MIRROR SYMMETRY ASPECTS FOR COMPACT G_2 MANIFOLDS

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Abstract. The present paper deals with mirror symmetry aspects of compact “barely” G_2 manifolds, that is, G_2 manifolds of the form (CY \times S^1)/\mathbb{Z}_2. We propose that the mirror of any barely G_2 manifold is another barely manifold constructed as a fibration of the mirror of the CY base. Also, we describe the Joyce manifolds of the first kind as “barely” with an underlying CY which is self-mirror with h^{1,1} = h^{2,1} = 19. We propose that the mirror of a Joyce space of the first kind is another Joyce space of the first kind. We also suggest that this self-mirror CY family is dual to K3 \times S^1 in the heterotic/M-theory sense. The Borcea-Voisin construction plays a significant role for showing this. As a spin-off we conclude that no 5-brane instantons are present in compactifications of eleven dimensional supergravity over Joyce manifolds of the first kind.

1. Introduction

A Calabi-Yau manifold is a Kähler 2n-manifold X with vanishing first Chern class and admits a Ricci-flat metric. Such manifolds come equipped with a nowhere vanishing holomorphic (n,0)-form \Omega their holonomy group is SU(n) or a subgroup. Harvey and Lawson [12] showed that Re(\Omega) is a calibration on X. The corresponding calibrated submanifolds in X are called special Lagrangian n-folds. The moduli space of special Lagrangian submanifolds is expected to play a role in explaining the mirror symmetry for Calabi-Yau (CY) manifolds.

CY manifolds are target spaces of (2,2) supersymmetric sigma models, and for this reason they are physically relevant. The infinitesimal symmetries of the models include, at classical level, two copies of N = 2 supersymmetry algebras without central charge. The quantization procedure becomes very difficult for them, but is expected to give an (2,2) super-conformal quantum field theory with central extension (or non vanishing central charge). These theories are representations of the (2,2) superconformal algebra.

The mirror symmetry problem has many aspects to be covered completely. But we can emphasize some aspects. The (2,2) superconformal algebra contains the usual Virasoro generators (L_n, \overline{L}_n) and the (2,2) super-symmetry generators (G^+_n, G^-_n). It also contains two U(1) currents (J_n, \overline{J}_n) which are introduced to make the algebra to close. These currents play a significant role in the mirror symmetry problem. The transformation
\begin{align}
L_n &\to L_n, \quad J_n \to -J_n, \quad G^+_n \to G^+_n,
\end{align}
is an automorphism of the (2,2) algebra and is called left mirror automorphism. The analogous definition hold for right mirror automorphism, and when it affects to both sectors it will be a target space automorphism. The transformation (1.1) just reverse the sign of the U(1) charge of a given state corresponding to J.

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The mirror automorphism has several important consequences. An important set of operators of $(2, 2)$ superconformal theories are are marginal operators, which are the ones for which the conformal weight sums $h + \bar{h} = 2$. These operators can be used to deform a given conformal theory to a nearly conformal field theory without changing the central charge. But in order to obtain a continuous family of conformal field theories one should consider truly marginal operators, that is, operators which are still marginal after deformation. It follows that the effect of these operators is to induce deformations of a CY preserving the CY property. These deformations are known to come in two types: deformations of the Kähler structure and of the complex structure, and are captured by cohomology groups $h^{1,1}$ and $h^{2,1}$ for 3-folds. This is a beautiful link between abstract conformal field theory aspects and the geometry of the target manifold [30].

In fact, the two different types of marginal operators differ only in the sign of the $U(1)$ charge. This means that the automorphism generate a transformation

$$h^{1,1} \leftrightarrow h^{2,1}$$

between the Hodge numbers of the CY. In other words, it corresponds to a change in the Euler signature $\chi = 2(h^{1,1} - h^{2,1})$ and therefore it result in a topologically different CY manifold. This is uncomfortable asymmetry which implies that is not possible to reconstruct the target space geometry from sigma model data. In order to avoid it it was postulated that CY manifolds come in pairs with the Hodge numbers related by (1.2) such that the resulting superconformal theories are isomorphic. Such manifolds will be called mirror to each other.

After this hypothesis was introduced several examples satisfying (1.2) were constructed in [32]. Nevertheless (1.2) is a necessary condition for two manifolds to form mirror pairs, but is not sufficient. Two manifolds are truly mirror pairs if they correspond to the *isomorphic* conformal field theories and these result in a relation involving 3-point correlation functions from both sides. Apparently the first explicit realization of mirror pairs were found in [31]. The idea was to find a group of automorphism of the conformal theory which is not an automorphism of the underlying CY space. This automorphism will automatically generate a mirror pair.

An alternative definition of mirror symmetry comes from Kaluza-Klein reductions. If one consider, for instance, compactification of IIA superstrings on a background of the form $M_4 \times K_6$ to 4-dimensions, then the condition of $N = 1$ supersymmetry for the low dimensional theory implies that $K_6$ is Calabi-Yau. The change of sign of the $U(1)$ charge reverse the GSO projection. This situation is analogous of a T-duality transformation on a two torus in which IIA superstring is mapped to IIB and viceversa. Therefore if IIA and IIB theories are compactified to $D = 4$ over two mirror Calabi-Yau three-folds then the resulting 4-dimensional theories are expected to be isomorphic.

An interesting question is if any CY manifold has a mirror and if the mirror transformation is somehow related to T-duality. The analysis of II compactifications including instanton corrections lead to the SYZ conjecture which states that if a CY three fold has a mirror, then both manifolds are locally a special Lagrangian $T^3$ fibration and are related to each other by a T-duality acting on each coordinate of the $T^3$ [25]. But a complete proof of this conjecture is still lacking.

The mirror symmetry conjecture was subsequently generalized to other type of manifolds. The “generalized” mirror conjecture [24] states that if there is an ambiguity in determining the topological properties of the target manifold in a sigma model, then there exists a dual manifold resolving the ambiguity. Another possible definition is that a pair of manifolds $(X, \overline{X})$ is a mirror pair if
compactifying IIA and IIB supergravities over them gives isomorphic low dimensional theories \[3\]. The natural problem is to understand how the topological invariants of mirror manifolds are related, i.e., to find an analog of the relation \[1^{2}\] for any kind of mirror set.

The present paper deals with mirror symmetry aspects of \(G_2\) holonomy manifolds \[1\]. These are 7-dimensional and are characterized by two \(G_2\) equivariant 3-form \(\varphi\) and 4-form \(*\varphi\) which are both closed. As CY manifolds, these are Ricci flat and Harvey and Lawson, \[12\] showed that \(\varphi\) and \(*\varphi\) are calibrations on \(M\). The corresponding calibrated sub-manifolds in \(M\) are called associative 3-folds and co-associative 4-folds, respectively. We will focus on compact \(G_2\) holonomy manifolds. Compactness is required for Kaluza-Klein matters, in order to obtain a discrete Kaluza-Klein spectrum.

