Vibration isolation with passive linkage mechanisms

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Abstract This study is to present a novel way to achieve superior passive vibration isolation by employing a specially designed and compact linkage mechanism. The proposed anti-vibration system has beneficial nonlinear inertia, inspired by swinging motion of human arms, and is constructed with an adjustable nonlinear stiffness system inspired by animal or human leg skeleton. It is shown with comprehensive theoretical analysis and consequently validated by a series of well-designed experiments that the nonlinear stiffness, nonlinear damping and nonlinear inertia of the proposed system are very helpful for significantly reducing resonant frequency and enhancing damping effect in a beneficial nonlinear way. This results in excellent vibration isolation performance with lower resonant frequency and resonant peak of faster decay rate. This study provides an innovative solution to a cost-efficient vibration control demanded in various engineering systems.

Keywords Nonlinear vibration isolation · X-shaped mechanisms · Nonlinear damping · Nonlinear inertia · Nonlinear stiffness

1 Introduction

For engineering vibration control, it is very challenging to achieve satisfactory vibration isolation at relatively low frequencies (e.g., below 5 Hz) [1]. Usually, a smaller resonant frequency of an isolation system can be obtained by applying smaller stiffness, which further results in better vibration isolation. However, a minimum stiffness requirement is often encountered due to the loading capacity. Meanwhile, larger damping coefficient is often applied to suppress resonance peak, which may consequently result in a deteriorated vibration transmission at relatively high frequencies. Although passive vibration isolation is often more preferable than active vibration control in practice [2], traditional linear passive vibration control theory cannot solve the mentioned problems above [3–6].

To overcome disadvantages of linear vibration isolation systems, nonlinearities have been explored in the literature for designing various nonlinear stiffness and nonlinear damping systems [3–9]. In [7–9], the nonlinearity due to oblique springs can enable the system to demonstrate a high static stiffness but with a
low resonant frequency. However, strict parameter settings required for achieving desired nonlinearities in many existing nonlinear systems may lead to difficulties in practical implementation, and small interferences could yield diverging outcomes [10]. Moreover, a reliable vibration isolation system should circumvent strong nonlinearity existing in the system, such as bifurcation (e.g., jumping) or chaos, etc. [11, 12].

Active vibration control is extensively studied with advanced actuators and velocity or displacement feedback control schemes [13, 14]. However, active approaches not only require excessive energy [15, 16] but also significantly increase system complexity or maintenance cost [17, 18]. Vibration suppression can also be done via design of active or passive inertia units [19, 20]. For example, tuned inertia dampers are known to be working well with auxiliary mass introduced to a primary structure [21–23], and the working frequency of such kind of vibration absorbers can be adjusted with the auxiliary mass/inertia. If the inertia effect can be much larger than their mass, the inertia elements would be very effective. Thus, it might be preferable in lightweight applications such as automotive and aerospace applications.

To employ nonlinear benefits in passive vibration control, a bio-inspired passive X-shaped structure/mechanism is recently proposed, which can achieve beneficial nonlinear stiffness and damping characteristics in vibration isolation and suppression, with adjustable QZS property and loading capacity, and can be well applied to various engineering scenario [24–29]. Preliminary results about the X-shaped mechanism as a passive isolator showed that the X-shaped mechanism is an effective way for providing adjustable nonlinear stiffness and damping properties, which are beneficial to vibration control and easy to circumvent potential instability or jumping phenomena via parameter design. Moreover, the X-shaped mechanism demonstrates good loading capacity with excellent static stability and easiness in implementation.

In this paper, a new compact bio-inspired anti-vibration platform with a specially designed nonlinear inertia unit (BIAVP-NI) is proposed, which is based on the X-shaped mechanism and to mimic the human body leg and arm structures and muscle motion to investigate novel vibration isolation performance. Previous studies indicate that human body shows excellent vibration isolation performance to prevent the brain from being subjected to excessive shocks and vibrations [30–35]. The legs movement with bended knees result in a geometrical nonlinearity (Fig. 1), which contributes to reduce the impact during human walking [24, 25, 35]. The highly flexible and strong muscle also shows beneficial nonlinear effects and acted as an absorber to suppress vibration [36]. The motion of the upper body with swing arms help to balances the center of mass and provide excellent stability during walking and jumping [37–40].

Inspired by the principle of human body in vibration isolation, the proposed BIAVP-NI has a X-shaped supporting mechanism, coupled with a small embedded horizontal X-shaped mechanism. This is to mimic human leg skeleton, and nonlinear tension and compression muscles and tendons. Meanwhile, a compact rotational unit is designed for simulating body rotation in walking. The schematic diagram is shown in Fig. 2.

One of the most important issue in engineering practice is how to realize those identified beneficial nonlinear stiffness, damping and inertia together within a single compact design. This paper is to present such a very compact prototype with new arrangement of springs and rotation units and new realization of damping mechanisms through a compactly embedded small X-shaped mechanism. The new design can automatically avoid potential negative stiffness generated by the large X-shaped mechanism, and the coupled inertia and damping effects are even better in vibration suppression.

Therefore, in the proposed anti-vibration system (Fig. 2a), a larger X-shaped supporting mechanism is used to support the payload object; a rotational unit driven by a rack and pinion mechanism is installed within the supporting mechanism in a much more compact way; a small X-shaped mechanism equipped with springs and dampers is horizontally installed into one layer of the X-shaped supporting mechanism. To ensure adjustability in different vibration isolation scenarios of engineering practices, structural parameters of the proposed BIAVP-NI are all tunable due to the flexibility of linkage connections. For example, the installment position of each joint can be easily changed with different rod holes such that the rod length and connection angle would be changed immediately, and pre-suppression or extension of the installed springs can also be changed to adjust the
assembly angle or working position, etc. All those changes would lead to the adjustment of the overall equivalent stiffness, damping and inertia effect (referring to the detailed parametric influence session). The adjustability of the inertia of the BIAVP-NI can be achieved by changing rotational mass or associated rod length, and it can also be done through tuning the structural parameters of the large X-shaped mechanism due to the nonlinear coupling relationship. The coupling nonlinearity between the inertia unit and the X-shaped mechanism presents new nonlinear dynamics for understanding. Compared with the linear inertia directly driven by a rack and pinion, the proposed BIAVP-NI has advantages in nonlinear characteristics and isolation performance.

Due to the new designs, BIAVP-NI has the following advantages. Firstly, the BIAVP-NI has high-static but low-dynamic stiffness. This can ensure loading capacity and quasi-zero-stiffness (QZS) for vibration isolation, allowing the BIAVP-NI to have an ultra-low resonant frequency and effectively isolate low frequency vibrations. Secondly, a nonlinear inertia unit (Fig. 2b) in specially introduced and horizontally installed within the X-shaped supporting mechanism, which can further reduce the resonant frequency. Due to the nonlinear inertia characteristics with different rotational velocity in extension and compression (Fig. 3), the fluctuation of the supporting force can be reduced to improve the stability and suppress vibration transmission. Thirdly, the BIAVP-NI has ideal favorable nonlinear damping characteristics. The system has larger damping effect for low frequency and large amplitude, but smaller damping effect for high frequency and small amplitude. This nonlinear damping quickly attenuates vibration and avoids increasing vibration at high frequencies.

The theoretical analyses and comparison studies will show that the rotational unit coupled with the
X-shaped mechanism in the proposed system can obviously enhance the nonlinear coupling characteristics between the translational and rotational directions and thus lead to much better isolation performance consequently. Although similar X-shaped structures are adopted, the modeling and analysis results of the proposed BIAVP-NI cannot be simply generalized from existing results in [24, 25]. The inertia unit of this BIAVP-NI is much better designed with a compact structure, compared to the previous work in [28, 29], with much bigger inertia effect and easier implementation.

