On the Oscillation of Fractional Order Emden-Fowler q-Difference Equations

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Abstract

In this article, we study the oscillatory behavior of fractional order Emden - Fowler q-difference equations of the form

\[ D_q \left[ r(t) \left( c D_q^\alpha z(t) \right) \right] + \phi(t) |x(\sigma(t))|^{\gamma - 1} x(\sigma(t)) = 0, \quad t \geq t_0, \]

where \( z(t) = x(t) + p(t)x(t - \tau) \), \( c D_q^\alpha \) denotes the Caputo q-fractional derivative of order \( \alpha \), \( 0 < \alpha \leq 1 \). Using the generalized Riccati technique, new oscillation criteria are established.

Key words: Oscillation, Fractional differential equations, Neutral, q- Calculus.

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1. Introduction

Fractional differential equations can be found in extensive range of many different subject areas. There are different concepts of fractional derivatives such as Riemann - Liouville and Caputo fractional derivatives are widely used. The Caputo fractional derivatives are based on integral expressions and gamma functions which are nonlocal. Fractional theory and its applications are mentioned many papers and monographs, we refer [1, 12, 15, 17, 24, 26, 27, 28, 29, 33].

Quantum calculus received a great attention and most of the published work has been interested in some problems of q-difference equations. The study of q - difference equations have been initiated by Jackson [22]. The paper of Carmichael [14]. See [2, 3, 4, 5, 7, 8, 9, 10, 11, 14, 16, 20, 22, 23, 30] and the references cited therein. The Emden-Fowler equations have been considered one of the important classical objects in the theory of differential equations. This type of equations has variety of
interesting physical applications occurring in astrophysics and atomic physics. See [6, 13, 18, 19, 21, 25, 31, 32, 34, 35] and the references cited therein. Li et al. [25] considered the Emden-Fowler neutral delay differential equation of the form

\[(r(t)(x(t) + p(t)x(t - \tau)))' + q(t)x^\gamma(\sigma(t)) = 0.\]

In [6], the researchers investigated the oscillatory behavior of second-order Emden-Fowler neutral delay differential equations of the form

\[((r(t)(x(t) + p(t)x(t - \tau)))')^\alpha + q(t)x^\gamma(\sigma(t)) = 0.\]

In this paper, we investigate the following fractional order Emden-Fowler neutral delay q-difference equation

\[D_q \left[r(t) \left( c D_q^\alpha z(t) \right) \right] + \phi(t) |x(\sigma(t))|^{\gamma-1} x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1}\]

where \(z(t) = x(t) + p(t)x(t - \tau)\), \(c D_q^\alpha\) denotes the Caputo q-fractional derivative of order \(\alpha, 0 < \alpha \leq 1\). We assume the following conditions throughout this paper without mentioning that

(A1) \(\gamma \in \mathbb{R}\), where \(\mathbb{R}\) is the set of all ratios of odd positive integers;

(A2) \(r \in C([t_0, \infty); (0, \infty)), p, \phi \in C([t_0, \infty); \mathbb{R}), 0 \leq p(t) < 1, \phi(t) \geq 0\), and \(q\) is a not identically zero for large \(t\);

(A3) \(\tau, \sigma \in C([t_0, \infty), \mathbb{R}), \tau(t) \leq t, \sigma(t) \leq t\).

By a solution of (1) we mean a nontrivial function \(x\) satisfying (1) for \(t \geq t_x \geq t_0\). In the sequel, we assume that solutions of (1) exist and can be continued indefinitely to the right. A solution of (1) is called oscillatory if it has arbitrarily on \([t_x, \infty)\); Otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

There is no work done on the oscillation of q-fractional Emden-Fowler equation. Our main aim of this paper is to establish new oscillation criteria for (1) by using generalized Riccati technique method.

2. Preliminaries

We state some definitions and fundamental results on quantum fractional calculus, see [9, 23, 30] and the references cited therein.

