PROJECTIONS AND INJECTIVE OBJECTS IN SYMMETRIC CATEGORICAL GROUPS

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Dedicated to the memory of Prof. V. K. Bentkus

Categorical rings (called also 2-rings) were introduced in [4]. Categorical modules (called also 2-modules) over a categorical ring were introduced in [7]. Categorical modules and symmetric categorical groups [1] are examples of abelian 2-categories studied in [2] (for other examples of abelian 2-categories see [5]). Projective objects in the framework of symmetric categorical groups were introduced in [1] have an obvious generalization to the case of abelian 2-categories. Hence by duality one can also talk on injective objects in any abelian 2-category. Moreover in [1] the authors conjectured that the abelian 2-category of symmetric categorical groups have enough projective objects. In the course of our work [6] we noted that the 2-category of symmetric categorical groups have enough projective and injective objects. Moreover this statement is a trivial consequences of the cohomological description of abelian 2-category of symmetric categorical groups obtained first in [8]. Using base change argument this result also yields that the 2-category of categorical modules over any categorical ring have enough projective and injective objects.

Quite recently these results were announced in [3] but with wrong proofs (Lemmata 3 and 11 of loc. cit. are both wrongs). Here we give our original proofs.

A groupoid enrich category $\mathcal{T}$ is a 2-category such that any 2-arrow is invertible. If $\mathcal{T}$ is a groupoid enrich category then we use the word "morphism" for 1-morphisms and we use the word "track" for 2-morphisms. We let $\text{Ho}(\mathcal{T})$ be the corresponding homotopy category. If $f, g : A \to B$ are morphism in $\mathcal{T}$, then we say that $f$ is homotopic to $g$ if there exists a track from $f$ to $g$. Let $\text{Ho}(\mathcal{T})$ be the corresponding homotopy category. Thus objects of $\text{Ho}(\mathcal{T})$ are the same as of $\mathcal{T}$, while morphisms in $\text{Ho}(\mathcal{T})$ are homotopic classes of morphisms in $\mathcal{T}$.

Symmetric categorical groups form a groupoid enrich category which is denoted by $\text{SymCatGr}$. In particular one can form the homotopy category of the 2-category of symmetric categorical groups. We will denote the corresponding homotopy category by $\text{Ho}$. This category has the following nice description which follows from the classical results of H.X.Sinh [8].

First we fix some notations. If $\mathcal{S}$ is a symmetric categorical group then we let $\pi_0(\mathcal{S})$ and $\pi_1(\mathcal{S})$ denote respectively the abelian group of connected components of $\mathcal{S}$ and the abelian group of all automorphisms of the neutral object of $\mathcal{S}$. For symmetric categorical groups $\mathcal{S}_1$ and $\mathcal{S}_2$ we have a groupoid (in fact a symmetric categorical group [1]) $\text{Hom}(\mathcal{S}_1, \mathcal{S}_2)$.

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describe \( \pi_i(\text{Hom}(S_1, S_2)) \), \( i = 0, 1 \) we need to introduce the category \( \mathfrak{T}_{\text{types}} \). Objects of the category \( \mathfrak{T}_{\text{types}} \) are triples \( A = (A_0, A_1, \alpha) \), where \( A_i \) is an abelian group, \( i = 0, 1 \) and 
\[
\alpha \in \text{hom}(A_0/2A_0, A_1) = \text{hom}(A_1, 2A_1)
\]
Here for an abelian group \( A \) we set 
\[
2A = \{ a \in A | 2a = 0 \}
\]
A morphism \( f \) from \( A = (A_0, A_1, \alpha) \) to \( B = (B_0, B_1, \beta) \) in \( \mathfrak{T}_{\text{types}} \) is given by a pair \( f = (f_0, f_1) \), where \( f_0 : A_0 \to B_0 \) and \( f_1 : A_1 \to B_1 \) are homomorphisms of abelian groups such that \( \beta f_0 = f_1 \alpha \).

Let \( S \) be a symmetric categorical group. We put 
\[
type(S) := (\pi_0(S), \pi_1(S), k_S)
\]
where \( k_S \) is the homomorphism induced by the commutativity constraints in \( S \). Both categories \( \text{Ho} \) and \( \mathfrak{T}_{\text{types}} \) are additive and the functor 
\[
type : \text{Ho} \to \mathfrak{T}_{\text{types}}
\]
is additive. According to [8] the functor \( \text{type} \) is full, essentially surjective on objects and the kernel of \( \text{type} \) (morphisms which goes to zero) is a square zero ideal of \( \text{Ho} \). It follows then that the functor \( \text{type} \) reflects isomorphisms and induces a bijection on the isomorphism classes of objects. More precisely, for any symmetric categorical groups \( S_1 \) and \( S_2 \) one has a short exact sequence of abelian groups
\[
0 \to \text{Ext}(\pi_0(S_1), \pi_1(S_2)) \to \pi_0(\text{Hom}(S_1, S_2)) \to \mathfrak{T}_{\text{types}}(\text{type}(S_1), \text{type}(S_2)) \to 0
\]
Furthermore one has also an isomorphism of abelian groups
\[
\pi_1(\text{Hom}(S_1, S_2)) \cong \text{hom}(\pi_0(S_1), \pi_1(S_1))
\]
These facts greatly simplifies to work with symmetric categorical groups.

