CLASSIFYING COMPLEMENTS FOR HOPF ALGEBRAS AND LIE ALGEBRAS

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Abstract. Let \( A \subseteq E \) be a given extension of Hopf (respectively Lie) algebras. We answer the classifying complements problem (CCP) which consists of describing and classifying all complements of \( A \) in \( E \). If \( H \) is a given complement then all the other complements are obtained from \( H \) by a certain type of deformation. We establish a bijective correspondence between the isomorphism classes of all complements of \( A \) in \( E \) and a cohomological type object \( \mathcal{H}A^2(H, A \mid \langle \rhd, \lhd \rangle) \), where \( \langle \rhd, \lhd \rangle \) is the matched pair associated to \( H \). The factorization index \( [E : A]^f \) is introduced as a numerical measure of the (CCP). For two \( n \)-th roots of unity we construct a \( 4n^2 \)-dimensional Hopf algebra whose factorization index over the group algebra is arbitrary large.

Introduction

Let \( \mathcal{C} \) be the category of groups, Lie algebras, Hopf algebras, etc. and \( A \subseteq E \) a given subobject of \( E \) in \( \mathcal{C} \). A subobject \( H \) of \( E \) is called a complement of \( A \) in \( E \) (or an \( A \)-complement of \( E \)) if \( E \) can be written as a 'product' of \( A \) and \( H \) such that \( A \) and \( H \) have 'minimal intersection' in \( E \); the meaning of 'product' and 'minimal intersection' depends on the given category \( \mathcal{C} \). We denote by \( [E : A]^f \) the cardinal of the (possibly empty) isomorphism classes of all \( A \)-complements of \( E \) and we call it the factorization index of \( A \) in \( E \). A natural question arises:

Classifying complements problem (CCP): Let \( A \subseteq E \) be a given subobject of \( E \) in \( \mathcal{C} \). If an \( A \)-complement of \( E \) exists, describe explicitly, classify all \( A \)-complements of \( E \) and compute the factorization index \( [E : A]^f \).

To start with, let \( \mathcal{C} = \mathcal{G}r \) be the category of groups. An \( A \)-complement of a group \( E \) is a subgroup \( H \leq E \) such that \( E = AH \) and \( A \cap H = \{1\} \). An \( A \)-complement of \( E \), if exists, is not necessarily unique. The basic example is the following: consider \( S_3 \) to be the symmetric group viewed as a subgroup in \( S_4 \) by considering 4 to be a fixed point. Then, the factorization index \( [S_4 : S_3]^f = 2 \). For more details on the group case we refer the reader to [3]. Hence, we expect to obtain non-trivial results for the (CCP) whose difficulty depends on the category \( \mathcal{C} \) as well as on the extension \( A \subseteq E \) in \( \mathcal{C} \).

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The aim of this paper is to give the answer to the (CCP) if \( \mathcal{C} \) is the category of Hopf (respectively Lie) algebras. There exists a general principle: \( H \) is an \( A \)-complement of \( E \) in a given category \( \mathcal{C} \) if and only if \( E \cong A \bowtie H \), where \( A \bowtie H \) is a 'bicrossed product' in the category \( \mathcal{C} \) associated to a 'matched pair' between the objects \( A \) and \( H \). This principle becomes a theorem when \( \mathcal{C} \) is the category of groups or groupoids [4], algebras [5], Hopf algebras [15], Lie groups or Lie algebras [13], locally compact quantum groups [19], multiplier Hopf algebras [6]. Let \( H \) be a given \( A \)-complement of \( E \). Hence, there exists a canonical isomorphism \( A \bowtie H \cong E \) in \( \mathcal{C} \). Now, the description and the classification part of the (CCP) is obtained from the following subsequent question: describe and classify all objects \( H \) in \( \mathcal{C} \) such that there exists an isomorphism \( A \bowtie H \cong A \bowtie H \) in \( \mathcal{C} \). This can also be viewed as a descent type question: the classification of all \( A \)-complements of \( E \) needs a parallel theory similar to what is called the classification of forms in the classical descent theory [12], [18].

As we will see, the theoretical answers to the (CCP) for Hopf algebras and Lie algebras follow a common path which might generate a method of tackling this problem in different other categories. In order to describe and then classify all the complements of a given extension \( A \subseteq E \) it is enough to know only one complement \( H \). All the other \( A \)-complements of \( E \) are obtained from \( H \) by a certain type of deformation which involves some special maps \( r : H \rightarrow A \) associated to the canonical matched pair \((A, H, \lhd, \rhd)\), called deformation maps. To each deformation map \( r : H \rightarrow A \) we associate a new \( A \)-complement denoted by \( H_r \) and, conversely, for any \( A \)-complement \( H \) there exists a deformation map \( r : H \rightarrow A \) of the canonical matched pair \((A, H, \lhd, \rhd)\) such that \( H \cong H_r \).

The paper is organized as follows. In Section 1 we shall review the Majid’s bicrossed product [14] associated to a matched pair of Hopf algebras \((A, H, \lhd, \rhd)\). Section 2 offers the answer to the (CCP) problem for Hopf algebras. The answer will be given in three steps. In Theorem 2.6 a new type of deformation of a given Hopf algebra \( H \) is introduced. This deformation \( H_r \) is a new Hopf algebra called the \( r \)-deformation of \( H \) and is associated to an arbitrary matched pair of Hopf algebras \((A, H, \rhd, \lhd)\) and to a deformation map \( r : H \rightarrow A \) in the sense of Definition 2.3. Furthermore, \( H_r \) is an \( A \)-complement of the bicrossed product \( A \bowtie H \). Now, let \( A \subseteq E \) be an extension of Hopf algebras \( H \) and \( A \) a given \( A \)-complement of \( E \). The description of all \( A \)-complements of \( E \) is given in Theorem 2.8: any other \( A \)-complement \( H \) of \( E \) is isomorphic as a Hopf algebra with an \( H_r \), for some deformation map \( r : H \rightarrow A \) of the canonical matched pair \((A, H, \lhd, \rhd)\) associated to \( H \). Finally, Theorem 2.5 provides the classification of \( A \)-complements of \( E \): there exists a bijection between the isomorphism classes of all \( A \)-complements of \( E \) and a cohomological type object \( \mathcal{H}A^2(H, A \mid (\rhd, \lhd)) \) and this bijection is explicitly described. In particular, we obtain the formula for computing the factorization index of a given extension \( A \subseteq E \) of Hopf algebras: \(|E : A|^f = |\mathcal{H}A^2(H, A \mid (\rhd, \lhd))|\). In Section 3 we shall construct an example of a Hopf algebra extension of a given factorization index in Theorem 3.2. This is the extension \( k[C_n] \subseteq H_{4n^2, \omega, \omega'} \), where \( H_{4n^2, \omega, \omega'} \) is a \( 4n^2 \)-dimensional quantum group associated to two distinct \( n \)-th roots of unity \( \omega \) and \( \omega' \). In Section 4 we give the answer to the (CCP) in the case when \( \mathcal{C} \) is the category of Lie algebras. The main result of the section is Theorem 4.5: if \( g \) is a Lie subalgebra of \( \Xi \) and \( h \) is a fixed \( g \)-complement of \( \Xi \), then the isomorphism classes of all \( g \)-complements of \( \Xi \) are
parameterized by a certain cohomological object denoted by $\mathcal{HA}^{2}(h,g|(\triangleright,\triangleleft))$ which is explicitly constructed.

1. Preliminaries

Unless explicitly specified otherwise, all coalgebras, Hopf algebras, Lie algebras, tensor products, homomorphisms, and so on are over a commutative ring $k$. For a coalgebra $C$, we use Sweedler’s $\Sigma$-notation: $\Delta(c) = c(1) \otimes c(2)$, $(I \otimes \Delta)\Delta(c) = c(1) \otimes c(2) \otimes c(3)$ (summation understood). A Hopf subalgebra of a Hopf algebra $E$ is a Hopf algebra inclusion $A \subseteq E$ that splits $k$-linearly. Let $A$ and $H$ be two Hopf algebras. $H$ is called a right $A$-module coalgebra if $H$ is a coalgebra in the monoidal category $\mathcal{M}_{A}$ of right $A$-modules, i.e. there exists $\triangleleft : H \otimes A \to H$ a morphism of coalgebras such that $(H, \triangleleft)$ is a right $A$-module. Similarly, $A$ is a left $H$-module coalgebra if $A$ is a coalgebra in the monoidal category of left $H$-modules.

