A general approach to deriving diagnosability results of interconnection networks

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\textbf{ABSTRACT}
We generalise an approach to deriving diagnosability results of various interconnection networks in terms of the popular $g$-good-neighbour and $g$-extra fault-tolerant models, as well as mainstream diagnostic models such as the PMC and the MM* models. As demonstrative examples, we show how to follow this constructive, and effective, process to derive the $g$-extra diagnosabilities of the hypercube, the $(n, k)$-star, and the arrangement graph. These results agree with those achieved individually, without duplicating structure independent technical details. Some of them come with a larger applicable range than those already known, and the result for the arrangement graph in terms of the MM* model is new.

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\section{1. Introduction}

A rapid and consistent technical progress in computing technology has made multi-processor systems a reality, where an inter-processor communication enabling network structure plays a central role. It is unavoidable that some of the processing nodes in such a system become faulty, potentially disabling its integrity. To cope with such a situation, it is mandatory to develop technology to identify, and then correct/replace, faulty nodes in these network systems in order to restore its normal operation. For obvious reasons, one would want to have a self-diagnosable system where the computing nodes are able to detect faulty ones themselves. The maximum number of faulty nodes that can be so identified in an interconnection system is called its diagnosability. An ideal system should come with a large diagnosability, indicating a greater fault tolerance in the sense that, even this many processing nodes fail, they can still be identified and corrected so that the normal functionality of the system can be restored. The process of deriving such diagnosability results helps us to choose a fault-tolerant, thus sustainable, interconnection networks to meet our daily needs. It hence has attracted much attention in the research community over an extended period of time.

Naturally, the diagnosability of a network depends on its topological structure, the intended fault-tolerant model, and the diagnostic model deemed appropriate. The topology of an interconnection network system is usually modelled with a connected graph $G(V, E)$, where $V$, the set of vertices of $G$, represents a collection of processing nodes; and $E$, the set of edges, the connection between pairs of nodes in such a system. Many network topologies have been suggested and studied, including
such influential graph structures as the hypercube [2] and many of its variants, the star graph [3], the bubble-sort graph [4], the arrangement graph [5], and the \((n, k)\)-star graph [6].

By a \textit{neighbour} of a vertex, \(v\), in a graph \(G\), we mean a vertex \(u\) such that \((u, v)\) is an edge in \(G\); and, the \textit{degree} of a vertex, \(v\), is simply the number of its neighbours in \(G\). We also say \(G\) is \(r\)-\textit{regular} if the degrees of all its vertices equal \(r\). Clearly, there is no way to tell if a vertex is faulty, if all its neighbours are. Such a vertex then cannot be used to judge the faulty status of any of its neighbours. The diagnosability of a graph \(G\) is thus no more than \(\delta(G)\), the minimum degree of any vertex in \(G\) [7–9], which is, unfortunately, neither theoretically interesting nor practically satisfactory. On the other hand, since this scenario that all the neighbours of every node (vertex) could be faulty, as implied by an \textit{unrestrictive fault-tolerant model}, is highly unlikely, several more sophisticated and realistic fault-tolerant models have since been suggested. Let a \textit{faulty set} be a collection of vertices \(F \subseteq V\), which are effectively removed from \(G\). Such a faulty set is \textit{conditional faulty} [10–12] when every vertex, faulty or not, has at least one \textit{fault-free} neighbour in the \textit{survival graph} of \(G - F\). For \(g \geq 0\), a faulty set is \textit{g-good-neighbour faulty} [13] when every fault-free vertex has at least \(g\) fault-free neighbours in the survival graph; and, a faulty set is \textit{g-extra faulty} [14,15] when every component in the survival graph contains at least \(g + 1\) vertices.

There is also an “edge version” for both the \(g\)-good-neighbour, and the \(g\)-extra, faulty sets. For example, a set of edges \(F\) in a connected graph \(G\) is called an \textit{g-good-neighbour edge-cut} if the survival graph \(G - F\) is disconnected and every fault-free vertex has at least \(g\) fault-free neighbours [16].

We will focus on the vertex version of these two notions in this paper. For various properties of the edge versions of various faulty sets, associated results, and their relationship with their vertex related cousins, readers are referred to [16].

The often adopted \textit{comparison diagnostic model}, i.e. the \(MM^*\) diagnostic model [17], places a restriction on the MM model [18,19], so that every processing node, acting as a \textit{testing node}, sends a test message to each and every pair of its distinct neighbours, referred to as the \textit{tested nodes}, and then compares their responses. The fault status of the system can then be determined, and the faulty nodes identified, based on the comparison results so obtained. The PMC model [9] is another popular diagnostic model where every node sends a test message to each of its neighbours and obtain the final diagnostic information based on the received testing results. Various efficient algorithms to identify such faulty sets have also been proposed in, e.g. [17,20–22]. With both the \(MM^*\) and the PMC models, it is assumed that, when a testing node is faulty, responses from those tested nodes will be unreliable. On the other hand, the BMG model [23] assumes that, under such circumstances, the response from a tested node is always fault free, even if it is faulty; and uncertain when the tested node is fault free.

A collection of all such test results obtained with a diagnostic model is called a \textit{syndrome} of the diagnosis in \(G\). A subset \(F \subseteq V\) is said to be \textit{allowed for a syndrome} [20], or \textit{compatible with a syndrome} [12], if it can be generated when all the vertices in \(F\) are faulty and all those in \(V \setminus F\) are fault free. Since faulty testing nodes lead to unreliable results, as observed in [18,19], two faulty sets may be compatible with the same syndrome, thus making such a faulty set unidentifiable, and the diagnostic process fallible. This observation leads to the notion of a graph being \(t\)-\textit{diagnosable} [12,17]: when up to \(t\) faulty vertices in \(G\) can be identified. And the \textit{diagnosability} of a graph \(G\), denoted by \(t(G)\), is defined to be the maximum number of faulty vertices that \(G\) can guarantee to identify in terms of this diagnostic model.

Because of their important role in both network theory and practice, many diagnosability results have appeared in literature. Recent examples include \(g\)-extra diagnosability results for the hypercube [15], the arrangement graph [24], the bubble-sort graph [25], the \((n, k)\)-star graph [26], and the hierarchical cubic network [27], all in terms of both PMC and/or \(MM^*\) models. We notice that much of the \textit{ad hoc} derivation details, as reported in those papers devoted to different structures are essentially shared among themselves, and even with the results on the \(g\)-good-neighbour diagnosability derived for these structures [13,28–32]. While we have benefited greatly from studying many of these earlier results, we believe that it has reached a point that such a practice is no longer desirable. Indeed, such a ‘wheel’ should be reused, but not reinvented every time.
In particular, we realise that many notions related to diagnosability are independent of both fault-tolerant, and diagnostic, models. Thus, much of the reasoning behind the derivation of the diagnosability of a specific network structure under several fault-tolerant models, e.g. the $g$-good-neighbour and the $g$-extra models, are essentially the same for a specific diagnostic model such as the PMC or the MM* model. We also observe that, because of the relationships among various fault-tolerant models, results applicable to one model might follow from existing ones pertinent to another. Furthermore, some of the recently developed proof techniques can also play a role in shortening mechanical proofs of related diagnosability results. We thus believe it is time for us to generalise such common and mechanical parts, separate them from the structure dependent analysis, and investigate their applicability so that future research in this important and active area, of a more creative nature, could focus on the important issues related to structure, fault tolerant, and diagnostic, models, but not on mundane derivation details. We do notice that several results of such a general and summarising nature have already appeared in [28,32–35]. On the other hand, some of them carry a restriction of an existential nature, assuming the existence of a certain property, thus computationally expensive, and may not be effectively applicable.

In this paper, we will continue an effort that we started in [10,34], focused on the conditional fault-tolerant model, and in [28], focused on the $g$-good-neighbour fault-tolerant model, by following a constructive approach to explore, expose, and summarise such a general, commonly shared, and effectively applicable, diagnosability derivation process for both the $g$-good-neighbour and the $g$-extra models. We will also demonstrate the applicability of this process to derive several diagnosability results for various network structures in terms of the more recently suggested $g$-extra fault-tolerant model, under both the PMC and the MM* diagnostic models.

The rest of this paper proceeds as follows: In Section 2, after presenting basic notions, we provide a general derivation to an existing result between various notions of diagnosability, and derive several related results, which were justified separately in [25], to set the stage for Section 3, where we summarise a general process of deriving diagnosability results shared by both the $g$-good-neighbour and the $g$-extra fault-tolerant models, in terms of either the PMC, or the MM*, model. We demonstrate the value, and applicability, of this general process by deriving the $g$-extra diagnosability of the hypercube graph in Section 4, that of the $(n,k)$-star graph in Section 5, and that of the arrangement graph in Section 6. We conclude this paper in Section 7.

2. Relationships among fault-tolerant models

Let $G(V,E)$ represent an interconnection network, and let $M$ stand for a certain fault-tolerant model, an $M$-faulty set of $G$ is a set, $F \subset V$, consistent with $M$. For example, $F(\subset V)$ is a $g$-extra faulty set if every component in $G-F$ contains at least $g+1$ vertices. $G$ is called $M$-diagnosable in terms of a diagnostic model $D$, if $G$ is diagnosable for each and every $M$-faulty set of size at most $t$ in $D$, where $D$ refers to either the PMC model or the MM* model in the rest of this paper.

Let $F_1$ and $F_2$ be two distinct $M$-faulty sets, $F_1 \subset V(G)$ and $F_2 \subset V(G)$, the pair $(F_1,F_2)$ is distinguishable in $G$ if and only if they are not compatible with the same syndrome, thus identifiable. They are indistinguishable if they are compatible with some syndrome. Then, as originally suggested in [9] and summarised later in [12, Lemma 5], $t_M(G,D)$, the $M$-diagnosability of $G$, in terms of a diagnostic model $D$, equals the maximum number $t$ such that, for all the distinct $M$-faulty set pairs $(F_1,F_2)$, such that $F_1 \subset V, F_2 \subset V$, and $|F_1| \leq t, |F_2| \leq t$, $(F_1,F_2)$ is distinguishable in terms of $D$.

We notice that all the diagnosability notions related to existing fault-tolerant models, including, unrestricted, conditional, $g$ -good-neighbour, and $g$-extra, use this maximum restriction. We assume that all the $M$-diagnosability notions that we discuss in this paper also satisfy this requirement.

Thus, the diagnosability problem, of determining this measurement of $t_M(G,D)$ for a given graph $G(V,E)$ in terms of a fault-tolerant model $M$ under a diagnostic model $D$, really comes down to a decision problem in graph theory: Are two $M$-faulty sets of $V$ distinguishable under $D$?
In this regard, the following result specifies a necessary and sufficient condition of two faulty sets being distinguishable under the MM* model, where \( F_1 \Delta F_2 \) stands for \((F_1 \setminus F_2) \cup (F_2 \setminus F_1)\), i.e. the symmetric difference of \( F_1 \) and \( F_2 \). We notice that this result has nothing to do with either the involved fault-tolerant model \( M \), or the size of such a faulty set.

**Theorem 2.1 ([17]):** Let \( G(V, E) \) be a graph, and let \( F_1 \) and \( F_2 \) be two distinct subsets of \( V \), \( F_1 \) and \( F_2 \) are distinguishable under the MM* model if and only if at least one of the following three conditions is satisfied:

- there are two distinct vertices \( v \) and \( w \) in \( V \setminus (F_1 \cup F_2) \) and there is a vertex \( x \) in \( F_1 \setminus F_2 \) such that \((v, w, x)\) is a path in \( G \);
- there are two distinct vertices \( v \) and \( x \) in \( F_1 \setminus F_2 \) and there is a vertex \( w \) in \( V \setminus (F_1 \cup F_2) \) such that \((v, w, x)\) is a path in \( G \); and
- there are two distinct vertices \( v \) and \( x \) in \( F_2 \setminus F_1 \) and there is a vertex \( w \) in \( V \setminus (F_1 \cup F_2) \) such that \((v, w, x)\) is a path in \( G \).

One of the two symmetric scenarios of Case 1, and Case 2, of Theorem 2.1 are demonstrated in Figure 1, while Case 3 is symmetric to Case 2.

The conditions associated with the PMC model are somewhat simpler, as expected, since in the MM* model we test pairs of neighbours, but not just individual neighbours as in the PMC model. The involved situation is demonstrated in Figure 2.

**Theorem 2.2 ([20]):** Let \( G(V, E) \) be a graph. For any two distinct subsets \( F_1 \) and \( F_2 \) of \( V \), \( F_1 \) and \( F_2 \) are distinguishable under the PMC model if and only if there exist a vertex \( u \) in \( V \setminus (F_1 \cup F_2) \) and another vertex \( v \) in \( F_1 \setminus F_2 \) such that \((u, v)\) is an edge of \( G \).

To study the relationships between fault-tolerant models, we start with a general result, where the gist of the proof was used in [25, Proposition 2.2] to derive several results of a similar nature, then apply this result to derive this collection of specific results to demonstrate the fact that the reasoning behind the same diagnostic issues among different fault-tolerant models are indeed shared.

**Theorem 2.3:** Let \( G \) be a connected graph, and let \( t_{M_1}(G, D) \) and \( t_{M_2}(G, D) \) be \( M_1 \)-diagnosability, and \( M_2 \)-diagnosability of \( G \) in \( D \), respectively. If every \( M_2 \)-faulty set in \( G \) is also a \( M_1 \)-faulty set in \( G \), then \( t_{M_1}(G, D) \leq t_{M_2}(G, D) \).

![Figure 1](image1.png) **Figure 1.** The MM* distinguishability.

