A Comparison between the Zero Forcing Number and the Strong Metric Dimension of Graphs

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January 15, 2014

Abstract

The zero forcing number, $Z(G)$, of a graph $G$ has recently become an interesting graph parameter studied in its own right since its introduction by the “AIM Minimum Rank-Special Graphs Work Group”. The strong metric dimension, $sdim(G)$, of a graph $G$ is the minimum among cardinalities of all strong resolving sets: $S \subseteq V(G)$ is a strong resolving set if for any $u, v \in V(G)$, there exists an $x \in S$ such that either $u$ lies on an $x-v$ geodesic or $v$ lies on an $x-u$ geodesic. In this paper, we prove that $Z(G) \leq sdim(G)$ when $G$ is a tree or a unicyclic graph, and we characterize trees $T$ attaining $Z(T) = sdim(T)$. It is easy to see that $sdim(T + e) - sdim(T)$ can be arbitrarily large for a tree $T$; we prove that $sdim(T + e) \geq sdim(T) - 2$ and show that the bound is sharp. We conclude with the open problem of seeking a refinement to $Z(G) \leq sdim(G) + 3r(G)$, an inequality proven in the final section and where $r(G)$ is the cycle rank of $G$.

Key Words: zero forcing number, strong metric dimension, tree, unicyclic graph

2010 Mathematics Subject Classification: 05C12, 05C50, 05C05, 05C38

1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, undirected, and connected graph of order $|V(G)| \geq 2$. The path cover number, $P(G)$, of $G$ is the minimum number of vertex disjoint paths, occurring as induced subgraphs of $G$, that cover all the vertices of $G$. The degree $\deg_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to the vertex $v$ in $G$; a leaf (or pendant) is a vertex of degree one. We denote the number of leaves of $G$ by $\sigma(G)$. For $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph induced by $S$. The distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of the shortest path in $G$ between $u$ and $v$. We omit $G$ when ambiguity is not a concern.

The notion of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced by the aforementioned “AIM group” in \cite{1} to bound the minimum rank of graphs. Let each vertex of a graph $G$ be given one of two colors, dubbed “black” and “white” by convention. Let $S$ denote the (initial) set of black vertices of $G$. The color-change rule converts the color of a vertex from white to black if the white vertex $u_2$ is the only white neighbor of a black vertex $u_1$; we say “$u_1$ forces $u_2$” in this case. The set $S$ is said to be a zero forcing set of $G$ if all vertices of $G$ will be turned black after finitely many applications of the color-change rule. The zero forcing number, $Z(G)$, of $G$ is the minimum of $|S|$, as $S$ varies over all zero forcing sets of $G$. 

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Since its introduction by the “AIM group”, zero forcing number has become a graph parameter studied for its own sake, as an interesting invariant of a graph. For example, for discussions on the number of steps it takes for a zero forcing set to turn the entire graph black (the graph parameter has been named the iteration index or the propagation time of a graph), see [3] and [12]. In [13], a probabilistic interpretation of zero forcing in graphs is introduced. It’s also noteworthy that physicists have independently studied the zero forcing parameter, referring to it as the graph infection number, in conjunction with the control of quantum systems (see [3], [1], and [19]).

A vertex \(x \in V(G)\) resolves a pair of vertices \(u, v \in V(G)\) if \(d(u, x) \neq d(v, x)\). A vertex \(x \in V(G)\) strongly resolves a pair of vertices \(u, v \in V(G)\) if \(u\) lies on an \(x - v\) geodesic or \(v\) lies on an \(x - u\) geodesic. A set of vertices \(W \subseteq V(G)\) (strongly) resolves \(G\) if every pair of distinct vertices of \(G\) is (strongly) resolved by some vertex in \(W\); then \(W\) is called a (strong) resolving set of \(G\). For an ordered set \(W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)\) of distinct vertices, the metric representation of \(v \in V(G)\) with respect to \(W\) is the \(k\)-vector \(D_G(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))\). The metric dimension of \(G\), denoted by \(\text{dim}(G)\), is the minimum among cardinalities of all resolving sets of \(G\). The strong metric dimension of \(G\), denoted by \(\text{sdim}(G)\), is the minimum among cardinalities of all strong resolving sets of \(G\).

