Approach to Solving Quasiclassical Equations with Gauge Invariance

Priya Sharma

Received: 1 September 2019 / Accepted: 30 June 2020 / Published online: 22 July 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
Quasiclassical equations with manifest gauge invariance are discussed in the context of unconventional singlet superconducting states in the static limit. Deviations of the quasiclassical propagator from its equilibrium solutions in the presence of magnetic fields and Hall terms are analysed in terms of a “small” parameter and a formulation developed to first order in “small”. A modified quasiclassical propagator is defined to this order that is a solution of a new gauge-invariant Eilenberger-like equation with a normalization condition. A Riccati parametrization with manifest gauge invariance is proposed. Riccati equations are derived to leading order in “small” that are directly applicable to superconducting systems in the presence of magnetic fields.

Keywords Quasiclassical theory · Gauge invariance · Superconductivity

1 Introduction
The quasiclassical theory [1] is established as a powerful method to describe superconducting systems in the presence of external perturbations that vary slowly on the scale of the Fermi energy and Fermi wavelength. The central object of the theory is the quasiclassical Green’s function for quasiparticles travelling with Fermi velocity, \( v_F \) along ballistic trajectories defined by their direction \( \hat{v}_F \). This propagator, represented as a Nambu matrix, is given by solutions of the Eilenberger equation [2], along with a normalization condition that picks the physical solution. In the presence of magnetic fields, additional forces on the quasiparticles such as the Lorentz force appear as driving terms in the quasiclassical equation and have been included in a gauge-invariant formulation by Kita [3]. The augmented Eilenberger equations have been applied to study vortex core charging, Hall currents and vortex lattices in superconductors [3–5].
The quasiclassical equation (Eilenberger as well as augmented versions) is a differential equation of the first order and usually, the solutions must be found numerically. In addition, the pair potential or the superconducting order parameter and self-energies (which are functions of the propagator) need to be solved for self-consistently. This is a nonlinear problem that is much simplified by a parametrization of the quasiclassical propagator in terms of coherence amplitudes that are solutions of a differential equation of Riccati form [6]. This Riccati parametrization transforms the Eilenberger equation for the matrix propagator to a scalar differential equation for the Riccati amplitudes. The solution then reduces to that of an initial value problem to a scalar differential equation. It is known that the Riccati parametrization of the quasiclassical propagator leads to a stable and fast numerical method to solve the Eilenberger equations.

In the presence of external magnetic fields, the quasiclassical propagator is found by solving the augmented Eilenberger equations. However, there is no simple normalization condition for the augmented quasiclassical propagator and physical solutions have to be picked by imposing conservation laws and their physical behaviour at zero fields or large energies. In this paper, I derive a normalization condition for a modified propagator and a Riccati parametrization for the augmented Eilenberger equations to leading order in a “small” parameter in the static limit.

2 Augmented Quasiclassical Theory in the Static Limit

The Keldysh formalism [7] is used to describe dynamical phenomena in quasiclassical theory. The information about the spectrum of quasiparticles is carried by the Retarded and Advanced parts of the quasiclassical Green’s function, \( \hat{g}^{RA} \). The quasiparticle distribution functions are described by the Keldysh part, \( \hat{g}^K \). In the following section, I focus on the retarded part and the derivation may be extended to advanced and Keldysh parts later.

Kita [3] derives a quasiclassical equation that retains a manifest gauge-invariance with respect to the space-time arguments of the quasiclassical Green’s function, \( \hat{g} \). In the static limit, dropping the superscript \( R \) for the retarded part, this equation for \( \hat{g}^R \) is:

\[
[\varepsilon \hat{\tau}_3 - \hat{\sigma}, \hat{g}]_o + i \mathbf{v}_F \cdot \partial_{R} \hat{g} + \left( \frac{i}{2} \frac{e}{c} (\mathbf{v}_F \times \mathbf{H}) \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{i}{2} e \mathbf{v}_F \cdot \mathbf{E} \frac{\partial}{\partial \varepsilon} \right) \{ \hat{\tau}_3, \hat{g} \}_o = 0 ,
\]

