A Gauge Field Model of Modal Completion

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Abstract Perceptual completion of figures is a basic process revealing the deep architecture of low level vision. In this paper, a complete gauge field Lagrangian is proposed which couples the retinex equation with neurogeometrical models and solves the problem of modal completion, i.e. the pop up of the Kanizsa triangle. Euler–Lagrange equations are derived by variational calculus and numerically solved. Plausible neurophysiological implementations of the particle and field equations are discussed and a model of the interaction between LGN and visual cortex is proposed.

1 Introduction

Perceptual completion is a low level visual process studied for more than a century, starting from the pioneers of the phenomenology of the Gestalt [65]. The psychologist Gaetano Kanizsa introduced in [34] a number of stunning examples of images allowing to clearly perceive the phenomenon of pop up of illusory figures. For example in Fig. 1, a triangle with curved boundaries is perceived out of the three Pacman inducers. Kanizsa called this pop up effect "modal completion" because the illusory figure and its boundaries are really perceived with the modality of vision, while the three Pacman are completed to discs with "a-modal completion" meaning that they remain invisible, partially masked by the triangle.

Note that the solid triangle appearing in the centre of the figure is perceived as darker than the background. With reverse contrast of the Pacman inducers, the triangle would appear brighter than the background [34]. This perceived contrast difference is essential to allow the visibility of the triangle with respect to the background that has the same physical luminance in the image. Then the problem of the triangle reconstruction involves both the completion (segmentation) of the triangular shape and the filling-in of the figure with the right contrast. In the present research, we would like to face both the geometric reconstruction and the contrast balancing problem.

A starting point to afford the task of modal completion is to consider illusory boundaries [34,51,54] and their neural correlates (see [26,60]). A number of mathematical models have been proposed on this topic. The celebrated model of elastica has been introduced by Mumford in [46] to take into account curvilinear illusory boundaries. Williams and Jacobs proposed a stochastic version of completion fields [66]. Recent models of boundary completion are based on the neurogeometrical structure of the visual cortex, and they show a strong explicative power of perceptual completion (see also Sect. 2.2 below). The first geometrical models were introduced by Hoffman [28,29] and Koenderink, [37], who proposed to use fibre bundles and Lie groups for the description of the cortex. The word neurogeometry was introduced by Petitot and Tondut in [49] who described the functional architecture of the visual cortex as a fibre bundle and proposed an efficient model of boundary completion. In 2003, Citti and Sarti [13] and Petitot [50] interpreted the whole fibre bundle as the group of position and orientation with a sub-Riemannian metric. This point of view has been further developed in [14,56]. This metric allows to reconstruct rectilinear or curved illusory boundaries. Other models of boundary completion have been developed in the same space, or in Lie groups [4,6,18,19,27,31,53,66,68]. While applied to Kanizsa triangle, these models correctly complete illusory
Fig. 1 The Kanizsa triangle with *curved* boundaries. Note the pop up of the illusory *triangle* out of the three Pacman inducers.

boundaries, but do not perform the filling-in of the image (see for example Fig. 2 left).

The problem of modal completion of both boundaries and figures together has been much less covered in literature. In [47] modal completion has been achieved by non-linear functional minimization by means of combinatorial techniques. In [57,58], a technique was proposed to construct the Kanizsa triangle by minimization of an area functional measured with respect to a metric induced by the image. In both models [47] and [57], a complete boundary/figure reconstruction was provided but a correct filling-in of figures with the perceived brightness is still missing.

In [14], the process of boundaries completion was described by means of two coupled equations: a first equation which generate the metric of the space, and a second one, which diffuses in the metric just defined.

In this paper, we further develop this idea, and we introduce a formal field theory of low level vision, able to face the problem of modal completion. The theory will couple two models of low level vision: a neurogeometrical model of boundary completion and the celebrated retinex model for filling-in. In particular, we will show that it is possible to integrate the retinex model of [23] with the neurogeometrical approach of [14] and thus propose a new model of modal completion based on complete contrast invariance. The two equations will act as a particle and a field term of a complete gauge field theory.

The paper is organised as follows: In Sect. 2, we recall the main properties of the retinex algorithm and the neurogeometrical model and reinterpret them with instruments of gauge field theory. In Sect. 3, we couple the models by introducing a complete gauge field Lagrangian. The corresponding Euler Lagrange equations are calculated by variational calculus. In Sect. 3.4, the Euler Lagrange equations are numerically solved, and a plausible neural implementation of the model is proposed and discussed in Sect. 3.5. In Sect. 4, we provide results on the pop up of the Kanizsa figure. In Sect. 5, we present the conclusion. Some technical aspects and notations are recalled in the Appendix.

2 The Retinex Algorithm and the Neurogeometrical Model

In this section, we recall the main properties of two low level vision models: the retinex model and a neurogeometrical model. The retinex algorithm has been inspired by the functionality of the retina in detecting image gradients and implementing contrast invariance. The second one has been inspired by the ability of the cortex to detect and complete boundaries. We will provide here a short description of the two processes, stressing the similarity of the mathematical instruments adopted by both.
2.1 A Mathematical Interpretation of the Retinex Algorithm

The celebrated retinex model has been introduced in [39, 40] to explain brightness perception, i.e. the phenomenon causing a grey patch to appear brighter when viewed against a dark background, and darker when viewed against a bright background. After its introduction, this model has inspired a wide range of improvements and new models have been proposed [9, 32, 36, 41, 43, 52].

A given grey scale image \( I : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \) can be decomposed in two components, the reflectance \( f \), which is the amount of reflected light per unit of incident light, and the luminance \( s \), which coincides with the incident light per unit area:

\[
I(x, y) = f(x, y)s(x, y). \tag{2.1}
\]

The visual system has the ability to recognise the images irrespectively of the illumination conditions. Hence, we can identify the reflectance \( f \) with the perceived image. The retinex algorithm explains this phenomenon, taking in input the image \( I \) and recovering the reflectance \( f \) after imposing differential constraint on the illumination \( s \). The original algorithm was expressed in terms of a random walk. Subsequent implementations of the model were based on the Poisson equation ([30, 36, 45], see also [48]). In [30], the authors proposed a physically based algorithm, assuming that the luminance varies slowly on the image, so that

\[
\Delta \log(s) = 0.
\]

Consequently the reflectance \( f \) of an image \( I \) satisfies

\[
\Delta \log f = \Delta \log I. \tag{2.2}
\]

The retinal image \( \log I \) and the perceived one \( \log f \) are not the same, but they differ for a harmonic function \( \log(s) \), which depends on the boundary data. In [45], the authors proved that the original retinal algorithm can be expressed by the same equation with Neumann boundary condition. Hence, the algorithm recovers the initial image, up to a luminance factor (see Fig. 3).