**Definition 2.1** Assume \(\nu \geq 0, q \in (0, 1)\), and a function \(g\) is defined on the interval \([0, 1]\). The fractional order q-integral of the Riemann-Liouville type is \((I_q^0 g)(t) = g(t)\)
and \((I_q^\nu g)(t) = \int_0^t \frac{(t-qs)^{(\nu-1)}}{\Gamma_q(\nu)} g(s) dq s, \quad \nu > 0, \quad t \in [0, 1]\), where
\[
\Gamma_q(\nu) = \frac{(1-q)^{(\nu-1)}}{(1-q)\nu-1}, \quad q \in (0, 1)
\]
and satisfies the relation: \(\Gamma_q(\nu + 1) = [\nu]_q \Gamma_q(\nu)\), with
\[
[\nu]_q = \frac{q^\nu - 1}{q - 1}, \quad (1-q)^{(0)} = 1, \quad (1-q)^{(n)} = \prod_{k=0}^{n-1} (1-q^{k+1}),
\]
\(n \in \mathbb{N}\). In general, if \(\alpha \in \mathbb{R}\), then
\[
(1-q)^{(\alpha)} = \prod_{i=0}^{\infty} \frac{(1-q^{i+1})}{(1-q^{i+\alpha+1})}.
\]

Now, we define the q-derivative of a real valued function \(g\) as
\[
D_q g(t) = \frac{g(t) - g(qt)}{(1-q)t}, \quad t \neq 0, \quad D_q g(0) = \lim_{n \to \infty} \frac{g(sq^n) - g(0)}{sq^n}, s \neq 0, \text{ for } q \in (0, 1).
\]

**Definition 2.2** The Riemann-Liouville type of the fractional q-derivative of the order \(\nu \geq 0\) is defined by \((D_q^\nu g)(t) = g(t)\) and
\[
(D_q^\nu g)(t) = (D_q^{[\nu]} I_q^{[\nu]-\nu} g)(t), \quad \nu > 0
\]
where \([\nu]\) is the smallest integer greater than or equal to \(\nu\).

**Definition 2.3** The Caputo type of the fractional q-derivative of order \(\nu \geq 0\) is defined by
\[
(^c D_q^\nu g)(t) = (I_q^{[\nu]-\nu} D_q^{[\nu]} g)(t), \quad \nu > 0
\]
where \([\nu]\) is the smallest integer greater than or equal to \(\nu\).

**Definition 2.4** For any \(x, y > 0\)
\[
B_q(x, y) = \int_0^1 t^{(x-1)}(1-qt)^{(y-1)} dq t
\]
is called q-beta function and we recall the relation
\[
B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.
\]
Lemma 2.5 Assume $\nu, \gamma \geq 0$ and let $g$ be a function defined on the interval $[0,1]$. Then

$$(i) (I_q^\nu I_q^\gamma g)(t) = (I_q^{\nu + \gamma} g)(t) \quad \text{and} \quad (ii) (D_q^\nu I_q^\nu g)(t) = g(t).$$

Lemma 2.6 Let $\nu > 0$. Then, the following result holds:

$$(I_q^\nu D_q^\nu g)(t) = g(t) - \sum_{k=0}^{[\nu]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k g)(0).$$

Lemma 2.7 Assume $\nu \geq 0$ and $n \in \mathbb{N}$. Then, the following equality holds:

$$\left( (I_q^\nu D_q^n g)(t) = D_q^n I_q^\nu g(t) - \sum_{k=0}^{[\nu]-1} \frac{t^{\nu+n+k}}{\Gamma_q(\nu - n + k)} (D_q^k g)(0) \right).$$

Lemma 2.8 For $\nu \in \mathbb{R}^+$, $\rho \in (-1, \infty)$, the following is valid:

$$I_q^\nu ((x-a)^{\rho}) = \frac{\Gamma_q(\rho+1)}{\Gamma_q(\nu+\rho+1)} (x-a)^{(\nu+\rho)}, \quad 0 < a < x < b.$$  

