mid \mathcal{F}_{864}) = 0.1316\Phi\left(\frac{R_{865} + 0.00052}{0.0266}\right) + 0.5627\Phi\left(\frac{R_{865} + 0.00178}{0.0093}\right) + 0.3057\Phi\left(\frac{R_{865} + 0.01932}{0.0169}\right)$$

The corresponding predictive density can be seen in Figure 7.

Value at risk at $\alpha = 95\%$ is estimated at $-0.0174$, with expected shortfall of $-0.0299$. The subsequently observed return is $R_{865} = -0.0062$, which we can see lies on a region of high density, and therefore is somewhat plausible.

We can also look at building a portfolio of the same assets looking two steps into the future, at $t = 866$. The minimum variance portfolio in this case
yields an expected return $\mu^{(2)}_{\text{MVP}} = -0.011$, with associated standard deviation $\sigma^{(2)}_{\text{MVP}} = 0.0101$. We use the distribution assumptions for $Y_{866}$, its expected value and covariance matrix to estimate optimal weights to look once again to increasing our return by building an efficient portfolio with same target return $\mu^* = 0.007$ as before:

$$w^{(2)}_{\text{EFF}} = (0.0402, 0.6174, 0.7554, -0.4130)^T$$

Using parameter estimates and (14), the conditional distribution of $R_{866}$ is a mixture of $3^2 = 9$ components:

$$F(R_{866} \mid F_{864}) = 0.0173 \Phi \left( \frac{R_{866} - 0.0170}{0.0285} \right) + 0.0741 \Phi \left( \frac{R_{866} - 0.0190}{0.0268} \right) + 0.0402 \Phi \left( \frac{R_{866} - 0.0252}{0.0276} \right) + 0.0741 \Phi \left( \frac{R_{866} - 0.0086}{0.0101} \right) + 0.3166 \Phi \left( \frac{R_{866} - 0.0069}{0.0096} \right) + 0.1720 \Phi \left( \frac{R_{866} - 0.0051}{0.0096} \right) + 0.0402 \Phi \left( \frac{R_{866} - 0.0098}{0.0206} \right) + 0.1720 \Phi \left( \frac{R_{866} - 0.0040}{0.0187} \right) + 0.0935 \Phi \left( \frac{R_{866} + 0.0077}{0.0196} \right)$$

Figure 7: Conditional one-step predictive density of $R_{865}$, with VaR at 95% (dashed line) and observed return (dotted line) highlighted.
The conditional distribution of $R_{866}$ can be seen in Figure 8. We notice, as expected, an increase in the standard deviation of the distribution, $\sigma_{\text{EFF}}^{(2)} = 0.0177$ with respect to $\sigma_{\text{EFF}} = 0.0177$. Overall, the two shapes in Figure 7 and 8 look similar, however the observed return $R_{866}$ is not in a high density region of its predictive distribution, as it actually exceeds the expectations. VaR is now estimated at $-0.021$, with expected shortfall equal to $-0.0315$.

6 Comparing VAR, MVAR and DCC

Dynamic Conditional Correlation models (DCC, Engle 2002) are a class of multivariate GARCH models in which conditional correlations between elements of a vector series are time dependent. In particular, given the conditional covariance matrix of the model at time $t$, $H_t$, each entry $h_{ij}$ of the matrix is modelled as a univariate GARCH model. DCC models are used in finance to predict behavior of vector time series in which the assets are correlated and heteroskedastic, thanks to the fact that the conditional covariance matrix of a predictor is always fully specified.

In his model, Engle (2002) states that correlation is based (i.e. conditional) on information known the previous period, and that correlation matrices of multi-period forecasts are similarly defined. According to this, each time point
produces a different conditional correlation matrix. A similar argument can be presented for MAR and MVAR models, except we work with conditional covariance matrices, rather than correlations. The conditional covariance matrix clearly depends on past observations, and one unique conditional covariance matrix is produced at each time point, except for some limiting cases. As seen in Section 3, this also applies to one and multi-step forecasts. For this reason, we consider a comparison between MVAR and DCC to be appropriate.

We compare here the performance of modelling the data in Section 5 with an MVAR model and a DCC model. We also add a comparison with a fitted vector autoregressive model of order 3 (VAR(3)). We use a rolling-window setup for comparison of the density forecasts.

