Multidegree for bifiltered $D$-modules

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Abstract

In commutative algebra, E. Miller and B. Sturmfels defined the notion of multidegree for multigraded modules over a multigraded polynomial ring. We apply this theory to bifiltered modules over the Weyl algebra $D$. The bifiltration is a combination of the standard filtration by the order of differential operators and of the so-called $V$-filtration along a coordinate subvariety of the ambient space defined by M. Kashiwara. The multidegree we define provides a new invariant for $D$-modules. We investigate its relation with the $L$-characteristic cycles considered by Y. Laurent. We give examples from the theory of $A$-hypergeometric systems $M_A(\beta)$ defined by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky. We consider the $V$-filtration along the origin. When the toric projective variety defined from the matrix $A$ is Cohen-Macaulay, we have an explicit formula for the multidegree of $M_A(\beta)$.

Introduction

We consider finite type modules over the Weyl algebra

$$D = \mathbb{C}[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n].$$

It is classical to endow $D$ with the filtration by the order in $\partial_1, \ldots, \partial_n$, which we call the $F$-filtration, and to endow a $D$-module $M$ with a good $F$-filtration. For instance that leads to the notion of the characteristic variety, which is the support of $\text{gr}^F(M)$, and to the characteristic cycle. M. Kashiwara introduced another kind of filtration, the $V$-filtration along a smooth subvariety $Y$ of $\mathbb{C}^n$. Then one has the notion of a good $(F,V)$-bifiltration (c.f. [12]), and we can also consider intermediate filtrations $L$ between $F$ and $V$ as developed by Y. Laurent in his theory of slopes (c.f. [11]). This leads to $L$-characteristic varieties (the support of $\text{gr}^L(M)$) and $L$-characteristic cycles.

Exploring that theory with homological methods, M. Granger, T. Oaku and N. Takayama considered $(F,V)$-bifiltered free resolutions of finite type $D$-modules in [8, 15]. More precisely, dealing with local analytic $D$-modules, they can define minimal bifiltered free resolutions. That provides invariants attached to a bifiltered module: the ranks, also called Betti numbers, and the shifts appearing in the minimal resolution. In the category of modules over the global
Weyl algebra, \((F, V)\)-bifiltered free resolutions still can be considered, but the minimality no longer makes sense.

Our main purpose in this paper is to introduce a new invariant, the multidegree, derived from the Betti numbers and shifts arising from any bifiltered free resolution of a \((F, V)\)-bifiltered \(D\)-module. It will be independent of the good bifiltration, i.e. a chosen presentation of the module. We will relate this invariant to the \(L\)-characteristic cycles.

To achieve this, we use the theory of \(K\)-polynomial and multidegree, as was developed by E. Miller and B. Sturmfels in [13]. The multidegree is a generalization of the usual degree in projective geometry; it is defined for finite type multigraded modules over a polynomial ring. After reviewing this theory in Section 1, we adapt it first to \(F\)-filtered \(D\)-modules in Section 2. We obtain the notion of multidegree for a \(F\)-filtered \(D\)-module, which is independent of the good filtration. This multidegree is a monomial \(mT^d\) with \(m \in \mathbb{N}\); we interpret \(m\) and \(d\) as a generic multiplicity and a generic codimension respectively.

Then we adapt the theory of multidegree to \((F, V)\)-bifiltered \(D\)-modules in section 3. The multidegree is an element of \(\mathbb{Z}[T_1, T_2]\), denoted by \(C_{F, V}(M; T_1, T_2)\), homogeneous in \(T_1, T_2\). Its degree \(d\) has to be fixed because of the non-positivity of the multigrading considered: if \(Y\) is the origin in \(\mathbb{C}^n\), \(d\) is the codimension of the \(V\)-homogenization module \(R_V(M)\). Using a proof in [12], we can show that \(C_{F, V}(M; T_1, T_2)\) is an invariant attached to the module, independently of the good filtration.

In section 4, we assume a strong regularity condition on the \((F, V)\)-bifiltered module, which we call a nicely bifiltered module. We prove that in the holonomic case, this condition implies that the module has no slopes along \(Y\). Then we show that the multidegree of such a module almost only depends on the \(L\)-characteristic cycle of the module, with \(L\) an intermediate filtration close to \(F\) or close to \(V\). Let us note here that we have to deal with some codimensions which may alter the link between multidegree and \(L\)-characteristic cycle: the codimension of the module \(R_V(M)\) may not be equal to that of \(\text{gr}^L(M)\).

Finally, we use the theory of hypergeometric systems to provide interesting examples in section 5. We consider the hypergeometric module \(M_A(\beta)\) introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in [5], in the case where \(Y\) is the origin in \(\mathbb{C}^n\). We take \(Y\) to be the origin in \(\mathbb{C}^n\). In that case the problems about codimensions described above does not remain, and the multidegree only depends on the \(L\)-characteristic cycle if \(M_A(\beta)\) is nicely bifiltered. Let \(\text{vol}(A)\) denotes the normalized volume of the convex hull of the set \(\{0, a_1, \ldots, a_n\}\) in \(\mathbb{R}^d\). Let us assume that the closure in \(\mathbb{P}^n\) of the variety defined by \(I_A\) is Cohen-Macaulay. Then for generic parameters \(\beta\) (or for all parameters if \(I_A\) is homogeneous), niceness holds and we have:

\[
C_{F, V}(M_A(\beta); T_1, T_2) = \text{vol}(A) \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j}.
\]

We give examples, computed with the computer algebra systems Singular [10] and Macaulay2 [9].
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1 Multidegree for modules over a commutative polynomial ring

1.1 Review of the theory

Let us give a review of the theory of $K$-polynomials and multidegrees in the commutative setting. Let $S = k[x_1, \ldots, x_n]$ with $k$ a field. A multigrading on $S$ is given by a homomorphism of abelian groups $\deg : \mathbb{Z}^n \to \mathbb{Z}^d$ with, denoting by $e_1, \ldots, e_n$ the canonical base of $\mathbb{Z}^n$, $\deg(e_i) = a_i \in \mathbb{Z}^d$. Identifying the set of monomials of $S$ with $\mathbb{N}^n$, we have $\deg(x_1^{\alpha_1} \ldots x_n^{\alpha_n}) = \sum a_i \alpha_i$, and $S$ becomes a multigraded ring over $\mathbb{Z}^d$.

Let $M = \bigoplus_{a \in \mathbb{Z}^d} M_a$ be a multigraded $S$-module of finite type. For $b \in \mathbb{Z}^d$, let us denote by $S[b]$ the module $S$ endowed with the multigrading such that for any $a \in \mathbb{Z}^d$, $S[b]_a = S_{a-b}$. A multigraded free module is a module isomorphic to $\bigoplus_{i=1}^r S[b_i]$, with $b_1, \ldots, b_r \in \mathbb{Z}^d$.

Take a multigraded free resolution, i.e. a multigraded exact sequence

$$0 \to \mathcal{L}_3 \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to M \to 0,$$

with $\mathcal{L}_i$ a multigraded free module.

**Definition 1.1.** For $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$, the $K$-polynomial of $S[b]$ is defined by

$$K(S[b]; T_1, \ldots, T_d) = T_1^{b_1} \cdots T_d^{b_d} \in \mathbb{Z}[T_1, \ldots, T_d, T_1^{-1}, \ldots, T_d^{-1}].$$

For $b_1, \ldots, b_r \in \mathbb{Z}^d$, The $K$-polynomial of $\mathcal{L} = \bigoplus_{j=1}^r S[b_j]$ is defined by

$$K(\mathcal{L}; T_1, \ldots, T_d) = \sum_j K(S[b_j]; T_1, \ldots, T_d) \in \mathbb{Z}[T_1, \ldots, T_d, T_1^{-1}, \ldots, T_d^{-1}].$$

Then the $K$-polynomial of $M$ is defined by

$$K(M; T) = \sum_i (-1)^i K(\mathcal{L}_i; T_1, \ldots, T_d) \in \mathbb{Z}[T_1, \ldots, T_d, T_1^{-1}, \ldots, T_d^{-1}].$$

**Proposition 1.1** ([13], Theorem 8.34). The definition of $K(M; T_1, \ldots, T_d)$ does not depend on the multigraded free resolution.

If we substitute $T_1, \ldots, T_d$ by $1-T_1, \ldots, 1-T_d$ in $K(M; T_1, \ldots, T_d)$, we get a well-defined power series in $\mathbb{Z}[T_1, \ldots, T_d]$. We then consider the total degree in $T_1, \ldots, T_d$. 

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Definition 1.2. We denote by \( C(M; T_1, \ldots, T_d) \in \mathbb{Z}[T_1, \ldots, T_d] \) the sum of the terms whose total degree equals \( \text{codim} M \) in \( K(M; 1 - T_1, \ldots, 1 - T_d) \). This is called the multidegree of \( M \).

Remind that the module \( M \) defines an algebraic cycle \( \sum m_i Z_i \), where \( Z_i \), defined by ideals \( p_i \), are the irreducible components of \( \text{rad}(\text{ann} M) \) and \( m_i \) is the multiplicity of \( M_{p_i} \). It turns out that the multidegree depends only on the algebraic cycle.

Proposition 1.2 ([13], Theorem 8.53). If \( p_1, \ldots, p_k \) are the maximal dimensional associated primes of \( M \), then

\[
C(M; T_1, \ldots, T_d) = \sum_i (\text{mult}_{p_i} M)C(S/p_i; T_1, \ldots, T_d).
\]

\( S \) is said to be positively multigraded if moreover for any \( b \in \mathbb{Z}^d \), we have \( \dim_S S_b < \infty \). In that case we can consider the Hilbert series

\[
H(M; T_1, \ldots, T_d) = \sum_{b \in \mathbb{Z}^d} (\dim_k M_b)T_1^{b_1} \ldots T_d^{b_d} \in \mathbb{Z}[[T_1, \ldots, T_d]].
\]

If \( b = (b_1, \ldots, b_d) \in \mathbb{Z}^d \), let us denote by \( T^b \) the product \( T_1^{b_1} \ldots T_d^{b_d} \).

Proposition 1.3. Let \( S \) be positively multigraded. Then

1.

\[
H(M; T_1, \ldots, T_d) = \frac{K(M; T_1, \ldots, T_d)}{\Pi(1 - T^n)}
\]

2. If \( M \neq 0 \), then \( C(M; T_1, \ldots, T_d) \neq 0 \), moreover \( C(M; T_1, \ldots, T_d) \) is the sum of the non-zero terms of least total degree in \( K(M; 1 - T_1, \ldots, 1 - T_d) \).