The rest of this paper is organized as follows. The structure of the BIAVP-NI and mathematical modeling are given in Sect. 2. Section presents the nonlinear inertia properties with parametric analysis. The displacement transmissibility is obtained through the Harmonic Balance Method (HBM) in Sect. 4. Section 5 and Sect. 6 will discuss the nonlinear transient response and sweep frequency response, respectively. Some comparison studies are discussed in Sect. 7. Experimental validation is presented in Sect. 8, followed by a conclusion.

2 Modeling

The lumped parameter element model of BIAVP-NI is shown in Fig. 4a. The object $M_1$ is supported by a X-shaped mechanism. A rotational unit as shown in introduced between the two horizontal joints, which is driven by rack and pinion mechanism and the radius of the gear is $r$. The rotational disc has uniform quality with the mass of $M_2$ and radius $R$. The rotational displacement of the disc is $\theta$. A smaller X-shaped mechanism is installed as shown in Fig. 4b. This is to mimic the elastic function due to muscles or tendons with X-way nonlinearities. The rod lengths of the X-shaped supporting mechanisms are denoted by $l$ and $l'$, respectively. The vertical spring has a stiffness $k$ and the vertical damper has a coefficient $c$. The angles $\alpha$ and $\beta$ are as shown. The absolute displacements of mass $M_1$ and the base are denoted by $y$ and $z$.

The deflection of the spring and damper $x$ can be expressed as

$$
\left( \frac{l' \sin \beta + \frac{x'}{2n'}}{2} \right)^2 + \left( \frac{l' \cos \beta - \frac{x'}{2}}{l} \right)^2 = l^2
$$

where $n$ and $n'$ are the layer numbers of the larger and smaller X-shaped supporting mechanism. Use $x'$ to denote the horizontal displacement of a joint in the X-shaped supporting mechanism, which is given by

$$
\left( \frac{y-z}{2n} + l \sin z \right)^2 + \left( l \cos z - \frac{x'}{2} \right)^2 = l^2
$$

The relation among $x'$, $x$ and $\theta$ can be given by (referring to Fig. 4b)

$$
x' = 2l \cos z - 2\sqrt{l^2 - \left(l \sin z + \frac{y}{2n} \right)^2}
$$

$$
x = 2l' \cos \beta - 2\sqrt{l'^2 - \left(l' \sin \beta + \frac{x'}{2n'} \right)^2}
$$
\[
\theta = \sqrt{\left(\frac{l}{r} \cos \alpha + \frac{x'}{2r}\right)^2 - 1} - \sqrt{\left(\frac{l}{r} \cos \alpha\right)^2 - 1} \quad (5)
\]

where
\[
\hat{y} = y - z \quad (6)
\]

The kinetic energy of the BIAVP-NI can be written as
\[
T = \frac{1}{2} (M_1 + M_2) \hat{y}^2 + \frac{1}{2} \left(\frac{1}{2} M_2 R^2\right) \hat{\theta}^2 \quad (7)
\]

where the angular velocity \( \hat{\theta} \) is
\[
\hat{\theta} = \left(\frac{\partial \theta}{\partial \hat{y}}\right) \left(\frac{\partial \hat{y}}{\partial \partial t}\right) \quad (8)
\]

The potential energy of the BIAVP-NI is given by
\[
V = \frac{1}{2} k_l (x + l_s)^2 + \frac{1}{4} k_n (x + l_s)^4 + (M_1 + M_2)(y + y_s)g \quad (9)
\]

where \( y_s \) is the displacement after the loading of the mass. More practically, the spring stiffness is supposed to satisfy \( f = k_l(\cdot) + k_n(\cdot)^3 \). The spring compression is denoted by \( l_s \). Apply the Lagrange principle for the equation of motion
\[
\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = -D \quad (10)
\]

The dissipated energy \( D \) is
\[
D = c \dot{\hat{y}} \quad (11)
\]

where \( c \) is the damping coefficient. For the term \( L \)
\[
L = T - V \quad (12)
\]

It is obtained by
\[
L = \frac{1}{2} (M_1 + M_2) \hat{y}^2 + \frac{1}{4} M_2 R^2 \hat{\theta}^2 \left(\frac{\partial \theta}{\partial \hat{y}}\right)^2 - \frac{1}{2} k_l (x + l_s)^2 - \frac{1}{4} k_n (x + l_s)^4 - (M_1 + M_2)(y + y_s)g \quad (13)
\]

Thus, the equation of motion for the BIAVP-NI can be obtained as
\[
(M_1 + M_2) \ddot{y} + \frac{1}{2} M_2 R^2 \left(\frac{\partial \theta}{\partial \hat{y}}\right)^2 \ddot{\hat{y}} - \frac{1}{2} M_2 R^2 \frac{\partial \theta}{\partial \hat{y}} \frac{\partial^2 \theta}{\partial \hat{y}^2} \ddot{\hat{y}}^2 + \left[k_l (x + l_s) + k_n (x + l_s)^3\right] \frac{\partial x}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \hat{y}} + (M_1 + M_2)g = -c \frac{\partial x}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \partial t} \quad (14)
\]

Define several nonlinear functions \( f_1, f_2, f_3 \) and \( f_4 \) as
where substituting (3) to (5) into (15) to (18), the equation of motion is given by

\[ f_1(y) = \frac{R^2}{2} \left( \frac{\partial \theta}{\partial y} \right)^2 \]  
(15)

\[ f_2(y) = \frac{R^2}{2} \frac{\partial^2 \theta}{\partial y^2} \frac{\partial^2 \theta}{\partial y} \]  
(16)

\[ f_3(y) = \left[ k_1 + k_n(x + l_o)^2 \right] (x + l_o) \frac{\partial^2 y}{\partial y^2} \]  
(17)

\[ f_4(y) = \frac{\partial x}{\partial y} \]  
(18)

Substituting (15) to (18) into (14) then the equation of motion of the BIAVP-NI is

\[(M_1 + M_2 + M_2 f_1(y)) \dot{y} - M_2 f_2(y) \ddot{y} + f_3(y) + c f_4(y) \ddot{y} + (M_1 + M_2) g = -(M_1 + M_2) \ddot{z} \]  
(19)

where substituting (3) to (5) into (15) to (18), \( f_1, f_2, f_3 \) and \( f_4 \) are given by

\[ f_1(y) = \frac{R^2 \psi_1(y)}{8r^2 n^2 (\varphi^2 - \psi_1^2(y))} \]  
(20)

\[ f_2(y) = \frac{R^2 \varphi^2 \psi_1(y)}{16r^2 n^3 (\varphi^2 - \psi_1^2(y))^2} \]  
(21)

\[ \psi_2(y) = 4 \left( 2 \sqrt{r^2 - (\dot{y}/2n + l \sin \varphi)^2} - 2l \cos \varphi \right) / r^2 \]  
(25)

\[ \psi_3(y) = l' \sin \beta \left( \cos \varphi - \sqrt{r^2 - (\dot{y}/2n + l \sin \varphi)^2} \right) / n' \]  
(26)

Replacing the \( f_1, f_2, f_3 \) and \( f_4 \) by \( F_1, F_2, F_3 \) and \( F_4 \) in Eq. (19), the equation of motion is given by

\[ \psi_1(y) \psi_3(y) \left( 2 \sqrt{l^2 - \psi_1^2(y)} - 2l \cos \beta + l \right) \left[ k_1 + k_n \left( 2 \sqrt{l^2 - \psi_1^2(y)} - 2l \cos \beta + l \right) \right] \]

\[ \frac{f_3(y)}{n \sqrt{l^2 - \psi_1^2(y)} / \sqrt{l^2 - \psi_1^2(y)}} \]  
(22)

\[ f_4(y) = \frac{\psi_1(y) \psi_3(y)}{mn \sqrt{l^2 - \psi_1^2(y)} / \sqrt{l^2 - \psi_1^2(y)}} \]  
(23)

where

\[ \psi_1(y) = \dot{y}/2n + l \sin \varphi \]  
(24)

\[ [M_1 + M_2 + M_2 F_1(y)] \dot{y} - M_2 F_2(y) \ddot{y} + F_3(y) + c F_4(y) \dot{y} + (M_1 + M_2) g \]

\[ = -(M_1 + M_2) \dot{z} \]  
(31)

Clearly, there are four nonlinear items. \( M_2 F_1(y) \) is nonlinear equivalent mass including the nonlinear inertia. \( M_2 F_2(y) \) is a nonlinear force additionally introduced by the rotational unit. \( F_3(y) \) is the nonlinear stiffness of the system, and \( c F_4(y) \) is the nonlinear damping term which are both incurred by the
X-shaped mechanisms including the complicated nonlinear coupling effects.