The sigma model analysis of mirror \(G_2\) manifolds was performed in \[24\]-\[21\]. There was shown that if one consider a sigma model with \(N = 1\) supersymmetry over a \(G_2\) manifold the effect of the calibrations \(\varphi\) and \(*\varphi\) is to add two new operators \(\Phi\) and \(X\) with spin 3/2 and 2 to the \(N=1\) generators \(T\) and \(G\), and also two more operators \(K\) and \(M\) of spin 2 and 5/2 in order the algebra close. This is called extended supersymmetry algebra. But the important thing is that \(\Phi\) and \(X\) generate themselves a new \(N=1\) superconformal sub-algebra with central charge 7/10, which corresponds to the tri-critical Ising minimal model. By use of this, the authors of \[24\] were able to classify the highest weight states of the algebra in terms of the tri-critical Ising highest weight and the eigenvalue of the remaining stress energy tensor. Also, they identified the marginal deformations of the theory and showed that the physical moduli space has dimension \(b_3 + b_2\). This is different than dimension geometrical moduli space dimension for \(G_2\) manifolds, which is \(b_3\). This discrepancy is due to the physical freedom to add a closed two form to add a phase to the action, which has no geometric analog.

In fact, the physical moduli space dimension \(b_3 + b_2\) is in agreement with the results of \[28\], where it was shown that IIA and IIB compactifications over the same compact \(G_2\) holonomy space give ”apparently” the same field theory content, namely \(b_2 + 1\) scalar multiplets and \(b_3\) vector multiplets. Guided by this fact, the authors of \[28\] raised the possibility that both compactifications are equivalent, in other words, that \(G_2\) manifolds are mirror to themselves. Some examples realizing these were suggested in \[2\], but also a counterexample. But there is a subtlety in three dimensions, which is a duality transformation taking scalar into vectors and vice versa. Therefore the analysis of \[28\] does not collect manifolds with the same \(b_2\) and \(b_3\), but instead those with the same value of \(b_2 + b_3\), as predicted by \[24\].

Besides these developments, the problem of mirror symmetry for \(G_2\) manifolds is less understood than for the CY. Two manifolds with the same \(b_2 + b_3\) value are not necessarily mirror, the condition to give isomorphic physics should stronger than that, just as in the CY case. An interesting analysis of mirror \(G_2\) manifolds was made in \[3\], where it was suggested that \(G_2\) manifolds admitting a mirror should possess four cycles \(C_4\) which satisfy the condition \(b_2^2(C_4) + b_3(C_4) = 7\). The only known example is the four torus \(T^4\) and therefore this suggest that mirror pairs are locally \(T^4\) fibrations. This statement will be the analogous of the SYZ conjecture for \(G_2\) manifolds. But its validity relies in our ability to prove that there are no other solutions \(C_4\) of the equation \(b_2^2(C_4) + b_3(C_4) = 7\), and this is still an open question.

\[1\]A more mathematical discussion about this subject can be found in the recent paper \[5\].
An important discovery is also the topological side for both $(2, 2)$ superconformal models over CY and $N=1$ sigma models over $G_2$ manifolds. In [27] the authors considered conformal field theories for which the stress tensor $T_{\mu\nu}$ is BRST-trivial, i.e., is of the form $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$, being $Q$ a nilpotent operator $Q^2 = 0$. This theories will be automatically topological and the topological conformal algebra between the operators $T, G$ and $Q$ was worked out explicitly [27]. The result is an conformal algebra without central charge which is related to the $N=2$ superconformal algebra by a redefinition of the stress energy tensor as $T \rightarrow T + \frac{1}{2}\partial J$. This redefinition is called a twist. In similar lines, the authors of [24] found a redefinition of the operators $X$ which has the effect of switch the central charge to zero. This remarkable result is an strong hint for the existence of a topolgical field theory associated to $G_2$ manifolds, which was worked out in [22]-[23] in more detail.

Another relevant branch is the presence of associative and coassociative submanifolds, which are the calibrated submanifolds of $G_2$ holonomy manifolds. As is well known, eleven dimensional supergravity compactified in a $G_2$ manifold gives $N = 1$ supersymmetry in the low effective theory. But supergravity also contains solitons breaking the supersymmetries of the theory, unless certain restrictions are satisfied [33]. These restrictions have been found to be equivalent to the presence of the calibrated submanifolds inside the compactification $G_2$ space. In other words, calibrated submanifolds of $G_2$ manifolds are supersymmetric cycles [33].

The present paper is organized as follows. In §2 we describe the K3 surfaces together with equivalent heterotic/M-theory compactifications over $K3 \times S^1$ CY spaces and identify the CY manifold involved in this duality. This CY turns to be important in our further discussions. In §3 we discuss general aspects of compact $G_2$ manifolds, in particular calibrated submanifolds and almost Calabi-Yau structures inside them. In §4 we review the construction of Joyce manifolds of the first kind and we present some aspects of M-theory/heterotic string dualities over $G_2$/CY manifolds and also of mirror symmetry for $G_2$ manifolds. We propose that the evidence found in [28] that the mirror map leave inert the betti numbers of the Joyce manifold is because they are fibrations over CY with zero Euler number, which are in some sense "protected" from topology change under the complex/sympletic map. We generalize the discussion to general "barely" $G_2$ manifolds. Also, we show that for compactification of Joyce spaces of the first kind no 5-brane instanton appears.

2. COMPACT SELF-DUAL METRICS ON K3

K3 surfaces play an important role in several string dualities and in the Joyce construction of compact $G_2$ holonomy manifolds. For these reasons, we discuss them first, together with the mentioned dualities.

2.1. Hyperkähler metrics over K3. As we stated in the introduction, Calabi-Yau spaces are 2m-dimensional with holonomy in $SU(m)$. By another side, hyperkähler spaces are by definition 4n-dimensional with holonomy in $Sp(n)$. Both spaces admit a Ricci-flat Kähler metric. The dimension $D = 4$ is special because it corresponds to $n = 1$ and $m = 2$ and by the isomorphism $Sp(1) \simeq SU(2)$, it provides a link between both cases. If the manifold is compact and $c_1 = 0$, then the Yau proof of the Calabi conjecture imply that it admits a unique Ricci flat Kähler metric [9]. In $D = 4$ this metric will be simultaneously Calabi-Yau and hyperkähler.

The curvature of the hyperkähler 4-metrics is always self-dual

$$R_{abcd} = *R_{abcd} = \frac{1}{2} \epsilon_{abcdef} R^{ef}_{cd},$$
and this implies Ricci-flatness, i.e., $R_{ij} = 0$ where $R_{ij}$ is the Ricci tensor of the metric. Besides, there exists a vielbein basis $e^i$ in which they are expressed as $g_{ij} = \delta_{ij} e^i \otimes e^j$ and for which the hyperkähler triplet
\[
(2.1) \quad \mathcal{J}_1 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad \mathcal{J}_2 = e^1 \wedge e^3 + e^4 \wedge e^2, \quad \mathcal{J}_3 = e^1 \wedge e^4 + e^2 \wedge e^3
\]
is closed.