The assertion 1 is [13], Theorem 8.20, and the assertion 2 follows from [13], Claim 8.54 and Exercise 8.10.

1.2 Genericity

Let \( S = k[\lambda_1, \ldots, \lambda_p][x_1, \ldots, x_n] \) be multigraded by \( \deg x_i = a_i \in \mathbb{Z}^d \) and \( \deg \lambda_i = 0 \). We consider \( \lambda_1, \ldots, \lambda_p \) as parameters and study the behaviour of the \( K \)-polynomial under the specialization.

Let \( K = \text{Frac}(k[\lambda_1, \ldots, \lambda_p]) \). Let \( M = S^r/N \) be a multigraded finite type \( S \)-module. For \( c \in k^p \), let

\[
M^c = \frac{S}{(\lambda_1 - c_1, \ldots, \lambda_p - c_p)} \otimes M,
\]

considered as a multigraded \( k[x_1, \ldots, x_n] \)-module. We are going to state that if \( c \) is generic, then \( K(K \otimes M; T) = K(M^c; T) \). More precisely, we shall describe the exceptional values of \( c \) in terms of Gröbner bases.

Let \( < \) be a well-ordering on \( \mathbb{N}^p \times \{1, \ldots, r\} \), such that for any \( \alpha, \beta, \delta \in \mathbb{N}^p \) and \( i, i' \in \{1, \ldots, r\} \), we have

\[
(\alpha, i) < (\beta, i') \Rightarrow (\alpha + \delta, i) < (\beta + \delta, i'),
\]
and let \( <' \) be the well-ordering on \( \mathbb{N}^p \times \mathbb{N}^n \times \{1, \ldots, r\} \) defined by

\[
(\alpha, \beta, i) <' (\alpha', \beta', i') \iff \begin{cases}
(\beta, i) < (\beta', i') \\
((\beta, i) = (\beta', i') \text{ and } \alpha <_{\text{lex}} \alpha')
\end{cases}
\]

Let \( P_1, \ldots, P_s \) be a Gröbner base of \( N \). For \( 1 \leq i \leq s \), \( q_i(\lambda) \in k[\lambda] \) denotes the leading coefficient, with respect to \(<\), of the image of \( P_i \) in \( \mathbb{K} \otimes S \). For \( P \in k[x]^r \) or \( P \in k[x]^s \), we denote by \( \text{Exp}_< P \in \mathbb{N}^n \times \{1, \ldots, r\} \) the leading exponent of \( P \) with respect to \(<\).

**Proposition 1.4** ([13], Propositions 6 and 7). 1. \( P_1, \ldots, P_s \) is a Gröbner base of \( \mathbb{K} \otimes N \).

2. Let \( c \in k^n \) such that \( c \notin \bigcup_i \{ q_i = 0 \} \). Then \( P_1(c), \ldots, P_s(c) \) is a Gröbner base of \( N^c \) and \( \text{Exp}_< \mathbb{K} \otimes N = \text{Exp}_< N^c \).

**Proposition 1.5.** Let \( c \in k^n \) such that \( c \notin \bigcup_i \{ q_i = 0 \} \). Then \( K(\mathbb{K} \otimes M; T) = K(M^c; T) \). Consequently \( C(\mathbb{K} \otimes M; T) = C(M^c; T) \).

This follows from Proposition [13] and from [13], Theorem 8.36 which asserts that the \( K \)-polynomial remains the same when taking the initial module with respect to any well-ordering.

## 2 Multidegree for \( F \)-filtered \( D \)-modules

Let \( D = \mathbb{C}[x_1, \ldots, x_n](\partial_1, \ldots, \partial_n) \) be the Weyl algebra. A vector \((u, v) \in \mathbb{Z}^p \times \mathbb{Z}^n\) is called an admissible weight vector for \( D \) if for all \( i, u_i + v_i \geq 0 \). For \( P = \sum_{\alpha, \beta} a_{\alpha, \beta}(x) x^\alpha \partial^\beta \in D \), we define

\[
\text{ord}((u, v))(P) = \max_{\{\alpha, \beta) | a_{\alpha, \beta} \neq 0\}}(\sum u_i \alpha_i + \sum v_i \beta_i).
\]

We then define an increasing filtration by \( F_d^{(u, v)}(D) = \{ P \in D, \text{ord}^F(P) \leq d \} \) with \( d \in \mathbb{Z} \).

In this section we consider only the weight vector \((0, 1)\); we will simply denote the associated filtration by \((F_d(D))_{d \in \mathbb{N}}\), called the \( F \)-filtration. We have \( \text{gr}^F(D) \simeq C[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] \).

Let \( M \) be a \( D \)-module. An \( F \)-filtration of \( M \) is an exhausting increasing filtration \( (F_d(M))_{d \in \mathbb{N}} \) compatible with the \( F \)-filtration of \( D \). For \( \nu = (n_1, \ldots, n_r) \in \mathbb{N}^r \), let us denote by \( D^\nu[n] \) the module \( D^\nu \) endowed with the \( F \)-filtration such that \( F_d(D^\nu[n]) = \bigoplus_{|\lambda| \leq d} F_{d-|\lambda|}(D) \). If \( N \) is a submodule of \( D^\nu \), we endow \( D^\nu[n]/N \) with the quotient filtration, i.e.

\[
F_d \left( \frac{D^\nu[n]}{N} \right) = \frac{F_d(D^\nu[n]) + N}{N}.
\]

We say that a filtration \( F_d(M) \) is good if \( M \) is isomorphic as an \( F \)-filtered \( D \)-module to a module of the type \( D^\nu[n]/N \).

Let us take a filtered free resolution

\[
0 \to D^{\nu^0}[n^{(0)}] \to \cdots \to D^{\nu^1}[n^{(1)}] \to D^{\nu}[n^{(0)}] \to M \to 0
\]

Its existence can be proved in the same way as [8], Theorem 3.4, forgetting the minimality.
Definition 2.1. The $K$-polynomial of $D^r[n]$ is defined by

$$K_F(D^r[n]; T) = \sum_i T^{n_i} \in \mathbb{Z}[T].$$

The $K$-polynomial of $M$ is defined by

$$K_F(M; T) = \sum_i (-1)^i K_F(D^r_i[n^{(i)}]; T) \in \mathbb{Z}[T].$$

Proposition 2.1. The definition of $K_F(M; T)$ does not depend on the filtered free resolution.

Proof. Let $R = \text{gr}^F(D)$, and for $n = (n_1, \ldots, n_r)$, $R^r[n] = \oplus_{i=1}^r R[n_i]$. By grading the filtered free resolution we get a graded free resolution over the commutative ring $R$:

$$0 \to R^r_1[n^{(1)}] \to \cdots \to R^r_i[n^{(i)}] \to R^r_0[n^{(0)}] \to \text{gr}^F(M) \to 0.$$

The $K$-polynomial is unchanged. Then apply Proposition 1.1. \qed

Definition 2.2. We denote by $C_F(M; T)$ the term of least degree in $T$ in $K_F(M; 1 - T)$. This is the multidegree of $M$ with respect to $F$.

Proposition 2.2. $C_F(M; T)$ does not depend on the good filtration.

Proof. Again we argue by grading. We have $C_F(M; T) = C(\text{gr}^F(M); T)$. Let $K = \text{Frac}(\mathbb{C}[x])$. We have $C(\text{gr}^F(M); T) = C(K \otimes \text{gr}^F(M); T)$. The graded ring $K \otimes \text{gr}^F(D)$ is a positively graded ring. Hence the $K$-polynomial is equal to the numerator of the Hilbert series, by Proposition 1.3. The multidegree is of the form $mT^d$ with $d = \text{codim} K \otimes \text{gr}^F(M)$ (unless it is 0), and $m$ is the multiplicity of $K \otimes \text{gr}^F(M)$ along the maximal ideal $\xi_1, \ldots, \xi_n$. We can show that this data is independent of the good filtration in the same way as [7], Remark 12 and Proposition 25. \qed

Let us give some interpretation. We have $C_F(M; T) = mT^d$. For $x_0 \in \mathbb{C}^n$, the graded $\mathbb{C}[\xi]$-module $(\text{gr}^F(M))^{x_0}$ is defined as in the section 1.2.

Proposition 2.3. 1. $m$ and $d$ are equal respectively to the multiplicity and the codimension of the graded $\mathbb{C}[\xi]$-module $\text{gr}^F(M)^{x_0}$ for $x_0$ generic. Let us denote by $\pi : T^* \mathbb{C}^n \to \mathbb{C}^n$ the canonical projection. $d$ is equal to the codimension of the variety $\text{char} M \cap \pi^{-1}(x_0)$ for $x_0$ generic.

2. If moreover $M$ is holonomic, then $m = \text{rank} M = \dim_k K \otimes \text{gr}^F(M)$.

Proof. 1. This is Proposition 1.5.

2. In the holonomic case, $K \otimes \text{gr}^F(M)$ is finite dimensional over $k$, and we have

$$\dim_k K \otimes \text{gr}^F(M) = H(K \otimes \text{gr}^F(M); T)|_{T=1}.$$

The result follows, by using Proposition 1.3. \qed
3 Multidegree for \((F, V)\)-bifiltered \(D\)-modules

Now set \(D = \mathbb{C}[x_1, \ldots, x_n, t_1, \ldots, t_p][\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{t_1}, \ldots, \partial_{t_p}]\). We will endow it with the \(F\)-filtration. We introduce the \(V\)-filtration along \(t_1 = \cdots = t_p = 0\). This is the filtration defined by assigning the weight vector \((0, -1, 0, 1)\) to the set of variables \((x, t, \partial_x, \partial_t)\). We denote this filtration by \(V_k(D)\) for \(d, k \in \mathbb{Z}\).

Then we have the \((F, V)\)-bifiltration on \(D\) defined by \(F_{d,k}(D) = F_d(D) \cap V_k(D)\) for \(d, k \in \mathbb{Z}\). For \(n, m \in \mathbb{Z}^r\), let us denote by \(D^r[n][m]\) the module \(D^r\) endowed with the bifiltration such that

\[
F_{d,k}(D^r[n][m]) = \bigoplus_{i=1}^r F_{d-n_i, k-m_i}(D).
\]

A quotient \(D^r[n][m]/N\) is endowed with the bifiltration \(F_{d,k}(D^r[n][m]/N) = (F_{d,k}(D^r[n][m]) + N)/N\).

