We develop a finite temperature mean field theory in the path integral picture for an extremely dilute system of interacting Fermions in a plane. In the limit of short ranged interaction, the system is shown to undergo a phase transition to a superconducting regime. Unlike the well known BCS transition for metals, this phase transition is found to be very sensitive to the two dimensional nature of the problem. A quantitative estimate of this sensitivity is carried out by repeating the analysis at non-zero thickness. The validity of the mean field results are also proved using renormalization group techniques.

1 Introduction

It is well known from the theory of superconductivity[1], that the low energy effective theory for electrons in metals is susceptible to a BCS phase transition at $T = 0$. The low energy theory which involves only the modes close to the Fermi surface ($K_F$) turns out (in any dimension $D \geq 2$), to be an effective one dimensional theory that describes a large number ($N$) species of Fermions satisfying a linear energy momentum dispersion relation. The inter-electron interaction for this theory turns out to be short ranged and attractive[2, 3]. The one dimensional nature of the effective theory and the linear energy momentum dispersion relation, are key ingredients that make the theory asymptotically free, while the attractive nature of the inter-electron potential is responsible for the formation of the gap in the energy spectrum. The effective one dimensional nature of the BCS theory is related to the fact that in considering modes close to the Fermi sea, (a natural cut off for the theory), one finds that all the directions
in momentum space tangential to \( K_F \) are degenerate, in the sense that the energy depends only on the directions normal to the Fermi sea. Hence one naturally obtains a large degeneracy, denoted by the number \( N \), which turns out to be the area of the Fermi surface measured in units of the cut off. Moreover, this dimensional reduction to one dimension can be carried out in a manner which is insensitive to the true dimensionality of the problem which is one of the reasons behind the universality of the BCS transition\[^2, 1, 4\].

The attractive inter-electron potential is also generic to all metallic systems in \( D \geq 2 \). This can be best understood by appealing to renormalization group (RG) arguments. RG analysis shows that due to coulomb screening all long ranged interactions (repulsive or otherwise) behave in a screened or short ranged manner. If one now decomposes the short ranged interaction into its various angular momentum components, then the RG flows for the various components couple to each other. The structure of these coupled equations suggests that it is always possible to find some value of \( l \) (the angular momentum), for which the effective coupling for that sector will flow to negative values. The couplings that do run negative, i.e. become attractive, lead to a BCS transition, while the others renormalize to zero. This is the so called Kohn - Luttinger effect\[^5, 6, 7\]. The Kohn - Luttinger effect, and the dimensional reduction to one dimension make the BCS phenomenon truly universal, irrespective of the true physical dimensionality of the metallic systems or the microscopic details of the inter-electron interaction.

Recently there has been much theoretical work towards understanding the behavior of two dimensional dilute electron - hole systems\[^8, 9\]. Whether or not such systems can undergo a phase transition to a conducting or superconducting regime, and the dependence on the true physical dimensions of such possible phase transitions is the object of much speculation, as these systems arise naturally in the context of high \( T_c \) superconductors. The physical simplifications mentioned before in the context of the usual BCS theory do not apply to these dilute systems, where the physical momentum scale \( \Lambda >> K_F \). For such systems it does not make sense to linearize the energy - momentum dispersion relation around \( K_F \) and reduce to a one dimensional problem. We present here the analytical study of such a dilute system of electrons, interacting through an attractive short ranged potential. We carry out the study at finite temperature in the absence of impurities or disorder, and find that such systems undergo a phase transition to a superconducting state for arbitrarily small values of the attractive coupling. Moreover we find that they do so in a manner which is extremely sensitive to the two dimensionality of the problem.

We will start with a short review of the quantum mechanical two body problem involving an attractive delta function potential in two dimensions, as it is an excellent toy model that encapsulates all the essential features of the corresponding many body problem, e.g. the formation of bound states and asymptotic freedom. We shall then focus on the second quantized version of the problem, which will be analyzed in the path integral formalism. This will be followed by an exploration of the sensitivity of the BCS like phase transition to two dimensionality. In a separate section we shall justify the use of mean field
theory in the context of this problem using renormalization group techniques.

2 Short Ranged Attraction in two Space Dimensions and Asymptotic Freedom

The Two body problem:
Let us consider a system of two particles interacting through an attractive short ranged potential in two dimensions. This two body system has an ultraviolet divergence, and the subtleties associated with the renormalization of theories which have one marginal parameter. The Schroedinger equation for this system (in momentum space) is:

\[
\left( \frac{p^2}{2m} - E \right) \Psi_\Lambda(p) - g(\Lambda) \rho_\Lambda(p) \int \rho_\Lambda(q) \Psi_\Lambda(q) [d^2q] = 0
\]

Note: \([d^2q]\) is a short hand for \(\frac{d^2q}{\pi^2}\).