For $\rho = 0$, $a = 0$, applying q- integration by parts, we get

$$(I_q^\nu 1)(x) = \frac{1}{\Gamma_q(\nu+1)} x^{(\nu)}.$$

3. Main Results

In this section, we establish some new oscillation criteria for (1). In the following for convenience. We denote

$$Z(t) := x(t) + p(t)x(t - \tau),$$
$$e(t) := r(\sigma(t)) \int_1^t \frac{c^D_q a(\sigma(s))}{r(\sigma(s))} d_q s,$$
$$R(t) := \int_{t_0}^t \frac{1}{r(s)} d_q s,$$
$$\delta(t) := \int_{\rho(t)}^\infty \frac{1}{r(s)} d_q a(s).$$

Theorem 3.1 Assume that $^cD_q^a(p(t)) \geq 0$, and there exists $\rho \in c^1_q([t_0, \infty), \mathbb{R})$ such that $\rho(t) \geq t$, $^cD_q^a(\rho(t)) > 0$, $\sigma(t) = \rho(t) - t$. If for all sufficiently large $t_1$, and for
all constants $M > 0$, $L > 0$ one has,
\[\int_0^\infty \left[ R^\gamma (\sigma(t))(1 - p(\sigma(t)))^\gamma \phi(t) - \left[ \frac{\gamma}{\epsilon(t)} M^{1-\gamma}(cD_q^\alpha(\sigma(t)))R^{\gamma-1}(\sigma(t)) \right] \right] d_q t = \infty\] (2)
and
\[\int_0^\infty \left[ \phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \delta^\gamma(t) - \left[ \frac{\gamma}{\epsilon(t)} M^{1-\gamma}(cD_q^\alpha(\rho(t))) \right] \right] d_q t = \infty\] (3)
then (1) is oscillatory.

Proof: Suppose to the contrary that $x$ is a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t) > 0$ for all large $t$. The case of $x(t) < 0$ can be considered by the same method.
From (1) we can easily obtain that there exists a $t_1 \geq t_0$ such that
Case I:

\[ Z(t) > 0, cD_q^\alpha(Z(t)) > 0, D_q \left[ r(t)^\gamma D_q^\alpha(Z(t)) \right] \leq 0 \] (4)

Case II:

\[ Z(t) > 0, cD_q^\alpha(Z(t)) < 0, D_q \left[ r(t)^\gamma D_q^\alpha(Z(t)) \right] \leq 0 \] (5)

In case I holds. We have that $\sigma(t) \leq t$

\[ r(t)^\gamma D_q^\alpha(Z(t)) \leq r(\sigma(t))^\gamma D_q^\alpha(\sigma(t)), \quad t \geq t_1 \] (6)

From the definition of $Z$, we have

\[ z(t) = x(t) + p(t)x(t - \tau) \]
\[ x(t) \geq (1 - p(t))z(t) \] (7)

Define,

\[ W(t) = R^\gamma(\sigma(t)) \frac{r(t)^\gamma D_q^\alpha(Z(t))}{(Z(\sigma(t)))^\gamma}, \quad t \geq t_1. \] (8)

Then $W(t) > 0$ for $t \geq t_1$. From (1), (7) and (8), we obtain

\[ D_q W(t) \leq \frac{[\gamma] R^{\gamma-1}(\sigma(t))^\gamma D_q^\alpha(\sigma(t)) r((t))^\gamma D_q^\alpha(Z((t)))}{r(\sigma(t))} + R^\gamma(\sigma(t)) \]
By (6), (9) and $^cD_q^\alpha(\sigma(t)) > 0$, we get

$$D_q W(t) \leq \frac{[\gamma] R^{r-1}(\sigma(t))^cD_q^\alpha(\sigma(t)) r(\sigma(t))^cD_q^\alpha(Z(\sigma(t)))}{r(\sigma(t)) (z(\sigma(t))^\gamma)} - R^\gamma(\sigma(t)) \frac{\phi(t)(1 - p(\sigma(t)))^\gamma Z^\gamma(\sigma(t))}{(Z(\sigma(qt)))^\gamma} \tag{9}$$

since $x(t)$ is positive and increasing, we see that $x(t) > x(qt)$ and this implies that

$$- \frac{1}{x^2(t)} > - \frac{1}{x(t)(x(qt))}.$$  

$$D_q W(t) \leq \frac{[\gamma] R^{r-1}(\sigma(t))^cD_q^\alpha(\sigma(t)) r(\sigma(t))^cD_q^\alpha(Z(\sigma(t)))}{r(\sigma(t)) (z(\sigma(t))^\gamma)} - R^\gamma(\sigma(t)) \phi(t)(1 - p(\sigma(t)))^\gamma Z^\gamma(\sigma(t)). \tag{10}$$