For a given object \( A \) of the category \( \mathfrak{T}_{\text{types}} \) we choose a symmetric categorical group \( H(A) \) such that \( \text{type}(H(A)) = A \). Such object exist and is unique up to equivalence. Moreover, for any morphism \( f : A \to B \) we choose a morphism of symmetric categorical groups \( H(f) : H(A) \to H(B) \), such that \( \text{type}(H(f)) = f \). The reader must be aware that the assignments \( A \to H(A), \ f \to H(f) \) does NOT define a functor \( \mathfrak{T}_{\text{types}} \to \text{Ho} \).

Recall that [1] a morphism \( F : S_1 \to S_2 \) of symmetric categorical groups is called essentially surjective (resp. faithful) if it is epimorphism on \( \pi_0 \) (resp. monomorphism on \( \pi_1 \)). A symmetric categorical group \( S \) is called projective provided for any essentially surjective functor \( F : S_1 \to S_2 \) and a morphism \( G : S \to S_2 \) there exist a morphism \( L : S \to S_1 \) and a track from \( FL \to G \). Dually a symmetric categorical group \( S \) is called injective provided for any faithful functor \( F : S_1 \to S_2 \) and a morphism \( G : S_1 \to S \) there exist a morphism \( L : S_2 \to S \) and a track from \( LF \to G \).

We can develop same sort of language in the category \( \mathfrak{T}_{\text{types}} \). A morphism \( f = (f_0, f_1) \) in \( \mathfrak{T}_{\text{types}} \) is essentially surjective if \( f_0 \) is epimorphism of abelian groups. Moreover an object
\( \mathcal{P} \) in \( \text{Ttypes} \) is projective of for any essentially surjective morphism \( f : \mathcal{A} \to \mathcal{B} \) in \( \text{Ttypes} \) the
induced map
\[
\text{Ttypes}(\mathcal{P}, \mathcal{A}) \to \text{Ttypes}(\mathcal{P}, \mathcal{B})
\]
is surjective.

Dually, a morphism \( f \) in \( \text{Ttypes} \) is faithful provided \( f_1 \) is injective and an object \( \mathbb{I} = (I_0, I_1, i) \) of \( \text{Ttypes} \) is injective if for any faithful morphism \( f : \mathcal{A} \to \mathcal{B} \) in \( \text{Ttypes} \) the induced map
\[
\text{Ttypes}(\mathcal{B}, \mathbb{I}) \to \text{Ttypes}(\mathcal{A}, \mathbb{I})
\]
is surjective.

It is clear that a morphism \( F : \mathbb{S}_1 \to \mathbb{S}_2 \) of symmetric categorical groups is essentially surjective (resp. faithful) iff \( \text{type}(F) : \text{type}(\mathbb{S}_1) \to \text{type}(\mathbb{S}_2) \) is so in \( \text{Ttypes} \).

For an abelian group \( M \) we introduce two objects in \( \text{Ttypes} \):
\[
l(M) := (M, M/2M, id_{M/2M}) \quad r(M) = (2M, M, id_M)
\]

**Lemma 1.** i) If \( M \) is an abelian group and \( \mathcal{A} = (A_0, A_1, \alpha) \) is an object in \( \text{Ttypes} \), then one has following functorial isomorphisms of abelian groups
\[
\text{Ttypes}(l(M), \mathcal{A}) = \text{hom}(M, A_0),
\]
\[
\text{Ttypes}(\mathcal{A}, r(M)) = \text{hom}(A_1, M).
\]

ii) For any free abelian group \( P \) the object \( l(P) \in \text{Ttypes} \) is projective in \( \text{Ttypes} \), dually for any divisible abelian group \( Q \) the object \( r(Q) \in \text{Ttypes} \) is injective.

iii) For any free abelian group \( P \) the symmetric categorical group \( H(l(P)) \) is projective symmetric categorical group and dually for any divisible abelian group \( Q \) the triple \( r(Q) \) is injective.

**Proof.** i) and ii) are obvious. Let \( F : \mathbb{S}_1 \to \mathbb{S}_2 \) be an essentially surjective morphism of symmetric categorical groups and \( G : H(l(P)) \to \mathbb{S}_2 \) be a morphism of symmetric categorical groups. Apply the functor \( \text{type} \) to get a essentially surjective morphism \( \text{type}(F) : \text{type}(\mathbb{S}_1) \to \text{type}(\mathbb{S}_2) \) in \( \text{Ttypes} \) and a morphism \( \text{type}(G) : l(P) \to \text{type}(\mathbb{S}_2) \) in \( \text{Ttypes} \). Since \( \pi_0(F) : \pi_0(\mathbb{S}_1) \to \pi_0(\mathbb{S}_2) \) is an epimorphism of abelian groups, \( F \) is a free abelian group the homomorphism \( \pi_0(G) : P \to \pi_0(\mathbb{S}_2) \) has a lifting to the homomorphism \( P \to \pi_0(\mathbb{S}_1) \) Since \( P \) is free abelian it follows from the exact sequence (1) that for \( i = 0, 1 \) one has an isomorphism
\[
\pi_0(\text{Hom}(H(l(P)), \mathbb{S}_i)) \cong \text{Ttypes}(l(P), \text{type}(\mathbb{S}_i)) \cong \text{hom}(P, \pi_0(\mathbb{S}_i))
\]
Take a morphism \( L : H(l(P)) \to \mathbb{S}_1 \) of symmetric categorical groups which corresponds to the homomorphism \( P \to \pi_0(\mathbb{S}_1) \). By our construction one has an equality \( \text{type}(FL) = \text{type}(G) \), which imply that the class of \( FL \) and of \( G \) in \( \pi_0(\text{Hom}(H(l(P)), \mathbb{S}_1)) \) are the same. Thus there exist a track from \( FL \) to \( G \). This shows that \( H(l(P)) \) is a projective symmetric categorical group. a dual argument works for injective objects. 