The equation above is regularized by the presence of \(\rho_\Lambda(p) = \Theta(|p| < \Lambda)\). In the absence of the regularization, the equation is ill defined, because the ground state energy of the system is not bounded below. This is an artifact of the scale invariance of the equation which implies that if \(E\) is an eigenvalue, then so is \(sE\), where \(s\) is a scaling parameter. Hence the bound state energy can be either 0, or \(-\infty\). We can make sense of this scenario, by formulating the basic dynamical equation with a cut-off, as above, and ask the question as to how the dimensionless coupling constant \(g\) depend on \(\Lambda\), such that all the physical quantities have a finite limit as \(\Lambda \to \infty\). Solving for \(\Psi_\Lambda(p)\) gives:

\[
\Psi_\Lambda(p) = A_\Lambda \left( \frac{A_\Lambda}{\frac{p^2}{2m} - E} \right) \rho_\Lambda(p)
\]

Where \(A_\Lambda = g(\Lambda) \int \rho_\Lambda(p) \Psi_\Lambda(p) [d^2p]\). Putting the expression for \(\Psi_\Lambda(p)\) in the equation for \(A_\Lambda\), we have:

\[
g^{-1}(\Lambda) - \int \frac{\rho^2_\Lambda(p)}{\frac{p^2}{2m} - E} [d^2p] = 0
\]

This expression for \(g^{-1}(\Lambda)\) is log-divergent. The situation can be remedied if we choose \(g(\Lambda)\) in the following way:

\[
g^{-1}(\Lambda) = \int \rho^2_\Lambda(p) \left( \frac{1}{\frac{p^2}{2m}} + \frac{\mu^2}{2m} \right) [d^2p]
\]

With this choice of \(g\), Eqn(3) above has a finite limit as \(\Lambda \to \infty\), and the resulting theory is asymptotically free. The essence of renormalization here is the trading of the bare coupling constant \(g\) in favor of the number \(\mu\), which has the dimensions of momentum. This number is the true physical parameter.
of the theory, which describes the strength of the interaction between the two particles.

The nature of the divergence, does not change if we make the transition from the two body problem, to the corresponding many body problem of non-relativistic particles in a plane. Indeed, it can be shown, that the divergence present at the level of the mean field theory of the many body problem is the only divergence that requires renormalization. In other words, renormalizing the mean field theory is enough to cast the problem in a finite form. Moreover renormalization group arguments (which we present in the last section) show that the predictions of the mean field analysis for this problem are qualitatively correct. In the next section we will generalize the two body Hamiltonian given above to the corresponding many body case, and recast the problem of evaluating the thermodynamic partition function for this Hamiltonian in the language of path integrals. In the next section we shall use this path integral point of view to construct the mean field theory for the problem.

The many body problem:

The generalization of the Hamiltonian given in Equation (1) to a second quantized language is,

$$H = \int [d^D r] \Psi^{\dagger} a_1(r) [-\nabla^2] \Psi a_1(r) - \alpha(\Lambda) \int [d^D r] \Psi^{\dagger} a_2(r) \Psi a_2(r) \Psi a_1(r)$$

We are keeping the dimensionality of space ($D$) arbitrary for the moment. This Hamiltonian can be written in a manner more suited to a mean field analysis, by using the following two component spinors,

$$\Psi = \left( \begin{array}{c} \psi_\uparrow(r) \\ \psi_\downarrow(r) \end{array} \right)$$

In terms of $\Psi$, the Hamiltonian can be written as,

$$H = \int d^D(r) \left[ \Psi^{\dagger} \left( -\nabla^2 \right) \sigma_3 \Psi - \alpha \Psi^{\dagger} \left( \begin{array}{cc} 0 & \psi_\uparrow \psi_\downarrow \\ 0 & 0 \end{array} \right) \Psi \right]$$

The composite operator $\psi_\uparrow(r) \psi_\downarrow(r)$, and it’s Hermitian conjugate appearing in the Hamiltonian above are the creation and annihilation operators for the quasi particles of the theory, which are scalars in the present case. The interesting physical features of the theory, such as the formation of a gap in the spectrum are related to the fact that $\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0$ in the ground state. Hence we would like to reformulate the problem in a language where these composite operators are the chief dynamical variables. We shall do it below in the finite temperature limit and in the path integral picture.

