Algebraic representations of von Neumann algebras

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Abstract

An (algebraic) extended bilinear Hilbert semispace $H^\mp_a$ is proposed as being the natural representation space for the algebras of von Neumann. This bilinear Hilbert semispace has a well defined structure given by the representation space $\text{Repsp}(GL_n(L_\pi \times L_\nu))$ of an (algebraic) complete bilinear semigroup $GL_n(L_\pi \times L_\nu)$ over the product of sets of completions characterized by increasing ranks.

This representation space is a $GL_n(L_{\pi}^{(nr)} \times L_{\nu}^{(nr)})$-bisemimodule $M_R^{(nr)} \otimes M_L^{(nr)}$, decomposing into subbisemimodules according to the pseudounramified or pseudoramified conjugacy classes of $GL_n(L_{\pi}^{(nr)} \times L_{\nu}^{(nr)})$, and is in one-to-one correspondence with its cuspidal representation according to the Langlands global program.

In this context, towers of von Neumann subbisemialgebras on graded bilinear Hilbert subsemispaces, of which structures are these subbisemimodules, are constructed algebraically which allows to envisage the classification of the factors of von Neumann from an algebraic point of view.
Introduction

The first essential step of this paper consists in building up a bilinear mathematical frame for the representations of the von Neumann algebras in such a way that the most convenient representation space be essentially an extended bilinear Hilbert semispace characterized by a non-orthogonal basis.

Considering that the representation space of a von Neumann algebra must be the enveloping algebra [13] of the Hilbert (semi)module on which this von Neumann algebra acts, an extended bilinear Hilbert semispace is then proposed whose Hilbert bisemimodule constitutes the searched enveloping semialgebra [30]: this constitutes the content of chapter 1 [29].

In this perspective, an algebraic (real) extended bilinear Hilbert semispace $H^\pm_a$ and an analytic (complex) extended bilinear Hilbert semispace $H^\pm_h$ are constructed and proved to be the natural representation spaces for the algebras of elliptic operators. In this context, semialgebras and bisemialgebras of von Neumann on the spaces $H^\pm_a$ and $H^\pm_h$ are introduced according to the general treatment of semistructures and bisemistructures introduced in [30].

The generation of algebraic bilinear Hilbert semispaces is related to the bilinear Eisenstein cohomology which constitutes the algebraic pillar of the bilinear global program of Langlands introduced in [29]. More concretely, we are interested in the representation space Repsp($GL_n(L_\pi \times L_v)$) of a bilinear general semigroup over the product $(L_\pi \times L_v)$ of sets of pseudoramified real completions, at infinite archimedean places, whose ranks (or degrees) are integers modulo $N$ in such a way that:

- $GL_n(L_\pi \times L_v)$ has the Gauss bilinear decomposition into the product of subgroups of diagonal matrices by the subgroups of upper and lower unitriangular matrices;

- $GL_n(L_\pi \times L_v) = T^i_n(L_\pi) \times T_n(L_v)$ has for representation space the tensor product $(M_R \otimes M_L)$ of a right $T^i_n(L_\pi)$-semimodule $M_R$ by a left $T_n(L_v)$-semimodule $M_L$ such that $M_L$ (resp. $M_R$) decomposes into $T_n(L_v_i)$-subsemimodules $M_{v_i}$ (resp. $T^i_n(L_\pi_i)$-subsemimodules $M_{\pi_i}$) according to the left (resp. right) archimedean places $v_i$ (resp. $\pi_i$) of $L_v$ (resp. $L_\pi$) and so that the set of left (resp. right) subsemimodules $M_{v_i}$ (resp. $M_{\pi_i}$) corresponds to the set of left (resp. right) conjugacy classes of $GL_n(L_\pi \times L_v)$.
The bilinear Eisenstein cohomology (semi)group is the cohomology of the Shimura bisemivariety given by

$$\partial \overline{S}_{G_{R \times L}} = P_n(L_{\tau^i} \times L_v^i) \setminus GL_n(L^+_R \times L^+_L)/GL_n((\mathbb{Z}/N\mathbb{Z})^2)$$

where

- $P_n(L_{\tau^i} \times L_v^i)$ is a bilinear parabolic subsemigroup over the product, right by left, of sets of irreducible real completions having a rank $N$;

- $GL_n((\mathbb{Z}/N\mathbb{Z})^2)$ is a bilinear arithmetic subsemigroup constituting the representation of the tensor product of Hecke operators and having a representation in a Hecke bilattice;

- $L^+_R$ and $L^+_L$ are symmetric (real) algebraic (semi)fields.

Then, the bilinear Eisenstein cohomology

$$H^{2j}(\partial \overline{S}_{G_{R \times L}}, \tilde{M}^{2j}_R \otimes \tilde{M}^{2j}_L) \simeq \text{Repsp}(GL_{2j}(L_{\tau} \times L_v)), \quad 2j \leq n,$$

of the Shimura bisemivariety $\partial \overline{S}_{G_{R \times L}}$ has coefficients in the (bisheaf) $\tilde{M}^{2j}_R \otimes \tilde{M}^{2j}_L$ over the $GL_{2j}(L_{\tau} \times L_v)$-bisemimodule $(M^{2j}_R \otimes M^{2j}_L)$ and is in bijection with the representation space of the complete bilinear algebraic semigroup $GL_{2j}(L_{\tau} \times L_v)$.

Furthermore, the complete reducibility of $\text{Repsp}(GL_{2n}(L_{\tau} \times L_v))$ induces the decomposition of the bilinear Eisenstein cohomology into (irreducible) two-dimensional bilinear Eisenstein cohomologies.

On the other hand, the analytic pillar of the global program of Langlands is given by the cuspidal representation of the coefficients of the bilinear Eisenstein cohomology in terms of products, right by left, of global elliptic semimodules which are (truncated) Fourier series over $IR$ whose number of terms corresponds to the number of conjugacy classes of the general bilinear semigroup $GL_{2j}(L_{\tau} \times L_v)$.

The Eisenstein and analytic de Rham cohomologies are considered and recalled to be isomorphic [21] from which it results that (bi)semialgebras of von Neumann on the algebraic and analytic bilinear Hilbert semispaces $H^\pm_a$ and $H^\pm_h$ are isomorphic:

$$\mathcal{M}_{r,L}(H^\pm_a) \simeq \mathcal{M}_{r,L}(H^\pm_h).$$
The action of a (differential) bioperator \( (T_R^D \otimes T_L^D) \in \mathcal{M}_{R \times L}(H^\infty) \) of rank \((m \times m)\), with \(m \leq n\), (associated with a principal \(GL_m(\mathbb{R} \times \mathbb{R})\)-bundle) on the \((n \times n)\)-dimensional \((\text{bissemi})\text{sheaf} \) \( (\widetilde{\mathcal{M}}_R \otimes \widetilde{\mathcal{M}}_L) \) consists in mapping \((\widetilde{\mathcal{M}}_R \otimes \widetilde{\mathcal{M}}_L) \) into the corresponding \((\text{bissemi})\text{sheaf} \) \( (\widetilde{\mathcal{M}}_{R[n]} \otimes \widetilde{\mathcal{M}}_{L[n]}) \) shifted into \((m \times m)\) dimensions such that \((\widetilde{\mathcal{M}}_{R[n]} \otimes \widetilde{\mathcal{M}}_{L[n]}) \) decomposes into subbissimesheaves according to:

- the pseudoramified conjugacy classes \( g_R(i) \times g_L(i) \), \(1 \leq i \leq q\), of \( GL_n(L_\pi \times L_v) \) where "i" denotes a global residue degree;

or

- according to the pseudounramified conjugacy classes \( \gamma_R(i) \times \gamma_L(i) \), \(1 \leq i \leq q\), of the pseudounramified bilinear semigroup \( GL_n(L^{nr}_\pi \times L^{nr}_v) \) over sets \( L^{nr}_v \) and \( L^{nr}_\pi \) of pseudounramified completions;

in such a way that:

- \((\widetilde{\mathcal{M}}_{R[n]} \otimes \widetilde{\mathcal{M}}_{L[n]}) \) be the coefficient system of the shifted bilinear Eisenstein cohomology \( H^{2j-2k}(\partial S_G_{R \times L[n]} \to \widetilde{\mathcal{M}}_{L[n]}^{2j} \otimes \widetilde{\mathcal{M}}_{L[n]}^{2j}) \) where

\[
\partial S_G_{R \times L[n]} = \mathcal{P}_{n[m]} \left( (L_\pi^s \otimes \mathbb{R}) \times (L_v^s \otimes \mathbb{R}) \right)
\]

\[
\setminus GL_{n[m]} \left( (L_R^+ \otimes \mathbb{R}) \times (L_L^+ \otimes \mathbb{R}) \right) \slash GL_{n[m]} \left( \mathbb{Z} / N \mathbb{Z} \right)^2 \otimes \mathbb{R}^2
\]

is the shifted Shimura bissemivariety;

- \((\widetilde{\mathcal{M}}_{R[n]}^{2j} \otimes \widetilde{\mathcal{M}}_{L[n]}^{2j}) \) decomposes into shifted subbissimesheaves according to the pseudoramified or pseudounramified conjugacy classes of \( GL_{2j}(L^{nr}_\pi \times L^{nr}_v) \) in such a way that the pseudoramified conjugacy classes correspond to the cosets of \( GL_{n[m]} \left( (L_R^+ \otimes \mathbb{R}) \times (L_L^+ \otimes \mathbb{R}) \right) \slash GL_{n[m]} \left( \mathbb{Z} / N \mathbb{Z} \right)^2 \otimes \mathbb{R}^2 \) .

As in the unshifted case, the shifted bilinear Eisenstein cohomology decomposes into direct sum of completely irreducible orthogonal or nonorthogonal shifted bilinear Eisenstein cohomologies.

Taking into account the decomposition of the complete algebraic (resp. analytic) \( GL_n(L^{nr}_\pi \times L^{nr}_v) \)-bissemimodule \( (M^{nr}_R \otimes M^{nr}_L) \) (resp. \( (M^{s, nr}_R \otimes M^{s, nr}_L) \)) into subbissemimodules according to its pseudounramified or pseudoramified conjugacy classes, the complete algebraic (resp. analytic) extended bilinear Hilbert semispace \( \mathcal{H}_a^{\mp, (nr)} \) (resp. \( \mathcal{H}_h^{\mp, (nr)} \) )
also decomposes into bilinear subsemispaces $H^+(a_{nr})_i$ (resp. $H^+(h_{nr})_i$), $1 \leq i \leq q$, or according to sums of bilinear subsemispaces:

$$H^+(a_{nr})_i = \bigoplus_{\nu=1}^{i} H^+(a_{nr})_{\nu} \quad \text{(resp. } H^+(h_{nr})_i = \bigoplus_{\nu=1}^{i} H^+(h_{nr})_{\nu}) \, , \quad 1 \leq i \leq q \, ,$$

or

$$H^+_a \{i\} = \bigoplus_{j=1}^{i} H^+_a (j) \quad \text{(resp. } H^+_h \{i\} = \bigoplus_{j=1}^{i} H^+_h (j)) \, .$$

So, towers of sums of embedded bilinear Hilbert subsemispaces

$$H^+_a \{1\} \subset \cdots \subset H^+_a \{i\} \subset \cdots \subset H^+_a \{q\} \, ,$$

(resp. $H^+_h \{1\} \subset \cdots \subset H^+_h \{i\} \subset \cdots \subset H^+_h \{q\} \, ,$

$$H^+_a \{1\} \subset \cdots \subset H^+_a \{i\} \subset \cdots \subset H^+_a \{q\} \, ,$$

or $H^+_a \{i\} = \bigoplus_{j=1}^{i} H^+_a (j)$ (resp. $H^+_h \{i\} = \bigoplus_{j=1}^{i} H^+_h (j)$).

And towers of sums of pseudounramified or pseudoramified von Neumann sub(bi)-semialgebras can be generated according to:

$$M_{R \times L} \{ H^+_a (nr) \{1\} \} \subset \cdots \subset M_{R \times L} \{ H^+_a (nr) \{i\} \} \subset \cdots \subset M_{R \times L} \{ H^+_a (nr) \{q\} \} \, .$$

Then, the discrete spectrum $\sigma(T^D_R \otimes T^D_L)$ of a (differential) bioperator $(T^D_R \otimes T^D_L) \in \mathbb{M}_{R \times L}(H^+_a (nr))$ is obtained throughout the morphism from the von Neumann bisemia-
gebra $\mathbb{M}_{R \times L}(H^+_a (nr))$ to the set of von Neumann subbisemialgebras $[\mathbb{M}_{R \times L}(H^+_a (nr)) \{i\}]$, defined on the set of pseudounramified or pseudoramified bilinear Hilbert subsemispaces $H^+_a (nr) \{i\}$ characterized by a diagonal metric associated with an orthonormal bilinear basis.

If the cuspidal representation space of the $GL_n(I^{(nr)} \times L^{(nr)})$-bisemimodule $(M^R (nr) \otimes M^L (nr))$ is taken into account, the corresponding set of eigenbifunctions of the differential bioperator $(T^D_R \otimes T^D_L)$ is given, according to the Langlands program, by the global elliptic subbisemimodules which are products, right by left, of (truncated) Fourier series (over $IR$) whose number of terms correspond to the number of archimedean places associated with the considered algebraic intermediate finite number (semi)fields.

In this context, the classification of the factors of von Neumann can be envisaged from the algebraic frame developed in this paper.
In correspondence with the introduction of pseudoramified bilinear Hilbert semispaces $H^\pm_a$ and of towers of embedded bilinear Hilbert subsemispaces, pseudounramified bilinear Hilbert semispaces $H^\text{nr}_a$ can be defined as well as towers of embedded bilinear pseudounramified Hilbert subsemispaces.

So, if “$i$” labels an algebraic intermediate (semi)field or the associated archimedean completion, $\mathbb{M}_{R,L}(H^\text{nr}_a(i))$ will refer to a factor of type $I_i$ while if “$j$” denotes an algebraic internal dimension, $\mathbb{M}_{R,L}(H^\pm_a(j))$, $1 \leq j \leq N$, will be a hyperfinite subfactor of type $\Pi_{1j}$ [23], [24], where $N$ is the order of a global inertia subgroup.

So, our main proposition can finally be stated as follows:

1. On the pseudounramified bilinear Hilbert semispace $H^\text{nr}_a$, there are $q$ factors of type $I_i$, $1 \leq i \leq q \leq \infty$ where “$i$” denotes a global residue degree.

2. On the bilinear Hilbert subsemispace $H^\text{in}_a[L_{v_1} \times L_{v_1}]$ restricted to the representation of the bilinear parabolic subsemigroup $P_n(L_{v_1} \times L_{v_1})$, there are $N$ subfactors of type $\Pi_{1j}$, where $j$ denotes an internal algebraic dimension.

The upper subfactor $\Pi_{1N}$ is the hyperfinite factor $\Pi_1$.

3. On the tensor products $H^\text{nr}_a(i) \otimes H^\text{in}_a(N)$, there are $q$ pseudoramified factors of type $\Pi_{\infty}$, $1 \leq i \leq q \leq \infty$, noted $\mathbb{M}_{R,L}(H^\text{nr}_a(i) \otimes H^\text{in}_a(N))$ where $i$ denotes a global residue degree.

4. On the tensor products $H^\text{nr}_a(\infty) \otimes H^\pm_a(i)$, $1 \leq i \leq N$, the factors of type $\Pi_{\infty}$ are defined.

1 Bilinear semigroups and bilinear Hilbert semispaces

The aim of this chapter is to introduce a sufficiently general mathematical frame for the representations of the von Neumann algebras. As the “representation” of a $k$-algebra $M_L$ over a number field $k$ of characteristic zero proceeds from its enveloping algebra, the most natural representation space for the von Neumann algebras will be an extended Hilbert semispace of bilinear nature which must then correspond to the representation space of the $k$-algebra $M_L$ in a linear Hilbert space $\mathcal{H}$.

If the representation space of a von Neumann algebra is assumed to be non commutative, its (algebraic)-geometric structure will then be of Riemann type and composed of the tensor product of a pair of faithfully projective isomorphic $k$-semimodules leading to an extended bilinear Hilbert semispace by projection of one of these semimodules on its copy.
Notations: \( R, L \) means " \( R \) " or " \( L \) " for "right" or "left";
\[ \times_{(D)} \] means a diagonal \( (\times_{D}) \) or complete \( (\times) \) product.

**Definition 1.1 Enveloping algebra:** Let \( M_{R,L} \) be a \( k \)-algebra considered as a finitely generated, projective and faithful right (resp. left) \( k \)-module. Its enveloping algebra is given by \( M^e = M_R \otimes_k M_L \) where \( M_R \) (resp. \( M_L \)) is a right (resp. left) \( k \)-module viewed as the opposite algebra of \( M_L \) (resp. \( M_R \)) [13]. If the homomorphism \( E_{h_{R,L}} : M^e \to \text{End}_k(M_{R,L}) \) is an isomorphism, then the \( k \)-algebra \( M_{R,L} \) is called an Azumaya algebra.

If \( M_{R,L} \) is a faithfully projective right (resp. left) \( k \)-module of dimension \( n \), then \( M_{R,L} \simeq k^n \) and we have that [6], [18], [37]:
\[ M^e \simeq \text{End}_k(M_{R,L}) \simeq \text{End}_k(k^n) \simeq M_n(k) \]
where \( M_n(k) \) is the ring of matrices of order \( n \) over \( k \).

The homomorphism \( E_{R,L} : M_{R,L} \to M_n(k) \) is called a \( n \)-dimensional representation of \( M_{R,L} \) [2].

**Definition 1.2 Symmetric algebraic extension field:** Let \( k \) be a number field of characteristic 0 and \( L^+ \) (resp. \( L \)) denote a finite real (resp. complex) extension of \( k \). A real (resp. complex) algebraic extension field \( L^+ \) (resp. \( L \)) will be said symmetric if it is composed of the set of positive (resp. complex) simple roots, noted \( L^+_R \) (resp. \( L_L \)), in one-to-one correspondence with the set of negative (resp. complex conjugate) simple roots, noted \( L^-_R \) (resp. \( L^-_L \)), such that to each positive (resp. complex) simple root \( x^+_L \in L^+_L \) (resp. \( x_L \in L_L \)) corresponds a symmetric negative (resp. complex conjugate) simple root \( x^-_R \in L^-_R \) (resp. \( x_R \in L_R \)). Geometrically, \( L_L \) is then localized in the upper halfspace and \( L_R \) in the lower half space. \( L^+_L \) (resp. \( L_L \)) and \( L^+_R \) (resp. \( L_R \)) are then respectively left and right semifields, i.e. commutative division left and right semirings.

\( L_L \) and \( L_R \) are semifields because they are abelian semigroups with respect to the addition and are endowed with associative multiplication and distributive laws.

**Definition 1.3 Completions associated with finite algebraic extensions:** The equivalence classes of the real completions of \( L^+_L \) (resp. \( L^+_R \)), obtained by an isomorphism of compactification of the corresponding extensions, are the left (resp. right) infinite places of \( L^+_L \) (resp. \( L^+_R \)) and are noted \( v = \{v_1, \cdots, v_i, \cdots, v_q\} \) (resp. \( \overline{v} = \{\overline{v}_1, \cdots, \overline{v}_i, \cdots, \overline{v}_q\} \)).

Similarly, the equivalence classes of the complex completions of \( L_L \) (resp. \( L_R \)), obtained by an isomorphism of compactification of corresponding finite extensions, are the left (resp. right infinite places of \( L^-_L \) (resp. \( L^-_R \)) and are noted \( w = \{w_1, \cdots, w_i, \cdots, w_q\} \) (resp. \( \overline{w} = \{\overline{w}_1, \cdots, \overline{w}_i, \cdots, \overline{w}_q\} \)).
right) infinite complex places of \( L_L \) (resp. \( L_R \)) and are noted \( \omega = \{ \omega_1, \ldots, \omega_i, \ldots, \omega_q \} \) (resp. \( \overline{\omega} = \{ \overline{\omega}_1, \ldots, \overline{\omega}_i, \ldots, \overline{\omega}_q \} \)).

