Describing classical spin Hamiltonians as automata: a new complexity measure

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We introduce a new complexity measure of classical spin Hamiltonians in which they are described as automata. Specifically, we associate a classical spin Hamiltonian to the formal language consisting of pairs of spin configurations and the corresponding energy, and classify this language in the Chomsky hierarchy. We prove that the language associated to local one-dimensional (1D) classical spin Hamiltonians is deterministic context-free, and the one associated to the two-dimensional (2D) case is context-sensitive. It follows that the Ising model without fields is easy / hard if defined on a 1D / 2D lattice, in contrast to the computational complexity of its ground state energy problem, which is easy / hard (namely in $\mathcal{P}$ / NP-complete) if defined on a planar / non-planar graph. We also prove that only highly non-physical spin Hamiltonians, namely totally unbounded ones, correspond to Turing machines. Our work puts classical spin Hamiltonians at the same level as automata, and paves the road toward a rigorous comparison of universal spin models and universal Turing machines.

1. INTRODUCTION

Statistical physics and computational complexity theory have had strong ties for decades [1, 2]. Traditionally, the complexity of classical spin Hamiltonians has been studied from the point of view of the computational complexity of the ground state energy problem (GSE), which is defined as follows: Given a Hamiltonian and an energy value, is there a spin configuration whose energy is below that value? This problem is naturally in NP, as the spin configuration witnesses the case in which it is a yes-instance, and in some “simple” cases it is already NP-hard [3]. For the most paradigmatic classical spin model, the Ising model without fields, the GSE is easy (i.e. in $\mathcal{P}$) if defined on a planar graph, and hard (i.e. NP-complete) if defined on a non-planar graph [4] [5].

Another complexity measure for Hamiltonians is the Kolmogorov complexity of the Hamiltonian, which is defined as the length of the shortest algorithm that produces a description of the Hamiltonian as its output [6–8]. This description of a Hamiltonian could be fed to a universal Turing machine, which could then simulate, e.g., the time evolution that it generates. This complexity measure is however uncomputable.

In this paper, we take a different route to measuring the complexity of a classical spin Hamiltonian. We propose to describe a classical spin Hamiltonian as an automaton that sends spin configurations to energies, and we classify what type of automaton this is. Equivalently, we associate a classical spin Hamiltonian to the formal language consisting of all pairs of spin configurations and their corresponding energy, and classify this language in the Chomsky hierarchy. In other words, we put classical spin models and automata at the same level, and then borrow the classification of automata to infer a complexity classification of classical spin Hamiltonians.

Our main result is that, on the one hand, classical spin Hamiltonians with physically local 1, 2, ..., and $k$-body interactions (where $k$ is fixed) on a one-dimensional (1D) chain — called local 1D spin Hamiltonians — correspond to deterministic context-free languages. On the other hand, these Hamiltonians in dimensions higher than one, for example on a two-dimensional (2D) lattice — local 2D spin Hamiltonians — correspond to context-sensitive languages. The same is true for 1D spin Hamiltonians where the local interaction term involves a fixed number of spins ($k$) but these need not be physically close to each other — called unbounded 1D spin Hamiltonians (see Fig. 1). In addition, we show that only highly non-physical spin Hamiltonians, namely those without any structure — called totally unbounded spin Hamiltonians — correspond to Turing machines (see Fig. 2 for a classification in the Chomsky hierarchy).

**Theorem 1** (Main result – informal). (i) The language associated to local 1D spin Hamiltonians is deterministic context-free.

(ii) The language associated to local 2D spin Hamiltonians, and to unbounded 1D spin Hamiltonians, is context-
sensitive.

(iii) The language associated to totally unbounded spin Hamiltonians is recursively enumerable.

Statements (i) and (ii) also hold if the local hamiltonian term is unspecified, i.e. it is given as part of the language, as we will see. These results apply, in particular, to the Ising model without fields, and show that the boundary between ‘easy’ and ‘hard’ (more precisely: deterministic context-free and context-sensitive) is given by whether the lattice is 1D or 2D. This is a different threshold than for the GSE, which is given by whether the graph is planar or non-planar, corresponding to the complexity class \( \mathbf{P} \) (for polynomial time) and \( \mathbf{NP} \)-complete (where \( \mathbf{NP} \) stands for non-deterministic polynomial time; Table 1). Since the complexity measures are different—in particular, our measure involves the full Hamiltonian, not only its behavior around the ground state—, it is not surprising that the thresholds are different.

The Ising model without fields defined on a graph

| Hamiltonian                  | Formal languages | Automata   |
|------------------------------|------------------|------------|
| Totally unbounded spin       | \( L_0 \) Recursive enumerable | Turing machines |
| Hamiltonian or Unbounded 1D  | \( L_1 \) Context-sensitive | Linear bounded automata |
| Hamiltonian                  | \( L_2 \) Context-free | Pushdown automata |
| Hamiltonian                  | \( L_3 \) Regular | Finite-state automata |

FIG. 2. (Middle column) The Chomsky hierarchy \( L_0 \supset L_1 \supset L_2 \supset L_3 \) together with the intermediary class deterministic context-free languages (DCFL). (Right column) Each level of the Chomsky hierarchy is associated to the class of automata that recognizes it. (Left column) In this work we put classical spin models at the same level as formal languages and automata, and find that local 1D spin Hamiltonians correspond to DCFL, and local 2D spin Hamiltonians, and unbounded 1D spin Hamiltonians, correspond to context-sensitive languages.

‘the 1D Ising model’ will refer to the set of maps

\[
(s_1, \ldots, s_n) \mapsto \sum_{i=1}^{n-1} s_i s_{i+1}
\]

for all \( n \) (here \( s_i \in \{1, -1\} \)). What is characteristic of the 1D Ising model, thus, is the form of the local Hamiltonian \( h(s_i, s_{i+1}) = s_i s_{i+1} \). Intuitively, this local Hamiltonian will be encoded in the transition rules of the head of the automaton, and will determine ‘how complicated’ this automaton is. The automaton will run (that is, compute the energy) for any system size.

This point can also be seen as follows: If the system size were fixed, the input–output relations (that is, the set of spin configurations and their corresponding energy) would be a finite list. In this case, using an automaton to recognize this list would be unnecessary—they would all correspond to the simplest kind of automaton (the finite-state automata, see Fig. 2), and our complexity measure would be meaningless. Equivalently, the corresponding formal language would have a finite number of elements, so it would be trivial.

Finally, note that describing certain objects in condensed matter physics as automata has been considered before. For example, certain 1D quantum Hamiltonians, called matrix product operators, as well as certain 1D quantum states, called matrix product states, are described as finite complex weighted automata in [9]. In this work we focus on describing classical spin Hamiltonians as automata. Similarly, the quantum Kolmogorov complexity of a quantum state was defined and studied in [10].

This paper is organized as follows. In Section 2 we present the preliminaries about formal languages and automata theory needed for this work. In Section 3 we prove that the lan-
we prove that unbounded 1D
The Kleene star creates the set of all strings over the given
work we denote the set of integers from 1 to
| A
| alphabet, including
ε alphabet, including

In Section 2.3.2, Notation and definitions (Section
concatenation is an operation defined on their Kleene stars:

\[ \Sigma_q \times \Sigma_p^* \rightarrow \{ \gamma \omega \mid \gamma \in \Sigma_q, \omega \in \Sigma_p^* \} \]

The empty string \( \varepsilon \) acts as the identity element of concatenation

\[ a\varepsilon = \varepsilon a = a. \]

2. PRELIMINARIES

In this section we introduce notation and definitions (Section
Throughout this paper, ‘language’ stands for ‘formal lan-

2.2. Pushdown automata

Here we briefly review the notion of a pushdown autom-
Definition 2 (Pushdown automaton). A pushdown automaton

\[ M = (Q, \Sigma, \Gamma, Z, \delta, q_0, F) \]

where

- \( Q \) is a finite set of states
- \( \Sigma \) is a finite set called input alphabet
- \( \Gamma \) is a finite set called stack alphabet
- \( Z \in \Gamma \) is the initial stack symbol
- \( \delta \) is a finite subset of \((Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma) \times (Q \times \Gamma)\) called transition relation
- \( q_0 \in Q \) is the start state, and
- \( F \subseteq Q \) is the set of accept states.

A PDA can be thought of as a combination of a tape on
which inputs are written as strings over \( \Sigma \), a machine head
which can take any of the states in \( Q \) and move along the tape
from beginning to end, reading one symbol at a time, and an
infinite amount of last-in-first-out stack memory accessible to
the head. A computation of the PDA consists of a number of
discrete steps. At each step the head may read a symbol from
the tape, advance to the next symbol, and may pop the symbol
at the top of the stack. Depending on the tape symbol \( t \), stack
symbol \( s \), and state of the head \( q \), the head transitions into a
state \( q' \), and pushes a string \( \gamma \in \Gamma^* \) on top of the stack. The
allowed transitions are given by \( \delta \) and need not be unique—
for a given combination \((q, t, s)\) there may be multiple possible
transitions, from which one is chosen. This property is known as
non-determinism. Additionally, transitions may not read
the current tape symbol and only operate on the stack; this is
called an \( \varepsilon \)-transition.

Initially the stack only contains the symbol \( Z \) and the head
is in state \( q_0 \), positioned over the first input symbol. If there
exists a sequence of transitions such that the head enters a state
in \( F \) after reading the entire input, the automaton accepts the
input. Otherwise the input is rejected. The set of all accepted

2.1. Notation and definitions

In this subsection we review a number of standard defini-
tions commonly used in automata theory. Throughout this
work we denote the set of integers from 1 to \( n \) by \([n] := {1, 2, \ldots, n}\).

For a given natural \( q \geq 1 \), an alphabet \( \Sigma_q \) is a finite set
with cardinality \( |\Sigma_q| = q \). The elements of an alphabet are
called symbols and are arbitrary—we only require that there
are \( q \) different symbols. The \( n \)-fold Cartesian product of \( \Sigma_q \) is
denoted

\[ \Sigma_q^n = \underbrace{\Sigma_q \times \cdots \times \Sigma_q}_{n \text{ times}}. \]

A string \( \alpha \) over an alphabet \( \Sigma_q \) is a finite sequence of symbols
in \( \Sigma_q \), namely \( \alpha \in \Sigma_q^n \) for some \( n \). Here \( n \) is the length
of the string \( \alpha \), denoted \(|\alpha|\). For example, if \( b \) is a symbol in
the alphabet \( \Sigma_q \), i.e. \( b \in \Sigma_q \), then the \( n \)-fold repetition of \( b \) is
denoted \( b^n \in \Sigma_q^n \). We additionally define \( \varepsilon \) to be the empty
string, \(|\varepsilon| = 0\), and in particular we define \( \Sigma_q^0 = \{\varepsilon\} \) for any
alphabet \( \Sigma_q \).