(1)

where the gauge-transformed quasiclassical propagator for quasiparticles with momentum \( \mathbf{p} = p_F \mathbf{\hat{p}} \) and energy \( \varepsilon \) at position \( \mathbf{R} \) is \( \hat{g}(\mathbf{p}, \mathbf{R}, \varepsilon) \). \( \hat{\sigma} \) is the gauge-transformed quasiclassical self-energy. Quasiparticles moving with a velocity \( \mathbf{v} = v_F \mathbf{\hat{p}} \) are in a magnetic field \( \mathbf{H} = \nabla \times \mathbf{A} \) and electric field \( \mathbf{E} \). \( e, c \) are the constants electron charge and speed of light, respectively. \( \hat{\tau}_{1,2,3} \) are the Pauli matrices in Nambu space. The \( \circ \)-product is the folding product

\[
\hat{a} \circ \hat{b}(\mathbf{\hat{p}}, \mathbf{R}, \varepsilon, \varepsilon') = \int \frac{d\varepsilon_1}{2\pi} \hat{a}(\mathbf{\hat{p}}, \mathbf{R}, \varepsilon, \varepsilon_1) \hat{b}(\mathbf{\hat{p}}, \mathbf{R}, \varepsilon_1, \varepsilon') .
\]

(2)
\( \delta \) is defined \([3]\) as \( \frac{\partial}{\partial \mathbf{R}} \) along the Nambu diagonal elements and \( \frac{\partial}{\partial \mathbf{R}} \pm 2i e_c A(\mathbf{R}) \) with the + sign operating on the 21 Nambu element and − sign on the 12 Nambu element of the operand matrix.

In the static limit, \( \mathbf{\hat{g}} = \mathbf{\hat{g}}(\mathbf{\hat{p}}, \mathbf{R}, \mathbf{\epsilon}) \) and the \( \circ \) product reduces to matrix multiplication in Nambu indices. I will drop the \( \circ \)-product notation assuming matrix multiplication for the rest of this paper.

Equation (1) is invariant under the gauge-transformation \( A \to A + \nabla \chi; \ \Phi \to \Phi - \frac{\chi}{\mathbf{\hat{g}}} \) of the magnetic vector potential and electric potential, respectively, with \( \mathbf{\hat{g}} \to e^{i \frac{\chi}{\mathbf{\hat{g}}}} \mathbf{\hat{g}} e^{-i \frac{\chi}{\mathbf{\hat{g}}}} \mathbf{\hat{g}} \) for \( \chi = \chi(\mathbf{R}) \). Consider a system with a two-dimensional Fermi surface in the plane perpendicular to \( \mathbf{H} \). \( \mathbf{v}_F = \mathbf{v}_{F \perp} \) and \( (\mathbf{v}_F \times \mathbf{H}) \cdot \partial / \partial \mathbf{p}_{\perp} = 0 \). Therefore,

\[
(\mathbf{v}_F \times \mathbf{H}) \cdot \frac{\partial}{\partial \phi_p} = \mathbf{v}_F \frac{H}{p_F} \frac{\partial}{\partial \phi_p}, \tag{3}
\]

where \( \phi_p \) is the angle in the plane of the Fermi surface i.e., \( \mathbf{p} = p_F(\mathbf{\hat{p}}, \phi_p) \) and

\[
\frac{i}{2} \frac{e}{c} (\mathbf{v}_F \times \mathbf{H}) \cdot \frac{\partial}{\partial \phi_p} \{ \mathbf{\hat{e}}_3, \mathbf{\hat{g}} \} = i \omega_c \frac{\partial}{\partial \phi_p} \{ \mathbf{\hat{e}}_3, \mathbf{\hat{g}} \}. \tag{4}
\]

Here, \( \omega_c \equiv \frac{i e H}{2 m^* c} \), where \( m^* \) is the effective mass of the quasiparticle. Define an operator

\[
\mathbf{\hat{\Omega}} \equiv i \omega_c \frac{\partial}{\partial \phi_p} + \frac{i}{2} (\mathbf{v}_F \cdot \mathbf{E}) \frac{\partial}{\partial \mathbf{\epsilon}}. \tag{5}
\]

