CORRELATION FUNCTIONS FROM TWO-DIMENSIONAL STRING WARD IDENTITIES

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ABSTRACT
We rederive the $w_\infty$ Ward identities, starting from the existence of trivial linearized gauge invariances, and using the method of canceled propagators in the operator formalism. Recursion relations for certain classes of correlation functions are derived, and these correlation function are calculated exactly. We clarify the relation of these results with another derivation of the Ward identities, which relies directly on charge conservation. We also emphasize the importance of the kinematics of canceled propagators in ensuring that the Ward identities are non-trivial. Finally, we sketch an extension of Ward identities to open strings.
Introduction

Recently there has been considerable interest in a model of bosonic strings propagating in two-dimensional space-time. This model arises upon the third quantization of two-dimensional quantum gravity coupled to one massless scalar field. The model has been solved exactly in the discretized approach implemented through large-$N$ matrix quantum mechanics [1,2]. Now it seems important to understand this exact solubility in the more conventional continuum approach to the sum over surfaces — the Liouville gravity [3]. In the conformal gauge the theory reduces to a sigma model which describes strings propagating in two dimensions [4]. One of the coordinates is the original massless scalar, while the other, $\phi$, arises from the conformal factor of the world sheet metric. There is a dilaton background linear in $\phi$ which breaks translation invariance. Thus, the two coordinates enter on a different footing and the model is not Lorentz invariant. However, as explained in refs. [5,6], the model possesses an infinite symmetry, which is isomorphic to the area-preserving diffeomorphisms (a wedge subalgebra of $w_\infty$). The symmetry charges, which are constructed from the discrete states [7,8,9,10], are expected to provide an explanation for the exact solubility of the model.

The symmetry structure found in the continuum approach strongly resembles the symmetries present in the matrix model solution [11]. A complete equivalence is still missing because in the matrix model the Liouville potential ("tachyon background") is turned on, while the continuum treatment has mostly dealt with the free field theory where the potential has been set to zero. The free field approach is sufficient to study the so-called bulk amplitudes [12,13], which obey an appropriate energy sum rule. The bulk amplitudes are believed to determine indirectly all the correlation functions of the theory [14,15]. Attempts to study the effects of the Liouville potential on the symmetry structure have not yet produced conclusive results [16]. In this paper we will restrict ourselves to the bulk amplitudes and to the free field theory where the potential is turned off, and the $w_\infty$ symmetry structure is completely transparent.

After the discovery of the infinite number of conserved charges, a natural step is to derive the Ward identities based on maintaining the charge conservation inside correlation
functions. In ref. [19] various sources of local charge non-conservation on the world sheet were examined, and it was shown that the requirement of global charge conservation leads to recursion relations among correlation functions. These recursion relations were used to calculate the bulk tachyon amplitudes in a few simple steps. In other words, the $w_\infty$ symmetry Ward identities allow one to calculate a class of formidable multiple integrals, which are generalizations of the Dotsenko-Fateev integrals [20], with hardly any work. The existence of the recursion relations is due to the fact that the $w_\infty$ charges generally act on the massless “tachyons” non-linearly, i.e. a charge acting on a state with $n$ particles produces a state with $m \leq n$ particles. In refs. [10,21] the Ward identities were derived, and their non-linear structure was explained, in a somewhat different way. Each tree level Ward identity was understood as the vanishing of a sum of products of correlation functions which arise when a sphere is pinched into two spheres in all possible ways. In ref. [21] the Ward identity was identified with the master equation of Batalin-Vilkovisky quantization [22].

The aim of this paper is to elucidate the relation between the methods of refs. [10,19,21], and to speculate on the general circumstances under which Ward identities provide non-trivial constraints on correlation functions. We also sketch extension to open string theories.

Perhaps, the simplest statement of the Ward identities is through the observation [5,23] that the theory possesses states that are both pure gauge and identically zero. The vertex operator for such a state is

$$\{Q_{BRST}, cW_{J,m}\} = 0$$  \hspace{1cm} (1)$$

and formally carries the physical ghost number $n_{gh} = 2$. It vanishes identically because the zero-picture current $cW_{J,m}$ is BRST invariant. A special feature of the theory under consideration is the presence of an infinite number of such BRST invariant zero-picture currents of ghost number 1. In general, linearized closed string gauge invariance assumes the form

$$\delta \Psi = \{Q_{BRST}, \lambda\}$$  \hspace{1cm} (2)$$

where the ghost numbers of $\Psi$ and $\lambda$ are 2 and 1 respectively. Therefore, there is an infinite number of gauge parameters, $\lambda = cW_{J,m}$, for which the linearized gauge invariance is trivial. In fact, as stated in refs. [5,23], the presence of such trivial gauge transformations

\footnote{For another approach to constraining the correlation functions, see refs. [17,18].}
guarantees that there are discrete states of ghost number 2 which cannot be gauged away. While at a general momentum there are enough gauge invariances to gauge away all the oscillator states, at the discrete momenta carried by $cW_{j,m}$ some gauge invariances become trivial, eq. (1), and there appear physical oscillator states that cannot be gauged away. Thus, in a theory with generally continuous momenta, the presence of symmetry charges seems intrinsically connected with the presence of discrete states, which are physical only at special discrete momenta. In higher-dimensional theories the only known cases of this phenomenon occur at zero momentum. Some familiar examples are the conservation of charge and the associated extra photon state at zero momentum, and the conservation of momentum and the extra graviton-dilaton states at zero momentum.

