On the existence of the Møller wave operator for wave equations with small dissipative terms.

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Abstract

The aim of this short note is to reconsider and to extend a former result of K. Mochizuki [Moc76], [MN96] on the existence of the scattering operator for wave equations with small dissipative terms.

Contrary to the approach used by Mochizuki we construct the wave operator explicitly in terms of the parametrix construction obtained by a (simplified) diagonalization procedure, cf. [Yag97]. The method is based on ODE techniques.

These considerations are part of a larger project and the idea is taken from [Wir] and generalized to $x$-dependent coefficients.

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We consider the Cauchy problem

$$\Box u + b(t, x)u_t = 0, \quad u(0, \cdot) = u_1, \quad D_t u(0, \cdot) = u_2$$

(1)

with $b \in L_1(\mathbb{R}, L_\infty(\mathbb{R}^n)) \cap L_\infty(\mathbb{R}^{1+n})$. We restrict our calculations to spaces of dimension $n \geq 2$. The modification to obtain results also for $n = 1$ are obvious.

We denote by $E = \dot{H}^1 \times L_2$ the energy space. We prove that in the energy space $(u, D_t u)$ converges to the local energy of a solution $\tilde{u}$ of the free wave equation $\Box \tilde{u} = 0$. 

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As usual we use the representation $\dot{H}^1 = |D|^{-1}L_2$, $|D|^{-1}$ being the Riesz potential operator of order 1, inverse to the operator $|D| = \sqrt{-\Delta}$. In our calculations we use this isomorphism $\dot{H}^1 \simeq L_2$ to restrict ourselves to calculations in $L_2$-space.

**Theorem 1.** Assume $b \in L_1(\mathbb{R}, L_\infty(\mathbb{R}^n)) \cap L_\infty(\mathbb{R}^{1+n})$.

There exist isomorphisms $W_\pm : E \to E$ of the energy space such that for $u = u(t, x)$ the solution of (1) to data $(u_1, u_2) \in E$ and for $(\tilde{u}_1, \tilde{u}_2) = W_\pm(u_1, u_2)$ and $\tilde{u}$ the solution of the free wave equation $\Box \tilde{u} = 0$ to data $\tilde{u}(0, \cdot) = \tilde{u}_1$, $D_t \tilde{u}(0, \cdot) = \tilde{u}_2$ the asymptotic relation

$$|| (u, D_t u) - (\tilde{u}, D_t \tilde{u}) ||_E \to 0 \quad \text{as} \quad t \to \pm \infty$$

holds.

We subdivide the proof in some steps and construct these wave operators explicitly in terms of the solution representation.

Let $U = (|D|\tilde{u}, D_t \tilde{u})^T$. Then $U$ satisfies the equation

$$D_t U = \left( |D| - \frac{1}{|D|} \right) U + \left( ib(t, x) \right) U.$$ 

The first matrix operator maps $\dot{H}^1 \to L_2$ while due to our assumptions the second one maps $L_2 \to L_2$. We will understand the first operator as closed unbounded operator on $L_2$ with domain $\dot{H}^1$ and solve in a first step the corresponding evolution equation. In a second step we understand the second operator as small perturbation of the first one.

We diagonalize the first matrix operator. Therefore we use

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and consider $U^{(0)} = M^{-1} U$. We get

$$D_t U^{(0)} = M^{-1} \left( |D| - \frac{1}{|D|} \right) MU^{(0)} + M^{-1} \left( ib(t, x) \right) MU^{(0)} = DU^{(0)} + B(t, x) U^{(0)},$$

where $D$ is the diagonal operator

$$D = \begin{pmatrix} |D| & \frac{1}{|D|} \\ \frac{1}{|D|} & -|D| \end{pmatrix}$$
and $B(t, x) \in L_1(\mathbb{R}, L_\infty(\mathbb{R}^n))$ is a matrix. Multiplication by this matrix defines a bounded operator on $L_2$ with $\|B(t, x)\|_{2 \to 2} = \|B(t, \cdot)\|_\infty \in L_1(\mathbb{R})$.

We start by solving the operator valued evolution equation

$$D_tE_0(t, s) = \mathcal{D}E_0(t, s), \quad E_0(s, s) = I : L_2 \to L_2.$$ 

We look for a solution from the evolution space

$$E_0(\cdot, s) \in C(\mathbb{R}, L_2 \to L_2) \cap C^1(\mathbb{R}, L_2 \to \dot{H}^{-1}).$$

This solution is given by the fundamental solution corresponding to the wave equation

$$E_0(t, s) = e^{i(t-s)\mathcal{D}} = \left( e^{i(t-s)|\mathcal{D}|} e^{-i(t-s)|\mathcal{D}|} \right).$$

The operator $E_0$ is unitary as well as $U_0(t, s)$, obtained by a similarity transform of $M E_0(t, s) M^{-1}$ with the (canonical) isomorphism $E \simeq L_2$. The operator $U_0(t)$ is the unitary solution operator of the homogeneous wave equation in the energy space.