The Kleene star of an alphabet \( \Sigma_q \), denoted \( \Sigma_q^* \), is defined as

\[ \Sigma_q^* = \bigcup_{i \geq 0} \Sigma_q^i. \]

The Kleene star creates the set of all strings over the given
alphabet, including \( \varepsilon \).
strings is the language recognized by the PDA. The set of all languages recognized by PDAs are the context-free languages, or $L_2$.

A less powerful form of the PDA is the deterministic pushdown automaton.

**Definition 3 (Deterministic pushdown automaton).** A deterministic pushdown automaton (DPDA) is defined as a PDA with one additional restriction: for any given instantaneous description of the automaton ($q \in Q, t \in \Sigma, s \in \Gamma$) there is at most one possible transition.

While the PDA can take multiple paths of transitions and accepts if any of them reach an accept state after consuming the input, the DPDA only follows one fixed sequence of transitions for a given input. The languages which are recognized by PDAs are the deterministic context-free languages (DCFL), which are a proper subset of $L_2$.

2.3. Turing machines and linear bounded automata

Let us now review a more powerful type of automaton, the Turing machine, to later restrict to the less powerful variant of linear bounded automata.

**Definition 4 (Turing machine).** A Turing machine (TM) $M$ is formally described by an 8-tuple

$$M = (Q, \Sigma, \Gamma, \sqcup, \delta, q_0, q_f, q_r)$$

where

- $Q$ is a finite set of states
- $\Sigma$ is a finite set called input alphabet
- $\Gamma$ is a finite set with $\Sigma \subset \Gamma$, called tape alphabet
- $\sqcup \in \Gamma$ is called the blank symbol
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function, where $L, R$ are symbols for left, right
- $q_0 \in Q$ is the start state
- $q_f \in Q$ is the accept state, and
- $q_r \in Q$ is the reject state.

As the PDA, a Turing machine can be imagined as a machine with a finite number of internal states and operating on a tape of symbols. The first major difference with PDA is that, while in PDA the head can only move to the right, in a TM the head can move left or right. The second major difference is the ability to overwrite the tape: While PDAs can only read the tape, the TM can also replace the symbol it just read. It is thus natural that the tape on which a TM operates is infinite, namely both sides of the input string are padded with endless strings of blank symbols $\sqcup$.

To summarize, the TM works as follows: for a given internal state $q \in Q$ and tape symbol $a \in \Gamma$, it overwrites $a$ with a symbol $b \in \Gamma$, enters a new state $p \in Q$, and chooses a direction $d \in \{L, R\}$ in which to move the head. In terms of the transition function this is succinctly expressed as

$$\delta(q, a) = (p, b, d).$$

Once the TM enters the accept or reject state, it is never allowed to leave it. In this case, the automaton is said to halt, and accept or reject this input, respectively. Formally, this means that for all $a \in \Gamma$ there exist $b, c \in \Gamma$ and $d, d' \in \{L, R\}$, such that

$$\delta(q_f, a) = (q_f, b, d)$$
$$\delta(q_r, a) = (q_r, c, d').$$

Note that the TM need not halt: a computation can also loop forever without ever entering the accept or reject state. The set of languages recognized by Turing machines are the recursively enumerable languages, or $L_0$.

One further interesting type of automaton is the linear bounded automaton (LBA), which is obtained by restricting the tape accessible to the Turing machine to be a linear function of the length of the input. Because of the linear speedup theorem [12], this is equivalent to requiring the length of the tape to be exactly the same as the length of the input. That is, the automaton can only overwrite the input, i.e. it has no blank spaces to the left or right of the input to work with.

**Definition 5 (Linear bounded automaton).** A linear bounded automaton (LBA) is defined as a TM with one additional restriction: the length of the tape accessible to the automaton is equal to the length of the input. Formally, this means that there exist special symbols $\hat{s}_L, \hat{s}_R$ at the left and right ends of the input, respectively, which the LBA cannot move past or overwrite. That is, for all $q \in Q$ there exist $p, r \in Q$ such that

$$\delta(q, \hat{s}_L) = (p, \hat{s}_L, R)$$
$$\delta(q, \hat{s}_R) = (r, \hat{s}_R, L).$$

The languages accepted by LBAs are the context-sensitive languages, or $L_1$. These are a proper subset of $L_0$.

3. 1D SPIN HAMILTONIANS AS AUTOMATA

In this section we define 1D classical spin Hamiltonians (Section 3.1), associate formal languages to them (Section 3.2), and show that they are deterministic context-free (Section 3.3).

3.1. Definition of 1D spin Hamiltonians

In this subsection we provide a rigorous definition of 1D classical spin Hamiltonians (Definition 6). Let us first fix some notation and conventions.

A discrete classical spin variable can take $q \in \mathbb{N}$ distinct values, which are called levels and labelled $1, \ldots, q$. In order to make a connection to automata, we let these $q$ levels define the alphabet $\Sigma_q$.

We denote a configuration of $n$ $q$-level spins as a tuple consisting of the string of $q$-level spins, $S^{(n)}$, and the length $n$:

$$(S^{(n)}, n) = (s_1 s_2 \ldots s_r, n) \in \Sigma_q^n \times \mathbb{N}.$$
For example, a spin configuration of four 2-level spins (with \( \Sigma_2 = \{1, 2\} \)) is denoted \((S(4), 4) = (1212, 4)\). Note that in 1D \( n \) is redundant, as the string itself already has a well-defined length, but this becomes meaningful when generalizing the construction to higher dimensions, as will be apparent in Section 4. In addition, for a given spin configuration \((S(n), n)\) we denote substrings of adjacent spins starting at spin \( s_i \) and ending at \( s_j \) as

\[
S_{[i,j]} := s_i s_{i+1} \ldots s_{j-1} s_j.
\]

Note that for substrings we omit the superscript \((n)\) to keep notation simple.

We are now ready to define 1D classical spin Hamiltonians with physically local interactions. Intuitively, \( k \) will denote the maximum range of the interaction (see below for the precise definition). Here we consider locality in the physical sense, meaning that these \( k \) spins are required to be ‘physically’ adjacent, which will translate to adjacency in the spin configuration. This is to be contrasted with locality in the computer science sense, in which one would require that interactions be at most between \( k \) spins—in other words: that every term in the Hamiltonian has arity at most \( k \), which we will consider in Section 6.1. Locality in the physical sense is thus more restrictive than in the computer science sense.

**Definition 6** (Local 1D Hamiltonian). Let \( q > 1, k \geq 1 \) be fixed natural numbers. Let

\[
D_{1D} = \bigcup_{n \geq 1} \{(S(n), n) \mid S(n) \in \Sigma_q^n\}
\]

be the set of all 1D \( q \)-level spin configurations. A local 1D Hamiltonian \( H \) with at most \( k \)-body interactions is a map

\[
H : D_{1D} \to \mathbb{Q},
\]

\[
(S(n), n) \mapsto \sum_{i=1}^{n} h(S_{[i-k+1,i]}).
\]

In addition, \( h : \Sigma_q^k \to \mathbb{Q} \) is called the local Hamiltonian.

In words, \( H \) takes a spin configuration of \( n \) \( q \)-level spins and maps it to an energy value [13], which is obtained by summing up the local Hamiltonians for each substring of \( k \) adjacent spins. The details of the interaction will be contained in the definition of \( h \). Note that this definition of \( h \) includes 1-, 2-, ..., and up to \( k \)-body interactions—hence the formulation of “at most” \( k \)-body interactions.

Note that in Eq. (2) the sum starts from \( i = 1 \), causing \( h \) to be evaluated on substrings with initial indices \( i-k+1 < 1 \). The way this is to be understood depends on the type of boundary conditions considered. For periodic boundary conditions the spin indices loop around

\[
s_j := s_{j+n} \quad \forall \ j < 1,
\]

whereas for open boundary conditions we identify spins \( s_j \) for \( j < 1 \) with special symbol \( x \notin \Sigma_q \). The local Hamiltonian

\[
h : (\Sigma_q \cup \{x\})^k \to \mathbb{Q}
\]

is extended so that spins of symbol \( x \) do not contribute to the energy. Note that this is just a notational choice to keep Eq. (2) simple. In the remainder of this Section, in Section 4 and Section 5 we will consider open boundary conditions; in Remark 21 we will briefly describe how to obtain the same results for periodic boundary conditions.

To illustrate this definition, we now describe the 1D Ising model with fields in this framework.

**Example 7** (The 1D Ising model). Consider the 1D Ising model with fields with homogeneous coupling strength 1 and magnetic field 1,

\[
H(\sigma_1, \ldots, \sigma_n) = \sum_{i=1}^{n} \sigma_i + \sum_{i=2}^{n} \sigma_{i-1} \sigma_i,
\]

where \( \sigma_i \in \{-1, 1\} \). In terms of Definition 6, this is described by \( k = 2 \), \( q = 2 \), \( s_i \in \{0, 1\} \cup \{x\} \), and

\[
h(s_{i-1} s_i) = \begin{cases} (-1)^{s_i} & \text{if } s_{i-1} = x, \\ (-1)^{s_i} + (-1)^{s_{i-1}s_i} & \text{otherwise.} \end{cases}
\]

### 3.2. The language associated to 1D spin Hamiltonians

In this subsection we associate a formal language to the definition of a 1D spin Hamiltonian that we just presented. Intuitively, the language associated to a Hamiltonian will contain the set of all pairs consisting of the spin configuration and its corresponding energy. Since the system size is not fixed, this will give rise to a language with an infinite number of strings.

There is obviously much freedom in the transformation of \((S(n), H(S(n)))\) to a string that belongs to a formal language. Since we want to study the complexity of the resulting formal language, it is only natural to choose the transformation that gives rise to the simplest possible language. Here we will choose one obvious encoding, in which we associate one spin to a fixed number of symbols, express the energy in unary, and introduce a constant overhead of string symbols [14].