![Figure 2](image2.png) **Figure 2.** The PMC distinguishability.
Proof: Assume that $t_{M_1}(G, D) = t$, and let $V_1$ (respectively, $V_2$) be the collection of all the $M_1$- (respectively, $M_2$-) faulty sets $F$ in $G$ such that $|F| \leq t$. Then $V_2 \subseteq V_1$ by assumption. Let $F_1$ and $F_2$ be two distinct sets, where $F_1, F_2 \in V_2$, then $F_1, F_2 \in V_1$.

Since $|F_1|, |F_2| \leq t$, and they are both $M_1$-faulty sets, by the assumed maximum restriction, $G$ is $M_1$ $t$-diagnosable in terms of $D$. By definition, $(F_1, F_2)$ is distinguishable in $G$ in terms of the diagnostic model $D$. Since both $F_1$ and $F_2$ are also $M_2$-faulty sets, by assumption of this result, $G$ is also $M_2$ $t$-diagnosable in terms of $D$. Finally, again, by the maximum restriction, $t_{M_2}(G, D)$ is the maximum value for which $G$ is $t_{M_2}$ $t$-diagnosable in $D$. Hence $t_{M_2}(G, D) \geq t = t_{M_1}(G, D)$.

The following result motivates most of the work along this line of diagnosability research since we naturally would like an interconnection network to be more fault-tolerant, i.e. to have a larger diagnosability.

Corollary 2.4 ([25, Proposition 2.1]): Let $G$ be a system, and let $t(G, D)$ be the unrestricted diagnosability of $G$, and let $t_M(G, D)$ be the $M$-diagnosability of $G$, satisfying the maximum restriction, then, $t(G, D) \leq t_M(G, D)$.

Proof: Since any $M$-faulty set as associated with $t_M(G, D)$ is immediately an unrestricted faulty set, the result now follows Theorem 2.3.

The following two results, generalising Corollary 2.4, are intuitively true, since a stronger restriction often leads to a larger fault tolerance. Notice that with both the 0-good-neighbour faulty sets and the 0-extra faulty sets, no restriction is placed on them, thus both 0-good-neighbour and 0-extra diagnosability reduce to the traditional unrestricted diagnosability [13, Section 4], i.e. the vertex connectivity of the involved graph. Hence, such more general notions, when $g \geq 1$, do lead to a more generous characterisation of the fault-tolerant properties of a graph. We will assume $g \geq 1$ in the rest of this paper, unless explicitly pointed out otherwise.

Corollary 2.5 ([25, Proposition 2.1]): Let $G$ be a system, and let $t_g(G, D)$ be the associated $g$-good-neighbour diagnosability of $G$, then $t_g(G, D) \leq t_{g'}(G, D), 0 \leq g \leq g'$.

Proof: By definition, the $g$-good-neighbour diagnosability satisfies the maximum restriction. Moreover, if a fault-free vertex has at least $g'$ fault-free neighbours, it must have at least $g$ such neighbours when $0 \leq g \leq g'$. Hence, a $g'$-good-neighbour faulty set is a $g$-good-neighbour faulty set, and the result follows Theorem 2.3.

Corollary 2.6 ([25, Proposition 2.2]): Let $G$ be a system, and let $\tilde{t}_g(G, D)$ be the $g$-extra diagnosability of $G$, then $\tilde{t}_g(G, D) \leq \tilde{t}_{g'}(G, D), 0 \leq g \leq g'$.

Proof: By definition, the $g$-extra diagnosability satisfies the maximum restriction. Moreover, if any component in $G-F$ contains at least $g'+1$ vertices, then such a component must contain at least $g+1$ vertices when $0 \leq g \leq g'$. The result now follows Theorem 2.3.

The following two results show that the $g$-good-neighbour diagnosability of a graph is an upper bound of its $g$-extra diagnosability, except when $g = 1$. In particular, in Section 5, we will show how to make use of Corollary 2.7 to derive an upper bound of the $g$-extra diagnosability of the $(n, k)$-star graph with an existing $g$-good-neighbour diagnosability result for the same graph.

Corollary 2.7 ([25, Theorem 2.2]): Let $G$ be a system, and let $g \geq 0$, then $\tilde{t}_g(G, D) \leq t_g(G, D)$.
Proof: Assume that $F$ is a $g$-good-neighbour faulty set in $G$, then, every fault-free vertex has at least $g$ fault-free neighbours, thus, any component in $G - F$ must have at least $g + 1$ fault-free vertices, i.e. $F$ is also a $g$-extra faulty set of $G$. And the result follows from Theorem 2.3. □

Corollary 2.8 ([25, Theorem 2.3]): Let $G$ be a system, then $\tilde{t}_1(G, D) = t_1(G, D)$.

Proof: By Corollary 2.7, $\tilde{t}_1(G, D) \leq t_1(G, D)$. On the other hand, let $F$ be a 1-extra faulty set of $G$, i.e. every component of $G - F$ contains at least two vertices, then each fault-free vertex in $G - F$ has at least one fault-free neighbour, thus, $F$ is also a 1-good-neighbour faulty set. The result now follows Theorem 2.3. □

Notice that, for $g \geq 2$, it does not generally hold that $t_g(G, D) \leq \tilde{t}_g(G, D)$, since a $g$-extra faulty set is not necessarily a $g$-good-neighbour faulty set of $G$. For example, if $F$ contains a path with $g + 1$ vertices, it is certainly not a $g$-good-neighbour faulty set of $G$. Indeed, the forthcoming Theorems 6.7 and 6.10 show that, for the arrangement graph, its 2-good-neighbour diagnosability is strictly larger than its 2-extra diagnosability.

As we will discuss later in the paper, when deriving diagnosability, the case of $g = 1$ is technically challenging as far as the MM* model is concerned, and often tedious. Examples include [25, Lemma 5.2], [36, Claim A.1], [35, Theorem 2], [28, Lemma 4.2], and [26, Lemma 4.6]. On the other hand, since the notion of $g$-good-neighbour fault-tolerant model was suggested earlier than that of $g$-extra diagnosability, more results for the former model have already been achieved, thus available when seeking related results in terms of the $g$-extra fault-tolerant model. Corollary 2.8 becomes valuable in this regard, and Theorem 4.9, Corollary 5.1, and Theorem 6.3 provide examples of its application.

It is important to point out that the notion of the 1-good-neighbour conditional diagnosability is not a straightforward generalisation of the conditional diagnosability. Indeed, the notion of the 1-good-neighbour conditional diagnosability is less restrictive in the sense that the related 1-good-neighbour faulty set only requires that a non-faulty vertex have at least one non-faulty neighbour, while a conditional faulty set requires that any vertex, faulty or not, have at least one non-faulty neighbour. Thus, a conditional faulty set is immediately a 1-good-neighbour faulty set, but the other direction is not necessarily true. As a result, the 1-good-neighbour diagnosability of a graph is a lower bound of its conditional diagnosability, which also naturally follows from the general result, i.e. Theorem 2.3.

Corollary 2.9 ([25,37]): Let $G$ be a system, and let $t_c(G, D)$ be its conditional diagnosability, then $t_1(G, D) \leq t_c(G, D)$.

Proof: Every conditional faulty set of a system $G$ is a 1-good-neighbour faulty set of $G$, since if every vertex in $V(G)$ has a fault-free neighbour, every fault-free vertex in $G - F$ has at least one fault-free neighbour. The result now follows from Theorem 2.3. □

For example, for $n \geq 4$, $k \in [3, n)$, the 1-good-neighbour diagnosability of the $(n, k)$-star graph is $n + k - 2$ [28, Theorem 5.3], while its conditional diagnosability is $n + 2k - 5$ [34, Corollary 4.1].

By Corollary 2.8, we immediately have the following result.

Corollary 2.10: Let $G$ be a system, $\tilde{t}_1(G, D) \leq t_c(G, D)$.

We would like to point out that, in establishing many of the results later in this paper, we require that the fault-tolerant model be in the format that only a fault-free vertex has a property in the survival graph. We notice that this ‘fault-free’ requirement is consistent with both the $g$-good-neighbour and the $g$-extra fault-tolerant models, although not with the conditional fault-tolerant model as pointed out earlier. This is certainly not a surprise as the concept of conditional fault-tolerant model predates both the $g$-good-neighbour and $g$-extra models.
3. A general process of deriving diagnosability results

Recall that an $M$-faulty set is just a faulty vertex set $F$ in a graph $G(V, E)$, related to a certain fault-tolerant model $M$. An $M$-faulty set $F$ is also an $M$-cut if $G - F$ is disconnected. For example, a $g$-extra faulty set, $F$, is also a $g$-extra cut if $G - F$ is disconnected, where every connected component contains at least $g + 1$ vertices. Although an $M$-faulty set does not need to be an $M$-faulty cut, the construction of such an $M$-faulty cut turns out to be a crucial step to derive an $M$-diagnosability result, especially its upper bound.

The size of a minimum $M$-faulty cut of a graph $G$, on the other hand, is referred to as its $M$-connectivity, denoted by $\kappa_M(G)$. The $M$-connectivity of a graph depends on its topology, and is often tedious and challenging to derive, but it plays a critical role in deriving the lower bound of the related diagnosability of a graph. Many results to this regard have appeared in literature. Readers are referred to [38] for the result of the $g$-good neighbour connectivity, $g \in [1, 2]$, of the arrangement graph, and its connection to its $g$-good-neighbour diagnosability, $g \in [1, 2]$, in [28]. The $g$-good neighbour connectivity, $g \in [0, n - k]$, of the $(n, k)$-star graph is given in [39, Theorem 9], and its connection to its $g$-good-neighbour diagnosability is explored in [28]. A general relationship between $g$-good-neighbour connectivity and its $g$-good-neighbour diagnosability is also discussed in [28,33,35]. Moreover, the 2-extra connectivity of the bubble-sort graph is derived in [25, Theorem 3.2], and its connection to its $g$-extra diagnosability is given in [25, Theorem 5.2]. The $g$-extra connectivity of the arrangement graphs, $g = 1, 2$, and an asymptotic result for the general case, are given in [40], and the 3-extra connectivity in [41]; and the $g$-extra diagnosability results of the arrangement graph, $g \in \{1, 2, 3\}$, are presented in [24]. The $g$-extra connectivity result of the $(n, k)$-star graphs appeared in [39], and their associated $g$-extra diagnosability is recently derived in [26]. It is not surprising that all these connectivity results are structure dependent, and tedious to derive.

We will first show how to come up with an upper bound for the $M$-diagnosability of a graph $G$ via a common construction, when $M$ refers to either the $g$-good-neighbour fault-tolerant model or the $g$-extra fault-tolerant model; and then show how to derive a lower bound of its $M$-diagnosability once its $M$-connectivity result is available. These two bounds could lead to a tight one when and if they agree with each other.

3.1. Upper bound result derivation

To show that $t$ is an upper bound of $t_M(G, D)$, i.e. $t_M(G, D) \leq t$, we only need to show that, for some pair of distinct $M$-faulty sets, $F_1, F_2$, $|F_1| \leq t + 1, |F_2| \leq t + 1, V \setminus (F_1 \cup F_2) \neq \emptyset$, $(F_1, F_2)$ is indistinguishable in $G$ according to $D$.

Let $G(V, E)$ be a connected graph, and $v \in V$, we use $N(v)$ to denote the set of neighbours of $v$ in $G$, i.e. $N_G(v) = \{ w : (v, w) \in E \}$. Let $S \subset V(G)$, we use $N_G(S)$ to denote the open neighbourhood of vertices in $S$, i.e. all the neighbours of vertices of $S$ in $G$, excluding those in $S$; and use $N_G^c(S)$ to denote the closed neighbourhood of vertices in $S$, that is, $N_G(S) \cup S$. We will drop the subscript $G$ when the context is clear. A usual upper bound construction for both the $g$-good-neighbour and the $g$-extra fault-tolerant models, as shown in Figure 3, is to select an non-empty set $Y (\subset V)$, let $F_1 = N(Y)$, and $F_2 = N^c(Y)$, such that $V \neq F_2$ and both $F_1$ and $F_2$ are $M$-faulty sets.

![Figure 3. A usual M-faulty set construction.](image-url)
Since \( F_1 \cup F_2 = F_2, F_1 \Delta F_2 = Y \), and \( F_1 = N(Y) \), there cannot be an edge connecting a vertex outside \( F_1 \cup F_2 (= N^c(Y)) \neq V(G) \), by assumption, and any vertex in \( F_1 \Delta F_2 (= Y) \). Thus, \((F_1, F_2)\) is indistinguishable in PMC by Theorem 2.2, and MM* by Theorem 2.1. Finally, since \(|F_2| = |N^c(Y)|\), and \(|F_1| = |N(Y)| < |N^c(Y)|\), by the maximum restriction assumption, \( t_M(G, D) \leq |N^c(Y)| - 1 \). We summarise the above discussion into the following result.

**Proposition 3.1:** Let \( G(V, E) \) be a connected graph, \( M \) stand for either the g-good-neighbour or the g-extra fault-tolerant model, and let \( Y \subseteq V \). If both \( N(Y) \) and \( N^c(Y) \) are \( M \)-faulty sets, and \( V \setminus N^c(Y) \neq \emptyset \), then \( t_M(G, D) \leq |N^c(Y)| - 1 \).

We notice that, since \( F_1 (= N(Y)) \) is an \( M \)-faulty cut, \( |N(Y)| \geq \kappa_M(G) \). In particular, we would have \( t_M(G, D) \leq t = (|Y| + \kappa_M(G)) - 1 \), if \( N(Y) \) happens to be a minimum \( M \)-faulty cut.

**Corollary 3.2:** Let \( G(V, E) \) be a connected graph, \( M \) stand for either the g-good-neighbour or the g-extra fault-tolerant model, and let \( Y \subseteq V \). If both \( N(Y) \) and \( N^c(Y) \) are \( M \)-faulty sets, \( N(Y) \) is a minimum \( M \)-faulty cut of \( G \), and \( V \setminus N^c(Y) \neq \emptyset \), then \( t_M(G, D) \leq |Y| + \kappa_M(G) - 1 \).