The finite temperature features of the theory can be probed by calculating the thermodynamic partition function $Tr[e^{-\beta H}]$ for the Hamiltonian $H$ given above in (5). Now for any normal ordered Fermionic Hamiltonian one can reduce the problem of calculating the partition function to the evaluation of a
Path integral over Grassmann variables, using the prescription,
\[ Z = Tr[e^{-\beta H}] = \int e^{-\beta \int_0^1 dt \int d^D r (\Psi(r) \partial_t \Psi(r) - H(\Psi(r), \Psi^\dagger(r)))} D[\Psi^\dagger] D[\Psi] \]  

(8)

Here it is understood, that the arguments of H appearing in the path integral are the Grassmann variables corresponding to the Fermion creation and annihilation operators appearing in Equation (5). This functional integral can be rewritten in the following way.

\[ Z = \int e^{-\beta \int_0^1 dt \int d^D r (\frac{1}{\beta} \partial_t - \nabla^2) \sigma_3 \Psi(r) - 2g^2 \psi^\dagger(r) \psi(r) \psi_\uparrow(r) \psi_\downarrow(r)} D[\Psi^\dagger] D[\Psi] D[\phi] \]  

(9)

The functional appearing in the exponential is to be thought of as the action \( S \) of an interacting theory. Here \( \phi \) is an auxiliary complex scalar field. To see that this is the correct action, we can integrate out the scalar fields, and recover an effective action for the Fermi fields which is the action appearing in (8), i.e.

\[ \int D[\phi] D[\phi^\dagger] e^{-S} = e^{-S_{Fermi}} \]  

(10)

where

\[ S_{Fermi} = \beta \int_0^1 dt \int d^D r (\frac{1}{\beta} \partial_t - \nabla^2) \sigma_3 \Psi(r) - 2g^2 \psi^\dagger(r) \psi(r) \psi_\uparrow(r) \psi_\downarrow(r) \]  

(11)

It is now obvious that \( S_{Fermi} \) is the action generated by the Hamiltonian in (5), when translated to the path integral picture using (8) if we identify \( 2g^2 \) with the coupling constant \( \alpha(\Lambda) \) appearing in (5). A comparison between equations (7) and (9) makes it obvious that \( \phi \) describes the quasi bosons of the theory, i.e \( \phi = \psi_\downarrow \psi_\uparrow \). Reformulation of the theory in terms of the composite operator describing the quasi scalars can now be accomplished, by integrating out the Fermi fields appearing in (9). This produces the effective action for the complex scalar field.

\[ S_{Eff}[\phi, \phi^\dagger, \beta] = \beta \frac{1}{2} \int d^D r |\phi|^2 - Tr \ln \left[ 1 + \frac{g}{(\frac{1}{\beta} \partial_t - \nabla^2) \sigma_3} \left( \begin{array}{c} 0 \\ \phi \end{array} \right) \right] \]  

(12)

Transforming to Fourier space, we have;

\[ Tr \ln (1 + \frac{g}{(\frac{1}{\beta} \partial_t - \nabla^2) \sigma_3} \left( \begin{array}{c} 0 \\ \phi \end{array} \right)) = \int [d^D k] \ln \Pi_n (1 + \frac{\beta^2 g^2 |\phi|^2}{(2n + 1)^2 \pi^2 + |\phi|^2 k^2}) \]  

(13)

The product on the left hand side is over the (odd) Matsubara frequencies. The evaluation of this trace can be carried out as follows; let

\[ A = \ln \Pi_n (1 + \frac{x^2}{(2n + 1)^2 \pi^2 + y^2}) \]  

(14)
Where \( x = \beta g|\phi| = \beta \Delta \), and \( y = \beta k^2 \). Now:

\[
\frac{d}{dx^2} A = \Sigma_{\infty} \frac{1}{(2n+1)^2 \pi^2 + \beta^2 E^2} = 2\Sigma_{\infty} \frac{1}{(2n-1)^2 \pi^2 + \beta^2 E^2}
\]  
(15)

In the equation above, \( E = (k^4 + \Delta^2)^{1/2} \). Now recalling the identity

\[
\tanh\left(\frac{\pi x}{2}\right) = 4\pi \frac{1}{\Sigma_{\infty}} \frac{1}{(2n-1)^2 + x^2}
\]  
(16)

we get

\[
\partial_{xx} A = \frac{1}{2\beta E} \tanh\left(\frac{\beta E}{2}\right)
\]  
(17)

\[
A = \frac{\beta}{2} \int_0^{g^2|\phi|^2} \int_0^{E(x)} \frac{\tanh\left(\frac{\beta E(x)}{2}\right)}{E} dx^2
\]  
(18)

Where \( E(x) = (\Delta^2 + \omega^2(k))^{1/2}, \omega(k) = k^2 \).

**Mean field theory:** Mean field theory corresponds to the saddle point of the non-linear scalar field theory described by (12). To get the saddle point equations, it is sufficient to consider a constant value for the scalar field \( \phi_c \).

Using the form given in (18) for the functional determinant, the effective action for a constant value of the scalar field is,

\[
S_{E_{ff}}[\phi_c] = \beta \left(\frac{|\phi_c|^2}{2}\right) - \frac{1}{2} \int \int g^2|\phi|^2 \frac{\tanh\left(\frac{\beta E}{2}\right)}{E} d^D k dx^2 = F
\]  
(19)

Where \( F \) denotes the free energy of the system.