We first introduce a rescaling of the Hamiltonian so that its image is in the natural numbers. The rescaling of the Hamiltonian (as given in Definition 6) is accomplished by replacing \( h \) by

\[
h'(\alpha) = \lambda h(\alpha) + \Lambda,
\]

where \( \lambda \in \mathbb{Q} \) is the greatest common divisor of all values in the image of \( h \), and \( \Lambda \in \mathbb{Z} \) is given by

\[
\Lambda = -\min_{\alpha \in \Sigma_q^k} h(\alpha).
\]

Note that \( \lambda \) and \( \Lambda \) are clearly computable, since the image of \( h \), and respectively \( \Sigma_q^k \), are finite sets. Note also that the image of \( h' \) now lies in \( \mathbb{N} \). The rescaling of a Hamiltonian \( H \) as given by Definition 6 is then defined as

\[
H' : D_{1D} \to \mathbb{N},
\]

\[
(S(n), n) \mapsto \sum_{i=1}^{n} h'(S_{[i-k+1,i]}).
\]
Definition 8 (Unary encoding). The unary encoding $u$ is a map from $\mathbb{N}$ to strings over a single symbol alphabet $\{\circ\}$

$$u : \mathbb{N} \to \{\circ\}^*$$

$$n \mapsto \circ^n, \quad n \geq 1$$

$$0 \mapsto \varepsilon.$$ 

In words, $u$ maps a number $n$ to the string consisting of $n$ times the symbol $\circ$, that is, $\circ^n$. It is important to note that $\circ$ is not part of the spin alphabet, $\circ \notin \Sigma_q$.

Definition 9 (Language of a 1D Hamiltonian). Let $H$ be the 1D Hamiltonian defined as in Definition 6, rescaled as in (4). The language associated to $H$, denoted $L_{\text{1D}}$, is defined as

$$L_{\text{1D}} = \{S^{(n)} \vdash u(H'(S^{(n)}, n)) \Downarrow \mid (S^{(n)}, n) \in \mathcal{D}_{\text{1D}}\}$$

where $\vdash, \Downarrow \notin \Sigma_q$ are delimiting symbols.

The symbol $\vdash$ serves as a separator between the spin configuration and the energy, while $\Downarrow$ marks the end of the string. Note that $|$ has the usual meaning of ‘such that’.

3.3. Classifying the language of 1D spin Hamiltonians

We now classify the language of 1D spin Hamiltonians in the Chomsky hierarchy.

Theorem 10 (1D Hamiltonians). $L_{\text{1D}}$ (Definition 9) is deterministic context-free.

To prove this theorem, we will use the well-known pumping lemma for regular languages.

Lemma 11 (Pumping lemma for regular languages [11]). Let $L$ be a regular language. Then the following property holds: There exists a pumping length $p \geq 1$ such that for any strings $\alpha, \beta, \gamma$ with $|\alpha\beta\gamma| \geq p$, there exist strings $\omega, \mu, \nu$ such that $\beta = \omega\mu\nu$, $\mu \neq \varepsilon$, and for all $j \geq 0$, the string $\alpha\omega\mu^j\nu\gamma \in L$.

Proof of Theorem 10. We first prove that $L_{\text{1D}}$ is not regular by Lemma 11. For a given $p$ we choose three strings

$$\alpha = S^{(n)} \vdash, \quad \beta = u(H'(S^{(n)}, n)) \quad \gamma = \Downarrow$$

such that $S^{(n)}$ is a string of spins with $H'(S^{(n)}, n) \geq p$. This ensures that the string

$$\alpha\beta\gamma = S^{(n)} \vdash \circ \circ \circ \cdots \circ \circ \Downarrow$$

is an element of $L_{\text{1D}}$ with $|\beta| \geq p$. It is now clear that any choice of $\omega\mu\nu = \beta$ with $j \neq 1$ will result in a malformed string $\alpha\omega\mu^j\nu\gamma \notin L_{\text{1D}}$, since the spin configuration and the energy no longer match. Therefore $L_{\text{1D}} \notin L_3$.

Now we prove that $L_{\text{1D}}$ is in DCFL by explicitly constructing a DPDA that accepts $L_{\text{1D}}$. In terms of Definition 3 the automaton is described by

- States $Q = \{a, b, c, f\}$
- Input alphabet $\Sigma = \Sigma_q \cup \{\circ, \vdash, \Downarrow\}$
- Stack alphabet $\Gamma = \{T_\omega \mid \omega \in (\Sigma_q \cup \{x\})^{k-1} \cup \{\circ, Z\}\}$
- Transition relation $\delta$ as shown in Fig. 3
- Start state $a$
- Initial stack symbol $Z$
- Accepting states $F = \{f\}$

A visualisation of the DPDA can be seen in Fig. 3. The main idea of this construction is the following. We encode the last $k - 1$ spins read from the tape in a single symbol $T_{S_{[k-1]}}$ on top of the stack to indicate that no spins have been read yet. Each transition reads the spin $S_{[i]}$ from the tape and the previous $k - 1$ spins $S_{[i-k+1]}$ from top of the stack, pushing back the unary energy from their local interaction and $T_{S_{[k-1]}}$ on top. The total energy is successively built up on the stack, while the topmost symbol always corresponds to the previously read spins. This continues until the delimiter $\vdash$ is encountered. (e): The energy string in the tape is compared to the energy string in the stack, and the accepting state $f$ is reached only if their length matches. We have omitted all transitions that lead the automaton to reject.

The code that simulates this DPDA is attached to this manuscript, in the file called dpda_simulator.py.

Note that a trivial Hamiltonian (e.g. a Hamiltonian that maps every spin configuration to the same value, $H' = 0$) is a particular case of Definition 6, and it corresponds to a regular language (cf. Fig. 2). To see this, note that in this...
The language of the 1D Ising model is recognized by a DPDA. The tape initially contains the spin configuration (written in some appropriate alphabet) and an energy value written in unary. This input string is an element of $L_{\text{Ising}}$ if and only if the energy is precisely $u(H'(S(n), n))$. The head of the PDA reads the state of the first two spins, calculates the energy $h'(s_1, s_2)$, and stores it on the stack in unary. It then proceeds similarly with the second and third spins and so on until the last pair. Finally, it compares the calculated energy stored on the stack with the value written on the tape and accepts if and only if they coincide.

Hamiltonian the spin configuration is independent of the energy, from which it follows that the pumping lemma for regular languages (Lemma 11) does apply. Recognizing this language reduces to verifying that the input obeys the pattern of a spin configuration followed by delimiters. This requires no stack and can therefore be done with a finite-state automaton.

**Example 12 (1D Ising model with fields revisited).** The language corresponding to the 1D Ising model with fields (Example 7), denoted $L_{\text{Ising}}^f$, is recognized by the DPDA of Fig. 4.

### 3.4. Language of 1D spin Hamiltonians with binary encoding

We now consider a binary encoding of the energy, instead of a unary one, and prove that the resulting language, $L_{1D}^\text{binary}$, is no longer deterministic context-free (Proposition 15). This illustrates that the classification in the Chomsky hierarchy of a language $L_H$ associated to a Hamiltonian $H$ depends on this association, $H \leftrightarrow L_H$, obviously. On the other hand, since it is a complexity classification, it makes sense to associate the language that gives rise to the minimal possible such complexity. In this respect, the unary encoding of the energy gives rise to a smaller complexity than the binary one, and thus justifies our standard choice of the unary encoding.

**Definition 13 (Binary encoding).** The binary encoding $b$ is the map

$$b : \mathbb{N} \rightarrow \{0, 1\}^*$$

$$n \mapsto b(n) = c_1 \ldots c_m$$

where $m = \lceil \log_2(n) \rceil$ and $n = \sum_{i=1}^m c_i 2^{i-1}$.

Note that we are slightly abusing notation, as we use $\{0, 1\}$ as symbols of an alphabet (first line of the definition), and as integers (second line of the definition).

**Definition 14 (Binary language of a 1D Hamiltonian).** Let $H$ be the 1D Hamiltonian defined as in Definition 6, rescaled as in (4). The binary language associated to $H$, denoted $L_{1D}^\text{binary}$, is defined as

$$L_{1D}^\text{binary} = \{ s^{(n)} \rightarrow b(H'(S^{(n)}, n)) \perp \mid (S^{(n)}, n) \in D_{1D} \}$$

where $\rightarrow, \perp \notin \Sigma_q$ are delimiting symbols.

Note that the symbols 0, 1 $\notin \Sigma_q$.

**Proposition 15 (1D Hamiltonians, binary).** $L_{1D}^\text{binary}$ (Definition 14) is context-sensitive.

To show that $L_{1D}^\text{binary}$ is not context-free we will use Ogden’s lemma.

**Lemma 16 (Ogden’s lemma [15]).** Let $L$ be a context-free language. Then there exists an integer $p \geq 1$ such that for any string $\eta \in L$ with $|\eta| \geq p$ the following holds: For any way of selecting $p$ or more of the symbols in $\eta$, there is a decomposition $\eta = \alpha \beta \gamma \mu \nu$ such that

(i) $\beta \mu$ contains at least one of the selected symbols,

(ii) $\beta \gamma \mu$ contains at most $p$ selected symbols, and

(iii) $\alpha \beta^j \gamma \mu^j \nu \in L$ for all $j \geq 0$.

**Proof of Proposition 15.** First we prove that $L_{1D}^\text{binary}$ is not context-free. So assume $L_{1D}^\text{binary}$ is context-free, and that $H$ is non-trivial, i.e. $H' \neq 0$. For any $p \geq 1$, there exists an element $\eta \in L_{1D}^\text{binary}$ such that the binary representation of the energy, $b(H'(S(n), n))$, contains the symbol 1 at least $p$ times. These 1 symbols will be the selected symbols of the statement of Lemma 16, and we claim that for any decomposition $\eta = \alpha \beta \gamma \mu \nu$ for which condition (i) holds, condition (iii) does not hold. So, by (i), $\eta = \alpha \beta \gamma \mu \nu$ includes at least one symbol 1 in $\beta \mu$. Now, the string $\alpha \beta^j \gamma \mu^j \nu$ with $j > 1$ has symbols 1 placed in a more significant digit of the binary representation, and thus the corresponding energy is exponentially larger than that in $\eta$. On the other hand, adding at most $jn$ spins results in a spin configuration whose energy (according to the Hamiltonian $H'$) only increases by a linear amount in $j$, because $k$ is bounded. It follows that (iii) is not fulfilled. Therefore (i) and (iii) can not be fulfilled simultaneously and $L_{1D}^\text{binary} \notin L_2$.

To prove that $L_{1D}^\text{binary}$ in $L_1$ we explicitly construct an LBA recognizing the language in Appendix A.

### 4. 2D SPIN HAMILTONIANS AS AUTOMATA

In this section we define 2D spin Hamiltonians (Section 4.1), associate a formal language to them (Section 4.2), show that they are context-sensitive (Section 4.3), and discuss the extension to other lattices (Section 4.4).
4.1. Definition of 2D spin Hamiltonians

In order to define 2D spin Hamiltonians we need to introduce some more notation. The total number of spins in the input is denoted as a product of the form $n = n_in_r$, where $n_r$ is the number of rows in the spin lattice, and $n_c$ the number of columns. The entire 2D configuration of $n$ spins is denoted $(S^{(n,n_c)}, n_r, n_c)$. A double subindex $s_{i,j}$ indicates the spin at the $i$th row and the $j$th column. In the string $S^{(n,n_c)}$, this spin will be labeled $s_{n_r+j}$. Strings describing a rectangle in the spin lattice spanned by the corners $s_{i,j}$ and $s_{i',j'}$ are denoted $S_{[i,j][i',j']} = S_{i,j} S_{i,j+1} \cdots S_{i,j'} S_{i+1,j} S_{i+1,j+1} \cdots S_{i',j'}$. The entire 2D configuration of $n$ spins is the number of rows in the spin lattice, and $n_c$ is the number of columns. Let $S_i$ be the symbol for the absence of a spin, and is used to model open boundary conditions. A rescaled lattice spanned by the corners $s_{i,j}$ corresponds to the $i$th row.

