Fractional-order derivatives in cosmological models of accelerated expansion

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Abstract – In this brief review, we present the results of the fractional differential approach in cosmology in the context of the exact models of cosmological accelerated expansion obtained by several authors to date. Most of these studies are devoted to the problem of introducing fractional derivatives or fractional integrals into the classical General Relativity (GR). There are several observational and theoretical motivations to investigate the modified or alternative theories of GR. Among other things, we cover General Relativity modified by a phenomenological approach dealing with fractional calculus. At the same time, a sufficiently large number of exact solutions of the cosmological equations modified by this approach were obtained. Some of these models may be especially relevant in the light of solving the problem of late accelerated expansion of the universe. These studies are largely motivated by rapid progress in the field of observational cosmology that now allows, for the first time, precision tests of fundamental physics on the scale of the observable Universe. The purpose of this review is to provide a reference tool for researchers and students in cosmology and gravitational physics, as well as a self-contained, comprehensive and up-to-date introduction to the subject as a whole.

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1 Introduction

The well-known cosmological observations, including a Type Ia supernova, cosmic microwave background radiation, and large-scale structure, strongly indicate that our universe is undergoing a phase of late accelerated expansion [1]–[9].

The fast rising of the number of publications over the past two decades, devoted to the mysterious late cosmological acceleration, led to a radical revision of cosmological models. However, two main directions of such modifications should be distinguished. First of all, it has been developed a lot of models with different exotic forms of matter, often called Dark Energy (DE), which satisfies the equation of state (EoS) \( w = p/ \rho < -1/3 \). Different types of the DE models have been proposed, such as the cosmological constant [10], quintessence [11, 12], phantom [13–15], tachyon [16, 17], k-essence [18–20], Chaplygin gas [21], quintom [22], holographic dark energy [23] and Yang–Mills fields [24, 25], etc.

An alternative approach to the problem of the late accelerated expansion of the Universe is associated with the consideration of numerous modifications of the theory of gravity itself. Various modifications include multidimensional theory, brane world models, teleparallel theory and many others [26]–[32]. Moreover, we could recall the broad class of modified gravity, such as \( f(R) \), \( f(G) \), \( f(R, G) \), \( F(R, T) \), etc., which produce adequate gravitational alternatives for DE [33–40]. Nevertheless, the mystery of the late cosmological acceleration is still one of the main problems of modern cosmology.

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At the same time, taking into account the cosmological inflation paradigm of the very early Universe, any reliable cosmological model should consist of at least two periods of accelerated expansion. Therefore, it is interesting to consider any cosmological models that are able to more or less realistically describe the entire evolution from the beginning to the present day [41]-[43].

Beyond such modifications, a certain phenomenological approach in theoretical cosmology can be noted in connection with the problem of late cosmological acceleration. For example, in Refs. [44]-[45], the massive gravity theory is considered as a modification of GR and include massive gravitons. This theory has a rich phenomenology, such as explaining the accelerated expansion of the universe without invoking dark energy, and have attracted much attention recently [46].

A class of phenomenological models based on the ideas of fractional calculus seems to be interesting and attractive in its attempts to describe the periods of accelerated expansion. The cosmological equations describing the dynamics of a homogeneous and isotropic Universe are the systems of ordinary differential equations. Modifying the set of cosmological equations in this approach, one should avoid the conflict between the integer dimension of (pseudo) Riemann space-time of GR and fractional order of derivatives in the modified equations. As it emphasized in Refs. [47, 48], there are two different methods of such modification towards the fractional derivative cosmology. The simplest method is the Last Step Modification (LSM) method, in which the Einstein’s GR equations are replaced with the fractional analogous. In other words, the substitution $\partial_t \rightarrow D_t^\alpha$, where $D_t^\alpha$ is a derivative of fractional order $\alpha$, should be done after the field equations for a specific geometry have been derived. The fundamentalist methodology could be the First Step Modification (FSM), in which one starts by constructing fractional derivative geometry. The intensively developed approach to modification of the main cosmological equations and non-conservative systems of Lagrangian dynamics on the basis of a variational principle for the action of a fractional order (Fractional Action-Like Variational Approach FALVA) developed in [49]-[52] represents one of the possible version of intermediate modification (Intermediate Step Approach, ISA) mentioned in [47].

The main part of articles, which have been published by now, devotes to the fractional-order derivatives cosmology on the bases of FSM, ISA, LSM and FALVA. In ISA and FALVA, the set of cosmological equations can be readily obtained by the direct replacing derivatives or using a variational principle for the Einstein-Hilbert action. In any case it is easy to generalize masters equations of the field theories to the case of fractional derivatives. Therefore, one could easily obtain the fractional-differential analogue of the Friedmann equations even in the case of involving some physical field as a source of gravitation in frameworks of ISA as well as on the basis of FALVA.

The paper is organized as follows. In Section 2, we briefly review the cosmological models with fractional-order derivatives. In Section 3, we address the fractional Einstein-Hilbert action cosmological models. In Section 4, we make some conclusive remarks. Finally, in Appendix A, we present a brief information concerning the definitions and some features of fractional derivatives and integrals.

2 Studying cosmological models with fractional-order derivatives

Since the appearance of the first cosmological models with fractional derivatives, this direction of research has not received proper development. Consider what we have here by now.
2.1 Cosmological models with fractional derivatives

Presumably the first cosmological models with equations containing fractional derivatives were proposed and discussed in [49]. In order to avoid a conflict between the occurrence of fractional derivatives in the Friedmann equations and classical definition of a tensor in the Einstein equation, based on the integer order derivative in tensor law of transformation, the quoted author has repeated the well known derivation of the Friedmann equations for a dust from the classical approach (see, for instance, [51], [53]) and than has replaced all integer derivatives with its fractional analog. For the spatially flat Universe, the Friedmann equations with a cosmological term are written down in [49] as follows:

\[
(D_t^\alpha a(t))^2 = (A_1 (G \rho)^\alpha + B (\Lambda c^2)^\alpha) a^2,
\]

\[
D_t^\alpha (D_t^\alpha a(t)) = -(A_2 (G \rho)^\alpha - B (\Lambda c^2)^\alpha) a,
\]

where \(a(t)\) is a scale factor of the Friedmann-Robertson-Walker (FRW) line element,

\[
ds^2 = dt^2 - a^2(t) (dr^2 + \xi^2(r) d\Omega^2),
\]

where \(\xi(r) = \sin r, r, \sinh r\) for the sign of space curvature \(k = +1, 0, -1\), consequently. The occurrence of \(\alpha\) - degrees in the right-hand side of equations (1) is caused, probably, by dimensional reasons. Then the author of paper [49] obtained the following solution to this set in a static case \(a = a_0 = constant\):

\[
(G \rho)^\alpha = \frac{C_1}{t^{2\alpha}}, \quad (\Lambda c^2)^\alpha = \frac{C_2}{t^{2\alpha}}.
\]

