SYMPELIC PbW TABLEAUX AND DEGENERATE RELATIONS

GEORGE BALLA

ABSTRACT. We define a set of PBW-semistandard tableaux that are in a weight preserving bijection with the set of monomials corresponding to integral points in the Feigin-Fourier-Littelmann-Vinberg polytope for highest weight modules of the symplectic Lie algebra. We then show that these tableaux parametrize bases of the homogeneous coordinate rings of the original and the PBW degenerate complete symplectic flag varieties. From this construction, we provide explicit degenerate relations that generate the defining ideal of the PBW degenerate complete symplectic flag variety. These relations consist of type A degenerate Plücker relations and a set of degenerate linear relations that we obtain from De Concini’s linear relations.

INTRODUCTION

Let $G$ be a simple, simply connected algebraic group over the field $\mathbb{C}$ and $\mathfrak{g}$ the corresponding Lie algebra. For a dominant, integral weight $\lambda$, let $V_{\lambda}$ be the corresponding simple $\mathfrak{g}$-module, and $\nu_{\lambda} \in V_{\lambda}$ a highest weight vector. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a Cartan decomposition and $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ the Borel subalgebra. For $\lambda$ regular, the complete flag variety $\mathcal{F}_{\lambda}$ is defined to be the closure of the $G$-orbit through a highest weight line: $\mathcal{F}_{\lambda} = G[\nu_{\lambda}] \hookrightarrow \mathbb{P}(V_{\lambda})$. Another realisation of this variety is through the quotient $G/B$, where $B$ is a Borel subgroup. On the other hand, one has $V_{\lambda} = \mathcal{U}(\mathfrak{n}^-)\nu_{\lambda}$, where $\mathcal{U}(\mathfrak{n}^-)$ is the universal enveloping algebra of $\mathfrak{n}^-$. There exists a degree filtration $\mathcal{U}(\mathfrak{n}^-)_s = \text{span}\{x_1 \cdots x_l : x_i \in \mathfrak{n}^-, l \leq s\}$ on $\mathcal{U}(\mathfrak{n}^-)$. This filtration in turn induces the filtration $F_s = \mathcal{U}(\mathfrak{n}^-)_s\nu_{\lambda}$ on $V_{\lambda}$, called the PBW filtration. The associated graded space is $F_0 \oplus_{s \geq 1} F_s/F_{s-1}$, which will be denoted by $V_{\lambda}^\alpha$ (see [8] and [9]). This graded space has a structure of $\mathfrak{g}^a$-module where $\mathfrak{g}^a$ is a Lie algebra which is a semi-direct sum of $\mathfrak{b}$ and an abelian ideal $(\mathfrak{n}^-)^a$ (see [10]). Let $G^a$ be a Lie group corresponding to $\mathfrak{g}^a$. Let $\nu_{\lambda}^\alpha$ be the image of $\nu_{\lambda}$ in $V_{\lambda}^\alpha$. The PBW degenerate flag variety is defined to be $\mathcal{F}_{\lambda}^\alpha := G^a[\nu_{\lambda}^\alpha] \hookrightarrow \mathbb{P}(V_{\lambda}^\alpha)$ (see [10]). Feigin in [10], studied the variety $\mathcal{F}_{\lambda}^\alpha$ in type $A$ when $G = \text{SL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. In order to show that this variety is a flat degeneration of the original variety $\mathcal{F}_{\lambda}$, he defined the PBW-semistandard tableaux which label bases of the homogeneous coordinate rings of both varieties. Let us review what these tableaux are. For a type $A_n$ dominant, integral weight $\lambda$, written as a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, consider the corresponding Young diagram $Y_{\lambda}$ (English convention). A type $A_n$ PBW-semistandard tableau of shape $\lambda$ is the filling of $Y_{\lambda}$ with entries from $\{1, \ldots, n+1\}$ such that the following three conditions are satisfied. First of all, in each column, each entry less than the length of that column is at row position equal to that entry (or in short, at its position). Secondly, every entry not at its position should be greater than all entries below it in any given column. And finally, for every entry in each column apart from the first column, there should be a greater or equal entry in the column to the left and in the same row or in a row below. We refer to the last condition as PBW-semistandardness.

Now consider type $C$, with $G = \text{Sp}_{2n}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. We consider the complete symplectic flag variety, which will be denoted by $\text{Sp}\mathcal{F}_{2n}$ and its PBW degeneration $\text{Sp}\mathcal{F}_{2n}^\alpha$. Let $\mathbb{C}[\text{Sp}\mathcal{F}_{2n}]$ and $\mathbb{C}[\text{Sp}\mathcal{F}_{2n}^\alpha]$ denote the (multi-)homogeneous coordinate rings of $\text{Sp}\mathcal{F}_{2n}$ and $\text{Sp}\mathcal{F}_{2n}^\alpha$. 

respectively. The first goal of this paper is to define a set of PBW-semistandard tableaux for type $C_n$, and to show that they label weighted bases of both $\mathbb{C}[\text{Sp}_2 F_{2n}]$ and $\mathbb{C}[\text{Sp}_2 F_{2n}^a]$. Let $\lambda$ be a type $C_n$ dominant, integral weight, written again as a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$. For $i \in \{1, \ldots, n\}$, let $\bar{i} := 2n + 1 - i$.

We define a symplectic (or type $C_n$) PBW-semistandard tableau to be a filling of the corresponding Young diagram $Y_\lambda$ with entries in the set $\{1 < \cdots < n < \pi < \cdots < \bar{1}\}$ such that not only the conditions for the type $A$ PBW-semistandard tableaux are satisfied, but also the following extra condition. For every element $i \in \{1, \ldots, n\}$ in any column, if the element $\bar{i}$ exists in the same column, then the position of $\bar{i}$ should be above that of $i$, whenever $i$ is less than the length of the column. We call this extra condition the \textit{PBW-symplectic condition}.

We would like to note that several symplectic tableaux already exist, for example those of De Concini [5], Hamel and King [14], Kashiwara and Nakashima [16], King [17], and Proctor [21]. The main difference between these tableaux and our tableaux is the PBW-semistandardness condition and in some cases, the PBW-symplectic condition (see Subsection 2.3 for a brief comparison). We prove

\textbf{Theorem} (Theorem 4.7). The symplectic PBW-semistandard tableaux index a basis of $\mathbb{C}[\text{Sp}_2 F_{2n}]$.

Feigin, Finkelberg and Littelmann showed in [11] that $\text{Sp}_2 F_{2n}^a$ is a flat degeneration of $\text{Sp}_2 F_{2n}$. It therefore follows naturally that our tableaux also label a basis for $\mathbb{C}[\text{Sp}_2 F_{2n}]$ (see Theorem 3.20). We would also like to discuss a correspondence between our tableaux and certain bases of the modules $V_\lambda$ and $V_\lambda^a$. In 2011, Feigin, Fourier and Littelmann in [8] and [9] defined the Feigin-Fourier-Littelmann-Vinberg polytopes that parametrize monomial bases for highest weight original and PBW degenerate simple modules for a Lie algebra $g$ in types $A_n$ and $C_n$ respectively. Bases arising this way are called FFLV bases. We prove that one has a weight preserving bijection between the FFLV basis for the symplectic modules $V_\lambda$ and $V_\lambda^a$ and the symplectic PBW-semistandard tableaux (see Theorem 2.14). It is worth noting that Young [23] was the first to introduce (semi-)standard Young tableaux to provide a basis for the irreducible polynomial representations of the general linear group and for the irreducible representations of symmetric groups. On the other hand, standard monomial theory was begun by Hodge [15], who used Young theory to give a basis of the homogeneous coordinate ring for flag varieties. The same theory has been further developed through the work of different authors (see for example, [5], [18], [19], [20], …).

At this point we would like to step back and discuss briefly one of the very important tools in our proof of Theorem 4.7; namely, the symplectic degenerate relations. Feigin in [10] defined the PBW degenerate Plücker relations (quadratic relations) and proved that they generate the defining ideal of the PBW degenerate flag variety in type $A$, a result which has led to many other results on understanding this variety. Since $\text{Sp}_2 F_{2n}^a$ is point-wise contained in the type $A_{2n-1}$ PBW degenerate complete flag variety (see [11]), it follows that Feigin’s degenerate relations are also satisfied on $\text{Sp}_2 F_{2n}^a$. We call these the \textit{symplectic degenerate quadratic relations} and denote them by $R_{A_{2n}}$. On the other hand, De Concini [5] defined linear relations while showing that his symplectic standard tableaux index a basis for $\mathbb{C}[\text{Sp}_2 F_{2n}]$. We call these the \textit{symplectic linear relations}, which will be denoted by $S_{A_{2n}}$. In his proof, he also used quadratic relations, which implies that these quadratic and linear relations generate the defining ideal of $\text{Sp}_2 F_{2n}$, since they provide a straightening law for $\mathbb{C}[\text{Sp}_2 F_{2n}]$. Note that Chirivi and Maffei in [4] and in [3] with Littelmann, gave a general framework for these defining equations for flag varieties
corresponding to a Lie algebra $\mathfrak{g}$ of any type. We now obtain degenerate relations from the symplectic linear relations, which we call \textit{symplectic degenerate linear relations} and denote them by $S_{(I_2,I_1)}^a$ (see Definition 4.1 for a full description). We obtain a fundamental result about the defining ideal of $\text{Sp}\mathcal{F}_{2n}$, which is the second and final goal of this paper. Let $I^a$ be the ideal generated by the relations $S_{(I_2,I_1)}^a$ and $R_{(1,2),I}^{1,a}$. For example, for $n = 2$, the ideal $I^a$ is generated by the symplectic degenerate quadratic relations

$$R_{(1,2),I}^{1,a} := X_{1,2}^a X_1^a + X_{2,1}^a X_2^a - X_{1,2}^a X_2^a,$$

and the symplectic degenerate linear relation $S_{(1,1)}^{a} := X_{1,1}^a + X_{2,2}^a$. We prove

\textbf{Theorem} (Theorem 4.8). The ideal $I^a$ is a prime defining ideal of $\text{Sp}\mathcal{F}_{2n} \hookrightarrow \mathbb{P}(V_{\lambda})$.

In a forthcoming work, we will further extend the following known type $A$ results to the symplectic setup: the work of Bossinger, Lamboglia, Mincheva and Mohammadi [1] on computing toric degenerations arising from tropicalization of flag varieties, and the work of Fang, Feigin, Fourier and Makhlin [6], in which they define a maximal cone of the tropical flag variety and identify several facets corresponding to linear degenerations ([2]). In the same spirit, we are also computing some first examples of tropical symplectic Grassmann varieties following [22].

This paper is organised as follows. In Section 1, we recall results on the FFLV basis for the symplectic Lie algebra. In Section 2, we define the symplectic PBW-semistandard tableaux and establish the bijection between them and the symplectic FFLV basis. We show that the symplectic PBW-semistandard tableaux label a basis for the homogeneous coordinate ring of $\text{Sp}\mathcal{F}_{2n}$ in Section 3. In Section 4, we give the definition of the symplectic degenerate relations and use them to show that the symplectic PBW-semistandard tableaux label a basis for the homogeneous coordinate ring of $\text{Sp}\mathcal{F}_{2n}$. We also prove here that the ideal generated by the symplectic degenerate relations is the defining ideal of $\text{Sp}\mathcal{F}_{2n}$.

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1. Preliminaries; Representation theory

In this section, we recall the description of the corresponding simple original and PBW degenerate modules for the symplectic Lie algebra and the FFLV basis as studied in [9].

1.1. The symplectic Lie algebra; a quick description. All information in this subsection can be found in [13]. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\mathfrak{sp}_{2n} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a Cartan decomposition, $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ the Borel subalgebra and let $R^+$ be the set of positive roots of $\mathfrak{sp}_{2n}$. For each $\alpha \in R^+$, fix a non zero element $f_\alpha \in \mathfrak{n}_{-\alpha}^\perp$. Let $\alpha, w_i$ with $i = 1, \ldots, n$ be the simple roots and
the fundamental weights respectively. All positive roots of $\mathfrak{sp}_{2n}$ can be divided into two sets namely:

\[
\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \quad 1 \leq i \leq j \leq n,
\]

\[
\alpha_{i,j}^- = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \alpha_{n-1} + \ldots + \alpha_j, \quad 1 \leq i \leq j < n, \quad i + j < 2n,
\]

where $\alpha_{i,n} = \alpha_{i,n}^-$. Henceforth, we will sometimes, when we consider it convenient, use the short forms:

\[
\alpha_i = \alpha_{i,i}, \quad \alpha_i^- = \alpha_{i,1}, \quad f_{i,j} = f_{\alpha_{i,j}} \quad \text{and} \quad f_{i,j}^- = f_{\alpha_{i,j}^-}.
\]

The formulas for the root vectors, $f_\alpha \in \mathfrak{n}_-\alpha$ of $\mathfrak{sp}_{2n}$ are explicitly given in [9], and we recall them below, with a slight modification of notation to suit our notation used here, in that we write $\bar{i}$ instead of $2n + 1 - i$:

\[
f_{i,j} = \begin{cases}
E_{j+1,i} - E_{\bar{j},2n-j}^\bar{i}, & 1 \leq i \leq j < n, \\
E_{j+1,i} + E_{\bar{j},2n-j}^\bar{i}, & j \geq n, i + j < 2n, \\
E_{\bar{i},i}, & 1 \leq i \leq n,
\end{cases}
\]

where $E_{j,i}$ is the matrix with zeros everywhere except for the entry 1 in the $j$-th row and $i$-th column.

