A NOTE ON THE VON WEIZSÄCKER THEOREM

STEFAN TAPPE

Abstract. The von Weizsäcker theorem states that every sequence of non-negative random variables has a subsequence which is Cesàro convergent to a nonnegative random variable which might be infinite. The goal of this note is to provide a description of the set where the limit is finite. For this purpose, we use a decomposition result due to Brannath and Schachermayer.

1. Introduction

Komlós’s theorem states that every $L^1$-bounded sequence has a subsequence which is Cesàro convergent to a finite limit; see [5], and also the refined results in [1] and [3, Thm. 5.2.1]. The paper [6] was dealing with the question whether the $L^1$-boundedness can be dropped. Its main result states that every nonnegative sequence in $L^0_+$ has a subsequence which is Cesàro convergent, but the limit can be infinite. More precisely, we have the following result.

1.1. Theorem (von Weizsäcker). Let $(\xi_n)_{n \in \mathbb{N}} \subset L^0_+$ be a sequence of nonnegative random variables. Then there exist a subsequence $(\xi_{n_k})_{k \in \mathbb{N}}$ and a nonnegative random variable $\xi : \Omega \rightarrow [0, \infty]$ such that the following statements are true:

(a) For every further subsequence $(n_k)_{k \in \mathbb{N}}$ and every permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$ the sequence $(\xi_{n_k \pi(l)})_{l \in \mathbb{N}}$ is almost surely Cesàro convergent to $\xi$.

(b) There exists an equivalent probability measure $Q \approx P$ such that the sequence $(\xi_{n_k} 1_{\{\xi < \infty\}})_{k \in \mathbb{N}}$ is $L^1(Q)$-bounded.

We refer to [6] and [3, Thm. 5.2.3]. Let $(\xi_{n_k})_{k \in \mathbb{N}}$ be a subsequence as in Theorem 1.1. Note that the limit $\xi$ can be infinite. In this note we will provide a description of the sets $\{\xi < \infty\}$ and $\{\xi = \infty\}$. In [6] it is already indicated that $\{\xi < \infty\}$ should be the largest subset on which $(\xi_{n_k})_{k \in \mathbb{N}}$ is $L^1(Q)$-bounded for some equivalent probability measure $Q \approx P$. However, à priori it is not clear whether such a set exists. We will approach this problem by looking at sets which are bounded in probability, without performing a measure change. For this purpose, we will use a decomposition result from [2] which states that for every convex subset $C \subset L^0_+$ there exists a partition $\{\Omega_0, \Omega_u\}$ such that $C|_{\Omega_0}$ is bounded in probability and $C$ is hereditarily unbounded in probability on $\Omega_u$. We refer to Section 2 for the precise definitions. The set $\Omega_b$ is characterized as the largest subset on which the convex set $C$ is bounded in probability.

Now, we define the convex hulls $C, \bar{C} \subset L^0_+$ as

\begin{equation}
C := \text{conv} \{\xi_{n_k} : k \in \mathbb{N}\} \quad \text{and} \quad \bar{C} := \text{conv} \{\xi_{n_k} : k \in \mathbb{N}\},
\end{equation}

where

$$\bar{\xi}_{n_k} := \frac{1}{k} \sum_{l=1}^{k} \xi_{n_l} \quad \text{for each } k \in \mathbb{N},$$

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and denote by \( \{ \Omega_b, \Omega_n \} \) and \( \{ \bar{\Omega}_b, \bar{\Omega}_n \} \) the corresponding partitions according to the decomposition result from [2]. Note that \( \bar{C} \subset C \), because

\[ \{ \xi_n : k \in \mathbb{N} \} \subset C. \]

Our result reads as follows.

1.2. **Proposition.** We have \( \{ \xi < \infty \} = \Omega_b = \bar{\Omega}_b \) and \( \{ \xi = \infty \} = \Omega_n = \bar{\Omega}_n \) up to \( \mathbb{P} \)-null sets.