Definition 17 (Local 2D Hamiltonian). Let $q > 1, k \geq 1$ be fixed natural numbers. Let

$$D_{2D} = \left\{ (S^{(n,n_c)}, n_r, n_c) \mid S^{(n,n_c)} \in \Sigma_q^{n_c} \right\}$$

be the set of 2D spin configurations. A local 2D Hamiltonian $H$ with at most $k$-body interactions is a map $H : D_{2D} \rightarrow \mathbb{Q}$

$$h(S^{(n,n_c)}, n_r, n_c) \mapsto \sum_{i=1}^{n_c} \sum_{j=1}^{n_c} h(S_{[i-k+1,j][i-k+1,j]})$$

In addition, $h : \Sigma_q^k \rightarrow \mathbb{Q}$ is called the local Hamiltonian.

4.2. The language associated to 2D spin Hamiltonians

We now associate a formal language to 2D spin Hamiltonians. As in the 1D case, the local interaction is extended with the special symbol $x$, which stands for “the absence of a spin” and is used to model open boundary conditions. A rescaled $H'$ is defined analogously as before. Namely first $h'$ is defined analogously as (3), and

$$H' : D_{2D} \rightarrow \mathbb{N}$$

$$(S^{(n,n_c)}, n_r, n_c) \mapsto \sum_{i=1}^{n_c} \sum_{j=1}^{n_c} h'(S_{[i-k+1,j][i-k+1,j]})$$

Namely we rescale the local interaction $h$ so that its image is in $\mathbb{N}$; this results in $h'$. The unary encoding is used exactly as before.

Definition 18 (Language of a 2D Hamiltonian). Let $H$ be a local 2D Hamiltonian as defined in Definition 17, rescaled as in (5). The language associated to $H$, denoted $L_{2D}$, is defined as

$$L_{2D} = \{ S_1 \cdots S_n \rightarrow \vdash H'(S^{(n,n_c)}, n_r, n_c)) \perp (S^{(n,n_c)}, n_r, n_c) \in D_{2D} \}$$

where $\rightarrow, \vdash, \perp \notin \Sigma_q$ are delimiting symbols.

The new delimiter $\rightarrow$ is used to indicate the end of a row of spins.

4.3. Classifying the language of 2D spin Hamiltonians

Now we classify $L_{2D}$ in the Chomsky hierarchy.

Theorem 19 (2D Hamiltonians). $L_{2D}$ (Definition 18) is context-sensitive.

Intuitively, the jump in complexity from $L_1$ to $L_2$ is due to the fact that the distance between interacting spins is unbounded in the input string. For example, spin $s_{1,1}$ interacts with $s_{2,1}$, but the input string is one-dimensional and presented as $s_{1,1}, \ldots, s_{1,n_r}, s_{2,1}, \ldots$. Since the number of columns $n_c$ is unbounded, $s_{1,1}$ and $s_{2,1}$ are arbitrarily separated in the input string.

To prove the theorem, we will use the pumping lemma for context-free languages.

Lemma 20 (Pumping lemma for context-free languages [11]). Let $L$ be a context-free language. Then the following property holds: There exists a pumping length $p \geq 1$ such that for any string $\eta \in L$ with $|\eta| \geq p$, there exists a decomposition $\eta = \alpha\beta\gamma\delta$ with $|\beta\delta| \geq 1$ and $|\beta\gamma| \leq p$, such that $\alpha\beta^\prime\gamma\delta^\prime \in L$ for all $j \geq 0$.

Note that the difference to the pumping lemma for regular languages is that in this case there are two substrings $\beta$ and $\mu$ which are pumped simultaneously.

Proof of Theorem 19. First we prove that $L_{2D}$ is not context-free by exploiting Lemma 20. We do so by showing that for any choice of pumping length $p$ there exists an element of $L_{2D}$ that does not fulfill the condition stated in the lemma. Specifically, for $p \geq 1$, choose any element $\eta \in L_{2D}$ with number of spin columns $n_c > p$ and rows $n_r \geq 3$. Now, for any decomposition $\eta = \alpha\beta\gamma\delta$ as stated by the lemma, the middle part can at most span across two of the rows, due to $|\beta\gamma| \leq p < n_c$. Pumping the string with any choice of $j \neq 1$ will therefore cause at least two rows to be of unequal length. It follows that $\alpha\beta^p\gamma\delta^p \notin L_{2D}$, and therefore $L_{2D} \notin L_2$.

To prove that $L_{2D} \in L_1$ we explicitly construct an LBA in Appendix B.

□

The code that simulates this LBA is also attached to this manuscript, in the file called lba_simulator.py.

Remark 21 (Periodic boundary conditions). Both Theorems 10 and 19 hold for periodic instead of open boundary conditions. In the 1D case the DPDA must be augmented to carry the state of the first $k-1$ spins all the way through the computation to compute their interactions with the last $k-1$ spins. This is achieved by extending the stack alphabet such that both the last $k-1$ spins, as well as the first $k-1$ spins read can be stored in a single symbol $T_{S_1 \cdots S_k}$ on top of the stack. On the last transition before comparing the energies, the energies of the boundary-crossing interactions are evaluated and pushed all at once.

For the 2D case the collecting spin step needs to be adjusted. Instead of adding the $x$ symbol to the internal string representation of the interaction when the edges of the lattice are encountered, the automaton instead moves its head to the
4.4. Extensions to other lattices

We now show that Theorem 19 is robust to a number of changes in the definition of the Hamiltonian (Definition 17) — in particular, to other lattice shapes, and also when extended to higher dimensional lattices.

First, the proof of Theorem 19 can also be applied to Hamiltonians on other periodic 2D arrangements of spin interactions, as shown in Fig. 5 for a triangular lattice. To do so the tape alphabet can be extended by the special symbol $x$ which is then used to embed the lattice into a rectangular grid. The interaction range $k$ and the interaction map $h$ can then be chosen to reproduce the original interactions on this new lattice, and language as well as automaton can be constructed with the methods discussed earlier.

Second, we sketch the extension of the construction of the LBA for 2D spin Hamiltonians (Appendix B) to higher dimensional lattices. Consider a $d$-dimensional spin lattice, with a local Hamiltonian acting on spins in a hypercube with $k^d$ elements. The same steps as in the 2D case can be used here, with the difference that the spin configuration is a hypercube. The checking of the row lengths is expanded to verify uniform length across each dimension of the hypercube, and the spin collecting step likewise navigates through these extra dimensions. Both these changes can be handled by introducing further states, transitions, and tape symbols, but require no additional space on the tape, so the automaton is still an LBA. By the same pumping arguments as before, the resulting language can also not be context-free, making it a context-sensitive language.

5. LOCAL HAMILTONIAN ENCODED IN THE INPUT

In the previous sections, every spin Hamiltonian $H$ was defined by the uniform “action” on all spins of a given local Hamiltonian $h$. For any $H$, $h$ was fixed. In this section, we consider the case in which $h$ is specified as part of the input. The only requirement will be that the interaction range of $h$ be bounded, and that $h$ be chosen from a finite set. As before, we first consider the 1D case (Section 5.1) and then the 2D case (Section 5.2).

5.1. 1D spin Hamiltonians with unfixed local Hamiltonian

Here we define 1D spin Hamiltonians with unfixed local Hamiltonian (Definition 22), associate a language to them (Definition 23), and show that this language is deterministic context-free (Corollary 24).

We consider the case in which the local Hamiltonian can be chosen from a finite set, $\{h_1, \ldots, h_p\}$, for some fixed $p \geq 1$. Which local Hamiltonian is applied, centered around every spin, is given as part of the input. Namely, for every system size $n$, $H$ will also pick local Hamiltonians

$$h_{I_1}, \ldots, h_{I_n},$$

where each index $I_j \in [p]$ specifies the local Hamiltonian that is picked “centered around spin $j$” (in fact $j$ will be at the right end of the interaction). The collection of all these indices is denoted $I = (I_1, \ldots, I_n)$, where we omit the superscript $(n)$ to keep notation simple.

**Definition 22 (Unfixed local 1D Hamiltonian).** Let $q > 1, k \geq 1, p \geq 1$ be fixed natural numbers. Let

$$\mathcal{F}_{1D} = \bigcup_{n \geq 1} \{ (I, S^{(n)}), n | I \in [p]^n, S^{(n)} \in \Sigma_q^n \}$$

be the set of all 1D spin configurations with unfixed local Hamiltonians. An unfixed 1D local Hamiltonian $H$ with at most $k$-body interactions is a map

$$H : \mathcal{F}_{1D} \rightarrow \mathbb{Q}$$

$$(I, S^{(n)}, n) \mapsto \sum_{i=1}^n h_{I_i}(S_{i-(k+1)i}),$$

where $h_1, \ldots, h_p : \Sigma_q^k \rightarrow \mathbb{Q}$ are the local Hamiltonians.

Recall that $h_i$ is a local interaction interaction in the physical sense (see [5]), meaning that it assigns energy values to subconfiguration $S^{(k)}$ of $k$ adjacent spins.

The rescaling of this Hamiltonian is done analogously to Eq. (3), with the only difference that indices now run over the finite set $[p]$. Namely, we choose $\lambda$ as the greatest common divisor among the images of all $h_j$, and

$$\lambda = \min \min_{j \in [p], \alpha \in \Sigma_q^k} h_j(\alpha).$$

The rescaled local interaction is thus

$$h_j'(\alpha) = \lambda(h_j(\alpha) + \lambda),$$

where each index $I_j \in [p]$ specifies the local Hamiltonian that is picked “centered around spin $j$” (in fact $j$ will be at the right end of the interaction). The collection of all these indices is denoted $I = (I_1, \ldots, I_n)$, where we omit the superscript $(n)$ to keep notation simple.

**Definition 22 (Unfixed local 1D Hamiltonian).** Let $q > 1, k \geq 1, p \geq 1$ be fixed natural numbers. Let

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$$H : \mathcal{F}_{1D} \rightarrow \mathbb{Q}$$

$$(I, S^{(n)}, n) \mapsto \sum_{i=1}^n h_{I_i}(S_{i-(k+1)i}),$$

where $h_1, \ldots, h_p : \Sigma_q^k \rightarrow \mathbb{Q}$ are the local Hamiltonians.