1.2. The PBW degeneration. Consider the increasing degree filtration on the universal enveloping algebra, $\mathcal{U}(\mathfrak{n}^{-})$:

\[
\mathcal{U}(\mathfrak{n}^{-})_s = \text{span}\{x_1 \cdots x_l : x_i \in \mathfrak{n}^{-}, l \leq s\}. \tag{1.1}
\]

For a dominant integral weight $\lambda = m_1 \omega_1 + \ldots + m_n \omega_n$, let as usual, $V_\lambda$ be the corresponding simple highest weight $\mathfrak{sp}_{2n}$-module with a highest weight vector $\nu_\lambda$. It is known that $V_\lambda = \mathcal{U}(\mathfrak{n}^{-}) \nu_\lambda$, therefore, the filtration (1.1) induces an increasing degree filtration $F_s$ on $V_\lambda$:

\[F_s = \mathcal{U}(\mathfrak{n}^{-})_s \nu_\lambda.\]

This filtration is called the PBW filtration. Let us denote the associated graded space by $V^a_\lambda$, one has:

\[V^a_\lambda = \bigoplus V^a_\lambda(s) = \bigoplus_{s \geq 0} F_s / F_{s-1}.\]

Elements of $V^a_\lambda(s)$ are said to be homogeneous of PBW-degree $s$. The graded space $V^a_\lambda$ has a structure of $\mathfrak{g}^a$-module where $\mathfrak{g}^a$ is a semi-direct sum of the Borel subalgebra $\mathfrak{b}$ and an abelian ideal $(\mathfrak{n}^{-})^a$, which is isomorphic to $\mathfrak{n}^{-}$ as a vector space. The Lie algebra $\mathfrak{g}^a$ is a PBW degeneration of $\mathfrak{g}$ (see [10]). For the highest weight vector $\nu_\lambda$ in $V_\lambda$, we denote by $\nu_\lambda^a$ its image in $V^a_\lambda$.

1.3. The symplectic FFLV basis. Here we recall results due to Feigin, Fourier and Littelmann in [9]. Our results on the symplectic PBW-semistandard tableaux strongly rely on these results. In order to describe fully the basis for $V_\lambda$, we recall first the notion of the symplectic Dyck path. The indexing set for the roots is $J = \{1, \ldots, n, n-1, \ldots, 1\}$ with the usual order: $1 < \ldots < n < n-1 < \ldots < 1$.

**Definition 1.1.** A symplectic Dyck path is a sequence $p = (p(0), \ldots, p(k))$, $k \geq 0$, of positive roots satisfying the conditions:

(i) the first root $p(0) = \alpha_i$ for some $1 \leq i \leq n$, i.e. it is simple.

(ii) the last root is either simple or the highest root of a symplectic subalgebra, i.e. $p(k) = \alpha_j$ or $p(k) = \alpha^-_j$ for some $1 \leq j < n$. 


(iii) the elements in between satisfy the recursion rule: If \( p(s) = \alpha_{p,q} \) with \( p, q \in J \), then the next element in the sequence is of the form either \( p(s+1) = \alpha_{p,q+1} \) or \( p(s+1) = \alpha_{p+1,q} \); where \( x+1 \) denotes the smallest element in \( J \) which is bigger than \( x \).

**Example 1.2.** For \( \mathfrak{sp}_6 \), the roots can be arranged in form of a triangle. The Dyck paths are the ones starting at a simple root and ending at one of the edges following the directions indicated by the arrows.

\[
\begin{align*}
\alpha_{1,1} &\rightarrow \alpha_{1,2} \rightarrow \alpha_{1,3} \rightarrow \alpha_{1,\overline{2}} \rightarrow \alpha_{1,\overline{1}} \\
\alpha_{2,2} &\rightarrow \alpha_{2,3} \rightarrow \alpha_{2,\overline{2}} \\
\alpha_{3,3} &
\end{align*}
\]

**Definition 1.3.** Denote by \( \mathbb{D} \) the set of all Dyck paths. For a dominant, integral weight \( \lambda = \sum_{i=1}^{n} m_i \omega_i \), the symplectic Feigin-Fourier-Littelmann-Vinberg (FFLV) polytope \( P(\lambda) \subset \mathbb{R}^{n_\mathbb{Z}_0} \) is the polytope \( P(\lambda) = \{(f_{\alpha})_{\alpha>0}, \forall \alpha \in \mathbb{D} \} \), such that:

\[
\begin{align*}
s_{p(0)} + \ldots + s_{p(k)} \leq m_i + \ldots + m_j, \quad &\text{if} \quad p(0) = \alpha_i, \quad p(k) = \alpha_j, \\
s_{p(0)} + \ldots + s_{p(k)} \leq m_i + \ldots + m_n, \quad &\text{if} \quad p(0) = \alpha_i, \quad p(k) = \alpha_{\overline{j}}, \\
s_{p(i)} \geq 0, &\text{for} \quad 0 \leq i \leq k.
\end{align*}
\]

**Example 1.4.** Consider the Dyck paths in Example 1.2.

Here we have \( \lambda = m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3 \), so \( P(\lambda) \subset \mathbb{R}^{9}_{\geq 0} \) is the polytope defined by all points \((s_{1,1}, s_{1,2}, s_{1,3}, s_{1,\overline{1}}, s_{2,1}, s_{2,2}, s_{2,3}, s_{2,\overline{1}}, s_{3,3})\) satisfying all the inequalities arising from all Dyck paths as seen in Definition 1.3 above. As an illustration, for the Dyck path corresponding to the green arrows, one has: \( s_{1,1} + s_{1,2} + s_{2,2} + s_{2,3} + s_{3,3} \leq m_1 + m_2 + m_3 \), as one of the inequalities.

Let \( S(\lambda) \) be the set of integral points in \( P(\lambda) \). For a multi-exponent \( s = (s_\beta)_{\beta > 0}, s_\beta \in \mathbb{Z}_{\geq 0}, \) let \( f^s \) be the element:

\[
f^s = \prod_{\beta \in R^+} f^{s_\beta}_\beta \in S(n^-),
\]

where \( S(n^-) \) denotes the symmetric algebra of \( n^- \). Recall the highest weight vector \( \nu_\lambda \in V_\lambda \) and its image \( \nu_\lambda^a \) in \( V^a_\lambda \).

**Theorem 1.5.** ([9]) The elements \( \{f^s \nu_\lambda^a, s \in S(\lambda)\} \) form a basis of \( V^a_\lambda \) and \( \{f^s \nu_\lambda, s \in S(\lambda)\} \) form a basis of \( V_\lambda \) (after fixing a total order on the root vectors \( f_\beta \)).

In what follows, we will refer to the basis \( \{f^s \nu_\lambda, s \in S(\lambda)\} \) as the symplectic FFLV basis. We end this section by stating the following result.

**Lemma 1.6.** ([9]) For any two dominant, integral and regular weights \( \lambda \) and \( \mu \), there exist homomorphisms of modules:

\[
V_{\lambda+\mu} \leftrightarrow V_\lambda \otimes V_\mu, \quad \nu_{\lambda+\mu} \mapsto \nu_\lambda \otimes \nu_\mu \quad \text{and} \quad V^a_{\lambda+\mu} \leftrightarrow V^a_\lambda \otimes V^a_\mu, \quad \nu^a_{\lambda+\mu} \mapsto \nu^a_\lambda \otimes \nu^a_\mu.
\]
2. THE SYMPLECTIC FFLV BASIS - PBW TABLEAUX CORRESPONDENCE

In this section we define a set of PBW-semistandard tableaux which are in a one-to-one correspondence to the basis described above. We explicitly construct the corresponding maps, first for fundamental weights and then we later generalise to any dominant, integral weight. The tableaux we define here take entries in \( N := \{1, \ldots, n, \bar{n}, \ldots, \bar{1}\} \), with the usual order: \( 1 < \ldots < n < \bar{1} < \ldots < \bar{n} \).

Remark 2.1. In general, to a dominant, integral weight \( \lambda = \sum_{k=1}^{n} m_k \omega_k \), we assign a partition \( \lambda = (m_1 + m_2 + \ldots + m_n, m_2 + \ldots + m_n, \ldots, m_n) \).

2.1. The case of fundamental weights. To a fundamental weight \( \lambda = \omega_k \) for \( 1 \leq k \leq n \), associate according to Remark 2.1, a partition \( \lambda = (1, \ldots, 1) \). The Young diagram of such a partition is just a single column of length \( k \). Below we describe a filling of these columns to give us what we term symplectic PBW tableaux.

Definition 2.2. For a partition \( \lambda = (1, \ldots, 1), 1 \leq k \leq n \), the symplectic PBW tableau \( T_{\lambda} \) is the filing of the corresponding Young diagram \( Y_{\lambda} \) with numbers \( T_i \in N \) such that:

(i) if \( T_i \leq k \), then \( T_i = i \),
(ii) if \( i_1 < i_2 \) and \( T_{i_1} \neq T_{i_2} \), then \( T_{i_1} > T_{i_2} \) and
(iii) if there exists \( i, i' \) with \( T_i = i \) and \( T_{i'} = \bar{i} \), then \( i' < i \), whenever \( i < k \).

Example 2.3. For \( N = \{1, 2, 3, 3, 3, 3\} \) with \( \lambda = (1, 1, 1) \), all the possible symplectic PBW tableaux are:

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & T & T & T & T & T & 2 \\
2 & 2 & 3 & 2 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}
\]

Definition 2.4. Let \( T_{\lambda} \) be a symplectic PBW-semistandard tableau of shape \( \lambda \), \( N^+ := \{i \in \{1, \ldots, n\} : i \in T_{\lambda}\} \) and \( N^- := \{j \in \{1, \ldots, n\} : \bar{j} \in T_{\lambda}\} \). Then the symplectic weight of \( T_{\lambda} \) is given by:

\[
wt(T_{\lambda}) := \sum_{i \in N^+} \varepsilon_i - \sum_{j \in N^-} \varepsilon_j
\]

For an operator \( f_{i,j} := f_\alpha \), the symplectic weight is \( wt(f_{i,j}) := -\varepsilon_i - \varepsilon_j \), and for the product \( f^s = \prod_{\alpha \geq 0} f_\alpha^{s_\alpha} \), the symplectic weight is:

\[
wt(f^s) := \sum_{\alpha : f_\alpha \in f^s} s_\alpha \cdot wt(f_\alpha),
\]

and for an assignment \( f^s \cdot t_{\lambda} \), we have:

\[
wt(f^s \cdot t_{\lambda}) = wt(f^s) + wt(t_{\lambda}).
\]

We also have \( wt(t_{\lambda}) = wt(\nu_{\lambda}) \).

Let \( SyP_{\lambda} \) be the set of all elements \( f^s \cdot \nu_{\lambda} \) with \( f^s = \prod_{\alpha > 0} f_\alpha^{s_\alpha} \), for \( s \in S(\lambda) \), and let \( SyT_{\lambda} \) be the set of all symplectic PBW tableaux as established above. We prove the following result:
Proposition 2.5. For $\lambda$ a fundamental weight, the set $\text{SyP}_\lambda$ is in a weight preserving one-to-one correspondence with the set $\text{SyT}_\lambda$.

Proof. Define the map:

$$\theta_1 : \text{SyP}_\lambda \rightarrow \text{SyT}_\lambda, \quad f^s \cdot \nu_\lambda \mapsto f^s \cdot t_\lambda,$$

where $t_\lambda$ here stands for a highest weight single column tableau of length $k$, filled with numbers $1, \ldots, k$, each number appearing at its position. The operators $f_{i,j}$ appearing in $f^s$ act each at a position $i$ according to the following:

$$f_{i,j} \cdot \begin{array}{|c|} \hline \end{array} i = \begin{cases} \begin{array}{|c|} \hline \end{array} j & \text{if } k \leq j \leq n, \\ \begin{array}{|c|} \hline \end{array} j & \text{if } \frac{n-1}{k-1} \leq j \leq \frac{n-1}{k}, \end{cases}$$

with $n + 1 = \overline{n}$. Let $f^s = f_{i_1,j_1} \cdots f_{i_s,j_s} \in \text{SyP}_\lambda$, then we have $1 \leq i_1 < \ldots < i_s \leq k$, and $\overline{1} \geq j_1 > \ldots > j_s \geq k$. Since we have $i_1 \neq \ldots \neq i_s$, then we have that the operators each act at a different position once. We also have:

$$f_{i_1,j_1} \cdot i_1 > \cdots > f_{i_s,j_s} \cdot i_s,$$

according to (2.1). We are left to show condition (iii) of Definition 2.2. For an entry $m \in T_\lambda$ with $m < k$, we need to check that if $\overline{m}$ exists in $T_\lambda$, then its position is above that of $m$.

Consider $f_{i_1,j_1} \cdots f_{i_s,j_s} \cdot \begin{array}{|c|} \hline \end{array} m$. Assume there exists $j_p \in \{j_1, \ldots, j_s\}$ such that $j_p = \overline{m}$. If $m \in \{i_1, \ldots, i_s\}$, then we have $f_{i_1,j_1} \cdots f_{i_s,j_s} \cdot \begin{array}{|c|} \hline \end{array} m = \overline{m}$. Hence $m$ will not appear in the resulting tableau. In case $m \notin \{i_1, \ldots, i_s\}$, then we have $f_{i_p,\overline{m}} \cdot \begin{array}{|c|} \hline \end{array} i_p = \overline{m}$ at position $i_p$. But $i_p < m$, so $\overline{m}$ is above $m$, and we are done.

