GAIOTTO’S LAGRANGIAN SUBVARIETIES VIA DERIVED SYMPLECTIC GEOMETRY

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To Alexander Alexandrovich Kirillov with gratitude and admiration

ABSTRACT. Let $Bun_G$ be the moduli space of $G$-bundles on a smooth complex projective curve. Motivated by a study of boundary conditions in mirror symmetry, D. Gaiotto associated to any symplectic representation of $G$ a Lagrangian subvariety of $T^*Bun_G$. We give a simple interpretation of (a generalization of) Gaiotto’s construction in terms of derived symplectic geometry. This allows to consider a more general setting where symplectic $G$-representations are replaced by arbitrary symplectic manifolds equipped with a Hamiltonian $G$-action and with an action of the multiplicative group that rescales the symplectic form with positive weight.

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1. STATEMENT OF THE RESULT

We will use the language of derived stacks. Throughout, a ‘stack’ means a ‘derived Artin stack over $k = \mathbb{C}$’ in the sense of [GR] and [PTVV]. We write $B^\infty G = \text{pt}/G$ for the classifying stack of a group $G$. We fix a smooth complex projective variety $X$ and let $K_X$ denote the canonical bundle. We write $G$ for an algebraic group and $Bun_G(X)$, resp. $Higgs_G(X)$, for the stack of $G$-bundles, resp. Higgs bundles, on $X$. One has a canonical isomorphism $Bun_G(X) \cong \text{Map}(X, BG)$, where $\text{Map}(X, Z)$ denotes a mapping stack that classifies morphisms $X \to Z$.

Given a $\mathbb{G}_m$-stack $\mathcal{Y}$ and a $\mathbb{G}_m$-bundle $L \to X$, there is an associated bundle $Y_L := \mathcal{Y} \times_{\mathbb{G}_m} L$. Let $\text{Sect}_X(Y_L)$ be the stack of sections of the projection $Y_L \to X$. By definition, we have $\text{Sect}_X(Y_L) = \{\text{Id}_X\} \times_{\text{Map}(X, X)} \text{Map}(X, Y_L)$. The $\mathbb{G}_m$-action on the first factor of $\mathcal{Y} \times L$ descends to a $\mathbb{G}_m$-action along the fibers of $Y_L \to X$. This induces a natural $\mathbb{G}_m$-action on $\text{Sect}_X(Y_L)$.

Remarks 1.1. Let $L \to X$ be a $\mathbb{G}_m$-bundle and $L$ an associated line bundle on $X$.

(i) We will abuse the notation and write $Y_L$ for $Y_L$.

(ii) For a $\mathbb{G}_m$-stack $\mathcal{Y}$, there is a canonical isomorphism $Y_L \cong \mathcal{Y} / \mathbb{G}_m \times_{BG_m} X$, where we have used the map $X = L/GL_m \to BG_m = pt/\mathbb{G}_m$ that classifies $L$.

(iii) For a $(G \times \mathbb{G}_m)$-stack $\mathcal{Y}$, we will often use natural identifications $(\mathcal{Y}/G)_L = (\mathcal{Y} \times L)/(G \times \mathbb{G}_m) = (Y_L)/G$.

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Let $M$ be a smooth symplectic algebraic manifold equipped with a $G \times \mathbb{G}_m$-action such that the action of the group $G = G \times \{1\}$ on $M$ is Hamiltonian and the symplectic 2-form has weight $\ell \geq 1$ with respect to the action of $\mathbb{G}_m = \{1\} \times \mathbb{G}_m$. Assume that there exists a line bundle $K_X^{1/\ell}$, an $\ell$-th root of $K_X$, and fix a choice of $K_X^{1/\ell}$.

Following Gaiotto, [Ga], we consider the stack $\text{Sect}_X(M_{K_X^{1/\ell}}/G)$. This stack classifies pairs $(P, s)$, where $P$ is a $(G \times \mathbb{G}_m)$-bundle on $X$ and $s : P \to M \times K_X^{1/\ell}$ is a $(G \times \mathbb{G}_m)$-equivariant morphism that intertwines the natural projections $P \to X$ and $M \times K_X^{1/\ell} \to X$. Here $K_X^{1/\ell}$ denotes the $\mathbb{G}_m$-bundle obtained from $K_X$ by removing the zero section. The group $G$ acts on $M \times K_X^{1/\ell}$ through its action on the first factor and $\mathbb{G}_m$ acts diagonally.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. The group $G \times \mathbb{G}_m$ acts on $\mathfrak{g}^*$, where $G$ acts by the coadjoint action and $\mathbb{G}_m$ acts by dilations. The symplectic 2-form on $M$ being of weight $\ell$, the moment map $\mu : M \to \mathfrak{g}^*$ intertwines, for any $t \in \mathbb{G}_m$, the $t$-action on $M$ with dilution by $t^\ell$ on $\mathfrak{g}^*$. It follows that $\mu$ gives a well defined morphism $M_{K_X^{1/\ell}} \to \mathfrak{g}^*_K X$, of stacks over $X$. Therefore, there is an induced morphism

$$\mu_{\text{Sect}} : \text{Sect}_X(M_{K_X^{1/\ell}}/G) \longrightarrow \text{Sect}_X(\mathfrak{g}^*_K X/G).$$ (1.2)

We now specialize to the case where $X = \Sigma$ is a smooth projective curve and $G$ is reductive. In such a case, we have $\text{Sect}_\Sigma(\mathfrak{g}^*_K G) \cong \text{Higgs}_G(\Sigma) \cong T^*\text{Bun}_G(\Sigma)$. Let $T^*\text{Bun}_G(\Sigma)^{reg}$ be an open substack of $T^*\text{Bun}_G(\Sigma)$ that corresponds to the Higgs bundles whose only automorphisms lie in the center. It is known that $T^*\text{Bun}_G(\Sigma)^{reg}$ is a smooth variety that comes equipped with a natural symplectic 2-form $\omega$.

Theorem 1.3. The map $\mu_{\text{Sect}}$ is Lagrangian, specifically, the 2-form $\mu_{\text{Sect}}^*(\omega)$ vanishes on the preimage of $T^*\text{Bun}_G(\Sigma)^{reg}$.

The above result was discovered by Gaiotto [Ga] in the linear case, i.e. in the special case where $M$ is a symplectic representation of $G$. In this case, $\mathbb{G}_m$ acts on $M$, a symplectic vector space, by dilations and the symplectic form on $M$ has weight 2.

One of the goals of this paper is to show that Theorem 1.3 is a simple consequence of some very general results of derived symplectic geometry.

