Correspondence between winding numbers and skin modes in non-hermitian systems

Kai Zhang,1,2, * Zhesen Yang,1,2, * and Chen Fang1, †

1Beijing National Laboratory for Condensed Matter Physics and Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
2University of Chinese Academy of Sciences, Beijing 100049, China

We establish exact relations between the winding of “energy” (eigenvalue of Hamiltonian) on the complex plane as momentum traverses the Brillouin zone with periodic boundary condition, and the presence of “skin modes” with open boundary condition in non-hermitian systems. We show that the nonzero winding with respect to any complex reference energy leads to the presence of skin modes, and vice versa. We also show that both the nonzero winding and the presence of skin modes share the common physical origin that is the non-vanishing current through the system.

I. INTRODUCTION

Some systems that are coupled to energy or particle sources or drains, or driven by external fields can be effectively modeled Hamiltonians having non-hermitian forms.1–6. For example, one may add a diagonal imaginary part in a band Hamiltonian for electrons to represent the effect of finite quasiparticle lifetime.7–10. One may also introduce an imaginary part to the dielectric constant in Maxwell equations to represent metallic conductivity in a photonic crystal.11–15. As non-hermitian operators in general have complex eigenvalues, the eigenfunctions of Schrödinger equations are no longer static, but decay or increase exponentially in amplitude with time.16,17

A topic in recent condensed-matter research is the study of topological properties in band structures, which are generally given by the wave functions, not the energy, of all occupied bands (or more generally, a group of bands capped from above and below by finite energy gaps).18–22. In non-hermitian systems, however, one immediately identifies a different type of topological numbers in bands, given by the phase winding of the “energy” (eigenvalue of Hamiltonian), not the wave functions, in the Brillouin zone (BZ).23,24. This winding number, together with several closely related winding numbers if other symmetries are present, give topological classification that is richer than that of their hermitian counterparts.17,24–28. Besides winding in energy in complex plane, another unique phenomenon recently proposed in non-hermitian systems is the presence of “skin modes” in open systems.29–33. A typical spectrum of open hermitian system consists of a large number of bulk states, and, if at all, a small number of edge states, and as the system increases in size $L$, the numbers of the bulk and of the edge states increase as $L^d$ and $L^{d-n}$ respectively, where $d$ is the dimension and $0 < n \leq d$. However, in certain non-hermitian systems, a finite fraction, if not all, of eigenstates are concentrated on one of the edges. These skin modes decay exponentially away from the edges just like edge states, but their number scales as the volume ($L^d$), rather than the area, of the system.

In this Letter, we show an exact relation between the new quantum number, i.e., the winding number of energy with periodic boundary, and the existence of skin modes with open boundary, for any one-band model in one dimension. To do this, we first extend the one-band Hamiltonian with finite-range hopping $H(k)$ to a holomorphic function $H(z) = P_n(z)/z^m$ ($n, m > 0$), where $P_n(z)$ is an $(n+m)$-polynomial, and the Brillouin zone maps to unit circle $|z| = 1$ (or $z = e^{ik}$). The image of the unit circle under $H(z)$ is the spectrum of the system with periodic boundary, and generally, it forms a loop on the complex plane, $\mathcal{L}_{BZ} \subseteq \mathbb{C}$. Then we show that as long as $\mathcal{L}_{BZ}$ has finite interior, or roughly speaking encloses finite area, skin modes appear as eigenstates with open boundary condition; but when $\mathcal{L}_{BZ}$ collapses into a curve having no interior on the complex plane, the skin modes disappear. In other words, skin modes with open boundary appear if and only if there be $E_b \in \mathbb{C}$ with respect to which $H(z)$ has nonzero winding. Finally, we show that the winding of the periodic boundary spectrum, and hence the presence of skin modes with open boundary, are related to the total direct current of the system. We prove that if the current vanishes for all possible state distribution functions $n(H, H^*)$, the winding and the skin modes also vanish, and vice versa. The relations we establish among nonzero winding, presence of skin modes and non-vanishing current are summarized in Fig. 1. These results are extended to higher dimensions and more bands.

II. HAMILTONIAN AS HOLOMORPHIC FUNCTION

We start with an arbitrary one-band tight-binding Hamiltonian in one dimension, only requiring that hoppings between $i$ and $j$ sites only exist within a finite range $-m \leq i - j \leq n$.

$$H = \sum_{i,j} t_{i-j} |i\rangle \langle j| = \sum_{k \in \mathcal{BZ}} H(k) |k\rangle \langle k|,$$n$,$$ (1)$

where $H(k) = \sum_{r=-m,\ldots,n} t_r (e^{ik})^r$ is the Fourier transformed $t_r$ ($t_0$ being understood as the onsite potential). For periodic boundary condition, we have $0 \leq k < 2\pi$,
and $e^{ik}$ moves along the unit circle on the complex plane. For future purposes, we define $z := e^{ik}$, and consider $z$ as a general point on the complex plane. Therefore for each Hamiltonian $H(k)$, we now have a holomorphic function

$$H(z) = t_{-m}z^{-m} + \cdots + t_n z^n = \frac{P_{m+n}(z)}{z^m},$$

(2)

where $P_{m+n}(z)$ is a polynomial of order $m+n$. $H(z)$ has one composite pole at $z = 0$, the order of which is $m$, and has $m+n$ zeros, i.e., the zeros of the $(m+n)$-polynomial. Along any oriented loop $C$ and any given reference point $E_b \in C$, one can define the winding number of $H(z)$

$$w_{C,E_b} := \frac{1}{2\pi} \oint_{C} \frac{d}{dz} \arg[H(z) - E_b]dz.$$

(3)

Specially, for $C = BZ$, $w_{C,E_b}$ is the winding of the phase of $H(z) - E_b$ along BZ, considered as a new topological number unique to non-hermitian systems. Complex analysis relates the winding number of any complex function $f(z)$ to the total number of zeros and poles enclosed in $C$, that is,

$$w_{C,E_b} = N_{zeros} - N_{poles},$$

(4)

where $N_{zeros,poles}$ is the counting of zeros (poles) weighted by respective orders. See Fig. 2(a,b) for the pole, the zeros and the winding of $L_{BZ}$ for a specific Hamiltonian. In fact, we always have $N_{poles} = m$, so that the winding number is determined by the number of zeros of $P_{m+n}(z) - z^m E_b$ that lie within the unit circle. As we will see later, the advantage of extending the Hamiltonian into a holomorphic function lies in exactly this relation between the winding numbers and the zeros.