### 3 Effects of nonlinearities

Beneficial structural nonlinearities of inertia, stiffness and damping are important characteristics that affects the effect of vibration reduction. The nonlinear damping has been fully investigated in previous studied [28, 29]. In this section, in-depth understanding about the influence of structural parameters on the nonlinear inertia and nonlinear stiffness will be developed, focusing on the two nonlinear functions $F_1(\dot{y})$ and $F_2(\dot{y})$.

#### 3.1 Nonlinear inertia and inertia-incurred nonlinear force

The nonlinear inertia is referred to the nonlinear equivalent mass, given by $M_2 F_1(\dot{y})$, i.e., $F_1(\dot{y})$. Define an indicator

$$\xi = \frac{R}{r}$$

Substituting the coefficients $\eta_0$ to $\eta_3$ into (27),

$$F_1(\dot{y}) = \frac{\xi^2 (\tan \alpha)^2}{8n^2} + \frac{\xi^2 \tan \alpha}{8n^3 l (\cos \alpha)} \dot{y}^3$$

$$+ \frac{\xi^2 (3 \sin^2(\alpha) + 1)}{16l^2 n^4 (\cos \alpha)^6} \dot{y}^2$$

$$+ \frac{\xi^2 (\sin^3(\alpha) + \sin(\alpha))}{16l^3 n^5 (\cos \alpha)^8} \dot{y}^3$$  \(33\)

From Eq. (33), the function $F_1(\dot{y})$ has relation with parameters $n$, $l$, $\xi$ and $\alpha$. An analysis of the parametric influence is thus conducted. By default, the parameter setting is given by $n = 2$, $l = 100$ (mm), $\xi = 10$ and $\alpha = 45$ (deg). The results are shown in Fig. 6.

Although the mass of the rotational unit is a constant, the equivalent mass increases basically from a smaller value during compression to a bigger value during extension. Meanwhile, the changing trend is more nonlinear for smaller layer number $n$, smaller rod length $l$, bigger ratio $\xi$ or bigger angle $\alpha$. 

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**Fig. 5** The original nonlinear functions and Taylor series expanded functions

(a) Comparisons between $f_1(\dot{y})$ and $F_1(\dot{y})$

(b) Comparisons between $f_2(\dot{y})$ and $F_2(\dot{y})$

(c) Comparisons between $f_3(\dot{y})$ and $F_3(\dot{y})$

(d) Comparisons between $f_4(\dot{y})$ and $F_4(\dot{y})$
The equivalent mass around $\dot{y} = 0$ for small vibration amplitude is basically the same for different rod length $l$. However, it is obviously changed for different $n$, $\zeta$ or $\alpha$ around $\dot{y} = 0$.

The equivalent mass tends to be a constant in $\dot{y}$ for smaller $\zeta$ (or $\alpha$) or bigger $n$ (or $l$) for a smaller $\dot{y}$. These imply that a structural optimization is needed for a specific application.

The equivalent mass is significantly increased for special parameter setting when the vibration amplitude is bigger as shown and the increase is even more in extension than in compression, demonstrating a special nonlinear property.

Due to the nonlinear effect, the equivalent mass can be increased several–ten times bigger than the original mass $M_2$ with appropriate parameter settings. This can therefore be used to significantly change the resonant frequency of the system for better vibration isolation performance.

The equivalent mass is presented as $M_2 F_1 (\dot{y}) \ddot{y}$, which is a nonlinear inertial force item in the equations of motion. For convenience, $F_x$ represents the nonlinear inertial force $M_2 F_1 (\dot{y}) \ddot{y}$. Assuming the displacement, velocity and acceleration as

$$\dot{y} = a \sin(\omega t)$$

$$\ddot{y} = a \omega \cos(\omega t)$$

$$\dddot{y} = -\omega^2 a \sin(\omega t)$$

The acceleration can then be written as

$$\dddot{y} = -\omega^2 a \cos(\omega t) = -\omega^2 \ddot{y}$$

Thus, the nonlinear inertial force $F_x$ is given by

$$F_x = M_2 F_1 (\ddot{y}) \dddot{y} = -\omega^2 M_2 F_1 (\dddot{y})$$

The nonlinear force $F_x$ is a function of $F_1 (\ddot{y})$, and thus a function of those related structural parameters including $n$, $\alpha$, $l$ and $\zeta$. To understand the force $F_x$, the following analysis is conducted. The default parameter setting is set as: $n = 2$, $l = 100$ (mm), $\zeta = 10$, $\alpha = 45$ (deg). The vibration amplitude $a$ is 30 (mm). The nonlinear force $F_x$ in a vibration cycle is shown in Fig. 7.
From Fig. 7, the nonlinear force $F_x$ demonstrates very special nonlinear influence with different parameter settings. It can be concluded that,

1. The values of the nonlinear force $F_x$ are varying with different parameter settings and also different in compression and extension states; larger $n$ or $l$, or smaller $\alpha$ or $n$ lead to a much smaller value (close to zero) of the nonlinear force $F_x$, and on the contrary the values can be very large;

2. The nonlinear force $F_x$ is asymmetric and it is relatively much bigger in extension than in compression, which would be special and beneficial nonlinear property to be employed in practice.

For more understanding, further discussions can be referred to Sect. 7.3.

3.2 Stiffness nonlinearities

The nonlinear stiffness coefficient of the bio-inspired anti-vibration platform with nonlinear inertia is given by $f_3(\dot{y})$. The loading force $F_l$ with the compression of $y_\Lambda$ is given by

$$F_l(y_\Lambda) = |f_3(-y_\Lambda)|$$  \hspace{1cm} (39)

With Eqs. (23), (25) and (27), it can be obtained that

$$F_l(y_\Lambda) = \left| \frac{\psi_1(-y_\Lambda)\psi_3(-y_\Lambda)\left(2\sqrt{l^2 - \psi_3^2(-y_\Lambda)} - 2l\cos \beta + l_t\right)}{nnl\sqrt{\beta - \psi_1^2(-y_\Lambda)}\sqrt{l^2 - \psi_3^2(-y_\Lambda)}} \left[k_n + k_l\left(2\sqrt{l^2 - \psi_3^2(-y_\Lambda)} - 2l\cos \beta + l_t\right)\right]^2 \right|$$  \hspace{1cm} (40)
Consider $n = 2$, $n' = 2$ as example. Then, the loading force above is shown in Fig. 8.

Figure 8a indicates that with compression the stiffness of the system is becoming smaller from a larger value, to zero and further to negative values, and the loading force is increasing at the beginning and becoming smaller after a maximum value, irrespectively of linear or nonlinear springs; (b) a nonlinear spring can enhance the loading capacity and quasi-zero-stiffness area, compared to a linear spring. These demonstrate very special nonlinear properties of this bio-inspired anti-vibration platform and also a special adjustable stiffness property, which can be well exploited in engineering practices.

