The split property for locally covariant quantum field theories in curved spacetime

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January 13, 2015

Abstract

The split property expresses the way in which local regions of spacetime define subsystems of a quantum field theory. It is known to hold for general theories in Minkowski space under the hypothesis of nuclearity. Here, the split property is discussed for general locally covariant quantum field theories in arbitrary globally hyperbolic curved spacetimes, using a spacetime deformation argument to transport the split property from one spacetime to another. It is also shown how states obeying both the split and (partial) Reeh–Schlieder properties can be constructed, providing standard split inclusions of certain local von Neumann algebras. Sufficient conditions are given for the theory to admit such states in ultrastatic spacetimes, from which the general case follows. A number of consequences are described, including the existence of local generators for global gauge transformations, and the classification of certain local von Neumann algebras.

1 Introduction

In relativistic physics, one expects that spacelike separated local spacetime regions should constitute independent subsystems. The simplest expression of this in quantum field theory (QFT) is Einstein causality, which requires that observables localized in spacelike separated regions commute and are therefore commensurable. Algebraic quantum field theory [23] offers various strengthened criteria for statistical independence of observables at spacelike separation (see [31] [32] for reviews) of which the split property has turned out to be particularly deep and fruitful. For the most part the split property has been studied in Minkowski space, while in curved spacetime results have related to particular linear field theories [34] [15]. In this paper we establish the split property in general globally hyperbolic

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spacetimes, within the framework of locally covariant QFT \cite{8} and subject to additional conditions described below.

To set the scene, we briefly recall the definition of the split property in Minkowski space. In the algebraic framework \cite{23} one considers a net of $C^\ast$-algebras $A(O)$ indexed by open bounded regions of Minkowski space. These algebras share a common unit, and (among other axioms) are isotonous, i.e., $O_1 \subset O_2$ implies that $A(O_1) \subset A(O_2)$. Let $\omega$ be a state on the $C^\ast$-algebra $A$ generated by all the $A(O)$, thereby inducing a GNS representation $\pi$ on Hilbert space $\mathcal{H}$ with GNS vector $\Omega$. In this representation we may form local von Neumann algebras by taking double commutants, $\mathcal{R}(O) = \pi(A(O))''$. Clearly, whenever $O_1 \subset O_2$, there is an inclusion $\mathcal{R}(O_1) \subset \mathcal{R}(O_2)$ of von Neumann algebras; following \cite{17}, the inclusion is said to split if there is a type I von Neumann factor $\mathcal{N}$ such that $\mathcal{R}(O_1) \subset \mathcal{N} \subset \mathcal{R}(O_2)$. That is, $\mathcal{N}$ has trivial centre, and is isomorphic as a von Neumann algebra to the algebra of all bounded operators on some (not necessarily separable) Hilbert space \cite[Prop. 2.7.19]{8}. The state $\omega$ is said to have the split property if such inclusions split for all $O_1, O_2$.

The relationship with statistical independence arises as follows. Suppose the net of local algebras obeys Einstein causality, so that algebras of causally disjoint regions commute elementwise. If $O_1$ and $O_3$ are causally disjoint and a region $O_2$ exists so that the inclusion $\mathcal{R}(O_1) \subset \mathcal{R}(O_2)$ is split, then $\mathcal{R}(O_1)$ and $\mathcal{R}(O_3)$ enjoy a high degree of statistical independence: the algebra they generate is isomorphic to their $W^\ast$-tensor product, and thus any normal states $\varphi_1$ and $\varphi_3$ on $\mathcal{R}(O_1)$ and $\mathcal{R}(O_3)$ can be extended to a normal product state $\varphi$ obeying $\varphi(A_1 A_3) = \varphi_1(A_1) \varphi_3(A_3)$ for $A_i \in \mathcal{R}(O_i)$ ($i = 1, 3$).

Originally conjectured by Borchers, the split property was first proved for free fields by Buchholz \cite{9}. Subsequently, it was established for general models \cite{10} under suitable hypotheses of nuclearity, which controls the growth of the localized state space with energy. As the nuclearity criterion is closely linked to the thermodynamic properties of the theory \cite{14, 12}, it is expected to hold for many theories of physical interest. In particular, it is satisfied by free fields and even for countably many free fields provided that the spectrum of masses obeys suitable conditions \cite{14}.

Our approach to the split property in curved spacetimes is similar in spirit to Sanders’ work on the Reeh–Schlieder property \cite{29}: the existence of a state with the desired properties on the given spacetime is deduced by deforming to a spacetime on which such a state is known (or assumed) to exist. (In the Reeh–Schlieder case, the states obtained are not generally cyclic for all local algebras, and so what is proved is a partial Reeh–Schlieder property.) Indeed, for linear fields, related arguments appear in \cite{33} and are also used in the proof of the split property \cite{34, 15}. In these cases, the existence of states with the Reeh–Schlieder or split property was proved for ultrastatic spacetimes and used to deduce similar results in more general spacetimes. A novelty of our specific approach is that we rephrase the deformation arguments for the split and partial Reeh–Schlieder properties into a common language, allowing streamlined proofs running in close analogy. Indeed, we will give a combined result on states obeying both the split and partial Reeh–Schlieder properties, thus yielding standard split inclusions \cite{17}.

The paper is structured as follows: in section 2 we describe the relevant geometrical
background, in particular introducing the concept of a \textit{regular Cauchy pair}, and also recall the main ideas needed from locally covariant QFT \cite{8}. Section \(3\) contains our main results. In the Reeh–Schlieder case, this reproduces results from \cite{29}; the interest here is that the proof runs in close analogy to that of the split property, and that the split and Reeh–Schlieder properties can hold simultaneously. Section \(4\) describes sufficient conditions for the existence of states with the Reeh–Schlieder and split properties in connected ultrastatic spacetimes. As every connected spacetime in our category can be deformed to such a spacetime, this establishes conditions for our results to hold in generality. Nonetheless, our deformation arguments hold even for disconnected spacetimes and we give an example of a state over a disconnected spacetime with the (full) Reeh–Schlieder property.

By way of outlook, a number of applications of the split property are described in Section \(3\). These include the statistical independence at spacelike separation, the existence of local generators of global gauge transformations (established in the Minkowski space case in \cite{16}) and the identification of local algebras as the unique hyperfinite type \(III_1\) factor, up to a tensor product with an abelian algebra. However there are numerous additional directions that can be explored, and in general the split property brings a much more detailed set of tools to bear on the general analysis of QFT in curved spacetimes than has so far been available.

2 Preliminaries

2.1 The category \textit{Loc} and spacetime deformation

Locally covariant quantum field theory \cite{8} describes QFT on a category of globally hyperbolic spacetimes \textit{Loc}. Fixing a spacetime dimension \(n \geq 2\), objects of \textit{Loc} are quadruples \(M = (\mathcal{M}, g, o, t)\) where \(\mathcal{M}\) is a smooth paracompact orientable nonempty \(n\)-manifold with finitely many connected components, \(g\) is a smooth time-orientable metric of signature \(+−\ldots−\) on \(\mathcal{M}\), \(o\) and \(t\) are choices of orientation and time-orientation respectively\footnote{The orientation (resp., time-orientation) is conveniently represented as a choice of one of the connected components of the nowhere-zero smooth \(n\)-forms (resp., \(g\)-timelike 1-forms) on \(\mathcal{M}\).}, so that the spacetime \(M\) is globally hyperbolic. That is, \(M\) has no closed causal curves and the intersections \(J^+_M(p) \cap J^-_M(q)\) of the causal future of \(p\) with the causal past of \(q\) is compact (including the possibility of being empty) for any pair of points \(p, q \in \mathcal{M}\). A morphism between two objects \(M = (\mathcal{M}, g, o, t)\) and \(M' = (\mathcal{M}', g', o', t')\) of \textit{Loc} is any smooth embedding \(\psi : \mathcal{M} \to \mathcal{M}'\) that is isometric, preserves the (time)orientation (i.e., \(\psi^*g' = g, \psi^*o' = o, \psi^*t' = t\)) and has a causally convex image. If the image contains a Cauchy surface of \(\mathcal{N}\), \(\psi\) will be described as a \textit{Cauchy morphism}.

We will often consider open causally convex subsets of \(\mathcal{M}\) with finitely many mutually causally disjoint components; the set of all such sets will be denoted \(\mathcal{O}(\mathcal{M})\). Suppose \(M = (\mathcal{M}, g, o, t) \in \textit{Loc}\), and that \(O \in \mathcal{O}(\mathcal{M})\) is nonempty. Then \(M|_O := (O, g|_O, o|_O, t|_O)\), i.e., \(O\) regarded as a spacetime in its own right with the induced metric and causal structures
from $M$, is an object of $\text{Loc}$, and the inclusion map $O \hookrightarrow M$ induces a morphism $\iota_{M,O} : M|_O \rightarrow M$.