III. GENERALIZED BRILLOUIN ZONE

In Ref. 29, it is shown that energy spectrum of one-band non-hermitian systems with open boundary may deviate drastically from that with periodic boundary, due to the presence of skin modes. Furthermore, in Ref. 29, the authors introduce a new concept of generalized Brillouin zone to signify the difference between periodic and open boundary: instead of evaluating $H(z)$ along BZ, the open-boundary energy spectrum is recovered as one evaluates $H(z)$ on another closed loop called GBZ as $L$ goes to infinity. The GBZ is determined by the equation

$$GBZ := \{z||H_m^{-1}(H(z))| = |H_{m+1}^{-1}(H(z))|\},$$

(5)

where $H_i^{-1}(E)$’s satisfying $|H_i^{-1}(E)| \leq |H_{i+1}^{-1}(E)|$ are the $m+i$ branches of the inverse function of $H(z)$. (In Ref. 34, $m = n$ is assumed, and we extend the results to $m \neq n$ cases in appendix A.) We emphasize that using GBZ, one can compute the open boundary spectrum of systems of large or infinite size by solving some algebraic equations such as Eq. (5), a process we sketch using the following steps. To begin with, one finds the inverse functions of $H(z)$, and orders them in ascending amplitude, thus obtaining $H_i^{-1}(E)$, where $i = 1, \ldots, m + n$ because the $P_{m+n}(z) - E z^m$ is an order $m + n$-polynomial of $z$. Then, as there are two variables (Re(E), Im(E)) in Eq. (5), by codimension counting its solution on the complex plane forms one or several close loops, which are nothing but the open boundary energy spectrum. Finally, one substitutes these solutions back into $H_i^{-1}(E)$. It is noted that if we are only interested in the spectrum, we may stop at the second last step, but we need GBZ in order to articulate some of our key results.

With GBZ thus defined, we state our central result (for proof see appendix B): GBZ is a single loop in complex plane that encloses the origin and exactly $m$ zeros of $P_{m+n}(z) - E z^m$ for arbitrary $E \in \mathbb{C}$. This seemingly technical result has following consequences. First, this means within GBZ the total number of zeros and poles (weighted by respective orders) cancel, so that the winding of $H(z) - E$ vanishes. Next, the arbitrariness of $E$ ensures that GBZ is invariant under a shift of energy origin in the complex plane $H(z) \rightarrow H_z - E_b$. Combining these two points, we see that the image of GBZ under $H(z)$ on the complex plane, denoted by $L_{GBZ}$, has zero winding with respect to any $E_b \in \mathbb{C}$, or symbolically, $w_{GBZ,E_b} = 0$.

Therefore, we finally see that the open-boundary spectrum of $H(z)$ cannot be a circle or eclipse like the periodic-boundary counterpart, and it cannot even be form a loop enclosing any finite area, because in that case one can choose $E_b$ inside that area so that the winding of $L_{GBZ}$ with respect to $E_b$ is nonzero. The only possibility is that $L_{GBZ}$ collapses into a curve as shown in Fig. 2(d). In this specific example ($m = n = 1$ and see caption for parameters), we plot the GBZ in Fig. 2(c) and $L_{GBZ}$ in Fig. 2(d) as $z$ moves counterclockwise along GBZ. We see that while GBZ is more or less a circle, its image $L_{GBZ}$ keeps “back-stepping” itself: except for a few turning and branching points, any point in $L_{GBZ}$ has two or an even number of pre-images in the GBZ, so
GBZ contains the unit circle, and we define \( U \) as the region inside GBZ but outside BZ (colored in red); (ii) as in Fig. 3(b1), GBZ is contained in the unit circle, and we define \( V \) as the region outside GBZ but inside BZ (colored in blue); (iii) as in Fig. 3(c1), one part of GBZ is outside and another part inside the unit circle. For case-(i), pick \( z_0 \in U \) and \( E_0 = H(z_0) \). \( z_0 \) is then a zero of \( H(z_0) - E_0 \), and from Eq.(6), we know there are exactly \( m \) zeros inside GBZ, so inside BZ there are at most \( m - 1 \) zeros, and from Eq.(4) we have \( w_{BZ, E_0} < -1 \neq 0 \) [see example in Fig. 3(a2)]. For case-(ii), pick \( z_0' \in V \) and \( E_0' = H(z_0') \), then use similar arguments to see \( w_{BZ, E_0'} > 1 \neq 0 \) [see example in Fig. 3(b2)]. We postpone the proof for case-(iii) to appendix C, but mention here that for \( z_0 \in U \) and \( z_0' \in V \), the periodic-boundary spectrum \( \mathcal{L}_{BZ} \), taking the shape of a fish [see Fig. 3(c2)], has opposite windings with respect to \( E_0 \) and \( E_0' \).

V. WINDING NUMBERS, SKIN MODES AND DIRECT CURRENT

From the above results, we see that if and only if \( \mathcal{L}_{BZ} \) does not enclose any \( E_0 \in \mathbb{C} \), then the skin modes do not exist. When this is the case, \( \mathcal{L}_{BZ} \) always “back-steps” itself just like \( \mathcal{L}_{GBZ} \), or more precisely, along \( \mathcal{L}_{BZ} \), for any small segment \( \delta H \) centered at some \( E \), there must be another segment \( -\delta H \) centered at exactly the same \( E \). What is the physical meaning of this condition? We show that this is equivalent to the absence of total direct current with periodic boundary. To define the current, we assume that the particles have some charge (taken to be unity), so the total direct current can be derived as \( J = \sum_k n_k v_k = \sum_k n_k H'(k) dk \), where \( n_k \) is some distribution function. Now we make a general physical assumption that \( n_k \) only depends on the “energy” of the state, that is \( n_k = n(H(k), H^*(k)) \), but does not depend on \( k \) explicitly. (Here \( n \) depends on both the real and the imaginary parts of \( H(k) \), so is not necessarily holomorphic.) For example, the Bose distribution \( n_k = (e^{\text{Re}[H(k)]/k_B T} - 1)^{-1} \) satisfies such a condition. When the curve \( \mathcal{L}_{BZ} \) has no interior, we have

\[
J = \int_0^{2\pi} n(H, H^*) \frac{dH(k)}{dk} dk = \int_{\mathcal{L}_{BZ}} n(H, H^*) dH = 0, \tag{7}
\]

that is, the total direct current vanishes. In appendix D, we prove the inverse statement that if there is any \( E_0 \in \mathbb{C} \) with respect to which \( H(z) \) has nonzero winding, then one can always find some \( n(H, H^*) \neq 0 \) such that \( J \neq 0 \). This equivalence is intuitively understood: if a direct current is driven through a ring, then as one cuts open the ring, the charge starts concentrating on one end of the open chain, being pushed by the driving voltage.
VI. DISCUSSION

So far we have established the reciprocal relations shown in Fig. 1 for one-band model in one dimension. Some of the results may be extended to the cases of more bands and/or higher dimensions. For example, in d-dimension, one should consider a multi-variable holomorphic function $H(z_1, z_2, \ldots, z_d) : \mathbb{C}^d \to \mathbb{C}$, where $z_j := e^{ik_j}$, and the general spectrum of $H(z_1, \ldots, z_d)$ is in general a continuum on the complex plane. Are there skin modes when we have open boundary along $0 < l \leq d$ directions, but periodic boundary along the other $d-l$ directions? We have two conjectures for two extreme cases: (i) if $l = d$, that is, if all directions are open, skin modes vanish if and only if each component direct current vanishes for arbitrary $n(H, H^*)$; and (ii) if $l = 1$, that is, if only one direction is open, the skin modes vanish if and only if the entire spectrum of $H(z_1, \ldots, z_d)$ collapse into a curve having no interior. The “only if” part of (i) and the “if” part of (ii) are trivial, but the other parts seem not quite so.

