THE DECOMPOSITION OF CERTAIN ABSTRACT-INDUCED MODULES OVER REDUCTIVE ALGEBRAIC GROUPS

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Abstract. Let $G$ be a connected reductive algebraic group over an algebraically closed field $F$, and $B$ be a Borel subgroup of $G$. Let $k$ be another field. We determine the composition factors of the abstract induced module $M(\theta) = kG \otimes_{kB} \theta$ (here $kH$ is the group algebra of $H$ over the field $k$ and $\theta$ is a character of $B$ over $k$) when $\text{char } F = p$ is positive and $\text{char } k \neq p$ for a general character $\theta$. In particular, when $\theta$ is trivial, we give the composition factors of $M(\text{tr})$ for any $F$ and $k$.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $F$. Let $k$ be another field. In this paper we study the abstract representations of $G$ over $k$. Firstly we assume that $F = \overline{F}_q$. According to a theorem of Borel and Tits (see [2]), we know that except the trivial representation, all other irreducible representations of $kG$ are infinite-dimensional if $G$ is a semisimple algebraic group over $\overline{F}_q$ when $\text{char } k \neq \text{char } F_q$. Let $G_{q^a}$ be the set of $F_{q^a}$-points of $G$, then $G = \bigcup G_{q^a}$. Thus the abstract representations of $G$ are closely related to the representations of finite reductive groups. Around 2013, Nanhua Xi provided a new and fundamental method to study the abstract representations of $G$ over $k$ by taking the direct limit of the finite-dimensional representations of $G_{q^a}$ and got many interesting results (see [12]). In particular, he showed that the infinite Steinberg module is irreducible when $\text{char } k = 0$ or $\text{char } F$. Later, Ruotao Yang removed this restriction on char $k$ and proved the irreducibility of Steinberg module for any field $k$. Recently, A.Putman and A.Snowden showed that when $F$ is an infinite field, then the Steinberg representation of $G$ is always irreducible (see [10]). Contrast to the proofs of Xi and Yang which make essential use of the fact that $F_q$ is a union of finite fields, the arguments of [10] can deal with any infinite field $F$.

As Joseph Bernstein [11] notes, the way to make an advance in representation theory is to find a way to construct representations and practically our only tool is the induction functor. So at the beginning of the study of abstract representations
of algebraic groups, we need to construct a lot of representations. Now let $T$ be a maximal torus and $B$ be a Borel subgroup containing $T$. Let $\theta$ be a character of $T$, which can be regarded as a character of $B$ by letting $U$ acts trivially on $\theta$. Motivated by the famous Verma module in the representations of complex Lie algebras, we study the abstract-induced module $M(\theta) = kG \otimes_{kB} \theta$ in this paper. Inspired by Xi’s results in [12], we construct the $kG$-modules $E(\theta)_J$ (see Section 2 for its definition), which are the subquotient modules of $M(\theta)$. The following is the main theorem (see Theorem 6.1) of this paper.

**Theorem 1.1.** Let $F$ be a field of characteristic $p > 0$ and $k$ be another field with $\text{char } k \neq p$. Then for each character $\theta$ of $T$, all $kG$-modules $E(\theta)_J$ are irreducible and pairwise non-isomorphic.

By this theorem, we determine all the composition factors of the induced module $M(\theta)$. In particular, we obtain a large class of abstract infinite-dimensional irreducible $kG$-modules. Hence we can also define the principal representation category $\mathcal{O}(G)$ as in [3]. Moreover, the Alvis-Curtis duality of infinite type (see [9]) can be generalized on the Grothendieck group $K_0(\mathcal{O}(G))$ of $\mathcal{O}(G)$.

It is well known that the flag variety $G/B$ plays a very important role in the representation theory. Now we just regard $G/B$ as a quotient set and consider the vector space $k[G/B]$, which has a basis of the left cosets of $B$ in $G$. Thus $k[G/B]$ is a $kG$-module, which is isomorphic to $M(\text{tr})$. We call $k[G/B]$ the permutation module on the flag variety $G/B$ and we will decompose this module for any fields $F$ and $k$ in the last section. In particular, the Steinberg module is the socle of $k[G/B]$. As A.Putman and A.Snowden introduced in [11] Section 1.1.4, the Steinberg module has show its importance in representation theory, number theory and algebraic K-theory. Since the flag variety is a common and crucial object in representation theory and algebraic geometry, we believe that our decomposition of $k[G/B]$ will also have much more applications.

This paper is organized as follows: Section 2 contains some notations and preliminary results. In particular, we study the properties of the subquotient module $E(\theta)_J$ of $M(\theta)$. In Section 3, we list some properties of the unipotent groups $U$ and introduce and study the self-enclosed subgroup of $U$, which is useful in the later discussion. In Section 4, we study certain modules over unipotent groups which generalize the results of [10] Proposition 6.7. Section 5 gives the nonvanishing property of the augmentation map. Combining the results in Section 4 and Section 5, we prove the irreducibility of $E(\theta)_J$ in Section 6 under some restrictions on the characteristics of $F$ and $k$. Thus, we get all the composition factors of $M(\theta)$. In the last section, we will prove that all the $kG$-modules $E(\text{tr})_J$ are irreducible for any fields $F$ and $k$. 
2. Notations and Preliminary Results

Let $G$ be a connected reductive algebraic group over an algebraically closed field $F$, e.g. $G = GL_n(F)$. Let $B$ be an Borel subgroup, and $T$ be a maximal torus contained in $B$, and $U = R_u(B)$ be the unipotent radical of $B$. We identify $G$ with $G(F)$ and do likewise for the various subgroups of $G$ such as $B, T, U \cdots$. We denote by $\Phi = \Phi(G; T)$ the corresponding root system, and by $\Phi^+$ (resp. $\Phi^-$) the set of positive (resp. negative) roots determined by $B$. Let $W = N_G(T)/T$ be the corresponding Weyl group. We denote by $\Delta = \{\alpha_i \mid i \in I\}$ the set of simple roots and by $S = \{s_i := s_{\alpha_i} \mid i \in I\}$ the corresponding simple reflections in $W$. For each $\alpha \in \Phi$, let $U_\alpha$ be the root subgroup corresponding to $\alpha$ and we fix an isomorphism $\varepsilon_\alpha : \overline{F}_q \rightarrow U_\alpha$ such that $\varepsilon_\alpha(c)t^{-1} = \varepsilon_\alpha(\alpha(t)c)$ for any $t \in T$ and $c \in \overline{F}_q$. For any $w \in W$, let $U_w$ (resp. $U_w'$) be the subgroup of $U$ generated by all $U_\alpha$ with $w(\alpha) \in \Phi^-$ (resp. $w(\alpha) \in \Phi^+$).

Now let $k$ be another field and all the representations are over $k$. Let $\overline{T}$ be the set of characters of $T$. Each $\theta \in \overline{T}$ can be regarded as a character of $B$ by the homomorphism $B \rightarrow T$. Let $\mathbb{k}_\theta$ be the corresponding $B$-module. We are interested in the induced module $M(\theta) = kG \otimes_k B \mathbb{k}_\theta$. Let $1_\theta$ be a fixed nonzero element in $\mathbb{k}_\theta$. We abbreviate $\xi \xi = x \otimes 1_\theta \in M(\theta)$ for $x \in G$.