There is a useful canonical form for objects of $\text{Loc}$. Objects of the form $(\mathbb{R} \times \Sigma, g, t \wedge w, t)$ where (a) $(\Sigma, w)$ is an oriented $(n-1)$-manifold, (b) $\partial / \partial t$ is future-directed according to $t$, where $t$ is the coordinate corresponding to the first factor of the Cartesian product $\mathbb{R} \times \Sigma$, and (c) the metric splits as

$$g = \beta dt \otimes dt - h_t,$$

where $\beta \in C^\infty(\mathbb{R} \times \Sigma)$ is strictly positive and $t \mapsto h_t$ is a smooth choice of (smooth) Riemannian metrics on $\Sigma$, are said to be in standard form. Every leaf $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface of the spacetime. The structure theorem for $\text{Loc}$ [19, §2.1] is:

**Proposition 2.1.** Supposing that $M \in \text{Loc}$, let $\Sigma$ be a smooth spacelike Cauchy surface of $M$ with induced orientation $w$, and let $t_* \in \mathbb{R}$. Then there is a $\text{Loc}$-object $M_{st} = (\mathbb{R} \times \Sigma, g, t \wedge w, t)$ in standard form and an isomorphism $\rho : M_{st} \rightarrow M$ in $\text{Loc}$ such that each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface of $M_{st}$, and $\rho(t_*, \cdot)$ is the inclusion of $\Sigma$ in $M$.

Here, the induced orientation $w$ of the Cauchy surface $\Sigma$ in $M = (M, g, o, t)$ is the unique orientation such that $o|_{\Sigma} = t|_{\Sigma} \wedge w$. Proposition 2.1 is a slight elaboration of results due to Bernal and Sánchez (see particularly, [3] Thm 1.2] and [2] Thm 2.4), which were previously long-standing folk-theorems.

Methods for deforming one globally hyperbolic spacetime into another go back to the work of Fulling, Narcowich and Wald [21], in which the existence of Hadamard states on ultrastatic spacetimes was used to deduce their existence on general globally hyperbolic spacetimes. As first recognized in [36], the same idea can be used to great effect in locally covariant QFT. The fundamental spacetime deformation result can be formulated as follows (see [19] Prop. 2.4).

**Proposition 2.2.** Two spacetimes $M, N$ in $\text{Loc}$ have oriented-diffeomorphic Cauchy surfaces if and only if there exists a chain of Cauchy morphisms in $\text{Loc}$ forming a diagram

$$M \xleftarrow{\iota} P \xrightarrow{\rho} I \xleftarrow{\iota} F \xrightarrow{\delta} N.$$

**Proof.** For later use, we sketch some details needed in the forward implication; see [19], Prop. 2.4 for the full proof. Assume without loss that $M$ and $N$ are in standard form with $M = (\mathbb{R} \times \Sigma, g_1, o, t_1)$ and $N = (\mathbb{R} \times \Sigma, g_2, o, t_2)$, where $o = t_1 \wedge w = t_2 \wedge w$ for some orientation $w$ of $\Sigma$.

Given any reals $t_1 < t'_1 < t'_2 < t_2$, one may construct a metric $g$ of the form (1), so that

- $g = g_1$ on $P = (-\infty, t_1) \times \Sigma$ and $g = g_2$ on $F = (t_2, \infty) \times \Sigma$
- on $(-\infty, t'_2) \times \Sigma$ every $g$-timelike vector is $g_1$-timelike
- on $(t'_1, \infty) \times \Sigma$ every $g$-timelike vector is $g_2$-timelike.

The idea for constructing such a metric is described in [21]; the argument is simplified and given in more detail in [19] Prop. 2.4. Choosing $t$ so that $\partial / \partial t$ is future-directed, the spacetime $I := (\mathbb{R} \times \Sigma, g, o, t)$ is globally hyperbolic, because every inextendible $g$-timelike
2.2 Regular Cauchy pairs

We will be interested in some particular subsets of Cauchy surfaces defined as follows.

**Definition 2.3.** Let \( M \in \text{Loc} \). A regular Cauchy pair \((S,T)\) in \( M \) is an ordered pair of subsets of \( M \), that are nonempty, open, relatively compact subsets of a common smooth spacelike Cauchy surface of \( M \) in which \( T \) has nonempty complement, and so that \( S \subset T \).

There is a preorder on regular Cauchy pairs so that \((S_1,T_1) \prec (S_2,T_2)\) if and only if \( S_2 \subset D_M(S_1) \) and \( T_1 \subset D_M(T_2) \).

These conditions ensure that \( D_M(S) \) and \( D_M(T) \) are open and casually convex, and hence elements of \( \mathcal{O}(M) \). Here, for any subset \( U \) of \( M \), \( D_M(U) \) denotes the Cauchy development, consisting of all points \( p \) in \( M \) with the property that all inextendible piecewise-smooth causal curves through \( p \) intersect \( U \). The preorder \( \prec \) is illustrated in Figure 1.

Two properties of regular Cauchy pairs will be needed.

**Lemma 2.4.** Let \( \psi : M \to N \) be a Cauchy morphism. Then a pair of subsets \((S,T)\) of \( M \) is a regular Cauchy pair if and only if \((\psi(S),\psi(T))\) is a regular Cauchy pair for \( N \).

**Proof.** This holds because the image of a Cauchy surface under a Cauchy morphism is again a Cauchy surface [19, Lem. A.2], and similar arguments show that the same is true for pre-images.

**Lemma 2.5.** Suppose that \( M \) takes standard form with underlying manifold \( \mathbb{R} \times \Sigma \), and that \((S,T)\) is a regular Cauchy pair in \( M \), lying in the surface \( \{t\} \times \Sigma \). Then there exists an \( \epsilon > 0 \) such that every Cauchy surface \( \{t'\} \times \Sigma \) with \( |t'-t| < \epsilon \) contains a regular Cauchy pair preceding \((S,T)\) with respect to \( \prec \) and also a regular Cauchy pair preceded by \((S,T)\).

\[\text{2 The preorder is not a partial order, because } (S_1,T_1) \prec (S_2,T_2) \prec (S_1,T_1) \text{ implies } D_M(S_1) = D_M(S_2) \text{ and } D_M(T_1) = D_M(T_2), \text{ but not necessarily } S_1 = S_2 \text{ and } T_1 = T_2.\]
Proof. It is clearly enough to treat the case \( t = 0, \ t' > 0 \), and we abuse notation by regarding \( S \) and \( T \) as subsets of \( \Sigma \). We assume without loss that \( \Sigma \) is connected (in the general case, the following construction is repeated for each component).

For each \( \tau \in \mathbb{R} \), the optical metric \([22]\) is defined by \( k_\tau = \beta^{-1} h_\tau \), thus forming a smoothly varying family of smooth Riemannian metrics on \( \Sigma \). We will regard \( \Sigma \) as a metric space with respect to the \( k \)-metric. Choose any \( \delta > 0 \) such that (a) \( S \) contains a closed ball of radius \( 2\delta \) about some \( p \in S \), (b) \( T \) contains the closed \( 3\delta \)-ball about \( S \) and (c) there exists a point \( q \in \Sigma \setminus T \) of distance at least \( 2\delta \) from \( T \), and the open \( \delta \)-ball about \( T \) is relatively compact\(^3\) and necessarily has nonempty exterior. Then define \( S_{\text{inner}} \) to be the open \( \delta \)-ball about \( p \), \( S_{\text{outer}} \) to be the open \( \delta \)-ball about \( S \), \( T_{\text{inner}} \) to be the open \( \delta \)-ball about \( S_{\text{inner}} \), and \( T_{\text{outer}} \) to be the open \( \delta \)-ball about \( T \). As \( T_{\text{outer}} \) is relatively compact, there is a constant \( K > 0 \) such that \( k_{0,\sigma}(u,u) \leq K k_{t,\sigma}(u,u) \) for all \( u \in T_{\sigma} \Sigma \), \( (\tau,\sigma) \in [0,\delta] \times T_{\text{outer}} \).

We set \( \epsilon = \min \{ \delta, \delta/\sqrt{K} \} \) and choose any \( t' \in (0, \epsilon) \).

