Isothermal and adiabatic magnetization processes of the spin-$1/2$ Heisenberg model on an isosceles triangular lattice

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In this study, we investigated the magnetic susceptibility, entropy, and isothermal magnetization curve of the spin-$1/2$ Heisenberg model on an isosceles triangular lattice using the orthogonalized finite-temperature Lanczos method. In addition, we investigated the adiabatic magnetization curve and magnetocaloric effect. We estimated these physical quantities with sufficient accuracy in the thermodynamic limit, except at low temperatures. We observed a $1/3$ magnetization plateau in the isothermal magnetization process, whereas the plateau was observed to have a slope in the adiabatic process. We showed that the magnetocaloric effect can be used to detect the signature of phase transitions. We believe that these results will be useful for understanding the magnetism of isosceles triangular lattice compounds through a comparison with experimental results in the future.

I. INTRODUCTION

Megagauss magnetic field generators have been studied for over half a century [1–6]. In recent years, the magnetization process of magnetic materials has been actively studied using megagauss magnetic field generators, and various quantum phase transitions have been successfully observed [7–15]. Most experiments using magnetic fields exceeding 100 T have been performed with pulse widths of a few to several tens of microseconds [1, 2]. Owing to the very narrow pulse width, the magnetization process is expected to be an adiabatic (isentropic) process rather than an isothermal process. In addition, several studies have observed the magnetocaloric effect (MCE) in magnetic compounds [8, 12, 16–18]. Therefore, theoretical study of the adiabatic magnetization process is important for understanding these experimental results.

The spin-$1/2$ Heisenberg model on isotropic and isosceles triangular lattices is a traditional model used in studies on magnetism. This model has been extensively investigated for several decades [19]. Several model compounds have been studied, and their magnetization processes at low temperatures have been observed to exhibit a $1/3$ magnetization plateau and various phase transitions because of the frustration and quantum effects [20–28]. In addition, in theoretical studies, magnetic-field-induced quantum phase transitions at zero temperature have been observed in the Heisenberg model on an isosceles triangular lattice (ITL) [20–31].

Very recently, $A_3$ReO$_5$Cl$_2$ ($A = \text{Ca, Ba, Sr}$) and spin-1/2 ITL compounds have been intensively studied [32–34]. Therefore, theoretical calculations of the isothermal and adiabatic magnetization curves and MCE in the ITL Heisenberg model are necessary for future experimental studies. However, the calculations of the adiabatic magnetization process for frustrated spin-1/2 systems are limited to approximately 20 sites using full exact diagonalization (FullED) [35]. Thus, numerical calculations with a larger size are necessary to estimate the physical quantities of the ITL compounds.

In this study, we investigated the magnetic properties of the spin-1/2 ITL with exchange interactions $J$ and $J'$, as shown in Fig. 1 using the orthogonalized finite-temperature Lanczos method (OFTLM) [36], which is an improved version of the standard finite-temperature Lanczos method (FTLM) [37, 38]. The OFTLM can be used to evaluate the adiabatic process with high accuracy because, unlike the standard FTLM, the value of the entropy is almost exact at low temperatures [39]. We first investigated the magnetic susceptibility and magnetic entropy of the ITL up to 36 sites under a zero magnetic field. Subsequently, we calculated the isothermal magnetization curves at finite temperatures and investigated the presence of the $1/3$ magnetization plateau. Finally, we calculated the adiabatic magnetization curves and the MCE. Consequently, we estimated the magnetic susceptibility, entropy, and isothermal magnetization curve of the ITL of the thermodynamic limit above certain temperatures. The $1/3$ magnetization plateau at $1 \geq J'/J \geq 0.5$ was observed in the isothermal magnetization process. In contrast, in the adiabatic process, the anomaly corresponding to the $1/3$ magnetization plateau was not flat but inclined. Regardless of the magnetic field, the magnetization did not reach saturation magnetization under the adiabatic process with finite entropy, but the temperature increased rapidly. Finally, we demonstrate that the magnetic phase boundaries can be determined from the MCE results. The results obtained using the OFTLM will be useful for understanding the magnetism of the ITL compounds via a comparison with experimental results in the future.

