PRIMITIVE AUTOMORPHISMS OF A SIMPLE ABELIAN VARIETY

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Abstract. We shall prove that an automorphism of a simple abelian variety is primitive if and only if it is of infinite order.

1. Introduction

This note provides a supplementary result (Theorem 1.1) of my talk at the sixty-first Algebra Symposium of Mathematical Society of Japan, held at Saga University on September 7–10, 2016. My talk there was based on my previous [Og16-2].

Throughout this note, the base field is assumed to be the complex number field \( \mathbb{C} \). Let \( M \) be a smooth projective variety of dimension \( m \geq 2 \) and \( f \in \text{Bir}(M) \). \( f \) is said to be imprimitive if there are a smooth projective variety \( B \) with \( 0 < \dim B < m \) and a dominant rational map \( \pi : M \dashrightarrow B \) with connected fibers such that \( \pi \) is \( f \)-equivariant, i.e., there is \( f_B \in \text{Bir}(B) \) satisfying \( \pi \circ f = f_B \circ \pi \). As \( \pi \) is just a rational dominant map, smoothness assumption of \( B \) is harmless by Hironaka resolution of singularities ([Hi64]). We say that \( f \) is primitive if it is not imprimitive.

The notion of primitivity is introduced by De-Qi Zhang [Zh09]. Note that if \( f \) is primitive, then \( \text{ord}(f) = \infty \). Indeed, otherwise, the invariant field \( \mathbb{C}(M)^f \) is of the same transcendental degree \( m \) as the rational function field \( \mathbb{C}(M) \). Thus we have \( \varphi \in \mathbb{C}(M)^f \setminus \mathbb{C} \) as \( m \geq 1 \). Then the Stein factorization of \( \varphi : M \dashrightarrow \mathbb{P}^1 \) is \( f \)-equivariant. \( f \) is then imprimitive as \( m \geq 2 \).

Assume that \( f \in \text{Aut}(M) \). The topological entropy \( h_{\text{top}}(f) \) of \( f \) is a fundamental quantity measuring the complexity of the orbit behaviour under \( f^n \) \( (n \geq 0) \). Let \( r_p \) be the spectral radius of \( f^*|H^{p,p}(M) \). Then, by Gromov-Yomdin’s theorem, \( h_{\text{top}}(f) \) satisfies

\[
0 \leq h_{\text{top}}(f) = \log \max_{0 \leq p \leq m} r_p(f)
\]

In this note, it is harmless to regard this formula as the definition of \( h_{\text{top}}(f) \) (See eg. [Og15] and references therein for details).

The aim of this note is to remark the following:

**Theorem 1.1.** Let \( A \) be a simple abelian variety of dimension \( m \geq 2 \) and \( f \in \text{Aut}(A) \). Then \( f \) is primitive if and only if \( \text{ord}(f) = \infty \). In particular, the translation automorphism \( t_a \) \((a \in A)\) defined by \( x \mapsto x + a \) is primitive if \( a \) is a non-torsion point of \( A \) with fixed zero. Moreover, if in addition \( A \) is of CM type, then \( A \) admits a primitive automorphism of positive entropy, possibly after replacing \( A \) by an isogeny.

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Here and hereafter, an abelian variety \( A = \mathbb{C}^m/\Lambda \) is said to be simple if \( A \) has no abelian subvariety \( B \) such that \( 0 < \dim B < \dim A \). A simple abelian variety \( A \) is called of CM type if the endomorphism ring \( E := \text{End}_{\text{group}}(A) \otimes \mathbb{Q} \) is a CM field with \( [E : \mathbb{Q}] = 2 \dim A \).

By definition, a field \( E \) is a CM field if \( E \) is a totally imaginary quadratic extension of a totally real number field \( K \). Note that if an abelian variety \( B \) is isogenous to a simple abelian variety of CM type, then so is \( B \) with the same endomorphism ring as \( A \). However, \( \text{Aut}_{\text{group}}(A) \not\cong \text{Aut}_{\text{group}}(B) \) in general (even for elliptic curves of CM type).

The "only if" part of Theorem \( 1.1 \) is clear as already remarked. Theorem \( 1.1 \) is a generalization of our earlier work [Og16-2, Theorem 4.3]. The last statement of Theorem \( 1.1 \) gives an affirmative answer to a question asked by Gongyo at the symposium. Our proof is a fairly geometric one based on works due to Amerik-Campana [AC13] and Bianco [Bi16] and is in some sense close to [Og16-3].