Extension of the relation between the direct current and the winding numbers in periodic boundary to multiple-band systems is straightforward. Now $H_{ab}(z)$ becomes a matrix function of $z := e^{ik}$, where $a, b = 1, \ldots, n$ label the orbitals. The direct current in this case becomes $J = \sum_{i=1}^n J_i$, where
\begin{equation}
J_i = \int_0^{2\pi} n(E_i, E_i^*) \frac{dE_i(k)}{dk} dk = \int_{\mathcal{L}_{i, BZ}} n(E_i, E_i^*) dE_i.
\end{equation}

While $J_i = 0$ implies $J = 0$, $J = 0$ does not necessitate $J_i = 0$ for each $i$. In fact, one part of the trajectory of $E_i(k)$ may be back-stepped by another part of the trajectory of $E_{i\neq i}(k)$ so that their contribution to $J$ cancel out. Therefore, $J = 0$ is equivalent to the collapse of the spectrum, not of each individual band, but of all bands, into a curve that has no interior. In more precise terms, $J = 0$ for arbitrary $n(E, E^*)$ if and only if for any $E_b \in \mathbb{C}$ and $E_b \notin \mathcal{L}_{i, BZ}$, the total winding number of all bands with respect to $E_b$ vanishes, or symbolically
\begin{equation}
\int_0^{2\pi} \frac{d \log \det[H(z) - E_b I_{n \times n}]}{dk} = 0.
\end{equation}

When there are additional conserved charges in the Hamiltonian, for example some spin component, we can simply replace the total current $J$ with the component current for each conserved charge $J_c$. At this point, we do not know exactly how the nonzero direct current or the winding numbers are related to the skin modes in multiband systems, but from physical intuition, we conjecture that $J \neq 0$ implies skin modes with open boundary, and vice versa.

Appendix A: Derivation of Eq.(5)

In this section, we first calculate the generalized Brillouin zone of a heuristic single-band model, and further give the general formal proof of Eq.(5) in the main text. Finally, we generalize the discussion to the two-band model with chiral symmetry.

1. Model

Consider a single-band model with the following real space Hamiltonian
\begin{equation}
\hat{H} = \sum_{i=1}^L t_{i+1} \hat{c}_i^\dagger \hat{c}_{i+1} + t_1 \hat{c}_1^\dagger \hat{c}_2 + t_2 \hat{c}_2^\dagger \hat{c}_3 + t_3 \hat{c}_3^\dagger \hat{c}_{i+3}, \quad (A1)
\end{equation}
and the corresponding eigenequation is
\[ H\Psi = E\Psi, \quad H = \begin{pmatrix}
0 & t_1 & t_2 & t_3 & 0 & \cdots & 0 & 0 \\
-1 & 0 & t_1 & t_2 & t_3 & \cdots & 0 & 0 \\
0 & 0 & -1 & 0 & t_1 & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & \cdots & 0 & t_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & t_1 \\
\end{pmatrix}, \quad \Psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\psi_5 \\
\psi_{L-1} \\
\psi_L \\
\end{pmatrix}. \quad (A2) \]

Before we proceed to solve Eq. (A2), we first review the procedure of exact solution. Recall the one-dimensional infinite square well problem in quantum mechanics. Although the translational symmetry is broken at the boundary of the well, we also need to solve the (translational invariance) Schrödinger equation \((\hat{p}^2/2m + V_0)|\phi\rangle = E|\phi\rangle\). For any given \(E\), there exist two linear independent plane wave solutions \(|k\rangle\) and \(|-k\rangle\). If their linear superposition \(|\phi\rangle = c_1|k\rangle + c_2|-k\rangle\) satisfy the boundary condition, we say \(|\phi\rangle\) is the eigenstate of the Hamiltonian with the corresponding eigenvalue \(E\). Back to Eq. (A2), we first notice that the eigenvalue equation can be separated to the bulk equation

\[ t_{-1}\psi_s - E\psi_{s+1} + t_1\psi_{s+2} + t_2\psi_{s+3} + t_3\psi_{s+4} = 0, \quad s = 1, 2, \ldots, L-4, \quad (A3) \]

and boundary equation

\[
\begin{align*}
-E\psi_1 + t_1\psi_2 + t_2\psi_3 + t_3\psi_4 &= 0, \\
t_{-1}\psi_{L-3} - E\psi_{L-2} + t_1\psi_{L-1} + t_2\psi_L &= 0, \\
t_{-1}\psi_{L-2} - E\psi_{L-1} + t_1\psi_L &= 0, \\
t_{-1}\psi_{L-1} - E\psi_L &= 0.
\end{align*} \quad (A4)\]

Here the bulk equation corresponds to the (translational invariance) Schrödinger equation, and the boundary equation refers to boundary condition in the one-dimensional infinite square well problem. Since the bulk equation has discrete translational symmetry, for a given \(E\), it has four linear independent eigenfunctions, which can be written as

\[ \Psi_i(E) = (z_i, z_i^2, \ldots, z_i^{L-1}, z_i^L), \quad i = 1, 2, 3, 4 \quad (A5) \]

where \(z_i\) satisfy the following characteristic polynomial equation for given \(E\)

\[ f(z_i, E) : = H(z_i) - E = t_{-1}/z_i + t_1z_i + t_2z_i^2 + t_3z_i^3 - E = 0 \quad (A6) \]

The solution of Eq. (A2) can be written as the linear superposition of \(\Psi_i(E)\) satisfying the boundary equation. To be more precise,

\[ \Psi(E) = c_1\Psi_1(E) + c_2\Psi_2(E) + c_3\Psi_3(E) + c_4\Psi_4(E) = (\psi_1, \psi_2, \ldots, \psi_{L-1}, \psi_L)^T, \quad (A7) \]

where

\[ \psi_n = \sum_{i=1}^{4} c_i z_i^n = c_1 z_1^n + c_2 z_2^n + c_3 z_3^n + c_4 z_4^n, \quad n = 1, \ldots, L \quad (A8) \]

and the solution of Eq. (A6) are ordered as follows

\[ |z_1| \leq |z_2| \leq |z_3| \leq |z_4|. \quad (A9) \]

Substituting Eq. (A8) to the boundary equation Eq. (A4), one can obtain the following matrix equation after an appropriate transformations

\[ H_B \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix} = \begin{pmatrix}
A(z_1) & A(z_2) & A(z_3) & A(z_4) \\
B_1(z_1)z_1^L & B_1(z_2)z_2^L & B_1(z_3)z_3^L & B_1(z_4)z_4^L \\
B_2(z_1)z_1^L & B_2(z_2)z_2^L & B_2(z_3)z_3^L & B_2(z_4)z_4^L \\
B_3(z_1)z_1^L & B_3(z_2)z_2^L & B_3(z_3)z_3^L & B_3(z_4)z_4^L \\
\end{pmatrix} \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix} = 0, \quad (A10) \]
where
\[
A(z_i) = -E z_i + t_1 z_i^2 + t_2 z_i^3 + t_3 z_i^4,
\]
\[
B_1(z_j) = t_2 + t_1 z_j - E/z_j^2 + t_{-1}/z_j^3,
\]
\[
B_2(z_j) = t_1 - E/z_j + t_{-1}/z_j^2,
\]
\[
B_3(z_j) = t_{-1}/z_j - E.
\]