The mean field (saddle point equations):

The equation for the saddle point is the equation for the extremum of the effective action, or the Free energy, hence we have from the form of \( S_{E_{ff}} \) above,

\[
\frac{\delta F}{\delta \phi_c} = 0 \Rightarrow 1 = g^2 \int d^D k \frac{\tanh\left(\frac{\beta (\Delta^2 + \omega^2(k))^{1/2}}{2}\right)}{(\Delta^2 + \omega^2(k))^{1/2}}
\]  
(20)

This is the familiar gap equation, and \( \Delta = |g|\phi_c \) has the physical interpretation of the gap in the energy spectrum. This equation is divergent in any dimension \( D \geq 2 \). The divergence can be made explicit by considering the zero temperature \((\beta \to \infty)\) limit, where it becomes,

\[
g^2 \int d^D k \frac{1}{(\Delta^2 + \omega^2(k))^{1/2}} = 1
\]  
(21)

In the case of \( D = 2 \), which is the critical dimension for the theory, this ultraviolet divergence is logarithmic, and it can be renormalized if we let \( g^{-2} \sim \int_0^\Lambda \frac{d\omega}{\sqrt{\omega^2 + \mu^2}} \), where \( \mu \) is a dimensional parameter that sets the scale for the
renormalized theory. After incorporating this renormalization, equation (20) above in (2+1) dimensions can be written in a manifestly finite form:

$$\int_0^{\infty} d\omega \frac{\tanh(\beta \sqrt{\omega^2 + \Delta^2}/2) - 1}{\sqrt{\omega^2 + \Delta^2}} = \ln\left(\frac{\Delta}{\mu}\right) = \lim_{\Lambda^2 \to \infty} \left[ \int_0^{\Lambda^2} \frac{d\omega}{\sqrt{\omega^2 + \mu^2}} - \int_0^{\Lambda^2} \frac{d\omega}{\sqrt{\omega^2 + \Delta^2}} \right]$$

Here \( \omega = k^2 \). In terms of the dimensional variables \( y = \frac{\Delta}{\mu} \) and \( \zeta = \beta \Delta \), the renormalized gap equation above becomes:

$$\ln(y) = G(\zeta) = 2 \int_0^{\infty} d\theta \left[ \frac{1}{1 + e^{\zeta \cosh \theta}} \right]$$

This is to be thought of as the equation of state for our system. The other dimensionless quantities like the temperature in the units of the binding energy \( x = \beta \mu = \frac{\Delta}{\mu} \) can be recovered from the parametric equation above. This equation is the same as the familiar gap equation for the BCS superconductor, and the numerical values for the physical quantities e.g. \( kT_c/\Delta \) predicted by the above equation of state are exactly equal to those of the usual BCS superconductor. Hence we observe, that in the absence of impurities, a dilute system of electrons, interacting through arbitrarily small, short - ranged attractive potentials, will undergo a phase transition to a superconducting stage.

Although the equations describing the phase transition are very reminiscent of the BCS transition for metals, they do in fact describe a different physical situation. This is evident if we note that the arguments that led to the mean field equation for this dilute system of electrons are very special to the two dimensional nature of the problem. Indeed the nature of the divergence of the gap equation itself is different, in higher dimensions, where the equation does not admit any solutions for values of the coupling constant less which are less than some critical value.

This is different from what happens in the theory of metals, at the BCS transition point. As mentioned before, in the case of metals, one deals with an effective theory of modes near the Fermi surface, which can be reduced to a one dimensional theory with a linear dispersion relation which leads to a similar logarithmic divergence. Moreover, this reduction is insensitive to the true dimensionality of the problem (the exceptional case being \( D = 1 \)) which makes the physics of the phase transition blind to the true dimensionality of the metal being studied.

It is clear from the discussion so far, that the model we are studying now, does not admit of these simplifications, because of the diluteness of the system. Since for our system, \( \frac{K_F}{\Lambda} \sim 0 \), where \( K_F \) is the Fermi momentum, and \( \Lambda \) is the typical energy scale of the problem, it is not meaningful to look for a theory for the modes close to the Fermi surface. This in turn ( as we pointed out above ) makes the phase transition depend critically on the two dimensionality of the model, which is an idealization. So we now probe the effects of a finite transverse spatial direction on the model.
3 The Analysis at Finite Thickness

In this section we ask the question as to how thin the system has to be for it to be considered two dimensional. To keep matters simple, let us investigate the situation at zero temperature. In particular we want to see the transition from the three dimensional case to the two dimensional one. Eqn (21) tells us that in dimensions greater than two, the strong and weak coupling phases of the system do not match smoothly, i.e. there is a critical coupling \( g^{-2} c \sim \Lambda^{D-2} \), below which the gap equation does not admit any solutions. This is indicative of a phase separation between the strong and weak coupling regimes. To understand the effects of a finite thickness \( L_3 \) on the system, we will address the following questions.

a: What is the correct renormalization prescription to use for \( g^{-2} \), for small but non zero values of the thickness \( L_3 \).

b: Given a finite thickness of the system \( L_3 \), and the fact that the system is in the strongly coupled phase, how should \( L_3 \) approach zero, so that the gap \( (\Delta) \) remains finite.