1.3. **Remark.** Let us mention some further consequences:

\begin{enumerate}
  \item The set \( \{ \xi < \infty \} \) is the largest subset on which the convex hull of \( (\xi_n)_{k \in \mathbb{N}} \) or \( (\bar{\xi}_n)_{k \in \mathbb{N}} \) is bounded in probability.
  \item The set \( \{ \xi < \infty \} \) is also the largest subset on which the sequence \( (\xi_n)_{k \in \mathbb{N}} \) or \( (\bar{\xi}_n)_{k \in \mathbb{N}} \) is \( L^1(\mathbb{Q}) \)-bounded for some equivalent probability measure \( \mathbb{Q} \approx \mathbb{P} \). This is in accordance with the findings in [3].
  \item The set \( \{ \xi < \infty \} \) is also the largest subset on which every sequence of convex combinations of \( (\xi_n)_{k \in \mathbb{N}} \) or \( (\bar{\xi}_n)_{k \in \mathbb{N}} \) has a convergent subsequence in the sense of weak convergence of their distributions.
\end{enumerate}

For more details, we refer to Section 2 and in particular Corollary [2.12], where we investigate when the limit \( \xi \) is almost surely finite.

2. **Proof of the result**

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. We denote by \( L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P}) \) the space of all equivalence classes of random variables, where two random variables \( X \) and \( Y \) are identified if \( \mathbb{P}(X = Y) = 1 \). We denote by \( L^0_+ = L^0_+(\Omega, \mathcal{F}, \mathbb{P}) \) the convex cone of all nonnegative random variables; that is, \( \mathbb{P}(X \geq 0) = 1 \). It is well-known that \( L^0 \) equipped with the translation invariant metric

\[ d(X, Y) = \mathbb{E}[|X - Y| \wedge 1], \quad X, Y \in L^0 \]

is a complete topological vector space. The induced convergence is just convergence in probability; that is, for a sequence \( (X_n)_{n \in \mathbb{N}} \subset L^0 \) and a random variable \( X \in L^0 \), we have \( d(X_n, X) \to 0 \) if and only if \( X_n \overset{P}{\to} X \). Furthermore, for every equivalent probability measure \( \mathbb{Q} \approx \mathbb{P} \) the translation invariant metric

\[ d\mathbb{Q}(X, Y) = \mathbb{E}_{\mathbb{Q}}[|X - Y| \wedge 1], \quad X, Y \in L^0 \]

induces the same topology.

2.1. **Definition.** A subset \( C \subset L^0 \) is called bounded in probability (or \( \mathbb{P} \)-bounded) if for every \( \varepsilon > 0 \) there exists \( M > 0 \) such that

\[ \sup_{X \in C} \mathbb{P}(|X| > M) < \varepsilon. \]

2.2. **Remark.** It is well-known that a subset \( C \subset L^0 \) is topologically bounded if and only if it is bounded in probability.

2.3. **Remark.** Note that a subset \( C \subset L^0 \) is bounded in probability if and only if the family of distributions \( \{ \mathbb{P} \circ X : X \in C \} \) is tight.

For a subset \( C \subset L^0 \) and an event \( B \in \mathcal{F} \) we agree on the notation

\[ C|_B := \{ X 1_B : X \in C \} \]

2.4. **Definition.** A subset \( C \subset L^0 \) is called hereditarily unbounded in probability (or hereditarily \( \mathbb{P} \)-unbounded) on a set \( A \in \mathcal{F} \) if for every \( B \in \mathcal{F} \) with \( B \subset A \) and \( \mathbb{P}(B) > 0 \) the set \( C|_B \) is not bounded in probability.
2.5. Definition. A subset $C \subset L^0$ is called $L^1(\mathbb{P})$-bounded if
\[ \sup_{X \in C} \mathbb{E}[|X|] < \infty. \]

The following two results which be useful for our analysis.