(A11)

The non-trivial solution \((c_1, c_2, c_3, c_4)\) requires
\[
\text{det}[H_B] = 0.
\]

(A12)

It is clear that the determinant of \(H_B\) contains \(4! = 24\) terms, and each term is a product of elements from different rows and columns of the matrix. Hence Eq. (A12) can be further expressed as
\[
\sum_{i \neq j \neq k \neq l = 1}^4 A(z_i) B_1(z_j) B_2(z_k) B_3(z_l) \times (z_j z_k z_l) = F_1(z, E) \times (z_2 z_3 z_4)^L + F_2(z, E) \times (z_1 z_3 z_4)^L + F_3(z, E) \times (z_1 z_2 z_4)^L + F_4(z, E) \times (z_1 z_2 z_3)^L = 0,
\]

(A13)

where the coefficient \(F_i(z, E)\) is a function of \(z = (z_1, z_2, z_3, z_4)\) and \(E\). Since both the degrees of \(A(z_i)\) and \(B_{1/2/3}(z_i)\) are finite and independent of \(L\), the leading term of Eq. (A13) can be ordered by
\[
(z_2 z_3 z_4)^L \geq (z_1 z_3 z_4)^L \geq (z_1 z_2 z_4)^L \geq (z_1 z_2 z_3)^L
\]

(A14)

according to Eq. (A27) in the thermodynamic limit. Hence in the \(L \to \infty\) limit, if \(|z_1(E)| < |z_2(E)|\), the only leading term of Eq. (A13) is \(F_1(z, E) \times (z_2 z_3 z_4)^L\), which requires \(F_1(z, E) = 0\). Since \(F_1(z, E)\) is a function of \(A(z, E)\) and \(B_{1/2/3}(z, E)\), the order of \(E\) in \(F_1(z, E)\) is independent of \(L\). This means there only exist finite solutions of \(E\) in \(F_1(z, E) = 0\). As a result, they can not form a continuous spectrum. On the other hand, if
\[
|z_1(E)| = |z_2(E)|,
\]

(A15)

there exist two leading terms in the \(L \to \infty\) limit, which implies
\[
\text{det}[H_B] = 0 \to F_1(z, E) \times (z_2 z_3 z_4)^L + F_2(z, E) \times (z_1 z_3 z_4)^L = 0.
\]

(A16)

In this case
\[
\frac{F_1(z, E)}{F_2(z, E)} = -\left(\frac{z_1}{z_2}\right)^L.
\]

(A17)

This means the order of \(E\) depends on \(L\), which will form a continuous band in the thermodynamic limit. The set of \(z\) satisfying Eq. (A15) is called generalized Brillouin zone (GBZ). From the above derivation, the GBZ condition \(|z_1(E)| = |z_2(E)|\) is related to the bulk Hamiltonian
\[
H(z) = t_{-1}/z + t_1 z + t_2 z^2 + t_3 z^3.
\]

(A18)

To be more precise, the order of the pole in \(H(z)\) determines the form of boundary matrix \(H_B\) in Eq. (A10), and finally determines the condition for the continuous band Eq. (A15). In the thermodynamic limit, as the image of GBZ on \(H(z)\) is labeled by \(L_{GBZ}\), and needs to satisfy
\[
|H_m^{-1}(L_{GBZ})| = |H_{m+1}^{-1}(L_{GBZ})|.
\]

(A19)

where \(H^{-1}(E)\) is the inverse function of \(H(z)\). Here we give numerical calculations (Fig. (4)) to support the above conclusions. Next we generalize the above procedure to a general single band Hamiltonian.
FIG. 4: (a) presents eigenvalues of Hamiltonian Eq. (A1) with the number of sites $L = 100$ and all parameters $t_{i=-1,1,2,3}$ equal to 1. Each eigenvalue corresponds to four solutions of Eq. (A18), which are ordered by absolute value and marked in red, orange, darker blue and darker green colors respectively. Then (b) shows that the first two solutions form the GBZ (gray continuous loop).

2. General case

Consider the following general single band real space Hamiltonian,

$$\hat{H} = \sum_{i,j=1}^{L} t_{i-j} \hat{c}_j^\dagger \hat{c}_i,$$  \hspace{1cm} (A20)

where the hopping parameters only exist in a finite range $-m \leq i - j \leq n$ and $L$ is the number of sites. For each site, the largest hopping range to left is $n$ and to right is $m$. The system reduces to Hermitian when $t_{i-j}$ equals to $t_{j-i}^\ast$. Similar to the derivation in the above section, this Hamiltonian can be divided into two parts: the bulk and the boundary. The bulk, ranging from $(n+1)_th$ site to $(L-m)_th$ site, maintains translational symmetry, while the boundary, including the remaining parts of two ends, has no longer the translation invariance. Starting from the eigenequation

$$H\Psi = E\Psi$$ \hspace{1cm} (A21)

we can solve the eigenequation from two parts, the bulk and boundary equations.

**Bulk equation:** The bulk equation is

$$t_{-m} \psi_s + t_{-m+1} \psi_{s+1} + \cdots + (t_0 - E) \psi_{s+m} + t_1 \psi_{s+m+1} + \cdots + t_n \psi_{s+m+n} = 0, \hspace{0.5cm} s = 1, ..., L - (m + n),$$  \hspace{1cm} (A22)

where $\psi_s$ denotes the $s_{th}$ component of wavefunction. For the sake of simplicity, the value of $t_0$ is usually taken as zero since it does not affect the eigenfunction. For a given $E$, the bulk equation has $m + n$ linear independent eigenfunctions, which can be written as

$$\Psi_i(E) = (z_i, z_i^2, ..., z_i^{L-1}, z_i^L), \hspace{0.5cm} i = 1, ..., m + n,$$  \hspace{1cm} (A23)

where $z_i$ satisfy the following characteristic polynomial equation for given $E$

$$f(z_i, E) := H(z_i) - E = \sum_{j=-m}^{n} t_{j} z_i^j - E = 0.$$  \hspace{1cm} (A24)

According to the linear superposition principle, the following wavefunction is also the eigenfunction of Eq. (A21)

$$\Psi(E) = \sum_{i=1}^{m+n} c_i \Psi_i(E)$$  \hspace{1cm} (A25)

$$= (\psi_1, \psi_2, ..., \psi_s, ..., \psi_{L-1}, \psi_L)^t,$$
\[ \psi_s = \sum_{i=1}^{m+n} c_i z_i^s, \quad s = 1, \ldots, L, \quad (A26) \]

and the solutions of Eq. (A24) are ordered as follows
\[ |z_1| \leq |z_2| \leq \ldots \leq |z_{m+n-1}| \leq |z_{m+n}|. \quad (A27) \]

