ON ALTERNATING SUMS OF BINOMIAL AND \(q\)-BINOMIAL COEFFICIENTS

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Abstract. In this paper we shall evaluate two alternating sums of binomial coefficients by a combinatorial argument. Moreover, by combining the same combinatorial idea with partition theoretic techniques, we provide \(q\)-analogues involving the \(q\)-binomial coefficients.

1. Introduction

Recall that the \(q\)-shifted factorials are given by

\[
(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a;q)_\infty = \lim_{n \to \infty} (a;q)_n = \prod_{i=0}^{\infty} (1 - aq^i)
\]

and the \(q\)-binomial coefficients are given by

\[
\begin{cases} 
\binom{n}{m} = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, & \text{if } n \geq m \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Evaluating alternating sums and differences involving the binomial coefficients and finding their \(q\)-analogues involving the \(q\)-binomial coefficients have been extensively studied throughout the years and there is a rich literature on the topic, see for instance \[9\ 4\ 5\ 6\ 9\ 10\ 11\ 12\]. A special case of a result by Andrews et al \[9\] states that

\[
\sum_k (-1)^k \binom{m+n}{m-kl} \geq 0 \quad \text{if } |m-n| \leq l
\]

with a corresponding \(q\)-analogue stating that

\[
\sum_k (-1)^k q^{k^2(a+b)+kl(b-a)} \binom{m+n}{m-kl}
\]

is a polynomial in \(q\) with nonnegative coefficients where \(m, n, l, a, b\) are nonnegative integers such that \(a + b < 2l\) and \(b - l \leq n - m \leq l - a\). The authors proved their results using integer partitions. Ismail et al \[11\] extended the previous results by considering among other things expressions of the form

\[
\sum_k \binom{m+n}{m-kl} \cos(kx)
\]

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and their \( q \)-analogues. Recently Guo and Zhang [10] gave combinatorial proofs for
a variety of alternating sums and differences of binomial and \( q \)-binomial coefficients
including
\[
\sum_{k=-\infty}^{\infty} (-1)^k \binom{2n}{n+2k} = 2^n \tag{1.1}
\]
and
\[
\sum_{k=-\infty}^{\infty} (-1)^k \binom{2n}{n+3k} = \begin{cases} 2 \cdot 3^{n-1}, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases} \tag{1.2}
\]
and their \( q \)-analogues
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} \binom{2n}{n+2k} = (-q; q^2)_n \tag{1.3}
\]
and
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+3k)/2} \binom{2n}{n+3k} = \begin{cases} 1, & \text{if } n = 0, \\ (1 + q^n) \frac{(q^n; q^n)_\infty}{(q; q)_\infty}, & \text{if } n \geq 1. \end{cases} \tag{1.4}
\]
respectively. In this note we will prove generalisations of the identities (1.1) and (1.2)
including \(-q\)-analogues. Recently Guo and Zhang [10] gave combinatorial proofs for
along with generalisations of their \( q \)-analogues (1.3) and (1.4). Our \( q \)-analogues are
expressed in terms of certain restricted integer partitions which we introduce now.

**Definition 1.** For any nonnegative integers \( N \) and \( M \) and any positive rational
numbers \( a \leq b \) let \( p_d(N, [a, b], M) \) denote the number of partitions of \( N \) into exactly
\( M \) distinct parts all of which are in the integer interval \([a, b]\). We assume that
\( p_d(-N, [a, b], M) = 0 \) if \( N \leq 0 \).

Further, to simplify the formulas, we introduce the following notation.

**Definition 2.** Let \( m \) and \( n \) be nonnegative integers and let \( a_{j,l} = p_d(j, [n, n+m], l) \).
For any positive integer \( k \) and any nonnegative integer \( r \) we let \( A_{n,m,k,r}(q) \) be the
polynomial in \( q \) given by:
\[
A_{n,m,k,r}(q) = A_{k,r}(q) = \sum_{l=0}^{\lfloor \frac{n+1-r}{k} \rfloor} \sum_{j=(kl+r)(n+\frac{1-r}{k})}^{\lfloor \frac{n+1-r}{k} \rfloor (kl+r-1)(n+\frac{1-r}{k})} a_{j,kl+r-1} q^j + \sum_{l=0}^{\lfloor \frac{n+1-r}{k} \rfloor} \sum_{j=(kl+r-1)(n+\frac{1-r}{k})}^{\lfloor \frac{n+1-r}{k} \rfloor (kl+r-1)(n+\frac{1-r}{k})} a_{j,kl+r} q^j
\]

2. The Results

**Theorem 1.** If \( m \) is an integer and \( n \) is a nonnegative integer, then
\[
\sum_{k=-\infty}^{\infty} (-1)^k \binom{2n+m}{n+2k} = 2^n \frac{\cos \frac{m\pi}{4}}{2}\cos \frac{m\pi}{4}
\]