The Ward identities follow after inserting eq. (1) into correlation functions [10,21]. As usual, the BRST anti-commutator can be re-written as a sum over the boundaries of the moduli space of a sphere with $n$ punctures where the sphere is pinched into two spheres with $m$ and $n - m$ punctures respectively. In the literature on string theory these boundaries of moduli space are sometimes referred to as the “canceled propagators”. The reason for this terminology is most apparent in the operator formalism which we will review below. In the 26-dimensional string theory, the “canceled propagator argument” is the statement that such boundary terms on moduli space typically vanish (at least in the context of an appropriate analytic continuation), because the momentum that flows through the pinch is off-shell. In the 2-dimensional string theory the situation is very different. As emphasized in ref. [19], there is an infinite set of special kinematical arrangements where the momentum that flows through the pinch is precisely on-shell. When this is the case, the “canceled propagator contribution” is non-vanishing and explicitly calculable. The Ward identity is nothing but the statement that the sum of all the canceled propagator contributions, which arise after the insertion of eq. (1) into a correlation function, vanishes. We emphasize that one can search for the non-trivial contributions to the Ward identity simply on the basis of studying the kinematics: whenever the momentum that flows through the canceled propagator corresponds to an on-shell state, one expects, and usually finds, a non-vanishing contribution.
1. Closed string Ward identities: derivation in the operator formalism

Before we proceed to explicit calculations, let us state our conventions. Following ref. [19] we set $\alpha' = 4$ so that $\langle X(z, \bar{z}) X(w, \bar{w}) \rangle = -2 \log |z - w|^2$ where $X$ is the $c = 1$ matter field and $\phi$ is the Liouville field. In these conventions the tachyon operators are given by

$$T_k^\pm (z, \bar{z}) = e^{ikX + \epsilon \pm \phi}(z, \bar{z})$$

$$\epsilon_\pm = -1 \pm k$$

and we define $V_k^\pm (z, \bar{z}) \overset{\text{def}}{=} c(z) \bar{c}(\bar{z}) T_k^\pm (z, \bar{z})$. We are interested in studying the bulk correlation functions of tachyons on a sphere,

$$\langle V_{k_1}^\pm \cdots V_{k_n}^\pm \rangle .$$

They satisfy the following conservation equations

$$\sum_{i=1}^n k_i = 0 \quad \sum_{i=1}^n \epsilon_i = -2 .$$

We will write the correlation functions more explicitly in two different ways. In the path integral formalism,

$$\langle V_{k_1}^\pm \cdots V_{k_n}^\pm \rangle = \langle V_{k_1}^\pm (+\infty) V_{k_2}^\pm (1) \int T_{k_3}^\pm \cdots \int T_{k_{n-1}}^\pm V_{k_n}^\pm (0) \rangle \quad (1.4)$$

where $\int T_k^\pm \overset{\text{def}}{=} \int d^2y T_k^\pm (y, \bar{y})$.

In the operator formalism,\(^2\)

$$\langle V_{k_1}^\pm \cdots V_{k_n}^\pm \rangle = \langle V_{k_1}^\pm | V_{k_2}^\pm (1) \Delta V_{k_3}^\pm (1) \Delta \cdots \Delta V_{k_{n-1}}^\pm (1) | V_{k_n}^\pm \rangle$$

$$+ \text{permutations} \quad (1.5)$$

where $|V_{k_n}^\pm \rangle \overset{\text{def}}{=} \lim_{z \to 0} V_{k_n}^\pm (z) |0\rangle$, $\langle V_{k_1}^\pm | \overset{\text{def}}{=} \lim_{z \to \infty} \langle 0 | V_{k_1}^\pm (z)$. The propagator, including the ghosts’ contribution, is

$$\Delta = \frac{b_0^+ b_0^-}{L_0 + \bar{L}_0} \Pi_{L_0, \bar{L}_0} \quad (1.6)$$

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\(^2\) See for example refs. [24,25] and references therein.
where \( b_0 \) def \( b_0 \pm \bar{b}_0 \), and the Virasoro generators include the ghost pieces. \( \Pi_{L_0, \bar{L}_0} \) is the projector onto states that satisfy \( L_0 = \bar{L}_0 \).

In ref. [5] it was shown that in the \( c = 1 \) closed string theory there exists an infinite number of conserved currents. The corresponding zero-picture ("fixed") vertex operators are

\[
\Omega^{(\bar{0})}_{J,m}(z, \bar{z}) = c(z)W_{J,m}(z, \bar{z}) \quad \text{and} \quad W_{J,m}(z, \bar{z}) = \Psi_{J+1,m}(z)\bar{\Omega}_{J,m}(\bar{z})
\]

and \( \bar{\Omega}^{(\bar{0})}_{J,m} \) which is built analogously. \( \Psi_{J+1,m}(z) \) are the gravitationally dressed primary fields of the \( c = 1 \) matter system whereas the \( \bar{\Omega}_{J,m}(\bar{z}) \) form the "Ground Ring" [5]. The current algebra is [5,6]

\[
W_{J_1,m_1}(z, \bar{z})W_{J_2,m_2}(w, \bar{w}) = \frac{2((J_1+1)m_2-(J_2+1)m_1)}{z-w}W_{J_1+J_2,m_1+m_2}(w, \bar{w}) .
\]

As shown in ref. [10], the associated charges are given by \(^3\)

\[
\mathcal{A}_{J,m} = \oint \frac{dz}{2\pi i} \Omega^z_{J,m}(z, \bar{z}) - \oint \frac{d\bar{z}}{2\pi i} \Omega^\bar{z}_{J,m}(z, \bar{z})
\]

where \( \Omega^z_{J,m}(z, \bar{z}) = W_{J,m}(z, \bar{z}) \), and \( \Omega^\bar{z}_{J,m}(z, \bar{z}) = c(z)\Psi_{J+1,m}(z)\bar{\Omega}_{J,m}(\bar{z}) \), where \( \bar{\Omega}_{J,m}(\bar{z}) = \bar{b}-1\bar{\Omega}_{J,m}(\bar{z}) \). The algebra of the charges is

\[
[\mathcal{A}_{J_1,m_1}, \mathcal{A}_{J_2,m_2}] = 2((J_1+1)m_2-(J_2+1)m_1)\mathcal{A}_{J_1+J_2,m_1+m_2} .
\]

