ELLIPTIC QUANTUM GROUPS AND BAXTER RELATIONS

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Abstract. We introduce a category $O$ of modules over the elliptic quantum group of $\mathfrak{sl}_N$ with well-behaved $q$-character theory. We construct asymptotic modules as analytic continuation of a family of finite-dimensional modules, the Kirillov–Reshetikhin modules. In the Grothendieck ring of this category we prove two types of identities: generalized Baxter relations in the spirit of Frenkel–Hernandez between finite-dimensional modules and asymptotic modules; three-term Baxter TQ relations of infinite-dimensional modules.

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Introduction

Fix $\mathfrak{sl}_N$ a special linear Lie algebra, $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ an elliptic curve, and $\hbar$ a complex number. Associated to this triple is the elliptic quantum group $E_{\tau, \hbar}(\mathfrak{sl}_N)$ introduced by G. Felder [17]. It is a Hopf algebroid (neither commutative nor co-commutative) in the sense of Etingof–Varchenko [14], so that the tensor product of two $E_{\tau, \hbar}(\mathfrak{sl}_N)$-modules is naturally endowed with a module structure. In this paper we study (finite- and infinite-dimensional) representations of the elliptic quantum group.

Suppose $\hbar$ is a formal variable. $E_{\tau, \hbar}(\mathfrak{sl}_2)$ is an $\hbar$-deformation [11] of the universal enveloping algebra of a Lie algebra $\mathfrak{sl}_2 \otimes R$, where $R$ is an algebra of meromorphic functions of $z \in \mathbb{C}$ built from the Jacobi theta function of the elliptic curve. For $\mathfrak{g}$ an arbitrary finite-dimensional simple Lie algebra, $E_{\tau, \hbar}(\mathfrak{g})$ is defined [40] to be a quasi-Hopf algebra twist of the affine quantum group $U_{\hbar}(L\mathfrak{g})$, an $\hbar$-deformation of the loop Lie algebra $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$. It admits a universal dynamical $R$-matrix in a completed tensor square, which provides solutions $R(z; \lambda) \in \text{End}(V \otimes V)$, for $V$ a suitable $E_{\tau, \hbar}(\mathfrak{g})$-module, to the quantum dynamical Yang–Baxter Equation:

$$R_{12}(z - w; \lambda + \hbar^{(3)}) R_{13}(z; \lambda) R_{23}(w; \lambda + \hbar^{(1)}) = R_{23}(w; \lambda) R_{13}(z; \lambda + \hbar^{(2)}) R_{12}(z - w; \lambda) \in \text{End}(V^{\otimes 3}).$$

Here $z, w$ are complex spectral parameters, $\lambda$ is the dynamical parameter lying in a Cartan subalgebra of $\mathfrak{g}$, the sub-indexes of $R$ indicate the tensor factors of $V^{\otimes 3}$ to be acted on, and the $\hbar^{(i)}$ are grading operators arising from the weight grading on $V$ by the Cartan subalgebra. See the comments following Eq. (1.1).
Such R-matrices $R(z; \lambda)$ appeared previously in face-type integrable models \cite{Baxter, Forrester}; for instance, the R-matrix of the Andrews–Baxter–Forrester model comes from two-dimensional irreducible modules of $\mathcal{E}_{\tau, h}(\mathfrak{sl}_2)$, as does the 6-vertex model from the affine quantum group $U_h(L\mathfrak{sl}_2)$. The definition of $\mathcal{E}_{\tau, h}(\mathfrak{sl}_N)$ in \cite{Faddeev}, by RLL exchange relations, is in the spirit of Faddeev–Reshetikhin–Takhtajan, originated from Quantum Inverse Scattering Method. We mention that elliptic R-matrices describe the monodromy of the quantized Knizhnik–Zamolodchikov equation associated with representations of affine quantum groups, e.g. \cite{Arnaudon, Cao, Felder, Reshetikhin}.

Recently Aganagic–Okounkov \cite{Aganagic} proposed the elliptic stable envelope in equivariant elliptic cohomology, as a geometric framework to obtain elliptic R-matrices. This was made explicit \cite{Cao} for cotangent bundles of Grassmannians, resulting in tensor products of two-dimensional irreducible representations of $\mathcal{E}_{\tau, h}(\mathfrak{sl}_2)$. The higher rank case of $\mathfrak{sl}_N$ was studied later by H. Konno \cite{Konno}.

Meanwhile, Nekrasov–Pestun–Shatashvili \cite{Nekrasov} from the 6d quiver gauge theory predicted the elliptic quantum group associated to an arbitrary Kac–Moody algebra, the precise definition of which (as an associative algebra) was proposed by Gauart–Toledano Laredo \cite{Gauart}. See also \cite{Toledano} in the context of quiver geometry.

We are interested in the representation theory of $\mathcal{E}_{\tau, h}(\mathfrak{g})$ with $\mathfrak{h} \subset \mathbb{C}$ generic. The formal twist constructions \cite{Etingof, Felder} from $U_h(\mathfrak{h})$ might reduce the problem to the representation theory of affine quantum groups, which is a subject developed intensively in the last three decades from algebraic, geometric and combinatorial aspects. However \textit{loc.cit.} involve formal power series of $\mathfrak{h}$ and infinite products in the comultiplication of $\mathcal{E}_{\tau, h}(\mathfrak{g})$. Some of these divergence issues was addressed \cite{Etingof} by Etingof–Moura, who defined a fully faithful tensor functor between representation categories of BGG type for $U_h(L\mathfrak{sl}_N)$ and $\mathcal{E}_{\tau, h}(\mathfrak{g})$. Towards this functor not much is known: its image, the induced homomorphism of Grothendieck rings, etc.

In this paper we study representations of $\mathcal{E}_{\tau, h}(\mathfrak{sl}_N)$ via the RLL presentation \cite{Faddeev} so as to bypass affine quantum groups, yet along the way we borrow ideas from the affine case. Compared to other works \cite{Baxter, Cao, Felder, Reshetikhin, Tarasov, Toledano, Varchenko}, our approach emphasizes more on the Grothendieck ring structure of representation category. It is a higher rank extension of a recent joint work with G. Felder \cite{Zhang}.

The presence of the dynamical parameter $\lambda$ is one of the technical difficulties of elliptic quantum groups. To resolve this, we need a commuting family of elliptic Cartan currents $\phi_j(z) \in \mathcal{E}_{\tau, h}(\mathfrak{sl}_N)$ for $j \in J := \{1, 2, \cdots, N - 1\}$. They act as difference operators on an $\mathcal{E}_{\tau, h}(\mathfrak{sl}_N)$-module $V$, and their matrix entries are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$ where $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{sl}_N$.

As in \cite{Zhang}, we impose the following triangularity condition: \footnote{In terms of the $K_j(z)$ from Eq \textit{i}, we have $\phi_j(z) = K_j(z + \ell_j h) K_{j+1}(z + \ell_j h)^{-1}$ where $\ell_j = (N - j - 1)/2$. These are elliptic deformations of diagonal matrices in $\mathfrak{sl}_N$.}

(i) there exists a basis of $V$, with respect to which the matrices $\phi_j(z)$ are upper triangular and their diagonal entries are independent of $\lambda$.

Our category $\mathcal{O}$ is the full subcategory of category BGG \cite{Felder} of $\mathcal{E}_{\tau, h}(\mathfrak{sl}_N)$-modules subject to Condition (i); see Definition \cite{Zhang} It is abelian and monoidal. It contains most of the modules in \cite{Baxter, Cao, Felder, Reshetikhin, Toledano}, although the proof is rather indirect. (We believe category $\mathcal{O}$ to be the image of the functor \cite{Etingof}.)

We extend the $q$-character of H. Knight \cite{Knight} and Frenkel–Reshetikhin \cite{Frenkel} to the elliptic case. The $q$-character of a module $V$ encodes the spectra of the $\phi_j(z)$, which are meromorphic functions of $z$ thanks to Condition (i). It distinguishes the isomorphism class $[V]$ in the Grothendieck ring $K_0(\mathcal{O})$, and embeds $K_0(\mathcal{O})$ in a commutative ring. Our main results are summarized as follows.
(ii) Proposition 4.10 on limit construction of infinite-dimensional asymptotic modules \( \mathcal{W}_{r,x} \), for \( r \in J \) and \( x \in \mathbb{C} \), from a distinguished family of finite-dimensional modules, the Kirillov–Reshetikhin modules.

(iii) Theorem 4.13 on generalized Baxter relations à la Frenkel–Hernandez [23]: the isomorphism class of any finite-dimensional module is a polynomial of the \( \mathcal{W}_{r,a} \) for \( r \in J \) and \( x, y \in \mathbb{C} \).

(iv) Corollary 5.2 relating an asymptotic module \( \mathcal{W} \) to a module \( D \) and tensor products \( S', S'' \) of asymptotic modules such that \( [D][\mathcal{W}] = [S'] + [S''] \).

The above results are known in category \( \mathcal{O}_{\text{HJ}} \) of Hernandez–Jimbo [36] for representations over a Borel subalgebra of an affine quantum group \( U_h(Lg) \). Category \( \mathcal{O}_{\text{HJ}} \) contains the modules \( L_{r,a}^\pm \) for \( a \in \mathbb{C} \) and \( r \) a Dynkin node of \( g \). The \( L_{r,a}^\pm \) are “prefundamental” in that their tensor products realize all irreducible objects of \( \mathcal{O}_{\text{HJ}} \) as sub-quotients, and they are not modules over \( U_h(Lg) \), which makes Borel subalgebras indispensable. The Grothendieck ring of \( \mathcal{O}_{\text{HJ}} \) is commutative.

(ii) is the asymptotic limit construction [36] of the \( L_{r,a}^\pm \) (iii) is the relation [23] between finite-dimensional modules and the \( L_{r,a}^\pm \). (iv) is either \( \mathcal{Q}^2 \)-system [16, 37] or \( \mathcal{Q}Q \)-system [24], as there are two choices of the modules \( D \) for \( \mathcal{W} = L_{r,a}^+ \).

Hernandez–Leclerc [37] interpreted the \( \mathcal{Q}Q \)-system [37] as cluster mutations of Fomin–Zelevinsky. They provided conjectural monoidal categorifications of infinite rank cluster algebras by certain subcategories of \( \mathcal{O}_{\text{HJ}} \).

In a quantum integrable system associated to \( U_h(Lg) \), the transfer-matrix construction defines an action of the Grothendieck ring \( K_0(\mathcal{O}_{\text{HJ}}) \) on the quantum space; to an isomorphism class \([V]\) is attached a transfer matrix \( t_V(z) \).

(iii) is one key step [23] in solving the conjecture of Frenkel–Reshetikhin [27] on the spectra of the quantum integrable system, which connects the eigenvalues of the \( t_V(z) \) to the \( q \)-character of \( V \) by the so-called Baxter polynomials [2]. These polynomials are eigenvalues of the \( t_{L_{r,a}^+} \) up to an overall factor [23]. In this sense the \( L_{r,a}^\pm \) have simpler structures than finite-dimensional modules, and the \( t_{L_{r,a}^+} \) are defined as Baxter Q operators, as an extension of earlier works of V. Bazhanov et al. [3, 4, 5] for \( g \) a special linear Lie (super)algebra. (iv) has as consequence the Bethe Ansatz Equations for the roots of Baxter polynomials [16, 24].

Recently category \( \mathcal{O}_{\text{HJ}} \) was studied for quantum toroidal algebras [1].

For elliptic quantum groups there are no obvious Borel subalgebras. Our idea is to replace the \( L_{r,a}^\pm \) over Borel subalgebras by the asymptotic modules \( \mathcal{W}_{d,a}^{(r)} \) (with a new parameter \( d \in \mathbb{C} \)) over the entire quantum group, which we now explain.

Let \( \theta(z) := \theta(z|\tau) \) be the Jacobi theta function. For \( r \in J \) a Dynkin node, \( a \in \mathbb{C} \) a spectral parameter, and \( k \) a positive integer, by [7, 51] there exists a unique finite-dimensional irreducible module \( W_{k,a}^{(r)} \) which contains a non-zero vector \( \omega \) (highest weight with respect to a triangular decomposition) such that:

\[
\phi_j(z)\omega = \omega \text{ if } j \neq r, \quad \phi_r(z)\omega = \frac{\theta(z + ah + kh)}{\theta(z + ah)} \omega.
\]

This is a Kirillov–Reshetikhin (KR) module, a standard terminology for affine quantum groups and Yangians once the \( \theta \) symbol is removed.

The core of this paper (Section 3) is analytic continuation with respect to \( k \). We modify the asymptotic limits \( L_{r,a} \) of Hernandez–Jimbo [36], as in [21, 55].

Firstly the existence of the inductive system \( (W_{k,a}^{(r)})_{k>0} \) in [36] relied on a cyclicity property of M. Kashiwara, Varagnolo–Vasserot and V. Chari, which is unavailable in the elliptic case. We reduce the problem to \( \mathcal{E}_{r,h}(\mathfrak{sl}_2) \) by counting “dominant
weights” in q-characters (Theorem 3.4), as in the proofs of T-system of KR modules over affine quantum groups by H. Nakajima [48] and D. Hernandez [34].

Secondly we express the matrix coefficients of any element of $\mathcal{E}_{r,N}(\mathfrak{sl}_N)$ acting on the $W_{k,a}^{(r)}$, viewed as functions of $k \in \mathbb{Z}_{>0}$, in products of the $\theta(kh + c)$ where $c \in \mathbb{C}$ is independent of $k$; see Lemma 1.8 In [50] these are polynomials in $k$ by induction. Our proof relies on the RLL comultiplication and is explicit.

$\theta(kh + c)$ being an entire function of $k$, we take $k$ in the matrix coefficients to be a fixed complex number $d$. This results in the asymptotic module $\mathcal{W}_{d,a}^{(r)}$ on the inductive limit $\lim_{\to} W_{k,a}^{(r)}$. The module $\mathcal{W}_{r,x}$ in (ii) is $\mathcal{W}_{x,0}^{(r)}$. All irreducible modules of category $\mathcal{O}$ are sub-quotients of tensor products of asymptotic modules.

For $\mathfrak{g}$ of general type (ii)–(iv) and their proofs can be adapted to affine quantum groups, whose asymptotic modules appeared in [53, Appendix], as well as Yangians [30, 31]. Borel subalgebras or double Yangians are not needed.

(ii)–(iv) were established for affine quantum general linear Lie superalgebras [54]; their proofs require more than q-characters as counting dominant weights is inefficient. It is interesting to consider elliptic quantum supergroups [29].

For elliptic quantum groups associated with other simple Lie algebras, one possible first step would be to derive the RLL presentation; see [33, 39] for Yangians.

R-matrix of Baxter–Belavin is governed by the vertex-type elliptic quantum group [10]. The equivalence [13] of representation categories between this elliptic algebra and $\mathcal{E}_{r,N}(\mathfrak{sl}_N)$, a Vertex-IRF correspondence, might give a representation theory meaning to the original Baxter Q operator of the 8-vertex model [2].

The paper is structured as follows. In Section 1 we review the theory of the elliptic quantum group associated to $\mathfrak{sl}_N$, and define category $\mathcal{O}$ of representations. We show that the q-character map is an injective ring homomorphism from the Grothendieck ring $K_0(\mathcal{O})$ to a commutative ring $\mathcal{M}_q$ of meromorphic functions. Then we present the q-character formula of finite-dimensional evaluation modules.

Section 2 is devoted to the proof of the q-character formula.

We derive in Section 3 basic facts on tensor products of KR modules (T-system, fusion) from the q-character formula. They are needed in Section 4 to construct the inductive system of KR modules and the asymptotic modules. We obtain a highest weight classification of irreducible modules in category $\mathcal{O}$. As a consequence, all standard irreducible evaluation modules of [31] are in category $\mathcal{O}$.

In Section 5 we establish the three-term Baxter TQ relations in $K_0(\mathcal{O})$, which are infinite-dimensional analogs of the T-system. These relations are interpreted as functional relations of transfer matrices in Section 4.

1. Elliptic quantum groups and their representations

Let $N \in \mathbb{Z}_{>0}$. We introduce a category $\mathcal{O}$ (abelian and monoidal) of representations of the elliptic quantum group attached to the Lie algebra $\mathfrak{sl}_N$, and prove that its Grothendieck ring is commutative, based on q-characters.

Fix a complex number $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$. Define the Jacobi theta function

$$\theta(z) = \theta(z|\tau) := -\sum_{j=-\infty}^{\infty} \exp \left( i\pi (j + \frac{1}{2})^2 \tau + 2i\pi (j + \frac{1}{2})(z + \frac{1}{2}) \right), \quad i = \sqrt{-1}.$$  

It is an entire function of $z \in \mathbb{C}$ with zeros lying on the lattice $\Gamma := \mathbb{Z} + \mathbb{Z}\tau$ and

$$\theta(z + 1) = -\theta(z), \quad \theta(z + \tau) = -e^{-i\pi\tau - 2i\pi z}\theta(z), \quad \theta(-z) = -\theta(z).$$

Fix a complex number $h \in \mathbb{C} \setminus (\mathbb{Q} + \mathbb{Q}\tau)$, which is the deformation parameter.
Let \( h \) be standard Cartan subalgebra of \( \mathfrak{sl}_N \); it is a complex vector space generated by the \( e_i \) for \( 1 \leq i \leq N \) subject to the relation \( \sum_{i=1}^{N} e_i = 0 \). Let \( C^N := \bigoplus_{i=1}^{N} C v_i \) and \( E_{ij} \in \text{End}_C(C^N) \) be the elementary matrices: \( v_k \mapsto \delta_{jk} v_i \) for \( 1 \leq i,j,k \leq N \).

Define the \( \text{End}_C(C^N \otimes C^N) \)-valued meromorphic functions of \( (z; \lambda) \in C \times h \) by:

\[
\text{R}(z; \lambda) = \sum_i E^{(2)}_{ii} + \sum_{i \neq j} \left( \frac{\theta(z) \theta(\lambda_{ij} - h)}{\theta(z + h) \theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(z + \lambda_{ij}) \theta(h)}{\theta(z + h) \theta(\lambda_{ij})} E_{ij} \otimes E_{ji} \right).
\]

In the summations \( 1 \leq i,j \leq N \), and \( \lambda_{ij} \in h^* \) sends \( \sum_{i=1}^{N} c_i e_i \in h \) to \( c_i - c_j \in C \).

By [17], \( \text{R}(z; \lambda) \) satisfies the quantum dynamical Yang-Baxter equation:

\[
\text{R}^{12}(z - w; \lambda + hh^{(1)}) \text{R}^{13}(z; \lambda) \text{R}^{23}(w; \lambda + hh^{(1)}) = \text{R}^{23}(w; \lambda) \text{R}^{13}(z; \lambda + hh^{(2)}) \text{R}^{12}(z - w; \lambda) \in \text{End}_C(C^N)^{\otimes 3}.
\]

If \( \text{R}(z; \lambda) = \sum_p c^p(x_p \otimes y_p) \) with \( x_p, y_p \in \text{End}_C(C^N) \), then

\[
\text{R}^{13}(z; \lambda + hh^{(2)}): u \otimes v_j \otimes w \mapsto \sum_p c^p(z + hh^{(2)}) x_p(u) \otimes v_j \otimes y_p(w).
\]

for \( u, w \in C^N \) and \( 1 \leq j \leq N \). The other symbols have a similar meaning.

Let \( M := M_h \) be the field of meromorphic functions of \( \lambda \in h \). It contains the subfield \( C \) of constant functions. A \( C \)-linear map \( \Phi \) of two \( M \)-vector spaces will sometimes be denoted by \( \Phi(\lambda) \) to emphasize the dependence on \( \lambda \).

1.1. **Algebraic notions.** Since the elliptic quantum groups will act on \( M \)-vector spaces via difference operators, which are in general not \( M \)-linear, we need to recall some basic constructions about difference operators. Our exposition follows largely [14], with minor modifications as in [21].

Define the category \( V \) as follows. An object is \( X = \oplus_{\alpha \in h} X[\alpha] \) where each \( X[\alpha] \) is an \( M \)-vector space and, if non-zero, is called a weight space of weight (or \( h \)-weight) \( \alpha \). Let \( \text{wt}(X) \subseteq h \) be the set of weights of \( X \). Write \( \text{wt}(v) = \alpha \) if \( v \in X[\alpha] \).

An action \( f: X \longrightarrow Y \) in \( V \) is an \( M \)-linear map which respects the weight gradings. Let \( V_h \) be the full subcategory of \( V \) consisting of \( X \) whose weight spaces are finite-dimensional \( M \)-vector spaces. ("ft" means finite type in [13].)

Viewed as subcategories of the category of \( M \)-vector spaces, \( V \) and \( V_h \) are abelian.

Let \( X,Y \) be objects of \( V \). Their **dynamical tensor product** \( X \otimes_h Y \) is constructed as follows. For \( \alpha, \beta \in h \), let \( X[\alpha] \otimes_Y Y[\beta] \) be the quotient of the usual tensor product of \( C \)-vector spaces \( X[\alpha] \otimes C Y[\beta] \) by the relation

\[
g(\lambda)(v \otimes w) = v \otimes g(\lambda + h\beta) w \quad \text{for } v \in X[\alpha], \ w \in Y[\beta], \ g(\lambda) \in M.
\]

Let \( \otimes \) denote the image of \( \otimes \) under the quotient. \( X[\alpha] \otimes_Y Y[\beta] \) becomes an \( M \)-vector space by setting \( g(\lambda)(v \otimes w) = \psi \otimes g(\lambda) w \). For \( \gamma \in h \), the weight space \( X[\alpha] \otimes Y[\beta] \gamma \) is then the direct sum of \( X[\alpha] \otimes Y[\gamma] \) with \( \alpha + \beta = \gamma \).

For \( \alpha, \beta \in h \), a \( C \)-linear map \( \Phi: X \longrightarrow Y \) is called a **difference map** of bi-degree \((\alpha, \beta)\) if it sends every weight space \( X[\gamma] \) to \( Y[\gamma + \beta \ominus \alpha] \), and if [13] §4.2:

\[
\Phi(g(\lambda)v) = g(\lambda + h\beta) \Phi(v) \quad \text{for } g(\lambda) \in M \text{ and } v \in X.
\]

Such a map can be recovered from its matrix as in the case of \( M \)-linear maps. Choose \( M \)-bases \( B, B' \) for \( X \) and \( Y \) respectively. Define the \( B' \times B \) matrix \( [\Phi] \) by taking its \((b', b)\)-entry \([\Phi]_{b'b}(\lambda) = \Phi_{b'b}(\lambda) \in M \), for \( b \in B \) and \( b' \in B' \), to be the coefficient of \( b' \) in \( \Phi(b) \). Then for any vector \( v = \sum_{b' \in B} b' \Phi(b')(\lambda) \) of \( X \) where \( g_b(\lambda) \in M \), we have

\[
\Phi(v) = \sum_{b' \in B'} b' \sum_{b \in B} [\Phi]_{b'b}(\lambda) \times g_b(\lambda + h\beta).
\]

\(^2\)Note that difference maps of bi-degree \((\alpha, \alpha)\) make sense for arbitrary \( M \)-vector spaces.
When $X = Y$, a difference map is also called a difference operator. To define its matrix, we always assume $B' = B$. By an algebra we mean a unital associative algebra over $\mathbb{C}$.

As in [13] Definition 4.1, an $\mathfrak{h}$-algebra is an algebra $A$, endowed with $\mathfrak{h}$-bigrading $A = \bigoplus_{\alpha, \beta \in \mathfrak{h}} A_{\alpha, \beta}$ which respects the algebra structure and is called the weight decomposition, and two algebra embeddings $\mu_1, \mu_2 : M \rightarrow A_{0,0}$ called the left and right moment maps, such that for $a \in A_{\alpha, \beta}$ and $g(\lambda) \in M$, we have
\[
\mu_1(g(\lambda))a = a\mu_1(g(\lambda - h\alpha)), \quad \mu_2(g(\lambda))a = a\mu_2(g(\lambda - h\beta)).
\]
Call $(\alpha, \beta)$ the bi-degree of elements in $A_{\alpha, \beta}$. A morphism of $\mathfrak{h}$-algebras is an algebra morphism preserving the moment maps and the weight decompositions.

