OMITTING TYPES IN LOGIC OF METRIC STRUCTURES

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Abstract. The present paper is about omitting types in logic of metric structures introduced by Ben Yaacov, Berenstein, Henson and Usvyatsov. While a complete type is omissible in a model of a complete theory if and only if it is not principal, this is not true for incomplete types by a result of Ben Yaacov. We prove that there is no simple test for determining whether a type is omissible in a model of a theory $T$ in a separable language. More precisely, we find a theory in a separable language such that the set of types omissible in some of its models is a complete $\Sigma^1_1$ set and a complete theory in a separable language such that the set of types omissible in some of its models is a complete $\Pi^1_1$ set. We also construct a complete theory and countably many types that are separately omissible, but not jointly omissible, in its models.

The Omitting Types Theorem is one of the most useful methods for constructing models of first-order theories with prescribed properties (see [23], [21], or any general text on model theory). It implies, among other facts, the following.

1. If $T$ is a theory in a countable language, then the set of all $n$-types realized in every model of $T$ is Borel in the logic topology on $S_n(T)$.

2. If $T$ is moreover complete, then any sequence $t_n$, for $n \in \omega$, of types each of which can be omitted in a model of $T$ can be simultaneously omitted in a model of $T$.

We note that the types $t_n$ in (2) are not required to be complete, but the theory $T$ is.

While in classical logic the criterion for a given type to be omissible in a model of a given theory applies regardless of whether the type is complete or not, the situation in logic of metric structures is a bit more subtle. The omitting types theorem ([3, §12] or [20, Lecture 4]) has the following straightforward consequences (see Proposition 1.7 for a proof of (3) and Corollary 3.4 for a proof of (4)).

3. If $T$ is a theory in a separable language of logic of metric structures, then the set of all complete $n$-types realized in every model of $T$ is Borel in the logic topology on $S_n(T)$.

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(4) If $T$ is moreover complete, then any sequence $t_n$, for $n \in \omega$, of complete types each of which can be omitted in a model of $T$ can be simultaneously omitted in a model of $T$.

Examples constructed by I. Ben-Yaacov (2) and T. Bice (7) demonstrate that omitting partial types in logic of metric structures is inherently more complicated than omitting complete types. Our results confirm this intuition and show that the problem is essentially intractable. An excellent reference for notions from the Descriptive Set Theory is [22]; see also the beginning of §2.

**Theorem 1.**

(5) There is a complete theory $T$ in a separable language such that the set of all types omissible over a model of $T$ is $\Pi^1_1$-complete.

(6) There is a theory $T$ in a separable language such that the set of all types omissible in a model of $T$ is $\Sigma^1_2$-complete.

(7) There is a separable structure $M$ in a separable language such that the set of all unary quantifier-free types realized in $M$ is a complete $\Sigma^1_1$ set.

**Proof.** (5) is proved in Theorem 2.5, (6) is proved in Theorem 2.6 and (7) is Proposition 2.4. □

If $T$ is a complete theory and $F$ is a set of types each one of which is omissible in a model of $T$, are all types in $F$ omitted in a single model of $T$?

The case when $F$ is finite easily reduces to the case when $F$ has two elements by an easy argument as in §1.1.2, and the latter case is Question 6.2. We have some information on the case when $F$ is countable.

**Theorem 2.** There is an example of a complete theory $T$ in a separable language and types $s_n$, for $n \in \omega$, such that for every $k$ there exists a model of $T$ that omits all $s_n$ for $n \leq k$ but no model of $T$ simultaneously omits all $s_n$.

**Proof.** This is proved in §5. □

Theorem 2 should be compared to a consequence of [4, Corollary 4.7], that under additional conditions any countable set of types that are not omissible in a model of a complete separable theory $T$ has a finite subset consisting of types that are not omissible in a model of $T$.

This work was motivated by potential applications to the classification problem of C*-algebras (see [12]). The negative results stated above fortunately do not present an obstruction to using omitting types theorem to construct interesting C*-algebras. This is because all the relevant sequences of types have an additional regularity property of being uniform (see 4) and the omissible uniform sequences of types have syntactic characterization analogous to one for complete types as well as one analogous to the first-order case (Theorem 4.2).
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1. Preliminaries

We assume that the reader is acquainted with logic of metric structures ([3], [20]). We strictly follow the outline of this logic given in [3]. In particular, all metric structures are required to have diameter 1 and all formulas are $[0, 1]$-valued. All function and predicate symbols are equipped with a fixed modulus of uniform continuity. Every structure is a complete metric space in which interpretations of functional and relational symbols respect this modulus. It is a straightforward exercise to see that our results apply to the modification of this logic adapted to operator algebras presented in [15].

In logic of metric structures there are two ways in which one can define theory of a structure $M$. We shall think of theory as a set of sentences, and accordingly set $\text{Th}(M) = \{ \phi : \phi^M = 0 \}$. Alternatively, one may consider the functional $\phi \mapsto \phi^M$ as the theory of $M$.

1.1. Conditions and types. A (closed) condition is an expression of the form $\phi(\vec{x}) = 0$ for a formula $\phi(\vec{x})$ and type is a set of conditions. It is realized in structure $M$ if there is a tuple $\vec{a}$ in $M$ of appropriate sort such that $\phi(\vec{a})^M = 0$ for all conditions $\phi(\vec{x}) = 0$ in $t$. A type $t$ is consistent if it is consistent with $T = \emptyset$. If free variables of every formula appearing in $t$ are included in $\{x_0, \ldots, x_n-1\}$ and $n$ is minimal with this property, we say that $t$ is an $n$-type.

1.1.1. Continuous functional calculus. Recall that every $L$-formula $\phi(\vec{x})$ has a modulus of uniform continuity and that the set $R_\phi$ of all possible values of $\phi$ in all $L$-structures is a compact subset of $\mathbb{R}$ ([3]). If $\phi(\vec{x})$ is a formula and $f$ is a continuous function on $R_\phi$ then $f(\phi(\vec{x}))$ is a formula.

Consider conditions of the form $\phi \in K$, where $K \subseteq \mathbb{R}$. Condition $\phi(\vec{x}) \in K$ is closed (open) if $K$ is closed (open, respectively). Two conditions $\phi(\vec{x}) \in K$ and $\psi(\vec{x}) \in L$ are equivalent if in every $L$-structure $A$ for every $\vec{a}$ of the appropriate sort one has $\phi(\vec{a})^A \in K$ if and only if $\psi(\vec{a})^A \in L$.

The following lemma will be used tacitly.

Lemma 1.1. Every open condition is equivalent to a condition of the form $\phi(\vec{x}) < 1$. Every closed condition is equivalent to one of the form $\phi(\vec{x}) = 0$.

Proof. Fix a condition $\psi(\vec{x}) \in K$. Since $R_\psi$ is compact, one can use Tietze extension theorem to find $f : R_\psi \rightarrow [0, 1]$ such that $f^{-1}(\{0\}) = K \cap R_\psi$ if $K$
is closed or $f^{-1}(\{1\}) = R_\psi \setminus K$ is $K$ is open. In either case $\phi = f(\psi)$ is as required. □

See also Lemma 3.1

1.1.2. Pairing types. The reader will excuse us for making some easy observations for future reference.

**Lemma 1.2.** If $t$ and $s$ are types over a consistent and complete theory then there are types $t \land s$ and $t \lor s$ such that for every $M \models T$ we have that

1. $M$ omits $t \lor s$ if and only if it omits both $t$ and $s$,
2. $M$ realizes $t \land s$ if and only if it realizes both $t$ and $s$.

**Proof.** We may assume that $t$ and $s$ are an $m$-type and a $k$-type, respectively, in disjoint sets of variables.

We may assume that $t$ consists of conditions $\phi_n(\bar{x}) = 0$ for $n \in \omega$ such that $T \models \phi_{n+1}(\bar{x}) \geq \phi_n(\bar{x})$ and that $s$ consists of conditions $\psi_n(\bar{y}) = 0$ for $n \in \omega$ such that $T \models \psi_{n+1}(\bar{y}) \geq \psi_n(\bar{y})$. Let $t \lor s = \{\min(\phi_n, \psi_n) : n \in \omega\}$ and $t \land s = \{\max(\phi_n, \psi_n) : n \in \omega\}$. These are both $m + k$-types, and they clearly satisfy the requirements (in case of $\lor$ we need to use the monotonicity assumption on conditions forming these types). □

1.1.3. Type $t_\omega$. Assume $T$ is a theory and $t = \{\phi_j(\bar{x}) = 0 : j \in \omega\}$ is an $n$-type omissible in a model of $T$. We shall assume $n = 1$ for simplicity.

To $t$ we associate the type $t_\omega$ in infinitely many variables $x_j$, for $j \in \omega$, consisting of all formulas of the form

1. $(t_\omega 1) \phi_j(x_n) \leq \frac{1}{n}$ for all $j \leq n$, and
2. $(t_\omega 2) \ d(x_j, x_{j+1}) \leq 2^{-j}$ for all $j \in \omega$.

We can think of $t_\omega$ as an $\omega$-type—an increasing union of $n$-types $t_n$, where $t_n$ is the restriction of $t_\omega$ to $x_j$, for $j < n$. The following is clear.

**Lemma 1.3.** Model $M$ realizes $t$ if and only if every (equivalently, some) dense subset $D$ of its universe includes a sequence that realizes $t_\omega$. □

We note that a slightly finer fact is true. If $D$ is an arbitrary dense subset of the universe of $M$ then we can consider $D^{<\omega}$ as a tree with respect to the end-extension. Let $T_{D,t}$ be the family of all $\bar{d} \in D^{<\omega}$ such that with $n$ being the length of $\bar{d}$ we have that $M \models t_n(\bar{d})$. Then $M$ realizes $t$ if and only if the tree $T_{D,t}$ is well-founded.

1.2. Spaces. For a fixed separable language $L$ we shall now introduce standard Borel spaces of formulas, structures, complete types, and incomplete types.
1.2.1. **Linear space of formulas.** For \( n \in \omega \) let \( \mathbb{F}_n(L) \) denote the set of formulas all of whose free variables are among \( \{x_0, \ldots, x_{n-1}\} \). If \( L \) is implicit from the context we shall write \( \mathbb{F}_n \) instead of \( \mathbb{F}_n(L) \). On \( \mathbb{F}_n(L) \) one considers the norm
\[
\|\phi\|_\infty = \sup_{M, \bar{a}} |\phi(\bar{a})^M|.
\]
Here \( M \) ranges over all \( L \)-structures, \( \bar{a} \) ranges over all \( n \)-tuples of elements in \( M \), and \( \phi(\bar{a})^M \) is the interpretation of \( \phi \) at \( \bar{a} \) in \( M \).

As pointed out in [3], if \( L \) is countable then \( \mathbb{F}_n(L) \) is separable with respect to this metric.

1.2.2. **Borel space of models.** One can consider structures of a fixed countable (discrete) language \( L \) as structures with universe \( \omega \). Thus the space of countable \( L \)-structures is equipped with the Cantor-set topology and a natural continuous action of the permutation group \( S_\infty \). This observation is a rich source of results on the interface between (classical first-order) model theory and descriptive set theory (see e.g., [18]). The space of metric structures of a fixed separable language \( L \) can be construed as a standard Borel space in more than one way. In [11] it was shown that every metric \( L \)-structure can be canonically extended to one whose universe is the Urysohn metric space. Borel space \( \mathcal{M}(L) \) of all \( L \)-structures obtained in this way is not convenient for our purposes. We consider the space that was essentially introduced in [5, p. 2]. Although this space was denoted \( \mathcal{M}(L) \) in [5], we use the notation \( \hat{\mathcal{M}}(L) \) to avoid conflict with [11]. Space \( \hat{\mathcal{M}}(L) \) is defined as follows.

For simplicity we consider the case when \( L \) has no predicate symbols. Let \( d_j \), for \( j \in \omega \), be a sequence of new constant symbols and let \( L^+=L\cup\{d_j: j\in\omega\} \). Let \( p_j \), for \( j \in \omega \), be an enumeration of a countable dense set of \( L^+ \)-terms closed under application of function symbols from \( L \). Space \( \hat{\mathcal{M}}(L) \) is the space of all functions
\[
\gamma: \omega^2 \to [0, 1]
\]
such that

(i) \( \gamma \) is a metric on \( \omega \),

(ii) \( \gamma \) respects the moduli of uniform continuity of all functions in \( L \).

In particular [13] (UC), p. 8] holds: if \( f \) is a function symbol with modulus of uniform continuity \( \Delta \) and \( i, j, i', j' \) are such that \( p_{j'} = f(p_j) \) and \( p_{i'} = f(p_i) \) then
\[
\gamma(i', j') < \Delta(\varepsilon) \text{ implies } \gamma(i, j) \leq \varepsilon.
\]

(An analogous condition holds for \( n \)-ary function symbols for \( n \geq 2 \).)

The set of \( \gamma \in [0, 1]^{\omega^2} \) satisfying (i) and (ii) is a closed subspace of the Hilbert cube, and \( \hat{\mathcal{M}}(L) \) is equipped with the induced compact metric topology. For \( \gamma \in \hat{\mathcal{M}}(L) \) we can consider the structure with universe \( \omega \) and metric given by \( \gamma \). This structure falls short of being an \( L \)-structure only because it
is incomplete, and the completion $M(\gamma)$ of such structure is a separable $L$-structure. Every complete separable metric $L$-structure $M$ is a completion of such countable structure. Also, every such $M$ has many different representations in $\hat{\mathcal{M}}(L)$.

One can modify $\hat{\mathcal{M}}(L)$ to accommodate the case when $L$ has predicate symbols. As a matter of fact, the version of space $\hat{\mathcal{M}}(L)$ in the case when $L$ has only predicate symbols was considered in [5, p. 2]; let us recall the details. For each $n$-ary predicate symbol $R$ add a function $\gamma_R: \omega^n \to [0,1]$ corresponding to the interpretation of $R$ in $M(\gamma)$. The straightforward details are omitted.