Let \( L_{v_i} \) (resp. \( L_{\overline{v}_i} \)) denote \( i \)-th basic real completion corresponding to the \( i \)-th left (resp. right) pseudoramified algebraic extension \( L^{+}_{L_i} \) (resp. \( L^{+}_{R_i} \)) of \( k \) and associated to the left (resp. right) place \( v_i \) (resp. \( \overline{v}_i \)). The other equivalent completions of \( v_i \) (resp. \( \overline{v}_i \)) are noted \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)), where \( m_i \in \mathbb{N} \) , \( m_i > 0 \), are increasing integers. ( \( m_i = 0 \) refers to the basic completion \( L_{v_i} \) (resp. \( L_{\overline{v}_i} \))).

It is assumed that the left (resp. right) pseudoramified completions \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)) are generated from an irreducible completion \( L_{v_i!} \) (resp. \( L_{\overline{v}_i!} \)) having a rank or degree equal to \( N \).

Then, the rank of the pseudoramified completions \( L_{v_i} \) (resp. \( L_{\overline{v}_i} \)) and \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)), corresponding to the degree of extension of the associated extension, is given by an integer modulo \( N \) according to:

\[
\begin{align*}
n_{iL} &= \lfloor L_{v_i, m_i} : k \rfloor = * + f_{v_i} \cdot N \simeq i \cdot N \\
(\text{resp. } n_{iR} &= \lfloor L_{\overline{v}_i, m_i} : k \rfloor = * + f_{\overline{v}_i} \cdot N \simeq i \cdot N )
\end{align*}
\]

where

- \( * \) denotes an integer inferior to \( N \);

- \( f_{v_i} \) (resp. \( f_{\overline{v}_i} \)), called a global class residue degree, is the degree of the corresponding pseudounramified completions \( L^{nr}_{v_i, m_i} \) (resp. \( L^{nr}_{\overline{v}_i, m_i} \)) given by

\[
\begin{align*}
\lfloor L^{nr}_{v_i, m_i} : k \rfloor &= f_{v_i} = i \quad (\text{resp. } \lfloor L^{nr}_{\overline{v}_i, m_i} : k \rfloor = f_{\overline{v}_i} = i )
\end{align*}
\]

So, the ranks or degrees of the pseudoramified completions \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)), \( 1 \leq i \leq q \), are integers modulo \( N , \mathbb{Z} / N \mathbb{Z} \).

Remark that the integer \( \sup(m_i) \) is interpreted as the multiplicity of the place \( v_i \) (resp. \( \overline{v}_i \)).

As the rank \( n_{iL} \) (resp. \( n_{iR} \)) of the completion \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)) is assumed to be a multiple of the integer \( N \), which is the rank of the irreducible subcompletion \( L_{v_i!} \) (resp. \( L_{\overline{v}_i!} \)), the completion \( L_{v_i, m_i} \) (resp. \( L_{\overline{v}_i, m_i} \)) will be cut into a set of \( i \) equivalent subcompletions \( L_{v_i', m_i} \) (resp. \( L_{\overline{v}_i', m_i} \)), \( 1 \leq i' \leq i \), of rank \( N \).

Finally, let

\[
\begin{align*}
L_v &= \{ L_{v_1}, \ldots, L_{v_i, m_i}, \ldots, L_{v_q, m_q} \} \\
(\text{resp. } L_{\overline{v}} &= \{ L_{\overline{v}_1}, \ldots, L_{\overline{v}_i, m_i}, \ldots, L_{\overline{v}_q, m_q} \} )
\end{align*}
\]
denote the set of real pseudoramified completions of $L_L^+$ (resp. $L_R^+$) with

$$L_{v_{i}} = \bigoplus_{m_i} L_{v_{i,m_i}}$$

(resp. $L_{\pi_{i}} = \bigoplus_{m_i} L_{\pi_{i,m_i}}$)

be their direct sum and let

$$L_{v}^{nr} = \{L_{v_{1}}^{nr}, \ldots, L_{v_{i,m_i}}^{nr}, \ldots, L_{v_{q,m_q}}^{nr} \}$$

(resp. $L_{\pi}^{nr} = \{L_{\pi_{1}}^{nr}, \ldots, L_{\pi_{i,m_i}}^{nr}, \ldots, L_{\pi_{q,m_q}}^{nr} \}$)

denote the corresponding set of real pseudounramified completions.

Similarly, let

$$L_{\omega} = \{L_{\omega_{1}}, \ldots, L_{\omega_{i,m_i}}, \ldots, L_{\omega_{q,m_q}} \}$$

(resp. $L_{\pi} = \{L_{\pi_{1}}, \ldots, L_{\pi_{i,m_i}}, \ldots, L_{\pi_{q,m_q}} \}$)

denote the set of complex pseudoramified completions of $L_L$ (resp. $L_R$) in such a way that the set $L_v$ (resp. $L_{\pi}$) of real completions covers the corresponding set $L_{\omega}$ (resp. $L_{\pi}$) of complex completions [29].

**Definition 1.4 Galois subgroups and inertia subgroups:** Let $\text{Gal}(L_{L,i}^+/k)$ (resp. $\text{Gal}(L_{R,i}^+/k)$) be the Galois subgroup of the pseudoramified extension $L_{L,i}^+$ (resp. $L_{R,i}^+$) and let $\text{Gal}(L_{L,i}^{nr,+}/k)$ (resp. $\text{Gal}(L_{R,i}^{nr,+}/k)$) denote the Galois subgroup of the corresponding pseudounramified extension $L_{L,i}^{nr,+}$ (resp. $L_{R,i}^{nr,+}$).

If $I_{L_{L,i}^+}$ (resp. $I_{L_{R,i}^+}$), denoting the global inertia subgroup of $\text{Gal}(L_{L,i}^+/k)$ (resp. $\text{Gal}(L_{R,i}^+/k)$), is the group of Galois automorphisms of the irreducible extension $L_{L,i}^+$ (resp. $L_{R,i}^+$) or the group of Galois inner automorphisms, then we have that

$$\text{Gal}(L_{L,i}^+/k)/I_{L_{L,i}^+} = \text{Gal}(L_{L,i}^{nr,+}/k)$$

(resp. $\text{Gal}(L_{R,i}^+/k)/I_{L_{R,i}^+} = \text{Gal}(L_{R,i}^{nr,+}/k)$)

such that the exact sequence:

$$0 \longrightarrow I_{L_{L,i}^+} \longrightarrow \text{Gal}(L_{L,i}^+/k) \longrightarrow \text{Gal}(L_{L,i}^{nr,+}/k) \longrightarrow 1$$

(resp. $0 \longrightarrow I_{L_{R,i}^+} \longrightarrow \text{Gal}(L_{R,i}^+/k) \longrightarrow \text{Gal}(L_{R,i}^{nr,+}/k) \longrightarrow 1$)

has kernel given by the global inertia subgroup $I_{L_{L,i}^+}$ (resp. $I_{L_{R,i}^+}$) associated to the place $v_i$ (resp. $\pi_i$).
If $m_i$ refers to the multiplicity of the left and right places $v_i$ and $\overline{v}_i$, then the left (resp. right) Galois group can be decomposed according to:

$$\Gal(L^+_L/k) = \bigoplus_{i=1}^q \bigoplus_{m_i} \Gal(L^+_{L_i,m_i}/k)$$

(resp. $\Gal(L^+_R/k) = \bigoplus_{i=1}^q \bigoplus_{m_i} \Gal(L^+_{R_i,m_i}/k)$).

1.5. Representation of the bilinear general semigroup: Let $L_v$ (resp. $L_{\overline{v}}$) be the set of pseudoramified real completions of $L^+_L$ (resp. $L^+_R$). Then, a bilinear general (or complete algebraic) semigroup over the product $L_v \times L_v$ can be defined as the product of the (semi)group $T^t_n(L_{\overline{v}})$ of lower triangular matrices of order $n$ over $L_{\overline{v}}$ by the (semi)group $T^t_n(L_v)$ of upper triangular matrices of order $n$ over $L_v$ according to [29]:

$$GL^t_n(L_v \times L_v) = T^t_n(L_{\overline{v}}) \times T^t_n(L_v)$$

such that:

a) $GL^t_n(L_{\overline{v}} \times L_v)$ has the bilinear Gauss decomposition:

$$GL^t_n(L_{\overline{v}} \times L_v) = [(D_n(L_{\overline{v}}) \times D_n(L_v))[UT^t_n(L_v) \times UT^t_n(L_{\overline{v}})]$$

where

- $D^t_n(\cdot)$ is the subgroup of diagonal matrices;

- $UT^t_n(\cdot)$ is the subgroup of unitriangular matrices.

b) $GL^t_n(L_{\overline{v}} \times L_v)$ has for representation space $\text{Repsp}(GL^t_n(L_{\overline{v}} \times L_v))$ given by the tensor product $M^R \otimes M^L$ of a right $T^t_n(L_{\overline{v}})$-semimodule $M^R$ localized in the upper half space by a left $T^t_n(L_v)$-semimodule $M^L$ localized in the lower half space.

c) the left (resp. right) conjugacy classes of $T^t_n(L_v)$ (resp. $T^t_n(L_{\overline{v}})$) correspond to the left (resp. right) places $v_i$ (resp. $\overline{v}_i$) of $L_v$ (resp. $L_{\overline{v}}$).

Similarly, $GL^t_n(L_{\overline{v}} \times L_v)$ has for representation space $\text{Repsp}(GL^t_n(L_{\overline{v}} \times L_v))$ given by the tensor product $M^R_{\overline{v}} \otimes M^L_v$ of a right pseudounramified $T^t_n(L_{\overline{v}})$-semimodule $M^R_{\overline{v}}$ by its left equivalent $T^t_n(L^nr_v)$-semimodule $M^L_v$.

Considering complete bilinear algebraic (semi)groups is justified by the fact that they “cover” their “linear” equivalents. Indeed, it was proved in [29] that a linear complete
algebraic group $GL_n(L_{\pi-v})$, with entries in $L_{\pi-v} \equiv L_\pi \cup L_v$ and representation space given by a vectorial space $V$ of dimension $n^2$, is covered by the bilinear complete algebraic semigroup $GL_n(L_\pi \times L_v)$, having as representation space the $GL_n(L_\pi \times L_v)$-bisemimodule $M_R \otimes M_L$, at the conditions given in [29].

On the other hand, let $M_{L_{\omega}}$ (resp. $M_{R_{\omega}}$) denote the representation space of $T_n(L_{\omega v})$ (resp. $T_n(L_{\pi_{\omega}})$) with entries in the sum $L_{\omega v}$ (resp. $L_{\pi_{\omega}}$) of real pseudoramified completions $L_{\omega v_i,m_i}$ (resp. $L_{\pi_{\omega_i,m_i}}$).

Then, $M_{L_{\omega}}$ (resp. $M_{R_{\omega}}$) is homomorphic to $M_L$ (resp. $M_R$) and decomposes into the direct sum of $T_n(L_{\omega v_i})$-subsemimodules $M_{v_i}$ (resp. $T_n(L_{\pi v_i})$-subsemimodules $M_{\pi v_i}$) according to:

$$M_{L_{\omega}} = \bigoplus_{i=1}^{q} \bigoplus_{m_i} M_{v_i,m_i}$$

such that:

a) each $T_n(L_{\omega v_i})$-subsemimodule $M_{v_i}$ (resp. $T_n(L_{\pi v_i})$-subsemimodule $M_{\pi v_i}$) of dimension $n$ constitutes a representative of the $i$-th conjugacy class of $T_n(L_v)$ (resp. $T_n(L_{\pi})$).

b) $M_{v_i}$ and $M_{\pi v_i}$, $1 \leq i \leq q$, has a rank given by:

$$n_i = i^n \cdot N^n = f_{v_i}^n \cdot N^n.$$

**Definition 1.6 Bisemimodules:** The bilinear tensor product between the right $T_n(L_{\pi})$-semimodule $M_R$ and the left $T_n(L_v)$-semimodule $M_L$ is given by [30]:

$$T_X : \{M_R, M_L\} \longrightarrow M_R \otimes M_L,$$

$$\{x_R, x_L\} \longrightarrow x_R \times x_L, \quad \forall x_R \in M_R, x_L \in M_L,$$

so that the pair $\{x_R, x_L\}$ of right and left points be mapped into the bipoint $x_R \times x_L$ characterized by a Riemannian signature [18]. $M_R \otimes M_L$ then is a $GL_n(L_{\pi} \times L_v)$-bisemimodule.

Similarly, the diagonal tensor product between the right and left semimodules $M_R$ and $M_L$ can be defined by

$$T_{X_D} : \{M_R, M_L\} \longrightarrow M_R \otimes_D M_L,$$

$$\{x_R, x_L\} \longrightarrow x_R \times_D x_L,$$

so that the “diagonal” bipoint $x_R \times_D x_L$ be characterized by a diagonal signature which can be Euclidian or not following that the metric be given by a diagonal unit matrix or by a diagonal matrix having diagonal elements taking values in the considered field.

$M_R \otimes_D M_L$ then is a $GL_n(L_{\pi} \times_D L_v)$-bisemimodule.
Definition 1.7  Bisemisheaves of rings:  We want to introduce the set of smooth differentiable (bi)functions on the $GL_n(L_τ \times L_v)$-bisemimodule $M_R \otimes M_L$, in such a way that these bifunctions are tensor products $φ_{G_R}(x_{gr}) \otimes φ_{G_L}(x_{gl})$ of smooth differentiable right functions $φ_{G_R}(x_{gr})$, $x_{gr} \in T^i_n(L_τ)$, on $M_R$, localized in the lower half space by symmetric smooth differentiable left functions $φ_{G_L}(x_{gl})$, $x_{gl} \in T^i_n(L_v)$, on $M_L$, localized in the upper half space.

As $GL_n(L_τ \times L_v)$ is partitioned into conjugacy classes, we have to take into account the bifunctions $φ_{G_i,mi_R}(x_{iR}) \otimes φ_{G_i,mi_L}(x_{iL})$ on the conjugacy class representatives $M_{\varphi,i,mi} \otimes M_{\varphi,mi}$. The set of smooth differentiable bifunctions $\{φ_{G_i,mi_R}(x_{iR}) \otimes φ_{G_i,mi_L}(x_{iL})\}_{i,mi}$ on the $GL_n(L_τ \times L_v)$-bisemimodule $M_R \otimes M_L$ is a bisemisheaf of rings noted $M_R \otimes M_L$ in such a way that this set of differentiable bifunctions are the (bi)sections of $\tilde{M}_R \otimes \tilde{M}_L$.

Indeed, $\tilde{M}_R$ (resp. $\tilde{M}_L$), having as sections the smooth differentiable functions $φ_{G_i,mi_R}(x_{iR})$ (resp. $φ_{G_i,mi_L}(x_{iL})$), is a semisheaf of rings because it is a sheaf of abelian semigroups $\tilde{M}_R(x_{iR})$ (resp. $\tilde{M}_L(x_{iL})$) for every right (resp. left) point $x_{iR}$ (resp. $x_{iL}$) of the topological semispace $M_R = \text{Reps}(T^i_n(L_τ))$ (resp. $M_L = \text{Reps}(T^i_n(L_v))$) where $\tilde{M}_R(x_{iR})$ (resp. $\tilde{M}_L(x_{iL})$) has the structure of a semiring.

The introduction of the bilinear Hilbert semispaces in the next section will concern the bisemisheaf of rings $\tilde{M}_R \otimes (D) \tilde{M}_L$ as well as the $GL_n(L_τ \times L_v)$-bisemimodule $M_R \otimes (D) M_L$, on which it is defined, but the developments will only bear on $M_R \otimes (D) M_L$ for the simplicity of the notations.

Definitions 1.8  a) External diagonal bilinear Hilbert semispaces $H^o_L$ and $H^o_R$:
Let $M_R \otimes (D) M_L$ be the diagonal $GL_n(L_τ \times D L_v)$-bisemimodule. Consider the projective linear mapping $p_L : M_R \otimes (D) M_L \rightarrow M_R(\alpha)/L$ projecting the $T^i_n(L_τ)$-semimodule $M_R$ on the $T^i_n(L_v)$-semimodule $M_L$. $M_R(\alpha)/L$ is a bisemimodule representable locally by the bilinear Hilbert scheme $\text{Hilb}_{S_R(\alpha)/S_L}$ (case $\tilde{M}_R(\alpha)/L$) [31].

If $M_R(\alpha)/L$ is endowed with an external scalar product $⟨φ_P, ψ⟩$ defined from $M_R(\alpha) \times_D M_L$ to $\mathbb{C}$, $∀ φ_P \in M_R(\alpha)$, $∀ ψ \in M_L$, this bisemimodule $M_R(\alpha)/L$ will be called a left external bilinear Hilbert semispace, noted $H^o_L$.

Similarly, if we consider the projective linear mapping $p_R : M_R \otimes (D) M_L \rightarrow M_R(\alpha)/R$ projecting the $T^i_n(L_v)$-semimodule $M_L$ on the $T^i_n(L_τ)$-semimodule $M_R$, we generate the bisemimodule $M_R(\alpha)/R$ representable locally by the bilinear Hilbert scheme $\text{Hilb}_{S_R(\alpha)/S_R}$.

Endowing $M_R(\alpha)/R$ with an external scalar product from $M_R(\alpha) \times_D M_R$ to $\mathbb{C}$, we shall get a right external bilinear Hilbert semispace noted $H^o_R$.

Notice that $H^o_L$ and $H^o_R$ are characterized by ortho(normal) basis.
b) **Internal diagonal bilinear Hilbert semispaces** $\mathcal{H}_a^-$ and $\mathcal{H}_a^+$: Let $B_L : M_{R(P)} \to M_L$ (resp. $B_R : M_{L(P)} \to M_R$) be a bijective linear isometric map from $M_{R(P)}$ (resp. $M_{L(P)}$) to $M_L$ (resp. $M_R$) mapping each covariant element of $M_{R(P)}$ (resp. $M_{L(P)}$) noted $M_{LR}$ (resp. $M_{RL}$) into a contravariant element of $M_L$ (resp. $M_R$).

Then, $B_L$ (resp. $B_R$) transforms the left (resp. right) external Hilbert semispace $\mathcal{H}_a^L$ (resp. $\mathcal{H}_a^R$) into the left (resp. right) internal bilinear Hilbert semispace $\mathcal{H}_a^+$ (resp. $\mathcal{H}_a^-$) in such a way that

a) the bielements of $\mathcal{H}_a^+$ (resp. $\mathcal{H}_a^-$) are bivectors, i.e. two confounded vectors;

b) each external scalar product of $\mathcal{H}_a^L$ (resp. $\mathcal{H}_a^R$) is transformed into an internal scalar product defined from $M_{LR} \times D M_L$ (resp. $M_{RL} \times D M_R$) to $\mathbb{C}$.

c) $\mathcal{H}_a^+$ and $\mathcal{H}_a^-$ are characterized by ortho(normal) basis.

c) **Extended external bilinear Hilbert semispaces** $H_a^0$ and $H_a^R$: If we consider on the non-Euclidian $GL_n(L_v \times L_v)$-bisemimodule $M_R \otimes M_L$ the projective linear mapping $p_L : M_R \otimes M_L \to M_{R(P)/cL}$ (“c” for complete), (resp. $p_R : M_R \otimes M_L \to M_{L(P)/cR}$) of the right (resp. left) semimodule $M_R$ (resp. $M_L$) on the left (resp. right) semimodule $M_L$ (resp. $M_R$), we get the non-Euclidian bisemimodule $M_{R(P)/cL}$ (resp. $M_{L(P)/cR}$).