From two $\mathfrak{h}$-algebras $A, B$ we construct their tensor product $\bigotimes B$ as follows. For $\alpha, \beta, \gamma \in \mathfrak{h}$, let $A_{\alpha, \beta} \otimes B_{\beta, \gamma}$ be $A_{\alpha, \beta} \otimes \mathbb{C} B_{\beta, \gamma}$ modulo the relation
\[
\mu_1^A(g(\lambda))a \otimes c b = a \otimes c \mu_1^B(g(\lambda))b \quad \text{for} \quad a \in A_{\alpha, \beta}, \quad b \in B_{\beta, \gamma}, \quad g(\lambda) \in M.
\]
$(A \otimes B)_{\alpha, \beta}$ is the direct sum of the $A_{\alpha, \beta} \otimes B_{\beta, \gamma}$ over $\beta \in \mathfrak{h}$. Multiplication in $A \otimes B$ is induced by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. The moment maps are given by ($\otimes$ denotes the image of $\otimes \mathbb{C}$ under the quotient $\otimes \mathbb{C} \rightarrow \mathfrak{h}$)
\[
\mu_1^{A \otimes B} : g(\lambda) \mapsto \mu_1^A(g(\lambda)) \otimes 1, \quad \mu_2^{A \otimes B} : g(\lambda) \mapsto 1 \otimes \mu_2^B(g(\lambda)) \quad \text{for} \quad g(\lambda) \in M.
\]

To an $\mathfrak{h}$-graded vector space one can attach naturally an $\mathfrak{h}$-algebra. Let $X$ be an object of $\mathcal{V}$. Let $D^X_{\alpha, \beta}$ denote the $\mathbb{C}$-vector space of difference operators $X \rightarrow X$ of bidegree $(\alpha, \beta)$. Then the direct sum $D^X := \oplus_{\alpha, \beta \in \mathfrak{h}} D^X_{\alpha, \beta}$ is a subalgebra of $\text{End}_\mathbb{C}(X)$. It is an $\mathfrak{h}$-algebra structure with the moment maps
\[
\mu_r(g(\lambda))v = g(\lambda)v, \quad \mu_l(g(\lambda))v = g(\lambda + h\alpha)v \quad \text{for} \quad v \in X[\alpha], \quad g(\lambda) \in M.
\]

Tensor products of difference operators are also difference operators. To be precise, let $X, Y$ be two objects of $\mathcal{V}$. Let $\Phi : X \rightarrow X$ and $\Psi : Y \rightarrow Y$ be difference operators of bi-degree $(\alpha, \beta)$ and $(\beta, \gamma)$ respectively. The $\mathbb{C}$-linear map
\[
X \otimes \mathbb{C} Y \rightarrow X \otimes Y, \quad v \otimes \mathbb{C} w \mapsto \Phi(v) \otimes \Psi(w)
\]
is easily seen to factorize through $X \otimes \mathbb{C} Y \rightarrow X \otimes Y$ and induces the $\mathbb{C}$-linear map $\Phi \otimes \Psi : X \otimes Y \rightarrow X \otimes Y$, which is shown to be a difference operator of bi-degree $(\alpha, \gamma)$. As in [13] Lemma 4.3], the following defines a morphism of $\mathfrak{h}$-algebras
\[
D^X \otimes D^Y \rightarrow D^{X \otimes Y}, \quad \Phi \otimes \Psi \mapsto \Phi \otimes \Psi.
\]

1.2. Elliptic quantum groups. For $1 \leq i, j, p, q \leq N$ let $R_{ij}^{pq}(z; \lambda)$ be the coefficient of $v_j \otimes v_q$ in $R(z; \lambda)(v_i \otimes v_j)$; it can be viewed as an element of $M$ after fixing $z \in \mathbb{C}$. The elliptic quantum group $\mathcal{E} := \mathcal{E}_{r, h}(\mathfrak{sl}_N)$ is an $\mathfrak{h}$-algebra generated by
\[
L_{ij}(z) \in \mathcal{E}_{r, \epsilon, \delta, \epsilon, \delta} \quad \text{for} \quad 1 \leq i, j \leq N
\]
subject to the dynamical RLL relation [14] §4.4: for $1 \leq i, j, m, n \leq N$,
\[
\sum_{p, q=1}^N \mu_1(R_{pq}^{ij}(z - w; \lambda)) L_{pq}(z)L_{ji}(w) \quad \mu_r(u_{pq}(z - w; \lambda)) L_{pq}(w)L_{ij}(z).
\]

\[\text{We use } \mathfrak{sl}_N, \text{ as in [13] [19], to emphasize that } \mathfrak{h} \text{ is the Cartan subalgebra of } \mathfrak{sl}_N. \text{ Other works [14] [15] use } g|_N, \text{ for the reason that the elliptic quantum determinant is not fixed to be 1.}\]
There is an ℓ-algebra morphism [14]:

\[(1.3) \quad \Delta : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E}, \quad L_{ij}(z) \mapsto \sum_{k=1}^{N} L_{ik}(z) \otimes L_{kj}(z) \text{ for } 1 \leq i, j \leq N\]

which is co-associative \((1 \otimes \Delta) \Delta = (\Delta \otimes 1) \Delta\) and is called the coproduct. For \(u \in \mathbb{C}\),

\[(1.4) \quad \Phi_u : \mathcal{E} \rightarrow \mathcal{E}, \quad L_{ij}(z) \mapsto L_{ij}(z + uh) \text{ for } 1 \leq i, j \leq N\]

extends uniquely to an ℓ-algebra automorphism (spectral parameter shift).

Strictly speaking, \(\mathcal{E}\) is not well-defined as an ℓ-algebra because of the additional parameter \(z\); this is resolved in [45] by viewing \(z, h\) as formal variables. In this paper we are mainly concerned with representations in which Eq. (1.2)–(1.4) make sense as identities of difference operators depending analytically on \(z\).

Let \(\mathfrak{S}\) be the group of permutations of \(\{1, 2, \ldots, N\}\). For \(1 \leq k \leq N\), let \(\mathfrak{S}^k\) be the subgroup of permutations which fix the last \(k\) letters. The \(k\)-th fundamental weight \(\varpi_k\) and elliptic quantum minor \(D_k(z)\) are defined by [51, Eq.(2.5)]:

\[(1.5) \quad \varpi_k := \sum_{i=1}^{k} \epsilon_i \in \ell, \quad \Theta_k(\lambda) := \prod_{N-k+1 \leq i < j \leq N} \theta(\lambda_{ij}) \in M^X,\]

\[(1.6) \quad D_k(z) := \frac{\mu_r(\Theta_k(\lambda))}{\mu_1(\Theta_k(\lambda))} \sum_{\sigma \in \mathfrak{S}} \text{sign}(\sigma) \prod_{i=N}^{N-k+1} L_{\sigma(i),i}(z + (N-i)h) \in \mathcal{E}.\]

Here \(\text{sign}(\sigma) \in \{\pm 1\}\) denotes the signature of the permutation \(\sigma\). We take the descending product over \(N \geq i \geq N-k+1\) in Eq. (1.6). Set \(\varpi_0 := 0\).

We shall need the following elements \(\tilde{L}_k(z)\) of \(\mathcal{E}_{k,\ell}\) as in [51, Eq.(4.1)]:

\[(1.7) \quad \tilde{L}_N(z) := L_{N \times N}(z), \quad \tilde{L}_k(z) = L_{k \times k}(z) \prod_{j=k+1}^{N} \frac{\mu_r(\theta(\lambda_{kj}))}{\mu_1(\theta(\lambda_{kj}))}.\]

**Theorem 1.1.** [51, Proposition 2.1] [15, Eq.(E.18)] \(D_N(z)\) is central in \(\mathcal{E}\) and group-like: \(\Delta(D_N(z)) = D_N(z) \otimes D_N(z)\).

The simple roots \(\alpha_i := \epsilon_i - \epsilon_{i+1}\) for \(1 \leq i < N\) generate a free abelian subgroup \(Q\) of \(\ell\), called the root lattice. Let \(Q_+, Q_-\) be submonoids of \(Q\) generated by the \(\alpha_i\), respectively. Define the lexicographic partial ordering \(<\) on \(\ell\) as follows:

\[
\alpha < \beta \text{ if } \beta - \alpha = n_1 \alpha_1 + \sum_{i=2}^{N} n_i \alpha_i \in Q \text{ with } n_i \in \mathbb{Z}_{>0}.
\]

This is weaker than the standard ordering: \(\alpha \leq \beta \text{ if } \beta - \alpha \in Q_+\).

**Corollary 1.2.** \(D_k(z)\) commutes with the \(L_{ij}(w)\) for \(N-k < i, j \leq N\) and \(\Delta(D_k(z)) - D_k(z) \otimes D_k(z)\) is a finite sum \(\sum_{\alpha} x_{\alpha} \otimes y_{\alpha}\) over \(\{\alpha \in \ell | -\varpi_{N-k} \prec \alpha\}\) where \(x_{\alpha}\) and \(y_{\alpha}\) are of bi-degree \((-\varpi_{N-k}, \alpha)\) and \((\alpha, -\varpi_{N-k})\) respectively.

The proof of the corollary is postponed to Section 2.1.

**1.3. Categories.** From now on unless otherwise stated vector spaces, linear maps and bases are defined over \(\mathbb{C}\). Let \(X\) be an object of \(\mathcal{V}_\varpi\). A representation of \(\mathcal{E}\) on \(X\) consists of difference operators \(L^X_{ij}(z) : X \rightarrow X\) of bi-degree \((\epsilon_i, \epsilon_j)\) for \(1 \leq i, j \leq N\) depending on \(z \in \mathbb{C}\) with the following properties:

\[\text{(M1) there exists a basis of } X \text{ with respect to which all the matrix entries of the difference operators } L^X_{ij}(z) \text{ are meromorphic functions of } (z, \lambda) \in \mathbb{C} \times \ell.\]

\[\text{(M2) Eq.(1.2) holds in } \mathbb{D}^X \text{ with } \mu_1, \mu_r \text{ being moment maps in } \mathbb{D}^X.\]

---

\(^4\text{This is called a representation of finite type in [19]. From condition (M1) it follows that the coefficients of the } L^X_{ij}(z) \text{ are meromorphic functions with respect to any basis of } X.\)
Call $X$ an $\mathcal{E}$-module. Property (M2) can be interpreted as an $h$-algebra morphism $\mathcal{E} \longrightarrow \mathbb{D}^X$ sending $L_{ij}(z) \in \mathcal{E}$ to the difference operator $L^X_{ij}(z)$ on $X$. Applying $\rho$ to the elements of Eqs. \[1.6\] - \[1.7\], one gets difference operators $D^X_{ij}(z)$, $L^X_{ij}(z)$ acting on $X$ bi-degree $(-\varpi_{N-1}, -\varpi_{N-1})$ and $(\epsilon_k, \epsilon_k)$ respectively. When no confusion arises, we shall drop the superscript $X$ from $L^X, D^X, L^X$ to simplify notations.

A morphism $\Phi : X \longrightarrow Y$ of $\mathcal{E}$-modules is a linear map which respects the $h$-gradings (so that $\Phi$ is a morphism in category $\mathcal{V}$) and satisfies $\Phi L^Y_{ij}(z) = L^Y_{ij}(z) \Phi$ for $1 \leq i, j \leq N$. The category of $\mathcal{E}$-modules is denoted by $\mathbf{Rep}$. It is a subcategory of $\mathcal{V}$, and is abelian, since the kernel and cokernel of a morphism of $\mathcal{E}$-modules, as $h$-graded $M$-vector spaces, are naturally $\mathcal{E}$-modules. 

**Definition 1.3.** [12 §4] $\tilde{O}$ is the full subcategory of $\mathbf{Rep}$ whose objects $X$ are such that $\text{wt}(X)$ is contained in a finite union of cones $\mu + \mathbb{Q}_-$ with $\mu \in h$.

For $X, Y$ objects in category $\tilde{O}$, the $L^X \otimes Y (z) := \sum_{k=1}^N L^X_{ik}(z) \otimes L^Y_{kj}(z)$ define a representation of $\mathcal{E}$ on $X \otimes Y$ which is easily seen to be in category $\tilde{O}$. So $\tilde{O}$ is a monoidal subcategory of $\mathcal{V}$. Similarly, $\tilde{O}$ is an abelian subcategory of $\mathbf{Rep}$.

**Definition 1.4.** [21 §2] An object in $\mathcal{F}_{\text{mer}}$ consists of a finite-dimensional vector space $V$ equipped with difference operators $D_l(z) : V \longrightarrow V$ of bi-degree $(-\varpi_{N-1}, -\varpi_{N-1})$ (see Footnote 2) for $1 \leq l \leq N$ depending on $z \in \mathbb{C}$ such that:

(M3) there exists an ordered basis of $V$ with respect to which the matrices of the difference operators $D_l(z)$ are upper triangular, the diagonal entries are non-zero meromorphic functions of $z \in \mathbb{C}$, and the off-diagonal entries are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times h$.

A morphism $\Phi : V \longrightarrow W$ in $\mathcal{F}_{\text{mer}}$ is a linear map commuting with the $D_l(z)$. (Namely, $\Phi D_l(z) = D_l(z) \Phi : V \longrightarrow W$ for $1 \leq l \leq N$. Here we add the superscripts $V, W$ in the $D_l(z)$ to indicate the space on which they act.)

For $V$ an object of $\mathcal{F}_{\text{mer}}$, the operators $D_l(z)$ being invertible because of the triangularity, one has a unique factorization of operators for $1 \leq l \leq N$:

\[ D_l(z) = K_N(z)K_{N-1}(z+h)K_{N-2}(z+2h) \cdots K_{N-l+1}(z+(l-1)h). \]

Notably $K_l(z) : X \longrightarrow X$ is a difference operator of bi-degree $(\epsilon_l, \epsilon_l)$. Property (M3) still holds if the $D_l(z)$ are replaced by the $K_l(z)$.

The forgetful functor from $\mathcal{F}_{\text{mer}}$ to the category of finite-dimensional vector spaces equips $\mathcal{F}_{\text{mer}}$ with an abelian category structure. (For a proof, we refer to [21 §2.1] where another characterization of category $\mathcal{F}_{\text{mer}}$ in terms of Jordan–Hölder series is given.) Let us describe its Grothendieck group $K_0(\mathcal{F}_{\text{mer}})$.

The multiplicative group $\mathbb{M}_c^X$ of non-zero meromorphic functions of $z \in \mathbb{C}$ contains a subgroup $\mathbb{C}^\times$ of non-zero constant functions. Let $\mathcal{M}$ be the quotient group of $(\mathbb{M}_c^X)^N$ by its subgroup formed of $(c_1, c_2, \cdots, c_N) \in (\mathbb{C}^\times)^N$ such that $c_1c_2 \cdots c_N = 1$. We show that $K_0(\mathcal{F}_{\text{mer}})$ has a $\mathcal{Z}$-basis indexed by $\mathcal{M}$.

For $f = (f_1(z), f_2(z), \cdots, f_N(z)) \in (\mathbb{M}_c^X)^N$, the vector space $\mathcal{M}$ with the following difference operators $D_l(z)$ is an object in category $\mathcal{F}_{\text{mer}}$ denoted by $\mathcal{M}_f$:

\[ g(\lambda) \mapsto g(\lambda - h \varpi_{N-1})f_N(z)f_{N-1}(z+h)f_{N-2}(z+2h) \cdots f_{N-l+1}(z+(l-1)h). \]

We have $K_l(z)g(\lambda) = g(\lambda + h\epsilon_l)f_l(z)$. As a consequence of (M3) in Definition 1.3 all irreducible objects of category $\mathcal{F}_{\text{mer}}$ are of this form.

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5 In other works [11] [12] [19] [24] [31] [32] [33] [34] [35] [36], a module $V$ is an $h$-graded $C$-vector space; morphisms of modules depend on the dynamical parameter $\lambda$, so do their kernel and cokernel; the abelian category structure is non trivial. The scalar extension gives a module $V \otimes_{\mathcal{M}} \mathcal{M}$ in the present situation. Since our modules and morphisms are $\mathcal{M}$-linear, the dependence of kernels and images on the dynamical parameter does not matter.
Lemma 1.5. Let $e, f \in (\mathbb{M}_\mathbb{C})^N$. The objects $\mathbb{M}_e$ and $\mathbb{M}_f$ are isomorphic in category $\mathcal{F}_{\text{mer}}$ if and only if $e, f$ have the same image under the quotient $(\mathbb{M}_\mathbb{C})^N \to \mathcal{M}$.

Proof. Write $e = (e_1(z), e_2(z), \ldots, e_N(z))$ and $f = (f_1(z), f_2(z), \ldots, f_N(z))$.

Sufficiency: assume $e_i(z) = f_i(z)c_i$ with $c_i \in \mathbb{C}^\times$ and $c_1c_2\cdots c_N = 1$. For $1 \leq l \leq N$, choose $b_l$ such that $c_l = e^{b_l}$. Set $b_N := -b_1 - b_2 - \cdots - b_{N-1}$. Then $e_{kn} = c_1^{b_1}c_2^{b_2}\cdots c_N^{b_N} = c_N$ and the following is a well-defined element of $\mathbb{M}_\mathbb{C}^\times$:

$$\varphi(x_1c_1 + x_2c_2 + \cdots + x_Nc_N) = e^{b_1x_1 + b_2x_2 + \cdots + b_Nx_N} \quad \text{for} \quad x_1, x_2, \ldots, x_N \in \mathbb{C}.$$  

Indeed $\varphi(\alpha + \beta) = \varphi(\alpha)\varphi(\beta)$ and $\varphi(xe_1 + xe_2 + \cdots + xe_N) = 1$ for $x \in \mathbb{C}$. Notably, $\varphi(\lambda + he_l) = \varphi(\lambda)\varphi(h\epsilon_l) = e^{b_l}\varphi(\lambda)$. So $\mathbb{M}_e \to \mathbb{M}_f$, $g(\lambda) \mapsto g(\lambda)\varphi(\lambda)$ is an isomorphism in category $\mathcal{F}_{\text{mer}}$.

Necessity: let $\Phi: \mathbb{M}_e \to \mathbb{M}_f$ be an isomorphism in category $\mathcal{F}_{\text{mer}}$. Set $\varphi(\lambda) := \Phi(1)$. Then $\varphi(\lambda) \in \mathbb{M}_\mathbb{C}^\times$. Applying $\Phi K_l(z) = K_l(z)\Phi$ to 1 we get

$$\varphi(\lambda + he_l)f_l(z) = \varphi(\lambda)e_l(z).$$  

So $\frac{e_l(z)}{f_l(z)} = \frac{\varphi_l(\lambda)}{\varphi_l(\lambda + he_l)}$, being independent of $z$, is a constant function $c_l \in \mathbb{C}^\times$. We have $e_l(z) = f_l(z)c_l$ and $\varphi(\lambda + he_l) = c_l\varphi(\lambda)$. It follows that $\varphi(\lambda) = \frac{\varphi(\lambda + he_1 + he_2 + \cdots + he_N) = c_1c_2\cdots c_N\varphi(\lambda)}{c_1c_2\cdots c_N}$. This implies $c_1c_2\cdots c_N = 1$. So $e$ and $f$ have the same image in $\mathcal{M}$. \hfill \Box

For each $f \in \mathcal{M}$, let us fix a pre-image $f' \in (\mathbb{M}_\mathbb{C})^N$ and set $\hat{\mathbb{M}}(f) := \mathbb{M}_f$. Then the isomorphism classes $\hat{\mathbb{M}}(f)$ for $f \in \mathcal{M}$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{F}_{\text{mer}})$. When no confusion arises, we identify an element of $(\mathbb{M}_\mathbb{C})^N$ with its image in $\mathcal{M}$.

Lemma 1.6. Let $V$ be in category $\mathcal{F}_{\text{mer}}$. Assume $B$ is an ordered basis of $V$ with respect to which the matrices of the difference operators $K_l(z)$ are upper triangular. Then for $b \in B$ and $1 \leq l \leq N$ there exist $\varphi_b(\lambda) \in \mathbb{M}_\mathbb{C}^\times$ and $f_{b,l}(z) \in \mathbb{M}_\mathbb{C}^\times$ such that

$$[K_l]_{bb}(z; \lambda) = f_{b,l}(z)\frac{\varphi_b(\lambda)}{\varphi_b(\lambda + he_l)}.$$  

Recall that $[K_l]_{bb}(z; \lambda)$ is the coefficient of $b$ in $K_l(z)b$. This lemma says that if the matrices of the $K_l(z)$ are upper triangular, then their diagonal entries must be of the form $f(z)h(\lambda)$, and the $h(\lambda)$ can be gauged away uniformly.

More precisely, the new basis $\{\varphi_b(\lambda)b \mid b \in B\}$ with the ordering induced from $B$ satisfies (M3) in Definition 1.3; the diagonal entry of $K_l(z)$ associated to $\varphi_b(\lambda)b$ is $f_{b,l}(z)$. This yields the following identity in the Grothendieck group $K_0(\mathcal{F}_{\text{mer}})$:

$$[V] = \sum_{b \in B} [\hat{\mathbb{M}}(f_{b,1}(z), f_{b,2}(z), \ldots, f_{b,N}(z))].$$  

Proof. Write $B = \{b_1 < b_2 < \cdots < b_m\}$. We proceed by induction on the dimension $m = \text{dim}(V)$. If $m = 1$, then there exist $f = (f_1(z), f_2(z), \ldots, f_N(z)) \in (\mathbb{M}_\mathbb{C})^N$ and an isomorphism $\Phi: \mathbb{M}_f \to V$ in category $\mathcal{F}_{\text{mer}}$. Let $\Phi(1) = \varphi(\lambda)b_1$. Then applying $\Phi K_l(z) = K_l(z)\Phi$ to 1 we obtain the desired identity

$$f_1(z)\varphi(\lambda) = [K_l]_{b_1,b_1}(z; \lambda)\varphi(\lambda + he_l).$$  

If $m > 1$, then the subspace $V'$ of $V$ spanned by $(b_1, b_2, \ldots, b_{m-1})$ is stable by the $K_l(z)$ and $D_l(z)$ by the triangularity assumption. So $V'$ is an object of category $\mathcal{F}_{\text{mer}}$ and we obtain a short exact sequence $0 \to V' \to V \to V/V' \to 0$. The rest is clear by applying the induction hypothesis to $V'$, $V/V'$, which have ordered bases $\{b_1 < b_2 < \cdots < b_{m-1}\}$ and $\{b_m + V'\}$ respectively. \hfill \Box

Definition 1.7. $\mathcal{O}$ is the full subcategory of $\mathcal{O}$ consisting of $\mathcal{E}$-modules $X$ such that $X[\mu]$ endowed with the action of the $\mathcal{D}_l(z)$ belongs to $\mathcal{F}_{\text{mer}}$ for all $\mu \in \text{wt}(X)$. 
The definition of $\tilde{\mathcal{O}}$ is standard as in the cases of Kac–Moody algebras \cite{[11]} and quantum affinizations \cite{[35]}. Definition \ref{L} is a special feature of elliptic quantum groups. It is meant to loose the dependence on the dynamical parameter $\lambda$. \footnote{For the elliptic quantum group associated to an arbitrary finite-dimensional simple Lie algebra, Gautam–Toledano Laredo \cite{[22]} \S2.3 defined a category of integrable modules on which the action of the elliptic Cartan currents, analogs of $D_k(z)$, is independent of $\lambda$. The asymptotic modules that we will construct in Section \ref{section} are not integrable.}

$\mathcal{O}$ is an abelian subcategory of $\widetilde{\mathcal{O}}$. For $X$ in category $\mathcal{O}$, Eq.\ref{E1.8} defines difference operators $K_l(z) : X \to X$ of bi-degree $(\epsilon_l, \epsilon_l)$ for $1 \leq l \leq N$. \footnote{\cite{[17] Definition 2.1}, a non-zero weight vector of a module $X$ in category $\tilde{\mathcal{O}}$ is called singular if it is annihilated by the $L_{ij}(z)$ for $1 \leq j < i \leq N$.}

Following \cite[Definition 2.1]{[7]}, a non-zero weight vector of a module $X$ in category $\tilde{\mathcal{O}}$ is called singular if it is annihilated by the $L_{ij}(z)$ for $1 \leq j < i \leq N$.

**Lemma 1.8.** Let $X$ be in category $\mathcal{O}$. If $v \in X$ is singular, then $K_i(z)v = \hat{L}_i(z)v$ for all $1 \leq i \leq N$.

**Proof.** Descending induction on $i$: for $i = N$ we have $K_N(z) = L_{NN}(z) = \hat{L}_N(z)$. Assume the statement for $i > N - t$ where $1 \leq t < N$. We need to prove the case $i = N - t$. Let $\alpha$ be the weight of $v$ and let $Y$ be the submodule of $X$ generated by $v$. By \cite[Lemma 2.3]{[7]}, $Y$ is linearly spanned by vectors of the form

$$L_{p_1q_1}(z_1)L_{p_2q_2}(z_2)\cdots L_{p_kq_k}(z_n)v$$

where $1 \leq p_l \leq q_l \leq N$ and $z_l \in \mathbb{C}$ for $1 \leq l \leq n$. So $\alpha + \epsilon_p - \epsilon_q \notin \text{wt}(Y)$ for $1 \leq p < q \leq N$, and any non-zero vector $\omega \in Y[\alpha]$ is singular. Apply $D_k(z)$ to $\omega$. At the right-hand side of Eq.(1.6) only the term $\sigma = 1$ is non-zero and equal to

$$\hat{L}_N(z)\hat{L}_{N-1}(z+h)\cdots \hat{L}_{N-t}(z+th)v$$

where $1 \leq N-t \leq N-1$. By \cite[Lemma 2.3]{[7]} we have $L_{N-1}(z+th)v = K_{N-1}(z+th)v$. We need to prove the case $i = N - t$, $\alpha = \epsilon_p - \epsilon_q$. For $1 \leq p < q \leq N$, and any non-zero vector $\omega \in Y[\alpha]$ is singular. Apply $D_k(z)$ to $\omega$.

$$D_{t+1}(z)v = \hat{L}_N(z)\hat{L}_{N-1}(z+h)\cdots \hat{L}_{N-t}(z+th)\hat{L}_{N-t}(z+th)v$$

where $1 \leq N-t \leq N-1$. By \cite[Lemma 2.3]{[7]} we have $L_{N-1}(z+th)v = K_{N-1}(z+th)v$. We need to prove the case $i = N - t$, $\alpha = \epsilon_p - \epsilon_q$. For $1 \leq p < q \leq N$, and any non-zero vector $\omega \in Y[\alpha]$ is singular. Apply $D_k(z)$ to $\omega$.

$$D_{t+1}(z)v = \hat{L}_N(z)\hat{L}_{N-1}(z+h)\cdots \hat{L}_{N-t}(z+th)\hat{L}_{N-t}(z+th)v$$

where $1 \leq N-t \leq N-1$. By \cite[Lemma 2.3]{[7]} we have $L_{N-1}(z+th)v = K_{N-1}(z+th)v$. We need to prove the case $i = N - t$, $\alpha = \epsilon_p - \epsilon_q$. For $1 \leq p < q \leq N$, and any non-zero vector $\omega \in Y[\alpha]$ is singular. Apply $D_k(z)$ to $\omega$.
Proposition 1.10. Let $X, Y$ be in category $O$. The $E$-module $X \otimes Y$ is also in category $O$ and $\chi_{\eta}(X \otimes Y) = \chi_{\eta}(X) \chi_{\eta}(Y)$.