The Laplace operator is at the basis of the well-known ability of the retinex algorithm to perform a filling-in since the image could be exactly recovered from its boundary value (see for example [23, 24]). However, it does not seem able to perform modal completion or to explain the phenomena of Kanizsa triangle (see Fig. 4).

Georgiev in [23] proposed a new interpretation of the retinex algorithm in terms of covariant derivatives and fibre bundles, which we will recall here. Indeed, the value of the perceived brightness \( f \) at every point is weighted by the luminance function \( s \). Hence, values of the function \( f \) at different points \((x, y)\) are not directly comparable, but comparison is
possible only after a given adaptation from one point to the other. The role of this adaptation is more clear in logarithmic variables, where we have \( \log(I) = \log(f) + \log(s) \). Hence when looking at \( \log(f) \), the space has a different origin \( \log(s) \) over each point. In order to differentiate this function, it is necessary to evaluate the difference of perceived brightness at different points. Hence, Georgiev introduced a space in which over each point \((x, y)\) it is defined a different copy of the group \((\mathbb{R}, \cdot)\), with a different weight \(s(x, y)\). This leads to a description of the image space as a fibre bundle extending to the contrast feature previous models of Hoffman [28, Koenderink [37], Zucker [68], Petitot [49], Citti and Sarti [14]. We explicitly recall that suitable instruments of differential geometry and in particular the notion of covariant derivative are necessary while considering functions with values in a fibre bundle (see [33]). Indeed, these instruments have been largely used in geometric models of the cortex (see for example [6,63], Chapter 9 in [20,22,38]). Here, we recall the instruments necessary for the interpretation of the retinex model given by Georgiev in [23].

By definition, a fibre bundle \((E, \pi, B)\) consists of two spaces and a function \(\pi\): the total space \(E\), the base space \(B\) and a projection \(\pi : E \rightarrow B\) (see [33]). As in the previously quoted models, the basis \(B\) of the fibre bundle is the retinal plane \((x, y)\), the total space \(E\) is \(\mathbb{R}^2 \times F\), and the fibre \(F\) is a group. In particular, this bundle is different from the classical tangent bundle, since \(F\) coincides with multiplicative group \((\mathbb{R}, \cdot)\) with null element \(s(x, y)\) (or, in logarithmic variables \((\mathbb{R}, \cdot)\) with origin \(\log(s)\)). Each fibre has its luminance coordinate, but luminance in different fibres are not related. The perceived luminance \(f\) is the coefficient of a section \(fs\) of this fibre bundle. We cannot compare the brightness of two points from different fibres because in the mathematical structure, there is no mapping that would produce that brightness. The adaptation which allows comparison of different values of \(f\) is called connection, and the derivative of \(f\) in this structure is the covariant derivative.

In order to clearly specify the expression of the covariant gradient in this case, we will express the equation as a minimum of a Dirichlet functional, which involves only first order terms. Indeed, the solution of Eq. (2.2) with Neumann condition at the boundary can be interpreted as the minimum of a functional:

\[
\tilde{F}(f) = \frac{1}{2} \int \left| \nabla \log(f) - \nabla \log(I) \right|^2 dxdy
\]

\[
= \frac{1}{2} \int \left| \frac{\nabla f}{f} - \frac{\nabla I}{I} \right|^2 dxdy.
\]

Calling

\[
\tilde{A} = \nabla I / I
\]

(2.3)

This numerator has now the familiar expression of a covariant derivative,

\[
\nabla f - \tilde{A} f
\]

so that Georgiev proposed to identify \(\tilde{A}\) with the connection. The functional in (2.5) is invariant with respect to the transformation

\[
f \rightarrow fs, \quad \tilde{A} \rightarrow \tilde{A} + \nabla s / s
\]

in the sense that it takes the same value on the couple \((f, \tilde{A})\) and \((fs, \tilde{A} + \nabla s / s)\). This property expresses the fact that the solution is defined up to the luminance factor \(s\). This factor will be interpreted as a gauge function. Note that setting

\[
\phi = \log f, \quad h = \log I,
\]

equations (2.1) and (2.2) simplify to

\[
h = \phi + \log(s), \quad \Delta \phi = \Delta h.
\]

while the functional in (2.3) becomes

\[
F(\phi) = \int |\nabla \phi - \nabla h|^2 dxdy.
\]

Since \(\nabla h = \nabla (\log(I)) = \nabla I / I = \tilde{A}\), then

\[
F(\phi) = \int |\nabla \phi - \tilde{A}|^2 dxdy
\]

(2.9)

while the transformations which leave the operator invariant become

\[
\phi \rightarrow \phi + k, \quad \tilde{A} \rightarrow \tilde{A} + \nabla k.
\]

(2.11)

### 2.2 A Neurogeometrical Model for Boundary Completion

Neurogeometric models of the visual cortex as fibre bundle were proposed by [28,49,68]. In this setting, completion models have been proposed by [4,14,18,66], and implementation with different instruments have been proposed by [7,31,55]. They are compatible with the phenomenological model of Mumford [46]. We recall here the neurogeometrical model of boundary completion proposed by [14], and put it in a different perspective. Indeed, we express the equations in term of a function \(\theta\), defined on the 2D plane, which attains at every point \((x, y)\) the value of the orientation of the level lines at that point. The model mimics the ability of simple cells to detect boundaries and level lines of images and to complete missing boundaries. This formulation of the problem will allow in the forthcoming sections to express a unitary Lagrangian model which takes into account boundaries completion and contrast invariance.
The retina can be modelled as a 2D plane, whose points will be denoted by \((x, y)\). Over each retinal point \((x, y)\), the primary visual cortex (V1) implements a whole fibre of cells, each one sensitive to a specific orientation \(\theta(x, y)\). Since the boundaries do not identify a direction, the angle belongs to \([0, \pi]\) = \(P^1\). Then the set of simple cells is identified with the 3D space \(R^2 \times P^1\). A visual stimulus is an image \(I\) defined on the 2D retinal plane. At every point \((x, y)\), the simple cell sensitive to the orientation \(\theta(x, y)\) of the gradient is maximally activated, namely to the value \(\bar{\theta}(x, y)\) such that

\[
\nabla I(x, y) = -\left(\sin(\bar{\theta}(x, y)), -\cos(\bar{\theta}(x, y))\right).
\]

Setting \(\tilde{H}(x, y, \bar{\theta}) = \frac{\nabla I(x, y)}{|\nabla I(x, y)|} + (\sin(\bar{\theta}), -\cos(\bar{\theta}))\), the set of activated cells defines a surface in the 3D cortical space \(R^2 \times S^1\)

\[\Sigma = \left\{ \tilde{H}(x, y, \bar{\theta}) = 0, |\nabla I(x, y)| > C \right\}. \tag{2.12}\]

The condition on the gradient of \(I\) is a threshold, which ensures that the function \(\tilde{H}\) is well defined around boundaries of the image. Since the level lines of an image are orthogonal to the gradient, a unitary tangent vector to the level lines is

\[\left(\cos(\bar{\theta}(x, y)), \sin(\bar{\theta}(x, y))\right), \tag{2.13}\]

where \((x, y, \bar{\theta}(x, y))\) belongs to the surface (Fig. 5).