Let $A \subseteq E$ be a Hopf subalgebra of $E$. A Hopf subalgebra $H \subseteq E$ is called a right complement of $A$ in $E$ (or a right $A$-complement of $E$) if the multiplication map $A \otimes H \to E$, $a \otimes h \mapsto ah$ is bijective. Similarly, $H$ is a left complement of $A$ in $E$ if the multiplication map $H \otimes A \to E$ is bijective. In the case that $E$ has a bijective antipode, then the concepts of right/left $A$-complement coincide [1, Proposition 3.1]. Throughout this paper by an $A$-complement we mean a right $A$-complement.

Let $H$ be a (right) $A$-complement of $E$: in this case we say that the Hopf algebra $E$ factorizes through $A$ and $H$. The bicrossed product of two Hopf algebras was introduced by Majid in [14, Proposition 3.12] under the name of double cross product. Let $A$ and $H$ be two Hopf algebras and $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$ two morphisms of coalgebras such that the following normalizing conditions hold for any $h \in H$, $a \in A$:

$$h \triangleright 1_{A} = \varepsilon_{H}(h)1_{A}, \quad 1_{H} \triangleright a = a, \quad 1_{H} \triangleleft a = \varepsilon_{A}(a)1_{H}, \quad h \triangleleft 1_{A} = h$$

(1)

We denote by $A \bowtie H$ the $k$-module $A \otimes H$ together with the multiplication:

$$(a \bowtie h) \cdot (c \bowtie g) := a(h_{(1)} \triangleright c_{(1)}) \bowtie (h_{(2)} \triangleleft c_{(2)})g$$

(2)

for all $a, c \in A$, $h, g \in H$, where we denoted $a \otimes h$ by $a \bowtie h$. The object $A \bowtie H$ is called the bicrossed product of $A$ and $H$ if $A \bowtie H$ is a Hopf algebra with the multiplication given by (2), the unit $1_{A} \bowtie 1_{H}$ and the coalgebra structure given by the tensor product of coalgebras. The next theorem provides necessary and sufficient conditions for $A \bowtie H$ to be a bicrossed product.

**Theorem 1.1.** Let $A$, $H$ be two Hopf algebras and $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$ two morphisms of coalgebras satisfying the normalizing conditions (1). The following statements are equivalent:

(1) $A \bowtie H$ is a bicrossed product;
(2) \((H,\triangleleft)\) is a right \(A\)-module coalgebra, \((A,\triangleright)\) is a left \(H\)-module coalgebra and the following compatibilities hold for any \(a, b \in A\), \(g, h \in H\).

\[
g \triangleright (ab) = (g(1) \triangleright a(1))((g(2) \triangleleft a(2)) \triangleright b) \tag{3}
\]

\[
(gh) \triangleleft a = (g \triangleleft (h(1) \triangleright a(1)))(h(2) \triangleleft a(2)) \tag{4}
\]

\[
g(1) \triangleleft a(1) \otimes g(2) \triangleright a(2) = g(2) \triangleleft a(2) \otimes g(1) \triangleright a(1) \tag{5}
\]

**Proof.** \((2) \Rightarrow (1)\) This is just [15, Theorem 7.2.2] or [11, Theorem IX.2.3]. \((1) \Rightarrow (2)\) Follows as a special case of [2, Theorem 2.4] if we consider \(f : H \otimes H \to A\) to be the trivial cocycle, i.e. \(f(g, h) = \epsilon_H(g)\epsilon_H(h)\). See also [2, Examples 2.5] for details. \(\square\)

A **matched pair** of Hopf algebras is a system \((A, H, \triangleleft, \triangleright)\), where \((H, \triangleleft)\) is a right \(A\)-module coalgebra, \((A, \triangleright)\) is a left \(H\)-module coalgebra such that the compatibility conditions \((1)\) and \((3)-(5)\) hold.

**Examples 1.2.** 1. Let \((A, \triangleright)\) be a left \(H\)-module coalgebra and consider \(H\) as a right \(A\)-module coalgebra via the trivial action, i.e. \(h \triangleleft a = \epsilon_A(a)h\). Then \((A, H, \triangleleft, \triangleright)\) is a matched pair of Hopf algebras if and only if \((A, \triangleright)\) is a left \(H\)-module algebra and the following compatibility condition holds

\[
g(1) \otimes g(2) \triangleright a = g(2) \otimes g(1) \triangleright a \tag{6}
\]

for all \(g \in H\) and \(a \in A\). In this case, the associated bicrossed product \(A \bowtie H = A\#H\) is the semi-direct (smash) product of Hopf algebras as defined by Molnar [17].

2. The fundamental example of a bicrossed product is the Drinfel’d double \(D(H)\). Let \(H\) be a finite dimensional Hopf algebra over a field \(k\). Then we have a matched pair of Hopf algebras \((\,^*H^*)_{\text{cop}}, H, \triangleleft, \triangleright)\), where the actions \(\triangleleft\) and \(\triangleright\) are defined by:

\[
h \triangleleft h^* := \langle h^*, S_1^{-1}(h(3))h(1)h(2) \rangle, \quad h \triangleright h^* := \langle h^*, S_1^{-1}(h(2))h(1) \rangle \tag{7}
\]

for all \(h \in H\) and \(h^* \in \,^*H^*\) ([11, Theorem IX.3.5]). The Drinfel’d double of \(H\) is the bicrossed product associated to this matched pair, i.e. \(D(H) = (\,^*H^*)_{\text{cop}} \bowtie H\).

3. Let \(k\) be a field of characteristic zero, \(g, h\) two Lie algebras and \(U(g), U(h)\) the corresponding universal enveloping algebras. Then there is a bijection between the matched pairs of Lie algebras \((g, h, \triangleright, \triangleleft)\) [15, Definition 8.3.1] and the matched pairs of Hopf algebras \((U(g), U(h), \triangleright, \triangleleft)\). The bijection is given such that there exists a Hopf algebra isomorphism \(U(g) \bowtie U(h) \cong U(g \bowtie h)\) ([16, Proposition 2.4]).

A bicrossed product \(A \bowtie H\) will be viewed as a left \(A\)-module via the restriction of scalars through the canonical inclusion \(i_A : A \to A \bowtie H\), \(i_A(a) = a \bowtie 1_H\), for all \(a \in A\). The next result is [15, Theorem 7.2.3]: Let \(A \subseteq E\) be a Hopf subalgebra and \(H\) a \(A\)-complement of \(E\). Then there exists a matched pair of Hopf algebras \((A, H, \triangleleft, \triangleright)\) such that the multiplication map

\[
m_E : A \bowtie H \to E, \quad m_E(a \bowtie h) = ah \tag{8}
\]

for all \(a \in A\) and \(h \in H\) is an isomorphism of Hopf algebras. The actions of the matched pair \((A, H, \triangleleft, \triangleright)\) are constructed as follows for all \(a \in A\), \(h \in H\):

\[
ha = (h(1) \triangleright a(1))(h(2) \triangleleft a(2))
\]

\]
From now on, the matched pair constructed in (8) will be called the canonical matched pair associated to the factorization of \( E \) through \( A \) and \( H \). The following is just the formal dual of the notion of central map:

**Definition 1.3.** Let \( A \) and \( H \) be two Hopf algebras. A coalgebra map \( r : H \to A \) is called cocentral if the following compatibility holds for all \( h \in H \):

\[
r(h_{(1)}) \otimes h_{(2)} = r(h_{(2)}) \otimes h_{(1)}
\] (9)

The set \( CoZ(H,A) \) of all cocentral maps is a group with respect to the convolution product. We denote by \( CoZ^1(H,A) \) the subgroup of \( CoZ(H,A) \) of all cocentral maps \( r : H \to A \) such that \( r(1_H) = 1_A \). Cocentral maps arise naturally from the following:

**Lemma 1.4.** Let \( A \) and \( H \) be two Hopf algebras. Then there exists a one to one correspondence between the set of all right \( H \)-comodule coalgebra maps \( f : H \to A \otimes H \) and the set of all cocentral maps \( r : H \to A \). The bijection \( (f \mapsto r_f, r \mapsto f_r) \) is given as follows:

\[
r_f := (\text{Id}_A \otimes \epsilon_H) \circ f, \quad f_r(h) = r(h_{(1)}) \otimes h_{(2)}
\] (10)

for all \( h \in H \).

**Proof.** Straightforward: here \( H \) and \( A \otimes H \) are viewed as right \( H \)-comodules via \( \Delta_H \).

For future use we state here the following:

**Lemma 1.5.** If a Hopf algebra \( E \) factorizes into two sub-bialgebras \( A \) and \( H \), then \( A \) and \( H \) are necessarily Hopf algebras.