**Boundary equation:** Consider the \( m + n \) boundary equations and substitute the solution Eq. (A26) into it, then one can obtain \( m \) constraint equations about \( \{\psi_1, \psi_2, \ldots, \psi_n\} \) and \( n \) limited equations about \( \{\psi_{L+n-1}, \psi_{L+n-2}, \ldots, \psi_L\} \), where \( L \) is the total number of the lattice sites. Open boundary means that there are no components of wave function beyond the two ends of the chain, namely, \( \psi_{1}\) = 0 and \( \psi_{L} = 0 \). In order to express the boundary equation, we first define
\[
T_s(\psi) := t_m \psi_s + t_{m+1} \psi_{s+1} + \cdots + (t_0 - E)\psi_{s+m} + t_1 \psi_{s+m+1} + \cdots + t_n \psi_{s+m+n}.
\quad (A28)
\]

Then the \( m + n \) boundary equations can be expressed as
\[
T_s(\psi) = 0, \quad s = -m + 1, -m + 2, \ldots, 0; \quad \quad (A29)
\]

with the open boundary condition
\[
\psi_{i<1} = \psi_{i>L} = 0 \quad (A30)
\]

In fact, the form of boundary equations ensures two things. There is always at least one term, of \( m + n \) terms in Eq. (A22), removed according to open boundary Eq. (A30). And we need to make sure that the term with coefficient \( E \) does not disappear. For example, \( T_{-m+1}(\psi) = -E \psi_1 + t_1 \psi_2 + \cdots + t_n \psi_{n+1} = 0 \) due to \( \psi_{<1} = 0 \) and \( T_{L-n}(\psi) = t_m \psi_{L-m} + t_{m+1} \psi_{L-m+1} + \cdots - E \psi_L = 0 \) due to \( \psi_{>L} = 0 \). According to Eq. (A26), these \( m + n \) boundary equations can be written as the following form:
\[
H_b \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ c_{m+1} \\ \vdots \\ c_{m+n} \end{pmatrix} = \begin{pmatrix} A_1(z_1) & A_1(z_2) & \cdots & A_1(z_{m+n}) \\ A_2(z_1) & A_2(z_2) & \cdots & A_2(z_{m+n}) \\ \vdots & \vdots & \ddots & \vdots \\ A_m(z_1) & A_m(z_2) & \cdots & A_m(z_{m+n}) \\ B_1(z_1) z_1^L & B_1(z_2) z_2^L & \cdots & B_1(z_{m+n}) z_{m+n}^L \\ \vdots & \vdots & \ddots & \vdots \\ B_m(z_1) z_1^L & B_m(z_2) z_2^L & \cdots & B_m(z_{m+n}) z_{m+n}^L \\ c_{m+1} & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \\ \vdots \\ c_{m+n} \end{pmatrix} = 0, \quad (A31)
\]

where \( z_i \) represents \( i_{th} \) roots of Eq. (A24), \( A_i(z_j) \) and \( B_i(z_j) \) are polynomials about \( z_j \) with finite order. The matrix elements \( A_i(z_j) \) can be obtained by substituting all the terms \( \psi_r \) of \( T_{-m+i}(\psi) \) into \( z_j^r \), likely, \( B_i(z_j) \) is obtained by substituting all the terms \( \psi_r \) of \( T_{L-m-n+i}(\psi) \) into \( z_j^{r-L} \). For example, \( A_1(z_j) = -E z_j + t_1 z_j^2 + \cdots + t_n z_j^{n+1} \) and \( B_1(z_j) = t_m z_j^{-m} + t_{m+1} z_j^{-m+1} + \cdots - E \). Notice that \( m, n \ll L \) and \( L \) representing the size of lattice tends to infinity in the thermodynamic limit. Nontrivial solutions of Eq. (A31) further require that
\[
\det[H_b] = 0. \quad (A32)
\]

The determinant of \( H_b \), a \( (m + n) \times (m + n) \) square matrix, is the summation of \( S_{m+n} = (m + n)! \) terms, and each term is the product of matrix elements belonging to different rows and columns. Therefore, each term contains the product of \( n \) different roots \( z_j^L \), and Eq. (A32) can be expressed as:
\[
\det[H_b] = F_1(z, E) \times (z_{m+1} z_{m+2} \cdots z_{m+n} z_{m+n})^L + F_2(z, E) \times (z_m z_{m+2} \cdots z_{m+n-1} z_{m+n})^L + \cdots = 0, \quad (A33)
\]
where $F_1$ and $F_2(z, E)$ are the coefficients of $(z_{m+1} \ldots z_{m+n} - z_{m+n-1} \ldots z_m)^L$ and $(z_{m+2} \ldots z_{m+n-1} \ldots z_m)^L$ respectively, and the subscript $i$ of $F_i(z, E)$ increases as the magnitude of $(z_i \ldots z_j)^L$ gradually decreases. In this way, $F_1(z, E)$ always corresponds to the term with the largest magnitude, namely, $(z_{m+1} \ldots z_{m+n-1} \ldots z_m)^L$, and so on. Based on the similar reasons with the previous part, the continuous band requires

$$|z_m(E)| = |z_{m+1}(E)|,$$

such that the leading order of Eq. (A33) is equivalent to

$$\left(\frac{z_{m+1}}{z_m}\right)^L = -\frac{F_1(z, E)}{F_2(z, E)}.$$  \hspace{1cm} (A35)

In fact, Eq. (A35) gives the constrain condition on $E$ because $z$ is the function of $E$. The exact solutions of $|z_m(E)| = |z_{m+1}(E)|$ give the spectrum in the thermodynamic limit and corresponding $z$ forms the generalized Brillouin zone. The right side of Eq. (A35) includes the detailed information about open boundary while the left side does not. It means that boundary conditions have little effect on the properties of the bulk if $N$ tends to infinity, but we need pay special attention to the fact that the above statements are based on the open boundary, it is no longer true for nonlocal boundary terms, such as periodic boundary.

3. Two-band chiral model

The above discussions can be naturally extended to multi-band cases, especially two-band Hamiltonian with chiral symmetry. Generically, the bulk Hamiltonian is written as

$$H(z) = \begin{pmatrix} 0 & p_{m+1}^{(1)}(z) \\ \frac{p_{m+n}^{(2)}(z)}{z_{m+2}} & 0 \end{pmatrix}.$$ \hspace{1cm} (A36)

The off-diagonal terms of the matrix are two different functions that are holomorphic at the entire complex plane except the origin. Next we show that the commonly used two-band tight-binding model can always mapped to the single-band model due to the presence of chiral symmetry. According to Eq. (A36), one can immediately write down the characteristic equation,

$$F(z, E) = \frac{P_{m+n}(z) - E^2 z^m}{z^m} = 0,$$ \hspace{1cm} (A37)

where $m = m_1 + m_2$ and $n = n_1 + n_2$, and this equation establishes the mapping between $z$ and $E$. Additionally, chiral symmetry ensures that the eigenvalues $(E, -E)$ always appear in pairs. Therefore, it comes to the conclusion that one can always take the base energy as $E^2$ in single-band model, which corresponds to the $\pm E$ in two-band chiral Hamiltonian, and they have the same characteristic equation that determines the generalized Brillouin zone. Then we can get the same conclusion as the single-band model

$$|z_m(E)| = |z_{m+1}(E)|,$$ \hspace{1cm} (A38)

where $m$ represents the multiplicity of the pole of characteristic equation Eq. (A37), and this conclusion is confirmed numerically in Fig. (7).