Simple cells are connected one to the other by the so-called cortico-cortical connectivity. This connectivity is strongly anisotropic, and a cell located at a point \((x, y)\) and sensitive to an orientation \(\theta(x, y)\) of the level lines of the image, mainly propagates in the direction tangent to the level lines. Hence, according to (2.13), the connectivity allows a propagation of the signal in the \(R^2 \times P^1\) along the integral curves of the vector fields

\[X_1 = \cos(\bar{\theta})\partial_x + \sin(\bar{\theta})\partial_y, \quad X_2 = \partial_\vartheta. \tag{2.14}\]

Propagation along the cortical connectivity seems to be at the base of the process of boundary completion. Indeed, the lifted surface \(\Sigma\) is not defined on the whole space, but only over the region where boundaries or level lines are detected. The joint action of orientation detection and cortical propagation along the vector fields completes the surface extending it on the set \(|\nabla I| < C\). In [14], it is shown that this propagation can be expressed as the solution of the minimal surface equation

\[X_1 \left(\frac{X_1 H}{\sqrt{|X_1 H|^2 + |X_2 H|^2}}\right)
+ X_2 \left(\frac{X_2 H}{\sqrt{|X_1 H|^2 + |X_2 H|^2}}\right) = 0, \text{ on } |\nabla I| < C \tag{2.15}\]

with internal boundary condition \(H = \tilde{H}\) on \(|\nabla I| = C\).

This last condition ensures that the existing boundaries are preserved, while the orientations of illusory boundaries or level lines are recovered as the 0 level set of the solution \(H\).

This model performs completion of boundaries, giving rise to illusory contours, and of level lines, giving rise to amodal completion, as in the case of the blind spot. But it is unable to perform filling-in when the level lines of the image are parallel to the missing or occluded regions, as in the case of modal completion of the Kanizsa triangle.

The surface \(\Sigma\), in (2.12), can be identified as the graph of a function \(\theta(x, y)\). It is then possible to express the equation in terms of the function \(\theta\) on the plane \((x, y)\). The problem of implicit functions in this setting has been studied in great generality by [12], and in the special case of this Lie group by [5] (see also, for the Heisenberg case [1]). The vector field \(X_1\) in (2.14) projects on the plane \((x, y)\) to the directional derivative in the direction \((\cos(\theta(x, y)), \sin(\theta(x, y)))\):

\[X_{1\theta} = \cos(\theta(x, y))\partial_x + \sin(\theta(x, y))\partial_y = \langle \nabla, \theta \rangle \cdot (\cos(\theta(x, y)), \sin(\theta(x, y)))\tag{2.16}\]

while the projection of the vector \(X_2\) on the same plane is 0. In analogy to (2.16), we will define a degenerate norm of a vector \(v\) as its projection on the same direction:

\[|v|_\theta = |\langle v, \theta \rangle| = |\langle v, (\cos(\theta), \sin(\theta)) \rangle|\tag{2.17}\]

Hence,

\[|\nabla \theta|_\theta = \left| \langle \nabla \theta, \theta \rangle \right| = |X_{1\theta} \theta|\]

The minimal surfaces equation (2.15) can be expressed as an equation for the graph \(\theta(x, y)\) and it becomes

\[X_{1\theta} \left(\frac{X_{1\theta}(x, y)}{\sqrt{|X_{1\theta}(x, y)|^2 + 1}}\right) = 0. \tag{2.18}\]
This equation can be interpreted as a second-order directional derivative, in the direction \((\cos(\theta), \sin(\theta))\). Formally computing the derivatives, the equation is equivalent to
\[
X_{1\theta}^2 \theta(x, y) = 0. \tag{2.19}
\]
The properties of the solution have been studied in \([5, 10]\). It is easy to check that the second-order equation (2.19) is the Euler-Lagrange equation of the Dirichlet functional
\[
\int |X_{1, \theta \theta}|^2 \, dx \, dy = \int |\nabla \theta(x, y)|^2 \, dx \, dy. \tag{2.20}
\]
This functional has been deeply studied in \([15, 59]\). Hence minima of this functional give rise to the same minimal graphs proposed in \([14]\) for boundary propagation (see also \([5]\) for a more detailed proof).

3 The Gauge Field Model

In this section, we will combine the two models in such a way to provide a solution to the modal completion problem. The two terms corresponding to the retinex and to the boundary completion will be coupled in such a way that the boundary completion term will propagate existing boundaries and the retinex model will recover the objective surface filling-in the new boundaries. Existing boundaries will be the known term for the completion problem and existing and new boundaries will be the known term for the retinex. A system of coupled equations will formalise the model. They will be derived as Euler–Lagrange equations of a suitable Lagrangian functional.

3.1 The Lagrangian

In Fig. 4, we showed that the retinex model is unable to fill-in the Kanizsa triangle. Analogously, the boundary completion models can recover at most boundaries of the triangle (see also the model of Williams and Jacobs \([66]\), or the model of \([67]\)). A previous model of Sarti, Malladi, Sethian \((57)\) does not reproduce the different darkness of the triangle and the background. In the following, we will couple the two models (retinex and neurogeometric) to allow both geometrical shape reconstruction and filling-in of the segmented figures with the perceived contrast. The task will be accomplished by considering the retinex model of Sect. 2.1 as the particle term and the cortical model of Sect. 2.2 as the field term of a classical gauge field theory. In this way, we will obtain an analogue of the classical theory of electromagnetism where both the particle and the fields are the unknowns of the problem. Indeed, instead of Eq. (2.9), we propose a complete Lagrangian, sum of three terms: a particle term, an interaction term and a field term. The Lagrangian will be written in terms of differential forms instead of the usual derivatives.

This choice allows us to obtain a model that is intrinsic, i.e. independent on the choice of any global or local reference system, exactly as for contemporary physical theories. Once this property is clearly established, for simplicity, we express the equations in term of the usual derivatives in the global Cartesian reference system.

The particle term is
\[
L_1 = \int |d\phi - dh|^2 \, dx \, dy \tag{3.1}
\]
and is directly inspired by the retinex model (2.9). It describes the reconstruction of the image from image boundaries. As described above, it implements the perceptual invariance respect to contrast.