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**Definition 22 (Unfixed local 1D Hamiltonian).** Let $q > 1, k \geq 1, p \geq 1$ be fixed natural numbers. Let

$$\mathcal{F}_{1D} = \bigcup_{n \geq 1} \{ (I, S^{(n)}), n | I \in [p]^n, S^{(n)} \in \Sigma_q^n \}$$

be the set of all 1D spin configurations with unfixed local Hamiltonians. An unfixed 1D local Hamiltonian $H$ with at most $k$-body interactions is a map

$$H : \mathcal{F}_{1D} \rightarrow \mathbb{Q}$$

$$(I, S^{(n)}, n) \mapsto \sum_{i=1}^n h_{I_i}(S_{i-(k+1)i}),$$

where $h_1, \ldots, h_p : \Sigma_q^k \rightarrow \mathbb{Q}$ are the local Hamiltonians.

Recall that $h_i$ is a local interaction interaction in the physical sense (see [5]), meaning that it assigns energy values to subconfiguration $S^{(k)}$ of $k$ adjacent spins.

The rescaling of this Hamiltonian is done analogously to Eq. (3), with the only difference that indices now run over the finite set $[p]$. Namely, we choose $\lambda$ as the greatest common divisor among the images of all $h_j$, and

$$\lambda = \min \min_{j \in [p], \alpha \in \Sigma_q^k} h_j(\alpha).$$

The rescaled local interaction is thus

$$h_j'(\alpha) = \lambda(h_j(\alpha) + \lambda),$$

where each index $I_j \in [p]$ specifies the local Hamiltonian that is picked “centered around spin $j$” (in fact $j$ will be at the right end of the interaction). The collection of all these indices is denoted $I = (I_1, \ldots, I_n)$, where we omit the superscript $(n)$ to keep notation simple.

**Definition 22 (Unfixed local 1D Hamiltonian).** Let $q > 1, k \geq 1, p \geq 1$ be fixed natural numbers. Let

$$\mathcal{F}_{1D} = \bigcup_{n \geq 1} \{ (I, S^{(n)}), n | I \in [p]^n, S^{(n)} \in \Sigma_q^n \}$$

be the set of all 1D spin configurations with unfixed local Hamiltonians. An unfixed 1D local Hamiltonian $H$ with at most $k$-body interactions is a map

$$H : \mathcal{F}_{1D} \rightarrow \mathbb{Q}$$

$$(I, S^{(n)}, n) \mapsto \sum_{i=1}^n h_{I_i}(S_{i-(k+1)i}),$$

where $h_1, \ldots, h_p : \Sigma_q^k \rightarrow \mathbb{Q}$ are the local Hamiltonians.

Recall that $h_i$ is a local interaction interaction in the physical sense (see [5]), meaning that it assigns energy values to subconfiguration $S^{(k)}$ of $k$ adjacent spins.

The rescaling of this Hamiltonian is done analogously to Eq. (3), with the only difference that indices now run over the finite set $[p]$. Namely, we choose $\lambda$ as the greatest common divisor among the images of all $h_j$, and

$$\lambda = \min \min_{j \in [p], \alpha \in \Sigma_q^k} h_j(\alpha).$$

The rescaled local interaction is thus

$$h_j'(\alpha) = \lambda(h_j(\alpha) + \lambda),$$
and the Hamiltonian $H'$ as

$$H'(I, S^{(n)}) = \sum_{i=1}^{n} h_i'(S_{(i-\hat{k}+1,i)}) .$$  \(7\)

Now we define the language associated to the Hamiltonian of Definition 22. We use alphabet $\Sigma_p = \{p\}$, whose symbols correspond to the indices of the $p$ different local Hamiltonians. We denote a string of spin symbols interleaved with a string of local Hamiltonian indices by

$$R^{(n)} = \bar{I}_1 s_1 \bar{I}_2 s_2 \ldots s_n \in (\Sigma_p \Sigma_q)^n .$$

**Definition 23 (Language of an unfixed local 1D Hamiltonian).** Let $H$ be a Hamiltonian as given in Definition 22, and $H'$ its rescaled version given by (7). The language associated to $H$, denoted $L^{\text{unfix}}_{1D}$, is defined as

$$L^{\text{unfix}}_{1D} = \{ R^{(n)} \uparrow u(H'(I, S^{(n)}), n) \downarrow \mid (I, S^{(n)}, n) \in T_{1D} \}$$

where $\uparrow, \downarrow \notin \Sigma_q$ are delimiting symbols.

We now prove that the unfixed local Hamiltonians do not change the complexity of $L_{1D}$, so that the complexity classification given by Theorem 10 remains true.

**Corollary 24 (Unfixed 1D Hamiltonian).** $L^{\text{unfix}}_{1D}$ (Definition 23) is deterministic context-free.

**Proof.** A construction of a DPDA accepting $L^{\text{unfix}}_{1D}$ is given in Fig. 6. For $k \geq 1$ and $q \geq 2$ it is formally described by

- States $Q = \{ a, b_1, b_2, e, f \}$
- Input alphabet $\Sigma = \Sigma_q \cup \Sigma_p \cup \{ \uparrow, \downarrow, \circ \}$
- Stack alphabet $\Gamma = \{ T_{i,o} \mid I_i \in \Sigma_q \text{ and } \omega \in (\{q\} \cup \{x\})^{k-1} \cup \{ \circ, Z \}$
- Transition relation $\delta$ as shown in Fig. 6
- Start state $a$
- Initial stack symbol $Z$
- Accepting state $f$

The automaton is similar to the previous construction for the fixed interaction case. The stack alphabet $T_{i,o}$ symbols are extended so that they do not only encode the previous $k-1$ spins, but also the last seen interaction map index. Using this information from the top of the stack the automaton chooses its transitions to push the corresponding amount of energy symbols on the stack for each interaction. The energy comparison step is the same as before.

The proof that $L_{1D}$ is not regular is identical to the one for the case of fixed couplings. \( \square \)

5.2. 2D spin Hamiltonians with unfixed local Hamiltonian

Here we present an analogous study for the 2D case: We define unfixed local 2D spin Hamiltonians (Definition 25), associate a language to them (Definition 26), and show that this language is context-sensitive (Corollary 27).

**Definition 25 (Unfixed local 2D Hamiltonian).** Let $q > 1, k \geq 1, p \geq 1$ be fixed natural numbers. Let $F_{2D} = \bigcup_{n,n \geq 1} \{(I, S^{(n,n)}, n, n) \mid I \in [p]^{n,n}, S^{(n,n)} \in \Sigma_q^n \}$ be the set of all 2D spin configurations with unfixed local Hamiltonians. An unfixed local 2D Hamiltonian $H$ with at most $k^2$-body interactions is a map $H : F_{2D} \rightarrow \mathbb{Q}$

$$(I, S^{(n,n)}, n, n) \mapsto \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij}(S_{[\hat{i}-\hat{k}+1,i] \times [\hat{j}-\hat{k}+1,j])} ,$$

where $h_1, \ldots, h_p : \Sigma_q^k \rightarrow \mathbb{Q}$ are the local Hamiltonians.

Similar to the 1D case, and generalising the 2D case of Section 4.1, we denote rows of spins interleaved with interaction map indices by $\hat{R}_r$,

$$\hat{R}_r = \bar{I}_1 s_1 \bar{I}_2 s_2 \ldots s_n \in (\Sigma_p \Sigma_q)^n \text{.}$$

**Definition 26 (Language of an unfixed 2D Hamiltonian).** Let $H$ be a Hamiltonian as given in Definition 25. The language associated to $H$, denoted $L^{\text{unfix}}_{2D}$, is defined as

$$L^{\text{unfix}}_{2D} = \{ \hat{R}_r \leftarrow \hat{R}_2 \leftarrow \ldots \leftarrow \hat{R}_n \leftarrow u(H'(I, S^{(n,n)}), n, n) \downarrow \mid (I, S^{(n,n)}, n, n) \in F_{2D} \}$$

where $\leftarrow, \downarrow \notin \Sigma_q$ are delimiting symbols.

We now show that the classification of $L^{\text{unfix}}_{2D}$ in the Chomsky hierarchy is the same as that of its fixed counterpart, $L_{2D}$ (Theorem 19).

**Corollary 27 (Unfixed 2D Hamiltonian).** $L^{\text{unfix}}_{2D}$ (Definition 26) is context-sensitive.

**Proof.** The proof for $L^{\text{unfix}}_{2D} \notin L_2$ is identical to the one for $L_{2D}$.

To prove that $L^{\text{unfix}}_{2D}$ is context-sensitive, we slightly modify the LBA given in Appendix B. During the collecting spins step, the interaction map index of the current interaction spin is also collected and considered when determining the energy of the interaction. Since this addition does not require additional space on the tape beyond the input, the resulting automaton is still an LBA and $L^{\text{unfix}}_{2D} \in L_1$. \( \square \)

6. UNBOUNDED INTERACTION RANGE

In this section we consider Hamiltonians with an unbounded interaction range. This means that the range of the interactions, called $k$ in the previous sections, is not upper bounded, i.e. grows with the system size $n$. We distinguish two cases: First, the case in which there is a local Hamiltonian which is still local in the computer science sense, but not in the physical sense (see the discussion before Definition 6), i.e. the local Hamiltonian has a fixed arity (Section 6.1); and the case in which the latter does not hold, i.e. there is no restriction on the local Hamiltonian (Section 6.2).
We first use the pumping lemma for context-free languages (Lemma 20). We consider the worst case in the sense.) Physically, the variables involved in $H$ can be as far from each other as possible, i.e. they are not local in the physical sense.

As in previous sections, we will first define these Hamiltonians (Definition 28), associate a formal language to them (Definition 29) and prove that they are context-sensitive (Proposition 30). We use the same complexity as for the 2D bounded case (Section 4). Indeed, the ultimate reason is also the same: the lack of upper bound on the distance between interacting spins will imply that the effective distance between interacting spins in the string of the language is unbounded, which implies that the corresponding language cannot be context-free.

Let us now define a spin Hamiltonian $H$ as a sum of local Hamiltonians $h$ of fixed arity $r$. (Note that we still refer to $h$ as a local Hamiltonian, although it is no longer local in the physical sense.) Physically, the variables involved in $h$ can be as far from each other as possible, i.e. we consider the worst case in which the physical range of the interaction $k$ is the whole system size $n, k = n$. Note that previously (Section 3, Section 4) we had $r = k$.