**Proof.** If we denote by \( S_E \) the antipode of \( E \), then the antipodes of \( A \) and \( H \) are given by the following formulæ:

\[
S_A(a) := (\text{Id}_A \otimes \epsilon_H) \circ S_E(a \otimes 1_H), \quad S_H(h) := (\epsilon_A \otimes \text{Id}_H) \circ S_E(1_A \otimes h)
\]

for all \( a \in A, h \in H \). Indeed, as \( S_E \) is the antipode for \( E \) we have:

\[
(a_{(1)} \otimes 1_H)S_E(a_{(2)} \otimes 1_H) = S_E(a_{(1)} \otimes 1_H)(a_{(2)} \otimes 1_H) = \epsilon_A(a)(1_A \otimes 1_H)
\]

for all \( a \in A \). By applying the algebra map \( \text{Id}_A \otimes \epsilon_H \) to the above identity we obtain that \( S_A \) is the antipode of \( A \). The formula for \( S_H \) follows in the same manner.

\[ \square \]

2. **Classifying complements for Hopf algebras**

Let \( A \subseteq E \) be a Hopf subalgebra of \( E \) and \( F(A,E) \) the (possibly empty) isomorphism classes of all \( A \)-complements of \( E \). The problem of existence of \( A \)-complements of \( E \) has to be treated ‘case by case’ for every given Hopf algebra extension, a computational part of it can not be avoided. This was the approach used in the similar problem at the level of groups, i.e. corresponding to the Hopf algebra extension \( k[A] \subseteq k[G] \), for two groups \( A \) and \( G \) with \( A \leq G \) (see [10] and the references therein). For example, if \( E = k[A_6] \) and \( A \) is a proper Hopf subalgebra, then \( F(A,k[A_6]) \) is the empty set. This is based on the fact that the alternating group \( A_6 \) has no proper factorizations [20].

In order to answer the (CCP) we need to introduce a few more concepts:
Definition 2.1. Let $A$ be a Hopf subalgebra of $E$. We define the factorization index of $A$ in $E$ as the cardinal of $\mathcal{F}(A, E)$ and it will be denoted by $|E : A|^f = |\mathcal{F}(A, E)|$. The extension $A \subseteq E$ is called rigid if $|E : A|^f = 1$.

We write down explicitly what a rigid extension of Hopf algebras $E/A$ means: $|E : A|^f = 1$ if and only if any two $A$-complements $H$ and $H'$ of $E$ are isomorphic as Hopf algebras. Equivalently, this can be restated as follows: if $E \cong A \bowtie H \cong A \bowtie H'$ (isomorphism of Hopf algebras and left $A$-modules), then $H \cong H'$. This is a Krull-Schmidt-Azumaya type theorem for bicrossed products of Hopf algebras.

Examples 2.2. 1. In most cases, for a given extension of Hopf algebras $A \subseteq E$ the factorization index $|E : A|^f$ is equal to 0 (i.e. there exists no $A$-complements of $E$) or 1. For instance, above we have shown in fact that $|k[A_6] : A|^f = 0$, for any proper Hopf subalgebra $A$ of the group Hopf algebra $k[A_6]$.

2. Let $E := A\#H$ be a semidirect product of two Hopf algebras in the sense of Example 1.2. Then the extension $A \subseteq A\#H$ is rigid. Indeed, since $A$ is a normal Hopf subalgebra of $E$, we obtain that any $A$-complement of $E$ is isomorphic to $E/A^+ E$.

3. Examples of extensions $E/A$ for which $|E : A|^f \geq 2$ are quite rare, which makes them tempting to identify. For instance, the extension $k[S_3] \subseteq k[S_4]$ has factorization index 2. We shall provide an elaborated way of constructing examples of Hopf algebra extensions $E/A$ of a given factorization index in Theorem 3.1.

Definition 2.3. Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. A unitary cocentral map $r \in CoZ^1(H, A)$ is called a deformation map of the matched pair $(A, H, \triangleright, \triangleleft)$ if the following compatibility holds for any $g, h \in H$:

$$r(h \triangleright r(g(1))) g(2) = r(h_{(1)}) (h_{(2)} \triangleright r(g))$$

(11)

Let $\mathcal{DM}(H, A | \triangleright, \triangleleft) \subseteq CoZ^1(H, A)$ be the set of all deformation maps of the matched pair $(A, H, \triangleright, \triangleleft)$. The trivial map $r : H \rightarrow A$, $r(h) = \varepsilon(h)1_A$ is of course a deformation map. We introduce the following:

Definition 2.4. Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Two deformation maps $r, R : H \rightarrow A$ are called equivalent and we denote this by $r \sim R$ if there exists $\sigma : H \rightarrow H$ an unitary automorphism of the coalgebra $H$ such that

$$\sigma((h \triangleright r(g(1))) g(2)) = (\sigma(h) \triangleleft R(\sigma(g(1)))) \sigma(g(2))$$

(12)

for all $g, h \in H$.

The theorem that gives the answer to the (CCP) for Hopf algebras is the following:

Theorem 2.5. (Classification of complements) Let $A$ be a Hopf subalgebra of $E$, $H$ an $A$-complement of $E$ and $(A, H, \triangleright, \triangleleft)$ the associated canonical matched pair. Then:

1. $\sim$ is an equivalence relation on $\mathcal{DM}(H, A | \triangleright, \triangleleft)$. We denote by $\mathcal{HA}^2(H, A | \triangleright, \triangleleft)$ the quotient set $\mathcal{DM}(H, A | \triangleright, \triangleleft)/ \sim$.

2. There exists a bijection between the isomorphism classes of all $A$-complements of $E$ and $\mathcal{HA}^2(H, A | \triangleright, \triangleleft)$. In particular, the factorization index of $A$ in $E$ is computed by
the formula:

\[ [E : A]^f = |\mathcal{H}A^2(H, A | (\triangleright, \triangleleft))| \]

We prove this theorem in two steps. First, we prove the following result where a new type of deformation of a given Hopf algebra \( H \) is introduced:

**Theorem 2.6. (Deformation of complements)** Let \( A \) be a Hopf subalgebra of \( E \), \( H \) an \( A \)-complement of \( E \) and \( r : H \to A \) a deformation map of the associated canonical matched pair \((A, H, \triangleright, \triangleleft)\).

(1) Let \( f_r : H \to A \otimes H \) be the coalgebra map defined for any \( h \in H \) by:

\[ f_r(h) = r(h_{(1)}) \otimes h_{(2)} \]

Then \( \mathbb{H} := \text{Im}(f_r) \) is an \( A \)-complement of \( E \cong A \bowtie H \).

(2) Let \( H_r := H \), as a coalgebra, with the new multiplication \( \bullet \) on \( H \) defined for any \( h, g \in H \) as follows:

\[ h \bullet g := (h \triangleleft r(g_{(1)})) g_{(2)} \quad (13) \]

Then \( H_r = (H_r, \bullet, 1_H, \Delta_H, \varepsilon_H) \) is a Hopf algebra with the antipode given by

\[ S : H_r \to H_r, \quad S(h) := S_{H_r}(h_{(2)}) \triangleleft (S_A \circ r)(h_{(1)}) \quad (14) \]

for all \( h \in H \), called the \( r \)-deformation of \( H \). Furthermore, \( H_r \cong \mathbb{H} \), as Hopf algebras.

**Proof.** (1) Without loss of generality, we can identify \( E = A \bowtie H \), since the multiplication map \( m_E : A \bowtie H \to E \) is a left \( A \)-linear Hopf algebra isomorphism. It follows from Lemma 1.4 that \( f_r : H \to A \otimes H \) is a unit preserving injective coalgebra map. Thus, \( \mathbb{H} = \text{Im}(f_r) \) is a subcoalgebra of \( E = A \bowtie H \). We will denote by \( f_r : H \to \mathbb{H} \) the coalgebra isomorphism induced by \( f_r \). We shall prove that \( \mathbb{H} \) is a sub-bialgebra of \( E = A \bowtie H \) and moreover, \( E \) factorizes through \( A \) and \( \mathbb{H} \). Indeed, using (11) it follows that \( \mathbb{H} \) is also a subalgebra of \( E \) since for any \( h, g \in H \) we have:

\[
(r(h_{(1)}) \triangleleft h_{(2)}) (r(g_{(1)}) \triangleleft g_{(2)}) = \frac{r(h_{(1)})(h_{(2)} \triangleright r(g_{(1)})) \triangleright (h_{(3)} \triangleleft r(g_{(2)})) g_{(3)}}{r((h_{(1)} \triangleleft r(g_{(1)})) g_{(2)} \triangleright (h_{(2)} \triangleleft r(g_{(3)}))) g_{(4)}} \quad (11) \]

\[ \equiv f_r((h \triangleleft r(g_{(1)})) g_{(2)}) \in \mathbb{H} \]