Metric dimension was introduced by Slater [20] and, independently, by Harary and Melter [11]. Applications of metric dimension can be found in robot navigation [15], sonar [20], combinatorial optimization [18], and pharmaceutical chemistry [5]. Strong metric dimension was introduced by Sebő and Tannier [18]; they observed that if \(W\) is a strong resolving set, then the vectors \(\{D_G(v|W)\mid v \in V(G)\}\) uniquely determine the graph \(G\) (also see [14] for more detail); whereas for a resolving set \(U\) of \(G\), the vectors \(\{D_G(v|U)\mid v \in V(G)\}\) may not uniquely determine \(G\). It is noted that determining the (strong) metric dimension of a graph is an NP-hard problem (see [10], [16]).

In this paper, we initiate a comparative study between the zero forcing number and the strong metric dimension of graphs. The zero forcing number and the strong metric dimension coincide for paths \(P_n\), complete graphs \(K_n\), complete bi-partite graphs \(K_{s,t}\) (\(s + t \geq 3\)), for examples; they are \(1, n - 1\), and \(s + t - 2\), respectively. The Cartesian product of two paths shows that zero forcing number can be arbitrarily larger than strong metric dimension; cycles \(C_n\) show that strong metric dimension can be arbitrarily larger than zero forcing number. We prove the sharp bound that \(Z(G) \leq \text{sdim}(G)\) when \(G\) is a tree or a unicyclic graph, and we characterize trees \(T\) attaining \(Z(T) = \text{sdim}(T)\). It is easy to see that \(\text{sdim}(T + e) = \text{sdim}(T)\) can be arbitrarily large for a tree \(T\); we prove that \(\text{sdim}(T + e) \geq \text{sdim}(T) - 2\) and show that the bound is sharp. In the final section, we show, for any graph \(G\) with cycle rank \(r(G)\), that \(Z(G) \leq \text{sdim}(G) + 3r(G)\) and pose an open problem pertaining to its refinement.

## 2 The zero forcing number and the strong metric dimension of trees

In this section, we show that \(Z(T) \leq \text{sdim}(T)\) for a tree \(T\), and we characterize trees \(T\) satisfying \(Z(T) = \text{sdim}(T)\). We first recall some results that will be used here.

**Theorem 2.1.** Let \(T\) be a tree. Then

(a) [1] \(Z(T) = P(T)\),

(b) [18] \(\text{sdim}(T) = \sigma(T) - 1\).

**Theorem 2.2.** [17] Let \(G\) be a graph with cut-vertex \(v \in V(G)\). Let \(V_1, V_2, \ldots, V_k\) be the vertex sets for the connected components of \(V(G) - \{v\}\), and for \(1 \leq i \leq k\), let \(G_i = \langle V_i \cup \{v\} \rangle\). Then \(Z(G) \geq |\sum_{i=1}^{k} Z(G_i)| - k + 1\).
The following terminology are defined for a graph $G$. A vertex of degree at least three is called a major vertex. A leaf $u$ is called a terminal vertex of a major vertex $v$ if $d(u, v) < d(u, w)$ for every other major vertex $w$. The terminal degree, $\text{ter}(v)$, of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ is an exterior major vertex if it has positive terminal degree. An exterior degree two vertex is a vertex of degree 2 that lies on a shortest path from a terminal vertex to its major vertex, and an interior degree two vertex is a vertex of degree 2 such that the shortest path to any terminal vertex includes a major vertex.

**Theorem 2.3.** [9] Let $T$ be a tree. Then

(a) $\dim(T) \leq Z(T)$,

(b) $\dim(T) = Z(T)$ if and only if $T$ has no interior degree two vertex and each major vertex $v$ of $T$ satisfies $\text{ter}(v) \geq 2$.

It is shown in [9] that $P(T) \leq \sigma(T) - 1$; this and Theorem 2.4 imply the following

**Theorem 2.4.** For any tree $T$, $Z(T) \leq \text{sdim}(T)$.

Next, we characterize trees $T$ satisfying $Z(T) = \text{sdim}(T)$.

**Theorem 2.5.** For any tree $T$, we have $Z(T) = \text{sdim}(T)$ if and only if $T$ has an interior degree two vertex on every $v_i - v_j$ path, where $v_i$ and $v_j$ are major vertices of $T$.