Now consider any smooth inextendible \( M \)-causal curve \( \gamma \), parameterized so that \( \gamma(\tau) = (\tau, \sigma(\tau)) \ (\tau \in \mathbb{R}) \), where \( \sigma \) is smooth and obeys \( k_\tau(\dot{\sigma}(\tau), \dot{\sigma}(\tau)) \leq 1 \) for all \( \tau \in \mathbb{R} \). If \( \sigma(0) \in T \), then \( \tau_* := \inf \{ \tau > 0: \sigma(\tau) \notin T_{\text{outer}} \} \) is strictly positive or \( +\infty \). We claim that \( \tau_* \geq \epsilon \): if not, we have \( k_0(\dot{\sigma}(\tau), \dot{\sigma}(\tau)) \leq K k_{\tau}(\dot{\sigma}(\tau), \dot{\sigma}(\tau)) \leq K \) for \( \tau \in [0, \tau_*] \) and

\[
\text{dist}(\sigma(0), \sigma(\tau_*)) \leq \sqrt{K} \tau_* < \sqrt{K} \epsilon \leq \delta,
\]

and hence \( \sigma(\tau_*) \in T_{\text{outer}} \), a contradiction. By similar reasoning, \( \text{dist}(\sigma(0), \sigma(\epsilon)) < \delta \).

Accordingly, if \( \sigma(0) \in S \) (resp., \( T \)) then \( \sigma(t') \in S_{\text{outer}} \) (resp., \( T_{\text{outer}} \)). Similarly, if \( \sigma(t') \in T \), it follows that \( \text{dist}(\sigma(0), \sigma(t')) < \delta \). Hence, if \( \sigma(t') \in S_{\text{inner}} \) (resp., \( T_{\text{inner}} \)) then \( \sigma(0) \in S \) (resp., \( T \)). These relationships establish

\[
\{t' \times S_{\text{outer}}, \{t' \times T_{\text{inner}} \} \prec \{0 \times S, \{0 \times T \} \prec \{t' \times S_{\text{inner}}, \{t' \times T_{\text{outer}} \}
\]

as required.\[\square\]

Remark 2.6. It follows immediately that, if finitely many regular Cauchy pairs \( (S_j, T_j) \) \( (1 \leq j \leq N) \) are specified in the Cauchy surface \( \{t\} \times \Sigma \), then every Cauchy surface \( \{t'\} \times \Sigma \) with \( t' \) sufficiently close to \( t \) contains, for each \( j \), a regular Cauchy pair preceding \( (S_j, T_j) \) and one that is preceded by it.

### 2.3 Locally covariant quantum field theory

The basic premise of locally covariant QFT \([8]\) is that a theory is given by a functor \( \mathcal{A} : \text{Loc} \to \text{C}^*\text{-Alg} \), where \text{C}^*\text{-Alg} is the category of unital \(*\)-algebras and injective unit-preserving \(*\)-homomorphisms\(^4\). This means that each spacetime \( M \) corresponds to a \text{C}^*-algebra \( \mathcal{A}(M) \), and that every morphism \( \psi : M \to N \) between spacetimes has a

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\(^3\)By the ‘\( \delta \)-ball about \( U \)’ we mean the open set of all points \( r \) with \( \text{dist}(r, U) < \delta \). The existence of a relatively compact \( \delta \)-ball about \( T \) follows from the existence of a compact exhaustion of \( \Sigma \) \([26]\) Prop. 4.76].

\(^4\)Other target categories are possible and frequently employed, for example the category \text{Alg} of unital \(*\)-algebras with injective unit-preserving \(*\)-homomorphisms.
corresponding C*-Alg-morphism \( \mathcal{A}(\psi) : A(M) \to A(N) \), subject to the requirement that \( \mathcal{A}(\text{id}_M) = \text{id}_{A(M)} \) and \( \mathcal{A}(\psi \circ \varphi) = \mathcal{A}(\psi) \circ \mathcal{A}(\varphi) \).

Given such a functor, a net of local algebras may be defined in each spacetime \( M \in \text{Loc} \) by setting \( A^{\text{kin}}(M; O) \) to be the image of the map \( A(\iota_{M; O}) \) for each nonempty \( O \in \mathcal{O}(M) \).

As described in [8], these local algebras obey suitable generalizations of the assumptions in the Araki–Haag–Kastler framework [23]. In particular, they are isotonous: if \( O_1 \subset O_2 \) then \( A^{\text{kin}}(M; O_1) \subset A^{\text{kin}}(M; O_2) \).

The additional assumptions we will use are that the theory is 

\textit{Einstein causal:} if \( O_1, O_2 \in \mathcal{O}(M) \) are causally disjoint (in the sense that no causal curve connects them), then \( A^{\text{kin}}(M; O_1) \) and \( A^{\text{kin}}(M; O_2) \) commute, and that the theory has the \textit{timeslice property:} if \( \psi : M \to N \) is Cauchy, then \( \mathcal{A}(\psi) \) is an isomorphism.

Definition 2.7. A locally covariant QFT is a functor \( \mathcal{A} : \text{Loc} \to C^*\text{-Alg} \) obeying Einstein causality and having the timeslice property.

The utility of the deformation result Proposition 2.2 arises because any chain of Cauchy morphisms such as (2) induces, by the timeslice property, an isomorphism

\[ \mathcal{A}(\delta) \circ \mathcal{A}(\gamma)^{-1} \circ \mathcal{A}(\beta) \circ \mathcal{A}(\alpha)^{-1} : A(M) \to A(N). \] \hspace{1cm} (5)

Although such isomorphisms are not canonical, owing to the many choices used in the construction, they often permit the transfer of properties and structures between the instantiations of the theory on \( M \) and \( N \).

The description just given encodes the algebraic aspects of the theory. To incorporate states as well, we first define a category \( C^*\text{-AlgSts} \) as follows. Objects of \( C^*\text{-AlgSts} \) are pairs \((A, S)\), where \( A \in C^*\text{-Alg} \) and \( S \) is a state space for \( A \), i.e., a convex subset of the set of all states on \( A \), that is closed under operations induced by \( A \).\hspace{1cm} (5)

A morphism in \( C^*\text{-AlgSts} \) between \((A, S)\) and \((B, T)\) is induced by any \( C^*\text{-Alg}\)-morphism \( \alpha : A \to B \) such that \( \alpha^* T \subset S \); as a slight abuse of notation we will often denote the \( C^*\text{-AlgSts}\)-morphism in the same way as its underlying \( C^*\text{-Alg} \) morphism.

A state space for a locally covariant QFT \( \mathcal{A} : \text{Loc} \to C^*\text{-Alg} \) is an assignment of state space \( S(M) \) to each \( A(M) \) \((M \in \text{Loc})\) so that \( X(M) = (A(M), S(M)) \) defines a functor \( X : \text{Loc} \to C^*\text{-AlgSts} \) for which each \( X(\psi) \) has underlying \( C^*\text{-Alg}\)-morphism \( A(\psi) \). We say that \( X \) obeys the timeslice axiom if \( X(\psi) \) is an isomorphism in \( C^*\text{-AlgSts} \) for all Cauchy morphisms \( \psi : M \to N \), which means that \( A(\psi)^* S(N) = S(M) \) (of course, \( A(\psi) \) is also an isomorphism because \( A \) obeys Definition 2.7). In this case \( X \) will be described as a \textit{locally covariant QFT with states}.

\footnote{That is, if \( \omega \in S \) and \( B \in A \) with \( \omega(B^*B) > 0 \), then the state \( \omega_B(A) := \omega(B^*AB)/\omega(B^*B) \) is also an element of \( S \).}
3 Main Results

3.1 The split property

The split property is defined as follows\(^6\)

**Definition 3.1.** Let \( \mathcal{A} : \text{Loc} \to \text{C*-Alg} \) be a locally covariant QFT and \( M \in \text{Loc} \). A state \( \omega \) on \( \mathcal{A}(M) \) is said to have the split property for a regular Cauchy pair \((S, T)\) if, in the GNS representation \((\mathcal{H}, \pi, \Omega)\) of \( \mathcal{A}(M) \) induced by \( \omega \), there is a type-I factor \( \mathfrak{N} \) such that

\[
\pi(A^{\text{kin}}(M; D_M(S)))'' \subset \mathfrak{N} \subset \pi(A^{\text{kin}}(M; D_M(T)))''.
\]

**Remark 3.2.** If \( \omega \) has the split property for \((S, T)\) then it does for every \((\tilde{S}, \tilde{T})\) with \((S, T) \prec (\tilde{S}, \tilde{T})\): for \( S \subset D_M(S) \) implies \( D_M(\tilde{S}) \subset D_M(S) \) and hence by isotony

\[
\pi(A^{\text{kin}}(M; D_M(\tilde{S})))'' \subset \pi(A^{\text{kin}}(M; D_M(S)))'' \subset \pi(A^{\text{kin}}(M; D_M(T)))''.
\]

Moreover, if \( O_i \in \mathcal{O}(M) \) obey \( O_1 \subset D_M(S), D_M(T) \subset O_2 \), then there is a split inclusion

\[
\pi(A^{\text{kin}}(M; O_1))'' \subset \mathfrak{N} \subset \pi(A^{\text{kin}}(M; O_2))''.
\]

by the same argument.

