Extremal boundedness of a variational functional in point vortex mean field theory associated with probability measures

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Abstract

We study a variational functional of Trudinger-Moser type associated with one-sided Borel probability measure. Its boundedness at the extremal parameter holds when the residual vanishing occurs. In the proof we use a variant of the Y.Y. Li estimate.

1 Introduction

The purpose of the present paper is to study the boundedness of a variational function concerning the mean field limit of many point vortices [21]. This limit takes the form

\[-\Delta v = \lambda \int_I \alpha \left( \frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) \text{ on } \Omega, \quad \int_\Omega v = 0, \quad (1.1)\]

where \(\Omega = (\Omega, g)\) is a compact and orientable Riemannian surface without boundary in dimension two, \(dx\) a volume element on \(\Omega\), and \(|\Omega|\) the volume of \(\Omega\): \(|\Omega| = \int_\Omega dx\). The unknown variable \(v\) stands for the stream function of the fluid and \(\mathcal{P} = \mathcal{P}(d\alpha)\) a Borel probability measure on \(I = [-1, 1]\), representing a deterministic distribution of the circulation of vortices.

Single circulation is described by \(\mathcal{P} = \delta_{+1}\). In this simplest case, equation (1.1) is sometimes called the mean field equation. Since Onsager’s pioneering work of statistical mechanics on two-dimensional equilibrium turbulence [17], there are numerous mathematical and physical references in this case (see, for instance, [21, 26] and the references therein). Also, the other model \(\mathcal{P} = (1 - \tau)\delta_{-1} + \tau\delta_{+1}, 0 < \tau < 1\), is concerned with signed vortices [9, 18]. Equation (1.1) is thus regarded as a generalization of these cases.

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There, the deterministic distribution of circulations is described by $P(d\alpha)$. Several works are already devoted to equation (1.1), particularly, when $P$ is atomic \cite{7, 10, 15, 16}, i.e.,

$$P = \sum_{i=1}^{N} b_i \delta_{\alpha_i}, \quad \alpha_i \in I, \quad b_i > 0, \quad \sum_{i=1}^{N} b_i = 1.$$ (1.2)

Actually, this model is equivalent to the Liouville system studied by \cite{4, 6, 23}. We note that L. Onsager himself arrived at (1.1) for (1.2), see \cite{8}.

Model (1.1) is the Euler-Lagrange equation of the functional

$$J_{\lambda}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \lambda \int_{I} \log \left( \int_{\Omega} e^{\alpha v} \right) P(d\alpha), \quad v \in E,$$

where

$$E = \{ v \in H^1(\Omega) \mid \int_{\Omega} v = 0 \}.$$

Hence it may be the first step to clarify its boundedness to study (1.1) for general $P(d\alpha)$. It is a kind of the Trudinger-Moser inequality,

$$\inf_{v \in E} J_{\lambda}(v) > -\infty. \quad (1.3)$$

In the atomic case of (1.2), the best constant of $\lambda$ for (1.3) is known \cite{23}. This result is originally described in the dual form of logarithmic HLS inequality (see \cite{15} for (1.3)). Taking the limit, we can detect the extremal parameter $\lambda = \bar{\lambda}$ for (1.3) to hold \cite{20}, that is,

$$\bar{\lambda} = \inf \left\{ \frac{8\pi P(K_{\pm})}{\left( \int_{K_{\pm}} \alpha P(d\alpha) \right)^2} \right\}, \quad K_{\pm} \subset I_{\pm} \cap \text{supp } P, \quad (1.4)$$

where

$$I_{+} = [0, 1], \quad I_{-} = [-1, 0]$$

and

$$\text{supp } P = \{ \alpha \in I \mid P(N) > 0 \text{ for any open neighborhood } N \text{ of } \alpha \}.$$

Thus we obtain

$$\lambda < \bar{\lambda} \Rightarrow \inf_{v \in E} J_{\lambda}(v) > -\infty$$

$$\lambda > \bar{\lambda} \Rightarrow \inf_{v \in E} J_{\lambda}(v) = -\infty.$$

The inequality

$$\inf_{v \in E} J_{\lambda}(v) > -\infty$$ (1.5)

however, is open, although (1.5) is the case if $P$ is atomic \cite{23}. Here we take the fundamental assumption

$$\text{supp } P \subset I_{+} \quad (1.6)$$
and approach the problem as follows. Namely, given $\lambda_k \uparrow \bar{\lambda}$, we have a minimizer $v_k \in E$ of $J_\lambda$, that is,

$$\inf_{v \in E} J_{\lambda_k}(v) = J_{\lambda_k}(v_k).$$

If

$$\limsup_{k \to \infty} J_{\lambda_k}(v_k) > -\infty \quad (1.7)$$

we have

$$J_{\lambda}(v) = \lim_{k \to \infty} J_{\lambda_k}(v) \geq \limsup_{k \to \infty} J_{\lambda_k}(v_k) > -\infty, \quad v \in E,$$

and hence (1.5) follows. In the case of

$$\sup_k \|v_k\|_{\infty} < +\infty,$$

inequality (1.7) is valid since $v = v_k$ is a solution to (1.1) for $\lambda = \lambda_k$.

Assuming the contrary, we use a result of [15] concerning the non-compact solution sequence $(\lambda_k, v_k)$ to (1.1). Regarding (1.6), we obtain

$$S \equiv \{x_0 \in \Omega \mid \text{there exists } x_k \in \Omega \text{ such that } x_k \to x_0 \text{ and } v_k(x_k) \to +\infty\} \neq \emptyset.$$

This blowup set $S$ is finite and there is $0 \leq s \in L^1(\Omega) \cap L^\infty_{\text{loc}}(\Omega \setminus S)$ such that

$$\nu_k \equiv \lambda_k \int_{I_+} \frac{ae^{av_k}}{J_{\lambda_k}} \mathcal{P}(d\alpha) \rightharpoonup \nu \equiv s + \sum_{x_0 \in S} u(x_0)\delta_{x_0} \in \mathcal{M}(\Omega) \quad (1.8)$$

with $n(x_0) \geq 4\pi$ for each $x_0 \in S$, where $\delta_{x_0}$ denotes the Dirac measure centered at $x_0$ and $\mathcal{M}(\Omega)$ is the space of measures identified with the dual space of $C(\Omega)$. Under these preparations our main result is stated as follows.

**Theorem 1.** Inequality (1.5) holds under the conditions (1.6) and $s = 0$ in (1.8).

Henceforth, we put

$$\alpha_{\min} = \inf_{\alpha \in \text{supp } \mathcal{P}} \alpha. \quad (1.9)$$

Here we have a note concerning the assumption made in the above theorem, that is, $s = 0$ in (1.8), which we call the residual vanishing. This condition is actually satisfied under a suitable assumption on $\mathcal{P}$.

**Proposition 1** ([25] Theorem 3). Residual vanishing, $s = 0$, occurs to the above $\{v_k\}$ if $\alpha_{\min} > 1/2$.

An immediate consequence is the following theorem.

**Theorem 2.** We have (1.5) under $\alpha_{\min} > 1/2$.

So far, there is no known inequality (1.5) for continuous $\mathcal{P}(d\alpha)$ except for Theorem 2.

Residual vanishing implies the following property used in the proof of Theorem 1.
Proposition 2 ([25] Lemma 3). If the residual vanishing occurs to the above \( \{v_k\} \) then it holds that

\[
\sharp S \leq 1, \quad \tilde{\lambda} = \frac{8\pi}{\left( \int_{I_+} \alpha \mathcal{P}(d\alpha) \right)^2}.
\] (1.10)

Theorem[2] contains the classical case of the Trudinger-Moser inequality for \( \mathcal{P} = \delta_+ \). We shall show a variant of Y.Y. Li’s estimate [11], which is the key of the proof of Theorem [11]. As we see later on, it takes the form that is weaker than the estimate shown for \( \mathcal{P} = \delta_+ \) in [11].

To state the result, let

\[
w_{k,\alpha}(x) = \alpha v_k(x) - \log \int_{\Omega} e^{\alpha v_k}, \quad k \in \mathbb{N}, \quad \alpha \in I_+ \setminus \{0\},
\] (1.11)

which satisfies

\[-\Delta w_{k,\alpha} = \alpha \lambda_k \int_{I_+} \beta \left( e^{w_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \quad \text{on } \Omega, \quad \int_{\Omega} e^{w_{k,\alpha}} = 1. \] (1.12)

Regarding (1.10), we put \( S = \{x_0\} \). There exists \( x_k \in \Omega \) such that

\[x_k \to x_0, \quad v_k(x_k) = \max_{\Omega} v_k, \quad w_{k,\alpha}(x_k) = \max_{\Omega} w_{k,\alpha}.
\]

Here we take an isothermal chart \((U_k, \Psi_k)\) satisfying

\[
\Psi_k(x_k) = 0 \in \mathbb{R}^2, \quad g = e^{\xi_k(X)}(dX_1^2 + dX_2^2), \quad \xi_k(0) = 0.
\]

Then it holds that

\[-\Delta_X w_{k,\alpha} = e^{\xi_k} \left( \alpha \lambda_k \int_{I_+} \beta \left( e^{w_{k,\beta}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \right) \quad \text{in } \Psi_k(U_k), \] (1.13)

where

\[w_{k,\alpha}(X) = w_{k,\alpha} \circ \Psi_k^{-1}(X).
\]

Henceforth, we shall write \( X \) by the same notation \( x \) for simplicity. Also, we do not distinguish any sequences with their subsequences. Under this agreement we have \( x_k = x_0 = 0 \). Moreover, there exists \( R_0 > 0 \) such that \( B_{3R_0} \subset \subset \Psi_k(U_k) \) and 0 is the maximizer of \( w_{k,\alpha} \) in \( B_{3R_0} \).

The estimate is now stated in the following proposition.

Proposition 3. Under the assumptions of Theorem [11] it holds that

\[w_{k,\alpha}(x) - w_{k,\alpha}(0) = -\alpha (\gamma_0 + o(1)) \log(1 + e^{w_{k,\alpha}(0)/2}|x|) + O(1)
\] (1.14)
as \( k \to \infty \) uniformly in \( x \in B_{R_0} \) and \( \alpha \in I_+ \setminus \{0\} \), where

\[
\gamma_0 = \frac{4}{\int_{I_+} \beta \mathcal{P}(d\beta)}.
\]
To conclude this section, we shall describe a sketch of the proof of Theorem 1. The first step is the blowup analysis. We put

$$
\tilde{w}_{k,\alpha}(x) = w_{k,\alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \to 0, \quad \tilde{w}_k(x) = \tilde{w}_{k,1}(x),
$$

and get

$$
- \Delta \tilde{w}_{k,\alpha} = \alpha (\tilde{f}_k - \tilde{\delta}_k \tilde{\zeta}_k), \quad \tilde{w}_{k,\alpha} \leq \tilde{w}_{k,\alpha}(0) \leq \tilde{w}_k(0) = 0 \quad \text{in } B_{R_0/\sigma_k},
$$

$$
\int_{B_{R_0/\sigma_k}} e^{\tilde{w}_{k,\alpha} + \tilde{\zeta}_k} \leq 1, \quad \int_{B_{R_0/\sigma_k}} \tilde{f}_k \leq \lambda_k \int_{I^+} \beta P(d\beta)
$$

for each $\alpha \in I^+ \setminus \{0\}$, where

$$
\tilde{f}_k = \lambda_k \int_{I^+} \beta e^{\tilde{w}_{k,\alpha} + \tilde{\zeta}_k} P(d\beta), \quad \tilde{\delta}_k = \frac{\sigma_k^2 \lambda_k \int_{I^+} \beta P(d\beta)}{\beta^2}, \quad \tilde{\zeta}_k(x) = \zeta(\sigma_k x).
$$

The compactness argument assures the existence of $\tilde{w} = \tilde{w}(x)$ and $\tilde{f} = \tilde{f}(x)$ such that

$$
\tilde{w}_k \to \tilde{w}, \quad \tilde{f}_k \to \tilde{f} \quad \text{in } C^2_{loc}(\mathbb{R}^2)
$$

and

$$
- \Delta \tilde{w} = \tilde{f} \neq 0, \quad \tilde{w} \leq \tilde{w}(0) = 0, \quad 0 \leq \tilde{f} \leq \bar{\lambda} \int_{I^+} \beta P(d\beta) \quad \text{in } \mathbb{R}^2
$$

$$
\int_{\mathbb{R}^2} e^{\tilde{w}} \leq 1, \quad \int_{\mathbb{R}^2} \tilde{f} \leq \bar{\lambda} \int_{I^+} \beta P(d\beta).
$$

Next we focus on the quantity

$$
\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}.
$$

Given a bounded open set $\omega \subset \mathbb{R}^2$, we have

$$
\int_{\omega} \left( \int_{\omega} e^{\tilde{w}_{k,\sigma} + \tilde{\zeta}_k} dx \right) P(d\beta) \leq 1,
$$

and hence there exists $\tilde{\zeta} = \tilde{\zeta}(d\beta) \in \mathcal{M}(I^+)$ such that

$$
\left( \int_{\omega} e^{\tilde{w}_{k,\sigma} + \tilde{\zeta}_k} dx \right) P(d\beta) \rightharpoonup \tilde{\zeta}(d\beta) \quad \text{in } \mathcal{M}(I^+).
$$

For this limit measure, we can show the absolute continuity with respect to $P$, equivalently, the existence of $\tilde{\psi} \in L^1(I^+, P)$ such that $0 \leq \tilde{\psi} \leq 1$ $P$-a.e. on $I_+$ and

$$
\tilde{\zeta}(\eta) = \int_{\eta} \tilde{\psi}(\beta) P(d\beta)
$$

for any Borel set $\eta \subset I_+$. Taking $R_j \uparrow +\infty$ and putting $\omega_J = B_{R_j}$, we have $\tilde{\zeta} \in \mathcal{M}(I^+)$ and $\tilde{\psi} \in L^1(I^+, P)$ such that

$$
0 \leq \tilde{\psi}(\beta) \leq 1, \quad P\text{-a.e. } \beta
$$

$$
0 \leq \tilde{\psi} \leq \tilde{\psi} \leq \tilde{\psi} \leq \cdots \to \tilde{\psi}(\beta), \quad P\text{-a.e. } \beta
$$

$$
\tilde{\zeta}(\eta) = \int_{\eta} \tilde{\psi}(\beta) P(d\beta) \quad \text{for any Borel set } \eta \subset I_+.
$$
by the monotonicity of \( \tilde{\psi} \) with respect to \( \omega \). Since it holds that

\[
\lambda \int_{I_+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \lim_{k \to \infty} \lambda_k \int_{I_+} \beta \left( \int_{\omega_j} e^{\tilde{w}_{k,\omega} + \tilde{\xi}_k} dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \tilde{f},
\]

we have

\[
\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} = \frac{\lambda}{2\pi} \int_{I_+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta)
\]

by the monotone convergence theorem. Furthermore, we use the Pohozaev identity and the behavior of \( \tilde{w} \) at infinity to obtain

\[
-\pi \tilde{\gamma}^2 = -2\lambda \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta).
\]

More precisely, the following property holds.

**Proposition 4.** It holds that

\[
\tilde{\gamma} = \frac{4}{\int_{I_+} \beta \mathcal{P}(d\beta)},
\]

in other words,

\[
\tilde{\psi} = 1 \quad \mathcal{P}\text{-a.e. on } I_+.
\]

Note that (1.16) and (1.17) are equivalent by (1.15) and (1.10). By virtue of Proposition 4, we can apply the method of [14] to prove Proposition 3. Finally, we use Proposition 3 and the another representation of \( J_{\lambda_k}(v_k) \) denoted by

\[
J_{\lambda_k}(v_k) = \frac{\lambda_k}{2} \left\{ \int_{I_+} (\tilde{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) \right. \\
+ \left. \int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{\tilde{w}_{k,\alpha}(x)} dx \right\}
\]

to show (1.17), where \( \tilde{w}_{k,\alpha} = \frac{1}{|\Omega|} \int_{\Omega} w_{k,\alpha} \).

This paper consists of five sections and Appendix. Sections 2 and 3 are devoted to the preliminary and the proof of Proposition 4 respectively. Then, we prove Proposition 3 in Section 4. The proof of Theorem 1 is provided in Section 5. An auxiliary lemma in Section 2 is shown in Appendix.

## 2 Preliminaries

We start with the following monotonicity properties.

