Nonequilibrium Thermodynamics of Quantum Friction

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Thermodynamic principles are often deceptively simple and yet surprisingly powerful. We show how a simple rule, such as the net flow of energy in and out of a moving atom under nonequilibrium steady state condition, can expose the shortcomings of many popular theories of quantum friction. Our thermodynamic approach provides a conceptual framework in guiding atom-optical experiments, thereby highlighting the importance of fluctuation-dissipation relations and long-time correlations between subsystems. Our results introduce consistency conditions for (numerical) models of nonequilibrium dynamics of open quantum systems.

Introduction.– Fluctuations have profound impact on physical reality, ranging from weak yet measurable forces all the way to structure formation in the universe. In the quantum realm, the existence of fluctuation-induced interactions was confirmed by pioneering [1, 2] and ensuing experiments with increasing accuracy and scope [3–13].

Many theoretical approaches have been designed to explain each distinct manifestation of these quantum fluctuations phenomena. However, a broader perspective is captured by the fluctuation-dissipation theorem (FDT): For an open system in equilibrium, this theorem expresses the detailed balance between incoming and outgoing power, ensuring that the system is in a state of maximal entropy [14]. When nonequilibrium conditions prevail, the description of quantum fluctuation-induced phenomena is remarkably more involved and, to the best of our knowledge, there are no general FDTs for nonequilibrium systems. Instead, a convenient assumption known as local thermal equilibrium (LTE) is often invoked [15]. This assumption significantly reduces the mathematical complexity of the problem and was broadly applied to the situation of temperature gradients between macroscopic bodies [16–20], atom-surface forces in thermal [21, 22] as well as mechanical [22, 23] nonequilibrium or under the influence of external driving fields [24], and for computing the radiation of a relativistic electron close to an interface [25]. However, the theoretical basis for LTE and the conditions in which it fails to apply are usually not so well discussed: First, under nonequilibrium conditions, detailed balance (which is implicitly contained in LTE) is broken and, second, LTE is known to often disregard back-action of the environment [26–28]. In quantitative terms, LTE was already proven to be insufficient in the context of atom-surface quantum friction, e.g., underestimating the force by roughly half [29] or misrepresenting other important mechanisms [30].

In the framework of nonequilibrium atom-surface interactions, other often used methods have their own strengths and shortcomings. For instance, the Born-Markov approximation (BM) [31, 32] or a perturbative treatment of the atomic level shift [33, 34] do not rely on equilibrium. However, with regards to back-action and memory effects, these methods can only partially capture the impact of the environment [35]. For quantum friction, they have been shown to lead to an incorrect velocity scaling [36, 37] or erroneously predict exponentially vanishing forces (see the discussion in Refs. [22, 35]).

In this Letter we address the deficiencies of these commonly used assumptions and approximations from a novel perspective, namely, the nonequilibrium thermodynamics of quantum friction. Even when the discrepancy between the approximate and the more carefully derived results might seem to be quantitatively marginal on the level of forces, the errors become manifest and easily identifiable when one applies the thermodynamic principles. In fact, neglecting the memory of the interaction or the long-time correlations between system and environment – as the BM and the LTE assumption do – can lead to non-existing thermodynamic instabilities, such as, in the case of quantum friction, an over-time increase to infinity of the internal energy of the atom. Our cure to this is the thermodynamic principle-enforced, self-consistent (back-action included) treatment of the relevant nonequilibrium quantum processes. This provides us with a benchmark to identify and explain why other approximate theories succeed or fail.

Physical Model.– We consider an atom moving at nonrelativistic velocity \( v \) along an axis of translational symmetry of one or an entire arrangement of several macroscopic objects with arbitrary cross-sectional shape. These objects are comprised of non-magnetic, reciprocal and spatially homogeneous materials. We focus on the fluctuation-induced interaction between the atomic electric dipole moment \( \mathbf{d} \) and the material-modified fluctuating electric field \( \mathbf{E} \) and demand that the atom’s center of mass approximately obeys a classical trajectory. This
implicitly includes the existence of an external “agent” driving the atom in such a way as to maintain uniform motion. We assume that the back-action of our total system, composed of the atom+field+matter, on the agent is sufficiently small compared to the force the agent exerts on the system to keep the atom moving at uniform velocity. Thus we can safely consider the inflow of energy to the moving atom from the agent separately from the outflow of energy from the atom to the field modified by the material. The back-action of the material-modified field on the atom appearing as quantum friction is of course included, it being the main character in the drama [38]. Finally, we assume zero temperature and an initial state factorized in the distant past [39].

