Fixed-time output tracking control for extended nonholonomic chained-form systems with state observers

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Abstract
This paper deals with the fixed-time tracking control problem of extended nonholonomic chained-form systems with state observers. According to the structure characteristic of such chained-form systems, two subsystems are considered to design controllers, respectively. First of all, using the fixed-time control theory, a controller is proposed to make the first tracking error subsystem converge to zero in bounded time independent initial state. Second, a state observer is proposed to estimate the unmeasurable states of the second subsystem. And the precise state estimation can be presented from the observer within finite time; moreover, the upper bound of time is a constant independent on the initial estimation error. Third, a fixed-time controller is designed to drive all states of the second chained-form subsystem to zero within pre-calculated time. Finally, the effectiveness of the proposed control scheme is validated by simulation results.

Keywords
Fixed-time, nonholonomic systems, state observers, tracking

Introduction
During the past decades, nonholonomic systems have drawn a lot of attention in the control community,¹–⁵ because it has great guiding significance for the practically engineering. Actually, it can be used to model many real systems, such as wheeled mobile robots, autonomous underwater vehicles undereactuated arm cranes, offshore-driven marine crane overhead vehicle system and free-floating space robots (see, for example, previous works⁶–¹⁴ and the references therein). However, controlling such systems is a big challenge. As the paper¹⁵ points out, we cannot stabilize the nonholonomic system to a point using smooth, or even continuous, static-state feedback approach. Therefore, the mature smooth nonlinear control law cannot be applied to this kind of systems, directly. With the continuous efforts of many researchers, several control methods have been proposed to realize this kind of nonholonomic systems that are asymptotically stable. Time-varying feedback control approaches have been proposed;¹⁶–¹⁸ nevertheless, the system state converges too slowly. Hence, researchers turned their attention to the possibility of achieving exponential (faster) convergence for nonholonomic systems, discontinuous feedback¹⁹,²⁰ and hybrid stabilization.²¹,²² With the further research on nonholonomic systems, the robustness issue of such systems with drift uncertainties, high-order nonholonomic and multi-intelligent systems, attract researchers’ attention (see previous works²³–²⁹ and references therein). However, the above control strategies only achieve asymptotic stability, which means we cannot obtain the convergence time in advance, although Zuo et al.²⁷–²⁹ proposed high-order multi-intelligence and control of the system for determining the convergence time, so that there is a faster convergence speed. In previous works,³⁰–³³ finite time control can stabilize systems within finite time, but convergence time was dependent on initial conditions. Polyakov³⁴ proposed fixed-time stability and got over this shortcoming. Compared with the stability problem of the nonholonomic control system, which has been researched deeply, the concern of tracking control problem is less.

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In fact, it is unclear whether the existing stabilization methods can be applied directly to the tracking problems of nonholonomic systems. Kanayama et al.\textsuperscript{35} proposed a stable tracking control strategy for an autonomous mobile robot with \(2\) degrees of freedom, but it is only suitable for solving local tracking problems. With the deepening of research, an adaptive visual servo tracking controller is designed\textsuperscript{36} to solve the nonholonomic motion constraints of mobile robots. It has been strictly proved that the tracking error converges to zero. Yan et al.\textsuperscript{37} proposed a robust motion control method based on equivalent input disturbance (EID) method; in this model, the position error control system is controlled by position control and trajectory tracking. Samson\textsuperscript{38} put forward global trajectory tracking controller for two-wheel-driven nonholonomic cart in Cartesian space. Ye\textsuperscript{39} applied backstepping idea to the design of the controllers. Chen et al.\textsuperscript{40} applied a finite-time servo tracking controller is designed\textsuperscript{36} to solve the nonholonomic motion constraints of mobile robots. It is worth noting that systems with chain forms and uncertainties are more common and widespread in practical engineering applications. For the extended chained form systems with incomplete information based on state observer, the fixed-time controller, and related proofs. Simulation results of the proposed control strategy are shown in section “Simulation results.” Finally, section “Conclusion” gives the conclusion.