**Theorem 2.** If \( m \) is a nonnegative integer and \( n \) is a positive integer, then
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} \binom{2n+m}{n+2k} = (-q; q^2)_n
\]
\[
\times \left( \sum_{l=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=2l(2n+4l)}^{\lfloor \frac{n+1}{2} \rfloor (2l+1)(2n+4l-2)} a_{j,l} q^l - \sum_{l=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=(2l+1)(2n+4l+2)}^{\lfloor \frac{n+1}{2} \rfloor (2l+1)(2n+4l+2)} b_{j,l} q^l \right),
\]
where

\[ a_{j,l} = p_d(j, [n + 1, n + m - 1], 4l) \text{ and } b_{j,l} = p_d(j, [n + 1, n + m - 1], 4l + 2). \]

**Theorem 3.** If \( n \) and \( m \) are positive integers, then

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2-k} \left[ \begin{array}{c} 2n + m \\ n + 2k \end{array} \right] = (-q^2; q^2)_n
\]

\[
\times \left\{ \sum_{l=0}^{\infty} \sum_{j=2l(2n+4l+1)}^{2l(2n+2m-4l-1)} a_{j,l} q^j + \sum_{l=0}^{\infty} \sum_{j=(4l+1)(n+2l)}^{(4l+1)(n-m-2l-1)} c_{j,l} q^j \\
- \left( \sum_{l=0}^{\infty} \sum_{j=(2l+1)(2n+4l+3)}^{(2l+1)(2n+2m-4l-3)} b_{j,l} q^j + \sum_{l=0}^{\infty} \sum_{j=(4l+1)(n+2l+2)}^{(4l+1)(n+m-2l+1)} d_{j,l} q^j \right) \right\},
\]

where

\[ a_{j,l} = p_d(j, [n + 1, n + m - 1], 4l), \quad c_{j,l} = p_d(j, [n + 1, n + m - 1], 4l - 1), \]
\[ b_{j,l} = p_d(j, [n + 1, n + m - 1], 4l + 2), \quad d_{j,l} = p_d(j, [n + 1, n + m - 1], 4l + 1). \]

**Theorem 4.** If \( m \) is an integer and \( n \) is a nonnegative integer, then

\[
\sum_{k=-\infty}^{\infty} (-1)^k \left( \begin{array}{c} 2n + m \\ n + 3k \end{array} \right) = \begin{cases} 1, & \text{if } n = m = 0, \\ 2 \cdot 3^{n-1} + \pi \cos \frac{m \pi}{6}, & \text{otherwise}. \end{cases}
\]

**Theorem 5.** Let \( m \) be a nonnegative integer and let \( n \) be a positive integer. Then

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2+3k} \left[ \begin{array}{c} 2n + m \\ n + 3k \end{array} \right] = \frac{(q^2; q^2)_n}{(q; q)_n} \left( A_{6,0}(q) - A_{6,3}(q) + A_{6,1}(q) - A_{6,4}(q) \right).
\]

**Remarks.** 1. As a consequence of the previous theorems, sums involving the partitions \( p_d(N, [a, b], M) \) and formulas involving \( A_{n,m,6;1}(1) \) will be obtained upon letting \( q \to 1 \). For instance, letting \( m = 4N + 2 \) and \( q \to 1 \) we find by combining Theorem 1 with Theorem 2 that

\[
\sum_{l=0}^{\infty} \sum_{j=2l(2n+4l+1)}^{2l(2n+2m-4l-1)} p_d(j, [n + 1, n + m - 1], 4l) = \\
\sum_{l=0}^{\infty} \sum_{j=(2l+1)(2n+4l+2)}^{(2l+1)(2n+2m-4l-2)} p_d(j, [n + 1, n + m - 1], 4l + 2)
\]

and letting \( m = 6N + 3 \) and \( q \to 1 \) we find by combining Theorem 4 with Theorem 5 that for all positive integer \( n \)

\[ A_{n,m,6;0}(1) - A_{n,m,6;3}(1) + A_{n,m,6;1}(1) - A_{n,m,6;4}(1) = 0. \]

2. Notice that for any integers \( m \) and \( n \), the binomial and \( q \)-binomial coefficients make the series in the previous theorems finite on both ends. However, if \( m, n \to +\infty \), then by virtue of the Jacobi triple product (see 2 [8]) the sum in Theorem 2 becomes:

\[
\lim_{m,n \to \infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} \left[ \begin{array}{c} 2n + m \\ n + 2k \end{array} \right] = \frac{(q^2; q^2)_\infty (q^4; q^4)_\infty}{(q; q)_\infty} = (-q^2; q^2)_\infty(q^2; q^4)_\infty
\]
and similarly the sums in Theorem 3 and Theorem 5 respectively become:

\[
\lim_{m,n \to \infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2-k} \left[ \frac{2n+m}{n+2k} \right] = \frac{1}{(q^2; q^4)_\infty}
\]

and

\[
\lim_{m,n \to \infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{9k^2+3k} \left[ \frac{2n+m}{n+3k} \right] = \frac{(q^3; q^4)}{(q; q)_\infty}
\]

3. We note further that the sums in Theorems 2, 3, and 5 are related to a finite sum version of a $\psi_2$ sum. Refer to \cite{2,8} for more details about the function $\psi_2$.