Therefore, \( \mathbb{H} \) is a sub-bialgebra of \( E \). Consider now the left \( A \)-linearization of \( f_r \), i.e. the map \( \varphi : A \otimes H \to A \otimes H \) given by \( \varphi(a \otimes h) = af_r(h) \), for all \( a \in A \) and \( h \in H \). Then \( \varphi \) is a bijection with the inverse given by: \( \varphi^{-1}(a \otimes h) = aS_A(r(h_{(1)})) \otimes h_{(2)}. \) Since \( \varphi \) decomposes as \( \varphi = m \circ (\text{Id}_{A} \otimes f_r) \), where \( m : A \otimes \mathbb{H} \to E \) is the multiplication map, it follows that \( m = \varphi \circ (\text{id}_{A} \otimes \tilde{f}_r^{-1}) \). Thus the multiplication map \( m : A \otimes \mathbb{H} \to E \) is bijective and hence \( E \) factorizes through \( A \) and \( \mathbb{H} \). Moreover, \( \mathbb{H} \) is necessarily a Hopf subalgebra of \( E \) by Lemma 1.5 and hence \( \mathbb{H} \) is an \( A \)-complement of \( E = A \bowtie H \), as needed.

(2) \( \tilde{f}_r : H \to \mathbb{H} \) is a unit preserving coalgebra isomorphism. Moreover, in the proof of part (1) we obtained that \( \tilde{f}_r(h) \tilde{f}_r(h) = \tilde{f}_r(h \bullet g) \), where \( \bullet \) is the multiplication on \( H \).
given by (13). Therefore, \( \tilde{f}_r : H_r \to \mathbb{H} \) is a bialgebra isomorphism between the Hopf algebra \( \mathbb{H} \) and \( H_r \). Thus, \( H_r \) is a Hopf algebra with the antipode given by (14).

\( \text{□} \)

**Remarks 2.7.**

1. Assume that in Theorem 2.6 the unitary cocentral map \( r : H \to A \) is the trivial one \( r(h) = \varepsilon_H(h)1_A \) or the right action \( \triangleleft : H \otimes A \to A \) on \( H \) is the trivial action, i.e. \( h \triangleleft a = \varepsilon_A(a)h \), for all \( h \in H \) and \( a \in A \). Then \( H_r = H \) as Hopf algebras. In general, the new Hopf algebra \( H_r \) may not be isomorphic to \( H \) as a Hopf algebra: an example will be provided in Theorem 3.2.

2. At this point we should notice that there are two other deformations of a given Hopf algebra in the literature. The first one was introduced by Drinfel’d [9]: the comultiplication of a Hopf algebra \( H \) is deformed using an invertible element \( R \in H \otimes H \), called twist, in order to obtain a new Hopf algebra \( H^R \). The dual case was defined by Doi [7]: the algebra structure of a Hopf algebra \( H \) was deformed using a Sweedler cocycle \( \tau : H \otimes H \to k \) as follows: let \( H_r = H \), as a coalgebra, with the new multiplication given by

\[
h \cdot g := \tau(h^{(1)}, g^{(1)}) h^{(2)} g^{(2)} \tau^{-1}(h^{(3)}, g^{(3)})
\]

for all \( h, g \in H \). Then \( H_r \) is a new Hopf algebra [7, Theorem 1.6] and among several applications we mention that the Drinfel’d double \( D(H) \) is a special case of this deformation [8].

Next we shall prove the converse of Theorem 2.6.

**Theorem 2.8.** (Description of complements) Let \( A \) be a Hopf subalgebra of \( E \), \( H \) an \( A \)-complement of \( E \) with the associated canonical matched pair \((A, H, \triangleright, \triangleleft)\) and let \( \mathbb{H} \) be an arbitrary \( A \)-complement of \( E \). Then there exists an isomorphism of Hopf algebras \( \mathbb{H} \cong H_r \), for some deformation map \( r : H \to A \) of the matched pair \((A, H, \triangleright, \triangleleft)\).

**Proof.** The multiplication map \( m_E : A \bowtie H \to E \) is a left \( A \)-linear Hopf algebra isomorphism and we denote its inverse by \( \nu \). In fact, without loss of generality, we can identify \( E = A \bowtie H \). We denote the multiplication map associated to the \( A \)-complement \( \mathbb{H} \) by \( m'_E : A \otimes \mathbb{H} \to E \) and its inverse by \( \mu \). Define \( f : H \to \mathbb{H} \) as the composition:

\[
f : H \xrightarrow{\iota} E \xrightarrow{\mu} A \otimes \mathbb{H} \xrightarrow{\varepsilon_A \otimes \text{Id}} \mathbb{H}
\]

(15)

Then, \( f \) is a unitary coalgebra isomorphism with the inverse \( f^{-1} \) given by the composition:

\[
f^{-1} : \mathbb{H} \xleftarrow{\iota} E \xrightarrow{\nu} A \otimes H \xrightarrow{\varepsilon \otimes \text{Id}} H
\]

(16)

The proof will be finished if we construct a deformation map \( r : H \to A \) of the canonical matched pair \((A, H, \triangleright, \triangleleft)\) such that \( f : H_r \to \mathbb{H} \) is an algebra map. This deformation map \( r : H \to A \) is given by the composition of the following maps:

\[
r : H \xrightarrow{\tilde{f}} \mathbb{H} \xleftarrow{\iota} E \xrightarrow{\nu} A \otimes H \xrightarrow{\text{Id} \otimes \varepsilon_H} A
\]

(17)

We shall prove this. We view \( \mathbb{H} \) as a right \( H \)-comodule along the coalgebra map \( f^{-1} : \mathbb{H} \to H \). Then, \( f : H \to \mathbb{H} \) is right \( H \)-colinear, since its inverse \( f^{-1} \) is. We denote by \( \tilde{f} : H \to A \otimes H \) the following composition:

\[
\tilde{f} : H \xrightarrow{f} \mathbb{H} \xleftarrow{\iota} E \cong A \bowtie H
\]
Then \( \tilde{f}: H \to A \otimes H \) is a unit-preserving right \( H \)-colinear and coalgebra map. It follows from Lemma 1.4 that there exists a unique unit-preserving cocentral map \( r: H \to A \) such that \( \tilde{f}(h) = r(h^{(1)}) \otimes h^{(2)} \), for all \( h \in H \) and moreover the map \( r \) is given by (17). Now, \( \text{Im}(\tilde{f}) = \text{Im}(f) = H \) is a Hopf subalgebra of \( E = A \bowtie H \), since \( H \) is an \( A \)-complement. Thus, for any \( h, g \in H \) we have that \( \tilde{f}(h)\tilde{f}(g) \in \text{Im}(\tilde{f}) \). Now, we have:

\[
\tilde{f}(h)\tilde{f}(g) = r(h^{(1)})(h^{(2)} \triangleright r(g^{(1)})) \bowtie (h^{(3)} \triangleleft r(g^{(2)}))g^{(3)}
\]

This element is of the form \( \tilde{f}(x) = r(x^{(1)}) \otimes x^{(2)} \), for some \( x \in H \) if and only if \( x = (h \triangleleft r(g^{(1)}))g^{(2)} \) and \( r \) is a descent map of the canonical matched pair \( (A, H, \triangleright, \triangleleft) \). Indeed, if we apply \( \varepsilon_A \otimes \text{Id}_H \) to the identity \( \tilde{f}(h)\tilde{f}(g) = \tilde{f}(x) \), we obtain the above formula for \( x \) while by applying \( \text{Id}_A \otimes \varepsilon_H \) to the formula \( \tilde{f}(h)\tilde{f}(g) = \tilde{f}((h \triangleleft r(g^{(1)}))g^{(2)}) \) we obtain that \( r: H \to A \) is a deformation map. Furthermore, in this case we have \( f(h)g = f((h \triangleleft r(g^{(1)}))g^{(2)}) = f(h \bullet g) \), that is \( f: H \cdot r \to H \) is an algebra map, as needed.