Let \(M\) be a \(D\)-module. A good bifiltration \((F_{d,k}(M))_{d \in \mathbb{N}, k \in \mathbb{Z}}\) is an exhaustive increasing bifiltration, compatible with the bifiltration \((F_{d,k}(D))\), such that \(M\) is isomorphic as a bifiltered module to a module of the type \(D^r[n][m]/N\).

**Proposition 3.1.** \(M\) admits a bifiltered free resolution, i.e. a bifiltered exact sequence

\[
0 \rightarrow D^{r_1}[n^{(1)}][m^{(1)}] \rightarrow \cdots \rightarrow D^{r_s}[n^{(s)}][m^{(s)}] \rightarrow D^{r_0}[n^{(0)}][m^{(0)}] \rightarrow M \rightarrow 0.
\]

We shall prove this proposition in a constructive way. For this purpose, let us introduce some Rees algebras. First, we have the Rees algebra with respect to the \(F\)-filtration (c.f. [3]):

\[
\mathcal{R}_F(D) = \bigoplus_d F_{d}(D)\tau^d.
\]

This is endowed with the \(V\)-filtration:

\[
V_k(\mathcal{R}_F(D)) = \bigoplus_{d \in \mathbb{N}} F_{d,k}(\mathcal{R}_F(D))\tau^d \text{ for } d \in \mathbb{N}.
\]

\(\mathcal{R}_F(D)\) is isomorphic to the \(\mathbb{C}\)-algebra generated by \(x_i, t_i, (\partial_x, \tau), (\partial_t, \tau), \tau\), subject to the relations \([\partial_x, \tau, x_1] = \tau\) and \([\partial_t, \tau, t_1] = \tau\), the commutators involving other pairs of generators being zero. This is a noetherian algebra. We will replace respectively the generators \(x_i, t_i, \partial_x, \partial_t, \partial_{x_i}, \partial_{t_i}, h\) by \(x_i, t_i, \partial_x, \partial_t, \partial_{x_i}, \partial_{t_i}, h\), thus we identify \(\mathcal{R}_F(D)\) with the \(\mathbb{C}\)-algebra, denoted \(D^{(h)}\), generated by \(x_i, t_i, \partial_x, \partial_t, \partial_{x_i}, \partial_{t_i}, h\), subject to the relations

\([\partial_x, x_1] = h \quad \text{and} \quad [\partial_t, t_1] = h\).

An admissible weight vector for \(D^{(h)}\) is a vector \((u, v, l) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}\) such that for any \(i, u_i + v_i \geq l\). A filtration is associated with such a vector by assigning it to the set of variables \((x, t, \partial_x, \partial_t, h)\). The filtration associated with \((u, v, l) = (0, -1, 0, 1, 0)\) gives the \(V\)-filtration. The bigraded ring \(\text{gr}^V(D^{(h)})\) is isomorphic to \(D^{(h)}\) endowed with the following multigrading:

\[
\text{deg}(x_i) = (0, 0), \quad \text{deg}(t_i) = (0, -1), \quad \text{deg}(h) = (1, 0),
\]

\[
\text{deg}(\partial_{x_i}) = (1, 0), \quad \text{deg}(\partial_{t_i}) = (1, 1).
\]
Let us denote $F_d(M) = \bigcup_k F_{d,k}(M)$. We associate with $M$ a $\mathcal{R}_F(D)$-module $\mathcal{R}_F(M) = \bigoplus_d F_d(M)\tau^d$, this is endowed with a $V$-filtration $V_k(\mathcal{R}_F(M)) = \bigoplus_d F_{d,k}(M)\tau^d$.

Conversely, there exists a dehomogenizing functor $\rho_F$ (see [8], where this functor is denoted by $\rho$), from the category of $V$-filtered graded $D(h)$-modules to the category of bifiltered $D$-modules. A $D(h)$-module is said to be $h$-saturated if the action of $h$ on this module is injective. [8], Proposition 3.6 states that the functors $\rho_F$ and $\mathcal{R}_F$ give an equivalence of categories between the category of $h$-saturated $D(h)$-modules with good $V$-filtrations and the category of $D$-modules with bifiltrations, and that moreover these functors are exact.

We have also the Rees algebra of $D$ with respect to $V$:

$$\mathcal{R}_V(D) = \bigoplus_{k \in \mathbb{Z}} V_k(D)\theta^k$$

This is endowed with the following filtration:

$$F_d(\mathcal{R}_V(D)) = \bigoplus_{k \in \mathbb{Z}} F_{d,k}(D)\theta^k \text{ for } d \in \mathbb{N}$$

$\mathcal{R}_V(D)$ is generated as a $\mathbb{C}$-algebra by $x, \partial_x, \theta^0, t_i\theta^{-1}, \partial_t, \theta, \theta$. Let us denote respectively those elements by $\tilde{x}_i, \tilde{\partial}_x, \tilde{t}_i, \tilde{\partial}_t, \tilde{\partial}_u, \tilde{\partial}_\theta, \tilde{\partial}_\theta$. The following lemma is clear.

**Lemma 3.1.** $\mathcal{R}_V(D)$ is isomorphic to the algebra $\mathbb{C}[\tilde{x}_i, \tilde{t}_i, \tilde{\partial}_x, \tilde{\partial}_t, \tilde{\partial}_\theta, \tilde{\partial}_\theta]$ subject to the relations $[\tilde{\partial}_x, \tilde{x}_i] = 1$ and $[\tilde{\partial}_t, \tilde{t}_i] = 1$ for any $i$.

The $F$-filtration is then given by assigning the weight vector $(0, 0, 0, 1, 1, 1)$ to the set of variables $(\tilde{x}_i, \tilde{t}_i, \tilde{\partial}_x, \tilde{\partial}_t, \tilde{\partial}_u, \tilde{\partial}_\theta)$.

Then the bigraded ring $\text{gr}^F(\mathcal{R}_V(D))$ is isomorphic to the commutative polynomial ring $\mathbb{C}[\tilde{x}_i, \tilde{t}_i, \tilde{\partial}_x, \tilde{\partial}_t, \tilde{\partial}_\theta, \tilde{\partial}_\theta]$ endowed with the following multigrading:

- $\deg(\tilde{x}_i) = (0, 0)$
- $\deg(\tilde{t}_i) = (0, -1)$
- $\deg(\theta) = (0, 1)$
- $\deg(\tilde{\partial}_x) = (1, 0)$
- $\deg(\tilde{\partial}_t) = (1, 1)$

Similarly, we define the Rees module associated with $M$ with respect to $V$:

$$\mathcal{R}_V(M) = \bigoplus_{k \in \mathbb{Z}} V_k(M)\theta^k$$

where $V_k(M) = \bigcup_d F_{d,k}(M)$. It admits an $F$-filtration

$$F_d(\mathcal{R}_V(M)) = \bigoplus_{k \in \mathbb{Z}} F_{d,k}(M)\theta^k$$

such that $\text{gr}^F(\mathcal{R}_V(M))$ is isomorphic to

$$\bigoplus_{d,k} F_{d,k}(M)\theta^k.$$

Conversely, as it has been stated before, there exists a dehomogenizing functor $\rho_V$, from the category of $F$-filtered graded $\mathcal{R}_V(D)$-modules to the category of bifiltered $D$-modules. A $\mathcal{R}_V(D)$-module is said to be $\theta$-saturated if the action of $\theta$ on this module is injective. The functors $\rho_V$ and $\mathcal{R}_V$ give an equivalence of categories between the category of $\theta$-saturated $\mathcal{R}_V(D)$-modules with good $F$-filtrations and the category of $D$-modules with good bifiltrations. Moreover these functors are exact.
Proof of Proposition 3.2. $\mathcal{R}_F(M)$ is a finite type $D^{(h)}$ module isomorphic as a $V$-filtered graded $D^{(h)}$ module to a quotient of $(D^{(h)})^r[n][m]$. A presentation of $\mathcal{R}_F(M)$ can be obtained by means of $F$-adapted Gröbner bases. By replacing $D$ by $D^{(h)}$ in [16], section 3, we can construct a $V$-adapted free resolution of $\mathcal{R}_F(M)$. Dehomogenizing this resolution provides a bifiltered free resolution of $M$.

We can use also the $V$-homogenization. Using [16], section 3, we construct a presentation of $\mathcal{R}_V(M)$. We take a bigraded free resolution of $\text{gr}_F \mathcal{R}_V(M)$, which can be lifted to a $F$-adapted resolution of $\mathcal{R}_V(M)$, as in [8], Proposition 2.7. Taking $\rho_V$ gives a bifiltered free resolution of $M$.

Definition 3.1. The $K$-polynomial of $D^r[n][m]$ with respect to $(F, V)$ is defined by

$$K_{F,V}(D^r[n][m]; T_1, T_2) = \sum_i T_1^m T_2^m \in \mathbb{Z}[T_1, T_2, T_2^{-1}].$$

The $K$-polynomial of $M$ with respect to $(F, V)$ is defined by

$$K_{F,V}(M; T_1, T_2) = \sum_i (-1)^i K_{F,V}(D^r[n][m]; T_1, T_2) \in \mathbb{Z}[T_1, T_2, T_2^{-1}].$$

Proposition 3.2. The definition of $K_{F,V}(M; T_1, T_2)$ does not depend on the bifiltered free resolution.

Proof of Proposition 3.2. A bifiltered free resolution of $M$ induces a bigraded free resolution of $\text{gr}_F \mathcal{R}_V(M)$. Thus $K_{F,V}(M; T_1, T_2) = K(\text{gr}_F \mathcal{R}_V(D); T_1, T_2)$ and we can apply Proposition [16].

Let $\mathbb{K} = \text{Frac}(\mathbb{C}[x_1, \ldots, x_n])$. Instead of $D$, we shall work with $\mathbb{K} \otimes D$. This has no influence on the bifiltration.

Definition 3.2. We denote by $C_{F,V}(M; T_1, T_2)$ the sum of the terms whose total degree in $T_1, T_2$ equals $\text{codim}(\mathbb{K} \otimes \text{gr}_F \mathcal{R}_V(M))$ in the expansion of $K_{F,V}(M; 1-T_1, 1-T_2)$. This is the multidegree of $M$ with respect to $(F, V)$.

Theorem 3.1. $C_{F,V}(M; T_1, T_2)$ does not depend on the good bifiltration.