From $D_q \left[ r(t)^cD_q^\alpha(Z(t)) \right] \leq 0$ and $^cD_q^\alpha(\sigma(t)) > 0$. We see that

$$\int_{t_1}^{t} (^cD_q^\alpha(Z(s))^cD_q^\alpha(\sigma(s)) d_q^\alpha s = Z(\sigma(t)) - Z(\sigma(t_1))$$

$$- \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q^{k+1}} (D_q^k(Z))(\sigma(t))(0) - Z(\sigma(t_1)) + \sum_{k=0}^{[\alpha]-1} \frac{t}{\Gamma_q^{k+1}} (D_q^k(Z))(\sigma(t))(0)$$

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\[
Z(\sigma(t_1)) + \int_{t_1}^{t} \left( c D_q^\alpha (Z(\sigma(s))^c D_q^\alpha (\sigma(s)) \right) d_q^s = Z(\sigma(t)) - \sum_{k=0}^{[a]-1} \frac{1}{\Gamma_{q,k+1}} t^K (D_q^K Z)(\sigma(t))(0) + t^K (D_q^K Z)(\sigma(t))(0)
\]

That is, \( Z(\sigma(t)) \geq \epsilon(t)^c D_q^\alpha Z(\sigma(t)) \). 

(11)

Thus by (10) and (11), we obtain

\[
D_q W(t) \leq \frac{[\gamma] R^{\gamma-1}(\sigma(t))(c D_q^\alpha(\sigma(t))) \rho(\sigma(t))^\gamma}{r(\sigma(t))} M^{1-\gamma} - \frac{R^\gamma(\sigma(t)(1 - p(\sigma(t))))^\gamma}{\epsilon(t)} \phi(t).
\]

(12)

Where \( M = Z(\sigma(t)). \) Integrating (12) from \( t_1 \) to \( t \). We get \( 0 < W(t). \)

\[
W(t) \leq W(t_1) - \int_{t_1}^{t} \frac{[\gamma] R^{\gamma-1}(\sigma(s))(c D_q^\alpha(\sigma(s))) \rho(\sigma(s))^\gamma}{\epsilon(t)} \phi(s)
\]

\[
- \frac{[\gamma] M^{\gamma-1}(c D_q^\alpha(\sigma(s))) R^{\gamma-1}(\sigma(s))}{\epsilon(t)}
\]

\[d_q^s \]  

(13)

Letting \( t \rightarrow \infty \) in (13) we get a contradiction with (2).

In case II holds. We define the function \( V \) by

\[
V(t) = \frac{r(t)^c D_q^\alpha Z(t)}{Z^\gamma(\rho(t))}, \quad t \geq t_1
\]

(14)

Then \( V(t) < 0 \) for \( t \geq t_1 \). Nothing \( D_q \left[ r(t)^c D_q^\alpha Z(t) \right] \leq 0 \). \( r(t)^c D_q^\alpha Z(t) \) is non-increasing so we have,

\[
r(s)^c D_q^\alpha Z(s) \leq r(t)^c D_q^\alpha Z(t), \quad s \leq t
\]

(15)

Dividing (15) by \( r(s) \)

\[
\frac{c D_q^\alpha Z(s)}{r(s)} \leq \frac{r(t)^c D_q^\alpha Z(t)}{r(s)}
\]
and integrating it from $\rho(t)$ to $l$ we obtain,

$$\int_{\rho(t)}^{l} \left( cD_q^a Z(s) \right) d_q^a s = Z(l) - Z(\rho(t)) - \sum_{K=0}^{[a]-1} \frac{l^K}{\Gamma(qK+1)} (D_q^k Z)(l)(0) - Z(\rho(t))$$

$$+ \sum_{K=0}^{[a]-1} \frac{(\rho t)^K}{\Gamma(qK+1)} (D_q^k Z)(\rho(t))(0)$$

$$= r(t)^c D_q^a Z(t) \int_{\rho(t)}^{l} \frac{1}{r(s)} d_q^a s.$$

$$Z(l) \leq Z(\rho(t)) + r(t)^c D_q^a Z(t) \int_{\rho(t)}^{l} \frac{1}{r(s)} d_q^a s.$$