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2. Proof of Theorem \( 1.1 \)

Let \( A \) be a simple abelian variety of dimension \( m \geq 2 \) and \( f \in \text{Aut}(A) \) such that \( \text{ord}(f) = \infty \). We first show that \( f \) is primitive.

The following two well-known propositions will be frequently used:

**Proposition 2.1.** Let \( V \) be a subvariety of \( A \) such that \( \dim V < m = \dim A \) and \( \tilde{V} \) is a Hironaka resolution of \( V \). Then \( \tilde{V} \) is of general type.

**Proof.** See [Ue75, Corollary 10.10]. \( \square \)

**Proposition 2.2.** Let \( M \) be a smooth projective variety of general type defined over a field \( k \) of characteristic 0. Then the birational automorphism group \( \text{Bir}(M/k) \) of \( M \) over \( k \) is a finite group.

**Proof.** By the Lefschetz principle, we may reduce to [Ue75, Corollary 14.3]. \( \square \)

**Lemma 2.3.** Let \( P \) be a very general closed point of \( A \). Then the \( \langle f \rangle \)-orbit \( \{ f^n(P) \mid n \in \mathbb{Z} \} \) of \( P \) is Zariski dense in \( A \).

**Proof.** As \( P \) is very general, \( f^n \) is defined at \( P \) for all \( n \in \mathbb{Z} \). By [AC13, Théorème 4.1], there is a smooth projective variety \( B \) and a dominant rational map \( \rho : A \to B \) such that \( \rho \circ f = \rho \) and \( \rho^{-1}(\rho(P)) \) is the Zariski closure of \( \langle f \rangle \)-orbit of \( P \). It suffices to show that \( \dim B = 0 \). In what follows, assume to the contrary that \( \dim B > 0 \), we derive a contradiction.

Let \( \eta \in B \) be the generic point in the sense of scheme and \( A_\eta \) be the fiber over \( \eta \). Then by Proposition 2.1 and specialization, a Hironaka resolution of each irreducible component of \( A_\eta \) is of general type over \( \mathbb{C}(B) \). By \( \rho \circ f = \rho \), \( f \) faithfully acts on \( A_\eta \) over \( \mathbb{C}(B) \). Thus, by Proposition 2.2 \( f^n = \text{id} \) on \( A_\eta \) for some positive integer \( n \). Thus \( f^n = \text{id} \) on \( A \), as the generic point \( \eta_A \) of \( A \) is in \( A_\eta \). This contradicts to \( \text{ord } f = \infty \). \( \square \)

The following general, useful proposition is due to Bianco:
Proposition 2.4. Let $X$ be a projective variety and $g \in \text{Bir}(X)$. Assume that $\pi : X \to B$ is a $g$-equivariant dominant rational map to a smooth projective variety $B$ with $\dim B < \dim X$. Assume that a Hironaka resolution $\tilde{X}_b$ of the fiber $X_b$ is of general type for a general closed point $b \in B$. Then for any very general closed point $P \in X$, the $(g)$-orbit $\{g^n(P) | n \in \mathbb{Z}\}$ of $P$ is never Zariski dense in $X$.

Proof. See [Bi16, Section 4]. See also [Og16-3, Remark 2.6] for a minor clarification. □

The next proposition completes the first part of Theorem 1.1.

Proposition 2.5. Let $A$ be a simple abelian variety of dimension $\geq 2$ and $f$ be an automorphism of $A$ of infinite order. Then $f$ is primitive.

Proof. Let $\pi : A \to B$ be an $f$-equivariant dominant rational map to a smooth projective variety $B$ with $\dim B < \dim A$ and with connected fibers. If $\dim B > 0$, then by Proposition 2.4 a Hironaka resolution $\tilde{A}_b$ of the fiber $A_b$ over $b \in B$ is of general type for general $b \in B$. Then, by Proposition 2.4, the $(f)$-orbit of a very general closed point $P \in A$ is not Zariski dense. This contradicts to Lemma 2.3. Thus $\dim B = 0$, i.e., $f$ is primitive. □

We shall show the last part of Theorem 1.1.