Now we define another map:

$$\theta_2 : \text{SyT}_\lambda \rightarrow \text{SyP}_\lambda, \quad [x_1, \ldots, x_s]^t \mapsto f^s \cdot \nu_\lambda = f_{i_1,j_1} \cdots f_{i_s,j_s} \cdot \nu_\lambda,$$

where $x_1, \ldots, x_s$ are elements not at their positions in the column of the tableau and the operator $f_{i_l,j_l}$ for $1 \leq l \leq s$ is obtained as:

$$f_{i_l,j_l} := \begin{cases} f_{i_l,x_l-1} & \text{if } \overline{n} \geq x_l > k, \\ f_{i_l,x_l} & \text{if } \overline{1} \geq x_l \geq \frac{n-1}{k-1}, \end{cases}$$

where $i_l$ is the position of $x_l$. We will show that this map is also well defined and injective. We have $\overline{1} \geq x_1 > \cdots > x_s > k$ and $1 \leq i_1 < \cdots < i_s \leq k$, and so each positive root $\alpha_{i_l,j_l}$ lies in some Dyck path with no two distinct roots lying in a common Dyck path. The corresponding point $(\ldots, s_{i_l,j_l}, \ldots)$ with $s_{i_l,j_l} = 1$ satisfies an inequality of the form: $\cdots + s_{i_l,j_l} + \cdots \leq 1$, therefore $f_{i_1,j_1} \cdots f_{i_s,j_s} \cdot \nu_\lambda \in \text{SyP}_\lambda$.

Now we will check that $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \text{id}$. Consider $\theta_1 \circ \theta_2([x_1, \ldots, x_s]^t) = \theta_1(f_{i_1,j_1} \cdots f_{i_s,j_s})$ with $f_{i_l,j_l}$ obtained as in (2.2) above. Then we have:

$$f_{i_l,j_l} \cdot \begin{array}{|c|} \hline \end{array} i_l = \begin{cases} x_l - 1 + 1 & \text{if } k \leq x_l \leq n, \\ x_l & \text{if } \frac{n-1}{k-1} \leq x_l \leq \frac{n-1}{k}, \end{cases}$$
therefore we have \( \theta_1(f_{i_1,j_1} \ldots f_{i_s,j_s} \cdot \nu_\lambda) = [x_1, \ldots, x_s]^t \Rightarrow \theta_1 \circ \theta_2 = \text{id} \). Now consider \( \theta_2 \circ \theta_1(f_{i_1,j_1} \ldots f_{i_s,j_s} \cdot \nu_\lambda) = \theta_2([x_1, \ldots, x_s]^t) \) with \( x_t \) obtained from \( f_{i_t,j_t} \) according to (2.1). Then we have:

\[
 f_{i_t,j_t} := \begin{cases} 
 f_{i_t,x_t+1-1} & \text{if } n \geq x_t > k, \\
 f_{i_t,x_t} & \text{if } 1 \geq x_t \geq n - 1,
\end{cases}
\]

therefore we have \( \theta_2([x_1, \ldots, x_s]^t) = f_{i_1,j_1} \ldots f_{i_s,j_s} \cdot \nu_\lambda \Rightarrow \theta_2 \circ \theta_1 = \text{id} \). We are now left with proving that the defined maps are weight preserving. For this we need to only show that the map:

\[
 \phi : \text{SyP}_\lambda \longrightarrow \text{SyT}_\lambda, \quad f^s \cdot \nu_\lambda \longmapsto f^s \cdot t_\lambda,
\]

is weight preserving, i.e. that \( \text{wt}(\phi(f^s \cdot \nu_\lambda)) = \text{wt}(f^s \cdot \nu_\lambda) \). For this we have: \( \text{wt}(\phi(f^s \cdot \nu_\lambda)) = \text{wt}(f^s \cdot t_\lambda) = \text{wt}(f^s) + \text{wt}(t_\lambda) = \text{wt}(f^s) + \text{wt}(\nu_\lambda) = \text{wt}(f^s \cdot \nu_\lambda) \). \( \square \)

2.2. The case of dominant weights.

**Definition 2.6.** Consider a partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0) \) corresponding to a dominant integral weight \( \lambda = \sum_{k=1}^{n} m_k \omega_k \). A symplectic PBW tableau, \( T_\lambda \) of shape \( \lambda \) is a filling of the corresponding Young diagram \( Y_\lambda \) with numbers \( T_{i,j} \in \mathcal{N} \) such that for \( \mu_j \), the length of the \( j \)-th column, we have:

(i) if \( T_{i,j} \leq \mu_j \), then \( T_{i,j} = i \).

(ii) if \( T_{i_1,j} \neq i_1 \), and \( i_2 > i_1 \), then \( T_{i_1,j} > T_{i_2,j} \).

(iii) if \( T_{i,j} = i \), and \( \exists i' \) such that \( T_{i',j} = i \), then \( i' < i \).

A symplectic PBW tableau is said to be PBW-semistandard if in addition, the following condition is satisfied:

(iv) for every \( j > 1 \) and every \( i, \exists i' \geq i \) such that \( T_{i',j-1} \geq T_{i,j} \).

**Example 2.7.** For \( \mathcal{N} = \{1, 2, 3, 4\} \), and \( \lambda = (2, 1) \) (i.e. \( \lambda = \omega_1 + \omega_2 \)), the set of all the 16 symplectic PBW-semistandard tableaux is the one given below:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

Denote by \( \text{SyST}_\lambda \) the set of all symplectic PBW-semistandard tableaux of shape \( \lambda \) on the set \( \mathcal{N} \) as above. In order to obtain the bijection of these tableaux with the symplectic FFLV basis for \( V_\lambda \), we introduce a total order on the operators \( f_{i,j} \) as seen in the following definition.

**Definition 2.8.** We say \( f_{i_1,j_1} > f_{i_2,j_2} \) if either \( i_1 < i_2 \) or \( i_1 = i_2 \) and \( j_1 > j_2 \). We now order our operators in the product \( f^s = \prod_{\alpha \geq 0} f^s_\alpha \) according to this order.

**Definition 2.9.** Define an assignment \( f^s \cdot t_\lambda \), where \( t_\lambda \) is the highest weight tableau of shape \( \lambda \), i.e. one with one’s in the first row, two’s in the second row, and so on. In this assignment, we begin with the smallest operator in the ordered product. An operator \( f_{i,j} \) acts at position \( i \) in column \( c \) whenever \( j \geq \mu_c \) where \( c \) is the first column from the left where this is true.

The assignment \( f^s \cdot t_\lambda \) then narrows down to the assignment \( f_{i,j} \cdot \boxed{i} \) of each operator \( f_{i,j} \) in the product \( f^s \) only once at position \( i \) in the best choice column \( c \) of \( t_\lambda \) according to the rule established in formula (2.1) in the proof of Proposition 2.5.
Remark 2.10. We want to point out that the assignment described in Definition 2.9 above is not a linear one, since each operator is just being assigned once at a single unique position by the help of the total order established above.

Also, if we have in the product $f^s$, a factor of the form: $f^1_{i_1,j_1} f^2_{i_2,j_2}$ and $i_1 = i_2$ but $j_1 > j_2$, then $f^1_{i_1,j_1} < f^2_{i_2,j_2}$ according to our total order, so we should apply $f^1_{i_1,j_1}$ first according to (2.9), and then we apply $f^2_{i_2,j_2}$ in the next column to the right. The point is that we don’t use the same operator in the same position of the same column more than once.

Example 2.11. For $\mathfrak{sp}_4$ and $\lambda = \omega_1 + \omega_2$, one writes down all the inequalities defining the polytope $P(\lambda) \subset \mathbb{R}^4$ and obtains all the 16 integral points in $P(\lambda)$ and from these points, one obtains the following set of monomials: $\{1, f_{11,1}, f_{12,2}, f_{12,22}, f_{12,12}, f_{11,11}, f_{11,11}, f_{12}^2, f_{12}^2, f_{11}^2, f_{11}^2, f_{12}^2, f_{11}^2, f_{11}^2, f_{12}^2, f_{11}^2, f_{11}^2\}$

each of them corresponding to the symplectic PBW-semistandard tableau appearing in the same position in the list of tableaux given in Example 2.7. For an illustration of how our assignment described in Definition 2.9 works, consider the second last monomial in the list above. Then one has:

$$f_{12} f_{11} f_{22} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = f_{12} f_{11} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = f_{12} \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$$

The resulting tableau is actually the second last one in the list of tableaux in Example 2.7.

Proposition 2.12. The map:

$$\phi : \text{SyP}_{\lambda} \rightarrow \text{SyST}_{\lambda}, \quad f^s \cdot \nu_{\lambda} \longmapsto f^s \cdot t_{\lambda},$$

where the assignment $f^s \cdot t_{\lambda}$ is the one described in Definition 2.9 is injective.

Proof. Let $f^s = f_{i_1,j_1} \cdots f_{i_s,j_s}$ be the ordered product, with $f_{i_1,j_1} \geq \cdots \geq f_{i_s,j_s}$. We begin ‘acting’ with the smallest operator $f_{i_s,j_s}$ in the first column from the left for which $j_s \geq \mu_1$. We then proceed to the next smallest one $f_{i_{s-1},j_{s-1}}$. If $i_{s-1} < i_s$ and $j_{s-1} > j_s$, then $f_{i_{s-1},j_{s-1}}$ also acts in the same column. Let $f_{i_s-k, j_s-k} \cdots f_{i_s, j_s}$ be the product of the operators which act in the same column. The result of this product satisfy all conditions of PBW tableaux defined on columns from Proposition 2.5. Now let $f_{i_{s-k-1}, j_{s-k-1}}$ be the next smallest entry for which $i_{s-k-1} \leq i_{s-k}$ and $j_{s-k-1} \leq j_{s-k}$. This operator then acts in the column next to the first one towards the right. Let us show that the resulting tableau lies in $\text{SyST}_{\lambda}$. If $\mu_1 \leq j_{s-k} \leq n$, then also $\mu_2 < j_{s-k-1} \leq j_{s-k} \leq \bar{n}$. So under our map, we have:

$$f_{i_{s-k}, j_{s-k}} \cdot \nu_{\lambda} \longmapsto j_{s-k} + 1 \quad \text{and} \quad f_{i_{s-k-1}, j_{s-k-1}} \cdot \nu_{\lambda} \longmapsto j_{s-k-1} + 1.$$  

We have $i_{s-k-1} \leq i_{s-k}$ and $j_{s-k-1} \leq j_{s-k} \Rightarrow j_{s-k-1} + 1 \leq j_{s-k} + 1$. If $\bar{n} - \bar{1} \leq j_{s-k} \leq \bar{1}$, then also $\mu_2 < j_{s-k-1} \leq j_{s-k} \leq \bar{1}$. Here again we have two cases:

(i) if $\mu_2 < j_{s-k-1} \leq n$ then under our map, we have:

$$f_{i_{s-k}, j_{s-k}} \cdot \nu_{\lambda} \longmapsto j_{s-k} \quad \text{and} \quad f_{i_{s-k-1}, j_{s-k-1}} \cdot \nu_{\lambda} \longmapsto j_{s-k-1} + 1.$$  

So we have $i_{s-k-1} \leq i_{s-k}$ and $j_{s-k-1} < j_{s-k} \Rightarrow j_{s-k-1} + 1 \leq j_{s-k}$.  

(ii) if $\bar{n} - \bar{1} \leq j_{s-k-1} \leq \bar{1}$ then under our map, we have:

$$f_{i_{s-k}, j_{s-k}} \cdot \nu_{\lambda} \longmapsto j_{s-k} \quad \text{and} \quad f_{i_{s-k-1}, j_{s-k-1}} \cdot \nu_{\lambda} \longmapsto j_{s-k-1}.$$  

So we have again $i_{s-k-1} \leq i_{s-k}$ and $j_{s-k-1} \leq j_{s-k}$. Since $j_{s-k-1}$ and $j_{s-k}$ are arbitrary, then all elements in the second column are dominated by elements from the first column. \qed
Proposition 2.13. The map:

\[ \pi : \text{SyST}_\lambda \rightarrow \text{SyP}_\lambda, \quad t_\lambda \mapsto f^s \cdot \nu_\lambda = \prod_{\alpha \geq 0} f^{s_\alpha_0} \cdot \nu_\lambda, \]

with the operators \( f_\alpha \) obtained as:

\[
f_\alpha := \begin{cases} 
  f_{d,p_d-1} & \text{if } \overline{\pi} \geq p_d > \mu_c, \\
  1 & \text{if } \mu_c \geq p_d \geq 1, \\
  f_{d,p_d} & \text{if } \overline{\Gamma} \geq p_d \geq n-1, 
\end{cases}
\]

(2.3)
is injective.

Proof. When \( \lambda = \omega_k \), a fundamental weight, then this is Proposition 2.5 above. Now for a tableau \( t_\lambda \) with at least two columns, consider any two arbitrary neighboured columns \( j_1 \) and \( j_2 \) in \( t_\lambda \), i.e. \( j_1 = j_2 - 1 \). Let \( \mu_{j_1} = l \) and such that \( \mu_{j_2} = s \), \( 1 \leq s \leq l \leq n \) with \( \{x_1, \ldots, x_l\} \) elements from \( j_1 \) and \( \{y_1, \ldots, y_s\} \) elements from \( j_2 \) satisfying the condition that for each \( y_k \), there exists \( x_m \) with \( m \geq k \) such that \( x_m \geq y_k \). Since all elements in columns \( j_1 \) and \( j_2 \) that appear in their positions are mapped to 1, it suffices to consider only those elements that are not at their positions. Let \( \{x_1, \ldots, x_{t_\lambda}\} \) be elements from \( j_1 \) all not at their positions and likewise \( \{y_1, \ldots, y_{r_\lambda}\} \) be elements from \( j_2 \) all not at their positions with \( 1 \leq t_1 < \cdots < t_k \leq l \) and \( 1 \leq r_1 < \cdots < r_k \leq s \). According to the definition of a symplectic PBW-semistandard tableau, we have that \( \{x_1, \ldots, x_{t_\lambda}\} \) and \( \{y_1, \ldots, y_{r_\lambda}\} \).