In dimension four, several explicit hyperkähler metrics are known, but they are all defined over non-compact manifolds. One of the best known examples is the Eguchi-Hanson gravitational instanton \cite{11}, which possess a complete hyperkähler metric given by
\[
(2.2) \quad g = \frac{r^2}{4} \left( 1 - (a/r)^4 \right) (d\theta + \cos \varphi dr)^2 + \left( 1 - (a/r)^4 \right)^{-1} dr^2 + \frac{r^2}{4} (d\varphi^2 + \sin^2 \varphi dr^2).
\]
This metric contains an $S^2$ sphere of radius $a$ at its tip. If the parameter $a$ tends to zero the result will be the hyperkähler flat metric on $\mathbb{C}^2/\mathbb{Z}_2$. In fact, the Eguchi-Hanson space is an ALE (asymptotically Euclidean space), which means that it approaches asymptotically to the Euclidean metric. The boundary at infinity is locally $S^3$. However, the situation is rather different in what regards its global properties. This can be seen by defining the new coordinate
\[
u^2 = r^2 \left( 1 - (a/r)^4 \right)
\]
for which the metric can be rewritten as
\[
(2.3) \quad g = \frac{\nu^2}{4} \left( d\theta + \cos \varphi dr \right)^2 + \left( 1 + (a/r)^4 \right)^{-2} du^2 + \frac{r^2}{4} (d\varphi^2 + \sin^2 \varphi dr^2).
\]
The apparent singularity at $r = a$ has been moved now to $u = 0$. Near the singularity, the metric looks like
\[
g \simeq \frac{\nu^2}{4} \left( d\theta + \cos \varphi dr \right)^2 + \frac{1}{4} du^2 + \frac{a^2}{4} (d\varphi^2 + \sin^2 \varphi dr^2),
\]
and, at fixed $\tau$ and $\varphi$, it becomes
\[
g \simeq \frac{\nu^2}{4} d\theta^2 + \frac{1}{4} du^2.
\]
This expression “locally” looks like the removable singularity of $\mathbb{R}^2$ that appears in polar coordinates. However, for actual polar coordinates, the range of $\theta$ covers from 0 to $2\pi$, while in spherical coordinates in $\mathbb{R}^4$, $0 \leq \theta < 4\pi$. This means that the opposite points on the geometry turn out to be identified and thus the boundary at infinity is the lens space $S^3/\mathbb{Z}_2$, which is the same boundary as for $B_4/\mathbb{Z}_2$.

To construct compact hyperkahler manifolds is more complicated, but there is an ingenious way to prove their existence \cite{20, 17}. The Eguchi-Hanson instanton plays a significant role in this proof. Consider a 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}_4$, where $\mathbb{Z}_4$ is generated by the canonical four lattice. Choose its flat metric and the hyperkähler triplet
\[
(2.4) \quad \mathcal{J}_1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \quad \mathcal{J}_2 = dx^1 \wedge dx^3 + dx^4 \wedge dx^2, \quad \mathcal{J}_3 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3,
\]
and identify the points of this torus by a $\mathbb{Z}_2$ action which reflects through the origin. This action preserves the triplet \cite{20} and has the fixed points $(r_1, r_2, r_3, r_4)$ with $r_i$ taking values 0 or 1/2. This gives a total of $4 \times 4 = 16$ fixed points. The corresponding singularities are of type $A_1$ and the holonomy of the resulting space is $\mathbb{Z}_2$. The Kummer construction of a compact hyperkähler metric
consist in excising a region of radius $R$ around all the $A_1$ singularities, which gives the topology of a ball $B_4/\mathbb{Z}_2$, and replacing it by a copy of an Eguchi-Hanson space. The modified metric around a singularity will be given by the expression (2.2) with the modification $a \rightarrow a \tau(|\vec{r} - \vec{r}_i|)$ where $\tau(x)$ is a smooth function which takes value 1 in the region $x \leq R_1$ and zero for $x > R_2$. Here $\vec{r}$ is a point in the manifold and $\vec{r}_i$ is the position of an $i$-th singularity. As a result it will be a smooth manifold $M$ and we have a map $\pi : M \rightarrow \mathbb{T}^4/\mathbb{Z}_2$ which is called the resolving map.

The reason for choosing the Eguchi-Hanson space is justified by its topological properties: it is an asymptotically flat self-dual metric with a natural $\mathbb{Z}_2$ action which matches the $\mathbb{Z}_2$ action on $\mathbb{T}^4/\mathbb{Z}_2$. Thus, although the imperfect matching in the boundary of $B_4/\mathbb{Z}_2$, the resulting metric is an approximation for a compact hyperkähler metric. It can be shown that the failure of the triplet (2.4) to be closed is of the order of $O(a^4/R^4)$. Clearly, we can be as close as we want to a hyperkähler metric by taking a sufficiently small value of $a$, but the limit $a \rightarrow 0$ will give our initial orbifold. However for small enough values of $a$ it is possible to deform it to a smooth structure $(M, g, J_i)$ with holonomy exactly $SU(2)$, [17]. This procedure is called a blowup, and it relies on deformation theory of singular complex manifolds. Thus compact hyperkähler metrics do exist, although nobody has found their explicit form.

It can be proved that for the compact metrics described above $b_1 = c_1 = 0$. As they are obviously Kähler, they are complex. Regular complex and compact surfaces are called K3 surfaces. It was proved by Kodaira that every K3 surface is a deformation of a non-singular quartic Kummer surface (2.5)

$$f = x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

in $\mathbb{CP}^3$ with homogeneous coordinates $(x_1, x_2, x_3, x_4)$, [15]. Hence K3 surfaces are Kähler and all diffeomorphic and the Yau proof implies that they admit an unique Ricci flat Kähler metric. This can be only the metric we described in this section.

2.2. Equivalent compactifications related to K3 surfaces. It is well known that eleven dimensional supergravity compactified on a circle $S^1$ gives the strong coupling limit of IIA theory. But if this theory is compactified on $S^1/\mathbb{Z}_2$ then the $\mathbb{Z}_2$ action takes out one of the two supersymmetries of the 10-dimensional theory. The two fixed points are two 10-dimensional hyperplanes. The anomaly cancellation on them requires the gauge group to be $E_8 \times E_8$, which arises from twisted sectors of the orbifold. It has been suggested that the resulting theory is heterotic string theory on $E_8 \times E_8$ [34].

The low energy limit of M-theory is eleven dimensional supergravity. If we go to seven dimensions by Kaluza-Klein reduction over $T^3$ the supersymmetries will be maximal. Compactifications over the orbifold $T^4/\mathbb{Z}_2$ break half of them and the same will occur in the K3 limit described in the previous section. It has been shown that the result of the compactification is an Einstein-Maxwell supergravity theory $D = 7$ with a three form coupled to the $D = 7$ supermembrane. There is evidence showing that this theory is the effective action of heterotic string on $T^3$ at strong coupling. Thus we have heterotic/M-theory equivalence over $T^3/K3$. Also IIA compactifications over $K_3 \times T^2$ have been conjectured to be equivalent to heterotic string theory over $T^0$ [35].

In view of these equivalences, it is natural to consider $K3 \times S^1$ compactifications of heterotic string and to investigate if there is a compactification from eleven dimensions to five giving the same theory [28]. The compactification manifolds should be six dimensional and the task is to identify
it. From the heterotic side it is obtained $N = 2$ supergravity in five dimensions coupled to 18 vector multiplets and 20 hypermultiplets. In order to obtain $N = 2$ supersymmetry from eleven dimensions, the internal space should be CY. After compactification over the CY it is obtained $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets, where $h^{1,1}$ and $h^{2,1}$ are the Hodge numbers of the unknown Calabi-Yau. Therefore the number of multiplets matches if and only if $h^{1,1} - 1 = 18$ and $h^{2,1} + 1 = 20$, in other words, if

$$h^{1,1} = h^{2,1} = 19.$$  

(2.6)

The constraint (2.6) implies that the Hodge numbers of the internal manifold are invariant under the mirror transformation (1.2), and that its Euler number $\chi = 2(h^{1,1} - h^{2,1})$ vanishes.