2. Derived symplectic geometry

Let $n$ be an integer and $Y$ a stack equipped with an $n$-shifted symplectic structure in the sense of [PTVV]. There is a notion of “Lagrangian structure” on a morphism $Z \to Y$, see [PTVV] §2.2] and [Ca]. One has the following result, where part (i) is [PTVV] Theorem 0.4], resp. part (ii) is [Ca] Theorem 2.10].

Theorem 2.1. Let $X$ be a smooth projective Calabi-Yau variety of dimension $d$. Then, one has:

(i) An $n$-shifted symplectic structure on a stack $Y$ gives rise to a natural $(n - d)$-shifted symplectic structure on $\text{Map}(X, Y)$.

(ii) A Lagrangian structure $f : Z \to Y$ gives rise to a natural Lagrangian structure on $\text{Map}(X, Z) \to \text{Map}(X, Y)$, the morphism of mapping stacks induced by $f$.

It was shown, see [PTVV] Corollary 2.6(2)], that part (i) of the theorem implies the following

Corollary 2.2. For any smooth projective Calabi-Yau variety $X$ of dimension $d$ the stack $\text{Higgs}_G(X)$ has a canonical $2(1 - d)$-shifted symplectic structure.

In the case where $X$ is a Fano variety suitable analogues of the statements of Theorem 2.1 were proved by Spaide [Sp], Theorem 3.3 and Theorem 3.5.
Below, we propose a modification of the above results that holds for more general, not necessarily Calabi-Yau, varieties $X$.

To this end, we recall some notions from derived algebraic geometry. For a (derived) stack $\mathcal{X}$, we will denote by $\text{QCoh}(\mathcal{X})$ the (unbounded) derived $\infty$-category of quasi-coherent sheaves on $\mathcal{X}$ (see, e.g. [GR] for a detailed account of this $\infty$-category). We will refer to objects of $\text{QCoh}(\mathcal{X})$ as “sheaves on $\mathcal{X}$.” Given $\mathcal{M} \in \text{QCoh}(\mathcal{X})$, we will denote by $\Gamma(\mathcal{X}, \mathcal{M}) = \text{Hom}(\mathcal{O}_X, \mathcal{M})$, the (derived) functor of global sections.

Let $f : Y \to X$ be a map of stacks and $\mathbb{L}_{Y/X} \in \text{QCoh}(Y)$ the relative cotangent complex of $f$. One has a sheaf

$$\tilde{A}_X^p(Y) := f_*(\wedge^p \mathbb{L}_{Y/X}) \in \text{QCoh}(\mathcal{X}),$$

of relative $p$-forms. There is also a sheaf $\tilde{A}_X^{p,cl}(Y) \in \text{QCoh}(\mathcal{X})$, of relative closed $p$-forms. The sheaf $\tilde{A}_X^{p,cl}(Y)$ comes equipped with a forgetful map $\tilde{A}_X^{p,cl}(Y) \to \tilde{A}_X^p(Y)$ which assigns to a closed $p$-form its underlying $p$-form (see [CPTVV] Sect. 1 or [GR] Vol. II, Chapter 9) for a discussion of relative differential forms). Note that in the derived setting, a closed $p$-form is a $p$-form equipped with additional closure data (as opposed to satisfying a condition).

We will use the following basic result about relative differential forms:

**Lemma 2.3.** Let

$$
\begin{array}{ccc}
Y_2 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_1 \\
\end{array}
$$

be a commutative square of stacks. Then, for each $i \geq 0$, there is a natural map

$$\phi_{i,cl} : g^* (\tilde{A}_X^{i,cl}(Y_1)) \to \tilde{A}_X^{i,cl}(Y_2).$$

Moreover, if the square is Cartesian and $\mathbb{L}_{Y_1/X_1}$ is perfect (more generally, it is sufficient to require $\mathbb{L}_{Y_1/X_1}$ be bounded below) then the map $\phi_{p,cl}$ is an isomorphism.

\[\square\]

**Definition.** Let $p : Y \to X$ be a map of stacks and $\mathcal{L}$ a line bundle on $\mathcal{X}$. We put

$$\mathcal{A}^i(Y/X; \mathcal{L}) := \Gamma(\mathcal{X}, \tilde{A}_X^i(Y) \otimes \mathcal{L}), \text{ and } \mathcal{A}^{i,cl}(Y/X; \mathcal{L}) := \Gamma(\mathcal{X}, \tilde{A}_X^{i,cl}(Y) \otimes \mathcal{L}).$$

(i) Assume the relative cotangent complex of $p : Y \to X$ is perfect. An $\mathcal{L}$-twisted $n$-shifted relative symplectic structure on $Y$ is a twisted relative closed 2-form $\omega \in \text{Hom}(k, \tilde{A}_X^{2,cl}(Y/X; \mathcal{L})(n))$ such that the underlying 2-form is nondegenerate, i.e. it induces an isomorphism

$$\mathbb{L}_{Y/X}^\vee \sim \mathbb{L}_{Y/X}[n] \otimes p^*(\mathcal{L}).$$

(ii) Assume that $p : Y \to X$ is equipped with an $\mathcal{L}$-twisted $n$-shifted relative symplectic structure and let $f : Z \to Y$ be a map of stacks with perfect relative cotangent complex. An ($\mathcal{L}$-twisted $n$-shifted) Lagrangian structure on $f$ is a nullhomotopy of $f^*(\omega) \in \text{Hom}(k, \tilde{A}_X^{2,cl}(Z/\mathcal{L})(n))$ such that the map

$$\mathbb{L}_{Z/X}^\vee \to \mathbb{L}_{Z/Y}[n - 1] \otimes (f \circ p)^*(\mathcal{L}),$$

induced by the nullhomotopy of the underlying 2-form, is an isomorphism.

The proposition below gives a preliminary version of our main construction. In Section 3, we will describe how to obtain relative twisted symplectic, resp. Lagrangian, structures from symplectic, resp. Lagrangian, structures of a fixed weight on a $\mathbb{G}_m$-stack.