Begin with the biggest element, which should be in column \( j_1 \), because otherwise, if it is in column \( j_2 \), it would not be dominated. Let \( \{x_1, \ldots, x_{t_\lambda}\} \) be the first \( z - 1 \) elements that lie in column \( j_1 \). Now assume the next element \( y_{r_\lambda} \) is in \( j_2 \). Then there must exist \( x_{t_\lambda+1} \) with \( t_{z+1} \geq r_\lambda \) such that \( x_{t_{z+1}} \geq y_{r_\lambda} \). If \( l < x_{t_{z+1}} \leq \overline{\pi} \), then \( s < y_{r_\lambda} \leq x_{t_{z+1}} \leq \overline{\pi} \), so we have \( f_{u_1,v_1} = f_{y_{r_\lambda},y_{r_\lambda}-1} \) and \( f_{u_2,v_2} = f_{x_{t_{z+1}},x_{t_{z+1}}-1} \) according to Equation (2.3). And the corresponding monomial is: \( f_{u_1,v_1} f_{u_2,v_2} = f_{r_\lambda,y_{r_\lambda}-1} f_{t_{z+1},x_{t_{z+1}}-1} \). The points \( (r_\lambda,y_{r_\lambda}-1) \) and \( (t_{z+1},x_{t_{z+1}}-1) \) lie on a symplectic Dyck path since \( t_{z+1} \geq r_\lambda \) and \( x_{t_{z+1}} \geq y_{r_\lambda} \) implies \( x_{t_{z+1}} \geq y_{r_\lambda} - 1 \). The corresponding point \( s = (0, \ldots, 0, s_{r_\lambda,y_{r_\lambda}-1}, 0, \ldots, 0, s_{t_{z+1},x_{t_{z+1}}-1}, 0, \ldots, 0) \) with \( s_{r_\lambda,y_{r_\lambda}-1} = 1 \) and \( s_{t_{z+1},x_{t_{z+1}}-1} = 1 \) satisfies the inequality:

\[ \cdots + s_{r_\lambda,y_{r_\lambda}-1} + \cdots + s_{t_{z+1},x_{t_{z+1}}-1} + \cdots \leq 2. \]

Therefore the monomial \( f_{u_1,v_1} f_{u_2,v_2} \in \text{SyP}_\lambda \). Moreover the monomial \( f_{u_2,v_2} = f_{t_{z+1},x_{t_{z+1}}-1} \) acts only in \( j_1 \), and not in \( j_2 \). If \( n - \overline{T} \leq x_{t_{z+1}} \leq \overline{T} \), then \( s < y_{r_\lambda} \leq x_{t_{z+1}} \leq \overline{T} \). We have two cases:

(i) if \( s < y_{r_\lambda} \leq \overline{\pi} \), then \( f_{u_1,v_1} f_{u_2,v_2} = f_{r_\lambda,y_{r_\lambda}-1} f_{t_{z+1},x_{t_{z+1}}-1} \) and the corresponding roots \( (r_\lambda,y_{r_\lambda}-1) \) and \( (t_{z+1},x_{t_{z+1}}-1) \) lie on a symplectic Dyck path since \( t_{z+1} \geq r_\lambda \) and \( x_{t_{z+1}} \geq y_{r_\lambda} \) implies \( x_{t_{z+1}} > y_{r_\lambda} - 1 \). Also the corresponding point \( s = (0, \ldots, 0, s_{r_\lambda,y_{r_\lambda}-1}, 0, \ldots, 0, s_{t_{z+1},x_{t_{z+1}}-1}, 0, \ldots, 0) \) with \( s_{r_\lambda,y_{r_\lambda}-1} = 1 \) and \( s_{t_{z+1},x_{t_{z+1}}-1} = 1 \) satisfies the inequality:

\[ \cdots + s_{r_\lambda,y_{r_\lambda}-1} + \cdots + s_{t_{z+1},x_{t_{z+1}}-1} + \cdots \leq 2. \]

(ii) if \( n - \overline{T} \leq y_{r_\lambda} \leq \overline{\pi} \), then \( f_{u_1,v_1} f_{u_2,v_2} = f_{r_\lambda,y_{r_\lambda}} f_{t_{z+1},x_{t_{z+1}}-1} \) and the corresponding roots \( (r_\lambda,y_{r_\lambda}) \) and \( (t_{z+1},x_{t_{z+1}}) \) lie on a symplectic Dyck path since \( t_{z+1} \geq r_\lambda \) and \( x_{t_{z+1}} \geq y_{r_\lambda} \). Also the corresponding point \( s = (0, \ldots, 0, s_{r_\lambda,y_{r_\lambda}}, 0, \ldots, 0, s_{t_{z+1},x_{t_{z+1}}}, 0, \ldots, 0) \) with \( s_{r_\lambda,y_{r_\lambda}} = 1 \) and \( s_{t_{z+1},x_{t_{z+1}}} = 1 \) satisfies the inequality:

\[ \cdots + s_{r_\lambda,y_{r_\lambda}} + \cdots + s_{t_{z+1},x_{t_{z+1}}} + \cdots \leq 2. \]
Since \( y_z \) was arbitrary, this means the product of monomials corresponding to the domination pairs lie in \( \text{SyP}_\lambda \). \qedhere

Theorem 2.14. Let \( \lambda = \sum_{k=1}^{n} m_k \omega_k \) be a highest weight and \( V_\lambda \) the corresponding highest weight \( \mathfrak{sp}_{2n} \)-module. Then the symplectic FFLV basis for \( V_\lambda \) is in a weight preserving one-to-one correspondence with the set \( \text{SyST}_\lambda \) of symplectic PBW-semistandard tableaux of shape \( \lambda \) with entries in \( \mathcal{N} \).

Proof. When \( \lambda \) is just a fundamental weight, then this is already dealt with in Proposition 2.5 above. Therefore it suffices to prove that for the maps \( \phi \) and \( \pi \) in Propositions 2.12 and 2.13 respectively, we have \( \phi \circ \pi = \pi \circ \phi = \text{id} \), where \( \text{id} \) is the identity map.

Let us begin with \( \phi \circ \pi \). We again consider two neighboured columns \( j_1 \) and \( j_2 \) with \( \mu_{j_1} \geq \mu_{j_2} \). We have elements not at their positions as before. As before, let \( \{ x_{i_1}, \ldots, x_{i_{\lambda_1}} \} \) be elements in the left column \( j_1 \), and \( y_{r_\lambda} \) the next element which is in \( j_2 \), the right-hand column, such that \( \exists x_{t_{x_\lambda+1}} \) with \( x_{t_{x_\lambda+1}} \geq y_{r_\lambda} \) and \( t_{z_1} \geq r_\lambda \). If \( \mu_{j_1} < x_{t_{x_\lambda+1}} \leq \overline{\pi} \), then we have \( \phi \circ \pi(T_\lambda) \). Let us begin with \( \pi \circ \phi \). We again consider two neighboured columns \( j_1 \) and \( j_2 \) with \( \mu_{j_1} \geq \mu_{j_2} \). We have elements not at their positions as before. As before, let \( \{ x_{i_1}, \ldots, x_{i_{\lambda_1}} \} \) be elements in the left column \( j_1 \), and \( y_{r_\lambda} \) the next element which is in \( j_2 \), the right-hand column, such that \( \exists x_{t_{x_\lambda+1}} \) with \( x_{t_{x_\lambda+1}} \geq y_{r_\lambda} \) and \( t_{z_1} \geq r_\lambda \). If \( \mu_{j_1} < x_{t_{x_\lambda+1}} \leq \overline{\pi} \), then we have \( \phi \circ \pi(T_\lambda) \).
(i) if \( \mu_{j_2} < j_{s-k-1} \leq n \), then:

\[
\pi \circ \phi(f_{i_{s-k-1}, j_{s-k-1}} f_{i_{s-k}, j_{s-k}} \cdot \nu_\lambda) = \pi((i_{s-k-1}, j_{s-k-1} + 1), (i_{s-k}, j_{s-k})),
\]

\[
= f_{i_{s-k-1}, j_{s-k-1}} f_{i_{s-k}, j_{s-k}} \cdot \nu_\lambda.
\]

(ii) if \( n - 1 \leq j_{s-k-1} \leq T \), then:

\[
\pi \circ \phi(f_{i_{s-k-1}, j_{s-k-1}} f_{i_{s-k}, j_{s-k}} \cdot \nu_\lambda) = \pi((i_{s-k-1}, j_{s-k-1}), (i_{s-k}, j_{s-k})),
\]

\[
= f_{i_{s-k-1}, j_{s-k-1}} f_{i_{s-k}, j_{s-k}} \cdot \nu_\lambda.
\]

So we have \( \pi \circ \phi(f^s) = f^s \), which completes the proof.

Now we are left with showing that this one-to-one correspondence is weight preserving. For this we need to only show that the map:

\[
\phi : \text{SyP}_{\lambda} \rightarrow \text{SyST}_{\lambda}, \quad f^s \cdot \nu_\lambda \mapsto f^s \cdot t_\lambda,
\]

is weight preserving, i.e. that \( \text{wt}(\phi(f^s \cdot \nu_\lambda)) = \text{wt}(f^s \cdot \nu_\lambda) \). For this we have: \( \text{wt}(\phi(f^s \cdot \nu_\lambda)) = \text{wt}(f^s \cdot \nu_\lambda) = \text{wt}(f^s \cdot \nu_\lambda) = \text{wt}(f^s \cdot \nu_\lambda) \).

2.3. A comparison with other existing tableaux. On the set \( \mathcal{N} \) as before, the usual semistandard Young tableaux are defined to be the filling of the numbers \( T_{i,j} \in \mathcal{N} \) into the Young diagram \( Y_\lambda \) for a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0) \) such that the numbers are strictly increasing down the columns and weakly increasing across the rows. Clearly, these tableaux can not be in one-to-one correspondence with the symplectic FFLV basis for the \( \mathfrak{sp}_{2n} \)-modules as it is with the symplectic PBW-semistandard tableaux.

On the other hand, the PBW-semistandard tableaux in type \( \mathfrak{a} \) are defined as follows:

**Definition 2.15** (Feigin, [10]). A type \( \mathfrak{a} \) PBW-semistandard tableau of shape \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_{2n-1} \geq 0) \) is a filling of the Young diagram \( Y_\lambda \) with numbers \( T_{i,j} \in \mathcal{N} \) satisfying the properties:

(i) if \( T_{i,j} \leq \mu_j \), then \( T_{i,j} = i \),

(ii) if \( i_1 < i_2 \) and \( T_{i_1,j} \neq i_1 \), then \( T_{i_1,j} > T_{i_2,j} \),

(iii) for any \( j > 1 \) and any \( i \) there exists \( i' \geq i \) such that \( T_{i',j-1} \geq T_{i,j} \).

If we extend this definition to type \( \mathfrak{c}_n \), namely by restricting to \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0) \), then the resulting tableaux are too many to correspond to the basis of the \( \mathfrak{sp}_{2n} \)-modules \( V_\lambda \) and \( V^a_\lambda \). So in this regard, the **PBW-symplectic condition** which is condition (iii) of Definition 2.6 is the sufficient condition to cut down this number to the right one.

**Example 2.16**. For \( \mathfrak{g} \) of type \( \mathfrak{h}_3 \), the full set PBW-semistandard tableaux restricted to \( \lambda = \omega_1 + \omega_2 \) (\( \lambda = (2, 1) \)) on the set \( \mathcal{N} = \{1, 2, 3, 4\} \) is the one given below:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

Note that these tableaux are different from semistandard Young tableaux. When we consider the PBW-symplectic condition, then we have to drop the last four tableaux from the above.
list. This way, we are able to recover all the 16 PBW-semistandard tableaux corresponding to 
\( \lambda = \omega_1 + \omega_2 \) for \( g \) of type \( C_2 \) as seen in Example 2.7.

As will be seen in the following section, the symplectic standard tableaux of De Concini in [5] are different from our symplectic PBW-semistandard tableaux. The symplectic semistandard tableaux of Hamel and King [14], King [17], Kashiwara and Nakashima [16] and Proctor [21] all yield semistandard Young tableaux when restricted to type \( A_{n-1} \), i.e., if entries are taken from the set \( \{1, \ldots, n\} \). Hence they are different from the symplectic PBW-semistandard tableaux since the restriction of these in the same way does not yield semistandard Young tableaux.

3. The complete symplectic flag variety, symplectic relations and a basis for the homogeneous coordinate ring

In this section we describe the complete symplectic flag variety and show that the symplectic PBW-semistandard tableaux label a basis for its homogeneous coordinate ring.

3.1. Flag varieties; a brief description. Let \( G \) be a simple, simply connected algebraic group with the corresponding Lie algebra \( g \). As before, we have a Cartan decomposition \( g = n^+ \oplus h \oplus n^- \). We know that \( V_\lambda \) has a structure as a \( G \)-module with highest weight vector \( \nu_\lambda \). Hence we have an action of \( G \) on the projectivization \( \mathbb{P}(V_\lambda) \). The flag variety \( \mathcal{F}_\lambda \) from this point of view can be understood as the closure of the \( G \)-orbit of the highest weight line:

\[
\mathcal{F}_\lambda = \overline{G[\nu_\lambda]} \hookrightarrow \mathbb{P}(V_\lambda).
\]

Let \( \lambda \) be any dominant integral weight of \( g \). Assuming \( (\lambda, \omega_i) = 0 \) if and only if \( f_{\alpha_i} \in \mathfrak{p} \), the Lie algebra corresponding to \( P \), a parabolic subgroup of \( G \), then each variety \( \mathcal{F}_\lambda \) is as well isomorphic to the quotient \( G/P \) of \( G \) by the parabolic subgroup \( P \) leaving \( C_{\nu_\lambda} \) invariant. This is the generalized/partial flag variety. In particular, when \( \lambda \) is also regular, then the flag variety \( \mathcal{F}_\lambda \) is isomorphic to \( G/B \), where \( B \subset P \) is a Borel subgroup, and this is then called the complete/full flag variety.

