Odd Entries in Pascal’s Trinomial Triangle

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Abstract. The $n$th row of Pascal’s trinomial triangle gives coefficients of $(1 + x + x^2)^n$. Let $g(n)$ denote the number of such coefficients that are odd. We review Moshe’s algorithm for evaluating asymptotics of $g(n)$ – this involves computing the Lyapunov exponent for certain $2 \times 2$ random matrix products – and then analyze further examples with more terms and higher powers of $x$.

Before discussing trinomials, let us recall well-known results for binomials. Define $f(n)$ to be the number of odd coefficients in $(1 + x)^n$. Let $N$ denote a uniform random integer between 0 and $n - 1$, then $f(N)$ has “typical growth” $\approx n^{1/2}$ in the sense that

$$E(\ln(f(N))) \sim \frac{1}{2} \ln(n)$$

as $n \to \infty$; equivalently,

$$\lim_{n \to \infty} \frac{1}{n \ln(n)} \sum_{k=0}^{n-1} \ln(f(k)) = \frac{1}{2} = 0.5.$$

Also $f(N)$ has “average growth” $\approx n^{\ln(3/2)/\ln(2)}$ in the sense that

$$\ln(E(f(N))) \sim \frac{\ln(3/2)}{\ln(2)} \ln(n)$$

as $n \to \infty$; equivalently,

$$\lim_{n \to \infty} \frac{1}{\ln(n)} \ln \left( \frac{1}{n} \sum_{k=0}^{n-1} f(k) \right) = \frac{\ln(3/2)}{\ln(2)} = 0.5849625007211561814537389....$$

The latter value is larger since most of the 1s in Pascal’s binomial triangle, modulo 2, are concentrated in relatively few rows. Exact results are available, due to Trollope [1] & Delange [2] for $E(\ln(f(N)))$ and Stein [3] & Larcher [4] for $\ln(E(f(N)))$; an overview of the subject is found in [5]. Our interest here is solely in first-order approximations.

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Moshe [6] introduced an algorithm for evaluating such asymptotics. Define \( P(i, j) \) to be the \((i, j)\)th entry of the triangle mod 2:

\[
\begin{pmatrix}
P(0, 0) \\
P(1, 0) & P(1, 1) \\
P(2, 0) & P(2, 1) & P(2, 2) \\
P(3, 0) & P(3, 1) & P(3, 2) & P(3, 3) \\
P(4, 0) & P(4, 1) & P(4, 2) & P(4, 3) & P(4, 4)
\end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

All values of \( P(i, j) \) in the upper right portion are 0s. Define \( o = 0 \), which serves as a placeholder. Let \( \ell = 1 \),

\[ A_0(j) = P(o, j), \quad 0 \leq j < \ell \]

and observe that the vector \( A_0 = (1) \). Constructing additional \( \ell \)-vectors \( A_1, A_2, \ldots, A_{m-1} \), if required, is one aspect of the algorithm. It is mandatory that \( A_i(0) = 1 \) always. For each \( 0 \leq s \leq 1, 0 \leq t \leq 1 \), let

\[ b^{s, t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell \]

which is obtained by replacing \((o, j)\) in the expression for \( A_0 \) by \((2o + s, 2j + t)\). Observe that the four vectors

\[ b^{0,0}_0 = (1), \quad b^{1,0}_0 = (1), \quad b^{0,1}_0 = (0), \quad b^{1,1}_0 = (1) \]

encompass only \((1) = A_0 \) and \((0) \). No “refinement” of \( b^{s, t}_0 \) is hence necessary (this will be clarified later) and we let

\[ B^{s, t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell. \]

Constructing additional \( \ell \)-vectors \( B^{s, t}_1, B^{s, t}_2, \ldots, B^{s, t}_{m-1} \), if required, is another aspect of the algorithm. We have

\[ B^{0,0}_0 = A_0, \quad B^{1,0}_0 = A_0, \quad B^{0,1}_0 = 0, \quad B^{1,1}_0 = A_0 \]

and this completes the iterative portion of the algorithm. Thus \( m = 1 \). Define \( m \times m \) matrices via

\[ D_s(i, j) = \# \{ t : B^{s, t}_j = A_i \} \]

and observe that \( D_0 = (1), D_1 = (2) \). Define \( m \)-vectors via

\[ e_i(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases} \quad w(i) = \# \{ j : A_i(j) = 1 \} \]
and observe that $e_0 = (1), w = (1)$. Let the binary expansion of a positive integer $n$
be $\sum_{r=0}^{k-1} n_r 2^r$ with $0 \leq n_r < 2$ and $n_{k-1} = 1$. Moshe [6]
proved the following formula:

$$f(n) = \# \{ j : P(n, j) = 1 \} = w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} e_0$$

which provides a useful check that $D_0, D_1, w$ are correct.

One consequence is a well-known recursive formula for $f(n)$. Writing binary ex-
pressions as $n = n_{k-1} n_{k-2} \ldots n_0 0$, we see

$$2n = n_{k-1} n_{k-2} \ldots n_1 n_0 0, \quad 2n + 1 = n_{k-1} n_{k-2} \ldots n_1 n_0 1$$

hence

$$f(2n) = w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} D_0 e_0$$

$$= w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} e_0$$

$$= f(n),$$

$$f(2n + 1) = w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} D_1 e_0$$

$$= 2w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} e_0$$

$$= 2f(n).$$

All $D_0$ matrices exhibited in this paper satisfy $D_0 e_0 = e_0$, therefore even arguments
are easy. Odd arguments are harder since $D_1 e_0$ is not as predictable.

Another consequence involves the growth rates $E(\ln(f(N)))$ and $\ln(E(f(N)))$. Let
us work with the latter first. Order the complex eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$
of the matrix $D_0 + D_1$ so that $\mu_1$ has maximum modulus; therefore [6]

$$\frac{\ln(E(f(N))))}{\ln(n)} \to \frac{\ln(\mu_1)}{\ln(2)} - 1 = \frac{\ln(3/2)}{\ln(2)}$$

as $n \to \infty$. Working with $E(\ln(f(N)))$ is more complicated. In this scalar case, it is

$$E(\ln(f(N))) = E \left( \sum_{r=0}^{k-1} \ln(D_{N_r}) \right) = \left( \frac{1}{2} \ln(1) + \frac{1}{2} \ln(2) \right) k = \frac{\ln(2)}{2} \left[ \frac{\ln(n)}{\ln(2)} \right]$$

and so

$$\frac{E(\ln(f(N)))}{\ln(n)} \to \frac{1}{2}$$

as $n \to \infty$. But commutativity fails for random matrix products, in general, and
a Lyapunov exponent-based approach will be presented in section 0.8 to deal with
this issue.
0.1. Trinomials I. Define \( g(n) \) to be the number of odd coefficients in \((1 + x + x^2)^n\). Properties of \( g(n) \) and Pascal’s trinomial triangle are given in [7, 8]. Define \( P(i, j) \) to be the \((i, j)\)th entry of the triangle mod 2:

\[
\begin{pmatrix}
P(0,0) & P(0,1) \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) \\
P(2,0) & P(2,1) & P(2,2) & P(2,3) & P(2,4) & P(2,5) \\
P(3,0) & P(3,1) & P(3,2) & P(3,3) & P(3,4) & P(3,5) & P(3,6) & P(3,7)
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}.
\]

All values of \( P(i, j) \) in the upper right portion are 0s. Let \( \ell = 2 \),

\[ A_0(j) = P(o, j), \quad 0 \leq j < \ell \]

and observe that the vector \( A_0 = (1, 0) \). For each \( 0 \leq s \leq 1, 0 \leq t \leq 1 \), let

\[ b_0^{s,t}(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell \]

which is obtained by replacing \((o, j)\) in the expression for \( A_0 \) by \((2o + s, 2j + t)\). Observe that the four vectors

\[ b_0^{0,0} = (1, 0), \quad b_0^{1,0} = (1, 1), \quad b_0^{0,1} = (0, 0), \quad b_0^{1,1} = (1, 0) \]

contain a nonzero vector \((1, 1) \neq A_0 \). Let \( A_1 = b_0^{1,0} = (1, 1) \). No “refinement” of \( b_0^{s,t} \) is necessary (this will be clarified soon) and we let

\[ B_0^{s,t}(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell. \]