We are now ready to prove Theorem 2.5:

**The proof of Theorem 2.5.** It follows from Theorem 2.8 that in order to classify all \( A \)-complements of \( E \) we can consider only \( r \)-deformations of \( H \), for various deformation maps \( r: H \to A \). Let \( r, R: H \to A \) be two deformation maps. As the coalgebra structure on \( H_r \) and \( H_R \) coincide with the one of \( H \), we obtain that the Hopf algebras \( H_r \) and \( H_R \) are isomorphic if and only if there exists \( \sigma: H \to H \) a unitary coalgebra isomorphism such that \( \sigma: H_r \to H_R \) is also an algebra map. Taking into account the definition of the multiplication on \( H_r \) given by (13) we obtain that \( \sigma \) is an algebra map if and only if the compatibility condition (12) of Definition 2.4 holds, i.e. \( r \sim R \). Hence, \( r \sim R \) if and only if \( \sigma: H_r \to H_R \) is an isomorphism of Hopf algebras. Thus we obtain that \( \sim \) is an equivalence relation on \( \mathcal{D}\mathcal{M}(H, A \mid (\triangleright, \triangleleft)) \) and the map

\[
\mathcal{H}\mathcal{A}^2(H, A \mid (\triangleright, \triangleleft)) \to \mathcal{F}(A, E), \quad \tau \mapsto H_R
\]

where \( \tau \) is the equivalence class of \( r \) via the relation \( \sim \), is well defined and a bijection between sets. This finishes the proof. \( \square \)

### 3. Examples

In this section we shall provide an example of a Hopf algebra extension \( A \subseteq E \) whose factorization index is arbitrary large. For a positive integer \( n \) we denote by \( U_n(k) = \{ \omega \in k \mid \omega^n = 1 \} \) the cyclic group of \( n \)-th roots of unity in \( k \) and by \( \nu(n) = |U_n(k)| \) the order of \( U_n(k) \). \( C_n \) will be the cyclic group of order \( n \) generated by \( c \) or \( d \) (if we consider two copies of \( C_n \)) and \( k \) will be a field of characteristic \( \neq 2 \). Let \( A := H_4 \) be the Sweedler’s 4-dimensional Hopf algebra generated by \( g \) and \( x \) subject to the relations:

\[
g^2 = 1, \quad x^2 = 0, \quad xg = -gx
\]

with the coalgebra structure given such that \( g \) is a group-like element and \( x \) is \((1, g)\)-primitive. [1, Proposition 4.3] proves that there exists a bijective correspondence between the set of all matched pairs \((H_4, k[C_n] \triangleleft, \triangleright)\) and the group \( U_n(k) \) such that the matched
pair \((H_4, k[C_n], \langle, \triangleright \rangle)\) associated to \(\omega \in U_n(k)\) is given as follows: \(\langle : k[C_n] \otimes H_4 \to k[C_n]\) is the trivial action and \(\triangleright : k[C_n] \otimes H_4 \to H_4\) is defined by:
\[
c^i \triangleright g = g, \quad c^i \triangleright x = \omega^i x, \quad c^i \triangleright gx = \omega^i gx
\]
for all \(i = 0, 1, \ldots, n - 1\). We denote by \(H_{4n, \omega}\) the bicrossed product \(H_4 \bowtie k[C_n]\) associated to this matched pair: \(H_{4n, \omega}\) is the \(4n\)-dimensional quantum group generated by \(g, x\) and \(c\) subject to the relations:
\[
g^2 = c^n = 1, \quad x^2 = 0, \quad xg = gx, \quad cg = gc, \quad cx = \omega xc
\]
with the coalgebra structure given such that \(g\) and \(c\) are group-like elements and \(x\) is \((1, g)\)-primitive. A \(k\)-basis in \(H_{4n, \omega}\) is given by \(\{c^i, gc^i, xc^i, gc^ix \mid i = 0, \ldots, n - 1\}\).

Let \(\xi\) be a generator of the group \(U_n(k)\). In what follows we will construct a family of matched pairs of Hopf algebras \((k[C_n], H_{4n, \xi}, \triangleleft, \triangleright)\) such that the Hopf algebra \(H_{4n, \xi} \triangleleft \triangleright\) will appear as an \(r\)-deformation of \(H_{4n, \xi}\).

**Theorem 3.1.** Let \(k\) be a field of characteristic \(\neq 2\), \(n\) a positive integer, \(\xi\) a generator of \(U_n(k)\), \(t \in \{0, 1, \ldots, v(n) - 1\}\) and \(C_n = \langle d \mid d^n = 1 \rangle\) the cyclic group of order \(n\). Then:

1. For any \(t \in \{0, 1, \ldots, v(n) - 1\}\) there exists a matched pair \((k[C_n], H_{4n, \xi}, \triangleleft, \triangleright)\), where \(\triangleright : H_{4n, \xi} \otimes k[C_n] \to k[C_n]\) is the trivial action and the right action \(\triangleleft : H_{4n, \xi} \otimes k[C_n] \to H_{4n, \xi}\) is given for any \(i, k = 0, 1, \ldots, n - 1\) by:
\[
c^i \triangleleft d^k = c^i, \quad (gc^i) \triangleleft d^k = gc^i, \quad (xc^i) \triangleleft d^k = \xi^{lk} xc^i, \quad (gxc^i) \triangleleft d^k = \xi^{lk} gxc^i
\]

2. The deformation maps associated to the matched pair \((k[C_n], H_{4n, \xi}, \triangleleft, \triangleright)\) are the algebra maps defined for any \(p \in \{0, 1, \ldots, n - 1\}\) as follows:
\[
r_p : H_{4n, \xi} \to k[C_n], \quad r_p(g) = 1, \quad r_p(c) = d^p, \quad r_p(x) = 0
\]

Furthermore, the \(r_p\)-deformation of \(H_{4n, \xi}\) is \(H_{4n, \xi} \triangleleft \triangleright\), i.e. \((H_{4n, \xi})_{r_p} = H_{4n, \xi} \triangleleft \triangleright\).

**Proof.** (1) The compatibility condition (5) is trivially fulfilled since \(\triangleright\) is the trivial action and \(k[C_n]\) is cocommutative. Moreover, (4) becomes:
\[
(yz) \triangleleft a = (y \triangleleft a_{(1)}) (z \triangleleft a_{(2)})
\]
for all \(y, z \in H_{4n, \xi}\) and \(a \in k[C_n]\). Since we have:
\[
c^i \triangleleft d^k = c^i, \quad g \triangleleft d^k = g, \quad x \triangleleft d^k = \xi^{lk} x
\]
then it is straightforward to see that (20) indeed holds. The fact that \(\triangleleft : H_{4n, \xi} \otimes k[C_n] \to H_{4n, \xi}\) is a coalgebra map is just a routinely computation. Finally, we only need to prove that the action \(\triangleleft\) respects the relations in \(k[C_n]\), respectively \(H_{4n, \xi}\). For instance, we have:
\[
xc^i \triangleleft d^n = (xc^i \triangleleft d^{n-1}) \triangleleft d = \xi^{l(n-1)} xc^i \triangleleft d = \xi^{ln} xc^i = xc^i
\]
\[
xg \triangleleft d^k = (x \triangleleft d^k)(g \triangleleft d^k) = \xi^{lk} gx = -\xi^{lk} gx = -gx \triangleleft d^k
\]
\[
xc \triangleleft d^k = (c \triangleleft d^k)(x \triangleleft d^k) = \xi^{lk} xc = \xi^{lk} \xi^t xc = \xi^t xc \triangleleft d^k
\]
Proving that the rest of the compatibilities also hold is a routinely check.
(2) Let \( r : H_{4n,\xi} \to k[C_n] \) be a deformation map. By applying (9) for \( xc^i \) and \( gxc^i \), where \( i = 0, 1, \ldots, n - 1 \) we obtain:

\[
\begin{align*}
  r(e^i) \otimes xc^i + r(xc^i) \otimes e^i &= r(xc^i) \otimes e^i + r(gc^i) \otimes xc^i \\
  r(gc^i) \otimes gxc^i + r(gxc^i) \otimes c^i &= r(gxc^i) \otimes g^i + r(c^i) \otimes gxc^i
\end{align*}
\]

Hence, it follows that \( r(xc^i) = r(gxc^i) = 0 \) and \( r(c^i) = r(gc^i) \) for all \( i = 0, 1, \ldots, n - 1 \). In particular we have \( r(g) = 1 \) and \( r(x) = 0 \). Moreover, since \( r \) is also a coalgebra map then \( r(c) \) is a grouplike element from \( k[C_n] \). Consider \( r(c) = dp \), for some \( p = 0, 1, \ldots, n - 1 \).

