A Numerical Scheme Based on Semi-Static Hedging Strategy

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Abstract

In the present paper, we introduce a numerical scheme for the price of a barrier option when the price of the underlying follows a diffusion process. The numerical scheme is based on an extension of a static hedging formula of barrier options. For getting the static hedging formula, the underlying process needs to have a symmetry. We introduce a way to “symmetrize” a given diffusion process. Then the pricing of a barrier option is reduced to that of plain options under the symmetrized process. To show how our symmetrization scheme works, we will present some numerical results applying (path-independent) Euler-Maruyama approximation to our scheme, comparing them with the path-dependent Euler-Maruyama scheme when the model is of the Black-Scholes, CEV, Heston, and \((\lambda)-SABR\), respectively. The results show the effectiveness of our scheme.

Keywords: Numerical Scheme for the Barrier Options, Put-Call Symmetry, Static Hedging, Stochastic Volatility Models.

1 Introduction

In financial practice, the pricing (and hedging) of barrier type derivatives becomes more and more important. In the Black-Scholes environment, some analytic formulas are available in Merton (1973).

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If the underlying process is a diffusion process which is more complicated than a Geometric Brownian Motion, it will not be able, basically, to rely on anymore an analytic formula. Instead one should resort to some numerical analysis. There is a problem, however, arising from the path-dependence of the pay-off function. Gobet (2000) pointed out that the weak convergence rate against the time-discretization gets worse compared with the standard path-independent pay-off cases due to the failure in the observation of hitting between two time steps. He showed that the weak order of Euler-Maruyama approximation is $\frac{1}{2}$, which is much slower than the standard case where the order is 1.

In the present paper, we will introduce a new numerical scheme where the pricing (and hedging) of barrier options are reduced to that of plain (path-independent) ones. The scheme is based on an observation made by Carr and Lee (2009) which we will refer to as “arithmetic put-call symmetry” (APCS). In the Black-Scholes economy, it is well-recognized that the reduction is possible due to the reflection principle (see Imamura (2011)). The put-call symmetry is an extension of the reflection principle, with which a semi-static hedge is still possible.

There are two keys in our scheme;

1. For a given diffusion $X$ and a real number $K$, we can find another diffusion $\tilde{X}$ which satisfies the APCS at $K$ (see section 2.3). We call this procedure “symmetrization”.

2. For $T > 0$, the expectations $E[f(X_T)1_{\{\tau > T\}}]$ and $E[f(\tilde{X}_T)1_{\{\tilde{\tau} > T\}}]$ coincides, where $\tau$ and $\tilde{\tau}$ are the first hitting time at $K$ of $X$ and $\tilde{X}$, respectively.

We do not anymore regard the equation for semi-static hedging but just a relation to calculate the expectation of the diffusion with a Dirichlet boundary condition in terms of those without boundary conditions. In other words, the pricing and hedging are reduced to path-independent ones, where many stable techniques are available. In this paper, we will present some numerical results of applying (path-independent) Euler-Maruyama (EM) approximation to our scheme, comparing them with the path-dependent EM under Constant Elasticity of Volatility (CEV) models (Cox (1973)) including as a special case the Black-Scholes (BS) model, and stochastic volatility models of Heston’s (Heston (1993)) and $(\lambda)$-SABR (Hagan and Woodward (2002); Henry-Labordere (2005)).

This paper consists of two parts. In the first part, the discussion of our new scheme is concentrated on one-dimensional diffusion models, while the latter part deals with applications to the stochastic volatility models. Mathematically, the first part is somehow self-contained,
while one may think the latter part to be dependent on the result in Akahori and Imamura (2012). The fact is that we have found, in advance of Akahori and Imamura (2012), through numerical experiments how it should be applied to stochastic volatility models (see Imamura, Ishigaki, Kawagoe, and Okumura (2012)). In anyway, the main aim of the present paper is to introduce the new scheme and to report numerical results which show the effectiveness of the scheme. In order to ensure the consistency of the experiments, we present detailed descriptions.

The paper is organized as follows. In Section 2 we recall the APCS and how it is applied to the pricing and semi-static hedging of barrier options. In section 2.2 we give a sufficient condition shown by Carr and Lee (2009) under which APCS holds. In Section 2.3, we introduce a way to “symmetrize” a given diffusion process. We then show that by using the symmetrized process which satisfies APCS, the pricing of barrier option is reduced to that of two plain options. In section 3, we give numerical examples under our symmetrized approximation method. The results of the path-wise EM scheme (in section 3.1) and our new scheme are given when the underlying asset price process follows CEV with the volatility elasticity $\beta = 1$ which is nothing but a BS model and other elasticities. From Section 4, we discuss applications to stochastic volatility models. The symmetrized method is also applicable to the stochastic volatility models where the underlying price process and its volatility follows a (degenerate) 2-dimensional diffusion process. In section 4.1, we give numerical results under Heston model and $\lambda$-SABR model. In section 4.2, we show that the symmetrization scheme also works for the pricing of double barrier option. One will find that our scheme overwhelms the path-wise EM in all numerical results.

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2 The Put-Call Symmetry Method for One Dimensional Diffusions

2.1 Arithmetic Put-Call Symmetry

Let $X$ be a real valued diffusion process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ which satisfies the usual conditions. For fixed $K > 0$, we say that arithmetic put-call symmetry (APCS) at $K$ holds for $X$ if the following equation is satisfied:

$$E[G(X_t - K) \mid X_0 = K] = E[G(K - X_t) \mid X_0 = K],$$

for any bounded measurable function $G$ and $t \geq 0$. The APCS at $K$ is alternatively defined to be the equivalence in law between $X_t - X_{t \wedge \tau}$ and $X_{t \wedge \tau} - X_t$ for any $t \geq 0$, where $\tau$ is the first hitting time of $X$ to $K$.

Intuitively, the APCS means the following. For every path of $X$ which crosses the level $K$ and is found at time $t$ at a point below $K$, there is a “shadow path” obtained from the reflection with respect to the level $K$ which exceeds this level at time $t$, and these two paths have the same probability. For one-dimensional Brownian motion, APCS holds for any $K > 0$ since the reflection principle holds.

In Carr and Lee (2009), the APCS, or more precisely, PCS, is applied to the pricing and semi-static hedging of barrier options. Semi-static hedging means replication of the barrier contract by trading European-style claims at no more than two times after inception.