The remainder of this paper is organized as follows. In Sec. II we describe the ITL model. In Sec. III we describe the FTLM and OFTLM. In Sec. IV we describe the results of the magnetic susceptibility, entropy, isothermal and adiabatic magnetization curves, and MCE of the ITL, and discuss the magnetic properties. Finally, a summary is provided in Sec V.

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several ITL compounds have been investigated to study the model at isotropic triangular lattice. In the present study, we investigated the temperature properties of various lattice models [39–51]. We set $J = 1$, this model becomes the one-dimensional Heisenberg chain, whereas at $J = 0$, this model becomes the isotropic triangular lattice. In the present study, we investigated the model at $J' = 0.25, 0.5, 0.75$, and 1 because several ITL compounds have $J \geq J'$. In this section, we describe the OFTLM and the calculation of physical quantities using this method. The partition function $Z(T, h)$ of the canonical ensemble at the temperature $T$ in the magnetic field $h$ is expressed as follows:

$$Z(T, h) = \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{\text{sat}}^{(m)} - 1}{R} \sum_{i=0}^{M_{L}-1} e^{-\beta E_{i,m}(h)},$$

where $N_{\text{sat}}^{(m)}$ is the dimension of the Hilbert subspace with $S_{\text{tot}}^{z} = m$ in $\mathcal{H}$, $\beta$ is the inverse temperature $1/T$ ($k_B = 1$), and $E_{i,m}(h)$ is the eigenenergy of the Hilbert subspace with $S_{\text{tot}}^{z} = m$ in $\mathcal{H}$, as a function of $h$. As $\sum_{i} S_{i}^{z}$ is a conserved quantity, $E_{i,m}(h)$ is expressed as

$$E_{i,m}(h) = E_{i,m} - mh,$$

where $E_{i,m}$ is the eigenenergy of the Hilbert subspace with $S_{\text{tot}}^{z} = m$ at $h = 0$, and the second term corresponds to the Zeeman term. We define the order of $\{E_{i,m}\}$ as $E_{0,0} \leq E_{1,0} \leq E_{2,0} \leq \cdots \leq E_{N_{z}-1,0}$.

Using the standard FTLM, the partition function $Z(T, h)$, as shown in Eq. 2, is approximated as follows:

$$Z(T, h)_{\text{FTL}} = \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{\text{sat}}^{(m)} R}{R} \sum_{r=1}^{M_{L}-1} \sum_{j=0}^{N_{z}} e^{-\beta \epsilon_{j,m}^{(r)}(h)} |\langle V_{r,m} | \psi_{j,m}^{(r)} \rangle|^2,$$

where $R$ is the number of random samplings of the FTLM, $M_{L}$ is the dimension of the Krylov subspace, $|V_{r,m}\rangle$ is a normalized random initial vector with $S_{\text{tot}}^{z} = m$, and $|\psi_{j,m}^{(r)}(h)\rangle$ are the eigenvectors (eigenvalues) in the $M_{L}$-th Krylov subspace with $S_{\text{tot}}^{z} = m$. Similar to Eq. 3, $\epsilon_{j,m}^{(r)}(h)$ is expressed as $\epsilon_{j,m}^{(r)}(h) = \epsilon_{j,m}^{(r)} - mh$.

In the OFTLM, we first calculate several low-lying exact eigenvectors $|\Psi_{i,m}\rangle$ with $N_{V}$ levels ($E_{0,0} \leq E_{1,0} \leq \cdots \leq E_{N_{V}-1,0}$). We then calculate the following modulated random vector:

$$|V'_{r,m}\rangle = \left[ I - \sum_{i=0}^{N_{V}-1} |\Psi_{i,m}\rangle \langle \Psi_{i,m}| \right] |V_{r,m}\rangle,$$  

with normalization

$$|V'_{r,m}\rangle = \frac{|V'_{r,m}\rangle}{\sqrt{|V'_{r,m}\rangle |V'_{r,m}\rangle}}.$$  

Note that $|V'_{r,m}\rangle$ is orthogonal to the states $|\Psi_{i,m}\rangle$ for $i \in \{0, 1, \cdots, N_{V} - 1\}$. The partition function of the OFTLM is obtained using $|V'_{r,m}\rangle$ as an initial vector, as follows:

$$Z(T, h)_{\text{OFTL}} = \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{\text{sat}}^{(m)} - N_{V}}{R} \sum_{r=1}^{M_{L}-1} \sum_{j=0}^{N_{V}-1} e^{-\beta \epsilon_{j,m}^{(r)}(h)} |\langle V'_{r,m} | \psi_{j,m}^{(r)} \rangle|^2 + \sum_{i=0}^{N_{V}-1} e^{-\beta E_{i,m}(h)}.$$  