Proof. As before we take the Rees algebra with respect to $V$. We get

$$\mathcal{R}_V(\mathbb{K} \otimes D) \simeq \mathbb{K}[[\hat{t}_i, \theta]] \langle \hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \hat{\alpha}_{i_3} \rangle.$$

and

$$A := \text{gr}_F(\mathcal{R}_V(\mathbb{K} \otimes D)) \simeq \mathbb{K}[[\hat{t}_i, \theta]] \langle \hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \hat{\alpha}_{i_3} \rangle.$$

The ring $A$ is bigraded as follows:

$$\text{deg}(\hat{t}_i) = (0, -1), \quad \text{deg}(\theta) = (0, 1), \quad \text{deg}(\hat{\alpha}_{i_1}) = (1, 0), \quad \text{deg}(\hat{\alpha}_{i_1}) = (1, 1).$$

This is not a positive grading since $\mathbb{K}[[\hat{t}_i, \theta]]$ is infinite over $\mathbb{K}$. Let

$$\tilde{M} = \mathbb{K} \otimes \text{gr}_F(\mathcal{R}_V(M)).$$

A bifiltered free resolution of $M$ induces a bigraded free resolution of $\tilde{M}$, thus $K_{F,V}(M; T_1, T_2) = K(\tilde{M}; T_1, T_2, T_2^{-1}).$

Let us endow $M$ with another good bifiltration $(F_{d,k}(M))_{d,k}$. We denote by $M'$ the module $M$ endowed with this bifiltration. In view of Proposition [15] it is sufficient to prove
• \( \text{rad(ann}\hat{M}) = \text{rad(ann}\hat{M}') \)

• For any prime ideal \( p \) of \( A \), \( \text{mult}_p \hat{M} = \text{mult}_p \hat{M}' \).

To prove these two assertions, we argue exactly in the same way as in the proof of Proposition 1.3.2 of [12]. For the convenience of the reader, we give here the details.

We shall also use the behaviour of dimensions and multiplicities in short exact sequences.

Lemma 3.2 ([7], Proposition 24). Let

\[
0 \to E \to F \to G \to 0
\]

be an exact sequence of finite type \( A \)-modules, and let \( p \) be a prime ideal of \( A \). Then

1. \( \dim F_p = \max(\dim E_p, \dim G_p) \).
2. If \( \dim E_p = \dim G_p \), then \( \text{mult}_p F = \text{mult}_p E + \text{mult}_p G \).
   If \( \dim E_p < \dim G_p \), then \( \text{mult}_p F = \text{mult}_p G \).
   If \( \dim E_p > \dim G_p \), then \( \text{mult}_p F = \text{mult}_p E \).

We will follow the proof of [12] and indicate at each step how to prove:

Claim 1. \( \text{rad(ann}\hat{M}) \subset \text{rad(ann}\hat{M}') \),

Claim 2. \( \text{mult}_p \hat{M} \geq \text{mult}_p \hat{M}' \) if \( \dim \hat{M}_p = \dim \hat{M}'_p \).

First, since \( F_{d,k}(M) \) and \( F'_{d,k}(M) \) are good bifiltrations, there exist \( d_0, k_0 \in \mathbb{N} \) such that for any \( d, k \), \( F_{d,k}(M) \subset F'_{d+d_0,k+k_0}(M) \). Let us denote by \( M'' \) the module \( M \) endowed with the bifiltration \( (F'_{d+d_0,k+k_0}(M))_{d,k} \). The algebraic cycle associated with \( \hat{M}' \) is equal to the algebraic cycle associated with \( \hat{M}'' \). Thus we can suppose \( F'_{d,k}(M) \subset F_{d,k}(M) \).

Let us introduce the Rees algebra \( \mathcal{R}(D) \) with respect to the bifiltration \( F, V \), i.e,

\[
\mathcal{R}(D) = \bigoplus_{d,k} F_{d,k}(D) \tau^d \theta^k.
\]

This is isomorphic to the \( \mathbb{C} \)-algebra generated by \( x_i, t_i, \tau, \partial_x, \tau, \partial_t, \tau \theta, \tau \) and \( \theta \), subject to the relations \( [\partial_x, \tau, x_i] = \tau \) and \( [\partial_t, \tau \theta, t_i, \theta^{-1}] = \tau \). This is a noetherian algebra.

We define also the Rees module \( \mathcal{R}(M) = \bigoplus_{d,k} F_{d,k}(M) \tau^d \theta^k \). We have

\[
\text{gr}^F(\mathcal{R}_V(M)) \simeq \frac{\mathcal{R}(M)}{\tau \mathcal{R}(M)}.
\]

Let us suppose moreover that there exists \( r \geq 1 \) such that for any \( d, k \), \( F'_{d,k}(M) \subset F_{d,k}(M) \subset F'_{d+r,k}(M) \). Let \( F''_{d,k}(M) = F_{d,k}(M) \cap F'_{d+1,k}(M) \). We have

\[
F'_{d,k}(M) \subset F''_{d,k}(M) \subset F'_{d+1,k}(M) \quad \text{and} \quad F_{d-r+1,k}(M) \subset F''_{d,k}(M) \subset F_{d,k}(M).
\]
By induction on $r$ we can suppose $r = 1$, i.e. $\tau R(M) \subset R(M') \subset R(M)$. Then we have the following exact sequences of $\text{gr}^{F} R_{V}(D)$-modules of finite type:

$$
0 \to \tau R(M) \to R(M') \to R(M) \to 0
$$

$$
0 \to R(M') \to R(M) \to R(M') \to 0.
$$

After tensorizing by $K$, we deduce $\text{rad}(\text{ann} \tilde{M}) = \text{rad}(\text{ann} \tilde{M}')$. Then using Lemma 3.2, we get $\text{mult}_P \tilde{M} = \text{mult}_P \tilde{M}'$.

Let $F''_{d,k}(M) = F_{d,k}(M) \cap (\bigcup_i F'_{i,k}(M))$. We have:

$$
\mathcal{R}(M'') = \mathcal{R}(M) \cap (\bigcup_i \tau^{-i}\mathcal{R}(M')).
$$

Let $L_j = \mathcal{R}(M) \cap (\bigcup_{0 \leq i \leq j} \tau^{-i}\mathcal{R}(M'))$. This is an ascending chain of finite type sub-modules of $\mathcal{R}(M)$. Hence it is stationary and there exists an integer $r \geq 0$ such that

$$
\mathcal{R}(M'') = \mathcal{R}(M) \cap \tau^{-r}\mathcal{R}(M').
$$

In particular $\mathcal{R}(M'')$ is of finite type and $F''_{d,k}(M)$ is a good filtration. We have $\tau^r \mathcal{R}(M'') \subset \mathcal{R}(M') \subset \mathcal{R}(M'')$, i.e. we are in the situation of the previous paragraph. This implies $\text{rad}(\text{ann} \tilde{M}'') = \text{rad}(\text{ann} \tilde{M}')$ and $\text{mult}_P \tilde{M}'' = \text{mult}_P \tilde{M}'$.

On the other hand, we have a canonical injection

$$
\frac{\mathcal{R}(M'')}{\tau \mathcal{R}(M'')} \to \frac{\mathcal{R}(M)}{\tau \mathcal{R}(M)}.
$$

Then $\text{rad}(\text{ann} \tilde{M}'') \subset \text{rad}(\text{ann} \tilde{M})$, and Claim 1 is proved. From this canonical injection, we deduce Claim 2 by using Lemma 3.2.

4 Nicely bifiltered $D$-modules

In this section we consider a bifiltered $D$-module satisfying the following condition:

**Definition 4.1.** Let $M$ be a $D$-module endowed with a good bifiltration. We say that the bifiltration is nice if for any $d, k$,

$$
\left(\bigcup_{d'} F_{d',k}(M)\right) \cap \left(\bigcup_{k'} F_{d,k'}(M)\right) = F_{d,k}(M).
$$

In such a case, we say that $M$ is nicely bifiltered.

**Definition 4.2.** Let $N$ be a bigraded $\text{gr}^{V}(D^{(b)})$-module. $N$ is said to be $h$-saturated if the map $N \to N$ sending $m$ to $hm$ is injective.

Let $N$ be a bigraded $\text{gr}^{F}(R_{V}(D))$-module. $N$ is said to be $\theta$-saturated if the map $N \to N$ sending $m$ to $\theta m$ is injective.

**Lemma 4.1.** The following are equivalent:

1. $M$ is nicely bifiltered,
2. $\text{gr}^V(\mathcal{R}_F(M))$ is $h$-saturated,

3. $\text{gr}^\theta(\mathcal{R}_V(M))$ is $\theta$-saturated.

Proof. By definition, 2) and 3) are equivalent to the following: $\forall d, k, F_{d+1,k}(M) \cap F_{d,k+1}(M) \subset F_{d,k}(M)$. By 2), Lemma 1.1, this is equivalent to 1). □

**h-saturatedness and Gröbner bases.** Let us give a criterion for $h$-saturatedness using Gröbner bases. Using the preceding lemma, that leads to a criterion for the niceness of a bifiltration. Let in this paragraph $D^{(h)} = \mathbb{C}[x_1, \ldots, x_n](\partial_1, \ldots, \partial_n, h)$. It is graded by setting for any $i$, $\deg x_i = 0$, $\deg \partial_i = 1$ and $\deg h = 1$.

Let $<$" be a well-order on $\mathbb{N}^{2n}$, compatible with sums. Then we define a well-order $<'$ on $\mathbb{N}^{2n+1}$ by

$$(\alpha, \beta, k) <' (\alpha', \beta', k') \iff$

$$\begin{cases}
|\beta| + k < |\beta'| + k' \\
\text{or } |\beta| + k = |\beta'| + k' \text{ and } |\beta| < |\beta'| \\
\text{or } |\beta| + k = |\beta'| + k', |\beta| = |\beta'| \text{ and } (\alpha, \beta) <" (\alpha', \beta').
\end{cases}$$

This is a well-order on the monomials of $D^{(h)}$ adapted to the $F$-filtration. To deal with submodules of $(D^{(h)})^r$, we define a well-ordering $< \text{ on } \mathbb{N}^{2n+1} \times \{1, \ldots, r\}$ by

$$(\alpha, \beta, k, i) < (\alpha', \beta', k', i') \iff \begin{cases}
(\alpha, \beta, k) <' (\alpha', \beta', k') \\
\text{or } (\alpha, \beta, k) = (\alpha', \beta', k') \text{ and } i < i'.
\end{cases}$$

Note that if $(\alpha, \beta, k, i) \geq (\alpha', \beta', k', i')$ and $|\beta| + k = |\beta'| + k'$, then $k \leq k'$. If $P \in W^r$, we denote by $\text{in}(P)$ the leading monomial of $P$.