Following refs. [10,21], we will generate the Ward identities by inserting eq. (1) into correlation functions. Then the Ward identities for tachyon correlation functions assume the form

\[
\langle \{Q_{BRST}, cW_{J,m}\} V_{k_1}^\pm \cdots V_{k_n}^\pm \rangle = 0 .
\]

More explicitly, in the operator formalism we have

\[
0 = \langle V_{k_1}^\pm | \{Q_{BRST}, cW_{J,m}\}(1) \Delta V_{k_2}^\pm (1) \Delta \cdots \Delta V_{k_{n-1}}^\pm (1) | V_{k_n}^\pm \rangle + \text{permutations} .
\]

Having written the Ward identity in this form, we can now apply the "canceled propagator argument". As already mentioned in the introduction, on general grounds a BRST anti-commutator can be rewritten as a sum over the boundaries of the moduli space of a sphere

\(^3\) We follow the conventions of ref. [10], section 4, for the orientation of the contour integrals.
with \( n \) punctures where the sphere is pinched into two spheres with \( p \) and \( n - p \) punctures respectively. In the operator formalism this is obtained explicitly by commuting \( Q_{BRST} \) through each of the propagators \( \Delta \). First observe that

\[
[Q_{BRST}, \Delta] = \Pi_{L_0,\bar{L}_0} b_0^- = \sum_i |\Phi_i\rangle\langle \Phi_i| b_0^-
\]  
(1.13)

where the sum is over a complete set of states that satisfy \((L_0 - \bar{L}_0)|\Phi_i\rangle = 0\). Thus, \( Q_{BRST} \) can literally cancel a propagator, replacing it with an insertion of \( b_0^- \). The conjugate states \(|\Phi_i\rangle\) and \(|\bar{\Phi}_i\rangle\) satisfy \( \langle \Phi^j|\Phi_i\rangle = \delta^j_i \) (for continuous spectrum the Kroenecker symbol is replaced by the Dirac delta function). Each pair of conjugate states has their ghost numbers add up to 6, their momenta \( k \) add up to 0, and their energies \( \epsilon \) add up to \(-2\). For instance, the state conjugate to \(|V_k^\pm\rangle\) is

\[
|\tilde{V}_k^\pm\rangle = c\partial c c\bar{\partial}c T^\pm_k(0)|0\rangle.
\]  
(1.14)

It is also convenient to define the states \(|\tilde{\Phi}_i^\pm\rangle \equiv b_0^- |\tilde{\Phi}_i^\pm\rangle\) which have ghost number 3 and are annihilated by \( b_0^- \) \([10,21]\). In the Batalin-Vilkovisky quantization these are to be thought of as “anti-tachyons” \([21]\).

Now, commuting \( Q_{BRST} \) in eq. (1.12) with the vertices and propagators until it annihilates against the vacua, one gets a sum of “canceled propagator contributions”. Each one has the form

\[
\sum_i \langle V_{k_1}|V_{k_2}\Delta\cdots\Delta V_{k_p}|\Phi_i\rangle\langle \tilde{\Phi}_i^p|V_{k_{p+1}}\Delta c W_{J,m}\Delta \cdots V_{k_{n-1}}|V_{k_n}\rangle,
\]  
(1.15)

where \(|\tilde{\Phi}_i^p\rangle = b_0^- |\Phi_i^p\rangle\). By ghost number counting, \(|\Phi_i\rangle\) must have the physical ghost number 2, and, therefore, \(|\tilde{\Phi}_i\rangle\) carries ghost number 3. Also, the energy and momentum of \(|\Phi_i\rangle\) are completely determined because the correlation functions obey the sum rules eq. (1.3). It may happen that at this energy and momentum there is no physical state. Then the usual “canceled propagator argument” applies and we conclude that eq. (1.15) vanishes. In this theory there is a class of cases, however, where there are physical states contributing to the sum over \( i \). Here we will only analyze the situations where the energy and momentum that flow through the “canceled propagator” obey the tachyon dispersion relation. Then the sum over \( i \) collapses to one non-vanishing term involving the tachyon states.

Note that eq. (1.15) can be interpreted as the pinching of a sphere with \( n \) punctures into two spheres with \( p \) and \( n - p \) punctures respectively. The amplitude corresponding
to one of these spheres is perfectly conventional, with each vertex operator carrying ghost number 2. The other amplitude is unusual, since the current operator $cW_{J,m}$ carries ghost number 1, while $|\tilde{\Phi}_i\rangle$ carries ghost number 3. In section 2.2 we will show that this amplitude can be interpreted as a matrix element of the charge operator $\mathcal{A}_{J,m}$. For the correlation functions involving a current we introduce the notation

$$\langle \tilde{V}_k V_{k_1} \cdots V_{k_n} cW_{J,m} \rangle = \sum_P \langle \tilde{V}_k | V_{k_1} \Delta cW_{J,m} \Delta \cdots V_{k_n-1} | V_{k_n} \rangle .$$ (1.16)

The permutations $P$ involve all the vertex operators, except for $\tilde{V}$ which has to be kept as an out-state (in the leftmost position). Since $\tilde{V}$ and $cW$ are both anti-commuting, this definition guarantees that all the permutations contribute with the same sign. In general, in dealing with correlators involving anti-commuting vertex operators, their ordering becomes important, and one may have to appeal to the operator formalism in order to construct their proper definition.