In the static case (v = 0), it can be shown that such a dynamical system equilibrates at late times [40]. For atomic velocities v ≠ 0, however, the state of the system can deviate from the global equilibrium condition [41]. Also, for finite coupling strength, system and environment are inseparably intertwined and the assumption that equilibrium ensues locally is not warranted. Yet, dissipation (e.g. in the material) leads to finite correlation times between system and environment establishing irreversibility in the interaction. When the different irreversible processes balance, the dynamics of the system becomes stationary and it reaches a nonequilibrium steady state (NESS) [42]. Such a state is thermodynamically characterized by existence of a non-vanishing current of energy, sourced by the external drive [43] and compensating for all different forms of losses in the system [Eq. (1)]. For the atomic subsystem the NESS requires a balance between incoming \( P_{\text{in}} \) and outgoing power \( P_{\text{out}} \) from and to the material-modified vacuum, respectively. If otherwise, the atomic energy would be changing continuously contradicting stationarity. In the following, in lieu of a rigorous proof of the existence of the NESS [44], we provide an explicit late-time solution for a specific model [see Eq. (1)] and show that the anticipated power balance \( P_{\text{in}} = P_{\text{out}} \) holds, but only under certain conditions. This supplies a physical reasoning for its existence in more general contexts.

Moving at constant velocity, the atom’s internal degrees of freedom are in continuous exchange of energy, translational and angular momentum with the surrounding material-modified quantum field. In the stationary limit for linear systems, these processes can be described in terms of the 3D-Langevin equation [45],

\[
\frac{\ddot{d}(t) + \omega_0^2 \dot{d}(t)}{\alpha_0 \omega_0^2} + 2 \int_0^\infty \text{d} \tau \gamma(\tau, v) \cdot \dot{d}(t - \tau) = \xi(t, v),
\]

where \( \alpha_0 \) is the atomic static polarizability and \( \omega_0 \) the bare resonance frequency of the dominant dipole transition. The quantum Langevin force and the dissipative memory kernel, respectively, can be written as [30]

\[
\begin{align*}
\dot{\xi}(t, v) &= \int \frac{\text{d} \omega}{2\pi} \int \frac{\text{d} q}{2\pi} \tilde{E}_0(q, \mathbf{R}_a, \omega) e^{-i\omega q^2 t}, \\
\gamma(t, v) &= \int \frac{\text{d} \omega}{2\pi} \int \frac{\text{d} q}{2\pi} \frac{G_\alpha(q, \mathbf{R}_a, \omega)}{\omega_q} e^{-i\omega_q t}. 
\end{align*}
\]

Here, \( q \) is the component of the radiation’s wave vector in the direction of motion, while \( \mathbf{R}_a \) is the atom’s position in the plane orthogonal to it. We also defined the Doppler-shifted frequency as \( \omega_q = \omega \pm q v \). The atomic system is driven by the fluctuations of the field in absence of the atom, \( \tilde{E}_0 \). The dispersion as well as dissipation mechanisms are encoded in the Green tensor \( G \) with \( G_\alpha = (G - G_0^0)/(2i) \). \( G \) solves the Maxwell equations with appropriate boundary conditions and hence incorporates the material properties, the translational symmetry of our system, and ensures the causality of the interaction [40]. Consequently, \( G_\alpha \) is a Hermitian positive semidefinite matrix for \( \omega > 0 \), while a stationary and a causal dynamics of the dipole implies that \( \gamma(\omega, v) \) must be positive definite. Since without the moving atom the system is in equilibrium, the field \( \tilde{E}_0 \) must satisfy the FDT

\[
\langle \tilde{E}_0(q, \omega) \tilde{E}_0(q', \omega') \rangle = h(2\pi)^2 \text{sgn}(\omega) G_\alpha(q, \mathbf{R}_a, \omega) \delta(\omega + \omega') \delta(q + q'),
\]

where \( \text{sgn}(\omega) \) is the sign-function and \( \delta(x) \) the Dirac delta. Hereafter we consider the symmetric quantum average, i.e. \( \langle AB \rangle = \langle BA \rangle /2 \). It has the advantage of rendering all quantum averages real [47]. Equation (1) is solved in Fourier domain as \( \hat{d}(\omega, v) = \hat{a}(\omega, v) \cdot \hat{\xi}(\omega, v) \) by means of the dressed and velocity-dependent atomic polarizability \( \hat{a}(\omega, v) \) (see Ref. [48] for details). Physically, the latter contains spontaneous emission [49], dispersion and dissipation due to the presence of the material [50]. The correlation matrix of the Langevin force becomes stationary and real in the steady state, i.e. \( \langle \xi(t, v) \xi(t', v) \rangle \equiv \hbar \nu(t, t', v) + \hbar \nu(\tau, v) (\tau = t - t') \). Moreover, the quantum noise is colored:

\[
\nu(\omega, v) = \int \frac{\text{d} q}{2\pi} \text{sgn}(\omega_q) G_\alpha(q, \mathbf{R}_a, \omega_q^2). 
\]