It follows that

\[ B_0^{0,0} = A_0, \quad B_0^{1,0} = A_1, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_0 \]

but we are not yet done (because of \( A_1 \)). Let

\[ b_1^{s,t}(j) = P(2(2o + s) + 1, 2(2j + t) + 0), \quad 0 \leq j < \ell \]

which is obtained by replacing \((o, j)\) in the expression for \( b_0^{1,0} = A_1 \) by \((2o + s, 2j + t)\). Observe that the four vectors

\[ b_1^{0,0} = (1, 0), \quad b_1^{1,0} = (1, 0), \quad b_1^{0,1} = (1, 0), \quad b_1^{1,1} = (0, 1) \]

contain a nonzero vector \((0, 1) \neq A_0, A_1 \). Setting \( A_2 = b_1^{1,1} \), however, violates the mandate that \( A_i(0) = 1 \) always. We thus refine the definition of \( b_1^{s,t} \):

\[ B_1^{s,t}(j) = \begin{cases} 
P(2(2o + s) + 1, 2(2j + t) + 0) & \text{if } (s, t) \neq (1, 1), \\
P(2(2o + s) + 1, 2(2(j + c) + t) + 0) & \text{if } (s, t) = (1, 1) 
\end{cases} \]
where $c = 1$, which corresponds to shifting one step to the right. It follows that

$$B_i^{0,0} = A_0, \quad B_1^{1,0} = A_0, \quad B_1^{0,1} = A_0, \quad B_1^{1,1} = A_0$$

and this completes the iterative portion of the algorithm. Thus $m = 2$. The definitions of $m \times m$ matrices

$$D_0 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$ 

and $m$-vectors $e_0 = (1, 0)$, $w = (1, 2)$ are as before. Likewise,

$$g(n) = \# \{ j : P(n, j) = 1 \} = w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} e_0$$

as before, where $n_{k-1} n_{k-2} \ldots n_1 n_0$ is the binary expansion of $n$.

It is easy to see that $g(2n) = g(n)$. From

$$2n + 1 = n_{k-1} n_{k-2} \ldots n_1 n_0 1, \quad D_1 e_0 = e_0 + e_1,$$

$$4n + 1 = n_{k-1} n_{k-2} \ldots n_1 n_0 01, \quad D_0 D_1 e_0 = 3e_0,$$

$$4n + 3 = n_{k-1} n_{k-2} \ldots n_1 n_0 11, \quad D_1 D_1 e_0 = 3e_0 + e_1$$

we reproduce Sillke’s result [9] that

$$g(4n + 1) = 3g(n), \quad g(4n + 3) = g(2n + 1) + 2g(n).$$

Also, the maximal eigenvalue $\mu_1$ of $D_0 + D_1$ is $1 + \sqrt{5}$; therefore

$$\frac{\ln(E(g(N)))}{\ln(n)} \to \frac{\ln(1 + \sqrt{5})}{\ln(2)} - 1 = \frac{\ln(\varphi)}{\ln(2)} = 0.6942419136306173017387902...$$

as $n \to \infty$, where $\varphi$ is the Golden mean. This constant is not new: see [10].

### 0.2. Quadrinomials.

Define $g_3(n)$ to be the number of odd coefficients in $(1 + x + x^2 + x^3)^n$. This extends our earlier definitions $f = g_1$ and $g = g_2$. Define $P(i, j)$ to be the $(i, j)^{th}$ entry of Pascal’s quadrinomial triangle mod 2:

$$\begin{pmatrix} P(0, 0) & P(0, 1) & P(0, 2) \\ P(1, 0) & P(1, 1) & P(1, 2) & P(1, 3) & P(1, 4) & P(1, 5) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$ 

All values of $P(i, j)$ in the upper right portion are 0s. Let $\ell = 3$,

$$A_0(j) = P(o, j), \quad 0 \leq j < \ell$$
and observe that $A_0 = (1, 0, 0)$. For each $0 \leq s \leq 1$, $0 \leq t \leq 1$, let

$$b^{s,t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j \leq \ell$$

and observe that the four vectors

$$b^{0,0}_0 = (1, 0, 0), \quad b^{1,0}_0 = (1, 1, 0), \quad b^{0,1}_0 = (0, 0, 0), \quad b^{1,1}_0 = (1, 1, 0)$$

contain a nonzero vector $(1, 1, 0) \neq A_0$. Let $A_1 = b^{1,0}_0 = (1, 1, 0)$. No refinement of $b^{s,t}_0$ is necessary and we let

$$B^{s,t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j \leq \ell.$$ 

It follows that

$$B^{0,0}_0 = A_0, \quad B^{1,0}_0 = A_1, \quad B^{0,1}_0 = 0, \quad B^{1,1}_0 = A_1$$

but we are not yet done. Let

$$b^{s,t}_1(j) = P(2(2o + s) + 1, 2(2j + t) + 0), \quad 0 \leq j < \ell$$

which is obtained by replacing $(o, j)$ in the expression for $b^{1,0}_0 = A_1$ by $(2o + s, 2j + t)$. Observe that

$$b^{0,0}_1 = (1, 0, 0), \quad b^{1,0}_1 = (1, 0, 1), \quad b^{0,1}_1 = (0, 0, 0), \quad b^{1,1}_1 = (0, 0, 0)$$

contain a nonzero vector $(1, 0, 1) \neq A_0, A_1$. Let $A_2 = b^{1,0}_1 = (1, 0, 1)$. No refinement of $b^{s,t}_1$ is necessary and we let

$$B^{s,t}_1(j) = P(2(2o + s) + 1, 2(2j + t) + 0), \quad 0 \leq j < \ell.$$ 

It follows that

$$B^{0,0}_1 = A_0, \quad B^{1,0}_1 = A_2, \quad B^{0,1}_1 = A_0, \quad B^{1,1}_1 = 0$$

but we are not yet done. Let

$$b^{s,t}_2(j) = P(2(2o + s) + 1, 2(2j + t) + 0) + 0, \quad 0 \leq j < \ell$$

which is obtained by replacing $(o, j)$ in the expression for $b^{1,0}_1 = A_2$ by $(2o + s, 2j + t)$. Observe that

$$b^{0,0}_2 = (1, 1, 0), \quad b^{1,0}_2 = (1, 0, 1), \quad b^{0,1}_2 = (0, 0, 0), \quad b^{1,1}_2 = (1, 0, 1)$$
and this completes the iterative portion of the algorithm. Thus 

\[ B_{2}^{s,t}(j) = P(2(2o + s) + 1) + 1, 2(2j + t) + 0) + 0, \quad 0 \leq j < \ell. \]

It follows that

\[ B_{2}^{0,0} = A_1, \quad B_{2}^{1,0} = A_2, \quad B_{2}^{0,1} = 0, \quad B_{2}^{1,1} = A_2 \]

and this completes the iterative portion of the algorithm. Thus \( m = 3 \). The definitions of \( m \times m \) matrices

\[
D_0 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}
\]

and \( m \)-vectors \( e_0 = (1, 0, 0) \), \( w = (1, 2, 2) \) are as before. Likewise,

\[ g_3(n) = \# \{ j : P(n, j) = 1 \} = w^TD_{n_{k-1}}D_{n_{k-2}} \cdots D_{n_1}D_{n_0}e_0 \]

as before, where \( n_{k-1}n_{k-2} \ldots n_1n_0 \) is the binary expansion of \( n \).