3.2. The complete symplectic flag variety; general description. Now we consider \( G = \text{Sp}_{2n} \). Let \( W \) be a \( 2n \)-dimensional vector space over \( \mathbb{C} \) with a fixed basis \( \{w_1, \ldots, w_{2n}\} \). We know that such a vector space admits a non degenerate skew symmetric bilinear form (non degenerate symplectic form). Following [11], let us fix a symplectic form \( \langle \ , \ \rangle \) defined by:

\[
\langle w_i, \omega_i \rangle = 1 \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad \langle w_i, w_j \rangle = 0 \quad \text{for all} \quad 1 \leq i, j \leq n, j \neq i,
\]

where as before, \( i = 2n + 1 - i \). The matrix of this symplectic form is given by

\[
M_s := \begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & 1 & & \\
& & & -1 & \\
& & & & -1
\end{pmatrix}.
\]

Recall that an isotropic subspace of a symplectic vector space is a subspace on which the symplectic form identically vanishes. For \( W \) as above, all the isotropic subspaces have dimension of at most \( n \). Hence for \( 1 \leq k \leq n \), the symplectic Grassmannian \( \text{SpGr}(k, 2n) \) is the quotient of \( \text{Sp}_{2n} \) by a maximal parabolic subgroup and it is known to coincide with the variety of isotropic \( k \)-dimensional subspaces of \( W \).
We consider the case $\text{Sp}_{2n}/B$, where $B \subset P$ is a Borel subgroup. This is the complete symplectic flag variety which we denote by $\text{Sp}F_{2n}$ and it coincides with the variety whose points are the full flags

$$\{U_1, \ldots, U_n, \text{ rank } U_i = i\}$$

with $U_i \in \text{SpGr}(i, 2n)$. This variety is also referred to as the isotropic flag variety as in [5]. Let $\mathbb{C}[\text{Sp}F_{2n}]$ denote the coordinate ring of $\text{Sp}F_{2n}$.

3.3. The Plücker embedding. Consider the irreducible fundamental $\text{Sp}_{2n}$-module $V_{\omega_k}$ of highest weight $\omega_k$. We have $V_{\omega_1} \simeq \mathbb{C}^{2n}$ and the canonical embedding,

$$V_{\omega_k} \hookrightarrow \bigwedge^k \mathbb{C}^{2n}, \quad \omega_k \mapsto w_1 \wedge \cdots \wedge w_k.$$ 

Since we do not have isomorphism, we should be able to describe the image of $V_{\omega_k}$ under this embedding.

For $i \in \{1, \ldots, n\}$, let $\tilde{i} := 2n + 1 - i$. For $J = (j_1 < \cdots < j_k) \subset \{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\}$, let $U_k \subset \text{SpGr}(k, 2n)$ such that $U_k = \text{span}(w_{j_1}, \ldots, w_{j_k})$. Consider the Plücker embedding

$$\text{SpGr}(k, 2n) \hookrightarrow \mathbb{P}\left(\bigwedge^k \mathbb{C}^{2n}\right), \quad \text{span}(w_{j_1}, \ldots, w_{j_k}) \mapsto [w_{j_1} \wedge \cdots \wedge w_{j_k}].$$

Let $X_J \in V^*_{\omega_k}$ be the corresponding Plücker coordinate. Notice that these Plücker coordinates are $k \times k$ minors of the $2n \times k$ matrix representing the subspaces $U_k$. From this, it can be seen that the image of $\text{SpGr}(k, 2n)$ is fully characterised by minors. Therefore the isotropic condition on the elements $U_k \in \text{SpGr}(k, 2n)$ translates naturally into a condition on these minors. That is to say, which kind of minors are permitted? This is the subject of the next subsection.

3.4. Reverse-admissible minors and their correspondence with the symplectic PBW tableau columns. Following [5], we consider now the variety $\mathcal{V}$ whose points over $\mathbb{C}$ are the $m$-th tuples, $(v_1, \ldots, v_m)$ of vectors in $W$ such that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i, j \leq m$, where $\langle \ , \ , \rangle$ is the symplectic form defined above. The variety $\mathcal{V}$ is therefore equivalently the variety of $2n \times m$ matrices $M$ with coefficients in $\mathbb{C}$ such that $M'M_aM = 0$.

Denote by $A$ the homogeneous coordinate ring of $\mathcal{V}$. Let $L := (i_k, \ldots, i_1|j_1, \ldots, j_k)$ with $1 \leq k \leq m$ be the $k \times k$ minor of the matrix $M$ where $(i_1, \ldots, i_k)$ are the row indices while $(j_1, \ldots, j_k)$ are the column indices. Therefore we have $1 \leq i_1, \ldots, i_k \leq \bar{1}$ and $1 \leq j_1, \ldots, j_k \leq m$. For what will follow, let us introduce a partial ordering $\leq$ on the subsets of $\{1, \ldots, n\}$ of equal length $k$ as follows. Given two such sets $L = \{l_1 < \cdots < l_k\}$ and $J = \{j_1 < \cdots < j_k\}$, we say that $L \leq J$ if $l_1 \leq j_1, \ldots, l_k \leq j_k$ with equality if and only if $l_1 = j_1, \ldots, l_k = j_k$.

Let $I_1, I_2 \subset \{1, \ldots, n\}$ be such that $I_1 := \{x_1, \ldots, x_t\}$ and $I_2 := \{y_1, \ldots, y_{k-t}\}$ for some $0 \leq t \leq k$, then the minor $L$ can be written as $L = (I_2, I_1|j_1, \ldots, j_k)$. Let $\Gamma := I_1 \cap I_2 = \{\gamma_1, \ldots, \gamma_\lambda\}$. Define $\tilde{I}_1 := I_1 \setminus \Gamma = \{a_1, \ldots, a_{\lambda-1}\}$ and $\tilde{I}_2 := I_2 \setminus \Gamma = \{b_1, \ldots, b_{k-t-\lambda}\}$, then the minor $L = (I_2, I_1|j_1, \ldots, j_k)$ can be put back in the first form by the following formula:

$$\tilde{I}_1 := (I_2, I_1|j_1, \ldots, j_k) = (\tilde{b}_1, \ldots, \tilde{b}_{k-t-\lambda}, a_{\lambda-1}, \ldots, a_1, \gamma_\lambda, \gamma_{\lambda-1}, \ldots, \gamma_1, \gamma_{k-1}, \gamma_k).$$ (3.1)

We call the minor on the right hand side of Equation 3.1 the computed minor corresponding to $(I_2, I_1|j_1, \ldots, j_k)$. In other words, $I_1$ corresponds to entries in $\{1, \ldots, n\}$ and $I_2$ corresponds to
entries in \( \{\pi, \ldots, \mathbf{T}\} \). From now on, we will often switch between these two notations depending on the situation, and when we write \( L \), we refer to any of the two notations. The following definition gives the set of minors permitted by De Concini in [5].

**Definition 3.1.** A minor \( (I_2, I_1 | j_1, \ldots, j_k) \) is called admissible if there exists a subset \( T \subset \{1, \ldots, n\} \backslash (I_1 \cup I_2) \) with \( |T| = |\Gamma| \) and \( T > \Gamma \).

**Proposition 3.2.** [5, Proposition 2.2] In the ring \( A \), the coordinate ring of the variety \( V \), any minor can be expressed as a linear combination of admissible minors of the same size and involving the same columns.

To find a connection of the variety \( V \) to \( \text{Sp}\mathcal{F}_{2n} \), the complete symplectic flag variety, we recall a few more results from [5]. The isotropic Stiefel variety \( W_{k,n} \) is the open set in \( V \) whose points over \( \mathbb{C} \) are the \( k \)-th tuples of vectors \( (v_1, \ldots, v_k) \) in \( W \) such that \( (v_1, \ldots, v_k) \) span an isotropic free direct summand of rank equal to \( \min(n, k) \).

**Proposition 3.3.** [5, Proposition 4.4] The complement of \( W_{k,n} \) in \( V \) has codimension \( \geq 2 \).

**Corollary 3.4.** [5, Corollary 4.6] Let \( A' \) be the ring of global polynomial functions on \( W_{k,n} \), then \( A' = A \), where \( A \) is the coordinate ring of \( V \).

Also there is a natural morphism \( g : W_{n,n} \to \text{Sp}\mathcal{F}_{2n} \) given by \( g((v_1, \ldots, v_n)) = \{U_{(v_1)} \subset U_{(v_1, v_2)} \subset \ldots \subset U_{(v_1, \ldots, v_n)}\} \), where \( U_{(v_1, \ldots, v_t)} = \{\text{linear span of } v_1, \ldots, v_t\} \) for some \( t \) vectors \( v_1, \ldots, v_t \) in \( W \).

**Proposition 3.5.** [5, Proposition 4.2] The morphism \( g : W_{n,n} \to \text{Sp}\mathcal{F}_{2n} \) is a principal B bundle, where B is the Borel subgroup of upper triangular elements in \( GL(n) \).

Proposition 3.5 implies that we actually have \( \text{Sp}\mathcal{F}_{2n} = W_{n,n}/B \). This and Corollary 3.4 imply that \( \mathbb{C}[\text{Sp}\mathcal{F}_{2n}] \) is a sub-ring of \( A \), i.e. it is the ring of invariants in \( A \) under the group action of \( B \) on \( W \). Right canonical minors are those with \( i \)'s on the \( i \)-th columns i.e. minors of the form \( (i_k, \ldots, i_1 | 1, \ldots, k) \). These are all we need to work with in \( \mathbb{C}[\text{Sp}\mathcal{F}_{2n}] \) (see [5, Theorem 4.8]). We will therefore restrict to these minors, in that we will write \( (i_1, \ldots, i_k) \) instead of \( (i_k, \ldots, i_1 | 1, \ldots, k) \) and \( (I_2, I_1) \) instead of \( (I_2, I_1 | 1, \ldots, k) \).

Now we would like to find a connection of these minors to our symplectic PBW-semistandard tableaux. For this, we choose a different set of minors and we call them reverse-admissible. In this regard, maintaining the same notation as above, we would like to give the following definition.

**Definition 3.6.** A right canonical minor \( (I_2, I_1) \) is called reverse-admissible if there exists a subset \( T \subset \{1, \ldots, n\} \backslash (I_1 \cup I_2) \) with \( |T| = |\Gamma| \) and \( T < \Gamma \).

**Proposition 3.7.** In the ring \( \mathbb{C}[\text{Sp}\mathcal{F}_{2n}] \), any minor can be expressed as a linear combination of reverse-admissible minors of the same size and involving the same columns.

To prove this proposition, we first recall Proposition 1.8 of [5], and a modified version of Definition 1.4 of [5] which gives a total ordering on the set of right canonical minors.

**Proposition 3.8.** Let \( (I_2 \cup \Gamma, I_1 \cup \Gamma) \) be a fixed minor of size \( k \leq n \), Then on \( \text{Sp}\mathcal{F}_{2n} \), the following relations hold.

\[
(I_2 \cup \Gamma, I_1 \cup \Gamma) = (-1)^{|\Gamma|} \sum_{\Gamma' \subset \Gamma \cap \{I_1 \cup I_2 \cup \Gamma\} \neq \emptyset} (I_2 \cup \Gamma', I_1 \cup \Gamma').
\] (3.2)
Definition 3.9. Given two $k \times k$ minors $L = (l_1, \ldots, l_k)$ and $J = (j_1, \ldots, j_k)$, we say that $L$ is smaller than $J$ if $l = (l_1 + \cdots + l_k) < (j_1 + \cdots + j_k) =: \nu_J$ and if $\nu_L = \nu_J$, then the last non zero entry of the vector $L - J$ is positive.

Proof of Proposition 3.7. The proof is in principle similar to the proof of Proposition 2.2 of [5]. We will therefore adapt the same proof here. Consider a minor $(I_2, I_1)$ which is not reverse-admissible. We will show that $(I_2, I_1)$ can be written as a linear combination of minors of the same size that are smaller in the total ordering $\leq$ of Definition 3.9. Clearly, we only need to consider the case $\Gamma = I_1 \cap I_2 \neq \emptyset$. Now let $\Gamma = \{\gamma_1, \ldots, \gamma_t\}$. Choose $1 \leq h_0 \leq t$ minimally such that there exists a tuple $T \subset \{1, \ldots, n\} \setminus (I_1 \cup I_2)$ of length $t - h_0$ with

$$T < (\gamma_{h_0+1}, \ldots, \gamma_t).$$

Choose $T_{h_0+1} = \{\lambda_{h_0+1}, \ldots, \lambda_t\}$ maximal (with respect to the partial order $\leq$) among those $T$. Choose $b \in \{h_0 + 1, \ldots, t\}$ maximally such that

$$(\lambda_{h_0+1}, \ldots, \lambda_b) < (\gamma_{h_0}, \ldots, \gamma_{b-1}),$$

or set $b = h_0$ if no such $b$ exists. Now define $T' := (\gamma_{h_0}, \ldots, \gamma_b)$. Recall the subsets of $\{1, \ldots, n\}$; $I_1 = I_1 \cap \Gamma$ and $I_2 = I_2 \cap \Gamma$. Applying Relation (3.2) to $T'$, taking $F = \Gamma \setminus \hat{T}$, we find:

$$(I_2, I_1) = (-1)^{b-h_0+1} \sum_{\Gamma' : \Gamma' \cap (I_1 \cup I_2) = \emptyset} (I_2 \cup F \cup \Gamma', I_1 \cup F \cup \Gamma'),$$

with $|\hat{T}| = |\Gamma'|$. For any $\Gamma' = \{\gamma_{h_0}' \prec \cdots \prec \gamma_b'\}$ appearing on the right-hand side of (3.3), the sum $\nu$ defined in Definition 3.9 has the same value which it takes for $(I_2, I_1)$. We claim now that for every such $\Gamma'$, we have $\gamma_b' < \gamma_b$. We will assume the contrary that $\gamma_b' \leq \gamma_b$. Now since $\gamma_b' \subset \{1, \ldots, n\} \setminus (I_1 \cup I_2)$ and $\lambda_{b+1} > \gamma_b$ (by the maximality of $b$), the maximality of $T_{h_0+1}$ implies $\gamma_b' \leq \lambda_b$. Now suppose by induction that $\gamma_e' \leq \lambda_e$, for all $h_0 + 1 < e \leq b$, then $\gamma_{f-1}' < \gamma_f' \leq \lambda_f < \gamma_{f-1}$ and if $f - 1 \leq h_0 + 1$, the maximality of $T_{h_0+1}$ implies that $\gamma_{f+1}' < \lambda_{f+1}$, if $f - 1 = h_0$ we have $\gamma_{h_0}' < \gamma_{h_0}$. In particular $\gamma_e' < \gamma_e$ for all $h_0 \leq e \leq b$. This then implies that we have

$$(\gamma_{h_0}', \ldots, \gamma_b', \lambda_{b+1}, \ldots, \lambda_t) < (\gamma_{h_0}, \ldots, \gamma_t)$$

component-wise and $\{\gamma_{h_0}', \ldots, \gamma_b', \lambda_{b+1}, \ldots, \lambda_t\} \subset \{1, \ldots, n\} \setminus (I_1 \cup I_2)$, which contradicts the minimality of $h_0$. Thus we have $\gamma_e' > \gamma_b$ and this together with what has been noted about the sum $\nu$, implies that each minor appearing on the right-hand side of (3.3) is smaller than $(I_2, I_1)$ in the total ordering $\leq$, which proves the proposition.