FIG. 1. Lattice structure of the ITL with exchange interactions, $J$ and $J'$. The solid and thin lines represent $J$ and $J'$, respectively. We set $J = 1$. The black circles represent the sites with a spin. The pink, red, and blue dashed quadrangles represent the clusters of $N = 27$, $N = 30$, and $N = 36$, respectively, used in the OFTLM with periodic boundary conditions, where $N$ is the number of sites.
Similarly, in the OFTLM, the energy $E(T, h)_{\text{OFTL}}$, magnetization $M(T, h)_{\text{OFTL}}$, magnetic susceptibility $\chi(T)_{\text{OFTL}}$, and magnetic entropy $S_m(T, h)_{\text{OFTL}}$ are obtained as follows:

$$E(T, h)_{\text{OFTL}} = \frac{1}{Z(T, h)_{\text{OFTL}}} \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{st}^{(m)} - N_V}{R}$$

$$M(T, h)_{\text{OFTL}} = \frac{1}{Z(T, h)_{\text{OFTL}}} \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{st}^{(m)} - N_V}{R}$$

$$\chi(T)_{\text{OFTL}} = \frac{1}{T Z(T, h = 0)_{\text{OFTL}}} \sum_{m=-M_{\text{sat}}}^{M_{\text{sat}}} \frac{N_{st}^{(m)} - N_V}{R}$$

$$S_m(T, h)_{\text{OFTL}} = \frac{E(T, h)_{\text{OFTL}}}{T} - \ln Z(T, h)_{\text{OFTL}}.$$

The last terms in Eqs. (7), (8), (9), and (10) are exact values, which are more accurate than those obtained using the standard FTLM, particularly at low temperatures. Therefore, using the OFTLM, we could evaluate the finite-size effects more accurately. For subspaces with large $S^J_{\text{tot}}$, all the eigenvalues can be calculated using FullED because $N_{st}^{(m)}$ is small. Therefore, we use the OFTLM for small $m$ and FullED for large $m$. The conditions of the calculation are listed in Table I for $N = 27$, Table II for $N = 30$, and Table III for $N = 36$. We note that $R$, $M_L$, and $N_V$ can be dependent on $m$ in the OFTLM, but we maintain them constant in the present study. Hereafter, the method combining the OFTLM and FullED is simply called OFTLM for simplicity.

**Table I. Conditions of the calculation for the $N = 27$ cluster.**

| $m$ | $N_{st}^{(m)}$ | method | $R$ | $M_L$ | $N_V$ |
|-----|---------------|--------|-----|------|------|
| 27/2 | 1            | Exact  | –   | –    | –    |
| 25/2 | 27           | FullED| –   | –    | –    |
| 23/2 | 351          | FullED| –   | –    | –    |
| 21/2 | 2925         | FullED| –   | –    | –    |
| 19/2 | 17550        | FullED| –   | –    | –    |
| 17/2 | 80730        | OFTLM | 30  | 100  | 6    |
| 15/2 | 296010       | OFTLM | 30  | 100  | 6    |
| 13/2 | 888030       | OFTLM | 30  | 100  | 6    |
| 11/2 | 2220075      | OFTLM | 30  | 100  | 6    |
| 9/2  | 4668825      | OFTLM | 30  | 100  | 6    |
| 7/2  | 8436285      | OFTLM | 30  | 100  | 6    |
| 5/2  | 13037895     | OFTLM | 30  | 100  | 6    |
| 3/2  | 17383860     | OFTLM | 30  | 100  | 6    |
| 1/2  | 20058900     | OFTLM | 30  | 100  | 6    |

