Commutators of Singular Integral Operators Related to Magnetic Schrödinger Operators

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Abstract. Let $A := -\nabla i\vec{a} \cdot \nabla i\vec{a} + V$ be a magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$, $n \geq 2$, where $\vec{a} = (a_1, \ldots, a_n) \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. In this paper, we show that for a function $b$ in Lipschitz space $\text{Lip}_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, 1)$, the commutator $[b, V^{1/2}A^{-1/2}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $p, q \in (1, 2)$ and $1/p - 1/q = \alpha/n$.

Key Words: Commutator, Lipschitz space, the sharp maximal function, magnetic Schrödinger operator, Hölder inequality.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

Let $b$ be a locally integrable function on $\mathbb{R}^n$ and $T$ be a linear operator. For a suitable function $f$, the commutator is defined by $[b, T]f = bT(f) - T(bf)$. It is well known that Coifman, Rochberg and Weiss [3] proved that $[b, T]$ is a bounded operator on $L^p$ for $1 < p < \infty$ if and only if $b \in \text{BMO}(\mathbb{R}^n)$, when $T$ is a Calderón-Zygmund operator. Janson [4] proved that the commutator $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $1 < p < q < \infty$, if and only if $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ with $\alpha = (1/p - 1/q)n$, where the Lipschitz space $\text{Lip}_\alpha(\mathbb{R}^n)$ consists of the functions $f$ satisfying

$$
\|f\|_{\text{Lip}_\alpha} := \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty, \quad 0 < \alpha < 1.
$$

Furthermore, Lu, Wu and Yang studied the boundedness properties of the commutator $[b, T]$ on the classical Hardy spaces when $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ in [12].

In recent years, more scholars pay attention to the boundedness of the commutators $[b, T]$ when $T$ are the singular integral operators associated with the Schrödinger operator

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(cf. [1, 6–11]). When the potential $V$ satisfies the weaker condition, the operator $T$ may not be a Calderón-Zygmund operator. In this paper we focus on the boundedness of the commutators $[b, T]$ when $T$ are the singular integral operators associated with the magnetic Schrödinger operator based on the research in [5] and [16].

Consider a real vector potential $\vec{a}=(a_1,\cdots,a_n)$ and an electric potential $V$. In this paper, we assume that

$$a_k \in L^2_{\text{loc}}(\mathbb{R}^n), \quad \forall k=1,\cdots,n,$$

$$0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Let $L_k = \partial / \partial x_k - ia_k$. We adopt the same notation as in [5] and define the sesquilinear form $Q$ by

$$Q(f,g) := \sum_{k=1}^n \int_{\mathbb{R}^n} L_k f \overline{L_k g} \, dx + \int_{\mathbb{R}^n} V f g \, dx,$$

with domain

$$D(Q) := \{ f \in L^2(\mathbb{R}^n) : L_k f \in L^2(\mathbb{R}^n), k \in 1,\cdots,n, \sqrt{V} f \in L^2(\mathbb{R}^n) \}.$$

It is known that $Q$ is closed and symmetric. So the magnetic Schrödinger operator $A$ is a self-adjoint operator associated with $Q$.

The domain of $A$ is given by

$$D(A) = \{ f \in D(Q), \exists g \in L^2(\mathbb{R}^n) \text{ such that } Q(f,\varphi) = \int_{\mathbb{R}^n} g \overline{\varphi} \, dx, \forall \varphi \in D(Q) \},$$

and $A$ is formally given by the following expression

$$Af = \sum_{k=1}^n L_k^* L_k f + V f$$

or $A=-((\nabla - i\vec{a})\cdot(\nabla - i\vec{a}))+V$, where $L_k^*$ is the adjoint operator of $L_k$. For $k=1,\cdots,n$, the operators $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are called the Riesz transforms associated with $A$. Moreover, it was proved in [14] that for each $k=1,\cdots,n$, the Riesz transform $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$. Namely, there exists a constant $C > 0$ such that

$$\| V^{1/2} A^{-1/2} f \|_{L^p(\mathbb{R}^n)} + \sum_{k=1}^n \| L_k A^{-1/2} f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq 2.$$