Proof. Clearly $X \otimes Y$ is in category $\tilde{O}$. Let us verify Property (M3) of Definition 1.3. The idea is almost the same as that of [21, Prop.3.9], which in turn followed [27, 2.4]. For $\alpha, \beta \in \mathfrak{h}$, let us choose ordered bases $(v^i_\alpha)_{1 \leq i \leq p_\alpha}$ and $(w^j_\beta)_{1 \leq j \leq q_\beta}$ for $X[\alpha]$ and $Y[\beta]$ respectively satisfying (M3). Note that $(v^i_\alpha \otimes w^j_\beta)_{\alpha, \beta, i, j}$ forms a basis $B$ of $X \otimes Y$. Choose a partial order $\preceq$ on $B$ with the property:

(a) $v^i_\alpha \otimes w^j_\beta \preceq v^r_\alpha \otimes w^s_\beta$ if $i \leq r$ and $j \leq s$;
(b) $v^i_\alpha \otimes w^j_\beta \prec v^i_\alpha \otimes w^j_\beta$ if $\gamma < \alpha$ and $\beta < \delta$.

For $1 \leq k \leq N$, by Corollary 1.2, $D_k^{X \otimes Y}(z)(v^i_\alpha \otimes w^j_\beta) = D_k^X(z)v^i_\alpha \otimes D_k^Y(z)w^j_\beta + Z$ where $Z$ is a finite sum of vectors in $X[\gamma + \varpi_{N-k} + \eta] \otimes Y[\delta - \varpi_{N-k} - \eta]$ for $\eta \in \mathfrak{h}$ such that $-\varpi_{N-k} \prec \eta$. So the ordered basis $B$ induces an upper triangular matrix for $D_k^{X \otimes Y}(z)$ whose diagonal entry associated to $v^i_\alpha \otimes w^j_\beta$ is the product of those associated to $v^i_\alpha$ and $w^j_\beta$. This implies (M3) for the weight spaces $(X \otimes Y)[\alpha]$ with bases $B \cap (X \otimes Y)[\alpha]$ and the multiplicative formula of $q$-characters as well. □

For $f(z) \in M_\alpha^\mathbb{C}$ and $\alpha \in \mathfrak{h}$ we make the simplifications

$$f(z) := (f(z), \cdots, f(z); 0), \quad e^\alpha := (1, \cdots, 1; \alpha) \in M_w.$$

Definition 1.11. Let $1 \leq i, k \leq N$ such that $i \neq N$. Set $\ell_k := \frac{N-k-1}{2}$. For $a \in \mathbb{C}$, define the following elements of $M_w$:

$$A_{i,a} := (1, \cdots, 1, \underbrace{\theta(z + (a - \ell_i)h), \cdots, \theta(z + (a - \ell_i)h)}_{i-1}, \underbrace{\theta(z + (a - \ell_i - 1)h), \cdots, \theta(z + (a - \ell_i + 1)h)}_{N-i-1}, 1, \cdots, 1; \alpha_i);$$

$$\Psi_{k,a} := \underbrace{\theta(z + (a - \ell_k)h), \cdots, \theta(z + (a - \ell_k)h)}_{k}, 1, \cdots, 1; \alpha \varpi_k);$$

$$Y_{k,a} := \underbrace{\theta(z + (a - \ell_k + \frac{1}{2})h), \cdots, \theta(z + (a - \ell_k + \frac{1}{2})h)}_{k}, 1, \cdots, 1; \varpi_k);$$

$$[k] := (\frac{\theta(u + h)\theta(u - h)}{\theta(u)^2}, \cdots, \frac{\theta(u + h)\theta(u - h)}{\theta(u)^2}, \theta(u)\theta(u), 1, \cdots, 1; \epsilon_k |_{u = -z + ah}].$$

$A_{i,a}, Y_{k,a}$ and $\Psi_{k,a}$ are elliptic analogs of generalized simple roots, fundamental $\ell$-weight [27] and prefundamental weight [36]. Set $c_{ij} := 2\delta_{ij} - \delta_{i,j+1}$ and $Y_{0,a} = \Psi_{0,a} := 1$. Then (in the products $1 \leq j \leq N$)

$$Y_{k,a} = \frac{\Psi_{k,a + \frac{1}{2}}}{\Psi_{k,a - \frac{1}{2}}}, \quad A_{i,a} = \prod_j \frac{\Psi_{j,a + \frac{\epsilon_j}{2}}}{\Psi_{j,a - \frac{\epsilon_j}{2}}} = Y_{i,a - \frac{1}{2}}Y_{i,a + \frac{1}{2}} \prod_{j=1}^N Y_{j-1,a}.$$

The interplay of $A, \Psi$ is the source of the three-term Baxter’s Relation ([5,32]) in category $O$. Note that $A, Y$ can also be written in terms of $[k]$.

$$A_{i,a} = \prod_{j=1}^\infty Y_{i,a - \frac{1}{2}}, \quad Y_{k,a} = \prod_{j=1}^k Y_{k,a - \frac{1}{2}},$$

$$\Psi_{N,a} = \theta(z + (a + \frac{1}{2})h), \quad Y_{N,a} = \frac{\theta(z + (a + 1)h)}{\theta(z + ah)}.$$
1.4. Vector representations. Let $V := \oplus_{i=1}^{N} M v_i$ with $h$-grading $V[e_i] = M v_i$.
Rewriting Eq.(1.1) in the form of Eq.(1.2), we obtain an $\mathcal{E}$-module structure on $V$:

$$L_{ij}(z)v_k = \sum_{l=1}^{N} \frac{\theta(z + h)}{\theta(z)} R_{il}^{jk}(z; \lambda)v_l.$$ 

The factor $\frac{\theta(z + h)}{\theta(z)}$ is used to simplify the q-character; see Eq.(1.12).

If $i \leq N - k + 1$, since $L_{pq}(z)v_i = 0$ for all $N \geq p > q > N - k$, only the term $\sigma = 1d$ in Eq.(1.6) survives and

$$D_k(z)v_i = \frac{\mu_2(\Theta_k(\lambda))}{\mu_1(\Theta_k(\lambda))} \prod_{j=N}^{N-k+1} L_{jj}(z + (N - j)h)v_i = g_k^i(z; \lambda)v_i,$$

$$g_k^i(z; \lambda) = \prod_{j>N-k} \frac{\theta(\lambda_{ij} + h)}{\theta(\lambda_{ij})} \quad \text{for } i \leq N - k, \quad g_k^{N-k+1}(z; \lambda) = \frac{\theta(z + kh)}{\theta(z + (k - 1)h)}.$$ 

If $i > N - k + 1$, then $L_{N-k+1,i}(z)v_{N-k+1} = \frac{\theta(h)\theta(z + \lambda_{N-k+1,i})}{\theta(z)\theta(\lambda_{N-k+1,i})}v_i$. Let us perform a change of basis (see [45, Eq.(E.2)]):

$$\tilde{v}_1 := v_i \prod_{l > i} \theta(\lambda_{il} + h) \in V[e_i].$$

After a direct computation, we obtain:

$$D_k(z)\tilde{v}_1 = \tilde{v}_1 \times \begin{cases} 1 & \text{for } i \leq N - k, \\ \left(\frac{\theta(z + kh)}{\theta(z + (k - 1)h)}\right) & \text{for } i > N - k. \end{cases}$$

The basis $\{\tilde{v}_1 < \tilde{v}_2 \cdots < \tilde{v}_N\}$ of $V$ satisfies Property (M3) of Definition 1.4 so $V$ is in category $\mathcal{O}$. For $a \in \mathcal{C}$, let $V(a)$ be the pullback of $V$ by the spectral parameter shift $\Phi_a$ in Eq.(1.13). Naturally $V(a)$ is in category $\bar{O}$; it is called a vector representation. Combining with Eq.(1.8) we have:

$$\chi_a(V(a)) = [1] + [2] + \cdots + [N].$$

1.5. Highest weight modules. Let $X$ be in category $\mathcal{O}$. A non-zero weight vector $v \in X[\alpha]$ is called a highest weight vector if it is singular and $L_k(z)v = f_k(z)v$ for $1 \leq k \leq N$; here the $f_k(z) \in M_{\alpha}^\mathbb{C}$. Call $(f_1(z), f_2(z), \cdots, f_N(z); \alpha) \in M_{\alpha}$ the highest weight of $v$; by Lemma 1.8 it belongs to $w_\alpha(X)$ if $X$ is in category $\mathcal{O}$.

If there is a highest weight vector $v \in X[\alpha]$ of $X$ which also generates the whole module, then $X$ is called a highest weight module; see [7] Definition 2.1. In this case, by [2] Lemma 2.3], $X[\alpha] = M v$ and $w_\alpha(X) \subset \alpha + Q_{-}$, so the highest weight vector is unique up to scalar product. This implies that $X$ admits a unique irreducible quotient. The highest weight of $v$ is also called the highest weight of $X$; it is of multiplicity one in $\chi_a(X)$ if $X$ is in category $\mathcal{O}$.

All irreducible modules in category $\mathcal{O}$ are of highest weight.

By [7], Theorem 2.8: two irreducible highest weight modules in category $\mathcal{O}$ are isomorphic if and only if their highest weights are identical in $M_{\alpha}$: all singular vectors of an irreducible highest weight module in category $\mathcal{O}$ are proportional. It follows that the q-characters distinguish irreducible modules in category $\mathcal{O}$.

Let $\mathcal{R}$ be the set of $d \in M_{\alpha}$ which appears as the highest weight of an irreducible module in category $\mathcal{O}$. For $d \in \mathcal{R}$, let us fix an irreducible module $S(d)$ in category $\mathcal{O}$ of highest weight $d$. Let $R_{a} \ (\text{resp. } R_{d})$ be the set of $d \in \mathcal{R}$ such that $S(d)$ is one-dimensional (resp. finite-dimensional).

We shall need the completed Grothendieck group $K_{\mathcal{O}}(\mathcal{O})$. Its definition is the same as that in [57] §3.2: elements are formal sums $\sum_{d \in \mathcal{R}} c_d[S(d)]$ with integer coefficients $c_d \in \mathbb{Z}$ such that $\gamma_{\mathcal{O}}S(d)[a]_{d} = \alpha$ is in category $\mathcal{O}$; addition is the usual
one of formal sums. As in the case of Kac–Moody algebras [11], the multiplicity $m_{d,X}$ of $S(d)$ in any object $X$ of category $O$ is well-defined due to Definition 1.7 and $[X] := \sum_{d} m_{d,X} [S(d)]$ belongs to $K_{0}(O)$. In the case $X = S(d)$ the right-hand side is simply $[S(d)] = \delta_{d,e}$ for $e \in \mathcal{R}$.

By Proposition 1.10 $K_{0}(O)$ is endowed with a ring structure with multiplication $[X][Y] = [X \otimes Y]$ for $X, Y$ in category $O$. Together with Definition 1.9 we obtain

**Corollary 1.12.** The assignment $[X] \mapsto \chi_{0}(X)$ defines an injective morphism of rings $\chi_{0} : K_{0}(O) \rightarrow \mathcal{M}_{w}$. In particular, $K_{0}(O)$ is commutative.

Let $O_{fd}$ be the full subcategory of $O$ consisting of finite-dimensional modules. It is abelian and monoidal. Its Grothendieck ring $K_{0}(O_{fd})$ admits a $\mathbb{Z}$-basis $[S(d)]$ for $d \in \mathcal{R}_{fd}$, and is commutative as a subring of $K_{0}(O)$.

By Proposition 1.10 $S(d) \otimes S(e)$ admits an irreducible sub-quotient $S(de)$, so the three sets $\mathcal{R} \supset \mathcal{R}_{fd} \supset \mathcal{R}_{0}$ are sub-monoids of $\mathcal{M}_{w}$.

**Lemma 1.13.** Let $d = ((f_{k}(z))_{1 \leq k \leq N}; \mu) \in \mathcal{M}_{w}$.

(i) Suppose $d \in \mathcal{R}$. Then for $1 \leq k \leq N$ we have

$$\frac{f_{k}(z)}{f_{k+1}(z)} = \sum_{l=1}^{n} (a_{l} - b_{l}) \frac{\theta(z + a_{l}h)}{\theta(z + b_{l}h)} \mu_{k,k+1} = \sum_{l=1}^{n} (a_{l} - b_{l})$$

for certain $a_{1}, a_{2}, \ldots, a_{n}, b_{1}, b_{2}, \ldots, b_{n} \in \mathbb{C}$ and $c \in \mathbb{C}$.\n
(ii) If $d \in \mathcal{R}_{fd}$, then (i) holds and after a rearrangement of the $a_{i}, b_{i}$ we have $a_{i} - b_{i} \in \mathbb{Z}_{\geq 0} + h^{-1} \mathbb{N}$ for all $l$.

(iii) $d \in \mathcal{R}_{0}$ if and only if (ii) holds with $a_{i} - b_{i} \in h^{-1} \mathbb{N}$ for all $l$.

**Proof.** (i) and (iii) are essentially [19] Theorems 6 & 9, which can be proved as in [21] Theorem 4.1 by replacing $L_{\pm \ldots, L_{+-}}$ therein with $L_{k,k+1}, L_{k+1,k}$. (ii) comes from either [7] Theorem 5.1 or [21] Corollary 4.6. □

As examples $Y_{N,a}, \Psi_{N,a} \in \mathcal{R}_{0}$. Call an $e$-weight $e \in \mathcal{M}_{w}$ dominant (resp. rational) if $e = \mathbf{m} \mathbf{n}$ where $d \in \mathcal{R}_{0}$ and $\mathbf{m}$ is a product of the $Y_{i,a}$ (resp. the $\Psi_{i,a}\Psi_{i,a}^{\dagger}$) with $a, b \in \mathbb{C}$ and $1 \leq i \leq N$. Lemma 1.13 implies that all elements of $\mathcal{R}_{fd}$ (resp. $\mathcal{R}$) are dominant (resp. rational).

**Theorem 1.14.** [7] $\mathcal{R}_{fd}$ is the set of dominant $e$-weights.

**Proof.** It suffices to prove $Y_{n,a} \in \mathcal{R}_{fd}$ for $1 \leq n \leq N$. Note that $V(w)$ and $\gamma$ from [7] Eq.(1.19)] correspond to our $V(-\frac{\mu}{2}) \otimes S(\frac{\theta(z-w)}{\theta(z-w-h)})$ and $-h$. Let us rephrase [7] Theorem 4.4 in terms of the $V$ by replacing $z, w$ in loc. cit. with $-ah, z$.

The $\mathcal{E}$-module $V(a) \otimes V(a+1) \otimes \cdots \otimes V(a+n-1)$ admits an irreducible quotient $S$ which contains a singular vector $\omega$ of weight $\varpi_{n}$ such that $L_{k}(\omega) = \Lambda_{k}(\omega)$ for $1 \leq k \leq N$ (set $\delta_{k \leq n} = 1$ if $1 \leq k \leq n$ and $\delta_{k \leq n} = 0$ if $n < k \leq N$):

$$\Lambda_{k}(\omega) = \frac{\theta(z + (a + 1)h)}{\theta(z + ah)} \frac{\theta(z + (a + 1)h)}{\theta(z + (a + n - \delta_{k \leq n})h)}, \quad g_{k}(\lambda) \in M^{\mathbb{C}}.$$  

As a sub-quotient of tensor products of vector representations, $S$ belongs to category $O$. By Lemma 1.6 the $g_{k}(\lambda)$ can be gauged away, and the highest weight of $S$ is $\Lambda_{N}(\omega)Y_{n,a}^{n-1} \mathbf{w}_{n,a} \in \mathcal{R}_{fd}$. This implies $Y_{n,a}^{n-1} \mathbf{w}_{n,a} \in \mathcal{R}_{fd}$. □

A sharp difference from the affine case [36] Theorem 3.11] is that category $O$ does not admit prefundamental modules, i.e. $\Psi_{r,a} \notin \mathcal{R}$ if $r < N$. One might want to introduce a larger category with well-behaved $q$-character theory, so that modules of highest weight $\Psi_{r,a}$ exist. For this purpose, the finite-dimensionality of weight spaces should be dropped because of [19] Theorem 9. The recent work [6] on representations of affine quantum groups is in this direction.
1.6. Young tableaux and q-character formula. Let \( \mathcal{P} \) be the set partitions with at most \( N \) parts, i.e. \( N \)-tuples of non negative integers \( (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N) \). To such a partition we associate a Young diagram

\[
Y_\mu := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq N, 1 \leq j \leq \mu_i\},
\]

and the set \( \mathcal{R}_\mu \) of Young tableaux of shape \( Y_\mu \). We put the Young diagram at the northwest position so that \((i, j) \in Y_\mu\) corresponds to the box at the \( i \)-th row (from bottom to top) and \( j \)-th column (from right to left). By a tableau we mean a function \( T : Y_\mu \rightarrow \{1 < 2 < \cdots < N\} \) weakly increasing at each row (from left to right) and strictly increasing at each column (from top to bottom).

For \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N) \in \mathcal{P} \) and \( a \in \mathbb{C} \), we have the dominant \( c \)-weight

\[
\theta_{\mu, a} := \begin{pmatrix}
\theta(z + (a + \mu_1)h) & \theta(z + (a + \mu_2)h) & \cdots & \theta(z + (a + \mu_N)h) \\
\theta(z + ah) & \theta(z + ah) & \cdots & \theta(z + ah)
\end{pmatrix}.
\]

The associated irreducible module in category \( \mathcal{O}_{\text{id}} \) is denoted by \( S_{\mu, a} \).

**Theorem 1.15.** Let \( \mu \in \mathcal{P} \) and \( a \in \mathbb{C} \). For the \( \mathcal{E}_{\tau, h}(sl_N) \)-module \( S_{\mu, a} \) we have

\[
\chi_q(S_{\mu, a}) = \sum_{T \in \mathcal{R}_\mu, (i, j) \in Y_\mu} T(i, j)_{i+j-i} \in \mathcal{M}_t.
\]

For \( \nu = (1 \geq 0 \geq \cdots \geq 0) \), we have \( S_{\nu, a} \cong V(a) \), and Eq. (1.13) specializes to the \( q \)-character formula in Section 1.4. As an illustration of the theorem, let \( N = 3 \) and \( \mu = (2 \geq 1 \geq 0) \). Pictorially \( \mathcal{R}_\mu \) consists of:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 \\
1 & 3 & 2 & 2 & 3 & 3 & 2 & 3 \\
\end{array}
\]

The fourth tableau gives rise to the term \( 211, 121, 112 \) in \( \chi_q(S_{\mu, a}) \).

**Remark 1.16.** Theorem 1.15 is an elliptic analog of the \( q \)-character formula for affine quantum groups [26 Lemma 4.7]. In principle it can be deduced from the functor of Gautam–Toledano Laredo [32, §6]. This is a functor from finite-dimensional representations of affine quantum groups to those of elliptic quantum groups (including our \( S_{\mu, a} \)), and it respects affine and elliptic \( q \)-characters.

The proof of Theorem 1.15 will be given in Section 2.4. It is in the spirit of [26], based on small elliptic quantum groups of Tarasov–Varchenko [51].

2. Small elliptic quantum group and evaluation modules

The aim of this section is to prove Corollary 1.2 and Theorem 1.15.

Recall that \( h \) is the \( \mathbb{C} \)-vector space generated by the \( \epsilon_i \) for \( 1 \leq i \leq N \) subject to the relation \( \epsilon_1 + \epsilon_2 + \cdots + \epsilon_N = 0 \). For \( 1 \leq k \leq N \), define the \( \mathbb{C} \)-vector space \( h_k \) to be the quotient of \( h \) by \( \epsilon_1 = \epsilon_2 = \cdots = \epsilon_{N-k} = 0 \). (By convention \( h_N = h \).) The quotient \( h \twoheadrightarrow h_k \) induces an embedding \( \mathbb{M}_{h_k} \hookrightarrow \mathbb{M} \).

Let \( \mathcal{E}_k \) (resp. \( \mathcal{E}_k \)) be the \( h \)-algebra (resp. \( h_k \)-algebra) generated by the \( L_{ij}(z) \) for \( N - k < i, j \leq N \) subject to Relation (1.2), with summations \( N - k < p, q \leq N \). (This makes sense because the \( R_{ij}^p(z; \lambda) \) for \( N - k < i, j, p, q \leq N \) belong to \( \mathbb{M}_{h_k} \).)

The following defines an \( h_k \)-algebra morphism

\[
\Delta_k : \mathcal{E}_k \rightarrow \mathcal{E}_k \otimes \mathcal{E}_k, \quad L_{ij}(z) \mapsto \sum_{p=N-k+1}^N L_{ip}(z) \otimes L_{pj}(z).
\]

One has natural algebra morphisms \( \mathcal{E}_k \rightarrow \mathcal{E}_k^0 \rightarrow \mathcal{E} \) sending \( L_{ij}(z) \) to itself; the second is an \( h \)-algebra morphism. \( D_1(z), D_2(z), \cdots, D_k(z) \) from Eq. (1.13) are well-defined in \( \mathcal{E}_k^0 \) and \( \mathcal{E}_k \). Their images in \( \mathcal{E} \) are the first \( k \) elliptic quantum minors.
2.1. Proof of Corollary 1.2. The \( \mathfrak{h} \)-algebra with coproduct \( (\mathcal{E}_\alpha, \Delta_\alpha) \) is isomorphic to the usual elliptic quantum group \( \mathcal{E}_{r,h}(\mathfrak{g}_k) \); here we view \( \mathfrak{h} \) as a Cartan subalgebra of \( \mathfrak{g}_k \) so that \( \mathcal{E}_{r,h}(\mathfrak{g}_k) \) is an \( \mathfrak{h} \)-algebra. Under this isomorphism, by Eq. (1.14), \( D_k(z) \in \mathcal{E}_\alpha \) corresponds to the \( k \)-th elliptic quantum minor of \( \mathcal{E}_{r,h}(\mathfrak{g}_k) \).

So Theorem 1.1 can be applied to \( (\mathcal{E}_\alpha, D_k(z), \Delta_\alpha) \) and then to the algebra morphism \( \mathcal{E}_k \longrightarrow \mathcal{E} \). The first statement of the corollary is obvious, and the second is based on the fact that for \( i, j > N - k \) the difference \( \Delta - \Delta_k \) at \( L_{ij}(z) \) is a finite sum over \( \alpha \in \mathfrak{h} \) of elements in \( \mathcal{E}_{\alpha,1} \otimes \mathcal{E}_{\alpha,1} \) with \( \Delta_N - k + 1 < \alpha \) and so \( \epsilon_i, \epsilon_j < \alpha \). \( \square \)

We believe \( 0 \neq \alpha + \omega_{N-k} \in \mathbb{Q}_+ \) in Corollary 1.2 as in [51] [7] and [53] [3].

2.2. Small elliptic quantum group of Tarasov–Varchenko [51]. Let us define the linear form \( \lambda_i \in \mathfrak{h}^* \) of taking \( i \)-th component for \( 1 \leq i \leq N \):

\[
x_1 \epsilon_1 + x_2 \epsilon_2 + \cdots + x_N \epsilon_N \mapsto x_i - \frac{1}{N} (x_1 + x_2 + \cdots + x_N).
\]

The linear form \( \lambda_{ij} \) of Section 1 is \( \lambda_i - \lambda_j \). For \( \gamma \in \mathfrak{h} \) and \( 1 \leq i, j \leq N \), set \( \gamma_i := \lambda_i(\gamma) \) and \( \gamma_{ij} := \gamma_i - \gamma_j \) as complex numbers. We hope this is not to be confused with the previously defined vectors \( \lambda_i \in \mathfrak{h}^* \) and \( \epsilon_i, \epsilon_j, \omega_{\alpha} \in \mathfrak{h} \).

Following [51] [3], let \( \mathbb{M}_2 \) be the ring of meromorphic functions \( f(\lambda(1), \lambda(2)) \) of \( \lambda(1), \lambda(2) \in \mathfrak{h} \oplus \mathfrak{h} \) whose location of singularities in \( \lambda(1) \) does not depend on \( \lambda(2) \) and vice versa. For brevity, we write \( f(\lambda(1)) \) or \( f(\lambda(2)) \) instead of \( f(\lambda(1), \lambda(2)) \) if the function does not depend on the other variable.