**Proposition 2.1.** For any $\theta \in \overline{T}$, we have the isomorphism $\text{End}_{kG}(M(\theta)) \simeq \mathbb{k}$ as $\mathbb{k}$-algebras. In particular, the $kG$-module $M(\theta)$ is indecomposable.

**Proof.** Let $\varphi \in \text{End}_{kG}(M(\theta))$ which is determined by $\varphi(1_\theta)$. Set $\xi = \varphi(1_\theta)$ and it is easy to see that $\xi \in M(\theta)^U$ and $t\xi = \theta(t)\xi$ for any $t \in T$. Using the Bruhat decomposition, we have

$$M(\theta) = \sum_{w \in W} \mathbb{k}U\tilde{w}1_\theta,$$

where $\tilde{w}$ is a fixed representative of $w \in W$. Now let $\xi$ be the following expression

$$\xi = \sum_{w \in W} \sum_{x \in U} a_{x,w}x\tilde{w}1_\theta, \quad a_{x,w} \in \mathbb{k}.$$

Noting that for an element $x \in U$ with $x \neq \text{id}$ (the neutral element of $U$), the $T$-orbit of $x$ has infinitely many elements. Thus $\xi$ has to be the form $\xi = \sum_{w \in W} a_w \tilde{w}1_\theta$.

However, we have $\xi \in M(\theta)^U$ and then $\xi = a1_\theta$ for some $a \in \mathbb{k}$ which completes the proof. \qed

For each $i \in I$, let $G_i$ be the subgroup of $G$ generated by $U_{\alpha_i}, U_{-\alpha_i}$, and set $T_i = T \cap G_i$. For $\theta \in \overline{T}$, define the subset $I(\theta)$ of $I$ by

$$I(\theta) = \{i \in I \mid \theta|_{T_i} \text{ is trivial}\}.$$

The Weyl group $W$ acts naturally on $\overline{T}$ by

$$(w \cdot \theta)(t) := \theta^{w}(t) = \theta(\tilde{w}^{-1}tw).$$
for any \( \theta \in \hat{\mathcal{T}} \).

Let \( J \subset I(\theta) \), and \( G_J \) be the subgroup of \( G \) generated by \( G_i, i \in J \). We choose a representative \( \tilde{w} \in G_J \) for each \( w \in W_J \) (the standard parabolic subgroup of \( W \)). Thus, the element \( w1_\theta := \tilde{w}1_\theta \) (\( w \in W_J \)) is well-defined. For \( J \subset I(\theta) \), we set

\[
\eta(\theta)_J = \sum_{w \in W_J} (-1)^{\ell(w)}w1_\theta,
\]
and let \( M(\theta)_J = kG\eta(\theta)_J \) the \( kG \)-module which is generated by \( \eta(\theta)_J \).

For \( w \in W \), denote by \( \mathcal{R}(w) = \{ i \in I \mid ws_i < w \} \). For any subset \( J \subset I \), we set

\[
X_J = \{ x \in W \mid x \text{ has minimal length in } xW_J \};
\]
\[
Y_J = \{ w \in X_J \mid \mathcal{R}(ww) = J \}.
\]

We have the following proposition.

**Proposition 2.2.** [\textsuperscript{[B Proposition 2.5]}] For any \( J \subset I(\theta) \), the \( kG \)-module \( M(\theta)_J \) has the form

\[
M(\theta)_J = \sum_{w \in X_J} kU\tilde{w}\eta(\theta)_J = \sum_{w \in X_J} kU_{w,w^{-1}}\tilde{w}\eta(\theta)_J.
\]

In particular, the set \( \{ uw\tilde{w}\eta(\theta)_J \mid w \in X_J, u \in U_{w,w^{-1}} \} \) forms a basis of \( M(\theta)_J \).

For the convenience of later discussion, we give some details about the expression of the element \( s_iu_i\tilde{w}\eta(\theta)_J \), where \( u_i \in U_{\alpha_i, \{ \text{id} \}} \) (the neutral element of \( U \)) and \( w \) satisfies that \( \ell(ww_J) = \ell(w) + \ell(w_J) \), where \( w_J \) is the longest element in \( W_J \). For each \( u_i \in U_{\alpha_i, \{ \text{id} \}} \), we have

\[
s_iu_i\tilde{s}_i^{-1} = f_i(u_i)s_ih_i(u_i)g_i(u_i),
\]
where \( f_i(u_i), g_i(u_i) \in U_{\alpha_i, \{ \text{id} \}} \), and \( h_i(u_i) \in T_i \) are uniquely determined. Moreover if we regard \( f_i \) as a morphism on \( U_{\alpha_i, \{ \text{id} \}} \), then \( f_i \) is a bijection. The following lemma is very useful in the later discussion. Its proof can be found in [B Proposition 2.5] and we omit it.

**Lemma 2.3.** Let \( u_i \in U_{\alpha_i, \{ \text{id} \}} \), with the notation above, we have

(i) If \( ww_J \leq s_iww_J \), then \( s_iu_i\tilde{w}\eta(\theta)_J = s_i\tilde{w}\eta(\theta)_J \).

(ii) If \( s_iw \leq w \), then \( s_iu_i\tilde{s}_i\eta(\theta)_J = \theta^{s_iw}(h_i(u_i))f_i(u_i)s_i\tilde{s}_i\eta(\theta)_J \).

(iii) If \( w \leq s_iw \) but \( s_iww_J \leq ww_J \), then \( s_iu_i\tilde{s}_i\eta(\theta)_J = \theta(t)(f_i(u_i) - 1)\tilde{w}\eta(\theta)_J \), where \( t \in T \) satisfies that \( s_i\tilde{w} = w\tilde{s}jt \) for some \( \tilde{s}_j \in G_j \) (thus \( \theta(t) \) is determined).

Now we define the most critical \( kG \)-module \( E(\theta)_J \) in this paper, which is a subquotient of \( M(\theta) \). For \( J \subset I(\theta) \), set

\[
E(\theta)_J = M(\theta)_J/M(\theta)_J',
\]
where \( M(\theta)_J' \) is the sum of all \( M(\theta)_K \) with \( J \subset K \subset I(\theta) \). We denote by \( C(\theta)_J \) the image of \( \eta(\theta)_J \) in \( E(\theta)_J \).
The argument about $E(\theta)_J$ in [6] Section 2 is also valid. We list some results here without proof. For $J \subset I(\theta)$, we set

$$Z_J = \{ w \in X_J \mid J'(ww_J) \subset J \cup (I \setminus I(\theta)) \}.$$  

The following proposition gives a basis of $E(\theta)_J$.

**Proposition 2.4.** [6] Proposition 2.7] For $J \subset I(\theta)$, we have

$$E(\theta)_J = \sum_{w \in Z_J} kU_{wJ} \bar{w}C(\theta)_J.$$  

In particular, the set $\{ \bar{w}C(\theta)_J \mid w \in Z_J, u \in U_{wJ} \}$ forms a basis of $E(\theta)_J$.