3. Proof of Theorem 1

Suppose first that $n > 0$ and $m \geq 0$. Following Guo and Zhang \cite{10}, throughout let $S = \{a_1, \ldots, a_{2n}, a_{2n+1}, \ldots, a_{2n+m}\}$ be a set of $2n + m$ elements. Let

- $\mathcal{F} = \{A \subseteq S : \#A \equiv n \pmod{2}\}$,
- $\mathcal{G} = \{A \subseteq S : \#(A \cap \{a_{2i-1}, a_{2i}\}) = 1 \text{ for all } i = 1, \ldots, n\}$,
- $\mathcal{H} = \{A \in \mathcal{F} : \#(A \cap \{a_{2i-1}, a_{2i}\}) \neq 1 \text{ for some } i = 1, \ldots, n\}$.

For simplicity of notation, if $A \in S$, we let $A' = A \cap \{a_{2n+1}, \ldots, a_{2n+m}\}$.

We define a map

\[\text{sgn} : \mathcal{F} \to \{-1, 1\}, \quad \text{sgn}(A) = (-1)^{\#A-n}.\]

Then it is clear that

\[
\sum_{k=-\left\lfloor \frac{n+m}{2} \right\rfloor}^{\left\lceil \frac{n+m}{2} \right\rceil} (-1)^k \left( \frac{2n+m}{n+2k} \right) = \sum_{A \in \mathcal{F}} \text{sgn}(A).
\]

Furthermore, if $A \in \mathcal{H}$ let $i_A$ be the minimum index $i$ such that $\#(A \cap \{a_{2i-1}, a_{2i}\}) \neq 1$. Next we define a map

\[\text{inv} : \mathcal{H} \to \mathcal{H}\]

\[\text{inv}(A) = \begin{cases} A \cup \{a_{2i_A-1}, a_{2i_A}\} & \text{if } \#(A \cap \{a_{2i_A-1}, a_{2i_A}\}) = 0 \\ A \setminus \{a_{2i_A-1}, a_{2i_A}\} & \text{if } \#(A \cap \{a_{2i_A-1}, a_{2i_A}\}) = 2. \end{cases}\]

Then it is easily seen that the function $\text{inv}$ is an involution satisfying $\text{sgn}(\text{inv}(A)) = -\text{sgn}(A)$ and therefore we have

\[
\sum_{A \in \mathcal{F}} \text{sgn}(A) = \sum_{A \in \mathcal{G}} \text{sgn}(A) = \sum_{\#A' \equiv 0 \pmod{2}} 1 - \sum_{\#A' \equiv 2 \pmod{4}} 1
\]

\[
= 2^n \left( \sum_{l=0}^{\left\lfloor \frac{m}{4} \right\rfloor} \binom{m}{4l} - \sum_{l=0}^{\left\lceil \frac{m-2}{4} \right\rceil} \binom{m}{2+4l} \right).
\]
Then the case \( m = 0 \) follows easily from the identities (6.1) and (3.2). Suppose now that \( m \geq 1 \). Note that by the formulas (6.1) and (3.2) it suffices to show that
\[
\sum_{l \geq 0} \binom{m}{4l} - \sum_{l \geq 0} \binom{m}{4l + 2} = 2^m \cos \frac{m\pi}{4}.
\]
By a well-known result, see Gould [9] and Benjamin et al [5], we have
\[
\sum_{l \geq 0} \binom{m}{4l} = \frac{1}{4} \sum_{k=0}^{3} (1 + e^{i\frac{2\pi k}{4}})^m
\]
and
\[
\sum_{l \geq 0} \binom{m}{4l + 2} = \frac{1}{4} \sum_{k=0}^{3} e^{-2k+i\frac{m\pi}{4}} (1 + e^{i\frac{2\pi k}{4}})^m,
\]
from which it follows that
\[
\sum_{l \geq 0} \binom{m}{4l} - \sum_{l \geq 0} \binom{m}{4l + 2} = 2^m \cos \frac{m\pi}{4},
\]
as desired. Suppose next that \( m < 0 \) and let \( M = -m = 4s - r \) with \( 0 \leq r < 4 \). Then with the help of the previous case we have
\[
\sum_{k=0}^{\infty} (-1)^k \binom{2n + m}{n + 2k} = \sum_{k=0}^{\infty} (-1)^k \binom{2n - 4s + r}{n - 2s + 2(k + s)}
\]
\[
= (-1)^s \sum_{k=0}^{\infty} (-1)^k \binom{2(n - 2s) + r}{(n - 2s) + 2k}
\]
\[
= (-1)^s 2^{n-2s+\frac{r}{4}} \cos \frac{\pi r}{4}
\]
\[
= 2^{n-4s} \cos \frac{\pi M}{4},
\]
as desired. Further, suppose that \( n = 0 \) and \( m \geq 4 \). Then
\[
\sum_{k=0}^{\infty} (-1)^k \binom{m}{2k} = \sum_{k=0}^{\infty} (-1)^k \binom{4 + (m - 4)}{2 + 2(k - 1)}
\]
\[
= - \sum_{k=0}^{\infty} (-1)^k \binom{4 + (m - 4)}{2 + 2k}
\]
\[
= -2^{m-4} \cos \frac{(m - 4)\pi}{4}
\]
\[
= 2^m \cos \frac{m\pi}{4}.
\]
Finally it is easy to check the cases \( n = 0 \) and \( m = 1, 2, 3 \). This completes the proof.
4. Proof of Theorem 2