It is easy to see that \( g_3(2n) = g_3(n) \). From

\[
2n + 1 = n_{k-1}n_{k-2} \ldots n_1n_01, \quad D_1e_0 = 2e_1,
\]
\[
4n + 1 = n_{k-1}n_{k-2} \ldots n_1n_001, \quad D_0D_1e_0 = 4e_0,
\]
\[
4n + 3 = n_{k-1}n_{k-2} \ldots n_1n_011, \quad D_1D_1e_0 = 2e_2,
\]
\[
8n + 1 = n_{k-1}n_{k-2} \ldots n_1n_0001, \quad D_0D_0D_1e_0 = 4e_0,
\]
\[
8n + 3 = n_{k-1}n_{k-2} \ldots n_1n_0011, \quad D_0D_1D_1e_0 = 2e_1,
\]
\[
8n + 5 = n_{k-1}n_{k-2} \ldots n_1n_0101, \quad D_1D_0D_1e_0 = 8e_1,
\]
\[
8n + 7 = n_{k-1}n_{k-2} \ldots n_1n_0111, \quad D_1D_1D_1e_0 = 4e_2
\]

we deduce that

\[ g_3(8n + 1) = g_3(4n + 1), \quad g_3(8n + 3) = g_3(2n + 1), \]
\[ g_3(8n + 5) = 4g_3(2n + 1), \quad g_3(8n + 7) = 2g_3(4n + 3). \]

Also, the maximal eigenvalue \( \mu_1 \) of \( D_0 + D_1 \) is 3; it is interesting that the same constant

\[
\frac{\ln(\mathbb{E}(g_3(N)))}{\ln(n)} \to \frac{\ln(3/2)}{\ln(2)}
\]

appears here as for \( g_1(N) \).
0.3. Trinomials II. Define $h_3(n)$ to be the number of odd coefficients in $(1 + x + x^3)^n$. This extends our earlier definition $g = h_2$. Define $P(i,j)$ to be the $(i,j)$th entry of the associated triangle mod 2:

\[
\begin{pmatrix}
P(0, 0) & P(0, 1) & P(0, 2) \\
P(1, 0) & P(1, 1) & P(1, 2) & P(1, 3) & P(1, 4) & P(1, 5)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

All values of $P(i,j)$ in the upper right portion are 0s. Let $\ell = 3$, 

\[A_0(j) = P(o,j), \quad 0 \leq j < \ell\]

and observe that $A_0 = (1, 0, 0)$. For each $0 \leq s \leq 1$, $0 \leq t \leq 1$, let

\[b^{s,t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell\]

and observe that the four vectors

\[b^{0,0}_0 = (1, 0, 0), \quad b^{0,1}_0 = (1, 0, 0), \quad b^{0,1}_0 = (0, 0, 0), \quad b^{1,1}_0 = (1, 1, 0)\]

contain a nonzero vector $(1, 1, 0) \neq A_0$. Let $A_1 = b^{1,1}_0 = (1, 1, 0)$. No refinement of $b^{s,t}_0$ is necessary and we let

\[B^{s,t}_0(j) = P(2o + s, 2j + t), \quad 0 \leq j < \ell.\]

It follows that

\[B^{0,0}_0 = A_0, \quad B^{1,0}_0 = A_0, \quad B^{0,1}_0 = 0, \quad B^{1,1}_0 = A_1\]

but we are not yet done. Let

\[b^{s,t}_1(j) = P(2(2o + s) + 1, 2(2j + t) + 1), \quad 0 \leq j < \ell\]

which is obtained by replacing $(o,j)$ in the expression for $b^{1,1}_0 = A_1$ by $(2o + s, 2j + t)$. Observe that

\[b^{0,0}_1 = (1, 0, 0), \quad b^{1,0}_1 = (1, 1, 1), \quad b^{0,1}_1 = (1, 0, 0), \quad b^{1,1}_1 = (0, 1, 0)\]

contain nonzero vectors $(1, 1, 1), (0, 1, 0) \neq A_0, A_1$. Let $A_2 = b^{1,0}_1 = (1, 1, 1)$. Setting $A_3 = b^{1,1}_1$, however, violates the mandate that $A_i(0) = 1$ always. We thus refine the definition of $b^{s,t}_1$:

\[B^{s,t}_1(j) = \begin{cases}
P(2(2o + s) + 1, 2(2j + t) + 1) & \text{if } (s,t) \neq (1,1), \\
P(2(2o + s) + 1, 2(2(j + c) + t) + 1) & \text{if } (s,t) = (1,1)
\end{cases}\]
where $c = 1$. It follows that

$$B_1^{0,0} = A_0, \quad B_1^{1,0} = A_2, \quad B_1^{0,1} = A_0, \quad B_1^{1,1} = A_0$$

but we are not yet done. Let

$$b_2^{s,t}(j) = P(2(2(2o + s) + 1) + 1, 2(2j + t) + 0) + 1), \quad 0 \leq j < \ell$$

which is obtained by replacing $(o, j)$ in the expression for $b_1^{1,0} = A_2$ by $(2o + s, 2j + t)$. Observe that

$$b_2^{0,0} = (1, 1, 0), \quad b_2^{1,0} = (1, 0, 1), \quad b_2^{0,1} = (1, 0, 0), \quad b_2^{1,1} = (0, 0, 1)$$

contain nonzero vectors $(1, 0, 1), (0, 0, 1) \neq A_0, A_1, A_2$. Let $A_3 = b_2^{1,0} = (1, 0, 1)$. Setting $A_4 = b_2^{1,1}$, however, violates the mandate that $A_4(0) = 1$ always. We thus refine the definition of $b_2^{s,t}$:

$$B_2^{s,t}(j) = \begin{cases} 
P(2(2(2o + s) + 1) + 1, 2(2j + t) + 0) + 1) & \text{if } (s, t) \neq (1, 1), \\
P(2(2(2o + s) + 1) + 1, 2(2(j + c) + t) + 0) + 1) & \text{if } (s, t) = (1, 1)
\end{cases}$$

where $c = 2$. It follows that

$$B_2^{0,0} = A_1, \quad B_2^{1,0} = A_3, \quad B_2^{0,1} = A_0, \quad B_2^{1,1} = A_0$$

but we are not yet done. Let

$$b_3^{s,t}(j) = P(2(2(2o + s) + 1) + 1, 2(2(2j + t) + 0) + 0) + 1), \quad 0 \leq j < \ell$$

which is obtained by replacing $(o, j)$ in the expression for $b_2^{1,0} = A_3$ by $(2o + s, 2j + t)$. Observe that

$$b_3^{0,0} = (1, 1, 0), \quad b_3^{1,0} = (1, 1, 0), \quad b_3^{0,1} = (0, 0, 0), \quad b_3^{1,1} = (1, 0, 1)$$

encompass only $A_1, A_3$ and 0. No refinement of $b_3^{s,t}$ is hence necessary and we let

$$B_3^{s,t}(j) = P(2(2(2o + s) + 1) + 1, 2(2(j + c) + t) + 0) + 0) + 1), \quad 0 \leq j < \ell.$$ 

It follows that

$$B_3^{0,0} = A_1, \quad B_3^{1,0} = A_1, \quad B_3^{0,1} = 0, \quad B_3^{1,1} = A_3$$

and this completes the iterative portion of the algorithm. Thus $m = 4$. The definitions of $m \times m$ matrices

$$D_0 = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
and $m$-vectors $e_0 = (1, 0, 0, 0)$, $w = (1, 2, 3, 2)$ are as before. Likewise,

$$h_3(n) = \# \{ j : P(n, j) = 1 \} = w^T D_{n_{k-1}} D_{n_{k-2}} \cdots D_{n_1} D_{n_0} e_0$$

as before, where $n_{k-1} n_{k-2} \cdots n_1 n_0$ is the binary expansion of $n$.

It is easy to see that $h_3(2n) = h_3(n)$. Omitting details, we deduce that

$$h_3(4n + 1) = 3h_3(n), \quad h_3(8n + 3) = h_3(2n + 1) + 4h_3(n),$$

$$h_3(16n + 7) = h_3(8n + 3) + h_3(2n + 1) + 3h_3(n),$$

$$h_3(16n + 15) = 2h_3(8n + 7) + h_3(2n + 1) - 2h_3(n).$$

Also, the eigenvalues of $D_0 + D_1$ have minimal polynomial $\xi^4 - 3\xi^3 - 2\xi^2 + 2\xi + 4$ and

$$\frac{\ln(E(h_3(N)))}{\ln(n)} \to 0.7274509132400228143266172...$$

as $n \to \infty$.