**Proposition 2.5.** [6] Proposition 2.8] Let $\theta_1, \theta_2 \in \hat{T}$ and $K_1 \subset I(\theta_1), K_2 \subset I(\theta_2)$. Then $E(\theta_1)_{K_1}$ is isomorphic to $E(\theta_2)_{K_2}$ as $kG$-modules if and only if $\theta_1 = \theta_2$ and $K_1 = K_2$.

3. **Self-enclosed Subgroup**

This section contains some preliminaries and properties of unipotent groups, which is useful in later discussion (especially in Section 7). As before, let $U$ be the unipotent radical of a Borel subgroup $B$. For any $w \in W$, we set

$$\Phi_w^- = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \}, \quad \Phi_w^+ = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+ \}.$$  

Now assume $\Phi_w^- = \{ \beta_1, \beta_2, \ldots, \beta_k \}$ and $\Phi_w^+ = \{ \gamma_1, \gamma_2, \ldots, \gamma_l \}$ for a given $w \in W$ and, we denote

$$U_w = U_{\beta_1} U_{\beta_2} \cdots U_{\beta_k} \quad \text{and} \quad U'_w = U_{\gamma_1} U_{\gamma_2} \cdots U_{\gamma_l}.$$  

The following properties are well known (see [3]).

(a) For $w \in W$ and $\alpha \in \Phi$ we have $\bar{w}U_\alpha \bar{w}^{-1} = U_{w(\alpha)}$;

(b) $U_w$ and $U'_w$ are subgroups and $\bar{w}U'_w \bar{w}^{-1} \subset U$;

(c) The multiplication map $U_w \times U'_w \to U$ is a bijection;

(d) Each $u \in U_w$ is uniquely expressible in the form $u = u_{\beta_1} u_{\beta_2} \cdots u_{\beta_k}$ with $u_{\beta_i} \in U_{\beta_i}$;

(e) (Commutator relations) Given two positive roots $\alpha$ and $\beta$, there exist a total ordering on $\Phi^+$ and integers $c_{\alpha\beta}^{mn}$ such that

$$[\varepsilon_{\alpha}(a), \varepsilon_{\beta}(b)] := \varepsilon_{\alpha}(a)\varepsilon_{\beta}(b)\varepsilon_{\alpha}(a)^{-1}\varepsilon_{\beta}(b)^{-1} = \prod_{m, n > 0} \varepsilon_{m \alpha + n \beta}(c_{\alpha\beta}^{mn} a^m b^n)$$

for all $a, b \in \mathbb{F}_q$, where the product is over all integers $m, n > 0$ such that $m \alpha + n \beta \in \Phi^+$, taken according to the chosen ordering.

Now we denote by $\Phi^+ = \{ \delta_1, \delta_2, \ldots, \delta_m \}$. For an element $u \in U$, we have $u = x_1 x_2 \cdots x_m$ with $x_i \in U_{\delta_i}$. If we choose another order of $\Phi^+$ and write $\Phi^+ = \{ \delta'_1, \delta'_2, \ldots, \delta'_m \}$, we get another expression of $u$ such that $u = y_1 y_2 \cdots y_m$ with $y_i \in U_{\delta'_i}$. If $\delta_i = \delta'_i = \alpha$ is a simple root, by the commutator relations of
root subgroups, we get $x_i = y_j$ which is called the $U_\alpha$-component of $u$. Noting
that the simple roots are $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and each $\gamma \in \Phi^+$ can be written
as $\gamma = \sum_{i=1}^{n} k_i \alpha_i$, we denote by $ht(\gamma) = \sum_{i=1}^{n} k_i$ the hight of $\gamma$. It is easy to see
that $\prod_{ht(\gamma) \geq s} U_\gamma$ is a subgroup of $U$ for any fixed integer $s \in \mathbb{N}$ by the commutator
relations of root subgroups.

Given an order $"\prec"$ on $\Phi^+$ and we list all the positive root as $\delta_1, \delta_2, \ldots, \delta_m$ with
respect to this order. For any $u \in U$, we have a unique expression in the form
$u = u_1 u_2 \ldots u_m$ with $u_i \in U_{\delta_i}$. Let $X$ be a subset of $U$, we denote by

$$X \cap \prec U_{\delta_k} = \{u_k \in U_{\delta_k} \mid \text{there exits } u \in X \text{ such that } u = u_1 u_2 \ldots u_k \ldots u_m\}.$$ 

It is easy to see that $X \cap U_{\delta_k} \subseteq X \cap \prec U_{\delta_k}$. Now let $H$ be a subgroup of $U$ and we
say that a subgroup $H$ is self-enclosed with respect the order $"\prec"$ if

$$H \cap \prec U_{\delta_k} = H \cap U_{\delta_k} \text{ for any } k = 1, 2, \ldots, m.$$ 

If $H$ is self-enclosed with respect to any order on $\Phi^+$, we say that $H$ is a self-enclosed
subgroup of $U$.

Let $H$ be a self-enclosed subgroup of $U$. For each $\gamma \in \Phi^+$, we set $H_\gamma = H \cap U_\gamma$. Then we have

$$H = H_{\delta_1} \cdot H_{\delta_2} \ldots \cdot H_{\delta_m}.$$ 

For $w \in W$, set $H_w = H \cap U_w$. Then it is easy to see that $H_w$ is also a self-enclosed
subgroup and we have $H_w = \prod_{\gamma \in \Phi^+} H_\gamma$.

**Example 3.1.** Suppose $\mathbb{F} = \overline{\mathbb{F}}_q$ and $\{\delta_1, \delta_2, \ldots, \delta_m\}$ are all the positive roots such
that $ht(\delta_1) \leq ht(\delta_2) \leq \ldots \leq ht(\delta_m)$. Assume that $U$ is defined over $\overline{\mathbb{F}}_q$ and let $U_{q^a}$
be the set of $\overline{\mathbb{F}}_{q^a}$-points of $U$. Given $a_1, a_2, \ldots, a_m \in \mathbb{N}$ such that $a_i$ is divisible by
$a_j$ for any $i < j$, set

$$H = U_{\delta_1, q^{a_1}} \cdot U_{\delta_2, q^{a_2}} \ldots \cdot U_{\delta_m, q^{a_m}}.$$ 

Then it is not difficult to check that $H$ is a self-enclosed subgroup of $U$.

Now let $H$ be a subgroup of $U$. Let $V = U_{\beta_1} \cdot U_{\beta_2} \ldots U_{\beta_k}$, which is also a
subgroup of $U$. We denote by

$$U = \bigcup_{x \in L} xV, \quad U = \bigcup_{y \in R} V y$$

where $L$ (resp. $R$) is a set of the left (right) coset representatives of $V$ in $U$. Then
we define the following two sets:

$$H_V = \{v \in V \mid \text{there exists } u \in H \text{ such that } u = x v \text{ for some } x \in L\},$$

$$vH = \{v \in V \mid \text{there exists } u \in H \text{ such that } u = v y \text{ for some } y \in R\}.$$ 

Using the commutator relations, it is not difficult to have the following proposition.
Proposition 3.2. With the notation above, let $H$ be a self-enclosed subgroup of $U$, then we have

$$H \cap V = H V = V H = H \cap V$$

for any subgroup $V$ of $U$ which has the form $U_{\beta_1} U_{\beta_2} \ldots U_{\beta_k}$ for some positive roots $\beta_1, \beta_2, \ldots, \beta_k$.