Then, the conclusion is followed that the density of matter and cosmological constant decrease as \(1/t^2\) if \(G\) and speed of light in vacuum \(c\) remains constant, but if the density of matter and \(\Lambda\) remain constant, then the following holds: \(G \sim 1/t^2\) and \(c \sim 1/t\). The solution of equations (1), mentioned as an illustration of the method of [49], in the form \(a(t) = a_0 E_{1,\alpha}(Ct^\alpha)\), where

\[
E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} t^k \frac{1}{\Gamma(ak + \beta)}
\]

is the so-called Mittag-Leffler two-parametric function [54], is not confirmed by any calculations.

It is interesting to note that for \(k = 0\) and in the absence of matter and the cosmological constant, that is if \(\rho = 0\) and \(\Lambda = 0\), equations (1) are reduced to

\[
(D_t^\alpha a(t)) = 0, \quad D_t^\alpha (D_t^\alpha a(t)) = 0,
\]

which for \(\alpha = 1\) gives the obvious result: \(a(t) = a_0 = constant\), and interval (2) is reduced to those of the Minkowski space. If \(\alpha \neq 1\), then the substitution of the zero constant (that is the derivative \(D_t^\alpha a(t)\)) from the first equation of (3) into the second one results in identity irrespectively to the definition of fractional derivative, RLD or CD. The solution of the first equation in (3) for CD, as well as in the case of integer derivative, is equal to constant, but for RLD its partial solution is zero, and the general solution depends on time: \(a(t) \sim t^{\alpha-1}\) for \(0 < \alpha \leq 1\) (see [54], p. 216). This circumstance and also the fact, that initial conditions for equations with CD should be expressed by means of integer derivative, instead of fractional one, as in the case of RLD, frequently decline a choice of definition for the benefit of CD. In general considerations, we shall not specify a type of fractional derivative so long as it will be possible.
It is useful to note that in [47] the fractional-derivatives analogue of Friedmann equations are written down in other form, namely:

\[ 3[k + (D^a_t a)^2] = \kappa \rho a^2, \quad a^3 D^a_t p = D^a_t [a^3(\rho + p)], \]

where we use \( \rho \) for the energy density and \( p \) for the pressure. The second equation represents the energy-momentum conservation law \( (T_{ij}^\mu = 0) \) for the perfect fluid in spacetime \( (2) \) with the integer-order derivatives replaced by its fractional analogues. In the assumption of \( k = 0 \) and the power-law dependence of \( a, p \) and \( \rho \) on the time: \( a = Ct^m, \quad p = At^n, \quad \rho = Bt^r \), the relations between \( m, r, n, \alpha, B \) and barometric coefficient \( w \) in the equation of state \( p = w\rho \) have been found in paper [48].

However, the paper [55] has already been done using the naive approach method [47], in which the Riemann curvature tensor and the Einstein tensor are defined by the inconceivable Christoffel symbols containing fractional (of order \( 0 < \alpha \leq 1 \)) derivatives of metrics coefficients,

\[ \Gamma^\mu_{\nu\lambda}(\alpha) = \frac{1}{2} g^{\mu\rho} \left( \partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} + \partial_\rho g_{\nu\lambda} \right), \]

where \( \partial_\nu^\alpha \) is a fractional derivative (A.2) with respect to \( x^\nu \). So it allowed to write down the fractional analogous for the Einstein equation,

\[ R_{\mu\nu}(\alpha) - \frac{1}{2} g_{\mu\nu} R(\alpha) = \frac{8\pi G}{c^4} T_{\mu\nu}(\alpha), \]

and the geodesic equation,

\[ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(\alpha) \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0. \]

Unfortunately, the author of [55] only shows that linearized equations in the limit \( \alpha \to 1 \) are reduced to the usual equations with the integer derivatives for the gravitational waves and the Newtonian potential. That could be predicted from the very beginning due to the definition of a fractional derivative. Let us note, that an attempt to prove such fractional derivative replacement in Christoffel symbols, such as [55], is undertaken in [47] on the basis of fractional-differential geometry, that would be related to FSM formalism. However, the author of [55] also mentioned that it is so far not clear what such geometry should be. Nevertheless, an attempt of construction of fractional geometry with the help of fractional coordinate transformations \( dx^i = D^a_i x^i dx^a \) is undertaken for the flat two-dimensional space in [47].

Much more advanced and proved results concerning FSM formalism were obtained by S. Vacaru (see [56]-[58] and the bibliography therein). In those works, the results of construction of the fractional theory of gravitation for the space - time of fractional (not integer) dimension are obtained. The author sees one of the simplest motivation for application of fractional differential calculus in the theory of gravitation in an opportunity to avoid singularities of the curvature tensor of physical meaning due to the completely different geometrical and physical solutions of the fundamental equations. Besides, it is noted that models of fractional order are more adapted to the description of processes with memory, branching and hereditarity, than those of integer order. The result of the application of the method developed by the author of nonholonomic deformations to cosmology was the construction of new classes of cosmological models [57].

2.2 Fractional derivatives cosmology with a scalar field

First, the naive (or LSM) approach to the fractional derivative cosmological models of a scalar field has been considered in Ref. [48]. Following LSM, the substitution of fractional
In the case of (10), one has \( q \), the following basic equations of the model can be derived:

\[
D_t^\alpha (D_t^\alpha \phi) + 3 \left( \frac{D_t^\alpha a}{a} \right) D_t^\alpha \phi + \frac{dV(\phi)}{d\phi} = 0 ,
\]

\[
(D_t^\alpha a)^2 + k = \frac{8\pi G}{3} \left[ \frac{1}{2} (D_t^\alpha \phi)^2 + V(\phi) \right] a^2 + \frac{\Lambda}{3} a^2 ,
\]

\[
D_t^\alpha (D_t^\alpha a) = -\frac{8\pi G}{3} \left( (D_t^\alpha \phi)^2 - V(\phi) \right) a + \frac{-\Lambda}{3} a .
\]

As known, among three equations of standard GR cosmology (when \( \alpha = 1 \)) only two equations are independent. In the case \( \alpha \neq 1 \), all three equations of (10) - (13) are generally independent due to the modified Leibniz rule for the fractional derivative \[54\]. Because the solution of the nonlinear fractional equations (10) - (13) is much complicated compared to their classical prototype, an example of an exact solution to the equations (10) - (12) is given in Ref. [48] to demonstrate the existence of exact solutions for such a model.