**Problem statement**

Consider the following extended nonholonomic chained-form system

\[
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_1 u_1 \\
\vdots &\quad & \\
\dot{x}_{n-1} &= x_n u_1 \\
\dot{x}_n &= u_2 \\
\end{aligned}
\]

where \([u_1, u_2]^T \in \mathbb{R}^2\) can be viewed as the velocity input of the kinematics model\textsuperscript{35} and \([\tau_1, \tau_2]^T \in \mathbb{R}^2\) is the generalized torque input.\textsuperscript{15} \([x_1, x_2]^T \in \mathbb{R}^2\) is the measured output vector, and \([x_1, x_2, \ldots, x_n]^T\) denotes the unavailable state vector. \(f_i(x, t) \in \mathbb{R}\) and \(g_i(x, t) \neq 0 (i \in \mathbb{R})\) for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}(i = 1, 2)\) are two measurable smooth nonlinear functions. Suppose that the expected trajectory \([x_{1d}, x_{2d}]^T \in \mathbb{R}^2\). The output tracking error can be expressed as follows

\[
e = [e_1, e_2]^T = [x_1 - x_{1d}, x_2 - x_{2d}]^T
\]

In this paper, the control method is to design controller \([\tau_1, \tau_2]^T \in \mathbb{R}^2\) such that the output tracking error (equation (2)) converges to zero in fixed time. Through some manipulations, we can obtain that the tracking error satisfies following differential equations

\[
\begin{aligned}
\dot{e}_1 &= \ddot{x}_1 - \ddot{x}_{1d} \\
\dot{e}_2 &= \ddot{x}_2 - \ddot{x}_{2d} \\
\vdots &\quad & \\
\dot{e}_{n-1} &= x_n u_1 \\
\dot{e}_n &= u_2 \\
\end{aligned}
\]

**Assumption 1.** The kinematics speed \([u_1, u_2]^T \in \mathbb{R}^2\) is measurable. \(\forall t \in \mathbb{R}, [x_{1d}, x_{2d}]^T \in \mathbb{R}^2\) exist nonzero \(n\)-order derivative at any time \(t\).

**Lemma 1.** According to Leibniz formula, if the functions \(u = u(x)\) and \(v = v(x)\) have \(n\)-order derivatives at point \(x\), the \(n\)-order derivative of \(u \cdot v\) can be expressed as follows

\[
(uv)^{(n)} = u^{(n)} v + nu^{(n-1)}v^{(1)} + \ldots + n(n-1) \cdots (n-k+1)v^{(k)} + \ldots + u v^{(n)}
\]
which can also be referred to
\[
(uv)^n = \sum_{k=0}^{n} C_k^n u^{a-k} v^k
\]

Lemma 2. Consider the following second-order system\(^{33}\)
\[
\begin{cases}
\dot{y}_1 = y_2 \\
\dot{y}_2 = f(y) + g(y)u
\end{cases}
\]
where \([y_1, y_2]^T \in \mathbb{R}^2\) is the state vector, \(f(y), g(y) \neq 0\) are the smooth real vector fields, \(u\) is the control input. System (4) is fixed-time stable when the control law was designed as
\[
u = -\frac{1}{g(y)} \left[ f(y) + \alpha_1 \left( \frac{1}{2} + \frac{m_1}{2n_1} - \frac{1}{2} \right) \right. \\
\left. \operatorname{sign}(|y_1| - 1)y_1 + \frac{1}{2} \frac{m_1}{n_1} \right. \\
\left. + \operatorname{sat}(\beta_1 \frac{m_1}{n_1} y_2, \bar{y}) + \alpha_2 \frac{1}{2} \left( \frac{m_1}{n_1} \right) \right.
\]
where the positive constants \(\alpha_1, \alpha_2, \beta_1, \beta_2\) and positive odd integers \(m_1, m_2, n_1, n_2, p_1, p_2, q_1, q_2\) satisfying
\[
m_1 > n_1, \quad m_2 > n_2, \quad p_1 < q_1, \quad p_2 < q_2, \quad (m_1 + n_1)/2, \quad (p_1 + q_1)/2, \quad (m_2 + n_2)/2, \quad (p_2 + q_2)/2\]
are positive odd integers. \(\operatorname{sign}(x)\) is the sign function, and \(\bar{y} > 0\) is a threshold parameter. \(s\) denotes the sliding mode surface, which is constructed as
\[
s = y_2 + \alpha_1 y_1  \\
\frac{1}{2} \frac{m_1}{n_1} \operatorname{sign}|y_1| - 1y_1 + \beta_1 y_1
\]
And the saturation function was defined as
\[
\operatorname{sat}(x, y) = \begin{cases}
x & \text{if } |x| < y \\
\operatorname{sign}(x) & \text{if } |x| \geq y
\end{cases}
\]