The next term describes the interaction between particle and the gauge field \(A\):
\[
L_2 = \int |d\phi - A|^2 \, dx \, dy. \tag{3.2}
\]
In Sect. 3, the gauge field \(A\) described only the existing boundaries, hence it was a priori given, and had the value of \(\nabla h\). Here, we propose a more general model where the field \(A\) is unknown and describes both the existing and illusory boundaries. This term \(L_2\) acts on \(\phi\), and it expresses the reconstruction of the image from the old and new boundaries explaining perceptual figure completion, by keeping contrast invariance properties. At the same time, this term acts on the unknown vector field \(A\), which will be forced to have the direction of \(\nabla \phi\). Hence, it will tend to be orthogonal to the existing boundaries or level lines.

Finally, we impose a field term to the one of classical fields theories:
\[
\tilde{L}_3 = \int |dA|^2 \, dx \, dy. \tag{3.3}
\]
We need to choose here a suitable norm, which expresses the propagation of existing contours and allows the creation of subjective ones. Hence, it will be defined suitably modifying the neurogeometrical model presented in Sect. 2.2. The main difference is that here we minimise the differential of the whole field \(A\) instead of the angle \(\theta\) alone. Since \(A\) is orthogonal to the existing level lines, the direction of propagation \((\cos(\theta(x, y)), \sin(\theta(x, y))\) will be obtained normalising the vector \(A = (-A_y, A_x)\):
\[
(\cos(\theta(x, y)), \sin(\theta(x, y))) = \frac{A}{\sqrt{A_x^2 + A_y^2}} = \frac{(-A_y, A_x)}{\sqrt{A_x^2 + A_y^2}}.
\]
Accordingly we will use the norm (2.17), with the value \(\theta = \theta_A\) inherited by the vector field:
Let us explicitly compute the expression of the square of the norm:

$$|v|^2_{\mathcal{A}} = \left| \langle v, (\cos(\theta(x,y)), \sin(\theta(x,y))) \rangle \right| = \frac{(v, \tilde{A})}{\sqrt{A_x^2 + A_y^2}}.$$ 

Equivalently, if we call

$$G_{\mathcal{A}} = (g^{ij})_{i,j=1,2} = \begin{pmatrix} A_x^2 & -A_x A_y \\ -A_x A_y & A_y^2 \end{pmatrix},$$

then the square of the norm is formally the norm associated to the matrix $G = (g^{ij})$. In this setting, $G$ is not invertible. On the contrary, in the Riemannian setting, $G$ is invertible, and it has the role of the inverse of the metric of the space. When needed, we assume to introduce a small perturbation which makes its determinant non zero:

$$G_{\epsilon \mathcal{A}} = g^{ij}_{\epsilon} = \begin{pmatrix} A_x^2 + \epsilon^2 A_y^2 & -A_x A_y(1-\epsilon^2) \\ -A_x A_y(1-\epsilon^2) & A_y^2 + \epsilon^2 A_x^2 \end{pmatrix},$$

We refer formula (6.1) in the Appendix for the precise definition of differential of a 1-form. Here, we only recall that the differential of $\tilde{A}$ is independent of the norm chosen, and it is the usual curl operator:

$$d \tilde{A} = \nabla \times \tilde{A} = \partial_y A_x - \partial_x A_y.$$ 

Note that $\tilde{A}$ is a 1-form so that $d \tilde{A}$ is a 2-form. In $\mathbb{R}^2$ a 2-form is identified with a 0-form, i.e. with a scalar function. The resulting functional $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ is then

$$\mathcal{L}(h, \phi, \tilde{A}) = \int |d\phi - dh|^2 \, dx \, dy + \int |d\phi - \tilde{A}|^2 \, dx \, dy + \int |d\tilde{A}|^2_{\mathcal{A}} \, dx \, dy.$$ 

### 3.2 Existence of Minima

The functional (3.6) can be represented as

$$\mathcal{L}(h, \phi, \tilde{A}) = \int J(\phi, \tilde{A}, d\phi, d\tilde{A}) \, dx \, dy$$

after calling

$$J(\phi, \tilde{A}, d\phi, d\tilde{A}) = |d\phi - dh|^2 + |d\phi - \tilde{A}|^2 + |d\tilde{A}|^2_{\mathcal{A}}.$$ 

Then the functional $\mathcal{L}(h, \phi, \tilde{A})$ falls within the general theory developed in [25] for studying minima of functionals of calculus of variation. If we assume that the given function $h$ satisfies

$$\int |dh|^2 \, dx \, dy < +\infty,$$

the domain of the functional is the class of functions $(\phi, \tilde{A})$ defined on an open set $\Omega$ such that

$$\int |d\phi|^2 \, dx \, dy + \int |d\tilde{A}|^2_{\mathcal{A}} \, dx \, dy < +\infty.$$ 

(3.7)

This condition defines a Sobolev space $S(\Omega)$, adapted to the considered metric.

We first remark the following property of the functional

**Remark 3.1** The functional $\mathcal{L}(h, \phi, \tilde{A})$ is weakly lower semicontinuous in the space $S(\Omega)$, since for every sequence $(\phi_j, \tilde{A}_j)$ weakly convergent in $S(\Omega)$ to a limit $(\phi, \tilde{A})$ the following condition is satisfied:

$$\mathcal{L}(h, \phi, \tilde{A}) \leq \liminf \mathcal{L}(h, \phi_j, \tilde{A}_j).$$

This is a general property satisfied by the norm of the space. Now we can apply Theorem 2.2 in [25], and deduce the following existence theorem

**Theorem 3.2** The functional $\mathcal{L}(h, \phi, \tilde{A})$, being weakly lower semicontinuous in the Sobolev space $S(\Omega)$, has a minimum.

According to [25], Remark 1.2, if a couple $(\phi, \tilde{A})$ minimises the functional (3.6) without boundary conditions, then it is a solution of the Neumann problem associated to the Euler Lagrange equation (see formulas (3.8a, 3.8b) below).

### 3.3 The Euler Lagrange Equation

In this subsection, we will compute the minimizer of the total Lagrangian functional $\mathcal{L}$. We refer to the Appendix for notation and explicit computation. The Euler Lagrange equation of the functional defined in (3.6) becomes

$$\Delta\phi = \frac{1}{2} (\Delta h + \text{div}(\tilde{A})).$$ 

(3.8a)

$$d^2_d \tilde{A} = -\nabla_\phi \phi + \tilde{A}.$$ 

(3.8b)

with Neumann boundary condition:

$$\langle \nabla \phi, v \rangle_G = 0, \quad \langle \nabla \tilde{A}, v \rangle_G = 0.$$ 

(3.9)

These equations have the meaning inherited by the corresponding terms of the functional: the first term is the particle equation that takes the two boundary terms (the rescaled Laplacian $\Delta h$ of the original image and the contribution
div(\vec{A}) generated by the gauge field \( \vec{A} \) and performs a reconstruction of the image by filling-in objects. Note that the two terms \( L_1 \) and \( L_2 \) which generalise the retinex functional give rise to a unique particle equation.