**Definition 4.3.** Let $P_1, \ldots, P_s$ be a Gröbner base of a homogeneous submodule $N \subset (D^{(h)})^r$. Such a base is called minimal if

$$\forall i, \, \text{Exp}P_i \notin \bigcup_{j \neq i} \{\text{Exp}P_j + \mathbb{N}^{2n+1}\}.$$

**Proposition 4.1.** The following assertions are equivalent:

1. $(D^{(h)})^r/N$ is $h$-saturated.

2. For any minimal homogeneous Gröbner base $P_1, \ldots, P_s$ of $N$, for any $i$, $h$ does not divide in $P_i$.

3. There exists a minimal homogeneous Gröbner base $P_1, \ldots, P_s$ of $N$, such that for any $i$, $h$ does not divide in $P_i$.

Proof. Let us prove 1) $\Rightarrow$ 2) Let $P_1, \ldots, P_s$ be a minimal homogeneous Gröbner base of $N$. Suppose that there exists $i$ such that $h$ divides in $P_i$. Then $h$ divides $P_i$ by the definition of $<$. By $h$-saturatedness, $P_i/h \in N$. Thus

$$\text{Exp}\frac{P_i}{h} \in \bigcup_{j \neq i} \{\text{Exp}P_j + \mathbb{N}^{2n+1}\}.$$
then
\[ \text{Exp}P_i = \text{hExp} \frac{P_i}{h} \in \bigcup_{j \neq i} (\text{Exp}P_j + h^{2n+1}), \]
which contradicts the minimality.

2) \( \Rightarrow \) 3) is obvious. Let us show 3) \( \Rightarrow \) 1). Let \( P \in (D(h))^r \) homogeneous such that \( hP \in N \). We shall show that \( P \in N \). By division, \( hP = \sum Q_i P_i \) with for any \( i \), \( Q_i \in D(h) \) homogeneous, \( \deg(Q_i P_i) = \deg(hP) \), and \( \ord^F(Q_i P_i) \leq \ord^F(hP) \).

Let us suppose that there exists \( i \) such that \( h \) does not divide \( Q_i \). Then \( \ord^F Q_i = \deg Q_i \). Since \( h \) does not divide \( P_i \), we have \( \ord^F P_i = \deg P_i \). Then
\[ \ord^F(Q_i P_i) = \ord^F(Q_i) + \ord^F(P_i) = \deg Q_i + \deg P_i = \deg(hP). \]
But
\[ \ord^F(Q_i P_i) \leq \ord^F(hP) < \deg(hP), \]
a contradiction. Thus for any \( i \), \( h \) divides \( Q_i \) and \( P = \sum(Q_i/h)P_i \in N \). □

We shall make a link between the \((F, V)\)-multidegree and the theory of slopes of Y. Laurent, c.f. [11]. We consider intermediate filtrations \( L \) between \( F \) and \( V \), denoted by \( pF + qV \) with \( p > 0 \), \( q > 0 \), defined by
\[ L_r(D) = \sum_{d + kq \leq r} F_{d, k}(D). \]

Similarly we endow \( M \) with the \( L \)-filtration \( L_r(M) = \sum_{d + kq \leq r} F_{d, k}(M) \), which is a good filtration since taking a bifiltered free presentation
\[ D^{r_1}[n^{(1)}][m^{(1)}] \to D^{r_0}[n^{(0)}][m^{(0)}] \to M \to 0, \]
we see that \( \text{gr}^L(M) \) is isomorphic to a quotient of \( \text{gr}^L(D^{r_0}[n^{(0)}] + qm^{(0)}) \).

On the other hand, since \( \text{gr}^V(M) \) is isomorphic to a quotient of \( \text{gr}^V(D^{r_0}[m^{(0)}])[n^{(0)}] \), it is endowed with a natural \( F \)-filtration. Similarly, \( \text{gr}^F(M) \) is isomorphic to a quotient of \( \text{gr}^F(D^{r_0}[n^{(0)}]) [m^{(0)}] \), and it is endowed with a natural \( V \)-filtration.

In [2], we considered also the bigraded module
\[ \text{bigr}(M) = \bigoplus_{d, k} \frac{F_{d, k}(M)}{F_{d, k-1}(M) + F_{d-1, k}(M)} \]
over the ring \( \text{bigr}(D) \simeq \text{gr}^V(\text{gr}^F(D)) \simeq \text{gr}^F(\text{gr}^V(D)) \).

**Lemma 4.2.** If \( M \) is nicely bifiltered, we have
\[ \text{bigr}(M) \simeq \text{gr}^V(\text{gr}^F(M)) \simeq \text{gr}^F(\text{gr}^V(M)). \]

**Proof.** For the sake of simplicity, we suppose that \( n^{(0)} = m^{(0)} = 0 \) and consider \( M = D^r/N \). We have
\[ F_{d, k}(M) = \frac{F_{d, k}(D^r) + N}{N}, \quad F_d(M) = \frac{F_d(D^r) + N}{N}, \quad V_k(M) = \frac{V_k(D^r) + N}{N}. \]
The niceness assumption is equivalent to the following:
\[ \forall d, k, \quad (F_d(D^r) + N) \cap (V_k(D^r) + N) \subset F_{d, k}(D^r) + N. \quad (2) \]
We have \( \text{gr}^V(M) = \text{gr}^V(D')/\text{gr}^V(N) \) with

\[
\text{gr}^V(N) = \bigoplus_k \frac{V_k(D') \cap N + V_{k-1}(D')}{V_{k-1}(D')},
\]

We naturally define

\[
F_d(\text{gr}^V(N)) = \bigoplus_k \frac{F_{d,k}(D') + V_{k-1}(D')}{V_{k-1}(D')},
\]

Thus we have

\[
\text{gr}^F \text{gr}^V(N) = \bigoplus_{d,k} \frac{(F_{d,k}(D') + V_{k-1}(D')) \cap (V_k(D') \cap N + V_{k-1}(D'))}{(F_{d-1,k}(D') + V_{k-1}(D')) \cap (V_k(D') \cap N + V_{k-1}(D'))}.
\]

This is included in

\[
\text{gr}^F \text{gr}^V(D') = \bigoplus_{d,k} \frac{F_{d,k}(D')}{F_{d-1,k}(D') + F_{d,k-1}(D')}
\]

via the map

\[
(F_{d,k}(D') + V_{k-1}(D')) \cap (V_k(D') \cap N + V_{k-1}(D')) \rightarrow F_{d,k}(D') + V_{k-1}(D') \rightarrow F_{d,k}(D').
\]

Hence

\[
\text{gr}^F \text{gr}^V(M) = \frac{\text{gr}^F \text{gr}^V(D')/\text{gr}^F \text{gr}^V(N)}{F_{d,k}(D') \cap (V_k(N) + V_{k-1}(D')) + F_{d-1,k}(D') + F_{d,k-1}(D')}.
\]

On the other hand,

\[
\text{bigr}_{d,k}(M) = \frac{F_{d,k}(M)}{F_{d-1,k}(M) + F_{d,k-1}(M)}
\]

\[
= \frac{F_{d,k}(D') + N}{F_{d-1,k}(D') + F_{d,k-1}(D') + N}
\]

\[
= \frac{F_{d,k}(D') \cap (F_{d-1,k}(D') + F_{d,k-1}(D') + N)}{F_{d,k}(D') + F_{d,k-1}(D') + N \cap F_{d,k}(D')}
\]

We have to show

\[
F_{d-1,k}(D') + F_{d,k-1}(D') + N \cap F_{d,k}(D') = F_{d,k}(D') \cap (V_k(N) + V_{k-1}(D')) + F_{d-1,k}(D') + F_{d,k-1}(D'). \tag{3}
\]

The inclusion \( \subseteq \) is obvious. On the other hand,

\[
F_{d,k}(D') \cap (V_k(N) + V_{k-1}(D')) \subseteq (F_{d,k-1}(D') + N) \cap F_{d,k}(D') \quad \text{(using (2))}
\]

\[
\subseteq F_{d,k-1}(D') + N \cap F_{d,k}(D'),
\]

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which proves (3).

We have showed that $\text{bigr}(M) \simeq \text{gr}^F(\text{gr}^V(M))$, and by exchanging the role of $F$ and $V$ we show that $\text{bigr}(M) \simeq \text{gr}^V(\text{gr}^F(M))$.

Note also that under the niceness assumption, the module $\text{bigr}(N)$ is identified with a submodule of $\text{bigr}(D^r)$ such that $\text{bigr}(M) \simeq \text{bigr}(D^r)/\text{bigr}(N)$. □

**Lemma 4.3** ([17], Lemma 2.1.6). For $\epsilon > 0$ small enough,

\[ \text{gr}^V(\text{gr}^F(M)) \simeq \text{gr}^L(M) \quad \text{with} \quad L = F + \epsilon V, \]

and

\[ \text{gr}^F(\text{gr}^V(M)) \simeq \text{gr}^L(M) \quad \text{with} \quad L = V + \epsilon F. \]

It is known that for any $L$, $\text{gr}^L(M)$ defines an algebraic cycle independent of the good filtration (the proof is almost the same as for the $F$-filtration). The variety defined by the annihilator of $\text{gr}^L(M)$ is denoted by $\text{char}_L(M)$. Remember that $\mathbb{K}$ denotes the fraction field of $\mathbb{C}[x]$. The module $\mathbb{K} \otimes \text{gr}^L(M)$ also defines an algebraic cycle independent of the good filtration.

**Proposition 4.2.** If $M$ is nicely bifiltered, we have

\[ K_{F,V}(M;T_1,T_2) = K(\text{bigr}(M);T_1,T_2) = K(\text{gr}^L(M);T_1,T_2) \]

with $L = V + \epsilon F$ or $L = F + \epsilon V$ with $\epsilon > 0$ small enough. Here $\text{gr}^L(M)$ is considered as a bigraded module.

**Proof.** Under this assumption, any bifiltered free resolution of $M$ induces a bigraded free resolution of $\text{bigr}M$ (see [2], Theorem 1.1, forgetting the minimality). Thus $K_{F,V}(M;T_1,T_2) = K(\text{bigr}M;T_1,T_2)$. But by Lemma 1.12 and Lemma 4.3, $\text{bigr}(M) \simeq \text{gr}^L(M)$. □

**Remark 4.1.** The multidegree $C_{F,V}(M;T_1,T_2)$ has total degree

\[ d = \text{codim} \mathbb{K} \otimes \text{gr}^F(\mathcal{R}_V(M)), \]

by definition. On the other hand, since the multigrading on $\mathbb{K} \otimes \text{bigr} D$ is positive, we know that the first non-zero terms in the expansion of $K_{F,V}(M;1-T_1,1-T_2)$ have total degree equal to

\[ d' = \text{codim} (\mathbb{K} \otimes \text{bigr} M). \]

Thus $d \leq d'$. If $d < d'$, then $C_{F,V}(M;T_1,T_2) = 0$. We will see in the next section non trivial cases in which $d = d'$.