**Definition 28** (Unbounded 1D Hamiltonian of fixed arity). Let $q > 1, r ≥ 2$ be fixed natural numbers. Let $D_{1D}$ be given by (1). An unbounded 1D Hamiltonian of fixed arity $H$ is a map

$$H : D_{1D} \rightarrow \mathbb{Q}$$

$$(S^{(α)}, n) \mapsto \sum_{0 < i_1 < i_2 < \cdots < i_n \leq n} h(s_{i_1}, s_{i_2}, \ldots, s_{i_n}). \quad (9)$$

In addition, $h : \Sigma_q^r \rightarrow \mathbb{Q}$ is called the local Hamiltonian.

In order to define the associated language $L_{1D}^{\text{unbound}}$, we rescaled $H$ to $H'$ analogously as in (4).

**Definition 29** (Language of an unbounded 1D Hamiltonian). Let $H$ be the unbounded 1D spin Hamiltonian of Definition 28, rescaled analogously as in (4). The language associated to $H$, denoted $L_{1D}^{\text{unbound}}$, is defined as

$$L_{1D}^{\text{unbound}} = \{S^{(α)} \mapsto h(H'(S^{(α)}, n)) \downarrow \mid \{S^{(α)}, n\} \in D_{1D} \}$$

where $\uparrow, \downarrow \notin \Sigma_q$ are delimiting symbols.

We now show that an unbounded $k$ does increase the complexity of the language.

**Proposition 30** (Unbounded 1D Hamiltonians). $L_{1D}^{\text{unbound}}$ (Definition 29) is context-sensitive.

**Proof.** We first use the pumping lemma for context-free languages (Lemma 20) to prove that $L_{1D}^{\text{unbound}} \notin \mathcal{L}_Z$. Let us assume that the Hamiltonian is not trivial, $H' \neq 0$. For a given pumping length $p$ we choose $\omega \in L_{1D}^{\text{unbound}}$ such that

$$\omega = \theta^p \circ \cdots \circ \downarrow,$$

where $\theta \in \Sigma_q^r$ such that $h'(\theta) > 0$. We now consider all possible decompositions of $\omega = αβγμ ν$ with $|βγμ| ≤ p$ and $|βμ| > 0$:

(i) $β$ or $μ$ contains one of the delimiters $\uparrow$ or $\downarrow$. The pumping lemma is clearly violated in this case, as any $j \neq 1$ leads to an amount of delimiters that does not conform with $L_{1D}^{\text{unbound}}$. In particular this implies that any choice of decomposition that fulfills the lemma must have $β$ and $μ$ which are purely substrings of the spin configuration $θ'$ or of the energy $\circ \cdots \circ$.

(ii) One of $β$ or $μ$ is empty and the other is a substring of $θ'$, or both are substrings of $θ'$. If we were to duplicate any particular spin of the configuration, the energy of the configuration would increase with at least $p$, since the new spin interacts with all $p$ substrings $θ$ of the configuration. Therefore, any possible choice of $β$ and $μ$ with $j \neq 1$ leads to $αβγμ ν \notin L_{1D}^{\text{unbound}}$, since the energy no longer matches.

(iii) One of $β$ or $μ$ is empty and the other is a substring of $\circ \cdots \circ$, or both are substrings of $\circ \cdots \circ$. Any choice of $j \neq 1$ leads to $αβγμ ν \notin L_{1D}^{\text{unbound}}$, since the energy is changed without changing the spin configuration.
(iv) $\beta$ is a substring of $\theta^p$ and $\mu$ is a substring of $\phi \ldots \phi$, both of which are non-empty. By the same argument as in (i), when replacing $\beta$ with $\beta'$ the energy increases by at least $(j-1)p$. On the other hand, duplicating $\mu$ with the same $j$ only adds $(j-1)|\mu|$ diamonds to the string. Due to $|\beta| \geq p$ and $|\beta| > 0$ it follows that $|\mu| < p$ and consequently $(j-1)|\mu| < (j-1)p$.

Since $\alpha \beta \gamma \mu \nu \not\in L_{\text{total}}^{\text{unbound}}$ for $j > 1$ in all cases, $L_{\text{total}}^{\text{unbound}} \not\subseteq L_1$.

To prove that $L_{\text{total}}^{\text{unbound}} \in L_1$ we explicitly construct an LBA accepting the language in Appendix C. \hfill \Box

6.2. Totally unbounded Hamiltonians

We now consider Hamiltonians which are not local in the physical or computer science sense — we call these Hamiltonians totally unbounded. To define these Hamiltonians it no longer makes sense to use local Hamiltonians $H$. Instead, we view the Hamiltonian as a family of functions $\{H_n\}_n$ for all system sizes $n$. We will only consider the case in which this family admits a computable description, i.e. where there exists a Turing machine such that on input $n$ it produces a description of $H_n$. We now give a formal definition of such Hamiltonians, as well as the language associated to them.

**Definition 31 (Totally unbounded Hamiltonian).** Let $q > 1$ be a fixed natural number. A totally unbounded Hamiltonian $H$ is a family of functions

$$H = \{H_n : \Sigma_q \to \mathbb{Q}\}_n.$$

Furthermore, there is a Turing machine $C$ that, when given input $n$, it produces a description $d$ of $H_n$,

$$C : n \mapsto d(H_n).$$

In addition, there exist $\lambda \in \mathbb{Q}$ and $\Lambda \in \mathbb{Q}$ such that $H'_n := \lambda(H_n + \Lambda)$ is a map

$$H'_n : \Sigma^n_q \to \mathbb{N}$$

for all $n$.

Note that in the second part of Definition 31 we are asking for the existence of a rescaling ($\lambda$) and shift ($\Lambda$) independent of the system size.

**Definition 32 (Language of a totally unbounded Hamiltonian).** Let $H$ be a totally unbounded Hamiltonian as given in Definition 31. The language associated to $H$, denoted $L_{\text{total}}$, is defined as

$$L_{\text{total}} = \{S^{(n)} \vdash u(H'_n(S^{(n)})) \perp \mid S^{(n)} \in \Sigma^n_q\}$$

where $\vdash$, $\perp \not\in \Sigma_q$ are delimiting symbols.

In the following we prove that this language is in $L_0$, for the obvious reason that the very description of the Hamiltonian involves a Turing machine itself.

**Proposition 33 (Totally unbounded Hamiltonians).** $L_{\text{total}}$ (Definition 32) is recursively enumerable.

**Proof.** To show that $L_{\text{total}} \not\subseteq L_1$, note that any automaton recognizing $L_{\text{total}}$ necessarily includes the Turing machine $C$ (Definition 31) as a subroutine. The tape space this TM needs need not be bounded by a linear function of the size of the input, i.e. it need not be an LBA.

That $L_{\text{total}} \not\subseteq L_0$ follows from the definition of the language: we require that the description of $H$ be computable. Explicitly, we claim there is a TM $M$ that takes as input the string $S^{(n)} \vdash u(E_n) \perp$ where $E_n$ is a natural number, and accepts if and only if $E_n = H'_n(S^{(n)})$. In addition, $M$ rejects all other strings, including ill-formed ones. This TM works as follows: first it counts the length of the spin configuration $S^{(n)}$, i.e. $n$. Then it uses the TM $C$ as a subroutine (Definition 31) in order to obtain $d(H_n)$. The latter could for example be a list of the values of spin configurations of length $n$ and their corresponding energy, where the spin configurations are ordered e.g. lexicographically. Now $M$ reads $S^{(n)}$ and looks for the corresponding energy value in the list, i.e. $H_n(S^{(n)})$. Since by assumption there exist $\lambda, \Lambda$ with the properties given in Definition 31, $M$ uses them to compute $H'_n(S^{(n)})$. Finally, $M$ compares $H'_n(S^{(n)})$ with $E_n$, and accepts if and only if they are identical. \hfill \Box

7. CONCLUSIONS AND OUTLOOK

We have introduced a new complexity measure for classical spin Hamiltonians in which these are described as automata. More precisely, we have associated a formal language to classical spin Hamiltonians and have classified this language in the Chomsky hierarchy. Our main results are that:

(i) The language associated to local 1D spin Hamiltonians is deterministic context-free (Theorem 10), even if the local Hamiltonian is given as part of the input (Corollary 24).

(ii) The language associated to local 2D spin Hamiltonians is context-sensitive (Theorem 19), even if the local Hamiltonian is part of the input (Corollary 27). The same classification holds for the language associated to unbounded 1D spin Hamiltonians (Proposition 30).

(iii) The language associated to totally unbounded Hamiltonians is recursively enumerable (Proposition 33).

In addition, we have shown that the language associated to bounded 1D spin Hamiltonians where the energy is written in binary is context-sensitive (Proposition 15).

We thus see that for ‘natural’ spin models, that is, local ones, the threshold in hardness occurs at the boundary between a 1D and a 2D lattice. In particular, this implies that the language associated to the Ising model without fields is deterministic context-free or context-sensitive if it is defined on a 1D or 2D lattice, respectively. This is in contrast to the
computational complexity of its ground state energy problem, which is in P or NP-complete if defined on a planar or non-planar graph, respectively (Table 1).

In any case, all of these natural spin models correspond to automata which are weaker than Turing machines. Only highly non-physical spin models — namely, totally unbounded Hamiltonians — correspond to Turing machines.

This work puts classical spin Hamiltonians and automata at the same level (by describing one as the other), and thereby opens the door to transfer results from one field to the other. As a first result of this cross-fertilisation, we have borrowed the Chomsky hierarchy of formal languages to derive a complexity classification of classical spin Hamiltonians. But this has sown the ground for many other results: An important one would be a rigorous comparison between universal spin models [16] and universal Turing machines. (Note that translationally invariant universal classical spin Hamiltonians have also been shown to exist [17]). For example, universal spin models allow for a full characterisation [16, Theorem 15], but universal Turing machines do not, due to Rice’s Theorem — thus, the two notions of universality do not seem to be equivalent. Another fascinating consequence involves exploring the reach of universality in classical spin Hamiltonians.

Another line of research concerns the quantum case, where one would consider quantum spin Hamiltonians. It would be interesting to associate formal languages to them and to classify their complexity. Note that a complexity classification of the ground state energy problem for local qubit Hamiltonian problems has been provided in [18]. Note also that for quantum spin Hamiltonians, the phenomenon of universality has also been shown to hold [19], as well as for translationally invariant versions thereof [20, 21].