Considering further the models followed from the intermediate approach (or ISA) in [48], the main equations have been derived from the variational principle for the Einstein-Hilbert action according to ADM formalism in cosmology (see, e.g., [59]). For this purpose, the derivatives over time in the Einstein-Hilbert action \( S_{EH} \equiv \int L_{EH} \, dt \) for FRW model of the Universe, filled with a real homogeneous scalar field \( \phi(t) \),

\[
ds^2 = N(t)^2 dt^2 - a^2(t)(dr^2 + \xi^2(r)d\Omega^2),
\]

where \( N \) is a lapse function, have been replaced with its fractional (of order \( \alpha \)) analogous. The result is as follows:

\[
S_{EH} = \int dt N \left[ \frac{3}{8\pi G} \left( \frac{-a(D_t^\alpha a)^2}{N^2} + ka \right) + a^3 \left( \frac{(D_t^\alpha \phi)^2}{2N^2} - V(\phi) \right) \right] .
\]

where \( V(\phi) \) is a potential of the field. Variational problem with fractional derivatives for functions \( q_j(t) \) in the action \( S[q_j](t) = \int_c^d L(q_j(t), tD_t^\alpha q_j(t), tD_d^\beta q_j(t)) dt \) on the interval \([c, d]\) yields the modified Euler-Lagrange equations [51], [60]:

\[
\frac{\partial L}{\partial q_j} + tD_d^\alpha \left( \frac{\partial L}{\partial (D_t^\alpha q_j)} \right) + cD_t^\beta \left( \frac{\partial L}{\partial (tD_d^\beta q_j)} \right) = 0 .
\]

In the case of (10), one has \( q_j(t) = \phi(t), N(t) \) and \( a(t) \). Therefore for \( L_{EH} \) from (10) and (11), the following basic equations of the model can be derived:

\[
tD^\alpha (a^3 D_t^\alpha \phi) - a^3 \frac{dV(\phi)}{d\phi} = 0 ,
\]

\[
(D_t^\alpha a)^2 + k = \frac{8\pi G}{3} \left[ \frac{1}{2} (D_t^\alpha \phi)^2 + V(\phi) \right] a^2 + \frac{\Lambda}{3} a^2 ,
\]

\[
2tD^\alpha (a D_t^\alpha a) + (D_t^\alpha a)^2 - k = 8\pi G \left( (D_t^\alpha \phi)^2 - V(\phi) \right) a^2 - \Lambda a^2 ,
\]

where \( 0D_t^\alpha \equiv D_t^\alpha \) and \( tD_d^\beta \equiv tD^\beta \).

In the case of the limit \( (\alpha \to 1) \), one has \( D_t^1 \to \frac{d}{dt}, tD^1 \to -\frac{d}{dt} \), and the set of fractional cosmological equations [12] - (14) tends to the classical set of the Friedmann...
and scalar field equations:

\[ 2\ddot{a} + \frac{a^2}{a^2} + \frac{k}{a^2} = -8\pi G \left( \frac{\dot{\phi}^2}{2} - V(\phi) \right) + \Lambda, \]

\[ \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) + \frac{\Lambda}{3}, \]

which had to be expected from the very beginning.

### 2.3 The recent studies in fractional derivatives gravity and cosmology

As it has been noted in Ref. [61], there are several models considered two limited aspects of gravity with fractional derivatives without embedding them in a fundamental theory: Newtonian gravity and cosmology. The author of this paper emphasizes that the theories with fractional derivatives do not have Lorentz symmetry and they are technically difficult due to the presence of fractional derivatives. We would like to draw the attention of researchers who are interested by scalar theories with fractional derivatives and scalar theories with fractional d’Alembertian to this paper and to “an unconventional review” [62].

In Ref. [63], Newton’s potential was derived from an ad hoc fractional Poisson equation. Following the hypothesis that the matter distribution of galaxies behaves as a fractal medium with non-integer dimension, and solving a Poisson equation with fractional Laplacian, one can describe the properties of such matter distribution [64].

In the paper [105], the fractional dark energy model with the accelerated expansion of the Universe driven by a non-relativistic gas with a non-canonical kinetic term. It is shown that the inverse momentum operator appears in fractional quantum mechanics and can be expressed through inverse of the Riesz fractional derivative.

In recent paper [65], a toy model for extending the Friedmann equations of relativistic cosmology with fractional derivatives is considered. The cosmological consequences of replacing the integer time derivatives in the Friedmann equations with fractional derivatives (that is using LSM) are explore. This simple approach allowed them to write down the fractional Friedmann equations as follows:

\[ \left( \frac{D^\gamma a}{D\tau^\gamma} \right)^2 = \kappa a^2 \left( \frac{8\pi G \rho + \Lambda c^2}{3} \right), \]

\[ \frac{D^\gamma}{D\tau^\gamma} \left( \frac{D^\gamma a}{D\tau^\gamma} \right) = \kappa a \left( -\frac{4\pi G \rho - \Lambda c^2}{3} \right), \]

for a pressure-less flat Universe. The constant \( \kappa \), with dimensions of time\(^2(1-\gamma)\) has been introduced into the fractional Friedmann equations in order to have dimensional coherence. The left-hand side of equation (19) is written as such since in general \( D^\gamma D^\gamma \neq D^{2\gamma} \). It is easy to see the obvious difference between the sets of Friedmann equations (18)-(19) and (1). Then the authors introduced Milgrom’s acceleration constant \( a_0 \) as a fundamental physical quantity for the description of gravitational phenomena at cosmological scales. With this and since the velocity of light \( c \) and Newton’s gravitational constant \( G \) are also fundamental, it follows from the dimensional analysis that \( \kappa = A(a_0/c)^{2(\gamma-1)} \), where \( A \) is a dimensionless constant. Due to the fractional order of the derivative, one could adapt
the cosmographic parameters as follows. For example, the fractional Hubble parameter can be defined as:

\[ H^* = \frac{1}{a} \frac{D^\gamma a}{Dt^\gamma} \]  

(20)

In a matter dominated Universe, where the dark energy density parameter \( \Omega_\Lambda = 0 \), the following ansatz is proposed:

\[ a = a_1 t^n, \]  

(21)

where \( a_1 \) is a constant, and using the rules of fractional derivative for a power law given in Appendix A, the fractional Hubble parameter \( H^* \) is given by:

\[ H^* = \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} t^{-\gamma}. \]  

(22)

As the standard Hubble parameter for the scale factor (21) is \( H = n t^{-1} \), one can get

\[ \frac{H^*}{H} = \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} t^{1-\gamma} = \frac{\Gamma(n+1)}{\Gamma(n+1)} \left( \frac{H}{n} \right)^{\gamma-1}. \]  

(23)

Finally, the equation for the Hubble parameter is obtained as follows

\[ H = \frac{a_0 t}{c} t^{1-\gamma} \Omega_M^{1/2(\gamma-1)}, \]  

(24)

where \( \Omega_M \) stands for the matter density. Moreover, in the definition of the fractional cosmographic parameters \( q^*, j^* \) and \( s^* \), the standard Hubble parameter \( H \) is used. With the use of the LSM technique, \( H \) is used for the simplicity.