### 0.4. Quintinomials

Define $g_4(n)$ to be the number of odd coefficients in $(1 + x + x^2 + x^3 + x^4)^n$. Define $P(i, j)$ to be the $(i, j)^{th}$ entry of Pascal’s quintinomial triangle mod 2:

$$\begin{pmatrix}
P(0,0) & P(0,1) & P(0,2) & P(0,3) \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) \\
P(1,4) & P(1,5) & P(1,6) & P(1,7)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

Let $\ell = 4$ and $A_0 = (1, 0, 0, 0)$. We simply summarize:

$$A_1 = b_0^{1,0} = (1, 1, 1, 0), \quad A_2 = b_0^{1,1} = (1, 1, 0, 0);$$

$$B_0^{0,0} = A_0, \quad B_0^{1,0} = A_1, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_2;$$

$$A_3 = b_1^{1,0} = (1, 1, 1, 1);$$

$$B_1^{0,0} = A_2, \quad B_1^{1,0} = A_3, \quad B_1^{0,1} = A_0, \quad B_1^{1,1} = A_0;$$

$$B_2^{0,0} = A_0, \quad B_2^{1,0} = A_0, \quad B_2^{0,1} = A_0, \quad B_2^{1,1} = A_0;$$

$$B_3^{0,0} = A_2, \quad B_3^{1,0} = A_2, \quad B_3^{0,1} = A_2, \quad B_3^{1,1} = A_2;$$

$$D_0 = \begin{pmatrix}
1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad D_1 = \begin{pmatrix}
0 & 1 & 2 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 0
\end{pmatrix}$$

and $w = (1, 3, 2, 4)$.

The eigenvalues of $D_0 + D_1$ have minimal polynomial $\xi^4 - \xi^3 - 6\xi^2 - 4\xi - 16$ and

$$\frac{\ln(E(g_4(N)))}{\ln(n)} \to 0.7896418505307685639015472...$$

as $n \to \infty$. 
0.5. Trinomials III. Define \( h_4(n) \) to be the number of odd coefficients in \((1 + x + x^4)^n\). Define \( P(i, j) \) to be the \((i, j)\)th entry of the associated triangle mod 2:

\[
\begin{pmatrix}
P(0,0) & P(0,1) & P(0,2) & P(0,3) \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) \\
& P(1,4) & P(1,5) & P(1,6) \\
& & P(1,7)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \( \ell = 4 \) and \( A_0 = (1, 0, 0, 0) \). We simply summarize:

\[
A_1 = b_1^{1,0} = (1, 0, 1, 0); \\
B_0^{0,0} = A_0, \quad B_0^{1,0} = A_1, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_0; \\
A_2 = b_0^{1,0} = (1, 1, 0, 0), \quad A_3 = b_1^{1,0} = (1, 1, 1, 1); \\
B_1^{0,0} = A_2, \quad B_1^{1,0} = A_3, \quad B_1^{0,1} = 0, \quad B_1^{1,1} = A_2; \\
A_4 = b_2^{1,0} = (1, 1, 1, 0); \\
B_2^{0,0} = A_0, \quad B_2^{1,0} = A_4, \quad B_2^{0,1} = A_0, \quad B_2^{1,1} = A_0; \\
A_5 = b_3^{1,0} = (1, 0, 0, 1); \\
B_3^{0,0} = A_2, \quad B_3^{1,0} = A_5, \quad B_3^{0,1} = A_2, \quad B_3^{1,1} = A_2; \\
A_6 = b_4^{1,0} = (1, 0, 1, 1); \\
B_4^{0,0} = A_2, \quad B_4^{1,0} = A_6, \quad B_4^{0,1} = A_0, \quad B_4^{1,1} = A_2; \\
A_7 = b_5^{1,0} = (1, 1, 0, 1); \\
B_5^{0,0} = A_0, \quad B_5^{1,0} = A_0, \quad B_5^{0,1} = A_0, \quad B_5^{1,1} = A_7; \\
B_6^{0,0} = A_2, \quad B_6^{1,0} = A_7, \quad B_6^{0,1} = A_0, \quad B_6^{1,1} = A_5; \\
B_7^{0,0} = A_0, \quad B_7^{1,0} = A_2, \quad B_7^{0,1} = A_2, \quad B_7^{1,1} = A_4;
\]

\[
D_0 = \begin{pmatrix}
1 & 0 & 2 & 0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad D_1 = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

and \( w = (1, 2, 2, 4, 3, 2, 3, 3) \).

The eigenvalues of \( D_0 + D_1 \) have minimal polynomial \( \xi^5 - 3\xi^4 - 2\xi^2 - 8\xi + 8 \) and

\[
\frac{\ln(E(h_4(N)))}{\ln(n)} \to 0.7362115557393079316549209... \]

as \( n \to \infty \).
0.6. Sextinomials. Define $g_5(n)$ to be the number of odd coefficients in $(1 + x + \ldots + x^4 + x^5)^n$. Define $P(i, j)$ to be the $(i, j)^{\text{th}}$ entry of Pascal’s sextinomial triangle mod 2:

$$
\begin{pmatrix}
P(0,0) & P(0,1) & \ldots & P(0,4) \\
P(1,0) & P(1,1) & \ldots & P(1,4) & P(1,5) & \ldots & P(1,9)
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Let $\ell = 5$ and $A_0 = (1, 0, 0, 0, 0)$. We simply summarize:

$$
A_1 = b_0^{1,0} = (1, 1, 1, 0, 0);
B_0^{0,0} = A_0, \quad B_0^{1,0} = A_1, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_1;
A_2 = b_1^{0,0} = (1, 1, 0, 0, 0);
B_1^{0,0} = A_2, \quad B_1^{1,0} = A_1, \quad B_1^{0,1} = A_0, \quad B_1^{1,1} = A_1;
A_3 = b_1^{1,0} = (1, 0, 0, 1, 0);
B_2^{0,0} = A_0, \quad B_2^{1,0} = A_3, \quad B_2^{0,1} = A_0, \quad B_2^{1,1} = 0;
A_4 = b_2^{1,0} = (1, 1, 0, 1, 1);
B_3^{0,0} = A_0, \quad B_3^{1,0} = A_4, \quad B_3^{0,1} = A_0, \quad B_3^{1,1} = A_3;
A_5 = b_2^{0,0} = (1, 0, 1, 0, 0);
B_4^{0,0} = A_5, \quad B_4^{1,0} = A_3, \quad B_4^{0,1} = A_2, \quad B_4^{1,1} = A_3;
B_5^{0,0} = A_2, \quad B_5^{1,0} = A_3, \quad B_5^{0,1} = 0, \quad B_5^{1,1} = A_3;
$$

$$
D_0 = \begin{pmatrix}
1 & 1 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad D_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

and $w = (1, 3, 2, 2, 4, 2)$.

The eigenvalues of $D_0 + D_1$ have minimal polynomial $\zeta^6 - 4\zeta^5 + \zeta^4 - \zeta^3 + 8\zeta^2 + 11\zeta + 8$ and

$$
\frac{\ln(E(g_5(N)))}{\ln(n)} \to 0.8194694621655401465959376\ldots
$$
as $n \to \infty$. 