In fact, there exists a large family of CY constructed as K3 fibrations [7, 29] which include examples satisfying (2.6). To construct them one, consider a K3 surface $X$ with an holomorphic 2-form $J$ together with an action $\sigma$ which acts by $\sigma^*(J) = -J$. The action $\sigma$ has a set of fixed points $\Sigma$ which has been classified by Nikulin [20]. This classification states that $\Sigma$ is a disjoint union of smooth curves in $X$ and there are three possibilities: either $\Sigma$ is empty, or $\Sigma = C_1 \cup C'_1$ where $C_1$ and $C'_1$ are both elliptic curves or $\Sigma = C_g + E_1 + \cdots + E_k$ where $C_g$ is a curve of genus $g$ and $E_i$ are rational curves.

Now consider an elliptic curve $E$ and an involution $(-1)$ which changes the sign of its coordinates, then the 6 dimensional quotient

$$X_6 = \frac{K3 \times E}{(\sigma, -1)}$$

possess a fixed point set $S_0$ consisting in four copies of $\Sigma$ and the quotient $(X \times E)/(\sigma, -1)$ has $A_1$-singularities along $S_0$ [7, 29]. These singularities can be blown up in order to give a smooth CY manifold, as for K3 manifolds. The resulting CYs are called Borcea-Voisin 3-folds and their Hodge numbers are given by

$$h^{1,1}(Y) = 11 + 5n - n', \quad h^{2,1}(Y) = 11 + 5n' - n,$$

(2.8)

where $n$ is the number of components of the singular set $\Sigma$ of a the K3 surface and $n'$ is the sum of the genus of all these components.

In principle the Borcea-Voisin family (2.7) is very large, but if one is looking for an specific CY with the Hodge numbers given in (2.6) then $n = n' = 2$. This means that $\Sigma$ should composed by two components of genus 1. Such components should be two 2-torus. Although this is not enough to determine the elliptic curve $E$, this deduction shows that CY manifolds satisfying (2.6) exist and is reasonable to suppose that they are included in this subfamily. In fact $K3 \times S^1$ arise by smoothing the orbifold $T^4/Z_2 \times S^1$. Compactification of heterotic string over such orbifold will give the same number of supersymmetries than compactifications over the orbifold $(T^6/Z_2 \otimes Z_2)$ if the two $Z_2$ acts over the whole $T^6$. All these facts suggest the following:

**Conjecture** The CY dual to $K3$ in the heterotic/M-theory sense is obtained as a quotient of the form (2.7) in which the elliptic curve $E$ is a 2-torus and the action $\sigma$ over $K3$ has a singular set which is the union of two 2-tori.
If the statement above is correct, then we are in the second case in the Nikulin classification. In fact, the dual of this compactification has also been considered in \([1]\) and, as far as we understand, our conjecture is in agreement with that reference.

### 3. \(G_2\) Manifolds

#### 3.1. Generalities

We now turn our attention to \(G_2\) holonomy manifolds. We review their basic properties, the reader can find more information in Harvey and Lawson, \([12]\).

The group \(G_2\) can be considered as the group of automorphisms of the imaginary octonions. The octonions \(O = \mathbb{H} \oplus i\mathbb{H} = \mathbb{R}^8\) constitute an 8-dimensional division algebra and are obtained from the quaternions \(\mathbb{H}\) using the Cayley-Dickson process. This algebra is generated by \(<1, i, j, k, l, li, lj, lk>\), where \(i, j, k\) are the pure quaternion units. The imaginary octonions \(\text{Im}O = \mathbb{R}^7\) are naturally equipped with the cross product operation \(\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7\) defined by \(u \times v = \text{im}(u \bar{v})\). Then one can define the exceptional Lie group \(G_2\) as the linear automorphisms of \(\text{Im}O\) preserving this cross product operation, i.e. \(u \times v = gu \times gv\) if \(g \in G_2\). Alternatively, \(G_2\) is the subgroup of \(GL(7, \mathbb{R})\) which fixes a particular 3-form \(\varphi_0 \in \Omega^3(\mathbb{R}^7)\) given below. Denote \(e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)\), then

\[
G_2 = \{ A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}. \tag{3.1}
\]

**Definition 3.1.** A smooth 7-manifold \(M^7\) has a \(G_2\) structure if its tangent frame bundle reduces to a \(G_2\) bundle. Equivalently, \(M^7\) has a \(G_2\) structure if there is a 3-form \(\varphi \in \Omega^3(M)\) such that at each \(x \in M\) the pair \((T_x(M), \varphi(x))\) is isomorphic to \((T_0(\mathbb{R}^7), \varphi_0)\). We call \((M, \varphi)\) a manifold with \(G_2\) structure.

A \(G_2\) structure \(\varphi\) on \(M^7\) gives an orientation \(\mu \in \Omega^7(M)\) on \(M\), and \(\mu\) determines a metric \(g = g_\varphi = \langle , \rangle\) on \(M\), and a cross product structure \(\times\) on the tangent bundle of \(M\) defined as

\[
\langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi] / \mu.
\]

\[
\varphi(u, v, w) = \langle u \times v, w \rangle.
\]

where \(i_v = v \iota\) be the interior product with a vector \(v\).

**Definition 3.2.** A manifold with \(G_2\) structure \((M, \varphi)\) is called a \(G_2\) manifold if the holonomy group of the Levi-Civita connection (of the metric \(g_\varphi\)) is a subgroup of \(G_2\). Equivalently, \((M, \varphi)\) is a \(G_2\) manifold if \(\varphi\) is parallel with respect to the metric \(g_\varphi\), \(\nabla_{g_\varphi}(\varphi) = 0\); which is equivalent to \(d\varphi = 0\), \(d(*_{g_\varphi}\varphi) = 0\). This implies that at each point \(x_0 \in M\) there is a chart \((U, x_0) \to (\mathbb{R}^7, 0)\) on which \(\varphi\) equals to \(\varphi_0\) up to second order term, i.e. on the image of \(U\) \(\varphi(x) = \varphi_0 + O(|x|^2)\).

We can paraphrase this definition by saying that for \(G_2\) holonomy manifolds there exist locally a 7-vein basis such that \(g_\varphi = \delta_{ab} e^a \otimes e^b\) and for which the 3-form \((3.1)\) and its dual are closed, being now \(e^{ijk} = e^i \wedge e^j \wedge e^k\).
3.2. **Calibrated submanifolds.** The reduction of the holonomy of a given manifold from $SO(7)$ to $G_2$ implies the existence of a covariantly constant spinor $\eta$, i.e., an spinor satisfying $D\eta = 0$ where $D$ is the spin connection of the seven manifold. This is a well known feature in Kaluza-Klein compactifications. For 11-supergravity compactified over a 7-manifold, the numbers of supersymmetries of the resulting 4-dimensional theory is equal to the number of Killing spinor of the internal manifold. In particular, if the holonomy of the 7-manifold is $G_2$, then the number of supersymmetries is one. But eleven dimensional supergravity contains membrane solitons. These solitons break the supersymmetries of the theory except if certain restrictions are satisfied [33]. These have been found to be equivalent to the presence of calibrated submanifolds inside the compactification $G_2$ space. There are two type of calibrated submanifolds in a $G_2$ holonomy manifold namely, associative and coassociative submanifolds.