For the rest of the proof we will denote this map by \( r_p \). As \( v \) is the trivial action then the compatibility condition (11) simplifies to:

\[
r_p\left((y \triangleleft r(z_{(1)})) z_{(2)}\right) = r_p(y)r_p(z)
\]

(21) for all \( y, z \in H_{4n,\xi} \). By applying (21) for \( c^i \) and \( c^j \), where \( i, j = 0, 1, \ldots, n - 1 \) we get \( r_p(c^{i+j}) = r_p(c^i)r_p(c^j) \). Hence, we have

\[
r_p(c^i) = r_p(gc^i) = d^{ip}
\]

(22) for all \( i = 0, 1, \ldots, n - 1 \). Now by using (22) and the fact that \( r_p(xc^i) = r_p(gxc^i) = 0 \), for any \( i = 0, 1, \ldots, n - 1 \) we can easily prove that \( r_p : H_{4n,\xi} \to k[C_n] \) is an algebra map. Finally, we are left to prove that (21) holds. For instance we have:

\[
\begin{align*}
  r_p\left((gc^i \triangleleft d_p c^j) c^j\right) &= r_p\left((gc^i \triangleleft d_p c^j) c^j\right) = r_p(gc^i+j) = r_p(gc^i)r_p(c^j) \\
  r_p\left((xc^i \triangleleft d_p c^j) c^j\right) &= r_p\left((xc^i \triangleleft d_p c^j) c^j\right) = r_p(\xi_p c^j x^c c^i) = 0 = r_p(xc^i)r_p(c^j) \\
  r_p\left((y \triangleleft r_p((xc^i)_{(1)}))(xc^i)_{(2)}\right) &= r_p\left((y \triangleleft r_p((xc^i)_{(1)}))(xc^i)_{(2)}\right) = r_p\left((y \triangleleft r_p((xc^i)_{(1)}))(xc^i)_{(2)}\right) + r_p\left((y \triangleleft r_p((gc^i)))(xc^i)\right) \\
  &= 0 = r_p(y)r_p(xc^i)
\end{align*}
\]

for all \( i, j = 0, 1, \ldots, n - 1 \) and \( y \in H_{4n,\xi} \). By a straightforward computation it can be seen that (21) also holds for the remaining elements of the \( k \)-basis of \( H_{4n,\xi} \).

Now, the algebra structure of \((H_{4n,\xi})_r_p\) is given by (13). Thus, in \((H_{4n,\xi})_r_p\) we have:

\[
\begin{align*}
  g \bullet g &= \left( g \triangleleft r_p(g) \right) g = \left( g \triangleleft 1 \right) g = g^2 = 1 \\
  x \bullet x &= \left( x \triangleleft r(x) \right) x = x^2 = 0 \\
  c^{n-1} \bullet c &= \left( c^{n-1} \triangleleft r_p(c) \right) c = \left( c^{n-1} \triangleleft c^p \right) c = c^{n-1} c = c^n = 1 \\
  g \bullet x &= \left( g \triangleleft r(x) \right) + \left( g \triangleleft r(g) \right) x = gx = -xg = -(x \triangleleft r(g)) g = -x \bullet g \\
  c \bullet x &= \left( c \triangleleft r(x) \right) + \left( c \triangleleft r(g) \right) x = cx = \xi^t xc = \xi^{t-I_p} (\xi^{I_p} xc) \\
  &= \xi^{t-I_p} (x \triangleleft r_p(c)) c = \xi^{t-I_p} x \bullet c
\end{align*}
\]

This shows that \((H_{4n,\xi})_r_p = \xi^{t-I_p} H_{4n,\xi} \). \(\square\)

The bicrossed product \( k[C_n] \rhd^I H_{4n,\xi} \) associated to the matched pair from Theorem 3.1 is the Hopf algebra generated by \( g, x, c \) and \( d \) subject to the relations:

\[
\begin{align*}
  g^2 &= c^n = d^n = 1, \quad x^2 = 0, \quad cg = gc, \quad cd = dc, \quad gd = dg, \\
  xg &= -gx, \quad cx = \xi^t xc, \quad xd = \xi^t dx
\end{align*}
\]
with the coalgebra structure given such that \( g, c, d \) are group-like elements and \( x \) is a \((1, g)\)-primitive element. We denote by \( H_{4n^2, \xi, t, l} \) this family of quantum groups, for any \( l, t \in \{0, 1, \ldots, \nu(n) - 1\} \) and \( \xi \) a generator of order \( \nu(n) \) of the group \( U_n(k) \). In what follows we view \( H_{4n^2, \xi, t, l} \) as a Hopf algebra extension of the group algebra \( k[C_n] = k\langle d \mid d^n = 1 \rangle \).

In this context, \( H_{4n^2, \xi} \) is a \([C_n]\)-complement of \( H_{4n^2, \xi, t, l} \).

Before stating the main result of this section we recall from [1, Theorem 4.10] the number of types of isomorphisms of Hopf algebras \( H_{4n, \omega} \), where \( \omega \in U_n(k) \); we denote this number by \( \# H_{4n, \omega} \). If \( \nu(n) = p_1^{\nu_1} \cdots p_r^{\nu_r} \) is the prime decomposition of \( \nu(n) = |U_n(k)| \) then we have:

\[
\# H_{4n, \omega} = \left\{ \begin{array}{ll}
(\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_r + 1), & \text{if } \nu(n) \text{ is odd} \\
\alpha_1(\alpha_2 + 1)\cdots(\alpha_r + 1), & \text{if } \nu(n) \text{ is even and } p_1 = 2
\end{array} \right. \quad (23)
\]

The main result of this section now follows: it computes the factorization index of the extension \( k[C_n] \subseteq H_{4n^2, \xi, t, l} \).

**Theorem 3.2.** Let \( k \) be a field of characteristic \( \neq 2 \), \( n \) a positive integer, \( \xi \) a generator of \( U_n(k) \) and \((k[C_n], H_{4n, \xi^{\nu(n)-1}}, \triangleleft, \triangleright)\) the matched pair where \( \triangleright \) is the trivial action and \( \triangleleft \) is given by (19) for \( l = 1 \). Then:

1) \( (H_{4n, \xi^{\nu(n)-1}})_p = H_{4n, \xi^{\nu(n)-1}}\) for all \( p = 0, 1, \ldots, \nu(n) - 1 \). Thus, any \( H_{4n, \xi^p} \) appears as a deformation of \( H_{4n, \xi^{\nu(n)-1}} \), for some deformation map \( r_p \).

2) Assume that \( \nu(n) \) is odd and \( \nu(n) = p_1^{\nu_1} \cdots p_r^{\nu_r} \) is the prime decomposition of \( \nu(n) \). Then we have \( (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_r + 1) \) non-isomorphic deformations of \( H_{4n, \xi^{\nu(n)-1}} \) and thus \( [H_{4n^2, \xi, t, 1} : k[C_n]]^f = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_r + 1) \).

3) Assume that \( \nu(n) \) is even and \( \nu(n) = 2^{\nu_1}p_2^{\nu_2} \cdots p_r^{\nu_r} \) is the prime decomposition of \( \nu(n) \). Then we have \( \alpha_1(\alpha_2 + 1)\cdots(\alpha_r + 1) \) non-isomorphic deformations of \( H_{4n, \xi^{\nu(n)-1}} \) and thus \( [H_{4n^2, \xi, t, 1} : k[C_n]]^f = \alpha_1(\alpha_2 + 1)\cdots(\alpha_r + 1) \).

**Proof.** 1) It follows by applying Theorem 3.1 for \( l = 1 \) and \( t = \nu(n) - 1 \). As any \( H_{4n, \xi^p} \) appears as a deformation of \( H_{4n, \xi^{\nu(n)-1}} \) via some deformation map \( r_p \), the last two statements are just easy consequences of (23).

4. **Classifying complements for Lie algebras**

Let \( \mathfrak{g} \subseteq \Xi \) be a Lie subalgebra of \( \Xi \). A Lie subalgebra \( \mathfrak{h} \) of \( \Xi \) is called a complement of \( \mathfrak{g} \) in \( \Xi \) (or a \( \mathfrak{g} \)-complement of \( \Xi \)) if \( \Xi = \mathfrak{g} + \mathfrak{h} \) and \( \mathfrak{g} \cap \mathfrak{h} = \{0\} \). In this case we say that the Lie algebra \( \Xi \) factorizes through \( \mathfrak{g} \) and \( \mathfrak{h} \). Related to these concepts, the bicrossed product associated to a matched pair of Lie algebras was introduced in [14]. We collect here some basic facts: for more details we refer the reader to [14], [15, Chapter 8] or [16].