Our self-consistent treatment of the system [Eq. (1)] describes the connection between field fluctuations and dipole fluctuations via the relation

\[
\langle \hat{d}(\omega) \hat{d}(\omega') \rangle = 2\pi \hbar \sum(\omega, v) \delta(\omega + \omega'),
\]

where \( \sum(\omega, v) = a(\omega, v) \nu(\omega, v) a^\dagger(\omega, v) \) is positive semidefinite for all \( \omega \) because of the properties of all involved matrices [48]. The relations in Eqs. (4) and (5) generalize the FDT to the NESS and lead to previously reported results on quantum friction [29, 30, 51].
Nonequilibrium Thermodynamics.—We now examine the thermodynamic implications of Eqs. [4] and [5]. The “in” and the “out” parts of the moving atom’s energy flow per unit of time are

\[
P_{\text{in}} = \langle \dot{\mathbf{x}}(t,v) \cdot \mathbf{d}(t) \rangle,  \tag{6a}
\]

\[
P_{\text{out}} = 2 \int_{0}^{\infty} d\tau \langle \dot{\mathbf{d}}(t) \cdot \gamma(t,v) \cdot \dot{\mathbf{d}}(t-\tau) \rangle,  \tag{6b}
\]

which yield a change in energy $E$ of the atom given by $\dot{E} = P = P_{\text{in}} - P_{\text{out}}$. Using Eq. (1), we can show that $P = 0$ in the NESS (see Ref. [18]), i.e. there is no net energy flow in or out of the system since

\[
P_{\text{in}} = P_{\text{out}} = 2 \int_{0}^{\infty} d\omega \frac{\hbar \omega}{2\pi} \text{Tr} [\hat{G}_{3}(\omega)G_{3}(\omega,v)],  \tag{7}
\]

where, similarly to $G_{3}$, we defined $\alpha_{3} = (\omega - \alpha^{\dagger})/(2i)$ and “Tr” takes the trace of the resulting matrix.

A few comments are in order. First, $P_{\text{in/out}}$ is positive since $\alpha_{3}(\omega,v)$ is positive definite for $\omega \geq 0$ [48]. Notably, within our initial assumptions, the previous results hold for any (non-relativistic) velocity and arbitrary functional frequency-behavior of the memory kernel. In particular, the damping $\gamma$ needs not be Ohmic and it can contain any physical resonance of the system.

Second, a vanishing power is equivalent to the condition $\dot{\mathbf{d}} \cdot \mathbf{E} = 0$ in the NESS, where $\mathbf{E}$ is the total field acting on the moving dipole. This allows us to formulate a relation between the (mechanical) frictional force $F_{\text{fric}}$ and the total power radiated from the particle into the environment $P_{\text{rad}}$ [48] [52]. We have $P_{\text{rad}} = -vF_{\text{fric}}$ where

\[
P_{\text{rad}} = 2\text{Tr} \int_{0}^{\infty} d\omega \int \frac{d\mathbf{q}}{2\pi} \sum_{\mathbf{R}_{\alpha}} \hat{G}^{T}(-\omega_{\mathbf{q}}^{},v)G_{3}(\mathbf{q},\mathbf{R}_{\alpha},\omega)  \tag{8}
\]

with “$T$” the transpose of a matrix. Here, $\hat{G}$ is the atomic power spectrum tensor defined in previous work and for our system it has a form very similar to $\Sigma$ [36] [48]. The expression for $F_{\text{fric}}$ is instead recovered by replacing $\omega \rightarrow q$ in the previous integrand [48]. The identity $P_{\text{rad}} = P_{\text{ext}}$ (power delivered by the external agent) is the counterpart to $P_{\text{out}} = P_{\text{in}} (P = 0)$, but puts the accent on the balance between the total mechanical power (work per unit time performed by the external agent balancing the frictional force) and the electromagnetic energy dissipated in the environment per unit time. The relation between $P_{\text{rad}}$ and $F_{\text{fric}}$ offers an alternative, more general perspective on the irreversible flow of energy (accompanied by the production of entropy) through the system [53], since it does not require a specific model for the atom’s internal degrees of freedom [36] [48].

Third, $P = 0$ implies that the total energy $E$ corresponding to the atom’s internal dynamics is constant [54]. From Eqs. (1) - (5), $E$ can be written as an integral over positive frequencies of the spectral density [48] [45]

\[
E(\omega, v) = \frac{\hbar}{2\pi} \omega_{0}^{2} + \omega_{v}^{2} \text{Tr} \left[ \frac{\hat{\Sigma}(\omega,v)}{\alpha_{0}} \right] \geq 0.  \tag{9}
\]