For more detailed, if the structural parameters are not properly set for a given payload when using a linear spring, a negative stiffness might be quickly incurred in the compression range beyond the limit (> 50 mm) as shown in Fig. 8b where the loading force can drop down shortly. To ensure a stable working range, the correct loading range is thus limited before point B. However, when the cubic nonlinear coefficients $f = k_1x + k_nx^3$ are adopted, the effective loading range can be expanded to point C and the quasi-zero-stiffness range is also obviously enlarged consequently. This shows that a nonlinear spring in practice can actually enhance the loading capacity and quasi-zero-stiffness property with this bio-inspired anti-vibration mechanism. This is a very special benefit or advantage which is not seen in other existing nonlinear vibration systems.

4 Displacement transmissibility

The equation of motion can be rewritten as

$$
[M_1 + M_2(\varepsilon_0 + \varepsilon_1y + \varepsilon_2y^2 + \varepsilon_3y^3)]\ddot{y} + M_2(\lambda_4 + \lambda_6y + \lambda_8y^2 + \lambda_3y^3)^2\dddot{y} + (\lambda_0 + \lambda_1y + \lambda_2y^2 + \lambda_3y^3)\ddot{y} + (M_1 + M_2)g = -M_1\dddot{\varepsilon}
$$

where the base excitation $z = z_0\cos(\omega t + \phi)$. The displacement transmissibility $T_d$ can be obtained as

$$
T_d = \frac{\sqrt{a^2 + z_0^2 + 2az_0\cos\phi}}{z_0}
$$

The displacement transmissibility can be obtained with the harmonic balance method, which is determined by several structural factors ($\alpha$, $\beta$, $\zeta$, $n$, $n'$, $k_1$, $c$, $M_1$ and $M_2$).

4.1 Parametric influence

The effect of the leverage ratio $\zeta$ is shown in Fig. 9. Clearly, a larger ratio leads to a smaller resonant frequency, a faster decay rate and a lower transmissibility over all frequencies above the resonant frequency, showing a better isolation performance in
a larger frequency range. Therefore, the leverage ratio can be designed to tune the performance of the system in practice.

However, the resonant peak is increased with the increase of the leverage/rod ratio $\zeta$. This can be solved by tuning other structural parameters which will be discussed later.

The effect of the stiffness $k$ is shown in Fig. 10a. A smaller stiffness $k$ leads to smaller resonant frequency and peak value. This is absolutely good for vibration isolation. However, in norm case the equivalent stiffness should be strong enough to satisfy a desired loading capacity. It is also noted that, a change of stiffness $k$ gives almost no effect on transmissibility of frequencies higher than 50 Hz.
Figure 10b shows the effect of damping $c$. An increase of damping $c$ leads to a smaller resonant peak value with a smaller increase of the decay rate but almost no influence at high frequencies more than 50 Hz. This demonstrates a very beneficial nonlinear damping effect, which is expected to be more influential around resonant frequency but little at other frequencies as discussed in [4–6, 24]. Moreover, when the damping coefficient $c$ is much smaller, an anti-resonant peak can be seen around 10 Hz, which can potentially be employed in practice too.

Figure 11 shows the effect of different mass $M_1$ and $M_2$. The resonant frequency is slightly decreased for a larger payload mass $M_1$ and $M_2$. This is consistent with traditional knowledge. But it can be seen that the increase of the rotational mass $M_2$ results in a much bigger change in the resonant frequency than the payload mass $M_1$. This confirms that the rotational unit does contribute a nonlinear inertial effect beneficial to vibration isolation.

Figure 12 shows the influence from angles $\alpha$ and $\beta$. Clearly, a smaller value of angle $\alpha$ or $\beta$ can produce a much smaller resonant frequency. This is very beneficial to isolation performance improvement. However, the resonant peak values are slightly increased, indicating a slight decrease in the equivalent damping. Moreover, the smaller angles also result in a more compact of the structure, which might be expected in practice due to space limitation. It should be noted that the change of angle $\beta$ has no influence on transmissibility of high frequency but it is not for angle $\alpha$. Therefore, to improve high frequency transmissibility, angle $\alpha$ can be tuned.

The effect of the layer number $n$ and $n'$ are shown in Fig. 13. It shows that, increasing layer number $n$ or $n'$ can both reduce the resonant frequency significantly and the high frequency transmissibility can be further reduced by increasing layer number $n$. However, the resonant peaks are slightly increased with the increase of layer numbers including a decrease of the equivalent damping. Moreover, the layer number $n'$ has no influence on high frequency vibration.

4.2 A summary of the parametric influence

In a summary, the transmissibility of the BIAVP-NI is significantly determined by the following parameters: angles $\alpha$, $\beta$, ratio $\zeta$, layers $n$, $n'$, stiffness $k$, damping $c$ and payload mass $M_1$ and properties of the rotational unit $M_2$. The parameter influence is summarized in Table 1.

The boxplots are used to quantify the effects of parameters variation on resonant frequency and peak value, which are shown in Fig. 14. It is clear to see that structural layers have a greater impact on the resonant frequency. Meanwhile, increasing the damping can greatly reduce the system resonance peak.

Compared to traditional vibration isolation systems, the nonlinear characteristics of the proposed BIAVP-NI can be summarized as follows:

1. The system has quasi-zero resonant frequency by tuning structural parameters;
2. A larger leverage ratio $\zeta$, smaller $k$, smaller angles $\alpha$ or $\beta$, bigger $M_2$, larger layer number $n$ and $n'$ can significantly reduce the resonant frequency;
3. The leverage ratio $\zeta$, angle $\alpha$, layer number $n$ can be used to reduce high frequency transmissibility obviously;
Table 1  Summary of parameter influence: ↓ decreased; ↑ increased; – no change

| Parameters                              | Resonant frequency | Resonant peak value | Anti-resonant frequency | Anti-resonant peak value | High frequency transmissibility |
|-----------------------------------------|--------------------|---------------------|--------------------------|--------------------------|-------------------------------|
| Increase of X-shaped structure          | Assembly angle α   | ↑                   | ↓                        | –                        | ↑                             |
| related parameters                      | Assembly angle β   | ↑                   | ↓                        | –                        | –                             |
| Layer number n                          | ↓                  | ↑                   | –                        | ↓                        | –                             |
| Layer number n’                         | ↓                  | ↑                   | ↑                        | –                        | –                             |
| Payload mass M₁                         | ↓                  | ↑                   | –                        | ↓                        | –                             |
| Spring stiffness k                       | ↑                  | ↑                   | –                        | ↑                        | –                             |
| Damping c                                | –                  | ↓                   | –                        | –                        | –                             |
| Increase of inertia related             | Rotational mass M₂ | ↓                  | ↑                        | ↑                        | –                             |
| structural parameters                   | Length ratio ξ     | ↓                  | ↑                        | –                        | ↓                             |

Fig. 14  Influence of parameter parameters variation on resonance

Fig. 15  Definitions of nonlinear damped free vibration responses specifications
The damping effect of the method in this proposed mechanism is a very beneficial non-linear one, compared to many others [4–6, 24]; there is a beneficial anti-resonance, which can be employed for some special vibration control in practice; a nonlinear stiffness in practice can actually enhance the vibration isolation performance compared to a linear spring, due to the special nonlinearity of the proposed system.

5 Nonlinear transient response

To particularly demonstrate the transient response under shocks, dynamic simulation is further conducted with different parameters. An impulse force is applied on the upper platform in order to investigate the damped free vibration response of the system. The equation of motion is solved by the Runge–Kutta method using solver ode45 in MATLAB.