**Definition 3.3.** Let $(M, \varphi)$ be a $G_2$ manifold. A 4-dimensional submanifold $X \subset M$ is called **coassociative** if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called **associative** if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w)$$

As they solve the conditions for unbroken symmetry of [33], associative and coassociative submanifolds are sometimes called supersymmetric 3- or 4-cycles, respectively. D-branes wrapping these cycles will be supersymmetric.

3.3. **Relation with CY manifolds.** The mirror map for CY relate deformations of the complex structure of one manifold to deformations of the symplectic structure of the mirror and vice versa. The question is how the mirror map acts on $G_2$ manifolds. It could be interesting to consider $G_2$ manifolds which are fibrations over CY three-folds, and to see how the mirror map on the internal CY affect the entire 7-manifold. Some examples of this situation are the barely $G_2$ manifolds, which we will consider in the following sections.

For a $G_2$ manifold it was shown in [3] that, similar to the definition (3.3) of $\chi$, one can also define a tangent bundle 2-form $\psi$, which is just the cross product of $M$.

**Definition 3.4.** Let $(M, \varphi)$ be a $G_2$ manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

$$\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle$$

Also, let $(M^7, \varphi, \Lambda)$ be a $G_2$ manifold with a non-vanishing oriented 2-plane field $\Lambda$. One can view $(M^7, \varphi)$ as an analog of a symplectic manifold, and the 2-plane field $\Lambda$ as an analog of a complex structure taming $\varphi$. This is possible because $\Lambda$ along with $\varphi$ gives the associative/complex bundle splitting $T(M) = E_{\varphi, \Lambda} \oplus V_{\varphi, \Lambda}$. Now, a choice of a non-vanishing unit vector field $\xi \in \Omega^0(M, TM)$, gives a codimension one distribution $V_\xi := \xi^\perp$ on $M$ with interesting structures induced from $\varphi$.

**Definition 3.5.** $(X^6, \omega, \text{Re } \Omega, J)$ is called an **almost Calabi-Yau manifold**, if $X$ is a Riemannian manifold with a non-degenerate 2-form $\omega$ (i.e. $\omega^3 = \text{vol}(X)$) which is co-closed, and $J$ is a metric invariant almost complex structure which is compatible with $\omega$, and $\text{Re } \Omega$ is a closed non-vanishing 3 form. Furthermore, when $J$ is integrable, $\omega$ is closed and $\text{Re } \Omega$ is co-closed we call this a Calabi-Yau manifold.
Then the following theorem can be proved [1].

**Theorem 3.6.** [1] Let \((M, \varphi)\) be a \(G_2\) manifold, and \(\xi\) be a unit vector field which comes from a codimension one foliation on \(M\), then \((X_\xi, \omega_\xi, \Omega_\xi, J_\xi)\) is an almost Calabi-Yau manifold with \(\varphi|_{X_\xi} = \text{Re} \Omega_\xi\) and \(\ast \varphi|_{X_\xi} = \ast \omega_\xi\). Furthermore, if \(L_\xi(\varphi)|_{X_\xi} = 0\) then \(d\omega_\xi = 0\), and if \(L_\xi(\ast \varphi)|_{X_\xi} = 0\) then \(J_\xi\) is integrable; when both of these conditions are satisfied then \((X_\xi, \omega_\xi, \Omega_\xi, J_\xi)\) is a Calabi-Yau manifold.

The main idea behind the formalism presented above is to use \(\chi\) and \(\psi\) on the \(G_2\) manifold to obtain the complex and symplectic structures on CY manifolds inside the \(G_2\) manifold. Theorem 3.6 implies that both complex and symplectic structure of the CY-manifold \(X_\xi\) are determined by \(\varphi\). Moreover, the choice of \(\xi\) can give rise to very different complex structures on \(X_\xi\) (i.e. \(SU(2)\) and \(SU(3)\) structures). So if we assume that \(\xi \in \Omega^0(M, V)\) and \(\xi' \in \Omega^0(M, E)\) are two unit vector fields, and let \(X_\xi\) and \(X_\xi'\) are pages of the corresponding codimension one foliations then using theorem 3.6 we showed that one can obtain two CY manifolds with different complex structures and called them “dual” in that sense.

### 3.4. A simple example: \(T^7\).

As an application of the previous notions let us consider the simplest compact \(G_2\) holonomy manifold, namely \(T^7\), [1]. Although this is a trivial example, several compact manifolds with \(G_2\) are obtaining by orbifolds of the 7-torus by perturbing them to smooth \(G_2\) holonomy metrics. We expect that many features of this example are preserved after permuting to a smooth \(G_2\) metric. So, let us consider the calibration 3-form \((3,1)\) for \(T^7\), which is given as

\[
\varphi = e^{127} + e^{136} + e^{145} + e^{235} + e^{426} + e^{347} + e^{567}.
\]

Note that this form is different than the 3-form \(\varphi\) used in [1]. The reason to use other coordinates is to make our calculations compatible with the ones in [19]. From [4], we have the decomposition \(T(M) = E \oplus V\), where \(E = \{e_1, e_2, e_7\}\) and \(V = \{e_3, e_4, e_5, e_6\}\). Now, if we reduce this along \(\xi = e_4\), then \(V_\xi = \langle e_1, \ldots, e_4, \ldots, e_7 \rangle\) and the induced symplectic form is \(\omega_\xi = -e^{15} + e^{26} - e^{37}\), and the induced complex structure is

\[
J_\xi = \begin{pmatrix}
e_1 & \mapsto & e_5 \\
e_2 & \mapsto & -e_6 \\
e_3 & \mapsto & e_7
\end{pmatrix}
\]

and the complex valued \((3,0)\) form is \(\Omega_\xi = (e^1 - ie^5) \wedge (e^2 + ie^6) \wedge (e^3 - ie^7)\).

On the other hand, if we choose \(\xi' = e_7\), then \(V_{\xi'} = \langle e_1, \ldots, e_6 \rangle\) and the symplectic form is \(\omega_{\xi'} = e^{12} + e^{34} + e^{56}\) and the complex structure is

\[
J_{\xi'} = \begin{pmatrix}
e_1 & \mapsto & -e_2 \\
e_3 & \mapsto & -e_4 \\
e_5 & \mapsto & -e_6
\end{pmatrix}
\]

Also \(\Omega_{\xi'} = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)\).

In the expressions of \(J\)’s the basis of associative bundle \(E\) is indicated by bold face letters to indicate the different complex structures on \(T^6\). If we choose \(\xi\) from the coassociative bundle \(V\) we get the complex structure which decomposes the 6-torus as \(T^3 \times T^3\). On the other hand if we choose
ξ from the associative bundle E then the induced complex structure on the 6-torus corresponds to the decomposition as $T^2 \times T^4$.

**Remark** Notice that we are finding a mirror pair for which the complex structures are induced from the same calibration 3-form $\varphi$ in a $G_2$ manifold (with a technical condition that the vector fields $\xi$ and $\xi'$ be deformations of each other). In this sense they are dual to each other. This torus example is of course trivial and it is well-known that $h^{(1,1)} = h^{(2,1)} = 9$ for $T^6$. But in the next section we will see that something analogous happens in other compact $G_2$ manifolds constructed by Joyce.

