Uncertain Curve Simplification

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Abstract

We study the problem of polygonal curve simplification under uncertainty, where instead of a sequence of exact points, each uncertain point is represented by a region, which contains the (unknown) true location of the vertex. The regions we consider are disks, line segments, convex polygons, and discrete sets of points. We are interested in finding the shortest subsequence of uncertain points such that no matter what the true location of each uncertain point is, the resulting polygonal curve is a valid simplification of the original polygonal curve under the Hausdorff or the Fréchet distance. For both these distance measures, we present polynomial-time algorithms for this problem.

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1 Introduction

In this paper, we investigate the topic of curve simplification under uncertainty. There are many classical algorithms dealing with curve simplification with different distance metrics; however, it is typically assumed that the locations of points making up the curves are known precisely, which is often not ideal when modelling real-life data. An obvious example highlighting the necessity of taking uncertainty into account comes with GPS data, where each measured location comes with inherent uncertainty due to the physical characteristics of the measurement. Curve simplification is often used as a first step to reduce the noise-to-signal ratio in the trajectory data before applying other algorithms or when storing large amounts of data. In both cases modelling uncertainty could reduce the error introduced by simplifying imprecise measurements while maintaining a short, efficient representation of the data.

There is a large volume of foundational previous work in the area of curve simplification [1], including work on vertex-constrained simplification, such as the well-known algorithms by Ramer and by Douglas and Peucker [16, 33] using the Hausdorff distance, by Imai and Iri [22] using either the Hausdorff or the Fréchet distance, by Agarwal et al. [3] using the Fréchet distance, and various improvements and related approaches [7, 8, 10, 15, 20, 21, 30, 35]. In particular, the basic approach of the Imai–Iri algorithm involves computing the shortcut graph, which captures all the possible simplifications of a curve, and then finding a shortest path through the graph in terms of the number of edges from the start node to the end node, thus finding the simplification with fewest edges. We adapt this approach in our work to the setting with uncertainty.
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Figure 1 (a) An uncertain curve modelled with convex polygons and a potential realisation. (b) A valid simplification under the Hausdorff distance with the threshold $\varepsilon$: for every realisation, the subsequence is within Hausdorff distance $\varepsilon$ from the full sequence. (c) An invalid simplification under the Hausdorff distance with the threshold $\varepsilon$: there is a realisation for which the subsequence is not within Hausdorff distance $\varepsilon$ from the full sequence.

Table 1 Running time of our approach in each setting. For indecisive points, $k$ is the number of options per point. For convex polygons, $k$ is the number of vertices.

|                      | Indecisive | Disks   | Line segments | Convex polygons |
|----------------------|------------|---------|---------------|-----------------|
| Hausdorff distance   | $O(n^3k^3)$ | $O(n^3)$ | $O(n^3)$      | $O(n^3k^3)$     |
| Fréchet distance     | $O(n^3k^3)$ | $O(n^3)$ | $O(n^3)$      | $O(n^3k^3)$     |

There have recently been some advances in the study of uncertainty in computational geometry, including work on maximising and minimising various measures on uncertain points [23, 24, 25, 27, 29], triangulations [11, 28, 36], visibility in uncertain polygons [14], moving points [18], and, more recently, by Ahn et al. [5], and, more recently, by Buchin et al. [9, 32] on various minimisation and maximisation variants involving curve similarity with the Fréchet distance under uncertainty, as well as other work in combinations of curve analysis and uncertainty [6, 12, 13]. To our knowledge, there is no previous work tackling curve simplification under uncertainty.

In this paper, we adopt the locational model for the uncertain points: we know that each point exists, but we do not know its precise location. It can be represented as a discrete set of points, of which only one is the true location; we say that this model uses indecisive points. We also use imprecise points, modelled as a compact continuous set, such as disks, line segments, or convex polygons; again, the true location is one (unknown) point from the set. An uncertain curve is a sequence of uncertain points of the same kind. A realisation of an uncertain curve is a precise polygonal curve obtained by taking one point from each uncertain point.

In this paper, we solve the following problem, illustrated in Figure 1: given an uncertain curve as a sequence of $n$ uncertain points, find the shortest subsequence of the uncertain points of the curve such that for any realisation of the curve, the corresponding realisation of the subsequence is a valid simplification of that realisation. We present a family of efficient algorithms for this problem under both the Hausdorff and the Fréchet distance, with the uncertain points modelled as indecisive points, as disks, as line segments, and as convex polygons, shown in Table 1.
2 Preliminaries

Denote $[n] := \{1, 2, \ldots, n\}$ for any $n \in \mathbb{N}$. Given two points $p, q \in \mathbb{R}^2$, denote their Euclidean distance with $\|p - q\|$. 

Denote a sequence of points in $\mathbb{R}^2$ with $\pi = \langle p_1, \ldots, p_n \rangle$. For only two points $p, q \in \mathbb{R}^2$, we also use $pq$ instead of $\langle p, q \rangle$. Denote a subsequence of a sequence $\pi$ from index $i$ to $j$ with $\pi[i : j] = \langle p_i, p_{i+1}, \ldots, p_j \rangle$. This notation can also be applied if we interpret $\pi$ as a polygonal curve on $n$ vertices (of length $n$). It is defined by linearly interpolating between the successive points in the sequence and can be seen as a continuous function, for $i \in [n - 1]$ and $\alpha \in [0, 1]$:

$$\pi(i + \alpha) = (1 - \alpha)p_i + \alpha p_{i+1}.$$ 

We also introduce the notation for the order of points along a curve. Let $p := \pi(a)$ and $q := \pi(b)$ for some $a, b \in [1, n]$. Then $p \prec q$ iff $a < b$, $p \preceq q$ iff $a \leq b$, and $p \equiv q$ iff $a = b$. 

Note that we can have $p = q$ for $a \neq b$ if the curve intersects itself.

Finally, given points $p, q, r \in \mathbb{R}^2$, define the distance from $p$ to the segment $qr$ as

$$d(p, qr) := \min_{t \in qr} \|p - t\|.$$ 

An uncertainty region $U \subset \mathbb{R}^2$ describes a possible location of a true point: it has to be inside the region, but there is no information as to where exactly. We use several uncertainty models, so the regions $U$ are of different shape. An indecisive point is a form of an uncertain point where the uncertainty region is represented as a discrete set of points, and the true point is one of them: $U = \{p^1, \ldots, p^k\}$, with $k \in \mathbb{N}^+$ and $p_i \in \mathbb{R}^2$ for all $i \in [k]$. Imprecise points are modelled with uncertainty regions that are compact continuous sets. In particular, we consider disks and polygonal closed convex sets. We denote a disk with the centre $c \in \mathbb{R}^2$ and the radius $r \in \mathbb{R}^+$ as $D(c, r)$. Formally, $D(c, r) := \{p \in \mathbb{R}^2 \mid \|p - c\| \leq r\}$. Define a polygonal closed convex set (PCCS) as a closed convex set with bounded area that can be described as the intersection of a finite number of closed half-spaces. Note that this definition includes both convex polygons and line segments (in 2D). Given a PCCS $U,$ let $V(U)$ denote the set of vertices of $U,$ i.e. vertices of a convex polygon or endpoints of a line segment.

We call a sequence of uncertainty regions an uncertain curve: $\Pi = (U_1, \ldots, U_n)$. If we pick a point from each uncertainty region of $\Pi$, we get a polygonal curve $\pi$ that we call a realisation of $\Pi$ and denote it with $\pi \in \Pi$. That is, if for some $n \in \mathbb{N}^+$ we have $\pi = \langle p_1, \ldots, p_n \rangle$ and $U_i = U_1, \ldots, U_n$, then $\pi \in \Pi$ if and only if $p_i \in U_i$ for all $i \in [n]$.

Suppose we are given a polygonal curve $\pi = \langle p_1, \ldots, p_n \rangle$, a threshold $\varepsilon \in \mathbb{R}^+$, and a curve built on the subsequence of vertices of $\pi$ for some set $I = \{i_1, \ldots, i_\ell\} \subseteq [n]$, i.e. $\sigma = \langle p_{i_1}, \ldots, p_{i_\ell} \rangle$ with $i_j < i_{j+1}$ for all $j \in [\ell - 1]$ and $\ell \leq n$. We call $\sigma$ an $\varepsilon$-simplification of $\pi$ if for each segment $\langle p_{i_j}, p_{i_{j+1}} \rangle$, we have $\delta(\langle p_{i_j}, p_{i_{j+1}} \rangle, \pi[i_j : i_{j+1}]) \leq \varepsilon$, where $\delta$ denotes some distance measure, e.g. the Hausdorff or the Fréchet distance.

The Hausdorff distance between two sets $P, Q \subset \mathbb{R}^2$ is defined as

$$d_H(P, Q) := \max \left\{ \sup_{p \in P} \inf_{q \in Q} \|p - q\|, \sup_{q \in Q} \inf_{p \in P} \|p - q\| \right\}.$$ 

For two polygonal curves $\pi$ and $\sigma$ in $\mathbb{R}^2$, since $\pi$ and $\sigma$ are closed and bounded, we get

$$d_H(\pi, \sigma) = \max \left\{ \max_{p \in \pi} \min_{q \in \sigma} \|p - q\|, \max_{q \in \sigma} \min_{p \in \pi} \|p - q\| \right\}.$$ 

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1 We use $a := b$ to denote assignment, $a \equiv b$ for equivalent quantities in definitions or to point out equality by earlier definition, and $a = b$ in other contexts. We also use $\equiv$, but its usage is always explained.
The Fréchet distance is often described through an analogy with a person and a dog walking along their respective curves without backtracking, where the Fréchet distance is the shortest leash needed for such a walk. Formally, consider a set of reparametrisations $\Phi_t$ of length $\ell$, defined as continuous non-decreasing surjective functions $\phi : [0, 1] \rightarrow [1, \ell]$. Given two polygonal curves $\pi$ and $\sigma$ of lengths $m$ and $n$, respectively, we can define the Fréchet distance as

$$d_F(\pi, \sigma) \overset{\text{def}}{=} \inf_{\alpha \in \Phi_m, \beta \in \Phi_n} \max_{t \in [0, 1]} \| \pi(\alpha(t)) - \sigma(\beta(t)) \|.$$

We refer to the pair of reparametrisations as an alignment. We often consider the Fréchet distance between a curve $\pi = \langle p_1, \ldots, p_n \rangle$ and a line segment $p_1p_n$, for some $n \in \mathbb{N}, n \geq 3$. In this setting the alignment can be described in a more intuitive way; see also Figure 2.