Note that \( \vec{A} = (A_x, A_y) \) is a vector, hence the equation (3.8b) for \( \vec{A} \) is indeed a system. In the same equation, the directional gradient is defined as

\[
\nabla_A \phi = G \nabla \phi,
\]

where \( G \) is defined in (3.4).

The term \( d^*_{\vec{A}} d \vec{A} \) is a 2-form, which can be identified with \( \nabla \times \text{curl}(\vec{A}) \) (see Appendix A) and precisely

\[
d^*_{\vec{A}} d \vec{A} = \nabla \times \text{curl}(\vec{A}) = -(\nabla_A (\partial_y A_y - \partial_x A_x))_x \, dx
+ (\nabla_A (\partial_x A_y - \partial_y A_x))_y \, dy.
\]

Here \( A_x \) denotes the \( x \) component of \( \vec{A} \), not the derivative. In Appendix A, we provide its explicit expression as sum of four terms, \( L_{\vec{A}} \), \( T_x(\vec{A}) \), \( T_y(\vec{A}) \), \( a \) (see 6.2):

\[
d^*_{\vec{A}} d \vec{A} = \begin{cases}
L_{\vec{A}} A_x - \partial_y a + T_x(\vec{A}), \\
L_{\vec{A}} A_y - \partial_x a + T_y(\vec{A}).
\end{cases}
\]

Precisely, \( L_{\vec{A}} \) is the second-order part of the operator, and is a directional operator associated to the considered metric. \( T_x(\vec{A}) \) and \( T_y(\vec{A}) \) are advection terms, with coefficients depending on the metric, while we will see that it is possible to choose \( a = 0 \).

The field equation on \( \vec{A} \) propagates the gradient of the image in the subriemannian metric, and allows the recovering of existing and subjective boundaries. We explicitly note that the equation is of second order in the variable \( \vec{A} \). In general, functionals of higher order are necessary to obtain completion of curved boundaries (as in the model of elastica [46]). However, here the field \( \vec{A} \) has the role of approximating the gradient of the image, following the Lagrangian \( L_2 \). Hence, its second derivatives express third derivatives of the image function.

### 3.3.1 Invariance Properties and Choice of the Gauge

This system of differential equations seems difficult to discretize. However, (as in all gauge theories) we will see that the functional is invariant with respect to the choice of the gauge. We will select a suitable choice of the gauge which simplifies the system. Precisely, we will prove the following proposition:

**Proposition 3.3** The functional \( \mathcal{L} \) is invariant with respect to the transformations:

\[
h \rightarrow h' = h + f, \quad \phi \rightarrow \phi' = \phi + f, \quad \vec{A} \rightarrow \vec{A}' = \vec{A} + df.
\]

Indeed,

\[
d\phi' - dh' = d\phi - dh,
\]

\[
d\phi' - \vec{A}' = d\phi - \vec{A},
\]

\[
d\vec{A}' = d\vec{A} + df = d\vec{A}
\]

since \( d^2 f = \text{curl} df = 0 \). This implies that the functional assumes the same values on \((h, \phi, \vec{A}) \) and \((h', \phi', \vec{A}') \).

\[
\mathcal{L}(h, \phi, \vec{A}) = \mathcal{L}(h', \phi', \vec{A}').
\]

Since the equation is invariant with respect to the choice of the gauge \( f \), we can freely choose it, and the choice of the gauge will not affect the value of the Lagrangian. Therefore, we will make the choice which decouples and simplifies the system, imposing \( a = 0 \). This is expressed by the following proposition

**Proposition 3.4** It is possible to perform a choice of the gauge \( f \) such that \( a = 0 \) in the system (3.10b)

\[
\begin{align*}
L_{\vec{A}} A_x + T_x(\vec{A}) & = 0, \\
L_{\vec{A}} A_y + T_y(\vec{A}) & = 0.
\end{align*}
\]

Since \( \vec{A} = \vec{A}' - df \), the expression of \( a \) reduces to

\[
a = g^{12} \partial_x (A'_y - \partial_y f) + g^{22} \partial_y (A'_x - \partial_x f)
+ g^{11} \partial_x (A'_x - \partial_x f) + g^{12} \partial_y (A'_x - \partial_x f).
\]

where \( f \) is an arbitrarily chosen gauge function. To obtain \( a = 0 \), we choose the function \( f \) as a solution of

\[
\begin{align*}
g^{22} \partial_y y f & + g^{11} \partial_x x f + 2g^{12} \partial_x y f = g^{12} \partial_x A'_y
+ g^{22} \partial_y A'_x + g^{11} \partial_x A'_x + g^{12} \partial_y A'_x.
\end{align*}
\]

that is a second-order subriemannian differential equation.

With this choice of the gauge, the second-order term of the system reduces to the simpler form

\[
d^*_{\vec{A}} d \vec{A} = \begin{cases}
L_{\vec{A}} A_x + T_x(\vec{A}) \\
L_{\vec{A}} A_y + T_y(\vec{A}).
\end{cases}
\]

where

\[
L_{\vec{A}} f = \frac{A_x^2 \partial_{xx} f - 2A_y A_x \partial_{xy} f + A_y^2 \partial_{yy} f}{A_x^2 + A_y^2}
\]

\[
T_x(\vec{A}) = \partial_x g^{11} \partial_x A_x + \partial_x g^{12} \partial_y A_x + \partial_x g^{21} \partial_x A_y + \partial_y g^{12} \partial_x A_y,
\]

\[
T_y(\vec{A}) = \partial_y g^{21} \partial_x A_y + \partial_y g^{22} \partial_y A_y + \partial_y g^{11} \partial_x A_y + \partial_y g^{12} \partial_y A_x.
\]

In conclusion, we can rewrite the Euler Lagrange equation (3.8a, 3.8b) in the form

\[
\begin{cases}
\Delta \phi = \frac{1}{2} (\Delta h + \text{div}(\vec{A})) \\
L_{\vec{A}} A + T(\vec{A}) = -\nabla_A \phi + \vec{A}.
\end{cases}
\]

### 3.3.2 Nonlinearity of the Equation

Let us explicitly note that (3.11) is a system of quasilinear differential equations, meaning that the system is linear in the second-order term, and the nonlinearity is in the first order.
terms only. We will focus here on the second Eq. (3.11), and look for a solution. We will first approximate the metric (3.4) with the Riemannian approximation (3.5), so that in Eq. (3.11), we replace the each coefficient $g^{ij}$ with the Riemannian approximation $\hat{g}^{ij}$. A typical way for studying a non-linear equation is to linearize it, meaning that we choose an initial approximated solution $\tilde{A}_0$, and iteratively find a better and better solution. A starting point for the procedure is the solution of the vector Laplace equation

\[ \Delta \tilde{A}_0 = \nabla \phi. \]  