Space $\hat{\mathcal{M}}(L)$ is similar to the space of separable C*-algebras $\hat{\Gamma}$ introduced in [17]. Although $\hat{\mathcal{M}}(L)$ is different from the Borel space of $L$-structures $\mathcal{M}(L)$ defined in [11], these two spaces are equivalent in the sense of [17]. The proof of this fact is analogous to the proof given in [11 §3] for the case of C*-algebras.

Special case of the following lemma in case of C*-algebras was proved in [17, Proposition 5.1]. Proof of the general case is virtually identical.

Lemma 1.4. Assume $L$ is a separable language. The function from $\hat{\mathcal{M}}(L)$ to the space of $L$-theories that associates the theory of $M$ to $M$ is Borel. □

Proof of the following lemma is a straightforward computation (similar to one in the appendix of [17]).

Lemma 1.5. If $T$ is an $L$-theory then the set of $\gamma \in \hat{\mathcal{M}}(L)$ such that $M(\gamma) \models T$ is Borel. □

1.2.3. Compact spaces of theories and types. Consider space $\mathbb{F}_n(L)$ as in §1.2.1. On the space of linear functionals on $\mathbb{F}_n(L)$ we consider the topology of pointwise convergence (i.e., the logic topology, also known as the weak*-topology). Every $L$-structure $M$ defines a linear functional on $\mathbb{F}_0$ by the evaluation of sentences, $\phi \mapsto \phi^M$. Such functionals are consistent $L$-theories.

Since $\mathbb{F}_0$ is normed by $\|\phi\|_\infty = \sup_{M,\bar{a}} |\phi(\bar{a})|^M$ (see §1.2.1), this functional has norm 1. By the compactness theorem ([3]) the space of complete, consistent $L$-theories is compact in logic topology, see [3].

Via the interpretation map $(M,\phi) \mapsto \phi^M$ the spaces $\mathcal{M}(L)$ and $\mathbb{F}_0(L)$ are in (nonlinear) duality.

Then an $L$-model $M$ and an $n$-tuple $a_i$, for $i < n$, in $M$ define by interpretation a linear functional $tp_M(\bar{a})$ by
\[
,tp_M(\bar{a})(\phi(\bar{x})) = \phi(\bar{a})^M.
\]
Again, by compactness the space $S_n(T)$ of all consistent complete $n$-types is compact in the logic topology.

We shall tacitly use completeness theorem for logic of metric structures whenever convenient ([6]).
1.2.4. Metric on the space of complete types over a complete theory. Let \( T \) be a complete \( L \)-theory. Following ([3, p. 44]) on the space \( S_n(T) \) of complete \( n \)-types over \( T \) we define metric \( d \) by

\[
d(t, s) = \inf \{ \max_{i<n} d(a_i, b_i) : \text{there exist } M \models T \text{ and } \bar{a} \text{ and } \bar{b} \text{ in } M \}
\]

such that \( M \models t(\bar{a}) \) and \( M \models s(\bar{b}) \).

(Since both types and \( T \) are complete, the triangle inequality is satisfied.)

We denote the set of realizations of type \( t \) in model \( M \) by \( t(M) \).

Given an \( n \)-type \( t \) over a theory \( T \) and a new \( n \)-tuple of constants \( \bar{c} \), we let \( T_{t/c} \) denote the theory in language \( L \cup \{ \bar{c} \} \) obtained by extending \( T \) with axioms asserting that \( \bar{c} \) realizes \( t \). More precisely, one adds all conditions of the form \( \phi(\bar{c}) = 0 \) to \( T \), where \( \phi(\bar{x}) = 0 \) is a condition in \( t \). One can iterate this definition and name realizations of more than one type, as in the proof of Lemma 1.6 below.

For a fixed \( n \) the set of types \((T, t)\) where \( T \) is a complete theory and \( t \) is a complete type over \( T \) is endowed with a Polish topology as follows. We identify each pair \((T, t)\) with the complete theory \( T_{t/c} \). Each theory obtained in this way is complete, and every complete theory in the language obtained by extending the language of \( T \) by adding constants \( \bar{c} \) is equal to \( T_{t/c} \) for some pair \((T, t)\).

**Lemma 1.6.** For every \( n \) and \( \varepsilon \geq 0 \) the set \( \{(r, s) : d(r, s) > \varepsilon \} \subseteq S_n(T)^2 \) is open in the logic topology.

**Proof.** Fix types \( t \) and \( s \) such that \( d(t, s) > \varepsilon \). This is equivalent to stating that

\[
T_{t/\bar{c}, s/\bar{d}} \models d(\bar{c}, \bar{d}) > \varepsilon.
\]

Then by compactness (or by [6]) there exists a finite subset \( T_0 \) of \( T_{t/\bar{c}, s/\bar{d}} \) such that \( T_0 \models d(\bar{c}, \bar{d}) > \varepsilon \). This condition defines a logic open neighbourhood \( U \) of \((t, s)\) in \( S_n(T) \) such that \( d(r, s) > \varepsilon \) for all \((r, s) \in U \). \qed

1.2.5. The compact metric space of incomplete types. Let \( T \) be a (not necessarily complete) theory in a separable language. An \( n \)-type \( t \) in \( T \) is a countable set of conditions ([14,11], but we can also identify it with the set of all complete types extending it. This set is closed (and therefore compact) in the logic topology. Fix separable language \( L \), \( n \geq 1 \) and an \( L \)-theory \( T \). For closed \( K \subseteq S_n(T) \) (considering \( s \in S_n(T) \) as the set of conditions) the set

\[
t_K = \bigcap_{s \in K} s.
\]

is a type that includes \( T \). If \( s \not\in K \), then there exists a condition \( \phi(\bar{x}) < \varepsilon \) such that \( \phi(\bar{x}) \leq \varepsilon/2 \) belongs to in \( s \) but not to any type belonging to \( K \). Therefore no type \( s \not\in K \) extends \( t_K \) and we have (considering types as sets of conditions)

\[
K = \{ s \in S_n(T) : s \supseteq t_K \}.
\]
We can therefore identify the space $S^{-n}_n(T)$ of not necessarily complete types over $T$ with the exponential space of $S_n(T)$, with its compact metric topology given by the Hausdorff metric.

1.3. Omitting complete types.

1.3.1. Principal types. Assume $T$ is a complete theory and $t$ is an $n$-type in the signature of $T$. As in §1.2.5 we identify $t$ with the set of all complete types extending $t$. This is a closed set in logic, and therefore also in metric, topology. An $n$-type $t$ is principal (or isolated) if for every $\varepsilon > 0$ the set

$$B_\varepsilon(t) = \{ s \in S_n(T) : \inf \{ d(t', s) : t' \in S_n(T), t \subseteq t' \} < \varepsilon \}$$

is not nowhere dense in the logic topology (with respect to $T$). If a type is not principal, then the proof of the omitting types theorem given in [3, §12] shows that $T$ has a separable model omitting $T$. If $t$ is principal and complete, then $\varepsilon$-balls as in the definition can be chosen so that they form a decreasing chain. The intersection of this chain gives a realization of $t$ in every model of $T$. This paper is about the case not covered by these observations: principal, but not complete, types over a complete theory in a separable language.

1.3.2. The set of omissible complete types is Borel. A type is omissible in a model of $T$ if and only if there exists a countable metric space with interpretations for all symbols in $L$ whose completion is a model of $T$ (it is not literally a model since its universe is not necessarily a complete metric space) such that no Cauchy sequence of its elements has a limit that realizes $t$ (or in the other words, no subset of this set realizes type $t_\omega$ as defined in §1.1.3). This condition is $\Sigma^1_2$, but by the following well-known result under additional assumptions it is Borel.

In the following proposition we consider the logic topology on the space of all complete theories in a fixed separable language $L$. For $n \in \omega$ consider the space of pairs $(T, t)$ where $T$ is an $L$-theory and $t$ is a complete $n$-type over $T$ with respect to the logic topology defined before Lemma 1.6.

**Proposition 1.7.** For every $n \in \omega$ the following sets are Borel.

1. The set of all pairs $(T, t)$ such that $T$ is a complete theory and $t$ is a complete $n$-type realized in every model of $T$.

2. The set of all pairs such that $T$ is a theory and $t$ is an $n$-type realized in some model of $T$.

**Proof.** (1) By the Omitting Types Theorem ([3, §12], [20], or Corollary §3.4) type $t$ has to be realized in every model of $T$ if and only if it is principal (principal types were defined in §1.3.1).

Since the logic topology is second countable, expressing the fact that $B_\varepsilon(t)$ is nowhere dense requires only quantification over a countable set. It therefore suffices to show that the set $\{ s : d(t, s) \geq \varepsilon \}$ is Borel, and this is done in Lemma 1.6 below.
(2) By the compactness theorem ([3, Theorem 5.8]) the condition that a type $t$ is consistent with theory $T$ is finitary, and therefore Borel. □

Completeness assumption on types in Proposition 1.7 is necessary by Theorem 1.

1.4. A test for elementary equivalence. We include a general result needed in §5. A subset $Y$ of a metric space is $\varepsilon$-dense if for every point $x \in X$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$.

Lemma 1.8. Assume $A$ and $B$ are structures of language $L$ such that for every finite $L_0 \subseteq L$ and every $\varepsilon > 0$ there are $\varepsilon$-dense substructures $A_0$ and $B_0$ of $L_0$-reducts of $A$ and $B$, respectively, which are isomorphic. Then $A$ and $B$ are elementarily equivalent.

Proof. By [3, Proposition 6.9] every formula can be uniformly approximated by formulas in prenex normal form. It will therefore suffice to show that every formula in prenex normal form has the same value in $A$ and $B$. Let $\phi$ be an $L$-sentence in prenex normal form

$$\sup_{x_0} \inf_{x_1} \ldots \sup_{x_{2n-2}} \inf_{x_{2n-1}} \psi(\bar{x})$$

where $\psi(\bar{x})$ is quantifier-free and let $L_0 \subseteq L$ be a finite subset consisting only of symbols that appear in $\phi$. For $\delta > 0$ fix $\varepsilon > 0$ small enough so that perturbing variables in $\bar{x}$ by $\leq \varepsilon$ does not change the value of $\psi(\bar{x})$ by more than $\delta/2$. It is then straightforward to check that

$$|\phi^A - \phi^{A_0}| \leq \delta/2 \quad \text{and} \quad |\phi^B - \phi^{B_0}| \leq \delta/2.$$

Since $\delta > 0$ was arbitrary and $A_0$ and $B_0$ are isomorphic, we conclude that $\phi^A = \phi^B$. Since $\phi$ was arbitrary, we conclude that $A$ and $B$ are elementarily equivalent. □

Lemma 1.8 can also be proved by using EF-games ([20]) and it admits a number of yet unexplored possibilities for generalizations. For example, one could define a variant of Gromov–Hausdorff distance on structures in a given language $L$ and show that for any fixed sentence $\phi$ the computation of $\phi$ is continuous with respect to this metric. See also [13, Corollary 2.1].

2. Complexity of spaces of types

From now on, all types are assumed to be partial and consistent. Following [3] we write $r-s$ for $\max(0, r-s)$. Recall that a subset $A$ of a Polish space $X$ is $\Pi^1_1$ if it is a complement of a continuous image of a Borel subset of a Polish space. A $\Pi^1_1$-set is $\Pi^1_1$-complete if for every $\Pi^1_1$ subset $B$ of a Polish space $Y$ there exists a continuous $f: Y \to X$ such that $B = f^{-1}(A)$. For more information see [22].

Whenever we say that type $t$ is omissible in a model of $T$ it is assumed to be consistent with $T$. 
2.1. **Basic complexity results.** Recall that the space $S_n(T)$ of complete $n$-types over a complete theory $T$ is a compact metric space (§1.2.3) and that the space $S^*_n(T)$ of not necessarily complete $n$-types over $T$ is identified with the compact metric space of its closed subsets (§1.2.5).

**Lemma 2.1.** Assume $M$ is a separable model in a separable language and $n \in \omega$.

1. The set of all complete $n$-types realized in $M$ is $\Sigma^1_1$.
2. The set of all not necessarily complete $n$-types realized in $M$ is $\Sigma^1_1$.

**Proof.**

(1) Let $\hat{M}$ be the space of $L$-models (§1.2.2). For $M \in \hat{M}$ let $d(M, j)$, for $j \in \omega$, be its universe. Define $f: \hat{M} \times (\omega^\omega)^n \to S_n(T) \cup \{\ast\}$ by $f(M, x(0), \ldots, x(n-1)) = \ast$ if sequence $\{d(M, x(i)(j)) : j \in \omega\}$ is not Cauchy for some $i < n$. Otherwise, we let $f(M, x(0), \ldots, x(n-1))$ be the limit of types of $x(0)(j), \ldots, x(n-1)(j)$, for $j \in \omega$, in $M$. This function is Borel and the image of $\{M\} \times (\omega^\omega)^n$ is the set of $n$-types realized in $M$.

(2) The set $A \subseteq S^*_n(T)$ of all types omitted in $M$ is $\Pi^1_1$. We need to show that the set $\{K \in S^*_n(T) : K \subseteq A\}$ is also $\Pi^1_1$. This is standard but we include an argument for the convenience of the reader. The set

$$Z = \{(x, K) \in S_n(T) \times S^*_n(T) : x \in K\}$$

is closed and $K \subseteq A$ if and only if $\forall x)((x, K) \in Z \to x \in A)$, giving the required $\Pi^1_1$ definition. □

If $T$ has a prime model then a type is omissible in a model of $T$ if and only if it is omitted in its prime model, and we have an immediate corollary to (2) above.

**Corollary 2.2.** If $T$ is a complete theory in a separable language with a prime model then the set of types omissible in a model of $T$ is $\Pi^1_1$. □

Note that the set of types realized in $M$ is always dense in the logic topology and therefore if $M$ does not realize all types then this set is not separable. Compare the following with Proposition 1.7.