Now we consider the special case that $F$ is a field of positive characteristic $p$. In this case, it is well known that all the finitely generated subgroups of $U$ are finite $p$-groups. We have the following lemma.

Lemma 3.3. Let $X$ be a finite set of $U$, then there exists a finite $p$-group $H$ of $U$ such that $H \supseteq X$ and $H$ is self-enclosed.

Proof. Denote by $\Phi^+ = \{ \delta_1, \delta_2, \ldots, \delta_m \}$ such that $\text{ht}(\delta_1) \leq \text{ht}(\delta_2) \leq \cdots \leq \text{ht}(\delta_m)$. For each $1 \leq k \leq m$, we denote by $X_k = X \cap U_{\delta_k}$. Let $H_1$ be the subgroup of $U_{\delta_1}$ which is generated by $X_1$. Now we define the subgroup $H_k$ by recursive step. Suppose that $H_1, H_2, \ldots, H_{k-1}$ are defined, we set

$$Y_k = \langle H_1, H_2, \ldots, H_{k-1} \rangle \cap U_{\delta_k}$$

and let $H_k$ be the subgroup of $U_{\delta_k}$ which is generated by $X_k$ and $Y_k$. Now we have a series of subgroups $H_1, H_2, \ldots, H_m$ and then we set

$$H = \langle H_1, H_2, \ldots, H_m \rangle$$

which is a finitely generated subgroup of $U$. Thus $H$ is a finite $p$-subgroup of $U$, which contains $X$ by its construction. Moreover, it is not difficult to check that $H$ is a self-enclosed of $U$ using the commutator relations of root subgroups.

\[\square\]

4. Certain Modules over Unipotent Groups

In this section, we study certain modules over unipotent groups. The setting and arguments follow [10, Section 6]. In this section, let $F$ be a field of characteristic $p > 0$. Let $k$ be another field with $\text{char } k \neq p$. We let $U$ be a smooth connected unipotent group over $F$ equipped with an action of $G_m$. For $t \in G_m$ and $g \in U$, denote the action of $t$ on $g$ by $t^g$. An action of $G_m$ on $U$ is defined to be positive if the weights of the action of $G_m$ on $\text{Lie}(U)$ are positive. Let $X(G_m)$ be the set of characters of $G_m$ over $k$. The set of all maps $G_m \to k$ can be made into a vector space $V$ over $k$. It is well known that $X(G_m)$ is a linearly independent subset of $V$. The main result of this section is the following.

Proposition 4.1. Let $U$ be a smooth connected unipotent group over an infinite field $F$ of positive characteristic $p$ equipped with a positive $G_m$-action. Let $k$ be
another field with char \( k \neq p \). Let \( \theta_1, \theta_2, \ldots, \theta_n \in X(\mathbb{G}_m) \) and \( M \) be a \( k[U \times \mathbb{G}_m] \)-module such that

\[
M = k[U]m_1 + k[U]m_2 + \cdots + k[U]m_n,
\]

where \( tm_i = \theta_i(t)m_i \) for any \( t \in \mathbb{G}_m \). Let \( N \subset M \) be a submodule such that

\[
\theta_1(t) \sum_{j_1} a_{j_1} g_{j_1} \cdot m_1 + \theta_2(t) \sum_{j_2} a_{j_2} g_{j_2} \cdot m_2 + \cdots + \theta_n(t) \sum_{j_n} a_{j_n} g_{j_n} \cdot m_n \in N
\]

for any \( t \in \mathbb{G}_m \). Then we have

\[
\sum_{j_1} a_{j_1} m_1 + \sum_{j_2} a_{j_2} m_2 + \cdots + \sum_{j_n} a_{j_n} m_n \in N.
\]

Proposition 4.1 is inspired by [10, Proposition 6.7]. The proof is also similar and we introduce some notations and give some preliminary results before we give the proof. For a subset \( S \) of \( U \) and an additive subgroup \( a \) of \( k \), define \( U(S, a) \) to be the subgroup generated by \( \{s^a | s \in S, a \in a \} \). For a group \( H \) and a \( \mathbb{Z}[H] \)-module \( M \), let \( M_H \) denote the \( H \)-coinvariants of \( M \), i.e., the largest quotient of \( M \) on which \( H \)-acts trivially. By the same arguments of [10, Proposition 6.6], we have the following key lemma.

**Lemma 4.2.** Let \( U \) be a smooth connected unipotent group over an infinite field \( F \) of positive characteristic \( p \) equipped with a positive \( \mathbb{G}_m \)-action. Let \( k \) be another field with char \( k \neq p \). Let \( S \) be a finite set of \( U \) and \( a \) be an infinite additive subgroup of \( F \). Let \( M \) be a \( k[U(S, a)] \)-module and \( m \in M \) be nonzero. Then there exists an infinite additive subgroup \( e \) of \( a \) such that the image of \( m \) in \( M_{U(S, e)} \) is nonzero.

**Proof of Proposition 4.1.** Replacing \( M \) by \( M/N \), we can assume that \( N = 0 \). Let \( S = \bigcup_{k=1}^n g_{j_k} \). Let \( b \) be a nonzero additive subgroup of \( F \). We denote by \( \equiv \) the equality in the \( U(S, b) \)-coinvariants of \( M \). Let \( t \in b \) be a nonzero element, since the elements \( t g_{j_k} \in U(S, b) \) act trivially on these coinvariants, we have \( t g_{j_k} \cdot m_k \equiv m_k \) for all \( k \) and \( j_k \). Thus we see that

\[
\theta_1(t) \sum_{j_1} a_{j_1} m_1 + \theta_2(t) \sum_{j_2} a_{j_2} m_2 + \cdots + \theta_n(t) \sum_{j_n} a_{j_n} m_n
\]

maps to 0 in \( M_{U(S, b)} \). Without lost of generality, we can assume that \( \theta_1, \theta_2, \ldots, \theta_n \) are different from each other. Noting that \( M_{U(S, b)} \) can also be a \( k[U \times \mathbb{G}_m] \)-module. Using the linear independence of \( \theta_1, \theta_2, \ldots, \theta_n \), it is not difficult to see that each \( \sum_{j_k} a_{j_k} m_k \) maps to 0 in \( M_{U(S, b)} \). In particular, we see that

\[
\sum_{j_1} a_{j_1} m_1 + \sum_{j_2} a_{j_2} m_2 + \cdots + \sum_{j_n} a_{j_n} m_n
\]
maps to 0 in $M_{U(S,b)}$ for all nonzero $b$. Then we use Lemma 4.2 (applied with $a = F$) and get that
\[
\sum_{j_1} a_{j_1} m_1 + \sum_{j_2} a_{j_2} m_2 + \cdots + \sum_{j_n} a_{j_n} m_n = 0.
\]
The proposition is proved.

\[\square\]

**Remark 4.3.** Proposition 4.1 is no longer true when $F = k$. For example, we consider $G = SL_2(F)$. Let $T$ be the diagonal matrices and $B$ be the upper triangular matrices in $SL_2(F)$. For each $c \in F^*$, we set $h(c) = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \in T$.