The following is a slightly revised version of an earlier result, by adding the necessary assumption that \( V \setminus N^c(H) \neq \emptyset \).

**Corollary 3.3 ([32, Theorem 3.3]):** Let \( G(V, E) \) be a connected graph. If there is a connected sub-graph \( H \) of \( G \) with \( |H| = g + 1 \) such that \( N(H) \) is a minimum g-extra cut of \( G \), and \( V \setminus N^c(H) \neq \emptyset \), then \( \tilde{\kappa}_g(G, D) \leq \tilde{\kappa}_g(G) + g \).

**Proof:** By assumption of this result, \( N(H) \) is a g-extra faulty set. Consider a component \( C \) of \( V \setminus N^c(H) \), i.e. \( V(C) \subseteq V \setminus N^c(H) \). By assumption, \( V(C) \neq \emptyset \). Clearly, \( V(C) \subseteq V \setminus N(H) \). By the assumption that \( N(H) \) is a g-extra faulty set, \( C \) contains at least \( g + 1 \) vertices in \( V \setminus N(H) \). Since none of these vertices belong to \( H \), \( N^c(H) \) is also a g-extra faulty set. By assumption of this result, and Corollary 3.2, \( \tilde{\kappa}_g(G, D) \leq |H| + \tilde{\kappa}_g(G) - 1 = \tilde{\kappa}_g(G) + g \).

We notice that Corollary 3.3, although would derive a lower upper bound of the g-extra diagnosability of a graph, indeed its lowest upper bound in light of the forthcoming Corollaries 3.10 and 3.16, makes an existence assumption on a subset \( H \), which is clearly not computationally feasible to check. Hence, this result is not effectively applicable. Theorems 1 and 2 in [41], on the relationship between the g-good-neighbour connectivity and the associated diagnosability, also share such a flavor. Such a concern leads to the following result.

**Corollary 3.4:** Let \( G(V, E) \) be a connected graph, and let \( Y \subseteq V \). If \( N(Y) \) is an \( M \)-faulty set, where \( M \) stands for either the g-good-neighbour or the g-extra fault-tolerant model, and \( V \setminus N^c(Y) \neq \emptyset \), then \( t_M(G, D) \leq |N^c(Y)| - 1 \).

**Proof:** In observing the proof of Corollary 3.3, we are left to show that \( N^c(Y) \) is also a g-good-neighbour faulty set, when \( N(Y) \) is. Let \( u \in V \setminus N^c(Y) \), then \( u \in V \setminus N(Y) \). By definition, \( u \) has at least \( g \) neighbours in \( V \setminus N(Y) \). Again, since \( u \notin N(Y) \), none of its neighbours could be in \( Y \). In other words, all such at least \( g \) neighbours of \( u \) are outside \( N^c(Y) \), thus, by definition, \( N^c(Y) \) is also a g-good-neighbour faulty set. The result now follows Proposition 3.1.

It is important to point out that Proposition 3.1 and all its corollaries fail to apply to the conditional fault tolerant model, which requires any vertex, faulty or not, have at least one fault-free neighbour: Just consider \( u \in Y \), any of its neighbours belongs to either \( Y \) or \( N(Y) \) (\( \subseteq N^c(Y) \)). Since \( N(Y) \) is a conditional faulty set, such a fault-free vertex \( u (\notin N(Y)) \) must have a fault-free neighbour in \( Y \), since it could
not have a neighbour in \( V \setminus N^c(Y) \). On the other hand, since such a \( u (\in N^c(Y)) \) does not have a neighbour outside \( N^c(Y), N^c(Y) \) could not be a conditional faulty set, since this faulty vertex \( u (\in N^c(Y)) \) does not have a fault-free neighbour in terms of the faulty set \( N^c(Y) \). Indeed, a different construction was made use of in deriving an upper bound for the conditional diagnosability of the hypercube structure [12, Lemma 11], and that for Cayley graphs generated by transposition trees [8, Theorem 3]. This is part of the reason that we imposed the ‘fault-free’ restriction on the fault tolerant models that we study in this paper, where only a fault-free vertex has to have a property in the survival graph. In particular, the aforementioned faulty vertex \( u \), as a member of the conditional faulty set \( N^c(Y) \), would not need to have a neighbour outside \( N^c(Y) \).

The proof of Proposition 3.1 is rather constructive in its nature. For a construction of such a pair of \( g \)-good-neighbour faulty sets, \((N(Y), N^c(Y))\), in the arrangement graph, readers are referred to [38, Theorem 3.1]. For a similar construction for the \((n, k)\)-star graph, readers are referred to [26,28], for an example in the bubble-sort graph, readers are referred to [25, Lemma 3.4].

Recall that \( t_g(G, D) \) denotes the \( g \)-good-neighbour diagnosability of a graph \( G \) in terms of a diagnostic model \( D \), which could be either the PMC model or the MM* model. Both the \((n, k)\)-star graph [42], denoted by \( S_{n,k} \), \( n \geq 2, k \in [1, n) \), and the arrangement graph [5], denoted by \( A_{n,k} \), \( n \geq 2, k \in [1, n) \), are well studied interconnection networks. Their respective \( g \)-good-neighbour diagnosability has been obtained in [28]; and, in this paper, we will study the \( g \)-extra diagnosability of these two structures in Section 5, and Section 6, respectively. To start, we provide two specific upper bound results for their \( g \)-good-neighbour diagnosabilities, which we will make use of later in deriving the upper bound of the respective \( g \)-extra diagnosability of the \((n, k)\)-graphs and the arrangement graphs, with the help of Corollary 2.7.

**Theorem 3.5 ([28, Theorem 5.2]):** For \( n \geq 4, k \in [2, n), g \in [0, n - k] \), \( t_g(S_{n,k}, D) \leq n + g(k - 1) - 1 \).

**Theorem 3.6 ([38, Theorem 4.3]):** For \( n \geq 3, k \in [2, n), g \in [0, n - k) \), \( t_g(A_{n,k}, D) \leq (n - k)(g + 1)(k - 1) + 1 \).

### 3.2. Lower bound result derivation under the PMC model

To show that \( t \) is a lower bound of \( t_M(G, D) \), i.e. \( t_M(G, D) \geq t \), we need to show that, for any two distinct \( M \)-faulty sets, \( F_1, F_2, |F_1| \leq t, |F_2| \leq t, V \setminus (F_1 \cup F_2) \neq \emptyset \), \( (F_1, F_2) \) is distinguishable in \( G \) according to \( D \).

The following result is always used in deriving diagnosability results, as far as we know. On the other hand, it turns out that this work-horse result does not depend on either the diagnostic model, or the fault-tolerant model, as long as they are consistent with the ‘fault-free’ restriction that we imposed earlier.

**Proposition 3.7:** Let \( G(V, E) \) be a connected graph, \( M \) stand for a fault-tolerant model, and let \( F_1, F_2 \subset V \). If both \( F_1 \) and \( F_2 \) are \( M \)-faulty sets in \( G \), so is \( F_1 \cap F_2 \).

**Proof:** We show that if, \( M \), as a property, holds for any vertex in both \( G - F_1 \) and \( G - F_2 \), then it also holds for any vertex in \( G - (F_1 \cap F_2) \).

We notice that \( G - (F_1 \cap F_2) = (G - (F_1 \cup F_2)) \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1) \). Let \( u \in V \setminus (F_1 \cup F_2) \), since \( u \) is in \( V \setminus F_1, M \) holds on \( u \) by assumption. Now, let \( u \in F_1 \setminus F_2 \), since \( M \) holds on \( G - F_2 \), by definition, \( M \) holds on such a \( u \), as well. The last case when \( u \in F_2 \setminus F_1 \) can be similarly argued. Hence, \( M \) holds on any such a vertex \( u \), and \( F_1 \cap F_2 \) is indeed an \( M \)-faulty set.

The following result is just one of its many specific applications.

**Corollary 3.8:** Let \( G(V, E) \) be a connected graph. If both \( F_1 \) and \( F_2 \) are \( g \)-extra faulty sets, so is \( F_1 \cap F_2 \).
**Proof:** Let $C$ be a component in $F_1 \setminus F_2$. Since it is outside of $F_2$, a $g$-extra faulty set, it must have at least $g + 1$ fault-free vertices outside $F_2$, thus outside $F_1 \cap F_2$. The case of $C$ being in $F_2 \setminus F_1$ can be similarly argued. Finally, let $C$ be a component in $V \setminus (F_1 \cup F_2)$, it is outside both $F_1$ and $F_2$, by definition, it has to contain at least $g + 1$ fault-free vertices outside $F_1 \cup F_2$, thus outside $F_1 \cap F_2$, as well. Hence, $F_1 \cap F_2$ is also a $g$-extra faulty set. ■

The structure independent part of the general process of deriving a lower bound of $M$-diagnosability of a graph $G$ in terms of the PMC model can be summarised with the following result.

**Proposition 3.9:** Let $G(V,E)$ be a connected graph, $\kappa_M(G)$ be the $M$-connectivity of $G$, and let $F_1, F_2$ be a pair of distinct $M$-faulty sets such that $|F_1|, |F_2| \leq t$, and $V \setminus (F_1 \cup F_2) \neq \emptyset$. Without loss of generality, assuming $F_1 \setminus F_2 \neq \emptyset^3$, if $t \geq |F_1 \setminus F_2| + \kappa_M(G)$ leads to a contradiction, $t_M(G,\text{PMC}) \geq t$.

**Proof:** Assume that $(F_1, F_2)$ is indistinguishable in terms of the PMC model. By Theorem 2.2, let $u$ be a vertex in $V \setminus (F_1 \cup F_2)$, $u$ is not adjacent to any vertex in $F_1 \Delta F_2$. Since $G$ is connected, there is a path between any vertex in $V \setminus (F_1 \cup F_2)$ and another one in $F_1 \setminus F_2$, which then has to go through a vertex in $F_1 \cap F_2$. Hence, $F_1 \cap F_2$ must be a cut. By Proposition 3.7, $F_1 \cap F_2$ is an $M$-faulty cut, thus $|F_1 \cap F_2| \geq \kappa_M(G)$. Hence,

$$t \geq |F_1| = |F_1 \setminus F_2| + |F_1 \cap F_2| \geq |F_1 \setminus F_2| + \kappa_M(G).$$

If the above inequality leads to a contradiction, by the arbitrary assumption on $F_1$ and $F_2$, any such pair of $M$-faulty sets, $(F_1, F_2)$, must be distinguishable in $G$. By the maximum restrictive assumption, $t_M(G, \text{PMC}) \geq t$. ■

The value of the following result regarding a lower bound of the $g$-extra diagnosability of a graph $G$ is that, once its $g$-extra connectivity is available, it is effectively applicable via the condition of $|V| > 2(\tilde{k}_g(G) + g)$, which does depend on $G$.

**Corollary 3.10:** Let $G(V,E)$ be a connected graph, and let $\tilde{k}_g(G)$ be its $g$-extra connectivity, $g \geq 1$. If $|V| > 2(\tilde{k}_g(G) + g)$, then $\ell_g(G, \text{PMC}) \geq \tilde{k}_g(G) + g$.

**Proof:** Let $F_1$ and $F_2$ be a pair of distinct $g$-extra faulty sets, $|F_1| \leq t = \tilde{k}_g(G) + g$, and $|F_2| \leq t = \tilde{k}_g(G) + g$. Assume that $(F_1, F_2)$ is indistinguishable in terms of the PMC diagnostic model. By the assumption of this result,

$$|V \setminus (F_1 \cup F_2)| = |V| - |F_1 \cup F_2| = |V| - (|F_1| + |F_2| - |F_1 \cap F_2|) = |V| - (|F_1| + |F_2|) + |F_1 \cap F_2| \geq |V| - (|F_1| + |F_2|) \geq |V| - 2(\tilde{k}_g(G) + g) > 0.$$

Hence, $V \setminus (F_1 \cup F_2) \neq \emptyset$.

Since $G$ is connected, and, by the indistinguishable assumption, any path between a vertex in $V \setminus (F_1 \cup F_2)$ and another in $F_1 \Delta F_2$ has to go through a vertex in $F_1 \cap F_2$. Hence, $F_1 \cap F_2$ is a cut. Since both $F_1$ and $F_2$ are assumed to be $g$-extra faulty sets, so is $F_1 \cap F_2$ by Corollary 3.8. Thus, $F_1 \cap F_2$ is a $g$-extra cut, and $|F_1 \cap F_2| \geq \tilde{k}_g(G)$.

On the other hand, as $F_1 \neq F_2$, without loss of generality, assume that $F_1 \setminus F_2 \neq \emptyset$. Since $F_1 \setminus F_2$ is a cut, so is $F_2$. Let $C$ be a component such that $V(C) \cap (F_1 \setminus F_2) \neq \emptyset$. By the assumption that $F_1$ and $F_2$ are indistinguishable in terms of the PMC model, and Theorem 2.2, no vertex in $V \setminus (F_1 \cup F_2)$ is adjacent to another vertex in $F_1 \setminus F_2$. Hence, $V(C) \subseteq F_1 \setminus F_2$. By the assumed $g$-extra nature of $F_2$, since $V(C) \cap F_2 = \emptyset$, $|F_1 \setminus F_2| \geq |V(C)| \geq g + 1$. 

Corollary 3.13 ([28, Theorem 5.3]): Corollary 2.5, showing the routine arithmetic shows that
Proof: Theorem 3.12 ([43]):

\[ t \geq |F_1 \setminus F_2| + \tilde{\kappa}_g(G) \]
\[ \tilde{\kappa}_g(G) \geq t \geq |F_1 \setminus F_2| + \tilde{\kappa}_g(G) \geq (g + 1) + \tilde{\kappa}_g(G), \]
by Proposition 3.3, \( t \tilde{g}(G, \text{PMC}) \geq \tilde{\kappa}_g(G) + g. \)

Almost the same argument establishes the following result, as we notice that, under the \( g \)-good-neighbour circumstances, \( F_1 \setminus F_2 \) contains at least \( g + 1 \) vertices since any vertex in \( F_1 \setminus F_2 \) has at least this many neighbours because of the \( g \)-good-neighbour nature of \( F_2 \), and all such vertices belong to \( F_1 \setminus F_2 \) because of the indistinguishable assumption and Theorem 2.2.