**Lemma 2.1.** For \( \alpha \in I_+ \), we have

\[
\frac{d}{d\alpha} w_{k,\alpha}(0) \geq 0 \quad \text{and} \quad \int_{\Omega} e^{\alpha v_k} \geq 0.
\]

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It follows from (2.1) that which contradicts

Proof. We calculate

\[
\frac{d}{d\alpha} w_{k,\alpha}(0) = v_k(0) - \frac{\int_{\Omega} v_k e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \geq v_k(0) - \frac{\int_{\{v_k>0\}} v_k e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \geq v_k(0) \left( 1 - \frac{\int_{\{v_k>0\}} e^{\alpha v_k}}{\int_{\Omega} e^{\alpha v_k}} \right) \geq 0
\]

for \( k, \alpha \in I_+ \), recalling that 0 is the maximizer of \( v_k \), and

\[
\frac{d}{d\alpha} \int_{\Omega} e^{\alpha v_k} = \int_{\Omega} v_k e^{\alpha v_k} = \int_{\Omega} v_k (e^{\alpha v_k} - 1) \geq 0
\]

by using \( \int_{\Omega} v_k = 0 \) and \( s(e^{\alpha s} - 1) \geq 0 \) which is true for \( s \in \mathbb{R} \) and \( \alpha \geq 0 \). \( \square \)

Henceforth, we put 

\[ w_k(x) = w_{k,1}(x). \]

It follows from (2.1) that

\[ w_{k,1}(0) = \max_{\alpha \in I_+} w_{k,\alpha}(0). \quad (2.3) \]

The following lemma is the starting point of our blowup analysis.

**Lemma 2.2.** For every \( \alpha \in I_+ \setminus \{0\} \), it holds that

\[ w_{k,\alpha}(0) = \max_{\bar{B}_{\rho_k}} w_{k,\alpha} \to +\infty. \]

Proof. Since \( e^{w_{k,\alpha}(0)} = e^{\alpha v_k(0)} / \int_{\Omega} e^{\alpha v_k} \geq |\Omega|^{-1} e^{\alpha w_{k,1}(0)} \) for \( \alpha \in I_+ \setminus \{0\} \), it suffices to show that \( w_k(0) = w_{k,1}(0) \to +\infty \).

Residual vanishing and (2.2) imply \( \int_{\Omega} e^{v_k} \to +\infty \), and then the local uniform boundedness of \( v_k \) in \( \Omega \setminus \{x_0\} \) shows that \( w_k \to -\infty \) locally uniformly in \( \Omega \setminus \{x_0\} \). Hence if \( \lim_{k \to \infty} w_k(0) = \lim_{k \to \infty} \|w_k\|_{L^\infty(\Omega)} < +\infty \) then \( \lim_{k \to \infty} \int_{\Omega} e^{w_k} = 0 \), which contradicts \( \int_{\Omega} e^{w_k} = 1 \). \( \square \)

We put

\[ \tilde{w}_{k,\alpha}(x) = w_{k,\alpha}(\sigma_k x) + 2 \log \sigma_k, \quad \sigma_k = e^{-w_k(0)/2} \to 0, \quad \tilde{w}_k(x) = \tilde{w}_{k,1}(x). \]

The last notation is consistent under the agreement of \( w_k = w_{k,1} \). For each \( \alpha \in I_+ \setminus \{0\} \) we have

\[
-\Delta \tilde{w}_{k,\alpha} = \alpha (\tilde{f}_k - \tilde{\delta}_k e^{\tilde{\xi}_k}), \quad \tilde{w}_{k,\alpha} \leq \tilde{w}_{k,\alpha}(0) \leq \tilde{w}_k(0) = 0 \quad \text{in } B_{R_0/\sigma_k}, \quad \text{(2.4)}
\]

\[
\int_{B_{R_0/\sigma_k}} e^{\tilde{w}_{k,\alpha} + \tilde{\xi}_k} \leq 1, \quad \int_{B_{R_0/\sigma_k}} \tilde{f}_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta), \quad \text{(2.5)}
\]

where

\[
\tilde{f}_k = \lambda_k \int_{I_+} \beta e^{\tilde{w}_{k,\alpha} + \tilde{\xi}_k} \mathcal{P}(d\beta), \quad \tilde{\delta}_k = \frac{\sigma_k^2 \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta)}{|\Omega|}, \quad \tilde{\xi}_k(x) = \xi(\sigma_k x). \quad \text{(2.6)}
\]

We shall use a fundamental fact of which proof is provided in Appendix.
Lemma 2.3. Given \( f \in L^1 \cap L^\infty(\mathbb{R}^2) \), let
\[
z(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log \frac{|x-y|}{1+|y|} dy.
\]
Then, it holds that
\[
\lim_{|x| \to +\infty} \frac{z(x)}{\log |x|} = \gamma \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} f.
\]

The following lemma is also classical (see [19] p. 130).

Lemma 2.4. If \( \phi = \phi(x) \) is a harmonic function on the whole space \( \mathbb{R}^2 \) such that
\[
\phi(x) \leq C_1(1 + \log |x|), \quad x \in \mathbb{R}^2 \setminus B_1
\]
then it is a constant function.

Now we derive the limit of (2.4)-(2.5).

Proposition 5. It holds that
\[
\tilde{\omega}_k \to \tilde{\omega}, \quad \tilde{f}_k \to \tilde{f}, \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2)
\]
for \( \tilde{\omega} = \tilde{\omega}(x) \) and \( \tilde{f} = \tilde{f}(x) \) satisfying
\[
- \Delta \tilde{\omega} = \tilde{f} \not\equiv 0, \quad \tilde{\omega} \leq \tilde{\omega}(0) = 0, \quad 0 \leq \tilde{f} \leq \tilde{\lambda} \int_{I_+} \beta P(d\beta)
\]
\[
\int_{\mathbb{R}^2} e^{\tilde{\omega}} \leq 1, \quad \int_{\mathbb{R}^2} \tilde{f} \leq \tilde{\lambda} \int_{I_+} \beta P(d\beta).
\]
In addition, it holds that
\[
\tilde{\omega}(x) \geq -\tilde{\gamma} \log(1 + |x|) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|y|}{1+|y|} dy
\]
for any \( x \in \mathbb{R}^2 \), where
\[
\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}.
\]

Proof. We have
\[
\tilde{\omega}_{k,\beta}(x) = \beta \tilde{\omega}_k(x) + (w_{k,\beta}(0) - w_k(0))
\]
for any \( \beta \in I_+ \setminus \{0\} \), and also
\[
\tilde{\omega}_k \leq \tilde{\omega}_k(0) = 0, \quad w_{k,\beta}(0) \leq w_k(0), \quad \beta \in I_+ \setminus \{0\}
\]
by (2.3). Hence \( \tilde{f}_k = \tilde{f}_k(x) \) satisfies
\[
0 \leq \tilde{f}_k(x) \leq \lambda_k \int_{I_+} \beta P(d\beta) \cdot \sup_{B_{\sigma_k}} e^{\tilde{\omega}_k} \quad \text{in } B_{R_0/\sigma_k}.
\]
Fix $L > 0$ and decompose $\bar{w}_k$, $k \gg 1$, as $\bar{w}_k = \bar{w}_{1,k} + \bar{w}_{2,k} + \bar{w}_{3,k}$, where $\bar{w}_{k,j}$, $j = 1, 2, 3$, are the solutions to

\[-\Delta \bar{w}_{1,k} = \bar{f}_k \geq 0 \quad \text{in } B_L, \quad \bar{w}_{1,k} = 1 \quad \text{on } \partial B_L\]

\[-\Delta \bar{w}_{2,k} = -\bar{\delta}_k e^{\tilde{\xi}_k} \leq 0 \quad \text{in } B_L, \quad \bar{w}_{2,k} = 0 \quad \text{on } \partial B_L\]

\[-\Delta \bar{w}_{3,k} = 0 \quad \text{in } B_L, \quad \bar{w}_{3,k} = \bar{w}_k - 1 \quad \text{on } \partial B_L.\]

First, there exists $C_{2,L} > 0$ such that

$$1 \leq \bar{w}_{1,k} \leq C_{2,L} \quad \text{on } B_L.$$  \hspace{1cm} (2.14)

Next it follows from $\bar{\delta}_k \to 0$ that

$$-\frac{1}{2} \leq \bar{w}_{2,k} \leq 0 \quad \text{on } B_L.$$  \hspace{1cm} (2.15)

Finally, we have

$$\bar{w}_{3,k} \leq -1 \quad \text{on } B_L$$

by $\bar{w}_k \leq 0$. Hence $\bar{w}_{3,k} = \bar{w}_{3,k}(x)$ is a negative harmonic function in $B_L$. Then the Harnack inequality yields $C_{3,L} > 0$ such that

$$\bar{w}_{3,k} \geq -C_{3,L} \quad \text{in } B_{L/2}.$$  \hspace{1cm} (2.16)

We thus end up with

$$\frac{1}{2} - C_{3,L} \leq \bar{w}_k \leq \bar{w}_k(0) = 0 \quad \text{in } B_{L/2},$$  \hspace{1cm} (2.17)

and then the standard compactness argument assures the limit (2.7)-(2.8) thanks to (2.13). Also (2.16) implies

$$-\Delta \bar{\bar{w}} = \bar{f}, \quad -\Delta \bar{\bar{z}} = -\bar{f}, \quad \bar{\bar{w}} \leq \bar{w}(0) = 0 \quad \text{in } \mathbb{R}^2$$

$$\bar{\bar{z}}(x) \leq (\bar{\bar{\gamma}} + 1) \log |x|, \quad x \in \mathbb{R}^2 \setminus B_R$$

for some $R > 0$ by (2.16). Hence we obtain $\bar{u} \equiv \bar{w} + \bar{z} \equiv \text{constant}$ by Lemma 2.4. Since $\bar{w}(0) = 0$ it holds that

$$\bar{w}(x) = -\bar{\bar{z}}(x) + \bar{\bar{z}}(0).$$  \hspace{1cm} (2.17)
Now we note
\[
\tilde{z}(x) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|x| + |y|}{1 + |y|} dy
\]
\[
\leq \log(1 + |x|) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} = \tilde{\gamma} \log(1 + |x|)
\]
by \(\tilde{f} \geq 0\). Hence, \(\tilde{w}(x) \geq -\tilde{\gamma} \log(1 + |x|) + \tilde{z}(0)\), and the proof is complete. \(\square\)

To study \(\tilde{\gamma}\) in (2.10), let
\[
\mathcal{B} = \{\beta \in \text{supp} \ P \mid \limsup_{k \to \infty} (w_{k,\beta}(0) - w_k(0)) > -\infty\}. \tag{2.18}
\]
From the proof of Proposition 5, it follows that if \(P(\mathcal{B}) = 0\) then \(\tilde{f} \equiv 0\), a contradiction. Hence \(P(\mathcal{B}) > 0\), and the value
\[
\beta_{\text{inf}} = \inf_{\beta \in \mathcal{B}} \beta \tag{2.19}
\]
is well-defined. Then we find
\[
\mathcal{B} = I_{\text{inf}} \cap \text{supp} P \tag{2.20}
\]
by the monotonicity (2.1), where
\[
I_{\text{inf}} = \begin{cases} \left[\beta_{\text{inf}}, 1\right] & \text{if } \beta_{\text{inf}} \in \mathcal{B}, \\ (\beta_{\text{inf}}, 1] & \text{if } \beta_{\text{inf}} \notin \mathcal{B}. \end{cases}
\]

**Lemma 2.5.** \(\beta_{\text{inf}} \tilde{\gamma} > 2\).

*Proof.* By the definition, every \(\beta \in \mathcal{B}\) admits a subsequence such that \(\tilde{w}_{k,\beta}(0) - w_k(0) = O(1)\). We recall that \(\tilde{w}_{k,\beta}\) satisfies (2.4) for \(\alpha = \beta\), i.e.,
\[
- \Delta \tilde{w}_{k,\beta} = \beta (- \Delta \tilde{w}_k) = \beta (f_k - \delta_k e^{\xi_k}).
\]
From the argument developed for the proof of (2.7)-(2.9), we have \(\tilde{w}_\beta = \tilde{w}_\beta(x) \in C^2(\mathbb{R}^2)\) such that
\[
\tilde{w}_{k,\beta} \to \tilde{w}_\beta \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^2). \tag{2.21}
\]
The limit \(\tilde{w}_\beta\) satisfies
\[
- \Delta \tilde{w}_\beta = \beta \tilde{f}, \quad \tilde{w}_\beta \leq \tilde{w}_\beta(0) \leq 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{\tilde{w}_\beta} \leq 1
\]
and
\[
\tilde{w}_\beta(x) \geq -\beta \tilde{\gamma} \log(1 + |x|) + \frac{\beta}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|y|}{1 + |y|} dy \tag{2.22}
\]
with \(\tilde{f} = \tilde{f}(x)\) given in Proposition 5.

If \(\beta_{\text{inf}} \in \mathcal{B}\), we take \(\beta = \beta_{\text{inf}}\). Since \(\tilde{f} \in L^1 \cap L^\infty(\mathbb{R}^2)\) and \(\int_{\mathbb{R}^2} e^{\tilde{w}_\beta} < +\infty\), we obtain the lemma by (2.22).

If \(\beta_{\text{inf}} \notin \mathcal{B}\), we take \(\beta_j \in \mathcal{B}\) in \(\beta_j \downarrow \beta_{\text{inf}}\) and apply (2.22) for \(\beta = \beta_j\). Since
\[
\frac{\beta_j}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(y) \log \frac{|y|}{1 + |y|} dy = O(1), \quad \int_{\mathbb{R}^2} e^{\tilde{w}_{\beta_j}} \leq 1
\]
there is \(\varepsilon_0 > 0\) independent of \(j\) such that \(\beta_j \tilde{\gamma} \geq 2 + \varepsilon_0\), and then we obtain the lemma. \(\square\)
Given a bounded open set $\omega \subset \mathbb{R}^2$, we have

$$\int_{I_+} \left( \int_{\omega} e^{\tilde{w}_{k, \beta} + \tilde{\xi}} \, dx \right) \mathcal{P}(d\beta) \leq 1.$$ 

Hence it holds that

$$\left( \int_{\omega} e^{\tilde{w}_{k, \beta} + \tilde{\xi}} \, dx \right) \mathcal{P}(d\beta) \xrightarrow[k \to \infty]{} \tilde{\zeta}^\omega(d\beta) \quad \text{in} \quad \mathcal{M}(I_+). \quad (2.23)$$

Now we shall show that the limit measure $\tilde{\zeta}^\omega = \tilde{\zeta}^\omega(d\beta) \in \mathcal{M}(I_+)$ is absolutely continuous with respect to $\mathcal{P}$.

**Lemma 2.6.** There exists $\tilde{\psi}^\omega \in L^1(I_+, \mathcal{P})$ such that $0 \leq \tilde{\psi}^\omega \leq 1$ $\mathcal{P}$-a.e. on $I_+$ and

$$\tilde{\zeta}^\omega(\eta) = \int_{\eta} \tilde{\psi}^\omega(\beta) \mathcal{P}(d\beta)$$

for any Borel set $\eta \subset I_+$.

**Proof.** Let $\eta \subset I_+$ be a Borel set and $\varepsilon > 0$. Then each compact set $K \subset \eta$ admits an open set $J \subset I_+$ such that

$$K \subset \eta \subset J, \quad \mathcal{P}(J) \leq \varepsilon + \mathcal{P}(K).$$

Now we take $\varphi \in C(I_+)$ satisfying

$$\varphi = 1 \quad \text{on} \quad K, \quad 0 \leq \varphi \leq 1 \quad \text{on} \quad I_+, \quad \text{supp} \varphi \subset J.$$

Then (2.23) implies

$$\tilde{\zeta}^\omega(K) = \int_{K} \tilde{\zeta}^\omega(d\beta) \leq \int_{I_+} \varphi(\beta) \tilde{\zeta}^\omega(d\beta)$$

$$= \lim_{k \to \infty} \int_{I_+} \varphi(\beta) \left( \int_{\omega} e^{\tilde{w}_{k, \beta} + \tilde{\xi}} \, dx \right) \mathcal{P}(d\beta) \leq \int_{I_+} \varphi(\beta) \mathcal{P}(d\beta)$$

$$\leq \int_{J} \mathcal{P}(d\beta) = \mathcal{P}(J) \leq \varepsilon + \mathcal{P}(\eta),$$

and therefore

$$0 \leq \tilde{\zeta}^\omega(\eta) = \sup\{\tilde{\zeta}^\omega(K) \mid K \subset \eta : \text{compact} \} \leq \varepsilon + \mathcal{P}(\eta).$$

This shows the absolute continuity of $\tilde{\zeta}^\omega$ with respect to $\mathcal{P}$. □

We take $R_j \uparrow +\infty$ and put $\omega_j = B_{R_j}$. From the monotonicity of $\tilde{\psi}^\omega$ with respect to $\omega$, there exist $\tilde{\zeta} \in \mathcal{M}(I_+)$ and $\tilde{\psi} \in L^1(I_+, \mathcal{P})$ such that

$$0 \leq \tilde{\psi}(\beta) \leq 1, \quad \mathcal{P}\text{-a.e.} \beta$$

$$0 \leq \tilde{\psi}_{\omega_1}(\beta) \leq \tilde{\psi}_{\omega_2}(\beta) \leq \cdots \to \tilde{\psi}(\beta), \quad \mathcal{P}\text{-a.e.} \beta$$

$$\tilde{\zeta}(\eta) = \int_{\eta} \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set} \ \eta \subset I_+.$$
First, (2.7) implies
\[
\lambda \int_{1+} \beta \tilde{\psi}^\omega (\beta) \mathcal{P}(d\beta) = \lim_{k \to \infty} \lambda_k \int_{1+} \beta \left( \int_{\omega_j} e^{\tilde{w}_k,\beta + \tilde{\xi}_k} \, dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \tilde{f}.
\]
Then we obtain
\[
\tilde{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f} = \frac{\lambda}{2\pi} \int_{1+} \beta \tilde{\psi}(\beta) \mathcal{P}(d\beta)
\]
by the monotone convergence theorem.