It can be described as a sequence of locations on the line segment to which the vertices of the curves are matched, $\langle s_2, \ldots, s_{n-1} \rangle$, where $s_i \in [1, 2]$ for all $i \in \{2, \ldots, n-1\}$ and $s_i \leq s_{i+1}$ for all $i \in \{2, \ldots, n-2\}$. To see that, assign $s_1 := 1$ and $s_n := 2$ and construct a helper reparametrisation $\phi : [0, 1] \rightarrow [1, n]$, defined as $\phi(t) = (n-1) \cdot t + 1$ for any $t \in [0, 1]$. Construct another reparametrisation $\psi : [1, n] \rightarrow [1, 2]$, defined as

$$\psi(t) = \begin{cases} 
  s_{[t]} : (1 - t + [t]) + s_{[t]+1} \cdot (t - [t]) & \text{if } t \in [1, n), \\
  s_n & \text{if } t = n.
\end{cases}$$

Note that $\phi$ and $\psi \circ \phi$ satisfy the definition of reparametrisations for $\pi$ and $p_1p_n$, respectively.

We also define an alignment between a curve and a line segment for the Hausdorff distance (see Figure 3). It represents the map from the curve to the line segment, where each point on the curve is mapped to the closest point on the line segment. It is given by a sequence $\langle s_1, \ldots, s_n \rangle$, where $s_i \in [1, 2]$ for all $i \in [n]$, such that $p_1p_n(s_i) = \arg \min_{p' \in p_1p_n} \| p' - p_i \|$. In other words, $p_1p_n(s_i)$ is the closest point to $p_i$ for all $i \in [n]$; as we show in Section 4.1 the Hausdorff distance is realised as the distance between $p_i$ and $p_1p_n(s_i)$ for some $i \in [n]$. Therefore, establishing such an alignment and checking that $\| p_1p_n(s_i) - p_i \| \leq \varepsilon$ for all $i \in [n]$ allows us to check that $d_H(\pi, p_1p_n) \leq \varepsilon$ for some $\varepsilon \in \mathbb{R}_{\geq 0}$.

We are discussing the following problem: given an uncertain curve $\mathcal{U} = \langle U_1, \ldots, U_n \rangle$ with $n \in \mathbb{N}, n \geq 3$, and $U_i \in \mathbb{R}^2$ for all $i \in [n]$, and the threshold $\varepsilon \in \mathbb{R}_{\geq 0}$, find a minimal-length subsequence $\mathcal{U}' = \langle U_{i_1}, \ldots, U_{i_\ell} \rangle$ of $\mathcal{U}$ with $\ell \leq n$, such that for any realisation $\pi \in \mathcal{U}$, the corresponding realisation $\pi' \in \mathcal{U}'$ forms an $\varepsilon$-simplification of $\pi$ under some distance measure $\delta$. We solve this problem both for the Hausdorff and the Fréchet distance for uncertainty modelled with indecisive points, line segments, disks, and convex polygons.
3 Overview of the Approach

In this section, we present the short description of our approach in different settings. We work out the details and show correctness in Sections 4–6.

On the highest level, we use the shortcut graph. Each uncertain point of an uncertain curve $U = \{U_1, \ldots, U_n\}$ corresponds to a vertex. An edge connects two vertices $i$ and $j$ if and only if the distance between any realisation of $U[i : j]$ and the corresponding line segment from $U_i$ to $U_j$ is below the threshold. The path with the least edges from vertex 1 to vertex $n$ then corresponds to the simplification using least uncertain points. So, we construct the shortcut graph and find the shortest path between two vertices. The key idea is that we find shortcuts that are valid for all realisations, so any sequence of shortcuts can be chosen. We discuss this in Section 4.

In order to construct the shortcut graph, we need to check whether an edge should be added to the graph, i.e. whether a shortcut is valid. The approach is different for the Hausdorff and the Fréchet distance and for each uncertainty model. For the first and the last uncertain point of the shortcut, we state in Section 4 that there are several critical pairs of realisations that need to be tested explicitly, and then for any other pair of realisations, we know that the distance is also below the threshold. Testing each pair corresponds to finding the distance between a precise line segment and any realisation of an uncertain curve; we show the simple procedures to do this in Section 4.

4 Shortcut Testing: Intermediate Points

In this section, we discuss testing a single shortcut where we fix the realisations of the first and the last uncertain point. We start by showing some basic facts about the Hausdorff and the Fréchet distance in the precise setting, and then we use them to design simple algorithms for testing shortcuts in the uncertain settings. We answer the following problem.

Problem 1. Given an uncertain curve $U = \{U_1, \ldots, U_n\}$ on $n \in \mathbb{N}$, $n \geq 3$ uncertain points in $\mathbb{R}^2$, as well as realisations $p_1 \in U_1$, $p_n \in U_n$, check if the largest Hausdorff or Fréchet distance between $U$ and its one-segment simplification is below a threshold $\varepsilon \in \mathbb{R}^{\geq 0}$ for any realisation with the fixed start and end points, i.e. for $\delta := d_H$ or $\delta := d_F$, verify

$$\max_{\pi \in U, \pi(1) = p_1, \pi(n) = p_n} \delta(\pi, p_1 p_n) \leq \varepsilon.$$ 

4.1 Hausdorff Distance

We start by showing some useful facts about the Hausdorff distance in the precise setting. We then solve Problem 1 for $\delta := d_H$.

Lemma 2. Given $n \in \mathbb{N}^{\geq 0}$ and a precise curve $\pi = \{p_1, \ldots, p_n\}$ with $p_i \in \mathbb{R}^2$, for all $i \in [n]$, we have that for any $q \in p_1 p_n$ with $q \prec p_n$, there is some $i \in [n - 1]$ such that $s := \arg\min_{r \in p_1 p_n} \|p_i - r\|$, $t := \arg\min_{r \in p_1 p_n} \|p_{i+1} - r\|$, and $s \preceq q \prec t$.

Proof. Assume this is not the case and pick a point $q \in p_1 p_n \setminus p_n$ that forms a counterexample. We now have for all $i \in [n - 1]$ and the definitions of $s$ and $t$ given above, that $s \preceq q \implies t \preceq q$. Clearly, for $i = 1$ we have $s \equiv p_1$ and so $s \preceq q$. By induction on $i$, we can conclude that for all $i \in [n]$, $\|p_i - r\| \preceq q$. In particular, as $\arg\min_{r \in p_1 p_n} \|p_n - r\| = p_n$, this means that $p_n \preceq q$. However, we picked $q \prec p_n$. This is a contradiction, so the lemma holds. ▶
Lemma 3. Given four points $a, b, c, d \in \mathbb{R}^2$ forming segments $ab$ and $cd$, the highest distance from one segment to the other is achieved at an endpoint:

$$\max_{p \in ab} d(p, cd) = \max \{ d(a, cd), d(b, cd) \}.$$ 

Proof. As $a, b \in ab$, trivially we get $\max_{p \in ab} d(p, cd) \geq \max \{ d(a, cd), d(b, cd) \}$, so it remains to show that $\max_{p \in ab} d(p, cd) \leq \max \{ d(a, cd), d(b, cd) \}$. Consider two sets $S_1 := \{ p \mid \|p\| \leq \varepsilon \}$ and $S_2 := cd$, with $\varepsilon := \max \{ d(a, cd), d(b, cd) \}$. Take their Minkowski sum:

$$S := \{ p + q \mid p \in S_1, q \in S_2 \} \defeq \{ p + q \mid \|p\| \leq \varepsilon, q \in cd \} = \{ r \mid \|r - q\| \leq \varepsilon, q \in cd \} \defeq \{ r \mid \min_{q \in cd} \|r - q\| \leq \varepsilon \} = \{ r \mid d(r, cd) \leq \varepsilon \}.$$

Note that both sets are convex: $S_1$ is a disk and $S_2$ is a line segment. Then their Minkowski sum $S$ is also convex. By definition of $S$ and $\varepsilon$, we have $a, b \in S$. By definition of a convex set, we conclude that $ab \in S$, so $\max_{p \in ab} d(p, cd) \leq \max \{ d(a, cd), d(b, cd) \}$, and the statement of the lemma holds.

Lemma 4. Given $n \in \mathbb{N}^+$, for any precise curve $\pi = (p_1, \ldots, p_n)$ with $p_i \in \mathbb{R}^2$ for all $i \in [n]$, we have

$$d_H(\pi, p_1 p_n) = \max_{i \in [n]} d(p_i, p_{i+1}).$$

Proof. Recall the definition of Hausdorff distance in this setting:

$$d_H(\pi, p_1 p_n) = \max \left\{ \max_{p \in \pi} \min_{q \in p_1 p_n} \|p - q\|, \max_{q \in p_1 p_n} \min_{p \in \pi} \|p - q\| \right\}.$$

We first show that $\max_{q \in p_1 p_n} \min_{p \in \pi} \|p - q\| \leq \max_{p \in \pi} \min_{q \in p_1 p_n} \|p - q\| = \varepsilon$. We do a case distinction on $q \in p_1 p_n$ and show that for all $q$, we have $\min_{p \in \pi} \|p - q\| \leq \varepsilon$.

- $q = p_n$. Note $p_n \in \pi$, so $\min_{p \in \pi} \|p - q\| = 0 \leq \varepsilon$.
- $q < p_n$. Using Lemma 2, we can find $i \in [n - 1]$ and the corresponding $s$ and $t$ such that $s < q < t$. As $\max_{p \in \pi} d(p, p_{i+1}) = \varepsilon$, $d(p_i, p_{i+1}) = \|p_i - s\| \leq \varepsilon$ and $d(p_{i+1}, p_{i+1}) = \|p_{i+1} - t\| \leq \varepsilon$. But then also $d(s, p_{i+1}) \leq \|s - p_i\| \leq \varepsilon$ and $d(t, p_{i+1}) \leq \|t - p_{i+1}\| \leq \varepsilon$. By Lemma 3, we conclude that $\max_{r \in st} d(r, p_{i+1}) \leq \varepsilon$. As $s < q < t$, we have $q \in st$, so $d(q, p_{i+1}) \leq \varepsilon$. This covers all cases, so indeed for all $q \in p_1 p_n$, $\min_{p \in \pi} \|p - q\| \leq \varepsilon$, and hence we conclude $\max_{q \in p_1 p_n} \min_{p \in \pi} \|p - q\| \leq \max_{p \in \pi} \min_{q \in p_1 p_n} \|p - q\|$. We can derive

$$d_H(\pi, p_1 p_n) = \max \left\{ \max_{p \in \pi} \min_{q \in p_1 p_n} \|p - q\|, \max_{q \in p_1 p_n} \min_{p \in \pi} \|p - q\| \right\} \defeq \max_{p \in \pi} \min_{q \in p_1 p_n} \|p - q\| \defeq \max_{i \in [n-1]} d(p_i, p_{i+1}) \{\text{Lemma 3}\} = \max_{i \in [n-1]} \max \{d(p_i, p_{i+1}), d(p_{i+1}, p_{i+1})\} \{\text{Lemma 3}\} = \max_{i \in [n]} d(p_i, p_{i+1})$.\n
Indecisive points.