Let $V = F^2$ be the natural representation of $SL_2(F)$ and denote by $v_1 = (1, 0)^t$, $v_2 = (0, 1)^t$ the natural basis of $F^2$. Define $\theta : T \to F^*$ by $\theta(h(c)) = c$. Let $\varphi$ be the homomorphism such that $\varphi(h(c)) = v_1$. Thus $V$ is a quotient module of $M(\theta)$, which is isomorphic to $M(\theta)/\ker \varphi$. It is easy to check that for any $x \in F$, we have
\[
1_\theta + \begin{pmatrix} 1 & x + 1 \\ 0 & 1 \end{pmatrix} s_1 - \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} s_1 \in \ker \varphi.
\]
However $1_\theta \not\in \ker \varphi$ and thus Proposition 4.1 is not valid generally when $F = k$.

### 5. Nonvanishing Property of the Augmentation Map

Fix a character $\theta \in \hat{T}$ and a subset $J \subset I(\theta)$. By Proposition 2.4, we have
\[
E(\theta)_J = \bigoplus_{w \in Z_J} k U_{w,jw^{-1}} wC(\theta)_J 
\]
as $k$-vector space. For each $w \in Z_J$, we denote by
\[
\mathcal{P}_w : E(\theta)_J \to kW_{w,jw^{-1}} wC(\theta)_J 
\]
the projection of vector spaces and by
\[
\epsilon_w : kW_{w,jw^{-1}} wC(\theta)_J \to k
\]
the augmentation map (restricted on $w$) which takes the sum of the coefficients with respect to the natural basis, i.e., for $\xi = \sum_{x \in U_{w,jw^{-1}}} a_x x wC(\theta)_J$, we set $\epsilon_w(\xi) = \sum_{x \in U_{w,jw^{-1}}} a_x$. Now we denote by
\[
\epsilon = \bigoplus_{w \in Z_J} \epsilon_w \mathcal{P}_w : E(\theta)_J \to k|Z_J|
\]
the augmentation map of $E(\theta)_J$. 
When considering the irreducibility of Sternberg module, the nonvanishing property of the augmentation map is very crucial (see [13, Lemma 2.5] and [10, Proposition 1.6]). In this section, we show that the nonvanishing property also holds for the general augmentation map $\epsilon$ defined above. Firstly we have a lemma as following.

**Lemma 5.1.** Let $\xi \in E(\theta)_J$ be a nonzero element. Then there exists $g \in G$ such that $\varPsi_g(g\xi)$ is nonzero.

**Proof.** By Proposition 2.4, we let $\xi \in E(\theta)_J$ with the following expression

$$\xi = \sum_{w \in Z_J} \sum_{x \in U_{wJw^{-1}}} a_{w,x} x \hat{w} C(\theta)_J.$$  

Then there exists one $h \in W$ with minimal length such that $a_{h,x} \neq 0$ for some $x \in U_{wJw^{-1}}$, which implies that $\varPsi_h(\xi)$ is nonzero. When $h = e$, the lemma is proved. Now suppose that $\ell(h) \geq 1$, so there is a simple reflection $s$ such that $\sigma = sh < h$. Without lost of generality, we can assume that $a_{h,id} \neq 0$. We claim that either $\varPsi_\sigma(\hat{s} \xi)$ is nonzero or $\varPsi_\sigma(s \hat{y} \xi)$ is nonzero for some $y \in U_s$.

If $\varPsi_\sigma(\hat{s} \xi) = 0$, then there exists an element $v \in Z_J$ satisfies the condition

$$(\clubsuit) \quad sv \notin Z_J \quad \text{and} \quad \varPsi_\sigma(\hat{s} \hat{v} C(\theta)_J) \neq 0.$$  

The subset of $Z_J$ whose elements satisfy this condition is also denoted by $\clubsuit$. Thus $\varPsi_\sigma(\hat{s} \xi) = 0$ tells us that

$$\varPsi_\sigma(\hat{s} \cdot \varPsi_h(\xi)) + \varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(\xi)) = 0.$$  

In particular, we get $\varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(\xi)) \neq 0$. Since $U$ is infinite, we can choose an element $y \in U_s$ such that all the $U_s$-component of $yx$ with $a_{h,x} \neq 0$ is nontrivial. Then $\varPsi_\sigma(\hat{s} \cdot \varPsi_h(y \xi)) = 0$ by Lemma 2.3 (ii). On the other hand, for $v \in \clubsuit$ and $a_{v,x} \neq 0$, we have

$$\hat{s} y x \hat{v} C(\theta)_J = \hat{s} y x y^{-1} \hat{s}^{-1} \hat{s} \hat{v} C(\theta)_J,$$  

which implies that $\varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(y \xi)) \neq 0$. Indeed if we write

$$\varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(\xi)) = \sum_{x \in U_{wJw^{-1}}} b_{\sigma,x} x \hat{\sigma} C(\theta)_J \neq 0,$$  

it is not difficult to see that

$$\varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(\eta \xi)) = \sum_{x \in U_{wJw^{-1}}} b_{\sigma,x} (\hat{s} y x y^{-1} \hat{s}^{-1} \hat{s} \hat{v} C(\theta)_J$$  

which is also nonzero. Therefore

$$\varPsi_\sigma(\hat{s} y \xi) = \varPsi_\sigma(\hat{s} \cdot \sum_{v \in \clubsuit} \varPsi_v(\eta \xi)) \neq 0.$$
By the argument above, we can do induction on the length of $h$ and thus the lemma is proved. \hfill \box

Now we give the nonvanishing property of the augmentation map of $E(\theta)_J$, which is very useful in the later discussion.

**Proposition 5.2.** Let $\xi \in E(\theta)_J$ be a nonzero element. Then there exists $g \in G$ such that $\epsilon(g\xi)$ is nonzero.

**Proof.** By Lemma 5.1 we can assume that $\mathcal{P}_e(\xi)$ is nonzero. For

$$\xi = \sum_{w \in Z_J} \sum_{x \in U_{w,J}} a_{w,x} x v C(\theta)_J \in E(\theta)_J,$$

we say that $\xi$ satisfies the condition $\triangledown_h$, if $\sum_{x \in U_{h}^e} a_{e,x} \neq 0$ for some $h \in W_J$. We prove the following claim: if $\xi$ satisfies the condition $\triangledown_h$ for some $h \in W_J$, then there exists $g \in G$ such that $\epsilon_e \mathcal{P}_e(g\xi)$ is nonzero.

We prove this claim by induction on the length of $h$. If $h = e$, then it is obvious that $\epsilon_e \mathcal{P}_e(\xi)$ is already nonzero. We assume that the claim is valid for any $h \in W_J$ with $\ell(h) \leq m$. Now let $h \in W_J$ with $\ell(h) = m + 1$ such that $\sum_{x \in U_{h}^e} a_{e,x} \neq 0$.