We then have, applying Proposition 1.12:

**Theorem 4.1.** The multidegree $C_{F,V}(M;T_1,T_2)$ only depends on $\text{codim} \mathbb{K} \otimes \text{gr}^F(\mathcal{R}_V(M))$ and on the algebraic cycle defined by $\mathbb{K} \otimes \text{gr}^L(M)$ with $L = V + \epsilon F$ or $L = F + \epsilon V$ with $\epsilon > 0$ small enough.

Let us recall some geometric meaning related to the $L$-filtration. Let $X = \mathbb{C}^{n+p}$, $Y = \{ t = 0 \} \subset X$ and $\Delta = T_1 \cap X$ the conormal bundle. We have $\text{gr}^V(D) \simeq \mathcal{O}(T^* \Delta)$, e.f. [11]. Let $\pi : T^* \Delta \to Y$ be the canonical projection.

By Proposition 1.13, $C_{F,V}(\mathbb{K} \otimes \text{gr}^L(M);T_1,T_2) = C_{F,V}(\text{gr}^L(M)^y;T_1,T_2)$ for $y \in Y$ generic. This depends only on the algebraic cycle on $\pi^{-1}(y)$ defined by
$\text{gr}^{F}(M)^{y}$ for $y$ generic. $d'$ is equal to the codimension of $\text{char}_{L}(M) \cap \pi^{-1}(y) \subset \pi^{-1}(y)$, for $y$ generic.

For any $L$, we have $\text{gr}^{F}(D) \simeq \text{gr}^{F}(\text{gr}^{V}(D))$ thus $\text{gr}^{F}(D)$ is a bigraded ring.

Following the theory of Y. Laurent, we say that $M$ has no slopes along $Y$ if for any $L$, the ideal $\text{rad}(\text{ann} \text{gr}^{L}(M))$ (defining $\text{char}_{L}(M)$) is bihomogeneous. The following means that niceness of the bifiltration is a strong regularity condition.

**Proposition 4.3.** If $M$ is a nicely bifiltered holonomic $D$-module, then $M$ has no slopes along $Y$.

**Proof.** As before, we identify $R_{V}(D)$ with $D[\theta]$. Let us take a bifiltered free presentation

$$D^{*}[n][m] \xrightarrow{\phi} D^{*}[m] \rightarrow 0,$$  \hspace{1cm} (4)

with $\phi_{1}(e_{i}) = P^{(i)} = \sum_{j} P^{(i)}_{j} e_{j}$, and let $N = \text{Im} \phi_{1}$. For the sake of simplicity, we have assumed $n^{(0)} = m^{(0)} = 0$. This induces a bigraded free resolution

$$\text{gr}^{s}D[\theta][n][m] \xrightarrow{\phi} \text{gr}^{s}D[\theta][m] \rightarrow \text{gr}^{s}D[\theta][n] \rightarrow 0.$$  

Using the lifting ([8], Proposition 2.7), we can suppose that the presentation (4) is minimal, in the sense that the elements $\phi_{1}(e_{i})$ form a minimal set of generators of $\text{Ker} \phi_{1}$.

Let us introduce some notations in order to determine $\phi_{1}(e_{i})$. If $P = \sum a_{\nu,\mu}(x, \theta) t^{\nu} \partial_{\theta}^{\mu} \in V_{k}(D)$, we define

$$H^{V}_{k}(P) = \sum a_{\nu,\mu}(x, \theta) t^{\nu} \partial_{\theta}^{\mu} \text{ord}^{k-|\nu|-|\mu|} \in D[\theta],$$

and $H^{V}(P) = H^{V}_{\text{ord}^{r}}(P)$, the $V$-homogenization of $P$. Similarly if $P = \sum P_{j} e_{j} \in \oplus V_{m_{j}}(D)$, we define $H^{V}_{m}(P) = \sum H^{V}_{m_{j}}(P_{j}) e_{j} \in (D[\theta])^{r}.$

Now if $P = \sum a_{\beta}(x, t, \theta) \partial_{\theta}^{\beta} \in F_{d}(D[\theta])$, we define

$$\sigma_{d}^{F}(P) = \sum a_{\beta}(x, t, \theta) \partial_{\theta}^{\beta} \in \text{gr}^{s}D[\theta],$$

and $\sigma^{F}(P) = \sigma_{\text{ord}^{r}}^{F}(P)$. Similarly if $P = \sum P_{j} e_{j} \in \oplus F_{m_{j}}(D[\theta])$, we define $\sigma_{m}^{F}(P) = \sum \sigma_{m_{j}}^{F}(P_{j}) e_{j} \in \text{gr}^{s}D[\theta].$

We have

$$\phi_{1}(e_{i}) = \sigma_{m}^{F}(H_{m}(P)).$$

For $P = \sum_{\nu,\beta,\mu} a_{\nu,\beta,\mu}(x) t^{\nu} \partial_{\theta}^{\beta} \partial_{t}^{\mu}$, let us define the Newton polygon by

$$P(\nu,\beta,\mu) = \{ (|\nu|-|\mu|, |\beta|+|\mu|) \in \mathbb{N}^{2} : \text{ord}^{r}P \geq (|\nu|-|\mu|, |\beta|+|\mu|) \}.$$  

We say that $P(\nu,\beta,\mu)$ is trivial if it is equal to a translate of $(-\mathbb{N}) \times (-\mathbb{N})$. 

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For $1 \leq i \leq s$, let $J(i)$ be the set of integers $1 \leq j \leq r$ such that
\begin{itemize}
  \item $\text{ord}_F P_j^{(i)} = n_i$,
  \item $\text{ord}_V P_j^{(i)} = m_i$,
  \item $P(P_j^{(i)})$ is trivial.
\end{itemize}

We claim that for any $i$, the set $J(i)$ is non-empty. Otherwise, $\theta$ would divide $\varphi_1(e_i)$. By $\theta$-saturatedness, $\varphi_1(e_i)/\theta$ would belong to $\text{gr}^F \mathcal{R}_V N$, thus the presentation (4) would not be minimal.

Then $\text{bigr}_N$ is generated by the elements
$$
\sum_{j \in J(i)} \sigma F (P_j^{(i)}) e_j
$$
for $1 \leq i \leq s$. Let $L$ be an intermediate filtration. We have
$$
\sigma_L (P(i)) = \sum_{j \in J(i)} \sigma_L (P_j^{(i)}) e_j = \sum_{j \in J(i)} \sigma F (P_j^{(i)}) e_j.
$$

Thus for any $L$,
$$
\text{bigr}_N \subset \text{gr}^L N. \tag{5}
$$

If $M$ is a $\text{gr}^L (D)$-module, we denote by $\text{supp}_L M$ the zero-set of the annihilator of $M$. By [19], Theorem 1.1 and [17], Theorem 2.2.1 (valid for any $L$), $\text{char}_L (M) = \text{supp}(\text{gr}_L (M))$ is pure of dimension $n + p$ for any $L$. Since $\text{bigr}_N = \text{gr}^F \text{gr}^V (N) = \text{gr}^L (N)$ for $L$ close to $V$, then $\text{supp}(\text{bigr}_M)$ is pure of dimension $n + p$.

By (5), we have for any $L$, $\text{char}_L M \subset \text{supp}(\text{bigr}_M)$, thus $\text{char}_L M$ is the union of some irreducible components of $\text{supp}(\text{bigr}_M)$. The irreducible components are bihomogeneous (a bihomogeneous module admits a bihomogeneous primary decomposition), so $\text{char}_L M$ is bihomogeneous.

5 Examples from the theory of hypergeometric systems

Let $D = \mathbb{C}[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$. We consider the $A$-hypergeometric $D$-module $M_A(\beta) = D/H_A(\beta)$. This is a holonomic system associated with a $d \times n$ integer matrix $A$ and $\beta_1, \ldots, \beta_d \in \mathbb{C}$ as follows. We suppose that the abelian group generated by the columns $a_1, \ldots, a_n$ of $A$ is equal to $\mathbb{Z}^d$. Let $I_A$ be the ideal of $\mathbb{C}[\partial_1, \ldots, \partial_n]$ generated by the elements $\partial^u - \partial^v$ with $u, v \in \mathbb{N}^n$ such that $A.u = A.v$. The hypergeometric ideal $H_A(\beta)$ is the ideal of $D$ generated by $I_A$ and the elements $\sum_{j \neq i} a_{i,j} x_j \partial_j - \beta_i$ for $i = 1, \ldots, d$. The hypergeometric modules were introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in [5]; their holonomicity (in the general case) was proved by A. Adolphson in [1].

We endow $M$ with the quotient $F$-filtration and the quotient $V$-filtration with respect to $x_1 = \cdots = x_n = 0$.

Let us assume that the abelian group generated by the rows of $A$ contains a vector $w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^n$. That is equivalent to the fact that the semigroup generated by the columns of $A$ is pointed. By applying the weight
vector $W = (-w, w)$ to $(x, \partial)$, we get a grading on $D$. The hypergeometric module $M_A(\beta)$ is homogeneous w.r.t. to $W$.

Our first topic is to strengthen the correspondence between $C_{F,V}(M_A(\beta); T_1, T_2)$ and $C(\bigr M_A(\beta); T_1, T_2)$, i.e. to prove that the modules $\bigr M_A(\beta)$ and $gr^F(M_A(\beta))$ have the same codimension if $M_A(\beta)$ is nicely bifiltered.

The codimension of a finite type $D$-module $M$ is by definition the codimension of $gr^F(M)$, that does not depend on the good $F$-filtration. In fact we can make the weight vector vary as well.

**Proposition 5.1** ([17], pp. 65-66). Let $(u, v) \in \mathbb{N}^{2n}$ be a weight vector such that for all $i$, $u_i + v_i > 0$. Endow $M$ with a good $(u, v)$-filtration. Then $\text{codim}(gr^{(u,v)}(M)) = \text{codim} M$.

We have an analogous statement for $D^{(h)}$-modules, proved in the same way. Let $(u, v, t) \in \mathbb{N}^{2n+1}$ such that for all $i$, $u_i + v_i + t_i > t$. Then $\text{gr}^{(u,v,t)}(D^{(h)})$ is commutative.