In a second step we construct the solution to $D_t - \mathcal{D} - B(t, x)$. Therefore let

$$\mathcal{R}(t, s) = E_0(s, t) B(t, x) E_0(t, s)$$

(as concatenation of (bounded) operators on $L_2$) and

$$\mathcal{Q}(t, s) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{R}(t_1, s) \int_s^{t_1} \mathcal{R}(t_2, s) \ldots \int_s^{t_{k-1}} \mathcal{R}(t_k, s) dt_k \ldots dt_1$$

in the sense of Bochner integrals. The matrix operator $\mathcal{Q}(t, s)$ solves the Cauchy problem

$$D_t \mathcal{Q}(t, s) - \mathcal{R}(t, s) \mathcal{Q}(t, s) = 0, \quad \mathcal{Q}(s, s) = I : L_2 \to L_2.$$ 

Using $\mathcal{Q}(t, s)$ we can express the fundamental solution to the diagonalized system. Let $\mathcal{E}(t, s) = E_0(t, s) \mathcal{Q}(t, s)$. Then we obtain

$$D_t(\mathcal{E}_0 \mathcal{Q}) = (D_t \mathcal{E}_0) \mathcal{Q} + \mathcal{E}_0(D_t \mathcal{Q}) = \mathcal{D} \mathcal{E}_0 \mathcal{Q} + \mathcal{E}_0 \mathcal{R}(t, s) \mathcal{Q} = \mathcal{D} \mathcal{E}_0 \mathcal{Q} + B(t, x) \mathcal{E}_0 \mathcal{Q}$$

and $\mathcal{E}_0(s, s) \mathcal{Q}(s, s) = I$. Thus $\mathcal{E}(t, s)$ is the desired fundamental solution.
Hence $M\mathcal{E}(t, s)M^{-1}$ is related to the operator

$$U(t, s) : E \ni (u(s), D_t u(s)) \mapsto (u(t), D_t u(t)) \in E$$

for solutions $u$ to $\Box u + b(t, x) u_t = 0$.

We estimate the norm of this operator. We do this step by step. At first we have

$$||\mathcal{E}_0(t, s)|| = 1.$$ 

The next estimate is

$$||\mathcal{R}(t, s)|| \leq ||B(t, \cdot)||_\infty \in L_1(\mathbb{R})$$

which will be used to estimate $Q(t, s)$. We use the following statement

$$\left| \int_s^t r(t_1, s) \int_s^{t_1} r(t_2, s) \cdots \int_s^{t_{k-1}} r(t_k, s) dt_k \cdots dt_1 \right| \leq \frac{1}{k!} \left( \int_s^t |r(\tau, s)| d\tau \right)^k.$$ 

This can be proved using induction over $k$. Combined with the series representation of $Q$ we get

$$||Q(t, s) - I|| \leq \sum_{k=1}^\infty \frac{1}{k!} \left( \int_s^t ||B(\tau, \cdot)||_\infty d\tau \right)^k$$

$$= \exp \left\{ \int_s^t ||B(\tau, \cdot)||_\infty d\tau \right\} - 1 \leq C$$

and therefore

$$||\mathcal{E}(t, s)|| \leq C.$$ 

Integrability of $||B(t, \cdot)||_\infty$ implies further $Q(t, s) \to I$ as $t, s \to \infty$.

**Remark 1.** The constant in this estimate can be larger than 1. This is due to the fact that we have not required the condition $b(t, x) \geq 0$.

**Remark 2.** We are interested in the Møller wave operator $W_+$. This operator can be understood as a limit in the following sense. We consider data $(u_1, u_2)$ from the energy space and apply the solution operator $U(t, 0)$. Then we go back to the initial line using the solution operator of the homogeneous problem $U_0(-t)$. This gives data to the homogeneous wave equation which produce a solution coinciding with $u$ at the time level $t$. Now we let $t \to \infty$.

$$W_+ = \lim_{t \to \infty} U_0(-t) U(t, s).$$
If this limit exists in some sense, we have constructed the first Møller wave operator. The second one, $W_-$ will be obtained by replacing $t$ by $-t$. In our case we will see that these limits exists as strong limits in the operator norm.