In equilibrium ($v = 0$) and in the weakly coupled limit ($\alpha_{0} \rightarrow 0$) $E \rightarrow \hbar \omega_{0}/2$ as expected, while a finite coupling effectively modulates its value [40]. Deviating from equilibrium ($v \neq 0$), the energy becomes an even function of the velocity and at the leading order in $\alpha_{0}$ we have

\[
E(0, v) \propto \alpha_{0} \varepsilon(v) \neq 0,  \tag{10}
\]

where $\varepsilon$ is a function of velocity with $\varepsilon(0) = 0$ [45]. Equation (10) is thermodynamically related to the stationary energy flow through the atom in the NESS and highlights two important aspects of our analysis: On the one hand, low frequencies (long-time correlations) play an important role in correctly capturing the nonequilibrium physics of the system. On the other hand, in equilibrium, $E$ vanishes for $\omega \rightarrow 0$, in agreement with the FDT and with a thermodynamically consistent description of a dissipative atomic system at $T = 0$. In contrast, assuming local equilibrium enforces $E(0, v) = 0$ for all atomic velocities (see also Fig. 1). Similarly, within the BM or a related perturbative treatment, even at $v = 0$, $E$ approaches a nonzero constant for $\omega \rightarrow 0$, whose value depends on the involved dissipative mechanisms and might be related to the initial state preparation [36] [48]. This means then that, although in different ways, both the LTE and the BM descriptions misrepresent the low frequency contributions to the system’s dynamics. Specifically for our system, Eqs. (9), (10) and the expressions for $F_{\text{fric}}$ [45] imply that an adequate description of the nonequilibrium process requires at least $\mathcal{O}(\alpha_{0}^{2})$. Consequently, the thermodynamical consistency and/or the accuracy of results that address the frictional process to first order in the atomic polarizability can be questionable and must be interpreted with care, depending on the specific approach being employed as well as on the dissipative mechanisms at work in the system. For instance, previous work has shown that, despite the LTE assumption for quantum friction can be justifiable to some extend at orders $\mathcal{O}(\alpha_{0})$ for a particle dynamics that allows for strong intrinsic dissipation (e.g. for metallic nanoparticles), it fails when radiation-induced damping prevails and back-action is relevant [29] [51].

Finally, it is important to underline that despite its direct appeal, the result $P = 0$ is technically non-trivial to realize. It could only be achieved with careful “bookkeeping” of the system’s full roto-translational spectrum of correlations taking the back-action from the environment fully into account [Eq. (5)]. Any deviation from this complete self-consistency can lead to thermodynamical instabilities. This is indeed the case for the LTE approach, which amounts to replacing $\gamma(\omega, v) \rightarrow \omega \text{sgn}(\omega) \gamma(\omega, v)$ in Eq. (5). It effectively neglects the Doppler-shift of the radiation in the evaluation of the sign-function in $\gamma(\omega, v)$ and breaks the total power balance, contradicting the sta-
tionarity condition for NESS. In this case we have [48]

\[ P^{\text{LTE}} = 2 \int_0^{\infty} \frac{d\omega}{2\pi} \hbar \omega \text{Tr} \left[ \left\{ \nu(\omega, v) - \omega \gamma(\omega, v) \right\} \alpha_\gamma(\omega, v) \right] \]

\[ \equiv P^{\text{LTE}}_{\text{in}} - P^{\text{LTE}}_{\text{out}} \neq 0. \]  

(11)

This is the thermodynamic evidence that not including nonequilibrium backaction in perturbative approaches or simplifying assumptions can lead to glaring mistakes. In contrast, nonequilibrium dynamics with self-consistent backaction is fully guaranteed from the thermodynamic principles we invoke.

**Fluctuation-Dissipation-Inequality.**—Equation (11) shows that the relation between the matrices \( \nu(\omega, v) \) and \( \gamma(\omega, v) \) gives a measure of the impact of nonequilibrium onto the system. If we define \( \bar{G}_\gamma(q, R_a, \omega) = \text{sgn}(\omega) G_\gamma(q, R_a, \omega) \) and use the identity \( \text{sgn}(x)[\text{sgn}(x) \pm 1] = 2\theta(\pm x) \), we can write

\[ \nu(\omega, v) \pm \omega \gamma(\omega, v) = \int \frac{d\theta}{\pi} \theta(\pm \omega_q) \bar{G}_\gamma(q, R_a, \omega^+) \]  

(12)

which is Hermitian and positive semidefinite for all values of \( q \) and \( \omega \). We can then conclude that for our system \( P^{\text{LTE}} \geq 0 \) for all velocities and colors of the noise. Also, using the Loewner order [55], in accordance with the fluctuation-dissipation inequality put forward in Ref. [56], we can write \( \nu(\omega, v) \geq |\omega \gamma(\omega, v)| \). This indicates that the fluctuations of the field \( \nu \) are always equal or exceed the hypothetical ground state fluctuations assigned to the given dissipation \( \langle \omega \gamma \rangle \) [56]. The matrix \( \nu(\omega, v) - \omega \gamma(\omega, v) \) only goes to zero either for \( v = 0 \) restoring the equilibrium FDT, or asymptotically for frequencies \( \omega \gg v/\lambda \), where \( \lambda \) is a length scale characterizing the geometry and the optical response of the corresponding materials. In agreement with the behavior of the energy spectral density [Eq. (10)], the largest deviations occur at low frequencies \( \omega \ll v/\lambda \), emphasizing once again their connection to the nonequilibrium dynamics of our system. Physically, this shows that simply using the equilibrium FDT neglects the interaction energy that corresponds to correlation times larger than \( \lambda/v \) (of the order of nanoseconds for typical values). These correlations are an inalienable part of the system interacting with its environment and an important feature of nonequilibrium settings. The fluctuation-dissipation inequality quantifies this mismatch and the complete description of the system requires a more careful treatment by means of the generalized FDT [Eq. (4)].