We have $h = \sigma s$ for some $s \in R(h)$. Firstly we consider the element $s \cdot \mathcal{P}_e(\xi)$. Since $U_{w,J} = U_{h}^e U_h = U_{h}^e U_{e}^s U_s$, each element $x \in U_{w,J}$ has a unique expression $x = x_h^e x_s^e x_s$. When $x_s^e x_s \neq id$ and $x_s = id$, we get $x_s^e \neq id$. For the case $x_s \neq id$, using Lemma 2.3 (iii), we have

$$s x v C(\theta)_J = x_h^e x_s^e x_s v C(\theta)_J = \lambda x_h^e x_s^e (f(x_s) - 1) C(\theta)_J$$

for some $\lambda_x \in k$, where $x_h^e = sx_h^{-1} s^{-1}$ and $f(x_s) \in U_s$. Therefore $s \cdot \mathcal{P}_e(\xi)$ satisfies the condition $\triangledown_s$. Hence we consider $s\xi$ and if $s\xi$ satisfies the condition $\triangledown_s$, then $\epsilon_e \mathcal{P}_e(s\xi)$ is nonzero and we are done. Otherwise there exists at least an element $v \in Z_J$ satisfies that

$$\mathcal{B}_v : sv \notin Z_J \text{ and } \mathcal{P}_e(s v C(\theta)_J) \neq 0.$$
Thus if we write
\[ P_e(\dot{s} \cdot \sum_{v \in \Delta} \mathcal{P}_e(\xi)) = \sum_{x \in U_w \cap U'_w} b_x x C(\theta)_J, \]
it is not difficult to see that
\[ P_e(\dot{s} \cdot \sum_{v \in \Delta} \mathcal{P}_e(y \xi)) = \sum_{x \in U_w \cap U'_w} b_x (\dot{s} y x y^{-1} \dot{s}^{-1}) C(\theta)_J. \]
Thus \( P_e(\dot{s} \cdot \sum_{v \in \Delta} \mathcal{P}_e(y \xi)) \) satisfies the condition \( \trianglelefteq_\sigma \) since \( P_e(\dot{s} \cdot \sum_{v \in \Delta} \mathcal{P}_e(\xi)) \) satisfies the condition \( \trianglelefteq_\sigma \). Thus \( \epsilon_e P_e(\dot{s} y \xi) \) is nonzero by inductive hypothesis.

In conclusion we have proved our claim by induction on the length of \( h \in W_J \).

6. Principal Representation Category

With the previous preparations in Section 3 and Section 4, the main result of this section is as following.

**Theorem 6.1.** Let \( F \) be a field of positive characteristic \( p \) and \( k \) be another field with \( \text{char } k \neq p \). For each \( \theta \in \hat{T} \), then all \( kG \)-modules \( E(\theta)_J (J \subset I(\theta)) \) are irreducible and pairwise non-isomorphic. In particular, the \( kG \)-module \( M(\theta) \) has exactly \( 2^{|I(\theta)|} \) composition factors, each occurring with multiplicity one.

**Proof.** We show that any nonzero submodule \( M \) of \( E(\theta)_J \) contains \( C(\theta)_J \), and hence \( M = E(\theta)_J \). In particular, all \( E(\theta)_J \) are irreducible for any \( J \subset I(\theta) \). Let \( \xi \in M \) be a nonzero element with the following expression
\[ \xi = \sum_{w \in Z_J} \sum_{x \in U_{wJw^{-1}}} a_{w,x} x \dot{w} C(\theta)_J \in M. \]

By Proposition 5.2 we can assume that \( \epsilon(\xi) \neq 0 \). By \cite{10} Proposition 3.9, there exists a one-parameter subgroup \( \sigma : \mathbb{G}_m \to T \) that acts positively on \( U \). Therefore using Proposition 4.1 we have
\[ \sum_{w \in Z_J} \sum_{x \in U_{wJw^{-1}}} a_{w,x} \dot{w} C(\theta)_J \in M. \]

In particular, we see that
\[ M \cap \sum_{w \in Z_J} k \dot{w} C(\theta)_J \neq 0. \]
Thus it is not difficult to see that
\[ M \cap \sum_{w \in Z_{J} \cap W_{\theta}} \mathbb{k}wC(\theta)_{J} \neq 0 \]
by some discussion about the \( T \)-eigenvectors. Noting that the argument of \cite[Lemma 3.8, 3.9]{6} is still valid in our general setting, so all \( E(\theta)_{J} \) are irreducible and pairwise non-isomorphic by Proposition \ref{2.5}. The theorem is proved. \( \square \)

**Remark 6.2.** (a) The above theorem is not true for general \( \theta \) when \( F = \mathbb{k} \). However when \( F = \mathbb{k} = \overline{\mathbb{F}}_{q} \), Theorem \ref{3.8} is also valid when \( \theta \) is antidominant (see \cite[Theorem 4.1]{6}). Moreover, the paper \cite{6} has showed that \( M(\theta) \) has such a composition series if and only if \( \theta \) is antidominant (see \cite[Theorem 5.1, 5.2]{6}.
(b) When \( \theta \) is trivial, we will prove that all the \( \mathbb{k}G \)-modules \( E(\text{tr})_{J} \) \( (J \subset I) \) are irreducible in Section 7 (see Theorem \ref{7.1}). This generalizes the results of \cite[Theorem 3.1]{4} and \cite[Theorem 4.1]{5}.
(c) I conjecture that Theorem \ref{5.1} still holds for any fields \( F, \mathbb{k} \) whenever \( \text{char} \ F \neq \text{char} \ \mathbb{k} \). So the methods deal with the case \( \text{char} \ F = 0 \) need to be developed.

From now on in this section, let \( F \) be a field of characteristic \( p > 0 \) and \( \mathbb{k} \) be another field with \( \text{char} \ \mathbb{k} \neq p \). As in \cite{8}, we introduce a category \( \mathcal{O}(G) \) called principal representation category. It is the full subcategory of \( \mathbb{k}G \)-Mod such that any object \( M \) in \( \mathcal{O}(G) \) is of finite length and its composition factors are \( E(\theta)_{J} \) for some \( \theta \in \widehat{T} \) and \( J \subset I(\theta) \). Thus \( \mathcal{O}(G) \) is an abelian category which is also noetherian and artinian. The paper \cite{8} gave some evidences to show that \( \mathcal{O}(G) \) is a highest weight category in the sense of Cline, Parshall and Scott (see \cite{7}) when \( F = \overline{\mathbb{F}}_{q} \) and \( \text{char} \ \mathbb{k} = 0 \). So is this conjecture established in the more general case?