From $E_0(0,t)E(t,0) = Q(t,0)$ it seems natural to ask whether $\lim_{t \to \infty} Q(t,0)$ exists in $L_2 \to L_2$. Therefore we consider the difference

$$Q(t,0) - Q(s,0) = \sum_{k=1}^{\infty} i^k \left[ \int_0^t R(t_1,0) \int_0^{t_1} R(t_2,0) \ldots \int_0^{t_{k-1}} R(t_k,0) dt_k \ldots dt_1 
- \int_s^t R(t_1,0,\xi) \int_0^{t_1} R(t_2,0) \ldots \int_0^{t_{k-1}} R(t_k,0) dt_k \ldots dt_1 \right]$$

$$= \sum_{k=1}^{\infty} i^k \int_s^t R(t_1,0) \int_0^{t_1} R(t_2,0) \ldots \int_0^{t_{k-1}} R(t_k,0) dt_k \ldots dt_1.$$

If we apply $\| \cdot \|_{2 \to 2}$ on both sides and use the same statement as above to estimate the integrals we get

$$\|Q(t,0) - Q(s,0)\|_{2 \to 2} \leq \sum_{k=1}^{\infty} \int_s^t \|B(t_1,\cdot)\|_{\infty} \frac{1}{(k-1)!} \left( \int_0^{t_1} \|B(\tau,\cdot)\|_{\infty} d\tau \right)^{k-1} dt_1$$

$$\leq \int_s^t \|B(t_1,\cdot)\|_{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \int_0^{t_1} \|B(\tau,\cdot)\|_{\infty} d\tau \right)^k dt_1$$

$$= \int_s^t \|B(t_1,\cdot)\|_{\infty} \exp \left\{ \int_0^{t_1} \|B(\tau,\cdot)\|_{\infty} d\tau \right\} dt_1 \to 0$$

as $t, s \to \infty$ from the integrability of $\|B(t,\cdot)\|_{\infty}$.

Thus the limit exists in $L_2 \to L_2$ and we can define

$$\tilde{W}_+ = \lim_{t \to \infty} M Q(t,0) M^{-1}.$$

**Remark 3.** By the construction it follows that

$$W_+ = \lim_{t \to \infty} U_0(-t)U(t,0)$$

is related to $\tilde{W}_+$ via the isomorphism $E \simeq L_2$. 

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The transpose $\mathbf{Q}^{-T}$ of the inverse of $\mathbf{Q}(t, s)$ satisfies the related equation

$$D_t - \mathbf{Q}^{-T}(t, s) + R^T(t, s)\mathbf{Q}^{-T}(t, s) = 0, \quad \mathbf{Q}^{-T}(s, s) = \mathbf{I}.$$ 

Thus we can estimate $\mathbf{Q}^{-T}$ in a similar style as $\mathbf{Q}$, especially we can prove that

$$\lim_{t \to \infty} \mathbf{Q}^{-1}(t, s)$$

exists.

The same argumentation is valid for $t \to -\infty$ and defines the second operator $\mathbf{W}_-$. Especially the scattering operator $\mathbf{S} = \mathbf{W}_+\mathbf{W}^{-1}_-$ exists.

**Corollary 1.** Under the assumptions of Theorem 1 exists the scattering operator $\mathbf{S} : E \to E$ and is invertible.

We can also estimate the rate of convergence in Theorem 1.

**Corollary 2.** Under the assumptions of Theorem 1 it holds

$$ ||(u, D_t u) - (\tilde{u}, D_t \tilde{u})||_E \lesssim ||(u_1, u_2)||_E \int_t^\infty ||b(\tau, \cdot)||_\infty d\tau.$$ 

**Proof.** The statement follows directly from

$$ \mathbf{Q}(\infty, 0) - \mathbf{Q}(t, 0) = \sum_{k=1}^{\infty} t^k \int_t^\infty R(t_1, 0) \int_0^{t_1} R(t_2, 0) \ldots \int_0^{t_{k-1}} R(t_k, 0) dt_k \ldots dt_1.$$ 

and

$$ ||\mathbf{Q}(\infty, 0) - \mathbf{Q}(t, 0)||_{2 \to 2} \leq \int_t^\infty ||B(t_1, \cdot)||_\infty \exp \left\{ \int_0^{t_1} ||B(t, \cdot)||_\infty dt \right\} dt_1 \lesssim \int_t^\infty ||b(\tau, \cdot)||_\infty d\tau,$$

where $\mathbf{Q}(\infty, s) = \lim_{t \to \infty} \mathbf{Q}(t, s).$ \qed

For our considerations it was essential that we had *not* to introduce a subdivision of the phase space into zones. This enables us to give a definition of the wave operator $\mathbf{W}_+(\xi)$ globally in the phase variable $\xi$.

For a more general treatment including a subdivision of the phase space we refer to [Wir] and the modified scattering result discussed there.

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1^In the usual matrix sense.
Figure 1: Short overview on operators involved in this note.

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