**Table II. Conditions of the calculation for the $N = 30$ cluster.**

| $m$ | $N_{st}^{(m)}$ | method | $R$ | $M_L$ | $N_V$ |
|-----|---------------|--------|-----|------|------|
| 15  | 1             | Exact  | –   | –    | –    |
| 14  | 30            | FullED| –   | –    | –    |
| 13  | 435           | FullED| –   | –    | –    |
| 12  | 4060          | FullED| –   | –    | –    |
| 11  | 27405         | FullED| –   | –    | –    |
| 10  | 142506        | OFTLM | 30  | 100  | 6    |
| 9   | 593775        | OFTLM | 30  | 100  | 6    |
| 8   | 2035800       | OFTLM | 30  | 100  | 6    |
| 7   | 5852925       | OFTLM | 30  | 100  | 6    |
| 6   | 14307150      | OFTLM | 30  | 100  | 6    |
| 5   | 30045015      | OFTLM | 30  | 100  | 6    |
| 4   | 54627300      | OFTLM | 30  | 100  | 6    |
| 3   | 86493225      | OFTLM | 30  | 100  | 6    |
| 2   | 119759850     | OFTLM | 30  | 100  | 6    |
| 1   | 145422675     | OFTLM | 30  | 100  | 6    |
| 0   | 155117520     | OFTLM | 30  | 100  | 6    |

**IV. RESULTS AND DISCUSSION**

**A. Magnetic susceptibility and entropy**

Figure 2 shows the results of the magnetic susceptibility $\chi(T)$ [2(a), 2(b), 2(c), 2(d)] and magnetic entropy $S_m(T)$ [2(e), 2(f), 2(g)] at $h = 0$ for $J' = 0.25, 0.5, 0.75$, and 1 at $N = 27, 30$, and 36. The shaded regions shown in Fig. 2 indicate the standard errors of the OFTLM using the jackknife technique [52]. In $\chi(T)$, the errors are almost maintained within the line width, whereas in $S_m(T)$, they are sufficiently small compared with the line width. Therefore, these results make the finite-size effects apparent. At $N = 27$, in any $J'$, $\chi(T)$ diverges at $T \to 0$, and $S_m(T)$ remains a finite value. This is because the total magnetization $S^J_{\text{tot}}$ of the ground states is not zero, but $\pm 1/2$, $\chi(T)$ at $J' = 0.25$, as shown in Fig. 2(a), has a maximum value at $T \sim 0.6$. As $J'$ increases, the position of the maximum value decreases to $T \sim 0.4$, as shown in Fig. 2(d). This peak
at \( T \sim 0.4 \) is consistent with the results of a previous study on high-temperature series expansions \[5, 28\]. As shown in Fig. 2 for \( T > 0.2 \), \( \chi(T) \) and \( S_m(T) \) are almost independent of size \( N \). Therefore, the positions of the peak of \( \chi(T) \) are expected to hardly change, even in the thermodynamic limit. Thus, the values of the exchange interactions \((J, J')\) of the model compounds can be estimated by comparing \( \chi(T) \) obtained by using the OFTLM and the experimental results for \( T > 0.2 \). In the case where \( S_m(T)/N > 0.1 \), almost no size dependence is observed. This suggests that the adiabatic magnetization process discussed in Sec. IV.C also has almost no size dependence for \( S_m(T)/N > 0.1 \).

### B. Isothermal magnetization process

In this subsection, we report the results of the ITL isothermal magnetization process using the OFTLM. Figure 4 shows the isothermal magnetization curves for \( J' = 0.25, 0.5, 0.75, \) and 1 at \( N = 27, 30, \) and 36. As the numerical errors are sufficiently small compared with the line width in Fig. 3, they are not shown. Here, we first discuss the finite-size effect. At \( T = 0.2 \), almost no size effect is observed, as shown in Figs. 3(c), 3(f), 3(i), and 3(l). At \( T = 0.1 \), there is a slight size dependence comparable to the line width at low fields, as shown in Figs. 3(b), 3(e), and 3(h). At \( T = 0.05 \), a size dependence exists at low magnetic fields, as shown in Figs. 3(a), 3(d), 3(g), and 3(j). These results suggest that the magnetization curve in the thermodynamic limit can be estimated with good accuracy at \( T \geq 0.1 \) using the OFTLM for \( N = 36 \). From Figs. 3(c), 3(h), and 3(k), we expect that the 1/3 magnetization plateau exists even in the thermodynamic limit at \( T \leq 0.1 \) for \( J' \geq 0.5 \). In addition, in a previous study, at \( T = 0 \), the 1/3 plateau was expected to be observed even at \( J' \sim 0.3 \) \[51\]. Therefore, we propose that the model compounds of the ITL with a very narrow 1/3 plateau or without a 1/3 plateau have \( J' < 0.5 \). Furthermore, by simultaneously comparing the calculated magnetization curve and susceptibility with those of the model compounds, the exchange interactions \((J, J')\) can be estimated more accurately.