First, we consider a window from the first observation on January 2nd 2016 to October 10th 2019, thus including 766 observations (roughly 90% of the available data). We use this to estimate MVAR, DCC and VAR models, and derive one and two steps density forecasts for the three models for October 11th and October 12th, and calculate forecasting accuracy using their respective observed values. We then move the window forward by one day. The window would now contain 766 observations from from the first observation on January 3rd 2016 to October 11th 2019. The procedure is repeated until the window contains the most up to date observation on January 28th 2020, which allows a one step forecast. Therefore, we obtain 100 one step density forecasts, and 99 two step density forecasts.

Comparison of density forecasts is usually done using scoring rules. Here we compare the three models in terms of some strictly proper scoring rules: Continuous Ranked Probability Score (CRPS, see for instance Gneiting and Raftery 2007), logarithmic score (LogS, Good 1952), and the Dawid-Sebastiani score (Dawid and Sebastiani 1999, DSS). In all cases, the "preferred" model is the one which minimises the score.

Given an observation \(x\) and the associated forecast distribution \(F\), CRPS is defined mathematically as:

\[
CRPS(F, x) = \int_{\mathbb{R}} (F(y) - I(y \geq x))^2 \, dy
\]

(15)

where \(I(\cdot)\) is the indicator function assuming value 1 when the argument \(y > x\) is true, and 0 otherwise. CRPS is a measure of discrepancy between the forecast CDF, \(F\), and the empirical CDF of the observation \(x\). Notice that it was purposely chosen not to involve factor models in the comparison due to the small number of assets considered.

LogS is calculated as the logarithm of the probability estimate for the ob-
servation with respect to its forecasting distribution:

\[
\text{LogS} (F, x) = \log f (x) .
\]  

(16)

Given the mean \( \mu \) and the variance \( \sigma^2 \) of the predictive distribution, DSS is calculated as

\[
\text{DSS} (F, x) = -\log \sigma^2 - \frac{1}{\sigma^2} (x - \mu)^2
\]  

(17)

A DCC-GARCH(1,1) was found to be the best model for the return series assuming multivariate normal innovations. For this, and for the VAR(3) model, the same rolling window procedure as for MVAR is performed. For each forecast of each model, we calculate CRPS, LogS and DSS, and take the average score for comparison. Results can be seen in Tables 1 and 2:

| Model                  | CRPS   | LogS   | DSS    |
|------------------------|--------|--------|--------|
| MVAR(2; 3, 2, 1)       | 0.004805 | -3.310742 | -8.439584 |
| DCC-GARCH(1,1)         | 0.004845 | -3.326115 | -8.490107 |
| VAR(3)                 | 0.005022 | -3.024682 | -7.887240 |

Table 1: Average scores for one step density forecasts.

| Model                  | CRPS   | LogS   | DSS    |
|------------------------|--------|--------|--------|
| MVAR(2; 3, 2, 1)       | 0.004805 | -3.310742 | -8.439584 |
| DCC-GARCH(1,1)         | 0.004845 | -3.326115 | -8.490107 |
| VAR(3)                 | 0.005022 | -3.024682 | -7.887240 |

Table 2: Average scores for two step density forecasts.

From this comparison, it appears that the only significant differences between MVAR and DCC-GARCH, in terms of forecast accuracy, are in the CRPS for the one step predictor (first column of Table 1), in which on average MVAR outperforms DCC-GARCH, and for DSS in the two step predictor (last column of Table 2), where DCC-GARCH outperforms MVAR instead. However, neither method is objectively better than the other. On the other hand, we notice that the VAR model is far behind in terms of forecasting accuracy, and therefore may not be suitable for predicting portfolio returns. We conclude that our method for portfolio optimisation with MVAR models may be a valid alternative to a widely accepted method such as DCC-GARCH, while it clearly outperforms the standard VAR model.
7 Conclusions

The paper presents an innovative way of using mixture vector autoregressive models for portfolio optimisation. The method consists in deriving analytically predictive distributions of future observations, and use the conditional covariance matrix, together with modern portfolio theory, to build an efficient portfolio and obtain a distribution for future returns. We have seen in fact that, assuming multivariate normal distributions for mixture components, the conditional predictive distribution of the portfolio return at a future horizon $h$ itself follows a (univariate) mixture of $g^h$ normal components, depending on observation up to the present.