In more detail, we have the following; if $X$ satisfies APCS at $K$, then for any bounded measurable $f$ and $T > 0$,

$$E[f(X_T)I_{\{\tau > T\}}] = E[f(X_T)I_{\{X_T > K, \tau > T\}}]$$

$$= E[f(X_T)I_{\{X_T > K\}}] - E[f(X_T)I_{\{X_T > K, \tau \leq T\}}],$$

where

$$\tau = \inf\{t \geq 0 : X_T \leq K\}. \quad (1)$$

By APCS of $X$, we see that

$$E[f(X_T)I_{\{X_T > K, \tau \leq T\}}] = E[E[f(X_T)I_{\{X_T > K\}} \mid \mathcal{F}_\tau]I_{\{\tau \leq T\}}]$$

$$= E[E[f(2K - X_T)I_{\{X_T < K\}} \mid \mathcal{F}_\tau]I_{\{\tau \leq T\}}].$$

Hence we obtain the following equation:

$$E[f(X_T)I_{\{\tau > T\}}] = E[f(X_T)I_{\{X_T > K\}}]$$

$$- E[f(2K - X_T)I_{\{X_T < K\}}]. \quad (2)$$

1 They defined PCS as the equality of the distribution between $\frac{X_T}{X_0}$ under $\mathbb{P}$ and $\frac{X_T}{X_0}$ under $\mathbb{Q}$, where $\frac{\mathbb{Q}}{\mathbb{P}} = \frac{X_\tau}{X_0}$. 
Of the equation (2), the left-hand-side reads the price of a barrier option written on $X$, whose pay-off is $f$, knocked out at $K$, and the right-hand-side is the price of a combination of two plain-vanilla options.

Here is a description of the hedging strategy of a barrier option implied from the right-hand-side of (2):

1. Hold a plain-vanilla options whose pay-off is $f(X_T)$ if the price at the maturity is less than $K$, and is nothing if the price at the maturity is greater than $K$. Moreover short-sell a plain-vanilla options whose pay-off is $f(2K - X_T)$ if the price at the maturity is greater than $K$, and is nothing if the price at the maturity is less than $K$.

2. Keep the above position until the price hits the barrier $K$. If the price never hits $K$ until the maturity, the pay-off is $f(X_T)$.

3. If the price hits $K$, clear both plain-vanilla positions at the hitting time. Indeed, the value of two options are exactly the same at $\tau$.

2.2 APCS of diffusion process

Let $X$ be a solution to the following one-dimensional stochastic differential equation (SDE) driven by a Brownian motion $W$ in $\mathbb{R}$,

\[dX_t = \sigma(X_t)dW_t + \mu(X_t)dt. \tag{3}\]

Here we assume the following hypotheses:

(H1) $\sigma : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$ are locally bounded measurable functions such that the linear growth condition is satisfied, ie, for a constant $C$, $|\sigma(x)| + |\mu(x)| \leq C(1 + |x|)$ for any $x \in \mathbb{R}$.

(H2) The following condition is satisfied;

$$\sigma(y) \neq 0 \iff \sigma^{-2} \text{ is integrable in a neighborhood of } y.$$  

Then we have the following result on the uniqueness of the solution to (3):

Theorem 2.1 (Theorem 4, Engelbert and Schmidt 1985). Under (H2), there exists a unique (in law) solution satisfying SDE (3).

Moreover, by the linear growth condition (H1), the unique (in law) solution will not explode in finite time. Carr and Lee (2009) gave a sufficient condition for a solution of (3) to satisfy PCS at $0 \in \mathbb{R}$. The following Proposition is essentially a corollary to Theorem 3.1 in Carr and Lee (2009).
Proposition 2.2. If the coefficients further satisfy the following conditions:

\[ \sigma(x) = \varepsilon(x)\sigma(2K - x) \quad (x \in \mathbb{R} \setminus \{K\}), \]

for a measurable \( \varepsilon : \mathbb{R} \to \{-1, 1\} \) and

\[ \mu(x) = -\mu(2K - x) \quad (x \in \mathbb{R} \setminus \{K\}), \]

then APCS at \( K \) holds for \( X \).

Proof. By the uniqueness in law, it is sufficient to show that \( (X_{t \wedge \tau} - (X_t - X_{t \wedge \tau}))_{t \geq 0} \) solves the SDE (3). By the assumptions (4) and (5), we obtain that

\[ X_{t \wedge \tau} - (X_t - X_{t \wedge \tau}) = X_{t \wedge \tau} - \int_{t \wedge \tau}^{t} \sigma(X_s)dW_s - \int_{t \wedge \tau}^{t} \mu(X_s)ds \]

\[ = X_{t \wedge \tau} - \int_{t \wedge \tau}^{t} \varepsilon(X_s)\sigma(2K - X_s)dW_s \]

\[ + \int_{t \wedge \tau}^{t} \mu(2K - X_s)ds \]

\[ = X_{t \wedge \tau} - \int_{t \wedge \tau}^{t} \varepsilon(X_s)\sigma(X_{\tau} - (X_s - X_{\tau}))dW_s \]

\[ + \int_{t \wedge \tau}^{t} \mu(X_{\tau} - (X_s - X_{\tau}))ds. \]

We set \( W'_t = W_{t \wedge \tau} - \int_{t \wedge \tau}^{t} \varepsilon(X_s)dW_s \). Since we obtain that

\[ \langle W' \rangle(t) = \langle W \rangle(t) \]

\[ = t, \]

\( W' \) is a Brownian motion (cf. Ikeda and Watanabe [1989] Chapter II, Theorem 6.1.). Therefore we see that

\[ X_{t \wedge \tau} - (X_t - X_{t \wedge \tau}) = X_0 + \int_{0}^{t} \sigma(X_{s \wedge \tau} - (X_s - X_{s \wedge \tau}))dW'_s \]

\[ + \int_{0}^{t} \mu(X_{s \wedge \tau} - (X_s - X_{s \wedge \tau}))ds. \]

Hence APCS at \( K \) holds. \( \square \)

2.3 Symmetrization of Diffusion Processes

We introduce a way to “symmetrize” a given diffusion to satisfy APCS. By using this symmetrized process satisfying APCS, the pricing of barrier options is reduced to that of plain options.
We start with a diffusion process $X$ given as a unique solution of SDE (3). We do not assume that the coefficients have the symmetric conditions (4) and (5). We then construct another diffusion $\tilde{X}$ that satisfies APCS at $K$ in the following way. Put

$$\tilde{\sigma}(x) := \begin{cases} \sigma(x) & x > K \\ \sigma(2K - x) & x \leq K, \end{cases}$$

(6)

$$\tilde{\mu}(x) := \begin{cases} \mu(x) & x > K \\ -\mu(2K - x) & x \leq K, \end{cases}$$

(7)

and consider the following SDE:

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t) dW_t + \tilde{\mu}(\tilde{X}_t) dt.$$  

(8)

Again by Theorem 2.1, there is a unique (in law) solution $\tilde{X}_t$. Then we obtain the following result:

**Theorem 2.3.** It holds that

$$\mathbb{E}[f(X_T) I_{\{\tau > T\}}] = \mathbb{E}[f(\tilde{X}_T) I_{\{\tilde{X}_T > K\}}] - \mathbb{E}[f(2K - \tilde{X}_T) I_{\{\tilde{X}_T < K\}}].$$