If the $2n + m$ elements of the set $S$ are integers and $A \subseteq S$ then we define the weight of $A$ by $\|A\| = \sum_{a \in A} a$. From the well-known fact, see Andrews [1],

$$\binom{zq; q}{N} = \sum_{j=0}^{N} \binom{N}{j} (-1)^j z^j q^{(j+1)/2}$$

we conclude that

$$\sum_{A \subseteq \{1, \ldots, n\} \atop \# A = k} q^{\|A\|} = \left[ \binom{n}{k} q^{(k+1)/2} \right].$$

Now let

$$S = \left\{ i - \frac{2n + 1}{2} : i = 1, \ldots, 2n + m \right\} = \left\{ \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \pm \frac{2n - 1}{2}, \frac{2n + 1}{2}, \ldots, \frac{2n + 2m - 1}{2} \right\},$$

that is,

$$a_{2i-1} = -\frac{2i - 1}{2}, a_{2i} = \frac{2i - 1}{2} \quad \text{for } i = 1, \ldots, n \text{ and } a_{2n+j} = \frac{2n + 2j - 1}{2} \quad \text{for } j = 1, \ldots, m.$$  

Then the function $\text{inv}$ defined above is weight preserving and therefore we have

$$\sum_{A \subseteq \mathcal{F}} \text{sgn}(A) q^{\|A\|} = \sum_{A \subseteq \mathcal{S} \atop \# A \equiv 0 \pmod{4}} q^{\|A\|} - \sum_{A \subseteq \mathcal{G} \atop \# A \equiv 2 \pmod{4}} q^{\|A\|}$$

As to the left-hand-side of the relation (7.1), we have

$$\sum_{A \subseteq \mathcal{F}} \text{sgn}(A) q^{\|A\|} = \sum_{k = -\lfloor \frac{n + m}{2} \rfloor}^{\lfloor \frac{n + m}{2} \rfloor} \sum_{A \subseteq \{1, \ldots, 2n + m\} \atop \# A = n + 2k} \text{sgn}(A) q^{\|A\|}$$

$$= \sum_{k = -\lfloor \frac{n + m}{2} \rfloor}^{\lfloor \frac{n + m}{2} \rfloor} \sum_{A \subseteq \{1, \ldots, 2n + m\} \atop \# A = n + 2k} (-1)^k q^{\|A\| + (n + 2k)(-\frac{2n + 1}{2})}$$

$$= \sum_{k = -\lfloor \frac{n + m}{2} \rfloor}^{\lfloor \frac{n + m}{2} \rfloor} (-1)^k q^{-\frac{2n + 1}{2}(n + 2k)} \sum_{A \subseteq \{1, \ldots, 2n + m\} \atop \# A = n + 2k} q^{\|A\|},$$

which with the help of the identity (4.1) gives

$$\sum_{A \subseteq \mathcal{F}} \text{sgn}(A) q^{\|A\|} = \sum_{k = -\lfloor \frac{n + m}{2} \rfloor}^{\lfloor \frac{n + m}{2} \rfloor} (-1)^k q^{-\frac{2n + 1}{2}(n + 2k)} \left[ \frac{2n + m}{n + 2k} \right] q^{(n + 2k + 1)/2}$$

(4.3)

$$= \sum_{k = -\lfloor \frac{n + m}{2} \rfloor}^{\lfloor \frac{n + m}{2} \rfloor} (-1)^k q^{\frac{(2k+1)^2 - 2n^2}{2n + 2k}} \left[ \frac{2n + m}{n + 2k} \right].$$

As to the first sum on the right-hand-side of the relation (7.1), notice first that the least sum and the largest sum into exactly 4l distinct parts belonging to the set $\{(2n + 1)/2, \ldots, (2n + 2m - 1)/2\}$ are respectively

$$\frac{2n + 1}{2} + \ldots + \frac{2n + 2(4l - 1) + 1}{2} = 2l(2n + 4l)$$
and \[ \frac{2n + 2(m - 4l) + 1}{2} + \ldots + \frac{2n + 2m - 1}{2} = 2l(2n + 2m - 4l). \]