We are now ready to generalise the setting to include imprecision. We first show that the straightforward setting with indecisive points permits an easy solution using Lemma 4.

Lemma 5. Given \( n, k \in \mathbb{N}^0 \), \( n \geq 3 \), for any indecisive curve \( U = \{U_1, \ldots, U_n\} \) with \( U_i = \{p_{i1}^1, \ldots, p_{ik}^i\} \) for all \( i \in [n] \) and \( p_j^i \in \mathbb{R}^2 \) for all \( i \in [n], j \in [k] \), and given some \( p_1 \in U_1 \) and \( p_n \in U_n \), we have

\[
\max_{\pi \in U, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n) = \max_{i \in [2, \ldots, n-1]} \max_{j \in [k]} d(p_j^i, p_1 p_n).
\]

Proof. Assume the setting of the lemma statement. Derive

\[
\max_{\pi \in U, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n) \quad \{\text{Lemma 4}\}
\
= \max_{\pi \in U, \pi(1) = p_1, \pi(n) = p_n} \max_{i \in [n]} d(\pi(i), p_1 p_n) \quad \{\text{Def. } \in\}
\
= \max_{i \in [2, \ldots, n-1]} \max_{p \in U_i} d(p, p_1 p_n) \quad \{\text{Def. indecisive point}\}
\
= \max_{i \in [2, \ldots, n-1]} \max_{j \in [k]} d(p_j^i, p_1 p_n),
\]

as was to be shown.

Note that this means that when the start and end realisations are fixed, we can test that a shortcut is valid using the lemma above in time \( O(nk) \) for a shortcut of length \( n \).

Disks.

We proceed to present the way to test shortcuts for fixed realisations of the first and the last points when the imprecision is modelled using disks. In the next arguments the following well-known form of a triangle inequality is useful.

Lemma 6. Given a metric space \( (X, d) \) and a non-empty subset \( S \subset X \), \( S \neq \emptyset \), for any \( x, y \in X \),

\[
d(x, S) \leq d(x, y) + d(y, S).
\]

Proof. Pick some \( z' \in S \) and \( x, y \in X \). By the triangle inequality, \( d(x, z') \leq d(x, y) + d(y, z') \), so

\[
d(x, S) \overset{\text{def}}{=} \inf_{z \in S} d(x, z) \leq d(x, z') \leq d(x, y) + d(y, z'),
\]

and this holds for any choice of \( z' \). Therefore, we conclude

\[
d(x, S) \leq d(x, y) + \inf_{z \in S} d(y, z) \overset{\text{def}}{=} d(x, y) + d(y, S).
\]

Corollary 7. For any \( p, q \in \mathbb{R}^2 \) and a line segment \( ab \) on \( a, b \in \mathbb{R}^2 \),

\[
d(p, ab) \leq \|p - q\| + d(q, ab).
\]
8 Uncertain Curve Simplification

We now state the result for disks.

\textbf{Lemma 8.} Given \( n \in \mathbb{N}^+, n \geq 3, \) for any imprecise curve modelled with disks \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) with \( U_i = D(c_i, r_i) \) for all \( i \in [n] \) and \( c_i \in \mathbb{R}^2, r_i \in \mathbb{R}^+ \) for all \( i \in [n] \), and given some \( p_1 \in U_1 \) and \( p_n \in U_n \), we have

\[
\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n) = \max_{i \in \{2, \ldots, n-1\}} \left( d(c_i, p_1 p_n) + r_i \right).
\]

\textbf{Proof.} Assume the setting of the lemma. As before, we derive

\[
\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n)
= \max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} \max_{i \in [n]} d(\pi(i), p_1 p_n)
= \max_{i \in \{2, \ldots, n-1\}} \max_{p \in U_i} d(p, p_1 p_n).
\]

It remains to show that \( \max_{p \in U_i} d(p, p_1 p_n) = d(c_i, p_1 p_n) + r_i \) for any \( i \in \{2, \ldots, n-1\} \).

Firstly, pick \( p' := \arg\max_{p \in U_i} d(p, p_1 p_n) \). Note that by Corollary 7, \( d(p', p_1 p_n) \leq ||p' - c_i|| + d(c_i, p_1 p_n) \). Furthermore, as \( p' \in U_i \), by definition of \( U_i \), we have \( ||p' - c_i|| \leq r_i \). Thus, \( \max_{p \in U_i} d(p, p_1 p_n) \leq d(c_i, p_1 p_n) + r_i \), and it remains to show the inequality in the other direction.

Now pick a point \( q' := \arg\min_{q \in p_1 p_n} ||q - c_i|| \), so that \( d(c_i, p_1 p_n) = ||q' - c_i|| \). Draw the line through \( c_i \) and \( q' \) and pick the point \( p' \) on that line on the boundary of \( U_i \) on the opposite side of \( q \) w.r.t. \( c_i \). Clearly, \( ||q' - c_i|| = r_i \) and \( q' = \arg\min_{q \in p_1 p_n} ||q' - q'|| \). Thus,

\[
d(p', p_1 p_n) = ||p' - q'|| = ||q' - c_i|| + ||p' - c_i|| = d(c_i, p_1 p_n) + r_i.
\]

Note that \( p' \in U_i \), so we conclude \( \max_{p \in U_i} d(p, p_1 p_n) \geq d(c_i, p_1 p_n) + r_i \). Hence, the statement of the lemma holds.

Once again, note that this lemma allows us to test a shortcut in a straightforward manner, in time \( \mathcal{O}(n) \) for a shortcut of length \( n \).

\textbf{Polygonal closed convex sets.}

\textbf{Lemma 9.} Given \( n, k \in \mathbb{N}^+, n \geq 3, \) for any imprecise curve modelled with PCCSs \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) with \( U_i \subset \mathbb{R}^2 \) and \( V(U_i) = \{ p_1^i, \ldots, p_k^i \} \) for all \( i \in [n] \), and given some \( p_1 \in U_1 \) and \( p_n \in U_n \), we have

\[
\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n) = \max_{i \in \{2, \ldots, n-1\}} \max_{v \in V(U_i)} d(v, p_1 p_n).
\]

\textbf{Proof.} Assume the setting of the lemma. As before, derive

\[
\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_H(\pi, p_1 p_n)
= \max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} \max_{i \in [n]} d(\pi(i), p_1 p_n)
= \max_{i \in \{2, \ldots, n-1\}} \max_{p \in U_i} d(p, p_1 p_n).
\]
We now turn our attention to the Fréchet distance. In this section, we do not show results as before, this lemma gives us a simple way to test the shortcut with fixed realisations of the points.

The idea is that in the precise case we can always align greedily as we move along the line segment. In this case, we also need to find the realisation for each indecisive point that follows from a well-known fact shown e.g. by Guibas et al. [21, Lemma 8]; it can also be seen as specialisation of the indecisive point case to or of the disk case to \( r = 0 \).

### 4.2 Fréchet Distance

We now turn our attention to the Fréchet distance. In this section, we do not show results for the Fréchet distance in the precise setting. For extra intuition, we show Algorithm 1 which follows from a well-known fact shown e.g. by Guibas et al. [21] Lemma 8; it can also be seen as specialisation of the indecisive point case to \( k = 1 \) or of the disk case to \( r = 0 \).

#### Indecisive points.

The idea is that in the precise case we can always align greedily as we move along the line segment. In this case, we also need to find the realisation for each indecisive point that makes for the ‘worst’ greedy choice.

\[ \max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_F(\pi, p_1 p_n) \leq \varepsilon \iff \text{CheckFréchetInd}(\mathcal{U}, p_1, p_n, n, k, \varepsilon) = \text{True}. \]

\[ \textbf{Algorithm 1} \] Testing a shortcut on a precise curve with the Fréchet distance.

\begin{algorithm}
\begin{algorithmic}[1]
\Require \( \pi = (p_1, \ldots, p_n), n \in \mathbb{N}^{>0}, \forall i \in [n]: p_i \in \mathbb{R}^2, \varepsilon \in \mathbb{R}^{>0} \)
\Function{CheckFréchetPrecise}{\( \pi, n, \varepsilon \)}
\State \( s_1 := 1 \)
\For{i \in \{2, \ldots, n-1\}}
\State \( S_i := \{ t \in [s_{i-1}, 2] \mid || p_t - p_1 p_n(t)|| \leq \varepsilon \} \)
\If{\( S_i = \emptyset \)}
\State \Return \text{False}
\Else\State \( s_i := \min S_i \)
\EndIf
\EndFor
\State \Return \text{True}
\EndFunction
\end{algorithmic}
\end{algorithm}

To show that the claim holds, it remains to show that for any PCCS \( U \) and a line segment \( ab \) it holds that \( \max_{p \in U} d(p, ab) = \max_{v \in V(U)} d(v, ab) \). Firstly, as \( V(U) \subset U \), we immediately have \( \max_{p \in U} d(p, ab) \geq \max_{v \in V(U)} d(v, ab) \). Consider any \( p \in U \). We will show that there is some \( v \in V(U) \) such that \( d(v, ab) \geq d(p, ab) \), thus completing the proof. We do a case distinction on \( p \).

- \( p \in V(U) \). Then pick \( v := p \), and we are done.
- \( p \notin V(U) \), but \( p \) is on the boundary of \( U \). Consider the vertices \( v, w \in V(U) \) with \( p \in vw \).

Using Lemma 3 we note
\[ \max_{q \in vw} d(q, ab) = \max \{ d(v, ab), d(w, ab) \}. \]

W.l.o.g. suppose \( d(v, ab) \geq d(w, ab) \). Then for \( v \) indeed we have \( d(v, ab) \geq d(p, ab) \).

- \( p \) is in the interior of \( U \) (cannot occur for line segments). Find the point \( q' := \arg \min_{q \in ab} || p - q || \), so \( d(p, ab) = ||p - q'|| \). Draw the line through \( p \) and \( q' \); let \( p' \) be the point on that line on the boundary of \( U \) on the opposite side of \( q' \) w.r.t. \( p \). Clearly, \( q' = \arg \min_{q \in ab} || p' - q || \), so \( d(p', ab) > d(p, ab) \). Then we can find a vertex \( v \in V(U) \) as in the previous cases, yielding \( d(v, ab) \geq d(p', ab) > d(p, ab) \).

