Scharnhorst effect for a general two-parameter lagrangian density

F. A. Barone*
Centro Brasileiro de Pesquisas Físicas Dr. Xavier Sigaud,
150, Urca, 22290-180 Rio de Janeiro, RJ, Brazil

C. Farina†
Instituto de Física, UFRJ, Caixa Postal 68528, Rio de Janeiro RJ, 21945-970, Brasil
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We discuss how the propagation of electromagnetic fields described by a general two-parameter lagrangian, which contains the Euler-Heisenberg effective lagrangian and the Born-Infeld lagrangian as particular cases, is affected by a pair of parallel plates that impose boundary conditions in the quantized field. We consider three different setups, namely: (i) two perfectly conducting plates; (ii) two infinitely permeable plates and (iii) a pair of plates in which one of them is a perfect conductor and the other has an infinite magnetic permeability.

In contrast to the classical vacuum, the quantum vacuum is not insensitive to external agents, but it behaves as an active medium, where virtual processes occur giving rise to real physical phenomena. For instance, the vacuum permittivity and permeability will be affected if we consider the radiation field constrained by two parallel and infinitely conducting plates. As a consequence, the velocity of light is changed. This phenomenon, known as the Scharnhorst effect [1], can be explained as follows: since the classical field interacts with the radiation field through the fermionic loop any change in the radiation field modes, as those imposed by the presence of the plates, will have some influence on the classical fields. Assuming that the plates do not impose any boundary condition (BC) to the fermionic field, the Scharnhorst effect turns to be a two-loop effect. Scharnhorst concluded that the speed of light propagating perpendicularly to the plates (and inside of them) is greater than the speed of light in unconstrained vacuum. Also, for propagation parallel to the plates, he found that the speed of light is unaltered. Scharnhorst’s result was reobtained by G. Barton [2] who used a much simpler technique, based on the Euler-Heisenberg effective lagrangian [8].

In the previous equation $\sigma$ and $\beta$ are real constants (two parameters) and $\mathcal{F} = (1/2)(\mathbf{B}^2 - \mathbf{E}^2)$ and $G^2 = (\mathbf{E} \cdot \mathbf{B})^2$ are the only gauge invariant Lorentz scalars. Classical corrections to the speed of light introduced by general two-parameter lagrangians had been considered in other papers for situations without boundary conditions [3, 4].

In particular, if we take $\sigma = (e^4)/(360\pi^2 m^4)$ and $\beta = \tau^2 = (7 e^4)/(360\pi^2 m^4)$ in equation (2), we reobtain the Euler-Heisenberg lagrangian [3] (we brought back the constants $\hbar$ and $c$)

$$\mathcal{L}_{EH} = \mathcal{L}_0 + \Delta \mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{2E_1}\left[\frac{1}{4}(\mathbf{E}^2 - \mathbf{B}^2)^2 + \frac{7}{4}(\mathbf{E} \cdot \mathbf{B})^2\right], \quad (3)$$

where $\alpha = e^2/4\pi\hbar c \simeq 1/137$ is the fine structure constant, $E_1 = (45\pi^3 m c^2)/(2\alpha^2 \lambda_C^3)$ is a constant with dimensions of electric field and $\lambda_C = 2\pi\hbar/mc$ is the Compton wavelength of the electron.

Another situation of special interest is a Born-Infeld like lagrangian density

$$\mathcal{L} = E_0^2\left[1 - \left(1 + \frac{\mathcal{F}}{E_0} + \frac{G^2}{E_0^2}\right)^{1/2}\right], \quad (4)$$

where $c$ is a real dimensionless constant and $E_0$ is an unknown constant with dimension of an electric field. For $c = -1$ we have the usual Born-Infeld lagrangian [3]. The Born-Infeld lagrangian was introduced for the first

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* e-mail: fbarone@cbpf.br
† e-mail: farina@if.ufrj.br
time in order to describe the dynamics of the classical electromagnetic field, avoiding some kinds of problems presented by the Maxweel theory. Some years later, it reappeared in a completely different context. It is worth noting that the Born-Infeld action was obtained as a low energy action to the vector modes of opened strings theory.