0.7. Septinomials. Define \( g_6(n) \) to be the number of odd coefficients in \((1 + x + \cdots + x^5 + x^6)^n \). Define \( P(i, j) \) to be the \((i, j)^{th}\) entry of Pascal’s septinomial triangle mod 2:

\[
\begin{pmatrix}
P(0,0) & P(0,1) & \cdots & P(0,5) \\
P(1,0) & P(1,1) & \cdots & P(1,5) & P(1,6) & P(1,7) & \cdots & P(1,11)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \( \ell = 6 \) and \( A_0 = (1, 0, 0, 0, 0, 0) \). We simply summarize:

\[
A_1 = b_0^{1,0} = (1, 1, 1, 0, 0, 0), \quad A_2 = b_0^{1,1} = (1, 1, 1, 0, 0, 0);
\]

\[
B_0^{0,0} = A_0, \quad B_0^{1,0} = A_1, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_2; \quad A_3 = b_1^{0,0} = (1, 1, 0, 0, 0, 0);
\]

\[
B_1^{0,0} = A_3, \quad B_1^{1,0} = A_3, \quad B_1^{0,1} = A_3, \quad B_1^{1,1} = A_3; \quad A_4 = b_1^{1,0} = (1, 1, 1, 1, 1, 0);
\]

\[
B_2^{0,0} = A_3, \quad B_2^{1,0} = A_4, \quad B_2^{0,1} = A_3, \quad B_2^{1,1} = A_3; \quad A_5 = b_2^{0,0} = (1, 1, 1, 1, 1, 1);
\]

\[
B_3^{0,0} = A_0, \quad B_3^{1,0} = A_0, \quad B_3^{0,1} = A_0, \quad B_3^{1,1} = A_0; \quad A_6 = b_2^{1,0} = (1, 1, 1, 1, 1, 1);
\]

\[
B_4^{0,0} = A_2, \quad B_4^{1,0} = A_5, \quad B_4^{0,1} = A_3, \quad B_4^{1,1} = A_0; \quad A_7 = b_3^{0,0} = (1, 1, 1, 1, 1, 1);
\]

\[
B_5^{0,0} = A_2, \quad B_5^{1,0} = A_2, \quad B_5^{0,1} = A_2, \quad B_5^{1,1} = A_2; \quad A_8 = b_3^{1,0} = (1, 1, 1, 1, 1, 1);
\]

\[
D_0 = \begin{pmatrix}
1 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad D_1 = \begin{pmatrix}
0 & 0 & 0 & 2 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 2 \\
0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

and \( w = (1, 4, 3, 2, 5, 6) \).

The eigenvalues of \( D_0 + D_1 \) have minimal polynomial \( \xi^6 - \xi^5 - 2\xi^4 - 28\xi^3 + 16\xi + 64 \) and

\[
\frac{\ln(E(g_6(N))))}{\ln(n)} \rightarrow 0.8317963967344406899938931... \]

as \( n \rightarrow \infty \).
0.8. Lyapunov Exponents. All $D_0$ matrices shown in this paper satisfy $\text{rank}(D_0^q) = 1$ for some positive integer $q$; further, there is a change of coordinates under which $D_0^q$ is transformed into the matrix whose $(0,0)^{th}$ entry is 1 and all of whose other entries are 0. That is, there is an invertible $m \times m$ matrix $Q$ with $Q^{-1}D_0^qQ = e_0e_0^T$. Define $D'_0 = Q^{-1}D_0Q$, $D'_1 = Q^{-1}D_1Q$ and let $z_0, z_1, \ldots, z_{k-2}, z_{k-1}$ denote a sequence of independent random coin tosses (heads=1 and tails=0 with equal probability). The Lyapunov exponent corresponding to random products of $D'_0$ and $D'_1$:

$$\lambda = \lim_{k \to \infty} \frac{1}{k} \ln \left\| D'_0 D'_1 \cdots D'_{z_{k-2}} D'_{z_{k-1}} \right\|$$

exists almost surely. It turns out that $\lambda/\ln(2)$ is precisely what we seek to characterize typical growth rates of $g_i(N)$ and $h_j(N)$.

Let $\chi(q^0)$ denote the set of all finite binary words $z$ with no subwords $0^q$ and with rightmost digit 1. $0^1$ means 0; $0^2$ means 00; $0^3$ means 000.) Let $\ell(z)$ denote the length of $z$. For example, 1111 is the only word of length 4 in $\chi(0)$; 0101, 0111, 1011, 1101, 1111 are the only words of length 4 in $\chi(00)$; the set $\chi(000)$ additionally contains 0011 and 1001. It is natural to sort the elements of $\chi(q^0)$ in terms of increasing length.

Write $z = z_0z_1\ldots z_{k-2}z_{k-1}$ and $D'_z = D'_0z_1 \cdots D'_{z_{k-2}} D'_{z_{k-1}}$. By $D'_z(0,0)$ is meant the upper left corner entry of $D'_z$.

Extending earlier work by Pincus [11] and Lima & Rahibe [12], Moshe [13] proved that

$$\lambda = \frac{1}{2^{q+1}(2^q - 1)} \sum_{z \in \chi(q^0)} \frac{1}{2^{\ell(z)}} \ln |D'_z(0,0)|.$$ 

This series is attractive, but computationally difficult since the number of words in $\chi(q^0)$ of length $k$ grows exponentially with increasing $k$. Summation of the series, coupled with Wynn’s $\varepsilon$-process for accelerating convergence, serves as our primary method for calculating $\lambda$. Our secondary method is based on the cycle expansion method applied to a corresponding Ruelle dynamical zeta function [14, 15, 16, 17].

In the case of the first trinomial $(1 + x + x^2)^n$, we have $q = 1$,

$$Q = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad D'_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D'_1 = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}$$

hence [6]

$$\lambda = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{2^k} \ln |(D'_1)^k(0,0)| = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{2^k} \ln \left( \frac{2^{k+2} - (-1)^k}{3} \right)$$

$$= 0.4299474333424527201146970...$$

$$= \ln(1.537176717823579495901403...)$$
Odd Entries in Pascal's Trinomial Triangle

and

$$\frac{E(\ln(g_2(N)))}{\ln(n)} \rightarrow \frac{\lambda}{\ln(2)} = 0.6202830299260946960737425...$$

as $n \to \infty$. This is smaller than $\ln(\varphi)/\ln(2) = 0.694...$, as discussed earlier.

In the case of the quadrinomial $(1 + x + x^2 + x^3)^n$, we have $q = 2$,

$$Q = \begin{pmatrix} 1 & -2 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_0' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_1' = \begin{pmatrix} 4 & -4 & -6 \\ 0 & 2 & 1 \\ 2 & -4 & -4 \end{pmatrix}$$

hence

$$\lambda = \frac{1}{24} \sum_{z \in \chi^{(00)}} \frac{1}{q^t(z)} \ln |D'_z(0, 0)| = 0.34657359...$$

and $\lambda/\ln(2) = 0.49999999... < 0.584...$. We conjecture that $\lambda/\ln(2)$ equals 1/2 (the typical growth rate for binomials) and prove this to be true in section [0.12]. The cycle expansion converges slowly in this case, therefore Moshe's technique is helpful here.

In the case of the second trinomial $(1 + x + x^3)^n$, we have $q = 2$,

$$Q = \begin{pmatrix} 1 & -3 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_0' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_1' = \begin{pmatrix} 3 & -6 & -2 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & -3 & -2 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

hence $\lambda = 0.45454538229305...$ and $\lambda/\ln(2) = 0.65577036889316... < 0.727...$. A related example appears in [13].

In the case of the quintinomial $(1 + x + x^2 + x^3 + x^4)^n$, we have $q = 2$,

$$Q = \begin{pmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_0' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_1' = \begin{pmatrix} 5 & -10 & -8 & 4 \\ 1 & -1 & -2 & -2 \\ 1 & -3 & -2 & 4 \\ 0 & 1 & 0 & -2 \end{pmatrix}$$

hence $\lambda = 0.504253705692...$ and $\lambda/\ln(2) = 0.727484320552... < 0.789...$. 