**Lemma 3.3.** Suppose \( \psi : M \to N \) is a Cauchy morphism and let \( \mathcal{A} \) be a locally covariant QFT. A state \( \omega_N \) on \( \mathcal{A}(N) \) has the split property for a regular Cauchy pair \((\psi(S), \psi(T))\) if and only if \( \mathcal{A}(\psi)^*\omega_N \) has the split property for \((S, T)\). (As \( \mathcal{A}(\psi) \) is an isomorphism, this implies that \( \omega_M \) is split for \((S, T)\) if and only if \( (\mathcal{A}(\psi)^{-1})^*\omega_M \) is split for \((\psi(S), \psi(T))\).)

**Proof.** Let \( \omega_M = \mathcal{A}(\psi)^*\omega_N \) and write \((\mathcal{H}_{\omega}, \pi_{\omega}, \Omega_{\omega})\), where \( \ast = M \) or \( N \), for the corresponding GNS representations. As \( \mathcal{A}(\psi) \) is an isomorphism there is a unitary \( U : \mathcal{H}_{\omega_M} \to \mathcal{H}_{\omega_N} \) so that \( U\Omega_{\omega_M} = \Omega_{\omega_N} \) and

\[
U\pi_{\omega_M}(A) = \pi_{\omega_N}(\mathcal{A}(\psi)A)U, \quad (A \in \mathcal{A}(M)).
\]

Consequently, \( \pi_{\omega_N}(A^{\text{kin}}(N; \psi(O)))'' = U\pi_{\omega_M}(A^{\text{kin}}(M; O))''U^{-1} \) for any nonempty \( O \in \mathcal{O}(M) \), and as \( U\mathfrak{N}U^{-1} \) is a type-I factor if and only if \( \mathfrak{N} \), the result follows.

We now present our first deformation result on the split property.

**Theorem 3.4.** Suppose \( \mathcal{A} \) is a locally covariant QFT. Let \( M, N \in \text{Loc} \) have oriented-diffeomorphic Cauchy surfaces and suppose \( \omega_N \) is a state on \( \mathcal{A}(N) \) that has the split property for all regular Cauchy pairs in \( N \). Given any regular Cauchy pair \((S_M, T_M)\) in \( M \), there is a chain of Cauchy morphisms between \( M \) and \( N \) inducing an isomorphism \( \nu : \mathcal{A}(M) \to \mathcal{A}(N) \) such that \( \nu^*\omega_N \) has the split property for \((S_M, T_M)\). Consequently (by Remark 3.2) if \( O_i \in \mathcal{O}(M) \) are such that \( O_1 \subset D_M(S_M), D_M(T_M) \subset O_2 \), then there is a split inclusion of the form (8) in the GNS representation of \( \nu^*\omega_N \).

\(^6\)This definition directly generalizes that used in Minkowski space, but differs from the condition studied in [7] and discussed briefly at the end of this section.
Proof. Assume without loss of generality (by Proposition 2.1 and Lemma 2.4) that \( \mathbf{M} \) is in standard form \( \mathbf{M} = (\mathbb{R} \times \Sigma, g_{\mathbf{M}}, \sigma, t_{\mathbf{M}}) \) and that \( S_{\mathbf{M}} \) and \( T_{\mathbf{M}} \) lie in the Cauchy surface \( \{ t_M \} \times \Sigma \) for some \( t_M \in \mathbb{R} \). By Lemma 2.5 there exists \( t_s > t_M \) and a regular Cauchy pair \( (S_s, T_s) \) in \( \{ t_s \} \times \Sigma \) such that \( (S_s, T_s) \prec (S_{\mathbf{M}}, T_{\mathbf{M}}) \), where \( \prec \) indicates the preorder with respect to the causal structure of \( \mathbf{M} \).

Now we may also assume without loss of generality that \( \mathbf{N} \) is also in standard form \( \mathbf{N} = (\mathbb{R} \times \Sigma, g_{\mathbf{N}}, \sigma, t_{\mathbf{N}}) \). As \( (S_s, T_s) \) is also a regular Cauchy pair for \( \mathbf{N} \), there exists \( t_N > t_s \) and a regular Cauchy pair \( (S_N, T_N) \) in \( \{ t_N \} \times \Sigma \) such that \( (S_N, T_N) \prec (S_s, T_s) \).

We now construct a metric \( \gamma \) using Prop. 2.2 choosing the values \( t_1, t_2, t_2, t_3 \) so that \( t_M < t_1 < t_2 < t_3 < t_2 < t_N \), and thus creating an interpolating globally hyperbolic spacetime \( \mathbf{I} \) and a chain of Cauchy morphisms \( \mathbf{I} \). The key point is that \( (S_N, T_N) \prec (S_s, T_s) \) and \( (S_s, T_s) \prec (S_M, T_M) \) and hence \( (S_N, T_N) \prec (S_M, T_M) \). To see this, consider any inextendible \( g \)-timelike curve \( \gamma \) through \( S_M \). In the region \( t \leq t_s \) this is also an \( g_{\mathbf{M}} \)-timelike curve and intersects \( S_s \), because \( S_M \subset D_M(S_s) \). Thus \( S_M \subset D_I(S_s) \). Similarly, if \( \gamma \) is an inextendible \( g \)-timelike curve through \( T_s \), it is \( g_{\mathbf{M}} \)-timelike in \( \mathbf{M} \) and intersects \( T_M \), so \( S_s \subset D_I(S_M) \). This shows that \( (S_s, T_s) \prec (S_M, T_M) \); one proves \( (S_N, T_N) \prec (S_s, T_s) \) in the same way.

As \( \omega_N \) has the split property for \( (S_N, T_N) \) in \( \mathbf{N} \), it follows (applying Lemma 3.3 twice) that \( (A(\delta) \circ A(\gamma)^{-1})^* \omega_N \) has the split property for \( (S_N, T_N) \), as a regular Cauchy pair in \( \mathbf{I} \), and hence for \( (S_M, T_M) \), again as a regular Cauchy pair in \( \mathbf{I} \), because \( (S_N, T_N) \prec (S_M, T_M) \). Two further applications of Lemma 3.3 show that \( (A(\beta) \circ A(\alpha)^{-1})^*(A(\delta) \circ A(\gamma)^{-1})^* \omega_N = \nu^* \omega_M \) has the split property for \( (S_M, T_M) \) in \( \mathbf{M} \).

Remark 3.5. The result may be extended as follows. Suppose finitely many regular Cauchy pairs \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) (\( 1 \leq j \leq N \)), lying in a common Cauchy surface of \( \mathbf{M} \) are given. Owing to Remark 2.6 the values \( t_s \) and \( t_N \) in the proof above may be chosen so that there are Cauchy pairs \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) and \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) (\( 1 \leq j \leq N \)) lying in the hypersurfaces \( \{ t_s \} \times \Sigma \) and \( \{ t_N \} \times \Sigma \) respectively so that \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \prec (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \prec (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \prec (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) for each \( 1 \leq j \leq N \) and hence \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \prec (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) for a common interpolating metric. Then the state \( \nu^* \omega_N \) has the split property for each of the pairs \( (S_{\mathbf{M}}^{(j)}, T_{\mathbf{M}}^{(j)}) \) (\( 1 \leq j \leq N \)).

For theories with states \( \mathcal{X} = (\mathcal{A}, \mathcal{S}) : \text{Loc} \to \text{C}^*\text{-AlgSts} \), we may say a little more. First, if the state \( \omega_N \) in the hypotheses of Theorem 3.4 belongs to the state space \( \mathcal{S}(\mathbf{N}) \), then the induced state obeys \( \nu^* \omega_N = \omega_N \), as a result of the timeslice property for \( \mathcal{X} \) and the fact that \( \nu \) arises from a chain of Cauchy morphisms. Much more follows if each \( \mathcal{S}(\mathbf{M}) \) consists of mutually locally quasi-equivalent states on \( \mathcal{A}(\mathbf{M}) \), in which case we describe \( \mathcal{X} \) as obeying local quasi-equivalence. This condition requires that for every spacetime \( \mathbf{M} \), relatively compact \( O \subset \mathbf{M} \) and states \( \omega_i \in \mathcal{S}(\mathbf{M}) \) (\( i = 1, 2 \)), the GNS representations \( (\mathcal{H}_{\omega_1}, \pi_{\omega_1}, \Omega_i) \) restrict to quasi-equivalent representations of \( \mathcal{A}^{\text{kin}}(\mathbf{M} ; O) \), i.e., there is an isomorphism of von Neumann algebras \( \beta : \pi_{\omega_1}(\mathcal{A}^{\text{kin}}(\mathbf{M} ; O))'' \to \pi_{\omega_2}(\mathcal{A}^{\text{kin}}(\mathbf{M} ; O))'' \) such that \( \beta \circ \pi_{\omega_2}(A) = \pi_{\omega_1}(A) \) for all \( A \in \mathcal{A}^{\text{kin}}(\mathbf{M} ; O) \).\footnote{An equivalent definition of quasi-equivalence is that the sets of states on \( \mathcal{A}^{\text{kin}}(\mathbf{M} ; O) \) induced by density matrices on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) coincide [6, Thm 2.4.26].}
state space is provided by the Hadamard states on the Weyl algebra of the Klein–Gordon field [35]. We have:

**Lemma 3.6.** If state \( \omega_1 \) has the split property for regular Cauchy pair \((S, T)\) in \( M \) and \( \omega_2 \) is locally quasi-equivalent to \( \omega_1 \), then \( \omega_2 \) also has the split property for \((S, T)\).