Example 3.10. Consider $n = 4$, $k = 4$, $I_1 = \{1, 2\}$ and $I_2 = \{1, 2\}$. Then $\{1, 2, 3, 4\} \setminus (I_1 \cup I_2) = \{3, 4\}$. The minor $(I_2, I_1)$ is not reverse-admissible in the sense of Definition 3.6. We have $\Gamma = \{1, 2\}$, so $h_0 = 2$. With this, we get $b = 2$, giving us $\Gamma = \{2\}$. Moreover we have $I_1 = I_2 = \emptyset$. Also $F = \Gamma \setminus \hat{T} = \{1\}$. So substituting into Equation (3.3), we get:

$$\{(1, 2), (1, 2)\} = (-1)^4 \left[ (\emptyset \cup \{1\} \cup \{3\}, \emptyset \cup \{1\} \cup \{3\}) + (\emptyset \cup \{1\} \cup \{4\}, \emptyset \cup \{1\} \cup \{4\}) \right],$$

$$= -((1, 3), (1, 3)) - (\{1, 4\}, \{1, 4\}).$$

Computing all the minors according to Equation (3.1), Equation (3.4) above becomes:

$$\{(\mathcal{I}, 2, \mathcal{I}, 1) = -(\mathcal{I}, 3, \mathcal{I}, 1) - (\mathcal{I}, 4, \mathcal{I}, 1).$$
We would now like to note that the set of computed reverse-admissible \(k \times k\) minors for \(1 \leq k \leq n\) is in a one-to-one correspondence with the length \(k\)-columns of symplectic PBW tableaux that we have defined earlier in Section 2, up to a reordering of the indices.

To move from any minor \((I_2, I_1)\) to a PBW tableau column, we compute the minor according to Equation (3.1), and then put every entry which is less than or equal to \(k\) at its position in the column of length \(k\), and then every other entry should be filled in such a way that it is bigger than entries below it. For example, the PBW tableau columns corresponding to the computed minors \((\overline{2}, 3, \Gamma, 1), (\overline{3}, 3, \Gamma, 1),\) and \((\overline{4}, 4, \Gamma, 1)\) are respectively the tableaux:

\[
\begin{bmatrix}
1 & 2 \\
\overline{1} & T \\
2 & 3 \\
\overline{3} & 4 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 \\
\overline{1} \\
2 \\
\overline{4} \\
\end{bmatrix}
\]

Moreover we can also move from the PBW tableau columns to the corresponding pairs \((I_2, I_1)\).

To do this, we put every element \(i\) for which \(\overline{i}\) belongs to our tableau in \(I_2\), and all other elements which appear in the tableau with out bars are put in \(I_1\). We prove the following result.

**Proposition 3.11.** The reverse-admissible \(k \times k\) minors are in a weight preserving bijection with the symplectic PBW tableau columns of length \(k\).

**Proof.** We first show that the tableaux corresponding to the reverse-admissible minors \((I_2, I_1)\) satisfy the conditions of Definition 2.2. Conditions (i) and (ii) are clearly always satisfied up on reordering the indices appearing in the computation as described above. It remains to verify condition (iii), namely, we want to show that whenever we have a pair \((i, \overline{i})\) with \(i < k\) in the computed minor, then after re-ordering to satisfy (i) and (ii), the position of \(\overline{i}\) is above that of \(i\). Recall that if \(\Gamma = I_1 \cap I_2 = \{\gamma_1, \ldots, \gamma_\lambda\}\), then we have that \((I_2, I_1)\) is reverse-admissible if we can find \(T \subset \{1, \ldots, n\}\) \(\backslash (I_1 \cup I_2)\) with \(T = \{\nu_1, \ldots, \nu_\lambda\}\) i.e. \(|T| = |\Gamma|\) and \(\nu_1 \leq \gamma_1, \ldots, \nu_\lambda \leq \gamma_\lambda\). We take \(T\) to be the maximal such set. Consider the computation of \((I_2, I_1)\), \(L = (b_1, \ldots, b_{k-\lambda}, a_1, \ldots, a_1, \gamma_\lambda, \ldots, \gamma_1, \gamma_1)\). We are going to describe how to fill in a column. We put each \(\gamma_i\) at position \(\nu_i\), each \(\gamma_i\) at position \(\gamma_i\), each \(a_i\) at position \(a_i\), and the \(b_i\)'s at the remaining spots in a descending order from top to bottom. This implies that \(\gamma_i\) is above \(\gamma_i\) since \(\nu_i \leq \gamma_i\) for all \(1 \leq i \leq k\) and \(T \cap \overline{I}_1 = \emptyset\), and hence the resulting column is a symplectic PBW tableau column. For the other direction, assume we are given a symplectic PBW column. For all \(\gamma_i\) in the column tableau, put \(i\) in \(I_2\), and put the rest of the indices in \(I_1\). Also, for all \((i_1, \ldots, i_\lambda)\) for which we have \((\overline{i}_1, \ldots, \overline{i}_\lambda)\) in the column, let \(j_i\) be the position of \(\gamma_i\) for all \(1 \leq i \leq \lambda\). The tableau being a symplectic PBW tableau implies \(j_i < j_{i_t}\) for all \(1 \leq t \leq \lambda\). Also we note that \(j_i \in \{1, \ldots, n\}\) \(\backslash (I_1 \cup I_2)\) and hence the set \(\{j_1, \ldots, j_\lambda\}\) is the minimal set with the required properties. Hence \((I_2, I_1)\) is reverse-admissible. This gives the bijection. The fact that this bijection is weight preserving follows directly. ∎

### 3.5. Defining ideal of the complete symplectic flag variety.

Consider the embeddings

\[
\text{Sp}\mathcal{F}_{2n} \hookrightarrow \prod_{k=1}^{n} \text{SpGr}(k, 2n) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}\left(\bigwedge^{k} \mathbb{C}^{2n}\right).
\]

Consider the polynomial ring \(\mathbb{C}[X_{j_1}, \ldots, j_d]\) generated by elements \(X_{j_1}, \ldots, j_d\), \(d = 1, \ldots, k, 1 \leq k \leq n\) and \(1 \leq j_1 < \cdots < j_d \leq \overline{\gamma}\). We want to be able to describe the defining ideal of \(\text{Sp}\mathcal{F}_{2n}\) under
the above embedding. Let $\mathcal{F}_{2n}$ denote the type $\mathfrak{A}_{2n-1}$ flag variety. For an algebraic variety $X$, let $\mathcal{I}(X)$ denote the vanishing ideal of this variety. We have the following lemma.

**Lemma 3.12.** For any $\lambda$ of the form $\lambda = \sum_{k=1}^{n} m_k \omega_k$, we have the inclusion $\mathcal{I}(\mathcal{F}_{2n}) \subset \mathcal{I}(\text{Sp} \mathcal{F}_{2n})$.

**Proof.** This follows from [11], Corollary 3.2 and the fact that for any two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \subset \mathcal{V}_2$, one has $\mathcal{I}(\mathcal{V}_2) \subset \mathcal{I}(\mathcal{V}_1)$, via the inclusion reversing property of ideals and algebraic varieties. \hfill \square

**Definition 3.13.** Let $L, J \subset \{1, \ldots, n, \pi_1, \ldots, \pi_1\}$ be two sequences of length $p$ and $q$ respectively, with $n \geq p \geq q \geq 1$. Suppose $L = \{l_1, \ldots, l_p\}$, with $l_1 < \cdots < l_p$ and $J = \{j_1, \ldots, j_q\}$ with $j_1 < \cdots < j_q$ after rearrangement, we have the Plücker relation

$$R_{L,J}^i := X_L X_J - \sum_{1 \leq r_1 < \cdots < r_p \leq p} X_{t_1} X_{t_2},$$

(3.5)

where $L'$ and $J'$ are obtained from $L$ and $J$ by interchanging $t$-tuples $(l_{r_1}, \ldots, l_{r_p})$ and $(j_{r_1}, \ldots, j_{r_p})$ in $L$ and $J$ respectively, while maintaining the order in which they appear. Following the notation of Proposition 3.7, we have the symplectic linear relation

$$S_{(1, 1)} := X_{(1, 1)} - (-1)^{|\Gamma'|} \sum_{\Gamma' : \Gamma' \cap \{1, 1\} = \emptyset} X_{(1, 1, \Gamma', 1, \cup \Gamma')},$$

(3.6)

In both Expressions (3.5) and (3.6), the equality

$$X_{j_1 \sigma(1), \ldots, j_d \sigma(d)} = (-1)^\sigma X_{j_1, \ldots, j_d},$$

for all $d = 1, \ldots, k$, $1 \leq k \leq n$ and $1 \leq j_1 < \cdots < j_d \leq \mathfrak{T}$, is assumed.

**Remark 3.14.** We use relation $S_{(1, 1)}$ to replace any element $X_{(1, 1)}$ corresponding to a non reverse-admissible minor which shows up in the summands of (3.5) by a linear combination of elements corresponding to reverse-admissible minors. So in the end we have quadratic relations but this time only among the reverse-admissible minors.

Let $I$ be the ideal generated by the symplectic relations $R_{L,J}^i$ and $S_{(1, 1)}$.

**Theorem 3.15 ([5]).** The ideal $I$ is the defining ideal of $\text{Sp} \mathcal{F}_{2n}$. It is a prime ideal.

**Proof.** It follows from Lemma 3.12, that the relations $R_{L,J}^i$ are satisfied on the complete symplectic flag variety since they are satisfied on the type $\mathfrak{A}_{2n-1}$ flag variety according to Lemma 1, p. 132, [12]. The relations $S_{(1, 1)}$ come from Equation 3.3 from the proof of Proposition 3.7 so they are clearly satisfied. The work of De Concini in [5] imply that the ideal $I$ is the defining ideal of $\text{Sp} \mathcal{F}_{2n}$. This is true because he used exactly these quadratic relations and the linear relations to show that his symplectic standard tableaux index a basis for the respective homogeneous coordinate ring. The ideal $I$ is prime since $\text{Sp} \mathcal{F}_{2n}$ is irreducible. \hfill \square
3.6. A basis for the homogeneous coordinate ring of the complete symplectic flag variety. One has the following about the homogeneous coordinate ring of $\text{Sp}F_{2n}$

$$\mathbb{C}[x_{j_1, \ldots, j_d}] / I = \mathbb{C}[\text{Sp}F_{2n}] = \bigoplus_{\lambda \in P^+} \mathbb{C}[\text{Sp}F_{2n}]_{\lambda} \simeq \bigoplus_{\lambda \in P^+} V^*_{\lambda},$$

where the multiplication $V^*_{\lambda} \otimes V^*_{\mu} \to V^*_{\lambda+\mu}$ is induced by the existence of the injective homomorphism of modules $V_{\lambda+\mu} \to V_{\lambda} \otimes V_{\mu}$. The isomorphism $\mathbb{C}[\text{Sp}F_{2n}]_{\lambda} \simeq V^*_{\lambda}$ is given by the Borel-Weil theorem. We want to describe a basis for these rings. To do this, we first introduce some more tools.

**Definition 3.17.** For a sequence $L = (l_1, \ldots, l_k)$, $1 \leq k \leq n$, the PBW-degree of $L$ is given by the formula:

$$\deg L = \#\{r : l_r > k\}. \quad (3.7)$$

We define the PBW-degree of the variable $X_L$ to be the PBW-degree of the sequence $L$. The PBW-degree of the minor $(I_2, I_1)$ is the PBW-degree of its computation as given in Equation (3.1).