**Definition 5.1.** Let $M$ be a graded $D^{(h)}$-module of finite type. Endow $M$ with a good $(u, v, t)$-filtration. We define $\text{codim} M = \text{codim}(\text{gr}^{(u,v,t)}M)$. This depends neither on the good filtration nor on the weight vector $(u, v, t)$.

Finally, since $\text{gr}^V(D^{(h)}) \simeq D^{(h)}$, we define in the same way the codimension of a $\text{gr}^V(D^{(h)})$-module of finite type.

We adopt the following notation. If $P = \sum a_\beta(x)\partial_x^{\beta} \in F_d(D)$, we define $H_\beta(P) = \sum a_\beta(x)\partial_x^{\beta}h^{d-|\beta|} \in D^{(h)}$, and the $F$-homogenization $H(P) = H_{\text{ord}^F(P)}(P)$. If $I$ is an ideal of $D$, let $H(I)$ be the ideal of $D^{(h)}$ generated by the elements $H(P)$ such that $P \in I$. We have $RF(M) = D^{(h)}/H(I)$. Similarly we define the $V$-homogenization, denoted by $H^V(P) \in D[\theta]$ and $H^V(I) \subset D[\theta]$.

**Proposition 5.2.** Let $M = D/I$ be a $W$-homogeneous nicely bifiltered $D$-module. Then the modules $M$, $\text{gr}^F(RF(M))$, $\text{gr}^V(RF(M))$ and $\bigr M$ all have the same codimension.

**Proof.** First, we prove that

\[ \text{codim}(RF(M)) = \text{codim} M. \]

Let $<$ be a well-order on $\mathbb{N}^{2n}$ (the monomials of $D$) adapted to $F$, i.e. for any $\alpha, \alpha', \beta, \beta', |\beta| < |\beta'| \Rightarrow (\alpha, \beta) < (\alpha', \beta')$. We derive from it a well-order $<'$ on $\mathbb{N}^{2n+1}$ (the monomials of $(D^{(h)})$) in the following way:

\[
\langle \alpha, \beta, k \rangle <' \langle \alpha', \beta', k' \rangle \quad \text{iff} \quad \begin{cases} |\beta| + k < |\beta'| + k' \\ \text{or} \quad |\beta| + k = |\beta'| + k' \\ \text{and} \quad (\alpha, \beta) < (\alpha', \beta') \end{cases}
\]

which is adapted to the $F$-filtration. Let $P_1, \ldots, P_n$ be a Gröbner base of $I$ with respect to $<$. Then $H(P_1), \ldots, H(P_n)$ is a Gröbner base of $H(I)$ with respect to $<'$ (use the Buchberger criterion). We have $\sigma^F(H(P_i)) = \sigma^F(P_i) \in \mathbb{C}[x, \zeta]$, thus $\text{codim}(\text{gr}^F(RF(M))) = \text{codim}(\text{gr}^F(M))$.

Now, we prove that

\[ \text{codim}(\text{gr}^V(RF(M))) = \text{codim} M. \]
The module $R_F(M)$ is bihomogeneous with respect to the weight vectors $(-w, w, 0)$ and $(0, 1, 1)$. Let $\mu = \max(w_i - 1) \in \mathbb{N}$ and

$$\Lambda = (-1, 1, 0) - (-w, w, 0) + \mu(0, 1, 1) = (w - 1, (1 + \mu)1 - w, \mu) \in \mathbb{N}^{2n+1}.$$

Using the bihomogeneity, a $V$-adapted base of $H(N)$ is also adapted to $\Lambda$, so $\text{gr}^\Lambda (R_F(M)) = \text{gr}^V (R_F(M))$. Then

$$\text{codim} \text{gr}^V (R_F(M)) = \text{codim} \text{gr}^{(0,1,0)} \text{gr}^V (R_F(M)) \quad \text{(by definition)}$$

$$= \text{codim} \text{gr}^{(0,1,0)} \text{gr}^\Lambda (R_F(M))$$

$$= \text{codim} \text{gr}^{\Lambda+\epsilon(0,1,0)} (R_F(M)) \quad \text{with} \ \epsilon > 0,$$

by [17, Lemma 2.1.6], which proves our assertion since $\Lambda + \epsilon(0, 1, 0) \in \mathbb{N}^{2n+1}$.

Next, let us see that

$$\text{codim} (\text{gr}^F (R_V(M))) = \text{codim} M.$$

We will slightly modify the problem using the niceness assumption. We can endow $\text{gr}^F(D) \simeq \mathbb{C}[x, \xi]$ with a filtration with respect to the weight vector $(-1, 1)$, which we still call the $V$-filtration. The module $\text{gr}^F(M) \simeq \text{gr}^F(D)/\text{gr}^F(I)$ is naturally endowed with the quotient $V$-filtration. In the same way as in the proof of Lemma [12] we have

$$\text{gr}^F (R_V(M)) = R_V (\text{gr}^F (M)).$$

Thus we are reduced to show $\text{codim} (R_V (\text{gr}^F (M))) = \text{codim} M$. As before, let $\mu = \max(w_i - 1)$ and define $\Lambda = V - (-w, w) + \mu(0, 1) \in \mathbb{N}^{2n}$. We have a ring isomorphism

$$R_V (\text{gr}^F(D)) \simeq \text{gr}^F(D)[\theta] \simeq R_\Lambda (\text{gr}^F(D)),$$

and $R_V (\text{gr}^F(M)) \simeq R_\Lambda (\text{gr}^F(M))$ above this ring isomorphism. Next,

$$\text{codim} R_\Lambda (\text{gr}^F(M)) = \text{codim} \Lambda \text{gr}^F(M)$$

$$= \text{codim} \text{gr}^{F+\epsilon \Lambda}(M)$$

$$= \text{codim} M.$$

Finally, we show that

$$\text{codim} (\text{bigr} M) = \text{codim} (M).$$

We have $\text{bigr} M \simeq \text{gr}^V \text{gr}^F(M)$, by Lemma [12].

Taking again $\Lambda = V - (-w, w) + \mu(0, 1)$, the assertion follows from $\text{gr}^V \text{gr}^F(M) = \text{gr}^\Lambda \text{gr}^F(M) = \text{gr}^{F+\epsilon \Lambda}(M)$.

**Remark 5.1.** If $M_A(\beta)$ is nicely bifiltered, then we have

$$C_{F,V}(M_A(\beta); T_1, T_2)|_{T_2=0} = (\text{rank} M_A(\beta))T_1^a.$$

Indeed, a bifiltered free resolution induces a $F$-filtered free resolution, thus $K_F(M; T_1) = K_{F,V}(M; T_1, T_2)|_{T_2=1}$, so $K_F(M; 1 - T_1) = K_{F,V}(M; 1 - T_1, 1 - T_2)|_{T_2=0}$, and by the Proposition above, we have $\text{codim} \text{gr}^F(R_V(M_A(\beta))) = \text{codim} M = \text{codim} \text{gr}^F(M_A(\beta)) = n$. We conclude by using Proposition [2.3].

Let us note for $1 \leq i \leq d$, $(Ax_i) = \sum_{j} a_{i,j} x_j \xi_j \in \text{gr}^F(D)$. 

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Lemma 5.1. If $\text{gr}^F(\mathbb{C}[\partial]/I_A)$ is Cohen-Macaulay, then $(Ax\xi)_1, \ldots, (Ax\partial)_d$ is a regular sequence in $\text{gr}^F(D/DIA)$.

Proof. By [13], Proposition 7.5, $\dim(\mathbb{C}[\partial]/I_A) = d$. Using Proposition 5.1, we get $\dim(\text{gr}^F(D/DIA)) = n + d$. But $\dim(\mathbb{C}[x, \xi]/(Ax\xi + \text{gr}^F(I_A))) = n$ by [18], proof of Proposition 3.8. The results follows from the Cohen-Macaulay assumption.

5.1 The homogeneous case

We suppose moreover that the columns of $A$ lie in a common hyperplane, i.e. $(1, \ldots, 1)$ belongs to the $\mathbb{Q}$-row span of $A$. Then $I_A$ is homogeneous for the weight vector $(1, \ldots, 1)$ and $M_A(\beta)$ is $V$-homogeneous.

Lemma 5.2. $M_A(\beta)$ is nicely bifiltered.

Indeed, $M_A(\beta)$ is $V$-homogeneous, thus $\mathbf{R}_F(M_A(\beta))$ is also $V$-homogeneous, thus $\text{gr}^V\mathbf{R}_F(M_A(\beta)) \simeq \mathbf{R}_F(M_A(\beta))$ is $h$-saturated. Then apply Lemma 4.1.

Lemma 5.3. Let $R = \text{bigr}D$ and $M$ be a finite type bigraded $R$-module. Let $P \in R$ be bihomogeneous of degree $(d, k)$. If $P$ is a non zero-divisor on $M$ then

1. $K_{F,V}(M/PM; T_1, T_2) = (1 - T_1^dT_2^k)K_{F,V}(M; T_1, T_2)$ and
2. $C_{F,V}(M/PM; T_1, T_2) = (dT_1 + kT_2)C_{F,V}(M; T_1, T_2)$.

Proof. Let us prove 1). If $N$ is a bigraded $R$-module, let $S_{d,k}(N)$ be the bigraded module defined by $(S_{d,k}(N))_{d', k'} = N_{d'' - d', k'' - k'}$. In particular, $S_{d,k}(D^e[n][m]) = D^e[n + d.1][m + k.1]$. A bigraded free resolution

$\cdots \to L_1 \to L_0 \to M \to 0$

of $M$ induces a bigraded free resolution

$\cdots \to S_{d,k}(L_1) \to S_{d,k}(L_0) \to S_{d,k}(M) \to 0$

of $S_{d,k}(M)$. We have a bigraded exact sequence

$0 \to S_{d,k}(M) \xrightarrow{P} M \xrightarrow{M/PM} 0$.

Then taking the cone of the morphism of resolutions $S_{d,k}(L) \xrightarrow{P} L$ gives a resolution

$\cdots \to S_{d,k}(L_1) \oplus L_2 \to S_{d,k}(L_0) \oplus L_1 \to L_0 \to M/PM \to 0$

of $M/PM$. Then 1) follows, and 2) follows from 1).

Let us denote by $\text{vol}(A)$ the normalized volume of the convex hull in $\mathbb{R}^d$ of the set $\{0, a_1, \ldots, a_n \}$. The normalization means that the set $[0, 1] \times \cdots \times [0, 1] \subset \mathbb{R}^d$ has volume $d!$. 