**Proof.** Let \( \mathfrak{N} \) be the type I factor obeying (8) and let \( \beta : \pi_{\omega_1}(A^{\text{kin}}(M; D_M(T)))'' \to \pi_{\omega_2}(A^{\text{kin}}(M; D_M(T)))'' \) be the isomorphism induced by local quasi-equivalence, obeying \( \beta \circ \pi_{\omega_1} = \pi_{\omega_2} \) on \( A^{\text{kin}}(M; D_M(T)) \). In particular, \( \beta \) restricts to an isomorphism of \( \pi_{\omega_1}(A^{\text{kin}}(M; D_M(S)))'' \to \pi_{\omega_2}(A^{\text{kin}}(M; D_M(S)))'' \). Then \( \beta(\mathfrak{N}) \) is a type I factor, and clearly obeys \( \pi_{\omega_2}(A^{\text{kin}}(M; D_M(S)))'' \subset \beta(\mathfrak{N}) \subset \pi_{\omega_2}(A^{\text{kin}}(M; D_M(T)))'' \).

As an immediate consequence (just as was argued for the Klein–Gordon theory in [34]):

**Theorem 3.7.** Suppose \( X = (\mathcal{A}, \mathcal{S}) : \text{Loc} \to \text{C}^*\text{-AlgSts} \) is a locally covariant QFT with states obeying local quasi-equivalence. Let \( M, N \in \text{Loc} \) have oriented-diffeomorphic Cauchy surfaces and suppose \( \omega_N \in \mathcal{S}(N) \) has the split property for all regular Cauchy pairs in \( N \). Then every state \( \omega_M \in \mathcal{S}(M) \) obeys the split property for all regular Cauchy pairs in \( M \). Consequently, if \( O_1 \subset \mathcal{O}(M) \) are such that \( O_1 \subset D_M(S), D_M(T) \subset O_2 \), for a regular Cauchy pair \((S, T)\) in \( M \), then there is a split inclusion of the form (8) in the GNS representation induced by any state of \( \mathcal{S}(M) \).

**Proof.** For each regular Cauchy pair \((S_M, T_M)\) of \( M \), Theorem 3.4 shows the existence of some state in \( \mathcal{S}(M) \) having the split property for \((S_M, T_M)\), and hence by Lemma 3.6 and local quasi-equivalence of \( X \), the same is true for all states of \( \mathcal{S}(M) \).

### 3.2 Partial Reeh–Schlieder results

As already mentioned, our result on the split property was inspired by Sanders’ partial analogue of the Reeh–Schlieder theorem [29]. The original Reeh–Schlieder theorem [28] establishes that the Minkowski vacuum vector is cyclic for all local algebras, and consequently separating for all local algebras for regions with nonempty causal complement. The results of [29] demonstrate the existence of states with partial Reeh–Schlieder properties: given a spacetime region in \( M \), one may find (suitably regular) states that are cyclic for the corresponding local algebra, on the assumption that \( M \) can be deformed to a spacetime that admits a (suitably regular) state enjoying the full Reeh–Schlieder property of being cyclic for all local algebras.

The introduction of regular Cauchy pairs allows for a streamlined proof of Sanders’ result, which we give for completeness. More significantly, we combine this proof with that of our result on the split property to demonstrate the existence of states obeying both the split and Reeh–Schlieder properties, which give so-called standard split inclusions [17].

The properties we will consider are given as follows. Terminology differs from [29].

**Definition 3.8.** Let \( \mathcal{A} : \text{Loc} \to \text{C}^*\text{-Alg} \) be a locally covariant QFT and \( M \in \text{Loc} \). A state \( \omega \) on \( \mathcal{A}(M) \) is said to have the Reeh–Schlieder property for a regular Cauchy pair \((S, T)\)
if, in the GNS representation \((\mathcal{H}, \pi, \Omega)\) of \(\mathcal{A}(M)\) induced by \(\omega\), the GNS vector \(\Omega\) is cyclic for \(\pi(\mathcal{A}^{\text{kin}}(M; D_M(S)))^\prime\) and separating for \(\pi(\mathcal{A}^{\text{kin}}(M; D_M(T)))^\prime\). For brevity, we will sometimes say that \(\omega\) is Reeh–Schlieder for \((S, T)\). If \(O \in \mathcal{O}(M)\) and \(\Omega\) is both cyclic and separating for \(\pi(\mathcal{A}^{\text{kin}}(M; O))^\prime\), we will say that \(\omega\) has the Reeh–Schlieder property for \(O\).

Note that we regard the separation condition as part of the Reeh–Schlieder property, which turns out to expedite the proofs below. See Corollary 3.13 for a formulation involving only cyclicity as a hypothesis.

**Remark 3.9.** If a vector is separating for an algebra, it is separating for any subalgebra thereof; if it is cyclic for an algebra, it is cyclic for any algebra of which it is a subalgebra. Thus it is clear that if \(\omega\) has the Reeh–Schlieder property for \((S, T)\) then it does for every \((\tilde{S}, \tilde{T})\) with \((\tilde{S}, \tilde{T}) \prec (S, T)\). Moreover, if \(O \in \mathcal{O}(M)\) is such that \(D_M(S) \subset O \subset D_M(T)\), then the GNS vector of \(\omega\) is both cyclic and separating for \(\pi(\mathcal{A}^{\text{kin}}(M; O))^\prime\), i.e., \(\omega\) is Reeh–Schlieder for \(O\). Note that the separating property is defined at the level of the represented algebras. If \(\omega\) induces a faithful GNS representation, we would have the stronger property that \(\omega(A^\ast A) = 0\) for \(A \in \mathcal{A}^{\text{kin}}(M; O)\) implies \(A = 0\).

**Lemma 3.10.** Let \(A\) be a locally covariant QFT. Let \((S, T)\) be a regular Cauchy pair in \(M \in \text{Loc}\) and suppose \(\psi : M \to N\) is Cauchy. A state \(\omega_N\) on \(\mathcal{A}(N)\) is Reeh–Schlieder for a regular Cauchy pair \((\psi(S), \psi(T))\) if and only if \(\mathcal{A}(\psi)^\ast \omega_N\) is Reeh–Schlieder for \((S, T)\). (As \(\mathcal{A}(\psi)\) is an isomorphism, this implies that \(\omega_M\) is Reeh–Schlieder for \((S, T)\) if and only if \((\mathcal{A}(\psi)^{-1})^\ast \omega_M\) is Reeh–Schlieder for \((\psi(S), \psi(T))\).)

**Proof.** As in the proof of Lemma 3.3 we set \(\omega_M = \mathcal{A}(\psi)^\ast \omega_N\), and infer the existence of a unitary \(U : \mathcal{H}_{\omega_M} \to \mathcal{H}_{\omega_N}\) so that \(U \Omega_{\omega_M} = \Omega_{\omega_N}\) and \(\pi_{\omega_N}(\mathcal{A}^{\text{kin}}(N; \psi(O))^\prime) = U \pi_{\omega_M}(\mathcal{A}^{\text{kin}}(M; O))^\prime U^{-1}\) for \(O \in \mathcal{O}(M)\). Consequently, \(\omega_{\omega_N}\) is cyclic (resp., separating) for \(\pi_{\omega_N}(\mathcal{A}^{\text{kin}}(N; \psi(O))^\prime)\) if and only if \(\Omega_{\omega_M}\) is cyclic (resp., separating) for \(\pi_{\omega_M}(\mathcal{A}^{\text{kin}}(M; O))^\prime\). 

An analogue of Theorem 3.4 now gives a partial Reeh–Schlieder result.