So, letting \[ a_{j,l} = p_d \left( j, \left[ \frac{2n + 1}{2}, \frac{2n + 2m - 1}{2} \right], 4l \right) = p_d(j, [n + 1, n + m - 1], 4l), \]

it is easily checked that

\[
\sum_{\substack{A \in G \\mod{1}}} \sum_{\# A' \equiv 0 \mod{4}} 1 = \prod_{i=1}^{n} (q + \frac{2i-1}{2}) \times \sum_{l=0}^{\frac{m}{2}} \sum_{j=2l(2n+4l)} a_{j,l} q^j
\]

(4.4)

\[
= q^{-\frac{n+2}{2}} (-q; q^4)_n \sum_{j=2l(2n+4l)} a_{j,l} q^j.
\]

As to the second sum on the right-hand-side of the relation (7.1) we use the same remark as before with 4l replaced by 4l + 2 and \(a_{j,l}\) replaced by

\[ b_{j,l} = p_d \left( j, \left[ \frac{2n + 1}{2}, \frac{2n + 2m - 1}{2} \right], 4l + 2 \right) = p_d(j, [n + 1, n + m - 1], 4l + 2) \]

to obtain

\[
\sum_{\substack{A \in G \\mod{1}}} \sum_{\# A' \equiv 2 \mod{4}} 1 = \prod_{i=1}^{n} (q + \frac{2i-1}{2}) \times \sum_{l=0}^{\frac{m}{2}} \sum_{j=(2l+1)(2n+4l+2)} b_{j,l} q^j
\]

(4.5)

\[
= q^{-\frac{n+2}{2}} (-q; q^4)_n \sum_{j=(2l+1)(2n+4l+2)} b_{j,l} q^j.
\]

Then the desired formula follows by combining the relation (7.1) with the relations (4.3), (4.4), and (4.5).

5. Proof of Theorem 3

We proceed as in the proof of Theorem [2]. Let

\[ S = \{ i - (n + 1) : i = 1, \ldots, 2n + m \} = \{ \pm 1, \ldots, \pm n, 0, n + 1, n + 2, \ldots, n + m - 1 \}, \]

that is,

\[ a_{2i-1} = -i, a_{2i} = i \] for \( i = 1, \ldots, n \), \( a_{2n+1} = 0 \), and \( a_{2n+j} = n+j-1 \) for \( j = 2, \ldots, m \).

Then

\[
\sum_{A \in \mathcal{F}} \sgn(A) q^{\|A\|} = \sum_{k=-\left\lfloor \frac{m+n}{2} \right\rfloor}^{\left\lfloor \frac{m+n}{2} \right\rfloor} (-1)^k q^{k^2-k-\frac{n(n+1)}{4}} \left[ \frac{2n + m}{n + 2k} \right].
\]

(5.1)

Let as before

\[ a_{j,l} = p_d(j, [n + 1, n + m - 1], 4l), \] and \( b_{j,l} = p_d(j, [n + 1, n + m - 1], 4l + 2) \).
However, because of the presence of 0 among the elements of $S$ we shall also take into account partition into exactly $4l - 1$ (nonzero) parts and therefore we let 
\[ c_{j,l} = p_d(j, [n + 1, n + m - 1], 4l - 1) \quad \text{and} \quad d_{j,l} = p_d(j, [n + 1, n + m - 1], 4l + 1). \]

We have
\[
q^{\|A\|} = \prod_{i=1}^{n}(q^{-i} + q^{i}) \]
for all $\# A \equiv 0 \pmod{4}$.

\[ \sum_{l=0}^{m-3} \sum_{j=2l(2n+4l+1)} a_{j,l}q^j + \sum_{l=0}^{m-2} \sum_{j=(4l+1)(n+2l)} c_{j,l}q^j \]
for all $\# A \equiv 0 \pmod{4}$.

Then the desired formula follows by combining the relations (5.1), (5.2), and (5.3) with the relation (6.1).

6. Proof of Theorem 4

The case $n = m = 0$ is clear. Suppose now that $n > 0$ and $m \geq 0$. Extending definitions from Guo and Zhang [10], throughout let $S = \{a_1, \ldots, a_{2n}, a_{2n+1}, \ldots, a_{2n+m}\}$ be a set of $2n + m$ elements and let

\[ \mathcal{F} = \{ A \subseteq S : \# A \equiv n \pmod{3} \}, \]

\[ \mathcal{G} = \{ A \subseteq \mathcal{F} : \# (A \cap \{a_1, a_{2i+1}\}) \not\in \{i - 1, i + 2\} \quad \text{for all} \quad i = 1, \ldots, n - 1 \}. \]