Furthermore, in [5] Duong and Yan proved that the commutators $[b, V^{1/2} A^{-1/2}]$ and $[b, L_k A^{-1/2}]$ are bounded on $L^p$ for $1 < p \leq 2$, that is, there exists a constant $C > 0$ such that

$$\| [b, V^{1/2} A^{-1/2}] f \|_{L^p(\mathbb{R}^n)} + \| [b, L_k A^{-1/2}] f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}, \quad \text{where } b \in \text{BMO}(\mathbb{R}^n).$$
See also Shen’s result in [15] for $L^p$-boundedness of singular integral operators related to the magnetic Schrödinger operator, which is different from the operators $L_k A^{-1/2}$ and $V^{1/2} A^{-1/2}$. Recently, D. Y. Yang in [16] has proven that for $k \in \{1, \ldots, n\}$, the commutators $[b, L_k A^{-1/2}]$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p - 1/q = \alpha/n$, where $b \in \text{Lip}_\alpha(\mathbb{R}^n)$. Inspired by [5] and [16], the purpose of this paper is to study the boundedness of commutator $[b, V^{1/2} A^{-1/2}]$ with a function $b$ in the Lipschitz space $\text{Lip}_\alpha(\mathbb{R}^n), \alpha \in (0, 1)$.

We are now in a position to give our main result, which will be proven in the next section.

**Theorem 1.1.** Let $\alpha \in (0, 1)$, $p, q \in (1, 2]$ with $1/p - 1/q = \alpha/n$. Assume that $b \in \text{Lip}_\alpha(\mathbb{R}^n)$. Then the commutator $[b, V^{1/2} A^{-1/2}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

## 2 Proof of Theorem 1.1

In this section, we adopt the method in [16] to prove Theorem 1.1. Firstly, we begin with the sharp maximal function $M^\#_A$ established in [13]. For any $f \in L^p(\mathbb{R}^n), p \in [1, \infty)$, the sharp maximal function $M^\#_A$ associated with the semigroup $\{e^{-tA}\}$ is given by

$$M^\#_A(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - e^{-tsA} f(y)| dy,$$

where $r_B$ is the radius of the ball $B$ and $t_B := r_B^2$.

**Lemma 2.1.** Let $p \in (1, \infty)$. There exists a positive constant $C_p$ such that for all $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_p \|M^\#_A(f)\|_{L^p(\mathbb{R}^n)}.$$ 

**Proof of Theorem 1.1.** Now, we prove the boundedness of the commutator $[b, V^{1/2} A^{-1/2}]$ in Theorem 1.1. Let $(V^{1/2} A^{-1/2})^* = A^{-1/2} V^{1/2}$ denote the adjoint operator of $V^{1/2} A^{-1/2}$. By duality, for given $p, q \in (1, 2]$ with $1/p - 1/q = \alpha/n$, it suffices to prove $[b, A^{-1/2} V^{1/2}]$ is bounded from $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. To obtain the conclusion, it suffices to prove that there exists a constant $C$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M^\#_A([b, A^{-1/2} V^{1/2}] f)(x) \leq C \|b\|_{\text{Lip}_\alpha(\mathbb{R}^n)} [M_{2,q}(A^{-1/2} V^{1/2} f)(x) + M_{2,q}(f)(x)],$$

where for $r \in [2, n/\alpha)$ and any suitable function $f$,

$$M_{r,q}(f)(x) := \sup_{x \in B} \frac{1}{|B|^{1-\alpha/n}} \left( \frac{1}{|B|} \int_B |f(y)||^r dy \right)^{1/r}. \quad (2.2)$$

In fact, assume that (2.1) holds. For $b \in \text{Lip}_\alpha(\mathbb{R}^n)$ and each $N \in \mathbb{N}$, define $b_N := \min\{N, |b|\} \text{sgn}(b)$. Then we conclude that $b_N \in L^\infty(\mathbb{R}^n)$ and $\|b_N\|_{\text{Lip}_\alpha(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_\alpha(\mathbb{R}^n)},$ where $C$ is a constant. Moreover, it has been proved in Theorem 1.1 of [14] that $V^{1/2} A^{-1/2}$
is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p \leq 2$. So $A^{-1/2}V^{1/2}$ is bounded on $L^s(\mathbb{R}^n)$ for $s \in [2, \infty)$, then we see that for all $f \in C_c^\infty(\mathbb{R}^n)$, $[b_N, A^{-1/2}V^{1/2}](f) \in L^s(\mathbb{R}^n)$ and

$$\| [b_N, A^{-1/2}V^{1/2}](f) \|_{L^s(\mathbb{R}^n)} \leq N \|f\|_{L^s(\mathbb{R}^n)}.$$