Similarly to [5], on the other hand, we have the following lemma, where \((r, \theta)\) denotes the polar coordinate in \(\mathbb{R}^2\).

**Lemma 2.7.** We have
\[
\lim_{r \to +\infty} r \tilde{w}_r = 0, \quad \lim_{r \to +\infty} \tilde{w}_\theta = 0
\]
uniformly in \(\theta\).

**Proof.** From (2.15) and (2.17), it follows that
\[
\tilde{w}_r(x) = -\tilde{\gamma} - \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y \cdot (x - y)}{|x - y|^2} \tilde{f}(y) \, dy,
\]
\[
\tilde{w}_\theta(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y \cdot (x - y)}{|x - y|^2} \tilde{f}(y) \, dy, \quad \tilde{y} = (y_2, -y_1).
\]
Hence it suffices to show
\[
\lim_{|x| \to +\infty} I_1(x) = \lim_{|x| \to +\infty} I_2(x) = 0,
\]
where
\[
I_1(x) = \int_{|x - y| > |x|/2} \frac{|y|}{|x - y|} \tilde{f}(y) \, dy, \quad I_2(x) = \int_{|x - y| \leq |x|/2} \frac{|y|}{|x - y|} \tilde{f}(y) \, dy.
\]
Since \(\tilde{f} \in L^1(\mathbb{R}^2)\), we have \(\lim_{|x| \to +\infty} I_1(x) = 0\) by the dominated convergence theorem.

Next, (2.7) implies
\[
I_2(x) = \lim_{k \to \infty} \int_{|x - y| \leq |x|/2} \frac{|y|}{|x - y|} \left( \lambda_k \int_{1+} \beta e^{\tilde{w}_k,\beta + \tilde{z}_k(y)} \mathcal{P}(d\beta) \right) \, dy
\]
\[
= \tilde{\lambda} \lim_{k \to \infty} \int_{[\beta_{\text{inf}}, 1]} \beta \left( \int_{|x - y| \leq |x|/2} \frac{|y|}{|x - y|} e^{\tilde{w}_k,\beta + \tilde{z}_k(y)} \, dy \right) \mathcal{P}(d\beta),
\]
recalling (3.13) and (3.14). Now we use (3.13), (3.14) and (2.7) with (3.14), to confirm
\[
\tilde{w}_{k,\beta}(x) \leq \beta \tilde{w}_k(x) = \beta (\tilde{\bar{z}}(x) + \tilde{z}(0)) + o(1)
\]
as \(k \to \infty\), locally uniformly in \(x \in \mathbb{R}^2\). Hence it holds that
\[
0 \leq I_2(x) \leq C_4 \int_{|x - y| \leq |x|/2} \frac{|y|}{|x - y|} \int_{[\beta_{\text{inf}}, 1]} e^{-\beta \tilde{z}} \mathcal{P}(d\beta) \, dy.
\]
Then (2.16) and Lemma 2.5 imply
\[
0 \leq I_2(x) \leq C_5|x|^{-(1+\varepsilon_0)} \int_{|x-y| \leq |x|/2} \frac{dy}{|x-y|} \leq C_6|x|^{-\varepsilon_0}
\]
with some \(\varepsilon_0 > 0\), where we have used
\[
|x - y| \leq \frac{|x|}{2} \Rightarrow \frac{1}{2} |x| \leq |y| \leq \frac{3}{2} |x|.
\]
Hence \(\lim_{|x| \to +\infty} I_2(x) = 0\) follows.

The Pohozaev identity
\[
R \int_{\partial B_R} \frac{1}{2} |\nabla u|^2 - u_x^2 \, ds = R \int_{\partial B_R} A(x)F(u) \, ds - \int_{B_R} 2A(x)F(u) + F(u)(x \cdot \nabla A(x)) \, dx \tag{2.25}
\]
is valid to \(u = u(x) \in C^2(\overline{B_R})\) satisfying
\[
-\Delta u = A(x)F'(u) \quad \text{in} \ B_R, \tag{2.26}
\]
where \(F \in C^1(\mathbb{R}), \ A \in C^1(\overline{B_R}),\) and \(ds\) denotes the surface element on the boundary. By this identity and Lemma 2.7 we obtain the following fact.

**Lemma 2.8.** It holds that
\[
\int_{I_+} \tilde{\psi}(\beta)\mathcal{P}(d\beta) = \left( \int_{I_+} \phi_0(\beta)\tilde{\psi}(\beta)\mathcal{P}(d\beta) \right)^2, \tag{2.27}
\]
where
\[
\phi_0(\beta) = \frac{\beta}{\int_{I_+} \alpha\mathcal{P}(d\alpha)}. \tag{2.28}
\]

**Proof.** We apply (2.25) for (2.26) to (2.4), \(\alpha = 1\), where \(u = \tilde{w}_k\) and
\[
F(\tilde{w}_k) = \lambda_k \int_{I_+} e^{\tilde{\psi}_k(\beta)}\mathcal{P}(d\beta) - \tilde{\delta}_k \tilde{w}_k, \quad A(x) = e^{\tilde{\xi}_k(x)} = e^{\xi_k(\sigma_kx)}.
\]
It follows that
\[
R \int_{\partial B_R} \frac{1}{2} |\nabla \tilde{w}_k|^2 - (\tilde{w}_k)_x^2 \, ds = -2\lambda_k \int_{I_+} \left( \int_{B_R} e^{\tilde{\psi}_k(\beta) + \tilde{\xi}_k} \, dx \right) \mathcal{P}(d\beta) \tag{2.29}
\]
\[
+ R\lambda_k \int_{I_+} \left( \int_{\partial B_R} e^{\tilde{\psi}_k(\beta) + \tilde{\xi}_k} \mathcal{P}(d\beta) - R\tilde{\delta}_k \int_{\partial B_R} \tilde{w}_k e^{\tilde{\xi}_k} \, ds \right)
- \sigma_k \cdot \lambda_k \int_{I_+} \left( \int_{B_R} e^{\tilde{\psi}_k(\beta) + \tilde{\xi}_k} (x \cdot \nabla \xi_k(\sigma_kx)) \, dx \right) \mathcal{P}(d\beta)
+ \tilde{\delta}_k \int_{B_R} 2\tilde{w}_k e^{\tilde{\xi}_k} + \sigma_k \tilde{w}_k \tilde{w}_k e^{\tilde{\xi}_k} (x \cdot \nabla \xi_k(\sigma_kx)) \, dx.
\]
for $\tilde{\delta}_k$ defined by (2.6).

For every $R > 0$, the last three terms on the right-hand side of (2.29) vanish as $k \to \infty$, because of $\tilde{\delta}_k \to 0$ and $\sigma_k \to 0$. For the second term we argue similarly as in the proof of Lemma 2.7, while the conclusion of Lemma 2.7 is applicable to the left-hand side on (2.29). We thus end up with

$$-\pi \tilde{\gamma}^2 = -2\bar{\lambda} \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) \quad (2.30)$$

by sending $k \to \infty$ and then $R \to +\infty$.

Combining (2.30), (2.24), and the value $\bar{\lambda}$ given in (1.10), we obtain (2.27)-(2.28).

Let

$$I^0(\psi) = \int_{I_+} \phi_0(\beta) \psi(\beta) \mathcal{P}(d\beta)$$

$$\mathcal{C}_d = \{ \psi \mid 0 \leq \psi(\beta) \leq 1 \ \mathcal{P}\text{-a.e. on } I_+ \text{ and } \int_{I_+} \psi(\beta) \mathcal{P}(d\beta) = d \}$$

and $\chi_A$ be the characteristic function of the set $A$. The following lemma is a variant of the result of [13].

**Lemma 2.9.** For each $0 < d \leq 1$, the value $\sup_{\psi \in \mathcal{C}_d} I^0(\psi)$ is attained by

$$\psi_d(\beta) = \chi_{\{\phi_0 > s_d\}}(\beta) + c_d \chi_{\{\phi_0 = s_d\}}(\beta) \quad (2.31)$$

with $s_d$ and $c_d$ defined by

$$s_d = \inf \{ t \mid \mathcal{P}(\{ \phi_0 > t \}) \leq d \}$$

$$c_d \mathcal{P}(\{ \phi_0 = s_d \}) = d - \mathcal{P}(\{ \phi_0 > s_d \}), \quad 0 \leq c_d \leq 1. \quad (2.32)$$

Furthermore, the maximizer is unique in the sense that $\psi_m = \psi_d \ \mathcal{P}\text{-a.e. on } I_+$ for any maximizer $\psi_m \in \mathcal{C}_d$.

**Proof.** Fix $0 < d \leq 1$. Given $\psi \in \mathcal{C}_d$, we compute

$$\int_{I_+} \phi_0(\psi_d - \psi) \mathcal{P}(d\beta) = \int_{\{\phi_0 > s_d\}} \phi_0(\psi_d - \psi) \mathcal{P}(d\beta)$$

$$+ s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0 \psi \mathcal{P}(d\beta)$$

$$\geq s_d \int_{\{\phi_0 > s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta)$$

$$+ s_d \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \phi_0 \psi \mathcal{P}(d\beta) \quad (2.33)$$

$$\geq s_d \int_{\{\phi_0 > s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta)$$

$$+ \int_{\{\phi_0 = s_d\}} (\psi_d - \psi) \mathcal{P}(d\beta) - \int_{\{\phi_0 < s_d\}} \psi \mathcal{P}(d\beta) \quad (2.34)$$

$$= s_d \int_{I_+} (\psi_d - \psi) \mathcal{P}(d\beta) = 0.$$
which means that \( \psi_d \) is the maximizer.

The equalities hold in (2.33) and (2.34) if and only if \( \psi \) is the maximizer, and so we shall derive the conditions that the former is true. The first condition is that \((\phi_0 - s_d)(\psi_d - \psi) = 0\) \(\mathcal{P}\)-a.e. on \(\{\phi_0 > s_d\}\), so that

\[
\psi = \psi_d \quad \mathcal{P}\text{-a.e. on } \{\phi_0 > s_d\}
\] (2.35)

by the monotonicity of \(\phi_0\) and \(\psi_d \geq \psi\) on \(\{\phi_0 > s_d\}\). The second one is that

\[
(s_d - \phi_0)\psi = 0 \quad \mathcal{P}\text{-a.e. on } \{\phi_0 < s_d\}
\] (2.36)

by the monotonicity of \(\phi_0\) and \(\psi \geq 0\). The uniqueness follows from (2.35)-(2.36) and \(\psi_d, \psi \in \mathcal{C}_d\).

3 Proof of Proposition 4

For the purpose, we assume the contrary, that is, \(\tilde{\psi} \in \mathcal{C}_d\) for some \(0 < d < 1\). We shall prove Proposition 4 by contradiction.

Since \(\tilde{\psi} = \tilde{\psi}(\beta)\) satisfies (2.27), it holds that

\[
d = \int_{I_+} \tilde{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2.
\]

Lemma 2.9 and (2.28) yield

\[
d = \mathcal{P}(\{\phi_0 > s_d\}) + c_d \mathcal{P}(\{\phi_0 = s_d\}) \leq \left( \frac{\int_{I_+} \psi_d(\beta) \beta \mathcal{P}(d\beta)}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2 (3.1)
\]

for \(\psi_d = \psi_d(\beta)\) defined by (2.31)-(2.32). By the monotonicity of \(\phi_0 = \phi_0(\beta)\), there exists the unique element \(\beta_d \in I_+\) such that

\[
\phi_0(\beta_d) = s_d,
\]

and then (3.1) reads

\[
d = \mathcal{P}((\beta_d, 1]) + c_d \mathcal{P}(\{\beta_d\}) \leq \left( \frac{\int_{(\beta_d, 1]} \beta \mathcal{P}(d\beta) + c_d \beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2. (3.2)
\]

Here we introduce

\[
H(\tau) = \mathcal{P}((\beta_d, 1]) + \tau \mathcal{P}(\{\beta_d\}) - \left( \frac{\int_{(\beta_d, 1]} \beta \mathcal{P}(d\beta) + \tau \beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_+} \beta \mathcal{P}(d\beta)} \right)^2.
\]

It follows from (1.4) and (1.10) that

\[
H(0) \geq 0, \quad H(1) \geq 0.
\] (3.3)
Moreover, we have either \( c_d = 0 \) or \( c_d = 1 \) if \( \mathcal{P}(\{\beta_d\}) > 0 \). In fact, since

\[
H''(\tau) = \text{const.} = -2 \left( \frac{\beta_d \mathcal{P}(\{\beta_d\})}{\int_{I_d} \beta \mathcal{P}(d\beta)} \right)^2 < 0
\]

by \( \mathcal{P}(\{\beta_d\}) > 0 \), it holds that \( H(\tau) > 0 \) for \( 0 < \tau < 1 \) by (3.3). On the other hand, \( H(c_d) \leq 0 \) by (3.2).

We now claim

\[
\tilde{\psi} = \psi_d = \chi_{I_d} \quad \mathcal{P}\text{-a.e. on } I_+,
\]

where

\[
I_d = \begin{cases} [\beta_d, 1] & \text{if } \mathcal{P}(\{\beta_d\}) = 0 \text{ or if } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 1 \\ (\beta_d, 1] & \text{otherwise (i.e., } \mathcal{P}(\{\beta_d\}) > 0 \text{ and } c_d = 0). \end{cases}
\]

First, we assume that \( \mathcal{P}(\{\beta_d\}) = 0 \). Then, \( H(\tau) = H(0) \) for \( \tau \in [0, 1] \). In this case, the equality holds in (3.2) by (3.3), and thus

\[
d = \left( \int_{I_d} \phi_0(\beta) \psi_d(\beta) \mathcal{P}(d\beta) \right)^2 = \left( \int_{I_d} \phi_0(\beta) \tilde{\psi}(\beta) \mathcal{P}(d\beta) \right)^2,
\]

which means \( \tilde{\psi} = \psi_d \) \( \mathcal{P}\text{-a.e. on } I_+ \) by the uniqueness of Lemma 2.9. Note that the integrands are non-negative. It is clear that \( \psi_d = \chi_{I_d} \) \( \mathcal{P}\text{-a.e. on } I_+ \). Next we assume that \( \mathcal{P}(\{\beta_d\}) > 0 \). Then, we use (3.2)-(3.3) to obtain \( H(c_d) = 0 \), which again implies that the equality holds in (3.2), and hence

\[
\tilde{\psi} = \psi_d = \begin{cases} \chi_{[\beta_d, 1]} & \text{if } c_d = 1 \\ \chi_{(\beta_d, 1]} & \text{if } c_d = 0. \end{cases}
\]

The claim (3.4) is established.

Property (3.4) is actually refined as follows, recall (2.20), i.e.,

\[
\mathcal{B} = I_{\inf} \cap \text{supp } \mathcal{P}.
\]

**Lemma 3.1.** \( \tilde{\psi} = \chi_{I_{\inf}} \) \( \mathcal{P}\text{-a.e. on } I_+ \).

**Proof.** There are the following six possibilities:

(i) \( \beta_d < \beta_{\inf} \)  (ii) \( \beta_d > \beta_{\inf} \)

(iii) \( \beta_d = \beta_{\inf}, I_d = (\beta_d, 1] \) and \( \beta_{\inf} \notin I_{\inf} \)

(iv) \( \beta_d = \beta_{\inf}, I_d = [\beta_d, 1] \) and \( \beta_{\inf} \notin I_{\inf} \)

(v) \( \beta_d = \beta_{\inf}, I_d = [\beta_d, 1] \) and \( \beta_{\inf} \in I_{\inf} \)

(vi) \( \beta_d = \beta_{\inf}, I_d = (\beta_d, 1] \) and \( \beta_{\inf} \notin I_{\inf} \)

The lemma is clearly true for the cases (v)-(vi), and thus it suffices to prove \( \mathcal{P}(I_d \setminus I_{\inf}) = 0, \mathcal{P}(I_{\inf} \setminus I_d) = 0 \) and \( \mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) = 0 \) for the cases (i), (ii) and (iii)-(iv), respectively.