Lemma 3. Consider the following system\(^{9}\)
\[\ddot{x} = f(\dot{x}), \quad f(0) = 0, \quad \ddot{x} \in \mathbb{R}^n\] (6)
Assuming that there exists a continuous function \(V(\dot{x}) : U \rightarrow \mathbb{R}\) such that the following conditions hold:
1. \(V(\dot{x})\) is positive definite.
2. There are real numbers \(\bar{c} > 0, \bar{\alpha} \in (0, 1)\) and an open neighborhood \(U_0 \in U\) of the origin such that \(V(\dot{x}) + \bar{c} \bar{\alpha}^{\bar{\alpha}} \dot{x}(\dot{x}) \leq 0, \dot{x} \in U_0\setminus\{0\}\).

Then, the origin is a finite-time stable equilibrium of the above system (6). If \(U = U_0 = \mathbb{R}^n\), the origin is a globally finite-time stable equilibrium of system (6).

Lemma 4. Assume that continuous real-valued functions \(V_1\) and \(V_2\) are homogeneous with regard to \(v\) of degrees \(l_1\) and \(l_2\), respectively\(^{4}\) and \(V_1\) is the positive definite. For each \(x \in \mathbb{R}^n\), the following inequality holds
\[
\min_{z: T_0(z) = 1} V_2(z) \left| V_1(x) \right|^\frac{1}{l_1} \leq V_2(z) \leq \max_{z: T_0(z) = 1} V_2(z) \left| V_1(x) \right|^\frac{1}{l_2}
\]
(7)

Definition 1. Consider the following differential equation system\(^{34}\)
\[
\dot{x}(t) = f(x(t)), \quad x(0) = x_0
\]
If the origin of system (8) is globally finite-time stable with bounded settling-time function \(T(x_0)\), and \(\exists T_{\max} > 0, \text{ such that } T(x_0) < T_{\max}\), it is said to be fixed-time stable equilibrium point.

Main result
In this section, we consider to design controllers \(\tau_1\) and \(\tau_2\), respectively. First, we separate system (3) to two subsystems as follows
\[
\begin{cases}
\dot{\tilde{y}}_1 = \tilde{y}_1 - \tilde{x}_{1d} \\
\dot{\tilde{y}}_2 = f_1(x, t) + g_1(x, t)\tau_1
\end{cases}
\]
(9)
and
\[
\begin{cases}
\dot{\tilde{x}} = \dot{\tilde{y}}_2 - \tilde{x}_{2d} \\
\dot{\tilde{y}}_3 = u_1 \tilde{x}_3 \\
\vdots \\
\tilde{x}_{n-1} = x_n u_1 \\
\tilde{x}_n = u_2 \\
\dot{u}_2 = f_2(x, t) + g_2(x, t)\tau_2
\end{cases}
\]
(10)
Let \(e_1 = E_{11},\ u_1 = \tilde{x}_{1d}\) and \(E_{12} = E_{12}\), then system (9) can be rewritten as
\[
\begin{cases}
\dot{E}_{11} = E_{12} \\
E_{12} = f_1(x, t) + g_1(x, t)\tau_1 - \tilde{x}_{1d}
\end{cases}
\]
(11)
And then we’re going to transform subsystem (10) for simplifying analysis process: if we design a controller \(\tau_1\) to drive \([E_{11}, E_{12}]^T\) converge to zero in finite time \(T_1\), which means \(u_1 = \tilde{x}_{1d}\) as \(t \geq T_1\), then we can substitute \(x_{1d}\) of \(u_1\) into equation (10), and let \(E_{11} = e_2, E_{22} = \tilde{e}_2, \ldots, E_{2n} = e_{2(n-1)}, E_{2(n+1)} = e_{2(n)}\), we obtain
\[
\dot{e}_2 = e_2, e_2 = x_2 - x_{2d}
\]
(12)
According to Lemma 1, we can obtain
\[ x_2^{(n)} = (x_3 u_1)^{(n-1)} \]
\[ = \sum_{k_1=0}^{n-1} C_{k_1}^{(n-1-k_1)} x_3^{(k_1)} \]
\[ = C_{n-1}^{0} u_1^{(n-1)} x_3 + \sum_{k_1=1}^{n-1} C_{k_1}^{(n-1-k_1)} (x_4 \cdot u_1)^{(k_1-1)} \]
\[ = C_{n-1}^{0} u_1^{(n-1)} x_3 + \sum_{k_1=1}^{n-1} C_{k_1}^{(n-1-k_1)} \left( u_1^{(k_1-1)} x_4 + \sum_{k_2=1}^{k_1-1} C_{k_2}^{(k_1-k_2)} u_1^{(k_2-1)} x_5 + \sum_{k_3=1}^{k_2-1} C_{k_3}^{(k_3)} (x_6 \cdot u_1)^{(k_3-1)} \right) \]
\[ : \]
\[ = C_{n-1}^{0} u_1^{(n-1)} x_3 + \sum_{k_1=1}^{n-1} C_{k_1}^{(n-1-k_1)} \left( u_1^{(k_1-1)} x_4 + \sum_{k_2=1}^{k_1-1} C_{k_2}^{(k_1-k_2)} \left( \cdots (u_1^{(k_2-1)} x_n \right) + C_{k_3-1}^{1} u_1^{(k_3-2)} u_2 + C_{k_3-3}^{2} u_1^{(k_3-3)} \right) \]