**Lemma 2.3.** If $T$ is a (not necessarily complete) theory in a separable language, then the set of all types omissible in a model of $T$ is $\Sigma^1_2$.

**Proof.** This is a consequence of (2) of Lemma 2.1. □

The following result (cf. Question 6.1) was inspired by [7].

**Proposition 2.4.** There is a separable language $L$ and a separable $L$-model $M$ such that the set of quantifier-free unary types realized in $M$ is a complete $\Sigma^1_1$ set.

**Proof.** Language $L$ has only one unary predicate symbol $f$, interpreted as a 1-Lipshitz function. Identify $\omega^\omega$ with $[0,1] \setminus \mathbb{Q}$ and consider it with a complete metric such that the identity map from $\omega^\omega$ into $[0,1]$ is 1-Lipshitz. Fix a closed $X \subseteq \omega^\omega \times \omega^\omega$. Consider $X$ with the max-metric induced from $\omega^\omega$. 


and interpret $f$ as the projection to the $x$-axis. Then $f$ is 1-Lipshitz by the choice of metric on $\omega^\omega$.

The only atomic formulas in $L$ are $f(x)$ and $d(x, y)$. If $\phi(x)$ is a quantifier-free formula with only one free variable, then the only atomic subformulas of $\phi$ are of the form $f(x)$ and $d(x, x)$. The latter is identically equal to 0, and therefore the quantifier-free type of an element of $M$ is completely determined by its projection to the $x$-axis. Choosing $X$ so that its projection is a complete analytic set completes the proof. □

2.2. Theory of the Baire space. Let $L_N$ be a language with a single sort $D_1$. The intended interpretation of $D_1$ is $\omega^\omega \sqcup \omega^\omega$. Language $L_N$ is equipped with the following.

1. Constant symbols for all elements of $\omega^{<\omega}$ (we shall identify $t \in \omega^{<\omega}$ with the corresponding constant).
2. Unary function symbols $f_k$ for $k \in \omega$.

The interpretation of each $f_k$ is required to be 1-Lipschitz. Theory $T_N$ is the theory of $L_N$-model $N$ described as follows. The universe of $N$ is the set $\omega^{<\omega} \sqcup \omega^\omega$.

The metric on $\omega^{<\omega} \sqcup \omega^\omega$ is the standard Baire space metric,

$$d(s, t) = 1/(\Delta(s, t) + 1).$$

If $s \subseteq t$ then $d(s, t) = 1/(|s| + 1)$. For $s \in \omega^{<\omega}$ we write $|s| = n$ if $s \in \omega^n$, and for $k \leq |s|$ we denote the initial segment of $s$ with length $k$ by $s \upharpoonright k$.

For $k \in \omega$ function $f_k$ is interpreted as

$$f_k(s) = \begin{cases} s \upharpoonright k & \text{if } k \leq |s| \\ s & \text{if } k > |s|. \end{cases}$$

Clearly $\omega^{<\omega}$ is a dense subset of $D_1^M$ which is closed under all $f_k$.

2.2.1. Well-foundedness. Every model $N$ of $T_N$ has a dense $F_\sigma$ set

$$T_N = \{a \in D_1^N : f_k(a) = a \text{ for some } k\}.$$ 

With the ordering defined by $a \subseteq b$ if and only if $a = f_k(b)$ for some $k$ this set is a tree of height $\omega$. Moreover, elements of $N \setminus T_N$ are in a natural bijective correspondence to branches of this tree, because $f_k(x) = f_k(y)$ for all $k$ implies $d(x, y) = 0$ and therefore $x = y$.

Note that $N$ has a dense subset consisting of elements that are interpretations of constant symbols. Therefore every model $N$ of $T_N$ has $N$ as an elementary submodel, hence $N$ is the prime model of $T_N$.

2.2.2. Type $s_0$ and the standard model. Let us describe a unary type $s_0$ in the expanded language $L_N \cup \{h\}$ such that the only model of the theory of an appropriate expansion of $M$ to $L_N$ omitting $s_0$ is the standard model itself.

To $L_N$ we add the following.

3. Unary function symbol $h$. 


Fix an enumeration \( s_n \), for \( n \in \omega \), of \( \omega^{<\omega} \). Then \( h \) is interpreted as follows (here \( \langle n \rangle \) denotes an element of \( \omega^{<\omega} \) of length 1 whose only digit is \( n \)).

\[
h(x) = \begin{cases} 
  s_n, & \text{if } x = \langle n \rangle \text{ for some } n \\
  f_1(x), & \text{otherwise}.
\end{cases}
\]

Then \( h \) is 3-Lipschitz because \( d(x, y) \leq 1/3 \) implies \( h(x) = f_1(x) = h(y) \). Let

\[
\psi := \sup_{x, y} \inf_{h(y)} d(h(y)).
\]

Then \( \psi^N = 0 \) since \( h \) is a surjection from \( \mathcal{N} \) onto \( \omega^{<\omega} \) and \( \omega^{<\omega} \) is a dense subset of \( \mathcal{N} \). Let \( \mathcal{M} \) be any other model of \( \mathcal{T}_N \). Since \( \mathcal{N} \) is a prime model of this theory, we can identify it with an elementary submodel of \( \mathcal{M} \). There is then \( x \in \mathcal{M} \setminus \mathcal{N} \) such that \( f_1(x) \neq x \) and therefore \( \text{dist}(x, \text{ran}(h^M)) = \text{dist}(x, \mathcal{N}) > 0 \). We conclude that \( \psi \in \mathcal{T}_N \).

Let \( s_0(x) \) be the type consisting of the following conditions.

\[
(4) \quad d(h(x), \langle n \rangle) = 1, \text{ for } n \in \omega.
\]

Then every finite subset of \( s_0 \) is realized in \( \mathcal{N} \) by a large enough \( \langle m \rangle \). On the other hand, \( h[\mathcal{N}] = \{\langle m \rangle : m \in \omega \} \) and therefore \( \mathcal{N} \) omits \( s_0 \). Let \( \mathcal{T}_0 \) be the theory of the expanded model. Since it includes \( \phi_h \), the interpretation of \( h \) in every model of \( \mathcal{T}_0 \) is an injection and therefore every model of \( \mathcal{T}_0 \) which omits \( s_0 \) has \( \omega^{<\omega} \sqcup \omega^\omega \) as the interpretation of \( D_1 \) and is therefore isometrically isomorphic to \( \mathcal{N} \).

### 2.3. \( \Pi_1^1 \)-complete

Fix a complete theory \( \mathcal{T} \) in a separable language. The set of (not necessarily complete) \( n \)-types omissible in a model of \( \mathcal{T} \) is \( \Sigma_2^1 \), by counting quantifiers.

**Theorem 2.5.** There is a complete theory \( \mathcal{T}_2 \) in a separable language \( L_2 \) such that the space of all 2-types \( t \) omissible in a model of \( \mathcal{T}_2 \) is \( \Pi_1^1 \)-complete.

**Proof.** Let \( L_2 \) be a two-sorted language with sorts \( D_1 \) and \( D_2 \), including \( L_N \). The intended interpretation of \( D_2 \) is the space \( T \) of all subtrees of \( \omega^{<\omega} \). Language \( L_2 \) is equipped with the following

\[
(5) \quad \text{constant symbols } S_n, \text{ for } n \in \omega, \text{ for all finite-width subtrees of } \omega^{<\omega} \text{ all of whose branches have eventually zero value.}\]

\[
(6) \quad \text{Binary predicate symbol Elm of sort } D_1 \times D_2.
\]

Let \( T \) denote the space of all subtrees of \( \omega^{<\omega} \). The interpretation of Elm is required to be 1-Lipschitz. Theory \( \mathcal{T}_2 \) is theory of the \( L_2 \)-model \( \mathcal{N}_2 \) described as follows. The universe of \( \mathcal{N}_2 \) is the set \( \omega^{<\omega} \sqcup \omega^\omega \sqcup T \) and \( \mathcal{N}_2 \) is an extension of the \( L_N \)-model \( \mathcal{N} \) as described in [2.2]

Metric on \( T \) is defined as (let \( \delta(a, b) = \min\{k : a \cap k^{\leq k} \neq b \cap k^{\leq k}\} \))

\[
d(a, b) = 1/(\delta(a, b) + 1).
\]

\(^1\)These constant symbols are included only for convenience. Their addition is of no consequence since \( T \) is compact.
Note that $D_2^{\mathcal{K}_2}$ is a compact metric space in which the interpretations of constant symbols from (5) form a countable dense set.

In order to interpret predicate Elm we introduce an auxiliary function $\ell: \omega^{<\omega} \to \omega$ via

$$\ell(t) = \max(|t| \cup \text{range}(t)).$$

Now define Elm on $\omega^{<\omega} \times \mathcal{T}$ via

$$\text{Elm}(t, S) = \begin{cases} 0, & \text{if } t \in S \\ 1/\ell(t), & \text{if } t \not\in S. \end{cases}$$

The predicate Elm is Lipshitz because $d(S, T) \leq 1/k$ and $d(s, t) \leq 1/k$ implies that $\text{Elm}(s, S) \leq 1/m$ iff $\text{Elm}(t, T) \leq 1/m$ for all $m \leq k$. By continuity we have $\text{Elm}(x, S) = 0$ for all $x \in \omega^\omega$ and all $S \in \mathcal{T}$.

Let $\mathcal{T}_2$ be the theory of $\mathcal{N}_2$. Just like in the case of $\mathcal{N} (\mathbb{R}, \leq)$ definable elements of $\mathcal{N}_2$ form a dense subset, and therefore $\mathcal{N}_2$ is a prime model of $\mathcal{T}_2$. By Corollary 2.2 the set of types omissible in a model of $\mathcal{T}$ is $\Pi^1_1$.

Let $S \in \mathcal{T}$. We let $t^S$ be the partial type in $x, y$ of sort $D_1 \times D_2$ consisting of the following conditions

1. $\text{Elm}(f_k(x), y) = 0$ for all $k$.
2. $\text{Elm}(t, y) = 0$ if $t \in S$ and $\text{Elm}(t, y) = 1/\ell(t)$ if $t \not\in S$, for all $t \in \omega^{<\omega}$.
3. $d(S_n, y) = \varepsilon_n$, where $\varepsilon_n = d(S_n, S)$, for all $n$.
4. $|d(f_k(x), y) - 1/(k + 1)| = 0$.

A realization of $t^S$ is a pair $(b, c)$ such that $c$ is a tree whose standard part $c^0 = c \cap \omega^\omega$ satisfies $d(c^0, S) = 0$. Hence $c^0 = S$ and $\{f_k(b) : k \in \omega\} \cap \omega^{<\omega}$ is included in $S$. Clearly map $\mathcal{T} \ni S \mapsto t^S \in S_2^-(\mathcal{T}_2)$ (see §2.5) is continuous.

Since the set of well-founded trees in $\mathcal{T}$ is $\Pi^1_1$-complete, it only remains to check that $t^S$ is omissible in a model of $\mathcal{T}_2$ if and only if $S$ is well-founded.

If $S$ is well-founded then the standard model $\mathcal{N}_2$ of $\mathcal{T}_2$ omits $t^S$. This is because if $(b, a)$ realizes $t^S$ then $a = S$, and therefore $b \in \omega^\omega$ has to be a ‘true’ branch of $S$.

Now assume $S$ is ill-founded and let $b$ be its branch. Let $N$ be a model of $\mathcal{T}_2$. Then $\mathcal{N}_2$ is an elementary submodel of $N$ and since $\mathcal{N}_2$ realizes $t^S$ by $(b, S)$, so does $N$. \hfill \Box

2.4. $\Sigma^1_3$-complete. The following theorem is logically incomparable with Theorem 2.5 since its conclusion stronger but the theory $\mathcal{T}_3$ in it is not required to be complete.

**Theorem 2.6.** There is a theory $\mathcal{T}_3$ in a separable language $L_3$ such that the space of all 1-types $t$ omissible in a model of $\mathcal{T}_3$ is $\Sigma^1_k$-complete.

**Proof.** By Lemma 1.2 it will suffice to show that the space of all triples $(\mathcal{T}, t, s)$ where $\mathcal{T}$ is a consistent $L$-theory and $t$ and $s$ are 1-types consistent with $\mathcal{T}$ and simultaneously omissible in a model of $\mathcal{T}$ is $\Sigma^1_k$-complete. Type $s$ will be $s_0$ as defined in §2.2.2.
Let $\mathcal{T}^2$ denote the space of all subtrees of $\omega^{<\omega} \times \omega^{<\omega}$. For $R \in \mathcal{T}^2$ and $x \in \omega^\omega$ let

$$R_x = \{ s \in \omega^{<\omega} : (s, x \upharpoonright |s|) \in R \}$$

be the projection of $R$. Then the subspace of all $R \in \mathcal{T}^2$ such that for some $x$ the tree $R_x$ is well-founded is a complete $\Sigma^1_1$ set (see [22]).

The language $L_3$ is a three-sorted language with sorts $D_1$, $D_2$ and $D_3$ which includes $L_2$ from Theorem [2.5] and function symbol $h$ as interpreted in [2.2.2]. The intended interpretation of $D_3$ is $\mathcal{T}^2$. In addition to (1)–(5) in $L_3$ have the following.

(11) constant symbol $c$ of sort $D_1$.
(12) constant symbols $R_n$, for $n \in \omega$, for all finite-width subtrees of $\omega^{<\omega} \times \omega^{<\omega}$ all of whose branches have eventually zero value.
(13) Predicate Elm is extended to sort $D_1 \times D_1 \times D_3$.

Theory $T_3$ is the theory of the $L_3$-model $N_3$ described as follows. Its universe is equal to $\omega^{<\omega} \sqcup \omega^\omega \sqcup \mathcal{T} \sqcup \mathcal{T}^2$ and it includes model $N_1$ as defined in the proof of Theorem [2.5].