The damped free vibration response of the system is investigated in time domain (Fig. 15). The period of oscillation $T$ is defined as the time between two peaks. Since the signal is periodic, it is calculated as follows:

$$T = \frac{t_n - t_1}{n}$$  \hspace{1cm} (43)

where $t_n$ is the time at which the $n$th peak occurs.

$$\delta = \frac{1}{n} \log \left( \frac{y_1}{y_n} \right)$$  \hspace{1cm} (44)

where $y_n$ is the displacement at the $n$th peak. The settling time $t_s$ is the time required for the response curve to reach and stay within a $\pm y_\delta$ range about the final value.

A 100 N impulse force is applied on the upper platform in order to investigate the nonlinear damped free vibration response of the system. The nonlinear free vibration of the upper platform is calculated with different parameters, namely length ratio $\xi$, layer number $n, n'$, assembly angle $\alpha, \beta$, mass $M_1, M_2$, spring stiffness $k$ and damping $c$.

The nonlinear damped free vibration responses with different rods’ length ratios $\xi$ are shown in Fig. 16.

The increasing of the rods’ length ratios $\xi$ results in two nonlinear effects on the BIAVP-NI. Firstly, the oscillating period $T$ are enhanced, which indicates the decreasing of the resonant frequency BIAVP-NI. Secondly, the logarithmic decreasing $\delta$ is increased and settling time $t_s$ is reduced, which shows advantages in suppressing the vibration. Therefore, the performance of the BIAVP-NI is enhanced by increasing the rods’ length ratios $\xi$. According to the definition of the $\xi$ in Eq. 32, the ratio can be increased by using larger $R$ and smaller $r$. However, the maximin value of $R$ is limited in order to meet the requirements of compact design. Meanwhile, the value of $r$ should be large enough to ensure that the rack pinion has enough gear teeth in engineering practice.

The increment of the layer number $n$ and $n'$ show the similar trend in the changing of the nonlinear damped free vibration responses of the BIAVP-NI (Fig. 17). The oscillating period $T$ is enhanced due to the decreasing of the resonant frequency. Meanwhile, the logarithmic decreasing $\delta$ is increased as a result of decreasing of the equivalent stiffness. It shows that it is better to use more layer to avoid harmful impact on
the payload during the impulse force excitation. However, the settling time $t_s$ is increased in consequence of the falling equivalent damping.

The variation value of the spring stiffness $k$ shows significant effect on the nonlinear damped free vibration responses of the BIAVP-NI (Fig. 18). The oscillating period $T$ is reduced by using larger spring stiffness $k$, however, which will lead to an obvious increment of the settling time $t_s$. Therefore, it is better to use a relatively small spring stiffness $k$ to suppress the vibration.
It is clear from Fig. 19 that the variation of the damping \( c \) shows little effect on oscillating period \( T \). However, larger damping \( c \) help to suppress the vibration amplitude, which results in the increase of the logarithmic decreasing \( \delta \) and the decrease of the settling time \( t_s \).

Both assembly angle \( \alpha \) and \( \beta \) show similar effects on the response of the BIAVP-NI (Fig. 20). The
The variation of assembly angle \( a \) shows little effect on the oscillating period \( T \). However, the oscillating period is extended by using smaller assembly angle \( b \).

Figure 21 shows the nonlinear damped free vibration responses of the system with different mass. It is shown that the mass \( M_1 \) has little effect in every aspect and the oscillating period is extended by using smaller assembly angle \( b \).
of the response, which provides a guaranteed vibration control performance regardless of the variation of the payload mass $M_1$. Meanwhile, the enhancement of the mass $M_2$ leads to an increasing of the logarithmic decreasing $\delta$ and a shortened settling time $t_s$, which advantages in suppressing the vibration.

In a summary, the nonlinear damped free vibration responses of the system are significantly determined by the structural parameters. Table 2 is given to summarize the discussions above.

6 Sweep-frequency response

To investigate the system responses under a sweep frequency excitation, a harmonic excitation is used with the increasing frequency from 0 to 6 Hz in 30 s. The dynamic response of the system is calculated using the Runge–Kutta method. The Hilbert-Huang transform is used to analysis the instantaneous frequency responses. Meanwhile, a typical BIAVP-NI system with the default parameters (“Appendix 2”) is used as the control group, the influence of parameters on the dynamic performance of the system are studied and analyzed through the frequency-time spectrum of the responses.

The dynamic behaviors of the BIAVP-NI with default parameters are shown in Fig. 22. As the excitation frequency increases, the response amplitude first increases, then decreases, which is shown in Fig. 22. The turning point is consistent with the resonance frequency. This result can be shown more
Fig. 24  Frequency-time domain responses with different layer number

(a) $n = 2$, $n' = 3$

(b) $n = 3$, $n' = 2$

(c) $n = 2$, $n' = 4$

(d) $n = 4$, $n' = 2$

Fig. 25  Frequency-time domain responses with different spring stiffness $k$

(a) $k = 50$ (N/mm)

(b) $k = 150$ (N/mm)
clearly through frequency-time domain spectrum in Fig. 22b. The frequency gradually increases with time and the maximum instantaneous energy is reached at the resonance frequency.

The nonlinear dynamic responses of the system with different length ratios are shown in Fig. 23. The enhancement of $\zeta$ results in a slightly lower resonance frequency, which indicates that the resonant frequency of the system can be reduced by increasing the length ratio $\zeta$. However, there is no obvious effect on the amplitude of the resonance peak with the length ratio increased. It is a significant feature of the BIAVP-NI that an ultra-low resonant frequency can be obtained with a larger length ratio $\zeta$ but no influence is incurred on the amplitude of resonance peak. This feature shows advantage in engineering applications, which need to reduce the resonant frequency of the system without sacrificing the loading capacity (smaller stiffness) or increasing the mass, such as vehicle suspension.

The increment of layer number $n$ and $n'$ show the similar changing trend in the dynamic responses of the BIAVP-NI (Fig. 24). The resonant frequency is reduced due to the decreasing of the equivalent stiffness. However, the amplitude of the resonance peak is increased substantially as a result of the reduction of the equivalent damping. It shows that it is better to use more layers to reduce the resonant frequency and increase the damping as the same time to suppress the resonance peaks around resonant frequency.

In Fig. 25, as the enhancement of the spring stiffness $k$, the resonant frequency is increased. It is important to see that although the amplitude of the resonance peak is increased as the increasing of the spring stiffness $k$, the amplitude in ultra-low frequency range is reduced. Thus, low spring stiffness has a beneficial effect on reducing the resonant frequency and resonance amplitude. However, there is a limitation of the minimum spring stiffness $k$ in consideration of the loading capacity and the maximum deformation of the spring in engineering applications.

Figure 26 shows that the variation of the damping $c$ has little effect on resonant frequency. Larger damping $c$ help to suppress the resonance amplitude.

The variation of the assembly angle $\alpha$ and $\beta$ lead to similar effects on the dynamic response of the BIAVP-NI (Fig. 27). The resonant frequency is reduced with smaller assembly angle $\alpha$ and $\beta$ as a consequence of the reduction of the equivalent stiffness. The amplitude of the resonance is increased due to the decreasing of the equivalent damping. Thus, it is better to use smaller assembly angle $\alpha$ and $\beta$ to suppress the vibration and increase the damping at the same time.

The dynamic responses of the system with different mass are shown in Fig. 28. It shows that the enhancement of the mass $M_1$ results in the decreasing of the resonant frequency. The vibration amplitude is increased due to the relatively smaller damping comparing to the mass. On the other hand, the increasing of the mass $M_2$ also reduce the resonant frequency of the system. However, it has little effect on the amplitude of the resonance. This is a novel
nonlinear feature of the BIAVP-NI and never been studied before. The increasing of the nonlinear rotational unit mass $M_2$ reduces the resonant frequency without increasing the resonance amplitude, which has advantages in suppressing the vibration in both ultra-low frequency and high frequency.

To summarize the characteristics of the BIAVP-NI mentioned above, the results of the parameters influences are shown in Table 3.