(9)

**Proof.** Since $\tilde{\sigma}$ and $\tilde{\mu}$ satisfy the condition (4) and (5), APCS at $K$ holds for $\tilde{X}$ by Proposition 2.2. Then the equation (2) is valid for $\tilde{X}$. Moreover, by the definition of $\tilde{\sigma}$ and $\tilde{\mu}$, we have $\sigma(x) = \tilde{\sigma}(x)$ and $\mu(x) = \tilde{\mu}(x)$ for $x < K$. Therefore by the uniqueness in law of the SDE, we have that $\{X_t\}_{t \leq \tau} = \{\tilde{X}_t\}_{t \leq \tau}$ pathwisely. Then we see that $\tau = \tilde{\tau}$ where $\tilde{\tau} = \inf\{t > 0 : \tilde{X}_t \leq K\}$. Hence we have

$$\mathbb{E}[f(\tilde{X}_T) I_{\{\tilde{\tau} > T\}}] = \mathbb{E}[f(X_T) I_{\{\tau > T\}}].$$

2.4 Important Remark

We do not anymore regard (9) as an equation for semi-static hedging but a relation to give a numerical scheme to calculate the expectation of the diffusion with a Dirichlet boundary condition in terms of those without boundary conditions. The former is very difficult while the latter is rather easier using rapidly developing technique from numerical finance. In the following sections, we will present some results of numerical examples to show the effectivity of the new scheme.
3 Numerical Experiments for One-Dimensional Models

3.1 The Euler-Maruyama Scheme

Here we briefly recall the Euler-Maruyama scheme for a diffusion process given as a solution to SDE (3). Fix $T > 0$. For $n \geq 1$, we set a subdivision of the interval $[0, T]$

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = T,$$

where $t_k := \frac{kT}{n}$, for $0 \leq k \leq n$, and we denote this net by $\Delta_n$.

The Euler-Maruyama scheme is a general method for numerically solving (3) by a discretized stochastic process which is given by

$$X^n_{t_{k+1}} = X^n_{t_k} + \sigma(X^n_{t_k})(t_{k+1} - t_k) + \mu(X^n_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad (10)$$

$k = 0, 1, 2, \cdots, n - 1$, and for $t_k < t < t_{k+1}$, $X_t$ is given by an interpolation. The approximating process $(X^n_T)$ is simulated by using independent quasi-random Gaussian variables for the increments $(W_{t_{k+1}} - W_{t_k})_{0 \leq k \leq n-1}$.

We rely on the following result;

**Theorem 3.1** (Theorem 3.1, Yan [2002]). If the set of discontinuous points of $\sigma$ and $\mu$ is countable, then the Euler scheme (10) converges weakly to the unique weak solution of SDE (3) as $n \to \infty$.

From now on, in addition to (H1) and (H2), we assume that $\sigma$ and $\mu$ are piece-wise continuous.

3.1.1 Path-wise Method

Since the convergence is in the space of probability measures on continuous functions, we see that this algorithm can also be used to simulate a path-dependent functional of the process; in particular, $f(X_T)I_{\{\tau > T\}}$, where $f$ is a (bounded) continuous function and $\tau$ is the first hitting time defined by (1). The functional is approximated by $f(X^n_T)I_{\{\tau^n > T\}}$, where $\tau^n := \inf\{t_k : X^n_{t_k} \leq K \}$ is the discretized first hitting time to $K$. Then the expectation $E[f(X_T)I_{\{\tau > T\}}]$ is approximated with a Monte-Carlo algorithm by

**Method 1. (Path-wise EM scheme)**

$$\frac{1}{M} \sum_{i=1}^{M} f(X^n_T(\omega_i))I_{\{\tau^n(\omega_i) > T\}}. \quad (11)$$
By the strong law of large numbers, (11) converges to $E[f(X^n_T)I_{\{\tau^n > T\}}]$ as $M$ goes to infinity. Moreover, as the index $n$ of the net $\Delta_n$ goes to infinity, $E[f(X^n_T)I_{\{\tau^n > T\}}]$ converges to $E[f(X_T)I_{\{\tau > T\}}]$. According to Gobet (2000), the following convergence rate was given:

**Theorem 3.2** (Theorem 2.3, Gobet (2000)). Assume that $\sigma$ and $\mu$ are in $C^\infty_b$, $\sigma$ is bounded below from zero and a solution is non-explosion. Then for a bounded measurable function $f$ such that $d(\text{supp} f, K) > 0$, there is a constant $C$ such that

$$|E[f(X^n_T)I_{\{\tau^n > T\}}] - E[f(X_T)I_{\{\tau > T\}}]| < C\frac{1}{\sqrt{n}}.$$  

### 3.1.2 Put-Call Symmetry Method

Let $\tilde{X}$ be a solution with coefficients $\tilde{\sigma}$ and $\tilde{\mu}$ given by (6) and (7), and $(\tilde{X}^n_T)$ be the discretized Euler-Maruyama process with respect to the net $\Delta_n$. Namely,

$$\tilde{X}^n_{t_{k+1}} = \tilde{X}^n_{t_k} + (\tilde{\sigma}(\tilde{X}^n_{t_k})(t_{k+1} - t_k) + \tilde{\mu}(\tilde{X}^n_{t_k})(W_{t_{k+1}} - W_{t_k}))I_{\{\tilde{X}^n_{t_k} > K\}} + (\tilde{\sigma}(2K - \tilde{X}^n_{t_k})(t - t_k) - \tilde{\mu}(2K - \tilde{X}^n_{t_k})(W_t - W_{t_k}))I_{\{\tilde{X}^n_{t_k} \leq K\}}$$

for $k = 0, 1, 2, \cdots, n - 1$. With an interpolation, $\tilde{X}^n_t$ for $t_k \leq t \leq t_{k+1}$ is obtained as well. Since the set of the discontinuous in the coefficients is Lebesgue measure zero, $\tilde{X}^n$ also converges weakly to $\tilde{X}$ by Theorem 3.1.

Combining Theorem 2.3 and 3.1 we may rely on the following algorithm; the expectation $E[f(X_T)I_{\{\tau > T\}}]$ is approximated with Monte-Carlo algorithm by

**Method 2.** (Put-Call symmetry method)

$$\frac{1}{M} \sum_{i=1}^{M} \left\{ f(\tilde{X}^n_T(\omega_i))I_{\{\tilde{X}^n_T(\omega_i) > K\}} - f(2K - \tilde{X}^n_T(\omega_i))I_{\{\tilde{X}^n_T(\omega_i) \leq K\}} \right\}. $$

(12)

As $M$ goes to $\infty$, (12) converges to

$$E[f(\tilde{X}^n_T)I_{\{\tilde{X}^n_T > K\}}] - E[f(2K - \tilde{X}^n_T)I_{\{\tilde{X}^n_T \leq K\}}].$$

(13)

By the weak convergence of $X^n$, (13) converges to

$$E[f(X_T)I_{\{X_T > K\}}] - E[f(2K - X_T)I_{\{X_T \leq K\}}].$$
as $\varepsilon \to \infty$. However, we don’t know the exact rate of convergence in
this algorithm since the coefficients are inevitably non-smooth at $K^2$.