By calculating the \( n \)-order derivative of \( x_2 \), system (10)

\[ \begin{align*}
E_{21} &= E_{22} \\
E_{22} &= E_{23} \\
\vdots & \\
E_{2(n-1)} &= E_{2n} \\
E_{2n} &= f(x, x_d, t) + g(x, x_d, t) \tau_2 - x_2^{(n)}
\end{align*} \tag{13} \]

where

\[ \begin{align*}
f(x, x_d, t) &= x_1^{(n)} x_3 + \sum_{k_1=1}^{n-1} C_{k_1}^{1} x_1^{(n-k_1)} x_3 + \sum_{k_2=1}^{k_1-1} C_{k_2}^{2} x_1^{(k_1-k_2)} \left( \cdots (x_1^{(k_2-1)} x_n \right) + C_{k_3-1}^{3} x_1^{(k_3-2)} u_2 + C_{k_3-3}^{4} x_1^{(k_3-3)} \right) \]

and

\[ g(x, x_d, t) = \sum_{k_1=1}^{n-1} C_{k_1}^{1} x_1^{(n-k_1)} \left( \sum_{k_2=1}^{k_1-1} C_{k_2}^{2} x_1^{(k_1-k_2)} \left( \cdots (C_{k_3-1}^{3} x_1^{(k_3-2)} \right) \right) \]

\( (E_{22}, E_{23}, \ldots, E_{2n})^T \) is unmeasured.

Next, we will give the primary design conclusion using the two-step switching control method.

- **Step 1.** We design a control law \( \tau_1 \) to achieve \([E_{11}, E_{12}]^T \in \mathbb{R}^2\) converge to zero in fixed time \( T_1 \). According to Lemma 2, subsystem (11) will converge to zero within bounded time that is not dependent on initial state when we construct controller \( \tau_1 \) as follows

\[ \tau_1 = -\frac{1}{g_1(x, t)} \left[ f_1(x, t) - \bar{x}_1^d + \gamma_1 \left( \frac{1}{2} + \frac{b_1}{2a_1} + \left( \frac{1}{2} - \frac{1}{2} \right) \right) \text{sign}(|E_{11}| - 1) \right] \]

\[ + \text{sat} \left( \eta_1 \frac{m_1}{n_1} E_{11}^{1-1} E_{12} + \gamma_2 \frac{1}{2} + \left( \frac{1}{2} - \frac{1}{2} \right) \right) \text{sign}(|E_{12}| - 1) \right] \]

\[ \text{sat}(a, b) = \begin{cases} 
\frac{1}{2} + \left( \frac{1}{2} - \frac{1}{2} \right) \text{sign}(|E_{11}| - 1)^T \in \mathbb{R}^2 \end{cases} \]
Therefore, $u_t = \dot{x}_{1d}$ as $t > t_1$, and then go to Step 2.