The pinching of the sphere represented in eq. (1.15) is conformally equivalent to the situation where $p$ tachyon vertex operators collide with $cW_{J,m}$, which is the language used in ref. [19]. Using the o.p.e., all the colliding operators are replaced by insertion of a single physical operator into the conventional amplitude. In eq. (1.15) this is the state $|\Phi_i\rangle$ that flows through the pinch.

Taking into account the sum over all permutations in eq. (1.12), the final form of the Ward identity is [21]

$$0 = \sum_{I,L} \langle V_{i_1} \cdots V_{i_I} \Phi \rangle \langle \tilde{\Phi} V_{l_1} \cdots V_{l_L} cW_{J,m} \rangle$$ (1.17)

where the sum is over all possible partitions of the vertex operators $V$ into two sets $I$ and $L$ with $I$ and $L$ elements respectively.

Now, as we remarked earlier, there are cases when the kinematics restricts $\Phi$ to be on the tachyon mass shell. Let us analyze the energy and momentum conservation laws for the correlator

$$\langle \tilde{\Phi} V^+_{k_1} \cdots V^+_{k_n} cW_{J,m} \rangle .$$ (1.18)

If $\tilde{\Phi}$ carries momentum $P$ and energy $E$, we obtain

$$m + P + \sum_{i=1}^n k_i = 0$$

$$J + E + \sum_{i=1}^n (-1 + k_i) = -2 .$$ (1.19)
For the number of particles $n = J - m + 1$, it follows that $E = -1 + P$, which is the dispersion relation of a positive chirality tachyon. Remarkably, this applies for arbitrary momenta $k_i$, which is a consequence of the two-dimensional kinematics! Therefore, whenever $n = J - m + 1$, eq. (1.17) receives a contribution from the canceled propagator with $\tilde{\Phi} = \tilde{V}_P^+$, $\Phi = V_{-P}^+$. Similarly, if we flip all the chiralities, then the same situation arises for $n = J + m + 1$. It is not hard to see that these two cases are the only non-trivial amplitudes of type (1.18) involving tachyons of generic (not discrete) momenta.

2. Correlation functions from the Ward identities

Now that we have reviewed the structure of the Ward identities in the two-dimensional closed string theory, and shown that they can give rise to non-trivial constraints, we will write them out explicitly. In other words, we will first explicitly calculate the unconventional correlators of eq. (1.16). After substituting their values, the Ward identities (1.17) become linear relations among the physical tachyon correlators, which were derived in ref. [19]. We will focus on the tachyon correlators of type $(N, 1)$ and $(1, N)$, which are the only ones that do not vanish for generic momenta. These correlators are given by the multiple integrals

$$A_{N,1}(k_1, \ldots, k_N) = \langle V_{-k}(-\infty)V_{k_1}(1)T_{k_2}^+ \cdots T_{k_{N-1}}^+ V_{k_N}(0) \rangle = \prod_l \int d^2 z_l \prod_i |z_i - 1|^{4s_{i1}}|z_i|^{4s_{i0}} \prod_{i<j} |z_i - z_j|^{4s_{ij}} \quad (2.1)$$

where $l, i$ and $j$ run from 2 to $N - 1$, and

$$s_{ij} = k_i k_j - \epsilon_i^+ \epsilon_j^+ = -1 + k_i + k_j \quad (2.2)$$

The integrals of eq. (2.1) are generalizations of the integrals calculated in ref. [20] by contour deformation techniques. We will find that the integrals (2.1) can be calculated via simple recursion relations which follow from the $w_\infty$ Ward identities. This is a mathematical result which does not seem to be well known.

To calculate the amplitude of eq. (1.16), we will observe that it can be interpreted as the matrix element of the charge operator $A_{j,m}$ between a $n$-tachyon state and a 1-tachyon
state of the form $|V_p^+\rangle$. More explicitly, we will show that

\[ \langle \tilde{V}_p^+(\infty) c(w) W_{J,m}(w, \bar{w}) V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle = \] 

\[ = \langle V_{k_1}^+ | A_{J,m} V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle \]

where the charge operator acts on all tachyons to its right.

We will also present a different method of calculation of of eq. (1.16), which relies directly on the Ward identities.

2.1 Action of the $w_\infty$ Charges on Tachyons

As a first step, we need to consider the kinematics of the action of the charge $A_{J,m}$ on $n$ tachyons [19,10]. Obviously, the analysis is the same as for the correlators of eq. (1.16). Just from such kinematical considerations, we conclude [19] that the charge $A_{m+n-1,m}$ annihilates $l$ $T^+$ tachyons of generic momenta, for $l < n$. The first non-trivial action of this charge is on $n$ $T^+$ tachyons, producing only one $T^+$ tachyon [19]:

\[ A_{m+n-1,m} V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ = F_{n,m}(k_1, \cdots, k_n) V_{k}^+(0) \]  

(2.4)

where $k = \sum k_i + m$. A similar formula holds for $A_{-m+n-1,m}$ acting on $T^-$ tachyons.

The situation is more complex if one or more tachyons carry one of the discrete momenta. For example, let us consider the action of the charges $A_{m+n-1,m}$ on vertex operators $V_{s/2}^+$, with $s$ positive integers. For $n = 1$ the action is correctly determined by eq. (2.4). However, for $n > 1$, the charge does not annihilate the vertex operator. Instead, it produces a discrete state in the “semi-relative” cohomology [10]. This discrete state is not separately annihilated by $b_0$ and $\bar{b}_0$, but is only annihilated by $b_0^-$. We will not analyze in detail such action of the charges in this paper.