Metric $d$ on $\mathcal{T}^2$ defined as

$$d(R, S) = \inf \{ 1/k : R \cap (k^{<k})^2 = S \cap (k^{<k})^2 \}$$

turns $\mathcal{T}^2$ into a compact metric space.

Now define Elm on $\omega^{<\omega} \times \omega^{<\omega} \times \mathcal{T}^2$ via

$$\text{Elm}(s, t, R) = \begin{cases} 0, & \text{if } (s, t) \in R \\ 1/(\max(\ell(s), \ell(t))), & \text{if } (s, t) \notin R \end{cases}$$

(the case when $|s| \neq |t|$ seems ok). The extended predicate Elm is Lipshitz because $d(S, T) \leq 1/k$, $d(s_1, t_1) \leq 1/k$ and $d(s_2, t_2) \leq 1/k$ implies that $\text{Elm}(s_1, s_2, S) = 1/m$ iff $\text{Elm}(t_1, t_2, T) = 1/m$ for all $m \leq k$. By continuity we have $\text{Elm}(x, y, S) = 0$ for all $x, y \in \omega^\omega$ and all $S \in \mathcal{T}$.

Let $T_3$ be the theory of $N_3$. Note that $T_3$ is not complete only because it provides no information on interpretation of constant $c$. For a tree $R \in \mathcal{T}^2$ type $t^R(x)$ consists of the following conditions.

(11) $\text{Elm}(f_k(x), f_k(c), R) = 0$ for all $k \in \omega$.
(12) $|d(f_k(x), x) - 1/(k + 1)| = 0$.

Condition (12) assures that $x \in \omega^\omega$ and condition (11) assures that $x$ is a branch of $R_c$.

Again, the map $\mathcal{T}^2 \ni R \mapsto t^R \in S_2^-(T_3)$ (see [1.2.5]) is clearly continuous. We claim that $t^R$ and $s$ are simultaneously omissible if and only if there exists real $a \in \omega^\omega$ such that $R_a$ is well-founded. If there is such a real, then the model of $T_3$ obtained by interpreting $c$ as $a$ omits both $t^R$ and $s$. Now assume there is no such real and let $N$ be a model of $T_3$ in which $s$ is omitted. Then by the choice of $s = s_0$ (2.2.2) the reduct of $N$ to $L_N$ is isometrically isomorphic to $\mathcal{N}$. If $a$ is the interpretation of $c$ in $N$, then $c \in \omega^\omega$. Therefore the tree $R_c$ is ill-founded, and $t^R$ is realized. \hfill \Box
Our study of generic models is motivated by potential applications to operator algebras (see [12], [13], [14] and [15]). Results related to our results from [13] were obtained in [16] and [17], similarly inspired by Keisler’s classic [18]. Both of these papers study a version of Keisler’s forcing adapted to the infinitary version of the logic of metric structures. In the classical situation a type (complete or partial) is omitted in the generic model if and only if it is omissible. In the context of logic of metric structures this statement remains true for complete types (by [19] or Proposition 3.2) but not for partial types (see Corollary 5.3). There are several good sources for model-theoretic forcing in the context of logic of metric structures ([8], [10], [4], [19, Appendix A]). Since the present paper is a companion to [14] meant to be self-contained and accessible to non-logicians, we include some of the basics for the reader’s convenience.

3.0.1. Syntax. Natural definition of the provability relation $\vdash$ is given in [6], where a completeness theorem was proven. For a theory $T$, new constants $\bar{d}$ and conditions $\phi(\bar{x}) < \varepsilon$ and $\psi(\bar{d}) < \delta$ of the same type we have

$$T \cup \{\phi(\bar{x}) < \varepsilon\} \vdash \psi(\bar{x}) < \delta$$

if and only if for every model $M$ of $T$ and every $n$-tuple $\bar{a}$ of elements of $M$ of the same type as $\bar{d}$ one has that $\phi(\bar{a})^M < \varepsilon$ implies $\psi(\bar{a})^M < \delta$.

3.0.2. Forcing notion. Fix a (not necessarily complete) theory $T$ in language $L$ and a set of $L$-formulas $\Sigma$. We postulate a simplifying assumption that $L$ has a single sort with a single domain of quantification. If this is not the case, forcing can be modified by adding an infinite supply of constants for every domain of quantification. We omit straightforward details.

Fix a set $\Sigma$ of $L$-formulas with the following closure properties.

$(\Sigma_1)$ $\Sigma$ includes all quantifier free formulas,

$(\Sigma_2)$ it is closed under taking subformulas and under the change of variables,

$(\Sigma_3)$ if $k \in \omega$ and $\phi_i(\bar{x})$, for $0 \leq i < k$, are in $\Sigma$ and $f: \mathbb{R}^k \to [0, \infty)$ is a continuous function then $f(\phi_0(\bar{x}), \ldots, \psi_{k-1}(\bar{x}))$ is in $\Sigma$.

The most interesting cases are when $\Sigma$ is the set of all quantifier-free formulas and when $\Sigma$ is the set of all formulas. Forcing notion $\mathbb{P}_{T,\Sigma}$ defined below is similar to the ones defined and discussed in detail in [4], [10] and [13], but we sketch definitions for the reader’s convenience.

Let $d_j$, for $j \in \omega$, be a sequence of new constant symbols and let $L^+ = L \cup \{d_j : j \in \omega\}$. If $F$ is an $n$-tuple of natural numbers, then $\bar{d}_F$ denotes the $n$-tuple $(d_i : i \in F)$. Conditions in $\mathbb{P}_{T,\Sigma}$ correspond to open conditions as defined in [14, 1.1.1]. They are triples

$$p = (\psi^p, F^p, \varepsilon^p)$$
(we shall write \((\psi, F, \varepsilon)\) whenever \(p\) is clear from the context) where \(\psi\) is an \(n\)-ary formula in \(\Sigma\), \(F\) is an \(n\)-tuple of natural numbers, \(\varepsilon > 0\), and \(\psi(d_F) < \varepsilon\) is a condition consistent with \(T\). Note that \(\psi(d_F)\) is an \(L^+\)-sentence. We shall write \(\bar{d}^p\) instead of \(d_F\). The ordering on \(\mathbb{P}_{T, \Sigma}\) is defined by

\[
p \geq q \text{ if } F^p \subseteq F^q \text{ and } T \cup \{ \psi^q(d^q) < \varepsilon^q \} \vdash \psi^p(d^p) < \varepsilon^p.
\]

If \(p \geq q\) then we say that \(q\) extends \(p\) or that \(q\) is stronger than \(p\). By Lemma 1.1 every condition is equivalent to some \(p\) such that \(\varepsilon^p = 1\). Conditions \(p\) and \(q\) are incompatible, \(p \not\perp q\), if no condition extends both \(p\) and \(q\).

In the terminology of [21] and [13], if \(\Sigma\) consists of all quantifier-free formulas then \(\mathbb{P}_{T, \Sigma}\) is the Robinson forcing, also known as finite forcing. If \(\Sigma\) consists of all formulas, then \(\mathbb{P}_{T, \Sigma}\) is the infinite forcing. In the latter case, we shall write \(\mathbb{P}_T\) for \(\mathbb{P}_{T, \Sigma}\).

We shall identify condition \(p = (\psi, F, \varepsilon)\) in \(\mathbb{P}_{T, \Sigma}\) with the expression \(\psi(d_F) < \varepsilon\) and use notations \(T + p\) and \(T \cup \{ \psi(d_F^p) < \varepsilon \}\) interchangeably.

A recap of the standard forcing terminology ([21], [25]) is in order. Subset \(G\) of \(\mathbb{P}_{T, \Sigma}\) is a filter if every two elements of \(G\) have a common extension in \(G\). A subset \(D\) of \(\mathbb{P}_{T, \Sigma}\) is dense if every \(q \in \mathbb{P}_{T, \Sigma}\) has an extension in \(D\). It is dense below some \(p \in \mathbb{P}_{T, \Sigma}\) if every \(q \leq p\) has an extension in \(D\). If \(F\) is a family of dense subsets of \(\mathbb{P}_{T, \Sigma}\) then a filter \(G\) is \(F\)-generic if \(G \cap D \neq \emptyset\) for all \(D \in F\). A straightforward diagonalization argument shows that one can always find an \(F\)-generic filter if \(F\) is countable.

For a formula \(\phi(d_F)\) in \(\Sigma\) and \(\varepsilon > 0\) the set

\[
D_{\phi(d_F), \varepsilon} = \{ p \in \mathbb{P}_{T, \Sigma} : (\exists r \in \mathbb{R}) T + p \vdash |\phi(d_F) - r| < \varepsilon \}
\]

is dense in \(\mathbb{P}_{T, \Sigma}\) since every \(p\) can be extended to a condition of the form \(\max(|\psi^p|, |\phi(d_F) - r|)\) for some \(r\). The fact that \(D_{\phi(d_F), \varepsilon}\) is dense in \(\mathbb{P}_{T, \Sigma}\) for all \(\phi\) and \(\varepsilon > 0\) even if \(\phi \notin \Sigma\) is also true (it follows from Cohen’s forcing lemma, see [21] or [25]), but we shall not need it. If \(L\) is separable then \(G\) meets all dense sets of the form \(D_{\phi_j(d_F), \varepsilon}\) if and only if it meets all dense sets of the form \(D_{\phi_j(d_F), 1/k}\) where \(\phi_j(x)\), for \(j \in \omega\), is a set of formulas dense in \(d_\infty\) metric and \(k \in \omega\).

**Lemma 3.1.** If \(T\) is theory in a separable language then \(\mathbb{P}_{T, \Sigma}\) has a countable dense subset, and is therefore equivalent to the standard forcing for adding a Cohen real.

**Proof.** This is a standard continuous functional calculus trick ([11.1.1]). By fixing a countable uniformly dense family of ‘propositional connectives’ of the form \(f : [0, 1]^n \to [0, 1]\) and considering only formulas built from the language of \(T\) and constants \(d_j\), for \(j \in \omega\), using these connectives, one obtains a countable set of formulas \(D\) dense in the uniform metric

\[
d_\infty(\phi, \psi) = \sup_{M, \bar{a}} (|\phi(\bar{a}) - \psi(\bar{a})|^M)
\]

where \(M\) ranges over all models of \(T\) and \(\bar{a}\) ranges over all tuples in \(M\) of the appropriate sort. For each \(m \in \omega\) let \(f_m(t) = \max(t - 1/m, 0)\). Consider
the set $C$ of all conditions of the form $f_m(\phi(\bar{d})) < 1/n$, for $m, n$ in $\omega$ and $\phi \in D$. We claim that this countable set is dense in $P_{T, \Sigma}$. Take a condition $\psi(\bar{d}) < \varepsilon$. Fix $n > 1/\varepsilon$ and let $\phi(\bar{x}) \in D$ be such that $d_{\infty}(\phi, \psi) < 1/(2n)$. Then $f_{2n}(\phi(\bar{d})) < 1/(2n)$ is a condition in $C$ stronger than $\psi(\bar{d}) < \varepsilon$. □

A formula is an $\exists$-formula if it is of the form

$$\sup_x \inf_y \psi(\bar{x}, \bar{y}, \bar{z})$$

where $\psi$ is quantifier-free. Note that in the following we do not need to assume that $T$ is complete.

**Proposition 3.2.** Assume $T$ is an $L$-theory and $\Sigma$ is a set of $L$-formulas satisfying $(\Sigma 1)$–$(\Sigma 3)$. Then there is a family $F$ of dense subsets of $P_{T, \Sigma}$ such that to an $F$-generic filter $G$ one can associate an $L^+$-structure $M_G$ satisfying the following.

1. $M_G$ has the interpretations of $\{d_j : j \in \omega\}$ as a dense subset.
2. If $\Sigma$ is the set of all $L$-formulas then $M_G \models T$.
3. If $\Sigma$ is the set of all quantifier-free $L$-formulas and $T$ is $\exists$-axiomatizable then $M_G \models T$.

If $L$ is separable, then $F$ can be chosen to be countable.

**Proof.** 1. In addition to meeting all $D_{\phi(d), \varepsilon}$ as defined above we need $G$ to meet other dense sets such as

$$E_{\phi(d, x)} = \{p \in P_{T, \Sigma} : T + p \vdash \inf_x \phi(d, x) \geq r \}$$

or $(\exists j)T + p \vdash \phi(d_F, d_j) < r$} as well as dense sets $D_{d, i, j, k}$, $D_{p, j}$, $E_{\phi(d_F, x), k}$ defined below. It is straightforward to check that each of these sets is dense and that separability of $L$ implies there are only countably many relevant dense sets of this form (details are very similar to the ones given in [8], [10], [4], [19, Appendix A]).

We shall construct a countable metric model with the universe $M_G^0 = \{d_j : j \in \omega\}$ such that for every $p \in P_T$ we have

$$p \models \psi^p(d^p) < \varepsilon^p.$$ Model $M_G$ will be metric completion of this countable model.

On $M_G^0$ define metric $d$ as follows.

$$d(d_i, d_j) = r \iff \{p \in P_T : T + p \vdash |d(d_i, d_j) - r| < \varepsilon\} \in G \text{ for all } \varepsilon > 0.$$ Since for all $i, j$ and $k > 0$ the set

$$D_{d, i, j, k} = \{p \in P_{T, \Sigma} : (\exists r)T + p \vdash |d(d_i, d_j) - r| < 1/k\}$$

is dense, if $G$ is a sufficiently generic filter, then this defines a metric on $M_G^0$.

For every $k$-ary predicate symbol $P$ in $L$ one defines interpretation of $P$ as a function from $(M_G^0)^k$ to $\mathbb{R}$ in an analogous manner. The universe of $M$ is the metric completion of $M_G$. For every function symbol $f$ in the language of $T$ one can now define an interpretation of $f$ as a function from $(M_G^0)^k$ (where $k$ is the arity of $f$) into $M$. All predicates and functions obtained in
this way are uniformly continuous since the uniform continuity modulus is built into the language, and we can therefore continuously extend them to predicates and functions on $M$.