The authors of Ref. [65] then apply a fitting procedure to the SN Ia data to estimate the unknown order values of the fractional derivative and fractional cosmographic parameters. It is noted that such a simple approach could explain the current accelerated expansion of the universe without using the dark energy component. Further, the best fit results for the fractional derivative model with three free parameters are reported. These three parameters are the fractional derivative order \( \gamma \), the matter density parameter \( \Omega_M \) and the power for the scale factor \( n \). They are presented with their corresponding errors for the initial values as follows: \( n = 2.6, \gamma = 2.1 \) and \( \Omega_M = 4.5 \). It is interesting that with the mean values mentioned above, the Hubble constant \( H_0 \) has the following numerical value \( H_0 = 66.95 \, km/s \cdot Mpc \), which is in a great agreement with the value reported by Planck [66].

3 Fractional action cosmology

3.1 From FALVA to FAC

First of all, we have to mention papers [51, 52], in which the approach to the dynamical field theories in general and to the theory of gravitation in particular is developed on the basis of the variational principle, formulated by the author, for the action of fractional order (the so-called FALVA). In this approach in the framework of ISA, the action functional integral \( S_L[q] \) for the Lagrangian \( L(\tau, q(\tau), \dot{q}(\tau)) \) can be written as the fractional integral (A.5):

\[ S_L[q_i] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} L(\tau, q_i(\tau), \dot{q}_i(\tau))(t - \tau)^{\alpha-1} d\tau. \]  

(25)
At fixed $t$ it is the Stieltjes integral with integrating function
\[ g_t(\tau) = \frac{1}{\Gamma(1 + \alpha)}[t^\alpha - (t - \tau)^\alpha], \]
having the following scale property: $g_{\mu t}(\mu \tau) = \mu^\alpha g_t(\tau)$, $\mu > 0$. Then $q_t(\tau)$ satisfies the modified (or fractional) Euler-Lagrange equation:
\[
\frac{\partial L}{\partial q_i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 1 - \alpha t - \tau \frac{\partial L}{\partial \dot{q}_i} \equiv F_i, \quad i = 1, 2, \ldots, n; \quad \tau \in (0, t),
\]
where the dot over a symbol stands for the first derivative with respect to time $\tau$, $F_i$ is the modified decaying force of "friction", that is the general expression for non-conservative force. In the article [67], time $\tau$ is treated as the intrinsic (proper) time, and $t$ is the observer time. The authors of Refs. [51, 52, 68] stated that at $\tau \to \infty$ one has $F_i = 0$, and provided some examples of application FALVA to the Riemann geometry and perturbed cosmological models. However, these articles do not use FALVA directly for the gravitational action of $S_G$, written according to (25), but try to take into account the influence of the fractional order in action (25) on the Friedman equations through the perturbed and time-dependent classical gravitational constant. Starting with the Lagrangian $L = g_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j$, the modified geodesic equation has been obtained as:
\[
\ddot{x}^i + \frac{\alpha - 1}{T} \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,
\]
where $\Gamma^i_{jk}$ are the usual Christoffel symbols, and $T = t - \tau$. The second term here is interpreted as a dissipative force, which infinitely increases as $\tau \to t$ for $\alpha \neq 1$ and under condition of fixing future time $t$. Using non-relativistic approximation, the time variation of Newton’s gravitational constant and, as a results, the perturbation of the gravitational constant have been found as $\Delta G = \frac{3(1 - \alpha) \dot{a}}{4\pi \rho T \dot{a}}$. After that, the so-called effective gravitational constant $G_{eff} = G + \Delta G$ can be substituted into the standard Friedman equations:
\[
\frac{1}{a^2} (\dot{a}^2 + k) = \frac{8\pi G_{eff}}{3} \rho + \frac{\Lambda}{3}, \quad \frac{\ddot{a}}{a} = - \frac{4\pi G_{eff}}{3} (\rho + 3p) + \frac{\Lambda}{3},
\]
where the cosmological constant $\Lambda$ equals to zero or $\Lambda = (\beta/t)(\dot{a}/a)$ [70], where $G_{eff} = G = const$. Then, the solutions of equations (28) are studied for several barotropic EoS $p = w \rho$ [70, 71].

As it noted in Ref. [48], using the canonical parameter transform $s = g_t(\tau)$ in equation (27), one can reduce it to the usual geodesic equation:
\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0,
\]
where the over dots stand for derivatives with respects to $s$. So that means that equations (28) could be obtained without applying FALVA but by replacing the parameter $s = g_t(\tau)$. The same remarks could be address to Ref.[72], in which the modified cosmological equations are obtained using the periodic weight function $g_t(\tau)$ in the generalized action (25) as follows:
\[
S_L[q_t] = \int_{t_0}^{t} L(\tau, q_t(\tau), \dot{q}_t(\tau)) \exp(-\chi \sin(\beta \tau))d\tau.
\]
The geodesic equation could be obtained by variation of (30) as

$$\ddot{x}^i - \beta \chi \cos(\beta \tau) \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0.$$  

The second term was again considered here as perturbation of the gravitational constant $\Delta G = \frac{3\chi \beta \cos(\beta \tau) H}{4\pi \rho}$. Nevertheless, the weight function in this case is defined by $\frac{dg_l(t)}{d\tau} = \exp(-\chi \sin(\beta t))$, and the transform $s = g_l(t)$ in (29) leads to the same equation.