(3.12)

Of course this is only an approximated solution $\tilde{A}_0$, but we can recover a better one $\tilde{A}_1$ as a solution of

\[ d^\ast A_0 \, d \tilde{A}_1 = \nabla_x \phi + \tilde{A}_0, \]  

(3.13)

using the approximated Riemannian operator associated to $A_0$. Recall that $\nabla_{\tilde{A}_0} = G \nabla$. From here we start an iteration:

\[ d^\ast A_1 \, d \tilde{A}_2 = \nabla_{\tilde{A}_1} \phi + \tilde{A}_1, \cdots d^\ast_{A_{\ell-1}} \, d \tilde{A}_j = \nabla_{A_{\ell-1}} \phi + \tilde{A}_j. \]  

(3.14)

At each step, we get a better approximation of the solution, moreover the sequence has a limit $\tilde{A} = \lim_{j \to +\infty} \tilde{A}_j$. Passing to the limit in the previous expression, we will get

\[ d^\ast A \, d \tilde{A} = \nabla \phi + \tilde{A}, \]  

so that the limit provides a solution of the non-linear equation. Precisely, this construction can be expressed by the following proposition:

**Proposition 3.5** For $\phi$ fixed, the solution of the non-linear equation (3.11) can be obtained as the limit of a sequence $(\tilde{A}_j)$ defined by recourence by the Eqs. (3.12) and (3.14).

**Proof** We give a sketch of the proof, which follows the same lines of [42] and [11]. First we note that the solution $(\tilde{A}, \phi)$ is bounded for [44]. As a consequence, for the special structure of our system, the operator which associates to $\tilde{A}_j$ the solution $\tilde{A}$ of $d^\ast A_j \, d \tilde{A} = \nabla_{A_j} \phi + \tilde{A}$ is a compact operator from the set $C^{1,\alpha}(\Omega, \mathbb{R}^3)$ of functions with H"older continuous first derivatives. This implies that the sequence $(\tilde{A}_j)$ has a subsequence converging to the minimum of the functional, which is unique, since it is convex. This ensures that from each subsequence $(\tilde{A}_{j_k})$ of $(\tilde{A}_j)$ we can extract a convergent subsequence, and that all these subsequences have with the same limit. This ensures that the sequence itself $(\tilde{A}_j)$ is convergent to the minimum. \hfill \qedsymbol

### 3.4 Numerical Approximation and Solution Computation

We have seen that an approximated solution can be found solving iteratively linear differential equation. Hence, the implementation of the algorithm consists in solving sequentially the system of coupled differential equations (3.11) with Neumann boundary conditions. A suitable discretization of the spatial domain is provided. We consider a rectangular uniform grid $(x, y)$ with points $(x_i, y_j) = (i \Delta x, j \Delta y)$. Following standard notation, we denote by $f^{ij}$ the value of the function $f(x, y)$ at the grid point $(x_i, y_j)$.

We assume that the field $\tilde{A}$ is uniformly 0, and start to solve the particle equation (3.11), which reduces to

\[ \Delta \phi = \frac{1}{2} \Delta h. \]  

(3.15)

It is a standard Poisson equation since $\Delta h$ is known, given $h$ the original image.

Equation (3.15) is approximated by finite differences by

\[ (D_{xx} + D_{yy}) \phi^{ij} = \frac{1}{2} (D_{xx} + D_{yy}) h^{ij}, \]  

(3.16)

where $D$ is the standard centred finite difference operator. To find $\phi^{ij}$, the linear system can be solved by standard linear solvers for sparse matrices.

Then we solve the field equation (3.11) to perform boundary propagation.

As we explained in the previous section, this equation will be solved iteratively by subsequent linearizations. First we compute $\tilde{A}_0$ using Eq. (3.12)

\[ \Delta \tilde{A}_0 = \nabla \phi, \]  

(3.17)

then we iterate and compute $\tilde{A}_1$ as a solution of

\[ L_{\tilde{A}_0} \tilde{A}_1 + \tilde{T}(\tilde{A}_1) = \nabla \phi + \tilde{A}_1. \]  

(3.18)

The vector equation (3.17) can be approximated by finite differences:

\[ \begin{cases} (D_{xx} + D_{yy}) A^{ij}_{0y} = \phi^{ij}_x \\ (D_{xx} + D_{yy}) A^{ij}_{0x} = \phi^{ij}_y \end{cases}, \]  

(3.19)

where $\phi^{ij}_x = D_x \phi^{ij}$ and $\phi^{ij}_y = D_y \phi^{ij}$ are known terms, since $\phi^{ij}$ has been computed in the previous step.

When applied to the Kanizsa inducers, the solution $\tilde{A}_0$ is visualised in Fig. 7, where the triangle inducers have been manually selected (Fig. 6).

Equation (3.18) is a linear degenerate equation since $L_{\tilde{A}_0}$ is a directional Laplacian. If we approximate the matrix $G_{\tilde{A}_0}$ with its Riemannian approximation $G_{\tilde{A}_0}$, it becomes elliptic, providing the missing regularity. Then the linear elliptic differential equation

\[ L_{\tilde{A}_0} \tilde{A}_1 + \tilde{T}(\tilde{A}_1) = \nabla \phi + \tilde{A}_1 \]  

(3.20)
This is a version of the retinex equation that is able to reconstruct the original image together with the subjective surface. In Fig. 9 (left), the forcing term \( \frac{1}{2}(\Delta h + \text{div}(A_1)) \) of the particle equation is visualised while in Fig. 9 (right) the solution \( \phi \) is shown.

### 3.5 A Neural Implementation in LGN and Cortex

The modal completion problem clearly has a psychophysical origin, and the proposed model is phenomenological more than neurophysiological. Note that both the neurogeometrical completion model and the retinex model are biologically inspired, but that a proved neurophysiological implementation is still missing. Nevertheless a possible cortical implementation will be presented.

In order to further support our model, we will discuss how the different terms of the Lagrangian can be implemented in neurophysiological structures.

We recall that the visual signal is first elaborated by the retina whose receptive profiles are well modelled by the classical Laplacian of the Gaussian (see [16]):

\[
\Delta G_\sigma = \Delta e^{-x^2/2\sigma^2},
\]

where \( \Delta \) is the standard Laplacian. Always in [16], it is observed that the same receptive profiles are found in the Lateral Geniculate Nucleus (LGN), which is a copy of the retina but strictly in contact with the visual cortex. The action of these receptive fields on the visual signal can be represented (see for example [49]) as the output of the neural cell

\[
\Delta G_\sigma * \log I = \Delta h,
\]

where \( G_\sigma * \log I \) is a smoothed version of the \( \log I \) and the logarithmic function is due to the nonlinearity of the cell response.