The numerical results in the next section, however, may imply that
the convergence rate of Put-Call symmetry method is better than that
of the path-wise EM scheme. To prove this conjecture would be a very
interesting mathematical challenge.

3.2 Numerical Results

In this section, we give numerical examples using method 1 (path-
wise EM method) and method 2 (Put-Call symmetry method) under
Black-Scholes model and other CEV models. Let us consider the value
of barrier call option with strike price $S$ and knockout barrier $K$.

3.2.1 Black-Scholes Model

The underlying price process of Black-Scholes model is the unique
solution of the following SDE:

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

for $r, \sigma \geq 0$. Then the value of barrier option is accurately-calculable
since the joint distribution of Brownian motion and the hitting time
of Brownian motion to a point is computable by using the reflection
principle. The exact option price is given by the following;

$$E[(X_T - S)^+ I_{\{\tau > T\}}] = V_{\text{call}}(X_0) - \left(\frac{K}{X_0}\right)^{\frac{2r}{\sigma^2}} V_{\text{call}}\left(\frac{K^2}{X_0}\right),$$

where

$$V_{\text{call}}(x) = xe^{rT} (1 - \Phi(d_+(x))) - S (1 - \Phi(d_-(x))),$$

$$d_\pm(x) = \frac{\log\left(\frac{x}{S}\right) - (r \pm \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}},$$

and $\Phi$ is the distribution function of the standard normal distribution.

Fix a maturity time $T > 0$. Tables 1-4 give simulation results for
the value of down-and-out call option $E[(X_T - S)^+ I_{\{\tau > T\}}]$ under
the path-wise Euler-Maruyama method (EM) and the Put-Call symmetry
method (PCM). We take $[X_0 = 100, S = 95, K = 90, T = 1]$, and

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2 There are many results on the rate of convergence when $\sigma$ and $\mu$ are smooth. For example, when $\sigma$ and $\mu$ are in $C^4_p$, the space of functions such that 4-th derivative exists and have a polynomial growth, we have

$$|E[g(X_T^n)] - E[g(X_T)]| = O(n),$$

for any $g \in C^4_p$ (See Kloeden and Platen (2011), pp. 476).
In the PCM, we symmetrize the functions $\mu(x) = rx$ and $\sigma(x) = \sigma x$ at $K$.

The number of simulation trials is set equal to the cube of the number of time steps for the Euler discretization. The errors in the last two columns are calculated as

$$\frac{|\text{EM(PCM)} - \text{true option price}|}{\text{true option price}}.$$ 

One sees that, in the experiments, the Put-Call symmetry method always beats the path-wise EM method.

### 3.2.2 CEV Model

Here the underlying price process is a solution of the following SDE;

$$dX_t = rX_t dt + \sigma X_t^\beta dW_t,$$

(15)

for $r, \sigma \geq 0$ and $\beta \geq \frac{1}{2}$. We take $0.75$ in the experiments.

Tables 5-6 are simulation results for down-and-out call options with EM and PCM. We set parameters to $[X_0 = 100, S = 95, K = 90, \beta = 0.75, \sigma = 0.45, T = 1]$, and

Table 5: $r = 0$,
Table 6: $r = 0.02$,

in the experiments. For CEV model, we do not have an analytic formula. So, as a benchmark, we used numerical results by the path-wise Euler-Maruyama scheme where the number of time steps for the Euler discretization is 5,000 and that of a Monte-Carlo simulation is 50,000,000. Note that since we are calculating the prices of down-and-out call options, we do not need to care about the singularity at $x = 0$ in the SDE.

### 4 Put-Call Symmetry Method Applied to Stochastic Volatility Models

In this section, we slightly extend the put-call symmetry method to apply it to stochastic volatility models which are described by two-dimensional SDE. Theoretical backgrounds of the extension is given in Akahori and Imamura (2012).
Table 1: Black-Scholes model
\( X_0 = 100, S = 95, K = 90, \sigma = 0.2, r = 0, T = 1, \) option price = 8.17140

| No. of simulation trials | No. of time steps | EM   | PCM   | EM error(%) | PCM error(%) |
|--------------------------|-------------------|------|-------|-------------|--------------|
| 1000                     | 10                | 8.881| 7.816 | 8.7         | 4.4          |
| 8000                     | 20                | 9.183| 8.172 | 12.4        | 0.0          |
| 27000                    | 30                | 8.992| 8.250 | 10.0        | 1.0          |
| 64000                    | 40                | 8.880| 8.175 | 8.7         | 0.0          |
| 125000                   | 50                | 8.804| 8.190 | 7.7         | 0.2          |
| 216000                   | 60                | 8.692| 8.137 | 6.4         | 0.4          |
| 343000                   | 70                | 8.697| 8.127 | 6.4         | 0.5          |
| 512000                   | 80                | 8.671| 8.171 | 6.1         | 0.0          |
| 729000                   | 90                | 8.672| 8.207 | 6.1         | 0.4          |
| 1000000                  | 100               | 8.597| 8.135 | 5.2         | 0.4          |

Table 2: Black-Scholes model
\( X_0 = 100, S = 95, K = 90, \sigma = 0.2, r = 0.02, T = 1, \) option price = 9.31138

| No. of simulation trials | No. of time steps | EM   | PCM   | EM error(%) | PCM error(%) |
|--------------------------|-------------------|------|-------|-------------|--------------|
| 1000                     | 10                | 10.953| 9.821 | 17.6        | 5.5          |
| 8000                     | 20                | 10.050| 9.165 | 7.9         | 1.6          |
| 27000                    | 30                | 10.090| 9.226 | 8.4         | 0.9          |
| 64000                    | 40                | 9.952| 9.258 | 6.9         | 0.6          |
| 125000                   | 50                | 9.974| 9.302 | 7.1         | 0.1          |
| 216000                   | 60                | 10.033| 9.389 | 7.7         | 0.8          |
| 343000                   | 70                | 9.911| 9.298 | 6.4         | 0.1          |
| 512000                   | 80                | 9.885| 9.353 | 6.2         | 0.4          |
| 729000                   | 90                | 9.839| 9.306 | 5.7         | 0.1          |
| 1000000                  | 100               | 9.811| 9.309 | 5.4         | 0.0          |
Table 3: Black-Scholes model
$X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.5$, $r = 0$, $T = 1$, option price = 9.37170

| No. of simulation trials | No. of time steps | EM  | PCM  | EM error(%) | PCM error(%) |
|--------------------------|------------------|-----|------|-------------|--------------|
| 1000                     | 10               | 15.981 | 9.521 | 70.5       | 1.6          |
| 8000                     | 20               | 14.455 | 9.742 | 54.2       | 4.0          |
| 27000                    | 30               | 13.074 | 9.126 | 39.5       | 2.6          |
| 64000                    | 40               | 12.837 | 9.479 | 37.0       | 1.1          |
| 125000                   | 50               | 12.281 | 9.251 | 31.0       | 1.3          |
| 216000                   | 60               | 11.942 | 9.231 | 27.4       | 1.5          |
| 343000                   | 70               | 11.838 | 9.307 | 26.3       | 0.7          |
| 512000                   | 80               | 11.750 | 9.450 | 25.4       | 0.8          |
| 729000                   | 90               | 11.549 | 9.392 | 23.2       | 0.2          |
| 1000000                  | 100              | 11.443 | 9.319 | 22.1       | 0.6          |