(2) Assume $\Sigma$ consists of all $L$-formulas, hence $\mathbb{P}_{T, \Sigma}$ is $\mathbb{P}_T$. We claim that $p \models \psi(d^p) < \varepsilon^p$ for every $p \in \mathbb{P}_T$. This is proved by induction on the complexity of formula $\psi$. The atomic case is immediate from the definition. Assume that the claim is true for all proper subformulas of $\psi$. If $\psi$ is formed by applying a continuous function to other formulas, the claim is immediate (details are similar to those provided in [4] or [13]).

If $p$ is a condition such that $\psi^p$ is of the form $\inf_x \phi(x)$ for some $\phi$, then the set
\[ D_{p,1} = \{ q : \psi^q < \varepsilon^q \mid \psi^p < \varepsilon^p \} \]
is dense, and therefore it is forced that $M_0 \models \psi^p < \varepsilon^p$.

Now assume $\psi$ is of the form $\sup_x \phi(x)$ for some $\phi$. Then for every condition $q$ extending $p$ and every $j \in F^q$ we have that $T + q \models \phi(d_j) < \varepsilon^p$. Moreover, the set
\[ D_{p,2} = \{ q : \psi^q < \varepsilon^q \mid \psi^p < \varepsilon^p - \delta \} \]
is dense in $\mathbb{P}_T$. Therefore $\mathbb{P}_T$ forces that there exists $\delta > 0$ such that for all $j$ the generic theory proves that $\phi(d_j) < \varepsilon^p - \delta$. By the continuity of $\phi^p$ in $M_G$, the conclusion follows.

This concludes proof that the generic model $M_G$ is a model of $\mathbb{P}_T$.

(3) Assume $\Sigma$ includes all quantifier-free formulas. The argument from the proof of (2) shows that for every quantifier-free formula $\phi(\bar{x}, \bar{y})$ such that $\sup_x \inf_y \phi(\bar{x}, \bar{y}) = 0$ is in $T$ for every $F$ and $k > 0$ the set
\[ E_{\phi(\bar{d}_F, x), k} = \{ q \in \mathbb{P}_{T, \Sigma} : (\exists j) T + q \models \phi(\bar{d}_F, d_j) < 1/k \} \]
is dense. If $G$ intersects $E_{\phi(\bar{d}_F, x), k}$ for a $d_{\infty}$-dense set of $\phi$ then $M_G$ satisfies all $\forall \exists$-consequences of $T$. Therefore (3) follows.

Type $t(\bar{x})$ is **non-principal** ([1.3.1]) if there exists $\varepsilon > 0$ such that its metric $\varepsilon$-neighbourhood is nowhere dense in the logic topology. This is equivalent to stating that for every condition $p(\bar{x})$ there exists a stronger condition $q(\bar{x})$ such that in every model $M$ of $T$, every tuple $\bar{a}$ in $M$ of the appropriate sort satisfies (with $t(M) = \{ \bar{b} \in M : M \models t(\bar{b}) \}$) $\text{dist}(\bar{a}, t(M)) \geq \varepsilon$. Note that this also applies to incomplete types, when identified with closed subsets of the space of complete types. If $\Sigma$ is a set of $L$-formulas satisfying (\Sigma1)–(\Sigma3) we say that a type $t(\bar{x})$ is **$\Sigma$-non-principal** if for every condition $p(\bar{x})$ in $\mathbb{P}_{T, \Sigma}$ there exists a stronger condition $q(\bar{x})$ in $\mathbb{P}_{T, \Sigma}$ such that in every model $M$ of $T$, every tuple $\bar{a}$ in $M$ of the appropriate sort satisfies (with $t(M) = \{ \bar{b} \in M : M \models t(\bar{b}) \}$) $\text{dist}(\bar{a}, t(M)) \geq \varepsilon$.

**Proposition 3.3.** Assume $T$ is a complete $L$-theory and $\Sigma$ is a set of $L$-formulas satisfying (\Sigma1)–(\Sigma3). If $t$ is a $\Sigma$-non-principal type then there is
a family \( F_t \) of dense subsets of \( \mathbb{P}_T \) such that if \( G \) is \( \mathbb{P}_t \)-generic then \( M_G \) omits \( t \). If \( L \) is countable then we can choose \( F_t \) to be countable. \( \square \)

Proposition 3.3 implies that a complete type over a complete theory \( T \) is omissible if and only if \( \mathbb{P}_T \) forces that \( M_G \) omits it ([312] or [20 Lecture 4]). Recall that \( \text{cov}(\mathcal{M}) \) denotes the minimal number of sets of first category required to cover the real line (\( \square \)).

**Corollary 3.4.** If \( T \) is a complete theory in a separable language, \( \kappa < \text{cov}(\mathcal{M}) \), and \( t_\gamma \) for \( \gamma < \kappa \) is a set of complete non-principal types over \( T \), then \( T \) has a separable model that omits all \( t_\gamma \).

**Proof.** By Proposition 3.2 and Proposition 3.3 \( \mathbb{P}_T \) forces that \( M_G \) is a model of \( T \) which omits each \( t_\gamma \). Pick a transitive model \( N \) of a large enough fragment of ZFC containing \( T \) and all \( t_\gamma \) for \( \gamma < \kappa \) such that the cardinality of \( N \) is \( \kappa \). Countable dense subset of \( \mathbb{P}_T \) defined in Lemma 3.1 is included in \( N \). Since \( \kappa < \text{cov}(\mathcal{M}) \) and \( \mathbb{P}_T \) is equivalent to the Cohen forcing (\( \square \)) we can choose a filter \( G \subseteq \mathbb{P}_T \) that meets all \( \kappa \) dense subsets of \( \mathbb{P}_T \) that belong to \( N \). We claim that \( M_G \) omits each \( t_\gamma \). This is because the assertion that \( M_G \) omits a fixed type is \( \Pi^1_1 \) and therefore absolute between \( N[G] \) and the universe. \( \square \)

### 3.1 Strong homogeneity of \( \mathbb{P}_{T, \Sigma} \).

Cohen forcing is *homogeneous* in the sense that for any two conditions \( p \) and \( q \) there exists an automorphism \( \Phi \) such that \( \Phi(p) \) is compatible with \( q \). Therefore \( \mathbb{P}_{T, \Sigma} \) is also homogeneous by Lemma 3.1. We shall prove a refinement of this fact in case when \( T \) is complete, showing that automorphism \( \Phi \) can be chosen to preserve the relevant logical structure. Recall that \( S_\infty \) denotes the group of permutations of \( \omega \). To a permutation \( h \in S_\infty \) we associate an automorphism \( \alpha_h \) of \( \mathbb{P}_{T, \Sigma} \) which sends \( d_j \) to \( d_{h(j)} \) for all \( j \in \omega \). More explicitly,

\[
\alpha_h((\psi(x), F, \varepsilon)) := (\psi(x), h(F), \varepsilon).
\]

**Lemma 3.5.** Assume \( T \) is a complete \( L \)-theory and \( \Sigma \) is a set of \( L \)-formulas satisfying (\( \Sigma 1 \))–(\( \Sigma 3 \)). For any two conditions \( p_1 \) and \( p_2 \) in \( \mathbb{P}_{T, \Sigma} \) there is \( h \in S_\infty \) such that \( p_1 \) and \( \alpha_h(p_2) \) are compatible.

**Proof.** Let \( p_j \) be \( (\psi_j, F_j, \varepsilon_j) \) for \( j = 1 \) and \( j = 2 \). We shall write \( d(j) \) and \( x(j) \) for \( d_{F_j} \) and \( x_{F_j} \), respectively. By Lemma 3.4 we may assume \( \varepsilon_1 = \varepsilon_2 = \varepsilon \). Since \( T \) is complete we have \( T \vdash \inf_{\bar{x}(j)} \psi_j(\bar{x}(j)) < \varepsilon \) for \( j = 1,2 \) and therefore

\[
T \vdash \max(\inf_{\bar{x}(1)} \psi_1(\bar{x}(1)), \inf_{\bar{x}(2)} \psi_2(\bar{x}(2))) < \varepsilon.
\]

Let \( h \) be such that \( h[F_2] \) is disjoint from \( F_1 \). Then

\[
q = (\max(\psi_1(\bar{d}(1)), \psi_2(\bar{d}_{h(F_2)})), F_1 \cup h(F_2), \varepsilon)
\]

is a condition in \( \mathbb{P}_{T, \Sigma} \) which extends both \( p_1 \) and \( h(p_2) \). \( \square \)

In the following \( M^0_G \) denotes the countable dense submodel of \( M_G \) as defined in the proof of Proposition 3.2.
Corollary 3.6. Assume $T$ is a complete $L$-theory and $\Sigma$ is a set of $L$-formulas satisfying (Σ1)–(Σ3). If $\Theta(x,y)$ is a statement of ZFC with parameters in the ground model, then $P_{T,\Sigma}$ either forces $\Theta(M_G, M_G^0)$ or it forces $¬\Theta(M_G, M_G^0)$.

Proof. Fix a condition $p$ which decides $\Theta(M_G, M_G^0)$. If $q$ is any other condition then by Lemma 3.5 there exists an $h \in S_\infty$ such that $\alpha_h(p)$ is compatible with $q$. But $\alpha_h$ is an automorphism of $P$ that sends $M_G^0$ to itself and $M_G$ to itself, and therefore $\alpha_h(p)$ forces $\Theta(M_G, M_G^0)$ if and only if $p$ does and $\alpha_h(p)$ forces $¬\Theta(M_G, M_G^0)$ if and only if $p$ does. This implies that every condition in $P$ decides $\Theta(M_G, M_G^0)$ the same way that $p$ does. \qed

3.2. Omitting types in generic model $M_G$. Recall that a model $M$ of a complete theory $T$ is atomic if the set of realizations of principal types is dense in $M$ (see [3] p. 79). This is equivalent to every element of $M$ having a principal type. Recall that $P_T$ denotes the 'infinite' forcing of the form $P_{T,\Sigma}$, the case when $\Sigma$ is the set of all formulas of the language of $T$.

Lemma 3.7. If $T$ is a complete theory in a separable language that has an atomic model $N$ then $P_T$ forces that $M_G \equiv N$.

Proof. It suffices to prove that for every type $t$ we have that it is omissible in a model of $T$ if and only if $P_T$ forces that $M_G$ omits $t$. We claim that $P_T$ forces that for every $\varepsilon > 0$ and every $n$ the tuple $d_1, \ldots, d_n$ is forced to be within $\varepsilon$ of an $n$-tuple realizing a principal type, for every $\delta > 0$ (we are considering the max metric on $N^n$ for definiteness).

Fix $\delta$ and $F$ and a condition $p$ such that $F^p \supseteq \{i : i \leq n\}$. Then $N \models \inf_{x} \psi^p(x) < \varepsilon^p$, and therefore for some $n$-tuple $\bar{a}$ in $N$ we have that $N \models \psi^p(\bar{a}) < \varepsilon^p$. Since $\bar{a}$ realizes a principal type $t$ in $N$, we can find a formula $\phi$ such that in every model $M$ of $T$ for all $\bar{x}$ in $M$ we have that $\phi(\bar{x}) < \delta$ implies there is an $n$-tuple $\bar{c}$ in $M$ satisfying $t$ such that $d(\bar{c}, \bar{x}) < \delta$. Then the condition $\max(\phi, \psi^p) < \min(\varepsilon^p, \delta)$ extends $p$ and forces that $d^F$ is within $\delta$ of a realization of $t$ in $M_G$.

Therefore it is forced that a dense subset of $n$-tuples in $M_G$ realize a principal type. Since a limit of principal types is principal, $M_G$ is forced to be atomic. \qed

Lemma 3.7 can be recast as the assertion that if $T$ has an atomic model then every omissible type is forced to be omitted in the generic model. The assumption that $T$ has an atomic model cannot be dropped from this assertion (Corollary 5.3). The following lemma assumes some proficiency in the forcing language (21).

Lemma 3.8. If $T$ is a theory in a separable language then the set of all types forced by $P_T$ to be omitted in the generic model $M_G$ is a $\Pi^1_1$-set.

Moreover, for every type $t$ we have that $P_T$ either forces $t$ is realized in $M_G$ or it forces that $t$ is omitted in $M_G$. 
Proof. The set of names $\dot{h}$ for a function from $\omega \to \omega$ can be identified with a Borel subset of $\mathbb{P}_0 \times \omega^2$, where $\mathbb{P}_0$ is a fixed countable dense subset of $\mathbb{P}_T$ as in Lemma 3.1.

By Lemma 1.3 type $t$ is forced to be omitted if and only if $t_\omega$ (see §1.1.3) is forced to be omitted by every subsequence of the generic sequence $\{d_j(G) : j \in \omega\}$. This is equivalent to saying that for every name $\dot{h}$ for a function from $\omega \to \omega$, for every $p \in \mathbb{P}_T$ there exists $q \leq p$ and $n$ such that the following holds.

$q$ decides $\dot{h}(i)$ for $i \leq n$ and this $n$-tuple is not an initial segment of a sequence satisfying $t_\omega$.

The latter condition is Borel. Since all quantifiers, except one on $\dot{h}$, range over a countable set this set is $\Pi^1_1$.

The last sentence is an immediate consequence of Corollary 3.6. □

3.3. Forcing with ‘certifying structures’. Let $T$ be a not necessarily complete theory, let $\Sigma$ be a set of formulas in the language of $T$ satisfying closure properties (1)–(3) as in §3.0.2 and let $\mathfrak{M}$ be some nonempty set of models of $T$. Forcing $\mathbb{P}_{T,\Sigma,\mathfrak{M}}$ is defined as follows. Conditions are triples $p = (\psi^p, F^p, \varepsilon^p)$ (we shall write $(\psi, F, \varepsilon)$ whenever $p$ is clear from the context) where $\psi$ is an $n$-ary formula, $F$ is an $n$-tuple of natural numbers, $\varepsilon > 0$, and $\psi(d^F) < \varepsilon$ is a condition satisfied in some model from $\mathfrak{M}$. We shall write $d^p$ instead of $d^F$.