Next, we will show that one can write a map $H^{1,1}(X_\beta) \to H^{2,1}(X_\alpha)$, where $\{\alpha, \beta\}$ are orthonormal vector fields on the $G_2$ 7-torus which give underlying $T^3 \times T^3$ and $T^2 \times T^2$ Calabi-Yau decompositions respectively. Let also $\Omega^{2,1}(TX_\alpha)$ and $\Omega^{1,1}(TX_\beta)$ be the $(2,1)$ and $(1,1)$ forms on the 3-tori $X_\alpha, X_\beta$, which are generated by complex coordinates $dz_i \wedge dz_j \wedge d\overline{z}_k$ and $dw_i \wedge d\overline{w}_j$. By using Proposition 6 in [4], one can construct a natural correspondence between $\Omega^{2,1}(TX_\beta)$ and $\Omega^{1,1}(TX_\alpha)$.

Let $w_i$ be complex coordinates on $T^2 \times T^4$ and $z_i$ be complex coordinates on $T^3 \times T^3$. Using the complex structures on $T^3 \times T^3$ and $T^2 \times T^2$ we can write $dz_i$ and $dw_i$ in terms of the local coordinates as follows:

$$dz_1 = dx_1 - idx_5, \quad dz_2 = dx_2 + idx_6, \quad dz_3 = dx_3 - idx_7.$$  

$$dw_1 = dx_1 + idx_2, \quad dw_2 = dx_3 + idx_4, \quad dw_3 = dx_5 + idx_6.$$  

One can see that $Re(dz_i \wedge dz_j \wedge d\overline{z}_k)$ and $Im(dz_i \wedge dz_j \wedge d\overline{z}_k)$ of $T^3 \times T^3$ can be written in terms of its $Re \Omega$ and $Im \Omega$ as follows:

$$Re(dz_i \wedge dz_j \wedge d\overline{z}_k) = (\partial/\partial x)_\omega(Re \Omega) \wedge e^a + (\partial/\partial y)_\omega(Re \Omega) \wedge e^b.$$  

$$Im(dz_i \wedge dz_j \wedge d\overline{z}_k) = (\partial/\partial x)_\omega(Im \Omega) \wedge e^a + (\partial/\partial y)_\omega(Im \Omega) \wedge e^b.$$  

Here $e^a + ie^b = dz_k$ and $dx + idy = dz_i$ for the complex coordinate $z_i$, where $i \neq l$ and $j \neq l$. $dz_k = da + idb$, and let $dz_i = dx + idy$ with $l \neq i, j$

Then by Proposition 6, in [4], on $X_\alpha$ the following hold

$$Re \Omega_\alpha = \omega_\beta \wedge \beta^# + Re \Omega_\beta.$$  

By plugging in $\omega_\beta \wedge \beta^# + Re \Omega_\beta$ for $Re \Omega_\alpha$ and $\alpha_\omega (\ast \omega_\beta) - (\alpha_\omega Im \Omega_\beta) \wedge \beta^#$ for $Im \Omega_\alpha$ one can get

$$Re(dz_i \wedge dz_j \wedge d\overline{z}_k) = (\partial/\partial x)_\omega(\omega_\beta \wedge \beta^# + Re \Omega_\beta) \wedge e^a + (\partial/\partial y)_\omega(\omega_\beta \wedge \beta^# + Re \Omega_\beta) \wedge e^b.$$  

$$Im(dz_i \wedge dz_j \wedge d\overline{z}_k) = (\partial/\partial x)_\omega(\alpha_\omega (\ast \omega_\beta) - (\alpha_\omega Im \Omega_\beta) \wedge \beta^#) \wedge e^a + (\partial/\partial y)_\omega(\alpha_\omega (\ast \omega_\beta) - (\alpha_\omega Im \Omega_\beta) \wedge \beta^#) \wedge e^b,$$

and writing $\omega_\beta, Re \Omega_\beta$ and $Im \Omega_\beta$ in terms of $dw_i \wedge d\overline{w}_j$ gives the required correspondence.
4. Dualities related to $G_2$ manifolds

In a previous section we described certain compactifications related to CY-manifolds with zero Euler number (see below (2.6)). These spaces are special because they are the ones which can be “dual” to $G_2$ holonomy spaces in the M-theory/heterotic sense. We should be more precise in this point. Compactification of D=11 supergravity over $G_2$ holonomy manifolds gives abelian gauge fields and non chiral matter. Instead compactification of heterotic string over CY spaces give chiral 4 dimensional supergravity with non-abelian gauge fields. Hence these compactifications can not be equivalent in general. But if the Euler number of the CY is zero, non chiral matter is obtained from the heterotic side. Moreover, there is an anomaly free condition which states that $\text{Tr } R^2 - \text{Tr } F^2$ is cohomologous to zero and this broke the gauge group to a subgroup. These subgroup can be further reduced by Wilson lines. It is reasonable to suppose that, by choosing a suitable CY, this group can be reduced to an abelian subgroup. The field strength $F$ would take values in a abelian subgroup of $E_8 \times E_8$ or $SO(32)$, so it can be assumed that it is $U(1)^{16}$. Under so limited circumstances, both compactifications could be equivalent. In the abelian case, the resulting theory from CY compactifications is $D = 4$ N=1 supergravity coupled to 16 N=1 vector multiplets and $(h_{1,1} + h_{2,1} + 1) N = 1$ scalar multiplets. Compactification of eleven dimensional supergravity over the $G_2$ manifold gives $b_2$ vector multiplets and $b_3$ scalar multiplets, being $b_2$ and $b_3$ the second and the third Betti number of the $G_2$ holonomy manifold. The equivalence can exist only if

\begin{equation}
(4.1) \quad b_2 = 16, \quad b_3 = h_{1,1} + h_{2,1} + 1.
\end{equation}

Fortunately, there exist spaces realizing the condition (4.1). For instance, for threefolds satisfying (2.6) it is deduced from (4.1) that $b_2 = 16$ and $b_3 = 39$, and $G_2$ holonomy spaces with these Betti numbers do exist \cite{13}. Nevertheless, even in the zero cohomology, there are obstructions to be anomaly free, so this equivalence can fail. This can be avoided by further compactifying to $D = 3$ it is obtained a new equivalence condition

\begin{equation}
(4.2) \quad b_2 + b_3 = h_{1,1} + h_{2,1} + 17,
\end{equation}

which is invariant under the mirror transformation (1.2). In particular, for CY-manifolds satisfying (2.6), there also exist several compact $G_2$ for which (4.2) is satisfied.

Other interesting dualities relating $G_2$ manifolds are those related to II compactifications. Let us recall that for compact $G_2$ manifold the two independent Betti numbers are $b_2$ and $b_3$, the other ones are related to them by the relations $b_5 = b_2$, $b_4 = b_3$ and $b_1 = b_7 = 1$. If IIA supergravity is compactified over any of such manifolds the result will be $b_2 + 1$ vector multiplets and $b_3$ scalar multiplets. For IIB compactifications the result is $b_2 + 1$ scalar multiplets and $b_3$ vector multiplets. Naively, this indicate that the Betti numbers of mirror $G_2$ manifolds are related by $b_2 + 1 \leftrightarrow b_3$. But in three dimensions scalars and vector multiplets are related by a duality transformation. Thus this result only implies that manifolds with the same $b_2 + b_3$ are collected together. This fact was noticed already in \cite{24} and \cite{13}. Also in \cite{2} there analyzing the spectrum of both theories in orbifolds of the form $T^7/Z_2 \oplus Z_2$, it was found that T-duality interchange IIA/IIB theories with (without) discrete torsion to IIB/IIA without (with) discrete torsion. In many cases, the Betti numbers are inert. But also there were found certain examples with different Betti numbers giving rise to (apparently) the same compactification \cite{2}.