Letting $l \to \infty$

$$0 \leq Z(\rho(t)) + r(t)^c D_q^a Z(\delta(t)), \quad t \geq t_1$$

That is,

$$\frac{r(t)^c D_q^a Z(t)(\delta(t))}{Z(\rho(t))} \geq - \frac{Z(\rho(t))}{Z(\rho(t))}$$

$$\frac{r(t)^c D_q^a Z(t)(\delta(t))}{Z(\rho(t))} \geq -1.$$

Thus,

$$-r(t)^c D_q^a Z(t) \left[ (-r(t)^c D_q^a Z(t)) \right]^{[a]-1} (\delta^\gamma(t)) \leq 1.$$

So by $(-r(t)^c D_q^a Z(t)) > 0$ and (14) we have,

$$\frac{-V(t)L^{[a]-1}\delta^\gamma(t)}{V(t)\delta^\gamma(t)} \leq \frac{1}{V(t)\delta^\gamma(t)}$$

$$L^{[a]-1} \leq \frac{1}{V(t)\delta^\gamma(t)}$$

$$\frac{1}{L^{[a]-1}} \leq V(t)\delta^\gamma(t) \leq 0, \quad t \geq t_1 \quad (16)$$
Where \( L = -r(t)^c D_q^a Z(t) \). Differentiating \((14)\), we get

\[
D_q V(t) = \frac{D_q \left[ r(t)^c D_q^a Z(t) \right] Z^\gamma(t) - r(t)^c D_q^a Z(t) D_q Z(t) Z^\gamma(t) D_q \left[ Z^\gamma(t) \right]}{Z^\gamma(t) Z^\gamma(t) Z^\gamma(t) Z^\gamma(t)}.
\]

\[
D_q V(t) = -\phi(t) \left[ \frac{x^\gamma(t)}{Z^\gamma(t)} \right] - \frac{[\gamma] r(t)^c D_q^a Z(t) Z^\gamma(t) Z^\gamma(t) D_q Z(t) D_q Z(t) Z^\gamma(t) D_q \left[ Z^\gamma(t) \right]}{Z^\gamma(t) Z^\gamma(t) Z^\gamma(t) Z^\gamma(t)}.
\]

Nothing that \( c D_q^a \rho(t) \geq 0 \).

Hence by \((14)\) and \((17)\) we get

\[
D_q V(t) \leq -\phi(t) \left[ \frac{x^\gamma(t)}{Z^\gamma(t)} \right] - \frac{[\gamma] r(t)^c D_q^a Z(t) Z^\gamma(t) Z^\gamma(t) D_q Z(t) D_q Z(t) Z^\gamma(t) D_q \left[ Z^\gamma(t) \right]}{Z^\gamma(t) Z^\gamma(t) Z^\gamma(t) Z^\gamma(t)}.
\]

Thus by \((14)\) and \((17)\) we get

\[
D_q V(t) \leq -\phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma.
\]

that is,

\[
D_q V(t) + \phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \leq 0, \quad t \geq t_1.
\]
and multiply (18) by $\delta^\gamma(t)$ and integrating from $t_1$ to $t$ implies that

$$
\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) + [\gamma] \int_{t_1}^t \frac{1}{r(\rho(s))} D_q(\rho(s)) \delta^{\gamma-1}(s)V(s) d_q s \\
+ \int_{t_1}^t \left( \frac{1}{1 + p(\rho(s))} \right)^\gamma \phi(s) \delta^\gamma(s) d_q s \leq 0. \tag{19}
$$

Therefore it follows from (16) and (19) that

$$
\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) - \left[ \frac{1}{1 + p(\rho(s))} \right] \phi(s) \delta^\gamma(s) d_q s \\
\delta^\gamma(t)V(t) - \delta^\gamma(t_1)V(t_1) = \left[ \frac{1}{1 + p(\rho(s))} \right] \phi(s) \delta^\gamma(s) d_q s.
$$

Letting $t \to \infty$ in the above inequality by (3). We get a contradiction with (16).