2.6. Lemma. [2] Lemma 2.3] Let $C \subset L^0_+\bar{\mathbb{P}}$ be a convex subset of $L^0_+\bar{\mathbb{P}}$. Then there exists a partition $\{\Omega_u, \Omega_b\}$ of $\Omega$ into disjoint sets $\Omega_u, \Omega_b \in \mathcal{F}$, unique up to $\mathbb{P}$-null sets, such that:
\begin{enumerate}
  \item $C|_{\Omega_u}$ is bounded in probability.
  \item $C$ is hereditarily unbounded in probability on $\Omega_u$.
\end{enumerate}

In particular, for every event $B \in \mathcal{F}$ such that $C|_B$ is bounded in probability, we have $B \subset \Omega_b$ up to $\mathbb{P}$-null sets.

2.7. Lemma. [1] Prop. 1.16] Let $K \subset L^0_+\bar{\mathbb{P}}$ be a subset. Then the following statements are equivalent:
\begin{enumerate}
  \item $(i)$ $K$ is bounded in probability.
  \item $(ii)$ There exists an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ such that $K$ is $L^1(\mathbb{Q})$-bounded.
\end{enumerate}

2.8. Remark. The implication $(i) \Rightarrow (ii)$ also follows from [2] Lemma 2.3.3. There it is even shown that one can find such an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ with bounded Radon-Nikodym density.

Now, let $(\xi_{nk})_{k \in \mathbb{N}}$ be a subsequence as in Theorem 1.1. Furthermore, let $C, \bar{C} \subset L^0_+\bar{\mathbb{P}}$ be the convex sets given by (1.1). We denote by $\{\Omega_u, \Omega_b\}$ and $\{\Omega_u, \Omega_b\}$ the corresponding partitions according to Lemma 2.6.

2.9. Lemma. We have $\Omega_b \subset \Omega_b$ up to $\mathbb{P}$-null sets.

Proof. Since $\bar{C} \subset C$ and $\bar{C}|_{\Omega_b}$ is bounded in probability, the set $\bar{C}|_{\Omega_b}$ is also bounded in probability, and hence we have $\Omega_b \subset \Omega_b$. \qed

2.10. Lemma. We have $\Omega_b \subset \{\xi < \infty\}$ up to $\mathbb{P}$-null sets.

Proof. We define the sequence $(\bar{\mu}_k)_{k \in \mathbb{N}}$ of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $\bar{\mu}_k := \mathbb{P} \circ (\xi_{nk} \mathbb{1}_{\Omega_b})$ for each $k \in \mathbb{N}$. Since $\bar{C}|_{\Omega_b}$ is $\mathbb{P}$-bounded, by Remark 2.8, the sequence $(\bar{\mu}_k)_{k \in \mathbb{N}}$ is tight. By Prohorov’s theorem there is a subsequence $(\bar{\mu}_{n_k})_{k \in \mathbb{N}}$ such that $\bar{\mu}_{n_k} \stackrel{w}{\rightarrow} \mu$ for some probability measure $\mu$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. On the other hand, since $\xi_{nk} \mathbb{1}_{\Omega_b} \stackrel{a.s.}{\rightarrow} \xi \mathbb{1}_{\Omega_b}$, we have $\mu = \mathbb{P} \circ (\xi \mathbb{1}_{\Omega_b})$. Therefore, it follows that $\mathbb{P}$-almost surely $\xi < \infty$ on $\Omega_b$. \qed

2.11. Lemma. We have $\{\xi < \infty\} \subset \Omega_b$ up to $\mathbb{P}$-null sets.

Proof. By part (b) of Theorem 1.1 there exists an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ such that $(\xi_{nk} \mathbb{1}_{\{\xi < \infty\}})_{k \in \mathbb{N}}$ is $L^1(\mathbb{Q})$-bounded. Therefore, by Lemma 2.7 the set $C|_{\{\xi < \infty\}}$ is $\mathbb{P}$-bounded, completing the proof. \qed

Now, the proof of Proposition 1.2 is a consequence of Lemmas 2.9, 2.11 and Remark 1.3 follows by additionally taking into account Lemma 2.7 and Prohorov’s theorem.