We define a map
\[ \text{sgn} : \mathcal{F} \to \{-1, 1\}, \quad \text{sgn}(A) = (-1)^{\frac{\#A - n}{3}}. \]

Then it is clear that
\[
\sum_{k=-\left\lfloor \frac{n+m}{3} \right\rfloor}^{\left\lfloor \frac{n+m}{3} \right\rfloor} (-1)^k \binom{2n+m}{n+3k} = \sum_{A \in \mathcal{F}} \text{sgn}(A). \]
Furthermore, if \( A \in \mathcal{F} \setminus \mathcal{G} \) let \( i_A \) be the minimum index \( i \) such that \( \#(A \cap \{a_1, a_{2i+1}\}) \in \{i - 1, i + 2\} \). Letting
\[
A' = A \Delta \{a_1, \ldots, a_{2i_A+1}\} = (A \cup \{a_1, \ldots, a_{2i_A+1}\}) \setminus (A \cap \{a_1, \ldots, a_{2i_A+1}\}),
\]
ote{that it is easily seen that \( \#A' = \#A \pm 3 \). Next we define a map
\[\text{inv}: \mathcal{F} \setminus \mathcal{G} \to \mathcal{F} \setminus \mathcal{G}, \quad \text{inv}(A) = A'\]
as follows:
- \( a_1 \in A'' \) if and only if \( a \not\in A \),
- \( a_{2j}, a_{2j+1} \in A'' \) if \( a_{2j}, a_{2j+1} \not\in A \) \((j = 1, \ldots, i_A)\);
- \( a_{2j}, a_{2j+1} \not\in A'' \) if \( a_{2j}, a_{2j+1} \in A \) \((j = 1, \ldots, i_A)\);
- \( a_{2j} \in A'' \) and \( a_{2j+1} \not\in A'' \), if \( a_{2j} \in A \) and \( a_{2j+1} \not\in A \) \((j = 1, \ldots, i_A)\);
- \( a_{2j} \not\in A'' \) and \( a_{2j+1} \in A'' \), if \( a_{2j} \not\in A \) and \( a_{2j+1} \in A \) \((j = 1, \ldots, i_A)\);
- \( a_k \in A'' \) if and only if \( a_k \in A \) \((2i_A + 2 \leq k \leq 2n + m)\).

Observing that \( \#A'' = \#A' \), we have from the previous note that \( \#A'' = \#A \pm 3 \). Further, it is clear that \( A'' \in \mathcal{F} \setminus \mathcal{G} \) and that the function inv is an involution satisfying \( \text{sgn}(\text{inv}(A)) = -\text{sgn}(A) \). Therefore, we have
\[
(6.2) \quad \sum_{A \in \mathcal{F}} \text{sgn}(A) = \sum_{A \in \mathcal{G}} \text{sgn}(A) + \sum_{A \in \mathcal{F} \setminus \mathcal{G}} \text{sgn}(A) = \sum_{A \in \mathcal{G}} \text{sgn}(A).
\]
We now evaluate \( \sum_{A \in \mathcal{G}} \text{sgn}(A) \). We claim that if \( A \in \mathcal{G} \), then
\[
\#(A \cap \{a_1, \ldots, a_{2i_A+1}\}) \in \{i - 1, i, i + 1, i + 2\},
\]
from which the claim follows since the cases \( i - 1 \) and \( i + 2 \) are excluded by definition. For simplicity of notation, if \( A \in \mathcal{S} \), we let
\[
A_1 = A \cap \{a_1, \ldots, a_{2n-1}\}, \quad A_2 = A \cap \{a_{2n}, \ldots, a_{2n+m}\},
\]
\[\mathcal{G}_1 = \{A \in \mathcal{G} : \#A_1 = n\}, \text{ and } \mathcal{G}_2 = \{A \in \mathcal{G} : \#A_1 = n - 1\}.
\]
By the previous claim we have \( \#A_1 \in \{n - 1, n\} \), which combined with the identity \( (6.2) \) yields
\[
(6.3) \quad \sum_{A \in \mathcal{F}} \text{sgn}(A) = \sum_{A \in \mathcal{G}_1} \text{sgn}(A) + \sum_{A \in \mathcal{G}_2} \text{sgn}(A).
\]
Furthermore, we clearly have
\[
(6.4) \quad \sum_{A \in \mathcal{G}_1} \text{sgn}(A) = \sum_{A \in \mathcal{G}_1 \cap \{a_1, \ldots, a_{2i_A+1}\}} 1 - \sum_{A \in \mathcal{G}_1 \setminus \{a_1, \ldots, a_{2i_A+1}\}} 1
\]
and
\[
(6.5) \quad \sum_{A \in \mathcal{G}_2} \text{sgn}(A) = \sum_{A \in \mathcal{G}_2 \cap \{a_1, \ldots, a_{2i_A+1}\}} 1 - \sum_{A \in \mathcal{G}_2 \setminus \{a_1, \ldots, a_{2i_A+1}\}} 1.
\]
Moreover, if \( A \in \mathcal{G} \), then there are three possible choices for \( A \cap \{a_{2i}, a_{2i+1}\} \) for all \( i = 1, \ldots, n - 1 \) and there is exactly one possible choice for \( A \cap \{a_1\} \), implying that
\[
(6.6) \quad \#\mathcal{G}_1 = \#\mathcal{G}_2 = 3^{n-1}.
\]
Then from the relations (6.4) and (6.6) we derive