Note that \( \{ \mathbf{\hat{e}}_3, \mathbf{\hat{g}} \} \) is diagonal in Nambu space. Using Eqs. (4), (1) becomes

\[
[\epsilon \mathbf{\hat{e}}_3 - \mathbf{\hat{\sigma}}, \mathbf{\hat{g}}]_{ll} + i \mathbf{v}_F \cdot \nabla \mathbf{g}_{ll} + \mathbf{\hat{\Omega}} \{ \mathbf{\hat{e}}_3, \mathbf{\hat{g}} \} = 0 \quad ; \quad l = 1, 2 \tag{6}
\]

for the Nambu diagonal part (subscripts denoting Nambu indices) and

\[
[\epsilon \mathbf{\hat{e}}_3 - \mathbf{\hat{\sigma}}, \mathbf{\hat{g}}]_{jk} + i \mathbf{v}_F \cdot \nabla \mathbf{g}_{jk} \pm 2 \frac{e}{c} \mathbf{v}_F \cdot \mathbf{A} \mathbf{g}_{jk} = 0 \quad ; \quad j \neq k ; jk = 12, 21 \tag{7}
\]

for the Nambu off-diagonal part.

The augmented quasiclassical propagator is the solution to the Kita equation (1) along trajectories parallel to \( \mathbf{v}_F \).

### 3 Expansion in a Small Parameter

In the absence of magnetic fields,

\[
[\epsilon \mathbf{\hat{e}}_3 - \mathbf{\hat{\sigma}}_0, \mathbf{\hat{g}}_0] + i \mathbf{v}_F \cdot \nabla \mathbf{\hat{g}}_0 = 0 \tag{8}
\]

recovering the Eilenberger equation. For a singlet order parameter with even pairing symmetry, the spin structure of \( \mathbf{\hat{g}}, \mathbf{\hat{\sigma}}, \mathbf{\hat{A}} \) is trivial and spin scalar. For the equilibrium case,
the normalization condition. \( \hat{1} \) is a unit vector in Nambu space. \( \Delta \) and \( \Delta^* \) are related by particle-hole conjugation \( \Delta(\p, \R, \epsilon) = \Delta(\p, \R, -\epsilon)^* \). The equilibrium solution is

\[
\hat{g}_0 = -\pi \frac{\epsilon \hat{\tau}_3 - \hat{\Delta}}{\sqrt{[\hat{\Delta}]^2 - \epsilon^2}} \equiv \zeta (\epsilon \hat{\tau}_3 - \hat{\Delta}) .
\]

(10)

The self-energies vary on the scale of the gap (magnitude of the order parameter) \( \sim \Delta \). Now, if we assume that the vector potential varies slowly on the scale of the Fermi wavelength, viz., the quasiclassical length scale, then it follows that the driving term in the Kita equation \( \propto \hat{\Omega} \) and the Kita derivative term \( \propto \frac{\epsilon}{\Delta} \frac{\epsilon}{\Delta} \frac{\epsilon}{\Delta} \) can be treated as “small” and \( \hat{g} \) expanded in this “small” parameter. “small” \( \sim \frac{h_0}{\Delta} , \frac{2e}{c} \frac{\epsilon}{\Delta} \). Let

\[
\hat{g} = \hat{g}_0(\text{small}^0) + \hat{g}_1(\text{small}^1) .
\]

(11)

Linearizing Eqs. (6 and 7), the diagonal and off-diagonal parts to \( \mathcal{O}(\text{small}^0) \) give the Eilenberger equation in the absence of fields. To \( \mathcal{O}(\text{small}^1) \),

\[
[\epsilon \hat{\tau}_3 - \hat{\Delta}, \hat{g}_1]_{ii} - [\hat{\sigma}_1, \hat{g}_0]_{ii} + i \nabla_F \cdot \nabla_0 (\hat{g}_1)_{ii} + \hat{\Omega} [\hat{\tau}_3, \hat{g}_0]_{ii} = 0 .
\]

(12)

Here, \( \hat{\sigma} = \hat{\sigma}_0(\text{small}^0) + \hat{\sigma}_1(\text{small}^1) = \hat{\Delta} + \hat{\sigma}_1 \). With the definition in Eq. (10), \( \hat{\Omega} ([\hat{\tau}_3, \hat{g}_0]) = 2\hat{\Omega}(\zeta \epsilon) \hat{1} \). Therefore, to \( \mathcal{O}(\text{small}^1) \), the diagonal part of the equation for \( \hat{g} \) is

\[
[\epsilon \hat{\tau}_3 - \hat{\Delta}, \hat{g}_1]_{ii} - [\hat{\sigma}_1, \hat{g}_0]_{ii} + (i \nabla_F \cdot \nabla_0 \hat{g}_1)_{ii} + 2\hat{\Omega}(\zeta \epsilon) = 0
\]