**Proposition 2.4.** Let $X$ be a smooth projective variety of dimension $d$ and $Y, Z$ a pair of stacks.

\[\square\]
Proof. Following [PTVV], we consider the evaluation map
\[ \text{Sect}_X(Y) \times X \xrightarrow{ev} Y, \]
a map of stacks over \( X \). By Lemma 2.3 there is a pull-back morphism in QCoh(\( X \)):
\[ ev^*: A^{2,cl}_X(Y) \otimes_{\mathcal{O}_X} K_X \to A^{2,cl}(\text{Sect}_X(Y)) \otimes_k K_X. \]
Using an integration map \( \int_X : \Gamma(X, K_X) \to k[-d] \) provided by Serre duality, one obtains a map
\[ (\text{Id} \times \int_X) \circ ev^*: A^{2,cl}(Y/X; K_X) \to A^{2,cl}(\text{Sect}_X(Y)). \]

Now, the same argument as in [PTVV] shows that if the twisted 2-form \( \omega \) on \( Y \) is nondegenerate then so is the 2-form
\[ \omega_{\text{Sect}} := (\text{Id} \times \int_X) \circ ev^*)(\omega). \]
This proves part (i) of Proposition 2.4. The proof of part (ii) is obtained by similarly tweaking the proof of [Ca] Theorem 2.10. \( \square \)

Remarks 2.5. (i) The same proof works in a more general setting where \( X \) is any strictly \( \mathcal{O} \)-compact stack in the sense of [PTVV] Definition 2.1 equipped with a line bundle \( K_X \) and a map \( \int_X : \Gamma(X, K_X) \to k[-d] \) that induces a perfect pairing as in [PTVV] Definition 2.4. For instance, one can take \( X \) be any proper Gorenstein (derived) scheme.

(ii) It is tempting to try to develop a formalism of ‘derived hyper-Kähler geometry’, at least a notion of ‘derived twistor space’. One could then consider an analogue of Proposition 2.4 as well as analogues of various results below, with a hyper-Kähler target \( Y \) and hyper-Lagrangian structures \( Z \to Y \).

3. Equivariance and Twistings

Let \( Y \) be a \( \mathbb{G}_m \)-stack. Given an integer \( m \), let \( Y^{(m)} \) denote the \( \mathbb{G}_m \)-stack with the same underlying stack as \( Y \) and the \( \mathbb{G}_m \)-action given by precomposition with the homomorphism \( \mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^m \). The space of (closed) \( p \)-forms on the \( \mathbb{G}_m \)-stack \( Y \) carries a natural \( \mathbb{Z} \)-grading, to be referred to as ‘weight’. Thus, one can consider \( n \)-shifted symplectic structures on \( Y \) of weight \( m \).

Given a \( \mathbb{G}_m \)-stack \( Z \), we say that \( f \) is a map from \( Z \) to \( Y \) of weight \( m \) if \( f \) is a \( \mathbb{G}_m \)-equivariant map \( Z \to Y^{(m)} \). Heuristically, a map \( f : Z \to Y \) has weight \( m \) if \( f(tz) = t^m f(z) \) for all \( t \in \mathbb{G}_m \).

Definition. Fix an \( n \)-shifted symplectic structure on \( Y \) of weight \( m \). This gives, for each \( \ell \geq 1 \), an \( n \)-shifted symplectic structure on \( Y^{(\ell)} \) of weight \( m\ell \).

(i) An equivariant Lagrangian structure is an equivariant map \( f : Z \to Y \), of \( \mathbb{G}_m \)-stacks, equipped with a nullhomotopy, in the space of closed 2-forms on \( Z \) of weight \( m \), of the pullback of the \( n \)-shifted symplectic form, satisfying a non-degeneracy condition.

(ii) An equivariant Lagrangian structure \( f : Z \to Y^{(\ell)} \) will be called a Lagrangian structure of weight \( \ell \).

Let \( X \) be a smooth projective variety of dimension \( d \) (or, more generally, a derived stack with a twisted orientation of degree \( d \) as in Remark 2.5). Fix \( m \in \mathbb{Z} \) and a choice, \( K^{1/m} \), of an \( m \)-th root of the line bundle \( K_X \) on \( X \).
Lemma 3.1. Let $Y$ be a $\mathbb{G}_m$-stack equipped with an $n$-shifted symplectic form of weight $m \geq 1$ with respect to the $\mathbb{G}_m$-action. Let $\mathcal{L}$ be a line bundle on $X$ and $L$ the corresponding $\mathbb{G}_m$-torsor. Then the stack $Y_L \to X$ carries an $\mathcal{L}^\otimes m$-twisted relative $n$-shifted symplectic structure of weight $m$.

Proof. Let $\lambda : X \times B\mathbb{G}_m \to B\mathbb{G}_m$ be the map classifying the line bundle $\mathcal{L} \otimes \mathcal{O}(-1)$. We have a diagram with cartesian squares:

$$
\begin{array}{ccc}
Y_L & \to & Y_L/\mathbb{G}_m \\
\downarrow & & \downarrow \\
X & \to & X \times \mathbb{G}_m \\
\downarrow & & \downarrow \\
X \times B\mathbb{G}_m & \overleftarrow{p_X \times \lambda} & X \times B\mathbb{G}_m
\end{array}
$$

By Lemma 2.3, we get an isomorphism

$$\tilde{A}_X^{2,cl}(Y_L/\mathbb{G}_m) \simeq (p_X \times \lambda)^* (\tilde{A}_X^{2,cl}(X \times Y/\mathbb{G}_m)).$$

In particular, the sheaf of weight $m$ relative closed 2-forms on $Y_L$ is given by

$$\tilde{A}_X^{2,cl}(Y_L)^{(m)} \simeq \mathcal{L}^\otimes (m) \otimes \tilde{A}_X^{2,cl}(Y)^{(m)}.$$  \ \ (3.2)

By adjunction, we obtain a map

$$\text{twist}_L : \tilde{A}_X^{2,cl}(Y)^{(m)} \to \Gamma(X, \tilde{A}_X^{2,cl}(Y_L)^{(m)} \otimes \mathcal{L}^\otimes m).$$

Thus, an $n$-shifted symplectic form of weight $m$ on $Y$ gives an $\mathcal{L}^\otimes m$-twisted relative closed 2-form of weight $m$ on $Y_L$. Moreover for a $\mathbb{G}_m$-equivariant Lagrangian map $f : Z \to Y$, functoriality of twist$_L$ induces a relative isotropic structure on $f_L : Z_L \to Y_L$. Now, to see that the twisted relative closed 2-form on $Y_L$ is nondegenerate (resp. that $f_L$ is Lagrangian), it suffices to check this locally on $X$. Thus, we can assume that $L$ is the trivial line bundle in which case the statement is manifest.

The following is one of the main results of the paper.

Theorem 3.3. Let $Y$ be a $\mathbb{G}_m$-stack equipped with an $n$-shifted symplectic form of weight $m \geq 1$. Then, one has:

(i) The stack $\text{Sect}_X(Y_{K_X^{1/m}})$ has a natural $(n-d)$-shifted symplectic structure of weight $m$.