Expanding in order $E_0^{-2}$ we have

$$\tilde{\mathcal{L}} \equiv \frac{1}{2}(E^2 - B^2) + \frac{1}{2}E_0^2 \left[ \left( \frac{1}{4}(E^2 - B^2)^2 - \gamma(E \cdot B) \right) \right],$$  

(5)

which has the same structure of with $\sigma = 1/(8E_0^2)$ and $\beta = -\gamma 1/(2E_0^2)$.

In order to compute the Scharnhorst effect for our general lagrangian density, we shall employ the technique developed by G. Barton. It is based on the fact that a correction $\Delta L \sim \frac{\gamma}{4}$ developed by G. Barton. It is based on the fact that $\langle \Delta L \rangle$ is given, for $\Delta \bar{e}$ written as in Eq. (2), by

$$P_i = \frac{1}{4\pi} \frac{\partial}{\partial E_i} \Delta \mathcal{L} = \frac{1}{4\pi} \left[ 4\sigma(E^2 - B^2)E_i + 2\beta(E \cdot B)B_i \right],$$  

(6)

$$M_i = \frac{1}{4\pi} \frac{\partial}{\partial B_i} \Delta \mathcal{L} = \frac{1}{4\pi} \left[ -4\sigma(E^2 - B^2)B_i + 2\beta(E \cdot B)E_i \right].$$  

(7)

The main step in Barton’s technique consists in substituting, in expressions (6) and (7), the electromagnetic fields $E$ and $B$ by a sum of two fields, a classical field and a quantized one: $E \rightarrow e + \tilde{E}$, $B \rightarrow b + \tilde{B}$. Here, $e$ and $b$ are classical fields that describe the propagating wave while $\tilde{E}$ and $\tilde{B}$ are, from now on, quantum field operators, that will be taken as free fields, except by the fact that their modes are altered by the presence of the material plates. In this approximation, it is correct to take the expansion of the quantum fields in terms of the creation and annihilation operators, as usually done in QED. Substituting these expressions for the fields into (6) and (7), taking the vacuum expectation value, and maintaining only the relevant linear terms in the classical fields $e$ and $b$, we obtain

$$\langle P_i \rangle \| = \frac{1}{4\pi} \left[ 4\sigma \langle (E^2 - B^2) \| \delta_{ij} + 2\langle E_i E_j \rangle \| + 2\beta \langle B_i B_j \rangle \| \rangle \right] e_j =: \chi_{ij}^{(c)} e_j,$$  

(8)

$$\langle M_i \rangle \| = \frac{1}{4\pi} \left[ 4\sigma \langle -(E^2 - B^2) \| \delta_{ij} + 2\langle B_i B_j \rangle \| \rangle + 2\beta \langle E_i E_j \rangle \| \rangle \right] b_j =: \chi_{ij}^{(m)} b_j.$$

(9)

where the symbol $\langle ... \rangle \|$ means that we are considering the desired boundary conditions, we used the fact that $\langle E_i B_j \rangle \| = 0$ and also defined the electric and magnetic vacuum polarizabilities $\chi_{ij}^{(c)}$ and $\chi_{ij}^{(m)}$. From the quantities $\chi_{ij}^{(c)}$ and $\chi_{ij}^{(m)}$, we obtain the electric permittivity $\epsilon_{ij} = \delta_{ij} + \Delta \epsilon_{ij}$ and magnetic permeability $\mu_{ij} = \delta_{ij} + \Delta \mu_{ij}$

$$\epsilon_{ij} = \delta_{ij} + 4\pi \chi_{ij}^{(c)},$$  

(10)

$$\mu_{ij} = \delta_{ij} + 4\pi \chi_{ij}^{(m)}.$$

(11)

Corrections in $\epsilon_{ij}$ and $\mu_{ij}$ will give rise to a variation in the refraction index $n = (\epsilon_\mu)^{1/2}$, given in first order by $\Delta n = 1/2(\Delta \epsilon + \Delta \mu)$. This variation implies a velocity of light $c'$ given by $c' = c/(n + \Delta n) \equiv (1 - \Delta n/n)(c/n) = 1 - \Delta n$, where we used that the free vacuum refraction index $n$ (without boundary conditions) is equal to unity (in our unity system $c = 1$).

With all previous results we can obtain the influence in the speed of light due to the presence of two infinite parallel plates, assuming the dynamics of the electromagnetic fields described by the Maxwell lagrangian density plus the two-parameter correction (2).