**Theorem 3.2** Assume that $\sqrt[\gamma]{D^\alpha q}(\rho(t)) \geq 0$, and there exists $\rho \in c^{1}(\{t_0, \infty\}, \mathbb{R})$, such that $\rho(t) \geq t$, $\sqrt[\gamma]{D^\alpha q}(\rho(t)) > 0$, $\sigma(t) = \rho(t) - \tau$. If for all sufficiently large $t_1$, and for all constants $M > 0$ such that (2) holds and

$$
\int_{t_1}^{\infty} \left[ \phi(t) \frac{1}{1 + p(\rho(t))} \right]^\gamma \delta^{\gamma+1}(t) d_q t = \infty. \tag{20}
$$

then (1) is oscillatory.

Proof: Suppose to the contrary that $x$ is a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t) > 0$ for all large $t$. The case of $x(t) < 0$ can be considered by the same method. From (1) we can easily obtain that there exists a $t_1 \geq t_0$ such that (4) or (5) holds. If (4) holds. Proceeding as in the proof of Theorem (3.1). We obtain a contradiction with (2). If case II holds. We proceed as in the proof of Theorem (3.1) then we get (16) and (18). Multiplying (18) by $\delta^\gamma(t)$,

$$
\delta^{\gamma+1}(t) D_q V(t) + \delta^{\gamma+1}(t) \phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \leq 0, \quad t \geq t_1.
$$
and integrating from \( t_1 \) to \( t \) implies that

\[
\delta^{\gamma+1}(t)V(t) - \delta^{\gamma+1}(t_1)V(t_1) + [\gamma + 1] \int_{t_1}^{t} \frac{1}{r(\rho(s))} \frac{c}{D_q^\alpha(\rho(t))} \delta^{\gamma}(s)V(s)d_q s + \int_{t_1}^{t} \phi(s) \left( \frac{1}{1 + p(\rho(s))} \right)^\gamma \delta^{\gamma+1}(s)d_q s \leq 0. \tag{21}
\]

In view of (16) we have,

\[
-V(t)\delta^{\gamma+1}(t) \leq \frac{1}{L_{\gamma-1}} \delta(t) < \infty, \quad t \to \infty,
\]

and consider

\[
\int_{t_1}^{t} -\frac{1}{r(\rho(s))} \frac{c}{D_q^\alpha(\rho(s))} \delta^{\gamma+1}(s)V(s) \frac{1}{\delta(s)} d_q s \leq \frac{1}{L_{\gamma-1}} \delta(s) \int_{t_1}^{t} \frac{c}{D_q^\alpha(\rho(s))} \frac{1}{r(\rho(s))} d_q s \leq \frac{1}{L_{\gamma-1}} \int_{t_1}^{t} \frac{1}{r(u)} d_q s.
\]

Therefore by the above inequality letting \( t \to \infty \) in (21). We obtain

\[
\int_{t_1}^{\infty} \phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \delta^{\gamma+1}(t)d_q t \leq \infty,
\]

which contradiction (20).

**Theorem 3.3** Assume that \( cD_q^\alpha(\rho(t)) \geq 0 \), and there exists \( \rho \in \mathcal{C}_q^1([t_0, \infty), \mathbb{R}) \), such that \( \rho(t) \geq t, \ cD_q^\alpha(\rho(t)) > 0, \sigma(t) = \rho(t) - \tau \). If for all sufficiently large \( t_1 \), and for all constants \( M > 0 \) such that (2) holds and

\[
\int_{t_1}^{\infty} \frac{1}{r(v)} \int_{t_1}^{v} \phi(u) \left( \frac{1}{1 + p(\rho(u))} \right)^\gamma \delta^{\gamma}(u)d_q u \ d_q v = \infty, \tag{22}
\]

then (1) is oscillatory.