If we endow $M_{R(P)/cL}$ (resp. $M_{L(P)/cR}$) with a complete external bilinear form defined from $M_{R(P)} \times M_L$ (resp. $M_{L(P)} \times M_R$) to $\mathbb{C}$, we get a left (resp. right) extended external bilinear Hilbert semispace noted $H_a^0$ (resp. $H_a^R$) characterized by a non-Euclidian geometry and a non-orthogonal basis.

d) **Extended internal bilinear Hilbert semispaces** $H_a^+$ and $H_a^-$: The left (resp. right) extended external bilinear Hilbert semispace $H_a^L$ (resp. $H_a^R$) can be transformed into the left (resp. right) extended internal bilinear Hilbert semispace $H_a^+$ (resp. $H_a^-$) by means of a bijective linear isometric map $B_L$ (resp. $B_R$) from $M_{R(P)}$ (resp. $M_{L(P)}$) into $M_L$ (resp. $M_R$).

The complete external bilinear form of $H_a^0$ (resp. $H_a^0$) is then transformed into a complete internal bilinear form of $H_a^+$ (resp. $H_a^-$).
2 Cohomologies and representation spaces of algebras of operators

We are interested in the cohomology of compact spaces [8]. So, the most evident algebraic cohomology of compact spaces is the Eisenstein cohomology which is based upon the Borel-Serre compactification of the lattice space attached to an arithmetic group $\Gamma$. The Eisenstein cohomology classes were assumed to be represented by differential forms which are Eisenstein series [21], [22], [34].

Definition 2.1 The Shimura bismivariety: Referring to the linear lattice space $X = GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, [6], [7], a bilinear complex lattice bismispace can be introduced by:

$$X_{SR\times L} = GL_n(L_R^{(C)} \times L_L^{(C)})/GL_n((\mathbb{Z} / N \mathbb{Z})^2)$$

where

- $GL_n((\mathbb{Z} / N \mathbb{Z})^2)$ is a bilinear arithmetic semigroup over squares of integers modulo $N$;

- $GL_n(L_R^{(C)} \times L_L^{(C)})$ is a bilinear algebraic semigroup with entries in the product $(L_R^{(C)} \times L_L^{(C)})$ of complex symmetric (semi)fields associated with $(L_R \times L_L)$.

The boundary $\partial X_{SR\times L}$ of the compactified bismispace $X_{SR\times L}$ corresponds to the boundary of the Borel-Serre compactification and is given by:

$$\partial X_{SR\times L} = GL_n(L_{Rd}^+ \times L_{ld}^+)/GL_n((\mathbb{Z} / N \mathbb{Z})^2) \approx GL_n(L_{\tau} \times L_v)$$

where $L_{Rd}^+$ and $L_{ld}^+$ are real compact semifields generated from $L_R^+$ and $L_L^+$.

The double coset decomposition $\partial S_{GR\times L}$ of the boundary $\partial X_{SR\times L}$ of the compactified lattice bismispace corresponds to a Shimura bismivariety and is given by:

$$\partial S_{GR\times L} = P_n(L_{R\tau} \times L_{v1}) \setminus GL_n(L_{Rd}^+ \times L_{ld}^+)/GL_n((\mathbb{Z} / N \mathbb{Z})^2) \approx GL_n(L_{\tau} \times L_v)$$

where

- the subgroup $GL_n((\mathbb{Z} / N \mathbb{Z})^2)$ constitutes the representation of the coset representatives of the tensor product $T_R(n; q) \otimes T_L(n; q)$ of Hecke operators [29];
• \( P_n(L_{v_1}) \) is the standard parabolic subsemigroup over the set \( L_{v_1} = \{ L_{v_1}^1, \cdots, L_{v_1}^{i_{m_i}}, \cdots, L_{v_1}^{q_{m_q}} \} \) of irreducible completions \( L_{v_1} \) having a rank \( N \). \( P_n(L_{\pi} \times L_{v_1}) \) is then a bilinear parabolic subsemigroup constituting the smallest connected pseudoramified normal bilinear subsemigroup of \( GL_n(L_{\pi} \times L_v) \) and representing the \( n \)-fold product \( I_{L_{\pi_i}} \times I_{L_{v_i}} \) of global inertia subgroups.

The double coset decomposition \( \partial S_{G_{R \times L}} \), corresponding to a Shimura bisemivariety and restricted to the lower (resp. upper) half space, becomes:

\[
\partial S_{G_R} = P_n(L_{\pi^1}) \setminus T_n^d(L_{R_d}^+) / T_n^d(\mathbb{Z} / N \mathbb{Z}) \\
\text{(resp.} \quad \partial S_{G_L} = P_n(L_{v^1}) \setminus T_n(L_{L_d}^+) / T_n(\mathbb{Z} / N \mathbb{Z}) \text{)}.
\]

**Proposition 2.2** The (bi)cosets of the bilinear quotient semigroup \( GL_n(L_{\pi}^+ \times L_{L_d}^+) / GL_n((\mathbb{Z} / N \mathbb{Z})^2) \) coincide with the conjugacy classes of the general bilinear semigroup \( GL_n(L_{\pi} \times L_v) \) with respect to the smallest connected pseudoramified normal bilinear subsemigroup given by the bilinear parabolic subsemigroup \( P_n(L_{\pi} \times L_{v^1}) \).

**Sketch of the proof:** According to 1.5, the conjugacy classes of \( GL_n(L_{\pi} \times L_v) \) are in one-to-one correspondence with the (bi)places of \( L_{\pi} \times L_v \). And, on the other hand, the bilinear subsemigroup \( GL_n((\mathbb{Z} / N \mathbb{Z})^2) \) is a representation of the tensor product of Hecke operators such that the \( i \)-th (bi)coset representative of \( GL_n((\mathbb{Z} / N \mathbb{Z})^2) \) corresponds to the biplace \( \pi_i \times v_i \) of \( L_{\pi} \times L_v \).

**Proposition 2.3** The bilinear cohomology (semi)group of the Shimura bisemivariety

\[
\partial S_{G_{R \times L}} = P_n(L_{\pi^1} \times L_{v^1}) \setminus GL_n(L_{R_d}^+ \times L_{L_d}^+) / GL_n((\mathbb{Z} / N \mathbb{Z})^2)
\]

has its coefficient system given by the bisemisheaf \( (\widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}) \) and is given by the bilinear Eisenstein cohomology:

\[
H^{2j}(\partial S_{G_{R \times L}}, \widetilde{M}_R^{2j} \otimes \widetilde{M}_L^{2j}) \simeq \text{Repsp}(GL_{2j}(L_{\pi} \times L_v)), \quad 2j \leq r,
\]

which:

• is in bijection with the representation space \( \text{Repsp}(GL_{2j}(L_{\pi} \times L_v)) \) of the bilinear general semigroup \( GL_{2j}(L_{\pi} \times L_v) \);

• decomposes according to the conjugacy classes of \( GL_{2j}(L_{\pi} \times L_v) \).
Proof:

1. It was demonstrated in \[29\] that the bilinear Eisenstein cohomology \(H^n(\partial S_{G_R \times L}, \widetilde{M}^{2j}_R \otimes \widetilde{M}^{2j}_L)\) is in bijection with the representation of the bilinear general semigroup \(GL_{2j}(L_{\tau} \times L_{\nu})\); this results from the fact that the Eisenstein bilinear cohomology can be deduced from the Weil bilinear algebra of the Lie bilinear nilpotent semialgebra.

2. As the bicosets of \(\partial S^{(2j)}_{G_R \times L} = P_{2j}(L_{\tau} \times L_{\nu}) \setminus GL_{2j}(L^+_{R_d} \times L^+_L) / GL_{2j}(\mathbb{Z} / N \mathbb{Z}^2)\) coincide with the conjugacy classes of \(GL_{2j}(L_{\tau} \times L_{\nu})\), we have that the bilinear Eisenstein cohomology decomposes according to:

\[
H^{2j}(\partial S_{G_R \times L}, \widetilde{M}^{2j}_{R_L} \otimes \widetilde{M}^{2j}_{L_L}) \simeq \bigoplus_{i=1}^q \bigoplus_{m_i} (\widetilde{M}^{2j}_{L_{i_{L_{m_i}}}} \otimes \widetilde{M}^{2j}_{L_{i_{L_{m_i}}}}).
\]

To each \(T_{2j}(L_{v_i})\)-subsemimodule \(M^{2j}_{v_i}\) (resp. \(T^{*}_{2j}(L_{\nu_i})\)-subsemimodule \(M^{2j*}_{\nu_i}\)) is associated a weight \(\lambda_{L_{i}}\) (resp. \(\lambda_{R_{i}}\)) characterizing the \(i\)-th left (resp. right) Hecke sublattice. Indeed, there exists the surjective morphism:

\[
i_{M_{L,R}} : M^{2j}_{L,R} \longrightarrow \Lambda_{L,R}
\]

from the \(T_{2j}(L_{v_i})\)-semimodule \(M^{2j}_{L_{i}}\) (resp. \(T^{*}_{2j}(L_{\nu_i})\)-semimodule \(M^{2j*}_{\nu_i}\)) into the \(T_{2j}((\mathbb{Z} / N \mathbb{Z}))\)-semimodule \(\Lambda_{L}\) (resp. \(T^{*}_{2j}((\mathbb{Z} / N \mathbb{Z}))\)-semimodule \(\Lambda_{R}\)) which is a left (resp. right) Hecke lattice decomposing according to the conjugacy classes of \(T_{2j}(L_{v})\) (resp. \(T^{*}_{2j}(L_{\nu})\))

\[
\Lambda_{L} = \bigoplus_{i=1}^q \bigoplus_{m_i} \Lambda_{L_{i,m_i}} \quad \text{(resp. } \Lambda_{R} = \bigoplus_{i=1}^q \bigoplus_{m_i} \Lambda_{R_{i,m_i}} \text{)}
\]

where \(\Lambda_{L_{i,m_i}}\) (resp. \(\Lambda_{R_{i,m_i}}\)) is the \(i\)-th left (resp. right) Hecke sublattice having multiplicity \(\text{sup}(m_i)\).

Indeed, on each left (resp. right) weight \(\lambda_{L_{i}}\) (resp. \(\lambda_{R_{i}}\)), which is a character of \(\text{Rep}(T_{2j}(L_{v_i}))\) (resp. \(\text{Rep}(T^{*}_{2j}(L_{\nu_i}))\)), there is the action of the Weyl semigroup \(W_{L}\) (resp. \(W_{R}\)) given by:

\[
\phi(s_{i_L}) = w_{i_L} \lambda_{L_{i}} \quad \text{(resp. } \phi(s_{i_R}) = w_{i_R} \lambda_{R_{i}} \text{)}
\]

where

- \(\phi(s_{i_L})\) (resp. \(\phi(s_{i_R})\)) is a left (resp. right) Hecke character;

- \(w_{i_L} \in W_{L}, w_{i_R} \in W_{R}\)
The left (resp. right) action of the Weyl group $W_L$ (resp. $W_R$) consists in generating the multiplicities of the Hecke sublattices $\Lambda_{Li}$ (resp. $\Lambda_{Ri}$) to which correspond the subsemimodules $M_{v_i,m_i}$ (resp. $M_{\overline{v}_i,m_i}$).

**Corollary 2.4** The general bilinear Eisenstein cohomology is characterized by the Künneth isomorphism:

\[ H^E_{R \times (D) L} : H^{2j}(\partial S_{G_R}, \tilde{M}^{2j}_R) \times (D) H^{2j}(\partial S_{G_L}, \tilde{M}^{2j}_L) \sim H^{2j}(\partial S_{G_R \times L}, \tilde{M}^{2j}_R \otimes (D) \tilde{M}^{2j}_L). \]

**Sketch of proof:** this is equivalent to defining the diagonal or complete product between a right and a left linear Eisenstein cohomology semigroup.

**Definition 2.5** Complete reducibility of $GL_{2n}(L_v \times L_v)$ [29]: Let

\[ n_L = 1_{1L} + \cdots + 1_{kL} + \cdots + 1_{\ell L} + \cdots + 1_{nL} \]

(resp. $n_R = 1_{1R} + \cdots + 1_{kR} + \cdots + 1_{\ell R} + \cdots + 1_{nR}$)

be a left (resp. right) partition of $n_L$ (resp. $n_R$) labeling the irreducible representations of $\mathbb{T}_{2nL}(L_v)$ (resp. $\mathbb{T}_{2nR}(L_v)$).

Then,

1. $\text{Rep}(GL_{2n-2_1+\cdots+2_\ell+\cdots+2_n}(L_v \times L_v)) = \bigoplus_{2_1=2}^{2n} \text{Rep}(GL_{2\ell}(L_v \times L_v))$

   constitutes a completely reducible orthogonal bilinear representation of $GL_{2n}(L_v \times L_v)$;

2. $\text{Rep}(GL_{2nR \times L}(L_v \times L_v))$

   \[ = \bigoplus_{2_\ell R=2_\ell L=2}^{2n} \text{Rep}(GL_{2\ell R \times L}(L_v \times L_v)) \bigoplus \text{Rep}(T_{2kR}(L_v) \times T_{2\ell L}(L_v)), \]

   where $GL_{2\ell R \times L}$ is another notation for $G_{2\ell}$, constitutes a completely reducible nonorthogonal bilinear representation of $GL_{2n}(L_v \times L_v)$.

**Proposition 2.6** Let $\tilde{M}^{2n}_L$ (resp. $\tilde{M}^{2n}_R$) be a 2n-dimensional semisheaf on the $\mathbb{T}_{2n(L_v)}$-semimodule (resp. $\mathbb{T}_{2n}(L_\overline{\nu})$-semimodule).

Let $\partial S_{G_{2n-2_1+\cdots+2_\ell+\cdots+2_n}}$ and $\partial S_{G_{2nR \times 2nL}}$ denote respectively a completely reducible orthogonal and nonorthogonal Shimura bisemivariety instead of $\partial S_{G_{R \times L}}$.

Then, the 2n-th bilinear Eisenstein cohomologies decompose into direct sums of completely irreducible orthogonal and nonorthogonal bilinear Eisenstein cohomologies according to:
\[ H^{2n}(\partial S^{P_{2n}}_{G_{2n}}(\partial S^{P_{2n}}_{G_{2n}})\otimes D\tilde{M}^n_L) \]
\[ = \bigoplus_{\ell_R=\ell_L} H^2(\partial S^{P_{2\ell_R}\ell_L}_{G_{2\ell_R}}, M^2_{R,2\ell_L}\otimes M^2_{L,2\ell_L}) \]
\[ \simeq \text{Repsp}(GL_{2n}(L_{\tau} \times L_v)). \]

\[ H^{2n}(\partial S^{P_{2n}\times 2n}_{G_{2n}\times 2n}) \]
\[ = \bigoplus_{\ell_R=\ell_L} H^2(\partial S^{P_{2\ell_R\times 2\ell_L}}_{G_{2\ell_R}}, M_{R,2\ell_L}^{2\ell_R}\otimes M_{L,2\ell_L}^{2\ell_R}) \]
\[ \bigoplus_{k_R\neq \ell_L} H^{2k_R\times 2\ell_L}(\partial S^{P_{2k_R\times 2\ell_L}}_{G_{2k_R}}, M_{R,2\ell_L}^{2k_R}\otimes M_{L,2\ell_L}^{2\ell_L}) \]
\[ \simeq \text{Repsp}(GL_{2n}\times L_{\tau} \times L_v). \]

where \( \tilde{M}^{2\ell_L} \) is a semisheaf on the \( T_{2\ell_L}(L_v) \)-semimodule.

Proof:

1. The completely reducible orthogonal and nonorthogonal Shimura bisemivarieties are given respectively by:
\[ \partial S^{P_{2n}}_{G_{2n}} = \bigoplus_{\ell_R=\ell_L} \partial S^{P_{2\ell_R\ell_L}}_{G_{2\ell_R}}(L_{\tau_R} \times L_{\tau_L}) \setminus GL_{2\ell_R}(L_{\tau_R}^\times \times L_{\tau_L}^\times)/(Z/NZ)^2 \]

and by:
\[ \partial S^{P_{2n}\times 2n}_{G_{2n}\times 2n} = \bigoplus_{\ell_R=\ell_L} \partial S^{P_{2\ell_R\times 2\ell_L}}_{G_{2\ell_R}} \bigoplus_{k_R\neq \ell_L} \partial S^{P_{2k_R\times 2\ell_L}}_{G_{2k_R}}. \]

2. The decomposition of the \( 2n \)-th bilinear Eisenstein cohomology into completely irreducible two-dimensional bilinear Eisenstein cohomologies results from its bijection with \( \text{Repsp}(GL_{2n}(L_{\tau} \times L_v)) \) or with \( \text{Repsp}(GL_{2n}\times L_{\tau} \times L_v) \) according to definition 2.5.

3. Every two-dimensional Eisenstein bilinear cohomology decomposes with respect to the places in \( (L_{\tau} \times L_v) \) according to one-dimensional components:
\[ H^2(\partial S^{P_{2\ell_R\times 2\ell_L}}_{G_{2\ell_R}}, M_{R,2\ell_L}^{2\ell_R}\otimes M_{L,2\ell_L}^{2\ell_R}) \simeq \bigoplus_{i=1}^{q} \bigoplus_{m_i} (M_{\tau_i,m_i}^{2\ell_R} \otimes M_{v_i,m_i}^{2\ell_L}). \]
Definition 2.7  Cuspidal representation in terms of global elliptic semimodules: The decomposition of the Eisenstein bilinear cohomology into one-dimensional irreducible components needs a cuspidal automorphic representation in terms of global elliptic bisemimodules.

Assume that \( f_L \) is a normalized eigenform (of a Hecke operator), holomorphic in the Poincare upper half plane \( H \) in \( \mathcal{O} \), and defined in \( \{ \text{Im}(z_L) > 0 \} \). \( f_L \), expanded in formal power series \( f_L = \sum_{i=1}^{q} a_i q^i \), where \( q_L = e^{2\pi i z_L} \), \( z_L \in \mathcal{O} \), is a cusp form of the space \( S_L(N) \) and is an eigenvector of the Hecke operators \( T_{qL} \) for \( q \nmid N \) and \( U_{qL} \) for \( q \mid N \) where \( N \) is a positive integer. The Fourier coefficients \( a_i \) are eigenvalues \( c(i, f_L) \) of Hecke operators such that \( c(i, f_L) \) generate the ring of integers \( \theta_L \) which leads to consider \( S_L(N) \) as a \( \theta_L \)-algebra.

The ring of endomorphisms acting on \( S_L(N) \) composed of dual cusp forms \( \phi \) as a \( \mathcal{O} \)-algebra.

Definition 2.8 The decomposition group:

\[
\sum_{i} \sum_{m_i} \phi(s_{iR,L})_{i,m_i} q_{iR,L}^i
\]

where \( \sum_i \) runs over the one-dimensional sections of \( \widehat{M}^L_{R,L} \) and where \( \sum_m \) runs over the ideals of the decomposition group \( D_{i^2} \) of the biplace \( \overline{v_i} \times v_i \).

Then, the space \( S_{R,L}(\phi_{R,L}) \) of global elliptic \( G_{s_{R,L}} \)-semimodules \( \phi_{R,L}(s_{R,L}) \) is included into the space \( S_{R,L}(N) \) of cusp forms \( f_{R,L} : S_{R,L}(\phi_{R,L}) \leftrightarrow S_{R,L}(N) \) implying that \( f_{R,L} \simeq \phi_{R,L}(s_{R,L}) \).

Definition 2.8  The decomposition group: The ring of endomorphisms acting on the global elliptic \( G_{s_{R,L}} \)-semimodules included into weight two cusp forms is generated over \( \mathbb{Z}/N \mathbb{Z} \) by the Hecke operators \( T_{qR,L} \) for \( q \nmid N \) and \( U_{qR,L} \) for \( q \mid N \). The coset representatives of \( U_{qL} \) are upper triangular and are given by the integral matrices \( \left( \begin{array}{cc} 1 & b_N \\ 0 & q_N \end{array} \right) \) while the coset representatives of \( U_{qL} \) are lower triangular and are given by the matrices \( \left( \begin{array}{cc} 1 & 0 \\ b_N & q_N \end{array} \right) \). For a general integer \( r = a \cdot d \), we would have respectively the integral matrices.
Proposition 2.10 According to the Langlands bilinear global program [29] and proposition 2.6, every two-dimensional Eisenstein bilinear cohomology is in bijection with a global elliptic semimodule $\phi_{R,L}(s_{R,L})$, i.e. $\phi_{q_{|q^2}}$ are such that they are coefficients of the global elliptic semimodule $\phi_{q_{|q^2}}$. Then, $GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2})$ corresponds to a Gauss decomposition of the class “$q^2_N$”.