### 7 Performance comparison

The proposed nonlinear inertia system has better vibration isolation performance both in resonance situation and in the high frequency range than many others. To demonstrate this advantage, some comparisons are discussed in this section.

#### 7.1 A benchmark linear-inertia system

Figure 29 shows a benchmark linear-inertia vibration isolation system, whose parameter setting is adopted with the same one in “Appendix 2”. In order to focus on the effects of the inertia, the payload mass $M_1$ in the linear inertia system is supported by the same X-shaped mechanisms.

For the modelling of the system in Fig. 29, the kinetic energy and potential energy are given by

$$T = \frac{1}{2}(M_1 + M_2)y^2 + \frac{1}{2} \left( \frac{1}{2} M_2 R^2 \right) \theta^2$$

(45)

where

$$\theta = (y - z)/2r$$

(46)

and
Fig. 28 Frequency-time domain responses with different mass

Table 3 Effects of the increase parameters on dynamic responses: ↓ decreased; ↑ increased; – no change

| Increase of the parameters | Resonant frequency | Resonance amplitude | Ultra-low frequency amplitude |
|----------------------------|--------------------|---------------------|-----------------------------|
| Length ratio \( \zeta \)   | ↓                  | –                   | –                           |
| Layer \( n \)              | ↓                  | ↑                   | ↑                           |
| Layer \( n' \)             | ↓                  | ↑                   | ↑                           |
| Stiffness \( k \)          | ↑                  | ↑                   | ↓                           |
| Damping \( c \)            | –                  | ↓                   | –                           |
| Angle \( \alpha \)         | ↓                  | ↑                   | ↑                           |
| Angle \( \beta \)          | ↓                  | ↑                   | ↑                           |
| Mass \( M_1 \)             | ↓                  | ↑                   | ↑                           |
| Mass \( M_2 \)             | ↓                  | –                   | –                           |
\[
V = \frac{1}{2} k(x + l_{h})^2 + (M_1 + M_2)(y + y_i)g \tag{47}
\]

With the dissipated energy as
\[
D = c\dddot{x} \tag{48}
\]

The equation of motion is
\[
\left( M_1 + M_2 + M_2 \frac{\xi^2}{8} \right) \dddot{y} + k F_3(\dddot{y}) + c F_4(\dddot{y}) \dddot{y} + (M_1 + M_2) g = -M_1 \dddot{\dddot{z}} \tag{49}
\]

7.2 Comparison of linear and nonlinear inertia

The term \( \left( M_1 + M_2 + M_2 \frac{\xi^2}{8} \right) \) in Eq. (49) is the equivalent mass, while it is \([M_1 + M_2 + M_2 F_1(\dddot{y})]\) in Eq. (31) for the proposed anti-vibration system of this study. Note that,
\[
M_1 + M_2 + M_2 F_1(\dddot{y}) = M_1 + M_2 + M_2 \left( \frac{\xi^2 (\tan \alpha)^2}{8n^2} + \frac{\xi^2 \tan \alpha}{8n^2 \tan \alpha} \dddot{y} + \frac{\xi^2 (3 \sin^2(x) + 1)}{16l^2 n^4 (\cos \alpha)^6} \dddot{y}^2 \right) + M_2 \left( \frac{\xi^2 (\sin^3(x) + \sin x)}{16l^2 n^5 (\cos \alpha)^8} \right) \dddot{y}^3 \tag{50}
\]

A comparison is shown in Fig. 30. It shows that, the equivalent mass is always a constant due to its linear property while the equivalent mass of the proposed system is a nonlinear function of the vibration displacement \(\dddot{y}\). As discussed before, this nonlinear equivalent mass can lead to several benefits to vibration isolation much better than a constant one.

7.3 Further understanding of the inertia incurred conservative force

The inertia-incurred nonlinear force is a unique feature comparing to a linear-inertia system. For convenience, \( F_c \) represents the inertia-incurred nonlinear force \( M_2 F_2(\dddot{y}) \dddot{y}^2 \). It is usually expected that in a vibration cycle the supporting force during compression and extension of the isolation structure should be as stable as possible instead of a significant change. A comparison is thus presented to show the effect of the nonlinear force \( F_c \), as shown in Fig. 31.
It can be seen that the proposed system can have much smaller vibration and thus more stable interactive force during an extension and compression cycle, compared with the linear system, due to the special inertia-incurred nonlinear force. This nonlinear feature particularly reveals the advantageous benefit that is offered by the proposed nonlinear inertia design of this study.

7.4 Performance comparison

The vibration isolation performance comparison between the BIAVP-NI and the system with/without linear inertia with different parameters $k, c, M_1, M_2, \alpha, \beta, n, n', \xi$. These structural parameters can be divided into two categories: inertia related parameters ($\xi, M_2$) and X-shaped mechanism related ($k, c, M_1, \alpha, \beta, n, n'$).

Firstly, inertia-related parameters are investigated with $\xi$ varying from 4 to 8 and $M_2$ from 5 to 15 kg. Other parameters remain unchanged, which are listed in “Appendix 2”. The vibration isolation performances of the systems are evaluated by the displacement transmissibility and shown in Fig. 32 (Red: Without inertia. Black: Linear inertia. Blue: Nonlinear inertia).

The conclusions can be drawn from Fig. 32,

1. The system with nonlinear inertia provides lower resonant frequency and resonant amplitude. This implies a much better vibration isolation performance over all frequency range compared with the linear-inertia system and the “non-inertia” system.

2. Larger ratio $\xi$ leads to the decreasing of the resonant frequency in both of the systems with linear and nonlinear inertia. This is due to the enhancement of the equivalent mass according to the analysis in Sect. 3.1.

3. The resonant amplitude increased with larger ratio $\xi$ in the linear inertia system while there is no significant change of the resonant amplitude in the nonlinear inertia system. This feature shows the advantage of the nonlinear inertia system, which achieved ultra-low resonant frequency with a stabilized resonant situation.

Moreover, vibration isolation performance comparisons with different X-mechanism related parameters ($k, c, M_1, \alpha, \beta, n, n'$) are shown in Fig. 33.

From Fig. 33, significant differences can be seen as follows.

The BIAVP-NI can have the lowest resonant frequency due to the nonlinear inertia effect with the same parameter setting. A lower resonant amplitude is achieved comparing with the system with/without linear inertia, which is tunable with the adjustment of the structural parameters.

The nonlinear inertia system has advantages in both low and high frequency range. This overcomes the performance degradation of the system with linear inertia.

8 Experiments

8.1 The experimental prototypes

The prototype, shown in Fig. 34, is composed by a 2-layer X-shaped supporting mechanism with rod length $l = 0.08 \text{ m}$ and $l' = 0.042 \text{ m}$. The initial assemble angle $\alpha$ is set at 30 degree and $\beta$ at 45 degree. With the
mass $M_2$ (from 0 kg to 0.4 kg), a rotational disc (0.2 kg) is used as the nonlinear inertia attached to the X-shaped supporting mechanism. The upper platform consists of two aluminum plates, with four rubber mounts between them. The weight of the plate is about 0.4 kg and its size is $0.25 \times 0.2 \times 0.003$ m. The total mass of the payload $M_1$ is about 1.8 kg to 3.2 kg. The weight of each rod is about 0.005 kg. Four springs with a stiffness of 250 N/m each in parallel are assembled in the small X-shaped mechanism.

Fig. 33 Displacement transmissibility with different X-shaped structural parameters

(a) Effect of stiffness $k$, 80N/mm (Dash), 100N/mm (Solid), 120N/mm (Dot).

(b) Effect of damping $c$, 90Ns/m (Dash), 110Ns/m (Solid) (Dot).

(c) Effect of angle $\alpha$, 40 degree (Dash), 45 degree (Solid), 50 degree (Dot).

