Experimental Observation of Localized Modes in a Dielectric Square Resonator

S. Bittner,1,2 E. Bogomolny,3 B. Dietz,1,* M. Miski-Oglu,1 and A. Richter1,†

1Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany
2Laboratoire de Photonique Quantique et Moléculaire, CNRS UMR 8537, Institut d’Alembert FR 3242, École Normale Supérieure de Cachan, F-94235 Cachan, France
3Université Paris-Sud, CNRS, LPTMS, UMR 8626, Orsay, F-91405, France

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We investigated the frequency spectra and field distributions of a dielectric square resonator in a microwave experiment. Since such systems cannot be treated analytically, the experimental studies of their properties are indispensable. The momentum representation of the measured field distributions shows that all resonant modes are localized on specific families of classical trajectories of the square billiard. Based on these observations a semiclassical model was developed. It shows excellent agreement with all but a single class of measured field distributions that will be treated separately.

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Introduction.— Dielectric microresonators are used for a wide range of applications, e.g., as microlasers, sensors or building blocks for optical circuits [1, 2]. Most investigations focus on cavities with circular [3] and deformed circular or polygonal shapes. The former can exhibit high quality factors or directional lasing emission [4,5] whereas the latter are of interest, firstly, for applications like filters [6,7] and, secondly, because the crystal structure of some materials naturally implies such a resonator geometry [8,9]. Here we concentrate on the simplest one, a dielectric square resonator, which cannot be handled analytically even though the classical hard-wall square billiard is integrable. A variety of models for its description has been proposed [10–20] that are based on similar assumptions, as, e.g., that the modes associated with the prominent resonances are localized along classical trajectories that impinge at the boundaries with an angle of incidence equal or close to 45°, like the diamond orbit. Similar results were obtained for pentagonal [18] and hexagonal [7, 10, 11, 21] resonators. However, there is experimental evidence that modes are also localized on other types of trajectories [22–24]. Furthermore, the existing models were so far only applied to subsets of resonances and to cases with a specific refractive index, and never compared to measured field distributions. In order to investigate the exact nature of the modal localization in dielectric square resonators and to develop a comprehensive model, experiments with a macroscopic microwave resonator were performed and are reported in this letter. Both the measured frequency spectrum and the measured near-field distributions in configuration and momentum space were investigated. The latter evidenced that the resonant modes are localized on trajectories with specific momentum vectors. We will introduce a semiclassical model that allows to label all modes with quantum numbers and their symmetry class. It shows excellent agreement with the experimental findings for all types of modes and for a wide range of the effective refractive index.

Experiment.— A sketch of the experimental setup is shown in Fig. 1. A ceramic plate made of alumina (Al2O3) with sharp corners and edges and refractive index n1 = 3.10 was used as microwave resonator. The side length was a = 297.30 mm and the thickness was b = 8.27 mm, which was small compared to the wavelengths, λ ≈ 30–60 mm, used in the experiments. Accordingly, the resonator is treated as a two-dimensional (2d) system by introducing the effective refractive index n_{eff} [25], which is between 1.5–2.5 in the frequency range f = 5.5–10.0 GHz considered here. It was placed atop a 120 mm thick foam [26] with a refractive index n_2 = 1.02 close to that of air to realize an effectively levitated resonator. Two wire antennas labeled 1 and 2 were positioned vertically to the resonator below and above it and connected to a vectorial network analyzer (VNA) [27]. The VNA measured the complex transmission amplitude S_{21}(f) between the antennas, where

* dietz@ikp.tu-darmstadt.de
† richter@ikp.tu-darmstadt.de

FIG. 1. Sectional drawing of the experimental setup (not to scale). The alumina plate is placed atop a foam with refractive index n_1 ≈ 1. The two vertical wire antennas protruding from coaxial rf cables are connected to a VNA (adapted from [20]).
| | | |
|---|---|---|
| [Image 57x660 to 296x740] | | |

FIG. 2. Measured frequency spectrum in semilogarithmic scale. The inset indicates the positions of the antennas at the corners of the square resonator.

$|S_{21}(f)|^2 = P_{2,\text{out}}/P_{1,\text{in}}$ is the ratio of the power coupled out of and into the resonator via antennas 2 and 1, respectively. An example of a measured frequency spectrum is shown in Fig. 2. The spectrum features a multitude of resonances with typical quality factors in the range of $Q = 200$–2000. Both transverse magnetic (TM) and transverse electric (TE) modes with the magnetic, respectively, electric field parallel to the plane of the resonator were excited. The polarization was determined using the procedure outlined in Ref. [28]. In the following, only the TM modes are considered since our setup is less sensitive to TE modes. In the frequency range $\approx 5.5$–8.5 GHz a series of roughly equidistant resonances is observed. In Refs. [16]–[22] these were generally associated with the diamond periodic orbit family. In addition, there are many other resonances yielding the overall complicated structure. The corresponding modes are localized on various families of classical trajectories as will be shown below.