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Appendix A: An LBA that recognizes the binary language of 1D spin Hamiltonians

Here we present an LBA that recognizes the binary language of 1D spin Hamiltonians $L_{1D}^{\text{binary}}$, thereby showing that $L_{1D}^{\text{binary}} \in L_1$ and completing the proof of Proposition 15. This LBA will be based on the one of Appendix B. Its specifications are given by

- States $Q = Q_{\text{validate}} \cup Q_{\text{collect}} \cup Q_{\text{compare}} \cup \{\text{Accept, Reject}\}$
- Input alphabet $\Sigma = \Sigma_q \cup \{\perp, \circ, \dag\}$
- Tape alphabet $\Gamma = \Gamma' \cup \{\frac{1}{2}L, \frac{1}{2}R\}$ where $\Gamma' = \Sigma \cup \Sigma_q' \cup \{\perp\}$
- Blank Symbol $\perp \in \Gamma$
- Left and right endmarker $\frac{1}{2}L, \frac{1}{2}R \in \Gamma$
- Transition function $\delta$
- Start state $\text{Start} \in Q_{\text{validate}}$
- Accept state $\text{Accept} \in Q$
- Reject state $\text{Reject} \in Q$

To present this LBA, we need to define binary addition $\oplus$ as

$$\oplus : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$$

$$(\alpha, \beta) \mapsto b(b^{-1}(\alpha) + b^{-1}(\beta))$$

where $b^{-1}$ is the inverse of the binary encoding given in Definition 13,

$$b^{-1} : \{0, 1\}^* \rightarrow \mathbb{N}$$

$$c_1 \ldots c_m \mapsto \sum_{i=1}^{m} c_i 2^{i-1}.$$

As in the construction for 2D Hamiltonians we split the presentation of the LBA in several steps. The first two steps validate the input to ensure there are no unexpected symbols and collect interacting spins to determine the amount of energy associated to each local interaction. Both these steps are already discussed in Appendix B for the 2D case, so we refer there for an explicit implementation. Instead we focus on the third step, which compares the energy of the local interactions to the energy on the tape.

The energy comparison step consists of the states

$$Q_{\text{compare}} = \{\text{CheckEnergy}\} \cup \{\text{SubtractBinary}_\alpha \mid \alpha \in \{0, 1\}^i \text{ and } 0 \leq i < \lfloor \log_2 \max_{\beta \in \Sigma_q' \cup \Sigma_q''} h'(\beta) \rfloor\},$$

and the transitions given in Table 2. For each local interaction, the collecting spin step transitions into $\text{SubtractBinary}_\alpha$, where $\alpha$ is the binary representation of the energy of the interaction. The $\text{SubtractBinary}_\alpha$ states then perform binary subtraction. The energy in the state of the head is subtracted bitwise from the energy on the tape, adding carried bits back into $\alpha$ by binary addition. If there is no more energy on the tape to subtract, the automaton rejects. Otherwise, after finishing subtracting it transitions into $\text{NextInteraction} \in Q_{\text{collect}}$ to consider the next interaction.

Once all interactions have been considered, the automaton transitions into $\text{CheckEnergy}$. The tape is checked for the symbol 1, which corresponds to leftover energy. If there is no energy left on the tape the automaton accepts, otherwise the input is rejected.

Appendix B: An LBA that recognizes the language of 2D spin Hamiltonians

Here we construct an LBA that recognizes $L_{2D}$, thereby proving that $L_{2D} \in L_1$ and completing the proof of Theorem 19. The flowchart for this LBA is presented in Fig. 7. We first present the basic properties of this LBA and then present the main steps in Appendix A 1, 2, 3 and 4.

The LBA is described by

- States $Q = Q_{\text{rows}} \cup Q_{\text{validate}} \cup Q_{\text{collect}} \cup Q_{\text{compare}} \cup \{\text{Accept, Reject}\}$
- Input alphabet $\Sigma = \Sigma_q \cup \{\perp, \circ, \dag\}$
- Tape alphabet $\Gamma = \Gamma' \cup \{\frac{1}{2}L, \frac{1}{2}R\}$ where $\Gamma' = \Sigma \cup \Sigma_q' \cup \Sigma_q'' \cup \{\perp\}$
- Blank Symbol $\perp \in \Gamma$
- Left and right endmarker $\frac{1}{2}L, \frac{1}{2}R \in \Gamma$
- Transition function $\delta$ as specified by Tables 3 to 6
- Start state $\text{CountFirstRow} \in Q_{\text{rows}}$
- Accept state $\text{Accept} \in Q$
- Reject state $\text{Reject} \in Q$

First note that the tape alphabet of the automaton not only includes the spin alphabet $\Sigma_q$, but also $\Sigma_q'$ and $\Sigma_q''$. These are copies of the spin alphabet where each spin symbol is marked by a prime or double prime, and they are used by the automaton to earmark selected spins in the configuration. This is used to navigate the spin configuration during the computation.
A 2). For every spin interaction, the automaton reads all interacting spins and calculates the energy of the interaction (Appendix A 3). It overwrites the corresponding amount of energy symbols ‡ from the tape, and rejects if there are none left to overwrite. Afterwards the tape is scanned for any leftover ‡ (Appendix A 4), and if there are none it accepts.

Further note that for the sake of simplicity we split the transition table and set of states into four different parts as depicted in Fig. 7. In the following we go over every part and explain the workings of the automaton in detail.

1. Checking the row length

In the first step it is ensured that the rows of the spin configuration given in the input are all of equal length. This is achieved by counting the spins in every row in parallel and rejecting if one row is either longer or shorter than the first. This step is implemented via the states

$$Q_{row} = \{ \text{CountFirstRow, AdvanceRow, CountRow, BackToStart, CompareLengths} \}$$

as well as the transitions given in Table 3. Note that some combinations of current state and symbol are omitted in this table, as we go by the convention that any such omitted transitions lead to the Reject state.

The automaton initially starts in state CountFirstRow, marking the first spin of the configuration with a prime. It transitions into AdvanceRow which moves the head to the next row of spins, and then CountRow which marks the first unprimed spin it encounters there with a prime. This continues until one spin of every row has been marked and ‡ is read, at which point BackToStart is entered, which moves the head to the beginning of the input and starts over with CountFirstRow. This way, with each cycle between these states, one spin of every row is marked with a prime.

Once all spins in the first row have been marked, CountFirstRow encounters the first row delimiting symbol, at which point it enters the state CompareLengths and the entire input is scanned for unprimed spin symbols. If any are encountered, the input is rejected as the rows are not of uniform length. Additionally, if at any point during the previous transition cycles there were no spins left to prime in any row, the input is also rejected. Otherwise, the right endmarker ‡ at the end of the input is encountered, and the automaton moves on to the next phase, where the input is checked for unexpected symbols.

2. Validating the input

In this step the automaton checks the input for unexpected symbols and rejects if any are found. For example, an energy symbol ‡ in the spin configuration part of the input, or delimiters in the wrong places would cause the input to be rejected. This step is implemented via the states

$$Q_{validate} = \{ q_1, q_2, q_3, q_4, q_{1a}, q_{2a} \}$$

and the transitions are given by Table 4. Starting from the end of the input the automaton moves its head to the beginning and checks for unexpected symbols, by keeping track of the last symbol encountered in its state. Further, all the marked spins from the previous step are overwritten with their unmarked counterparts.

After encountering the left endmarker ‡ at the beginning of the input, the automaton has been validated against unexpected symbols. The automaton moves to the next step by again marking the first spin in the configuration with a prime and transitioning into the state RowEnd.
This is repeated until the interaction spin in the interaction row is encountered. Once the interaction spin is reached, the head encounters the left endmarker \( \perp \) it transitions to Return, because the subconfiguration extends beyond the input configuration.

After returning to the end of the current row the process repeats. This continues until MarkSpin\(_{k-1}\) encounters the interaction spin \( s \)—distinguished by being marked with a prime—at which point the automaton transitions to CollectSpins\(_{k-1,k-1}\).

The purpose of the CollectSpins\(_{a,i}\) states is to visit all the spins in the subconfiguration on the tape and store them in \( \alpha \), building a representation of the subconfiguration in the internal state of the automaton. The index \( i \) keeps track of how many more rows there are to visit. While \( i > 0 \), \( j > 0 \) the head moves to the left, skipping over double primed spins and prepending any unmarked spins it reads to \( \alpha \), decrementing \( i \) for each spin collected in this manner. For \( i = 0 \) the head continues moving to the left, but stops collecting spins. Whenever the row delimiter \( \perp \) is encountered \( \alpha \) is prepended by the string \( x \), \( j \) is decremented, and \( i \) is set to \( k \).

In the cases where \( i \neq 0 \) the string \( x \) is non-empty and corresponds to \( i \) spins missing from the current row, otherwise \( x^0 = \varepsilon \) and \( \alpha \) remains unchanged. As explained in earlier sections, this serves to model open boundary conditions in the cases where the subconfiguration extends beyond the input configuration.

This step ends once both \( i = 0 \) and \( j = 0 \), \( \alpha \) now containing \( k^2 \) spins collected from \( k \) consecutive rows. The automaton transitions to the state SubtractEnergy\(_{\perp} \), and proceeds to the energy comparison step. The index of the state corresponds to the energy of the square interaction. If instead the head encounters the left endmarker before collecting all spins it transitions to SubtractEnergy\(_{\perp} \), which accounts for \( i \) spins from the current row, as well as \( j \) complete rows missing.

### 3. Collecting interacting spins

This step determines the energy associated to the interaction of a \( k \)-by-\( k \) square subconfiguration of the input and is alternated with the energy comparison step until all interactions have been considered. It is implemented via the states

\[
Q_{\text{collect}} = \{ \text{RowEnd}, \text{Return} \} \cup \{ \text{MarkSpin}_i \mid 0 \leq i < k \}
\]

\[
\cup \{ \text{NextRow}_i \mid 0 \leq i < k \}
\]

\[
\cup \{ \text{CollectSpins}_{a,i} \mid 0 \leq i \leq k \text{ and } 0 \leq j < k \}
\]

and \( \alpha \in \{ \varepsilon \} \cup \{ \Sigma \}^{k^2-i-j} \).

where the indices \( i, j, \alpha \) are used to index a multiplicity of states. The transitions of this step are given in Table 5.

Continuing from the previous step the head of the automaton starts to the right of a spin marked with a prime, this spin marks the bottom right corner of the square interaction which is currently being determined. We refer to this spin in the following as the interaction spin, and similarly to the row it is part of as the interaction row. All other spins are unmarked. Beginning in the state RowEnd the head moves to the right until it reaches the end of the interaction row, signalled by either of the delimiters \( \perp \) or \( \perp \), where it switches to the state MarkSpin\(_{k-1}\).

Next the automaton tries to identify which spins fall into the interacting square subconfiguration. This is accomplished by marking the rightmost unmarked spin of \( k \) rows with a double prime, beginning with the interaction row and continuing left. This is repeated until the interaction spin in the interaction row is encountered. Once the interaction spin is reached, the spins to the right of the subconfiguration in each row will be marked with a double prime making it possible to find the subconfiguration spins by skipping over the marked ones.