The special examples in which the Betti numbers stands unchanged under mirror symmetry correspond to the so called ”Joyce manifolds of the first kind”, which were the only examples known
when the previous analysis was done. It could be interesting to understand the geometric property which "protects" them under the mirror transformation. The formalism presented in the previous section and a description of the Joyce space will be very important for this purpose.

4.1. Joyce $G_2$ manifolds of first kind. These are compact 7-manifolds with holonomy $G_2$. An intuitive idea of their construction comes from compactifications of eleven dimensional SUGRA to $d = 4$ with orbifolds of $T^7$ as internal spaces. When eleven dimensional supergravity is compactified over $T^7$ the number of supersymmetries of the four dimensional theory is $N = 8$. A $Z_2$ action will kill two supersymmetries and we will have $N = 4$. A further quotient by a new $Z_2$ commuting with the first one will kill again half of the supersymmetries, so the number will be $N = 2$. Still another $Z_2$ action will reduce the supersymmetry to $N = 1$, which is the number of supersymmetries obtained from compact $G_2$ manifolds. This suggest the possibility of constructing compact $G_2$ manifolds by starting with a quotient of $T^7$ by a suitable $Z_2 \oplus Z_2 \oplus Z_2$ group, and blowing up conveniently the singular set. This ideas has certain analogy with the $K3$ case.

In fact, this idea was implemented effectively in [13]. A group $Z_2 \oplus Z_2 \oplus Z_2$ affecting the seven coordinates of the torus is $\Gamma = \{ \alpha, \beta, \gamma \}$, being

$$\alpha(x_1, \ldots, x_7) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),$$

(4.3)

$$\beta(x_1, \ldots, x_7) = (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7),$$

$$\gamma(x_1, \ldots, x_7) = (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7),$$

with $b_1$, $b_2$, $c_1$, $c_3$, and $c_5$ some constants in the interval $\{0, \frac{1}{2}\}$. It is not difficult to check that the transformations $\alpha$, $\beta$ and $\gamma$ are $Z_2$ actions and commute thus, one can blowup their singular set successively [19]. Consider first $T^7/\alpha$. An inspection of (4.3) shows that $\alpha$ affect the first four coordinates of $T^7$ and leave the other three inert. This means that resolving the singular set of $T^7/\{\alpha\}$ is equivalent to resolving $T^4/\{\alpha\} \times T^3$, where $T^3$ is the torus parameterized by the coordinates $x_5$, $x_6$ and $x_7$. By replacing balls $B^4/Z_2$ around the singularities by copies of the Eguchi-Hanson space and making a blowup we will obtain $K3 \times T^3$ as a resolution.

The next task is to find the effect of $\beta$ on $K3 \times T^3$, which will follow from the effect on $T^4/\{\alpha\} \times T^3$. From (4.3) it is seen that the coordinate $x_7$ is rigid under the action of $\beta$. Let us parameterize the remaining 6-torus $T^6$ with the coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $z_3 = x_5 + ix_6$ and their complex conjugates $\bar{z}_i$. The Kähler form of $T^6$ is $\mathcal{J}_6 = dz_1 \wedge d\bar{z}_1$ and $\beta^* (\mathcal{J}_6) = \mathcal{J}_6$. Thus $\beta$ acts holomorphically on such torus. In similar fashion, $\gamma$ acts anti-holomorphically. Let us decompose $T^6 = T^4 \times T^2$, where the 4-torus is parameterized by $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ and the 2-torus is parameterized by $(z_3, \bar{z}_3)$. Then the $\beta$-action on $T^2$ is $(-1)$. Also, the $\beta$ action over the Kähler two form $\mathcal{J}_1 + i\mathcal{J}_2 = dz_1 \wedge d\bar{z}_2$ of $T^4$ is $(-1)$. After resolving the singular set of $\alpha$, the torus $T^4$ becomes $K3$ and the two-form $dz_1 \wedge d\bar{z}_2$ descends to a holomorphic two form $\mathcal{J}$ over $K3$. Then the $\beta$ action over $T^4$ descends to an action $\sigma$ on $K3$ that acts on $\mathcal{J}$ as $\sigma(\mathcal{J}) = -\mathcal{J}$. In conclusion, $\beta$ descends on $K3 \times T^2$ to the automorphism $(\sigma, -1)$ as for the Borcea-Voisin case (2.7). The corresponding elliptic curve is $E = T^2$. Thus the resolution of $T^7/\{\alpha, \beta\}$ is a trivial bundle $X_6 \times T^1$, where $X_6$ is the resolution of (2.7) with $E = T^2$.

The final step is to find the effect of $\gamma$ over $X_6 \times T^1$. As we have seen, $\gamma$ is anti-holomorphic on the Kähler form $\mathcal{J}_6$ of $T^6$ and descends to an anti-holomorphic action over $X_6$. The calibration form
\( \varphi \) of \( X_6 \times T^1 \) is

\[
\varphi = dx_7 \wedge J_6' + \text{Re}(\Omega')
\]

\[\ast \varphi = dx_7 \wedge (Im \Omega') + \frac{1}{2} J_6' \wedge J_6' \]

where \( J_6' \) is the Kähler form of the Borcea-Voisin threefold and \( \Omega' \) its complex closed three form.

As \( \gamma \) acts as a negation on \( x_7 \) we conclude from the \( \gamma \)-invariance of \( \varphi \) and \( \ast \varphi \) that \( \gamma(J_6') = -J_6' \) and that \( \gamma(\Omega') = M' \). Therefore \( \gamma \) descends to an anti-holomorphic action on the CY threefold and as a negation on \( S^1 \). Furthermore, from (4.3) it follows that the fibers corresponding to \( x_7 = 0 \) and \( x_7 = \frac{1}{2} \) are \( \gamma \)-invariant. The singular set of \( X_6 \times T^1 \) is located in these fibers and consists on the 3-tori or the free \( \mathbb{Z}_2 \)-quotient of 3-tori descending from the fixed \( T^3 \)'s elements of \( \gamma, \alpha \gamma, \beta \gamma, \) or \( \alpha \beta \gamma \) on \( T^7 \). The neighbor of the singularities is of the form \( T^3 \times B^4 / \mathbb{Z}_2 \) and the resolution is by blowups as in the Kummer case. The result are the Joyce spaces of the first kind, which therefore are fibrations over Borcea-Voisin CY three folds [19].

4.2. A mirror pair inside a \( G_2 \) space. We have seen in a previous section that \( T^7 \) with its flat \( G_2 \) structure \( \varphi \) induces a mirror pair of CYs, which is the torus \( T^6 \) decomposed as \( T^4 \times T^2 \) and as \( T^3 \times T^3 \). This provide a mapping between the complex and sympletic structure required by mirror symmetry and was obtained by reduction along an associative and coassociative cycle respectively. This example is rather simple, because \( T^6 \) is mirror to itself.

Nevertheless, we can show that something analogous happens for Joyce manifolds of the first kind, this issue has also been analyzed in [5]. As they are locally fibrations over Borcea-Voisin 3-folds obtained by resolving the singularities of orbifolds of \( T^6 \), it is plausible to guess that the CY base spaces have \( (h_1, h_2) = (19, 19) \). Note that if this were the case, the base of the fibration is protected again the complex/sympletic mapping, that is, this mapping does not change the topology of the manifold.