Definition 2.1. [51] The small elliptic quantum group \( \varepsilon := \varepsilon(\mathfrak{g}_N) \) is the algebra with generators \( \mathbb{M}_2 \) and \( t_{ij} \) for \( 1 \leq i, j \leq N \) and subject to relations: \( \mathbb{M}_2 \) is a subalgebra; for \( f(\lambda(1), \lambda(2)) \in \mathbb{M}_2 \) and \( 1 \leq i, j, k, l \leq N \),

\[
t_{ij} f(\lambda(1), \lambda(2)) = f(\lambda(1) + h \epsilon_i, \lambda(2) + h \epsilon_j) t_{ij}, \quad t_{ij} t_{ik} = t_{ik} t_{ij},
\]

\[
t_{ik} t_{jk} = \frac{\theta(\lambda_{ij}^1 - h) - \theta(\lambda_{ij}^2 - h)}{\theta(\lambda_{ik}^1 + h)} t_{jk} t_{ik} \quad \text{for } i \neq j,
\]

\[
t_{ij} t_{kl} - \frac{\theta(\lambda_{ik}^1 - h)}{\theta(\lambda_{ik}^1 + h)} t_{kl} t_{ij} = \frac{\theta(\lambda_{ij}^1 + \lambda_{kl}^2) - \theta(-h)}{\theta(\lambda_{ik}^1) \theta(\lambda_{ij}^2)} t_{ij} t_{kl} \quad \text{for } i \neq k \text{ and } j \neq l.
\]

Here \( \lambda_{ij}^1 = \lambda_i^1 - \lambda_j^1 \) and \( \lambda_{ij}^2 = \lambda_i^2 - \lambda_j^2 \).

\( \varepsilon \) is equipped with an \( \mathfrak{h} \)-algebra structure: elements of \( \mathbb{M}_2 \) are of bi-degree \((0,0)\); \( t_{ij} \) is of bi-degree \((\epsilon_j, \epsilon_i)\); the moment maps are given by

\[
\mu_1(g(\lambda)) = g(\lambda(1)), \quad \mu_2(g(\lambda)) = g(\lambda(2)).
\]

Let \( X \) be an object of \( \mathcal{V}_\mathfrak{h} \). A representation \( \rho \) of \( \varepsilon \) on \( X \) is a morphism of \( \mathfrak{h} \)-algebras \( \rho : \varepsilon \longrightarrow \mathbb{D}^X \) such that for \( f(\lambda(1), \lambda(2)) \in \mathbb{M}_2 \) and \( v \in X[\gamma] \),

\[
\rho(f(\lambda(1), \lambda(2))) : v \mapsto f(\lambda, \lambda + h \gamma)v.
\]

A morphism of two representations \( (\rho, X) \) and \( (\sigma, Y) \) is a morphism \( \Phi : X \longrightarrow Y \) in \( \mathcal{V}_\mathfrak{h} \) such that \( \Phi \rho(t_{ij}) = \sigma(t_{ij}) \Phi \) for \( 1 \leq i, j \leq N \). Let \( \text{rep} \) be the category of \( \varepsilon \)-modules. The following result is [51] Corollary 3.4].

Corollary 2.2. Let \( (\rho, X) \) be a representation of \( \varepsilon \) on \( X \). Then for \( a \in \mathbb{C} \),

\[
L_{ij}(z) \mapsto \frac{\theta(z + ah + \lambda_{ij}^1)}{\theta(z + ah)} \rho(t_{ji})
\]

defines a representation of \( \mathcal{E} \) on \( X \), called the evaluation module \( X(a) \).
There is a flip of the subscripts $i, j$ because the bi-degrees of $L_{ij}$ and $t_{ij}$ are flips of each other. See also [51, Eq.(3.6)] where $T_{ij}(u)$ comes from $t_{ji}$. $X \mapsto X(a)$ defines a functor $ev_a : \text{rep} \to \text{Rep}$. Let $\mathcal{F}$ be the full subcategory of \text{rep} whose objects are finite-dimensional $\epsilon$-modules $X$ with $X(x)$ being in category $\mathcal{O}$. Then $ev_a$ restricts to a functor of abelian categories $\mathcal{F} \to \mathcal{O}_{fd}$, and induces an injective morphism of Grothendieck groups $K_0(\mathcal{F}) \to K_0(O_{fd})$.

For $1 \leq k \leq N$, define $\hat{t}_k \in \mathfrak{e}$ in the same way as Eq. (17). Then

$$\hat{t}_N(z) := t_{NN}, \quad \hat{t}_k(z) = \prod_{j=k+1}^{N} \frac{\mu_\nu(\theta(\lambda_k))}{\mu_\mu(\theta(\lambda_k))};$$

Let $\mu \in \mathfrak{h}$. There exists a unique (up to isomorphism) irreducible $\epsilon$-module $V_\mu$ with the property: $V_\mu$ admits a non-zero vector $v$ of weight $\mu$ such that $\hat{t}_k v = v$, $\hat{t}_j v = 0$ for $1 \leq i, j, k \leq N$ and $j < k$; it is called standard in [51, §4]. Let $L_\mu$ denote the complex irreducible module over the simple Lie algebra $\mathfrak{sl}_N$ of highest weight $\mu$. For $v \in \mathfrak{h}$, let $d_\mu[v] = \dim_{\mathbb{C}} L_\mu[v]$ where $L_\mu[v]$ is the weight space of weight $v$.

**Theorem 2.3.** [51, Theorem 5.9] The $\epsilon$-module $V_\mu$ is finite-dimensional if and only if $\mu_{kj} \in \mathbb{Z}_{\geq 0} + h^{-1}\Gamma$ for $1 \leq i < j \leq N$. If $\tilde{\mu} \in \mathfrak{h}$ is such that $\mu_{ij} - \tilde{\mu}_{ij} \in h^{-1}\Gamma$ and $\tilde{\mu}_{ij} \in \mathbb{Z}_{\geq 0}$ for $i < j$, then $\dim V_\mu[\mu + \gamma] = d_\mu[\tilde{\mu} + \gamma]$ for $\gamma \in \mathbb{Q}_-$. In the theorem $\tilde{\mu}$ is uniquely determined by $\mu$ since $\mathbb{Z} \cap h^{-1}\Gamma = \{0\}$. Such an $\epsilon$-module $V_\mu$ is in category $\mathcal{O}$. Indeed, the evaluation module $V_\mu(a)$ is irreducible in category $\mathcal{O}$ of highest weight

$$\left( \frac{\theta(z + (\mu_1 + a)h)}{\theta(z + ah)}, \frac{\theta(z + (\mu_2 + a)h)}{\theta(z + ah)}, \ldots, \frac{\theta(z + (\mu_N + a)h)}{\theta(z + ah)} \right);$$

One checks that such an $\epsilon$-weight is dominant. So $V_\mu(a)$ is in category $\mathcal{O}_{fd}$ by Theorem 1.13. The character $\chi(V_\mu)$ of $V_\mu$ is $\sum_\gamma d_\mu[\tilde{\mu} + \gamma] e^{\mu + \gamma} \in \mathcal{M}_\epsilon$.

The isomorphism classes $[V_\mu]$ where $\mu \in \mathfrak{h}$ and $\mu_{ij} \in \mathbb{Z}_{\geq 0} + h^{-1}\Gamma$ for $i < j$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{F})$, and $[V_\mu] \mapsto \chi(V_\mu)$ extends uniquely to a morphism of abelian groups $\chi : K_0(\mathcal{F}) \to \mathcal{M}_\epsilon$, which is injective thanks to the linear independence of characters of irreducible representations of the simple Lie algebra $\mathfrak{sl}_N$.

### 2.3. Category $\mathcal{O}'_{fd}$
We are going to prove Theorem 1.13 by induction on $N$. The idea is to view the irreducible $\mathcal{E}$-module $S_{\mu,a}$ as an $E_N$-module and to apply the induction hypothesis. For this purpose, we need to adapt carefully the definitions of finite-dimensional module category $\mathcal{O}_{fd}$ and its q-characters in Section 1.3 to $E_N$. To distinguish with $\mathcal{E}$ and to simplify notations, we shall add a prime (instead of the index $N - 1$) to objects related to $E_{N-1}$. Notably $\mathfrak{h}' := \mathfrak{h}_{N-1}$.

We define category $\mathcal{O}'_{fd}$. An object is a finite-dimensional $\mathfrak{h}$-graded vector space $X$ (viewed as an object of category $\mathcal{V}_{\mathfrak{h}}$) endowed with difference operators $L^X_{ij}(z) : X \to X$ of bi-degree $(\epsilon_i, \epsilon_j)$ for $2 \leq i, j \leq N$ depending on $z \in \mathbb{C}$ such that:

- (M1') there exists a basis of $X$ with respect to which the matrix entries of the difference operators $L^X_{ij}(z)$ are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$;
- (M2') $L_{ij}(z) \to L^X_{ij}(z)$ defines an $\mathfrak{h}$-algebra morphism $E_{N-1} \to \mathbb{D}^X$;
- (M3') $X$ admits an ordered weight basis with respect to which the matrices of the difference operators $D^X_{ij}(z)$ for $1 \leq l < N$ are upper triangular and their diagonal entries are non-zero meromorphic functions of $z \in \mathbb{C}$.

A morphism in category $\mathcal{O}'_{fd}$ a linear map $\Phi : X \to Y$ such that $\Phi L^Y_{ij}(z) = L^X_{ij}(z)\Phi$ for $2 \leq i, j \leq N$. Category $\mathcal{O}'_{fd}$ is an abelian subcategory of $\mathcal{V}_{\mathfrak{h}}$.

The $\mathfrak{h}$-algebra morphism $E_{N-1} \to \mathcal{E}$ induces restriction functor $\mathcal{O}_{fd} \to \mathcal{O}'_{fd}$.

---

8 The $t_n$ are slightly different from the $\hat{t}_n$ in [51, Eq.(4.1)]. Yet they play the same role.
Let $X$ be in category $\mathcal{O}_{\text{ld}}$. Eq. (1.8) defines difference operators $K^X_l(z) : X \longrightarrow X$ of bi-degree $(\epsilon_l, \epsilon_l)$ for $2 \leq l \leq N$. Condition (M3') implies that for each weight $\alpha$, the weight space $X[\alpha]$ admits an ordered basis $B_\alpha$ with respect to which the matrix of $K^X_l(z)$ is upper triangular and has as diagonal entries $f_{b,l}(z) \in \mathbb{M}_b^e$ for $b \in B_\alpha$. Following Definition 1.9 we define the q-character of $X$ to be

$$\chi^*_\alpha(X) = \sum_{\alpha' \in \text{wt}(X)} \sum_{b \in B_\alpha} (1, f_{b,2}(z), f_{b,3}(z), \ldots, f_{b,N}(z); \alpha) \in \mathcal{M}_t.$$  

It is independent of the choice of the bases $B_\alpha$, as one can use category $\mathcal{F}_{\text{mer}}$ to characterize the $f_{b,l}(z)$; see the comments after Lemma 1.6.

**Remark 2.4.** Let $X$ be in category $\mathcal{O}_{\text{ld}}$, viewed as an object of category $\mathcal{O}'_{\text{ld}}$. Then $\chi^*_\alpha(X)$ is obtained from $\chi_\alpha(X)$ by replacing each e-weight $g$ of the $\mathcal{E}$-module $X$ with $g'$; here for $g = (g_1(z), g_2(z), \ldots, g_N(z); \alpha) \in \mathcal{M}_t$ we define

$$g' = (1, g_2(z), g_3(z), \ldots, g_N(z); \alpha) \in \mathcal{M}_t.$$  

Reciprocally, if $X$ is an irreducible $\mathcal{E}$-module in category $\mathcal{O}_{\text{ld}}$ of highest weight $(e_1(z), e_2(z), \ldots, e_N(z); \alpha) \in \mathcal{R}_{\text{ld}}$, then $\chi_\alpha(X)$ can be recovered from $\chi^*_\alpha(X)$. Indeed, since the $N$-th elliptic quantum minor is central, by Schur Lemma, it acts on $X$ as a scalar. Each e-weight $(f_1(z), f_2(z), \ldots, f_N(z); \beta)$ of the $\mathcal{E}$-module $X$ is determined by the last $N$ components of $\chi^*_\alpha(X)$ as follows:

$$e_1(z+(N-1)h)e_2(z+(N-2)h)\cdots e_N(z) = f_1(z+(N-1)h)f_2(z+(N-2)h)\cdots f_N(z).$$  

The highest weight theory in Section 1.5 carries over to category $\mathcal{O}'_{\text{ld}}$ since $\hat{L}_k(z) \in \mathcal{E}_{N-1}^h$ for $2 \leq k \leq N$. Irreducible objects in $\mathcal{O}'_{\text{ld}}$ are characterized by their highest weight, and the q-character map is an injective morphism from the Grothendieck group $K_0(\mathcal{O}'_{\text{ld}})$ to the additive group $\mathcal{M}_t$. Let $\mathcal{P}'$ be the set of partitions with at most $N-1$ parts ($\nu_2 \geq \nu_3 \geq \cdots \geq \nu_N$). For such a partition and for $c, a \in \mathbb{C}$,

$$\begin{pmatrix} 1, \frac{\theta(z + (a + \nu_2)h)}{\theta(z + ah)} & \cdots & \frac{\theta(z + (a + \nu_N)h)}{\theta(z + ah)} \end{pmatrix} \cdot \nu_1 + \sum_{j=2}^N \nu_j e_j,$$

is the highest weight of an irreducible $\mathcal{E}_{N-1}^h$-module in category $\mathcal{O}'_{\text{ld}}$, which is denoted by $S_{\nu,c,a}^h$. As in Section 1.6, $\nu$ is identified with its Young diagram $Y_\nu$. Let $\mathcal{B}'_\nu$ be the set of Young tableaux $Y_\nu \longrightarrow \{2 < 3 < \cdots < N\}$ of shape $\nu$.

**Lemma 2.5.** Assume that Theorem 1.7 is true for $\mathcal{E}_{\text{LD}}(sl(N-1))$-modules. Then for $\nu \in \mathcal{P}'$ and $c, a \in \mathbb{C}$, the q-character of the $\mathcal{E}_{N-1}^h$-module $S_{\nu,c,a}^h$ is

$$\chi^*_\nu(S_{\nu,c,a}^h) = e^{c\nu_1} \sum_{T \in \mathcal{B}'_\nu} \prod_{(i,j) \in Y_\nu} T(i,j)_{+1,i-1} \in \mathcal{M}_t.$$  

**Proof.** We shall need $\mathcal{E}_{N-1}$-modules which are $\mathfrak{h}'$-graded $\mathbb{M}_b$-vector spaces; similar category of finite-dimensional modules and q-characters are defined, based on the $\mathfrak{h}'$-algebra isomorphism $\mathcal{E}_{\text{LD}}(sl(N-1)) \cong \mathcal{E}_{N-1}$ in Section 2.4.

For $\nu := (\nu_2 \geq \nu_3 \geq \cdots \geq \nu_N) \in \mathcal{P}'$ and $a \in \mathbb{C}$, there exists a unique (up to isomorphism) irreducible $\mathcal{E}_{N-1}$-module, denoted by $S_{\nu,a}^h$, which contains a non-zero vector $\omega$ of $h'$-weight $\nu_2 e_2 + \nu_3 e_3 + \cdots + \nu_N e_N$ such that

$$L_{ij}(z)\omega = 0, \quad \hat{L}_k(z)\omega = \frac{\theta(z + (a + \nu_k)h)}{\theta(z + ah)} \omega,$$

for $2 \leq i, j, k \leq N$ with $j < i$. We endow the $\mathbb{M}$-vector space $X := \mathbb{M} \otimes \mathbb{M}_a S_{\nu,a}^h$ with an $\mathcal{E}_{N-1}$-module structure in category $\mathcal{O}'_{\text{ld}}$.  

---
Let $w$ be a non zero weight vector in $S'_{\nu,a}$. Its $\mathfrak{h}'$-weight is written uniquely in the form $(\nu_2 \epsilon_2 + \nu_3 \epsilon_3 + \cdots + \nu_N \epsilon_N) + (x_2 \alpha_2 + x_3 \alpha_3 + \cdots + x_{N-1} \alpha_{N-1}) \in \mathfrak{h}'$, where $x_j \in \mathbb{Z}_{\geq 0}$. Define the $\mathfrak{h}$-weight of $g(\lambda) \otimes M_w$, for $g(\lambda) \in M^\times$, to be

$$(\alpha_1 + \nu_2 \epsilon_2 + \nu_3 \epsilon_3 + \cdots + \nu_N \epsilon_N) + (x_2 \alpha_2 + x_3 \alpha_3 + \cdots + x_{N-1} \alpha_{N-1}) \in \mathfrak{h},$$

and define the action of $L_{ij}(z)$ for $2 \leq i, j \leq N$ by the formula

$$L_{ij}(z)(g(\lambda) \otimes M_w) = g(\lambda + h_{ij} z) \otimes M_w.$$ 

(M1')–(M2') are clear from the $E_{N-1}$-module structure on $S'_{\nu,a}$. Choose an ordered weight basis $B$ of $S'_{\nu,a}$ over $M_{\mathfrak{h}'}$ such that the matrices of $D_k(z)$ for $1 \leq k < N$ are upper triangular and their diagonal entries belong to $M_{\mathfrak{h}'}^\times$. Then the ordered basis $\{1 \otimes M_w, b \in B\} =: B'$ of $X$ satisfies (M3'). So $X$ is in category $O'_{fd}$.

The matrices of $D_k(z)$ with respect to the basis $B'$ of $X$ and the basis $B$ of $S'_{\nu,a}$ are the same. So $\chi'_{\nu}(X)$ up to a normalization factor $e^{c_{\nu}}$, is equal to the $q$-character of the $E_{\nu}(sl_{N-1})$-module $S_{\nu,a}$. The latter is given by Eq. \ref{Eq:chi'.}

$X$ has a unique (up to scalar) singular vector and is of highest weight, so it is irreducible. A comparison of highest weights shows that $X \cong S'_{\nu,c,a}$. \hfill \Box

Fix $\mu \in \mathscr{P}$ a partition with at most $N$ parts. Given a tableau $T \in \mathscr{B}_{\mu}$, by deleting the boxes $\begin{array}{|l|} \hline \\ \end{array}$ in $T$, we obtain a Young diagram $T^{-1}(\{2, 3, \ldots, N\})$ with at most $N - 1$ rows, which corresponds to a partition in $\mathscr{P}'$, denoted by $\nu_T$. Let $W_{\mu}$ be the set of all such $\nu_T$ with $T \in \mathscr{B}_{\mu}$. For $\nu \in W_{\mu}$, define $c_{\nu}$ to be the cardinal of the finite subset $Y_{\mu} \setminus Y_{\nu}$ of $\mathbb{Z}^2$.

Again take the example $N = 3$ and $\mu = (2 \geq 1)$ after Theorem \ref{Thm:40}. The eight tableaux in $\mathscr{B}_{\mu}$, with $\begin{array}{|l|} \hline \\ \end{array}$ deleted give four Young diagrams and partitions

$\begin{array}{|l|} \hline \\ \end{array} = (1), \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \end{array} = (2), \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \end{array} = (1 \geq 1), \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \end{array} = (2 \geq 1).$

The corresponding integers $c_{\nu}$ are $2, 1, 1, 0$.

Lemma 2.6. Let $\mu \in \mathscr{P}$ and $a \in \mathbb{C}$. In the Grothendieck group $K_0(O'_{fd})$:

$$[S_{\mu,a}] = \sum_{\nu \in W_{\mu}} [S'_{\nu,c_{\nu,a}}].$$

Proof. Let $\mathfrak{c}'$ be the subalgebra of $\mathfrak{c}$ generated by $M_2$ and the $t_{ij}$ for $2 \leq i, j \leq N$. One can define similar abelian category $F'$ of $\mathfrak{c}'$-modules (which are $\mathfrak{h}$-graded $M$-vector spaces) equipped with:

$\begin{itemize}
  \item[(a)] the evaluation functor $ev_{\mu}' : F' \rightarrow O'_{fd}$ from $\mathfrak{c}'$-modules to $\mathfrak{c}^b_{N-1}$-modules;
  \item[(b)] the injective character map $\chi : K_0(F') \rightarrow M_4$ from the $\mathfrak{h}$-grading.
\end{itemize}$

Theorem \ref{Thm:2.3} applied to the $\mathfrak{h}'$-algebra $\mathfrak{c}_{\tau,\mathfrak{h}}(sl_{N-1})$, from the scalar extension in the proof of Lemma \ref{Lem:2.3} one obtains an irreducible object $V'_{\nu,c}$ in category $F'$ for $\nu = (\nu_2 \geq \nu_3 \geq \cdots \geq \nu_N) \in \mathscr{P}'$ and $c \in \mathbb{C}$ with the following properties:

$\begin{itemize}
  \item[(c)] $V'_{\nu,c}$ admits a non-zero vector $v$ of weight $\alpha_1 + \nu_2 \epsilon_2 + \nu_3 \epsilon_3 + \cdots + \nu_N \epsilon_N$ and $t_{ij} v = 0$ for $2 \leq i, j, k \leq N$ and $i < j$;
  \item[(d)] $\chi(V'_{\nu,c})$ is equal to the character of the irreducible $\mathfrak{sl}_{N-1}$-module of highest weight $\alpha_1 + \nu_2 \epsilon_2 + \nu_3 \epsilon_3 + \cdots + \nu_N \epsilon_N$; here $\mathfrak{sl}_{N-1}$ is the parabolic Lie subalgebra of $\mathfrak{sl}_N$ (with the same Cartan algebra $\mathfrak{h}$) associated to the simple roots $\alpha_2, \alpha_3, \cdots, \alpha_{N-1}$.
\end{itemize}$

By comparing highest weight we observe that $ev_{\mu}'(V'_{\nu,c}) \cong S'_{\nu,c,a}$ in category $O'_{fd}$.
Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N) \in P$. Set $\overline{\mu} := \mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \cdots + \mu_N \epsilon_N$. Then $S_{\mu, a} \cong ev_a(V_{\overline{\mu}})$ in category $\mathcal{O}_d$. By diagram chasing

$$
\begin{array}{ccc}
K_0(\mathcal{O}_{fd}) & \xrightarrow{ev_a} & K_0(\mathcal{F}) \\
\text{Res} & \downarrow & \text{Res} \\
K_0(\mathcal{O}_{fd}) & \xrightarrow{ev_a'} & K_0(\mathcal{F}')
\end{array}
$$

Lemma 2.6 is equivalent to the character identity $\chi(V_{\overline{\mu}}) = \sum_{\nu \in \mathcal{W}_\mu} \chi(V_{\nu, c_{\nu}})$. Since the left-hand side (resp. the right-hand side) is the character of a representation of $\mathfrak{sl}_N$ by Theorem 2.3 (resp. of $\mathfrak{sl}_{N-1}^\prime$ by (d)), this identity is a consequence of the branching rule for representations of the reductive Lie algebras $\mathfrak{sl}_N \supset \mathfrak{sl}_{N-1}$. □

2.4. **Proof of Theorem 1.15.** We proceed by induction on $N$. For $N = 1$ and $\mu = (n)$, since $S_{\mu, a}$ is one-dimensional, its q-character is equal to its highest weight

$$
\left( \frac{\theta(z + (a + n)h)}{\theta(z + ah)} \right)^{n \epsilon_1} = \prod_{j=1}^{n} \left[ \frac{1}{1 + j - i - 1} \right].
$$

Suppose $N > 1$. By Lemma 2.5 the induction hypothesis in the case of $N - 1$ gives the q-character formula for all the $\mathcal{E}_{N-1}^h$-modules $S_{\mu, c_{\nu}}$ where $\nu \in \mathcal{P}$ and $c \in \mathbb{C}$. So the q-character $\chi_q'(S_{\mu, a})$ of the $\mathcal{E}_{N-1}^h$-module $S_{\mu, a}$ is known by Lemma 2.6.

Since $S_{\mu, a}$ is an irreducible $\mathcal{E}$-module in category $\mathcal{O}_{fd}$, by Remark 2.4, $\chi_q'(S_{\mu, a})$ can be recovered from $\chi_q'\left(S_{\mu, a}\right)$ by Lemma 2.6. Since $\mathcal{B}_a$ is the disjoint union of the $\mathcal{B}_a$ for $\nu \in \mathcal{W}_\mu$, it suffices to check that for each $e$-weight $(m^1_T(z), m^2_T(z), \cdots, m^N_T(z); a)$ at the right-hand side of Eq. 1.13, where $T \in \mathcal{B}_a$, the following product

$$
m^T(z) := m^1_T(z + (N - 1)h)m^2_T(z + (N - 2)h) \cdots m^N_T(z)
$$

is the eigenvalue of scalar action of $\mathcal{D}_N(z)$ on $S_{\mu, a}$. Notice first that

$$
\prod_{p=1}^{N} \frac{\theta(z + (a + N - p + \mu_p)h)}{\theta(z + (a + N - p)h)} = \prod_{(i,j) \in Y_a} \frac{\theta(z + (a + j - i + N)h)}{\theta(z + (a + j - i + N - 1)h)}.
$$

By Eq. 1.12, each box $[i, j]$ contributes to $\frac{\theta(z + (a + N)h)}{\theta(z + (a + N - 1)h)}$, so the right-hand side of the identity is exactly $m^T(z)$. By Remark 2.4 the left-hand side is the scalar of $\mathcal{D}_N(z)$ acting on $S_{\mu, a}$. This completes the proof of Theorem 1.15. □

3. **Kirillov–Reshetikhin modules**

We study certain irreducible $\mathcal{E}$-modules via q-characters.

Fix $a \in \mathbb{C}$. For $k \in \mathbb{C}$ and $1 \leq r \leq N$, define the asymptotic e-weight

$$
w_{k,a}^{(r)} := \Psi_{r,a+k} \Psi_{r,a}^{-1} \in \mathcal{M}_w.
$$

Assume $k \in \mathbb{Z}_{\geq 0}$. We identify $k \mathbb{X}_a$ with the partition $(k \geq k \geq \cdots \geq k)$ where $k$ appears $r$ times. Then $w_{k,a}^{(r)} = Y_{r,a+1} \cdots Y_{r,a+2} Y_{r,a+k-\frac{1}{2}} = \theta_{k \mathbb{X}_a, a}$ by Eqs. 1.9–1.10, and the finite-dimensional irreducible $\mathcal{E}$-module $S(w_{k,a}^{(r)})$ in category $\mathcal{O}_{fd}$ is denoted by $W_{k,a}^{(r)}$ and called **Kirillov–Reshetikhin module** (KR module).