Proposition 2.9 The eigenvalues $\lambda_{\pm}(q^2_N, b^2_N)$ of the coset representatives $GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2})$ of $U_{q_{|q^2}}\times U_{q_{|q^2}}$ are such that they are coefficients of the global elliptic $G_{q_{|q^2}}$-semimodules $\phi_{q_{|q^2}}$, i.e. $\phi_{q_{|q^2}} \equiv \lambda_{\pm}(q^2_N, b^2_N)$. Then, the one-dimensional components of the global elliptic semimodule $\phi_{q_{|q^2}}$ are one-dimensional semitori localized respectively in the upper and in the lower half space and characterized by radii given by $r(q^2_N, b^2_N) = (\lambda_{\pm}(q^2_N, b^2_N) - \lambda_{\pm}(q^2_N, b^2_N))/2$.

Proof: The eigenvalues of $GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2}) = \alpha_{q^2_N} \cdot D_{q^2_N}$, are

$$\lambda_{\pm}(q^2_N, b^2_N) = \frac{1 + b^2_N + q^2_N}{2} \pm \sqrt{(1 + b^2_N + q^2_N)^2 - 4q^2_N}$$

and verify

$$\text{Trace}(GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2})) = 1 + b^2_N + q^2_N,$$

and

$$\det(GL_2((\mathbb{Z}/N\mathbb{Z})_{|q^2})) = \lambda_{\pm}(q^2_N, b^2_N) \cdot \lambda_{\pm}(q^2_N, b^2_N) = q^2_N.$$

Proposition 2.10 According to the Langlands bilinear global program [29] and proposition 2.6, every two-dimensional Eisenstein bilinear cohomology is in bijection with a global elliptic $G_{q_{|q^2}}$-bisemimodule $\phi_{R}(s_{R}) \otimes \phi_{L}(s_{L})$:

$$H^2((\partial S_{G_{q_{|q^2}}^{\ell_{R},\ell_{L}}} M_{\ell_{R}}^{\ell_{R},\ell_{L}}), \mathbb{M}_{\ell_{R}}^{\ell_{R},\ell_{L}}) \cong \bigoplus_{q=1}^{\varphi} \bigoplus_{m_i} \left( \mathbb{M}_{\ell_{R}}^{\ell_{R},\ell_{L}} \otimes \mathbb{M}_{\ell_{L}}^{\ell_{L},\ell_{L}} \right) \cong \phi_{R}(s_{R}) \otimes \phi_{L}(s_{L})$$

$$= \sum_{i=1}^{q} \sum_{m_i} \lambda_{\pm}(i^2_{N_i}, m^2_{i}) e^{-2\pi i(i)x} \otimes \sum_{i=1}^{q} \sum_{m_i} \lambda_{\pm}(i^2_{N_i}, m^2_{i}) e^{2\pi i(i)x}$$
in such a way that the \( i \)-th bisection on the \( GL_2(\mathbb{L}_N \times \mathbb{L}_v) \)-subbisimodule \( (M_{R,L}^2 \otimes M_{L}^2) \) in \( H^{2\xi} (\partial S_{\mathbb{G}_{2,R}^2}) \mathbb{M}^2_{R,L} \otimes \mathbb{M}^2_{L} \) be in one-to-one correspondence with the set of \( m_i \) biterms \( \{ \lambda_+ (i \gamma, m_i^2) e^{-2\pi i x} \times \lambda_-(i \gamma, m_i^2) e^{2\pi i x} \}_{m_i} \) of the global elliptic bisimodule \( \phi_R(s_R) \otimes \phi_L(s_L) \).

So, the global elliptic bisimodule constitutes the cuspidal representation of the Eisenstein bilinear cohomology.

**Definition 2.11**  The semialgebra of elliptic operators \( \text{Ell}_{R,L}(\mathbb{M}_{R,L}) \) is the semialgebra of linear differential operators \( D_{R,L} \) defined on the space \( \Gamma_{R,L}(\mathbb{M}_{R,L}) \) of smooth sections \( s_{R,L} \) of \( \mathbb{M}_{R,L} \) and having their principal symbol \( \sigma(D_{R,L}) \) invertible [3].

**Proposition 2.12**  The bilinear Hilbert semispace \( H^\pm_a \) is the natural representation space for the semialgebra of elliptic operators.

**Proof:** Taking into account the \( B_L \circ p_L \) (resp. \( B_R \circ p_R \)) map as introduced in definitions 1.8, the bisimisheaf \( \mathbb{M}_{R,L} \otimes (\mathbb{D}) \mathbb{M}_L \) on the \( GL_2(\mathbb{L}_N \times \mathbb{L}_v) \)-bisimodule \( M_R \otimes (\mathbb{D}) M_L \) will be transformed into an extended internal or internal left (resp. right) Hilbert bisimisheaf according to:

\[
B_L \circ p_L : \quad \mathbb{M}_R \otimes (\mathbb{D}) \mathbb{M}_L \longrightarrow \mathbb{M}_{LR} \otimes (\mathbb{D}) \mathbb{M}_L \equiv \mathbb{M}_L ,
\]

\[
B_R \circ p_R : \quad \mathbb{M}_R \otimes (\mathbb{D}) \mathbb{M}_L \longrightarrow \mathbb{M}_{RL} \otimes (\mathbb{D}) \mathbb{M}_R \equiv \mathbb{M}_R .
\]

Consequently, \( \text{Ell}_R(\mathbb{M}_R) \otimes (\mathbb{D}) \text{Ell}_L(\mathbb{M}_L) \) acting on \( \mathbb{M}_{LR} \otimes (\mathbb{D}) \mathbb{M}_L \) (resp. \( \mathbb{M}_{RL} \otimes (\mathbb{D}) \mathbb{M}_R \)) will be an algebra of bioperators (or a semialgebra of operators) acting on an extended internal or internal left (resp. right) bilinear Hilbert semispace \( H^+_a \) (resp. \( H^-_a \)) or \( H^+_a \) (resp. \( H^-_a \)) and will be noted:

\[
[\text{Ell}_R(\mathbb{M}_R) \otimes (\mathbb{D}) \text{Ell}_L(\mathbb{M}_L)](H^\pm_a)
\]

or \( [\text{Ell}_R(\mathbb{M}_R) \otimes (\mathbb{D}) \text{Ell}_L(\mathbb{M}_L)](H^\pm_a) \).

On the other hand, a semialgebra of operators \( \text{Ell}_{R,L}(\mathbb{M}_{R,L}) \) acting on \( H^\pm_a \) or \( H^\pm_a \) will be given by \( \text{Ell}_{R,L}(\mathbb{M}_{R,L})(H^\pm_a) \) or \( \text{Ell}_{R,L}(\mathbb{M}_{R,L})(H^\pm_a) \) in such a way that \( \text{Ell}_R(\mathbb{M}_R) \) (resp. \( \text{Ell}_L(\mathbb{M}_L) \)) be a semialgebra of right (resp. left) elliptic linear operators acting on the set of sections of the semisheaf \( \mathbb{M}_R \) (resp. \( \mathbb{M}_L \)) over the \( G_R(\mathbb{L}_N) \) (resp. \( G_L(\mathbb{L}_v) \)) -semimodule \( M_R \) (resp. \( M_L \)) of \( H^\pm_a \) or \( H^\pm_a \), where \( G_R \) (resp. \( G_L \)) is another notation for \( T^l_n \) (resp. \( T^r_n \)).

Taking into account the considerations given about the enveloping algebras in definition 1.1, it then becomes clear that the extended bilinear Hilbert semispace \( H^\pm_a \) is the natural representation space for the bisimialgebra and the semialgebra of elliptic operators.
Definitions 2.13  

a) Semialgebra of bounded operators: If \( \mathcal{L}^{B}_{R,L}(\widetilde{M}_{R,L}) \) denotes the semialgebra of right (resp. left) operators on the semisheaf \( \widetilde{M}_{R,L} \) over the \( G_{R,L}(\pi) \)-semimodule \( M_{R,L} \), then the semialgebra of right (resp. left) self-adjoint bounded operators \( T_{R,L} \) on \( H_{a}^{\pm} \) or \( \mathcal{H}_{a}^{\mp} \) will be given by: \( \mathcal{L}^{B}_{R,L}(H_{a}^{\pm}) \) and \( \mathcal{L}^{B}_{R,L}(\mathcal{H}_{a}^{\mp}) \), while the bisemialgebra of self-adjoint bounded operators on \( H_{a}^{\mp} \) and on \( \mathcal{H}_{a}^{\pm} \) will be: \( \left( \mathcal{L}^{B}_{R} \otimes \mathcal{L}^{B}_{L}(H_{a}^{\pm}) \right) \) and \( \left( \mathcal{L}^{B}_{R} \otimes \mathcal{D} \mathcal{L}^{B}_{L}(\mathcal{H}_{a}^{\pm}) \right) \) such that the right and left self-adjoint bounded operators \( T_{R,L} \in \mathcal{L}^{B}_{R,L} \) act respectively on the right and left semisheaves of \( H_{a}^{\mp} \) and \( \mathcal{H}_{a}^{\pm} \).

b) A weight on a semialgebra \( \mathcal{L}^{B}_{R,L}(H_{a}^{\pm}) \) is given by the positive bilinear form \( (T_{R}s_{i_{R}},s_{i_{L}}) \) or \( (s_{i_{l_{R}}},T_{L}s_{i_{L}}) \) which is a map from \( \mathcal{L}^{B}_{R,L}(\widetilde{M}_{LR} \times \widetilde{M}_{L}) \) into \( \mathcal{C} \) for every section \( s_{i_{l_{R}}} \in \widetilde{M}_{LR} \) and \( s_{i_{L}} \in \widetilde{M}_{L} \).

Similarly, a weight on a bisemialgebra \( \left( \mathcal{L}^{B}_{R} \otimes \mathcal{L}^{B}_{L}(H_{a}^{\pm}) \right) \) will be given by the positive bilinear form \( (T_{R}s_{i_{l_{R}}},T_{L}s_{i_{L}}) \) which is a map from \( \left( \mathcal{L}^{B}_{R}(\widetilde{M}_{LR}) \times \mathcal{L}^{B}_{L}(\widetilde{M}_{L}) \right) \) into \( \mathcal{C} \) for all \( T_{R,L} \in \mathcal{L}^{B}_{R,L} \).

Definition 2.14  

Complex analytic semivariety: Let \( \overline{X}_{S_{R}} \) (resp. \( \overline{X}_{S_{L}} \)) denote the right (resp. left) complex semispace compactified from \( X_{S_{R}} = GL_{n}(L_{R})/GL_{n}(\mathbb{Z}/N \mathbb{Z}) \) (resp. \( X_{S_{L}} = GL_{n}(L_{L})/GL_{n}(\mathbb{Z}/N \mathbb{Z}) \)) being the right (resp. left) complex (lattice) analytic semivariety introduced in section 2.1 and covered by \( \partial \overline{X}_{S_{R}} \) (resp. \( \partial \overline{X}_{S_{L}} \)) or by \( \partial \overline{S}_{G_{R}} \) (resp. \( \partial \overline{S}_{G_{L}} \)).

Let \( \widetilde{M}_{R,L} \) be an analytic semisheaf on \( \overline{X}_{S_{R}} \) (resp. \( \overline{X}_{S_{L}} \)).

Then, the analytic de Rham cohomology \( H^{*}(\overline{X}_{S_{R,L}},\widetilde{M}_{R,L}) \) can be computed through the analytic de Rham complex taking into account that:

Lemma 2.15  

There is an isomorphism between the (algebraic) Eisenstein cohomology \( H^{*}(\partial \overline{S}_{G_{R,L}},\widetilde{M}_{R,L}) \) and the analytic de Rham cohomology \( H^{*}(\partial \overline{X}_{S_{R,L}},\widetilde{M}_{R,L}) \).

Proof: Indeed, the isomorphism between the following two de Rham cohomologies of \( \Omega^{*} \)-smooth differential forms with respect to \( \partial \overline{S}_{G_{R,L}} \) and \( \overline{X}_{S_{R,L}} \) [20], [8], [12]:

\[
H^{*}(\Omega^{*}_{\partial \overline{S}_{G_{R,L}}}) \simeq H^{*}(\Omega^{*}_{\overline{X}_{S_{R,L}}})
\]

leads naturally to the following isomorphism:

\[
H^{*}(\partial \overline{S}_{G_{R,L}},\widetilde{M}_{R,L}) \simeq H^{*}(\overline{X}_{S_{R,L}},\widetilde{M}_{R,L}).
\]

Definition 2.16  

Analytic bilinear Hilbert semispaces: From the complete (resp. diagonal) bilinear tensor [30] product between the right and left analytic semisheaves \( \widetilde{M}_{R} \),
and \( \widetilde{M}^s_L \), we can construct a left (resp. right) analytic bisemisheaf \( \widetilde{M}^s_{L(D)} \) (resp. \( \widetilde{M}^s_{R(D)} \)) of a left (resp. right) analytic bilinear Hilbert semispace \( H^+_h \) or \( \mathcal{H}^+_h \) (resp. \( H^-_h \) or \( \mathcal{H}^-_h \)) in complete analogy with which was done in definition 1.8.

**Proposition 2.17**  The analytic bilinear Hilbert semispace \( H^\pm_h \) is the natural representation space for the (bi)semialgebras of elliptic operators: \( \text{Ell}_{R,L}(\widetilde{M}^s_{R,L}) \) and \( (\text{Ell}_R(\widetilde{M}^s_R) \otimes_{(D)} \text{Ell}_L(\widetilde{M}^s_L)) \).

**Proof:** This results from definitions 2.12 and 1.1.

**Definitions 2.18**

a) **Serre-Swan theorem:** Let \( \widetilde{M}^{\text{top}}_{R,L} = C(X_{R,L}) \) be the semialgebra of continuous functions on a compact (semi)space \( X_{R,L} \). We shall denote by \( \text{VEC}(X_{R,L}) \) the category of complex vector bundles over \( X_{R,L} \) and \( P(\widetilde{M}^{\text{top}}_{R,L}) \) the category of finitely generated projective right (resp. left) semimodules \( P^{\text{top}}_{R,L} \) over \( \Gamma(\text{VEC}(X_{R,L})) \).

Then, the Serre-Swan theorem asserts that the categories \( \text{VEC}(X_{R,L}) \) and \( P^{\text{top}}_{R,L}(\Gamma(\text{VEC}(X_{R,L}))) \) are equivalent [36].

b) **The bisemialgebra** \( C(X_R \times_{(D)} X_L) \): Let \( x_{R,L} \) be a right (resp. left) point of the right (resp. left) semialgebra \( \widetilde{M}^{\text{top}}_{R,L} \).

The complete (resp. diagonal) tensor product between the right and left semialgebras \( \widetilde{M}^{\text{top}}_R \) and \( \widetilde{M}^{\text{top}}_L \) can be defined by:

\[
T^\text{top}_X : \{ \widetilde{M}^{\text{top}}_R, \widetilde{M}^{\text{top}}_L \} \rightarrow \widetilde{M}^{\text{top}}_R \otimes_{(D)} \widetilde{M}^{\text{top}}_L,
\]

\[
\{ x_R, x_L \} \rightarrow x_R \times_{(D)} x_L,
\]

so that the bipoint \( x_R \times_{(D)} x_L \) be characterized by a complete (resp. diagonal) signature. \( \widetilde{M}^{\text{top}}_R \otimes_{D} \widetilde{M}^{\text{top}}_L \) is then a finitely generated bisemialgebra.

c) **Topological bilinear Hilbert semispace:** By application of the \((B_L \circ p_L)\) (resp. \(B_R \circ p_R\)) linear map, the bisemialgebra \( \widetilde{M}^{\text{top}}_R \otimes_{(D)} \widetilde{M}^{\text{top}}_L \) can be transformed into an extended internal or internal left (resp. right) topological Hilbert bisemisheaf \( \widetilde{M}^{\text{top}}_{L,D} \) (resp. \( \widetilde{M}^{\text{top}}_{R,D} \)) which becomes an extended internal or internal left (resp. right) topological bilinear Hilbert semispace \( H^\pm_{\text{top}} \) or \( \mathcal{H}^\pm_{\text{top}} \) if it is endowed with a complete or a diagonal bilinear form with values in \( \mathcal{C} \).

**Proposition 2.19**  The extended internal and internal left (resp. right) topological bilinear Hilbert semispaces \( H^\pm_{\text{top}} \) and \( \mathcal{H}^\pm_{\text{top}} \) are \( C^*\)-(bi)semialgebras.
**Proof:** By definition $\tilde{M}_{R,L}^{\text{top}}$ is a right (resp. left) semialgebra $C(X_{R,L})$ of continuous sections $s_{R,L}^{\text{top}}(X_{R,L})$ on $X_{R,L}$.

Now, the bisemialgebra $\tilde{M}_{L(D)}^{\text{top}}$ or $\tilde{M}_{R(D)}^{\text{top}}$ is an involutive bisemialgebra over $\mathcal{C}$ of continuous bifunctions $s_{R}^{\text{top}}(X_{R}) \otimes (D) s_{L}^{\text{top}}(X_{L})$. Indeed, the involution, which must be taken into account, is a bilinear map transforming $H_{\text{top}}^{+}$ or $H_{\text{top}}^{+}$ (resp. $H_{\text{top}}^{-}$ or $H_{\text{top}}^{-}$) into $H_{\text{top}}^{+}$ or $H_{\text{top}}^{-}$ (resp. $H_{\text{top}}^{+}$ or $H_{\text{top}}^{-}$).

Recall the composition of maps:

$$B_{L} \circ p_{L} : \tilde{M}_{R}^{\text{top}} \otimes (D) \tilde{M}_{L}^{\text{top}} \longrightarrow \tilde{M}_{R}^{\text{top}} \otimes (D) \tilde{M}_{R}^{\text{top}} \equiv \tilde{M}_{L(D)}^{\text{top}};$$

$$B_{R} \circ p_{R} : \tilde{M}_{R}^{\text{top}} \otimes (D) \tilde{M}_{L}^{\text{top}} \longrightarrow \tilde{M}_{L}^{\text{top}} \otimes (D) \tilde{M}_{R}^{\text{top}} \equiv \tilde{M}_{R(D)}^{\text{top}},$$

as introduced in definitions 1.8.

So, the bilinear map:

$$(p_{L}^{-1} \circ B_{L}^{-1}) \otimes (D) (B_{R} \circ p_{R}) : \tilde{M}_{L(D)}^{\text{top}} \longrightarrow \tilde{M}_{R(D)}^{\text{top}},$$

$$(B_{R} \circ p_{R}) \otimes (D) (p_{L}^{-1} \circ B_{L}^{-1}) : \tilde{M}_{R(D)}^{\text{top}} \longrightarrow \tilde{M}_{L(D)}^{\text{top}},$$

transforms the bisemialgebra $\tilde{M}_{L(D)}^{\text{top}}$ (resp. $\tilde{M}_{R(D)}^{\text{top}}$) into the bisemialgebra $\tilde{M}_{R(D)}^{\text{top}}$ (resp. $\tilde{M}_{L(D)}^{\text{top}}$) which corresponds to an antilinear involution transforming the left (resp. right) bilinear Hilbert semispace $H_{\text{top}}^{+}$ or $H_{\text{top}}^{+}$ (resp. $H_{\text{top}}^{-}$ or $H_{\text{top}}^{-}$) into the right (resp. left) involuted bilinear Hilbert semispace $H_{\text{top}}^{-}$ or $H_{\text{top}}^{-}$ (resp. $H_{\text{top}}^{+}$ or $H_{\text{top}}^{+}$).