A matched pair of Lie algebras is a quadruple \((\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)\), where \( \mathfrak{g}, \mathfrak{h} \) are Lie algebras, \( \mathfrak{g} \) is a left \( \mathfrak{h} \)-Lie module under \( \triangleright : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \), \( \mathfrak{h} \) is a right \( \mathfrak{g} \)-Lie module under \( \triangleleft : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{h} \) and the following compatibilities hold for all \( a, b \in \mathfrak{g}, x, y \in \mathfrak{h} \):

\[
x \triangleright [a, b] = [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a
\]

\[
[x, y] \triangleleft a = [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)
\]
The fact that \( g \) is a left \( h \)-Lie module under \( \triangleright : h \otimes g \to g \) and \( h \) is a right \( g \)-Lie module under \( \triangleleft : h \otimes g \to h \) can be written explicitly as follows:

\[
x, y \triangleright a = x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a), \quad x \triangleleft [a, b] = (x \triangleleft a) \triangleleft b - (x \triangleleft b) \triangleleft a
\] (26)

The following is [15, Proposition 8.3.2]: If \((g, h, \triangleright, \triangleleft)\) is a matched pair of Lie algebras then the direct sum \( g \oplus h \) together with the bracket defined by:

\[
[a \oplus x, b \oplus y] = ([a, b] + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a)
\] (27)

for all \( a, b \in g, x, y \in h \) is a Lie algebra, called the bicrossed product of \( g \) and \( h \), and will be denoted by \( g \bowtie h \). The Lie algebra \( h \cong \{0\} \oplus h \) is a complement of \( g \cong g \oplus \{0\} \) in the bicrossed product \( g \bowtie h \). Conversely, if \( h \) is a \( g \)-complement of \( \Xi \), then there exists a matched pair of Lie algebras \((g, h, \triangleright, \triangleleft)\) such that the corresponding bicrossed product \( g \bowtie h \) is isomorphic as a Lie algebra with \( \Xi \). The actions of the matched pair \((g, h, \triangleright, \triangleleft)\) arises from the unique decomposition:

\[
[x, a] = x \triangleright a \oplus x \triangleleft a
\] (28)

for all \( a \in g, x \in h \). The matched pair constructed in (28) will be called the canonical matched pair associated to the \( g \)-complement \( h \) of \( \Xi \).

For a Lie subalgebra \( g \) of \( \Xi \) we denote by \( \mathcal{F}(g, \Xi) \) the isomorphism classes of \( g \)-complements of \( \Xi \). The factorization index of \( g \) in \( \Xi \) is defined as \( [\Xi : g]_f := |\mathcal{F}(g, \Xi)| \).

**Definition 4.1.** Let \((g, h, \triangleright, \triangleleft)\) be a matched pair of Lie algebras. A \( k \)-linear map \( r : h \to g \) is called a deformation map of the matched pair \((g, h, \triangleright, \triangleleft)\) if the following compatibility holds for any \( x, y \in h \):

\[
r([x, y]) - [r(x), r(y)] = r(y \triangleleft r(x) - x \triangleleft r(y)) + x \triangleright r(y) - y \triangleright r(x)
\] (29)

We denote by \( \mathcal{D}M(h, g | (\triangleright, \triangleleft)) \) the set of all deformation maps of the matched pair \((g, h, \triangleright, \triangleleft)\). The right hand side of (29) measures how far a deformation map is from being a Lie algebra map. Using this concept the following deformation of a given Lie algebra is proposed:

**Theorem 4.2.** Let \( g \) be a Lie subalgebra of \( \Xi \), \( h \) a given \( g \)-complement of \( \Xi \) and \( r : h \to g \) a deformation map of the associated canonical matched pair \((g, h, \triangleright, \triangleleft)\).

1. Let \( f_r : h \to \Xi = g \oplus h \) be the \( k \)-linear map defined for any \( x \in h \) by:

\[
f_r(x) = r(x) \oplus x
\]

Then \( h := \text{Im}(f_r) \) is a \( g \)-complement of \( \Xi \).

2. \( h_r := h \), as a \( k \)-module, with the new bracket defined for any \( x, y \in h \) by:

\[
[x, y]_r := [x, y] + x \triangleleft r(y) - y \triangleleft r(x)
\] (30)

is a Lie algebra called the \( r \)-deformation of \( h \). Furthermore, \( h_r \cong h \), as Lie algebras.
Moreover, it is straightforward to see that

\[
\tilde{g} = \{(r(x) \oplus x) \mid x \in \mathfrak{h}\}
\]

is a Lie subalgebra of \( \Xi = \mathfrak{g} \Join \mathfrak{h} \). Indeed, for all \( x, y \in \mathfrak{h} \) we have:

\[
[r(x) \oplus x, r(y) \oplus y] = (r(x), r(y)) + x \triangleright r(y) - y \triangleright r(x) \oplus (x, y) + x \triangleleft r(y) - y \triangleleft r(x)
\]

Moreover, it is straightforward to see that \( \mathfrak{g} \cap \tilde{h} = \{0\} \) and \( a \oplus x = (a-r(x)) \oplus (r(x) \oplus x) \in \mathfrak{g} + \tilde{h} \). Therefore, \( \tilde{h} \) is a \( \mathfrak{g} \)-complement of \( \Xi \).

(2) We denote by \( \tilde{f}_r \) the \( k \)-linear isomorphism from \( \mathfrak{h} \) to \( \tilde{h} \) induced by \( f_r \). We will prove that \( \tilde{f}_r \) is also a Lie algebra map if we consider on \( \mathfrak{h} \) the bracket given by (30). Indeed, for any \( x, y \in \mathfrak{h} \) we have:

\[
\tilde{f}_r([x, y]) = \tilde{f}_r([x, y] + x \triangleleft r(y) - y \triangleleft r(x)) = r([x, y] + x \triangleleft r(y) - y \triangleleft r(x)) + [x, y] + x \triangleleft r(y) - y \triangleleft r(x)
\]

Therefore, \( \tilde{h} \) is a Lie algebra and the proof is now finished.

We are now able to describe all complements of a Lie subalgebra \( \mathfrak{g} \) of \( \Xi \).

**Theorem 4.3.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \Xi \), \( \mathfrak{h} \) a given \( \mathfrak{g} \)-complement of \( \Xi \) with the associated canonical matched pair of Lie algebras \((\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)\). Then \( \tilde{h} \) is a \( \mathfrak{g} \)-complement of \( \Xi \) if and only if there exists an isomorphism of Lie algebras \( \tilde{h} \cong \tilde{h}_r \), for some deformation map \( r : \mathfrak{h} \to \mathfrak{g} \) of the matched pair \((\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)\).

**Proof.** Let \( \tilde{h} \) be an arbitrary \( \mathfrak{g} \)-complement of \( \Xi \). Since \( \Xi = \mathfrak{g} \Join \mathfrak{h} = \mathfrak{g} \Join \tilde{h} \) we can find four \( k \)-linear maps:

\[
u : \mathfrak{h} \to \mathfrak{g}, \quad \triangleright : \mathfrak{h} \to \tilde{h}, \quad \triangleleft : \tilde{h} \to \mathfrak{g}, \quad \triangleright : \tilde{h} \to \mathfrak{h}
\]

such that for all \( x \in \mathfrak{h} \) and \( y \in \tilde{h} \) we have:

\[
x = u(x) \oplus v(x), \quad y = t(y) \oplus w(y)
\]

By applying (31) for \( x = w(y) \in \mathfrak{h} \), \( y \in \tilde{h} \), we get:

\[-t(y) \oplus y = w(y) = u(w(y)) \oplus v(w(y))
\]

Therefore, by the unique decomposition in a direct sum, we obtain \( v(w(y)) = y \) and \( u(w(y)) = -t(y) \), for all \( y \in \tilde{h} \). In the same manner it can be proved that \( w(v(x)) = x \) and \( t(v(x)) = -u(x) \), for all \( x \in \mathfrak{h} \). In particular, we proved that \( v : \mathfrak{h} \to \tilde{h} \) is a \( k \)-linear isomorphism. We denote by \( \tilde{v} : \mathfrak{h} \to \mathfrak{g} \Join \mathfrak{h} \) the composition:

\[
\tilde{v} : \mathfrak{h} \to \tilde{h} \to \Xi = \mathfrak{g} \Join \mathfrak{h}
\]
More precisely, we have \( \tilde{v}(x) = v(x) = -u(x) + x \), for all \( x \in \mathfrak{h} \). We denote \( r := -u \) and we will prove that \( r \) is a deformation map and \( \tilde{h} \cong h_r \). Indeed, \( \tilde{h} = \text{Im}(v) = \text{Im}(\tilde{v}) \) is a Lie subalgebra of \( \Xi = g \bowtie \mathfrak{h} \) and therefore we have:

\[
(r(x) \oplus x, r(y) \oplus y) = (r(x), r(y)) + x \triangleright r(y) - y \triangleright r(x) \oplus
\]

\[
(x, y) + x \triangleleft r(y) - y \triangleleft r(x)
\]

for some \( z \in \mathfrak{h} \). Thus, we obtain:

\[
r(z) = [r(x), r(y)] + x \triangleright r(y) - y \triangleright r(x), \quad z = [x, y] + x \triangleleft r(y) - y \triangleleft r(x) \quad (32)
\]