(ii) For any Lagrangian structure $f : Z \to Y$, of weight $\ell$, the map $\text{Sect}_X(Z_{K_X^{1/m}}) \to \text{Sect}_X(Y_{K_X^{1/m}})$, induced by $f$, has a natural Lagrangian structure of weight $\ell$.

Proof. Put $\mathcal{L} = K_X^{1/m}$, and let $L \to X$ be the corresponding $\mathbb{G}_m$-torsor. By Lemma 3.1, we have that $Y_L \to X$ has a $K_X$-twisted relative $n$-shifted symplectic structure of weight $m$. By Proposition 2.4, we obtain an $(n-d)$-shifted symplectic structure on $\text{Sect}_X(Y_L)$, resp. Lagrangian structure, on $\text{Sect}_X(Z_L) \to \text{Sect}_X(Y_L)$. Moreover, since the maps

$$\text{Sect}_X(Y_L) \leftarrow \text{Sect}_X(Y_L) \times X \to Y_L$$

are $\mathbb{G}_m$-equivariant, the corresponding symplectic structure has weight $m$. The required statements now follow from an observation that, for any $\mathbb{G}_m$-stack and a $\mathbb{G}_m$-bundle $L \to X$, one has natural isomorphisms of $\mathbb{G}_m$-stacks $\text{Sect}_X(Y_L^{(m)}) \simeq \text{Sect}_X(Y_L^{(m)})$.

We apply the above result to get a description of the symplectic structure on cotangent stacks to mapping stacks.

Proposition 3.4. Let $Y = T^*[n]Z$ be the shifted cotangent stack with its $n$-shifted symplectic structure of weight 1. In this case, there is a natural isomorphism of $(n-d)$-shifted symplectic stacks $\text{Sect}_X(Y_{K_X}) \simeq T^*[n-d] \text{Map}(X, Z)$. 

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Proof. The symplectic form on $T^*[n]Z$ is given by the deRham differential of the canonical $n$-shifted 1-form on $T^*[n]Z$. Therefore, it will suffice to construct an isomorphism of derived stacks $\text{Sect}_X(Y_{KX}) \simeq T^*[n-d]\text{Map}(X, Z)$ such that the transgression of the canonical 1-form is the canonical 1-form.

Recall that given a stack $W$ together with a quasi-coherent sheaf $\mathcal{E} \in \text{QCoh}(W)$, we can form the “total space of $\mathcal{E}$” as the stack $T(\mathcal{E})$ defined as follows. A map from a test scheme $S$ to $T(\mathcal{E})$ is a map $f : S \to W$ together with a section of $f^*(\mathcal{E})$. For instance, the stack $T^*[n]Z$ is the total space of the sheaf $\mathbb{L}_Z[n]$ on $Z$ and the canonical 1-form on $T^*[n]Z$ is given by the image of the section obtained from the identity map on $T^*[n]Z$ along

$$p^*\mathbb{L}_Z[n] \to \mathbb{L}_{T^*[n]Z}[n],$$

where $p : T^*[n]Z \to Z$ is the projection map.

The projection map $p : T^*[n]Z \to Z$ gives a map $f : Y_{KX} \to Z \times X$. In fact, by construction, $Y_{KX}$ is the total space of the sheaf $\mathbb{L}_Z[n] \boxtimes K_X$ on $Z \times X$. In particular, we have a section of $\mathbb{L}_{Y_{KX}/X}[n] \otimes K_X$ given by the image of the canonical section of $f^*(\mathbb{L}_Z[n] \boxtimes K_X)$ along the natural map

$$f^*(\mathbb{L}_Z[n] \boxtimes K_X) \to \mathbb{L}_{Y_{KX}/X}[n] \otimes K_X.$$

Moreover, the map $f$ induces the map

$$g : \text{Sect}_X(Y_{KX}) \to \text{Map}(X, Z),$$

together with a section of $ev^*(\mathbb{L}_{Y_{KX}/X}[n] \otimes K_X)$, where

$$ev : \text{Sect}_X(Y_{KX}) \times X \to \text{Sect}_X(Y_{KX})$$

is the evaluation map. Integrating along $X$, we obtain a section of $\pi_* (ev^*(\mathbb{L}_{Y_{KX}/X}[n] \otimes K_X)) \simeq g^*(\mathbb{L}_{\text{Map}(X,Z)}[n-d])$. This gives the desired map of derived stacks

$$h : \text{Sect}_X(Y_{KX}) \to T^*[n-d]\text{Map}(X, Z),$$

which is easily seen to be an isomorphism. Moreover, by construction, the pullback of the canonical 1-form on $T^*[n-d]\text{Map}(X, Z)$ along $h$ is identified with the transgression of the canonical 1-form on $T^*[n]Z$, as desired. \qed

In addition to equivariant symplectic structures, we will also need to consider equivariant Calabi-Yau structures.

Definition. Let $S$ be a stack with a $\mathbb{G}_m$-action. A $d$-Calabi-Yau structure of weight $m$ on $S$ is a map

$$\Gamma(S, \mathcal{O}_X) \to \mathbb{C}[-d]$$

of weight $m$ satisfying the nondegeneracy condition of [PTV] Definition 2.4. Equivalently, such a structure is given by a map of quasi-coherent sheaves on $BG_m$

$$\pi_* (\mathcal{O}_{S/\mathbb{G}_m}) \to \mathbb{C}(m)[-d],$$

where $\pi : S/\mathbb{G}_m \to BG_m$ is the projection map.