The output of LGN cells is propagated via the horizontal connectivity in the LGN. Since this connectivity is isotropic...
Fig. 8 The $x$ and $y$ components of the gauge field $A_1$ related to the Kanizsa triangle inducers. $A_1$ is an approximation of the field $A$, which is a solution of the gauge field equation (3.7b).

Fig. 9 Left The forcing term $\frac{1}{2}(\Delta h + \text{div}(A_1))$ of the particle equation. Right The reconstructed Kanizsa triangle as the solution $\phi$ of the particle equation.

(see for example [64]), it can be modelled with the fundamental solution

$$\Gamma(x, y) = \log \left( |(x, y)| \right)$$

of the 2D Laplacian operator. LGN horizontal connectivity with strength $\Gamma(x, y)$ acts linearily on the feedforward input $h$, giving a total contribution

$$\phi = \frac{1}{2} (\Gamma \ast \Delta h).$$

This is exactly the solution of the Laplacian equation (3.15) of the particle term, implementing the reconstruction of the image from the boundaries. Note that the action of receptive profiles $\Delta h(x, y)$ and the one of LGN horizontal connectivity $\Gamma(x, y)\ast$ is dual in a differential sense.

The gauge field equation in $\vec{A}$ performs boundary propagation and we will conjecture now how it is implemented at the cortical level. Simple cells perform stimulus differentiation $\nabla \phi$ that is propagated by horizontal connectivity in the direction of the stimulus orientation [3,8]. For this reason, horizontal connectivity can be modelled by the fundamental solution of the vector Laplacian $\vec{\Gamma}(x, y)$ and the total connectivity excited by the stimulus can be accounted as

$$\vec{A}_0 = \vec{\Gamma} \ast \nabla \phi.$$

This is equivalent to say that $\vec{A}_0$ is a solution of equation (3.17). Now, the feedforward output of simple cells $\nabla \phi$ is propagated by the excited connectivity $\vec{A}_0$ generating the distribution $\vec{A}_1$, the solution of Eq. (3.18):

$$L_{\vec{A}_0} \vec{A}_1 + T(\vec{A}) = \nabla \phi.$$

We have shown in [14] (see also [17]) that $\vec{A}_1$ is the field tangent to the perceptual association fields measured by Field, Heyes and Hess in [21], and it is cortically implemented by means of horizontal connectivity propagation.

Finally, the forcing term $\vec{A}$ in the particle equation can be interpreted as the feedback of the cortical processing to LGN, showing the strength of the gauge field theory in coupling the activity of different physiological layers. Equation (3.22) is again the retinex equation but with the feedback from V1 that takes into account illusory boundaries.
4 Results

In this section, we test the algorithm on a number of classical Gestalt images and visualise the results.

We first consider the classical Kanizsa triangle with curved boundaries (see Fig. 1), and its completion shown in Fig. 9. The boundaries are correctly completed, and the retinex part of the algorithm gives the pop up effect of the triangle. This result has been obtained after one iteration alternatively solving the particle and the field equations. In this case and in all the following examples, we observed that one iteration is sufficient to obtain a satisfactory solution from the perceptual point of view.

The same effect is reproduced on a Kanizsa pentagon (see Fig. 10) and on a non-symmetric figure (see Fig. 11).

Then we test the algorithm on varying the contrast polarity of the image. In Fig. 12, we show that the algorithm complete the Kanizsa triangle when simply reverting the contrast. The triangle surface appears here brighter than both Pacman and background.

In Fig. 13, we consider a Kanizsa square with black and white Pacman on a grey background, in such a way to have two positive and two negative contrasts with respect to the background. In this case, phenomenological experiments show that modal completion is still present but associated to a reduced pop up effect (see [62]). The algorithm performs correctly modal completion and the pop up effect in this case is less evident, in good agreement with psychophysical evidence.

In Fig. 14 the Pacman angles are varied, so that to produce completed triangles with different boundary curvatures.

We consider here a couple of images (Figs. 15 and 16) which clarify the problem of fragmentation of subjective boundaries and surfaces. In these cases, an observer is not able to perceive inflections in the emergent contours (see [61]). If the Pacman inductor is misaligned, the figure "fragments". This phenomenon is well reproduced by the completion results of our model.

In the Koffka cross images (Figs. 17 left and 18 left) we considered as inducers the gradient of the termination of the existing lines (Figs. 17 middle and 18 middle). If the lines are closeby, a disc is reconstructed (Fig. 17 right), while a square is obtained if the lines are far apart (Fig. 18 right).
Fig. 12 The Kanizsa triangle with inverted contrast polarity (left) and its completion via our algorithm (right).

Fig. 13 The Kanizsa square with two positive and two negative contrasts polarity with respect to the background (left) and its completion via our algorithm (right).

Fig. 14 Different Kanizsa triangles by varying the Pacman angles (above) and their completion via this algorithm (below).
Fig. 15 The Kanizsa Triangle with misaligned inducers (left), and the output of our algorithm (right): the completed figure is fragmented, which is in good agreement with the experimental findings that one is not able to perceive inflections in the emergent contours.

Fig. 16 The Kanizsa square with misaligned inducers (left), and the output of this algorithm (right): it is obtained a fragmented figure which is compatible with experimental evidence (see [61]).

Fig. 17 The Koffka cross image (left). Since in this model we consider only the role of the gradient in the reconstruction of subjective boundaries, and not the orthogonal component, we consider as inducers the gradient of the termination of the existing lines (middle image). The reconstruction via this algorithm is shown in the right image.

Finally we show an example of self occluding image, the famous paisley Kanizsa image. The result clearly shows the modal completion of the ‘head’ over the “tails” of the paisley figure. The modally completed heads appear brighter and induce a depth effect (Fig. 19).

Note that an important question raised up in the contemporary phenomenological debate deals with the nature of modal and amodal completion. Particularly Kellman and Shipley [35] claimed a substantial identity of the processes underlying modal and amodal completion. In the opposite Anderson
et al. argued against the 'identity hypothesis' observing a fundamental difference between modal and amodal completion [2]. They observed that when pure modal completion is present, like in case of camouflage, apparent depth is consistent with inferred contrast, while in presence of modal and amodal completion (object occlusion), like in the Kanizsa triangle, depth is not necessarily consistent with inferred contrast and depends on other geometrical constraints. Of course, we are very prudent to enter in this delicate and important debate, but nevertheless, we cannot avoid to observe that results obtained applying our field model to Kanizsa triangle with opposite contrast polarity seem to to be compatible with the Anderson et al. position. If we look at Figs. 9 and 12, where the Kanizsa triangle with opposite contrast polarity has been reconstructed, it is easy to check that the triangle appears above the Pacman in both contrast situations. In this experiment, the depth is invariant with respect to contrast polarity change, corroborating the idea that geometrical constraints are more important than contrast to set up depth order.