This covers all the cases, so the statement holds. –

As before, this lemma gives us a simple way to test the shortcut with fixed realisations of the first and the last points in time \( O(nk) \) for a shortcut of length \( n \) and PCCSs with \( k \) vertices.
Algorithm 2 Testing a shortcut on an indecisive curve with the Fréchet distance.

| Require: $\mathcal{U} = \{U_1, \ldots, U_n\}$, $n, k \in \mathbb{N}^0$, $\forall i \in [n]: U_i = \{p_{1}^{i}, \ldots, p_{n}^{i}\}$, $\forall i \in [n], j \in [k]: p_{t}^{i} \in \mathbb{R}^2$, $\varepsilon \in \mathbb{R}^0$, $p_1 \in U_1$, $p_n \in U_n$
| 1: function CHECKFRECHETIND($\mathcal{U}, p_{1}, p_{n}, n, k, \varepsilon$)
| 2: $s_1 := 1$
| 3: for $i \in \{2, \ldots, n-1\}$ do
| 4: $T_i := \emptyset$
| 5: for $j \in [k]$ do
| 6: $S_{ij}^{i} := \{t \in [s_{i-1}, 2] \mid \|p_{t}^{i} - p_{1}p_{n}(t)\| \leq \varepsilon\}$
| 7: if $S_{ij}^{i} = \emptyset$ then
| 8: return False
| 9: $T_i := T_i \cup S_{ij}^{i}$
| 10: $s_i := \max T_i$
| 11: return True

Proof. First, assume that $\max_{\pi \in \mathcal{U}, \pi(1) = p_{1}, \pi(n) = p_{n}, \delta_{F}(\pi, p_{1}p_{n})} \leq \varepsilon$. In the algorithm, we compute some set $S_{ij}^{i}$ for each $p_{t}^{i}$ and then pick one value from it and add it to $T_i$; from $T_i$ we then pick a single value as $s_i$. So, $s_i \in S_{ij}^{i}$ for some $j_i \in [k_i]$, on every iteration $i \in \{2, \ldots, n-1\}$. Consider a realisation $\pi \in \mathcal{U}$ with $\pi(1) = p_1$, $\pi(n) = p_n$, and $\pi(i) \equiv p_{t_i}^{i}$ for every $i \in \{2, \ldots, n-1\}$, where $j_i$ is chosen as the value corresponding to $s_i$. Then we know $\delta_{F}(\pi, p_{1}p_{n}) \leq \varepsilon$. So, there is an alignment that can be given as a sequence of $n$ positions, $t_i \in [1, 2]$, such that $\|\pi(i) - p_{1}p_{n}(t_i)\| \leq \varepsilon$ and $t_i \leq t_{i+1}$ for all $i$. The alignment is established by interpolating between the consecutive points on the curves, as discussed in Section 2.

We now show by induction that $s_i \leq t_i$ for all $i$. For $i = 2$, we get, for the chosen $j_2$, $s_2 := \min\{t \in [1, 2] \mid \|p_{t}^{2} - p_{1}p_{n}(t)\| \leq \varepsilon\}$. As we have $t_2 \in \{t \in [1, 2] \mid \|p_{t}^{2} - p_{1}p_{n}(t)\| \leq \varepsilon\}$, we get $s_2 \leq t_2$. Now assume the statement holds for some $i$, then for $i + 1$ we get $s_{i+1} := \min\{t \in [s_i, 2] \mid \|p_{t}^{i+1} - p_{1}p_{n}(t)\| \leq \varepsilon\}$; we can rephrase this so that

$$s_{i+1} \overset{\text{def}}{=} \min\{t \in [1, 2] \mid \|p_{t}^{i+1} - p_{1}p_{n}(t)\| \leq \varepsilon\} \cap [s_i, 2].$$

So, there are two options.

- $s_{i+1} = s_i$. Then we know $s_{i+1} = s_i \leq t_i \leq t_{i+1}$.
- $s_{i+1} > s_i$. Then we can use the same argument as for $i = 2$ to find that $s_{i+1} \leq t_{i+1}$.

Now we know that for every $i$, $t_i \in S_{j_i}^{i}$ for the choice of $j_i$ described above. Therefore, for any $p_{t_i}^{i+1}$ there is always a realisation prefix such that any valid alignment has $t_{i+1} \geq s_i$; as we know that there is a valid alignment for every realisation, we conclude that every $S_{ij}^{i}$ is non-empty. Thus, the algorithm returns True.

Now assume that the algorithm returns True. Consider any realisation $\pi \in \mathcal{U}$. We claim that there is a valid alignment, described with a sequence of $t_i \in [1, 2]$ for $i \in \{2, \ldots, n-1\}$, such that $s_{i-1} \leq t_i \leq s_i$ and $\|p_{1}p_{n}(t_i) - \pi(i)\| \leq \varepsilon$. Denote the realisation $\pi \overset{\text{def}}{=} (p_1, p_2, p_{j_2}^{i}, \ldots, p_{j_{i-1}}^{i-1}, p_n)$, so the sequence $(j_2, \ldots, j_{n-1})$ describes the choices of the realisation. Consider the set $S_{j_i}^{i}$ for any $i \in \{2, \ldots, n-1\}$. We know that it is non-empty, otherwise the algorithm would have returned False. We claim that we can pick $t_i = \min S_{j_i}^{i}$ for every $i$. By definition, $S_{j_i}^{i} \subseteq [1, 2]$ and $\|p_{1}p_{n}(t_i) - \pi(i)\| \leq \varepsilon$. We also trivially get that $s_{i-1} \leq t_i$. Finally, note that $t_i \in T_i$, and $s_i := \max T_i$, so $t_i \leq s_i$.

This argument shows that $t_i \leq t_{i+1}$ for every $i$, and that $\|p_{1}p_{n}(t_i) - \pi(i)\| \leq \varepsilon$. Therefore,
\[d_F(\pi, p_1p_n) \leq \varepsilon.\] As this works for any realisation with \(\pi(1) \equiv p_1\) and \(\pi(n) \equiv p_n\), we conclude \(\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_F(\pi, p_1p_n) \leq \varepsilon.\)

**Disks.**

To show the generalisation to disks, it is helpful to reframe the problem as that of disk stabbing for appropriate disks. We demonstrate some useful facts first.

> **Lemma 11.** Given a disk \(D_1 := D(c,r)\) with \(c \in \mathbb{R}^2\), \(r \in \mathbb{R}^{\geq 0}\), a threshold \(\varepsilon \in \mathbb{R}^{>0}\), and a point \(p \in \mathbb{R}^2\), define \(D_2 := D(c,\varepsilon-r)\). We have

\[
\max_{p' \in D_1} \|p-p'\| \leq \varepsilon \iff p \in D_2.
\]

**Proof.** First, assume \(p \in D_2 \overset{\text{def}}{=} \{s \in \mathbb{R}^2 \mid \|s-c\| \leq \varepsilon-r\}\); thus, we know \(\|p-c\| \leq \varepsilon-r\). Take \(q := \arg\max_{p' \in D_1} \|p-p'\|\). Then \(q \in D_1 \overset{\text{def}}{=} \{s \in \mathbb{R}^2 \mid \|s-c\| \leq r\}\), so \(\|q-c\| \leq r\). Then by the triangle inequality,

\[
\|p-q\| \leq \|p-c\| + \|q-c\| \leq \varepsilon-r + r = \varepsilon.
\]

Now assume that \(p \notin D_2 \overset{\text{def}}{=} \{s \in \mathbb{R}^2 \mid \|s-c\| \leq \varepsilon-r\}\). Then \(\|p-c\| > \varepsilon-r\). Consider a point \(q\) on the line \(pc\) on the boundary of \(D_1\), so that \(c\) is between \(p\) and \(q\) on the line. Note that \(q \in D_1\), so

\[
\max_{p' \in D_1} \|p-p'\| \geq \|p-q\| = \|p-c\| + \|q-c\| > \varepsilon-r + r = \varepsilon,
\]

completing the proof.

We can now generalise the previous statement to talk about distance to line segments.

> **Lemma 12.** Given a disk \(D_1 := D(c,r)\) with \(c \in \mathbb{R}^2\), \(r \in \mathbb{R}^{\geq 0}\), a threshold \(\varepsilon \in \mathbb{R}^{>0}\), and a line segment \(pq\) with \(p,q \in \mathbb{R}^2\), define \(D_2 := D(c,\varepsilon-r)\). We have

\[
\max_{p' \in D_1} d(p',pq) \leq \varepsilon \iff pq \cap D_2 \neq \emptyset.
\]

**Proof.** First, assume \(pq \cap D_2 \neq \emptyset\). Take \(t \in pq \cap D_2\). Consider an arbitrary point \(s \in D_1\). By Lemma 11 we know that \(\|t-s\| \leq \varepsilon\); so also \(d(s,pq) \overset{\text{def}}{=} \min_{q' \in pq} \|q'-s\| \leq \|t-s\| \leq \varepsilon\). As this holds for arbitrary \(s \in D_1\), we conclude \(\max_{p' \in D_1} \min_{q' \in pq} \|p'-q'\| \leq \varepsilon\).

Now assume that \(\max_{p' \in D_1} d(p',pq) \leq \varepsilon\). Take \(s := \arg\max_{p' \in D_1} \min_{q' \in pq} \|p'-q'\|\) and \(t := \arg\min_{q' \in pq} \|s-q'\|\). In disks it is easy to see that the furthest point of a disk from a line segment is positioned in a way that the centre of the disk is on the line through the point of the disk and the closest point of the line segment, so in our case \(c \in st\). Then \(\|t-s\| = \|t-s\| + \|s-c\| \leq \varepsilon-r\), so indeed \(t \in D_2\), and \(pq \cap D_2 \neq \emptyset\).

> **Lemma 13.** Given \(n \in \mathbb{N}^{>0}\) and \(\varepsilon \in \mathbb{R}^{>0}\), for any imprecise curve modelled with disks \(\mathcal{U} = \{U_1,\ldots,U_n\}\) with \(U_i = D(c_i,r_i)\) for all \(i \in [n]\) and \(c_i \in \mathbb{R}^2\), \(r_i \in \mathbb{R}^{\geq 0}\) for all \(i \in [n]\), and given some \(p_1 \in U_1\) and \(p_n \in U_n\), we have, using Algorithm 3

\[
\max_{\pi \in \mathcal{U}, \pi(1) = p_1, \pi(n) = p_n} d_F(\pi, p_1p_n) \leq \varepsilon \iff \text{CHECKFréchetDisks}(\mathcal{U}, p_1, p_n, n, \varepsilon) = \text{True}.
\]
Polygonal closed convex sets.