We can directly check that our previous discussion is true. The 3-folds in consideration arise as resolutions of \( T^6 \) divided by the \( \alpha \) and \( \beta \) actions of (4.3), which are commuting \( \mathbb{Z}_2 \) actions. The first \( \mathbb{Z}_2 \) generator possess a singular set consisting in sixteen copies of \( T^2 \). But these copies are interchanged by the second generator, leaving eight invariant torus. The same argument is true for the second generator so the total number of \( T^2 \) copies is 16. For \( T^6 \) we have \( (h_1, h_2) = (3, 3) \) and any of these \( T^2 \) copies add 1 to the Hodge numbers. The result is finally \( (h_1, h_2) = (19, 19) \). Thus, the Borcea-Voisin manifolds presented in the previous section are of zero Euler number and satisfy the constraint (2.6).

In fact, we can make another computation of the Hodge numbers (2.7) giving the same result. From (2.8) we see that the calculation of the Hodge numbers is related to find the components of the singular set \( \Sigma \) of the \( \sigma \) of (2.7), which acts over the K3 surface. An inspection of (4.3) shows that the \( \beta \) action over \( T^4 \) has 4-fixed points, but they are interchanged by the \( \alpha \) action and thus there are essentially two components. This means that \( n' = 2 \) in (2.8). Also, in \( T^6 \) the neighborhood of the singularities is of the form \( T^2 \), and these tori are disjoint. It means that \( \Sigma = T^2 \cup T^2' \). Thus \( n = 2 \) and we obtain from (2.8) that

\[
h^{1,1}(Y) = h^{2,1}(Y) = 19.
\]
So, we have checked that the calculation is correct. Thus, we have found a mirror CY structure inside Joyce manifolds of the first kind. Note that this result does not depend on the election of the constants $b_i$ and $c_i$ of the actions (4.3).

Let us go back to our original motivation, which is to understand why "apparently" mirror symmetry leave the betti numbers of the Joyce spaces inert. As the underlying CY base is protected under the complex/sympletic mapping, if the fibration were trivial, the same will occur with the entire 7-manifold. We suggest that the situation still holds after performing the quotient by $\gamma$ and perturbing the metric around the singularities. This is an plausibility argument only. We also suggest that this situation is generalized to barely $G_2$ manifolds, which we describe next.

4.3. Joyce spaces as “barely” $G_2$ manifolds. Summing up our discussion at this point, Joyce spaces of the first kind arise as resolutions of the singular set of

$$X_7 = \frac{(X_6 \times S^1)}{\mathbb{Z}_2}$$

being the $\mathbb{Z}_2$ action identified with $\gamma$ and $X_6$ the CY 3-fold described in (2.7). $\gamma$ is an anti-holomorphic action on the $X_6$ threefold and as a negation on $x_7$. Such kind of $G_2$ manifolds are called “barely” $G_2$ manifolds. In other words barely $G_2$ manifolds arise as resolutions of orbifolds of the form

$$X_7 = \frac{(CY \times S^1)}{\mathbb{Z}_2},$$

being CY a Calabi-Yau manifold. The action $\mathbb{Z}_2$ is given by $(\sigma, -1)$ being $\sigma$ a real structure, which acts as an isometry of the CY for which

$$\sigma^*(\Omega) = \Omega, \quad \sigma^*(\mathcal{J}) = -\mathcal{J},$$

where $\Omega$ is the closed holomorphic three form and $\mathcal{J}$ the Kähler two form.

Several aspects of barely $G_2$ manifolds are already investigated and can be applied for Joyce manifolds. The associative three cycles of barely $G_2$ manifolds fall into two types [36]. The first are of the form

$$\Sigma_{hol} = \frac{(\Sigma_2 \times S^1)}{\mathbb{Z}_2}$$

where $\Sigma_2$ is a holomorphic cycle on the CY which is mapped to $-\Sigma_2$ by the action of $\sigma$. If $\Sigma_2$ is rational then $\Sigma$ is a rational 3-sphere. The other associative submanifolds are

$$\Sigma_r = \frac{(\Sigma^+)}{\mathbb{Z}_2}$$

where $\Sigma^+$ is a Lagrangian 3-cycle in the CY manifold. These are mapped to $-\Sigma^+$ by the action of $\sigma$.

Also, in M-theory compactifications on such “barely” backgrounds there are no 5-brane instantons [36]. Therefore we conclude that compactifications of eleven dimensional supergravity over Joyce spaces of the first kind give no 5-brane instantons. This implies that contribution to the superpotential is given in terms of the associative cycles, i.e, $W = W(\Sigma_{hol}) + W(\Sigma_r)$. Other aspects of strings propagating in barely $G_2$ spaces have been also considered for instance, in [38] and [40].
Finally, for barely $G_2$ manifolds we have that $b_2 + b_3 = h^{1,2} + h^{1,1} + 1$. Notice that this relation is invariant under the complex/symplectic map $h^{1,2} \leftrightarrow h^{1,1}$ of the underlying CY.

5. Interpretation

Our aim is now to interpret the presented results in the context of mirror symmetry. We suggest that the reason for which the mirror map apparently leave the betti numbers of the Joyce spaces of the first kind inert [28] is that they are fibrations over CY 3-folds which are ”protected” under the action complex/symplectic map. In other words, this map does not affect the topology of the underlying CY base. We generalize this suggestion to ”barely” $G_2$ manifolds by supposing that for any barely $G_2$ manifold fibered over a CY $X_6$ the natural mirror candidate is another barely $G_2$ manifold fibered over the mirror $X'_6$ of $X_6$. In fact the sum $b_2 + b_3 = h^{1,2} + h^{1,1} + 1$ will be the same in both cases. Although this is not conclusive evidence we propose that, as $X_6 \times S^1$ and $X'_6 \times S^1$ are mirror 7-manifolds, the mirror property is (approximately) preserved after dividing by a $\mathbb{Z}_2$ action and perturbing the resulting orbifolds to smooth $G_2$ holonomy metrics.

We should make some comments about this suggestion. First, the mirror $G_2$ pairs found till the moment are not exact. The resulting theories agree up to certain extent, but they are not completely isomorphic [2]. Second, our suggestion should be understood as a classical one for the following reason. The untwisted sector of II strings propagating on such manifolds contains massless states consisting $b_2 + b_3$ chiral multiplets. But has been shown in [38] the appearance of additional massless states in the twisted sector. These states were interpreted in terms of quantum effects [38]. Classically, if the anti-holomorphic action has no fixed points on the CY, one does not expect new massless states to appear. Thus, our statement make sense only as classical one.

There are other compact examples of $G_2$ holonomy manifolds found by Kovalev, [16], which are constructed by gluing asymptotically cylindrical $G_2$ manifolds along the boundary. It sounds reasonable to suppose that the resulting manifold can be also described as some kind of CY fibration. This leads to the following question:

**Question** For any given compact $G_2$ manifold, does there exist an open subset which is of the form Calabi-Yau $\times$ (open interval)?

An affirmative answer to this question will imply that compact mirror $G_2$ manifolds admit a description in terms of mirror CY manifolds. In any case, we feel that a deep study of the submanifolds inside compact $G_2$ manifolds is worthy.

As a future investigation it will be interesting to repeat the analysis performed in [38] for the barely $G_2$ manifolds presented here and for those presented in [39]. We also would like to make a more precise definition of mirror symmetry for $G_2$ manifolds, perphaps using the topological versions presented in [22]-[23]. We leave this for a future investigation.

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