We define a preorder $\leq$ on $\mathbb{P}_{T,\Sigma}$ via

$p \geq q$ if $F^p \subseteq F^q$ and $T \cup \{\psi^q(d^q) < \varepsilon^q\} \vdash \psi^p(d^p) < \varepsilon^p$.

Therefore models in $\mathfrak{M}$ only serve to ‘certify’ conditions in $\mathbb{P}_{T,\Sigma,\mathfrak{M}}$. If $T$ is a complete theory, then every condition consistent with $T$ is certified in every model of $T$ and $\mathbb{P}_{T,\Sigma,\mathfrak{M}}$ is isomorphic to $\mathbb{P}_{T,\Sigma}$ regardless of the choice of $\mathfrak{M}$.

Proofs of the following lemma and proposition are analogous to the proofs of Lemma 3.1 and Proposition 3.2 and are therefore omitted.

Lemma 3.9. If $T$ is a theory in a separable language then $\mathbb{P}_{T,\Sigma,\mathfrak{M}}$ has a countable dense subset, and is therefore equivalent to the Cohen forcing. □

Proposition 3.10. Assume $T$ is a (not necessarily complete) $L$-theory and that either $\Sigma$ consists of all $L$-formulas or that $T$ is $\forall \exists$-axiomatizable and $\Sigma$ includes all quantifier-free formulas. Then forcing $\mathbb{P}_{T,\Sigma,\mathfrak{M}}$ generically adds a model $M_G$ of $T$ which has the interpretations of $\{d_j : j \in \omega\}$ as a dense subset. □

4. Uniform sequences of types

Fix a separable language $L$ throughout this section. For $m \in \omega$ a sequence $t_n$ of $m$-ary types is uniform if there are $m$-ary formulas $\phi_i(\bar{x})$ for $i \in \omega$ such that

$$t_n(\bar{x}) = \{\phi_i(\bar{x}) \geq 2^{-n} : i \in \omega\}$$
for every \( n \) and all \( \phi_i \) have the same modulus of uniform continuity.

If \( t_n \) and \( \phi_i \) are as above, then, since \( \phi_i \) have the same modulus of uniform continuity,

\[
\psi(\bar{x}) := \inf_{i \in \omega} \phi_i(\bar{x})
\]

is an \( L_{\omega_1, \omega} \) formula (see [4]) whose interpretation is uniformly continuous in every \( L \)-structure \( A \) and \( \sup_{\bar{x}} \psi(\bar{x})^A = 0 \) if and only if \( A \) omits all \( t_n \) for \( n \in \omega \). We shall not be using this observation (although Theorem 4.2 may be related to omitting types theorems of [4] stated in terms of \( L_{\omega_1, \omega} \)).

For simplicity of notation in the following we consider a single-sorted language.

**Lemma 4.1.** Assume \( t_n \), for \( n \in \omega \), is a uniform sequence of \( m \)-types in \( L \). If \( M \) is an \( L \)-structure then the set

\[
\{ \bar{a} \in M^m : \text{for all } n, \ t_n \text{ is not realized by } \bar{a} \text{ in } M \}
\]

is a closed subset of \( M^m \). In particular, if \( D \) is a dense subset of \( M \), then all \( t_n \) are omitted in \( M \) if and only if none of them is realized by any \( m \)-tuple of elements of \( D \).

**Proof.** Assume \( t_n \) is realized in \( M \) by a tuple \( \bar{a} \). We shall find an open neighbourhood of \( \bar{a} \) all of whose elements realize \( t_{n+1} \). We have \( \phi_m(\bar{a}) \geq 2^{-n} \) for all \( m \). Since all \( \phi_m \) have the same modulus of uniform continuity we can fix \( \delta > 0 \) small enough so that for every \( \bar{b} \) such that \( \max(d(a_i, b_i)) < \delta \) for all \( i \) implies \( |\phi_m(\bar{b}) - \phi_m(\bar{a})| < 2^{-n-1} \). Every such tuple \( \bar{b} \) realizes \( t_{n+1} \).

The last sentence of lemma follows immediately. \( \square \)

Syntactic characterization of the omissible uniform sequences of types given below is analogous to the syntactic characterization of complete omissible types given in [3]. As Itaï Ben Yaacov and Todor Tsankov pointed out, the set \( X \) of complete types extending a type in a uniform sequence of types is metrically open ([1.2.3]) and therefore by a standard argument (see [3, §12] or [20, Lecture 4]) types in \( X \) are simultaneously omissible iff \( X \) is meager in the logic topology ([1.2.3]). We spell out details of the proof below since we will need a similar argument in case when theory \( T \) is not necessarily complete in Theorem 4.4. Principal types were defined in §1.3.1.

**Theorem 4.2.** Assume \( T \) is a complete theory in a separable language \( L \). If for every \( m \in \omega \) we have a uniform sequence of types

\[
t_n^m = \{ \phi_j^m(\bar{x}) \geq 2^{-n} : j \in \omega \}, \text{ for } n \in \omega,
\]

then the following are equivalent.

1. None of the \( t_n^m \), for \( m, n \in \omega \), is principal.
2. \( T \) has a model omitting all \( t_n^m \), for all \( n \in \omega \).
3. There are no \( \delta > 0, m \in \omega \), and condition \( \psi(\bar{x}) < \epsilon \) such that \( T \vdash \inf_{\bar{x}} \psi(\bar{x}) < \epsilon \) and \( T + \psi(\bar{x}) < \epsilon \vdash \phi_j^m(\bar{x}) \geq \delta \) for every \( j \in \omega \).
Proof. To see that (1) implies (2) assume that none of the $t^n_m$ for $n \in \omega$ is principal. By Proposition 3.3 and Lemma 4.1 one produces a model of $\mathbf{T}$ that omits all $t^n_m$.

Now assume (1) fails and let $m$ and $n$ be such that $t^n_m$ is principal. Since all $\phi^n_j$ have the same modulus of uniform continuity we can find $\delta > 0$ such that $\max_i d(x_i, y_i) < \delta$ implies that $|\phi^n_j(\bar{x}) - \phi^n_j(\bar{y})| \leq 2^{-n-1}$ for all $j$. Let $\psi(\bar{x})$ be a formula such that in a sufficiently saturated model $M$ of $\mathbf{T}$ we have that (denoting the realization of $t^n_m$ is $M$ by $t^n_m(M)$)

$$|\psi(\bar{a})^M - \text{dist}(\bar{a}, t^n_m(M))| < \delta.$$ 

Since $t^n_m$ is a consistent type, this implies that $\mathbf{T} \vdash \inf_x \psi(\bar{x}) < \delta$. It also implies that $\mathbf{T} + \psi(\bar{x}) < \delta \vdash \phi^n_j(\bar{x}) \geq 2^{-n} - 2^{-n-1} \geq 2^{-n-1}$ for all $j$ and therefore (3) fails.

Now assume (3) fails. Then we have condition $\psi(\bar{x}) < \varepsilon$ and $\delta > 0$ such that $\mathbf{T} \vdash \inf_x \psi(\bar{x}) < \varepsilon$ and $\mathbf{T} + \psi(\bar{x}) < \varepsilon \vdash \phi^n_j(\bar{x}) \geq \delta$ for all $j$. If $2^{-n} < \delta$ then clearly every model of $\mathbf{T}$ realizes $t^n_m$, hence (2) fails. \qed

For a separable language $\mathbf{L}$ compact metric topology on the space of $\mathbf{L}$-theories and compact metric topology on the space of $\mathbf{L}$-types were defined in \textbf{1.2.3}. The space of sequences of $\mathbf{L}$-types is considered with the product Borel structure. We have an analogue of Proposition \textbf{1.7}

**Corollary 4.3.** For every $n$ the following sets are Borel.

1. The set of uniform sequences of $n$-types.
2. The set of all pairs $(\mathbf{T}, (t_j : j \in \omega))$ such that $\mathbf{T}$ is a complete theory and $(t_n : n \in \omega)$ is a uniform sequence of types omissible in a model of $\mathbf{T}$.
3. The set of all pairs such that $\mathbf{T}$ is a theory and $(t_j : j \in \omega)$ is a uniform type realized in some model of $\mathbf{T}$.

**Proof.** (1) is clear from the definition.

(2) and (3) follow from Theorem \textbf{4.2} and the proof of Proposition \textbf{1.7}. \qed

We state an extension of Theorem \textbf{4.2} to the case when theory $\mathbf{T}$ is not necessarily complete.

**Theorem 4.4.** Assume $\mathbf{T}$ is a not necessarily complete theory in a separable language. If for every $m \in \omega$ we have a uniform sequence of types

$$t^n_m = \{\phi^n_j(\bar{x}) \geq 2^{-n} : j \in \omega\}, \text{ for } n \in \omega,$$

then the following are equivalent.

1. $\mathbf{T}$ has a model omitting all $t^n_m$, for all $m$ and $n$ in $\omega$.
2. There are no $\delta > 0$, finite $F \subseteq \omega$ and conditions $\psi_m(\bar{x}) < \varepsilon$ for $m \in F$ such that $\mathbf{T} \vdash \inf_x \min_{m \in F} \psi_m(\bar{x}) < \varepsilon$ and $\mathbf{T} + \psi_m(\bar{x}) < \varepsilon \vdash \phi^n_j(\bar{x}) \geq \delta$ for every $j \in \omega$. 
Proof. By Theorem 1.2, it suffices to show that $T$ satisfies (2) if and only if it can be extended to a complete theory that still satisfies (2). Only the direct implication requires a proof and the proof is analogous to the proof in the first order case. Let $\theta_k$, for $k \in \omega$, enumerate a countable dense set of $L$-sentences. We shall use continuous functional calculus (\cite{1.1}). Recall that by Lemma 1.1 for a closed interval $V \subseteq \mathbb{R}$ we have that $\psi$ is innocuous. By repeating this successively for all $k \leq n$ we claim that there exists a closed interval $U_{nk} \subseteq \mathbb{R}$ of diameter at most $2^{-n}$ such that

$$T_n \Vdash \theta_k \in U_{nk}$$

and $T_n$ still satisfies (2). Assume that $T_n$ as required is chosen and fix $k \leq n + 1$. Let $\mathcal{V}$ be a finite cover of $U_{nk}$ by closed intervals of diameter $\leq 2^{-n+1}$. We claim that there exists $V \in \mathcal{V}$ such that $T_n \cup \{\theta_k \in V\}$ (identified with a theory by using the closed case of Lemma 1.1) still satisfies (2). Assume otherwise. By the ‘open’ case of Lemma 1.1 for every $V \in \mathcal{V}$ there are a condition of the form $\psi_V(\bar{x}) < 1$, $k(V)$, and $m(V)$ such that $T_n \Vdash \inf_{\bar{x}} \psi_V(\bar{x}) < 1$ and for every $v \in V$ we have $T_n + \psi_V(\bar{x}) < 1 \Vdash \phi_j^{m(V)}(\bar{x}) \geq 2^{-k(V)}$ for all $j \in \omega$. Since $\mathcal{V}$ is finite, we have that

$$\theta' = \min_{V \in \mathcal{V}} \text{dist}(\theta_k, V)$$

is an $L$-sentence and $T_n \Vdash \theta' = 0$, and therefore with $\psi(\bar{x}) = \min_{v \in V} \psi_V(\bar{x})$ we have that $T_n \Vdash \inf_{\bar{x}} \psi(\bar{x}) < 1$. With $k = \max_{V \in \mathcal{V}} k(V)$ we also have $T_n + \psi(\bar{x}) \Vdash \phi_j^{m(V)}(\bar{x}) \geq 2^{-k}$, contradicting the assumption that $T_n$ satisfies (2).

We can therefore find $V \in \mathcal{V}$ such that adding condition $\theta_k \in V$ to $T$ is innocuous. By repeating this successively for all $k \leq n$ we obtain $T_{n+1}$ as required.

Once all $T_n$ are constructed, theory $T_\infty = \bigcup_n T_n$ is a complete theory that satisfies (2), as required. \qed

Unlike Theorem 1.2, the second clause of Theorem 1.4 allows for the possibility that $T$ has models that omit sequence $t_n^m$, for $n \in \omega$, for every $m$ separately, but no single model of $T$ omits these types. Since $T$ is not assumed to be complete this possibility cannot be ruled out.

4.1. Uniform sequences of types and forcing. A class $\mathcal{M}$ of models is uniformly definable by a sequence of types if there is a set of sequences of uniform types $\langle t_n^m : n \in \omega \rangle$, for $m \in I$ such that $A$ is in $\mathcal{M}$ if and only if it omits all of these types.

If $\Sigma$ is the set of all formulas of the language of $T$ then we denote $\mathbb{P}_{T, \Sigma, \mathcal{M}}$ by $\mathbb{P}_{T, \mathcal{M}}$.

**Proposition 4.5.** Assume $\mathcal{M}$ is a nonempty class of models of a theory $T$. If $t_n$, for $n \in \omega$, is a uniform sequence of types that are omitted in every model in $\mathcal{M}$ then $\mathbb{P}_{T, \mathcal{M}}$ forces that $M_G$ omits all $t_n$. 


Proof. Let $D$ denote the set of interpretations of constants $\{d_j : j \in \omega\}$ in $M_G$. Assume that some condition $p$ forces that a tuple $F \subseteq F^p$ realizes $t_n$ for some $n \in \omega$. But there is $M \in \mathcal{M}$ and a tuple $\vec{a}$ in $M$ of the appropriate sort such that $M \models \psi^p(\vec{a}) < \varepsilon^p$. Since $M$ omits $t_n$, we can extend $p$ to a condition that decides that $F$ does not satisfy some condition in $t_n$, contradicting our assumption on $p$. Proposition now follows by Lemma 4.4. \qed

Theorem 4.2 and Proposition 4.5 are potentially useful because some of the most important properties of C*-algebras are uniformly definable by a sequence of types. This includes nuclearity, nuclear dimension, decomposition rank, and being TAF, AF or UHF ([9], [14]). These types are particularly simple and we include a straightforward technical sharpenings of Proposition 4.5 with an eye to potential applications.