**Corollary 3.11:** Let \( G(V, E) \) be a connected graph, and let \( \kappa_g(G) \) be its \( g \)-good-neighbour connectivity. If \( |V| > 2(\kappa_g(G) + g) \), then \( t_g(G, \text{PMC}) \geq \kappa_g(G) + g. \)

As a demonstration of the value of Proposition 3.9 and its corollaries, we provide an alternative derivation of the \( g \)-good-neighbour diagnosability of the \((n, k)\)-start graph by starting with the following connectivity result.

**Theorem 3.12 ([43]):** Let \( \kappa_g(G) \) be the \( g \)-good-neighbour connectivity of \( G \), for \( n \geq 3, k \in [2, n), g \in [0, n - k], \kappa_g(S_{n,k}) = n + g(k - 2) - 1. \)

The following result follows directly from Corollary 3.11, and provides a concrete instance of Corollary 2.5, showing the \( g \)-good-neighbour diagnosability of the \((n, k)\)-star graph increases monotonically in terms of \( g \).

**Corollary 3.13 ([28, Theorem 5.3]):** Let \( t_g(G) \) be the \( g \)-good-neighbour diagnosability of \( G \), for \( n \geq 4, k \in [2, n), g \in [0, n - k], t_g(S_{n,k}, \text{PMC}) = n + g(k - 1) - 1. \)

**Proof:** Routine arithmetic shows that \( |V(S_{n,k})| = n!/(n-k)! > 2[n + g(k - 1) - 1] = 2[\kappa_g(S_{n,k}) + g] [28, Theorem 5.2] \), when \( n \geq 4, k \in [2, n) \). Notice that this condition leads to a restriction on the parameters \( n \) and \( k \). Thus, by Corollary 3.11, \( t_g(S_{n,k}, \text{PMC}) \geq n + g(k - 1) - 1, \) which is actually a tight bound, thanks to Theorem 3.5.

We remark that the above process is much shorter, and cleaner, than the original one, as shown in [28].

### 3.3. Lower bound result derivation under the MM* model

The process of deriving a lower bound result of the \( M \)-diagnosability in terms of the MM* diagnostic model is essentially the same as that for the PMC model, except that we also need to show that no isolated vertex exists in the non-empty \( V(G) \setminus (F_1 \cup F_2) \), where \( (F_1, F_2) \) is the pair of indistinguishable \( M \)-faulty sets that we use to construct the desired contradiction as required in Proposition 3.9. The reason for this additional requirement, the ‘isolation condition’ henceforth, is that, in this MM* case, if \( u \) is isolated in \( V(G) \setminus (F_1 \cup F_2) \), it can be adjacent to some vertex in \( F_1 \triangle F_2 \). Then, \( F_1 \cap F_2 \) would not be a cut, thus the reasoning as we followed earlier in establishing Proposition 3.9 is no longer applicable.

**Proposition 3.14:** Let \( G(V, E) \) be a connected graph, \( \kappa_M(G) \) be the \( M \)-connectivity of \( G \), and let \( F_1, F_2 \) be two \( M \)-faulty sets, such that \( |F_1|, |F_2| \leq t \), and the non-empty set of \( V \setminus (F_1 \cup F_2) \) contains no isolated vertex. Assume that \( F_1 \setminus F_2 \neq \emptyset \), if \( t \geq |F_1 \setminus F_2| + \kappa_M(G) \) leads to a contradiction, then \( t_M(G, \text{MM}^*) \geq t. \)

**Proof:** Assume that \( (F_1, F_2) \) is indistinguishable in \( G \) in terms of the MM* diagnostic model. Let \( u \in V \setminus (F_1 \cup F_2) \), by the assumed ‘isolation condition’, \( u \) is not isolated, thus adjacent to another vertex
Assume that $w \in V \setminus (F_1 \cup F_2)$. By the indistinguishable assumption and Theorem 2.1, $u$ is not adjacent to any vertex in $F_1 \Delta F_2$. Thus, $F_1 \cap F_2$ is a cut by the assumption that $G$ is connected. Together with Proposition 3.7, $F_1 \cap F_2$ is a $M$-faulty cut. Hence, $|F_1 \cap F_2| \geq \kappa_M(G)$.

By assumption of this result,

$$t \geq |F_1| = |F_1 \setminus F_2| + |F_1 \cap F_2| = |F_1 \setminus F_2| + \kappa_M(G).$$

If the above inequality leads to a contradiction, then, any such pair of $M$-faulty sets, $(F_1, F_2)$, must be distinguishable in $G$. By the assumed maximum restriction, $t_M(G, \text{MM}^*) \geq t$. ■

It turns out that this additional isolation condition is not needed when $g \geq 2$ for both $g$-good-neighbour and $g$-extra fault tolerant models as shown in the following Corollaries 3.15 and 3.16, respectively.

**Corollary 3.15:** Let $G(V, E)$ be a connected graph, and let $\kappa_g(G), g \geq 2$, be the $g$-good-neighbour connectivity of $G$. If $|V| > 2(\kappa_g(G) + g)$, then $t_g(G, \text{MM}^*) \geq \kappa_g(G) + g$.

**Proof:** Let $F_1, F_2$ be two $g$-good-neighbour faulty sets, $|F_1| \leq t = \kappa_g(G) + g$, $|F_2| \leq t = \kappa_g(G) + g$. Assume that $(F_1, F_2)$ is indistinguishable.

The same argument as made in proving Corollary 3.10 shows that the condition of $|V| > 2(\kappa_g(G) + g)$ implies that $V \setminus (F_1 \cup F_2) \neq \emptyset$.

Let $w$ be a vertex in $V \setminus (F_1 \cup F_2)$. Since $F_1$ (respectively, $F_2$) is a $g$-good-neighbour faulty set, $w$ has at least $g$ ($\geq 2$) neighbours outside $F_1$ (respectively, $F_2$). Since $F_1$ and $F_2$ are indistinguishable, $w$ will have at most one neighbour in $F_2 \setminus F_1$ (respectively, $F_1 \setminus F_2$). Thus, it has at least $g - 1$ ($\geq 1$) neighbour(s) in $V \setminus (F_1 \cup F_2)$. In other words, $w$ could not be isolated in $F_1 \cup F_2$.

The same argument as made in proving Corollary 3.11 shows that $|F_1 \setminus F_2| \geq g + 1$. The result now follows from Proposition 3.14 since the assumption that

$$\kappa_g(G) + g = t \geq |F_1 \setminus F_2| + \kappa_g(G) \geq g + 1 + \kappa_g(G)$$

would lead to a contradiction, showing that such a pair of $g$-good-neighbour faulty sets must be distinguishable, and the result follows from the ‘maximum restriction’. ■

**Corollary 3.16:** Let $G(V, E)$ be a connected graph, and let $\kappa_g(G), g \geq 2$, be the $g$-extra connectivity of $G$. If $|V| > 2(\tilde{\kappa}_g(G) + g)$, then $\tilde{t}_g(G, \text{MM}^*) \geq \tilde{\kappa}_g(G) + g$.

**Proof:** Let $F_1, F_2$ be two $g$-extra faulty sets, $|F_1| \leq t = \tilde{\kappa}_g(G) + g$, $|F_2| \leq t = \tilde{\kappa}_g(G) + g$. Assume that $(F_1, F_2)$ is indistinguishable. Let $w \in V \setminus (F_1 \cup F_2)$ and let $C$ the component that contains $w$. By the indistinguishable nature of $F_1$ and $F_2$, $|V(C) \cap (F_1 \setminus F_2)| \leq 1$, and $|V(C) \setminus (F_2 \setminus F_1)| \leq 1$. Furthermore, by the $g$-extra assumption on both $F_1$ and $F_2$, $C$ contains at least $g + 1 \geq 3$ vertices outside $F_1$ and $F_2$. Hence, $C$ would have to contain at least another vertex in $V \setminus (F_1 \setminus F_2)$. Thus, $w$ is not isolated, either.

The same argument as made in proving Corollary 3.10 shows that $|F_1 \setminus F_2| \geq g + 1$. The rest of the proof is the same as that for the above result. ■

We give an example in this regard as follows:

**Theorem 3.17 ([38]):** For $n \geq 8, \kappa_2(A_{n,2}) = 4n - 12$; and, for $k \in [3, n - 5] \cup \{n - 2, n - 1\}, \kappa_2(A_{n,k}) = (3k - 2)(n - k) - 2$.

**Corollary 3.18 ([28]):** For $n \geq 7, k \in [4, n - 1], t_2(A_{n,k}, \text{MM}^*) = (3k - 2)(n - k)$. 
**Proof:** Routine arithmetic shows that \(|V(A_{n,k})| = n!/(n-k)! > 2[(3k-2)(n-k)]| [28, Theorem 4.4]|, when \(n \geq 7, k \in [4, n-1]\). By Corollary 3.15 and Lemma 3.17, \(t_2(A_{n,k}, MM^*) \geq (3k-2)(n-k)\), which is actually a tight bound by Theorem 3.6, taking \(g = 2\).

We give another example, where we have to enforce the isolation condition when \(g = 1\). We start with the important connectivity result.

**Theorem 3.19 ([38]):** For \(n \geq 3, n \neq 4, k \in [2, n]\), \(\kappa_1(A_{n,k}) = (2k-1)(n-k) - 1\). And \(\kappa_1(A_{4,2}) = \kappa_1(A_{4,3}) (= \kappa_1(S_4)) = 4\).

The following provides the needed no-isolation-vertex result.

**Lemma 3.20 ([28]):** Let \(F_1, F_2\) be two distinct 1-good-neighbour conditional cuts of \(A_{n,k}, n \geq 6, k \in [5, n-1]\), or \(n \geq 11, k \in [10, n]\), \(|F_1|, |F_2| \leq (2k-1)(n-k)\), such that \(V \setminus (F_1 \cup F_2) \neq \emptyset\). Then, \(V \setminus (F_1 \cup F_2)\) contains no isolated vertices.

We are now ready to achieve the following lower bound result.

**Corollary 3.21 ([28]):** For \(n \geq 6, k \in [5, n-1]\), or \(n \geq 11, k \in [10, n]\), \(t_1(A_{n,k}, MM^*) = (2k-1)(n-k)\).

**Proof:** Let \(F_1, F_2\) be two distinct 1-good-neighbour faulty sets, \(|F_1|, |F_2| \leq t = (2k-1)(n-k)\), and assume that \((F_1, F_2)\) is indistinguishable in terms of the MM* model.

Routine arithmetic shows that \(|V(A_{n,k})| = n!/(n-k)! > 2[(2k-1)(n-k)] = 2[k_2(A_{n,k}) + 1]|, when \(n \geq 5, k \in [2, n]\). Thus, \(|V(A_{n,k})/(F_1 \cup F_2)|\) does not contain an isolated vertex by Lemma 3.20.

Without loss of generality, \(F_1 \setminus F_2 \neq \emptyset\). By an argument similar to that made in Corollary 3.10, \(|F_1 \setminus F_2| \geq 2\). Since \((2k-1)(n-k) = t \geq |F_1 \setminus F_2| + \kappa_1(G) \geq (2k-1)(n-k) - 1 + 2 = (2k-1)(n-k) + 1\), \(t\) is a contradiction, by Proposition 3.14, \(t_1(A_{n,k}, MM^*) \geq (2k-1)(n-k)\), which is again a tight bound by Theorem 3.6, taking \(g = 1\).

To recapitulate, a uniform construction of an appropriate faulty cut in a graph \(G\) leads to an upper bound of both the \(g\)-good-neighbour, and \(g\)-extra diagnosability of \(G\), in terms of both the PMC model and the MM* model, as shown in Corollary 3.4. On the other hand, the size of a minimum \(g\)-extra faulty cut of \(G\), referred to as its \(g\)-extra connectivity, plays a critical role to derive a lower bound of its \(g\)-extra diagnosability, as shown in Corollary 3.10 for the PMC model, and in Corollary 3.16 for the MM* model. The same situation arises for the \(g\)-good-neighbour diagnosability, as shown in Corollary 3.11 for the PMC case, and in Corollary 3.15 for the MM* case.

When, and if, these two bounds agree, we will obtain the exact bound of the desired \(M\) diagnosability of the graph \(G\) in terms of a certain diagnostic model \(D\). In the rest of this paper, we will apply this general process to derive the \(g\)-extra diagnosabilities of the hypercube, the \((n,k)\)-star, and the arrangement graph.

### 4. The \(g\)-extra diagnosability of the hypercube graph

The hypercube, \(Q_n\), of Harary et al. [2] is perhaps one of the most studied, also the simplest, interconnection networks, with commercial applications [44,45]. It is \(n\)-regular, both vertex and edge transitive, with small diameter. Several hypercube variants have also been suggested, including augmented cubes, crossed cubes, enhanced cubes, folded cubes, moebius cubes, twisted cubes, and (generalised) exchanged cubes. Many algorithms have been designed to run on these hypercube based architectures to solve realistic issues in applications. As a recent example, Bcube, a general hypercube-based structure, was suggested in [46] as a network structure to support reconﬁgurable modular data centres.
The \(g\)-extra diagnosability of the hypercube structure, in terms of both the PMC and the MM* models, have been derived earlier [15,47,48] by following a structure dependent derivation process. As an opening example, we will show how to follow the general process that we discussed in the previous section to derive the \(g\)-extra diagnosability of the hypercube structure, denoted by \(Q_n\), for \(n \geq 4\), and \(g \in [1, n - 3]\). We will also explore its \(g\)-extra diagnosability for a wider range of \(n\), making use of some recent results on its \(g\)-extra connectivity.