Recall that $M_{2,\alpha}$ is bounded from $L^s(\mathbb{R}^n)$ to $L^l(\mathbb{R}^n)$ with $s \in (2, n/\alpha)$ and $1/s - 1/t = \alpha/n$, see Chanillo [2]. By this fact together with $1/q' - 1/p' = \alpha/n$, Lemma 2.1 and (2.1), we have that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\| [b_N, A^{-1/2}V^{1/2}](f) \|_{L^{q'}(\mathbb{R}^n)} \leq C \|M_{2,\alpha}([b_N, A^{-1/2}V^{1/2}](f))\|_{L^{q'}(\mathbb{R}^n)}$$

$$\leq C \|b_N\|_{\text{Lip}_a(\mathbb{R}^n)} \|M_{2,\alpha}(A^{-1/2}V^{1/2}f)(x)\| + M_{2,\alpha}(f)(x)\|_{L^{q'}(\mathbb{R}^n)}$$

$$\leq C \|b\|_{\text{Lip}_a(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)}.$$

A standard argument together with the Fatou lemma then implies that for all $f \in L^{q'}(\mathbb{R}^n)$,

$$[b, A^{-1/2}V^{1/2}](f) \in L^{p'}(\mathbb{R}^n)$$

and

$$\| [b, A^{-1/2}V^{1/2}](f) \|_{L^{p'}(\mathbb{R}^n)} \leq C \|b\|_{\text{Lip}_a(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)},$$

which imply Theorem 1.1.

Now, we prove (2.1) is valid. For any $f \in L^{q'}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, choose a ball $B := B(x_B, r_B) = \{y \in \mathbb{R}^n : |x_B - y| < r_B\}$ which contains $x$. Set $T := A^{-1/2}V^{1/2}$. Let $f_1 = f\chi_{2B}$ and $f_2 := f - f_1$. Then we have the following decompositions:

$$[b, T]f = (b - b_B)Tf - T((b - b_B)f_1) - T((b - b_B)f_2)$$

and

$$e^{-tsA}([b, T]f) = e^{-tsA}((b - b_B)Tf) - e^{-tsA}T((b - b_B)f_1) - e^{-tsA}T((b - b_B)f_2)$$

for any function $f$ and ball $B$, where

$$f_B := \frac{1}{|B|} \int_B f(z) \, dz, \quad t_B = r_B^2.$$
Therefore,
\[
\frac{1}{|B|} \int_B |(I-e^{-tA})[b,T]f(y)|dy 
\leq \frac{1}{|B|} \int_B |(b-b_B)Tf(y)|dy + \frac{1}{|B|} \int_B |T((b-b_B)f_1)(y)|dy 
\leq \frac{1}{|B|} \int_B |e^{-tA}((b-b_B)Tf)(y)|dy + \frac{1}{|B|} \int_B |e^{-tA}T((b-b_B)f_1)(y)|dy 
\leq \frac{1}{|B|} \int_B |(I-e^{-tA})T((b-b_B)f_2)(y)|dy 
\]
\[
= I + II + III + IV + V.
\]
Firstly, we get
\[
|b(y) - b_B| = \left| b(y) - \frac{1}{|B|} \int_B b(z)dz \right| = \left| \frac{1}{|B|} \int_B (b(y) - b(z))dz \right| 
\leq \frac{1}{|B|} \int_B |y-z|^a \frac{|b(y) - b(z)|}{|y-z|^a}dz 
\leq \|b\|_{\text{Lip}_a(\mathbb{R}^n)} \left| \frac{1}{|B|} \int_B |y-z|^a dz \right| 
\leq C|B|^{a/n} \|b\|_{\text{Lip}_a(\mathbb{R}^n)}. \tag{2.3}
\]
For I, by the Hölder inequality and (2.3), we have
\[
I = \frac{1}{|B|} \int_B |(b-b_B)Tf(y)|dy 
\leq \left( \frac{1}{|B|} \int_B |b(y) - b_B|^2dy \right)^{1/2} \left( \frac{1}{|B|} \int_B |Tf(y)|^2dy \right)^{1/2} 
\leq C\|b\|_{\text{Lip}_a(\mathbb{R}^n)} M_{2,a}(Tf)(x) = C\|b\|_{\text{Lip}_a(\mathbb{R}^n)} M_{2,a}(A^{-1/2}V^{1/2}f)(x).
\]
For II, using the Hölder inequality again and the $L^2(\mathbb{R}^n)$-boundedness of $T$, if follows that
\[
II = \frac{1}{|B|} \int_B |T((b-b_B)f_1)(y)|dy 
\leq \left( \frac{1}{|B|} \int_B |T((b-b_B)f_1)(y)|^2dy \right)^{1/2} 
\leq C\left( \frac{1}{|B|} \int_{2B} |b(y) - b_B|^2|f(y)|^2dy \right)^{1/2} 
\leq C\|b\|_{\text{Lip}_a(\mathbb{R}^n)} \left| \frac{1}{|B|} \int_{2B} |f(y)|^2dy \right|^{1/2} 
\leq C\|b\|_{\text{Lip}_a(\mathbb{R}^n)} M_{2,a}(f)(x).
\]
To estimate III, it follows from [5] that the kernel $p_t(y,z)$ of $e^{-tA}$ satisfies that for all $t > 0$ and almost all $y,z \in \mathbb{R}^n$,