**Theorem 3.11.** Suppose \(A\) is a locally covariant QFT. Let \(M, N \in \text{Loc}\) have oriented-diffeomorphic Cauchy surfaces and suppose \(\omega_N\) is a state on \(\mathcal{A}(N)\) that is Reeh–Schlieder for all regular Cauchy pairs. Given any regular Cauchy pair \((S_M, T_M)\) in \(M\), there is a chain of Cauchy morphisms between \(M\) and \(N\) inducing an isomorphism \(v : \mathcal{A}(M) \to \mathcal{A}(N)\) such that \(v^\ast \omega_N\) has the Reeh–Schlieder property for \((S_M, T_M)\). Consequently, if \(O \in \mathcal{O}(M)\) is relatively compact with nontrivial causal complement, there is a state (formed in the same way) on \(\mathcal{A}(M)\) with the Reeh–Schlieder property for \(O\).

**Proof.** The first part of the argument is identical to that of Theorem 3.4 except that we replace \(\prec\) by \(\succ\), and ‘split’ by ‘Reeh–Schlieder’ on every occasion, and use Lemma 3.10 and Remark 3.9 in place of Lemma 3.3 and Remark 3.2. For the last part, choose any

\[\text{Sanders [29]}\] uses this term for cyclicity alone.

\[\text{Note the reversal of order relative to Remark 3.2.}\]
smooth spacelike Cauchy surface $\Sigma$ intersecting $O$; then there certainly exist open relatively compact subsets $S$ and $T$ of $\Sigma$ so that $(S, T)$ is a regular Cauchy pair with $D_M(S) \subset O \subset D_M(T)$, and we apply the first part of the result along with Remark 3.9.

**Remark 3.12.** For exactly the same reason as in Remark 3.5, Theorem 3.11 may be extended to yield a state that has the Reeh–Schlieder property simultaneously for finitely many regular Cauchy pairs specified in a common Cauchy surface of $M$.

The following result reproduces the main statement of [29, Thm 4.1].

**Corollary 3.13.** Let $A$ be a locally covariant QFT and assume the geometric hypotheses of Theorem 3.11. Suppose that $\omega_N$ has the property that its GNS vector is cyclic for each $\pi_{\omega_N}(A_{\text{kin}}(N; O)^{\prime\prime})$ indexed by a relatively compact $O \in \mathcal{O}(N)$ with nontrivial causal complement. Then the conclusions of Theorem 3.11 hold.

**Proof.** We need only prove that $\omega_N$ is Reeh–Schlieder for all regular Cauchy pairs $(S_N, T_N)$ of $N$. By hypothesis, the GNS vector $\Omega_{\omega_N}$ is cyclic for $\pi_{\omega_N}(A_{\text{kin}}(N; D_N(S_N)^{\prime\prime})^{\prime\prime}$, so we need only prove that it is separating for $\pi_{\omega_N}(A_{\text{kin}}(N; D_N(T_N)^{\prime\prime})^{\prime\prime}$. Choose any relatively compact $O \in \mathcal{O}(N)$ contained in the causal complement of $T_N$ (so $O$ itself also has non-trivial causal complement), whereupon $\Omega_{\omega_N}$ is cyclic for $\pi_{\omega_N}(A_{\text{kin}}(N; O)^{\prime\prime}$, and hence separating for (any subalgebra of) $\pi_{\omega_N}(A_{\text{kin}}(N; O)^{\prime}$. By Einstein causality, this includes $\pi_{\omega_N}(A_{\text{kin}}(N; D_N(T_N)))$ and its weak closure.

For a theory with states $\mathcal{X} : \text{Loc} \rightarrow C^*-\text{AlgSts}$, we may argue further that the state $\nu^*\omega_N$ belongs to $S(M)$. If one assumes that each $S(M)$ is a full local-equivalence class then further conclusions on the existence of states that are Reeh–Schlieder for arbitrary globally hyperbolic regions of $M$ may be obtained – see [29], which also discusses various applications of these results.

We have emphasized that the proofs of Theorems 3.4 and 3.11 run in close analogy. Indeed, they may be combined.

**Theorem 3.14.** Assume the hypotheses of Theorem 3.4. If, in addition, $\omega_N$ is Reeh–Schlieder for all regular Cauchy pairs in $N$, then the state $\nu^*\omega_N$ has both the Reeh–Schlieder and split properties for $(S_M, T_M)$.

**Proof.** We combine the proofs of Theorems 3.4 and 3.11. The value $t_*$ may be chosen so that $\{t_*\} \times \Sigma$ contains regular Cauchy pairs $(S, T)$ and $(S, T_*)$ with

$$\{t_*\} \times \Sigma$$

while $t_N > t_*$ may be chosen so that $\{t_N\} \times \Sigma$ contains regular Cauchy pairs $(S_N, T_N)$ and $(S_N, T_*)$ such that

$$(S_N, T_*) \prec_N (S_N, T_N), \quad (S_*, T) \prec_N (S_N, T_N).$$

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Constructing the interpolating metric as in the proof of Theorem 3.4, the orderings (10) and (11) hold with \( \prec_M \) and \( \prec_N \) replaced by \( \prec_I \), and we may deduce
\[
(N_S, N_T) \prec_I (S_M, T_M) \prec_I (S_N, T_N).
\]

Now \( \omega_N \) has the Reeh–Schlieder property for \( (S_N, T_N) \) and is split for \( (N_S, N_T) \) in \( N \), and hence the same is true in \( I \) for \( (A(\delta) \circ A(\gamma)^{-1})^*\omega_N \). By (12) and Remarks 3.2 and 3.9, \( (A(\delta) \circ A(\gamma)^{-1})^*\omega_N \) is both Reeh–Schlieder and split for \( (S_M, T_M) \), again as a regular Cauchy pair in \( I \). Hence \( \nu^*\omega_M \) is both Reeh–Schlieder and split for \( (S_M, T_M) \) in \( M \).

Remark 3.15. This result also extends to the case of finitely many Cauchy pairs in a common Cauchy surface.

3.3 Standard split inclusions and applications

In the situation of Theorem 3.14, let \( \tilde{S} \) be an open subset of the Cauchy surface containing \( S \) and \( T \) such that \( \tilde{S} \subset T \setminus S \), whereupon \( (\tilde{S}, T) \) is a regular Cauchy pair lying in a common Cauchy surface with \( (S, T) \). Applying Remark 3.15 the construction of \( \nu \) may be arranged so that \( \omega = \nu^*\omega_N \) has the Reeh–Schlieder and split properties for both \( (S, T) \) and \( (\tilde{S}, T) \).

Writing \( (\mathcal{H}, \pi, \Omega) \) for the corresponding GNS representation, we define
\[
\mathcal{R}_U = \pi(\mathcal{A}^\text{kin}(M; D_M(U)))''
\]
where \( U \) is any of \( S, \tilde{S}, T \). So far, we have \( \mathcal{R}_S \subset \mathcal{R} \subset \mathcal{R}_T \) and that \( \Omega \) is cyclic for \( \mathcal{R}_S \) (hence also for \( \mathcal{R} \) and \( \mathcal{R}_T \)). Moreover \( \Omega \) is cyclic for \( \mathcal{R}_{\tilde{S}} \), and therefore also for \( \mathcal{R}_T \cap \mathcal{R}'_{\tilde{S}} \) (using Einstein causality and causal disjointness of \( S \) and \( \tilde{S} \)). Accordingly \( \Omega \) is separating for \( \mathcal{R}_S \) (because it is cyclic for a subalgebra of \( \mathcal{R}'_{\tilde{S}} \) as well as \( \mathcal{R}_T \) and its subalgebra \( \mathcal{R}_T \cap \mathcal{R}'_{\tilde{S}} \). In summary, the inclusion \( \mathcal{R}_S \subset \mathcal{R}_T \) is split, and \( \Omega \) is cyclic and separating for each of \( \mathcal{R}_S, \mathcal{R}_T \) and \( \mathcal{R}_T \cap \mathcal{R}_{\tilde{S}} \). In the terminology of [17], the triple \( (\mathcal{R}_S, \mathcal{R}_T, \Omega) \) is, therefore, a standard split inclusion.

Excluding a trivial situation in which \( \mathcal{R}_T = \mathbb{C}1 \) (which can only arise if the GNS space \( \mathcal{H} \) is one-dimensional) it follows that both \( \mathcal{R}_S \) and \( \mathcal{R}_T \) are properly infinite von Neumann algebras with separable preduals, and the Hilbert space \( \mathcal{H} \) is infinite-dimensional and separable [17, Prop. 1.6].