Let \(K_2\) be the complete graph with two vertices 0 and 1; and let \(\square\) be the Cartesian product, \(Q_n\), \(n (\geq 2)\), can be defined as follows:

\[
\begin{align*}
Q_1 &= K_2, & \text{for all } n \geq 2, \\
Q_n &= K_2 \square Q_{n-1}.
\end{align*}
\]

Thus, a vertex of \(Q_n\), \(u\), can be represented as an \(n\)-bit binary string: \((u_0, u_1, \ldots, u_{n-1})\), where, for all \(i \in [0, n-1]\), \(u_i \in \{0, 1\}\). Clearly, \(Q_n\) contains \(2^n\) vertices, and two vertices of \(Q_n\) are adjacent to each other if and only if their corresponding binary strings differ in exactly one position.

We first seek an upper bound of \(\hat{t}_g(Q_n, D)\), the \(g\)-extra diagnosability of \(Q_n\), \(n \geq 4\), in terms of a diagnostic model, \(D\), which could be either the PMC model or the MM* model, through the usual construction, originally suggested in [16, Theorem 4.3].

Let \(Y\) be a star graph \(K_{1, g}\), \(g \in [0, n - 3]\), \(n \geq 4\), consisting of \(g + 1\) vertices \(u_0, u_1, \ldots, u_g\) such that

\[
\begin{align*}
|N(Y)| &= g(n - 1) - \left(\frac{g}{2}\right) + (n - g) \\
&= \frac{1}{2}(g + 1)[2(n - 1) - g] + 1, \quad \text{and,} \\
|N^c(Y)| &= |N(Y)| + |Y| = \frac{1}{2}(g + 1)[2(n - 1) - g] + 1 + (g + 1) \\
&= \frac{1}{2}(g + 1)(2n - g) + 1.
\end{align*}
\]

For example, in a \(Q_4\), as shown in Figure 4, if we choose \(K_{1, 2}\), consisting of \(u_0 = 0000\) and \(u_1 = 0010\), then \(Y = \{u_0, u_1\}\). Since \(u_0\) is also adjacent to \(0001, 0100, 1000\), and \(u_1\) is adjacent to three other vertices: \(0110, 0011, 1011\), \(N(Y)\) contains these six vertices, while \(N^c(Y)\) contains two more vertices in \(Y\). Indeed, Equations (1) and (2) return 6 and 8, respectively.

We need to show that \(N(Y)\) is a \(g\)-extra faulty set, i.e. every component in \(Q_n - N(Y)\) contains at least \(g + 1\) vertices, and \(|V(Q_n)| > |N^c(Y)|\), so that we can apply Corollary 3.4 to obtain

\[
\hat{t}_g(Q_n, D) \leq |N^c(Y)| - 1 = \frac{1}{2}(g + 1)(2n - g).
\]

To this regard, we decompose \(Q_n\) to \(Q_{n-1}^L\) and \(Q_{n-1}^R\), such that \(Q_{n-1}^L\) (respectively, \(Q_{n-1}^R\)) contains vertices \((0, a_1, \ldots, a_{n-1})\) (respectively, \((1, a_1, \ldots, a_{n-1})\)) where, for all \(i \in [1, n - 1]\), \(a_i \in \{0, 1\}\). It is clear that, both \(Q_{n-1}^L\) and \(Q_{n-1}^R\) are isomorphic to \(Q_{n-1}\), and each vertex in \(Q_{n-1}^L\) is associated with a unique vertex in \(Q_{n-1}^R\). In particular, each of the \(g + 1\) vertices, \(u_i, i \in [0, g]\), in \(Y \subseteq V(Q_{n-1}^L)\) has a unique neighbour \(u'\) in \(Q_{n-1}^R\). For example, as shown in Figure 4, \(u_0' = 1000\), and \(u_1' = 1010\).
Theorem 4.1 ([16]): Let $G$ be an $r$-regular graph. If $G$ is super $p$-vertex-connected of order $q$, then the restricted vertex connectivity of order $q + 1$ is at least $p + 1$.

Since, for $n \geq 4$ and $k \in [1, n - 2]$, $Q_n$ is super $(kn - k(k + 1)/2)$-vertex connected of order $k - 1$ [50], we immediately have the following result, which first appeared in [51].
Corollary 4.2 ([16]): Let \( n \geq 4 \) and \( k \in [1, n - 2] \), the restricted vertex connectivity of order \( k \) of \( Q_n \) is \( [kn - k(k + 1)/2] + 1 \).

Setting \( k = g + 1 \), we have the following result, after simplification.

Corollary 4.3: Let \( n \geq 4 \) and \( g \in [0, n - 3] \), \( \kappa_g(Q_n) = \frac{1}{2}(g + 1)[2(n - 1) - g] + 1 \).

We are now ready to derive the \( g \)-extra diagnosability of \( Q_n \) with the PMC model.

Theorem 4.4: Let \( n \geq 4 \) and \( g \in [1, n - 3] \), \( \tilde{\kappa}_g(Q_n, \text{PMC}) = \frac{1}{2}(g + 1)(2n - g) \).

Proof: By Equation (3), and Corollary 3.10, we only need to prove \( 2^n > (g + 1)(2n - g) \), which holds if

\[ 2^n > (n - 2)(2n - 1). \]

It is certainly true when \( n \geq 2 \). \( \blacksquare \)

The above result slightly generalises the one achieved in [15, Theorem 3.11], when \( n \geq 4 \), \( g \in [0, n - 4] \).

Given a graph \( G(V, E) \), and let \( Y \subseteq V \), the vertex boundary number of \( Y \) is simply \( |N(Y)| \), denoted by \( b_v(H; G) \); and the minimum \( k \)-boundary number of \( G \) is defined as the minimum boundary number of all its subgraphs with order \( k \), denoted by \( b_v(k; G) \). The relationship between \( \kappa_g(Q_n) \), the \( g \)-extra connectivity of \( Q_n \), and \( b_v(k; Q_n) \) is recently explored in [52]. As a result, \( \kappa_g(Q_n) \) is derived for a much bigger range of \( g \in [0, 3n - 7] \).

Theorem 4.5 ([52]):

\[
\kappa_g(Q_n) = \begin{cases} 
-\frac{1}{2}(g + 1)^2 + \left(n - \frac{1}{2}\right)(g + 1) + 1 & \text{if } n \geq 5, g \in [0, n - 4] \\
-\frac{1}{2}(n - 2)^2 + \left(n - \frac{1}{2}\right)(n - 2) + 1 & \text{if } n \geq 5, g \in [n - 3, n] \\
-\frac{1}{2}(g + 1)^2 + \left(2n - \frac{3}{2}\right)(g + 1) - n^2 + 2 & \text{if } n \geq 7, g \in [n + 1, 2n - 5] \\
-\frac{1}{2}(2n - 3)^2 + \left(2n - \frac{3}{2}\right)(2n - 3) - n^2 + 2 & \text{if } n \geq 7, g \in [2n - 4, 2n - 1], \text{ and,} \\
-\frac{1}{2}(g + 1)^2 + \left(3n - \frac{7}{2}\right)(g + 1) - 3n^2 + 4n + 2 & \text{if } n \geq 9, g \in [2n, 3n - 7].
\end{cases}
\]

We notice that, for \( n \geq 5, g \in [0, n] \), this extended \( \kappa_g(Q_n) \) agrees with the result as shown in Corollary 4.3. This extended \( g \)-extra connectivity result has also been used to derive the following \( g \)-extra diagnosability result for \( Q_n \).

Theorem 4.6 ([48]): Let \( n \geq 9, g \in [0, 3n - 7] \), then \( \tilde{\kappa}_g(Q_n, \text{PMC}) = \kappa_g(Q_n) + g \).

Similarly, by Equation (3), and Corollary 3.16, we have the following \( g \)-extra diagnosability result for \( Q_n \) in terms of the MM* model.

Corollary 4.7: Let \( n \geq 4 \) and \( g \in [2, n - 3] \), \( \tilde{\kappa}_g(Q_n, \text{MM*}) = \frac{1}{2}(g + 1)(2n - g) \).
Theorem 4.8 ([50]): If \( n \geq 4 \) with \( k \in [1, n - 1] \), then \( Q_n \) is super \( kn - k(k+1)/2 \)-vertex connected of order \( k - 1 \).

In other words, let \( F \subseteq V(Q_n) \), \( |F| \leq kn - k(k+1)/2 \), then either \( Q_n - F \) is connected; or \( Q_n - F \) contains a large component and all the smaller components contains at most \( k - 1 \) vertices. Equivalently, if all such smaller components contain, in particular, any of them contains, at least \( k \) vertices, then \( |F| \geq [kn - k(k+1)/2] + 1 \).

Now let \( F_1, F_2 \) be any distinct 1-extra faulty sets such that \( |F_1|, |F_2| \leq 5 \). By assumption, since \( |F_1 \cup F_2| \leq 10 \), \( V(Q_4) \setminus (F_1 \cup F_2) \neq \emptyset \). Let \( w \) be any vertex in \( Q_4 - (F_1 \cup F_2) \), it is shown in [47, Proposition 3.8] that \( w \) is not isolated in \( Q_4 - (F_1 \cup F_2) \). If \( (F_1, F_2) \) is not distinguishable, by Theorem 2.1, \( w \) is not adjacent to any vertex in \( F_1 \Delta F_2 \). Since \( Q_4 \) is connected, \( F_1 \cap F_2 \) must be a cut. Furthermore, by assumption, both \( F_1 \) and \( F_2 \) are 1-extra faulty sets, so is \( F_1 \cap F_2 \) by Corollary 3.8. Let \( C, |C| \geq 2 \), be a minimum component of \( Q_4 - (F_1 \cap F_2) \). Since \( Q_4 - (F_1 \cap F_2) \) is disconnected, by Lemma 4.8, taking \( n = 4, k = 2 \), to have such a component \( C \) containing at least two vertices, \( |F_1 \cap F_2| \geq 6 \). On the other hand, since \( F_1 \neq F_2 \), without loss of generality, assume that \( |F_1 \setminus F_2| \geq 1 \), then \( 6 \leq |F_1 \cap F_2| = |F_1| - |F_1 \setminus F_2| \leq 4 \), which is a contradiction. Hence, it must be the case that \( (F_1, F_2) \) is distinguishable, namely, \( \tilde{t}_1(Q_4) \geq 5 \).
Theorem 4.9: \( \tilde{t}_1(Q_3, \text{MM}^*) = 3, \tilde{t}_1(Q_4, \text{MM}^*) = 5, \) and, for \( n \geq 5 \) and \( g \in [1, n-3] \), \( \tilde{t}_g(Q_n, \text{MM}^*) = \frac{1}{2}(g+1)(2n-g) \).

The above result as shown in Theorem 4.9 agrees with the one achieved in \([47]\), where \( n \geq 5, g \in [1, \frac{n-1}{4}] \), and we notice that \( n-3 > \frac{n-1}{4} \) when \( n \geq 4 \).

We would like to point out that, the general \( g \)-extra connectivity result for the hypercube structure, as stated in Theorem 4.5, can also be used to derive the following \( g \)-extra diagnosability result for \( Q_n \) under the MM* model, by applying the aforementioned general derivation process.

Theorem 4.10: \( \tilde{t}_1(Q_3, \text{MM}^*) = 3, \tilde{t}_1(Q_4, \text{MM}^*) = 5, \) and, for \( n \geq 5, g \in [0, 3n-7] \), \( \tilde{t}_g(Q_n, \text{MM}^*) = \tilde{k}_g(Q_n) + g \), where \( \tilde{k}_g(Q_n) \) is given in Theorem 4.4 with the proper and respective range of \( g \).

Proof: Beside the special cases when \( n \leq 4 \), and \( g = 1 \), by Corollaries 4.3, 3.3, the usual upper bound construction, together with the justification as given in \([48, \text{Lemma } 3.1]\), we have

\[
\tilde{t}_g(Q_n, \text{MM}^*) \leq \tilde{k}_g(Q_n) + g.
\]

It is also a routine check to verify that, for all the cases, \( |V(Q_n)| = 2^n > 2(\tilde{k}_g(Q_n) + g) \). Hence, by Corollary 3.16, for all \( g \geq 2 \),

\[
\tilde{t}_g(Q_n, \text{MM}^*) \geq \tilde{k}_g(Q_n) + g.
\]

This completes the proof of this result.

5. The \( g \)-extra diagnosability of the \((n, k)\)-star graph

The star graph, denoted by \( S_n \), was proposed in [3] as an attractive alternative to the hypercube structure when used as an interconnection network. For comparison between the hypercube and the star graph, readers are referred to [42,53]. However, the requirement that the number of vertices in the star graph be \( n! \) results in a large size gap between \( S_n \) and \( S_{n+1} \). To address this scalability issue, the \((n, k)\)-star graph was suggested in [6], which brings in a flexibility in choosing its size, while preserving many attractive properties of the star graph, including vertex symmetry. The \((n, k)\)-star graph has been well studied in the literature, including its fault-tolerant properties, e.g. [10,26,28,39,43,54].

Let \( \langle n \rangle \) stand for \( \{1, 2, \ldots, n\} \), \( V(S_{n,k}), n \geq 2, k \in [1, n] \), is simply the collection of all the \( k \)-permutations taken out of \( \langle n \rangle \), thus, \( |V| = n!/(n-k)! \). Let \( u = [p_1, \ldots, p_k] \), \( v = [q_1, \ldots, q_k] \), \( (p, q) \in E(S_{n,k}) \) either, for some \( i \in [2, k] \), \( v \) can be obtained from \( u \) by swapping \( p_1 \) and \( p_i \) (i-edge); or, for some \( e \in \langle n \rangle \setminus \{p_1, \ldots, p_k\} \), \( v \) can be obtained from \( u \) by replacing \( p_1 \) with \( e \) (1-edge). Thus, \( S_{n,k} \) is an \( n-1 \) regular graph, containing exactly \( [(n-1)n!]/[2(n-k)!] \) edges.