$$p_t(y,z) \leq (4\pi t)^{-\frac{n}{2}} \exp \left( -\frac{|y-z|^2}{4t} \right). \quad (2.4)$$

Let $g := (b - b_B) Tf$. By (2.4), the formula equation of $e^{-|x|} \leq C|x|^{-N}$ and the conclusion of I, for any $y \in B$,

$$|e^{-t_B A} g(y)| \leq \int_{\mathbb{R}^n} |p_{t_B}(y,z)g(z)| dz \leq \int_{\mathbb{R}^n} |p_{t_B}(y,z)||g(z)| dz$$

$$\leq \int_{|y-z| < 2t_B} t_B^\frac{n}{2} \exp \left( -\frac{|y-z|^2}{4t_B} \right) |g(z)| dz$$

$$= \sum_{k=0}^\infty \int_{2^k t_B^{1/2} < |y-z| < 2^{k+1} t_B^{1/2}} t_B^\frac{n}{2} \exp \left( -\frac{|y-z|^2}{4t_B} \right) |g(z)| dz$$

$$\leq C \left[ \frac{1}{|B|} \int_{B} |g(z)| dz + \sum_{k=0}^\infty \frac{1}{t_B^N} \int_{2^k t_B^{1/2} < |y-z| < 2^{k+1} t_B^{1/2}} \left( \frac{t_B}{|y-z|^2} \right)^N |g(z)| dz \right]$$

$$\leq C \left[ ||b||_{Lip_\alpha(\mathbb{R}^n)} M_{2,n}(A^{-1/2} V^{1/2} f)(x) \right.$$  

$$+ \sum_{k=0}^\infty \frac{1}{2^{k(2N-n)}} ||b||_{Lip_\alpha(\mathbb{R}^n)} M_{2,n}(A^{-1/2} V^{1/2} f)(x) \bigg]$$

$$\leq C ||b||_{Lip_\alpha(\mathbb{R}^n)} M_{2,n}(A^{-1/2} V^{1/2} f)(x),$$

where $N > n/2$. So for III, it is easy for us to get

$$III = \frac{1}{|B|} \int_{B} |e^{-t_B A} ((b - b_B) Tf)(y)| dy$$

$$= \frac{1}{|B|} \int_{B} |e^{-t_B A} g(y)| dy$$

$$\leq C ||b||_{Lip_\alpha(\mathbb{R}^n)} M_{2,n}(A^{-1/2} V^{1/2} f)(x).$$

For IV, for all locally integrable functions $f$ and $x \in \mathbb{R}^n$, let $M(f)(x)$ be the Hardy-Littlewood maximal function as follow:

$$M(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_{B} |f(y)| dy.$$
By (2.4), the conclusion of II, the Hölder inequality and the \( L^2(\mathbb{R}^n) \)-boundedness of \( T \) and \( M \), we conclude that

\[
IV = \frac{1}{|B|} \int_B |e^{-tA} T((b-b_B)f_1)(y)| dy \\
\leq C \frac{1}{|B|} \int_B M[T((b-b_B)f_1)](y) dy \\
\leq C \left[ \frac{1}{|B|} \int_B \{ M[T((b-b_B)f_1)](y) \}^2 dy \right]^{1/2} \\
\leq C \left[ \frac{1}{|B|} \int_B |T((b-b_B)f_1)(y)|^2 dy \right]^{1/2} \\
\leq C \left[ \frac{1}{|B|} \int_{2B} |(b(y)-b_B)f(y)|^2 dy \right]^{1/2} \\
\leq C |b|_{\text{Lip}^\alpha(\mathbb{R}^n)} M_{2,\alpha}(f)(x).
\]