**Definitions 2.20 a) K-functor of Kasparov [25]:** We are now interested in extensions of the bisemialgebra $\tilde{M}_{R,L}^{\text{top}}$. Let $L_{R,L}^{B}(\tilde{M}_{R,L}^{\text{top}})$ denote the semialgebra of bounded operators on $\tilde{M}_{R,L}^{\text{top}}$, and let $\mathcal{K}_{R,L}$ be the ideal of compact operators.

The set of extension classes of $\mathcal{K}_{R,L}$ by $L_{R,L}^{B}(\tilde{M}_{R,L}^{\text{top}})$, noted $\text{Ext}(L_{R,L}^{B}(\tilde{M}_{R,L}^{\text{top}}), \mathcal{K}_{R,L})$, is an abelian semigroup naturally isomorphic to $\text{Ext}(X_{R,L})$ as developed by Brown, Douglas and Fillmore [10], [11].

In connection with the work of Atiyah [4], [5], G.G. Kasparov constructed a general $K$-functor $K_{*}K_{*}(\tilde{M}_{R,L}^{\text{top}}, L_{R,L}^{B})$, special cases of which are the ordinary cohomological $K$-functor $K_{*}(\tilde{M}_{R,L}^{\text{top}})$ and the homological $K$-functor $K_{*}(L_{R,L}^{B})$.

Especially interesting is the case where the $C^{*}$-semialgebras $\tilde{M}_{R,L}^{\text{top}}$ and $L_{R,L}^{B}$ are equipped with the continuous action of a locally compact semigroup $G_{R,L}^{\text{c}}$. This allows to define an abelian group $KK_{R,L}^{G_{R,L}, G_{R,L}^{\text{c}}}(\tilde{M}_{R,L}^{\text{top}}, L_{R,L}^{B})$ [26].

**b) Bisemialgebra of bounded operators:** Considering the $C^{*}$-bisemialgebra $\tilde{M}_{L(D)}^{\text{top}}$ (resp. $\tilde{M}_{R(D)}^{\text{top}}$), the bisemialgebra of bounded operators on it will be $(L_{R}^{B} \otimes (D) L_{L}^{B})(\tilde{M}_{L(D)}^{\text{top}})$.
(resp. \((\mathcal{L}_R^B \otimes_D \mathcal{L}_L^B)(\widehat{\mathcal{M}}_{\text{top}}^L)\)) or \((\mathcal{L}_R^B \otimes_D \mathcal{L}_L^B)(H_{\text{top}}^\mp)\) (resp. \((\mathcal{L}_R^B \otimes_D \mathcal{L}_L^B)(H_{\text{top}}^\mp)\)) if we envisage their actions on the extended (resp. diagonal) bilinear Hilbert semispace \(H_{\text{top}}^\mp\) (resp. \(H_{\text{top}}^\mp\)).

3 Von Neumann semialgebras and bisemialgebras

Definitions 3.1  a) Norm topology of bounded operators: Let \((\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_{\text{top}}^+)\) be the bisemialgebra of bounded operators acting from the topological extended bilinear Hilbert semispace \(H_{\text{top}}^+\) into itself.

Then, the norm topology for an operator \(T_R \otimes T_L \in \mathcal{L}_R^B \otimes \mathcal{L}_L^B\) will be defined by

\[
\|T_R \otimes T_L\| = \sup \left( \|T_R s_{L_R}^{\text{top}} \otimes T_L s_{L_L}^{\text{top}}\| / \|s_{L_R}^{\text{top}} \otimes s_{L_L}^{\text{top}}\| \right)
\]

for every section \(s_{L_R}^{\text{top}} \in \widehat{\mathcal{M}}_{L_R}^{\text{top}}\) and \(s_{L_L}^{\text{top}} \in \widehat{\mathcal{M}}_{L_L}^{\text{top}} \subset H_{\text{top}}^+\), since, if \(\mathcal{L}_L^B(H_{\text{top}}^+\)) is the semialgebra of left bounded operators acting on the semisheaf \(\widehat{\mathcal{M}}_{L_L}^{\text{top}}\) of \(H_{\text{top}}^+\), the norm topology for a left bounded operator \(T_L\) is given by

\[
\|T_L\| = \sup \left( \|T_L s_{L_L}^{\text{top}}\| / \|s_{L_L}^{\text{top}}\| \right).
\]

b) An involution on the operator \(T_{R,L}\) is defined by

\[
i_R : T_L \longrightarrow T_L^\dagger = T_R,
\]

\[
i_L : T_R \longrightarrow T_R^\dagger = T_L,
\]

such that \((T_R^\dagger s_{L_R}^{\text{top}}, T_L^\dagger s_{L_L}^{\text{top}}) = (T_R s_{L_R}^{\text{top}}, T_L s_{L_L}^{\text{top}})\) making \(T_R\) and \(T_L\) self-adjoint.

Definitions 3.2  Bisemialgebras of von Neumann on extended bilinear Hilbert semispaces:  a) A right (resp. left) semialgebra of von Neumann \(\mathbb{M}_{R,L}(H_{\text{top}}^\mp)\) in the topological extended bilinear Hilbert semispace \(H_{\text{top}}^\mp\) is an involutive subalgebra of \(\mathcal{L}_R^B(H_{\text{top}}^\mp)\) having a closed norm topology [19].

Similarly, a semialgebra of von Neumann \(\mathbb{M}_{R,L}(H_{\text{top}}^\mp)\) in \(H_{\text{top}}^\mp\) is an involutive subsemialgebra of Ell_{R,L}(\widehat{\mathcal{M}}_{R,L}^a) having a closed norm topology.

b) A bisemialgebra of von Neumann \(\mathbb{M}_{R \times L}(H_{\text{top}}^\mp)\) in \(H_{\text{top}}^\mp\) is an involutive subbisemialgebra of \((\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_{\text{top}}^\mp)\) having a closed norm topology [19].

c) A bisemialgebra of von Neumann \(\mathbb{M}_{R \times L}(H_{\text{top}}^\mp)\) in the algebraic extended bilinear Hilbert semispace \(H_{\text{top}}^\mp\) is also an involutive subbisemialgebra of \((\mathcal{L}_R^B \otimes \mathcal{L}_L^B)(H_{\text{top}}^\mp)\) having a closed norm topology.
Proposition 3.3  Let $\mathcal{M}_{R,L}(H^\pm_a)$ and $\mathcal{M}_{R\times L}(H^\pm_a)$ be respectively a semialgebra and a bisemialgebra of von Neumann on the algebraic bilinear extended Hilbert semispace $H^\pm_a$.

Let $\mathcal{M}_{R,L}(H^\pm_h)$ and $\mathcal{M}_{R\times L}(H^\pm_h)$ be respectively a semialgebra and a bisemialgebra of von Neumann on the analytic bilinear extended Hilbert semispace $H^\pm_h$.

Then, we have the isomorphisms:

$$i_{\mathcal{M}_{R,L}^a - \mathcal{M}_{R,L}^h} : \mathcal{M}_{R,L}(H^\pm_a) \longrightarrow \mathcal{M}_{R,L}(H^\pm_h),$$
$$i_{\mathcal{M}_{R\times L}^a - \mathcal{M}_{R\times L}^h} : \mathcal{M}_{R\times L}(H^\pm_a) \longrightarrow \mathcal{M}_{R\times L}(H^\pm_h).$$

Proof: This results immediately from lemma 2.15.

Proposition 3.4  There exists an isomorphism

$$i_{\mathcal{M}_{R,L}^{\{\text{top}\} h} - \mathcal{M}_{R,L}^{\{\text{top}\} h}} : \mathcal{M}_{\{\text{top}\} h}^{\{\text{top}\} h} \longrightarrow \mathcal{M}_{R,L}^{\{\text{top}\} h} (H^\pm_{\{\text{top}\} h})$$

between an algebra of von Neumann $\mathcal{M}_{\{\text{top}\} h}$ on a linear Hilbert space $h_{\{\text{top}\} h}$ [26] and a semialgebra of von Neumann $\mathcal{M}_{R,L}^{\{\text{top}\} h} (H^\pm_{\{\text{top}\} h})$ on the extended bilinear Hilbert semispace $H^\pm_{\{\text{top}\} h}$.

Proof: Let $V_{\{\text{top}\} h}$ be a compact manifold of class $C^0$ (resp. $C^\infty$) associated with $M_{R \otimes (D)} \mathcal{M}_L$ in the sense of [29] and let $A_{\{\text{top}\} h}$ be the corresponding stellar algebra of $C^0$ (resp. $C^\infty$) functions on $V_{\{\text{top}\} h}$ with values in $\mathcal{C}$.

Then, a Fredholm module on $A_{\{\text{top}\} h}$ is essentially given by the involutive representation $\Pi_{\{\text{top}\} h}$ of $A_{\{\text{top}\} h}$ in a linear Hilbert space $h_{\{\text{top}\} h}$ and by a self-adjoint operator $F$.

Furthermore, an algebra of von Neumann $\mathcal{M}_{\{\text{top}\} h}$ in a linear Hilbert space $h_{\{\text{top}\} h}$ is an involutive subalgebra of bounded operators $\mathcal{L}(h_{\{\text{top}\} h})$ from $h_{\{\text{top}\} h}$ to $h_{\{\text{top}\} h}$ such that $\mathcal{M}_{\{\text{top}\} h}$ be $\sigma(\mathcal{L}(h_{\{\text{top}\} h}), \mathcal{L}(h_{\{\text{top}\} h}^*))$ closed.

Now, it is clear that there is a one-to-one correspondence between:

a) a Fredholm module on $A_{\{\text{top}\} h}$ and a subsemialgebra of $\mathcal{L}_{R,L}^B (H^\pm_{\{\text{top}\} h})$ since the extended bilinear Hilbert semispace $H^\pm_{\{\text{top}\} h}$ can be considered as a representation space of the linear Hilbert space $h_{\{\text{top}\} h}$ covered by $H^\pm_{\{\text{top}\} h}$ [29].

b) the weak topological condition of closeness of $\sigma(\mathcal{L}(h_{\{\text{top}\} h}), \mathcal{L}(h_{\{\text{top}\} h}^*))$ and the condition of closed norm topology of $\mathcal{L}_{R,L}^B (H^\pm_{\{\text{top}\} h})$ since $\mathcal{L}(h_{\{\text{top}\} h}^*)$ is the dual of $\mathcal{L}(h_{\{\text{top}\} h})$. 
As there is an isomorphism between a Fredholm module on $A_{\text{top}}^{\{h\}}$ and a subspace of $\mathcal{L}(h_{\{h\}})$, we have the announced isomorphism $i$

$$M_{\text{top}}^{\{h\}} \rightarrow M_{R,L}^{\{h\}} (H^{\pm}_{\text{top}})$$.

3.5. Shifted actions of differential bioperators on the representation spaces of bilinear semigroups:

1. Let $T_{R,L}^{D_{m}} \in \mathbb{M}_{R,L}(H_{a}^{\pm})$ be a right (resp. left) differential linear operator of rank $m$ (i.e. operating on $m$ variables) of the semialgebra of von Neumann $\mathbb{M}_{R,L}(H_{a}^{\pm})$. This operator $T_{R,L}^{D_{m}}$ (resp. $T_{L}^{D_{m}}$), noted in condensed form $T_{R}^{D_{m}}$ (resp $T_{L}^{D_{m}}$), is assumed to be associated with the action of a $T_{m}(IR)$-semigroup structure (resp. a $T_{m}(IR)$-semigroup structure) on the right (resp. left) $n$-dimensional semisheaf $\tilde{M}_{R}$ (resp. $\tilde{M}_{L}$) of the extended bilinear Hilbert semispace $H_{a}^{+}$, with $m \leq n$. Recall [3] that a $T_{m}(IR)$-semigroup structure (resp. a $T_{m}(IR)$-semigroup structure) on $\tilde{M}_{R}$ (resp. $\tilde{M}_{L}$) means a principal $T_{m}(IR)$-bundle (resp. a $T_{m}(IR)$-bundle) on $\tilde{M}_{R}$ (resp. $\tilde{M}_{L}$).

2. Similarly, $(T_{R}^{D} \otimes T_{L}^{D})$ will denote the tensor product of the right and left differential operators ($T_{R}^{D}$ and $T_{L}^{D}$) acting on the semisheaf $(\tilde{M}_{R} \otimes \tilde{M}_{L})$ such that $(T_{R}^{D} \otimes T_{L}^{D}) \in \mathbb{M}_{R \times L}(H_{a}^{\pm})$ be associated with a principal $GL_{m}(IR \times IR) = T_{m}(IR) \times T_{m}(IR)$-bundle on $(\tilde{M}_{R} \otimes \tilde{M}_{L})$.

3. Let $(T_{R}^{D} \otimes T_{L}^{D})$ be the tensor product of a right and a left linear differential operator of rank $m$ such that the action of $(T_{R}^{D} \otimes T_{L}^{D})$ be associated with a $GL_{m}(IR \times IR)$-principal bundle on the semisheaf $(\tilde{M}_{R} \otimes \tilde{M}_{L})$ over the $GL_{n}(L_{\pi} \times L_{v})$-bismodule $(M_{R} \otimes M_{L})$. Then, the action of $(T_{R}^{D} \otimes T_{L}^{D})$ on $(\tilde{M}_{R} \otimes \tilde{M}_{L})$ is equivalent to:

   (a) consider the mapping

   $$T_{R}^{D} \otimes T_{L}^{D} : \tilde{M}_{R} \otimes \tilde{M}_{L} \longrightarrow \tilde{M}_{R[m]} \otimes \tilde{M}_{L[m]}$$

   from the semisheaf $(\tilde{M}_{R} \otimes \tilde{M}_{L})$ over the $GL_{n}(L_{\pi} \times L_{v})$-bismodule $(M_{R} \otimes M_{L})$ to the semisheaf $(\tilde{M}_{R[m]} \otimes \tilde{M}_{L[m]})$ over the $GL_{n[m]}((L_{\pi} \otimes IR) \times (L_{v} \otimes IR))$-bismodule $(M_{R[m]} \otimes M_{L[m]})$ such that $(\tilde{M}_{R[m]} \otimes \tilde{M}_{L[m]})$ be a semisheaf shifted into $(m \times m)$ dimensions.

   (b) consider a shift into $(m \times m)$ dimensions of the functional representation space $\text{FRep}(GL_{n}(L_{\pi} \times L_{v}))$ of the general bilinear semigroup $GL_{n}(L_{\pi} \times L_{v})$ leading to the homomorphism:

   $$T_{R}^{D} \otimes T_{L}^{D} : \text{FRep}(GL_{n}(L_{\pi} \times L_{v})) \longrightarrow \text{FRep}(GL_{n[m]}((L_{\pi} \otimes IR) \times (L_{v} \otimes IR)))$$.
where \( \text{FRepsp}(GL_{n[m]}((L_{\mathbf{R}} \otimes IR) \times (L_v \otimes IR))) \), denoting the functional representation space of \( GL_{n}[L_{\mathbf{R}} \times L_v] \) shifted into \((m \times m)\) dimensions, is the shifted bisemisheaf \((\widetilde{M}_{R_{n[m]}} \otimes \widetilde{M}_{L_{n[m]}})\) on the bisemigroupoid \( GL_{n[m]}((L_{\mathbf{R}} \otimes IR) \times (L_v \otimes IR)) \) and is equal to:

\[
\text{FRepsp}(GL_{n[m]}((L_{\mathbf{R}} \otimes IR) \times (L_v \otimes IR))) = \text{AdFRepsp}(GL_{m}(IR \times IR)) \times \text{FRepsp}(GL_{n}(L_{\mathbf{R}} \otimes L_v))
\]

in such a way that [32]

- \( \text{AdFRepsp}(GL_{m}(IR \times IR)) \), being the adjoint functional representation space of \( GL_{m}(IR \times IR) \), corresponds to the action of \((T_R^D \times T_L^D)\);

- \( \text{FRepsp}(GL_{n}(L_{\mathbf{R}} \otimes L_v)) \), being the functional representation space of \( GL_{n}(L_{\mathbf{R}} \times L_v) \), correspond to the bisemisheaf \((\widetilde{M}_{R} \otimes \widetilde{M}_{L})\).

4. Similarly, the shifting “action” of \((T_R^D \otimes T_L^D)\) on functional representation space of the bilinear subsemigroup \( GL_n((\mathbb{Z}/N\mathbb{Z})^2) \) would be:

\[
T_R^D \otimes T_L^D : \text{FRepsp}(GL_n((\mathbb{Z}/N\mathbb{Z})^2)) = \text{FRepsp}(D_{n}((\mathbb{Z}/N\mathbb{Z})^2) \times [UT_{n}^{t}(\mathbb{Z}/N\mathbb{Z}) \times UT_{n}(\mathbb{Z}/N\mathbb{Z})]) \rightarrow \text{FRepsp}(GL_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes IR^2)) = \text{FRepsp}(D_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes IR^2)) \times [UT_{n[m]}^{t}((\mathbb{Z}/N\mathbb{Z}) \otimes IR^2) \times UT_{n[m]}((\mathbb{Z}/N\mathbb{Z}) \otimes IR^2)]
\]

where:

- \( \text{FRepsp}(D_{n[m]}((\mathbb{Z}/N\mathbb{Z})^2 \otimes IR^2)) \) is the functional representation space of the subgroup of integer diagonal matrices of order \( n \) shifted into \( m \) dimensions.

- \( \text{FRepsp}(UT_{n[m]}((\mathbb{Z}/N\mathbb{Z}) \otimes IR)) \) is the functional representation space of the subgroup of integer unitriangular matrices shifted in \( m \) dimensions.

5. And, the functional representation space of bilinear parabolic subsemigroup \( P_n(L_{\mathbf{R}} \times L_v) \) would also be shifted into \((m \times m)\) dimensions under the action of \((T_R^D \otimes T_L^D)\)
according to:

\[ T_R^D \otimes T_L^D : \]
\[ \text{FRepsp}(P_n((L_{\pi^1} \times L_{\omega^1})) = \text{FRepsp}(D_n(L_{\pi^1} \times L_{\omega^1}) \times [UT^t_n(L_{\pi^1}) \times UT_n(L_{\omega^1})]) \]
\[ \quad \longrightarrow \text{FRepsp}(P_{n|m}((L_{\pi^1} \otimes IR) \times (L_{\omega^1} \otimes IR))) \]
\[ = \text{FRepsp}(D_{n|m}((L_{\pi^1} \otimes IR) \times (L_{\omega^1} \otimes IR)) \times [UT^t_{n|m}(L_{\pi^1} \otimes IR) \times UT_{n|m}(L_{\omega^1} \otimes IR))] \]

6. On the other hand, referring to section 1.5, \( GL_n(L_{\pi}^{nr} \times L_{\nu}^{nr}) \) has \( GL(M_{R}^{nr} \otimes M_{L}^{nr}) \equiv \Gamma_R \times \Gamma_L \) for bilinear (semi)group of automorphisms and has for pseudounramified conjugacy classes the biclasses \( \gamma(i)_R \times \gamma(i)_L \), \( 1 \leq i \leq q \), if the set of fixed bielements is the smallest normal bilinear subsemigroup \( P_n(L_{\pi}^{nr} \times L_{\nu}^{nr}) \) of \( M_{R}^{nr} \otimes M_{L}^{nr} \). This smallest normal bilinear subsemigroup of \( GL_n(L_{\pi}^{nr} \times L_{\nu}^{nr}) \) is the \( n \)-dimensional equivalent of the product, right by left, of the global inertia subgroups \( I_{L_{\pi_i}} \times I_{L_{\nu_i}} \) of degree \( N^2 = 1 \) as introduced in definition 1.4 [27].