The $Y_{i,a+m}$ (resp. the $A_{i,a+m}$) for $1 \leq i \leq N$ (resp. $1 \leq i < N$) and $m \in \frac{1}{2} \mathbb{Z}$ are linearly independent in the abelian group $\mathcal{M}_w$, and generate the subgroup $\mathcal{P}_a$ (resp. $\mathcal{Q}_a$) and the submonoid $\mathcal{P}_a^+$ (resp. $\mathcal{Q}_a^+$). The inverses of these submonoids are denoted by $\mathcal{P}_a^-$ and $\mathcal{Q}_a^-$ respectively. By Eq. 1.13 and Eq. 1.10,

$$
\text{wt}_{e}(S_{\mu, a}) \subset \theta_{\mu, a} \mathcal{Q}_a^- \subset \mathcal{P}_a \quad \text{for} \quad \mu \in P.
$$
Indeed, let $T_\mu \in \mathcal{B}_\mu$ be such that the associated monomial in Eq. (1.13) is $\theta_{\mu,a}$. Then for $S \in \mathcal{B}_\mu$, we must have $S(i,j) \ge T_\mu(i,j)$ for all $(i,j) \in Y_\mu$.

Following [25] [6], we call $f \in \mathcal{P}_a$ right negative if the factors $Y_{s,a+m}$ with $1 \le i < N$ appearing in $f$, for which $m \in \frac{1}{2}\mathbb{Z}$ is minimal, have negative powers.

**Lemma 3.1.** [25] Let $e, f \in \mathcal{P}_a$. If $e, f$ are right negative, then so is $ef$.

All elements in $Q^{-}$ different from $1$ are right-negative by Eq. (1.9).  

**Lemma 3.2.** Let $k \in \mathbb{Z}_{>0}$ and $1 \le r < N$.

1. For $1 \le l \le k$, $w_{k,a}^{(r)}A_{r,a+1}^{-1}A_{r,a+2}^{-1} \cdots A_{r,a+l-1}^{-1}$ is an $e$-weight of $W_{k,a}^{(r)}$ of multiplicity one in $\chi_q(W_{k,a}^{(r)})$.

2. An $e$-weight of $W_{k,a}^{(r)}$ different from those in (1) and from $w_{k,a}^{(r)}$ must belong to $w_{k,a}^{(r)}A_{r,a}^{-1}A_{r,a-1}^{-1}Q^{-}$ for certain $1 \le s < N$ with $s = r \pm 1$.

3. Any $e$-weight of $W_{k,a}^{(r)}$ is either $w_{k,a}^{(r)}$ or right negative.

**Proof.** The Young diagram $Y_{s,a}$ is a rectangle of $r$ rows and $k$ columns. For (1)–(2) the proof of [51] Lemma 3.4] works by applying Theorem [1.15 to $W_{k,a}^{(r)} \cong S_{\mathbb{Z}_{\ge a},r-a,t}$.

For (3), $w_{k,a}^{(r)}A_{r,a}^{-1}$ is right negative, and so is any element of $w_{k,a}^{(r)}A_{r,a}^{-1}Q_{-}$.

For $1 \le r < N$ and $k, a \in \mathbb{C}$, define as in [24] [§4.3] and [54] Remark 3.2] for $k, t \in \mathbb{Z}_{>0}$, then $d_{k,a}^{(r,t)} \in \mathcal{R}_{d,t}$ and set $D_{k,a}^{(r,t)} := S(d_{k,a}^{(r,t)})$.

**Lemma 3.3.** Let $1 \le r < N$ and $k, m \in \mathbb{Z}_{>0}$.

1. The dominant $e$-weights of $W_{k+m-1,1}^{(r)} \otimes W_{k,0}^{(r)}$ and $W_{k-1,1}^{(r)} \otimes W_{k+m,0}^{(r)}$ are $w_{k+m-1,1}^{(r)}w_{k,0}^{(r)}$ and $w_{k+m-1,1}^{(r)}w_{k,0}^{(r)}A_{r,1}^{-1}A_{r,2}^{-1} \cdots A_{r,l}^{-1}$ for $1 \le l \le k$.

2. The module $W_{k-1,1}^{(r)} \otimes W_{k+m,0}^{(r)}$ is irreducible.

**Proof.** For (1), one can copy the last two paragraphs of the proof of [22] Theorem 4.1, since the right-negativity property of KR modules in the elliptic case (Lemma 3.2) is the same as in the affine case. Let $T$ be the tensor product module of (2). Suppose $T$ is not irreducible. Then there exists $1 \le l \le k - 1$ such that $T$ admits an irreducible sub-quotient $S \cong S(d_1)$ where by Eq. (1.15):

$$d_1 := w_{k-1,1}^{(r)}w_{k+m,0}^{(r)} \prod_{j=1}^{l} A_{r,j}^{-1} \frac{\Psi_{s,r+l}^{(-1)} \Psi_{s,l+1}^{(r)}}{\Psi_{s+l}^{(-1)} \Psi_{s,r}^{(r)}} \quad \Psi_{s+l}^{(-1)} \Psi_{s,r}^{(r)} \quad \Psi_{s,r+l}^{(-1)} \Psi_{s,l+1}^{(r)}$$

Set $\mu := \omega(d_1)$. The weight space $S[\mu - \alpha_r]$ is non-zero since the $\Psi_s$ do not cancel in $d_1$, and its possible $e$-weights are $d_1A_{r,l}^{-1}d_1A_{r,l+1}$ since $S$ is a sub-quotient of $W_{k-1,1+l}^{(r)} \otimes W_{k-1,1+l}^{(r)} \otimes (\otimes_{s=r \pm 1} W_{s,r}^{(r)})$. If $d_1A_{r,l}^{-1}$ is an $e$-weight of $S$, then

$$\Psi_{s,r+1}^{(-1)} \Psi_{s,l+1}^{(r)} \Psi_{s,r}^{(-1)} \Psi_{s,l}^{(r)}$$

which contradicts with the $q$-characters of KR modules in Lemma [5.2]. So $k > l + 1$ and $S[\mu - \alpha_r] = \mathbb{M}v \neq 0$. Let $\omega$ be a highest weight vector of $S$. Then $p := \mu_{r,r+1} = 2k - 2l + m - 1$, $L_{r,r+1}(\omega) = A(z; \lambda)v$, $L_{r+1,r}(\omega) = B(z; \lambda)w$.
for some meromorphic functions $A, B$ of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$. For $1 \leq i \leq N$, let $g_i(z) \in \mathcal{M}_\omega$ be the $i$-th component of $d_i \in \mathcal{M}_\omega$. Then $L_{ii}(z)\omega = g_i(z)\varphi_i(\lambda)\omega$ for certain $\varphi_i(\lambda) \in \mathcal{M}_\omega$ by Eq. (1.12). Set $h(z) := \frac{\varphi_i(z)}{g_{r+1}(z)}$. We have
\[
h(z) = \frac{\theta(z + (k - \ell_r)h)\theta(z + (k + m - \ell_r)h)}{\theta(z)\theta(z + (l + 1 - \ell_r)h)} = \frac{\theta(z - w_1)\theta(z - w_2)}{\theta(z - w_3)\theta(z - w_4)},
\]
where $w_1 := (t_r - k)h$, $w_2 := (t_r - k - m)h$ and so on. Applying Eq. (1.12) with $(i, j) = (r + 1, r) = (n, m)$ to $\omega$, as in the proof of [21] Theorem 4.1, we obtain
\[
\left(\frac{\theta(z - w + \lambda_{r,r+1} + ph)\theta(h)}{\theta(z - w + h)\theta(\lambda_{r,r+1} + ph)}\right)\varphi_{r+1}(z)g_{r+1}(z)g_r(w) = \frac{\theta(z - w + \lambda_{r,r+1})\theta(h)}{\theta(z - w + h)\theta(\lambda_{r,r+1})}g_{r+1}(w)g_r(z)
\]
\[
\times \varphi_r(z + h \epsilon_{r+1})\varphi_{r+1}(\lambda) = \frac{\theta(z - w)\theta(\lambda_{r,r+1} + h)}{\theta(z - w + h)\theta(\lambda_{r,r+1})}B(w; \lambda)A(z; \lambda + h \epsilon_r).
\]
Multiplying both sides by $\frac{\theta(z - w + h)\theta(\lambda_{r,r+1} + ph)}{g_{r+1}(w; \lambda)g_r(z)}$, and noticing $g_{r+1}(z) = \frac{\theta(z - w_1)}{\theta(z - w_3) - \theta(z - w_4)}$, one can evaluate $w$ at $w_1$ and $w_2$ to obtain identities of meromorphic functions of $(z, \lambda)$:
\[
\tilde{A}(z; \lambda)\chi_i(\lambda) = \frac{\theta(z - w_1 + \lambda_{r,r+1})}{\theta(z - w_1)}f(\lambda)h(z) \quad \text{for } i = 1, 2.
\]
Here we set $A(\lambda) := \varphi_r(\lambda + h \epsilon_{r+1})\varphi_{r+1}(\lambda)$ and
\[
\tilde{A}(z; \lambda) := \frac{A(z; \lambda + h \epsilon_r)}{g_{r+1}(z)}, \quad \chi_i(\lambda) := \frac{B(w_i; \lambda)}{g_{r+1}(w_i)}, \quad f(\lambda) := \frac{\theta(h)\varphi_r(\lambda + h \epsilon_r)}{\theta(\lambda_{r,r+1} + h)}.
\]
Since $f(\lambda)h(z) \neq 0$, we have $\chi_i(\lambda) \neq 0$ and so
\[
\frac{\theta(z - w_1 + \lambda_{r,r+1})}{\theta(z - w_1)}\frac{\theta(z - w_2)}{\theta(z - w_3)}\frac{\theta(z - w_4)}{\theta(z - w_3)} = \frac{\theta(z - w_2 + \lambda_{r,r+1})\theta(z - w_1)}{\theta(z - w_3)\theta(z - w_4)},
\]
on as non-zero meromorphic functions of $(z, \lambda)$. This forces $w_1 - w_2 = mh \in \mathbb{Z} + \mathbb{Z} \tau$, which certainly does not hold. This proves (3).

**Theorem 3.4.** For $1 \leq r < N$, $t \in \mathbb{Z}_{>0}$ and $k > 0$, we have the following identities in the Grothendieck ring of category $\mathcal{O}_d$:
\[
(3.16) \quad [D_{kr+k+1}] + [W_{kr+1}] [W_{kr+1}] = [W_{k+k}] [W_{k}];
\]
\[
(3.17) \quad [D_{kr+k}] [W_{k+1,t+1}] = [D_{kr+k+1}] [W_{k+1,t+1}] + [D_{kr+k}] [W_{k+1,t+1}].
\]

**Proof.** Set $T := W_{k+1,t+1} \otimes W_{k,0}^r$ and $d := W_{k+1,t+1}^r \otimes W_{k,0}^r$. Then $S := S(d)$ is an irreducible sub-quotient of $T$ and by Eqs. (3.4)–(3.5):
\[
d = W_{k+1,t+1}^r \otimes W_{k,0}^r, \quad d^{(t,r)}_{k,k+1} = A_{r+1}^{-1}A_{r+1}^{-1} \cdots A_{r+1}^{-1}d.
\]
Set $m = t+1$ in Lemma 3.3. Then $S \cong W_{k+1,t+1}^r \otimes W_{k+1,t+1,0}^r$, and there is exactly one dominant e-weight (counted with multiplicity) in $w_t(S) \setminus w_t(S)$, namely $d^{(t,r)}_{k,k+1}$. This proves Eqs. (3.16), which implies after taking spectral parameter shifts
\[
[D_{kr+k}] = [W_{k+1,t+1}][W_{k}^r] - [W_{k+1,t+1}][W_{k+2,t+1}^r],
\]
\[
[D_{kr+k+1}] = [W_{k+1,t+1}][W_{k+2,t+1}^r] - [W_{k+1,t+1}][W_{k+2,t+1}^r] + [W_{k+1,t+1}][W_{k+2,t+1}^r].
\]
Eq. (3.17) becomes a trivial identity involving only KR modules.

**D.**

$D_{kr+1}^{(t,r)}$ is special in the sense of [18] as it contains only one dominant e-weight. For $t = 0$, we have $D_{k,k+1}^{(0)} \cong W_{k+1}^{(r-1)} \otimes W_{k+1}^{(r+1)}$ by showing that the tensor product is special as in [18], and Eq. (3.16) is the T-system of KR modules.
Corollary 3.5. Let \(1 \leq r < N\), \(a \in \mathbb{C}\) and \(k, l \in \mathbb{Z}_{\geq 0}\).

1. \(d_{k,a}^{(r)} A_{r,a-1}^{-1} A_{r,a-1}^{-1} \cdots A_{r,a-1}^{-1} A_{r,a+l-1}^{-1} \in \text{wt}_k(D_{k,a}^{(r)})\) for \(1 \leq l \leq t\).

2. Any \(e\)-weight of \(D_{k,a}^{(r)}\) different from those in (1) and from \(d_{k,a}^{(r,d)}\) belongs to \(d_{k,a}^{(r,t)}(A_{r,a-k-1}^{-1} A_{r,a-k-1}^{-1} \cdots A_{r,a-k-1}^{-1} Q_a^-)\) for certain \(1 \leq s < N\) with \(s = r \pm 1\).

Proof. This comes from Lemma 3.2 and Theorem 3.4. \(\square\)

Lemma 3.6. Let \(1 \leq r < N\) and \(t \in \mathbb{Z}_{\geq 0}\). There is a short exact sequence

\[
0 \longrightarrow D_{1,a}^{(r,t)} \longrightarrow W_{t+1,a-1}^{(r)} \otimes W_{1,a-2}^{(r)} \longrightarrow W_{t+2,a-2}^{(r)} \longrightarrow 0
\]

of \(\mathcal{E}\)-modules in category \(\mathcal{O}_{\mathcal{H}}\).

Proof. Let \(T\) and \(S\) be the second and third terms above (zero excluded). Let \(\omega_1, \omega_2\) be highest weight vectors of \(W_{t+1,a-1}^{(r)}\) and \(W_{1,a-2}^{(r)}\) respectively. Then \(\omega_1 \otimes \omega_2\) is a highest weight vector of \(T\) and generates a sub-module \(S\). There is a surjective morphism of modules \(T \longrightarrow S\), the kernel of which is \(D_{1,a}^{(r,t)}\) by Eq. (8.10) (one applies a spectral parameter shift \(\Phi_{a-2}\) to the equation with \(k = 1\)). This is the desired short exact sequence.

Suppose \(T \neq T'\). Then \([T] = [S]\) or \([T] = [D_{1,a}^{(r,t)}]\). By comparing highest weights, we have \([T'] = [S]\). So the weight space \(T'[t+2] = -\alpha_r\) is one-dimensional. Corollary 3.2 applied to \(W_{t+1,a-1}^{(r)} \cong S_{t+1}[a] = 0\), one finds \(g(\lambda) \in \mathbb{M}^x\) such that \(L_{t+1,a-1}(z) \omega_1 = \omega_1\) and (set \(b := a - \ell_r - 1\))

\[
L_{r,r+1}(z) \omega_1 = \frac{\theta(z + (b + t)h + \lambda_r, r+1)}{\theta(z + bh)} \omega_1', \quad L_{r,r}(z) \omega_1 = \frac{\theta(z + (b + t + 1)h)}{\theta(z + bh)} g(\lambda) \omega_1
\]

where \(0 \neq \omega_1'\) is of weight \((t + 1)\varpi_r - \alpha_r\). Similarly \(L_{r+1,r}(z) \omega_2 = \omega_2\) and

\[
L_{r,r+1}(z) \omega_2 = \frac{\theta(z + (b - 1)h + \lambda_r, r+1)}{\theta(z + (b - 1)h)} \omega_2'
\]

with \(\omega_2' \neq 0\) of weight \(\varpi_r - \alpha_r\). Since \(\omega_1, \omega_2\) are highest weight vectors, we have

\[
L_{r,r+1}(z) (\omega_1 \otimes \omega_2) = L_{r,r+1}(z) \omega_1 \otimes L_{r+1,r}(z) \omega_2 + L_{rr}(z) \omega_1 \otimes L_{r,r+1}(z) \omega_2
\]

\[
= \left( \frac{\theta(z + (b + t)h + \lambda_r, r+1)}{\theta(z + bh)} \omega_1' \right) \otimes \omega_2 + \frac{\theta(z + (b + t + 1)h)}{\theta(z + bh)} g(\lambda) \omega_1 \otimes \left( \frac{\theta(z + (b - 1)h + \lambda_r, r+1)}{\theta(z + (b - 1)h)} \omega_2' \right)
\]

Setting \(z = -(b + t + 1)h\) we obtain \(\omega_1' \otimes \omega_2' \in T'\), and so \(\omega_1 \otimes \omega_2' \in T'\). The weight space \(T'[t+2] = -\alpha_r\) is at least two-dimensional, a contradiction. \(\square\)

Lemma 3.6 is inspired by [57, 5.3]: to transform identities in the Grothendieck group into exact sequences by restriction to \(\mathfrak{sl}_2\). For more generally, we have the short exact sequences in category \(\mathcal{O}_{\mathcal{H}}\) by [21] Proposition 4.3, Corollary 4.5: \(\square\)

\[
0 \longrightarrow D_{k+1}^{(r,t)} \longrightarrow W_{k+1}^{(r)} \otimes W_{k,0}^{(r)} \longrightarrow W_{k+1}^{(r)} \otimes W_{k+1,0}^{(r)} \longrightarrow 0,
\]

\[
0 \longrightarrow D_{k+1}^{(r,0)} \otimes W_{k-1,0}^{(r)} \longrightarrow D_{k+1}^{(r+1)} \otimes W_{k+1,0}^{(r)} \longrightarrow D_{k+1}^{(r+1)} \otimes W_{k+1,0}^{(r)} \longrightarrow 0.
\]

These exact sequences hold for affine quantum (super)groups [22, 54]. In the super case the proof is more delicate since Lemma 5.2 (3) fails.

\(\end{document}\)
4. Asymptotic representations

We construct infinite-dimensional modules in category $\mathcal{O}$ as inductive limits $(k \to \infty)$ of the KR modules $W_{k,a}^{(r)}$ for fixed $1 \leq r < N$ and $a := \ell_r$.

The general strategy follows that of Hernandez–Jimbo [36]:

(i) produce an inductive system of vector spaces $W_{0,a}^{(r)} \subseteq W_{1,a}^{(r)} \subseteq W_{2,a}^{(r)} \subseteq \cdots$; 
(ii) prove that the matrix entries of the $L_{ij}(z)$ are good functions of $k \in \mathbb{Z}_{\geq 0}$; 
(iii) define the module structure on the inductive limit of (i).

Step (i) is done in Lemma 4.2 Step (ii) in Lemma 4.8 and Step (iii) in Proposition 4.18

We shall see that the proofs in each step are different from [36].

As in [36], for $k > l > m$, set $Z_{kl} := W_{k-l,a+l}^{(r)} \cong S_{(k-l)\mathbb{Z},\ell}$, and fix a highest weight vector $\omega_{kl} \in Z_{kl}$. By Eq. (4.17), we have for $1 \leq i \leq r < j \leq N$:

$$L_{ii}(z)\omega_{kl} = \frac{\theta(z + kh)}{\theta(z + lh)} \prod_{q=r+1}^{N} \frac{\theta(\lambda_{i} + (k-l+1)h)}{\theta(\lambda_{i} + h)}, \quad L_{jj}(z)\omega_{kl} = \omega_{kl}. 
$$

Note that $Z_{k0} = W_{k,0}^{(r)}$, and we simply write $\omega_{k0} := \omega_k$.

**Lemma 4.1.** Let $t > k > l > m$. There exists a unique morphism of $\mathcal{E}$-modules

$$\mathcal{G}^{t}_{k,m} : Z_{kl} \otimes Z_{lm} \longrightarrow Z_{km}$$

such that $\mathcal{G}^{t}_{k,m}(\omega_{kl} \otimes \omega_{lm}) = \omega_{km}$. Moreover the following diagram commutes:

$$\begin{align*}
Z_{tk} \otimes Z_{kl} \otimes Z_{lm} & \xrightarrow{\mathcal{G}^{t}_{k,m} \otimes \text{id}} Z_{kl} \otimes Z_{lm} \\
\text{id} \otimes \mathcal{G}^{t}_{m,k} & \xrightarrow{\mathcal{G}^{t}_{m,k} \otimes \text{id}} Z_{km} \\
Z_{tk} \otimes Z_{km} & \xrightarrow{\mathcal{G}^{t}_{m,k}} Z_{lm}.
\end{align*}$$

**Proof.** (Uniqueness) Let $F,G$ be two such morphisms and let $X$ be the image of $F - G$. Then $\omega_{km} \notin X$. If $X \neq 0$, then $X$ has a highest weight vector $\nu \neq 0$, which is proportional to $\omega_{km}$ by the irreducibility of $Z_{km}$, a contradiction. So $X = 0$ and $F = G$. The commutativity of (4.18) is proved in the same way.

(Existence) Let $b \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$. By Lemma 5.6 there exists a surjective $\mathcal{E}$-linear map $W_{n-1,b+1}^{(r)} \otimes W_{1,b}^{(r)} \longrightarrow W_{n,b}^{(r)}$. An induction on $n$ shows that the $\mathcal{E}$-module $W_{1,b+n-1}^{(r)} \otimes W_{1,b+n-2}^{(r)} \otimes \cdots \otimes W_{1,b+1}^{(r)} \otimes W_{1,b}^{(r)}$ can be projected onto $W_{n,b}^{(r)}$.

Setting $(n,b) := (k-m,a+m)$ we obtain a surjective $\mathcal{E}$-linear map

$$g : Z_{k,k-1} \otimes Z_{k-1,k-2} \otimes \cdots \otimes Z_{m+2,m+1} \otimes Z_{m+1,m} =: T \twoheadrightarrow Z_{km}.$$

Taking $(n,b)$ to be $(k-l,a+l)$ and $(l-m,a+m)$, we project the first $k-l$ and the last $l-m$ tensor factors of $T$ onto $Z_{kl}$ and $Z_{lm}$ respectively. The tensor product of these projections gives $f : T \twoheadrightarrow Z_{kl} \otimes Z_{lm}$. Since $\omega_{kl} \otimes \omega_{lm}$ and $\omega := \omega_{k+1} \otimes \omega_{k-1} \otimes \cdots \otimes \omega_{m+1} \otimes \omega_{m+1,m} \in T$ are highest weight vectors of the same $\epsilon$-weight, by surjectivity of $g$, one can assume $f(\omega) = \omega_{kl} \otimes \omega_{lm}$ and $\omega(\omega) = \omega_{km}$. It suffices to prove that $g$ factorizes through $f$, and so $g = \mathcal{G}^{t}_{k,m}f$. Set $Y := \ker(f)$ and $Z := \ker(g)$. The image of $g$ being irreducible, $Z$ is a maximal submodule of $T$. Since $\omega \notin Y + Z$, we have $Y + Z = Z$ and $Y \subseteq Z$. $\square$

We need two special cases of the $\mathcal{G}$: for $k > l$ and $t - 1 > l$,

$$\mathcal{F}_{k,l} = \mathcal{G}^{t}_{k,0} : Z_{kl} \otimes W_{l,a}^{(r)} \longrightarrow W_{k,a}^{(r)}, \quad \mathcal{F}_{l,l} = \mathcal{G}^{t+1}_{l,1} : Z_{l,t+1} \otimes Z_{l+1,l} \longrightarrow Z_{l}.$$

As in [36] §4.2, for $k > l$ define the restriction map

$$F_{k,l} : W_{l,a}^{(r)} \longrightarrow W_{k,a}^{(r)}, \quad v \mapsto \mathcal{F}_{k,l}(\omega_{kl} \otimes v).$$
It is a difference map of bi-degree \(((l-k)\omega_r, 0)\).

Applying (4.18) with \(t > k > l > 0\) to \(\omega_k \otimes \omega_l \otimes W_{t,a}^{(r)}\) gives \(F_{t,k}F_{k,l} = F_{t,l}\). So \((W_{t,a}^{(r)}, F_{t,l})\) is an inductive system of vector spaces. \(\square\)

Applying (4.18) with \(k > l + 1 > 0\) to \(\omega_{k+1} \otimes Z_{l,a}^{(r)} \otimes W_{t,a}^{(r)}\), we obtain

\[
\mathcal{F}_{k,l}(\mathcal{G}_{k,l}(\omega_{k+1} \otimes v) \otimes w) = F_{k,l}\mathcal{F}_{l+1,l}(v \otimes w) \quad \text{for } v \otimes w \in Z_{l+1,l} \otimes W_{t,a}^{(r)}.
\]

**Lemma 4.2.** The linear maps \(F_{k,l}\) are injective.

**Proof.** Assume \(K := \ker(F_{k,l}) \neq 0\); it is a graded subspace of \(W_{t,a}^{(r)}\). Choose \(\mu \in \text{wt}(K)\) such that \(\mu + \alpha_i \notin \text{wt}(K)\) for all \(1 \leq i < N\) and fix \(0 \notin \mu \in K\).

We show that \(w\) is a singular vector, so \(w \in M\omega_l\) and \(\omega_l \in K\), a contradiction. It suffices to prove that \(L_{ji}(z)w \in K\) for all \(1 \leq i < j \leq N\); this implies \(L_{ji}(z)w = 0\) because by assumption on \(\mu\) the weight space \(K[\mu + \epsilon_i - \epsilon_j]\) vanishes.