In order to estimate the term \( V \), we need the following Proposition 2.1 and Lemma 2.2.

**Proposition 2.1** (cf. Proposition 3.1 in [5]). Fix \( s > 0 \). Let \( A = -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V \) be the magnetic Schrödinger operator. Then for any \( m \in \mathbb{N} \), there exist positive constants \( C \) and \( c \) such that

\[
\int_{2^m \sqrt{t} \leq |x-y| < 2^{m+1} \sqrt{t}} \left( |V^{1/2} p_s(x,y)|^2 + \sum_{k=1}^n |L_k p_s(x,y)|^2 \right) dx \leq C s^{-n} (2^n \sqrt{t})^{n-2} \exp \left( -\frac{2^m t}{cs} \right)
\]

for all \( s > 0 \) and \( y \in \mathbb{R}^n \).

**Lemma 2.2.** For a real vector potential \( \vec{a} = (a_1, \ldots, a_n) \) and an electric potential \( V \), we assume that \( a_k \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n) \), \( \forall k = 1, \ldots, n \), \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). Then the composite operator \( (I - e^{-tA}) A^{-1/2} V^{1/2}, t > 0 \), which is the adjoint operator of \( V^{1/2} A^{-1/2}(I - e^{-tA}) \), has an associated kernel \( K_t^s(y,z) \) which satisfies

\[
\sum_{m=0}^\infty 2^m (2^m \sqrt{t})^{n/2} \left( \int_{2^m \sqrt{t} \leq |y-z| < 2^{m+1} \sqrt{t}} |K_t^s(y,z)|^2 dy \right)^{1/2} \leq C < \infty, \quad z \in \mathbb{R}^n. \tag{2.5}
\]

**Proof of Lemma 2.2.** Firstly, we need to compute the \( K_t^s(y,z) \) of \( (I - e^{-tA}) A^{-1/2} V^{1/2} \). Observe that

\[
A^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-sA} \frac{ds}{\sqrt{s}}
\]
then we have
\[
(I - e^{-iA})A^{-1/2}V^{1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-sA}V^{1/2} \frac{ds}{\sqrt{s}} - \frac{1}{\sqrt{\pi}} \int_0^\frac{\pi}{2} e^{-(s+i)A}V^{1/2} \frac{ds}{\sqrt{s}}
\]
\[
\leq \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-sA}V^{1/2} \frac{ds}{\sqrt{s}} - \frac{1}{\sqrt{\pi}} \int_0^\frac{\pi}{2} e^{-sA}V^{1/2} \frac{ds}{\sqrt{s}}
\]
\[
= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-sA}V^{1/2} \frac{ds}{\sqrt{s}} - \frac{1}{\sqrt{\pi}} \int_0^\frac{\pi}{2} e^{-sA}V^{1/2} \frac{ds}{\sqrt{s}}
\]
\[
= \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right) e^{-sA}V^{1/2} ds.
\]
Therefore,
\[
|K_t^r(y, z)| = \left| \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right) p_s(y, z) V^{1/2} ds \right|.
\]
By Minkowski’s inequality, we have
\[
\left( \int_{2^n \sqrt{T} \leq |y-z| < 2^{n+1} \sqrt{T}} |K_t^r(y, z)|^2 dy \right)^{1/2}
\]
\[
\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \left| \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right| \left( \int_{2^n \sqrt{T} \leq |y-z| < 2^{n+1} \sqrt{T}} |p_s(y, z) V^{1/2}|^2 dy \right)^{1/2} ds.
\]
Together with Proposition 2.1, this gives
\[
\sum_{m=0}^{2^n} \left( \int_{2^n \sqrt{T} \leq |y-z| < 2^{n+1} \sqrt{T}} |K_t^r(y, z)|^2 dy \right)^{1/2}
\]
\[
\leq C \sum_{m=0}^{2^n} \left( \int_{2^n \sqrt{T} \leq |y-z| < 2^{n+1} \sqrt{T}} |p_s(y, z) V^{1/2}|^2 dy \right)^{1/2} ds
\]
\[
\leq C \sum_{m=0}^{2^n} \left( \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right) \left( \int_{2^n \sqrt{T} \leq |y-z| < 2^{n+1} \sqrt{T}} |p_s(y, z) V^{1/2}|^2 dy \right)^{1/2} ds
\]
\[
= C \sum_{m=0}^{2^n} \int_t^1 \left( \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right) s^{-n/2} (2^n \sqrt{T})^{n-1} \exp \left( -\frac{2^m t}{2cs} \right) ds
\]
\[
+ C \sum_{m=0}^{2^n} \int_t^\infty \left( \frac{1}{\sqrt{s}} - \frac{\chi(s > t)}{\sqrt{s-t}} \right) s^{-n/2} (2^n \sqrt{T})^{n-1} \exp \left( -\frac{2^m t}{2cs} \right) ds
\]
\[
:= I_1 + I_2.
\]
We first estimate the term $I_1$. Note that $\chi_{s > t} \equiv 0$ for $s < t$. This, together with the fact
that $\omega^\beta e^{-\omega t} \leq C$ for any $\omega, \beta > 0$, shows