(d) Effect of angle $\beta$, 55 degree (Dash), 60 degree (Solid), 65 degree (Dot).

(e) Effect of angle $M_2$, 250 kg (Dash), 300 kg (Solid), 350 kg (Dot).

(f) Effect of angle $n$, $n = 3$ (Dash), $n = 2$ (Solid), $n = 1$ (Dot).

(g) Effect of angle $n'$, $n' = 3$ (Dash), $n' = 2$ (Solid), $n' = 1$ (Dot).
The prototype with a payload is mounted on a platform connected to a shaker vertically. The test results are presented.

8.2 Vibration isolation performance

A random excitation with frequency from 0.1 Hz to 100 Hz is applied to the base and the response of the payload on the upper platform is obtained by the acceleration sensor. The results are presented in Fig. 35. The acceleration amplitude of the upper platform is approximately up to 10% of that of the base.
excitation in vibration energy. This indicates an excellent isolation performance.

In order to further analyze the vibration isolation performance of the BIAVP-NI, the displacement transmissibility of the experimental prototype is investigated. The transmissibility \( H(\omega) \) can be obtained by

\[
H(\omega) = \frac{P_{zy}(\omega)}{P_{zz}(\omega)}
\]

where \( P_{zy}(\omega) \) and \( P_{zy}(\omega) \) are power spectral density and cross-spectral density of the excitation signal and excitation-response signal, respectively. With different payload mass \( M_1 \) and rotational unit \( M_2 \), the results are shown in Fig. 36.

It can be seen from Fig. 36 that the resonant frequency occurs at ultra-low frequency about 1.1 Hz. The transmissibility is smaller than 0 dB when the frequency is bigger than about 3 Hz. The BIAVP-NI shows a very good isolation performance in a broad frequency range subject to random excitation. The transmissibility reaches the maximum isolation around \(-50 \) dB after 10 Hz.

The second peak in the transmissibility is about 65 Hz with a peak around \(-30 \) dB, which is the modal frequency of the mechanism due to horizontally swinging. This peak can be easily suppressed by using damping materials in engineering applications.

The effect of payload mass \( M_1 \) are investigated. A larger payload mass \( M_1 \) leads to a better vibration isolation performance. However, the resonant peak is increased due to the reduction of the damping ratio as discussed before. Comparing Fig. 36a–b, the amplitude of the second peak is suppressed by using larger rotational unit \( M_2 \), which shows the effect of the rotational unit in the high frequency vibration suppression.

To further study the effects of the rotational unit, a comparison experiment is presented with different rotational unit \( M_2 \) and two different payload masses.

The results in Fig. 37 indicate that a larger rotational unit \( M_2 \) can always result in a smaller resonant frequency and thus a better vibration isolation. These are consistent with the theoretical analysis before.

Further studies about the amplitude nonlinearity under different frequency and amplitude excitation are presented for more validation. The results are shown in Fig. 38.

Comparing the results in Fig. 38a, b, the excitation amplitude is increased 100% (from 1 m/s² to 2 m/s² under the same frequency at 5 Hz). The response, however, is changed from 0.275 m/s² to 0.433 m/s², which only increased by 57%. The same phenomenon happens with larger frequency at 10 Hz in Fig. 38c, d, the excitation amplitude is increased from 2 m/s² to 4 m/s². The response amplitude is 0.392 m/s² to 0.706 m/s², which is 80% increasement of the response comparing to the 100% excitation enhancement. It is obvious that the novel nonlinear feature of the BIAVP-NI has the advantage in isolating the large amplitude vibration, comparing to the traditional linear vibration isolation systems. This excellent nonlinear feature, consistently with theoretical analysis, can be used in a variety of industries practices.

8.3 Transient response

Considering shocks or impacts, the system with different rotational units are tested for further
Fig. 38 Harmonic excitation and response

(a) Excitation amplitude 1m/s², frequency 5Hz

(b) Excitation amplitude 2m/s², frequency 5Hz

(c) Excitation amplitude 2m/s², frequency 10Hz

(d) Excitation amplitude 4m/s², frequency 10Hz

Fig. 39 Transient excitation and transient response in time domain

(a) Excitation amplitude 4m/s², $M_2 = 0.4$kg

(b) Excitation amplitude 4m/s², $M_2 = 0.2$kg

(c) Excitation amplitude 2m/s², $M_2 = 0.4$kg

(d) Excitation amplitude 2m/s², $M_2 = 0.2$kg
understanding. Impulse excitations can be exerted to the base of the prototype to investigate a damped free vibration. The excitations and the responses are measured by accelerometers and shown in time domain (Fig. 39).

From Fig. 39a, b, the responses all have a reasonable convergence rate within 1 s. In Fig. 39c, d, with smaller excitation amplitude, the vibration amplitudes are reduced to zero in less than 0.3 s.

In order to further study the variations in the frequency and amplitude of excitations and responses over time, the Hilbert-Huang transform is used to obtain instantaneous dynamic behaviors (Figs. 40, 41, 42 and 43).

It is clear to see from the Hilbert spectrum of the excitations, unlike the random vibration generated by a shaker, the excitation is a damped free vibration, which has a specific frequency and amplitude. There are two frequency ranges where the excitation amplitude is large. One of them is between 1 and 10 Hz, which are coupled vibration frequencies between the prototype and its associated connections (including 4 supporting guides and the base structures). The other frequency range is 40 Hz and above, which is caused by the impact on the base which is the other elastic platform supported from the bottom during the test.

Comparing the amplitude of the excitation and response in Figs. 40, 41, 42 and 43, the vibration is
greatly suppressed. Since the response amplitude and frequency are not only related to resonant frequency of the prototype in the vertical direction, but also related to the excitation amplitude/direction and the overall structures, the resulting response peaks demonstrates all those associated frequencies. For example, Fig. 42a indicates clearly a vibration peak around 1 Hz subject to the given impact excitation, and also shows some other vibration frequencies due to the structural dynamics. However, overall it can be seen that all vibrations can quickly disappear in a very short time period with smaller amplitudes in all cases. This clearly demonstrates the potential damping effect of the prototype subject to impact excitations, which is absolutely expected in engineering practice.

9 Conclusions

In this paper, a new passive and compact anti-vibration platform is proposed by exploiting an X-shaped linkage mechanism, inspired by the human body anti-vibration mechanism including leg skeleton and arm swing, and the resulting design can achieve beneficial nonlinearities in stiffness, damping and inertia all in the same system. The advantages of the BIAVP-NI are fully discussed with the theoretical

![Fig. 42](image)

**Fig. 42** Transient excitation and response in time–frequency domain (Excitation amplitude 2 m/s², \(M_2 = 0.4\)kg)

![Fig. 43](image)

**Fig. 43** Transient excitation and response in time–frequency domain (Excitation amplitude 2 m/s², \(M_2 = 0.2\)kg)
The following conclusions can be drawn.

1. The BIAVP-NI presents a desired high-static-low-dynamic-stiffness (HSLDS) nonlinear characteristic. This nonlinear property allows an adjustable and higher payload but presents a lower resonant frequency. A nonlinear spring in practice can even enhance the HSLDS property as shown.

2. The effect of nonlinear inertia due to the nonlinear coupling effect with the X-shaped mechanism can further reduce the resonant frequency of the system and enhance the stability of the mass centre with less fluctuation in supporting force subject to external base vibration excitation. Several structural parameters can be used to tune the performance of the nonlinear inertia effect, which offers the flexibility of such a special linkage mechanism in engineering practice.

3. The transient responses and sweep frequency responses indicate that the BIAVP-NI has beneficial nonlinear damping characteristics and increasing the damping coefficients can enhance the nonlinear damping effect around resonant frequency but have little effect on high frequency transmissibility, demonstrating a beneficial nonlinear damping effect with such a special embedded X-shaped structure design.