(i) Assume \( \mathcal{P}(I_d \setminus I_{\inf}) > 0 \). Then,

\[
\tilde{\psi}(\beta) = 0 \quad \text{for } \beta \in I_d \setminus I_{\inf},
\]

(3.5)
by the definitions of $I_{\inf}$ and $\tilde{\psi}$. Note that $\tilde{w}_{k,\beta} \to -\infty$ locally uniformly in $\mathbb{R}^2$ for $\beta \in I_d \setminus I_{\inf}$. On the other hand, $\tilde{\psi}(\beta) = 1$ for some $\beta \in I_d \setminus I_{\inf}$ by (3.4), which contradicts (3.5).

(ii) Assume $\mathcal{P}(I_{\inf} \setminus I_d) > 0$. Then,
\[
\tilde{\psi}(\beta) = 0 \quad \text{for } \mathcal{P}\text{-a.e. } \beta \in I_{\inf} \setminus I_d
\]
by (3.4). On the other hand, $\tilde{\psi}(\beta) > 0$ for any $\beta \in I_{\inf} \setminus I_d$ by the definitions of $I_{\inf}$ and $\tilde{\psi}$, and by the convergence (2.21), which contradicts (3.6).

(iii) If $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) > 0$ then $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 0$ by (3.4) and $I_d = (\beta_d, 1]$. On the other hand, $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) > 0$ by $\beta_{\inf} \in I_{\inf}$ as shown for the case (ii) above, a contradiction.

(iv) If $\mathcal{P}(\{\beta_d\}) = \mathcal{P}(\{\beta_{\inf}\}) > 0$ then $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 1$ by (3.4) and $I_d = [\beta_d, 1]$. On the other hand, $\tilde{\psi}(\beta_d) = \tilde{\psi}(\beta_{\inf}) = 0$ by $\beta_{\inf} \notin I_{\inf}$ as shown for the case (i) above, a contradiction.

Since the equality holds in (3.5) as shown above, it follows from (3.4) and Lemma 3.1 that
\[
\mathcal{P}(I_{\inf}) > 0, \quad \frac{\mathcal{P}(I_{\inf})}{\int_{I_{\inf}} \beta \mathcal{P}(d\beta)}^2 = \frac{1}{\int_{I_d} \beta \mathcal{P}(d\beta)}^2
\] (3.7)

**Lemma 3.2.** For every $R > 0$ and $\alpha \in I_+ \setminus I_{\inf}$, it holds that
\[
\lim_{k \to \infty} \int_{B_{R,\alpha}^k} e^{w_{k,\alpha}} = 0,
\]
where $\sigma_{k,\alpha} = e^{-w_{k,\alpha}(0)/2}$.

**Proof.** Fix $R > 0$ and $\alpha \in I_+ \setminus I_{\inf}$. Putting
\[
\tilde{w}_{(1)}^{(k)}(x) = w_{k,\alpha}(\sigma_{k,\alpha} x) + 2 \log \sigma_{k,\alpha},
\]
we have
\[
\int_{B_{R,\alpha}^k} e^{w_{k,\alpha}} = \int_{B_R} e^{\tilde{w}_{(1)}^{(k)}}, \quad \tilde{w}_{(1)}^{(k)} \leq \tilde{w}_{(1)}^{(k)}(0) = 0 \quad \text{in } B_R,
\]
and therefore it suffices to show
\[
\tilde{w}_{(1)}^{(k)} \to -\infty \quad \text{locally uniformly in } B_R \setminus \{0\}.
\]
If this is not the case, then there exist $C_1 > 0$ and $r_1 > 0$ such that
\[
\max_{B_R \setminus B_{r_1}} \tilde{w}_{(1)}^{(k)} \geq -C_1
\]
for $k \gg 1$. Since there exists $y_k \in B_R \setminus B_{r_1}$ such that
\[
\tilde{w}_{(1)}^{(k)}(y_k) = \max_{B_R \setminus B_{r_1}} \tilde{w}_{(1)}^{(k)},
\]
it holds that
\[
\tilde{w}_{(1)}^{(k)}(y_k) - \tilde{w}_{(1)}^{(k)}(0) \geq -C_1
\]
for \( k \gg 1 \), and thus
\[
{w_k(\sigma_{k,\alpha}y_k) - w_k(0) = (\tilde{w}_k^{(1)}(y_k) - \tilde{w}_k^{(1)}(0))/\alpha \geq -C_1/\alpha}
\]
(3.8)
for \( k \gg 1 \). On the other hand, we have \( \beta \in I_{inf} \) satisfying
\[
\lim_{L \to +\infty} \lim_{k \to +\infty} \int_{B_{\frac{1}{2}L}} e^{w_k,\beta} = 1
\]
(3.9)
by the definitions of \( I_{inf} \) and \( \tilde{\psi} \), and by the convergence shown in the proof of Lemma 2.5.

Now, we introduce
\[
\tilde{w}_k^{(2)}(x) = w_k,\beta(\mu_k x + \sigma_{k,\alpha}y_k) + 2 \log \mu_k, \quad \mu_k = e^{-w_k,\beta(\sigma_{k,\alpha}y_k)/2}.
\]
Since \( \beta \in I_{inf} \), there exists \( C_2 > 0 \) such that
\[
w_k(0) \leq w_k,\beta(0) + C_2
\]
(3.10)
for \( k \gg 1 \). Moreover, it follows from (3.8) and (3.10) that
\[
w_k(0) - w_k,\beta(\sigma_{k,\alpha}y_k) \leq C_3
\]
(3.11)
for \( k \gg 1 \), where \( C_3 = \beta C_1/\alpha + C_2 \). Noting (3.11) and
\[
w_k(0) = \sup_{\alpha \in I_{+}} \sup_{x \in \Omega} w_k,\alpha(x),
\]
we find
\[
-\Delta \tilde{w}_k^{(2)} = \beta \lambda_k \tilde{\xi}_k^{(2)} \int_{I_{+}} \tau \left(e^{\tilde{w}_k^{(2)} - \frac{\mu_k}{\Omega}}\right) \mathcal{P}(d\tau) \quad \text{in } B_{\frac{1}{2}L}
\]
(3.12)
\[
\tilde{w}_k^{(2)}(0) = 0, \quad \tilde{w}_k^{(2)}(x) \leq w_k(0) - w_k,\beta(\sigma_{k,\alpha}y_k) \leq C_3 \quad \text{in } B_{\frac{1}{2}L}
\]
for any \( \tau \in I_{+} \)
\[
\int_{B_{\frac{1}{2}L}} e^{\tilde{w}_k^{(2)} + \tilde{\xi}_k^{(2)}} \leq 1 \quad \text{for any } \tau \in I_{+},
\]
where
\[
\tilde{w}_k^{(2)}(x) = w_k,\tau(\mu_k x + \sigma_{k,\alpha}y_k) + 2 \log \mu_k, \quad \tilde{\xi}_k^{(2)}(x) = \xi_k(\mu_k x + \sigma_{k,\alpha}y_k).
\]
Noting \( \sigma_{k,\alpha}y_k \to 0, \xi_k(0) = 0 \) and the smoothness of \( \xi_k \), we perform the compactness argument, similarly to the proof of Proposition 5, to obtain \( \tilde{w}_k^{(2)}, \tilde{f}_k^{(2)} \in C^2(\mathbb{R}^2) \) and \( \tilde{C}_4 \) such that
\[
\tilde{w}_k^{(2)} \rightarrow \tilde{w}^{(2)}, \quad \lbrack \text{r.h.s. of (3.12) \rbrack} \rightarrow \tilde{f}^{(2)} \quad \text{in } C^2_{loc}(\mathbb{R}^2)
\]
and
\[
-\Delta \tilde{w}^{(2)} = \tilde{f}^{(2)}, \quad \tilde{w}^{(2)} \leq C_3 \quad \text{in } \mathbb{R}^2
\]
\[
\tilde{w}^{(2)}(0) = 0, \quad \int_{\mathbb{R}^2} e^{\tilde{w}^{(2)}} \leq 1, \quad \int_{\mathbb{R}^2} \tilde{f}^{(2)} + \|\tilde{f}^{(2)}\|_{L^{\infty}(\mathbb{R}^2)} \leq \tilde{C}_4.
\]
Therefore, there exist $\ell_1 > 0$ and $0 < \delta \ll 1$ such that
\[ \int_{B_{\mu_k \ell_1(\sigma_k,\alpha y_k)}} e^{w_{k,\beta}} \geq 2\delta \]
for $k \gg 1$.

On the other hand, (3.9) admits $\ell_2 > 0$ satisfying
\[ \int_{B_{\sigma_k \ell_2}} e^{w_{k,\beta}} \geq 1 - \delta \]
for $k \gg 1$. In addition,
\[ |\sigma_{k,\alpha y_k}| - \mu_k \ell_1 - \sigma_k \ell_2 \geq \sigma_{k,\alpha}(r_1 - \frac{\mu_k}{\sigma_{k,\alpha}} \ell_1 - \frac{\sigma_k}{\sigma_{k,\alpha}} \ell_2) \geq \frac{\sigma_{k,\alpha} r_1}{2} \]
for $k \gg 1$ since $\sigma_k \sigma_{k,\alpha} \to 0$, $\mu_k \sigma_{k,\alpha} \to 0$ by $\alpha \notin I_{\text{inf}}$ and (3.11). Hence there holds
\[ B_{\mu_k \ell_1(\sigma_{k,\alpha} y_k)} \cap B_{\sigma_k \ell_2} = \emptyset \]
for $k \gg 1$. Combining (3.13)-(3.15) shows
\[ 1 = \int_\Omega e^{w_{k,\beta}} \geq \int_{B_{\mu_k \ell_1(\sigma_{k,\alpha} y_k)} \cup B_{\sigma_k \ell_2}} e^{w_{k,\beta}} \geq 2\delta + (1 - \delta) = 1 + \delta > 1 \]
for $k \gg 1$, a contradiction.

**Lemma 3.3.** There are no $\mathcal{P}$-measurable sets $K_1, K_2 \subset I_+$ satisfying
\[ \begin{cases} \mathcal{P}(K_i) > 0 (i = 1, 2), & \mathcal{P}(K_1 \cap K_2) = 0 \\ \frac{\mathcal{P}(K_i)}{(\int_{I_+} \beta \mathcal{P}(d\beta))^2} = \frac{1}{(\int_{I_+} \beta \mathcal{P}(d\beta))^2} (i = 1, 2). \end{cases} \]

**Proof.** Assume that there exist $\mathcal{P}$-measurable sets $K_1, K_2 \subset I_+$ satisfying (3.16), and put
\[ a_i = \mathcal{P}(K_i), \quad b_i = \int_{I_+} \beta \mathcal{P}(d\beta) \] for $i = 1, 2$.
so that
\[ a_i = b_i^2 \] for $i = 1, 2$.

On the other hand, (1.4) and (1.10) show
\[ \frac{1}{(\int_{I_+} \beta \mathcal{P}(d\beta))^2} \leq \frac{\mathcal{P}(K_1 \cup K_2)}{(\int_{K_1 \cup K_2} \beta \mathcal{P}(d\beta))^2} = \frac{\mathcal{P}(K_1) + \mathcal{P}(K_2)}{(\int_{I_+} \beta \mathcal{P}(d\beta) + \int_{I_+} \beta \mathcal{P}(d\beta))^2} \]
or
\[ \frac{a_1 + a_2}{(b_1 + b_2)^2} \geq 1. \]

Hence we have
\[ b_1^2 + b_2^2 \geq (b_1 + b_2)^2, \]
which is impossible since $b_i > 0 (i = 1, 2)$.
Lemma 3.4. There exists $C_5 > 0$, independent of $k \gg 1$, such that

$$\sup_{\alpha \in I_+} \sup_{x \in B_{2R_0}} \{w_{k,\alpha}(x) + 2 \log |x|\} \leq C_5 (3.17)$$

for $k \gg 1$.

Proof. The proof is divided into four steps.

**Step 1.** Assume the contrary, that is, there exist $\alpha_k \in I_+$ and $x_k \in \bar{B}_{R_0}$ such that

$$M_k \equiv w_{k,\alpha_k}(x_k) + 2 \log |x_k| = \sup_{\alpha \in I_+} \sup_{x \in B_{2R_0}} \{w_{k,\alpha}(x) + 2 \log |x|\} \to +\infty.$$  

We have

$$w_{k,\alpha_k}(x_k) = M_k - 2 \log |x_k| \geq M_k - 2 \log R_0 \to +\infty$$

$$\ell_k \equiv e^{w_{k,\alpha_k}(x_k)/2} \cdot |x_k| = \frac{e^{M_k/2}}{2} \to +\infty.$$  

For any $x \in B_{|x_k|/2}(x_k)$, $\alpha \in I_+$ and $k$, it holds that

$$w_{k,\alpha}(x) - w_{k,\alpha_k}(x_k) = (w_{k,\alpha}(x) + 2 \log |x|) - (w_{k,\alpha_k}(x_k) + 2 \log |x_k|) + 2 \log \frac{|x_k|}{|x|} \leq 2 \log 2.$$  

We put

$$\hat{w}_k(x) = w_{k,\alpha_k}(\tau_k x + x_k) + 2 \log \tau_k, \quad \tau_k = e^{-w_{k,\alpha_k}(x_k)/2},$$

and get

$$-\Delta \hat{w}_k = \hat{f}_k - \hat{d}_k e^{\hat{\xi}_k} \quad \text{in } B_{\ell_k}$$

$$\hat{w}_{k,\beta} \leq 2 \log 2 \quad \text{in } B_{\ell_k} \quad \text{for any } \beta \in I_+ \text{ and } k$$

$$\hat{w}_k(0) = 0, \quad \int_{B_{\ell_k}} e^{\hat{w}_{k,\beta} + \hat{\xi}_k} \leq 1 \quad \text{for any } \beta \in I_+ \text{ and } k,$$

where

$$\hat{f}_k(x) = \alpha_k \lambda_k \int_{I_+} \beta e^{w_{k,\beta}(\tau_k x + x_k) + 2 \log \tau_k} P(d\beta), \quad \hat{d}_k = \frac{\alpha_k \lambda_k \tau_k^2 \int_{I_+} \beta P(d\beta)}{|\Omega|},$$

$$\hat{w}_{k,\beta}(x) = w_{k,\beta}(\tau_k x + x_k) + 2 \log \tau_k, \quad \hat{\xi}_k(x) = \xi_k(\tau_k x + x_k).$$

The compactness argument, similarly to the proof of Proposition 5 admits $\hat{\omega}, \hat{f} \in C^2(\mathbb{R}^2)$ and $C_6 > 0$ such that

$$\hat{\omega}_k \to \hat{\omega}, \quad \hat{f}_k \to \hat{f} \quad \text{in } C^2_{loc}(\mathbb{R}^2)$$  

and

$$-\Delta \hat{\omega} = \hat{\delta} \neq 0, \quad \hat{\omega} \leq 2 \log 2 \quad \text{in } \mathbb{R}^2,$$

$$\hat{\omega}(0) = 0, \quad \int_{\mathbb{R}^2} e^{\hat{\omega}} \leq 1, \quad \|\hat{f}\|_{L^\infty(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \hat{f} \leq C_6.$$  

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Note that
\[ \alpha_0 = \lim_{k \to \infty} \alpha_k \neq 0 \tag{3.21} \]
by the Liouville theorem. Since \( \hat{f} \in L^1 \cap L^\infty(\mathbb{R}^2) \), the function
\[ \hat{z}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(y) \log \frac{|x-y|}{1+|y|} \, dy \tag{3.22} \]
is well-defined, and satisfies
\[ \frac{\hat{z}(x)}{\log |x|} \to \hat{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f} \text{ as } |x| \to +\infty \tag{3.23} \]
by Lemma 2.3. Similarly to the proof of Proposition 5, we see
\[ \hat{w}(x) = -\hat{z}(x) + \hat{z}(0), \tag{3.24} \]
\[ \hat{w}(x) \geq -\hat{\gamma} \log(1+|x|) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} \, dy, \]
for any \( x \in \mathbb{R}^2 \), where
\[ \hat{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}. \]

**Step 2.** We introduce
\[ \hat{B} = \{ \beta \in \text{supp } P \mid \limsup_{k \to \infty} (w_{k,\beta}(x_k) - w_{k,\alpha_k}(x_k)) > -\infty \} \tag{3.25} \]
and put
\[ \hat{\beta}_{\text{inf}} = \inf_{\beta \in \hat{B}} \beta. \tag{3.26} \]
Note that \( P(\hat{B}) > 0 \), and so \( \hat{\beta}_{\text{inf}} \) is well-defined, since \( \hat{f} \neq 0 \) as in (3.20).

In this step, we shall show
\[ \frac{\hat{\beta}_{\text{inf}} \hat{\gamma}}{\alpha_0} > 2, \tag{3.27} \]
where \( \alpha_0 \) is as in (3.21).