In this context, the action of \((T_R^D \otimes T_L^D)\) on \((\tilde{M}_{R}^{nr} \otimes \tilde{M}_{L}^{nr})\), associated with the principal \( GL_m(\mathbb{R} \times \mathbb{R})\)-bundle on \((\tilde{M}_{R}^{nr} \otimes \tilde{M}_{L}^{nr})\) with group \( GL_m(\mathbb{R} \times \mathbb{R})\), leads to envisage that the bilinear semigroupoid \( GL_{n|m}((L_{\pi}^{nr} \otimes IR) \times (L_{\nu}^{nr} \otimes IR)) \), shifting in \( (m \times m) \) dimensions, has \( GL(\tilde{M}_{R}^{nr} \otimes \tilde{M}_{L}^{nr}) \equiv \Gamma^{[m]}_R \times \Gamma^{[m]}_L \) for bilinear semigroup of shifted automorphisms and has for pseudounramified conjugacy classes the biclasses \( (\gamma^{[m]}(i)_R \times \gamma^{[m]}(i)_L) \) shifted in \( (m \times m) \) dimensions, if the set of shifted fixed bielements corresponds to the smallest normal bilinear subsemigroup \( P_{n|m}((L_{\pi}^{nr} \otimes IR) \times (L_{\nu}^{nr} \otimes IR)) \), i.e. the bilinear pseudounramified parabolic subsemigroup.

The shifted pseudounramified conjugacy biclasses \( (\gamma^{[m]}(i)_R \times \gamma^{[m]}(i)_L) \) are in one-to-one correspondence with their unshifted equivalents \( (\gamma(i)_R \times \gamma(i)_L) \) because the bilinear subsemigroup \( (\Gamma^{[m]}_R \times \Gamma^{[m]}_L) \) of automorphisms shifting in \( (m \times m) \) real dimensions results from the principal \( GL_m(\mathbb{R} \times \mathbb{R})\)-bundle on \((\tilde{M}_{R}^{nr} \otimes \tilde{M}_{L}^{nr})\) and corresponds to the \( (m \times n) \)-dimensional representation of the product, right by left, of the differential Galois semigroups of the algebraic extensions \( L_{R}^{nr,+} \) and \( L_{L}^{nr,+} \).

7. \( GL_n(L_{\pi} \times L_{\nu}) \) has for bilinear subsemigroup of automorphisms \( \text{Pra}\Gamma_R \times \text{Pra}\Gamma_L \) and has for pseudounramified conjugacy classes the biclasses \( g(i)_R \times g(i)_L \) if the set of fixed bielements is of dimension \( N > 1 \) with respect to the basis of \( M_R \otimes M_L \). These fixed bielements of \( g(i)_R \times g(i)_L \) correspond to the product, right by left, of completions of degrees equal to \( N > 1 \).

Similarly, \( GL_{n|m}((L_{\pi} \otimes IR) \times (L_{\nu} \otimes IR)) \) has for bilinear subsemigroup of shifted
automorphisms \((\text{Pra} \Gamma^{|m|}_R \times \text{Pra} \Gamma^{|m|}_L)\) and has for shifted pseudoramified conjugacy classes the biclasses \((g^{|m|}(i)_R \times g^{|m|}(i)_L)\).

As \((\text{Pra} \Gamma^{|m|}_R \times \text{Pra} \Gamma^{|m|}_L)\) is the bilinear subsemigroup \(GL(\widetilde{M}_{R_{n[n]}} \otimes \widetilde{M}_{L_{n[n]}})\) of automorphisms shifting in \((m \times m)\) real dimensions with respect to the biaction of \((T^D_R \otimes T^D_L)\) on \((\widetilde{M}_R \otimes \widetilde{M}_L)\), associated with the \(GL_m(\mathbb{R} \times \mathbb{R})\)-principal bundle introduced in 3.), it is clear that the shifted pseudoramified conjugacy biclasses \((g^{|m|}(i)_R \times g^{|m|}(i)_L)\) are in one-to-one correspondence with the unshifted pseudoramified conjugacy biclasses \(g(i)_R \times g(i)_L)\).

**Proposition 3.6** The action of the differential bioperator \((T^D_R \otimes T^D_L)\) of rank \((m \times m)\), associated with a principal \(GL_m(\mathbb{R} \times \mathbb{R})\)-bundle on the \((n \times n)\)-dimensional pseudo\((un)\)-ramified bisemisheaf \((\widetilde{M}^{nr}_{R_{m[n]}} \otimes \widetilde{M}^{nr}_{L_{m[n]}})\), consists in mapping \((\widetilde{M}^{nr}_{R_{\oplus}} \otimes \widetilde{M}^{nr}_{L_{\oplus}})\) into \((\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}})\) shifted into \((m \times m)\) dimensions:

\[
T^D_R \otimes T^D_L : \quad \widetilde{M}^{nr}_{R_{\oplus}} \otimes \widetilde{M}^{nr}_{L_{\oplus}} \longrightarrow (\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}})
\]

such that:

a) \(\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}}\) decomposes into shifted pseudoramified subbisemisheaves according to the shifted pseudoramified conjugacy biclasses \(g^{|m|}_R(i) \times g^{|m|}_L(i)\) of the bisemigroupoid \(GL_{n[n]}((L_T \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))\) and with respect to the shifted automorphisms \(\text{Pra} \Gamma^{|m|}_R \times \text{Pra} \Gamma^{|m|}_L\) of \(GL_{n[n]}((L_T \otimes \mathbb{R}) \times (L_v \otimes \mathbb{R}))\) as follows:

\[
\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}} = \bigoplus_{i=1}^{q} \bigoplus_{m_i} (\widetilde{M}^{nr}_{R_{n[n]}}(i) \otimes \widetilde{M}^{nr}_{L_{n[n]}}(i))
\]

where the integer \(q\) is related to the dimension \((q \cdot N)^n\) of the algebraic basis of \(\widetilde{M}^{nr}_{R_{n[n]}(q)}\) and \(\widetilde{M}^{nr}_{L_{n[n]}(q)}\), i.e. to the number of Galois automorphisms.

b) \(\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}}\) decomposes into shifted pseudounramified subbisemisheaves according to the shifted pseudounramified conjugacy biclasses \(\gamma^{|m|}_R(i) \times \gamma^{|m|}_L(i)\) of \(GL_{n[n]}((L^T_{nr} \otimes \mathbb{R}) \times (L^v_{nr} \otimes \mathbb{R}))\) as follows:

\[
\widetilde{M}^{nr}_{R_{n[n]}} \otimes \widetilde{M}^{nr}_{L_{n[n]}} = \bigoplus_{i=1}^{q} \bigoplus_{m_i} (\widetilde{M}^{nr}_{R_{n[n]}}(i) \otimes \widetilde{M}^{nr}_{L_{n[n]}}(i))
\]

where the integer \(q\), i.e. the global class residue degree \(f_{\text{res}} = q\) (see definition 1.3), refers to the algebraic dimension \(q^n\) of \(\widetilde{M}^{nr}_{R_{n[n]}(q)}\) and \(\widetilde{M}^{nr}_{L_{n[n]}(q)}\).
Proof:

1. The shifted bisemisheaf \((\widetilde{M}_{R_{n|m}}^{(nr)} \otimes \widetilde{M}_{L_{n|m}}^{(nr)})\) is a biobject of the derived category \(D(\widetilde{M}_R \otimes \widetilde{M}_L, IR \otimes IR)\).

2. The algebraic dimension \((q \cdot N)^{n}\) of \(M_{R_{n|m}}(q)\) and of \(M_{L_{n|m}}(q)\) corresponds to the number of Galois automorphisms while the algebraic dimension \((q \cdot N)^{m}\) corresponds to the number of shifted automorphisms.

3. The pseudounramified algebraic dimension \(q^n\) is such that \(q\) corresponds to the number of archimedean places of the semifields \(L_L^+\) and \(L_R^+\).

\[\blacksquare\]

Definition 3.7  **Pseudoramified and pseudounramified algebraic dimensions:**

Until now, two kinds of algebraic dimensions have emerged:

a) the “pseudoramified” algebraic dimensions \(i^n N^n\), referring to the Galois extension degrees being multiples of \(N > 1\).

The shifted pseudoramified algebraic dimensions \(i^m \cdot N^n\) referring to the dimensions of the \(m\)-dimensional representations of the differential Galois subgroups;

b) the pseudounramified algebraic dimension \(i^n\) referring to the \(n\)-th powers of the global residue degree \(i\).

The pseudounramified algebraic dimension \(i^m\) referring to the dimensions of the \(m\)-dimensional representations of the corresponding differential Galois subgroups.

Consider for example the left \(T_n(L_v)\)-subsemimodule \(M_{v_i} \subset M_L\) (see section 1.5) having a rank \(n_i = i^n \cdot N^n = f_{v_i}^n \cdot N^n\). Then, the pseudoramified algebraic dimension of \(M_{v_i}\) is equal to its rank \(n_i = i^n \cdot N^n\).

Note that the geometric dimension of the \(T_n(L_v)\)-subsemimodule \(M_{v_i}\) is equal to “\(n\)”. So, the geometric and algebraic dimensions generally do not coincide.

Proposition 3.8  **Let \(\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)}\) denote the pseudo(un)ramified bisemisheaf over the real \(GL_n(L_{\mathbb{R}}^{(nr)} \times L_v^{(nr)})\)-bisemimodule \((M_R^{(nr)} \otimes M_L^{(nr)})\) isomorphic to its analytic counterpart \((M^{(an)}_R \otimes M^{(an)}_L)\).

Let \((T^D_R \otimes T^D_L)\) be a differential bioperator acting on \((\widetilde{M}_R^{(nr)} \otimes \widetilde{M}_L^{(nr)})\) and transforming them into the corresponding shifted bisemisheaves \((\widetilde{M}^{(nr)}_{R_{n|m}} \otimes \widetilde{M}^{(nr)}_{L_{n|m}})\).

Then, the bisemimodules \((M_R^{(nr)} \otimes M_L^{(nr)})\) as well as their shifted counterparts \((M_{R_{n|m}}^{(nr)} \otimes M_{L_{n|m}}^{(nr)})\) are characterized by the following ranks or algebraic dimensions:
a) the pseudounramified bisemimodule $M_{R}^{nr} \otimes M_{L}^{nr}$ has for algebraic dimension $d = \sum_{i=1}^{d} i^{n_{2}}$;

b) pseudounramified shifted bisemimodule $M_{R[n]}^{nr} \otimes M_{L[n]}^{nr}$ has for algebraic dimension $d = \sum_{i=1}^{d} i^{n_{2}}$ and for shifted algebraic dimension $d_{s} = \sum_{i=1}^{q} i^{m_{2}}$;

c) pseudoramified bisemimodule $M_{R} \otimes M_{L}$ has for algebraic dimension $d = \sum_{i=1}^{d} (i \cdot N)^{n_{2}}$;

d) pseudoramified shifted bisemimodule $M_{R[m]} \otimes M_{L[n,m]}$ has for algebraic dimension $d = \sum_{i=1}^{d} (i \cdot N)^{n_{2}}$ and for shifted algebraic dimension $d_{s} = \sum_{i=1}^{q} (i \cdot N)^{m_{2}}$.

Proof: This results from sections 3.5 and 3.6 and from [30].

Proposition 3.9 Under the “action” of the bioperator $(T_{D}^{P} \otimes T_{L}^{D})$ of rank $(m \times m)$, the Shimura bisemivariety
\[ \overline{\partial S}_{G_{R \times L}} = P_{n}(L_{V}^{+} \times L_{V}^{+}) \setminus GL_{n}(L_{Rd}^{+} \times L_{Ld}^{+})/GL_{n}((\mathbb{Z}/N\mathbb{Z})^{2}) \]
is shifted into $(m \times m)$ dimensions according to:
\[ T_{R}^{P} \otimes T_{L}^{D} : \overline{\partial S}_{G_{R \times L}} \longrightarrow \overline{\partial S}_{G_{R \times L,n[m]}} \]
where $\overline{\partial S}_{G_{R \times L,n[m]}}$ is the shifted Shimura bisemivariety given by:
\[ \overline{\partial S}_{G_{R \times L,n[m]}} = P_{n[m]}((L_{v1} \otimes \mathbb{R}) \times (L_{v2} \otimes \mathbb{R})) \setminus GL_{n[m]}((L_{Rd}^{+} \otimes \mathbb{R}) \times (L_{Ld}^{+} \otimes \mathbb{R}))/GL_{n[m]}((\mathbb{Z}/N\mathbb{Z})^{2} \otimes \mathbb{R}^{2}) . \]

Proposition 3.10 The bilinear cohomology semigroup of the Shimura bisemivariety $\overline{\partial S}_{G_{R \times L}}$ is shifted under the action of the differential bioperator $(T_{R}^{P} \otimes T_{L}^{D})$ of rank $(m \times m)$ according to:
\[ T_{R}^{P} \otimes T_{L}^{D} : H^{2j}(\overline{\partial S}_{G_{R \times L}}) \otimes H^{2j}(\overline{\partial S}_{G_{R \times L,n[m]}}) \longrightarrow H^{2j-2k}(\overline{\partial S}_{G_{R \times L,n[m]}} \otimes \overline{\partial S}_{G_{R \times L,n[m]}}) \]
in such a way that the shifted bilinear Eisenstein cohomology decomposes according to the bicosets of the quotient bisemigroupoid
\[ GL_{2j[2k]}((L_{v} \otimes \mathbb{R}) \times (L_{v} \otimes \mathbb{R}))/GL_{2j[2k]}((\mathbb{Z}/N\mathbb{Z})^{2} \otimes \mathbb{R}^{2}) \]
as follows:

\[ H^{2j-2k}(\partial \overline{\mathcal{S}}_{G \times L, n[m]}, \overline{M}^{2j}_{R_{2j}[2k]_{\Omega}} \otimes \overline{M}^{2j}_{L_{2j}[2k]_{\Omega}}) = \bigoplus_{i=1}^{q} \bigoplus_{m_i} (\overline{M}^{2j}_{R_{2j}[2k]}(i;m_i) \otimes \overline{M}^{2j}_{L_{2j}[2k]}(i;m_i)) \]

where \( m_i \) refers to the multiplicity of the shifted subbimodule \( (M^{2j}_{R_{2j}[2k]}(i;m_i) \otimes M^{2j}_{L_{2j}[2k]}(i;m_i)) \).

**Proof:** According to Proposition 2.3 and the Langlands bilinear global program developed in [29] and in [32], we have that

\[ H^{2j}(\partial \overline{\mathcal{S}}_{G \times L}, \overline{M}^{2j}_{R_{\Omega}} \otimes \overline{M}^{2j}_{L_{\Omega}}) \approx \text{FRep}_\text{sp}(GL_{2j}(L_{\Omega} \times L_{v_{\Omega}})) \]

\[ = \bigoplus_{i=1}^{q} \bigoplus_{m_i} (\overline{M}^{2j}_{R_{2j}[2k]}(i;m_i) \otimes \overline{M}^{2j}_{L_{2j}[2k]}(i;m_i)) \]

Then, the shifted bilinear Eisenstein cohomology verifies:

\[ H^{2j-2k}(\partial \overline{\mathcal{S}}_{G \times L, n[m]}, \overline{M}^{2j}_{R_{2j}[2k]_{\Omega}} \otimes \overline{M}^{2j}_{L_{2j}[2k]_{\Omega}}) \approx \text{FRep}_\text{sp}(GL_{2j}[2k](L_{\Omega} \otimes IR) \times (L_{v_{\Omega}} \otimes IR)) \]

\[ = \bigoplus_{i=1}^{q} \bigoplus_{m_i} (\overline{M}^{2j}_{R_{2j}[2k]}(i;m_i) \otimes (\overline{M}^{2j}_{L_{2j}[2k]}(i;m_i)) \]

such that:

\[ \text{FRep}_\text{sp}(GL_{2j}[2k](L_{\Omega} \otimes IR) \times (L_{v_{\Omega}} \otimes IR)) = \overline{M}^{2j}_{R_{2j}[2k]}(i;m_i) \otimes \overline{M}^{2j}_{L_{2j}[2k]}(i;m_i)) \]

**Proposition 3.11** Let us fix the integers

\[ 1 \leq \ell_{R,L} \leq j , \quad 1 \leq k_R \leq j \quad \text{and} \quad 1 \leq u_{R,L} \leq k , \quad 1 \leq v_R \leq k \]

with the condition that \( m \leq n \).

Then, the shifted bilinear Eisenstein cohomology decomposes into the direct sum of completely irreducible orthogonal or nonorthogonal shifted bilinear Eisenstein cohomologies according to:

- \( H^{2j-2k}(\partial \overline{\mathcal{S}}_{G \times L, n[m]}, \overline{M}^{2j}_{R_{2j}[2k]_{\Omega}} \otimes D \overline{M}^{2j}_{L_{2j}[2k]_{\Omega}}) \)

\[ = \bigoplus_{i=1}^{q} \bigoplus_{m_i} \bigoplus_{\ell_R=\ell_L}^{\ell} \bigoplus_{u_{R,L}}^{u_R} H^{2j-2k}(\partial \overline{\mathcal{S}}_{G \times L, n[m]}, \overline{M}^{2j}_{R_{2j}[2k]_{\Omega}}(i;m_i) \otimes \overline{M}^{2j}_{L_{2j}[2k]_{\Omega}}(i;m_i)) \]

\[ = \text{FRep}_\text{sp}(GL_{2j} = 2_1 + \cdots + 2_j \otimes (L_{\Omega} \otimes IR) \times (L_{v_{\Omega}} \otimes IR)) \]

- \( H^{2j-2k}(\partial \overline{\mathcal{S}}_{G \times L, n[m]}, \overline{M}^{2j}_{R_{2j}[2k]_{\Omega}} \otimes \overline{M}^{2j}_{L_{2j}[2k]_{\Omega}}) \)
Proposition 2.6 and according to [32].

The shifted cohomology semigroup in complete analogy with the unshifted case developed in Definition 3.12 Solvable bilinear Hilbert semispaces:

\[
\begin{align*}
\sum_{i=1}^{q} \bigoplus_{m_i} \bigoplus_{\ell_R} \bigoplus_{u_{R,L}} H^{2i_R-2u_R} (\partial \bar{S}_{G_{R,L};n(m)} \otimes \bar{M}^{2}_{R} (i; m_i))
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{q} \bigoplus_{m_i} \bigoplus_{k_R} \bigoplus_{v_{R,L}} H^{2k_R-2v_R} (\partial \bar{S}_{G_{R,L};n(m)} \otimes \bar{M}^{2}_{R} (i; m_i))
\end{align*}
\]

\[
\begin{align*}
= \text{FRep}^p(GL_{2J_{X,L}[2k]}((L_\pi \otimes IR) \times (L_v \otimes IR))
\end{align*}
\]

where \( \bar{M}^{2}_{2u_L}(i; m_i) \) is a two-dimensional shifted functional representation space over the \( T_{2u_L}(L_v, IR) \)-semimodule.

**Proof:** This proposition introduces the complete reducibility of the bilinear Eisenstein shifted cohomology semigroup in complete analogy with the unshifted case developed in proposition 2.6 and accordingly to [32].

**Definition 3.12** Solvable bilinear Hilbert semispaces:

1. Let \( \bar{M}^{(nr)}_R \otimes \bar{M}^{(nr)}_L = \{ \bar{M}^{(nr)}_{\pi, m_i} \otimes \bar{M}^{(nr)}_{\nu, m_i} \}_{i=1}^{q} \) be the bisemisheaf of differentiable bifunctions

\[
\begin{align*}
\bar{M}^{(nr)}_{\pi, m_i} \otimes \bar{M}^{(nr)}_{\nu, m_i} \equiv \phi_{G_i, m_i} (x_{i_R}) \otimes \phi_{G_i, m_i} (x_{i_L})
\end{align*}
\]

over the \( GL_n(L^{(nr)}_\pi \times L^{(nr)}_v) \)-bisemimodule \( M^{(nr)}_R \otimes M^{(nr)}_L \) in such a way that \( \bar{M}^{(nr)}_R \otimes \bar{M}^{(nr)}_L \) is an extended internal (pseudounramified) bilinear Hilbert semispaces \( H_a^{+(nr)} \) according to definitions 1.8.