**Remark 3.18.** We can obtain the PBW-degree of $(I_2, I_1)$ directly from the subsequences $I_1$ and $I_2$ without first computing the minor. For this we use the formula:

$$\text{deg}(I_2, I_1) = |I_2| + \#\{i \in I_1 : i > k\}. \quad (3.8)$$

To see that the degrees given in Equations (3.7) and (3.8) agree, we only need to consider the PBW-degree of the computed minor $L$ of $(I_2, I_1)$. Indeed from Equation (3.1), we have that:

$$\deg L = |\tilde{I}_2| + |\Gamma| + \#\{z : a_z \in \tilde{I}_1, a_z > k\} + \#\{z : \gamma_z \in \Gamma, \gamma_z > k\},$$

where $\gamma_z \in \Gamma, \gamma_z > k$,

$$= |I_2| + |\Gamma| + \#\{i \in I_1, i > k\} - \#\{z : \gamma_z \in \Gamma, \gamma_z > k\} + \#\{z : \gamma_z \in \Gamma, \gamma_z > k\},$$

$$= |I_2| - \#\{i \in I_1, i > k\},$$

$$= \text{deg}(I_2, I_1).$$

We prove the following fundamental lemma.

**Lemma 3.19.** Following the notation of Proposition 3.7, the PBW-degree of each of the summands appearing on the right hand side of

$$X_{(I_2, I_1)} = (-1)^{|\Gamma|} \sum_{\Gamma' : |\Gamma'\cap\{I_1\cup I_2\}|=0 \text{ and } |\Gamma'|=|\tilde{\Gamma}|} X_{(I_2\cup F \cup \Gamma', \tilde{I}_1\cup F \cup \Gamma')}, \quad (3.9)$$

is greater or equal to the PBW-degree of the term $X_{(I_2, I_1)}$ on the left hand side, whenever $(I_2, I_1)$ is not reverse-admissible.

**Proof.** We claim that $|I_2| = |\tilde{I}_2 \cup F \cup \Gamma'|$ for every $\Gamma'$. Indeed one has

$$|\tilde{I}_2 \cup F \cup \Gamma'| = |I_2 \backslash (\Gamma \cup \tilde{\Gamma} \cup \Gamma')| = |I_2 \backslash \tilde{\Gamma} \cup \Gamma'| = |I_2| - |\tilde{\Gamma}| + |\Gamma'| = |I_2|, \quad (3.10)$$

since $|\tilde{\Gamma}| = |\Gamma'|$. Now we will show that

$$\#\{i \in \tilde{I}_1 \cup F \cup \Gamma' : i > k\} \geq \#\{i \in I_1 : i > k\}. \quad (3.11)$$

But we know that $\#\{i \in I_1 : i > k\} = \#\{i \in \tilde{I}_1 \cup F \cup \tilde{\Gamma} : i > k\}$. Therefore proving the Inequality (3.11) reduces to showing that:

$$\#\{i \in \tilde{I}_1 \cup F \cup \Gamma' : i > k\} \geq \#\{i \in \tilde{I}_1 \cup F \cup \tilde{\Gamma} : i > k\}.$$

This in turn reduces to showing that:
\[
\#\{i \in \Gamma' : i > k\} \geq \#\{i \in \bar{\Gamma} : i > k\}.
\]

In fact from the proof of Proposition 3.7, we know that the maximum element in \(\bar{\Gamma}\) is \(\gamma_b\). We claim that \(\gamma_b < k\). For \((I_2, I_1)\) non reverse-admissible, recall the set \(T = T_{h_0+1}\). Claim: for all \(i < \gamma_b\), \(i \in T \cup I_1 \cup I_2\). Assume this was not true, then \(T \cup \{i\} < (\gamma_{h_0}, \ldots, \gamma_t)\), which contradicts the minimality of \(h_0\). Set \(M = \{1, \ldots, \gamma_b\}\). Then by the claim, and as \(T \cap (I_1 \cup I_2) = \emptyset\),

\[
\gamma_b = |M| = |M \cap T| + |M \cap (I_1 \cup I_2)| \leq |T| + |I_1 \cup I_2| < |\Gamma| + |I_1 \cup I_2| = k.
\]

This together with (3.10) implies the lemma. \(\square\)

Now we are ready to prove that our symplectic PBW-semistandard tableaux index a basis for \(\mathbb{C}[\text{Sp} F_{2n}]\). For a symplectic PBW-semistandard tableau \(T \in \text{SyST}_\lambda\) of shape \(\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)\), we associate the monomial:

\[
X_T = \prod_{j=1}^{\lambda_1} X_{T_{1,j}, \ldots, T_{\mu_j,j}} \in V_\lambda^*.
\]

We prove the following result:

**Theorem 3.20.** The elements \(X_T, T \in \text{SyST}_\lambda\), form a basis of \(\mathbb{C}[\text{Sp} F_{2n}]_\lambda\).

**Proof.** From Theorem 2.14, we have that \(#\{T : T \in \text{SyST}_\lambda\} = \dim V_\lambda\), so it remains to show that the elements \(X_T, T \in \text{SyST}_\lambda\) span \(\mathbb{C}[\text{Sp} F_{2n}]_\lambda\). For this it suffices to consider any two symplectic PBW tableaux columns \(L\) and \(J\) of length \(p\) and \(q\) respectively, with \(p \geq q\) and whose product is not semistandard. We apply Relation (3.5), to express the product \(X_L X_J\) as a sum of products \(X_{L^{(i)}} X_{J^{(i)}}\), that is:

\[
X_L X_J = \sum_i X_{L^{(i)}} X_{J^{(i)}}.
\]

Moreover after exchanging, it may happen that for one of the variables \(X_{L^{(i)}}\) or \(X_{J^{(i)}}\), the corresponding \(L^{(i)}\) or \(J^{(i)}\) is no longer a symplectic PBW tableau column, that is to say, the corresponding minor \((I_2, I_1)^{(i)}\) is not reverse-admissible. In this case, we apply Relation (3.6) to replace such a variable with a sum of variables corresponding to reverse-admissible minors. Now from Lemma 3.19 and from the proof of Proposition 4.12 of [10], we see that

\[
\deg X_{L^{(i)}} + \deg X_{J^{(i)}} > \deg X_L + \deg X_J.
\]

Therefore in \(\mathbb{C}[\text{Sp} F_{2n}]_\lambda\), any \(X_T\) with \(T \notin \text{SyST}_\lambda\), can be written as a linear combination of \(X_{T'}\) with the sum of PBW-degrees of \(X_{T'}\) bigger than that of \(X_T\). This implies the claim since the sum of PBW-degrees of fixed shaped tableaux is bounded from above, and hence our theorem is proved. \(\square\)

4. **The PBW degenerate complete symplectic flag variety, symplectic degenerate relations and a basis for the homogeneous coordinate ring**

In this section we describe the PBW degenerate complete symplectic flag variety following [11]. We then provide our own results on a basis for the homogeneous coordinate ring of this degenerate variety labelled by our tableaux. At the end we obtain the defining ideal of \(\text{Sp} F_{2n}^d\).
4.1. PBW degenerate flag varieties; a brief description. Let \( G^a \) be a Lie group corresponding to the PBW degenerate Lie algebra \( g^a \). Let us briefly describe the Lie group \( G^a \). Let \( G_a \) be the additive group of the field \( \mathbb{C} \) and let \( M = \dim n^- \). The Lie group \( G^a \) is a semidirect product \( G_a^M \times B \) of the normal subgroup \( G_a^M \) and the Borel subgroup \( B \). For any dominant, integral weight \( \lambda \), there exist induced \( g^a \)- and \( G^a \)-module structures on \( V^a_\lambda \). The group \( G^a \) therefore acts on \( \mathbb{P}(V^a_\lambda) \), the projectivization of \( V^a_\lambda \). The PBW degenerate flag variety is defined to be the closure of the orbit of the action of \( G^a \) on the highest weight line, that is to say:

\[
\mathcal{F}^a_\lambda = \overline{G^a[\nu^a_\lambda]} \to \mathbb{P}(V^a_\lambda)
\]

(see [10]).

For \( g = sl_{n+1} \) (type \( A_n \)), we have \( \mathcal{F}_{\omega_k} \cong \mathcal{F}^a_{\omega_k} \), for all \( k = 1, \ldots, n \), where \( \omega_k = \lambda \). This is true because all the fundamental weights \( \omega_k \) are co-minuscule in type \( A \), and hence the radical corresponding to each \( \omega_k \) is abelian (see [7]). So the PBW degenerate flag variety in type \( A \) is isomorphic in general. In fact, the only exception is the case \( k = n \), because \( \omega_n \) is the only co-minuscule weight of \( g \).

4.2. Degenerate complete symplectic flag variety. Recall the vector space \( W \) with the fixed basis \( \{ w_1, \ldots, w_{2n} \} \). Let \( SpGr^a(k, 2n) \) denote the symplectic degenerate Grassmannian variety and let \( Gr(k, 2n) \) denote the usual Grassmannian. Let \( W = W_{k,1} \oplus W_{k,2} \oplus W_{k,3} \), where \( W_{k,1} = \text{span}(w_1, \ldots, w_k) \), \( W_{k,2} = \text{span}(w_{k+1}, \ldots, w_{2n-k}) \) and \( W_{k,3} = \text{span}(w_{2n-k+1}, \ldots, w_{2n}) \). Let \( pr_{1,3} \) denote the projection \( pr_{1,3} : W \to W_{k,1} \oplus W_{k,3} \), that is to say,

\[
pr_{1,3}(x_1, \ldots, x_{2n}) = (x_1, \ldots, x_k, 0, \ldots, 0, x_{2n-k+1}, \ldots, x_{2n}).
\]

Then \( SpGr^a(k, 2n) = \{ U \in Gr(k, 2n) \mid pr_{1,3}(U) \text{ is isotropic} \} \) (see [11]).

For a dominant, integral and regular weight \( \lambda \) denote by \( Sp\mathcal{F}^a_{2n} \) the degenerate complete symplectic flag variety. Denote by \( pr_i : W \to W \) the projections along \( w_i \), i.e.,

\[
pr_i(\sum_{j=1}^{2n} c_j w_j) = \sum_{j \neq i} c_j w_j.
\]

Then \( Sp\mathcal{F}^a_{2n} \) is naturally embedded into the product \( \prod_{i=1}^n SpGr^a(i, 2n) \) of degenerate symplectic Grassmannians. This means that we have the tower of embeddings:

\[
Sp\mathcal{F}^a_{2n} \hookrightarrow \prod_{i=1}^n SpGr^a(i, 2n) \to \mathbb{P}(V^a_\lambda).
\]

The image of these embeddings is equal to the sub-variety formed by the collections \( (U_i)^n_{i=1} \), \( U_i \in SpGr^a(i, 2n) \) satisfying the conditions \( pr_{i+1} U_i \subset U_{i+1} \), \( i = 1, \ldots, n-1 \) (see [11]).

4.3. Symplectic degenerate relations. One has two kinds of degenerate relations; the linear ones and quadratic ones. These relations live in the polynomial ring \( \mathbb{C}[X^a_{j_1, \ldots, j_d}] \) in variables \( X^a_{j_1, \ldots, j_d} \), \( d = 1, \ldots, k, 1 \leq k \leq n \) and \( 1 \leq j_1 < \cdots < j_d \leq \tilde{T} \).

**Definition 4.1.** Recall the notation from Proposition 3.7. The degenerate linear relations are

\[
S^a_{\{1,2,3\}} := X^a_{\{1,2,3\}} - (-1)^{|\Gamma'|} \sum_{\Gamma': \Gamma' \cap \{1,2,3\} = \emptyset} X^a_{\{1,2,3\} \cup \Gamma'},
\]

where the terms are obtained by picking up the minimum PBW-degree terms from the relations \( S_{\{1,2,3\}} \) in (3.6) and introducing a superscript “a”. The degenerate quadratic relations are
obtained from the relations $R^{a}_{i,j}$ by picking out the lowest PBW-degree terms and introducing a superscript \("a\) . Therefore we have

$$R^{a}_{i,j} := X_{i}^{a}X_{j}^{a} - \sum_{1 \leq r_{1} < \cdots < r_{t} \leq p} X_{L}^{a}X_{J}^{a}, \tag{4.2}$$

labelled by the numbers $p, q$ with $1 \leq q \leq p \leq n$, by an integer $t$, $1 \leq t \leq q$, and by sequences $L = (l_{1}, \ldots, l_{p})$, $J = (j_{1}, \ldots, j_{q})$ which are subsets of $\{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$.

**Example 4.2.** For $k = n$, we have $S_{(l_{2}, l_{1})} = S_{(l_{2}, l_{1})}^{a}$ up to a superscript, the relations $S_{(l_{2}, l_{1})}$ are homogeneous with respect to the PBW degree in this case. This is exactly because we have the isomorphism $\operatorname{SpGr}(n, 2n) \simeq \operatorname{SpGr}^{a}(n, 2n)$.

**Example 4.3.** For $g = \mathfrak{sp}$, the symplectic degenerate relations for $\mathcal{S}^{a}_{4}$ (PBW degenerate complete symplectic flag variety containing full flags of length 2) are:

$$R^{a}_{(1, 2), (2)} := X_{1}^{a}X_{2}^{a} + X_{2}^{a}X_{1}^{a}, \qquad R^{a}_{(1, 2), (1)} := X_{1}^{a}X_{2}^{a} + X_{2}^{a}X_{1}^{a},$$

$$R^{a}_{(1), (2)} := X_{1}^{a}X_{2}^{a} + X_{2}^{a}X_{1}^{a} - X_{1}^{a}X_{2}^{a} - X_{2}^{a}X_{1}^{a}, \quad R^{a}_{(1, 2), (2)} := X_{1}^{a}X_{2}^{a} - X_{2}^{a}X_{1}^{a} + X_{2}^{a}X_{2}^{a} - X_{1}^{a}X_{2}^{a},$$

and the linear relation $S^{a}_{(1), (2)} := X_{1}^{a}X_{2}^{a} + X_{2}^{a}X_{2}^{a} - X_{1}^{a}X_{2}^{a}$.