Lemma 14. Given $n, k \in \mathbb{N}_{>0}$ and $\epsilon \in \mathbb{R}_{>0}$, for any imprecise curve modelled with PCCSs $U = \{U_1, \ldots, U_n\}$ with $U_i \subset \mathbb{R}^2$ and $V(U_i) = \{p_{i1}^1, \ldots, p_{ik_i}^i\}$ for all $i \in [n]$, and given some $p_1 \in U_1$ and $p_0 \in U_n$, we have, using Algorithm 3

$$\max_{\pi : t, \pi(1) = p_1, \pi(n) = p_0} d_F(\pi, p_1p_n) \leq \epsilon \iff \text{CHECKFréchetPCCS}(U, p_1, p_n, n, k, \epsilon) = \text{True}.$$ 

Proof. As we have shown in Lemma 9, it suffices to test the vertices of a PCCS to establish that the distance from every point to the line segment is below the threshold. It remains to show that the extreme alignment (in terms of ordering) for the Fréchet distance is also achieved at a vertex. This case then becomes identical to the indecisive points case, so we can use Lemma 10 to show correctness.

Consider an arbitrary point $t \in U_i$ and let $s$ be the earliest point in the $\epsilon$-disk around $t$ that is on $pq$. Clearly, if $t$ is in the interior of $U_i$, then we can take any $t'$ on the line through
Algorithm 4 Testing a shortcut on an imprecise curve modelled with PCCSs with the Fréchet distance.

Require: $U = (U_1, \ldots, U_n)$, $n, k \in \mathbb{N}^+ \setminus \{0\}$, $\forall i \in [n] \forall j \in [k]: p_i^j \in \mathbb{R}^2$, $\varepsilon \in \mathbb{R}^+ \land p_i \in U_1$, $p_n \in U_n$

1: function CheckFréchetPCCS($U, p_1, p_n, n, k, \varepsilon$)
2:   $s_1 := 1$
3:   for $i \in \{2, \ldots, n-1\}$ do
4:       $T_i := \emptyset$
5:       for $j \in [k]$ do
6:           $S_j^i := \{ t \in [s_{i-1}, 2] \mid \| p_i^j - p_1 p_n(t) \| \leq \varepsilon \}$
7:           if $S_j^i = \emptyset$ then
8:               return False
9:           $T_i := T_i \cup \min S_j^i$
10:      $s_i := \max T_i$
11:  return True

Figure 3 Illustration for the computation in Lemma 14.

$t$ parallel to $pq$ and get the corresponding $s'$ with $s \prec s'$. So, assume $t$ is on the boundary of $U_i$. Suppose that $t \in uv$ with $u, v \in V(U_i)$. Rotate and translate the coordinate plane so that $pq$ lies on the $x$-axis. Derive the equation for the line containing $uv$, say, $y' = kx' + b$. First consider $k = 0$, so the line containing $uv$ is parallel to the line containing $pq$. In this case, clearly, moving along $uv$ in the direction coinciding with the direction from $p$ to $q$ increases the $x$-coordinate of point of interest, so moving to a vertex is optimal. Now assume $k > 0$. If $k < 0$, reflect the coordinate plane about $y = 0$. Geometrically, it is easy to see (Figure 3) that the coordinate of interest can be expressed as

$$x = x' - \sqrt{\varepsilon^2 - y'^2} = \frac{y' - b}{k} - \sqrt{\varepsilon^2 - y'^2}.$$  

We want to maximise $x$ by picking the appropriate $y'$. We take the derivative:

$$\frac{dx}{dy'} = \frac{1}{k} + \frac{y'}{\sqrt{\varepsilon^2 - y'^2}}.$$  

We can equate it to 0 to find the critical point of the function. Simplifying, we find

$$y'_0 = -\frac{\varepsilon}{\sqrt{k^2 + 1}}.$$  

We can check that for $y' < y'_0$, the value of the derivative is negative, and for $y' > y'_0$ it is positive, so at $y' = y'_0$ we achieve a local minimum. There are no other critical points. Therefore, to maximise $x$, we want to move as far as possible in either direction, away from the local minimum. Since we are limited to the line segment $uv$, the maximum is clearly achieved at one of the segment endpoints.
5 Shortcut Testing: All Points

In the previous section, we have covered testing a shortcut, given that the first and the last points are fixed. Here we remove that restriction.

**Problem 15.** Given an uncertain curve $\mathcal{U} = (U_1, \ldots, U_n)$ on $n \in \mathbb{N}$, $n \geq 3$ uncertain points in $\mathbb{R}^2$, check if the largest Hausdorff or Fréchet distance between $\mathcal{U}$ and its one-segment simplification is below a threshold $\varepsilon \in \mathbb{R}^+$ for any realisation, i.e. for $\delta = d_H$ or $\delta = d_F$, verify $\max_{\pi \in \mathcal{U}} \delta(\pi, p_1 p_n) \leq \varepsilon$.

We first show how this can be done for indecisive points, both for $\delta = d_H$ and $\delta = d_F$.

**Lemma 16.** Given $n, k \in \mathbb{N}^0$, $n \geq 3$, and $\delta = d_H$ or $\delta = d_F$, for any indecisive curve $\mathcal{U} = (U_1, \ldots, U_n)$ with $U_i = \{p_i^1, \ldots, p_i^k\}$ for all $i \in [n]$ and $p_i^j \in \mathbb{R}^2$ for all $i \in [n], j \in [k]$, we have

$$\max_{\pi \in \mathcal{U}} \delta(\pi, (\pi(1), \pi(n))) = \max_{a \in [k]} \max_{b \in [k]} \max_{\pi \in \mathcal{U}, \sigma(1) \equiv p_i^a, \sigma(n) \equiv p_i^b} \delta(\pi, p_i^a p_i^b).$$

**Proof.** We can derive

$$\max_{\pi \in \mathcal{U}} \delta(\pi, (\pi(1), \pi(n))) = \max_{p_i \in U_1 \ldots p_n \in U_n} \delta((p_1, \ldots, p_n), p_1 p_n)$$

$$\max_{p_i \in U_1 \ldots p_n \in U_n, p_2 \in U_2 \ldots p_{n-1} \in U_{n-1}} \delta((p_1, \ldots, p_n), p_1 p_n)$$

as was to be shown.

That is to say, for either Hausdorff or Fréchet distance we can simply test the shortcut using the corresponding procedure from Lemma 5 or Lemma 10, and do so for each combination of the start and end points. We can then test an indecisive shortcut of length $n$ overall in time $O(k^2 \cdot nk) = O(nk^3)$.

We now proceed to show the approach for disks and polygonal closed convex sets. The procedure is the same for the Hausdorff and the Fréchet distance, but differs between disks and PCCSs, since disks have some convenient special properties.

5.1 Disks

We start by stating some useful observations.

**Observation 17.** Suppose we are given two non-degenerate disks $D_1 := D(p_1, r_1)$ and $D_2 := D(p_2, r_2)$ with $D_1 \nsubseteq D_2$ and $D_2 \nsubseteq D_1$. We make the following observations. (See Figure 4.)

- There are exactly two outer tangents to the disks, and the convex hull of $D_1 \cup D_2$ consists of an arc from $D_1$, an arc from $D_2$, and the outer tangents.
Assume the lines of the outer tangents intersect. When viewed from the intersection point, the order in which the tangents touch the disks is the same, i.e. either both first touch \(D_1\) and then \(D_2\), or the other way around. If the lines are parallel, the same statement holds when viewed from points on the tangent lines at infinity.

To see that the second observation is true, note that the distance from the intersection point to the tangent points of a disk is the same for both tangent lines. These observations mean that we can restrict our attention to the area bounded by the outer tangents and define an ordering in the resulting strip.

**Definition 18.** Given two distinct non-degenerate disks \(D_1 := D(p_1, r_1)\) and \(D_2 := D(p_2, r_2)\), consider a strip defined by the lines that form the outer tangents to the disks. Assume we have two circular arcs \(O_1, O_2\) that intersect both tangents and lie inside the strip. Define \(s_1\) and \(v_1\) to be the points where one of the tangents touches \(D_1\) and \(D_2\), respectively, and let \(t_1\) and \(u_1\) be the points where \(O_1\) and \(O_2\) intersect that tangent, respectively. Define the order on the tangents from \(D_1\) to \(D_2\), so \(s_1 \prec v_1\). Define points \(s_2, t_2, u_2, v_2\) similarly for the other tangent. We say that \(O_2\) is to the right of \(O_1\) if either \(t_1 = u_1\) for \(i \in \{1, 2\}\) and the radius of \(O_2\) is larger than that of \(O_2\); or if otherwise \(t_i \leq u_i\) for \(i \in \{1, 2\}\) and \(O_1\) and \(O_2\) do not properly intersect. We say that \(O_2\) is to the left of \(O_1\) if either \(t_i = u_i\) for \(i \in \{1, 2\}\) and the radius of \(O_1\) is smaller than that of \(O_2\); or if otherwise \(u_i \leq t_i\) for \(i \in \{1, 2\}\) and \(O_1\) and \(O_2\) do not properly intersect. (See Figure 4 for a visual interpretation.)

We are now ready to state the main result for the Hausdorff distance.

**Lemma 19.** Given \(n \in \mathbb{N}^{\geq 0}\), \(n \geq 3\), for any imprecise curve modelled with disks \(U = \{U_1, \ldots, U_n\}\) with \(U_i = D(c_i, r_i)\) for all \(i \in [n]\) and \(c_i \in \mathbb{R}^2\), \(r_i \in \mathbb{R}^{\geq 0}\) for all \(i \in [n]\), and assuming \(U_1 \neq U_n\), we have

\[
\max_{\pi \in \mathcal{P}} d_{\text{H}}(\pi, (\pi(1), \pi(n))) \leq \varepsilon
\]

if and only if both of the following are true:

- \[
\max_{\pi \in \mathcal{P}} \max_{\pi(1) = s, \pi(n) = u} d_{\text{H}}(\pi, st) \leq \varepsilon,
\]

where \(s, u \in U_1\), \(t, v \in U_n\), and \(st\) and \(uv\) are the outer tangents to \(U_1 \cup U_n\); and

- for each \(i \in \{2, \ldots, n-1\}\), the right arc of the disk \(D(c_i, \varepsilon - r_i)\) bounded by the intersection points with the tangent lines is to the right of the right arc of \(U_1\) and the left arc of the disk \(D(c_i, \varepsilon - r_i)\) is to the left of the left arc of \(U_n\).
Proof. Assume the right side of the lemma statement holds. First of all, as we have \( \max_{\pi \in \mathcal{U}} d_H(\pi, \sigma) \leq \varepsilon \), Lemma 1 shows that for all \( i \in \{1, \ldots, n\} \), we have \( d(c_i, st) + \varepsilon \leq \varepsilon \), or \( d(c_i, st) \leq \varepsilon - r_i \), so \( st \) stab each disk \( D(c_i, \varepsilon - r_i) \). We can draw a similar conclusion for \( uv \). Therefore, each disk \( D(c_i, \varepsilon - r_i) \) crosses the entire strip bounded by the tangent lines, with the intersection points splitting it into the left and the right circular arcs. We can thus apply Definition 18 to these arcs, as stated in the lemma.