(13)

and the off-diagonal part (to \( \mathcal{O}(\text{small}^1) \)) is

\[
[\epsilon \hat{\tau}_3 - \hat{\Delta}, \hat{g}_1]_{jk} - [\hat{\sigma}_1, \hat{g}_0]_{jk} + (i \nabla_F \cdot \nabla_0 \hat{g}_1)_{jk} \pm \frac{2e}{c} (\nabla_F \cdot \A_0) \hat{g}_0 |_{jk} = 0 .
\]

(14)

An expansion in such a “small” parameter was used by Kita [8] to estimate the induced electric field and Hall coefficient in conventional superconductors in response to an applied magnetic fields. I employ this expansion to now derive an equation for a new \( \hat{g} \), that is gauge-invariant to \( \mathcal{O}(\text{small}^1) \).

### 4 Equation for a Modified \( \hat{g} \)

\( \hat{g} \) is given by a gauge-invariant Eq. (1) with a normalization \( \hat{\nu} \equiv \hat{g} \circ \hat{\nu} \) given by [1]

\[
[\epsilon \hat{\tau}_3 - \hat{\sigma}, \hat{\nu}] + i \nabla_F \cdot \nabla_R \hat{\nu} = \{ \hat{g}, \hat{\nu} \} (\hat{\nu}) .
\]

(15)

Now, to \( \mathcal{O}(\text{small}^1) \), the driving term is given by \(-4\hat{\Omega}(\zeta \epsilon) \hat{g}_0 \). Linearizing \( \hat{\nu} \), let

\( \hat{\nu} \) Springer
\[ \dot{\nu} = \dot{\nu}_0(\text{small}^0) + \dot{\nu}_1(\text{small}^1) \]
\[ \dot{\nu}_0 = \hat{g}_0 \circ \hat{g}_0 = -\pi^2 \hat{l} ; \dot{\nu}_1 = \hat{g}_0 \circ \hat{g}_1 + \hat{g}_1 \circ \hat{g}_0 \]

using the equilibrium condition (9) and keeping terms to \( O(\text{small}^1) \) using Eqs. (11). Equation (15) is trivially satisfied to \( O(\text{small}^0) \). To \( O(\text{small}^1) \),

\[ [\varepsilon \hat{r}_3 - \hat{\Delta}, \dot{\nu}_1] - [\hat{\sigma}_1, \dot{\nu}_0] + i \nu_F \cdot \hat{\sigma}(\dot{\nu}_0 + \dot{\nu}_1) = -4\hat{\Omega}(\varepsilon)\hat{g}_0 . \]  

Using Eq. (10), to \( O(\text{small}^1) \),

\[ i \nu_F \cdot \hat{\sigma} \rightarrow i \nu_F \cdot \nabla \dot{\nu}_1 . \]  

Equation (17) simplifies to

\[ \zeta^{-1}[\hat{g}_0, \dot{\nu}_1] + i \nu_F \cdot \nabla \dot{\nu}_1 = -4\hat{\Omega}(\varepsilon)\hat{g}_0 . \]  

Multiplying from the left and right by \( \hat{g}_0 \) and adding the two resulting equations, I get

\[ i \nu_F \cdot \nabla \{ \hat{g}_0, \dot{\nu}_1 \} = 8\pi^2 \hat{\Omega}(\varepsilon) \]
\[ \Rightarrow \{ \hat{g}_0, \dot{\nu}_1 \} \equiv N(\varepsilon, \phi, s) . \]

where \( \hat{s} \parallel \dot{\nu}_F \) is a coordinate along the quasiclassical trajectory. Here, I assume the equilibrium solution to be the homogenous solution. While this assumption is valid only in the limit of the homogenous state, viz., far away from vortices for applied fields \( H \ll H_c^2 \), it leads to an elegant result that effects the Riccati parametrization, as I show shortly. Now, define

\[ \hat{g}_1 = G_0 \hat{l} + \delta \hat{g} , \]

where \( \delta \hat{g} \propto \hat{r}_1, \hat{r}_2, \hat{r}_3 \) only. Then \( \dot{\nu}_1 \) as given by Eq. (16) is expressed as

\[ \dot{\nu}_1 = 2G_0 \hat{g}_0 + \{ \hat{g}_0, \delta \hat{g} \} , \]

and

\[ \{ \hat{g}_0, \dot{\nu}_1 \} = -2\pi^2 \hat{g}_1 + 2\hat{g}_0 \circ \hat{g}_1 \circ \hat{g}_0 . \]