By the definition of \( \hat{B} \), for every \( \beta \in \hat{B} \), there exists a subsequence such that
\[ \hat{w}_{k,\beta}(0) = w_{k,\beta}(x_k) - w_{k,\alpha_k}(x_k) = O(1). \]
It follows from (3.18) that
\[ -\Delta \hat{w}_{k,\beta} = \frac{\beta}{\alpha_k} (\hat{f}_k - \hat{\delta}_k e^{\hat{\xi}_k}) \text{ in } B_{E_k}. \]
We repeat the procedure developed in Step 1 to obtain \( \hat{w}_\beta = \hat{w}_\beta(x) \in C^2(\mathbb{R}^2) \) satisfying
\[ \hat{w}_{k,\beta} \to \hat{w}_\beta \text{ in } C^2_{\text{loc}}(\mathbb{R}^2), \]
\[ -\Delta \hat{w}_\beta = \frac{\beta}{\alpha_0} \hat{f}, \quad \hat{w}_\beta \leq \hat{w}_\beta(0) \leq 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{\hat{w}_\beta} \leq 1, \]
and
\[ \hat{w}_\beta(x) \geq -\frac{\beta}{\alpha_0} \hat{\gamma} \log(1+|x|) + \frac{\beta}{2\pi \alpha_0} \int_{\mathbb{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} \, dy, \tag{3.28} \]
where \( \hat{f} = \hat{f}(x) \) is the limit function in (3.19).

If \( \hat{\beta}_{\text{inf}} \in \mathcal{B} \) then we take \( \hat{\beta} = \hat{\beta}_{\text{inf}} \), and obtain (3.27) by (3.28) and

\[
\hat{f} \in L^1 \cap L^\infty(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} e^{\hat{w}_\beta} \leq 1.
\]

If \( \hat{\beta}_{\text{inf}} \not\in \mathcal{B} \) then we take \( \hat{\beta}_j \in \mathcal{B} \) satisfying \( \hat{\beta}_j \downarrow \hat{\beta}_{\text{inf}} \), and obtain \( \varepsilon_1 > 0 \) independent of \( j \) such that \( \hat{\beta}_j \hat{\gamma} \geq 2 + \varepsilon_1 \), using (3.28) for \( \hat{\beta} = \hat{\beta}_j \) and

\[
\frac{\hat{\beta}_j}{2\pi} \int_{\mathbb{R}^2} \hat{f}(y) \log \frac{|y|}{1+|y|} \, dy = O(1), \quad \int_{\mathbb{R}^2} e^{\hat{w}_{\beta_j}} \leq 1,
\]

and thus (3.27) is shown.

**Step 3.** Given a bounded open set \( \omega \subset \mathbb{R}^2 \), it holds that

\[
\int_{I_+} \left( \int_{\omega} e^{\hat{w}_{k,\hat{\beta}}} \, dx \right) \mathcal{P}(d\beta) \leq 1,
\]

and hence

\[
\int_{\omega} e^{\hat{w}_{k,\hat{\beta}}} \, dx \mathcal{P}(d\beta) \rightarrow \hat{\zeta}^\omega(d\beta) \quad \text{in } \mathcal{M}(I_+)
\]

for some \( \hat{\zeta}^\omega \in \mathcal{M}(I_+) \). Similarly to the proof of Lemma 2.6 we see that there exists \( \hat{\psi}^\omega \in L^1(I_+, \mathcal{P}) \) such that \( 0 \leq \hat{\psi}^\omega \leq 1 \) \( \mathcal{P} \)-a.e. on \( I_+ \) and

\[
\hat{\zeta}^\omega(\eta) = \int_{\eta} \hat{\psi}^\omega(\beta) \mathcal{P}(d\beta)
\]

for any Borel set \( \eta \subset I_+ \). We take \( R_j \uparrow +\infty \) and put \( \omega_j = B_{R_j} \). From the monotonicity of \( \hat{\psi}^\omega \) with respect to \( \omega \), there exist \( \hat{\zeta} \in \mathcal{M}(I_+) \) and \( \hat{\psi} \in L^1(I_+, \mathcal{P}) \) such that

\[
0 \leq \hat{\psi}(\beta) \leq 1, \quad \mathcal{P}\text{-a.e. } \beta
\]

\[
0 \leq \hat{\psi}^{\omega_j}(\beta) \leq \hat{\psi}^{\omega_2}(\beta) \leq \cdots \to \hat{\psi}(\beta), \quad \mathcal{P}\text{-a.e. } \beta
\]

\[
\hat{\zeta}(\eta) = \int_{\eta} \hat{\psi}(\beta) \mathcal{P}(d\beta) \quad \text{for any Borel set } \eta \subset I_+.
\]

It follows from (3.19) that

\[
\alpha_0 \lambda \int_{I_+} \beta \hat{\psi}^{\omega_j}(\beta) \mathcal{P}(d\beta) = \lim_{k \to \infty} \alpha_0 \lambda k \int_{I_+} \beta \left( \int_{\omega_j} e^{\hat{w}_{k,\hat{\beta}}} \, dx \right) \mathcal{P}(d\beta) = \int_{\omega_j} \hat{f},
\]

and thus we obtain

\[
\hat{\gamma} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f} = \frac{\alpha_0 \lambda}{2\pi} \int_{I_+} \beta \hat{\psi}(\beta) \mathcal{P}(d\beta), \quad (3.29)
\]

sending \( j \to \infty \).

Complying the proof of Lemma 2.7 one can show that

\[
\lim_{r \to +\infty} ru_r = -\hat{\gamma}, \quad \lim_{r \to +\infty} \hat{u}_\theta = 0, \quad (3.30)
\]
by using (3.22), (3.24), (3.19), (3.25)-(3.26), (3.23), (3.27) and the property, derived from (3.19), (3.24) and (3.21), that
\[ \hat{w}_{k,\beta}(x) = \beta \alpha_k \hat{w}_k(x) \]
\[ \leq \beta \alpha_k \hat{w}_k(x) = \beta \alpha_0 (\hat{z}(x) - \hat{z}(0)) + o(1) \]
as \( k \to \infty \), locally uniformly in \( x \in \mathbb{R}^2 \), for any \( \beta \in I_+ \), where \((r, \theta)\) denotes the polar coordinate in \( \mathbb{R}^2 \). Then, following the proof of Lemma 2.8, we use the Pohozaev identity (2.25), (3.30), (3.29) and the value \( \bar{\lambda} \) given in (1.10) to obtain
\[ \int_{I_+} \hat{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{I_+} \hat{\phi}_0(\beta) \hat{\psi}(\beta) \mathcal{P}(d\beta) \right)^2, \quad (3.31) \]
where
\[ \hat{\phi}_0(\beta) = \frac{\sqrt{\alpha_0}}{\int_{I_+} \alpha \mathcal{P}(dx)} \beta. \]

Step 4. In this final step, we shall show that there exists a \( \mathcal{P} \)-measurable set \( J \subset I_+ \) such that \( \tilde{\psi} = \chi_J \mathcal{P} \)-a.e. on \( I_+ \) and that
\[ \mathcal{P}(J) > 0, \quad \mathcal{P}(J \cap I_{inf}) = 0, \quad \frac{\mathcal{P}(J)}{(\int_J \beta \mathcal{P}(d\beta))^2} = \frac{1}{\left( \int_{I_+} \beta \mathcal{P}(d\beta) \right)^2}. \quad (3.32) \]
The proof of the lemma is reduced to showing (3.32) since (3.7) and (3.32) do not occur simultaneously by Lemma 3.3.

Noting that \( \tilde{\psi} = \chi_{I_{inf}} \mathcal{P} \)-a.e. on \( I_+ \), recall Lemma 3.1 and that
\[ B_{\gamma_k R}(x_k) \cap B_{\sigma_k R} = \emptyset, \quad \int_{\Omega} e^{w_k} = 1 \]
for any \( k \gg 1, \beta \in I_+ \setminus \{0\} \) and \( R > 0 \), we find
\[ 0 \leq \tilde{\psi} + \hat{\psi} \leq 1 \quad \mathcal{P} \text{-a.e. on } I_+, \]
and thus
\[ \hat{\psi} = 0 \quad \mathcal{P} \text{-a.e. on } I_{inf}. \quad (3.33) \]
We put
\[ \hat{I} = I_+ \setminus I_{inf} \]
and see from (3.31) and (3.33) that
\[ \hat{d} = \int_{\hat{I}} \hat{\psi}(\beta) \mathcal{P}(d\beta) = \left( \int_{\hat{I}} \hat{\phi}_0(\beta) \hat{\psi}(\beta) \mathcal{P}(d\beta) \right)^2 > 0. \quad (3.34) \]
Let
\[ \hat{\mathcal{C}}(\psi) = \int_{\hat{I}} \hat{\phi}_0(\beta) \hat{\psi}(\beta) \mathcal{P}(d\beta) \]
\[ \hat{\mathcal{C}} = \{ \psi \mid 0 \leq \hat{\psi}(\beta) \leq 1 \mathcal{P} \text{-a.e. on } \hat{I} \text{ and } \int_{\hat{I}} \hat{\psi}(\beta) \mathcal{P}(d\beta) = \hat{d} \}. \]
Noting the monotonicity of $\hat{\phi}_0 = \hat{\phi}_0(\beta)$ and complying the proof of Lemma \ref{lem:2.9}, we can show the following properties:

(a) The value $\sup_{\psi \in \hat{I}} \hat{I}(\psi)$ is attained by

$$\psi^*(\beta) = \chi_{\\{\hat{\phi}_0 > \hat{s}\}} \cap \hat{I}(\beta) + \hat{c} \chi_{\\{\hat{\phi}_0 = \hat{s}\}}(\beta)$$

with $\hat{s}$ and $\hat{c}$ defined by

$$\hat{s} = \inf \{t \mid \mathcal{P}(\{\hat{\phi}_0 > t\} \cap \hat{I}) \leq \hat{d}\}$$
$$\hat{c} \mathcal{P}(\{\hat{\phi}_0 = \hat{s}\}) = \hat{d} - \mathcal{P}(\{\hat{\phi}_0 > \hat{s}\} \cap \hat{I}), \quad 0 \leq \hat{c} \leq 1.$$

(b) The uniqueness holds in the sense that if $\psi_m \in \hat{C}$ is the maximizer then $\psi_m = \psi^* \mathcal{P}$-a.e. on $\hat{I}$.

Following the argument to show (\ref{eq:3.4}), which is developed in the first part of the present section, and using (\ref{eq:1.4}), (\ref{eq:1.10}), (\ref{eq:3.33}) and properties (a)-(b), we find that there exists $\hat{\beta} \in I_+$ such that

$$\hat{\psi} = \chi_{\hat{J}} \mathcal{P} \text{-a.e. on } \hat{I}, \quad (3.35)$$

where

$$\hat{J} = \begin{cases} [\hat{\beta}, 1] \setminus I_{\inf} & \text{if } \mathcal{P}(\{\hat{\beta}\}) = 0 \text{ or if } \mathcal{P}(\{\hat{\beta}\}) > 0 \text{ and } \hat{c} = 1 \\ (\hat{\beta}, 1] \setminus I_{\inf} & \text{otherwise (i.e., } \mathcal{P}(\{\hat{\beta}\}) > 0 \text{ and } \hat{c} = 0). \end{cases}$$

Note that either $\hat{c} = 0$ or $\hat{c} = 1$ holds if $\mathcal{P}(\{\hat{\beta}\}) > 0$. Consequently, (\ref{eq:3.34}) and (\ref{eq:3.35}) yield

$$\frac{\mathcal{P}(\hat{J})}{(\int_j \beta \mathcal{P}(d\beta))^2} = \frac{\alpha_0}{(\int_{I_+} \alpha \mathcal{P}(d\alpha))^2} \leq \frac{1}{(\int_{I_+} \alpha \mathcal{P}(d\alpha))^2},$$

which implies

$$\frac{\mathcal{P}(\hat{J})}{(\int_j \beta \mathcal{P}(d\beta))^2} = \frac{1}{(\int_{I_+} \alpha \mathcal{P}(d\alpha))^2},$$

by $\alpha_0 \leq 1$ and (\ref{eq:1.10}). Hence (\ref{eq:3.32}) is shown for $J = \hat{J}$.

**Lemma 3.5.** There exist $t \in (0, 1)$ and $C_7 > 0$ such that

$$\sup_{\partial B_r} w_{k, \alpha} \leq C_7 + t \inf_{\partial B_r} w_{k, \alpha} - 2(1 - t) \log r$$

for any $r \in \{2r', R_0\}$, $r' \in (0, R_0/2]$, $\alpha \in I_+$, and $k \gg 1$, where $t$ and $C_7$ are independent of $r$, $r'$, $R_0$, $\alpha$ and $k \gg 1$.

**Proof.** We comply \cite{12}. Fix $r \in \{2r', R_0\}$ and $r' \in (0, R_0/2]$, and put

$$z_{k, \alpha}(x) = w_{k, \alpha}(rx) + 2 \log r$$

for $\alpha \in I_+$ and $k$. Then it holds that

$$-\Delta z_{k, \alpha} = \alpha \lambda_k e^{\xi_k(rx)} \int_{I_+} \beta \left( e^{z_{k, \beta}} - \frac{1}{\|\Omega\|} \right) \mathcal{P}(d\beta) \text{ in } B_2 \setminus \overline{B_{1/2}}.$$  \hfill (3.36)
It follows from Lemma 3.4 that
\[
z_{k,\alpha}(x) = (w_{k,\alpha}(rx) + 2 \log(r|x|)) - 2 \log |x| \leq C_5 + 2 \log 2 \tag{3.37}
\]
for any \(x \in B_2 \setminus B_{1/2}, \alpha \in I_+\) and \(k \gg 1\). Thus there exists \(C_8 > 0\), independent of \(r, r', R_0, \alpha\) and \(k \gg 1\), such that
\[
\left| |r.h.s. of (3.36)| \right| \leq \lambda_k \sup_{B_{2r_0}} e^{\xi_k} \left( \frac{1}{|\Omega|} + \sup_{\beta \in I_+ \setminus B_2 \setminus B_{1/2}} e^{z_{k,\beta}} \right)
\leq \lambda_k \sup_{B_{2r_0}} e^{\xi_k} \left( \frac{1}{|\Omega|} + 4 e^{C_5} \right) \leq C_8 \quad \text{in} \ B_2 \setminus B_{1/2} \tag{3.38}
\]
for \(\alpha \in I_+\) and \(k \gg 1\) by (3.37).

Let \(z'_{k,\alpha} = z_{k,\alpha}'(x)\) be the unique solution to
\[
- \Delta z'_{k,\alpha} = \alpha \lambda_k e^{\xi_k(rx)} \int_{I_+} \beta \left( e^{z_{k,\beta}} - \frac{1}{|\Omega|} \right) P(d\beta) \quad \text{in} \ B_2 \setminus B_{1/2},
\]
\[z'_{k,\alpha} = 0 \quad \text{on} \ \partial(B_2 \setminus B_{1/2}).\]

The elliptic regularity and (3.38) admit \(C_9 > 0\), independent of \(r, r', R_0, \alpha\) and \(k \gg 1\), such that
\[
|z'_{k,\alpha}| \leq C_9 \quad \text{in} \ B_2 \setminus B_{1/2} \tag{3.39}
\]
for \(\alpha \in I_+\) and \(k \gg 1\). Here we introduce
\[
h_{k,\alpha}(x) = C_{10} + (z'_{k,\alpha}(x) - z_{k,\alpha}(x)), \quad C_{10} = C_5 + 2 \log 2 + C_9
\]
in view of (3.37) and (3.39). The maximum principle assures that \(h_{k,\alpha} = h_{k,\alpha}(x)\) is the non-negative harmonic function on \(B_2 \setminus B_{1/2}\), and then the Harnack inequality admits a universal constant \(t \in (0, 1)\) such that
\[
t \sup_{\partial B_1} h_{k,\alpha} \leq \inf_{\partial B_1} h_{k,\alpha}
\]
or
\[
t \sup_{\partial B_1} (z'_{k,\alpha} - z_{k,\alpha}) \leq (1 - t)C_{10} + \inf_{\partial B_1} (z'_{k,\alpha} - z_{k,\alpha}) \tag{3.40}
\]
for \(\alpha \in I_+\) and \(k \gg 1\). Combining (3.39) and (3.40) shows
\[
-t C_9 - t \inf_{\partial B_1} z_{k,\alpha} \leq (1 - t)C_{10} + C_9 - \sup_{\partial B_1} z_{k,\alpha},
\]
which means the lemma for \(C_7 = (1 + t)C_9 + (1 - t)C_{10}\).