The methodology was tested both on a simulated and a real dataset. For the latter, we compared performance of MVAR with the widely used dynamic conditional correlation model, which uses multivariate GARCH to estimate conditional correlations, and with the VAR model, using a rolling-window forecasting scheme. In particular, forecasting accuracy was assessed using three strictly proper scoring rules, averaged over the number of forecasts. In terms of minimum variance portfolios, the conclusion was that the MVAR and DCC-GARCH have similar performance on the analysed datasets, suggesting MVAR may be considered a valid alternative to DCC-GARCH. Furthermore, it was seen that MVAR outperformed VAR.

There are various ways in which the methodology could be extended. One possibility is to employ the GMVAR model (Kalliovirta et al. 2016) in place of the MVAR model. GMVAR has the useful property that the mixing weights depend on past values of the process. On the other hand, the region for the autoregressive parameters of GMVAR is restricted to a subset of that of MVAR. Also, MVAR and GMVAR have different dynamics and stationary distributions. So the two classes of models complement each other.

Another possible extension of our method is to incorporate factor models, in order to be able to model a large number of possibly correlated assets at a time. In addition, distribution assumptions other than normal can be made on the innovation terms. For example, considering a distribution with heavy tails might need a smaller number of components to fit the data. However, estimation would become much more complicated, and numerical algorithms would be required.
A Property of the multivariate normal distribution

At several places we use the standard result that any linear combination of the elements of a random vector from the multivariate normal distribution is univariate normal. More specifically, let $X$ be a random vector of length $m$ following a multivariate normal distribution with mean $\mu$ and covariance matrix $\Sigma$. Let $a$ be a constant vector of the same length as $X$. Then

$$a^TX = \sum_{j=1}^{m} a_j X_j \sim N(a^T \mu, a^T \Sigma a)$$ (18)

B Derivation of the representation of $y_{t+2}$ and its characteristic function

We here derive the analytic expression for the predictor $y_{t+2}$ when information up to time $t$ is available. We also derive its characteristic function, to show that the distribution of such predictor is a mixture of $g^2$ normal distributions. Proof is simply the multivariate version of the proof in Boshnakov (2009). Let $Z_t \in \{1, \ldots, g\}$ be the allocation random variable defined in Section 2, and assume $z_{t+2} = k, z_{t+1} = l$ at times $t + 2$ and $t + 1$. We have that

$$y_{t+2} = \mu_{t+2,k} + \Omega_t^{1/2} \varepsilon_{t+2,k}$$

$$= \mu_{t+2,k} - \Theta_{k,1} Y_{t+1} + \Theta_{k,1} \mu_{t+1} + \Omega_t^{1/2} \varepsilon_{t+2,k}$$

$$= (\mu_{t+2,k} - \Theta_{k,1} Y_{t+1} + \Theta_{k,1} \mu_{t+1} + \Theta_{k,1} \Omega_t^{1/2} \varepsilon_{t+1,l} + \Omega_t^{1/2} \varepsilon_{t+2,k})$$

$$= \mu_{t+2,k,l} + \Theta_{k,1} \Omega_t^{1/2} \varepsilon_{t+1,l} + \Omega_t^{1/2} \varepsilon_{t+2,k}$$ (19)

where $\varepsilon_{t+h,k}$ is the innovation term associated with the $k^{th}$ component.

We want an expression that does not contain $Y_{t+1}$. Hence, we rewrite $\mu_{t+2,k,l}$
as

\[ \mu_{t+2,k,l} = \mu_{t+2,k} - \Theta_{k,1}Y_{t+1} + \Theta_{k,1}\mu_{t+1,l} \]

\[ = \Theta_{k,0} + \sum_{i=1}^{p} \Theta_{k,i}Y_{t+2-i} - \Theta_{k,1}Y_{t+1} + \Theta_{k,1} \left( \Theta_{l,0} + \sum_{i=1}^{p} \Theta_{l,i}Y_{t+1-i} \right) \]