Proof: Suppose to the contrary that \( u \) is a nonoscillatory solution of (1). Without loss of generality we may assume that \( x(t) > 0 \) for all large \( t \). The case of \( x(t) < 0 \) can be considered by the same method.

From (1) we can easily obtain that there exists a \( t_1 \geq t_0 \) such that (4) or (5) holds.

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If (4) holds. Proceeding as in the proof of Theorem (3.1), we obtain a contradiction with (2).

If case II holds. We proceed as in the proof of Theorem (3.1) then we get (15).

Dividing (15) by \( r(s) \)

\[
cD_q^\alpha Z(s) \leq \frac{r(t)cD_q^\alpha Z(t)}{r(s)}.
\]

and integrating it from \( \rho(t) \) to \( l \), letting \( l \to \infty \) yields.

\[
Z(\rho(t)) - Z(l) \geq -r(t)cD_q^\alpha Z(t) \int_{\rho(t)}^{\infty} \frac{1}{r(s)} ds
\]

\[
\geq -r(t)cD_q^\alpha Z(t)\delta(t)
\]

\[
Z(\rho(t)) - Z(l) \geq a\delta(t).
\]

by (1) We have

\[
D_q \left[ -r(t)cD_q^\alpha Z(t) \right] = \phi(t)x^\gamma(\sigma(t)).
\]

Noticing that \( cD_q^\alpha P(t) \geq 0 \), We see that \( cD_q^\alpha x(t) \leq 0 \) for \( t \geq t_1 \), so by \( \sigma(t) \leq \rho(t) - \tau \), We get

\[
\frac{x(\sigma(t))}{Z(\rho(t))} = \frac{x(\sigma(t))}{x(\rho(t)) + p(\rho(t))x(\rho(t) - \tau)} \geq \frac{1}{1 + p(\rho(t))}.
\]

Hence we obtain

\[
D_q \left[ -r(t)cD_q^\alpha Z(t) \right] \geq a^\gamma \phi(t) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \delta^\gamma(t).
\]

Integrating the above inequality from \( t_1 \) to \( t \), we have

\[
-r(t)cD_q^\alpha Z(t) \geq -r(t_1)cD_q^\alpha Z(t_1) + a^\gamma \int_{t_1}^{t} \phi(u) \left( \frac{1}{1 + p(\rho(t))} \right)^\gamma \delta^\gamma(u) d_q u.
\]

\[
\geq a^\gamma \int_{t_1}^{t} \phi(u) (1 + P(\rho(t)))^\gamma \delta^\gamma(u) d_q u.
\]
Integrating the above inequality from $t_1$ to $t$, we obtain

$$z(t_1) - Z(t) \geq a^\gamma \int_{t_1}^{t} \frac{1}{r(v)} \int_{t_1}^{v} \phi(u) \left( \frac{1}{1 + p(\rho(u))} \right)^\gamma \delta^\gamma(u) d_q u \ d_q v.$$ 

Which contradicts (22).

4. Conclusion

In this paper, we have obtained some oscillation results for the fractional order Emden-Fowler quantum difference equation using generalized Riccati technique. In this paper is q-analog of [6]. Our results are new.

References

[1] Abbas S, Benchohra M and N’Guerekata GM, Topics in Fractional Differential Equations, Springer, New york, (2012).

[2] Abdallh B, On the oscillation of q-fractional difference equations, Advances in difference equations, 252, 2017, 01-11.

[3] Abreu L, Sampling theory associated with q-difference equations of the Sturm-Liouville type, J. Phys. A, 38(48), 2005, 10311-10319.

[4] Adams CR, On the linear ordinary q-difference equation, Ann. Math., 30, 1928, 195-205.

[5] Agarwal RP, Certain fractional q-integrals and q-derivatives, Proc. Camb. Philos. Soc. 66, 1969, 365-370.

[6] Agarwal RP, Bohner M, Li T and Zhang C, Oscillation of second order Emden-Fowler neutral delay differential equations, Annali di Matematica, 193, 2014, 1861-1875.