**Proof.** ($\Rightarrow$) Suppose that there exist a pair of major vertices, say $v_1$ and $v_2$, in $T$ such that no interior degree two vertex lies in the $v_1 - v_2$ path (i.e., $v_1v_2 \in E(T)$). We consider two disjoint subtrees $T_1, T_2 \subseteq T$ such that $v_1 \in V(T_1)$, $v_2 \in V(T_2)$, $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{v_1v_2\}$. By Theorem 2.4 $P(T_1) \leq \sigma(T_1) - 1$ and $P(T_2) \leq \sigma(T_2) - 1$. So, $P(T) = P(T_1) + P(T_2) \leq \sigma(T_1) + \sigma(T_2) - 2 = \sigma(T) - 2$, i.e., $Z(T) \leq \text{sdim}(T) - 1$.

($\Leftarrow$) We will induct on $m(T)$, the number of major vertices of the tree $T$. If $m(T) = 0$, then $Z(T) = 1 = \text{sdim}(T)$; if $m(T) = 1$, then $Z(T) = P(T) = \sigma(T) - 1 = \text{sdim}(T)$. Suppose the statement holds for all trees $T$ with $2 \leq m(T) \leq k$. Let $x$ be a degree 2 vertex lying between two major vertices $u$ and $v$ of a tree $T$ with $m(T) = k + 1$. Let $\ell$ and $r$ be the two edges of $T$ incident with $x$, and denote by $T_{\ell}$ ($T_r$, resp.) the subtree of $T - r$ ($T - \ell$, resp.) containing $x$. Clearly, $T$ is the vertex sum of $T_\ell$ and $T_r$ at the vertices being labeled $x$. The induction hypothesis applies to $T_\ell$ and $T_r$, since each has at most $k$ major vertices; thus, $Z(T_\ell) = \sigma(T_\ell) - 1$ and $Z(T_r) = \sigma(T_r) - 1$. Now by Theorem 2.4 $Z(T) \geq (Z(T_\ell) + Z(T_r)) - 1 = (\sigma(T_\ell) - 1 + \sigma(T_r) - 1) - 1 = \sigma(T) - 1 = \text{sdim}(T)$; thus, by Theorem 2.4 $Z(T) = \text{sdim}(T)$.

**Remark 2.6.** Notice $\dim(T) \leq Z(T) \leq \text{sdim}(T)$ by Theorem 2.3(a) and Theorem 2.4, where the equalities are characterized by Theorem 2.3(b) and Theorem 2.4.

3 The zero forcing number and the strong metric dimension of unicyclic graphs

A graph is unicyclic if it contains exactly one cycle. Notice that a connected graph $G$ is unicyclic if and only if $|E(G)| = |V(G)|$. By $T + e$, we shall mean a unicyclic graph obtained from a tree $T$ by attaching the edge $e$ joining two non-adjacent vertices of $T$. In this section, we show that $Z(G) \leq \text{sdim}(G)$ for a unicyclic graph $G$ and the bound is sharp. We first recall some results that will be used here.

We say that $x \in V(G)$ is maximally distant from $y \in V(G)$ if $d_G(x, y) \geq d_G(z, y)$, for every $z \in N_G(x) = \{v \in V(G) \mid xv \in E(G)\}$. If $x$ is maximally distant from $y$ and $y$ is maximally distant from $z$, then we say that $x$ and $y$ are mutually maximally distant and denote this by $x$ MMD $y$. It is pointed out in [10] that if $x$ MMD $y$ in $G$, then any strong resolving set of $G$ must contain either $x$ or $y$. Noting that any two distinct leaves of a graph $G$ are MMD, we have the following
Observation 3.1. For any connected graph \( G \), all but one of the \( \sigma(G) \) leaves must belong to any strong resolving set of \( G \).

Theorem 3.2. Let \( G \) be a connected graph of order \( n \geq 2 \). Then

(a) \( Z(G) - 1 \leq Z(G + e) \leq Z(G) + 1 \) for \( e \in E(G) \), where \( G \) denotes the complement of \( G \),

(b) \( \text{sdim}(G) = 1 \) if and only if \( G = P_n \).

Proposition 3.3. Let \( T \) be a tree of order at least three. Then \( \text{sdim}(T + e) \geq \text{sdim}(T) - 2 \) for \( e \in E(T) \), and the bound is sharp.