Theorem 3.5. Let $S$ be a $\mathbb{G}_m$-stack with a $d'$-Calabi-Yau structure of weight $m$. Let $X$ be a smooth projective variety of dimension $d$ (or more generally, a derived stack with a twisted orientation $K_X$ of degree $d$ as above) together with a choice of $K_X^{1/m}$. Then:

(i) The stack $\tilde{X} := X \times_{BG_m} S/\mathbb{G}_m$ has a natural $(d + d')$ Calabi-Yau structure of weight $m$, where the map $X \to BG_m$ classifies the line bundle $K_X^{1/m}$.
(ii) Given an $n$-shifted symplectic stack $Y$, there is a natural $\mathbb{G}_m$-equivariant equivalence of $(n-d-d')$-shifted symplectic stacks of weight $m$

$$\text{Map}(\tilde{X}, Y) \simeq \text{Sect}_X(\text{Map}(S, Y)_{K_{X}^{1/m}}).$$

Proof. We have the Cartesian square of stacks

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & S/G_m \\
\downarrow & & \downarrow \pi \\
X & \longrightarrow & B\mathbb{G}_m
\end{array}$$

Therefore, by base change, we have

$$\Gamma(\tilde{X}, \mathcal{O}_X) \simeq \Gamma(X, l^*\pi_*(\mathcal{O}_S/\mathbb{G}_m)).$$

The desired Calabi-Yau structure on $\tilde{X}$ is then given as the composition of Calabi-Yau structures on $S$ and $X$:

$$\Gamma(X, l^*\pi_*(\mathcal{O}_S/\mathbb{G}_m)) \to \Gamma(X, l^*(\mathbb{C}(m)[d'])) \to \Gamma(X, K_X[d']) \to \mathbb{C}[d + d'].$$ 

Now, we have isomorphisms

$$\text{Map}(\tilde{X}, Y) \simeq \text{Sect}_X(\text{Map}_X(\tilde{X}, Y \times X)) \simeq \text{Sect}_X(\text{Map}(S, Y)_{K_{X}^{1/m}}),$$

which by construction of the Calabi-Yau structure on $\tilde{X}$ are compatible with the $(n-d-d')$-shifted symplectic structures of weight $m$. \qed

4. THE CASE OF $G$-BUNDLES

For any stack $\mathcal{Y}$ and an integer $n$, the $n$-shifted cotangent stack $T^*[n]\mathcal{Y}$ comes equipped with a natural $n$-shifted symplectic form, see [PTVV Proposition 1.21] and also [Ca2]. This 2-form has weight 1 with respect to the $\mathbb{G}_m$-action on $T^*[n]\mathcal{Y}$ by dilations along the fibers of the cotangent bundle. The zero section $\mathcal{Y} \to T^*[n]\mathcal{Y}$ has a natural Lagrangian structure.

One has a canonical isomorphism $\mathfrak{g}^*/G \simeq T^*[1]BG$, which provides the stack $\mathfrak{g}^*/G$ with a natural 1-shifted symplectic structure of weight 1.

In what follows, it will be convenient to have another description of this 1-shifted symplectic stack as a mapping stack. Recall that an $Ad$-invariant nondegenerate symmetric bilinear form $\kappa$ on $\mathfrak{g}$ gives a 2-shifted symplectic structure on the stack $BG$. Now, let $S = \widehat{BG_a}$, the formal completion of $BG_a$ at a point, with its natural $\mathbb{G}_m$ action. We have that $\Gamma(S, O_S) \simeq \mathbb{C}[\epsilon]$, where $|\epsilon| = 1$ and the map $\mathbb{C}[\epsilon] \to \mathbb{C}[-1]$, given by $\epsilon \mapsto 1$ gives $S$ a 1-Calabi-Yau structure of weight 1. We then have:

**Lemma 4.1.** There is a canonical isomorphism of 1-shifted symplectic stacks of weight 1

$$\text{Map}(S, BG) \simeq T^*[1]BG.$$ 

Proof. We have a $\mathbb{G}_m$-equivariant isomorphism of derived stacks $\text{Map}(S, BG) \simeq T[-1]BG \simeq \mathfrak{g}/G$. Recall that the 2-shifted symplectic structure on $BG$ is given by the image of an $Ad$-invariant symmetric bilinear form $\kappa$ on $\mathfrak{g}^*$ under the natural map

$$\left(\bigoplus_{i \geq 0} \text{Sym}^{2+i}(\mathfrak{g}^*)[-2 - 2i]\right)^G \to \mathcal{A}^{2,cl}(BG).$$

Unraveling the definitions, we have that the composite map

$$\left(\bigoplus_{i \geq 0} \text{Sym}^{2+i}(\mathfrak{g}^*)[-2 - 2i]\right)^G \to \mathcal{A}^{2,cl}(BG) \to \mathcal{A}^{2,cl}(\mathfrak{g}/G)[-1]$$

...
factors through the map
\[(\oplus_{p+q \geq 0} \Omega^p(\mathfrak{g}) \otimes \mathbb{C} \text{Sym}^q(\mathfrak{g}^*)[2-p-2q]))^G \to \mathcal{A}^{2 \text{cl}}(\mathfrak{g}/G),\]
where the differential in the complex on the left is given by the sum of the internal differential and the deRham differential on \(\mathfrak{g}\). Thus, we obtain that the only nonzero component of the 1-shifted symplectic structure on \(\mathfrak{g}/G\) is given by the image of \(\kappa\) along the map
\[\text{Sym}^2(\mathfrak{g}^*) \to \Omega^1(\mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathcal{O}_G).\]
It follows that the \(\mathbb{G}_m\) equivariant identification \(\mathfrak{g}/G \simeq \mathfrak{g}^*/G\) induced by \(\kappa\) upgrades to an isomorphism of 1-shifted symplectic stacks of weight 1. \(\square\)

The map \(0/G \to \mathfrak{g}^*/G\), induced by the imbedding \(\{0\} \hookrightarrow \mathfrak{g}^*\), may be identified with the zero section \(1: \mathbb{B}G \to T^*[1]BG\).

Let \(M\) be a smooth symplectic variety equipped with a Hamiltonian \(G\)-action. It was observed by Calaque [Ca], that the map \(M/G \to \mathfrak{g}^*/G\), induced by the moment map \(\mu : M \to \mathfrak{g}^*\), has a natural Lagrangian structure. Hence, from Theorem 3.3 in the special case where \(Y = \mathfrak{g}^*/G\) and \(n = 1\) we deduce the following result.

**Corollary 4.2.** (i) For any \(m \geq 1\), the stack \(\text{Sect}_X(\mathfrak{g}^*_{K_X/m}/G)\) has a canonical \((1-d)\)-shifted symplectic structure of weight \(m\).

(ii) For a smooth symplectic \(G \times \mathbb{G}_m\)-variety \(M\) such that the action of the group \(G\) is hamiltonian and the symplectic 2-form has weight \(\ell \geq 1\) with respect to the \(\mathbb{G}_m\)-action, the map \(\text{Sect}_X(M_{K_X/m}) \to \text{Sect}_X(\mathfrak{g}^*_{K_X/m}/G)\), induced by the moment map \(M \to \mathfrak{g}^*\), has a natural Lagrangian structure of weight \(\ell\).