There is also a unitary \( W : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) with the properties
\[
W A B' W^{-1} = A \otimes B' \quad (A \in \mathcal{R}_S, \ B' \in \mathcal{R}'_T)
\]
\[
W \mathcal{R}_S W^{-1} = \mathcal{R}_S \otimes 1_{\mathcal{H}}
\]
\[
W \mathcal{R}_T W^{-1} = 1_{\mathcal{H}} \otimes \mathcal{R}'_T
\]
\[
W \mathcal{R}'_T W^{-1} = \mathcal{B}(\mathcal{H}) \otimes \mathcal{R}_T
\]
\[
\text{To bring out the main point: } \Omega \text{ is a faithful normal state on } \mathcal{R}, \text{ which is therefore countably decomposable, and hence (by virtue of being a type I factor) isomorphic to } \mathcal{B}(\mathcal{K}) \text{ where } \mathcal{K} \text{ has countable dimension [24, 7.6.46]. That is, } \mathcal{R} \text{ is of type } I_\infty. \text{ As } \Omega \text{ is cyclic for } \mathcal{R}, \text{ separability of } \mathcal{H} \text{ follows.}
and we may take $\mathcal{N}$ to be the ‘canonical type I’ factor
\[ \mathcal{N} = W^{-1} (\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{H}}) W. \] (15)

It is conventional to denote the split inclusion by $\Lambda = (\mathcal{R}_S, \mathcal{R}_T, \Omega)$.

As is well-known, various consequences follow from this situation (see, e.g., [23, §V.5]). We give some representative applications.

**Statistical independence**  The algebras $\mathcal{R}_S$ and $\mathcal{R}_T$ are statistically independent in the $W^*$-sense\(^{11}\) because any pair of normal states $\varphi_S$ and $\varphi_T$ on these algebras with respective density matrices $\rho_S$ and $\rho_T$ induces a normal state $\varphi$ with density matrix $\rho = W^{-1} \rho_S \otimes \rho_T W$ so that
\[ \varphi(AB') = \text{Tr} \rho AB' = \text{Tr} ((\rho_S A) \otimes (\rho_T B')) = \varphi_S(A)\varphi_T(B') \] (16)
for $A \in \mathcal{R}_S, B' \in \mathcal{R}_T'$.

**Strictly localized states**  States of the form $\Psi = W^{-1} \psi \otimes \Omega (\psi \in \mathcal{H}, \|\psi\| = 1)$ may be regarded as states strictly localized in $D_M(T)$, because
\[ \langle \psi | B' \Psi \rangle = \langle \psi \otimes \Omega | (1_{\mathcal{H}} \otimes B')\psi \otimes \Omega \rangle = \langle \Omega | B'\Omega \rangle \] (17)
for all $B' \in \mathcal{R}_T'$.

**Local implementation of gauge symmetries**  In locally covariant QFT, the global gauge group of a theory $\mathcal{A}$ may be identified with the automorphism group $\text{Aut}(\mathcal{A})$, the group of natural isomorphisms of $\mathcal{A}$ with itself \(^{12}\).

Suppose that the state $\omega_N$ is gauge invariant in the sense that $\omega_N \circ \zeta_N = \omega_N$ for all $\zeta \in \text{Aut}(\mathcal{A})$, where $\zeta_N$ is the component of natural transformation $\zeta$ in spacetime $\mathcal{N}$. Then $\omega = \nu^{\dagger} \omega_N$ is gauge invariant, $\omega \circ \zeta_M = \omega_M$ for all $\zeta \in \text{Aut}(\mathcal{A})$, by naturality and the definition of $\nu$, so the GNS representation carries a unitary implementation $\zeta \mapsto U(\zeta)$ of the gauge group $\text{Aut}(\mathcal{A})$ under which $\Omega$ is fixed. Then we may define
\[ U_\Lambda(\zeta) = W^{-1} (U(\zeta) \otimes 1_{\mathcal{H}}) W, \] (18)
which provides a second representation of $\text{Aut}(\mathcal{A})$, implemented by unitaries belonging to $\mathcal{N} \subset \mathcal{R}_T$, with
\[ U_\Lambda(\zeta)AB'U_\Lambda(\zeta)^{-1} = W^{-1} \left(U(\zeta)AU(\zeta)^{-1} \otimes B'\right) W = U(\zeta)AU(\zeta)^{-1}B' \] (19)
for $A \in \mathcal{R}_S, B' \in \mathcal{R}_T'$. In other words, $U_\Lambda$ is a local representation of the gauge group on $\mathcal{R}_S$, leaving the commutant of $\mathcal{R}_T$ fixed. The representation is strongly continuous (with respect to a given topology on $\text{Aut}(\mathcal{A})$) if and only if $U$ is, and this construction produces local generators for the gauge group and thus a local current algebra \(^{16}\). In principle this discussion could be developed further to incorporate geometric symmetries of the Cauchy surface (cf. \(^{11}\)) by modifying the construction of the interpolating spacetimes to ensure that the isometry is preserved throughout, and starting from an invariant state on $\mathcal{N}$.

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\(^{11}\)See \[31\] for discussion of the relation between $C^*$- and $W^*$-senses of statistical independence.
Suppose $T$ can be approximated from within by subsets $S_k \subset T$ so that each $(S_k, S_{k+1})$ is a regular Cauchy pair, $T = \bigcup_{k \in \mathbb{N}} S_k$ and

$$\mathcal{R}_T = \bigvee_{k \in \mathbb{N}} \mathcal{R}_{S_k}. \quad (20)$$

(This inner continuity would be expected if, for example, the von Neumann algebras are generated by a system of fields, cf. [10]; alternatively, it might be imposed as an additivity assumption.) Then because each inclusion $\mathcal{R}_{S_k} \subset \mathcal{R}_{S_{k+1}}$ is split there is an increasing family of type I factors $\mathcal{N}_k$ so that

$$\mathcal{R}_T = \bigvee_{k \in \mathbb{N}} \mathcal{N}_k \quad (21)$$

and as $\mathcal{H}$ is separable, $\mathcal{R}_T$ is seen to be hyperfinite. If, in addition, the factors appearing in the central decomposition of $\mathcal{R}_T$ are known to be of type $\text{III}_1$, as would happen given a suitable scaling limit [1, Thm 16.2.18] (based on [20]) then $\mathcal{R}_T$ is isomorphic to the unique hyperfinite $\text{III}_1$ factor [24] (up to a tensor product with the centre of $\mathcal{R}_T$).

Now consider the situation of a theory with states $\mathcal{X} = (\mathcal{A}, \mathcal{S}) : \text{Loc} \rightarrow \text{C}^*\text{-AlgSts}$ obeying local quasi-equivalence, and so that $\omega_N \in \mathcal{S}(M)$. Then the state $\omega$ discussed above lies in $\mathcal{S}(M)$ and the GNS representation $\mathcal{H}, \tilde{\pi}, \tilde{\Omega}$ of any state $\tilde{\omega} \in \mathcal{S}(M)$ restricts to quasi-equivalent representations of $\mathcal{A}^{\text{kin}}(M; \mathcal{D}_M(S))$ and $\mathcal{A}^{\text{kin}}(M; \mathcal{D}_M(T))$. As already mentioned, the corresponding von Neumann algebras $\mathcal{R}_S, \mathcal{R}_T$ are split, though the GNS vector is not necessarily a standard vector. However, some elements of the discussion above hold true as a result of the quasi-equivalence: for instance, the Hilbert space $\tilde{\mathcal{H}}$ is separable (cf. the proof of [3, Thm 2.4.26]) and $\mathcal{R}_T$ contains unitaries implementing the global gauge group on $\mathcal{R}_S$, and leaving $\mathcal{R}'_T$ fixed. Of course, the type of the local von Neumann algebras is preserved, because they are isomorphic.

Further applications of the split property to the issue of independence of local algebras can be found in [31, 32]. A weaker condition than the split property, namely intermediate factoriality, is studied in [33], where various consequences are derived. The interpretative framework for quantum systems described by funnels of type $I_\infty$ factors has recently been addressed in [13]. Finally, we comment on the version of the split property used in [7] in a discussion of the tensorial structure of locally covariant QFTs. This differs from ours in that the type I von Neumann factor is required to lie between the $C^*$-algebras $\mathcal{A}^{\text{kin}}(M; O)$ and $\mathcal{A}(M)$ for connected $O \in \mathcal{O}(M)$ with compact closure, rather than between the von Neumann algebras of nested relatively compact regions in suitable representations. An additional continuity requirement is also imposed in [7]. While it seems likely that one could at least partly address this version of the split property with our deformation argument, we will not do this here. Alternatively one could investigate whether the results of [7] hold under the version of the split property established here.

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12Terminology in these references differs in some respects from ours, which follows [17]; refs. 31 32 refer to a pair $(\mathcal{R}_1, \mathcal{R}_2)$ as split if $\mathcal{R}_1 \subset \mathcal{R} \subset \mathcal{R}'_2$ for some type I factor $\mathcal{R}$. 