For convenience, let us assume that the plates are parallel to the $\hat{X}\hat{Y}$ plane, with one of them located at $z = 0$ and the other one at $z = L$. Then, let us consider a wave propagating perpendicularly to the plates, that is, in the $\hat{z}$ direction, and a wave propagating in a direction parallel to the plates, for instance, in the $\hat{x}$ direction. In both cases, there are two possible polarizations, and in each case, we must take appropriately the electric and magnetic susceptibilities from equations (13), in order to compute the changes in the refractive index, as follows.

1. Polarization in the $\hat{y}$ direction:

$$\text{e} = e_2 \hat{y} \quad \implies \quad \Delta \epsilon = \Delta \epsilon_{22},$$  

$$\text{b} = b_3 \hat{z} \quad \implies \quad \Delta \mu = \Delta \mu_{33};$$

$$\Delta n = 4\sigma \left( \langle E_2 E_2 \rangle \| + \langle B_3 B_3 \rangle \| \right) + \beta \left( \langle B_2 B_2 \rangle \| + \langle E_3 E_3 \rangle \| \right).$$

(13)

2. Polarization in the $\hat{z}$ direction:

$$\text{e} = e_3 \hat{z} \quad \implies \quad \Delta \epsilon = \Delta \epsilon_{33},$$  

$$\text{b} = b_2 \hat{y} \quad \implies \quad \Delta \mu = \Delta \mu_{22};$$

$$\Delta n = 4\sigma \left( \langle E_3 E_3 \rangle \| + \langle B_2 B_2 \rangle \| \right) + \beta \left( \langle B_3 B_3 \rangle \| + \langle E_2 E_2 \rangle \| \right).$$

(15)
• Propagation perpendicular to the plates

(k = ±|k|z)

1. Polarization in the \( \hat{x} \) direction:

\[
\begin{align*}
\mathbf{e} &= e_1 \hat{x} \\
\mathbf{b} &= b_2 \hat{y}
\end{align*}
\]

\[
\Delta \varepsilon = \Delta \varepsilon_{11} \quad \Delta \mu = \Delta \mu_{22} \ .
\]

\[
\Delta n = 4\sigma \left( \langle E_1 E_1 \rangle + \langle B_2 B_2 \rangle \right) + \beta \left( \langle B_1 B_1 \rangle + \langle E_2 E_2 \rangle \right) .
\]

(17)

2. Polarization in the \( \hat{y} \) direction:

\[
\begin{align*}
\mathbf{e} &= e_2 \hat{y} \\
\mathbf{b} &= b_1 \hat{x}
\end{align*}
\]

\[
\Delta \varepsilon = \Delta \varepsilon_{22} \quad \Delta \mu = \Delta \mu_{11} \ .
\]

\[
\Delta n = 4\sigma \left( \langle E_2 E_2 \rangle + \langle B_1 B_1 \rangle \right) + \beta \left( \langle B_2 B_2 \rangle + \langle E_1 E_1 \rangle \right) .
\]

(19)

From now on we shall restrict ourselves to three distinct boundary conditions:

(i) Two conducting plates-(CC)

This configuration refers to two parallel and perfectly conducting plates. The field correlators, submitted to these boundary conditions, are given by [14]

\[
\begin{align*}
\langle 0 | E_1^2(x) | 0 \rangle &= \langle 0 | E_2^2(x) | 0 \rangle \\
&= -\langle 0 | B_2^2(x) | 0 \rangle = \frac{\pi^2}{48 L^4} \left( F(\theta) - \frac{1}{15} \right) ,
\end{align*}
\]

where we defined

\[
\begin{align*}
\rho &= \frac{\pi^2}{24 L^3 5} \\
\sigma &= \frac{\pi^2}{24 L^3 5} .
\end{align*}
\]

and compute the energy density of the electromagnetic field in the vacuum state, \( \rho = (1/2)\langle 0 | \mathbf{E}^2 + \mathbf{B}^2 | 0 \rangle \), for the three considered boundary conditions, which can be done with the aid of correlators (20), (21) and (22):

\[
\begin{align*}
\rho^{CC} &= \rho^{PP} = -\frac{\pi^2}{24 L^3 5} \frac{1}{3} \\
\rho^{CP} &= -\frac{7}{8} \rho^{CC} .
\end{align*}
\]