Appendix B: GBZ encircles the same number of zeros and poles

In this section, we prove that GBZ is a single loop in complex plane that encircles the origin and exactly $m$ zeros of $H(z) - E_b = 0$ for arbitrary $E_b \in \mathbb{C}$, and give numerical supports for this statement.

Here we focus on the cases where GBZ is a simple closed loop and characteristic equation (Hamiltonian in single-band cases) has the form $(P_{m+n}(z) - E_b z^m)/z^m$. Assume that there is a simple closed loop $C_z$ in complex plane with $(\text{Re}[z], \text{Im}[z])$ as the coordinate system, which can be mapped into the set $\mathcal{L}_{C_z}$ living in $(\text{Re}(E), \text{Im}(E))$ space (complex energy plane) by $H(z)$. If choose the point $E$ in complex energy plane, then one can find corresponding zeros of $H(z) - E = 0$ in the $(\text{Re}(z), \text{Im}(z))$ space. We first propose the following lemma before coming to the final proof.
**Lemma:** If there exists a smooth path $E_s$ (in the complex plane) that is controlled by parameter $s$ and has no intersections with the set $L_{C_z}$, then any two points connected through this path $E_s$ have the same number of zeros of $H(z) - E_s = 0$ within the area surrounded by $C_z$.

This lemma will be frequently used, and we have proved it rigorously.

**proof:** The zeros of Hamiltonian are determined by $P_{m+n}(z) - z^m E_b = 0$. Defining two analytic functions of $z$ as

$$
\begin{align*}
g_1(z) &= P_{m+n}(z) - z^m E_1 \\
g_2(z) &= P_{m+n}(z) - z^m E_2
\end{align*}
$$

(B1)

where $E_2$ is located in the neighborhood of $E_1$ and expressed as $E_2 = E_1 + \delta$ with $|\delta|$ tends to zero. Let both $E_1$ and $E_2$ do not belong to the set $L_{C_z}$. There always exist a small enough $\delta$ to ensure $z_c \in C_z$ satisfies the following inequation

$$
|g_1(z_c)| > |g_2(z_c) - g_1(z_c)| = |\delta z_c^m|
$$

(B2)

Then it follows from Rouche’s theorem that $g_1(z)$ possess the same number of zeros as that of $g_2(z)$ in the region enclosed by $C_z$.

By constantly using the trick of neighboring $\delta$, one can obtain a smooth path $E_s$ that connects $E_\infty$ with $E_b$. There are two cases, one is that $E_s$ does not intersect with the set $L_{C_1}$, such that $g_1(z) = P_{m+n}(z) - z^m E_\infty = 0$ and $g_2(z) = P_{m+n}(z) - z^m E_b = 0$ have the same number of zeros inside the region surrounded by $C_z$. This case just corresponds to case 1 illustrated in Fig. (5), in which $C_z$ is taken as GBZ. In another case, the path $E_s$ has interactions with $L_{C_1}$, just like case 2 in Fig. (5)($C_z$ is taken as GBZ), one can always find at least one path where zeros inside and outside GBZ exchange with each other and the total number of zeros inside GBZ is invariant.

For a given base energy $E_b$, the holomorphic function defined on the complex plane except the origin, $H(z) - E_b$, possess $m + n$ zeros expressed as $z = z_{1,2,...,m+n}$ and $m \delta$, order pole at the origin of complex. The form of complex function is that

$$
t_{m} + t_{m+1} z + \cdots + t_{1} z^{m-1} + t_{1} z^{m+1} + \cdots + t_{n} z^{m+n} = E
$$

(B3)

has asymptotic behavior that $m$ roots verge to zero and the remaining roots tend to infinity as $E$ tends to infinity. Here notice that the number $m$ of zeros located in the region bounded by $C_z$ is the same as multiplicity of pole if taking $E$ as $E_\infty$. 

**FIG. 5:** An illustration to the proof. The $z^{A/B}_z$ curves represent the loci of $z$ roots(zeros) of $P_{m+n}(z) - z^m E_b$. As parameter $s$ changes from 0 to 1, the energy changes from $E_\infty$ to arbitrary selected base energy $E_b$, and the corresponding zeros move continuously from $z_0^{A/B}$ to $z_1^{A/B}$. There are two different cases. In one case, the locus of $E_s$ has no intersections with the set $L_{GBZ}$, hence the corresponding zeros will not enter or escape from GBZ so that the total number of zeros is invariant during the process. In another case, there are intersections between $E_s$ and the set $L_{GBZ}$, and the same number of zeros will passed through and entered into GBZ, thus ensuring that the total number of zeros within GBZ is unchanged.
FIG. 6: (a) represents the energy spectrum of non-Hermitian system determined by Hamiltonian Eq. (B5) under open boundary condition with parameters \( \{t_3, t_2, t_1, t_{-1}, t_{-2}\} = \{1/2, 1, 1/2, 1/2, 2\} \) and the number of site \( L = 110 \), and the green arrows denote the direction of energy band as \( z \) clockwise along GBZ depicted in (c); (b) shows the energy spectrum with periodic boundary and the color gradually turns from black to red as \( k \) change from 0 to \( 2\pi \). Given three base energy \( E_1 = 0 \) in red color, \( E_2 = 1.2i \) in purple color and \( E_3 = 2 + i \) in blue color, and their corresponding zeros are colored with the same color on the \( z \) plane; In Fig.(c), gray unit circle and yellow loop indicate GBZ of the system with periodic and open boundary respectively, and the cross notation denotes the pole with multiplicity two. Here only the first two, with smaller absolute values, of the five zeros are depicted; (d) presents the distribution of five roots of Eq. (B5) as \( E \) varying uniformly from \(-10 - 10i\) to \( 10 + 10i\). The red and pink denote the first two roots, and blue, purple and black regions denote last three roots respectively, which also shows that GBZ(yellow curve) is exactly the boundary between them.

Now we have two conclusion: (i:) Two points \( E_{s=0} \) and \( E_{s=1} \), connected by smooth curve \( E_s \), have the same number of zeros within simple closed loop \( C_z \); (ii:) The number of zeros of \( P_{m+n}(z) - E_\infty = 0 \) is the same as the multiplicity of the pole. Combining this two points and taking \( E_{s=0} \) as \( E_\infty \), the conclusion is obtained that for arbitrary base energy \( E_b \) connected to \( E_\infty \) via the smooth path \( E_s \), the number of zeros is the same as the multiplicity of the pole inside \( C_z \).