$$I_1 = C \sum_{m=0}^{\infty} 2^m \int_0^t \frac{1}{\sqrt{s}} \frac{\chi(s>t)}{\sqrt{s-t}} |s^{-n/2}(2^m \sqrt{t})^{n-1}| \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$= C \sum_{m=0}^{\infty} 2^m \int_0^t |s^{-(n+1)/2}(2^m \sqrt{t})^{n-1}| \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$\leq C \sum_{m=0}^{\infty} 2^m \int_0^t \left( \frac{2^m t}{s} \right)^{n+1} \frac{2^m t}{s} ds$$

$$\leq C \sum_{m=0}^{\infty} 2^{-m+1} t \int_0^t ds \leq C' < \infty.$$

Consider the term $I_2$. Observe that, for $s > t$, then $\chi_{s>t} = 1$. A direct calculation shows that

$$\left| \frac{1}{\sqrt{s}} - \frac{\chi(s>t)}{\sqrt{s-t}} \right| = \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-t}} = \frac{1}{\sqrt{s\sqrt{s-t}}} \times |\sqrt{s-t} - \sqrt{s}|$$

$$= \frac{1}{\sqrt{s\sqrt{s-t}}} \times \frac{t}{\sqrt{s-t} + \sqrt{s}} = \frac{t}{s\sqrt{s-t}}$$

Substituting the above into the term $I_1$, we obtain

$$I_2 = C \sum_{m=0}^{\infty} 2^m \int_0^t \frac{1}{\sqrt{s}} \frac{\chi(s>t)}{\sqrt{s-t}} |s^{-n/2}(2^m \sqrt{t})^{n-1}| \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$\leq C \sum_{m=0}^{\infty} 2^m \int_0^t \frac{t}{s\sqrt{s-t}} |s^{-(n+1)/2}(2^m \sqrt{t})^{n-1}| \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$= C \sum_{m=0}^{\infty} 2^m \int_0^t \frac{2^m t}{s\sqrt{s-t}} \left( \frac{2^m t}{s} \right)^{n+1} \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$+ C \sum_{m=0}^{\infty} 2^m \int_0^t \frac{2^m t}{s\sqrt{s-t}} \left( \frac{2^m t}{s} \right)^{n+1} \exp \left( -\frac{2^m t}{2cs} \right) ds$$

$$:= I_{1,1} + I_{1,2}.$$

Because that $t < s \leq 2t$, so we can get

$$\left( \frac{2^m t}{s} \right)^{n+1} \leq 2^{m(n-1)}, \quad \exp \left( -\frac{2^m t}{2cs} \right) \leq \exp \left( -\frac{2^m t}{4c} \right), \quad \frac{t}{s\sqrt{s}} < t^{-1/2}.$$
Therefore,

\[ II_{1,1} = C \sum_{m=0}^{\infty} 2^m \int_{t}^{\infty} \frac{t}{\sqrt{s-t}} \left( \frac{2^m}{s} \right)^{\frac{n}{2}} \exp \left( -\frac{2^m}{2cs} \right) ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^m \int_{t}^{\infty} \frac{t^{-1/2}}{\sqrt{s-t}} 2^{m(n-1)} \exp \left( -\frac{2^m}{4c} \right) ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^m \exp \left( -\frac{2^m}{4c} \right) t \int_{t}^{\infty} \frac{1}{\sqrt{s-t}} ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^m \exp \left( -\frac{2^m}{4c} \right) \leq C' < \infty. \]