\[
\sum_{A \in G_1} \text{sgn}(A) = 3^{n-1} \left( \sum_{l=0}^{\lfloor \frac{m+1}{6} \rfloor} \binom{m+1}{6l} - \sum_{l=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{m+1}{6l+3} \right)
\]

and from the relations (6.5) and (6.6) we derive

\[
\sum_{A \in G_2} \text{sgn}(A) = 3^{n-1} \left( \sum_{l=0}^{\lfloor \frac{m+1}{6} \rfloor} \binom{m+1}{6l} - \sum_{l=0}^{\lfloor \frac{m-3}{6} \rfloor} \binom{m+1}{6l+4} \right).
\]

Substituting (6.7) and (6.8) in the formula (6.3) gives

\[
\sum_{A \in F} \text{sgn}(A) = 3^{n-1} \left( \sum_{l=0}^{\lfloor \frac{m+1}{6} \rfloor} \binom{m+1}{6l} - \sum_{l=0}^{\lfloor \frac{m-2}{6} \rfloor} \binom{m+1}{6l+3} \right) + 3^{n-1} \left( \sum_{l=0}^{\lfloor \frac{m+1}{6} \rfloor} \binom{m+1}{6l+1} - \sum_{l=0}^{\lfloor \frac{m-3}{6} \rfloor} \binom{m+1}{6l+4} \right).
\]

Then the case \( m = 0 \) follows easily from the identities (6.1) and (6.9). Suppose now that \( m \geq 1 \). By a well-known result, see Gould [9] and Benjamin et al [5], we have

\[
\sum_{l \geq 0} \binom{m}{6l} - \sum_{l \geq 0} \binom{m}{6l+3} = 2 \cdot 3^{\frac{m-1}{2}} \cos \left( \frac{m+1}{6} \pi \right),
\]

and

\[
\sum_{l \geq 0} \binom{m}{6l+1} - \sum_{l \geq 0} \binom{m}{6l+4} = 2 \cdot 3^{\frac{m-1}{2}} \cos \left( \frac{m-1}{6} \pi \right),
\]

which combined with (6.9) yields

\[
\sum_{A \in F} \text{sgn}(A) = 3^{n-1} \left( 2 \cdot 3^{\frac{m}{6}} \cos \frac{m \pi}{6} \right) = 2 \cdot 3^{n-1} + \frac{3^{m \pi}}{6},
\]

as desired. Suppose next that \( m < 0 \) and let \( M = -m = 6s - r \) with \( 0 \leq r < 6 \). Then with the help of the above we have

\[
\sum_{k=-\infty}^{\infty} \binom{2n + m}{n + 3k} = \sum_{k=-\infty}^{\infty} \binom{2n - 6s + r}{n - 3s + 3(k + s)} = (-1)^s \sum_{k=-\infty}^{\infty} (-1)^k \binom{2(n - 3s) + r}{n - 3s + 3k} = (-1)^s 2 \cdot 3^{n-3s-1} \cos \frac{\pi r}{6} = 2 \cdot 3^{n-1} \cos \frac{\pi M}{6},
\]
as desired. Further, suppose that \( n = 0 \) and \( m \geq 6 \). Then

\[
\sum_{k=\infty}^{\infty} \binom{m}{3k} = \sum_{k=\infty}^{\infty} \left( \frac{6 + (m - 6)}{3 + 3(k - 1)} \right) = -\sum_{k=\infty}^{\infty} \left( \frac{6 + (m - 6)}{3 + 3k} \right) = -2 \cdot 3^{3-1+\frac{m-6}{2}} \cos \left( \frac{(m - 6)\pi}{6} \right) = 2 \cdot 3^{-1+\frac{m}{2}} \cos \left( \frac{m\pi}{6} \right).
\]

Finally it is easy to check the cases \( n = 0 \) and \( m \in \{1, 2, 3, 4, 5\} \). This completes the proof.