\( \square \)
Proposition 2. The 2-category of symmetric categorical groups have enough projective and injective objects.

Proof. Let $S$ be a symmetric categorical group. Choose a free abelian group $P$ and epimorphism of abelian groups $f_0 : P \to \pi_0(S)$. By Lemma 1 it has a unique extension to a morphism $f = (f_0, f_1) : l(P) \to \text{type}(S)$ which is essentially surjective. Since $P$ is a free abelian group, we have the isomorphism (3), which show that there exist a morphism of symmetric categorical groups $H(l(P)) \to S$ which realizes $f_0$ on the level of $\pi_0$. Clearly this morphism does the job.

Dually, choose a monomorphism $g_1 : \pi_1(S)_1 \to Q$ with divisible abelian group $Q$. By Lemma 1 it has the unique extension as a morphism $g : \text{type}(S) \to r(Q)$ which is faithful by the construction. Since $\pi_1(r(Q)) = Q$ is injective object in the category of abelian groups by the short exact sequence (1) we have

$$\pi_0(\text{Hom}(S, H(r(Q)))) \cong \text{Types}(\text{type}(S), r(Q)) \cong \text{hom}(\pi_1(S), Q)$$

which shows that $g$ can be realized as a morphism of symmetric categorical groups and we get the result. □

Proposition 3. Let $\mathbb{R}$ be a categorical group. Then the category of categorical right $\mathbb{R}$-modules have enough projective and injective objects.

Proof. By Yoneda Lemma for symmetric categorical groups the categorical ring $\mathbb{R}$ considered as a right $\mathbb{R}$-module is projective and from this fact one easily deduces the statement on projective objects. For injectivity we consider the 2-functor $\text{Hom}(\mathbb{R}, -)$ from the 2-category of symmetric categorical groups to the 2-category of categorical right $\mathbb{R}$-modules. It is a right 2-adjoint to the forgetful 2-functor. Since the forgetful functor is exact it follows that the 2-functor $\text{Hom}(\mathbb{R}, -)$ takes injective objects to injective ones. Let $M$ be a categorical left $\mathbb{R}$-module. Choose a faithful morphism $M \to Q$ in the 2-category of symmetric categorical groups with injective symmetric categorical group $Q$. Apply now the 2-functor $\text{Hom}(\mathbb{R}, -)$. It follows from the isomorphism (2) that $\text{Hom}(\mathbb{R}, M) \to \text{Hom}(\mathbb{R}, Q)$ is a faithful morphism of right $\mathbb{R}$-modules. By the same reasons the obvious morphism $M \to \text{Hom}(\mathbb{R}, M)$ is also faithful. Taking the composite we obtain a faithful morphism $M \to \text{Hom}(\mathbb{R}, Q)$ and hence the result. □

Note that the proof of the last statement is essentially the same as it was for classical rings. The same is also true for the following result and because of this we omit the proof. Recall that if $\mathcal{F}$ is an additive 2-category and $M$ is an object in $\mathcal{F}$ then one has the categorical ring $\text{Hom}(M)$ (compare [2],[7]).

Proposition 4. If $\mathcal{F}$ is a 2-abelian category which posses a small projective generator $P$, then $\mathcal{F}$ is 2-equivalent to the category of right categorical modules over the categorical ring $\text{Hom}(P, P)$.

Consider the following symmetric categorical group $\mathbb{H}$. Objects of the groupoid $\mathbb{H}$ are integers. If $n \neq m$ then there is no morphism from $n$ to $m$, $n, m \in \mathbb{Z}$. The group of automorphisms of $n$ is the cyclic group of order two with generator $\epsilon_n$, $n \in \mathbb{Z}$. The monoidal
functor is induced by the addition of integers. The associativity and unite constants are identity morphisms, while the commutativity constraint $n + m \to m + n$ equals to $\epsilon_{n+m}$. By our construction $\mathbb{H}$ is a small projective generator in the 2-category of symmetric categorical groups. Hence we obtained the following important fact.

**Proposition 5.** The 2-category of symmetric categorical groups is 2-equivalent to the category of right categorical modules over the categorical ring $\text{Hom}(\mathbb{H}, \mathbb{H})$.

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