Suppose \(j > r\). If \(1 \leq p \leq N\) and \(p \neq j\), then \((k-l)\omega_r + \epsilon_p - \epsilon_j \notin \text{wt}(Z_{kl})\) by Theorem 1.15. It follows that for \(v \in W_{t,a}^{(r)}\) we have in \(Z_{kl} \otimes W_{t,a}^{(r)}\),

\[
L_{ji}(z)(\omega_k \otimes v) = L_{ji}(z)\omega_k \otimes L_{ji}(z)v = \omega_k \otimes L_{ji}(z)v.
\]

It follows that \(L_{ji}(z)K \subset K\) because of the commutativity:

\[
L_{ji}(z)F_{k,l} = F_{k,l}L_{ji}(z) \quad \text{for } 1 \leq i, j \leq N \text{ with } j > r.
\]

Suppose \(j \leq r\). For \(p > r\) since \(\geq j > r\) we have \(L_{ji}(z)w \in K\) and so \(L_{ji}(z)w = 0\). For \(p \leq r\), by Theorem 1.15 \(L_{ji}(z)\omega_k = 0\) if \(p \neq j\). This implies

\[
L_{ji}(z)(\omega_k \otimes w) = L_{ji}(z)\omega_k \otimes L_{ji}(z)w = \frac{\theta(z+kh)}{\theta(z+lh)}g(\lambda)(\omega_k \otimes L_{ji}(z)w)
\]

certain \(g(\lambda) \in \mathbb{M}^\times\). Applying \(\mathcal{F}_{k,l}\) we obtain \(F_{k,l}L_{ji}(z)w = 0\), as desired. \(\square\)

In what follows, \(k, l\) denote positive integers, while \(i, j, m, n, p, q, s, t, u, v\) the integers between 1 and \(N\) related to the Lie algebra \(\mathfrak{g}_N\).

**Lemma 4.3.** For \(k > l\) and \(1 \leq i \leq N\) we have

\[
K_{i}(z)F_{k,l} = \frac{\theta(z+kh)}{\theta(z+lh)}F_{k,l}K_{i}(z).\tag{4.21}
\]

**Proof.** We compute \(D_i(z)(\omega_k \otimes v)\) for \(v \in W_{t,a}^{(r)}\) based on the coproduct of Corollary 1.2. If \(-\omega_{N-k} \prec \alpha\) then \(\alpha + \omega_{N-k} \notin \mathbb{Q}^-\) and \((k-l)\omega_k + \alpha + \omega_{N-k} \notin \text{wt}(Z_{kl})\).

The extra terms \(x_{\omega_k} \otimes y_{\alpha_i}\) in the coproduct do not contribute, and so \(D_i(z)(\omega_k \otimes v) = D_i(z)\omega_k \otimes D_i(z)v\). By Eq. (4.18) similar identity holds when \(D_i(z)\) is replaced by \(K_{i}(z)\), because \(K_{i}(z)\omega_k = \left(\frac{\theta(z+kh)}{\theta(z+lh)}\right)^{\delta_{i,r}}\omega_k\) is independent of \(\lambda\). Applying \(\mathcal{F}_{k,l}\) to the new identity involving \(K_{i}(z)\), we obtain Eq. (4.21). \(\square\)

From now on up to Corollary 1.4, we shall always fix integers \(j, p\) with condition \(1 \leq j \leq r < p \leq N\). For \(k > l\), introduce \(\omega_{k,l}^j \in Z_{kl}\) by Corollary 2.26

\[
L_{ji}(z)\omega_k = \frac{\theta(z+(k-1)h+\lambda_{jp})}{\theta(z+lh)}\omega_{k,l}^j.
\]

Indeed \(\omega_{k,l}^j = t_{jp}\omega_k \otimes \omega_l\) in the evaluation module \(Z_{kl} \cong V_{(k-l)\omega_r}(l)\). Since \(Y_{(k-l)\omega_r}\) is a rectangle, \(M\omega_{k,l}^j\) is the weight space of weight \((k-l)\omega_r + \epsilon_p - \epsilon_j\).

\[\text{In the affine case} \ [\text{Eq.(4.26)}] \text{ the structure map comes from the stronger fact that } Z_{kl} \otimes Z_{lm} \text{ is of highest weight with } Z_{km}, \text{ being the irreducible quotient.}\]
Lemma 4.4. In the $\mathcal{E}$-module $Z_{k,l}$ we have $\omega_{kl}^p \neq 0$ and
\[
L_{pq}(z)\omega_{kl}^p = -\omega_{kl}^p \frac{\theta(z + l + h - \lambda_{pq})\theta((l - k + 1)h)(h)}{\theta(z + l + h)\theta(\lambda_{pq})\theta(\lambda_{pq} + h)} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)}.
\]
The product is taken over integers $q$ such that $r + 1 \leq q \leq N$ and $q \neq p$.

Proof. The weight grading on $Z_{k,l} = S_{(k-1)\varpi_r,\lambda}$ indicates $t_{jq}\omega_{kl}^p = g(\lambda)\omega_{kl}$ for certain $g(\lambda) \in \mathbb{M}$. The last relation of Definition 22 with $a = d = j$ and $c = b = p$ applied to the highest weight vector $\omega_{kl}$, the second term vanishes and
\[
\frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + (k - 1)h)} \cdot g(\lambda) = \frac{\theta((l - k + 1)h)(h)}{\theta(\lambda_{jq})\theta(\lambda_{jq} + (k - l)h)} \prod_{q > r} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)}.
\]
This implies $\omega_{kl}^p \neq 0$. Conclude from $L_{pq}(z)\omega_{kl}^p = \frac{\theta(z + l - \lambda_{pq})}{\theta(z + l + h)} g(\lambda) \omega_{kl}$. □

Lemma 4.5. Let $k > 1$. In the $\mathcal{E}$-module $Z_{k,l+1} \otimes Z_{l+1,1}$ we have
\[
L_{pq}(z) \left( a_{lp}^{(l)}(k;\lambda;\omega_{k,l+1}^p \otimes \omega_{l+1,1}^p) - \omega_{k,l+1}^p \otimes \omega_{l+1,1}^p \right) = 0 \quad \text{where}
\]
\[
a_{lp}^{(l)}(k;\lambda) := \frac{\theta((k - l + 1)h)(\lambda_{jq} - h)}{\theta(h)\theta(\lambda_{jq})} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)}.
\]
Furthermore $G_{k,l}^{(l)}(k;\lambda;\omega_{k,l+1}^p \otimes \omega_{l+1,1}^p) - \omega_{k,l+1}^p \otimes \omega_{l+1,1}^p = 0$.

Proof. We compute $L_{pq}(z)(\omega_{k,l+1}^p \otimes \omega_{l+1,1}^p) = \sum_{q=1}^{N} L_{pq}(z)\omega_{k,l+1}^p \otimes L_{jq}(z)\omega_{l+1,1}^p$.

Since $\omega_{l+1,1}$ is a highest weight vector, the terms with $q > j$ vanish. The weight of $L_{jq}(z)\omega_{k,l+1}^p$ is $(k - l + 1)\varpi_r + \epsilon_q - \epsilon_j$, which does not belong to $\text{wt}(Z_{k,l+1})$ for $q < j$. So only the term $q = j$ survives. By Lemma 4.4
\[
L_{pq}(z)(\omega_{k,l+1}^p \otimes \omega_{l+1,1}^p) = L_{pq}(z)\omega_{k,l+1}^p \otimes L_{jj}(z)\omega_{l+1,1}^p
\]
\[
= - \frac{\theta((k - l + 1)h)(\lambda_{jq} - h)}{\theta(h)\theta(\lambda_{jq})} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)} \omega_{k,l+1}^p
\]
\[
\otimes \frac{\theta((l + 1)h)}{\theta(z + l + h)} \prod_{q > r} \frac{\theta(\lambda_{jq} + 2h)}{\theta(\lambda_{jq} + h)} \omega_{l+1,1}^p
\]
\[
= - \frac{\theta((k - l + 1)h)(\lambda_{jq} - h)}{\theta(h)\theta(\lambda_{jq})} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)} (\omega_{k,l+1}^p \otimes \omega_{l+1,1}^p).
\]

Similar arguments lead to:
\[
L_{pq}(z)(\omega_{k,l+1} \otimes \omega_{l+1,1}^p) = L_{pp}(z)\omega_{k,l+1} \otimes L_{pq}(z)\omega_{l+1,1}^p
\]
\[
= - \frac{\theta((k - l + 1)h)(\lambda_{jq} - h)}{\theta(h)\theta(\lambda_{jq})} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + 2h)}{\theta(\lambda_{jq} + h)} (\omega_{k,l+1} \otimes \omega_{l+1,1}^p).
\]

$a_{lp}^{(l)}(k;\lambda + h\epsilon_j)$ is the ratio of the two coefficients of $\omega_{kl}^p \otimes \omega_{l+1,1}^p$ above, which is easily seen to be independent of $z$. For the last identity, let $x$ be the vector in the argument of $G_{k,l}$. Then both $G_{k,l}(x)$ and $\omega_{kl}^p$ belong to the one-dimensional weight space of weight $(k - l)\varpi_r + \epsilon_j - \epsilon_p$. These two vectors are proportional, the first is annihilated by $L_{pq}(z)$, while the second is not. So $G_{k,l}(x) = 0$. □

Corollary 4.6. Let $k > 1$. In the $\mathcal{E}$-module $Z_{k,l}$ we have
\[
L_{kp}(z)\omega_{kl} = G_{k,l}(\omega_{k,l} \otimes \omega_{l+1,1}^p) \times b_{kp}^{(l)}(k;\lambda;\lambda)
\]
\[
b_{kp}^{(l)}(k;\lambda;\lambda) := \frac{\theta((k - l + 1)h + \lambda_{kp})\theta((k - l)h)}{\theta(h)\theta(\lambda_{kp})\theta(\lambda_{kp} + h)} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)}.
\]
Proof. The idea is similar to [43, Lemma 7.6]. We compute \( L_{jp}(z)(\omega_{k,l+1} \otimes \omega_{q+1,l}) \).

As in the proof of Lemma 4.5 only two terms survive:

\[
L_{jp}(z)(\omega_{k,l+1} \otimes \omega_{q+1,l}) = L_{jp}(z)\omega_{k,l+1} \otimes L_{jp}(z)\omega_{q+1,l} + L_{jp}(z)\omega_{k,l+1} \otimes L_{pp}(z)\omega_{q+1,l}
\]

\[
= \frac{\theta(z + kh)}{\theta(z + (l + 1)h)} \prod_{q > r} \frac{\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)} \omega_{k,l+1} \otimes \frac{\theta(z + lh + \lambda_{jp})}{\theta(z + lh)} \omega_{q+1,l}
\]

\[
+ \frac{\theta(z + (k - 1)h + \lambda_{jp})}{\theta(z + (l + 1)h)} \omega_{k,l+1} \otimes \omega_{q+1,l}
\]

\[
= e_{jp}^{(l)}(k, z; \lambda) (\omega_{k,l+1} \otimes \omega_{q+1,l}^p) + \frac{\theta(z + kh + \lambda_{jp})}{\theta(z + (l + 1)h)} (\omega_{q+1,l+1}^{jp} \otimes \omega_{q+1,l}).
\]

Here \( e_{jp}^{(l)}(k, z; \lambda) \) is the following meromorphic function of \((k, z, \lambda) \in \mathbb{C} \times \mathbb{C} \times \mathfrak{h}\):

\[
\frac{\theta(z + kh)\theta(z + lh + \lambda_{jp})\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)}.
\]

Set \( x := a_{jp}^{(l)}(k, \lambda)(\omega_{k,l+1} \otimes \omega_{q+1,l}^p) - \omega_{q+1,l} \otimes \omega_{q+1,l+1} \), which is in the kernel of \( \mathfrak{h}_{k,l} \) by Lemma 4.5. It follows that for any \( g(\lambda) \in \mathbb{M} \) we have

\[
L_{jp}(z)\omega_{kl} = L_{jp}(z)\mathfrak{h}_{k,l}(\omega_{k,l+1} \otimes \omega_{q+1,l}) = \mathfrak{h}_{k,l}(L_{jp}(z)(\omega_{k,l+1} \otimes \omega_{q+1,l}) + g(\lambda)x).
\]

Let us fix \( g(z; \lambda) := \frac{\theta(z + kh + \lambda_{jp})}{\theta(z + (l + 1)h)} \). Then \( L_{jp}(z)(\omega_{k,l+1} \otimes \omega_{q+1,l}) + g(\lambda)x \) is proportional to \( \omega_{k,l+1} \otimes \omega_{q+1,l}^p \) and \( L_{jp}(z)\omega_{kl} = \mathfrak{h}_{k,l}(\omega_{k,l+1} \otimes \omega_{q+1,l}^p) \times b_{jp}^{(l)}(k, z; \lambda) \) where

\[
b_{jp}^{(l)}(k, z; \lambda) = e_{jp}^{(l)}(k, z; \lambda) + g(z; \lambda)a_{jp}^{(l)}(k; \lambda)
\]

\[
b(k, z; \lambda) := \frac{\theta(z + kh)\theta(z + lh + \lambda_{jp})\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)} \prod_{r < q \neq p} \frac{\theta(\lambda_{jq} + (k - l)h)}{\theta(\lambda_{jq} + h)}.
\]

\[b(k, z; \lambda)\]

viewed as an entire function of \( k \), satisfies the same double periodicity as \( \theta(kh)\theta(kh + z + \lambda_{jp} - (l + 1)h) \). One checks that \( b(l, z; \lambda) = 0 \). This implies

\[b(k, z; \lambda) = \theta(kh + z - h + \lambda_{jp})\theta(kh - lh)f(z; \lambda)
\]

where \( f(z; \lambda) \) is a meromorphic function of \((z; \lambda) \in \mathbb{C} \times \mathfrak{h}\) independent of \( k \). Now setting \( kh = -z \), we obtain \( f(z; \lambda) = \frac{1}{\theta(z + h)\theta(h)} \). \( \square \)

Corollary 4.7. Let \( 1 \leq i, j \leq N \) with \( j \leq r \). For \( k - 1 > l \) and \( x \in W_{l,a}^{(r)} \):

\[
L_{ji}(z)F_{k,l}(x) = F_{k,l} \frac{\theta(z + kh)}{\theta(z + lh)} \mu_1 \left( \prod_{j = r + 1}^{N} \frac{\theta(\lambda_{jq} + (k - l + 1)h)}{\theta(\lambda_{jq} + h)} \right) L_{ji}(x)
\]

\[
+ F_{k,l+1} \mathcal{F}_{l+1,1} \left( \sum_{p = r + 1}^{N} \omega_{l+1,d} \otimes \mu_1 \left( b_{jp}^{(l)}(k, z; \lambda) \right) L_{pi}(x) \right).
\]

Proof. Consider \( L_{ji}(z)F_{k,l}(x) = \mathcal{F}_{l,1} \left( \sum_{p = r + 1}^{N} L_{jp}(z)\omega_{kl} \otimes L_{pi}(z)x \right) \). As in the proof of Lemma 4.5, \( L_{jp}(z)\omega_{kl} = 0 \) if \( p \notin \{ j, r + 1, r + 2, \cdots, N \} \). For \( p = j \), we obtain the first row of Eq. (4.22), while for \( r < p \leq N \), Corollary 4.6 and Eq. (4.19) with \( v = \omega_{l+1,d} \) give the second row. \( \square \)
Fix weight bases $\mathcal{B}_l$ of $W_{l,a}^{(r)}$ for $l > 0$ uniformly so that $F_{k,l}(\mathcal{B}_l) \subseteq \mathcal{B}_k$.

We view $b_{ji}^{(l)}(c, z; \lambda)$ in Corollary 4.6 as a meromorphic function of $(c, z, \lambda) \in \mathbb{C}^2 \times \mathbb{h}$. For $1 \leq i < j \leq N$, $l > 0$ and $c, z \in \mathbb{C}$, define $L^{(l)}_{ji}(c, z) : W_{l,a}^{(r)} \rightarrow W_{l+1,a}^{(r)}$:

$$L^{(l)}_{ji}(c, z)x = F_{l+1,1}L_{ji}(z)x = \frac{\theta(z + (\gamma_j + \delta_{ij} - 1)\mathbb{h} + \lambda_{ji})}{\theta(z)}F_{l+1,1}t_{ij}x \quad \text{for } j > r,$$

$$L^{(l)}_{ji}(c, z)x = \frac{\theta(z + ch)}{\theta([zh])} \prod_{q=r+1}^{N} \frac{\theta(\lambda_{jq} + (c + \gamma_{jq} + \delta_{ij} - \delta_{iq})\mathbb{h})}{\theta(\lambda_{jq} + (l + \gamma_{jq} + \delta_{ij} - \delta_{iq})\mathbb{h})}F_{l+1,1}L_{ji}(z)x + \sum_{p=r+1}^{N} b_{ji}^{(l)}(c, z; \lambda + (l + \gamma_{jq} + \delta_{ij} - \delta_{iq})\mathbb{h})F_{l+1,1}l_{ji}(z)x \quad \text{for } j \leq r.$$

Here $x \in W_{l,a}^{(r)}[\gamma + l\mathbb{h},r]$ and $\delta_{ij}$ is the usual Kronecker symbol. Corollary 4.2 applied to the evaluation module $W_{l,a}^{(r)} \cong V_{\pi_r}(0)$ indicates that for $b' \in B_{l+1}$ and $b \in B_l$:

$$L^{(l)}_{ji}(c, z)$$

is a difference map of bi-degree $(\epsilon_j - \pi_r, \epsilon_i)$. Its matrix entry $[L^{(l)}_{ji}]_{b'b}(c, z; \lambda)$ is a meromorphic function of $(c, z, \lambda) \in \mathbb{C}^2 \times \mathbb{h}$. Moreover, $\theta(z)\theta(z + \mathbb{h})[L^{(l)}_{ji}]_{b'b}(c, z; \lambda)$ is entire on $(c, z)$ for generic $\lambda$.

As a unification of Eqs. (4.20) and (4.22), we have

$$L_{ji}(z)x = F_{k,l+1}L^{(l)}_{ji}(k, z)$$

for $k > l + 1$.

For $k \in \mathbb{Z}_{>0}$ and $z \in \mathbb{C}$ let $\Xi(c; k, z)$ be the set of entire functions $F(c)$ of $c \in \mathbb{C}$ with the following double periodicity:

$$F(c + \mathbb{h}^{-1}) = (-1)^{k}F(c), \quad F(c + \mathbb{h})^{-1} = (-1)^{k}e^{-k\pi i} - 2k\tau \mathbb{h} - 2k\pi \mathbb{h}F(c).$$

A typical example is $\theta(ch)^{-1} \theta(ch + z)$. Such a function is called homogeneous. If $f(c), g(c) \in \Xi(c; k, z)$, then we write $f(c) \approx g(c)$.

Note that $\Xi(c; k, z)[\Xi(c; k', z')] \subseteq \Xi(c; k + k', z + z')$.

**Lemma 4.8.** Let $b \in B_l$ be of weight $\gamma + l\mathbb{h},r$ and $b' \in B_{l+1}$. For $j > r$ the matrix entry $[L^{(l)}_{ji}]_{b'b}(c, z; \lambda)$ is independent of $c$. For $j \leq r$ as entire functions of $c$

$$[L^{(l)}_{ji}]_{b'b}(c, z; \lambda) \approx \theta(z + ch) \prod_{q=r+1}^{N} \theta(\lambda_{jq} + (c + \gamma_{jq} + \delta_{ij} - \delta_{iq})\mathbb{h}).$$

Moreover, $\theta(z)[L^{(l)}_{ji}]_{b'b}(c, z; \lambda)$ is an entire function of $(c, z)$ for generic $\lambda$.

**Proof.** In the case $j > r$, Corollary 4.2 applied to $W_{l,a}^{(r)} \cong S_{l+1,a}$, the matrix entry is of the form $\theta(z + (\gamma_j + \delta_{ij} - 1)\mathbb{h} + \lambda_{ji})/\theta(z)$ for $g_1(\lambda)$ of $g_1(\lambda) \in \mathbb{M}$. Assume $j \leq r$. By Corollary 4.3 the matrix entry is of the form $E(c; z, \lambda)g_2(\lambda)$, where $g_2(\lambda) \in \mathbb{M}$ and $E(c; z, \lambda)$ is an entire function of $(c, z, \lambda) \in \mathbb{C} \times \mathbb{C} \times \mathbb{h}$. As functions of $z, c$ resp., we have

$$E(c, z; \lambda) \in \Xi(z; 2c, (c + l + \gamma_{jq} + \delta_{ij} - 1)h + \lambda_{ji}),$$

$$E(c, z; \lambda) \in \Xi(c; N - r + 1, (\lambda_{jq} + (\gamma_{jq} + \delta_{ij} - \delta_{iq})\mathbb{h}) + z).$$

On the other hand, for $k > l + 1$ we have by Corollary 4.2 and Eq. 4.28, $F_{k,l+1}L^{(l)}_{ji}(k, z)b = L_{ji}(z)x = F_{k,l}L^{(l)}_{ji}(k, z)b = (\mathbb{h} + \lambda_{ji})^{(l)}F_{k,l}t_{ij}b$. 
The right-hand side as a function of $z$ is regular at $z = -\hbar$, so is any of the coefficients of the left-hand side $E(k, z; \lambda)$. This forces $E(k, -\hbar; \lambda) = 0$ and $E(c, z; \lambda) = \theta(z + \hbar)\theta(z + (c + \gamma_{ij} + \delta_{ij} - 1)\hbar + \lambda_{ij})D(c; \lambda)g_{3}(\lambda)$ where $g_{3}(\lambda) \in M$ and $D(c; \lambda)$ is an entire function of $(c, \lambda)$. Applying the double periodicity with respect to $c$ once more, we obtain the desired result.

**Lemma 4.9.** Let $f(c)$ be a homogeneous entire function. If $f(k) = 0$ for infinitely many integers $k$, then $f(c)$ is identically zero.

**Proof.** By definition the homogeneous entire function $f(c)$, if non-zero, can be written as a product of theta functions $\theta(ch + z)$. Since $h \notin \mathbb{Q} \cap \mathbb{R}$, each of these theta functions of $c$ can not have zeroes at infinitely many integers.

Let $W_{\infty}$ be the inductive limit of the inductive system $(W^{(r)}_{l,a}, F_{k,l})$ of vector spaces (over $M$), with the $F_{i} : W^{(r)}_{l,a} \to W_{\infty}$ for $l > 0$ being the structural maps.

From now on fix $d \in \mathbb{C}$. A vector $0 \neq w \in W_{\infty}$ is of weight $dx_{r} + \gamma$ if there exist $l > 0$ and $w' \in W^{(r)}_{l,a}[dx_{r} + \gamma]$ such that $w = F_{i}(w')$. The weight grading is independent of the choice of $l$ because $F_{k,l}$ sends $W^{(r)}_{l,a}[dx_{r} + \gamma]$ to $W^{(r)}_{k,a}[dx_{r} + \gamma]$. Let $W_{d}^{\infty}$ denote the resulting object of $\mathcal{V}$. By construction $\text{wt}(W_{d}^{\infty}) \leq dx_{r} + \mathcal{Q}_{\infty}$, and $F_{i} : W^{(r)}_{l,a} \to W_{d}^{\infty}$ is a difference map of bi-degree $((l - d)dx_{r}, 0)$.

Let $\gamma \in \mathcal{Q}_{\infty}$. The injective maps $F_{k,l}$ together with Theorems [13, 23] imply that $\dim(W^{(r)}_{k,a}[dx_{r} + \gamma]) = d_{k,a}[dx_{r} + \gamma]$, as $k \to \infty$, converges to an integer which is exactly $\dim(W_{d}^{\infty}[dx_{r} + \gamma])$. So $W_{d}^{\infty}$ is an object of $\mathcal{V}_{\mathcal{K}}$. Our goal is to make $W_{d}^{\infty}$ into an $\mathcal{E}$-module in category $\mathcal{O}$ with favorable $q$-character.

For $1 \leq i, j \leq N$ and $z \in \mathbb{C}$ with $\theta(z) \neq 0$, the $\mathcal{L}^{(l)}_{ji}(d, z)$ constitute a morphism of inductive system of $\mathbb{C}$-vector spaces:

\[
\begin{align*}
W^{(r)}_{l,a} &\xrightarrow{F_{r,i}} W^{(r)}_{l+1,a} & &\text{for } l' > l. \\
W^{(r)}_{l,a} &\xrightarrow{F_{r+1,j}} W^{(r)}_{l+1,a}
\end{align*}
\]

Indeed, the matrix entries of $F_{r+1,j} \mathcal{L}^{(l)}_{ji}(c, z)$ and $\mathcal{L}^{(r)}_{ji}(c, z)F_{r,i}$, as difference maps $W^{(r)}_{l,a} \to W^{(r)}_{l+1,a}$, are homogeneous entire functions of $c$ with the same double periodicity by Lemma 1.8 and are equal at all integers $c$ larger than $l + 1$ by Eq. (1.23). By Lemma 1.9 these two maps coincide for all $c \in \mathbb{C}$. Define

\[
\mathcal{L}^{\delta}_{ji}(z) := \lim_{l \to \infty} \mathcal{L}^{(l)}_{ji}(d, z) \in \text{Hom}_{\mathbb{C}}(W_{\infty}^{d}, W_{\infty}^{d}).
\]

For $x \in W_{\infty}[dx_{r} + \gamma]$ with $x = F_{i}(x')$ and $x' \in W^{(r)}_{l,a}[dx_{r} + \gamma]$, we have

\[
(4.24) \quad \mathcal{L}^{\delta}_{ji}(z)x = F_{i+1} \mathcal{L}^{(l)}_{ji}(d, z)x'.
\]

The difference maps $\mathcal{L}^{(l)}_{ji}(d, z)$ and $F_{i+1}$ are of bi-degree $(\epsilon_{j} - dx_{r}, \epsilon_{i})$ and $((l + 1 - d)dx_{r}, 0)$ respectively. So $\mathcal{L}^{\delta}_{ji}(z)$ is a difference operator of bi-degree $(\epsilon_{j}, \epsilon_{i})$.