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4 Ultrastatic spacetimes

In this section we comment briefly on sufficient conditions for a locally covariant QFT to admit a state obeying both the split and (full) Reeh–Schlieder properties on the class of connected ultrastatic spacetimes, i.e., those spacetimes \( \mathcal{N} \in \text{Loc} \) in standard form \( \mathcal{N} = (\mathbb{R} \times \Sigma, dt \otimes dt - h, \sigma, t) \) where \( h \) is a fixed Riemannian metric on \( \Sigma \), which is assumed connected. As every connected spacetime \( \mathcal{M} \in \text{Loc} \) has Cauchy surfaces oriented-diffeomorphic to those of an ultrastatic spacetime (indeed, by virtue of [27] one may even demand that the metric \( h \) is complete), such conditions would enable the results of Section 3 to apply nontrivially to any connected \( \mathcal{M} \in \text{Loc} \).

Let \( \mathcal{N} \) be connected and ultrastatic, as defined above. Then \( \mathcal{N} \) admits a one-parameter group of time translations \( T_\tau : (t, \sigma) \mapsto (t+\tau, \sigma) \) and hence automorphisms \( \mathcal{A}(T_\tau) \) of \( \mathcal{A}(\mathcal{M}) \). Our first assumption is that \( \mathcal{A}(\mathcal{N}) \) admits a faithful ground state \( \omega_\mathcal{N} \) for the time translations \( \mathcal{A}(T_\tau) \). That is, (a) \( \omega_\mathcal{N} \) is a time-translationally invariant state, \( \mathcal{A}(T_\tau)^* \omega_\mathcal{N} = \omega_\mathcal{N} \) for all \( \tau \in \mathbb{R} \), and (b) the unitary implementation \( U(\tau) \) of \( \mathcal{A}(T_\tau) \) in the GNS representation \( (\mathcal{H}_{\omega_\mathcal{N}}, \pi_{\omega_\mathcal{N}}, \Omega_{\omega_\mathcal{N}}) \) induced by \( \omega_\mathcal{N} \), which obeys \( U(\tau) \pi_{\omega_\mathcal{N}}(A) U(\tau)^{-1} = \pi_{\omega_\mathcal{N}}(A(T_\tau)A) \) and \( U(\tau) \Omega_{\omega_\mathcal{N}} = \Omega_{\omega_\mathcal{N}} \), has a positive generator, i.e., \( U(\tau) = e^{iH_\tau} \) with positive self-adjoint operator \( H \). In the case of a theory with states \( (\mathcal{A}, \mathcal{S}) \), one would also assume that \( \omega_\mathcal{N} \in \mathcal{S}(\mathcal{N}) \). (If \( \zeta \in \text{Aut}(\mathcal{A}) \) is a global gauge transformation, we have \( \zeta_\mathcal{N} \circ \mathcal{A}(T_\tau) = \mathcal{A}(T_\tau) \circ \zeta_\mathcal{N} \) by naturality, and as \( \zeta_\mathcal{N} \) is an isomorphism, \( \zeta_\mathcal{N}^* \omega_\mathcal{N} \in \mathcal{S}(\mathcal{N}) \) is also a ground state. Hence, if there is a unique ground state in \( \mathcal{S}(\mathcal{N}) \), it is automatically gauge invariant.)

The second assumption is needed for the Reeh–Schlieder property. Defining the local von Neumann algebras \( \mathcal{R}(O) := \pi_{\omega_\mathcal{N}}(\mathcal{A}_{\text{kin}}(\mathcal{N}; O))^\prime \) for nonempty \( O \in \mathcal{O}(\mathcal{N}) \), we assume the weak timelike tube criterion

\[
\left( \bigcup_{\tau \in \mathbb{R}} \mathcal{R}(T_\tau O) \right)^{''} = \mathcal{R}(\mathcal{N})
\]  

(22)

holds for any nonempty \( O \in \mathcal{O}(\mathcal{N}) \) (the right-hand side is of course \( \mathcal{B}(\mathcal{H}_{\omega_\mathcal{N}}) \) in the case that \( \omega_\mathcal{N} \) is pure).\(^{13}\) This condition was established by Borchers in general Wightman theories in Minkowski space \(^{14}\) and (in suitable representations) for linear fields in stationary spacetimes by Strohmaier \(^{30}\).\(^{14}\) Given this condition, it then holds immediately that \( \Omega \) is cyclic for every \( \pi_{\omega_\mathcal{N}}(\mathcal{A}_{\text{kin}}(\mathcal{N}; O)) \) with nonempty \( O \in \mathcal{O}(\mathcal{N}) \) and so satisfies the hypotheses of Corollary \(^{3, 13}\) See, e.g., Borchers’ version \(^{3, \text{Thm 1}}\) of the Reeh–Schlieder theorem \(^{28}\). It seems reasonable that the timelike tube criterion holds on connected ultrastatic spacetimes for general theories of interest.

For the split property, we assume additionally that \( \Omega_{\omega_\mathcal{N}} \) obeys a suitable nuclearity criterion. Let \( O \in \mathcal{O}(\mathcal{N}) \) be nonempty and denote \( \mathcal{R}(O) := \pi_{\omega_\mathcal{N}}(\mathcal{A}_{\text{kin}}(\mathcal{N}; O))^{''} \). We say that \( \omega_\mathcal{N} \) obeys the nuclearity criterion for \( O \) if the maps \( \Xi_\beta : \mathcal{R}(O) \rightarrow \mathcal{H}_{\omega_\mathcal{N}} \) given for \( \beta > 0 \) by \( \Xi_\beta(A) = e^{-\beta H} A \Omega_{\omega_\mathcal{N}} \) are nuclear. That is, for each \( \beta \) there is a countable

\(^{13}\)E.g., this condition is fulfilled if the \( \mathcal{A}_{\text{kin}}(\mathcal{N}; T_\tau O) \) (\( \tau \in \mathbb{R} \)) generate a dense subspace of \( \mathcal{A}(\mathcal{N}) \).

\(^{14}\)An alternative proof of the Reeh–Schlieder theorem on ultrastatic spacetimes, based on antilocality of fractional powers of the Laplace operator, is given in \(^{33}\).
decomposition $Ξ_β(\cdot) = \sum_i \psi_i \varphi_i(\cdot)$ for vectors $\psi_i \in \mathcal{H}_{\omega_N}$ and bounded linear functionals $\varphi_i$ on $\mathcal{H}(O)$ such that $\sum_i \|\psi_i\| \|\varphi_i\|$ is finite, whereupon we write $\|Ξ_\beta\|_1$ for the infimum of this sum over all possible decompositions – a quantity called the 
\textit{nuclearity index}. Using [11 \text{ Prop. 17.1.4}] (which is abstracted from [10]), one easily sees that if $(S, T)$ is a regular Cauchy pair in $\mathcal{N}$ and $\omega_N$ obeys nuclearity for $D_N(T)$ with the corresponding nuclearity index obeying $\|Ξ_\beta\|_1 \leq e^{(\beta_0/\beta)^n}$ for some fixed $n > 0$, $\beta_0 > 0$ and all $\beta \in (0, 1)$, then $\omega_N$ has the split property for $(S, T)$ with $\Omega_{\omega_N}$ as a cyclic and separating vector. In the Minkowski space theory, nuclearity conditions of this type are closely related to good thermodynamic properties such as the existence of KMS states [12], so again there is good reason to believe that they should hold for theories of interest. In ultrastatic spacetimes, nuclearity was established for the Klein–Gordon field in [34] and for Dirac fields in [15].

In summary, there is good reason to believe that physically well-behaved locally covariant theories should admit states satisfying the Reeh–Schlieder and split properties in connected ultrastatic spacetimes, and hence that the results of Section 3 apply nontrivially to yield states with the split and partial Reeh–Schlieder properties in general connected globally hyperbolic spacetimes.

The question of whether Reeh–Schlieder and split states can be expected in general disconnected ultrastatic spacetimes would seem to require more detailed information concerning $\mathcal{A}$. Our deformation arguments work equally well for disconnected spacetimes, however, and one can certainly find states on disconnected spacetimes that are sufficiently entangled across the various components that they have the Reeh–Schlieder property. For example, suppose $\omega_M$ has the full Reeh–Schlieder property on a connected spacetime $M$, and let $O \in \mathcal{O}(M)$ have multiple components. Then the restriction of $\omega_M$ to $\mathcal{A}(M|O)$ has the full Reeh–Schlieder property on this disconnected spacetime. In this situation the ‘behind the moon’ aspect of the Reeh–Schlieder property is brought into sharp relief: the moon need not even be in the same spacetime component as the experimenter!
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