In the case of the third trinomial $(1 + x + x^4)^n$, we have $q = 2$,

$$Q = \begin{pmatrix} 1 & -2 & -2 & 0 & -1 & -2 & -1 & -1 \\ 0 & 1 & 0 & -2 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
hence $\lambda = 0.45759385431410$ and $\lambda/\ln(2) = 0.66016838436022... < 0.736....$

In the case of the sextinomial $(1 + x + \cdots + x^4 + x^5)^n$, we have $q = 3$,

$$Q = \begin{pmatrix}
1 & -1 & -2 & -2 & -1 & -2 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}, \quad D_0' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad D_1' = \begin{pmatrix}
3 & 0 & -2 & -8 & -4 & -2 \\
0 & 1 & 0 & -1 & -4 & -2 \\
0 & 0 & 1 & 0 & -1 & -2 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

hence $\lambda = \frac{1}{112} \sum_{z \in \chi(000)} \frac{1}{2^{\ell(z)}} \ln |D_2'(0,0)| = 0.5344481528...$

and $\lambda/\ln(2) = 0.7710456996... < 0.819....$

In the case of the septinomial $(1 + x + \cdots + x^5 + x^6)^n$, we have $q = 3$,

$$Q = \begin{pmatrix}
1 & -1 & -2 & -2 & -1 & -2 & -1 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0
\end{pmatrix}, \quad D_0' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad D_1' = \begin{pmatrix}
6 & -8 & -8 & -10 & -4 & -6 & -4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

hence $\lambda = 0.53765282...$ and $\lambda/\ln(2) = 0.77566905... < 0.831....$

Let us conclude by mentioning relevant sequences in Sloane’s online encyclopedia. A001316 is $f(n)$; A000120 is $\ln(f(n))/\ln(2)$; A006046 is $\sum_{k<n} f(k)$; A000788 is $\sum_{k<n} \ln(f(k))/\ln(2)$. A071053 is $g(n)$; A134659 is $\sum_{k<n} g(k)$. A134660 is $g_3(n)$; A036555 is $\ln(g_3(n))/\ln(2)$. A134661 is $h_3(n)$. A134662 is $h_4(n)$. A007318, A027907, A008287, A035343, A063260, A063265 are Pascal’s triangles associated with $(1 + x + \cdots + x^r + x^r)^n$ for $r = 1, \ldots, 6$; A038717 and A134663 are likewise for $(1 + x + x^3)^n$ and $(1 + x + x^4)^n$. It is well-known that $\ln(f(n))/\ln(2)$ is the number of 1s in the binary expansion of $n$, but scarcely noticed that $\ln(g_3(n))/\ln(2)$ is the number of 1s in the binary expansion of $3n$. 
0.9. Pascal’s Rhombus. The sequence of polynomials giving Pascal’s trinomial triangle arises from the first-order recurrence

\[ p_n(x) = (1 + x + x^2)p_{n-1}(x), \quad p_0(x) = 1. \]

Pascal’s rhombus [18], by contrast, arises from the second-order recurrence

\[ p_n(x) = (1 + x + x^2)p_{n-1}(x) + x^2p_{n-2}(x), \quad p_1(x) = 1 + x + x^2, \quad p_0(x) = 1. \]

In this addendum, we perform the same analysis as in the preceding. Define \( u(n) \) to be the number of odd coefficients in \( p_n(x) \). Define \( P(i, j) \) to be the \((i, j)\)th entry of the associated “rhombus” mod 2:

\[
\begin{pmatrix}
P(0,0) & P(0,1) & P(0,2) & P(0,3) \\
P(1,0) & P(1,1) & P(1,2) & P(1,3) \\
P(2,0) & P(2,1) & P(2,2) & P(2,3) \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

The objects \( A_i \) and \( B_i \) are no longer \( \ell \)-vectors but \( 2 \times \ell \) matrices. It is mandatory that \( A_i(0,0) = 1 \) or \( A_i(1,0) = 1 \) (or both) for every \( i \). Let \( \ell = 4 \) and

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.
\]

We simply summarize:

\[
A_1 = b_0^{0,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = b_0^{1,0} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix};
\]

\[
B_0^{0,0} = A_1, \quad B_0^{1,0} = A_2, \quad B_0^{0,1} = 0, \quad B_0^{1,1} = A_0;
\]

\[
A_3 = b_1^{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad A_4 = B_1^{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};
\]

\[
B_1^{0,0} = A_0, \quad B_1^{1,0} = A_2, \quad B_1^{0,1} = A_3, \quad B_1^{1,1} = A_4; \quad B_2^{0,0} = A_1, \quad B_2^{1,0} = A_0, \quad B_2^{0,1} = A_1, \quad B_2^{1,1} = A_0; \quad B_3^{0,0} = A_4, \quad B_3^{1,0} = A_1, \quad B_3^{0,1} = A_4, \quad B_3^{1,1} = A_1; \quad B_4^{0,0} = A_3, \quad B_4^{1,0} = A_1, \quad B_4^{0,1} = A_4, \quad B_4^{1,1} = 0;
\]

\[
D_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.
\]
and \( w = (1, 1, 2, 0, 0) \). The maximal eigenvalue \( \mu_1 \) of \( D_0 + D_1 \) is \((3 + \sqrt{17})/2\); therefore
\[
\frac{\ln(E(u(N)))}{\ln(n)} \to \frac{\ln((3 + \sqrt{17})/4)}{\ln(2)} = 0.8325063835804514437981667... 
\]
as \( n \to \infty \). To compute \( \lim_{n \to \infty} E(\ln(u(N))) / \ln(n) \) via Moshe’s technique requires a binary word \( z \) for which the product \( D_z \) satisfies \( \text{rank}(D_z) = 1 \). No such word \( z \) exists, therefore our primary method is inapplicable here. Our secondary method gives \( \lambda = 0.57331379313... \), hence \( \lambda / \ln(2) = 0.82711696622... \).

We mention the Fibonacci polynomials
\[
p_n(x) = xp_{n-1}(x) + p_{n-2}(x), \quad p_1(x) = x, \quad p_0(x) = 1
\]
and that the number \( v(n) \) of odd coefficients in \( p_n(x) \) is the \( n^{th} \) term of Stern’s sequence [19, 20, 21]
\[
v(2n + 1) = v(n), \quad v(2n) = v(n) + v(n - 1).
\]
It follows that
\[
D_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad w = (1, 0);
\]
\[
\frac{\ln(E(v(N)))}{\ln(n)} \to \frac{\ln(3/2)}{\ln(2)}, \quad \frac{E(\ln(v(N)))}{\ln(n)} \to \frac{\lambda}{\ln(2)} = 0.571613901650254...
\]
as \( n \to \infty \). The estimate \( \lambda = 0.396212564297744... \) is new (as far as we know), but the challenge of computing \( \lambda \) was posed long ago [22].

Relevant sequences in Sloane’s online encyclopedia are A059319 for \( u(n) \), A059317 for Pascal’s rhombus, A002487 for \( v(n) \) and A049310 for “Fibonacci’s rhombus”. We recall Glaisher’s theorem that \( f(n) \) is the number of odd binomial coefficients of the form \((n \choose m)\), \( 0 \leq m \leq n \); this is mirrored by Carlitz’s theorem that \( v(n) \) is the number of odd binomial coefficients of the form \((n-m \choose m)\), \( 0 \leq 2m \leq n \).

0.10. Extreme Values. The function \( f(n) \) has “maximum growth” \( \approx n^1 \) in the sense that
\[
\limsup_{n \to \infty} \frac{1}{\ln(n)} \ln(f(n)) = 1.
\]
To evaluate the limit superior, we note that
\[
f(n) < f(2^k - 1) = 2^k \quad \text{for all } n < 2^k - 1
\]
for each \( k = 0, 1, 2, \ldots \). No other disjoint subsequence possesses this property and

\[
\lim_{k \to \infty} \frac{\ln(2^k)}{\ln(2^k - 1)} = 1.
\]

The functions \( g(n) \) and \( u(n) \) likewise satisfy

\[
\limsup_{n \to \infty} \frac{1}{\ln(n)} \ln(g(n)) = \limsup_{n \to \infty} \frac{1}{\ln(n)} \ln(u(n)) = 1.
\]

In the case of \( g(n) \), we have

\[
g(n) < \begin{cases} 
   g(2k-1) = \frac{2^{k+2} - (-1)^k}{3} & \text{for all } n < 2^k - 1, \\
   g(3 \cdot 2^{k+1} - 1) = 3 \cdot 2^{k+3} + (-1)^k & \text{for all } n < 3 \cdot 2^{k+1} - 1, \\
   g(11 \cdot 2^{k+1} - 1) = 3(2^{k+3} + (-1)^k) & \text{for all } n < 11 \cdot 2^{k+1} - 1
\end{cases}
\]

for each \( k = 0, 1, 2, \ldots \). No other disjoint subsequence possesses this property and

\[
\lim_{k \to \infty} \frac{\ln\left(\frac{2^{k+2} - (-1)^k}{3}\right)}{\ln(2^k - 1)} = \lim_{k \to \infty} \frac{\ln(3(2^{k+3} + (-1)^k))}{\ln(11 \cdot 2^{k+1} - 1)} = 1.
\]