We can also define the Alvis-Curtis duality of infinite type as in \cite{9}. Let \( K_{0}(\mathcal{O}(G)) \) be the Grothendieck group of \( \mathcal{O}(G) \). Thus we call the following functor \( \mathbb{D}_{G} : K_{0}(\mathcal{O}(G)) \rightarrow K_{0}(\mathcal{O}(G)) \) Alvis-Curtis duality of infinite type which is defined by
\[ \mathbb{D}_{G} = \sum_{J \subset I} (-1)^{|J|} \mathcal{R}^{J} \mathcal{R}_{J}, \]
where \( \mathcal{R}_{J} \) is Harish-Chandra restriction and \( \mathcal{R}^{J} \) is Harish-Chandra induction defined in \cite[Section 3]{9}. For \( J \subset I \), let
\[ \sigma(J) = \{ i \in I \mid s_{i} = w_{0}s_{j}w_{0} \text{ for some } j \in J \} \]
which is an involution on the set \( I \). Fixed \( \theta \in \widehat{T} \) and for \( J \subset I(\theta) \), we introduce an operation \( \mathbb{D}_{\theta} \) on \( J \) by
\[ \mathbb{D}_{\theta}(J) = \sigma(I(\theta)) \setminus \sigma(J) = I(\theta^{w_{0}}) \setminus \sigma(J). \]
With these notations we can also get the following theorem whose proof is the same as \cite[Theorem 4.2]{9}.
Theorem 6.3. For \( \theta \in \hat{T} \), \( J \subset I(\theta) \), we have
\[
\mathcal{D}_G([E(\theta, J)]) = [E(\theta w_0)_{\mathcal{D}_G(J)}].
\]

It is easy to check that \( \mathcal{D}_G(\mathcal{D}_G(J)) = J \) for \( J \subset I(\theta) \). Then \( \mathcal{D}_G \circ \mathcal{D}_G \) is the identity functor on \( K_0(\mathcal{O}(G)) \). This duality is a morphism on \( K_0(\mathcal{O}(G)) \). Does there exist a functor \( \mathcal{D}_G : \mathcal{O}(G) \to \mathcal{O}(G) \) such that \([\mathcal{D}_G] = \mathcal{D}_G\)?

7. Permutation Module on Flag Varieties

One important and interesting case is that \( \theta \) is trivial. In such case, we call \( k[\mathcal{G}/\mathcal{B}] = \mathcal{G} \otimes_k \mathcal{B}_{tr} \) the permutation module on the flag variety \( \mathcal{G}/\mathcal{B} \). The flag varieties are very important in the representations of reductive algebraic groups. Moreover, the decomposition \( k[\mathcal{G}/\mathcal{B}] \) may have many applications in other areas such as algebraic geometry and number theory. We simply denote \( E(\mathcal{G}) \) by \( E \) and \( C(\mathcal{G}) \) by \( C \).

The following lemma is easy to get but very useful in our discussion later.

Lemma 7.2. Let \( G \) be a finite abelian \( p \)-group with a direct product \( G = H \times K \). Let \( H' \) be another subgroup of \( G \) such that \( |H| = |H'| \). Then \( H'K = 0 \) or \( G \).

For \( H \) a self-enclosed subgroup of \( \mathcal{U} \), denote by \( H_\gamma = H \cap U_\gamma \) as before for each \( \gamma \in \Phi^+ \). Noting \( \Phi^+ = \{ \delta_1, \delta_2, \ldots, \delta_m \} \), we have
\[
H = H_{\delta_1} H_{\delta_2} \ldots H_{\delta_m}.
\]

Let \( H_w = H \cap U_w \). Then we have \( H_w = \prod_{\gamma \in \Phi^+} H_\gamma \) and \( H_w = \prod_{\gamma \in \Phi^+} H_\gamma \). The following two lemmas are very crucial in the later proof of Theorem 7.1.

Lemma 7.3. Assume that \( \text{char } k = p > 0 \) and let \( M \) be a nonzero \( kG \)-submodule of \( E_J \). Then there exists an element \( w \in Y_J \) and a finite \( p \)-subgroup \( X \) of \( U_{w,J,w-1} \) such that \( XwC_J \in M \).

Proof. Let \( \xi \) be a nonzero element of \( M \) which has the form
\[
\xi = \sum_{w \in Y_J} \sum_{x \in U_{w,J,w-1}} a_{w,x} xwC_J \in E_J.
\]
By Lemma 3.3, there exists a self-enclosed finite $p$-subgroup $V$ of $U$, which contains all $x \in U_{w_jw^{-1}}$ with $w \in Y_j$ and $a_{w,x} \neq 0$. Then we have
\[ kV \xi \subset \bigoplus_{w \in Y_j} kV_{w_jw^{-1}w}wC_j \]
as $kV$-modules. Since $(kV \xi)^V \neq 0$ by Proposition 26] and noting that
\[ ( \bigoplus_{w \in Y_j} kV_{w_jw^{-1}w}wC_j)^V \subset \bigoplus_{w \in Y_j} kV_{w_jw^{-1}w}wC_j, \]
there exists a nonzero element
\[ \eta = \sum_{w \in Y_j} a_w V_{w_jw^{-1}w}wC_j \in kV \xi \subset M. \]
Set $A(\eta) = \{ w \in Y_j \mid a_w \neq 0 \}$. If $|A(\eta)| = 1$, the lemma is proved.

Now we assume that $|A(\eta)| \geq 2$. Denote by $\Phi(\eta) = \bigcup_{w \in A(\eta)} \Phi_{w_jw^{-1}}$. We give an order on $\Phi(\eta) = \{ \gamma_1, \gamma_2, \ldots, \gamma_d \}$ such that $ht(\gamma_1) \leq ht(\gamma_2) \leq \cdots \leq ht(\gamma_d)$. We can choose a root $\gamma_s$ with $s$ maximal such that $\gamma_s \notin \bigcap_{w \in A(\eta)} \Phi_{w_jw^{-1}}$. We choose an element $y \in U_{\gamma_s \setminus V_{\gamma_s}}$ and let $H$ be a self-enclosed finite $p$-subgroup of $U_{\gamma_s \setminus U_{\gamma_{s+1}} \setminus \ldots \setminus U_{\gamma_d}}$ such that $H$ contains $V_{\gamma_s} \cap V_{\gamma_{s+1}} \cap \ldots \cap V_{\gamma_d}$ and $y$. Let $X$ be the subgroup of $U$ which is generated by $H$ and $V$. Then it is easy to check that $X$ is a self-enclosed subgroup of $U$. Denote by $\Omega_1$ a set of left coset representatives of $V_{\gamma_s} \cap V_{\gamma_{s+1}} \cap \ldots \cap V_{\gamma_d}$ in $H$. For the $w \in Y_j$ such that $\gamma_s \notin \Phi_{w_jw^{-1}}$, we have
\[ \Omega_1 V_{w_jw^{-1}}wC_j = X_{w_jw^{-1}}wC_j. \]
For the $w \in Y_j$ such that $\gamma_s \notin \Phi_{w_jw^{-1}}$, we have
\[ \Omega_1 V_{w_jw^{-1}}wC_j = 0 \]
since $\text{char } k = p$. Then we get
\[ \eta' = \Omega_1 \eta = \sum_{w \in Y_j} b_w X_{w_jw^{-1}}wC_j, \]
which satisfies that $|A(\eta')| < |A(\eta)|$, where $A(\eta') = \{ w \in Y_j \mid b_w \neq 0 \}$. Thus by the induction on the cardinality of $A(\eta)$, the lemma is proved.

\[ \square \]

**Lemma 7.4.** Assume that $\text{char } F = \text{char } k = p > 0$ and let $M$ be a nonzero $kG$-submodule of $E_F$. If there exists a finite $p$-group $X$ of $U_{w_jw^{-1}}$ such that $\Omega s wC_j \in M$, where $sw \in Y_j$ and $sw > w$ (which implies that $w \in Y_j$), then there exists a finite $p$-group $H$ of $U_{w_jw^{-1}}$ such that $HwC_j \in M$.