Proof. Since \( \sigma(T) - 2 \leq \sigma(T + e) \leq \sigma(T) \), the desired inequality follows from Theorem 2.1(b) and Observation 3.1. For the sharpness of the bound, let \( T \) be the “comb” with \( k \geq 4 \) exterior major vertices (see Figure 1). Then \( \text{sdim}(T) = \sigma(T) - 1 = k + 1 \). Since \( \{ \ell_i \mid 1 \leq i \leq k - 1 \} \) forms a strong resolving set for \( T + e \), \( \text{sdim}(T + e) \leq k - 1 = \text{sdim}(T) - 2 \); thus \( \text{sdim}(T + e) = \text{sdim}(T) - 2 \).

Figure 1: Unicyclic graph \( T + e \) satisfying \( \text{sdim}(T + e) = \text{sdim}(T) - 2 \)

Remark 3.4. We note that \( \text{sdim}(T + e) - \text{sdim}(T) \) can be arbitrarily large. For example, suppose that \( T = P_n \) and \( T + e = C_n \); then \( \text{sdim}(T) = 1 \) and, as noted in [10], \( \text{sdim}(C_n) = \left[ \frac{n}{2} \right] \).

Theorem 2.4, Theorem 3.2(a), and Proposition 3.3 imply that \( Z(T + e) \leq \text{sdim}(T + e) + 3 \). We will show that, in fact, \( Z(T + e) \leq \text{sdim}(T + e) \).

As defined in [2], a partial \( n \)-sun is the graph \( H_n \) obtained from \( C_n \) by appending a leaf to each vertex in some \( U \subseteq V(C_n) \), and a segment of \( H_n \) refers to any maximal subset of consecutive vertices in \( U \). By a generalized partial \( n \)-sun, we shall mean a graph obtained from \( C_n \) by attaching a finite, and not necessarily equal, number of leaves to each vertex \( v \in V(C_n) \). See Figure 2.

Figure 2: A partial 6-sun and a generalized partial 6-sun

Theorem 3.5. [17] Let \( H_n \) be a partial \( n \)-sun with segments \( U_1, U_2, \ldots, U_t \). Then

\[
Z(H_n) = \max \left\{ 2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil \right\}.
\]

Corollary 3.6. Let \( H_n \) be a partial \( n \)-sun. Then \( Z(H_n) \leq \text{sdim}(H_n) \).
Proof. The formula in Theorem 6.5 implies that $Z(H_n) \leq \lceil \frac{n}{2} \rceil$. Considering MMD vertices, it’s clear that $sdim(H_n) \geq sdim(C_n)$ and, as noted in (16), $sdim(C_n) = \lceil \frac{n}{2} \rceil$. □

Following [2], for a given unicyclic graph $G$, a vertex $v \in V(G)$ is called an appropriate vertex if at least two components of $G - v$ are paths; a vertex $\ell \in V(G)$ is called a peripheral leaf if $\deg_{G}(\ell) = 1$, $\ell u \in E(G)$, and $\deg_{G}(u) = 2$ (whereas $\deg_{G}(u) \leq 2$ in [2]). The trimmed form of $G$ is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible. Further, define $sdim(G) = sdim(G_{1}) + sdim(G_{2})$ (additivity of $sdim$ over disjoint components), when $G$ is the disjoint union of $G_{1}$ and $G_{2}$. This is a natural extension of the (original) definition of $sdim$ for a connected graph; it is needed for the inductive arguments to come.

Remark 3.7. [17] Let $G$ be a unicyclic graph. Then

(a) for an appropriate vertex $v$ in $G$, $Z(G - v) - 1 = Z(G)$;

(b) for an isolated path $P$ in $G$, $Z(G - V(P)) + 1 = Z(G)$;

(c) for a peripheral leaf $\ell$ in $G$, $Z(G - \ell) = Z(G)$.

Lemma 3.8. Let $G$ be a unicyclic graph, and let $C$ be the unique cycle in $G$.

(a) If $v$ is an appropriate vertex in $G$ such that $v \notin V(C)$, then $sdim(G - v) - 1 \leq sdim(G)$.

(b) If $P$ is an isolated path in $G$, then $sdim(G - V(P)) + 1 = sdim(G)$.

(c) If $\ell$ is a peripheral leaf in $G$, then $sdim(G - \ell) = sdim(G)$.

Proof. Let $\mathcal{M}_{H}(x) = \{y \in V(H) : y \text{ MMD } x\}$.