Table 4: Black-Scholes model
$X_0 = 100$, $S = 95$, $K = 90$, $\sigma = 0.5$, $r = 0.02$, $T = 1$, option price = 10.02470

| No. of simulation trials | No. of time steps | EM  | PCM  | EM error(%) | PCM error(%) |
|--------------------------|------------------|-----|------|-------------|--------------|
| 1000                     | 10               | 15.488 | 9.688 | 54.5       | 3.4          |
| 8000                     | 20               | 14.687 | 9.540 | 46.5       | 4.8          |
| 27000                    | 30               | 14.065 | 10.341 | 40.3       | 3.2          |
| 64000                    | 40               | 13.472 | 10.191 | 34.4       | 1.7          |
| 125000                   | 50               | 13.012 | 9.779 | 29.8       | 2.4          |
| 216000                   | 60               | 12.981 | 10.257 | 29.5       | 2.3          |
| 343000                   | 70               | 12.707 | 9.991 | 26.8       | 0.3          |
| 512000                   | 80               | 12.391 | 9.916 | 23.6       | 1.1          |
| 729000                   | 90               | 12.418 | 10.098 | 23.9       | 0.7          |
| 1000000                  | 100              | 12.235 | 10.025 | 22.0       | 0.0          |
Table 5: CEV model

\[ X_0 = 100, S = 95, K = 90, \beta = 0.75, \sigma = 0.45, r = 0, T = 1, \]

benchmark of option price = 7.50095

| No. of simulation trials | No. of time steps | EM   | PCM  | EM error(%) | PCM error(%) |
|-------------------------|-------------------|------|------|-------------|--------------|
| 1000                    | 10                | 7.781| 7.068| 3.8         | 5.7          |
| 8000                    | 20                | 7.997| 7.504| 6.6         | 0.1          |
| 27000                   | 30                | 7.805| 7.397| 4.1         | 1.4          |
| 64000                   | 40                | 7.758| 7.379| 3.5         | 1.6          |
| 125000                  | 50                | 7.730| 7.412| 3.1         | 1.2          |
| 216000                  | 60                | 7.733| 7.407| 3.1         | 1.2          |
| 343000                  | 70                | 7.714| 7.422| 2.9         | 1.0          |
| 512000                  | 80                | 7.691| 7.423| 2.6         | 1.0          |
| 729000                  | 90                | 7.680| 7.414| 2.4         | 1.1          |
| 1000000                 | 100               | 7.654| 7.414| 2.1         | 1.1          |

Table 6: CEV model

\[ X_0 = 100, S = 95, K = 90, \beta = 0.75, \sigma = 0.45, r = 0.02, T = 1, \]

benchmark of option price = 8.82718

| No. of simulation trials | No. of time steps | EM   | PCM  | EM error(%) | PCM error(%) |
|-------------------------|-------------------|------|------|-------------|--------------|
| 1000                    | 10                | 9.418| 8.918| 6.7         | 1.0          |
| 8000                    | 20                | 9.349| 8.986| 5.9         | 1.8          |
| 27000                   | 30                | 9.242| 8.791| 4.7         | 0.4          |
| 64000                   | 40                | 9.193| 8.772| 4.1         | 0.6          |
| 125000                  | 50                | 9.109| 8.760| 3.2         | 0.8          |
| 216000                  | 60                | 9.089| 8.751| 3.0         | 0.9          |
| 343000                  | 70                | 9.063| 8.742| 2.7         | 1.0          |
| 512000                  | 80                | 9.009| 8.722| 2.1         | 1.2          |
| 729000                  | 90                | 9.027| 8.745| 2.3         | 0.9          |
| 1000000                 | 100               | 8.995| 8.722| 1.9         | 1.2          |
A generic stochastic volatility model is given as follows:

\[
\begin{align*}
    dX_t &= \sigma_{11}(X_t, V_t) dW_t + \mu_1(X_t, V_t) \, dt \\
    dV_t &= \sigma_{21}(V_t) dW_t + \sigma_{22}(V_t) dB_t + \mu_2(V_t) \, dt,
\end{align*}
\]

where \( W \) and \( B \) are mutually independent (1-dim) Wiener processes,

\[
\sigma(x, v) = \begin{pmatrix} \sigma_{11}(x, v) & 0 \\ \sigma_{21}(v) & \sigma_{22}(v) \end{pmatrix}
\]

and \( \mu(x, v) = (\mu_1(x, v), \mu_2(v)) \) are continuous functions on \( \mathbb{R}^2 \). Here we simply assume that \( \sigma \) and \( \mu \) are sufficiently regular (not so irregular) to allow a unique weak solution in \( (16) \). The independence of \( V \) against \( S \) plays an important role in applying our scheme. In fact, thanks to the property, we may simply work on the symmetrization with respect to the reflection \((x, y) \mapsto (2K - x, y)\). Let us be more precise. Let \((X, V)\) be a 2-dimensional diffusion process given as a (weak) unique solution of SDE \((16)\), and \(\tau\) be the first passage time of \(X\) to \(K\). We note that \(\tau\) is not dependent on \(V\). We say that arithmetic put-call symmetry at \(K\) holds for \((X, V)\) if

\[
(X_t, V_t)_{1\{\tau \leq t\}} \overset{d}{=} (2K - X_t, V_t)_{1\{\tau \leq t\}}
\]

for any \(t > 0\).

Mathematically, we rely on the following result from Akahori and Imamura (2012).

**Proposition 4.1 (Akahori and Imamura (2012)).** If the coefficients satisfy the following conditions;

\[
\begin{align*}
    \sigma_{11}(x, v) &= -\sigma_{11}(2K - x, v), \\
    \mu_1(x, v) &= -\mu(2K - x, v),
\end{align*}
\]

for \((x, v) \in (\mathbb{R} \setminus \{K\}) \times \mathbb{R}\), then APCS at \(K\) holds for \((X, V)\).