Using the known \( \hat{g}_0 \) (10) and Eq. (20) which gives the anti-commutator \( \{ \hat{g}_0, \dot{\nu}_1 \} \propto \hat{l} \) in this limit,

\[ N(\varepsilon, \phi, s) = -2\pi^2 G_0 - 2\pi^2 G_0 = -4\pi^2 G_0 \]
\[ \Rightarrow G_0 = -\frac{1}{4\pi^2} N(\varepsilon, \phi, s) ! \]

Using the definition (21), the normalization condition for \( \hat{g}_0 \), and the result above, some further algebra gives

\[ \{ \hat{g}_0, \delta \hat{g} \} = 0 . \]
I now define a new $\tilde{G}$ thus

$$\tilde{g} = \hat{g}_0 + \hat{G}_0 \hat{1} + \delta \hat{g} = \hat{G}_0 \hat{1} + \tilde{G}$$  \hspace{1cm} (26)$$

and evaluate $\tilde{G} \circ \tilde{G}$ to find

$$\tilde{G} \circ \tilde{G} \to -\pi^2 + \{\hat{g}_0, \delta \hat{g}\} = -\pi^2 \hat{1}.$$  \hspace{1cm} (27)$$
to $O(\text{small}^4)$.

Summarizing, I separate the part of $\tilde{g}$ that is $\propto \hat{1}$, viz., $\tilde{g} = \hat{G}_0 \hat{1} + \tilde{G}$ and define a modified $\tilde{G}$ that is traceless and satisfies the usual normalization condition.

$\tilde{G}$ is given by an equation:

$$[\varepsilon \hat{\tau}_3 - \hat{\sigma}, \tilde{G}] + iv_F \cdot \delta \hat{G}_0 \hat{1} + iv_F \cdot \delta \tilde{G} + \hat{\Omega}(\hat{\tau}_3, \hat{G}_0 \hat{1} + \tilde{G}) = 0$$  \hspace{1cm} (28)$$

with normalization condition

$$\tilde{G} \circ \tilde{G} = -\pi^2 \hat{1}.$$  \hspace{1cm} (29)$$

Each term in Eq. (28) can be rigorously shown to be invariant under the transformation $A \to A + \nabla \chi(\mathbf{R})$; $\hat{g} \to e^{i \frac{\tau_3}{2} x(\mathbf{R}) \tau_3} \hat{g} e^{-i \frac{\tau_3}{2} x(\mathbf{R}) \tau_3}$, i.e., Eq. (28) is a gauge-invariant equation for $\tilde{G}$.

### 5 Solutions Near a Boundary

As an illustrative exercise, consider now the solutions of the equation for $G$ near a boundary. Let the boundary be defined at $x = 0$. The solution for $G$ is sought in the half-space $x > 0$. To $O(\text{small}^2)$, denote $\varepsilon \hat{\tau}_3 - \hat{\sigma} = \varepsilon \hat{\tau}_3 - \hat{\Lambda}'$. For a real order parameter with even pairing and singlet symmetry, let $\hat{\Lambda}' = \hat{\Lambda}' \hat{1}$. The traceless part of Eq. (28) may be formulated in terms of an equation for $\tilde{G} = \tilde{G}_1 \hat{\tau}_1 + \tilde{G}_2 \hat{\tau}_2 + \tilde{G}_3 \hat{\tau}_3$ thus:

$$\frac{1}{2} v_F \cdot \nabla \tilde{G} = \hat{B} \tilde{G} + \hat{C}$$  \hspace{1cm} (30)$$

where $\hat{G} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3)^T$ and

$$\hat{B} = \begin{pmatrix} 0 & \varepsilon' & i \Delta' \\ -\varepsilon' & 0 & 0 \\ -i \Delta' & 0 & 0 \end{pmatrix} ; \quad \hat{C} = \begin{pmatrix} \frac{-v}{c} (v_F \cdot A) \varepsilon & 0 \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (31)$$