Then, the \( i \)-th class \( \{ \bar{M}^{(nr)}_{\pi, m_i} \otimes \bar{M}^{(nr)}_{\nu, m_i} \}_{m_i} \) of \( \bar{M}^{(nr)}_R \otimes \bar{M}^{(nr)}_L \) corresponds to the extended internal bilinear Hilbert subsemispaces \( H_a^{+(nr)}(i) \); so that we get the towers

\[
\begin{align*}
H_a^{+(nr)}(1) \subset \cdots \subset H_a^{+(nr)}(i) \subset \cdots \subset H_a^{+(nr)}(q),
H_a^{+(nr)}(1) \subset \cdots \subset H_a^{+(nr)}(i) \subset \cdots \subset H_a^{+(nr)}(q),
\end{align*}
\]

of embedded pseudounramified and pseudoramified bilinear Hilbert subsemispaces.

Taking into account the isomorphism between the algebraic and analytic bilinear Hilbert semispaces \( H_a^{+} \) and \( H_h^{+} \), corresponding towers of embedded analytic bilinear Hilbert subsemispaces can also be envisaged:

\[
\begin{align*}
H_h^{+(nr)}(1) \subset \cdots \subset H_h^{+(nr)}(i) \subset \cdots \subset H_h^{+(nr)}(q),
H_h^{+(nr)}(1) \subset \cdots \subset H_h^{+(nr)}(i) \subset \cdots \subset H_h^{+(nr)}(q).
\end{align*}
\]
2. Let $\widetilde{M}^\text{nr}_{R,L_{\oplus}} \otimes_{(D)} \widetilde{M}^\text{nr}_{L_{\oplus}} = \bigoplus_{i=1}^{q} (\widetilde{M}^\text{nr}_{v_i,m_i} \otimes_{(D)} \widetilde{M}^\text{nr}_{v_i,m_i})$ be the decomposition of the bisemifield over the $GL_n(I_{\oplus}^{nr} \times L_{v_i}^{nr})$-bisemimodule $M^\text{nr}_{R,L_{\oplus}} \otimes_{(D)} M^\text{nr}_{L_{\oplus}}$. Then, the algebraic pseudounramified extended (resp. diagonal) bilinear Hilbert semispace $H^+_{a_{\oplus}}$ (resp. $\mathcal{H}^{+,nr}_{a_{\oplus}}$) decomposes according to:

$$H^+_{a_{\oplus}} = \bigoplus_{i=1}^{q} H^+_{a_{\oplus}}(i) \quad \text{(resp. } \mathcal{H}^{+,nr}_{a_{\oplus}} = \bigoplus_{i=1}^{q} \mathcal{H}^{+,nr}_{a_{\oplus}}(i))$$

where $\widetilde{M}^\text{nr}_{v_i,m_i} \otimes \widetilde{M}^\text{nr}_{v_i,m_i} \simeq H^+_{a_{\oplus}}(i)$.

So, we can construct a tower of direct sums of embedded algebraic pseudounramified extended (resp. diagonal) bilinear Hilbert subsemispaces

$$H^+_{a_{\oplus}}\{1\} \subset \cdots \subset H^+_{a_{\oplus}}\{i\} \subset \cdots \subset H^+_{a_{\oplus}}\{q\}$$

such that:

- $H^+_{a_{\oplus}}\{q\} \equiv H^+_{a_{\oplus}} = \bigoplus_{\nu=1}^{q} H^+_{a_{\oplus}}(\nu)$,

- $H^+_{a_{\oplus}}\{i\} = \bigoplus_{\nu=1}^{i} H^+_{a_{\oplus}}(\nu)$,

- $\mathcal{H}^{+,nr}_{a_{\oplus}}\{i\} = \bigoplus_{\nu=1}^{i} \mathcal{H}^{+,nr}_{a_{\oplus}}(\nu)$,

refer respectively to the $q$-th, $i$-th and $i$-th state of $H^+_{a_{\oplus}}$, $H^+_{a_{\oplus}}\{i\}$ and $\mathcal{H}^{+,nr}_{a_{\oplus}}\{i\}$.

3. Considering the isomorphism between the algebraic and analytic bisemimodules $(M^\text{nr}_{R} \otimes_{(D)} M^\text{nr}_{L})$ and $(M^\text{s, nr}_{R} \otimes_{(D)} M^\text{s, nr}_{L})$, a tower of direct sums of embedded analytic pseudounramified extended (resp. diagonal) bilinear Hilbert subsemispaces also exists:

$$H^+_{h_{\oplus}}\{1\} \subset \cdots \subset H^+_{h_{\oplus}}\{i\} \subset \cdots \subset H^+_{h_{\oplus}}\{q\}$$

and

$$\mathcal{H}^{+,nr}_{h_{\oplus}}\{1\} \subset \cdots \subset \mathcal{H}^{+,nr}_{h_{\oplus}}\{i\} \subset \cdots \subset \mathcal{H}^{+,nr}_{h_{\oplus}}\{q\}$$

such that:

- $H^+_{h_{\oplus}}\{q\} \equiv H^+_{h_{\oplus}} = \bigoplus_{\nu=1}^{q} H^+_{h_{\oplus}}(\nu)$,
\[ H_{h}^{+,nr} \{i\} = \bigoplus_{\nu=1}^{i} H_{h}^{+,nr}(\nu) , \]

\[ \mathcal{H}_{h}^{+,nr} \{i\} = \bigoplus_{\nu=1}^{i} \mathcal{H}_{h}^{+,nr}(\nu) , \]

refer respectively to the \( q \)-th, \( i \)-th and \( i \)-th state of \( H_{h}^{+,nr}, H_{h}^{+,nr} \) and \( \mathcal{H}_{h}^{+,nr} \).

4. If the decomposition of the pseudoramified bisemisheaf \( \widetilde{M}_{RL} \otimes_{(D)} \widetilde{M}_{L} \) over the \( GL_n(L_{\tau} \times L_{v}) \)-bisemimodule \( M_{RL} \otimes_{(D)} M_{L} \) is envisaged, then the algebraic pseudoramified bilinear Hilbert semispaces \( H_{a}^{+}, H_{h}^{+} \) and \( \mathcal{H}_{a}^{+}, \mathcal{H}_{h}^{+} \) decompose according to:

\[ H_{a}^{+} \{i\} = \bigoplus_{j=1}^{i} H_{a}^{+}(j) ; \]

\[ \mathcal{H}_{a}^{+} \{i\} = \bigoplus_{j=1}^{i} \mathcal{H}_{a}^{+}(j) ; \quad 1 \leq j \leq i , \]

where \( 1 \leq i \leq q \). This leads to towers of direct sums of embedded algebraic and analytic pseudoramified bilinear Hilbert subsemispaces, i.e. towers of states of these bilinear Hilbert semispaces \( H_{a}^{+}, H_{h}^{+}, \mathcal{H}_{a}^{+} \) and \( \mathcal{H}_{h}^{+} \):

\[ H_{a}^{+}(1) \subset \cdots \subset H_{a}^{+} \{i\} \subset \cdots \subset H_{a}^{+} \{q\} , \]

\[ H_{h}^{+}(1) \subset \cdots \subset H_{h}^{+} \{i\} \subset \cdots \subset H_{h}^{+} \{q\} , \]

\[ \mathcal{H}_{a}^{+}(1) \subset \cdots \subset \mathcal{H}_{a}^{+} \{i\} \subset \cdots \subset \mathcal{H}_{a}^{+} \{q\} , \]

\[ \mathcal{H}_{h}^{+}(1) \subset \cdots \subset \mathcal{H}_{h}^{+} \{i\} \subset \cdots \subset \mathcal{H}_{h}^{+} \{q\} , \]

where \( H_{a}^{+}(1) \equiv H_{a}^{+}(1) \).

The towers of embedded bilinear Hilbert subsemispaces lead to consider that these bilinear Hilbert semispaces are “solvable” and thus graded.

**Definition 3.13 Projectors:** (a) Let

\[ H_{a}^{+,nr}(1) \subset \cdots \subset H_{a}^{+,nr}(i) \subset \cdots \subset H_{a}^{+,nr}(q) , \]

\[ H_{a}^{+}(1) \subset \cdots \subset H_{a}^{+}(i) \subset \cdots \subset H_{a}^{+}(q) \]
be the two towers of embedded pseudounramified and pseudoramified bilinear Hilbert
subsemispaces introduced in section 3.12.

Then, the following projectors:

\[ P_{i_{\text{R} \times \text{L}}}^{\text{fac}({\text{nr}})} : H_a^{+,\text{nr}}(q) \longrightarrow H_a^{+,\text{nr}}(i) , \quad \forall 1 \leq i \leq q , \]

\[ P_{i_{\text{R} \times \text{L}}}^{\text{fac}} : H_a^{+}(q) \longrightarrow H_a^{+}(i) , \]

can be introduced, as it is done classically, in such a way that:

- \( P_{i_{\text{R} \times \text{L}}}^{\text{fac}({\text{nr}})} \) projects \( H_a^{+,\text{nr}}(q) \) onto the \( i \)-th pseudounramified bilinear Hilbert subsemispace \( H_a^{+,\text{nr}}(i) \);

- \( P_{i_{\text{R} \times \text{L}}}^{\text{fac}} \) projects \( H_a^{+}(q) \) onto the \( i \)-th pseudoramified bilinear Hilbert subsemispace \( H_a^{+}(i) \).

(b) Let \( H_{a_{\oplus}}^{+} \) be an extended bilinear Hilbert semispase decomposing according to:

\[ H_{a_{\oplus}}^{+} = \bigoplus_{i=1}^{q} H_{a}^{+} \{i\} \text{ such that } H_{a}^{+} \{i\} = \bigoplus_{\nu=1}^{i} H_{\nu}^{+} \]

or

\[ H_{a_{\oplus}}^{+}\text{nr} = \bigoplus_{i=1}^{q} H_{a}^{+}\text{nr} \{i\} \text{ such that } H_{a}^{+}\text{nr} \{i\} = \bigoplus_{j=1}^{i} H_{a}^{+}\text{nr} \{j\} . \]

Then, we can define the (bi)projectors of states:

\[ P_{i_{\text{R} \times \text{L}}}^{\text{nr}} : H_{a_{\oplus}}^{+} \longrightarrow H_{a_{\oplus}}^{+}\text{nr} \{i\} , \quad 1 \leq i \leq q , \]

\[ P_{i_{\text{R} \times \text{L}}}^{+} : H_{a_{\oplus}}^{+} \longrightarrow H_{a_{\oplus}}^{+} \{i\} , \]

mapping \( H_{a_{\oplus}}^{+}\text{nr} \) respectively into its closed extended bilinear subsemispace \( H_{a_{\oplus}}^{+}\text{nr} \{i\} \) which is the \( i \)-th (bisemi)state.

The (bi)projectors \( P_{i_{\text{R} \times \text{L}}}^{\text{nr}} \) and \( P_{i_{\text{R} \times \text{L}}}^{+} \) are idempotent (bi)operators in such a way that the mappings they generate are inverse deformations (of Galois representations) as proved by the author elsewhere [31].

**Proposition 3.14** The operator \( T_{R_{\text{R} \times \text{L}}}^{D}(\text{Pra} \Gamma_{R_{\text{R} \times \text{L}}})^{[m]} \) (resp. \( T_{R_{\text{R} \times \text{L}}}^{D}(\Gamma_{R_{\text{R} \times \text{L}}})^{[m]} \)) is a random operator decomposing into a set of operators \( \{T_{R_{\text{R} \times \text{L}}}^{D}(\gamma_{R_{\text{R} \times \text{L}}}(i)))\}_{i} \) (resp. \( \{T_{R_{\text{R} \times \text{L}}}^{D}(\gamma_{R_{\text{R} \times \text{L}}}(i)))\}_{i} \))

\[ \forall 1 \leq i \leq q \] according to the shifted pseudoramified (resp. pseudounramified) conjugacy classes of \( GL_{a[m]}(L_{a}^{+} \otimes \mathbb{R}) \) associated with the \( T_{m}^{(l)}(\mathbb{R}) \)-principal bundle.
Proof: Indeed, according to section 3.5, a random operator \( T_{R,L}^D(\text{Pra}_{m,L}^{[m]}(\ell)) = \{ T_{R,L}^D(g^{[m]}_{R,L}(i)) \}_{i=1}^q \) (resp. \( T_{R,L}^D(\gamma_{R,L}^{[m]}(i)) \) ), acting on an extended bilinear Hilbert semispace \( H_a^{\mp (nr)} \), is a set \( \{ T_{R,L}^D(g^{[m]}_{R,L}(i)) \}_{i=1}^q \in \{ \mathbb{M}_{R,L}(H_a^{\mp (nr)}(i)) \} \) (resp. \( \{ T_{R,L}^D(\gamma_{R,L}^{[m]}(i)) \}_{i=1}^q \in \{ \mathbb{M}_{R,L}(H_a^{\mp (nr)}(i)) \} \) ) such that the bilinear form:

\[
\begin{align*}
t_R(\ell, m) &= (T_R^D(g_R^{[m]}(\ell)) e_R^\ell, e_L^m) \\
\text{(resp. } t_R(\mu, \nu) &= (T_R^D(\gamma_R^{[m]}(\mu)) e_R^\mu, e_L^\nu) )
\end{align*}
\]

or

\[
\begin{align*}
t_L(\ell, m) &= (e_R^\ell, T_L^D(g_L^{[m]}(m)) e_L^m) \\
\text{(resp. } t_L(\mu, \nu) &= (e_R^\mu, T_L^D(\gamma_L^{[m]}(\nu)) e_L^\nu) )
\end{align*}
\]

be measurable.

\( \mathbb{M}_{R,L}(H_a^{\mp (i)}) \) (resp. \( \mathbb{M}_{R,L}(H_a^{\mp (nr)}(i)) \) ) is a von Neumann subsemialgebra relative to bounded operators on a closed connected subsemispace \( H_{a}^{\mp (nr)}(i) \) of \( H_a^{\mp (nr)} \) referring to the \( i \)-th conjugacy class of \( GL_n(L_{\mu}^{(nr)} \times L_{\nu}^{(nr)}) \).

These considerations are made in complete analogy with what is known for random operators on linear Hilbert (semi)spaces [9].

Proposition 3.15 1) Let \( T_{R,L}^D(g_{R,L}^{[m]}(u)) \) and \( T_{R,L}^D(g_{R,L}^{[m]}(v)) \) be two right or left random operators such that \( u < v \). Then, the random operator \( T_{R,L}^D(g_{R,L}^{[m]}(v)) \) is an “extension” of the random operator \( T_{R,L}^D(g_{R,L}^{[m]}(u)) \) corresponding to a difference of conjugacy classes \( (v - u) \).

2) Let \( T_{R,L}^D(\gamma_{R,L}^{[m]}(o)) \) and \( T_{R,L}^D(\gamma_{R,L}^{[m]}(p)) \) be two right or left random operators such that \( o < p \). Then, \( T_{R,L}^D(\gamma_{R,L}^{[m]}(p)) \) is an “extension” of \( T_{R,L}^D(\gamma_{R,L}^{[m]}(o)) \) corresponding to a difference of conjugacy classes \( (p - o) \).

Definition 3.16 Towers of pseudoramified and pseudounramified von Neumann subsemialgebras: (a) In connection with the definition 3.13 introducing towers of direct sums of embedded bilinear Hilbert subsemispaces, we shall define here towers of sums of random operators:

\[
T_{R,L}^{D(m)}(g_{R,L}(i)) = \bigoplus_{j=1}^i T_{R,L}^{D(m)}(g_{R,L}(j))
\]

(resp. \( T_{R,L}^{D(m)}(\gamma_{R,L}(i)) = \bigoplus_{j=1}^i T_{R,L}^{D(m)}(\gamma_{R,L}(j)) \)).
such that
\[ T_{R,L}^{\gamma} (g_{R,L}^{m}\{i\}) \in \mathbb{M}_{R,L}(H_{a}^{\pm}\{i\}) , \quad 1 \leq i \leq q , \]
(resp. \[ T_{R,L}^{\gamma} (\gamma_{R,L}^{m}\{i\}) \in \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{i\}) , \]
where \( \mathbb{M}_{R,L}(H_{a}^{\pm}\{i\}) \) (resp. \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{i\}) \)) is the pseudoramified (resp. pseudounramified) von Neumann subsemialgebra of the \( i \)-th state referring to the \( i \)-th sum of random operators.

So, a tower of pseudoramified and pseudounramified von Neumann subsemialgebras of states can be introduced by:
\[ \mathbb{M}_{R,L}(H_{a}^{\pm}\{1\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_{a}^{\pm}\{i\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_{a}^{\pm}\{q\}) , \]
(resp. \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{1\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{i\}) \subset \cdots \subset \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{q\}) \),
such that
\[ \mathbb{M}_{R,L}(H_{a}^{\pm}\{i\}) = \bigoplus_{j=1}^{i} \mathbb{M}_{R,L}(H_{a}^{\pm}(j)) \]
(resp. \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{i\}) = \bigoplus_{j=1}^{i} \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}(j)) \)).

(b) Similarly, on the towers
\[ H_{h}^{\pm,\text{nr}}(1) \subset \cdots \subset H_{h}^{\pm,\text{nr}}(i) \subset \cdots \subset H_{h}^{\pm,\text{nr}}(q) \]
and \( H_{h}^{\pm}(1) \subset \cdots \subset H_{h}^{\pm}(i) \subset \cdots \subset H_{h}^{\pm}(q) \)
of analytic Hilbert subsemispaces introduced in definition 3.12, the corresponding towers of pseudounramified and pseudoramified von Neumann subsemialgebras will be given by:
\[ \mathbb{M}_{R,L}(H_{h}^{\pm,\text{nr}}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_{h}^{\pm,\text{nr}}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_{h}^{\pm,\text{nr}}(q)) , \quad 1 \leq i \leq q \leq \infty , \]
and by
\[ \mathbb{M}_{R,L}(H_{h}^{\pm}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H_{h}^{\pm}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H_{h}^{\pm}(q)) . \]

**Proposition 3.17** Let \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{(nr)}}) \) be the von Neumann semialgebra of bounded self-adjoint operators on the smooth extended bilinear Hilbert semiaspace \( H_{a}^{\pm,\text{(nr)}} \).

Let \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{(nr)}}\{i\}) \) be the von Neumann subsemialgebra of random operators on the closed smooth **extended** bilinear subsemispace \( H_{a}^{\pm,\text{nr}}\{i\} \) and let \( \mathbb{M}_{R,L}(H_{a}^{\pm,\text{nr}}\{i\}) \) be the corresponding von Neumann subsemialgebra on the closed smooth **internal** diagonal bilinear subsemispace \( H_{a}^{\pm,\text{nr}}\{i\} \).
Then, the discrete spectrum $\sigma(T^D_{R,L})$ of an operator $T^D_{R,L} \in \mathfrak{M}_{R,L}(H^{\mp,(nr)}_a)$ is obtained by the morphism:

$$i^a_{\{i\}_R,L} \circ i^a_{\{i\}_R,L} : \mathfrak{M}_{R,L}(H^{\mp,(nr)}_a) \longrightarrow \mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i,$$

$$T^D_{R,L} \longrightarrow \sigma(T^D_{R,L})$$

where $i^a_{\{i\}_R,L}$ and $i^a_{\{i\}_R,L}$ are given by:

$$i^a_{\{i\}_R,L} : \mathfrak{M}_{R,L}(H^{\mp,(nr)}_a) \longrightarrow \mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i,$$

$$i^a_{\{i\}_R,L} : [\mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i \longrightarrow \mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i.$$

**Proof:** First remark that $\mathfrak{M}_{R,L}(H^{\mp,(nr)}_a)$ is a non-abelian von Neumann semialgebra since the extended bilinear Hilbert semispace $H^{\mp,(nr)}_a$ constitutes the enveloping (semi)algebra of the semimodule $M^{(nr)}_L$ (resp. $M^{(nr)}_{LR}$).