**Remark 4.4.** For an intuition behind the way we obtain $S^{a}_{L}$ as in Definition 4.1, consider an element $U_{2} \subset \mathbb{C}^{6}$, generated by the vectors $u = a_{1, i}w_{1} + a_{2, i}w_{2} + a_{3, i}w_{3} + a_{4, i}w_{4} + a_{5, i}w_{5} + a_{6, i}w_{6}$ and $v = a_{1, j}w_{1} + a_{2, j}w_{2} + a_{3, j}w_{3} + a_{4, j}w_{4} + a_{5, j}w_{5} + a_{6, j}w_{6}$. We want to describe the criterion for $U_{2}$ to be in $\operatorname{SpGr}(2, 6)$. Recall the projection $pr_{1, 3}$. Applying it to $u$ and $v$ we have:

$$pr_{1, 3}(u) = a_{1, i}w_{1} + a_{2, j}w_{2} + a_{5, i}w_{5} + a_{6, i}w_{6} \quad \text{and} \quad pr_{1, 3}(v) = a_{1, j}w_{1} + a_{2, i}w_{2} + a_{5, j}w_{5} + a_{6, j}w_{6}.$$ Then $pr_{1, 3}(U_{2})$ is isotropic if and only if $pr_{1, 3}(u)^{T}M_{4}pr_{1, 3}(v) = 0$, i.e.,

$$\begin{pmatrix}
    a_{1, i} & a_{2, j} & 0 & 0 & a_{5, i} & a_{6, i}
\end{pmatrix}
\begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & -1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 & 0 & 0 \\
    -1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    a_{1, j} \\
    a_{2, i} \\
    0 \\
    0 \\
    a_{5, j} \\
    a_{6, j}
\end{pmatrix} = 0,$$

$$-a_{6, i}a_{1, j} - a_{5, i}a_{2, j} + a_{2, i}a_{5, j} + a_{1, i}a_{6, j} = 0,$$

which leads to the degenerate symplectic linear relation $X_{1}^{a} + X_{2}^{a} = 0$ (or $X_{1}^{a} + X_{2}^{a} = 0$) which is the relation $S^{a}_{(1), (2)}$ obtained by picking out the terms of minimal PBW degree from the corresponding relation $S^{a}_{(1), (2)} := X_{1}^{a} + X_{2}^{a}$.

**Lemma 4.5.** The symplectic degenerate relations $R^{a}_{i,j}$ and $S^{a}_{(l_{2}, l_{1})}$ are homogeneous with respect to the PBW degree.

**Proof.** This follows directly from Definition 4.1 since the terms in $R^{a}_{i,j}$ and $S^{a}_{(l_{2}, l_{1})}$ are those of minimal PBW degrees picked from the relations $R^{a}_{i,j}$ and $S^{a}_{(l_{2}, l_{1})}$ respectively. \(\square\)

### 4.4. A basis for the homogeneous coordinate ring

Let $\mathbb{C}[\mathcal{S}^{a}_{2n}]$ denote the homogeneous coordinate ring of the PBW degenerate symplectic complete flag variety. Then one has

$$\mathbb{C}[\mathcal{S}^{a}_{2n}] = \bigoplus_{\lambda \in P^{+}} \mathbb{C}[\mathcal{S}^{a}_{2n}]_{\lambda} \simeq \bigoplus_{\lambda \in P^{+}} (V^{a}_{\lambda})^{*},$$
where the multiplication $(V^*_{\lambda})^* \otimes (V^*_{\mu})^* \to (V^*_{\lambda+\mu})^*$ for any two weights $\lambda$ and $\mu$ is implied by the existence of the injective homomorphism of $\mathfrak{g}^*$-modules, $V^*_{\lambda+\mu} \hookrightarrow V^*_{\lambda} \otimes V^*_{\mu}$ according to Lemma (1.6). For the isomorphism $\mathbb{C}[\mathcal{S}P^a_{2n}] \simeq (V^*_{\lambda})^*$, see [11], Theorem 8.2 which is an analogue of the Borel-Weil theorem for the PBW degenerate module $V^*_{\lambda}$.

We have the elements $X^a_{j_1,\ldots,j_d} \in (V^*_{\lambda})^*$.

**Proposition 4.6.** The symplectic degenerate relations $R^{L_{a,1}}_{a,1,1}$ and $S^{a}_{(1,2,1,1)}$ are both zero in $\mathbb{C}[\mathcal{S}P^a_{2n}]$.

**Proof.** We claim that the elements $X^a_{j_1,\ldots,j_d}$ satisfy relations $R^{L_{a,1}}_{a,1,1}$ and $S^{a}_{(1,2,1,1)}$ in $\bigoplus_{\lambda}(V^*_{\lambda})^*$. We know already that the elements $X^a_{j_1,\ldots,j_d}$ satisfy relations $R^{L_{a,1}}_{a,1,1}$ and $S^{a}_{(1,2,1,1)}$ in the algebra $\bigoplus_{\lambda} V^*_\lambda$ (in other words, these relations vanish). We also know that the relations $R^{L_{a,1}}_{a,1,1}$ and $S^{a}_{(1,2,1,1)}$ are lowest degree terms with respect to the PBW grading. Since the algebra $\bigoplus_{\lambda}(V^*_{\lambda})^*$ is PBW-graded, the claim follows. Also the proposition follows since we have the isomorphism $\mathbb{C}[\mathcal{S}P^a_{2n}] \simeq \bigoplus_{\lambda}(V^*_{\lambda})^*$.

Recall $\text{SyST}_\lambda$, the set of all symplectic PBW-semistandard tableaux of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$. To each $T \in \text{SyST}_\lambda$, associate the element $X^a_T = \prod_{j=1}^{\lambda_1} X^a_{T_{1,j},\ldots,T_{p,j}} \in (V^*_{\lambda})^*$, and call such an element, the symplectic PBW-semistandard monomial. We prove the following result.

**Theorem 4.7.** The elements $X^a_T$, $T \in \text{SyST}_\lambda$, form a basis of $\mathbb{C}[\mathcal{S}P^a_{2n}]_\lambda$.

**Proof.** From Theorem 1.5 and Theorem 2.14, we know that $\dim V^*_{\lambda} = \# \{ T : T \in \text{SyST}_\lambda \}$. It therefore remains to prove that the elements $X^a_T$, $T \in \text{SyST}_\lambda$ span $\mathbb{C}[\mathcal{S}P^a_{2n}]_\lambda$. From Proposition 4.6, we know that the elements $X^a_{j_1,\ldots,j_d}$ satisfy relations $R^{L_{a,1}}_{a,1,1}$ and $S^{a}_{(1,2,1,1)}$ in $\mathbb{C}[\mathcal{S}P^a_{2n}]$. We are going to therefore use these relations to write the element $X^a_T$ for $T$ not semistandard as a linear combination of elements $X^a_{T'}$ with $T'$ semistandard. For this, we first follow [10] to define an order on the set of PBW tableaux of shape $\lambda$. Say that $T^{(1)} > T^{(2)}$ if there exists $i_0, j_0$ such that $T^{(1)}_{i_0,j_0} > T^{(2)}_{i_0,j_0}$ and $T^{(1)}_{i,j} = T^{(2)}_{i,j}$ if $(j > j_0, i = i_0)$ or $(j = j_0, i > i_0)$.

Since the condition of PBW-semistandardness is defined between any two neighbouring columns, we can reduce the proof to any two arbitrary neighbouring columns. Supposing we are given two arbitrary such columns $L$ and $J$ that form a symplectic PBW tableau that is not PBW-semistandard. We are going to first use the degenerate quadratic relations $R^{L_{a,1}}_{a,1,1}$ to obtain terms corresponding to smaller PBW tableaux. In fact, let $L = (l_1,\ldots,l_p)$ and $J = (j_1,\ldots,j_q)$ with $p \geq q$. From the proof of Proposition 4.12 of [10], we have that the term

\[ X^a_{l_1,\ldots,l_p} X^a_{j_1,\ldots,j_q} \]

is present in the relation

\[ R^{L_{a,1}}_{(l_1,\ldots,l_p),(j_1,\ldots,j_q)} \]

and that all the other terms correspond to smaller PBW tableaux with respect to the order “$>$” on the set of PBW tableaux. The only thing that remains is to show that all resulting tableaux will be symplectic as well. For this, we use the symplectic degenerate linear relations to replace the smaller PBW tableaux with even smaller ones. Indeed, let $L'$ be a non symplectic column that appears after the exchanging. Then from Lemma 3.19, the term $X^a_{L'}$ is among the terms with minimal PBW degree in $S_{L'}$. This means that the term $X^a_{L'}$ is present in the
relation $S_{\gamma_b}^a$ since this relation is obtained by picking out terms of minimal PBW degree from $S_{L'}^a$. We can therefore use the relation $S_{L'}^a$ to replace terms corresponding to non symplectic columns. It now remains to show that the new columns are smaller with respect to the order “>”. Recall the definition of relations $S_{L'}^a$ and $S_{L'}^a$. From Lemma 3.19, we know that $\gamma_b < k$. Hence we see that the PBW degree goes up only when $\gamma_b' > k$. Therefore since we are using relations $S_{L'}^a$, it suffices to consider the case $\gamma_b' \leq k$. For any given term $X_{L''}^a$ in $S_{L'}^a$ apart from $X_{L'}^a$, and for a corresponding sequence $L''$, let $f$ be the position of $\gamma_b'$ after rearranging the entries to form a PBW tableau column. Clearly we need to begin comparing the entries of the columns $L'$ and $L''$ starting from position $f$ downwards. To see this, recall that since $\gamma_b' \leq k$, then $f = \gamma_b'$. This implies that the entry at position $f$ in $L'$ is different from $f$ since $\gamma_b' \in \{1, \ldots, n\} \setminus (I_1 \cup I_2)'$ with $L' = (I_2, I_1)'$. Let $L'_f$ denote the entry at position $f$ in PBW tableau column $L'$. We have $L'_f > f = \gamma_b'$. Moreover all entries below position $f$ (if any), are pairwise equal in $L'$ and $L''$. This implies that $L' > L''$. This proves the claim and hence the theorem follows.

4.5. The defining ideal for the degenerate complete symplectic flag variety. Let $I^a \subset \mathbb{C}[X_{j_1, \ldots, j_d}^a]$ be the ideal generated by the symplectic degenerate relations $R_{L',J}^{t,a}$ and $S_{I_{12},I_1}^a$. The following is the major statement of this paper.

**Theorem 4.8.** The ideal $I^a$ is the defining ideal of $\text{SpF}_{2n}^a \hookrightarrow \mathbb{P}(V_{\lambda}^a)$.

**Proof.** From Theorem 4.7, we see that the relations $R_{L',J}^{t,a}$ and $S_{I_{12},I_1}^a$ in $I^a$ are enough to express every monomial in Plücker coordinates as a linear combination of symplectic PBW-semistandard monomials (i.e. these relations provide a straightening law for $\mathbb{C}[\text{SpF}_{2n}^a]$). Following the idea of the proof of Theorem 7 in [2], this implies that the ideal $I^a$ is the defining ideal of $\text{SpF}_{2n}^a$ since otherwise, it would imply that the symplectic PBW-semistandard monomials are not a basis for $\mathbb{C}[\text{SpF}_{2n}^a]$. □

**Remark 4.9.** From Theorem 4.8, we can now write down the homogeneous coordinate ring of $\text{SpF}_{2n}^a$ as a quotient of the polynomial ring $\mathbb{C}[X_{j_1, \ldots, j_d}^a]$ by the ideal $I^a$, i.e.

$$\mathbb{C}[\text{SpF}_{2n}^a] = \mathbb{C}[X_{j_1, \ldots, j_d}^a]/I^a \simeq \bigoplus_{\lambda \in P^+} (V_{\lambda}^a)^*.$$  

**Corollary 4.10.** The ideal $I^a$ is a prime ideal of the polynomial ring $\mathbb{C}[X_{j_1, \ldots, j_d}^a]$.

**Proof.** This follows directly from Theorem 4.8 and the fact that the variety $\text{SpF}_{2n}^a$ is irreducible (see [11], Corollary 5.6). □

**Remark 4.11.** As noted in the introduction, Feigin, Finkelberg and Littelmann in [11], proved that $\text{SpF}_{2n}^a$ is a flat degeneration of $\text{SpF}_{2n}$. We would like to give a formulation of this result in terms of the results of this paper. Let $s$ be a variable. We follow [10] to define an algebra $Q^s$ over the ring $\mathbb{C}[s]$ as a quotient of the polynomial ring $\mathbb{C}[s][X_{j_1, \ldots, j_d}, \ d = 1, \ldots, k]$ by the ideal $I^s$ generated by quadratic relations $R_{L',J}^{t,s}$ and linear relations $S_{I_{12},I_1}^s$ which are $s$-deformations of the relations $R_{L',J}^t$ and $S_{I_{12},I_1}$. Let $R_{L',J}^t = \sum_i X_L^{(i)} X_J^{(i)}$ and $S_{I_{12},I_1} = \sum_i X_{(I_{12},I_1)}^{(i)}$, then:

$$R_{L',J}^{t,s} = s^{-\min_i(\deg L^{(i)} + \deg J^{(i)})} \sum_i s^{\deg L^{(i)} + \deg J^{(i)}} X_L^{(i)} X_J^{(i)},$$

$$S_{I_{12},I_1}^s = s^{-\min_i((I_{12},I_1)^{(i)})} \sum_i s^{\deg (I_{12},I_1)^{(i)}} X_{(I_{12},I_1)^{(i)}}.$$
We have $Q^s/(s) \simeq \mathbb{C}[SpF_{2n}]$, and $Q^s/(s - u) \simeq \mathbb{C}[SpF_{2n}]$ for $u \neq 0$. Moreover following Theorem 4.7, one checks that the elements $X_T, T \in \text{SyST}_\lambda, \lambda \in P^+$ form a $\mathbb{C}[s]$ basis of $Q^s$, hence showing that $Q^s$ is $\mathbb{C}[s]$ free.

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