First suppose that the disks \( U_1 \) and \( U_n \) do not intersect. Then for any line segment from \( U_1 \) to \( U_n \) and any disk \( D' \coloneqq D(c_i, \varepsilon - r_i) \), we exit \( D' \) after exiting \( U_1 \) and enter \( D' \) before entering \( U_n \). Hence, for any line \( pq \) with \( p \in U_1 \) and \( q \in U_n \) and any \( i \in \{2, \ldots, n-1\} \), we can find a point \( w \in pq \cap D' \); this means, as stated in Lemma 12, that indeed \( \max_{\pi \in \mathcal{U}} d(w', pq) \leq \varepsilon \). As this holds for all disks and any choice of \( p \) and \( q \), we conclude that \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \).

Now assume that the disks \( U_1 \) and \( U_n \) intersect. If we consider the line segments \( pq \) with \( p \in U_1 \), \( q \in U_n \), we end up in the previous case if either \( p \notin U_1 \cap U_n \) or \( q \notin U_1 \cap U_n \). So assume that the segment \( pq \) lies entirely in the intersection \( U_1 \cap U_n \). However, it can be seen that for each disk \( D' \coloneqq D(c_i, \varepsilon - r_i) \), the left boundary of the intersection is to the right of the left boundary of the disk, and the right boundary of the intersection is to the left of the right boundary of the disk; hence \( pq \subset U_1 \cap U_n \). Therefore, we have \( \max_{\pi \in \mathcal{U}} d_{\pi}(w', pq) \leq \varepsilon \), and so also in this case \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \).

We now assume that the right side of the lemma statement is false and show that then \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) > \varepsilon \). If \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) > \varepsilon \), then immediately \( \max_{\pi \in \mathcal{U}} d_{\pi}(\pi, \langle \pi(1), \pi(n) \rangle) > \varepsilon \). Same holds for \( uv \). So, assume those statements hold; then it must be that for at least one intermediate disk the arcs do not lie to the left or to the right of the arcs of the respective disks. Assume this is disk \( i \), so the disk \( D' \coloneqq D(c_i, \varepsilon - r_i) \). W.l.o.g. assume that the right arc of the disk does not lie entirely to the right of the right arc of \( U_1 \). The argument for the left arc w.r.t. \( U_n \) is symmetric.

There must be at least one point \( p' \) on the right arc of \( U_1 \) that lies outside of \( D' \). Assume for now that \( U_1 \) and \( U_n \) are disjoint. Then a line segment \( p'q \) for any \( q \in U_n \) does not stab \( D' \), so \( \max_{\pi \in \mathcal{U}} d_{\pi}(w', pq) > \varepsilon \), and so \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) > \varepsilon \). If \( U_1 \) and \( U_n \) intersect, then either \( p' \) is outside of the intersection and of \( D' \) and there is a point \( q \in U_n \) such that \( p'q \) does not stab \( D' \); or we can pick the degenerate line segment \( p'p' \), as \( p' \in U_1 \cap U_n \), and so \( p'p' \) also does not stab \( D' \). In either case, we conclude that \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) > \varepsilon \). \( \blacksquare \)

It is also worth noting that the case of \( U_1 = U_n \) is similar to how we treat the intersection \( U_1 \cap U_n \) above; however, our definition for the ordering between two disks does not apply. So, if \( U_1 = U_n \), then \( \max_{\pi \in \mathcal{U}} d_H(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \) if and only if \( U_1 \subseteq D(c_i, \varepsilon - r_i) \) for all \( i \in \{2, \ldots, n-1\} \).

Similarly, we state the following for the Fréchet distance.

**Lemma 20.** Given \( n \in \mathbb{N}^+, n \geq 3 \), for any imprecise curve modelled with disks \( \mathcal{U} = \{U_1, \ldots, U_n\} \) with \( U_i = D(c_i, r_i) \) for all \( i \in [n] \) and \( c_i \in \mathbb{R}^2 \), \( r_i \in \mathbb{R}^{>0} \) for all \( i \in [n] \), and assuming \( U_1 \neq U_n \), we have

\[
\max_{\pi \in \mathcal{U}} d_F(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon
\]

if and only if both of the following are true:

\[
\max \left\{ \max_{\pi \in \mathcal{U}} d_F(\pi, st), \max_{\pi \in \mathcal{U}} d_F(\pi, uv) \right\} \leq \varepsilon,
\]

where \( s, u \in U_1 \) and \( t, v \in U_n \), and \( st \) and \( uv \) are the outer tangents to \( U_1 \cup U_n \); and
for each $i \in \{2, \ldots, n-1\}$, the right arc of the disk $D(c_i, \varepsilon - r_i)$ bounded by the intersection points with the tangent lines is to the right of the right arc of $U_1$ and the left arc of the disk $D(c_i, \varepsilon - r_i)$ is to the left of the left arc of $U_n$.

Proof. First assume that $\max_{x \in U} d_F(\pi_i, (\pi(1), \pi(n))) \leq 1$. As $d_F(\pi, \sigma) \leq d_H(\pi, \sigma)$ for any curves $\pi, \sigma$, this also means that $\max_{x \in U} d_H(\pi_i, (\pi(1), \pi(n))) \leq 1$. Furthermore, immediately we get that $\max_{x \in U} d_H(\pi, st) \leq 1$, and the same for $uv$. Together with Lemma 19 this yields the right side of the lemma.

Now assume that the right side holds. As in Lemma 19, we know that the disks cross the entire strip and that Definition 18 applies. It remains to show that for any line segment $pq$ with $p \in U_1, q \in U_n$, there is a valid alignment that maintains the correct ordering and bottleneck distance, assuming it exists for every realisation for $st$ and $uv$. Consider a valid alignment established for $st$ and $uv$, so the sequence of points $a_i$ on $st$ and $b_i$ on $uv$ that are mapped to $U_i$. As we showed in Lemma 11, we can always find such points for each individual $U_i$, and as we know that the Fréchet distance is below the threshold for $st$ and $uv$, there is such a valid alignment, i.e. we know that $a_i \preceq a_{i+1}$ and $b_i \preceq b_{i+1}$ for all $i \in [n-1]$.

First suppose that the disks $U_1$ and $U_n$ do not intersect. Consider the region $R$ bounded by the outer tangents and the disk arcs that are not part of the convex hull of $U_1 \cup U_n$. We connect, for each $i \in \{2, \ldots, n-1\}$, $a_i$ to $b_i$ with a geodesic shortest path in $R$. We claim that for any line segment $pq$ defined above, the intersection points of the shortest paths with the segment give a valid alignment, yielding $\max_{x \in U, x \in (1) \equiv p, x(n) \equiv q} d_F(\pi, pq) \leq 1$. As the choice of $pq$ was arbitrary, this will complete the proof.

To show that the alignment is valid, we need to show that the order is correct and that the distances fall below the threshold. First consider the case where the geodesic shortest path for point $i$ does not touch the boundary formed by arcs of region $R$. In this case, it is simply a line segment $a_i b_i$. Note that by definition $a_i, b_i \in D(c_i, \varepsilon - r_i)$; as disks are convex, also $a_i b_i \subset D(c_i, \varepsilon - r_i)$; thus, the intersection point $p'_i$ of $pq$ with $a_i b_i$ is in $D(c_i, \varepsilon - r_i)$, so by Lemma 11 $\max_{w \in U_1} \|p'_i - w\| \leq 1$. Furthermore, note that $a_i \preceq a_{i+1}$ and $b_i \preceq b_{i+1}$; thus, the line segments $a_i b_i$ and $a_{i+1} b_{i+1}$ cannot cross, so also $p'_i \preceq p'_{i+1}$.

Now w.l.o.g. consider the case where the geodesic shortest path for point $i$ touches the arc of $U_1$. The geodesic shortest paths do not cross: on the path from $a_i$ (or $b_i$) to the arc they form a tangent to the arc, thus for $a_i \preceq a_{i+1}$ the tangent point for $a_i$ comes before that of $a_{i+1}$ when going along the arc from $s$ to $u$. So, just as in the previous case, these line segments cannot cross. Having reached the arc, both shortest paths will follow it, as otherwise the path would not be a shortest path; thus, the arcs do not cross, either. Finally, a path from the previous case does not touch any path that touches the arc boundary of $R$ by definition. Finally, note that the condition that we have established on the right arcs of disks being to the right of the right arc of $U_1$ (and symmetric for the left arcs and $U_n$) means that the geodesic shortest paths that touch the arc boundary of $R$ stay within the respective disks $D(c_i, \varepsilon - r_i)$. Thus, we have established that for all $i$ we have $p'_i \preceq p'_{i+1}$ and $\max_{w \in U_1} \|p'_i - w\| \leq 1$, concluding the proof for disjoint $U_1$ and $U_n$.

Finally, consider the case where $U_1$ intersects $U_n$. Above we used geodesic paths within the region $R$. However, when $U_1$ intersects $U_n$, $R$ consists of two disconnected regions. Observe that one region contains $a_i$ and the other contains $b_i$. To connect $a_i$ with $b_i$ we use the geodesic from $a_i$ to the intersection point of the two inner boundaries of $U_1$ and $U_n$ that is in the same region of $R$, the geodesic from $b_i$ to the other intersection point of the inner boundaries, and join these two by a line segment between the intersection points. Any line segment from a point in $U_1$ to a point in $U_n$ crosses these paths in order, just like in the previous case. If the line segment goes through the intersection, note that any point in the
intersection is close enough to all the intermediate objects, as the intersection is the subset of each disk. So, any point in the intersection can be chosen to establish the trivially in-order alignment to all the intermediate objects.

Again, in the case that \( U_1 = U_n \), we can see that \( \max_{\pi \in U} d_F(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \) if and only if \( U_1 \subseteq D(c_i, \varepsilon - r_i) \) for all \( i \in \{2, \ldots, n - 1\} \).