Similarly to Refs. [71] - [72], the number of fractional action cosmological models are considered in articles [73] - [76], where the models are followed again from the fractional action applied to the classical Lagrangian of a curve defined by $L = (1/2)g_{ij} \dot{x}^i \dot{x}^j$, where $g_{ij}$ is the metric tensor. The modified Friedmann equations are used in the form:

$$H^2 + \frac{2(\alpha - 1)}{T_1} H + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \quad \dot{H} - \frac{(\alpha - 1)}{T_1} H - \frac{k}{a^2} = -4\pi G(\rho + p), \tag{31}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter, and $T_1 = t - \tau$. Using equations (31), a varying gravitational coupling constant, the model of dark energy in this paradigm and relevant cosmological parameters are obtained.

It is important to note that, strictly speaking, the action (30) and its generalizations in Refs. [73] - [76] are not the fractional ones contrary to the action functional (25) which is really a fractional integral (A.5). Therefore, using the concept of fractional calculus of variations (or the fractional action-like variational approach, FALVA), the fractional action cosmology should deal with fractional weight function as it has been proposed in [77] - [82]. Indeed, as it noted in Ref. [48], it would be more correct to apply FLAVA directly to the gravitational fractional action functional $S_G$ trying to modify the Friedmann equations in this way. Later on, all models on the basis of such modifications of the cosmological equations were called "Fractional Action Cosmology" or FAC. In FAC, the action integral $S_L[q]$ in (25) with Lagrangian density $L(t', q_i(t'), \dot{q}_i(t'))$ is represented as a fractional Riemann-Stieltjes integral: $S^\alpha_L[q_i] = \Gamma^{-1}(\alpha) \int_t^{t'} L(t', q_i(t'), \dot{q}_i(t'))(t - t')^{\alpha - 1} dt'$ with the integrating function $g_i(t') = \Gamma^{-1}(\alpha) |t' - t|^\alpha$. This approach realizes the space scaling concepts of Mandelbrot to define the scaling in fractional time as $d^\alpha t = \pi^{\alpha/2} \Gamma^{-1}(\alpha/2) |t|^{\alpha - 1} dt$, where $0 < \alpha < 1$.

### 3.2 FAC Models

A lot of the FAC models could be derived using a variational principle for the modified fractional Einstein-Hilbert action with a varying cosmological term $\Lambda$, that is $S^\alpha_{EH} = M^2_p \int \sqrt{-g} g_{\nu}(t) d^4x (R - 2\Lambda)/2$, where $M^2_p = 8\pi G$ is reduced Planck mass. First of all, one can suppose that the matter content of the universe is minimally coupled to gravity. Then, the total action of the system is $S^\alpha_{total} = S^\alpha_{EH} + S_m$. Here, the effective action for matter $S_m$ can be represented either by $S^\alpha_m = \int \mathcal{L}_m \sqrt{-g} g_{\nu}(t) d^4x$, which follows from the fractional matter action similar to (25), or by the usual expression for the matter action with standard measure $S_m = \int \mathcal{L}_m \sqrt{-g} d^4x$.

#### 3.2.1 FAC Models with a fractional matter action

The first option is realized in Refs. [48] - [84]. Using definition (25), the modified fractional effective action in a spatially flat Friedmann-Robertson-Walker interval [2], that is $ds^2 =$
\( N(t)^2 dt^2 - a^2(t) \delta_{ik} dx^i dx^k \), where \( N \) is the lapse function and \( a(t) \) is a scale factor, can be given by a fractional integral as [48]:

\[
S_{\text{eff}} = \frac{1}{\Gamma(\alpha)} \int_0^t N \left[ \frac{3}{8\pi G} \left( \frac{a^2 \ddot{a}}{N^2} + \frac{a \dot{a}^2}{N^2} - \frac{a \dot{a} \dot{N}}{3} - \frac{\Lambda a^3}{3} \right) + a^3 \mathcal{L}_m \right] (t - \tau)^{\alpha - 1} d\tau, \tag{32}
\]

where \( \alpha \in (0, 1) \), and \( \mathcal{L}_m \) is the Lagrangian density of matter represented by the energy density \( \rho \) and pressure \( p \). Where such a choice yields the following modified continuity equation,

\[
\dot{\rho} + 3 \left( H + \frac{1 - \alpha}{3t} \right) (\rho + p) = 0, \tag{33}
\]

and the set of the following modified Friedmann equations

\[
\begin{align*}
3H^2 + \frac{3(1 - \alpha)}{t} H &= \rho + \Lambda, \tag{34} \\
2\dot{H} + 3H^2 + \frac{2(1 - \alpha)}{t} H + \frac{(1 - \alpha)(2 - \alpha)}{t^2} &= -p + \Lambda, \tag{35}
\end{align*}
\]

where the gravitational constant \( 8\pi G = 1 \), and \( H(t) = \dot{a}/a \) is the Hubble parameter. In the standard FRW cosmology of GR, the continuity equation for a perfect fluid is the energy conservation law for matter which is followed from the Bianchi identity. Therefore, the continuity equation (33) could be derived from the field equations (34), (35). These equations also yield the modified continuity equation (32) in the case \( \alpha \neq 1 \), but only if the following equation is valid [84]:

\[
\dot{H} + 3H^2 - \frac{2(4 - \alpha)}{t} H - \frac{(1 - \alpha)(2 - \alpha)}{t^2} = \frac{t\dot{\Lambda}}{1 - \alpha}. \tag{36}
\]

This final form of the equation is obtained after dividing by \( (1 - \alpha) \neq 0 \). Therefore, in the limit of the standard cosmology of general relativity with \( \alpha = 1 \), the equation preceding the equation (36) uniquely leads to the constant cosmological term \( \Lambda \), and the set of equations (34), (35) becomes the usual Friedmann equations.

Several exact models have been obtained on the basis of main FAC equations (34)-(36) in Refs. [48, 84] under the assumption of a vacuum-like state of matter that fills the universe, or on the basis of a rather general ansatz for the dynamical cosmological term. In any case, the behavior of the models demonstrate a significant difference from the corresponding standard models. Obviously, this is followed from the fractional nature of the action functional.

From equations (34) and (35), the effective EoS \( w_{\text{eff}} = p_{\text{eff}}/\rho_{\text{eff}} \), where \( p_{\text{eff}} = p - \Lambda \) and \( \rho_{\text{eff}} = \rho + \Lambda \), has been proposed as

\[
w_{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} - \frac{1 - \alpha}{2(tH)} + \frac{(1 - \alpha)(2 - \alpha)}{2(tH)^2} \frac{1 - \alpha}{(tH)}, \tag{37}
\]

and coincides with the standard expression \( w_{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} \) in the limit \( \alpha \to 1 \).