In the case of \( u(n) \), we have

\[
u(n) < \begin{cases} 
   u(5) = 6 & \text{for all } n < 5, \\
   u(37) = 45 & \text{for all } n < 37, \\
   u(2^k - 1) = \frac{5 \cdot 2^{k+3} + (-1)^k}{3} & \text{for all } n < 2^k - 1, \\
   u(5 \cdot 2^{k+1} - 1) = \frac{5 \cdot 4^{k+2} + 12k + 1}{3} & \text{for all } n < 5 \cdot 2^{k+1} - 1, \\
   u(2 \cdot 4^{k+2} - 7) = \frac{5 \cdot 4^{k+2} + 12k + 1}{3} & \text{for all } n < 2 \cdot 4^{k+2} - 7
\end{cases}
\]

for each \( k = 0, 1, 2, \ldots \). No other disjoint subsequence possesses this property and

\[
\lim_{k \to \infty} \frac{\ln\left(\frac{5 \cdot 2^{k+3} + (-1)^k}{3}\right)}{\ln(5 \cdot 2^{k+1} - 1)} = \lim_{k \to \infty} \frac{\ln\left(\frac{5 \cdot 4^{k+2} + 12k + 1}{3}\right)}{\ln(2 \cdot 4^{k+2} - 7)} = 1.
\]

By contrast, the function \( v(n) \) satisfies [20]

\[
\limsup_{n \to \infty} \frac{1}{\ln(n)} \ln(v(n)) = \frac{\ln(\phi)}{\ln(2)}
\]
where \( \varphi = (1 + \sqrt{5})/2 \). Let \( a_0 = 0, a_1 = 1, a_j = a_{j-1} + a_{j-2} \) denote the Fibonacci sequence. We have

\[
v(n) < \begin{cases} 
  v\left(\frac{2(4^k - 1)}{3}\right) = a_{2k+1} & \text{for all } n < \frac{2(4^k - 1)}{3}, \\
  v\left(\frac{4(4^k - 1) - 1}{3}\right) = a_{2k+2} & \text{for all } n < \frac{4(4^k - 1) - 1}{3},
\end{cases}
\]

for each \( k = 0, 1, 2, \ldots \). No other disjoint subsequence possesses this property and

\[
\lim_{k\to\infty} \frac{\ln(a_{2k+1})}{\ln\left(\frac{2(4^k - 1)}{3}\right)} = \lim_{k\to\infty} \frac{\ln(a_{2k+2})}{\ln\left(\frac{4(4^k - 1) - 1}{3}\right)} = \frac{\ln(\varphi)}{\ln(2)}.
\]

Our analyses in this addendum are aided by the \( D_0, D_1, w \) matrices from earlier sections. “Minimum growth”, defined with limit superior replaced by limit inferior, is not as interesting for three of the cases since

\[
f(2^k) = 2, \quad g(2^k) = 3, \quad v(2^k - 1) = 1
\]

always. The remaining case, \( u(n) \), resists all attempts at simplification.

0.11. Variability. We shall be very brief here. Let \( N \) denote a uniform random integer between 0 and \( n - 1 \). Kirschenhofer \[23\] proved that \( f(N) \) has “typical dispersion” \( \approx n^{\ln(2)/4} \) in the sense that

\[
\text{Var}(\ln(f(N))) \sim \frac{\ln(2)}{4} \ln(n)
\]

as \( n \to \infty \); equivalently,

\[
\lim_{n\to\infty} \frac{1}{\ln(n)} \left[ \frac{1}{n} \sum_{k=0}^{n-1} \ln(f(k))^2 - \left( \frac{1}{n} \sum_{k=0}^{n-1} \ln(f(k)) \right)^2 \right] = \frac{\ln(2)}{4}.
\]

In fact, this is related to a generalized Lyapunov exponent of order two corresponding to random products of \( D_0 = (1) \) and \( D_1 = (2) \). We defer further study of such quantities to a later paper.

Define \( \psi = (5 + \sqrt{17})/4 \) for convenience. As another example, \( v(N) \) has “average dispersion” \( \approx n^{\ln(\psi)/\ln(2)} \) in the sense that

\[
\ln(\text{Var}(v(N))) \sim \frac{\ln(\psi)}{\ln(2)} \ln(n)
\]
as \( n \to \infty \); equivalently,

\[
\lim_{n \to \infty} \frac{1}{\ln(n)} \ln \left[ \frac{1}{n} \sum_{k=0}^{n-1} v(k)^2 - \left( \frac{1}{n} \sum_{k=0}^{n-1} v(k) \right)^2 \right] = \frac{\ln(\psi)}{\ln(2)}.
\]

More details on the latter result can be found in [24, 25, 26, 27, 28, 29, 30]. In fact, \( \psi \) is the leading eigenvalue of \( \frac{1}{2}(\mathcal{D}_0 \otimes \mathcal{D}_0 + \mathcal{D}_1 \otimes \mathcal{D}_1) \), where \( \otimes \) is the direct or Kronecker product of matrices

\[
\mathcal{D}_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{D}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};
\]

thus

\[
\mathcal{D}_0 \otimes \mathcal{D}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \mathcal{D}_1 \otimes \mathcal{D}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

In the same way,

\[
\frac{\ln(\text{Var}(f(N))))}{\ln(n)} \to \frac{\ln(5/2)}{\ln(2)}, \quad \frac{\ln(\text{Var}(g(N))))}{\ln(n)} \to \frac{\ln(\xi)}{\ln(2)}
\]

where \( \xi = 2.813... \) has minimal polynomial \( \xi^3 - 2\xi^2 - 3\xi + 2 \) and

\[
\frac{\ln(\text{Var}(u(N))))}{\ln(n)} \to \frac{\ln(\eta)}{\ln(2)},
\]

where \( \eta = 3.194... \) has minimal polynomial \( 4\eta^7 - 8\eta^6 - 25\eta^5 + 22\eta^4 + 24\eta^3 + 16\eta^2 + \eta - 2 \).

0.12. Proof of Conjecture. Since the Lyapunov exponent is defined as an almostsure limit:

\[
\lambda = \lim_{k \to \infty} \frac{1}{k} \ln \left( \| D_{z_0} D_{z_1} \cdots D_{z_{k-2}} D_{z_{k-1}} \| \right)
\]

we may assume that \( z_0 = 1 \). Every binary word \( z \) thus looks like

\[
1 0^{j_0} 1 0^{j_1} 1 0^{j_2} \cdots 1 0^{j_{n-2}} 1 0^{j_{n-1}}
\]

where each \( j_i \geq 0 \). We introduce a rewording of \( \mathcal{D}_z \):

\[
\mathcal{D}_{j_0} \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_{n-2}} \mathcal{D}_{j_{n-1}} = \left( \mathcal{D}_1 \mathcal{D}_0^{j_0} \right) \left( \mathcal{D}_1 \mathcal{D}_0^{j_1} \right) \left( \mathcal{D}_1 \mathcal{D}_0^{j_2} \right) \cdots \left( \mathcal{D}_1 \mathcal{D}_0^{j_{n-2}} \right) \left( \mathcal{D}_1 \mathcal{D}_0^{j_{n-1}} \right)
\]
and note that the probability associated with $\tilde{D}_{j_i}$ is $1/2^{j_i+1}$. Clearly

$$\lambda = \lim_{n \to \infty} \frac{1}{2n} \ln \| \tilde{D}_{j_0} \tilde{D}_{j_1} \cdots \tilde{D}_{j_{n-2}} \tilde{D}_{j_{n-1}} \|$$

since the mean length of $\tilde{D}_{j_i}$ is $1 + \sum_{j \geq 0} j/2^{j+1}$. Also

$$\tilde{D}_0 = D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad \tilde{D}_1 = D_1 D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{D}_j = D_1 D_0^j = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

for all $j \geq 2$. Hence the rewording actually consists of only three matrices $\tilde{D}_0$, $\tilde{D}_1$, $\tilde{D}_2$ with probabilities $1/2$, $1/4$, $1/4$. Further, the initial row of each matrix is zero, thus we may consider only the lower-right $2 \times 2$ submatrix:

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \alpha_1 \beta_1^T,$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = M,$$

$$\begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 4 & 4 \end{pmatrix} = \alpha_2 \beta_2^T.$$

Let $p_1 = 1/2$, $p_2 = 1/4$ denote the weights corresponding to $\alpha_1 \beta_1^T$, $\alpha_2 \beta_2^T$ and $q = 1/4$ denote the weight corresponding to $M$. The case of two $2 \times 2$ matrices, one with rank 1 and the other with rank 2, was solved in [11, 12]. Our case involves three matrices, two with rank 1 and one with rank 2, as well as the scaling factor $1/2$ due to rewording. Generalizing, we obtain

$$\lambda = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=0}^{\infty} p_i p_j q^k \ln (\beta_j^T M^k \alpha_i).$$

Observe that

$$\beta_j^T M^k \alpha_i = \begin{cases} 2 & \text{if } i = 1, j = 1 \\ 2^2 & \text{if } i = 1, j = 2 \\ 2^{2k} & \text{if } i = 2, j = 1 \\ 2^{2(k+1)} & \text{if } i = 2, j = 2 \end{cases}$$
and therefore
\[
\frac{2\lambda}{\ln(2)} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{4^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{2}{4^k} + \frac{1}{8} \sum_{k=0}^{\infty} \frac{2k}{4^k} + \frac{1}{16} \sum_{k=0}^{\infty} \frac{2(k+1)}{4^k} = 1,
\]
as was to be shown. An independent proof of the conjecture was found by Thomas Doumenc.

Let \(\#(n)\) denote the number of 1s in the binary expansion of \(n\). We know that
\[
\#(n) = \ln(f(n))/\ln(2)
\]
and
\[
\sum_{k=0}^{n} \#(k) \sim \frac{1}{2 \ln(2)} n \ln(n)
\]
as \(n \to \infty\). A corollary of our proof is that
\[
\sum_{k=0}^{[n/3]} \#(3k) \sim \frac{1}{2 \ln(2)} \frac{n}{3} \ln(n)
\]
because \(\#(3n) = \ln(g_3(n))/\ln(2)\). The formulas
\[
\sum_{k=0}^{[n/3]} \#(3k+1) \sim \frac{1}{2 \ln(2)} \frac{n}{3} \ln(n) \sim \sum_{k=0}^{[n/3]} \#(3k+2)
\]
follow similarly, that is, counting binary 1s is (on average) independent of ternary residue. We wonder whether simpler proofs of this fact can be found.

References

[1] J. R. Trollope, An explicit expression for binary digital sums, *Math. Mag.* 41 (1968) 21–25; MR0233763 (38 #2084).

[2] H. Delange, Sur la fonction sommatoire de la fonction “somme des chiffres”, *Enseign. Math.* 21 (1975) 31–47; MR0379414 (52 #319).

[3] A. H. Stein, Exponential sums of sum-of-digit functions, *Illinois J. Math.* 30 (1986) 660–675; MR0857218 (89a:11014).

[4] G. Larcher, On the number of odd binomial coefficients, *Acta Math. Hungar.* 71 (1996) 183–203; MR1397551 (97e:11026).

[5] S. R. Finch, Stolarsky-Harborth constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 145–151; MR2003519 (2004i:00001).
Odd Entries in Pascal’s Trinomial Triangle

[6] Y. Moshe, The distribution of elements in automatic double sequences, *Discrete Math.* 297 (2005) 91–103; MR2159434 (2006b:11023).

[7] L. Euler, On the expansion of the power of any polynomial \(1+x+x^2+x^3+x^4+\cdots\), http://arxiv.org/abs/math.HO/0505425.

[8] R. C. Bollinger, Extended Pascal triangles, *Math. Mag.* 66 (1993) 87–94; MR1212524 (94a:11025).

[9] T. Sillke, Odd trinomials, http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/trinomials.

[10] S. Wolfram, Statistical mechanics of cellular automata, *Rev. Mod. Phys.* 55 (1983) 601–644; also Theory and Applications of Cellular Automata, World Scientific, 1986, pp. 7–50; MR0709077 (85d:68057).

[11] S. Pincus, Strong laws of large numbers for products of random matrices, *Trans. Amer. Math. Soc.* 287 (1985) 65–89; MR0766207 (86i:60087).

[12] R. Lima and M. Rahibe, Exact Lyapunov exponent for infinite products of random matrices, *J. Phys. A* 27 (1994) 3427–3437; MR1282183 (95d:82004).

[13] Y. Moshe, Random matrix products and applications to cellular automata, *J. d’Analyse Math.* 99 (2006) 267–294; MR2279553.

[14] R. Mainieri, Zeta function for the Lyapunov exponent of a product of random matrices, *Phys. Rev. Lett.* 68 (1992) 1965–1968; chao-dyn/9301001.

[15] R. Mainieri, Cycle expansion for the Lyapunov exponent of a product of random matrices, *Chaos* 2 (1992) 91–97; MR1158540 (93e:82029).

[16] J. L. Nielsen, Lyapunov exponent for products of random matrices (1997), available at http://chaosbook.org/extras/.

[17] Z.-Q. Bai, On the cycle expansion for the Lyapunov exponent of a product of random matrices, *J. Phys. A* 40 (2007) 8315–8328.

[18] J. Goldwasser, W. Klostermeyer, M. Mays and G. Trapp, The density of ones in Pascal’s rhombus, *Discrete Math.* 204 (1999) 231–236; MR1691871 (2000b:05008).

[19] I. Urbiha, Some properties of a function studied by de Rham, Carlitz and Dijkstra and its relation to the (Eisenstein-)Stern’s diatomic sequence, *Math. Commun.* 6 (2001) 181–198; MR1908338 (2003f:11018).
[20] N. J. Calkin and H. S. Wilf, Binary partitions of integers and Stern-Brocot-like
trees, unpublished manuscript (1998).

[21] B. Reznick, $\sum_{j < 2^k} v(j) = (3^k + 1)/2$ goes back to Stern (1858), private com-
munication.

[22] P. Chassaing, G. Letac and M. Mora, Brocot sequences and random walks in
SL(2, $\mathbb{R}$), Probability Measures on Groups. VII, Proc. 1983 Oberwolfach conf.,
ed. H. Heyer, Lect. Notes in Math. 1064, Springer-Verlag, 1984, pp. 36–48; MR0772400 (86g:60012).

[23] E. Makover and J. McGowan, The length of closed geodesics on random Riemann
surfaces, arXiv:math/0504175v1 [math.DG].

[24] B. Reznick, $\sum_{j < 2^k} v(j)^2 = 5 \sum_{j < 2^{k-1}} v(j)^2 - 2 \sum_{j < 2^{k-2}} v(j)^2 - 1$, private com-
munication.

[25] P. Kirshenhofer, On the variance of the sum of digits function, Number-
Theoretic Analysis: Vienna 1988-89, ed. E. Hlawka and R. F. Tichy, Lect. Notes in Math. 1452, Springer-Verlag, 1990, pp. 112–116; MR1084640 (92f:11103).

[26] R. Artuso, P. Cvitanović and B. G. Kenny, Phase transitions on strange irrational
sets, Phys. Rev. A 39 (1989) 268–281; MR0978321 (89k:28003).

[27] P. Cvitanović, Circle maps: irrationally winding, From Number Theory to
Physics, Proc. 1989 Les Houches conf., ed. M. Waldschmidt, P. Moussa, J.
M. Luck and C. Itzykson, Springer-Verlag, 1992, pp. 631–658; MR1221112
(94d:58097).

[28] P. Contucci and A. Knauf, The phase transition of the number-theoretical spin
chain, Forum Math. 9 (1997) 547–567; MR1457137 (98j:82011).

[29] P. Cvitanović, K. Hansen, J. Rolf and G. Vattay, Beyond the periodic orbit
theory, Nonlinearity 11 (1998) 1209–1232; MR1644377 (99g:58102).

[30] G. Alkauskas, The moments of Stern diatomic sequence, manuscript in prepa-
ration.

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