- **Step 2.** In this step, we are going to give a proof of the state vector $[E_{21}, E_{22}, \ldots, E_{2n}]^T \in R^p$ of subsystem (13) can be stabilized to zero in a fixed time. Due to $[E_{22}, E_{23}, \ldots, E_{2n}]^T \in R^{p-1}$ is the unmeasured system state, only $E_{21} \in R$ can be measured. The state observer is constructed as

$$
\dot{z}_1(t) = z_2(t) - r_1 \theta(t)|E_{21} - z_1(t)|^{\alpha_1}\text{sign}(E_{21} - z_1(t)) - t_1(1 - \theta(t))|E_{21} - z_1(t)|^{\alpha_2}\text{sign}(E_{21} - z_1(t))
$$
$$
\vdots
$$
$$
\dot{z}_i(t) = z_{i+1}(t) - r_i \theta(t)|E_{21} - z_1(t)|^{\alpha_1}\text{sign}(E_{21} - z_1(t)) - t_i(1 - \theta(t))|E_{21} - z_1(t)|^{\alpha_2}\text{sign}(E_{21} - z_1(t))
$$
$$
\vdots
$$
$$
\dot{z}_n(t) = - r_n \theta(t)|E_{21} - z_1(t)|^{\alpha_1}\text{sign}(E_{21} - z_1(t)) - t_n(1 - \theta(t))|E_{21} - z_1(t)|^{\alpha_2}\text{sign}(E_{21} - z_1(t))
$$

(16)

The observer variables $z_i(t) (i = 1, 2, \ldots, n)$ are estimated for $E_{2i}(t) (i = 1, 2, \ldots, n)$. $\lambda_i$, $\varphi_i$, $r_i$, $t_i$ and $\theta(t)$ are system parameters, which can be chosen to meet the following three conditions:

1. The exponents $\lambda_i$, $\varphi_i$ are selected to satisfy $\lambda_i = (i + 1)\lambda - i$, $\varphi_i = (i + 1)\varphi - i$, where $\lambda \in (1 - \omega_1, 1)$ and $\varphi \in (1, 1 + \omega_2)$, where $\omega_1$, $\omega_2$ are sufficient small positive real numbers.
2. The coefficients $r_i$ and $t_i (i = 1, 2, \ldots, n)$ are assigned such that Hurwitz matrices $A$ and $B$ are as follows

$$
A = \begin{bmatrix}
-r_1 & 1 & 0 & \cdots & 0 \\
-2r_1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-r_{n-1} & 0 & 0 & \cdots & 1 \\
-r_n & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
-1 & 0 & 0 & \cdots & 0 \\
-t_1 & 0 & 0 & \cdots & 0 \\
-t_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-t_{n-1} & 0 & 0 & \cdots & 1 \\
-t_n & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
$$

3. The function $\theta(t)$ is defined as

$$
\theta(t) = \begin{cases} 
0 & \text{if } t < T_{sw} \\
1 & \text{if } t > T_{sw}
\end{cases}
$$

where $T_{sw}$ is the switch time.

Based on the designed observer above, the theorem is given as follows

**Theorem 1.** To use the state observer (equation (16)), $E_2$ can be estimated accurately within finite time $t_{sw}$; the upper bound $T_{es}$ exists such that $t_{es} < T_{es}$, and $T_{es}$ is independent of the initial state.

**Proof.** The observer variable $z_i(t) (i = 1, 2, \ldots, n)$ is estimated for $E_{2i}(t) (i = 1, 2, \ldots, n)$, hence we can define estimation error as $\tilde{e}_i(t) = z_i(t) - E_{2i}(t) (i = 1, 2, \ldots, n)$, and the estimation error system can be expressed as

$$
\begin{align*}
\dot{\tilde{e}}_1 &= \tilde{e}_2 - r_1 \theta(t)\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} - t_1(1 - \theta(t))\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} \\
\vdots & \vdots \\
\dot{\tilde{e}}_i &= \tilde{e}_{i+1} - r_i \theta(t)\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} - t_i(1 - \theta(t))\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} \\
\vdots & \vdots \\
\dot{\tilde{e}}_n &= - r_n \theta(t)\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} - t_n(1 - \theta(t))\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)}
\end{align*}
$$