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Theorem 5.1. If $\mathbb{C}[\partial]/I_A$ is homogeneous and Cohen-Macaulay, then for any $\beta \in \mathbb{C}^d$ we have

$$C_{F,V}(M_A(\beta); T_1, T_2) = \text{vol}(A) \sum_{j=d}^{n} \binom{n-d}{j-d} T_1^j T_2^{n-j},$$

Proof. By Proposition [4,2] Proposition [5,2] and Lemma [4,2] $C_{F,V}(M_A(\beta); T_1, T_2)$ is equal to the sum of the terms of least degree in $K_{F,V}(\text{bigr } M_A(\beta); 1-T_1, 1-T_2)$, and by Lemma [1,2] we have

$$\text{bigr } M_A(\beta) \simeq \text{gr } F \text{gr } V(M_A(\beta)) = \text{gr } F(M_A(\beta)).$$

When $\mathbb{C}[\xi]/I_A$ is Cohen-Macaulay, $(Ax_1, \ldots, (Ax_d)$ form a regular sequence in $\mathbb{C}[x, \xi]/I_A$, and $\text{gr } F(H_A(\beta))$ is generated by $I_A$ and $(Ax_1, \ldots, (Ax_d)$, by Lemma [5,1] and [17], Theorem 4.3.8. Using Lemma [5,3] repeatedly, we get

$$C_{F,V}(\text{gr } F(M_A(\beta)); T_1, T_2) = T_1^1 C_{F,V}(\mathbb{C}[x, \xi]/I_A; T_1, T_2).$$

But $C_{F,V}(\mathbb{C}[x, \xi]/I_A; T_1, T_2) = C_F(C[\xi]/I_A; T_1, T_2)$ since $I_A \subset \mathbb{C}[\xi]$. Let $R = \mathbb{C}[\xi]$, $P(T_1, T_2) = K_{F,F}(R/I_A; T_1, T_2)$ and $Q(T) = K_{F,F}(R/I_A; T)$. Consider a graded free resolution

$$0 \to R^s[n^{(\delta)}] \to \cdots \to R^n[n^{(0)}] \to R/I_A \to 0$$

of $R/I_A$. Then we have a bigraded free resolution

$$0 \to R^s[n^{(\delta)}][n^{(\delta)}] \to \cdots \to R^n[n^{(0)}][n^{(0)}] \to R/I_A \to 0$$

of $R/I_A$. We deduce that $P(T_1, T_2) = Q(T_1 T_2)$. We have $Q(1-T) = b_{n-d} T^{n-d} + O(n-d+1)$, with $b_{n-d} = \text{deg}(R/I_A) 
eq 0$, and $O(n-d+1)$ denotes a polynomial of valuation greater than $n-d$. By [16], Chapter 6, Theorem 2.3, $\text{deg}(R/I_A) = \text{vol}(A)$. We have

$$P(1-T_1, 1-T_2) = Q((1-T_1)(1-T_2)) = Q(1-(T_1 + T_2 - T_1 T_2)) = b_{n-d} (T_1 + T_2)^{n-d} + O(n-d+1) = b_{n-d} \left( \sum_{i=0}^{n-d} \binom{n-d}{i} T_1^i T_2^{n-d-i} \right) + O(n-d+1),$$

from which the statement follows. \(\square\)

To compute the multidegree in the following examples, we used the computer algebra systems Singular [10] and Macaulay2 [9].

Example 1. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Then $I_A$ is generated by $\partial_1 \partial_1 - \partial_2^2$. For all $\beta$, $C_{F,V}(M_A(\beta); T_1, T_2) = 2T_1^3 + 2T_2^2 T_2$. 

Example 2. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. Then $I_A$ is generated by $\partial_2 \partial_1 - \partial_3^2, \partial_1 \partial_4 - \partial_2 \partial_3, \partial_1 \partial_3 - \partial_2^2$. For all $\beta$, $C_{F,V}(M_A(\beta); T_1, T_2) = 3T_1^4 + 6T_1^3 T_2 + 3T_1^2 T_2^2$. 

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Let us give homogeneous non-Cohen-Macaulay examples from the book \cite{book}. Using Proposition \ref{prop} repeatedly, we can establish the existence of a stratification of the space of the parameters \(\beta_1, \beta_2\) by the multidegree. In the following two examples, this stratification equals the stratification by the holonomic rank.

**Example 3.** Let \(A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}\). Then \(I_A\) is generated by \(\partial_2\partial_1^2 - \partial_3, \partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_3^2 - \partial_2^2\partial_4, \partial_3^2\partial_3 - \partial_2^2\). For \((\beta_1, \beta_2) \neq (1, 2)\), we have
\[
C_{F,V}(M_A(\beta); T_1, T_2) = 4T_1^4 + 8T_1^3T_2 + 4T_1^2T_2^2.
\]
For \((\beta_1, \beta_2) = (1, 2)\), we have
\[
C_{F,V}(M_A(\beta); T_1, T_2) = 5T_1^4 + 12T_1^3T_2 + 10T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.
\]

**Example 4.** Let \(A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 7 \end{pmatrix}\). Then \(I_A\) is generated by \(\partial_2\partial_4 - \partial_3^2, \partial_1\partial_3 - \partial_2\partial_3\). Let \(E = \{(2, 10), (2, 12), (3, 19)\}\). For \((\beta_1, \beta_2) \notin E\), we have
\[
C_{F,V}(M_A(\beta); T_1, T_2) = 9T_1^5 + 27T_1^4T_2 + 27T_1^3T_2^2 + 9T_1^2T_2^3.
\]
For \((\beta_1, \beta_2) \in E\), we have
\[
C_{F,V}(M_A(\beta); T_1, T_2) = 10T_1^5 + 32T_1^4T_2 + 37T_1^3T_2^2 + 19T_1^2T_2^3 + 5T_1T_2^4 + T_2^5.
\]

### 5.2 The inhomogeneous case

Following arguments in the book \cite{book}, we extend Theorem \ref{thm} in the inhomogeneous case, for generic parameters \(\beta\).

**Theorem 5.2.** Assume that \(C[\partial,h]/H(I_A)\) is Cohen-Macaulay. Then for generic \(\beta\), the module \(M_A(\beta)\) is nicely bifiltered and
\[
C_{F,V}(M_A(\beta); T_1, T_2) = \text{vol}(A) \sum_{j=d}^{n} \binom{n-d}{j} T_1^j T_2^{n-j}.
\]
Here, the assumption is that the closure of the variety defined by \(I_A\) in the projective space \(\mathbb{P}^n\) is Cohen-Macaulay.

**Proof.** First, note that the \(C[\partial,h]\)-module \(C[\partial,h]/H(I_A)\) and the \(\text{gr}^F(C[\partial])\)-module \(\text{gr}^F(C[\partial])/\text{gr}^F(I_A)\) have same codimension and same projective dimension. Thus by \cite{book}, Corollary 19.15, the Cohen-Macaulayness of the former is equivalent to that of the latter.

Also,
\[
C(\text{gr}^F(C[\partial])/\text{gr}^F(I_A); T) = C(C[\partial,h]/H(I_A); T) = \text{deg}(C[\partial,h]/H(I_A)) T^{n-d}
\]
and again by \cite{book}, Chapter 6, Theorem 2.3, \(\text{deg}(C[\partial,h]/H(I_A)) = \text{vol}(A)\).

For generic \(\beta\), by \cite{book}, Theorem 3.1.3 (with \(w = (1, \ldots, 1)\)), and \cite{book}, Theorem 2.5,
\[
H^V(H_A(\beta)) = D[\partial]H^V(I_A) + \sum_i D[\partial](Ax\partial_i - \beta_i).
\]
By Lemma 5.1, because of the Cohen-Macaulay assumption, \((Ax\xi)_1, \ldots, (Ax\xi)_d\) is a regular sequence in \(gr^F(D[\theta])/gr^F(I_A) = gr^F(D[\theta])/gr^F(H^V(I_A))\). That implies that \(H^V(I_A)\) and \(((Ax\partial)_1, \beta_i)\), form an \(F\)-involutive base of \(H^V(H_A(\beta))\) (see [17], Proposition 4.3.2). Then

\[
gr^F(H^V(H_A(\beta))) = gr^F(D[\theta])gr^F(H^V(I_A)) + \sum_i gr^F(D[\theta])(Ax\partial)_i.
\]

Thus \(gr^F(H^V(H_A(\beta)))\) is generated by elements independent of \(\theta\); this implies that \(gr^F(R_V(M_A(\beta)))\) is \(\theta\)-saturated (consider the graduation given by the degree in \(\theta\)), which is equivalent to niceness by Lemma 4.1

We have again \(\text{bigr}M_A(\beta) \simeq gr^F gr^F(M_A(\beta))\). With same arguments as above, we show that \(gr^F gr^F(H_A(\beta))\) is generated by \(gr^F(I_A)\) and \((Ax\xi)_i\) for generic \(\beta\). We conclude the computation of the multidegree as in the proof of Theorem 5.1

To finish, let us give examples in the inhomogeneous case.

**Example 5.** Let \(A = \begin{pmatrix} 0 & 1 & 3 \\ 4 & 3 & 2 \end{pmatrix}\). Then \(I_A\) is generated by \(\partial_1^3\partial_2^2 - \partial_1^1\partial_2^1\). The ring \(\mathbb{C}[\partial, h]/H(I_A)\) is Cohen-Macaulay. For any \(\beta\), \(M_A(\beta)\) is nicely bifiltered and \(C_{F,V}(M_A(\beta); T_1, T_2) = 12T_1^3 + 12T_1^2T_2\).

**Example 6.** Let \(A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\). Then \(I_A\) is generated by \(\partial_2^2\partial_1^1 - \partial_1^3, \partial_1\partial_2 - \partial_2\partial_3, \partial_1\partial_2^2 - \partial_2\partial_4, \partial_1^2\partial_3 - \partial_2^1\partial_4\). The ring \(\mathbb{C}[\partial, h]/H(I_A)\) is not Cohen-Macaulay. For \(\beta\) generic, \(M_A(\beta)\) is nicely bifiltered and

\[
C_{F,V}(M_A(\beta); T_1, T_2) = 6T_1^3 + 12T_1^2T_2 + 6T_2^2.
\]

We could check that the couple \(\beta = (-1, 2)\) is exceptional. In that case \(M_A(\beta)\) is also nicely bifiltered and we have

\[
C_{F,V}(M_A(\beta); T_1, T_2) = 7T_1^3 + 16T_1^2T_2 + 12T_1^2T_2^2 + 4T_1T_2^3 + T_2^4.
\]

Let us remark that in Examples 1–6, the formula of Theorems 5.1 and 5.2 holds for generic \(\beta\), sometimes without the Cohen-Macaulay assumption.

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