### C. Adiabatic magnetization process

In experiments using pulsed magnetic fields with pulse widths of a few to several tens of microseconds, which have been conducted extensively in recent years \[1, 2\], the magnetization process is not an isothermal process but an adiabatic process because of the very narrow pulse width. In this subsection, we investigate the adiabatic magnetization process of ITL.

Figure 5 shows the adiabatic magnetization curves at \( N = 27, 30, \) and 36 using the OFTLM. The temperature curves under the adiabatic magnetization process, which correspond to the MCE, are also shown. The magnetization curves and temperature curves were calculated at \( S_m/N = 0.075, 0.1, \) and 0.2. Here, we first discuss the finite-size effect. At \( S_m/N = 0.2 \), almost

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**TABLE III. Conditions of the calculation for the \( N = 36 \) cluster.**

| \( m \) | \( N^{(m)} \) | method | \( R \) | \( M_L \) | \( N_V \) |
|--------|------------|--------|------|------|------|
| 15     | 1          | Exact  | –    | –    | –    |
| 16     | 9          | FullED| –    | –    | –    |
| 17     | 18         | FullED| –    | –    | –    |
| 18     | 27         | FullED| –    | –    | –    |
| 19     | 36         | FullED| –    | –    | –    |
| 20     | 45         | FullED| –    | –    | –    |
| 21     | 54         | FullED| –    | –    | –    |
| 22     | 63         | FullED| –    | –    | –    |
| 23     | 72         | FullED| –    | –    | –    |
| 24     | 81         | FullED| –    | –    | –    |
| 25     | 90         | FullED| –    | –    | –    |
| 26     | 100        | FullED| –    | –    | –    |
| 27     | 110        | FullED| –    | –    | –    |
| 28     | 120        | FullED| –    | –    | –    |
| 29     | 130        | FullED| –    | –    | –    |
| 30     | 140        | FullED| –    | –    | –    |
| 31     | 150        | FullED| –    | –    | –    |
| 32     | 160        | FullED| –    | –    | –    |
| 33     | 170        | FullED| –    | –    | –    |
| 34     | 180        | FullED| –    | –    | –    |
| 35     | 190        | FullED| –    | –    | –    |
| 36     | 200        | FullED| –    | –    | –    |

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**FIG. 2.** Temperature dependence of the magnetic susceptibility \( \chi(T) \) (a–d) and magnetic entropy \( S_m(T) \) (e–h) per site for the ITL with \( N = 27, 30, \) and 36 at \( J' = 0.25, 0.5, 0.75, \) and 1, obtained using the OFTLM. The shaded regions indicate the standard errors of the method using the jackknife technique.
FIG. 3. Isothermal magnetization process of the ITL with \( N = 27, 30, \) and 36 at \( J' = 0.25 \) (a–c), \( J' = 0.5 \) (d–f), \( J' = 0.75 \) (g–i), and \( J' = 1 \) (j–l) for \( T = 0.05, 0.1, \) and 0.2, obtained by using the OFTLM.

no finite-size effect is observed, as shown in Figs. (c), (f), (i), and (l). At \( S_m/N = 0.1 \), in the magnetization curves, almost no finite-size effect is observed, as shown in Figs. (b), (e), (h), and (k); however, in the temperature curves, particularly at \( J' = 0.5 \), a size dependence is observed. At \( S_m/N = 0.075 \), particularly at \( J' \leq 0.5 \), the size dependence of the magnetization curves and temperature curves is observed.

In the isothermal process, the magnetization curves have a 1/3 plateau for \( J' \geq 0.5 \), whereas in the adiabatic process, the anomaly corresponding to the 1/3 plateau is not flat but inclined, as shown in Figs. (d), (e), (g), (h), (j), and (k). The temperature curves have maxima around the center of the region showing this anomaly. This is explained as follows: because the 1/3 plateau state with the up-up-down structure has a threefold degeneracy and an energy gap in the thermodynamic limit, the entropy has minima at the center of the plateau in the isothermal process. Consequently, the temperature has maxima in the adiabatic process. As the temperature is not constant in the adiabatic process, the magnetization curve is not completely flat around \( M/M_{\text{sat}} = 1/3 \).