We now specialize to the case where \(\Sigma = X\) is a smooth projective curve. The stack of Higgs bundles on \(\Sigma\) is defined as \(\text{Higgs}_G(\Sigma) := \text{Map}(\Sigma_{\text{Dol}}, \mathbb{B}G)\), where \(\Sigma_{\text{Dol}}\) is the Dolbeault stack, see [PTVV]. Since \(d = \dim \Sigma = 1\), the stack \(\text{Higgs}_G(\Sigma)\) is equipped with a 0-shifted symplectic structure, by [PTVV, Corollary 2.6(2)].

**Lemma 4.3.** There are natural isomorphisms of 0-shifted symplectic stacks
\[\text{Map}(\Sigma_{\text{Dol}}, \mathbb{B}G) \simeq \text{Sect}_\Sigma(\mathfrak{g}_K^*/G) \simeq T^*\text{Bun}_G(\Sigma).\]

**Proof.** By definition, \(\Sigma_{\text{Dol}}\) is identified with \(X \times_{\mathbb{B}G_m} S\), where the map \(X \to \mathbb{B}G_m\) classifies \(K_X\).

Moreover, by construction of the Calabi-Yau structure in Theorem 3.5(i), this isomorphism gives an isomorphism of 1-CY stacks. The first, resp. second, isomorphism of the lemma then follows from Theorem 3.5(ii), resp. Proposition 3.4. \(\square\)

Using the above lemma, from Corollary 4.2 we deduce

**Theorem 4.4.** Let \(M\) be a smooth symplectic \(G \times \mathbb{G}_m\)-variety such that the action of the group \(G\) is hamiltonian and the symplectic 2-form has weight \(\ell \geq 1\) with respect to the \(\mathbb{G}_m\)-action. Then, the map
\[\text{Sect}_\Sigma(M_{K_X/m}/G) \to \text{Sect}_\Sigma(\mathfrak{g}_K^*/G) \simeq T^*\text{Bun}_G(\Sigma),\]
induced by the moment map \(M \to \mathfrak{g}^*\), has a natural Lagrangian structure of weight \(\ell\).

To complete the proof of Theorem 1.3 one observes that, on the locus \(T^*\text{Bun}_G(\Sigma)^{\text{reg}}\) where \(T^*\text{Bun}_G(\Sigma)\) is a smooth variety, the 0-shifted symplectic 2-form is nothing but the standard symplectic 2-form \(\omega\) on \(T^*\text{Bun}_G(\Sigma)^{\text{reg}}\) in the ordinary sense. Similarly, if \(\Lambda\) is a smooth variety and a map \(f : \Lambda \to T^*\text{Bun}_G(\Sigma)^{\text{reg}}\) has a Lagrangian structure then one has \(f^*\omega = 0\). Thus, Theorem 1.3 follows from Theorem 4.4.
5. Additional comments and speculations

5.1. A generalization of Gaiotto’s argument. In the linear case, an ‘infinite dimensional’ approach to Theorem [3] is explained in [Ga]. Gaiotto’s approach is based on a standard differential geometric interpretation of $Bun_G(\Sigma)$ as a quotient of an infinite dimensional space of $\partial$-connections by a gauge group. It was suggested to us by Gaiotto that the argument in [Ga] can be adapted to the more general, nonlinear setting of Theorem [3] as follows. Below, we assume that $\ell = 2$, for simplicity.

Fix a principal $C^\infty$-bundle $P \xrightarrow{G} \Sigma$ and let $Conn_{\partial}(P)$ be (an infinite dimensional) space of $\partial$-connections on $P$. Further, let $\text{Sect}_{\Sigma,C^\infty}(M_{K_X^1/2} \times G P)$ be (an infinite dimensional) space of $C^\infty$-sections of an associated bundle $M_{K_X^1/2} \times G P \rightarrow \Sigma$. Let $z \in \text{Sect}_{\Sigma,C^\infty}(M_{K_X^1/2} \times G P)$ be such a section and $A \in Conn_{\partial}(P)$ a $\partial$-connection. Then $\nabla_A z$, a covariant derivative of $z$ with respect to $A$, is a $C^\infty$-section of $z^*T_M \otimes K_X^{1/2} \otimes \Omega_{\Sigma}^{0,1}$, where $T_M$ stands for the holomorphic tangent sheaf on $M$ and $\Omega_{\Sigma}^{0,q}$ is the sheaf of $C^\infty$ differential forms on $\Sigma$ of type $(p,q)$. Further, let $\lambda_M = i_{eu_M}\omega_M$, where $\omega_M$ is the (holomorphic) symplectic form on $M$ and $eu_M$ is the Euler field that generates the $\mathbb{G}_m$-action on $M$. Thus, $z^*\lambda_M$ is a $C^\infty$-section of $z^*T_M \otimes K_X^{1/2}$. Using the canonical pairing $\left\langle -, - \right\rangle$ of holomorphic vector fields and holomorphic 1-forms on $M$, we obtain a $C^\infty$-section $\left\langle \nabla_A z, z^*\lambda_M \right\rangle$ of the sheaf $K_X^{1/2} \otimes \Omega_{\Sigma}^{0,1} \otimes K_X^{1/2} = K_X \otimes \Omega_{\Sigma}^{0,1} = \Omega_{\Sigma}^{1,1}$.

In the above setting, the role of the potential from [Ga formula (2.3)] is played by a function on $\text{Sect}_{\Sigma,C^\infty}(M_{K_X^1/2} \times G P) \times Conn_{\partial}(P)$ defined by the formula

$$W(z, A) = \int_{\Sigma} \left\langle \nabla_A z, z^*\lambda_M \right\rangle. \quad (5.1)$$

To prove that the map $\mu_{\text{Sect}}$ in Theorem [3] is Lagrangian we show, by a calculation similar to the one in [Ga Appendix A], that (5.1) is a generating function (aka ‘Lagrange multiplier’) for $\text{Sect}_{\Sigma}(M_{K_X^1/2}/G)$.