This behaviour is implemented via the MarkSpin, NextRow, and Return states. MarkSpin moves the head to the left, skipping over double primed spins until encountering an unmarked one. This spin is then marked with a double prime and the state changes to NextRow, which continues to the left until encountering a delimiter signalling the start of another row, at which point the state transitions to MarkSpin\(_{k-1}\).

The index \( i \) keeps track of how many more rows need to be marked. Once MarkSpin\(_0\) marks a spin the automaton transitions into Return and moves back to the start of the current row instead of continuing. Similarly, if NextRow encounters the left endmarker \( \perp \) it transitions to Return, because the subconfiguration extends beyond the input configuration.

### 4. Comparing energy

This step determines if the energy of the spin configuration and the energy given as part of the input match. It is alternated with the collecting spins step until all interactions have been considered, at which point the tape is checked for any remaining energy and the automaton either accepts or rejects the input. It is implemented via the states

\[
Q_{\text{compare}} = \{ \text{NextInteraction}, \text{CheckEnergy}, \text{CleanUp}, \text{ClearRow} \}
\]

\[
\cup \{ \text{SubtractEnergy}_i \mid 0 \leq i \leq \max h'(\beta), \beta \in \Sigma_\alpha \}
\]

and the transitions given in Table 6.

Continuing from the previous step the automaton is in the state SubtractEnergy\(_i\), where \( i \) corresponds to the energy determined from the previously collected spins. This energy is next subtracted from the energy written on the tape as part of the input. To achieve this the head moves to the right, skipping...
TABLE 5. Transitions of an LBA for gathering all spins involved in an interaction of a 2D spin Hamiltonian. The states with indices in this table correspond to multiple different states as indexed by $0 \leq i, j < k$, or by a string $a \in \{\text{RowEnd}\} \cup \{\text{MarkSpin}\} \cup \{\text{NextRow}\} \cup \{\text{CollectSpins}\} \cup \{\text{CleanUp}\} \cup \{\text{NextInteraction}\}$. By making use of the prime spin alphabets $\Sigma_{\hat{a}}$, the automaton navigates the lattice to find all spins in a k-by-k square and stores them in its internal state as a. Open boundary conditions are handled by prepending $a$ with the symbol $x$, indicating missing spins, when the edges of the lattice are encountered. Once all interacting spins have been collected, the LBA transitions into a state $\text{SubtractEnergy}_i$, where $i$ is the energy of this particular interaction, and continues with the transitions in Table 6.

Over spins and delimiters. Once the head reaches the energy section of the tape, the automaton begins overwriting energy symbols $\diamond$ by the blank symbol $\perp$ one by one, decrementing $i$ each time. If the end of input symbol $\perp$ is read then there are no more $\diamond$ on the tape to overwrite, meaning the energy written on the tape does not match the energy of the spin configuration, causing the automaton to reject the input.

As soon as $\text{SubtractEnergy}_0$ is reached, exactly as many energy symbols as determined in the spin collection step have been overwritten. The head now moves back to the left, skipping over everything until the left endmarker is read. The automaton transitions to $\text{ClearUp}$, which moves the head to the right until the interaction spin is encountered. The double primes on the marked spins from the previous step are removed. Once the interaction spin is read, it is replaced with a regular unprimed spin and the head transitions to $\text{NextInteraction}$ to determine the next interaction spin. It moves to the right until encountering a spin $x$ which is marked with a prime as the new interaction spin, and the automaton transitions to the state $\text{RowEnd}$, restarting the spin collection step. If it encounters a double primed spin it instead transitions to $\text{ClearRow}$. $\text{ClearRow}$ removes the remainder of the double primes in the row and then transitions to $\text{MarkSpin}_{k-1}$ to restart the spin collection. If the end of the spin configuration is encountered by $\text{NextInteraction}$, all spins have already been considered as interaction spins and the automaton transitions to $\text{CheckEnergy}$.

$\text{CheckEnergy}$ tests if there are any energy symbols $\diamond$ remaining on the tape. The head is moved to the right skipping over everything until $\diamond$ or $\perp$ is read. If $\diamond$ is read, there are still energy symbols remaining on the tape, meaning that the energy given as part of the input does not match the energy of the given spin configuration and the automaton rejects the input. If instead $\perp$ is read, there are no $\diamond$ left on the tape, meaning the input energy and the spin configuration energy match exactly and the automaton accepts. This concludes our construction of a linear bounded automaton which recognizes $L_{2D}$, proving $L_{2D} \in L_1$.

APPENDIX C: An LBA that recognizes the language of unbounded 1D spin Hamiltonians

Here we present an LBA that recognizes the language of unbounded 1D spin Hamiltonians $\Sigma_{\text{unbound}}^{1D}$, thereby showing that $L_{\text{unbound}}^{1D} \in L_1$ and completing the proof of Proposition 30. This LBA will be based on the one of Appendix B. Its specifications are given by

- States $Q = Q_{\text{validate}} \cup Q_{\text{evaluate}} \cup \{\text{Accept}, \text{Reject}\}$
- Input alphabet $\Sigma = \Sigma_0 \cup \{\text{Left, Right}\}$
- Tape alphabet $\Gamma = \Gamma' \cup \{\underline{L}, \underline{R}\}$ where $\Gamma' = \Sigma \cup (\underline{\text{RowEnd}}, \Sigma_0^\emptyset) \cup \{|\}$
- Blank Symbol $\perp \in \Gamma$
- Left and right endmarker $\underline{L}, \underline{R} \in \Gamma$

TABLE 6. Transitions of an LBA for comparing the energy of a spin configuration as calculated by the automaton with the energy given in the input. The states $\text{SubtractEnergy}_i$ overwrite $i$ energy symbols from the tape, where the $i$ corresponds to the energy of a particular interaction. If there are no more symbols left to overwrite, or if there are still some left on the tape after all interactions have been considered, the input is rejected. Conversely, if the amount of symbols overwritten after all interactions is exactly the amount of symbols initially in the input, the input is accepted.
• Transition function $\delta$
• Start state $\text{Start} \in Q_{\text{validate}}$
• Accept state $\text{Accept} \in Q$
• Reject state $\text{Reject} \in Q$

Note that here we let the tape alphabet contain $r$ distinguishable copies $\Sigma_q^{(i)}$ of the spin alphabet $\Sigma_q$. As in the 2D construction, they are used to earmark certain positions in the spin configuration.

Similar to the construction in Appendix B, we split the states and transitions of the automaton across two steps. The first step serves to validate the input, ensuring that there are no unexpected symbols. This is functionally identical to the validating step presented in Appendix A.2, so we refer there for a concrete implementation.

The second step considers all groups of interacting spins and compares their energy with what is given in the input. It is implemented by the states

$$Q_{\text{evaluate}} = \{\text{PrepareCollection, CheckEnergy}\}$$

\[ \cup \{\text{SelectSpin}_i | 0 \leq i \leq r\} \]

\[ \cup \{\text{ClearSpin}_i | 0 < i \leq r\} \]

\[ \cup \{\text{CollectSpins}_s | 0 \leq i \leq r \text{ and } s \in \Sigma_q^{(i)}\} \]

\[ \cup \{\text{SubtractEnergy}_s | 0 \leq i \leq \max h(\beta)\} \]

and the transitions given in Table 7. It is initiated after the string has been validated by the first step, and begins with the head in state $\text{SelectSpin}$, positioned at the leftmost spin.

In the evaluating step all combinations of $r$ interacting spins are considered sequentially. The $\text{PrepareCollection}$, $\text{SelectSpin}_i$, and $\text{ClearSpin}_i$ states serve to select the next set of interacting spins. This is accomplished by overwriting them on the tape with the corresponding symbol from the duplicated spin alphabets $\Sigma_q^{(i)}$. Once $r$ spins have been marked this way, the head reads the spin configuration in the $\text{CollectSpins}_s$ states, where $s$ is used to store the interacting spins as part of the internal state of the head.

Once all $r$ spins are collected, the automaton transitions to the $\text{SubtractEnergy}_s$ states, specifically $\text{SubtractEnergy}_{h(\beta)}$. In these states the head moves to the energy written on the tape and overwrites a total of $h(\beta)$ energy symbols from the tape with $\perp$. If there are no $\perp$ left to overwrite at any point the automaton rejects, since the energy of the spin configuration does not match the energy on the tape. Otherwise the procedure is repeated for the next interaction of $r$ spins.

After considering all interactions the head transitions into the state $\text{CheckEnergy}$. The tape is scanned for any leftover $\perp$, and if there are none, the automaton accepts. Otherwise it rejects the input.

| $q \in Q$ | $a \in \Gamma$ | $\delta(q, a)$ |
|-----------|--------------|--------------|
| $\text{SelectSpin}_i, i > 0$ | $s \in \Sigma_q$ | $(\text{SelectSpin}_{i-1}, *, R)$ |
| $\text{SelectSpin}_i, r > i > 0$ | $\perp$ | $(\text{ClearSpin}_{i-1}, *, L)$ |
| $\text{SelectSpin}_i$ | $* \in \Gamma$ | $(\text{PrepareCollection}, *, L)$ |
| $\text{ClearSpin}_i$ | $\perp$ | $(\text{CheckEnergy}, *, R)$ |
| $\text{CollectSpins}_s, i > 0$ | $s \in \Sigma_q$ | $(\text{CollectSpins}_{s-1}, *, R)$ |
| $\text{CollectSpins}_s, i > 0$ | $s(0) \in \Sigma_q^{(i)}$ | $(\text{CollectSpins}_{s-1}, s(0), R)$ |
| $\text{CollectSpins}_s$ | $* \in \Gamma$ | $(\text{SubtractEnergy}_{h(\beta)}, *, R)$ |
| $\text{SubtractEnergy}_s, i > 0$ | $* \in \Gamma \cup \{\perp, \perp\}$ | $(\text{SubtractEnergy}_s, *, R)$ |
| $\text{SubtractEnergy}_s, i > 0$ | $\perp$ | $(\text{SubtractEnergy}_{s-1}, \perp, R)$ |
| $\text{SubtractEnergy}_s, i > 0$ | $\perp$ | $(\text{Reject}, \perp, L)$ |
| $\text{CheckEnergy}$ | $\perp$ | $(\text{CheckEnergy}, \perp, L)$ |
| $\text{CheckEnergy}$ | $\perp$ | $(\text{Reject}, *, R)$ |

**Table 7. Transitions of an LBA for evaluating if the energy of a given spin configuration of an unbounded 1D Hamiltonian matches the energy written on the tape. Starting at the leftmost spin in the state SelectSpin, each group of $r$ interacting spins is considered separately. For each of these interactions the corresponding amount of energy is overwritten from the tape. If the sum of the energies from all interactions matches the energy on the tape, the automaton accepts, otherwise it rejects the input.**