5 Conclusions

In this paper, we made the effort to construct a formal field theory of low level vision. Contemporary instruments of field theory based on gauge invariance have been used to introduce a complete Lagrangian with its particle, interaction and field terms. The Lagrangian couples two well-known models for brightness and boundary propagation: the retinex and the neurogeometrical models. Particularly, the problem of modal completion of illusory figures is faced, and it is shown how the Euler–Lagrange field equations well represent the process of constitution of the Kanizsa examples of modal completion with curved boundaries. In addition, the model can be considered as a plausible model for the interaction between different structures of the visual systems, particularly regarding the coupling between the activity of LGN and the visual cortex. The gauge Lagrangian formulation seems to be strong enough to describe both the feedforward and the feedback processes of low level vision by keeping the desired invariance.
6 Appendix

We rapidly recall here the definition of differential in the Riemannian setting, in the special case where $\det(G)$ is a constant. We will call $G = g^{-1}$ since $G$ plays the role of inverse of the metric. Then the Riemannian scalar product is

$$\langle v, w \rangle_g = \langle g v, w \rangle.$$ 

If $a$ is a function then we will denote $da$ the usual differential, whose components are $\nabla a = \langle \partial_x a, \partial_y a \rangle:
\begin{align*}
da = \partial_x a \, dx + \partial_y a \, dy.
\end{align*}

The gradient of the function $a$ in the metric $g$ is defined as

$$\nabla_g a = G \nabla a = \begin{pmatrix} g^{11} \partial_x a & g^{12} \partial_y a \end{pmatrix}.$$ 

In the sequel, we will denote its components as $((\nabla_g a)_x, (\nabla_g a)_y)$. The Laplacian is expressed as

$$\Delta_g a = \text{div}(\nabla_g a).$$

If $\vec{A} = A_x \, dx + A_y \, dy$, then

$$d\vec{A} = \text{curl}(\vec{A}) = \langle \partial_x A_y - \partial_y A_x \rangle \, dx \wedge dy$$

and the Laplacian is not the Laplacian of the two components in general but can be expressed in terms of the $d$ and $d^*$ operators, which we will now define. Since $d\vec{A}$ is a 2-form, first recall that for a general 2-form $\beta = b dx \wedge dy$,

$$d^* \beta = - (\nabla_g b)_x \, dx + (\nabla_g b)_y \, dy.$$ 

Having defined $d^* \vec{A}$, we will now explicitly compute the expression of $d^*d\vec{A}$ in a metric $G$.

**Proposition 6.1** $d^*d\vec{A}$ can be expressed as

$$d^*d\vec{A} = \begin{cases} L_g A_x - \partial_x a + T_x(\vec{A}) \\ L_g A_y - \partial_y a + T_y(\vec{A}) \end{cases},$$

where

$$L_g A_i = \langle \nabla_g \partial_x A_i, \nabla_g \partial_y A_i \rangle_y + \langle \nabla_g \partial_y A_i, \nabla_g \partial_x A_i \rangle_x$$

$$T_x(\vec{A}) = \partial_x g^{11} \partial_x A_x + \partial_x g^{12} \partial_y A_x + \partial_x g^{21} \partial_x A_y + \partial_x g^{22} \partial_y A_y$$

$$T_y(\vec{A}) = \partial_y g^{11} \partial_x A_y + \partial_y g^{12} \partial_y A_y + \partial_y g^{21} \partial_x A_x + \partial_y g^{22} \partial_y A_x$$

$$a = g^{12} \partial_x A_y + g^{22} \partial_y A_y + g^{11} \partial_x A_x + g^{12} \partial_y A_x.$$ 

(6.2)

By definition,

$$d^*d\vec{A} = \nabla_g^\perp \text{curl}(a) = - \langle \nabla_g (\partial_x A_y - \partial_y A_x) \rangle_x \, dx$$

(6.3)

This is a 1-form with components

$$d^*d\vec{A} = \begin{cases} \langle \nabla_g (\partial_x A_x - \partial_y A_y) \rangle_y \\ \langle \nabla_g (\partial_y A_x - \partial_x A_y) \rangle_x \end{cases}$$

(6.4)

For future use, it will be simpler to keep distinct the second-order terms of the Laplacian from the first order ones. Hence we call, for $i = x, y$

$$L_g A_i = \langle \nabla_g \partial_x A_i, \nabla_g \partial_y A_i \rangle_y + \langle \nabla_g \partial_y A_i, \nabla_g \partial_x A_i \rangle_x.$$ 

The expression (6.4) then becomes

$$d^*d\vec{A} = \begin{cases} L_g A_x - \partial_x a + T_x(\vec{A}) + L_g A_y - \partial_y a + T_y(\vec{A}) = 0 \end{cases}$$

(6.5)

Recalling that $[\nabla_g, \partial_x] = [\nabla_g, \partial_y]$, we get

$$L_g A_x + [\partial_x, \langle \nabla_g \rangle_x] A_x + [\partial_y, \langle \nabla_g \rangle_y] A_y - \partial_x (\langle \nabla_g A_x \rangle_x + \langle \nabla_g A_y \rangle_y) = 0$$

$$d^*d\vec{A} = \begin{cases} L_g A_x + [\partial_x, \langle \nabla_g \rangle_x] A_x + [\partial_y, \langle \nabla_g \rangle_y] A_y - \partial_x (\langle \nabla_g A_x \rangle_x + \langle \nabla_g A_y \rangle_y) = 0 \end{cases}$$

This concludes the proof of the proposition, since

$$\begin{align*}
[\partial_x, \langle \nabla_g \rangle_x] A_x + [\partial_y, \langle \nabla_g \rangle_y] A_y &= \partial_x g^{11} \partial_x A_x + \partial_y g^{12} \partial_y A_x + \partial_x g^{12} \partial_y A_y + \partial_y g^{22} \partial_y A_y \\
[\partial_y, \langle \nabla_g \rangle_y] A_y + [\partial_x, \langle \nabla_g \rangle_x] A_x &= \partial_y g^{22} \partial_y A_y + \partial_x g^{11} \partial_x A_x + \partial_y g^{12} \partial_x A_x + \partial_x g^{12} \partial_y A_x \\
\langle \nabla_g A_x \rangle_x + \langle \nabla_g A_y \rangle_y &= g^{12} \partial_x A_y + g^{22} \partial_y A_y + g^{11} \partial_x A_x + g^{12} \partial_y A_x.
\end{align*}$$

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