**Lemma 3.6.** There exist \(\varepsilon_*, R_*, C_\iota > 0\) and \(C_i > 0\) \((i = 11, 12)\) such that
\[
w_{k,\alpha}(0) + C_{11} \inf_{\partial B_r} w_{k,\alpha} + 2(1 + C_{11}) \log r \leq C_{12} \tag{3.41}
\]
for any \(r \in (0, R_*], \alpha \in [\beta_{\infty} - \varepsilon_*, 1]\) and \(k \gg 1\), where \(\varepsilon_*, R_*, C_{11}\) and \(C_{12}\) are independent of \(r, \alpha\) and \(k \gg 1\).
Proof. At first, we note that there exists \( \delta = \delta(\mathcal{P}, I_{\text{int}}) > 0 \) such that

\[
\beta_{\text{int}} = (1 + \delta) \frac{\int_{I_{\text{int}}} \beta \mathcal{P}(d\beta)}{2\mathcal{P}(I_{\text{int}})}
\]
since \( \beta_{\text{int}} > \int_{I_{\text{int}}} \beta \mathcal{P}(d\beta)/(2\mathcal{P}(I_{\text{int}})) \) by Lemma 2.3, Lemma 3.1 and 3.7. We put

\[
D = \frac{2}{\delta}
\]
and introduce the auxiliary function

\[
P_{k,\alpha}(r) = w_{k,\alpha}(0) + \frac{D}{2\pi r} \int_{\partial B_r} w_{k,\alpha} ds + 2(1 + D) \log r
\]
inspired by \([3, 22]\). Since

\[
\frac{d}{dr} \left( \frac{1}{2\pi r} \int_{\partial B_r} w_{k,\alpha} ds \right) = \frac{1}{2\pi r} \int_{\partial B_r} \frac{\partial w_{k,\alpha}}{\partial \nu} ds,
\]

it holds that

\[
\frac{dP_{k,\alpha}}{dr}(r) \leq \frac{D\lambda_k}{2\pi r} Q_{k,\alpha}(r), \quad (3.42)
\]
for \( r \in (0, R_0] \) and \( \alpha \in I_+ \), where \( \nu \) is the outer unit normal vector and

\[
Q_{k,\alpha}(r) = \frac{4\pi(1 + D)}{D\lambda_k} + \frac{1}{|\Omega|} \int_{B_r} e^{\xi_k} dx \cdot \int_{I_+} \beta \mathcal{P}(d\beta) - \alpha \int_{I_{\text{int}}} \beta \left( \int_{B_r} e^{w_{k,\alpha} + \xi_k} dx \right) \mathcal{P}(d\beta).
\]

Given \( \varepsilon > 0 \) whose range is determined later on, there exists \( R_\varepsilon = R_\varepsilon(\mathcal{P}, \Omega) > 0 \) such that

\[
\frac{1}{|\Omega|} \int_{B_{R_\varepsilon}} e^{\xi_k} dx \cdot \int_{I_+} \beta \mathcal{P}(d\beta) \leq \varepsilon \quad (3.43)
\]
for any \( k \). We may assume that \( R_\varepsilon \) is monotone increasing in \( \varepsilon \). We also have \( L_\varepsilon > 0 \), independent of \( r \) and \( k \), such that

\[
\int_{I_{\text{int}}} \beta \left( \int_{B_r} e^{w_{k,\alpha} + \xi_k} dx \right) \mathcal{P}(d\beta) \geq \int_{I_{\text{int}}} \beta \mathcal{P}(d\beta) - \varepsilon \quad (3.44)
\]
for any \( r \geq \sigma_k L_\varepsilon \) and \( k \gg 1 \) by the definition of \( \tilde{\psi} \), Lemma 3.1 and the convergence \([2, 21]\). We may assume that \( L_\varepsilon \) is monotone decreasing in \( \varepsilon \). It is clear that

\[
4\pi(1 + D) \leq \frac{4\pi(1 + D)}{D\lambda_k} + \varepsilon \quad (3.45)
\]
for \( k \gg 1 \). Properties \([3.13]-[3.15]\) imply

\[
Q_{k,\alpha}(r) \leq 2\varepsilon + \frac{4\pi(1 + D)}{D\lambda} - (\beta_{\text{int}} - \varepsilon) \left( \int_{I_{\text{int}}} \beta \mathcal{P}(d\beta) - \varepsilon \right) \quad (3.46)
\]
for any \( r \in [\sigma_k L_\varepsilon, R_\varepsilon] \), \( \alpha \in [\beta_{\text{int}} - \varepsilon, 1] \) and \( k \gg 1 \).
We now examine the range of $\varepsilon$ such that the right-hand-side of (3.46) is non-positive. It follows from (1.10) and (3.7) that

$$4\pi(1 + D) \frac{D\lambda}{D} = (1 + 1/D) \frac{\left(\int_{\inf} \beta \mathcal{P}(d\beta)\right)^2}{2P(\inf)}.$$  (3.47)

We use (3.7), (3.47) and $D = 2/\delta$ to obtain

$$[\text{r.h.s. of (3.46)}] = -\varepsilon^2 + \left\{2 + \left(1 + \frac{1 + \delta}{2P(\inf)} \right) \int_{\inf} \beta \mathcal{P}(d\beta) \right\} \varepsilon - \frac{\delta \left(\int_{\inf} \beta \mathcal{P}(d\beta)\right)^2}{4P(\inf)},$$  

and therefore, there exists $\varepsilon^* = \varepsilon^*(P, \inf) > 0$ such that

$$[\text{r.h.s. of (3.46)}] \leq 0$$  (3.48)

for any $0 < \varepsilon < \varepsilon^*$.

Noting that $Q_{k,\alpha}(r)$ is independent of $\varepsilon$, we organize (3.42), (3.46) and (3.48), so that $P'_{k,\alpha}(r) \leq 0$ for any $r \in [\sigma_k L_\varepsilon, R_\varepsilon]$, $\alpha \in [\inf - \varepsilon^*, 1]$ and $k \gg 1$. This implies

$$\sup_{0 < r \leq R_\varepsilon} P_{k,\alpha}(r) = \sup_{0 < r \leq \sigma_k L_\varepsilon} P_{k,\alpha}(r)$$  (3.49)

for $\alpha \in [\inf - \varepsilon^*, 1]$ and $k \gg 1$, where $R_\varepsilon = R_{\alpha}$ and $L_\varepsilon = L_{\alpha}$. Using $w_{k,\alpha} \leq \tilde{w}_{k,\alpha}(0) \leq \bar{w}_k(0)$ valid for any $\alpha \in I_+$, we estimate $P_{k,\alpha}$ by

$$P_{k,\alpha}(r) = (1 + D)w_{k,\alpha}(0) + \frac{D}{2\pi r} \int_{\partial B_r} (w_{k,\alpha} - w_{k,\alpha}(0)) ds + 2(1 + D) \log r \leq (1 + D)(w_{k,\alpha}(0) + 2 \log \sigma_k L_\varepsilon) \leq 2(1 + D) \log L_\varepsilon$$  (3.50)

for $r \in (0, \sigma_k L_\varepsilon]$, $\alpha \in [\inf - \varepsilon^*, 1]$ and $k \gg 1$.

Finally, we obtain $C_{11} = D = 2/\delta$ and $C_{12} = 2(1 + D) \log L_\varepsilon = 2(1 + 2/\delta) \log L_\varepsilon$ by (3.49), (3.50) and $[\text{l.h.s. of (3.41)}] \leq P_{k,\alpha}(r)$, provided that $\varepsilon_\varepsilon$ and $R_\varepsilon$ are given above.

We are now in a position to prove Proposition 4.

**Proof of Proposition 4**: Fix $\alpha_0 \in I_+$ such that

$$\begin{cases} 
\alpha_0 \in \max\{\alpha_{\min}, \inf - \varepsilon^*\} & \text{if $\inf > \alpha_{\min}$} \\
\alpha_0 = \alpha_{\min} > \inf - \varepsilon^* & \text{if $\inf = \alpha_{\min}$}
\end{cases}$$

recall that $\inf$, $\varepsilon^*$ and $\min$ are as in (2.19), Lemma 3.6 and (1.9), respectively. Note that $\mathcal{P}(I_\varepsilon \setminus \inf) > 0$ and that $\inf = (\inf, 1]$ and $\mathcal{P}(\{\alpha_{\min}\}) > 0$ if $\inf = \alpha_{\min}$. It follows from Lemma 3.2 and the uniform boundedness of $\xi_k$ that

$$\lim_{k \to \infty} \int_{B_{\sigma_k, \alpha_0} r} e^{w_{k,\alpha_0} + \xi_k} = 0$$  (3.51)
for any $R > 0$, where $\sigma_{k,\alpha_0} = e^{-w_{k,\alpha_0}(0)/2}$. In addition, the residual vanishing, the uniform boundedness of $\xi_k$ and the monotonicity \[\tag{2.2}\] imply

$$\lim_{k \to \infty} \int_{B_{2R_0} \setminus B_{R_0}} e^{w_{k,\alpha_0} + \xi_k} = 0, \quad \lim_{k \to \infty} \int_{\Omega \setminus \Psi^{-1}(B_{2R_0})} e^{w_{k,\alpha_0}} = 0,$$

where $R_*$ is as in Lemma 3.6.

Next, we shall prove

$$\lim_{k \to \infty} \int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} e^{w_{k,\alpha_0} + \xi_k} = 0. \quad \tag{3.53}$$

For any $r = |x| \in [\sigma_{k,\alpha_0}, R_*$, we calculate

$$w_{k,\alpha_0}(x) \leq \sup_{\partial B_r} w_{k,\alpha_0} \leq C_7 + t \inf_{\partial B_r} w_{k,\alpha_0} - 2(1 - t) \log r \leq C_7 + \frac{t}{C_{11}} \{-w_{k,\alpha_0}(0) - 2(1 + C_{11}) \log r + C_{12}\} - 2(1 + s) \log r + C_{13},$$

using Lemmas 3.5-3.6 where $s = t/C_{11}$, $C_{13} = C_7 + tC_{12}/C_{11}$.

Hence it holds that

$$\int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} e^{w_{k,\alpha_0} + \xi_k} \leq \sup_{B_{R_0}} e^{\xi_k} \cdot e^{C_{13} - sw_{k,\alpha_0}(0)} \int_{B_{R_*} \setminus B_{\sigma_{k,\alpha_0}}} |x|^{-2(1 + s)} \, dx \leq C_{14} e^{-2sw_{k,\alpha_0}(0)} \int_{1}^{\infty} r^{-(1 + 2s)} \, dr \to 0$$

as $k \to \infty$, where

$$C_{14} = 2\pi C_{12} \sup_k \sup_{B_{R_0}} e^{\xi_k}.$$

Consequently, \[\tag{3.51}-\tag{3.53}\] yield

$$\lim_{k \to \infty} \int_{\Omega} e^{w_{k,\alpha_0}} = 0,$$

which is impossible since $\int_{\Omega} e^{w_{k,\alpha_0}} = 1$ for any $k$. The proof is complete.

We conclude this section with the following proposition.

**Proposition 6.** Under the assumptions of Theorem 7 it holds that

$$\alpha_{\min} > \frac{1}{2} \int_{I_+} \beta \mathcal{P}(d\beta) = \frac{2}{\gamma}, \quad \tag{3.54}$$

**Proof.** It suffices to show that $\hat{\beta}_{\inf} = \alpha_{\min}$. Indeed, if this is the case, \[\tag{3.54}\] follows from Lemma 2.2 and \[\tag{1.16}. Since $\hat{\beta}_{\inf} \geq \alpha_{\min}$ is obvious, we assume the contrary, $\hat{\beta}_{\inf} > \alpha_{\min}$. Then it holds that supp $\tilde{\psi} \subset [\hat{\beta}_{\inf}, 1]$ by the definitions of $\beta_{\inf}$ and $\psi$, and thus we obtain $\mathcal{P}(\alpha_{\min}, (\beta_{\inf} + \alpha_{\min})/2) > 0$ and $\tilde{\psi} = 0 \mathcal{P}$-a.e. on $[\alpha_{\min}, (\hat{\beta}_{\inf} + \alpha_{\min})/2]$. However, this is impossible by \[\tag{1.14}].
4 Proof of Proposition 3

Henceforth, we put
\[ \bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w. \]

Let \( G = G(x, y) \) be the Green function:
\[ -\Delta_s G(\cdot, y) = \delta_y - \frac{1}{|\Omega|} \quad \text{in } \Omega, \quad \int_{\Omega} G(x, y) dx = 0. \]

We begin with the following lemma.

Lemma 4.1. It holds that
\[ w_{k,\alpha} - \bar{w}_{k,\alpha} \to \alpha \bar{\lambda} \left( \int_{I_{1+}} \beta P(d\beta) \right) G(\cdot, x_0) \text{ in } C^2_{\text{loc}}(\Omega \setminus \{x_0\}). \tag{4.1} \]

For every \( \omega \subset \subset \Omega \setminus \{x_0\} \), there exists \( C_{1,\omega} > 0 \), independent of \( k \) and \( \alpha \), such that
\[ \text{osc}_{\omega} w_{k,\alpha} \equiv \sup_{\omega} w_{k,\alpha} - \inf_{\omega} w_{k,\alpha} \leq C_{1,\omega}. \tag{4.2} \]

Proof. Since
\[ w_{k,\alpha}(x) - \bar{w}_{k,\alpha} = \alpha \bar{\lambda} \int_{I_{1+}} \beta e^{w_k(\cdot, y)} P(d\beta) \]
and
\[ \bar{\lambda} \int_{I_{1+}} \beta e^{w_k(\cdot, y)} P(d\beta) \to \bar{\lambda} \left( \int_{I_{1+}} \beta P(d\beta) \right) \delta_{x_0} \]
by (1.8) with \( s = 0 \) and \( S = \{x_0\} \), recalling Proposition 2, we have
\[ w_k \to \bar{w}_k \text{ locally uniformly in } \Omega \setminus \{x_0\}. \]
Then the standard argument of elliptic regularity implies (4.1) and (4.2).

We decompose \( w_k \) as \( w_k = w_k^{(1)} + w_k^{(2)} + w_k^{(3)} \), using the solutions \( w_k^{(1)} \), \( w_k^{(2)} \) and \( w_k^{(3)} \) to
\begin{align*}
- \Delta w_k^{(1)} &= g_k \quad \text{in } B_{2R_0}, \quad w_k^{(1)} = 0 \quad \text{on } \partial B_{2R_0} \\
- \Delta w_k^{(2)} &= h_k \quad \text{in } B_{2R_0}, \quad w_k^{(2)} = 0 \quad \text{on } \partial B_{2R_0} \\
- \Delta w_k^{(3)} &= 0 \quad \text{in } B_{2R_0}, \quad w_k^{(3)} = w_k \quad \text{on } \partial B_{2R_0},
\end{align*}

where
\begin{align*}
g_k &= g_k(x) \equiv \lambda_k \int_{I_{1+}} \beta e^{w_k(\cdot, x) + \xi_k(x)} P(d\beta) \\
h_k &= h_k(x) \equiv -\lambda_k \int_{I_{1+}} \beta P(d\beta) e^{\xi_k(x)}.
\end{align*}
By the elliptic regularity there exists $C_2 > 0$ independent of $k$ such that

$$-C_2 \leq w_k^{(2)} \leq 0 \quad \text{in } B_{2R_0}.$$  

By the maximum principle and Lemma 4.1, we also have $C_3 > 0$ independent of $k$ such that

$$\text{osc}_{B_{2R_0}} w_k^{(3)} \leq C_3.$$  

Thus it holds that

$$w_k(x) - w_k(0) = w_k^{(1)}(x) - w_k^{(1)}(0) + O(1) \quad (4.3)$$  

as $k \to \infty$ uniformly in $x \in B_{2R_0}$.

Let $G_0 = G_0(x,y)$ be the another Green function defined by

$$-\Delta_x G_0(\cdot,y) = \delta_y \quad \text{in } B_{2R_0}, \quad G_0(\cdot,y) = 0 \quad \text{on } \partial B_{2R_0}.$$  

Then it holds that

$$w_k^{(1)}(x) - w_k^{(1)}(0) = \int_{B_{2R_0}} (G_0(x,y) - G_0(0,y))g_k(y)dy \quad (4.4)$$  

for $x \in B_{2R_0}$. We have, more precisely,

$$G_0(x,y) = \begin{cases} 
\Gamma(|x - y|) - \Gamma\left(\frac{|y|}{2R_0}|x - \bar{y}|\right), & y \neq 0, \ y \neq x \\
\Gamma(|x|) - \Gamma(2R_0), & y = 0, \ y \neq x 
\end{cases}$$  

using the fundamental solution and the Kelvin transformation:

$$\Gamma(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|}, \quad \bar{y} = \left(\frac{2R_0}{|y|}\right)^2 y,$$  

which implies

$$G_0(x,y) - G_0(0,y) = \frac{1}{2\pi} \log \frac{|y|}{|x-y|} - \frac{1}{2\pi} \log \frac{|\bar{y}|}{|x-\bar{y}|}$$  

for $y \in B_{2R_0}$ satisfying $y \neq x$ and $y \neq 0$.