On the basis of Proposition 4.1 we construct another diffusion \((\tilde{X}, V)\) that satisfies APCS at \(K\) in a totally similar way as the one dimensional case, and we obtain a static hedging formula corresponding to Theorem 2.3.

**Proposition 4.2.** Let \(K > 0\) and put

\[
\begin{align*}
    \tilde{\sigma}_{11}(x, v) &= \begin{cases} 
    \sigma_{11}(x, v) & x \geq K, \\
    -\sigma_{11}(2K - x, v) & x < K
    \end{cases}, \\
    \tilde{\mu}_1(x, v) &= \begin{cases} 
    \mu_1(x, v) & x \geq K, \\
    -\mu_1(2K - x, v) & x < K
    \end{cases}.
\end{align*}
\]
and let \( \tilde{X} \) be the unique (weak) solution to
\[
d\tilde{X}_t = \tilde{\sigma}_{11}(\tilde{X}_t, V_t)dW_t + \tilde{\mu}_1(\tilde{X}_t, V_t)dt,
\]
where \( V \) is the solution to \([16]\). Then, it holds for any bounded Borel function \( f \) and \( t > 0 \) that
\[
E[f(X_t)1_{\{X_t>K\}}1_{\{\tau_K>t\}}]
= E[f(\tilde{X}_t)1_{\{\tilde{X}_t>\tilde{K}\}}] - E[f(2K - \tilde{X}_t)1_{\{\tilde{X}_t<K\}}],
\tag{19}
\]
where \( X \) is the solution to \([16] \) with \( X_0 > K \).

Proof. Omitted.

4.1 Numerical Results on Single Barrier Options under Stochastic Volatility Models

In this section we give numerical examples of the price of a single barrier option under Heston’s and SABR type stochastic volatility models, using numerical method based on \([25]\).

The Euler-Maruyama scheme of the solution of SDE \([16] \) with respect to the net \( \Delta_n = \{t_0, t_1, \cdots, t_n\} \) is given by the following;
\[
X^n_{tk+1} = X^n_{tk} + \sigma_{11}(X^n_{tk}, V^n_{tk})(W^n_{tk+1} - W^n_{tk}) + \mu_{11}(X^n_{tk}, V^n_{tk})(t_{k+1} - t_k),
\]
\[
V^n_{tk+1} = V^n_{tk} + \sigma_{21}(V^n_{tk})(W^n_{tk+1} - W^n_{tk}) + \mu_{21}(V^n_{tk})(B^n_{tk+1} - B^n_{tk})
+ \sigma_{22}(V^n_{tk})(t_{k+1} - t_k),
\]
for \( k = 0, 1, 2, \cdots , n - 1 \). With an interpolation, \( X_t \) for \( t_k < t \leq t_{k+1} \) is obtained as well. Here \( W \) and \( B \) denotes two independent 1-dimensional Brownian motions. The increments \( W^n_{tk+1} - W^n_{tk} \) and \( B^n_{tk+1} - B^n_{tk} \) are simulated by quasi-random independent Gaussian variables.

The underlying price process of Heston model is given as follows;
\[
\begin{align*}
dX_t &= rX_tdt + \sqrt{V_t}X_tdW_t, \\
dV_t &= \kappa(\theta - V_t)dt + \nu\sqrt{V_t}(\rho dW_t + \sqrt{1-\rho^2dB_t})
\end{align*}
\tag{20}
\]
for \( r, \kappa, \theta, \nu > 0 \) and \(-1 \leq \rho \leq 1 \). Then the symmetrized path \( \tilde{X} \) is constructed as a solution to the following SDE;
\[
\begin{align*}
d\tilde{X}_t &= \left(r\tilde{X}_tI_{\{\tilde{X}_t>\tilde{K}\}} - r(2K - \tilde{X}_t)I_{\{\tilde{X}_t<K\}}\right)dt \\
&\quad + \left(\sqrt{\tilde{V}_t}\tilde{X}_tI_{\{\tilde{X}_t>\tilde{K}\}} - \sqrt{\tilde{V}_t}(2K - \tilde{X}_t)I_{\{\tilde{X}_t<K\}}\right)dW_t,
\end{align*}
\]
where \( V \) is the solution of SDE \([20]\).
The underlying asset price of $\lambda$-SABR model is described as
\[
\begin{align*}
\frac{dX_t}{X_t} &= rdt + \sqrt{V_t} \, dB_t, \\
\frac{dV_t}{V_t} &= \lambda(\theta - V_t)dt + \nu V_t(\rho dB_t + \sqrt{1-\rho^2} \, dW_t) 
\end{align*}
\]
for $r, \lambda, \theta, \nu > 0$, $\beta \geq 1$ and $-1 \leq \rho \leq 1$. Then the symmetrized process $\tilde{X}$ is given by the following SDE:
\[
\frac{d\tilde{X}_t}{\tilde{X}_t} = \left( r\tilde{X}_t I_{\{\tilde{X}_t > K\}} - r(2K - \tilde{X}_t)I_{\{\tilde{X}_t < K\}} \right)dt + \left( V_t\tilde{X}_t^{\beta} I_{\{\tilde{X}_t > K\}} - V_t(2K - \tilde{X}_t)^{\beta}I_{\{\tilde{X}_t < K\}} \right) \, dW_t, 
\]
where $V$ is the solution of SDE (21).

Tables 7 - 10 below are simulation results of the price of a single barrier call option under Heston’s and $\lambda$-SABR model, respectively.

We set the parameters as $X_0 = 100$, $V_0 = 0.03$, $K = 95$, $H = 90$, $\theta = 0.03$, $r = 0$, $T = 1$, $\kappa = 1$, $\rho = -0.7$, $\nu = 0.03$ in Heston model (Table 7 and Table 8), and $X_0 = 100$, $V_0 = 0.5$, $S = 95$, $K = 90$, $\theta = 0.03$, $r = 0$, $T = 1$, $\beta = 0.75$, $\lambda = 1.0$, $\rho = -0.7$, $\nu = 0.3$ in $\lambda$-SABR model (Table 9 and Table 10), and

in the experiments. Benchmark is given in the same setting of Section 3.2.2. We again observe the superiority of our scheme.

4.2 Application to Pricing Double Barrier Options under the Stochastic Volatility Models

Fix $K, K' > 0$. Let us consider a double barrier option knocked out if price process $X$ hit either the boundary $K$ or $K + K'$. The price of a double barrier option with payoff function $f$ and barriers $K$ and $K + K'$ is given by $E[f(X_T)I_{\{\tau(K,K+K')>T\}}]$, where $\tau(K,K+K')$ is the first exit time of $X$ from $(K, K + K')$. In a similar way as the static hedging formula of a single barrier option, we obtain a static hedging formula if the price process satisfies APCS both at $K$ and $K + K'$.