For the purposes of this illustration, let the magnitude of the order parameter $\Delta'$ be unaltered by the boundary and be fixed at its equilibrium value. The homogeneous differential equation, $\hat{B} \hat{\Lambda}_i = \lambda_i \hat{\Lambda}_i$ can be solved with $\lambda_i$ and $\hat{\Lambda}_i$ being the eigenvalues and eigenvectors of $\hat{B}$. The eigenvalue $\lambda_0 = 0$ corresponds to the bulk equilibrium solution with $\hat{\Lambda}_0 = \hat{g}_0$. The eigenvalues $\lambda_{\pm} = \pm \sqrt{\Delta' 2 - \varepsilon' 2}$ correspond to eigenvectors of the form $\Lambda_{\pm} \propto (\lambda_{\pm} - \varepsilon - i \Delta')^T e^{2 \lambda_{\pm} x / v_s}$. The solution for the modified $\tilde{G}$ near the boundary is constructed from the homogeneous solution as $\tilde{G} = \alpha(x) \hat{\Lambda}_i e^{2 \lambda_{\pm} x / v_s}$. The factor $\alpha(x) \propto \int e^{2 \lambda_{\pm} x / v_s} (v_F \cdot A) dx$ and reflects the
gauge-invariant form of \( \mathcal{G} \). The physical solutions are the ones with the sign of \( \lambda_{\pm} \) chosen so that they decay into the bulk over a length scale set by the coherence length. The effect of the additional driving terms modifies the off-diagonal part of the modified \( \mathcal{G} \) close to the boundary. The part of Eq. (28) that is \( \propto \hat{1} \) yields an equation for \( \mathcal{G}_0 \),

\[
i_{F} \cdot \mathcal{G}_0 + 2\tilde{\Omega}(\zeta \varepsilon) - 2\frac{e}{c} v_F \cdot A \zeta \Delta = 0.
\]

(32)

The solution \( \mathcal{G}_0 \), along with \( \mathcal{G}_3 \) give the current density \( j_y = \int \frac{d\varepsilon}{4\pi} \langle ev_y g^K \rangle \) where the \( \langle ... \rangle \) refers to the Fermi surface average and \( g^K = (g^R - g^A)f(\varepsilon) \) is the Nambu particle element of the Keldysh version of the quasiclassical propagator and \( f(\varepsilon) \) is the Fermi distribution function. Since \( \hat{g}_R, A = \mathcal{G}_{R, A} \hat{G}_0 + \hat{G}_{R, A} \) by the definition (26), \( \mathcal{G}_0 \) and \( \mathcal{G}_3 \) give spontaneous currents at the boundary for such systems. The energy integral gives a vanishing boundary current for the homogenous order parameter used in this illustration.

A rigorous calculation would self-consistently include gap suppression near the boundary and is beyond the scope of this paper. For more complicated order parameters and geometries, such a calculation could be significantly simplified by a parametrization of \( \mathcal{G} \) in terms of coherence amplitudes or Riccati amplitudes.

6 Parametrization of \( \hat{\mathcal{G}} \)

With the normalization condition (29), \( \hat{\mathcal{G}} \) can be parametrized thus:

\[
\hat{\mathcal{G}} = \frac{-i\pi}{1 - \gamma \tilde{\gamma}} \left( \begin{array}{c} 1 + \gamma \tilde{\gamma} \\ -2\gamma \\ -1 - \gamma \tilde{\gamma} \end{array} \right).
\]

(33)

To \( \mathcal{O}(small^1) \), \( \hat{\mathcal{G}} \to \hat{g}_0 \) and \( \gamma, \tilde{\gamma} \) are the usual Riccati amplitudes. Using the parametrization (33) in Eq. (28), I derive a set of equation for the parameters \( \gamma, \tilde{\gamma} \) that are Riccati equations:

\[
\begin{align*}
2(\bar{\epsilon} + \frac{c}{e} (v_F \cdot A))\gamma + \bar{\Delta}' - \bar{\Delta}' \gamma^2 + iv_F \cdot \nabla \gamma &= 0 \\
2(\bar{\epsilon} - \frac{c}{e} (v_F \cdot A))\tilde{\gamma} - \bar{\Delta} - \Delta' \tilde{\gamma}^2 - iv_F \cdot \nabla \tilde{\gamma} &= 0
\end{align*}
\]

(34)