Here we take \( C_z \) as GBZ, then, for all \( E \in \mathbb{C} \setminus \mathcal{L}_{GBZ} \), there always exist at least one path connecting \( E_\infty \) with \( E \) such that \( P_{m+n}(z) - zmE \) has \( m \) zeros inside GBZ as \( E_\infty \) does. Thus the winding number of energy spectrum with open boundary is calculated

\[
w_{GBZ,E_b} = \frac{1}{2\pi i} \oint_{C_z} \frac{H'(z)}{H(z) - E_b} \, dz = N_{zeros} - N_{poles} = 0 \tag{B4}
\]

where \( N_{zeros} \) denotes the number of zero points and \( N_{poles} \) indicates the multiplicity of the pole.

### 1. Example 1

Here we take a special single-band model to further confirm numerically the main conclusions, and the bulk Hamiltonian is shown as:

\[
H(z) = \sum_{i=-2}^{3} t_i z^i, \tag{B5}
\]

where \( z \) is complex variable and \( t_i \) denotes the hopping parameter. Then transform Eq.(B5) into the form, \((t_3 z^5 + t_2 z^4 + t_1 z^3 - E_b z^2 + t_{-1} z + t_{-2})/z^2 = 0\). We fix the other coefficients, then let \( E_b \) arbitrarily take values in the complex plane except for the energy spectrum \( \mathcal{L}_{GBZ} \), and always get five zeros, two of which are inside GBZ and the rest are
outside it, which is illustrated in Fig.(6). It further comes to a conclusion that if and only if GBZ is the boundary of the region composed of the first two solutions ordered by absolute value, the winding regarding to any choices of base energy $E_b$ vanish.

In Fig.(6), We have chosen three representative base energy $E_b$, which are marked in different colors, and the winding with respect to base energy $E_1, E_2, E_3$ equal to $-2, -1, 0$ respectively under periodic boundary conditions, as well as all of which equal to zero with open boundary. It is worth noting that even if the spectrum surrounds the $E_1$(red point) in Fig. 6(a), the winding of the energy is still zero. The reason is that as $z$ moves clockwise along GBZ, its image $L_{GBZ}$ keeps back-stepping itself and has no interior. These following significant conclusions have been confirmed numerically: (i:) Once we have determined the multiplicity $m$ of the pole of $H(z)$, GBZ always contains $m$ zeros accordingly, regardless of how the base energy takes values on the complex plane except for $L_{GBZ}$. (ii:) If $z$ directionally circles $BZ$, the corresponding $L_{BZ}$ will surround a finite area, while if $z$ moves along GBZ, $L_{GBZ}$ always wraps around itself and contains zero area, which refers to the collapse from $L_{BZ}$ to $L_{GBZ}$. (iii:) GBZ is precisely the boundary between the first $m$ zeros and the last $n$ zeros.

2. Example 2

Consider a two-band Hamiltonian with respect to chiral symmetry in real-space representation, which is shown as:

$$H = \sum_{i=1}^{L} t_1 a_i^\dagger b_i + t_2 a_{i+1}^\dagger b_i + t_3 a_i^\dagger b_{i+1} + w_1 b_{i+1} a_i + w_2 b_i a_{i+2} + w_3 b_i a_{i+3}$$  \hspace{1cm} (B6)

with the number of unit cell $L$ and two degrees per unit cell. Then the corresponding bulk Hamiltonian is written as:

$$H(z) = \begin{pmatrix} 0 & t_2 z_1 + t_3 z_2^2 \\ \frac{1}{z} & 0 \end{pmatrix} ,$$  \hspace{1cm} (B7)

in which the complex variable $z$ is expressed as $e^{ik}$ under periodic boundary conditions. The characteristic equation is written as $F(z, E) = \det(H(z) - EI_{2 \times 2}) = \frac{F_{m+n}(z) - E^2 z^m}{z^n} = 0$, with $m = 2$ and $n = 4$. According to the arguments in appendix A, the GBZ is constructed by $|z_m(E)| = |z_{m+1}(E)|$, which has been confirmed numerically in Fig.(7). Form Fig.(7), one can extract the following informations: (i:) GBZ is formed by $z_2$ and $z_3$, which are roots of characteristic equation with regards to eigenvalues of $H$, and this ensures that GBZ contains the same number $m$ of zeros and

FIG. 7: (a) illustrates the eigenvalues of non-Hermitian system determined by Hamiltonian Eq.(B7) under open boundary condition with parameters $\{t_1, t_2, t_3, w_1, w_2, w_3\} = \{1, 4, 1, 1, 1, \frac{1}{2}\}$ and size $L = 75$, and the red point denotes two-degenerate edge state protected by chiral symmetry. The arrows represents the evolution orientation of eigenvalues as $z$ anticlockwise along the GBZ; (b) shows the GBZ in orange color and BZ in dashed gray color, and the roots $z_i$, obtained by substituting eigenvalues into $H^{-1}(E)$, are ordered in ascending amplitude and marked in different colors.
poles. (ii:) $E$ and $-E$ always correspond to the same roots due to chiral symmetry, hence the two-part connected bulk spectrum always circulates around itself and encloses zero area as $z$ varies along the GBZ, and their behavior is symmetric about the origin. So one can always map two-band chiral model to a single-band Hamiltonian.

**Appendix C: The Proof for case-(iii)**

Here we give a rigorous proof for the case (iii) in main text. In this case, one part of GBZ outside and another part inside the unit circle(BZ), then we define the region inside GBZ but outside BZ as $U$, the region inside BZ but outside GBZ as $V$, and the region inside both GBZ and BZ as $W$. Before the proof, we must display two facts, one is that GBZ must enclose the origin, and another is that GBZ formed by $H^{-1}(L_{GBZ})$ and $H^{-1}_{m+1}(L_{GBZ})$ always encircles the first $m$ zeros that have ordered by absolute value.

Then we first prove that for any choice of base energy $E_b$, the roots of $H(z) - E_b = 0$ can not appear in $U$ and $V$ regions at the same time. A obvious fact is that the magnitude of $z_0'$ inside $V$ is always less than that of $z_0$ inside $U$ region. Assuming that there exist at least two zeros $z_0$ and $z_0'$ correspond to the same base energy $E_b$, then it will happen that for the $E_b$, the root $z_0'$ with smaller absolute value does not belongs to GBZ but the root $z_0$ with larger absolute value belongs to. This distinctly contradicts the facts we have displayed. Hence the assumption is not true, that is to say, there are no two roots at most correspond to the same $E_b$, i.e., the roots of $H(z) - E_b = 0$ can not appear in $U$ and $V$ regions at the same time for any $E_b$. Furthermore, it comes to the conclusion that the other roots of base energy corresponding to $z_0'$ may only appear in region $W$ but not in region $U$.

The next steps are the same as in case (i) and (ii). Pick $z_0 \in U$ and $E_0 = H(z_0)$, then $z_0$ is a zero of $H(z) - E_0$, likely pick $z_0' \in V$ and $E_0' = (z_0')$, then $z_0'$ is a zero of $H(z) - E_0$. Here we notice that $E_0 \neq E_0'$ that has been proved above. With the fact that GBZ encloses $m$ zeros, therefore in case (iii) there are always $E_0$ and $E_1$ such that $w_{BZ,E_0} < -1$ and $w_{BZ,E_0'} > 1$, and $E_0$ is not equal to $E_0'$. Here the case (iii) has been rigorously proved.