(a) Denote the connected components of $G - v$ by $G_{1}$ (with $C \subseteq G_{1}$) and $T_{1} \ldots T_{k}$ ($k \geq 2$), of which $T_{1}$ and $T_{2}$ (and possibly more trees) are isolated paths; let $u$ denote the sole neighbor of $v$ in $V(G_{1})$. Let $S$ be a minimum strong resolving set of $G$. Let $L$ denote the set of leaves in $G - G_{1}$. By Observation 3.8 $0 \leq |L - S| \leq 1$. If $|L - S| = 0$, then $S \cup \{u\}$ forms a strong resolving set for $G - v$, since a geodesic between any $\ell \in L$ and any $x \in V(G_{1})$ necessarily passes through $u$; thus we have $sdim(G - v) - 1 \leq sdim(G)$. So, suppose $|L - S| = 1$, and let $\ell_{0} \in L - S$. Since $L \cup \{\ell_{0}\}$ strongly resolves the complement of $G_{1}$ in $G - v$, it suffices to prove the following.

Claim. $S \cap V(G_{1})$ strongly resolves $G_{1}$.

Proof of Claim. Let $x, y \in V(G_{1})$ be strongly resolved by $\ell \in L \cap S$; we will show that $x$ and $y$ are strongly resolved by some $z \in V(G_{1})$. If $x$ or $y$, say $x$, does not lie on $C$, then there must exist a leaf $\ell' \in V(G_{1}) \cap S$ which strongly resolves $x$ and $y$, and we are done. So, suppose both $x$ and $y$ lie on $C$. Let $u'$ denote the vertex on $C$ which is closest to $u$. There must exist a $w \in V(G_{1})$ satisfying $w$ MMD $\ell_{0}$ and such that $d(u', w')$ equals the diameter of $C$; here $w'$ denotes the vertex on $C$ which is closest to $w$. This $w$ lies in $S$, since $\ell_{0} \notin S$. Notice that $x$ and $y$ together lie on the same one of the two semi-circles defined by $u'$ and $w'$; otherwise, $u' - x$ geodesic does not contain $y$ and $u' - y$ geodesic does not contain $x$; the relevance here being that a geodesic from $\ell \in L$ to either $x$ or $y$ must pass through $u'$. Thus, without loss of generality, we may assume $u' - y$ geodesic contains $x$. Then, a $w' - x$ geodesic, hence also a $w - x$ geodesic, contains $y$. It follows that $w \in S \cap V(G_{1})$ strongly resolves $x$ and $y$. □

(b) This follows from the fact $sdim(P) = 1$ and the additivity of $sdim$ over disjoint components.

(c) Since $\ell$ is a peripheral leaf in $G$, there exists a vertex $u \in V(G)$ such that $\ell u \in E(G)$ with $\deg_{G}(u) = 2$. Let $G' = G - \ell$. Since $\mathcal{M}_{G}(u) = \emptyset$ and $\mathcal{M}_{G'}(u) = \mathcal{M}_{G}(\ell)$, $sdim(G - \ell) = sdim(G)$.
Remark 3.9. Let $G$ be a unicyclic graph, and let $C$ be the unique cycle of $G$.

(a) For an appropriate vertex $v \in V(G)$, $sdim(G) - sdim(G - v)$ can be arbitrarily large. If $G$ is a unicyclic graph as in (a) of Figure 3 then $sdim(G) = \left\lceil \frac{m}{2} \right\rceil + k - 1$ and $sdim(G - v) = k + 1$.

(b) There exists $G$ such that, for an appropriate vertex $v \in V(C)$, $sdim(G - v) = sdim(G) + 2$. If $G$ is a unicyclic graph as in (b) of Figure 3 then $sdim(G) = 6$ (the solid vertices form a minimum strong resolving set of $G$) and $sdim(G - v) = 8$.

![Figure 3: Unicyclic graph $G$ and an appropriate vertex $v \in V(G)$](image)

Lemma 3.10. Let $H$ be a generalized partial $n$-sun. Then $Z(H) \leq sdim(H)$.

Proof. It’s clear that our claim holds for a $H$ which has only one major vertex. Thus, we may assume that $H$ contains at least two major vertices. Let $H^0$ be a maximal partial $n$-sun contained in $H$; then $Z(H^0) \leq sdim(H^0)$ by Corollary 3.9. For $i \geq 0$, let $H^{i+1}$ denote the graph obtained as the vertex sum of a $P_2$ with $H^i$ at a major vertex of $H^i$, so that $H = H^k$ for some $k \geq 0$. By the choice of $H^0$, we have $sdim(H^{i+1}) = sdim(H^i) + 1 \geq Z(H^i) + 1 \geq Z(H^{i+1})$ for each $0 \leq i \leq k - 1$, where the left inequality is given by the induction hypothesis.