A uniform sequence of types $t_n = \{\phi_j(\vec{x}) \geq 2^{-n} : j \in \omega\}$, for $n \in \omega$, is universal if every $\phi_j(\vec{x})$ is of the form $\inf_y \psi(\vec{y}, \vec{x})$ for some quantifier-free formula $\psi(\vec{y}, \vec{x})$. (By Lemma 1.1 condition of the form $\inf_y \psi(\vec{y}, \vec{x}) \geq 2^{-n}$ is equivalent to a condition of the form $\sup_y \psi'(\vec{y}, \vec{x}) = 0$, and the latter form is commonly recognized as universal.) The following theorem probably follows from the results from [4] but we include it for the reader’s convenience.

Proposition 4.6. Assume $\mathcal{M}$ is a nonempty class of models of a theory $\mathbf{T}$ and let $\Sigma$ be the set of all quantifier-free formulas of the language of $\mathbf{T}$. If $t_n$, for $n \in \omega$, is a uniform sequence of universal types that are omitted in every model in $\mathcal{M}$ then $\mathbb{P}_{\mathbf{T}, \Sigma, \mathcal{M}}$ forces that $M_G$ omits all $t_n$.

Proof. Let $G \subseteq \mathbb{P}_{\mathbf{T}, \Sigma, \mathcal{M}}$ be generic and let $D$ be as in the proof of Proposition 4.5. Assume that some condition $p$ forces that a tuple $F \subseteq F^p$ realizes $t_n$ for some $n \in \omega$. But there is $M \in \mathcal{M}$ and a tuple $\vec{a}$ in $M$ of the appropriate sort such that $M \models \psi^p(\vec{a}) < \varepsilon^p$. Since $M$ omits $t_n$, there exists a condition $\inf_y \psi(\vec{y}, \vec{x}) \geq 2^{-n}$ in $t_n$ such that $\psi$ is quantifier-free and $\inf_y \psi(\vec{y}, \vec{a})^M < 2^{-n} - \varepsilon$ for some $\varepsilon > 0$. If $F'$ is disjoint from $F$ and of cardinality $|\vec{y}|$, by the open case of Lemma 1.1 we have that and $\psi(\vec{d}_F, \vec{d}_F) < 2^{-n} - \varepsilon$ is equivalent to a condition in $\mathbb{P}_{\mathbf{T}, \Sigma, \mathcal{M}}$. Then $M$ certifies that this condition is compatible with $p$, and it decides that $d_F$ does not realize $t_n$. Proposition now follows by Lemma 4.4. \qed

If $\mathcal{M}$ is a class of L-models then $M \in \mathcal{M}$ is existentially closed for $\mathcal{M}$ if whenever $N \in \mathcal{M}$ is such that $M$ is isomorphic to a submodel of $N$, $\vec{a} \in M$, and $\phi(\vec{x}, \vec{y})$ is a quantifier-free $L$-formula then

$$\inf_{\vec{y}} \phi(\vec{a}, \vec{y})^M = \inf_{\vec{y}} \phi(\vec{a}, \vec{y})^N.$$

Existentially closed C*-algebras and $\Pi_1$ factors were studied in [19] and [13], respectively.

Corollary 4.7. Assume $\mathbf{T}$ is an $\forall \exists$-axiomatizable theory in a separable language and $t_n$, for $n \in \omega$, is a uniform sequence of universal types. If the class $\mathcal{M}$ of all models of $\mathbf{T}$ that omit all $t_n$, for $n \in \omega$, is nonempty then it contains a model that is existentially closed for $\mathcal{M}$. \qed
Proof. By [19, Lemma A.4] the generic model is in \( \mathcal{M} \), and by [19, Lemma A.7] it is existentially closed. \( \square \)

5. Simultaneous omission of types

We prove Theorem 2 by constructing an example of a separable complete theory \( T \) and types \( s_n \), for \( n \in \omega \), such that for every \( k \) there exists a model of \( T \) that omits all \( s_n \) for \( n \leq k \) but no model of \( T \) simultaneously omits all \( s_n \).

In what follows type \( s_0 \) will be denoted by \( t \). We shall prove that all other types \( s_n \), for \( n > 0 \), are simultaneously omissible in a single model \( M \) of \( T \). As a matter of fact, we shall first define \( M \). Let \( \text{rank}(T) \) denote the rank of a well-founded tree \( T \) and let \( \rho_T \) denote the rank function on \( T \). Hence \( \text{rank}(T) = \sup_{t \in T} \rho_T(t) \). We write \( \text{rank}(T) = \infty \) if \( T \) is ill-founded.

For a sequence \( T_i \) for \( i \in \omega \) of trees we denote their disjoint sum by \( \bigoplus T_i \). Thus \( \text{rank}(\bigoplus T_i) = \sup_i \text{rank}(T_i) \). We write \( \bigoplus \omega T \) for \( \bigoplus T_i \) if all \( T_i \) are equal to \( T \).

If \( S \) and \( T \) are trees then \( S \triangleright T \) denotes the tree obtained by adding \( \omega \) copies of \( T \) to every node of \( S \). Formally, we identify \( S \) and \( \bigoplus \omega T \) with trees of finite sequences from a large enough set and with \( s \triangleright t \) denoting the concatenation of \( s \) and \( t \) we let (assuming that \( S \) and \( \bigoplus T \) are disjoint)

\[
S \triangleright T = \{ s \triangleright t : s \in S, t \in \bigoplus \omega T \}
\]

with the natural end-extension ordering. Then

\[
\text{rank}(S \triangleright T) = \text{rank}(S) + \text{rank}(T).
\]

For a countable ordinal \( \alpha \) let \( T_\alpha \) denote the tree of all strictly decreasing sequences of ordinals \( < \alpha \):

\[
T_\alpha = \{ s : n \to \alpha : n \in \omega, s(i) > s(i + 1) \text{ for } 0 \leq i < n - 1 \}
\]

ordered by the extension. Hence \( T_\alpha \) is a well-founded tree of rank \( \alpha \).

Also let \( T_{\alpha, \omega} \) be the ‘wider’ version of \( T_\alpha \) defined as (a function \( s : n \to \alpha \times \omega \) is identified with a pair of functions \( s_0 : n \to \alpha \) and \( s_1 : n \to \omega \)):

\[
T_{\alpha, \omega} = \{ s : n \to \alpha \times \omega : n \in \omega, s_0(i) > s_0(i + 1) \text{ for } 0 \leq i < n - 1 \}
\]

This is also a well-founded tree of rank \( \alpha \). For every node \( t \) in \( T_{\alpha, \omega} \) and every immediate successor \( s \) of \( t \) there are infinitely many immediate successors \( s' \) of \( t \) such that there is an automorphism of \( T_{\alpha, \omega} \) swapping \( s \) and \( s' \).

Consider language \( L_t \) in the logic of metric structures that includes the language \( L_N \) of Baire space as defined in \[22\] and has unary predicate symbols \( P_{i,j} \) for \( i, j \in \omega \). For an ordinal \( \alpha \) and

\[
\hat{k} = (k_s \in \omega, \text{ for } s \in T_{\alpha, \omega}),
\]

define a model \( M = M(\alpha, \hat{k}) \) of \( L_t \) as follows. The underlying set is the tree \( T_{\alpha, \omega} \). Function symbols \( f_k \) for \( k \in \omega \) are interpreted as in \[22\] by \( f_k(a) = b \) if \( b \) is the unique element of the \( k \)-th level below \( a \) if there is such
b or $f_k(a) = a$ otherwise. Predicates $P_{i,j}$ are interpreted as follows (here $t^-$ stands for the immediate predecessor of $t \in T_{\alpha,\omega}$ and $t = (t_0, t_1)$)

$$P_{i,j}(t) = \begin{cases} 0, & \text{if } |t| = i \text{ and } j - k_t^- = t_1(i - 1) \\ 1, & \text{otherwise.} \end{cases}$$

Therefore $k_t$ is the immediate successors of $t$ assume all colours $P_{i,j}$ for $i = |t|$ and $j \geq k_t$, but no other colours.

Fix a countable indexed family $\bar{k}^i$ for $i \in \omega$ of functions from $T_{\omega,\omega}$ into $\omega$ such that every function from a finite subset of $T_{\omega,\omega}$ into $\omega$ is extended by some $\bar{k}^i$. We define model $M$ to be

$$\bigoplus M(\omega, \bar{k}^n) \sim T_{\omega}.$$ 

The bottom part of $M$ is $\bigoplus M(\omega, \bar{k}^n)$ and the top part of $M$ consists of copies of $T_{\omega}$ added to all nodes of the bottom part. Bottom part is taken with all of its structure, in particular predicates $P_{i,j}$ are interpreted in the natural way. For all $i, j$ and all nodes $t$ of the top part we let $P_{i,j}(t) = 1$.

Then each $P_{i,j}$ has interpretation that is $i$-Lipschitz.

Let $s_0(x)$ be the 1-type of an infinite branch of the tree underlying this model. That is, it consists of all conditions of the form $d(x, f_m(x)) = 1/n$ for $n \in \omega$.

Type $s_m(x,y)$ is realized in the canonical model $M$ by any pair $a,b$ such that $a$ is a terminal node of the bottom part of $M$ which belongs to its $m$th level and $b$ is its immediate successor.

In symbols, for $m > 0$ type $s_m(x,y)$ asserts the following.

(1) $x$ is on the $m$th level of the tree: $d(f_m(x), x) = 0$ and $d(f_{m-1}(x), x) = 1/m$.

(2) On every level of the tree there is a node above $x$. This can be expressed by formulas $\phi_k(x)$ for $m < k$ stating that there exists $z$ on the $k$th level such that $f_m(z) = x$.

(3) $|y| = m + 1$ and $x = f_m(y)$: $d(f_m(y), x) = 0$, $d(x, y) = 1/m$ and $d(f_{m+1}(y), y) = 0$.

(4) $P_{m+1,n}(y) = 1$ for all $m,n$.

We shall prove that $s_m(x,y)$ is generically omissible. The following is the key lemma towards this goal. The set of the realizations of $P_{m,n}$ is clopen in every model because $P_{m,n}(y) = 0$ is equivalent to $P_{m,n}(y) < 1/2$.

In the following $P_T$ denotes the forcing defined in §3

**Lemma 5.1.** Assume $\phi(\bar{x}, y, z) < \varepsilon$ is a condition in $P_T$ such that, together with $T$, it implies the following statements.

(a) $P_{m,n}(y) = 0$ for some $m$ and $n$,

(b) $z$ is an immediate successor of $y$.

Then this condition can be extended to a condition that in addition implies
(c) $P_{m+1,k}(z) = 0$ for some $k$.

Proof. We can assume that $\phi$ is in the prenex normal form since such formulas are uniformly dense by Proposition 6.9]. Since $\phi(\bar{x}, y, z) < \varepsilon$ is a consistent condition, we can find a tuple $\bar{a}, b, c$ in $M$ that realizes it. Let

$$\varepsilon' := (\varepsilon - \phi(\bar{a}, b, c)^M)/3.$$ 

Let $\psi_j(\bar{x}, y, z, t)$, for $j < m$, be the list of all atomic subformulas of $\phi(\bar{x}, y, z)$. Thus $\phi(\bar{x}, y, z)$ is of the form (variables in $\psi_j$ suppressed for readability)

$$Qt_1Qt_2 \ldots Qt_l f(\psi_0, \ldots, \psi_{m-1})$$

for some continuous function $f$. Since the interpretation of $f(\psi_0, \ldots, \psi_{m-1})$ is uniformly continuous and its modulus of continuity does not depend on the interpretation, we can find $\delta > 0$ such that changing values of all variables occurring in any $\psi_j$ by $< \delta$ affects the change of the value of $\phi(\bar{x}, y, z)$ by $< \varepsilon'$. Let $l > 1/\delta$ be such that all pairs $i, j$ for which predicate $P_{i,j}$ occurs in some $\psi_j(\bar{x}, y, z)$ satisfy $\max(i, j) < l$. By increasing $l$ we may also assume that all projection functions $f_i$ occurring in some $\psi_j(\bar{x}, y, z)$ satisfy $i < l$ and that $\bar{a}, b, c$ belong to one of the first $l$ levels of $M$. Let $L^-$ be the reduct of $L_t$ to the language containing only $P_{i,j}$ for $\max(i, j) < l$ and $g_i$ for $i < l$.

Let $M \upharpoonright l$ be the submodel of the $L^-$-reduct of $M$ consisting only of its first $l$ levels.

If $\bar{d}$ is any node in the intersection of $M \upharpoonright l$ and the bottom part of $M$, then its rank in $M$ is at least $\omega$. Moreover, for any two such nodes $\bar{d}$ and $\bar{d}'$ on the same level of $M$ the trees $\{e \in M \upharpoonright l : d \sqsubseteq e\}$ and $\{e \in M \upharpoonright l : d' \sqsubseteq e\}$ are isomorphic. Whether an isomorphism between these trees extends to an automorphism of $M \upharpoonright l$ depends only on whether the $P_{i,j}$ labels are matched.