**Proposition 4.3** (Akahori and Imamura (2012)). If $X$ satisfies APCS at both $K$ and $K + K'$, then for any bounded Borel function $f$ and $T > 0$, we have
\[
\begin{align*}
E[f(X_T)I_{\{\tau(K,K+K')>T\}}] &= \sum_{n \in \mathbb{Z}} E[f(X_T - 2nK')I_{[K+2nK',K+(2n+1)K']}](X_T) \\
&\quad - \sum_{n \in \mathbb{Z}} E[f(2K - (X_T - 2nK'))I_{[K+(2n-1)K',K+2nK']}(X_T)],
\end{align*}
\]

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Table 7: Heston model

\[ X_0 = 100, \ V_0 = 0.03, \ K = 95, \ H = 90, \ \theta = 0.03, \ r = 0, \ T = 1, \ \kappa = 1, \ \rho = -0.7, \ \nu = 0.03, \ \text{benchmark of option price} = 7.92706 \]

| No. of simulation trials | No. of time steps | EM   | PCM   | EM error(%) | PCM error(%) |
|-------------------------|-------------------|------|-------|-------------|--------------|
| 1000                    | 10                | 8.638| 7.953 | 9.0         | 0.3          |
| 8000                    | 20                | 8.761| 8.167 | 10.5        | 3.0          |
| 27000                   | 30                | 8.466| 7.932 | 6.8         | 0.1          |
| 64000                   | 40                | 8.477| 8.017 | 6.9         | 1.1          |
| 125000                  | 50                | 8.366| 7.892 | 5.5         | 0.4          |
| 216000                  | 60                | 8.301| 7.877 | 4.7         | 0.6          |
| 343000                  | 70                | 8.246| 7.875 | 4.0         | 0.7          |
| 512000                  | 80                | 8.273| 7.902 | 4.4         | 0.3          |
| 729000                  | 90                | 8.221| 7.875 | 3.7         | 0.7          |
| 1000000                 | 100               | 8.212| 7.871 | 3.6         | 0.7          |

Table 8: Heston model

\[ X_0 = 100, \ V_0 = 0.03, \ K = 95, \ H = 90, \ \theta = 0.03, \ r = 0.02, \ T = 1, \ \kappa = 1, \ \rho = -0.7, \ \nu = 0.03, \ \text{benchmark of option price} = 9.15602 \]

| No. of simulation trials | No. of time steps | EM   | PCM   | EM error(%) | PCM error(%) |
|-------------------------|-------------------|------|-------|-------------|--------------|
| 1000                    | 10                | 10.308| 9.192 | 12.6        | 0.4          |
| 8000                    | 20                | 9.828| 9.197 | 7.3         | 0.5          |
| 27000                   | 30                | 9.572| 8.953 | 4.5         | 2.2          |
| 64000                   | 40                | 9.674| 9.133 | 5.7         | 0.3          |
| 125000                  | 50                | 9.632| 9.134 | 5.2         | 0.2          |
| 216000                  | 60                | 9.552| 9.093 | 4.3         | 0.7          |
| 343000                  | 70                | 9.525| 9.096 | 4.0         | 0.7          |
| 512000                  | 80                | 9.524| 9.135 | 4.0         | 0.2          |
| 729000                  | 90                | 9.498| 9.116 | 3.7         | 0.4          |
| 1000000                 | 100               | 9.454| 9.106 | 3.2         | 0.5          |
Table 9: \(\lambda\)-SABR model
\(X_0 = 100, V_0 = 0.5, S = 95, K = 90, \theta = 0.03, r = 0, T = 1, \beta = 0.75, \lambda = 1.0, \rho = -0.7, \nu = 0.3,\) benchmark of option price = 6.59534

| No. of simulation trials | No. of time steps | EM     | PCM    | EM error(%) | PCM error(%) |
|-------------------------|-------------------|--------|--------|-------------|--------------|
| 1000                    | 10                | 6.643  | 6.478  | 0.7         | 1.8          |
| 8000                    | 20                | 6.708  | 6.591  | 1.7         | 0.1          |
| 27000                   | 30                | 6.701  | 6.584  | 1.6         | 0.2          |
| 64000                   | 40                | 6.671  | 6.565  | 1.1         | 0.5          |
| 125000                  | 50                | 6.668  | 6.568  | 1.1         | 0.4          |
| 216000                  | 60                | 6.672  | 6.581  | 1.2         | 0.2          |
| 343000                  | 70                | 6.669  | 6.585  | 1.1         | 0.2          |
| 512000                  | 80                | 6.671  | 6.597  | 1.1         | 0.0          |
| 729000                  | 90                | 6.655  | 6.579  | 0.9         | 0.2          |
| 1000000                 | 100               | 6.646  | 6.576  | 0.8         | 0.3          |

Table 10: \(\lambda\)-SABR model
\(X_0 = 100, V_0 = 0.5, S = 95, K = 90, \theta = 0.03, r = 0.02, T = 1, \beta = 0.75, \lambda = 1.0, \rho = -0.7, \nu = 0.3,\) benchmark of option price = 8.71005

| No. of simulation trials | No. of time steps | EM     | PCM    | EM error(%) | PCM error(%) |
|-------------------------|-------------------|--------|--------|-------------|--------------|
| 1000                    | 10                | 9.493  | 8.779  | 9.0         | 0.8          |
| 8000                    | 20                | 9.081  | 8.582  | 4.3         | 1.5          |
| 27000                   | 30                | 9.106  | 8.723  | 4.5         | 0.2          |
| 64000                   | 40                | 9.029  | 8.656  | 3.7         | 0.6          |
| 125000                  | 50                | 9.007  | 8.683  | 3.4         | 0.3          |
| 216000                  | 60                | 9.008  | 8.710  | 3.4         | 0.0          |
| 343000                  | 70                | 8.988  | 8.707  | 3.2         | 0.0          |
| 512000                  | 80                | 8.940  | 8.670  | 2.6         | 0.5          |
| 729000                  | 90                | 8.923  | 8.671  | 2.4         | 0.4          |
| 1000000                 | 100               | 8.929  | 8.680  | 2.5         | 0.3          |
Of the formula (22), the left-hand-side is the price of a barrier option, and the right-hand-side is an infinite series of the prices of plain-vanilla options. It means that a double barrier option can be hedged by infinite plain-vanilla options. Practically, the series should be approximated by finite terms. In our numerical scheme, however, finite sum approximation is not necessary as we will explain later in Remark 4.5.

We give a numerical scheme of a double barrier option under stochastic volatility model by using the symmetrized process which satisfies APCS at both $K$ and $K'$. The scheme is summarized as Proposition 4.4.