\[ = \Theta_{k,0} + \Theta_{k,1}\Theta_{l,0} - \Theta_{k,1}Y_{t-1} + \Theta_{k,1}Y_{t+1} + \sum_{i=2}^{p} \Theta_{k,i}Y_{t+2-i} \]

\[ + \Theta_{k,1} \sum_{i=1}^{p} \Theta_{l,i}Y_{t+1-i} \]

\[ = \Theta_{k,0} + \Theta_{k,1}\Theta_{l,0} + \sum_{i=1}^{p-1} \Theta_{k,i+1}Y_{t+1-i} + \Theta_{k,1} \sum_{i=1}^{p} \Theta_{l,i}Y_{t+1-i} \]

\[ = \Theta_{k,0} + \Theta_{k,1}\Theta_{l,0} + \sum_{i=1}^{p-1} (\Theta_{k,i+1} + \Theta_{k,1}\Theta_{l,i})Y_{t+1-i} + \Theta_{k,1}\Theta_{l,p}Y_{t+1-p} \]

And therefore we have the expression for \( Y_{t+2} \)

\[ Y_{t+2} = \Theta_{k,0} + \Theta_{k,1}\Theta_{l,0} + \sum_{i=1}^{p-1} (\Theta_{k,i+1} + \Theta_{k,1}\Theta_{l,i})Y_{t+1-i} + \Theta_{k,1}\Theta_{l,p}Y_{t+1-p} \]

\[ + \Theta_{k,1}\Omega_{k,1/2}^{1/2}e_{t+1,l} + \Omega_{k,1}^{1/2}z_{t+2,k} \]

We deduce that, given observed \( z_{t+2}, z_{t+1} \):

\[ \mathbb{E}[Y_{t+2} \mid Z_{t+2}, Z_{t+1}, F_{t}] = \Theta_{k,0} + \Theta_{k,1}\Theta_{l,0} + \sum_{i=1}^{p-1} (\Theta_{k,i+1} + \Theta_{k,1}\Theta_{l,i})Y_{t+1-i} + \Theta_{k,1}\Theta_{l,p}Y_{t+1-p} \]

\[ \text{Cov}(Y_{t+2} \mid Z_{t+2}, Z_{t+1}, F_{t}) = \Theta_{k,1}\Omega_{k,1} + \Omega_{k} \]

We now need to derive the characteristic function for the predictor. Recall the characteristic function for the multivariate normal distribution and \( Y_{t+1} \) can be written as

\[ \varphi_{t+1} \equiv \mathbb{E}\left[ e^{is^T Y_{t+1}} \mid F_{t} \right] = \mathbb{E} \left[ \sum_{k=1}^{g} \pi_k e^{is^T \mu_{t+1,k}} \varphi_k(\Omega_{k}^{1/2}) \right] \]
It follows that, for $Y_{t+2}$, we have

$$\varphi_{t+2}(s) \equiv E \left[ e^{is^{T}Y_{t+2}} \mid F_t \right] = E \left[ E \left( e^{is^{T}Y_{t+2}} \mid z_{t+2}, z_{t+1}, F_t \right) \mid F_t \right]$$

$$= E \left[ e^{is^{T}Y_{t+2}} \mid F_t \right]$$

$$= \sum_{k,l=1}^{g} \pi_k \pi_l e^{i s^{T} \mu_{t+2,k,l}} \varphi_1(\Theta_{k,1} \Omega_{l}^{1/2} s) \varphi_2(\Omega_{k}^{1/2} s)$$

Thus, the conditional distribution of $Y_{t+2}$ given $F_t$ is a mixture of $g^2$ components with mixing weights $\pi_k \pi_l$. For a normal mixture, we also have that:

$$\varphi_1(\Theta_{k,1} \Omega_{l}^{1/2} s) \varphi_2(\Omega_{k}^{1/2} s) = e^{\Theta_{k,1,1} \Omega_{k,1}^{1/2} \Omega_{l}^{1/2} s} = e^{\Theta_{k,1,1} \Omega_{k,1}^{T} + \Omega_{k}}$$

which shows that the conditional distribution of the two-step predictor is a mixture of Normals with means $\mu_{t+2,k,l}$ and covariance matrices $\Theta_{k,1} \Omega_{l} \Theta_{k,1}^{T} + \Omega_{k}$.

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