7. Proof of Theorem 5

Let

\[
S = \{i - n : i = 1, \ldots, 2n + m\} = \{\pm 1, \ldots, \pm (n - 1), 0, n, n + 1, \ldots, n + m\},
\]

with

\[
a_1 = 0, a_{2i} = -i, a_{2i+1} = i \text{ for } i = 1, \ldots, n - 1 \text{ and } a_{2n+j} = n + j \text{ for } j = 0, \ldots, m.
\]

Then the function inv defined above is weight preserving and therefore we have

\[
\sum_{A \in F} \text{sgn}(A) q^{\|A\|} = \sum_{A \in \mathcal{G}_1} \text{sgn}(A) q^{\|A\|} + \sum_{A \in \mathcal{G}_2} \text{sgn}(A) q^{\|A\|}
\]

\[
= \sum_{\#A_2 \equiv 0 \pmod{6}} q^{\|A\|} - \sum_{\#A_2 \equiv 3 \pmod{6}} q^{\|A\|} + \sum_{\#A_2 \equiv 1 \pmod{6}} q^{\|A\|} - \sum_{\#A_2 \equiv 4 \pmod{6}} q^{\|A\|}.
\]

(7.1)

As to the left-hand-side of the relation (7.1), we have

\[
\sum_{A \in F} \text{sgn}(A) q^{\|A\|} = \sum_{k=-\infty}^{\infty} \sum_{A \subseteq S \#A=n+3k} \text{sgn}(A) q^{\|A\|}
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{A \subseteq \{1, \ldots, 2n+m\} \#A=n+3k} (-1)^k q^{\|A\| - n(n+3k)}
\]

\[
= \sum_{k=-\infty}^{\infty} (-1)^k q^{-n(n+3k)} \sum_{A \subseteq \{1, \ldots, 2n+m\} \#A=n+3k} q^{\|A\|},
\]
which with the help of the identity \(1.1\) gives

\[
\sum_{A \in F} \sgn(A)q^{\|A\|} = \sum_{k=-\lfloor \frac{n+3}{2} \rfloor}^{\lfloor n+3k \rfloor} (-1)^k q^{-n(n+3k)} \left[ \frac{2n + m}{n + 3k} \right] q^{k(n+3k+1)}
\]

(7.2)

\[
= \sum_{k=-\lfloor \frac{n+3}{2} \rfloor}^{\lfloor n+3k \rfloor} (-1)^k q^{\frac{9k^2+3k+2}{2}n} \left[ \frac{2n + m}{n + 3k} \right].
\]

To evaluate the sum \(\sum_{A \in \mathcal{G}_1} q^{\|A\|}\) in the relation (7.1), we shall consider partitions into exactly 6l distinct parts belonging to the set \(\{n, \ldots, n + m\}\) and moreover, because of the presence of 0 among the elements of \(S\), we shall also take into account partition into exactly 6l - 1 distinct (nonzero) parts belonging to the set \(\{n, \ldots, n + m\}\). Notice that the least sum and the largest sum into exactly 6l distinct parts belonging to the set \(\{n, \ldots, n + m\}\) are respectively

\[n + (n + 1) + \ldots + (n + 6l - 1) = 3l(2n + 6l - 1)\]

and

\[(n + m) + (n + m - 1) + \ldots + (n + m - 6l + 1) = 3l(2n + 2m - 6l + 1)\]

and similarly the least sum and the largest sum into exactly 6l - 1 distinct parts belonging to the set \(\{n, \ldots, n + m\}\) are \((6l - 1)(n+3l - 1)\) and \((6l - 1)(n + m - 3l + 1)\) respectively. Then it is easily checked that

\[
\sum_{\# A_2 \equiv 0 \pmod{6}} q^{\|A\|} = \prod_{i=1}^{n-1} \left( q^{-i} + q^i + q^0 \right)
\]

\[
\times \left( \sum_{l=0}^{\lfloor \frac{n+3}{2} \rfloor} \sum_{j=6l(n+m-6l-2)}^{n(n+m-6l-2)} a_{j,6l}q^j + \sum_{l=0}^{\lfloor \frac{6l-1}{2} \rfloor} \sum_{j=(6l-1)(n+6l-2)}^{n(n+m-6l-2)} a_{j,6l-1}q^j \right)
\]

(7.3)

\[
= q^{n^{2-n}} \left( \frac{q^3; q^3}{(q; q)_{n-1}} A_{6,0}(q) \right).
\]

Similarly, we obtain

(7.4)

\[
\sum_{\# A_2 \equiv 3 \pmod{6}} q^{\|A\|} = q^{n^{2-n}} \left( \frac{q^3; q^3}{(q; q)_{n-1}} A_{6,3}(q) \right),
\]

(7.5)

\[
\sum_{\# A_2 \equiv 1 \pmod{6}} q^{\|A\|} = q^{n^{2-n}} \left( \frac{q^3; q^3}{(q; q)_{n-1}} A_{6,1}(q) \right),
\]

and

(7.6)

\[
\sum_{\# A_2 \equiv 4 \pmod{6}} q^{\|A\|} = q^{n^{2-n}} \left( \frac{q^3; q^3}{(q; q)_{n-1}} A_{6,4}(q) \right).
\]

Then the desired formula follows by combining the relation (7.1) with the formulas (7.2), (7.3), (7.4), (7.5), and (7.6).
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