[7] Al-Salam WA, q-analogues of cauchy’s formulas, Proc. Am. Math. Soc, 17, 1966, 616-621.

[8] Al-Salam WA, Some fractional q-integrals and q-derivatives, Proc. Edinb. Math. Soc., 15(2), 1966/1967, 135-140.

[9] Annaby MH and Mansour ZS, q-fractional calculus and equations, Lecture Notes in Mathematics, Springer-Verlag, Berlin, (2012).

[10] Askey R, The q-gamma and q-beta functions, Appl. Anal., 8(2), 1978/1979, 125-141.
[11] Atici FM and Eloe P, Fractional q-calculus on a time scale, J. Nonlinear Math. Phys., 14(3), 2007, 341-352.

[12] Bagley RL and Torvick PJ, A theoretical basis for the application of fractional calculus to visco elasticity, J. Rheology, 27, 1983, 201-210.

[13] Berkovich LM, The generalized Emden-Fowler equation, Sym. Nonlinear Math. Phys, 1, 1997, 155-163.

[14] Carmichael RD, The general theory of linear q-difference equations, Am. J. Math., 34, 1912, 147168.

[15] Chen DX, Oscillatory behavior of a class of fractional differential equations with damping, UPB Scientific bulletin, Series A, 75, 2013, 107-118.

[16] De Sole A and Kac V, On Integral Representations of q- Gamma and q-Beta functions, Rend. Mat. Acc. Lincei, 9, 2005, 11-29.

[17] Diethelm K, The Analysis of Fractional Differential Equations, Springer, Berlin, (2010).

[18] Domoshnitsky A and Koplatadze R, On asymptotic behavior of solutions of generalized Emden-Fowler differential equations with delay argument, Abstract and Applied Analysis, 2004, 1-13.

[19] Došlá Z, and Marini M, On super-linear Emden-Fowler type differential equations, Journal of Mathematical Analysis and Applications, 416, 2014, 497-510.

[20] Ernst T, The history of q-Calculus and a new method, (2000).

[21] Fowler RH, Further studies of Emden’s and similar differential equations, Quarterly Journal of Mathematics, 2 (1), 1931, 259-288.

[22] Jackson FH, On q-definite integrals, Q. J. Pure Appl. Math., 41, 1910, 193-203.

[23] Kac V and Cheung P, Quantum Calculus, Springer, New York, (2002).

[24] Kilbas AA, Srivastava HM and Trujillo JJ, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, Elsevier Science, Amsterdam, 204, 2006.

[25] Li T, Han Z, Zhang C and Sun S, On the oscillation of second order Emden-Fowler neutral delay differential equations, J Appl Math, 37, 2011, 601-610.

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[26] Mainardi F, Fractional Calculus and Waves in Linear Visco elasticity, Imberical College Press, London, (2010).

[27] Milici C and Draganescu G, Introduction to Fractional Calculus, Lambert Academic Publishing, (2016).

[28] Miller K.S and Ross B, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, Newyork, (1993).

[29] Podlubny I, Fractional Differential Equations, Academic Press, SanDiego, (1999).

[30] Rajkovic PM, Marinkovic SD and Stankovic MS, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math., 1, 2007, 311323.

[31] Sadhasivam V, Nagajothi N, and Deepa M, Oscillation of third-order neutral delay mixed type Emden-Fowler differential equations, International Journal of Research and Analytical Reviews, 5(3), 2018, 25-31.

[32] Saker SH, Oscillation Theory of Delay Differential and Difference Equations, VDM Verlag Dr.Muller Aktienge sellschaft and Co, USA, (2010).

[33] Samko SG, Kilbas AA and Marichev OI, Fractional Integrals and Derivatives, Gordon and Breach Science Publishers, Yverdon, (1993).

[34] Wong JSW, On the generalized Emden-Fowler equation, SIAM Rev, 17, 1975, 339-360.

[35] Wu Y, Yu Y, Zhang J and Xiao J, Oscillation criteria for second order Emden-Fowler functional differential equations of neutral type, Journal of Inequalities and Applications, 328, 2016, 01-11.

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