The simplest example of exact solution to equations (34)–(36) has been found for the quasi-vacuum EoS of matter: \( w = -1 \). From equation (33), it follows that \( \rho(t) = \rho_0 = \frac{\dot{a}}{a} \).
constant and \( p = -\rho = -\rho_0 \) similarly to the standard GR cosmology. Then, equations (34)-(36) can be easily solved for the Hubble parameter,

\[
H = \frac{C_\alpha}{t} + H_0 t \frac{1 - \alpha}{2},
\]

where \( C_\alpha = \frac{(1 - \alpha)(2 - \alpha)}{3 - \alpha} \), \( H_0 \) is a positive constant of integration, and the scale factor of the universe:

\[
a = a_0 t^{C_\alpha} \exp \left( \frac{3 - \alpha}{2} H_0 t \frac{3 - \alpha}{2} \right).
\]

Moreover, a class of exact models is obtained using the solutions of equation (36) followed to the assumption that cosmological term \( \Lambda(t) \) is a known function of time. With the help of the following substitution

\[
x = \ln(t/t_0) \Leftrightarrow t = t_0 \exp(x); \quad Y(t) = tH(t),
\]

where \( t_0 > 0 \) is a constant, equation (36) can be rewritten as follows

\[
Y'(x) - (9 - 2\alpha)Y(x) + 3Y^2(x) - (1 - \alpha)(2 - \alpha) = \frac{t_0^2}{1 - \alpha} e^{2x} \Lambda'(x),
\]

where the prime denotes the derivative with respect to \( x \). Taking into account the structure of this equation, it could be assumed that there exists a class of solutions with the cosmological term satisfied the following equation:

\[
\Lambda'(x) = \frac{1 - \alpha}{t_0^2} e^{-2x} \left( k_1 Y'(x) + k_2 Y(x) + k_3 Y^2(x) + k_4 \right),
\]

where \( k_i \) are arbitrary constants. Finally, some examples of exact solutions with the phenomenological functions \( \Lambda(t) \), widely discussed in the literature (see, e.g. Refs. [88, 89]) are obtained using ansatz

\[
\dot{\Lambda} = (1 - \alpha) \left[ k_1 \frac{\dot{H}}{H} + (k_1 + k_2) \frac{H}{t^2} + k_3 \frac{H^2}{t} + k_4 \right].
\]

Several exact solutions for the FAC models were proposed in Ref. [81] for different values of \( k_i \).

### 3.2.2 FAC Models with a standard matter action

The case of the standard matter action is studied in Refs. [85]-[87]. Using the fractional variational procedure in a spatially flat FRW metric for the total action of

\[
S^{\alpha}_{EH} = -\frac{1}{8\pi G} \Gamma(\alpha) = 1 \text{ for the sake of simplicity.}
\]

However, the continuity equation can be obtained

\[
\dot{\rho} + 3H(\rho + p) = 0,
\]

where again \( 8\pi G \Gamma(\alpha) = 1 \).
expressing the standard energy conservation law for a perfect fluid. Therefore, it can be mentioned that the perturbed continuity equation is not a specific property of the FAC, while it almost always arises in many modifications of the theory of gravity. The main idea of Ref. [85] is to keep the usual form of the continuity equation within the FAC. As one can see, this aim was achieved using the concept of fractional order for the action functional only in relation to the gravitational sector. In addition, it was also proposed to write the system of basic equations in such a way that the effective $\Lambda$ - term could be considered as a kinematically induced (by the Hubble parameter) cosmological term. It was shown on a specific example that a model based on this proposal can lead to some fairly realistic modes of expansion of the Universe.

Using Eqs. (45) and (45), one can obtain the EoS of matter as follows:

$$w_m = \frac{p}{\rho} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} - \frac{1}{2(tH)^2} \frac{(1 - \alpha)(2 - \alpha)}{1 + \frac{1}{(tH)^2} - \frac{\Lambda}{3H^2}}.$$  \hspace{1cm} (47)$$

Moreover, it can be shown that these equations yield the continuity equation (46) in the case $\alpha \neq 1$, only if

$$\frac{d}{dt} \left( t^{\alpha-1} \Lambda \right) = \frac{3(1 - \alpha)}{t^{2-\alpha}} \left[ \dot{H} - \frac{2(2 - \alpha)}{t} H \right].$$  \hspace{1cm} (48)$$

This equation can be solved in quadratures as

$$\Lambda(t) = \Lambda_0 t^{1-\alpha} + 3(1 - \alpha) \left[ \frac{H(t)}{t} - (2 - \alpha) t^{1-\alpha} \int t^{\alpha-3} H(t) dt \right],$$ \hspace{1cm} (49)$$

where $\Lambda_0$ is a constant of integration. Substituting Eq. (49) into the model equations (45), (45), the following set of equations:

$$3\dot{H}^2 = t^{1-\alpha} \rho_{eff},$$ \hspace{1cm} (50)$$

$$2\dot{H} + 3\dot{H}^2 - \frac{1}{t} \frac{\dot{H}}{H} + \frac{(1 - \alpha)(2 - \alpha)}{t^2} = -t^{1-\alpha} p_{eff},$$ \hspace{1cm} (51)$$

were obtained. Here the effective energy density and pressure are represented by

$$\rho_{eff} = \rho + \Lambda_{eff}, \quad p_{eff} = p - \Lambda_{eff},$$ \hspace{1cm} (52)$$

where $\Lambda_{eff} = \Lambda_0 - 3(1 - \alpha)(2 - \alpha) \int t^{\alpha-3} H(t) dt$. The last equation supposes that the effective cosmological term consist of the cosmological constant $\Lambda_0$ and the induced cosmological term.

As shown in Ref. [85], a broad class of exact solutions to the equation (48) in the case $w_m \neq -1$ can be obtained using the following ansatz for the cosmological term $\Lambda(t)$:

$$\left[ e^{-(1 - \alpha)x} \Lambda(x) \right]' = \frac{3(1 - \alpha)}{t^2_0} e^{(\alpha - 3)x} \left[ c_1 Y'(x) + c_2 Y(x) + c_3 + F(x) \right],$$ \hspace{1cm} (53)$$

where $x = \ln(t/t_0), Y(t) = t H(t)$, $c_i$ and $t_0 > 0$ are constants, and $F(x)$ is an arbitrary smooth function. Applying (53) to Eq. (48), the following expression for the Hubble parameter was obtained:

$$H(t) = \frac{1}{t} \left[ \frac{H_0 L}{K} - \frac{M}{L} + \frac{t L/K}{F(t)t^{L/K} - 1} dt \right],$$ \hspace{1cm} (54)$$
where $H_0$ is an integration constant. The generic character of the generating function $F(t)$ provides a lot of opportunities in constructing the exact models. So, in the simplest case $F(t) \equiv 0$, the scale factor of the model follows hybrid law of evolution:

$$a(t) = a_0 t^{-M/L} \exp \left\{ H_0 t L / K \right\} \tag{55}$$

where $K = 1 - c_1$, $L = 5 - 2\alpha + c_2$, $M = c_3$. Using the same approach, a number of exact solutions were obtained in Ref. [86].