Since

$$\frac{2}{3} \leq \frac{|\bar{y}|}{|x-\bar{y}|} \leq 2, \quad x \in B_{R_0}, \ y \in B_{2R_0} \setminus \{0\},$$  

and since

$$0 \leq \int_{B_{2R_0}} g_k \leq \lambda_k \int_{I_+} \beta \mathcal{P}(d\beta) \cdot \sup_{B_{R_0}} e^{\xi_k} = O(1),$$  

we end up with

$$\int_{B_{2R_0}} (G_0(x,y) - G_0(0,y))g_k(y)dy$$

$$= \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x-y|}dy + O(1) \quad (4.5)$$  

as $k \to \infty$ uniformly in $x \in B_{R_0}$. 

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Consequently, (4.3)-(4.5) yield

\[
wk(x) - wk(0) = \frac{1}{2\pi} \int_{B_{2R_0}} g_k(y) \log \frac{|y|}{|x-y|} dy + O(1)
\]

as \(k \to \infty\) uniformly in \(x \in B_{R_0}\). This means

\[
\tilde{w}_k(x) = \frac{1}{2\pi} \int_{B_{2R_0}/\sigma_k} \sigma_k^2 g_k(y) \log \frac{|y|}{|\sigma_k x-y|} dy + O(1)
\]

as \(k \to \infty\) uniformly in \(x \in B_{R_0}/\sigma_k\), where \(\tilde{f}_k = \tilde{f}_k(y)\) is as in (2.6).

Let \(\tilde{\gamma}_k\) be as in (1.16), and put

\[
\tilde{\gamma}_k = \frac{1}{2\pi} \int_{B_{2R_0}/\sigma_k} \tilde{f}_k.
\]

(4.7)

To employ the argument of [14], we prepare the following lemma with which \(\tilde{\gamma}_k\) and \(\tilde{\gamma}\) are connected.

**Lemma 4.2.** It holds that

\[
\lim_{k \to \infty} \tilde{\gamma}_k = \tilde{\gamma}.
\]

(4.8)

**Proof.** From (2.6), \(\int_{\Omega} e^{w_k} = 1\), \(\lambda_k \uparrow \bar{\lambda}\) and (1.10), it follows that

\[
\tilde{\gamma}_k = \frac{1}{2\pi} \int_{B_{2R_0}/\sigma_k} \tilde{f}_k \leq \frac{\lambda_k}{2\pi} \int_{I_\varepsilon} \beta \mathcal{P}(d\beta) \leq \tilde{\gamma}
\]

for any \(k\). On the other hand, given \(\varepsilon > 0\), we have \(L_\varepsilon > 0\) such that

\[
\liminf_{k \to \infty} \tilde{\gamma}_k \geq \liminf_{k \to \infty} \left( \frac{1}{2\pi} \int_{B_{L_\varepsilon}} \tilde{f}_k \right) \geq \tilde{\gamma} - \varepsilon
\]

by (2.7), (2.10) and (1.16).

□

**Lemma 4.3.** For every \(0 < \varepsilon \ll 1\), there exist \(\tilde{R}_\varepsilon \geq 2\) and \(C_{4,\varepsilon} > 0\) such that

\[
\tilde{w}_k(x) \leq - (\tilde{\gamma}_k - \varepsilon) \log |x| + C_{4,\varepsilon}
\]

for \(k \gg 1\) and \(x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_\varepsilon}\).

**Proof.** By (4.8) and (2.7), given \(0 < \varepsilon \ll 1\), we can take \(\tilde{R}_\varepsilon \geq 2\) such that

\[
\frac{1}{2\pi} \int_{B_{R_\varepsilon/3}} \tilde{f}_k \geq \tilde{\gamma}_k - \varepsilon/3
\]

for \(k \gg 1\). It follows from (4.6) that

\[
\tilde{w}_k(x) = K_1^k(x) + K_2^k(x) + K_3^k(x) + O(1), \quad k \to \infty
\]

(4.12)
uniformly in \( x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_e} \), where

\[
K_1^k(x) = \frac{1}{2\pi} \int_{B_{\tilde{R}_e}/2} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy
\]

\[
K_2^k(x) = \frac{1}{2\pi} \int_{B_{|x|/2}(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy
\]

\[
K_3^k(x) = \frac{1}{2\pi} \int_{B(x)} \tilde{f}_k(y) \log \frac{|y|}{|x-y|} dy
\]

for \( B'(x) = B_{2R_0/\sigma_k} \setminus (B_{R_e/2} \cup B_{|x|/2}(x)) \).

Since

\[
\frac{|y|}{|x-y|} \leq 2 \frac{|y|}{|x|}, \quad y \in B_{R_e/2}, \quad x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_e},
\]

there exists \( C_{5,\varepsilon} > 0 \) independent of \( k \gg 1 \) and \( x \) such that

\[
K_1^k(x) \leq \frac{1}{2\pi} (\log \tilde{R}_e - \log |x|) \int_{B_{R_e/2}} \tilde{f}_k \leq C_{5,\varepsilon} - (7_k - \varepsilon/3) \log |x| \quad (4.13)
\]

for \( k \gg 1 \) and \( x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_e} \) by (4.11). We also have

\[
\frac{|y|}{|x-y|} \leq 3, \quad y \in B_{2R_0/\sigma_k} \setminus B_{|x|/2}(x),
\]

and hence

\[
K_2^k(x) \leq \log 3 \int_{B'(x)} \tilde{f}_k \leq \frac{\log 3}{2\pi} \|\tilde{f}_k\|_{L^1(B_{2R_0/\sigma_k})}
\]

\[
\leq \frac{\lambda_k \log 3}{2\pi} \int_{I_+} \beta \mathcal{P}(d\beta) \cdot \sup_{B_{\tilde{R}_e}} e^{\xi_k} \quad (4.14)
\]

for \( k \gg 1 \) and \( x \in B_{R_0/\sigma_k} \setminus B_{\tilde{R}_e} \).

Now we take

\[
D_1(x) = B_{1/|x|}(x), \quad D_2(x) = B_{|x|/2}(x) \setminus B_{1/|x|}(x).
\]

Since

\[
|y| < |x| + 1/|x|, \quad y \in D_1(x)
\]

and

\[
\frac{|y|}{|x-y|} \leq \frac{3}{2} |x|^2, \quad y \in D_2(x), \quad x \in \mathbb{R}^2 \setminus B_{\sqrt{2}}
\]
we have
\[K_k^2(x) = \frac{1}{2\pi} \int_{D_1(x) \cup D_2(x)} f_k(y) \log \frac{|y|}{|x-y|} dy \]
\[\leq \frac{1}{2\pi} \int_{D_1(x)} f_k(y) \log \frac{|x| + 1/|x|}{|x-y|} dy + \frac{2 \log |x| + \log(3/2)}{2\pi} \int_{D_2(x)} f_k \]
\[\leq \frac{\|f_k\|_{L^\infty(D_1(x))}}{2\pi} \int_{D_1(x)} \log \frac{1}{|x-y|} dy + \frac{\log(3/2)}{2\pi} \int_{D_1(x)} f_k \]
\[+ \frac{2 \log |x| + \log(3/2)}{2\pi} \int_{D_2(x)} f_k \]
\[\leq \frac{\|f_k\|_{L^\infty(D_1(x))}}{2\pi} \int_{D_1(x)} \log \frac{1}{|x-y|} dy + \frac{\log(3/2)}{2\pi} \int_{D_1(x)} f_k \]
\[+ \frac{\log |x|}{\pi} \int_{D_1(x)} f_k \leq C_0 + \frac{2\varepsilon}{3} \log |x| \quad (4.15)\]

for some $C_0 > 0$ independent of $x \in B_{R_0/\sigma_k} \setminus B_{\xi_k}$, $k \gg 1$, and $\varepsilon$.

Here, the last inequality of $(4.15)$ follows from $(4.11)$ and $(4.8)$. Properties $(4.12)$-$(4.15)$ imply $(4.10)$. \hfill \Box

Lemma 4.4. It holds that
\[\int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) \log |y| dy = O(1) \quad \text{as } k \to \infty. \quad (4.16)\]

Proof. By $(4.53)$ and $(4.3)$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that
\[-\alpha_{\min}(\gamma_k - \varepsilon_0/2) \leq -(2 + 3\delta_0) \quad (4.17)\]

for $k \gg 1$. Let
\[\tilde{R}_0 = \tilde{R}_{\varepsilon_0/2} \quad (4.18)\]

for $\tilde{R}$, as in Lemma 4.3 with $\varepsilon = \varepsilon_0/2$. Then, by $(2.11)$-$(2.12)$, $(4.10)$ and $(4.17)$, we obtain $C_{7,\varepsilon_0} > 0$ such that
\[\tilde{f}_k(y) = \lambda_k \int_{I_{\tilde{R}}} \beta e^{\tilde{w}_k(y)} + \tilde{\xi}_k(y) \mathcal{P}(d\beta) \]
\[\leq \lambda_k \int_{I_{\tilde{R}}} \beta e^{\tilde{w}_k(y)} + \tilde{\xi}_k(y) \mathcal{P}(d\beta) \]
\[\leq \lambda_k \int_{I_{\tilde{R}}} \exp \left[-\beta \{(\gamma_k - \varepsilon_0/2) \log |y| - C_{4,\varepsilon_0} \} + \sup_{B_{2R_0}} \xi_k \right] \mathcal{P}(d\beta) \]
\[\leq C_{7,\varepsilon_0} |y|^{-(2 + 3\delta_0)} \quad (4.19)\]

for $k \gg 1$ and $y \in B_{R_0/\sigma_k} \setminus B_{\xi_k}$.

Therefore, we obtain $C_{8,\varepsilon_0,\delta_0} > 0$ independent of $k \gg 1$ such that
\[\int_{B_{2R_0/\sigma_k}} \tilde{f}_k(y) \log |y| dy \leq \|\tilde{f}_k\|_{L^\infty(B_{2R_0/\sigma_k})} \int_{B_{R_0}} \log |y| dy \]
\[+ C_{7,\varepsilon_0} \int_{\mathbb{R}^2 \setminus B_{R_0}} |y|^{-(2 + 3\delta_0)} \log |y| dy \leq C_{8,\varepsilon_0,\delta_0}\]
for \( k \gg 1 \), which means \( \text{(4.10)} \).

**Lemma 4.5.** There exists \( \delta_0 > 0 \) such that
\[
\tilde{w}_k(x) = -\tilde{\gamma}_k \log |x| + O(1) \quad \text{as} \quad k \to \infty
\]
uniformly in \( x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}} \).

**Proof.** Let \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) satisfy \( \text{(4.17)} \) and consider
\[
\tilde{\gamma}_k'(x) = \frac{1}{2\pi} \int_{B_{|x|/2}} \tilde{f}_k
\]
for \( x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}} \) and \( k \gg 1 \). Since \( \text{(4.19)} \) holds, there exists \( C_{\delta_0, \varepsilon_0, \delta_0} > 0 \) such that
\[
0 \leq \tilde{\gamma}_k - \tilde{\gamma}_k'(x) \leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}}} \tilde{f}_k
\]
\[
\leq \frac{1}{2\pi} C_{\delta, \varepsilon_0} \int_{B_{2R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}}} |y|^{-(2+3\delta_0)} dy
\]
\[
\leq C_{\delta, \varepsilon_0, \delta_0} (\log \sigma_k)^{-3}
\]
for \( x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}} \) and \( k \gg 1 \).

Similarly we have
\[
\left| \int_{B_{2R_0/\sigma_k} \setminus B_{|x|/2}} \tilde{f}_k(y) \log \frac{1}{|x - y|} dy \right|
\]
\[
= \int_{(B_{2R_0/\sigma_k} \setminus B_{|x|/2}) \cap \{|y - x| \leq 1\}} \tilde{f}_k(y) \log \frac{1}{|x - y|} dy
\]
\[
+ \int_{(B_{2R_0/\sigma_k} \setminus B_{|x|/2}) \cap \{|y - x| > 1\}} \tilde{f}_k(y) \log |x - y| dy
\]
\[
\leq C_{\delta, \varepsilon_0} \left\{ (|x| - 1)^{-2(2+3\delta_0)} \int_{B_1} \log \frac{1}{|y|} dy \right. \\
\left. + \int_{(R^2 \setminus B_{|x|/2}) \cap \{|y - x| > 1\}} |y|^{-(2+3\delta_0)} \log |x - y| dy \right\} \equiv I.
\]
Since
\[
|y|^{-\delta_0} \log |x - y| \leq |y|^{-\delta_0} \log(|x| + |y|) \leq |y|^{-\delta_0} \log(3|y|)
\]
for
\[
y \in (R^2 \setminus B_{|x|/2}) \cap \{|y - x| > 1\}, \quad x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k)^{1/\delta_0}}
\]
and \( k \gg 1 \), we have \( C_{10, \delta_0} > 0 \) such that
\[
|y|^{-(2+3\delta_0)} \log |x - y| \leq C_{10, \delta_0} |y|^{-2(1+\delta_0)}
\]
for \( (x, y) \) in \( \text{(4.23)} \) with \( k \gg 1 \). Hence we have \( C_{11, \varepsilon_0, \delta_0} > 0 \) such that
\[
I \leq C_{11, \varepsilon_0, \delta_0} (\log \sigma_k)^{-2}.
\]
Now we see from (4.10) and (4.20) that

\[
\begin{align*}
|\tilde{w}_k(x) + \gamma_k'(x) \log |x| &\leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k}} \hat{f}_k(y) \log |y| \, dy \\
&+ \frac{1}{2\pi} \left| \int_{B_{2R_0/\sigma_k} \setminus B_{\varepsilon/2}} \hat{f}_k(y) \log |x - y| \, dy \right| \\
&+ \frac{1}{2\pi} \int_{B_{\varepsilon/2}} \hat{f}_k(y) \log \frac{|x|}{|x - y|} \, dy + O(1) \\
&\leq \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k} \setminus B_{\varepsilon/2}} \tilde{f}_k(y) |\log |y|| \, dy + \frac{\log 2}{2\pi} \| \tilde{f}_k \|_{L^1(B_{2R_0/\sigma_k})} \\
&+ \frac{1}{2\pi} \int_{B_{2R_0/\sigma_k} \setminus B_{\varepsilon/2}} \tilde{f}_k(y) \log \frac{1}{|x - y|} \, dy + O(1)
\end{align*}
\]

for \( x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}} \). Therefore, it holds that

\[
|\tilde{w}_k(x) + \tilde{\gamma}_k(x) \log |x|| = O(1) \quad \text{as} \quad k \to \infty \tag{4.25}
\]

by (4.10), (4.22) with (4.24), and the uniform \(L^1\) boundedness of \(\tilde{f}_k\). Then (4.21) and (4.25) imply

\[
|\tilde{w}_k(x) + \tilde{\gamma}_k \log |x|| \leq (\tilde{\gamma}_k - \tilde{\gamma}_k'(x)) \log |x| + |\tilde{w}_k(x) + \tilde{\gamma}_k'(x) \log |x|| \\
\leq C_{9,\varepsilon_0,\delta_0}(\log \sigma_k^{-1})^{-3} \log(\sigma_k^{-1} R_0) + O(1) = O(1) \quad \text{as} \quad k \to \infty
\]

for \( x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}} \). \( \square \)

Now we complete the proof of Proposition 3.

**Proof of Proposition 3.** We take \( \delta_0 \) and \( \bar{R}_0 \) as in (4.17) and (4.18), respectively. First, (2.7) and (4.10) imply

\[
|\tilde{w}_k(x) + \tilde{\gamma}_k \log(1 + |x|)| \leq |\tilde{w}_k(x)| + |\tilde{\gamma}_k \log(1 + |x|)| \\
\leq C_{12}, \quad x \in B_{\bar{R}_0}, \tag{4.26}
\]

while Lemma 4.3 means

\[
|\tilde{w}_k(x) + \tilde{\gamma}_k \log(1 + |x|)| \leq C_{13}, \quad x \in B_{R_0/\sigma_k} \setminus B_{(\log \sigma_k^{-1})^{1/\delta_0}} \tag{4.27}
\]

where \( k \gg 1 \).