(26)

Now, with the previous results, we can show that for the considered cases

\[
\bar{c}' = 1 - \frac{2}{3} (4\sigma + 2\beta) \rho ,
\]

which is the “magic formula” for the two-parameter lagrangian density \( \mathcal{L} = \mathcal{L}_0 + \Delta \mathcal{L} \), where \( \Delta \mathcal{L} \) is given by equation (2).
Choosing coefficients $\sigma$ and $\beta$ to give the Euler-Heisenberg lagrangian \[3\), we get for the speed of light perpendicular to the plates:

\[
(c')_{\perp}^{CC} = (c')_{\perp}^{PP} = 1 + \frac{11\pi^2}{2\sqrt{3}\pi^2} \frac{\alpha^2}{(mL)^4} > 1 , \tag{28}
\]

\[
(c')_{\perp}^{CP} = 1 - \frac{7\pi^2}{8\sqrt{3}\pi^2} \frac{\alpha^2}{(mL)^4} < 1 , \tag{29}
\]

where $\lambda_C = 2\pi/m$ is the Compton wavelength. These results, which are particular cases of ours, are in perfect agreement with those found in literature \[11\,12\].

Taking $\sigma$ and $\beta$ to give the lagrangian density of the Born-Infeld type \[5\), we have

\[
(c')_{\perp}^{CC} = (c')_{\perp}^{PP} = 1 + \frac{\pi^2}{2\sqrt{3}} \frac{1}{(1 - \gamma) (E_0L)^2} , \tag{30}
\]

\[
(c')_{\perp}^{CP} = 1 - \frac{7\pi^2}{2\sqrt{3}} \frac{1}{(1 - \gamma) (E_0L)^2} . \tag{31}
\]

The magic formula for this Born-Infeld-like lagrangian density reads (for the averaged velocity of light)

\[
\bar{c'} = 1 - \frac{3}{1 - \gamma} \frac{\rho}{E_0^2} \tag{32}
\]

For a lagrangian density with the general form \[2\), we showed that the speed of light for a propagation parallel to the plates isn’t changed. For a propagation perpendicular to the plates, the boundary conditions CC and PP give the same variation for the speed of light. For the CP case, the variation in the speed of light has an opposite sign compared with the results found for the other two ones. In fact, if we take into account equation \[27\), the vacuum energy densities for the three considered boundary conditions, this result is expected. For a Born-Infeld like lagrangian density, given by \[5\) with a generic coefficient $\gamma$, we can verify according to \[31\) or \[31\), the following results.

If we take a factor $\gamma < 1$ we will have

\[
\gamma < 1 \implies (c')_{\perp}^{CC} = (c')_{\perp}^{PP} > 1 , \quad (c')_{\perp}^{CP} < 1 . \tag{32}
\]

In the cases where $\gamma > 1$, we obtain

\[
\gamma > 1 \implies (c')_{\perp}^{CC} = (c')_{\perp}^{PP} < 1 , \quad (c')_{\perp}^{CP} > 1 . \tag{33}
\]

However, a very peculiar result occurs when we take $\gamma = 1$: if we take $\gamma = 1$ we will not have, in the considered approximation (order $1/E_0^2$), any change in the speed of light due to the presence of the plates. Then, we have

\[
\gamma = 1 \implies (c')_{\perp}^{CC} = (c')_{\perp}^{PP} = (c')_{\perp}^{CP} = 1 . \tag{34}
\]

This is an interesting result because, even if we do not take into account quantum effects, the lagrangian density \[4\) with $\gamma = 1$ will give a null variation for the speed of light \[6\,7\].

The fact that we obtained values for the speed of light greater than $c = 1$ is not in disagreement with causality, since these are phase velocities, and all results presented here are valid only for low frequencies. In order to elucidate this point, one must consider the propagation of a wave-packet and analyse the speed of the wave front.

As a last comment, we would like to emphasize that Barton’s technique is well suited for calculating variations in the speed of light due to several other reasons, as for example: thermal effects, other kinds of boundary conditions, etc. All one has to do is to consider the appropriate electromagnetic field correlators.

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