**Appendix D: Prove the inverse statement of Eq.(7)**

In this section we prove the statement that if there is any $E_b \in \mathcal{C}$ with respect to which $H(z)$ has nonzero winding, then one can always find some $n(H, H^*) \neq 0$ such that $J \neq 0$. The natural extension of the definition of total current from Hermitian systems is that

$$J = \int_{0}^{2\pi} n(H, H^*) \frac{dH(k)}{dk} dk = \oint_{L_{BZ}} n(H, H^*) dH. \quad \text{(D1)}$$

If the interior area of $L_{BZ}$ is nonzero, it means that $L_{BZ}$ is composed of one or several close loops. One can always find the base energy $E_b$ surrounded by one closed loop $L'_{BZ} \subseteq L_{BZ}$. Here we denote the interior area of $L'_{BZ}$ as $S(L'_{BZ})$, and we have

$$S(L'_{BZ}) \neq 0 \quad \text{(D2)}$$

Hence we can always define the distribution function $n(H, H^*)$ as follows

$$\begin{align*}
\left\{ \begin{array}{ll}
n(H, H^*) = \frac{1}{2\pi i} (H - H^*), & H \in L'_{BZ} \\
n(H, H^*) = 0, & H \in L_{BZ} \setminus L'_{BZ}
\end{array} \right. \quad \text{(D3)}
\end{align*}$$

such that

$$J = \oint_{L_{BZ}} n(H, H^*) dH = \oint_{L'_{BZ}} \text{Im}(H) dH, \quad \text{(D4)}$$

obviously, the imaginary part of which is zero. Then the total current becomes

$$J = \oint_{L_{BZ}} \text{Im}(H) d\text{Re}(H) = S(L'_{BZ}) \neq 0. \quad \text{(D5)}$$

The finite area enclosed by $L_{BZ}$ ensures the existence of the current under periodic boundary conditions, and which collapse into skin modes with periodic boundary. Here the inverse statement of Eq.(7) has been proven.
These two authors contributed equally

Electronic address: cfang@iphy.ac.cn

1 E. Persson, I. Rotter, H.-J. Stöckmann, and M. Barth, Phys. Rev. Lett. 85, 2478 (2000), URL https://link.aps.org/doi/10.1103/PhysRevLett.85.2478.

2 A. Volya and V. Zelevinsky, Phys. Rev. C 67, 054322 (2003), URL https://link.aps.org/doi/10.1103/PhysRevC.67.054322.

3 I. Rotter, Journal of Physics A: Mathematical and Theoretical 42, 153001 (2009), URL https://doi.org/10.1088%2F1751-8121%2FF92%2F153001.

4 Y. Choi, S. Kang, S. Lim, W. Kim, J.-R. Kim, J.-H. Lee, and K. An, Phys. Rev. Lett. 104, 153601 (2010), URL https://link.aps.org/doi/10.1103/PhysRevLett.104.153601.

5 S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, Nature Physics 7, 971 (2011), URL https://doi.org/10.1038/nphys2106.

6 F. Reiter and A. S. Sørensen, Phys. Rev. A 85, 032111 (2012), URL https://link.aps.org/doi/10.1103/PhysRevA.85.032111.

7 H. Shen and L. Fu, Phys. Rev. Lett. 121, 026403 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.026403.

8 H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, M. Soljačić, and F. Nori, Phys. Rev. Lett. 121, 086803 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.086803.

9 M. Papaj, H. Isobe, and L. Fu, Phys. Rev. B 99, 201107 (2019), URL https://link.aps.org/doi/10.1103/PhysRevB.99.201107.

10 V. Kozii and L. Fu, Non-hermitian topological theory of finite-lifetime quasiparticles: Prediction of bulk fermi arc due to exceptional point (2017), 1708.05841.

11 S. Longhi, Laser & Photonics Reviews 3, 243 (2009), URL onlinelibrary.wiley.com/doi/pdf/10.1002/lpor.200810055.

12 K. G. Zloshchastiev, in 2016 9th International Kharkiv Symposium on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves (MSMW) (2016), pp. 1–4.

13 D. L. Sounas and A. Ali, Nature Photonics 11, 774 (2017), URL https://doi.org/10.1038/s41566-017-0051-x.

14 R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Nature Physics 14, 11 EP (2018), URL https://doi.org/10.1038/nphys4323.

15 K. Y. Blokh, D. Leykam, M. Lein, and F. Nori, Nature Communications 10, 580 (2019), URL https://doi.org/10.1038/s41467-019-08397-6.

16 D. C. Brody, Journal of Physics A: Mathematical and Theoretical 47, 035305 (2013), URL https://doi.org/10.1088%2F1751-8121%2F47%2F3%2F035305.

17 Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.031079.

18 M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010), URL https://link.aps.org/doi/10.1103/RevModPhys.82.3045.

19 X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011), URL https://link.aps.org/doi/10.1103/RevModPhys.83.1057.

20 B. A. BERNEVIG and T. L. Hughes, Topological Insulators and Topological Superconductors (Princeton University Press, 2013), stu - student edition ed., ISBN 9780691151755, URL http://www.jstoro.com/stable/j.ctt19cc2gc.

21 C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016), URL https://link.aps.org/doi/10.1103/RevModPhys.88.035005.

22 N. P. Armitage, E. J. Mele, and A. Vishwanath, Rev. Mod. Phys. 90, 015001 (2018), URL https://link.aps.org/doi/10.1103/RevModPhys.90.015001.

23 H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. 120, 146402 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.120.146402.

24 T. E. Lee, Phys. Rev. Lett. 116, 133903 (2016), URL https://link.aps.org/doi/10.1103/PhysRevLett.116.133903.

25 D. Leykam, K. Y. Blokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. 118, 040401 (2017), URL https://link.aps.org/doi/10.1103/PhysRevLett.118.040401.

26 C. Yin, H. Jiang, L. Li, R. Lü, and S. Chen, Phys. Rev. A 97, 052115 (2018), URL https://link.aps.org/doi/10.1103/PhysRevA.97.052115.

27 X. Song, Journal of Physics Communications 2, 035043 (2018), URL https://doi.org/10.1088%2F2399-6528%2F2ab46a.

28 A. Ghatak and T. Das, Journal of Physics: Condensed Matter 31, 263001 (2019), URL https://doi.org/10.1088%2F1361-648X%2Fab1b3.

29 Y. Yoon, C. H. Lee, and R. Thomale, Phys. Rev. Lett. 121, 086803 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.086803.

30 S. Yao and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.086803.

31 S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.136802.

32 F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.026808.

33 F. K. Kunst and V. Dwivedi, Phys. Rev. B 99, 245116 (2019), URL https://link.aps.org/doi/10.1103/PhysRevB.99.245116.

34 K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019), URL https://link.aps.org/doi/10.1103/PhysRevLett.123.066404.