For a better understanding of the proof of eq. (2.3) we need to review some of the results from refs. [19,10]. Consider first the action of the charge $A_{1/2, -1/2}$ on two $T^+$ tachyons of generic momenta:

\[ A_{1/2, -1/2} V_{k_1}^+ \int T_{k_2}^+ = \left[ \oint \frac{dz}{2\pi i} W_{1/2, -1/2}(z, \bar{z}) - \oint \frac{d\bar{z}}{2\pi i} c(z) \Psi_{1/2, -1/2}(z) \bar{X}_{1/2, -1/2}(z) \right] \cdot 

\[ \cdot V_{k_1}^+(0, 0) \int d^2w T_{k_2}^+(w, \bar{w}) \]  

(2.5)

\[ \text{A similar equation holds for the } T^- \text{ tachyons.} \]
where the contour $\gamma$ encloses 0. To get a non-zero result from the action of the holomorphic part of the charge, it is necessary that

$$W_{1, -\frac{1}{2}}(z, \bar{z})V_{k_1}^+(0, 0) = \frac{1}{z}F_{2, -\frac{1}{2}}(k_1, k_2)V_{k_1+k_2-\frac{1}{2}}^+(0, 0) + \ldots \quad (2.6)$$

The only contributions to the residue come from the region where $z$ and $w$ approach each other and 0. The integral for $F_{2, -\frac{1}{2}}(k_1, k_2)$ was calculated in ref. [19] using the results of ref. [26]

$$F_{2, -\frac{1}{2}}(k_1, k_2) = 2\pi(2k_1 + 2k_2 - 1)\frac{\Gamma(1 - 2k_1)\Gamma(1 - 2k_2)}{\Gamma(2k_2)}\frac{\Gamma(2k_1 + 2k_2 - 1)}{\Gamma(2 - 2k_1 - 2k_2)} \quad (2.7)$$

Also, in ref. [10] it was shown that the anti-holomorphic part of the charge gives a contribution proportional to $\bar{z}^s$ with $s > -1$. This is not singular enough to be relevant and, therefore, the anti-holomorphic part of the charge does not contribute in this case. One could anticipate this on general grounds because this part of the charge carries (left,right) ghost numbers equal to $(1, -1)$. The conservation of these ghost numbers forbids the physical tachyon from appearing in the O.P.E.

In an analogous way one can compute the action of the charge $A_{m,m}$ on one $T^+_k$ tachyon (see ref. [19]). Using these results and the $w_\infty$ algebra of charges, in ref. [19] the action of the charge $A_{m+n-1,m}$ on $n T^+$ tachyons was determined. The final result for eq. (2.4) is

$$F_{n,m}(k_1, \ldots, k_n) = 2\pi^{n-1}(n!)k_2 \frac{\Gamma(2k)}{\Gamma(1-2k)} \prod_{i=1}^n \frac{\Gamma(1-2k_i)}{\Gamma(2k_i)} \quad (2.8)$$

where $k = m + \sum_{i=1}^n k_i$ and $n \geq 1$. A similar formula obviously holds for $A_{-m+n-1,m}$ applied to $n$ generic $T^-$ vertex operators. Note that an explicit evaluation of this formula would require performing $n-1$ integrals, a very difficult task, which is avoided here thanks to the algebraic structure of the model.

### 2.2 Calculation of the correlators involving a current

In this section we will explicitly calculate the correlators of eq. (1.16). We will do it in two different ways. The first method involves the proof of eq. (2.3) and then the use of eqs. (2.4) and (2.8). The second method is more direct: it relies on the application of the Ward identities to the correlators involving a current.
Let us start by proving eq. (2.3). As we have seen in the previous section, the action of the charge is completely characterized by the O.P.E. of the holomorphic and anti-holomorphic currents with some number of physical vertex operators. We will assume that the following O.P.E. holds:

\[
W_{J,m}(w, \bar{w}) V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ = \frac{1}{w} F V^+_k(0) + \ldots .
\] (2.9)

We will now show that the R.H.S. and the L.H.S. of eq. (2.3) are equal. Consider the R.H.S.:

\[
\langle V_{-k} \mid A_{J,m} V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle = (2.10)
\]

\[
= \langle V_{-k} \mid \left( \oint_{\gamma} \frac{dz}{2\pi i} W_{J,m}(z, \bar{z}) - \oint_{\gamma} \frac{d\bar{z}}{2\pi i} \Omega_{J,m}^z(z, \bar{z}) \right) V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle
\]

where the contour of integration \(\gamma\) surrounds the point \(z = 0\). From the previous section, we already know, for example by ghost number conservation, that the anti-holomorphic part of the charge, when acting on generic tachyons, gives zero. Then, using eq. (2.9), one gets

\[
\langle V_{-k} \mid A_{J,m} V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle = F .
\] (2.11)

Consider now the L.H.S. of eq. (2.3):

\[
\langle \tilde{V}_{-k}^+(\infty) c(w) W_{J,m}(w, \bar{w}) V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle = (2.12)
\]

\[
= \langle V_{-k}^+ \mid \left( \oint_{\gamma} \frac{dz}{2\pi i} z b(z) + \oint_{\gamma} \frac{d\bar{z}}{2\pi i} \bar{b}(\bar{z}) \right) \cdot c(w) W_{J,m}(w, \bar{w}) V_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_n}^+ \rangle
\]

where we have written \(b_0\) as \(\oint_{\gamma} \frac{dz}{2\pi i} z b(z)\), and \(\gamma\) is the contour of integration that wraps around the point \(z = +\infty\). Now we may deform the contour so that it surrounds all the other points, i.e. \(z = w, z = y_i\) and \(z = 0\).