Let $A$ be a simple abelian variety of CM type of dimension $m \geq 2$. We write $E := \text{End}_{\text{group}}(A) \otimes \mathbb{Q}$. Then by definition, $E$ is a totally imaginary quadratic extension of a totally real number field $K$ with $[K : \mathbb{Q}] = m \geq 2$. First we make $A$ explicit up to isogeny. As $E$ is a totally imaginary field with $[E : \mathbb{Q}] = 2m$, there are exactly $2m$ different complex embeddings $\varphi_i : E \to \mathbb{C}$ ($1 \leq i \leq 2m$) such that $\varphi_{2m-i} = \overline{\varphi_i}$. Here $-$ is the complex conjugate of $\mathbb{C}$. Note that there are exactly $2^m \cdot m!$ ways of numberings $I$ of the embeddings here. Choosing one such numbering $I$, we consider the embedding:

$$\varphi_I := (\varphi_1, \varphi_2, \cdots, \varphi_m): E \to \mathbb{C}^m; \ a \mapsto (\varphi_1(a), \varphi_2(a), \ldots, \varphi_m(a)) .$$

Let $O_E$ (resp. $O_K$) be the integral closure of $\mathbb{Z}$ in $E$ (resp. in $K$). Then

$$B_I := \mathbb{C}^m / \varphi_I(O_E)$$

is an abelian variety and $A$ is isogenous to $B_I$ for some numbering $I$ (See eg. [AR07, Chapter I, Section 3]).

From now, we shall prove that the abelian variety $B := B_I$ admits an automorphism of positive entropy.

Definition 2.6. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $\mathbb{Z}$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ and $\mathbb{Z}^x$ be the unit group of the ring $\mathbb{Z}$. A real algebraic integer is an element of $\mathbb{Z} \cap \mathbb{R}$. A real algebraic integer $\alpha$ is called a Pisot number if $\alpha > 1$ and $|\alpha'| < 1$ for all Galois conjugates $\alpha' \neq \alpha$ of $\alpha$ over $\mathbb{Q}$. A Pisot number $\alpha$ is called a Pisot unit if $\alpha \in \mathbb{Z}^x$.

Then, by [BDGPS92, Theorem 5.2.2], we have

Theorem 2.7. For any real number field $L$, there is a Pisot unit $\alpha \in L$ such that $L = \mathbb{Q}(\alpha)$.

As $K$ is (totally) real, there is then a Pisot unit $\alpha$ such that $K = \mathbb{Q}(\alpha)$. Consider the linear automorphism of $\mathbb{C}^m$ defined by:

$$\tilde{f}_\alpha : \mathbb{C}^d \to \mathbb{C}^d \ ; \ (z_1, z_2, \ldots, z_m) \mapsto (\varphi_1(\alpha)z_1, \varphi_2(\alpha)z_2, \ldots, \varphi_m(\alpha)z_m) .$$
As $\alpha$ is a unit in $O_K$ (hence in $O_E$), so are $\varphi_i(\alpha)$ in $\varphi_i(O_E)$. Thus $\tilde{f}_\alpha(\varphi_i(O_E)) = \varphi_i(O_E)$ by the definition of $\varphi_i$. Hence $\tilde{f}_\alpha$ descends to an automorphism $f_\alpha$ of $B$. We set $f := f_\alpha$.

As $K$ is totally real, regardless of $I$, we have
\[
\{\varphi_i(\alpha) | 1 \leq i \leq m\} = \{\alpha := \alpha_1, \alpha_2, \ldots, \alpha_m\}.
\]
Here the right hand side is the set of all Galois conjugates of $\alpha$ over $Q$. By the construction of $f$ from $\tilde{f}_\alpha$, the left hand side set also coincides with the set of eigenvalues of $f_\alpha|H^0(B, \Omega_B^1)^*$, and therefore, coincides with the set of eigenvalues of $f^*|H^0(B, \Omega_B^1)$. As $B$ is an abelian variety, we have
\[
H^{1,1}(B) = H^0(B, \Omega_B^1) \otimes \overline{H^0(B, \Omega_B^1)}.
\]
Here $\overline{H^0(B, \Omega_B^1)}$ is the complex conjugate of $H^0(B, \Omega_B^1) \subset H^1(B, \mathbb{Z}) \otimes \mathbb{C}$. As $\alpha$ is real, it follows that $\alpha^2$ is an eigenvalue of the action of $f$ on $H^{1,1}(B)$. Hence
\[
h_{\text{top}}(f) \geq r_1(f) \geq \alpha^2 > 1.
\]
Here the last inequality follows from the fact that $\alpha > 1$. Thus $f$ is of positive entropy. In particular, $\text{ord}(f) = \infty$. Therefore, $f$ is primitive as well by the first part of Theorem 1.1.

This completes the proof of Theorem 1.1.

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