To this end, observe that an infinitesimal variation of $z$ is given by a section $\dot{z}$ of $z^*T_M \otimes K_X^{1/2}$. The corresponding variation of the $(1,1)$-form $\left\langle \nabla_A z, z^*\lambda_M \right\rangle$ reads

$$\frac{\delta}{\delta z} \left\langle \nabla_A z, z^*\lambda_M \right\rangle(\dot{z}) = \bar{\partial}(\nabla_A \dot{z}, z^*\lambda_M) + (d\lambda_M)(\nabla_A z, \dot{z}),$$

where the operator $\bar{\partial}$ that appears in the first summand on the right is the Dolbeault differential $\bar{\partial} : \Omega_{\Sigma}^{1,0} \rightarrow \Omega_{\Sigma}^{0,1}$. Using that $d\lambda_M = \omega_M$ and that, on $\Omega_{\Sigma}^{1,0}$, one has $\bar{\partial} = d$, we find that the variation of (5.1) equals

$$\frac{\delta W}{\delta z}(\dot{z}) = \int_{\Sigma} d(\nabla_A \dot{z}, z^*\lambda_M) + \int_{\Sigma} \omega_M(\nabla_A z, \dot{z}).$$

The first summand on the right vanishes by Stokes’ theorem. Hence, the form $\omega_M$ being non-degenerate, we deduce that the equation $\frac{\delta W}{\delta z}(\dot{z}) = 0$ holds for all $\dot{z}$ if and only if $\nabla_A z = 0$, that is, if and only if the section $z$ is holomorphic with respect to the complex structure on $M_{K_X^1/2} \times G P$ determined by the $\bar{\partial}$-connection $A$.

Next, let $\dot{A}$ be an infinitesimal variation of $A$. Then, it is easy to check that $\frac{\delta W}{\delta A}(\dot{A}) = \mu_{\text{Sect}}(z, A)(\dot{A})$, proving that $W$ is a generating function for $\text{Sect}_{\Sigma}(M_{K_X^1/2}/G)$.

Remark 5.2. Let $eu$, resp. $eu_{\text{Sect}}$, be the Euler vector field on $T^*Bun_G(\Sigma)$, resp. $\text{Sect}_{\Sigma}(M_{K_X^1/2}/G)$, that generates the $\mathbb{G}_m$-action. Recall that $\omega = d\lambda$ where $\lambda = i_{eu}\omega$ is the Liouville 1-form on $T^*Bun_G(\Sigma)$.
The map $\mu_{\text{Sect}}$ in (1.2) being of weight $\ell$, one finds:

$$\mu^*_{\text{Sect}}(\lambda) = \mu^*_{\text{Sect}}(i_{\text{eu}}\omega) = \frac{1}{\ell} \cdot i_{\text{eu}}\mu^*(\omega).$$

It follows, as has been observed by Hitchin [HI], that Theorem 4.4 is equivalent to the equation $\mu^*_{\text{Sect}}(\lambda) = 0$.

5.2. Relation to the global nilpotent cone. Let $B$ be a Borel subgroup of $G$, so $G/B$ is the flag variety. The symplectic form on $T^*(G/B)$ has weight 1 and the moment map $\mu : T^*(G/B) \to \mathfrak{g}^*$ is the Springer resolution $T^*(G/B) \to N$, where $N \subset \mathfrak{g}^*$ is the nilpotent cone. The stack $\text{Sect}_\Sigma(N/G)_{K_{\Sigma}}$ can be identified with $N_{\Sigma}$, the global nilpotent cone in $T^*\text{Bun}_G(\Sigma)$. Further, the stack $\text{Sect}_\Sigma(T^*(G/B)_{K_{\Sigma}}/G)$ can be identified with $T^*\text{Bun}_B(\Sigma)$. Explicitly, writing $n$ for the nilradical of Lie $B$, the stack $T^*\text{Bun}_B(\Sigma)$ classifies triples $(P, \sigma, \phi)$, where $P$ is a $G$-bundle on $\Sigma$, $\sigma : \Sigma \to P/B$ is a section, i.e. a reduction of $P$ to a $B$-bundle, and $\phi : P \times_B n \to (P \times_B n) \otimes K_{\Sigma}$ is a Higgs field. Assume that the genus of the curve $\Sigma$ is greater than 1. Then, the derived stacks $T^*\text{Bun}_G(\Sigma)$ and $T^*\text{Bun}_B(\Sigma)$ are concentrated in homological degree 0, i.e. they are actually non-derived stacks. The stack $N_{\Sigma}$ is not concentrated in homological degree 0, and one can consider $N_{\Sigma}^{\text{classical}}$, its non-derived counterpart, which is an ordinary substack of $T^*\text{Bun}_G(\Sigma)$.

The map $(P, \sigma, \phi) \mapsto (P, \phi)$, that forgets reduction of the structure group, may be identified with the composition

$$\mu_{\text{Sect}} : T^*\text{Bun}_B(\Sigma) \xrightarrow{\pi_1} N_{\Sigma} \xrightarrow{\pi_2} T^*\text{Bun}_G(\Sigma).$$

The map $\mu_{\text{Sect}}$ has a Lagrangian structure by Corollary 4.2. One can show that the map $\pi_2$ has a natural coisotropic structure in the sense of [MS]. However, this coisotropic structure is easily seen to be not Lagrangian.

On the other hand, it was shown in [Gi] that, for any field extension $K/k$, the map $\pi_1^{\text{classical}} : T^*\text{Bun}_B(\Sigma)(\text{Spec } K) \to N_{\Sigma}^{\text{classical}}(\text{Spec } K)$, of $K$-points of the corresponding non-derived stacks, is surjective. This result was used in [Gi] to prove that $N_{\Sigma}^{\text{classical}}$ is (as opposed to its derived analogue) a Lagrangian substack of $T^*\text{Bun}_G(\Sigma)$ in the sense explained in loc cit.

More generally, let $\tilde{Y} \to Y$ be a $(G \times \mathbb{G}_m)$-equivariant symplectic resolution such that $Y$ is affine, the $\mathbb{G}_m$-action on $Y$ is a contraction to a unique $\mathbb{G}_m$-fixed point and, moreover, the symplectic form on $\tilde{Y}$ has weight $m \geq 1$. Then, we have $k[\tilde{Y}] = k[Y]$, so the Poisson bracket on the algebra $k[Y]$ provides $Y$ with a $(G \times \mathbb{G}_m)$-equivariant Poisson structure. Also, the moment map $\tilde{Y}/G \to \mathfrak{g}^*/G$ factors through $Y/G$. Therefore, there is a chain of induced maps $\text{Sect}_\Sigma((\tilde{Y}/G)_{K_{\Sigma}^m}) \xrightarrow{\pi_1} \text{Sect}_\Sigma(Y_{K_{\Sigma}^m}/G) \xrightarrow{\pi_2} T^*\text{Bun}_G(\Sigma)$ such that $\pi_2 \circ \pi_1 = \mu_{\text{Sect}}$. The map $\mu_{\text{Sect}}$ has a Lagrangian structure, by Theorem 4.4. Again, one can show that the map $\pi_2 : \text{Sect}_\Sigma(Y_{K_{\Sigma}^m}/G) \to T^*\text{Bun}_G(\Sigma)$ has a natural coisotropic structure.