At finite thickness, the non-relativistic dispersion relation reads \( \omega(k) = k^2 + (\frac{2\pi n}{L_3})^2 \), where \( n \) takes on integer values. So the gap equation at finite thickness is:

\[
|g|^2 \sum_n \int [d^2k][dk_0] \frac{1}{(k^2 + (\frac{2\pi n}{L_3})^2 + \Delta^2 + k_0^2)} = 1 \quad (24)
\]

\( k_0 \)'s are the Matsubara frequencies which take on continuous values in \( (-\infty, \infty) \) at \( T = 0 \).

Summation of the series:

The series given by \( S = \sum_{n} -f(i\omega_n) \), where \( f(i\omega_n) = \frac{-1}{(k^2-(i\omega_n)^2)^2 + k_0^2 + \Delta^2} \) and \( \omega_n = (\frac{2\pi n}{L_3}) \), can be evaluated by considering the following contour integral:

\[
I = \oint \frac{dz}{2\pi i} f(z)n(z), n(z) = \frac{1}{e^{L_3z} - 1} \quad (25)
\]

If the contour of integration is taken to be the infinite circle centered at the origin, then the integral vanishes, i.e. \( \sum \text{Res}(f(z)n(z)) = 0 \). The poles of the integrand are at \( z = i\omega_n \), and at the four roots of \( ((k^2 - z^2)^2 + k_0^2 + \Delta^2) \). Let us denote the residues at the last four poles by \( R_i \). The residues at the poles labelled by the integer \( n \) are \( \frac{1}{L_3} f(i\omega_n) \). Hence \( S = L_3 \sum_{i} R_i \). Summing the series by computing the four residues, the gap equation at finite thickness reads as:

\[
g^{-2} = \int [d^2k][dk_0] \frac{L_3}{4\pi i b} \left[ \frac{\coth[(a - ib)^{1/2}L_3/2]}{(a - ib)^{1/2}} - \frac{\coth[(a + ib)^{1/2}L_3/2]}{(a + ib)^{1/2}} \right] \quad (26)
\]

Here \( a = k^2 \) and \( b = (k_0^2 + \Delta^2)^{1/2} \).

The question now is to chose a suitable renormalization prescription for the coupling constant. To do this we must first understand the nature of the divergence in the equation. As \( \coth(x) \sim 1 \), for \( x >> 1 \), equation (26) tells us that
the nature of the divergence is linear for any finite value of $L_3$ and logarithmic
for $L_3 = 0$. Hence at a finite thickness, choosing $g^{-2} \sim \int[d^2 k] \frac{1}{(k^2 + \mu^2)^{1/2}}$, i.e.
trading the dimensionless coupling constant for the dimensional parameter $\mu$
should make the problem manifestly finite.

To carry out the renormalization, we shall isolate the divergent part of the
integral appearing in the gap equation (27). Recalling that the divergence in
the integral remains the same when we let $L_3 \to \infty$, i.e. let the hyperbolic
cotangent go to 1, we can add and subtract, this divergent amount from the
integral and rewrite it as:

$$
g^{-2} = \int \frac{[d^2 k][dk_0]}{4 \pi ib} \left[ \frac{\coth((a - ib)^{1/2} L_3/2)}{(a - ib)^{1/2}} - \frac{1}{(a - ib)^{1/2}} - ((a - ib) \to (a + ib)) \right] +
+ \frac{1}{(a - ib)^{1/2}} - ((a - ib) \to (a + ib)) \right]$$

Carrying out the integration over momenta (which are now finite), we have:

$$
g^{-2} = \int \frac{[dk_0]}{2ib} \arg \left[ \frac{1 - e^{-(ib)^{1/2} L_3}}{1 - e^{-(ib)^{1/2} L_3}} \right] + L_3 \int \frac{[dk_0]}{b^{1/2}} (28)
$$