To obtain quantitative insight, it is interesting to consider the case of an atom moving at a distance \( z_a \sim \lambda \) close to a planar interface separating vacuum from an infinite half-space composed of a typical Ohmic dissipative and spatially local material (Fig. 1) [55]. For this geometry, the analytic expression for the Green tensor is known [49]. Since \( v/z_a \) is usually in the material’s Ohmic

![FIG. 1. Spectral energy for an atom moving parallel to a planar interface (solid line) and respective LTE result (dashed). We employ the Drude model, where \( r_{\text{TM}} = \omega_a^2 [\omega_{\text{sp}}^2 - \omega^2 - i\Gamma \omega]^{-1} \) with \( \omega_{\text{sp}} \) the surface plasmon-polariton resonance and \( \Gamma \) the associated damping. We set \( v = 10^{-3}c \), \( z_a = 1 \) nm and use parameters for gold [57]. Inset: Fluctuation-dissipation inequality and the asymptote of Eq. (13) (dashed).](attachment:figure1.png)
for given functional characteristics using numerical optimization procedures [62]. In nonequilibrium setups, this is a particularly complicated problem since one is mostly concerned with vector-valued quantities and a complex resonance structure that can lead to numerical obstacles [63]. Also, due to the lack of analytical solutions, one has to rely on limiting scenarios as well as more general properties based on the system’s symmetries for validating the obtained result. Power balance and the described inequality hence serve as a benchmark for such nonequilibrium calculations. Additionally, due to the extensive efforts in controlling atomic systems (see also Refs. [64]-[66] in addition to the above), the principles and methodology presented here can be used for experimentally understanding and probing nonequilibrium fluctuation theorems [53] and entropy production in nonequilibrium situations [111][68]. In particular, this means that we can provide a general proof of what is often found case by case based on partially justifiable assumptions. Experimentally, when signatures of quantum friction are detected, our criteria can be used to ascertain and discriminate whether it truly originates from nonequilibrium quantum fluctuations.

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Notice that the Born-Markov approximation and its thermodynamical implications were recently under scrutiny also concerning the proper treatment of the coupling between system and bath, the so-called “local versus global” approaches to master equations [69, 70].

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For a more general treatment of the material properties in the context of quantum friction see, e.g., Refs. [50, 82].

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SUPPLEMENTAL MATERIAL

The nonequilibrium power flux

The 3D-quantum Langevin equation [Eq. (1) of the main text], describing the atom’s internal dynamics, is solved by \( \dot{\mathbf{d}}(\omega, v) = \alpha(\omega, v) \cdot \dot{\mathbf{\xi}}(\omega, v) \) (stationary solution), where

\[
\alpha(\omega, v) = \alpha_B(\omega) [1 - \alpha_B(\omega) \Delta(\omega, v)]^{-1}
\]

is the dressed velocity-dependent polarizability. The scalar function \( \alpha_B(\omega) = \alpha_0 \omega_0^2 / (\omega_0^2 - \omega^2) \) is the atomic bare polarizability and, using the Kramers-Kronig relations, we have defined

\[
\Delta(\omega, v) = \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{\omega(\omega', v) + i\omega(\omega, v)}{\omega - \omega'} = \int \frac{d\omega'}{2\pi} G(q, R_a, \omega_q^+) \]

with \( \mathcal{P} \) the Cauchy principal value. The integration goes over the whole real axis if not indicated otherwise. The properties of the Green tensor yield some important relations: \( G(-q, R_a, \omega) = G^T(q, R_a, \omega) \) and \( G^*(q, R_a, \omega) = \bar{G}(q, R_a, -\omega) \). They imply that \( \alpha(-\omega, v) = \alpha(\omega, v) \) and \( \alpha(\omega, -\omega) = \alpha(-\omega, -\omega) \) as well as the identity

\[
\alpha_\alpha(\omega, v) = \int \frac{dh}{2\pi} \bar{\alpha}(\omega, v) \bar{G}_\alpha(q, R_a, \omega_q^+) \bar{\alpha}^\dagger(\omega, v) = \omega \bar{\alpha}(\omega, v) \gamma(\omega, v) \bar{\alpha}^\dagger(\omega, v)
\]

where, in analogy to \( G_\alpha \), we defined \( \alpha_\alpha = (\alpha - \alpha^\dagger) / 2(\mathcal{I}) \). Since \( \gamma \) is positive semidefinite so is \( \alpha_\alpha > 0 \). Our self-consistent description also leads to the matrix \( \gamma \) and to the definition of \( \Sigma \) which are also positive semidefinite (see its definition and the discussion around Eq. (5) of the main text).