\footnote{In the affine case, the matrix entries of analogs of $\mathcal{L}^{(l)}_{ji}(k, z)$ are Laurent polynomials of $e^{k\hbar}$. Hernandez–Jimbo [52] proved this by using elimination theorems of $q$-characters and then took the limit $e^{k\hbar} \to 0$ as $k \to \infty$ to obtain modules over Borel subalgebras of affine quantum groups. Later in [53, 55] an elementary proof of polynomiality was given based on $\mathfrak{s}\mathfrak{l}_{2}$-representation theory, which by taking limit $e^{k\hbar} \to e^{\hbar}$ as $k \to \infty$ (with $d \in \mathbb{C}$ a new parameter) resulted in modules over affine quantum groups. Here we adapt the second approach to the elliptic case.}
Proposition 4.10. \((W^d_{\infty}, \mathcal{L}^d_{ji}(z))\) is an \(\mathcal{E}\)-module in category \(\mathcal{O}\). Moreover,\(^{(4.25)}\)

\[
\chi_q(W^d_{\infty}, \mathcal{L}^d_{ji}(z)) = W^{(r)}_{d,a} \times \lim_{k \to \infty} (W^{(r)}_{k,a})^{-1} \chi_q(W^{(r)}_{k,a}).
\]

**Proof.** We need to prove Conditions (M1)–(M3) of Section 1.3. First (M1) follows from Eq. (4.24) and from the comments before Eq. (4.23). To prove (M2), let \(x \in W^d_{\infty} [d\varpi_r + \gamma] \) and \(x' \in W^{(r)}_{l,a} [d\varpi_r + \gamma] \) such that \(x = F(x')\). We assume \(l \) so large that \(W^d_{\infty} [d\varpi_r + \gamma] \) and \(W^{(r)}_{l,a} [d\varpi_r + \gamma] \) have the same dimension.

**Step I: Proof of (M2).** We need to show that for \(1 \leq i, j, m, n \leq N\)

\[
\sum_{p, q} R_{mn}^{pq}(z - w; \lambda + (\epsilon_i + \epsilon_j + d\varpi_r + \gamma + h)\mathcal{L}^d_{pq}(z)\mathcal{L}^d_{ji}(w)x
\]

\[
= \sum_{s, t} R_{jt}^{st}(z - w; \lambda)\mathcal{L}^d_{nq}(w)\mathcal{L}^d_{ma}(z)x \in W^d_{\infty}.
\]

Here at the right-hand side we have used \(R_{mn}^{pq}(z; \lambda) = R_{mn}^{pq}(z; \lambda + c\epsilon_p + h\epsilon_q)\) to move \(R\) to the left. By Eq. (4.24) it is enough to prove the equation:

\[
\sum_{p, q} R_{mn}^{pq}(z - w; \lambda + (\epsilon_i + \epsilon_j + c\varpi_r + \gamma + h)\mathcal{L}^d_{pq}(z)\mathcal{L}^d_{ji}(w)x
\]

\[
= \sum_{s, t} R_{jt}^{st}(z - w; \lambda)\mathcal{L}^d_{nq}(w)\mathcal{L}^d_{ma}(z)x \in W^{(r)}_{l+2, a}.
\]

Let \(A_1(c, z, w)\) and \(A_2(c, z, w)\) denote the left-hand side and the right-hand side of this equation without \(x'\). These are difference maps \(W^{(r)}_{l,a} \to W^{(r)}_{l+2, a}\) of bi-degree \((c_m + c_n - 2\varpi_r, \epsilon_i + \epsilon_j)\), as \(R_{mn}^{pq} \neq 0\) implies \(c_m + c_n = c_p + c_q\).

**Claim 1.** For \(b \in B_1\) of weight \(l\varpi_r + \gamma\) and \(b' \in B_{l+2}\), as entire functions of \(c\),

\[
[A_1]_{\nu b}(c, z, w; \lambda) \approx [A_2]_{\nu b}(c, z, w; \lambda).
\]

This is divided into four cases. For simplicity let us drop \(b', b, z, w, \lambda\) from \(A_1, A_2\).

**Case 1.1:** \(m, n > r\). \(A_1(c)\) and \(A_2(c)\) are independent of \(c\) by Lemma 1.8.

**Case 1.2:** \(m, n \leq r\). At the left-hand side of Eq. (4.26) we have \(\{p, q\} = \{m, n\}\) and so \(R_{mn}^{pq}\) is independent of \(c\). At the right-hand side \(\{s, t\} = \{i, j\}\). Therefore

\[
A_1(c) \approx \theta(ch + z) \prod_{u > r} \theta(\lambda_{pu} + (c + \gamma_{pu} + \delta_{ip} - \delta_{iu} + \delta_{jp} - \delta_{ju} - \delta_{qp} + \delta_{qu})h)
\]

\[
\times \theta(ch + w) \prod_{v > r} \theta(\lambda_{qv} + (c + \gamma_{qv} + \delta_{iq} - \delta_{iv} + \delta_{jq} - \delta_{jv})h),
\]

\[
A_2(c) \approx \theta(ch + w) \prod_{u > r} \theta(\lambda_{uu} + (c + \gamma_{uu} + \delta_{tn} - \delta_{tu} + \delta_{sn} - \delta_{su} - \delta_{mn} + \delta_{mu})h)
\]

\[
\times \theta(ch + z) \prod_{v > r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{sm} - \delta_{sv} + \delta_{sm} - \delta_{tv})h).
\]

These formulas are deduced from Lemma 1.8. One needs to take into account the shifts of \(\gamma, \lambda\). For example at the left-hand side of Eq. (4.26), the term \(\mathcal{L}^d_{ji}\) (resp. \(\mathcal{L}_{pq}\)) shifts \(\gamma\) (resp. \(\lambda\)) by \(\epsilon_j - \epsilon_i\) (resp. \(h\epsilon_i\)). The right-hand sides of these two formulas lie in \(\Xi(c; 2 + 2N - 2r, e)\) with \(e \in \mathbb{C}\) independent of the choices of \(p, q, s, t\).

**Case 1.3:** \(m \leq r < n\). At the right-hand side \(\{s, t\} = \{i, j\}\) and

\[
A_2(c) \approx \theta(ch + z) \prod_{u > r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{sm} - \delta_{sv} + \delta_{tm} - \delta_{tv})h)
\]

\[
\approx \theta(ch + z) \prod_{v > r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{tm} - \delta_{tv} + \delta_{jm} - \delta_{jv})h).
\]
The last term is independent of $s, t$. On the other hand $A_1(c) = E(c) + F(c)$ where $E, F$ correspond to $(p, q) = (m, n)$ and $(p, q) = (n, m)$ respectively and so:

\[ E(c) \approx \frac{\theta(f - h)}{\theta(f)} \theta(ch + z) \prod_{u \geq r} \theta(\lambda_{mu} + (c + \gamma_{mu} + \delta_{im} - \delta_{iu} + \delta_{jm} - \delta_{ju} + \delta_{nu})h), \]

\[ F(c) \approx \frac{\theta(f + z - w)}{\theta(f)} \theta(ch + w) \prod_{v \geq r} \theta(\lambda_{mv} + (c + \gamma_{mv} + \delta_{jm} - \delta_{jv} + \delta_{im} - \delta_{uw})h). \]

Here $f := ch + \lambda_{mn} + (\gamma_{mn} + \delta_{im} - \delta_{in} + \delta_{jm} - \delta_{jn})h$. We observe easily that $A_2(c) \approx E(c) \approx F(c)$ and so $A_1(c) \approx A_2(c)$ are homogeneous.

Case 1.4: $n \leq r < m$. This is parallel to the third case.

**Claim 2.** In Claim 1 equality holds for $c = k \in \mathbb{Z}_{>l+2}$.

Let us apply $F_{k,l+2}$ to Eq.(4.29) with $c = k$ and $x' = b$. By Eq.(4.29):

\[ F_{k,l+2} \cdot \mathcal{L}^{(l+1)}_{aj}(k, z) \mathcal{L}^{(l)}_{aj}(k, w) = L_{p_l}(z) F_{k,l+1} \mathcal{L}^{(l)}_{aj}(k, w) x' = L_{p_l}(z) L_{l_j}(w) F_{k,l}, \]

and similarly $F_{k,l+2} \mathcal{L}^{(l+1)}_{at}(d, z) \mathcal{L}^{(l)}_{at}(d, z)b = L_{at}(w) L_{at}(z) F_{k,b}$. We obtain the defining relation $\text{RLL} = \text{LLR}$ of the $\mathcal{E}$-module $W_{l+1}^e$ applied to the vector $F_{k,l+1}(b)$. Since $F_{k,l+2}$ is injective, Eq.(4.29) holds for $c = k$ and $x' = b$. This proves Claim 2.

Together with Lemma 4.9 we obtain equality in Claim 1 for all $c \in \mathbb{C}$. This proves Eq.(4.26).

**Step II.** Let $1 \leq i \leq N$. We have by Eqs.(4.6) and (4.26):

\[ T_{i}(z)x = \frac{\Theta_i(\lambda)}{\Theta_i(\lambda + (d x_r + \gamma)h)} F_{i,l + i} \mathcal{G}^{(l)}_i(d, z)x'. \]

Here $\mathcal{G}^{(l)}_i(c, z) = \sum_{\sigma \in S_l} T_{\sigma}(c, z)$ and $T_{\sigma}(c, z) : W_{l+1}^{(l)} \rightarrow W_{l+1}^{(l)}$ for $\sigma \in S_l$ is

\[ \text{sign}(\sigma) \mathcal{L}^{(l+1)}_{(N),N}(c, z) \mathcal{L}^{(l+1)}_{(N-1),N-1}(c, z + h) \cdots \mathcal{L}^{(l)}_{(N-i+1),N-i+1}(c, z + (i - 1)h). \]

Each $T_{\sigma}(c, z)$ is a difference map of bi-degree $(-\omega_{N-i} - i \omega_r, -\omega_{N-i})$. Define the meromorphic function of $(c, z) \in \mathbb{C}^2$ (note that $l$ is fixed):

\[ g(c, z) = 1 \text{ if } i < N + 1 - r, \quad g(c, z) = \prod_{p=N-i+1}^{r} \frac{\theta(z + (N - p + c)h)}{\theta(z + (N - p + l)h)} \text{ otherwise.} \]

**Claim 3.** For $b \in \mathcal{B}_{l}$ of weight $l \omega_r + \gamma$ and $b' \in \mathcal{B}_{l+1}$, as entire functions of $c, z \in \mathbb{C}$,

\[ [T_{\sigma}]_{b, b'}(c, z; \lambda) \approx g_{b}(c, z) \Theta_i(\lambda + (c \omega_r + \gamma)h). \]

The idea is the same as Claim 1, based on Lemma 4.8. If $N - i + 1 > r$, then $T_{\sigma}^{(l)}(c, z; \lambda), \Theta_i(\lambda + (c \omega_r + \gamma)h)$ are independent of $c$, and we are done.

Assume $N - i + 1 \leq r$. By Eq.(4.5) and Lemma 4.8

\[ \Theta_i(\lambda + (c \omega_r + \gamma)h) \approx \prod_{p=N-i+1}^{r} \prod_{u=r+1}^{N} \theta(\lambda_{pu} + (c + \gamma_{pu})h), \]

\[ [T_{\sigma}]_{b, b'}(c, z; \lambda) \approx \prod_{p=N-i+1}^{r} \prod_{u=r+1}^{N} \theta(z + (N - p + c)h) \prod_{u=r+1}^{N} \theta(\lambda_{pu} + (c + \gamma_{pu} + 1)h). \]

Here $\lambda(\rho) = \lambda(h + \sum_{u=r+1}^{N} c_{u})$ and so $\lambda(\rho) = \lambda_{pu} - h$ for $p \leq r < u$. The case $\sigma = \text{Id}$ in Claim 3 is now obvious. It remains to show $[T_{\sigma}]_{b, b'}(c, z; \lambda) \approx [T_{\sigma'}]_{b, b'}(c, z; \lambda)$ for all $\sigma, \sigma' \in S_l$. One can assume $\sigma' = \sigma s_j$ where $s_j = (j, j+1)$ is a simple transposition with $N - i + 1 \leq j < N - 1$. Let us define

\[ p := \sigma(j + 1), \quad q := \sigma(j), \quad l' := l + i + j - 1 - N, \quad w := z + (N - j)h \]
Then we have the decomposition of difference maps

\[ T_\sigma(c, z) = \text{sign}(\sigma)A(c, z)U_{pq}(c, w)B(c, z), \]

\[ T_{\sigma'}(c, z) = \text{sign}(\sigma)A(c, z)U_{qp}(c, w)B(c, z). \]

The difference maps \( A, B, U \) are defined by (descending order in the products)

\[ A(c, z) = \prod_{u=N}^{j+2} \mathcal{L}_{\sigma(u), u}^{(j+1-2u-N)}(c, z + (N - u)\hbar) : W_{l+2, a}^{(r)} \rightarrow W_{l+1-1, a}^{(r)}. \]

\[ B(c, z) = \prod_{u=j-1}^{N-i+1} \mathcal{L}_{\sigma(u), u}^{(j-1-2u-N)}(c, z + (N - u)\hbar) : W_{l, a}^{(r)} \rightarrow W_{l-1, a}^{(r)}. \]

\[ U_{pq}(c, w) = \mathcal{L}_{p,j+1}^{q+1}(c, w - \hbar)\mathcal{L}_{q}^{(l')}(c, w) : W_{l'+a}^{(r)} \rightarrow W_{l'+1, a}^{(r)}. \]

Flipping \( p, q \) one gets \( U_{qp} \). Now \( [T_\sigma]_{\sigma'}(c, z; \lambda) \approx [T_{\sigma'}]_{\sigma}(c, z; \lambda) \) is a consequence of the following claim.

**Claim 4.** For \( y \in \mathcal{B}_t \) of weight \( t'\omega_r + \eta \) and \( y' \in \mathcal{B}_{t'+2} \), as entire functions of \( c \),

\[ [U_{pq}]_{y'y}(c, w; \lambda) \approx [U_{qp}]_{y'y}(c, w; \lambda). \]

If \( p, q \leq r \), then by Lemma 4.8 (setting \( \eta' = \eta + \epsilon_j - \epsilon_q \) and \( \lambda' = \lambda + \hbar c_j + 1 \))

\[ [U_{pq}]_{y'y}(c, w; \lambda) \approx \theta(w + (c - 1)\hbar) \prod_{u=r+1}^{N} \theta(\lambda_{pu} + (c + \eta_{pu} + \delta_{p,j+1} - \delta_{u,j+1})\hbar) \]

\[ \times \theta(w + ch) \prod_{v=r+1}^{N} \theta(\lambda'_{qv} + (c + \eta_{qv} + \delta_{jq} - \delta_{jv})\hbar). \]

We have \( U_{pq}'(c, w; \lambda) \in \mathfrak{E}(c; 2N - 2r + 2, e) \) with \( e = e(p, q) \) symmetric on \( p, q \). So \( [U_{pq}]_{y'y}(c, w; \lambda) \approx [U_{qp}]_{y'y}(c, w; \lambda) \).

The other cases of \( p, q \) are proved in the same way as in Claim 1.

**Step III: Proof of (M3).** Let \( k > l + i \). Notice that \( \mathcal{D}_i(z)\omega_{kl} = g_i(k, z)\omega_{kl} \). From the proof of Lemma 4.3 and from Eqs. 4.23 and 4.27 we get

\[ g_i(k, z)F_k \mathcal{D}_i(z)x' = \mathcal{D}_i(z)F_k \mathcal{D}_i(z)x' = F_k \mathcal{D}_i(z)x' = \Theta_i(\lambda + (k\omega_r + \gamma)\hbar) \mathcal{E}_i^{(l)}(k, z)x'. \]

Applying \( F_k \) to this identity and multiplying \( \Theta_i(\lambda + (k\omega_r + \gamma)\hbar) \) we have

\[ \Theta_i(\lambda + (k\omega_r + \gamma)\hbar)g_i(k, z)F_k \mathcal{D}_i(z)x' = \Theta_i(\lambda)F_{l+i} \mathcal{E}_i^{(l)}(k, z)x' \]

for \( k > l + i \). Both sides after taking coefficients with respect to a basis of \( W_{\infty}^d [d\omega_r + \gamma] \) can be viewed as entire functions of \( k \in \mathbb{C} \), and they satisfy the same double periodicity by Claim 3. By Lemma 4.9 the above identity holds for all \( k \in \mathbb{C} \). Taking \( k = d \), by Eq. 4.27, we obtain \( \mathcal{D}_i(z)x = g_i(d, z)F_k \mathcal{D}_i(z)x' \).

Let \( B \) be a basis of \( W_{l+2}^{(r)} [l\omega_r + \gamma] \) satisfying the upper triangular property of (M3). Then so does the basis \( F_1(B) \) of \( W_{\infty}^d [d\omega_r + \gamma] \). The \( \mathcal{E} \)-module \( W_{\infty}^d \) is in category \( \mathcal{O} \). The diagonal entry of \( \mathcal{D}_i(z) \) associated to \( F_1(x') \in W_{\infty}^d \) for \( x' \in B \) is equal to that of \( \mathcal{D}_i(z) \) associated to \( x' \in W_{l+2}^{(r)} \) multiplied by \( g_i(d, z) \). The q-character formula in Eq. 4.25 follows from the explicit formula of \( g_i(d, z) \). \( \square \)

**Question 4.11.** Let \( F(c) \) be a finite sum of homogeneous entire functions. If \( F(k) = 0 \) for infinitely many integers \( k \), then is \( F(c) \) identically zero?
If the answer to this question is affirmative, then the proof of Proposition 4.10 can be largely simplified: Claims 1, 3 and 4 are not necessary. \footnote{In the affine case, by Footnote 11 the situation is much easier: a Laurent polynomial vanishing at infinitely many integers must be zero; see [53, \S2].}

Remark 4.12. By Lemma 4.18 \( W^d_\infty \cong \mathcal{W}(0) \) with \( \mathcal{W} \) an \( \epsilon \)-module of character
\[
\lim_{k \to \infty} e^{(d-k)\pi \iota} \chi(W^{(r)}_{k,a}),
\]
so it is in the image of the functor [12, Proposition 4.1]. By Lemma 4.12 and its proof, \( \mathcal{W} \) contains a unique highest weight vector up to scalar. Let \( Q \) be the quotient of standard Verma module \( M_{d \infty,1} \) in [51, Proposition 4.7] by \( t_{a+1,a}^{\nu \infty},1 \) for \( a \neq r \). Then \( \mathcal{W} \) is the contragradient module to \( Q \) in [51, \S6]. It is interesting to have a direct proof of \( \mathcal{W}(0) \) being in category \( \mathcal{O} \).

For \( x \in \mathbb{C} \) let \( \mathcal{W}^{(r)}_{d,x} \) be the pullback of \( W^d_\infty \) by \( \Phi_{x-a} \) in Eq. (4.14); it is called asymptotic module. Set \( \mathcal{W}^{(N)}_{d,x} := S(w^{(N)}_{d,x}) \) and \( \mathcal{W}^{(s)}_{s,x} := \mathcal{W}_{s,0} \) for \( 1 \leq s \leq N \).

**Corollary 4.13.**
(i) \( \mathcal{R} \) is the set of rational \( \epsilon \)-weights.
(ii) For any \( \mathcal{E} \)-module \( M \) in category \( \mathcal{O} \), we have \( \text{wt}_e(M) \subset \mathcal{R} \).
(iii) For \( d, x \in \mathbb{C} \) and \( 1 \leq r \leq N \) we have in \( \mathcal{M}_t \) and \( \mathcal{K}_0(\mathcal{O}) \) respectively
\[
\chi_q(\mathcal{W}^{(r)}_{d,x}) = w^{(r)}_{d,x} \times \chi_q(\mathcal{W}^{(r)}_{0,x}), 
[\mathcal{W}^{(r)}_{d,0}] [\mathcal{W}^{(r)}_{0,x}] = [\mathcal{W}^{(r)}_{d-x,x}][\mathcal{W}^{(r)}_{x,0}].
\]

**Proof.** (iii) comes from Eq. (4.25), as in the proof of [21, Theorem 3.11]. \( w^{(r)}_{d,x} \) as a highest weight of \( \mathcal{W}^{(r)}_{d,x} \) belongs to \( \mathcal{R} \). Together with Lemma 1.13, we obtain (i). In (ii) one may assume \( M \) irreducible. Then \( M \) is a sub-quotient of a tensor product of asymptotic modules. Since \( \epsilon \)-weights of an asymptotic module are rational, we conclude from the multiplicative structure of \( \epsilon \)-characters in Proposition 1.10.

In Section 2.2 the evaluation module \( V_\mu(x) \) is an irreducible highest weight module in category \( \mathcal{O} \). Its highest weight is easily shown to be rational.

**Corollary 4.14.** \( V_\mu(x) \) is in category \( \mathcal{O} \) for \( \mu \in \mathfrak{h} \) and \( x \in \mathbb{C} \).

Finite-dimensional modules in category \( \mathcal{O} \) are related to the asymptotic modules by *generalized Baxter relations* in the sense of Frenkel–Hernandez [23, Theorem 4.8]; see [21, Corollary 4.7] and [55, Theorem 5.11] for a closer situation.

**Theorem 4.15.** Let \( V \) be a finite-dimensional \( \mathcal{E} \)-module in category \( \mathcal{O} \). Then
\[
(4.29) \quad |V| = \sum_{j=1}^{\dim V} [S(d_j)] \times m_j
\]
in a fraction ring of the Grothendieck ring of \( \mathcal{O} \). Here \( d_j \in \mathcal{R}_0 \) and \( m_j \) is a product of the \( \frac{[\mathcal{W}^{(r)}_{d,x}]}{[\mathcal{W}^{(r)}_{y,x}]} \) with \( x, y \in C \) and \( 1 \leq r < N \).

**Proof.** The idea is the same as [23]. Since the \( \epsilon \)-character map is injective, one can replace isomorphism classes with \( \epsilon \)-characters. \( \chi_q(V) \) is the sum of its \( \epsilon \)-weights, the number of which is \( \dim V \). By Corollary 4.13 any \( \epsilon \)-weight \( \mathbf{e} \) is of the form \( \mathbf{d} \prod_{x,y} \Psi_{x,y} = \chi_q(S(d)) \prod_{x,y} \frac{[\mathcal{W}^{(r)}_{d,x}]}{[\mathcal{W}^{(r)}_{y,x}]} \), where \( d \in \mathcal{R}_0 \) and the product is over \( 1 \leq r < N \) and \( x, y \in \mathbb{C} \). This proves Eq. (4.29) in terms of \( \epsilon \)-characters.