These Riccati equations are invariant under the transformation \( \gamma \to \gamma e^{2i\varepsilon \chi(R)} ; \tilde{\gamma} \to \tilde{\gamma} e^{-2i\varepsilon \chi(R)} ; \Delta' \to \Delta' e^{2i\varepsilon \chi(R)} ; \bar{\Delta}' \to \bar{\Delta}' e^{-2i\varepsilon \chi(R)} ; A \to A + \nabla \chi(R) \). \( \Delta', \bar{\Delta}' \) transform as the off-diagonal parts of \( \hat{\mathcal{G}} \) transform. Equation (34) can be solved for \( \gamma, \tilde{\gamma} \). \( \hat{\mathcal{G}} \) can be constructed from \( \gamma, \tilde{\gamma} \). The full gauge-transformed \( \hat{g} \) to \( \mathcal{O}(small^1) \) is given by \( \hat{g} = \mathcal{G}_0 \hat{1} + \hat{\mathcal{G}} \).
7 Discussion

Generally, solutions of the quasiclassical equation are complicated and have to be computed numerically. The parametrization of a gauge-invariant $\mathcal{G}$ transforms the gauge-invariant quasiclassical partial differential equation for the Nambu matrix propagator into a pair of Riccati equations for scalar amplitudes. Further, these Riccati equations can be transformed into an initial value problem that is numerically stable. The scheme outlined in this paper augments this simplification with manifest gauge invariance. The resulting equations are intuitive, but have been derived rigorously and are a nontrivial result outlined in this paper.

The driving term in Eq. (1) captures the contribution to $g^K$ from the broken particle-hole symmetry effected by the applied field. To $O(\text{small})$, this contribution, $g_0$, comes only from the effect of this symmetry breaking on the homogenous equilibrium state. Separation of this part renders $\mathcal{G}$ traceless and this can be Riccati parametrized for homogenous ground states. To leading order in $O((p_F\xi_0)^{-1})$ (the expansion parameter for quasiclassical theory), the effect of the driving terms enters the Keldysh part of the gauge-invariant quasiclassical propagator, while the Retarded and Advanced parts can still be parametrized in terms of $\gamma^{RA}$ and $\tilde{g}^{RA}$. This is the main result of this paper. The driving terms in Eq. (1) are of $O((p_F\xi_0)^{-1})$ and drop out of the Eilenberger equation in standard quasiclassical theory. But the effect of these terms is growing in importance in recent sophisticated geometries such as dirty layered superconducting wafers in magnetic fields and disordered interfaces of superconductors with magnetic materials. The quasiparticle distribution is modified via these terms while the density of states remains unaffected to this order. This gives rise to additional contributions to transport and non-equilibrium effects that are driven by the breaking of particle-hole symmetry effected by the applied field. As shown in this paper, these effects can still be calculated numerically via solutions to scalar Riccati equations, as has been customarily done in standard quasiclassical theory, but now via a modified $\mathcal{G}$.

The inclusion of applied fields in addition to disorder and finite size effects has become essential in the exploration of chiral order parameters and potentially topologically physics. A numerically robust scheme to work out a quasiclassical theory in a gauge-invariant form is invaluable in this context. Such calculations have been done recently [9, 10] using the Riccati method as an effective numerical tool with suitable choices of gauge. This paper extends this convenience in a rigorous way (at least to $O((p_F\xi_0)^{-1})$) to include manifest gauge-invariance.

Acknowledgements I acknowledge financial support from the Department of Science and Technology, Government of India: Scheme No. SR/WOS-A/PM-4/2016.

References

1. A.I. Larkin, Y.N. Ovchinnikov, in Nonequilibrium Superconductivity, vol. 493, ed. by D.N. Langenberg, A.I. Larkin (Elsevier, Amsterdam, 1986)
2. Gert Eilenberger, Zeitschrift für Physik 214, 195–213 (1968)
3. Takafumi Kita, Phys. Rev. B 64, 054503 (2001)
4. Hikaru Ueki, Marie Ohuchi, Takafumi Kita, J. Phys. Soc. Jpn. 87, 044704 (2018)
5. Wataru Kohno, Hikaru Ueki, Takafumi Kita, J. Phys. Soc. Jpn. 85, 083705 (2016)
6. Nils Schopol, arXiv:cond-mat/9804064
7. L.V. Keldysh, Soviet Phys. JETP 20, 1018 (1965)
8. Takafumi Kita, Phys. Rev. B 79, 024521 (2009)
9. P. Holmwall, A.B. Vorontsov, M. Fogelstrom, T. Lofwander Nature. Communications 9, 2190 (2018)
10. Xin Wang, Zhiqiang Wang, Catherine Kallin, Phys. Rev. B 98, 094501 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.