The field distributions inside the resonator were measured with the scanning antenna technique (cf. [30]–[32]), i.e., the receiving antenna was moved around on the top surface of the resonator on a Cartesian grid with spatial resolution $\Delta_x = \Delta_y = a/150 \approx 2$ mm. Here, the $(x,y)$–plane is chosen parallel to that of the resonator and the $z$–direction perpendicular to it (cf. Fig. 1). The transmission amplitude is proportional to the $z$–component of the electric field vector, $E_z$, at the position $(x,y)$ of the antenna, so that for a TM mode the measurement of $S_{21}(x,y,f_{\text{res}})$ at the resonance frequency $f_{\text{res}}$ yields the corresponding field distribution $E_z(x,y)$, denoted as the measured wave function (WF) $\Psi_{\text{exp}}(x,y)$ in the following.

Analysis of Wave Functions.— Several measured WFs are shown in Figs. 3(a)–(c). All are of a simple and clear structure reminiscent of that observed for integrable systems. Similar WFs were calculated numerically in Refs. [14]–[19]. Note that in some examples perturbations of the patterns around the position of the emitting antenna [e.g., in the center of Fig. 3(c)] are observed. They originate from the direct transmission between the two antennas.

A better understanding of the localization of WFs is achieved by considering their spatial Fourier transforms (FTs) $\tilde{\Psi}_{\text{exp}}(k_x,k_y)$. Those corresponding to the WFs in Figs. 3(a)–(c) are depicted in Figs. 3(e)–(g). They exhibit a clear localization at eight specific momentum vectors $n_{\text{eff}} \vec{k} = (k_x, k_y)$ pointing from the origin to the bright spots as indicated by the white lines. Such a localization is observed for all measured WFs. Note that in each panel the modulus of the momentum vectors, i.e., the radius of the white circles, equals $n_{\text{eff}} \vec{k}$, where $k = 2\pi f/c$ with $c$ the speed of light in vacuum. Thus, the calculation of $\Psi_{\text{exp}}(k_x,k_y)$ allows for the direct experimental determination of the effective refractive index [30]. The eight momentum vectors correspond to a single family of classical trajectories that are specified by their angles of incidence $\alpha_{\text{inc}}$ [see Fig. 3(e)] and $\pi/2 - \alpha_{\text{inc}}$ as shown in Fig. 4. Here, $\alpha_{\text{inc}} \in [0^\circ, 45^\circ]$ is defined as $\alpha_{\text{inc}} = \min\{\alpha_x, \alpha_y\}$, and $\alpha_{x,y} = \arctan[\text{Re}(k_y)/\text{Re}(k_x)]$ are the angles of incidence on the vertical and horizontal boundaries of the square (cf. Fig. 4), respectively. In Figs. 3(e)–(g) it has the values $\alpha_{\text{inc}} = 39.1^\circ$, $39.0^\circ$, and $33.3^\circ$, respectively. Note that only modes with $\alpha_{\text{inc}} \geq \alpha_{\text{crit}} = \arcsin(1/n_{\text{eff}})$ are observed experimentally [30].

Ray-based Model.— A semiclassical model can be readily deduced from the localization of the WFs on classical trajectories with the same angles of incidence observed in Fig. 3. The classical trajectories depicted in Fig. 4 are quantized in the same manner as those in a Fabry-Perot cavity, that is, by imposing a phase matching condition after one roundtrip on a plane wave which propagates with wave vector $n_{\text{eff}} \vec{k} = (k_x, k_y)$,

\[
\begin{align*}
\exp\{2ik_xa\}r^2(\alpha_x) &= \exp\{2\pi im_x\} \\
\exp\{2ik_ya\}r^2(\alpha_y) &= \exp\{2\pi im_y\}.
\end{align*}
\]