### 3.3 Testing FAC

As assumed in Ref. [87] in order to determine the evolution of FAC model, the following effective cosmological term widely discussed in the literature (see, e.g., Refs. [88, 89]) could be used

$$\Lambda_{eff} = \beta H^2. \tag{56}$$

where $\beta = \frac{3}{2} m H_0^{1-\alpha}$, $m, H_0$ are constants, and $\alpha \in (0, 1)$. In terms of the dimensionless cosmic time $\tau = H_0 t$, the following solution to the given FAC was obtained

$$H(\tau) = H_0 \left[ \frac{1 - \alpha}{m \tau^{2-\alpha}} + \frac{m - (1 - \alpha)}{m} \right], \tag{57}$$

Thus, in the far future, the Hubble parameter (56) tends the following value

$$H_\infty \equiv H(t \to \infty) = \left( \frac{m - 1 + \alpha}{m} \right) H_0,$$

so that $\Lambda_0 = \beta H_\infty^2$. From equation (56), the scale factor of this model can be found as

$$a(\tau) = a_0 \exp \left[ \left( 1 - \frac{1 - \alpha}{m} \right) \tau - \frac{1}{m \tau^{1-\alpha}} + \frac{2 - \alpha - m}{m} \right], \tag{58}$$

where $a_0 = a(\tau = 1)$. Because the red shift $z$ is defined as $1 + z = a_0 / a(\tau)$, one can obtain it from equation (58) as

$$z = \exp \left[ \frac{1}{m \tau^{1-\alpha}} - \left( 1 - \frac{1 - \alpha}{m} \right) \tau - \frac{2 - \alpha - m}{m} \right] - 1. \tag{59}$$

The deceleration parameter $q = -a^2 \ddot{a} / a^2 = -1 - \dot{H} / H^2$ in FAC is defined just as in the standard cosmology, since it is a cosmography parameter of the model, and can be represented as

$$q(\tau) = -1 + \frac{m(1 - \alpha)(2 - \alpha) \tau^{1-\alpha}}{[1 - \alpha + (m - 1 + \alpha) \tau^{2-\alpha}]^2}. \tag{60}$$

In the paper [87], this model was subjected primarily to theoretical diagnostics based on cosmographic parameters and the so-called $Om$ diagnostics [69]. This made it possible to analyze the behavior of the model for various values of its main parameters. An even more important result of the cited article lies in the numerical estimates of the model parameters obtained from observational data.

Since this model contains three independent parameters ($\alpha$, $m$ and $H_0$), the observational constraints on all these parameters can be done using 28 data points of $H(z)$ in the redshift range $0.07 \leq z \leq 2.3$ [90, 91]. The observational data consist of measurements of the Hubble parameter $H_{obs}(z_i)$ at redshifts $z_i$, with the corresponding one standard deviation uncertainties $\sigma_{H_{obs}}$ is well consistent with the observation result from Planck+WP [66] : $H_0 = 67.3 \pm 1.2 \, \text{km s}^{-1} \text{Mpc}^{-1}$. 
Then, using the best fit values of the main cosmographic parameters for the Union 2.1 SNIa data from Table II in Ref. [92],

\[ H_0 = 69.97^{+0.42}_{-0.41} \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad q_0 = -0.5422^{+0.0718}_{-0.026}, \quad r_0 = 0.5762^{+0.4478}_{-0.3528}. \] (61)

the following values can be obtained: \( \alpha \approx 0.926, \quad m \approx 0.174, \) with the same accuracy as the cosmographic parameters given by (61).

### 3.4 Some recent studies of FAC

In Refs. [93, 94], the FRW cosmology characterized by a scale factor obeying different independent types of fractional differential equations was studied, and both types of fractional operators: the Riemann-Liouville fractional integral and the Caputo fractional derivative were considered. The solutions for such models are given in terms of Mittag-Leffler and generalized Kilbas-Saigo-Mittag-Leffler functions.

In Ref. [95] the wormholes solutions based on FAC are studied, and the cosmic dynamics in the presence of wormhole in closed FRW universe are discussed. As it found, the cosmic acceleration with traversable wormhole may be realized without the need of exotic matter unless the scale factor of the universe obeys a power law dominated by a negative fractional parameter which is constrained from SNe Ia data.

Fractional action cosmology with variable order parameter was constructed in Ref. [96], where a large number of cosmological equations are obtained depending on the mathematical type of the fractional order parameter. This idea results on a number of cosmological scenarios and their dynamical consequences. It was observed that the used fractional cosmological formalism is able to create a large family of solutions and offers new features not found in the standard formalism and in many fundamental research papers.

The late-time evolution of a flat FRW model of the universe in the context of a non-minimal fractional cosmology characterized by fractional weight in time was studied in paper [97]. It was shown that due to the conformal coupling between the scalar field and gravity, a negative time-dependent quadratic potential and provided that the scale factor of the universe is related to the scalar field through a power-law ansatz, the universe is oscillating with time. Moreover, the oscillating behavior of the EoS parameter can be realized around -1 by crossing the phantom divide line an infinite number of times.

The short communication [98] proposed an original approach based on a generalized fractional integral operator which mixes the Riemann-Liouville and the Erdelyi-Kober integrals in one single operator usually “the Glaeske-Kilbas-Saigo fractional”. This generalized fractional integral is defined by

\[ aI^{\alpha,\beta,\gamma} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} e^{-\beta t} e^{-\gamma(x-t)} dt, \]

where \((\alpha, \beta, \gamma)\) are constants. A number of non-singular gravitational fields are obtained without using extra-dimensions, and some examples is provided to show that these gravitational fields hold many motivating features in space-time physics.