It is easy to see and well known that the connectivity of the \((n, k)\)-star graph is \( n-1 \) \([6, \text{Theorem } 9]\). Thus, by Corollary 2.4, it is immediate that, for \( g \geq 1 \), \( t_g(S_{n,k}) \geq t_0(S_{n,k}) = n-1 \), and \( \tilde{t}_g(S_{n,k}) \geq \tilde{t}_0(S_{n,k}) = n-1 \).

Let \( H_{ij}, i \in [1, n] \), be the collection of all the vertices of \( S_{n,k} \), where the corresponding \( k \)-permutation ends with \( p_k = i \), and let \( S_{n,k}^i \) be the subgraph of \( S_{n,k} \) with its vertex set being \( H_i \); it is easy to see and well known that \( S_{n,k}^i \) is isomorphic to \( S_{n-1,k-1} \). Moreover, every vertex in \( S_{n,k}^i \) has a unique neighbour in \( S_{n,k}^{i'} \), if \( i \neq i' \), and for each pair of \( S_{n,k}^i \) and \( S_{n,k}^{i'} \), there are exactly \( (n-2)!/(n-k)! \) independent edges connecting their respective vertices between. Readers are referred to [6,39] for more details.

For example, \( [1, 2] \) (represented as 12) is a vertex in \( S_{4,2} \), as shown in Figure 5, where \( (1, 2), (3, 2) \) and \( (1, 2), (4, 2) \) are both 1-edges, and \( (1, 2), (2, 1) \) is a 2-edge. Clearly, \( H_2 = \{[1, 2], [3, 2], [4, 2]\} \), which are the vertices of \( S_{4,2}^1 \). Moreover, the vertex \([1, 2]\) has a unique neighbour \([2, 1]\) in \( S_{4,2}^1 \), and this 2-edge \( ([1, 2], [2, 1]) \) is a unique one between \( S_{4,2}^1 \) and \( S_{4,2}^2 \).

Issues related to the \( g \)-good-neighbour diagnosability of the \((n, k)\)-star graph have been addressed in \([28,29]\). Its \( g \)-extra diagnosability has also been derived recently in \([26]\) by following a structure
dependent approach. It turns out that these results on g-extra diagnosability of the \((n, k)\)-star graph also follow the general result that we have derived in the previous section, thus all the structure independent technical details are unnecessary, and could be spared.

Since it has already been proved that \(t_1(S_{n,k}, D) = n + k - 2\) \([28, \text{Theorem } 5.3]\) (PMC) and \([29]\) (MM*), by Corollary 2.8, we immediately have the following result.

**Corollary 5.1:** Let \(n \geq 4, k \in [2, n), \tilde{t}_1(S_{n,k}, D) = n + k - 2.\)

We now move forward to the cases of \(g \geq 2.\) By Corollary 2.7 and Theorem 3.5, we obtain the following upper bound result for the \(g\)-extra diagnosability of the \((n, k)\)-star graph.

**Lemma 5.2:** For \(n \geq 4, k \in [2, n), g \in [0, n - k], \tilde{t}_g(S_{n,k}, D) \leq n + g(k - 1) - 1.\)

We again make use of the super-connectedness property of the \((n, k)\)-star graph to derive the \(g\)-extra connectivity of various graphs \([24,26]\), then its \(g\)-extra diagnosability. We start with the following observation.

**Theorem 5.3 ([39, Theorem 8]):** Let \(n \geq 4, k, r\) be positive integers such that \(k \in [2, n)\) and \(r \in [1, n - k + 1].\) If \(T\) is a set of vertices of \(S_{n,k}\) such that \(|T| \leq n + (r - 1)k - 2r\), then \(S_{n,k} - T\) is either connected or has a large component and small components with at most \(r - 1\) vertices in total.

Thus, for \(r \in [1, n - k + 1], (n, k)\)-star graphs are super \(n + (r - 1)k - 2r\)-vertex connected of order \(r - 1.\) Taking \(r = g + 1,\) if we want to have a component, beside the larger one, in a disconnected survival graph \(S_{n,k} - T,\) which contains at least \(g + 1\) vertices, \(g \in [0, n - k],\) we have to remove at least \(n + g(k - 2) - 1\) vertices, i.e.

\[\tilde{t}_g(S_{n,k}) \geq n + g(k - 2) - 1.\]

The above lower bound is actually tight. Indeed, as suggested in \([39]\), let \(Y = \{u_1, u_2, \ldots, u_{g+1}\},\)

where \(u_j = [j, n - k + 2, \ldots, n], j \in [1, g + 1].\) \(Y\) is well defined since \(g + 2 \leq n - k + 2 \equiv g \leq n - k.\)

By definition, each vertex \(u_j\) has \(k - 1\) neighbours through \(i\)-edges, \(i \in [2, k],\) thus \((g + 1)(k - 1)\) distinct \(i\)-neighbours in total, since each \(u_j, j \in [1, g + 1],\) starts with a distinct symbol. Moreover, since a total of \(k + g\) symbols have occurred in \(Y,\) there are \(n - k - g\) distinct neighbours of vertices in \(Y\) through \(1\)-edges by switching \(j \in [1, g + 1]\) with \(p \in [g + 2, n - k + 1].\) There are thus \(n - k - g\) neighbours of all the vertices in \(Y.\) As a result, \(|N(Y)| = (g + 1)(k - 1) + (n - k - g) = n + g(k - 2) - 1.\)

We notice that all these vertices in \(Y\) belong to \(S_{n,k}^0,\) and, all of the \(n + g(k - 2) - 1\) neighbours as contained in \(N(Y),\) except \(g + 1\) of them, also belong to \(S_{n,k}^0.\) For \(j \in [1, g + 1], u'_j,\) each of these \(g + 1\) neighbours that do not belong to \(S_{n,k}^0,\) belongs to \(S_{n,k}^{0,j}\) respectively. It is clear that such a neighbour \(u'_j (\equiv [n, n - k + 2, \ldots, j])\) is obtained from \(u_j (\equiv [j, n - k + 2, \ldots, n])\) by swapping \(j\) with \(n.\)

Let \(F = N(Y),\) and, for all \(j \in [1, n], F_j = H_j \cap F,\) we have that, for \(j \in [1, g + 1], |F_j| = 1,\) for \(j \in [g + 2, n - 1], |F_j| = 0,\) and \(|F_n| = n + g(k - 2) - 2.\) Thus, although \(S_{n,k}^0 - F\) is disconnected, for all \(j \in [1, n - 1], S_{n,k}^{0,j} - F_j\) is connected. In particular, since, e.g. \(|F_{g+2}| = 0,\) when \(n - k \geq 3,\) for all \(j \in [1, n - 1],\) \(|F_{g+2}| = 0,\) there are at least \(\lceil \frac{n-2}{n-k} \rceil \geq (n - 2) \geq 2\) independent edges connecting \(S_{n,k}^{0,j}\) and \(S_{n,k}^{0,j+2}.\)

Thus, for all \(j \in [1, n - 1],\) and \(S_{n,k}^{0,j} - F_j\) belongs to the large component \(Z_1.\) Notice that, when \(n > k \geq 3 > 2, g \leq n - k < n - 2, |[j, F_j] = 0| = n - g - 2 \geq 1.\) Then \(|Z_1| \geq |H_{g+2}| = (n - 1)!/(n - k)! \geq 3(n - 1) > n - 2 > g + 1.\)

Moreover, each vertex in \(S_{n,k}^{0,j} - F\) is adjacent to a unique neighbour in \(S_{n,k}^{0,j+1} - F,\) thus not with any \(u'_j, j \in [1, g + 1].\) Therefore, when we remove \(N(Y)\) from \(S_{n,k},\) the survival graph, \(S_{n,k} - N(Y),\) contains a large component, \(Z (\equiv S_{n,k} - N(\tilde{t}_g(Y)), Z_1 \subseteq Z,\) and a small one, \(Y,\) both containing at least \(g + 1\) vertices.
Hence, $N(Y)$ is indeed a $g$-extra cut of $S_{n,k}$, and $\tilde{k}_g(S_{n,k}) \leq n + g(k - 2) - 1$. Combining with the aforementioned lower bound result, we have the following $g$-extra connectivity result for $S_{n,k}$.

**Corollary 5.4:** Let $n \geq 4$, $k \in [3, n)$, $g \in [1, n - k]$, $\tilde{k}_g(S_{n,k}) = n + g(k - 2) - 1$.

We are now ready to derive the following general result.

**Theorem 5.5:** Let $n \geq 4$, $k \in [3, n)$, $g \in [1, n - k]$. Then, $\tilde{\kappa}_g(S_{n,k}, \text{PMC}) = n + g(k - 1) - 1$.

**Proof:** Since $n \geq 4$, and $k \geq 3$, $g \leq n - 3$, and $k \leq n - 1$, we have that

$$|V| - 2(n + g(k - 1) - 1) \geq n(n - 1)(n - 2) - 2[n + (n - 4)(n - 2) - 1]$$

$$= n^3 - 5n^2 + 20n - 14 > 0.$$  

The result now holds by Corollary 3.10, Corollary 5.4, and Lemma 5.2. ■

We notice that the above result agrees with that obtained in [26, Theorem 4.3] with essentially the same ranges for $n$, $k$ and $g$.

We also have the following result by Corollaries 3.16, 5.1, which provides the $g = 1$ case, Corollary 5.4, Lemma 5.2, and the routine checking as made in the proof of the above result.

**Theorem 5.6:** Let $n \geq 4$, $k \in [3, n)$, $g \in [1, n - k]$, $\tilde{\kappa}_g(S_{n,k}, \text{MM}^*) = n + g(k - 1) - 1$.

We notice that the above result also agrees with that obtained in [26, Theorem 4.7], where $n \geq 6$, $k \in [3, n - 3]$, and $g \in [1, \min(k - 2, \frac{n-k+1}{2})]$.

It is worth pointing out that Theorem 5.3 was recently restated in [26, Lemma 3.3] with a shorter proof, and an alternative, structure dependent, derivation was made in [26] to obtain the $g$-extra diagnosability of the $S_{n,k}$ in terms of both the PMC and the MM$^*$ models, where much structure independent details, as we summarised in Sections 2 and 3, could be spared.

It also holds that, since we have shown $N(Y)$ is a $g$-extra faulty set, and $|N^c(Y)| = n + g(k - 1)$, we can also apply Corollary 3.4 to obtain Lemma 5.2.

Incidentally, since $S_{n,n-1}$ is isomorphic to the star graph [6, Lemma 4], and $S_{n,n-2}$ is isomorphic to the alternating group network [55], the $g$-extra diagnosability results of these latter two graphs immediately follow.

Finally, we comment that, when taking $g = 0$ in both Theorems 5.5 and 5.6, we have that

$$\tilde{\kappa}_0(S_{n,k}, \text{PMC}) = \tilde{\kappa}_0(S_{n,k}, \text{MM}^*) = n - 1,$$

i.e. the unrestricted diagnosability, or the vertex connectivity, of $S_{n,k}$, as expected.

### 6. The $g$-extra diagnosability of the arrangement graph

The arrangement graph is another alternative structure suggested in [5] to address the scalability issue as associated with the star graph [3]. This class of graphs also preserve many nice properties of the star graph such as vertex and edge symmetry, hierarchical and recursive structure, and simple shortest path routing. It has also drawn a considerable amount of attention with its various fault-tolerant properties [28,34,39,40,56–58].

The vertex set of an arrangement graph, denoted by $A_{n,k}$, $n \geq 2$, $k \in [1, n)$, is also the collection of all the $k$-permutations taken out of $\{n\} = \{1,2,\ldots,n\}$, and two vertices are adjacent to each other if and only if they differ in exactly one position. $A_{n,k}$ thus also contains exactly $n!/((n - k)!)$ vertices. Let $u (= [p_1, p_2, \ldots, p_k])$ be a vertex of $A_{n,k}$, we can get $u'$, a neighbour of $u$, by replacing $p_i$, $i \in [1, k]$, with
any of the \( n-k \) symbols that does not occur in \( u \). Thus, \( A_{n,k} \) is a regular graph where the degree of all its vertices equals \( k(n-k) \), which is also its connectivity [5].

Recall that \( H_i, i \in [1,n] \), collects all the vertices where the corresponding \( k \)-permutation ends with \( p_k = i \), and let \( A^i_{n,k} \) be the sub-graph of \( A_{n,k} \) restricted on \( H_i, i \in [1,n] \), it is also well known that \( A^i_{n,k} \) is isomorphic to \( A_{n-1,k-1} \). Each vertex in \( A^i_{n,k} \) is adjacent to exactly \( n-k \) neighbours, one each in a different \( A^j_{n,k}, j \neq i \), and, for each pair of \( A^i_{n,k} \) and \( A^j_{n,k} \), there are exactly \( (n-2)!/(n-k-1)! \) independent edges connecting them. Readers are referred to [5,40] for more detailed discussion of the structural properties of the arrangement graph.

Let \( x, y \) be two vertices in a graph \( G(V,E) \), we use \( d(x,y) \) to denote the distance of, i.e. the length of a shortest path between, \( x \) and \( y \) in \( V \) in terms of \( E \). Recall that \( N(u) \) stands for the neighbours of \( u \), we find the following common neighbour result useful.