(17)

1. when $t < T_{sw}$, which means $\theta(t) = 0$, system (17) can be rewritten as

$$
\begin{align*}
\dot{\tilde{e}}_1 &= \tilde{e}_2 - t_1\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} \\
\vdots & \vdots \\
\dot{\tilde{e}}_i &= \tilde{e}_{i+1} - t_i\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)} \\
\vdots & \vdots \\
\dot{\tilde{e}}_n &= - t_n\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)}
\end{align*}
$$

(18)

Consider a Lyapunov function $V_1(\tilde{e}, \varphi) = \xi^T(\tilde{e})G\xi(\tilde{e})$, where $\xi(\tilde{e}) = [\tilde{e}_1^{1|\text{sign}(\tilde{e}_1)}, \tilde{e}_2^{1|\text{sign}(\tilde{e}_1)}, \ldots, \tilde{e}_n^{1|\text{sign}(\tilde{e}_1)}]^T$, and $G$ is a positive symmetric matrix satisfying $B^T G + GB = -Q$, where $Q$ is positive matrix. If $\varphi = 1$, system (18) will change into $\dot{\tilde{e}} = \tilde{B}\tilde{e}$. According to the above statement, matrix $B$ is Hurwitz; consequently, $\dot{\tilde{e}}(t)$ is asymptotically stable. The derivative of $V_1(\tilde{e}, 1)$ is as follows

$$
V_1(\tilde{e}, 1) = \tilde{e}^T (B^T G + GB) \tilde{e} = - \tilde{e}^T Q \tilde{e} \leq 0
$$

$V_1(\tilde{e}, 1) = 0$ if and only if $\tilde{e} = 0$. When $\tilde{e} = 0$, we can obtain that $\dot{\tilde{e}} = 0$, which implies that there is a sufficient small constant $\omega_2$, for all $\varphi \in (1, 1 + \omega_2)$; $V_1(\tilde{e}, 1) \leq 0$ also means that $V_i(\tilde{e}, \varphi) \leq 0$.

Based on the definition, we can prove that the Lyapunov function $V_i(\tilde{e}, \varphi)$ is homogeneous of degree $\sigma_1 = 2$ and its derivative $\dot{V}_i(\tilde{e}, \varphi)$ is homogeneous of degree $\sigma_2 = 1 + \varphi$ with respect to the same weights $\delta_i = \varphi - (j - 1)$. According to Lemma 4, we can establish the following inequality

$$
\dot{V}(\tilde{e}, \varphi) \leq - \lambda(\tilde{e}, \varphi)(V(\tilde{e}, \varphi))^{\frac{\sigma_1}{\sigma_2}}
$$

(19)
Due to $\lim_{\varphi \to 1} \chi(\varphi, \varphi) = \lambda_{\min}(Q)/\lambda_{\min}(G)$, and there is a small positive constant $\omega_2$, such that for all $\varphi \in (1, 1 + \omega_2)$, the inequality $\chi(\varphi, \varphi) > \lambda_{\min}(Q)/2\lambda_{\min}(G)$ holds. Therefore, we can obtain

$$\dot{V}(\varphi, \varphi) \leq -\frac{\lambda_{\min}(Q)}{2\lambda_{\min}(G)} (V(\varphi, \varphi))^{\gamma + 1}$$

(20)

2. As $t > T_{sw}$, which means $\theta(t) = 1$, and system (17) becomes

$$\begin{align*}
\dot{\bar{x}}_1 &= \bar{x}_2 - r_1|\bar{x}_1|^{\alpha_1}\text{sign}(\bar{x}_1) \\
\dot{\bar{x}}_2 &= \bar{x}_1 - r_2|\bar{x}_1|^{\alpha_2}\text{sign}(\bar{x}_1) \\
\dot{\bar{x}}_3 &= -\bar{x}_2 - r_3|\bar{x}_1|^{\alpha_3}\text{sign}(\bar{x}_1)
\end{align*}$$

(21)