### 5.2 Non-intersecting PCCSs

Suppose the regions are modelled by convex polygons. Consider first the case where the interiors of \( U_1 \) and \( U_n \) do not intersect, so at most they share a boundary segment.

> **Observation 21.** Given an uncertain curve modelled by convex polygons \( U = \langle U_1, \ldots, U_n \rangle \) with the interiors of \( U_1 \) and \( U_n \) not intersecting, note:

- There are two outer tangents to the polygons \( U_1 \) and \( U_n \), and the convex hull of \( U_1 \cup U_2 \) consists of a convex chain from \( U_1 \), a convex chain from \( U_n \), and the outer tangents.
- Let \( C_i \) be the convex chain from \( U_i \) that is not part of the convex hull for \( i \in \{1, n\} \). Then for \( \delta := d_H \) or \( \delta := d_F \),

\[
\max_{\pi \in U} \delta(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \iff \max_{\pi \in U, \pi(1) \in C_1, \pi(n) \in C_n} \delta(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon.
\]

To see that the second observation is true, note that one direction is trivial. In the other direction, note that any line segment \( pq \) with \( p \in U_1 \), \( q \in U_n \) crosses both \( C_1 \) and \( C_n \), say, at \( p' \in C_1 \) and \( q' \in C_n \). We know that there is a valid alignment for \( p'q' \), both for the Hausdorff and the Fréchet distance; we can then use this alignment for \( pq \). See Figure 5.

We claim that we can use the following procedure to check \( \max_{\pi \in U} d_H(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon \).

1. Triangulate the region \( R \) bounded by two convex chains \( C_1 \) and \( C_n \) and the outer tangents.
2. For each line segment \( st \) of the triangulation with \( s \in C_1 \), \( t \in C_n \), and for either \( \delta := d_H \) or \( \delta := d_F \), check that \( \max_{\pi \in U, \pi(1) \equiv s, \pi(n) \equiv t} \delta(\pi, st) \leq \varepsilon \).

First of all, observe that we can compute a triangulation, and that every triangle has two points from one convex chain and one point from the other chain (see Figure 5). If all three points were from the same chain, then the triangle would lie outside of \( R \). Now consider some line segment \( pq \) with \( p \in C_1 \), \( q \in C_n \). To complete the argument, it remains to show that the checks in step 2 mean that also \( \max_{\pi \in U, \pi(1) \equiv p, \pi(n) \equiv q} \delta(\pi, pq) \leq \varepsilon \). Observe that the triangles span across the region \( R \), so when going from one tangent to the other within \( R \) we cross all the triangles. Therefore, we can order them, in the order of occurrence on such a path, from 1 to \( k \). Denote the alignment established on line \( j \in [k] \) with the sequence of \( a_{ij} \), for \( i \in [n] \); this alignment can be established both for \( \delta := d_H \) and \( \delta := d_F \). We can then establish polygonal curves \( A_i := \langle a_{1i}, \ldots, a_{ni} \rangle \); clearly, they all stay within \( R \). We claim that for any line segment \( pq \) defined above, it is possible to establish a valid alignment from...
intersection points of $pq$ and $A_i$. We do this separately for the Fréchet and the Hausdorff distance.

**Lemma 22.** Given a set of curves $A := \{A_2, \ldots, A_{n-1}\}$ in $R$ described above for $\delta := d_H$ and a line segment $pq$ with $p \in C_1$, $q \in C_n$, we have $\max_{\pi \in U, \pi(1) = p, \pi(n) = q} d_H(\pi, pq) \leq \varepsilon$.

**Proof.** Note that $pq$ crosses each $A_i$ at least once. We can take any one crossing for each $i$ and establish the alignment. Consider such a crossing point $p_i'$. It falls in some triangle bounded by a segment from either $C_1$ or $C_n$ and two line segments that contain points $a_i^j$ and $a_i^{j+1}$ for some $j \in [k]$. We know, using Lemma 9, that $\max_{w \in U_1} \|a_i^j - w\| \leq \varepsilon$ and $\max_{w \in U_1} \|a_i^{j+1} - w\| \leq \varepsilon$. Consider any point $w' \in U_i$. Then, using Lemma 3 with $\varepsilon := d := w'$, we find that $\|w' - p_i'\| \leq \varepsilon$. Therefore, also $\max_{w \in U_1} \|p_i' - w\| \leq \varepsilon$; using Lemma 9, we conclude that indeed $\max_{\pi \in U, \pi(1) = p, \pi(n) = q} d_H(\pi, pq) \leq \varepsilon$. ▶

For the Fréchet distance, we can use the same argument to show closeness; however, we need more care to establish the correct order for the alignment to be valid.

**Lemma 23.** Given a set of curves $A := \{A_2, \ldots, A_{n-1}\}$ in $R$ described above for $\delta := d_F$ and a line segment $pq$ with $p \in C_1$, $q \in C_n$, we have $\max_{\pi \in U, \pi(1) = p, \pi(n) = q} d_F(\pi, pq) \leq \varepsilon$.

**Proof.** Compared to Lemma 22 instead of taking any intersection point of $pq$ with each $A_i$, we take the last intersection point.

We need to show, first of all, that curves $A_i$ and $A_{i+1}$ do not cross for any $i \in [n-1]$. Note that each curve $A_i$ crosses each triangle once, so it suffices to show that a segment $a_i^j a_i^{j+1}$ does not cross $a_{i+1}^j a_{i+1}^{j+1}$. Indeed, as $a_i^j \preceq a_{i+1}^j$ and $a_i^{j+1} \preceq a_{i+1}^{j+1}$, these line segments cannot cross.

Now consider, for each $i \in \{2, \ldots, n-1\}$, the polygon $P_i$ bounded by $C_1$, $A_i$, and the corresponding segments of the outer tangents. With the previous statement, it is easy to see that $P_2 \subseteq P_3 \subseteq \cdots \subseteq P_{n-1}$. Assume this is not the case, so some $P_i \not\subseteq P_{i+1}$. Then there is a point $z \in P_i$, but $z \notin P_{i+1}$. The point $z$ falls into some triangle with lines $j$ and $j+1$. In this triangle, it means that $z$ is between $C_1$ and $a_i^j a_i^{j+1}$, but not between $C_1$ and $a_{i+1}^j a_{i+1}^{j+1}$. However, as these segments do not cross, this would imply that $a_{i+1}^j \prec a_i^j$, but then the check in step 2 would not pass for line $j$.

Consider the points at which the line segment $pq$ leaves the polygons $P_i$ for the last time. From the definition it is obvious that $p \in P_i$, for all $i \in \{2, \ldots, n-1\}$, so this is well-defined. Clearly, due to the subset relationship, the order of such points $p_i'$ is correct, i.e., $p_i' \preceq p_{i+1}'$. Furthermore, each such $p_i' \in A_i$, so using the arguments of Lemma 22, we can show that also the distances are below $\varepsilon$. Thus, we conclude that indeed $\max_{\pi \in U, \pi(1) = p, \pi(n) = q} d_F(\pi, pq) \leq \varepsilon$.

The proofs of Lemmas 22 and 23 show us how to solve the problem for two convex polygons with non-intersecting interiors. We can also use them directly for the case of line segments that do not intersect except at endpoints.

**Corollary 24.** Given $n \in \mathbb{N}^\geq 0$, $n \geq 3$, for any imprecise curve modelled with line segments $\mathcal{U} = \{U_1, \ldots, U_n\}$ with $U_i = p_i^1 p_i^2 \subseteq \mathbb{R}^2$ for all $i \in [n]$, given a threshold $\varepsilon \in \mathbb{R}^>0$, and given that $U_1 \cap U_n \subseteq \{p_1^1, p_n^2\}$, and assuming that the triangles $p_1^1 p_n^2 p_i^1$ and $p_1^2 p_n^1 p_i^2$ form a triangulation of the convex hull of $U_1 \cup U_n$, we have

$$\max_{\pi \in \mathcal{U}} \delta(\pi, (\pi(1), \pi(n))) \leq \varepsilon$$
if and only if
\[
\max_{\pi \in \mathcal{U}} \max_{p \in p_1} \max_{\tau \in \tau_1} \delta(\pi_{1,1}, p_{1,1}^{(1)}), \\
\max_{\pi \in \mathcal{U}} \max_{p \in p_1} \max_{\tau \in \tau_1} \delta(\pi_{1,2}, p_{1,2}^{(1)}), \\
\max_{\pi \in \mathcal{U}} \max_{p \in p_1} \max_{\tau \in \tau_1} \delta(\pi_{1,2}, p_{1,2}^{(2)}) \leq \varepsilon.
\]

We should note that in this particular case it is not necessary to use a triangulation, so we can get rid of the second term; also in the previous proofs a convex partition could work instead, but a triangulation is easier to define.

5.3 Intersecting PCCSs

We proceed to discuss the situation where the interiors of $U_1$ and $U_n$ intersect, or where line segments $U_1$ and $U_n$ cross. The argument is the same for both $\delta := d_H$ and $\delta := d_F$, but it is easier to treat line segments and convex polygons separately.

Line segments.

Assume line segments $U_1 \equiv p_1^1 p_1^2$ and $U_n \equiv p_n^1 p_n^2$ cross; call their intersection point $s$. Then we can use Corollary \[24\] separately on pairs of $\{p_1^1, p_1^2\} \times \{p_n^1, p_n^2\}$. Clearly, together this will cover the entire set of realisations of $pq$ with $p \in U_1$, $q \in U_n$, thus completing the checks.

▶ Lemma 25. Given $n \in \mathbb{N}^+ > 0$, $n \geq 3$, for any imprecise curve modelled with line segments $U = \{U_1, \ldots, U_n\}$ with $U_i = p_i^1 p_i^2 \subset \mathbb{R}^2$ for all $i \in [n]$, given a threshold $\varepsilon \in \mathbb{R}^+ > 0$, we can check for both $\delta := d_H$ and $\delta := d_F$, using procedures above, that
\[
\max_{\pi \in \mathcal{U}} \delta(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon.
\]

Convex polygons.

Convex polygons whose interiors intersect can be partitioned along the intersection lines, so into a convex polygon $R := U_1 \cap U_n$ and two sets of polygons $P_1 := \{P_1^1, \ldots, P_1^k\}$ and $P_n := \{P_n^1, \ldots, P_n^k\}$ for some $k, \ell \in \mathbb{N}^+ > 0$. Just as for line segments, we can look at pairs from $P_1 \times P_n$ separately. The pairs where $R$ is involved are treated later. Consider some $(P, Q) \in P_1 \times P_n$. Note that $P$ and $Q$ are convex polygons with a convex cut-out, so the boundary forms a convex chain, followed by a concave chain. We need to compute some convex polygons $P'$ and $Q'$ with non-intersecting interiors that are equivalent to $P$ and $Q$, so that we can apply the approaches from Section 5.2.