The morphism

$$i^a_{\{i\}_R,L} : \mathfrak{M}_{R,L}(H^{\mp,(nr)}_a) \longrightarrow \mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i,$$

$$T^D_{R,L}(\text{Pra} \Gamma^{[m]}_{R,L}) \longrightarrow [T^D_{R,L}(\gamma^{[m]}_{R,L}\{i\}),]_i$$

(transp. $T^D_{R,L}(\Gamma^{[m]}_{R,L}) \longrightarrow [T^D_{R,L}(\gamma^{[m]}_{R,L}\{i\})]_i$),

transforms the bounded operator $T^D_{R,L}(\text{Pra} \Gamma^{[m]}_{R,L})$ (resp. $T^D_{R,L}(\Gamma^{[m]}_{R,L})$) into the set $[T^D_{R,L}(\gamma^{[m]}_{R,L}\{i\})]_i$ (resp. $[T^D_{R,L}(\gamma^{[m]}_{R,L}\{i\})_i$) of bounded operators (i.e. sums of random operators acting on closed subsemispaces $M_{LR,L}\{i\}$ whose sums of enveloping subsemispaces are $H^{\mp,(nr)}_a\{i\}$).

On the other hand, the isomorphism $i^a_{\{i\}_R,L}$

$$i^a_{\{i\}_R,L} : [\mathfrak{M}_{R,L}(H^{\mp,(nr)}_a\{i\})_i \longrightarrow \mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i,$$

transforms the non-abelian von Neumann subsemialgebra $\mathfrak{M}_{R,L}(H^{\mp,(nr)}_a\{i\})$ into the abelian or diagonal von Neumann subsemialgebra $\mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})$ of sums of random operators. $[\mathfrak{M}_{R,L}(\mathcal{H}^{\mp,(nr)}_a\{i\})_i$, is then an algebra of the sum of random operators acting on diagonal enveloping subsemialgebras $(\mathcal{H}^{\mp,(nr)}_a\{i\})$. $\sigma(T^D_{R,L})$ (resp. $\sigma_{nr}(T^D_{R,L})$) is thus the pseudoramified (resp. pseudounramified) spectrum of the bounded operator $T^D_{R,L}$. 

\[\square\]
Corollary 3.18 Let $\mathbb{M}_{R\times L}(H^+_a\{n\})$ be the von Neumann bisemialgebra of bounded bioperators $T^D_R \otimes T^D_L$ on $H^+_a\{n\}$ and let $\mathbb{M}_{R\times L}(H^+_a\{i\})$ be the $i$-th corresponding von Neumann subbisemialgebra of the sum of random bounded bioperators on $H^+_a\{i\}$. If $\mathbb{M}_{R\times L}(H^+_a\{i\})$ is the $i$-th von Neumann diagonal subbisemialgebra of random diagonal bioperators $T^D_R\{i\} \otimes_D T^D_L\{i\}$ on $H^+_a\{i\}$, then the discrete spectrum $\sigma(T^D_R \otimes T^D_L)$ of $(T^D_R \otimes T^D_L) \in \mathbb{M}_{R\times L}(H^+_a\{n\})$ is obtained by the morphism:

$$
i^a_{\{i\}R\times L} \circ i^a_{\{i\}R\times L} : \mathbb{M}_{R\times L}(H^+_a\{n\}) \longrightarrow [\mathbb{M}_{R\times L}(H^+_a\{i\})]_i,$$

$$T^D_{R\times L} \longrightarrow \sigma(T^D_{R\times L}),$$

where $T^D_{R\times L}$ is the condensed notation for $T^D_R \otimes T^D_L$.

$\mathbb{M}_{R,L}(H^+\{n\})$ then corresponds to a solvable (bi)semialgebra.

Proof: This corollary is an extension of the preceding proposition to the bioperator $(T^D_R(\Gamma_R) \otimes T^D_L(\Gamma_L))$.

3.19. Shifted global pseudounramified (resp. pseudoramified) elliptic bsemimodules: Referring to proposition 3.6, the action of the differential bioperator $(T^D_R \otimes T^D_L)$ on the bisemisheaf $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ over the $GL_n(\mathbb{L}^\{n\} \times L^\{n\})$-bsemimodule $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ consists in mapping it into the shifted bisemisheaf $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ over the $GL_{n|m}(\mathbb{L}^\{n\} \otimes \mathbb{L}^\{l\})$-bsemimodule $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ such that $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ decomposes into “$q$” bsemisheaves.

But, according to proposition 2.10 referring to the Langlands global program introduced in [29], there is a bijection between the $GL_n(\mathbb{L}^\{n\} \times L^\{n\})$-bsemimodule $(\widehat{M}^\{n\} \otimes \widehat{M}^\{l\})$ and its cuspidal counterpart given by the global pseudounramified (resp. pseudounramified) elliptic $G_{sR\times L}$-bsemimodule:

$$\text{ELLIP}^\{n\}_{R\times L}(n,q) = \sum_{i=1}^{q} \lambda(n,i,m_i)e^{-2\pi i iz} \otimes \sum_{i=1}^{q} \lambda(n,i,m_i)e^{2\pi i iz}, \quad z \in \mathbb{R}^n,$$

(resp. $\text{ELLIP}^\{n\}_{R\times L}(n,q) = \sum_{i=1}^{q} \lambda_{nr}(n,i,m_i)e^{-2\pi i iz} \otimes \sum_{i=1}^{q} \lambda_{nr}(n,i,m_i)e^{2\pi i iz}$).
such that we have the commutative diagram:

\[
\begin{array}{ccc}
\widetilde{M}_R^{nr} \otimes \widetilde{M}_L^{nr} & \overset{T_R^D \otimes T_L^D}{\longrightarrow} & \widetilde{M}_{R_{n[m]}}^{nr} \otimes \widetilde{M}_{L_{n[m]}}^{nr} \\
\downarrow & & \downarrow \\
\text{ELLIP}^{nr}_{R \times L}(n, q) & \overset{T_R^D \otimes T_L^D}{\longrightarrow} & \text{ELLIP}^{nr}_{R \times L}(n[m], q) \\
\downarrow & & \downarrow \\
\text{ELLIP}_{R \times L}(n, q) & \overset{T_R^D \otimes T_L^D}{\longrightarrow} & \text{ELLIP}_{R \times L}(n[m], q) \\
\uparrow & & \uparrow \\
\widetilde{M}_R \otimes \widetilde{M}_L & \overset{T_R^D \otimes T_L^D}{\longrightarrow} & \widetilde{M}_{R_{n[m]}} \otimes M_{L_{n[m]}} \\
\end{array}
\]

where \( \text{ELLIP}^{nr}_{R \times L}(n[m], q) \) (resp. \( \text{ELLIP}^{nr}_{R \times L}(n[m], q) \)) is the shifted global pseudounramified (resp. pseudoramified) elliptic \((G_{sR} \otimes IR) \times (G_{sL} \otimes IR)\)-bisemimodule.

As an application of proposition 3.17, we suggest the following proposition [32].

**Proposition 3.20** The shifted global pseudounramified (resp. pseudoramified) \( n \)-dimensional elliptic bisemimodule

\[
\text{ELLIP}^{nr}_{R \times L}(n[m], q) = \text{ELLIP}^{nr}_R(n[m], q) \otimes \text{ELLIP}^{nr}_L(n[m], q)
\]

(resp. \( \text{ELLIP}_{R \times L}(n[m], q) = \text{ELLIP}_R(n[m], q) \otimes \text{ELLIP}_L(n[m], q) \)),

gives rise to (or is functorially equivalent to) the eigenbivalue equation of the \( i \)-th (bi)states:

\[
(T_R^D \otimes T_L^D)(\text{ELLIP}^{nr}_R(n, i) \otimes \text{ELLIP}^{nr}_L(n, i)) = E^{nr}_R\{n, i\} \times E^{nr}_L\{n, i\} \cdot (\text{ELLIP}^{nr}_R(n, i) \otimes \text{ELLIP}^{nr}_L(n, i)) , \quad 1 \leq i \leq q ,
\]

(resp. \( (T_R^D \otimes T_L^D)(\text{ELLIP}_{R \times L}(n, i)) = E_R\{n, i\} \times E_L\{n, i\} \cdot (\text{ELLIP}_{R \times L}(n, i)) , \quad 1 \leq i \leq q \).

**Proof:**

1. The shifted global pseudounramified elliptic bisemimodule \( \text{ELLIP}^{nr}_{R \times L}(n[m], q) \) generates the eigenbivalue equation:

\[
\text{ELLIP}^{nr}_{R \times L}(n[m], i) = (E^{nr}_{R \times L}\{n, i\})(\text{ELLIP}^{nr}_{R \times L}(n, i))
\]

which can be rewritten according to [32]:

\[
(T_R^D \otimes T_L^D)(\text{ELLIP}^{nr}_{R \times L}(n, i)) = (E^{nr}_R\{n, i\} \times E^{nr}_L\{n, i\})(\text{ELLIP}^{nr}_{R \times L}(n, i))
\]
where the right (resp. left) eigenvalue $E^r_R\{n,i\}$ (resp. $E^l_L\{n,i\}$) corresponds to a sum over the $i$ first pseudounramified algebraic classes of shifts into $m$ dimensions of the Hecke characters $\lambda_{nr}(n,\nu,m_\nu)$ (resp. $\lambda_{nr}(n,\nu,m_\nu)$) i.e. to infinitesimal generators of the considered Lie algebra, $1 \leq \nu \leq i$.

2. The bisemialgebra of von Neumann $\mathcal{M}_{R\times L}(H^{\pm, nr}_h)$ can then be considered as a solvable bisemialgebra generating a tower of sums of pseudounramified von Neumann subbisemialgebras according to definition 3.16. On the other hand, the set of pseudounramified eigenbivalues of $(T^D_R \otimes T^D_L)$ forms an embedded sequence:

$$E^r_R\{n,1\} \cdot E^l_L\{n,1\} \subset \cdots E^r_R\{n,i\} \cdot E^l_L\{n,1\} \subset \cdots E^r_R\{n,q\} \cdot E^l_L\{n,q\}$$

in one-to-one correspondence with the set of embedded eigenbifunctions given by the product, right by left, of the truncated Fourier series at “$i$” terms:

$$\text{ELLIP}^n_{R\times L}(n,i) = \sum_{\nu=1}^i \sum_{m_\nu} \lambda_{nr}(n,\nu,m_\nu) e^{-2\pi i \nu z} \otimes \sum_{\nu=1}^i \sum_{m_\nu} \lambda_{nr}(n,\nu,m_\nu) e^{2\pi i \nu z}, \ z \in \mathbb{R}^n.$$

3. The proof was given for the “pseudounramified” case, taking into account that the “pseudoramified” case can be handled similarly. \hfill \blacksquare

**Proposition 3.21** The discrete spectrum $\sigma(T^D_{R\times L})$ of $(T^D_{R\times L}) \in \mathcal{M}_{R\times L}(H^{\pm, (nr)}_h)$ and the discrete spectrum $\sigma^\alpha(T^D_{R\times L})$ of $(T^D_{R\times L}) \in \mathcal{M}_{R\times L}(H^{\pm, (nr)}_a)$ are isomorphic (and often equal).

**Proof:** Consider the commutative diagram:

$\begin{array}{ccc}
\mathcal{M}_{R\times L}(H^{\pm, (nr)}_h) & \xleftarrow{\pm^a_{R\times L}} & \mathcal{M}_{R\times L}(H^{\pm, (nr)}_a) \\
\downarrow & & \downarrow \quad i_{R\times L} \\
\left[\mathcal{M}_{R\times L}(H^{\pm, (nr)}_h)\right]_i & \xleftarrow{i^a_{R\times L}} & \left[\mathcal{M}_{R\times L}(H^{\pm, (nr)}_a)\right]_i \\
\downarrow & & \downarrow \\
\left[\mathcal{M}_{R\times L}(H^{\pm, (nr)}_h)\right]_i & \xleftarrow{i^a_{R\times L}} & \left[\mathcal{M}_{R\times L}(H^{\pm, (nr)}_a)\right]_i \\
\end{array}$

where

- the isomorphism $i^a_{R\times L}$ has been introduced in proposition 3.3;
• the morphisms \( i^h_{\{i\}_{R,L}} \) and \( i^a_{\{i\}_{R,L}} \) result from the decomposition of \( \mathbb{M}_{R \times L}(H^\mp_{h}(nr)) \) and of \( \mathbb{M}_{R \times L}(H^\mp_{a}(nr)) \) into sums of pseudounramified or pseudoramified subbisemialgebras (see definition 3.16).

From the isomorphism \( i^a_{D_{R,L}} \), it results that the discrete spectrum \( \sigma(T^D_{R,L}) \) of \( T^D_{R,L} \in \mathbb{M}_{R \times L}(H^\mp_{h}(nr)) \) and the discrete spectrum \( \sigma^a(T^D_{R,L}) \) of \( T^D_{R,L} \in \mathbb{M}_{R \times L}(H^\mp_{a}(nr)) \) are isomorphic. So, we get the thesis.

3.22. Factors of von Neumann:

• We are now interested in the classification of the factors of von Neumann, i.e. in von Neumann algebras having trivial centers (reduced to \( \mathbb{C} \)). According to definition 3.16, we see that two types of towers of von Neumann sub(bi)semialgebras have been introduced:

  – the first referring to \textbf{pseudounramified (algebraic) classes} of the bilinear Hilbert semispaces \( H^\mp_{a,nr} \) (or \( H^\mp_{h,nr} \)) on which they have been defined;

  – the second referring to \textbf{pseudoramified (algebraic) classes} of the bilinear Hilbert semispaces \( H^\mp_{a} \) (or \( H^\mp_{h} \)).

So, the classification of factors of von Neumann will be based on these two types of towers of von Neumann sub(bi)semialgebras on bilinear Hilbert (sub)semispaces which are associated with Hecke sublattices as developed in proposition 2.3 (proof). As a result, the dimensions of the factors of von Neumann will directly refer to Hecke sublattices.

• The bilinear Hilbert semispaces \( H^\mp_{a} \), isomorphic to \( H^\mp_{h} \), constituting the natural representation spaces of the von Neumann (bi)semialgebras, were supposed to be pseudoramified in the sense that the \( GL_n(L_{\pi} \times L_{v}) \)-bisemimodule \( (M_{R_L} \otimes M_{L_{v}}) \) is pseudoramified. That is to say that the \( T_n(L_{v_{i}}) \)-subsemimodule \( M_{v_{i}} \) (as well as \( M_{\pi_{v}} \)) has a rank given by \( n_{i} = i^{n} \cdot N^{n} \) (see section 1.5).

On the other hand, the corresponding pseudounramified \( T_n(L^{nr}_{v_{i}}) \)-subsemimodule \( M^{nr}_{v_{i}} \) would have a rank \( n_{i}^{nr} = i^{n} \) according to [29], which allows to envisage the introduction of pseudounramified bilinear Hilbert subsemispaces, noted \( H^{nr}_{a}(i) \), as it was defined in section 3.12.
Proposition 3.23 (Classification of (bi)factors of von Neumann with respect to algebraic dimensions)

1. **Type I**: on the pseudounramified bilinear Hilbert semispaces \( H^a_{nr} \), there are \( q \) factors \( \mathbb{M}_{R,L}(H^a_{nr}(i)) \) of type \( I_i \), \( 1 \leq i \leq q \leq \infty \), where \( i \) denotes a global residue degree.

2. **Type II**:
   - **II\(_1\)**: on the bilinear Hilbert subsemispace \( H^+_{a,[L_{vi}]} \) restricted to the representation space of the bilinear parabolic subsemigroup \( P_n(L_{vi} \times L_{v_1}) \), there are \( N \) factors \( \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(i)) \), \( 1 \leq i \leq N \), of type \( II_{1i} \), where \( i \) denotes an internal algebraic dimension corresponding to the number of automorphisms of the global inertia subgroup.
   - The factor \( \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(N)) \) is the factor of type \( II_1 \).

3. **Type II\(_\infty\)**:
   - on the tensor products \( H^a_{nr}(\infty) \otimes H^+_{a,[L_{vi}]}(N) \), there are \( q \) pseudounramified factors \( \mathbb{M}_{R,L}(H^a_{nr}(\infty) \otimes H^+_{a,[L_{vi}]}(N)) \) of type \( II_{\infty} \), \( 1 \leq i \leq q \leq \infty \), where \( i \) denotes a global residue degree.

4. **Type II\(_\infty\)**: on the tensor products \( H^a_{nr}(\infty) \otimes H^+_{a,[L_{vi}]}(j) \), \( 1 \leq j \leq N \), the factors \( \mathbb{M}_{R,L}(H^a_{nr}(\infty)) \otimes \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(j)) \) are defined.

**Proof:**

1. As there are \( q \) conjugacy classes of the pseudounramified bilinear Hilbert semispaces \( H^a_{nr} \), there are \( q \) “pseudounramified” factors \( \mathbb{M}_{R,L}(H^a_{nr}(i)) \) in the tower:
   \[
   \mathbb{M}_{R,L}(H^a_{nr}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H^a_{nr}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H^a_{nr}(q))
   \]
   as introduced in sections 3.12, 3.13 and 3.16.

   So, there are \( q \) factors of type \( I_i \), \( 1 \leq i \leq q \leq \infty \) with minimal projections.

2. If we consider the \( N \) internal conjugacy classes of the bilinear parabolic semigroup \( P_n(L_{vi} \times L_{v_1}) \) corresponding to the (shifted) intermediate inner automorphisms of the global inertia subgroups \( I_{L_{v_1}} \) having an order \( N \), we can introduce on \( H^+_{a,[L_{vi}]} \) a tower of inner hyperfinite subfactors [23], [24]:
   \[
   \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(1)) \subset \cdots \subset \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(i)) \subset \cdots \subset \mathbb{M}_{R,L}(H^+_{a,[L_{vi}]}(N))
   \]
   in such a way that:
• the index \([\mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(i)) : \mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(1))] = i\) of the \(i\)-th hyperfinite subfactor with respect to the first hyperfinite subfactor is the internal algebraic dimension (see section 3.7).

• the upper hyperfinite subfactor \(\mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(N))\) is the hyperfinite factor of type II\(1\) having an index \(N\) and corresponding to the order of the global inertia subgroup \(I_{L_{v_i}}\).  

Indeed, if we take into account proposition 2.9, the Hecke characters on sublattices associated with \(\mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(i))\), \(1 \leq i \leq N\), must take values in the interval \([0, 1]\): they then correspond to the continued dimensions [17] of the classes of the projectors of the subfactors of type II\(1\) of von Neumann algebras on a linear Hilbert semispace.

3. As on the pseudounramified bilinear Hilbert subsemispaces \(H_a^{nr}(i)\), pseudounramified factors \(\mathbb{M}_{R,L}(H_a^{nr}(i))\) of type I\(i\) are defined and as, on the bilinear Hilbert subsemispace \(H_a^{\pm,\text{in}}(N)\), a factor of type II\(1\) is defined, it is evident that, on their tensor products \(H_a^{nr}(i) \otimes H_a^{\pm,\text{in}}(N)\), pseudoramified factors of type II\(i\), characterized by minimal projections, \(1 \leq i \leq q \leq \infty\), can be defined, the factor of type II\(1\) “ramifying” the pseudounramified factors I\(i\).

4. And, then, the classical factors of Araki-Woods [1], [15] of type II\(\infty\) correspond to the factors \(\mathbb{M}_{R,L}(H_a^{nr}(i = \infty)) \otimes \mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(j))\), \(1 \leq j \leq N\), where

- \(\mathbb{M}_{R,L}(H_a^{nr}(i = \infty))\) is the pseudounramified factor of type I\(\infty\);

- \(\mathbb{M}_{R,L}(H_a^{\pm,\text{in}}(j))\) is the hyperfinite subfactor of type II\(I_j\).

**Corollary 3.24** The equivalent of a factor of type III\(\lambda\) can be obtained by considering the cross product of the factor II\(\infty\) by a subgroup of automorphisms of it [1], [15].

**Proof:** Indeed, a factor \(M_\lambda\) of type \(M_\lambda\) [28] is isomorphic to the cross product of a factor “\(N\)” of type II\(\infty\) by Aut “\(N\)” [14], [16], [33].
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