By applying \( r \) to the second part of (32) we get:

\[
r(z) = r([x, y]) + r(x \triangleleft r(y)) - r(y \triangleleft r(x)) = [r(x), r(y)] + x \triangleright r(y) - y \triangleright r(x)
\]

Therefore, \( r \) is a deformation map of the matched pair \((g, \mathfrak{h}, \triangleright, \triangleleft)\). Moreover, we have:

\[
[v(x), v(y)] = v(z) = v([x, y] + x \triangleleft r(y) - y \triangleleft r(x)) = v([x, y]_{\mathfrak{h}})
\]

that is, \( v : \mathfrak{h} \rightarrow \tilde{h} \) is a Lie algebra map and the proof is now finished. \( \square \)

In order to classify all complements we need to introduce the following:

**Definition 4.4.** Let \((g, \mathfrak{h}, \triangleright, \triangleleft)\) be a matched pair of Lie algebras. Two deformation maps \( r, R : \mathfrak{h} \rightarrow g \) are called equivalent and we denote this by \( r \sim R \) if there exists \( \sigma : \mathfrak{h} \rightarrow \mathfrak{h} \) a \( k \)-linear automorphism of \( \mathfrak{h} \) such that for any \( x, y \in \mathfrak{h} \):

\[
\sigma([x, y]) - [\sigma(x), \sigma(y)] = \sigma(x) \triangleleft R(\sigma(x)) - \sigma(x \triangleleft r(y)) - \sigma(y) \triangleleft R(\sigma(x)) + \sigma(y \triangleleft r(x)) \quad (33)
\]

As a conclusion of this section we obtain the answer to the (CCP) for Lie algebras:

**Theorem 4.5.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \Xi \), \( \mathfrak{h} \) a \( \mathfrak{g} \)-complement of \( \Xi \) and \((\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)\) the associated canonical matched pair. Then:

1. \( \sim \) is an equivalence relation on \( D.M(\mathfrak{h}, \mathfrak{g} \mid (\triangleright, \triangleleft)) \). We denote by \( \mathcal{H}_A^2(\mathfrak{h}, \mathfrak{g} \mid (\triangleright, \triangleleft)) \) the quotient set \( D.M(\mathfrak{h}, \mathfrak{g} \mid (\triangleright, \triangleleft)) / \sim \).

2. There exists a bijection between the isomorphism classes of all \( \mathfrak{g} \)-complements of \( \Xi \) and \( \mathcal{H}_A^2(\mathfrak{h}, \mathfrak{g} \mid (\triangleright, \triangleleft)) \). In particular, the factorization index of \( \mathfrak{g} \) in \( \Xi \) is computed by the formula:

\[
[\Xi : \mathfrak{g}]^f = |\mathcal{H}_A^2(\mathfrak{h}, \mathfrak{g} \mid (\triangleright, \triangleleft))|
\]

**Proof.** It follows from Theorem 4.3 that in order to classify all \( \mathfrak{g} \)-complements of \( \Xi \) it is enough to consider only \( r \)-deformations of \( \mathfrak{h} \), for various deformation maps \( r : \mathfrak{h} \rightarrow \mathfrak{g} \). Now let \( r, R : \mathfrak{h} \rightarrow \mathfrak{g} \) be two deformation maps. As \( h_r = h_R := h \) as \( k \)-spaces, we obtain that the Lie algebras \( h_r \) and \( h_R \) are isomorphic if and only if there exists \( \sigma : h_r \rightarrow h_R \) a \( k \)-linear isomorphism which is also a Lie algebra map. Using (30) we obtain that \( \sigma \) is a Lie algebra map if and only if the compatibility condition (33) holds, i.e. \( r \sim R \). Hence,
r ∼ R if and only if \( σ : \mathfrak{h}_r \to \mathfrak{h}_R \) is an isomorphism of Lie algebras. Thus we obtain that ∼ is an equivalence relation on \( \mathcal{D}\mathcal{M}(\mathfrak{h}, \mathfrak{g} \mid (\langle , \rangle)) \) and the map

\[
\mathcal{H}\mathcal{A}^2(\mathfrak{h}, \mathfrak{g} \mid (\langle , \rangle)) \to \mathcal{F}(\mathfrak{g}, \Xi), \quad \mathcal{F} \mapsto \mathfrak{h},
\]

is a bijection between sets, where \( \mathcal{F} \) is the equivalence class of \( r \) via the relation ∼. □

**Example 4.6.** Let \( k \) be a field of \( \text{char}(k) \neq 2 \) and \( \Xi \) the 4-dimensional Lie algebra with \( \{e_1, e_2, f_1, f_2\} \) as a basis and the bracket given by:

\[
[e_1, e_2] = 2e_1, \ [e_1, f_1] = e_2, \ [e_2, f_2] = 2f_1
\]

Let \( \mathfrak{g} \) be the Lie subalgebra of \( \Xi \) with basis \( \{e_1, e_2\} \). Then \( [\Xi : \mathfrak{g}]^f = 2 \).

Indeed, let \( \mathfrak{h} \) be the abelian Lie algebra of dimension 2 with basis \( \{f_1, f_2\} \). Then \( \mathfrak{h} \) is a \( \mathfrak{g} \)-complement of \( \Xi \) with the associated canonical matched \( (\mathfrak{g}, \mathfrak{h}, \langle, \rangle) \) given as follows:

\[
f_1 \triangleright e_1 := -e_2, \quad f_1 \triangleleft e_2 := -2f_1
\]

It is straightforward to see that \( r_c : \mathfrak{h} \to \mathfrak{g} \) given by \( r(f_1) := 0, r(f_2) := cf_2 \), for some \( c \in k \) is a deformation map of the matched pair \( (\mathfrak{g}, \mathfrak{h}, \langle, \rangle) \). Furthermore, the \( r_c \)-deformation of \( \mathfrak{h} \) has the bracket \( [f_1, f_2]_{r_c} := -2cf_1 \). As this is not an abelian Lie algebra for \( c \neq 0 \), it follows that \( \mathfrak{h}_{r_c} \) is not isomorphic to \( \mathfrak{h} \). Since there are only two types of Lie algebras of dimension 2 we obtain that \( [\Xi : \mathfrak{g}]^f = 2 \).

5. Outlooks and open problems

In this paper we solve the (CCP) for the category of Lie algebras and Hopf algebras respectively. The common tool used is the bicrossed product construction. All the results proven above can serve as a model for obtaining similar theories for the (CCP) in other categories \( \mathcal{C} \) where the bicrossed product was introduced, such as: (co)algebras, \( C^* \)-algebras or von Neumann algebras, Lie groups, locally compact groups or locally compact quantum groups, groupoids or quantum groupoids, multiplier Hopf algebras, etc. Another direction for further inquiry is given by the following three open questions related to the results of this work:

**Question 1:** Let \( \tau : H \otimes H \to k \) be a normalized Sweedler cocycle and \( H_\tau \) be Doi’s \([7]\) deformation of the Hopf algebra \( H \). Does there exist a Hopf algebra \( A \), a matched pair of Hopf algebras \( (A, H, \langle, \rangle) \) and a deformation map \( r : H \to A \) such that \( H_\tau = H_r \), where \( H_r \) is the \( r \)-deformation of \( H \) in the sense of Theorem 2.6?

Having in mind Theorem 2.5 it is natural to ask:

**Question 2:** Does there exist an example of an extension of finite dimensional Hopf algebras \( A \subset E \) having an infinite factorization index \( [E : A]^f \) ?

We have approached the (CCP) for right \( A \)-complements of a given extension \( A \subset E \) of Hopf algebras. The same theory can be developed for left \( A \)-complements. If \( E \) has a bijective antipode then \( H \) is a right \( A \)-complement if and only if \( H \) is a left \( A \)-complement ([1, Proposition 3.1]); i.e. in this case \( [E : A]^f = [E : A]^l \), where \( [E : A]^f \) (resp. \( [E : A]^l \)) denotes the factorization index corresponding to right (resp. left) \( A \)-complements. In general, we ask the following:
Question 3: Does there exist an example of an extension $A \subset E$ of Hopf algebras such that $[E : A]_f \neq [E : A]_l$?

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