We can choose a Lyapunov function $V_2(\bar{x}, \gamma) = \zeta(\bar{x})M \zeta(\bar{x})$, where $\bar{x} = [\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n]^T$. $\zeta(\bar{x}) = [\zeta_1, \zeta_2, \ldots, \zeta_n]$, positive symmetric matrix $M$ satisfying $A^T M + MA = -N$, where $N$ is positive matrix. Analogously, it is not difficult to prove that system (21) converges for a finite time. The derivative of Lyapunov function with respect to time $V_2(\bar{x}, \gamma)$ satisfies the following inequality

$$\dot{V}(\bar{x}, \gamma) \leq -\frac{\lambda_{\min}(N)}{2\lambda_{\min}(M)} (V(\bar{x}, \gamma))^{\gamma + 1}$$

(22)

Since $\gamma < 1$, the Lyapunov function (equation (22)) will converge to zero within finite time. System (21) can converge to zero within finite time, and the upper bound of time is

$$t_c = \frac{2\lambda_{\min}(M)}{\lambda_{\min}(N)} \left( 1 - \frac{\varphi - 1}{\varphi - 4} \lambda_{\min}(N) T_{sw} \right)^{1/\gamma} + T_{sw}$$

(23)

According to equation (23), we can conclude that the upper bound of the estimated time depends only on the design parameters $\gamma$, $\varphi$, $T_{sw}$, $M$, $N$. This implies that the estimated time has nothing to do with the initial estimation error. Moreover, it can be obtained in advance.

The proof is completed.

Next, design a fixed-time convergent control law of the second subsystem as

$$\tau_2 = -\frac{1}{g(x, x_{dA}, t)} \left[ f(x, x_{dA}, t) - x^{(n)}_{2d} + \sum_{i=1}^{n} l_i |E_{2d}(t)|^\alpha_i \right]$$

(24)

The exponents $\alpha_i, \beta_i (i = 1, 2, \ldots, n)$ satisfy conditions:

$$\alpha_i \in (0, 1) (i = 1, 2, \ldots, n), \alpha_{i-1} = \alpha_i \alpha_{i+1}/(2\alpha_{i+1} - \alpha_i) (i = 2, 3, \ldots, n)$$

and $\alpha_n = \alpha$ where $\alpha \in (1 - \omega_3, 1)$, $\omega_3$ is a small positive constant.

$$\beta_i > 1 (i = 1, 2, \ldots, n), \beta_{i+1} = \beta_i \beta_{i+1} + (2\beta_i - 1) (i = 2, 3, \ldots, n)$$

and $\beta_1 = 1$, $\beta_n = \beta$, where $\beta \in (1, 1 + \omega_4)$, $\omega_4$ is a small positive constant.

The coefficients $l_1, l_2 (i = 1, 2, \ldots, n)$ are assigned such that matrices $C$ and $D$ are Hurwitz

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -l_1 & -l_2 & -l_3 & \cdots & -l_n \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -L_1 & -L_2 & -L_3 & \cdots & -L_n \end{bmatrix}$$

Based on the designed controller above, the convergence time of system (13) is described in the following theorem.

**Theorem 2.** The state of the second subsystem $E_{21}, E_{22}, \ldots, E_{2n} \in \mathbb{R}^n$ converges to zero within a fixed time $t_2$

$$t_2 < T_2 = \frac{\lambda_{\max}(X)}{\lambda_{\min}(Y)} \alpha + \frac{\lambda_{\max}(\tilde{X})}{\lambda_{\min}(Y)} \beta \frac{1}{\beta - 1}$$

(25)

where $X, \tilde{X}$ are symmetric positive definite matrices and satisfy the following equations

$$C^T X + CX = Y, D^T \tilde{X} + D\tilde{X} = \tilde{Y}$$

$Y, \tilde{Y}$ are positive definite matrices.

**Proof.** Starting from $t_{gt}, z_i = E_{2i} (i = 1, 2, \ldots, n)$, consequently, after $t = t_{gt}$, the controller (equation (24)) is activated. Hence, we can use $z_i$ replace unmeasured state $E_{2i}$. Based on Theorem 1 in the work by Basin et al.,45 the proposed fixed-time control law (equation (24)) can drive all states $E_{2i} (i = 1, 2, \ldots, n)$ to zero within a fixed time. For any initial state of system (13), the total convergence time is less than $T_{es} + T_2$. The proof of this theorem is completed.