As before, just by ghost number counting, the anti-holomorphic contribution to eq. (2.12) vanishes. For the holomorphic integral, we get

\[
\langle V_{-k}^+ \mid \oint_{\gamma} \frac{dz}{2\pi i} z F V_k^+(0) \rangle = F .
\] (2.13)
Thus, we have shown that eq. (2.3) holds. From eq. (2.4) it immediately follows that
\[
\langle \tilde{V}_{-k}^+ V_{k_1}^+ \cdots V_{k_n}^+ cW_{m+n-1,m} \rangle = F_{n,m}(k_1, \ldots, k_n) 
\]
(2.14)
where \( k = m + \sum_{i=1}^n k_i \) and \( F_{n,m} \) is given by eq. (2.8).

Now, we give another derivation of eq. (2.14), relying directly on the existence of trivial gauge invariances. We start from the Ward identities
\[
\langle \tilde{V}_{-k}^+ \{ Q_{BRST}, cW_{J_1,m_1} \} V_{k_1}^+ \cdots V_{k_n}^+ cW_{J_2,m_2} \rangle = 0 
\]
(2.15)
where the kinematical constraints imply that
\[
n = J_1 + J_2 - m_1 - m_2 + 1, \quad k = m_1 + m_2 + \sum_{i=1}^n k_i . 
\]
(2.16)

We carefully define the correlator (2.15) in the operator formalism, summing over the permutations of all the operators except for \( \tilde{V}_{-k}^+ \), which is kept in the leftmost position. Retaining all the cancelled propagator contributions, we find the following recursion relation
\[
2((J_1 + 1)m_2 - (J_2 + 1)m_1) \langle \tilde{V}_{-k}^+ V_{k_1}^+ \cdots V_{k_n}^+ cW_{J_1,J_2,m_1+m_2} \rangle = 
\]
(2.17)
\[
= \sum_{\mathcal{I}, \mathcal{L}} \langle \tilde{V}_{-k}^+ V_{k_1}^+ \cdots V_{k_n}^+ cW_{J_1,m_1} V_{k_1}^+ \cdots V_{k_n}^+ cW_{J_2,m_2} \rangle 
- \sum_{\mathcal{R}, \mathcal{S}} \langle \tilde{V}_{-k}^+ V_{r_1}^+ \cdots V_{r_R}^+ cW_{J_2,m_2} V_{r_1}^+ \cdots V_{r_R}^+ cW_{J_1,m_1} \rangle . 
\]

Let us explain the notation. \( \sum_{\mathcal{I}, \mathcal{L}} \) means the sum over all possible partitions of the \( n \) tachyons into two sets \( \mathcal{I} \) and \( \mathcal{L} \) containing \( I = J_1 - m_1 \) and \( L = J_2 - m_2 + 1 \) elements respectively. Similarly, \( \sum_{\mathcal{R}, \mathcal{S}} \) means the sum over all possible partitions of the \( n \) tachyons into two sets \( \mathcal{R} \) and \( \mathcal{S} \) containing \( R = J_2 - m_2 \) and \( S = J_1 - m_1 + 1 \) elements respectively. In each term in the sums, the momentum \( k' \) is determined by the momentum conservation. Finally, the \( w_\infty \) structure constant on the left-hand side arises from the 3-point function of two currents and one “anti-current” [21].

The remarkable property of the recursion relations (2.17) is that they only involve correlation functions of type (1.16), i.e. those containing a single insertion of a current. In order to solve these recursion relations, we need an input. An example of the necessary
input is all the three-point functions and at least one four-point function. Here we may use the direct calculations from ref. [19], which imply that

\[
\langle \tilde{V}^+_{-k_{1}-p} cW_{p,p} V^+_{k_1} \rangle = 2(p + k_1) \frac{\Gamma(2p+2k_1)}{\Gamma(1-2p-2k_1)} \frac{\Gamma(1-2k_1)}{\Gamma(2k_1)}
\]

\[
\langle \tilde{V}^+_{-k_{1}-k_2+\frac{1}{2}} cW_{\frac{1}{2},-\frac{1}{2}} V^+_{k_1} V^+_{k_2} \rangle = 2\pi(2k_1 + 2k_2 - 1) \frac{\Gamma(2k_1+2k_2-1)}{\Gamma(2-2k_1-2k_2)} \frac{\Gamma(1-2k_1)}{\Gamma(2k_1)} \frac{\Gamma(1-2k_2)}{\Gamma(2k_2)} .
\]

Let us illustrate the use of the relations (2.17) by a simple example, taking \((J_1, m_1) = \left(\frac{1}{2}, \frac{1}{2}\right)\) and \((J_2, m_2) = (p, p)\). Writing out eq. (2.17) explicitly in this case, we find

\[
(4p + 1) \langle \tilde{V}_{-k_{1}}^+ V_{k_2}^+ V_{k_2}^+ cW_{p+p-1} \rangle = \langle \tilde{V}_{-k_{1}}^+ cW_{\frac{1}{2},-\frac{1}{2}} V_{k_2}^+ V_{k_2}^+ p \rangle \langle \tilde{V}_{-k_{2}}^+ cW_{p+p} V_{k_1}^+ \rangle
\]

\[
+ \langle \tilde{V}_{-k_{1}}^+ cW_{\frac{1}{2},-\frac{1}{2}} V_{k_2}^+ V_{k_2}^+ p \rangle \langle \tilde{V}_{-k_{2}}^+ cW_{p+p} V_{k_1}^+ \rangle
\]

where, by kinematics, \(k = k_1 + k_2 + p - \frac{1}{2}\). Substituting eq. (2.18), we immediately find

\[
\langle \tilde{V}_{-k_{1}}^+ V_{k_2}^+ V_{k_2}^+ cW_{p+p-1} \rangle = 4\pi k \frac{\Gamma(2k)}{\Gamma(1-2k)} \frac{\Gamma(1-2k_1)}{\Gamma(2k_1)} \frac{\Gamma(1-2k_2)}{\Gamma(2k_2)} .
\]