The first integral in the equation above is finite while the second one encodes
the linear divergence of the problem. It diverges as $\sqrt{k_0}$, and since $k_0$ has
the dimensions of $k^2$, it’s divergence is linear in the momentum cutoff. So the
appropriate renormalization prescription to use in this problem would be to let
$g^{-2} = L_3 \int \frac{L_3}{(k^2 + \mu^2)^{1/2}}$. So the renormalized gap equation now is:

$$
L_3 \left[ \sqrt{\frac{\mu}{\Delta}} f(\Lambda^2/\mu) - f(\Lambda^2/\Delta) \right] = G(L_3', \Lambda^2/\Delta) (29)
$$

Here $f(x) = \int_0^a \frac{dx}{(x^2 + 1)^{1/4}}$, $L_3' = \sqrt{\Delta} L_3$, and $G(x, \Lambda^2/\Delta) = \int_{\Lambda^2/\Delta}^{\Delta} \frac{dk_0}{\sqrt{k_0^2 + 1}} \arg \left[ \frac{1 - e^{-\sqrt{(k_0^2 + 1)^{1/2}}}}{1 - e^{-\sqrt{-i(k_0^2 + 1)^{1/2}}}} \right]$. The above equation tells us that:

$$
\sqrt{\frac{\Delta}{\mu}} = \frac{1}{G(L_3', \Lambda^2/\Delta) / L_3 f(\Lambda^2/\mu) + f(\Lambda^2/\Delta) / f(\Lambda^2/\mu))} (30)
$$

In the limit $\Lambda \to \infty$, $G$ remains finite, and the ratio of the two linearly divergent
integrals approaches unity, and we are left with a result that is insensitive to the
finite size, i.e. $\Delta = \mu$. So we see that for any finite value of $L_3$, the behavior of the
system is the same as that of a truly three dimensional system in the strongly
coupled phase. This is indeed signalled by the renormalization prescription used
above i.e $g^{-2} \sim \Lambda$, which is precisely of the order of the critical coupling in the
full three dimensional theory.
The interesting dependence on the finite size is in the opposite direction, i.e. in the approach towards two dimensionality. So let us now ask the question, how should $L_3$ approach zero (when the cutoff is removed) so that $\sqrt{2/\mu}$ remains finite? To answer this question, we revert back to Equation (26), and change the renormalization prescription for $g$. We let $g$ be renormalized in the way it would be for a truly two dimensional theory, i.e. $g^{-2} = \frac{1}{4\pi} \ln(\frac{2\Lambda^2}{\mu})$, and estimate the allowed range of values for $L_3$, which is now thought of as a function of the cutoff. Expanding the hyperbolic cotangents in equation (26) to first order in $L_3$, we get,

$$\frac{1}{4\pi} \ln(\frac{2\Lambda^2}{\mu}) = \int_{\Lambda} [d^2k] [dk_0] \frac{1}{k^4 + k_0^2 + \Delta^2} + \frac{1}{3} L_3^2 \int_{\Lambda} [d^2k] [dk_0] \frac{1}{\sqrt{k_0^2 + \Delta^2}}$$

or, in terms of dimensionless quantities,

$$\ln(\frac{\Delta}{\mu}) = C L_3^2 \Lambda^2 \int_{0}^{\Lambda^2/\Delta} \frac{dx}{\sqrt{x^2 + 1}}$$

Here $C$ denotes a positive number (whose precise value is unimportant). Hence from (31) above, we have,

$$\Lambda L_3 = \frac{\ln(\frac{\Delta}{\mu})}{\sqrt{C \ln(\frac{2\Lambda^2}{\Delta})}}$$

The equation above has the expected behavior, i.e. $L_3 \to 0$, as $\Lambda \to \infty$. $\Lambda^{-1}$ can be thought of as the measure of the lattice spacing in two dimensions. Hence we see that the theory will behave like a two dimensional theory even for a finite thickness, as long as the transverse thickness, $L_3$ measured in units of the lattice spacing in the two dimensions is small. An estimate of this smallness is provided in the equation above.

## 4 Justification of the mean field theory, a renormalization group analysis

In the previous sections we carried out the mean field analysis of a system of non-relativistic Fermions interacting through a short ranged attractive potential, by constructing the corresponding Landau-Ginsburg theory. At the level of the mean field theory, we identified a logarithmic divergence in the problem, which was removed by a renormalization of the coupling constant. The mean field analysis also predicted the existence of a gap in the spectrum, for arbitrarily small values of the attractive coupling. So the natural question that arises is that how can one justify the validity of the mean field approximation, which leads to these non-trivial consequences? In other words, we know of several examples where the predictions of mean field theory are qualitatively wrong; the $D = 1,$
Heisenberg chain at half filling is a good example of this; examples such as these provide us with a warning that motivate a surer justification of the mean field analysis. In this section here, we shall carry out a one loop renormalization group analysis of the theory, and show that the BCS instability predicted by the mean field analysis is also implied by the renormalization group flow equations, and this lends substance to the qualitative nature of the predictions of the mean field theory.