Now, we arrive at our main result.

Theorem 3.11. If $G$ is a unicyclic graph, then $Z(G) \leq sdim(G)$.

Proof. Assume $Z(G) > sdim(G)$ for some unicyclic graph $G$. By trimming as much as possible, but NOT trimming at any vertex lying on the unique cycle $C$ of $G$, we arrive at a generalized partial $n$-sun $H \subseteq G$. We descend from the given $G$ to $H$ by, for each trim at an allowed vertex $x$ of $G$, discarding all components of $G' - x$ except the connected component $G''$ containing $C$. Let $G' - x = G'' + T_1 + \ldots + T_m$, where $+$ denotes disjoint union. Remark 3.10 and Lemma 3.11 imply $Z(G'') + \sum_{i=1}^{m} Z(T_i) > sdim(G'') + \sum_{i=1}^{m} sdim(T_i)$.

Since $Z(T_i) \leq sdim(T_i)$ for each tree $T_i$ by Theorem 2.3, inequality (1) implies $Z(G'') > sdim(G'')$. Through this process of “descent”, we eventually reach $Z(H) > sdim(H)$, which is the desired contradiction to Lemma 3.10.

Remark 3.12. There exists a unicyclic graph $G$ satisfying $Z(G) = sdim(G)$. Let $C$ be the unique cycle of $G$: let $V(C) = \{u_i \mid 1 \leq i \leq 2k\}$, $E(C) = \{u_iu_{i+1} \mid 1 \leq i \leq 2k - 1\} \cup \{u_1u_{2k}\}$, $ter(u_2) = 0$, and $ter(u_{2j-1}) = 1$, where $1 \leq j \leq k$ and $k \geq 2$ (see Figure 4). Then $Z(G) = k$ by Theorem 3.11 and $sdim(G) = k$: (i) $sdim(G) \geq k$ since $u_j$ MMD $u_{j+k}$ for each $j \in \{1, 2, \ldots k\}$; (ii) $sdim(G) \leq k$ since $\{\ell_{2j-1} \mid 1 \leq j \leq k\}$ forms a strong resolving set for $G$. 

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4 A concluding thought

The cycle rank $r(G)$ of a connected graph $G$ is defined as $|E(G)| - |V(G)| + 1$. In the preceding sections, we have provided sharp bounds (relating $Z(G)$ and $sdim(G)$) when $r(G)$ equals 0 or 1; now, we offer a rough bound which, notably, places no restriction on $r(G)$.

**Proposition 4.1.** Let $G$ be a connected graph with cycle rank $r(G)$. Then $Z(G) \leq sdim(G) + 3r(G)$.

**Proof.** Let $T$ be a spanning tree of $G$ obtained through the deletion of $r = r(G)$ edges of $G$. We have $Z(G) \leq Z(T) + r \leq sdim(T) + r$, where the left and right inequalities are respectively given by Theorem 3.2(a) and Theorem 2.4. Since the removal of an edge $e$ from $G$ results in at most two more leaves in $G - e$, we have $\sigma(T) \leq 2r + \sigma(G)$. Since $sdim(T) = \sigma(T) - 1$ by Theorem 2.1(b), we have $Z(G) \leq 2r + \sigma(G) - 1 + r$. Since $\sigma(G) - 1 \leq sdim(G)$ by Observation 3.1, we obtain $Z(G) \leq sdim(G) + 3r$. \qed

**Question.** What is the best $k$ such that $Z(G) \leq sdim(G) + k \cdot r(G)$ for any connected graph $G$?

We conjecture $0 < k < 1$, as suggested by the following example.

**Example.** Let $G = P_s \square P_s$ be the Cartesian product of $P_s$ with itself, where $s \geq 2$. Then $Z(G) = s$ (see [1]) and $sdim(G) = 2$. Notice that $r(G) = (s - 1)^2$. So, $Z(G) = sdim(G) + \frac{s - 2}{(s - 1)^2} \cdot r(G)$. See Figure 5 when $s = 3$, where the solid vertices in (a) of Figure 5 form a minimum zero forcing set for $G$ and the solid vertices in (b) of Figure 5 form a minimum strong resolving set for $G$.

![Figure 5](image)

**Figure 5:** $Z(P_3 \square P_3) = 3$ and $sdim(P_3 \square P_3) = 2$

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