Now, using eq. (2.17) with \(J_1 = m_1 + 1\) and \(J_2 = m_2 + 1\), we obtain the formula for correlation functions of \(cW_{m+2,m}\), and so on. This recursive procedure is equivalent to that used in ref. [19] in the language of \(w_\infty\) charges acting on states. Here, following refs. [10,21], we succeeded in reformulating the recursion relations in terms of correlation functions. Repeatedly using the recursion relations, we once again arrive at eq. (2.14).

### 2.3 Calculation of tachyon correlators via the Ward identities

We can find all the correlation functions \(A_{N,1}\), given by equation (2.1), from the following Ward identity

\[
\langle \{Q_{BRST}, cW_{-m,m}\} V^+_{k_1} \cdots V^+_{k_N} V^-_{-\frac{1}{2}} \rangle = 0 \quad (2.21)
\]

where \(N \geq 3\), and the kinematics fixes \(m = 1 - \frac{N}{2}\). Summing over all the non-vanishing canceled propagators, we find

\[
0 = \sum_{i=1}^{N} \langle \tilde{V}_{-k_{p(1)}}^+ V_{k_{p(2)}}^+ \cdots V_{k_{p(N-1)}}^+ cW_{-m,m} \rangle \langle V_{k_1}^+ V_{k_2}^+ V_{-\frac{1}{2}}^- \rangle
\]

\[
+ \langle \tilde{V}_{-r}^+ V_{-\frac{1}{2}}^- cW_{-m,m} \rangle \langle V_{r}^- V_{k_1}^+ V_{k_2}^+ \cdots V_{k_{N-1}}^+ V_{k_N}^+ \rangle \quad (2.22)
\]
where \( p \) is the set of the first \( N \) strictly positive integers except for \( i \), and \( p(j) \) is the \( j^{th} \) element of the set. The momenta of the intermediate states, \( r = \frac{1-N}{2} \) and \( k = 1 - \frac{N}{2} + \sum_{j=1}^{N-1} k_{p(j)} \), have been fixed using the momentum and energy conservation laws. The form of the Ward identity in eq. (2.22) is identical to that in ref. [19], although our starting point here was somewhat different. While ref. [19] relied explicitly on the charge conservation, here we instead used the technique of inserting a pure gauge state which is identically zero.

Now we use eq. (2.14), where \( F_{n,m} \) is given by eq. (2.8), to substitute the explicit expressions for the correlators involving the current. Thus, we obtain a linear relation expressing the tachyon \( N+1 \)-point function in terms of the tachyon three-point function,

\[
[(N-2)!]^2(N-1) \cdot A_{N,1}(k_1, \ldots, k_N) = \sum_{i=1}^{N} F_{N-1,m}(k_{p(1)}, \ldots, k_{p(N-1)}) \cdot A_{2,1}(k, k_i).
\]

Using the momentum conservation equation \( 2 \sum_{i=1}^{N} k_i = (N-1) \), one gets

\[
F_{N-1,m}(k_{p(1)}, \ldots, k_{p(N-1)}) = \pi^{N-2}(N-1)!(1-2k_i) \prod_{l=1}^{N} \frac{\Gamma(1-2k_l)}{\Gamma(2k_l)}.
\]

The three-point function \( A_{2,1} \) contains no integrations. Therefore, it is independent of the momenta and is normalized as \( A_{2,1} = 1 \). Now eq. (2.23) gives

\[
[(N-2)!]^2(N-1) \cdot A_{N,1}(k_1, \ldots, k_N) = \pi^{N-2}(N-1)! \prod_{l=1}^{N} \frac{\Gamma(1-2k_l)}{\Gamma(2k_l)}.
\]

from which we recover the correct answer [13,15]

\[
A_{N,1}(k_1, \ldots, k_N) = \pi^{N-2} \frac{\prod_{l=1}^{N} \Gamma(1-2k_l)}{(N-2)!}.
\]

If we change the normalization of the positive chirality tachyons as in ref. [19], setting \( T_k^+ = \frac{\Gamma(2k)}{\Gamma(1-2k)} \exp(ikX + \epsilon_+ \phi) \), we get

\[
A_{N,1}(k_1, \ldots, k_N) = \pi^{N-2} \frac{\prod_{l=1}^{N} \Gamma(1-2k_l)}{(N-2)!}.
\]

in agreement with eq. (38) of ref. [19].
3. Remarks on the open string case

In this section we will sketch an extension of the Ward identities to theories that include both closed and open strings.

As emphasized in ref. [21], the essential feature of the pure closed string case is that a tube, pinched by a BRST charge, is replaced by insertion of $\Phi_i$ on one side and $\tilde{\Phi}^i = b_0^- \Phi^i$ on the other. In the operator formalism, the remaining insertion of $b_0^-$ simply follows from eq. (1.13). Thus, if $\Phi_i$ is annihilated by $b_0^-$, so is $\tilde{\Phi}^i$, i.e. both belong to the semi-relative cohomology [10]. The resulting structure bears a strong resemblance to the Batalin-Vilkovisky quantization scheme [22], where $\Phi_i$ and $\tilde{\Phi}^i$ are to be thought of as field and anti-field vertex operators [21]. Since the sum of their ghost numbers is 5, the corresponding field and anti-field couplings, $\alpha^i$ and $\tilde{\alpha}_i$, carry opposite space-time statistics. From the structure of the canceled propagator contributions, it follows that the tree-level free energy $F$ satisfies the master equation [21]

$$\sum_i \frac{\partial F}{\partial \alpha^i} \frac{\partial F}{\partial \tilde{\alpha}_i} = 0 \ . \quad (3.1)$$

This equation is a nice way to formalize the content of all the tree-level closed string Ward identities.