**Lemma 6.1 ([24, Lemma 3]):** Let \( u, v \) be two vertices in \( A_{n,k}, n \geq k-1 \), then

\[
|N(u) \cap N(v)| = \begin{cases} 
0 & \text{if } d(u,v) \geq 3, \\
2 & \text{if } d(u,v) = 2 \text{ and } n \geq k + 2, \\
1 & \text{if } d(u,v) = 2 \text{ and } n = k + 1 \text{ and,} \\
 n-k-1 & \text{if } d(u,v) = 1.
\end{cases}
\]

Figure 6 shows \( A_{4,2} \), where, e.g. [1, 2] (represented as 12), a vertex of \( A^2_{4,2} \), has two neighbours, [1, 3], a vertex of \( A^3_{3,2} \), and [1, 4], a vertex of \( A^4_{4,2} \). There are thus clearly two independent edges (\((1, 2), (1, 3)), ((4, 2), (4, 3))\) between \( A^2_{4,2} \) and \( A^3_{4,2} \). It is also clear that 12 and 13 share exactly one common neighbour, i.e. 14; 12 and 34 share two common neighbours, 14 and 32; while 12 and 31 share no neighbours, as mandated by Lemma 6.1.

Issues related to the \( g \)-good-neighbour diagnosability of the arrangement graphs have been addressed in [28,59]. The \( g \)-extra diagnosability, \( g \in \{1, 2, 3\} \), of the arrangement graph in terms of the PMC model has also been derived in [24] by following a structure dependent approach. We will show that this diagnosability result, and that in terms of the MM* model, also naturally follow the general process that we have described in the previous section.

Since it has been proved in [28, Theorem 4.5] that for \( n \geq 5, k \in [2, n), t_1(A_{n,k}, \text{PMC}) = (2k - 1)(n-k), \) and [28, Theorem 4.7] that for \( n \geq 6, k \in [5, n-1), \) or \( n \geq 11, k \in [10, n), t_1(A_{n,k}, \text{MM*}) = (2k - 1)(n-k), \) the following results immediately follow Corollary 2.8.

![Figure 6. A_{4,2}](image)
**Theorem 6.2:** For \( n \geq 5, k \in [2, n) \), \( \tilde{t}_1(A_{n,k}, \text{PMC}) = (2k - 1)(n - k) \).

We notice that the above result slightly generalises the one obtained in [24, Theorem 4], where \( n \geq 5, k \in [3, n - 2] \).

**Theorem 6.3:** For \( n \geq 6, k \in [5, n - 1] \), or \( n \geq 11, k \in [10, n) \), \( \tilde{t}_1(A_{n,k}, \text{MM*}) = (2k - 1)(n - k) \).

We now move to the case when \( g = 2 \), starting with the following super-connectedness property of the arrangement graphs.

**Theorem 6.4 ([40]):** Let \( n \geq 8, k \in [2, n - 5] \), and let \( T \) be a subset of the vertices of \( A_{n,k} \) such that \( |T| \leq (3k - 2)(n - k) - 4 \). Then \( A_{n,k} - T \) is either connected or has a large component and small components with at most two vertices in total unless \( k = 2 \) and \( |T| = 4n - 12 \), in which case \( A_{n,k} - T \) could have a large component and a 4-cycle.

It is thus clear, by [16, Theorem 4.2], that if we want to end up with a survival graph, \( A_{n,k} - T \), where every component, beside the largest one, contains at least three vertices, \( |T| \) has to be at least \((3k - 2)(n - k) - 3 \). It is also shown in [40] that such a bound is tight\(^6\). Hence, we have the following result.

**Corollary 6.5:** Let \( n \geq 8, k \in [3, n - 5] \), \( \tilde{v}_2(A_{n,k}) = (3k - 2)(n - k) - 3 \).

It is easy to see that, when \( n \geq 4, k \in [3, n - 5] \),

\[
n!/(n - k)! \geq n(n - 1)(n - 2) > 2[(3(n - 5) - 2)(n - 3) - 1] > 2[(3k - 2)(n - k) - 1].
\]

Thus, by Corollaries 3.10 (PMC), and 3.16 (MM*), we have achieved the following lower bound results for 2-extra diagnosability for the arrangement graph.

**Corollary 6.6:** Let \( n \geq 8, k \in [3, n - 5] \), \( \tilde{t}_2(A_{n,k}, D) \geq (3k - 2)(n - k) - 1 \).

To derive an upper bound for \( \tilde{v}_2(A_{n,k}) \), we notice that, by Corollary 2.6, \( \tilde{t}_2(A_{n,k}, D) \leq t_2(A_{n,k}, D) \), where \( D \) refers to either the PMC or the MM* model. On the other hand, we have achieved the following result earlier.

**Theorem 6.7 ([28]):** For \( n \geq 7, k \in [4, n - 1] \), \( t_2(A_{n,k}, D) = (3k - 2)(n - k) \).

Hence, we have the following result, which apparently is not a tight upper bound in light of Corollary 6.6.

**Corollary 6.8:** For \( n \geq 7, k \in [4, n - 1] \), \( \tilde{t}_2(A_{n,k}, D) \leq (3k - 2)(n - k) \).

Indeed, the \( g \)-extra fault-tolerant model is not as demanding as the \( g \)-good-neighbour fault-tolerant model, as characterised in Corollary 2.7. We now follow the guidance of Proposition 3.1 to construct a tight upper bound.

Consider the following length 2 path, \( Y_1 (= (u, v, w)) \) in \( A_{n,k}^k, k \in [4, n - 2] \), where \( u = [1, 2, 3, \ldots, k] \), \( v = [1, k + 1, 3, \ldots, k] \), and \( w = [k + 2, k + 1, 3, \ldots, k] \). Since there do not exist common neighbours of \( u, v \) and \( w \), to identify \( N(Y) \), out of \( 3k(n - k) \) neighbours of \( u, v \) and \( w \), we need to (1) remove those in \( Y \), (2) remove those neighbours shared by both \( u \) and \( v \), (3) those shared by both \( v \) and \( w \), and (4) those shared by \( u \) and \( w \).

Beside the fact that \( Y_1 = [u, v, w] \), \( u \) and \( w \) have exactly two neighbours: \( v \) and \( [k + 2, 2, 3, \ldots, k] \), it is clear that,
\begin{itemize}
\item $N(u,v) = \{1,p_2,3,\ldots,k\}$, where $p_2 \in [k+2,n]$. Thus, $|N(u,v)| = n-k-1$, consistent with Lemma 6.1. In other words, there are $n-k-1$ neighbours shared by both $u$ and $v$. They are $[1,k+2,3,\ldots,k], [1,k+4,3,\ldots,k],$ ..., and $[1,n,3,\ldots,k]$, all falling into $A^k_{n,k}$.
\item $N(v,w) = \{p_1,k+1,3,\ldots,k\}$, where $p_1 \in [2] \cup [k+3,n]$. Thus, $|N(v,w)| = n-k-1$, also consistent with Lemma 6.1. Those neighbours are $[2,k+1,3,\ldots,k], [k+3,k+1,3,\ldots,k],$ ..., and $[n,k+1,3,\ldots,k]$, also falling into $A^k_{n,k}$.
\end{itemize}

We also notice that both $u$ and $w$ are neighbours of $v$, each counted once as a neighbour of $v$, and $v$ is a neighbour of both $u$ and $w$ in $Y_1$, counted once as a neighbour of both $u$ and $w$. Thus, by the Principle of Inclusion and Exclusion,

$$|N(Y_1)| = |N(u)| + |N(v)| + |N(w)|$$

$$- |N(u,v) + N(v,w) + N(u,w)| + |N(u,v,w)|$$

$$= [3k(n-k) - 4] - [2(n-k-1) + 1]$$

$$= (3k - 2)(n-k) - 3. \quad (4)$$

Thus, $|N^c(Y_1)| = |N^c(Y_1)| + |Y_1| = (3k - 2)(n-k)$.

We now proceed to show that $N(Y_1)$, clearly a vertex cut, is indeed a 2-extra faulty set. In particular, we show that $A^i_{n,k} \cap N(Y_1)$ contains two components, $Y_1$ and another, larger, component, referred to as $Z_1$ in the later discussion, both containing at least 3 vertices. To this regard, we observe that, out of the $k(n-k)$ neighbours of $u$, $\{(i,p_i,\ldots,k) : i \in [1,k], p_i \in [k+1,n]\}$, $(k-1)(n-k)$ of them, including $u (= [1,k+1,3,\ldots,k])$, fall into $A^k_{n,k}$, when taking $i \in [1,k-1]$, and the other $n-k$ of them, $\{(1,2,\ldots,p_k) : p_k \in [k+1,n]\}$, fall into $A^i_{n,k} j \in [k+1,n]$.

Moreover, out of the $k(n-k)$ neighbours of $v$, $\{(i,k+1,p_i,\ldots,k) : i \in [1,k], p_i \in [2] \cup [k+2,n]\}$, $(k-1)(n-k)$ of them, including both $u (= [1,2,\ldots,k])$ and $v (= [1,k+1,3,\ldots,k])$, fall into $A^k_{n,k}$, when taking $i \in [1,k-1]$, and the other $n-k$ of them, $\{(1,k+1,\ldots,p_k) : p_k \in [2] \cup [k+2,n]\}$, fall into $A^i_{n,k} j \in [2] \cup [k+2,n]$.

Finally, out of the $k(n-k)$ neighbours of $w$, $\{(i,k+2,k+1,p_i,\ldots,k) : i \in [1,k], p_i \in [1,2] \cup [k+3,n]\}$, $(k-1)(n-k)$ of them, including $v (= [k+2,k+1,3,\ldots,k])$, fall into $A^k_{n,k}$, when taking $i \in [1,k-1]$, and the other $n-k$ of them, $\{(k+2,k+1,\ldots,p_k) : p_k \in [1,2] \cup [k+3,n]\}$, fall into $A^i_{n,k} j \in [1,2] \cup [k+3,n]$.

As discussed earlier, we also know that each of $N(u,v)$ and $N(v,w)$ contains $n-k-1$ vertices, and $N(u,w)$ contains $n-k-1$ vertices, and $N(u,w)$ contains $n-k-1$ vertices, and $N(u,w)$ contains $n-k-1$ vertices, and $N(u,w)$ contains $n-k-1$ vertices.

To summarise, let $F = N(Y_1)$, and for all $i \in [1,n]$, let $F_i = H_i \cap F$, we have that for all $i \in [3,k-1], |F_i| = 0$, $|F_i| = |F_{k+1}| = 1$, $|F_2| = |F_{k+2}| = 2$, for all $i \in [k+3,n], |F_i| = 3$, and $|F_k| = (3k-5)(n-k) - 3^8$. Thus, for all $i \in [1,n], i \neq k, |F_i| = 3$, it follows that, for all $i,j \in [1,n] \setminus \{k\}, |F_i| + |F_j| \leq 6$. Assuming $4 \leq k \leq n-2$, since there are $(n-2)!/(n-k-1)! \geq (n-2)(n-3) \geq 6^9$ independent edges between $A^i_{n,k}$ and $A^j_{n,k}, k \neq j$, each such $A^i_{n,k} - F_k$ is a part of one large component, $Z_1$, via such independent edges. Moreover, let $u \in A^k_{n,k} - F_k$, it has $k \geq 2$ unique outside neighbours, none being a neighbour of a vertex in $Y_1 \subset H_k$. Hence, all vertices in $A^k_{n,k} - F$ belong to this large component $Z_1$, as well.

It is clear that $|Y_1| = 3$, and, by assumption, for at least one $i \in [3,k-1], |F_i| = 0$, thus, $|V(H_i)| = (n-1)!/(n-k)! \geq (n-1)(n-2)(n-3) \geq 60 > 3$. It is thus also clear that $Z_1$, containing at least one such $A^i_{n,k}$, contains more than three vertices. Therefore, $N(Y_1)$ is indeed a 2-extra faulty set of $A_{n,k}, k \in [4,n-2]$. Hence, $k^2(A_{n,k}) \leq |N(Y_1)| = (3k-2)(n-k) - 3$, verifying Corollary 6.5.

Finally, when $k \in [4,n-2]$,

$$|V(A_{n,k})| = n!/(n-k)! \geq n(n-1)(n-2)(n-3) > (3n-2)(n-k) - 3(n-2)(n-4)$$

$$\geq (3k-2)(n-k) = |N^c(Y_1)|.
By Corollary 3.4, we have the following result.

**Corollary 6.9:** For \( n \geq 6, k \in [4, n - 2] \), \( \tilde{t}_2(A_{n,k}, D) \leq (3k - 2)(n - k) - 1 \).

Combining Corollaries 6.6 and 6.9, we have the following tight bound result of the 2-extra diagnosability of the arrangement graphs.

**Theorem 6.10:** Let \( n \geq 8, k \in [3, n - 5] \), \( \tilde{t}_2(A_{n,k}, D) = (3k - 2)(n - k) - 1 \).

The above result, when \( D \) refers to the PMC model, agrees with that as shown in [29, Theorem 5], where \( n \geq 6, k \in [4, n - 2] \).

We comment that, if we use a 3-cycle, \( Y_1 = (u, v, w) \), instead of the length 2 path \( Y_1 \), where \( u = [1, 2, \ldots, k] \), \( v = [k + 1, 2, \ldots, k] \), and \( w = [k + 2, 2, \ldots, k] \), let

\[
C = \{[k + 3, 2, \ldots, k], \ldots, [n, 2, \ldots, k]\},
\]

\[
D = \{[1, p_i, \ldots, k] : i \in [2, k], p_i \in [k + 1, n]\},
\]

\[
E = \{[k + 1, p_i, \ldots, k] : i \in [2, k], p_i \in [1] \cup [k + 2, n]\}, \text{ and}
\]

\[
F = \{[k + 2, p_i, \ldots, k] : i \in [2, k], p_i \in [1, k + 1] \cup [k + 3, n]\},
\]

we would have \( N(u) = C \cup D, N(v) = C \cup E, N(w) = C \cup F, \) and \( N(u, v) = N(v, w) = N(u, w) = N(u, v, w) = C \).