Finally, we estimate the term \( II_{1,2} \). Since \( s > 2t \), we have that \( \sqrt{s-t} > \sqrt{s}/2 \). Hence,

\[ II_{1,2} = C \sum_{m=0}^{\infty} 2^m \int_{t}^{\infty} \frac{t}{\sqrt{s-t}} \left( \frac{2^m}{s} \right)^{\frac{n}{2}} \exp \left( -\frac{2^m}{2cs} \right) ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^m \int_{t}^{\infty} \frac{t^{-1/2}}{\sqrt{s-t}} \left( \frac{2^m}{s} \right)^{\frac{n}{2}} \exp \left( -\frac{2^m}{2cs} \right) ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^m \int_{t}^{\infty} \frac{t^{-1/2}}{\sqrt{s-t}} \left( \frac{s}{2^m} \right)^{\frac{n}{2}} \ exp \left( -\frac{2^{m+1}}{2cs} \right) ds \]

\[ \leq C \sum_{m=0}^{\infty} 2^{-2m} t \int_{t}^{\infty} \frac{1}{s} \frac{1}{s} ds \leq C' < \infty. \]

Combining the estimates of \( I_1, II_{1,1}, II_{1,2} \) we obtain (2.5). Hence, the proof of (2.5) is complete. Now, we will start the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let us consider the term \( V \). We have that

\[ (V^{1/2} A^{-1/2}(I-e^{-tA})) = (I-e^{-tA}) A^{-1/2} V^{1/2}. \]

So the kernel \( K_t^*(y,z) \) of the operator \( (V^{1/2} A^{-1/2}(I-e^{-tA})) = (I-e^{-tA}) A^{-1/2} V^{1/2} \) satisfies the following estimate

\[ \sum_{m=0}^{\infty} 2^m (2^m)^{n/2} \left( \int_{2^m \sqrt{t} \leq |y-z| < 2^{m+1} \sqrt{t}} |K_t^*(y,z)|^2 dz \right)^{1/2} \leq C < \infty, \]

where \( C \) is a constant independent of \( t \) and \( y \).

Finally, from the fact that \( |y-z| \geq r_B \) for any \( y \in B, z \notin 2B \), the conclusion of (2.3), Lemma
2.2 and the Hölder inequality, the term \( V \) is dominated as follows,

\[
V = \frac{1}{|B|} \int_B \left[(I-e^{-t\Delta})T((b-b_B)f_2)(y)\right]dy
\]

\[
\leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n \setminus (2B)} |K_{1_B}^*(y,z)|((b(z) - b_B)f(z))|dzdy
\]

\[
\leq \frac{1}{|B|} \int_B \sum_{m=0}^{\infty} \left( \int_{2^m r_B \leq |y-z| < 2^{m+1} r_B} |K_{1_B}^*(y,z)|((b(z) - b_B)||f(z)||dz \right)dy
\]

\[
\leq C \frac{1}{|B|} \int_B \sum_{m=0}^{\infty} |2^{m+1} B|^{a/n} \|b\|_{\text{Lip}_a(\mathbb{R}^n)} \left( \int_{2^m r_B \leq |y-z| < 2^{m+1} r_B} |K_{1_B}^*(y,z)|^2 dz \right)^{1/2}
\]

\[
\times \left( \int_{2^{m+1} B} |f(z)|^2 \right)^{1/2} dy
\]

\[
\leq C \sup_{m \geq 0} 2^{-m} \|b\|_{\text{Lip}_a(\mathbb{R}^n)} M_{2,a}(f)(x)
\]

\[
\leq C \|b\|_{\text{Lip}_a(\mathbb{R}^n)} M_{2,a}(f)(x).
\]

Combining the estimates from I to V, we see that (2.1) holds, which completes the proof of Theorem 1.1.

\[\square\]

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