Here, $r(\alpha)$ denotes the Fresnel reflection coefficient for angle of incidence $\alpha$, and $m_{x,y} = 0, 1, 2, \ldots$ are the $x$ and $y$ quantum numbers [19]. Models taking into account only the diamond periodic orbit [15]–[20] are recovered in the limit $\alpha_{\text{inc}} \approx 45^\circ$. The model WFs $\Psi_{\text{mod}}$ corresponding to our ansatz are superpositions of eight plane waves with momentum vectors $(\pm k_x, \pm k_y)$ and $(\pm k_y, \pm k_x)$. The associated amplitudes depend on the symmetry class of the mode. There are altogether six different symmetry classes in the square resonator [15]–[23]. We label the modes by $(m_x, m_y, s_1s_2)$ with $s_{1,2} \in \{+, -\}$. Here $s_1 = +1$ ($s_2 = +1$) when $\Psi_{\text{mod}}$ is symmetric, and $s_1 = -1$ ($s_2 = -1$) when $\Psi_{\text{mod}}$ is antisymmetric with respect to the diagonal $x = y$ ($x = -y$). The model WFs $\Psi_{\text{mod}}$ and their symmetries with respect to the horizontal and vertical axes are listed in Table I. The $(m_x, m_y, -+)$ and $(m_x, m_y, +-) \mod$ modes are degenerate due to symmetry reasons [37]. The model furthermore predicts that this is also the case for the $(m_x, m_y, +)$ and $(m_x, m_y, -)$ modes. In practice, however, they have slightly differing resonance frequencies due to their distinct features at the corners, where the $(+-)$ modes have minimal whereas the $(++)$ modes have maximal intensity [30]. To identify the model WF which corresponds to
FIG. 3. (Color online) Examples of measured wave functions for (a) 6.835 GHz, (b) 7.615 GHz, and (c) 6.869 GHz with color scale given in (d), and (e)–(g) the corresponding momentum distributions [with color scale in (h)]. They are identified with the model wave functions (i) TM(16, 20, −−) with 84.1% overlap, (j) TM(20, 25, −+) with 67.1% overlap, and (k) TM(14, 22, ++) with 68.0% overlap, respectively [color scale in (l)].

TABLE I. Symmetry classes, quantum numbers and wave functions. The first column denotes the symmetry with respect to the diagonals, the second column is the symmetry with respect to both the horizontal and vertical axis, the third column is the parity of \( m_x + m_y \), the fourth column the parity of \( m_x \) and \( m_y \) (which is the same for ++ and −− modes but different for +− and −+ modes), and the fifth column is the corresponding model wave function.

| diag. sym. | horiz./vert. sym. | parity of \( m_x + m_y \) | parity of \( m_x, m_y \) | model wave function |
|------------|-------------------|--------------------------|--------------------------|--------------------|
| ++         | +                 | even                     | even                     | \( \Psi_{mod}(x, y) = \cos(k_x x) \cos(k_y y) + \cos(k_y x) \cos(k_x y) \) |
| ++         | −                 | even                     | odd                      | \( \Psi_{mod}(x, y) = \sin(k_x x) \sin(k_y y) + \sin(k_y x) \sin(k_x y) \) |
| −−         | +                 | even                     | even                     | \( \Psi_{mod}(x, y) = \cos(k_x x) \cos(k_y y) - \cos(k_y x) \cos(k_x y) \) |
| −−         | −                 | even                     | odd                      | \( \Psi_{mod}(x, y) = \sin(k_x x) \sin(k_y y) - \sin(k_y x) \sin(k_x y) \) |
| +−         | none              | odd                      | none                     | \( \Psi_{mod}(x, y) = \sin(k_x x) \cos(k_y y) + \cos(k_y x) \sin(k_x y) \) |
| −+         | none              | odd                      | none                     | \( \Psi_{mod}(x, y) = \sin(k_x x) \cos(k_y y) - \cos(k_y x) \sin(k_x y) \) |

a given measured WF \( \Psi_{expt} \) we calculated the overlap integral \( |C|^2 = | \langle \Psi_{expt}(f) | \Psi_{mod}(m_x, m_y, s_1, s_2) \rangle |^2 \) for several trial functions \( \Psi_{mod} \). For isolated resonances, a typical overlap of \( |C|^2 = 50-80\% \) indicates that the corresponding model WF is the correct one, while the overlaps with model WFs not related to \( \Psi_{expt} \) are usually less than 3\%. Three examples of model WFs are shown in the bottom row of Fig. 3. The overlap of the WF \( \Psi_{mod}(16, 20, −−) \) in Fig. 3(i) with the measured WF in Fig. 3(a) is 84.1\%, that of \( \Psi_{mod}(20, 25, −+) \) in Fig. 3(j) with the WF in Fig. 3(b) is 67.1\%, and that of \( \Psi_{mod}(14, 22, ++) \) in Fig. 3(k) with the WF in Fig. 3(c) is 68.0\%. Indeed, each measured WF with \( m_x \neq m_y \) could be unambiguously identified with one model WF. Accordingly, the resonant modes can be labeled by quantum numbers and allocated to a symmetry class as is the case for integrable systems.