From the previous expressions, we can write the power flowing into the atomic subsystem due to fluctuations, \( P_{in} = \langle \mathbf{\xi}(t, v) \cdot \dot{\mathbf{d}}(t) \rangle \), as follows

\[
P_{in} = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} (-i\omega') e^{-(i\omega + \omega')t} \times \text{Tr} \left[ \bar{\alpha}^\dagger(\omega', v) \mathbf{\xi}(\omega', v) \mathbf{\xi}(\omega, v) \right]
= \int \frac{d\omega}{2\pi} i\hbar \omega \text{Tr} \left[ \bar{\alpha}^\dagger(\omega, v) \nu(\omega, v) \right]
= 2 \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \text{Tr} \left[ \nu(\omega, v) \alpha_\alpha(\omega, v) \right]
\]

where we used that \( \text{Tr}[A^\dagger] = \text{Tr}[A] \) for any matrix \( A \) and \( \nu(-\omega, v) = \nu^\dagger(\omega, v) \). Similarly, the power leaving the atomic subsystem due to dissipation reads

\[
P_{out} = 2 \int_0^\infty \frac{d\tau}{\pi} \text{Tr} \left[ \gamma^T(\tau, v) \dot{\mathbf{d}}(t) \mathbf{d}(t - \tau) \right]
= 2 \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \int \frac{d\omega'}{2\pi} \times \text{Tr} \left[ \bar{G}_\alpha(q, R_a, \omega_q^+) \alpha(\omega, v) \nu(\omega, v) \alpha^\dagger(\omega, v) \right]
= 2 \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \int \frac{d\omega'}{2\pi} \times \text{Tr} \left[ \alpha(-\omega, -\omega) \bar{G}_\alpha(q, R_a, \omega_q^+) \alpha^\dagger(-\omega, -\omega) \nu(\omega, v) \right]
= 2 \int_0^\infty \frac{d\omega}{2\pi} (-\hbar) \text{Tr} \left[ \nu(\omega, v) \alpha_\alpha(-\omega, -\omega) \right].
\]

Since \( \alpha_\alpha(-\omega, -v) = -\alpha_\alpha(\omega, v) \) it follows that

\[
P_{in} = P_{in} = P_{in}^{LTE}.
\]

The self-consistency of our treatment is central for obtaining the previous result. A deviation from it can lead to a steady nonzero power transfer to the atomic system. Even if this value is small, it will be accumulating over time and the consequences of making artificial assumptions on the underlying statistics of the interaction can be dramatic. An example is the impact of the local thermal equilibrium (LTE) assumption. This approach does not modify \( \langle \mathbf{\xi}(\omega, v) \mathbf{\xi}(\omega', v) \rangle \) effectively leading to

\[
P_{in} = P_{in}^{LTE}.
\]

Instead, \( P_{out} \) is intimately related to the nonequilibrium relation in Eq. (5). The LTE approach assumes the dipole correlations to fulfill the equilibrium FDT leading to

\[
\langle \dot{\mathbf{d}}(\omega) \dot{\mathbf{d}}(\omega') \rangle_{LTE} = 2\pi \hbar \text{sgn}(\omega) \alpha_\alpha(\omega, v) \delta(\omega + \omega').
\]

In some cases even the dependence of the polarizability on the velocity is ignored [22]. The LTE assumption then modifies the outgoing power as follows

\[
P_{out}^{LTE} = 2 \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \text{Tr} \left[ \text{sgn}(\omega \omega) \omega_\gamma(\omega, v) \alpha_\alpha(\omega, v) \right] \neq P_{out}.
\]

Equation [22], eventually leads to an imbalance of the total power \( P_{LTE} = P_{in} - P_{out} \), i.e.

\[
P_{LTE} = 2 \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega \times \text{Tr} \left[ \left\{ \nu(\omega, v) - \text{sgn}(\omega) \omega_\gamma(\omega, v) \right\} \alpha_\alpha(\omega, v) \right]
\]

which is in general positive. Indeed, using the fluctuation-dissipation inequality (see the main text), the previous integrand contains the trace of two positive semidefinite matrices, which is always positive or zero.

At low velocity and at the leading order in \( \alpha_0 \), the previous expression takes the form

\[
P_{LTE} \sim \frac{\hbar}{\pi} \alpha_0^2 \omega^4 \int \frac{dq}{2\pi} \int \frac{dq'}{2\pi} \langle q^2 - 2qq' \rangle \times \text{Tr} \left[ \bar{G}_\alpha(q, R_a, 0) \cdot \bar{G}_\alpha(q', R_a, 0) \right].
\]
where the prime indicates a derivative with respect to frequency.