To compare with [23, Theorem 4.8], one imagines that for \( 1 \leq r < N \) and \( x \in \mathbb{C} \) there existed a *positive prefundamental module* \( L^+_{r,x} \) in category \( \mathcal{O} \) with \( \epsilon \)-character
\[
\chi_q(L^+_{r,x}) = \Psi_{r,x} \times \chi(L^+_{r,0}) \text{ as in [23, Theorem 4.1]. Then}\]
\[
\frac{[\mathcal{W}^{(r)}_{d,x}]}{[\mathcal{W}^{(r)}_{y,x}]} = \frac{[\mathcal{W}^{(r)}_{d,x}]}{[\mathcal{W}^{(r)}_{y,x}]} \text{. Note that}\]
\[
\text{the q-character of } \mathcal{W}^{(r)}_{0,x} \text{ in Eq. (4.28) is different from its character.}
\]
Example 4.16. Let $N = 3$. Consider the vector representation $V$ of Section 1.3.

\[
egin{align*}
\mathbf{1} &= \frac{\Psi_{1,\frac{3}{2}}}{\Psi_{1,\frac{3}{2}}}, \\
\mathbf{2} &= \frac{\Psi_{1,\frac{1}{2}}}{\Psi_{1,\frac{1}{2}}}, \\
\mathbf{3} &= \frac{\Psi_{1,\frac{1}{2}}}{\Psi_{1,0}}, \\
[\mathbf{V}] &= \left[ \frac{\Psi_{1,\frac{3}{2}}}{\Psi_{1,\frac{3}{2}}} \right] + \left[ \frac{\Psi_{1,\frac{1}{2}}}{\Psi_{1,\frac{1}{2}}} \right] + \left[ \frac{\Psi_{1,\frac{1}{2}}}{\Psi_{1,0}} \right].
\end{align*}
\]

Example 4.17. Let us construct the $E_{r,a}(\mathfrak{sl}_2)$-module $\mathcal{W}_{\lambda}$ from [19, Theorem 3]. In loc.cit., set $\eta = -\frac{1}{2}h$, $\lambda = \lambda_{12}$ and $(a, b, c, d) = (L_{11}, L_{12}, L_{21}, L_{22})$. For $\Lambda \in \mathbb{Z}_{>0}$, consider the evaluation module $L_{\Lambda}((\Lambda - 1)\eta)$ with bases $(e_k)_{0 \leq k \leq \Lambda}$. Note that $k$ indicates the basis vectors, while $\Lambda$ the integer parameter of a KR module. Let us make a change of basis (the second product is empty if $k = 0$)

\[
v_k := e_k \prod_{i=1}^{\Lambda} \frac{\theta(\lambda + (i-k)\lambda)}{\theta(\lambda)} \times \prod_{j=1}^{k} \frac{\theta(\lambda - j\lambda)}{\theta((\Lambda - k + j)\lambda)} \quad \text{for } 0 \leq k \leq \Lambda.
\]

Tensoring $L_{\Lambda}((\Lambda - 1)\eta)$ with the one-dimensional module of highest weight $\frac{\theta(w+\Delta h)}{\theta(w)}$, we obtain another irreducible module $V_{\lambda}$ with basis $(v_k = v_0 \otimes 1)_{0 \leq k \leq \Lambda}$; here to follow loc.cit. $w$ denotes $z$. We have $w(v_k) = (\Lambda - k)e_1 + ke_2$ and

\[
\begin{align*}
a(w)v_k &= \frac{\theta(w + (\Lambda - k)\lambda)}{\theta(w)} \frac{\theta(\Lambda + (\Lambda - k + 1)\lambda)}{\theta(\lambda)} v_k, \\
b(w)v_k &= \frac{\theta(w + \lambda + (\Lambda - k - 1)\lambda)}{\theta(w)} \frac{\theta((\Lambda - k)\lambda)}{\theta(\lambda)} v_{k+1}, \\
c(w)v_k &= \frac{\theta(w - \lambda + (k - 1)\lambda)}{\theta(w)} \frac{\theta(k\lambda)}{\theta(\lambda)} v_{k-1}, \\
d(w)v_k &= \frac{\theta(w + k\lambda)}{\theta(w)} \frac{\theta(\Lambda - k + 1)\lambda)}{\theta(\lambda)} v_{k}.
\end{align*}
\]

We have $t_{12}v_k = \frac{\theta(k\lambda)}{\theta(\lambda)} v_{k-1}$ and $v_0$ is of highest weight $w^{(1)}_{\lambda,0}$, so $V_{\lambda} \simeq W^{(1)}_{\lambda,0}$. The bases $(v_k)$ trivialize the inductive system $(V_{\lambda})$ because the inductive maps commute with $t_{12}$ by Eq. (4.20). For $\Lambda \in \mathbb{C}$, the above formulas define an $E_{r,h}(\mathfrak{sl}_2)$-module structure on $\otimes_{k=0}^{\Lambda} v_k$, with $w(v_k) = (\Lambda - k)e_1 + ke_2$. This is the desired $\mathcal{W}_{\lambda}$.

General formulas for the $E_{r,a}(\mathfrak{sl}_N)$-module $\mathcal{W}_{\lambda}$ can be found in [7, §3.4].

5. BAXTER TQ RELATIONS

We derive three-term relations in the Grothendieck ring $K_0(\mathcal{O})$ for the asymptotic modules. For $1 \leq r < N$ and $k, x, t \in \mathbb{C}$, by Corollary 4.13 $d^{(r,t)}_{k,x} \in \mathbb{R}$ and is the highest weight of an irreducible module $D^{(r,t)}_{k,x}$ in category $\mathcal{O}$.

Call a complex number $c \in \mathbb{C}$ generic if $c \not\in \frac{1}{2}\mathbb{Z} + \frac{1}{2}(\mathbb{Z} + \mathbb{Z} \tau)$. This condition is equivalent to $Q_a \cap Q_{a+c} = \{1\}$ for all $a \in \mathbb{C}$.

Theorem 5.1. Let $1 \leq r < N$, $t \in \mathbb{Z}_{>0}$ and $k, a, b \in \mathbb{C}$ with $k$ generic. Then

\[
\chi_q(D^{(r,t)}_{k,a}) = d^{(r,t)}_{k,a} (1 + \sum_{l=1}^{N} A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1}) \prod_{s=r \pm 1} \chi_q(w^{(s)}_{0,a-k-\frac{1}{2}})
\]

and $D^{(r,0)}_{k,a} \cong \otimes_{s=r \pm 1} w^{(s)}_{k,a-k-\frac{1}{2}}$.

Proof. Set $x := a - k - \frac{1}{2}$. Define $d := d^{(r,t)}_{k,a}$ and for $1 \leq l \leq t$:

\[
m_l := m_0 A_{r,a}^{-1} A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1}, \quad m_0 := d \prod_{j=1}^{t} w^{(r)}_{k+j-\frac{1}{2},x}.
\]
By Eqs. (1.9) and (3.14)–(3.15), we have for $0 \leq l \leq t$:
\[
m_l = \prod_{j=t+1}^{l} \frac{\Psi_{r,a+j}}{\Psi_{r,x}} \times \prod_{j=1}^{l} \frac{\Psi_{r,a+j-2}}{\Psi_{r,x}} \times \prod_{s=r+1}^{l} \frac{\Psi_{s,a+l-\frac{1}{2}}}{\Psi_{s,x}}.
\]

Let us introduce the tensor products for $0 \leq l \leq t$,
\[
S^l := (\bar{\otimes}_{j=t+1}^{l} \Psi_{r,k+j+\frac{1}{2},a}) \otimes (\bar{\otimes}_{j=1}^{l} \Psi_{r,k+j-\frac{1}{2},x}) \otimes (\bar{\otimes}_{s=r+1}^{l} \Psi_{s,k+l-\frac{1}{2},x}),
\]
\[
T := D_{k,a}^{(r,t)} \otimes (\bar{\otimes}_{j=1}^{l} \Psi_{r,k+j-\frac{1}{2},x}).
\]

Eq. (5.30) is equivalent to $\chi_q(T) = \sum_{l=0}^{t} \chi_q(S^l)$ in view of Eq. (4.28).

Given two elements $\chi = \sum f \epsilon_f$ and $\chi' := \sum f' \epsilon_f$ of $M_t$, we say that $\chi$ is bounded above by $\chi'$ if $\epsilon_f \leq \epsilon_f'$ for all $f \in M_w$. When this is the case, $\chi'$ is bounded below by $\chi$. If $\chi$ is bounded below and above by $\chi'$, then $\chi = \chi'$.

**Claim 1.** The $S^l$ are irreducible. In particular, $D_{k,a}^{(r,0)} \simeq \bar{\otimes}_{s=r+1} S_{k,a}$.

Fix $0 \leq l \leq t$. Let $S' := S(S^l)$. For $n \in \mathbb{Z}_{>0}$, set
\[
S'_n := (W_{n,x})^{\otimes l} \otimes (\bar{\otimes}_{s=r+1} W_{n,x}^{(r)}),
\]
\[
s'_n := (w_{n,x}^{(r)})^l \times \prod_{s=r+1}^{l} w_{n,x}^{(s)}.
\]

By Lemma 4.2 any e-weight $s'_n \in \mathcal{P}_x$ of $S'_n$ different from $s'_n$ is right negative. So $S'_n$ is irreducible. Viewing $S'_n$ as an irreducible sub-quotient of
\[
S^l \otimes (\bar{\otimes}_{j=t+1}^{l} \Psi_{n-k-j+\frac{1}{2},a}) \otimes (\bar{\otimes}_{j=1}^{l} \Psi_{n-k-j-\frac{1}{2},a}) \otimes (\bar{\otimes}_{s=r+1}^{l} \Psi_{a+l-\frac{1}{2}}),
\]
we have $e = e' \prod_{j=1}^{l} e_j \prod_{s=r+1}^{l} e_s$ where $m(e'), w_{n-k-j+\frac{1}{2},a} e_j$ for $l < j \leq t$, $w_{n-k-j-\frac{1}{2},a} e_j$ for $1 \leq j \leq l$, and $w_{a} e_s$ are e-weights of the corresponding tensor factors. By Lemma 4.2 and Proposition 4.10,
\[
e, e' \in \mathcal{Q}_x^-, \quad e_j, e_s \in \mathcal{Q}_x^{-}\]

Since $a - x = k + \frac{1}{2}$ is generic, $\mathcal{Q}_x^+ \cap \mathcal{Q}_x^- = \{1\}$ and so $e = e'$. The normalized q-character of $S^l$ is bounded below by that of $S'_n$ for all $n \in \mathbb{Z}_{>0}$. On the other hand, viewing $S'$ as an irreducible sub-quotient of $S^l$ and applying Eq. (4.25) to $S^l$, we see that the normalized q-character of $S^l$ is bounded above by the limit of that of $S'_n$ as $n \to \infty$. Therefore $S^l \simeq S'$ is irreducible.

**Claim 2.** For $1 \leq l \leq t$, we have $dA_{r,a+1}^{l-1} \cdots A_{r,a+l-1}^{l-1} \in \mathrm{wt}_{e}(D_{k,a}^{(r,t)})$. It follows that $m_l \in \mathrm{wt}_{e}(T)$.

Let us view the KR module $W_{t,a}^{(r,t)}$ as an irreducible sub-quotient of
\[
D_{k,a}^{(r,t)} \otimes (\bar{\otimes}_{s=r+1}^{l} \Psi_{a}^{(s)}).
\]

By Lemma 4.2, $w_{t,a}^{(r,t)} A_{r,a}^{l-1} \cdots A_{r,a+l-1}^{l-1} \in \mathrm{wt}_{e}(W_{t,a}^{(r,t)})$. The $A_{r,a+j}^{l-1}$ must arise from $\mathrm{wt}_{e}(D_{k,a}^{(r,t)})$ instead of any of the $\mathrm{wt}_{e}(W_{t,a}^{(r,s)})$ with $s \neq r$.

For $0 \leq j, l \leq t$, since $\mathrm{wt}_{e}(S^l) \subset m_l \mathcal{Q}_x^-$ and $m_j \in m_l \mathcal{Q}_x$, we have $m_j \in \mathrm{wt}_{e}(S^l)$ if and only if $l = j$. Therefore, all the $S^l$ appear as irreducible sub-quotients of $T$, and they are mutually non-isomorphic. So $\chi_q(T)$ is bounded below by $\sum_{l=0}^{t} \chi_q(S^l)$.

**Claim 3.** $\chi_q(D_{k,a}^{(r,t)})$ is bounded above by
\[
\chi_q(D_{k,a}^{(r,t)}) \leq \sum_{l=1}^{t} \prod_{s=r+1}^{l} \chi_q(W_{s,x}^{(s)}).
\]
Let $\kappa$, $t$ be generic and $\ell \geq 0$ be the subset of $\ell$ such that:

$$\ell = \{1, A_{r,a}^{-1}A_{r,a+1}^{-1}, \ldots, A_{r,a}^{-1}A_{r,a+1}^{-1} \}$$

is uniquely determined by $\ell$. The coefficient of $df$ in $\chi_q(M_{k,a}^{(r,t)})$ is bounded above by that of $\prod_{s=r+1} f(s)$ in $\prod_{s=r+1} \chi_q(\mathcal{W}_{0,x}^{(s)})$. This proves the claim.

It follows from Claim 3 that $\chi_q(T)$ is bounded above by

$$\mathcal{M}_0(1 + \sum_{l=1}^{t} A_{r,a}^{-1}A_{r,a+1}^{-1} \cdots A_{r,a+l-1}^{-1}) \prod_{s=r+1}^{t} \chi_q(\mathcal{W}_{0,x}^{(s)}) \times \prod_{j=1}^{t} \prod_{s=r+1}^{t} \chi_q(\mathcal{W}_{0,x}^{(r)}) = \sum_{i=1}^{t} \chi_q(S_i).$$

Since “bounded below” also holds, we obtain the exact formula for $\chi_q(T)$, which implies Eq.\((5.30)\). This completes the proof of the theorem. \hfill \Box

Claim 1 is in the spirit of [23, Theorem 4.11], and Claim 3 [37, Eq.\((6.14)\)], [24 §4.3] and [54, Theorem 3.3], the main difference being the non-existence of prefundamental modules. If both $k, t$ are generic, then $\chi_q(D_{k,a}^{(r,t)})$ is obtained from the right-hand side of Eq.\((5.30)\) by replacing $\sum_{l=1}^{t}$ therein with $\sum_{l=1}^{\infty}$.

**Corollary 5.2.** Let $k \in \mathbb{C}$ be generic and $1 \leq r < N$. In $K_0(\mathcal{O})$ holds

\[(5.31)\quad [D_{k,k}^{(r,1)}] [\mathcal{W}_{k,k+\frac{r}{2}}] = [\mathcal{W}_{r,k+\frac{r}{2}}] \prod_{s=r+1}^{t} [\mathcal{W}_{s,k+1}] + [\mathcal{W}_{r,k+\frac{r}{2}}] \prod_{s=r+1}^{t} [\mathcal{W}_{s,k}].\]

**Proof.** From Eq.\((5.30)\) and the injectivity of the q-character map we obtain

\[(5.32)\quad [D_{k,a}^{(r,1)}] [\mathcal{W}_{a-b+t-1,b}] = [D_{k,k+a+t-1}^{(r,0)}] [\mathcal{W}_{a-b-t+1,b}] + [D_{k,a}^{(r,1)}] [\mathcal{W}_{a-b+t,b}]\]

for $a, b \in \mathbb{C}$ and $t \in \mathbb{Z}_{>0}$. Eq.\((5.31)\) is the special case $(t, a, b) = (1, k + \frac{r}{2}, 0)$ of this identity in view of the tensor product decomposition of $D_{k,a}^{(r,t)}$ in Theorem 5.1. \hfill \Box

**Eq.\((5.32)\)** can be viewed as a generic version of Eq.\((5.17)\).

### 6. Transfer matrices and Baxter operators

We have obtained three types of identities Eq.\((4.28)\), Eq.\((4.29)\), and Eq.\((5.31)\) in the Grothendieck ring $K_0(\mathcal{O})$. These are viewed as universal functional relations [3] in the sense that when specialized to quantum integrable systems they imply functional relations of transfer matrices. In this section, we study one such example, with the quantum space being a tensor product of vector representations [38].

Fix $\ell := N\kappa$ with $\kappa \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_\ell \in \mathbb{C} \setminus \Gamma$. Set $I := \{1, 2, \ldots, N\}$. Let $I'_0$ be the subset of $I'$ formed of $\ell$ such that $\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_{i+\ell} = 0 \in \mathfrak{h}$. Upon identification $\ell := v_1 \otimes v_2 \otimes \cdots \otimes v_{\ell}$, the weight space $\mathcal{V}^{[0]}$ has basis $I'_0$.

Let $D_{a}^{(z)}$ be the set of formal sums $\sum_{o \in \mathcal{B}} p^{o} T_o f_a(z; \lambda)$ such that: the $f_a(z; \lambda)$ are meromorphic functions of $(z, \lambda) \in \mathbb{C} \times \mathfrak{h}$; the set $\{\alpha : f_a \neq 0\}$ is contained in a...
finite union of cones \( \nu + Q_{\mathbb{C}} \) with \( \nu \in \mathfrak{h} \). Make \( \mathbb{D}_p \) into a ring: addition is the usual form of formal sums; multiplication is induced from

\[
\rho^\alpha T_{\alpha} f(z;\lambda) \times \rho^\beta T_{\alpha + \beta} g(z;\lambda) = \rho^{\alpha + \beta} T_{\alpha + \beta} f(z;\lambda + \hbar b) g(z;\lambda).
\]

As in \([20, 21]\), we construct a ring morphism \([X] \mapsto t_X(z)\) from \( K_0(\mathcal{O}) \) to the ring \( M(I_0^I; \mathbb{D}_p) \) of \( I_0^I \times I_0^I \) matrices with coefficients in \( \mathbb{D}_p \). (We think of \( M(I_0^I; \mathbb{D}_p) \) as a ring of formal difference operators on \( V^\otimes \{0\} \).)

Let \( X \) be an object of category \( \mathcal{O} \). To \( L_{ij} \in I_0^I \) we associate

\[
L_{ij}^X(z) := L_{i_1j_1}(z + a_1)L_{i_2j_2}(z + a_2)\cdots L_{i_\ell j_\ell}(z + a_\ell) \in (DX)_{0,0}.
\]

Since \( (DX)_{0,0} \subseteq \text{End}_\mathbb{C}(X) \), one can take trace of \( L_{ij}^X(z) \) over weight spaces of \( X \).

**Definition 6.1.** The transfer matrix associated to an object \( X \) in category \( \mathcal{O} \) is the matrix \( t_X(z) \in M(I_0^I; \mathbb{D}_p) \) whose \( (i,j) \)-th entry for \( L_{ij} \in I_0^I \) is

\[
\sum_{\alpha \in \text{wt}(X)} \rho^\alpha T_{\alpha} \times \text{Tr}_{X|\alpha} \left( L_{ij}^X(z)|_{X|\alpha} \right) \in \mathbb{D}_p.
\]

Almost all of the results and comments in \([21, \S 5]\) hold true after slight modification in our present situation. In the following, we focus on the modification of \( D \) and \( \mathcal{O} \).

The transfer matrix associated to the one-dimensional module of highest weight \( g(z) = \prod_{i=1}^N g(z + a_i) \) is the scalar matrix \( \Pi z \in \mathfrak{g} \) is the scalar matrix \( \prod_{i=1}^N \overline{g}(z + a_i) \).

For \( 1 \leq r \leq N \) and \( x \in \mathbb{C} \), consider the \( \mathcal{E} \)-module \( \mathcal{W}_{r,x} := \mathcal{W}_{r,x} \otimes S(\theta(z - t_r h)) \) in category \( \mathcal{O} \). By Lemma \([18, 35]\), the matrix entries of the difference operators \( L_{ij}(z) \) for \( 1 \leq i,j \leq N \), with respect to any basis of \( \mathcal{W}_{r,x} \), are entire functions of \( z \in \mathbb{C} \).

**Definition 6.2.** The \( r \)-th Baxter \( Q \)-operator for \( 1 \leq r \leq N \) is defined to be

\[
Q_r(u) := t_{\overline{\mathcal{W}}_{r+1}^\prime}(z)\big|_{z=0} \quad \text{for } u \in \mathbb{C}.
\]

Since \( \overline{\mathcal{W}}_{r+1}^\prime = S(\theta(z + (x + \frac{1}{r}) h)) \) is one-dimensional, \( Q_N(z) = \prod_{i=1}^N \theta(z + a_i + \frac{1}{r} h) \).

Let \( 1 \leq r < N \). Then \( Q_r(z) = \rho^{\frac{1}{r} h - \frac{1}{r} m} T_{-\alpha} \frac{Q(z)}{Q(z)} \) is a power of \( r \)-th \( \alpha \)-representation, \( 1 \leq i < N \). The leading term \( \frac{Q_0(z)}{Q(z)} \) of \( Q(z) \) is invertible.

Indeed \( \frac{Q_0(z)}{Q(z)} \) is the scalar matrix \( \prod_{i=1}^N \theta(a_i) \in M(I_r^I; \mathbb{C}) \), which is invertible because \( \theta(a_i) \neq 0 \) by assumption. (One can prove furthermore that with respect to certain order on \( I_0^I \), the matrix \( \frac{Q_0(z)}{Q(z)} \) is upper triangular, whose entries are meromorphic functions of \( z \in \mathbb{C} \times \mathfrak{h} \) and entire on \( \mathbb{C} \).) Therefore \( Q_r(z) \in \text{GL}(I_0^I; \mathbb{D}_p) \).

Similarly one can show that \( t_{\overline{\mathcal{W}}_{r+1}^\prime}(z) \) is invertible for \( x \in \mathbb{C} \).

**Proposition 6.3.** Let \( X, Y \) be in category \( \mathcal{O} \) and let \( x, u \in \mathbb{C} \).

(i) \( t_{\overline{\mathcal{W}}_{X}^\prime}(z) = t_X(z + uh) \).

(ii) \( t_X(z) t_Y(z) = t_{X \otimes Y}(z) \).

(iii) \( t_{\overline{\mathcal{W}}_{r,u}}(z + uh) = t_{\mathcal{W}_{r,u}}(z + uh) t_{\mathcal{W}_{r,u}}(z) \).

(iv) \( t_X(z) t_Y(u) = t_Y(u) t_X(z) \).

In (iv), we replace one of the \( u \) in Eq.\((6.33)\) with \( w \) to define the multiplication. It is proved as in \([25, \text{Theorem 5.3}]\): the commutativity of transfer matrices is a consequence of the commutativity of the Grothendieck ring \( K_0(\mathcal{O}) \). The standard proof by using the Yang–Baxter equation \([2]\) would require braiding in category \( \mathcal{O} \), whose existence is not clear.

(iii) and the fact that \( t_X(z) \) only depends on the isomorphism class \([X] \) of \( X \) imply that \([X] \mapsto t_X(z)\) is a ring homomorphism \( t_r : K_0(\mathcal{O}) \longrightarrow M(I_0^I; \mathbb{D}_p) \). Applying
Let \( z \) be a zero of \( Q_r(z) \) and \( t_{\mathcal{W}_r, o}(z) \). We have

\[
\frac{t_{\mathcal{W}_r, o}(z)}{t_{\mathcal{W}_r, o}(z)} = \frac{Q_r(z + xh)}{Q_r(z)} \tag{6.35}
\]
as in \[21\] Theorem 5.6 (i). Now applying \( \text{tr}_p \) to Eq. (4.29), we obtain

**Corollary 6.4.** Let \( V \) be a finite-dimensional \( \mathcal{E} \)-module in category \( \mathcal{O} \). Then in Eq. (4.29) replacing \( V \), \( S(d_j) \) and the \( \frac{[\mathcal{W}_r, o]}{[\mathcal{W}_r]} \) with \( t_V(z) \), \( t_{S(d_j)}(z) \) and \( \frac{Q_r(z + xh)}{Q_r(z + bh)} \) respectively, we obtain an identity in \( M(U'_0 \otimes \mathbb{D}_p) \).

This forms the generalized Baxter relations for transfer matrices. If the fundamental modules \( L_{r,a} \) before Example 1.16 existed, then we would have defined alternatively the \( r \)-th Baxter operator \( Q^{\mathcal{R}}_p(z) = t_{L_{r,a}}(z) \) as a real transfer matrix \[23\] §5.5 and so \( \frac{Q_r(z + xh)}{Q_r(z + bh)} \) based on \( \frac{[\mathcal{W}_r, o]}{[\mathcal{W}_r]} = \frac{[L_{r,a}]}{[L_{r,a}]} \).

As an illustration of the corollary, let us be in the situation of Example 4.16: (iii) and take the inverse of \( t \) to Eq. (4.28) we obtain (iii). Replace \((x, z, \ell \in \mathbb{C}) \) to Eq. (5.31), divide both sides by the second term, and then perform a change of variable \(\frac{Q_r(z)}{Q_r(z + bh)} \); then as in \[24\] Eq. (5.16):

\[
X_k^{(r)}(w) \frac{Q_r(w)}{Q_r(w - h)} = 1 + \frac{Q_r(w + h)}{Q_r(w - h)} \prod_{s = r + 1}^\ell \frac{Q_s(w + h)}{Q_s(w)}.
\]

This forms three-term Baxter TQ relations for transfer matrices, where

\[
X_k^{(r)}(w) = t_{D_k^{(r,1)}}(w) \prod_{s = r + 1}^\ell t_{\mathcal{W}_s, o}(w - (k + \frac{1}{2})h).
\]

By Eq. (6.36), \( X_k^{(r)}(z) \in M(U'_0 \otimes \mathbb{D}_p) \) is independent of the choice of generic \( k \in \mathbb{C} \).

In the homogeneous case \( a_1 = a_2 = \cdots = a_\ell = a \), the entries of the matrix \( Q_r(z) \), as entire functions of \( z \), in general do not satisfy the uniform double periodicity of \[21\] Theorem 5.6 (ii)). By “uniform” we mean the multipliers with respect to \( z + 1 \) and \( z + \tau \) only depend on \((a, z, \ell \in \mathbb{C}) \). This is because the transfer matrix construction in \[21\] is based on a slightly different elliptic quantum group; see Footnote 9.

We follow \[23\] §5 to derive the Bethe Ansatz equations from Eq. (6.36). Let \( u \) be a zero of \( Q_r(z) \). Suppose \( X_k^{(r)}(z), Q_r(z - h), Q_s(z + \frac{1}{2}h) \) for \( s \neq r + 1 \) have no poles at \( z = u \). (This is a genericity condition.) Then as in \[24\] Eq. (5.16)):

\[
\frac{Q_r(u + h)}{Q_r(u - h)} \prod_{s = r + 1}^\ell \frac{Q_s(u - \frac{1}{2}h)}{Q_s(u + \frac{1}{2}h)} = -1.
\]

To compare with \[23\], we can assume furthermore that eigenvalues of \( Q_r(z) \) are of the form \( p = h^{-1} \sum_{i=1}^d \theta(z - u_{r,i}) \) based on \[21\] Remark 5.8. Then

\[
p^d \prod_{i=1}^d \frac{\theta(u_{r,i} + h - u_{s,i})}{\theta(u_{r,i} - h - u_{s,i})} \prod_{s = r + 1}^\ell \frac{\theta(u_{r,i} + \frac{1}{2}h - u_{s,i})}{\theta(u_{r,i} + \frac{1}{2}h - u_{s,i})} = -1
\]

for \( 1 \leq k \leq d_r \).

We remark that similar Bethe Ansatz equations for \( \mathcal{E} \) appeared in \[38\] Eq. (3.45).

For affine quantum groups and toroidal \( \mathfrak{g}_k \), the genericity condition of Bethe Ansatz equations has been dropped in \[15\] [16].

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