We claim that we can simply take the convex hull of $P$ and $Q$ to obtain $P'$ and $Q'$. Clearly, the resulting polygons will be convex. Also, the concave chains of $P$ are bounded by points $s$ and $t$ and are replaced with the line segment $st$; same happens for $Q$ with point $u$ and $v$. The points $s, t, u, v$ are points of intersection of original polygons $U_1$ and $U_n$, so they lie on the boundary of $R$, and their order along that boundary can only be $s, t, u, v$ or $s, t, v, u$. Thus, it cannot happen that $st$ crosses $uv$, and it cannot be that $uv$ is in the interior of the convex hull of $P$, as otherwise $R$ would not be convex. Hence, the interiors of $P'$ and $Q'$ cannot intersect, so they satisfy the necessary conditions.

Finally, we need to show that the solution for $(P', Q')$ is equivalent to that for $(P, Q)$. One direction is trivial, as $P \subseteq P'$ and $Q \subseteq Q'$; for the other direction, consider any line segment that leaves $P$ through the concave chain. In our approach, we test the lines starting

Assume line segments $U_1 \equiv p_1^1 p_1^2$ and $U_n \equiv p_n^1 p_n^2$ cross; call their intersection point $s$. Then we can use Corollary \[24\] separately on pairs of $\{p_1^1, p_1^2\} \times \{p_n^1, p_n^2\}$. Clearly, together this will cover the entire set of realisations of $pq$ with $p \in U_1$, $q \in U_n$, thus completing the checks.

▶ Lemma 25. Given $n \in \mathbb{N}^+ > 0$, $n \geq 3$, for any imprecise curve modelled with line segments $U = \{U_1, \ldots, U_n\}$ with $U_i = p_i^1 p_i^2 \subset \mathbb{R}^2$ for all $i \in [n]$, given a threshold $\varepsilon \in \mathbb{R}^+ > 0$, we can check for both $\delta := d_H$ and $\delta := d_F$, using procedures above, that
\[
\max_{\pi \in \mathcal{U}} \delta(\pi, \langle \pi(1), \pi(n) \rangle) \leq \varepsilon.
\]
in $s$ and $t$; the established alignments are connected into paths. The paths $A_i$ do not cross
st. So, any alignment in the region of $CH(P \cup Q) \setminus (P \cup Q)$ can also be made in the region
$CH(P' \cup Q') \setminus (P' \cup Q')$. So, this approach yields valid solutions for all pairs not involving $R$.

Now consider the pair $(R, R)$. A curve may now consist of a single point, so the approach
for the Fréchet and the Hausdorff distance is the same: all the points of $U_i$ need to be close
enough to all the points of $R$. To check that, observe that the pair of points $p \in U_i$ and $q \in R$
that has maximal distance has the property that $p$ is an extreme point of $U_i$ in direction $pq$ and
$q$ is an extreme point of $R$ in direction $pq$. So, it suffices, starting at the rightmost point
of $U_i$ and leftmost point of $R$ in some coordinate system, to then rotate clockwise around
both regions keeping track of the distance between tangent points. Note that only vertices
need to be considered, as the extremal point cannot lie on an edge. Finally, any other pair
that involves $R$ is covered by the stronger case of $(R, R)$: for any line we can align every
intermediate object to any point in $R$.

\begin{lemma}
Given $n \in \mathbb{N}^>$, $n \geq 3$, for any imprecise curve modelled with convex polygons $U = \{U_1, \ldots, U_n\}$ with $U_i \subset \mathbb{R}^2$ for all $i \in [n]$ and $V(U_i) = \{p^1_i, \ldots , p^k_i\}$ for all $i \in [n]$, $k \in \mathbb{N}^>$, given a threshold $\varepsilon \in \mathbb{R}^>$, we can check for both $\delta := d_H$ and $\delta := d_F$, using
procedures above, that
\[\max_{\pi \in U} \delta(\pi, (\pi(1), \pi(n))) \leq \varepsilon.\]
\end{lemma}

\section{Combining Steps}

In Sections 4 and 5 we have established correctness of the routines that can be used to check
if a shortcut is valid under either the Hausdorff distance or the Fréchet distance. In this
section, we summarise the approach, discuss the shortcut graph, and analyse the running
times.

\begin{lemma}
Given $n \in \mathbb{N}^>$, for any uncertain curve modelled with indecisive points, disks,
or PCCSs $U = \{U_1, \ldots, U_n\}$, and given a threshold $\varepsilon \in \mathbb{R}^>$, and fixing either $\delta := d_H$ or
$\delta := d_F$, if we can check in time $T$ for any pair $i,j \in [n]$, $i < j$ that
\[\max_{\pi \in U[i:j]} \delta(\pi, (\pi(1), \pi(j-i+1))) \leq \varepsilon,\]
then in time $O(Tn^2)$ we can find the shortest index subsequence $I \subseteq [n]$ with $|I| = \ell$ such
that for all $j \in [\ell]$, 
\[\max_{\pi \in U[I(j):I(j+1)]} \delta(\pi, (\pi(1), \pi(I(j+1)+1))) \leq \varepsilon.\]
\end{lemma}

\begin{proof}
The approach is simple: construct a graph $G := (V,E)$ with $V := \{v_1, \ldots, v_n\}$ and
$(v_i, v_j) \in E$ if and only if $\max_{\pi \in U[i:j]} \delta(\pi, (\pi(1), \pi(j-i+1))) \leq \varepsilon$. Clearly, this takes $O(Tn^2)$
time. Any path in the graph from $v_1$ to $v_n$ gives a subsequence for which the condition in
the statement of the lemma holds; there are no simplifications that would not correspond to
such a path; thus, finding the shortest path in $G$ using e.g. BFS in time $O(n^2)$ indeed yields
the answer.
\end{proof}

It is easy to see that the result of the lemma is exactly the problem we were trying to solve:
obtaining a single simplification such that no matter which realisation of the curve is chosen,
the resulting realisation of the simplification is valid.

\[\triangleright\]
We now proceed to recap the methods for checking the shortcuts. For indecisive points, one can test all combinations for the first and the last point of the shortcut, as in Lemma \ref{lem:shortcuts_tests} and for each such combination do the testing either for the Hausdorff or the Fréchet distance, as in Lemmas \ref{lem:testing_points} and \ref{lem:testing_disks}.

For imprecise points modelled with disks, it suffices to test the outer tangents and check some extra conditions on the intermediate disks, as in Lemmas \ref{lem:testing_disks} and \ref{lem:testing_disks}. For the outer tangents, the testing can be done using the approaches of Lemmas \ref{lem:testing_tangents} and \ref{lem:testing_tangents}.

For imprecise points modelled with line segments, one can split the first and the last one into regions if they cross, as in Lemma \ref{lem:partitioning_line_segments} and apply Corollary \ref{cor:testing_line_segments} to each pair. The testing of the outer tangents can be done using Lemmas \ref{lem:testing_disks} and \ref{lem:testing_tangents} for the Hausdorff and the Fréchet distance, respectively.

Finally, for imprecise points modelled with convex polygons, we again split the first and the last one into regions if their interiors intersect, as in Lemma \ref{lem:partitioning_convex_polygons} and apply Lemmas \ref{lem:testing_convex_polygons} and \ref{lem:testing_convex_polygons}. To test each shortcut with the fixed endpoints, we can again use Lemmas \ref{lem:testing_points} and \ref{lem:testing_disks}.

Having constructed the graph, we can find the shortest path through it from vertex corresponding to \( U_1 \) to that corresponding to \( U_n \), as discussed in Lemma \ref{lem:shortest_path}.

\begin{theorem}
We can solve the problem of finding the shortest vertex-constrained simplification of an uncertain curve, such that for any realisation the simplification is valid, both for the Hausdorff and the Fréchet distance, and for uncertainty modelled using indecisive points, disks, line segments, or convex polygons in time shown in Table \ref{tab:combinations}.
\end{theorem}

\begin{proof}
Correctness of the approaches has been shown before. For the running time, observe that we need \( O(n^2 T) \) time in any setting, due to the shortcut graph construction.

For indecisive points, when testing a shortcut we do \( O(nk) \)-time testing for \( O(k^2) \) combinations of starting and ending points, where \( k \) is the number of options per point.

For disks, we do a linear number of constant-time checks and two linear-time checks, getting \( T \in O(n) \).

For line segments, we also do two (three) linear-time checks per part; two line segments can be split into at most two parts each, so we repeat the process four times. Either way, we get \( T \in O(n) \).

Finally, for convex polygons, assume the complexity of each polygon is at most \( k \). Assume the partitioning resulting from two intersecting polygons yields \( \ell_1 \) and \( \ell_2 \) parts for the first and the second polygon, respectively. Denote the two polygons \( P \) and \( Q \) and the resulting parts with \( P_1, \ldots, P_{\ell_1} \) and \( Q_1, \ldots, Q_{\ell_2} \), respectively. Suppose part \( P_i \) has complexity \( k_i \) and part \( Q_j \) has complexity \( k'_j \), so \( |V(P_i)| = k_i \) and \( |V(Q_j)| = k'_j \) for some \( i \in [\ell_1], j \in [\ell_2] \). We know that every vertex of the original polygons occurs in a constant number of parts, so \( \sum_{i=1}^{\ell_1} k_i \in O(k) \) and \( \sum_{j=1}^{\ell_2} k'_j \in O(k) \); we also know \( \ell_1 + \ell_2 \in O(k) \). We consider all pairs from \( P \) and \( Q \), and for each pair we triangulate and do the checks on the triangulation. The triangulation can be done in time \( O((k_i + k'_j) \cdot \log(k_i + k'_j)) \), yielding \( O(k_i + k'_j) \) lines, each of which is tested in time \( O(nk) \). The testing dominates, so we need \( O((k_i + k'_j) \cdot nk) \) time.

We are interested in

\[
\sum_{i=1}^{\ell_1} \sum_{j=1}^{\ell_2} O((k_i + k'_j) \cdot nk) = O(nk) \cdot \sum_{i=1}^{\ell_1} \sum_{j=1}^{\ell_2} O(k_i + k'_j) = O(nk^3) .
\]

So, \( T \in O(nk^3) \) both for the Fréchet and the Hausdorff distance.
\end{proof}
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