Consider now the specific case of an atom moving along the \(\mathbf{x}\)-direction in front of planar metallic interface. For atom-surface separations \(z_a\) smaller than the metal’s plasma wavelength (typically \(\sim 100\ \text{nm}\) or larger [57]), the Green tensor is dominated by its scattered part evaluated in the near-field limit. For a plane we have

\[
\mathcal{G}_\mathcal{A}(q, \mathbf{R}_a, \omega) \equiv \mathcal{G}(p_x, z_a, \omega) \\
\sim \int \frac{dp_x}{2\pi} \frac{p}{2\epsilon_0} r^{\text{TM}}(\omega) e^{-2pz_a} \Pi_x \Pi_-
\]

(26)

with \(\epsilon_0\) the vacuum permittivity and \(r^{\text{TM}}\) the transverse magnetic reflection coefficient. The vectors \(\Pi_x = \mathbf{z} \mp ip/p\) describe the near-field polarization, where \(p\) is the component of the wave vector parallel to the surface \((q = p_x\) and \(p = |p| = \sqrt{p_x^2 + p_y^2}\)), and \(\mathbf{z}\) the unit vector orthogonal to the surface. For simplicity, we describe the metal using the spatially local Drude dielectric function

\[
\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\Gamma\omega},
\]

(27)

where \(\omega_p\) is the plasma frequency and \(\Gamma\) the metal’s dissipation rate. In the near-field limit, the reflection coefficient can then be written as

\[
r^{\text{TM}}(\omega) = \frac{\epsilon(\omega) - 1}{\epsilon(\omega) + 1} = \frac{\omega_p^2}{\omega_{sp}^2 - \omega^2 - i\Gamma\omega},
\]

(28)

where we also defined the surface plasmon-polariton frequency \(\omega_{sp} = \omega_p/\sqrt{2}\) [57]. At low frequencies, we have \(\text{Im}(r^{\text{TM}}(\omega)) \sim 2\epsilon_0\omega\rho\), where \(\rho = \Gamma/\omega_p^2\) is the metal’s resistivity. Upon using the near-field expression of the Green tensor in Eq. (25), the power evaluates to

\[
P^{\text{LTE}} \sim 45\hbar v^4 \frac{\alpha_0^2 \rho^2}{(2\pi)^3 (2z_a)^{10}} > 0,
\]

(29)

i.e. a positive total power flux that tends to constantly increase the internal energy of the atom [23]. The previous value would have been even larger for a velocity-independent polarizability.

### Power and frictional force

The balance between \(P_{\text{in}}\) and \(P_{\text{out}}\) is equivalent to the condition that in the NESS \((\dot{\mathbf{d}} \cdot \mathbf{E}) = 0\). Physically, this is equivalent to saying that the total power transferred to or dissipated within the atom must vanish in the steady-state. Proceeding as in Ref. [30] one can show that the condition \((\dot{\mathbf{d}} \cdot \mathbf{E}) = 0\) implies

\[
\lim_{t_i \to \infty} \lim_{t \to \infty} \text{Re} \left( \frac{2i}{\pi} \int_0^\infty dq \int_0^{t-i} d\tau e^{-i\omega\tau} \int \frac{d\omega}{2\pi} \right. \\
\times \left. \text{Tr} \left[ \partial_t \mathcal{C}(t, t - \tau) \cdot \mathcal{G}_\mathcal{A}(q, \mathbf{R}_a, \omega) \right] \right) = 0
\]

(30)

where \(\tau = t - t'\) and \(x_a(t)\) is the atomic trajectory. In the previous expression, \(\mathcal{C}(t, t') = \langle \dot{\mathbf{d}}(t) \dot{\mathbf{d}}(t') \rangle\) is the dipole correlation matrix defined as in Refs. [29, 30]. Notice that, contrary to what was used in the main text, in this approach the usual (non-symmetric) quantum average is considered, i.e. \((A\mathcal{B}) \equiv \langle A\mathcal{B} \rangle\).

Using that \(\partial_t = \partial_\tau\), we always have that in the NESS,

\[
\partial_\tau \mathcal{C}(t, t - \tau) \xrightarrow{\text{NESS}} \mathcal{C}(t, t - \tau)
\]

(31)

In the limit \(-t_i, t \to \infty\), Eq. (30) leads to the expression

\[
2 \int_0^\infty dq \int \frac{d\omega}{2\pi} \omega q \text{Tr} \left[ \mathcal{S}^{\text{T}}(-\omega q, v) \mathcal{G}_\mathcal{A}(q, \mathbf{R}_a, \omega) \right] = 0
\]

(32)

where we used that the trace of the product of two Hermitian matrices is real. The previous relation can be rewritten as follows: \(P_{\text{rad}} = -v F_{\text{fric}}\), where

\[
F_{\text{fric}} = -2 \int_0^\infty dq \int \frac{d\omega}{2\pi} q \text{Tr} \left[ \mathcal{S}^{\text{T}}(-\omega q, v) \mathcal{G}_\mathcal{A}(q, \mathbf{R}_a, \omega) \right]
\]

(33)

is the frictional force \((-v F_{\text{fric}}\) is the work per unit of time preformed by the external agent) and

\[
P_{\text{rad}} = 2 \int_0^\infty dq \int \frac{d\omega}{2\pi} \omega q \text{Tr} \left[ \mathcal{S}^{\text{T}}(-\omega q, v) \mathcal{G}_\mathcal{A}(q, \mathbf{R}_a, \omega) \right]
\]

(34)

defines the electromagnetic power dissipated (radiated) into the environment [52].