The logistics for carrying out the renormalization group analysis for our model are going to be as follows; we shall start with a two dimensional, free, scale invariant theory, which we will define with a fundamental cut-off \( \Lambda \). The Gaussian action is:

\[
S_0 = \int [dk_0] \int [d\theta] \int_0^\Lambda k [dk] [\Psi^\dagger(k_0k)(ik_0 - k^2)\Psi(k_0k)]
\]

The action is invariant under the scaling transformation \( k \rightarrow sk, k_0 \rightarrow s^2k_0, \Psi \rightarrow s^{-3}\Psi \).

Next we shall introduce various scale invariant perturbations around the free action, and study the evolutions of these perturbations under the RG transformations. To generate the RG transformations, we shall split up the phase space (the k space) into ‘high’ and ‘low’ degrees of freedom, the high and low degrees of freedom being defined as, \( \Psi_{\text{high}} = [\Psi(k), \text{for } (\Lambda/s \leq k \leq \Lambda) \text{ and } 0 \text{ otherwise}] \), \( \Psi_{\text{low}} = [\Psi(k), \text{for } (0 \leq k \leq \Lambda/s) \text{ and } 0 \text{ otherwise }] \). \( s \) is a real positive number greater than one. Following this we’ll proceed to integrate out the high degrees of freedom, and get an effective theory for the low modes. This process will generate the RG flow equations for the various couplings that perturb the free action. Of special interest to us is the flow corresponding to the four point interaction, as that encodes the possibility or (lack of) a BCS type of instability in the system.

Before we proceed to carry out the renormalization group transformation for the four point function, it is worth pointing out that unlike the usual BCS theory, we are not building an effective theory for the modes close to the Fermi surface, indeed we are interested in the scenario, where the system is sufficiently dilute such that \( \frac{\pi}{\Lambda k_F} \ll 1 \). In the case of the BCS theory, since the physically interesting degrees of freedom are the modes close to the Fermi surface, one is justified in linearizing the energy momentum dispersion relation around the Fermi energy and keeping the contribution only from the direction normal to the Fermi surface, one is justified in linearizing the energy momentum dispersion relation around the Fermi energy and keeping the contribution only from the direction normal to the Fermi surface, as these correspond the degrees of freedom of significance to the RG transformations. The degeneracy of the angular directions, and the linearization of the dispersion relation allows one to reduce the theory to a 1 + 1 dimensional theory with a relativistic dispersion relation, involving \( N \) species of Fermions, where \( N \) is a large number proportional to the area of the Fermi surface measured in units of the cutoff. These generic arguments carry through in any dimension greater than or equal to two, which is an explanation for the fact that all metallic systems, irrespective of the number of space dimensions face the BCS instability in the limit of \( T \rightarrow 0 \).
In the case we are interested in such simplifications are absent, although the BCS instability is present, however in a way, which is special to the two dimensional nature of the problem.

To see this, let us start with a generic four point coupling;

\[ S_{\text{int}} = \int \Pi_i [d\vec{k}_i][dk_{0i}]U(\vec{k}_i)\Psi^4(4)\Psi^4(3)\Psi^2(2)\Psi^2(1)\delta(\vec{k})\delta(k_{0}) \]  

(35)

In the above interaction the number 1...4 denote the momenta and Matsubara frequencies. U is the four point coupling function, which in general will have a dependence on the momenta. The delta functions ensure than the momenta and the Matsubara frequencies corresponding to 1 and 2 equal those of 3 and 4. All the momentum integrals run between 0 and \( \Lambda \). If we expand the four point function in a Taylor series, only the constant part \( U_0 \) turns out to be of importance, as the terms with dependence on the momenta, correspond to irrelevant couplings, i.e, only the constant part leads to a four point interaction invariant under the scaling defined above.

If we now split up the fields into the high and low modes, then the one loop corrections to the coupling constant are given by the three diagrams above (fig 1), where the internal loop momenta correspond to the high degrees of freedom, i.e \( \Lambda/s \leq \) loop momenta \( \leq \Lambda \). i.e.

\[ \delta U_0 = u_0^2 \int_{-\infty}^{\Lambda} dk_0 \int_{\Lambda/s}^{\Lambda} [d^2 k] \int d\theta \frac{1}{(ik_0-k^2)(ik_0-(k+Q)^2)} + \frac{1}{(ik_0-k^2)(ik_0-(k+Q')^2)} - \frac{1}{2 (ik_0-k^2)(ik_0-(P-k)^2)} \]  

(36)

It is now obvious that the first two diagrams are identically equal to zero as the \( k_0 \) integrals vanish for these diagrams. Both the poles are on the same half of the complex \( k_0 \) plane because \((k + Q)^2 \) and \((k + Q')^2 \) being quadratic functions are always positive for all values of momentum transfer. Hence we can always close the contour in the other half plane and this makes these integrals vanish. The third integral however is non zero, and produces a flow, which is given by;

\[ \beta = \frac{dU_0}{dt} = \frac{-U_0^2}{4\pi} \]  