**Proof.** Using Lemma 3.3, we can assume that $X$ is a self-enclosed subgroup of $U_{w_jw^{-1}}$. Since $U_{w_jw^{-1}} = U_s(U_{w_jw^{-1}})^s$, we can write $X = X_n V$, where $V = X \cap (U_{w_jw^{-1}})^s$ is also a self-enclosed subgroup of $(U_{w_jw^{-1}})^s$. Thus we have $X = \ldots$
In the following, we will prove that if \( Y \subseteq X \) and \( sw < wCJ \in M \) for some finite subset \( Y \) of \( U_s \) and a self-enclosed subgroup \( V \) of \( (U_{w,jw^{-1}})^s \), then there exists a finite \( p \)-group \( H \) of \( U_{w,jw^{-1}} \) such that \( HwCJ \in M \). Without lost of generality, we can assume that the subset \( Y \) contains the neutral element of \( U_s \).

For each \( u \in U_\alpha \setminus \{ id \} \), we have

\[
\hat{s}u\hat{s} = f_\alpha(u)h_\alpha(u)\hat{s}g_\alpha(u)
\]

where \( f_\alpha(u), g_\alpha(u) \in U_\alpha \) and \( h_\alpha(u) \in T \) are unique determined. Then

\[
\hat{s}u\hat{s}wCJ = f_\alpha(u)h_\alpha(u)\hat{s}g_\alpha(u)s^{-1}wCJ.
\]

Without lost of generality, we can assume that the group \( V \) contains enough elements such that

\[
g_\alpha(u)s^{-1}wCJ = \hat{s}^{-1}wCJ
\]

for any \( u \in Y \setminus \{ id \} \). Indeed, denote by

\[
G_\alpha(X) = \{ g_\alpha(u) \in U_\alpha \mid u \in Y \setminus \{ id \} \}
\]

and let \( H \) be a self-enclosed subgroup which contains \( G_\alpha(X) \) and \( \hat{s}^{-1}V\hat{s} \). Then \( H_{w,jw^{-1}} = H \cap U_{w,jw^{-1}} \) is also a self-enclosed subgroup which contains \( \hat{s}^{-1}V\hat{s} \). Then we can consider \( \hat{Y} \hat{s}H_{w,jw^{-1}}\hat{s}^{-1} \) instead of \( \hat{Y} \hat{V} \) at the beginning. Therefore we have

\[
\hat{s}u\hat{s}wCJ = f_\alpha(u)h_\alpha(u)\hat{s}wCJ = f_\alpha(u)h_\alpha(u)Vh_\alpha(u)^{-1}wCJ,
\]

which implies that

\[
\hat{s}XwCJ = \hat{s}V\hat{s}^{-1}wCJ + \sum_{u \in Y \setminus \{ id \}} f_\alpha(u)h_\alpha(u)Vh_\alpha(u)^{-1}wCJ.
\]

Now we denote

\[
\Phi_{w,jw^{-1}} \cup \Phi_{w,jw^{-1}s} = \{ \beta_1 = \alpha, \beta_2, \ldots, \beta_m \}
\]

with \( \text{ht}(\beta_1) \leq \text{ht}(\beta_2) \leq \cdots \leq \text{ht}(\beta_m) \). Since \( sw \in Y_j \) and \( sw > w \), we have \((U_{w,jw^{-1}})^s \neq U_{w,jw^{-1}} \) by [5 Corollary 2.2]. Thus we can choose a maximal integer \( r \) such that \( \beta_r \not\in \Phi_{w,jw^{-1}} \cap \Phi_{w,jw^{-1}s} \) and \( \beta_j \in \Phi_{w,jw^{-1}} \cap \Phi_{w,jw^{-1}s} \) for \( j > r \). When \( \beta_r \in \Phi_{w,jw^{-1}} \setminus \Phi_{w,jw^{-1}s} \), using Lemma 3.3 and Lemma 7.2 we can choose certain subgroup \( \Omega_k \) of \( U_{\beta_k} \) for each \( r \leq k \leq m \) such that

\[
\Omega_r \Omega_{r+1} \cdots \Omega_m f_\alpha(u)h_\alpha(u)Vh_\alpha(u)^{-1}wCJ = 0
\]

and \( \Omega_r \Omega_{r+1} \cdots \Omega_m \hat{s}V\hat{s}^{-1}wCJ = \Omega wCJ \) for some finite subgroup \( \Omega \) of \( U_{w,jw^{-1}} \). Then the lemma is proved in this case.

When \( \beta_r \in \Phi_{w,jw^{-1}} \setminus \Phi_{w,jw^{-1}s} \), also by Lemma 3.3 and Lemma 7.2 we can choose certain subgroup \( \Gamma_k \) of \( U_{\beta_k} \) for each \( r \leq k \leq m \) such that there exists at least one
$u \in Y \setminus \{\text{id}\}$ satisfies that
\[ \Gamma_r \Gamma_{r+1} \ldots \Gamma_m f_\alpha(u) h_\alpha(u) V h_\alpha(u)^{-1} s w C_J = f_\alpha(u) I_s w C_J, \]
where $\Gamma$ is some finite subgroup of $(U_{w, w^{-1}})^s$. On the other hand, these groups $\Gamma_k$ also make $\Gamma_r \Gamma_{r+1} \ldots \Gamma_m s V \hat{s}^{-1} w C_J = 0$. So we get $\sum_{x \in F} x I_s w C_J \in M$ for some set $F$ with $|F| < |Y|$ and some finite subgroup $\Gamma$ of $(U_{w, w^{-1}})^s$. Hence by the same discussion as before, we could get another element $\sum_{y \in F'} y I'_s w C_J \in M$ for some set $F'$ with $|F'| < |F|$ and some finite subgroup $\Gamma'$ of $(U_{w, w^{-1}})^s$. Finally, we get an element $H sw C_J \in M$ for some finite subgroup $H$ of $(U_{w, w^{-1}})^s$. Thus we have $H^s w C_J \in M$ and the lemma is proved.

$\square$

**Proof of Theorem 7.1.** We just need to consider the case $\text{char } F = \text{char } k = p > 0$ by the previous discussion. Let $M$ be a nonzero $kG$-submodule of $E_J$. Combining Lemma 7.3 and Lemma 7.4, there exists a finite $p$-subgroup $H$ of $U_{w, J}$ such that $H C_J \in M$. Similar to the arguments of [13, Lemma 2.5], we see that the sum of all coefficients of $w J x C_J$ in terms the basis $\{uC_J \mid u \in U_{w, J}\}$ is zero when $x$ is not the neutral element of $U_{w, J}$. So if we write
\[ \xi = w J H C_J = \sum_{x \in U_{w, J}} a_x x C_J, \]
we have $\sum_{x \in U_{w, J}} a_x = (-1)^{l(w, J)}$ which is nonzero. We consider the $kU_{w, J}$-module generated by $\xi$, and then using [10] Proposition 4.1, we see that $C_J \in M$. Therefore $M = E_J$, which implies the irreducibility of $E_J$ for any $J \subset I$. All the $kG$-modules $E_J$ are pairwise non-isomorphic by Proposition 2.5 and thus the theorem is proved.

$\square$

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