The results of this study would present an alternative and effective way to achieving beneficial nonlinearities and passive vibration isolation systems for many engineering practices.

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Declarations

Conflict of interests The authors declare that they have no conflict of interest.

Appendix 1

\[ e_0 = \frac{R^2 (\tan \alpha)^2}{8r^2 n^2} \]

\[ e_1 = \frac{R^2 \tan \alpha}{8lr^2 n^3 (\cos \alpha)^3} \]

\[ e_2 = \frac{R^2 (3 \sin^2(\alpha) + 1)}{16l^2 r^4 (\cos \alpha)^6} \]

\[ e_3 = \frac{R^2 (\sin^3(\alpha) + \sin(\alpha))}{16l^3 r^5 (\cos \alpha)^8} \]

\[ e_4 = -\frac{R^2 \sin \alpha}{16n^3 r^2 l (\cos \alpha)^2} \]

\[ e_5 = \frac{R^2 (3 \sin^2(\alpha) + 1)}{32l^2 r^4 (\cos \alpha)^3} \]

\[ e_6 = -\frac{3R^2 \sin(\alpha) \left[ (\sin \alpha)^6 - (\sin \alpha)^4 - (\sin \alpha)^2 + 1 \right]}{32l^3 r^5 (\cos \alpha)^6} \]

\[ e_7 = \frac{R^2 \left( 5(\sin \alpha)^7 - 14(\sin \alpha)^4 + 8(\sin \alpha)^2 + 1 \right)}{64(\cos \alpha)^7 l^4 r^6} \]

\[ \lambda_0 = \frac{\tan \alpha \cdot \tan \beta}{\eta \eta} l_s \]

\[ \lambda_1 = \frac{2l \cos \alpha (\sin \alpha)^2 \left( l' \cos \beta (\sin \beta)^2 - I_s / 2 \right) + l' l_n' \sin(\beta) (\cos \beta)^3}{2l l_n' n^2 (\cos \alpha \cos \beta)^3} \]
\[ \lambda_2 = \frac{3 \sin \left( n' l \cos z \left( l' \left( \sin \beta \right)^2 \cos \beta + l_s/2 \right) \left( \cos \beta \right)^2 + l'^2 n'' \sin \beta \left( \cos \beta \right)^4/2 - \sin \beta \left( l \sin x \cos z \right)^2 \left( l \cos \beta + l_s/2 \right) \right)}{4Pl'^2 n'^3 \left( \cos x \cos \beta \right)^5} \]  

\[ \lambda_3 = \frac{-8l' \left( \cos \beta \right)^2 - 5/4 \left( \cos x \right)^3 + 16l' \left( \cos \beta \right)^2 - 5/4 \left( \cos z \right)^5}{16l'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]  

\[ + \frac{-8l' \left( \cos \beta \right)^2 + l'^2 n'' \left( \cos x \right)^5 + 2l' \left( \cos \beta \right)^3 + l'^2 \left( \cos \beta \right)^2 + 2 \left( \cos \beta \right)^2 - 2 - 5l' \left( \cos \beta \right)^2 - 5l' \left( \cos \beta \right)^2 \left( \cos x \right)^3 - 2l' \left( \cos \beta \right)^3 \left( \cos x \right)^3 - 2l' \left( \cos \beta \right)^3 \left( \cos x \right)^3 \left( \cos z \right)^3}{16l'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]  

\[ + \frac{-10 \left( \cos \beta \right)^3 - l' \left( \cos \beta \right)^3 - l' \left( \cos \beta \right)^3 - l_s/2 \right) \left( \cos x \right)^2 \left( \cos \beta \right)^5 \left( \cos \beta \right)^2 \left( \cos x \right)^2 \left( \cos \beta \right)^2 \left( \cos z \right)^2 + 5l' \left( \cos \beta \right)^2 \left( \cos x \right)^2 \left( \cos \beta \right)^2 \left( \cos x \right)^2 \left( \cos z \right)^2}{16l'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]

\[ \lambda_4 = \frac{\tan z \tan \beta}{n'n'} \]  

\[ \lambda_5 = \frac{\sin \beta \left( \cos \beta \right)^2 l' n' + \left( \sin z \right)^2 \cos \beta}{2ll' \left( \cos \beta \right)^2 \left( \cos x \cos \beta \right)^5} \]  

\[ \lambda_6 = \frac{3 \sin z \left[ \left( \cos \beta \right)^4 \sin \beta l'^2 n' + \left( \cos \beta \right)^2 \left( \cos z \cos z \right)^2 \right]}{8Pl'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]  

\[ \lambda_7 = \frac{l'^2 n''^2 \left[ 5 - 4 \left( \cos \beta \right)^2 \right] \left[ l' \left( \cos \beta \right)^2 \cos \beta + 1 \cos \beta \right] \left( \cos \beta \right)^4 + l'^2 l' \left( \cos \beta \right)^2 \left( \cos \beta \right)^4}{16l'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]  

\[ + \frac{l'^2 \left[ 5 - 4 \left( \cos \beta \right)^2 \right] \left( \cos \beta \right)^2 \left( \cos \beta \right)^4 + 6l' \left( \cos \beta \right)^2 \left( \cos \beta \right)^4 \left( \sin z \right)^2 \sin \beta}{16l'^3 n'^4 \left( \cos x \cos \beta \right)^7} \]

**Table 4** Default parameters of the example

| Names             | Symbols | Values | Units |
|------------------|---------|--------|-------|
| Mass             | $M_1$   | 300 kg |       |
| Rotational inertia | $M_2$   | 10 kg  |       |
| Stiffness        | $k$     | 100 N/mm |     |
| Damping          | $c$     | 500 Ns/m |     |
| Rod length       | $l$     | 0.1 m  |       |
| Rod length $\dot{l}$ | $\dot{l}$ | 0.06 m |       |
| Rod length $l_s$ | $R$     | 0.06 m |       |
| Rod length $\gamma$ | $\gamma$ | 0.006 m |       |
| Angle $\alpha$   | $\alpha$ | 45 deg |       |
| Angle $\beta$    | $\beta$  | 60 deg |       |
| Layers number $n$| $n$     | 2 –    |       |
| Layers number $n'$| $n'$    | 2 –    |       |
Appendix 3

See Table 5.

### Table 5 Parameters of the prototypes

| Names                  | Symbols | Values | Units |
|------------------------|---------|--------|-------|
| Mass                   | $M_1$   | 1.8 to 3.2 kg |
| Rotational inertia     | $M_2$   | 0 to 0.4 kg |
| Stiffness              | $k$     | 1 N/mm  |
| Rod length $l$         | $l$     | 0.08 m   |
| Rod length $l'$        | $l'$    | 0.042 m  |
| Rod length $l_0$       | $R$     | 0.035 m  |
| Rod length $r$         | $r$     | 0.006 m  |
| Angle $\alpha$         | $\alpha$| 30 deg  |
| Angle $\beta$          | $\beta$ | 45 deg  |
| Layers number $n$      | $n$     | 2 -     |
| Layers number $n'$     | $n'$    | 2 -     |

Appendix 4

See Figs. 44, 45, 46, 47, 48, 49.

**Fig. 44** Nonlinear dynamic responses versus time with different length ratio $\xi$
**Fig. 45** Nonlinear dynamic responses versus time with different layer number $n$.

**Fig. 46** Nonlinear dynamic responses versus time with different spring stiffness $k$. 
Fig. 47 Nonlinear dynamic responses versus time with different damping $c$

![Graph showing nonlinear dynamic responses with different damping values $c$.]  

Fig. 48 Nonlinear dynamic responses versus time with different assembly angle

![Graph showing nonlinear dynamic responses with different assembly angles $\alpha$ and $\beta$.]
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