Now we put

\[
\tilde{w}_k^+(x) = -\tilde{\gamma}_k \log |x| + C_{14} + \frac{C_{7,\varepsilon_0}}{9\delta_0} |x|^{-3\delta_0} \\
\tilde{w}_k^-(x) = -\tilde{\gamma}_k \log |x| - C_{14} - \frac{1}{4} |x|^2 \delta_k \sup_{B_{2R_0}} e^{\tilde{\gamma}_k}
\]

for \( C_{14} = 1 + \max\{C_{12}, C_{13}\} \) and \( k \gg 1 \), recalling (2.7), and let

\[
A_k = B_{(\log \sigma_k^{-1})^{1/\delta_0}} \setminus \bar{B}_{\bar{R}_0}.
\]

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Then (4.19) implies
\[- \Delta \tilde{w}_k^+ = C_{\gamma,\epsilon_0} |x|^{-(2+3\delta_k)} \geq \tilde{f}_k - \delta_k e^{\xi_k} \quad \text{in } A_k\]
\[\tilde{w}_k^+ \geq \tilde{w}_k \quad \text{on } \partial A_k.\]

Next, we have
\[- \Delta \tilde{w}_k^- = -\delta_k \sup_{B_{1\delta_k}} e^{\xi_k} \leq \tilde{f}_k - \delta_k e^{\xi_k} \quad \text{in } A_k\]
\[\tilde{w}_k^- \leq \tilde{w}_k \quad \text{on } \partial A_k.\]

Since \(-\Delta \tilde{w}_k = \tilde{f}_k - \delta_k e^{\xi_k}\) in \(A_k\), it follows from the maximum principle that
\[\tilde{w}_k^- \leq \tilde{w} \leq \tilde{w}_k^+ \quad \text{in } A_k. \quad (4.28)\]

Using
\[\left| \frac{1}{4} |x|^2 \delta_k \right| \leq C_{15}, \quad x \in B_{R_0/\sigma_k}\]
and
\[\left| \frac{C_{\gamma,\epsilon_0}}{9 \delta_k^3} |x|^{-3\delta_k} \right| \leq C_{16}, \quad x \in A_k,\]
we obtain
\[|\tilde{w}_k(x) + \tilde{\gamma}_k \log |x|| \leq C_{14} + \max\{C_{15}, C_{16}\}, \quad x \in A_k \quad (4.29)\]
for \(k \gg 1\).

Properties (4.26), (4.29), (1.16) and (4.38) imply (1.14) for \(\alpha = 1,\)
\[w_k(x) - w_k(0) = -\left( \frac{4}{f^{(1)}P(\mathcal{d}\beta)} + o(1) \right) \log(1 + e^{w_k(0)/|x|}) + O(1).\]

The other case of \(\alpha\) follows from the relation \((w_{k,\alpha}(x) - w_{k,\alpha}(0)) = \alpha(w_k(x) - w_k(0))\), and the proof is complete.

5 Proof of Theorem 1

We begin with the following lemma.

Lemma 5.1. It holds that
\[w_{k,\alpha}(0) = w_k(0) + O(1) \quad \text{as } k \to \infty \quad (5.1)\]
uniformly in \(\alpha \in [\alpha_{\min}, 1] \).

Proof. By the monotonicity (2.11), we have only to show
\[w_{k,\alpha_{\min}}(0) = w_k(0) + O(1). \quad (5.2)\]
As shown in the previous section, estimate (1.14) is equivalent to
\[w_{k,\alpha}(x) - w_{k,\alpha}(0) = -\alpha \tilde{\gamma}_k \log(1 + e^{w_k(0)/|x|}) + O(1), \quad (5.3)\]
where $\tilde{\gamma}_k$ is as in (4.7). Since $\alpha_{\min}\tilde{\gamma}_k \geq 2 + 3\delta_0$ for $k \gg 1$ by (4.17), we use (5.3) to get

$$\int_{B_{R_0}} e^{w_{k,\alpha_{\min}}} = O(1) \cdot e^{w_{k,\alpha_{\min}}(0)} \int_{B_{R_0}} (1 + e^{w_k(0)/2|x|})^{-\alpha_{\min}\tilde{\gamma}_k} dx$$

$$= O(1) \cdot e^{w_{k,\alpha_{\min}}(0)} \int_{B_{R_0/\sigma_k}} (1 + |x|)^{-\alpha_{\min}\tilde{\gamma}_k} dx$$

$$\leq O(1) \cdot e^{w_{k,\alpha_{\min}}(0)} \int_{\mathbb{R}^2} (1 + |x|)^{-(2+3\delta_0)} dx.$$  (5.4)

If (5.2) fails then (5.4) and Lemma 4.1 imply that $w_{k,\alpha_{\min}} \to -\infty$ uniformly in $\Omega \setminus B_{R_0/2}$, and therefore we conclude $\int_{\Omega} e^{w_{k,\alpha_{\min}}} \to 0$ as $k \to \infty$, which contradicts $\int_{\Omega} e^{w_{k,\alpha_{\min}}} = 1$. 

**Lemma 5.2.** It holds that

$$\limsup_{k \to \infty} \int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) > -\infty.$$  (5.5)

**Proof.** Fix $x' \in \partial B_{R_0/2}$. Then it holds that

$$\bar{w}_{k,\alpha} = w_{k,\alpha}(x') + O(1) = w_{k,\alpha}(0) - \alpha \tilde{\gamma}_k \log(1 + e^{w_k(0)/2|x'|}) + O(1)$$

$$= \left(1 - \frac{\alpha \tilde{\gamma}_k}{2}\right) w_k(0) + O(1)$$  (5.6)

by (1.11), (1.2), (5.3) and (5.1). Since $\tilde{\gamma} \geq \tilde{\gamma}_k$ and $\int_{I_+} (1 - \alpha \tilde{\gamma}/4) \mathcal{P}(d\alpha) = 0$ by (4.9) and (1.16), respectively, it follows that

$$\int_{I_+} (\bar{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) = 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha \tilde{\gamma}_k}{4}\right) \mathcal{P}(d\alpha) + O(1)$$

$$= 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha \tilde{\gamma}}{4}\right) \mathcal{P}(d\alpha) + \frac{1}{2} w_k(0) (\tilde{\gamma} - \tilde{\gamma}_k) \int_{I_+} \alpha \mathcal{P}(d\alpha) + O(1)$$

$$\geq 2w_k(0) \int_{I_+} \left(1 - \frac{\alpha \tilde{\gamma}}{4}\right) \mathcal{P}(d\alpha) + O(1) = O(1),$$

where we have used (5.6) and (5.1) in the first equality. 

**Lemma 5.3.** It holds that

$$J_{\lambda_k}(v_k) = \frac{\lambda_k}{2} \int_{I_+} \left(\bar{w}_{k,\alpha} + \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}}\right) \mathcal{P}(d\alpha).$$  (5.7)

**Proof.** By (1.11) and $\int_{\Omega} v_k = 0$, we have

$$\frac{1}{2} \int_{\Omega} |\nabla v_k|^2 = \frac{1}{2\alpha^2} \int_{\Omega} |\nabla w_{k,\alpha}|^2$$  (5.8)

and

$$\bar{w}_{k,\alpha} = -\log \left(\int_{\Omega} e^{\alpha v_k} dx\right),$$  (5.9)
respectively. Multiplying (1.12) by \( w_{k,\alpha} \) and using
\[
\int_{\Omega} \nabla w_{k,\alpha}^2 = \alpha \lambda_k \int_{\Omega} w_{k,\alpha} \left( \int_{I_+} \beta \left( e^{w_{k,\alpha}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\beta) \right) dx
\]
we get
\[
\int_{\Omega} |\nabla w_{k,\alpha}|^2 = \alpha \lambda_k \int_{\Omega} \int_{I_+} \beta e^{w_{k,\beta}} \mathcal{P}(d\beta) \, dx
\]
\[
= \alpha^2 \lambda_k \int_{I_+} \int_{I_+} w_{k,\beta} e^{w_{k,\beta}} \mathcal{P}(d\beta) \, dx
\]
\[
= \alpha^2 \lambda_k \int_{I_+} \int_{I_+} w_{k,\beta} e^{w_{k,\beta}} \mathcal{P}(d\beta) \, dx - \alpha^2 \lambda_k \int_{I_+} \tilde{w}_{k,\beta} \mathcal{P}(d\beta). \tag{5.10}
\]
We combine (5.8)-(5.10) with \( \mathcal{P}(I_+) = 1 \) to obtain
\[
J_{k}(v_k) = \frac{1}{2} \int_{\Omega} |\nabla v_k|^2 - \lambda_k \int_{I_+} \log \left( \int_{\Omega} e^{\alpha v_k} \right) \mathcal{P}(d\alpha)
\]
\[
= \frac{1}{2} \int_{I_+} \frac{1}{\alpha^2} \int_{\Omega} |\nabla w_{k,\alpha}|^2 \mathcal{P}(d\alpha) + \lambda_k \int_{I_+} \tilde{w}_{k,\alpha} \mathcal{P}(d\alpha)
\]
\[
= \frac{\lambda_k}{2} \int_{I_+} \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}} \mathcal{P}(d\alpha) - \frac{\lambda_k}{2} \int_{I_+} \tilde{w}_{k,\alpha} \mathcal{P}(d\alpha)
\]
\[
+ \lambda_k \int_{I_+} \tilde{w}_{k,\alpha} \mathcal{P}(d\alpha) = \frac{\lambda_k}{2} \int_{I_+} \left( \tilde{w}_{k,\alpha} + \int_{\Omega} w_{k,\alpha} e^{w_{k,\alpha}} \right) \mathcal{P}(d\alpha).
\]
The proof is complete. \( \square \)

We now prove Theorem 1 in the following.

**Proof of Theorem 1.** We shall show that (1.7) holds. To this end we apply \( \int_{\Omega} e^{w_{k,\alpha}} = 1 \) in (5.1) and get
\[
J_{k}(v_k) = \frac{\lambda_k}{2} \left\{ \int_{I_+} (\tilde{w}_{k,\alpha} + w_{k,\alpha}(0)) \mathcal{P}(d\alpha) \right. \\
+ \left. \int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx \right\}
\]
Hence the proof of (1.7) is reduced to showing
\[
\int_{I_+} \mathcal{P}(d\alpha) \int_{\Omega} (w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} dx = O(1), \tag{5.11}
\]
thanks to (5.3).

To show (5.11), we take \( x' \in \Psi_k^{-1}(\partial B_{R_0}/2) \). Then, (111, 112, 131, 131, 3.54) and (4.8) imply, uniformly in \( x \in \Omega \setminus \Psi_k^{-1}(B_{R_0}) \) and \( \alpha \in [\alpha_{\min}, 1] \), that
\[
(w_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{w_{k,\alpha}(x)} = e^{O(1)} \left( O(1) + w_{k,\alpha}(x') - w_{k,\alpha}(0) \right) e^{w_{k,\alpha}(x')}
\]
\[
= e^{O(1)} \left( O(1) - \alpha \tilde{\gamma}_k \log \left( 1 + e^{w_k(0)/2|x'|} \right) \right) e^{w_{k,\alpha}(0) - \alpha \tilde{\gamma}_k \log \left( 1 + e^{w_k(0)/2|x'|} \right)}
\]
\[
= e^{O(1)} \left( O(1) - \frac{\alpha \tilde{\gamma}_k}{2} w_k(0) \right) e^{-\left( \frac{\alpha \tilde{\gamma}_k}{2} - 1 \right) w_k(0)} = o(1).
\]
Hence it follows that
\[ \int_{\{\alpha_{\min}\}}^{} P(d\alpha) \int_{\Omega \setminus \Phi^{-1}(B_{R_0})} (u_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{u_{k,\alpha}(x)} dx = o(1). \] (5.12)

Finally, (5.3), (5.1), (3.54) and (4.8) imply
\[ \int_{B_{R_0}} (u_{k,\alpha}(x) - w_{k,\alpha}(0)) e^{u_{k,\alpha}(x)+\xi_1(x)} dx \]
\[ = -e^{u_{k,\alpha}(0)+O(1)} \int_{B_{R_0}} e^{(u_{k,\alpha}(x)-w_{k,\alpha}(0))+\xi_1(x)} \log \left(1 + e^{u_{k,\alpha}(0)/2|x|} \right) dx + O(1) \]
\[ = -e^{u_{k,\alpha}(0)+O(1)} \int_{B_{R_0}} \frac{\log(1 + |x|)}{\left(1 + e^{u_{k,\alpha}(0)/2}|x| \right)^{\alpha_k}} dx + O(1) \]
\[ = -e^{u_{k,\alpha}(0)-w_k(0)+O(1)} \int_{B_{R_0}} \frac{\log(1 + |x|)}{(1 + |x|)^{\alpha_k}} dx + O(1) = O(1) \] (5.13)

uniformly in \( \alpha \in [\alpha_{\min}, 1] \). Then (5.11) follows from (5.12) and (5.13). \( \square \)

A Proof of Lemma 2.3

Given \( K > 0 \), we put
\[ I_1(x) = \int_{D_1} \frac{\log |x-y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy \]
\[ I_{2,K}(x) = \int_{D_{2,K}} \frac{\log |x-y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy \]
\[ I_{3,K}(x) = \int_{D_{3,K}} \frac{\log |x-y| - \log(1 + |y|) - \log |x|}{\log |x|} f(y) dy, \]

where
\[ D_1 = D_1(x) = \{ y \in \mathbb{R}^2 \mid |y-x| \leq 1 \} \]
\[ D_{2,K} = D_{2,K}(x) = \{ y \in \mathbb{R}^2 \mid |y-x| > 1, \ |y| \leq K \} \]
\[ D_{3,K} = D_{3,K}(x) = \{ y \in \mathbb{R}^2 \mid |y-x| > 1, \ |y| > K \}. \]

Then it holds that
\[ \frac{z(x)}{\log |x|} - \gamma = \frac{1}{2\pi} (I_1(x) + I_{2,K}(x) + I_{3,K}(x)). \]

We have only to show that each \( \varepsilon > 0 \) admits \( K_{\varepsilon} \) and \( L_{\varepsilon} > 0 \) such that
\[ |I_1(x)| + |I_{2,K_{\varepsilon}}(x)| + |I_{3,K_{\varepsilon}}(x)| \leq \varepsilon \] (A.1)

for all \( x \in \mathbb{R}^2 \setminus B_{L_{\varepsilon}}. \)

Since
\[ \frac{\log(1 + |y|) + \log |x|}{\log |x|} \leq \frac{\log(2 + |x|) + \log |x|}{\log |x|} \leq 3, \quad x \in \mathbb{R}^2 \setminus B_2, \ y \in D_1(x), \]
we have

$$|I_1(x)| \leq 3 \int_{D_1(x)} f(y)dy - \frac{1}{\log |x|} \int_{D_1(x)} f(y) \log |x-y|dy$$

$$\leq 3 \int_{D_1(x)} f(y)dy - \frac{\|f\|_\infty}{\log |x|} \int_{B_1} \log |y|dy \to 0 \quad (A.2)$$

uniformly as $|x| \to +\infty$, recalling $f \in L^1 \cap L^\infty(\mathbb{R}^2)$.

Next, we have

$$\left| \frac{\log |x-y| - \log(1+|y|) - \log |x|}{\log |x|} \right| \leq \frac{1}{\log |x|} \left\{ \log(1 + K) + \left| \log \frac{|x-y|}{|x|} \right| \right\}$$

for $x \in \mathbb{R}^2 \setminus B_2$ and $y \in D_{2,K}(x)$, and thus

$$|I_{2,K}(x)| \leq \frac{1}{\log |x|} \int_{D_{2,K}(x)} \left\{ \log(1 + K) + \left| \log \frac{|x-y|}{|x|} \right| \right\} f(y)dy \quad (A.3)$$

for $x \in \mathbb{R}^2 \setminus B_2$. From

$$\frac{1}{2 + |x|} \leq \frac{|x-y|}{1 + |y|} \leq 1 + |x|, \quad x \in \mathbb{R}^2, \quad |y-x| \geq 1,$$

we derive

$$\left| \frac{\log |x-y| - \log(1+|y|) - \log |x|}{\log |x|} \right| \leq 3, \quad x \in \mathbb{R}^2 \setminus B_2, \quad |y-x| \geq 1$$

to obtain

$$|I_{3,K}(x)| \leq 3 \int_{D_{3,K}(x)} f(y)dy \leq 3 \int_{\mathbb{R}^2 \setminus B_K} f(y)dy \quad (A.4)$$

for $x \in \mathbb{R}^2 \setminus B_2$.

Recalling $0 \leq f \in L^1(\mathbb{R}^2)$, let $\epsilon_0 > 0$ be given. From (A.4), there exists

$$K_0 > 0$$

such that

$$|I_{3,K}(x)| \leq \epsilon_0$$

for all $K \geq K_0$ and $x \in \mathbb{R}^2 \setminus B_2$. Next, by (A.3) any $K > 0$ admits $L_K > 0$ such that

$$|I_{2,K}(x)| \leq \epsilon_0$$

for all $x \in \mathbb{R}^2 \setminus B_{L_K}$, and therefore

$$|I_{2,K_0}(x)| + |I_{3,K_0}(x)| \leq 2\epsilon_0 \quad (A.5)$$

for all $x \in \mathbb{R}^2 \setminus B_{L_{K_0}}$.

Thus we obtain (A.1) by (A.2) and (A.5).

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