(37)
Here \( t = -\ln s \), and the solution to the flow equation is,

\[
U_0(t) = \frac{U_0(0)}{1 + \frac{1}{4\pi} U_0(0)}
\]  

(38)

Thus we find that for an attractive microscopic coupling, (corresponding to \( U_0(0) < 0 \)), a BCS instability is inevitable. Hence the divergence of the coupling constant that showed up in the mean field analysis is not an artifact of the mean field approximation, but is indeed a true signal of the onset of a BCS transition.

We would like to emphasize the difference of this coupling constant renormalization with its counterpart in the theory of metals. In metallic systems, the constraint that all momenta lie very close to the Fermi surface imposes a stringent dependence of the coupling constant on the transverse momenta, i.e. on the directions orthogonal to the radial directions in momentum space. Moreover the only couplings that contribute to the flow are the ones for which \( \vec{k}_1 = -\vec{k}_2 \), and \( \vec{k}_3 = -\vec{k}_4 \), and \( U_0 = U_0(\vec{k}_1, \vec{k}_3) \), i.e. it is a function of an angle. When this function is decomposed into its various angular momentum components, the contribution from the first two diagrams (which are non zero, as the dispersion relation is linear in the momentum) couple the flows corresponding to the various angular momentum sectors (the Kohn - Luttinger effect\[^{[5, 6, 7]}\]), in a manner that drive (for certain values of the angular momentum) couplings which might have been repulsive to negative values. These couplings then lead to the BCS instability according to the flow equation (37) given above.

Hence it is clear that the system we are presently interested in is different from the usual BCS superconductor, although it also possesses a similar ground state. But there is no analogue of the Kohn Luttinger effect for this system, however a coupling that is attractive to start with will be driven by the RG flow to produce a BCS like ground state. Moreover, the arguments and the analysis we given above are very special to the two dimensional case as was shown in the previous section.

5 Conclusion:

We conclude that a dilute, two dimensional system of electrons, interacting through a short ranged attractive potential will undergo a phase transition to a superconducting regime. This phase transition is different from the usual BCS transition in several respects.

a: The phenomena described here is very special to the two dimensionality of the problem.

b: The phase transition is from a semiconducting / insulating phase to a superconducting regime.

c: The diluteness of the system plays a very significant role in the analysis, as that is what warrants the use of the quadratic dispersion relation, for which the critical dimension is \( d = 2 \).

On a speculative note, we would like to emphasize that these phenomena may be of relevance towards understanding the behavior of high \( T_c \) superconductors.\[^{[8, 13]}\]
In dimensions greater than two we find that this kind superconductivity will not arise for arbitrarily small values of the electron - electron coupling, and that the coupling will have to be of greater than some critical value $g_{cr}$ for the phase transition to take place. If one allows for a finite thickness of the material being studied, then we find that there are stringent bounds on the allowed range of thickness if the phenomena characteristic of two dimensions is to survive; in fact we find that the square of the thickness can exceed the lattice spacing in two dimensions only by logarithmic amounts.

Acknowledgement: We thank G.Krishnaswami and Y.Shapir for useful discussions. This work was supported in part by the US Department of Energy, Grant No. DE-FG02-91ER40685

References

[1] Abrikosov, Gorkov, and Dzyaloshinshi, Methods of Quantum Field Theory in Statistical Mechanics (Dover, New York, 1963).

[2] R.Shankar, Rev of Modern Physics 66, 129 (1994).

[3] J.Polchinski, Effective Field Theory and the Fermi Surface, in Proceedings of the 1992 Theoretical Advanced Studies Institute in Elementary Particle Physics, World Scientific, Singapore, 1992.

[4] J.Feldman, J.Magnen, and V.Rivasseau, Helv. Phys. Acta 66, 498.

[5] J.M.Luttinger, Phys. Rev 119, 1153 (1960).

[6] J.M.Luttinger, Phys. Rev 121, 942 (1961).

[7] M.A.Baranov, A.V.Chubukov, and M.Kagan, Int. J. Mod. Phys 6, 2471 (1992).

[8] E. Abrahams, S. V. Kravchenko, and M. P. Sarachik, (2000), cond-mat/0006055.

[9] P.Phillips, Nature 406, 687 (2000).

[10] K.Huang, Quarks Leptons and Gauge Fields (World Scientific, Singapore, 1982).

[11] K.S.Gupta and S.G.Rajeev, Phys Rev D 48, 5940 (1993).

[12] R.J.Henderson and S.G.Rajeev, J. Math. Phys 39, 749 (1998).