Importantly, in all these results the expression for the power spectrum tensor \(\mathcal{S}\) are left unspecified and therefore they do not rely on any specific model for the atom’s internal dynamics. For the case considered in the main text, \(\mathcal{S}\) has the same expression as \(\mathcal{S}\) where, however, the sign-function appearing in Eq. (4) is replaced by \(2\theta(\omega^+_q)\), with \(\theta(x)\) the Heaviside function [31].

### Atomic Steady-state energy

The equivalence of \(P_{\text{in}}\) and \(P_{\text{out}}\) leaves the energy of the atomic subsystem constant. It can be written as

\[
E = \lim_{t \to \infty} \frac{\langle \dot{\mathbf{d}}(t) \cdot \dot{\mathbf{d}}(t') \rangle + \omega_n^2 \langle \mathbf{d}(t) \cdot \dot{\mathbf{d}}(t') \rangle}{2\alpha_0 \omega_n^2}
\]

(35)
where we highlighted the connection to the correlation function. At late times, we already found that
\[
\hat{d}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dq}{2\pi} \alpha(\omega, v) \mathcal{E}_0(q, R_a, \omega^+_q)
\]
and similarly for its time derivative. Using the expression derived in the main text and the ones above, we have
\[
E = \text{Tr} \int \frac{d\omega}{2\pi} \frac{h}{2} \frac{\omega_a^2 + \omega^2}{2 \omega_a^2} \frac{\alpha(\omega, v) \nu(\omega, v) \alpha^\dagger(\omega, v)}{\alpha_0} \alpha_0

= \int_0^\infty \frac{d\omega}{2\pi} \frac{h}{2} \frac{\omega_a^2 + \omega^2}{\omega_a^2} \text{Tr} \left[ \frac{\Sigma(\omega, v)}{\alpha_0} \right]
\]
(37)
obtaining the definition for \( E(\omega, v) \) given in Eq. (9). In the last expression we used the properties of the involved matrices discussed above, which also allow us to say that \( E \) (and therefore \( E \)) is an even function of \( v \).

In equilibrium \((v = 0)\), we recover the FDT and then
\[
E(\omega, 0) = \hbar \frac{\omega_a^2 + \omega^2}{\omega_a^2} \text{Tr} \left[ \frac{\text{Im}\alpha(\omega)}{\alpha_0} \right],
\]
(38)
where \( \text{Im}\alpha(\omega, 0) = \text{Im}\alpha(\omega) \). \( E \) then takes the form previously obtained in the literature \[32\]. For a generic atomic system, since \( \alpha_I(0) = 0 \), we have that \( E(0, 0) = 0 \) as reported in the main text. Notice that in equilibrium, the BM approximation leads to a dipole correlation function given in terms of a (multi-)exponentially decay function \[36\] \[78\] \[79\]. For a single resonance \( \omega_a \), \( C_{\text{BM}}(\tau) \approx \langle \hat{d}\hat{d}\rangle e^{-i\omega_a \tau - \gamma_a |\tau|} \), where \( \gamma_a \geq 0 \) is related to the dissipative atom’s dynamics. From Eq. \[35\], we have
\[
E_{\text{BM}}(\omega, 0) \rightarrow \langle \hat{d}^2 \rangle \frac{\omega_a^2 + \omega^2}{2\pi\alpha_0} \frac{\gamma_a}{(\omega - \omega_a)^2 + \gamma_a^2}
\]
(39)
(a multi-exponential decay leads to a similar expression).

Therefore \( E_{\text{BM}}(0, 0) \neq 0 \) and, since \( \langle \hat{d}^2 \rangle \propto \alpha_0 \), its value only depends on the dissipative mechanism and the resonance. Usually, however, \( \gamma_a = \mathcal{O}(\alpha_0) \) (e.g. for radiation damping) leading to the same behavior for \( E(0, 0) \).

For \( v \neq 0 \) neither \( \alpha \) nor \( \nu \) in Eq. \[37\] are vanishing for \( \omega = 0 \). Therefore \( \Sigma(0, v) \propto \alpha_0 \) with a prefactor which is even in the velocity, leading to Eq. (10) of the main text. In general, the functional behavior of \( E \) on \( v \) depends on how the velocity relates to the other system’s characteristic scales (e.g. \( \omega_a \) and/or \( \omega_{\text{sp}} \)), featuring a non-resonant and a resonant regime \[36\]. Similar to the quantum frictional force, the energy changes from a power law to an exponential behavior (vanishing for decreasing \( v \)) as a function of the velocity.