A significantly different approach to the application of fractional derivatives in cosmology from the above approach is presented in Ref. [99], where quantization is studied in terms of the fractional derivative of the cosmological coupling theory of the non-minimal derivative, namely, the Fab Four John theory. Its Hamiltonian version is the problem of fractional powers of momenta. Moreover, this problem is solved using the so-called conformable fractional derivative [100], which leads to the Wheeler-DeWitt equation of the
second order. It has been shown that a wide range of scale factor solutions are possible, including a bouncing solution.

The prospect of using such local fractional derivatives as Conformable Fractional Derivatives (CFD) in cosmology and astrophysics is confirmed by work [101], where the fractional equation of an isothermal gas sphere is written and solved. to conformable fractional isothermal gas spheres

In this study, the fractional form of isothermal Lane-Emden for the CFD isothermal gas sphere is considered using the power series method and obtain a recurrence relation for the power series coefficient. In order to evaluate the fractional parameter impact on the configuration of the stars, the physical parameters of the isothermal gas sphere are derived and determined for the neutron stars.

One more new approach to FAC is presented in the recent article [102], where a brief summary of fractional quantum mechanics is given in order to motivate towards fractional quantum cosmology. A model of stiff matter in a spatially flat homogeneous and isotropic universe is investigated and discussed. A new quantum cosmological solution, where fractional calculus implications are explicit, is presented and then contrasted with the corresponding standard quantum cosmology setting.

4 Conclusion

Summing up, we can conclude that in this short review we have presented the main results of studies on cosmological models with fractional derivatives aimed at attempts to explain the accelerated expansion of the universe outside the hypothesis of the existence of exotic types of matter. Indeed, there are not too many such attempts and the results of these attempts are not so significant, which is primarily due to the difficulties in justifying the admissibility of the use of fractional calculus in the cosmology and gravity. This is indeed a problem, despite some success and advances of fractional analysis in several fields of science such as engineering, biology and so on. Despite this, based on the achievements of the application of fractional analysis to the problems of gravity, certain really successful results in fractional cosmology have been obtained, presented in this review. It was noted that such studies have certain achievements and perspectives, possibly related to the new definition of fractional derivatives, such conformal fractional derivatives and some others. We hope that this review will be useful in discussing this problem and the results obtained will find their worthy application to subsequent cosmological studies.

A Fractional derivatives and integrals

Today there are more than two dozen definitions of the fractional derivative [54]. In physical and technical applications of fractional differential calculus, the Riemann-Liouville derivative (RLD), the Caputo derivative (CD) and some others are most applicable.

Such derivatives are defined by means of analytical continuation of the Cauchy formula for the multiple integral of integer order as a single integral with a power-law core into the field of real order $\mu > 0$:

$$cI_x^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_c^x f(t)(x-t)^{\mu-1}dt.$$  \hspace{1cm} (62)

The Riemann-Liouville derivative of fractional order $\alpha \geq 0$ of function $f(x)$ is defined as
the integer order derivative of the fractional-order integral (1):

\[ D_x^n f(x) \equiv D_x^n I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_c^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \]  

(63)

where \( D_x^n \equiv d^n/dx^n \), \( n = [\alpha] + 1 \). This definition corresponds to the so-called left derivative, frequently denoted as \( L_x f(x) \). For the limit \( \alpha = 1 \), this definition gives \( df(x)/dx \). For example, the left RLD of \( x^k \) for \( \alpha \leq 1, c = 0 \) equals:

\[ D_x^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}. \]  

(64)

For \( \alpha = 1 \), one has the usual result: \( D_x^1 x^k = kx^{k-1} \). The interesting feature of RLD is that RLD of non-zero constant \( C_0 \) does not equal zero, but for \( \alpha \leq 1 \) it equals \( D_x^\alpha C_0 = C_0 x^{-\alpha}/\Gamma(1-\alpha) \). The right RLD is defined similarly to (2) on the interval \([c, d]\):

\[ _rD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dx} \right)^n \int_x^d \frac{f(t)}{(t-x)^{\alpha-n+1}} dt \]  

(65)

It should be emphasized again that the Riemann-Liouville fractional integral of order \( \alpha \) is defined by

\[ I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt, \]  

(66)

and has a memory kernel.

One needs to be aware that according to the formulas of addition of orders, the following holds (see [5], p.161):

\[ D_x^\alpha D_x^\beta f(x) = D_x^{\alpha+\beta} f(x) - \sum_{j=1}^{\alpha} D_x^{\beta-j} f(c+) \frac{(x-c)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \]

that is \( D_x^\alpha D_x^\beta f(x) \neq D_x^{\alpha+\beta} f(x) \), if only not all derivatives \( D_x^{\beta-j} f(c+) \) at the beginning of the interval are equal to zero. That is why \( D_x^\alpha D_x^\beta f(x) \neq D_x^{2\alpha} f(x) \) in the general case. Generalizing the Laplace operator in the equation for Newtonian gravitational potential, the author of [47] wrongly doubles the order of the repeated fractional derivative. The authors of [103] have avoided this mistake, having written down the Laplacian \( \Delta^\alpha \) as:

\[ \Delta^\alpha u = \frac{1}{r^{2\alpha}} D_r^{2\alpha} (r^{2\alpha} D_r^\alpha u) + \frac{\Gamma^2(\alpha+1)}{r^{2\alpha} \sin^{2\alpha} \theta} \frac{\partial}{\partial \theta} (\sin^\alpha \theta \frac{\partial u}{\partial \theta}) + \frac{\Gamma^2(\alpha+1)}{r^{2\alpha} \sin^{2\alpha} \theta} \frac{\partial^2 u}{\partial \phi^2}. \]

One can note one more property of the fractional derivative expressed in modification of the Leibniz rule (see [51], p.162):

\[ D_x^\alpha [f(x)g(x)] = \sum_{k=0}^\infty \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} D_x^{\alpha-k} f(x) D_x^k g(x), \]  

(67)

which becomes the usual rule as \( \alpha = n \). It can be represented as the integral over the order of fractional derivative:

\[ D_x^\alpha [f(x)g(x)] = \int_{-\infty}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(\mu+1)\Gamma(\alpha+1-\mu)} D_x^{\alpha-\mu} f(x) D_x^\mu g(x) d\mu. \]
These rules of fractional differentiation can lead to an essential modification to the cosmological models with fractional derivatives.

However, there are other definitions of fractional derivatives that differ significantly from those given above. For example, the definitions of fractional derivatives as fractional powers of derivative operators are provided in [104]. For this end, the Taylor series and Fourier series are used to define fractional power of self adjoint derivative operator.

Recently, the authors of [100] have defined a new well-behaved simple fractional derivative called "conformable" fractional derivative depending just on the basic limit definition of the derivative.

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