Furthermore, at high magnetic fields, the magnetization \( M \) does not reach the saturation magnetization \( M_{\text{sat}} \). The state with \( M = M_{\text{sat}} \) has all spins aligned in the magnetic field direction; thus, the entropy is zero. In the adiabatic process, the entropy is constant (non-zero), and \( M \) never reaches \( M_{\text{sat}} \), but the temperature increases rapidly, as shown in Fig. (l) Notably, in the experiment with a high magnetic field and a pulse width of a few microseconds, regardless of the magnitude of the magnetic field, \( M \) does not reach \( M_{\text{sat}} \) unless
FIG. 4. Adiabatic magnetization curve and MCE of the ITL with $N = 27$, 30, and 36 at $J' = 0.25$ (a–c), $J' = 0.5$ (d–f), $J' = 0.75$ (g–i), and $J' = 1$ (j–l) for $S_m(T)/N = 0.075$, 0.1, and 0.2, obtained by using the OFTLM.

the temperature is sufficiently low.

D. Temperature and magnetic field dependence of the magnetic entropy

Figure 5 shows the magnetic entropy $S_m$ as a function of temperature $T$ and magnetic field $h$ for the ITL with $N = 36$ calculated using the OFTLM. In the low-temperature region $T \leq 0.05$, vertical streaks are visible owing to the finite-size effect.

For $J' \geq 0.5$, the temperature curves at $S_m/N = \frac{1}{3} \ln 2 (\sim 0.0866)$ have maxima, as indicated by the white arrows in Figs. 5(b), 5(c), and 5(d). These maxima are derived from the 1/3 plateau as described in Sec. IV C. Therefore, such temperature maxima, if experimentally obtained in the MCE measurements, would suggest the presence of a magnetization plateau.

As shown by the red arrow in Fig. 5(a), there is a sharp drop and rise in the temperature under the isentropic process for $S_m/N < 0.1$ at $J' = 0.25$ around $h = 2.5$. This phenomenon of a sudden temperature change around the critical magnetic field corresponds to the divergence of the magnetic Grüneisen ratio $\Gamma_H = \frac{1}{T} \frac{\partial T}{\partial H}|_{S_m}$ at a quantum critical point [32, 42, 54, 55]. Similarly, at $J' = 0.75$, a rapid temperature change is observed around $h = 2.5$. This anomaly would indicate the signature of a quantum phase transition [31].

We believe that these results can be compared with those obtained experimentally in the future.
FIG. 5. Magnetic entropy $S_m/\text{N}$ per site as a function of temperature $T$ and magnetic field $h$ for the ITL with $N = 36$ using the OFTLM. (a) $J' = 0.25$. (b) $J' = 0.5$. (c) $J' = 0.75$. (d) $J' = 1$.

V. SUMMARY

Inspired by the recent development of pulsed magnetic field generators [1, 2] and the experimental results of ITL compounds [32–34], we investigated the magnetic susceptibility, magnetic entropy, isothermal and adiabatic magnetization curves, and the MCE of the ITL using the OFTLM.

We obtained almost size-independent results with $T \geq 0.2$ for the magnetic susceptibility and $T \geq 0.1$ for the isothermal magnetization curve. The $1/3$ magnetization plateau was observed at $J' \geq 0.5$ in the isothermal magnetization process. By comparing our results for the magnetic susceptibility and isothermal magnetization curve with the experimental results, we could quantitatively determine the exchange interactions ($J$ and $J'$) of the ITL compounds.

In the adiabatic magnetization process, the anomaly corresponding to the $1/3$ plateau was not flat but inclined. This is because the entropy of the $1/3$ plateau state was lower. We also obtained the magnetic entropy as a function of the temperature and magnetic field for the ITL with $N = 36$. In other words, we obtained the temperature of the adiabatic (isentropic) process as a function of the magnetic field, which corresponds to the MCE. We observed an anomaly in the temperature at $J' = 0.75$ around $h = 1.0$, which indicates the signature of a quantum phase transition. We believe that our results will be useful for understanding the experimental results of MCE in the future.

We would like to emphasize that the OFTLM is useful not only for isothermal processes but also for adiabatic processes. We hope that our study will motivate further theoretical and experimental investigations of the ITL in the future.

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