Using Proposition 1.2 we can characterize when the limit $\xi$ is almost surely finite, and when it is almost surely infinite. As a consequence of the next result, the limit is almost surely finite if and only if every sequence of convex combinations has a convergent subsequence in the sense of weak convergence of their distributions.

2.12. Corollary. The following statements are equivalent:
\begin{enumerate}
  \item $(i)$ We have $\xi < \infty$ almost surely.
(ii) The set $C$ is bounded in probability.
(iii) The set $\bar{C}$ is bounded in probability.
(iv) There exists an equivalent probability measure $Q \approx P$ such that the sequence $(\xi_n)_{n \in \mathbb{N}}$ is $L^1(Q)$-bounded.
(v) There exists an equivalent probability measure $Q \approx P$ such that the sequence $(\xi_n)_{n \in \mathbb{N}}$ is $L^1(Q)$-bounded.
(vi) For every sequence $(\eta_n)_{n \in \mathbb{N}} \subseteq C$ there is a subsequence $(\eta_{n_k})_{k \in \mathbb{N}}$ such that $(P \circ \eta_{n_k})_{k \in \mathbb{N}}$ converges weakly.
(vii) For every sequence $(\bar{\eta}_n)_{n \in \mathbb{N}} \subseteq \bar{C}$ there is a subsequence $(\bar{\eta}_{n_k})_{k \in \mathbb{N}}$ such that $(P \circ \bar{\eta}_{n_k})_{k \in \mathbb{N}}$ converges weakly.

Proof. This is an immediate consequence of Proposition 1.2, Lemma 2.7 and Prohorov’s theorem. □

2.13. Corollary. The following statements are equivalent:

(i) We have $\xi = \infty$ almost surely.
(ii) The set $C$ is hereditarily unbounded in probability.
(iii) The set $\bar{C}$ is hereditarily unbounded in probability.

Proof. This is an immediate consequence of Proposition 1.2. □

As an example, let us consider a sequence $(\xi_n)_{n \in \mathbb{N}} \subseteq L^0_1$ of independent, identically distributed random variables. By the strong law of large numbers for every subsequence $(n_k)_{k \in \mathbb{N}}$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$ the sequence $(\xi_{\pi(n_k)})_{k \in \mathbb{N}}$ is almost surely Cesàro convergent to the constant $\xi = E[\xi_1] \in [0, \infty]$. Furthermore, the sequence $(\xi_n, 1_{\xi < \infty})_{n \in \mathbb{N}}$ is $L^1(P)$-bounded. Consequently, we see that in the von Weizsäcker theorem (Theorem 1.1) we can take the original sequence $(\xi_n)_{n \in \mathbb{N}}$ – that is we do not have to pass to a subsequence $(\xi_{n_k})_{k \in \mathbb{N}}$ – and the limit $\xi$ is a constant. Furthermore, we have either $\xi < \infty = \Omega$ or $\xi < \infty = \emptyset$ up to $P$-null sets. More precisely, the following statements are true:

1. We have $\xi < \infty$ almost surely if and only if $\xi_1 \in L^1$.
2. We have $\xi = \infty$ almost surely if and only if $\xi_1 \notin L^1$.

Defining the convex hulls $C, \bar{C} \subseteq L^0_1$ as

$$C := \text{conv} \{\xi_n : n \in \mathbb{N}\} \quad \text{and} \quad \bar{C} := \text{conv} \{\bar{\xi}_n : n \in \mathbb{N}\},$$

by Corollary 2.12 the following statements are equivalent:

(i) We have $\xi_1 \in L^1$.
(ii) The set $C$ is bounded in probability.
(iii) The set $\bar{C}$ is bounded in probability.

Moreover, by Corollary 2.13 the following statements are equivalent:

(i) We have $\xi_1 \notin L^1$.
(ii) The set $C$ is hereditarily unbounded in probability.
(iii) The set $\bar{C}$ is hereditarily unbounded in probability.

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Ludwig Maximilian University of Munich, Department of Mathematics, Theresienstr. 39, 80333 Munich, Germany

E-mail address: tappe@math.lmu.de