We claim that for any tuple $\bar{p}, d, e$ in $M \upharpoonright l$ of the same sort as $\bar{x}, y, z$ we have that

$$|\phi(\bar{p}, d, e)^M|l - \phi(\bar{p}, d, e)^M| < \varepsilon'.$$

Let us prove this. For every tuple $\bar{q}$ in $M$ such that $\bar{p}, d, e, \bar{q}$ is of the same sort as $\bar{x}, y, z$ (where $\bar{t}$ are variables occurring freely in formulas $\psi_j$ but not in $\phi$), we have $d((t_i, f_i(t_i))) < \delta$ for all $i$. Since $f_i(t_i) \in M \upharpoonright l$, and of course $\bar{p}, d,$ and $e$ are in $M \upharpoonright l$, this means that for every choice of values in $M$ for variables in the body of $\phi$ there is a choice of values for these variables in $M \upharpoonright l$ at the distance $< \delta$. Now (1) follows easily by the argument from the first few lines of [16] Lemma 1.8.

Let us write

$$N_p := M(\omega, \bar{k}^p) \sim T_\omega,$$

considered as a submodel of $M$. These submodels of $M$ are hereditary (at least when turned upside down) as the sets of successors of a node $s \in N_p$ in $M$ and $N_p$ coincide.

Recall that we have previously fixed $\bar{a}, b, c$ such that $\phi(\bar{a}, b, c)^M < \varepsilon$. Since $P_{m+1,n}(b) = 0$, $b$ belongs to the bottom part of $M$. Let $r$ be such that $b \in N_r$. 


Now take a look at $M \upharpoonright l$. We have that $c$ is an immediate successor of $b$. If $P_{m+1,j}(c) = 0$ for some $j < l$, then all of the above work was unnecessary and we can extend condition $\phi$ as required so that it implies $f_{m+1}(z) = z$, $f_m(z) = y$ and $P_{m+1,j}(z) = 0$.

Now assume a weaker condition, that $P_{m+1,j}(e) = 0$ for some $j$ and some immediate successor $e$ of $b$. Then $b$ is not a terminal node of the bottom part of $M$. We can choose $j > m$ and consider an automorphism of $M \upharpoonright l$ that swaps $c$ with an immediate successor $e$ of $b$ satisfying $P_{m+1,j}(e) = 0$ in $M$. Then we can extend condition $\phi$ as required so that it implies $f_{m+1}(z) = z$, $f_m(z) = y$ and $P_{m+1,j}(z) = 0$.

It only remains to consider the case when none of the immediate successors of $b$ in $M \upharpoonright l$ satisfies any of the $P_{m+1,j}$ for $j < l$. Choose $p$ such that $k^p$ and $k^r$ agree everywhere except that $k^p_0 > l$. The map $\theta$ that swaps $N_p \upharpoonright l$ and $N_r \upharpoonright l$ and keeps all other $N_q \upharpoonright l$ fixed does not move any of the labelled nodes and is therefore an automorphism of $M \upharpoonright l$. By applying (11) twice we obtain

$$|\phi(\bar{a}, b, c)^M - \phi(\theta(\bar{a}), \theta(b), \theta(c))^M| < 2\varepsilon$$

and in particular $\phi(\theta(\bar{a}), \theta(b), \theta(c)) < \varepsilon$. Now we can choose $j > k^p_0 > l$ and an immediate successor $c'$ of $\theta(b)$ such that $c'$ and $\theta(c)$ have the same $L^-$-type over $\bar{a}$ and $b$ but $P_{m+1,j}(c') = 0$. Therefore the condition (assuming $\varepsilon < 1$ for simplicity)

$$\max(\phi(\bar{x}, y), P_{m+1,j}(z)) < \varepsilon$$

is satisfied in $M$ by $\theta(\bar{a}), \theta(b), c'$. Now we can extend our condition as required and complete the proof. \qed

**Lemma 5.2.** For every $m$ type $s_m(x, y)$ is generically omissible. That is, $\mathbb{P}_T$ forces that $s_m(x, y)$ is omitted in generic model $M_G$.

**Proof.** Let $\psi(\bar{x}, y, z) < \varepsilon$ be a condition. It will suffice to find an extension of this condition which implies one of the following:

(a) For some $k$ there is no extension of $y$ to the $k$-th level.

(b) For some $m$ and $n$ it implies $P_{m+1,n}(z) = 0$.

(c) $z$ is not an immediate successor of $y$.

Since $\psi(\bar{x}, y, z) < \varepsilon$ is a consistent condition, we can find a tuple $\bar{a}, b, c$ in $M$ that realizes it. If $b$ has the property that for some $k$ the $k$-th level of the tree is empty above it, then we can easily extend our condition to a condition satisfying (a). Similarly, if $c$ is not an immediate successor of $b$ then we can easily extend our condition to one that satisfies (c).

We may therefore assume that in every interpretation of our condition all levels of the tree above the interpretation of $z$ are nonempty. We first extend the condition to decide the level $m$ to which $y$ belongs and that $z$ is an immediate successor of $y$. By applying Lemma 5.1 at most $m$ times we can extend our condition by adding variables $z_0, \ldots, z_{m-1} = z$ so that $P_{m+1,n(i)}(z_i) = 0$ for $n(i), i < m$. This implies $P_{m+1,n}(z) = 0$ for some $n$. \qed
Fix \( l \in \omega \). Let \( M(\omega, k, l) \) be the submodel of \( M(\omega, \bar{k}) \) obtained by pruning all nodes \( s \) that have no extensions to the \( l \)th level. We define model \( M(l) \) to be
\[
\bigoplus_n M(\omega, \bar{k}^n, l)^{-T_\omega}.
\]
Then \( M(l) \) omits \( s_m \) for all \( m < l \) because the bottom part of \( M(l) \) has no terminal nodes.

For all \( l, k \in \omega \) and every finite subset \( L_0 \) of language \( L \) the \( L_0 \)-reducts of \( M(l) \upharpoonright k \) and \( M \upharpoonright k \) consisting of the first \( k \) levels of the corresponding trees are isomorphic. They are also \( 1/k \)-dense in \( L_0 \)-reducts of \( M(l) \) and \( M \) respectively. Lemma \[\text{L5}\] therefore implies that every \( M(l) \) is elementarily equivalent to \( M \).

**Proof of Theorem 2.** We shall prove that \( T \) and types \( s_n \), for \( n \in \omega \) as defined above are such that \( T \) is complete and such that for every \( k \) there exists a model of \( T \) that omits all \( s_n \) for \( n \leq k \) but no model of \( T \) simultaneously omits all \( s_n \).

Assume \( N \) is an \( L_T \)-model elementarily equivalent to \( M \). Then by elementarity every ‘named’ node in \( N \) has the property that arbitrarily high levels of \( N \) above this node are nonempty. Now assume \( N \) omits \( s_m(x, y) \) for all sufficiently large \( m \). Then the ‘bottom part’ of \( N \) (consisting of all nodes labelled by some \( P_{m,j} \)) is ill-founded and therefore \( N \) realizes \( t(x) \). Therefore Lemma \[\text{L2}\] implies that \( N \) omits all \( s_m \) and realizes \( t \).

Since \( M(l) \) omits \( t \) and all \( s_m \) for \( m < l \), and it is a model of \( T \) by the above, this concludes the proof.

**Corollary 5.3.** There exists a complete theory \( T \) in a separable language and a type \( t(x) \) which is omissible in a model of \( T \) but \( P_T \) forces that \( MG \) realizes \( t \).

**Proof.** Types that are forced to be omitted in the generic model are simultaneously omitted in a sufficiently generic model. Therefore the existence of such type is an immediate consequence of Theorem 2. Moreover, type \( t \) from its proof clearly has this property.

\[\square\]

**6. Concluding remarks**

The original motivation for this study came from model-theoretic study of C*-algebras. The answers to some of the most prominent open problems in the theory of C*-algebras depend on whether nuclear C*-algebras can be constructed in a novel way.

While C*-algebras are axiomatizable in logic of metric structures, essentially none of the important classes of C*-algebras is axiomatizable \([15]\). C*-algebras which are UHF, AF \([9]\), nuclear, of finite nuclear dimension, of finite decomposition rank, or TAF \([14]\) are uniformly definable by a sequence of universal types. Therefore Theorem \[\text{L2}\] Theorem \[\text{L4}\] Proposition \[\text{L5}\] Proposition \[\text{L6}\] and Corollary \[\text{L7}\] open the possibility of constructing C*-algebras with prescribed first-order properties in these classes.
According to [8] and [10, Definition 4.12], a type $t(\bar{x})$ is metrically principal over a theory $T$ (we consider the case when $L$ is the fragment consisting of all finitary sentences) if and only if for every $\delta > 0$ the type $t^{\delta}(\bar{x})$, asserting that every finite subset of $t$ is realizable by an $n$-tuple within $\delta$ of $\bar{x}$, is principal.

For example, type $t$ defined in the proof of Proposition 2.5 is metrically principal over $T$ if the tree $S$ has height $\omega$. This is because $t_{i/n}$ is realized by any node of $S$ that is not and end-node. This gives an example of an omissible metrically principal type. A simple argument shows that a complete metrically principal type cannot be omissible.

Some fundamental questions about omitting types in logic of metric structures not resolved in the present paper are the following.

Question 6.1. Is there a separable model $M$ in a separable language such that the set of types realized in $M$ is a complete $\Sigma^1_1$-set?

Question 6.2. Are there a complete theory $T$ in a separable language and two types $t$ and $s$ each of which is omissible in a model of $T$ that are not simultaneously omissible in a model of $T$?

If both $t$ and $s$ are complete, then the negative answer easily follows from the Omitting Types Theorem of [3] (see also Corollary 3.4). We don’t even know what is the answer to Question 6.2 with the additional assumption that one of the types $t$ and $s$ is complete. By the types pairing argument (Lemma 1.2) Question 6.2 has a positive answer if and only if there are a complete theory $T$ and $n \geq 2$ such that there are $n$ types $s_0, \ldots, s_{n-1}$ omissible in a model of $T$ that are not simultaneously omissible in a model of $T$. Thus in a sense Theorem 2 is the sharpest result of this form that falls short of answering Question 6.2.

Question 6.3. Is there a complete theory $T$ in a separable language such that the set of types omissible in a model of $T$ is $\Sigma^1_2$-complete? More generally, what are possible complexities of the set of types omissible in a model of such $T$?

Important notions of the first-order model theory, such as stability and $\omega$-stability, can be expressed in terms of cardinalities of sets of types. This carries over to logic of metric structures (see [3], [15, §5]). Complexity of the set of not necessarily complete types over a given theory may give some information about theories in logic of metric structures. In particular, it is plausible that in case when $T$ is natural theory (e.g., theory of a C*-algebra) the set of types omissible in a model of $T$ is Borel.

References

[1] T. Bartoszynski and H. Judah. *Set theory: on the structure of the real line*. A.K. Peters, 1995.
[2] I. Ben Yaacov. Definability of groups in $\aleph_0$-stable metric structures. *J. Symbolic Logic*, 75:817–840, 2010.

[3] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov. Model theory for metric structures. In Z. Chatzidakis et al., editors, *Model Theory with Applications to Algebra and Analysis, Vol. II*, number 350 in London Math. Soc. Lecture Notes Series, pages 315–427. Cambridge University Press, 2008.

[4] I. Ben Yaacov and J. Iovino. Model theoretic forcing in analysis. *Annals of Pure and Applied Logic*, 158(3):163–174, 2009.

[5] I. Ben Yaacov, A. Nies, and T. Tsankov. A lopez-escobar theorem for continuous logic. *arXiv preprint arXiv:1407.7102*, 2014.

[6] I. Ben Yaacov and A. P. Pedersen. A proof of completeness for continuous first-order logic. *Journal of Symbolic Logic*, 75(1):168–190, 2010.

[7] T. Bice. A brief note on omitting partial types in continuous model theory. preprint, 2012.

[8] X. Caicedo and J. Iovino. Omitting uncountable types and the strength of $[0, 1]$-valued logics. *Annals of Pure and Applied Logic*, 165(6):1169–1200, 2014.

[9] K. Carlson, E. Cheung, I. Farah, A. Gerhardt-Bourke, B. Hart, L. Mezuman, N. Squeira, and A. Sherman. Omitting types and AF algebras. *Arch. Math. Logic*, 53:157–169, 2014.

[10] C. Eagle. Omitting types for infinitary $[0, 1]$-valued logic. *Annals of Pure and Applied Logic*, 165:913–932, 2014.

[11] G.A. Elliott, I. Farah, V. Paulsen, C. Rosendal, A.S. Toms, and A. Törnquist. The isomorphism relation of separable C*-algebras. *Math. Res. Letters*, 20:1071–1080, 2013.

[12] I. Farah. Logic and operator algebras. In *Proceedings of the Seoul ICM*. 2014. arXiv:1404.4978.

[13] I. Farah, I. Goldbring, B. Hart, and D. Sherman. Existentially closed II$_1$ factors. arXiv preprint arXiv:1310.5138, 2013.

[14] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter. Model theory of nuclear C*-algebras. 2014.

[15] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras II: Model theory. *Israel J. Math.*, 201:477–505, 2014.

[16] I. Farah and S. Shelah. Rigidity of continuous quotients. *J. Math. Inst. Jussieu*, to appear. arXiv preprint arXiv:1401.6689.

[17] I. Farah, A.S. Toms, and A. Törnquist. The descriptive set theory of C*-algebra invariants. *Int. Math. Res. Notices*, 22:5196–5226, 2013. Appendix with C. Eckhardt.

[18] S. Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[19] I. Goldbring and T. Sinclair. On Kirchberg's embedding problem. arXiv:1404.1861, 2014.

[20] B. Hart. *Continuous model theory and its applications*. 2012. Course notes, available at http://www.math.mcmaster.ca/~bradd/courses/math712/index.html.

[21] W. Hodges. *Building models by games*. Courier Dover Publications, 2006.

[22] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate texts in mathematics*. Springer, 1995.

[23] H. J. Keisler. Forcing and the omitting types theorem. In M. Morley, editor, *Studies in Model Theory*, volume 8 of *Studies in Mathematics*, pages 96–133. Math. Assoc. Amer., 1973.

[24] K. Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland, 1980.

[25] R. Schindler. *Set theory: exploring independence and truth*. Springer, 2014.