Now we will show how to include the open string couplings in the master equation. The structure of canceled open string propagators is different from the closed string case, and we need to take this into account. In open string theory, the propagator is

$$\Delta_{open} = \frac{b_0^-}{L_0} \ . \quad (3.2)$$

Therefore, we get

$$\{Q_{BRST}, \Delta_{open}\} = 1 = \sum_i |\phi_i\rangle \langle \phi^i| \ . \quad (3.3)$$

Thus, unlike in the closed string case, there is no extra $b$ insertion. The difference is obvious geometrically: when a tube is pinched, there is a left-over modulus corresponding to the twist angle of the pinch; when a strip is pinched, there are no left-over moduli. Thus, a pinched strip is replaced by insertion of pairs of states that are simply the conjugates of each other. For instance, the state conjugate to $ct^\pm_k(0)|0\rangle$ is

$$c\partial c t^\pm_{-k}(0)|0\rangle \quad (3.4)$$
where the open string tachyon vertex operators are

\[ t^\pm_k = e^{\pm \left( i k X + \epsilon_{\pm} \phi \right)} , \]
\[ \epsilon_{\pm} = -1 \pm k . \]

(3.5)

In general, if \( \phi_i \) has ghost number \( n \), then \( \phi^i \) has ghost number \( 3 - n \), so that they have opposite statistic. Thus, in the open string case, it is tempting to interpret \( \phi^i \) as the anti-field vertex operator of \( \phi_i \). The corresponding coupling constants, \( \beta^i \) and \( \beta_i \), can then be thought of as a space-time field and its anti-field. Note that, if \( b_0|\phi_i\rangle = 0 \), then \( b_0|\phi^i\rangle \neq 0 \). Thus, in the open string Batalin-Vilkovisky scheme we cannot impose the condition that all states are annihilated by \( b_0 \), i.e. we have to include the whole absolute cohomology.\(^5\)

Now that we have discussed the structure of the canceled open string propagators, we can write down the generalization of the classical master equation that should apply to the theory of coupled open and closed strings,

\[ \sum_i \frac{\partial F}{\partial \alpha^i} \frac{\partial F}{\partial \tilde{\alpha}_i} + \sum_j \frac{\partial F}{\partial \beta^j} \frac{\partial F}{\partial \tilde{\beta}_j} = 0 . \]

(3.6)

We expect this equation to summarize all the Ward identities that arise on a sphere and on a disc.

On a more detailed level, the Ward identities again follow from the existence of trivial gauge invariances. Now, there are two types of gauge invariances. In the open string sector, we have

\[ \delta \Psi_{open} = [Q_{BRST}, \lambda_{open}] \]

(3.7)

where the physical ghost number of \( \Psi_{open} \) is 1. There are also the closed string gauge invariances of eq. (2). Thus, there are two types of constraints on physical amplitudes of open and closed strings. The first type follows from replacing one of the open string vertex operators by

\[ [Q_{BRST}, O_{J,m}] = 0 \]

(3.8)

where \( O_{J,m} \) are the open string ground ring operators of ghost number 0 (they are analogous to the chiral ground ring of the closed string). The second type amounts to studying the effects of the closed string symmetries. Here, as in the previous chapters, we replace one of the closed string vertex operators by \( \{Q_{BRST}, cW_{J,m}\} = 0 \).

\(^5\) Recall, for comparison, that in the closed string sector the \( b_0^- = 0 \) condition could be consistently imposed.
In both cases we commute $Q_{BRST}$ through the propagators, retaining the canceled propagator contributions whose sum has to vanish. The analysis of kinematics, which is almost as simple as in the pure closed string case, once again reveals many possibilities of on-shell canceled propagators. The resulting Ward identities are formally summarized in eq. (3.6). We expect these equations to provide stringent constraints on the correlation functions, and to be a helpful tool in their calculations. We will leave the detailed calculations for future work.\(^6\)

4. Discussion

One may wonder, which elements of the structure of the Ward identities are general, and which are special to the two-dimensional model. As emphasized in refs. [21,10], the structure of the canceled propagator contributions is in general such that the pinch is replaced by insertions of field and anti-field vertex operators. We have argued that the same conclusion applies to models which include open strings. This simple analysis implies a general connection between string Ward identities and Batalin-Vilkovisky quantization. However, the physical content of the Ward identities may be quite trivial. In ref. [19], and in this paper, it was shown that this is definitely not the case in the two-dimensional model. Here, because of the presence of the discrete states that generate the $w_\infty$ symmetry, and due to the special kinematics, the Ward identities imply powerful recursion relations among the physical correlation functions. These relations are, in fact, a valuable practical tool that renders seemingly formidable calculations easily doable.

In higher dimensions, because there is no infinite on-shell symmetry such as $w_\infty$, the Ward identities are only expected to establish the Poincaré invariance of the on-shell amplitudes. One may hope, however, that the Ward identities have more interesting manifestations off-shell [10,21]. This question needs to be explored further.

\(^6\) Bershadsky and Kutasov have recently informed us that, using somewhat different methods, they obtained constraints on the correlation functions that, among other things, reproduce their results in ref. [27] (see also ref. [28]).
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