There is only one class of modes whose structure cannot be associated with a single model WF. An example, \( \Psi_{expt}(8.581 \text{ GHz}) \), is shown in Fig. 5(b). The measured WF has an overlap of 62.4\% with \( \Psi_{mod}(28, 28, ++) \), which is comparable to the values of the cases shown
FIG. 4. Two examples of trajectories (black and white thin arrows) belonging to a family of orbits with the same angles of incidence. The associated momentum vector \( n_{\text{eff}} \mathbf{k} = (k_x, k_y) \) (thick black arrows) corresponds to an angle of incidence of \( \alpha_{x,y} \) on the vertical and horizontal boundaries of the square billiard, respectively.

Furthermore, its momentum distribution (not shown) exhibits a localization at \( \alpha_{\text{inc}} = 45^\circ \), i.e., one along the trajectories from the family of the diamond orbit, like the model mode. However, the measured WF does not exhibit the chessboard structure observed for the model WF in Fig. 4(d). The frequency spectrum in Fig. 5(a) shows another resonance nearby at 8.591 GHz. The two resonances are reasonably well isolated and exhibit only a small spectral overlap. The measured WF associated with the second resonance is shown in Fig. 5(c) and can be identified with the model mode \( \text{TM}(28,30,++) \) [see Fig. 3(e)] with an overlap of 59.8\%. Thus, the symmetry class associated with both resonances is the same. Its structure also shows slight deviations from that of the model WF. The overlaps between the measured WFs and the model WFs associated with the other one are, respectively, \( |\langle \Psi_{\text{expt}}(8.581\;\text{GHz})|\Psi_{\text{mod}}(26,30,++) \rangle|^2 = 10.2\% \) and \( |\langle \Psi_{\text{expt}}(8.5891\;\text{GHz})|\Psi_{\text{mod}}(28,28,++) \rangle|^2 = 11.1\% \). Thus, evidently there is a non-negligible coupling between the two resonances, and consequently more than one model WF is needed to account for their structures. It should be noted that this cannot be attributed to the weak overlap of the two resonances in the frequency spectrum. Note, that the same effect has been observed in the numerical studies \[36\]. Indeed, the respective superpositions

\[
|\Psi_{\text{sup}}(f)\rangle = \\
\langle \Psi_{\text{mod}}(28,28,++)|\Psi_{\text{expt}}(f)\rangle |\Psi_{\text{mod}}(28,28,++) \rangle + \langle \Psi_{\text{mod}}(26,30,++)|\Psi_{\text{expt}}(f)\rangle |\Psi_{\text{mod}}(26,30,++) \rangle
\] (2)

agree well with the measured WFs [see Figs. 5(f) and (g), respectively], as also confirmed by the corresponding overlaps \( |\langle \Psi_{\text{expt}}(8.581\;\text{GHz})|\Psi_{\text{sup}}(8.581\;\text{GHz}) \rangle|^2 = 72.5\% \) and \( |\langle \Psi_{\text{expt}}(8.591\;\text{GHz})|\Psi_{\text{sup}}(8.591\;\text{GHz}) \rangle|^2 = 70.9\% \). This situation is exemplary for all modes localized on \( \alpha_{\text{inc}} = 45^\circ \), i.e., with \( m_x = m_y \) and ++ symmetry, and their neighboring modes [i.e., \((m_x - 2,m_x + 2,++)\)].

The physical reason is that the \((m_x,m_x,++)\) and the \((m_x - 2,m_x + 2,++)\) model modes are closer in frequency than any other model modes having the same symmetry class and their mutual interaction thus becomes important. This issue will be discussed elsewhere in more detail. A similar effect for the modes with \( m_x = m_y \) was observed in Ref. \[14\], which was, however, attributed to the influence of the coupling to a waveguide.

**Conclusions.**— We have measured the frequency spectrum and near-field distributions of a dielectric square resonator and demonstrated that the resonant states are localized on a single family of generally non-closed classical trajectories each. We developed a simple but efficient semiclassical model that describes all the measured modes of the dielectric square resonator very well and
allows to label them unambiguously with quantum numbers. Note that to our knowledge this was possible hitherto only for dielectric resonators with circular shape \cite{14}. Furthermore, we showed that the model works well for a large range of the effective refractive index ($n_{\text{eff}} \approx 1.5$–2.5), evidencing the general validity of its use. Only for a single class of modes, i.e., those localized on trajectories with angle of incidence 45°, a small coupling between neighboring model modes of the same symmetry needs to be taken into account. Such a coupling effect has so far not been discussed for dielectric square resonators and the investigation of the underlying mechanism is of interest. A future project is the modeling of the far-field distributions which are important for microlaser applications. Furthermore, the generalization of the model to other polygonal shapes like hexagons would be of great practical and theoretical interest.

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