Set
\[
\hat{\sigma}_{11}(x, v) = \sum_{n \in \mathbb{Z}} \sigma_{11}(x - 2nK', v)I_{[K + 2nK', K + (2n + 1)K']}(x) - \sum_{n \in \mathbb{Z}} \sigma_{11}(2K - (x - 2nK'), v)I_{[K + (2n - 1)K', K + 2nK']}(x), \tag{23}
\]
and let $\hat{\mu}_1$ be the unique (weak) solution to
\[
d\hat{X}_t = \hat{\sigma}_{11}(\hat{X}_t, V_t)dW_t + \hat{\mu}_1(\hat{X}_t, V_t)dt,
\]
where $V$ is the solution to SDE (16). Then, it holds for any bounded Borel function $f$ and $t > 0$ that
\[
E[f(X_t)1_{\{\tau_{(K, K')} > t\}}] = \sum_{n \in \mathbb{Z}} E[f(\hat{X}_t - 2nK')I_{[K + 2nK', K + (2n + 1)K']}(\hat{X}_t)] - \sum_{n \in \mathbb{Z}} E[f(2K - (\hat{X}_t - 2nK'))I_{[K + (2n - 1)K', K + 2nK']}(\hat{X}_t)]. \tag{25}
\]

Proof. This is an easy consequence of Proposition 4.3. □

Remark 4.5. The infinite series of the right hand side in (23) and (24) is expressed by the following;
\[
\text{(the right hand side of (23))}
\begin{align*}
\sigma(x - \lfloor \frac{x - K}{K} \rfloor K', v) & \quad \text{if } \lfloor \frac{x - K}{K} \rfloor \equiv 0 \pmod{2}, \\
-\sigma(2K - (x - \lfloor \frac{x - K}{K} \rfloor - 1)K', v) & \quad \text{if } \lfloor \frac{x - K}{K} \rfloor \equiv 1 \pmod{2},
\end{align*}
\]
and

\[
\text{(the right hand side of (24))}
= \left\{ \begin{array}{ll}
\mu(x - \left[\frac{x-K}{K'}\right]K', v) & \text{if } \left[\frac{x-K}{K'}\right] \equiv 0 \mod 2, \\
-\mu(2K - (x - (\left[\frac{x-K}{K'}\right] - 1)K'), v) & \text{if } \left[\frac{x-K}{K'}\right] \equiv 1 \mod 2.
\end{array} \right.
\]

Therefore the discretized process of \((\hat{X}, V)\) by Euler-Maruyama scheme can be simulated without approximating the infinite series by finite sums. Similarly, we have

\[
\text{(the right hand side of (25))}
= E \left[ f(\hat{X}_t[\left[\frac{\hat{X}_t-K}{K'}\right]K', V_t)I_{\left[\frac{\hat{X}_t-K}{K'}\right] \equiv 0 \mod 2} \right]
- f(2K - (\hat{X}_t - (\left[\frac{\hat{X}_t-K}{K'}\right] - 1)K'), V_t)I_{\left[\frac{\hat{X}_t-K}{K'}\right] \equiv 1 \mod 2} \right].
\]

Therefore Put-Call symmetry method is available for the pricing of a barrier option.

Table 11 and Table 12 below are numerical results of the pricing of a double barrier call option under Heston model and \(\lambda\)-SABR model, respectively. We take

Table 11: \(X_0 = 100, V_0 = 0.03, S = 95, K + K' = 115, K = 85, \theta = 0.03, r = 0.02, T = 1, \kappa = 1, \rho = -0.7, \nu = 0.03,\)

Table 12: \(X_0 = 100, V_0 = 0.3, S = 95, K + K' = 110, K = 90, \theta = 0.3, r = 0.02, T = 1, \beta = 0.75, \lambda = 1, \rho = -0.7, \nu = 0.3,\)

in the experiments. Benchmark is given by the same setting of Section 3.2.2.

We still see that the put-call symmetry method beats the path-wise EM.

5 Concluding Remark

The new scheme, which is based on the symmetrization of diffusion process, is, though not theoretically, experimentally proven to be more effective than the path-wise Euler-Maruyama approximation scheme. The scheme is also applicable to stochastic volatility models including Heston’s and SABR type.
Table 11: Heston model

$X_0 = 100$, $V_0 = 0.03$, $S = 95$, $K + K' = 115$, $K = 85$, $\theta = 0.03$, $r = 0.02$, $T = 1$, $\kappa = 1$, $\rho = -0.7$, $\nu = 0.03$, benchmark of option price $= 1.40319930$

| No. of simulation trials | No. of time steps | EM | PCM | EM error(%) | PCM error(%) |
|--------------------------|------------------|----|-----|-------------|--------------|
| 1000                     | 10               | 2.987 | 1.671 | 112.869     | 19.1         |
| 8000                     | 20               | 2.368 | 1.498 | 68.759      | 6.7          |
| 27000                    | 30               | 2.144 | 1.588 | 52.785      | 13.2         |
| 64000                    | 40               | 2.045 | 1.475 | 45.770      | 5.1          |
| 125000                   | 50               | 1.921 | 1.402 | 36.903      | 0.1          |
| 216000                   | 60               | 1.876 | 1.453 | 33.662      | 3.6          |
| 343000                   | 70               | 1.820 | 1.411 | 29.728      | 0.6          |
| 512000                   | 80               | 1.792 | 1.438 | 27.733      | 2.5          |
| 729000                   | 90               | 1.765 | 1.411 | 25.791      | 0.6          |
| 1000000                  | 100              | 1.744 | 1.416 | 24.281      | 0.9          |

Table 12: $\lambda$-SABR model

$X_0 = 100$, $V_0 = 0.3$, $S = 95$, $K + K' = 110$, $K = 90$, $\theta = 0.3$, $r = 0.02$, $T = 1$, $\beta = 0.75$, $\lambda = 1$, $\rho = -0.7$, $\nu = 0.3$, benchmark of option price $= 2.46950606$

| No. of simulation trials | No. of time steps | EM | PCM | EM error(%) | PCM error(%) |
|--------------------------|------------------|----|-----|-------------|--------------|
| 1000                     | 10               | 3.779 | 2.451 | 53.017      | 0.8          |
| 8000                     | 20               | 3.427 | 2.566 | 38.768      | 3.9          |
| 27000                    | 30               | 3.164 | 2.442 | 28.126      | 1.1          |
| 64000                    | 40               | 3.037 | 2.489 | 22.997      | 0.8          |
| 125000                   | 50               | 2.955 | 2.514 | 19.640      | 1.8          |
| 216000                   | 60               | 2.915 | 2.480 | 18.036      | 0.4          |
| 343000                   | 70               | 2.875 | 2.481 | 16.438      | 0.5          |
| 512000                   | 80               | 2.838 | 2.478 | 14.906      | 0.4          |
| 729000                   | 90               | 2.806 | 2.464 | 13.631      | 0.2          |
| 1000000                  | 100              | 2.779 | 2.465 | 12.540      | 0.2          |
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