**Simulation results**

In this section, we will verify the validity of our proposed tracking controller with MATLAB simulations. In the following simulation, the system is divided into two subsystems (11) and (13). We will simulate the two subsystems, respectively. For the first subsystem, we assume that $E_{11}(0) = 0.9, E_{12}(0) = 0.8$, and we select the parameters of $\tau_1$ as $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1, \beta_1 = \beta_2 = \beta_3 = \beta_4 = 5$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Before simulating the second subsystem, we should first
verify the validity of the state observer. For the estimation error system (17), here we consider the fourth-order situation:

\[
\begin{align*}
\dot{e}_1 &= \dot{e}_2 - r_1 \theta(t)|\dot{e}_1|^{\alpha_1} \text{sign}(\dot{e}_1) - t_1(1 - \theta(t))|\dot{e}_1|^{\alpha_1} \text{sign}(\dot{e}_1) \\
\dot{e}_2 &= \dot{e}_3 - r_2 \theta(t)|\dot{e}_1|^{\alpha_2} \text{sign}(\dot{e}_1) - t_2(1 - \theta(t))|\dot{e}_1|^{\alpha_2} \text{sign}(\dot{e}_1) \\
\dot{e}_3 &= -r_3 \theta(t)|\dot{e}_1|^{\alpha_3} \text{sign}(\dot{e}_1) - t_3(1 - \theta(t))|\dot{e}_1|^{\alpha_3} \text{sign}(\dot{e}_1) \\
\dot{e}_4 &= -r_4 \theta(t)|\dot{e}_1|^{\alpha_4} \text{sign}(\dot{e}_1) - t_4(1 - \theta(t))|\dot{e}_1|^{\alpha_4} \text{sign}(\dot{e}_1)
\end{align*}
\]

(26)

We select the switch time \(T_{sw}\) as 2, the exponents of observer are selected as \(\lambda_1 = 0.6\), \(\lambda_2 = 0.4\), \(\lambda_3 = 0.2\), \(\lambda_4 = 0\) and \(\varphi_1 = 1.4\), \(\varphi_2 = 1.6\), \(\varphi_3 = 1.8\), \(\varphi_4 = 2\). The coefficients are chosen as \(t_1 = 18\), \(t_2 = 50\), \(t_3 = r_3 = 50\), \(t_4 = 50\). And then, for the second subsystem, we are going to give simulation for the following third-order tracking error system:

\[
\begin{align*}
\dot{E}_{21} &= E_{22} \\
\dot{E}_{22} &= E_{23} \\
\dot{E}_{23} &= f(x, x_d, t) + g(x, x_d, t)\tau_2 - x_{2d}^{(3)}
\end{align*}
\]

Choosing the parameters \(\alpha_1 = 0.826\), \(\alpha_2 = 0.864\), \(\alpha_3 = 0.905\), \(\beta_1 = 1.235\), \(\beta_2 = 1.167\), \(\beta_3 = 1.105\). The coefficients of \(\tau_2\) are assigned as \(l_1 = l_2 = l_3 = 20\), \(l_1 = 20\), \(L_2 = 30\), \(L_3 = 20\).

Based on the above parameters, we will simulate the convergence process of the two subsystems and the state observer, and draw the following figures.

According to Figure 1, we can see the tracking error \(e_1\) converges to zero in a finite time \(t_1 < 7s\). For the designed state observer of the second subsystem, it can be seen from Figure 2 that the estimated error is stable to zero after 6s. Defining \(N\) as an identity matrix, and using equation (23), we can calculate the upper bound of estimation time \(T_{es} = 104.273\). After \(t = T_{es}\), the controller (equation (24)) is activated; from Figure 3, we can see the total convergence time \(t_{es} + t_2 = 16.3s\), approximately, does not exceed the bound time \(T_{es} + T_2 = 104.273 + 81.6 = 185.873\).

**Conclusion**

The fixed-time tracking control problem of extended nonholonomic \(n\)-order chained-form systems with incomplete information is considered in this article. Based on the special structure of chain-form system, the system is divided into two subsystems to design the controller based on fixed-time stability control theory and state observer methodology. Compared with the traditional control methods, fixed-time control has its own unique advantages: The convergence time can be calculated in advance and does not depend on the initial state. This has brought great convenience to the practical engineering application.
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