**Question 5.4.** Is $\text{Sect}_\Sigma(Y_{K_{\Sigma}^m}/G)^{\text{classical}}$, a non-derived counterpart of $\text{Sect}_\Sigma(Y_{K_{\Sigma}^m}/G)^{\text{classical}}$, isotropic in the sense of [Gi], specifically, is it possible to partition $\text{Sect}_\Sigma(Y_{K_{\Sigma}^m}/G)^{\text{classical}}$ as a disjoint union of substacks such that the pull-back of the symplectic 2-form on $T^*\text{Bun}_G(\Sigma)$ to each of these substacks vanishes?

5.3. Hamiltonian reduction. Let $M$ be a stack equipped with a 0-shifted symplectic structure and with a Hamiltonian $G$-action with moment map $\mu$. The stack $\mu^{-1}(0)/G$, a stacky Hamiltonian reduction of $M$, comes equipped with a canonical 0-shifted symplectic structure. On the other hand, let $\Lambda_1 = 0/G \to \mathfrak{g}^*/G$ be the map induced by the imbedding $\{0\} \hookrightarrow \mathfrak{g}^*$ and $\Lambda_2 = M/G \to \mathfrak{g}^*/G$ be the map induced by $\mu$. One has a natural isomorphism

$$\Lambda_1 \times_{\mathfrak{g}^*/G} \Lambda_2 = 0/G \times_{\mathfrak{g}^*/G} M/G \cong \mu^{-1}(0)/G.$$ 

(5.5)
Recall that the stack $g^*/G$ has the canonical 1-shifted symplectic structure and each of the two maps $\Lambda_i \to g^*/G$, $i = 1, 2$, has a Lagrangian structure, cf. §4. Further, according to [PTVV Theorem 0.5], for any stack $\mathcal{Y}$ equipped with an $n$-shifted symplectic structure and a pair $\Lambda_i \to \mathcal{Y}$, $i = 1, 2$, of Lagrangian structures, the stack $\Lambda_1 \times \mathcal{Y} \Lambda_2$ has a natural $(n-1)$-shifted symplectic structure. Therefore, the stack $0/G \times g^*/G \ M/G$ comes equipped with a 0-shifted symplectic structure. It was shown by Calaque [Ca] that the isomorphism in (5.5) respects the 0-shifted symplectic structures.

Next, we fix a smooth projective curve $\Sigma$ and let $K = K_{\Sigma}$. The stack $g^*_K = T^*Bun_G(\Sigma)$, a global counterpart of $g^*/G$, has the 0-shifted symplectic structure of weight 1. Also, the Lagrangian structure on the map $0/G \to g^*/G$ induces, for any $\ell$, a weight $\ell$ Lagrangian structure $\text{Sect}_\Sigma((0/G)_{K^{1/\ell}}) \to \text{Sect}_\Sigma((g^*/G)_K)$. The latter Lagrangian structure corresponds, via the isomorphisms $T^*Bun_G(\Sigma) \cong g^*_K/G$ and $Bun_G(\Sigma) \cong \text{Sect}_\Sigma((0/G)_K)$, to an obvious Lagrangian structure on the zero section $Bun_G(\Sigma) \to T^*Bun_G(\Sigma)$. (We have used here that for any variety $\mathcal{Y}$ equipped with a trivial $G_m$-action and any $G_m$-bundle $L$ on $\Sigma$, one has $\text{Sect}_\Sigma(\mathcal{Y}_L) = \text{Map}(\Sigma, \mathcal{Y})$, in particular, we have $\text{Sect}_\Sigma((0/G)_K) = \text{Map}(\Sigma, BG) = Bun_G(\Sigma)$.)

Now, let $M$ be a symplectic manifold equipped with a $(G \times G_m)$-action such that the symplectic 2-form has weight $\ell \geq 1$ and the $G$-action is Hamiltonian. One has canonical isomorphisms

$$\text{Sect}_\Sigma((0/G)_{K^{1/\ell}}) \times_{T^*Bun_G(\Sigma)} \text{Sect}_\Sigma((M/G)_{K^{1/\ell}}) \cong \text{Sect}_\Sigma((0/G)_{K^{1/\ell}}) \times_{g^*_K/G} (M/G)_{K^{1/\ell}} \cong \text{Sect}_\Sigma((\mu^{-1}(0)/G)_{K^{1/\ell}}).$$

Here, the fiber product on the left involves the maps (5.5), which has a weight $\ell$ Lagrangian structure, by Theorem 4.4. Thus, according to [PTVV Theorem 0.5], the fiber product of Lagrangians on the left of (5.6) has a $(−1)$-shifted symplectic structure. On the other hand, the 0-shifted symplectic structure on $\mu^{-1}(0)/G$ induces, by Theorem 3.3(i), a $(−1)$-shifted symplectic structure of weight $\ell$ on $\text{Sect}_\Sigma((\mu^{-1}(0)/G)_{K^{1/\ell}})$, the stack on the right of (5.6). One can check that the composite isomorphism in (5.6) respects the $(−1)$-shifted symplectic structures described above.

Let $\mathcal{X}$ be a stack and assume there is a line bundle $K_{\mathcal{X}}^{1/2}$, a square root of the dualizing complex of $\mathcal{X}$. In [Pr], Pridham shows that an $(−1)$-shifted symplectic structure on $\mathcal{X}$ gives rise to a canonical self-dual quantization of $K_{\mathcal{X}}^{1/2}$. Moreover, associated with that quantization, there is a constructible complex on $\mathcal{X}$ of vanishing cycles. Therefore, one might expect that, in the setting of the previous paragraph, the stack $\text{Sect}_\Sigma((\mu^{-1}(0)/G)_{K^{1/\ell}})$ comes equipped (perhaps, under some additional assumptions) with a natural constructible complex of vanishing cycles.

The linear case, where $\ell = 2$ and $M$ is a linear symplectic representation of $G$, has been considered in the physics literature in the framework of Coulomb branches for 3-dimensional gauge theories, cf. [Ga] and references therein. The special case where $M = E \oplus E^*$ is a direct sum of a pair of dual representations of $G$ is simpler than the general case. In that case, the geometry of $\text{Sect}_\Sigma((\mu^{-1}(0)/G)_{K^{1/2}})$ can be reduced, in a sense, to the geometry of $\text{Sect}_\Sigma(E_{K^{1/2}})$. Such a reduction allows to avoid the use of vanishing cycles. A mathematical theory of Coulomb branches in the case $M = E \oplus E^*$ was developed by H. Nakajima [Na], cf. also [BFN].

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