It is clear that \(|C| = n - k - 2\), and \(|D| = |E| = |F| = (k - 1)(n - k)\). By the Principle of Inclusion-Exclusion, we have the following:

\[
|N(Y_1')| = |N(u)| + |N(v)| + |N(w)|
- |N(u, v)| + |N(v, w)| + |N(u, w)| - |N(u, v, w)|
= 3|C| + (|D| + |E| + |F|) - 3|C| + |C|
= |C| + |D| + |E| + |F| = (n - k - 2) + 3(k - 1)(n - k)
= (3k - 2)(n - k) - 2 > |N(Y_1)|.
\]

Thus, for the case of \( g = 2 \), a length 2 path provides a smaller upper bound of \( g \)-extra diagnosability as compared with a 3-cycle, in light of Proposition 3.1.

When moving towards the case of \( g = 3 \), we notice the following \( g \)-extra connectivity result appears in [24, Lemma 6], which appeared earlier in [41].

**Theorem 6.11 ([24]):** For \( n \geq 6, k \in [3, n - 3] \) or \( k \in [4, n - 2] \), \( \tilde{k}_3(A_{n,k}) = 4(k - 1)(n - k) - 4 \).

Since, when \( n \geq 6, k \in [3, n - 3] \), we have

\[
n!/(n - k)! \geq n(n - 1)(n - 2) \geq 2(4(n - 4)(n - 3) - 1) + 4 \geq 2(4(k - 1)(n - k) - 1) + 4,
\]

by Corollary 3.10 (PMC), and Corollary 3.16 (MM*), we immediately have the following lower bound results for the 3-extra diagnosability.

**Corollary 6.12:** Let \( n \geq 6, k \in [3, n - 3] \), \( \tilde{t}_3(A_{n,k}, D) \geq 4(k - 1)(n - k) - 1 \).

To the best of our knowledge, no results regarding \( t_3(A_{n,k}) \) exist. Thus, we cannot use Corollary 2.7 to get even an estimate of the upper bound of \( \tilde{t}_3(A_{n,k}, D) \). We now follow the construction as discussed in Section 3.1 to seek such an upper bound, making use of an example originally suggested in proving [24, Theorem 3].
Thus, in [29, Theorem 6], where \( u = [1, 2, 3, \ldots, k] \), \( v = [k + 1, 2, 3, \ldots, k] \), \( w = [k + 1, k + 2, 3, \ldots, k] \) and \( x = [1, k + 2, 3, \ldots, k] \). Again, by Lemma 6.1, no vertex could be a neighbour of all the three vertices in \( Y_1 \). To identify \( N(Y_2) \), out of \( 4k(n - k) \) neighbours of \( u, v \) and \( w \), we also need to 1) remove those in \( Y_2 \), and 2) remove those neighbours shared by both \( u \) and \( v \), \( v \) and \( w \), \( u \) and \( w \), and also by \( w \) and \( x \).

Similar to the analysis made to derive \( N(Y_1) \), we can find out that \( |N(u, v)| = |N(v, w)| = N(w, x)| = |N(x, u)| = n - k - 1 \). Considering that each of the four vertices in \( Y_2 \) is a neighbour of two other vertices in \( Y_2 \), we have that, again by the Principle of Inclusion-Exclusion,

\[
|N(Y_2)| = 4k(n - k) - 8 - 4(n - k - 1) = 4[(k - 1)(n - k) - 1].
\]

Thus, \( N^c(Y_2)| = 4(k - 1)(n - k) \).

We comment that, if we use a length 3 path, \( Y'_2 = (u, v, w, x) \), in the construction, since the distance between \( u \) and \( x \) is 3, none of their neighbours could be shared by \( Y_2 \). As a result, for \( k \leq n \),

\[
|N(Y'_2)| = (4k - 3)(n - k) - 3 > 4(k - 1)(n - k) - 4 = |N(Y_2)|.
\]

Thus, for the case of \( g = 3 \), a 4 cycle is a better choice as compared with a length 3 path.

We proceed to show that \( A_{n,k} - N(Y_2) \) contains two components, a large component \( Z_2 \), and \( Y_2 \), both containing at least 4 vertices.

Let \( F = N(Y_2) \), and for all \( i \in [1, n] \), let \( F_i = H_i \cap F_1 \), we have also found, through an analysis similar to the case of \( g = 2 \), that, for all \( i \in [3, k] \), \( |F_i| = 0 \), \( |F_1| = |F_2| = |F_{k+1}| = |F_{k+2}| = 2 \), \( |F_k| = (4k - 2)(n - k) - 4 \), and, for all \( i \in [k + 3, n] \), \( |F_i| = 4 \). Thus, for all \( i \in [1, n] \), \( i \neq k \), \( |F_i| \leq 4 \), it follows that, for all \( i, j \in [1, n] \setminus \{k\} \), \( |F_i| + |F_j| \leq 8 \).

Again, assuming that \( 4 \leq k \leq n - 2 \), since \( n \geq 7 \), there are \((n - 2)!/(n - k - 1)! \geq (n - 2)(n - 3)(n - 4) \geq 60 > 8 \) independent edges between \( A^i_{n,k} \) and \( A^j_{n,k} \), \( k \notin \{i, j\} \), all such \( A^i_{n,k} - F_i \) are connected into one large component \( Z'_2 \), via such independent edges. Moreover, let \( u \in A^i_{n,k} - F_k \), it has \( n - k \geq 2 \) unique outside neighbours, none being a neighbour of a vertex in \( Y_2 \subset V(H_k) \). Hence, all vertices in \( A^i_{n,k} - F \) belong to the same component containing \( Z'_2 \), forming a large component \( Z_2 \).

It is clear that \( |Y_2| = 4 \), and, by assumption, for \( i \in [3, k] \), \( |F_i| = 0 \), thus, \( |V(H_i)| = (n - 1)!/(n - k)! \geq (n - 1)(n - 2)(n - 3) \geq 120 > 4 \), as \( n \geq 7 \). It is thus also clear that \( Z_2 \), which contains at least two such \( A^i_{n,k} \)'s, also contains more than four vertices. Therefore, \( N(Y_2) \) is indeed a 3-extra faulty set of \( A_{n,k} \), \( k \in [4, n - 2] \). Finally, when \( k \in [4, n - 2] \),

\[
|V(A_{n,k})| = n!(n - k)! \geq n(n - 1)(n - 2)(n - 3) > 4(n - 3)(n - 4) \\
> 4(k - 1)(n - k) > |N^c(Y_2)|.
\]

Hence, by Corollary 3.4,

\[
\tilde{t}_3(A_{n,k}, D) \leq |N^c(Y_2)| - 1 = |N(Y_2)| + |Y_2| - 1 = 4(k - 1)(n - k) - 1.
\]

Together with Corollary 6.12, we have the following result.

**Theorem 6.13:** Let \( n \geq 7 \), \( k \in [4, n - 3] \), \( \tilde{t}_3(A_{n,k}, D) = 4(k - 1)(n - k) - 1 \).

The PMC version of the above result agrees, with a slightly smaller range, with that as obtained in [29, Theorem 6], where \( n \geq 6 \), \( k \in [3, n - 3] \).

It is well known that \( A_{n,1} \) is isomorphic to \( K_n \), the complete graph with \( n \) vertices. We notice that both the \( g \)-good-neighbour and \( g \)-extra diagnosability of \( K_n \), thus \( A_{n,1} \), have been derived in [60, Theorem 11] in terms of the PMC and the MM* model. We also notice that, since \( A_{n, n-1} \) is isomorphic to the star graph [6, Lemma 4], and \( A_{n, n-2} \) is isomorphic to the alternating group graph [61], the \( g \)-extra diagnosability results of these latter two graphs immediately follow. For example, Theorems 6.2, and 6.3 do agree with Corollary 5.1, when taking \( k = n - 1 \).
Incidentally, by Corollary 2.7, Theorems 3.6 and 6.13, we have the following range for $t_3(A_{n,k})$, the 3-good-neighbour diagnosability of $A_{n,k}$.

**Corollary 6.14:** For $n \geq 6$, $k \in [3, n - 3]$, 

$$4(k - 1)(n - k) - 1 \leq t_3(A_{n,k}, D) \leq 4(k - 1)(n - k) + n - k.$$ 

7. **Concluding remarks**

In this paper, we explored general relationships among various ‘fault-free’ fault-tolerant models, where only fault-free vertices are to satisfy the required properties, and discussed the connection between such fault-tolerant models, and the diagnosability notion, consistent with the ‘maximum restriction’ requirement. We then generalised a uniform process that we can effectively apply to derive diagnosability results of various interconnection networks under the $g$-good-neighbour model and the $g$-extra fault-tolerant models, in terms of mainstream diagnostic models such as the PMC and the MM* models.

As demonstrating examples, we showed how to apply such a general process to obtain $g$-extra diagnosability results for the hypercube, the $(n,k)$-star graph, and the arrangement graph. These results agree with those achieved individually, without duplicating structure independent technical details. Some of these results come with a larger range of application, and the result for the arrangement graph, for $g = 3$, in terms of the MM* model is new. It is clear that such a general process can be applied to other interconnection networks to obtain their diagnosability results, assuming the associated connectivity result of such a graph is available, and appropriate construction can be identified for the fault-tolerant models.

As future research topics, beside studying other interconnection structures under those existing fault-tolerant models in light of this general process, we would also look into other appropriate fault-tolerant models, and the feasibility of applying this general process to derive various fault-tolerant properties under such alternative models.

Beside the fact that the lower bound of an $M$-diagnosability result directly depends on the connectivity property related to the fault-tolerant model $M$, the upper bound part, i.e. the construction of a pair of appropriate indistinguishable faulty sets, certainly depends on such a model, as well. Although the upper bound construction fits well with both the $g$-good-neighbour and the $g$-extra fault-tolerant models, it might not with other models. Beside the conditional fault-tolerant model related example as we gave in Section 3.1, as another example, a set of vertices $F$ in a connected non-complete graph $G$ is called a *cyclic vertex-cut* if the survival graph $G - F$ is disconnected and at least two components in the survival graph contain a cycle [16]. It is straightforward to come up with a notion of a $g$-cyclic faulty set when at least two components contain a cycle of length at least $g \geq 3$. Various related cyclic connectivity results have been achieved for the hypercube, the star graph, and other Cayley graphs generated by a transposition tree [16, Chapter 4]. On the other hand, it is clear that the $(N(Y), N^c(Y))$ construction that we used in this paper does not fit in the context of this alternative fault-tolerant model since although, taking $Y$ as a $g$-cycle, $N(Y)$ could be a $g$-cyclic faulty set, i.e. the associated survival graph consists of at least two components, each containing a cycle, $N^c(Y)$ might not be, since it includes $Y$, where a cycle resides.

Therefore, to derive the $M$-diagnosability of an interconnection network $G$ for a given fault-tolerant model $M$ and a diagnostic model $D$, we need to derive the $M$ connectivity of $G$ to get the lower bound of such a diagnosability, and choose a pair of appropriate indistinguishable $M$-faulty sets for $G$ in terms of $D$ to establish its upper bound.

**Notes**

1. The original notion of a good-neighbour edge-cut of order $m$ as coined in [16] is that a set of edges $T$ in a connected graph $G$ is called a good-neighbour edge-cut of order $m$ if $G - F$ is disconnected and every vertex in $G - F$ has degree at least $m$. 

2. This is where the ‘fault-free’ restriction is needed.

3. Since $F_1 \neq F_2$, either $F_1 \setminus F_2 \neq \emptyset$ or $F_2 \setminus F_1 \neq \emptyset$.

4. For a given pair of supposedly indistinguishable pair of distinct $M$-faulty set, $F_1, F_2$, if $F_1 \setminus F_2 = \emptyset$, then $F_1 \subset F_2$.

- If both $F_1$ and $F_2$ are $g$-good-neighbour faulty sets, then when $F_1 \subset F_2$, any vertex, $u$, in $V(G) \setminus (F_1 \cup F_2)$ must have at least $g$ neighbouring vertices outside $F_2$, thus such a vertex $u$ has at least $g \geq 1$ neighbours in $V(G) \setminus (F_1 \cup F_2)$, i.e. it is not isolated. This would lead to a simpler argument as given in Proposition 3.9.

- If both $F_1$ and $F_2$ are $g$-extra faulty sets, then when $F_1 \subset F_2$, any component in $V(G) \setminus (F_1 \cup F_2)$ must have at least $g + 1$ vertices $F_2$, thus any vertex $u$ in $V(G) \setminus (F_1 \cup F_2)$ has at least one neighbour in $V(G) \setminus (F_1 \cup F_2)$, i.e. it is not isolated, either.

Thus, we can assume that $F_1 \setminus F_2 \neq \emptyset$. Moreover, since $F_1 \neq F_2$, if $F_1 \setminus F_2 \neq \emptyset$, then $F_2 \setminus F_1 \neq \emptyset$.

5. We comment that it is this line of reasoning that requires $d(u, w) = 2$, and $k \leq n − 2$, by Lemma 6.1, $u$ and $w$ share exactly two common neighbours, one of them being $v$. Hence, $u, v$ and $w$ share no common neighbours, as $v$ cannot be its own neighbour.

6. We comment that $\sum_{i=1}^{n} |F_i| = (3k − 2)(n − k) − 3$, as expected.

7. We notice that, if we require $k \in [3, n − 2]$, we would have $n \geq k + 1 \geq 4$, and there would have $(n − 2)(n − 3) = 6$ independent edges between. On the other hand, if $k \in [4, n − 1]$, then $n \geq k + 1 = 5$, and there would be $(n − 2)(n − 3)(n − 4)$ independent edges between, which also leads to six. This is why we have to require $k \in [4, n − 2]$. Therefore, $n \geq 7$.

8. It is based on this analysis that we set $k \geq 4$, hence $n \geq 7$.

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